"Old" Conformal Bootstrap in the AdS/CFT Context

Boris L. Altshuler

Theoretical Physics Department, P. N. Lebedev Physical Institute, 53 Leninsky Prospect, Moscow, 119991, Russia

Abstract: The Schwinger-Dyson equations for Green functions and 3-vertexes in ladder approximation used 50 years ago in the earlier theory of conformal bootstrap are written on the language of Witten bubble and triangle one-loop diagrams in frames of the AdS/CFT correspondence. Simple spectral equations for conformal dimensions of primary and composite operators are obtained in the toy model of the interacting bulk scalar fields.

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1E-mail addresses: baltshuler@yandex.ru & altshul@lpi.ru
1 Introduction

This paper was inspired by the work \cite{1} that Igor Klebanov has reported in the Lebedev Institute in 2017. In \cite{1} (see also \cite{2}), among other interesting things, the expressions for spectra of conformal dimensions in one-dimensional Sachdev-Ye-Kitaev and in $d$-dimensional field theory models were obtained from the field theory ”bootstrap” equations for Green function $G(x_1, x_2)$ and vertex $\Gamma(x_1, x_2, x_3)$ expressed symbolically in most general form as:

\[
G(x_1, x_2) = \int G(x_1, x) \Sigma(x, y | G, \Gamma) G(y, x_2) dx dy;
\]

\[
\Gamma(x_1, x_2, x_3) = M^\Delta(x_1, x_2, x_3 | G, \Gamma),
\]

where $\Sigma$ is quantum self-energy and $M^\Delta$ is a triangle 3-gamma diagram - both constructed from the same $G$ and $\Gamma$ that stand on the LHS in (1). Bootstrap equation $\Gamma = M^\Delta$ in (1), that actually is a sort of Bethe-Salpeter equation, was considered in \cite{1} in the limit $x_3 \to \infty$, we also follow this way in Sec. 5.

Another background of the paper are works \cite{3} where bubble Witten diagrams were calculated in case of triple bulk interactions, and \cite{4} where it is shown that Witten diagrams with ”extremal” vertexes and bulk Green functions being replaced by the bulk Wightman functions are easily calculable, and that conformal IR divergences may be absorbed into ”bare” bulk coupling constant giving the finite Witten diagrams expressed through norm-invariant coupling constant.

Actually Eq-s (1) are just a conventional Schwinger-Dyson equations written in planar approximation when renormalization constants are put equal to zero. This may be called the zero-Lagrangian approach applied by Sakharov in his quantum induced theory of gravity \cite{5}. The attempt to follow this way in the AdS context was made in \cite{6}, \cite{7} where UV-finite induced gravitational and gauge coupling constants were calculated. Also in Sec. 5.C ”Principle of quantum self-consistency” in \cite{6} the exactly solvable toy model of zero-dimensional field theory bootstrap was presented.

The field theory bootstrap equations (1) are actually used in the mean-field theory of superconductivity, in dynamical symmetry breaking pioneered in \cite{8} and in many other aspects. They were applied in the Migdal-Polyakov
"old" conformal bootstrap theory \cite{9}, \cite{10} developed in \cite{11} - \cite{14} - see for example \cite{15} and references therein.

"Old" conformal bootstrap should not be confused with most popular nowadays conformal bootstrap based on the demand of crossing symmetry of the operator product expansions - see pioneer papers \cite{16}, \cite{17}, \cite{18} and modern reviews \cite{19}, \cite{20}. This non-Lagrangian conformal bootstrap is in a sense close to the "nuclear democracy" bootstrap based upon the demand of $S$-matrix crossing symmetry \cite{21}.

The general motivation of all bootstrap efforts is a hope to calculate in this way the values of the fundamental constant, now introduced in the theory \textit{ad hoc}. The goal of this paper is to demonstrate on the toy model that "hunting for numbers" of conformal dimensions may be successful when "old" conformal bootstrap is considered in the AdS/CFT context.

In Sec.2 familiar expressions used in the bulk of the paper are summed up. Sec. 3 describes self-energy bootstrap of three primary scalar fields when primary conformal correlators are equated to the 2-point bubble Witten diagrams; this permits to fix fields' bulk coupling constant and their scaling dimensions. In Sec. 4 simple expression is obtained for the four-point amplitude that serve a kernel in the Bethe-Salpeter type equation for the composite conformal operator considered in Sec. 5. The bootstrap equating of the tree vertex diagram and triangle one-loop Witten diagram is investigated in Sec. 5. The spectral equation for the scaling dimension of the composite operator is a result of calculations. In Conclusion the possible directions of future work are outlined.

2 Preliminaries

We work in $AdS_{d+1}$ in Poincare Euclidean coordinates $Z = \{z_0, \vec{z}\}$, where AdS curvature radius $R_{AdS}$ is put equal to one:

$$ds^2 = \frac{dz_0^2 + d\vec{z}^2}{z_0^2},$$

and consider bulk scalar fields. Bulk field $\phi(X)$ of mass $m$ is dual to boundary conformal operator $O_{\Delta_+}(\vec{x})$ or to its "shadow" operator $O_{\Delta_-}(\vec{x})$ with scaling dimensions
\[
\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}.
\] (3)

We take definition of the scalar field bulk-to-boundary operator and of the corresponding conformal correlator like in [3], [4]:

\[
K_\Delta(Z; \vec{x}) = \lim_{x_0 \to 0} \left[ \frac{G_{BB}^\Delta(Z, X)}{(x_0)^\Delta} \right] = C_\Delta \left[ \frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right]^{\Delta},
\] (4)

\[C_\Delta = \frac{\Gamma(\Delta)}{2\pi^{d/2} \Gamma(1 + \Delta - \frac{d}{2})},\]

and:

\[
< O_\Delta(\vec{x})O_\Delta(\vec{y}) > = \lim_{x_0 \to 0; y_0 \to 0} \left[ \frac{G_{BB}^\Delta(Y, X)}{(x_0 y_0)^\Delta} \right] = \frac{C_\Delta}{P_{xy}} P_{xy} = |\vec{x} - \vec{y}|^2 \] (5)

Bulk-to-bulk scalar field Green function \(G_{BB}^\Delta(X, Y)\) is given by the Kallen-Lehmann-split representation (see e.g. [3], [4], [22] and references therein):

\[
G_{BB}^\Delta(X, Y) = C_\Delta \left( \frac{\xi}{2} \right)^\Delta F \left( \frac{\Delta}{2}, \frac{\Delta + 1}{2}, \Delta - \frac{d}{2} + 1; \xi^2 \right) =
\]

\[
= \int_{-\infty}^{+\infty} \frac{\nu^2 d\nu}{\pi [\nu^2 + (\Delta - \frac{d}{2})^2]} \cdot \int K_{\frac{d}{2} + i\nu}^\Delta(X; \vec{x}_c) K_{\frac{d}{2} - i\nu}^\Delta(Y; \vec{x}_c) d^d \vec{x}_c,
\] (6)

\[
\xi = \frac{2 x_0 y_0}{x_0^2 + y_0^2 + (\vec{x} - \vec{y})^2}.
\]

However in this paper we, following [4] and [23] - [26], [6], make essential simplification with changing in Witten diagrams bulk-to-bulk Green function (6) to Wightman function \(\tilde{G}_\Delta(X, Y)\) which is a difference of residues of poles \(\nu = \pm i(\Delta - d/2)\) in the RHS of (6):

\[
\tilde{G}_\Delta(X, Y) = G_{BB}^{\Delta} - G_{d-\Delta}^{BB} = (d - 2\Delta) \int K_\Delta(X; \vec{x}_c) K_{d-\Delta}(Y; \vec{x}_c) d^d \vec{x}_c. \] (7)
The choice of sign in this definition of $\bar{G}$ is essential. We took the sign like in [23] - [26], [6] where tadpole $\bar{G}_\Delta(X, X)$ gave the correct sign of the UV-finite one-loop quantum potential.

In calculation of the tadpole Witten diagrams in [23] - [26], [6] the substitution $G$ ([6]) to $\bar{G}$ ([7]) could be justified by postulating that instead of the bulk functional integral the ratio of two bulk functional integrals $f$ or different boundary conditions is considered, see general analysis in [27]. In our case those are two b.c. corresponding to UV ($z^\Delta -$) and IR ($z^\Delta +$) asymptotics of scalar field at $z \to 0$ ($\Delta_\pm$ see in [3]). However it is not clear how to justify this substitution when in the Witten diagram there are more than one bulk Green functions. Thus, in the present work, the substitution $G$ ([6]) to $\bar{G}$ ([7]) is just a toy trick that allows to carry out calculations to the simple result and to demonstrate in this way the possible effectiveness of the ”old” bootstrap approach in the AdS/CFT context.

We shall also need expression for AdS/CFT tree vertex [28], [3], [4]; $g_{123}$ is a coupling constant of three bulk scalar fields:

$$
\Gamma_{\Delta_1, \Delta_2, \Delta_3}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = g_{123} \int K_{\Delta_1}(X; \vec{x}_1) K_{\Delta_2}(X; \vec{x}_2) K_{\Delta_3}(X; \vec{x}_3) dX =
$$

$$
= g_{123} \frac{B(\Delta_1, \Delta_2, \Delta_3)}{P_{12}^{\delta_{12}} P_{13}^{\delta_{13}} P_{23}^{\delta_{23}}},
$$

where

$$
\delta_{12} = \frac{\Delta_1 + \Delta_2 - \Delta_3}{2}; \quad \delta_{13} = \frac{\Delta_1 + \Delta_3 - \Delta_2}{2}; \quad \delta_{23} = \frac{\Delta_2 + \Delta_3 - \Delta_1}{2},
$$

$$
B(\Delta_1, \Delta_2, \Delta_3) = \frac{\pi^{d/2}}{2} \left( \prod_{i=1}^{3} \frac{C_{\Delta_i}}{\Gamma(\Delta_i)} \right) \cdot \Gamma \left( \frac{\Sigma \Delta_i - d}{2} \right) \cdot \Gamma(\delta_{12}) \Gamma(\delta_{13}) \Gamma(\delta_{23}).
$$

Also some well known [29], [11], [12], [3], [4] conformal integrals will be used below:

$$
\int \frac{d^d \vec{y}}{P_{1y}^{\Delta_1} P_{2y}^{\Delta_2} P_{3y}^{\Delta_3}} \frac{\Sigma \Delta_i = d}{P_{12}^{\Delta_1 - \Delta_3} P_{13}^{\Delta_2 - \Delta_3} P_{23}^{\Delta_1 - \Delta_2}},
$$

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and
\[ \int \frac{d^d \vec{y}}{P_{1y}^{\Delta_1} P_{2y}^{\Delta_2}} = \frac{A(\Delta_1, \Delta_2, d - \Delta_1 - \Delta_2)}{P_{12}^{\Delta_1 + \Delta_2 - \frac{d}{2}}}, \quad (12) \]
where
\[ A(\Delta_1, \Delta_2, \Delta_3) = \frac{\pi^{d/2} \Gamma(d - \Delta_1) \Gamma(d - \Delta_2) \Gamma(d - \Delta_3)}{\Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\Delta_3)}. \quad (13) \]

We will also need integral (12) when \( \Delta_1 + \Delta_2 = d, \Delta_1 \neq \Delta_2 \neq d/2 \) (the derivation of this formula is elementary in momentum space, see e.g. in [12]):
\[ \int \frac{d^d \vec{y}}{P_{1y}^{\Delta_1} P_{2y}^{d-\Delta}} = \frac{\pi^d}{d} \frac{\Gamma(d - \frac{d}{2}) \Gamma(\frac{d}{2} - \Delta)}{\Gamma(\Delta) \Gamma(d - \Delta)} \cdot \delta^{(d)}(\vec{x}_1 - \vec{x}_2), \quad \Delta \neq \frac{d}{2}; \quad (14) \]
and the divergent integral analyzed in detail in [3]:
\[ \int \frac{d^d \vec{y}}{P_{1y}^{\Delta} P_{2y}^{d-\Delta}} = \frac{\pi^d}{d} \frac{\Gamma(d - \frac{d}{2}) \Gamma(\frac{d}{2} - \Delta)}{\Gamma(\Delta) \Gamma(d - \Delta)} \cdot \frac{1}{P_{12}^{\frac{d}{2}}}, \quad (15) \]
(we used here (12) and (13) for \( \Delta_1 = \Delta_2 = d/2 \)).

In (15) \( \Gamma(0) \) is written symbolically, its dimensional regularization is presented in [3]. The point is that, like it was observed in [4], this divergence reduces in final expressions for Witten diagrams if the ”extremal” vertexes are considered and the norm-invariant squared coupling constant is used. Definition of these notions see below.

Throughout the paper three bulk scalar fields \( \phi(X), \psi(X) \) and \( \sigma(X) \) are considered, with their bulk interaction \( L_{int} = g_{\phi\psi\sigma} \phi(X) \psi(X) \sigma(X) \). These bulk fields correspond to the boundary conformal operators \( O_\phi, O_\psi, O_\sigma \) with corresponding conformal dimensions \( \Delta_\phi, \Delta_\psi \) and \( \Delta_\sigma \) of their conformal correlators (5).

\( g_{\phi\psi\sigma} \) is the ”bare” coupling constant, that, according to [4], must be renormalized. Namely in Witten diagrams \( g_{\phi\psi\sigma} \) squared must be changed to the norm-invariant:
\[ g_{\phi\psi\sigma}^2 = g_{\phi\psi\sigma}^2 \cdot a_{\phi\psi\sigma}^{UV} = g_{\phi\psi\sigma}^2 \cdot \frac{B^2(\Delta_\phi, \Delta_\psi, \Delta_\sigma)}{C_{\Delta_\phi} C_{\Delta_\psi} C_{\Delta_\sigma}}, \quad (16) \]
here last equality is the definition of $a^{UV}_{\phi\psi\sigma}$; $B(\Delta_1, \Delta_2, \Delta_3)$ and $C_\Delta$ are given in (10) and (4).

In [4] it is observed that Witten diagrams become easily calculable in case three-point vertexes are ”extremal” [30]. In our case this means that conformal dimensions corresponding to fields $\phi, \psi, \sigma$ obey:

$$\Delta_\phi = \Delta_\psi + \Delta_\sigma.$$  \hspace{1cm} (17)

In this case norm-invariant coefficient $a^{UV}_{\phi\psi\sigma}$ (16) includes $\Gamma^2(0)$ in numerator and this infinity is absorbed in ”bare” $g^2_{\phi\psi\sigma}$. Then, as it is shown in [4], Witten diagrams expressed through renormalized squared coupling constant $g^2_{\phi\psi\sigma}$ (16) are finite and are extremely simple.

## 3 AdS bubble analogy of the Green function bootstrap

Let us consider the following AdS/CFT version of the first of the field theory symbolic Eq-s (1) written for the boundary conformal correlator (5) of field $\phi$ whereas bubble (”self-energy” of $\phi$) is built of fields $\psi, \sigma$:

$$<O_\phi(\vec{x}_1) O_\phi(\vec{x}_2)> = \tilde{M}_{2pt \text{ bubble}}^{\Delta_\phi},$$ \hspace{1cm} (18)

where LHS is given in (5), and in calculation of Witten bubble diagram in the RHS we follow [3] with substitution of Wightman functions (7) instead of bulk-to-bulk Green functions (6). Thus bootstrap Eq. (18) looks as:

$$\frac{C_{\Delta_\phi}}{P_{12}} = g^2_{\phi\psi\sigma} \int K_{\Delta_\phi}(X; \vec{x}_1) \tilde{G}_{\Delta_\psi}(X,Y) \tilde{G}_{\Delta_\sigma}(X,Y) K_{\Delta_\phi}(Y; \vec{x}_2) dX dY. \hspace{1cm} (19)$$

Actually this is nothing but first of symbolic equations (1) written for bulk Green function (6) where ”external” points $x_1, x_2$ are put on the AdS boundary; then $G$ in the LHS of (11) becomes correlator (5) in the LHS of (19), whereas two $G$ on the RHS of (11) become bulk-to-boundary operators (4) in the RHS of (19) 2.

Taking $\tilde{G}(X,Y)$ from (7) and performing two bulk integrals (see (8)-(10)) gives for integral in the RHS of (19):

2I am grateful to Ruslan Metsaev for this observation.
\[(d - 2\Delta_\psi)(d - 2\Delta_\sigma) \int d\vec{x}_a d\vec{x}_b \int K_{\Delta_\phi}(X; \vec{x}_1) K_{\Delta_\psi}(X; \vec{x}_a) K_{\Delta_\sigma}(X; \vec{x}_b) dX \cdot\]

\[
\cdot \int K_{d-\Delta_\psi}(Y; \vec{x}_a) K_{d-\Delta_\sigma}(Y; \vec{x}_b) K_{\Delta_\psi}(Y; \vec{x}_a) K_{\Delta_\sigma}(Y; \vec{x}_b) dY = \]

\[
= \int d\vec{x}_a d\vec{x}_b \left[ \frac{(d - 2\Delta_\psi)(d - 2\Delta_\sigma) B(\Delta_\phi, \Delta_\psi, \Delta_\sigma) B(\Delta_\phi, d - \Delta_\psi, d - \Delta_\sigma)}{P_{12}^{\delta_{1a}} P_{1b}^{\delta_{1b}} P_{2a}^{\delta_{2a}} P_{2b}^{\delta_{2b}} P_{ab}^{(2)}} \right],
\]

where

\[
\delta_{1a} = \delta_{2b} = \frac{\Delta_\phi + \Delta_\sigma - \Delta_\psi}{2}; \quad \delta_{1b} = \delta_{2a} = \frac{\Delta_\phi + \Delta_\psi - \Delta_\sigma}{2};
\]

\[
\delta_{ab}^{(1)} = \frac{\Delta_\psi + \Delta_\sigma - \Delta_\phi}{2}; \quad \delta_{ab}^{(2)} = \frac{2d - \Delta_\phi - \Delta_\psi - \Delta_\sigma}{2}.
\]

Conformal integrals in (20) may be taken with the help of standard formulas (11) - (15). Following [4] we use in (20) instead of \(g^2_{\phi\psi\sigma}\) norm-invariant squared coupling constant \(g^2_{\phi\psi\sigma} = g^2_{\phi\psi\sigma} a_{\phi\psi\sigma}^{UV}\) (16). Then bootstrap equation (19) takes the form:

\[
\frac{C_{\Delta_\phi}}{P_{12}^{\Delta_\phi}} = g^2_{\phi\psi\sigma} C_{\Delta_\phi} C_{\Delta_\psi} C_{\Delta_\sigma} \frac{B(\Delta_\phi, d - \Delta_\psi, d - \Delta_\sigma)}{B(\Delta_\phi, \Delta_\psi, \Delta_\sigma)} \cdot
\]

\[
\cdot \frac{A(\delta_{1b}, \delta_{2b}, d - \Delta_\psi) A(d/2, d/2, 0)}{P_{12}^{\Delta_\phi}},
\]

expressions for \(C\), \(B\) and \(A\) see in [4], [10], [13], [15].

In case of validity of the "extremal" equality (17) between the conformal dimensions \(\Delta_\phi, \Delta_\psi, \Delta_\sigma\) the infinity - \(\Gamma(0)\) in \(A(d/2, d/2, 0)\) in the RHS of (22) is reduced with the similar infinity in \(B(\Delta_\phi, \Delta_\psi, \Delta_\sigma)\). This important observation in [4] permitted to obtain the finite sensible answers.

Also the validity of (17) leads to the reduction of practically all \(\Gamma\)-functions in the RHS of (22). The remarkable simplicity of final expressions for Witten
diagrams in case of the "extremal" relation between conformal dimensions was also observed in [4].

Here are the final forms of the bubble bootstrap equation (19) (or (22)) for the correlation function of field $\phi$, and of similar to (19) equations for the correlation functions corresponding to fields $\psi, \sigma$ when bubbles are formed of fields $(\phi, \sigma)$ and $(\phi, \psi)$ correspondingly.

$$\frac{C_{\Delta\phi}}{P_{12}^{\Delta\phi}} = \frac{g_{\phi\psi\sigma}^2}{P_{12}^{\Delta\phi}} \frac{C_{\Delta\psi}}{P_{12}^{\Delta\psi}} \frac{F(\Delta_{\phi})}{F(\Delta_{\psi})} \cdot F(\Delta_{\sigma}) \frac{F(\Delta_{\phi})}{F(\Delta_{\sigma})},$$

where

$$F(\Delta_i) = \frac{\Gamma(\Delta_i) \Gamma(d - \Delta_i)}{\Gamma(\Delta_i - \frac{d}{2}) \Gamma(\frac{d}{2} - \Delta_i)} \quad (i = \phi, \psi, \sigma).$$

Thus Eq. (23) fixes the norm-invariant coupling constant (16) (in units $R_{AdS}^{(d-5)}$):

$$g_{\phi\psi\sigma}^2 = 1,$$

and (24)-(26), (17) give a system determining dimensions $\Delta_i$:

$$F(\Delta_{\phi}) = F(\Delta_{\psi}) = F(\Delta_{\sigma}); \quad \Delta_{\phi} = \Delta_{\psi} + \Delta_{\sigma}.$$  

Eq-s (27), (28) are the main result of this Section.

Function $F(\Delta)$ (26) for even $d$ is a polynomial of degree $d$, whereas for odd $d$ it is expressed in terms of trigonometric functions. Here are functions $F(\Delta)$ for $d = 1, 2, 3, 4$:

$$F_{d=1}(\Delta) = \frac{(\Delta - 1/2) \cos \pi \Delta}{\sin \pi \Delta}; \quad F_{d=4} = (\Delta - 1)(\Delta - 2)^2(\Delta - 3);$$

$$F_{d=2}(\Delta) = -(\Delta - 1)^2; \quad F_{d=3}(\Delta) = \frac{(\Delta - 1)(\Delta - 3/2)(\Delta - 2) \cos \pi \Delta}{\sin \pi \Delta}. $$

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Spectral Eq-s (28) possess finite number of roots for even $d$ and infinite number of roots for odd $d$. However if the condition of unitary bound $(d-2) < 2 \Delta_- < d$, $\Delta_-$ see in (3), is imposed the number of solutions of (28) is limited. Let us look at two simple solutions of (28).

The first one is valid for any dimension $d$:

$$\Delta_\phi = \frac{2d}{3}, \quad \Delta_\psi = \Delta_\sigma = \frac{d}{3}.$$  \hspace{1cm} (30)

It means that three bulk fields $\phi, \psi, \sigma$ are actually one and the same self-interacting bulk field of $m^2 = \Delta(\Delta - d) = -2d^2/9$ (in units $R_{\text{AdS}}^{-2}$), that is $\Delta_\phi = \Delta_\psi$ and $\Delta_\psi = \Delta_\sigma = \Delta_-$ for this $m^2$ in (3). Unitary bound for solution (30) is valid for $d < 6$.

Second solution is obtained from (28), (29) in four dimensions, $d = 4$:

$$\Delta_\phi = \frac{13}{5}, \quad \Delta_\psi = \frac{7}{5}, \quad \Delta_\sigma = \frac{6}{5}.$$  \hspace{1cm} (31)

In this case there are two bulk scalar fields of mass squared $m_\phi^2 = m_\psi^2 = -91/25$ ($\Delta_\phi = \Delta_+, \Delta_\psi = \Delta_-$ in (3)), and $m_\sigma^2 = -84/25$ (that is $\Delta_\sigma = \Delta_-$ in (3) for this $m^2$). Unitary bound condition ($1 < \Delta_- < 2$ for $d = 4$) is satisfied for these values of $\Delta_\psi$ and $\Delta_\sigma$.

In both solutions (30), (31) dimensions $\Delta_\phi$ and $\Delta_\psi$ correspond to conjugate conformal operators, that is:

$$\Delta_\phi = d - \Delta_\psi; \quad \Delta_\sigma = d - 2 \Delta_\psi,$$  \hspace{1cm} (32)

equality for $\Delta_\phi$ follows from (17).

When (32) is valid system (28) comes to single spectral equation for $\Delta_\psi$:

$$F(\Delta_\psi) = F(2 \Delta_\psi),$$  \hspace{1cm} (33)

where $F(\Delta)$ is given in (26) or (29). Surely Eq. (33) has solutions (30) and (31), in particular $\Delta_\psi = 2/3$ for $d = 2$ and $\Delta_\psi = 4/3$ or $7/5$ for $d = 4$. For $d = 1$ and $d = 3$ (33) and (29) give following spectral equations:

$$d = 1 : \quad \tan^2 \pi \Delta_\psi = \frac{1}{4 \Delta_\psi - 1};$$  \hspace{1cm} (34)

$$d = 3 : \quad \tan^2 \pi \Delta_\psi = \frac{3(\Delta_\psi - 1)(2 \Delta_\psi + 1)}{(4 \Delta_\psi - 3)(2 \Delta_\psi - 1)}.$$
For \( d = 1 \) there is only one solution (30) (that is \( \Delta_\psi = \Delta_\sigma = 1/3, \Delta_\phi = 2/3 \)) satisfying unitary bound condition \(-1/2 < \Delta_\psi < 1/2\). Whereas for \( d = 3 \) unitary bound \( 1/2 < \Delta_\psi < 3/2 \) is valid for two solutions of (32) - (34): \( \Delta_\psi = \Delta_\sigma = 1, \Delta_\phi = 2 \) and another one: \( \Delta_\psi \approx 1.24; \Delta_\sigma \approx 0.52; \Delta_\phi \approx 1.76 \).

4 Four-point correlators

In the next section the AdS/CFT analogy of the Bethe-Salpeter field theory bootstrap equation (second symbolic eq. in (1)) will be considered. In this section the Kernel of this equation is calculated. The Kernel is the connected one-channel tree Witten diagram for the four-point function \( \vec{x}_1 \vec{x}_2 \rightarrow \vec{x}_a \vec{x}_b \):

\[
\tilde{M}_{4pt\, tree}^{\psi\phi|\phi\psi}(\vec{x}_1, \vec{x}_2, \vec{x}_a, \vec{x}_b) = <O_\psi(\vec{x}_1) O_\phi(\vec{x}_2) O_\phi(\vec{x}_a) O_\psi(\vec{x}_b)>, \tag{35}
\]

where interaction is carried out by means of the field \( \sigma(X) \) in the "t-channel" \( \vec{x}_1 \vec{x}_a \rightarrow \vec{x}_2 \vec{x}_b \), and bulk Green function \( G_{BB}^{\Delta_\sigma}(X, Y) \) (6) is changed to corresponding Wightman function \( \tilde{G}^{BB}_{\Delta_\sigma}(X, Y) \) (7):

\[
\tilde{M}_{4pt\, tree}^{\psi\phi|\phi\psi}(\vec{x}_1, \vec{x}_2, \vec{x}_a, \vec{x}_b) = g_{\phi\psi}^2 \cdot \]

\[
\cdot \int dX\, dY\, K_{\Delta_\psi}(X; \vec{x}_1) K_{\Delta_\phi}(X; \vec{x}_a) \tilde{G}_{\Delta_\sigma}(X, Y) K_{\Delta_\phi}(Y; \vec{x}_2) K_{\Delta_\psi}(Y; \vec{x}_b). \tag{36}
\]

The subsequent computation algorithm is obvious:
- to substitute in (36) \( \tilde{G}_{\Delta_\sigma}(X, Y) \) from (7);
- to perform two bulk integrals using (8)-(10);
- to substitute "extremal" equality (11) and then to perform boundary conformal integral using (11) and (13);
- instead of the "bare" squared coupling constant \( g_{\phi,\psi,\sigma}^2 \) to insert the norm-invariant one (16).

Omitting these elementary steps we present the simple final expressions for three amplitudes \( \tilde{M}_{4pt\, tree} \), symbolically: \( \psi\phi \rightarrow \phi\psi, \phi\phi \rightarrow \psi\psi, \psi\psi \rightarrow \phi\phi \). The first one is deciphered in (35), two others are correlators

\[
\tilde{M}_{4pt\, tree}^{\phi\phi|\psi\psi}(\vec{x}_1, \vec{x}_2, \vec{x}_a, \vec{x}_b) = <O_\phi(\vec{x}_1) O_\phi(\vec{x}_2) O_\psi(\vec{x}_a) O_\psi(\vec{x}_b)>,
\]
\[ \tilde{M}_{4\text{pt tree}}^{\psi\phi\phi\psi}(\vec{x}_1, \vec{x}_2, \vec{x}_a, \vec{x}_b) = O_\psi(\vec{x}_1) O_\psi(\vec{x}_2) O_\phi(\vec{x}_a) O_\phi(\vec{x}_b), \]

where in every case interaction is carried out by the field \( \sigma(X) \) in "t-channel".

Finally:

\begin{align*}
\tilde{M}_{4\text{pt tree}}^{\psi\phi\phi\psi}(\vec{x}_1, \vec{x}_2, \vec{x}_a, \vec{x}_b) &= g^2 \phi \psi \sigma \cdot \Delta \Delta \psi \
&\quad \cdot \{ K_\psi(X; \vec{x}_1) \tilde{G}_\Delta(\vec{X}, Y) K_\phi(Y; \vec{x}_2) \tilde{G}_\Delta(Y, Z) K_\Delta(\vec{Z}; \vec{x}_3) \tilde{G}_\Delta(Z, X) \}.
\end{align*}

It is seen that these three 4-point amplitudes differ only in the last factor in denominators in (37)-(39).

## 5 AdS/CFT vertex "old" bootstrap

Let us look at the 3-point correlator

\[ \Gamma_{\Delta_\psi, \Delta_\phi, \Delta_3}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = O_\psi(\vec{x}_1) O_\phi(\vec{x}_2) O_{\Delta_3}(\vec{x}_3), \]

where \( O_\psi, O_\phi \) are primary conformal operators considered in previous sections and \( O_{\Delta_3} \) is a composite conformal operator. Bootstrap equation proposed below give the spectrum of conformal dimension \( \Delta_3 \).

In the analogy with second symbolic field theory bootstrap equation in (1) the equality of 3-point tree vertex (40) to the corresponding three-point one-loop triangle Witten diagram is proposed as a candidate for AdS/CFT vertex bootstrap equation:

\[ \Gamma_{\Delta_\psi, \Delta_\phi, \Delta_3}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \tilde{M}_{3\text{pt triangle}}^{\Delta_\psi, \Delta_\phi, \Delta_3}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = g^2 \phi \psi \sigma g_{\psi \psi \phi \phi} \int dX dY dZ \cdot \]

\[ \cdot \{ K_\psi(X; \vec{x}_1) \tilde{G}_\Delta(\vec{X}, Y) K_\phi(Y; \vec{x}_2) \tilde{G}_\Delta(Y, Z) K_\Delta(\vec{Z}; \vec{x}_3) \tilde{G}_\Delta(Z, X) \}. \]

Here \( g_{\psi \psi \phi \phi} \) is a bulk coupling constant of two primary fields \( \phi(X), \psi(X) \) and bulk scalar field corresponding to composite conformal operator \( O_{\Delta_3} \); it
will reduce in final expressions since \( \Gamma_{\Delta_{\psi}, \Delta_{\phi}, \Delta_{3}}(\vec{x}_1, \vec{x}_2, \vec{x}_3) \) in the LHS of (41) includes it (see (8)).

Taking again \( \tilde{G} \) from (7), using (8)-(10) to perform three integrals over bulk coordinates \( X, Y, Z \), and (11), (13), (17) to perform integral over one of three boundary points we obtain from (41) following Bethe-Salpeter equation:

\[
\Gamma_{\Delta_{\psi}, \Delta_{\phi}, \Delta_{3}}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \int \tilde{M}_{\psi \phi}(\vec{x}_1, \vec{x}_2, \vec{x}_a, \vec{x}_b) \cdot (d - 2\Delta_{\phi})(d - 2\Delta_{\psi}) \Gamma_{d - \Delta_{\phi}, d - \Delta_{\psi}, \Delta_{3}}(\vec{x}_a, \vec{x}_b, \vec{x}_3) d\vec{x}_a d\vec{x}_b,
\]

where vertexes \( \Gamma \) are given in (8)-(10) and Kernel \( \tilde{M}_{\psi \phi}(\vec{x}_1, \vec{x}_2) \) - in (37).

Integrals over \( \vec{x}_a, \vec{x}_b \) that arise after these substitutions in (42) can’t be taken with elementary formulas (11), (12). However, as it was shown in [1], spectral equation for \( \Delta_{3} \) may be obtained from Bethe-Salpeter equation in the limit \( \vec{x}_3 \to \infty \). In this limit, according to (8), both sides of (42) are proportional to \( \vec{x}_3 \), and two boundary integrals in the RHS of (42) may be performed using (12).

After these manipulations, with account of (8)-(10), (12), (13), (37) and ”extremal” equation (17), Bethe-Salpeter equation (42) comes to:

\[
\frac{B(\Delta_{\psi}, \Delta_{\phi}, \Delta_{3})}{P_{12}^{\Delta_{\psi} + \Delta_{\phi} - \Delta_{3}/2}} = g_{\psi \phi \sigma}^{\ast 2} \frac{B(d - \Delta_{\psi}, d - \Delta_{\phi}, \Delta_{3}) (d - 2\Delta_{\phi})(d - 2\Delta_{\psi})}{P_{12}^{\Delta_{\psi} + \Delta_{\phi} - \Delta_{3}/2}} \cdot C_{\Delta_{\phi}} C_{\Delta_{\psi}} A\left(\Delta_{\psi}, \delta_{1}, \frac{\Delta_{\sigma} + \Delta_{3}}{2}\right) A\left(\Delta_{\psi}, \frac{d + \Delta_{\sigma} - \Delta_{3}}{2}, \delta_{2}\right),
\]

where

\[
\delta_{1} = \frac{2d - \Delta_{\phi} - \Delta_{\psi} - \Delta_{3}}{2}, \quad \delta_{2} = \frac{d - \Delta_{\psi} - \Delta_{\phi} + \Delta_{3}}{2}.
\]

Substitution of \( C, B, A \) from (4), (10), (13) and \( g_{\psi \phi \sigma}^{\ast 2} = 1 \) from (27) gives the final form of the bootstrap equation (43) for conformal dimension \( \Delta_{3} \) of the composite operator for given conformal dimensions \( \Delta_{\phi}, \Delta_{\psi}, \Delta_{\sigma} \) of primary operators:
\[
\frac{\Gamma\left(\Delta - \frac{\Delta_\psi}{2}\right) \Gamma\left(\frac{d}{2} - \Delta + \frac{\Delta_\psi}{2}\right)}{\Gamma\left(\Delta + \frac{\Delta_\psi}{2}\right) \Gamma\left(\frac{d}{2} - \Delta - \frac{\Delta_\psi}{2}\right)} = \frac{\Gamma(\Delta_\psi) \Gamma\left(\frac{d}{2} - \Delta_\psi\right)}{\Gamma(\Delta_\phi) \Gamma\left(\frac{d}{2} - \Delta_\phi\right)}. \tag{44}
\]

Is not it curious that for \(\Delta_\psi = d/4, \Delta_\sigma = d/2, \Delta_\phi = \Delta_\psi + \Delta_\sigma = 3d/4\) Eq. (44) coincides, up to the factor \(-3\), with spectral formula (4.14) \(g(h) = 1\) in \([1]\) (\(\Delta_3\) here is \(h\) in \([1]\))? However in our case \(\Delta_\sigma = d/2\) is forbidden by spectral equations (28).

In case primary conformal operators \(O_\phi, O_\psi\) are conjugate, that is in case of validity of relations (32) between dimensions \(\Delta_\phi, \Delta_\psi\) and \(\Delta_\sigma\), (44) simplifies:

\[
\frac{\Gamma\left(\Delta_\psi - \frac{\Delta_3}{2}\right) \Gamma\left(\Delta_\psi - \frac{d}{2} + \frac{\Delta_3}{2}\right)}{\Gamma\left(d - \Delta_\psi - \frac{\Delta_3}{2}\right) \Gamma\left(\frac{d}{2} - \Delta_\psi + \frac{\Delta_3}{2}\right)} = \frac{\Gamma(\Delta_\psi) \Gamma\left(\frac{d}{2} - \Delta_\psi\right)}{\Gamma(d - \Delta_\psi) \Gamma\left(\frac{d}{2} - \Delta_\psi\right)}. \tag{45}
\]

The analysis of spectral equation (45) is beyond the scope of this paper. We just note that for any \(\Delta_\psi\) among solutions of (45) there are \(\Delta_3 = 0\) and \(\Delta_3 = d\), that corresponds to \(\Delta_\pm\) (3) of the massless bulk scalar field.

\section{Conclusion}

The immediate generalization of the considered toy model of scalar fields would be to look at the AdS/CFT bootstrap equations (19), (41) for more physical models including fermions, gauge fields and gravity. Generalization of (19), (41) to higher spins would be also interesting.

The role of ”extremal” relation (17) must be specially investigated. ”Extremal” Witten diagrams proved to be calculated in the elementary way, that permitted to obtain simple spectral equations (28), (44) from the AdS/CFT bootstrap equations (19), (41). But the question remains: if these bootstrap equations make sense for the ”non-extremal” Witten diagrams?

Surely the ”toy” replacing in Witten diagrams of bulk-to-bulk Green functions (6) to Wightman functions (7) is questionable. Thus the task for future is to look at (19), (41) when not \(\tilde{G}(X,Y)\) but \(G(X,Y)\) are used in these expressions.

It would be important to find links between ”old” conformal bootstrap considered in this paper in the AdS context and most popular nowadays
"OPE-expansion" conformal bootstrap. The unification of these two approaches supposedly may be based on the crucial word "decoupling" when only limited number of the low dimension conformal operators contribute in the OPE expansions in crossing channels, and correspondingly contribute to the bootstrap equations of type (19), (41). Decoupling of high dimensional operators is studied in [31] where also the options of "hard-wall" and "soft-wall" breaking of conformal symmetry in AdS space are considered.

And perhaps most promising would be to study spontaneous breakdown of conformal symmetry in frames of the bootstrap in the AdS context. The possible fundamental role of the spontaneously broken conformal symmetry was advocated by 't Hooft [32]. Here, in the language of the present paper, this means that bulk-to-boundary and bulk-to-bulk functions entering (19) or (41) must be taken on the background of the two-branes Randall-Sundrum model ("hard-wall" breaking of the AdS conformal symmetry) or on the background with correspondingly modified metric (2) ("soft-wall" breaking of the AdS conformal symmetry). However calculation of Witten diagrams in these models is a hard problem and it is not clear how in these cases to obtain visible spectral equations similar to formulas (28) (or (33)) and (44) (or (45)) of this paper.

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