GABOR FRAME BOUND OPTIMIZATIONS

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Abstract. We study sharp frame bounds of Gabor systems over rectangular lattices for different windows. In some cases we obtain optimality results for the square lattice, while in other cases the lattices optimizing the frame bounds and the condition number are different. Also, in some cases optimal lattices do not exist at all and a degenerated system is optimal.

1. Introduction

We aim to find extremal lattices for the spectral bounds of Gabor systems with specific windows. The quantities which we aim to optimize are the lower and upper frame bound, as well as their ratio, which is the condition number of the associated frame operator. We study the cases provided in [35], where Janssen computed sharp spectral bounds for Gabor frames over rectangular lattices of the form $a\mathbb{Z} \times b\mathbb{Z}$, and for several different window functions and $(ab)^{-1} \in \mathbb{N}$ and we will build upon his work. Our findings are for rectangular lattices of integer density and for different windows. The main results can be summed up as follows:

- **Hyperbolic secant**: the square lattice optimizes the lower and upper frame bound simultaneously and, hence, also the condition number.
- **Cut-off exponentials**: we find that a lattice optimizing the frame bounds does not exist for cut-off exponentials supported on $[0,1/b]$. For support $[0,2/b]$, optimal lattices may or may not exist, depending on the lattice density and which quantity we seek to optimize. If they exist, then they depend on the density and on the decay parameter of the exponential.
- **One-sided exponential**: the optimizing lattices for the condition number and the frame bounds are different from each other and depend on the (over)sampling rate.
- **Two-sided exponentials**: each quantity has a unique optimizer. They depend on the (over)sampling rate and they all differ from each other.

Until now, Gaussians were the only family of window functions for which optimality results have been known [4], [11], [15]. We see that many of the situations are quite different from what we know about the Gaussian case. Therefore, our results emphasize that in general we may not expect that an optimal lattice exists at all (for the frame bounds or the condition number) and that a lattice optimizing one quantity does not necessarily optimize the others.

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2020 Mathematics Subject Classification. primary: 42C15; secondary: 26A06, 33B10.

Key words and phrases. exponential functions, frame bounds, Gabor frame, hyperbolic functions, lattice.

Markus Faulhuber was supported by the Austrian Science Fund (FWF) project TAI6. Irina Shafkulovska was supported by the Austrian Science Fund (FWF) project P33217.
Indeed, assume that the lower frame bound $A$ and the upper frame bound $B$ each have a unique critical point, so there exists a unique optimizing lattice. The critical points of the condition number $B/A$ need to fulfill

$$\left(\frac{B}{A}\right)' = 0 \iff \frac{A'}{A} = \frac{B'}{B}.$$  

If the critical points of $A$ and $B$ do not coincide, then no critical point of $B/A$ can coincide with either of the other two. As can be seen from the example of the two-sided exponential, this is not necessarily a pathology of the window not being in the modulation space $M^1(\mathbb{R})$, also known as Feichtinger’s algebra $S_0(\mathbb{R})$. However, the situation for the hyperbolic secant in the rectangular case is comparable to the Gaussian case (see also Appendix B).

This work is structured as follows. We present our results in Section 2. In Section 3 we settle the notation and in Section 4 we recall how to compute exact Gabor frame bounds for integer (over)sampling. As we will need a lot of identities and properties of hyperbolic functions, we go over these in Section 5 for the reader’s convenience. Also, we explain briefly what we call interval estimates, as these will be needed throughout this work. The proofs of our results then follow in Section 6 for the hyperbolic secant, Section 7 for the cut-off exponentials, Section 8 for the one-sided exponential and, finally, in Section 9 we present the proofs for the two-sided exponential.

2. The results

We will now present an overview of the results and also provide some background information. The complete proofs are given in the later sections of this work.

2.1. Hyperbolic secant. The frame property of the hyperbolic secant and lattices of the form $a\mathbb{Z} \times b\mathbb{Z}$ has been studied by Janssen and Strohmer [38]. By relating its Zak transform to the one of the Gaussian (see Appendix B), they were able to show that

$$\mathcal{G}(\text{sech}, a\mathbb{Z} \times b\mathbb{Z}) \text{ is a frame } \iff (ab)^{-1} > 1.$$  

As the (properly scaled) hyperbolic secant is invariant under the Fourier transform, it follows by the general theory of symplectic matrices and metaplectic operators [20, 23, 21] that the two the Gabor systems

$$\mathcal{G}(\text{sech}(\pi t), a\mathbb{Z} \times b\mathbb{Z}) \quad \text{and} \quad \mathcal{G}(\text{sech}(\pi t), b\mathbb{Z} \times a\mathbb{Z})$$

are unitarily equivalent. In particular, they have the same frame bounds and the frame operator is invariant under rotations of the lattice by integer multiples of 90 degrees. Therefore, for any fixed density the square lattice has to be a local extremum for both frame bounds. Theorem 2.1 states that the square lattice is the global extremum for integer oversampling.

**Theorem 2.1.** Let $g(t) = \left(\frac{\pi}{2}\right)^{1/2} \text{sech}(\pi t)$ be the normalized hyperbolic secant and consider the rectangular lattice $\Lambda_{a,b} = a\mathbb{Z} \times b\mathbb{Z}$ of density $n$, i.e. $(ab)^{-1} = n$, with $2 \leq n \in \mathbb{N}$. By $A(a,b)$ and $B(a,b)$ we denote the sharp lower and upper frame bound of the Gabor system $\mathcal{G}(g, \Lambda_{a,b})$, respectively. Then, we have that

$$A \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right) \geq A(a,b) \quad \text{and} \quad B \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right) \leq B(a,b),$$

with equality if and only if $a = b = \frac{1}{\sqrt{n}}$, i.e., if the lattice is the square lattice of density $n$. 

We remark that in the case of critical sampling, i.e., $(ab)^{-1} = 1$, we obtain
\[ B(1, 1) \leq B(a, b) < \infty, \]
with equality if and only if $a = b = 1$. This can be deduced from results on Gaussian Gabor systems. Since we have a finite upper frame bound, the lower frame bound must vanish identically in this case, as imposed by the Balian-Low theorem.

2.2. Cut-off exponentials. The frame set of this class of functions is not known and so we focus only on the cases from [35]. In these situations we also know that we have a frame. We consider the following families of functions
\[ g_m(t) = C_{b, \gamma} e^{-\gamma t} \chi_{[0, m/b]}(t), \quad m \in \{1, 2\}, \gamma \geq 0, \quad C_{b, \gamma} \text{ chosen such that } \|g_m\|_2 = 1. \]
The support $[0, m/b]$ is chosen to be compatible with computations on the adjoint lattice.

2.2.1. The case $m = 1$. We treat $\gamma = 0$ first, which results in a study of the box function.

**Proposition 2.2.** The Gabor system $G(\sqrt{b} \chi_{[0, 1/b]} a\mathbb{Z} \times b\mathbb{Z})$ with $(ab)^{-1} = n \in \mathbb{N}$ is a tight frame with bounds $A = B = n$.

This case may not be particularly interesting as we, more or less, only take $n$ copies of translates of the (dilated) Fourier basis. However, our computations cover this case which is why we mention it here. A general treatise of the frame set of the indicator function has been carried out by Dai and Sun [8] (see [37] for first results by Janssen).

This particular family of windows is actually adapted to the lattice and we may perform an analysis depending on the decay parameter $\gamma > 0$.

**Proposition 2.3.** Let $A(\gamma)$ and $B(\gamma)$ be the frame bounds of the Gabor system $G(g_1, a\mathbb{Z} \times b\mathbb{Z})$ with $(ab)^{-1} = n \in \mathbb{N}$. Then, for any fixed lattice (hence also fixed $n$), the behavior of the frame bounds with respect to the decay parameter $\gamma > 0$ is

\[ A(\gamma) < n, A'(\gamma) < 0 \quad \text{and} \quad B(\gamma) > n, B'(\gamma) > 0. \]

The last two results combined show that the exponential decay inside the box $[0, 1/b]$ actually yields worse frame bounds and condition numbers than only using the box function. Furthermore, the frequency localization is (up to a constant) not better than $1/\omega$ (see (7.1)).

**Proposition 2.4.** Let $A(a)$ and $B(a)$ be the frame bounds of the Gabor system $G(g_1, a\mathbb{Z} \times b\mathbb{Z})$ with $(ab)^{-1} = n \in \mathbb{N}$. Then, for fixed $n$ and fixed decay parameter $\gamma > 0$, the behavior of the frame bounds with respect to the lattice parameter $a$ is

\[ A(a) < n, A'(a) < 0 \quad \text{and} \quad B(a) > n, B'(a) > 0. \]

The proposition implies in particular that there is no optimal (rectangular) lattice in this case. The optimal system is degenerated ($a \to 0, b \to \infty$) and the window tends to a Dirac delta. We conclude with a property of the frame condition number for these systems.

**Proposition 2.5.** The frame condition number of the Gabor system $G(g_1, a\mathbb{Z} \times b\mathbb{Z})$ with $(ab)^{-1} = n \in \mathbb{N}$ is independent of $n$ and always given by

\[ \frac{B}{A} = e^{2a \gamma}. \]
2.2.2. The case \( m = 2 \). We treat the case \( \gamma = 0 \), i.e., the box function, separately again.

**Proposition 2.6.** The Gabor system \( \mathcal{G}(\sqrt{b/2} \chi_{[0,2/[b]} , a\mathbb{Z} \times b\mathbb{Z}) \) with \((ab)^{-1} = n \in \mathbb{N}\) is never a frame. The frame bounds are, independently of \( n \), \( A = 0 \) and \( B = 2 \).

So, the behavior of the box function is again not particularly interesting in this case (and covered by [8] anyways), but our computations provide this result as well. Let us turn to the more interesting case \( \gamma > 0 \) now. We start with results for fixed lattices and optimization with respect to the decay parameter.

**Proposition 2.7.** Let \( A(\gamma) \) and \( B(\gamma) \) be the frame bounds of the Gabor system \( \mathcal{G}(g_2, a\mathbb{Z} \times b\mathbb{Z}) \) with \((ab)^{-1} = n \in \mathbb{N}\). Then, for any fixed lattice (hence also fixed \( n \)), the behavior of the frame bounds with respect to the decay parameter \( \gamma > 0 \) is as follows:

(i) The lower frame bound \( A(\gamma) \) has a unique maximum which depends on \( n \) and \( a = 1/(bn) \). The unique maximum is attained for the unique positive solution of the equation

\[
-1 + \frac{1}{\gamma a} - \coth(\gamma a) = n \coth\left(\frac{\sqrt{\gamma}a}{2}\right) \text{sech}(\sqrt{\gamma}na).
\]

(ii) For \( n \in \{1, 2\} \), we have \( B(\gamma) > 2 \) and \( B'(\gamma) > 0 \).

(iii) For \( 3 \leq n \in \mathbb{N} \), we have that \( B \) has a local maximum and a local minimum, which may be global (depending on \( n \) and \( a \)). Denoting the point of the local maximum and minimum by \( \gamma^* \) and \( \gamma_* \), respectively, we have

\[
\gamma^* < \frac{\text{arccosh}\left(\frac{1+\sqrt{5}}{2}\right)}{\gamma n} < \gamma_*.
\]

If \( B(\gamma_*) < 2 \), then the minimum is global. Moreover, we have \( B'(\gamma) > 0 \) for all \( \gamma > \gamma_* \).

From the computations in the proofs we will see that the parameters \( \gamma \) and \( a \) are exchangeable.

**Proposition 2.8.** Let \( A(a) \) and \( B(a) \) be the frame bounds of the Gabor system \( \mathcal{G}(g_2, a\mathbb{Z} \times b\mathbb{Z}) \) with \((ab)^{-1} = n \in \mathbb{N}\). Then, for any fixed \( n \) and fixed decay parameter \( \gamma \), the behavior of the frame bounds with respect to the lattice parameter \( a > 0 \) is as follows:

(i) The lower frame bound \( A(a) \) has a unique maximum which depends on \( n \) and \( \gamma \). The unique maximum is attained for the unique positive solution of the equation

\[
-1 + \frac{1}{\gamma a} - \coth(\gamma a) = n \coth\left(\frac{\sqrt{\gamma}a}{2}\right) \text{sech}(\gamma na).
\]

(ii) For \( n \in \{1, 2\} \), we have \( B(a) > 2 \) and \( B'(a) > 0 \).

(iii) For \( 3 \leq n \in \mathbb{N} \), we have that \( B \) has a local maximum and a local minimum, which may be global (depending on \( n \) and \( \gamma \)). Denoting the point of the local maximum and minimum by \( a^* \) and \( a_* \), respectively, we have

\[
a^* < \frac{\text{arccosh}\left(\frac{1+\sqrt{5}}{2}\right)}{\gamma n} < a_*. \]

If \( B(a_*) < 2 \), then the minimum is global. Moreover, we have \( B'(a) > 0 \) for all \( a > a_* \).

We close again with a property of the frame condition number.
Proposition 2.9. The frame condition number of the Gabor system $G(m_1, a\mathbb{Z} \times b\mathbb{Z})$ with $(ab)^{-1} = n \in \mathbb{N}$ is dependent of $n$ and given by

$$B/A = e^{2\gamma \gamma} \coth \left( \frac{\gamma n a}{2} \right)^2.$$ 

The condition number is minimal if and only if

$$\gamma a = \frac{n}{n + \sqrt{n^2 + 4}} - \log(2).$$

We see that optimality for the decay parameter $\gamma$ depends on the lattice parameter $a$, and vice versa, as well as on the (over)sampling rate $n$.

2.3. One-sided exponential. The one-sided exponential function is given by

$$g(t) = \sqrt{2\gamma} e^{-\gamma t} \chi_{R_+}(t), \quad \gamma > 0.$$ 

We note that it is known that the Gabor system $G(g, a\mathbb{Z} \times b\mathbb{Z})$ is a frame whenever $(ab)^{-1} \geq 1$ [35, Sec. 4] (see also [27]). Note that $g$ is not continuous (so $g \notin M^1(\mathbb{R})$), so having a frame at critical sampling rate is possible. Also, the one-sided exponential plays a central role when it comes to finding examples, other than the Gaussian, of zero-free ambiguity functions [25].

Proposition 2.10. Consider the Gabor system $G(g, a\mathbb{Z} \times b\mathbb{Z})$ with $(ab)^{-1} = n \in \mathbb{N}$. For fixed $n$, the frame condition number is minimal only for the lattice $(a_0/n)\mathbb{Z} \times (1/a_0)\mathbb{Z}$ with

$$a_0 = \frac{\text{arcsinh}(n)}{\gamma}.$$ 

The lower frame bound has a unique maximizing lattice $(a_+/n)\mathbb{Z} \times (1/a_+)\mathbb{Z}$ with

$$a_+ < \frac{\text{arcsinh}(n)}{\gamma}.$$ 

The upper frame bound has a unique minimizing lattice $(a^-/n)\mathbb{Z} \times (1/a^-)\mathbb{Z}$ with

$$a^- > \frac{\text{arcsinh}(n)}{\gamma}.$$ 

2.4. Two-sided exponential. Lastly, consider the two-sided exponential, which is given by

$$g(t) = \sqrt{2\gamma} e^{-\gamma |t|}, \quad \gamma > 0.$$ 

The Gabor system $G(g, a\mathbb{Z} \times b\mathbb{Z})$ is a frame whenever $(ab)^{-1} > 1$ [36, Sec. 5]. Note that the case of critical sampling is excluded this time. This is a consequence of the Balian-Low theorem as $g \notin M^1(\mathbb{R})$. We also refer to [37] where windows with certain convexity properties on $\mathbb{R}_+$ have been studied, and the two-sided exponential falls into this class.

Proposition 2.11. Consider the Gabor system $G(g, a\mathbb{Z} \times b\mathbb{Z})$ with $(ab)^{-1} = n \in \mathbb{N}$. For fixed $n$, the frame condition number is minimal only for the lattice $(a_0/n)\mathbb{Z} \times (1/a_0)\mathbb{Z}$ where $a_0$ is contained in some open interval depending on $n$:

$$a_0 \in \frac{1}{\gamma} I_n \subset \frac{1}{\gamma} \left( \frac{\eta_2}{2}, 2\eta_n \right), \quad \eta_n = 2 \text{arccosh}(n).$$ 

The lower frame bound has a unique maximum, depending on the oversampling rate $2 \leq n \in \mathbb{N}$ ($A = 0$ for $n = 1$). The maximizing lattice $(a_+/n)\mathbb{Z} \times (1/a_+)\mathbb{Z}$ satisfies $a_+ < 1/\gamma I_n$.

The upper frame bound has a unique minimum, depending on the oversampling rate $n \in \mathbb{N}$. The minimizing lattice $(a^-/n)\mathbb{Z} \times (1/a^-)\mathbb{Z}$ satisfies $a^- < 1/\gamma I_n$.

Analytically we show that $a^- < a_0 < a_+$, and for $n \geq 4$ we have $a^- < 2 \text{arccosh}(n)/\gamma < a_+$. 


3. Notation and preliminaries

We briefly settle the setting. Our notation is more or less the same as in the textbook of Gröchenig [23] and the reader familiar with Gabor systems and frames may simply skip this section. For two functions $f, g \in L^2(\mathbb{R})$, the inner product and induced norm are given by

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t) \overline{g(t)} \, dt \quad \text{and} \quad \|f\|_2^2 = \langle f, f \rangle.$$  

Here $\overline{g}$ is the complex conjugation of $g$. The Fourier transform of a (suitable) function $f$ is

$$\mathcal{F} f(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} \, dt.$$  

The operator $\mathcal{F}$ extends, of course, to a unitary operator on $L^2(\mathbb{R})$. After having settled the notation, we come to defining the central objects of this work. We consider Gabor systems for the Hilbert space $L^2(\mathbb{R})$ over rectangular lattices:

$$\mathcal{G}(g, \Lambda_{a,b}) = \{ \pi(\lambda) g \mid \lambda \in \Lambda_{a,b} \}, \quad \text{with} \quad \Lambda_{a,b} = a \mathbb{Z} \times b \mathbb{Z}.$$  

Here $\pi(\lambda)$ is a unitary operator on $L^2(\mathbb{R})$, usually called a time-frequency shift by $\lambda \in \mathbb{R}^2$;

$$\pi(\lambda)g(t) = M_\omega T_x g(t) = g(t - x) e^{2\pi i \omega t}, \quad \lambda = (x, \omega).$$  

The operators $T_x$ and $M_\omega$ are the familiar translation and modulation operator, respectively. They only commute up to a phase factor, which will be of importance in the next section:

$$M_\omega T_x = e^{2\pi i \omega x} T_x M_\omega. \quad (3.1)$$  

A Gabor system is a frame if and only if the frame inequality is fulfilled

$$A \|f\|_{L^2}^2 \leq \sum_{\lambda \in \Lambda_{a,b}} |\langle f, \pi(\lambda) g \rangle|^2 \leq B \|f\|_{L^2}^2, \quad \forall f \in L^2(\mathbb{R}), \quad (3.2)$$  

for some positive constants $0 < A \leq B < \infty$, which we call frame bounds. Note that the middle expression is derived by sampling the short-time Fourier transform $V_g f$ on the lattice:

$$V_g f(x, \omega) = \langle f, \pi(\lambda) g \rangle = \int_{\mathbb{R}} f(t) \overline{g(t - x)} e^{-2\pi i \omega t} \, dt.$$  

Being a frame usually requires some redundancy of the system. The number $(ab)^{-1}$ yields the density of the lattice, which is the average number of lattice points per unit area. In the context of Gabor systems, we also speak of the (over)sampling rate. Comparably to the Nyquist rate for band-limited functions, we need $(ab)^{-1} \geq 1$. The best achievable constants in (3.2) are the spectral bounds of the associated Gabor frame operator which is given by

$$S_{g, \Lambda_{a,b}} f = \sum_{\lambda \in \Lambda_{a,b}} \langle f, \pi(\lambda) g \rangle \pi(\lambda) g.$$  

In the sequel, we will suppress the index and simply write $S$ for the frame operator. In case of a Gabor frame, any element in our Hilbert space has a stable expansion of the form

$$f = \sum_{\lambda \in \Lambda_{a,b}} c_\lambda \pi(\lambda) g,$$

with $(c_\lambda) \in \ell^2(\Lambda_{a,b})$ (see also [32], [41]).
We recall the notion of the frame set of a window function $g$ [24] (see also [12]) for rectangular lattices. The rectangular (or reduced) frame set of $g \in L^2(\mathbb{R})$ is given by
\[ \mathcal{F}_{a,b}(g) = \{ \Lambda_{a,b} = a\mathbb{Z} \times b\mathbb{Z} \mid \mathcal{G}(g, \Lambda_{a,b}) \text{ is a frame} \} . \]
A necessary condition on $\Lambda_{a,b}$ to be contained in $\mathcal{F}_{a,b}$ of any function in the Hilbert space $L^2(\mathbb{R})$ is given by the density theorem for Gabor systems (see also [29]) and can be summed up in the following way:
\[ \mathcal{F}_{a,b}(g) \subset \{ \Lambda_{a,b} \mid (ab)^{-1} \geq 1 \} . \]
Assuming that $g \in S_0(\mathbb{R})$, which is known as Feichtinger’s algebra [16], [17] (see also [3], [30]), it is known that the frame set is open and contains a neighborhood of $0$ [24]. This means that for sufficiently high oversampling rate $(ab)^{-1}$, Gabor frames exist for any window in $S_0(\mathbb{R})$. The case of density 1 is called critical density and only at this level Gabor orthonormal bases exist. However, for windows in Feichtinger’s algebra the Balian-Low theorem ([1], [39], see also [23, Chap. 8.3]) shows that there exist no Gabor frames at critical density. For the hyperbolic secant Janssen and Strohmer [38] could show that the rectangular frame set is the largest possible;
\[ \mathcal{F}_{a,b}(\text{sech}) = \{ \Lambda_{a,b} \mid (ab)^{-1} > 1 \} . \]
There are other functions for which we now know the rectangular frame set. Among these are totally positive functions of finite type which all possess a full rectangular frame set [28]. Note that the hyperbolic secant belongs to this class of functions as well as the two-sided exponential and both are also in $S_0(\mathbb{R})$. Despite the fact they belong to the same function spaces, they exhibit very different optimality properties. Another example is the indicator function of a finite interval and its frame set has an extremely complicated structure [7, 8]. The first and still only window for which a full characterization of Gabor frames has been known is the Gaussian window [40], [45], [46]. For more general density results we refer to [31]. For more details on frames and Gabor systems we refer to the textbooks of Christensen [6] and Gröchenig [23] and for related results we also refer to the textbook of Folland [20].

4. Computing optimal Gabor frame bounds

After having determined the frame set of a function $g$, a natural follow-up question is on the optimality of a lattice $\Lambda_{a,b} \in \mathcal{F}_{a,b}(g)$ of fixed density $n$. Several things can be meant by that. We could look for a maximizer of the lower bound $A(a,b)$, a minimizer of the upper bound $B(a,b)$ or a minimizer of the condition number $\kappa(a,b) = B(a,b)/A(a,b)$. Neither the existence nor the uniqueness of optimizing pairs $(a,b)$ is an obvious matter. Even in the uniqueness case, the optimal lattices will not necessarily be the same for all quantities. There are several theoretical ways to investigate the behavior of the frame bounds. In the case of integer or rational density, a popular method is the Zak transform (see Appendix [19]). Alternatively, one can turn to duality theory. We present a reduced form and refer to the results of Daubechies, Landau and Landau [10], Janssen [34], Ron and Shen [41] and Wexler and Raz [48]. For a more thorough treatise of the topic we refer to [26], where we also find the following statement:

The concept of a frame expresses a strong form of completeness and already carries a stability result within its definition. The complementary concept asks for the linear independence of
time-frequency shifts, which leads to Riesz sequences. A system \( \mathcal{G}(g, \Lambda_{a,b}) \) is called a Riesz sequence for \( L^2(\mathbb{R}) \) if and only if

\[
A \|c\|_2^2 \leq \left\| \sum_{\lambda \in \Lambda_{a,b}} c_\lambda \pi(\lambda) g \right\|_2^2 \leq B \|c\|_2^2, \quad \forall c \in \ell^2(\Lambda_{a,b}).
\]

If \( G(g, \Lambda_{a,b}) \) is a frame and a Riesz sequence at the same time, then it is called a Riesz basis, which only exist if \((ab)^{-1} = 1\), i.e., at critical sampling density.

**Theorem** (Duality theorem). Let \( g \in L^2(\mathbb{R}) \) and \( \Lambda_{a,b} \subseteq \mathbb{R}^2 \) a rectangular lattice. Then, the following conditions are equivalent:

(i) \( \mathcal{G}(g, \Lambda_{a,b}) \) is a frame for \( L^2(\mathbb{R}) \).

(ii) \( \mathcal{G}(g, \Lambda_{a,b}^\circ) \) is a Riesz sequence for \( L^2(\mathbb{R}) \).

Above \( \Lambda_{a,b}^\circ \) denotes the adjoint or symplectic dual lattice. It may be characterized by commutation relations for time-frequency shifts:

\[
\Lambda_{a,b}^\circ = \{ \lambda^\circ \in \mathbb{R}^2 \mid \pi(\lambda^\circ) \pi(\lambda) = \pi(\lambda) \pi(\lambda^\circ) \quad \forall \lambda \in \Lambda_{a,b} \}.
\]

Note that the definition above gives at first only a set, but it turns out that this set is again a lattice. The adjoint of a rectangular lattice \( \Lambda_{a,b} \) can be written in terms of the lattice itself:

\[
\Lambda_{a,b}^\circ = \Lambda_{1/b,1/a} = (ab)^{-1} \Lambda_{a,b}.
\]

The reader more familiar with the dual lattice \( \Lambda_{a,b}^\perp \) may simply think of the dual lattice as a 90 degrees rotated version of the dual lattice.

For what follows, we need to introduce the Gramian operator \( T_{g, \Lambda_{a,b}} \), where we will again suppress the index in the sequel. It is given by

\[
T = \langle \pi(\lambda_1^g) g, \pi(\lambda_2^g) g \rangle, \quad \lambda_1^g, \lambda_2^g \in \Lambda_{a,b}^\circ.
\]

We note that the frame operator is a composition of the two adjoint operators

\[
C : L^2 \to \ell^2, \quad D : \ell^2 \to L^2
\]

\[
f \mapsto (\langle f, \pi(\lambda) g \rangle), \quad (c_\lambda) \mapsto \sum_{\lambda \in \Lambda_{a,b}} c_\lambda \pi(\lambda) g.
\]

In contrast to the frame operator \( S = DC \), the Gramian is the composition \( T = CD \) on the adjoint lattice. This leads to the following result, which has been obtained by Janssen [34].

**Proposition** (Gramian and frame operator). Let \( g \in L^2(\mathbb{R}) \) and assume \( 0 \leq A \leq B < \infty \) (note that \( A = 0 \) is included). Then, the following are equivalent:

(i) The frame operator satisfies \( AI_{L^2} \leq S \leq BI_{L^2} \).

(ii) The Gramian operator satisfies \( AI_{\ell^2} \leq (ab)^{-1} T \leq BI_{\ell^2} \).

We will explicitly compute the entries of the Gramian operator as already carried out by Janssen [35]. We use that \( \Lambda_{a,b}^\circ \) is explicitly given by \((1/b)\mathbb{Z} \times (1/a)\mathbb{Z}\). For a general window \( g \) the entries of \( T \) are thus explicitly given by

\[
c_{(k,l), (k',l')} = \langle M_{l/a} T_{k/b} g, M_{l'/a} T_{k'/b} g \rangle, \quad k, k', l, l' \in \mathbb{Z}.
\]

By using that \( T_z \) and \( M_z \) are unitary and by their commutation relations [34], we obtain

\[
c_{(k,l), (k',l')} = e^{-2\pi i k (l'-l)/(ab)} \langle g, M_{l'-l/a} T_{(k'-k)/b} g \rangle.
\]

First, we observe that the phase factor equals 1 if \((ab)^{-1} \in \mathbb{N}\). Next, we see that the entries of \( T \) then actually only depend on the differences \( k' - k \) and \( l' - l \). The Gramian operator \( T \) maps double sequences onto double sequences and is a double bi-infinite matrix, which is
constant along the diagonals in \( k' - k \) and \( l' - l \), hence it is a Laurent operator, which is the extension of a Toeplitz matrix to the infinite-dimensional setting (see [5]). In this case, the spectral bounds of the operator can be computed by the essential infimum and supremum of a Fourier series where the coefficients are the entries from the diagonals. We illustrate the process for a simple Laurent operator \( L : \ell^2(Z) \to \ell^2(Z) \):

\[
L = \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & c_0 & c_1 & c_2 & c_3 \\
\ddots & c_1 & c_0 & c_1 & c_2 \\
\ddots & c_2 & c_1 & c_0 & c_1 \\
\ddots & c_3 & c_2 & c_1 & \ddots
\end{pmatrix}
\quad \text{and} \quad F_L(t) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k t}.
\]

The norm of the operator \( L \) and the (inverse of the) norm of its inverse \( L^{-1} \) are obtained from the essential supremum and infimum of the related Fourier series \( F_L \):

\[
\|L\|_{\text{op}} = \underset{t}{\text{ess sup}} F_L(t) \quad \text{and} \quad \|L^{-1}\|_{\text{op}}^{-1} = \underset{t}{\text{ess inf}} F_L(t).
\]

We re-discover the result of Janssen [35], namely that the frame bounds are thus given by

\[
A = \underset{(x,\omega)}{\text{ess inf}} (ab)^{-1} \sum_{k,l \in \mathbb{Z}} V_{g,g}(\frac{k}{b}, \frac{l}{a}) e^{2\pi i (kx+\omega)}
\]

and

\[
B = \underset{(x,\omega)}{\text{ess sup}} (ab)^{-1} \sum_{k,l \in \mathbb{Z}} V_{g,g}(\frac{k}{b}, \frac{l}{a}) e^{2\pi i (kx+\omega)}.
\]

This provides an alternative to the Zak transform approach (see Appendix B). In [35], the series given by (4.1) are explicitly computed for 6 different windows. We built upon those computations and determine the critical points with respect to the lattice parameters. The case of the Gaussian and rectangular lattices has already been treated by Faulhuber and Steinerberger in [15], where optimality of the square lattice (in all three senses) under all lattices of integer density has been shown. This was followed by an optimality result for the upper bound and the hexagonal lattice [11] (see also [42]) and lastly Bétermin, Faulhuber and Steinerberger [4] proved the optimality of the lower bound for the hexagonal lattice (for even oversampling), which also implies the optimality of the hexagonal lattice for the condition number. In the following sections we will see that this does not seem to be a general pattern.

5. Preparations

5.1. The hyperbolic functions. Before we begin with the proofs, we recall the well-known hyperbolic functions as they will be omnipresent throughout the calculations. They are the analogues of the ordinary trigonometric functions and fulfill similar identities. For the reader’s convenience, we not only recall them, but also provide some easy bounds which we will need later on. Although we only observe them with strictly positive arguments, their maximal
domain is the complex plane. We write for \( x > 0 \)

\[
\cosh(x) = \frac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!},
\]

\[
\sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!},
\]

\[
\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = 1 - 2 \left( e^{2x} + 1 \right)^{-1},
\]

\[
\coth(x) = \frac{\cosh(x)}{\sinh(x)} = 1 + 2 \left( e^{2x} - 1 \right)^{-1}.
\]

A wealth of identities, similar to the trigonometric functions, holds. We will frequently use

\[
\cosh(x)^2 - \sinh(x)^2 = 1,
\]

\[
\tanh(x)^2 + \text{sech}(x)^2 = 1,
\]

\[
\coth(x)^2 - \text{csch}(x)^2 = 1.
\]

The series expansions of \text{exp}, \cosh, \sinh consist of strictly positive terms and converge quite fast. It is not difficult to see that \cosh, \sinh and \tanh are strictly monotonically increasing, whereas the others are strictly monotonically decreasing. Truncating the series gives lower bounds for \cosh, \sinh and \tanh. In the latter case, we use the second equality for \tanh and truncate the series of the exponential function. Hence, in the same manner we obtain upper bounds for their reciprocals: \text{sech}, \text{csch} and \text{coth}. For further bounds, estimates of the type

\[
\frac{e^x}{2} < \cosh(x) < \frac{e^{x+1}}{2} < e^x
\]

are usually good enough. We mention these rather simple facts because the proofs which follow require evaluating and bounding these functions multiple times at various points. To verify the estimates, it would often suffice to sum the first 10 terms of the above given expansions. Of course, the reader is also welcome to use their preferred software. For numerical checks, we used Mathematica [50] (with a precision to the 24th digit), which evaluates these functions to arbitrary precision. Still, our arguments are elementary and can also be checked by hand, but the bookkeeping is delicate at several points and makes the proofs non-trivial.

We will frequently encounter the following functions and use their monotonicity properties.

In some of our calculations we will use the symbols \( \nearrow \) and \( \searrow \) to denote that an expression is increasing or decreasing, respectively. The following properties will be useful for our proofs:

(I) \( x \tanh(x) \) is strictly monotonically increasing as the product of two positive, strictly monotonically increasing functions. Also, as

\[
\lim_{x \to \infty} \tanh(x) = 1
\]

the expression behaves like \( x \mapsto x \). Moreover, we have the fact that the fixed point of \coth, which solves the equation \( x \tanh(x) = 1 \), lies in the open interval \((1.19, 1.2)\).

(II) \( x \coth(x) \) is unbounded, strictly monotonically increasing on \((0, \infty)\) and can be continuously extended at 0 by setting the value to 1 (see Figure [I]). First of all, we observe

\[
\lim_{x \to 0^+} x \coth(x) = \lim_{x \to 0^+} x + \lim_{x \to 0^+} \left( \frac{e^{2x} - e^0}{2x} \right)^{-1} = 0 + \frac{1}{\exp'(0)} = 1.
\]

\[1\text{The Mathematica notebook is provided on arxiv.org as an ancillary file.}\]
For the monotonicity, we compute the first derivative. For all \( x > 0 \) holds
\[
\frac{\partial}{\partial x} [x \coth(x)] = \coth(x) - x \csch(x)^2 = \csch(x)^2 (\sinh(x) \cosh(x) - x)
\]
\[
= \csch(x)^2 \left( \frac{1}{2} \sinh(2x) - x \right) = \csch(x)^2 \frac{1}{2} \sum_{k=1}^{\infty} \frac{(2x)^{2k+1}}{(2k+1)!} > 0
\]

(III) \((x \coth(x))^2 - (x \coth(x))\) is unbounded, strictly monotonically increasing and tends to 0 as \( x \) approaches 0. Indeed, \( p(t) = t^2 - t = t(t - 1) \) possesses these properties on \((1, \infty)\). The claim follows from the properties of \( x \coth(x) \).

(IV) \( x \csch(x) \) is strictly monotonically decreasing. We have \( \lim_{x \to 0} x \csch(x) = 1 \) and \( \lim_{x \to \infty} x \csch(x) = 0 \) (see Figure 1). Equivalently, \( \sinh(x)/x \) is strictly monotonically increasing, tends to 1 as \( x \) tends to 0 and is unbounded as \( x \) increases. One can easily verify the asymptotics by the series expansion of \( \sinh \):
\[
\frac{\sinh(x)}{x} = \frac{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}}{x} = \frac{\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1)!}}{x} = \frac{\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1)!}}{1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k+1)!}} \to 0, \quad x \to 0.
\]

(V) \( x^2 \csch(x) \) is strictly monotonically increasing on \((0, 1.2]\). Equivalently, \( \sinh(x)/x^2 \) is strictly monotonically decreasing in this interval.
\[
\frac{\partial}{\partial x} \frac{\sinh(x)}{x^2} = \frac{\cosh(x)x^2 - 2x \sinh(x)}{x^4} = \frac{\sinh(x)}{x^3} (x \coth(x) - 2).
\]

Due to the monotonicity of \( x \coth(x) \) and its value at 1.2, the derivative is strictly negative on \((0, 1.2]\), proving the claim.

(VI) \( \sinh(x) < x + x^3 \) on \((0, 1.2]\). Indeed,
\[
\frac{\partial}{\partial x} [\sinh(x) - x - x^3] = \cosh(x) - 1 - 3x^2,
\]
\[
\frac{\partial^2}{\partial x^2} [\sinh(x) - x - x^3] = \sinh(x) - 6x = x \left( \frac{\sinh(x)}{x} - 6 \right).
\]

Due to the monotonicity of \( \sinh(x)/x \), by evaluating at 1.2, one sees that we are dealing with a strictly concave function. Further, \( \cosh(0) - 1 - 3 \cdot 0^2 = 0 \), so \( \sinh(x) - x - x^3 \) is strictly monotonically decreasing. It takes the value 0 at 0, so the inequality holds on the entire interval \((0, 1.2]\).

(VII) We will need the following property only for integers, but the result holds for positive real numbers. For \( x > 0 \), we set
\[
\rho(x) = \frac{2 \arccosh(x)}{x}.
\]
The expression \( \rho(x) \) is strictly monotonically decreasing on \([2, \infty)\) (see Figure 1).

\[
\frac{\partial}{\partial n} \rho(x) = -\frac{2 \operatorname{arccosh}(x)}{x^2} + \frac{2}{x} \cdot \frac{1}{(2x^2-1)^{1/2}} = \frac{1}{x} \left( \frac{2}{(2x^2-1)^{1/2}} - \rho(x) \right).
\]

This derivative is negative if and only if

\[
\left( 1 + \frac{1}{x^2-1} \right) < \operatorname{arccosh}(x)^2.
\]

This can be easily verified for \( n = 2 \), and for all other holds

\[
\left( 1 + \frac{1}{n^2-1} \right) < \left( 1 + \frac{1}{2^2-1} \right) < \operatorname{arccosh}(2)^2 < \operatorname{arccosh}(n)^2.
\]

Figure 1. The graphs indicating the asymptotic behavior of \( x \coth(x) \), \( x \csch(x) \) and \( \rho(x) \).

Besides the various elementary identities involving the derivatives or squares of the hyperbolic functions, which can all be found in [22], we will also use the following equation [2, eq. 86]

\[
(5.2) \quad x \sum_{k=-\infty}^{\infty} \operatorname{sech}(kx)^2 = 2 + \frac{2}{x} \sum_{k=0}^{\infty} \operatorname{csch} \left( \left( k + \frac{1}{2} \right) \frac{1}{x} \right)^2.
\]

This and many other equations in [2] illuminate the deeper connection of the hyperbolic functions to elliptic functions and elliptic integrals, which also appear to be closely linked to Gabor frames with the Gaussian window (see [35], compare also [14]).

5.2. Interval estimate. We will often speak of interval estimates. By that we mean bounding an expression consisting of monotonic functions (or otherwise well-known bounded functions) on an interval \([x_0, y_0]\) by the worst possible value. For example, \( \tanh \) is strictly monotonically increasing and \( \operatorname{csch} \) is strictly monotonically decreasing, both strictly positive. Then \( \operatorname{csch}(t) - \operatorname{csch}(2t) > 0 \) and

\[
\tanh(t) \left( \operatorname{csch}(t) - \operatorname{csch}(2t) \right) < \tanh(y_0) \left( \operatorname{csch}(x_0) - \operatorname{csch}(2y_0) \right), \quad t \in (x_0, y_0).
\]

which gives us as a rough upper bound (if the end point is 0 or \( \infty \), we would take the limit, presuming it exists). The arrows indicate the monotonicity properties we applied for the estimate. Of course, the sign of each factor also goes into the overall estimate.

The interval might be too long, though. Therefore, we will split it into sub-intervals (see Figure 2) and do the estimate there. In that case, we will either provide the explicit points where we split, or divide the interval into sub-intervals of appropriate lengths.
2.5
3.0
3.5
4.0
4.5
5.0
0.05
0.10
0.15
0.20
0.25

\text{tanh}(t) (\text{csch}(t) - \text{csch}(2t))

\text{estimate: (2,5)}

\text{estimate: (2,3)}

\text{estimate: (3,5)}

Figure 2. Estimate of the above type for \( \text{tanh}(t)(\text{csch}(t) - \text{csch}(2t)) \) on the interval \((2, 5)\). In order to get more appropriate estimates we may use the sub-intervals \((2, 3)\) and \((3, 5)\).

6. The hyperbolic secant window

In this section we will prove Theorem 2.1, which states that the lower frame bound, upper frame bound and condition number of the Gabor system \( \mathcal{G}(\sqrt{\pi/2} \text{sech}(\pi t), a\mathbb{Z} \times b\mathbb{Z}) \) are optimal if and only if \( a = b \), assuming \( 2 \leq (ab)^{-1} \in \mathbb{N} \). We begin with a transformation of the problem. We use the formula provided by Janssen [35, eq. (7.8)] (with the additional normalizing factor \( \pi/2 \)) to express the frame bounds. To simplify the expressions, we set

\[ f_A(t) = t \sum_{k=0}^{\infty} \text{sech}(\alpha_k t)^2 \quad \text{and} \quad f_B(t) = t \sum_{k=0}^{\infty} \text{csch}(\alpha_k t)^2, \quad t \in (0, \infty), \]

where \( \alpha_k = \pi(k + 1/2) \). Note that the series defining \( f_A \) and \( f_B \) are starting at \( k = 0 \). We use the following change of variables, which is respecting the fact that \( (ab)^{-1} = n \):

\[ (a, b) \mapsto \left( \frac{a}{n}, \frac{b}{n} \right) \]

The exact frame bounds can now be expressed by (compare [35, eq. (7.8)])

\[ A\left(\frac{a}{n}, \frac{b}{n}\right) = \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \frac{a}{n} \text{sech}^2\left(\pi\left(k + \frac{1}{2}\right) \frac{a}{n}\right) + \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \frac{1}{n} \text{sech}^2\left(\pi\left(k + \frac{1}{2}\right) \frac{1}{n}\right) - n, \]

\[ B\left(\frac{a}{n}, \frac{b}{n}\right) = \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \frac{a}{n} \text{sech}^2\left(\pi k \frac{a}{n}\right)^2 + \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \frac{1}{n} \text{sech}^2\left(\pi k \frac{1}{n}\right)^2 - n, \]

where we used the symmetry of \( \text{sech} \) and identity (5.2). The above series converge locally uniformly, hence the functions involved are infinitely differentiable as each term is already infinitely differentiable. The quantity \( \eta \) describes the lattice geometry and we have a square lattice if \( \eta = n^{1/2} \). We note that optimizing the bounds gets more and more difficult as the density grows, since the curves get flatter and flatter as indicated in Figure [3].
Our strategy is the following: From the algebraic structure of the problem, it follows immediately that $\eta = n^{1/2}$ gives a critical point, which by (6.1) shows that the square lattice is critical in the set of rectangular lattices. Then, we will prove certain monotonicity properties of the involved terms and their derivatives to exclude further critical points.

Determining the critical points is achieved by finding all values $\eta \in \mathbb{R}_+$ such that

\[
\frac{n}{\pi} \frac{\partial}{\partial \eta} A \left( \frac{\eta}{n}, \frac{1}{\eta} \right) = \frac{n}{\pi} f'_A \left( \frac{\eta}{n} \right) - \frac{1}{\eta} f'_A \left( \frac{1}{\eta} \right) = h_A \left( \frac{\eta}{n} \right) - h_A \left( \frac{1}{\eta} \right) = 0,
\]

\[
\frac{n}{\pi} \frac{\partial}{\partial \eta} B \left( \frac{\eta}{n}, \frac{1}{\eta} \right) = -\frac{n}{\pi} f'_B \left( \frac{\eta}{n} \right) + \eta f'_B \left( \eta \right) = -h_B \left( \frac{\eta}{n} \right) + h_B \left( \eta \right) = 0.
\]

Here we used the abbreviation $h_A(t) = t f'_A(t)$ and $h_B(t) = t f'_B(t)$.

Obviously, the expressions in (6.2) vanish for $\eta = n^{1/2}$, which corresponds to the square lattice of density $n$. We seek to show that there are no other critical points to prove Theorem 2.1.

So far, the lower and upper bound’s dependence on $f_*$ and $h_*$, as well as the structure of $f_*$ and $h_*$ seem to be nearly the same. Yet, the accumulation of derivatives of sech and csch secretly amplifies several differences (see also Figure 4).

6.1. The lower bound. In [2], we find the identity (abbreviated with our notation above)

\[
\pi f_A(t) + \pi f_A(t^{-1}) = 1,
\]

which implies the point symmetry $h_A(t) = h_A(t^{-1})$. Rather remarkably, we could have concluded (6.3) with only our knowledge about Gabor frames at critical density (a similar remark is given in [33]). Indeed, $A(\eta, 1/\eta) \equiv 0$, because of the Balian-Low theorem [11, 39] (see also [23 Chap. 8.4]).

We will prove that the following equivalence.

**Lemma 6.1.** For all $x, y \in (0, \infty)$ holds

\[
h_A(x) = h_A(y) \iff x = y \lor x = y^{-1}.
\]
Proof. One implication is rather simple: the case $x = y$ is trivial and the case $x = y^{-1}$ follows from the already established point symmetry of $h_A$. We start the proof of the other implication with the observation that due to the symmetry, it suffices to prove the strict monotonicity on $(1, \infty)$ to exclude further solutions for (6.4). We split our computations in two parts. The monotonicity is straightforward on the interval $(1, 0.03, \infty)$. On the part $(1, 1.04)$, we prove that $h_A'$ is strictly convex, i.e., $h_A''(t) > 0$ for all $t > 1$.

Monotonicity away from 1: As $h_A(t) = t f_A'(t)$, we start with computing derivatives of $f_A$:

$$f_A'(t) = \sum_{k=0}^{\infty} \alpha_k \tanh(\alpha_k t),$$

$$f_A''(t) = -4 \sum_{k=0}^{\infty} \alpha_k \tanh(\alpha_k t)$$
$$+ 4t \sum_{k=0}^{\infty} \alpha_k^2 \tanh(\alpha_k t)^2 - 2t \sum_{k=0}^{\infty} \alpha_k^2 \tanh(\alpha_k t)^4,$$

$$f_A'''(t) = 6 \sum_{k=0}^{\infty} \alpha_k^2 \tanh(\alpha_k t)^2 (2 \tanh(\alpha_k t)) - \tanh(\alpha_k t)^2$$
$$+ 8t \sum_{k=0}^{\infty} \alpha_k^3 \tanh(\alpha_k t)^2 \tanh(\alpha_k t) (2 \tanh(\alpha_k t) - \tanh(\alpha_k t)^2).$$

As a next step we compute the derivative of $h_A(t) = t f_A'(t)$.

$$h_A'(t) = \sum_{k=0}^{\infty} \tanh(\alpha_k t)^2 \left( 1 - 2 \alpha_k^2 t^2 \tanh(\alpha_k t)^2 + 4 \alpha_k^2 t^2 \tanh(\alpha_k t)^2 - 6 \alpha_k t \tanh(\alpha_k t) \right).$$

We split the expression in the brackets in two. The goal is to determine points which satisfy

$$1 - 2x^2 \tanh(x)^2 > 0 \quad \text{and} \quad 4x^2 \tanh(x)^2 - 6x \tanh(x) \geq 0.$$
The second inequality is fulfilled as long as \( x \) \( \tanh(x) \geq \frac{3}{2} \). The solution of \( x \) \( \tanh(x) = \frac{3}{2} \) lies in \((1.6218, 1.6219)\). For the first part, we observe
\[
\lim_{x \to 0} 1 - 2x^2 \sech(x)^2 = \lim_{x \to \infty} 1 - 2x^2 \sech(x)^2 = 1,
\]
\[
\frac{d}{dx} \left[1 - 2x^2 \sech(x)^2\right] = 4x \sech(x)^2(x \tanh(x) - 1).
\]
Recall that the fixed point of coth lies in \((1.19, 1.2)\), which yields the only critical point of \( x \to 1 - 2x^2 \sech(x)^2 \) for \( x > 0 \). This must yield the minimum. Therefore, it holds that
\[
1 - 2x^2 \sech(x)^2 > 1 - 2 \cdot 1.2^2 \cdot \sech(1.19)^2 > 0.1 > 0, \quad \forall x > 0.
\]
As \( \alpha_k t \geq \alpha_0 t \), for all \( t \geq 1.6219 \alpha_0^{-1} = \frac{3.2438}{\pi} \in (1.03, 1.04) \) we see that \( h_A'(t) > 0 \) for \( t > 1.03 \).

**Strict convexity close to 1:** Since \( h_A'(1) = 0 \), trying to find positive lower bounds for \( h_A' \) close to 1 is playing a losing game, so we turn to the second derivative of \( h_A \).

\[
h_A''(t) = 2 \sum_{k=0}^{\infty} \alpha_k \sech(\alpha_k t)^2 \times \left(8\alpha_k^2 t^2 \sech(\alpha_k t)^2 \tanh(\alpha_k t) - 5\alpha_k t \sech(\alpha_k t)^2 - 4 \tanh(\alpha_k t) + 10\alpha_k t \tanh(\alpha_k t)^2 - 4\alpha_k^2 t^2 \tanh(\alpha_k t)^3\right)
\]

We do a similar separation as above.

**Part 1:** Recall that \( \alpha_k \geq \frac{5}{2} \) for \( k \geq 0 \). For all \( t \in [1, 1.04] \) it holds that
\[
8\alpha_k^2 t^2 \sech(\alpha_k t)^2 \tanh(\alpha_k t) - 5\alpha_k t \sech(\alpha_k t)^2 = \alpha_k t \sech(\alpha_k t)^2 (8\alpha_k t \tanh(\alpha_k t) - 5) \geq \sech(\alpha_k t)^2 (8 \frac{5}{2} \tanh(\frac{5}{2}) - 5) > 6 \sech(\alpha_k t)^2 > 0,
\]
due to the monotonicity of \( x \tanh(x) \).

**Part 2:** We compute
\[
\alpha_k \sech(\alpha_k t)^2 \left( -4 \tanh(\alpha_k t) + 10\alpha_k t \tanh(\alpha_k t)^2 - 4\alpha_k^2 t^2 \tanh(\alpha_k t)^3\right)
\]
\[
= \alpha_k \sech(\alpha_k t)^2 \tanh(\alpha_k t) \left( -4\alpha_k^2 t^2 \tanh(\alpha_k t)^2 + 10\alpha_k t \tanh(\alpha_k t) - 4\right)
\]
\[
= 2\alpha_k \sech(\alpha_k t)^2 \tanh(\alpha_k t) p(\alpha_k t \tanh(\alpha_k t),
\]
where \( p(x) = -2x^2 + 5x - 2 = -2(x - 1)(x - 2) \). For \( x > 2 \), the polynomial \( p(x) \) is clearly negative and decreasing. However, for \( t \) close to 1 we have that \( x = \alpha_0 t \tanh(\alpha_0 t) \in (1/2, 2) \), where \( p \) is positive. More precisely, for all \( t \in (1, 1.04) \) it holds that
\[
\alpha_0 t \tanh(\alpha_0 t) \in \left(\frac{5}{2} \tanh(\frac{5}{2}), \frac{1.04 \pi}{2} \tanh(\frac{1.04 \pi}{2})\right) \subset (1.44, 1.52).
\]
The polynomial \( p \) is strictly positive and strictly monotonically decreasing on \((1.44, 1.52)\). All other terms land in the negative part of \( p \) as
\[
\alpha_k t \tanh(\alpha_k t) \geq \alpha_1 \tanh(\alpha_1) > 4 > 2.
\]
We recall that \( 0 < \tanh < 1 \), as well as that \( p \) is strictly monotonically decreasing and strictly negative on \((2, \infty)\). With a standard curve discussion, we see that \( p(x) > -3x^2 \) for all \( x \geq 1 \).
Finally, we note that one power of sech will be omitted in the estimate, as it is positive and has a minor impact here, but will be a significant weight later on. We estimate

\[
0 \geq 2\alpha_k \tanh(\alpha_k t) \sech(\alpha_k t) p(\alpha_k t \tanh(\alpha_k t)) \\
> 2\alpha_k \tanh^3(\alpha_k t) \sech(\alpha_k t) \left( -3\alpha_k^2 t^2 \right) \\
> \frac{6}{t} (\alpha_k t)^3 \sech(\alpha_k t) > -12(\alpha_k t)^3 e^{-\alpha_k t}.
\]

We now look for an upper bound of the expression \(x^3 e^{-x}\). We compute the derivative

\[
\frac{d}{dx}(x^3 e^{-x}) = (3-x)x^2 e^{-x},
\]

so the maximum is attained at \(x = 3\). We complete the estimate:

\[
0 \geq 2\alpha_k \tanh(\alpha_k t) \sech(\alpha_k t) \cdot p_1(\alpha_k t \tanh(\alpha_k t)) > -12 \cdot 27 \cdot e^{-3} > -16.14.
\]

To sum up, for all \(t \in (1, 1.04)\) we have

\[
\frac{1}{2} h''_A(t) \geq \sum_{k=0}^{\infty} 6\alpha_k \sech^2(\alpha_k t) - 16.14 \sum_{k=1}^{\infty} \sech(1.04 \alpha_k) \\
+ \alpha_0 t \tanh(\alpha_0 t) \sech(\alpha_0 t) \cdot p(\alpha_0 t \tanh(\alpha_0 t)) \\
\geq 6 \sum_{k=0}^{\infty} e^{-2\pi k t} e^{-\pi t} - 16.14 \sum_{k=1}^{\infty} 2 e^{-1.04 \pi k e^{-0.52 \pi}} \\
+ 1.44 \sech(\frac{1.04 \pi}{2})^2 p(1.52) \\
> 3 \cdot \frac{1}{\sinh(\pi t)} - 16.14 \frac{e^{-1.04 \pi}}{\sinh(0.52 \pi)} + 1.44 \cdot 0.14 \cdot 0.9792 \\
> 3 \cdot 0.07 - 16.14 \cdot \frac{0.94}{2.36} + 0.19 > 0.13 > 0.
\]

All in all, we have proven that \(h_A\) is strictly strictly monotonically increasing on \((1, \infty)\) and due to the symmetry it has to be monotonically decreasing on \((0, 1)\). Let now \(x, y \in (0, \infty)\) be two distinct points satisfying \(h_A(x) = h_A(y)\). Without loss of generality, we can assume \(x < y\). By the established monotonicity, \(x < 1 < y\), implying also \(1 < x^{-1}\). So, we can write

\[
h_A(\frac{1}{x}) = h_A(x) = h_A(y).
\]

The monotonicity on \((1, \infty)\) now implies \(x^{-1} = y\). \(\square\)

So, by (6.11) of the above lemma we have

\[
h_A(\frac{n}{n}) = h_A(\frac{1}{n}) \iff \frac{n}{n} = \frac{1}{n} \land \frac{n}{n} = \eta \iff \eta = \sqrt{n} \lor n = 1.
\]

The case of critical density \((n = 1)\) is trivial and \(A\) vanishes identically in this case. For all other densities \(n \in \mathbb{N}, \eta = \sqrt{n}\) is the only critical point. To conclude the statement on the lower frame bound, recall that for \((ab)^{-1} = n\), we used the map \((a, b) \mapsto (\eta/n, 1/\eta)\). Hence, the square lattice \(n^{-1/2}\mathbb{Z} \times n^{-1/2}\mathbb{Z}\) yields the unique maximum of \(A\) for density \(2 \leq n \in \mathbb{N}\).
6.2. The upper bound. Contrary to \( h_A \), the function \( h_B \) possesses no symmetries, but is strictly monotonically increasing. We will split \((0, \infty)\) in multiple intervals. There is no denial that there are far better estimates than the following ones. In fact, numerical inspection of the function \( \psi \) suggest \( \psi > 1 \), while we only show its strict positivity. However, the estimates here are very simple and easy to verify. Trying to improve them would likely make the conditions, under which the interval splitting is obtained, significantly more complicated. At least two cases are probably unavoidable, since the function under consideration \( \psi \) behaves rather differently for very small and for very big arguments. Our main strategy is similar: Show that

\[
h_B(x) = h_B(y) \iff x = y.
\]

This time, however, we show that \( h'_B(t) > 0 \) for all \( t \in (0, \infty) \), which then gives the proof. We begin with computing the derivatives of \( f_B \):

\[
f'_B(t) = \sum_{k=0}^{\infty} \text{csch}(\alpha_k t)^2 (1 - 2\alpha_k t \coth(\alpha_k t)),
\]

\[
f''_B(t) = -4 \sum_{k=0}^{\infty} \alpha_k \text{csch}(\alpha_k t)^2 \coth(\alpha_k t) + 4t \sum_{k=0}^{\infty} \alpha_k^2 \text{csch}(\alpha_k t)^2 \coth(\alpha_k t)^2
\]

\[
+ 2t \sum_{k=0}^{\infty} \alpha_k^2 \text{csch}(\alpha_k t)^4.
\]

Using the identity \( \text{csch}(x)^2 = \coth(x)^2 - 1 \) and \( h_B(t) = t f_B(t) \), we compute \( h'_B \).

\[
h'_B(t) = \sum_{k=0}^{\infty} \text{csch}(\alpha_k t)^2 \left( 1 - 2\alpha_k^2 t^2 + 6\alpha_k^2 t^2 \coth(\alpha_k t)^2 - 6\alpha_k t \coth(\alpha_k t) \right)
\]

\[
= \sum_{k=0}^{\infty} \text{csch}(\alpha_k t)^2 \psi(\alpha_k t),
\]

where \( \psi(x) = 1 - 2x^2 + 6x^2 \coth(x)^2 - 6x \coth(x) \). If we prove that \( \psi \) is strictly positive on \((0, \infty)\), then so is \( h'_B \).

**Part 1:** \( x > \log(\sqrt{21}) \). Due to the monotonicity of \exp, one can easily convince themselves that \( 1.5 < \log(\sqrt{21}) < 1.6 \). Since \coth is strictly monotonically decreasing, strictly greater than 1 and can be written as \( \coth(x) = 1 + 2/(\exp(2x) - 1) \), we estimate as follows:

\[
\coth(x) < 1 + \frac{2}{21 - 1} = 1.1, \quad \forall x > \log(\sqrt{21}),
\]

and for \( x > \log(\sqrt{21}) > 1.5 \),

\[
\psi(x) > 1 - 2x^2 + 6x^2 \cdot 1^2 - 6x \cdot 1.1 = 4x^2 - 6.6x + 1 = 4 \cdot (1.5 - 0.825)^2 - 1.7225 = 0.1 > 0.
\]

Notice that the same estimate does not work if the estimate is for \( \coth(x) < 1.2 \), as this corresponds to \( x > \log(\sqrt{11}) \). Close to zero, \( \coth(x) \) cannot be bounded, so we have to take into account the interaction with \( x \).

Before we continue with the remaining intervals, we reconsider the terms in \( \psi \) on \((0, 1.6)\). We rewrite it as

\[
\psi(x) = q_1(x) + 6q_2(x \coth(x)), \quad q_1(x) = 1 - 2x^2 \quad \text{and} \quad q_2(x) = x(x - 1).
\]
Clearly, $q_2$ is strictly monotonically decreasing on $(0, \infty)$, whereas $q_1$ is strictly monotonically increasing and strictly positive on $(1, \infty)$. The interval estimate of $\psi$ on an interval $(x_0, y_0) \subseteq (0, 1.6)$ will satisfy
\[ \psi(x) > q_1(y_0) + 6 q_2(x_0 \coth(x_0)) > 0, \]
if we choose $x_0$, $y_0$ well. We convince ourselves of the inequality with interval estimates on $(0, \sqrt{0.5})$, $(0.7, 1.01)$, $(1, 1.3)$ and $(1.29, 1.6)$. The intervals are chosen, so that there is a slight overlap between the cases and no point will be omitted.

The following interval estimates are exemplary for some of the later sections, where we will then omit such trivial calculations as the number of sub-intervals becomes rather large.

**Part 2:** $x \in (0, \sqrt{0.5})$.

\[ \psi(x) > q_1(\sqrt{0.5}) + 0 = 0. \]

**Part 3:** $x \in (0.7, 1.01)$.

The choice of the lower interval bound is due to $0.7 = \sqrt{0.49} < \sqrt{0.5}$. To that, $0.7 \cdot \coth(0.7) > 1.158$ determines the upper bound. We have
\[ \psi(x) > q_1(1.01) + 6 q_2(1.158) = 0.057584 > 0. \]

**Part 4:** $x \in (1, 1.3)$.

\[ \psi(x) > q_1(1.3) + 6 q_2(1.31) = 0.0566 > 0. \]

**Part 5:** $x \in (1.29, 1.6)$.

\[ \psi(x) > q_1(1.6) + 6 q_2(1.5) = 0.38 > 0. \]

This proves that $\psi$, hence $h_B'$ is strictly positive on $(0, \infty)$. We now consider what that means for the optimality of the upper bound $B$. We have established
\[ h_B(\frac{2}{n}) = h_B(\frac{1}{\eta}) \iff \frac{\eta}{n} = \frac{1}{\eta} \iff \eta = \sqrt{n}. \]

Furthermore, for all $\eta > \sqrt{n}$ we have
\[ \frac{\partial}{\partial \eta} B(\frac{2}{n}, \frac{1}{\eta}) > 0. \]

Due to the symmetry about $\sqrt{n}$, the square lattice $n^{-1/2}Z \times n^{-1/2}Z$ is the unique minimizing lattice of $B$ among rectangular lattices of density $n \in \mathbb{N}$.

As $A$ is maximized by the square lattice $n^{-1/2}Z \times n^{-1/2}Z$ and $B$ is minimized by the square lattice $n^{-1/2}Z \times n^{-1/2}Z$, also the condition number $\kappa = B/A$ has to be minimal in this case. In contrast to the Gaussian case, we do not know sufficient density conditions for the hyperbolic secant and general lattices to yield Gabor frames. So, before speculating about an optimality result for general lattices one should work out the lattice frame set of the hyperbolic secant. Nonetheless, we note that the conjecture of Strohmer and Beaver [47] was based on the radial symmetry of the Gaussian in the time-frequency plane and connections to sphere packings (see also [4]). By the characterization of radially symmetric ambiguity functions, which can only come from Gaussians and Hermite functions [20], it is clear that the hyperbolic secant does not possess a radial symmetry in the time-frequency plane.
7. Cut-off exponentials

7.1. Support 1/b. In this section we study frame bounds of cut-off exponentials with support in \([0, 1/b]\) (or any interval of length 1/b) for the lattice \(\Lambda_{a,b} = a\mathbb{Z} \times b\mathbb{Z}\) with density \((ab)^{-1} > 1\). In particular, the support of the function is tailored to the adjoint lattice. As the frame bounds are calculated using the adjoint lattice \(\frac{1}{b}\mathbb{Z} \times \frac{1}{a}\mathbb{Z}\), the support condition ensures that there is no influence from shifted copies of the window in the computations. Therefore, the arising double series simplify to simple series. We denote the cut-off exponential by

\[
g_{b,\gamma}(t) = C_{b,\gamma} e^{-\gamma t} \chi_{[0, 1/b]}(t),
\]

where \(C_{b,\gamma}\) is a normalizing factor, which is explicitly given by \(C_{b,\gamma} = \sqrt{2\gamma/(1 - e^{-2\gamma/b})}\). Note that, by using l’Hôpital’s rule, we get \(C_{b,0} = \sqrt{b}\) by taking the limit \(\gamma \to 0\). The study of the cut-off exponentials case is inspired by the examples in \[35\], Sec. 2. We also refer to \[9\], Sec. 3.4.4] where similar computations have been carried out. As the normalization differs from \[35\], Sec. 2], we carry out most of the computations. Recall that the spectral bounds of the frame operator \(S : L^2(\mathbb{R}) \to L^2(\mathbb{R})\) and the pre-Gramian operator \(T : \mathcal{E}^2(\Lambda_{a,b}^* \mathbb{Z}) \to \mathcal{E}^2(\Lambda_{a,b}^* \mathbb{Z})\) coincide up to the normalizing constant \((ab)^{-1}\).

For general window \(g\) with \(\text{supp}(g) \subset [0, 1/b]\), the entries of \(T\) are given by the inner products

\[
c_{(k,l),(k',l')} = \langle M_{l/a} T_{k/b} g, M_{l'/a} T_{k'/b} g \rangle, \quad k, k', l, l' \in \mathbb{Z}.
\]

By using that \(T_x\) and \(M_{\omega}\) are unitary and their commutation relations, we obtain

\[
c_{(k,l),(k',l')} = e^{-2\pi i k (l' - l)/(ab)} \langle g, M_{l'/a} T_{(k' - k)/b} g \rangle.
\]

As \(g\) is supported on \([0, 1/b]\) the expression vanishes whenever \(k \neq k'\). Also, the entries only depend on the difference \(l' - l \in \mathbb{Z}\) and not \(l\) and \(l'\) separately. For \((ab)^{-1} \in \mathbb{N}\), the phase factor in front of the inner product disappears and the operator is a Laurent operator. The (re-normalized) spectral bounds of \(T\) are thus given by

\[
A = \text{ess inf}_{\tau \in \mathbb{R}} \langle ab \rangle^{-1} \sum_{l \in \mathbb{Z}} c_l e^{2\pi i l \tau} \quad \text{and} \quad B = \text{ess sup}_{\tau \in \mathbb{R}} \langle ab \rangle^{-1} \sum_{l \in \mathbb{Z}} c_l e^{2\pi i l \tau},
\]

where \(c_l = c_{(0,0),(0,l)} = \langle g, M_{l/a} g \rangle = \mathcal{F}(|g|^2)(l/a)\). By using the Poisson summation formula and the commutation rules of the Fourier transform with (unitary) dilations, we obtain

\[
\sum_{l \in \mathbb{Z}} \mathcal{F}(|g|^2)(l/a) e^{2\pi i l \tau} = \sum_{k \in \mathbb{Z}} a \left| g(ak + a\tau) \right|^2.
\]

Getting back to the cut-off exponential \(g_{\gamma}\), the series describing the exact frame bounds turn out to be expressible by simple algebraic expressions of elementary functions. First, we compute the Fourier transform of \(|g_{b,\gamma}|^2\) evaluated at \(l/a\):

\[
(7.1) \quad c_l = \int_0^{1/b} C_{b,\gamma}^2 e^{-2\gamma t} e^{-2\pi i \frac{l}{a} t} dt = C_{b,\gamma}^2 \frac{a \left( 1 - e^{-2\pi i l} \right)}{2a\gamma + 2\pi il}.
\]

We set \(F(\tau) = \sum_{l \in \mathbb{Z}} c_l e^{2\pi i l \tau}\) and, hence,

\[
A = \text{ess inf}_{\tau \in \mathbb{R}} \langle ab \rangle^{-1} F(\tau) \quad \text{and} \quad B = \text{ess sup}_{\tau \in \mathbb{R}} \langle ab \rangle^{-1} F(\tau).
\]
We note that $F(\tau)$ is actually a Fourier series on the torus $T \cong [0,1]$ and the coefficients have been obtained from (7.1). Thus, for $b = 1$ and using the change of variables \( t \mapsto at \), we see that $F(\tau)$ is just the periodization (see Figure 5) \( \mathcal{P}_\tau \) of \( \tau \mapsto e^{-2\gamma a \tau} \chi_{[0,1]}(\tau) $:

\[
\mathcal{P}_\tau (e^{-2\gamma a \tau} \chi_{[0,1]}(\tau)) = \sum_{k \in \mathbb{Z}} \left( e^{-2\gamma a (\tau + k)} \right) = e^{-2\gamma a (\tau - \tau)}.
\]

Figure 5. Periodization of the function $\tau \mapsto e^{-2\gamma a \tau} \chi_{[0,1]}(\tau)$. The function is 1-periodic and can thus be developed into a Fourier series on the standard torus $T \cong [0,1]$.

We only need to adjust (7.1) a little bit to obtain

\[
(ab)^{-1} F(\tau) = C^2_{b,\gamma} (ab)^{-1} \sum_{l \in \mathbb{Z}} a \left( \frac{1 - e^{-2\pi i l}}{2a\gamma - 2\pi il} \right) e^{2\pi i l} = C^2_{b,\gamma} \frac{1 - e^{-\frac{2\pi}{b}}}{b} \sum_{l \in \mathbb{Z}} \frac{1 - e^{-2\gamma a}}{2a\gamma - 2\pi il} e^{2\pi i l}
\]

\[
= C^2_{b,\gamma} \frac{1 - e^{-\frac{2\pi}{b}}}{b - e^{-2\gamma a}} \mathcal{P}_\tau (e^{-2\gamma a \tau}).
\]

Obviously, for $a > 0$, $b > 0$ and $\gamma > 0$ we have $F(1) < F(\tau) < F(0)$ and, hence,

\[
A = C^2_{b,\gamma} \frac{1 - e^{-\frac{2\pi}{b}}}{b - e^{-2\gamma a}} e^{-2\gamma a} \quad \text{and} \quad B = C^2_{b,\gamma} \frac{1 - e^{-\frac{2\pi}{b}}}{b - e^{-2\gamma a}}.
\]

Assuming $(ab)^{-1} = n(\in \mathbb{N})$ and using the explicit expression for $C_{b,\gamma}$, this simplifies to

\[
(7.2) \quad A = n \frac{2\gamma a}{1 - e^{-2\gamma a}} e^{-2\gamma a} \quad \text{and} \quad B = n \frac{2\gamma a}{1 - e^{-2\gamma a}}.
\]

We may include the case of the box function ($\gamma = 0$) by a limiting procedure and by l’Hôpital’s rule we have $A = B = n$. As we always fix $(ab)^{-1}$, we usually have one free parameter to optimize the frame bounds. However, this time we consider a family of windows parametrized by $\gamma \geq 0$. We could also allow $\gamma \in \mathbb{R}$, but for $\gamma < 0$, we have the inequalities $F(0) < F(\tau) < F(1)$ and the frame bounds are

\[
A = n \frac{2\gamma a}{e^{-2\gamma a} - 1} \quad \text{and} \quad B = n \frac{2\gamma a}{e^{-2\gamma a} - 1} e^{-2\gamma a}, \quad \gamma < 0.
\]

By a simple flipping argument (see Figure 5) and a translation we may restrict to $\gamma \geq 0$. 


Figure 6. The family of windows \( g_{b,\gamma} \) for \( \gamma = 0 \) (black box function), \( \gamma = 2 \) (gray) and \( \gamma = -2 \) (gray, dashed). Clearly, the substitution \( \gamma \mapsto -\gamma \) does not change the frame bounds (just flip and translate all \( f \in L^2(\mathbb{R}^d) \)). The scaling of the axis has intentionally been omitted as we obtain the same (scaled) picture for any lattice parameter \( b \).

7.1.1. The lower frame bound. We fix the oversampling rate \( n \) and consider the lower frame bound \( A \) as a function of \( \gamma \). Taking the derivative yields the expression

\[
A'(\gamma) = -2an \frac{e^{2a\gamma}(2a\gamma - 1) + 1}{(e^{2a\gamma} - 1)^2} < 0.
\]

Verifying that the above expression is indeed negative is a simple exercise: to show that the numerator is positive, just observe that \( e^x > (1 + x) \), which is its linearization at 0. Thus, numerator and denominator are positive and there is a minus sign in front.

7.1.2. The upper frame bound. With a very similar observation we also obtain that

\[
B'(\gamma) = 2an \frac{e^{2a\gamma}(e^{2a\gamma} - 1) - 2a\gamma}{(e^{2a\gamma} - 1)^2} > 0.
\]

Figure 7. Frame bounds of the Gabor systems \( G(g_{\gamma}, aZ \times bZ) \) for \( (ab)^{-1} = n \in \{1, 3, 6\} \) and \( a = 1, b = 1/n \) and \( a = b = \sqrt{1/n} \). For \( \gamma = 0 \), we always have a tight frame with bounds \( A = B = n \). The upper frame bounds (black) are increasing with \( \gamma \) whereas the lower frame bounds (gray, dashed) are decreasing.
From Figure 7 one may get the impression that the bounds deteriorate slower for the square lattice than for rectangular lattices. This is, however, only the case as long as \( a > b \), which we deduce from (7.2).

7.1.3. The condition number. Finally, we easily observe that the condition number is
\[
\frac{B}{A} = e^{2a\gamma}.
\]
We give some concluding remarks for the parameters of the family of the window \( g_{b,\gamma} \). Fixing any rectangular lattice of density \( n \), we see that the box function is actually optimal within the family of localization windows \( g_{b,\gamma} \). Thus, the additional decay without additional smoothness in the time variable yields to even worse behavior in the frequency variable. Therefore, the frame condition is only made worse by increasing the decay parameter \( \gamma \).

From (7.2) it is obvious that the decay parameter \( \gamma \) and the lattice parameter \( a \) are exchangeable for our computations. So, on the other hand, for any fixed \( \gamma \) the lower bound decreases and the upper bound increases as \( a \) grows. For \( a \to 0 \) we obtain a tight frame. This may be interpreted in a distributional sense: For \( b \to \infty \) (\( \leftrightarrow a \to 0 \)), the function \( g_{b,\gamma} \) tends to (a multiple of) the Dirac delta. So, even though we use an infinitely wide step size in the frequency domain, we know the function at every instance in time. The frame operator is therefore (a multiple of) the identity operator. In particular, an optimal lattice in terms of the frame bounds (or the frame condition number) does not exist.

7.2. Support \( 2/b \). Similarly to the previous case, we study frame bounds of cut-off exponentials, but this time with support in \([0, 2/b]\) for the lattice \( \Lambda_{a,b} = a\mathbb{Z} \times b\mathbb{Z} \) of density \((ab)^{-1} > 1\).

Again, the adjoint lattice \( \frac{b}{2a} \mathbb{Z} \times \frac{a}{2b} \mathbb{Z} \) is used to compute the frame bounds. Thus, the support condition ensures that there is only influence from the left- and right-shifted copies of the (modulated) window(s) in the computations. The arising double series simplifies to a finite sum of 3 terms of simple series. We may use (more or less) the same notation as in the previous section and denote the window (see Figure 8) by
\[
g_{b/2,\gamma}(t) = C_{b/2,\gamma} e^{-\gamma t} \chi_{[0, 2/b]}(t),
\]
where \( C_{b/2,\gamma} \) is the normalizing factor, which is explicitly given by \( C_{b/2,\gamma} = \sqrt{2\gamma/(1 - e^{-4\gamma/b})} \).

We skip the computational details this time and refer to Janssen [35, Sec. 3] where the bounds have been computed explicitly. The only difference is that we again use the normalizing factor, to only compare windows of the same energy. The frame bounds are then
\[
A = C_{b/2,\gamma}^2 \frac{a(1 - e^{-2\gamma/b})(1 - e^{-\gamma/b})^2}{1 - e^{-2\gamma a}} e^{-2a\gamma} \quad \text{and} \quad B = C_{b/2,\gamma}^2 \frac{a(1 - e^{-2\gamma/b})(1 + e^{-\gamma/b})^2}{1 - e^{-2\gamma a}}.
\]
By using the explicit expression for \( C_{b/2,\gamma} \) and the fact that \((ab)^{-1} = n\), we get
\[
A = \frac{2\gamma a(1 - e^{-2\gamma n a})(1 - e^{-\gamma n a})^2}{(1 - e^{-2\gamma a})(1 - e^{-4\gamma n a})} e^{-2a\gamma} \quad \text{and} \quad B = \frac{2\gamma a(1 - e^{-2\gamma n a})(1 + e^{-\gamma n a})^2}{(1 - e^{-2\gamma a})(1 - e^{-4\gamma n a})}.
\]
For \( \gamma \to 0 \), the bounds are \( A = 0 \) and \( B = 2 \). For \( \gamma \to \infty \) we obtain \( A = 0 \) and \( B = \infty \). So, in both limiting cases we do not have a frame. This stands in contrast to the previous case, where we had a tight frame (orthonormal basis) with \( A = B = n \) for the integer lattice \( n^{-1/2}\mathbb{Z} \times n^{-1/2}\mathbb{Z} \) and \( \gamma = 0 \) (the box function).
7.2.1. \textit{The lower bound.} Our next goal is to determine optimizing parameters for the lower bound. We start with re-writing the expression as

\[ A = \gamma a \left( \frac{\coth(\gamma a)}{f(\gamma)} - 1 \right) \left( \frac{1 - \sech(\gamma na)}{h(\gamma)} \right) . \]

Determining critical points with respect to \( \gamma \) is equivalent to finding points where the sum of the logarithmic derivatives of \( f \) and \( h \) vanishes (because \( f \) and \( h \) are positive) or where:

\[ \frac{f'(\gamma)}{f(\gamma)} = -\frac{h'(\gamma)}{h(\gamma)}. \quad (7.3) \]

We compute

\[ \frac{f'(\gamma)}{f(\gamma)} = -a + \frac{1}{\gamma} - a \coth(\gamma a) = -a + \frac{1}{\gamma} \left( 1 - \gamma a \coth(\gamma a) \right), \]

which is a negative and decreasing function of \( \gamma \) (in fact it is less than \(-a\)). On the other side, we compute

\[ \frac{h'(\gamma)}{h(\gamma)} = na \coth(\gamma a) \sech(\gamma na). \]

This is the product of two positive, decreasing functions and hence positive and decreasing. It follows that the right-hand side of (7.3) is negative and increasing and therefore (7.3) has at most one solution. Considering the (directional) limits

\[ \lim_{x \to 0^+} \coth(x/2) \sech(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} \coth(x/2) \sech(x) = 0 \]

we see that (7.3) must have exactly one solution and so the lower bound has a unique maximum (see Figure 9). The optimal parameter \( \gamma \), which is the unique solution of (7.3) depends on the lattice geometry (the parameter \( a \) and hence also \( b \)) and the density \( n \) of the lattice. We note that, from a computational point of view, the decay parameter \( \gamma \) and the lattice parameter \( a \) are exchangeable. Hence, we also find that there is a unique lattice maximizing the lower frame bound. The lattice geometry, i.e., the parameter \( a \), depends on the density and the decay parameter \( \gamma \) and is (for any fixed \( n \) and \( \gamma \)), the unique solution of \( f'(a)/f(a) = -h'(a)/h(a) \).
7.2.2. The upper bound. We will repeat the same kind of calculations for the upper frame bound. We re-write it as

$$B = \gamma a (\coth(\gamma a) + 1) \left(1 + \text{sech}(\gamma na)\right).$$

Any critical points must satisfy the equation

$$\frac{\tilde{f}'(\gamma)}{\tilde{f}(\gamma)} = -\frac{\tilde{h}'(\gamma)}{\tilde{h}(\gamma)}.$$  \hspace{1cm} (7.4)

The left-hand side of (7.4) is independent of $n$ and is given by

$$\frac{\tilde{f}'(\gamma)}{\tilde{f}(\gamma)} = a + \frac{1}{\gamma} - a \coth(\gamma a) = a + a \left(\frac{1}{\gamma a} - \coth(\gamma a)\right),$$

which is positive and decreasing (in fact it is less than $a$). The right-hand side is given by

$$\frac{\tilde{h}'(\gamma)}{\tilde{h}(\gamma)} = n \text{sech}(\gamma na) \tanh\left(\frac{\gamma na}{2}\right).$$

This function is positive and we may use the logarithmic derivative to check the sign of the derivative. We obtain, after some simplifications by using identities for hyperbolic functions,

$$\partial_\gamma \left[\log \left(n \text{sech}(\gamma na) \tanh\left(\frac{\gamma na}{2}\right)\right)\right] = na (\text{csch}(\gamma na) - \tanh(\gamma na)), $$

which has a unique 0. Since $-\tilde{h}'(\gamma)/\tilde{h}(\gamma)$ has only 1 critical point, we conclude that this is the unique global maximum by the asymptotic behavior. As the right-hand side of (7.4) decays by an exponential factor faster than the left-hand side, we conclude that, depending on $n$, (7.4) has either 0, 1 or 2 solutions (see Figure 10). We show that for $n \in \{1, 2\}$, there is no solution and for $n \geq 3$ we find 2 solutions: dividing (7.4) by $a > 0$ leaves us with the equation

$$1 + \frac{1}{a} - \coth(\gamma a) = n \text{sech}(\gamma na) \tanh\left(\frac{\gamma na}{2}\right),$$

which is a transcendental equation that can be solved numerically.
where the left-hand side is decreasing and at most 1 and where the right-hand side has a unique
global maximum. The point of the maximum is attained where the logarithmic derivative
vanishes. This brings us to solving the equation
\[ \text{csch}(\gamma na) = \tanh(\gamma na) \iff \cosh(\gamma na)^2 - \cosh(\gamma na) - 1 = 0. \]
The quadratic polynomial in \( \cosh \) has exactly one positive so-
lution, namely:
\[ \gamma na = \arccosh \left( \frac{1 + \sqrt{5}}{2} \right) \approx 1.06128 \ldots \]

![Figure 10. Illustration of the two sides of equation (7.5) for fixed \( a = 2 \) and \( n \in \{1, 2, 3, 4\} \). The illustration suggests that for any \( n \geq 3 \) we have two solutions in \( \gamma \) for equation (7.5).](image)

We set \( y = \gamma na \) and to imply the correctness of (7.5) we want to determine those \( n \) where
\[ n \text{ sech}(y) \tanh\left( \frac{y}{2} \right) > 1. \]
Note that the solution depends on the product \( \gamma na \) and so the optimal parameter \( \gamma \)
depends on the lattice geometry (the parameter \( a \)) and the lattice density (the parameter \( n \)). We evaluate \( \text{sech}(y) \tanh(y) \) close to its maximum, i.e., close to \( y = \arccosh \left( \frac{1 + \sqrt{5}}{2} \right) \). To simplify computations, we set \( y = 1 \), which is sufficiently close to the point of interest. After an elementary manipulation of the last inequality, this leaves us with solving
\[ n \tanh\left( \frac{1}{2} \right) > \cosh(1), \]
which could be done numerically, but also with comparable effort analytically. We want to
show that, for \( n \) sufficiently large,
\[ n \tanh\left( \frac{1}{2} \right) = n - \frac{2n}{e + 1} > e + e^{-1} = \cosh(1). \]
It is not difficult to establish \( e + 1 > 3.7 \) and \( 3.5 > e + e^{-1} \). So, it suffices to find \( n \) such that
\[ 2n - \frac{4n}{3.7} > 3.5, \]
which holds for all \( n \geq 4 \). Hence, for \( n \geq 4 \) the upper bound \( B \) (as a function of \( \gamma \)) has
two critical points. By the behavior of the bound, the smaller value yields a local maximum
and the larger value a minimum, which may be global, depending on \( n \). For \( n \in \{1, 2, 3\} \) we
evaluate both sides of (7.5) with Mathematica at \( \arccosh\left( \frac{1 + \sqrt{5}}{2} \right) \) and see that (7.4) does not
have a solution for \( n \in \{1, 2\} \) and two solutions for \( n = 3 \). Again, \( \gamma \) may be exchanged for \( a \).
So, for \( n \in \{0, 1\} \) we find that there is no optimal lattice minimizing \( B \) (the minimizing lattice degenerates) and for \( n \geq 3 \) there exist two critical lattices, one which is a local maximizer and the other a (local) minimizer, which is the only candidate for the global minimizer. As \( B(0) = 2 \), the minimizer is global if its value is less than 2.

7.2.3. The condition number. What is left to study is the condition number, which is simply given by the expression

\[
\frac{B}{A} = e^{2\gamma a} \coth \left( \frac{\gamma n a}{2} \right)^2.
\]

Taking the derivative with respect to \( \gamma \) and looking for critical points leads to the equation

\[
\frac{2a e^{2\gamma a} (e^{2n\gamma a} - n e^{n\gamma a} - 1)}{(e^{\gamma n} - 1)^3} = 0 \iff e^{2n\gamma a} - n e^{n\gamma a} - 1 = 0.
\]

Setting \( x = e^{n\gamma a} \) leaves us with solving a simple quadratic equation: \( x^2 - nx - 1 = 0 \). There is only one positive solution to this equation, which is \( x = (n + \sqrt{n^2 + 4})/2 \). Therefore, the critical points of the condition number are attained for

\[
\gamma = \frac{\log \left( n + \sqrt{n^2 + 4} \right) - \log(2)}{a n}.
\]

Figure 11. Left: Optimal rectangular lattice for the parameters \( \gamma = 2 \) and density \( n = 9 \). Right: curves of optimal parameters \((a, \gamma)\) for different lattice densities \( n \in \{1, 2, 3, 4, 5, 6\} \).

Note that we optimize with respect to the decay parameter \( \gamma \) and aim to find the best parameter for a given rectangular lattice. The parameter depends on the lattice geometry, as well as on the lattice density. Interestingly, the dependence on the lattice parameter \( a \) is the same as for the decay parameter \( \gamma \), as can easily be seen from (7.6). Hence, taking the
derivative with respect to $a$ and looking for critical points we only need to exchange $a$ and $\gamma$ in the above calculations. This yields that the only critical point is attained for

$$a = \frac{\log \left( n + \sqrt{n^2 + 4} \right) - \log(2)}{\gamma n}.$$ 

So, the optimal lattice depends on the decay parameter and the density. Combining the two results, we see that there is a curve in the space of parameters $(a, \gamma)$ describing the optimal decay and lattice conditions in dependence of the oversampling rate $n$ (see Figure 11):

$$a \gamma = \frac{\log \left( n + \sqrt{n^2 + 4} \right) - \log(2)}{n}.$$ 

Lastly, we easily deduce from (7.6) that $B/A$ is decreasing as a function of $n \in \mathbb{N}$ and that

$$\frac{B}{A} \to e^{2\gamma a} \quad \text{for} \quad n \to \infty.$$

So, for general $\gamma > 0$ and non-degenerate lattice ($a > 0$) the frame operator does not converge to a (multiple of) the identity operator, which however holds for any window $g \in S_0(\mathbb{R})$ [19]. This fact seems to be accompanied by the results in [18] (compare also [23, p. 132]). Also, we remark that exchanging the parameters $a$ and $\gamma$ comes from the algebraic expressions in the calculations and is not a dilation result obtained by applying a metaplectic operator.

8. The one-sided exponential

In this section we study frame bounds of one-sided exponentials. The lattice parameters here satisfy $(ab)^{-1} = n \in \mathbb{N}$, in particular, we do obtain a frame at the critical density. The function of interest is

$$g_\gamma(t) = \sqrt{2}\gamma e^{-\gamma t} \chi_{(0,\infty)}(t), \quad t \in \mathbb{R},$$

where $\gamma > 0$ is an additional dilation parameter. Considering the dependencies as presented in [35], we will express the lattice pair again as $(\eta/n, 1/\eta)$, $\eta > 0$. Then the frame bounds are illustrated in Figure 12 and given by

$$A_\gamma(\frac{\eta}{n}, \frac{1}{\eta}) = 2\gamma \eta \tanh(\frac{\eta}{2}) \operatorname{csch}(\frac{\eta}{n}) e^{-\gamma \frac{\eta}{n}} = A_1(\frac{2\gamma}{\eta}, \frac{1}{\eta}),$$

$$B_\gamma(\frac{\eta}{n}, \frac{1}{\eta}) = 2\gamma \eta \coth(\frac{\eta}{2}) \operatorname{csch}(\frac{\eta}{n}) e^{\gamma \frac{\eta}{n}} = B_1(\frac{2\gamma}{\eta}, \frac{1}{\eta}).$$

The formulas make it evident that it suffices to focus on the case $\gamma = 1$. A more sophisticated argument would be to use the interplay between the symplectic and the metaplectic group [20, 21, 23] to reduce to the case $\gamma = 1$.

8.1. The condition number. This time, we start with the condition number which has an obvious minimizing lattice. We easily compute

$$\kappa(\frac{\eta}{n}, \frac{1}{\eta}) = \left( \coth(\frac{\eta}{2}) e^{\frac{\eta}{n}} \right)^2,$$

which is minimal, if and only if $\sqrt{\kappa}$ is minimal. We compute the derivative of $\sqrt{\kappa}$:

$$\frac{\partial}{\partial \eta} \left( \coth(\frac{\eta}{2}) e^{\frac{\eta}{n}} \right) = \frac{1}{n} e^{\frac{\eta}{n}} \operatorname{csch}^2(\frac{\eta}{2}) \left( \sinh(\eta) - n \right).$$
The sign of the derivative is determined by sinh, which is a strictly monotonically increasing, unbounded, and attains 0 at 0. Therefore, the unique minimum of the condition number $\kappa$ is attained at $\eta = \arcsinh(n)$. Next, we will show that, this point lies between the maximizer of the lower and the minimizer of the upper bound.

8.2. The lower bound. For $\gamma = 1$, we start with computing the first derivative of the lower bound $A = A_1$ and show that it has a unique zero.

$$
\frac{\partial}{\partial \eta} A\left(\frac{n}{\eta}, \frac{1}{\eta}\right) = \tanh\left(\frac{n}{\eta}\right) e^{-\frac{n}{\eta}} \text{csch}\left(\frac{n}{\eta}\right) - \frac{n}{\eta} \tanh\left(\frac{n}{\eta}\right) e^{-\frac{n}{\eta}} \text{csch}\left(\frac{n}{\eta}\right) \coth\left(\frac{n}{\eta}\right)
$$

$$
= e^{-\frac{n}{\eta}} \text{csch}\left(\frac{n}{\eta}\right) \tanh\left(\frac{n}{\eta}\right) \left(1 + \eta \text{csch}\left(\frac{n}{\eta}\right) - \frac{1}{\eta} - \frac{n}{\eta} \coth\left(\frac{n}{\eta}\right)\right),
$$

where the last equality is due to the addition theorems. Furthermore, the function

$$
\eta \mapsto 1 + \eta \text{csch}(\eta) - \frac{n}{\eta} - \frac{n}{\eta} \coth\left(\frac{n}{\eta}\right), \quad n \in \mathbb{N},
$$

is a sum of strictly monotonically decreasing functions (see again Section 5.1). We evaluate this expression at the point $\eta = 0.5$, which gives

$$
1 + 0.5 \text{csch}(0.5) - \frac{0.5}{0.5} - \frac{0.5}{0.5} \coth\left(\frac{0.5}{0.5}\right) \geq 1 + 0.5 \cdot \text{csch}(0.5) - \frac{0.5}{0.5} - \frac{0.5}{0.5} \coth\left(\frac{0.5}{0.5}\right) > 0.
$$

On the other hand, the function is unbounded from below.

$$
\lim_{\eta \to \infty} \eta \text{csch}(\eta) - \frac{n}{\eta} - \frac{n}{\eta} \coth\left(\frac{n}{\eta}\right) = -\infty.
$$

Therefore, there exists $\eta_{A,n} > 0$ such that the lower bound $A\left(\frac{n}{\eta_{A,n}}, \frac{1}{\eta_{A,n}}\right)$ has a unique maximum in $\eta_{A,n}$. We check whether it is $\eta = \arcsinh(n)$, which we know optimizes the condition number:

$$
1 + \eta \text{csch}(\eta) - \frac{n}{\eta} - \frac{n}{\eta} \coth\left(\frac{n}{\eta}\right) = 1 + \frac{n}{\eta} - \frac{n}{\eta} - \frac{n}{\eta} \coth\left(\frac{n}{\eta}\right) = 1 - \frac{n}{\eta} \coth\left(\frac{n}{\eta}\right) < 0.
$$
It follows that $\eta_{A,n} < \arcsinh(n)$. It is also simple to see that $(\eta_{A,n})_{n\in\mathbb{N}}$ is strictly monotonically increasing: $\eta_{A,n}$ is the unique solution to

$$1 + \eta \operatorname{csch}(\eta) = \frac{\eta}{n} + \frac{n}{\eta} \coth\left(\frac{\eta}{n}\right).$$

The left-hand side is independent of $n$ whereas the right-hand side is strictly decreasing in $n$.

8.3. The upper bound. We deal with the upper bound in the same manner and set $B = B_1$.

$$\frac{\partial}{\partial \eta} B\left(\frac{\eta}{n}, \frac{1}{\eta}\right) = \frac{\eta}{n} \coth\left(\frac{\eta}{n}\right) e^{\frac{n}{\eta}} \operatorname{csch}\left(\frac{\eta}{n}\right) - \eta \frac{1}{2} \operatorname{csch}\left(\frac{\eta}{n}\right)^2 e^{\frac{n}{\eta}} \operatorname{csch}\left(\frac{\eta}{n}\right)$$

$$+ \frac{n}{\eta} \coth\left(\frac{\eta}{n}\right) \frac{n}{\eta} \operatorname{csch}\left(\frac{\eta}{n}\right) - \frac{n}{\eta} \coth\left(\frac{\eta}{n}\right) e^{\frac{n}{\eta}} \operatorname{csch}\left(\frac{\eta}{n}\right) \coth\left(\frac{\eta}{n}\right).$$

$$= e^{\frac{n}{\eta}} \operatorname{csch}\left(\frac{\eta}{n}\right) \coth\left(\frac{\eta}{n}\right) \left(1 - \eta \frac{\operatorname{csch}\left(\frac{\eta}{n}\right)^2}{2 \coth\left(\frac{\eta}{n}\right)} + \frac{n}{\eta} - \frac{n}{\eta} \coth\left(\frac{\eta}{n}\right)\right)$$

$$= e^{\frac{n}{\eta}} \operatorname{csch}\left(\frac{\eta}{n}\right) \coth\left(\frac{\eta}{n}\right) \left(1 - \eta \operatorname{csch}(\eta) + \frac{n}{\eta} - \frac{n}{\eta} \coth\left(\frac{\eta}{n}\right)\right),$$

where the last equality is due to the addition theorems. We again analyze the expression in the brackets. To begin with, $-x \operatorname{csch}(x)$ is a strictly increasing function. For the rest, we set

$$\psi(x) = 1 + x - x \coth(x).$$

This is also a strictly monotonically function. In fact, it is strictly concave.

$$\psi'(x) = 1 - \coth(x) + x \operatorname{csch}(x)^2,$$

$$\psi''(x) = \operatorname{csch}(x)^2 + \operatorname{csch}(x)^2 - 2x \operatorname{csch}(x)^2 \coth(x) = 2 \operatorname{csch}(x)^2 (1 - x \coth(x)) < 0.$$

This means that the first derivative is strictly decreasing. It is strictly positive, because

$$\lim_{x \to \infty} \psi'(x) = 1 - 1 + 0 = 0.$$

We now ensure that there is a critical point.

$$\lim_{\eta \to 0^+} 1 - \eta \operatorname{csch}(\eta) + \frac{n}{\eta} - \frac{n}{\eta} \coth\left(\frac{\eta}{n}\right) = 1 - 1 + 0 - 1 = -1 < 0,$$

and if we evaluate the same expression at $2n$, we get

$$1 - 2n \operatorname{csch}(2n) + \frac{2n}{n} - \frac{2n}{n} \coth\left(\frac{2n}{n}\right) \geq 3 - 2 \coth(2) - 2 \operatorname{csch}(2) > 0.$$

We could again check whether $\eta = \arcsinh(n)$ is a critical point. From the general theory, though, we already know that it has to be between $\eta_{A,n}$ and $\eta_{B,n}$ and we have already established that it is greater than $\eta_{A,n}$.

9. The two-sided exponential

The last section is dedicated to the two-sided exponential. At first sight, the bounds look quite similar to those already analyzed. However, this one has proven to be the most challenging to inspect. The current window function is

$$g_{\gamma}(t) = \sqrt{\gamma} e^{-\gamma|t|}, \quad t \in \mathbb{R},$$

where $\gamma > 0$ is an additional parameter. Unlike the one-sided exponential, the two-sided one is in Feichtinger’s algebra, so we do not have a frame a the critical density. It has been shown
in \([35]\) that we obtain a frame for densities larger than 1. With the same parametrization \((\eta/n, 1/\eta)\) as in the previous sections, the frame bounds are given by

\[
A_{\gamma}(\frac{\eta}{n}, \frac{1}{\eta}) = \tanh\left(\frac{\gamma}{2}\eta\right)\left(\frac{\gamma}{n}\csch\left(\frac{\gamma}{n}\eta\right) - \gamma\eta\csch(\gamma\eta)\right) = \frac{1}{n}A_1(\frac{2\eta}{n}, \frac{1}{\gamma\eta}),
\]

\[
B_{\gamma}(\frac{\gamma}{n}, \frac{1}{\gamma}) = \coth\left(\frac{\gamma}{2}\eta\right)\left(\frac{\gamma}{n}\coth\left(\frac{\gamma}{n}\eta\right) + \gamma\eta\csch(\gamma\eta)\right) = \frac{1}{n}B_1(\frac{2\eta}{n}, \frac{1}{\gamma\eta}).
\]

Just as for the one-sided exponential, we may simply reduce to the case that \(\gamma = 1\).

When one looks at the graphs of the lower or the upper bound for a fixed density in Figure 13 it seems obvious that there is a unique extremum. However, the first and second derivative of the concerned bound are somewhat difficult to analyze over \((0, \infty)\), as they are combinations of products and sums which exhibit different behaviors at different intervals. The visually “clear” situation is due to the interaction altogether, so dissecting each part separately leads to unsatisfactory estimates. Therefore, to determine the global extrema and to show their uniqueness, we will first exclude most of the positive half-axis through interval estimates (see Section 5.2 and Figure 14) as yielding potential global extrema. Then for the lower bound, upper bound and condition number we will establish certain concavity, convexity and log-convexity results, respectively, on the small remaining interval. These will allow us to conclude that the extremum is unique in each case. For the lower bound, the cases for small \(n\), which is \(3 \leq n \leq 10\), and in particular \(n = 3\), will need more attention at some points compared to large \(n\). The case \(n = 2\) needs to be handled completely separately, since it does not agree with the asymptotic positions of the extremal points. In the sequel, we will set

\[
f(t) = t\csch(t), \quad \forall t > 0,
\]

so as to have more compact expressions (the properties of \(f\) were discussed in Section 5.1).

9.1. The lower bound. The initial reference point will be \(\eta_n = 2\text{arccosh}(n)\). We will first compare the value attained here with the function’s values on \((0, \eta_n/2]\) and \([2\eta_n, \infty)\) in order to exclude global maxima in these intervals. Then we will show that there is a unique
maximum in \((\eta_n/2, 2\eta_n)\). By using (9.2) and the formulas in Section 5.1, we write the lower frame bound evaluated at \(\eta_n = 2\arccosh(n)\) as

\[
A\left(\frac{\eta_n}{n}, \frac{1}{\eta_n}\right) = (n^2 - 1)^{1/2} \left(f\left(\frac{\eta_n}{n}\right) - \frac{\eta_n}{2n(n^2 - 1)^{1/2}}\right) = (n^2 - 1)^{1/2} f\left(\frac{\eta_n}{n}\right) - \arccosh\left(\frac{n}{2}\right).
\]

Note that we omitted the factor \(1/n\) this time, which needs to be taken into account later.

9.1.1. The left cut-off. We will show that, for any \(3 \leq n \in \mathbb{N}\) the value of the lower bound achieved at \(\eta_n = 2\arccosh(n)\) is larger than any value attained in \((0, \eta_n/2]\). However, for small \(n\) we will later only be able to show a concavity result for slightly smaller intervals (depending on \(n\)). The needed intervals are given below and we use interval estimates, as presented in Section 5.2, to show that the maximum is still not attained on the mildly larger interval.

Using the expression from (9.1) for \(A\) evaluated at \(\eta_n\) in combination with the notation (9.2), we have for all \(\eta \leq \eta_n/2\) that

\[
\frac{1}{n} A\left(\frac{\eta_n}{n}, \frac{1}{\eta_n}\right) < \tanh(\eta_n/4) \left(1 - f(\eta_n/2)\right) = \left(1 - \frac{1}{\cosh(\eta_n/4)^2}\right)^{1/2} \left(1 - \frac{\eta_n}{2(n^2 - 1)^{1/2}}\right) = \left(1 - \frac{2}{n+1}\right)^{1/2} \left(1 - \frac{\eta_n}{2(n^2 - 1)^{1/2}}\right).
\]

Using (9.3) (now including the factor \(1/n\)), the desired inequality is

\[
\left(1 - \frac{2}{n+1}\right)^{1/2} \left(1 - \frac{\eta_n}{2(n^2 - 1)^{1/2}}\right) < \left(\frac{n^2 - 1}{n}\right)^{1/2} f\left(\frac{\eta_n}{n}\right) - \frac{\arccosh(n)}{n^2}
\]

\[
\iff \left(1 - \frac{2}{n+1}\right)^{1/2} - \frac{n^2 - 1}{n} f\left(\frac{\eta_n}{n}\right) < \arccosh\left(\frac{n}{2}\right) \left(1 - \frac{2}{n+1}\right)^{1/2} \left(1 - \frac{1}{(n^2 - 1)^{1/2}}\right) - \frac{1}{n^2}.
\]

We claim that the left-hand side is negative and the right-hand side is positive.

\[
\left(1 - \frac{2}{n+1}\right)^{1/2} - \frac{n^2 - 1}{n} f\left(\frac{\eta_n}{n}\right) < 0 \iff \frac{n}{n+1} - f\left(\frac{\eta_n}{n}\right) < 0.
\]
For all $n \geq 3$, it would suffice to show
\[
1 - \frac{3}{4n} < f\left(\frac{2n}{t}\right) \iff \left(1 - \frac{3}{4n}\right) n - \text{csch}\left(\frac{2n}{n}\right) < 0.
\]

The left side of the last inequality is strictly monotonically increasing. Recall the properties of $\rho(n) = \eta_n / n$ defined by (5.1) in Section 5.1.
\[
\frac{\partial}{\partial n} \left[ \left(1 - \frac{3}{4n}\right) \frac{n}{\eta_n} - \text{csch}\left(\frac{2n}{n}\right) \right]
= \frac{3}{4n^2} \frac{n}{\eta_n} + \left(1 - \frac{3}{4n}\right) \frac{n}{\eta_n} - \frac{n}{\eta_n} - \frac{2n}{n^2} \frac{\eta_n - 2n(n^2 - 1)^{-1/2}}{n^2} \text{csch}'\left(\frac{2n}{n}\right)
= \frac{3}{4n} + \frac{n}{\eta_n} - \frac{2n}{n^2} (n^2 - 1)^{-1/2} \frac{n^2}{n^2} \text{csch}'\left(\frac{2n}{n}\right) + 1 - \frac{3}{4n}.
\]

The only part whose positivity has not been proven stands in the brackets. Keeping in mind that $\eta_n / n \leq \eta_3 / 3 \leq 1.2$,
\[
\frac{n^2}{n^2} \text{csch}'\left(\frac{2n}{n}\right) + 1 - \frac{3}{4n} = 1 - \frac{3}{4n} - \frac{n^2}{n^2} \text{csch}'\left(\frac{2n}{n}\right) \frac{\eta_n}{n} \text{coth}\left(\frac{2n}{n}\right) - 1.
\]

The atomic parts of the expression are all positive, so in total, the expression is strictly monotonically increasing. Evaluating at $n = 3$ shows that the expression is always positive, therefore $\left(1 - \frac{3}{4n}\right) \frac{n}{\eta_n} - \text{csch}\left(\frac{2n}{n}\right)$ is strictly monotonically increasing in $n$. Due to the fact that $1/t - 1/\sinh(t) \to 0$, for $t \to 0$ (see Section 5.1) we have
\[
\lim_{n \to \infty} \left(1 - \frac{3}{4n}\right) \frac{n}{\eta_n} - \text{csch}\left(\frac{2n}{n}\right) = \lim_{n \to \infty} \frac{n}{\eta_n} - \text{csch}\left(\frac{2n}{n}\right) - \frac{3}{4n} = 0 - 0 = 0,
\]

implying that the desired expression is negative. For the other side of the initial inequality,
\[
\left(1 - \frac{2}{n+1}\right)^{1/2} \frac{1}{(n^2 - 1)^{1/2}} < \frac{1}{n^2} \iff \frac{1}{n+1} - \frac{1}{n^2} > 0.
\]

All in all, we have shown that the maximum of $A$ cannot be attained in the interval $(0, \eta_n/2]$. Now, as mentioned above, we will need to improve our estimates for $n = 3, \ldots, 10$ in order to show the necessary concavity/convexity later on.

- $(4 \leq n \leq 10)$: we compare the value of $\tanh(\eta/2)(1 - f(\eta))$ (which majorizes to the lower bound) at $\eta = 3.08$ with that of $A(\eta_n / n, 1/\eta_n) / n$. We see that there is no maximum in $(0, 3.08] \ni (0, \eta_n/2]$.
- $(n = 3)$: this case is a bit more subtle. Here, we perform an interval estimate on $(1.7, 3.3) \ni (\eta_3, 2, 3.3)$ with sub-intervals of length 0.3 to see that the global maximum does not lie in the interval $(0, 3.3]$.

9.1.2. The right cut-off. Similarly to the left cut-off, we will show that for $3 \leq n \in \mathbb{N}$ the lower bound $A$ does not assume its global maximum on $[2\eta_n, \infty)$. This is again achieved by comparing with the value at $\eta_n$ and with extra estimates for small $n$. By using the properties of $f(t)$, defined by (9.2), as explained in Section 5.1 and that $\tanh(\eta) < 1$, we have
\[
\frac{1}{n} A_1\left(\frac{3n}{n}, \frac{1}{n}\right) < f\left(\frac{2n}{n}\right) = 2\frac{n}{n} \text{csch}\left(\frac{2n}{n}\right) = \frac{2n}{n} \text{csch}\left(\frac{2n}{n}\right) \text{sech}\left(\frac{2n}{n}\right) = f\left(\frac{2n}{n}\right) \text{sech}\left(\frac{2n}{n}\right).
\]
As already introduced in Section [5.1], we set \( \rho(n) = \frac{2\arccosh(n)}{n} \). This leaves us with showing the following inequality:

\[
\frac{(n^2 - 1)^{1/2}}{n} f(\rho(n)) - \frac{\rho(n)}{2n} > f(\rho(n)) \text{sech}(\rho(n))
\]

\[
\iff \left( \frac{(n^2 - 1)^{1/2}}{n} - \text{sech}(\rho(n)) \right) \rho(n) \text{csch}(\rho(n)) > \frac{\rho(n)}{2n}
\]

\[
\iff \left( (n^2 - 1)^{1/2} - n \text{sech}(\rho(n)) \right) \text{csch}(\rho(n)) > \frac{1}{2}
\]

\[
\iff (n^2 - 1)^{1/2} \cosh(\rho(n)) - n > \frac{\sinh(2\rho(n))}{4}.
\]

We want to show that

\[
(n^2 - 1)^{1/2} \cosh(\rho(n)) - n - \frac{\sinh(2\rho(n))}{4} > 0.
\]

We first assume that \( n \geq 10 \) and bound \( \cosh \) from below and \( \sinh \) from above. For \( \sinh \), we keep in mind that \( 2\rho(n) \leq 2\rho(10) < 1.2 \) (as \( n \geq 10 \)), and apply the bound from Section [5.1]

\[
(n^2 - 1)^{1/2} \cosh(\rho(n)) - n - \frac{\sinh(2\rho(n))}{4} > (n^2 - 1)^{1/2} \left( 1 + \frac{\rho(n)^2}{2} \right) - n - \frac{2\rho(n)^2 + (2\rho(n))^3}{4}
\]

\[
= (n^2 - 1)^{1/2} - n + (n^2 - 1)^{1/2} \frac{\rho(n)^2}{2} - \rho(n) - 4\rho(n)^3.
\]

We claim that \( (n^2 - 1)^{1/2} - n > -\frac{1}{n} \) and that the remaining part dominates \( \frac{1}{n} \). We compute

\[
\frac{\partial}{\partial n} \left[ (n^2 - 1)^{1/2} - n + 1/n \right] = \left( 1 + \frac{1}{(n^2 - 1)} \right)^{1/2} - 1 - \frac{1}{n^2} < 0,
\]

where the inequality holds if and only if

\[
1 + \frac{1}{(n^2 - 1)} < \left( 1 + \frac{1}{n^2} \right)^{2} = 1 + \frac{2}{n^2} + \frac{1}{n^4} \iff n^2 < (2 + \frac{1}{n^2})(n^2 - 1).
\]

Since \( n^2 < 2(n^2 - 1) < (2 + 1/n^2)(n^2 - 1) \) holds for all \( 2 \leq n \in \mathbb{N} \), (*) is strictly monotonically decreasing. To that, considering it tends to 0 as \( n \) tends to infinity, it is strictly positive. We are left to prove

\[
\frac{1}{n} < \rho(n) \left( (n^2 - 1)^{1/2} \frac{\rho(n)}{2} - 1 - 4\rho(n)^2 \right)
\]

\[
= \frac{2\arccosh(n)}{n} \left( \frac{(n^2 - 1)^{1/2}}{n} \arccosh(n) - 1 - 4\rho(n)^2 \right)
\]

\[
\iff 1 < 2 \arccosh(n) \left( \frac{1}{n} \arccosh(n) \right)^{1/2} - 1 - 4\rho(n)^2.
\]

In summary, for \( n \geq 10 \), the expression in the brackets is strictly monotonically increasing. Evaluating at 10 shows that it is also positive for all \( n \geq 10 \). Therefore, the right-hand side of the above inequality is strictly monotonically increasing. Finally, the evaluation at \( n = 10 \) also shows the claim. For the initial \( 3 \leq n \leq 9 \) we evaluated the initial inequality to verify it. Again, the intervals where we look for the maximum need to be updated for \( n \leq 10 \).
\(4 \leq n \leq 10\): we ensure that there is no global maximum in \([4n/3, 2\eta_n]\). One can check this with a direct interval estimate for \(5 \leq n \leq 10\). In the case \(n = 4\) we split it in the intervals \([16/3, 5)\) and \([5, 2\eta_4)\).

\((n = 3)\): we divide \([4.8, 7.1) \supset [4.8, 2\eta_3]\) in sub-intervals of length 0.1 to show that there is no global maximum.

Summing up all the cases from the left cut-off and right cut-off, we are left to show that there is a unique (global) maximum in the interval (see also Figure 15)

\[(9.4) \quad I_n = (\max\{3.08, \frac{4n}{3}\}, \min\{4n/3, 2\eta_n\}), \quad 4 \leq n \in \mathbb{N} \quad \text{or} \quad I_n = (3.3, 4.8), \quad n = 3.\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure15.png}
\caption{Illustration of the intervals \(I_n\) for \(n \in \{3, 4, 7, 10, 11, 20, 40\}\). We have the intervals \(I_3 = (3.3, 4.8), I_n = (3.08, 4n/3)\) for \(4 \leq n \leq 10\) and \(I_n = (\eta_n/2, 2\eta_n)\) for \(n \geq 11\).}
\end{figure}

9.1.3. The concavity result. We fix \(n \geq 3\) and set \(\psi_n(\eta) = \psi(\eta) = f(\eta/n) - f(\eta)\), where \(f(t) = t \text{csch}(t)\) is still defined by (9.2). The lower frame bound can then be expressed as

\[\frac{1}{n} \sqrt[n]{A(\frac{\eta}{n}, \frac{1}{\eta})} = \tanh\left(\frac{\eta}{n}\right) \psi(\eta).\]

We want to show that \(A\) has a unique maximum in \((\eta_n/2, 2\eta_n)\). As \(A\) is strictly positive, we may perform the analysis for \(\log(A)\), which has the same critical points. We compute

\[
\frac{\partial^2}{\partial \eta^2} \left[ \log \left( \frac{1}{n} A\left(\frac{\eta}{n}, \frac{1}{\eta}\right) \right) \right] = \frac{\tanh''(\frac{\eta}{n}) \tanh'(\frac{\eta}{n}) - \tanh'(\frac{\eta}{n})^2}{4 \tanh'(\frac{\eta}{n})^2} + \frac{\psi''(\eta) \psi(\eta) - \psi'(\eta)^2}{\psi(\eta)^2}.
\]

Since, \(\tanh\) is positive, increasing and concave and \(\psi\) is positive (see Section 5.1), it suffices to show that \(\psi\) is concave in order to have the whole expression negative. We compute its second derivative, which we may compactly write as

\[\psi''(\eta) = \frac{1}{n^2} f''\left(\frac{\eta}{n}\right) - f''(\eta), \quad \eta \in (\eta_n/2, 2\eta_n).\]

So, we need to know some properties of \(f''(t)\). We compute

\[(9.5) \quad f''(t) = -\frac{2t \text{csch}(t)}{\text{coth}(t)^2} \left( \frac{1}{2} \tanh(t)^2 + \frac{\tanh(t)}{t} - 1 \right).\]

Clearly, \(-2t \text{csch}(t)/\text{coth}(t)^2 < 0\). So, in order to determine where \(f''(t) < 0\), we only need to know where

\[\frac{1}{2} \tanh(t)^2 + \frac{\tanh(t)}{t} - 1 > 0.\]
We compute the derivative of this expression:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \tanh(t)^2 + \frac{\tanh(t)}{t} - 1 \right) = \tanh(t) \operatorname{sech}(t)^2 + \frac{\operatorname{sech}(t)^2 - \tanh(t)}{t^2} = \tanh(t) \left( \operatorname{sech}(t)^2 + \frac{2}{t \sinh(2t)} - \frac{1}{t^2} \right).
\]

An issue which appears is that we have a difference of two unbounded functions (close to \(t = 0\)). Yet, it is not as uncontrolled as it may seem at a first glance:

\[
\begin{align*}
\frac{2}{t \sinh(2t)} - \frac{1}{t^2} &= \frac{1}{t^2} \sum_{k=0}^{\infty} \frac{(2t)^{2k}}{(2k+1)!} - \frac{1}{t^2} = \frac{t^{-2} \left( 1 - \frac{(2t)^{2 \cdot 0}}{0!} \right) - 4(2t)^{-2} \sum_{k=1}^{\infty} \frac{(2t)^{2(k+1)}}{(2k+1)!}}{1 + \sum_{k=1}^{\infty} \frac{(2t)^{2k}}{(2k+1)!}} \\
&= \frac{4 \sum_{k=0}^{\infty} \frac{(2t)^{2k}}{(2k+3)!}}{1 + \sum_{k=1}^{\infty} \frac{(2t)^{2k}}{(2k+1)!}} = \frac{4}{3} + 4 \sum_{k=1}^{\infty} \frac{(2t)^{2k}}{(2k+1)!}
\end{align*}
\]

We now take the limit for \(t \to 0^+\):

\[
\lim_{t \to 0^+} \frac{2}{t \sinh(2t)} - \frac{1}{t^2} = \lim_{t \to 0^+} \frac{4}{3} + 4 \sum_{k=1}^{\infty} \frac{(2t)^{2k}}{(2k+1)!} = \frac{2}{3}
\]

From the above calculation, we also get

\[
\operatorname{sech}(t)^2 - 4 \sum_{k=0}^{\infty} \frac{(2t)^{2k}}{(2k+3)!} < \operatorname{sech}(t)^2 + \frac{2}{t \sinh(t)} - \frac{1}{t^2}.
\]

The minor \(\operatorname{sech}(t)^2 - 4 \sum_{k=0}^{\infty} \frac{(2t)^{2k}}{(2k+3)!}\) is strictly monotonically decreasing on \((0, \infty)\) and positive at \(t = 0.5\). On the other hand, due to the monotonicity of \(f\), we have for all \(t > 1.1\)

\[
\frac{2}{t \sinh(2t)} - \frac{1}{t^2} < -\frac{1}{2t^2} \iff \frac{2}{t \sinh(2t)} < \frac{1}{2t^2} \iff f(2t) < \frac{1}{2}.
\]

To that, for all \(t > 1.2\), we have

\[
\operatorname{sech}(t)^2 + \frac{2}{t \sinh(2t)} - \frac{1}{t^2} < \operatorname{sech}(t)^2 - \frac{1}{4t^2} < 0,
\]

or equivalently, \(2 < \cosh(t)/t\). With a standard curve discussion, one sees that \(\cosh(t)/t\) is strictly increasing on \((1.2, \infty)\). In summary, the expression in brackets in (9.5), i.e.,

\[(9.6) \quad \frac{1}{2} \tanh(t)^2 + \tanh(t) \cdot \frac{1}{t} - 1\]

is strictly monotonically increasing on the interval \((0, 0.5)\) and strictly monotonically decreasing on the interval \((1.2, \infty)\). By l’Hôpital’s rule, the value at \(t = 0\), is given by

\[
\lim_{t \to 0^+} \left( \frac{1}{2} \tanh(t)^2 + \frac{\tanh(t)}{t} - 1 \right) = 0 + \left( \lim_{t \to 0^+} \frac{\operatorname{sech}(t)^2}{t} \right) - 1 = \frac{1}{2} - 1 = 0.
\]
This shows that the expression (9.1) is strictly positive on (0, 0.5]. We claim that this is preserved on the interval (0.5, 1.2). For a useful interval estimate (with the indicated monotonicities in (9.0)), one should split it (0.5,1.2) in sub-intervals of length 0.01. Finally, we check that the expression is still strictly positive at 1.6 and less than 0 at 1.61 by evaluating it at these points. Strict negativity on (1.61,∞) follows from the monotonicity.

Looking back at \( f'' \), we see that it is strictly negative on (0,1.6] and strictly positive on [1.61,∞). Now, recall that

\[
\eta \in (\eta_n/2, 2\eta_n) \quad \text{with} \quad \eta_n = 2 \arccosh(n).
\]

We observe that \( \eta_n/2 \geq \eta_3/2 > 1.76 \). As a consequence, we have that \( f''(\eta) > 0 \) for all \( \eta \geq \eta_n/2 > 1.61 \). Next, recall that \( \rho(x) = 2 \arccosh(x)/x \) defined in (5.1) is strictly decreasing for \( x > 2 \) (see Section 5.1). Also, \( \eta/n < 2\rho(n) \) and \( 2\rho(7) < 1.51 \). In particular, \( f''(\eta/n) < 0 \) for all \( \eta \leq 2\eta_n \) and \( n \geq 7 \). Therefore, \( \psi''(\eta) < 0 \) for all \( n \geq 7 \). All in all, setting \( \tilde{A}(\eta) = \frac{1}{n} A(B_n, \frac{1}{n}) \), we have shown, for \( n \geq 7 \), that:

- \( \tilde{A}(\eta_n) > \tilde{A}(\eta) \), for all \( \eta \in (0, \eta_n/2] \cup [2\eta_n, \infty) \).
- \( \log(\tilde{A}(\eta)) \) is concave on \( (\eta_n/2, 2\eta_n) \) and therefore has a unique maximum on this interval. As \( \log(\tilde{A}) \) and \( \tilde{A} \) have maxima at the same positions, it follows that \( \tilde{A} \) has a unique maximum in \( (\eta_n/2, 2\eta_n) \).

The fact that \( \eta/n \) needs to be less than 1.6 makes it necessary to shorten the intervals appropriately for \( 3 \leq n \leq 6 \). This is achieved by using the intervals defined in (9.1). To that, due to the position of the maximum, this does not work for \( n = 2 \).

All in all, for \( n \geq 3 \) the lower frame bound has a unique global maximum in \( \eta_{A,n} \in (\eta_n/2, 2\eta_n) \).

### 9.2. The upper bound

We carry on with the same tactic, i.e., compare the values at \( \eta_n/2 \), \( \eta_n \) and \( 2\eta_n \) and obtain a convexity result to see that the minimum lies in \( (\eta_n/2, 2\eta_n) \).

#### 9.2.1. The convexity

This time we start with the convexity of the bound, which actually holds on \( \mathbb{R}_+ \). It then follows from the asymptotic behavior that we have a unique minimum.

\[
\frac{1}{n} B\left(\frac{\eta_n}{2}, \frac{1}{n}\right) > 1 \left(\frac{2}{n} \coth\left(\frac{2}{n}\right) + 0\right) \to \infty, \quad \eta \to \infty,
\]

\[
\frac{1}{n} B\left(\frac{\eta_n}{2}, \frac{1}{n}\right) > \coth\left(\frac{2}{n}\right) (1 + 0) \to \infty, \quad \eta \to 0^+.
\]

We may already include the case \( n = 2 \) this time. For a fixed \( n \geq 2 \), we set

\[
h(\eta) = \frac{2}{n} \coth\left(\frac{2}{n}\right) + \eta \csch(\eta).
\]

We compute the first two derivatives of \( h \):

\[
h'(\eta) = \frac{1}{n} \coth\left(\frac{2}{n}\right) + \csch(\eta) - \eta \left(\frac{1}{n^2} \csch\left(\frac{2}{n}\right)^2 + \csch(\eta) \coth(\eta)\right)
= \frac{h(\eta)}{\eta} - \eta \left(\frac{1}{n^2} \csch\left(\frac{2}{n}\right)^2 + \csch(\eta) \coth(\eta)\right),
\]

\[
h''(\eta) = -2\left(\frac{1}{n^2} \csch\left(\frac{2}{n}\right)^2 + \csch(\eta) \coth(\eta)\right)
+ \eta \left(\frac{2}{n^2} \csch\left(\frac{2}{n}\right)^2 \coth(\eta) + \csch(\eta) \coth(\eta)^2 + \csch(\eta)^3\right)
= -2\left(\frac{1}{n^2} \csch\left(\frac{2}{n}\right)^2 + \csch(\eta) \coth(\eta)\right) + \eta \left(\frac{2}{n^2} \csch\left(\frac{2}{n}\right)^2 \coth(\eta) + \csch(\eta) + 2 \csch(\eta)^3\right).
\]
We apply the Leibniz rule and obtain
\[
\frac{\partial^2}{\partial \eta^2} \left[ \frac{1}{n} B_1 \left( \frac{\eta}{n}, \frac{\eta}{n} \right) \right] = \frac{2}{\eta} \coth\left( \frac{\eta}{2} \right) \csch\left( \frac{\eta}{2} \right)^2 h(\eta) - \frac{2}{\eta} \csch\left( \frac{\eta}{2} \right)^2 h'(\eta) + \coth\left( \frac{\eta}{2} \right) h''(\eta)
\]
\[
= \frac{1}{\eta} \coth\left( \frac{\eta}{2} \right) \csch\left( \frac{\eta}{2} \right)^2 \left( \frac{2}{n^2} \coth\left( \frac{\eta}{n} \right) + \eta \csch(\eta) \right)
- \csch\left( \frac{\eta}{2} \right)^2 \left( \frac{2}{n^2} \coth\left( \frac{\eta}{n} \right) + \csch(\eta) - \eta \left( \frac{1}{n^2} \csch\left( \frac{\eta}{n} \right)^2 + \csch(\eta) \coth(\eta) \right) \right)
+ \coth\left( \frac{\eta}{2} \right) \left( - 2 \left( \frac{1}{n^2} \csch\left( \frac{\eta}{n} \right)^2 + \csch(\eta) \coth(\eta) \right) + \eta \left( \frac{2}{n^2} \csch\left( \frac{\eta}{n} \right)^2 \coth\left( \frac{\eta}{n} \right) + \csch(\eta) + 2 \csch(\eta)^3 \right) \right).
\]

First, we deal with the parts independent of \(n\). For uniformity, all hyperbolic functions will be transformed to an argument \(\eta/2\) by using addition formulas provided in Section 5.1. Set
\[
T(\eta) = \csch\left( \frac{\eta}{2} \right)^2 \cdot \left( \frac{1}{2} \coth\left( \frac{\eta}{2} \right) \eta \csch(\eta) - \csch(\eta) + \eta \csch(\eta) \coth(\eta) \right)
+ \coth\left( \frac{\eta}{2} \right) \left( - 2 \csch(\eta) \coth(\eta) + \eta \csch(\eta) + 2\eta \csch(\eta)^3 \right).
\]

Then, this can be written as
\[
T(\eta) = \frac{1}{4} \csch\left( \frac{\eta}{2} \right)^2 \cdot \left( \eta \coth\left( \frac{\eta}{2} \right) \csch\left( \frac{\eta}{2} \right) \sech\left( \frac{\eta}{2} \right) - 2 \csch\left( \frac{\eta}{2} \right) \sech\left( \frac{\eta}{2} \right) \right)
\]
\[
+ \eta \csch\left( \frac{\eta}{2} \right) \sech\left( \frac{\eta}{2} \right)^2 \left( 2 \cosh\left( \frac{\eta}{2} \right)^2 - 1 \right)
- 2 \sinh\left( \frac{\eta}{2} \right) \cosh\left( \frac{\eta}{2} \right) \csch\left( \frac{\eta}{2} \right) \sech\left( \frac{\eta}{2} \right)^2 \left( 2 \cosh\left( \frac{\eta}{2} \right)^2 - 1 \right)
+ 2\eta \sinh\left( \frac{\eta}{2} \right) \cosh\left( \frac{\eta}{2} \right) \csch\left( \frac{\eta}{2} \right) \sech\left( \frac{\eta}{2} \right)
+ \eta \sinh\left( \frac{\eta}{2} \right) \cosh\left( \frac{\eta}{2} \right) \csch\left( \frac{\eta}{2} \right)^3 \sech\left( \frac{\eta}{2} \right)^3 \right)
= \frac{1}{4} \csch\left( \frac{\eta}{2} \right)^2 \cdot (\circ).
\]

We aim to show positivity of the expression (\circ), implying positivity of the entire part independent of \(n\). Using addition formulas for squares of hyperbolic functions provided in Section 5.1, we compute
\[
(\circ) = \left( \eta \csch\left( \frac{\eta}{2} \right)^2 - 2 \csch\left( \frac{\eta}{2} \right) \sech\left( \frac{\eta}{2} \right) + 2\eta \csch\left( \frac{\eta}{2} \right)^2 \sech\left( \frac{\eta}{2} \right)^2 \cosh\left( \frac{\eta}{2} \right)^2 - \eta \csch\left( \frac{\eta}{2} \right)^2 \sech\left( \frac{\eta}{2} \right)^2 \right.
\]
\[
- 4 \csch\left( \frac{\eta}{2} \right) \sech\left( \frac{\eta}{2} \right) \cosh\left( \frac{\eta}{2} \right)^2
+ 2 \csch\left( \frac{\eta}{2} \right) \sech\left( \frac{\eta}{2} \right) + 2\eta + \csch\left( \frac{\eta}{2} \right)^2 \sech\left( \frac{\eta}{2} \right)^2 \right)
\]
\[
= \csch\left( \frac{\eta}{2} \right)^2 \left( \eta + 2\eta - 4 \sinh\left( \frac{\eta}{2} \right) \cosh\left( \frac{\eta}{2} \right) + 2\eta \sinh\left( \frac{\eta}{2} \right)^2 \right)
\]
\[
= \csch\left( \frac{\eta}{2} \right)^2 \left( \eta \cosh(\eta) + 2\eta - 2 \sinh(\eta) \right)
\]
\[
= \eta \csch\left( \frac{\eta}{2} \right)^2 \left( \sum_{k=0}^{\infty} \frac{\eta^2}{(2k+1)!} + 2 - \sum_{k=0}^{\infty} \frac{2\eta^2}{(2k+1)!} \right) = \eta \csch\left( \frac{\eta}{2} \right)^2 \cdot \sum_{k=1}^{\infty} \frac{\eta^2}{(2k-1)!} > 0.
\]

So, the part independent of \(n\) is positive. If we can show the same for the remaining terms of the second derivative, then we have proven convexity of the bound on its entire domain. We
collect the terms dependent on $n$ and set

$$
T_n(\eta) = \frac{1}{n} \cdot \frac{\eta}{n} \coth\left(\frac{\eta}{n}\right) \csch\left(\frac{\eta}{n}\right)^2 \coth\left(\frac{\eta}{n}\right) - \frac{1}{n} \cdot \csch\left(\frac{\eta}{n}\right)^2 \coth\left(\frac{\eta}{n}\right) + \frac{\eta}{n^2} \cdot \csch\left(\frac{\eta}{n}\right)^2 \csch\left(\frac{\eta}{n}\right)^2
- \frac{1}{n} \cdot \coth\left(\frac{\eta}{n}\right) \csch\left(\frac{\eta}{n}\right)^2 + 2 \frac{n}{n^2} \cdot \coth\left(\frac{\eta}{n}\right) \csch\left(\frac{\eta}{n}\right)^2 \coth\left(\frac{\eta}{n}\right).
$$

This can be written as

$$
T_n(\eta) = \frac{1}{n} \cdot \csch\left(\frac{\eta}{n}\right)^2 \coth\left(\frac{\eta}{n}\right) \left(\frac{\eta}{n} \coth\left(\frac{\eta}{n}\right) - 1\right) + \frac{\eta}{n} \cdot \csch\left(\frac{\eta}{n}\right)^2 \csch\left(\frac{\eta}{n}\right)^2
+ \frac{\eta}{n^2} \cdot \coth\left(\frac{\eta}{n}\right) \csch\left(\frac{\eta}{n}\right)^2 \left(\frac{\eta}{n} \coth\left(\frac{\eta}{n}\right) - 1\right) > 0.
$$

All in all, the upper frame bound is strictly convex and tends to infinity when approaching the boundary of its domain, which is $\mathbb{R}_+$. Therefore, it has a unique minimizer $\eta_{B,n}$, as claimed.

We further show that $\eta_{B,n}$ is between $\eta_{n/2}$ and $2\eta_{n}$, where $\eta_{n} = 2 \operatorname{arccosh}(n)$ as before. Again, we compare the values at $\eta_{n/2}$, $\eta_{n}$ and $2\eta_{n}$:

\[
\frac{1}{n} B\left(\frac{\eta_{n}}{n}, \frac{2}{2n}\right) = \frac{n+1}{(n-1)^{2/3}} \eta_{n} \left( \frac{1}{n} \coth\left(\frac{\eta_{n}}{n}\right) + \frac{1}{(n-1)^{2/3}} \right) = \frac{n+1}{(n-1)^{2/3}} \eta_{n} \left( \frac{1}{n} \coth\left(\frac{\eta_{n}}{n}\right) + \frac{1}{(n-1)^{2/3}} \right),
\]

\[
\frac{1}{n} B\left(\frac{\eta_{n}}{n}, \frac{1}{2n}\right) = \frac{n}{(n-1)^{2/3}} \eta_{n} \left( \frac{1}{n} \coth\left(\frac{\eta_{n}}{n}\right) + \frac{1}{2(n-1)^{2/3}} \right) = \frac{n}{(n-1)^{2/3}} \eta_{n} \left( \frac{1}{n} \coth\left(\frac{\eta_{n}}{n}\right) + \frac{1}{2(n-1)^{2/3}} \right),
\]

\[
\frac{1}{n} B\left(\frac{2\eta_{n}}{n}, \frac{1}{2n}\right) = \frac{2n-1}{2(n-1)^{2/3}} \cdot 2 \eta_{n} \left( \frac{1}{n} \coth\left(\frac{2\eta_{n}}{n}\right) + \csch\left(4 \operatorname{arccosh}(n)\right) \right)
= \frac{2n-1}{n(n-1)^{2/3}} \cdot \eta_{n} \left( \frac{1}{n} \coth\left(\frac{2\eta_{n}}{n}\right) + \frac{1}{2n} \csch\left(\frac{2\eta_{n}}{n}\right) + \frac{1}{2n} \csch\left(\frac{2\eta_{n}}{n}\right) \right)
= \frac{2n-1}{n(n-1)^{2/3}} \left( \coth\left(\frac{\eta_{n}}{n}\right) \left(1 - \frac{1}{2} \cdot \sech\left(\frac{\eta_{n}}{n}\right)^2\right) + \frac{1}{2n} \right).
\]

9.2.2. The left cut-off. Because of the convexity in the entire domain, we only need to compare at $\eta_{n/2}$ and $\eta_{n}$ and do not need any interval estimates this time. We aim to show that

\[
B\left(\frac{\eta_{n}}{n}, \frac{1}{n}\right) < B\left(\frac{\eta_{n}}{n}, \frac{2}{2n}\right)
\]

\[
\iff 2 \coth\left(\frac{\eta_{n}}{n}\right) + \frac{1}{(n-1)^{2/3}} < \left( n + \frac{1}{n} \right) \coth\left(\frac{\eta_{n}}{n}\right) + \left( n + \frac{1}{n} \right) \csch\left(\frac{\eta_{n}}{n}\right) + \frac{n+1}{(n-1)^{2/3}}
\]

\[
\iff \left( 1 - \frac{\eta}{n} \right) \coth\left(\frac{\eta}{n}\right) < \left( 1 + \frac{\eta}{n} \right) \csch\left(\frac{\eta}{n}\right) + \frac{n+1}{(n-1)^{2/3}}
\]

\[
\iff \left( 1 - \frac{\eta}{n} \right)^2 \coth\left(\frac{\eta}{n}\right)^2 < \left( 1 + \frac{\eta}{n} \right)^2 \coth\left(\frac{\eta}{n}\right)^2 - 1 + \frac{\eta^2}{n} \csch\left(\frac{\eta}{n}\right)
\]

\[
\iff - \frac{\eta}{n} \coth\left(\frac{\eta}{n}\right)^2 < \frac{n+1}{n} \coth\left(\frac{\eta}{n}\right)^2 - \frac{\eta^2}{n} \csch\left(\frac{\eta}{n}\right)
\]

\[
\iff 4 \coth\left(\frac{\eta}{n}\right)^2 > \frac{(n+1)^2}{n} - \frac{n^3}{n^2-1} - \frac{2n}{n^2-1} \csch\left(\frac{\eta}{n}\right).
\]

The left-hand side is strictly greater than 4. For the right-hand side, we estimate

\[
\frac{(n+1)^2}{n} - \frac{n^3}{n^2-1} - \frac{2n}{n^2-1} \csch\left(\frac{\eta}{n}\right) < \frac{(n^2+2n+1)(n^2-1)-n^4}{n(n^2-1)} = \frac{2n^3-2n-1}{n(n^2-1)} = 2 - \frac{1}{n(n^2-1)}.
\]

This is clearly less than 2 and so the value at $\eta_{n/2}$ is less than the value at $\eta_{n}$. 
9.2.3. The right cut-off. We now compare the values at $\eta_n$ and $2\eta_n$. We aim to show that
\[
B\left(\frac{2n}{n}, \frac{1}{n}\right) < B\left(\frac{2n}{n}, \frac{1}{2n}\right)
\]
\[\iff\] \[
\coth\left(\frac{2n}{n}\right) + \frac{1}{2(n^2 - 1)^{1/2}} < \frac{2n^2 - 1}{n^2} \coth\left(\frac{2n}{n}\right) \left(1 - \frac{1}{2} \operatorname{sech}\left(\frac{2n}{n}\right)^2\right) + \frac{1}{2n^2 (n^2 - 1)^{1/2}}
\]
\[\iff\] \[
\coth\left(\frac{2n}{n}\right) \left(1 - 2 + \frac{1}{n^2} - (2 - \frac{1}{n^2})\left(1 - \frac{1}{2} \operatorname{sech}\left(\frac{2n}{n}\right)^2\right)\right) < (-1 + \frac{1}{2n}) \frac{1}{2n^2 (n^2 - 1)^{1/2}}.
\]

The right-hand side is clearly negative and greater than $-1$. We show that the left-hand side is also negative, but less than $-1$. This will then confirm the initial inequality. It suffices to observe the following:
\[-1 + \frac{1}{n^2} - (2 - \frac{1}{n^2})(1 - \frac{1}{2} \operatorname{sech}\left(\frac{2n}{n}\right)^2) < -1 + \frac{1}{n^2} - \frac{1}{2}(2 - \frac{1}{n^2}) = -2 + \frac{1}{2n^2} < -1.
\]

This confirms the claim for all $n \geq 2$. We remark already that, for the first $3 \leq n \leq 10$, we will use the intervals $I_n$ defined in (9.4) in order to obtain the needed concavity results for the condition number. It will suffice to evaluate the first derivative on the boundaries of $I_n$.

9.3. The condition number. Unlike for all other windows, things seem to get only more complicated with the condition number. It is given by
\[
\kappa\left(\frac{2}{n}, \frac{1}{n}\right) = \frac{B\left(\frac{2}{n}\right)}{A\left(\frac{2}{n}\right)} = \frac{2 \coth\left(\frac{2}{n}\right) + \eta \operatorname{csch}(\eta)}{\frac{2}{n} \operatorname{csch}\left(\frac{2}{n}\right) - \eta \operatorname{csch}(\eta)}
\]

We try to use maximally what we already know and to simplify the problem of determining an optimal lattice as much as possible. By the asymptotic behavior of $A$ and $B$, we know that $\kappa$ is unbounded towards 0 and $\infty$. Showing convexity would therefore ensure that we have a unique minimum. This turns out to be a rather difficult problem due to the algebraic nature of the condition number. We simplify the problem by showing log-convexity on $I_n$, defined by (9.7), and that there is no minimum outside. We compute
\[
\log(\kappa\left(\frac{2}{n}, \frac{1}{n}\right)) = -\log(A\left(\frac{2}{n}\right)) + \log(B\left(\frac{2}{n}\right)) = -\log(A\left(\frac{2}{n}, \frac{1}{n}\right)) - 2 \log(\tanh\left(\frac{2}{n}\right)) + \log(h(\eta)),
\]

where, for $3 \leq n \in \mathbb{N}$ fixed,
\[
h(\eta) = \frac{2}{n} \coth\left(\frac{2}{n}\right) + \eta \operatorname{csch}(\eta),
\]

is still defined by (9.7), just as in the proof of the convexity of the upper bound. We know that $\tanh(x)$ is log-concave for $x > 0$ and that the lower bound is log-concave on $I_n$. If we can show that $h$ is log-convex on $I_n$, then we have proven that $\kappa$ is log-convex, hence convex, on $I_n$. As $A$ does not have a maximum outside $I_n$ and $B$ does not possess a minimum there, the minimum in $I_n$ has to be global and, by convexity, unique. So, we want to show that
\[
\frac{\partial^2}{\partial \eta^2} \log(h(\eta)) > 0, \quad \eta \in I_n \iff h(\eta)h''(\eta) - h'(\eta)^2 > 0, \quad \eta \in I_n.
\]

We continue by using the explicit formulas for $h$, $h'$ and $h''$ and, then, further simplifying the expression by grouping terms and using the addition theorems for hyperbolic functions from Section 5.1. The computations need very good intuition of how to group the terms efficiently. This makes this part rather laborious and quite lengthy. However, as each step only needs elementary manipulations of hyperbolic functions, we omit the details at this point. The
interested reader may consult Appendix \[A\] for the missing details in (9.8). The result to further analyze is

\[
h(\eta)h''(\eta) - h'(\eta)^2 = \frac{\cosh(\frac{\eta}{n})^2}{n^2} \left( (\frac{\eta}{n})^2 \coth(\frac{\eta}{n})^2 - \cosh(\frac{\eta}{n})^2 + (\frac{\eta}{n})^2 \right) \\
+ \frac{4}{n} \coth\left(\frac{\eta}{n}\right) \csc(\eta) \left( \eta \left( \frac{\eta}{n} \coth\left(\frac{\eta}{n}\right) \coth(\eta) \right) + 2 \left( \frac{\eta}{n} \coth\left(\frac{\eta}{n}\right)^2 - 1 \right) \right) > 0 \\
+ \frac{2n}{\csc(\eta)} \left( \frac{2\csc(\eta)}{2} \left( \csc(\eta) - \frac{\eta}{n} \right) + \frac{2}{2} \coth(\eta) - \frac{2}{n} \coth\left(\frac{\eta}{n}\right) \right) \\
+ \csc(\eta)^2 \left( \frac{2n}{\csc(\eta)} \coth\left(\frac{\eta}{n}\right) \coth(\eta) + \eta^2 \csc(\eta)^2 \right) > 0.
\]

We will now go through the expressions (1) – (4) separately. We first analyze everything for \( n \geq 4 \) and then give the details for the case \( n = 3 \).

Expression (1): we use the change of variables \( t = \eta/n \) and subsequently the duplication formulas for hyperbolic functions from Section \[5.1\]. For \( t \in \frac{1}{n}I_n \) we want to show that

\[
0 < t^2 \coth(t)^2 - \cosh(t)^2 + t^2 \\
\iff \cosh(t)^2 < t^2 \frac{\cosh(t)^2 + \sinh(t)^2}{\sinh(t)^2} \\
\iff 1 < 4t^2 \frac{\cosh(t)^2 + \sinh(t)^2}{4 \sinh(t)^2 \cosh(t)^2} = 4t^2 \cosh(2t) \csc(2t)^2 = 4t^2 \cosh(2t) \csc(2t) \\
\iff 0 > \sinh(x) - x^2 \coth(x), \quad x \in \frac{2}{n}I_n.
\]

We observe that the asymptotic behavior of the last expression close to 0 is

\[
\lim_{x \to 0^+} (\sinh(x) - x^2 \coth(x)) = 0.
\]

To see when our inequality holds, we compute the derivative of the above expression:

\[
\frac{\partial}{\partial x} [\sinh(x) - x^2 \coth(x)] = \cosh(x) - 2x \coth(x) + x^2 \csc(x)^2 \\
< \cosh(x) - x \coth(x) - 1 + x^2 \csc(x)^2 \\
= \cosh(x)(1 - x \csc(x)) - (1 + x \csc(x))(1 - x \csc(x)) \\
= (1 - x \csc(x))(\cosh(x) - 1 - x \csc(x)).
\]

The expression \( \cosh(x) - 1 - x \csc(x) \) is strictly monotonically increasing as \( \cosh(x) \) is increasing and \( x \csc(x) \) is decreasing (see Section \[5.1\]) and its value at 0 is -1. Therefore, it has a unique zero \( x_0 \) and we can check that it is in the interval \((1.19, 1.2)\). To sum it up, this tells us that \( \sinh(x) - x^2 \coth(x) \) is strictly monotonically decreasing on \((0, x_0]\) and strictly monotonically increasing on \([x_0, \infty)\). As a consequence, \( \sinh(2t) - (2t)^2 \csc(2t) \) has its minimum at \( t_0 = x_0/2 \). In particular, for all \( t \in (0, 4/3) \) the function’s maximum is only attained
at one of the boundary points;
\[
\sinh(2t) - 4t^2 \coth(2t) < \max \{ \sinh(2 \cdot \frac{4}{3}) - 4 \cdot \frac{16}{3} \coth(2 \cdot \frac{4}{3}), 0 \} = 0,
\]
proving the initial inequality and that (1) is positive. Here we used the behaviour of \( x \coth(x) \) close to zero:
\[
\lim_{x \to 0^+} \sinh(x) - x^2 \coth(x) = 0 - \lim_{x \to 0^+} x \cdot x \coth(x) = -0 \cdot 1 = 0.
\]
This first expression, as well as the following one, are the reason why we had to improve the intervals \((\eta_n/2, 2\eta_n)\) to \(I_n\) for the first few \(4 \leq n \leq 10\) and why the case \(n = 3\) needs then to be treated separately.

**Expression (2):** Recall that \( t \coth(t) \) is strictly increasing for all \( t > 0 \) (see Section 5.1). Hence, for \( \eta/n = t \in (0, 4/3) \) we obtain the following estimate:
\[
\eta - \frac{2\eta}{n} \coth(\frac{\eta}{n}) \coth(\eta) \geq \eta - 2 \cdot \frac{4}{3} \coth(\frac{4}{3}) \coth(\eta).
\]
What we are left with is strictly monotonically increasing. We evaluate the right-hand side at \( \arccosh(11) \) to see that it is strictly positive for all \( n \geq 11 \) and \( \eta \in I_n \). For \( 4 \leq n \leq 10 \), we evaluate the right-hand side of the above inequality at \( \eta = 3.08 > \arccosh(n) \), which suffices for the positivity and explains the choice of the bounds of the interval \( I_n \) for \( n \geq 4 \).

**Expression (3):** For \( n \geq 4 \), \( \eta \in I_n \subset (\eta_n/2, 2\eta_n) \) and \( \eta_n = 2 \arccosh(n) \), we have \( \eta > 1 \) and
\[
\cosh(\eta) - \frac{2}{\eta} > n - \frac{4}{3} = 0.
\]

**Expression (4):** we want to show that
\[
\frac{1}{2} \eta \coth(\eta) - \frac{2}{n} \coth(\frac{\eta}{n}) > 0 \iff \frac{1}{2} \eta \coth(\eta) > \frac{2}{n} \coth(\frac{\eta}{n}).
\]
The left-hand side of the second inequality is bounded from below by
\[
\frac{1}{2} \eta \coth(\eta) > \frac{1}{2} \cdot 3.08 \coth(3.08) > 1.54,
\]
whereas its right-hand side is bounded from above:
\[
\frac{2}{n} \coth(\frac{\eta}{n}) < \frac{4}{3} \coth(\frac{4}{3}) < 1.53.
\]
Therefore, the original inequality holds. This concludes the estimate, i.e., for all \( n \geq 4 \),
\[
h(\eta) h''(\eta) - h'(\eta)^2 > 0, \quad \eta \in I_n.
\]

9.3.1. **The case \( n = 3 \).** For \( n = 3 \) and the interval \( I_3 = (3.3, 4.8) \) we make three claims:
\[
h \geq 1.6 \quad \text{and} \quad |h'| \leq 0.22 \quad \text{and} \quad h'' > 0.
\]
Towards the first two claims, we use interval estimates for subintervals of \( I_3 \) of length 0.01. This shows that \( h \geq 1.6 \) on \( I_3 \). For \( h'(\eta), \eta \in I_3 \), we observe that
\[
h'(\eta) = \frac{1}{3} \coth(\frac{\eta}{3}) + \coth(\eta) - \eta \cdot \frac{2}{3} \coth(\frac{\eta}{3})^2 - \eta \cdot \coth(\eta) \coth(\eta).
\]
To show \(-0.22 < h' < 0.22\), we use subintervals of length 0.01, again. For the third claim,
\[
\frac{9}{2} h''(\eta) = - \coth(\frac{\eta}{3})^3 - 9 \coth(\eta) \coth(\eta) + \eta \left( \frac{4}{3} \coth(\frac{\eta}{3})^2 \coth(\frac{\eta}{3}) + 9 \coth(\eta)^3 + \frac{9}{2} \coth(\eta) \right)
\]
\[
= \coth(\frac{\eta}{3})^2 \left( \frac{2}{3} \coth(\frac{\eta}{3}) - 1 \right) + 9 \coth(\eta) \left( \eta \coth(\eta)^2 + \frac{9}{2} - \coth(\eta) \right), \quad \eta \in I_3.
\]
The last estimate has not been established before. Yet, it is easy to see that
\[
\eta \coth(\eta)^2 + \frac{9}{2} - \coth(\eta) \geq 0 + \frac{3}{2} - \coth(3.3) > 0.64.
\]
Having proven the claims, we can estimate as follows.

\[
h(n)h''(n) - h'(n)^2 > 1.6h''(n) - 0.22^2 \\
= \frac{32}{9} \cosh\left(\frac{n}{2}\right)^2 \left(\frac{n}{2} \coth\left(\frac{n}{2}\right) - 1\right) \\
+ 3.2 \cosh(n) \left(\eta \cosh(n)^2 + \frac{\eta}{2} - \coth(n)\right) - 0.0484 \\
> \frac{32}{9} \frac{\eta}{2} \cosh\left(\frac{n}{2}\right) \cdot \cosh\left(\frac{n}{2}\right) \coth\left(\frac{n}{2}\right) - \frac{32}{9} \cdot 0.64 \cosh(n) - 0.0484 > 0.
\]

The final estimate can be confirmed by an interval estimate on subintervals of length 0.1. In conclusion, the condition number is strictly log-convex, hence convex, on (3, 4.8), where the minimum lies. Therefore, the minimum is unique on this interval and by what we know about the bounds A and B outside this interval, the minimum has to be global.

9.4. The case \(n = 2\). This case needs to be treated separately as the necessary adjustments would go beyond simple interval shrinkage. This case has an additional symmetry, as we use the arguments \(\eta/2\) and \(\eta/n\) now coincide. This allows us to manipulate the relevant functions more easily with the addition theorems for hyperbolic functions (see Section 3.1).

9.4.1. The lower bound. We start again with the lower bound.

\[
\frac{1}{n} A\left(\frac{2}{n}, \frac{1}{n}\right) = \frac{2}{n} \tanh\left(\frac{2}{n}\right) \left(\cosh\left(\frac{2}{n}\right) - 2 \cosh(n)\right) = \frac{2}{n} \tanh\left(\frac{2}{n}\right) \left(\cosh\left(\frac{2}{n}\right) - \cosh(\frac{2}{n}) \sech\left(\frac{2}{n}\right)\right) \\
= \frac{2}{n} \frac{\sinh\left(\frac{2}{n}\right)}{\cosh\left(\frac{2}{n}\right)} \cosh\left(\frac{2}{n}\right) (1 - \sech\left(\frac{2}{n}\right)) = \frac{n}{2} \sech\left(\frac{2}{n}\right) (1 - \sech\left(\frac{2}{n}\right)).
\]

To avoid all the factors of 1/2 which arise from derivatives, we optimize \(t \sech(t) (1 - \sech(t))\).

\[
\frac{\partial}{\partial t} [t \sech(t) (1 - \sech(t))] = \sech(t) (1 - \sech(t)) - t \sech(t) \tanh(t) (1 - \sech(t)) \\
+ t \sech(t) \sech(t) \tanh(t) \\
= \sech(t) \left( (1 - \sech(t)) (1 - t \tanh(t)) + t \tanh(t) \sech(t) \right) \\
= \sech(t) \left( 1 - \sech(t) - t \tanh(t) (1 - 2 \sech(t)) \right).
\]

Since the fixed point of \(\coth\) is in \([1.19, 1.2]\), the first derivative is strictly positive on \((0, 1.19]\). On the other hand, for all \(t > 1.4, 2 \sech(t) < 1\). Therefore,

\[
cosh(t) \cdot \frac{\partial}{\partial t} [t \sech(t) (1 - \sech(t))] < 1 - 0 - t \tanh(t) (1 - 2 \sech(t)), \quad t \geq 1.4,
\]

and the left-hand side of the above inequality is strictly monotonically decreasing. Now, by evaluating at \(t = 2.1\), we see that the right-hand side is negative there and, hence, we conclude that the left-hand side is strictly negative for all \(t \geq 2.1\). Taking into account that \(t = \eta/2\), this means that \(A\) has its maximum in \((2.38, 4.2]\). Notice that with the methods for \(n \geq 3\), we would have reduced it to only \((1.31, 7.91)\) at best (see Figure 10).
Finally, we show that $A'$ has only one zero in $(2.38, 4.2)$. We compute (keeping $t = \eta/2$)

$$\frac{\partial}{\partial t} \left[ 1 - \text{sech}(t) - t \tanh(t) + 2t \text{sech}(t) \tanh(t) \right]$$

$$= \text{sech}(t) \tanh(t) - \tanh(t) - t \text{sech}(t)^2 + 2t \tanh(t) \text{sech}(t) + 2t \text{sech}(t)^3 - 2t \text{sech}(t) \tanh(t)^2$$

$$= 3 \text{sech}(t) \tanh(t) - \tanh(t) - t \text{sech}(t)^2 + 2t \text{sech}(t) - 4t \text{sech}(t) \tanh(t)^2$$

$$= t \text{sech}(t)(2 - 3t \tanh(t)^2) + \text{sech}(t) \tanh(t) \left( \frac{3}{2} - \cosh(t) \right)$$

$$+ \text{sech}(t) \tanh(t) \left( \frac{3}{2} - t \text{csch}(t) - t \tanh(t) \right).$$

Considering the monotonicity of the hyperbolic functions, as well as their combinations with monomials, which we have already exploited numerous times, we get

$$2 - 3t \tanh(t)^2 < 0, \quad \frac{3}{2} - \cosh(t) < 0, \quad \frac{3}{2} - t \text{csch}(t) - t \tanh(t) < 0$$

with a simple interval estimate. All in all, the expression

$$1 - \text{sech}(t) - t \tanh(t) + 2t \text{sech}(t) \tanh(t)$$

is strictly monotonically decreasing on $(1.19, 2.1)$, so $t \text{sech}(t) (1 - \text{sech}(t))$ has exactly one critical point in this interval. By evaluating, one can now tell that it lies in the interval $(1.85, 1.87)$. Equivalently, we have a unique maximum for $A\left(\frac{\eta}{2}, \frac{1}{\eta}\right)$, which is assumed in the interval $(2 \cdot 1.85, 2 \cdot 1.87) = (3.7, 3.74)$.  

9.4.2. The upper bound. The convexity of the upper bound still stands as shown in Section 9.2.1. The minimum is therefore also unique. Since we have fixed $n = 2$, we can evaluate and also improve the interval for $\eta_{\text{B.z}}$. Indeed, by evaluating the first derivative, we can tell that the unique minimizer $\eta_{B,n}$ lies in $(3, 3.1)$.  

9.4.3. The condition number. What remains to be determined is the minimum of the condition number $\kappa$. The only possible interval where the derivative of $A$ and $B$ have the same sign is $(3, 3.74)$ (also recall equation (1.11) from the very beginning). We compute

$$\kappa\left(\frac{\eta}{2}, \frac{1}{\eta}\right) = \coth\left(\frac{\eta}{2}\right)^2 \frac{\coth\left(\frac{\eta}{2}\right) + \eta \text{csch}\left(\eta\right)}{\frac{\eta}{2} \text{csch}\left(\frac{\eta}{2}\right) - \eta \text{csch}\left(\eta\right)} = \coth\left(\frac{\eta}{2}\right)^2 \frac{\coth\left(\frac{\eta}{2}\right) + \text{csch}\left(\frac{\eta}{2}\right) \text{sech}\left(\frac{\eta}{2}\right)}{\text{csch}\left(\frac{\eta}{2}\right) - \text{csch}\left(\frac{\eta}{2}\right) \text{sech}\left(\frac{\eta}{2}\right)}$$

$$= \coth\left(\frac{\eta}{2}\right)^2 \left( \text{cosh}\left(\frac{\eta}{2}\right)^2 + 1 \right) \frac{1}{\text{cosh}\left(\frac{\eta}{2}\right) - 1}.$$
We change the variable to $t = \eta/2$ and show the strict log-convexity of each of the terms on the interval $(1.5, 1.87)$.

$$\frac{\partial^2}{\partial t^2} \left[ \log(\coth(t)^2) \right] = -2 \frac{\partial}{\partial t} \left[ \text{csch}(t) \text{sech}(t) \right] = 2 \left( \text{csch}(t)^2 + \text{sech}(t)^2 \right) > 0.$$  

Secondly,

$$\frac{\partial^2}{\partial t^2} \left[ \log(\cosh(t)^2 + 1) \right] = \frac{\partial}{\partial t} \left[ \frac{2 \sinh(2t)}{\cosh(2t) + 3} \right]$$

$$= \frac{4 \cosh(2t) (\cosh(2t) + 3) - 4 \sinh(2t) \sinh(2t)}{(\cosh(2t) + 3)^2}$$

$$= \frac{4 + 12 \cosh(2t)}{(\cosh(2t) + 3)^2} > 0.$$  

Finally,

$$-\frac{\partial^2}{\partial t^2} \left[ \log(\cosh(t) - 1) \right] = -\frac{\partial}{\partial t} \frac{\sinh(t)}{\cosh(t) - 1}$$

$$= -\frac{\cosh(t) (\cosh(t) - 1) - \sinh(t) \sinh(t)}{(\cosh(t) - 1)^2}$$

$$= \frac{\cosh(t)}{(\cosh(t) + 1)^2} > 0.$$  

In conclusion, the condition number is strictly log-convex on $(3, 3.74)$, so it assumes its unique minimum $\eta_{n,2}$ here.

9.5. Three optimal points. Lastly, we want to show that the locations of the optimum for the lower bound, upper bound and condition number differ from each other. As we already know that the optimum is unique in each case, it suffices to show that $A$ and $B$ do not assume their optima at the same point. By recalling (1.1), it is immediate that also $\kappa$ cannot have its optimum at one of these positions.

9.5.1. The lower bound. We are now in a position to determine the location of the maximum. We will show that $\eta_{A,n} > \eta_n$. It suffices to show that the first derivative of $A$ is positive at $\eta_n$. Since $f$ is strictly monotonically decreasing, we have

$$\frac{1}{n} \left. \frac{\partial}{\partial \eta} A \left( \frac{n}{n}, \frac{1}{n}, \frac{1}{n} \right) \right|_{\eta=\eta_n} = \frac{1}{2} \text{sech}(\frac{\eta_n}{2})^2 \left( f(\frac{\eta_n}{n}) - f(\eta_n) \right) + \text{tanh}(\frac{\eta_n}{2}) \left( \frac{1}{n} f'(\frac{\eta_n}{n}) - f'(\eta_n) \right)$$

$$> \text{tanh}(\frac{\eta_n}{2}) \left( \frac{1}{n} f'(\frac{\eta_n}{n}) - f'(\eta_n) \right).$$

This is strictly positive if and only if

$$0 < n^2 \coth(\frac{\eta_n}{2}) \tanh\left( \frac{\eta_n}{2} \right) \left( \frac{1}{n} f'(\frac{\eta_n}{n}) - f'(\eta_n) \right)$$

$$= n \text{csch}\left( \frac{\eta_n}{n} \right) \left( 1 - \frac{\eta_n}{n} \coth\left( \frac{\eta_n}{n} \right) \right) - n^2 \text{csch}(\eta_n) \left( 1 - \eta_n \coth(\eta_n) \right).$$

We make several observations. First of all, the addition formulas can be applied wherever the term $\eta_n = 2 \text{arcosh}(n)$ appears. Secondly, the parts involving $\eta_n/n$ are not as straightforward to analyze, so we transform the parameter to $t = \eta_n/n$ and use that it is strictly monotonically decreasing in $n \geq 2$ (recall that $\eta_n/n = \rho(n)$ from Section 5.1). We further
manipulate the right-hand side of the above inequality with the mentioned addition formulas and substitutions:

\[
0 < n \operatorname{csch}\left(\frac{n}{n}\right)(1 - \frac{n}{n} \operatorname{coth}\left(\frac{n}{n}\right)) - \frac{n^2}{2(n^2-1)^{1/2}} + \eta_n \cdot \frac{n^2}{2(n^2-1)^{1/2}} \cdot \frac{2n^2-1}{2(n^2-1)^{1/2}} \\
= n \operatorname{csch}(t)(1 - t \operatorname{coth}(t)) - \frac{1}{2}(1 + \frac{1}{n^2-1})^{1/2} + \eta_n \cdot \frac{2n^2-2+1}{4(n^2-1)} \\
= n \operatorname{csch}(t)(1 - t \operatorname{coth}(t)) + \frac{1}{2}(1 + \frac{1}{n^2-1})^{1/2} + \frac{n t}{2} + \frac{n t}{4(n^2-1)} \\
= nt\left(\frac{\operatorname{csch}(t)}{t} - \operatorname{csch}(t) \operatorname{coth}(t) + \frac{1}{2}\right) - \frac{1}{2}(1 + \frac{1}{n^2-1})^{1/2} + \frac{n t}{4(n^2-1)} \\
= (\infty).
\]

As usual, we decompose the expression \((\infty)\) into parts:

\[
\frac{\operatorname{csch}(t)}{t} - \operatorname{csch}(t) \operatorname{coth}(t) + \frac{1}{2} = \operatorname{csch}(t)^2 - \operatorname{csch}(t) \operatorname{coth}(t) + \frac{\cosh(t)^2 - 1}{2} \\
= \operatorname{csch}(t)^2 - 1 + \frac{\cosh(t)^2 - 2 \cosh(t) + 1}{2} \\
= \operatorname{csch}(t)^2 - 1 + \frac{\operatorname{csch}(t)^2 (\operatorname{csch}(t) - 1)^2}{2} \\
> \operatorname{csch}(t)^2 (\operatorname{csch}(t) - 1) \\
= \frac{\operatorname{csch}(t)}{t} - \operatorname{csch}(t)^2.
\]

We will now show that the last expression is strictly monotonically decreasing for all \(t > 0\);

\[
\frac{\partial}{\partial t} \left[ \frac{\operatorname{csch}(t)}{t} - \operatorname{csch}(t)^2 \right] = - \frac{\operatorname{csch}(t)}{t^2} - \frac{\operatorname{csch}(t) \operatorname{coth}(t)}{t} + 2 \operatorname{csch}(t)^2 \operatorname{coth}(t) \\
= - \operatorname{csch}(t)\left(\frac{1}{t} + \frac{\cosh(t)}{t} - 2 \operatorname{csch}(t) \operatorname{coth}(t)\right) \\
= - \frac{\operatorname{csch}(t)^2}{t^2}\left(\cosh(t)^2 - 1 + \frac{2t \sinh(t) \cosh(t)}{2} - 2t^2 \cosh(t)\right) \\
= - \frac{\operatorname{csch}(t)^2}{t^2}\left(\frac{1}{2} \cosh(2t) - \frac{1}{2} + \frac{t \sinh(2t)}{2} - 2t^2 \cosh(t)\right) \\
= - \frac{\operatorname{csch}(t)^2}{t^2}\left(\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} \cdot 2^k + \sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!} \cdot 2^{2k+2} - \sum_{k=0}^{\infty} \frac{2}{(2k)!} \cdot 2^{2k+2}\right) \\
= - \frac{\operatorname{csch}(t)^2}{t^2}\sum_{k=0}^{\infty} \frac{4 \cdot 2^{2k+2}}{(2k+2)!} \left(2^{2k-1} + 2^{2k-1}(k+1) - (2k+1)(k+1)\right) \\
= - \frac{\operatorname{csch}(t)^2}{t^2}\sum_{k=2}^{\infty} \frac{4 \cdot 2^{2k+2}}{(2k+2)!} \left((k+2)(2^{2k-1} - 2k - 1) + (2k + 1)\right).
\]

It is not hard to convince oneself of the positivity of

\[
(k+2)(2^{2k-1} - 2k - 1) + (2k + 1), \quad k \geq 2,
\]

through the behavior of \(2^{x-1} - x\) for \(x \geq 4\). The expression is clearly greater than 1 for \(x = 4\) and \(2^{x-1}\) is growing much faster than \(x\) for \(x > 4\). Therefore,

\[
\frac{\operatorname{csch}(\rho(n))}{\rho(n)} - \operatorname{csch}(\rho(n))^2 \geq \frac{\operatorname{csch}(\rho(2))}{\rho(2)} - \operatorname{csch}(\rho(2))^2 > 0, \quad \forall n \geq 2,
\]
and with the negative sign in front, the expression is negative. So, \( csh(t)/t - csh(t)^2 \) is decreasing for \( t > 0 \), as claimed. We go back to proving positivity of the expression \((\odot)\).

\[
(\odot) > nt \cosh(t)\left(\frac{1}{t} - \cosh(t)\right) - \frac{1}{2}\left(1 + \frac{1}{n^2-1}\right)^{1/2} \\
= 2 \arccosh(n) \left(\frac{\cosh(\rho(n))}{\rho(n)} - \cosh(\rho(n))^2\right) - \frac{1}{2}\left(1 + \frac{1}{n^2-1}\right)^{1/2}.
\]

We evaluate at \( n = 5 \) to verify the positivity of \((\odot)\) for all \( n \geq 5 \) by the monotonicity of all terms as indicated above. For the initial \( n = 2, 3, 4 \), we actually evaluate the derivative of \( A \) at \( \eta_n \) and conclude that

\[
\frac{\partial}{\partial \eta} A\left(\frac{2}{n}, \frac{1}{\eta}\right)\bigg|_{\eta=\eta_n} > 0
\]

for all \( n \geq 2 \). By the concavity result for \( A \) for \( n \geq 3 \), as well as the properties we established specifically for \( n = 2 \), we conclude that \( \eta_n < \eta_{A,n} \) for all \( n \geq 2 \).

9.5.2. The upper bound. We finalize by showing that \( \eta_{B,n} \neq \eta_{A,n} \). In fact, we show a stronger statement: \( \eta_{B,n} < \eta_n \) for \( n \geq 4 \). Through the additional analysis for \( n = 3 \) and \( n = 2 \), we have already established that \( \eta_{B,n} \neq \eta_{A,n} \).

Using the calculations for \( h \) as in Section 9.2 and \( \eta_n = \arccosh(n) \), we compute

\[
\frac{\partial}{\partial \eta} B\left(\frac{2}{n}, \frac{1}{\eta}\right)\bigg|_{\eta=\eta_n} = -\frac{1}{2} csh\left(\frac{\eta_n}{2}\right)\left(\frac{n}{n^2} \coth\left(\frac{\eta_n}{n}\right) + \eta_n \cosh(\eta_n)\right) \\
+ \coth\left(\frac{\eta_n}{2}\right)\left(\frac{1}{n} \coth\left(\frac{\eta_n}{2}\right) + \cosh(\eta_n)\right) \\
- \coth\left(\frac{\eta_n}{2}\right)\left(\eta_n \left(\frac{1}{n} \cosh\left(\frac{\eta_n}{2}\right) + \coth(\eta_n)\right)\right) \\
= -\frac{1}{2n(n-1)^{1/2}} \left(\frac{\eta_n}{n} \coth\left(\frac{\eta_n}{n}\right) + \frac{\eta_n}{2n(n^2-1)^{1/2}}\right) \\
+ \frac{n}{(n^2-1)^{1/2}} \left(\frac{1}{n} \coth\left(\frac{\eta_n}{2}\right) + \cosh(\eta_n)\right) \\
- \frac{n}{(n^2-1)^{1/2}} \left(\eta_n \left(\frac{1}{n} \cosh\left(\frac{\eta_n}{2}\right) + \coth(\eta_n)\right)\right) \\
= \frac{1}{2n(n^2-1)^{1/2}} \left(1 - \frac{n}{2n^2-1} - \eta_n \coth\left(\frac{\eta_n}{n}\right)\right) \\
+ \frac{n}{(n^2-1)^{1/2}} \left(\coth\left(\frac{\eta_n}{2}\right) - \frac{\eta_n}{n} \cosh\left(\frac{\eta_n}{2}\right) + \frac{\eta_n}{2n^2-1} + \frac{n}{4(n^2-1)}\right).
\]

If the last expression, and hence the derivative of the upper frame bound at \( \eta_n \), is positive, then a conclusion is that \( \eta_{B,n} \leq \eta_n \). We will be able to prove this for \( n \geq 4 \).

Notice that we have two groups, one with the factor \( 1/(n^2 - 1) \) and the other one with \( 1/(n^2 - 1)^{1/2} \). This plays a crucial part in the following estimate. We make two substitutions, namely \( \coth^2 = \cosh^2 - 1 \) and \( t = \eta_n/n \);

\[
\coth(t) - t \cosh(t)^2 - \frac{t}{2} + \frac{t}{4(n^2-1)} > \coth(t) - t \coth(t)^2 + \frac{t}{2}, \quad t > 0, \ n \geq 2.
\]

We improve this with a simple lower bound:

\[
\coth(t) - t \coth(t)^2 + \frac{t}{2} > \coth(t) - t \coth(t)^2 + \frac{t}{2} - \frac{t}{4n}, \quad t > 0.
\]
We will now study the properties of this new lower bound. We compute

\[
\frac{\partial^2}{\partial t^2} \left[ \coth(t) - t \coth(t)^2 + \frac{t}{2} - \frac{t}{14} \right] = \frac{\partial}{\partial t} \left[ -\csch(t)^2 - \coth(t)^2 + 2t \csch(t)^2 \coth(t) + \frac{3}{7} \right]
\]

\[
= \frac{\partial}{\partial t} \left[ -2 \csch(t)^2 - 1 + 2t \ \csch(t)^2 \coth(t) \right]
\]

\[
= 4 \csch(t)^2 \coth(t) + 2 \coth(t) \csch(t)^2
\]

\[
- 2t \csch(t)^4 - 4t \csch(t)^2 \coth(t)^2
\]

\[
= \csch(t)^4 (6 \cosh(t) \sinh(t) - 6t \coth(t)^2 + t \sinh(t)^2)
\]

\[
= \csch(t)^4 (3 \sinh(2t) - 3t(\cosh(2t) + 1) + t(\cosh(2t) - 1))
\]

\[
= \csch(t)^4 (6t + \sum_{k=1}^{\infty} \frac{(2t)^{2k+1}}{(2k+1)!} - 2t - \sum_{k=1}^{\infty} \frac{(2t)^{2k+1}}{(2k)!} - 4t)
\]

\[
= - \csch(t)^4 \sum_{k=1}^{\infty} 2k \frac{(2t)^{2k+1}}{(2k+1)!} < 0, \quad \forall t > 0.
\]

This shows that the expression \(\coth(t) - t \coth(t)^2 + \frac{t}{2} - \frac{t}{14}\) is strictly concave. To begin with, \(\coth(1.05) - 1.05 \coth(1.05)^2 + \frac{3}{14} > 0\). Additionally,

\[
\lim_{t \to 0^+} \coth(t) - t \coth(t)^2 + \frac{t}{2} - \frac{t}{14} = \lim_{t \to 0^+} t^2 \csch(t)^2 \frac{\cosh(t) \sinh(t) - t \cosh(t)^2}{t^2}
\]

\[
= \lim_{t \to 0^+} \sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!} t^{2k+1} - 2t - \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} t^{2k}
\]

\[
= \lim_{t \to 0^+} \sum_{k=1}^{\infty} \frac{2^{2k+1}(1-k)}{(2k+1)!} t^{2k-1} = 0,
\]

due to the convergence of the series. Altogether, this implies

\[
\coth(t) - t \coth(t)^2 + \frac{t}{2} > \frac{t}{14}
\]

on \((0, 1.05) \supseteq (0, \rho(n))\) for all \(n \geq 4\).

\[
\frac{1}{2(n^2-1)} (1 - \frac{\rho(n)}{2(n^2-1)^{1/2}} - \rho(n) \coth(\rho(n)))
\]

\[
+ \frac{1}{(n^2-1)^{1/2}} \left( \coth(\rho(n)) - \rho(n) \csch(\rho(n))^2 - \frac{\rho(n)}{2} + \frac{\rho(n)}{4(n^2-1)} \right)
\]

\[
> \frac{1}{2(n^2-1)} (1 - \frac{\rho(n)}{2(n^2-1)^{1/2}} - \rho(n) \coth(\rho(n)) + (n^2 - 1)^{1/2} \frac{\rho(n)}{7})
\]

\[
> \frac{1}{2(n^2-1)} (1 - \frac{\rho(n)}{2(n^2-1)^{1/2}} - \rho(n) \coth(\rho(n)) + (1 - \frac{1}{n^2})^{1/2} \frac{2 \arccosh(\rho(n))}{7}).
\]

By evaluating at \(n = 4\), we obtain
\[
\frac{1}{2(n^2-1)} \left(1 - \frac{\rho(n)}{2(n^2-1)^{1/2}} - \rho(n) \coth(\rho(n)) + (1 - \frac{1}{n^2})^{1/2} 2 \arccosh(n) \right)
\]
\[
> \frac{1}{2(n^2-1)} \left(1 - \frac{\rho(4)}{2(4^2-1)^{1/2}} - \rho(4) \coth(\rho(4)) + (1 - \frac{1}{4^2})^{1/2} 2 \arccosh(4) \right) > 0.
\]

Hence, \(\eta_{B,n} < \eta_h < \eta_{A,n}\) for all \(n \geq 4\).

For \(n = 2, 3\), \(\eta_{B,n} > \eta_h\), but we have already established that \(\eta_{B,2} < 3.2 < \eta_{A,2}\). As far as \(n = 3\) is concerned,

\[
\left. \frac{\partial}{\partial \eta} A\left(\frac{n}{3}, \frac{1}{\eta} \right) \right|_{\eta=3.9} > 0 \quad \text{and} \quad \left. \frac{\partial}{\partial \eta} B\left(\frac{n}{3}, \frac{1}{\eta} \right) \right|_{\eta=3.9} > 0,
\]

finally confirming that

\[\eta_{B,n} < \eta_{A,n}, \quad \forall n \geq 2, \ n \in \mathbb{N}.\]

**APPENDIX A. CALCULATIONS FOR (9.8)**

We will now provide the details of (9.8), which we omitted earlier. We set

\[H(\eta) = h(\eta)h''(\eta) - h'(\eta)^2,\]

where \(h(\eta) = \eta/n \coth(\eta/n) + \eta \csch(\eta),\ n \geq 4\), as defined by (9.7). We compute

\[
H(\eta) = h(\eta) \left( -2 \left( \frac{1}{n^2} \csch\left(\frac{2}{n} \right)^2 + \csch(\eta) \coth(\eta) \right) + \eta \left( \frac{2}{n^2} \csch\left(\frac{2}{n} \right)^2 \coth\left(\frac{2}{n} \right) + \csch(\eta) + 2 \csch(\eta)^3 \right) \right).
\]

Note that the underbraced expressions have a factor \(-2h(\eta)\) and \(2h(\eta)\), respectively, so they cancel out. We further have

\[
H(\eta) = \eta^2 \left( \frac{1}{n} \coth(\frac{2}{n}) + \csch(\eta) \right) \left( \frac{2}{n^2} \csch\left(\frac{2}{n} \right)^2 \coth\left(\frac{2}{n} \right) + \csch(\eta) + 2 \csch(\eta)^3 \right) - \frac{1}{n^2} \coth(\frac{2}{n})^2
\]
\[
- \csch(\eta)^2 - \frac{2}{n} \coth\left(\frac{2}{n} \right) \csch(\eta) - \frac{n^2}{n^4} \csch\left(\frac{2}{n} \right)^4 - \eta^2 \csch(\eta)^2 \coth(\eta)^2
\]
\[
- 2 \frac{n^2}{n^4} \csch\left(\frac{2}{n} \right)^2 \csch(\eta) \coth(\eta).
\]

We multiply out the remaining brackets and order by positive and negative sign to obtain

\[
H(\eta) = \frac{2n^2}{n^4} \csch\left(\frac{2}{n} \right)^2 \coth\left(\frac{2}{n} \right)^2 + \frac{2n^2}{n^4} \coth\left(\frac{2}{n} \right) \csch(\eta)^3 + \frac{n^2}{n^4} \coth\left(\frac{2}{n} \right) \csch(\eta)
\]
\[
+ \frac{2n^2}{n^4} \csch\left(\frac{2}{n} \right)^2 \coth\left(\frac{2}{n} \right) \csch(\eta) + 2n^2 \csch(\eta)^4 + \eta^2 \csch(\eta)^2
\]
\[
- \frac{1}{n^2} \coth\left(\frac{2}{n} \right)^2 - \csch(\eta)^2 - \frac{2}{n} \coth\left(\frac{2}{n} \right) \csch(\eta) - \frac{n^2}{n^4} \csch\left(\frac{2}{n} \right)^4
\]
\[
- \eta^2 \csch(\eta)^2 \coth(\eta)^2 - \frac{2n^2}{n^4} \csch\left(\frac{2}{n} \right)^2 \csch(\eta) \coth(\eta).
\]
We now group together terms where we find a common factor \(1/n^2\), or \(\text{csch}(\eta)^2\), or where we find \(2\eta/n^2\) \(\text{csch}(\eta/n)^2\) \(\text{csch}(\eta)\). We get

\[
H(\eta) = \frac{1}{n^2} \left( \frac{2\eta^2}{n^2} \text{csch}(\frac{2\eta}{n})^2 \coth(\frac{2\eta}{n})^2 - \frac{2\eta^2}{n^2} \text{csch}(\frac{2\eta}{n})^4 - \coth(\frac{2\eta}{n})^2 \right) + \frac{1}{n} \cot(\frac{2\eta}{n}) \text{csch}(\eta) (\eta^2 - 2) \\
+ \text{csch}(\eta)^2 \left( \frac{2\eta^2}{n} \cot(\frac{2\eta}{n}) \text{csch}(\eta) + 2\eta^2 \text{csch}(\eta)^2 + \eta^2 - 1 - \eta^2 \cot(\eta)^2 \right) \\
+ \frac{2\eta^2}{n} \cot(\frac{2\eta}{n})^2 \text{csch}(\eta) \left( \frac{2\eta}{n} \cot(\frac{2\eta}{n}) - \eta \cot(\eta) \right).
\]

We use the formula \(\text{coth}(x)^2 - \text{csch}(x)^2 = 1\) from Section 5.1. This gives us

\[
H(\eta) = \frac{1}{n^2} \left( \frac{2\eta^2}{n^2} \text{csch}(\frac{2\eta}{n})^2 \coth(\frac{2\eta}{n})^2 - \frac{2\eta^2}{n^2} \text{csch}(\frac{2\eta}{n})^4 - \coth(\frac{2\eta}{n})^2 \right) + \frac{1}{n} \cot(\frac{2\eta}{n}) \text{csch}(\eta) (\eta^2 - 2) \\
+ \text{csch}(\eta)^2 \left( \frac{2\eta^2}{n} \cot(\frac{2\eta}{n}) \text{csch}(\eta) + 2\eta^2 \text{csch}(\eta)^2 + \eta^2 - 1 - \eta^2 \cot(\eta)^2 \right) \\
+ \frac{2\eta^2}{n} \cot(\frac{2\eta}{n})^2 \text{csch}(\eta) \left( \frac{2\eta}{n} \cot(\frac{2\eta}{n}) - \eta \cot(\eta) \right) - \frac{2\eta^2}{n} \text{csch}(\eta) \left( \frac{2\eta}{n} \cot(\frac{2\eta}{n}) - \eta \cot(\eta) \right).\]

We re-group the terms in the first bracket, collect terms with factor \(1/n\) and pull out \(\text{coth}(\eta/n) \text{csch}(\eta)\) for this collection, and collect terms with factor \(2\eta/n\) and pull out \(\text{csch}(\eta)\), and lastly, collect terms with factor \(\text{csch}(\eta)^2\). We obtain

\[
H(\eta) = \frac{1}{n^2} \left( \text{csch}(\frac{2\eta}{n})^2 (\frac{2\eta^2}{n} \text{coth}(\frac{2\eta}{n})^2 - 1) + (\frac{2\eta^2}{n})^2 \text{csch}(\frac{2\eta}{n})^2 - 1 \right) \\
+ \frac{1}{n} \cot(\frac{2\eta}{n}) \text{csch}(\eta) \left( \eta^2 - 2 - \frac{2\eta^2}{n} \cot(\frac{2\eta}{n}) (\eta \cot(\eta) - \frac{2\eta}{n} \cot(\frac{2\eta}{n})) \right) \\
+ \frac{2\eta^2}{n} \text{csch}(\eta) (\eta \cot(\eta) - \frac{2\eta}{n} \cot(\frac{2\eta}{n})) \\
+ \frac{1}{n} \cot(\frac{2\eta}{n}) \text{csch}(\eta) \left( \eta^2 - 2 - \frac{2\eta^2}{n} \cot(\frac{2\eta}{n}) \cot(\eta) + 2(\frac{2\eta}{n})^2 \cot(\frac{2\eta}{n})^2 \right) \\
+ \frac{2\eta^2}{n} \text{csch}(\eta) (2 \cdot \frac{\eta}{n} \cot(\eta) - \frac{2\eta}{n} \cot(\frac{2\eta}{n}) - \frac{2\eta^2}{n} \text{csch}(\eta)) \\
+ \frac{2\eta^2}{n} \cot(\frac{2\eta}{n})^2 \text{csch}(\eta) \left( \eta^2 - 2 - \frac{2\eta^2}{n} \cot(\frac{2\eta}{n}) \cot(\eta) + 2(\frac{2\eta}{n})^2 \cot(\frac{2\eta}{n})^2 \right) \\
= \frac{\text{csch}(\frac{2\eta}{n})^2}{n^2} \left( (\frac{2\eta^2}{n} \text{coth}(\frac{2\eta}{n})^2 - 1) + (\frac{2\eta}{n})^2 - \sinh(\frac{2\eta}{n})^2 \right) \\
+ \frac{1}{n} \cot(\frac{2\eta}{n}) \text{csch}(\eta) \left( \eta (\eta - \frac{2\eta^2}{n} \cot(\frac{2\eta}{n}) \cot(\eta)) + 2 (\frac{2\eta}{n})^2 \cot(\frac{2\eta}{n})^2 - 1 \right) \\
+ \frac{2\eta^2}{n} \text{csch}(\eta) \left( \frac{2\eta \text{csch}(\eta)}{2} (\cosh(\eta) - \frac{2\eta}{n}) + \frac{\eta}{n} \cot(\eta) - \frac{2\eta^2}{n} \cot(\frac{2\eta}{n}) \right) \\
+ \text{csch}(\eta)^2 \left( \frac{2\eta^2}{n} \cot(\frac{2\eta}{n}) \text{csch}(\eta) + \eta^2 \text{csch}(\eta)^2 \right),
\]

which is the final expression which we use in [9,8].
Appendix B. The Zak transform of the hyperbolic secant and the Gaussian

Instead of the Gramian operator we may as well have used the Zak transform [51] to compute the frame bounds. There are several normalizations of the Zak transform. For a (suitable) function \( f \) we define its Zak transform by

\[
Z_\alpha f(x, \omega) = \alpha^{-1/2} \sum_{k \in \mathbb{Z}} f \left( \frac{x + k}{\alpha} \right) e^{2\pi i k \omega}, \quad \alpha > 0.
\]

The parameter \( \alpha \) can be seen as a (unitary) dilation of the function \( f \). The Zak transform is quasi-periodic on the lattice \( \mathbb{Z} \times \mathbb{Z} \). If we seek to study Gabor systems \( G(g, a\mathbb{Z} \times b\mathbb{Z}) \) with \((ab)^{-1} \in \mathbb{N}\), then the dilation parameter may be used to adjust the geometry of the lattice. It was already observed in [33] and [35] that for \((ab)^{-1} \in \mathbb{N}\) we have

\[
\text{(B.1) } (ab)^{-1} \sum_{k,l \in \mathbb{Z}} V_g \left( \frac{k}{a}, \frac{l}{b} \right) e^{2\pi i (kx + l\omega)} = \sum_{n=0}^{N-1} Z_{b\omega} \left( \frac{x + k}{n}, \omega \right)^2.
\]

Note that this time, \( V_g g \) is sampled on the dual lattice. We may thus have used the Zak transform to compute frame bounds for integer oversampling rate. As pointed out in [35], the approach involving \( V_g g \) seems somehow more natural. Nonetheless, we note the following: Let \( \varphi(t) = 2^{1/4} e^{-\pi t^2} \). Then (see also [33], [38])

\[
Z_{b^{-1/2}} \varphi(x, \omega) = (2b)^{1/4} \sum_{k \in \mathbb{Z}} e^{-\pi b(x-k)^2} e^{2\pi i k \omega} = (2b)^{1/4} e^{-\pi bx^2} \vartheta_3(\omega - ibx, ib),
\]

where \( \vartheta_3 \) is the classical Jacobi theta-3 function (see [49])

\[
\vartheta_3(z, \tau) = \sum_{k \in \mathbb{Z}} e^{\pi i k z} e^{-2\pi i k \tau}, \quad z \in \mathbb{C}, \tau \in \mathbb{H}.
\]

By \( \mathbb{H} \) we denote the Segal upper half space, i.e., \( \tau \in \mathbb{H} \) if and only if \( \Im(\tau) > 0 \). Accompanying \( \vartheta_3 \), we have the other Jacobi theta functions:

\[
\vartheta_2(z, \tau) = \sum_{k \in \mathbb{Z}} e^{\pi i (k+1/2)^2} e^{(2k+1)\pi i z} \quad \text{and} \quad \vartheta_4(z, \tau) = \sum_{k \in \mathbb{Z}} (-1)^k e^{\pi ik^2} e^{2k\pi i z}.
\]

Denoting by \( \phi(t) = \sqrt{\pi/2} \sech(\pi t) \), it was proven in [38] that

\[
Z_{b^{-1}} \phi(x, \omega) = \sqrt{\frac{\pi}{2b}} \vartheta_1'(0, ib) e^{-\pi bx^2} \frac{\vartheta_3(\omega - ibx, ib)}{\vartheta_4(\omega, ib) \vartheta_4(x, i/b)}
\]

\[
= \sqrt{\frac{\pi}{2b^2}} b^{3/4} \vartheta_1'(0, ib) \frac{Z_{b^{-1/2}} \phi(x, \omega)}{\vartheta_4(\omega, ib) \vartheta_4(x, i/b)},
\]

where \( \vartheta_1' \) denotes differentiation of \( \vartheta_1 \) with respect to \( z \). Note that whereas the parameter in the Zak transform of \( \phi \) is \( b^{-1} \), it is only \( b^{-1/2} \) in the Zak transform of the Gaussian in the formula above. This is due to the faster decay of the Gaussian. We remark that

\[
b^{3/4} \vartheta_1'(0, ib) = 2 \left( \det' \Delta_b \right)^{3/4},
\]

where \( \det' (\Delta_b) \) is the zeta-regularized determinant (the prime indicates the regularization) of the Laplace-Beltrami operator \( \Delta_b \) on the flat torus \( \mathbb{C}/(b^{-1} \mathbb{Z} \times ib\mathbb{Z}) \) (see [13], [33]).
We recall the following result from [15] on Jacobi theta functions.

**Theorem (Theta products).** Let \( t > 0 \) and consider the following products of Jacobi theta nulls, i.e., \( z = 0 \) in the Jacobi theta function:

\[
P_2(t) = \vartheta_2(0, it)\vartheta_2(0, i/t) \quad \text{and} \quad P_4(t) = \vartheta_4(0, it)\vartheta_4(0, i/t),
\]
as well as

\[
P_3(t) = \vartheta_3(0, it)\vartheta_3(0, i/t).
\]

Then, \( P_2 \) and \( P_4 \) are maximal if and only if \( t = 1 \). Moreover, \( P_2'(t) > 0, \ t < 1 \), \( P_4'(t) < 0, \ t > 1 \) and \( P_4'(t) > 0, \ t < 1 \), \( P_4'(t) < 0, \ t > 1 \).

The product \( P_3 \) is minimal if and only if \( t = 1 \) and

\[
P_3'(t) > 0, \ t < 1 \quad \text{and} \quad P_3'(t) < 0, \ t > 1.
\]

Turning back to the Gabor system \( G(\phi, 1/b\mathbb{Z} \times b\mathbb{Z}) \), we know that the lower frame bound vanishes. However, we may still show an optimality result for the upper frame bound. We have (see [33], [35])

\[
2 \vartheta_3(0, ib) \vartheta_4(0, ib) = \vartheta_4(0, i/b)\vartheta_3(0, ib)\vartheta_4(0, ib) - \vartheta_3(0, ib)^2 \vartheta_4(0, i/b) - \vartheta_4(0, i/b)\vartheta_3(0, ib)^2.
\]

Recall the Jacobi identity for the theta-2 and theta-4 functions (see [49]), which basically follow from the Poisson summation formula and the transformation behavior of translated and modulated Gaussians under the Fourier transform:

\[
\sqrt{-i\tau} \vartheta_2(0, \tau) = \vartheta_4(0, -1/\tau), \quad \tau \in \mathbb{H}.
\]

Also, \( \vartheta_1' \) can be written as the product of the other three Jacobi theta functions [49] and we obtain the following equality

\[
\sqrt{b} \vartheta_1'(0, ib) = \sqrt{b} \vartheta_2(0, ib)\vartheta_3(0, ib)\vartheta_4(0, ib) = \vartheta_4(0, i/b)\vartheta_3(0, ib)\vartheta_4(0, ib),
\]

which follows from the Poisson summation formula (see also [49] for the Jacobi identities).

Hence, the upper bound of the Gabor system \( G(\phi, 1/b\mathbb{Z} \times b\mathbb{Z}) \) simplifies to the theta product

\[
B(b) = \frac{\pi}{2} \left| \vartheta_3(0, ib)\vartheta_4(0, ib) \right|^2 = \frac{\pi}{2} \left| \vartheta_3(0, ib^{-1})\vartheta_3(0, ib) \right|^2 = \frac{\pi}{2} \vartheta_3(0, ib^{-1})^2 \vartheta_3(0, ib)^2.
\]

By the theorem on theta products, we see that \( B \) is minimal if and only if \( b = 1 \), i.e., if and only if the lattice is \( \mathbb{Z}^2 \) (minimality among rectangular lattices of density 1, see Figure [17]). We were actually hoping to also be able to use the theorem on theta products for other integer densities and for the lower frame bound. For density \( n = 2 \), starting from the results in [33], [35], we get the lower bound of \( G(\phi, 1/b\mathbb{Z} \times b/2\mathbb{Z}) \) (see also [6, Chap. 13.2], [23 Chap. 8.3]):

\[
A(b) = \sum_{k=0}^{1} \left| Z_{b^{-1}} \phi \left( \frac{1}{4} + \frac{k}{2}, \frac{1}{2} \right) \right|^2 = 2 \left| Z_{b^{-1}} \phi \left( \frac{1}{4}, \frac{1}{2} \right) \right|^2.
\]

The second equality is due the symmetry of the hyperbolic secant which reflects itself in the Zak transform as well (see [33]). By squaring the lower frame bound and using the explicit expression of the Zak transform of the hyperbolic secant \( \phi \), we obtain

\[
A(b)^2 = 4 \left| Z_{b^{-1}} \phi \left( \frac{1}{4}, \frac{1}{2} \right) \right|^4 = 8\pi^2 b^3 \vartheta_1'(0, ib)^4 \frac{\vartheta_3(\frac{1}{2} - \frac{b}{2}, ib)^4}{\vartheta_4(\frac{1}{2}, ib)\vartheta_4(\frac{1}{2} - \frac{b}{2}, ib)}.
\]
With some effort and the right manipulations, one gets the result that $A$, expressed as above, is maximal if and only if $b = \sqrt{2}$. The optimal rectangular lattice of density 2 is hence the square lattice $1/\sqrt{2}Z \times 1/\sqrt{2}Z$. However, due to the connection given by (B.1), the expressions involving the Zak transform become algebraically more and more complicated as $n$ grows. Despite some efforts, we were not able to come up with satisfying results by following the Zak transform approach and using properties of the Jacobi theta functions.

Figure 17. For the Gabor system $G(\phi, 1/bZ \times bZ)$ the upper bound is minimal if and only if $|Z_{b^{-1}} \phi(0,0)|^2$ is minimal. This happens only for $b = 1$, which yields a square lattice of density 1. For the Gabor system $G(\phi, 1/bZ \times b/2Z)$ the lower bound is maximal if and only if $2 |Z_{b^{-1}} \phi(1/4,1/2)|^2$ is maximal which happens only for $b = \sqrt{2}$. This yields the square lattice of density 2. Note that we use a logarithmic scale on the $b$-axis.

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