The Classical Polylogarithm

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## Contents

I  Motivation  

II  Review of Hodge-theory  
   II.1 Mixed Hodge structures  
   II.2 Variations  
   II.3 Everything over $\mathbb{R}$

III The Logarithm and the polylogarithm  
   III.1 The logarithm  
   III.2 The polylogarithm extension  
   III.3 Geometric origin of $\text{Log}(1)$

IV Explicit description of the polylog  
   IV.1 Rigidity  
   IV.2 The local system underlying pol  
   IV.3 Extensions of variations of Tate-Hodge structure  
   IV.4 Explicit shape of pol

V Cyclotomic elements, and special values  
   V.1 The splitting principle  
   V.2 Cyclotomic elements

VI $K$-theory  
   VI.1 Generalized cohomology  
   VI.2 Motivic cohomology

VII Motivic polylogarithm  
   VII.1 Geometric origin of $\text{Log}$ and pol  
   VII.2 The motivic splitting principle

VIII Zagier’s conjecture
The purpose of this series of lectures was to give an overview of the central ideas in the proof of the following

**Main Theorem.** Let $d \geq 2$, $j \geq 2$, let $\mu_0^d$ be the set of primitive $d$-th roots of unity in $F := \mathbb{Q}(\mu_d)$. There is a (unique) map of sets

$$\epsilon_j : \mu_0^d \rightarrow K_{2j-1}(F)_\mathbb{Q} := K_{2j-1}(F) \otimes \mathbb{Z} \mathbb{Q}$$

whose composition with the regulator to Deligne, or absolute Hodge cohomology

$$r_H : K_{2j-1}(F)_\mathbb{Q} \rightarrow \left( \bigoplus_{\sigma : F \hookrightarrow \mathbb{C}} \mathbb{C}/(2\pi i)^j \mathbb{Q} \right)^+$$

sends $\omega \in \mu_0^d$ to the element

$$(-Li_j(\sigma \omega))_\sigma = \left( -\sum_{k \geq 1} \frac{\sigma \omega^k}{k^j} \right)_\sigma.$$

Here, $+$ denotes the invariant part under the joint operation of complex conjugation on the set of embeddings of $F$ into $\mathbb{C}$, and on $\mathbb{C}/(2\pi i)^j \mathbb{Q}$.

The result is due to Beilinson ([B1], 7.1.5). The original proof, somewhat sketchy, is beautifully reviewed in [Neu]. It relies heavily on a result on the explicit shape of a construction called the “Loday symbol” in Deligne cohomology. This so-called “Crucial Lemma” ([Neu], II.2.4) was subsequently proved in [E], 3.9.

The proof given in the course of this series is different from the original one, and makes use of the classical, or cyclotomic polylogarithm. One of the great advantages of this approach is that the complicated calculations in Deligne cohomology are no longer necessary. In fact, the polylog enjoys a characteristic property called rigidity. One of the aims of the lectures was to emphasize the rôle of rigidity played in the explicit representation of the objects.

Let us remark that the Main Theorem admits an $l$-adic counterpart: [HW], Corollary 9.7. The statement was conjectured by Bloch and Kato ([BK], Conjecture 6.2), and here the only complete proof is via polylogarithms. That the talks concentrate on the Hodge theoretic aspects of the theory has to do with the speakers’ desire to make the objects as “visible” as possible – to their taste, this requirement is satisfied to a larger degree of satisfaction by the objects of Hodge theory rather than those of the étale world. The main strategy of proof, and the abstract concepts however admit immediate translations to the $l$-adic setting.

The main ideas of the proof of the Main Theorem, and its $l$-adic counterpart, appear already in the preprint [B2]. Since then, quite a lot of polylogarithmic
literature has been published. The speakers like to think of their talks, and in fact, of this abstract, as a guide through the literature. We hope that it will be useful particularly to those who are new to the field.

We would like to thank the organizers of the workshop, G. Frey, H. Gangl, and H.-G. Rück, for the invitation to Essen. To us, it meant a great opportunity to give a self-contained exposition of some central aspects of the theory.

I Motivation

In this talk, it was tried to indicate the main strategy of proof: Consider, for a number field $F$, the regulator

$$r_H : K_{2j-1}(F)_\mathbb{Q} \longrightarrow \left( \bigoplus_{\sigma : F \to \mathbb{C}} \mathbb{C}/(2\pi i)^j \mathbb{Q} \right)^+. \quad (*)$$

The zeroth step is to introduce the concept of “Yoneda extensions in categories of mixed sheaves” in order to reinterpret both sides of $(*)$. In lecture II, we shall define, for a smooth and separated $\mathbb{R}$-scheme $X$, a $\mathbb{Q}$-linear tensor category

$$\text{Var}(X/\mathbb{R})$$

of variations of mixed $\mathbb{Q}$-Hodge structure over $\mathbb{R}$ on $X$.

The reinterpretation of the right hand side of $(*)$ acquires the following shape:

**Proposition.** Let $j \geq 1$. There is a canonical isomorphism

$$\text{Ext}^1_{\text{Var}(F \otimes \mathbb{Q} / \mathbb{R})}(\mathbb{Q}(0), \mathbb{Q}(j)) \cong \left( \bigoplus_{\sigma : F \to \mathbb{C}} \mathbb{C}/(2\pi i)^j \mathbb{Q} \right)^+$$

where $+$ denotes the invariant part under the joint operation of complex conjugation on $\{F \to \mathbb{C}\}$, and on $\mathbb{C}/(2\pi i)^j \mathbb{Q}$.

According to the motivic folklore, there should be a $\mathbb{Q}$-linear tensor category of smooth mixed motivic sheaves $\mathcal{M}_M^\bullet(X)$ on any smooth and separated scheme $X$ over $\mathbb{Q}$, together with an exact tensor functor, called the *Hodge realization*

$$\text{real}_H : \mathcal{M}_M^\bullet(X) \longrightarrow \text{Var}(X \times \mathbb{Q} \times \mathbb{R}/\mathbb{R}).$$

There should be an isomorphism

$$\text{Ext}^1_{\mathcal{M}_M(\text{Spec } F)}(\mathbb{Q}(0), \mathbb{Q}(j)) \cong K_{2j-1}(F)_\mathbb{Q}$$

for $j \geq 1$ identifying the regulator $r_H$ with the morphism induced by $\text{real}_H$. This would give the sheaf theoretical reinterpretation of the left hand side of
In first approximation, the proof of the Main Theorem, and indeed, also of its \(l\)-adic counterpart, proceeds in two steps, corresponding to lectures II–V, and VI–VII respectively:

1. Construct
\[
\rho \circ \epsilon_{j} : \mu_{d}^{0} \to \text{Ext}^{1}_{\text{Var}(\mathbb{F} \otimes \mathbb{Q} / \mathbb{R})} (\mathbb{Q}(0), \mathbb{Q}(j))
\]
first.

   a) The construction is a priori sheaf theoretical, and uses concepts like Leray spectral sequences. The objects will be characterized by certain universal properties, one consequence of which will be the earlier mentioned rigidity.

   b) Via rigidity, it is possible to describe explicitly the objects defined by abstract nonsense. In particular, we get the formula of the Main Theorem for \(\rho \circ \epsilon_{j}(\omega)\), \(\omega \in \mu_{d}^{0}\).

   c) Again via rigidity, it is possible to show that the abstract construction of a) is “geometrically motivated”: the one-extensions \(\rho \circ \epsilon_{j}(\omega)\) occur as cohomology objects, with Tate coefficients, of certain \(F\)-schemes.

2. Because of the present non-availability of a sheaf theoretical machinery on the level of motives, step 1.a) cannot simply be imitated. However, it turns out that 1.c) admits a translation to \(K\)-theory, yielding the map
\[
\epsilon_{j} : \mu_{d}^{0} \to K_{2j-1}(F)_{\mathbb{Q}}.
\]
Its compatibility with the map \(\rho \circ \epsilon_{j}\) under the regulator is then a consequence of the definition.

Let us give a more detailed account of step 1. Again, we owe to Beilinson the insight that instead of treating the \(\rho \circ \epsilon_{j}(\omega)\), \(\omega \in \mu_{d}^{0}\), \(d, j \geq 2\) separately, one should construct one object containing all the information.

In lecture III, we are going to define, by some universal property, the logarithmic \((pro-)variation\) \(\text{Log}\) on \(\mathbb{G}_{m, \mathbb{R}}\). We have
\[
\text{Gr}^{W}_{\ast} \text{Log} = \prod_{j \geq 0} \mathbb{Q}(j),
\]
i.e., \(\text{Log}\) is a successive extension of \(\mathbb{Q}(0)\) by \(\mathbb{Q}(1)\) by \(\mathbb{Q}(2)\)... The polylogarithmic extension \(\text{pol}\) is a one-extension, in the category of variations on the \(\mathbb{R}\)-scheme \(U_{\mathbb{R}} := \mathbb{P}^{1}_{\mathbb{R}} \setminus \{0, 1, \infty\}\), of \(\mathbb{Q}(0)\) by the restriction of \(\text{Log}\):
\[
\text{pol} \in \text{Ext}^{1}_{\text{Var}(U_{\mathbb{R}} / \mathbb{R})} (\mathbb{Q}(0), \text{Log}_{U_{\mathbb{R}}}).
\]
Again, the definition is via a universal property. From it, we deduce rigidity, and are consequently able, in talk IV, to give an explicit description of \(\text{pol}\),
essentially in terms of (the inverse of) its period matrix. This description will then justify the name “polylogarithm” as the entries of this matrix are essentially given by the higher logarithms.

In lecture V, we establish another characteristic feature of our objects: the so-called splitting principle. Any \( \omega \in \mu_0^d \) induces an embedding

\[
i_\omega : \text{Spec } \mathbb{Q}(\mu_d) \hookrightarrow \mathbb{U}.
\]

**Theorem.** \( i_\omega^* \text{Log} \) is canonically split:

\[
i_\omega^* \text{Log} = \prod_{j \geq 0} \mathbb{Q}(j).
\]

Therefore, we may think of \( i_\omega^* \text{pol} \) as an element of

\[
\prod_{j \geq 1} \text{Ext}^1_{\text{Var}(\mathbb{Q}(\mu_d) \otimes \mathbb{R}/\mathbb{R})}(\mathbb{Q}(0), \mathbb{Q}(j)) = \prod_{j \geq 1} \left( \bigoplus_{\sigma} \mathbb{C}/(2\pi i)^j \mathbb{Q} \right)^+.
\]

From the explicit description of pol, it is straightforward to see that the \( j \)-th component of \( i_\omega^* \text{pol} \) equals, up to scaling, the element

\[
(-L_{ij}(\sigma \omega))_{\sigma}.
\]

In this very precise sense, all the \( r_{\text{H}\sigma j}(\omega), \omega \in \mu_0^d \) are interpolated by pol. The additional data is the action of the fundamental group of \( \mathbb{U}(\mathbb{C}) \) given by the local system underlying the variation pol. In fact, rigidity is formulated in terms of this local system.

In talk VI, we give a sketch of some of the technical ingredients for a suitable formalism of “relative K-theory”. Lecture VII establishes the geometric realization of \( \text{Log} \), and of pol in absolute and motivic cohomology.

## II Review of Hodge-theory

We assemble some facts from Hodge theory that are needed in the construction of the polylogarithm. Basically everything (including references) is contained in [BZ].

### II.1 Mixed Hodge structures

We are mostly interested in Hodge structures of Tate type, i.e., ones where only Hodge numbers \((n, n)\) for \( n \in \mathbb{Z} \) occur.

**Lemma II.1.1.** There is a natural isomorphism for \( n > 0 \)

\[
\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong \mathbb{C}/(2\pi i)^n \mathbb{Q}
\]
which assigns to $s \in \mathbb{C}$ the extension class of $E_s$ where $E_{s\mathbb{C}} = \mathbb{C}^2$ and $E_{s\mathbb{Q}} \subset \mathbb{C}^2$ is given by

$$
\begin{pmatrix}
1 & 0 \\
-\frac{s}{(2\pi i)^n} & 1
\end{pmatrix}
\begin{pmatrix}
\mathbb{Q} \\
(2\pi i)^n \mathbb{Q}
\end{pmatrix}.
$$

with weight and Hodge filtration in $\mathbb{C}^2$ given by

$$W_i = \begin{cases}
(0) & i < -2n, \\
(0) & -2n \leq i < 0, \\
(0) & 0 \leq i,
\end{cases} \quad F^p = \begin{cases}
(0) & p \leq -n, \\
(0) & -n < p \leq 0, \\
(0) & 0 < p.
\end{cases}$$

Proof. E.g. [J] Lemma 9.2 and Remark 9.3.a).

Examples:

1. In particular, we have

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong \mathbb{C}/2\pi i \mathbb{Q} \xrightarrow{\exp} \mathbb{C}^* \otimes \mathbb{Q}.$$

We reinterpret this relation by saying that we assign to each $z \in \mathbb{G}_m(\mathbb{C})$ a mixed Hodge structure called $\text{Log}^{(1)}(z)$, namely $E_{\text{log}(z)}$ in the notation of the lemma.

2. For $z \in \mathbb{C}^*$ we have a long exact sequence in MHS:

$$0 \to H^0(\mathbb{G}_m(\mathbb{C})) \xrightarrow{\Delta} H^1(\mathbb{G}_m(\mathbb{C}) \text{ rel } \{1\}) \to H^1(\mathbb{G}_m(\mathbb{C})) \to 0.$$

($H$ denotes the corresponding singular cohomology as mixed Hodge structure.) Put $G^{(1)}(z) = H^1(\mathbb{G}_m(\mathbb{C}) \text{ rel } \{1\})$. This is again an element in $\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), \mathbb{Q}(1))$. We will see in lecture III that $G^{(1)}(z) = \text{Log}^{(1)}(z)$.

II.2 Variations

Let $X$ be a smooth complex algebraic variety.

Definition II.2.1. A variation $\mathcal{V}$ of mixed Hodge structure on $X(\mathbb{C})$ consists of

- a locally constant sheaf $\mathcal{V}_\mathbb{Q}$ of $\mathbb{Q}$-vector spaces on $X(\mathbb{C})$,
- an increasing filtration $W_\ast$ of $\mathcal{V}_\mathbb{Q}$ by locally constant sheaves,
- a decreasing filtration $F^\ast$ of $\mathcal{V} = \mathcal{V}_\mathbb{Q} \otimes \mathbb{Q} \mathcal{O}_X$ by holomorphic subvector bundles,
such that for each $x \in X$, the data induce a mixed Hodge structure on $V_x$ and such that Griffith transversality holds. We denote the category $\text{Var}(X(\mathbb{C}))$. A variation is called unipotent if all $\text{Gr}^W V$ are constant on $X$, e.g., all variation of Tate Hodge structure are.

**Examples:** $\text{Log}(1)$ and $G(1)$ are unipotent variations on $G_m(\mathbb{C})$.

**Lemma II.2.2.**

$$\text{Ext}^1_{\text{Var}(X(\mathbb{C}))}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong \mathcal{O}_{\text{hol}}^*(X(\mathbb{C})) \otimes \mathbb{Q}.$$

**Proof.** E.g.,[W] IV Theorem 3.7. a)

There is a notion of admissible variations of Hodge structure. Rather than giving the definition, we give their main properties which will suffice for all that follows. We denote the corresponding category $\text{Var}(X)$ to stress its algebraic nature.

1. If $X$ is compact, then all variations are admissible.
2. Everything coming from geometry is admissible, e.g., $G(1)$ is.
3. If $U$ is an algebraic variety and $X$ a smooth proper compactification, then ([W] IV Theorem 3.7 b)):

$$\text{Ext}^1_{\text{Var}(U)}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong \{g \in \mathcal{O}_{\text{hol}}^*(U) \otimes \mathbb{Q} \mid g \text{ meromorphic on } X\}$$

$$= \mathcal{O}_{\text{alg}}^*(U) \otimes \mathbb{Q}.$$

4. If $V$ is admissible on $X$, then all $H^i(X, V)$ carry a canonical mixed Hodge structure. This is a deep result, due to Steenbrink-Zucker in the case of curves, and M. Saito in general.

5. If $Y \subset X$ is an immersion of smooth varieties of pure codimension $d$, $U = X \setminus Y$ and $V \in \text{Var}(X)$, then there is a natural long exact sequence in MHS:

$$\cdots \rightarrow H^{i-2d}(Y, V_Y(-d)) \rightarrow H^i(X, V) \rightarrow H^i(U, V_U) \rightarrow H^{i+1-2d}(Y, V_Y(-d)) \rightarrow \cdots.$$

**II.3 Everything over $\mathbb{R}$**

The reference for the following is [HW] Appendix A.2. Let $X$ be a smooth algebraic variety over $\mathbb{R}$. Then there is a continuous map $\iota : X(\mathbb{C}) \rightarrow X(\mathbb{C})$ given by complex conjugation on points. It induces a functor

$$\iota^* : \text{Var}(X(\mathbb{C})) \rightarrow \text{Var}(X(\mathbb{C}))$$

$$(V, W_*, F^*) \mapsto (\iota^* V, \iota^* W_*, \iota^* F^*).$$

**Definition II.3.1.** An admissible variation of mixed Hodge structure defined over $\mathbb{R}$ is a pair $(V, F_\infty)$ where $V \in \text{Var}(X_{\mathbb{C}})$ and $F_\infty : V \rightarrow \iota^* V$ is an involution, i.e., $F_\infty^{-1} = \iota^* F_\infty$. 

8
Lemma II.3.2. Let $U$ be a smooth variety over $\mathbb{R}$ and $X$ a smooth compactification. Then

$$\text{Ext}^1_{\text{Var}(U/R)}(\mathbb{Q}(0), \mathbb{Q}(1)) = \mathcal{O}_{\text{alg}}(U) \otimes \mathbb{Q}.$$

In particular, we need the case $X = \text{Spec} K \otimes \mathbb{Q} \mathbb{R}$ where $K$ is some finite extension of $\mathbb{Q}$. Note that then $X(\mathbb{C}) = \bigoplus_{\sigma: k \rightarrow \mathbb{C}} \text{point}$. Hence

$$\text{Ext}^1_{\text{Var}(X/R)}(\mathbb{Q}(0), \mathbb{Q}(n)) = \left( \bigoplus_{\sigma: K \rightarrow \mathbb{C}} \mathbb{C}/(2\pi i)^n \mathbb{Q} \right)^+$$

where $+\text{ denotes the invariant part under the joint operation of complex conjugation on } X(\mathbb{C}) \text{ and } \mathbb{C}/(2\pi i)^n \mathbb{Q}.$

### III The Logarithm and the polylogarithm

The aim of this lecture was to construct two things:

a) a (pro)-object $\text{Log}$ in $\text{Var}(\mathbb{G}_{m,R}/\mathbb{R})$ such that (at least)

$$\text{Gr}^W_{W} \text{Log} = \prod_{n \geq 0} \mathbb{Q}(n) \text{ and } \text{Log}_1 = \prod_{n \geq 0} \mathbb{Q}(n).$$

b) on $U_R = \mathbb{P}^1_{\mathbb{R}} \setminus \{0, 1, \infty\}$ an element

$$\text{pol} \in \text{Ext}^1_{\text{Var}(U_R/\mathbb{R})}(\mathbb{Q}(0), \text{Log}_U).$$

There are three possibilities to do this: explicitly (talk IV), geometrically (end of III and VII) and by a universal property. It is this last method that we describe first.

#### III.1 The logarithm

**Theorem III.1.1** (Chen, [BZ] 6.23). Let $X$ be a smooth algebraic variety over $\mathbb{C}$. Let $x \in X(\mathbb{C})$ and $\pi = \pi_1(X(\mathbb{C}), x)$. We denote $U = \mathbb{Q}[\pi]$ and its augmentation ideal $a$. Then the completion $\hat{U} = \lim U/a^n$ carries a (unique) mixed Hodge structure such that

- $\hat{U} \otimes \hat{U} \xrightarrow{\text{mult}} \hat{U}$ and unity : $\mathbb{Q}(0) \rightarrow \hat{U}$ are morphisms of mixed Hodge structures;
- there is an isomorphism of mixed Hodge structures

$$a/a^2 \xleftarrow{\pi_1(X(\mathbb{C}), x)^{ab}} \otimes \mathbb{Q} \cong H_1(X(\mathbb{C}), \mathbb{Q}).$$
Our example is $X = \mathbb{G}_m$, $x = 1$ and hence $\pi = \mathbb{Z}\gamma$ with a positively oriented loop around 0. Then $a$ is generated by $\gamma - 1$. We get $\hat{U} = \mathbb{Q}[e]$ where $e = \log \gamma$. Note that the latter element is defined in the completion. The mixed Hodge structure on $a/a^2 \cong H_1(\mathbb{C}^*, \mathbb{Q}) \cong \mathbb{Q}(1)$ is in fact pure. All in all

$$\hat{U} = \text{Sym}^* a/a^2 = \prod_{n \geq 0} \mathbb{Q}(n).$$

**Theorem III.1.2** (Hain-Zucker, [BZ] 7.19). Let $X$ be a smooth connected algebraic variety over $\mathbb{C}$. Then there is an equivalence of categories

$$\{\text{admissible unipotent variations on } X\} \overset{\cong}{\to} \{\text{V} \in \text{MHS} \text{ with an operation } \hat{U} \otimes V \to V \text{ which is a morphism of MHS.}\}$$

where we assign to a variation its monodromy representation on the stalk at $x$.

**Definition III.1.3** ([W] p.43). $\text{Gen}_x$, the generic variation based at $x$ is the variation corresponding to the representation $\hat{U} \otimes \hat{U} \to \hat{U}$ given by multiplication.

The generic variation has a universal property. Let

$$\Gamma : \text{MHS} \to \mathbb{Q}\text{-vector spaces}$$

be the global section functor, i.e., $\Gamma(H) = \text{Hom}_{\text{MHS}}(\mathbb{Q}(0), H) = W_0 H \cap F^0 H_C$.

**Proposition III.1.4.** The pair $(\text{Gen}_x, 1 \in \Gamma(\hat{U}))$ (pro)-represents the functor

$$\Gamma(a^*?) : \{\text{unipotent objects in Var}(X)\} \to \mathbb{Q}\text{-vector spaces.}$$

**Proof.** This follows immediately from the Theorem of Hain and Zucker. $\square$

We can now identify the object we were after:

**Definition III.1.5** ([W] p. 94). Let the logarithmic sheaf on $\mathbb{G}_m$ be $\text{Log} = \text{Gen}_1$. It has the above universal property with respect to the stalk at 1, which is $\prod_{i \geq 0} \mathbb{Q}(i)$. Let $\text{Log}^{(n)} = \text{Log}/W_{-2n-2} \text{Log}$ be the quotients of finite length.

In fact $\text{Log}$ is easily seen to be defined over $\mathbb{R}$.

### III.2 The polylogarithm extension

From the Leray spectral sequence for the composition of functors

$$\text{Hom}_{\text{Var}(\mathbb{G}_m/\mathbb{R})}(\mathbb{Q}(0), ?) = \text{Hom}_{\text{MHS}/\mathbb{R}}(\mathbb{Q}(0), \underline{H}^0(\mathbb{G}_m(\mathbb{C}), ?))$$

we get the short exact sequence

$$0 \to \text{Ext}^1_{\text{MHS}/\mathbb{R}}(\mathbb{Q}(0), \underline{H}^0(U(\mathbb{C}), \text{Log}_U)) \to \text{Ext}^1_{\text{Var}(U/\mathbb{R})}(\mathbb{Q}(0), \text{Log}_U) \to \text{Hom}_{\text{MHS}/\mathbb{R}}(\mathbb{Q}(0), \underline{H}^1(U, \text{Log}_U)) \to 0.$$
Lemma III.2.1.

\[
H^0(U, \Log_U) = H^0(\mathbb{G}_m, \Log) = 0, \\
H^1(U, \Log_U) = \mathbb{Q}(-1) \oplus \Log_1(-1) = \mathbb{Q}(-1) \oplus \prod_{k \geq 0} \mathbb{Q}(k-1).
\]

The map \( H^1(U, \Log_U) \to \Log_1(-1) \) is residue at the point 1.

Hence

\[
\Ext^1_{\text{Var}(U/\mathbb{R})}(\mathbb{Q}(0), \Log_U) = \Hom_{\text{MHS}/\mathbb{R}}(\mathbb{Q}(0), \mathbb{Q}(-1) \oplus \Log_1(-1)) = \mathbb{Q}.
\]

Definition III.2.2. We define the polylogarithmic extension

\[ \text{pol} \in \Ext^1_{\text{Var}(U/\mathbb{R})}(\mathbb{Q}(0), \Log_U) \]

as the preimage of 1 under the above identification. In terms of the group \( \Hom_{\text{MHS}/\mathbb{R}}(\mathbb{Q}(0), \mathbb{Q}(-1) \oplus \Log_1(-1)) \) it is given by \( 1 \mapsto e \otimes (2\pi i)^{-1} \).

III.3 Geometric origin of \( \Log^{(1)} \)

Recall the variation \( G^{(1)} \) with fibre \( H^1(\mathbb{G}_m(\mathbb{C}) \text{ rel } \{1\} \amalg \{z\}, \mathbb{Q}(1)) \) at \( z \in \mathbb{C}^* \). It is unipotent and admissible. At \( z = 1 \), the short exact sequence of mixed Hodge structures

\[
0 \to \mathbb{Q}(1)^2 / \Delta(\mathbb{Q}(1)) \to G^{(1)}_1 \to \mathbb{Q}(0) \to 0
\]

has a splitting by \( H^1(\mathbb{G}_m(\mathbb{C}) \text{ rel } \{1\}, \mathbb{Q}(1)) \cong \mathbb{Q}(0) \). This defines a global section of \( G^{(1)}_1 \). By the universal property of \( \Log \), there is a canonical morphism

\[ \phi : \Log \to G^{(1)} \]

compatible with the projection to \( \mathbb{Q}(0) \).

Proposition III.3.1 ([HW] Theorem 4.11).

\( \phi \) induces an isomorphism on \( \Log^{(1)} \).

Proof. The morphism factors for weight reasons. It is enough to check that it induces an isomorphism on the underlying local systems. Note that both objects are 2-dimensional. The image is at least one-dimensional. If it was indeed just one-dimensional, then \( G^{(1)} \) would be split and hence constant. So all we have to see is whether \( G^{(1)} \) has non-trivial monodromy. This is not hard to do explicitly.

IV Explicit description of the polylog

In this lecture, we use the abstract definition of the polylog to deduce its main characteristic property, the so-called rigidity principle. We then determine the explicit shape of pol.
IV.1 Rigidity

Denote by $\text{loc}(M)$ the category of local systems in finite dimensional $\mathbb{Q}$-vector spaces on a topological space $M$, and by

$$\text{For} : \text{Var}(X/\mathbb{R}) \rightarrow \text{loc}(X(\mathbb{C}))$$

the forgetful functor.

**Theorem IV.1.1** ([B2], 2.1, [W], III, Theorem 2.1). pol is uniquely determined by

$$\text{For}(\text{pol}) \in \text{Ext}^1_{\text{loc}(U(\mathbb{C}))}(\mathbb{Q}, \text{For}(\text{Log}_U)).$$

**Proof.** There is a commutative diagram of boundary morphisms in Leray spectral sequences

$$\begin{array}{c}
\text{Ext}^1_{\text{Var}(U/\mathbb{R})}(\mathbb{Q}(0), \text{Log}_U) \\ \text{For} \downarrow \\
\text{Ext}^1_{\text{loc}(U(\mathbb{C}))}(\mathbb{Q}, \text{For}(\text{Log}_U)) \\
\end{array} \longrightarrow \begin{array}{c}
\text{Hom}_{\text{MHS}/\mathbb{R}}(\mathbb{Q}(0), H^1(U, \text{Log}_U)) \\ \text{For} \\
\text{Hom}_{\mathbb{Q}}(\mathbb{Q}, H^1(U(\mathbb{C}), \text{For}(\text{Log}_U))) \\
\end{array}$$

By III.2, the upper horizontal map is an isomorphism. Since For is injective on the level of homomorphisms, we see that the left vertical map is injective, too.

Recall from Lemma III.2.1 that

$$H^1(U, \text{Log}_U) = \mathbb{Q}(-1) \oplus \text{Log}_1(-1) = \mathbb{Q}(-1) \oplus \prod_{k \geq 0} \mathbb{Q}(k - 1),$$

and that $H^0(U, \text{Log}_U) = 0$. It follows as in III.2 that the lower horizontal map of the diagram in the proof of the theorem is an isomorphism, and that it maps the class of $\text{For}(\text{pol})$ to the morphism

$$\mathbb{Q} \rightarrow H^1(U(\mathbb{C}), \text{For}(\text{Log}_U)) = \text{For} \left( \mathbb{Q}(-1) \oplus \prod_{k \geq 0} \mathbb{Q}(k - 1) \right),$$

$$1 \mapsto \frac{1}{2\pi i} \cdot e.$$
circle around $i$.

Define the following representation of $\bar{\pi}$:

\[
E := \langle 1 \rangle_{\mathbb{Q}} \oplus \langle e^k, k \geq 0 \rangle_{\mathbb{Q}},
\]

\[
\alpha_0 : 1 \mapsto 1,
\]

\[
e^k \mapsto e^k \cdot \exp(e).
\]

\[
\alpha_1 : 1 \mapsto 1 + e,
\]

\[
e^k \mapsto e^k.
\]

We get an extension of $\bar{\pi}$-modules, i.e., of local systems on $U(\mathbb{C})$

\[
0 \to \text{For}(\text{Log}_U) \to E \to \mathbb{Q} \to 0
\]

(recall from III.1 that $e = \log \alpha_0$).

From the remark following Theorem IV.1.1, one concludes:

**Proposition IV.2.1** ([B2], 2.1, [W], IV, Theorem 2.2). The class of the above extension equals $\text{pol}$. 

### IV.3 Extensions of variations of Tate-Hodge structure

We need to develop a language in which we can describe variations explicitly. The following will be crucial:

**Theorem IV.3.1.** Let $(\mathcal{V}_Q, W_*, \mathcal{F}^*) \in \text{Var}(X(\mathbb{C}))$ be a variation of THS (Tate-Hodge structure). Then the underlying bifiltered vector bundle

\[
(\mathcal{V}, W_*, \mathcal{F}^*)
\]

is canonically split. $\mathcal{V}$ and all $\mathcal{F}^p \mathcal{W}_{2p} \mathcal{V}$ are generated by global sections.

**Proof.** Since $\text{Gr}_p^S \cdot \text{Gr}_n^{W_*} \mathcal{V} = 0$ for $n \neq 2p$, we have

\[
\mathcal{V} = \bigoplus_p \mathcal{F}^p \mathcal{W}_{2p} \mathcal{V}.
\]

For any $p$, we have canonically

\[
\mathcal{F}^p \mathcal{W}_{2p} \mathcal{V} \xrightarrow{\sim} \text{Gr}_{W_*}^{2p} \mathcal{V},
\]

which is constant. 

This gives our recipe for describing variations of THS: Let $(\mathcal{V}_Q, W_*, \mathcal{F}^*)$ be one such.
1. Choose a basis of global sections of $V$ respecting the decomposition

$$V = \bigoplus_p \mathcal{F}^p W_{2p} V.$$ 

2. Express a basis of $Q$-rational flat sections respecting the weight filtration $W_* V_Q$ in the basis of 1.

The result will be a lower triangular matrix. Its entries will in general consist of multivalued functions since sections of $V_Q$ will usually only exist on the universal cover of $X(\mathbb{C})$.

Actually, we already applied this recipe: If $X$ is a point, Lemma II.1.1 determines one-extensions of $Q(0)$ by $Q(n)$. Lemma II.2.2 describes one-extensions of $Q(0)$ by $Q(1)$ for arbitrary $X$.

### IV.4 Explicit shape of pol

In order to write down a matrix describing pol in the sense of the previous section, we need to define some multivalued functions:

**Definition IV.4.1.**

$$L_i(t) := -\log(1 - t),$$

$$L_{i+1}(t) := \int_0^t \frac{L_i(s)}{s} ds, \quad k \geq 1,$$

$$\Lambda_k := \frac{1}{(-2\pi i)^k} \sum_{n=1}^k \frac{(-\log)^{k-n}}{(k-n)!} L_n.$$ 

Using IV.2.1, one proves:

**Lemma IV.4.2 ([W], IV, Lemma 3.3).**

$$f := 1 + \sum_{k=1}^{\infty} \Lambda_k \cdot e^k$$

is a global section of

$$\text{For}(\text{pol}) \otimes Q \mathcal{O}(\mathbb{U}(\mathbb{C})).$$

So in the basis of global sections $(f, e_0, e_1, \ldots)$, where

$$e_k : t \mapsto e^k \cdot \exp \left( \frac{\log(t)}{2\pi i} \cdot e \right) = e^k + \frac{\log(t)}{2\pi i} \cdot e^{k+1} + \ldots,$$
the rational structure is described by the following matrix $P$:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & \cdots \\
\frac{1}{2\pi i} \text{Li}_1 & \frac{1}{2\pi i} \log & 1 & 0 & 0 & \cdots \\
\frac{1}{(2\pi i)^2} \text{Li}_2 & \frac{1}{2\pi i} \left(\frac{1}{2\pi i} \log\right)^2 & -\frac{1}{2\pi i} \log & 1 & 0 & \cdots \\
\frac{1}{(2\pi i)^3} \text{Li}_3 & \frac{1}{2\pi i} \left(\frac{1}{2\pi i} \log\right)^3 & \frac{1}{2\pi i} \left(\frac{1}{2\pi i} \log\right)^2 & -\frac{1}{2\pi i} \log & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}
$$

We need to know that if we define $\mathcal{F}^0$ as the span of $\mathcal{F}^0(\text{Log}_U) = \langle e_0 \rangle$ and of $f$ (rather than $f +$ some non-zero global section of $\text{Log}_U$), then we get an admissible variation of THS on $U$. Modulo a shift of the filtrations, this is the content of [W], IV, Theorem 3.5:

**Theorem IV.4.3.** If we let $f$ be a section of $\mathcal{F}^0$, then the data define an admissible variation on $U$. Because of rigidity, it equals pol. Therefore, $P$ is the matrix describing pol in the sense of IV.3.

For the description of pol in $l$-adics, see [B2], 3.3, or [W], IV.4.

### V Cyclotomic elements, and special values

The talk given at the workshop concerned itself with two further properties of our objects: the *splitting principle*, and *norm compatibility*. Since the latter plays no strategic rôle in the proof of the Main Theorem, we refer to [W], pp. 224-226 for a detailed account.

#### V.1 The splitting principle

Splitting over roots of unity is a property of the logarithmic sheaf $\text{Log}$ rather than of pol. Let $\omega \in \mathbb{G}_m(\mathbb{C})_{\text{tors}}$, and consider the mixed Tate-Hodge structure $\omega^* \text{Log}$.

**Proposition V.1.1.** $\omega^* \text{Log}$ splits canonically:

$$
\omega^* \text{Log} = \prod_{j \geq 0} \mathbb{Q}(j).
$$

**Proof.** One can either employ the universal property III.1.4 of $\text{Log}$ to deduce a canonical isomorphism

$$
\text{Log} \xrightarrow{\sim} [n]^* \text{Log},
$$

where $[n] : \mathbb{G}_m \to \mathbb{G}_m, t \mapsto t^n$. Since $1^* \text{Log}$ is split, so is the fibre of $\text{Log}$ at any preimage of 1 under $[n]$.

Or use Lemma II.1.1, and the explicit description of $\text{Log}$. \qed
V.2 Cyclotomic elements

The splitting principle provides us with canonical projections

\[ \text{pr}_{\omega,j} : \omega^* \text{Log} \to \mathbb{Q}(j), \]

for any root of unity \( \omega \), and any \( j \geq 1 \). We get an induced map \( (\text{pr}_{\omega,j})_* \) on the level of Ext groups.

**Proposition V.2.1.** For any \( \omega \neq 1 \), we have

\[ (\text{pr}_{\omega,j})_* (\omega^* \text{pol}) = (-1)^j \text{Li}_j(\omega) \mod (2\pi i)^j \mathbb{Q} \]

in \( \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), \mathbb{Q}(j)) = \mathbb{C}/(2\pi i)^j \mathbb{Q} \).

**Proof.** Look at the matrix \( P(\omega)! \)

VI K-theory

The general motivic philosophy says (among other things):

**Conjecture VI.0.2** (Beilinson et. al.). For all varieties over \( \mathbb{Q} \), there is a universal cohomology theory \( X \mapsto h^*(X) \) with values in an abelian category \( \mathcal{M} \mathcal{M} \) (mixed motives) and a universal cohomology theory with values in \( \mathcal{M} \mathcal{M} \). There is also a universal absolute cohomology theory (motivic cohomology) such that the Leray spectral sequence gives short exact sequences

\[ 0 \to \text{Ext}^1_{\mathcal{M} \mathcal{M}}(\mathbb{Q}(0), h^{n-1}(X)(j)) \to H^n_{\mathcal{M}}(X, j) \to \text{Hom}_{\mathcal{M} \mathcal{M}}(\mathbb{Q}(0), h^n(X)(j)) \to 0. \]

Moreover, for smooth varieties \( X \), there should be natural isomorphisms

\[ H^i_{\mathcal{M}}(X, j) \cong \text{Gr}^j_i K_{2j-i}(X)_{\mathbb{Q}}. \]

This leads us to define

**Definition VI.0.3.** For smooth \( X \) over \( \mathbb{Z} \), we put \( H^i_{\mathcal{M}}(X, j) \cong \text{Gr}^j_i K_{2j-i}(X)_{\mathbb{Q}}. \)

We call this motivic cohomology of \( X \).

We need to extend this definition. We want

- relative motivic cohomology groups,
- motivic cohomology of certain singular varieties,
- localization sequences in this context.

We use the approach of Gillet and Soulé [GS] also used in [dJ] by de Jeu. Details can be found in [HW] Appendix B. The following is a quick and very imprecise overview.
VI.1 Generalized cohomology

We recall that the geometric realization functor induces an equivalence of categories between simplicial sets and CW-complexes up to homotopy. On the other hand, the ‘associated complex’ functor gives an equivalence of categories between simplicial abelian groups and cohomological complexes concentrated in negative degrees, both up to homotopy. We sheafify these notions for the Zariski-topology. By this we mean the big or small site of smooth schemes over a fixed smooth $\mathbb{Z}$-scheme $S_0$, equipped with the Zariski-topology.

**Definition VI.1.1** ([HW] B.1). A space $Y$ is a simplicial sheaf of sets for the Zariski-topology which is pointed by a map $\ast \to Y$. Here $\ast$ is the constant simplicial object associated to the constant sheaf $S \mapsto \{\ast\}$. A morphism $y : Y \to Y'$ is called weak equivalence if the induced morphisms on the sheafified homotopy groups are isomorphisms for all choices of base point.

Spaces form a closed model category in the sense of Quillen. This means that they behave ‘like topological spaces’. In particular, we can form suspensions $S_\ast$, cones, loop spaces and form a homotopy category by formally inverting weak equivalence.

If $X$ is a scheme, let $\tilde{X}$ be the constant simplicial object associated to $S \mapsto X(S) \cup \{\ast\}$ pointed by the disjoint copy of $\ast$.

**Definition VI.1.2** ([HW] B.1.1). A space $Y$ is constructed from schemes if all $Y_n$ are of the form $\ast \cup \text{scheme}$. We define generalized cohomology of $Y$ with coefficients in a space $A$ by

$$H^{-m}_{\text{gen}}(Y, A) = \left[ S^m Y, A \right]$$

for $m \geq 0$ where $[\cdot, \cdot]$ denotes morphisms in the homotopy category.

In particular, we use $A = K$, the sheafification of $S \mapsto \mathbb{Z} \times \mathbb{Z} \times \text{BGl}(S)$. We speak of $K$-cohomology.

**Proposition VI.1.3** (Brown-Gersten). If $Y = \tilde{X}$ for a scheme $X$ in the site, then $H^{-m}_{\text{gen}}(\tilde{X}, K) = K_m(X)$.

Cohomology of abelian sheaves, e.g., absolute Hodge cohomology, can also be written as generalized cohomology of a space $K(A)$. So one point of generalized cohomology is that it allows to treat $K$-groups and cohomology of abelian sheaves on an equal footing. Gillet has constructed Chern classes $ch_j : K \to K(A)$ for good $A$ like the complexes defining absolute Hodge cohomology.

VI.2 Motivic cohomology

Gillet and Soulé have constructed maps $\lambda^i : K \to K$ for $i \geq 1$ such that $H^i_{\text{gen}}(Y, K)$ is a turned into a $\lambda$-algebra, at least if $Y$ is constructed from schemes. (We are lying here! See [HW] B.2.) Hence we also have a $\gamma$-filtration on $K$-cohomology.
Definition VI.2.1. If $Y$ is constructed from schemes and $2j + i \leq 0$, let

$$H^j_{\text{H}}(Y, j) = \text{Gr}^j_0 H^i_{\text{gen}}(Y, \mathbb{K}) \mathbb{Q}.$$ 

The extension of the definition to spaces allows a lot of extra flexibility.

Examples:

1. If $T = \bigcup_{i \in I} C_i$ where all $C_i$ and all $\bigcap_{i \in I'} C_i$ for subsets $I'$ of $I$ are smooth over our base $S_0$, then we put $T^* = \text{cosk}^0(\bigcup_{i \in I} C_i / T)$ i.e.

$$T_0 = \bigcap_{i \in I} C_i \quad \text{and} \quad T_n = \bigcap_{i \in I_n} C_i$$

where $I_n$ runs through all $n+1$-tuples of elements in $I$. The big advantage of $T^*$ is that all its components are smooth. By adding a disjoint base point we turn this into a space $\tilde{T}^*$ constructed from schemes. We define $H^i_{\text{H}}(T^*, j) = H^i_{\text{H}}(\tilde{T}^*, j)$.

2. Let $T$ as before, $T \subset X$ where $X$ is also smooth. We put $H^i_{\text{H}}(X \text{ rel } T, j) = H^i_{\text{H}}(\text{Cone}(\tilde{T}^* \to \tilde{X}), j)$ where the cone is taken in the category of spaces. By definition it sits in a long exact sequence for relative cohomology.

Theorem VI.2.2 (Soulé, de Jeu,[HW] B.2.16). Let $T$ and $X$ be as in the example. Let $Z \subset X$ be smooth of pure codimension $d$. Suppose that $Z$ intersects all $\bigcap_{i \in I'} C_i$ transversally. Let $U = X \setminus Z$. Then there is a natural long exact sequence

$$\cdots \to H^{i-2d}_{\text{H}}(Z \text{ rel } T \cap Z, j-d) \to H^i_{\text{H}}(X \text{ rel } T, j) \to H^i_{\text{H}}(U \text{ rel } T \cap U, j) \to \cdots.$$ 

Moreover, it is compatible with the same sequence in absolute Hodge cohomology via the Chern class morphism.

VII Motivic polylogarithm

We now need mixed Hodge modules over $\mathbb{R}$ ([HW] Appendix A). The category is denoted $\text{MHM}(X/\mathbb{R})$. Whereas admissible variations are the Hodge theoretic version of locally constant sheaves, Hodge modules correspond to (perverse) constructible sheaves.

Definition VII.0.3. Let $X$ be a smooth variety over $\mathbb{R}$ and $Y \subset X$ with complement $j: U \to X$. We define absolute Hodge cohomology by

$$H^k_{\text{abs}}(X/\mathbb{R}, \mathbb{Q}(n)) = \text{Ext}^k_{\text{MHM}(X/\mathbb{R})}(\mathbb{Q}(0), \mathbb{Q}(n))$$

$$H^k_{\text{abs}}(X \text{ rel } Y/\mathbb{R}, \mathbb{Q}(n)) = \text{Ext}^k_{\text{MHM}(X/\mathbb{R})}(\mathbb{Q}(0), j!\mathbb{Q}(n)).$$

It can be shown that this agrees with Beilinson’s ad hoc version ([HW], Theorem A.2.7). Everything done in this talk translates immediately into the $l$-adic setting.
VII.1 Geometric origin of \( \text{Log and pol} \)

We now come to a quick tour through [HW]. We consider the following geometric situation with \( U = \mathbb{P}^1 \setminus \{0, 1, \infty\} \) and \( Z = 1 \times U \Delta, V = \mathbb{G}_{m, U} \setminus Z, Z^{(n)} = \mathbb{G}_{m, U} \setminus V^n: \)

\[
\begin{align*}
V \xrightarrow{p} & \mathbb{G}_{m, U} \leftarrow Z \\
\downarrow p & \quad \downarrow p^n \quad \downarrow p^n \\
U & \quad U & \quad U
\end{align*}
\]

Recall from lecture III that \( \text{Log}_U^{(1)} = G^{(1)} = R^1 p_* \mathcal{V}(1) \) where we now use the correct formulation in terms of Hodge modules. Hence

\[
\text{Log}_U^{(n)} = \text{Sym}^n \text{Log}_U^{(1)} = \text{Sym}^n R^1 p_* \mathcal{V}(1) = R^n p_* v^n Q(n)^{\text{sgn}}
\]

where we have to take the sign-eigenspace with respect to the operation of the symmetric group because the cup-product is anti-symmetric.

**Definition VII.1.1.** Let \( G^{(n)} = R^n p_* v^n Q(n)^{\text{sgn}}. \)

With this definition we have ([HW] §4)

\[
\begin{align*}
\text{Ext}^1_{\text{Var}(U/R)}(Q(0), G^{(n)}) = \text{Ext}^1_{\text{MHM}(\mathbb{G}_{m, U}/R)}(Q(0), R^n p^n v^n Q(n)^{\text{sgn}}) \\
&= \text{Ext}^1_{\text{MHM}(\mathbb{G}_{m, U}/R)}(Q(0), v^n Q(n)^{\text{sgn}}) \\
&= H^{n+1}_{\text{abs}}(\mathbb{G}_{m, U} \text{ rel } Z^{(n)}, Q(n))^{\text{sgn}}.
\end{align*}
\]

Note that the corresponding motivic cohomology groups \( H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m, U} \text{ rel } Z^{(n)}, Q(n))^{\text{sgn}} \) are also well-defined!!! Our main tool in the sequel is the **Residue sequence**.

**Proposition VII.1.2** ([HW] after 4.6 and 7.2). Let \( \overline{Z} = \mathbb{A}^1_U \setminus V. \) There are long exact sequences in motivic and absolute Hodge cohomology, which are also compatible under Chern classes:

\[
\cdots \to H^i_{\overline{Z}}(\mathbb{A}^1_U \text{ rel } \overline{Z}^{(n)}, j)^{\text{sgn}} \to H^i_{\overline{Z}}(\mathbb{G}_{m, U} \text{ rel } Z^{(n)}, j)^{\text{sgn}} \\
\to H^{i-1}_{\overline{Z}}(\mathbb{G}_{m, U} \text{ rel } Z^{(n-1)}, j - 1)^{\text{sgn}} \to H^{i+1}_{\overline{Z}}(\mathbb{A}^1_U \text{ rel } \overline{Z}^{(n)}, j)^{\text{sgn}} \to \cdots
\]

**Proof.** In the \( n = 2 \)-case we consider the localization sequences in relative cohomology for the two inclusions \( \mathbb{G}_{m, U} \subset A^2 \setminus (0, 0) \subset A^2. \) The effect of the sign-eigenspaces leads to the above form of the sequence. \( \square \)

**Definition VII.1.3.** The residue maps are given by the map from the residue sequence \( \text{res}_n : H^i_n(\mathbb{G}_{m, U} \text{ rel } Z^{(n)}, n) \to H^i_n(\mathbb{G}_{m, U} \text{ rel } Z^{(n-1)}, n). \)

**Lemma VII.1.4** ([HW] 7.3). \( H^i_{\overline{Z}}(\mathbb{A}^1_U \text{ rel } \overline{Z}^{(n)}, n) \cong H^{i-n}_{\mathbb{A}^1_U}(U, j). \)
There is an alternative description of the localization sequence in the case of absolute Hodge cohomology. The same constructions that led to the residue sequence, also lead to a sequence of variations on $U$ by using $H^i_U(\cdot) = R^i p_*(\cdot)$, namely

$$0 \to \mathbb{Q}(n)_U \to H^i_U(\mathbb{G}^n_{m,U} \text{ rel } \mathbb{Z}^{(n)}_n \text{sgn}) \to H^i_U(\mathbb{G}^{n-1}_{m,U} \text{ rel } \mathbb{Z}^{(n-1)}_n \text{sgn}) \to 0.$$ 

Note that $H^n_U(\mathbb{G}^n_{m,U} \text{ rel } \mathbb{Z}^{(n)}_n \text{sgn})$ is just a different way of writing $G^{(n)}$.

**Proposition VII.1.5** ([HW] 4.9, 4.8). The square

$$G^{(n)} \xrightarrow{\text{res}} G^{(n-1)} \cong \begin{array}{c} \downarrow \cong \\text{Sym}^n G^{(1)} \xrightarrow{\text{proj}} \text{Sym}^{n-1} G^{(1)} \end{array}$$

commutes. Hence the transition maps of $\text{Log}$ are of geometric origin. Moreover the residue sequence in absolute Hodge cohomology is the long exact sequence attached to the short exact sequence of sheaves above.

Up to now everything would have worked for a general base $S$ instead of $U$. But now we use the simple form of $U$. Its cohomology is Tate and by Borel’s theorem we understand the corresponding motivic cohomology and the regulator very well.

**Lemma VII.1.6** ([HW] 8.3). With $B = \text{Spec } \mathbb{Z}$, the following composition is bijective:

$$H^n_M(B,0) \xrightarrow{\text{res}} \bigoplus_{i=0,1} H^n_M(B,0) = H^n_M(U,1) \to H^2_M(\mathbb{G}^n_{m,U} \text{ rel } Z,1).$$

Call the inverse map res, the total residue.

**Theorem VII.1.7** ([HW] Corollary 8.8). We have a commutative square:

$$\lim_{\text{res at 1}} H^n_M(\mathbb{G}^n_{m,U} \text{ rel } Z^{(n)}_n \text{sgn}) \xrightarrow{\text{res}} H^1_{\text{abs}}(U_R, \text{Log}_U) \xrightarrow{\text{res at 1}} H^0_{\text{abs}}(B_{\mathbb{Z}}(0)).$$

The map res on the left is an isomorphism.

We now define $\text{pol}_M$, the motivic polylogarithm simply as $\text{res}^{-1}(1)$. By construction, $r_H \text{pol}_M = \text{pol}$.

### VII.2 The motivic splitting principle

Let $d \geq 2$ and $b$ prime to $d$. Let $C = \text{Spec } \mathbb{Z}[T]/\Phi_d(T)[1/d]$. It embeds canonically into $U$. It can be twisted by raising to the $b$-th power on $C$. Call the resulting embedding $i_b: C \to U$. The morphism $[d+1]$ on $\mathbb{G}^n_m$ (raising to the $d+1$-th power) respects $Z_C$. We can analyze the eigenvalues of this operation and find:
Proposition VII.2.1 ([HW] Lemma 9.3). There is a natural splitting
\[ H^{n+1}_M(\mathbb{G}_{m,C} \text{ rel } \mathbb{Z}^{(n)}_{C}, n) = \prod_{1 \leq i \leq n} H^1_M(C, i). \]
The splittings are compatible in the projective system and they are also compatible with the splitting in absolute Hodge cohomology induced by the splitting principle there.

We can now prove our main theorem from lecture I:

Definition VII.2.2. Let
\[ \epsilon : \{ \text{primitive } d\text{-th roots of unity} \} \to H^1_M(C, n) \]
\[ T^b \mapsto (-1)^{n-1} \frac{1}{n!} \text{n-component of } i_* \text{ pol}_M. \]

Clearly \( r_M \epsilon(\omega) = (-1)^{n-1} \frac{1}{n!} \text{n-component of } \omega^* \text{ pol} \) whose explicit value was computed in lecture V.

VIII Zagier's conjecture

This talk was devoted to the presentation of the main ideas of the article [BD].

The weak version of Zagier's conjecture, meanwhile a theorem of de Jeu's ([dJ]) concerns itself with the \( K \)-theory of number fields. There is a conjecture for any integer \( j \geq 1 \), and the \( j \)-th can only be formulated if the preceding ones are true.

Fix a number field \( F \). One wants to construct a \( \mathbb{Q} \)-vector space \( \mathcal{L}_j \), a map
\[ \{ \} : \mathbb{U}(F) = F^* \setminus \{ 1 \} \to \mathcal{L}_j, \]
a homomorphism
\[ d_j : \mathcal{L}_j \to \bigwedge^2 \left( \bigoplus_{l=1}^{j-1} \mathcal{L}_l \right), \]
and a monomorphism
\[ \varphi_j : \ker(d_j) \hookrightarrow K_{2j-1}(F)_\mathbb{Q}. \]
For \( j = 1 \), one defines \( \mathcal{L}_1 := F^* \otimes_{\mathbb{Z}} \mathbb{Q}, \)
\[ \{ x \}_1 := (1 - x) \otimes 1 \in \mathcal{L}_1, \]
\( d_1 := 0 \), and \( \varphi_1 \) as the isomorphism between \( F^* \otimes_{\mathbb{Z}} \mathbb{Q} \) and \( K_1(F)_\mathbb{Q} \).
For $j \geq 2$, let $\tilde{L}_j$ be the free $\mathbb{Q}$-vector space in the symbols $\{x\}_j$, $x \in F^* \setminus \{1\}$.

Define
\[
\tilde{d}_j : \tilde{L}_j \rightarrow L_{j-1} \otimes_{\mathbb{Q}} L_1 \rightarrow \bigwedge^2 \left( \bigoplus_{l=1}^{j-1} L_l \right)
\]
by sending the symbol $\{x\}_j$ to $\{x\}_{j-1} \wedge x$.

The conjecture predicts a map
\[
\tilde{\varphi}_j : \ker(\tilde{d}_j) \rightarrow K_{2j-1}(F)_{\mathbb{Q}}.
\]

It also predicts the explicit shape of the composition
\[
r_{H \circ \tilde{\varphi}_j} : \ker(\tilde{d}_j) \rightarrow \left( \bigoplus_{\sigma : F \to \mathbb{C}} \mathbb{C}/(2\pi i)^j \mathbb{R} \right)^+.
\]

If the conjecture holds, one sets
\[
L_j := \tilde{L}_j / \ker(\tilde{\varphi}_j).
\]

In the talk, it was explained, following [BD], section 2, that the conjecture follows from the motivic folklore, explained in lecture I, plus the existence of “pol $\in \mathcal{M} \mathcal{M}^*(U)$”. More precisely, if $S \in \ker(\tilde{d}_j)$, then $\tilde{\varphi}_j(S)$ is obtained by a linear combination of “coefficients” of the value of pol at points $x \in \mathcal{U}(F)$.

The material covered in lectures II-VII can be seen as a description of $\tilde{\varphi}_j$ on a very particular kind of elements of $\ker(\tilde{d}_j)$, namely those symbols concentrated on roots of unity.

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