Quark Mass Renormalization in the MS and RI schemes up to the NNLO order

Enrico Franco\textsuperscript{a} and Vittorio Lubicz\textsuperscript{b}

\textsuperscript{a} Dip. di Fisica, Università degli Studi di Roma “La Sapienza” and INFN, Sezione di Roma, P.le A. Moro 2, 00185 Roma, Italy.
\textsuperscript{b} Dip. di Fisica, Università di Roma Tre and INFN, Sezione di Roma, Via della Vasca Navale 84, I-00146 Roma, Italy

Abstract

We compute the relation between the quark mass defined in the minimal modified MS scheme and the mass defined in the “Regularization Invariant” scheme (RI), up to the NNLO order. The RI scheme is conveniently adopted in lattice QCD calculations of the quark mass, since the relevant renormalization constants in this scheme can be evaluated in a non-perturbative way. The NNLO contribution to the conversion factor between the quark mass in the two schemes is found to be large, typically of the same order of the NLO correction at a scale $\mu \sim 2$ GeV. We also give the NNLO relation between the quark mass in the RI scheme and the renormalization group-invariant mass.
1 Introduction

The values of quark masses are of great importance in the phenomenology of the Standard Model and beyond. For instance, bottom and charm quark masses enter significantly in the theoretical expressions of inclusive decay rates of heavy mesons, while the strange quark mass plays a crucial role in the evaluation of the $K \to \pi\pi$, $\Delta I = 1/2$, decay amplitude and of the CP-violation parameter $\epsilon'/\epsilon$.

In the Standard Model, quark and lepton masses are fundamental parameters. Quark masses, however, cannot be measured directly, since quarks do not appear as physical states and are confined into hadrons. Thus, for quarks the kinematical concept of on-shell mass is meaningless. The values of the quark masses then depend on precisely their definitions, which can be given in terms of a short distance mass in some renormalization scheme.

A popular definition of the quark mass is the $\overline{\text{MS}}$ mass. For each flavour of quarks, this quantity is a well defined, short distance running coupling, provided the relevant scale $\mu$ is chosen inside the perturbative region of QCD. An alternative, but equally convenient definition of the quark mass is the so called invariant quark mass. This has the major advantage of being both a scale and a scheme independent quantity. At present, the connection between the invariant mass and the $\overline{\text{MS}}$ mass is known up to four-loop order in QCD [1].

For heavy quarks only, the pole mass has also been considered and its relation to the $\overline{\text{MS}}$ mass has been computed up to two-loop order [2]. Since, however, for confined quarks there should not be any pole in the full propagator, the definition of the pole mass makes sense only in perturbation theory. On the other hand, the pole mass is affected by a renormalon ambiguity [3, 4], which prevent the possibility of precisely define its perturbative expansion. For this reason, and also because the pole mass can be only defined in the heavy quark case, a short distance definition of the quark mass, as the $\overline{\text{MS}}$ or the invariant quark masses, should be always preferred.

Despite the great theoretical effort which has been spent in the last few years to determine the values of quark masses, these quantities still remain among the least well known parameters in the Standard Model. This is especially true in the light quark sector, where the relative uncertainties on the up, down and strange quark masses are estimated to be as large as 50% [5].

Lattice QCD is in principle able to predict the mass of any quark by fixing, to its experimental value, the mass of a hadron containing a quark with the same flavour [6]. The quark mass that is directly determined in lattice simulations is the (short distance) bare lattice quark mass $m(a)$, where $a$ is the lattice spacing, the inverse of the UV cut-off. The conversion factor relating $m(a)$ to a continuum renormalized mass, $m(\mu)$, only depends on typical scales of the order of $\mu \sim a^{-1} \sim 2 - 4$ GeV. In this respect lattice QCD is unique, since QCD sum rules calculations of quark masses have to work at much smaller scales, where higher-order corrections and/or non-perturbative effects [7] may be rather large.
It has been observed that perturbative calculations, within the lattice regularization, suffer from large higher-order corrections \[8\]. This is due to the presence of tadpole-like diagrams, which are absent in continuum perturbation theory. Such corrections might then introduce a large uncertainty in the calculation of the renormalization constant relating the bare lattice quark mass \(m(a)\) to the renormalized mass \(m(\mu)\) in a continuum scheme. This constant is only known at one-loop order \[9, 10\], and its NLO renormalization group-improved version has been derived in ref. \[11\].

The uncertainties related to the perturbative calculation of the lattice renormalization constants can be completely avoided by using the non-perturbative renormalization program proposed in ref. \[12\]. Within this approach, the renormalization constants of lattice bare correlation functions, or couplings and masses, are computed in a non-perturbative way in the so-called “Regularization Invariant” (RI) scheme. When required, the connection between the renormalized quantities in the RI scheme and those defined in any other scheme, like the \(\overline{\text{MS}}\) one, can be then computed in continuum perturbation theory.

Recently, an extensive lattice calculations of light quark masses, with non-perturbatively determined renormalization constants in the RI scheme, has been performed in ref. \[13\]. The relation between the quark mass in the RI and \(\overline{\text{MS}}\) schemes has been computed at one loop in ref. \[12\]. This result, combined with the two-loop expansion of the QCD \(\beta\)-function and the quark mass anomalous dimension, has been then converted into a NLO renormalization group-improved determination of the quark masses.

At present, both the \(\beta\)-function and the mass anomalous dimension in the \(\overline{\text{MS}}\) scheme have been computed up to four loops, in refs. \[14\] and \[1\, 15\] respectively. Thus, the two-loop conversion factor \(R_m\), relating the quark masses in the RI and \(\overline{\text{MS}}\) schemes, is the only missing ingredient necessary to convert the non-perturbative lattice determinations of the quark masses to their \(\overline{\text{MS}}\) counterparts at the NNLO. Such a calculation is the main result of this paper. The factor \(R_m\) is given in eq. \([16]\) for \(N_c = 3\) and in the Landau gauge, which is the gauge usually adopted in RI-scheme lattice calculations. We find that the size of the NNLO corrections is quite large, of the order of 7% at a typical scale \(\mu \sim 2\) GeV, and comparable with the NLO contribution.

The paper is organized as follows. In sec. \[2\], we discuss the renormalization conditions for the quark mass, by referring in particular to the \(\overline{\text{MS}}\) and RI schemes. The perturbative expressions for the relevant renormalization constants and the ratio \(R_m\) are given in sec. \[3\]. In sec. \[4\] we discuss the evolution of the running quark mass, we introduce the invariant quark mass and express the result of our calculation as a relation between the invariant mass and the mass in the RI scheme.

\[1\] An explorative study can be found in ref. \[11\].
Quark mass renormalization schemes

The quark mass renormalization constant, \( Z_m(\mu) \), relates the bare quark mass \( m_0 \) to the renormalized mass \( m(\mu) \), defined at a scale \( \mu \) in a given scheme:

\[
m(\mu) = Z_m^{-1}(\mu) m_0
\]

This definition provides the relation between the values of quark masses in different renormalization schemes. By considering for definitiveness the \( \overline{\text{MS}} \) and RI schemes, one finds:

\[
R_m(\mu) = \frac{m_{\overline{\text{MS}}}(\mu)}{m_{\text{RI}}(\mu)} = \frac{Z_{\text{RI}}m(\mu)}{Z_{\overline{\text{MS}}}m(\mu)}
\]

Even though \( Z_{\overline{\text{MS}}}m(\mu) \) and \( Z_{\text{RI}}m(\mu) \) are separately divergent in the infinite cut-off limit, the ratio \( R_m \) is finite, being equal to the ratio of renormalized quark masses in different schemes.

The quark mass renormalization can be conveniently defined through a renormalization condition for the inverse bare quark propagator in momentum space, \( S_{0}^{-1}(p) \). In perturbation theory, \( S_{0}^{-1}(p) \) has the form:

\[
i S_{0}^{-1}(p) = \not{p} \Sigma_1(p) - m_0 \Sigma_2(p)
\]

where \( \Sigma_{1,2} \) are (bare) scalar quantities which depend also on the cut-off and the bare coupling constant, mass and gauge parameter, \( \alpha_0, m_0 \) and \( \xi_0 \). Once expressed in terms of renormalized parameters, \( \alpha_s, m \) and \( \xi \), the renormalization of the quark propagator still requires the renormalization of the quark field itself. Thus, the renormalized propagator can be written as:

\[
i S^{-1}(p) = Z_0 \left[ \not{p} \Sigma_1(p) - Z_m m \Sigma_2(p) \right]
\]

where \( \Sigma_{1,2} \) are the bare functions \( \Sigma_{1,2} \) expressed in terms of renormalized parameters.

Different renormalization prescriptions on \( \Sigma_{1,2} \) define the expressions of the quark field and mass renormalization constants, \( Z_q \) and \( Z_m \), in the corresponding schemes. In perturbation theory, these prescriptions can be conveniently expressed by giving the (finite) coefficients entering in the perturbative expansions of the renormalized quantities. Thus, for instance, for \( \Sigma_{1,2} \), up to \( O(\alpha_s^2) \), a renormalization scheme can be defined by:

\[
\lim_{m \to 0} \frac{1}{48} Z_q \text{Tr} \left[ \gamma_\mu \frac{\partial (\not{p} \Sigma_1(p))}{\partial p_\mu} \right]_{p^2 = -\mu^2} = 1 + \frac{\alpha_s}{(4\pi)} r_1 + \frac{\alpha_s^2}{(4\pi)^2} s_1 + \ldots
\]

\[
\lim_{m \to 0} \frac{1}{12} Z_q Z_m \text{Tr} \left[ \Sigma_2(p) \right]_{\xi = \xi^*} = 1 + \frac{\alpha_s}{(4\pi)} r_2 + \frac{\alpha_s^2}{(4\pi)^2} s_2 + \ldots
\]
where traces are taken over the color and spin indices.

The popular MS scheme \cite{16} amounts to require that, in dimensional regularization, the coefficients of the perturbative expansion of the renormalization constants only contain singular terms in $1/\epsilon$, where $D = 4 - 2\epsilon$ is the space-time dimension. Here we consider its standard modification, the \(\overline{\text{MS}}\) scheme \cite{17}. We choose to renormalize the gauge coupling constant $\alpha_s$ and the gauge parameter $\xi$ always in this scheme, independently on the renormalization condition adopted for the quark field and mass. A notable feature of the \(\overline{\text{MS}}\) scheme is that the renormalization constants of gauge-invariant operators and parameters, like the quark mass itself, are gauge independent. Consequently, the renormalized quark mass, in the \(\overline{\text{MS}}\) scheme, is gauge independent.

One of the advantages of the RI scheme is that the corresponding renormalization prescriptions can be stated in a non-perturbative way. In the case of $\Sigma_{1,2}$, for instance, one simply requires the left-hand sides of eqs. \(\{3\}\) to be equal to unity. In perturbation theory, this implies the vanishing of all the coefficients $r_i$ and $s_i$. The renormalization conditions in the RI scheme clearly depend on the choice of the gauge parameter $\xi^*$, and different choices define in fact different RI renormalization schemes. Throughout this paper we will present results for the RI scheme in a generic covariant gauge, even though a common choice in this context is the Landau gauge, corresponding to $\xi^* = 0$. Indeed, this choice is particularly convenient to implement in non-perturbative calculations of lattice QCD.

In numerical lattice simulations, the quark field renormalization condition of eq. \(\{3\}\) is not easy to implement, because of the presence of the continuum derivative. A common practice is to define a different quark field renormalization through the condition:

$$\lim_{m \to 0} \frac{1}{12} Z'_q \text{Tr} \left[ \Sigma_1 (p) \right]_{p^2 = -\mu^2} = 1$$

In the Landau gauge, the difference between the two definitions appears at order $O(\alpha^2_s)$, and it must be taken into account at the NNLO accuracy. For this reason, we have also computed the ratio of the two relevant renormalization constants and the result will be given in the next section.

In the calculation of $\Sigma_{1,2}$ an important check is provided by the vector chiral Ward identity, which relates these functions to the amputated Green functions of the vector current and scalar density respectively between external quark states, $\Gamma_{V\mu}(p)$ and $\Gamma_S(p)$. In particular, in the chiral limit:

$$\left[ \frac{\partial}{\partial p_\mu} \Sigma_1 (p) \right]_{m=0} = \Gamma_{V\mu}(p), \quad \left[ \Sigma_2 (p) = \Gamma_S(p) \right]_{m=0}$$

We have verified that the above equations are indeed satisfied at the level of bare Green functions. By requiring the Ward identities to be satisfied also by
renormalized quantities, one obtains the relations:

\[ Z_V = 1 \quad , \quad Z_m = Z_S^{-1} \]  

among the renormalization constants in a generic scheme. In the \( \overline{\text{MS}} \)-NDR scheme, the vector and axial Ward identities are unaffected by the minimal subtraction procedure. Thus, in this scheme, eqs. (8) are automatically satisfied. In the RI scheme, in order to satisfy the Ward identities, the following renormalization prescriptions on \( \Gamma_{V\mu} \) and \( \Gamma_S \) must be consistently applied:

\[
\lim_{m \to 0} \frac{1}{48} Z^\text{RI}_q \left( Z^\text{RI}_V \right)^{-1} \text{Tr} \left[ \gamma_\mu \Gamma_{V\mu} (p) \right]_{p^2 = -\mu^2} = 1
\]

\[
\lim_{m \to 0} \frac{1}{12} Z^\text{RI}_q \left( Z^\text{RI}_S \right)^{-1} \text{Tr} \left[ \Gamma_S (p) \right]_{p^2 = -\mu^2} = 1
\]

The above conditions can be also applied at finite values of the quark mass, provided a sufficiently large value of the renormalization scale is chosen, \( \mu^2 \gg m^2 \).

3 Results for the renormalization constants

In order to present the results of our calculation, we expand a generic renormalization constant \( Z \) as a series in the strong coupling constant:

\[ Z = 1 + \frac{\alpha_s}{4\pi} Z^{(1)} + \frac{\alpha_s^2}{(4\pi)^2} Z^{(2)} + \ldots \]  

Subsequently, each coefficient \( Z^{(i)} \) is expanded in inverse powers of \( \epsilon \):

\[ Z^{(i)} = \sum_{j=0}^{i} \left( \frac{1}{\epsilon} \right)^j Z_j^{(i)} \]  

By definition, all the coefficients \( Z_0^{(i)} \) vanish in the \( \overline{\text{MS}} \) scheme. In addition, by requiring that the ratio of renormalization constants in different schemes, namely \( Z^\text{RI} / Z^\overline{\text{MS}} \), is finite, one finds a set of useful identities between the \( Z_j^{(i)} \)’s in the two schemes. These observations, combined together, imply the following relations:

\[
\left( Z^\overline{\text{MS}} \right)_1^{(1)} = \left( Z^\text{RI} \right)_1^{(1)}
\]

\[
\left( Z^\overline{\text{MS}} \right)_2^{(2)} = \left( Z^\text{RI} \right)_2^{(2)}
\]

\[
\left( Z^\overline{\text{MS}} \right)_1^{(2)} = \left( Z^\text{RI} \right)_1^{(2)} - \left( Z^\text{RI} \right)_1^{(1)} \left( Z^\text{RI} \right)_0^{(1)}
\]
Thus, the renormalization constant in the \( \overline{\text{MS}} \) scheme can be completely derived from its counterpart in the RI scheme (but not vice-versa). One also finds that the perturbative expansion of the ratio \( Z_{\text{RI}}^{\text{RI}}/Z_{\text{MS}}^{\text{MS}} \) is simply given, up to two-loop order, by the finite coefficient of perturbative expansion of \( Z_{\text{RI}}^{\text{RI}} \):

\[
R_m = \frac{Z_{m}^{\text{RI}}}{Z_{m}^{\text{MS}}} = 1 + \frac{\alpha_s}{(4\pi)} \left( Z_{m}^{\text{RI}} \right)_{0}^{(1)} + \frac{\alpha_s^2}{(4\pi)^2} \left( Z_{m}^{\text{RI}} \right)_{0}^{(2)} + \ldots
\]  

(13)

We have calculated the quark field and mass renormalization constants, \( Z_q \) and \( Z_m \), in the \( \overline{\text{MS}} \) and RI schemes. Adopting naïve dimensional regularization, in a generic covariant gauge, we have computed the Feynman diagrams shown in fig. 1. Since the corresponding amplitudes can be expanded up to linear terms in the bare quark mass, this only requires the evaluation of one- and two-loop p-integrals, which we have computed by using the method of integration by parts of ref. [18]. The Ward identities of eqs. (7) have been used to check the correctness of our calculation. For the quark field renormalization constant, in the RI scheme, we find:

\[
\left( Z_{q}^{\text{RI}} \right)_{1}^{(1)} = \frac{(N_c^2 - 1)}{2N_c} (-\xi)
\]

\[
\left( Z_{q}^{\text{RI}} \right)_{0}^{(1)} = \frac{(N_c^2 - 1)}{4N_c} (-\xi)
\]

\[
\left( Z_{q}^{\text{RI}} \right)_{2}^{(2)} = \frac{(N_c^2 - 1)}{8N_c^2} \xi \left( -\xi + 3N_c^2 + 2\xi N_c^2 \right)
\]

(14)

\[
\left( Z_{q}^{\text{RI}} \right)_{1}^{(2)} = \frac{(N_c^2 - 1)}{16N_c^2} \left( -3 - 2\xi^2 - 22N_c^2 - 8\xi N_c^2 + \xi^2 N_c^2 + 4N_c n_f \right)
\]
\[
\left(Z_{n}^{\text{RI}}\right)^{(2)}_{0} = \frac{(N_{c}^{2} - 1)}{32 N_{c}^{2}} \left(1 - 6 \xi^2 - 115 N_{c}^{2} - 76 \xi N_{c}^{2} - 6 \xi^2 N_{c}^{2} + 48 \xi N_{c}^{2} \zeta_3 + 48 N_{c}^{2} \zeta_3 + 20 N_{c} n_{f}\right)
\]

where \(\zeta\) is the Riemann zeta function and \(\zeta_3 = 1.20206\ldots\). The quark mass renormalization constant, in the same scheme, is given by:

\[
(Z_{m}^{\text{RI}})^{(1)}_{0} = \frac{(N_{c}^{2} - 1)}{2 N_{c}} (-3)
\]

\[
(Z_{m}^{\text{RI}})^{(1)}_{0} = \frac{(N_{c}^{2} - 1)}{4 N_{c}} (-8 - 3 \xi)
\]

\[
(Z_{m}^{\text{RI}})^{(2)}_{0} = \frac{(N_{c}^{2} - 1)}{8 N_{c}^{2}} (-9 + 31 N_{c}^{2} - 4 N_{c} n_{f})
\]

\[
(Z_{m}^{\text{RI}})^{(2)}_{0} = \frac{(N_{c}^{2} - 1)}{48 N_{c}^{2}} (-135 - 54 \xi - 59 N_{c}^{2} + 54 \xi N_{c}^{2} + 20 N_{c} n_{f})
\]

\[
(Z_{m}^{\text{RI}})^{(2)}_{0} = \frac{(N_{c}^{2} - 1)}{96 N_{c}^{2}} (-75 - 144 \xi - 36 \xi^2 - 2645 N_{c}^{2} - 108 \xi N_{c}^{2} - 18 \xi^2 N_{c}^{2} + 288 \zeta_3 + 576 N_{c}^{2} \zeta_3 + 356 N_{c} n_{f})
\]

The corresponding values of renormalization constants in the \(\overline{\text{MS}}\) scheme can be obtained from eqs. (14) and (15) by using eq. (12). The result for \(Z_{m}^{\overline{\text{MS}}}\) is in agreement with the original calculation of ref. [19]. Notice that, as expected, \(Z_{m}^{\overline{\text{MS}}}\) turns out to be gauge-independent.

We have also calculated the ratio \(\Delta_{q}\) between the quark field renormalization constants \(Z_{q}^{\text{RI}}\) and \(Z_{q}^{'}\), the latter being defined in eq. (3). The result, expanded in series of the strong coupling constant, is:

\[
\Delta_{q} = \frac{Z_{q}^{\text{RI}}}{Z_{q}^{'}} = 1 + \frac{\alpha_s}{(4\pi)} \Delta_{q}^{(1)} + \frac{\alpha_s^2}{(4\pi)^2} \Delta_{q}^{(2)} + \ldots
\]

where we find:

\[
\Delta_{q}^{(1)} = \frac{(N_{c}^{2} - 1)}{4 N_{c}} \xi
\]

\[
\Delta_{q}^{(2)} = \frac{(N_{c}^{2} - 1)}{16 N_{c}^{2}} (3 - \xi^2 + 22 N_{c}^2 + 14 \xi N_{c}^2 + 4 \xi^2 N_{c}^2 - 4 N_{c} n_{f})
\]

Notice that, in the Landau gauge (\(\xi = 0\)), at a typical scale \(\mu \sim 2\) GeV, \(\Delta_{q}\) represents a tiny correction, smaller than 1%.

Finally, we discuss the results for the ratio \(R_{m}\), which provides the relation between the quark mass in the \(\overline{\text{MS}}\) and RI schemes at a fixed scale \(\mu\). This ratio is obtained by substituting the values of \(Z_{m}^{\text{RI}})^{(1)}_{0}\) and \(Z_{m}^{\text{RI}})^{(2)}_{0}\) in eq. (13). For
convenience, we present here its numerical expression as obtained for $N_c = 3$ in the Landau gauge:

$$R_{m}^{LAN}(\mu) = 1 - \frac{16}{3} \frac{\alpha_s(\mu)}{(4\pi)} - \left( \frac{1990}{9} - \frac{152}{3} \zeta_3 - \frac{89}{9} n_f \right) \frac{\alpha_s^2(\mu)}{(4\pi)^2} \quad (18)$$

Eq. (18) is the main result of this paper. Numerically, one finds that the size of the NNLO contribution to $R_{m}^{LAN}$, at a scale $\mu = 2$ GeV ($n_f = 4$), is about 7%. This represents a significant correction, comparable to the NLO contribution.

4 Quark mass evolution and the invariant quark mass

In this section we discuss the NNLO relations between quark masses in different renormalization schemes and at different renormalization scales. This discussion also provides us with the opportunity to introduce the so-called renormalization group-invariant quark mass, $\hat{m}$, and its perturbative relation with the quark mass in the RI scheme.

In a generic renormalization scheme, the renormalized quark mass obeys the following renormalization group equation:

$$\left[ \mu^2 \frac{d^2}{d\mu^2} + \frac{\gamma_m}{2} \right] m(\mu) = \left[ \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} + \beta_1(\alpha_s) \frac{\partial}{\partial \xi} + \frac{\gamma_m}{2} \right] m(\mu) = 0 \quad (19)$$

Both the $\beta$-function and the mass anomalous dimension in the $\overline{\text{MS}}$ scheme have been computed up to four loops in [14] and [1] [15] respectively. In the NNLO approximation we are considering here, we only need the corresponding expansions up to three loops. The QCD $\beta$-function is given by:

$$\frac{\beta(\alpha_s)}{4\pi} = \mu^2 \frac{d}{d\mu^2} \left( \frac{\alpha_s}{4\pi} \right) = - \sum_{i=0}^{\infty} \beta_i \left( \frac{\alpha_s}{4\pi} \right)^{i+2}$$

$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f$$

$$\beta_1 = \frac{34}{3} N_c^2 - \frac{10}{3} N_c n_f - \frac{(N_c^2 - 1) n_f}{N_c}$$

$$\beta_{2\overline{\text{MS}}} = \frac{2857}{54} N_c^3 + \frac{(N_c^2 - 1)^2}{4N_c^2} n_f - \frac{205}{36} \frac{(N_c^2 - 1) n_f}{N_c}$$

$$- \frac{1415}{54} N_c^2 n_f + \frac{11}{18} \frac{(N_c^2 - 1) n_f^2}{N_c} + \frac{79}{54} N_c n_f^2$$

The mass anomalous dimension in the $\overline{\text{MS}}$ scheme is:

$$\gamma(\alpha_s) = 2Z^{-1} \mu^2 \frac{d}{d\mu^2} Z = \sum_{i=0}^{\infty} \gamma^{(i)} \left( \frac{\alpha_s}{4\pi} \right)^{i+1}$$
\[ \hat{\gamma}_m^{(0)} = 3 \frac{N_c^2 - 1}{N_c} \]
\[ \hat{\gamma}_m^{(1)} = \frac{N_c^2 - 1}{N_c^2} \left( -\frac{3}{4} + \frac{203}{12} N_c^2 - \frac{5}{3} N_c n_f \right) \]
\[ \hat{\gamma}_m^{(2)} = \frac{N_c^2 - 1}{N_c^3} \left[ \frac{129}{8} - \frac{129}{8} N_c^2 + \frac{11413}{108} N_c^4 + n_f \left( \frac{23}{2} N_c - \frac{1177}{54} N_c^3 - 12 N_c \zeta_3 - 12 N_c^3 \zeta_3 \right) - \frac{35}{27} N_c n_f \right] \]

The evolution of the renormalized gauge parameters is given by:

\[ \beta_\xi(\alpha_s) = \frac{\mu^2}{\xi} \frac{d\xi}{d\mu^2} = -\sum_{i=1}^{\infty} \beta_\xi^{(i)} \left( \frac{\alpha_s}{4\pi} \right)^i \]
\[ \beta_\xi^{(0)} = -\frac{N_c}{2} \left( \frac{13}{3} - \xi \right) + \frac{2}{3} n_f \quad (22) \]

For completeness, we also present the coefficients of the perturbative expansion of the quark field anomalous dimension, up to the NLO. Using the results of eq. (14), we derive, in the \( \overline{\text{MS}} \) scheme and in a generic covariant gauge:

\[ \gamma_q^{(0)} = \frac{N_c^2 - 1}{N_c} \xi \]
\[ \gamma_q^{(1)} = \frac{N_c^2 - 1}{4 N_c^2} \left( 3 + 22 N_c^2 + 8 \xi N_c^2 + \xi^2 N_c^2 - 4 N_c n_f \right) \quad (23) \]

The evolution of the quark mass is determined by eq. (19). The solution is particularly simple in the \( \overline{\text{MS}} \) scheme, where the renormalized quark mass is gauge independent. In this case, it can be expressed in the form:

\[ m_{\overline{\text{MS}}} (\mu) = \frac{c_{\overline{\text{MS}}} (\mu)}{c_{\overline{\text{MS}}} (\mu_0)} m_{\overline{\text{MS}}} (\mu_0) \quad (24) \]

where:

\[ c_{\overline{\text{MS}}} (\mu) = \alpha_s (\mu) \overline{\gamma}_0 \left[ 1 + \frac{\alpha_s}{4\pi} (\overline{\gamma}_1 - \overline{\beta}_1 \overline{\gamma}_0) \right] + \frac{1}{2} \left( \frac{\alpha_s (\mu)}{4\pi} \right)^2 \left[ (\overline{\gamma}_1 - \overline{\beta}_1 \overline{\gamma}_0)^2 + \overline{\gamma}_2 + \overline{\beta}_1^2 \overline{\gamma}_0 - \overline{\beta}_1 \overline{\gamma}_1 - \overline{\beta}_2 \overline{\gamma}_0 \right] \quad (25) \]

with \( \overline{\beta}_i = \beta_i / \beta_0 \) and \( \overline{\gamma}_i = \gamma_i^{(i)} / (2 \beta_0) \).

By using eq. (2), we can convert eq. (24) into a relation between the RI quark mass at a scale \( \mu_0 \) and the \( \overline{\text{MS}} \) one at a different scale \( \mu \):

\[ m_{\overline{\text{MS}}} (\mu) = \frac{c_{\overline{\text{MS}}} (\mu)}{c_{\overline{\text{MS}}} (\mu_0)} \frac{Z_{\text{RI}} (\mu)}{Z_{\overline{\text{MS}}} (\mu_0)} m_{\text{RI}} (\mu_0) \quad (26) \]
This formula can be now easily interpreted by observing that the function:

$$c_{RI}(\mu) = \frac{Z_{MS}(\mu)}{Z_{RI}(\mu)} c_{MS}(\mu)$$ (27)

is the solution of eq. (19) in the RI scheme. So we can conventionally define a renormalization group-invariant mass as:

$$\hat{m} = \frac{m_{MS}(\mu)}{c_{MS}(\mu)} = \frac{m_{RI}(\mu_0)}{c_{RI}(\mu_0)}$$ (28)

The mass \( \hat{m} \) is a short-distance quantity, which is both scale and scheme independent. It can be conveniently used in phenomenological applications involving quark masses, in alternative to the MS quark mass.

The relation between \( \hat{m} \) and the MS quark mass has been computed in ref. [1]. For \( n_f = 3, 4, 5 \), the result has the form:

$$\hat{m}^{(3)} = \alpha_s(\mu)^{-4/9} \left[ 1 - \frac{\alpha_s(\mu) 290}{4\pi 81} - \frac{\alpha_s^2(\mu)}{(4\pi)^2} \left( \frac{259943}{13122} - \frac{80}{9} \zeta_3 \right) \right] m_{MS}(\mu)$$ (29)

$$\hat{m}^{(4)} = \alpha_s(\mu)^{-12/25} \left[ 1 - \frac{\alpha_s(\mu) 7606}{4\pi 1875} - \frac{\alpha_s^2(\mu)}{(4\pi)^2} \left( \frac{446305267}{21093750} - \frac{64}{5} \zeta_3 \right) \right] m_{MS}(\mu)$$ (30)

$$\hat{m}^{(5)} = \alpha_s(\mu)^{-12/23} \left[ 1 - \frac{\alpha_s(\mu) 7462}{4\pi 1587} - \frac{\alpha_s^2(\mu)}{(4\pi)^2} \left( \frac{344665349}{15111414} - \frac{400}{23} \zeta_3 \right) \right] m_{MS}(\mu)$$ (31)

We can use eq. (27) to state our results of eqs. (13)-(18) as a relation between the renormalization group-invariant mass \( \hat{m} \) and the mass \( m_{RI} \):

$$\hat{m}^{(3)} = \alpha_s(\mu)^{-4/9} \left[ 1 - \frac{\alpha_s(\mu)}{4\pi} \left( \frac{722}{81} + 2\xi \right) - \frac{\alpha_s^2(\mu)}{(4\pi)^2} \left( \frac{2521517}{13122} - \frac{536}{9} \zeta_3 + \frac{257}{81} \xi + \frac{11}{6} \xi^2 \right) \right] m_{RI}(\mu)$$ (32)

$$\hat{m}^{(4)} = \alpha_s(\mu)^{-12/25} \left[ 1 - \frac{\alpha_s(\mu)}{4\pi} \left( \frac{17606}{1875} + 2\xi \right) - \frac{\alpha_s^2(\mu)}{(4\pi)^2} \left( \frac{3819632767}{21093750} - \frac{952}{15} \zeta_3 + \frac{4163}{1875} \xi + \frac{11}{6} \xi^2 \right) \right] m_{RI}(\mu)$$ (33)

$$\hat{m}^{(5)} = \alpha_s(\mu)^{-12/23} \left[ 1 - \frac{\alpha_s(\mu)}{4\pi} \left( \frac{15926}{1587} + 2\xi \right) - \frac{\alpha_s^2(\mu)}{(4\pi)^2} \left( \frac{2559841211}{15111414} - \frac{4696}{69} \zeta_3 + \frac{1475}{1587} \xi + \frac{11}{6} \xi^2 \right) \right] m_{RI}(\mu)$$ (34)

We note that in the RI scheme both \( m_{RI} \) and \( c_{RI} \) depend on the gauge, but the dependence cancels out in the ratio \( \hat{m} \).
Acknowledgements

We warmly thank G. Martinelli for many interesting discussions. We acknowledge the M.U.R.S.T. and the INFN for partial support.

References

[1] J.A.M. Vermaseren, S.A. Larin and T. van Ritbergen, Phys. Lett. B405 (1997) 327, hep-ph/9703284.
[2] N. Gray, D.J. Broadhurst, W. Grafe and K. Schilcher, Z. Phys. C48 (1990) 673.
[3] I.I. Bigi, M.A. Shifman, N.G. Uraltsev and A.I. Vainshtein, Phys. Rev. D50 (1994) 2234, hep-ph/9402360.
[4] M. Beneke and V.M. Braun, Nucl. Phys. B426 (1994), 301, hep-ph/9402364.
[5] The Particle Data Group, R.M. Barnett et al., Phys. Rev. D54 (1996) 1.
[6] For a review of recent results, see V. Lubicz, talk given at the APCTP-ICTP Joint International Conference (AIJIC 97), Seoul, Korea, 26-30 May 1997, hep-ph/9707374.
[7] E. Gabrielli and P. Nason, Phys. Lett. B313 (1993) 430.
[8] G.P. Lepage and P.B. Mackenzie, Phys. Rev. D48 (1993), 2250, hep-lat/9209022.
[9] A. Gonzales Arroyo, F.J. Yndurain and G. Martinelli, Phys. Lett. B117 (1982) 437.
G. Martinelli and Y.C. Zhang, Phys. Lett. B123 (1983) 433.
H.W. Hamber and C.M. Wu, Phys. Lett. B133 (1983) 351.
[10] E. Gabrielli, G. Martinelli, C. Pittori, G. Heatlie and C.T. Sachrajda, Nucl. Phys. B362 (1991) 475.
A. Borrelli, C. Pittori, R. Frezzotti and E. Gabrielli, Nucl. Phys. B409 (1993) 382.
[11] C.R. Allton, C.T. Sachrajda, V. Lubicz, L. Maiani and G. Martinelli, Nucl. Phys. B349 (1991) 598.
[12] G. Martinelli, C. Pittori, C.T. Sachrajda, M. Testa and A. Vladikas, Nucl. Phys. B445 (1995) 81, hep-lat/9411013.
[13] V. Gimenez, L. Giusti, F. Rapuano and M. Talevi, EDINBURGH-97-15, Jan 1998, hep-lat/9801028.
[14] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, Phys. Lett. B400 (1997) 379, hep-ph/9701390.

[15] K.G. Chetyrkin, Phys. Lett. B404 (1997) 161, hep-ph/9703278.

[16] G. ’t Hooft, Nucl. Phys. B61 (1973) 455.

[17] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. D18 (1978) 3998.

[18] K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B192 (1981) 159.

[19] R.Tarrach, Nucl. Phys. B183 (1981) 384.