Separating Functions,
Spectral Graph Theory
and Locally Scalar
Representations in Hilbert Spaces

I. K. Redchuk †

† Institute of Mathematics of National Academy of Sciences of Ukraine,
Tereshchenkovska str., 3, Kiev, Ukraine, ind. 01601
E-mail: red@imath.kiev.ua

1 Separating Functions \( \rho \)

The present paper is dedicated to studying the connections of separating functions \( \rho \), introduced in \cite{1}, with locally scalar representations of graphs on the one hand, and with spectral graph theory on the other hand.

In the article \cite{2} the function \( P(S) \), attaching to each partially ordered set \( S \) a positive rational number, was introduced. In terms of function \( P \) the criterion of finite presentability and tameness of an arbitrary partially ordered set is formulated in \cite{2}: a partially ordered set \( S \) is finite presented (tame) iff \( P(S) < 4 \) \( (P(S) = 4) \). If partially ordered set \( S \) is the union of \( s \) disjoint chains (i. e. such that any two elements belonging to different chains are incomparable) and \( n_i \) is the number of elements in each chain, \( i = 1, s \), then \( P(S) = \rho(n_1, n_2, \ldots, n_s) = \sum_{i=1}^{s} \rho(n_i) = \sum_{i=1}^{s} 1 + \frac{n_i-1}{n_i+1}, n_i \in \mathbb{N} \). The list of all solutions of the equations \( \rho(n_1, n_2, \ldots, n_s) = 4 \) is:

\[
(1,1,1,1), \ (2,2,2), \ (1,3,3), \ (1,2,5).
\] (1)

Since the function \( \rho \) is increasing (the partial order on integer vectors \( (n_1, n_2, \ldots, n_s) \) is defined naturally), one may easily obtain from the list (1) the list of all solutions of the inequality \( \rho(n_1, n_2, \ldots, n_s) < 4 \):

\[
(1,2,2), \ (1,2,3), \ (1,2,4), \ (1,1,k), \ (l,m), \ (n),
\] (2)

where \( k, l, m, n \) are arbitrary natural numbers.

It is remarkable that the list (1) corresponds to all extended Dynkin graphs with one point of branching (the exact definition see in s. 4 of the present paper) \( \tilde{D}_4, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \): the components of vectors from the list (1) correspond to the number of vertices on each branch. In the same way the list (2) describes Dynkin graphs \( E_6, E_7, E_8, D_n, A_n \).
For algebras defined by generators and linear or polylinear relations (see \[4\]) the general formulas for dimensions can be obtained and criterions of finite dimensionality and polynomiality of growth can be also formulated in terms of the function $\rho$.

In \[1\] the natural generalization of the function $\rho$ was suggested. Given $m \in \mathbb{N}$, denote $V_m$ the set of unordered suits of $m$ nonnegative integers; then denote $V = \bigcup_{m \in \mathbb{N}} V_m$. Let for given $r \in \mathbb{R}$, $r \geq 4$ a recurrent number sequence $\{u_i\}$ is defined: $u_0 = 0$, $u_1 = 1$, $u_{i+2} = (r - 2)u_{i+1} - u_i$. Put

$$
\rho_r(0) = 0,
\rho_r(n) = 1 + \frac{u_{n-1}}{u_n+1}, \quad n \in \mathbb{N},
$$

(3)

and for $\overline{v} \in V$, $\overline{v} = (n_1, n_2, \ldots, n_s)$

$$
\rho_r(\overline{v}) = \sum_{i=1}^{s} \rho_r(n_i).
$$

(1)

It is easy to see that $\rho_4 \equiv \rho$.

The functions $\rho_r$ was exploited for describing the standard characters of locally scalar representations of certain types of graphs (see \[1\], \[5\]).

In \[1\] the following properties of the functions $\rho_r$ was obtained:

**Proposition 1.1.** For any $r > 4$

$$
\rho_r(n) = \frac{(\lambda + 1)(\lambda^n - 1)}{\lambda^{n+1} - 1},
$$

(4)

where $\lambda = \frac{r-2+\sqrt{r^2-4r}}{2}$.

**Proposition 1.2.** Given $r \in \mathbb{Z}$, the equation

$$
\rho_r(n_1, n_2, \ldots, n_s) = r
$$

(5)

has following solutions:

$$
(1,1,\ldots,1), \quad (2,2,\ldots,2), \quad (1,3,3,\ldots,3), \quad (1,2,5,5,\ldots,5).
$$

For any $r \in \mathbb{Q} \setminus \mathbb{Z}$ the equation (5) has no solutions.

Here we will give the simpler way of defining the functions $\rho_r$. Let us show that

$$
\rho_r(n + 1) = \frac{r}{r - \rho_r(n)}
$$

(6)

for any $n \in \mathbb{N} \cup \{0\}$.

\[1\]The given definition slightly differs from one in \[1\]: $\rho_r(n)$ here corresponds to $\rho_{r-4}$ in \[1\].
Indeed, by formula (1) we have \( r_{r-\rho_r(n)} = \frac{r}{r-(\lambda^n+1)(\lambda^{n-1})}; \) since \( r = \frac{(\lambda+1)^2}{\lambda} \) and \( \lambda \neq 1, \) the last equality reduces to form \( \frac{(\lambda+1)(\lambda^{n+1+1}-1)}{\lambda^{n+2+1}-1} = \rho_r(n+1), \) which required.

Further we will define the functions \( \rho_r \) by the formulas (3) for \( r \geq 1, r \in \mathbb{R} \) (the starting condition is the same: \( \rho(0) = 0. \) However for \( 1 \leq r < 4 \) we may obtain \( \lambda^{n+1} = 1 \) and \( \rho_r(n) \) will be not defined. The equality \( \lambda^{n+1} = 1 \) is equivalent to \( \lambda = \cos \frac{2\pi k}{n+1} + i \sin \frac{2\pi k}{n+1}, \; k = 1, n \) \( (\lambda \neq 1), \) which means that \( r = 4 \cos^2 \frac{\pi k}{n+1}, \; k = 1, n. \) When \( r = 4 \cos^2 \frac{\pi k}{n+1}, \; k = 1, n \) we will formally put \( \rho_r(n) = \infty, \) and all rational transformations with such \( \rho_r(n) \) we will do using the natural transition to the limit. In particular, if \( \rho_r(n) = \infty \) then \( \rho_r(n+1) = 0, \) which implies that for \( r = 4 \cos^2 \frac{\pi k}{n+1}, \; k = 1, n, \) the function \( \rho_r(m) \) is periodic with period \( n + 1. \)

It is easy to determine that the formulas (3) and (1) and the proposition 1.2 hold for such definition.

Let us point to one more case, where the functions \( \rho_r \) appear naturally. *-representations of *-algebras \( \mathcal{P}_{r, \alpha} = \mathbb{C}(p_1, p_2, \ldots, p_r \mid p_k^* = p_k, \sum_{k=1}^{r} = \alpha \epsilon) \) in \( H, \) where \( \epsilon \) is the identity of algebra, \( H \) is separable Hilbert space, are studied in [6]. One of the results of this work is such that if \( \Sigma_r \) is the set of those \( \alpha \in \mathbb{R} \) for which \( \mathcal{P}_{r, \alpha} \) has at least one representation and \( r \geq 4, \) then \( \Sigma_r = \{ \Lambda^1_r, \Lambda^2_r \mid \frac{r-\sqrt{r^2-4r}}{2}, \frac{r+\sqrt{r^2-4r}}{2}, r-\Lambda^1_r, r-\Lambda^2_r \}, \) where \( \Lambda^1_r, \Lambda^2_r \) are discrete sets, which can be defined recurrently:

\[
\begin{align*}
\Lambda^1_r &= \{0, 1 + \frac{1}{r-1}, 1 + \frac{1}{(r-2) - \frac{1}{r-1}}, \ldots, \\
&\quad 1 + \frac{1}{(r-2) - \frac{1}{(r-2) - \frac{1}{\ldots - \frac{1}{r-1}}}} \} \\
\Lambda^2_r &= \{1, 1 + \frac{1}{r-2}, 1 + \frac{1}{(r-2) - \frac{1}{r-2}}, \ldots, \\
&\quad 1 + \frac{1}{(r-2) - \frac{1}{(r-2) - \frac{1}{\ldots - \frac{1}{r-2}}}} \},
\end{align*}
\]

It is easy to see, that \( \Lambda^1_r = \{ \rho_r(2k) \}, \; \Lambda^2_r = \{ \rho_r(2k + 1) \}, \) i.e. \( \Sigma_r = \{ \{ \rho_r(k) \}, \frac{r-\sqrt{r^2-4r}}{2}, \frac{r+\sqrt{r^2-4r}}{2}, r-\Lambda^1_r, r-\Lambda^2_r \}, \{ r-\rho_r(k) \}, k \in \mathbb{N} \cup \{0\}^2 \).

In [6] on the categories \( \text{Rep} \mathcal{P}_{r, \alpha} \) of *-representations of algebras \( \mathcal{P}_{r, \alpha} \) it was determined functors \( \Phi^+ \) and \( \Phi^- \) (Coxeter functors), which gave an opportunity to describe all irreducible *-representations of algebras \( \mathcal{P}_{r, \alpha} \) (up to unitary equivalence) in points of discrete spectrum of the set \( \Sigma_r. \) There was also given the explicit (but rather complicated) formula for calculating \( \Phi^+ \) \( (\alpha) = (\Phi^+)^k(\alpha). \) Let us show that this formula in terms of the functions \( \rho_r \) has more simple form. To do that, we will need

\footnote{The introduction of the functions \( \rho_r \) was proposed by A. V. Roiter exactly for defining of these sets.}
Lemma 1.3. \( \rho_r(2k-1) = 1 + \frac{u_{k-1}}{u_k} \).

**Proof.** If \( r = 4 \) then \( \rho_r(2k-1) = 1 + \frac{2k-2}{2k} = 1 + \frac{k-1}{k} = 1 + \frac{u_{k-1}}{u_k} \).

If \( r > 4 \) then \( u_k = \frac{k-\lambda - k}{\lambda^2 - 1} = (\text{see } \square) \). Therefore, \( 1 + \frac{u_{k-1}}{u_k} = 1 + \frac{\lambda^{2k-2} - 1}{\lambda^2 - 1} = \rho_r(2k-1) \). The lemma is proved.

We will show now that

\[ \Phi^{+k}(\alpha) = \frac{r - \rho_r(2k-1)\alpha}{r - \rho_r(2k-1) - \alpha} \]

In \( \square \) it was obtained that \( \Phi^{+k}(\alpha) = 1 + \frac{a_{k+1}}{a_k} \), where \( a_k \) is defined recurrently: \( a_1 = 1 \), \( a_2 = r - 1 - \alpha, a_{k+2} = (r-2)a_{k+1} - a_k \). Simple induction gives \( a_k = u_{k+1} + (1-\alpha)u_k \). Thus,

\[ \Phi^{+k}(\alpha) = 1 + \frac{a_{k-1}}{a_k} = 1 + \frac{u_k + u_{k+1} - \alpha u_k}{u_{k+1} + u_k - \alpha u_k} = \frac{ru_k - \alpha(u_k + u_{k-1})}{ru_k - (u_k + u_{k-1})} = \frac{r - \rho_r(2k-1)\alpha}{r - \rho_r(2k-1) - \alpha}, \]

which required.

### 2 Spectra and indexes of graphs

Further (if the contrary is not indicated specially) all considering graphs suppose to be finite, connected and not containing loops and multiple edges.

Let \( G_v \) to be the set of vertices and \( G_e \) be the set of edges of graph \( G \). Determine the numeration on the vertices of graph \( G \): \( G_v = \{g_1, g_2, \ldots, g_n\} \), \( n \in \mathbb{N} \). Denote \( M(g_k) = \{g_i \mid g_i \text{ connected with } g_k\}, 1 \leq k, i \leq n \).

Matrix \( A_G = ||a_{ij}||_{i,j=1,n} \), where \( a_{ij} = \begin{cases} 1, & \text{if } g_i \in M(g_j) \\ 0, & \text{if } g_i \notin M(g_j) \end{cases} \) is called the adjacency matrix of graph \( G \).

Since matrix \( A_G \) is symmetrical, all its eigenvalues over \( \mathbb{C} \) are real. The linearly ordered set \( \sigma(G) = \{\lambda_{\min} = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n = \lambda_{\max}\} \) of eigenvalues of the matrix \( A_G \) is called the spectrum of graph \( G \), and the number \( \text{ind}(G) = \lambda_{\max} \) is the index of the graph \( G \).

Denote \( V_G \) — linear space over \( \mathbb{R} \), consisting of suits \( x = (x_i), \) identifying each vector \( x \in V_G \) with function \( x : G_v \to \mathbb{R}, x_i = x(g_i) \). Elements \( x \in V_G \) are called \( G \)-vectors (see \( \square \)). The adjacency matrix \( A_G \) can be considered as the matrix of certain linear operator in the natural basis in the space \( V_G \).

The spectral graph theory has been studied rather deeply (see \( \square \)). In particular, the following remarkable statement, known as Smith’s theorem, holds:

**Theorem 2.1.** \( \square \)

Let \( \lambda = \text{ind}(G) \) for connected graph \( G \). Then \( \lambda < 2 \) iff \( G \) is Dynkin graph \((A_n, D_n, E_6, E_7, E_8)\); \( \lambda = 2 \) iff \( G \) is extended Dynkin graph \((\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8)\).

Note, that if \( G \) is disconnected, then its index is equal to the maximal of the indexes of its connected components. Also the following statements hold (\( \square \), \( \circledast \)):

**Proposition 2.2.** For an arbitrary graph \( G \) the inequalities \( 1 \leq \text{ind}(G) \leq |G_v| - 1 \) hold.
Proposition 2.3. The index \( \lambda = \text{ind}(G) \) of an arbitrary graph \( G \) is a simple eigenvalue, iff graph \( G \) is connected, and in this case the eigenspace of \( V_G \), belonging to \( \lambda \), is spanned by a vector, whose coordinates are all positive.

This vector is called the principle eigenvector of \( G \). If \( \lambda = \text{ind}(G) \), \( A_G = ||a_{ij}|| \), then the condition “\( x = (x_1, x_2, \ldots, x_n) \) is a principal eigenvector of \( G \)” can be written in a form

\[
\lambda x_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i, j = 1, n. \tag{7}
\]

graph \( G \) is called bipartite, if \( G_v = \hat{G}_v \sqcup \hat{G}_v^c \), \( \hat{G}_v \cap \hat{G}_v = \varnothing \) and \( g \in \hat{G}_v \) implies \( M(g) \subseteq \hat{G}_v \) and conversely: \( h \in \hat{G}_v \) implies \( M(g) \subseteq \hat{G}_v^c \). The set \( \hat{G}_v \) is called the set of even vertices and the set \( \hat{G}_v^c \) is the set of odd vertices of \( G \) (see [5].) If the graph \( G \) is bipartite and at first we numerate add, and then — even vertices, then its adjacency matrix has the form

\[
A_G = \begin{bmatrix} O_1 & B \\ B^* & O_2 \end{bmatrix}
\]

where \( O_1, O_2 \) are quadratic zero matrices of the orders \( | \hat{G}_v | \) and \( | \hat{G}_v^c | \) respectively, \( B^* \) is a transposed matrix of \( B \).

Evidently, any tree is bipartite graph, and a cycle with \( n \) vertices is bipartite iff \( n \) is even. (Therefore, all Dynkin graphs and extended Dynkin graphs are bipartite, except \( \tilde{A}_{n-1} \) with odd number of vertices \( n \).)

3 Standard characters of star-shaped graphs

Let \( G \) be a bipartite graph, \( G_v = \hat{G}_v \sqcup \hat{G}_v^c \). Determine numeration of vertices of \( G \) such that \( \hat{G}_v = \{g_1, g_2, \ldots, g_p\} \), \( \hat{G}_v^c = \{g_{p+1}, g_{p+2}, \ldots, g_n\} \). Let (according to this numeration) \( y = (y_1, y_2, \ldots, y_n) \) is a principal eigenvector of \( G \), i.e. vector \( y \) satisfies the equalities (8):

\[
\lambda y_i = \sum_{j=1}^{n} a_{ij} y_j, \quad i, j = 1, n, \tag{8}
\]

where \( \lambda = \text{ind}(G) \). Vector \( y_\bullet = (y_1, y_2, \ldots, y_p, 0, 0, \ldots, 0) \) is the odd standard vector, and vector \( y_\circ = (0, 0, \ldots, 0, y_{p+1}, y_{p+2}, \ldots, y_n) \) is the even standard vector of \( G \). Since the principal eigenvector \( y \) is determined up to nonzero real multiplier, \( y_\bullet \) and \( y_\circ \) are also determined up to common for both vectors \( y_\bullet, y_\circ \) nonzero multiplier.

The Coxeter transformation in the space \( V_G \) is a linear transformation \( c = \sigma_{g_n} \cdots \sigma_{g_1} \), where \( (\sigma_{g_i}(x))_j = x_j \) for \( i \neq j \) and \( (\sigma_{g_i}(x))_i = -x_i + \sum_{j \mid g_j \in M(g_i)} x_j \). Denote \( \overset{\circ}{c} = \sigma_{g_p} \cdots \sigma_{g_1} \), i.e. \( c = \overset{\circ}{c} \overset{\circ}{c} \). Obviously, \( c^{-1} = \overset{\circ}{c} \overset{\circ}{c} \) and \( (\overset{\circ}{c})^2 = (\overset{\circ}{c})^2 = \text{id} \). Put \( c_t = \cdots \overset{\circ}{c} \overset{\circ}{c} \) for
Let $t > 0$, $c_t = \cdots \circ \circ \circ \circ$ for $t < 0$ and $c_0 = \text{id}$. For $G$-vectors $v$ and $w$ we denote $v \simeq w$, if $v$ and $w$ are linearly dependent.

In [11] the explicit formulas in terms of the functions $\rho_r$, indicating how odd and even standard vectors transform under the action of $c_t$, was obtained for certain graphs of the special type. Now we will prove an analogous statement for an arbitrary bipartite graph $G$.

**Proposition 3.1.** Let $G$ be a bipartite graph, $r = (\text{ind}(G))^2 \geq 4$, $y_\bullet$, $y_o$ is its odd and even standard vectors, $t \in \mathbb{N}$. Then the following hold:

\[
\begin{align*}
\sigma_{2t-1}(y_o) &\simeq (y_o + \frac{\sqrt{r}}{\rho_r(2t-1)} y_\bullet); \\
\sigma_{2t}(y_o) &\simeq (y_o + \frac{\rho_r(2t)}{\sqrt{r}} y_\bullet); \\
\sigma_{-2t+1}(y_o) &\simeq (y_o + \frac{\sqrt{r}}{\rho_r(2t+1)} y_o); \\
\sigma_{-2t}(y_o) &\simeq (y_o + \frac{\rho_r(2t)}{\sqrt{r}} y_o).
\end{align*}
\]

**Proof.** We will prove the statement by induction for the case $c_m(y_\bullet)$, where $m < 0$ (in the case $m > 0$ the proof is exactly the same up to “evenness”.)

For $t = -1$ we have $c_{-1}(y_\bullet) = c (y_\bullet) = c (y_1, \ldots, y_p, 0, \ldots, 0) = (y_1, \ldots, y_p, x_{p+1}, \ldots, x_n)$, where $x_k = \sum_{g_i \in \mathcal{M}(g_k)} y_i = \sqrt{r} y_k$, $k = 1, p$ (formulas (9)). Thus, $c_{-2}(y_\bullet) = (y_\bullet + \sqrt{r} y_o) \simeq y_\bullet + \frac{\sqrt{r}}{\rho_r(2)} y_o = \rho_r(2) y_\bullet$, since $\rho_r(2) = \frac{r}{r-1}$.

Assume, that the formulas (9) hold for all $m \leq 2t$. Then $c_{-(2m+1)}(y_\bullet) = c c_{-2t}(y_\bullet) \simeq c (y_\bullet + \frac{\rho_r(2t)}{\sqrt{r}} y_o) = c (y_1, \ldots, y_p, \sqrt{r} y_{p+1}, \ldots, \sqrt{r} y_n) = c (y_1, \ldots, y_p, x_{p+1}, \ldots, x_{n})$, where $x_k = \sum_{g_i \in \mathcal{M}(g_k)} y_i = \sqrt{r} y_k = \sqrt{r} y_{p+1} y_k$, therefore, $c_{-(2m+1)}(y_\bullet) = y_\bullet + \frac{\sqrt{r}}{\rho_r(2t+1)} y_o$.

Then, $c_{-(2t+2)}(y_\bullet) \simeq c (y_\bullet + \frac{\sqrt{r}}{\rho_r(2t+1)} y_o) = c (y_1, \ldots, y_p, \sqrt{r} y_{p+1}, \ldots, \sqrt{r} y_n, y_{p+1}) = (x_1, \ldots, x_p, \sqrt{r} y_{p+1}, \ldots, \sqrt{r} y_n)$, where $x_k = \sum_{g_i \in \mathcal{M}(g_k)} y_i - y_k = y_k (\frac{r}{\rho_r(2t+1)} - 1)$. Therefore, $c_{-(2t+2)}(y_\bullet) = (\frac{r}{\rho_r(2t+1)} - 1) y_\bullet + \frac{\sqrt{r}}{\rho_r(2t+1)} y_o \simeq y_\bullet + \frac{\rho_r(2t+1)}{\rho_r(2t+1) - r} \frac{\rho_r(2t+1)}{r} y_o = y_\bullet + \frac{\sqrt{r}}{r - \rho_r(2t+1)} y_o = y_\bullet + \frac{\rho_r(2t)}{\sqrt{r}} y_o$.

The proposition is proved.\(^3\)

Consider now the connection between standard vectors of a bipartite graph $G$ and locally scalar representations of this graph. Locally scalar representations of graphs were introduced and studied in [5]. We remind some notions from this work.

Let $\mathcal{H}$ be the category of Hilbert spaces, which objects are separable Hilbert spaces, and morphisms are bounded operators. **Representation $\pi$ of a graph $G$ in $\mathcal{H}$ attaches to each vertex**

\(^3\)These formulas were also obtained independently by V. L. Ostrovskiy [arxiv: math.RA/0509240] in the case of star-shaped graphs.
\( a \in G_v \) an object \( \pi(a) = H_a \in \text{Ob} \mathcal{H} \) and to each edge \( \gamma \in G_e \) connecting vertices \( a \) and \( b \) a pair of interadjoint linear operators \( \pi(\gamma) = \{\Gamma_{ab}, \Gamma_{ba}\} \), where \( \Gamma_{ab} : H_b \to H_a \). Denote \( A_g = \sum_{b \in M(g)} \Gamma_{gb}\Gamma_{bg}, b, g \in G_v \). A representation \( \pi \) is called \textit{locally scalar}, if all operators \( A_g \) are scalar, \( A_g = \alpha_g I_{H_g} \), where \( I_{H_g} \) is identity operator in space \( H_g \). Since operators \( A_g \) are bounded, \( \alpha_g \geq 0 \). \( G \)-vector \( f \), such that \( f(g) = \alpha_g f \) for all \( g \in G_v \), is called the \textit{character} of the representation, and \( G \)-vector \( d \), such that \( d(g) = \dim \pi(g) \) is the dimension of the representation \( \pi \). Note, that for given representation \( \pi \) its dimension is determined uniquely, but the character, in general, is not (it is determined uniquely on the \textit{support} \( G^\pi = \{a \in G_v \mid \pi(a) \neq 0\} \) of the representation.) A representation \( \pi \) is \textit{faithful}, if \( G^\pi = G_v \).

In \( G \) for an arbitrary graph \( G \) such categories are considered: a category \( \text{Rep}(G, \mathcal{H}) \) of representations of \( G \) in the category of Hilbert spaces, a category \( \text{Rep}(G) \) of locally scalar representations, its (full) subcategory \( \text{Rep}(G,d,f) \) of representations with fixed dimension \( d \) and character \( f \). In \( G \) for bipartite graph \( G \) are considered: a category \( \text{Rep}(G, \bigcup') \) — the union of categories \( \text{Rep}(G,d,f) \), such that \( f(g) > 0 \) for all \( g \in M(G^\pi) \); a category \( \text{Rep}_o(G, \bigcup') \subseteq \text{Rep}(G, \bigcup') \) — full subcategory of representations with character \( f \), for which \( (d,f) \), such that \( f(g) > 0 \) for \( g \in (M(G^\pi) \bigcup G^\pi) \bigcap G_v \), and an analogous category \( \text{Rep}_\bullet(G, \bigcup') \); also there was defined functor \( \Phi \), which is an equivalence of the category \( \text{Rep}_o(G, \bigcup') \), such that \( \Phi(f)_{i} = f_{i} \) for \( g_{i} \in G_v \), and if \( g_{i} \in G_v \), then \( \Phi(f)_{i} = d_{i} = 0 \), \( g_{i} \in M(G^\pi) \), and in another cases \( \Phi(f)_{i} = \sigma_i(f) \); and the analogous functor \( \Phi \) is defined. Also for any \( k \in \mathbb{N} \) functors \( \Phi_k = \Phi \circ \Phi \circ \cdots \Phi \) and \( \Phi_k = \Phi \circ \Phi \circ \cdots \Phi \) are constructed, such that \( d(\Phi_t(\pi)) = c_t(d(\pi)), t \in \mathbb{Z} \).

A nonnegative \( G \)-vector \( x \) is called \textit{regular}, if \( c_t(x) \) is nonnegative for any \( t \in \mathbb{Z} \), and \textit{singular} in the opposite case. Locally scalar representation \( \pi \) of a graph \( G \) is \textit{singular} (regular), if \( \pi \) is indecomposable, and \( d(\pi) \) is a singular (regular) vector. An object \( (\pi, f) \in \text{Ob} \text{Rep}(G, \bigcup') \) is \textit{singular}, if \( \pi \) is singular.

A group \( W \), generated by reflections \( \sigma_g \), is called the \textit{Weyl group}. A vector \( x \in V_G \) is called the (real) \textit{root}, if \( x = \omega a \) for certain \( a \in G_v \) and \( \omega \in W \), where \( \omega \) a is a \textit{simple root}, i. e. \( \omega(\alpha) = 1 \) and \( \pi(\alpha) = 0 \) for \( \alpha \neq a \).

The \textit{simplest object} in the category \( \text{Rep}(G, \bigcup') \) is a pair \((\Pi_g, \bar{f})\), such that \( \dim \Pi_g = \bar{g} \); \( \bar{f}(g) = 0 \) and \( \bar{f}(a) > 0 \) for \( a \in M(g) \).

**Theorem 3.2.** \( G \)

Each singular object of the category \( \text{Rep}(G, \bigcup') \) can be obtained as \( \Phi_m(\Pi_g, \bar{f}) \), where \( (\Pi_g, \bar{f}) \) is a simplest object \( (m \geq 0 \text{ for } g \in G_v \text{ and } m \leq 0 \text{ for } g \in \hat{G}_v) \). At that each faithful singular representation \( G \) corresponds (up to equivalence) to one singular object of \( \text{Rep}(G, \bigcup') \).

A character \( f \) of a representation is \textit{standard}, is \( f = c_t(g_o) \) or \( f = c_t(g_o) \), \( t \in \mathbb{Z} \). A representation \((\pi, f)\) with a standard character \( f \) is a \textit{standard representation}, and the corresponding object of the category \( \text{Rep}(G, \bigcup') \) is a \textit{standard object}.

The following statement is well-known (see., for instance, \( G \))
Lemma 3.3. If \( x \) is a real root, ther either \( x > 0 \), or \((-x) > 0\).

Let us prove the following

Proposition 3.4. If \( d \) is a positive real singular root of a graph \( G \), then there exists such \( t \in \mathbb{Z} \) that \( d = c_t(\overrightarrow{g}) \), \( g \in G_v \).

Proof. Let \( d \) be not a simple root (in the opposite case we have \( t = 0 \).) Then, let \( m \) be a minimal in absolute value integer number with property \( c_m(d) < 0 \). Let, for definiteness, \( m > 0 \). Then \( c_{m-1}(d) > 0 \) (lemma 3.3). Therefore, \( c_{m-1}(d) \) is a simle root (if \( c_{m-1}(d) \) has at least two positive coordinates, then it is clear, that coordinates, corresponding to the neighbour vertices, are zeros, and, applying the reflection in one of these positive coordinates, we obtain a contradiction with lemma 3.3). Thus, \( t = 1 - m \) and the proposition is proved.

Theorem 3.5. Let \( G \) be a bipartite graph with \( \text{ind}(G) \geq 2 \), \( d \) is a singular real root in \( G \). Then there exists a unique standard representation \( \pi \) with dimension \( d \).

Proof. Existence. Since \( d \) is a singular real root, there exists \( t \in \mathbb{Z} \), such that \( d = c_t(\overrightarrow{g}) \) (proposition 3.4). Therefore, allowing theorem 3.2 and proposition 3.1, we have for \( t \geq 0 \) \( \Phi_t(\pi \circ g, y) \) and for \( t \leq 0 \) \( \Phi_t(\pi \bullet g, y) \) is equal to \( (\pi, y) \), where \( y \) is a standard character, \( \dim \pi = d \).

Uniqueness. Formula (9) imply that if \( (\pi, f) \) is a standard object of the category \( \text{Rep}(G, \bigcup^{'}) \), then \( \Phi_t(\pi \circ g, y) \) is a standard object only if \( \pi \neq \Pi g \), where \( g \in G_v \), and \( \Phi_t(\pi \bullet g, y) \) is a standard object, only if \( \pi \neq \Pi g \), where \( g \in G_v \).

4 Indexes of star-shaped graphs

A path of length \( l \in \mathbb{N} \) on a graph \( G \) is an ordered sequence of vertices \((g_i, \ldots, g_{i+l})\), such that \( g_i \in M(g_{i+1}) \), \( k = 1, l \). Vertices \( g_i \) and \( g_{i+l} \) are called the beginning and the end of path respectively. A vertex \( g \in G_v \) of \( G \) is called the point of branching, if \( M(g) \geq 3 \). Star-shaped graph is a tree which has no more than one point of branching. For a star-shaped graph \( G \) the set \( G_v \) can be presented as

\[
G_v = B_0 \bigcup B_1 \bigcup B_2 \bigcup \ldots \bigcup B_s,
\]

where \( B_i \bigcap B_j = \emptyset, i, j = 0, s, B_0 = \{g_0\} \), and \( g_0 \) is a point of branching in \( G \) (if one exists) and \( g, h \in B_i \) for certain \( i \) iff the path of minimal length with the beginning in \( g \) and the end in \( h \) does not contain \( g_0 \). The sets \( B_i \) are branches of \( G \).

Further we will point at the direct connection of the separating functions \( \rho_r \) with indexes of star-shaped graphs.

First we need to prove the following lemma.

Lemma 4.1. Given real \( r \geq 1 \). Let \( \{v_n\} \) be a number sequence, defined recurrently: \( v_0 = 0 \), \( v_1 = 1 \), \( v_{n+2} = \sqrt{rv_{n+1} - v_n} \). Then \( \rho_r(n) = \sqrt[r]{\frac{v_n}{v_{n+1}}} \)
Proof. Induction by \( n \). \( \rho_r(0) = \sqrt{\frac{\mathbf{v}_0}{\mathbf{v}_1}} = 0 \). Assume \( \rho_r(n) = \sqrt{\frac{\mathbf{v}_n}{\mathbf{v}_{n+1}}} \). Then by formula (6) \( \rho_r(n + 1) = \frac{r}{r - \sqrt{\frac{\mathbf{v}_n}{\mathbf{v}_{n+1}}}} = \frac{\sqrt{\mathbf{v}_n}}{\sqrt{\mathbf{v}_{n+1}} - \mathbf{v}_n} = \frac{\sqrt{\mathbf{v}_{n+1}}}{\mathbf{v}_{n+2}}, \) which required.

Theorem 4.2. Let \( G_v = B_0 \bigcup B_1 \bigcup \ldots \bigcup B_s \) be the separation of a star-shaped graph by branches, \(|B_i| = n_i, i = 1, s\). Then \( \rho_r(n_1, n_2, \ldots, n_s) = r \), where \( r = (\text{ind}(G))^2 \).

Proof. Let \( B_0 = \{g_0\}, B_k = \{g_1^k, \ldots, g_{n_k}^k\}, k = 1, s \), and vertices in the branches \( B_k \) are numerated in such a way: \( M(g_1^k) = 1, g_i^k \in M(g_{i+1}^k), i = 1, n_k - 1, g_{n_k}^k \in M(g_0) \). Let \( y \) be a principal eigenvector of \( G \), \( y_i^k = y(g_i^k), y_0 = y(g_0), k = 1, s, i = 1, n_k \). Then the equations (7) have the form

\[
\begin{align*}
\lambda y_1^k &= y_2^k, \\
\lambda y_2^k &= y_1^k + y_3^k, \\
\lambda y_3^k &= y_2^k + y_4^k, \\
&\vdots \\
\lambda y_{n_k}^k &= y_{n_k - 1}^k + y_0, \\
\lambda y_0 &= y_{n_1}^k + \cdots + y_{n_k}^k,
\end{align*}
\]

(10)

where \( \lambda = \text{ind}(G) \).

The equations (10) for any branch \( B_k \) imply \( y_0 = v_{n_k+1} y_1^k \) and \( y_{n_k}^k = v_{n_k} y_1^k \), where \( \{v_i\} \) is a sequence, defined recurrently: \( v_0 = 0, v_1 = 1, v_{i+2} = \lambda v_{i+1} - v_i \). Then \( y_{n_k}^k = \frac{v_{n_k}}{v_{n_k+1}} y_0 \) for all \( k = 1, s \). Consider these relations and the equation (11) we obtain \( \lambda y_0 = \sum_{k=1}^s \frac{v_{n_k}}{v_{n_k+1}} y_0 \) and, taking into account that \( y_0 \neq 0 \) (proposition 2.3) and \( \lambda = \sqrt{r} \), we have \( \sqrt{r} = \sum_{k=1}^s \frac{v_{n_k}}{v_{n_k+1}} \). Then, by lemma 4.1 \( r = \sum_{k=1}^s \rho_r(n_k) = \rho_r(n_1, n_2, \ldots, n_s) \), which required.

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