Nonabelian sine-Gordon theory and its application to nonlinear optics

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Using a field theory generalization of the spinning top motion, we construct nonabelian generalizations of the sine-Gordon theory according to each symmetric spaces. A Lagrangian formulation of these generalized sine-Gordon theories is given in terms of a deformed gauged Wess-Zumino-Witten action which also accounts for integrably perturbed coset conformal field theories. As for physical applications, we show that they become precisely the effective field theories of self-induced transparency in nonlinear optics. This provides a dictionary between field theory and nonlinear optics.

Among many integrable equations, the sine-Gordon equation is one of the most well-known equation which finds countless applications in a wide range of physics due to its ubiquitous nature. In many cases, however, it is desirable to have a “generalized” sine-Gordon equation in order to accommodate more realistic physical systems. In this talk, I will show that such generalizations are indeed possible. They are constructed according to each coset $G/H$ and shown to provide a Lagrangian formulation of integrably perturbed coset conformal field theories. Surprisingly, when $G/H$ is restricted to a Hermitian symmetric space, the generalized sine-Gordon equation finds unexpected applications in nonlinear optics. In particular, when $G/H = SU(2)/U(1)$, it describes the two-level self-induced transparency (SIT), a phenomenon of anomalously low energy loss in coherent optical pulse propagation.

In order to gain physical insight as well as a handle for the generalization, we first introduce the SIT equation which in the sharp line limit is given by the Maxwell equation under the “slowly varying envelope approximation”

$$\bar{\partial}E + 2\beta P = 0 \quad \text{(1)}$$

and the optical Bloch equation

$$\partial D - E^* P - EP^* = 0$$
$$\partial P + 2i\xi P + 2ED = 0 \quad \text{(2)}$$

where $\xi = w - w_0$, $\partial \equiv \partial/\partial z$, $\bar{\partial} \equiv \partial/\partial \bar{z}$, $z = t - x/c$, $\bar{z} = x/c$ and $E$, $P$ and $D$ represent the electric field, the polarization and the population inversion respectively. Here, we will not concern about the details of nonlinear optics and for the sake of this talk, it suffices to say that Eqs.(1) and (2) are coupled nonlinear partial differential equations in 1+1-dimensional spacetime
with two complex fields $E$ and $P$ and one real field $D$ together with constant parameters $\beta$ and $\xi$. Note that the optical Bloch equation admits an interpretation as a spinning top equation like the corresponding magnetic resonance equations \cite{2}. To see this, denote real and imaginary parts of $E$ and $P$ by $E = E_R + iE_I$, $P = P_R + iP_I$. Then, the Bloch equation (2) can be expressed as

$$\partial \vec{S} = \vec{\Omega} \times \vec{S}$$

that is, it describes a spinning top motion where the electric dipole “pseudospin” vector $\vec{S} = (P_R, P_I, D)$ precesses about the “torque” vector $\vec{\Omega} = (2E_I, -2E_R, -2\xi)$. This clearly shows that the length of the vector $\vec{S}$ is conserved,

$$|\vec{S}|^2 = P_R^2 + P_I^2 + D^2 = 1,$$  

where the length equals unity due to the conservation of probability. In case $P_I = 0$, we may solve Eq.(3) for $P_R = -\sin 2\varphi$ and $D = \cos 2\varphi$. Then, Eq.(3) can be solved easily by taking $E = \partial \varphi$ which changes the Maxwell equation to the sine-Gordon equation. The Maxwell equation determines the strength of the torque vector along the $\bar{z}$-axis which agrees with the conventional mechanical interpretation of the sine-Gordon equation as a continuum limit of the infinite chain of coupled pendulum equations, i.e. it becomes a field theory generalization of a pendulum. This shows that the SIT equation, without the assumption $P_I = 0$, is a field theory generalization of a spinning top which includes the sine-Gordon theory as a special case.

The key observation leading to the nonabelianization of the sine-Gordon equation is that the constraint Eq.(4) can be solved generally in terms of an $SU(2)$ matrix potential variable $g$ by

$$
\begin{pmatrix}
D \\
P^* \\
-D
\end{pmatrix} = g^{-1}\sigma_3 g, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Then, the Bloch equation arises from the identity,

$$\partial(g^{-1}\sigma_3 g) = [g^{-1}\sigma_3 g, g^{-1}\partial g],$$

if we take

$$g^{-1}\partial g - R = \begin{pmatrix} i\xi & -E \\ E^* & -i\xi \end{pmatrix},$$

where $R$ is an arbitrary matrix commuting with $g^{-1}\sigma_3 g$ which will be determined later. Finally, the Maxwell equation becomes

$$\bar{\partial}(g^{-1}\partial g - R) = \begin{pmatrix} 0 & -\partial E \\ \partial E^* & 0 \end{pmatrix} = -\left[ \begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix} , i\begin{pmatrix} D \\
P^* \\
-D\end{pmatrix} \right] = \beta[\sigma_3, g^{-1}\sigma_3 g].$$
Thus, the SIT equation changes into a single nonlinear sigma model-type equation up to an undetermined quantity $R$.

Now, we show that $R$ is fixed when we consider a Lagrangian formulation of the SIT equation. In fact, we will construct a Lagrangian which is more general than the SIT case in terms of the gauged Wess-Zumino-Witten action as follows:

$$S = S_{WZW} + S_{\text{gauge}} - S_{\text{pot}}$$ (9)

where $S_{WZW}$ is the usual group $G$ Wess-Zumino-Witten action and $S_{\text{gauge}}$ is the gauging part,

$$S_{\text{gauge}} = \frac{1}{2\pi} \int \text{Tr}(-A\bar{A}g^{-1} + \bar{A}g^{-1}\partial g + Ag\bar{A}g^{-1} - A\bar{A})$$ (10)

which gauges the anomaly free vector subgroup $H \in G$. This gauged Wess-Zumino-Witten action $S_{WZW} + S_{\text{gauge}}$ has been identified as an action of $G/H$ coset conformal field theories [3]. The potential term $S_{\text{pot}}$ is added in such a way that the integrability of the model is preserved while the conformal symmetry is broken. A general construction of $S_{\text{pot}}$ is given by a triplet of Lie groups $F \supset G \supset H$ for every symmetric space $F/G$, where the Lie algebra decomposition $f = g \oplus k$ satisfies the commutation relations,

$$[g, g] \subset g, \ [g, k] \subset k, \ [k, k] \subset g.$$ (11)

We take $T$ and $\bar{T}$ as elements of $k$ and define $h$ as the simultaneous centralizer of $T$ and $\bar{T}$, i.e. $h = C_g(T, \bar{T}) = \{B \in g : [B, T] = 0 = [B, \bar{T}]\}$ with $H$ its associated Lie group. With these specifications, the potential $S_{\text{pot}}$ is given by

$$S_{\text{pot}} = \frac{\beta}{2\pi} \int \text{Tr} g^{-1} T.$$ (12)

The (classical) integrability can be demonstrated by expressing the equation of motion arising from the action (3) in a zero curvature form with a spectral parameter $\lambda$,

$$\delta g S = 0 \leftrightarrow [\partial + g^{-1}\partial g + g^{-1}Ag + \beta\lambda T, \partial + \bar{A} + \frac{1}{\lambda}g^{-1}\bar{T}g] = 0$$ (13)

Also, due to the absence of the kinetic terms, $A, \bar{A}$ act as Lagrange multipliers which result in the constraint equations;

$$(-\bar{A}g^{-1} + g\bar{A}g^{-1} - \bar{A})_{h} = 0 = (g^{-1}\partial g + g^{-1}Ag - A)_{h}$$ (14)

where the subscript $h$ denotes the projection to the subalgebra $h$. Explicit expressions of equations are given in Ref. [4] for various cases of symmetric spaces, especially for the type I symmetric spaces: $F/G = SO(n +
1)/SO(n), SU(n)/SO(n), SU(n + 1)/U(n), Sp(n)/U(n). These equations include the sine-Gordon equation as a special case thereby called as symmetric space sine-Gordon (SSSG) equations [7].

A couple of interesting cases of SSSG equations are in order in terms of a triplet of symmetric spaces ($F, G, H$),

$$(F, G, H) = (SU(4), SU(2) \times SU(2) \times U(1), SU(2))$$

The $U(1)$ factor can be decoupled consistently and the model describes an integrable perturbation of the minimal model in conformal field theory which itself can be defined by the $SU(2) \times SU(2)/SU(2)$-gauged WZW model [5]. If we choose

$$g = \begin{pmatrix} g_1 \in SU(2) & 0 \\ 0 & g_2 \in SU(2) \end{pmatrix}, \quad T = \bar{T} = \begin{pmatrix} 0 & i1_{2 \times 2} \\ i1_{2 \times 2} & 0 \end{pmatrix}, \quad (15)$$

then, the model describes the integrable deformation of the minimal conformal theory for the critical Ising model by the operator $\Phi_{(2,1)}$. This case has been called as a matrix sine-Gordon theory and its classical behavior and soliton solutions have been analyzed in [4]. Another interesting deformation arises with a choice

$$g = g_1 g_2 \in SU(2) \otimes SU(2), \quad T = \bar{T} = \sum_{a=1}^{3} L^a \otimes M^a \quad (16)$$

for $L^a(M^a)$ generators of $su(2)$ and this describes the deformation by the operator $\Phi_{(3,1)}$ [3].

$$(F, G, H) = (SO(5), SO(3) \times U(1), SO(2))$$

With the $U(1)$ decoupled, this case is known as the complex sine-Gordon equation which in turn accounts for the SIT equation [1] and [2] as explained below. These two cases of SSSG are special cases of the nonabelian Toda theory with $N = 1$ grading [3].

In case of compact symmetric spaces of type II, e.g. symmetric spaces of the form $G \times G/G$, the elements $g$ and $T$ take the form $g \otimes g$ and $T \otimes 1 - 1 \otimes T$ (and similarly for $\bar{T}$). In which case, the model becomes effectively equivalent to the case where $T, \bar{T}$ belong to the Lie algebra $\mathfrak{g}$. Thus the model is specified by the coset $G/H$ where $H$ is the stability subgroup of $T$ and $\bar{T}$ for $T, \bar{T} \in \mathfrak{g}$. In particular, when $G/H$ is further restricted to Hermitian symmetric space, a symmetric space equipped Hermitian structure, the SSSG equation finds a nice physical application, i.e. it becomes precisely the SIT equations for various atomic systems. In this case, the adjoint action of $T$
defines a complex structure on $G/H$. For example, in the simplest $CP^1$ case where $G/H = SU(2)/U(1) \approx CP^1$, we may choose $T = \bar{T} = i\sigma_3$ and fix the vector gauge invariance of the action \((\mathbb{R})\) by
\[ A = i\xi\sigma_3, \quad A = 0 \quad (17) \]
for a constant $\xi$. Such a gauge fixing is possible due to the flatness of $A, \bar{A}$.

Also, we parameterize the $2 \times 2$ matrix $g$ by
\[ g = e^{i\eta\sigma_3}e^{i\varphi(\cos \theta \sigma_1 - \sin \theta \sigma_2)}e^{i\eta\sigma_3} = \begin{pmatrix} e^{2i\eta \cos \varphi} & i\sin \varphi e^{i\theta} \\ i\sin \varphi e^{-i\theta} & e^{-2i\eta \cos \varphi} \end{pmatrix} \quad (18) \]

and identify $E, P$ and $D$ with $g$ through the relation
\[ g^{-1}\partial g + \xi g^{-1}T g - \xi T = \begin{pmatrix} 0 & -E \\ E^* & 0 \end{pmatrix}, \quad g^{-1}\bar{T} g = -i \begin{pmatrix} D & P \\ P^* & -D \end{pmatrix} \quad (19) \]
where we have imposed the constraint \([14]\). Then, it is a straightforward exercise to show that the zero curvature equation \([13]\) reduces to the SIT equation.

Other cases of Hermitian symmetric spaces are also associated with various multi-level SIT systems with resonant transitions and many new aspects of SIT arising from these identifications and their physical implications are explained in \([8]\).

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