DIFFERENCE OF COMPOSITION OPERATORS ON WEIGHT BERGMAN SPACES WITH DOUBLING WEIGHT

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ABSTRACT. In this paper, some characterizations for the compact difference of composition operators on Bergman spaces $A^p_\omega$ with doubling weight are given, which extend Moorhouse’s characterization for the difference of composition operators on the weighted Bergman space $A^2_\omega$.

Keywords: Bergman space, composition operator, difference.

1. INTRODUCTION

Let $\mathbb{D}$ be the the unit disc and $H(\mathbb{D})$ be the class of analytic functions on $\mathbb{D}$. Let $\varphi$ be an analytic self-map of $\mathbb{D}$. The map $\varphi$ induces a composition operator $C_\varphi$ on $H(\mathbb{D})$, which is defined by $C_\varphi f = f \circ \varphi$. We refer to [4, 22] for various aspects on the theory of composition operators acting on analytic function spaces.

A function $\omega : \mathbb{D} \to [0, \infty)$, integrable over $\mathbb{D}$, is called a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. For $0 < p < \infty$ and a radial weight $\omega$, the weighted Bergman space $A^p_\omega$ is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p_\omega}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty,$$

where $dA(z)$ is the normalized Lebesgue measure on $\mathbb{D}$. As usual, $A^p_\omega$ stands for the classical weighted Bergman space induced by the standard radial weight $\omega(z) = (1 - |z|^2)^\alpha$, where $-1 < \alpha < \infty$. $A^p_\omega$ equipped with the norm $\| \cdot \|_{A^p_\omega}$ is a Banach space for $1 \leq p < \infty$ and a complete metric space for $0 < p < 1$ with respect to the translation-invariant metric $(f, g) \mapsto \|f - g\|_{A^p_\omega}$.

For a radial weight $\omega$, we assume throughout the paper that $\tilde{\omega}(r) = \int_r^1 \omega(s) ds$ for all $0 \leq r < 1$. A radial weight $\omega$ belongs to $\tilde{\mathbb{D}}$ if there exists a constant $C = C(\omega) > 1$ such that

$$\tilde{\omega}(r) \leq C\tilde{\omega}(\frac{1 + r}{2})$$

for all $0 \leq r < 1$. If there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\tilde{\omega}(r) \geq C\tilde{\omega}(1 - \frac{1 - r}{K}), \quad 0 \leq r < 1,$
then we say that $\omega \in \tilde{D}$. We write $\mathcal{D} = \tilde{\mathcal{D}} \cap \tilde{\mathcal{D}}$. For some properties of these classes of weights, see [12, 13, 14, 15, 16, 17, 18] and the references therein.

Efforts to understand the topological structure of the space of composition operators in the operator norm topology have led to the study of the difference operator $C_\varphi - C_\psi$ of two composition operators induced by analytic self-maps $\varphi, \psi$ of $\mathbb{D}$. By Littlewood’s subordination principle, all composition operators, and hence all differences of two composition operators, are bounded on all Hardy space $H^p$ and weighted Bergman spaces $A^p_\omega$. Thus the question of when the operator $C_\varphi - C_\psi$ is compact naturally arises. Shapiro and Sundberg [23] raised and studied such a question on Hardy spaces, motivated by the isolation phenomenon observed by Berkson [1]. After that, such related problems have been studied between several spaces of analytic functions by many authors. See, for example, [6, 11, 24] on Hardy spaces and [2, 8, 9, 10, 19, 20, 21] on weighted Bergman spaces.

In 2005, Moorhouse [10] characterized the compact difference of composition operators on weighted Bergman spaces $A^2_\omega$ by angular derivative cancellation property. More precisely, she showed that $C_\varphi - C_\psi$ is compact on $A^2_\omega$ if and only if

$$\lim_{|z| \to 1} \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\psi(z)|^2} \right) \rho(\varphi(z), \psi(z)) = 0. \quad (1)$$

We remark here that this characterization has been extended not only to higher dimensional balls and polydisks, but also to general parameter $p$, see [2, 3, 9].

It is known that all composition operator and hence all differences of two composition operators, are bounded on $A^p_\omega$ for $\omega \in \tilde{\mathcal{D}}$ (see [15]). In this paper we extend Moorhouse’s characterization as well as some related properties to weighted Bergman spaces $A^p_\omega$, whenever $\omega \in \mathcal{D}$. The approach employed in the proof of the main results of this paper follows the guideline of [3, 7, 10], however a good number of steps cannot adapted straightforwardly and need substantial modifications.

Our main result (Theorem 12) is a characterization for compact combination of two composition operators. As a consequence we obtain that the Moorhouse’s characterization for compact difference [1] remains valid when $0 < p < \infty$ and $\omega \in \mathcal{D}$. According to this result, the compactness of $C_\varphi - C_\psi : A^p_\omega \to A^p_\omega$ depends neither on $p$ nor $\omega$, whenever $0 < p < \infty$ and $\omega \in \mathcal{D}$. The key ingredient for obtaining the previously mentioned results is the characterization of the $p$-Carleson measure for $A^p_\omega$.

The present paper is organized as follows. In Section 2, we give some notations and preliminary results which will be used later. In Sections 3, we devote to the question of when a given finite linear combination of composition operators is compact. Section 4 is devoted to show that the Moorhouse’s characterization for compact difference remains valid when $0 < p < \infty$ and $\omega \in \mathcal{D}$. We also obtain a characterization for a composition operator to be equal modulo compact operators to a linear combination of composition operators (see Theorem 14).

For two quantities $A$ and $B$, we use the abbreviation $A \lesssim B$ whenever there is a positive constant $C$ (independent of the associated variables) such that $A \leq CB$. We write $A \asymp B$, if $A \lesssim B \lesssim A$. 
2. PREREQUISITES

In this section we provide some basic tools for the proofs of the main results in this paper.

2.1. Pseudo-hyperbolic distance. We denote by $\sigma_z(w)$ the Möbius transformation on $\mathbb{D}$ that interchanges the points 0 and $z$. More explicitly,

$$\sigma_z(w) = \frac{z - w}{1 - wz}. $$

It is well known that $\sigma_z$ satisfies the following properties: $\sigma_z \circ \sigma_z(w) = w$, and

$$1 - |\sigma_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - w|}, \quad z, w \in \mathbb{D}. $$

For $z, w \in \mathbb{D}$, the pseudo-hyperbolic distance between $z$ and $w$ is defined by

$$\rho(z, w) = |\sigma_z(w)|. $$

It is also well known that the pseudo-hyperbolic metric have the following strong form of triangle inequality (see [5]):

$$\rho(z, w) \leq \frac{\rho(z, a) + \rho(a, w)}{1 + \rho(a, w)}$$

for all $a, z, w \in \mathbb{D}$. For $z \in \mathbb{D}$ and $r > 0$, the pseudo-hyperbolic disk at $z \in \mathbb{D}$ with radius $r \in (0, 1)$ is given by

$$\Delta(z, r) = \{w \in \mathbb{D} : \rho(z, w) < r\}. $$

Note that $\Delta(z, r)$ is Euclidean disk with center and radius given by

$$c = \frac{(1 - r^2)z}{1 - r^2|z|^2}, \quad t = \frac{1 - |z|^2}{1 - r^2|z|^2}r. $$

For $w \in \Delta(z, r)$, it is geometrically clear that

$$|c| - t \leq |w| \leq |c| + t. $$

Therefore,

$$\frac{(1 - |z|)(1 - r|z|)(1 - r)}{1 - r^2|z|^2} \leq 1 - |w| \leq \frac{(1 - |z|)(1 + r|z|)(1 + r)}{1 - r^2|z|^2}, $$

and $|w| \to 1$ uniformly in $w \in \Delta(z, r)$, as $|z| \to 1$.

2.2. Basic properties of weights. The following two lemmas contains basic properties of weights in the class $\hat{D}$ and $\tilde{D}$ and will be frequently used in the sequel. For a proof of the first lemma, see [12, Lemma 2]. The second one can be proved by similar arguments.

**Lemma A.** Let $\omega$ be a radial weight. Then the following statements are equivalent:

(i) $\omega \in \hat{D}$;

(ii) There exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that

$$\bar{\omega}(r) \leq C \left(\frac{1 - r}{1 - t}\right)^\beta \bar{\omega}(t), \quad 0 \leq r \leq t < 1;$$
(iii) There exist \( \gamma = \gamma(\omega) > 0 \) such that
\[
\int_D \frac{dA(z)}{|1 - \overline{z}\zeta|^{\gamma+1}} \approx \frac{\hat{\omega}(\zeta)}{(1 - |\zeta|)^\gamma}, \quad \zeta \in \mathbb{D}.
\]

**Lemma B.** Let \( \omega \) be a radial weight. Then \( \omega \in \mathcal{D} \) if and only if there exist \( C = C(\omega) > 0 \) and \( \alpha = \alpha(\omega) > 0 \) such that
\[
\hat{\omega}(t) \leq C \left( \frac{1 - t}{1 - r} \right)^\alpha \hat{\omega}(r), \quad 0 \leq r < t < 1.
\]

The following equivalent norm will be used in our proof, see [17, Lemma 5].

**Lemma C.** Let \( 0 < p < \infty, \omega \in \mathcal{D} \) and \( -\alpha < \gamma < \infty \), where \( \alpha = \alpha(\omega) > 0 \) is that of Lemma B. Then
\[
\int_D |f(z)|^p(1 - |z|^2)^\gamma \omega(z)dA(z) \approx \int_D |f(z)|^p(1 - |z|^2)^{\gamma-1}\hat{\omega}(z)dA(z), \quad f \in H(\mathbb{D}).
\]

The following estimate plays an important role in this paper and will be frequently used in the sequel.

**Lemma 1.** Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \) and \( \omega \in \mathcal{D} \). Then
\[
\left( \frac{1 - |z|}{1 - |\varphi(z)|} \right)^{\beta+1} \leq \frac{\omega(S(z))}{\omega(S(\varphi(z)))} \leq \left( \frac{1 - |z|}{1 - |\varphi(z)|} \right)^{\alpha+1},
\]
where \( \alpha = \alpha(\omega) \) and \( \beta = \beta(\omega) \) are that of Lemma B and Lemma A, respectively.

**Proof.** An application of Lemma A shows that
\[
\omega(S(z)) \approx \hat{\omega}(z)(1 - |z|) \quad \text{and} \quad \omega(S(\varphi(z))) \approx \hat{\omega}(\varphi(z))(1 - |\varphi(z)|).
\]
By Schwarz’s Lemma, we have
\[
|\varphi(z)| \leq \frac{c - 1}{c} + \frac{|z|}{c}, \quad \text{where} \quad c = \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}.
\]
By Lemmas A and B, we get
\[
\frac{\hat{\omega}(z)}{\hat{\omega}(\varphi(z))} = \frac{\hat{\omega}(z)}{\hat{\omega}(\frac{c - 1}{c} + \frac{|z|}{c})} \cdot \frac{\hat{\omega}(\frac{c - 1}{c} + \frac{|z|}{c})}{\hat{\omega}(\varphi(z))} \approx \left( \frac{1 - |z|}{1 - (\frac{c - 1}{c} + \frac{|z|}{c})} \right)^\alpha \left( \frac{1 - (\frac{c - 1}{c} + \frac{|z|}{c})}{1 - |\varphi(z)|} \right)^\beta
\]
\[
\approx \left( \frac{1 - |z|}{1 - |\varphi(z)|} \right)^\beta.
\]
and
\[
\frac{\tilde{\omega}(z)}{\tilde{\omega}(\varphi(z))} = \frac{\tilde{\omega}(z)}{\tilde{\omega}(z/c + |z|/c)} \cdot \frac{\tilde{\omega}(z/c + |z|/c)}{\tilde{\omega}(\varphi(z))} \\
\leq \left( \frac{1 - |z|}{1 - (z/c + |z|/c)} \right)^\beta \left( \frac{1 - (z/c + |z|/c)}{1 - |\varphi(z)|} \right)^\alpha \\
= \left( \frac{1 - |z|}{1 - |\varphi(z)|} \right)^\alpha.
\]

The proof is complete. □

**Lemma 2.** Let \( \omega \in \mathcal{D} \). If \( 0 < \lambda < \alpha(\omega) \), then \( \omega_\lambda(z) := \frac{\omega(z)}{(1 - |z|)^\lambda} \in \mathcal{D} \) and
\[
\tilde{\omega}_\lambda(z) = \frac{\tilde{\omega}(z)}{(1 - |z|)^\lambda}, \text{ for all } z \in \mathcal{D}.
\]

**Proof.** An integration by parts shows that
\[
\tilde{\omega}_\lambda(r) = \tilde{\omega}(r) + \lambda \int_r^1 \tilde{\omega}(t)(1 - t)^{\lambda - 1} dt.
\]
Therefore, by Lemmas A and B, we have
\[
\tilde{\omega}_\lambda(r) \geq \frac{\tilde{\omega}(r)}{(1 - r)^\lambda} + \lambda \int_r^1 \tilde{\omega}(t)(1 - t)^{\lambda - 1} dt \geq \frac{\tilde{\omega}(r)}{(1 - r)^\lambda}
\]
and
\[
\tilde{\omega}_\lambda(r) \leq \frac{\tilde{\omega}(r)}{(1 - r)^\lambda} + \lambda \int_r^1 \tilde{\omega}(t)(1 - t)^{\lambda - 1} dt \leq \frac{\tilde{\omega}(r)}{(1 - r)^\lambda}.
\]
Thus,
\[
\tilde{\omega}_\lambda(z) \approx \frac{\tilde{\omega}(z)}{(1 - |z|)^\lambda} \text{ for all } z \in \mathcal{D}.
\]
By Lemmas A and B, \( \omega_\lambda \in \mathcal{D} \). □

2.3. **Local estimates and test functions.** The following lemmas are crucial in our work and will be used in this paper.

**Lemma 3.** Let \( 0 < p < \infty \), \( \omega \in \mathcal{D} \) and \( 0 < r_1 < 1 \) be arbitrary. Denote \( \tilde{\omega}(\cdot) = \frac{\tilde{\omega}(\cdot)}{1-|\cdot|} \). Then there exists \( 0 < r_2 < 1 \) and a constant \( C = C(p, \omega, r_1, r_2) \) such that
\[
|f(z) - f(a)|^p \leq C \rho(z, a)^p \int_{\Delta(a, r_2)} |f(\zeta)|^p \tilde{\omega}(\zeta) dA(\zeta) / \omega(S(z))
\]
for all \( a \in \mathcal{D}, z \in \Delta(a, r_1) \) and \( f \in A_{\omega}^p \).
Proof. Let \( a \in \mathbb{D}, 0 < r_1 < 1, r := \frac{2r_1}{1+\eta}, \delta := \frac{2r}{1+\sigma}, r_2 := \frac{2\delta}{1+\sigma} \) and \( z \in \Delta(a, r_1) \) be fixed. Consider \( g_a := f \circ \sigma_a \). Then,

\[
|f(z) - f(a)|^p = |g_a(\sigma_a(z)) - g_a(0)|^p
\]

\[
= |g_a'(\eta)|^p|\sigma_a(z)|^p
\]

\[
= |\sigma_a(z)|^p \left| \frac{1}{2\pi} \int_{|\xi|=r} \frac{g_a(\xi)}{(\xi - \eta)^2} d\xi \right|^p
\]

for some \( \eta \) with \( |\eta| \leq |\sigma_a(z)| < r_1 \). Since \( |\xi| = \rho(\sigma_a(\xi), a) = r \), we get \( u := \sigma_a(\xi) \in \Delta(a, \delta) \). Thus

\[
|f(z) - f(a)|^p \leq \rho(a, z)^p \left( \frac{1}{2\pi} \int_{|\xi|=r} \left| \frac{g_a(\xi)}{(\xi - \eta)^2} \right| d\xi \right)^p
\]

\[
\leq \rho(a, z)^p \sup_{\xi \in \Delta(a, \delta)} |f(u)|^p. \tag{2}
\]

Using the subharmonicity of \( |f(u)|^p \), \( 1 - |u| \approx 1 - |\zeta| \) for \( \zeta \in \Delta(u, \delta) \), and

\[
\omega(S(\zeta)) \approx \omega(\zeta)(1 - |\zeta|) \approx \omega(\zeta)(1 - |\zeta|)^2,
\]

we get

\[
|f(u)|^p \leq \frac{1}{(1 - |u|^2)^2} \int_{\Delta(u, \delta)} |f(\zeta)|^p dA(\zeta)
\]

\[
\leq \int_{\Delta(u, \delta)} |f(\zeta)|^p \frac{\bar{\omega}(\zeta)}{\omega(S(\zeta))} dA(\zeta)
\]

\[
\leq \frac{1}{\omega(S(a))} \int_{\Delta(a, r_2)} |f(\zeta)|^p \bar{\omega}(\zeta) dA(\zeta), \tag{3}
\]

where we use the fact that \( \Delta(u, \delta) \subset \Delta(a, r_2) \) and

\[
\omega(S(a)) \approx \omega(S(\zeta)), \tag{4}
\]

for \( \zeta \in \Delta(a, r_2) \). Combining (2) and (3), we obtain

\[
|f(z) - f(a)|^p \leq C \rho(z, a)^p \frac{\int_{\Delta(a, r_2)} |f(\zeta)|^p \bar{\omega}(\zeta) dA(\zeta)}{\omega(S(a))}.
\]

The proof is complete. \( \square \)

By [25] Lemma 4.30, for all \( a, z, w \in \mathbb{D} \) with \( \rho(z, w) < r \) and any real \( s \), we have

\[
\left| 1 - \left( \frac{1 - \overline{az}}{1 - \overline{aw}} \right)^s \right| \leq C(s, r) \rho(z, w),
\]

and therefore, for all \( w, z, a \in \mathbb{D} \) with \( z \in \Delta(a, r) \) and any \( s > 0 \),

\[
\left| \frac{1}{1 - \overline{az}^s} - \frac{1}{1 - \overline{aw}^s} \right| \leq C(s, r) \rho(z, w) \left| \frac{1}{(1 - \overline{az})^s} \right|.
\]

Although the converse inequality does not hold, we have the following partial converse inequality (see [8] Theorem 2.8] or [21] Lemma 2.3]), which is crucial in the proof of the necessary part of Theorems 12 and 14.
Lemma D. Suppose \( s > 1 \) and \( 0 < r_0 < 1 \). Then there are \( N = N(r_0) > 1 \) and \( C = C(s, r_0) \) such that
\[
\left| \frac{1}{(1 - \overline{az})^s} - \frac{1}{(1 - \overline{aw})^s} \right| + \left| \frac{1}{(1 - t_N\overline{az})^s} - \frac{1}{(1 - t_N\overline{aw})^s} \right| \\
\geq C \rho(z, w) \frac{1}{|1 - \overline{az}|^s},
\]
for all \( z \in \Delta(a, r_0) \) with \( 1 - |a| < \frac{1}{2N} \), \( t_N = 1 - N(1 - |a|) \) and \( w \in \mathbb{D} \).

2.4. Carleson measure. Let \( \mu \) be a finite positive Borel measure on \( \mathbb{D} \). \( \mu \) is called a \( q \)-Carleson measure for \( A^p_\omega \) if the identity operator \( I_d : A^p_\omega \to L^q(d\mu) \) is bounded, i.e. there is a positive constant \( C > 0 \) such that
\[
\int_\mathbb{D} |f(z)|^q d\mu(z) \leq C \|f\|_{A^p_\omega}^q
\]
for any \( f \in A^p_\omega \). Also, \( \mu \) is called a vanishing \( q \)-Carleson measure if the identity operator \( I_d : A^p_\omega \to L^q(d\mu) \) is compact.

The characterization of (vanishing) \( q \)-Carleson measure for \( A^p_\omega \) has been solved for \( \omega \in \widehat{\mathbb{D}} \) \([13, 18]\). It is worth mentioning that the pseudohyperbolic disk is not the right one to describe the Carleson measure for \( A^p_\omega \) when \( \omega \in \mathbb{D} \), since for a fixed \( r > 0 \), the quantity \( \omega(\Delta(a, r)) \) may equal to zero for some \( a \) close to the boundary if \( \omega \in \mathbb{D} \) (see \([13]\)). However, if \( \omega \in \mathbb{D} \), we have the following characterization. The proof is similar with the proof of Theorem 2.1 in \([13]\). We give the proof here for completeness.

Theorem 4. Let \( \mu \) be a positive Borel measure on \( \mathbb{D} \), \( 0 < p < \infty \), \( \omega \in \mathbb{D} \) and \( 0 < r < 1 \). Then the following assertions hold:
(i) \( \mu \) is a \( p \)-Carleson measure for \( A^p_\omega \) if and only if
\[
\sup_{\omega \in \mathbb{D}} \frac{\mu(\Delta(a, r))}{\omega(S(a))} < \infty. \tag{5}
\]
(ii) \( \mu \) is a vanishing \( p \)-Carleson measure for \( A^p_\omega \) if and only if
\[
\lim_{|a| \to 1} \frac{\mu(\Delta(a, r))}{\omega(S(a))} = 0. \tag{6}
\]

Remark. In the above, \( \omega(S(a)) \) can be replaced by \( \omega(\Delta(a, r)) \) for any fixed \( r \in (0, 1) \) large enough.

Proof. (i) Assume first that \( \mu \) is a \( p \)-Carleson measure for \( A^p_\omega \). Consider the test functions
\[
f_\alpha(z) = \left( \frac{1 - |\alpha|^2}{1 - \overline{\alpha}z} \right)^{\frac{\gamma + 1}{\gamma}},
\]
where \( \gamma = \gamma(\omega) > 0 \) is chosen large enough. Then the assumption together with Lemma A yield
\[
\mu(\Delta(z, r)) \leq \int_{\Delta(z, r)} |f_\alpha(z)|^p d\mu(z) \leq \|f_\alpha\|_{A^p_\omega}^p \leq \omega(S(z)).
\]
Conversely, assume that (5) holds. By Fubini’s Theorem, Lemma C and the following well known estimate

\[ |f(z)|^p \leq \frac{1}{(1 - |z|^2)^2} \int_{\Delta(z,r)} |f(\zeta)|^p dA(\zeta), \quad z \in \mathbb{D}, \]

we have

\[
\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq \int_{\mathbb{D}} \left( \frac{1}{(1 - |z|^2)^2} \int_{\Delta(z,r)} |f(\zeta)|^p dA(\zeta) \right) d\mu(z)
\]

\[
= \int_{\mathbb{D}} |f(\zeta)|^p \frac{\mu(\Delta(\zeta, r))}{(1 - |\zeta|^2)^2} dA(\zeta)
\]

\[
\leq \int_{\mathbb{D}} |f(\zeta)|^p \frac{\omega(S(\zeta))}{(1 - |\zeta|^2)^2} dA(\zeta)
\]

\[
\leq \int_{\mathbb{D}} |f(\zeta)|^p \frac{\omega(\zeta)}{(1 - |\zeta|^2)^2} dA(\zeta)
\]

\[
\leq \|f\|_{A^p_\omega}^p.
\]

(ii) Assume first that \( \mu \) is a vanishing \( p \)-Carleson measure for \( A^p_\omega \). Following the proof of [13, Theorem 2.1(ii)], with Lemma B in hand, we get

\[
\lim_{|a| \to 1} \frac{\mu(\Delta(a, r))}{\omega(S(a))} = 0.
\]

Conversely, assume that (6) holds. Denote \( \mathbb{D}_s = \{ z \in \mathbb{D} : |z| < s \} \) and set

\[
d\mu_s(z) = \chi_{s \leq |z| < 1}(z)d\mu(z).
\]

We claim that (i) implies

\[
\|h\|_{L^q_\omega} \leq K_{\mu_s} \|h\|_{A^p_\omega}, \quad h \in A^p_\omega,
\]

where

\[
K_{\mu_s} = \sup_{a \in \mathbb{D}} \frac{\mu_s(\Delta(a, r))}{\omega(S(a))}.
\]

Following the proof of [13, Theorem 2.1(ii)], it remains to show that

\[
\lim_{s \to 1} K_{\mu_s} = \lim_{s \to 1} \left( \sup_{a \in \mathbb{D}} \frac{\mu_s(\Delta(a, r))}{\omega(S(a))} \right) = 0.
\]

Let \( t_r(s) = \frac{r - s}{r - 1} \). After an easy calculation, we get that \( \Delta(a, r) \cap (\mathbb{D} \setminus \mathbb{D}_s) \neq \emptyset \) if and only if \( |a| \geq t_r(s) \). It is easy to see that \( t_r(s) \) is continuous and increasing on \([r, 1)\), and \( \lim_{s \to 1} t_r(s) = 1 \). Thus,

\[
0 = \lim_{|a| \to 1} \sup_{|d| \geq t_r(s)} \frac{\mu(\Delta(a, r))}{\omega(S(a))} = \lim_{s \to 1} \sup_{|d| \geq t_r(s)} \frac{\mu(\Delta(a, r))}{\omega(S(a))}
\]

\[
\geq \lim_{s \to 1} \sup_{|d| \geq t_r(s)} \frac{\mu(\Delta(a, r)) \cap (\mathbb{D} \setminus \mathbb{D}_s))}{\omega(S(a))}
\]

\[
= \lim_{s \to 1} \sup_{a \in \mathbb{D}} \frac{\mu_s(\Delta(a, r))}{\omega(S(a))}.
\]

The proof is complete. \( \square \)
The connection between composition operators and Carleson measures comes from the standard identity

\[ \int_{\mathbb{D}} (f \circ \varphi)(z) \omega(z) dA(z) = \int_{\mathbb{D}} f(z) d\nu(z), \]

where \( \nu \) denotes the pullback measure defined by \( \nu(E) = \int_{\varphi^{-1}(E)} \omega(z) dA(z) \), for all Borel sets \( E \subset \mathbb{D} \). One can easily see from the above equality that \( C_\varphi : A^p_\omega \to A^p_\omega \) is bounded (compact) on \( A^p_\omega \) if and if \( \nu \) is a (vanishing \( p \)-Carleson measure) \( p \)-Carleson measure for \( A^p_\omega \).

The following result plays a fundamental role in this study. It can be proved by employing the method used by Moorhouse [10].

**Lemma 5.** Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \), \( \omega \in L^1(\mathbb{D}) \), and \( u \) to be a non-negative, bounded, measurable function on \( \mathbb{D} \). Define the measure \( \nu(E) = \int_{\varphi^{-1}(E)} u(z) \omega(z) dA(z) \) on all Borel subset \( E \) of \( \mathbb{D} \). If

\[ \lim_{|z| \to 1} u(z) \frac{1 - |z|}{1 - |\varphi(z)|} = 0, \]

then \( \nu \circ \varphi^{-1} \) is a vanishing \( p \)-Carleson measure for \( A^p_\omega \) and hence the inclusion map \( I_{p,\omega} : A^p_\omega \to L^p(\nu \circ \varphi^{-1}) \) is compact.

**Proof.** Fix \( r \in (0, 1) \). For \( a \in \mathbb{D} \), let

\[ \epsilon := \epsilon(a) = \sup_{z \in \varphi^{-1}(\Delta(a, r))} \frac{1 - |z|}{1 - |\varphi(z)|} u(z). \]

Using the Schwarz-Pick Theorem, one has

\[ \frac{1 - |z|}{1 - |\varphi(z)|} \leq \frac{1 - |\varphi(0)|}{1 - |\varphi(0)|} = C < \infty. \]

So that if \( \varphi(z) \in \Delta(a, r) \), then

\[ 1 - |z| \leq C(1 - |\varphi(z)|) \leq C \frac{(1 - |a|)(1 - r|a|)(1 + r)}{1 - r^2|a|^2}. \]

This implies that \( |z| \to 1 \) uniformly in \( z \in \varphi^{-1}(\Delta(a, r)) \) as \( |a| \to 1 \). By hypothesis \( \epsilon(a) \to 0 \) as \( |a| \to 1 \).

Now, fix \( 0 < \lambda < \min\{1, \alpha(\omega)\} \). Taking \( M \) to be an upper bound of \( u \), we have

\[ \nu \circ \varphi^{-1}(\Delta(a, r))) = \int_{\varphi^{-1}(\Delta(a, r))} u(z) \omega(z) dA(z) \]

\[ \leq \int_{\varphi^{-1}(\Delta(a, r))} \epsilon^4(1 - |\varphi(z)|)^4 u(z)^{1-\lambda} \omega(z) dA(z) \]

\[ \leq \epsilon^4 M^{1-\lambda} (1 - |a|)^4 \int_{\varphi^{-1}(\Delta(a, r))} \omega(z)^{(1-\lambda)} dA(z). \]
Denote \( \omega_\lambda = \frac{\omega(z)}{(1-|z|)^\lambda} \). By Lemma 2, we get \( \omega_\lambda \in D \). Therefore, \( C_\varphi : A^{p}_{\omega_\lambda} \to A^{p}_{\omega_\lambda} \) is bounded, that is

\[
(1 - |a|)^1 \int_{\varphi^{-1}(\Delta(a, r))} \frac{\omega(z)}{(1-|z|)^1} dA(z) \leq (1 - |a|)^1 \omega_\lambda(\Delta(a, r)) \times \frac{\overline{\omega}(a)(1 - |a|)^{1+\lambda}}{\omega_\lambda(1 - |a|)} \times \omega(\Delta(a, r)).
\]

Therefore

\[
\frac{\nu \circ \varphi^{-1}(\Delta(a, r))}{\omega(\Delta(a, r))} \leq e(a)^{1-\lambda}
\]

for all \( a \in \mathbb{D} \), and hence we conclude that \( \nu \circ \varphi^{-1} \) is a vanishing \( p \)-Carleson measure for \( A^p_\nu \). The proof is complete. \( \square \)

### 2.5. Angular Derivative

Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \). We say that \( \varphi \) has an angular derivative, denoted by \( \varphi'(\zeta) \in \mathbb{C} \), at \( \zeta \in \partial \mathbb{D} \) if \( \varphi \) has nontangential limit \( \varphi(\zeta) \in \partial \mathbb{D} \) such that

\[
\zeta \operatorname{lim}_{z \to \zeta} \frac{\varphi(z) - \eta}{z - \zeta} = \varphi'(\zeta),
\]

where \( \zeta \operatorname{lim} \) stands for the nontangential limit. We denote by \( F(\varphi) \) the set of all boundary points at which \( \varphi \) has finite angular derivatives. Note from the Julia-Carathéodory Theorem (see [4, Theorem 2.44]) that

\[
F(\varphi) = \left\{ \zeta \in \partial \mathbb{D} : d_{\varphi}(\zeta) := \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty \right\}.
\]

For \( \zeta \in F(\varphi) \), we call the vector

\[
\mathcal{D}(\varphi, \zeta) := (\varphi(\zeta), d_{\varphi}(\zeta)) \in \partial \mathbb{D} \times \mathbb{R}^+
\]

the first-order data of \( \varphi \) at \( \zeta \).

If \( \varphi \) and \( \psi \) are two analytic self-maps of the disk with finite angular derivative at \( \mathbb{D} \), we say that \( \varphi \) and \( \psi \) have the same first-order data at \( \zeta \) if \( \mathcal{D}(\varphi, \zeta) = \mathcal{D}(\psi, \zeta) \).

### 3. Linear Combination of Composition Operators

For a linear operator \( T : X \to Y \), the essential norm of \( T \), denoted by \( \|T\|_{e,X \to Y} \), is defined by

\[
\|T\|_{e,X \to Y} = \inf \{ \|T - K\|_{X \to Y} : K \text{ is compact from } X \text{ to } Y \}.
\]

It is obvious that the operator \( T \) is compact if and only if \( \|T\|_{e,X \to Y} = 0 \).

We have the following lower estimates for the essential norm of a linear combination of composition operators acting on Bergman spaces with doubling weight.
Lemma 6. Let $0 < p < \infty$ and $\omega \in \overline{D}$. Let $\varphi_1, \ldots, \varphi_n$ be finitely many analytic self-maps of $D$. Then there is a constant $C > 0$ and $\gamma = \gamma(\omega)$ is sufficiently large such that

$$\left\| \sum_{j=1}^{n} \lambda_j C_{\varphi_j} \right\|_{A_\omega^p}^p \geq C \limsup_{|\phi| \to 1} \left\| \sum_{j=1}^{n} \lambda_j (C_{\varphi_j} f_a) \right\|_{A_\omega^p}^p,$$

where $f_a(z) = \left( \frac{1 - |z|^2}{1 - \rho^2} \right)^{\frac{m+1}{\gamma}} \omega(S(\rho))^\frac{1}{\gamma}$.

Proof. Let $K$ be a compact operator on $A_\omega^p$. Consider the operator on $H(D)$ defined by

$$K_m(f)(z) = f(\frac{m}{m+1}z), \ m \in \mathbb{N}.$$

Denote $R_m = I - K_m$. It is easy to see that $K_m$ is compact on $A_\omega^p$ (see [15, Theorem 15]) and

$$\|K_m\|_{A_\omega^p} \leq 1, \ |R_m|_{A_\omega^p} \leq 2$$

for any positive integer $m$. Then we have

$$2 \left\| \sum_{j=1}^{n} \lambda_j C_{\varphi_j} - K \right\|_{A_\omega^p}^p \geq \|R_m \circ \left( \sum_{j=1}^{n} \lambda_j C_{\varphi_j} - K \right)\|_{A_\omega^p}^p \geq \sup_{\omega \in D} \|R_m \circ \left( \sum_{j=1}^{n} \lambda_j C_{\varphi_j} - K \right)(f_a)\|_{A_\omega^p}^p.$$

Since $K$ is compact, we can extract a sequence $\{a_i\} \subset D$ such that $|a_i| \to 1$ and $K f_{a_i}$ converges to some $f \in A_\omega^p$. So,

$$\left\| R_m \circ \left( \sum_{j=1}^{n} \lambda_j C_{\varphi_j} - K \right)(f_{a_i}) \right\|_{A_\omega^p}^p \geq \sup_{\omega \in D} \left\| R_m \circ \left( \sum_{j=1}^{n} \lambda_j C_{\varphi_j} - K \right)(f_{a_i}) \right\|_{A_\omega^p}^p.$$

Since $K_m$ is compact and $\sum_{j=1}^{n} \lambda_j C_{\varphi_j}$ is bounded on $A_\omega^p$, we have $K_m \circ \left( \sum_{j=1}^{n} \lambda_j C_{\varphi_j} \right)$ is compact on $A_\omega^p$. Therefore, letting $i \to \infty$ and then using Fatou’s Lemma as $m \to \infty$ in (7), we have

$$\left\| \sum_{j=1}^{n} \lambda_j C_{\varphi_j} - K \right\|_{A_\omega^p}^p \geq \limsup_{i \to \infty} \left\| \sum_{j=1}^{n} \lambda_j (C_{\varphi_j} f_{a_i}) \right\|_{A_\omega^p}^p.$$

Therefore,

$$\left\| \sum_{j=1}^{n} \lambda_j C_{\varphi_j} \right\|_{A_\omega^p}^p \geq C \limsup_{|\phi| \to 1} \left\| \sum_{j=1}^{n} \lambda_j (C_{\varphi_j} f_a) \right\|_{A_\omega^p}^p.$$
The proof is complete. □

For $M > 1$ and $\zeta \in \partial \mathbb{D}$, we denote by $\Gamma_{M,\zeta}$ the $\zeta$-curve consisting of points $|z - \zeta| = M(1 - |z|^2)$, the boundary of a non-tangential approach region with vertex at $\zeta$. We will use the notation “$\lim_{M \to \infty} \Gamma_{M,\zeta}$” to indicate a limit taken as $z \to \zeta$ along the standard leg of $\Gamma_{M,\zeta}$. The following result taken from [7].

**Lemma E.** Let $\varphi$ and $\psi$ be analytic self-maps of $\mathbb{D}$. Then the following equality

$$
\lim_{M \to \infty} \lim_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - \varphi(z)\psi(z)} = \begin{cases} 
1, & \text{if } \zeta \in F(\varphi) \text{ and } D(\varphi, \zeta) = D(\psi, \zeta) \\
0, & \text{otherwise}
\end{cases}
$$

holds for $\zeta \in F(\varphi)$.

We are now ready to establish a lower estimate for the essential norm of a general linear combination of composition operators acting on $A^p_\omega$ when $\omega \in \widehat{\mathbb{D}}$. Let $\varphi_1, \ldots, \varphi_n$ be finitely many analytic self-maps of $\mathbb{D}$. For $\varphi \in F(\varphi_i)$, we denote by $J(\varphi)$ the set of all indices $j$ for which $\zeta \in F(\varphi)$ and $\varphi_j$ and $\varphi$ have the same first-order data at $\zeta$.

**Theorem 8.** Let $0 < p < \infty$ and $\omega \in \widehat{\mathbb{D}}$. Let $\varphi_1, \ldots, \varphi_n$ be finitely many analytic self-maps of $\mathbb{D}$. Then there is a constant $C(p, \omega) > 0$ such that

$$
\left\| \sum_{j=1}^n \lambda_j C_{\varphi_j} \right\|_{e,A^p_\omega}^p \geq C \max_{1 \leq i \leq n} \left( \frac{1}{\omega(\zeta)\beta+1} \right)
$$

for all $\zeta \in \partial \mathbb{D}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. In case $\zeta \notin F(\varphi_i)$ the quantity inside the parenthesis above is to be understood as 0.

**Proof.** We denote $T := \sum_{j=1}^n \lambda_j C_{\varphi_j}$ and $f_a(z) = \frac{1 - |a|^2}{1 - \overline{a}z} \omega(S(a))^{-\frac{1}{p}}$, for $a \in \mathbb{D}$ and $\gamma$ is that of Lemma A. Fix any index $i$ such that $\zeta \in F(\varphi_i)$. We have $|\varphi_i(z)| \to 1$ as $z \to \zeta$ along any $\Gamma_{M,\zeta}$ which is a restricted $\zeta$-curve. So, by Lemma 6, we obtain

$$
\|T\|_{e,A^p_\omega} \geq \sup_M \left( \lim_{z \to \zeta} \|Tf_{\varphi_i(z)}\|_{A^p_\omega}^p \right).
$$

Meanwhile, note that

$$
\|Tf_{\varphi_i(z)}\|_{A^p_\omega}^p \geq \|Tf_{\varphi_i(z)}(z)\|^p \omega(S(z))
$$

$$
= \left| \sum_{j=1}^n \lambda_j \frac{1 - |\varphi_i(z)|^2}{1 - \varphi_i(z)\varphi_j(z)} \right|^p \omega(S(z)) \omega(S(\varphi_i(z))).
$$

Thus, applying Lemma E, Lemmas 1 and 6, we get the desired result. □

By Theorem 8, we immediately yield the following three corollaries for the compactness of linear combinations.
Corollary 9. Let $0 < p < \infty$ and $\omega \in \overline{D}$. Let $\varphi_1, \ldots, \varphi_n$ be finitely many analytic self-maps of $\mathbb{D}$. If $\sum_{j=1}^{n} \lambda_j C_{\varphi_j}$ is compact on $A^p_{\omega}$, then
\[
\sum_{\zeta \in F(\varphi)} \lambda_j = 0
\]
for all $\zeta \in \partial \mathbb{D}$ and $(\zeta, s) \in \partial \mathbb{D} \times \mathbb{R}_+$.

Corollary 10. Let $0 < p < \infty$ and $\omega \in \overline{D}$. Let $\varphi, \psi$ be analytic self-maps of $\mathbb{D}$. Suppose both $C_{\varphi}$ and $C_{\psi}$ are not compact on $A^p_{\omega}$. If $aC_{\varphi} + bC_{\psi}$ is compact on $A^p_{\omega}$, then the following statements hold:
(i) $a + b = 0$;
(ii) $F(\varphi) = F(\psi)$;
(iii) $D(\varphi, \zeta) = D(\psi, \zeta)$ for each $\zeta \in F(\varphi)$.

Corollary 11. Let $0 < p < \infty$ and $\omega \in \overline{D}$. Let $\varphi, \varphi_1, \ldots, \varphi_n$ be finitely many analytic self-maps of $\mathbb{D}$. If $C_{\varphi} - C_{\varphi_1} - C_{\varphi_2} - \cdots - C_{\varphi_n}$ is compact on $A^p_{\omega}$, then the following statements hold:
(i) $F(\varphi_1), \ldots, F(\varphi_n)$ are pairwise disjoint and $F(\varphi) = \bigcup_{j=1}^{n} F(\varphi_j)$
(ii) $D(\varphi, \zeta) = D(\varphi_j, \zeta)$ at each $\zeta \in F(\varphi_j)$ for $j = 1, \ldots, n$.

4. COMPACT DIFFERENCE AND FURTHER RELATED RESULTS

We have the following characterization for compact linear combinations of two composition operators.

Theorem 12. Let $0 < p < \infty$ and $\omega \in \overline{D}$. Suppose $\varphi$ and $\psi$ be analytic self-maps of $\mathbb{D}$. Then $\lambda_1 C_{\varphi} + \lambda_2 C_{\psi}$ is compact on $A^p_{\omega}$ if and only if either one of the following two conditions holds:
(i) Both $C_{\varphi}$ and $C_{\psi}$ are compact;
(ii) $\lambda_1 + \lambda_2 = 0$ and
\[
\lim_{|z| \to 1} \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\psi(z)|^2} \right) \rho(\varphi(z), \psi(z)) = 0. \tag{10}
\]

Proof. Suppose that $\lambda_1 C_{\varphi} + \lambda_2 C_{\psi}$ is compact on $A^p_{\omega}$. Note that if (i) fails, then one of $C_{\varphi}$ and $C_{\psi}$ is not compact on $A^p_{\omega}$. We may assume that both $C_{\varphi}$ and $C_{\psi}$ are not compact on $A^p_{\omega}$ and show (ii). By Corollary 10, we have $\lambda_1 + \lambda_2 = 0$ and hence we may assume that $\lambda_1 = 1$ and $\lambda_2 = -1$. We assume that (10) does not hold. Then there exists a sequence $\{z_n\} \subset \mathbb{D}$ with $|z_n| \to 1$ such that either
\[
a_n = \frac{1 - |z_n|}{1 - |\varphi(z_n)|} \rho(\varphi(z_n), \psi(z_n))
\]
or
\[
b_n = \frac{1 - |z_n|}{1 - |\psi(z_n)|} \rho(\varphi(z_n), \psi(z_n))
\]
does not converge to zero. By passing to a subsequence, we may assume that $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$ exist and that one is non-zero. Without loss of generality we may further assume that $a \neq 0$. Again by passing to a subsequence, we
may assume that \( c = \lim_{n \to \infty} |\varphi(z_n)| \) exist. Since \( a \neq 0 \), we have \( c = 1 \). Thus, we may assume that \( |z_n| \to 1 \), \( |\varphi(z_n)| \to 1 \) and \( a = \lim_{n \to \infty} a_n \) exists and non-zero. For \( u \in \mathbb{D} \), consider the test functions

\[
g_u(z) = \left( \frac{1 - |u|^2}{1 - \overline{u}z} \right)^{\frac{\beta + 1}{p}} \omega(S(u))^{-\frac{1}{p}},
\]

and

\[
h_u(z) = \left( \frac{1 - |u|^2}{1 - t_N \overline{u}z} \right)^{\frac{\beta + 1}{p}} \omega(S(u))^{-\frac{1}{p}}.
\]

It is easy to see that \( \|g_u\|_{A^p} = \|h_u\|_{A^p} \approx 1 \) and \( g_u \to 0 \), \( h_u \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( |u| \to 1 \). Therefore,

\[
\lim_{n \to \infty} \| (C_\varphi - C_\psi) g_{\varphi(z_n)} \|_{A^p} = 0
\]

and

\[
\lim_{n \to \infty} \| (C_\varphi - C_\psi) h_{\varphi(z_n)} \|_{A^p} = 0.
\]

Since

\[
\omega(S(z)) |f(z)|^p \lesssim \|f\|^p_{A^p}, \quad \text{for all} \quad f \in A^p_\omega,
\]

we have

\[
\lim_{n \to \infty} \omega(S(z_n)) \left( |g_{\varphi(z_n)}(\varphi(z_n)) - g_{\varphi(z_n)}(\psi(z_n))| + |h_{\varphi(z_n)}(\varphi(z_n)) - h_{\varphi(z_n)}(\psi(z_n))| \right) = 0.
\]

Then Lemma D yeids

\[
\lim_{n \to \infty} \frac{\omega(S(z_n))}{\omega(S(\varphi(z_n)))} \rho(\varphi(z_n), \psi(z_n))^p = 0.
\]

Therefore, by Lemma 1, we obtain that

\[
\lim_{n \to \infty} \left( \frac{1 - |z_n|}{1 - |\varphi(z_n)|} \right)^{\beta + 1} \rho(\varphi(z_n), \psi(z_n))^p = 0.
\]

Since the two sequences \( \left\{ \frac{1 - |z_n|}{1 - |\varphi(z_n)|} \right\} \) and \( \{\rho(\varphi(z_n), \psi(z_n))\} \) are both bounded. Thus, we obtain

\[
a = \lim_{n \to \infty} \left( \frac{1 - |z_n|}{1 - |\varphi(z_n)|} \right) \rho(\varphi(z_n), \psi(z_n)) = 0,
\]

which is a desired contradiction.

Conversely, we only have to prove (10) implies that \( C_\varphi - C_\psi \) is compact. Let \( \{f_k\} \) be an arbitrary bounded sequence in \( A^p_\omega \) such that \( f_k \to 0 \) uniformly on compact subsets of \( \mathbb{D} \). It suffices to show that

\[
\|(C_\varphi - C_\psi) f_k\|_{A^p} \to 0,
\]

as \( k \to \infty \). In order to prove this, give \( 0 < r < 1 \), we put

\[
E := \{z \in \mathbb{D} : \rho(\varphi(z), \psi(z)) < r\} \quad \text{and} \quad F := \mathbb{D} \setminus E.
\]
Then for each $k$, 
$$\| (C_\varphi - C_\psi)f_k \|_{A_p^\omega}^p = \int_D |f_k(\varphi(z)) - f_k(\psi(z))|^p \omega(z)dA(z)$$

$$= \int_E |f_k(\varphi(z)) - f_k(\psi(z))|^p \omega(z)dA(z) + \int_F |f_k(\varphi(z)) - f_k(\psi(z))|^p \omega(z)dA(z) \quad (11)$$

We first estimate the second term in the right-hand side of the equality (11). Let $\chi_F$ denote the characteristic function of $F$. Since $r\chi_F \leq \rho(\varphi, \psi)$, by (10), we get

$$\lim_{|z| \to 1} \chi_F(z) \left| \frac{1 - |z|}{1 - |\varphi(z)|} \right| + \left| \frac{1 - |z|}{1 - |\psi(z)|} \right| = 0.$$  

This, together with Lemma 5, yields

$$\int_E |f_k(\varphi(z)) - f_k(\psi(z))|^p \omega(z)dA(z)$$

$$\leq \int_D |f_k(\varphi(z))|^p \chi_F(z) \omega(z)dA(z) + \int_D |f_k(\psi(z))|^p \chi_F(z) \omega(z)dA(z)$$

$$:= \int_D |f_k(z)|^p \nu_1(z) + \int_D |f_k(z)|^p \nu_2(z) \to 0,$$

as $k \to \infty$, where

$$\nu_1(K) = \int_{\varphi^{-1}(K)} \chi_F(z) \omega(z)dA(z) \quad \text{and} \quad \nu_2(K) = \int_{\psi^{-1}(K)} \chi_F(z) \omega(z)dA(z),$$

for all Borel set $K \subset \mathbb{D}$.

Next, we estimate the first term in the right-hand side of the equality (11). Using Lemma 3, Fubini’s Theorem, inequality (4), Theorem 4 and Lemma C, we have

$$\int_E \rho(\varphi(z), (\psi(z)))^p \Delta(\varphi(z), \psi(z)) \frac{|f_k(\xi)|^p \omega(\xi)dA(\xi)}{\omega(S(\varphi(z)))} \omega(z)dA(z)$$

$$\leq r \int_D |f_k(\xi)|^p \frac{\int_S \omega(z)dA(z)}{\omega(S(\xi))} dA(\xi)$$

$$\leq r^p ||f_k||_{A_p^\omega}^p \| C_\varphi \|$$

Letting $r \to 0$, we get

$$\| (C_\varphi - C_\psi)f_k \|_{A_p^\omega} \to 0.$$  

The proof is complete. □

As a corollary, we obtain the following characterization for the operator $C_\varphi - C_\psi : A_p^\omega \to A_p^\omega$. The compactness of $C_\varphi - C_\psi$ on $A_p^\omega$ is independent of $p$ and $\omega$, whenever $\omega \in \mathcal{D}$. 


Corollary 13. Let $0 < p < \infty$ and $\omega \in \mathcal{D}$. Suppose $\varphi$ and $\psi$ are analytic self-maps of $\mathcal{D}$. Then the operator $C_\varphi - C_\psi : A_\omega^p \to A_\omega^p$ is compact if and only if

$$\lim_{|z| \to 1} \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\psi(z)|^2} \right) \rho(\varphi(z), \psi(z)) = 0.$$ 

In the rest of this section we assume that $\varphi_i : \mathcal{D} \to \mathcal{D}$ is analytic and $\varphi_i \neq \varphi_j$ if $i \neq j$. We define

$$F_i := \{ \zeta \in \partial \mathcal{D} : \varphi_i \text{ has a finite angular derivative at } \zeta \}$$

and

$$\rho_{ij}(z) := \frac{|\varphi_i(z) - \varphi_j(z)|}{1 - \overline{\varphi_i(z)} \varphi_j(z)}.$$ 

The proof of the following Theorem will be quite similar to the proof of Theorem 12, with a few added complications.

Theorem 14. Let $0 < p < \infty$ and $\omega \in \mathcal{D}$. Let $\varphi, \varphi_1, \ldots, \varphi_n$ be finitely many analytic self-maps of $\mathcal{D}$. Suppose that $C_{\varphi}, C_{\varphi_1}, \ldots, C_{\varphi_n}$ are not compact on $A_\omega^p$. Then the operator $C_\varphi - C_{\varphi_1} - \cdots - C_{\varphi_n} : A_\omega^p \to A_\omega^p$ is compact if and only if the following two conditions hold.

(i) $F = \bigcup_{j=1}^n F_j$ and $F_i \cap F_j = \emptyset$ if $i \neq j$ with $i, j \geq 1$;
(ii) 

$$\lim_{|z| \to \zeta} \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\varphi_j(z)|^2} \right) \rho(\varphi(z), \varphi_j(z)) = 0$$ 

for all $\zeta \in F(\varphi_j)$ for $j = 1, 2, \ldots, n$.

Proof. For the simplicity of notation, we put $T = \sum_{j=1}^n C_{\varphi_j}$. If $C_\varphi - T$ is compact on $A_\omega^p$, then by Corollary 11, (i) holds. Now, assume that (ii) fails. We will derive a contradiction.

Since (ii) fails, there exist $\zeta \in F(\varphi_j)$ for some $j$ and a sequence $\{ z_k \} \subset \mathcal{D}$ such that $z_k \to \zeta$ and

$$\lim_{k \to \infty} \rho(\varphi(z_k), \varphi_j(z_k)) \left( \frac{1 - |z_k|^2}{1 - |\varphi(z_k)|^2} + \frac{1 - |z_k|^2}{1 - |\varphi_j(z_k)|^2} \right) > 0.$$ 

By passing to a subsequence, we may assume that

$$a_k := \rho(\varphi(z_k), \varphi_j(z_k)) \frac{1 - |z_k|^2}{1 - |\varphi(z_k)|^2}$$

or

$$b_k := \rho(\varphi(z_k), \varphi_j(z_k)) \frac{1 - |z_k|^2}{1 - |\varphi_j(z_k)|^2}$$

does not converge to zero.

Without loss of generality, we assume that $a_k$ does not converge to zero. We take $g_k := g_{\varphi(z_k)}$ and $h_k := h_{\varphi(z_k)}$, for each $k$. Note that the two sequences both
\{\rho(\varphi(z_k), \varphi_j(z_k))\} and \{\frac{1-|z_k|^2}{1-|\varphi(z_k)|^2}\} are bounded. Thus, by passing to another subsequences if necessary, we may further assume that
\[\lim_{k \to \infty} \rho(\varphi(z_k), \varphi_j(z_k)) = c_1\] and
\[\lim_{k \to \infty} \frac{1-|z_k|^2}{1-|\varphi(z_k)|^2} = c_2,\]
for some constant \(c_1, c_2 > 0\) with \(c_1 \leq 1\).

Also, note that \(\zeta \notin F(\varphi_i)\) for \(i \neq j\). By the Julia-Caratheodory Theorem, we have
\[
\lim_{k \to \infty} \frac{1-|z_k|}{1-|\varphi_i(z_k)|} = 0, \quad i \neq j,
\]

\[
\begin{align*}
\lim_{k \to \infty} \omega(S(z_k))|g_k(\varphi_i(z_k))|^p & = \lim_{k \to \infty} \frac{\omega(S(z_k))}{\omega(S(\varphi_i(z_k)))} \frac{1-|\varphi(z_k)|^2}{1-|\varphi(z_k)\varphi_i(z_k)|}^{y+1} \\
& \leq \lim_{k \to \infty} \left( \frac{1-|z_k|}{1-|\varphi_i(z_k)|} \right)^{y+2} = 0.
\end{align*}
\]

\[
\begin{align*}
\lim_{k \to \infty} \omega(S(z_k))|h_k(\varphi_i(z_k))|^p & = \lim_{k \to \infty} \frac{\omega(S(z_k))}{\omega(S(\varphi_i(z_k)))} \frac{1-|\varphi(z_k)|^2}{1-t_N\varphi(z_k)\varphi_i(z_k)}^{y+1} \\
& \leq \lim_{k \to \infty} \left( \frac{1-|z_k|}{1-|\varphi_i(z_k)|} \right)^{\alpha+2} \left( \frac{1-|z_k|}{1-t_N|\varphi_i(z_k)|} \right)^{y+1} \\
& \leq \lim_{k \to \infty} \left( \frac{1-|z_k|}{1-|\varphi_i(z_k)|} \right)^{\alpha+2} = 0.
\end{align*}
\]

The same argument as in the proof of Theorem 12 yields
\[
\lim_{k \to \infty} \omega(S(z_k)) |(g_k(\varphi(z_k)) - (Tg_k)(z_k))|^p + |h_k(\varphi(z_k)) - (Th_k)(z_k)|^p = 0.
\]

Thus, the same argument as in the proof of Theorem 12 yields
\[
\lim_{k \to \infty} \left( \frac{1-|z_k|}{1-|\varphi(z_k)|} \right) \rho(\varphi(z_k), \varphi_j(z_k)) = 0,
\]
which is a desired contradiction.

Assume next that both (i) and (ii) hold. We will prove that \(C_{\varphi} - T\) is compact. The proof will be quite similar to the proof of Theorem 12. Define
\[
D_i := \left\{ z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} \geq \frac{1-|z|^2}{1-|\varphi(z)|^2}, \quad \text{for all } j \neq i \right\}
\]
for \(i = 1, \ldots, N\). Fix \(0 < r < 1\) and define
\[
E_i := \{ z \in D_i : \rho(\varphi(z), \varphi(z)) < r \} \quad \text{and} \quad E_i' := D_i \setminus E_i.
\]
By the proof of [10, Theorem 5], we get
\[
\lim_{|z| \to 1} \chi_{E_i'}(z) \left( \frac{1-|z|}{1-|\varphi(z)|} + \frac{1-|z|}{1-|\varphi_j(z)|} \right) = 0, \quad \text{for all } i, j, \tag{12}
\]
and

\[
\lim_{i \to 1} \chi_{E_i}(z) \frac{1 - |z|}{1 - |\varphi_j(z)|} = 0, \quad \text{whenever } i \neq j.
\]  

(13)

Now, let \( \{ f_k \} \) be a bounded sequence in \( A^p_\omega \) such that \( f_k \to 0 \) uniformly on compact subset of \( \mathbb{D} \). Since \( \mathbb{D} = \cup_{i=1}^n D_i \), we have

\[
\|(C_\varphi - T) f_k\|_{A^p_\omega}^p = \int_D |f_k \circ \varphi - \sum_{i=1}^n f_k \circ \varphi_i|^p \omega dA \leq \sum_{i=1}^n \int_{E_i} + \sum_{j \neq i} \int_{E_i}.
\]

Note, as in the proof of Theorem 12, that the second sum of the above tends to 0 as \( k \to \infty \), by equality (12) and Lemma 5. For the \( i \)-th term of the first sum, we have

\[
\int_{E_i} \lesssim \int_{E_i} |f_k \circ \varphi - f_k \circ \varphi_i|^p \omega dA + \sum_{j \neq i} \int_{E_i} |f_k \circ \varphi_j|^p \omega dA.
\]

Note from equality (13) and Lemma 5 that the second term of the above tends to 0 as \( k \to \infty \). Finally, from the proof of Theorem 12 we see that the first term of the above is dominated by \( r^p \). So, we conclude that

\[
\limsup_{k \to \infty} \|(C_\varphi - T) f_k\|_{A^p_\omega}^p \leq r^p.
\]

Letting \( r \to 0 \), we obtain \( \limsup_{k \to \infty} \|(C_\varphi - T) f_k\|_{A^p_\omega}^p = 0 \). The proof is complete. \( \square \)

Theorem 14 and Corollary 9 immediately yield the following characterization for a composition operator to be equal module compact operators to a linear combination of composition operators.

**Theorem 15.** Let \( 0 < p < \infty \) and \( \omega \in \mathbb{D} \). Let \( \varphi, \varphi_1, \ldots, \varphi_n \) be finitely many analytic self-maps of \( \mathbb{D} \). Suppose that \( C_\varphi, C_{\varphi_1}, \ldots, C_{\varphi_n} \) are not compact on \( A^p_\omega \). Let \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \setminus \{0\} \). Then the operator \( C_\varphi - \sum_{j=1}^n \lambda_j C_{\varphi_j} : A^p_\omega \to A^p_\omega \) is compact if and only if the following three conditions holds:

1. \( \lambda_1 = \ldots = \lambda_n = 1; \)
2. \( F = \bigcup_{j=1}^n F_j \) and \( F_i \cap F_j = \emptyset \) if \( i \neq j \) with \( i, j \geq 1; \)
3. \( \lim_{\varepsilon \to 0} \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\varphi_j(z)|^2} \right) \rho(\varphi(z), \varphi_j(z)) = 0 \)

for all \( \zeta \in F_j \) for \( j = 1, 2, \ldots, n. \)

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