Fixed-\(N\) Superconductivity:
The Exact Crossover from the Bulk to the Few-Electron Limit

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We use two truly canonical approaches to describe superconductivity in ultrasmall metallic grains: (a) a variational fixed-\(N\) projected BCS-like theory and (b) the exact solution of the model Hamiltonian developed by Richardson in context with Nuclear Physics. Thereby we obtain a description of the entire crossover from the bulk BCS regime (mean level spacing \(d \ll \text{bulk gap } \Delta\)) to the “fluctuation-dominated” few-electron regime \((d \gg \Delta)\). A wave-function analysis shows in detail how the BCS limit is recovered and how for \(d \gg \Delta\) pairing correlations become delocalized in energy space.

I. INTRODUCTION

When a system of (correlated) electrons is sufficiently small, the electronic spectrum becomes discrete. Such a discrete spectrum was directly measured for the first time by Ralph, Black and Tinkham (RBT) \cite{1,2}, for ultrasmall Al grains. This allowed them to study the nature of pairing correlations in a small superconductor in unprecedented detail. These experiments gave new actuality to an old and fundamental question: What is the lower size limit for superconductivity? Besides Anderson’s prediction \cite{3} that superconductivity breaks down once the single-particle mean level spacing \(d\) becomes larger than the (bulk) superconducting gap \(\Delta\), the finite size of a superconducting grain also manifests itself in its large charging energy, which effectively suppresses electron number fluctuation on the grain. Hence superconductivity on small grains has to be formulated in a manifestly canonical way.

After briefly introducing the experiments and a toy model which captures their essential features, we show how the entire crossover between the bulk BCS-like regime and the few-electron regime can be described theoretically by fixing the electron number on the grain and using either a projected BCS approach or, even better, a long-forgotten exact solution of the reduced BCS model Hamiltonian. Both the projected BCS approach and the exact solution enable us (i) to significantly improve previous g.c. upper bounds on ground state energies, \cite{4–7}, in the latter case by giving the exact result; (ii) to find in the crossover regime a remnant of the “break-down of superconductivity” obtained in g.c. studies, at which the condensation energy changes from being extensive to practically intensive; and (iii) to study this change by an explicit wave-function analysis, which shows in detail how the BCS limit is recovered for \(d \ll \Delta\), and how for \(d \gg \Delta\) pairing correlations delocalize in energy space.

A. Spectroscopic Gap in an Ultrasmall Superconducting Grain

In RBT’s experiments, an ultrasmall grain was used as central island in a single-electron transistor: it was connected via tunnel barriers to external leads and capacitively coupled to a gate, and its electronic spectrum determined by measuring the tunnel current through the grain as a function of transport voltage at a fixed temperature of 50mK. For a typical grain the radius was \(r \approx 5\text{nm}\), and the crude order-of-magnitude free-electron estimate \(d = 2\pi^2 \hbar^2 / (m k_F \text{Vol})\) for the mean level spacing near \(\varepsilon_F\) yields \(d \approx 0.5\text{meV}\). The grain’s charging energy was about \(E_C = \varepsilon^2 / 2C_{\text{total}} = 50\text{meV}\) and therefore much larger than all other energy scales, such as the Aluminum bulk gap \((\Delta \approx 0.4\text{meV})\), typical values of the transport voltage \((V \lesssim 2\text{mV})\) and the temperature.

The most remarkable feature of the experimental tunneling spectrum, shown in Fig. 1, is the presence of a clear spectroscopic gap for the grain with even electron number and its absence for the odd grain. This reveals the presence of pairing correlations: in even grains, all excited states involve at least two BCS quasi-particles and hence lie significantly above the ground state, whereas odd grains always have at least one quasi-particle and excitations need not overcome an extra gap.

The charging energy, being the largest energy scale of the system, strongly suppresses particle number fluctuations on the grain and hence the discrete energies measured in RBT’s experiments essentially correspond to the eigenspectrum of a grain with fixed electron number \(N\). We therefore consider below an ultrasmall grain completely isolated from the rest of the world, e.g. by infinitely thick oxide barriers. Our main goal will be to elucidate, within a canonical framework, the nature of the pairing correlations in an ultrasmall grain at \(T = 0\).
II. MODELLING AND FIRST APPROXIMATION

To investigate the influence of pairing correlations, on the excitation spectrum of an ultrasmall grain, we model the grain by a reduced BCS-Hamiltonian. It has been used before to describe small superconducting grains and was phenomenologically successful for $d \leq \Delta$, but probably is unrealistically simple for $d \gg \Delta$, for which it should rather be viewed as toy model:

$$H = \sum_{j=0,\sigma}^{N-1} \varepsilon_j c_{j\sigma}^\dagger c_{j\sigma} - \lambda d \sum_{j,j'=0}^{N-1} c_{j\sigma}^\dagger c_{j'\sigma} - c_{j'\sigma} c_{j\sigma}^\dagger. \quad (1)$$

The $c_{j\pm}$ create electrons in free time-reversed single-particle-in-a-box states $|j,\pm\rangle$, with discrete, uniformly spaced, doubly degenerate eigenenergies $\varepsilon_j = jd + \varepsilon_0$. The interaction scatters only time-reversed pairs of electrons within $\omega_D$ of $\varepsilon_F$. Its dimensionless strength $\lambda$ is related to the two material parameters $\Delta$ and $\omega_D$ via the bulk gap equation $\sinh(1/\lambda) = \omega_D/\Delta$. We chose $\lambda = 0.224$, close to that of Al. The level spacing $d$ determines the number $N = 2\omega_D/d$ of levels, taken symmetrically around $\varepsilon_F$, within the cutoff; electrons outside the cutoff remain unaffected by the interaction and are thus neglected throughout.

A. Grand-Canonical BCS Approach

The most direct and easiest theoretical approach to describing an ultrasmall grain simply uses the well-known grand-canonical (g.c.) variational BCS ansatz for the ground state of an even or odd grain (subscript $p = 0$ or 1, respectively):

$$\langle \text{BCS}\rangle_0 = \prod_j (u_j + v_j c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger) |\text{Vac}\rangle$$

$$\langle \text{BCS}\rangle_1 = c_{j_{\text{odd}}}^\dagger \prod_{j \neq j_{\text{odd}}} (u_j + v_j c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger) |\text{Vac}\rangle$$

$$(u_j^2 + v_j^2 = 1). \quad (2)$$

$v_j$ and $u_j$ are the amplitudes that level $j$ is doubly occupied or empty, respectively. Note that on an odd grain one electron necessarily is unpaired; to minimize its kinetic energy, it is put at the Fermi energy ($\varepsilon_{j\text{odd}} = \varepsilon_F$). Minimizing the energy expectation value $\mathcal{E}^{\text{GC}}_p = \langle \text{BCS}|\hat{H}|\text{BCS}\rangle (p = 0, 1)$ with respect to $u_j$ and $v_j$ yields the even and odd “gap equations” (at $T = 0$):

$$\frac{1}{\lambda} = d \sum_j \frac{1}{\sqrt{\varepsilon_j^2 + \Delta_0^2(d)}}, \quad \frac{1}{\lambda} = d \sum_{j \neq j_{\text{odd}}} \frac{1}{\sqrt{\varepsilon_j^2 + \Delta_1^2(d)}}. \quad (3)$$

These are solved for the even and odd pairing parameters $\Delta_0$ and $\Delta_1$ as a function of level-spacing $d$. Note that $j_{\text{odd}}$ is excluded from the odd sum.

As predicted in 1959 by Anderson, it turned out that above a critical level spacing the gap equation ceases to have a non-trivial solution: when the sample becomes too small, superconductivity breaks down. More precisely, the even and odd “gap equations” (at $T = 0$) are the amplitudes that level $j$ is doubly occupied or empty, respectively

$$\frac{1}{\lambda} = d \sum_j \frac{1}{\sqrt{\varepsilon_j^2 + \Delta_0^2(d)}}, \quad \frac{1}{\lambda} = d \sum_{j \neq j_{\text{odd}}} \frac{1}{\sqrt{\varepsilon_j^2 + \Delta_1^2(d)}}. \quad (3)$$

B. Fluctuations

While the g.c. BCS ansatz confirms Anderson’s prediction, it is, however, clear that the approach has two problems: (a) it inherently contains fluctuations of the particle number $N$ which are not present in the actual grain system due to its large charging energy; and (b) it fails to describe superconducting fluctuations for large level spacings $d \gg \Delta$, where it trivially yields $\Delta = 0$.

For bulk systems, the fluctuations in $N$ are negligibly small ($\delta N \sim \sqrt{N} \ll N$), and a more rigorous fixed-$N$ treatment would only correct the BCS ground state energy per electron by order $1/N$, which vanishes in the thermodynamic limit $d \to 0$. For ultrasmall systems, however, precisely such corrections are important and we will have to incorporate them.

On the other hand, pairing correlations (the redistribution of pairs from below to above $\varepsilon_F$) can lower the condensation energy below zero even when $\Delta_p = 0$. In this regime, pairing correlations are traditionally called “superconducting fluctuations”, which are evidently not captured adequately by the g.c. ansatz. Matveev and Larkin (ML) calculated the energy lowering to logarithmic order for $d \gg \Delta$ (i.e. $\Delta_p = 0$). However, they are known to become important already in the crossover regime $d \sim \Delta$.

To adequately describe an ultrasmall superconducting grain we therefore must go beyond standard BCS theory by (a) fixing the particle number $N$ and (b) incorporating superconducting fluctuations. These ingredients will allow us to describe the full crossover between the bulk system, which is dominated by BCS superconductivity, and the few-electron system, which mainly shows superconducting fluctuations. In particular we shall study how, at the crossing of the two energy scales $d$ and $\Delta$, the breakdown of superconductivity predicted by BCS theory is softened and how the fluctuations become dominant.
III. THE FULL CROSSOVER FROM SMALL $N$ TO THE BULK LIMIT

A. Fixed-$N$ Projection

Our first crossover study adapts a method developed by Dietrich, Mang and Pradal for shell models of nuclei with pairing interaction to the case of ultrasmall metallic grains. This “projected BCS” (PBCS) method also is a variational approach, but projects (before variation) the trial wave-function onto a fixed electron number $N = 2n_0 + p$ ($p$ is the parity):

$$|\text{PBCS}\rangle_0 = C \int_0^{2\pi} d\phi e^{-i\phi n_0} \prod_{j=0}^{N-1} \left( u_j + e^{i\phi} v_j c_j^+ c_{j'}^+ \right) |\text{Vac}\rangle, \quad (4)$$

(In the odd case $\prod_j$ again is replaced by $c_j^+ \prod_{j \neq j_0} c_j^+$.) The integral over $\phi$ performs the projection onto the fixed electron pair number $n_0$, and $C$ is a normalization constant ensuring $\langle 0|\text{PBCS}^\dagger \text{PBCS} | 0 \rangle = 1$. Again, the amplitudes $v_j$ and $u_j$ are found by minimizing the energy expectation value of the projected wave-function $E_p^{\text{PBCS}} = \langle p|\text{PBCS}^\dagger | H |\text{PBCS} | p \rangle$. While in the g.c. case the wave-function essentially can be described by a single parameter $\Delta_p$, in this case the minimization leads to a set of 2$n_0$ coupled non-linear equations which include projection integrals (for details see [11]). Following Ref. [12], we evaluate all integrals numerically (using fast Fourier transform routines).

In the limit $d \to 0$ at fixed $n_0d$, the PBCS theory reduces to the g.c. BCS theory of Ref. [3] (proving that the latter’s $N$-fluctuations become negligible in this limit): The projection integrals can then be approximated by their saddle point values [12]; at the saddle, the variational equations decouple and reduce to the BCS gap equation while the saddle point condition fixes the mean number of electrons to be $2n_0$. To check the opposite limit of $d \gg \Delta$ where $n_0$ becomes small, the so-called fluctuation-dominated regime, we compared the PBCS results for $E_0$ with exact results, finding agreement to within, say, 6% for $n_0 \leq 10$.1 This shows that superconducting fluctuations are automatically treated adequately in the PBCS approach.

The advantages of the PBCS method relative to the g.c. one are (a) the similarity of the trial wave-function to the BCS wave function, allowing it to capture the bulk limit and, on the other hand, (b) the increase of variational degrees of freedom from one ($\Delta$ in the g.c. BCS theory) to $N-1$ (one for each single particle level minus 1 for fixing $N$) due to the projection onto a fixed particle number. These additional degrees of freedom allow the method to also capture the superconducting fluctuations for $d \gg \Delta$. Because it works so well for $d \ll \Delta$ and $d \gg \Delta$, one might hope that it acceptably describes the crossover regime $d \sim \Delta$, too. There it really amounts to an uncontrolled approximation, whose quality will be checked against exact results in the next section.

B. Richardson’s Exact Solution

Since the publication of [11] we became aware of the fact that Richardson had shown already in the mid-60’s [13], also in the context of nuclear physics, that the ground state of the Hamiltonian (1) actually can (for non-degenerate $\epsilon_j$) be found exactly by solving a set of $n_0$ coupled algebraic non-linear equations.2

Richardson introduces electron pairs $b_j = c_j^+ - c_{j'}^+$, which have the commutation relation $[b_j, b_{j'}^\dagger] = \delta_{jj'}(1 - 2b_j^\dagger b_j)$, and exploits the fact that in the Hilbert space of non-singly-occupied states, the modified Hamiltonian

$$\hat{H} = \sum_j (2\epsilon_j b_j^\dagger b_j - \lambda d \sum_{j,j'} b_j^\dagger b_{j'}). \quad (5)$$

is equivalent to the $H$ of (1). Solving $\hat{H}$ exactly would be trivial if the $b$’s would represent true bosons. However, they actually are “hard-core bosons” instead. Richardson thus expresses the general ground state $|G\rangle^N$ of the system as

$$|G\rangle^N = \sum_{j_1 \neq \cdots \neq j_{n_0}} \varphi(j_1 \cdots j_{n_0}) b_{j_1}^\dagger \cdots b_{j_{n_0}}^\dagger |\text{Vac}\rangle, \quad (6)$$

where the sum is explicitly restricted to exclude double occupancy of pair states. The wave-function $\varphi$ is found by solving the many-body Schrödinger equation for $\varphi$. Richardson showed that the following ansatz works:

$$\varphi(j_1 \cdots j_{n_0}) \propto \sum_P \epsilon_{j_{k}} \prod_{k=1}^{n_0} \frac{1}{2 \epsilon_{j_k} - E_P(k)}. \quad (7)$$

1Because this error refers to the correlation energy, it is larger than the error ($< 1\%$) cited in [11] for the total ground state energy $E_0$.

2The numerical implementation of the exact solution is much easier than for the PBCS method, since it does not include any projections integrals.
Here \( \sum P \) represents the sum over all permutations of 
1, . . . , \( n_0 \), and the parameters \( E_k \) are the solution of the 
coupled algebraic equations

\[
\frac{1}{\lambda d} + \sum_{i \neq k}^{n_0} \frac{2}{E_i - E_k} = \sum_{j=1}^{2n_0} \frac{1}{2 \epsilon_j - E_k}, \quad k = 1 \ldots n_0. \tag{8}
\]

The total ground state energy is given by \( E_0^{\text{exact}} = \sum_{i=1}^{n_0} E_i \).

The discussion of Richardson’s exact solution in context with ultrasmall grains will be the subject of a forthcoming paper.

### C. Ground State Energies

Figure 2(a) shows the ground state condensation energies for both even and odd grains calculated with g.c. BCS method, the PBCS approach (for \( N \leq 600 \)) and Richardson’s exact solution. The result \( E_0^{\text{BCS}} \) is also shown for comparison. The g.c. curves suggest the aforementioned “breakdown of superconductivity” at some critical \( p \)-dependent level spacing \( d_p^{\text{GC}} \) above which \( E_0^{\text{GC}} = 0 \). In contrast, the \( E_0^{\text{PBCS}} \)'s (i) are significantly lower than the \( E_0^{\text{GC}} \)'s, thus the projection much improves the variational ansatz; and (ii) are negative for all \( d \), which shows that the system can always gain energy by allowing pairing correlations, even for arbitrarily large \( d \). The exact solution \( E_0^{\text{exact}} \) further improves the PBCS results, especially for intermediate level spacings. The PBCS results are evidently quite accurate for \( d \gg \Delta \) and, like the g.c. results, for \( d \ll \Delta \).

As anticipated in [7], the “breakdown of superconductivity” is evidently not as complete in the canonical as in the g.c. case. Nevertheless, some remnant of it does survive in \( E_0^{\text{PBCS}} \), since its behaviour, too, changes markedly at a \( p \) (and \( \lambda \)) dependent characteristic level spacing \( d_p^{\text{GC}} \) (\( < d_p^{\text{GC}} \)): it marks the end of bulk BCS-like behavior for \( d < d_p^{\text{GC}} \), where \( E_0^{\text{GC}} \) is extensive \((\sim 1/d)\), and the start of a fluctuation-dominated plateau for \( d > d_p^{\text{GC}} \). The standard heuristic interpretation [7] of the bulk BCS limit \( -\Delta^2/(2d) \) (which is indeed reached by \( E_0^{\text{PBCS}} \) for \( d \to 0 \)) hinges on the scale \( \Delta \): the number of levels strongly affected by pairing is roughly \( \Delta / d \) (those within \( \Delta \) of \( \varepsilon_F \)), with an average energy gain per level of \(-\Delta^2/2\).

To analogously interpret the \( d \) independence of \( E_0^{\text{GC}} \) in the fluctuation-dominated regime, we argue that the scale \( \Delta \) loses its significance – fluctuations affect all \( n_0 = \omega_D/d \) unblocked levels within \( \omega_D \) of \( \varepsilon_F \) (this is made more precise below), and the energy gain per level is proportional to a renormalized coupling \(-d\).

The exact results smear out the crossover even more than the PBCS results (\( E_0^{\text{exact}} \) lacks the kinks of \( E_0^{\text{PBCS}} \)), so much so that no sharply-defined crossover level spacing can be associated with \( E_0^{\text{exact}} \). However, the crossover scale evidently still is \( d \sim \Delta \). This can be confirmed by analyzing the functional dependence of the ground state energy on the coupling strength \( \lambda \): In the BCS limit, \( E_0 \approx -\Delta^2/(2d) \), where \( \Delta \) depends exponentially on \( \lambda \) [since \( \Delta = \omega_D \sinh(1/\lambda) \)]. In the fluctuation-dominated regime, however, perturbation theory in \( \lambda \) suffices and the correlation energy is roughly linear in \( \lambda \). For each \( d \), we thus fitted the numerical results for \( E_0^{\text{exact}}(d, \lambda) \), calculated for various \( \lambda \), to

\[
\frac{E_0(\lambda, d)}{E_0(\lambda_0, d)} = \alpha(d) \frac{\sinh(1/\lambda_0)^2}{\sinh(1/\lambda)^2} - \beta(d) \frac{\lambda}{\lambda_0}, \tag{9}
\]

a “phenomenological ansatz” which intends to capture the relative importance of the exponential or linear \( \lambda \) dependence in the coefficients \( \alpha(d) \) and \( \beta(d) \). The results, shown in Fig. 2(b), clearly show the crossover from the BCS-dominated regime \((\alpha > \beta)\) to the fluctuation-dominated regime \((\alpha < \beta)\).

### D. Wave Functions

Next we analyze the ground state wave-function, which can be characterized by

\[
C_j^2(d) = \langle \hat{c}_{j+}^\dagger \hat{c}_{j+} + \hat{c}_{j-}^\dagger \hat{c}_{j-} \rangle - \langle \hat{c}_{j+}^\dagger \hat{c}_{j+} \rangle \langle \hat{c}_{j-}^\dagger \hat{c}_{j-} \rangle, \tag{10}
\]

a set of correlators that measure the amplitude enhancement for finding a pair instead of two uncorrelated electrons in a single-particle niveau \(|j, \pm\rangle\). For all \( j \) of an uncorrelated state one has \( C_j = 0 \). For the g.c. BCS case \( C_j = u_j v_j \) and the \( C_j \)'s have a characteristic peak of width \( \sim \Delta \) around \( \varepsilon_F \), see Fig. 2(c), implying that pairing correlations are “localized in energy space”. For the BCS regime, both canonical methods produce \( C_j \)'s virtually identical to the g.c. case, vividly illustrating why the g.c. BCS approximation is so successful: not performing the canonical projection hardly affects the parameters \( v_j \) if \( d \ll \Delta \), but tremendously simplifies their calculation. However, in the fluctuation-dominated regime \( d \gtrsim \Delta \), the character of the wave-function changes: weight is shifted into the tails far from \( \varepsilon_F \), implying that pairing correlations be associated with unblocked levels within \( \omega_D \) of \( \varepsilon_F \). Thus

\[
|j, \pm\rangle \text{ come delocalized in energy space}.
\]

In the extreme case \( d \gg \Delta \), superconducting fluctuations are roughly equally strong for all interacting levels.

### E. Parity Effect

The parity effect predicted in the g.c. ensemble can be studied with a simple generalization [6] of the
above methods. Specifically, we shall study the parity-dependent “pair-breaking energy” \( \Omega_p \), i.e. the minimum energy required to break a pair by flipping a spin in an even or odd grain, defined as \( \Omega_0 = \frac{1}{2}(E_2 - E_0) \) and \( \Omega_1 = \frac{1}{2}(E_3 - E_1) \), where \( E_u \) denotes the energy of the lowest-lying state with \( u \) unpaired electrons with the same spin. (The pair-breaking energies can readily be measured in BRT’s experiments by applying a magnetic field, whose Zeeman energy favors the breaking of pairs.) For a non-interacting system and approximately also for large \( d \gg \Delta \), \( \Omega_0 \approx d/2 \) while \( \Omega_1 \approx d \). On the other hand, in the bulk limit \( \Omega_0 = \Omega_1 = \Delta \), since in the bulk \( E_{P+2} = E_{P} + 2\Delta \). The parity effect now states that as \( d \) increases from \( d \approx 0 \), the pairing correlations die faster for an odd than an even grain, causing \( \Omega_1 \) to initially decrease faster than \( \Omega_0 \). Since in the large-\( d \)-limit \( \Omega_1 > \Omega_0 \), the two energies must cross somewhere at \( d \sim \Delta \), as shown in Fig. 3(d). This crossing is a manifestation of the parity effect already predicted in the g.c. framework.

Remarkably, despite the crudeness and incorrect treatment of fluctuations of the g.c. method, it gives surprisingly good results for energy differences like the \( \Omega_p \). The fluctuations which it neglects seem to cancel substantially in energy differences such as \( \Omega_p \). This observation \textit{a posteriori} justifies the use of the g.c. method even for small grains, at least for rough calculations of energy differences.

\section*{IV. CONCLUSION}

In summary, the crossover from the bulk to the fluctuation-dominated regime can be captured in full using a fixed-\( N \) projected BCS ansatz, or even exactly using Richardson’s method. With increasing \( d \), the pairing correlations change from being strong and localized within \( \Delta \) of \( \varepsilon_F \), to being mere weak, energetically delocalized “fluctuations”; this causes the condensation energy to change from being \textit{extensive} to \textit{intensive} (modulus small corrections). Thus, the qualitative difference between “superconductivity” for \( d \ll \Delta \), and “fluctuations” for \( d \gtrsim \Delta \), is that for the former but not the latter, adding \textit{more} particles gives a \textit{different} condensation energy; for superconductivity, as Anderson put it, “more is different”.

\textit{Note added:}

After submission of this manuscript works by Dukelsky and Sierra \cite{16,17} used DMRG methods to numerically calculate the ground states of the presented Hamiltonian \cite{4}. Their results agree within a relative error of \( 10^{-4} \) with those obtained by Richardson’s exact method.

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FIG. 1. RBT’s experimental tunneling spectra. The distances between lines give the fixed-N excitation spectra of the same grain containing (a) an even and (b) an odd number of electrons, as function of magnetic field.

$$(\varepsilon - \varepsilon_F)/\Delta$$

FIG. 2. (a) The ground state correlation energies $E_p$ for even and odd systems (even: lower curves, odd: upper curves), calculated grand-canonically (GC), with PBCS and Richardson’s method (exact) as functions of $d/\Delta = 2\sinh(1/\lambda)/N$. (b) shows the coefficients $\alpha$ and $\beta$ of Eq. (9) as discussed in the text. The pairing amplitudes $C_j$ of Eq. (10) are shown in (c) for bulk (no symbols, dot-dashed), PBCS (“+” dashed) and exact results (“×”, solid) for $d = 1.09\Delta$ and $d = 4.34\Delta$. For $d = 0.27\Delta$, the PBCS and exact curves (not shown) are indistinguishable from the bulk curve. (d), finally, shows the results the pair-breaking energies $\Omega_{0/1}$ as calculated with g.c. BCS (dot-dashed) PBCS (dashed) and exactly (solid). (For clarity the graphs are offset by successively one unit each.)