The categorical origins of Lebesgue integration

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Abstract

We identify simple universal properties that uniquely characterize the Lebesgue $L^p$ spaces. There are two main theorems. The first states that the Banach space $L^p[0,1]$, equipped with a small amount of extra structure, is initial as such. The second states that the $L^p$ functor on finite measure spaces, again with some extra structure, is also initial as such. In both cases, the universal characterization of the integrable functions produces a unique characterization of integration. We use the universal properties to derive some of the basic elements of integration theory. We also state universal properties characterizing the sequence spaces $\ell^p$ and $c_0$, as well as the functor $L^2$ taking values in Hilbert spaces.

1 Introduction

Lebesgue integration is universally agreed to lie at the heart of analysis, yet its fundamental nature contrasts with the long and perhaps technical-seeming string of preliminaries that faces the student wishing to learn the basic definitions. For instance, a standard route to defining Lebesgue integrability and integration for functions on $\mathbb{R}$ involves (1) defining null sets and almost everywhere convergence, (2) defining step functions, (3) defining the set $L^{inc}$ of functions that are almost everywhere limits of some increasing sequence of step functions, (4) defining the set $L^1$ of differences of elements of $L^{inc}$, (5) defining integration for step functions, and (6) proving that the definition is consistent, before finally (7) extending the integral to $L^1$ by linearity and continuity, and (8) passing from $L^1$ to its quotient $L^1$. A different route constructs $L^1$ as the completion of the normed space of continuous functions; but for that, one must first develop the theory of integration in the continuous case.

One might therefore wish for a description of Lebesgue integration that is as simple and direct as the concept is fundamental. This paper presents two such theorems. One uniquely characterizes the space $L^1[0,1]$ and the operator $\int_0^1$. The other does the same on an arbitrary measure space. Both theorems entirely bypass steps (1)–(8).

The theorems characterize spaces of integrable functions uniquely up to isomorphism via a universal property. Specifically, they characterize them as initial objects in certain categories.

Let us recall this notion. An object $Z$ of a category $\mathcal{C}$ is initial if for each object $C$ of $\mathcal{C}$, there is exactly one map $Z \to C$ in $\mathcal{C}$. For example, the empty set is initial in the category of sets, the trivial group is initial in the category

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of groups, and \( \mathbb{Z} \) is initial in the category of rings with identity. A slightly less trivial example, closer in shape to those we will consider, is as follows. There is a category whose objects are triples \((X, x, r)\) consisting of a set \(X\), an element \(x \in X\) and a function \(r: X \to X\), and whose maps are functions preserving this structure. Its initial object is \((\mathbb{N}, 0, s)\), where \(s\) is the successor function \(n \mapsto n + 1\). Concretely, initiality means that for any set \(X\), element \(x\) and function \(r: X \to X\), there is a unique sequence \((x_n)_{n \geq 0}\) in \(X\) satisfying \(x_0 = x\) and \(x_{n+1} = r(x_n)\).

Any two initial objects of a category are isomorphic. Hence, any theorem stating that an object is initial in some category characterizes it uniquely up to isomorphism. Our first theorem does this for the Banach space \(L^p[0, 1]\) equipped with a small amount of extra structure. Our second theorem does it for the functor \(L^p\) from finite measure spaces to Banach spaces, again with some extra structure. Thus, both characterize \(L^p\) uniquely up to isometric isomorphism. And in both cases, the initiality of \(L^1\) leads swiftly to a unique characterization of the integration operator.

What is the point of these theorems?

First, they enable us to leapfrog all the customary preliminary definitions, directly characterizing Lebesgue integrability and Lebesgue integration.

Second, any theorem stating that some object is initial establishes uniqueness at two levels: the uniqueness up to isomorphism of the object itself, and the literal uniqueness of the map to any other object. Such a theorem is effectively a large family of uniqueness theorems, one for each object of the category. Generally, for any important mathematical object, one can ask: is it the only object enjoying the fundamental properties that it enjoys? If not, why do we use it rather than something else? Or if so, can we prove it? For example, theorems of Alesker, Artstein-Avidan and Milman answer these questions for the Fourier and Legendre transforms [3, 4], and the present work answers them for the \(L^p\) spaces and Lebesgue integration.

Third, the main theorems clarify conceptual dependencies. They show that, granted some general categorical language, the concepts of Lebesgue integrability and integration arise inevitably from little more than the concept of Banach space. Perhaps surprisingly, they arise automatically, without invoking any prior concept of area under the curve or antidifferentiation.

Fourth, the second main theorem, characterizing the \(L^p\) functors, provides a guide for the discovery of new theories of integration. A researcher seeking the right notion of integration in some new context (perhaps some new kind of function on a new kind of space) could follow the same template: decide what kind of spaces the integrable functions should form and what kind of functoriality should hold, formulate a universal property analogous to the one below, and find the functor satisfying it.

Finally, such theorems are important simply because the Lebesgue theory is important. There is no question of displacing the twin classical perspectives on integration, area under the curve and the inverse of differentiation. But the more perspectives the better, and here we provide a new one: its characterization by a universal property.

The deep theorems on integration and measure are particular to that subject, so there is no chance of deriving them using general categorical methods that would apply equally to other categories. The fact that (for instance) \(L^1[0, 1]\) is the initial object of a certain category does determine it uniquely, so all of
The map \( \gamma: L^1[0, 1] \oplus L^1[0, 1] \to L^1[0, 1] \) juxtaposes two functions and scales the domain by a factor of \( 1/2 \) (Figure 1). We prove:

**Theorem A**  
The initial object of \( \mathcal{A} \) is \((L^1[0, 1], I, \gamma)\).

Thus, for each object \((V, v, \delta)\) of \( \mathcal{A} \), there is a unique map \((L^1[0, 1], I, \gamma) \to (V, v, \delta)\) in \( \mathcal{A} \). Certain choices of \((V, v, \delta)\) are especially consequential. For example, there is an object of \( \mathcal{A} \) consisting of the ground field \( F \) (either \( \mathbb{R} \) or \( \mathbb{C} \)), together with \( 1 \in F \) and the arithmetic mean \( m: F \oplus F \to F \). We prove that the unique map \((L^1[0, 1], I, \gamma) \to (F, 1, m)\) in \( \mathcal{A} \) is the integration operator \( \int_0^1 \).

The universal characterization of the integrable functions therefore gives rise to a unique characterization of integration.

In fact, Theorem A is just the case \( p = 1 \) of a general result characterizing \((L^p[0, 1], I, \gamma)\) as the initial object of a category \( \mathcal{A}^p \) (Theorem 2.1). Here \( 1 \leq p < \infty \). When \( p = \infty \), the analogous result characterizes not \( L^\infty[0, 1] \) but the space \( C(\{0, 1\}^\mathbb{N}) \) of continuous functions on the Cantor set (Proposition 2.6).
The key to Theorem A is that integration on $[0, 1]$ is a continuous notion of mean, and means are built into the category $\mathcal{A} = \mathcal{A}^1$ via the definition of the norm on $V \oplus W$. To define $\mathcal{A}^p$ for $p > 1$, we simply replace the arithmetic mean in the definition of $\mathcal{A}$ by the power mean of order $p$.

Abstract characterizations are all well and good, but it is natural to want to realize $L^p[0, 1]$ as a function space. We do so in two senses. First, by an observation of Meckes (Proposition 2.4), the universal property of $L^1[0, 1]$ produces a canonical map $L^1[0, 1] \rightarrow C[0, 1]$, which in concrete terms maps $f \in L^1[0, 1]$ to the continuous function $x \mapsto \int_0^x f$. Differentiating $\int_0^x f$, we recover $f$ as a function (up to equality almost everywhere). In the opposite direction, the universal property of $\mathcal{C}((0, 1)^0)$ leads to a canonical map $C[0, 1] \rightarrow L^p[0, 1]$ for every $p$, which in concrete terms is the usual inclusion (Proposition 2.8).

Repeatedly exploiting the universal property of $L^p[0, 1]$, we derive further maps fundamental in analysis, including the canonical pairing

$$L^p[0, 1] \times L^q[0, 1] \rightarrow \mathbb{F}$$

for conjugate exponents $p$ and $q$ (Proposition 2.10).

The universal property of $L^p[0, 1]$ stems from the self-similarity of $[0, 1]$ (see ‘Related work’ below), and the same idea can be used to characterize $L^p(X)$ for other self-similar spaces $X$. Proposition 2.11 is a universal characterization of the sequence space $\ell^p$ (the case $X = \mathbb{N}$) for each $p \in [1, \infty)$, and also of the sequence space $c_0$.

The results described so far uniquely characterize the spaces concerned, but general categorical techniques actually construct them (Remark 2.12). So even if Lebesgue had never formulated his theory, categorical machinery would still construct its basic objects.

Section 3 concerns functions on an arbitrary finite measure space. The $L^1$ construction on measure spaces is functorial in two ways: contravariantly with respect to measure-preserving maps and covariantly with respect to embeddings. To combine the two, let $\text{Meas}$ be the category of finite measure spaces and measure-preserving partial maps. Then $L^1$ defines a functor $\text{Meas}^{op} \rightarrow \text{Ban}$.

Our second main theorem characterizes this functor $L^1$ together with, for each $X$, the function $I_X \in L^1(X)$ with constant value 1. We define a category $\mathcal{B}$ of pairs $(F, v)$ consisting of a functor $\text{Meas}^{op} \rightarrow \text{Ban}$ and an element $v_X \in F(X)$ for each $X$, subject to some simple axioms. Then:

**Theorem B** The initial object of $\mathcal{B}$ is $(L^1, I)$.

As for Theorem A, the universal characterization of the integrable functions produces a unique characterization of integration. Indeed, there is another object $(\widehat{F}, t)$ of $\mathcal{B}$, where $\widehat{F}$ is the constant functor whose value is the ground field and $t_X$ is the total measure of a measure space $X$. The unique map $(L^1, I) \rightarrow (\widehat{F}, t)$ in $\mathcal{B}$ is nothing but integration.

A variant of Theorem B uses the functoriality of $L^1$ with respect to embeddings only. Thus, $\text{Meas}^{op}$ is replaced by the category of measure spaces and embeddings, and $\mathcal{B}$ is replaced by a simpler category $\mathcal{B}_{\text{emb}}$. We prove that $(L^1, I)$ is also initial in $\mathcal{B}_{\text{emb}}$. As a consequence, we obtain the construction $(f, \mu) \mapsto f \, d\mu$ (Proposition 3.10). This in turn allows us, via the Radon–Nikodym theorem, to realize elements of the abstractly characterized space $L^1(X)$ as equivalence classes of functions on $X$.  

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At the cost of a further axiom, Theorem B can be generalized to any \( p \in [1, \infty) \). That is, \((L^p, I)\) is the initial object of a suitably defined category \( B^p \) (and, similarly, of \( B^p_{\text{gal}} \)), where \( B^1 = B \). This is Theorem 3.7. In the case \( p = 2 \), the functor \( L^2 \) takes values in the category of Hilbert spaces, and also possesses a slightly simpler universal property (Proposition 3.11).

Although Theorem B concerns arbitrary finite measure spaces and Theorem A concerns only the space \([0, 1]\), Theorem A is not a special case of Theorem B. Indeed, the hypotheses of Theorem A make no mention of the set \([0, 1]\), let alone its \( \sigma \)-algebra or Lebesgue measure on it, whereas the starting point for Theorem B is the category of measure spaces. To my knowledge, there is no reasonable way to derive Theorem A from Theorem B.

Related work This work arose from a universal characterization of the real interval by Freyd [8] (itself related to a characterization by Escardó and Simpson [6]). A topological variant of Freyd’s result, due to the author, runs as follows [11, Theorem 2.5]. A bipointed space is a topological space with an ordered pair of distinct, closed basepoints; an example is \([0, 1]\) with 0 and 1. Two bipointed spaces \( X \) and \( Y \) can be joined to form a new bipointed space \( X \vee Y \), identifying the second basepoint of \( X \) with the first of \( Y \); for example, \([0, 1] \vee [0, 1] \cong [0, 2] \). The theorem is that in the category of bipointed spaces \( X \) equipped with a basepoint-preserving continuous map \( X \to X \vee X \), the terminal object is \([0, 1]\) (topologized as usual) with the map \( \gamma : [0, 1] \to [0, 2] \). Our first main theorem is a kind of dual to this, using the same self-similarity of the interval to exhibit \( L^1[0, 1] \) as an initial object.

Algebraic approaches to integration go at least as far back as the 1965 work of Irving Segal [13]. For example, he proved (p. 432) that every commutative \( \mathbb{R} \)-algebra equipped with a linear functional \( \int \) satisfying certain axioms must be a dense subalgebra of \( L^\infty(X) \) for some measure space \( X \).

Convention We work throughout over a field \( F \), which is either \( \mathbb{R} \) or \( \mathbb{C} \).

2 Integration on \([0, 1]\)

Let \( p \in [1, \infty) \). In this section, we uniquely characterize the Banach space \( L^p[0, 1] \) together with two further pieces of data: the function \( I \in L^p[0, 1] \) taking constant value 1, and the juxtaposition map

\[ \gamma : L^p[0, 1] \oplus L^p[0, 1] \to L^p[0, 1] \]

defined on \( f, g \in L^p[0, 1] \) by

\[ (\gamma(f, g))(x) = \begin{cases} f(2x) & \text{if } x < 1/2, \\ g(2x - 1) & \text{if } x > 1/2 \end{cases} \]

(Figure 1).

For us, a map of Banach spaces is a linear contraction (map with norm \( \leq 1 \)), and \( \text{Ban} \) is the category of Banach spaces and maps between them. Thus, the isomorphisms in \( \text{Ban} \) are the isometric isomorphisms. For Banach spaces \( V \)
and \( W \), let \( V \oplus_p W \) denote the direct sum with norm
\[
\| (v, w) \|_p = \left( \frac{1}{2} (\| v \|_p^p + \| w \|_p^p) \right)^{1/p}.
\]
Let \( \mathcal{A}^p \) be the category whose objects are triples \((V, v, \delta)\) where \( V \) is a Banach space, \( v \in V \) with \( \| v \| \leq 1 \), and \( \delta: V \oplus_p V \to V \) is a map of Banach spaces satisfying \( \delta(v, v) = v \). The maps \((V', v', \delta') \to (V, v, \delta)\) in \( \mathcal{A}^p \) are the maps \( \theta: V' \to V \) in \( \text{Ban} \) that preserve the structure:
\[
\theta(v') = v, \quad \theta(\delta'(v_1', v_2')) = \delta(\theta(v_1'), \theta(v_2'))
\]
for all \( v_1', v_2' \in V' \). For example, the category \( \mathcal{A} \) of the Introduction is \( \mathcal{A}^1 \).

The map \( \gamma \) is an isometric isomorphism \( L^p[0, 1] \oplus_p L^p[0, 1] \to L^p[0, 1] \), and in particular, a contraction. Hence \((L^p[0, 1], I, \gamma)\) is an object of \( \mathcal{A}^p \).

**Theorem 2.1 (Universal property of \( L^p[0, 1] \))** Let \( 1 \leq p < \infty \). Then \((L^p[0, 1], I, \gamma)\) is the initial object of \( \mathcal{A}^p \).

**Proof** For \( n \geq 0 \), let \( E_n \) be the subspace of \( L^p[0, 1] \) consisting of the equivalence classes of step functions constant on each of the intervals \(((i - 1)/2^n, i/2^n)\) \((1 \leq i \leq 2^n)\). Write \( E = \bigcup_{n \geq 0} E_n \), which is the space of step functions whose points of discontinuity are dyadic rationals.

The assumption that \( p < \infty \) implies that \( E \) is dense in the set of all step functions on \([0, 1]\), which in turn is dense in \( L^p[0, 1] \); so \( E \) is dense in \( L^p[0, 1] \). It follows that \( L^p[0, 1] \) is the colimit (direct limit) of the diagram \( E_0 \to E_1 \to \cdots \) in \( \text{Ban} \) [5, Examples 2.24(h) and 2.26(g)]. Also note that \( \gamma \) restricts to an isomorphism \( E_n \oplus_p E_n \to E_{n+1} \) for each \( n \geq 0 \).

Now let \((V, v, \delta) \in \mathcal{A}^p \). We must show that there exists a unique map \((L^p[0, 1], I, \gamma) \to (V, v, \delta)\) in \( \mathcal{A}^p \).

**Uniqueness** Let \( \theta \) be a map \((L^p[0, 1], I, \gamma) \to (V, v, \delta)\) in \( \mathcal{A}^p \). Then \( \theta(I) = v \), which by linearity determines \( \theta|_{E_n} \) uniquely. Suppose inductively that \( \theta|_{E_n} \) is determined uniquely. Since \( \theta \) is a map in \( \mathcal{A}^p \), the square
\[
\begin{array}{ccc}
E_n \oplus_p E_n & \xrightarrow{\gamma} & E_{n+1} \\
\theta|_{E_n} \oplus \theta|_{E_n} & \downarrow & \theta|_{E_{n+1}} \\
V \oplus_p V & \xrightarrow{\delta} & V
\end{array}
\]
commutes. But \( \gamma: E_n \oplus_p E_n \to E_{n+1} \) is invertible, so \( \theta|_{E_{n+1}} \) is uniquely determined by \( \theta|_{E_n} \), completing the induction. Hence \( \theta \) is uniquely determined on the dense subspace \( E \) of \( L^p[0, 1] \), and so, as \( \theta \) is bounded, on \( L^p[0, 1] \) itself.

**Existence** For each \( n \geq 0 \), define a map \( \theta_n: E_n \to V \) in \( \text{Ban} \) as follows: \( \theta_0: E_0 = \mathbb{F} \to V \) is given by \( \theta_0(I) = v \) (and is a contraction because \( \| v \| \leq 1 \)), and inductively,
\[
\theta_{n+1} = \left( E_{n+1} \xrightarrow{\gamma^{-1}} E_n \oplus_p E_n \xrightarrow{\theta_n \oplus \theta_n} V \oplus_p V \xrightarrow{\delta} V \right).
\]
Using the axiom that $\delta(v,v) = v$, one checks that $\theta_{n+1}$ extends $\theta_n$ for each $n \geq 0$. Since $L^p[0,1]$ is the colimit of $E_0 \hookrightarrow E_1 \hookrightarrow \cdots$, there is a unique map $\theta: L^p[0,1] \to V$ such that $\theta|_{E_n} = \theta_n$ for each $n$.

It remains to prove that $\theta$ is a map $(L^p[0,1], I, \gamma) \to (V, v, \delta)$ in $\mathcal{A}^p$. First, $\theta(I) = \theta_0(I) = v$. Second, we must show that the lower square of the diagram

\[
\begin{array}{ccc}
E_n \oplus_p E_n & \xrightarrow{\gamma} & E_{n+1} \\
\theta_n \oplus \theta_n & \downarrow & \downarrow \\
L^p[0,1] \oplus_p L^p[0,1] & \xrightarrow{\gamma} & L^p[0,1] \\
\theta & \downarrow & \downarrow \\
V \oplus_p V & \xrightarrow{\delta} & V
\end{array}
\]

commutes. The upper square commutes trivially, the outer square commutes by definition of $\theta_{n+1}$, and the triangles commute by definition of $\theta$. Thus, the lower square commutes on the subspace $E_n \oplus_p E_n$ of $L^p[0,1] \oplus L^p[0,1]$, for each $n$. But

\[
\bigcup_{n \geq 0} E_n \oplus_p E_n = E \oplus_p E,
\]

and $E \oplus_p E$ is dense in $L^p[0,1] \oplus_p L^p[0,1]$ because $E$ is dense in $L^p[0,1]$ and $\oplus_p$ induces the product topology. Hence the lower square does commute.

Taking $p = 1$, we immediately obtain a characterization of integration. Let $m: \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ denote the arithmetic mean, $m(x,y) = \frac{1}{2}(x+y)$. Then $(\mathbb{F}, 1, m)$ is an object of $\mathcal{A}^1$.

**Proposition 2.2 (Uniqueness of integration)** The unique map

\[
(L^1[0,1], I, \gamma) \to (\mathbb{F}, 1, m)
\]

in $\mathcal{A}^1$ is the integration operator $\int_0^1$.

**Proof** By Theorem 2.1, it suffices to prove that $\int_0^1$ is a map in $\mathcal{A}^1$. This statement amounts to the linearity of $\int_0^1$ together with the properties

\[
\left| \int_0^1 f \right| \leq \int_0^1 |f|,
\]

\[
\int_0^1 I = 1,
\]

\[
\int_0^1 \gamma(f,g) = \frac{1}{2} \left( \int_0^1 f + \int_0^1 g \right)
\]

($f, g \in L^1[0,1]$).

There is an entirely elementary corollary, using no categorical language:
Corollary 2.3 \( f_0^1 \) is the unique bounded linear functional on \( L^1[0,1] \) such that \( f_0^1 1 = 1 \) and
\[
\int_0^1 f(x) \, dx = \frac{1}{2} \left( \int_0^1 f\left(\frac{x}{2}\right) \, dx + \int_0^1 f\left(\frac{x+1}{2}\right) \, dx \right)
\]
for all \( f \in L^1[0,1] \).

**Proof** This is just a restatement of Proposition 2.2, except that the hypothesis that \( f_0^1 \) is a contraction has been weakened to boundedness. That suffices because the uniqueness part of the proof of Theorem 2.1 uses only boundedness, not contractivity, of the maps in \( \mathscr{A}^p \). \( \square \)

The universal property of \( L^1[0,1] \) also produces integration on subintervals of \([0,1]\). The following result is due to Mark Meckes (personal communication).

Write \( C_*[0,1] \) for the Banach space of continuous functions \( F: [0,1] \to \mathbb{F} \) such that \( F(0) = 0 \), with the sup norm. Define \( i \in C_*[0,1] \) by \( i(x) = x \), and
\[
\kappa: C_*[0,1] \oplus C_*[0,1] \to C_*[0,1]
\]
by
\[
(\kappa(F,G))(x) = \begin{cases} 
\frac{1}{2} F(2x) & \text{if } 0 \leq x \leq \frac{1}{2}, \\
\frac{1}{2} (F(1) + G(2x-1)) & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}
\]
(\( F, G \in C_*[0,1] \)). Then \((C_*[0,1], i, \kappa)\) is an object of \( \mathscr{A}^1 \).

**Proposition 2.4 (Meckes)** The unique map \((L^1[0,1], I, \gamma) \to (C_*[0,1], i, \kappa)\) in \( \mathscr{A}^1 \) is the definite integration operator
\[
\int_0^x: L^1[0,1] \to C_*[0,1] \\
f \mapsto \int_0^x f.
\]

**Proof** By Theorem 2.1, it suffices to show that \( \int_0^x \) is a map in \( \mathscr{A}^1 \). This amounts to the linearity of integration together with the elementary facts that
\[
\left| \int_0^x f \right| \leq \int_0^1 |f|,
\]
\[
\int_0^x 1 = x,
\]
\[
\int_0^x \gamma(f,g) = \left( \kappa\left( \int_0^x f, \int_0^x g \right) \right)(x)
\]
for all \( f, g \in L^1[0,1] \) and \( x \in [0,1] \). \( \square \)

Theorem 2.1 uniquely characterizes \( L^1[0,1] \) as an abstract Banach space, but Proposition 2.4 allows us to realize its elements as equivalence classes of functions. Given an element \( \alpha \in L^1[0,1] \), first apply the map of Proposition 2.4 to obtain an element of \( C_*[0,1] \), then differentiate to obtain a function defined almost everywhere on \([0,1]\). This function is a representative of \( \alpha \), since every integrable function \( f \) satisfies \( f(x) = \frac{d}{dx} \int_0^x f \) for almost all \( x \).
Remarks 2.5  

i. The analogue of Theorem 2.1 for \( p = \infty \) is false. Let \( \oplus_{\infty} \) denote the direct sum with norm \( \|(v, w)\| = \max\{|v|, |w|\} \). Define \( \mathcal{A}^\infty \) analogously to \( \mathcal{A}^p \). Then by the same argument as for \( p < \infty \), the initial object of \( \mathcal{A}^\infty \) is the closure of \( E \) in \( L^\infty[0,1] \), together with \( I \) and \( \gamma \). This is not \( L^\infty[0,1] \); for example, \( E \) does not contain the indicator function \( I_{[0,1/3]} \).

ii. In Theorem 2.1, \([0,1]\) can equivalently be replaced by Cantor space \( \{0,1\}^\mathbb{N} \) with the probability measure in which the set of sequences beginning with \( n \) prescribed bits has measure \( 2^{-n} \). The measure-preserving surjection

\[
s: \{0,1\}^\mathbb{N} \to [0,1] \\
(x_0, x_1, \ldots) \mapsto \sum_{n=0}^\infty x_n 2^{-(n+1)}
\]

induces an isomorphism \( L^p[0,1] \cong L^p(\{0,1\}^\mathbb{N}) \) for each \( p \in [1, \infty] \). Under this isomorphism, \( \gamma \) corresponds to the map

\[
\gamma: L^p(\{0,1\}^\mathbb{N}) \oplus L^p(\{0,1\}^\mathbb{N}) \to L^p(\{0,1\}^\mathbb{N})
\]

defined by

\[
(\gamma(f, g))(x_0, x_1, \ldots) = \begin{cases} f(x_1, x_2, \ldots) & \text{if } x_0 = 0, \\ g(x_1, x_2, \ldots) & \text{if } x_0 = 1 \end{cases} \tag{1}
\]

(\( f, g \in L^p(\{0,1\}^\mathbb{N}), x_i \in \{0,1\} \)). Thus, \( (L^p(\{0,1\}^\mathbb{N}), I, \gamma) \) is initial in \( \mathcal{A}^p \) whenever \( p < \infty \), where \( I \) is the constant function 1 on \( \{0,1\}^\mathbb{N} \).

We now show that the initial object of \( \mathcal{A}^\infty \) consists of not bounded functions, but continuous functions, on the Cantor space rather than the interval. Give \( \{0,1\}^\mathbb{N} \) the product topology and \( C(\{0,1\}^\mathbb{N}) \) the sup norm. The map

\[
\gamma: C(\{0,1\}^\mathbb{N}) \oplus_{\infty} C(\{0,1\}^\mathbb{N}) \to C(\{0,1\}^\mathbb{N})
\]

defined by formula (1) is an isometric isomorphism, so \( (C(\{0,1\}^\mathbb{N}), I, \gamma) \) is an object of \( \mathcal{A}^\infty \).

**Proposition 2.6 (Universal property of functions on Cantor space)**

\( (C(\{0,1\}^\mathbb{N}), I, \gamma) \) is the initial object of \( \mathcal{A}^\infty \).

**Proof** For \( n \geq 0 \), let \( E_n \) be the subspace of \( C(\{0,1\}^\mathbb{N}) \) consisting of the functions \( f \) such that for all \( x = (x_0, x_1, \ldots) \) and \( y = (y_0, y_1, \ldots) \) in \( \{0,1\}^\mathbb{N} \),

\[
x_i = y_i \text{ for all } i < n \implies f(x) = f(y).
\]

Then \( E_0 \) is the linear span of \( I \) and \( E_{n+1} = \gamma(E_n \oplus E_n) \) for each \( n \geq 0 \). The argument used to prove Theorem 2.1 also shows that \( (C(\{0,1\}^\mathbb{N}), I, \gamma) \) is initial in \( \mathcal{A}^\infty \), as long as \( E = \bigcup_{n \geq 0} E_n \) is dense in \( C(\{0,1\}^\mathbb{N}) \). We show this now.

The topology on \( \{0,1\}^\mathbb{N} \) is metrized by \( d(x, y) = 2^{-\min(n: x_n \neq y_n)} \). For \( n \geq 0 \), define \( \pi_n: \{0,1\}^\mathbb{N} \to \{0,1\}^n \) by

\[
\pi_n(x) = (x_0, \ldots, x_{n-1}, 0, 0, \ldots).
\]
Thus, \(d(x, \pi_n(x)) \leq 2^{-n}\) for all \(x\). Now let \(f \in C(\{0,1\}^N)\); we prove that \(f \in \overline{E}\).

For each \(n \geq 0\), we have \(f \circ \pi_n \in E_n\) and
\[
\|f - f \circ \pi_n\|_\infty \leq \sup_{x,y : d(x,y) \leq 2^{-n}} |f(x) - f(y)|.
\]

But since \(\{0,1\}^N\) is compact, \(f\) is uniformly continuous, so the right-hand side converges to 0 as \(n \to \infty\). Hence \(f = \lim_{n \to \infty} f \circ \pi_n \in \overline{E}\), as required. \(\square\)

The next two results show how the universal property induces some standard inclusions between function spaces. First let \(1 \leq p \leq q < \infty\). For any Banach spaces \(V\) and \(W\), the identity map
\[
V \oplus_r W \to V \oplus_p W
\]
is a contraction, by the elementary fact that power means are increasing in their order \([9, \text{Theorem 16}]\). Hence \(\mathcal{A}_p\) is a subcategory of \(\mathcal{A}_r\), and in particular, \((L^r[0,1], I, \gamma)\) can be regarded as an object of \(\mathcal{A}_r\). By Theorem 2.1, there is a unique map \((L^r[0,1], I, \gamma) \to (L^p[0,1], I, \gamma)\) in \(\mathcal{A}_r\).

**Proposition 2.7** For \(1 \leq p \leq q < \infty\), the unique map \((L^r[0,1], I, \gamma) \to (L^p[0,1], I, \gamma)\) in \(\mathcal{A}_r\) is the inclusion \(L^r[0,1] \hookrightarrow L^p[0,1]\).

**Proof** Clearly the inclusion is one such map, and by Theorem 2.1, it is the only one. \(\square\)

Similarly, \(\mathcal{A}_p\) is a subcategory of \(\mathcal{A}_\infty\) for every \(p\), so \((L^p[0,1], I, \gamma)\) is an object of \(\mathcal{A}_\infty\). Hence by Proposition 2.6, there is a unique map \n((C(\{0,1\}^N), I, \gamma) \to (L^p[0,1], I, \gamma) (2)\)
in \(\mathcal{A}_\infty\). On the other hand, composition with the map \(s : \{0,1\}^N \to [0,1]\) of Remark 2.5(ii) defines a map \(s^* : C(\{0,1\}) \to C(\{0,1\}^N)\).

**Proposition 2.8** Let \(1 \leq p < \infty\). The composite of \(s^*\) with the unique map (2) in \(\mathcal{A}_\infty\) is the inclusion \(C[0,1] \hookrightarrow L^p[0,1]\).

**Proof** Let \(i : [0,1] \to \{0,1\}^N\) be any section of the surjection \(s\) (a choice of binary expansion of each element of \([0,1]\)). One easily checks that \(f \mapsto f \circ i\) is a map of the form (2) in \(\mathcal{A}_\infty\), so it is the unique such map. Hence the composite of \(s^*\) with (2) is the map \(C[0,1] \to L^p[0,1]\) given by \(g \mapsto g \circ s \circ i = g\). \(\square\)

This result relates the abstractly characterized space \(L^p[0,1]\) to the concrete function space \(C[0,1]\).

Next we use universal properties to construct the multiplication map
\[
L^p[0,1] \times L^q[0,1] \longrightarrow L^1[0,1] (3)
\]
for each \(p, q > 1\) such that \(\frac{1}{p} + \frac{1}{q} = 1\). Composing (3) with the integration operator \(\int_0^1 : L^1[0,1] \to \mathbb{F}\) (derived in Proposition 2.2), we obtain the standard pairing between \(L^p[0,1]\) and \(L^q[0,1]\).

For the rest of this section, write \(L^p\) as shorthand for \(L^p[0,1]\). For Banach spaces \(V\) and \(W\), write \(\text{Hom}(V, W)\) for the Banach space of all bounded linear maps from \(V\) to \(W\), with the operator norm.
To construct the multiplication map (3), we will give $\text{Hom}(L^q, L^1)$ the structure of an object of $\mathcal{A}_p$. By Theorem 2.1, this structure will induce a map $L^p \rightarrow \text{Hom}(L^q, L^1)$, or equivalently a map $L^p \times L^q \rightarrow L^1$. We will show that this is multiplication.

To give $\text{Hom}(L^q, L^1)$ the structure of an object of $\mathcal{A}_p$, first recall that we have already constructed the inclusion $j : L^q \hookrightarrow L^1$ (Proposition 2.7). This $j$ is a map in $\text{Ban}$, that is, an element of the closed unit ball of $\text{Hom}(L^q, L^1)$.

Now define a linear map

$$\Gamma : \text{Hom}(L^q, L^1) \otimes_p \text{Hom}(L^q, L^1) \rightarrow \text{Hom}(L^q, L^1)$$

as follows: for $\phi, \phi' \in \text{Hom}(L^q, L^1)$, let $\Gamma(\phi, \phi')$ be the composite

$$L^q \xrightarrow{\gamma^{-1}} L^q \oplus L^q \xrightarrow{\phi \oplus \phi'} L^1 \oplus L^1 \xrightarrow{\gamma} L^1.$$

**Lemma 2.9** ($\text{Hom}(L^q, L^1), j, \Gamma$) is an object of $\mathcal{A}_p$.

**Proof** It is immediate from the definitions that $\Gamma(j, j) = j$, so we only have to show that $\Gamma$ is a contraction. Let $g \in L^q$. Writing $\gamma^{-1}(g) = (g_1, g_2)$,

$$\|\Gamma(\phi_1, \phi_2)(g)\| = \|\gamma(\phi_1(g_1), \phi_2(g_2))\|$$

$$= \frac{1}{2}(\|\phi_1(g_1)\| + \|\phi_2(g_2)\|) \leq \frac{1}{2}(\|\phi_1\| \|g_1\| + \|\phi_2\| \|g_2\|) \leq \frac{1}{2}(\|\phi_1\|^p + \|\phi_2\|^p)^{1/p} (\|g_1\|^q + \|g_2\|^q)^{1/q} = \left(\frac{1}{2}(\|\phi_1\|^p + \|\phi_2\|^p)\right)^{1/p} \left(\frac{1}{2}(\|g_1\|^q + \|g_2\|^q)\right)^{1/q} = \|\phi_1, \phi_2\| \|g\|$$

where (4) holds because $\gamma : L^1 \otimes_1 L^1 \rightarrow L^1$ is an isometry, (5) is by Hölder’s inequality for vectors in $\mathbb{R}^2$, and (6) holds because $\gamma : L^1 \otimes_q L^q \rightarrow L^q$ is an isometry. Hence $\Gamma$ is a contraction, as claimed. □

By Theorem 2.1, then, there is a unique map

$$(L^p, L, \gamma) \rightarrow (\text{Hom}(L^q, L^1), j, \Gamma)$$

in $\mathcal{A}_p$. As a linear map $L^p \rightarrow \text{Hom}(L^q, L^1)$, it corresponds to a bilinear map

$$\square : L^p \times L^q \rightarrow L^1.$$

**Proposition 2.10** The map $\square : L^p \times L^q \rightarrow L^1$ is the product $(f, g) \mapsto f \cdot g$.

**Proof** By Hölder’s inequality for functions on $[0, 1]$, there is a linear contraction

$$\theta : L^p \rightarrow \text{Hom}(L^q, L^1), \quad f \mapsto f \cdot -. $$

By the uniqueness part of Theorem 2.1, it suffices to show that $\theta$ is a map of the form (7) in $\mathcal{A}_p$. Evidently $\theta(I) = j$. It remains to show that the square

$$\begin{array}{ccc}
L^p \oplus L^p & \xrightarrow{\gamma} & L^p \\
\begin{array}{c}
\theta \oplus \theta \\
\downarrow \\
\text{Hom}(L^q, L^1) \oplus \text{Hom}(L^q, L^1)
\end{array} & \xrightarrow{\Gamma} & \text{Hom}(L^q, L^1)
\end{array}$$

is commutative.

\[11\]
commutes, or equivalently that for all $f_1, f_2 \in L^p$,
\[
\Gamma(\theta(f_1), \theta(f_2)) = \theta(\gamma(f_1, f_2)).
\]
This in turn is equivalent to
\[
\gamma(f_1 \cdot g_1, f_2 \cdot g_2) = \gamma(f_1, f_2) \cdot \gamma(g_1, g_2)
\]
for all $f_1, f_2 \in L^p$ and $g_1, g_2 \in L^q$, which follows from the definition of $\gamma$. \qed

The universal property of $L^p[0,1]$ arises from the self-similarity of $[0,1]$, or more specifically, the isomorphism between $[0,1]$ and two copies of itself glued end to end. An analogous universal property is satisfied by $L^p(X)$ for other self-similar spaces $X$. For example, the isomorphism $\mathbb{N} \cong \{\ast\} \mathbb{N}$ gives rise to a universal property characterizing the sequence space $\ell^p$, as follows.

Let $p \in [1,\infty]$. For Banach spaces $V$ and $W$, denote by $V \oplus_p W$ the direct sum $V \oplus W$ with norm
\[
\|(v,w)\| = \begin{cases} 
(||v||^p + ||w||^p)^{1/p} & \text{if } p < \infty, \\
\max\{||v||, ||w||\} & \text{if } p = \infty.
\end{cases}
\]
Let $\mathcal{B}^p$ be the category of Banach spaces $V$ equipped with a map $\delta: F \oplus_p V \to V$ in $\textbf{Ban}$. The maps $(V'_{V'},\delta') \to (V,\delta)$ in $\mathcal{B}^p$ are the maps $\theta: V' \to V$ in $\textbf{Ban}$ such that $\theta \circ \delta' = \delta \circ (\text{id}_V \oplus \theta)$.

Define $\gamma: F \oplus_p \ell^p \to \ell^p$ by
\[
\gamma(a, (a_0, a_1, \ldots)) = (a, a_0, a_1, \ldots).
\] (8)
Then $\gamma$ is an isometric isomorphism, so $(\ell^p, \gamma)$ is an object of $\mathcal{B}^p$. Write $c_0$ for the Banach space of sequences in $F$ converging to 0, with the sup norm; then there is a map $\gamma: F \oplus_\infty c_0 \to c_0$ defined by (8), and $(c_0, \gamma)$ is an object of $\mathcal{B}^\infty$.

**Proposition 2.11 (Universal properties of sequence spaces)** $(\ell^p, \gamma)$ is the initial object of $\mathcal{B}^p$ for $1 \leq p < \infty$, and $(c_0, \gamma)$ is the initial object of $\mathcal{B}^\infty$.

**Proof** We assume that $p < \infty$, the case $p = \infty$ being very similar.

For $n \geq 0$, let $E_n \subseteq \ell^p$ be the subspace of sequences of the form $(a_0, \ldots, a_n, 0, 0, \ldots)$. Then $\bigcup_{n\geq 0} E_n$ is dense in $\ell^p$, so $\ell^p$ is the colimit of the diagram $E_0 \leftarrow E_1 \leftarrow \cdots$ in $\textbf{Ban}$. Moreover, $\gamma: F \oplus_p \ell^p \to \ell^p$ restricts to an isomorphism $F \oplus_p E_n \to E_{n+1}$ for each $n \geq 0$.

Let $(V,\delta) \in \mathcal{B}^p$. To prove the existence of a map $\theta: (\ell^p, \gamma) \to (V,\delta)$ in $\mathcal{B}^p$, we define $\theta_n: E_n \to V$ inductively by $\theta_0(a_0, 0, 0, \ldots) = \delta(a_0, 0)$ and
\[
\theta_{n+1} = \left( E_{n+1} \xrightarrow{\gamma^{-1}} F \oplus_p E_n \xrightarrow{\text{id}_F \oplus \theta_n} F \oplus_p V \xrightarrow{\delta} V \right).
\]
Then $\theta_{n+1}$ extends $\theta_n$ for each $n$, and by the colimit description of $\ell^p$, there is a unique map $\theta: \ell^p \to V$ extending every $\theta_n$. The rest of the proof is similar to that of Theorem 2.1, and further details are omitted. \qed

**Remark 2.12** Theorem 2.1 on $L^p[0,1]$, Proposition 2.6 on $C(\{0,1\}^\mathbb{F})$, and Proposition 2.11 on $\ell^p$ and $c_0$ are all instances of a general categorical theorem.
that not only proves the existence of these initial objects, but also constructs them.

Let $\mathscr{D}$ be a category. An algebra for an endofunctor $T: \mathscr{D} \to \mathscr{D}$ is an object $D$ of $\mathscr{D}$ together with a map $\delta: T(D) \to D$. A map of algebras $(D', \delta') \to (D, \delta)$ is a map $\theta: D' \to D$ in $\mathscr{D}$ such that $\theta \circ \delta' = \delta \circ T\theta$. A lemma of Lambek states that if $(D, \delta)$ is an initial algebra then $\delta$ is an isomorphism [10, Lemma 2.2].

Now suppose that $\mathscr{D}$ itself has an initial object, $Z$, that the diagram

\[ Z \xrightarrow{!} T(Z) \xrightarrow{T(\delta)} T^2(Z) \xrightarrow{T^2(\delta)} \cdots \]

has a colimit (where $!$ is the unique map $Z \to T(Z)$), and that this colimit is preserved by $T$; in other words, the canonical map

\[ \eta: \text{colim}_n T^n(Z) \to T\left(\text{colim}_n T^n(Z)\right) \]

is an isomorphism. A theorem of Adámek states that $(\text{colim}_n T^n(Z), \eta^{-1})$ is the initial $T$-algebra ([1, p. 591] or [2, Theorem 3.17]).

For example, let $\mathscr{D}$ be the category $\text{Ban}$, of pairs $(V, v)$ where $V \in \text{Ban}$ and $v \in V$ with $\|v\| \leq 1$; maps in $\text{Ban}$, are maps in $\text{Ban}$ preserving the chosen points. (This is the coslice category $\text{F/Ban}$.) Define $T: \mathscr{D} \to \mathscr{D}$ by

\[ T(V, v) = (V \oplus_p V, (v, v)) \]

Then the category of $T$-algebras is $\mathcal{A}^P$. Moreover, $(\mathcal{F}, 1)$ is initial in $\text{Ban}$, and the hypotheses of Adámek’s theorem hold, so the initial object of $\mathcal{A}^P$ is constructed as the colimit over $n \geq 0$ of the objects $T^n(\mathcal{F}, 1)$ of $\text{Ban}$. Concretely, $T^n(\mathcal{F}, 1) = (E_n, I)$, where $E_n$ is as defined in the proof of Theorem 2.1. Hence this categorical method constructs $L^p[0, 1]$ as the colimit of $E_0 \hookrightarrow E_1 \hookrightarrow \cdots$ in $\text{Ban}$. Similarly, Propositions 2.6 and 2.11 are also instances of Adámek’s theorem. In each of these results, the map denoted by $\gamma$ is an isometric isomorphism, and Lambek’s lemma explains why.

### 3 Integration on an arbitrary measure space

Our second main theorem characterizes the functor $L^p$ from measure spaces to Banach spaces, again by a simple universal property.

The measure on a measure space $X$ is written as $\mu_X$. Throughout, all measure spaces are understood to be finite ($\mu_X(X) < \infty$).

An embedding $i: Y \to X$ of measure spaces is an injection such that $B \subseteq Y$ is measurable if and only if $iB \subseteq X$ is measurable, and in that case, $\mu_Y(B) = \mu_X(iB)$. Measure spaces and embeddings form a category $\text{Meas}_{\text{emb}}$.

A measure-preserving partial map $(A, s): X \to Y$ is a measurable subset $A \subseteq X$ together with a measure-preserving map $s: A \to Y$. Here $A$ is given the unique measure space structure such that the inclusion $A \hookrightarrow X$ is an embedding. The composite of measure-preserving partial maps

\[ X \xrightarrow{(A, s)} Y \xrightarrow{(B, t)} Z \]

is $(s^{-1}B, u)$, where $u: s^{-1}B \to Z$ is defined by $u(x) = t(s(x))$. Measure spaces and measure-preserving partial maps form a category $\text{Meas}$.
For each $p \in [1, \infty]$, there is a functor

$$L^p : \text{Meas}^{\text{op}} \to \text{Ban},$$

defined on objects in the usual way and on maps as follows: given a measure-preserving partial map $(A, s): X \to Y$, the induced map $L^p(Y) \to L^p(X)$ is $g \mapsto (g \circ s)^X$, where $(g \circ s)^X$ denotes the composite $g \circ s : A \to \mathcal{F}$ extended by zero to $X$.

Any embedding $i: Y \to X$ determines a map $(iY, i^{-1}): X \to Y$ in $\text{Meas}$; note the change of direction. (In particular, every measurable subset $A$ of $X$ gives a map $(A, \text{id}_A): X \to A$.) Also, every measure-preserving map $s: X \to Y$ determines a map $(X, s): X \to Y$ in $\text{Meas}$. Thus, both $\text{Meas}^{\text{emb}}$ and the category $\text{Meas}^{\text{pres}}$ of measure spaces and measure-preserving maps can be viewed as subcategories of $\text{Meas}$:

$$\text{Meas}^{\text{emb}} \subseteq \text{Meas}^{\text{op}} \subseteq \text{Meas}.$$

Furthermore, every measure-preserving partial map factors canonically into maps of these two special types:

$$X \xrightarrow{(A, s)} Y = \left( X \xrightarrow{(A, \text{id}_A)} A \xrightarrow{(A, s)} Y \right).$$

This has two consequences. First, any functor $F : \text{Meas}^{\text{op}} \to \mathcal{C}$ to any category $\mathcal{C}$ is determined by its effect on objects, embeddings, and measure-preserving maps. For simplicity, we write $Fi$ for the map $F(iY, i^{-1}): F(Y) \to F(X)$ induced by an embedding $i: Y \to X$, and $Fs$ for the map $F(X, s): F(Y) \to F(X)$ induced by a measure-preserving map $s: X \to Y$. Second, a transformation between two functors $\text{Meas}^{\text{op}} \to \mathcal{C}$ is natural if and only if it is natural with respect to both embeddings and measure-preserving maps.

**Lemma 3.1** Let $F : \text{Meas}^{\text{op}} \to \mathcal{C}$ be a functor into a category $\mathcal{C}$. Let $s: X \to Y$ be a measure-preserving map and $B$ a measurable subset of $Y$. Write

$$s^{-1}B \xrightarrow{s'} B \xrightarrow{j} Y \xleftarrow{i} X$$

for the resulting inclusions $(i$ and $j$) and the restriction $s'$ of $s$. Then the square

$$\begin{array}{ccc}
F(s^{-1}B) & \xrightarrow{F(s')} & F(B) \\
F_i \downarrow & & \downarrow F_j \\
F(X) & \xrightarrow{F_s} & F(Y)
\end{array}$$

in $\mathcal{C}$ commutes.

Results of this type are known as Beck–Chevalley conditions [12, Section IV.9].
Table 2: Results leading up to the main theorem, Theorem 3.7. For example, Proposition 3.4 concerns two categories, \( \mathcal{V} \) and \( \mathcal{V}_{\mathrm{emb}} \), whose objects are vector-space-valued functors together with some extra data; it states that the initial object of both categories is \((\mathcal{S}, I)\), where \( \mathcal{S} \) denotes the simple functions.

**Proof** The square

\[
\begin{array}{ccc}
X & \xrightarrow{(s^{-1}B, \id_{s^{-1}B})} & B \\
\downarrow & & \downarrow \\
Y & \xrightarrow{(B, \id_B)} & Y
\end{array}
\]

in \( \text{Meas} \) commutes, and the result follows by functoriality of \( F \). \( \square \)

We prove our main theorem in three steps (Table 2), which informally are as follows. First, the universal vector space obtained from a measure space consists of the simple functions. Second, the universal normed vector space on a measure space consists of the a.e. equivalence classes of simple functions. Third, as the main theorem, the universal Banach space on a measure space consists of the integrable functions. Each step has two versions, according to whether one considers functoriality with respect to all measure-preserving partial maps or only the embeddings.

We now begin the first step. Two embeddings

\[
Y \xrightarrow{i} X \leftarrow \xleftarrow{j} Z
\]

of measure spaces are **complementary** if \( iY \cap jZ = \emptyset \) and \( iY \cup jZ = X \). Write \( \text{Vect} \) for the category of vector spaces. Let \( \mathcal{V}_{\mathrm{emb}} \) be the category of pairs \((F, v)\) consisting of a functor \( F: \text{Meas}_{\mathrm{emb}} \to \text{Vect} \) together with an element \( v_X \in F(X) \) for each measure space \( X \), satisfying the following axiom:

**I**  \( (Fi)(v_Y) + (Fj)(v_Z) = v_X \) for all pairs of complementary embeddings \( Y \xrightarrow{i} X \xleftarrow{j} Z \).

A map \((F', v') \to (F, v)\) in \( \mathcal{V}_{\mathrm{emb}} \) is a natural transformation \( \psi: F' \to F \) such that \( \psi_X(v'_X) = v_X \) for all \( X \).

When \( i: Y \to X \) is an embedding and \( i \) is understood, we write \( (Fi)(v_Y) \in F(X) \) as \( v_X^Y \). In this notation, axiom (I) states that \( v_Y^X + v_Z^X = v_X \).

For example, there is an object \((\mathcal{S}, I)\) of \( \mathcal{V}_{\mathrm{emb}} \) defined as follows. The functor \( \mathcal{S}: \text{Meas}_{\mathrm{emb}} \to \text{Vect} \) assigns to each measure space \( X \) the space of simple functions \( X \to \mathbb{F} \), and is defined on embeddings by extending simple functions by zero. The element \( I_X \in \mathcal{S}X \) is the function with constant value 1. Given an embedding \( i: Y \to X \), the simple function \( I_Y^X = (\mathcal{S}i)(I_Y) \) on
X is the indicator function of iY ⊆ X. Axiom (I) is the elementary fact that $I_X^X + I_Z^Z = I_X$ for any measurable partition $X = Y \sqcup Z$.

To incorporate functoriality with respect to all measure-preserving partial maps, let $\mathcal{Y}$ be the category of pairs $(F, v)$ consisting of a functor $F : \text{Meas}^{\text{op}} \to \text{Vect}$ together with an element $v_X \in F(X)$ for each measure space $X$, satisfying axiom (I) and:

(II) $(F s)(v_Y) = v_X$ for all measure-preserving maps $X \xrightarrow{s} Y$.

The maps in $\mathcal{Y}$ are defined as in $\mathcal{Y}_{\text{emb}}$.

The functor $\mathcal{S}$ on $\text{Meas}_{\text{emb}}$ extends naturally to a functor $\mathcal{S} : \text{Meas}^{\text{op}} \to \text{Vect}$, since the composite of a simple function $Y \to \mathbb{F}$ with a measure-preserving map $X \to Y$ is again simple. The pair $(\mathcal{S}, Meas^{\text{op}} \to \text{Vect}, I)$ is an object of $\mathcal{Y}$. By abuse of notation, we write $(\mathcal{S}, I)$ for both this object of $\mathcal{Y}$ and the object of $\mathcal{Y}_{\text{emb}}$ defined previously, which is its image under the evident forgetful functor $\mathcal{Y} \to \mathcal{Y}_{\text{emb}}$.

The following two lemmas establish basic properties of $\mathcal{Y}_{\text{emb}}$ and $\mathcal{Y}$.

**Lemma 3.2** Let $(F, v) \in \mathcal{Y}_{\text{emb}}$. Let $n \geq 0$ and let $(i_r : Y_r \to X)_{1 \leq r \leq n}$ be a family of embeddings with pairwise disjoint images. Then

$$v^X_{i_1 Y_1 \sqcup \cdots \sqcup i_n Y_n} = v^Y_{Y_1} + \cdots + v^Y_{Y_n}.$$ 

In particular, $v^X_\emptyset = 0$ for any measure space $X$.

**Proof** Put $Y = i_1 Y_1 \sqcup \cdots \sqcup i_n Y_n \subseteq X$. First we show that

$$v^Y = v^Y_{Y_1} + \cdots + v^Y_{Y_n}. \quad (9)$$

Axiom (I) applied to the embeddings $\emptyset \to \emptyset \leftarrow \emptyset$ gives $v_\emptyset + v_\emptyset = v_\emptyset$ and so $v_\emptyset = 0$, which is the case $n = 0$. The general case follows by induction, again using (I). This proves (9). Now the functoriality of $F : \text{Meas}_{\text{emb}} \to \text{Vect}$ gives

$$v^X_Y = (v^Y_{Y_1} + \cdots + v^Y_{Y_n})^X = v^X_{Y_1} + \cdots + v^X_{Y_n},$$

as required. The last part of the lemma is the case $n = 0$. \hfill \Box

**Lemma 3.3** Let $(F, v) \in \mathcal{Y}$. Let $s : X \to Y$ be a measure-preserving map and $B$ a measurable subset of $Y$. Then $(F s)(v^Y_B) = v^X_{s^{-1}B}$.

**Proof** Define $i$, $j$ and $s'$ as in Lemma 3.1. We have

$$(F s)(v^Y_B) = (F s)(F j)(v_B)$$

by definition of $v^Y_B$, and

$$(F s)(v^X_{s^{-1}B}) = (F i)(v^X_{s^{-1}B}) = (F i)(F s')(v_B)$$

by definition of $v^X_{s^{-1}B}$ and axiom (II). The result follows from Lemma 3.1. \hfill \Box

We now establish the universal properties of the simple functions. The proof implicitly uses the fact that a function on a measure space is simple if and only if its image is finite and each fibre is measurable.

**Proposition 3.4** (Universal properties of spaces of simple functions)

i. $(\mathcal{S}, I)$ is the initial object of $\mathcal{Y}_{\text{emb}}$.

ii. $(\mathcal{S}, I)$ is the initial object of $\mathcal{Y}$.

**Proof** For (i), let $(F, v) \in \mathcal{Y}_{\text{emb}}$. We show that there exists a unique map $\psi : (\mathcal{S}, I) \to (F, v)$ in $\mathcal{Y}_{\text{emb}}$.
**Uniqueness** Let $\psi: (\mathcal{F}, I) \to (F, v)$ in $\mathcal{K}_{emb}$. For each measure space $X$ and measurable $Y \subseteq X$, the naturality of $\psi$ with respect to the inclusion $i: Y \hookrightarrow X$ gives a commutative square

$$
\begin{array}{ccc}
\mathcal{F}(Y) & \xrightarrow{\psi_Y} & \mathcal{F}(X) \\
\psi_Y \downarrow & & \downarrow \psi_X \\
F(Y) & \xrightarrow{F_i} & F(X).
\end{array}
$$

Evaluating the square at $I_Y \in \mathcal{F}(Y)$ gives

$$
\psi_X(I_X Y) = (F_i)(\psi_Y(I_Y)) = (F_i)(v_Y) = v_X^Y.
$$

Hence $\psi_X$ is uniquely determined on indicator functions of measurable subsets of $X$, and therefore, by linearity, on all of $\mathcal{F}(X)$.

**Existence** For each measure space $X$ and $f \in \mathcal{F}(X)$, define

$$
\psi_X(f) = \sum_{c \in F} c v_{f^{-1}(c)}^X \in F(X). \quad (10)
$$

This sum has only finitely many nonzero summands, since if $c$ is not in the image of $f$ then $v_{f^{-1}(c)}^X = v_\emptyset^X = 0$ by Lemma 3.2.

To show that $\psi_X: \mathcal{F}(X) \to F(X)$ is linear, let $f, g \in \mathcal{F}(X)$. Then

$$
\psi_X(f + g) = \sum_{c} c v_{(f+g)^{-1}(c)}^X. \quad (11)
$$

But $(f + g)^{-1}(c) = \bigsqcup_{a, b: a + b = c} f^{-1}(a) \cap g^{-1}(b)$, so by Lemma 3.2,

$$
v_{(f+g)^{-1}(c)}^X = \sum_{a, b: a + b = c} v_{f^{-1}(a) \cap g^{-1}(b)}^X.
$$

Hence by (11),

$$
\begin{align*}
\psi_X(f + g) &= \sum_{a} a \sum_{b} v_{f^{-1}(a) \cap g^{-1}(b)}^X + \sum_{b} b \sum_{a} v_{f^{-1}(a) \cap g^{-1}(b)}^X \\
&= \sum_{a} av_{f^{-1}(a)}^X + \sum_{b} bv_{g^{-1}(b)}^X \\
&= \psi_X(f) + \psi_X(g), \quad (12)
\end{align*}
$$

where (12) is obtained by applying Lemma 3.2 to the disjoint unions

$$
f^{-1}(a) = \bigsqcup_b f^{-1}(a) \cap g^{-1}(b), \quad g^{-1}(b) = \bigsqcup_a f^{-1}(a) \cap g^{-1}(b).
$$

We have now shown that $\psi_X(f + g) = \psi_X(f) + \psi_X(g)$ for all $f, g \in \mathcal{F}(X)$. A similar argument shows that $\psi_X$ preserves scalar multiplication. Thus, $\psi_X$ is a linear map $\mathcal{F}(X) \to F(X)$.  

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Next we show that \( \psi_X \) is natural in \( X \in \text{Meas}_{\text{emb}} \). That is, given an embedding \( i : Y \to X \), we show that the square

\[
\begin{array}{ccc}
\mathcal{F}(Y) & \xrightarrow{\psi_Y} & \mathcal{F}(X) \\
\downarrow & & \downarrow \\
F(Y) & \xrightarrow{F_i} & F(X)
\end{array}
\]

commutes. By linearity, it suffices to check this on the indicator function \( I_B^Y \) of a measurable subset \( B \subseteq Y \). On the one hand, \( \psi_Y(I_B^Y) = v_B^Y \) by definition of \( \psi_Y \), so \( (F_i)(\psi_Y(I_B^Y)) = (v_B^Y)^X = v_B^X \). On the other, the extension by zero of \( I_B^Y \) to \( X \) is \( I_B^X \), and \( \psi_X(I_B^X) = v_B^X \). The square therefore commutes.

We have now shown that \( \psi = (\psi_X) \) defines a natural transformation \( \mathcal{F} \to F \). Moreover, \( \psi_X(I_X) = v_X \) for each measure space \( X \), by definition of \( \psi_X \). Hence \( \psi \) is a map \( (\mathcal{F}, I) \to (F, v) \) in \( \mathcal{Y}_{\text{emb}} \), completing the proof of (i).

To prove (ii), let \( (F, v) \in \mathcal{Y} \). By (i), there is a unique map \( \psi : (\mathcal{F}, I) \to (F, v) \) in \( \mathcal{Y}_{\text{emb}} \), and it suffices to prove that \( \psi \) is a map in \( \mathcal{Y} \). This reduces to showing that for every measure-preserving map \( s : X \to Y \), the naturality square

\[
\begin{array}{ccc}
\mathcal{F}(Y) & \xrightarrow{\psi_Y} & \mathcal{F}(X) \\
\downarrow & & \downarrow \\
F(Y) & \xrightarrow{Fs} & F(X)
\end{array}
\]

commutes. By linearity, it suffices to check this on the indicator function \( I_B^Y \) of a measurable subset \( B \subseteq Y \). On the one hand, \( \psi_X(I_B^Y \circ s) = \psi_X(I_{s^{-1}B}^X) = v_{s^{-1}B}^X \). On the other, \( (Fs)(\psi_Y(I_B^Y)) = (Fs)(v_B^Y) \). Lemma 3.3 concludes the proof. \( \square \)

This completes the first step towards the main theorem (Table 2). We now embark on the second.

Write \( \text{NVS} \) for the category of normed vector spaces and contractions. Let \( \mathcal{A}_p \) be the category of pairs \( (F, v) \) consisting of a functor \( F : \text{Meas}_{\text{emb}} \to \text{NVS} \) together with an element \( v_X \in F(X) \) for each measure space \( X \), satisfying axiom (I) and the following axioms:

(III) \( ||v_X|| \leq \mu_X(X)^{1/p} \) for all measure spaces \( X \);

(IV) \( ||(Fi)(u) + (Fj)(w)|| \leq (||u||^p + ||w||^p)^{1/p} \) for all pairs \( Y \xrightarrow{i} X \xleftarrow{j} Z \) of complementary embeddings, \( u \in F(Y) \), and \( w \in F(Z) \).

The maps in \( \mathcal{A}_p \) are defined as in \( \mathcal{Y}_{\text{emb}} \).

When \( p = 1 \), axiom (IV) is redundant, since the maps in \( \text{Ban} \) are contractions. When \( p = \infty \), the expression \( \mu_X(X)^{1/p} \) in (III) is interpreted as 0 if \( \mu_X(X) = 0 \) and 1 otherwise, and the right-hand side of the inequality in (IV) as \( \max(||u||, ||w||) \).

Let \( \mathcal{A} \) be the category defined in the same way, but replacing \( \text{Meas}_{\text{emb}} \) by \( \text{Meas}^{\infty} \) and requiring that each object \( (F, v) \) satisfies axioms (I)–(IV).

For example, an object \( (S^p, I) \) of \( \mathcal{A}_p \) and an object \( (S^p, I) \) of \( \mathcal{A} \) may be defined as follows (abusively using the same name for both). For a measure space
write $S^p(X)$ for the vector space of equivalence classes of simple functions on $X$ under equality almost everywhere, with the $L^p$ norm:

$$\|f\|_p = \left( \sum_{c \in F} |c|^p \mu_X(f^{-1}(c)) \right)^{1/p}$$

if $p < \infty$, and

$$\|f\|_\infty = \max \{ |c| : c \in \mathbb{F}, \mu_X(f^{-1}(c)) > 0 \}.$$  

(All but finitely many arguments in the sum and max are 0.) This construction defines functors

$$S^p : \text{Measemb} \to \text{NVS}, \quad S^p : \text{Meas}^{op} \to \text{NVS},$$

with the same functorial action as $\mathcal{J}$. They determine an object $S^p, I \in \text{N}^p_{\text{emb}}$ and an object $(S^p, I)$ of $\mathcal{N}^p$. Axioms (I)–(IV) are elementary properties of indicator functions and the $p$-norm. Equality holds in the inequalities (III) and (IV). Axiom (IV) is a consequence of the fact that the $p$-norm of a function on a disjoint union $Y \sqcup Z$ is determined in the evident way by the $p$-norms of its restrictions to $Y$ and $Z$.

**Lemma 3.5** Let $(F, v) \in \mathcal{N}^p_{\text{emb}}$. Then $\|v^X_Y\| \leq \mu_X(iY)^{1/p}$ for any embedding $i : Y \to X$ of measure spaces. In particular, $v^X_Y = 0$ whenever $Y$ is a measure-zero subspace of a measure space $X$.

**Proof** We have $\|v^X_Y\| = \|(Fi)(v_Y)\| \leq \|v_Y\|$ since $Fi$ is a contraction, and $\|v_Y\| \leq \mu_Y(Y)^{1/p} = \mu_X(iY)^{1/p}$ by axiom (III). $\square$

**Proposition 3.6 (Universal properties of spaces of simple functions)**

**Existence** First we prove that $\|\psi_X(f)\| \leq \|f\|_p$ for each measure space $X$ and $f \in \mathcal{J}(X)$. If $p < \infty$ then

$$\|\psi_X(f)\| = \left\| \sum_{c \in F} cv^X_{f^{-1}(c)} \right\|$$

$$\leq \left( \sum_{c \in F} |cv^X_{f^{-1}(c)}|^p \right)^{1/p}$$

$$\leq \left( \sum_{c \in F} |c|^p \mu_X(f^{-1}(c)) \right)^{1/p}$$

$$= \|f\|_p.$$
where (14) follows from the formula (10) for $\psi$, inequality (15) follows by induction from axiom (IV), (16) follows from Lemma 3.5, and (17) follows from (13). The proof for $p = \infty$ is similar. Hence $\|\psi_X(f)\| \leq \|f\|_p$, as claimed.

It follows that for each measure space $X$, there is a unique linear map $\phi_X$ such that the triangle

$$
\begin{array}{ccc}
\mathcal{S}(X) & \longrightarrow & S^p(X) \\
\psi_X & \downarrow & \psi_X \\
F(X) & \downarrow & \end{array}
$$

commutes (where the horizontal arrow is the quotient map), and that $\phi_X$ is a contraction. Moreover, $\phi_X$ is natural in $X$ because $\psi_X$ is, and $\phi_X(I_X) = \psi_X(I_X) = v_X$. Hence $\phi$ is a map $(S^p, I) \to (F, v)$ in $\mathcal{N}^p_{\text{emb}}$, completing the proof of (i).

Part (ii) follows from the fact that $\psi$ is natural with respect to measure-preserving maps, by Proposition 3.4(ii). □

We now come to the main theorem. Let $R^p_{\text{emb}}$ denote the full subcategory of $\mathcal{N}^p_{\text{emb}}$ consisting of the pairs $(F, v)$ such that $F$ takes values in Banach spaces, and similarly $\mathcal{N}^p_{\text{emb}}$ and $\mathcal{N}^p$ are defined in the same way as $\mathcal{N}^p_{\text{emb}}$ and $\mathcal{N}^p$ respectively, but replacing NVS by Ban.

We have already defined the functor $L^p: \text{Meas}^{\text{op}} \to \text{Ban}$, and we also write $L^p$ for the restricted functor $\text{Meas}^{\text{emb}} \to \text{Ban}$. These functors define an object $(L^p, I)$ of $R^p_{\text{emb}}$ and an object $(L^p, I)$ of $R^p$.

**Theorem 3.7 (Universal properties of the $L^p$ functors)** Let $1 \leq p \leq \infty$.

i. $(L^p, I)$ is the initial object of $R^p_{\text{emb}}$.

ii. $(L^p, I)$ is the initial object of $R^p$.

Theorem B of the Introduction is the case $p = 1$ of (ii).

**Proof** The inclusion $\text{Ban} \hookrightarrow \text{NVS}$ has a left adjoint, the completion functor $V \mapsto \overline{V}$. Given $(F, v) \in R^p_{\text{emb}}$, define $\overline{F}: \text{Meas} \to \text{Ban}$ by $\overline{F}(X) = \overline{F(X)}$, and regard $v_X \in F(X)$ as an element of $\overline{F(X)}$. It is routine to verify that $(\overline{F}, v) \in R^p_{\text{emb}}$ and that the forgetful functor $R^p_{\text{emb}} \to \mathcal{N}^p_{\text{emb}}$ has left adjoint $(F, v) \mapsto (\overline{F}, v)$.

Left adjoints preserve initial objects, so by Proposition 3.6, the initial object of $R^p_{\text{emb}}$ is $(\overline{S^p}, I)$. But $L^p(X)$ is the completion of $S^p(X)$ for each $X$ [7, Proposition 6.7 and Theorem 6.8], so the initial object of $R^p_{\text{emb}}$ is $(L^p, I)$.

This proves (i), and the same argument proves (ii). □

Theorem 3.7 characterizes the spaces $L^p(X)$ uniquely up to isometric isomorphism, since the maps in Ban are contractions.

We now focus on the case $p = 1$. As in the case of the interval, the universal characterization of the integrable functions yields a unique characterization of integration, as follows.

Write $\mathcal{F}: \text{Meas}^{\text{emb}} \to \text{Ban}$ for the functor that sends all measure spaces to the ground field $F \in \text{Ban}$ and all maps in $\text{Meas}^{\text{emb}}$ to $id_F$. For each measure space $X$, put $t_X = \mu_X(X) \in \mathcal{F}$. Then $(\mathcal{F}, t) \in R^1_{\text{emb}}$. 

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Proposition 3.8 (Uniqueness of integration) The unique map \((L^1, I) \to (\mathbb{F}, t)\) in \(B_{\text{emb}}^1\) is the family of operators

\[
\int_X = \left( \int_X : L^1(X) \to \mathbb{F} \right)_{\text{measures spaces } X}.
\]

Proof By Theorem 3.7(i), it suffices to show that \(\int_X\) is indeed a map \((L^1, I) \to (\mathbb{F}, t)\) in \(B_{\text{emb}}^1\). This reduces to the following standard properties of integration. First, whenever \(X\) is a measure space, \(\int_X\) is a map in \(B_{\text{emb}}^1\) by linearity of integration and the triangle inequality

\[
\left| \int_X f \right| \leq \int_X |f|
\]

\((f \in L^1(X))\). Second, \(\int_X\) defines a natural transformation \(L^1 \to \mathbb{F}\) because

\[
\int_X g^X = \int_Y g
\]

for any embedding of measure spaces \(Y \to X\) and \(g \in L^1(Y)\), where \(g^X\) denotes the extension by zero of \(g\) to \(X\). Finally,

\[
\int_X I_X = \mu_X(X)
\]

for every measure space \(X\). \(\square\)

Remark 3.9 Here we have treated \((\mathbb{F}, t)\) as an object of \(B_{\text{emb}}^1\). But the constant functor \(\mathbb{F} : \text{Meas}^{\text{op}} \to \text{Ban}\) together with the elements \(t_X \in \mathbb{F}\) also define an object \((\mathbb{F}, t)\) of \(B^1\). Theorem 3.7(ii) implies that the unique map \((L^p, I) \to (\mathbb{F}, t)\) in \(B_{\text{emb}}^1\) is in fact a map in \(B^1\). In concrete terms, this statement is the formula for integration under a change of variables:

\[
\int_Y g = \int_X g \circ s
\]

whenever \(s : X \to Y\) is a measure-preserving map and \(g \in L^1(Y)\).

Our next result relates the abstractly characterized spaces \(L^1(X)\) to actual spaces of functions. Of course, we cannot hope to evaluate an element of \(L^1(X)\) at a point of \(X\), since it is only an equivalence class of integrable functions. The best we can hope for is to be able to integrate it over any measurable subset of \(X\). That is, we would like to construct for each \(f \in L^1(X)\) the signed or complex measure \(f \mu_X = f d\mu_X\) defined by \((f \mu_X)(A) = \int_A f d\mu_X\). We now show that this construction arises naturally from the universal property of \(L^1\).

Write \(M(X)\) for the Banach space of finite signed measures (if \(\mathbb{F} = \mathbb{R}\)) or complex measures (if \(\mathbb{F} = \mathbb{C}\)) on the underlying \(\sigma\)-algebra of a measure space \(X\), with the total variation norm \(\nu \mapsto |\nu|(X)\). Any embedding \(i : Y \to X\) induces an isometry \(M(Y) \to M(X)\) that extends measures by zero; thus, \(M\) defines a functor \(\text{Meas}_{\text{emb}} \to \text{Ban}\). Together with the elements \(\mu_X \in M(X)\), it gives an object \((M, \mu)\) of \(B_{\text{emb}}^1\).
Proposition 3.10 The unique map \((L^1, I) \to (M, \mu)\) in \(B^1_{\text{emb}}\) has \(X\)-component

\[
L^1(X) \to M(X)  \\
f \mapsto f\mu_X
\]

for each measure space \(X\).

Proof For each \(X\), define \(\theta_X : L^1(X) \to M(X)\) by \(\theta_X(f) = f\mu_X\). By Theorem 3.7(i), it suffices to show that \(\theta\) defines a map \((L^1, I) \to (M, \mu)\) in \(B^1_{\text{emb}}\).

It is elementary that \(\theta\) is an isometry and that \(\theta_X(I_X) = \mu_X\), for each \(X\). So it only remains to prove that \(\theta_X\) is natural in \(X \in \text{Meas}_{\text{emb}}\).

Let \(i : Y \to X\) be an embedding. We must show that the square

\[
\begin{array}{ccc}
L^1(Y) & \xrightarrow{\theta_Y} & L^1(X) \\
\downarrow \theta_Y & & \downarrow \theta_X \\
M(Y) & \xrightarrow{\theta_X} & M(X)
\end{array}
\]

commutes, where both horizontal maps are extension by zero. Equivalently, writing \((\cdot)^X\) for the extension by zero to \(X\) of a function or measure on \(Y\), we must show that \(g^X\mu_X = (g\mu_Y)^X\) for all \(g \in L^1(Y)\). But this just states that

\[
\int_A g^X \, d\mu_X = \int_{A \cap Y} g \, d\mu_Y
\]

for all measurable \(A \subseteq X\), which is true. \(\square\)

There is a similar theorem in which \(M(X)\) is replaced by the subspace \(AC(X)\) of signed or complex measures absolutely continuous with respect to \(\mu_X\); the unique map \((L^1, I) \to (AC, \mu)\) in \(B^1_{\text{emb}}\) is \(f \mapsto f\mu_X\). This map is an isomorphism (the Radon–Nikodym theorem), but that does not seem to be an easy consequence of the universal property of \((L^1, I)\).

Finally, consider the case \(p = 2\). Theorem 3.7 characterizes \((L^2, I)\) as the initial object of \(B^2\), but there is an alternative characterization. Write \(\text{Hilb}\) for the category of Hilbert spaces and linear contractions. Let \(\mathscr{H}\) be the category of pairs \((F, v)\) consisting of a functor \(F : \text{Meas}_{\text{op}} \to \text{Hilb}\) and an element \(v_X \in F(X)\) for each measure space \(X\), subject to axioms (I)–(III) (with \(p = 2\) in (III)) and:

\[
(\text{IV}_\text{H}) \quad \langle (Fi)(v_Y), (Fj)(v_Z) \rangle = 0 \text{ whenever } Y \xrightarrow{i} X \xleftarrow{j} Z \text{ are embeddings with disjoint images.}
\]

Thus, the difference between the categories \(B^2\) and \(\mathscr{H}\) is that \(\text{Ban}\) has been replaced by \(\text{Hilb}\) and (IV) by (\(\text{IV}_\text{H}\)).

The functor \(L^2 : \text{Meas}_{\text{op}} \to \text{Hilb}\), together with the constant functions \(I_X \in L^2(X)\), defines an object \((L^2, I)\) of \(\mathscr{H}\). It is universal as such:

Proposition 3.11 (Universal property of the \(L^2\) functor) \((L^2, I)\) is the initial object of \(\mathscr{H}\).

Proof In the proof of Theorem 3.7, the only point where axiom (IV) was used was to prove the inequality (15), which in the case \(p = 2\) states that

\[
\left\| \sum_{c \in \mathbb{F}} cv_X^{-1}(c) \right\|^2 \leq \sum_{c \in \mathbb{F}} \left\| cv_X^{-1}(c) \right\|^2
\]

(18)
for any simple function $f$ on a measure space $X$ and any object $(F, v) \in \mathcal{B}$.
So, it suffices to prove that (18) also holds for any $(F, v) \in \mathcal{H}$. Indeed, recall
that if we write $i_c$ for the inclusion $f^{-1}(c) \hookrightarrow X$ then by definition, $v^X_{f^{-1}(c)} =
(F_i)(v_{f^{-1}(c)})$. Axiom (IV) therefore implies that the elements $v^X_{f^{-1}(c)}$ of $F(X)$
are orthogonal for distinct $c$, giving equality in (18).

\[\square\]

Acknowledgements I thank Mark Meckes for allowing me to include Proposition 2.4, and Ruijun Lin for helpful comments and corrections. This work was supported at different times by an EPSRC Advanced Research Fellowship and a Leverhulme Trust Research Fellowship.

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