CONSTRUCTION OF 2-SOLITONS WITH LOGARITHMIC DISTANCE FOR THE ONE-DIMENSIONAL CUBIC SCHRÖDINGER SYSTEM

YVAN MARTEL* AND TIÉN VINH NGUYÊN
CMLS, École Polytechnique, CNRS
91128 Palaiseau, France

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ABSTRACT. We consider a system of coupled cubic Schrödinger equations in one space dimension
\[
\begin{aligned}
id_t u + \partial_x^2 u + (|u|^2 + \omega |v|^2)u &= 0 \\
id_t v + \partial_x^2 v + (|v|^2 + \omega |u|^2)v &= 0
\end{aligned}
\]
(t, x) ∈ R × R,
in the non-integrable case 0 < ω < 1.

First, we justify the existence of a symmetric 2-solitary wave with logarithmic distance, i.e. a solution of the system satisfying
\[
\lim_{t \to +\infty} \left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} e^{it Q(-1/2 \log(\Omega t) - 1/4 \log \log t)} \\ e^{it Q(1/2 \log(\Omega t) + 1/4 \log \log t)} \end{pmatrix} \right\|_{H^1 \times H^1} = 0
\]
where \( Q = \sqrt{2} \sech \) is the explicit solution of \( Q'' - Q + Q^3 = 0 \) and \( \Omega > 0 \) is a constant. This result extends to the non-integrable case the existence of symmetric 2-solitons with logarithmic distance known in the integrable case \( \omega = 0 \) and \( \omega = 1 \) ([15, 33]). Such strongly interacting symmetric 2-solitary waves were also previously constructed for the non-integrable scalar nonlinear Schrödinger equation in any space dimension and for any energy-subcritical power nonlinearity ([20, 22]).

Second, under the conditions \( 0 < c < 1 \) and \( 0 < \omega < \frac{1}{2} c(c+1) \), we construct solutions of the system satisfying
\[
\lim_{t \to +\infty} \left\| \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} e^{it Q_c(-1/(c+1) \log(\Omega_c t))} \\ e^{it Q_c(1/(c+1) \log(\Omega_c t))} \end{pmatrix} \right\|_{H^1 \times H^1} = 0
\]
where \( Q_c(x) = cQ(cx) \) and \( \Omega_c > 0 \) is a constant. Such logarithmic regime of non-symmetric solitons does not exist in the integrable cases \( \omega = 0 \) and \( \omega = 1 \) and is still unknown in the non-integrable scalar case.

1. Introduction.

1.1. System of cubic Schrödinger equations. We consider the following one-dimensional focusing-focusing system of coupled cubic Schrödinger equations
\[
\begin{aligned}
id_t u + \partial_x^2 u + (|u|^2 + \omega |v|^2)u &= 0 \\
id_t v + \partial_x^2 v + (|v|^2 + \omega |u|^2)v &= 0
\end{aligned}
\]
(t, x) ∈ R × R (coupled NLS)

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* Corresponding author: Yvan Martel.
for \(u(t, x), v(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}\) and for any parameter \(0 < \omega < 1\). The initial
data \(u(0, x) = u_0(x), v(0, x) = v_0(x)\) is taken in \(H^1(\mathbb{R}) \times H^1(\mathbb{R})\). The Hamiltonian
system (coupled NLS) arises as a model for the propagation of the electrical field in
nonlinear optics. Such systems also appear to model the interaction of two Bose-
Einstein condensates in different spin states. See [1, 2, 32].

For \(\omega = 0\), the system (coupled NLS) simply reduces to two cubic focusing
Schrödinger equations without coupling (see [1, 32, 33])
\[
i \partial_t u + \partial_x^2 u + |u|^2 u = 0 \quad (t, x) \in \mathbb{R} \times \mathbb{R}.
\]
(cubic NLS)

For \(\omega = 1\), the system (coupled NLS) is called the Manakov system (see [1, 15, 32])
\[
\begin{align*}
i \partial_t u + \partial_x^2 u + (|u|^2 + |v|^2) u &= 0 \\
i \partial_t v + \partial_x^2 v + (|v|^2 + |u|^2) v &= 0.
\end{align*}
\]
(MS)

Both (cubic NLS) and (MS) are completely integrable. For \(0 < \omega < 1\), the system
is not known to be integrable.

It follows from standard arguments (see e.g. [3, 8]) that (coupled NLS) is locally
well-posed in \(H^1 \times H^1\). In this paper, we work in the framework of such \(H^1 \times H^1\)
solutions. Moreover, the system is invariant under the following symmetries:

- Phase: \(\gamma, \gamma' \in \mathbb{R}, \begin{pmatrix} u_0(x)e^{i\gamma} \\ v_0(x)e^{i\gamma'} \end{pmatrix} \mapsto \begin{pmatrix} u(t, x)e^{i\gamma} \\ v(t, x)e^{i\gamma'} \end{pmatrix};\)
- Scaling: \(\lambda > 0, \lambda \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}(\lambda x) \mapsto \lambda \begin{pmatrix} u \\ v \end{pmatrix}(\lambda^2 t, \lambda x);\)
- Space translation: \(\sigma \in \mathbb{R}, \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}(x + \sigma) \mapsto \begin{pmatrix} u \\ v \end{pmatrix}(t, x + \sigma);\)
- Galilean invariance: \(\beta \in \mathbb{R}, e^{i\beta x} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}(x) \mapsto e^{i\beta(x - \beta t)} \begin{pmatrix} u \\ v \end{pmatrix}(t, x - 2\beta t).\)

For \(H^1 \times H^1\) solutions, the following quantities are constant:

- Masses:
  \[M(u(t)) = \int_{\mathbb{R}} |u(t, x)|^2 dx = M(u_0), \quad M(v(t)) = \int_{\mathbb{R}} |v(t, x)|^2 dx = M(v_0);\]
- Energy:
  \[
  E(u(t), v(t)) = \frac{1}{2} \int_{\mathbb{R}} (|\partial_x u|^2 + |\partial_x v|^2) (t, x) dx
  - \frac{1}{4} \int_{\mathbb{R}} (|u|^4 + |v|^4 + 2\omega |u|^2 |v|^2) (t, x) dx
  = E(u_0, v_0);
  \]
- Momentum:
  \[
  J(u(t), v(t)) = 3 \int_{\mathbb{R}} \partial_x u(t, x) \bar{u}(t, x) dx + 3 \int_{\mathbb{R}} \partial_x v(t, x) \bar{v}(t, x) dx
  = J(u_0, v_0).
  \]

From the Gagliardo-Nirenberg inequality: \(||u||^4_{L^4} \lesssim ||u||^3_{L^2} ||\partial_x u||_{L^2}\) and standard
arguments, the system is globally well-posed in \(H^1 \times H^1\) (see e.g. [3, 28]).
Let $Q$ be the ground state, defined as

$$Q(x) = \frac{\sqrt{2}}{\cosh(x)}$$

unique (up to translation) $H^1$ solution of $Q'' - Q + Q^3 = 0$ on $\mathbb{R}$. Recall that (cubic NLS) admits solitary wave solutions, also called solitons, of the form

$$u(t, x) = e^{i\gamma + i\lambda x^2 + i\beta(x - \beta t)}Q_{\lambda}(x - \sigma - 2\beta t) \quad \text{with} \quad Q_{\lambda}(x) = \lambda Q(\lambda x)$$

where $\lambda > 0$, $\gamma, \sigma, \beta \in \mathbb{R}$. When $v = 0$ (or $u = 0$), the system (coupled NLS) simplifies into (cubic NLS), and thus we deduce soliton solutions of (coupled NLS):

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = \begin{pmatrix} e^{i\Gamma_1(t, x)}Q_{\lambda}(x - \sigma - 2\beta_1 t) \\ 0 \end{pmatrix}, \quad \Gamma_1(t, x) = \gamma_1 + \lambda_1^2 t + \beta_1(x - \beta_1 t)$$

and

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\Gamma_2(t, x)}Q_{\lambda}(x - \sigma - 2\beta_2 t) \end{pmatrix}, \quad \Gamma_2(t, x) = \gamma_2 + \lambda_2^2 t + \beta_2(x - \beta_2 t)$$

for any $\lambda_j > 0$, $\gamma_j, \sigma_j, \beta_j \in \mathbb{R}$ ($j = 1, 2$). By definition, a multi-solitary wave (or multi-soliton) is a solution behaving in large time as a sum of such single solitons. In this article, we focus on 2-solitons such that one solitary wave is carried by $u$ and the other one by $v$.

1.2. Previous results and motivation. Multi-solitons have been studied in the integrable case, i.e. for (cubic NLS) and (MS), as well as for some nearly integrable models; see [1, 7, 9, 13, 24, 32, 33]. From the inverse scattering theory, there are three types of 2-solitons for (cubic NLS):

(a) Two solitons with different velocities: as $t \to +\infty$, the distance between the solitons is of order $t$ ([33]).
(b) Double pole solutions: the two solitons have the same amplitude and their distance is logarithmic in $t$ ([24, 33]).
(c) Periodic 2-solitons: the two solitons have different amplitudes and their distance is a periodic function of time ([32, 33]).

More generally, the integrability theory deals with the case of $K$-solitary waves for any $K \geq 2$. Moreover, in the integrable case, it is known that multi-solitons have a pure soliton behavior for both $t \to +\infty$ and $t \to -\infty$ and describe the elastic interactions between solitons. For (MS), a trichotomy similar to (a)-(b)-(c) is studied formally and numerically in [31].

For non-integrable models, the study of multi-solitons is for now mostly limited to situations where solitons are decoupled asymptotically in large time. Consider first the scalar nonlinear Schrödinger equation

$$i \partial_t u + \Delta u + |u|^{p-1}u = 0, \quad u(0, x) = u_0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad \text{(NLS)}$$

in any space dimension $d \geq 1$ and for any energy subcritical power nonlinearity (i.e. $p > 1$ for $d = 1, 2$ and $1 < p < 1 + \frac{4}{d-2}$ for $d \geq 3$). This equation is known to be completely integrable only for $d = 1$ and $p = 3$, i.e. (cubic NLS). Define the ground state $Q$ as the unique radial positive $H^1$ solution (up to symmetries) of $\Delta Q - Q + Q^3 = 0$ in $\mathbb{R}^d$ (for more properties of the ground state, see [3, 10, 25, 30]) and $Q_{\lambda}(x) = \lambda^{\frac{d+2}{2}}Q(\lambda x)$, for any $\lambda > 0$. The existence of $K$-solitary waves for
Theorem 1.1. For any \( \beta_k > 0 \) and any two by two different \( \beta_k \in \mathbb{R}^d \), was established in [5, 17, 21].

Recently, the second author proved that the dynamics (b) is also a universal regime for (NLS), by constructing two symmetric solitary waves with logarithmic distance, [22]. The \( L^2 \) critical case \( (p = 1 + \frac{4}{d}) \), previously studied in [20], exhibits a specific blow-up behavior also related to symmetric 2-solitons with logarithmic distance in rescaled variables.

Turning back to the system (coupled NLS) in the non-integrable case, i.e. for \( 0 < \omega < 1 \), the existence of multi-solitary wave solutions corresponding to case (a)

\[
\lim_{t \to +\infty} \left\| u(t) - \sum_{k=1}^{K} e^{-i\tau_k(t,t')} Q_{\lambda_k} (\cdot - \sigma_k - 2\beta_k t) \right\|_{H^1} = 0
\]

for any \( \lambda_k > 0 \) and any two by two different \( \beta_k \in \mathbb{R}^d \), was established in [5, 17, 21].

A first goal of this paper is to justify the persistence of the regime (b) for the scalar (NLS) in \([20, 22]\).

More importantly, we investigate the question of the (non) persistence of the regime (c). Indeed, we exhibit a new logarithmic regime corresponding to non-symmetric 2-solitons with logarithmic distance which replaces the behavior (c). At the formal level, the system of parameters of the 2-solitons is no more integrable and periodic solutions disappear, see Remark 3. Another logarithmic regime then takes place (see Theorem 1.2 and Remark 2), which does not exist in the integrable cases \( \omega = 0 \) and \( \omega = 1 \). To our knowledge, such question is open for the scalar (NLS) in the non-integrable case (see Section 5).

1.3. Main results. First, we present the symmetric logarithmic regime.

**Theorem 1.1.** For any \( 0 < \omega < 1 \), there exists a solution \((u, v) \in C(\mathbb{R}, H^1 \times H^1)\) of (coupled NLS) such that

\[
\lim_{t \to +\infty} \left\| \left( u(t), v(t) \right) - \left( e^{i t} Q(\cdot - \frac{1}{2} \log(\Omega t) - \frac{1}{4} \log \log t), e^{i t} Q(\cdot + \frac{1}{2} \log(\Omega t) + \frac{1}{4} \log \log t) \right) \right\|_{H^1 \times H^1} = 0
\]

where \( \Omega > 0 \) is a constant depending on \( \omega \).

Note that as \( t \to +\infty \), the distance between the two solitary waves is asymptotic to

\[
y(t) = \log t + \frac{1}{2} \log \log t + \log \Omega.
\]

**Remark 1.** An analogous dynamics was constructed for (cubic NLS) in [24, 33] and for (NLS) in [20, 22].

Second, we construct for (coupled NLS) a new logarithmic dynamics of 2-solitary waves with different amplitude.

**Theorem 1.2.** For any \( 0 < c < 1 \) and \( 0 < \omega < \frac{1}{2} (c + 1) < 1 \), there exists a solution \((u, v) \in C(\mathbb{R}, H^1 \times H^1)\) of (coupled NLS) such that

\[
\lim_{t \to +\infty} \left\| \left( u(t), v(t) \right) - \left( e^{i c t^2} Q_c (\cdot - \frac{1}{(c+1)c} \log(\Omega t)), e^{i t} Q(\cdot + \frac{1}{4} \log(\Omega t)) \right) \right\|_{H^1 \times H^1} = 0
\]
where $\Omega_c > 0$ is a constant depending on $c$ and $\omega$.

In this case, as $t \to +\infty$, the distance between the two solitary waves is asymptotic to
\[
y_c(t) = \frac{1}{c} \log t + \frac{1}{c} \log \Omega_c.
\]
As mentioned before, such solution does not exist in the integrable cases and the analogous question for the non-integrable scalar equation (NLS) seems open. See Section 5.

**Remark 2.** The slight difference between the two regimes (1) and (2) is due to stronger interactions when the two solitary waves have equal amplitudes. We refer to Sections 4.2 and 2.3 for formal derivations of the regimes (1) and (2).

We believe that there is no other logarithmic regime for (coupled NLS). In support of this conjecture, we refer to the case of the generalized Korteweg-de Vries equation, for which existence of a logarithmic regime was proved in [23] and uniqueness (in the super-critical case) was established in [12].

The case $\frac{1}{2}c(c + 1) \leq \omega < 1$ in Theorem 1.2 is open (see step 1 of the proof of Proposition 1).

**Remark 3.** The dynamics of the distance between the two solitary waves is related to nonlinear interactions. A formal study (see notably [9, 13] and Chapter 4 in [32]) shows that the three behaviors (a), (b) and (c) are related to different solutions of
\[
\begin{align*}
\dot{\gamma} &= c_\gamma e^{-\sigma} \sin \gamma \\
\dot{\sigma} &= -c_\sigma e^{-\sigma} \cos \gamma
\end{align*}
\]
where $\gamma$ is the phase difference, $\sigma$ the relative distance and $c_\gamma, c_\sigma$ are constants.

For (cubic NLS), it holds $c_\gamma = c_\sigma > 0$. Denoting $Y = \sigma + i\gamma$, the resulting equation $\ddot{Y} = -c_\gamma e^Y$ is integrable and admits nontrivial solutions for which $\sigma$ is periodic.

**Remark 4.** The proofs of Theorems 1.1 and 1.2 follow the overall strategy of several previous articles on multi-solitons ([14, 16, 17, 18, 19, 20, 21, 22, 26]), particularly of [20, 22] which started the study of multi-solitons with logarithmic distance in a non-integrable setting. We focus on the proof of Theorem 1.2, which is more original in the construction of a suitable approximate solution and the determination of the asymptotic regime (see Remark 5).

See Section 5 for a comment on the introduction of a refined energy method.

1.4. **Notation and preliminaries.** For complex-valued functions $f, g \in L^2(\mathbb{R})$, we denote
\[
(f, g) = \Re \left( \int f \overline{g} \right).
\]
For $r$ a positive function of time, the notation $f(t, x) = O_H(r(t))$ means that there exists a constant $C > 0$ such that $\|f(t)\|_{H^1} \leq Cr(t)$.

For any $\lambda > 0$ and any function $f$, let
\[
f_\lambda(x) = \lambda f(\lambda x) \quad \text{and} \quad \Lambda f(x) = f(x) + xf'(x) = \partial_\lambda f_\lambda(x)_{\lambda=1}.
\]
Note the following relation which describes the asymptotics of $Q(x)$ as $x \to -\infty$,
\[
Q(x) = \kappa e^x - e^{2x} Q(x) \quad \text{on } \mathbb{R} \quad \text{where } \kappa = 2\sqrt{2}.
\]
Throughout this paper, we consider $\omega$ and $c$ such that
\[
0 < c \leq 1 \quad \text{and} \quad 0 < \omega < \frac{c(c + 1)}{2}.
\]
The linearization of (coupled NLS) around solitons involves the following operators:
\[ \mathcal{L}_+ = -\partial_x^2 + 1 - 3Q^2, \quad \mathcal{L}_- = -\partial_x^2 + 1 - Q^2, \quad \mathcal{L}_c = -\partial_x^2 + c^2 - \omega Q^2. \]
Recall the special relations ([29])
\[ \mathcal{L}_- Q = 0, \quad \mathcal{L}_+(\Lambda Q) = -2Q, \quad \mathcal{L}_+(Q') = 0, \quad \mathcal{L}_-(xQ) = -2Q'. \]
We will use the following properties of these operators.

**Lemma 1.3.** Assume (4).

(i) There exists \( \mu > 0 \) such that, for all \( z \in H^1 \),
\[
\langle \mathcal{L}_+ Rz, Rz \rangle + \langle \mathcal{L}_- Rz, Rz \rangle \geq \mu \|z\|_{H^1}^2 - \frac{1}{\mu} \left( \langle z, Q \rangle^2 + \langle z, xQ \rangle^2 + \langle z, i\Lambda Q \rangle^2 \right),
\]
\[
\langle \mathcal{L}_c z, z \rangle \geq \mu \|z\|_{H^1}^2.
\]
(ii) For any \( f \in L^2 \), there exists a unique solution \( u \in H^2 \) of \( \mathcal{L}_c u = f \). Moreover,
- If \( |f(x)| \lesssim e^{-\lambda|x|} \) for some \( \lambda > c \), then \( |u(x)| \lesssim e^{-c|x|} \).
- If \( |f(x)| \lesssim e^{-c|x|} \) then \( |u(x)| \lesssim (1 + |x|)e^{-c|x|} \).

**Proof.** (i) The coercivity properties of \( \mathcal{L}_+ \) and \( \mathcal{L}_- \) (here in the \( L^2 \) sub-critical case) are well-known facts (see e.g. [17, 29, 30]).

Let \( 0 < \rho < c \) be such that \( \omega = \frac{1}{2}\rho(\rho + 1) \). By [27] or direct computation, we see that the positive function \( Q^\rho \) satisfies \( \mathcal{L}_c Q^\rho = (c^2 - \rho^2)Q^\rho \). The coercivity property follows.

(ii) Let \( c \leq \lambda \leq 1 \). If \( \mathcal{L}_c u = f \) with \( |f(x)| \lesssim e^{-\lambda|x|} \) then \( -u'' + c^2u = g \) where \( g = f + \omega Q^2u \) also satisfies \( |g(x)| \lesssim e^{-\lambda|x|} \). The decay properties of \( u \) then follows from standard arguments. \(\square\)

The following result follows directly from Lemma 1.3.

**Lemma 1.4.**

(i) Assume \( 0 < c < 1 \). There exists a solution \( A \) of
\[
\mathcal{L}_c A = -A'' + c^2 A - \omega Q^2 A = c\kappa\omega e^{\kappa t}Q^2
\]
satisfying
\[
|A(x)| + |A'(x)| + |A''(x)| \lesssim Q_c(x) \quad \text{on} \quad \mathbb{R}. \tag{7}
\]
(ii) There exists a solution \( B \) of
\[
\mathcal{L}_1 B = -B'' + B - \omega Q^2 B = \kappa\omega e^{\kappa t}Q^2
\]
satisfying
\[
|B(x)| + |B'(x)| + |B''(x)| \lesssim (1 + |x|)Q(x) \quad \text{on} \quad \mathbb{R}. \tag{9}
\]

2. Approximate solution in the case \( 0 < c < 1 \).

2.1. Definition of the approximate solution. Consider \( C^1 \) time-dependent real-valued functions \( \sigma_1, \sigma_2, \gamma_1, \gamma_2, \beta_1, \beta_2, \) to be fixed later and set
\[
\sigma = \sigma_1 - \sigma_2, \quad \beta = \beta_1 - \beta_2, \quad \gamma = \gamma_1 - \gamma_2.
\]
Denote
\[
U = P + \varphi, \quad P(t,x) = Q_c(x - \sigma_1(t))e^{i\Gamma_1(t,x)}, \quad \varphi(t,x) = e^{-c\sigma(t)}A(x - \sigma_2(t))e^{i\Gamma_1(t,x)},
\]
\[
V = R, \quad R(t,x) = Q(x - \sigma_2(t))e^{i\Gamma_2(t,x)},
\]
where
\[
\Gamma_1(t,x) = c^2 t + \gamma_1(t) + \beta_1(t)x, \quad \Gamma_2(t,x) = t + \gamma_2(t) + \beta_2(t)x.
\]
Introduce the notation
\[ \partial_1 P = Q_c'(x - \sigma_1)e^{i\Gamma_1}, \quad x_1 P = (x - \sigma_1)P, \quad \Lambda_1 P = \Lambda Q_c(x - \sigma_1)e^{i\Gamma_1}, \]
\[ \partial_1 \varphi = e^{-c\sigma} A'(x - \sigma_2)e^{i\Gamma_1}, \quad x_2 \varphi = (x - \sigma_2)\varphi, \]
\[ \partial_2 R = Q'(x - \sigma_2)e^{i\Gamma_2}, \quad x_2 R = (x - \sigma_2)R, \quad \Lambda_2 R = \Lambda Q(x - \sigma_2)e^{i\Gamma_2}. \]

Define the approximate solution
\[ \mathbf{Z} = \begin{pmatrix} U \\ V \end{pmatrix} \quad \text{and set} \quad \mathbf{E}_\mathbf{Z} = \begin{pmatrix} \mathbf{E}_U \\ \mathbf{E}_V \end{pmatrix} = \begin{pmatrix} i\partial_1 U + \partial_1^2 U + (|U|^2 + \omega|V|^2) U \\ i\partial_2 V + (|V|^2 + \omega|U|^2) V \end{pmatrix}. \]

Lemma 2.1. It holds
\[ \begin{cases} \mathbf{E}_U = F - \bar{m}_1 \cdot \bar{M}_1 - \bar{m}_\varphi \cdot \bar{M}_\varphi \\ \mathbf{E}_V = G - \bar{m}_2 \cdot \bar{M}_2 \end{cases} \tag{10} \]
where
\[ \begin{cases} F = 3|P|^2 \varphi + 3|\varphi|^2 P + |\varphi|^2 \varphi - \omega e^{2c(x-\sigma_1)}|R|^2 P \\ G = \omega|P + \varphi|^2 R \end{cases} \tag{11} \]

and
\[ \bar{m}_1 = \begin{pmatrix} \dot{\beta}_1 + 2\beta_1 \\ \gamma_1 + \beta_1 \sigma_1 + \beta_1^2 \end{pmatrix}, \quad \bar{M}_1 = \begin{pmatrix} i\partial_1 P \\ P \end{pmatrix}, \]
\[ \bar{m}_\varphi = \begin{pmatrix} \dot{\beta}_1 - 2\beta_1 \\ \gamma_1 - \beta_1 \sigma_1 + \beta_1^2 + ic\sigma \end{pmatrix}, \quad \bar{M}_\varphi = \begin{pmatrix} i\partial_1 \varphi \\ \varphi \end{pmatrix}, \]
\[ \bar{m}_2 = \begin{pmatrix} \dot{\beta}_2 - 2\beta_2 \\ \gamma_2 + \beta_2 \sigma_2 + \beta_2^2 \end{pmatrix}, \quad \bar{M}_2 = \begin{pmatrix} i\partial_2 R \\ R \end{pmatrix}. \]

Proof. Using \( Q'_c - c^2 Q_c = Q_c^3 \) and (6), we compute
\[ i\partial_1 P + \partial_1^2 P + |P|^2 P = -\bar{m}_1 \cdot \bar{M}_1, \]
\[ i\partial_1 \varphi + \partial_1^2 \varphi + \omega|R|^2 (P + \varphi) = -\bar{m}_\varphi \cdot \bar{M}_\varphi + \omega|R|^2 \left[ Q_c(x - \sigma_1) - cke^{c(x-\sigma_1)} \right] e^{i\Gamma_1}. \]

Using (3), we obtain (10) for \( \mathbf{E}_U \) with \( F \) defined as in (11).

Similarly, the equation
\[ i\partial_2 R + \partial_2^2 R + |R|^2 R = -\bar{m}_2 \cdot \bar{M}_2 \]
implies (10) for \( \mathbf{E}_V \) with \( G \) defined as in (11).

2.2. Projection of the error terms. The soliton dynamics is expected to be determined by the following projections
\[ a = \frac{1}{2c} \langle F, \partial_1 P \rangle \quad \text{and} \quad b = \frac{1}{2} \langle G, \partial_2 R \rangle. \]
Using the relations $\langle \partial_1 P, x_1 P \rangle = \langle Q_c', x Q_c \rangle = -\frac{1}{2} \|Q_c\|_{L^2}^2 = -2c$ and $\langle \partial_2 R, x_2 R \rangle = -\frac{1}{2} \|Q\|_{L^2}^2 = -2$, we decompose $F$ and $G$ as follows
\begin{align}
F &= F^\perp - a x_1 P, \quad \langle F^\perp, \partial_1 P \rangle = 0 \\
G &= G^\perp - b x_2 R, \quad \langle G^\perp, \partial_2 R \rangle = 0
\end{align}
(12)
so that (10) rewrites
\begin{align}
\mathcal{E}_U &= F^\perp - \vec{m}_1^a \cdot \vec{M}_1 - \vec{m}_\varphi \cdot \vec{M}_\varphi \\
\mathcal{E}_V &= G^\perp - \vec{m}_2^b \cdot \vec{M}_2
\end{align}
(13)
with
\[
\vec{m}_1^a = \begin{pmatrix}
\dot{\sigma}_1 - 2\beta_1 \\
\dot{\beta}_1 + a
\end{pmatrix}
\quad \text{and} \quad
\vec{m}_2^b = \begin{pmatrix}
\dot{\sigma}_2 - 2\beta_2 \\
\dot{\beta}_2 + b
\end{pmatrix}.
\]
We compute the main order of these projections.

**Lemma 2.2.** Let $1 < \theta < \min \{ \frac{1}{2}, 2 \}$. It holds
\[
a = \alpha_c e^{-2c\sigma} + O(e^{-2c\theta\sigma}), \quad b = -c\alpha_c e^{-2c\sigma} + O(e^{-2c\theta\sigma})
\]
(14)
where
\[
\alpha_c = 4c^2 \omega \|e^{cx} Q\|_{L^2}^2 + \frac{1}{2} \langle L_c A, A \rangle > 0.
\]

**Remark 5.** The expression of the positive constant $\alpha_c$, relevant in the dynamics of the 2-soliton (see Section 2.3), suggests that even at the formal level, the introduction of the approximate solution $\left( \frac{1}{\sqrt{\theta}} \right)$ including the refined term $\varphi$ is necessary to determine correctly the non-symmetric logarithmic regime.

**Proof.** We start by proving the following estimates
\[
\int e^{2c(x-\sigma)} Q_c^2(x-\sigma) Q^2(x) dx = O(e^{-2c\theta\sigma}),
\]
(15)
\[
\int Q_c^2(x-\sigma) Q(x) Q'(x) dx = -c^3 \kappa^2 e^{-2c\sigma} \int e^{2cx} Q^2(x) dx + O(e^{-2c\theta\sigma}).
\]
(16)
Proof of (15). By (3) and the condition on $\theta$, we have
\[
e^{2c(x-\sigma)} Q_c^2(x-\sigma) Q^2(x) \lesssim e^{2c(x-\sigma)} Q^2(x) \lesssim e^{-2c\theta\sigma} e^{-2(1-c)x},
\]
and (16) follows.

Proof of (16). It follows from (3) that
\[
Q_c^2(x) = c^2 \kappa^2 e^{2cx} + O(e^{3cx} Q_c(x)),
\]
and so
\[
Q_c^2(x-\sigma) = c^2 \kappa^2 e^{-2c\sigma} e^{2cx} + O(e^{-2c\theta\sigma} e^{2c\sigma}).
\]
Thus
\[
\int Q_c^2(x-\sigma) Q(x) Q'(x) dx = c^2 \kappa^2 e^{-2c\sigma} \int e^{2cx} Q(x) Q'(x) dx + O(e^{-2c\theta\sigma}).
\]
and (16) follows by integration by parts.
Similarly as in the proof of (16), using (3) we observe equation (15) and in (7) and the condition on \( \theta \) that

\[
\langle G, \partial_2 R \rangle = \omega \int Q_c^2(x - \sigma)Q(x)Q'(x)dx + 2\omega e^{-c\sigma} \int Q_c(x - \sigma)A(x)Q(x)Q'(x)dx + \omega e^{-2c\sigma} \int A^2(x)Q(x)Q'(x)dx.
\]

Using also (15) and \( \kappa^2 = 8 \), we find

\[
a = \frac{e^{-2c\sigma}}{2} \left[ \kappa^2 e^{-c\sigma} \int e^{2c\sigma} Q^2(x)dx + \langle \mathcal{L}_c A, A \rangle \right] + O(e^{-2c\sigma})
\]

\[
= \alpha_c e^{-2c\sigma} + O(e^{-2c\sigma}).
\]

From the definition of \( G \), we have

\[
\langle L_c A, A' \rangle = -\omega \int Q^2(x)A(x)A'(x)dx = \omega \int A^2(x)Q(x)Q'(x)dx.
\]

For the first term, using \(- (Q'_c)'' + c^2 Q'_c = 3Q_c^2 Q'_c \) (obtained by differentiating the equation of \( Q_c \)) and the equation \( A \) in (6), we compute

\[
3 \int Q_c^2(x)Q'_c(x)A(x + \sigma)dx = \int Q'_c(x - \sigma)(-A''(x) + c^2 A(x))dx
\]

\[
= \omega \int Q'_c(x - \sigma) \left[ Q^2(x)A(x) + c\kappa e^{c\sigma} Q^2(x) \right] dx.
\]

Similarly as in the proof of (16), using (3) we observe

\[
\int Q'_c(x - \sigma)Q^2(x)A(x)dx = \kappa e^{-c\sigma} \int e^{c\sigma} Q^2(x)A(x)dx + O(e^{-c\sigma}),
\]

\[
\int Q'_c(x - \sigma)e^{c\sigma} Q^2(x)dx = \kappa e^{-c\sigma} \int e^{2c\sigma} Q^2(x)dx + O(e^{-c\sigma}).
\]

Moreover, it follows from (6) and the coercivity of the operator \( L_c \) that

\[
c\kappa \omega \int e^{c\sigma} Q^2(x)A(x)dx = \langle \mathcal{L}_c A, A \rangle > 0.
\]

Last, we check using the decay property of \( A \) in (7) and the condition on \( \theta \) that

\[
\int Q_c(x)Q'_c(x)A^2(x + \sigma)dx = O(e^{-c\sigma}), \quad \int Q'_c(x)A^3(x + \sigma)dx = O(e^{-c\sigma}).
\]

Using also (15) and \( \kappa^2 = 8 \), we find

\[
a = \frac{e^{-2c\sigma}}{2} \left[ \kappa^2 e^{-c\sigma} \int e^{2c\sigma} Q^2(x)dx + \langle \mathcal{L}_c A, A \rangle \right] + O(e^{-2c\sigma})
\]

\[
= \alpha_c e^{-2c\sigma} + O(e^{-2c\sigma}).
\]
On the other hand, using (6) and then integration by parts, it holds
\[ \langle L_c A, A' \rangle = \kappa \omega \int e^{cx} Q^2(x) A'(x) dx \]
\[ = -c^2 \kappa \omega \int e^{cx} Q^2(x) A(x) dx - 2c \kappa \omega \int e^{cx} Q(x) Q'(x) A(x) dx \]
\[ = -c \langle L_c A, A \rangle - 2c \kappa \omega \int e^{cx} Q(x) Q'(x) A(x) dx. \]

Thus, also using
\[ \int Q_c(x - \sigma) A(x) Q(x) Q'(x) dx = c e^{-c \sigma} \int e^{cx} Q(x) Q'(x) A(x) Q dx + O(e^{-c \sigma}) \]
and (16), we obtain \( b = -c \alpha e^{-2c \sigma} + O(e^{-2c \sigma}). \)

2.3. **Formal discussion.** Formally, the previous computations lead to the system
\[ \dot{\sigma}_1 = 2 \beta_1, \quad \dot{\beta}_1 = -\alpha_c e^{-2c \sigma}, \quad \sigma_2 = 2 \beta_2, \quad \dot{\beta}_2 = c \alpha_c e^{-2c \sigma}. \]
Recalling \( \sigma = \sigma_1 - \sigma_2 \) and \( \beta = \beta_1 - \beta_2 \), this gives
\[ \dot{\sigma} = -2(c + 1) \alpha_c e^{-2c \sigma}, \quad 2 \beta = \dot{\sigma}, \]
which admits the following solution
\[ \sigma(t) = \frac{1}{c} \log(\Omega_c t), \quad 2 \beta(t) = \frac{1}{c} \frac{\Omega_c}{e} e^{-c \sigma} \quad \text{where} \quad \Omega_c = [2c(c + 1) \alpha_c]^\frac{1}{2} > 0. \]

This justifies the existence of the regime (2) of Theorem 1.2. In particular, observe that the positive sign of the constant \( \alpha_c \) is responsible for the emergence of the special non-symmetric logarithmic regime. The phase parameters \( \gamma_1 \) and \( \gamma_2 \) are not essential for the dynamics and so we do not discuss them here.

2.4. **Decomposition around the approximate solution.** Let \( T_\infty \gg 1 \) to be fixed later and consider a solution \( (\hat{u}) \) of (coupled NLS) under the form
\[ \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} \text{ with } \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix}(T_\infty) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

Then, using the notation
\[ h(u, v) = (|u|^2 + \omega |v|^2) u \]
the function \( \begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} \) satisfies the system
\[ \begin{cases} i \partial_t \varepsilon + \partial_x^2 \varepsilon + h(U + \varepsilon, V + \eta) - h(U, V) + \mathcal{E}_U = 0 \\ i \partial_t \eta + \partial_x^2 \eta + h(V + \eta, U + \varepsilon) - h(V, U) + \mathcal{E}_V = 0 \end{cases} \]
(18)
The parameters \( \sigma_1, \sigma_2, \gamma_1, \gamma_2, \beta_1 \) and \( \beta_2 \) in the definition of \( (\hat{U}) \) are fixed by imposing the following orthogonality conditions
\[ \begin{cases} \langle \varepsilon, x_1 P \rangle = \langle \varepsilon, i \Lambda_1 P \rangle = \langle \varepsilon, i \partial_1 P \rangle = 0 \\ \langle \eta, x_2 R \rangle = \langle \eta, i \Lambda_2 R \rangle = \langle \eta, i \partial_2 R \rangle = 0 \end{cases} \]
(19)
and initial conditions
\[
\begin{align*}
\sigma_1(T_\infty) &= \frac{\sigma_\infty}{c+1}, \\
\sigma_2(T_\infty) &= \frac{c\sigma_\infty}{c+1}, \\
\beta_1(T_\infty) &= \frac{\beta_\infty}{c+1}, \\
\beta_2(T_\infty) &= \frac{c\beta_\infty}{c+1}, \\
\gamma_1(T_\infty) &= 0, \\
\gamma_2(T_\infty) &= 0,
\end{align*}
\] (20)
where \(\sigma_\infty\) is to be chosen later close to \(\frac{1}{c} \log(\Omega_c T_\infty)\) (see below (23)) and
\[
\beta_\infty = \frac{\Omega_c}{2c} e^{-c\sigma_\infty}.
\] (21)
Indeed, by a standard argument and the initial conditions (including \(\varepsilon(T_\infty) = \eta(T_\infty) = 0\)), the orthogonality conditions are equivalent to a first order differential system in the parameters \((\sigma_1, \sigma_2, \gamma_1, \gamma_2, \beta_1, \beta_2)\), which admits a unique local solution in the regime considered in this paper. See e.g. Lemma 2.7 in [4] for a detailed argument in the case of the (gKdV) equation, and Lemma 3.3 in the present paper for the corresponding estimates on the time derivatives of the parameters. For technical reasons, one can fix zero initial conditions on \(\gamma_1, \gamma_2\) as in (20), but the initial conditions on \(\sigma_1, \sigma_2, \beta_1\) and \(\beta_2\) have to depend on a parameter \(\sigma_\infty\) to be fixed later by a topological argument.

As in [20, 22, 26], the orthogonality conditions in (19) are related to (5). Using the conservation of masses and \(L^2\) sub-criticality, we avoid the modulation of the scaling parameters of the solitons (see [30] and the proof of Lemma 3.3).

3. Proof of Theorem 1.2.

3.1. Bootstrap bounds. Fix \(\theta_1, \theta_2\) and \(\theta_3\) such that \(1 < \theta_3 < \theta_2 < \theta_1 < \min \{\frac{1}{c}; 2\}\). Following Section 2.3, we work under the following bootstrap estimates, for \(1 \ll t \leq T_\infty\),
\[
\begin{align*}
\|\varepsilon\|_{H^1} + \|\eta\|_{H^1} &\leq t^{-\theta_1}, \\
|\beta - \frac{1}{2c t}| + |\beta_1 - \frac{1}{2c(c+1)t}| + |\beta_2 + \frac{1}{2(c+1)t}| &\leq t^{-\theta_3}, \\
|\sigma_1 - \frac{\log(\Omega_c t)}{c(c+1)}| + |\sigma_2 + \frac{\log(\Omega_c t)}{c+1}| &\leq t^{1-\theta_3}, \\
\left|\frac{e^{c\sigma}}{\Omega_c t} - 1\right| &\leq t^{1-\theta_2}.
\end{align*}
\] (22)
For consistency, the free parameter \(\sigma_\infty\) in (20) will have to be chosen such that
\[
\left|\frac{e^{c\sigma_\infty}}{\Omega_c T_\infty} - 1\right| \leq T_\infty^{1-\theta_2}.
\] (23)

Lemma 3.1. Let \(0 < c_1 \leq c_2\) and \(q \geq 0\). It holds, for \(\sigma > 1\),
\[
\int (1 + |x-\sigma|)^q e^{-c_1 |x-\sigma|} e^{-c_2 |x|} dx \lesssim \begin{cases} 
\sigma^{q+1} e^{-c_1 \sigma} & \text{if } c_1 = c_2 \\
\sigma^q e^{-c_1 \sigma} & \text{if } c_1 \neq c_2.
\end{cases}
\] (23)
Proof. We decompose
\[
\int (1 + |x - \sigma|)^q e^{-c_1|x - \sigma|} e^{-c_2|\sigma|} dx = e^{-c_1\sigma} \int_{-\infty}^{0} (1 + |x - \sigma|)^q e^{(c_1+c_2)x} dx
\]
\[+ e^{-c_1\sigma} \int_{0}^{\sigma} (1 + |x - \sigma|)^q e^{(c_2-c_1)x} dx + e^{c_1\sigma} \int_{\sigma}^{+\infty} (1 + |x - \sigma|)^q e^{(c_1+c_2)x} dx.\]
The result follows by integration.

Lemma 3.2. The following hold
\[
\|\partial_t P - i c^2 P\|_{L^2} \lesssim \left( |\dot{\gamma}_1| + |\dot{\beta}_1| |\sigma_1| + |\dot{\sigma}_1| \right),
\]
\[
\|\partial_t \varphi - i c^2 \varphi\|_{L^2} \lesssim \left( |\dot{\gamma}_1| + |\dot{\beta}_1| |\sigma_2| + |\dot{\sigma}_2| + |\dot{\sigma}| \right) e^{-c_\sigma},
\]
\[
\|\partial_t R - i R\|_{L^2} \lesssim \left( |\dot{\gamma}_2| + |\dot{\beta}_2| |\sigma_2| + |\dot{\sigma}_2| \right).
\]
Let $1 < \theta < \min \left\{ \frac{1}{2}; 2 \right\}$. The following hold
\[
\|F\|_{L^2} + \|F^\perp\|_{L^2} \lesssim e^{-2c_\sigma},
\]
\[
\|\partial_t F - ic^2 F\|_{L^2} + \|\partial_t F^\perp - ic^2 F^\perp\|_{L^2} \lesssim \left( |\dot{\gamma}_1| + |\dot{\beta}_1| |\sigma_1| + |\dot{\sigma}_1| + |\dot{\sigma}| \right) e^{-2c_\sigma},
\]
\[
\|G\|_{L^2} + \|G^\perp\|_{L^2} \lesssim e^{-c_\theta \sigma},
\]
\[
\|\partial_t G - i G\|_{L^2} + \|\partial_t G^\perp - i G^\perp\|_{L^2} \lesssim \left( |\dot{\gamma}_2| + |\dot{\beta}_2| |\sigma_2| + |\dot{\sigma}_2| + |\dot{\sigma}| \right) e^{-c_\theta \sigma}.\]

Proof. Estimates (24) are simple consequences of the definitions of $P$, $\varphi$ and $R$.

Proof of (25). Recall that $F(t, x) = F_1(t, x - \sigma_1(t))e^{i\Gamma_1(t, x)}$, where $F_1 = 3e^{-c_\alpha}Q_c^2 A(x + \sigma) + 3e^{-2c_\sigma}Q_c A^2(x + \sigma) + e^{-3c_\sigma} A^3(x + \sigma) - \omega e^{2c_\sigma} Q^2(x + \sigma)Q_c.$

Moreover, from (7) and Lemma 3.1, it holds
\[
\|Q_c^2 A(x + \sigma)\|_{L^2} + \|Q_c A^2(x + \sigma)\|_{L^2} \lesssim e^{-c_\sigma},
\]
and $\|e^{2c_\sigma} Q^2(x + \sigma)Q_c\|_{L^2} \lesssim e^{-2c_\sigma}$. (25)

Proof of (26). Note that
\[
\partial_t F - ic^2 F = i(\dot{\gamma}_1 + \dot{\beta}_1 \sigma_1) F + i\dot{\beta}_1 (x - \sigma_1) F - \sigma_1 \partial_x F_1(t, x - \sigma_1) e^{i\Gamma_1} + \partial_x F_1(t, x - \sigma_1) e^{i\Gamma_1},
\]
We see from the expression of $F_1$ and similar estimates that the following hold
\[
\| (\dot{\gamma}_1 + \dot{\beta}_1 \sigma_1) F_1 \|_{L^2} \lesssim (|\dot{\gamma}_1| + |\dot{\beta}_1| \sigma_1) e^{-2c_\sigma}, \quad \| x F_1 \|_{L^2} \lesssim e^{-2c_\sigma},
\]
\[
\| \partial_x F_1 \|_{L^2} \lesssim e^{-2c_\sigma}, \quad \| \partial_t F_1 \|_{L^2} \lesssim |\dot{\sigma}| e^{-2c_\sigma}.
\]
This proves estimate (26) for $F$.

Next, note that from the definition of $a$, we have
\[
\dot{a} = \frac{1}{2c} (\partial_t F - i c^2 F, \partial_t P) + \frac{1}{2c} (F, \partial_t \partial_t P - i c^2 \partial_t P).
\]

Thus, from the analogue of (24) for $\partial_t P$ and (25)-(26), we deduce
\[
|\dot{a}| \lesssim \left( |\dot{\gamma}_1| + |\dot{\beta}_1| \sigma_1| + |\dot{\sigma}_1| + |\dot{\sigma}| \right) e^{-2c_\sigma}.
\]

Estimate (26) for $F^\perp$ then comes from
\[
\partial_t F^\perp - ic^2 F^\perp = \partial_t F - ic^2 F + \dot{a} x_1 P + a \left[ \partial_t (x_1 P) - ic^2 (x_1 P) \right]
\]
and the analogue of (24) for $x_1 P$. 
Proof of (27). We rewrite $G(t, x) = G_2(x - \sigma_2(t))e^{i\Pi_2(t, \cdot)}$, where
\[ G_2 = \omega Q_2(x - \sigma)Q + 2\omega e^{-c\sigma}Q_2(x - \sigma)AQ + \omega e^{-2c\sigma}A^2 Q. \]
From Lemma 3.1 and the definition of $\theta$, we have
\[ \|Q_2^2(x - \sigma)Q\|_{L^2} \lesssim e^{-c\theta}, \quad \|Q_2(x - \sigma)AQ\|_{L^2} \lesssim e^{-c\sigma}. \]

Proof of (28). We have
\[ \partial_t G - iG = i(\dot{\gamma}_2 + \ddot{\beta}_2\sigma_2)G + \dot{\beta}_2(x - \sigma_2)G - \dot{\sigma}_2\partial_x G_2(t, x - \sigma_2)e^{i\Pi_2} + \partial_t G_2(t, x - \sigma_2)e^{i\Pi_2}. \]
As before, we use the following estimates to prove (28) for $G$
\[ \|\dot{\gamma}_2 + \ddot{\beta}_2\sigma_2)G\|_{L^2} \lesssim \|\dot{\gamma}_2\| + \|\ddot{\beta}_2\sigma_2\|e^{-c\theta}, \quad \|xG_2\|_{L^2} \lesssim e^{-c\theta}, \quad \|\partial_x G_2\|_{L^2} \lesssim |\dot{\sigma}|e^{-c\theta}. \]
The proof of (28) for $G^\perp$ follows from similar arguments and it is omitted. \qed

3.2. Modulation equations.

Lemma 3.3. Let $\theta_1 < \theta < \min \left\{ \frac{1}{2}, 2 \right\}$. It holds
\[ \|\langle \varepsilon, P \rangle\| \lesssim t^{-2} \log t, \quad \|\langle \eta, R \rangle\| \lesssim t^{-2\theta_1}, \quad (29) \]
\[ |\sigma_1 - 2\beta_1| + |\beta_2 - 2\beta_2| + |\gamma_1| + |\gamma_2| \lesssim t^{-\theta}, \quad (30) \]
\[ |\bar{m}_1| + |\bar{m}_2| + |\bar{m}_1^\perp| + |\bar{m}_2^\perp| \lesssim t^{-\theta}, \quad |\bar{m}_\varepsilon| \lesssim t^{-1}, \quad (31) \]
\[ |\dddot{\beta}_1 + a| + |\dddot{\beta}_2 + b| \lesssim t^{-1-\theta_1}. \quad (32) \]
Proof. Proof of (29). First, it follows from Lemma 3.1 and (22) that
\[ \|U\|_{L^2}^2 = \|Q_\varepsilon + e^{-c\sigma}A\varepsilon + \sigma\|_{L^2}^2 = \|Q_\varepsilon\|_{L^2}^2 + O(t^{-2} \log t). \]
We use the mass conservation for $u$ and $\varepsilon(T_\infty) = 0,$
\[ \|U + \varepsilon\|_{L^2}^2 = \|u(T_\infty)\|_{L^2}^2 = \|U(T_\infty)\|_{ L^2}^2 = \|Q_\varepsilon\|_{L^2}^2 + O(T_{\infty}^{-2} \log T_{\infty}), \]
and thus by (22),
\[ 2\langle \varepsilon, U \rangle = \|U + \varepsilon\|_{L^2}^2 - \|U\|_{L^2}^2 - \|\varepsilon\|_{L^2}^2 = O(t^{-2} \log t). \]
Last, using $|\langle \varepsilon, \varphi \rangle| \leq \|\varepsilon\|_{L^2}\|\varphi\|_{L^2} \lesssim t^{-1-\theta_1}$ and $2\langle \varepsilon, P \rangle = \langle \varepsilon, U \rangle - 2\langle \varepsilon, \varphi \rangle,$ we obtain $|\langle \varepsilon, P \rangle| \lesssim t^{-2} \log t$. The estimate on $\langle \eta, R \rangle$ follows directly from $\|\varepsilon\|_{L^2} = \|\varepsilon(T_\infty)\|_{L^2}.$

Proof of (30)-(31)-(32). We use the special choice of orthogonality conditions (19) as well as the relations (5). We refer to the proof of Lemma 7 in [20] for a similar argument. First, differentiating the second orthogonality in (19) and using (18),
\[ 0 = \frac{d}{dt}\langle \varepsilon, i\Lambda_1 P \rangle = -\langle i\partial_t \varepsilon, \Lambda_1 P \rangle + \langle \varepsilon, i\partial_t \Lambda_1 P \rangle \]
\[ = -\langle -\partial^2\varepsilon + c^2\varepsilon + h(U + \varepsilon, V + \eta) - h(U, V), \Lambda_1 P \rangle + \langle F, \Lambda_1 P \rangle - \langle \bar{m}_1 \cdot \bar{M}_1, \Lambda_1 P \rangle - \langle \bar{m}_\varepsilon \cdot \bar{M}_\varepsilon, \Lambda_1 P \rangle - \langle i\varepsilon, \partial_t (\Lambda_1 P) - i\varepsilon \Lambda_1 P \rangle. \]
We claim
\[ |\langle -\partial^2\varepsilon + c^2\varepsilon + h(U + \varepsilon, V + \eta) - h(U, V), \Lambda_1 P \rangle| \lesssim t^{-2} \log t. \quad (33) \]
Observe that
\[ h(U + \varepsilon, V + \eta) - h(U, V) = 2|U|^2\varepsilon + U^2\varepsilon + \omega |V|^2\varepsilon + 2\omega U\Re(V\bar{\eta}) + O(|\varepsilon|^2 + |\eta|^2). \]
By Lemma 3.1, \( \|V^2\Lambda_1 P\|_{L^2} \lesssim t^{-1}\log t \), \( \|U\Lambda_1 P\|_{L^2} \lesssim t^{-1} \), and thus

\[
\begin{align*}
|\langle -\partial_2^2\varepsilon + c^2\varepsilon + h(U + \varepsilon, V + \eta) - h(U, V), \Lambda_1 P \rangle & - \langle \varepsilon, -\partial_2^2(\Lambda_1 P) + c^2\Lambda_1 P + 3|P|^2\Lambda_1 P \rangle | \\
& \lesssim t^{-1}(\log t)(\|\varepsilon\|_{L^2}^2 + \|\eta\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2 + \|\eta\|_{L^2}^2) \lesssim t^{-1-\theta_1} \log t.
\end{align*}
\]

Using \( \mathcal{L}_+(AQ) = -2Q \) from (5) and \( \|\partial_2^2(\Lambda_1 P) - \partial_2^2(\Lambda_1 P)\|_{L^2} \lesssim |\beta_1| \lesssim t^{-1} \) (by analogy with the notation introduced in §2.1, we set \( \partial_2^2(\Lambda_1 P) = (AQ)^\nu(x-\sigma_1)e^{i\Gamma_1} \)) we see that

\[
\|[-\partial_2^2(\Lambda_1 P) + c^2\Lambda_1 P + 3|P|^2\Lambda_1 P] + 2c^2 P\|_{L^2} \lesssim t^{-1}.
\]

Thus, by (22) and (29), we obtain (33).

The estimate \( \|F, \Lambda_1 P\| \lesssim e^{-2c\sigma} \lesssim t^{-2} \) is clear from (25) and then (22). Next, using \( \langle P, \Lambda_1 P \rangle = \langle Q_c, AQ_c \rangle = \frac{1}{2}\|Q_c\|_{L^2}^2 = 2c \) and \( \langle iQ_c, AQ_c \rangle = \langle xQ_c, AQ_c \rangle = 0 \), we obtain

\[
-\langle \tilde{m}_1 \cdot \tilde{M}_1, \Lambda_1 P \rangle = -2c(\hat{\gamma}_1 + \hat{\beta}_1 \sigma_1 + \beta_1^2).
\]

Moreover, using Lemma 3.1,

\[
-\langle \tilde{m}_\varphi \cdot \tilde{M}_\varphi, \Lambda_1 P \rangle = -\langle \hat{\gamma}_1 + \hat{\beta}_1 \sigma_2 + \beta_2^2 \rangle \langle \varphi, \Lambda_1 P \rangle + \hat{\beta}_1 \langle x_2 \varphi, \Lambda_1 P \rangle
= \langle |\tilde{\gamma}_1| + |\tilde{\beta}_1| |\sigma_2| + \beta_2^2 \rangle O(\sigma^2 e^{-2c\sigma}).
\]

Last, using the analogue of (24) for \( \Lambda_1 P \), we have

\[
|\varepsilon, \partial_t(\Lambda_1 P) - ic^2\Lambda_1 P| \lesssim (|\tilde{\gamma}_1| + |\tilde{\beta}_1| |\sigma_1| + |\tilde{\sigma}_1 - 2\beta_1| + |\beta_1|)\|\varepsilon\|_{L^2}.
\]

The conclusion of these estimates is

\[
|\tilde{\gamma}_1| \lesssim t^{-2}\log t + t^{-1}|\tilde{\sigma}_1 - 2\beta_1| + |\tilde{\beta}_1| \log t.
\]

Proceeding similarly with the orthogonality condition \( \langle \eta, i\Lambda_2 R \rangle = 0 \), we check

\[
|\tilde{\gamma}_2| \lesssim t^{-\theta} + t^{-1}|\tilde{\sigma}_2 - 2\beta_2| + |\tilde{\beta}_2| \log t.
\]

Note that we again use \( \mathcal{L}_+(AQ) = -2Q \) and (29) for \( \eta \). The term \( t^{-\theta} \) comes from estimate of \( G \) in (27), which is to be compared with (25) for \( F \).

Next, differentiating the orthogonality conditions \( \langle \varepsilon, x_1 P \rangle = \langle \eta, x_2 R \rangle = 0 \), we find the relation \( \mathcal{L}_-(xQ) = -2Q' \) from (5) and last \( \langle \varepsilon, i\partial_1 P \rangle = \langle \eta, i\partial_2 R \rangle = 0 \), we find

\[
\begin{align*}
|\tilde{\sigma}_1 - 2\beta_1| & \lesssim t^{-1-\theta_1} \log t + t^{-1}(|\tilde{\gamma}_1| + |\tilde{\beta}_1| |\sigma_1| + |\tilde{\sigma}_2 - 2\beta_2|), \\
|\tilde{\sigma}_2 - 2\beta_2| & \lesssim t^{-1-\theta_1} \log t + t^{-1}(|\tilde{\gamma}_2| + |\tilde{\beta}_2| |\sigma_2| + |\tilde{\sigma}_1 - 2\beta_1|).
\end{align*}
\]

Note that for these estimates, we have also used \( \langle F, ix_1 P \rangle = 0 \) and \( \langle G, ix_2 R \rangle = 0 \).

Last, differentiating the orthogonality conditions \( \langle \varepsilon, i\partial_1 P \rangle = \langle \eta, i\partial_2 R \rangle = 0 \), we find the relation \( \mathcal{L}_+(Q') = 0 \) from (3) and \( \langle F^\perp, \partial_1 P \rangle = \langle G^\perp, \partial_2 R \rangle = 0 \), we check that

\[
|\tilde{\beta}_1 + a| + |\tilde{\beta}_2 + b| \lesssim t^{-1-\theta_1} + t^{-1}(|\tilde{\gamma}_1| + |\tilde{\sigma}_1 - 2\beta_1| + |\tilde{\gamma}_2| + |\tilde{\sigma}_2 - 2\beta_2|).
\]

The proof of (30)-(31)-(32) follows from the above estimates and (14).
3.3. Energy estimates. Let
\[ H(u, v) = \frac{1}{4} |u|^4 + \frac{1}{4} |v|^4 + \frac{\omega}{2} |u|^2|v|^2, \quad h(u, v) = (|u|^2 + \omega |v|^2)u. \]
and remark that
\[ d_1 H(U, V)(\varepsilon) = \frac{1}{2}(|U|^2 + \omega |V|^2)(U\varepsilon + \bar{U}\varepsilon) = \Re (h(U, V)\varepsilon), \]
\[ d_2 H(U, V)(\eta) = \frac{1}{2}(|V|^2 + \omega |U|^2)(V\eta + \bar{V}\eta) = \Re (h(V, U)\eta), \]
\[ d_1 h(U, V)(\varepsilon) = 2|U|^2 \varepsilon + U^2 \varepsilon + \omega |V|^2 \varepsilon, \quad d_2 h(U, V)(\eta) = \omega (V\bar{\eta} + \bar{V}\eta)U, \]
\[ \frac{1}{2} (\varepsilon, \eta)^T (d^2 h)(U, V)(\varepsilon, \eta) = 2\varepsilon \Re \langle U \varepsilon \rangle + U|\varepsilon|^2 + 2\omega \varepsilon \Re \langle V \bar{\eta} \rangle + \omega U|\eta|^2. \]
Consider the energy functional for \((\varepsilon, \eta)\)
\[ K(t, \varepsilon, \eta) = \frac{1}{2} \int \{ |\partial_x \varepsilon|^2 + |\partial_x \eta|^2 - 2[H(U + \varepsilon, V + \eta) - H(U, V)] \}
- d_1 H(U, V)(\varepsilon) - d_2 H(U, V)(\eta) \}
and the mass functionals for \(\varepsilon\) and \(\eta\)
\[ M = M_1 + M_2, \quad M_1(\varepsilon) = \frac{c^2}{2} \int |\varepsilon|^2, \quad M_2(\eta) = \frac{1}{2} \int |\eta|^2. \]
Let \(\chi : [0, +\infty) \rightarrow [0, +\infty)\) be a smooth non-increasing function satisfying \(\chi \equiv 1\) on \([0, \frac{1}{2}]\) and \(\chi \equiv 0\) on \([\frac{1}{2}, +\infty)\). Denote \(J = J_1 + J_2\) where, for \(j = 1, 2\),
\[ J_j(t, \varepsilon, \eta) = \beta_j \int [(\partial_x \varepsilon) \bar{\varepsilon} + (\partial_x \eta) \bar{\eta}] \chi_j \quad \text{where} \quad \chi_j(t, x) = \chi \left(\frac{|x - \sigma_j(t)|}{\log t}\right). \]
Last, we set
\[ S(t, \varepsilon, \eta) = \langle \varepsilon, F_1 \rangle + 2\beta(\varepsilon, \phi) + \langle \eta, G_1 \rangle \quad \text{where} \quad \phi = \partial_x \varphi - c\varphi. \]
Last, set
\[ W(t, \varepsilon, \eta) = K(t, \varepsilon, \eta) + M(t, \varepsilon, \eta) - J(t, \varepsilon, \eta) - S(t, \varepsilon, \eta). \]
We refer to [14, 16, 17, 19, 20, 22, 26] for similar energy functionals. However, the introduction of the correcting term \(S\) seems to be a previously unnoticed general improvement of the energy method in this context. See Section 5.

Under the bootstrap (22), we prove the following estimates.

Proposition 1. Let \(\theta_1 < \theta < \min\{\frac{1}{2}; 2\}\). It holds
\[ \|\varepsilon\|_{H^1}^2 + \|\eta\|_{H^1}^2 \lesssim W(t, \varepsilon, \eta) + Ct^{-2\theta}, \quad (34) \]
and
\[ \left| \frac{d}{dt} W(t, \varepsilon, \eta) \right| \lesssim t^{-1-2\theta}(\log t)^{-1}. \quad (35) \]

Proof of Proposition 1. Step 1. The coercivity property (34) is a consequence of the coercivity property around one solitary wave in Lemma 1.3, the orthogonality relations (19)-(29) and the positivity of \(L_c\). It also involves a localization argument similar to the proof of Lemma 4.1 in [19] for the scalar case.

Note that by (22),
\[ |J(t, \varepsilon, \eta)| \lesssim t^{-1}(\|\varepsilon\|_{H^1}^2 + \|\eta\|_{H^1}^2) \]

and by (25) and (27),

\[ |S(t, \varepsilon, \eta)| \lesssim t^{-\theta} (\|\varepsilon\|_{H^1} + \|\eta\|_{H^1}). \]

Next, we see that the following terms in the functional \( K \) are easily controlled

\[ \int (|P\varphi| + |\varphi|^2)\varepsilon|^2 + \int |U|\varphi||V\eta| \lesssim t^{-1} \left( \|\varepsilon\|^2_{H^1} + \|\eta\|^2_{H^1} \right). \]

Moreover, cubic and higher order terms in \( \varepsilon \) or \( \eta \) are of order \( t^{-\theta_1} \left( \|\varepsilon\|^2_{H^1} + \|\eta\|^2_{H^1} \right) \).

Therefore, we are reduced to consider the following two decoupled functionals

\[ W_1 = \int \{ |\partial_\varepsilon \varepsilon|^2 + c^2 |\varepsilon|^2 - |P|^2 |\varepsilon|^2 - 2|\Re(P\varepsilon)|^2 - \omega |\varepsilon|^2 |\varepsilon|^2 \}, \]

\[ W_2 = \int \{ |\partial_\varepsilon \varepsilon|^2 + |\varepsilon|^2 - |R|^2 |\varepsilon|^2 - 2|\Re(R\varepsilon)|^2 - \omega |\varepsilon|^2 |\varepsilon|^2 \}. \]

We focus on the coercivity property for \( W_1 \), the case of \( W_2 \) is similar.

Denote \( \Phi : \mathbb{R} \to \mathbb{R} \) an even function of class \( C^2 \) such that

\[ \Phi \equiv 1 \text{ on } [0, 1], \quad \Phi \equiv e^{-x} \text{ on } [2, +\infty], \quad e^{-x} \leq \Phi(x) \leq e^{-3x}, \quad \Phi' \leq 0 \text{ on } \mathbb{R}. \]

Let \( B > 1 \) and \( \Phi_B(x) = \Phi(x/B) \). We claim that for \( B \) large enough, there exists \( \mu_1 > 0 \), such that for any \( \tilde{\varepsilon} \) satisfying \( \langle \varepsilon, Q \rangle = \langle \tilde{\varepsilon}, xQ \rangle = \langle \tilde{\varepsilon}, i\Lambda Q \rangle = 0 \), and any \( \varepsilon \), it holds

\[ \mathcal{N}_1(\tilde{\varepsilon}) := \int \Phi_B \left\{ |\partial_\varepsilon \varepsilon|^2 + |\varepsilon|^2 - Q^2 |\varepsilon|^2 - 2|\Re(Q\varepsilon)|^2 \right\} \geq \mu_1 \int \Phi_B (|\partial_\varepsilon \varepsilon|^2 + |\varepsilon|^2), \]

\[ \mathcal{N}_2(\tilde{\varepsilon}) := \int \Phi_B \left\{ |\partial_\varepsilon \varepsilon|^2 + c^2 |\varepsilon|^2 - \omega Q^2 |\varepsilon|^2 \right\} \geq \mu_1 \int \Phi_B (|\partial_\varepsilon \varepsilon|^2 + |\varepsilon|^2). \]

Setting \( z = \tilde{\varepsilon} \Phi_B^2 \) and following the proof of Claim 8 in [19], the coercivity of \( \mathcal{N}_1 \) follows from (i) of Lemma 1.3 applied to the function \( z \). A similar localization argument, using the coercivity property of \( \mathcal{L}_c \) proves the estimate for \( \mathcal{N}_2(\tilde{\varepsilon}) \) without any orthogonality condition on \( \tilde{\varepsilon} \). This is where our proof needs the condition (4).

Using these estimates with \( \varepsilon \) and \( \tilde{\varepsilon} \) defined by the relations \( \varepsilon = c\tilde{\varepsilon} (c(x - \sigma_1)) e^{it_1} \) and \( \varepsilon = \tilde{\varepsilon}(x - \sigma_2)e^{it_2} \), using the orthogonality conditions (19) and the almost orthogonality relation (29), we obtain the estimate \( \|\varepsilon\|^2_{H^1} \lesssim W_1 + t^{-4}(\log t)^2 \).

**Step 2.** Time variation of the energy. Denote

\[ K_1 = h(U + \varepsilon, V + \eta) - h(U, V) - d_1 h(U, V)(\varepsilon) - d_2 h(U, V)(\eta), \]

\[ K_2 = h(V + \eta, \varepsilon) - h(V, U) - d_1 h(V, U)(\eta) - d_2 h(V, U)(\varepsilon), \]

so that

\[ K_1 = \frac{1}{2} (\varepsilon, \eta)^T (d^2 h)(U, V)(\varepsilon, \eta) + O(|\varepsilon|^3 + |\eta|^3), \]

\[ K_2 = \frac{1}{2} (\eta, \varepsilon)^T (d^2 h)(V, U)(\eta, \varepsilon) + O(|\varepsilon|^3 + |\eta|^3). \]

We prove the following estimate

\[ \frac{d}{dt} [K(t, \varepsilon, \eta)] = 2\beta_1 \langle \partial_\varepsilon U, K_1 \rangle + 2\beta_2 \langle \partial_\eta V, K_2 \rangle - c^2 \langle iU, \varepsilon \rangle - \langle iV, K_2 \rangle - \langle iD_\varepsilon K, \varepsilon \rangle - \langle iD_\eta K, \varepsilon \rangle + O(t^{-1-2\theta_1}(\log t)^{-1}). \] (36)

The time derivative of \( t \mapsto K(t, \varepsilon(t), \eta(t)) \) splits into three parts

\[ \frac{d}{dt} [K(t, \varepsilon, \eta)] = D_\varepsilon K(t, \varepsilon, \eta) + \langle D_\varepsilon K(t, \varepsilon, \eta), \partial_\varepsilon \rangle + \langle D_\eta K(t, \varepsilon, \eta), \partial_\eta \rangle, \]
where $D_t$ denotes the differentiation of $K$ with respect to $t$, and $D_\varepsilon, D_\eta$ denote the differentiation of $K$ with respect to, respectively, $\varepsilon$ and $\eta$. In particular, it holds $D_t K = -\langle \partial_t U, K_1 \rangle - \langle \partial V, K_2 \rangle$.

We claim
\[
\partial_t U = ic^2U - 2\beta_1 \partial_x U + O_{H^1}(t^{-\theta}),
\]
\[
\partial_t V = iV - 2\beta_2 \partial_x V + O_{H^1}(t^{-\theta}).
\] (37)

Indeed, from the definition of $U$
\[
\partial_t U = i\partial_x (U^2) - 2\beta_1 \partial_x U - (\partial_x - i)P + i(\dot{\gamma}_1 + \dot{\beta}_1 \gamma_1)P + i\dot{\beta}_1 x_1 P
\]
\[- (\partial_x - i)\partial_x \varphi + i(\dot{\gamma}_1 + \dot{\beta}_1 \varphi + i\varphi) + i\dot{\beta}_1 x_2 \varphi.
\]

Thus, using (30) and (32), we obtain (37) for $U$. The proof for $V$ is similar.

Using (37) and (22), we obtain
\[
D_t K(t, \varepsilon, \eta) = 2\beta_1 (\partial_x U, K_1) + 2\beta_2 (\partial_x V, K_2) - c^2 (iU, K_1) - (iV, K_2) + O(t^{-\theta - 2\theta_1}).
\]

Next, we observe
\[
D_t K(t, \varepsilon, \eta) = -\partial_x^2 \varepsilon - h(U + \varepsilon, V + \eta) + h(U, V)
\]
so that the equation of $\varepsilon$ in (18) rewrites
\[
\partial_t \varepsilon - D_\varepsilon K(t, \varepsilon, \eta) + \mathcal{E}_U = 0
\]
and thus
\[
\langle D_t K(t, \varepsilon, \eta), \partial_t \varepsilon \rangle = -i \langle D_\varepsilon K(t, \varepsilon, \eta), \mathcal{E}_U \rangle.
\]

Similarly,
\[
\langle D_\eta K(t, \varepsilon, \eta), \partial_\eta \varepsilon \rangle = -i \langle D_\eta K(t, \varepsilon, \eta), \mathcal{E}_V \rangle.
\]

We have proved (36).

**Step 3.** Time variation of the total mass. We claim
\[
\frac{d}{dt} [M(\varepsilon, \eta)] = c^2 \langle iU, K_1 \rangle + \langle iV, K_2 \rangle - \langle ic^2 \varepsilon, \mathcal{E}_U \rangle - \langle i\eta, \mathcal{E}_V \rangle.
\] (38)

By integration by parts, we have $\partial_t^2 \varepsilon = 0$, so $M(\varepsilon)$ is constant, and
\[
\frac{d}{dt} [M_1(\varepsilon)] = c^2 \langle \partial_t \varepsilon, \varepsilon \rangle = -c^2 \langle i\varepsilon, h(U + \varepsilon, V + \eta) - h(U, V) \rangle - c^2 \langle i\varepsilon, \mathcal{E}_U \rangle.
\]

We claim the following identity
\[
\langle iU, K_1 \rangle + \langle i\varepsilon, h(U + \varepsilon, V + \eta) - h(U, V) \rangle = 0.
\] (39)

Indeed, since $h(u, v)\pi$ is real, for all $\theta \in \mathbb{R}$, it holds
\[
\langle i(U + \theta \varepsilon), h(U + \theta \varepsilon, V + \theta \eta) \rangle = 0.
\]

Differentiating with respect to $\varepsilon$ and taking $\theta = 0$, we obtain
\[
\langle i\varepsilon, h(U, V) \rangle + i \langle U, d_1 h(U, V)(\varepsilon) \rangle + i \langle U, d_2 h(U, V)(\varepsilon) \rangle = 0.
\]

Moreover, with $\theta = 0$ and $\theta = 1$
\[
\langle iU, h(U, V) \rangle = 0, \quad \langle i(U + \varepsilon), h(U + \varepsilon, V + \eta) \rangle = 0.
\]

We see that (39) follows from combining these identities.

This yields $\frac{d}{dt} M_1 = c^2 \langle iU, K_1 \rangle - c^2 \langle i\varepsilon, \mathcal{E}_U \rangle$. Computing also $\frac{d}{dt} M_2$, we obtain (38).

**Step 4.** Time variation of the localized momentum. We claim
\[
\frac{d}{dt} [J(t, \varepsilon, \eta)] = 2\beta_1 (\partial_x U, K_1) + 2\beta_2 (\partial_x V, K_2) + O(t^{-1 - 2\theta_1}(\log t)^{-1}).
\] (40)
By direct computation, 
\[
\frac{d}{dt} [J_1(t, \varepsilon, \eta)] = \beta_1 \Im \int [(\partial_x \varepsilon) \chi + (\partial_x \eta) \eta_1] + \beta_1 \Im \int [(\partial_x \varepsilon) \chi + (\partial_x \eta) \eta] \partial_t \chi_1 \\
+ \beta_1 (\partial_t \varepsilon, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1) + \beta_1 (\partial_t \eta, 2\chi_1 \partial_x \eta + \eta \partial_x \chi_1).
\]
By (22) and (32), we have 
\[
\left| \beta_1 \Im \int [(\partial_x \varepsilon) \chi + (\partial_x \eta) \eta] \partial_t \chi_1 \right| \lesssim t^{-2} (\|\varepsilon\|^2_{H^1} + \|\eta\|^2_{H^1}) \lesssim t^{-2-2\theta_1}.
\]
By direct computations, 
\[
\partial_t \chi_1(t, x) = -\left[ \frac{\dot{\sigma}_j}{t} \frac{x - \sigma_j}{|x - \sigma_j|^2} + \frac{|x - \sigma_j|}{t|\log(t)|^2} \right] \chi' \left( \frac{|x - \sigma_j|}{\log(t)} \right)
\]
and so by (22), (30) and the properties of \( \chi', |\partial_t \chi_1| \lesssim t^{-1}(\log t)^{-1} \). It follows that 
\[
\left| \beta_1 \Im \int [(\partial_x \varepsilon) \chi + (\partial_x \eta) \eta] \partial_t \chi_1 \right| \lesssim t^{-2}(\log t)^{-1} (\|\varepsilon\|^2_{H^1} + \|\eta\|^2_{H^2}) \lesssim t^{-2-2\theta_1}(\log t)^{-1}.
\]
Next, using the equation (18) 
\[
\langle \partial_t \varepsilon, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle = -\langle \partial_x^2 \varepsilon, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle \\
- \langle h(U + \varepsilon, V + \eta) - h(U, V), 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle \\
- \langle \mathcal{E}_U, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle.
\]
Integrating by parts, we have 
\[
-\langle \partial_x^2 \varepsilon, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle = \int |\partial_x \varepsilon|^2 \partial_x \chi_1 - \frac{1}{2} \int |\varepsilon|^2 \partial_x^2 \chi_1.
\]
Since \( |\partial_x \chi_1| \lesssim (\log t)^{-1} \) and \( |\partial_x^2 \chi_1| \lesssim (\log t)^{-3} \), from (22), we have 
\[
|\beta_1 (\partial^2_x \varepsilon, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1)| \lesssim t^{-1-2\theta_1}(\log t)^{-1}.
\]
For the term containing \( \mathcal{E}_U \), we use (10), (22), (25) and (31), 
\[
\left| \beta_1 \langle \mathcal{E}_U, 2\chi_1 \partial_x \varepsilon + \varepsilon \partial_x \chi_1 \rangle \right| \lesssim t^{-1-\theta} \|\varepsilon\|_{H^1} \lesssim t^{-1-2\theta_1}(\log t)^{-1}.
\]
Then, we estimate, using \( |\partial_x \chi_1| \lesssim (\log t)^{-1} \), 
\[
|\langle h(U + \varepsilon, V + \eta) - h(U, V), \varepsilon \partial_x \chi_1 \rangle| \lesssim (\log t)^{-1} (\|\varepsilon\|^2_{H^1} + \|\eta\|^2_{H^2}) \lesssim t^{-2\theta_1}(\log t)^{-1}.
\]
Collecting the above estimates, we obtain 
\[
\frac{d}{dt} [J_1(t, \varepsilon, \eta)] = -2\beta_1 \chi_1 \partial_x \varepsilon, h(U + \varepsilon, V + \eta) - h(U, V) \\
- 2\beta_1 \chi_1 \partial_x \eta, h(V + \eta, U + \varepsilon) - h(V, U) + O(t^{-1-2\theta_1}(\log t)^{-1}).
\]
We complete the proof of (40) by showing the following 
\[
\langle \partial_x U, K_1 \rangle + \chi_1 \partial_x \varepsilon, h(U + \varepsilon, V + \eta) - h(U, V) \\
+ \chi_1 \partial_x \eta, h(V + \eta, U + \varepsilon) - h(V, U) = O(t^{-2\theta_1}(\log t)^{-1}).
\]
First, we prove the identity 
\[
\langle \partial_x U, K_1 \rangle + \langle \partial_x \varepsilon, h(U + \varepsilon, V + \eta) - h(U, V) \\
+ \langle \partial_x \eta, h(V + \eta, U + \varepsilon) - h(V, U) \rangle = 0.
\]
Indeed, we have
\[
\langle \partial_x u, h(u, v) \rangle + \langle \partial_x v, h(v, u) \rangle = \int \partial_x [H(u, v)] = 0.
\]
Applying this to \( u = U + \theta \varepsilon \) and \( v = V + \theta \eta \), we have that for all \( \theta \in \mathbb{R} \)
\[
\langle \partial_x (U + \theta \varepsilon), h(U + \theta \varepsilon, V + \theta \eta) \rangle + \langle \partial_x (V + \theta \eta), h(V + \theta \eta, U + \theta \varepsilon) \rangle = 0.
\]
Taking the derivative with respect to \( \theta \) at \( \theta = 0 \), we obtain
\[
\langle \partial_x \varepsilon, h(U, V) \rangle + \langle \partial_x \eta, h(V, U) \rangle + \langle \partial_x U, d_1 h(U, V)(\varepsilon) \rangle + \langle \partial_x U, d_2 h(U, V)(\eta) \rangle + \langle \partial_x V, d_1 h(V, U)(\eta) \rangle + \langle \partial_x V, d_2 h(V, U)(\varepsilon) \rangle = 0.
\]
Moreover, using the above identity with \( \theta = 0 \) and \( \theta = 1 \), we have
\[
\langle \partial_x (U + \varepsilon), h(U + \varepsilon, V + \eta) \rangle + \langle \partial_x (V + \eta), h(V + \eta, U + \varepsilon) \rangle = 0.
\]
Gathering these identities, we obtain (42).

We apply identity (42) to \( \chi_1^{\frac{3}{4}} U, \chi_1^{\frac{3}{4}} V, \lambda_1^{\frac{3}{4}} \varepsilon \) and \( \lambda_1^{\frac{3}{4}} \eta \). Recall that \( |\partial_x \chi_1| \lesssim (\log t)^{-1} \) and also note that by the definition of \( \chi \), \( |\chi_1 V| + (1 - \chi_1) |\partial_x U| \lesssim (\log t)^{-1} \). In particular, this shows that
\[
\langle |\chi_1^{\frac{3}{4}} U, K_1 \chi_1^{\frac{3}{4}} V, K_1 \rangle | - \langle |\chi_1^{\frac{3}{4}} U, \chi_1^{\frac{3}{4}} V, \chi_1 \rangle \rangle = O(t^{-2\theta_{\varepsilon}} (\log t)^{-1}),
\]
\[
\langle |\chi_1^{\frac{3}{4}} U, \chi_1^{\frac{3}{4}} V, \chi_1 \rangle | - \langle |\chi_1^{\frac{3}{4}} U, \chi_1^{\frac{3}{4}} V, \chi_1 \rangle \rangle = O(t^{-2\theta_{\varepsilon}} (\log t)^{-1}),
\]
and
\[
\langle |\chi_1^{\frac{3}{4}} U, \chi_1^{\frac{3}{4}} V, \chi_1 \rangle | - \langle |\chi_1^{\frac{3}{4}} U, \chi_1^{\frac{3}{4}} V, \chi_1 \rangle \rangle = O(t^{-2\theta_{\varepsilon}} (\log t)^{-1}).
\]
This proves (41) and then (40), the computations for \( J_2 \) being identical.

**Step 5.** Additional correction terms. We claim
\[
\frac{d}{dt} \langle S(t, \varepsilon, \eta) \rangle = -i(\partial_0 D_\varepsilon \mathbf{K} + c^2 \varepsilon), F^\perp + 2i \beta \phi - i(\partial_0 D_\eta \mathbf{K} + \eta), G^\perp \rangle + O(t^{-(1 + \theta_{\varepsilon} + \theta_1)}).
\]
(43)

We compute, using (18),
\[
\frac{d}{dt} \langle \varepsilon, F^\perp \rangle = -i(\partial_0 D_\varepsilon \mathbf{K} + c^2 \varepsilon), F^\perp \rangle - \langle \mathcal{E}_U, i F^\perp \rangle + \langle \varepsilon, \partial_t F^\perp - ic^2 F^\perp \rangle.
\]
From (12) and \( F^\perp e^{-i t^1}, i \in \mathbb{R} \), it follows that \( \langle \tilde{m}_\varepsilon \cdot \tilde{M}_1, i F^\perp \rangle = 0 \). One also observes that
\[
\langle \varepsilon, \partial_t F^\perp - ic^2 F^\perp \rangle = O(t^{-5} (\log t)^2) = O(t^{-(1 + 2\theta_{\varepsilon})}),
\]
where we have used (30) and (from Lemma 3.1 and the definitions of \( F^\perp \) and \( \varepsilon \))
\[
|\langle \varepsilon, F^\perp \rangle| + |\langle \partial_0 \varepsilon, F^\perp \rangle| \lesssim t^{-4} (\log t)^2.
\]
(44)

Since \( \langle F^\perp, i F^\perp \rangle = 0 \), it follows that \( \langle \mathcal{E}_U, i F^\perp \rangle = O(t^{-(1 + 2\theta_{\varepsilon})}) \). Last, it follows from (26), (30) and (22) that
\[
|\langle \varepsilon, \partial_t F^\perp - ic^2 F^\perp \rangle| \lesssim t^{-3 - \theta_{\varepsilon}} \lesssim t^{-1 - \theta_{\varepsilon} - \theta_{\varepsilon}}.
\]
Thus, using (13),
\[
\frac{d}{dt} \langle \varepsilon, F^\perp \rangle = -i(\partial_0 D_\varepsilon \mathbf{K} + c^2 \varepsilon), F^\perp \rangle + O(t^{-(1 + \theta_{\varepsilon} - \theta_{\varepsilon})}).
\]
From (28) and similar estimates, we also obtain
\[ \frac{d}{dt} \langle \eta, G^\perp \rangle = -i(D_\eta K + \eta) + O(t^{-1-\theta_1-\theta}). \]

Finally, we compute
\[ \frac{d}{dt} [2\beta \langle \varepsilon, i\phi \rangle] = 2\beta \langle \partial_t \varepsilon, i\phi \rangle + 2\beta \langle \partial_\varepsilon \varepsilon, i\phi \rangle + 2\beta \langle \varepsilon, i\partial_\varepsilon \phi \rangle. \]

The first term is estimated \( |\beta \langle \varepsilon, i\phi \rangle| \lesssim t^{-3} \|\varepsilon\|_{H^1} \lesssim t^{-3-\theta_1} \) using (32). Then, using (18),
\[ \langle \partial_\varepsilon \varepsilon, i\phi \rangle + \langle \varepsilon, i\partial_\varepsilon \phi \rangle = -i\langle D_\varepsilon K + c^2 \varepsilon, i\phi \rangle + \langle \mathcal{E}_U, \phi \rangle - i\varepsilon - i\partial_\varepsilon \phi - i\varepsilon^2 \phi. \]

From (44), \( |\beta \langle F^\perp, \phi \rangle| \lesssim t^{-5} (\log t)^2 \lesssim t^{-1-2\theta}. \) From (31), the expression of \( \bar{m}_1^a \cdot \bar{M}_1 \) and Lemma 3.1,
\[ |\beta \langle \bar{m}_1^a \cdot \bar{M}_1, \phi \rangle| \lesssim t^{-1} |\bar{m}_1^a| (|\langle \varepsilon, \phi \rangle| + |\langle x_1 \varepsilon, \phi \rangle|) \lesssim t^{-3-\theta} (\log t)^2 \lesssim t^{-1-\theta_1} \]

Next, from (30), the expression of \( \bar{m}_\varepsilon \cdot \bar{M}_\varepsilon \) and Lemma 3.1,
\[ |\beta \langle \bar{m}_\varepsilon \cdot \bar{M}_\varepsilon, \phi \rangle| \lesssim t^{-1} \left( |\bar{m}_1 + \bar{m}_2 + i| |\langle \phi, \phi \rangle| + |\bar{a} + \bar{a}_1| |\langle x_2 \phi, \phi \rangle| \right) \lesssim t^{-3-\theta} \lesssim t^{-1-\theta_1}. \]

Last, using (24) and (30),
\[ |\beta \langle i\varepsilon, \partial_\varepsilon \phi - i\varepsilon \phi \rangle| \lesssim t^{-3} |\varepsilon|_{L^2} \lesssim t^{-3-\theta_1}. \]

Estimate (43) is now proved.

**Step 6.** Conclusion. Combining the estimates (36), (38), (40), (43) and using the decompositions of \( \mathcal{E}_U \) and \( \mathcal{E}_V \) in (13), we have obtained
\[ \frac{d}{dt} \mathbf{W}(t, \varepsilon, \eta) = \langle i(D_\eta K + c^2 \varepsilon), \bar{m}_1^a \cdot \bar{M}_1 \rangle + \langle i(D_\eta K + c^2 \varepsilon), 2i\beta \phi + \bar{m}_\varepsilon \cdot \bar{M}_\varepsilon \rangle + \langle i(D_\eta K + \eta), \bar{m}_2^b \cdot \bar{M}_2 \rangle + O(t^{-1-2\theta_1} (\log t)^{-1}). \]

We claim
\[ |\langle i(D_\eta K + c^2 \varepsilon), \bar{m}_1^a \cdot \bar{M}_1 \rangle| \lesssim t^{-(1+\theta_1+\theta)}. \] (45)

Indeed, following the proof of (33), using Lemma 3.1, the relations (5), (22) and the third orthogonality condition in (19), it holds
\[ |\langle -\partial_\varepsilon^2 \varepsilon + c^2 \varepsilon - h(U + \varepsilon, V + \eta) + h(U, V), \partial_1 P \rangle| \lesssim t^{-(1+\theta_1)}, \]
\[ |\langle -\partial_\varepsilon^2 \varepsilon + c^2 \varepsilon - h(U + \varepsilon, V + \eta) + h(U, V), iP \rangle| \lesssim t^{-(1+\theta_1)}, \]
\[ |\langle -\partial_\varepsilon^2 \varepsilon + c^2 \varepsilon - h(U + \varepsilon, V + \eta) + h(U, V), ix_1 P \rangle| \lesssim t^{-(1+\theta_1)} \log t. \]

Thus, (45) follows from (31) and (32). Similarly,
\[ |\langle i(D_\eta K + \eta), \bar{m}_2^b \cdot \bar{M}_2 \rangle| \lesssim t^{-(1+\theta_1+\theta)}. \]

Finally, we remark that from the explicit expression of \( \bar{m}_\varepsilon \cdot \bar{M}_\varepsilon \) and (30)
\[ \|2i\beta \phi + \bar{m}_\varepsilon \cdot \bar{M}_\varepsilon \|_{H^1} \lesssim t^{-1-\theta}, \]
which implies by integration by parts and then (22)
\[ |\langle i(D_\eta K + c^2 \varepsilon), 2i\beta \phi + \bar{m}_\varepsilon \cdot \bar{M}_\varepsilon \rangle| \lesssim t^{-1-\theta_1-\theta}. \]

The proof of Proposition 1 is complete. \( \Box \)
3.4. Bootstrap argument.

**Proposition 2.** There exists \( T_0 > 1 \) large enough and for any \( T_\infty \geq T_0 \), there exists \( \sigma_\infty \) satisfying (23) such that the solution \( (\psi) \) of (coupled NLS) corresponding to initial data \( (\psi)(T_\infty) \) at \( t = T_\infty \) with parameters chosen as in (20)-(21) admits a decomposition (17)-(19) which satisfies (22) on \([0, T_\infty]\). Moreover, \(|\gamma_1| + |\gamma_2| \lesssim t^{1-\theta_1}\) on \([0, T_\infty]\).

**Proof.** For \( T_0 \) large enough, for any \( T_\infty \geq T_0 \) and any \( \sigma_\infty \) satisfying (23), we define \( T_* = T_\ast(T_\infty, \sigma_\infty) = \inf \{t \in [T_0, T_\infty] \mid (22) \text{ holds on } [t, T_\infty]\} \). We prove by contradiction that, provided \( T_\ast = T_0 \). We work only on the time interval \([T_\ast, T_\infty]\) on which the bootstrap estimates (22) hold.

First, we strictly improve the estimates of \( \varepsilon \) and \( \eta \) in (22). Indeed, integrating (35) on \([t, T_\infty]\) and using (34), it holds
\[
\|\varepsilon\|_{H^1}^2 + \|\eta\|_{H^1}^2 \lesssim t^{-2\theta_1} (\log t)^{-1},
\]
which strictly improves the estimate in (22) for large \( t \).

Next, we close the estimates on \( \beta_1, \beta_2 \) and \( \beta \) in (22). Using the estimate of \( \sigma \) in (22), (32), (14) and the expression of \( \Omega_c \), it holds
\[
\left| \dot{\beta}_1 + \frac{1}{2c(c+1)t^2} \right| \lesssim t^{1-\theta_2}.
\]
At \( T_\infty \), we remark that by (21) and (23),
\[
\beta_\infty - \frac{1}{2cT_\infty} \lesssim T^{-\theta_2} \quad \text{and so} \quad \left| \beta_1(T_\infty) - \frac{1}{2c(c+1)T_\infty} \right| \lesssim T^{-\theta_2}.
\]
Integrating on \([t, T_\infty]\) and using (20) for \( \beta_1 \), we obtain
\[
\left| \beta_1 - \frac{1}{2c(c+1)t} \right| \lesssim t^{-\theta_2},
\]
which strictly improves (22) for \( \beta_1 \) provided that \( t \) is large enough. Improving the estimate for \( \beta_2 \) (and then \( \beta \)) is similar.

Then, using (30), we find
\[
\left| \dot{\sigma}_1 - \frac{1}{c(c+1)t} \right| \lesssim t^{-\theta_2}.
\]
Integrating on \([t, T_\infty]\), using (20) and (23) we obtain
\[
\left| \sigma_1 - \log(\Omega_c t) \right| \lesssim t^{1-\theta_2},
\]
which strictly improves the estimate in (22). The estimate on \( \sigma_2 \) is improved similarly.

We only have to improve the estimate on \( \sigma \) to finish the bootstrap argument. This is where we need to argue by contradiction (see [5] for a similar argument). Using (30), (32) and (14), it holds, on the interval \([T_\ast, T_\infty]\),
\[
\left| \dot{\sigma} - 2\beta \right| \lesssim t^{-\theta_1} \quad \text{and} \quad \left| \ddot{\beta} + (1 + c)\alpha e^{-2\sigma} \right| \lesssim t^{-1-\theta_1}.
\]
Set \( g = \beta^2 - \frac{(1+c)\alpha e^{-2\sigma}}{2c} \), so that by the above estimates and (21) it holds
\[
\dot{g} = 2\beta^2 + (1 + c)\alpha e^{-2\sigma} = O(t^{-2-\theta_1}) \quad \text{and} \quad g(T_\infty) = 0.
\]
By integration on \([t, T_\infty]\), this yields
\[
\left| \beta^2 - \frac{(1+c)\alpha_c}{2c} e^{-2\sigma} \right| \lesssim t^{-1-\theta_1} \quad \text{and so} \quad \left| 2\beta - \frac{\alpha_c}{c} e^{-\sigma} \right| \lesssim t^{-\theta_1}.
\]
Define
\[
\zeta(t) = \frac{e^{\sigma t}}{\Omega_c} \quad \text{and} \quad \xi(t) = \left( \frac{\zeta(t)}{t} - 1 \right)^2.
\]
The previous estimates imply
\[
|\dot{\zeta}(t) - 1| \lesssim t^{1-\theta_1}.
\]
Assume for the sake of contradiction that for all \(\zeta_t \in [-1,1]\), the choice
\[
\zeta(T_\infty) = T_\infty + \zeta t^{2-\theta_2}
\]
leads to \(T_\ast \in (T_0, T_\infty]\). By a continuity argument, this means that the bootstrap estimates are reached at \(T_\ast\). Since all estimates in (22) except the one on \(\sigma\), have been strictly improved on \([T_\ast, T_\infty]\), this yields
\[
\left| \frac{e^{\sigma(T_\ast)}}{\Omega_c T_\ast} - 1 \right| = T_\ast^{1-\theta_2}.
\]
Following the argument of [5], we remark that for any \(t \in [T_\ast, T_\infty]\) satisfying (47), using (46) and \(\theta_2 < \theta_1\), it holds (taking \(T_0\) large enough)
\[
\dot{\zeta}(t) = 2(\dot{\zeta}(t) - 1)(\zeta(t) - t)t^{-2} - 2(\zeta(t) - t)^2 t^{-3} = -2 t^{1-2\theta_1}(1 + O(t^{\theta_2 - \theta_1})) < 0.
\]
This transversality condition implies that \(T_\ast\) is a continuous function of \(\sigma_\infty\) and thus
\[
\Phi: \zeta_t \in [-1,1] \mapsto T_\ast^{2-\theta_2}(\zeta(T_\ast) - T_\ast) \in \{-1,1\}
\]
is also a continuous function whose image is \(\{-1,1\}\), which is contradictory.

To complete the proof of Proposition 2, we observe that from (30), \(\gamma_1 + |\gamma_2| \lesssim t^{-\theta}\) holds on the interval \([T_0, T_\infty]\). Integrating and using (20), this gives the uniform estimate \(|\gamma_1| + |\gamma_2| \lesssim t^{1-\theta}\) on \([T_0, T_\infty]\).

3.5. End of the proof of Theorem 1.2 by compactness. We use Proposition 2 with \(T_\infty = n\), for any \(n \geq T_0\), to construct a sequence of solutions \((u_n) \in C([T_0, n], H^1 \times H^1)\) of (coupled NLS) such that, for some \(\delta > 0\), on \([T_0, n],\)
\[
\left\| \begin{pmatrix} u_n \\ \bar{v}_n \end{pmatrix} - \begin{pmatrix} e^{ic^2t} Q_c \left( -\frac{\log t}{c(c+1)} - \frac{\log \Omega_c}{c(c+1)} \right) \\ e^{it} Q \left( +\frac{\log t}{c+1} - \frac{\log \Omega_c}{c+1} \right) \end{pmatrix} \right\|_{H^1 \times H^1} \lesssim t^{-\delta}.
\]
Now, we adapt from [17] (in the scalar case) and from [11] (in the vector case), the following convergence result.

**Lemma 3.4.** There exists \((\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})\) such that up to a subsequence, as \(n \to \infty\)
\[
(\begin{pmatrix} u_n \\ v_n \end{pmatrix})(T_0) \to (\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}) \quad \text{weakly in} \quad H^1(\mathbb{R}) \times H^1(\mathbb{R})
\]
\[
(\begin{pmatrix} u_n \\ v_n \end{pmatrix})(T_0) \to (\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}) \quad \text{in} \quad H^s(\mathbb{R}) \times H^s(\mathbb{R}) \quad \text{for any} \quad 0 \leq s < 1.
\]
We consider \( (\frac{u}{v}) \) the solution of (coupled NLS) corresponding to initial data \( (\frac{u_0}{v_0}) \) at \( t = T_0 \). By \( H^1(\mathbb{R}) \times H^1(\mathbb{R}) \) boundedness and local well-posedness of Cauchy problem in \( H^s(\mathbb{R}) \times H^s(\mathbb{R}) \) for any \( 0 \leq s < 1 \) (see e.g. [3]), we have the continuous dependence of the solution on the initial data, so for all \( t \in [T_0, +\infty) \), as \( n \to \infty \),

\[
\left( \frac{u_n}{v_n} \right) (t) \to \left( \frac{u}{v} \right) (t) \text{ in } H^1(\mathbb{R}) \times H^1(\mathbb{R}),
\]

\[
\left( \frac{u_n}{v_n} \right) (t) \to \left( \frac{u}{v} \right) (t) \text{ in } H^s(\mathbb{R}) \times H^s(\mathbb{R}), \quad 0 \leq s < 1.
\]

Passing to the weak limit as \( n \to \infty \) in the uniform estimates (48), the solution \( (\frac{u}{v}) \) satisfies Theorem 1.2.

4. Sketch of the proof of Theorem 1.1.

4.1. **Approximate solution in the case** \( c = 1 \). In this case, the approximate solution and the solution are symmetric (i.e. \( u(t, x) = v(t, -x) \)) and thus have \( \sigma_1 = -\sigma_2 = \frac{\varphi}{2}, \beta_1 = -\beta_2 = \frac{\omega}{2} \) and \( \gamma_1 = \gamma_2 \). Using the same notation as in Sections 2 and 3, we define (the function \( B \) is introduced in Lemma 1.4)

\[
U = P + \varphi, \quad P(t, x) = Q(x - \sigma_1(t))e^{i\varphi(t, x)}, \quad \varphi(t, x) = e^{-\sigma(t)}B(x - \sigma_2(t))e^{i\varphi(t, x)},
\]

\[
V = R + \psi, \quad R(t, x) = Q(x - \sigma_2(t))e^{i\psi(t, x)}, \quad \psi(t, x) = e^{-\sigma(t)}B(x - \sigma_1(t))e^{i\psi(t, x)}.
\]

**Lemma 4.1.** It holds

\[
E_U = F - \bar{m}_1 \cdot \bar{M}_1 - \bar{m}_\varphi \cdot \bar{M}_\varphi,
\]

where

\[
F = 3|P|^2 \varphi + 3|\varphi|^2 P + |\varphi|^2 \varphi - \omega e^{2(x - \sigma_1)} |R|^2 P + \omega(2|R\psi| + |\psi|^2)P,
\]

and

\[
\bar{m}_1 = \begin{pmatrix} \dot{\varphi}_1 + \beta_1 \varphi_1 + \beta_1^2 \\ \sigma_1 - 2\beta_1 \\ \beta_1 \\ \end{pmatrix}, \quad \bar{M}_1 = \begin{pmatrix} i\dot{1}_1 P \\ P \\ x_1 P \\ \end{pmatrix},
\]

\[
\bar{m}_\varphi = \begin{pmatrix} \dot{\gamma}_1 + \beta_1 \sigma_2 + \beta_1^2 + i\varphi \\ \sigma_2 - 2\beta_1 \\ \beta_1 \\ \end{pmatrix}, \quad \bar{M}_\varphi = \begin{pmatrix} i\dot{1}_2 \varphi \\ \varphi \\ x_2 \varphi \\ \end{pmatrix}.
\]

We set

\[
a = \frac{1}{2} \langle F, \partial_1 P \rangle.
\]

**Lemma 4.2.** It holds

\[
a = \alpha \sigma e^{-2\sigma} + O(e^{-2\sigma}) \quad \text{where} \quad \alpha = 32\omega.
\]

**Proof.** From the expression of \( F \), one has

\[
\langle F, \partial_1 P \rangle = 3e^{-\sigma} \int Q^2(x)Q'(x)B(x + \sigma)dx + 3e^{-2\sigma} \int Q(x)Q'(x)B^2(x + \sigma)dx + e^{-3\sigma} \int \frac{Q'(x)}{2}B^3(x + \sigma)dx - \omega \int e^{2\sigma}Q(x)Q'(x)Q^2(x + \sigma)dx.
\]
From (9) and Lemma 3.1, the second and third terms in the right-hand side are bounded by $\sigma^3 e^{-4\sigma}$. The last term is bounded by
\[
\int e^{2x} Q^2(x) Q^2(x+\sigma)dx = e^{-2\sigma} \int e^{2x} Q^2(x-\sigma) Q^2(x)dx \lesssim e^{-2\sigma} \int Q^2(x-\sigma)dx \lesssim e^{-2\sigma}.
\]
For the first term, using $\mathcal{L}_+ Q' = 0$ and then (8), we compute
\[
3 \int Q^2(x) Q'(x) B(x+\sigma)dx = \int Q'(x-\sigma)(-B''(x) + B(x))dx = \omega \int Q'(x-\sigma) \left[ Q^2(x) B(x) + \kappa e^x Q^2(x) \right] dx.
\]
By Lemma 3.1, we have $\int |Q'(x-\sigma)| Q^2(x) B(x)|dx \lesssim e^{-\sigma}$.
We only have to compute $\int Q'(x-\sigma) e^x Q^2(x)dx$. First, we see
\[
\int_{x<0} Q'(x-\sigma) e^x Q^2(x)dx \lesssim e^{-\sigma} \int_{x<0} e^{4x} dx \lesssim e^{-\sigma},
\]
\[
\int_{x>\sigma} |Q'(x-\sigma)| e^x Q^2(x)dx \lesssim e^\sigma \int_{x>\sigma} e^{-2x} dx \lesssim e^{-\sigma}.
\]
Second, using (3)
\[
Q'(x-\sigma) = \kappa e^{x-\sigma} - e^{2x-2\sigma} Q(x-\sigma), \quad Q^2(x) = \kappa^2 e^{-2x} + O(e^{-3x} Q(x)),
\]
and thus
\[
\int_0^\sigma Q'(x-\sigma) e^x Q^2(x)dx = \kappa^3 \sigma e^{-\sigma} + O(e^{-\sigma}).
\]
In conclusion, $a = \omega \frac{4^4}{\pi} \sigma e^{-2\sigma} + O(e^{-2\sigma}) = 32 \omega \sigma e^{-2\sigma}$.

4.2. **Formal discussion for** $c = 1$. The previous computations leads us to
\[
\dot{\sigma} = -4 \alpha \sigma e^{-2\sigma}, \quad 2\beta = \dot{\sigma},
\]
for which the following function is an approximate solution
\[
\sigma_0(t) = \log t + \frac{1}{2} \log \log t + \log \Omega, \quad 2\beta_0(t) = \frac{1}{t} \text{ where } \Omega = \sqrt{4\alpha} = 8\sqrt{2} \sigma.
\]

4.3. **Bootstrap estimates in the case** $c = 1$. Fix $\theta_1$ such that $1 < \theta_1 < 2$. The following bootstrap estimates are used in this case: for $1 \ll t \leq T_\infty$,
\[
\left\{
\begin{array}{l}
\|\varepsilon\|_{H^1} + \|\eta\|_{H^1} \leq t^{-\theta_1},
\|\beta - 1\|_{H^1} \leq t^{-1}(\log t)^{-\frac{1}{2}},
\left|\frac{e^\sigma}{\Omega \sigma^2 t} - 1\right| \leq (\log t)^{-\frac{1}{2}},
\end{array}
\right.
\]
where $\sigma_\infty$ is to be chosen satisfying
\[
\left|\frac{e^{\sigma_\infty}}{\Omega \sigma_\infty^2 T_\infty} - 1\right| \leq (\log T_\infty)^{-\frac{1}{2}}.
\]
We refer to [20, 22] for similar bootstrap estimates.

The rest of the proof is similar to the one of Theorem 1.2 and we omit it.
5. Discussion. For (coupled NLS), with any coupling coefficient $0 < \omega < 1$, we have proved the existence of symmetric 2-solitary waves (Theorem 1.1) and of non-symmetric 2-solitary waves (Theorem 1.2) with logarithmic distance. Symmetric 2-solitons with logarithmic distance were already known in the literature for the integrable cases ($\omega = 0$ and $\omega = 1$) and in the scalar case (NLS). In contrast, the existence of non-symmetric 2-solitary waves with logarithmic distance is new. In particular, it does not hold for the integrable case where instead a periodic regime exists.

An interesting remaining open question is whether non-symmetric logarithmic 2-solitary waves exist for the non-integrable scalar (NLS). We conjecture that it is indeed the case, as long as $p \neq 3$. Indeed, the first step of the strategy used in this paper, i.e. the computation of an approximate solution involving the main interaction terms, works equally well for (NLS) as for (coupled NLS). We expect a logarithmic regime with oscillations. However, whereas (coupled NLS) enjoys two $L^2$ conservation laws, the scalar equation (NLS) enjoys only one, which does not seem sufficient for the energy method to apply in the context of two solitons with logarithmic distance without symmetry.

A more technical original aspect of this article is the introduction of a refinement of the energy method. In previous articles using approximate solutions in the context of error terms of order $t^{-k}$ (e.g. in [20, 22, 23]), the energy method induces a loss of time decay. Here, the additional correction term $S$ in Section 3.3 allows an estimate of the remainder $(\varepsilon \eta)$ directly related to the size of the error term $\left( \frac{E_U}{E_V} \right)$. We believe that this general observation will be useful elsewhere.

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E-mail address: yvan.martel@polytechnique.edu
E-mail address: tien-vinh.nguyen@polytechnique.edu