Regularities of the Gurtin-Pipkin equation

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Abstract

We study regularity of the solution $\theta$ to the Gurtin-Pipkin integral-differential equation of the first order in time. In particular, we prove that the ‘perturbation’ part, namely, the difference of $\theta$ and the solution to the corresponding wave equation is smoother than $\theta$.

1 Introduction

In several fields of physics such as heat transfer with finite propagation speed [2], systems with thermal memory [5], viscoelasticity problems [3], and acoustic waves in composite media [1], the integro-differential equations arise. We consider the equation of the first order in time

$$\theta_t(x, t) = \int_0^t k(t - s)\theta_{xx}(x, s)\, ds + f(x, t), \quad x \in (0, \pi), \quad t > 0,$$

(1)

with the Dirichlet boundary conditions and with the initial data $\theta(0, x) = \xi(x)$.

In the case $k(t) = \text{Const} = \alpha^2$ the equation (1) is, in a fact, an integrated wave equation. Indeed, differentiate (1) gives

$$\theta_{tt} = \alpha^2 \theta_{xx} + f_t(x, t), \quad \theta(x, 0) = \xi, \quad \theta_t(x, 0) = 0.$$

(2)

Thus, the wave equation is a special case of (1) and we will compare general regularity results with the regularity of the solutions to the wave equation.

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†The work was supported by Russian Foundation for Basic Research, RFBR Project 11-01-00790a and RFBR Project 11-01-00667a.
2 Fourier method and the Laplace transform

First, apply the Fourier method: we set \( \varphi_n = \sqrt{\frac{2}{\pi}} \sin nx \) and expand the solution, the RHS, and the initial data in series in \( \varphi_n \)

\[
\theta(x, t) = \sum_{1}^{\infty} \theta_n(t) \varphi_n(x), \quad \xi(x) = \sum_{1}^{\infty} \xi_n \varphi_n(x), \quad f(x, t) = \sum_{1}^{\infty} f_n(t) \varphi_n(x).
\]

The components \( \theta_n \) satisfy ordinary integral-differential equations

\[
\dot{\theta}_n(t) = -n^2 \int_{0}^{t} k(t-s) \theta_n(s) ds + f_n(t), \quad t > 0, \quad \theta_n(0) = \xi_n. \quad (3)
\]

Note that the solutions to this integral-differential is unique and continuous, what we can see by \( t \)-integration of the equation from 0 to \( t \). Indeed, we obtain a Volterra integral equation: with \( \varphi = \int f(t) dt \), we have

\[
\theta_n(t) - \xi_n = -n^2 \int_{0}^{t} \int_{0}^{\tau} k(\tau-s) \theta_n(s) ds + \varphi(t).
\]

Change order of integrations

\[
\theta_n(t) = -n^2 \int_{0}^{t} ds \theta_n(s) \int_{s}^{t} d\tau k(\tau-s) + \varphi(t) + \xi_n = -n^2 \int_{0}^{t} q(t-s) \theta_n(s) ds + \varphi(t) + \xi_n
\]

with

\[
q(s) = \int_{0}^{s} k(y) dy.
\]

Introduce the scale \( \mathcal{H}_s \), \( s \) is real, of the Hilbert spaces \( \mathcal{H}_s = \text{Dom}(A^{s/2}) \), where the operator \( A \) is \(-d^2/dx^2\) with the Dirichlet boundary conditions at 0 and at \( \pi \). A space \( \mathcal{H}_s \) is a subspace of the Sobolev space \( H^s \) and may be described in terms of the Fourier coefficients. Let the space \( l_s \) be the space of sequences \( \{c_n\} \) such that

\[
\sum_{1}^{\infty} |c_n|^2 n^{2s} < \infty.
\]

Then

\[
\mathcal{H}_s = \left\{ u(x) = \sum_{1}^{\infty} u_n \varphi_n(x) \mid \{u_n\} \in l_s \right\}.
\]
Consider also the space $H_{s,\varepsilon}$ of functions $g(x,t) = \sum_{1}^{\infty} f_n(t) \varphi_n(x)$ with the norm
\[ \| g \|_{H_{s,\varepsilon}}^2 = \sum_{1}^{\infty} \| e^{-2\varepsilon t} f_n \|_{L^2(0,\infty)}^2. \]

**Definition 1** The function $\theta(x,t) = \sum_{1}^{\infty} \theta_n(t) \varphi_n(x)$ is a solution to (1) in $H_{s,\varepsilon}$ if the functions $\theta_n$, satisfy the integral equation (4) and $\theta \in H_{s,\varepsilon}$ with $s \in \mathbb{R}$.

Let $H^2_{\varepsilon}$ denote the Hardy space in the right half plane $\mathbb{R}z > \varepsilon$. The Paley-Wiener theorem says that
\[ \| F \|_{H^2_{\varepsilon}}^2 = \int_{0}^{\infty} e^{-2\varepsilon t} |f(t)|^2 dt. \]

Here and in what follows will denote the Laplace image by the capital characters. Applying the Laplace Transform to (3) we find
\[ z \Theta_n(z) - \xi_n = -n^2 K(z) \Theta_n(z) + F_n(z) \]
or
\[ \Theta_n(z) = \frac{\xi_n + F_n(z)}{z + n^2 K(z)}. \]  

Denote the denominators in (5) by $G_n(z)$. The set $\Lambda$ of all zeros of $G_n(z)$ is called the spectrum of the equation (1).

Regularity of the Gurtin-Pipkin type equation is studied in [4] for several spatial variables, where under assumption that $k(t)$ is twice continuously differentiable it was shown, in particular, that $\Theta(x,t) \in C([0,T]; L^2(0,T))$.

Regularity of strong solutions has been studied in several works of V. Vlasov with the coauthors, see, e.g., [8] and the Sec.5 below. The spectrum of the equation is studied in [9],[8].

Let us describe the regularity of the solutions to the wave equation (2). Let $Q_T = (0, \pi) \times (0, T)$.

**Proposition 2** The solution to the (2) satisfy the following estimates: (i) Let $f = 0$. Then the Dalambert solution gives
\[ \| \theta \|_{H_{s,\varepsilon}} \prec \| \xi \|_{H_s}, \quad \| \partial_t \theta \|_{H_{s,\varepsilon}} \prec \| \xi \|_{H_{s+1}}, \]  
(ii) let $\xi = 0$. Then, see [10] the (generalized) solutions satisfy
\[ \| \partial_t \theta \|_{L^2(Q_T)} + \| \partial_x \theta \|_{L^2(Q_T)} \prec \| \xi \|_{H_1} + \| f_t \|_{L^2(Q_T)}. \]
If \( k(t) = \alpha^2 e^{-bt} \) (and \( K(z) = \alpha^2/(z + b) \)), then differentiation gives a damped wave equation
\[
\theta_{tt} = \alpha^2 \theta_{xx} - b\theta_t.
\] (8)

By \( \theta^0 \) we denote the solution to this equation with the initial data
\[
\theta(\cdot, 0) = \xi, \quad \theta_t(\cdot, 0) = 0.
\] (9)

This will be an unperturbed equation, see the Sect. 4.

In application, see, e.g.,[1], the kernels \( k(t) \) is a series of exponentials
\[
k(t) = \sum_{k=1}^{\infty} a_k e^{-b_k t}, \quad a_k \geq 0, \quad 0 \leq b_1 < b_2 < \cdots < b_k < \cdots.
\]

We can consider the following smoothness conditions
\[
\sum_{k=1}^{\infty} \frac{a_k}{b_k} < \infty, \tag{10}
\]
\[
\alpha^2 = \sum_{k=1}^{\infty} a_k < \infty, \tag{11}
\]
or
\[
\beta = \sum_{k=1}^{\infty} a_k b_k < \infty, \tag{12}
\]
or
\[
\gamma = \sum_{k=1}^{\infty} a_k b_k^2 < \infty. \tag{13}
\]

**Remark 3** These conditions maybe written as
\[
k \in L^1(0, \infty), \quad k \in C[0, \infty), \quad k, k' \in L^1(0, \infty),
k' \in C[0, \infty), \quad k, k', k'' \in L^1(0, \infty),
k'' \in C[0, \infty), \quad k, k', k'', k''' \in L^1(0, \infty).
\]

Write the asymptotic of \( K(z) \). The Laplace image of \( k(t) \) is
\[
K(z) = \sum_{k=1}^{\infty} \frac{a_k}{z + b_k}, \quad k(0) = \alpha = \sum_{k=1}^{\infty} a_k.
\]

Without loss of generality we can set \( \alpha = 1 \) if \( \alpha \) is finite.
Proposition 4 Let for a $\delta > 0$

$$|\arg z| < \pi - \delta,$$

Then for large $z$

(i) under (10)

$$K(z) = o(1),$$

(ii) under (11)

$$K(z) = \frac{1}{z} + o\left(\frac{1}{z}\right),$$

(iii) under (12)

$$K(z) = \frac{1}{z} - \frac{\beta}{z^2} + o\left(\frac{1}{z^2}\right),$$

(iv) under (13)

$$K(z) = \frac{1}{z} - \frac{\beta}{z^2} + \frac{\gamma}{z^3} + o\left(\frac{1}{z^3}\right).$$

The statement of this proposition follows from known results about Cauchy transform of a measure.

3 Regularity of the solution in the spatial variable

Here we prove the results about the regularity with respect to the $x$-variable, i.e., in terms of $H_s$ spaces.

**Theorem 5** Let (11) be true, $\{\xi_n\} \in \ell_s$ and $f \in L^2(0, \infty; H_s)$. Then for any $\varepsilon > 0$ the solution $\theta$ to (1) satisfy

$$\|\theta\|^2_{H_{s,\varepsilon}} \prec \|\xi\|^2_s + \|f\|^2_{H_{s,\varepsilon}}.$$  (15)

**Proof:**

**Lemma 6** The following estimates are fulfilled

$$|z/G_n(z)| \prec 1, \quad \Re z > \varepsilon$$

(16)

$$\|1/G_n\|_{L^2(\varepsilon-i\infty, \varepsilon+i\infty)} \prec 1.$$  (17)

The lemma implies by (5)

$$\int_0^\infty |e^{-et}\theta_n(t)|^2 \prec |\xi_n|^2 + \|e^{-et}f_n\|^2_{L^2(0,\infty)}$$

and then (15).
Proof of the lemma. Set for the simplicity $\varepsilon = 1$ Then for $z = 1 + iy$ and $\gamma_k = 1 + b_k$ we obtain

$$G_n(z) = (1 + iy) + n^2 \sum_{k=1}^{\infty} \frac{a_k \gamma_k}{\gamma_k^2 + y^2} - iyn^2 \sum_{k=1}^{\infty} \frac{a_k}{\gamma_k^2 + y^2}$$

Therefore

$$|G_n(z)|^2 \geq \left(1 + n^2 \frac{a_1 \gamma_1}{\gamma_1^2 + y^2}\right)^2 + y^2 \left(1 - n^2 \sum_{k=1}^{\infty} \frac{a_k}{\gamma_k^2 + y^2}\right)^2$$

Setting

$$s(y) = \sum_{k=1}^{\infty} \frac{a_k}{\gamma_k^2 + y^2},$$

we have

$$|G_n(z)|^2 \geq \left(1 + n^2 \frac{1}{1 + y^2}\right)^2 + y^2 \left(1 - n^2 s(y)\right)^2.$$ 

This gives (16).

Divide $[0, \infty)$ into three intervals

$$I_1 = [0, n/2], I_2 = [n/2, 3n/2], I_3 = [3n/2, /iy].$$

Write

$$\|1/G_n\|_{L^2([\varepsilon - i\infty, \varepsilon + i\infty])}^2 = \int_{-\infty}^{\infty} \frac{dy}{|G_n(y)|^2} = 2 \left[\int_{0}^{n/2} + \int_{n/2}^{2n} + \int_{2n}^{\infty} \frac{dy}{|G_n(y)|^2}\right] = 2(J_1 + J_2 + J_3).$$

1. Estimates on $I_1 = [0, n/2]$.

Evidently, $s(y)$ decreases and then on $[0, n/2]$ we have $s(n/2) < s(y) < s(0)$. Further, the series

$$n^2 s(y) = \sum_{k=1}^{\infty} \frac{a_k n^2}{\gamma_k^2 + n^2/4}$$

has the majorant $4 \sum_{k=1}^{\infty} a_k = 4 \alpha = 4$ and the terms of this series approaches to $4a_k$. Then

$$n^2 s(n/2) \to 4.$$

Take $n > n_0$ such that $n^2 s(n/2) \geq 2$ for $n/2 \geq n_0$. We obtain $n^2 s(y) > n^2 s(n/2) \geq 2$ and

$$\left(n^2 s(n/2) - 1\right)^2 \geq 1.$$

This gives

$$|G_n(z)|^2 \geq 1 + y^2 \left(n^2 s(n/2) - 1\right)^2 \geq 1 + y^2.$$
Estimate $J_1$.  

$$J_1 \leq \frac{1}{\int_{0}^{\infty} \frac{1}{1+y^2}} < 1. \quad (18)$$

2. Estimate $J_2$.

For $n/2 \leq y \leq 2n$ we have

$$|G_n(z)| \geq 1 + n^2(1 - n^2s(y)).$$

Consider the increasing variable $\xi = 1 - n^2s(y)$. Then

$$\xi' = -n^2s' = n^2 \sum_{1}^{\infty} \frac{2aky}{(\gamma_k^2 + y^2)^2} \geq n^2 \sum_{1}^{\infty} \frac{akn}{(\gamma_k^2 + n^2)^2}$$

$$\geq \frac{1}{n} \sum_{1}^{\infty} \frac{ak}{(\gamma_k^2/n^2 + 1)^2} \geq \frac{1}{n}.$$  

Indeed,

$$\frac{a_1}{(\gamma_1^2/n^2 + 1)^2} \leq \sum_{1}^{\infty} \frac{ak}{(\gamma_k^2/n^2 + 1)^2} \leq \sum_{1}^{\infty} ak.$$  

Now for $J_2$ we have "$nd\xi \approx dy$" and

$$J_2 \geq \int_{n/2}^{2n} \frac{dy}{1 + n^2(1 - n^2s(y))^2} = \int_{\xi(n/2)}^{\xi(2n)} \frac{nd\xi}{1 + n^2\xi^2} < \infty.$$  

3. Estimate $J_3$.

For $y \geq 2n$ we have

$$n^2s(y) \leq n^2s(2n) = \sum_{1}^{\infty} \frac{a_kn^2}{\gamma_k^2 + 4n^2} \leq \frac{1}{4} \sum_{1}^{\infty} a_k = \frac{1}{4}.$$  

Now

$$|G_m(z)| \geq 1 + y^2(1 - n^2s(y))^2 \geq 1 + y^2 \frac{9}{16},$$

and

$$J_3 \geq \int_{2n}^{\infty} \frac{dy}{1 + y^2} < 1.$$  

The theorem is proved.

**Remark 7** For the case $k(t) = 1$, and $f = 0$, i.e., for the wave equation we have

$$\Theta_n(z) = \frac{\xi_n}{z + n^2/z}, \quad \theta_n(t) = \xi_n \cos nt.$$  

We see that $\theta_n \notin L^2(0, \infty)$ and $e^{-\epsilon t}\theta_n \in L^2(0, \infty)$. In this sense Theorem 7 is sharp.
Theorem 8 Let (11) is true, \( \{ \xi_n \} \in \ell_s \) and \( f \in L^2(0, \infty; H_s) \). Then \( \theta(x, t) \) is an \( H_s \) valued continuous function:
\[
\| \theta(t) - \theta(t + t_0) \|_{H_s, \varepsilon} \to 0,
\]
as \( t_0 \to t_0 \).

Proof. The solutions \( \theta_n \) are continuous and the series in \( \theta_n \) has a majorant.

4 GP as a perturbation to the wave equation

Let us find regularity of the ‘perturbation’ \( \theta - \theta^0 \) of the solution to the wave equation. Recall that \( \theta^0 \) is the solution to the problem (8), (9). Let \( f(x, t) = 0 \) for simplicity and set
\[
K_0(z) = \frac{1}{z + \beta}, \quad G_0^0(z) = z + n^2 K_0(z), \quad D_n(z) = \frac{1}{G_n(z)} - \frac{1}{G_n^0(z)}.
\]
The solution to (11) has the form
\[
\Theta_n(z) = \frac{1}{G_n^0(z)} \xi_n + D_n(z)\xi_n = \Theta_n^0(z) + D_n(z)\xi_n.
\]

Theorem 9 Let (13) is true and \( f = 0 \). Then for \( s < 9/2 \)
\[
\| z^s D_n \|_{L^2(i\mathbb{R})} \prec n^{s-1} |\xi_n|.
\]

Proof:
If \( \beta = 0 \) the theorem is trivial: \( D_n = 0 \). Thus, we can assume \( \beta \neq 0 \) and then integrate the functions on the imaginary axis.

Lemma 10 For \( z = iy, y \to \infty \) and \( \beta \neq 0 \) we have
\[
|G_n(z)|^2 \asymp |G_n^0(z)|^2 \asymp \frac{1}{y^4} \left( y^2 (y^2 - n^2) \right)^2 + n^4 =: Q(y).
\]

Proof of the lemma. (14) implies
\[
G_n(iy) = iy + n^2 \left( \frac{1}{iy} - \frac{\beta}{y^2} + o \left( \frac{1}{y^2} \right) \right)
= \frac{i}{y^2} [y^3 - n^2 y + o(1)n^2] - \frac{n^2}{y^2} [\beta + o(1)].
\]
And the same is true for \( G_n^0 \). From here
\[
|G_n(y)|^2 = \frac{1}{y^4} \left[ y^3 - n^2 y + o(1)n^2 \right]^2 + \frac{n^4}{y^2} [\beta + o(1)]^2 \asymp \frac{1}{y^4} \left( y(y^2 - n^2) + o(1)n^2 \right)^2 + \frac{n^4}{y^2}.
\]
Now 
\[
\frac{|G_n(y)|^2}{Q(y)} = \frac{y(y^2 - n^2) + o(1)n^2}{(y^2(y^2 - n^2)) + n^4}
\]

Use the elementary inequalities 
\[(a + qb)^2 + b^2 \simeq a^2 + b^2\]
with, say, \(q < 1/2\). Setting 
\[a = y(y^2 - n^2), \ b = n^2\]
we complete the proof of the lemma.

Return to the proof of the theorem.

\[\|z^s D_n\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} n^4 \frac{y^{2s}|K(iy) - K_0(iy)|^2}{|iy + n^2 K(iy)|^2 |iy + n^2 K_0(iy)|^2} dy.\]

Use Lemma 10 to estimate the denominator and the condition (13) to estimate the numerator. By (14) we have 
\[K(iy) - K_0(iy) = O(1/n^3)\] and obtain
\[\|z^s D_n\|_{L^2(\mathbb{R})}^2 < n^4 \int_0^{\infty} \frac{y^{2s+2}}{[y^2(y^2 - n^2)^2 + n^4]^2} dy.\] (19)

For \(s < 9/2\) this integral converges. The main contribution gives the interval of the length \(O(n)\) centered at \(y = n\). Estimate it. Set
\[J = \int_{n/2}^{3n/2} \frac{y^{2s+2}}{[y^2(y^2 - n^2)^2 + n^4]^2} dy.\]

(this integral enters (19) with the factor \(n^4\)).

\[J \asymp n^{2s-2} \int_{n/2}^{3n/2} \frac{dy}{[(y^2 - n^2)^2 + n^2]^2} \asymp n^{2s-2} \int_{n/2}^{3n/2} \frac{dy}{[n^2(y - n)^2 + n^2]^2} \]
\[\asymp n^{2s-6} \int_{n/2}^{3n/2} \frac{dy}{[(y-n)^2 + 1]^2} \leq n^{2s-6} \int_{n/2}^{3n/2} \frac{dy}{[t^2 + 1]^2} = Cn^{2s-6}.\]

Thus, the interval of \((n/2, 3n/2)\) gives the contribution \(n^{2s-2}\). The theorem is proved.

This theorem give, of course, information about regularity of \(\theta - \theta^0\), for example

**Corollary 11**

\[e^{-\varepsilon t} [\theta(x, t) - \theta_0(x, t)] \in L^2(0, \infty; \mathcal{H}_{s+1}) \cap W^1_2(0, \infty; \mathcal{H}_s)\]
5 Estimate of the derivative with respect to time

Before now we have considered a weak solution. Here we present results about a strong solutions, as, e.g., in [8]. By the definition of the strong solution the equation (11) can be considered as an equality of elements of $L^2$ spaces.

Proposition 12

Let (12) is fulfilled, $f_t \in \mathcal{H}_{1,\gamma}$, $f(x,0) = 0$, and $\xi \in \mathcal{H}_2$. Then for any $\varepsilon > \gamma$

$$
\|e^{-\varepsilon t} \partial_t \theta\|^2_{L^2(0,\infty;L^2(0,\pi))} + \|e^{-\varepsilon t} \theta\|_{L^2((0,\infty);\mathcal{H}_2)} \prec \|e^{-\varepsilon t} \partial_t f\|_{L^2((0,\infty),\mathcal{H}_1)} + \|\xi\|_{\mathcal{H}_2}.
$$

In the right hand side we see, roughly speaking, the $L^2$ norm of $\partial_t \partial_x f$.

We can slightly strengthen this estimate with the $L^2$ norm of $\partial_t f$ in the right hand side. It is a sharp result in the sense that it close to the estimate (7).

Theorem 13

Let (11) is fulfilled, $f_t \in \mathcal{H}_{0,\gamma}$, $f(x,0) = 0$, and $\xi \in \mathcal{H}_s$. Then

$$
\|e^{-\varepsilon t} \partial_t \theta\|^2_{L^2(0,\infty;L^2(0,\pi))} + \|e^{-\varepsilon t} \theta\|_{L^2((0,\infty);\mathcal{H}_2)} \prec \|e^{-\varepsilon t} \partial_t f\|_{L^2((0,\infty),\mathcal{H}_0)} + \|\xi\|_{\mathcal{H}_2}.
$$

Proof:
(i) Let $F = 0$. Then

$$
\mathcal{L}[\theta'_n] = z\mathcal{L}[\theta_n] - \theta_n(0) = \left(\frac{z}{z + n^2 K(z)} - 1\right) \xi_n = -\frac{n^2 K(z)}{z + n^2 K(z)} \xi_n.
$$

$K(z)$ decreases in infinity,

$$
|K(1 + iy)| \prec \frac{1}{1 + |y|},
$$

Therefore we can use

$$
|\mathcal{L}[\theta'_n]| \prec n \left|\frac{K}{G_n}\right| |\xi_n| \prec n |\xi_n|,
$$

what gives by (17) the estimate

$$
\|e^{-\varepsilon t} \theta_t\| \prec \|\xi\|_{\mathcal{H}_{s+2}}. \tag{20}
$$

(ii). Let $\xi = 0$. Then, taking into account $f(x,0) = 0$, we obtain

$$
\mathcal{L}(\theta'_n) = \frac{z F_n(z)}{z + n^2 K(z)}.
$$
By (10) we have
\[ \| L(\theta'_{n}) \| \prec \| zF_{n}(z) \|. \]
Then
\[ \| e^{-\gamma t} \theta'_{n} \| \prec \| e^{-\gamma t} f'_{n} \|_{L^{2}(0,\infty)}. \]
and
\[ \| e^{-\gamma t} \theta_{t} \| \prec \| e^{-\gamma t} f' \|_{L^{2}(0,\infty;\mathcal{H})}. \] (21)
The theorem is proved.

Acknowledgements
The author grateful to Prof. V. V. Vlasov for the fruitful discussions.

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