Turbulence modulation in buoyancy-driven bubbly flows

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We present a Direct Numerical Simulation (DNS) study of buoyancy-driven bubbly flows in the presence of large scale driving that generates turbulence. On increasing the turbulence intensity: (a) the bubble trajectories become more curved, and (b) the average rise velocity of the bubbles decreases. We find that the energy spectrum of the flow shows a pseudo-turbulence scaling for length scales smaller than the bubble diameter and a Kolmogorov scaling for scales larger than the bubble diameter. We conduct a scale-by-scale energy budget analysis to understand the scaling behaviour observed in the spectrum. Although our bubbles are weakly buoyant, the statistical properties of our DNS are consistent with the experiments that investigate turbulence modulation by air bubbles in water.

Key words:

1. Introduction

The flow of suspension of deformable objects (bubbles or droplets) is omnipresent in a variety of natural and industrial processes (Mudde 2005; Balachandar & Eaton 2010; Risso 2018; Said 2019; Mathai et al. 2020). The presence of particles dramatically alters the rheological and thereby mixing properties of flows (Almeras et al. 2019; Alméras et al. 2015; Rosti et al. 2018; Rosti & Brandt 2018). A swarm of rising bubbles in an otherwise quiescent fluid, at moderate volume-fraction, generates pseudo-turbulence studied by several experiments and numerical simulations over the last three decades (Lance & Bataille 1991; Mudde 2005; Risso 2018; Mathai et al. 2020; Pandey et al. 2020).

A more complex but ubiquitous scenario is where large scale external stirring that generates turbulence is also present along with the bubbles (Deckwer 1992; Tabib et al. 2008; Mathai et al. 2020). In the absence of bubbles, a nonlinear
transfer of energy (maintaining constant energy flux) from forcing to dissipation range characterizes turbulence (Kolmogorov 1941; Frisch 1997; Pope 2012). How does the presence of bubbles modify this flow? The answer, in principle, depends on the ratio of the bubble diameter to the dissipation scale, the bubble volume fraction, and its density and viscosity contrast with the ambient fluid.

Experiments with large scale forcing that generates nearly homogeneous and isotropic flows, at large Reynolds number, show that the presence of bubbles dramatically alters the energy spectrum for scales smaller than the bubble diameter (Prakash et al. 2016; Almeras et al. 2017). Although the liquid velocity fluctuations have been well-characterized, an understanding of the energy transfer mechanisms remain mostly unexplored.

Direct Numerical Simulation (DNS) studies of bubbly flows have explored: a) buoyancy-driven flows that generate pseudo-turbulence or bubble induced agitation in the absence of external stirring (Bunner & Tryggvason 2002b,a; Roghair et al. 2011; Pandey et al. 2020; Ramadugu et al. 2020; Innocenti et al. 2021), b) modulation of turbulence by suspension of neutrally buoyant particles (Rosti et al. 2019; Yousefi et al. 2020), and c) Lagrangian investigations of an isolated bubble in the presence of external stirring (Loisy & Naso 2017). However, to the best of our knowledge, a numerical study designed to unravel the statistical properties of buoyancy-driven bubbly flows in presence of external stirring is still missing.

Most numerical studies are restricted to low or moderate Galilei numbers because extremely fine grids are required to fully resolve bubbles with high-density and viscosity contrasts (for e.g., air bubbles in water) (Cano-Lozano et al. 2016; Innocenti et al. 2021). Furthermore, the use of second-order finite-difference methods limits the range of Reynolds numbers accessible to these simulations (Canuto et al. 2012).

Fortunately, the DNS studies of buoyancy-driven bubbly flow have shown that the statistical properties of pseudo-turbulence such as the PDF of velocity fluctuations, the scaling of the energy spectrum, and the energy transfer mechanisms are universal and do not depend upon density and viscosity ratios (Pandey et al. 2020; Ramadugu et al. 2020; Innocenti et al. 2021). A key finding of these studies is the presence of energy flux from length scales corresponding to the bubble diameter to small scales. This has also been confirmed in a recent study on bubble-laden turbulent channel flow (Ma et al. 2021). Motivated by these findings, in this article, we investigate turbulence modulation in suspensions of weakly buoyant bubbles. Similar to the experiments, we characterize the flow in terms of the ‘bubblance’ parameter \( b = \Phi (V_0/u_0)^2 \), where \( \Phi \) is the bubble volume fraction, \( V_0 \) is the rise velocity of an isolated bubble in a quiescent fluid, and \( u_0 \) is the r.m.s. velocity of the turbulent flow in the absence of bubbles. The two extreme limits \( b = 0 \) and \( b = \infty \) correspond to pure fluid turbulence and buoyancy-driven bubbly flow, respectively.

2. Model

We simulate the Navier-Stokes (NS) equations with a surface tension force to investigate the suspension of bubbles. Since we are interested in studying the weakly buoyant regime, we invoke the Boussinesq approximation (Chandrasekhar...
Here \( \mathbf{u} \) is the velocity field, \( D_t \equiv \partial_t + \mathbf{u} \cdot \nabla \) is the material derivative, \( P \) is the pressure field, and \( \nu \) is the viscosity (assumed to be identical in the two phases). The two phases are distinguished using an indicator function \( c \) which is equal to 1 in the liquid and 0 inside bubble (Popinet 2018; Tryggvason et al. 2001). The buoyancy force \( \mathbf{F}^b \equiv 2A_B[c - c_a]g \), where \( c_a \) is the mean value of the indicator function, \( A_B \equiv (\rho_L - \rho_b)/(\rho_L + \rho_b) \) is the Atwood number, \( g \equiv -g\hat{z} \) is the acceleration due to gravity, \( \hat{z} \) is a unit vector along the vertical (positive \( z \)) direction, and \( \rho_L \) (\( \rho_b \)) is the fluid (bubble) density. The surface tension force is \( \mathbf{F}^\sigma = \sigma \kappa \hat{n} \), where \( \kappa \) is the local curvature of the bubble-front whose unit normal is \( \hat{n} \), and \( \sigma \) is the coefficient of the surface tension. Turbulence is generated using a large scale stirring force \( \mathbf{F}^s \).

For a detailed discussion on the Boussinesq approximation, we refer the reader to Appendix A. Experimentally small Atwood \( A_t \) (weakly buoyant regime) number flows can be realized in a mixture of oils (Shukla et al. 2019; Yi et al. 2021).

We use a pseudo-spectral method (Canuto et al. 2012) for the DNS of (2.1) in a periodic cube with each side of length \( L = 2\pi \). The bubbles are resolved using a front-tracking method. The same method had been earlier employed by us to investigate buoyancy-driven bubbly flows in absence of turbulent stirring (Pandey et al. 2020; Ramadugu et al. 2020). For a detailed discussion on the numerical implementation of front-tracking method to study a variety of multiphase flows, we refer the reader to Tryggvason et al. (2001); Popinet (2018).

For time-evolution, we use a second-order exponential time differencing scheme (Cox & Matthews 2002) for (2.1) and a second-order Runge-Kutta scheme to update the front. A substantial part of the computational effort is spent in resolving the front; DNS with the bubbles is four times slower than the one without them. The large-scale stirring force is implemented in Fourier space, i.e., \( \mathbf{F}^S = \varepsilon^s\hat{u} \sum_k |\hat{u}|^2 \) with \( |k| \leq k_{inj} \) (Machiels 1997; Petersen & Livescu 2010; Perlekar 2019), where \( \hat{u} \) is the Fourier transform of \( \mathbf{u} \) and \( k_{inj} = 2 \). This implementation ensures a constant rate of energy injection, \( \varepsilon^s \).

We discretize the simulation domain with \( N^3 \) collocation points, set the initial velocity field such that the corresponding energy spectrum \( E(k, t = 0) = \varepsilon^s k^4 \exp(-4k^2) \), and place \( N_b = 80 \) non-overlapping spherical bubbles of diameter \( d = 0.46 \) at random locations such that no two bubbles overlap.

The dimensionless numbers that characterize the flow are the Taylor-scale Reynolds number \( \text{Re}_t \equiv u_0 \lambda / \nu \), the Galilei number \( \text{Ga} \equiv \sqrt{2A_B gd^3} / \nu^2 \), the Bond number \( \text{Bo} \equiv 2A_B \rho_b gd^2 / \sigma \), and the bubbblance parameter \( b \equiv \Phi (V_0 / u_0)^2 \), where \( \Phi \equiv N_b (\pi / 6)(d/L)^3 \) is the volume fraction occupied by the bubbles, \( V_0 \approx 0.8 \) is the rise speed of a single bubble of diameter \( d \) in quiescent fluid, \( \lambda \equiv \sqrt{15\nu u_0^2 / \varepsilon^s} \) is the Taylor-microscale, \( u_0 \equiv \sqrt{2E/3} \) is the r.m.s. velocity in absence of bubbles, \( E \equiv \langle |\mathbf{u}|^2 \rangle / 2 \) is the average kinetic energy, we set the average density \( \rho_a = 1 \). The parameters used in our DNS are summarized in table 1. We conduct a grid-resolution study in Appendix B to show that our simulations are well resolved.
3. Results

In what follows, we first investigate the statistical properties of bubbles rising in the turbulent flow, we then investigate the statistical properties of the fluid velocity fluctuations. Although we study turbulence modulation in the presence of weakly buoyant bubbles, we show in the subsequent sections that the statistical properties of the flow are in qualitative agreement with experiments that typically have large density and viscosity contrast. Finally, we present the results for the spectral properties of the flow by using a scale-by-scale energy budget analysis.

3.1. Bubble trajectories and rise velocity

For every bubble, we monitor the time evolution of its center-of-mass $X_i(t)$ after every $\delta t = 0.08 \tau_\eta$ time interval, where $i$ denotes the bubble index, and $\tau_\eta = \sqrt{\nu/\epsilon^s}$ is the Kolmogorov dissipation time scale. From the bubble tracks, we obtain the center-of-mass velocity $V_i(t)$ and the acceleration $A_i(t)$ using centered, second-order, finite-differences.

The plots in figure (1a-b) show a representative snapshot of bubbles and isovorticity surfaces for Re$_d = 79$, $b = 0.35$ and Re$_d = 110$, $b = 0.13$, respectively. In figure (1c-d) we show a few typical trajectories for the same parameters. It is clear that higher Reynolds number and small ‘bubblance’ parameter corresponds to more complex trajectories. To quantify this behaviour we plot the probability distribution function (PDF) of the curvature $K \equiv |A \times V|/|V|^2$ in figure (1e). Consistent with the observation that the trajectories are more curved for larger Re$_d$, we find that the probability distribution function $P(K)$ is broader—has an exponential tail.

Note that, Bhatnagar et al. (2016) showed that the PDF, of curvature of trajectories of heavy inertial particles in homogeneous and isotropic turbulence, has a power-law tail with an exponent of $-5/2$. To the best of our knowledge no such results exists for bubbles.

Another consequence of large-scale turbulent stirring is that the average bubble rise velocity $U \equiv (1/N_b) \sum_{i=1}^{N_b} V_i(t) \cdot \hat{z}$ (see figure (1f)) increases with increasing $b$ (decreasing Re$_d$), where $\hat{z}$ represents temporal averaging.

In a recent study, Salibindla et al. (2020) show that the rise velocity of the bub-
Figure 1: Top panel: Representative steady-state snapshot of the bubbles and super-imposed iso-surfaces of the \( \omega_z \)-component of the vorticity field \( \omega_z = \hat{z} \cdot \nabla \times \mathbf{u} \) for \( \omega_z = \pm 3 \langle \omega_z^2 \rangle^{1/2} \) for (a) \( b = 0.35 \), and (b) \( b = 0.13 \). Middle panel: Typical trajectories of the center-of-mass of bubbles in a turbulent flow for (c) \( \text{Re}_d = 79, b = 0.35 \) (R1) and (d) \( \text{Re}_d = 110, b = 0.13 \) (R3). Bottom panel: (e) The PDF of the curvature \( K \) for different values of \( b \). (f) Plot showing that the bubble rise velocity increases with increasing \( b \) or decreasing \( \text{Re}_d \). We also show that \( U \) obtained directly from the trajectories and the estimate \( \varepsilon^K/(2Atg\Phi) \) are in excellent agreement.
bles can be enhanced by turbulence provided the velocity ratio \( \gamma \equiv (V_0^2/(\varepsilon^s d)^{2/3}) < 1 \). Our DNS (see Table 2) and the experiments that investigate turbulence modulation by bubbles (Lance & Bataille 1991; Prakash et al. 2016) have \( \gamma \gg 1 \).

Note that even for \( b = \infty \), the rise velocity of a bubble in a swarm is slightly smaller than the rise velocity of an isolated bubble due to bubble-wake interactions (Riboux et al. 2010). Using the definition of \( F_g \) and noting that \( \langle u_z \rangle = 0 \) in the Boussinesq regime, we obtain \( \Omega = 2A\Phi \Phi \) and verify it in figure (1f).

### 3.2. Pair Distribution Function

To understand the distribution of bubbles in the domain, following Bunner & Tryggvason (2002a), we define the pair distribution function,

\[
G[r, \cos(\theta)] = \frac{L^3}{N_b(N_b - 1)} \sum_{i=1}^{N_b} \sum_{j=1, j \neq i}^{N_b} \delta(r - X_{ij}, t),
\]

where \( \delta(\cdot) \) is the Dirac delta function, and \( X_{ij} = X_i - X_j \). In figure (2a), we sketch a bubble pair configuration to show the co-ordinate system used for evaluating (3.1). The plot of \( G[r, \cos(\theta)] \) for \( r = 2d \) and \( 4d \) is shown in figure (2b). At \( b = \infty \), we observe a peak in \( G[r, \cos(\theta)] \) for \( r \approx 2d \) and \( \cos(\theta) \approx 0 \) indicating a horizontal alignment of bubbles that are separated by a distance \( 2d \). Bubbles separated by distances, \( r \geq 4d \) are uniformly distributed. Our results are consistent with earlier numerical studies of pseudo-turbulence (Bunner & Tryggvason 2002a; Roghair et al. 2013). In contrast, as turbulence makes flow more isotropic, for \( b = 0.13 \) we find that \( G[r, \cos(\theta)] \) is uniform which indicates that the bubbles are uniformly distributed for all separations \( r \).

### 3.3. Average flow around a bubble

In this section, we study the average wake structure of the bubbles for different values of bubblance \( b \). At a given time \( t \), the velocity field in the center-of-mass frame of the bubble \( i \) is given by

\[
u_i^{CM}(\xi, t) = u(\xi, t) - V_i,
\]

where \( \xi \equiv x - X_i \) and \(-L/2 < (\xi_x, \xi_y, \xi_z) \leq L/2 \). The average flow around a bubble is then obtained by performing temporal averaging over every bubble as follows

\[
u^{CM}(\xi) = \frac{1}{N_b} \sum_{i=1}^{N_b} u_i^{CM}(\xi, t).
\]

In figure (3a,b) we plot the velocity streamlines of the average velocity field \( u^{CM}(\xi) \) for \( b = \infty \) (R0) and \( b = 0.13 \) (R3). Although the flow structure look qualitatively similar, we find that the bubble in the absence of large scale stirring

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**Table 2: Velocity ratio \( \gamma \) for our DNS runs R1 – 3**

| Runs | R1 | R2 | R3 |
|------|----|----|----|
| \( \gamma \) | 58.3 | 36.7 | 23.1 |

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Figure 2: (a) The separation vector $\mathbf{r} = X_i - X_j$, and the angle $\theta$ between the $X_{ij}$ and $\hat{z}$. The bubbles are represented as shaded ellipse. (b) The angular distribution function $G[r, \cos(\theta)]$ versus $\cos(\theta)$ for $r = 2d$, and $r = 4d$ in absence (presence) $b = \infty$ ($b = 0.13$) of turbulence. The area under the curve is normalized to unity for each $G[r, \cos(\theta)]$ curve.

Figure 3: The streamline plot of the average velocity field in the frame of bubble for (a) $b = \infty$ run R0 and (b) $b = 0.13$ run R3. The streamlines are colored according to $\mathbf{u}^\text{CM} \cdot \hat{z}$ is more ellipsoidal. This can be understood by noting that presence of stirring imposes stronger isotropy on the flow.

To quantify, the behavior of the average bubble wake, similar to the experiments (Risso et al. 2008; Almeras et al. 2017) we plot $v(\xi_z) \equiv \mathbf{u}^\text{CM}(0, 0, \xi_z) \cdot \hat{z}$ in figure (4) and find that it decays exponentially $v(\xi_z) \sim C \exp(-A \xi_z/d)$ in the wake region for all values of $b$. However, consistent with earlier observations, presence of stirring leads to a faster decay of the wake. Therefore, for small $b$ (or large $\text{Re}_d$) we
Figure 4: (a) The average bubble wake velocity $v(\xi_z)$ for run R0 ($b = \infty$), R1 ($b = 0.35$), and R3 ($b = 0.13$). (b) Same as (a), but in semi-log scale to highlight the exponential decay of the velocity field in the wake region. The dashed-dot line show the exponential fits $\sim \exp(-Az/d)$ to the data. We find $A = 0.67, 1.15$ and $1.6$ for R0, R1, and R3 respectively.

Figure 5: The PDF of the horizontal (a) and the vertical (b) component of the liquid velocity fluctuations for different values of $b$ [ $b = \infty$ (R0), $b = 0.35$ (R1), $b = 0.21$ (R2), $b = 0.13$ (R3), $b = 0$ (Re$_{c1} = 110$)]. The black dashed line indicates a Gaussian distribution, and the brown dash-dot line in panel (a) shows the exponential distribution. (Inset) Variance of the horizontal and vertical velocity fluctuations increases with an increase in the stirring intensity $1/b$.

expect (see next section) the velocity fluctuations to be similar to homogeneous, isotropic turbulence.

3.4. Liquid velocity fluctuations

The PDFs of the normalized horizontal and vertical liquid velocity fluctuation with varying $b$ are shown in figure (5). For $b = \infty$, our results agree with the earlier studies on pseudo-turbulence (Riboux et al. 2010; Risso 2016; Pandey et al. 2020): the PDF of the horizontal component shows exponential behaviour and the PDF of the vertical component has a Gaussian core and is positively skewed. The presence of external stirring dramatically alters the PDFs as they tend to a Gaussian distribution with decreasing $b$ (increasing Re$_{c1}$). Indeed, in figure (5a,inset) we verify that $\langle u_h^2 \rangle \sim \langle u_z^2 \rangle$ on decreasing $b$ confirming that the stirring makes the flow isotropic. This is consistent with earlier experimental observations on turbulent bubbly flows (Prakash et al. 2016; Almeras et al. 2017).
3.5. Energy spectrum

Earlier DNS studies (Roghair et al. 2011; Pandey et al. 2020; Innocenti et al. 2021) have only investigated the nature of the energy spectrum in the absence of large scale turbulent forcing. These studies, consistent with experiments, confirm the presence of a $k^{-3}$ scaling in the spectrum that appears because of the balance of net energy production in the wakes with viscous dissipation.

Experiments have investigated temporal spectrum of the Eulerian liquid velocity fluctuations in presence of a large scale stirring. They observe a Kolmogorov spectrum for frequencies smaller than the bubble frequency and a pseudo-turbulence scaling for higher frequencies (Lance & Bataille 1991; Prakash et al. 2016; Almeras et al. 2017).

Hence we expect that in our simulations we would find a Kolmogorov scaling, for wavenumbers $k < k_d$, with a crossover to pseudo-turbulence scaling for $k > k_d$, where $k_d = 2\pi/d$ is the wavenumber corresponding to the bubble diameter.

In figure (6), we plot the scaled energy spectrum for different values of $b$ ($Re_d$). As expected, we observe Kolmogorov scaling $E(k) \sim k^{-5/3}$ for $k < k_d$ and a pseudo-turbulence scaling $E(k) \sim k^{-3}$ for $k > k_d$. In figure (7a,b) we plot the compensated spectrum to highlight the region showing $-5/3$ and $-3$ scaling. Note that none of the scaling ranges are large enough to make an accurate determination of the scaling exponent possible.
3.6. Scale-by-scale energy budget and flux

To lay bare the mechanism by which bubbly turbulence emerge we study the scale-by-scale energy budget. Following Pope (2012) we define a low-pass filtered velocity field coarse-grained at scale $\ell = 2\pi/K$ as

$$u_K^<(x) \equiv \int \exp(iq \cdot x)G_K(q)\hat{u}(q)\,dq,$$

with $G_K(q) \equiv \exp\left(-\frac{\pi^2q^2}{24K^2}\right)$. (3.4)

Note that Frisch (1997); Pandey et al. (2020) use a sharp stepdown function as a filter: $G_K(q) = 1$ for $|q| \leq K$ and zero otherwise, whereas we use a smooth Gaussian filter (Pope 2012). In what follows, we use the symbol $(\cdot)_K$ to denote the filtering operation (Frisch 1997). In real space, this corresponds to

$$u_K^<(x) = \int G_\ell(r)u(x-r)\,dr,$$

with $G_\ell(r) = \left(\frac{6}{\pi\ell^2}\right)^{\frac{1}{2}} \exp\left(-\frac{6r^2}{\ell^2}\right)$ and $\ell \equiv 2\pi/K$.

Using the filtered velocity field, we obtain the following scale-by-scale energy budget equation from (2.1)

$$\Pi_K + \mathcal{F}_K^\sigma = -\mathcal{D}_K + \mathcal{F}_K^g + \mathcal{F}_K^s.$$

Here $\mathcal{F}_K^\sigma \equiv \langle u_K^< \cdot (F^\sigma)_K^< \rangle$ is the contribution from surface-tension forces, $\mathcal{F}_K^g \equiv \langle u_K^< \cdot (F^g)_K^< \rangle$ is the contribution from buoyancy, and $\mathcal{F}_K^s \equiv \langle u_K^< \cdot (F^s)_K^< \rangle$ is the contribution due to large-scale forcing. To obtain the contribution from the nonlinear term and viscous dissipation, following Eyink (1995); Borue & Orszag (1998); Pope (2012), we define a filtered version of the Reynolds stress tensor,

$$T_K^{\alpha\beta}(x) \equiv \langle u^\alpha u^\beta \rangle_K^< - \langle u^\alpha \rangle_K^< \langle u^\beta \rangle_K^<,$$

the rate-of-strain tensor

$$S_K^{\alpha\beta}(x) \equiv \frac{1}{2} \left[ \left( \partial_\alpha u^\beta \right)_K^< + \left( \partial_\beta u^\alpha \right)_K^< \right],$$

and the local nonlinear energy flux

$$\pi_K(x) \equiv -T_K^{\alpha\beta}S_K^{\alpha\beta}.$$ (3.8)

Using (3.4),(3.6),(3.7), and (3.8), we get the net nonlinear flux $\Pi_K \equiv \langle \pi_K \rangle$, and the viscous contribution to the budget $\mathcal{D}_K \equiv 2\nu \left( S_K^{\alpha\beta} S_K^{\alpha\beta} \right)$ which is always positive.

3.6.1. Scale-by-scale energy budget in the absence of bubbles: $b = 0$

In this case buoyancy makes no contribution to the fluxes and (3.5) simplifies to

$$\Pi_K = -\mathcal{D}_K + \mathcal{F}_K^s.$$ (3.9)

The plot in figure (8a) shows the energy budget for $b = 0$ (Re$_d = 110$). Since the stirring force is limited to small Fourier modes $k \leq k_{\text{inj}}$, $\mathcal{F}_K^s = \varepsilon^s$ is a constant for $K > k_{\text{inj}}$. The viscous contribution $\mathcal{D}_K$ is significant only for very large $K \geq k_g$. Hence, for intermediate values of $K$ in the inertial range ($k_{\text{inj}} < K < k_\eta$), the flux $\Pi_K = \mathcal{F}_K^s$ remains a constant. The four-fifth law of Kolmogorov and the Kolmogorov scaling, $E(k) \sim k^{-5/3}$, is a consequence of this constancy of flux (see, e.g., Frisch 1997, section 6.2). Because of the moderate Re$_d = 110$ used by
Figure 8: Scale-by-scale energy budget: plot of the energy flux $\Pi_K$, cumulative viscous dissipation $\mathcal{D}_K$, the surface tension contribution $\mathcal{F}_K$, the cumulative energy injected due to buoyancy $\mathcal{E}_K$, and the energy injected due to turbulent forcing $\mathcal{F}_s$ for $b = 0$ ($\text{Re}_d = 110$) (a), $b = \infty$ (b), $b = 0.35$ (c), and $b = 0.13$ (d). The black dashed line indicates log($K$) scaling. In (a-d) we normalize the ordinate by the viscous dissipation $\nu'$. In panel (a), (c) and (d) we mark the injection wavenumbers by a shaded region.

Figure 9: Log-log plot of $\left(\frac{K}{k_d}\right)|d(\mathcal{F}_K + \Pi_K)/dK|$ versus $K/k_d$ for different values of the bubbleance parameter $b$. Horizontal dashed lines represent $K^{-1}$ scaling.

us, the range of wavenumbers over which the flux is constant is very small. A significant range of constant flux is observed in very high $\text{Re}_d$ and large resolution DNS (Ishihara et al. 2009).
3.6.2. Scale-by-scale budget in the absence of stirring ($b = \infty$):

Next, in figure (8b) we study the other extreme, $b = \infty$. Stirring makes no contribution here. Energy injection by buoyancy forces happens around the scale of the bubble diameters, the flux due to buoyancy $\mathcal{F}^g_K$ becomes almost a constant for $K \gg k_d$. Hence for $K \gg k_d$ we obtain

$$\Pi_K + \mathcal{F}^\sigma_K = - \mathcal{D}_K + \mathcal{F}^g_K,$$

with $\mathcal{F}^g_K$ approximately a constant. By taking a derivative of both sides of (3.10) with respect to $K$ at $K = k$ we obtain

$$\frac{d(\Pi_K + \mathcal{F}^\sigma_K)}{dK} \bigg|_{K=k} = \nu k^2 E(k).$$

(3.11)

Our DNS shows that the net production $\Pi_K + \mathcal{F}^\sigma_K \sim \log(K)$ (Lance & Bataille 1991; Pandey et al. 2020). Although, taking derivative can enhance approximation errors, we directly confirm the scaling relation in figure (9). Generalizing Lance & Bataille (1991) argument if we now assume locality of net transfer then by dimensional analysis $d(\Pi_K + \mathcal{F}^\sigma_K)/dK \big|_{K=k} \sim k^{-1}$ follows. Substituting in (3.11) we obtain $E(k) \sim k^{-3} - \text{the spectrum of pseudo-turbulence}$ (Lance & Bataille 1991; Mercado et al. 2010; Prakash et al. 2016; Almeras et al. 2017; Bunner & Tryggvason 2002b; Roghair et al. 2011; Pandey et al. 2020; Ramadugu et al. 2020). Risso (2011) has shown that the same $k^{-3}$-spectrum can be obtained, under certain conditions, as a sum of localized random, statistically independent, bursts; which comes from localized velocity disturbances caused by the bubbles.

3.6.3. Scale-by-scale budget in the presence of both bubbles and stirring

In figure (8c,d) we plot the energy budget for the two intermediate cases with $b = 0.35$ and $b = 0.13$. For $K \ll k_d$ both the buoyancy force and the surface tension contribute very little to the flux. The viscous contribution is also very small as $k_d < k_y$, the dissipation wavenumber. Let us also assume that there is a scale separation between the stirring scale, $k_{\text{inj}}$ and $k_d$, with $k_{\text{inj}} \ll k_d$. Then for range of scales $k_{\text{inj}} < K < k_d$ the flux balance gives $\Pi_K = \mathcal{F}^s_K$, equal to a constant. Consequently we obtain $E(k) \sim k^{-5/3}$ for $k_{\text{inj}} < k < k_d$. Next we consider $K \gg k_d$; the net contribution from both stirring and buoyancy forces $\mathcal{F}^s_K + \mathcal{F}^g_K$ is almost a constant, hence we again obtain (3.11). Our DNS show that for both the bubble scaling, $b = 0.35$, and $0.13$, $\Pi_K + \mathcal{F}^\sigma_K \sim \log(K)$ (see figure (9)). Although their individual contribution to the energy budget does depend on $b$, in particular: for $b = 0.13$, $\Pi_K$ is larger than $\mathcal{F}^\sigma_K$; but for $b = 0.35$, $\Pi_K$ is smaller than $\mathcal{F}^\sigma_K$. Hence for both of these cases we obtain $E(k) \sim k^{-3}$ for $k > k_d$ and $E(k) \sim k^{-5/3}$ for $k < k_d$.

In Appendix C, we show that qualitatively similar results are obtained even by using a sharp filter instead of a Gaussian filter.

3.6.4. Spatial distribution of the nonlinear energy flux $\pi_K(x)$

For homogeneous and isotropic turbulence, for any $K$ in the inertial range, the net nonlinear flux $\Pi_K$ is positive, i.e., on average energy flows from small to large $K$ or from large to small spatial scales. Kraichnan (Kraichnan 1974; Eyink 1995) argued that the local nonlinear energy flux $\pi_K$ (3.8) satisfies the refined similarity hypothesis. Using DNS, Chen et al. (2003) verified this and showed that
the scaling exponents of the flux show multiscaling. The multiscale analysis of the flux is also crucial to model subgrid scale dissipation in large-eddy simulations (Meneveau & Katz 2000).

To the best of our knowledge, the spatial distribution of local energy flux in bubbly flows remains unexplored. How does the sign of this flux correlate with the bubbles? For example, is the flux pre-dominantly positive in the wake of a bubble? In the following discussion, we address this question by performing a multiscale analysis of the local nonlinear energy flux $\pi_K(x)$ with varying filtering scale $\ell \sim 1/K$.

In figure (10), we show a typical snapshot from the run with no external stirring, $b = \infty$. The position of the bubbles is shown by plotting the indicator function in the top panel. In the middle and bottom panel, we plot the local nonlinear flux $\pi_K$. In each panel, we use four different values for the filtering wavenumber $K/k_d = 0.6, 1.0, 1.4,$ and $2.2$, from left to right. Note that we use a Gaussian filter; therefore, a proper distinction between liquid and bubble phase can be made only for $K > k_d$. We make the following observations:

(i) In the front of the bubble, the energy is primarily transferred downscale, i.e., to scales smaller than $\ell \sim 1/K$.

(ii) Depending on the filtering scale, we observe both upscale and downscale transfer of energy in the wake of the bubble. For large $K$ (small $\ell$), downscale transfer of energy dominates the wake region, but there are also regions of upscale transfer.

(iii) On reducing the filter wavenumber $K$ (large $\ell$), we observe that the region of upscale transfer is enhanced in the aft region of the bubble. For the smallest filtering wavenumber $K/k_d = 0.6$, the front-aft region of the bubble has a similar structure but appears with opposite signs.

We can understand the fore-aft structure of the energy flux in the vicinity of a bubble in a straightforward manner. Consider a Stokesian spherical bubble with the same viscosity as ambient fluid rising in a quiescent flow; the stream function is given by the Hadamard-Rybczynski solution (Hadamard 1911; Rybczynski 1911; Clift et al. 1978):

$$\Psi(r, \theta) = \frac{V_0 r^2 \sin^2(\theta)}{2} \left\{ \begin{array}{ll}
1 - & \frac{5d}{8r} + \frac{d^3}{32r^3}, \quad \text{for } r \geq d/2, \\
1/4 - & \frac{4r^2}{d^2}, \quad \text{for } r < d/2.
\end{array} \right.$$  \hspace{1cm} (3.12)

The radial and the angular component of the velocity field are $u_r = \partial_\theta \Psi/r^2 \sin(\theta)$ and $u_\theta = -\partial_r \Psi/r \sin(\theta)$. Using (3.12), we calculate the nonlinear flux $\pi_K$ and plot it in figure (11) for four different values of the filtering wavenumber $K/k_d = 0.6, 1.0, 1.4,$ and $2.2$. There is a downscale energy transfer in the front and an upscale energy transfer at the back side of the bubble. Note that the net energy flux $\Pi_K$ is zero for the Hadamard-Rybczynski solution. Comparing figure (10) with figure (11), it seems that the spatial distribution of the energy flux comprises of a Hadamard-Rybczynski-like solution superimposed with turbulent fluctuations generated in the wake region of a rising bubble. Thus our multiscale analysis of the spatial energy flux provides a direct evidence that the net forward energy flux in figure (8b) is due to the bubble wakes.

The situation is more complex in presence of stirring, as now both the large scale forcing as well as the wake of the bubble creates complex spatio-temporal pattern
Figure 10: Buoyancy driven flow in absence of stirring \((b = \infty, R_0)\). The pseudocolor plot of the filtered indicator function \(c\) (top panel) and the local nonlinear flux \(\pi_K / \max(\pi_{k_d})\) (middle panel) in the \(y = L/2\) plane. Constant-\(\pi_K\) isosurfaces for \(|\pi_K| = 0.03 \max(\pi_{k_d})\) in a slab \(L \times 2d \times L\) around the \(y = L/2\) plane (bottom panel). The filter wavenumber (scale) is increased (decreased) from left to right \(K/k_d = 0.6, 1.0, 1.4\) and 2.2.

Figure 11: The space-dependent nonlinear flux \(\pi_K / \max(\pi_{k_d})\) in the \(y = L/2\) plane for the Hadamard-Rybczynski flow Eq. (3.12). The filter wavenumber (scale) is increased (decreased) from left to right \(K/k_d = 0.6, 1.0, 1.4\) and 2.2. The green line represents the bubble interface.

For \(\pi_K(x)\) with regions of downscale and upscale transfer (see figure (12)). In figure (13) we plot the PDF of \(\pi_K(x)\) with \(K = k_d\) for \(b = 0, 0.13\), and \(b = \infty\). For all the cases we observe that the PDF is positively skewed confirming a net positive flux of energy. The skewness of the PDF for \(b = 0\) is nearly 1.3 times larger than the \(b = \infty\), indicating presence of stronger inverse energy transfers in buoyancy driven bubbly flows in comparison to homogeneous, isotropic turbulence. This is further verified by noting that the skewness for \(b = 0.13\), where both stirring and buoyancy driven bubbles generate turbulence, is smaller than the case with \(b = 0\).
Figure 12: Buoyancy driven flow in presence of stirring ($b = 0.13, R3$). The pseudocolor plot of the filtered indicator function $c$ (top panel) and the local nonlinear flux $\pi_K/\max(\pi_{k_d})$ (middle panel) in the $y = L/2$ plane. Constant-$\pi_K$ isosurfaces for $|\pi_K| = 0.03\max(\pi_{k_d})$ in a slab $L \times 2d \times L$ around the $y = L/2$ plane (bottom panel). The filter wavenumber (scale) is increased (decreased) from left to right $K/k_d = 0.6, 1.0, 1.4$ and 2.2.

Figure 13: The PDF of the scaled nonlinear flux $\pi_K/\langle \pi_K^2 \rangle^{1/2}$ for different values of $b$, and with $K = k_d$. 
3.7. Total energy budget

Using (2.1) we obtain the steady-state the total energy budget equation as

\[ \varepsilon^g + \varepsilon^s = \varepsilon^v \]  \hspace{1cm} (3.13)

i.e., energy injected by buoyancy and stirring is dissipated by viscosity. Using table 1, (3.13) is easily verified.

In this section, we study the contribution to the total budget from each of the phases. The two phases are characterized by the indicator function \( c \) which takes value 1 in the liquid phase, 0 inside the bubble and an intermediate value at the interface. In a DNS of two-phase flows, usually, the interface is diffused over to 3–4 grid points. Thus, using \( c \) to distinguish the phases implies that the interface region contributes to both the phases. In order to avoid this conundrum, we construct a new indicator function \( c' \) such that the interface points are included inside the bubble. To construct \( c' \) we first initialize it to be the same as \( c \). The points which lie closest to \( c' = 1/2 \) contour are identified as bubble interface points. For points where \( c' < 1/2 \), \( c' \) is set to zero and it is unity outside. Next we set \( c' = 0 \) at all points that are within a distance of 0.16d from the interface points. This completes the procedure of generating an inflated region around each bubble.

Henceforth we shall use the term bubble to indicate the regions where \( c' = 0 \). Using \( c' \) we define the net injection and dissipation rates in the liquid as:

\[ \varepsilon^v_l = 2\nu \langle c' \mathbf{S} : \mathbf{S} \rangle, \]  \hspace{1cm} (3.14a)
\[ \varepsilon^g_l = \langle c' \mathbf{u} \cdot \mathbf{F}^g \rangle, \]  \hspace{1cm} (3.14b)
\[ \varepsilon^s_l = \langle c' \mathbf{u} \cdot \mathbf{F}^s \rangle. \]  \hspace{1cm} (3.14c)

The contribution from the bubble phase can be obtained by subtracting the contribution from the liquid phase from the total, for instance, dissipation rate in the bubble phase is \( \varepsilon^v_b = \varepsilon^v - \varepsilon^v_l \).

In figure (14a,b) we show the pseudocolor plot of the local viscous dissipation \( \varepsilon^v_{loc}(x) = 2\nu \mathbf{S} : \mathbf{S} \). For the case with no stirring, \( b = \infty \), the dissipation is strongly concentrated inside and in the wake of the bubbles, whereas when stirring is present, \( b = 0.13 \), strong dissipation is also observed in the liquid phase away from the bubbles.

In figure (15a) we look at the balance between energy injection and dissipation in each phase for the case of no stirring, \( b = \infty \). In the liquid phase, viscous dissipation \( \varepsilon^v_l \) far exceeds energy injected due to buoyancy \( \varepsilon^g_l \), whereas in the bubble phase the situation is reversed. Note that the overall viscous dissipation inside the bubble phase is larger than the overall dissipation in the liquid phase.

We can now summarise the flow of energy completely for the case of no stirring, \( b = \infty \). Buoyancy force injects energy at the scale of the bubbles, largely in the gas phase. A large fraction of this energy is dissipated within the bubble itself. Rest of it is transferred to the liquid phase by bubble-liquid interaction. Both the nonlinear flux and the flux due to the surface tension cascades this energy to smaller and smaller scales in the fluid. Energy dissipation happens in both the gas and liquid phase starting from the scale of bubble down to the smallest scales.

We next plot the injection and dissipation rates obtained for different phases for the case \( b = 0.13 \) in figure (15b). Here, we find that dominant energy injection is due to the stirring. This appears largely in the liquid phase. The net energy
dissipated in the liquid phase exceeds the energy injected by stirring due to the additional energy transfer from the bubble phase to the liquid phase. In the bubble phase energy is injected by the buoyancy forces. Most of this energy is dissipated in the bubble phase, but as pointed out above, a part of it is also transferred to the liquid phase.

4. Conclusion

We conduct a DNS study of buoyancy-driven bubbly flow in the presence of large-scale stirring. We investigate the statistical properties of the flow and compare our findings with the experiments. Our key results are summarised below:

(i) The rise velocity of a bubble in the suspension reduces, and the liquid velocity fluctuations are rendered isotropic on increasing the stirring intensity.

(ii) Consistent with experiments (Lance & Bataille 1991; Prakash et al. 2016), we find the energy spectrum shows a Kolmogorov scaling for \( k \ll k_d \) and a pseudo-turbulence scaling – \( E(k) \sim k^{-3} \) – for \( k \gg k_d \).

(iii) We rationalize the scaling observed in the energy spectrum by using a scale-by-scale energy budget analysis. For \( k \ll k_d \), energy flux is the dominant
energy transfer mechanism although viscous dissipation is effective for all scales \( k < k_d \). The balance of net production with viscous dissipation leads to the pseudo-turbulence scaling for \( k \gg k_d \).

We want to emphasize that although we study turbulence modulation by weakly buoyant bubbles, the statistical properties of the flow are in qualitative agreement with the experiments (Lance & Bataille 1991; Prakash et al. 2016; Salibindla et al. 2020). Therefore, we believe that the energy transfer mechanisms discussed in our study should also apply to the experimental scenario of high density and viscosity contrast; our previous study (Pandey et al. 2020) already verified this in the absence of stirring.

However, we expect that the details of the wake structure in the vicinity of the bubble would depend on the density and viscosity contrast. How relevant is this for the energy transfer mechanism that we have proposed remains to be investigated. We hope that our results will motivate further investigations in this direction.

Appendix A. Boussinesq approximated Navier-Stokes equations

In this section we derive the Boussinesq approximate equations (2.1) starting from the following multiphase Navier-Stokes equations (Pandey et al. 2020):

\[
\rho(c) D_t u = \nabla \cdot (\mu(c) S) + f^\sigma + f,
\]  
(A 1)

where the density field

\[
\rho(c) = \rho_f c + \rho_b (1 - c),
\]  
(A 2)

the dynamic viscosity field \( \mu(c) = \mu_f c + \mu_b (1 - c) \), \( \rho_f(\mu_f) \) is the density (viscosity) of fluid phase, \( \rho_b(\mu_b) \) is the density (viscosity) of the bubble phase, \( f^\sigma \) is the surface tension force, the external force \( f \equiv [(\rho(c)a - \rho(c)a), a] \), \( a \) is the acceleration, and in this section \( \langle \cdot \rangle \) denotes spatial averaging. Note that as we work with periodic boundaries, our choice of external force ensures that no net momentum is added to the flow.

We assume small density contrast \( (\Delta \rho \ll 1) \) and identical dynamical viscosity of the two phases \( (\mu_f/\mu_b = 1) \). Thus we invoke Boussinesq approximation, whereby \( \rho(c) \) on the left-hand side of (A 1) is replaced by the average density

\[
\rho_a \equiv \overline{\rho(c)} = (\rho_f - \rho_b)c + \rho_b \approx (\rho_f + \rho_b)/2.
\]  
(A 3)

The above assumptions drastically simplify (A 1) to give,

\[
D_t u = \nu \nabla^2 u + F^\sigma + F,
\]  
(A 4)

where \( F^\sigma = f^\sigma/\rho_a \) and \( F = f/\rho_a \). The above equation is identical to the Boussinesq equation (2.1) that we use. Next we derive the buoyancy and the turbulent stirring force in the Boussinesq regime.

Using the definitions (A 2) and (A 3) in \( F \) we get,

\[
F = \left[1 - \frac{(\rho_f - \rho_b)c_a}{\rho_a} \right] \left(a - \overline{a}\right) + \frac{(\rho_f - \rho_b)}{\rho_a} (c a - \overline{c a}).
\]  
(A 5)

When \( a = g \), the first term on the right hand side of (A 5) is zero and we obtain
the buoyancy force

\[ F^s = \frac{(\rho_f - \rho_b)}{\rho_a}(c - c_a)g \approx 2At(c - c_a)g. \tag{A 6} \]

On the other hand for turbulence stirring, we use an acceleration field with \( \bar{a} = 0 \). Therefore, (A 5) simplifies to:

\[ F^s = \left[ 1 - \frac{(\rho_f - \rho_b)c_a}{\rho_a} \right] a + \frac{(\rho_f - \rho_b)}{\rho_a}(c a - \bar{c}a). \tag{A 7} \]

In the Boussinesq regime, \( (\rho_f - \rho_b)/\rho_a \ll 1 \) and we get \( F^s = a \) to the leading order, i.e., the stirring force is applied irrespective of the phase or the indicator function. In the main manuscript we choose \( \rho_a = 1 \) everywhere.

### Appendix B. Resolution test

To study grid convergence, we conduct DNS of turbulent bubbly flows for our runs \( R1 \) and \( R3 \) with increasing grid-resolution \( N = 360, 512, \) and 720. The plot of the energy spectrum figure (16) clearly shows that even with \( N = 360 \), the inertial range as well as the \( k^{-3} \) scaling of pseudo-turbulence are well-captured. However, as expected, on increasing the grid-resolution the range of \( k^{-3} \) scaling obtained due the balance of net production with viscous dissipation extends. The departure from the \( k^{-3} \) scaling around \( k \approx k_{max} \) is an artifact of finite resolution.

### Appendix C. Energy budget using sharp filter

We now present the result of the scale-by-scale energy budget analysis obtained by using a sharp low-pass filter instead of the Gaussian filter. The low-pass filtered velocity field for a sharp filter is defined as (Frisch 1997; Verma 2019; Pandey et al. 2020):

\[ u^s_K(x) = \sum_{q \leq K} u_q \exp(iq \cdot x). \tag{C 1} \]

In figure (17a) we show the scale-by-scale budget obtained for the case \( b = \infty \) and in figure (17b) we plot the budget for \( b = 0.13 \). By comparing with figure (8), it is clear that the choice of filtering does not qualitatively change the scale-by-scale energy budget. Our observations are consistent with the recent finding.
Figure 17: Scale-by-scale energy budget: plot of the energy flux $\Pi_K$, cumulative viscous dissipation $\mathcal{D}_K$, the surface tension contribution $\mathcal{F}_K^\sigma$, the cumulative energy injected due to buoyancy $\mathcal{F}_K^g$, and the energy injected due to turbulent forcing $\mathcal{F}_K^s$ for (a) $b = \infty$, and (b) $b = 0.13$. In both the panels we normalize the ordinate by the viscous dissipation $\varepsilon^\nu$.

of Alexakis & Chibbaro (2020) who did a similar comparison for homogeneous, isotropic turbulence.

Author contributions
V.P. performed the simulations. All authors contributed equally to analysing data and reaching conclusions, and in writing the paper.

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Declaration of interests
The authors report no conflict of interest.

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