An Unexpected Boolean Connective

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Abstract. We consider a 2-valued non-deterministic connective \( \& \lor \) defined by the table resulting from the entry-wise union of the tables of conjunction and disjunction. Being half conjunction and half disjunction we named it platypus. The value of \( \& \lor \) is not completely determined by the input, contrasting with usual notion of Boolean connective. We call non-deterministic Boolean connective any connective based on multi-functions over the Boolean set. In this way, non-determinism allows for an extended notion of truth-functional connective. Unexpectedly, this very simple connective and the logic it defines, illustrate various key advantages in working with generalized notions of semantics (by incorporating non-determinism), calculi (by allowing multiple-conclusion rules) and even of logic (moving from Tarskian to Scottian consequence relations). We show that the associated logic cannot be characterized by any finite set of finite matrices, whereas with non-determinism two values suffice. Furthermore, this logic is not finitely axiomatizable using single-conclusion rules, however we provide a very simple analytic multiple-conclusion axiomatization using only two rules. Finally, deciding the associated multiple-conclusion logic is coNP-complete, but deciding its single-conclusion fragment is in P.

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So they cut him to pieces, wrote a thesis
A cranium of deceit, he’s prone to lie and cheat
It’s no wonder – a blunder from down under
Duckbill, watermole, duckmole!

Mr. Bungle, Platypus

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1. Introduction

This paper works as an overview of a series of concepts, results and techniques that have been yielding new insights in the analysis (and synthesis) of logics in recent years. The power of these methods is illustrated by establishing various uncommon properties of a very simple non-deterministic Boolean connective that we name platypus. There are two crucial ingredients to be explored.

On one hand, we depart from the traditional approach in logic of using (deterministic) semantics based on logical matrices, as proposed long ago by Lukasiewicz and followers, and adopt a generalization of the standard logic matrix semantics proposed in the beginning of this century by Avron and his collaborators [4], which allows the interpretation of the connectives to be based in multi-functions instead of simply functions. The central idea is that in a non-deterministic matrix (Nmatrix) a connective can non-deterministically pick from a set of possible values instead of its value being completely determined by the input values. This allows us to mix conjunction ($\land$) and disjunction ($\lor$) into a single connective ($\land\lor$) such that for each input it may choose from the values output by conjunction and disjunction with that same input, enlarging the Boolean world beyond truth-functionality. This is reminiscent of the platypus whose appearance mixes bird and mammal traits and has generously lent its name to $\land\lor$. Introducing the possibility of non-determinism has very powerful consequences. The most immediate advantage is that Nmatrices can finitely characterize logics that are not characterizable by (deterministic) matrices [5,14]. As we shall see, this is also true for the logic of $\land\lor$. The extra expressivity offered by non-determinism has also proven extremely valuable in obtaining recent compositional results in logic. Namely, in producing simple modular semantics for combined logics [22]. In particular yielding finiteness-preserving semantics of strengthenings of a given logic with a set of axioms [11], covering a myriad of examples in the literature and explaining the emergency of structures like twist-structures [24,29] or swap-like structures [17]. Further, non-determinism can also be used to give simple infectious semantics to a range of syntax-based relevant companions of a given many-valued logic [12].

On the other hand, we consider the symmetrical multiple-premises/multiple-conclusions notion of logic introduced by Scott [30], and also Shoesmith and Smiley [34], in the 1970s. This bilateralist view [8], generalizes the asymmetrical multiple-premises/single-conclusion approach of most modern logic, introduced as a mathematical object by Tarski and his followers. The gain in symmetry of expressive-power supports the effective development of analytic calculi for logics that could not even be finitely axiomatized before [10,23,34]. The internalization of case analysis in the derivation mechanism, yields nice proof-theoretical properties impossible in the single-conclusion setting. These advantages have been mostly neglected in logic itself, but have been well-appreciated, for instance, in providing constructive proofs in algebra [28]. A key aspect of multiple-conclusion consequence is that it can be used to study the single-conclusion fragments of logics. The logic of $\land\lor$ also allows
us to illustrate such advantages. Although it is not single-conclusion finitely axiomatizable, using the results in [23] we were not only able to axiomatize it in an automated way, but also guarantee that the obtained axiomatization is analytic, a crucial property from the proof-theoretical point of view. In [23] we also have shown how to get purely symbolic decision and proof-search procedures from analytic axiomatizations. From the compositionality point of view there are clear advantages in considering such calculi, involving no additional meta-language as in the (less pure) usual alternatives: sequent calculi, labeled tableaux or natural deduction. This internal view of logic is also directly associated with the fundamental notion of logic as a consequence operation. Notably, merging calculi for given logics precisely captures the mechanism for combining logics known as fibring [19,32], yielding the least logic on the joint language that extends them. Such perspective allows us to better isolate, study and tame the origins of interactions in combined logics.

We detail the structure of the paper highlighting the most relevant results in each section. In Sect. 2 we introduce the platypus connective: first as a multi-function and show that together with any basis for the clone of all Boolean functions it forms a basis for the clone of all Boolean multi-functions (Proposition 2.1); then, as a non-deterministic logical connective characterized by a finite Nmatrix and show that this would not be possible using a finite matrix (Theorem 2.4). In Sect. 3 we explore the divide between single- and multiple-conclusion settings in terms of axiomatizability. We show that platypus’ logic is not finitely axiomatizable by a finite set of single-conclusion rules (Theorem 3.2) and provide an analytic axiomatization using two multiple-conclusion rules (Theorem 3.3). In Sect. 4 we show that deciding the single-conclusion fragment of platypus’ logic is in \(P\) whilst seen as a multiple-conclusion logic it is \(coNP\)-complete (Theorem 4.1). We wrap up with Sect. 5, where we summarize the obtained results and open some doors for future work.

2. The Birth of Platypus

The study of classical propositional logic is deeply connected to the study of Boolean functions, that is, maps \(f : 2^n \rightarrow 2\) with \(2 = \{0, 1\}\). The pioneering work on the clones of Boolean functions by Post [25] provides a complete analysis of the semantical expressivity of every fragment of classical logic. What happens if we consider functions that may output multiple values?

2.1. From Functions to Multi-Functions

Multi-functions associate to each given input possibly more than one value. An \(n\)-ary Boolean multi-function\(^1\) is a map \(f : 2^n \rightarrow \wp(2) \setminus \{\emptyset\} = \{\{0\}, \{1\}, \{0, 1\}\}. Of course, if for every input \(\vec{x} \in 2^n\) we have that \(f(\vec{x})\) is a singleton, then \(f\) is also a (Boolean) function. Observe that there are \(2^{2^n}\) \(n\)-ary Boolean functions and \(3^{2^n}\) \(n\)-ary Boolean multi-functions. Hence, there

\(^1\)In this paper exclude the possibility of a multi-function outputting the empty set however there are situations where this option is desirable as we mention in the end of Sect. 3.
are 81 binary Boolean multi-functions, 16 of them are Boolean functions, we will give particular attention to one of the 65 that are not functions. The main actor in this paper, platypus, can be seen as a binary Boolean multi-function given by $\mathbb{X}(x, y) = \{x \land y, x \lor y\}$. Multi-functions can be easily represented as tables just like functions, check $\mathbb{X}$ in tabular form in Example 2.2. The multi-function $\mathbb{X}$ was considered in [5] in order to approximate the behaviour of a faulty AND gate, which responds correctly if the inputs are similar, and unpredictably otherwise. In this same sense $\mathbb{X}$ can also be seen as a faulty OR gate.

A clone is a set of functions, over some fixed set, closed by composition and containing every projection function ($\pi_n^i(x_1, \ldots, x_n) = x_i$ for every $n \in \mathbb{N}$ and $1 \leq i \leq n$). A set of functions generates (or is a basis for) a clone if this clone is exactly the smallest that contains that given set. This concept can be generalized to multi-functions once we fix a notion of composition. A natural possibility would be to see multi-functions as particular cases of relations. In that case, we have that, when composing given $n$-ary $f$ and $g_i$ for $1 \leq i \leq n$, for each input, the possible values output by the $g_i$'s are accumulated through the composition, yielding that

$$f(g_1(\bar{x}_1), \ldots, g_n(\bar{x}_n)) = \bigcup \left\{ f(\bar{y}) : \bar{y} \in \prod_{1 \leq i \leq n} g_i(\bar{x}_i) \right\}$$

As an alternative we may require that two terms representing the same formula must have the same value when being fed to the outer function and obtain

$$f(g_1(\bar{x}_1), \ldots, g_n(\bar{x}_n)) = \bigcup \left\{ f(\bar{y}) : \bar{y} \in \prod_{1 \leq i \leq n} g_i(\bar{x}_i), y_i = y_j \text{ if } g_i(\bar{x}_i) = g_j(\bar{x}_j) \right\}$$

Note that whenever $g_i$ for $1 \leq i \leq n$ are functions both notions coincide. However, if $g(0) = g(1) = \{0, 1\}$, $f(0, 0) = f(1, 1) = \{0\}$ and $f(0, 1) = f(1, 0) = \{1\}$ then the composition yields $f(g(x), g(x)) = \{0, 1\}$ with the first (more liberal) option and simply $f(g(x), g(x)) = \{0\}$ with the second.

The first interesting property of platypus is that, together with any basis for the clone of all Boolean functions, it forms a basis for the clone of all Boolean multi-functions. Moreover, this is the case regardless of the notion of composition between the two we mentioned above.

**Proposition 2.1.** Given any Boolean multi-function $f$, there are Boolean functions $g_0$ and $g_1$ such that $f = \mathbb{X}(g_0, g_1)$.

**Proof.** For each input $\bar{x} \in 2^n$ we have that either $f(\bar{x}) = \{0\}$, $f(\bar{x}) = \{1\}$ or $f(\bar{x}) = \{0, 1\}$. Easily, the result follows by letting for $i = 0, 1$

$$g_i(\bar{x}) = \begin{cases} \{i\} & \text{if } f(\bar{x}) = \{0, 1\} \\ f(\bar{x}) & \text{otherwise} \end{cases}$$
As $g_0$ and $g_1$ are functions the result of the composition is the same with both notions of composition we considered above.

This decomposition is not unique and the following equalities hold:

$$\forall (x, y) = \forall (y, x) = \forall (\pi_1^2(x, y), \pi_2^2(x, y)) = \forall (x \land y, x \lor y) = \{x, y\}.$$  

Furthermore, if we consider the presentation of $\forall$ given by the last equality, and extend $\forall$ to act over a finite set $I$ as $x\forall_I y = \{x, y\}$ and we add it to a basis for the clone of all functions over $I$, we obtain a basis for the clone of all multi-functions over $I$. That is, we can extend Proposition 2.1 to the statement that for every multi-function $f$ over a fixed set $I$ of size $n$, we have that there are functions over $I, g_1, \ldots, g_n$ such that $f = \forall(g_1, \forall(g_2, \forall(\ldots, \forall(g_{n-1}, g_n))))$. One might argue that a good alternative name\(^2\) for $\forall_I$ could be (non-deterministic) union or choice (over $I$).

### 2.2. Platypus as a Non-Deterministic Boolean Matrix

For the sake of readability, we start by quickly revisiting some of the basic concepts needed, for more details see [18,20,34,35]. A propositional signature is an indexed family $\Sigma = \{\Sigma^{(k)} : k \in \mathbb{N}\}$ where $\Sigma^{(k)}$ are the $k$-ary connectives. Given a set $X$, we denote by $L_\Sigma(X)$ the set of formulas written with connectives in $\Sigma$ from the elements of $X$. We consider fixed a (denumerable) set of propositional variables $P$. The propositional language associated to $\Sigma$ is $L_\Sigma(P)$. Along the paper, whilst considering languages over signatures named $\Sigma_x$ we shall denote $L_{\Sigma_x}(P)$ simply by $L_x$. Fixed a propositional language $L$, a (Scottian) logic is a $\triangleright \subseteq \phi(L) \times \phi(L)$ satisfying:

- **(O)** $\Gamma \triangleright \Delta$ if $\Gamma \cap \Delta \neq \emptyset$ (overlap).
- **(D)** $\Gamma \cup \Gamma' \triangleright \Delta \cup \Delta'$ if $\Gamma \triangleright \Delta$ (dilation).
- **(C)** $\Gamma \triangleright \Delta$ if $\Gamma \cup \Omega \triangleright \Omega \cup \Delta'$ for every partition $(\Omega, \Omega')$ of $L$ (cut).
- **(S)** $\Gamma^\sigma \triangleright \Delta^\sigma$ for any $\sigma : P \rightarrow L$ if $\Gamma \triangleright \Delta$ (substitution invariance).

We say that $\triangleright$ is finitary whenever $\Gamma \triangleright \Delta$ implies $\Gamma' \triangleright \Delta'$ for finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. It is well known that this is a generalization of the notion of Tarskian logic. Indeed, given a Scottian logic $\triangleright$, its single conclusion fragment $\triangleright_{\triangleright} =: \triangleright \cap (\phi(L) \times L)$ is a Tarskian consequence relation satisfying:

- **(R)** $\Gamma \vdash \varphi$ if $\varphi \in \Gamma$ (reflexivity).
- **(M)** $\Gamma \cup \Gamma' \vdash \varphi$ if $\Gamma \vdash \varphi$ (monotonicity).
- **(T)** $\Gamma \vdash \varphi$ if $\Delta \vdash \varphi$ and $\Gamma \vdash \psi$ for every $\psi \in \Delta$ (transitivity).
- **(S)** $\Gamma^\sigma \vdash \varphi^\sigma$ for any $\sigma : P \rightarrow L$ if $\Gamma \vdash \varphi$ (substitution invariance).

Furthermore, $\vdash$ is finitary whenever $\Gamma \vdash \varphi$ then $\Gamma' \vdash \varphi$ for some finite $\Gamma' \subseteq \Gamma$.

We say that $\triangleright$ is a multiple-conclusion companion of $\vdash_{\triangleright}$. Each Tarskian consequence relation $\vdash$ has potentially infinite multiple-conclusion companions. The smallest among these, denoted by $\triangleright_{\vdash}$, is defined as $\Gamma \triangleright_{\vdash} \Delta$ if and only if there is $\varphi \in \Delta$ such that $\Gamma \vdash \varphi$. For example, $\vdash$ may be finitary but have non finitary companions, but surely $\triangleright_{\vdash}$ is finitary. In an abstract sense any companion of $\vdash$ can be used to study $\vdash$, however there are advantages in

\(^2\)I thank Carlos Caleiro for suggesting platypus and thus steering me away from using such boring alternatives.
considering companions that are particularly well behaved. As we shall see, although \( \vdash \) is a faithful representation of \( \vdash \) in the multiple-conclusion setting, there are nicer ones that can be considered.

On an orthogonal direction, we will work also with a generalized notion of logical matrix. Along with multi-functions comes an ingenious extension of the standard notion of matrix semantics introduced in [4]. A \( \Sigma-N \) matrix is a tuple \( M = \langle V, \cdot M, D \rangle \) where \( V \) is the set of truth-values and \( D \subseteq V \) the set of designated values. Further, for each \( \odot \in \Sigma^{(n)} \), \( \odot M : V \to \wp(V) \setminus \{\emptyset\} \), interpreting \( \odot \) as a multi-function over \( V \) instead of a function as in the case of matrices. When complex formulas are interpreted over \( N \) matrices, the value is not completely determined by the values of the subformulas, instead, the multi-functions giving the interpretation of each connective are read nondeterministically, allowing the valuation to choose for each formula a different possible value. An \( M \)-valued \( v \) is a function \( v : L_\Sigma(P) \to V \) satisfying \( v(\odot(\varphi_1, \ldots, \varphi_k)) \in \odot M (v(\varphi_1), \ldots, v(\varphi_k)) \) for any \( k \)-place connective \( \odot \in \Sigma \). Just like in the usual matrix semantics, the logic characterized by an \( \Sigma-N \) matrix \( M \), \( \vdash_M \), is defined by \( \Gamma \vdash_M \Delta \) whenever \( v(\Gamma) \subseteq D \) implies \( v(\Delta) \cap D \neq \emptyset \) for every \( M \)-valuation \( v \). As intended, the set of formulas in the left is read conjunctively and the one in the right disjunctively. We denote simply by \( \vdash_M \) (instead of \( \vdash_M \)) the single-conclusion fragment of \( \vdash_M \), corresponding to the Taskian consequence relation defined by \( M \).

The usual logical matrix semantics is recovered when one considers an \( N \) matrix for which every connective is interpreted as a function. Furthermore, \( N \) matrix semantics preserves various fundamental properties of matrix semantics. Every partial valuation defined over a set closed for subformulas can be extended to a full valuation (analyticity). Furthermore, for finite \( N \) matrix \( M \), \( \vdash_M \) and \( \vdash_M \) are finitary, and deciding \( \vdash_M \) and \( \vdash_M \) is in \( \text{coNP} \) (see [5, 10, 23]).

Example. For \( \text{conn} \subseteq \{\land, \lor, \otimes\} \), let \( \Sigma_{\text{conn}} \) contain exactly the binary connectives in \( \text{conn} \), and \( B_{\text{conn}} = \langle 2, \cdot, \{1\} \rangle \) be the \( \Sigma_{\text{conn}}-N \) matrix where for each \( \odot \in \text{conn} \) its interpretation multi-function is described in tabular form as

\[
\begin{array}{c|c|c}
\land & \0 & \1 \\
0 & 0 & 0 \\
1 & 0 & 1
\end{array} 
\quad \otimes & \0 & \1 \\
0 & 0 & 0,1 \\
1 & 0,1 & 1
\quad \lor & \0 & \1 \\
0 & 0 & 1 \\
1 & 1 & 1
\]

For ease of notation we do not distinguish between the connective and its interpretation, and write \( \vdash_{\text{conn}} \) instead of \( \vdash_{B_{\text{conn}}} \), and \( B_{\otimes} \) and \( B_{\land \lor \otimes} \) instead of \( B_{\{\otimes\}} \) and \( B_{\{\land, \lor, \otimes\}} \).

Whenever \( \otimes \notin \text{conn} \) we have that \( B_{\text{conn}} \) is the well known Boolean matrix characterizing classical logic in the corresponding signature. For now, let us look closer at platypus’ logic \( \vdash_{\otimes} = \vdash_{B_{\otimes}} \) and its single-conclusion fragment \( \vdash_{\otimes} = \vdash_{B_{\otimes}} = \vdash_{B_{\otimes}} \).

As the inclusion arrows indicate, the interpretation of \( \land \) and \( \lor \) are contained in the interpretation of \( \otimes \). Meaning that if we use the tables of \( \land \) or \( \lor \) to interpret \( \otimes \) we obtain a strict subset of valuations. Yielding,

\[
\vdash_{\otimes} \subseteq t_{\otimes}(\vdash_{\land}) \cap t_{\otimes}(\vdash_{\lor})
\]
where $t_{\times}$ swaps every occurrence of $\land$ and $\lor$ by $\times$. This fact has an interesting immediate consequence: $\nrightarrow_{\times}$ inherits the property of being a right-inclusion logic from $\land$, and of being a left-inclusion logic from $\lor$. We say that a logic is of left-inclusion (right-inclusion) whenever $\Gamma \nrightarrow \Delta$ implies there are $\Gamma^\prime \subseteq \Gamma$ and $\Delta^\prime \subseteq \Delta$ such that $\text{var}(\Gamma) \subseteq \text{var}(\Delta)$ (var($\Gamma$) $\supseteq$ var($\Delta$)) where var($\varphi$) denotes the set of variables occurring in $\varphi$.

**Proposition 2.2.** If $\Gamma \nrightarrow_{\times} \Delta$ then $\text{var}(\Gamma^\prime) = \text{var}(\Delta^\prime)$ for some $\Gamma^\prime \subseteq \Gamma$, $\Delta^\prime \subseteq \Delta$.

*Proof.* It follows immediately from the previous observation and the fact that $\text{var}(t_{\times}(\Gamma)) = \text{var}(\varphi)$ for every $\varphi \in L_{\land} \cup L_{\lor}$.

It is well known that every logic $\nrightarrow$ is characterized by some family of matrices. For example, by the family of matrices $M$ satisfying $\nrightarrow \subseteq \nrightarrow_{M}$. Or even by of subset of these, the Lindenbaum bundle of $\nrightarrow$ formed by $M_{\Gamma} = \langle L_{\Sigma}(P), \cdot, \Gamma \rangle$ for $\Gamma \subseteq L_{\Sigma}(P)$ satisfying $\Gamma \not\in \Gamma$ with $\Gamma = L_{\Sigma}(P) \setminus \Gamma$ (see [8, 18, 34]). The same is true of course in the single-conclusion context. However can $\nrightarrow_{\times}$ or $\nrightarrow_{\times}$ be characterized by a single matrix? How could such semantics look like?

**Proposition 2.3.** There is a (deterministic) $\Sigma_{\times}$-matrix $M$ such that $\nrightarrow_{\times} = \nrightarrow_{M}$.

*Proof.* From [34, Thm. 15.2] we know that a logic $\nrightarrow$ is definable by a single matrix if and only if it permits cancellation. That is, if $\bigcup_{k} X_{k} \nrightarrow \bigcup_{k} Y_{k}$ and $\text{var}(X_{i} \cup Y_{j}) \cap \text{var}(X_{i} \cup Y_{j})$ for $i \neq j$, then there is $i$ such that $X_{i} \nrightarrow Y_{j}$. It is easy to see that $\nrightarrow_{\times}$ permits cancellation. Given $\mathbb{B}$-valuations $v_{i}$ such that $v_{i}(X_{i}) = \{1\}$ and $v_{i}(Y_{j}) = \{0\}$, we consider the partial valuation defined over $Z = \bigcup_{k} L_{\times}(\text{var}(X_{k} \cup Y_{k}))$ making $v(\varphi) = v_{k}(\varphi)$ for $\varphi \in L_{\times}(\text{var}(X_{k} \cup Y_{k}))$. As $Z$ is closed under taking subformulas it can be extended to a full $\mathbb{B}$-valuation $v$ showing that $\bigcup_{k} X_{k} \not\in_{\mathbb{B}} \bigcup_{k} Y_{k}$.

Therefore, $\nrightarrow_{\times}$, and hence also $\nrightarrow_{\times}$, can be characterized by a single matrix, however, as we will see, it cannot be a finite one. Given $\varphi \in L_{\times}$ let $nf(\varphi)$ be the smallest subformula $\psi$ of $\varphi$ such that $\varphi \in L_{\times}(\{\psi\})$. That is, $nf((p \times p) \times p) = nf(p \times p) = nf(p) = p$ and $nf((p \times q) \times (p \times q)) = nf(p \times q) = p \times q$. Let also

$\times^{0}(\varphi) = \varphi$

$\times^{n+1}(\varphi) = (\times^{n}(\varphi)) \times \varphi$

It is straightforward to see that for every formula $\varphi$, $\varphi \not\rightarrow_{\times} \text{nf}(\varphi)$. In particular $\varphi \not\rightarrow_{\times} p = \text{nf}(\varphi)$ for every $\varphi \in L_{\times}(\{p\})$. Every pair of formulas written in one variable are equivalent, however when two variables are present the situation dramatically changes and allows us to show the following result.

**Theorem 2.4.** There is no finite matrix $M$ such that $\nrightarrow_{\mathbb{B}} = \nrightarrow_{M}$ or $\nrightarrow_{\mathbb{B}} = \nrightarrow_{M}$.

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3This notion is usually presented over Tarskian consequence relations. In such setting, due to the asymmetry in the very notion of logic, the two notions are not symmetric. We will not enter in details here but just state that for positive logics $\nrightarrow$ where $L_{\Sigma}(P) \not\in \emptyset$ we have that $\nrightarrow$ is left-(right-)inclusion logic iff $\nrightarrow$ is. We point to [9, 12] for more details.
Proof. For each natural \( n \), let \( \varphi_n = (\wedge\wedge p)\wedge xq \). Any \( \mathbb{B} \)-valuation such that 
\[
v(\wedge^n p) = 1 \text{ for every } n \in \mathbb{N}, v(q) = 0, v((\wedge\wedge p)\wedge xq) = 1 \text{ and } v((\wedge\wedge p)\wedge xq) = 0 \text{ for } i \neq j \in \mathbb{N},\]
showing that \( \varphi_i \not\vdash \varphi_j \). Hence, \( \vdash \mathbb{B} \) splits \( L_{\Sigma}\{(p, q)\} \) in an infinite number of equivalence classes. Hence, \( \vdash \mathbb{B} \) is not locally tabular and hence by [14] the first part of the result stands. The second part follows immediately as any \( \mathbb{M} \) such satisfying \( \triangleright \mathbb{B} = \triangleright \mathbb{M} \) satisfies also \( \vdash \mathbb{B} = \vdash \mathbb{M} \). □

It is easy to see that \( p \triangleright \wedge x p \wedge x p \) and \( p\wedge x p \triangleright \wedge x p \) but \( (p\wedge x p)\wedge x p \wedge x q \) as is shown in the previous proposition. Hence, \( \triangleright \wedge x \) is not self-extensional as it lacks substitution by equivalents.

2.3. Abbreviations Over Nmatrices

As we have been seeing, and will be ever more clear along the remainder of this paper, \( \wedge x \) behaves in many different ways from its deterministic Boolean cousins. In fact, non-determinism opens the door to various unexpected phenomena. A notable difference regards considering connectives defined by abbreviation. As in the deterministic case, given an \( \Sigma \)-Nmatrix each formula \( \varphi \in L_{\Sigma}(P) \) (in \( n \) variables), defines an \( n \)-ary multi-function
\[
\varphi_M(x_1, \ldots, x_n) = \{v(\varphi(p_1, \ldots, p_n)) : v \text{ is } \mathbb{M}-\text{valuation, } v(p_i) = x_i, 1 \leq i \leq n\}
\]
Note that \( \varphi_M \) does not correspond to the composition of the interpretation of the connectives (as multi-functions) forming \( \varphi \) using any of the notions we mentioned in the previous subsection. Although it is closer to the second, it is still more restrictive. Given \( f \) defined by \( \varphi \) and \( g_i \) by \( \psi_i \), their composition is given by \( f \circ (g_1, \ldots, g_n) = \varphi(\psi_1, \ldots, \psi_n)_M \). Whenever composing \( f(g_1(h(x)), g_2(h(x))) \) we must guarantee that the values fed to \( f \), must come from values of \( g_1(h(x)) \) and \( g_2(h(x)) \) for the same values of \( h(x) \). Nonetheless, Proposition 2.1 still applies to this notion of composition of (expressible) multi-functions. Another particularity of definitions by abbreviation in the context of Nmatrices is that whenever two formulas determine the same function (instead of a multi-function) we have that they are logically interchangeable in every context. However when some input may output more than a value this is not necessarily the case. For example, \( \wedge x \) is given by a symmetric table, and indeed \( \wedge x(y, x) \) and \( \wedge x(x, y) \) define the same multi-function, we have that \( p\wedge x q \wedge x q\wedge x p \). This means that when working with Nmatrices we have to be careful about what we expect from a connective defined by abbreviation. Clearly, it may lose any connection with the connectives used to define it, as the possibility of independent choices offered by the non-determinism may break the relation between them in the resulting logic. This is a question that must be taken into account by any approach to the logics of Nmatrices using clones of multi-functions.

3. Axiomatizability

Both Tarskian and Scottian logics are associated with very natural notions of axiomatizability according to their type. A set of sound rules \( R \subseteq \vdash \) axiomatizes (or is a basis for) \( \vdash \) whenever \( \vdash \) is the closure of \( R \) under (\( R \)(\( M \)))(\( T \))(\( S \))
in which case we write \( \vdash_R = \vdash \). Analogously, set sound rules \( R \subseteq \triangleright \) axiomatizes (or is a basis for) \( \triangleright \) whenever \( \triangleright \) is the closure of \( R \) under \((O)(D)(C)(S)\), and in that case we write \( \triangleright_R = \triangleright \). These definitions fare well on the compositional front, as given two logics of compatible type axiomatized by sets of rules \( R_1 \) and \( R_2 \) (of according type) their fibring, the smallest logic (of the same type) in the combined language that contains both, is axiomatized by \( R_1 \cup R_2 \).

Crucially, in both cases the abstract properties defining each type of calculi correspond to the machinery of Hilbert-style calculi where derived consequences using a set of rules \( R \) are exactly the ones that hold in the logic axiomatized by \( R \). In the single-conclusion case, derivations are sequences where the application of a rule produces a new formula, and in the multiple-conclusion case the proofs take an arboreal shape since the application of the rules produces set of formulas, each corresponding to a child of the node where it was applied\(^4\). The second notion strictly generalizes the first as derivations using only single-conclusion rules coincide in both settings. If \( R \) is a set of single conclusion rules \( \triangleright_R = \triangleright_{R_1} \). For a formal definitions and illustrative examples of such derivations we point to \([10,23,34]\).

### 3.1. Single-Conclusion Rules Only

Rautenberg has shown in \([27]\) that every fragment of classical logic is finitely axiomatizable (using single-conclusion rules).

**Example.** Consider the following well known axiomatizations for the logics of the deterministic reducts of \( \mathbb{B}_{\land \lor \land} \). For \( \text{conn} \subseteq \{\land, \lor\} \), \( \vdash_{\text{conn}} = \vdash \text{conn} \) with

\[
R_{\land}^{sc} = \left\{ \frac{p \land q}{r_1^\land}, \frac{p \land q}{r_2^\land}, \frac{p}{p \land q} \right\}
\]

\[
R_{\lor}^{sc} = \left\{ \frac{p}{p \lor q}, \frac{p \lor q}{q \lor p}, \frac{p \lor (q \lor r)}{(p \lor q) \lor r} \right\}
\]

\[
R_{\land \lor}^{sc} = R_{\land}^{sc} \cup R_{\lor}^{sc} \cup \left\{ \frac{p \lor q}{p \lor (q \lor r)}, \frac{p \lor (q \land r)}{p \lor q}, \frac{p \lor (q \land r)}{p \lor r} \right\}
\]

We have shown in \([13]\) that the rules mixing \( \land \) and \( \lor \) are fundamental in the single-conclusion setting to capture the interaction between these connectives, contrasting with what happens in the multiple-conclusion setting, as we shall see latter on in this section. \(\triangle\)

How about \( \vdash \mathbb{X} \)? Can Rautenberg’s result be extended to Boolean Nmatrices? The next theorem shows that the answer is negative, but first a proposition giving an useful explicit syntactic characterization of \( \vdash \mathbb{X} \).

**Proposition 3.1.** \( \Gamma \vdash \mathbb{X} \varphi \) if and only if \( \varphi \in L_\mathbb{X}(\text{nf}(\Gamma)) \).

\(^4\)Rules with empty set of conclusion discontinue the branch of the node where it is applied. We have that \( \Gamma \triangleright_R \Delta \) whenever there is a \( R \)-derivation departing from \( \Gamma \) where the leaf of each non discontinued branch must be a formula in \( \Delta \).
Proof. From right to left, let \( v \) be an \( M_\times \)-valuation such that \( v(\Gamma) = \{1\} \). As \( \psi \vdash_\times \text{nf}(\psi) \) for every \( \psi \in \Gamma \) we get that \( v(\text{nf}(\Gamma)) = \{1\} \), and from \( p, q \vdash_\times p \wedge q \) we get that \( v(\psi) = 1 \) for \( \psi \in L_\times(\text{nf}(\Gamma)) \).

From left to right, we show, by induction on the structure of \( \varphi \), that if \( \varphi \notin L_\times(\text{nf}(\Gamma)) \) then there is a \( B_\times \)-valuation such that \( v(\Gamma) = \{1\} \) and \( v(\varphi) = 0 \). If \( \varphi = p \in P \) then \( v(\psi) = 0 \) iff \( \psi = p \) does the job, as every formula \( \Gamma \) must contain some variable different from \( p \). If \( \varphi = \varphi_1 \wedge \varphi_2 \) then there must be \( i \in \{1,2\} \) such that \( \varphi_i \notin L_\times(\Gamma) \). By induction hypothesis there is \( v \) such that \( v(\Gamma) = 1 \) and \( v(\varphi_i) = 0 \). Hence, we can define some \( v' \) that coincides with \( v \) on the set of subformulas of \( \Gamma \), and makes \( v'(\varphi_1 \wedge \varphi_2) = 0 \). \( \square \)

For \( n \in \mathbb{N} \), let

\[
R_n = \left\{ \varphi \vdash_\times \text{nf}(\{p\}), n_\times(\varphi) \leq n \right\}
\]

\[
R_\omega = \bigcup_{i < \omega} R_i
\]

\[
R^{sc}_\times = \left\{ \frac{p \cdot q}{p \wedge q} \right\} \cup R_\omega
\]

where \( n_\times(\varphi) \) is the number of occurrences of \( \times \) in \( \varphi \).

**Theorem 3.2.** \( \vdash_\times \) is axiomatized by \( R^{sc}_\times \) and it is not finitely axiomatizable.

**Proof.** That \( \vdash_\times = \vdash^{R^sc}_\times \) follows easily from Proposition 3.1 by observing that the rules in \( R_\omega \) are enough to produce \( \text{nf}(\Gamma) \) from \( \Gamma \) and \( \frac{p \cdot q}{p \wedge q} \) is enough to generate every formula in \( L_\times(\text{nf}(\Gamma)) \) from \( \text{nf}(\Gamma) \).

To see that \( \vdash_\times \) is not finitely axiomatizable, it is enough to show that \( \vdash_\times \) is the limit of an infinite strictly increasing sequence \( \vdash_n \) (see [35, Theorem 2.2.8]). Consider the logic \( \vdash^n \) axiomatized by \( \{ \frac{p \cdot q}{p \wedge q} \} \cup R_n \) for each \( n \in \mathbb{N} \). Clearly, \( \vdash_n \subseteq \vdash_{n+1} \subseteq \vdash_\times \) for each \( n \) and \( \vdash_\times \) is the limit of this sequence. It remains to show that \( \vdash_n \not\subseteq \vdash_{n+1} \). Let \( M_n = \langle \{a_0, \ldots, a_n, 1\}, \cdot, \cdot, \{1\} \rangle \) with

\[
\begin{align*}
\varphi_n(1, x) &= \varphi_n(x, 1) = \{1, x\} \\
\varphi_n(a_0, a_n) &= \varphi_n(a_n, a_0) = \{1, a_0\} \\
\varphi_n(a_i, a_j) &= \{a_{\max(0,\min(i,j)-1)} : i, j \neq \{0, n\}\}.
\end{align*}
\]

It is easy to check that \( p, q \vdash_{M_n} p \wedge q \) and \( \varphi^k(p) \vdash_{M_n} p \) for \( k \leq n \). To show that \( \varphi^{n+1}(p) \vdash_{M_n} \) \( p \) for \( k \leq n \), consider an \( M_n \)-valuation \( v \) such that \( v(p) = a_n \), \( v(p \wedge p) = a_{n-1} \), \ldots, \( v(\varphi^k(p)) = a_0 \) and \( v(\varphi^{n+1}(p)) = v(\varphi^n(p)) = 1 \). \( \square \)

### 3.2. Allowing Multiple-Conclusion Rules

The fact that for certain finite Nmatrices \( M, \vdash_M \) is non-finitely axiomatizable using single-conclusion rules is nothing that non-determinism can be blamed for. In [36], Wrόnski shown that \( \vdash_C \) for the (deterministic) \( \Sigma^*_C \)-matrix \( C = \{\{0,1,2\}, \cdot, \{2\}\} \) where \( \Sigma^*_C \) contains a single binary connective \( \cdot \) and

| \( \cdot \) | 0 | 1 | 2 |
|-------|---|---|---|
| 0     | 1 | 2 | 2 |
| 1     | 2 | 2 | 2 |
| 2     | 1 | 2 | 2 |
is not finitely (single-conclusion) axiomatizable. However, \(\vartriangleright_{\mathcal{C}}\) is axiomatized multiple-conclusion by the following 6 rules [23]:

\[
\begin{array}{ccc}
(p \cdot q) \cdot (p \cdot q) & q & q \cdot q \\
\frac{p \cdot p}{p, p \cdot q} & \frac{p \cdot q}{q \cdot q} & \frac{p \cdot p, q \cdot q}{p \cdot q}
\end{array}
\]

The first advantage of working in the multiple-conclusion setting is that every finite (deterministic) matrix is finitely axiomatizable [34, Theorem 19.12], whereas in the single-conclusion this fails already for matrices of size 3.

In [23] we shown that this result could be extended to every finite Nmatrix provided it is monadic, a reasonable expressiveness requirement. An Nmatrix is \(\mathcal{M} = (V, \cdot \mathcal{M}, D)\) is monadic if there is a set of formulas in one variable \(S \subseteq L_{\Sigma \{\{p\}\}}\) separating \(\mathcal{M}\), that is, such that for each entry where \(\varphi \in S\) there is \(\varphi \in S\) such that \(\varphi_{\mathcal{M}}(x) \subseteq D\) and \(\varphi_{\mathcal{M}}(y) \subseteq V \setminus D\), or \(\varphi_{\mathcal{M}}(x) \subseteq V \setminus D\) and \(\varphi_{\mathcal{M}}(y) \subseteq D\). In the 3-valued matrix \(\mathbb{C}\) this requirement is met with \(S = \{p, p \cdot p\}\) which allowed us to produce the axiomatization above.

Moreover, it is easy to check that for any signature \(\Sigma\) and Boolean Nmatrix \(\mathcal{M} = (2, \cdot \mathcal{M}, \{1\})\) we have that the set \(x \subseteq \mathcal{M}\) separates \(\mathcal{M}\). In this (Boolean) case, the general strategy to axiomatize \(\vartriangleright_{\mathcal{M}}\) introduced in [23] boils down to collecting the rules \(R_{\ominus, x}\) for each \(\ominus \in \Sigma^{(n)}\) and \(x \in 2^{\pi}\) where \(\mathcal{M}(x) \neq \{0, 1\} \neq \{0, 1\}\).

\[
\begin{aligned}
\text{if } \mathcal{M}(x) = \{0\}, & \text{ let } R_{\ominus, x} = \{p_i : x_i = 1\} \cup \{\ominus(p_1, \ldots, p_k)\} \\
\text{if } \mathcal{M}(x) = \{1\}, & \text{ let } R_{\ominus, x} = \{p_i : x_i = 1\} \cup \{\ominus(p_1, \ldots, p_k)\}
\end{aligned}
\]

Note that this axiomatization is completely modular on the connectives being considered. Furthermore, it is modular in the entries of the table defining each connective, a single rule is collected for each entry where \(\mathcal{M}(x)\) is a singleton. Yielding the following axiomatization of \(\vartriangleright_{\circ}\).

**Theorem 3.3.** \(\vartriangleright_{R_{\circ}} = \vartriangleright_{\circ}\) with

\[
R_{\circ} = \left\{ \frac{p, q, p \otimes q, \frac{p \otimes q}{p, q}}{\frac{p \otimes q}{p, q}} \right\}
\]

As a corollary we obtain that \(\circ\) is the mysterious connective introduced in [20, Exercise 4.31.5], by observing that a multiple-conclusion rule \(\Gamma_{\mathcal{A}}\) is equivalent to the metarule \(\Gamma_{\mathcal{A}} \cup \{\Gamma_i, B_i \vdash C : B_i \in \Delta\}\).

This recipe can be applied to the deterministic case too, yielding \(\vartriangleright_{\mu} = \vartriangleright_{R_{\circ}^\mu}\) (so \(\vartriangleright_{\lambda} = \vartriangleright_{R_{\circ}^\mu}\)) and \(\vartriangleright_{\vee} = \vartriangleright_{R_{\circ}^\vee}\) with \(R_{\circ}^\mu = \left\{ \frac{p, q, p \otimes q, \frac{p \otimes q}{p, q}}{\frac{p \otimes q}{p, q}} \right\}\). Using the modularity on the connectives immediately we obtain that, for \(\text{conn} \in \{\mu, \lambda, \vee\}\),
we have \( \triangleright_{\text{conn}} \Rightarrow R_{\text{mc}}^\triangleright \), with \( R_{\land,\lor}^\text{mc} = R_{\land,\lor}^\triangle \cup R_{\land,\lor}^\text{mc} \), \( R_{\land,\lor}^\text{sc} = R_{\land,\lor}^\text{mc} \cup R_{\land,\lor}^\text{mc} \) and \( R_{\land,\lor}^\text{mc} = R_{\land,\lor}^\text{mc} \cup R_{\land,\lor}^\text{mc} \).

**Example.** The following three multiple-conclusion derivations illustrate how multiple-conclusion derivations can be useful even if the set of conclusions is a singleton.

\[
p \triangleright \varpi^3(p) \quad \varpi^3(p) \triangleright \varpi \quad (p \land q) \varpi (q \land p) \triangleright \varpi \ p \land q
\]

The first derivation exemplifies how formulas in \( L_{\varpi}(\{p\}) \) can be derived from \( p \), using the rule \( r_{\varpi}^\text{sc} \) shared by \( R_{\varpi}^\text{sc} \) and \( R_{\varpi}^\text{mc} \). In the second we show how in this more expressive setting we can recover the infinite rules in \( R_\omega \) needed to axiomatize \( \triangleright \varpi \). In the second we show a valid rule in \( B_{\land,\lor} \) that will be useful later on.

Note that there are Nmatrices that are finitely axiomatizable using only single-conclusion rules. For example, the above strategy applied to the Boolean \( \Sigma_{\rightarrow} \)-Nmatrix \( B_{\rightarrow} \) where \( \Sigma_{\rightarrow} \) contains a single binary connective \( \rightarrow \) interpreted by the table

|   | 0 | 1 |
|---|---|---|
| 0 | 0,1| 0,1|
| 1 | 0 | 0,1|

yields the single rule of *modus ponens* \( R_{mp} = \{ \frac{p \rightarrow q}{p, p \rightarrow q} \} \). The fact that a multiple-conclusion logic is axiomatized by single-conclusion rules, implies that \( \triangleright_M = \triangleright_{R_{mp}} \) is the smallest companion of \( \triangleright_R \), \( \triangleright_{\triangleright_R} \). Note that \( \triangleright_{B_{\rightarrow}} \) is also not characterizable by finite matrix \([14,23]\).

There is still another advantage in considering multiple-conclusion rules. The result in \([14,23]\) extends the result in \([34]\) in yet another way, a big novelty on Hilbert-style calculi (that avoid the inclusion of meta-language in the derivation mechanism). The obtained calculi are *analytic*, in the sense that \( \Delta \triangleright_R \Gamma \) if and only if there is a \( R \)-derivation where only subformulas of \( \Gamma \) and \( \Delta \) appear. This allows for effective purely symbolic decision procedures for the logic and proof-search mechanism, see \([10,23]\).
3.3. Compositionality: Single- vs Multiple-Conclusion

As we mentioned above there are great advantages on the compositional-unity front in avoiding any meta-language. The smallest (single- or multiple-conclusion) logic that contains two (single- or multiple-conclusion) logics is axiomatized by joining axiomatizations of both logics. This allows us to control the desired interactions. The study the underlying compositional mechanisms can further enlighten us regarding the divide between single-/multiple-conclusion settings. In [13], by taking profit from the complete characterization of the Boolean clones (of functions) given by Post [25], we have shown that fragments of classical logic seen as a single-conclusion logics cannot in general be axiomatized by joining the single-conclusion axiomatizations of each of the connectives in that fragment. This sharply contrasts with what happens in multiple-conclusion, as is evident by the general recipe for axiomatizing Boolean Nmatrices presented above. The fact is that the single-conclusion fragment of the fibring of two multiple-conclusion logics differs from the fibring of their single-conclusion fragments. As shown in Example 3.2 (p ∧ q)X(q ∧ p) ∪ R^cωR^cω p ∧ q, however by observing the rules in R^cωR^cω one can sense that (p ∧ q)X(q ∧ p) ⊬ R^cωR^cω p ∧ q. How can we prove it?

Recent work (some still unpublished) on fibring semantics shines a new light on this phenomena. We will now give a glimpse over it. We say that an Nmatrix M = ⟨V, ·M, D⟩ is saturated whenever for every ⊬M-theory Γ there is a M-valuation ψ such that that designates v(ψ) ∈ D iff ψ ∈ Γ. Equivalently, M is saturated iff Γ ⊬M ψ for every ψ ∈ ∆ then Γ ⊭M ∆.

Given two Nmatrices M1 = ⟨V1, ·1, D1⟩ and M2 = ⟨V2, ·2, D2⟩ over disjoint signatures let M1 * M2 = ⟨V1, ·1, 12, D12⟩ where V12 = (D1 × D2) ∪ (V1 \ D1 × V2 \ D2), D12 = D1 ∪ D2 and

(⊙i)12(x) = {(y1, y2) ∈ V12 : yi ∈ ⊙i if ⊙i ∈ Σi}.

In [22] we have show that given two Nmatrices M1 = ⟨V1, ·1, D1⟩ and M2 = ⟨V2, ·2, D2⟩ over disjoint signatures their fibring is simply characterized by M1 * M2 whenever M1 and M2 are saturated. For every M = ⟨V, ·M, D⟩ let Mω = ⟨Vω, ·ω, Dω⟩ where ·ω interprets each connective component wise. As for any Nmatrix M we have that ⊬M = ⊬Mω and Mω is always saturated we obtain a general semantics for joining single-conclusion rules: if ⊬Mi = ⊬Di for i ∈ {1, 2} then ⊬R1 ⊔ R2 = ⊬R1 * R2.

We know that Bω ∞ is saturated, hence ⊬RωRω ∪ Rω = ⊬Bω ∞ * Bω ∞. It is not hard to show that Bω ∞ * Bω ∞ is isomorphic to ⟨ϕ(N), ψ∗, {N}⟩ where

Xψ∗Y = \{Z ∈ ϕ(N) : X \subseteq Z \subseteq N \setminus (X \cup Y)\}

\begin{align*}
Xψ\neg Y &= \begin{cases} 
\{N\} & \text{if } X = Y = N \\
ϕ(N) \setminus \{N\} & \text{otherwise}. 
\end{cases}
\end{align*}

Now we can finally show that indeed (p ∧ q)X(q ∧ p) ⊬RωRω p ∧ q. Clearly defining v(p) = v(q) = v(q ∧ p) = ϕ(q ∧ p) = ∅ and v((p ∧ q)X(q ∧ p)) = N makes v a partial Bω ∞ * Bω ∞-valuation defined over a set closed for subformulas

5Using 2ω ∞ ≈ ϕ(N) ≈ {N, 1} ∪ {0, X : X ⊆ N}. 
and hence the result follows. As a consequence we obtain that $B_X$ cannot be saturated. Furthermore, no finite saturated Nmatrix characterizes $\vdash_{X}$.

**Proposition 3.4.** There is no finite saturated Nmatrix $M$ such that $\vdash_{M} = \vdash_{X}$.

**Proof.** Let $\Gamma_n = \{p_i \& p_j : 0 \leq i \neq j \leq n\}$ and $\Delta_n = \{p_i : 0 \leq i \leq n\}$. We show that given $M = \langle V, \cdot_{M}, D \rangle$ such that $V \setminus D$ has at most $n$ elements and $\vdash_{X} \subseteq \vdash_{M}$ then $\Gamma_n \not\vdash_{M} \Delta_n$. Given $M$-valuation $v$ either if $v(p_i) \in D$ for some $0 \leq i < n$ or $v(p_i) = v(p_j) \in V \setminus D$ for some $i \neq j$. If the second case holds then since $p_i \& p_j \not\vdash_{X} p_k$ for $0 \leq i \leq n$ we conclude that $M$ is not saturated. $\square$

It is easy to see that given two Boolean Nmatrices over disjoint signatures their strict product corresponds to joining the operations in a single Nmatrix. If instead we consider the two $\Sigma_{X} \& \lor$-Nmatrices $B_{\in X} = \langle 2, \cdot_{\in}, \{1\} \rangle$ and $B_{\ell X} = \langle 2, \cdot_{\ell}, \{1\} \rangle$ where

| $\cdot_{\in}$ | 0 | 1 |
|--------------|---|---|
| 0            | 0,1,1 | 1 |
| 1            | 0,1,1 | 1 |

and ignore the restriction on the definition of strict product above demanding that the signatures are disjoint, we obtain that $B_X$ is isomorphic to $B_{\in X} \times B_{\ell X}$ where $(0,0)$ and $(1,1)$ are renamed 0 and 1, respectively. If we apply the axiomatization strategy introduced in the last subsection we obtain that $\vdash_{B_{\in X}}$ is axiomatized by the single rule $r_{\in X}^c$ and $\vdash_{B_{\ell X}}$ is axiomatized by the single rule $r_{\ell X}^c$. Hence, platypus’ logic is the multiple-conclusion fibring of the two logics axiomatized by each of the rules that axiomatize $\vdash_{X}$, and $B_X$ is the strict product of the semantics characterizing each of these rules. In general the strict product of two Nmatrices may end up with some entries with an empty set of values, corresponding to entries where each of the Nmatrices output incompatible values. There is a generalization of Nmatrices called PNmatrices where partiality is also allowed. We are working on a general description of fibred semantics that takes PNmatrices as semantical units.

### 4. Complexity

In this section we show how moving to the richer multiple-conclusion setting may be essential to capture an otherwise hidden behaviour of a certain connective. Can the problem of deciding the single-conclusion logic and some of its multiple-conclusion companion differ in complexity? It would not be hard to cook an artificial example where this is the case, however the logic of $X$ readily does the job.

**Theorem 4.1.** Deciding $\vdash_{X}$ is in $P$ and deciding $\vdash_{X}$ is coNP-complete.

**Proof.** The first part follows easily from Proposition 3.1: $\text{nf}(\Gamma)$ can be built from $\Gamma$ in polynomial time and its size at most as big as $\Gamma$. Deciding $\varphi \in$
\(L_{\vec{x}}(nf(\Gamma))\) can be done in polynomial time in the sum of the sizes of \(nf(\Gamma)\) and \(\varphi\).

For the second part we will give a polynomial reduction one of the standard problems known to be \(\text{NP}\)-complete to the problem of deciding \(\varphi_{\vec{x}}\). Given an instance of 3-sat, \(I = \{x_1^i \lor x_2^i \lor x_3^i : 1 \leq i \leq k\}\) where \(x_j^i\) are literals, that is \(x_j^i = p_j^i\) or \(x_j^i = \neg p_j^i\), let:

\[
\begin{align*}
\Gamma_I &= \{q_{x_1^i} \land (q_{x_2^i} \land q_{x_3^i}) : 1 \leq i \leq k\} \\
\Gamma_{\text{neg}} &= \{q_p \land q_{\neg p} : p \in \text{var}(I)\} \\
\Delta_{\text{neg}} &= \{q_{\neg p} \land q_p : p \in \text{var}(I)\}
\end{align*}
\]

The set \(\Gamma_I \cup \Gamma_{\text{neg}} \cup \Delta_{\text{neg}}\) can be produced in polynomial time from \(I\) and it is only polynomially larger than it. Given a valuation \(v\) over \(\mathbb{B}\) such that \(v(\Gamma_I) = \{1\}\) means that for each \(i\) at least one of \(q_{x_1^i}, q_{x_2^i}, q_{x_3^i}\) must have value \(1\). Furthermore, if \(v(\Gamma_{\text{neg}}) = \{1\}\) and \(v(\Delta_{\text{neg}}) = \{0\}\) means for every variable \(p\) appearing in \(I\) either \(q_{\neg p}\) or \(q_p\) has value \(1\) but not both. Hence, \(I\) is satisfiable if and only if \(\Gamma_I \cup \Gamma_{\text{neg}} \varphi_{\vec{x}} \Delta_{\text{neg}}\).

\(\square\)

Is this possible for some fragment of classical logic? The answer is no. Let \(\mathbb{B}\) be a Boolean \(\Sigma\)-matrix where \(\Sigma\) contain some arbitrary set of Boolean connectives. From the complete characterization of the complexity of deciding the single-conclusion fragments of classical logic in [7] we know that \(\vdash_{\mathbb{B}}\) is not \(\text{coNP}\)-complete if and only if the connectives in \(\Sigma\) are expressible using the constant functions 0 and 1, and only one of the following three binary connectives \(\{\land, \lor, \oplus\}\). Deciding \(\vdash_{\mathbb{B}}\) for all these non \(\text{coNP}\)-complete cases is in \(\text{P}\). Since \(\vdash_{\mathbb{B}}\) is a fragment of \(\triangleright_{\mathbb{B}}\) and deciding the latter is always in \(\text{coNP}\), we have that if \(\vdash_{\mathbb{B}}\) is \(\text{coNP}\)-complete then \(\triangleright_{\mathbb{B}}\) is also. The complexity of \(\triangleright_{01\land} (= \triangleright_{\Gamma_{01\land}})\) is in \(\text{P}\) hence if the connectives in \(\Sigma\) are expressible using 0, 1 and \(\land\) then \(\triangleright_{\mathbb{B}}\) is also in \(\text{P}\). Furthermore, we have that \(\Gamma \triangleright_{\mathbb{B}} \{\delta_1, \ldots, \delta_k\}\) is equivalent to \(\Gamma \vdash_{\mathbb{B}} \delta_1 \lor \ldots \lor \delta_k\) wherever \(\lor\) is expressible by the connectives in \(\Sigma\), and to \(\Gamma, \neg \delta_2, \ldots, \neg \delta_k \vdash_{\mathbb{B}} \delta_1\) wherever \(\neg\) is expressible by the connectives in \(\Sigma\). Easily we have that \(\neg x = p \oplus 1\). Hence, in all remaining cases deciding \(\triangleright_{\mathbb{B}}\) is polynomially reducible to either \(\vdash_{01\lor}\) or \(\vdash_{01\oplus}\) and, therefore, also in \(\text{P}\).

In [7] some (sub-polynomial) nuances between deciding the single-premise fragments of \(\vdash_{\mathbb{B}}\) are also studied. Most likely, such variations are also present in the divide \(\vdash_{\mathbb{B}}\) and \(\triangleright_{\mathbb{B}}\) but we leave such analysis for another occasion.

5. Conclusion

We have shown how incorporating non-determinism in semantics and moving towards a symmetrical view on logic, considering multiple-conclusion consequence relations, significantly widens the range of available tools. In particular, it allows for a compositional, bottom up, approach to the analysis of logics. Despite the recent advances there is still a long way to go to reach the depth and sophistication of what is known in the particular field of modal logics. We look to it as source of case studies and inspiration on where to go. We intend to improve on [22] and provide a completely general and modular semantics...
for combined logics by taking as semantical units PNmatrices, covering, of course, what is known for fusion of modal logics. We expect to take profit from such knowledge, and develop techniques as in [11], to provide insights on the semantics of strengthening of logics with sets of axioms depending on their syntactical structure, akin to what is known regarding Sahlqvist formulas.

The use of the extra expressivity of (P)Nmatrix semantics to capture relevant behaviours in many practical engineering or scientific contexts is still in its infancy. Nmatrices have been used already to give effective semantics to a big range of important non-classical logics [2,6,12,17]. For example, these structures can be used to deal with paraconsistent behaviour [15], to model how a processor deals with information from multiple-sources [5], or for reasoning about computation errors [3]. The fact that $\land$ can be seen as a defective $\land$ or $\lor$ suggests that (P)Nmatrices can be of value when reasoning about unreliable logic circuits [26,31]. Also, recently a natural interpretation of quantum states as valuations over Nmatrices was introduced [21].

There is still another front opened by considering multi-functions. Acknowledging the importance of the study of function clones in logic, we consider that the development of the theory of multi-function clones is an interesting path to follow. The fact that there are various possible notions of composition, relating to the differences between tree-automata, dag- and term-automata [1,16], complicates the question. However, the connections with such well established and studied structures looks promising.

We finish with a table summarizing some of the results obtained in this paper, illustrating the differences between the logic of $\land$ and the classical fragments with $\land$ and $\lor$. We can see that certain properties of $\land$ and $\lor$ are preserved (being inclusion logics for instance). However, others are not. Finite-valuedness in terms of matrices and finite axiomatizability using single-conclusion rules are lost. In terms of complexity, deciding the Tarskian logic defined by $\land$ is in the same complexity of each $\land$ and $\lor$ separately. However deciding its Scottian logic meets the maximum possible complexity of deciding a logic given by a finite Nmatrix, coinciding with the complexity of deciding any fragment of classical logic expressing both $\land$ and $\lor$. Lastly, it seems interesting to explore the relation of $\land$, and the process that gave rise to it, with the notion of meet-combination of logics introduced by Sernadas et al. in [33].

| Logic | Fin. Ax. | Fin. matrix | Complexity | Left-inclusion | Right-inclusion |
|-------|----------|-------------|------------|---------------|-----------------|
| $\vdash_\land$ | Y | Y | P | N | Y |
| $\vdash_\lor$ | Y | Y | P | Y | N |
| $\vdash_\land$ | N | N | P | Y | Y |
| $\vdash_\land \lor$ | Y | Y | coNP | N | N |
| $\vdash_\land$ | Y | Y | P | N | Y |
| $\vdash_\lor$ | Y | Y | P | N | Y |
| $\vdash_\land$ | Y | N | coNP | Y | Y |
| $\vdash_\land \lor$ | Y | Y | coNP | N | N |
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