FRACTIONAL LAPLACIAN, HOMOGENEOUS SOBOLEV SPACES AND THEIR REALIZATIONS

ALESSANDRO MONGUZZI, MARCO M. PELOSO, MAURA SALVATORI

Abstract. We study the fractional Laplacian and the homogeneous Sobolev spaces on $\mathbb{R}^d$, by considering two definitions that are both considered classical. We compare these different definitions, and show how they are related by providing an explicit correspondence between these two spaces, and show that they admit the same representation. Along the way we also prove some properties of the fractional Laplacian.

1. Introduction and statement of the main results

The goal of this paper is to clarify a point that in our opinion has been overlooked in the literature. Classically, the homogeneous Sobolev spaces and the fractional Laplacian are defined in two different ways. In one case, we consider the Laplacian $\Delta$ as densely defined, self-adjoint and positive on $L^2(\mathbb{R}^d)$, and following Komatsu [Kom66] it is possible to define the fractional powers $\Delta^{s/2}$ by means of the spectral theorem, $s > 0$. Denoting by $S$ the space of Schwartz functions, one observes that for $p \in (1, \infty)$, $\|\Delta^{s/2}\varphi\|_{L^p(\mathbb{R}^n)}$ is a norm on $S$. The homogeneous Sobolev space is the space $W^{s,p}$ defined as the closure of $S$ in such a norm. The second definition is modelled on the classical Littlewood–Paley decomposition of function spaces (see details below) and gives rise to an operator, that we will denote by $\tilde{\Delta}^{s/2}$, acting on spaces of tempered distributions modulo polynomials.

Below, we describe the latter approach and show how these two definitions are related to each other, by showing that they admit the same, explicit realization. We mention that the analysis involved by the fractional Laplacian $\Delta^s$ has drawn great interest in the latest years, beginning with the groundbreaking papers [CS07, DNPV12], see also the recent papers [Bou11, Bou13, BS19, Kwa17]. We also mention that we were led to consider this problem while working on spaces of entire functions of exponential type whose fractional Laplacian is $L^p$ on the real line, [MPS19].

Let $S$ denote the space of Schwartz functions, $S'$ denote the space of tempered distributions. We denote by $\mathbb{N}_0$ the set of nonnegative integers and by $P_k$ be the set of all polynomials of $d$-real variables of degree $\leq k$, with $k \in \mathbb{N}_0$. We also set $P_\infty = \bigcup_{k=0}^{+\infty} P_k$, the collection of all polynomials. We also write $\overline{\mathbb{N}} = \mathbb{N}_0 \cup \{+\infty\}$.

For $k \in \mathbb{N}$, we consider the equivalence relations in $S'$

$$f \sim_k g \iff f - g \in P_k$$

and we denote by $S'/P_k$ the space of tempered distribution modulo polynomials in $P_k$, that is, the space of the equivalence classes with respect the equivalence relation $\sim_k$. If $u \in S'$ we denote by...
\[ \left[ u \right]_k \text{ its equivalent class in } S'/P_k. \text{ For } k \in \mathbb{N}_0, \text{ we denote by } S_k \text{ the space of all Schwartz functions } \varphi \text{ such that } \]
\[ \int x^\gamma \varphi(x) \, dx = 0 \]

for any multi-index \( \gamma \in \mathbb{N}^d \) with \( |\gamma| \leq k \) and set \( S_\infty = \bigcap_{k=0}^{+\infty} S_k. \) Each space \( S_k, k \in \mathbb{N} \) is a subspace of \( S \) that inherits the same topology of \( S \). Moreover, it is easy to see that \( (S_k)' = S'/P_k. \)

For \( p \in [1, \infty] \) we denote by \( L^p(\mathbb{R}^d) \) (or, simply by \( L^p \)) the standard Lebesgue space on \( \mathbb{R}^d \) and we write
\[ \|f\|_{L^p}^p = \int |f(x)|^p \, dx, \]

if \( p < \infty \), and \( \|f\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{R}^d} |f(x)|. \)

If \( f \in L^1(\mathbb{R}^d) \) we define the Fourier transform
\[ \hat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} \, dx. \]

The Fourier transform induces a surjective isomorphism \( \mathcal{F} : S \to S \) and then it extends to a unitary operator \( \mathcal{F} : L^2 \to L^2 \) and to a surjective isomorphism \( \mathcal{F} : S' \to S' \). For \( u \in S' \), we shall also write \( \hat{u} \) to denote \( \mathcal{F}^{-1}u. \)

It is important here to observe that if \( \varphi \in S_\infty \), then for all \( \tau \in \mathbb{R}, \mathcal{F}(| \cdot |^{\tau} \varphi) \in S_\infty \) as well, see e.g. [Tri10, Gra14b].

1.1. The fractional Laplacian \( \mathcal{A}^{s/2} \) and the homogeneous Sobolev spaces \( \dot{L}^p_s \). We now recall the perhaps most standard definition of the fractional powers of the Laplacian and of the homogeneous Sobolev spaces. This material is classical and well known, see e.g. [Tri10, Gra14b].

We now have the following definition.

**Definition 1.1.** Let \( s > 0. \) For \( \varphi \in S_\infty, \) we define the fractional Laplacian of \( \varphi \) as
\[ \mathcal{A}^{s/2} \varphi = \mathcal{F}^{-1}(| \cdot |^{s} \hat{\varphi}). \]

The next lemma follows from the arguments in [Gra14b Chapter 1], or [Tri10 5.1.2].

**Lemma 1.2.** The operator
\[ \mathcal{A}^{s/2} : S_\infty \to S_\infty \]
is a surjective isomorphism with inverse
\[ \mathcal{I}_s \varphi = \mathcal{F}^{-1}(| \xi |^{-s} \hat{\varphi}). \]

Next, given \( [u] \in S'/P_\infty \) we define another distribution in \( S'/P_\infty \) by setting for all \( \varphi \in S_\infty \)
\[ \langle \mathcal{F}^{-1}(| \cdot |^{s} \hat{u}), \varphi \rangle := \langle u, \mathcal{F}(| \cdot |^{s} \varphi) \rangle, \]

where, with an abuse of notation we denote by \( \langle \cdot, \cdot \rangle \) also the pairing of duality between \( S_\infty \) and \( S'/P_\infty \). We remark that, clearly, the terms of both sides of (1) are independent of the choice of the representative in \( [u] \). The fact that (1) defines a well-defined distribution follows at once from Lemma 1.2.

\[ ^1 \text{We warn the reader that, we shall denote with the same symbol } \langle \cdot, \cdot \rangle \text{ different pairings of duality, such as } S \text{ and } S', S_\infty \text{ and } S'/P_\infty, L^p \text{ and } L^{p'}, \text{ etc. The actual pairing of duality should be clear from the context and there should not be any confusion.} \]
Then, we extend the definition of $\hat{\Delta}^{s/2}$ to $(S_\infty)' = S'/\mathcal{P}_\infty$.

**Definition 1.3.** We define the operator

$$\hat{\Delta}^{s/2} : S'/\mathcal{P}_\infty \to S'/\mathcal{P}_\infty$$

by setting, for any $[u] \in S'/\mathcal{P}_\infty$,

$$\hat{\Delta}^{s/2}[u] = F^{-1}(\cdot \cdot^s \hat{u}),$$

and we call it the fractional Laplacian (of order $s$) of $[u]$.

Since the term on the right hand side is independent of the choice of the representative in $[u]$, we may simply write $[\hat{\Delta}^{s/2}u]$ in place of $[\hat{\Delta}^{s/2}[u]]$.

**Remark 1.4.** We stress the fact that the terminology “homogeneous Laplacian” is non-standard, and that such operator is typically simply called fractional Laplacian. However, one of the main goals of the present paper is to consider the two standard different definitions of the fractional Laplacian and of the corresponding homogeneous Sobolev spaces and to study how they are related to each other. Thus, we need to distinguish them, and we do so by introducing such terminology.

For $1 < p < \infty$, we define $\hat{L}^p$ as the space of all elements in $S'/\mathcal{P}_\infty$ such that in every equivalence class there is a representative that belongs to $L^p$, that is,

$$\hat{L}^p = \{[u] \in S'/\mathcal{P}_\infty : \|u\|_{L^p} = \inf_{P \in \mathcal{P}_\infty} \|u + P\|_{L^p} < +\infty\}. \quad (3)$$

Clearly, if $[u] \in \hat{L}^p$, the representative of $[u]$ in $L^p$ is unique.

We observe that, since $S_\infty$ is dense in $L^p$ for all $p \in [1, +\infty)$, see Lemma 1.6 below, we have that $\hat{\Delta}^{s/2}u \in \hat{L}^p$, $p \in (1, \infty)$, if (and only if) the mapping

$$S_\infty \ni \psi \mapsto \langle u, \hat{\Delta}^{s/2}\psi \rangle$$

is bounded in the $L^{p'}$-norm. In fact, in this case, there exists a unique $g \in L^p$ such that $\langle u, \hat{\Delta}^{s/2}\psi \rangle = \langle g, \psi \rangle$ for all $\psi \in S_\infty$ and we set

$$\hat{\Delta}^{s/2}u = [g].$$

Next, notice that by definition $\hat{\Delta}^{s/2}P = 0$ for any polynomial $P$. The homogeneous Sobolev spaces $\hat{L}_s^p$ are defined as follows.

**Definition 1.5.** Let $s > 0$ and let $1 < p < \infty$. Then, the homogeneous Sobolev space $\hat{L}_s^p$ is defined as

$$\hat{L}_s^p = \{[f] \in S'/\mathcal{P}_\infty : \hat{\Delta}^{s/2}f \in \hat{L}^p\},$$

and we set

$$\|f\|_{\hat{L}_s^p}^p = \|\hat{\Delta}^{s/2}f\|_{\hat{L}^p}.$$

Notice that, because of (3), equation (4) defines a norm on $\hat{L}_s^p$. Thus, the homogeneous Sobolev space $\hat{L}_s^p$ is a space of equivalent classes in $S'/\mathcal{P}_\infty$. 
1.2. Fractional powers of $\Delta$ and the homogeneous Sobolev spaces $\dot{W}^{s,p}$. We now describe more in details the other definition of the homogeneous Sobolev spaces. As we mentioned, the operator $\Delta$ is densely defined, self-adjoint and positive on $L^2(\mathbb{R}^d)$, and following Komatsu [Kom66], it is possible to define the fractional powers $\Delta^{\alpha/2}$ for all $\alpha \in \mathbb{C}$, with $\text{Re} \alpha > 0$. Here we restrict our attention to the case $\alpha = s > 0$. For $\varphi \in \mathcal{S}$, by the spectral theorem we have that

$$\Delta^{s/2} \varphi = \mathcal{F}^{-1}(|\cdot|^s \varphi).$$

We now have a simple lemma to describe the basic properties of the action of $\Delta^{s/2}$ on $\mathcal{S}$.

Lemma 1.6. Let $s > 0$ and $p \in (1, \infty)$. The following properties hold:

(i) $\mathcal{S}_n$ is dense in $L^p$;
(ii) for every $\varphi \in \mathcal{S}$, $\Delta^{\alpha/2} \varphi \in L^p$;
(iii) $\|\Delta^{\alpha/2} \varphi\|_{L^p}$ is a norm on $\mathcal{S}$.

Proof. (i) is well-known and also easy to prove directly. For (ii), for $\varphi \in \mathcal{S}$, write $\Delta^{s/2} \varphi = \Delta^{s/2}(I + \Delta)^{-M}(I + \delta)^M \varphi$. If $M > s$ it is easy to check that $\Delta^{s/2}(I + \Delta)^{-M}$ is a Fourier multiplier satisfying Mihlin–Hörmander condition (see [Gra14a, Theorem 5.2.7]), hence bounded on $L^p, p \in (1, \infty)$. Since $(I + \delta)^M \varphi \in L^p$ for all $p$'s, (ii) follows. In order to prove (iii), we only need to observe that if $\varphi \in \mathcal{S}$ and $(I + \Delta)^M \varphi = 0$, then $\widehat{\varphi}$ is a distribution supported at the origin, hence $\varphi = 0$. \qed

Then, we have the following definition.

Definition 1.7. For $1 < p < \infty$ and $s > 0$ we define $\dot{W}^{s,p}$, also called the homogeneous Sobolev space as the closure of $\mathcal{S}$ with respect to the norm $\|\Delta^{s/2} \varphi\|_{L^p}$. More precisely, given the equivalent relation on the space of $L^p$-Cauchy sequences of Schwartz functions, $\{\varphi_n\} \sim \{\psi_n\}$ if $\Delta^{s/2}(\varphi_n - \psi_n) \to 0$ as $n \to +\infty$, and $\{\{\varphi_n\}\}$ denote the equivalent classes, then

$$\dot{W}^{s,p} = \left\{\{\varphi_n\} : \{\varphi_n\} \subseteq \mathcal{S}, \{\Delta^{s/2} \varphi_n\} \text{ is a Cauchy sequence in } L^p, \right. \left. \right\}$$

with $\left\|\{\varphi_n\}\right\|_{\dot{W}^{s,p}} = \lim_{n \to +\infty} \|\Delta^{s/2} \varphi_n\|_{L^p}. \quad (6)$

If $\{\Delta^{s/2} \varphi_n\}$ is a Cauchy sequences in $L^p$, and we denote by $f$ its limit, we set

$$\|f\|_{\dot{W}^{s,p}} = \lim_{n \to +\infty} \|\Delta^{s/2} \varphi_n\|_{L^p}.$$

We remark that it is equivalent to define $\dot{W}^{s,p}$ as the closure of the $C_c^\infty$ functions with compact support, space that we denote by $C_c^\infty$, with respect to the same norm $\|\Delta^{s/2} \varphi\|_{L^p}$.

Thus, we have described two different notions of the fractional Laplacian, $\dot{\Delta}^{s/2}$ and $\Delta^{s/2}$, that in particular are defined and taking values on completely different spaces. Both these notions can be considered as “classical”. Moreover, we have described two different scales $\dot{L}^{p}_s$ and $\dot{W}^{s,p}$ of homogeneous Sobolev spaces, for $s > 0$ and $p \in (1, \infty)$.

The main goals of this note are two. The first one is to describe the spaces $\dot{W}^{s,p}$ explicitly and obtain a realization of them as space of functions (see below). The second one is to show how the homogeneous Sobolev spaces $\dot{W}^{s,p}$ and $\dot{L}^{p}_s$ are related to each other, providing in fact an explicit one-to-one and onto correspondence between them.

Finally, observe that, if $\varphi \in \mathcal{S}$, then $\dot{\Delta}^{s/2} \varphi = [\Delta^{s/2} \varphi]_\infty$. 

1.3. The realization spaces $E^{s,p}$. Following G. Bourdaud [Bou88, Bou11, Bou13], if $k \in \mathbb{N}$ and $\hat{X}$ is a given subspace of $S'/\mathcal{P}_k$ which is a Banach space, such that the natural inclusion of $\hat{X}$ into $S'/\mathcal{P}_k$ is continuous, we call realization of $\hat{X}$ a subspace $E$ of $S'$ such that there exists a bijective linear map

$$R: \hat{X} \to E$$

such that $[R[u]] = [u]$ for every $[u] \in \hat{X}$. We endow $E$ of the norm given by $\|R[u]\|_E = \|[u]\|_{\hat{X}}$. Obviously, $E$ becomes a Banach space with such norm.

Thus, a realization of $L^p_k$ is a space $E$ of tempered distributions such that for each $f \in E$, the equivalent class modulo polynomials $[f]$ of $f$ is in $L^p_k$ and conversely, for each $[f] \in L^p_k$ there exists a unique $\hat{f} \in [f]$ such that $\hat{f} \in E$, and such correspondence is an isometry. Our first main result is that the spaces $E^{s,p}$ defined below are a realization of $\tilde{W}^{s,p}$, for $p \in (1, +\infty)$, $s > 0$. The spaces are also a realization of the spaces $\tilde{L}^p_k$, as was indicated already in [Bou88, Bou11], and here we give a proof of this latter fact. As a consequence, we obtain a precise correspondence between the spaces $\tilde{W}^{s,p}$ and $\tilde{L}^p_k$.

In order to describe such realization spaces $E^{s,p}$ we need to recall the definition of BMO and of the homogeneous Lipschitz spaces $\Lambda^s$, for $s > 0$. We begin with the latter one. For $s > 0$ we write $[s]$ to denote its integer part.

**Definition 1.8.** The difference operators $D^k_h$ of increment $h \in \mathbb{R}^d \setminus \{0\}$ and of order $k \in \mathbb{N}$ are defined as follows. If $k = 1$, we write $D_h$ in place of $D^1_h$, is $D_h f(x) = f(x + h) - f(x)$, and then inductively, for $k = 2, 3, \ldots$

$$D^k_h f(x) = D_h [D^{k-1}_h f](x).$$

For a non-negative integer $m$, we denote by $\mathcal{C}^m$ the space of continuously differentiable functions of order $m$.

We recall the following well-known facts, see [Gra14b, 6.3.1].

**Proposition 1.9.** (i) For $k = 2, 3, \ldots$ we have the explicit expression

$$D^k_h f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh).$$

(ii) If $f \in \mathcal{C}(\mathbb{R}^d)$ is such that $D^k_h f(x) = 0$ for all $x, h \in \mathbb{R}^d$, $h \neq 0$, then $f$ is a polynomial of degree at most $k - 1$, and conversely, $D^k_h f(x) = 0$ for all polynomials of degree $\leq k - 1$.

**Definition 1.10.** Let $\gamma > 0$. We define the homogeneous Lipschitz space $\Lambda^\gamma$ as

$$\Lambda^\gamma = \{ [f] \in \mathcal{C}/\mathcal{P}_\gamma : \|f\|_{\Lambda^\gamma} := \sup_{x,h \in \mathbb{R}, h \neq 0} \frac{|D^1_h f(x)|}{|h|^{\gamma}} < +\infty \}. $$

We remark that, for all $\gamma > 0$, if $[f]_{\gamma} \in \Lambda^\gamma$, then $f$ is a function of moderate growth, so that $\Lambda^\gamma \subset S'/\mathcal{P}_{\gamma}$. Moreover, $f$ is in $\mathcal{C}^{\lfloor \gamma \rfloor}$. For these and other properties of Lipschitz spaces, see e.g. [Kra83, Tri10, Gra14b].

Next, we introduce the Sobolev-BMO spaces.
Definition 1.11. The space BMO is the space of locally integrable functions, modulo constants such that

\[ \|f\|_{\text{BMO}} := \sup_{x \in \mathbb{R}^d} \sup_{r > 0} \frac{1}{|B_r|} \int_{B_r(x)} |f(y) - f_{B_r}| \, dy < +\infty. \]

where \( B_r(x) = B_r \) denotes the ball centered at \( x \) of radius \( r > 0 \), \(|B_r|\) its measure, and \( f_{B_r} \) the average of \( f \) over \( B_r(x) \).

For \( k = 1, 2, \ldots \), we define the Sobolev-BMO space as

\[ S_k(\text{BMO}) = \{ f \in \mathcal{S}/\mathcal{P}_k : \partial^\alpha f \in \text{BMO} \text{ for } |\alpha| = k \}, \]

and set

\[ \|f\|_{S_k(\text{BMO})} = \sum_{|\alpha| = k} \|\partial^\alpha f\|_{\text{BMO}}. \]

Definition 1.12. Given \( f \in \mathcal{S}' \), if there exists a sequence \( \{\varphi_j\} \subseteq \mathcal{S} \) such that

\begin{itemize}
  \item \( \varphi_j \to f \) in \( \mathcal{S}' \),
  \item \( \{\Delta^{s/2} \varphi_j\} \) is a Cauchy sequence in \( L^p(\mathbb{R}^d) \),
\end{itemize}

then we say that \( \Delta^{s/2} f \) is defined as the \( L^p \)-limit of \( \{\Delta^{s/2} \varphi_j\} \) and we set \( \|\Delta^{s/2} f\|_{L^p} = \lim_{n \to +\infty} \|\Delta^{s/2} \varphi_n\|_{L^p} \).

Remark 1.13. We observe that the definition is well-given since if \( \{\varphi_j\}, \{\psi_j\} \subseteq \mathcal{S} \), are two sequences such that \( \varphi_j, \psi_j \to f \) in \( \mathcal{S}' \) and \( \{\Delta^{s/2} \varphi_j\}, \{\Delta^{s/2} \psi_j\} \) are both Cauchy in \( L^p \), then for every \( \eta \in \mathcal{S}_\infty \),

\[ \langle \Delta^{s/2} \varphi_n - \Delta^{s/2} \psi_n, \eta \rangle = \langle \varphi_n - \psi_n, \Delta^{s/2} \eta \rangle \to 0 \]

as \( n \to \infty \). Since \( \mathcal{S}_\infty \) is dense in \( L^p \), \( \{\Delta^{s/2} \varphi_n\}, \{\Delta^{s/2} \psi_n\} \) have the same \( L^p \)-limit.

We are now ready to define the spaces \( E^{s,p} \) that we will show to be the realization spaces for \( \dot{W}^{s,p} \). For a sufficiently smooth function \( f \), we denote by \( P_{f;m;x_0} \) the Taylor polynomial of \( f \) of degree \( m \in \mathbb{N}_0 \) at \( x_0 \in \mathbb{R}^d \).

Definition 1.14. For \( s > 0 \) and \( p \in (1, +\infty) \), we define the spaces \( E^{s,p} \) as follows.

(i) Let \( 0 < s < \frac{d}{p} \), and let \( p^* \in (1, \infty) \) given by \( \frac{1}{p^*} = \frac{1}{p} - \frac{s}{d} \). Then, we define

\[ E^{s,p} = \{ f \in L^{p^*} : \|f\|_{E^{s,p}} := \|\Delta^{s/2} f\|_{L^p} < +\infty \}. \]

(ii) Let \( s > d/p, s - d/p \notin \mathbb{N} \) and let \( m = [s - d/p] \). Then, we define

\[ E^{s,p} = \{ f \in \dot{\Lambda}^{s-\frac{d}{p}} : P_{f;m;0} = 0, \|f\|_{E^{s,p}} := \|\Delta^{s/2} f\|_{L^p} < +\infty \}. \]

(iii) Let \( s - d/p \in \mathbb{N} \) and set \( m = [s - d/p] \). Let \( B \) be a fixed ball in \( \mathbb{R}^d \). Then, if \( m = 0 \),

\[ E^{s,p} = \{ f \in \text{BMO} : f_B = 0, \|f\|_{E^{s,p}} := \|\Delta^{s/2} f\|_{L^p} < +\infty \}, \]

while, if \( m \geq 1 \),

\[ E^{s,p} = \{ f \in S_m(\text{BMO}) \cap C^{m-1} : (i) P_{f;m-1;0} = 0, \ (ii) (\partial^\alpha f)_B = 0 \text{ for } |\alpha| = m, \]

\[ (iii) \|f\|_{E^{s,p}} := \|\Delta^{s/2} f\|_{L^p} < +\infty \}. \]
1.4. **Littlewood–Paley decomposition.** In order to state our second main result we need the Littlewood–Paley decomposition of $\hat{L}_s^p$, for whose details see e.g. to [Gra14b, 6.2.2]. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\text{supp} \eta \subseteq \{ \xi : 1/2 \leq |\xi| \leq 2 \}$, identically 1 on the annulus $\{ \xi : 1 \leq |\xi| \leq 3/2 \}$ and such that

$$\sum_{j \in \mathbb{Z}} \eta(2^{-j} \xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}. $$

For $j \in \mathbb{Z}$ we set $\eta_j = \eta(2^{-j} \cdot)$ and for $f \in \mathcal{S}'$ we define\footnote{Classically, these operators are denoted by $\Delta_j$. Given the recurrent appearance of $\Delta$ in this paper, we decided to use the notation $M_j$ instead.}

$$M_j f = \mathcal{F}^{-1}(\eta_j \hat{f}).$$

We observe that, due to the support conditions of the $\eta_j$'s, for all $j, k \in \mathbb{Z}$ we have that

- $M_j \lesssim (M_{j-1} + M_j + M_{j+1})M_j \lesssim 3M_j$;
- $M_k M_j = 0$ if $|k - j| > 1$.

Then, $M_j f = 0$ for any $f \in \mathcal{P}_\infty$, therefore the operators $M_j$ are well-defined on $\mathcal{S}'/\mathcal{P}_\infty$ and taking values in $\mathcal{S}'$, as it is immediate to check. Thus, for simplicity of notation, we write $M_j u$ instead of $M_j[u]_\infty$, and observe that

$$[u]_\infty = \sum_{j \in \mathbb{Z}} [M_j u]_\infty, \quad (7)$$

where the convergence is in $\mathcal{S}'/\mathcal{P}_\infty$. Then, we have the following characterization of the spaces $\hat{L}_s^p$, $\text{BMO} := \text{BMO} / \mathcal{P}_\infty$, and $\hat{\Lambda}^\gamma$, respectively.

For $1 < p < \infty$ and $s > 0$, $[f]_\infty \in \hat{L}_s^p$, if and only if $\left( \sum_{j \in \mathbb{Z}} (2^{js} |M_j f|)^2 \right)^{1/2} \in L^p$ and

$$\left\| \left( \sum_{j \in \mathbb{Z}} (2^{js} |M_j f|)^2 \right)^{1/2} \right\|_{L^p} \approx \| f \|_{L_s^p}. \quad (8)$$

It holds that $[f]_\infty \in \text{BMO}$ if and only if $\left( \sum_{j \in \mathbb{Z}} |M_j f|^2 \right)^{1/2} \in L^\infty$ and in this case

$$\sup_{x \in \mathbb{R}^d} \left( \sum_{j \in \mathbb{Z}} |M_j f|^2 \right)^{1/2} \|_{L^p} \approx \| f \|_{\text{BMO}}. \quad (9)$$

For $\gamma > 0$, we have that $[f]_{\gamma} \in \mathcal{S}'/\mathcal{P}_{\gamma}$, we have $[f]_{\gamma} \in \hat{\Lambda}^\gamma$ if and only if $\sup_{j \in \mathbb{Z}} 2^{j\gamma} \| M_j f \|_\infty < + \infty$ and we have

$$\sup_{j \in \mathbb{Z}} 2^{j\gamma} \| M_j f \|_\infty \approx \| f \|_{\hat{\Lambda}^\gamma}, \quad (10)$$

see e.g. [Gra14b, Theorem 6.3.6.]

1.5. **Statement of the main results.** Our first result describes explicitly the elements of $\hat{W}^{s,p}$, $s > 0$, $p \in (1, \infty)$.

**Theorem 1.** Let $s > 0$, $1 < p < \infty$, and let $\{ \Delta^{s/2} \varphi_n \}$ be a Cauchy sequence in $L^p$, with $\varphi_n \in \mathcal{S}$. Then, the following properties hold.

(i) If $0 < s < d/p$, then there exists a unique $g \in L^p$ such that $\varphi_n \to g$ in $L^p$ as $n \to + \infty$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{s}{d}$. 

(ii) If $d/p < s < 1$, then there exists a unique $g \in L^p$ such that $\varphi_n \to g$ in $L^p$ as $n \to + \infty$. 

(iii) If $s \geq 1$, then there exists a unique $g \in L^p$ such that $\varphi_n \to g$ in $L^p$ as $n \to + \infty$. 

(iv) If $s = 0$, then there exists a unique $g \in L^p$ such that $\varphi_n \to g$ in $L^p$ as $n \to + \infty$.
(ii) If $s - d/p = m \in \mathbb{N}_0$, then there exists a unique $g \in S_m(\text{BMO})$ such that $[\varphi_n]_m \rightarrow [g]_m$ in $S_m(\text{BMO})$ as $n \rightarrow +\infty$.

(iii) If $s > d/p$ and $s - d/p \notin \mathbb{N}$, then there exists a unique $g \in \mathcal{L}^{s-d/p}$ such that $[\varphi_n]_m \rightarrow [g]_m$ in $\mathcal{L}^{s-d/p}$ as $n \rightarrow +\infty$, where $m = |s - d/p|$.

In particular, $\dot{W}^{s,p}$ is a space of functions if $s < d/p$, while $\dot{W}^{s,p} \subseteq S'/P_m$, where $m = |s - d/p|$, if $s \geq d/p$.

**Theorem 2.** For $s > 0$ and $p \in (1, \infty)$, the spaces $E^{s,p}$ are realization spaces of $\dot{W}^{s,p}$. More precisely, the space $\dot{W}^{s,p}$ can be identified with a subspace of $S'/P_m$, where $m = |s - d/p|$, and for each $[u]_m \in \dot{W}^{s,p}$ there exists a unique $f \in [u]_m \cap E^{s,p}$ and such that

$$\|\Delta^{s/2} f\|_{L^p} = \|[u]_m\|_{\dot{W}^{s,p}}.$$ 

Recall that, given $B = B(x_0, r)$ a ball in $\mathbb{R}^d$, and an locally integrable function $f$, we denote by $f_B$ the average of $f$ over $B$.

**Theorem 3.** Let $s > 0$, $1 < p < \infty$. Then the following properties hold.

(i) If $0 < s < d/p$, given $[u]_\infty \in \hat{L}^p_s$ there exists a unique $f \in [u]_\infty$ and such that $f \in E^{s,p}$, and it is given by

$$f = \sum_{j \in \mathbb{Z}} M_j u,$$

where the convergence is in $L^p_s$.

(ii) If $s - d/p = m \in \mathbb{N}_0$, let $B = B(x_0, r)$ be any fixed ball in $\mathbb{R}^d$. Then, if $m = 0$, given $[u]_\infty \in \hat{L}^p_s$ consider $f \in [u]_\infty$ given by

$$f = f_0 + f_1 - (f_0 + f_1)_B := \sum_{j \leq 0} (M_j u - M_j u(0)) + \sum_{j \geq 1} M_j u - (f_0 + f_1)_B,$$  \hspace{1cm} (11)

where the first series converges uniformly on compact subsets of $\mathbb{R}^d$, while the second one in the BMO-norm. Then, $f \in \text{BMO}$ with $f_B = 0$, hence $f \in E^{s,p}$.

If $s - d/p = m \in \mathbb{N}$, $m \geq 1$, given $[u]_\infty \in \hat{L}^p_s$, let

$$f = f_0 + f_1 - (f_0 + f_1)_B := \sum_{j \leq 0} (M_j u - P_{M_j u, m, 0}) + \sum_{j \geq 1} M_j u - (f_0 + f_1)_B,$$  \hspace{1cm} (12)

where the both series converge uniformly on compact subsets of $\mathbb{R}^d$, together with all derivatives up to order $m - 1$. Then, $f \in E^{s,p}$, in particular, $f \in S_m(\text{BMO})$, $P_{f, m - 1; 0} = 0$, and $(\partial^\alpha f)_B = 0$ for $|\alpha| = m$.

(iii) If $s > d/p$ and $s - d/p \notin \mathbb{N}$, set $m = |s - d/p|$. Then given $[u]_\infty \in \hat{L}^p_s$ there exists a unique $f \in [u]_\infty$ such that $f \in E^{s,p}$, and it is given by

$$f = \sum_{j \in \mathbb{Z}} (M_j u - P_{M_j u, m, 0}),$$  \hspace{1cm} (13)

where the convergence is uniformly on compact sets in $\mathbb{R}^d$, together with all the derivatives up to order $m$, and in the $\mathcal{L}^{s-d/p}$-norm.

Finally, we provide the explicit isometric correspondence between the two homogeneous Sobolev spaces $\dot{W}^{s,p}$ and $\hat{L}^p_s$. 
Theorem 4. Let \( s > 0, 1 < p < \infty \) and let \( m = [s - d/p] \). Then, given any \([u]_\infty \in \tilde{L}^p_s\) there exists a unique \( f \in [u]_\infty \) such that \( f \in E^{s,p} \), so that \( f \in \dot{W}^{s,p} \), or \([f]_m \in \dot{W}^{s,p} \), resp., if \( 0 < s < d/p \) or \( s \geq d/p \), resp., with equality of norms.

Conversely, given \( f \in \dot{W}^{s,p} \) if \( 0 < s < d/p \), or \([f]_m \in \dot{W}^{s,p} \), if \( s \geq d/p \), resp., let \( f \) be its image in the realization space \( E^{s,p} \), then \([f]_\infty \in \tilde{L}^p_s \), with equality of norms.

2. Preliminary facts

2.1. The fractional integral operator \( I_s \). For \( 0 < s < d \), we consider the Riesz potential \( I_s \) defined for \( \varphi \in S \) as

\[
I_s \varphi = \mathcal{F}^{-1}(|\xi|^{-s} \hat{\varphi}) = \omega_{s,d}(|\cdot|^{-d-s} \ast \varphi),
\]

where \( \omega_{s,d} = \frac{\Gamma((d-s)/2)}{2^s \pi^{s/2} \Gamma(s/2)} \). Observe that for \( s \in (0, d) \), \(|\xi|^{-s} \) is locally integrable, so that if \( \varphi \in S \), \(|\xi|^{-s} \hat{\varphi} \in L^1 \) and \( I_s \) is well-defined.

For \( p \in (0, \infty) \) we denote by \( H^p(\mathbb{R}^d) \), or simply by \( H^p \), the classical Hardy space on \( \mathbb{R}^d \). Having fixed \( \Phi \in S \) with \( \int \Phi = 1 \), then

\[
H^p(\mathbb{R}^d) = \{f \in S' : f^\#(x) := \sup_{t>0} |f * \Phi_t(x)| \in L^p(\mathbb{R}^d)\},
\]

where

\[
\|f\|_{H^p} = \|f^\#\|_{L^p}.
\]

We recall that the definition of \( H^p \) is independent of the choice of \( \Phi \) and that, when \( p \in (1, \infty) \), \( H^p \) coincides with \( L^p \), with equivalence of norms. We also recall that \( S_\infty \) is dense in \( H^p \) for all \( p \in (0, \infty) \), see [Ste93, Ch.II,5.2]. For these and other properties of the Hardy spaces see e.g. [Ste93] or [Gra14b].

For the operator \( I_s \) the following regularity result holds, see [Ste93], [Gra14b], [Kra82] and in particular [Ada75] for (ii).

Proposition 2.1. Let \( s > 0 \). The operator \( I_s \), initially defined for \( \varphi \in S \), extends to a bounded operator

(i) if \( 0 < p < d/s \), then \( I_s : H^p \rightarrow H^p^s \), where \( \frac{1}{p^*} = \frac{1}{p} - \frac{s}{d} \);

(ii) if \( p = d/s \), then \( I_s : H^p \rightarrow \text{BMO} \);

(iii) if \( p > d/s \), then \( I_s : H^p \rightarrow \dot{A}^{s-d/p} \).

We also recall that the Riesz transforms are the operators \( R_j, j = 1, \ldots, d \), initially defined on Schwartz functions

\[
R_j f(x) = \mathcal{F}^{-1}(-i \frac{\xi_j}{|\xi|} \hat{f}),
\]

and that they satisfy the relation \( \sum_{j=1}^d R_j^2 = I \). Moreover, the \( R_j \) are bounded

\[
R_j : H^p \rightarrow H^p
\]

for all \( p \in (0, \infty) \), and \( j = 1, \ldots, d \).
3. PROOFS OF THE MAIN RESULTS

Our first task is to describe explicitly the elements of $\tilde{W}^{s,p}$, that is, the limits of $L^p$-Cauchy sequences of the form $\{\Delta^{s/2} \varphi_n\}$, with $\varphi_n \in \mathcal{S}$. We are going to use this simple lemma, of which we include the proof for sake of completeness.

**Lemma 3.1.** Let $1 < p < \infty$, $d/p < \nu < 1 + d/p$. Given $a \in \mathbb{R}^d \setminus \{0\}$, let and let $g_a(x) = |a - x|^{\nu-d} - |x|^{\nu-d}$. Then, $g_a \in L^{p'}$ and

$$\|g_a\|_{L^{p'}} \lesssim |a|^{\nu-d/p}.$$  

**Proof.** We first observe that $g_a \in L^{p'}_{\text{loc}}$ if and only if $\nu > d/p$, and then that $\|g_a\|_{L^{p'}} = C|a|^{\nu-d/p}$, where

$$Cp' = \int_0^\infty \int_S \frac{1}{\sigma^{d-\nu} r^{d-1}} \left| \frac{1}{r^{d-\nu}} - \frac{1}{r^{d-\nu}} \right|^{p'} d\sigma(x') r^{d-1} dr$$

$$= \int_0^\infty \frac{r^{d-1}}{r^{(d-\nu)p'}} \int_S \left| \frac{1}{\sigma^{d-\nu} r^{d-1}} - 1 \right|^{p'} d\sigma(x') dr$$

$$= \int_0^\infty \frac{r^{d-1}}{r^{(d-\nu)p'}} F(r) dr,$$

where $S$ denotes the unit sphere in $\mathbb{R}^d$, $\sigma$ the induced surface measure, and $a' = a/|a|$. Thus, since by rotation invariance, $C$ is independent of $a'$, we only need to show that $C$ is finite if and only if $0 < \nu < 1 + d/p$. Notice that $F(r)$ is finite when $\nu > d/p$, so we only need to check the integrability of $\frac{r^{d-1}}{r^{(d-\nu)p'}} F(r)$ as $r \to 0^+, \infty$. As $r \to 0^+$, $F(r) = \mathcal{O}(1)$ and $\int_0^\infty \frac{r^{d-1}}{r^{(d-\nu)p'}} < \infty$ if and only if $d/p < \nu$, which is the case. As $r \to \infty$, $F(r) \approx 1/s', so that

$$\int_2^\infty \frac{r^{d-1}}{r^{(d-\nu)p'}} F(r) dr \approx \int_0^\infty \frac{1}{r^{(d-\nu)p'-d+1+p'}} dr$$

which is finite if and only if $\nu - d/p < 1$. The conclusion now follows. \hfill $\Box$

**Proof of Thm. 1.** For $0 < s < d$, it is immediate to check that for $\varphi \in \mathcal{S}$ we have that $I_s \Delta^{s/2} \varphi = \varphi$. Therefore, by Prop. 2.4 (i) we have

$$\|\varphi\|_{L^{p^*}} = \|I_s \Delta^{s/2} \varphi\|_{L^{p^*}} \lesssim \|\Delta^{s/2} \varphi\|_{L^p}.$$  

Then, let $\{\varphi_n\} \subseteq \mathcal{S}$ be such that $\{\Delta^{s/2} \varphi_n\}$ is a Cauchy sequence in $L^p$. By the above estimate (15) it follows that

$$\|\varphi_m - \varphi_n\|_{L^{p^*}} \lesssim \|\Delta^{s/2} \varphi_m - \Delta^{s/2} \varphi_n\|_{L^p},$$

so that there exists $f \in L^{p^*}$ such that $\varphi_n \to f$ in $L^{p^*}$ as $n \to +\infty$. By Remark 13 this implies that $\Delta^{s/2} \varphi_n \to \Delta^{s/2} f$ in $L^p$. Moreover,

$$\|f\|_{L^{p^*}} = \lim_{n \to +\infty} \|\varphi_n\|_{L^{p^*}} \lesssim \lim_{n \to +\infty} \|\Delta^{s/2} \varphi_n\|_{L^p} = \|\Delta^{s/2} f\|_{L^p}. $$

This proves (i).

In order to prove case (ii) with $s = d/p$, let $\{\varphi_n\} \subseteq \mathcal{S}$ be such that $\{\Delta^{s/2} \varphi_n\}$ is a Cauchy sequence in $L^p$. Since $0 < s = d/p < d$, again we have that $I_s \Delta^{s/2} \varphi_n = \varphi_n$. By Prop. 2.4 (ii) it follows that

$$\|\varphi_m - \varphi_n\|_{\text{BMO}} \lesssim \|\Delta^{s/2} \varphi_m - \Delta^{s/2} \varphi_n\|_{L^p}.$$
Hence, there exists a unique \( g \in \text{BMO} \) such that \( \varphi_n \to g \) in \( \text{BMO} \), and it is such that \( \Delta^{s/2} g \in L^p \). For, given any \( \psi \in \mathcal{S}_\infty \) we have \( \Delta^{s/2} \psi \in \mathcal{S}_\infty \subseteq H^1 \) so that

\[
\langle g, \Delta^{s/2} \psi \rangle = \lim_{n \to +\infty} \langle \varphi_n, \Delta^{s/2} \psi \rangle = \lim_{n \to +\infty} \langle \Delta^{s/2} \varphi_n, \psi \rangle = \langle G, \psi \rangle, \tag{17}
\]

where \( G \) is the \( L^p \)-limit of \( \{ \Delta^{s/2} \varphi_n \} \). This shows that the mapping \( \psi \to \langle g, \Delta^{s/2} \psi \rangle \) is bounded in the \( L^{p'} \)-norm on a dense subset, and therefore \( \Delta^{s/2} g \) defines an element of \( L^p \), that is \( G \).

Let now \( s - d/p = m \in \mathbb{N} \) and let \( \{ \varphi_n \} \subseteq \mathcal{S} \) be such that \( \{ \Delta^{s/2} \varphi_n \} \) is a Cauchy sequence in \( L^p \). Using the identity

\[
|\Delta^{(s-1)/2} \nabla \varphi|^2 = \sum_{j=1}^{d} |R_j \Delta^{s/2} \varphi|^2, \tag{18}
\]

which is valid for all \( \varphi \in \mathcal{S} \), and the boundedness of the Riesz transforms, we see that \( \{ \Delta^{(s-1)/2} \partial_j \varphi_n \} \) is a Cauchy sequence in \( L^p \), for \( j = 1, \ldots, d \). If \( s - d/p = 1 \), we apply the previous argument to each of the sequences \( \{ \Delta^{(s-1)/2} \partial_j \varphi_n \} \), \( j = 1, \ldots, d \). If \( s - d/p = 2 \), the argument is analogous but simpler, given the identity \( \Delta^{s/2} = \Delta^{(s-2)/2} \Delta \), and in the general case we argue inductively.

We obtain that, for all multiindices \( \alpha \), with \( |\alpha| = m \), there exists \( g_\alpha \in \text{BMO} \) such that \( \partial_\alpha \varphi_n \to g_\alpha \) in \( \text{BMO} \) and \( \Delta^{(s-m)/2} g_\alpha \in L^p \). Using the fact that in \( \mathcal{S}' \)

\[
\partial_\alpha \varphi_n = \tilde{\partial}_\alpha \tilde{g}_\alpha \]

if \( \alpha + e_j = \beta + e_\ell \), where \( |\alpha| = |\beta| = e_j, e_\ell \) are the standard basis vectors, \( j, \ell \in \{ 1, \ldots, d \} \), and induction again, it is easy to see that there exists \( g \in S_m(\text{BMO}) \) such that \( \partial^n \varphi = g_\alpha \), for \( |\alpha| = m \). This implies that \( \varphi_n \to g \) in \( S_m(\text{BMO}) \), hence in particular in \( \mathcal{S}'/\mathcal{P}_m \). Let \( \psi \in \mathcal{S}_\infty \), so that we have \( \Delta^{s/2} \psi \in \mathcal{S}_\infty \subseteq H^1 \), so that

\[
\langle g, \Delta^{s/2} \partial^n \psi \rangle = \lim_{n \to +\infty} \langle \partial^n \varphi_n, \Delta^{s/2} \psi \rangle = \lim_{n \to +\infty} \langle \Delta^{s/2} \varphi_n, \partial^n \psi \rangle = \langle G, \partial^n \psi \rangle, \]

Hence, the mapping \( \eta \to \langle g, \Delta^{s/2} \eta \rangle \) is bounded in the \( L^{p'} \)-norm on \( \mathcal{S}_\infty \), that is, \( \Delta^{s/2} g \) defines an element of \( L^p \). This proves (ii).

Let now \( s > d/p \) and \( s - d/p \notin \mathbb{N} \). We assume first that \( d/p < s < 1 + d/p \). For \( \varphi \in \mathcal{S} \) and \( x \neq y \) we write

\[
\varphi(x) - \varphi(y) = \int \frac{e^{iy\xi} - e^{ix\xi}}{|\xi|^s} (\Delta^{s/2} \varphi)(\xi) \, d\xi = \int H(x, y, t) \Delta^{s/2} \varphi(t) \, dt,
\]

where

\[
H(x, y, t) = \mathcal{F}^{-1} \left( \frac{e^{iy\xi} - e^{ix\xi}}{|\xi|^s} \right)(t) = \gamma_{s,d}(|t - x|^{s-d} - |t - y|^{s-d}) \cdot \gamma_{s,d}(|t|^{s-d})
\]

By Lemma \( 3.1 \) it follows that \( H(x, y, \cdot) \in L^{p'} \) and that

\[
\|H(x, y, \cdot)\|_{L^{p'}} \lesssim |x - y|^{s-d/p}.
\]

Therefore,

\[
\|\varphi\|_{\dot{H}^{s-d/p}} \lesssim \|\Delta^{s/2} \varphi\|_{L^p}.
\]

Hence, given \( \{ \Delta^{s/2} \varphi_n \} \) is a Cauchy sequence in \( L^p \), there exists a unique \( g \in \dot{H}^{s-d/p} \) such that \( \varphi_n \to g \), as \( n \to +\infty \). We argue as before, using the fact that the dual space of \( H^q \) is \( \dot{H}^{d(1/q - 1)} \).
Corollary 3.3. Let \( q \) be given by \( 1/q = 1 - 1/p + s/d \), so that \( 0 < q < 1 \). Then \( \Lambda^{s-d/p} = (H^q)^* \), and given any \( \psi \in S' \), we have \( \Delta^{s/2}\psi \in S' \subseteq H^q \), so that
\[
\langle g, \Delta^{s/2}\psi \rangle = \lim_{n \to +\infty} \langle \varphi_n, \Delta^{s/2}\psi \rangle = \lim_{n \to +\infty} \langle \Delta^{s/2}\varphi_n, \psi \rangle = \langle G, \psi \rangle,
\]
where \( G \) is the \( L^p \)-limit of \( \{\Delta^{s/2}\varphi_n\} \). This shows that the mapping \( \psi \to \langle g, \Delta^{s/2}\psi \rangle \) is bounded in the \( L^p' \)-norm on a dense subset, and therefore \( \Delta^{s/2}g \) defines an element of \( L^p \), that is \( G \). We conclude that \( \Delta^{s/2}g \in L^p \) as well. Finally, if \( s - d/p \notin \mathbb{N} \) and \( s - d/p > 1 \), let \( m = [s - d/p] \), and \( |\alpha| = m \). Since, \( \|\Delta^{(s-m)/2}\partial^\alpha \varphi_n\|_{L^p} \leq \|\partial^\alpha \varphi_n\|_{L^p} \leq \|\Delta^{(s-m)/2}\partial^\alpha \varphi_n\|_{L^p} \), \( \{\Delta^{(s-m)/2}\partial^\alpha \varphi_n\} \) is a Cauchy sequence in \( L^p \). Using the first part we can find \( a_\alpha \in \Lambda^{s-m-d/p} \) such that \( \partial^\alpha \varphi_n \to a_\alpha \) in \( \Lambda^{s-m-d/p} \), for all \( \alpha \) with \( |\alpha| = m \). Hence, arguing as in (ii), there exists a unique \( g \in \Lambda^{s-d/p} \) such that \( \partial^\alpha g = a_\alpha \) for \( |\alpha| = m \). Hence, \( \varphi_n \to g \) in \( S'/P_m \), since \( \partial^\alpha \varphi_n \to \partial^\alpha g \) in \( \Lambda^{s-m-d/p} \) for \( |\alpha| = m \), \( \varphi_n \to g \) in \( \Lambda^{s-d/p} \). Finally, by \((19)\) it is now clear that \( \Delta^{s/2}g \in L^p \), and (iii) follows as well.

We state the characterization of \( \dot{W}^{s,p} \), that also provides the precise embedding result. It may be considered folklore by some, but to the best of our knowledge, there exists no explicit statement in the literature. From this, the proof of Thm. 2 is then obvious.

Corollary 3.2. For \( s > 0 \) and \( p \in (1, +\infty) \), the following properties hold.
(i) Let \( 0 < s < \frac{d}{p} \), and let \( p^* \in (1, +\infty) \) given by \( \frac{1}{p^*} = \frac{1}{p} - \frac{d}{s} \). Then,
\[
\dot{W}^{s,p} = \{f \in L^{p^*} : \|\Delta^{s/2}f\|_{L^p} < +\infty\},
\]
and
\[
\|f\|_{L^{p^*}} \leq \|f\|_{\dot{W}^{s,p}}.
\]
(ii) Let \( s - d/p \in \mathbb{N} \) and set \( m = [s - d/p] \). Then,
\[
\dot{W}^{s,p} = \{f \in S_m(\text{BMO}) : \|\Delta^{s/2}f\|_{L^p} < +\infty\},
\]
and
\[
\|f\|_{S_m(\text{BMO})} \leq \|f\|_{\dot{W}^{s,p}}.
\]
(iii) Let \( s > d/p \), \( s - d/p \notin \mathbb{N} \) and let \( m = [s - d/p] \). Then,
\[
\dot{W}^{s,p} = \{f \in \Lambda^{s-d/p} : \|\Delta^{s/2}f\|_{L^p} < +\infty\},
\]
and
\[
\|f\|_{\Lambda^{s-d/p}} \leq \|f\|_{\dot{W}^{s,p}}.
\]

In particular, \( \dot{W}^{s,p} \) is a space of functions if \( s < d/p \), while \( \dot{W}^{s,p} \subseteq S'/P_m \), where \( m = [s - d/p] \), if \( s \geq d/p \).

We also have the following characterization. We point out that if \( m \) is a negative integer, \( P_m = \emptyset \) and \( S'/P_m \) is \( S' \) itself, as well as \( S_m = S \).

Corollary 3.3. For \( s > 0 \) and \( p \in (1, +\infty) \), let \( m = [s - d/p] \). Then,
\[
\dot{W}^{s,p} = \{[f]_m \in S'/P_m : (i) \text{ there exists a sequence } \{\varphi_n\} \subseteq S \text{ such that } [\varphi_n]_m \to [f]_m \text{ in } S'/P_m; (ii) \text{ the sequence } \{\Delta^{s/2}\varphi_n\} \text{ is a Cauchy sequence in } L^p\}.
\]
Proof. We have shown that any element of \( \hat{W}^{s,p} \) is an element of \( S'/P_m \) satisfying the conditions (i) and (ii) above. Conversely, let \( \{f\}_m \in S'/P_m \) satisfy the conditions (i) and (ii). We need to show that for any \( f \in \{f\}_m \), \( \Delta^{s/2}f \) is a well-defined element of \( L^p \) — in fact the \( L^p \)-limit of the sequence \( \{\Delta^{s/2}\varphi_n\} \).

Let \( g = \lim_{n \to +\infty} \Delta^{s/2}\varphi_n \) and fix \( \psi \in S_\infty \). Then \( \Delta^{s/2}\psi \in S_\infty \subseteq S_m \) and we have
\[
\langle g, \psi \rangle = \lim_{n \to +\infty} \langle \Delta^{s/2}\varphi_n, \psi \rangle = \lim_{n \to +\infty} \langle \varphi_n, \Delta^{s/2}\psi \rangle = \lim_{n \to +\infty} \langle [\varphi_n], \Delta^{s/2}\psi \rangle = \langle [f], \Delta^{s/2}\psi \rangle .
\]
Therefore, we may set \( \Delta^{s/2}[f]_m = g \in L^p \). □

We also have the following, but significant observation.

**Corollary 3.4.** Let \( s > 0, m = [s - \frac{d}{2}] \). Then, for every \( P \in P_\infty \), \( \Delta^{s/2}P = 0 \).

**Proof.** We show that there exists sequences of Schwartz functions \( \{\varphi_n\} \) such that \( \{\Delta^{s/2}\varphi_n\} \) is a Cauchy sequence in \( L^2 \), while \( \varphi_n \to P \) in \( S' \). Suppose first \( s > \frac{d}{2} \) and \( s - \frac{d}{2} \notin \mathbb{N} \). Let \( \psi \in C^\infty_0 \) be nonnegative, identically 1 for \( |x| \leq 1 \) and with support in \( \{|x| \leq 2\} \). Define \( \psi (x) = \psi (\varepsilon x) \), and set \( \varphi_n = P \psi \frac{1}{n} \). Then, it is obvious that \( \varphi_n \to P \) in \( S' \), while if we assume momentarily that \( P = x^\alpha \), with \( |\alpha| = k \), and write
\[
\|\Delta^{s/2}\varphi_n\|_{L^p} = \int |\xi|^{2s} |\hat{P} \ast \hat{\psi}(1/n)\hat{\varphi}(\xi)|^2 \, d\xi = \int |\xi|^{2s} |\hat{\varphi}(\xi)|^2 \, d\xi = \frac{n^{s+2(d-k)}}{2} \int |x|^{2s} \hat{\psi}(x)^2 \, dx,
\]
which tends to 0 as \( n \to +\infty \) when \( s - \frac{d}{2} > k \), i.e. when \( k \leq [s - \frac{d}{2}] \).

Next, we assume that \( s = \frac{d}{2} \). Let \( \{\varphi_n\} \) be a sequence in \( S_\infty \) such that \( \Delta^{s/2}\varphi_n \to 0 \) in \( L^2 \). Then, since \( \varphi_n \in S_\infty \), \( \varphi_n = \mathcal{I}_{d/2} \Delta^{s/2}\varphi_n \), by Prop. 3.1 (ii), \( \varphi_n \to 0 \) in \( \text{BMO} \), hence to a constant. Finally, if \( s - \frac{d}{2} = m \in \mathbb{N} \), and we assume the conclusion valid for \( m - 1 \), we proceed inductively, using identity (18). If \( \Delta^{s/2}\varphi_n \to 0 \) in \( L^2 \), then \( \Delta^{s-1/2} \partial_j \varphi_n \to 0 \) in \( L^2 \) and therefore \( \partial_j \varphi_n \to 0 \) in \( P_{m-1} \). This proves the lemma.

We now turn to the proof of Thm. 3.

**Proof of Thm. 3.** (i) Let \( [u]_\infty \in \tilde{L}^s_p \) with \( s < d/p \), and set \( \varphi_n = \sum_{|j| \leq m} M_j u \). Then, \( \varphi_n \in S_\infty \) and using Cor. 3.2 (1) and the equality \( \hat{\Delta}^{s/2} \varphi = [\Delta^{s/2}\varphi]_\infty \) for functions \( \varphi \) in \( S \) we have that, for \( n < m \in \mathbb{N} \),
\[
\|\varphi_m - \varphi_n\|_{L^p} = \| \sum_{n < |j| \leq m} M_j u \|_{L^p} \lesssim \left\| \Delta^{s/2} \left( \sum_{n < |j| \leq m} M_j u \right) \right\|_{L^p} = \left\| \sum_{n < |j| \leq m} M_j u \right\|_{L^p} \approx \left\| \left( \sum_{k \in \mathbb{Z}} \left( 2^{ks} \left| M_k \sum_{m < |j| \leq m} M_j u \right| \right)^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{n-2 |k| \leq m+2} \left( 2^{ks} |M_k u| \right)^2 \right)^{1/2} \right\|_{L^p}.
\]

Since \( u \in \tilde{L}^s_p \), the equivalence of norms in (18) implies that \( \{\varphi_n\} \) is a Cauchy sequence in \( L^p \). Therefore, it follows that \( f = \sum_{j \in \mathbb{Z}} M_j u \), where the convergence is in \( L^p \). This proves (i)
We now prove (iii). Consider the series on the right hand side of (13). We claim that the series converges uniformly on compact sets in \( \mathbb{R}^d \), together with the series of partial derivatives up to order \( m := [s - d/p] \). Assuming the claim, and letting \( f \) denotes its sum, we clearly have that \( f \in C^m \) and \( P_{f;m,0} = 0 \). Moreover, letting \( \Phi_n = \sum_{|j| \leq n} (M_j u - P_{M_j u;m,0}) \), \( \varphi_n = \sum_{|j| \leq n} M_j u \), and using Cor. 3.2 we have:

\[
\left\| \Phi_n \right\|_{\dot{\Lambda}^{s-d/p}} = \left\| \varphi_n \right\|_{\dot{W}^{s,p}} \lesssim \left\| \varphi_n \right\|_{W^{s,p}} = \left\| \Delta^{s/2} \varphi_n \right\|_{L^p} \leq \left( \sum_{|k| \leq n+1} 2^{ks} |M_j u|^2 \right)^{1/2}_{L^p}.
\]

Hence, the series on the right hand side of (13) converges also in \( \dot{\Lambda}^{s-d/p} \) and uniquely determines \( f \). Then, the conclusion follows modulo the claim.

The following estimate holds true for \( f \in \dot{\Lambda}^\gamma \), \( \gamma \) not an integer, see [Kra83 Cor. 3.4],

\[
|f(x) - P_{f;m,y}(x)| \leq |x - y|^{\gamma-m} \left\| \nabla^m f \right\|_{\dot{\Lambda}^{\gamma-1}_1},
\]

where we set \( \nabla^m f = (\partial^m f)_{|\alpha| = m} \). Since \( s - m > d/p \), by Sobolev’s embedding theorem, Cor. 3.2 (iii), if \( |x| \leq r \) we have

\[
\left| \Phi_\ell(x) - \Phi_n(x) \right| \lesssim_r \left\| \nabla^m \Phi_\ell - \nabla^m \Phi_n \right\|_{\dot{\Lambda}^{s-m-d/p}} \lesssim \left\| \varphi_\ell - \varphi_n \right\|_{W^{s,p}} \\
\lesssim \left\| \frac{2^{ks} |M_k u|^2}{L^p} \right\|_{\dot{\Lambda}^{s-m-d/p}} \lesssim \left\| \varphi_\ell - \varphi_n \right\|_{W^{s,p}}.
\]

This shows that the series converges uniformly on compact sets. The same argument now easily applies to series of partial derivatives up to order \( m \), since

\[
|\partial^\alpha \varphi_\ell(x) - \partial^\alpha \varphi_n(x)| \lesssim_r \left\| \nabla^{m-|\alpha|} \partial^\alpha \varphi_\ell - \nabla^{m-|\alpha|} \partial^\alpha \varphi_n \right\|_{\dot{\Lambda}^{s-m-d/p}} \lesssim \left\| \varphi_\ell - \varphi_n \right\|_{W^{s,p}}.
\]

Therefore, \( \Phi_n \to f \), which is an element of \( [u]_{\infty} \), and this proves (iii).

In order to prove (ii), let \( s - d/p = m = 0 \), and consider the two series on the right hand side of (11), that is, \( \sum_{j \leq 0} (M_j u - M_j(0)) \), and \( \sum_{j \geq 1} M_j u \). We begin with the latter one. We observe that \( \sum_{j \geq 1} M_j u \) converges to an element of BMO := BMO / \( P_\infty \), say to \( f + P \), for some \( P \in \mathcal{P}_\infty \). However, its Fourier transform has support in \( \{ |\xi| \geq 1 \} \), hence it must be \( P = 0 \), so that \( \sum_{j \geq 1} M_j u = f_1 \) is a locally integrable function.

Next we show that the former series converge uniformly on compact subsets to a function \( f_0 \). For \( |x| \leq r \) we have

\[
|M_j u(x) - M_j u(0)| \lesssim_r \sup_{|x| \leq r} |M_j \nabla u(x)| \lesssim 2^{j(s-d/p)\|M_j u\|_{L^p}} \lesssim 2^{j\left( \int |M_k 2^{j-d/p} M_j u(x)|^2 \right)^{1/2} dx}^{1/p} \lesssim 2^{j\left( \int \sum_{|k-j| \geq 1} (2^k |M_k u(x)|)^2 \right)^{1/2} dx}^{1/p} \lesssim 2^{j \left\| [u]_\infty \right\|_{L^p}}.
\]
Since \( j \leq 0 \), the conclusion follows. Now set
\[
f = f_0 + f_1 - (f_0 + f_1)_B,
\]
and the case \( m = 0 \) is proved.

Let now \( s - d/p = m \geq 1 \) and consider the series on the right hand side of (12). For \( |x| \leq r \), arguing as before we have
\[
|M_j u(x) - P_{M_j u;0}(x)| \leq r \sup_{|x| \leq r} |M_j \nabla^{m+1} u(x)| \leq 2^{j(m+1+d/p)} \|M_j u\|_{L^p}.
\]
\[
\leq 2^j \left( \int \left( \sum_{k \in \mathbb{Z}} |M_k 2^{jd/p} M_j u(x)|^2 \right)^{p/2} dx \right)^{1/p}
\]
\[
\leq 2^{j(m+1)} \left( \int \left( \sum_{|k-j| \leq 1} (2^k |M_k u(x)|)^2 \right)^{p/2} dx \right)^{1/p}
\]
\[
\leq 2^{j(m+1)} \|u\|_{L^p}.
\]
Thus, the series converges uniformly on compact subsets. Clearly, the same argument applies to all derivatives \( \partial_x^\alpha \) with \( |\alpha| \leq m \). Then, the function \( f_0 \in C^m \). On the other hand, the series \( \sum_{j \geq 1} M_j u \) is such that, for \( |\alpha| \leq m - 1 \), recalling that \( j \geq 1 \),
\[
|\partial_x^\alpha M_j u(x)| \leq 2^{j(|\alpha|+d/p)} \|M_j u\|_{L^p} \leq \left( \int \left( \sum_{|j-k| \leq 1} |M_k (2^j |\partial_x^\alpha u|)^2 \right)^{p/2} dx \right)^{1/p}
\]
\[
\leq \left( \int \left( \sum_{|j-k| \leq 1} |M_k 2^{jd/p} M_j u|^2 \right)^{p/2} dx \right)^{1/p}
\]
\[
\leq 2^{-j} \left( \int \left( \sum_{k \geq 0} (2^{ks} |M_k u|)^2 \right)^{p/2} dx \right)^{1/p}.
\]
Thus, also the series \( \sum_{j \geq 1} M_j u \) converges uniformly on compact subsets, together with its partial derivatives \( \partial_x^\alpha \), with \( |\alpha| \leq m - 1 \). Finally, if \( |\alpha| = m \) we have
\[
\partial_x^\alpha (f_0 + f_1) = \sum_{j \leq 0} M_j (\partial_x^\alpha u) - M_j (\partial_x^\alpha u)(0) + \sum_{j \geq 1} M_j (\partial_x^\alpha u),
\]
and we can repeat the argument of the case \( m = 0 \).

Finally, we set \( f_0 + f_1 = f_2 \) and define
\[
f = f_2 - P_{f_2;m-1,0}(x) - \sum_{|\alpha| = m} (\partial_x^\alpha f_2)_B.
\]
It is now easy to see that \( f \in E^{s,p} \).

The proof of Theorem 4 is now obvious, and we are done.

4. Final remarks, and comparison with the work of Bourdaud

In [Bou88] Bourdaud proved a version of Thm. 3 and our work is inspired by his. The descriptions of the realization spaces are similar, but not identical. In particular we explicitly refer to the BMO space. Moreover, Bourdaud is mainly focused on the techniques of the Besov spaces, and the proof in the case of Sobolev spaces are different, and typically more involved. However, we point out that Bourdaud proved that the spaces \( E^{s,p} \) with \( s - d/p \notin \mathbb{N}_0 \) are the unique realization
of $\dot{L}_p^s$ that are dilation invariant, while, if $s - d/p \in \mathbb{N}_0$ there does not exist any dilation invariant realization of $\dot{L}_p^s$.

We mention that we have restrict ourselves to the case $p \in (1, \infty)$. The case of the Hardy spaces, that is, $p \in (0, 1]$ is also of great interest and we believe it is worth further investigation.

Finally, it would also be very interesting to study analogous properties for the homogeneous versions of Sobolev and Besov spaces in the sub-Riemannian setting, see the recent papers [PV18, BPTV19, BPV19a, BPV19b].

**Acknowledgements.** We warmly thank G. Bourdaud for several useful comments and remarks that improved the presentation of this paper.

**References**

[Ada75] David R. Adams, *A note on Riesz potentials*, Duke Math. J. **42** (1975), no. 4, 765–778. MR 458158

[Bou88] G. Bourdaud, *Réalisations des espaces de Besov homogènes*, Ark. Mat. **26** (1988), no. 1, 41–54. MR 948279

[Bou11] Gérard Bourdaud, *Realizations of homogeneous Sobolev spaces*, Complex Var. Elliptic Equ. **56** (2011), no. 10-11, 857–874. MR 2838225

[Bou13] ———, *Realizations of homogeneous Besov and Lizorkin-Triebel spaces*, Math. Nachr. **286** (2013), no. 5-6, 476–491. MR 3048126

[BPTV19] Tommaso Bruno, Marco M. Peloso, Anita Tabacco, and Maria Vallarino, *Sobolev spaces on Lie groups: embedding theorems and algebra properties*, J. Funct. Anal. **276** (2019), no. 10, 3014–3050. MR 3944287

[BPV19a] Tommaso Bruno, Marco M. Peloso, and Maria Vallarino, *Besov and Triebel–Lizorkin spaces on Lie groups*, ArXiv e-prints (2019).

[BPV19b] ———, *Potential spaces on Lie groups*, ArXiv e-prints (2019).

[BS19] Lorenzo Brasco and Ariel Salort, *A note on homogeneous Sobolev spaces of fractional order*, Ann. Mat. Pura Appl. (4) **198** (2019), no. 4, 1295–1330. MR 3987216

[CS07] Luis Caffarelli and Luis Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245–1260. MR 2354493

[DNPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573. MR 2944369

[GCRdF85] Jos´e Garcia-Cuerva and Jos´e L. Rubio de Francia , *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985, Notas de Matemática [Mathematical Notes], 104. MR 807149

[Gra14a] L. Grafakos, *Classical Fourier analysis*, third ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2014. MR 3243734

[Gra14b] ———, *Modern Fourier analysis*, third ed., Graduate Texts in Mathematics, vol. 250, Springer, New York, 2014. MR 3243741

[Kom66] H. Komatsu, *Fractional powers of operators*, Pacific J. Math. **19** (1966), 285–346. MR 0201985

[Kra82] Steven G. Krantz, *Fractional integration on Hardy spaces*, Studia Math. **73** (1982), no. 2, 87–94. MR 667967

[Kra83] ———, *Lipschitz spaces, smoothness of functions, and approximation theory*, Exposition. Math. **1** (1983), no. 3, 193–260. MR 782608

[Kwa17] Mateusz Kwaśnicki, *Ten equivalent definitions of the fractional Laplace operator*, Fract. Calc. Appl. Anal. **20** (2017), no. 1, 7–51. MR 3613319

[MPS19] A. Monguzzi, M. M. Peloso, and M. Salvatori, *Fractional Paley–Wiener and Bernstein spaces*, ArXiv e-prints (2019).

[PV18] Marco M. Peloso and Maria Vallarino, *Sobolev algebras on nonunimodular Lie groups*, Calc. Var. Partial Differential Equations **57** (2018), no. 6, Art. 150, 34. MR 3858833

[Ste93] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR 1232192

[Tri10] Hans Triebel, *Theory of function spaces*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 2010, Reprint of 1983 edition [MR0730762]. MR 3024598
Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano–Bicocca, Via R. Cozzi 55, 20126 Milano, Italy
E-mail address: alessandro.monguzzi@unimib.it

Dipartimento di Matematica, Università degli Studi di Milano, Via C. Saldini 50, 20133 Milano, Italy
E-mail address: marco.peloso@unimi.it
E-mail address: maura.salvatori@unimi.it