Representation Formula for Viscosity Solutions to Parabolic PDEs with Sublinear Operators

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Abstract

We provide a representation formula for viscosity solutions to a class of nonlinear second order parabolic PDE problem involving sublinear operators. This is done through a dynamic programming principle derived from [8]. The formula can be seen as a nonlinear extension of the Feynman–Kac formula and is based on the backward stochastic differential equations theory.

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1 Introduction

It is well known that viscosity solutions were conceived by Crandall and Lions (1982) in the framework of optimal control theory. The goal was to show well posedness of Hamilton–Jacobi–Bellman equations in the whole space, and to prove, via dynamic programming principle, the value function of a suitable optimal control problem being the unique solution.

When trying to extend viscosity methods to the analysis of second order parabolic partial differential equations, PDEs for short, and getting representation formulae, it appeared clear that some stochastic dynamics must be brought into play. Not surprisingly, this has been first done for stochastic control models. The hard work of a generation of mathematicians, [14, 15, 2, 13, 11, 18] among the others, allowed making effective dynamic programming approach to stochastic control problems.

Prompted by this body of investigations, a stream of research arose in the probabilistic community ultimately leading to the theory of backward stochastic differential equations, BSDEs for short, which was introduced by Pardoux and Peng in [20] (1990). Since then, it has attracted a great interest due to its connections with mathematical finance and PDEs, as well

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as with stochastic control. This theory has been in particular used to extend the classical Feynman–Kac formula, which establishes a link between linear parabolic PDEs and stochastic differential equations, SDEs for short, to semilinear and quasilinear equations, see for example [6, 7, 16]. See also [21] for a rather complete overview of the semilinear case.

For sake of clarity, let us consider the following semilinear parabolic PDE problem coupled with final conditions,

\[
\begin{aligned}
&\frac{1}{2} \left( \sigma \sigma^t, D_x^2 u \right)(t, x) + (\nabla_x u) b(t, x) \\
&\quad + \partial_t u(t, x) + f(t, x, u, \nabla_x u) = 0 \\
&u(T, x) = g(x),
\end{aligned}
\tag{1}
\]

then its viscosity solution can be written as \(u(t, x) = E \left( Y_t^{t,x} \right)\), where \(Y\) is given by the following system, called forward backward stochastic differential equation or FBSDE in short, which is made in turn of two equations, the first one is a SDE, and the second one a BSDE depending on the first one

\[
\begin{aligned}
X_s^{t,x} &= x + \int_t^s \sigma(r, X_r^{t,x}) dW_r + \int_t^s b(r, X_r^{t,x}) dr, \\
Y_s^{t,x} &= g \left( X_T^{t,x} \right) + \int_s^T f \left( r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x} \right) dr - \int_s^T Z_r^{t,x} dW_r,
\end{aligned}
\tag{2}
\]

As can be intuitively seen, the SDE takes care of the linear operator defined by \(\sigma\) and \(b\), also called the infinitesimal generators of the SDE, while the BSDE depends on \(f\) and \(g\). In other words, this extension of the Feynman–Kac formula basically does not modify the treatment of the second order linear operator with respect to the completely linear case.

Subsequently, Peng introduced in [22] (2006) the notion of \(G\)-expectation, a nonlinear expectation generated by a fully nonlinear second order operator \(G\) via its viscosity solutions. This work has originated an active research field, with relevant applications to Mathematical Finance.

Peng has improved this theory in several papers and has given a comprehensive account of it in the book [23], where he highlights the role of the so-called sublinear expectations, namely \(G\)-expectations generated by sublinear operators. Finally in [8], Peng provides representation formulae for viscosity solutions using these expectations. More precisely, given a sublinear operators \(G\) and the \(G\)-heat equation

\[
\begin{aligned}
&\partial_t u(t, x) + G \left( D_x^2 u(t, x) \right) = 0, \\
&u(T, x) = g(x),
\end{aligned}
\tag{2}
\]

he represents the viscosity solution as

\[u(t, x) = \sup_{\sigma \in A} E \left( g \left( x + \int_t^T \sigma_s dW_s \right) \right),\]
where $\mathcal{A}$ is a family of stochastic process associated to $G$ and $W$ is a Brownian motion. The key to prove it is a *dynamic programming principle* that we will illustrate in the paper, see section 2.2. We point out that here the novelty with respect to the Feynman–Kac formula is essentially given by the sublinearity of the operator $G$.

The purpose of this article is to apply a generalized version of the dynamic programming principle of [8] in order to give representation formulae of solutions to PDE problems of the type

$$
\begin{align*}
\partial_t u(t, x) &+ F(t, x, \nabla_x u, D^2_x u) + f(t, x, u, \nabla_x u) = 0, \quad t \in [0, T], x \in \mathbb{R}^N, \\
u(T, x) &= g(x),
\end{align*}
$$

where $F$ is a sublinear operator, with respect the third and the fourth argument. This problem is clearly a blend between (1) and (2), where the additional difficulty with respect to (1) is given by the sublinearity of the operator, while the generalization with respect to (2) is the dependence of $F$ on $(t, x)$ of and the presence of the term $f$.

This is hopefully just a first step to further extend the Feynman–Kac formula to problems with sublinear operators using a the BSDE theory in order to deal with general quasilinear problems. We also point out that there is close connection between this method and second order BSDEs, 2BSDEs for short. 2BSDEs were introduced by Cheridito, Soner, Touzi and Victoir in [4] (2007). Then, in 2011, Soner, Touzi and Zhang [29] provided a complete theory of existence and uniqueness for 2BSDEs under Lipschitz conditions. In those papers is also analyzed the connection between 2BSDEs and fully nonlinear PDEs. Among the subsequent developments of this theory we cite [26, 29, 9, 10, 17, 12] and in particular [25], which performs its analysis replacing the Lipschitz condition on $y$ with monotonicity as we do here.

This paper is organized as follows: in section 2 we make a preliminary study of the problem, analyzing the structure of the sublinear operator $F$ and developing a dynamic programming principle which is the core of our theory. Then, in section 3, we perform the essential part of our analysis, and obtain in this way our main results. In section 4 we summarily analyze the connection between 2BSDEs and our representation formula. The appendix at the end briefly gives some probability results we need, with a focus on the BSDE theory. We proceed setting the notation used in the paper.

**Notation**

We will work on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, \infty)}, \mathbb{P})$,

- $\mathcal{F}$ is a complete $\sigma$–algebra on $\Omega$;
- the stochastic process $\{W_t\}_{t \in [0, \infty)}$ will denote the $N$ dimensional Brownian motion under $\mathbb{P}$;
• \( \{ \mathcal{F}_t \}_{t \in [0, \infty)} \) is the filtration defined by \( \{ W_t \}_{t \in [0, \infty)} \) which respects the usual condition of completeness and right continuity;

• \( \{ W^t_s \}_{s \in [t, \infty)} := \{ W_s - W_t \}_{s \in [t, \infty)} \) is a Brownian motion independent from \( \{ W_s \}_{s \in [0, t]} \) by the strong Markov property;

• \( \{ \mathcal{F}^t_s \}_{s \in [t, \infty)} \) is the filtration generated by \( \{ W^t_s \}_{s \in [t, \infty)} \) which we assume respect the usual condition and is independent from \( \mathcal{F}_t \);

• we will say that a stochastic process \( \{ H_t \}_{t \in [0, \infty)} \) is adapted if \( H_t \) is \( \mathcal{F}_t \)-measurable for any \( t \in [0, \infty) \);

• we will say that a stochastic process \( \{ H_t \}_{t \in [0, \infty)} \) is progressively measurable, or simply progressive, if, for any \( T \in [0, \infty) \), the application that to any \( (t, \omega) \in [0, T] \times \Omega \) associate \( H_t(\omega) \) is measurable for the \( \sigma \)-algebra \( \mathcal{B}([0, T]) \times \mathcal{F}_T \);

• a function on \( \mathbb{R} \) is called cadlag if is right continuous and has left limit everywhere;

• a cadlag (in time) process is progressive if and only if is adapted;

• \( B_\delta(x) \) will denote an open ball centered in \( x \) with radius \( \delta \);

• for any Lipschitz continuous function \( f \) we will denote its Lipschitz constant as \( \text{Lip}(f) \);

• if \( A \in \mathbb{R}^{N \times M} \) then \( A^\dagger \) will denote its transpose and \( \sigma_A \) its spectrum;

• (Frobenius product) if \( A, B \in \mathbb{R}^{N \times M} \) then \( \langle A, B \rangle := \text{tr} (AB^\dagger) \) and \( |A| = \sqrt{\langle A, A \rangle} = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{M} A^2_{i,j}} \);

• \( \mathcal{S}_+^N \) is the space of all \( N \times N \) real valued symmetric matrices and \( \mathcal{S}_+^N \) is the subset of \( \mathcal{S}_+^N \) made up by the definite positive matrices.

2 Preliminaries

2.1 Sublinear Operators

We consider the space \( \mathbb{R}^N \times \mathcal{S}_+^N \) with the inner product

\[
((p, S), (p', S')) := \frac{1}{2} \langle S, S' \rangle + p^\dagger p'
\]

and the norm \( \|(p, S)\| := \sqrt{((p, S), (p, S))} \).
**Assumptions 2.1.** In this subsection we will concentrate on the study of continuous operators of the form

\[ F : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R} \]

such that the following properties hold true for any \((t, x) \in [0, T] \times \mathbb{R}^N, (p, S) \) and \((p', S') \) in \(\mathbb{R}^N \times \mathbb{S}^N\):

(i) *(Subadditivity)* \( F(t, x, p + p', S + S') \leq F(t, x, p, S) + F(t, x, p', S') \);

(ii) *(Positive Homogeneity)* If \( \delta \geq 0 \) then \( F(t, x, \delta p, \delta S) = \delta F(t, x, p, S) \);

(iii) *(Uniform Ellipticity)* Exists a constant \( \lambda > 0 \) such that, if \( S' \geq 0 \),

\[
F(t, x, p, S + S') - F(t, x, p, S) \geq \lambda |S'|;
\]

(iv) *(Lipschitz Continuity)* Exists a positive \( \ell \) such that, for any \( y \in \mathbb{R}^N \),

\[
|F(t, x, p, S) - F(t, y, p, S)| \leq \ell |x - y| \| (p, S) \|.
\]

We will usually refer to \( T \) as the terminal time of \( F \), since it will play the role of terminal time in the parabolic problems which we will deal with later.

The operators satisfying conditions (i) and (ii) are commonly known as *sublinear operators*. Notice that items (i) and (ii) imply convexity in the third and fourth arguments and, vice versa, convexity and (ii) imply (i).

The main result of this section is the following characterization theorem:

**Theorem 2.2.** Let \( F \) be as in assumptions 2.1 and \( K_F \) be the set of the elements \((b, a) \in C^0 \left([0, T] \times \mathbb{R}^N; \mathbb{R}^N\right) \times C^0 \left([0, T] \times \mathbb{R}^N; \mathbb{S}^N\right)\) such that, for any \((t, x, p, S) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N\),

\[
\frac{1}{2} \langle a(t, x), S \rangle + p^\dagger b(t, x) \leq F(t, x, p, S),
\]

\[\text{Lip}(b(t)) \leq 2\ell, \text{ Lip}(a(t)) \leq 2\sqrt{2\ell} \text{ and the eigenvalues of } a(t, x) \text{ are bigger than } 2\lambda. \] Then \( K_F \) is a non empty and convex set, and

\[
F(t, x, p, S) = \max_{(b, a) \in K_F} \frac{1}{2} \langle a(t, x), S \rangle + p^\dagger b(t, x)
\]

for any \((t, x, p, S) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N\). Furthermore for each \((b, a) \in K_F\) the linear operator

\[
(t, x, p, S) \mapsto \frac{1}{2} \langle a(t, x), S \rangle + p^\dagger b(t, x)
\]

has the same ellipticity conditions of \( F \).
To prove this we preliminarily need the followings two lemmas. The first one is an adaptation of [28, Lemma 1.8.14], which permit us to express the Hausdorff distance using support function, while the second one is just an adaptation of the Hahn–Banach theorem.

**Lemma 2.3.** Given, $A$ and $B$, two compact and convex subset of $\mathbb{R}^N \times \mathbb{S}^N$ we define the application $$h_A, h_B : \mathbb{R}^N \times \mathbb{S}^N \rightarrow \mathbb{R}$$ as the support functions of $A$ and $B$ respectively, that is to say $$h_A((p,S)) = \sup_{(p',S') \in A} ((p, S), (p', S'))$$ and $$h_B((p,S)) = \sup_{(p',S') \in B} ((p, S), (p', S'))$$ for any $(p, S) \in \mathbb{R}^N \times \mathbb{S}^N$. Then, for the Hausdorff distance $$d_H(A, B) := \max \left\{ \max_{(p,S) \in A} \min_{(p',S') \in B} \left\| (p, S) - (p', S') \right\|, \min_{(p,S) \in A} \max_{(p',S') \in B} \left\| (p, S) - (p', S') \right\| \right\},$$ we have that $$d_H(A, B) = \max_{\|p,S\|=1} |h_A((p,S)) - h_B((p,S))|.$$

**Lemma 2.4.** If $F$ is like in assumptions 2.1 then the set $$\mathcal{L}_{F(t,x)} := \{ L \in (\mathbb{R}^N \times \mathbb{S}^N)^* : L \leq F(t,x) \}$$ is non empty, compact and convex for any $(t, x) \in [0, T] \times \mathbb{R}^N$. Furthermore $F(t,x) = \max_{L \in \mathcal{L}_{F(t,x)}} L$.

**Proof of theorem 2.2.** By lemma 2.4 and the Riesz representation theorem we have that, for any $(t, x) \in [0, T] \times \mathbb{R}^N$, there exists the non empty convex and compact set $$K^x_t := \left\{ (b, a) \in \mathbb{R}^N \times \mathbb{S}^N : \frac{1}{2} (a, S) + p^b \leq F(t, x, p, S) \right\}.$$ for any $(p, S) \in \mathbb{R}^N \times \mathbb{S}^N$.

Given a $(t, x) \in [0, T] \times \mathbb{R}^N$ and $(\overline{b}, \overline{a}) \in K^x_t$ we define the function $$(b, a) : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}^N \times \mathbb{S}^N$$ such that $$(b, a)(s, y) := \begin{cases} (\overline{b}, \overline{a}), & \text{if } (s, y) = (t, x), \\ \arg \min_{(b,a) \in K^x_t} \| (\overline{b} - b, \overline{a} - a) \|, & \text{if } (s, y) \neq (t, x). \end{cases}$$
This function is well defined because is well known that the projection of a point onto a convex set, i.e. \( \text{arg min} \| (\overline{b} - b, \overline{a} - a) \| \), exists and is unique.

We will show that \((b, a) \in K_F\), since this yields that \(K_F\) is a non empty convex set (the convexity proof is trivial, hence we skip it) such that, thanks to the arbitrariness of the construction,

\[
F(t, x, p, S) = \max_{(b, a) \in K_F} \frac{1}{2} (a(t, x), S) + p^\dagger b(t, x)
\]

for any \((t, x, p, S) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times S^N\).

As a consequence of the definition and lemma 2.3 we have, for any \((s, y)\) in \([0, T] \times \mathbb{R}^N\),

\[
\| (b, a)(t, x) - (b, a)(s, y) \| = \min_{(b, a) \in K_F} \| (b(t, x) - b, a(t, x) - a) \|
\]

\[
\leq \max_{(b_1, a_1) \in K_F} \min_{(b_2, a_2) \in K_F} \| (b_1 - b_2, a_1 - a_2) \|
\]

\[
= \max_{(p, S) \in \mathbb{R}^N \times S^N : \| (p, S) \| = 1} | F(t, x, p, S) - F(s, y, p, S) |.
\]

Since \(|a| \leq \sqrt{2} \|(b, a)\|\) for any \((b, a) \in \mathbb{R}^N \times S^N\) and

\[
|a(r, z) - a(s, y)| \leq |a(t, x) - a(s, y)| + |a(r, z) - a(t, x)|
\]

the previous inequality yields that \(\text{Lip}(a(s)) \leq 2\sqrt{2} \ell\) for any \(s \in [0, T]\), and similarly that \(\text{Lip}(b(s)) \leq 2\ell\) for any \(s \in [0, T]\).

We now prove the ellipticity part of the statement and, as a consequence, that \(\sigma_{a(t, x)} \subset [2\lambda, \infty)^N\) for any \((t, x) \in [0, T] \times \mathbb{R}^N\), thus that \((b, a) \in K_F\).

Let \(\lambda\) be the ellipticity constants of \(F\) and

\[
L : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times S^N \to \mathbb{R}
\]

\[
(t, x, p, S) \mapsto \frac{1}{2} (a(t, x), S) + p^\dagger b(t, x),
\]

then, by its linearity, we only have to prove that for any \(S \in S^N_+\) and \((t, x)\) in \([0, T] \times \mathbb{R}^N\)

\[
L(t, x, 0, S) \geq \lambda |S|.
\]

(4)

Obviously we have, for any \(S \in S^N_+\) and \((t, x) \in [0, T] \times \mathbb{R}^N\),

\[
\lambda |S| \leq F(t, x, 0, 0) - F(t, x, 0, -S) \leq -L(t, x, 0, -S) = L(t, x, 0, S),
\]

hence (4). Finally, let \(q\) an element of \(\mathbb{R}^N\) and define \(Q := qq^\dagger\), which is an element of \(S^N_+\) such that

\[
|Q|^2 = \text{tr}(qq^\dagger qq^\dagger) = \text{tr}(q|q|^2q^\dagger) = |q|^4.
\]
Therefore (4) yields, for any \((t, x) \in [0, T] \times \mathbb{R}^N\),

\[
\lambda |q|^2 = \lambda |Q| \leq \frac{1}{2} \langle a(t, x), qq^\dagger \rangle = \frac{1}{2} q^\dagger a(t, x) q
\]

and the Rayleigh quotient formula proves that \(\sigma_{a(t,x)} \subset [2\lambda, \infty)^N\), concluding the proof.

\[\square\]

Remark 2.5. We point out that can be easily proved that for the previous theorem holds a converse. We have characterized \(F\) as the support function of a set of linear operators. Usually, to obtain representation formulas for viscosity solutions to a second order PDE with linear operator like (1), is useful to study a function \(\sigma\) such that \(\sigma^\dagger\) is the diffusion part of that operator, hence we will do something similar: if we define the application from \(\mathbb{S}^N_+\) to itself which associate via singular value decomposition the matrix \(a\) with its square root \(\sigma\) then it is well defined, as can be seen in [1, Section 6.5]. Moreover we know from [27, Lemma 2.1] that, on the space of matrices with eigenvalues equal or bigger than \(2\lambda\), this application is Lipschitz continuous with Lipschitz constant \(c_\lambda := \frac{1}{2 \sqrt{2c_\lambda}}\), therefore the application \((b, \sigma) \mapsto (b, \sigma^\dagger)\) that maps the set \(K_F\), which contains the \((b, \sigma)\) in \([0, T] \times \Omega \times \mathbb{R}^N \times \mathbb{S}^N_+\) such that \(\frac{1}{2} \langle \sigma^2(t, \omega, x), S \rangle + \int_0^T b(t, \omega, x) \leq F(t, x, p, S)\), into \(K_F\) is surjective and consequently

\[
F(t, x, p, S) = \max_{(b, \sigma) \in K_F} \frac{1}{2} \langle \sigma^2(t, x), S \rangle + \frac{1}{2} b^\dagger b(t, x).
\]

Our method to obtain representation formulas relies on a dynamic programming principle, which will be presented later and is based on a construction on a broader set than \(K_F\). This set, which we call \(\mathcal{A}_F\), is made up of the functions

\[
(b, \sigma) : [0, T] \times \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N \times \mathbb{R}^{N \times N}
\]

which are cadlag, i.e. right continuous and left bounded, on \([0, T]\) and such that, for any \((t, x, p, S) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N_+\) and \(\omega \in \Omega\),

\[
\frac{1}{2} \langle \sigma^2(t, \omega, x), S \rangle + \frac{1}{2} b^\dagger b(t, \omega, x) \leq F(t, x, p, S),
\]

\(\text{Lip}(b(t, \omega)) \leq 2\ell, \text{Lip}(\sigma(t, \omega)) \leq 2\sqrt{2c_\lambda}\ell\), the eigenvalues of \((\sigma \sigma)(t, \omega, x)\) belong to \([2\lambda, \infty)^N\) and \(\{(b, \sigma)(t, \omega)\}_{t \in [0, T]}\) is a progressive process. \(\mathcal{A}_F\) is obviously non empty, since it contains \(K_F\). For any stopping time \(\tau\) with value in \([0, T]\), an useful subset of \(\mathcal{A}_F\), which we will use later, is \(\mathcal{A}_F^\tau\), which consists of the \((b, \sigma)\) belonging to \(\mathcal{A}_F\) such that \(\{(b, \sigma)(\tau + t, \omega)\}_{t \in [0, \infty)}\) is progressive with respect to the filtration \(\{\mathcal{F}_t^\tau\}_{t \in [0, \infty)}\). Trivially \(\mathcal{A}_F^0 = \mathcal{A}_F\).
Remark 2.6. It is easy to see that, for any $p > 0$, the process $(b, \sigma)(s, 0)$ belongs to $L^p_N(t) \times L^p_N(t)$ for each $(b, \sigma) \in \mathcal{A}_F$ and $t \in [0, \infty)$, since the image of $[0, t] \times \Omega \times \{0\}$ under $(b, \sigma)$ is contained on a compact set for any $(b, \sigma) \in \mathcal{A}_F$ and $t \in [0, \infty)$.

2.2 Dynamic Programming Principle

The scope of this section is to provide the necessary instruments to prove the dynamic programming principle 3.10, which will be used to derive representation formulas for viscosity solutions to the parabolic problem which we will study later. The dynamic programming principle is, in this context, an instrument that permit us to break a stochastic trajectory in two or more parts. In particular for the problem (1) it means that

$$
\mathbb{E} \left( u(s, X^t_s) + \int_s^T f(r, X^s_r, Y^t_r, Z^s_r) dr \right) = \mathbb{E} \left( Y^t_s \right) = u(t, x),
$$

which is just a simple consequence of the uniqueness of the solutions to the FBSDE, while for the $G$–heat equation (2) this means that

$$
\sup_{\sigma \in \mathcal{A}_G} \mathbb{E} \left( u \left( s, x + \int_t^s \sigma(r) dW_r \right) \right) = \sup_{\sigma \in \mathcal{A}_G} \sup_{\sigma' \in \mathcal{A}_G} \mathbb{E} \left( g \left( x + \int_t^s \sigma(r) dW_r + \int_s^T \sigma'(r) dW_r \right) \right) = \sup_{\sigma \in \mathcal{A}_G} \mathbb{E} \left( g \left( x + \int_t^T \sigma(r) dW_r \right) \right) = u(t, x).
$$

The proof of (5) is contained in [8, Subsection 3.1]. This also intuitively explain why we ask to the elements of $\mathcal{A}_F$ to be cadlag in time. We point out that in [8] the authors ask to the elements of $\mathcal{A}_F$ to only be measurable in time, but for the analysis of the more general problem (3) in section 3 we will use right continuity.

The dynamic programming principle exposed in theorem 3.10 is a generalization of the one presented by Denis, Hu and Peng in [8] and can be obtained using a similar method, slightly adapting the proofs of [8, Lemmas 41–44]. Hence we will just present the results that are relevant for our analysis skipping the proofs.
First of all we notice that for any \((b, \sigma) \in A_F\) we can define an \(X_{(b, \sigma)}\) solution to the SDE \((b, \sigma)\) as in (22). However, to ease notations, we will usually omit the dependence of \(X\) from \((b, \sigma)\).

Given a sublinear operator \(F\) with a positive terminal time \(T\) as in assumptions 2.1, a stopping time \(\tau\) with value in \([0, T]\) and a measurable application \(\varphi\) from \([0, T] \times \mathbb{R}^N \times A_F\) into \(\mathbb{R}\), continuous in probability with respect to \(A_F\) and such that

\[
\varphi_{\tau}(\zeta) := \text{ess sup}_{(b, \sigma) \in A_F} E(\varphi_{\tau}(\zeta, b, \sigma) | \mathcal{F}_\tau).
\]

We assume that \(\varphi_t(x, b, \sigma)\) is \(\mathcal{F}^{\mathcal{K}_t}_\infty\)-measurable for any \(t \in [0, T]\), \(x \in \mathbb{R}^N\) and \((b, \sigma) \in A_F\), and, for any stopping time \(\tau\) with value in \([0, T]\) and \(\zeta\) in \(L^2(\Omega, \mathcal{F}_\tau; \mathbb{R}^N)\),

\[
\sup_{(b, \sigma) \in A_F} E(|\varphi_{\tau}(\zeta, b, \sigma)|) < \infty. \tag{6}
\]

In this section the function \(\Phi\) represents, roughly speaking, the viscosity solution, \(\zeta\) is the first part of a stochastic trajectory broken off at \(\tau\) (this is why we restrict \(\varphi_\tau\) on \(A_F|_{[\tau, T]}\)) and \(\varphi\) the function which we will use to build the viscosity solution.

**Lemma 2.7.** For each \((b_1, \sigma_1), (b_2, \sigma_2)\) in \(A_F\) and stopping time \(\tau\) with value in \([0, T]\) there exists an \((b, \sigma) \in A_F\) such that

\[
E(\varphi_{\tau}(\zeta, b, \sigma) | \mathcal{F}_\tau) = E(\varphi_{\tau}(\zeta, b_1, \sigma_1) | \mathcal{F}_\tau) \lor E(\varphi_{\tau}(\zeta, b_2, \sigma_2) | \mathcal{F}_\tau).
\]

Therefore there exists a sequence \(\{(b_i, \sigma_i)\}_{i \in \mathbb{N}}\) in \(A_F\) such that a.e.

\[
E(\varphi_{\tau}(\zeta, b_i, \sigma_i) | \mathcal{F}_\tau) \uparrow \Phi_{\tau}(\zeta).
\]

We also have

\[
E(|\Phi_{\tau}(\zeta)|) \leq \sup_{(b, \sigma) \in A_F} E(|\varphi_{\tau}(\zeta, b, \sigma)|) < \infty,
\]

and, for any stopping time \(\tau' \leq \tau\),

\[
E \left( \text{ess sup}_{(b, \sigma) \in A_F} E(\varphi_{\tau}(\zeta, b, \sigma) | \mathcal{F}_{\tau'}) \bigg| \mathcal{F}_{\tau'} \right) = \text{ess sup}_{(b, \sigma) \in A_F} E(\varphi_{\tau}(\zeta, b, \sigma) | \mathcal{F}_{\tau'}).
\]

**Remark 2.8.** To prove lemma 2.7 the randomness of the elements of \(A_F\) is crucial, this is the reason why we consider a set of stochastic process instead of a deterministic one.
To continue we need a density result on $A$ endowed with the topology of the $L^2$–convergence on compact set, which is to say that a sequence in $A$ converges to an element of $A$ if and only if it converges in $L^2([0, T] \times \Omega \times K)$ for any compact set $K \subset \mathbb{R}^N$.

**Lemma 2.9.** The set

$$
\mathcal{J}^t := \left\{ \alpha \in A : \alpha|_{[t, T]} = \sum_{i=0}^{n} \chi_{A_i} \alpha_i|_{[t, T]}, \text{ where } \{\alpha_i\}_{i=0}^{n} \subset A_i^t \right\}
$$

and $\{A_i\}_{i=0}^{n}$ is a $\mathcal{F}_t$–partition of $\Omega$ is dense in $A$ for any $t \in [0, T]$.

**Proof.** To prove this we will show that, fixed a $k \in \mathbb{N}$, we can approximate, in $L^2([0, T] \times \Omega \times B_k(0))$, any element of $A$ with an element of $\mathcal{J}^t$.

Preliminarily notice that by our assumption each element of $A$ can be approximated in $L^2([0, T] \times \Omega \times B_k(0))$ by a sequence of simple functions. We will denote with $\mathcal{B}([0, T] \times B_k(0))$ the Borel $\sigma$–algebra of $[0, T] \times B_k(0)$.

Furthermore, since the collection $\mathcal{I}$ of the rectangles $A \times B$ where $A \in \mathcal{F}_T$ and $B \in \mathcal{B}([0, T] \times B_k(0))$ is a $\pi$–system which contains the complementary of its sets and generate $\sigma(\mathcal{F}_T \times \mathcal{B}([0, T] \times B_k(0)))$, by [30, Dynkin’s lemma A1.3] each set in $\sigma(\mathcal{F}_T \times \mathcal{B}([0, T] \times B_k(0)))$, which is the smallest $d$–system containing $\mathcal{I}$, can be approximate by a finite union of sets in $\mathcal{I}$. Similarly, each set in $\mathcal{F}_T$ can be approximated by finite intersection and union of sets in $\mathcal{F}_T$ and $\mathcal{F}_T'$, since $\mathcal{F}_T = \sigma(\mathcal{F}_t, \mathcal{F}_T')$.

Therefore, fixed $(b, \sigma) \in A$, for any $\varepsilon > 0$ there exists a simple function $s_\varepsilon(t, \omega, x) = \sum_{i=1}^{n} \sum_{j=1}^{m} s_{ij}(t, x) \chi_{A_i}(\omega) \chi_{A'_j}(\omega)$ where $\{A_i\}_{i=1}^{n}$ and $\{A'_j\}_{j=1}^{m}$ are respectively a $\mathcal{F}_t$–partition and a $\mathcal{F}_T'$–partition of $\Omega$ and

$$
\mathbb{E} \left( \int_0^T \int_{B_k(0)} |(b, \sigma)(t, x) - s_\varepsilon(t, x)|^2 \, dx \, dt \right) < \varepsilon. \tag{7}
$$

Then, for each $A_i$ and $A'_j$ with $\mathbb{P}(A_i \cap A'_j) > 0$, there exists a $\omega_{ij} \in A_i \cap A'_j$ such that

$$
\int_0^T \int_{B_k(0)} \left| (b, \sigma)(t, \omega_{ij}, x) - s_{ij}(t, x) \right|^2 \, dx \, dt < \frac{\varepsilon}{\mathbb{P}(A_i \cap A'_j)}.
$$
otherwise we would have that
\[
E \left( \int_0^T \int_{B_k(0)} |(b, \sigma)(t, x) - s_\varepsilon(t, x)|^2 \, dx \, dt \right) \\
=E \left( \sum_{i=1}^n \sum_{j=1}^m \int_0^T \int_{B_k(0)} |(b, \sigma)(t, x) - s_i^j(t, x)|^2 \chi_{A_i} \chi_{A_i'} \, dx \, dt \right) \\
\geq E \left( \int_0^T \int_{B_k(0)} |(b, \sigma)(t, x) - s_i^j(t, x)|^2 \chi_{A_i} \chi_{A_i'} \, dx \, dt \right) \\
\geq \varepsilon,
\]
in contradiction with (7). Finally, let \( \omega_i^j \) be any elements of \( A_i \cap A_j' \) if \( \mathbb{P}(A_i \cap A_j') = 0 \) and \( (b_k^i, \sigma_k^i) := \sum_{j=1}^m (b, \sigma) \left( \omega_i^j \right) \chi_{A_j'} \). Then \( (b_k^i, \sigma_k^i) \in A _t \mathcal{F} \) and \( (b_k^i, \sigma_k^i) := \sum_{i=1}^n (b_k^i, \sigma_k^i) \chi_{A_i} \) is an element of \( \mathcal{J}_t \) satisfying
\[
E \left( \int_0^T \int_{B_k(0)} |(b, \sigma)(t, x) - (b_k^i, \sigma_k^i)(t, x)|^2 \, dx \, dt \right) < 4\varepsilon.
\]
This proves that \( \mathcal{J}_t \) is dense in \( \mathcal{A}_F \). \( \square \)

**Lemma 2.10.** For each \( t \in [0, T] \) and \( x \in \mathbb{R}^N \), \( \Phi_t(x) \) is deterministic. Furthermore
\[
\Phi_t(x) = \operatorname{ess sup}_{(b, \sigma) \in \mathcal{A}_F} E(\varphi_t(x, b, \sigma)|\mathcal{F}_t) = \operatorname{ess sup}_{(b, \sigma) \in \mathcal{A}_t \mathcal{F}} E(\varphi_t(x, b, \sigma)|\mathcal{F}_t). \tag{8}
\]

**Lemma 2.11.** We define the function
\[
u : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \\
(t, x) \mapsto \Phi_t(x)
\]
and assume that it is continuous. Then, for each stopping time \( \tau \) with value in \( [0, T] \) and \( \zeta \in L^2_N (\Omega, \mathcal{F}_\tau; \mathbb{R}^N) \), we have that \( \nu(\zeta) = \Phi_\tau(\zeta) \) a.e..

**Remark 2.12.** This lemma says, as a consequence of (8), that
\[
\operatorname{ess sup}_{(b, \sigma) \in \mathcal{A}_F} E(\varphi_t(\zeta, b, \sigma)|\mathcal{F}_t) = \operatorname{ess sup}_{(b, \sigma) \in \mathcal{A}_t \mathcal{F}} E(\varphi_t(\zeta, b, \sigma)|\mathcal{F}_t)
\]
for any \( t \in [0, T] \).
3 Parabolic PDEs with Sublinear Operators

We analyze now the following problem:

Problem 3.1. Let $T$ be a terminal time, $F$ a uniformly elliptic operator satisfying assumptions 2.1 and

$$f : [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \text{ and } g : \mathbb{R}^N \rightarrow \mathbb{R}$$

two continuous functions for which there exist two constants $\mu \in \mathbb{R}$ and $\ell \geq 0$ such that, for any $t \in [0, T]$, $x, x' \in \mathbb{R}^N$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^N$,

(i) $|g(x) - g(x')| \leq \ell |x - x'|$;
(ii) $|g(x)| \leq \ell (1 + |x|)$;
(iii) $|f(t, x, y, z) - f(t, x', y, z')| \leq \ell (|x - x'| + |z - z'|)$;
(iv) $|f(t, x, y, z)| \leq \ell (|x| + |y| + |z|)$;
(v) $(y - y')(f(t, x, y, z) - f(t, x, y', z)) \leq \mu |y - y'|^2$.

Find the solution $u$ to the parabolic PDE

$$\begin{cases}
\partial_t u(t, x) + F(t, x, \nabla_x u, D^2_x u) + f(t, x, u, \nabla_x u) = 0, & t \in (0, T), x \in \mathbb{R}^N, \\
u(T, x) = g(x),
\end{cases}$$

Remark 3.2. To ease notations, we can assume without loss of generality that the $\ell$ in problem 3.1 is the same as in assumptions 2.1. Since $F$ is continuous, we can also assume that, for any $(b, \sigma) \in \mathcal{A}_F$, $|b(t, 0)| \leq \ell$ and $|\sigma(t, 0)| \leq \ell$ for any $t \in [0, T]$. We also define the sets $\mathcal{L}_F$ whose elements are the operators $L_{(b, \sigma)}$ with $(b, \sigma) \in \mathcal{A}_F$ such that

$$L_{(b, \sigma)}(p, S) := \frac{1}{2} \langle \sigma^2(t, x), S \rangle + p^\dagger b(t, x).$$

As previously done with $\mathcal{A}_F$, we also define the set $\mathcal{L}_F := \mathcal{L}_F^0$.

Let us define what we mean with viscosity solution to problem 3.1. For a detailed overview of the viscosity solution theory we refer to [5].

Definition 3.3. Given an upper semicontinuous function $u$ we say that a function $\varphi$ is a super tangent to $u$ at $(t, x)$ if $(t, x)$ is a local maximizer of $u - \varphi$.

Similarly we say that a function $\psi$ is a sub tangent to a lower semicontinuous function $v$ at $(t, x)$ if $(t, x)$ is a local minimizer of $v - \psi$.  

Definition 3.4. An upper semicontinuous function $u$ is called a viscosity subsolution to problem 3.1 if, for any suitable $(t, x)$ and $C^{1,2}$ supertangent $\varphi$ to $u$ at $(t, x)$,

$$\partial_t \varphi(t, x) + F(t, x, \nabla_x \varphi, D^2 \varphi) + f(t, x, u, \nabla_x \varphi) \geq 0.$$ 

Similarly a lower semicontinuous function $v$ is called a viscosity supersolution to problem 3.1 if, for any suitable $(t, x)$ and $C^{1,2}$ subtangent $\psi$ to $v$ at $(t, x)$,

$$\partial_t \psi(t, x) + F(t, x, \nabla_x \psi, D^2 \psi) + f(t, x, u, \nabla_x \psi) \leq 0.$$ 

Finally a continuous function $u$ is called a viscosity solution to problem 3.1 if it is both a super and a subsolution to problem 3.1.

We derive from theorem B.2 that a comparison result holds true.

Theorem 3.5. Let $u$ and $v$ be respectively a subsolution and a supersolution to problem 3.1 satisfying polynomial growth condition. If $u|_{t=T} \leq v|_{t=T}$, then $u \leq v$ on $(0, T] \times \mathbb{R}^N$.

When $F$ is a linear operator it is known that the representation formula of its viscosity solution is built from a FBSDE, see (1). To adapt this method to our case we will use the dynamic programming principle 3.10.

Definition 3.6. Consider the FBSDE

$$\begin{align*}
X_t^{t, \zeta} &= \zeta + \int_t^s \sigma \left( r, X_r^{t, \zeta} \right) dW_r + \int_t^s b \left( r, X_r^{t, \zeta} \right) dr, \\
Y_t^{t, \zeta} &= g \left( X_T^{t, \zeta} \right) + \int_s^T f_{\sigma} \left( r, X_r^{t, \zeta}, Y_r^{t, \zeta}, Z_r^{t, \zeta} \right) dr \quad s \in [t, T],
\end{align*}$$

(9)

where $\zeta \in L^2 \left( \Omega, \mathcal{F}_t; \mathbb{R}^N \right)$, $(b, \sigma) \in \mathcal{A}_F$, the function $f_{\sigma}$ is defined as

$$f_{\sigma}(t, x, y, z) := f \left( t, x, y, z(\sigma(t, x))^{-1} \right),$$

for any $(t, x, y, z)$ in $[0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ and the functions $f$ and $g$ are as in the assumptions of problem 3.1. Thanks to the uniformly ellipticity condition we know that $f_{\sigma}$ is well defined and that the Lipschitz constant for its the fourth argument is $\ell \sqrt{N}$, but for simplicity we will assume that it is $\ell$, possibly increasing it.

Note that under this conditions the assumptions A.2 and A.6 hold for $X$ and $(Y, Z)$ respectively. We will call $(X, Y, Z)$ a solution to the FBSDE if $X$ is a solution to the SDE part of this system and $(Y^{t, \zeta}, Z^{t, \zeta})$ is a solution to the BSDE part for any $(t, \zeta) \in [0, T] \times L^2 \left( \Omega, \mathcal{F}_t; \mathbb{R}^N \right)$. Notice that, under our assumptions, there exists a unique solution to (9), thanks to theorems A.3 and A.7. Due to remark A.4 and proposition A.9, this is true even if $t$ is an a.e. finite stopping time.
Remark 3.7. Notice that the uniqueness property of the FBSDE imply that, for any $0 \leq t \leq r \leq s \leq T$,

$$\left( X_r^{r,X,t}, Y_r^{r,X,t}, Z_r^{r,X,t} \right) = \left( X_s^{s,X,t}, Y_s^{s,X,t}, Z_s^{s,X,t} \right).$$

This holds true even if $t$, $r$ and $s$ are stopping time.

Remark 3.8. We point out that since the elements of $\mathcal{L}_F^\tau$ and the solutions to SDEs $(b, \sigma)$ can be uniquely determined, except for the initial data of the SDEs, by an element of $\mathcal{A}_F^\tau$, we can uniquely link to each operator $L \in \mathcal{L}_F^\tau$ an $X_{(b,\sigma)}$. Moreover, for each problem, we can uniquely associate in the same way a solution of the FBSDE (9).

For the remainder of this section, we will simply write $Y$ to denote the second term of the triplet $(X,Y,Z)$ solution to the FBSDE defined in definition 3.6, for $(b,\sigma)$ that varies in $\mathcal{A}_F$. For simplicity we will omit the dependence of $Y$ and $Z$ from $X$ and $\sigma$ or, equivalently, from $(b,\sigma)$.

We will prove that $u(t,x) := \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}(Y_{t,x}^\tau)$ is a viscosity solution to the problem 3.1 breaking the proof in several steps.

**Proposition 3.9.** The function $u(t,x) := \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}(Y_{t,x}^\tau)$ is $\frac{1}{2}$-Hölder continuous in the first variable and Lipschitz continuous in the second one. Furthermore we have that there exists a constant $c$, which depends only on $\ell$, $\mu$ and $T$, such that

$$\mathbb{E}(|u(\tau,\zeta)|^2) \leq \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}\left(\left|Y_{\tau,\zeta}^\tau\right|^2\right) \leq c \left(1 + \mathbb{E}\left(|\zeta|^2\right)\right), \quad (10)$$

for any $t \in [0,T]$ and $\zeta \in L^2(\Omega,\mathcal{F}_t;\mathbb{R}^N)$.

We point out that this proposition permits us to use the results of section 2.2 on $u$. In particular $Y_{\tau}^\tau$, which is $\mathcal{F}_\tau^\tau$-measurable and therefore a.e. deterministic, has the same role of $\varphi$ in section 2.2. We already know that $Y_{\tau}^\tau$ is continuous in probability, thanks to our assumptions, theorems A.3 and A.7, furthermore we prove here that it satisfies (6) and the continuity of $u$, which is needed for lemma 2.11.

**Proof.** To prove our statement note that by the definition and the Jensen’s inequality

$$|u(t,x) - u(s,y)| = \left| \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}(Y_{t,x}^\tau) - \sup_{(b,\sigma) \in \mathcal{A}_F} \mathbb{E}(Y_{s,y}^\tau) \right|$$

$$\leq \sup_{(b,\sigma) \in \mathcal{A}_F} \left( \mathbb{E}\left(\left|Y_{t,x}^\tau - Y_{s,y}^\tau\right|^2\right) \right)^{\frac{1}{2}}$$

$$\leq \sup_{(b,\sigma) \in \mathcal{A}_F} \left( \mathbb{E}\left(\left|\frac{Y_{t,x}^\tau - Y_{s,y}^\tau}{\left|Y_{t,x}^\tau - Y_{s,y}^\tau\right|}\right|^2\right) \right)^{\frac{1}{2}}.$$
and
\[ |u(t, \zeta)| = \left| \sup_{(b, \sigma) \in \mathcal{A}_F} \mathbb{E} \left( Y_{t, \zeta}^{t, x} \right) \right| \leq \left( \sup_{(b, \sigma) \in \mathcal{A}_F} \mathbb{E} \left( \left| Y_{t, \zeta}^{t, x} \right|^2 \right) \right)^{\frac{1}{2}}, \]
for any \( t, s \in [0, T] \), \( x, y \in \mathbb{R}^N \) and \( \zeta \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^N) \). The statement is then a consequence of theorems A.3 and A.7.

We can now prove the dynamic programming principle for \( u \).

**Theorem 3.10 (Dynamic programming principle).** For any \((b, \sigma) \in \mathcal{A}_F\) we let \((\overline{Y}, \overline{Z})\) be the solution of the BSDE
\[
\overline{Y}_s = u(\tau, X_{t,x}^{t, x}) + \int_{s \wedge \tau}^\tau f_s(r, X_{r,x}^{t, x}, \overline{Y}_r, \overline{Z}_r) \, dr - \int_{s \wedge \tau}^\tau \overline{Z}_r \, dW_r, \tag{11}
\]
where \( s \in [t, T] \) and \( \tau \) is a stopping time with value in \([t, T] \). Then we have
\[
\sup_{(b, \sigma) \in \mathcal{A}_F} \mathbb{E} (\overline{Y}_t) = u(t, x).
\]

**Proof.** Fix \((\overline{b}, \overline{\sigma}) \in \mathcal{A}_F\) in (11) and define \(\overline{X} := X_{(b, \sigma)}\) and the subset of \(\mathcal{A}_F\)
\[
\mathcal{A}_F := \{ (b, \sigma) \in \mathcal{A}_F : (b, \sigma)(s) = (\overline{b}, \overline{\sigma}) \text{ for } s \in [t, \tau] \}.
\]
From lemma 2.11 we know that
\[
\operatorname{ess} \sup_{(b, \sigma) \in \mathcal{A}_F} \mathbb{E} (Y_{\tau, \overline{X}_{\tau}^{t, x}}^{t, x} | \mathcal{F}_\tau) = \operatorname{ess} \sup_{(b, \sigma) \in \mathcal{A}_F} \mathbb{E} (Y_{\tau, \overline{X}_{\tau}^{t, x}}^{t, x} | \mathcal{F}_\tau) = u(\tau, \overline{X}_{\tau}^{t, x})
\]
and lemma 2.7 yields the existence of a sequence \(\{(b_n, \sigma_n)\}_{n \in \mathbb{N}}\) in \(\mathcal{A}_F\) and a corresponding sequence \(\{Y_{n}\}_{n \in \mathbb{N}}\) such that
\[
\lim_{n \to \infty} \mathbb{E} (Y_{n, \overline{X}_{\tau}^{t, x}}^{t, x} | \mathcal{F}_\tau) = \operatorname{ess} \sup_{(b, \sigma) \in \mathcal{A}_F} \mathbb{E} (Y_{\tau, \overline{X}_{\tau}^{t, x}}^{t, x} | \mathcal{F}_\tau) = u(\tau, \overline{X}_{\tau}^{t, x}).
\]
Then, by theorem A.7 and the dominated convergence theorem, there exists a constant \(c\) such that
\[
\lim_{n \to \infty} \mathbb{E} \left( \left| Y_t - Y_{n,t}^{t, x} \right|^2 \right) \leq \lim_{n \to \infty} c \mathbb{E} \left( \left| u(\tau, \overline{X}_{\tau}^{t, x}) - Y_{n, \overline{X}_{\tau}^{t, x}}^{t, x} \right|^2 \right) = 0,
\]
hence, up to subsequences,
\[
\lim_{n \to \infty} \mathbb{E} (Y_{n,t}^{t, x}) = \mathbb{E} (\overline{Y}_t). \tag{12}
\]
Furthermore, thanks to theorem A.8, \(Y_{t}^{t, x} \leq \overline{Y}_t\) for any \((b, \sigma) \in \mathcal{A}_F\), which together with (12) implies that
\[
\sup_{(b, \sigma) \in \mathcal{A}_F} \mathbb{E} (Y_{t}^{t, x}) = \mathbb{E} (\overline{Y}_t). \tag{13}
\]
Therefore we can use the arbitrariness of \((\overline{b}, \overline{\sigma})\) to obtain our conclusion:
\[
\sup_{(b, \sigma) \in \mathcal{A}_F} \mathbb{E} (\overline{Y}_t) = \sup_{(b, \sigma) \in \mathcal{A}_F} \mathbb{E} (Y_{t}^{t, x}) = u(t, x).
\]
Now we proceed to show that $u$ is a viscosity subsolution. In order to do that, we need the following lemma:

**Lemma 3.11.** For any $t \in (0, T)$, let $L$ be an element of $\mathcal{L}_F^t$ and $(X, Y, Z)$ the solution to the FBSDE (9) associated to $L$ as in remark 3.8. If we define, for any $x \in \mathbb{R}^N$ and $s \in [t, T]$, $u_L(s, x) := \mathbb{E}(Y_s^{t, x})$ we have that, for any supertangent $\varphi$ to $u_L$ at $(t, x)$,

$$L(t, x, \nabla_x \varphi, D_x^2 \varphi) \geq -\partial_t \varphi(t, x) - f(t, x, u_L, \nabla_x \varphi).$$

**Proof.** We preliminarily denote by $(b, \sigma)$ the element of $\mathcal{A}_F^t$ associated to $L$ and point out that since $(b, \sigma)$, restricted in $[t, T]$, is progressive with respect to the $\sigma$-algebra $\{\mathcal{F}_s^t\}_{s \in [t, T]}$, so are $L, X^t$ and $Y^t$. They are therefore constants a.e. in $t$. As a consequence $u_L(t, x) = Y_t^{t, x}$ a.e. for any $x \in \mathbb{R}^N$. Given $x \in \mathbb{R}^N$ and a supertangent $\varphi$ to $u_L$ at $(t, x)$ we can assume without loss of generality that $u_L(t, x) = \varphi(t, x)$, so we suppose that, a.e.,

$$\partial_t \varphi(t, x) + L(t, x, \nabla_x \varphi, D_x^2 \varphi) + f_\sigma(t, x, u_L, \nabla_x \varphi) < 0 \quad (13)$$

and we will find a contradiction. Note that, as a consequence of the Blumenthal’s 0–1 law, $(13)$ is a deterministic inequality a.e.. By the definition of supertangent, there exists a $\delta \in (0, T - t)$ such that, for any $s \in [t, t + \delta]$ and $y \in B_\delta(x)$,

$$u_L(s, y) \leq \varphi(s, y), \quad (14)$$

hence we define the stopping time

$$\tau := (t + \delta) \wedge \inf \{ s \in [t, \infty) : |X_s^{t, x} - x| \geq \delta \}$$

and assume, possibly taking a smaller $\tau$, that

$$\partial_t \varphi\left(s \wedge \tau, X_{s \wedge \tau}^{t, x}\right) + L\left(s \wedge \tau, X_{s \wedge \tau}^{t, x}, \nabla_x \varphi, D_x^2 \varphi\right)$$

$$+ f_\sigma\left(s \wedge \tau, X_{s \wedge \tau}^{t, x}, \varphi, \nabla_x \varphi\sigma\right) < 0 \quad (15)$$

We point out that, by $(13)$ and lemma A.1, the previous inequality holds true on a set of positive measure for the $\chi_{(t, t+\delta)} dt \times d\mathbb{P}$ measure, thus $\tau > t$ on a set of positive measure.

Let $(\overline{Y}_s, \overline{Z}_s) := \left(Y_{s \wedge \tau}^{t, x}, Z_{s \wedge \tau}^{t, x}\right)$, which solve the BSDE

$$\overline{Y}_s = Y_{t \wedge \tau}^{t, x} + \int_{s \wedge \tau}^{\tau} f_\sigma\left(r, X_{r \wedge \tau}^{t, x}, \overline{Y}_r, \overline{Z}_r\right) dr - \int_{s \wedge \tau}^{\tau} \overline{Z}_r dW_r, \quad s \in [t, T],$$

and $(\hat{Y}_s, \hat{Z}_s) := \left(\varphi\left(s, X_{s \wedge \tau}^{t, x}\right), \left(\nabla_x \varphi\sigma\right)\left(s, X_{s \wedge \tau}^{t, x}\right)\right)$ which, by Itô’s formula, is solution to

$$\hat{Y}_s = \varphi\left(\tau, X_{\tau}^{t, x}\right) - \int_{s \wedge \tau}^{\tau} \hat{Z}_r dW_r$$

$$- \int_{s \wedge \tau}^{\tau} \left(\partial_t \varphi\left(r, X_{r \wedge \tau}^{t, x}\right) + L\left(r, X_{r \wedge \tau}^{t, x}, \nabla_x \varphi, D_x^2 \varphi\right)\right) dr,$$
By (14) we have that
\[ u_L(\tau, X^{t,x}_\tau) - \varphi(\tau, X^{t,x}_\tau) = Y^\tau_{\tau, X^{t,x}_\tau} - \varphi(\tau, X^{t,x}_\tau) \leq 0 \]
and (15) imply, thanks to theorem A.8, that $Y^{t,x}_t < \varphi(t, x)$ a.e., but this lead to a contradiction since we know that, by our assumptions, $\varphi(t, x) = Y^{t,x}_t$ a.e.. This conclude the proof.

**Proposition 3.12.** The function $u(t, x)$ is a continuous viscosity subsolution to the problem 3.1.

**Proof.** We know from proposition 3.9 that $u$ is continuous, thus we just have to prove the subsolution property to conclude the proof.

Let $L$ be an element of $L^r_F$ and $u_L$ as defined in lemma 3.11, then if $\varphi$ is a supertangent to $u_L$ in $(t, x)$ we have that, by the definition of $L^r_F$,
\[ F(t, x, \nabla_x \varphi, D^2_x \varphi) \geq L(t, x, \nabla_x \varphi, D^2_x \varphi) \geq -\partial_t \varphi(t, x) - f(t, x, u_L, \nabla_x \varphi), \]
therefore $u_L$ is a viscosity subsolution to the problem 3.1 at $(t, x)$. Thanks to the arbitrariness of $t$, $L$ and $x$ we then have that $u_L$ is a viscosity subsolution in $(t, x)$ for any $L \in L^r_F$, $x \in \mathbb{R}^N$ and $t \in (0, T)$. From remark 2.12 we have that
\[ \sup_{L \in L^r_F} u_L(t, x) = \sup_{(b, \sigma) \in A^r_F} \mathbb{E}\left(Y^{t,x}_t\right) = u(t, x), \]
therefore the family of functions $\{u_L\}$ is locally equibounded, thanks to proposition 3.9. Well known properties of viscosity solutions hence yield that
\[ \sup_{L \in L^r_F} u_L(t, x) = u(t, x) \]
is a viscosity subsolution for any $(t, x) \in (0, T) \times \mathbb{R}^N$.

We conclude this section with our main statement.

**Theorem 3.13.** The function $u(t, x) := \sup_{(b, \sigma) \in A_F} \mathbb{E}\left(Y^{t,x}_t\right)$ is the only viscosity solution to the problem 3.1 satisfying polynomial growth condition such that $u(T, x) = g(x)$ for any $x$ in $\mathbb{R}^N$.

**Proof.** The uniqueness is a consequence of theorem 3.5 and (10), hence we only have to show that $u$ is a viscosity solution.

From proposition 3.12 we know that $u$ is a continuous viscosity subsolution and it is easy to see that $u(T, x) = g(x)$ for any $x \in \mathbb{R}^N$, so we only need to prove the supersolution property of $u$. Fixed $(t, x)$ in $(0, T) \times \mathbb{R}^N$, let $\psi$ be a subtangent to $u$ in $(t, x)$ which we assume, without loss of generality, equal to $u$ at $(t, x)$ and $\delta$ a positive constant such that
\[ \psi(s, y) \leq u(s, y) \quad \text{for any } (s, y) \in [t, t + \delta] \times B_\delta(x). \quad (16) \]
We know, thanks to theorem 2.2, that there exists a continuous and deterministic $L \in L^F_F$ for which

$$F(t, x, \nabla_x \psi, D^2_x \psi) = L(t, x, \nabla_x \psi, D^2_x \psi)$$

and assume by contradiction

$$F(t, x, \nabla_x \psi, D^2_x \psi) = L(t, x, \nabla_x \psi, D^2_x \psi) > -\partial_t \psi(t, x) - f_\sigma(t, x, u, \nabla_x \psi \sigma).$$

Then, by continuity,

$$\partial_t \psi(s, y) + L(s, y, \nabla_x \psi, D^2_x \psi) > -f_\sigma(s, y, \psi, \nabla_x \psi \sigma)$$

for any $(s, y) \in [t, t + \delta] \times B_\delta(x)$, possibly taking a smaller $\delta$.

We denote with $(b, \sigma)$ and $(X, Y, Z)$, respectively, the element of $A$, which, to repeat, is continuous and deterministic, and the solution to the FBSDE (9) associated to $L$. We define the stopping time

$$\tau := (t + \delta) \land \inf \{ s \in [t, \infty) : |X^t_x - x| \geq \delta \},$$

let $(Y_s, Z_s)$ be the solution to the BSDE

$$\begin{align*}
Y_s &= u(\tau, X^\tau_x) + \int_{s \land \tau}^\tau f_\sigma(r, X^r_x, Y_r, Z_r) dr - \int_{s \land \tau}^\tau Z_r dW_r, \quad s \in [t, T]
\end{align*}$$

and $(\hat{Y}_s, \hat{Z}_s) := \left( \psi(\tau, X^\tau_x) \right)$, which, by Itô’s formula, is solution to

$$\begin{align*}
\hat{Y}_s &= \psi(\tau, X^\tau_x) - \int_{s \land \tau}^\tau \hat{Z}_r dW_r \\
&\quad - \int_{s \land \tau}^\tau \left( \partial_t \psi(r, X^r_x) + L(r, X^r_x, \nabla_x \psi, D^2_x \psi) \right) dr,
\end{align*}$$

$s \in [t, T].$

We know from the dynamic programming principle 3.10 that

$$\sup_{(b, \sigma) \in A_F} \mathbb{E}(Y_t) = u(t, x) = \psi(t, x),$$

but by (16) we have $u(\tau, X^\tau_x) \geq \psi(\tau, X^\tau_x)$, which together with (17) imply, thanks to theorem A.8, that $Y_t > \psi(t, x)$ a.e., contradicting (18). \(\square\)

4 Connection with 2BSDEs

In this section we will briefly present a different approach to second order BSDEs, 2BSDEs for short, using our dynamic programming principle. This is intended as a short survey and not as a complete analysis of the subject.
We start giving the classical formulation of 2BSDE. Following [29, 25] assume that \( \Omega := \{ \omega \in C ([0, T]; \mathbb{R}^N) : \omega_0 = 0 \} \) and let \( P_0 \) be the Wiener measure. Note that in this space the Brownian motion \( W \) is a projection, i.e. \( W_t(\omega) = \omega_t \). Denote with \( [W] \) the quadratic variation of the projection and with 
\[
\hat{a}_t := \lim_{\varepsilon \downarrow 0} \frac{[W]_t - [W]_{t-\varepsilon}}{\varepsilon}
\]
its variation. We will then denote with \( \mathcal{P}_W \) the set of the probability measures \( P \) such that \( [W] \) is absolutely continuous in \( t \) and \( \hat{a} \in \mathbb{S}^+_N \), \( P \)-a.e. In particular \( P_0 \in \mathcal{P}_W \) because \( [W]_t = tI_N \) and \( \hat{a}_t = I_N \), \( P_0 \)-a.e., where \( I_N \) is the \( N \times N \) identity matrix. Moreover, let \( \mathcal{P}_S \) be the subset of \( \mathcal{P}_W \) composed by the probability measures \( P^\alpha := P_0 \circ (X^\alpha)^{-1} \), where
\[
X^\alpha_t := \int_0^t \alpha^{1/2}_s dW_s, \quad P_0 \text{-a.e.,}
\]
and \( \alpha \) is a progressive process in \( \mathbb{S}^+_N \) such that, for two fixed \( \underline{a}, \overline{a} \) in \( \mathbb{S}^+_N \), \( \underline{a} \leq \alpha \leq \overline{a} \), \( P_0 \)-a.e.. It is then apparent a link between \( \mathcal{P}_S \) and the control set \( \mathcal{A}_F \). Now, given a function
\[
h : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^N \times D_h \to \mathbb{R},
\]
where \( D_h \) is a subset of \( \mathbb{R}^{N \times N} \) containing 0, define for any \( a \in \mathbb{S}^+_N \)
\[
f(t, \omega, y, z, a) := \sup_{\gamma \in D_h} \left( \frac{1}{2} \langle \gamma, a \rangle - h(t, \omega, y, z, \gamma) \right).
\]
Furthermore let \( \hat{f}(t, y, z) := f(t, y, z, \hat{a}_t) \),
\[
\mathcal{P}^2_h := \left\{ P \in \mathcal{P}_S : \mathbb{E}^P \left( \int_0^T |\hat{f}(t, 0, 0)|^2 \, dt \right) < \infty \right\}
\]
and \( \mathcal{P}^2_h(t, P) := \{ P' \in \mathcal{P}^2_h : P' = P \text{ on } \mathcal{F}_t \} \).
A pair of progressive processes \((Y, Z)\) is solution to the 2BSDE
\[
Y_t = \xi + \int_t^T \hat{f}(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s + K_T - K_t \quad (19)
\]
if, for any \( P \in \mathcal{P}^2_h \),
(i) \( Y_T = \xi \), \( P \)-a.e.;
(ii) the process \( K^P \) defined below has non decreasing path \( P \)-a.e.,
\[
K^P_t = Y_0 - Y_t - \int_0^t \hat{f}(s, Y_s, Z_s) \, ds + \int_0^t Z_s \, dW_s, \quad t \in [0, T], \ P \text{-a.e.;}
\]
(iii) the family \( \{ K^P, P \in \mathcal{P}_2^h \} \) satisfies the minimum condition
\[
K^P_t = \operatorname{ess} \inf_{P' \in \mathcal{P}_2^h(t,P)} \mathbb{E}^P \left( K^{P'}_T \mid \mathcal{F}_t \right), \quad \mathbb{P}\text{-a.e.} \tag{20}
\]

Under suitable conditions the 2BSDE (19) admits a unique solution and, if we denote with \( \{Y^P(r, \xi), Z^P(r, \xi)\} \) the solution to the BSDE
\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, r], \mathbb{P}\text{-a.e.,}
\]
it can be proved that \( Y_t = \operatorname{ess} \sup_{P' \in \mathcal{P}_2^h(t,P)} Y^{P'}_t(r, Y_r) \) for any \( r \in [t, T] \) and \( P \in \mathcal{P}_2^h \).

The last identity is a dynamic programming principle and can be seen as the connection between 2BSDE and our method.

Now we will show a different formulation of 2BSDEs, using controls instead of probability measures. Let \( \mathcal{A} \) be our control set, made up by the progressive processes in \( L^2([0, T] \times \Omega; \mathcal{B}) \), where \( \mathcal{B} \) is a Banach space, and, for any \( t \in [0, T] \),
\[
\mathcal{A}(t, \alpha) := \{ \alpha' \in \mathcal{A} : \alpha'_s = \alpha_s \text{ for any } s \in [0, t] \}.
\]
Then define the function
\[
f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{B} \to \mathbb{R}
\]
and assume that there exists a \( C > 0 \) such that
\[
|f(t, y, z, \alpha) - f(t, y, z, \alpha')| \leq C(1 + |y| + |z|)|\alpha - \alpha'|.
\]
We will also assume that, for each \( \alpha \in \mathcal{A} \), \( f_\alpha(t, y, z) := f(t, y, z, \alpha) \) satisfies assumptions \( A.6 \) uniformly with respect to \( \alpha \). We point out that these conditions are not intended to be minimal. We then have that the BSDEs
\[
Y^\alpha_t = \xi + \int_t^T f_\alpha(s, Y^\alpha_s, Z^\alpha_s) ds - \int_t^T Z^\alpha_s dW_s, \quad t \in [0, T],
\]
admirt a unique solution for any \( \alpha \in \mathcal{A} \). If we moreover require that, for any \( t \in [0, T] \), \( \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left( |Y^\alpha_t|^2 \right) < \infty \) (which can be achieved if, for example, \( B \) is compact), then by the dynamic programming principle 3.10,
\[
Y^\alpha_t := \operatorname{ess} \sup_{\alpha' \in \mathcal{A}(t, \alpha)} Y^\alpha_t = \operatorname{ess} \sup_{\alpha' \in \mathcal{A}(t, \alpha)} Y^\alpha_t \left( r, Y^\alpha_r \right), \quad \text{for any } 0 \leq t \leq r \leq T, \tag{21}
\]
where \( Y^\alpha \left( r, Y^\alpha_r \right) \) is solution to the BSDE \( \left( Y^\alpha_r, f_\alpha, r \right) \). It is easy to see that, for any \( \alpha \in \mathcal{A} \), \( Y^\alpha \) is a continuous progressive process in \( L^2 \) and \( Y^\alpha_T = \xi_\alpha \text{ a.e.} \).
Using the same arguments of [25] we that, for each \( \alpha \in \mathcal{A} \), there exist two progressive processes in \( L^2, Z^\alpha \) and \( K^\alpha \), such that \( K^\alpha \) is a continuous and increasing process in \( t \) with \( K^\alpha_0 = 0 \) and

\[
\overline{Y}_t^\alpha = \xi_\alpha + \int_t^T f_\alpha (s, \overline{Y}_s^\alpha, \overline{Z}_s^\alpha) \, ds - \int_t^T \overline{Z}_s^\alpha \, dW_s + K^\alpha_T - K^\alpha_t, \quad t \in [0, T].
\]

We also have that, as in (20),

\[
K^\alpha_t = \text{ess inf}_{\alpha' \in \mathcal{A}(t, \alpha)} \mathbb{E} \left( K^\alpha_T' \mid \mathcal{F}_t \right), \quad \text{a.e. for any } t \in [0, T], \alpha \in \mathcal{A}.
\]

If we let \( X \) be as in (9) and \((\overline{Y}, \overline{Z}, K)\) be the solution to the 2BSDE (we omit the dependence on the control set \( \mathcal{A}_F \))

\[
\overline{Y}_{s,t}^x = g(X_T^x) + \int_s^T f_\sigma (r, X_T^x, \overline{Y}_{r,t}^x, \overline{Z}_{r,t}^x) \, dr - \int_s^T \overline{Z}_{r,t}^x \, dW_r + K_{T,t}^x - K_{s,t}^x
\]

for any \( x \in \mathbb{R}^N \) and \( 0 \leq t \leq s \leq T \), we then have that the function \( u(t, x) := \overline{Y}_t^x \) is the viscosity solution to problem 3.1.

A Some Probability Results

Here we give some probability results we use in this paper.

**Lemma A.1.** Let \( \{U_t\}_{t \in [0, \infty)} \) be a cadlag process, then for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
\mathbb{P}(\{|U_t - U_s| < \varepsilon, \text{ for any } s \in [t, t+\delta]\}) > 0.
\]

**Proof.** Our argument is by contradiction. Assume that there exists an \( \varepsilon > 0 \) such that for any \( \delta > 0 \)

\[
\mathbb{P}(\{|U_t - U_s| < \varepsilon, \text{ for any } s \in [t, t+\delta]\}) = 0,
\]

which is equivalent to

\[
\mathbb{P}(\{|U_t - U_s| \geq \varepsilon, \text{ for any } s \in [t, t+\delta]\}) = 1.
\]

Let, for any positive integer \( n \),

\[
A_n := \left\{|U_t - U_s| \geq \varepsilon, \text{ for any } s \in \left[t, t + \frac{1}{n}\right]\right\},
\]

then \( A_n \subseteq A_k \) if \( k \leq n \) and

\[
A := \bigcap_{n=1}^{\infty} A_n = \left\{ \lim_{s \to t} |U_t - U_s| \geq \varepsilon \right\}.
\]

Since \( U \) is right continuous we know that \( \mathbb{P}(A) = 0 \) which contradicts our assumption, since \( \mathbb{P}(A) = \lim_{n \to \infty} \mathbb{P}(A_n) = 1. \)

\[
22
\]
Consider the SDE
\[ X^{t,\zeta}_s = \zeta + \int_t^s \sigma \left( r, X^{t,\zeta}_r \right) \, dW_r + \int_t^s b \left( r, X^{t,\zeta}_r \right) \, dr, \quad s \in [t, \infty), \] (22)
under the following assumptions:

**Assumptions A.2.** \( t \in [0, \infty), \zeta \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^N) \) and for the functions
\[ b : [0, \infty) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N \text{ and } \sigma : [0, \infty) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^{N \times M} \]
there exists a positive constant \( \ell \) such that a.e., for any \( r \in [0, \infty), x, y \in \mathbb{R}^N, \)
(i) \(|b(r, x) - b(r, y)| + |\sigma(r, x) - \sigma(r, y)| \leq \ell |x - y|; \)
(ii) \( \{b(\cdot, \cdot)(t, 0)\}_{t \in [0, \infty)} \) is a progressive process belonging to \( L^2([0, T] \times \Omega) \) for any \( T \in [0, \infty). \)

A solution to this SDE is a continuous progressive process \( X \) as in (22) such that \( X \in L^2([0, T] \times \Omega) \) for any \( T \in [0, \infty). \) The next theorem summarizes some SDE result given in [13].

**Theorem A.3.** Under assumptions A.2 there exists a unique solution to the SDE (22). Moreover, for any \( T \in [t, \infty) \), there exists a constant \( c_2 \), depending only on \( \ell \) and \( T \) such that
\[
\mathbb{E} \left( \sup_{s \in [t, T]} \left| X^{t,\zeta}_s - \zeta \right|^2 \right) \leq c_2 \mathbb{E} \left( \int_t^T (|b(s, 0)|^2 + |\sigma(s, 0)|^2) \, ds \right), 
\]
\[
\mathbb{E} \left( \sup_{s \in [t, T]} \left| X^{t,\zeta}_s \right|^2 \right) \leq c_2 \mathbb{E} \left( |\zeta|^2 + \int_t^T (|b(s, 0)|^2 + |\sigma(s, 0)|^2) \, ds \right), 
\]
\[
\mathbb{E} \left( \sup_{s \in [t, T]} \left| X^{t,\zeta}_s - X^{t,\zeta'}_s \right|^2 \right) \leq c_2 \mathbb{E} \left( |\zeta - \zeta'|^2 \right). 
\]

**Remark A.4.** The results obtained in this section hold even for SDEs with an a.e. finite stopping time \( \tau \) as starting time. In fact if for any \( \zeta \) in \( L^2(\Omega, \mathcal{F}_\tau; \mathbb{R}^N) \) we define
\[
\overline{b}(t, x) := b(t, x + \zeta)\chi_{\{\tau \leq t\}} \text{ and } \overline{\sigma}(t, x) := \sigma(t, x + \zeta)\chi_{\{\tau \leq t\}}, 
\]
then \( X^{\tau,\zeta} \) is solution of the SDE \( (b, \sigma) \) if and only if \( \overline{X}^{0,0} := X^{\tau,\zeta} - \zeta \) is solution of the SDE \( (\overline{b}, \overline{\sigma}) \). The claim can be easily obtained from this.

**Remark A.5.** By the strong Markov property, for any a.e. finite stopping time \( \tau \), the process \( \{W^\tau_t\}_{t \in [0, \infty)} \) := \( \{W^\tau_{t+t} - W^\tau_t\}_{t \in [0, \infty)} \) is a Brownian motion. Thus if \( b \) and \( \sigma \) are are progressive with respect to the filtration \( \{\mathcal{F}^\tau_t\}_{t \in [0, \infty)} \) then any solution to the SDE \( (b, \sigma) \) with initial data \( \tau + t \) and \( \zeta \in L^2(\Omega, \mathcal{F}_\tau; \mathbb{R}^N) \) is also progressive with respect to that filtration. In fact, in this case, the stochastic integral with respect to \( W^\tau_t \) is the same as the one with respect to \( W^\tau_{\tau+t} \).
A.1 Backward Stochastic Differential Equations

In this subsection we give some results on BSDEs used in our investigation. Most of them are well known and actually hold under more general assumptions. We refer to [19, 24, 21, 3] for their proofs.

We will work under the followings assumptions:

**Assumptions A.6.** Let $T \in [0, \infty)$, $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^M)$ and $f : [0, T] \times \Omega \times \mathbb{R}^M \times \mathbb{R}^{M \times N} \to \mathbb{R}^M$

a function which admits a positive constant $\ell$ and a real number $\mu$ such that a.e., for any $t \in [0, T]$, $y, y' \in \mathbb{R}^M$ and $z, z' \in \mathbb{R}^{M \times N},$

(i) $\{f(s, 0, 0)\}_{s \in [0, T]}$ is a progressive process belonging to $L^2([0, T] \times \Omega)$;

(ii) $|f(t, y, z)| \leq |f(t, 0, 0)| + \ell(1 + |y|)$;

(iii) $|f(t, y, z) - f(t, y, z')| \leq \ell |z - z'|$;

(iv) $(y - y')(f(t, y, z) - f(t, y', z)) \leq \mu |y - y'|^2$;

(v) $v \mapsto f(t, v, z)$ is continuous.

A solution to the BSDE $(\xi, f, T)$, where $\xi$ and $T$ have respectively the role of a final condition and a terminal time, is a pair $(Y, Z)$ of progressive processes belonging to $L^2([0, T] \times \Omega)$ such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad \text{for any } t \in [0, T]. \quad (23)$$

The followings are classical results of BSDE theory.

**Theorem A.7.** Under the assumptions A.6 the BSDE (23) has a unique solution $(Y, Z)$. Furthermore there exists a constant $c$, which depends on $T$, $\mu$ and $\ell$, such that

$$\mathbb{E}\left(\sup_{t \in [0, T]} |Y_t|^2 + \int_0^T |Z_t|^2 dt\right) \leq c \mathbb{E}\left(|\xi|^2 + \int_0^T |f(t, 0, 0)|^2 dt\right)$$

and, if $(Y', Z')$ is the solution to the BSDE $(\xi', f', T)$,

$$\mathbb{E}\left(\sup_{t \in [0, T]} |Y_t - Y'_t|^2 + \int_0^T |Z_t - Z'_t|^2 dt\right) \leq c \mathbb{E}\left(|\xi - \xi'|^2 + \int_0^T |f(t, Y'_t, Z'_t) - f'(t, Y'_t, Z'_t)|^2 dt\right).$$
Theorem A.8. Assuming $M = 1$, let $(Y,Z)$ be the solution to the BSDE $(\xi, f, T)$ under the assumptions A.6 and
\[ Y'_t = \xi' + \int_t^T V_s ds - \int_t^T Z'_s dW_s, \quad t \in [0,T], \]
where $\xi' \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, $Y', V \in L^2([0,T] \times \Omega)$ and $Z' \in L^2([0,T] \times \Omega)$. Suppose that $\xi \leq \xi'$ a.e. and $f(t, Y'_t, Z'_t) \leq V_t$ a.e. for the $dt \times d\mathbb{P}$ measure. Then, for any $t \in [0,T]$, $Y_t \leq Y'_t$ a.e.
If moreover $Y_0 = Y'_0$ a.e., then $Y_t = Y'_t$ a.e. for any $t \in [0,T]$, or in other words, whenever either $\mathbb{P}(|\xi - \xi'| > 0)$ or $f(s, Y'_s, Z'_s) < V_s$ for any $(y,z)$ in $\mathbb{R} \times \mathbb{R}^N$ on a set of positive $dt \times d\mathbb{P}$ measure, then $Y_0 < Y'_0$.

Proposition A.9. Let $(Y,Z)$ be the solution to the BSDE (23) and assume that there exists a stopping time $\tau$ such that $\tau \leq T$, $\xi$ is $\mathcal{F}_\tau$-measurable and $f(t,y,z) = 0$ on the set $\{\tau \leq t\}$. Then $Y_t = Y_{\tau \wedge t}$ a.e. and $Z_t = 0$ a.e. on the set $\{\tau \leq t\}$.

B Comparison Theorem

Consider the parabolic problem
\[ \partial_t u(t,x) + F(t, x, u, \nabla u, D^2 u) = 0, \quad t \in (0,T), x \in \mathbb{R}^N, \quad (24) \]
where $F$ is a continuous elliptic operator which admits, for any $t \in [0,T]$, $(x, r, p, S)$ and $(y, r', p', S')$ in $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S^N$, a $\mu \in \mathbb{R}$ and a positive constant $\ell$ such that
\begin{enumerate}
  \item \( |F(t, x, r, p, S) - F(t, x, r, p, S')| \leq \ell (1 + |x|^2) |S - S'| \);
  \item \( |F(t, x, r, p, S) - F(t, x, r, p', S)| \leq \ell (1 + |x|) |p - p'| \);
  \item \( |F(t, x, r, p, S) - F(t, y, r, p, S)| \leq \ell (1 + |x| + |y|) |x - y| \| (p, S) \| \);
  \item (Monotonicity) \( F(t, x, r, p, S) - F(t, x, r', p, S)) \leq \mu |r - r'|^2 \);
  \item the continuity of the function $r \mapsto F(t, x, r, p, S)$ is independent from the fourth variable.
\end{enumerate}

Notice that, given a compact set $K \subset \mathbb{R}^N \times \mathbb{R} \times S^N$, item (v) and the Heine–Cantor theorem yield the existence of a modulus of continuity $\omega_K$ such that, if $(x,r,S), (x',r',S) \in K$,
\[ |F(t, x, r, p, S) - F(t, x, r', p, S)| \leq \omega_K(|r - r'|) \quad (25) \]
for any $t \in [0,T]$ and $p \in \mathbb{R}^N$.

Here we give a comparison result, which is an adaptation of [23, Theorem C.2.3] for problem (24). To prove it we will need the following theorem, adaptation of [23, Theorem C.2.2] which can be proved similarly. Notice that condition (G) in [23] is replaced by (26).
Proof. We set $\phi(x) := (1 + |x|^2)^{\frac{\gamma}{2}}$,

$$\gamma > \mu + \ell \sup_{x \in \mathbb{R}^N} \left( (1 + |x|) \frac{\nabla \phi(x)}{\phi(x)} + (1 + |x|^2) \frac{D^2 \phi(x)}{\phi(x)} \right),$$

and assume that $u_1(t,x) := \frac{e^{-\gamma t}u(t,x)}{\phi(x)}$ and $u_2(t,x) := \frac{e^{-\gamma t}v(t,x)}{\phi(x)}$, where $c$ is such that both $|u_1|$ and $|u_2|$ converge uniformly to 0 as $|x| \to \infty$. Notice that

$$\nabla \phi(x) = c \frac{\phi(x)x}{1 + |x|^2} \text{ and } D^2 \phi(x) = \phi(x) \left( \frac{c}{1 + |x|^2} I + \frac{c(c-2)}{(1 + |x|^2)^2} x \otimes x \right),$$

therefore $\gamma$ is well defined. We also set the operators $F_1(t,x,r,p,S)$, given by

$$\frac{e^{-\gamma t}}{\phi(x)} F(t,x,e^{\gamma t}r\phi,e^{\gamma t} \left( r\nabla \phi + \phi \nabla p \right), e^{\gamma t} \left( rD^2 \phi + \nabla \phi \otimes p + p \otimes \nabla \phi + \phi S \right)), \text{ on } (0,T) \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N.$$
and \( F_2(t, x, r, p, S) \), given by
\[
-\frac{e^{-\gamma t}}{\phi(x)} F\left(\begin{array}{c}
-\gamma t r \phi \\
-\gamma t \left(r \nabla \phi + \phi p\right) \\
-\gamma t \left(r D^2 \phi + \nabla \phi \otimes p + p \otimes \nabla \phi + \phi S\right)
\end{array}\right).
\]

It is easy to check that, for \( i \in \{1, 2\} \), \( F_i \) is still continuous, elliptic, Lipschitz continuous in \( p \) and \( S \), that its monotonicity constant is
\[
\mu + \ell \sup_{x \in \mathbb{R}^N} \left( (1 + |x|) \left| \nabla \phi(x) \right| + (1 + |x|^2) \left| \frac{D^2 \phi(x)}{\phi(x)} \right| \right),
\]
i.e. is lower than \( \gamma \), and \( u_i \) is a viscosity subsolution to
\[
\partial_t u(t, x) - \gamma u(t, x) + F_i(t, x, u, \nabla u, D^2 u) = 0, \quad t \in (0, T), x \in \mathbb{R}^N.
\]

It can also be checked that, if \((x, r, S), (y, r, S)\) belong to a compact set \( K \subset \mathbb{R}^N \times \mathbb{R} \times S^N \), there exist a constant \( C_K \) and a modulus of continuity \( \tilde{\omega}_K \) bigger than \( \omega_K \) in (25) such that
\[
|F_i(t, x, r, p, S) - F_i(t, y, r, p, S)| \leq C_K (1 + |p|)|x - y| + \tilde{\omega}_K(|x - y|)
\]
for any \( i \in 1, 2, t \in [0, T] \) and \( p \in \mathbb{R}^N \).

Furthermore \( (u_1 + u_2)|_{t=0} \leq 0 \) and \( F_1(t, x, r, p, S) + F_2(t, x, r, p, S) = 0 \).

From these properties we have, for any \( r_i \in \mathbb{R}, p_i \in \mathbb{R}^N \) and \( S_i \in S^N \) such that \( r = r_1 + r_2 \geq 0 \), \( p_1 = -p_2 \) and \( S = S_1 + S_2 \leq 0 \),
\[
-\gamma r + F_1(t, x, r_1, p_1, S_1) + F_2(t, x, r_2, p_2, S_2)
\]
\[
= -\gamma r + F_1(t, x, r_1, p_1, S_1) + F_2(t, x, r_1, -p_1, -S_1)
\]
\[
+ (F_2(t, x, r_2, p_2, S_2) - F_2(t, x, r_2 - r, p_2, S_2 - S)) \frac{r}{r}
\]
\[
\leq -\gamma r + (F_2(t, x, r_2, p_2, S_2) - F_2(t, x, r_2 - r, p_2, S_2)) \frac{r}{r}
\]
\[
\leq -\gamma r + \gamma r = 0.
\]

As a consequence we have that all the conditions of theorem B.1 are satisfied, thus \( u_1 + u_2 \leq 0 \), or equivalently, \( u \leq v \) in \( (0, T) \times \mathbb{R}^N \).

\[
\square
\]

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