1 Background

- Earlier in 20th century logic proofs (deductions, derivations) were understood like this: Formula \( \varphi \) is derivable from axioms \( A : \ A \vdash \varphi \) iff 
  \[ (\exists \varphi_0, \cdots, \varphi_n) \varphi = \varphi_n \text{ and } (\forall k \leq n) \varphi_k \in A \text{ or } (\exists i, j < k) \varphi_j = \varphi_i \rightarrow \varphi_k, \]
  i.e. \( \varphi_k \) follows from \( \varphi_i \) and \( \varphi_j \) by the rule “modus ponens” (detachment).
  Other rules of inference could be included analogously.

- Proofs in the algebraic logic (boolean and relation algebras) were treated analogously with regard to the transitivity of “=”, instead of modus ponens.

- These definitions reflected traditional written linear presentation of mathematical proofs of new theorems via axioms, known theorems, suitable lemmas, etc.

- Corresponding proof systems in mathematical logic are usually referred to as Frege-Hilbert-Bernays-Tarski style calculi.

- Later came graph-theoretic interpretations leading to genuine structural proof theory.

- Corresponding basic proof systems are referred to as natural deduction (ND) and sequent calculus (SC) – both initiated by Gentzen and further developed by Prawitz resp. Schütte, et al. These proofs are usually presented in tree-like form, where branching points are determined by the conclusions of the rules involved. Moreover

  1. ND derivations contain single formulas, whereas SC ones contain finite collections thereof (called sequents).

  2. ND proofs have no axioms. However all assumptions shall be discharged according to special conditions on the threads.

- Both ND and SC allow normalizations (mutually different) making proofs more transparent and suitable for analysis.

2 Proof size

- Linear proofs admit tree-like interpretation, and v.v. Different nodes in tree-like proofs might correspond to identical formulas (“references”) \( \varphi_i, \varphi_j \) occurring in linear proofs \( A \vdash \varphi \) (sequent case is analogous). So passing to tree-like proofs might essentially increase the size of linear inputs. The
opposite direction is called proof compression. Actually we compress tree-like proofs into dag-like proofs (dag = directed acyclic graph) by merging different nodes labeled with identical formulas (sequents). Moreover

1. Proof compression in SC is easy. However, we can’t really control the size of resulting dag-like proofs, as there can be too many different (sub)sequents occurring in (even normal) proofs of given “small” conclusions.

2. In contrast, ND proofs contain single formulas thus being more appropriate for the size control. However, proof compression in ND is more involved.

3 Minimal logic

- In this work we consider basic ND of minimal purely implicational logic, NM →, having two standard rules of inferences

\[
\begin{array}{c}
\frac{\alpha}{\beta} \\
\vdots \\
\alpha \rightarrow \beta \\
\rightarrow I : \frac{\beta}{\alpha \rightarrow \beta}
\end{array}
\]

and auxiliary repetition rule

\[
\frac{\alpha}{\alpha}
\]

where \([\alpha]\) in \(\rightarrow I\) indicates that all \(\alpha\)-leaves occurring above \(\beta\)-node exposed are considered discharged assumptions.

**Definition 1 (minimal validity)** A given (whether tree- or dag-like) \(\text{NM}\rightarrow\)-deduction \(\partial\) proves its conclusion \(\rho\) (abbr.: \(\partial \vdash \rho\)) iff every maximal thread connecting the root labeled \(\rho\) with a leaf labeled \(\alpha\) is closed, i.e. it contains a \(\rightarrow I\) with conclusion \(\alpha \rightarrow \beta\) and discharged assumption \(\alpha\), for some \(\beta\). Now \(\rho\) is valid in minimal logic iff there exists a tree-like \(\text{NM}\rightarrow\)-deduction \(\partial\) that proves \(\rho\); such \(\partial\) is called a proof of \(\rho\).

**Remark 2** Tree-like constraint in the definition of validity is inessential, as any dag-like \(\partial\) can be unfolded into a tree-like \(\partial'\) by thread-preserving top-down recursion. Moreover, “\(\partial\) proves \(\rho\)” is deterministically verifiable in \(|\partial|\)-polynomial time, where \(|\partial|\) denotes the weight of \(\partial\).

**Definition 3** A given \(\text{NM}\rightarrow\)-deduction \(\partial\) with conclusion \(\rho\) is polynomial, resp. quasi-polynomial, if its weight (= total number of symbols) \(|\partial|\), resp. height \(h(\partial)\) plus total weight \(\phi(\partial)\) of distinct formulas occurring in \(\partial\), is polynomial in the weight of \(\rho\), \(|\rho|\). Note that \(|\partial|\) of quasi-polynomial \(\partial\) can be exponential in \(|\rho|\).

**Theorem 4 (Main Theorem)** Any given quasi-polynomial tree-like proof \(\partial \vdash \rho\) can be compressed into a polynomial dag-like proof \(\partial^* \vdash \rho\).
Proof. See GH1, 2 that presented desired *horizontal compression* of quasi-polynomial tree-like proofs into equivalent polynomial dag-like proofs having mutually different formulas on every horizontal level (see also Section 5 below).

4 Propositional complexity.

4.1 Case NP vs coNP

Lemma 5 *Any normal tree-like NM→-proof* \( \partial \) *of* \( \rho \) *whose height* \( h(\partial) \) *is polynomial in* \( |\rho| \) *is quasi-polynomial.*

Lemma 6 (GH3) *Let* \( P \) *be the Hamiltonian graph problem and* \( \rho_G \) *express that a given graph* \( G \) *has no Hamiltonian cycles. There exists a normal tree-like NM→-proof* \( \partial \) *of* \( \rho_G \) *such that* \( h(\partial) \) *is polynomial in* \( |G| \) *(and hence* \( |\rho_G| \)), *provided that* \( G \) *is non-Hamiltonian.*

Recall that the non-hamiltoniancy in question is coNP-complete. Hence Theorem 4 yields

**Corollary 7 (GH2, GH3)** NP = coNP *holds true.*

4.2 Case NP vs PSPACE

Recall that the minimal validity is PSPACE-complete. Let LM→ be Hudelmaier’s SC that is sound and complete for minimal logic.

**Theorem 8 (Hudelmaier)** *Any formula* \( \rho \) *is valid in minimal logic iff sequent* \( \Rightarrow \rho \) *is provable in LM→ by a quasi-polynomial tree-like derivation.*

**Lemma 9 (GH1)** *For any quasi-polynomial tree-like derivation of* \( \Rightarrow \rho \) *in LM→ there exists a quasi-polynomial tree-like proof* \( \partial \vdash \rho \) *in NM→.*

**Corollary 10 (GH2)** PSPACE ⊆ NP *and hence NP = PSPACE holds true.*

**Remark 11** *Using PSPACE-completeness of quantified boolean logic V. Sopin claimed to have obtained a partial result* PH = PSPACE.

5 More on Main Theorem

- First part of tree-to-dag horizontal compression

For any tree-like NM→ proof \( \partial \) *of* \( \rho \) *let* \( \partial' \in NM→ \) *be defined by bottom-up recursion on* \( h(\partial) \) *such that for any* \( n \leq h(\partial) \), *the* \( n^{th} \) *horizontal section of* \( \partial' \) *is obtained by merging all nodes with identical formulas occurring in the* \( n^{th} \) *horizontal section of* \( \partial \). The inferences in \( \partial' \) *are naturally inherited by the ones
in $\partial$. Obviously $\partial'$ is a dag-like (not necessarily tree-like anymore) deduction with conclusion $\rho$. Moreover $\partial'$ is polynomial as $|\partial'| \leq h(\partial) \times \phi(\partial)$. However, $\partial'$ need not preserve the local correctness with respect to basic inferences ($\rightarrow I$), ($\rightarrow E$), ($R$). For example, a compressed multipremise configuration

\[
(\rightarrow I, E) : \frac{\beta}{\alpha} \quad \frac{\gamma \rightarrow (\alpha \rightarrow \beta)}{\alpha \rightarrow \beta}
\]

that is obtained by merging identical conclusions $\alpha \rightarrow \beta$ of

\[
(\rightarrow I) : \frac{\beta}{\alpha \rightarrow \beta} \quad \text{and} \quad (\rightarrow E) : \frac{\gamma}{\alpha} \quad \frac{\gamma \rightarrow (\alpha \rightarrow \beta)}{\alpha \rightarrow \beta}
\]

is not a legitimate inference in $\text{NM} \rightarrow$.

To overcome this trouble we upgrade $\partial'$ to a modified deduction $\partial^\flat$ that separates such multiple premises using instances of the separation rule ($S$)

\[
(S) : \frac{n \text{ times}}{\alpha \rightarrow \beta}
\]

that is understood disjunctively: “if at least one premise is proved then so is the conclusion” (in contrast to ordinary inferences: “if all premises are proved then so are the conclusions”).

For example, $(\rightarrow I, E)$ as above should be replaced by this modified configuration in $\text{NM}^\flat \rightarrow = \text{NM} \rightarrow + (S)$

\[
(S) : \frac{\beta}{\alpha \rightarrow \beta} \quad (\rightarrow I) : \frac{\gamma}{\alpha} \quad \frac{\gamma \rightarrow (\alpha \rightarrow \beta)}{\alpha \rightarrow \beta}
\]

Such $\partial^\flat$ is a locally correct dag-like deduction in $\text{NM}^\flat$, with conclusion $\rho$. Moreover $\partial^\flat$ is polynomial, since its every ($S$)-free subdeduction at most doubles the weight of $\partial'$. However, we can’t claim that $\partial^\flat$ proves $\rho$ because arbitrary maximal dag-like threads in $\partial^\flat$ can arise by concatenating different segments of different threads in $\partial$, which can destroy the required closure condition.

To solve this problem we observe that $\partial^\flat$ satisfies certain conditions of coherence with respect to the set of threads, and continue our compression as follows.

- **Second part of tree-to-dag horizontal compression**

Here we prove weak ($S$)-elimination theorem showing that any coherent deduction $\partial^\flat$ is further compressible into a desired ($S$)-free subdeduction $\partial^\ast$. This part of compression (also called cleansing) is defined by nondeterministic bottom-up recursion on $h(\partial)$ while using as oracle the whole (possibly exponential) set of maximal threads. **This completes proof of Main Theorem.**

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1 see GH1, GH2 for details
6 More on dag-like provability

Formal verification of the assertion $\partial \vdash \rho$ is simple – whether for tree-like or, generally, dag-like $\partial$. Every node $x \in \partial$ is assigned, by top-down recursion, a set of assumptions $A(x)$ such that:

1. $A(x) := \{\alpha\}$ if $x$ is a leaf labeled $\alpha$,
2. $A(x) := A(y)$ if $x$ is the conclusion of $(R)$ with premise $y$,
3. $A(x) := A(y) \setminus \{\alpha\}$ if $x$ is the conclusion of $(\rightarrow I)$ with label $\alpha \rightarrow \beta$ and premise $y$,
4. $A(x) := A(y) \cup A(z)$ if $x$ is the conclusion of $(\rightarrow E)$ with premises $y, z$.

**Theorem 12** $\partial \vdash \rho \iff A(r) = \emptyset$ holds with respect to standard set-theoretic interpretations of “$\cup$” and “$\setminus$”, where $r$ is the root of $\partial$ with formula-label $\rho$. Moreover, problem $A(r) \neq \emptyset$ is solvable by a deterministic TM in $|\partial|$-polynomial time.

7 References

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