A Novel Algorithm to Solve for an Underwater Line Source Sound Field Based on the Theory of Global Matrix Coupled Modes and the Chebyshev–Tau Spectral Method

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Abstract

High-precision numerical sound field is the basis of underwater target detection, positioning and communication. A line source in the plane is a common type of sound source in computational ocean acoustics, and the exciting waveguide in a range-dependent marine environment is often structurally complicated. However, traditional algorithms often assume that the waveguide has a simple seabed boundary, and the line source is located at a horizontal range of 0 m, which is an ideal and rare situation in the actual ocean. In this paper, a novel algorithm is designed that can solve for the sound field generated by a line source at any position in a range-dependent ocean environment. The proposed algorithm uses the classic stepwise approximation to address the range dependence.
of the environment and uses the Chebyshev–Tau spectral method to solve for the horizontal wavenumbers and modes of approximately range-independent segments. After obtaining the modal information of flat segments, a global matrix is assembled to solve for the coupling coefficients of all segments, and finally, a complete sound field is synthesized. Numerical experiments using a robust numerical program we developed based on this algorithm verify the correctness and usability of the novel algorithm and software. Furthermore, a detailed analysis and test of the computational cost of this algorithm show that it is extremely efficient.

**Keywords:** Chebyshev–Tau spectral method, coupled modes, range-dependent, line source, computational ocean acoustics

1 Introduction

Continuous research on the theory of underwater sound propagation has led to various breakthroughs in the development and utilization of underwater acoustics. A particularly salient example is the development of computational ocean acoustics, which has long been a research field of great academic value and far-reaching significance in ocean acoustics and has achieved incredible advances in recent decades motivated by urgent needs for numerical sound fields and rapid advances in computer technology. Various computational ocean acoustic models based on, for example, rays, normal modes, the parabolic approximation and wavenumber integration have been applied to calculate sound fields [1]. The goal of computational ocean acoustics is to solve the Helmholtz equation derived from the wave equation; therefore, since different models make unique assumptions and apply different transformations to the Helmholtz equation, the scope of application and advantages/disadvantages of these models vary [2]. Nevertheless, Buckingham provided a comprehensive summary of various available ocean acoustic propagation models and asserted that the normal mode solution of the Helmholtz equation, including coupled modes, can generally provide a benchmark solution to the acoustic propagation problem [3].

The normal mode solution was proposed by Pekeris to calculate a homogeneous waveguide within an acoustic half-space [4]. In Pekeris’s branch cut approach, the sound field is expressed as a finite number of discrete normal modes and a continuous spectrum corresponding to the contour integral. However, due to the complexity of integrating a continuous spectrum and its minimal impact on the far field, traditional normal modes typically ignore the continuous spectrum. Alternatively, because classic normal modes can solve for only range-independent waveguides, Pierce and Milder introduced coupled modes into ocean acoustics to extend the solution scope of normal modes to range-dependent waveguides [5, 6]. Furthermore, Evans used a stepwise approximation to establish the commonly used line-source elliptic two-way
model in a planar geometric waveguide and developed the numerical program COUPLE [7–9].

Considering the range dependence of the stepwise approximation, Porter et al. studied energy conservation issues and proposed energy conservation correction schemes such as sound pressure matching, radial particle velocity matching and impedance matching [10]. The latest model, COUPLE07, is often used as a benchmark for testing other models and methods. However, due to the unreasonable normalization of the range, COUPLE may diverge in actual calculations. Accordingly, Luo and Yang proposed a global matrix coupled mode solution and established a two-way model to solve for the coupling coefficients [11–15]. This technique is highly computationally efficient and numerically stable and thus is expected to become a representative method of coupled modes. Consequently, this article partially adopts this method and further optimizes it.

The finite difference method is classically employed to solve for local normal modes, and its representative programs are KRAKEN and KRAKENC [16]. However, KRAKEN cannot accurately calculate the sound field in the case of strong absorption, and KRAKENC finds roots directly on the complex plane, resulting in poor stability, and some roots may be missed due to changes in media parameters. Moreover, Evans incorporated the Galerkin method to solve for local normal modes in COUPLE [9]. This method starts from the weak form of the modal equation, applies essential boundary conditions to eliminate the derivative term in the partial integration, and finally discretizes the problem into a generalized matrix eigenvalue problem. However, this approach encounters the following problems: first, the lower boundary of the waveguide must be a pressure release boundary, which is not universal in ocean environment; second, the basis/weight function of the Galerkin method must be constructed by solving a nonsingular Sturm–Liouville problem.

As a high-precision method for solving differential equations, the spectral method was introduced into computational ocean acoustics at the end of the twentieth century [17–19]. Using this method, our team has performed a series of studies to solve underwater acoustic propagation models in recent years [20–26]. In particular, we developed a normal mode solver named NM-CT based on the Chebyshev–Tau spectral method and provided the code in the open-source Ocean Acoustics Library (OALIB) [27]. Numerical simulations have shown that this solver achieves good robustness and accuracy [21, 28].

As a good simplification of the Helmholtz equation, the normal modes excited by a line source in the plane are also of considerable significance [1]. Hence, many mature numerical programs include the interface of the line source sound field [9, 16]. In this article, we employ the theory of global matrix coupled modes and the Chebyshev–Tau spectral method to design an efficient, accurate, stable and reliable algorithm to solve for the range-dependent waveguides of line source.
2 Model and Methodology

2.1 Normal modes of line source in the plane geometry

Considering a two-dimensional line source acoustic field in a plane geometric environment, the infinite harmonic line source is located at a horizontal range $x = x_s$ and a depth $z = z_s$. Let the acoustic pressure be denoted as $p \equiv p(x, z)$, and omit the time factor $\exp(-i\omega t)$, Then the acoustic governing equation (Helmholtz equation) can be written as [1]:

$$\frac{\partial^2 p}{\partial x^2} + \rho(z) \frac{\partial}{\partial z} \left( \frac{1}{\rho(z)} \frac{\partial p}{\partial z} \right) + \frac{\omega^2}{c^2(z)} p = -\delta(x - x_s)\delta(z - z_s)$$  \hspace{1cm} (1)

where $\omega = 2\pi f$, $f$ is the frequency of the acoustic source and $c(z)$ and $\rho(z)$ are the acoustic speed and density, respectively. Using the technique of the separation of variables, the acoustic pressure can be decomposed into:

$$p(x, z) = X(x) \psi(z)$$  \hspace{1cm} (2)

$\psi(z)$ in Eq. (2) satisfies the following modal equation:

$$\rho(z) \frac{d}{dz} \left( \frac{1}{\rho(z)} \frac{d\psi(z)}{dz} \right) + \left[ k^2(z) - k_x^2 \right] \psi(z) = 0$$  \hspace{1cm} (3a)

$$k(z) = (1 + i\eta\alpha)\omega/c(z), \quad \eta = (40\pi \log_{10} e)^{-1}$$  \hspace{1cm} (3b)

where $k(z)$ is also called the complex wavenumber, $\alpha$ is the attenuation coefficient in units of decibels (dB) divided by $\lambda$ (dB/\(\lambda\), where $\lambda$ is the wavelength), and $k_x$ is the horizontal wavenumber. Ignoring the contribution of the continuous spectrum, Eq. (3) has a set of solutions $\{ (k_{x,m}, \psi_m) \}_{m=1}^{\infty}$, where $\psi_m$ is also called the eigenmode. The eigenmodes of Eq. (3) should be normalized as follows:

$$\int_0^H \frac{\psi_m(z)\psi_n(z)}{\rho(z)} dz = \delta_{mn}, \quad m, n = 1, 2, \ldots$$  \hspace{1cm} (4)

where $H$ is the depth of the ocean and $\delta$ is the Kronecker delta function.

$X(x)$ in Eq. (2) is related only to the horizontal range $x$ and satisfies:

$$\frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = -\frac{\delta(x - x_s)\psi(z_s)}{\rho(z_s)}$$  \hspace{1cm} (5)

According to the knowledge of ordinary differential equations, the solution is:

$$X(x) = \frac{i}{2\rho(z_s)} \psi(z_s) \frac{\text{e}^{ik_x|x|}}{k_x}$$  \hspace{1cm} (6)
Finally, the fundamental solution of the 2D-Helmholtz equation can be approximated by:

\[ p(x, z) = \frac{i}{2\rho(z_s)} \sum_{m=1}^{\infty} \psi_m(z_s) \psi_m(z) e^{ik_{x,m}|x|} \]

(7)

For a range-independent marine environment in which the ocean floor is topped with sediment, \( \rho(z), c(z) \) and \( \alpha(z) \) are discontinuous at the interface \( z = h \). The ocean is divided into a discontinuous water column layer and a bottom sediment layer. The marine environmental parameters are defined in the water column and bottom sediment as:

\[
c(z) = \begin{cases} 
c_w(z), & 0 \leq z \leq h \\
c_b(z), & h \leq z \leq H \\
c_\infty, & z \geq H
\end{cases}
\]

(8a)

\[
\rho(z) = \begin{cases} 
\rho_w(z), & 0 \leq z \leq h \\
\rho_b(z), & h \leq z \leq H \\
\rho_\infty, & z \geq H
\end{cases}
\]

(8b)

\[
\alpha(z) = \begin{cases} 
\alpha_w(z), & 0 \leq z \leq h \\
\alpha_b(z), & h \leq z \leq H \\
\alpha_\infty, & z \geq H
\end{cases}
\]

(8c)

Boundary conditions should be imposed at the sea surface \( (z = 0) \) and the seabed \( (z = H) \), as should interface conditions at the discontinuous surface \( (z = h) \) between the water column and bottom sediment. The sea surface is usually set as the pressure release boundary:

\[ \psi(z = 0) = 0 \]

(9)

The seabed can be a pressure release boundary or a rigid seabed:

\[
\begin{align*}
\psi(z = H) &= 0 \\
\psi'(z = H) &= 0
\end{align*}
\]

(10a)

(10b)

In addition, the use of an acoustic half-space is quite common in underwater acoustic modeling [1]:

\[ \psi(H) + \frac{\rho_\infty}{\rho_b(H)\gamma_\infty} \psi'(H) = 0, \quad \gamma_\infty = \sqrt{k_x^2 - \frac{\omega^2}{c_\infty^2}} \]

(11)
Note that the modal normalization should add the $z \in [H, +\infty]$ at this time:

$$
\int_0^H \frac{\psi_m^2(z)}{\rho(z)} \, dz + \frac{\psi_m^2(H)}{2 \rho_\infty \gamma_\infty} = 1, \quad m = 1, 2, \ldots
$$

(12)

The interface conditions are defined as follows:

$$
\psi(h^-) = \psi(h^+)
$$

(13a)

$$
\frac{1}{\rho(z = h^-)} \frac{d\psi(h^-)}{dz} = \frac{1}{\rho(z = h^+)} \frac{d\psi(h^+)}{dz}
$$

(13b)

where the superscripts $-$ and $+$ indicate limits from above and below, respectively.

### 2.2 Chebyshev–Tau spectral method

The spectral method, which is derived from the weighted residual method [29], is a numerical technique similar to the finite difference method and the finite element method that achieves a high accuracy in solving differential equations [30]. According to the method employed to select the weight functions, the spectral method can be divided into the Galerkin-type spectral method and the spectral collocation method. The weight functions of the former take the same form as the basis functions, while those of the latter adopt the Kronecker delta function [31, 32]. Since the classic Galerkin-type spectral method often encounters difficulties when constructing basis functions that satisfy the boundary conditions, the Tau approach is widely used as a variant of the Galerkin method for differential equations featuring complex boundary conditions because the Tau method does not need to construct a set of basis functions that satisfy the boundary conditions [33]. Alternatively, the boundary conditions are usually transformed into the spectral space and then into the constraints on the spectral coefficients of the quantity to be solved. The Chebyshev polynomial and Legendre polynomial are the most common choices for the basis functions in the spectral method. The basis functions of the Chebyshev–Tau spectral method is Chebyshev polynomial, which is a cluster of smooth orthogonal polynomials.

In our previous research [20, 21], we comprehensively introduced the Chebyshev–Tau spectral method and its application to range-independent normal modes. We further developed the related NM-CT program, which is publicly available in the open-source library OALIB [27]. Similarly, we employ the Chebyshev–Tau spectral method in this study to solve for the local modes (Eq. (3)) in the range-independent segments of a range-dependent waveguide and further refine the method.

Since the Chebyshev polynomial $\{T_i(x)\}$ is defined on the interval $[-1, 1]$, when using the Chebyshev–Tau spectral method to solve the modal equation,
Eq. (3) needs to be scaled to \( x \in [-1, 1] \):

\[
\frac{4}{|\Delta h|^2} \rho(x) \frac{d}{dx} \left( \frac{1}{\rho(x)} \frac{d\psi(x)}{dx} \right) + k^2 \psi(x) = k_x^2 \psi(x), \quad x \in [-1, 1]
\]  

As the variable to be solved, the modal function \( \psi(x) \) needs to be transformed into the spectral space formed by the Chebyshev orthogonal polynomial \( \{T_i(x)\} \):

\[
\psi(x) \approx \sum_{i=0}^{N} \hat{\psi}_i T_i(x)
\]  

where \( \{\hat{\psi}_i\}_{i=0}^{N} \) denotes the spectral coefficients of \( \psi(x) \). Obviously, the larger \( N \) is, the more accurate this approximation. Due to the good properties of the Chebyshev polynomial, the following relations are easily derived [34]:

\[
\hat{\psi}'_i \approx \frac{2}{c_i} \sum_{j=i+1, j+i=\text{odd}}^{N} j \hat{\psi}_j, \quad c_0 = 2, c_{i>1} = 1 \quad \iff \quad \hat{\Psi}' \approx D_N \hat{\Psi} \tag{16a}
\]

\[
(\nabla \hat{\psi})_i \approx \frac{1}{2} \sum_{m+n=i}^{N} \hat{\psi}_m \hat{\psi}_n + \frac{1}{2} \sum_{|m-n|=i}^{N} \hat{\psi}_m \hat{\psi}_n \quad \iff \quad (\nabla \hat{\Psi}) \approx C_v \hat{\Psi} \tag{16b}
\]

Eq. (16a) illustrates the relationship between the spectral coefficients of a function and those of its derivative function. Likewise, Eq. (16b) indicates the relationship between the spectral coefficients of a product of two functions and the spectral coefficients of one of the functions, where the right-hand side is the matrix-vector representation of this relationship.

When using the Chebyshev–Tau spectral method to solve a differential equation, we should start with the weak form of the differential equation. The weak form of the modal equation in the Chebyshev spectral space is:

\[
\int_{-1}^{1} \left[ \frac{4}{|\Delta h|^2} \rho(x) \frac{d}{dx} \left( \frac{1}{\rho(x)} \frac{d\psi(x)}{dx} \right) + k^2 \psi(x) - k_x^2 \psi(x) \right] \frac{T_i(x)}{\sqrt{1-x^2}} dx = 0 \quad x \in (-1, 1), \quad i = 0, 1, \ldots, N - 2
\]  

By substituting Eq. (15) into Eq. (17) and considering Eq. (16), the modal equation can be discretized into the following matrix-vector form:

\[
\left( \frac{4}{|\Delta h|^2} C_\rho D_N C_{1/\rho} D_N + C_{k^2} \right) \hat{\Psi} = k_x^2 \hat{\Psi}
\]  

The above represents an obvious matrix eigenvalue problem; accordingly, boundary constraints must be added to the actual solution. For details regarding the discretization process, please see Eq. (29) in reference [21].
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For the waveguide in Eqs. (8) through (13), the modal equation (Eq. (3)) must be established in both the water column and the bottom sediment. In a range-independent waveguide, a single set of basis functions cannot span both layers since the Chebyshev polynomial in interface $h$ are not continuously differentiable. Thus, we use the domain decomposition strategy [35] in Eq. (3) and split the domain interval into two subintervals:

$$
\psi(z) = \begin{cases} 
\psi_w(z) = \psi_w(x) \approx \sum_{i=0}^{N_w} \hat{\psi}_{w,i}(x_w), & x_w = -\frac{2z}{h} + 1, \quad 0 \leq z \leq h \\
\psi_b(z) = \psi_b(x) \approx \sum_{i=0}^{N_b} \hat{\psi}_{b,i}(x_b), & x_b = -\frac{2z}{H-h} + \frac{H+h}{H-h}, \quad h \leq z \leq H
\end{cases}
$$

(19)

where $N_w$ and $N_b$ are the spectral truncation orders in the water column and bottom sediment, respectively, and $\{\hat{\psi}_{w,i}\}_{i=0}^{N_w}$ and $\{\hat{\psi}_{b,i}\}_{i=0}^{N_b}$ are the modal spectral coefficients in these two layers. Similar to Eq. (18), the modal equations in the water column and bottom sediment can be discretized into matrix-vector form:

$$
A \hat{\Psi}_w = k_x^2 \hat{\Psi}_w,
$$

(20a)

$$
B \hat{\Psi}_b = k_x^2 \hat{\Psi}_b,
$$

(20b)

where $A$ and $B$ are square matrices of order $(N_w+1)$ and $(N_b+1)$, respectively, and $\hat{\Psi}_w$ and $\hat{\Psi}_b$ are column vectors composed of $\{\hat{\psi}_{w,i}\}_{i=0}^{N_w}$ and $\{\hat{\psi}_{b,i}\}_{i=0}^{N_b}$, respectively. Since the interface conditions are related to both the water column and the bottom sediment, Eqs. (20a) and (20b) should be solved simultaneously as follows:

$$
\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \hat{\Psi}_w \\ \hat{\Psi}_b \end{bmatrix} = k_x^2 \begin{bmatrix} \hat{\Psi}_w \\ \hat{\Psi}_b \end{bmatrix}
$$

(21)

The boundary conditions and interface conditions in Eqs. (9)–(13) must also be expanded to the Chebyshev spectral space and explicitly added to Eq. (21). Rearranging and combining the modified rows by an elementary row transformation, Eq. (21) can be rewritten into the form of the following block matrix:

$$
\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} \hat{\Psi}_1 \\ \hat{\Psi}_2 \end{bmatrix} = k_x^2 \begin{bmatrix} \hat{\Psi}_1 \\ \hat{\Psi}_2 \end{bmatrix}
$$

(22)

where $L_{11}$ is a square matrix of order $(N_w + N_b - 2)$ and $L_{22}$ is a square matrix of order 4. Solving this mixed linear algebraic system can yield the horizontal wavenumbers and spectral coefficients of the eigenmodes $(k_x, \hat{\Psi}_w, \hat{\Psi}_b)$. For details regarding the treatment of the boundary conditions in Eqs. (9), (10) and (13), please see Eq. (38) in reference [21].

For the acoustic half-space boundary condition in Eq. (11), since $\gamma_\infty$ contains the eigenvalue $k_x$ to be solved, the elements of $L_{21}$ and $L_{22}$ on the left side of Eq. (22) contain $k_x$, so Eq. (22) is no longer a matrix eigenvalue problem and can be solved iteratively only by a root-finding algorithm. The biggest
drawback of root-finding algorithms is that they must make a reasonable initial guess concerning the eigenvalue being sought, $k_x$ \cite{19}. Since a prior estimate of $k_x$ is usually not available, many of the existing numerical programs following similar principles fail to converge to a specific root in some cases. To avoid the same problem when using the Chebyshev–Tau spectral method to solve for the waveguides with an acoustic half-space boundary condition, we consider a more clever approach here: we use $k_{z,\infty} = \sqrt{k_{\infty}^2 - k_x^2}$ to transform the modal equation and Eq. (11) as follows:

$$
\frac{d}{dz} \left( \frac{1}{\rho(z)} \frac{d\psi}{dz} \right) + \left[ k^2(z) - k_{\infty}^2 + k_{z,\infty}^2 \right] \psi = 0 \quad (23a)
$$

$$
\left( \frac{i\rho_{\infty}}{\rho_b(H)} \frac{d\psi(z)}{dz} \right) \bigg|_{z=H} + k_{z,\infty} \psi(H) = 0 \quad (23b)
$$

where $k_{\infty} = (1 + i\eta_{\alpha_{\infty}})\omega/c_{\infty}$ is a complex constant. Eq. (23a) can naturally be discretized into the following form:

$$
\left[ \mathbf{U} + k_{z,\infty}^2 \mathbf{I} \right] \hat{\boldsymbol{\Psi}} = \mathbf{0}, \quad \mathbf{U} = \mathbf{L} - k_{\infty}^2 \mathbf{I} \quad (24)
$$

However, due to the addition of Eq. (23b) including $k_{z,\infty}$, Eq. (24) finally becomes the following polynomial eigenvalue problem:

$$
\left[ \mathbf{U} + k_{z,\infty} \mathbf{V} + k_{z,\infty}^2 \mathbf{W} \right] \hat{\boldsymbol{\Psi}} = \mathbf{0} \quad (25)
$$

where $\mathbf{U}$ in Eq. (25) is not exactly identical to that in Eq. (24), as it has been modified by boundary conditions and interface conditions; nevertheless, we denote it as $\mathbf{U}$. In addition, $\mathbf{V}$ is a zero matrix of order $(N_w + N_b + 2)$ with only the last row corresponding to the boundary condition in Eq. (23b), and $\mathbf{W}$ is simply the identity matrix that has been modified by the boundary conditions. This polynomial eigenvalue problem can be efficiently solved by the $QZ$ algorithm; alternatively, it can be transformed into a general matrix eigenvalue problem using the following trick, although the scale of the matrices is doubled:

$$
\tilde{\mathbf{U}} \tilde{\boldsymbol{\Psi}} = k_{z,\infty}^2 \mathbf{V} \tilde{\boldsymbol{\Psi}},
$$

$$
\tilde{\mathbf{U}} = \begin{bmatrix} -\mathbf{V} & -\mathbf{U} \\ \mathbf{I} & 0 \end{bmatrix}, \quad \tilde{\mathbf{V}} = \begin{bmatrix} \mathbf{W} & 0 \\ 0 & \mathbf{I} \end{bmatrix}, \quad \tilde{\boldsymbol{\Psi}} = \begin{bmatrix} \mathbf{W} \hat{\boldsymbol{\Psi}} \\ \mathbf{I} \hat{\boldsymbol{\Psi}} \end{bmatrix} \quad (26b)
$$

It is necessary to take the inverse Chebyshev transform of the eigenvectors $\hat{\boldsymbol{\Psi}}_w$ and $\hat{\boldsymbol{\Psi}}_b$ to $[0, h]$ and $[h, H]$ by Eq. (15), respectively. The vectors $\mathbf{\Psi}_w$ and $\mathbf{\Psi}_b$ are stacked into a single column vector to form $\mathbf{\Psi}$. Then, Eq. (4) is used to normalize $\mathbf{\Psi}$, and finally, a set of modes $(k_x, \psi(x))$ is obtained.
3 Coupled Modes in Range-dependent Ocean

3.1 Line source at a horizontal range of \( x_s = 0 \)

For range-dependent marine environments, the classic solution is to divide the environment into many sufficiently short and narrow segments [7], such as stairs, as shown in Figure 1. We treat each segment as being range-independent. After the eigenmodes and horizontal wavenumbers of each segment are obtained, the continuity conditions at the \( J \) segments are used to couple the acoustic fields of all the segments, thereby obtaining the acoustic field of the entire waveguide.

When ignoring the contribution of the continuous spectrum, the acoustic field of the \( j \)-th segment can generally be written as:

\[
p_j(x, z) \approx \sum_{m=1}^{M} \left[ a_{jm}^j E_m^j(x) + b_{jm}^j F_m^j(x) \right] \psi_m^j(z), \quad j = 1, 2, \ldots, J
\]  

(27)

where \( M \) is the total number of normal modes needed to synthesize the acoustic field; \( \{a_{jm}^j\}_{m=1}^M \) and \( \{b_{jm}^j\}_{m=1}^M \) are the amplitudes of the forward and backward propagation modes, respectively, which are also called coupling coefficients in the \( j \)-th segment; and \( \psi_m^j(z) \) is the \( m \)-th eigenmode of the \( j \)-th segment. \( E_m^j(x) \) and \( F_m^j(x) \) are the normalized range functions and are defined as follows:

\[
E_m^j(x) = \exp \left[ \frac{i k_{x,m}^j (x - x_{j-1})}{c_{\infty}} \right]
\]  

(28)
\[ F^j_m(x) = \exp \left[ -ik^j_{x,m} (x - x_j) \right] \]  

(29)

where \( k^j_{x,m} \) is the horizontal wavenumber of the \( m \)-th modes in the \( j \)-th segment. For special cases when \( j = 1, x_{j-1} = x_1 \). It is worth noting that the definition of \( E^j_m(x) \) here is the same as that in COUPLE [9], whereas the definition of \( F^j_m(x) \) is not the same. Instead, the definition of \( F^j_m(x) \) in COUPLE is:

\[ F^j_m(x) = \exp \left[ -ik^j_{x,m} (x - x_{j-1}) \right] \]

(30)

This improvement was reported by Luo and Yang [11], who called this improved approach the direct global matrix coupled mode (DGMCM) algorithm. The existence of leakage modes may cause the value of \( F^j_m(x) \) defined in COUPLE to overflow, specifically for the leakage mode \( k^j_{x,m} = R + iI \), where \( R \) and \( I \) are the real and imaginary parts of \( k^j_{x,m} \), respectively, and \( I > 0 \). In Eq. (30), \( x - x_{j-1} > 0 \), so Eq. (30) contains \( \exp[I(x - x_j)] \). When \( I \) is large, using Eq. (30) may cause numerical overflow. In contrast, in Eq. (29), since the exponential part contains \( \exp[I(x - x_j)] \) at this time \( (x - x_j < 0) \), regardless of the value of \( I \) at this time, the value of \( F^j_m(x) \) is limited, which ensures that no numerical overflow occurs. In general, this improvement uses the left boundary to normalize the forward acoustic field and the right boundary to normalize the backward acoustic field, thereby guaranteeing the numerical stability of the calculation.

The method of coupling segments explicitly imposes two segment continuity conditions on the sides of the segments. The first segment condition requires that the acoustic pressure be continuous at the \( j \)-th side:

\[ p^{j+1}(x_j, z) = p^j(x_j, z) \]

(31)

The second segment condition requires that the radial velocity of the acoustic pressure be continuous at the \( j \)-th side:

\[ \frac{1}{\rho_{j+1}(z)} \frac{\partial p^{j+1}(x_j, z)}{\partial x} = \frac{1}{\rho_j(z)} \frac{\partial p^j(x_j, z)}{\partial x} \]

(32)

For the first segment condition, we have:

\[ \sum_{m=1}^{M} \left[ a^j_m E^j_m (x_j) + b^j_m F^j_m (x_j) \right] \psi^{j+1}_m(z) = \]

(33)

\[ \sum_{m=1}^{M} \left[ a^j_m E^j_m (x_j) + b^j_m F^j_m (x_j) \right] \psi^j_m(z) \]

Next, to the above equation, we apply the following operator:

\[ \int_0^H \frac{\psi^{j+1}_m(z)}{\rho_{j+1}(z)} \, dz \]
and we use the orthogonal normalization formula in Eq. (4) for the eigenmodes of the \((j + 1)\)-th segment, yielding the following:

\[
a_{j+1}^{\ell} + b_{j+1}^{\ell} F_{\ell}^{j+1} (x_j) = \sum_{m=1}^{M} \left[ a_m^j E_m^j (x_j) + b_m^j \right] \tilde{c}_{\ell m} \tag{34a}
\]

\[
\tilde{c}_{\ell m} = \int_{0}^{H} \frac{\psi_{\ell+1}^j (z) \psi_m^j (z)}{\rho_{j+1} (z)} dz, \quad \ell = 1, \ldots, M \tag{34b}
\]

The above formula can be easily written in the following matrix-vector form:

\[
a^{j+1} + F^{j+1} b^{j+1} = \tilde{C}^j (E^j a^j + b^j) \tag{35}
\]

Eq. (27) clearly reveals:

\[
\frac{1}{\rho_j} \frac{\partial p^j (x, z)}{\partial x} \approx \frac{1}{\rho_j} \sum_{m=1}^{M} k_{x,m}^j \left[ a_m^j E_m^j (x) - b_m^j F_m^j (x) \right] \psi_m^j (z) \tag{36}
\]

Therefore, for the second segment condition we have:

\[
\frac{1}{\rho_{j+1}} \sum_{m=1}^{M} k_{x,m}^{j+1} \left[ a_m^{j+1} - b_m^{j+1} F_m^{j+1} (x_j) \right] \psi_m^{j+1} (z) = \sum_{m=1}^{M} k_{x,m}^j \left[ a_m^j E_m^j (x_j) - b_m^j \right] \psi_m^j (z) \tag{37}
\]

Similarly, we apply the following operator to the above equation:

\[
\int_{0}^{H} (\cdot) \psi_{\ell+1}^j (z) dz
\]

and we use the orthogonal normalization formula in Eq. (4) for the eigenmodes of the \((j + 1)\)-th segment, yielding the following:

\[
a_{\ell}^{j+1} - b_{\ell}^{j+1} F_{\ell}^{j+1} = \sum_{m=1}^{M} \left[ a_m^j E_m^j (x_j) - b_m^j \right] \hat{c}_{\ell m} \tag{38a}
\]

\[
\hat{c}_{\ell m} = \frac{k_{x,m}^{j+1}}{k_{x,\ell}^j} \int_{0}^{H} \frac{\psi_{\ell+1}^j (z) \psi_m^j (z)}{\rho_{j} (z)} dz, \quad \ell = 1, \ldots, M \tag{38b}
\]

Then, the above formula can be naturally written in the following matrix-vector form:

\[
a^{j+1} - F^{j+1} b^{j+1} = \hat{C}^j (E^j a^j - b^j) \tag{39}
\]
Eqs. (35) and (39) can be combined into the following form:

\[
\begin{bmatrix}
a^{j+1} \\
b^{j+1}
\end{bmatrix} =
\begin{bmatrix}
R^j_1 & R^j_2 \\
R^j_3 & R^j_4
\end{bmatrix}
\begin{bmatrix}
a^j \\
b^j
\end{bmatrix}
\]

(40)

where

\[
R^j_1 = \frac{1}{2} \left( \tilde{C}^j + \hat{C}^j \right) E^j
\]

(41a)

\[
R^j_2 = \frac{1}{2} \left( \tilde{C}^j - \hat{C}^j \right)
\]

(41b)

\[
R^j_3 = \frac{1}{2} \left( F^{j+1} \right)^{-1} \left( \tilde{C}^j - \hat{C}^j \right) E^j
\]

(41c)

\[
R^j_4 = \frac{1}{2} \left( F^{j+1} \right)^{-1} \left( \tilde{C}^j + \hat{C}^j \right)
\]

(41d)

Finally, the segment condition at the acoustic source \(x_s = 0\) and the radiation condition at \(x \to \infty\) should also be considered. The segment condition at the acoustic source \(x_s = 0\) is:

\[
a^1 = s
\]

(42a)

\[
s_m = \frac{i}{2 \rho (z_s)} \psi^1_m(z_s) e^{ik_{1,m,x} x} , \quad m = 1, \ldots, M
\]

(42b)

For the radiation condition at \(x \to \infty\), \(b^J = 0\) is sufficient, where \(J\) denotes the total number of horizontal segments.

Combining the continuity conditions at the boundaries of the \(J\) segments with the boundary condition at the acoustic source and the radiation condition at infinity, the following system of linear algebraic equations can be assembled:

\[
\begin{bmatrix}
I & 0 & 0 \\
R^1_1 & R^1_2 & -I & 0 \\
R^1_3 & R^1_4 & 0 & -I \\
R^{j-2}_1 & R^{j-2}_2 & -I & 0 \\
R^{j-2}_3 & R^{j-2}_4 & 0 & -I \\
R^{J-1}_1 & R^{J-1}_2 & -I & 0 \\
R^{J-1}_3 & R^{J-1}_4 & 0 & -I
\end{bmatrix}
\begin{bmatrix}
a^1 \\
b^1 \\
a^2 \\
\vdots \\
b^{J-2} \\
a^{J-1} \\
b^J \\
a^J
\end{bmatrix}
= 
\begin{bmatrix}
s \\
0 \\
0 \\
\vdots \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

(43)

The global matrix formed by a line source is clearly a sparse band matrix whose order is \((2J - 1) \times M\); the upper bandwidth is \((2M - 1)\), and the lower bandwidth is \((3M - 1)\). These linear algebraic equations are then solved to obtain the coupling coefficients, and Eq. (27) is implemented to synthesize the sound pressure field of the entire waveguide.
Since $x_{j-1} = x_1$ when $j = 1$, in the first segment, $E_m^j(x)$ is normalized to the right side. For the leakage modes, when $\exp[I(x_1 - x)]$ is large, $E_m^j(x)$ triggers numerical overflow. To avoid this problem, the superposition principle is used to solve for the acoustic field. Substituting $a^j$ in Eq. (42) into Eq. (27):

$$p^1(x, z) = \frac{i}{2\rho(z_s)} \sum_{m=1}^{M} \psi^1_m(z_s) \psi^1_m(z) e^{ik_{x,m}x} + \sum_{m=1}^{M} b^1_m F^1_m(x) \psi^1_m(z)$$  \hspace{1cm} (44)

where the first term on the right side represents the range-independent acoustic field, and the second term represents the backscattered acoustic field caused by range dependence.

### 3.2 Generalized line source at any position

In the above environment, the line source must be located at $x_s = 0$. However, after a simple improvement, the sound source can be located at any horizontal range $x_s$ [11]. This improvement is valuable for solving the acoustic propagation where the line source is located on a slope, as in Figure 2. Suppose a line source is located at $(x_s, z_s)$, and a virtual boundary $j_s$ is introduced at the horizontal range $x_s$. All boundaries except $j_s$ still satisfy the boundary conditions of Eqs. (31) and (32). However, due to the existence of the sound source, the boundary conditions at the boundary $j_s$ need to be modified. The
sound fields in the $j_s$ and $j_s + 1$ segments can be expressed as:

$$
p^{j_s}(x, z) = \sum_{m}^{M} \left[ a^{j_{m}} E^{j_s}_{m}(x) + b^{j_{m}} F^{j_s}_{m}(x) \right] \psi^{j_{m}}(z) \tag{45a}$$

$$
p^{j_s+1}(x, z) = \sum_{m}^{M} \left[ a^{j_{m}+1} E^{j_{s}+1}_{m}(x) + b^{j_{m}+1} F^{j_{s}+1}_{m}(x) \right] \psi^{j_{m}+1}(z) \tag{45b}$$

Applying segment condition (31) at virtual boundary $j_s$ and taking into account $\psi^{j_{m}}(z) = \psi^{j_{m}+1}(z)$, $F^{j_{s}}_{m}(x_{j_{s}}) = 1$ and $E^{j_{s}+1}_{m}(x_{j_{s}}) = 1$, we can obtain:

$$
a^{j_{m}+1} + b^{j_{m}+1} F^{j_{s}+1}_{m} = a^{j_{m}} E^{j_s}_{m} + b^{j_{m}} \tag{46a}$$

$$
a^{j_{s}+1} + E^{j_{s}+1} b^{j_{s}+1} = E^{j_s} a^{j_{s}} + b^{j_{s}} \tag{46b}$$

This equation is consistent with the form of Eq. (35), except that the acoustic properties on both sides of boundary $j_{s}$ are completely consistent, resulting in $\tilde{C} = I$.

The integral of Eq. (5) yields the following:

$$
\frac{\text{d}X_{m}}{\text{d}x} \bigg|_{x_{s}} = -\frac{\psi^{j_{s}}(z_{s})}{\rho(z_{s})} \tag{47}$$

Considering the boundary condition and $k^{j_{s}}_{x,m} = k^{j_{s}+1}_{x,m}$, we can obtain:

$$
(ik^{j_{s}+1}_{x,m} a^{j_{m}+1} - ik^{j_{s}+1}_{x,m} b^{j_{m}+1} F^{j_{s}+1}_{m}) - (ik^{j_{s}}_{x,m} a^{j_{m}} E^{j_s}_{m} - ik^{j_{s}}_{x,m} b^{j_{m}}) = -\frac{\psi^{j_{s}}_{m}(z_{s})}{\rho(z_{s})} \tag{48a}$$

$$
a^{j_{s}+1} - E^{j_{s}+1} b^{j_{s}+1} = E^{j_{s}} a^{j_{s}} - b^{j_{s}} - s \tag{48b}$$

$$
s_{m} = -\frac{i}{\rho(z_{s})} \frac{\psi^{j_{s}}_{m}(z_{s})}{k^{j_{s}}_{x,m}}, \quad m = 1, 2, \cdots, M \tag{48c}$$

Eq. (48) also corresponds to (39). Because the acoustic properties on both sides of virtual boundary $j_{s}$ are exactly the same, $\tilde{C} = I$. Eqs. (46) and (48) demonstrate that the relationship of the coupling coefficients between the $j_{s}$-th and $(j_{s} + 1)$-th segments is:

$$
\begin{bmatrix}
a^{j_{s}+1} \\
b^{j_{s}+1}
\end{bmatrix} = \begin{bmatrix}
E^{j_{s}} & 0 \\
0 & (F^{j_{s}+1})^{-1}
\end{bmatrix} \begin{bmatrix}
a^{j_{s}} \\
b^{j_{s}}
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
1 \\
-1\frac{s}{2}
\end{bmatrix} (F^{j_{s}+1})^{-1} s \tag{49}
$$

Since the line source is located at $x_{s} \neq 0$, the boundary condition at $x = 0$ becomes the radiation condition $a^1 = 0$. At this time, the global matrix used
to solve for the coupling coefficients can be assembled as follows:

\[
\begin{bmatrix}
R_1^j & -I_0 & 0 & 0 \\
R_2^j & 0 & -I_0 & 0 \\
R_3^j & R_2^j & -I_0 & 0 \\
R_4^j & R_3^j & R_2^j & -I_0 \\
\end{bmatrix}
\begin{bmatrix}
E_j^s \\
0 \\
0 \\
(\mathbf{F}_j^{s+1})^{-1} \\
\end{bmatrix}
= \begin{bmatrix}
b_1^j \\
a_2^j \\
b_3^j \\
a_4^j \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
-1/2(\mathbf{F}_j^{s+1})^{-1}s \\
\end{bmatrix}
\]

The coupling coefficients can be obtained by solving this system of linear equations. Similarly, the global matrix formed by the line source is a sparse band matrix whose order is \((2J - 2) \times M\); the upper bandwidth is \((2M - 1)\), and the lower bandwidth is \((3M - 1)\).

### 4 Algorithm

Based on the above derivation, the proposed algorithm can be devised as follows:

**Input:** Data from the marine environment and the program parameters.

**Output:** The sound pressure field.

1. Set the parameters.
   The parameters include the frequency \(f\) and location of the line source \((x_s, z_s)\), the total depth of the ocean \(H\), the topography of the seabed, the number of acoustic profiles, and the specific information of each group of acoustic profiles. In addition, the parameters should also include the spectral truncation order \((N_w\) and \(N_b)\), the horizontal and vertical resolutions, and the phase velocity range used by the screening modes. If the lower boundary is the upper interface of an acoustic half-space, the speed \(c_\infty\), density \(\rho_\infty\) and attenuation \(\alpha_\infty\) of the half-space must also be specified.

2. Segment the marine environment based on the seabed topography and acoustic profiles.
   Stair-step discretization criteria have been established to accurately represent the smoothly varying bathymetry in numerical models. Jensen confirmed that the strictest segmentation criterion is \(\Delta r \leq \lambda/4\), where \(\lambda = \min\{c_w, c_b\}/f\) [36]. We suppose the entire waveguide is divided into \(J\) segments.

3. Apply the Chebyshev–Tau spectral method to solve for the horizontal wavenumbers and eigenmodes \((k_x, \psi)\) of \(J\) range-independent segments.
   It is worth emphasizing that the modal spectral coefficients \(\tilde{\Psi}\) obtained for \(J\) segments should be transformed to a uniform vertical resolution. Otherwise, \(\tilde{c}_{tm}\) in Eq. (35) and \(\hat{c}_{tm}\) in Eq. (39) cannot be calculated. The uniform vertical resolution is specified by the environment file in the
first step. This process exhibits natural parallelism because the solution processes of the segments are independent.

4. Select the appropriate modes according to the phase velocity interval, and denote the number of modes as $M$.

   If the numbers of modes selected among the $J$ segments are not equal, the maximum number of modes $M$ is adopted, and segments with fewer than $M$ modes are allowed to retain a small number of evanescent modes to ensure that the number of modes is identical among all $J$ segments.

5. Calculate the coupling submatrices $\{R^1_j\}_{j=1}^{J-1}$, $\{R^2_j\}_{j=1}^{J-1}$, $\{R^3_j\}_{j=1}^{J-1}$, and $\{R^4_j\}_{j=1}^{J-1}$. This step also exhibits natural parallelism.

6. According to whether $x_s$ is equal to 0, calculate $s$ in the boundary conditions, construct the global matrix of coupling coefficients according to Eq. (43) or Eq. (50), and solve the system of linear equations to obtain the coupling coefficients ($\{a^j\}_{j=1}^J$, $\{b^j\}_{j=1}^J$) of the $J$ segments.

7. Calculate the synthetic sound field.

   The sound pressure subfields of the segments are calculated according to Eq. (27), and the subfield in the first segment is corrected according to Eq. (44) if $x_s = 0$. The subfields of the $J$ segments are then spliced to form the sound pressure field of the entire waveguide.

5 Numerical Simulation

To verify the correctness of the algorithm and to test and demonstrate the developed code, the following ten numerical experiments are conducted. Here, we name the program developed based on the proposed algorithm SPEC. To present the acoustic field results, the transmission loss (TL) of the acoustic pressure is defined as $\text{TL} = -20 \log_{10}(|p|/|p_0|)$ in dB, where $p_0 = iH_0^{(1)}(k_s)/4$ is the acoustic pressure 1 m from the line source and $k_s$ is the wavenumber of the medium at the location of the line source. The TL field is often used in actual displays to compare and analyze sound fields [1]. For convenience, the spectral truncation orders in SPEC of all the examples are set to $N_w = N_b = 30$; in actual calculations, users can arbitrarily specify the spectral truncation orders in the input file of SPEC.

5.1 Analytical examples: simple wedge-shaped waveguide

Consider the ideal wedge-shaped waveguide shown in Figure 3(a), which is a primary benchmark problem of range-dependent waveguides. Both the sea surface and the seabed are pressure release boundaries. The line source is perpendicular to the plane and intersects the plane at $(x_s, z_s)$. This benchmark problem has been proposed and discussed at two consecutive conferences of the Acoustic Society of America (ASA), and Buckingham gave an analytical solution to this problem [37]. Here, we briefly introduce this problem. The
velocity potential in the water body is:

\[
\Phi (r, r', \theta, \theta') = \frac{i}{\theta_0} \sum_{m=1}^{M} \mathcal{J}_{\nu_m} (k r_<) \mathcal{H}_{\nu_m}^{(1)} (k r_> \sin(\nu_m \theta) \sin(\nu_m \theta')
\]

where \( r \) and \( r' \) are the distances from the receiver and line source to the apex of the wedge, respectively; \( \theta \) and \( \theta' \) are the angles to the depths of the receiver and source, respectively, measured about the apex; \( \theta_0 \) is the wedge angle; \( k \) is the wavenumber of the seawater; \( r_< = \min(r, r') \) and \( r_> = \max(r, r') \); and
\( J_{\nu_m}(\cdot) \) and \( H^{(1)}_{\nu_m}(\cdot) \) are the Bessel function and Hankel function of the first kind of order \( \nu_m \), respectively. The TL field is calculated by the following formula:

\[
TL = -20 \log_{10} \left| \frac{\Phi (r, r', \theta, \theta')}{\Phi_0 (1)} \right| \quad (52a)
\]

\[
\Phi_0 (R) = \frac{i}{4} H^{(1)}_0 (kR) \quad (52b)
\]

where \( R \) is the radial distance from the line source to the field point.

For example 1, Figure 4 shows the sound fields of the wedge-shaped waveguide calculated by the analytical solution and SPEC. To ensure the accuracy of the results, we divide the SPEC model into 800 horizontal segments; this segmentation fully satisfies the step approximation discretization criterion demonstrated by Jensen [36]. At the same time, to avoid numerical overflow in the calculation of the Bessel function and Hankel function (see Eq. (51)) that may be caused by a large number of modes (although this problem can be handled by the Debye approximation), the number of modes is uniformly taken as \( M = 6 \). Figure 4 clearly confirms that the SPEC results are almost perfectly consistent with the analytical solution regardless of the TL in the whole field or at a certain receiving depth.

![Sound fields of example 1 and the TL curves at a depth of 30 m calculated by analytical solution and SPEC.](image-url)
For example 2, the configuration in Figure 3(b) is the same as that in Figure 3(a). The main differences are that the line source in Figure 3(b) is located in the center of the oceanic wedge, and the maximum depth of the water column is twice as large as that in Figure 3(a). In this case, the analytical solution shown in Eqs. (51) and (52) still exists, as illustrated in Figure 5. In this example, the horizontal distance is divided into 1600 segments, and the number of modes is taken as $M = 12$. The SPEC solution shown in Figure 5(b) demonstrates excellent overall agreement with the analytical solution shown in Figure 5(a). The positions of the peaks and troughs are the same, with only slight occasional discrepancies between the heights of the interference peaks. Moreover, the absolute levels of the TL curves are in good agreement.

Figure 3(c) and Figure 3(d) illustrate wedge-shaped waveguides on a rigid seabed corresponding to Figure 3(a) and Figure 3(b), respectively. Except for the nature of the ocean bottom, these configurations are exactly the same as those in examples 1 and 2. In this case, there is still an analytical solution, and the expression is the same as Eq. (51a); however, the following usually holds for a rigid seabed:

$$\nu_m = \left( m - \frac{1}{2} \right) \frac{\pi}{\theta_0}, \quad m = 1, 2, \ldots$$  \hspace{1cm} (53)
Figure 6 shows the analytical solution of the wedge-shaped waveguide atop a rigid seabed and the SPEC-calculated sound field. The numbers of segments and modes in SPEC are the same as those in the case of a free seabed. Given the perfect agreement between the analytical and SPEC solutions, the same conclusion as that from example 1 can naturally be drawn with confidence.

Similarly, we calculate the rigid-bottomed waveguide when the line source is in the center of the wedge, and the results are shown in Figure 7. Again, the numbers of segments and modes in SPEC are the same as those in the case of a free seabed. As with the previous examples, Figure 7 demonstrates that the SPEC results still match the analytical solution very well, although certain differences are observed in areas far from the line source.

The above comparisons of the SPEC solution with the analytical solution clearly reflects the excellent accuracy achieved by SPEC, which is sufficient to verify the reliability of the proposed algorithm (regardless of whether the line source is classic or generalized) and the correctness of the code’s implementation. This stability and accuracy provide motivation to use this code for the numerical simulation of more practical ocean acoustic problems in future research.
5.2 Complex examples without analytical solution

5.2.1 Modified wedge-shaped ocean

As depicted in Figure 3(e), we consider two modifications to the ideal wedge-shaped waveguide, leading to some interesting conclusions. The first modification is to flatten the downward slope to the left of the line source in Figure 3(b). Please note that this modification involves only the half of the sound field to the left of the sound source. The same number of segments is used as in Figure 3(b), but since the maximum depth of the ocean is 200 m, the number of modes is taken as $M = 6$. For comparison, we show the analytical sound field and TL curves of Figure 3(b) in Figure 8. Remarkably, for the unmodified right half of the waveguide (from 4 km to 8 km), the sound field generated by the line source is exactly the same as that in Figure 3(b). This indicates that the uphill sound field excited by the line source is not affected by the downhill sound field; that is, the backscatter occurring in the downhill area has a negligible impact on the uphill area.

Similarly, the second modification is to flatten the upward slope to the right of the line source in Figure 3(b). Please note that this modification involves only the half of the sound field to the right of the sound source. The same numbers of segments and modes are used as in Figure 3(b). For comparison,
we show the analytical sound field and TL curves of Figure 3(b) in Figure 9. In contrast to the results of the first modification, we do not observe a similar phenomenon; that is, for the unmodified left half of the sound field (from 0 km to 4 km), the sound field generated by the line source does not match the analytical solution. This indicates that the downhill sound field excited by the line source is affected by the backscattering occurring in the uphill area, indicating that backscattering in the uphill area cannot be ignored because it is sufficient to induce a completely different behavior in the downhill sound field.

These two modifications not only demonstrate the characteristics of the sound field excited by the line source but also further explain that the proposed algorithm for a generalized line source is completely correct. Moreover, these examples verify the robustness of the SPEC program when dealing with line sources located exactly above changes in the seabed topography.

### 5.2.2 Penetrable slope and seamount problems

Consider the penetrable slope in the marine environment shown in Figure 10(a), which is a very common range dependency. The sound field in this
case does not have an analytical solution, so we illustrate the result of COU-
PLE for comparison. First, we consider the situation where the sound source is placed at $x_s = 0$. Figure 11 displays the sound fields calculated by COU-
PLE and SPEC. In the calculation, the number of coupled modes is $M = 5$, and the slope is divided into 200 segments. The sound fields confirm that the results of the two programs roughly match.

Next, we consider the situation where the sound source is placed on a slope. Because COUPLE cannot handle this situation, we visualize only the result of SPEC to demonstrate its functionality. We also show the case where the lower boundary of the waveguide is an acoustic half-space in Figure 10(b). The configurations of Figure 10(b) and Figure 10(a) are exactly the same; the only difference is that the lower boundary in the latter is taken as an acoustic half-space. Figure 12 shows the sound fields under the configurations of Figure 10(a) and Figure 10(b), where the range dependence is discretized into 200 flat segments. Figure 10(a) has 6 coupled modes, whereas Figure 10(b) has 3 coupled modes. These figures illustrate that SPEC can calculate the range-dependent waveguide with the lower boundary being an acoustic half-space. Figure 12 also shows the effects of different lower boundary conditions on the sound field.

**Fig. 9** Sound fields of example 6 and the TL curves at a depth of 30 m calculated by the analytical solution and SPEC.
Example 9 is a classic seamount problem whose configuration is shown in Figure 10(c). Similarly, we first consider the sound field where the sound source is located at $x_s=0, z_s=100$ m, and we compare the results with COUPLE. Figure 13 shows the sound fields calculated by COUPLE and SPEC when the same numbers of segments and coupled modes are used, confirming that the results of the two programs are in good agreement. In addition, here, we consider two different sound source depths above a slope: the sound sources are at the same horizontal distance of $x_s=2500$ m, but the depth is variable at $z_s=50$ m and $z_s=100$ m. The SPEC-calculated sound fields produced by these sound sources at two different depths are shown in Figure 14. The deeper sound source produces more energy that is distributed in the center of the waveguide, whereas less energy reaches the other side of the seamount.

Figure 10(d) presents a flat-topped seamount problem. First, we consider the situation when the sound source is located at $x_s=0$, $z_s=100$ m and the lower boundary is a pressure release boundary. Figure 15 shows that the results of SPEC and COUPLE are highly consistent when the same numbers of segments and coupled modes are used. In addition, we consider line sources located at different positions above the slope. Because the range dependence of this example is symmetrical, the positions of the line source are also symmetrical. Accordingly, from the SPEC results shown in Figure 16, the sound fields...
Fig. 11 Sound fields of example 7 calculated by COUPLE and SPEC (the line source is located at $x_s=0, z_s=36$ m).

Fig. 12 Sound fields of examples 7 and 8 calculated by SPEC.

excited by a symmetrical line source in a symmetrical range-dependent waveguide are also symmetrical. This outcome further verifies the correctness of the proposed algorithm and program.

It must be noted that when $x_s = 0$ m in COUPLE, $p_0$ is taken as $2\sqrt{2\pi k_s}$. In the examples for comparison with COUPLE, $p_0$ in SPEC also takes the above form instead of $p_0 = i\mathcal{H}^{(1)}_0(k_s)/4$. 
Fig. 13 Sound fields of example 9 calculated by COUPLE and SPEC (the line source is located at $x_s=0, z_s=100$ m).

Fig. 14 Sound fields of example 9 calculated by SPEC.

5.3 Analysis and Parallelization

5.3.1 Analysis

To compare the running time of the algorithm proposed in this paper with those of COUPLE, the test results of the example are listed in Table 1. The tests are performed on a Dell XPS8930 personal computer, and each program
is executed ten times. The running times listed in the table are average values. The compiler used is gfortran 7.5.0; all programs used for comparison are also compiled with this compiler. For the same experiments, when using identical numbers of segments and test functions, SPEC has much shorter running times than COUPLE, which directly demonstrates the efficiency of the devised algorithm.

**Table 1** Comparison between the running times of SPEC and COUPLE (unit: seconds).

| No. | COUPLE    | SPEC   |
|-----|-----------|--------|
| 7   | 22.573    | 1.830  |
| 9   | 13.150    | 1.074  |
| 10  | 28.338    | 5.118  |

From a computational cost perspective, the computational load of the algorithm proposed in this article is concentrated in the third and sixth steps. The former needs to solve $J$ dense matrix eigenvalue problems with a scale of $(N_w + N_b + 2)$ (see Eq. (22)), and the size of the matrix doubles with the introduction of an acoustic half-space boundary (see Eq. (26)). In addition, the
Fig. 16 Sound fields of example 10 calculated by COUPLE and SPEC (the line source is located at $x_s=0, z_s=100$ m).

latter needs to solve the system of linear equations of the band matrix with a scale of $(2J - 1) \times M$ (see Eq. (43)). Table 2 lists the running times of these two steps in comparison with the running time of the entire program. Obviously, these two steps compose the main computational load of the algorithm, and step 3 has a larger computational load than step 6.

Table 2 Comparison between the running times of the computationally intensive steps in SPEC (unit: seconds).

| No. | Step 3 | Step 6 | Total |
|-----|--------|--------|-------|
| 7   | 0.983  | 0.174  | 1.790 |
| 8   | 7.033  | 0.950  | 9.638 |
| 9   | 0.489  | 0.413  | 1.125 |
| 10  | 3.432  | 1.244  | 5.019 |

However, this is exactly why SPEC is more efficient than COUPLE. COUPLE uses the Galerkin method to solve the modal equation of a range-independent segment and first constructs a set of basis functions that
satisfy the boundary conditions. COUPLE then solves a nonsingular Sturm–Liouville problem and uses its solution as the basis functions. In SPEC, the Chebyshev–Tau spectral method directly uses Chebyshev polynomial as the basis functions without additional construction. Moreover, the Chebyshev–Tau spectral method needs only a simple spectral transformation to form the matrix eigenvalue problem, while the Galerkin method needs to implement many numerical integrations to form the generalized matrix eigenvalue problem. Furthermore, to solve for the coupling coefficients, COUPLE performs a large number of small matrix multiplications and solves a system of linear equations, while the global matrix formed by SPEC has a band shape, thereby presenting a problem that many mature algorithms can efficiently solve.

5.3.2 Parallelization

The main computational load of the proposed algorithm can naturally be distributed in parallel. We therefore try to optimize and accelerate SPEC using OpenMP multithreaded parallel technology. Table 3 lists the running times and speedup ratios of SPEC under different numbers of threads. In terms of the speedup, multithreaded acceleration achieves incredible results. Four threads can reach a speedup of almost 3, which greatly reduces the running time. This remarkable effect is of extraordinary significance, as this finding means that SPEC can easily run in parallel and achieve good accuracy on an inexpensive personal computer, thereby overcoming the shortcomings of coupled modes to a certain extent. The running time on four threads further demonstrates that SPEC is far more efficient than COUPLE in practical applications.

| No. serial | Number of Threads |
|------------|-------------------|
|            | 1 | 2 | 4 | 1 | 2 | 4 |
| 7          | 1.830 | 1.802(1) | 0.983 (1.83) | 0.599 (3.00) |
| 8          | 9.638 | 9.507(1) | 4.812 (1.98) | 2.501 (3.80) |
| 9          | 1.017 | 1.081(1) | 0.581 (1.86) | 0.351(3.08) |
| 10         | 5.118 | 5.815(1) | 3.374 (1.72) | 2.204 (2.64) |

6 Remarks and Conclusion

6.1 Remarks

From the above analysis, we can intuitively summarize the following advantages of the algorithm and program proposed in this article:
1. The proposed algorithm can solve the acoustic propagation problem of both classic line sources and generalized line sources in a planar coordinate system. At present, no open-source numerical software can solve for the sound field excited by a generalized line source at any position.

2. The proposed algorithm improves the method of normalizing the range solution and is unconditionally stable. This approach uses a global matrix to solve for all the coupling coefficients at once, therefore, the numerical program (SPEC) is stable and robust.

3. The global coupling matrices formed by the two types of line sources are band-shaped and sparse, so they can be solved efficiently.

4. The normal mode solver based on the Chebyshev–Tau spectral method is capable and can accurately solve for the lower boundary conditions of perfectly free, perfectly rigid boundaries and acoustic half-spaces. Therefore, SPEC does not need to assume an infinite density for the bottom layer in the input file (as COUPLE does) to simulate a perfectly rigid seabed, nor does it need numerical methods to find roots like KRAKEN.

5. The proposed algorithm is naturally parallel, so SPEC can be easily run in parallel on a personal computer.

6.2 Conclusion

In this article, we propose a numerical algorithm to solve for the sound field produced by a line source in the plane. This algorithm considers a range-dependent ocean environment and an infinite line source that can be located anywhere in the waveguide. The proposed algorithm is generally based on the coupled modes of stepwise approximation; that is, a number of narrow range-independent segments are used to approximate the complete waveguide. The proposed algorithm uses a global matrix to solve for the coupling coefficients of all segments at once. These coupling coefficients include two parts of forward and backward propagation, so this algorithm provides a fully two-way coupling solution. In terms of solving for local normal modes, the proposed algorithm employs a Chebyshev–Tau spectral method that is more efficient, accurate, capable and robust than classic algorithms. This method can flexibly deal with various complex acoustic profiles and seabed topographies, and no modes will be missed.

It is worth mentioning that even though the algorithm proposed in this article greatly improves the computational efficiency over the traditional coupled modes algorithm, from the perspective of the model itself, the coupled mode technique may still be slower than the methods based on rays and the parabolic approximation. Therefore, the proposed algorithm is suitable mainly for low-frequency and shallow-sea calculations or to provide accurate benchmark solutions for other models. For high-frequency, deep-sea and long-range sound fields, as analyzed in Section 5.3.1, the local eigensolutions that need to be solved increase sharply, and the scale of the global matrix also increases rapidly; hence, the proposed algorithm needs to be further optimized and expanded.
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