A Note on Terence Tao’s Paper “On the Number of Solutions to \( \frac{4}{p} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \)”

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**Abstract.** For the positive integer \( n \), let \( f(n) \) denote the number of positive integer solutions \( (n_1, n_2, n_3) \) of the Diophantine equation

\[
\frac{4}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}.
\]

For the prime number \( p \), \( f(p) \) can be split into \( f_1(p) + f_2(p) \), where \( f_i(p)(i = 1, 2) \) counts those solutions with exactly \( i \) of denominators \( n_1, n_2, n_3 \) divisible by \( p \).

Recently Terence Tao proved that

\[
\sum_{p < x} f_2(p) \ll x \log^2 x \log \log x.
\]

with other results. But actually only the upper bound \( x \log^2 x \log \log^2 x \) can be obtained in his discussion. In this note we shall use an elementary method to save a factor \( \log \log x \) and recover the above estimate.

1. Introduction

For the positive integer \( n \), let \( f(n) \) denote the number of positive integer solutions \( (n_1, n_2, n_3) \) of the Diophantine equation

\[
\frac{4}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}.
\]

Erdös and Straus conjectured that for all \( n \geq 2 \), \( f(n) > 0 \). It is still an open problem now although there are some partial results.

In 1970, R. C. Vaughan[2] showed that the number of \( n < x \) for which \( f(n) = 0 \) is at most \( x \exp(-c \log^{3/2} x) \), where \( x \) is sufficiently large and \( c \) is a positive constant.

Recently Terence Tao[1] studied the situation in which \( n \) is the prime number \( p \). He gave lower bound and upper bound for the mean value of \( f(p) \). Precisely, he split \( f(p) \) into \( f_1(p) + f_2(p) \), where \( f_i(p)(i = 1, 2) \) counts
those solutions with exactly $i$ of denominators $n_1, n_2, n_3$ divisible by $p$. He proved that

$$x \log^2 x \ll \sum_{p < x} f_2(p) \ll x \log^2 x \log \log x$$

(1)

and

$$x \log^2 x \ll \sum_{p < x} f_1(p) \ll x \exp\left(\frac{c \log x}{\log \log x}\right),$$

where $p$ denotes the prime number, $x$ is sufficiently large and $c$ is a positive constant. Then he conjectured that for $i = 1, 2$,

$$\sum_{p < x} f_i(p) \ll x \log^2 x.$$  

(2)

But actually Terence Tao[1] only proved

$$\sum_{p < x} f_2(p) \ll x \log^2 x \log \log^2 x,$$

(3)

since there was an error in his discussion. In this note we shall use an elementary method to save a factor $\log \log x$ and recover the upper bound in the right side of (1).

**Theorem.** Let $p$ denote the prime number. Then for sufficiently large $x$, we have

$$\sum_{p < x} f_2(p) \ll x \log^2 x \log \log x.$$

2. The proof of Theorem

**Lemma 1.** If $\varphi(n)$ is the Euler totient function, then

$$\varphi(n) \gg \frac{n}{g(n)}.$$

Here

$$g(n) = \prod_{p|n} (1 + \frac{1}{p}) = \sum_{d|n} \frac{\mu^2(d)}{d},$$

where $\mu(d)$ is the Möbius functions.

**Proof.** We know that

$$\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p}).$$
Then

\[ \varphi(n) = n \prod_{p|n(1 - \frac{1}{p^2})} \geq \frac{n}{g(n)} \prod_p (1 - \frac{1}{p^2}) \gg \frac{n}{g(n)}. \]

It is easy to see

\[ g(n) = \sum_{d|n} \frac{\mu^2(d)}{d}. \]

**Lemma 2.** If \( x \geq 1 \), then

\[ \sum_{x<n\leq 2x} \frac{1}{\varphi(n)} \ll 1. \]

**Proof.** By Lemma 1, we have

\[
\sum_{x<n\leq 2x} \frac{1}{\varphi(n)} \ll \sum_{x<n\leq 2x} \frac{g(n)}{n} = \sum_{x<n\leq 2x} \frac{1}{n} \sum_{d|n} \frac{\mu^2(d)}{d} = \sum_{d\leq 2x} \frac{\mu^2(d)}{d^2} \sum_{x<n\leq 2x} \frac{1}{n} = \sum_{d\leq 2x} \frac{\mu^2(d)}{d^2} \ll 1. \]

**Lemma 3.** Let \( p \) denote the prime number. Then the functions \( f_2(p) \) is equal to three times the number of triples \((a, b, c)\) of positive integers such that

\[(a, b) = 1, \quad c|a+b, \quad 4ab|p+c.\]

One can see Proposition 1.2 of [1].

By some transformation, Terence Tao[1] got

\[
\sum_{p<x} f_2(p) \ll \sum_{i=1}^x \sum_{j=\log_2 x}^{i \leq j \leq \log_2 x - i} \frac{1}{1 + \log_2 x - i - j} \cdot \sum_{2^i < a \leq 2^{i+1}} \sum_{2^j < b \leq 2^{j+1}} \frac{d(a+b)}{\varphi(a)\varphi(b)}.
\]
Here $d(n)$ is the divisor function. It is necessary to keep the condition $(a, b) = 1$.

Now we consider the estimate for the sum
\[
\sum_{V < a \leq 2V} \frac{1}{\varphi(a)} \sum_{W < b \leq 2W \atop (a, b) = 1} d(a + b),
\]

where $1 \leq V \leq W \leq x$.

Let
\[
S(a, W) = \sum_{W < b \leq 2W \atop (a, b) = 1} \frac{d(a + b)}{\varphi(b)},
\]

Then Lemma 1 yields that
\[
S(a, W) \ll \sum_{W < b \leq 2W \atop (a, b) = 1} d(a + b) \cdot \frac{g(b)}{b}
\]
\[
\ll \frac{1}{W} \sum_{W < b \leq 2W \atop (a, b) = 1} d(a + b)g(b)
\]
\[
= \frac{1}{W} \sum_{W + a < k \leq 2W + a \atop (k, a) = 1} d(k)g(k - a)
\]
\[
= \frac{1}{W} \sum_{W + a < r \leq 2W + a \atop (r, a) = 1} g(rl - a)
\]
\[
\leq \frac{2}{W} \sum_{r \leq \sqrt{2W + a} \atop (r, a) = 1} \sum_{W < n \leq 2W \atop n \equiv -a \pmod{r} \atop (n, a) = 1} g(n)
\]
\[
\ll \frac{1}{W} \sum_{r \leq \sqrt{2W + a} \atop (r, a) = 1} \sum_{W < n \leq 2W \atop n \equiv -a \pmod{r}} g(n)
\]
\[
\leq \frac{1}{W} \sum_{r \leq \sqrt{2W + a} \atop (r, a) = 1} \sum_{W < n \leq 2W \atop n \equiv -a \pmod{r}} g(n).
\]
Since \((r, a) = 1\), \(n \equiv -a \pmod{r} \implies (n, r) = 1\). Then

\[
\sum_{\substack{W < n \leq 2W \\ n \equiv -a \pmod{r} \\ (n, r) = 1}} g(n) = \sum_{\substack{W < n \leq 2W \\ n \equiv -a \pmod{r} \\ (n, r) = 1}} g(n)
\]

\[
= \sum_{\substack{W < n \leq 2W \\ n \equiv -a \pmod{r} \\ (n, r) = 1}} \sum_{d \mid n} \frac{\mu^2(d)}{d}
\]

\[
= \sum_{\substack{d \leq 2W \\ (d, r) = 1}} \frac{\mu^2(d)}{d} \sum_{\substack{W < n \leq 2W \\ n \equiv -a \pmod{r} \\ (n, r) = 1 \\ d \mid n}} 1
\]

\[
= \sum_{\substack{d \leq 2W \\ (d, r) = 1}} \frac{\mu^2(d)}{d} \sum_{\substack{W < k \leq \frac{2W}{d} \\ (k, r) = 1 \\ dk \equiv -a \pmod{r}}}} 1
\]

\[
\leq \sum_{\substack{d \leq 2W \\ (d, r) = 1}} \frac{\mu^2(d)}{d} \sum_{\substack{W < k \leq \frac{2W}{d} \\ k \equiv da \pmod{r}}}} 1
\]

where \(\tilde{d}\) is an integer such that \(\tilde{d}d \equiv 1 \pmod{r}\).

We have

\[
\sum_{\substack{W < k < \frac{2W}{d} \\ k \equiv da \pmod{r}}}} 1 \ll \frac{W}{dr} + 1.
\]

Thus

\[
\sum_{\substack{W < n \leq 2W \\ n \equiv -a \pmod{r}}} g(n) \ll \sum_{\substack{d \leq 2W \\ (d, r) = 1}} \frac{\mu^2(d)}{d}\left(\frac{W}{dr} + 1\right)
\]

\[
\leq \frac{W}{r} \sum_{d \leq 2W} \frac{\mu^2(d)}{d^2} + \sum_{d \leq 2W} \frac{\mu^2(d)}{d}
\]

\[
\ll \frac{W}{r} + \log 2W.
\]
It follows that
\[ S(a, W) \ll \frac{1}{W} \sum_{r \leq \sqrt{2W+a}} \left( \frac{W}{r} + \log 2W \right) \]
\[ \leq \frac{1}{W} \sum_{r \leq 2\sqrt{W}} \left( \frac{W}{r} + \log 2W \right) \]
\[ \ll \log 2W. \]

By Lemma 2, we have
\[ \sum_{V < a \leq 2V} \frac{1}{\varphi(a)} \sum_{W < b \leq 2W} \frac{d(a+b)}{\varphi(b)} \]
\[ \ll \sum_{V < a \leq 2V} \frac{1}{\varphi(a)} \cdot \log x \]
\[ \ll \log x. \]

Therefore
\[ \sum_{p < x} f_2(p) \ll x \log x \sum_{1 \leq i \leq \frac{1}{2} \log_2 x} \sum_{i \leq j \leq \log_2 x - i} \frac{1}{1 + \log_2 x - i - j}. \quad (6) \]

We have
\[ \sum_{1 \leq i \leq \frac{1}{2} \log_2 x} \sum_{i \leq j \leq \log_2 x - i} \frac{1}{1 + \log_2 x - i - j} \]
\[ \leq \sum_{1 \leq i \leq \frac{1}{2} \log_2 x} \sum_{1 \leq k \leq \log_2 x - 2i + 2} \frac{1}{h} \]
\[ \ll \sum_{1 \leq k \leq \frac{1}{2} \log_2 x} \log(\log_2 x - 2i + 4) \]
\[ \ll \sum_{1 \leq k \leq \frac{1}{2} \log_2 x + 1} \log(2k + 8) \]
\[ \ll \log x \log \log x. \]

So far the proof of Theorem is finished.
Similar discussion can yield
\[ \sum_{1 \leq i \leq \frac{1}{2} \log_3 x} \sum_{i \leq j \leq \log_2 x - i} \frac{1}{1 + \log_2 x - i - j} \gg \log x \log \log x. \]
In [1],
\[ \sum_{1 \leq i \leq \frac{1}{x} \log x} \sum_{i \leq j \leq \log x - i} \frac{1}{1 + \log x - i - j} \ll \log x \]
is proved, where a factor \( \log \log x \) is lost. From the above discussion, it seems reasonable to conjecture
\[ x \log^2 x \log \log x \ll \sum_{p < x} f_2(p). \] (7)

References

[1] Terence Tao, *On the number of solutions to* \( \frac{4}{p} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \), available at http://arxiv.org/abs/1107.1010

[2] R. C. Vaughan, *On a problem of Erdös, Straus and Schinzel*, Mathematica, 17(1970), 193-198.

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