Frank-Wolfe with Subsampling Oracle

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Abstract
We analyze two novel randomized variants of the Frank-Wolfe (FW) or conditional gradient algorithm. While classical FW algorithms require solving a linear minimization problem over the domain at each iteration, the proposed method only requires to solve a linear minimization problem over a small subset of the original domain. The first algorithm that we propose is a randomized variant of the original FW algorithm and achieves a $O(1/t)$ sublinear convergence rate as in the deterministic counterpart. The second algorithm is a randomized variant of the Away-step FW algorithm, and again as its deterministic counterpart, reaches linear (i.e., exponential) convergence rate making it the first provably convergent randomized variant of Away-step FW. In both cases, while subsampling reduces the convergence rate by a constant factor, the linear minimization step can be a fraction of the cost of that of the deterministic versions, especially when the data is streamed. We illustrate computational gains of the algorithms on regression problems, involving both $\ell_1$ and latent group lasso penalties.

1. Introduction

The Frank-Wolfe (FW) or conditional gradient algorithm (Frank & Wolfe, 1956; Jaggi, 2013) is designed to solve optimization problems of the form

$$\min_{x \in \mathcal{M}} f(x), \text{ with } \mathcal{M} = \text{conv}(\mathcal{A}),$$

where $\mathcal{A}$ is a (possibly infinite) set of vectors which we call atoms, and $\text{conv}(\mathcal{A})$ is its convex hull. The FW algorithm and variants have seen an impressive revival in recent years, due to their low memory requirements and projection-free iterations, which make them particularly appropriate to solve large scale convex problems, for instance convex relaxations of problems written over combinatorial polytopes (Zaslavskiy et al., 2009; Joulin et al., 2014; Vogelstein et al., 2015).

The Frank-Wolfe algorithm is projection-free, i.e. unlike most methods to solve (OPT), it does not require to compute a projection onto the feasible set $\mathcal{M}$. Instead, it relies on a linear minimization oracle over a set $\mathcal{A}$, written $\text{LMO}(\cdot, \mathcal{A})$, which solves the following linear problem

$$\text{LMO}(r, \mathcal{A}) \in \arg \min_{v \in \mathcal{A}} \langle v, r \rangle.$$ 

For some constraint sets, such as the nuclear or latent group norm ball (Jaggi et al., 2010; Vinyes & Obozinski, 2017), computing the LMO can be orders of magnitude faster than projecting. Another feature of FW that has greatly contributed to its practical success is its low memory requirements. The algorithm maintains its iterates as a convex combination of a few atoms, enabling the resulting sparse and low rank iterates to be stored efficiently. This feature allows the FW algorithm to be used in situations with a huge or even infinite number of features, such as architecture optimization in neural networks (Ping et al., 2016) or estimation of an infinite-dimensional sparse matrix arising in multi-output polynomial networks (Blondel et al., 2017).

Despite these attractive properties, for problems with a large number of variables or with a very large atomic set (or both), computing the full gradient and LMO at each iteration can become prohibitive. Designing variants of the FW algorithm which alleviate this computational burden would have a significant practical impact on performance.

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One recent direction to achieve this is to replace the LMO with a randomized linear oracle in which the linear minimization is performed only over a random sample of the original atomic domain. This approach has proven to be highly successful on specific problems such as structured SVMs (Lacoste-Julien et al., 2013) and $\ell_1$-constrained regression (Frandi et al., 2016), however little is known in the general case. Is it possible to design a FW variant with a randomized oracle that achieves the same convergence rate (up to a constant factor) as the non-randomized variant? Can this be extended to linearly-convergent FW algorithms (Lacoste-Julien & Jaggi, 2013; 2015; Garber & Hazan, 2015)? In this paper we give a positive answer to both questions and explore the trade-offs between subsampling and convergence rate.

Outline and main contribution. The main contribution of this paper is to develop and analyze two algorithms that share the low memory requirements and projection-free iterations of FW, but in which the LMO is computed only over a random subset of the original domain. In many cases, this results in large computational gains in computing the LMO which can also speed up the overall FW algorithm. In practice, the algorithm will run a larger number of cheaper iterations, which is typically more efficient for very large data sets (e.g. in a streaming model where the data does not fit in core memory and can only be accessed by chunks). The paper is structured as follows

- §2 describes the Randomized FW algorithm, proving a sublinear convergence rate.
- §3 describes “Randomized Away FW”, a variant of the above algorithm with linear convergence rate on polytopes. To the best of our knowledge this is the first provably convergent randomized version of the Away-steps FW algorithm.
- Finally, in §4 we discuss implementation aspects of the proposed algorithms and study their performance on lasso and latent group lasso problems.

Note that with the proven sub-linear rate of convergence for Randomized FW, the cost of the LMO is reduced by the subsampling rate, but this is compensated by the fact that the number of iteration required by RFW to reach same convergence guarantee as FW is itself multiplied by the sampling rate. Similarly the linear convergence rate in Randomized AFW does not theoretically show a computational advantage since the number of iterations is multiplied by the squared sampling rate, in our highly conservative bounds at least. Nevertheless, our numerical experiments show that randomized versions are often numerically superior to their deterministic counterparts.

1.1. Related work

Several references have focused on reducing the cost of computing the linear oracle. The analysis of (Jaggi, 2013) allows for an error term in the LMO, and so a randomized linear oracle could in principle be analyzed under this framework. However, this is not fully satisfactory as it requires the approximation error to decrease towards zero as the algorithm progresses. In our algorithm, the subsampling approximation error doesn’t need to decrease.

Lacoste-Julien et al. (2013) studied a randomized FW variant named block-coordinate FW in which at each step the LMO is computed only over a subset (block) of variables. In this case, the approximation error need not decrease to zero, but the method can only be applied to a restricted class of problems: those with block-separable domain, leaving out important cases such as $\ell_1$-constrained minimization. Because of the block separability, a more aggressive step-size strategy can be used in this case, resulting overall in a different algorithm.

Finally, Frandi et al. (2014) proposed a FW variant which can be seen as a special case of our Algorithm 1 for the Lasso problem, analyzed in (Frandi et al., 2016). Our analysis here brings three key improvements on this last result. First, it is provably convergent for arbitrary atomic domains, not just the $\ell_1$ ball (furthermore the proof in (Frandi et al., 2016) has technical issues discussed in Appendix C). Second, it allows a choice of step size that does not require exact line-search (Variant 2), which is typically only feasible for quadratic loss functions. Third, we extend our analysis to linearly-convergent FW variants such as the Away-step FW.

A different technique to alleviate the cost of the linear oracle was recently proposed by Braun et al. (2017). In that work, the authors propose a FW variant that replaces the LMO by a “weak” separation oracle and showed significant speedups in wall-clock performance on problems such as the video co-localization. This approach was combined with gradient sliding in (Lan et al., 2017), a technique (Lan & Zhou, 2016) that allows to skip the computation of gradients from time to time. However, for problems such as Lasso or latent group lasso, a randomized LMO avoids all full gradient computations, while the lazy weak separation oracle still requires it. Combining these various techniques is an interesting open question.
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Proximal coordinate-descent methods (Richtárik & Takáč, 2014) (not based on FW) have also been used to solve problems with a huge number of variables. They are particularly effective when combined with variable screening rules such as (Tibshirani et al., 2012; Fercq et al., 2015). However, for constrained problems they require evaluating a projection operator, which on some sets such as the latent group lasso ball can be much more expensive than the LMO. Furthermore, these methods require that the projection operator is block-separable, while our method does not.

**Notation.** We denote vectors with boldface lower case letters (i.e., \( \mathbf{x} \)), and sets in calligraphic letter (i.e., \( \mathcal{A} \)). We denote \( \text{clip}_{[0,1]}(s) = \max\{0, \min\{1, s\}\} \). Probability is denoted \( P \). The cardinality of a set \( \mathcal{A} \) is denoted \(|\mathcal{A}|\). For \( x^* \) a solution of \((\text{OPT})\), we denote \( h(x) = f(x) - f(x^*) \).

**Randomized vs stochastic.** We denote FW variants with randomness in the LMO randomized and reserve the name stochastic for FW variants that replace the gradient with a stochastic approximation, as in (Hazan & Luo, 2016).

## 2. Randomized Frank-Wolfe

In this section we present our first contribution, a FW variant that we name Randomized Frank-Wolfe (RFW). The method is detailed in Algorithm 1. Compared to the standard FW algorithm, it has the following two distinct features.

First, the LMO is computed over a random subset \( \mathcal{A}_t \subseteq \mathcal{A} \) of the original atomic set in which each atom is equally likely to appear, i.e., in which \( P(v \in \mathcal{A}_t) = \eta \) for all \( v \in \mathcal{A} \) (Line 2). For discrete sets this can be implemented simply by drawing uniformly at random a fixed number of elements at each iteration. The sampling parameter \( \eta \) controls the fraction of the domain that is considered by the LMO at each iteration. If \( \eta = 1 \), the LMO considers the full domain at each iteration and the algorithm defaults to the classical FW algorithm. However, for \( \eta < 1 \), the LMO only needs to consider a fraction of the atoms in the original dataset and can be faster than the FW LMO.

Second, because of this subsampling we can no longer guarantee that the atom chosen by the LMO is a descent direction and so it is no longer possible to use the “oblivious” (i.e., independent on the result of the LMO) \( 2/(2 + t) \) step-size commonly used in the FW algorithm. We provide two possible choices for this step-size: the first variant (Line 5) chooses the step-size by exact line search and requires to solve a 1-dimensional convex optimization problem. This approach is efficient when this sub-problem has a closed form solution, as it happens for example in the case of quadratic loss functions. The second variant does not need to solve this sub-problem, but in exchange requires to have an estimate of the curvature constant \( C_f \) (defined in next subsection). Note that in absence of an estimate of this quantity, one can use the bound \( C_f \leq \text{diam}(\mathcal{M})^2 L \), where \( L \) is the Lipschitz constant of \( \nabla f \) and \( \text{diam}(\mathcal{M}) \) is the diameter of the domain in euclidean norm.

**Gradient coordinate subsampling.** We note that the gradient of \( f \) only enters Algorithm 1 through the computation of the randomized LMO, and so only the dot product between the gradient and the subsampled atomic set are truly necessary. In some cases the elements of the atomic set have a specific structure that makes computing dot products particularly effective. For example, when the atomic elements are sparse, only the coordinates of the gradient that are in the support of the atomic set need to be evaluated. As a result, for sparse atomic sets such as the \( \ell_1 \) ball, the group lasso ball (also known as \( \ell_1/\ell_2 \) ball), or even the latent group lasso (Obozinski et al., 2011) ball, only a few coordinates of the gradient need to be evaluated at each iteration. The number of exact gradients that need to be evaluated will depend on both the sparsity of this atomic set and the subsampling rate. For example, in the case of the \( \ell_1 \) ball, the extreme atoms have a single nonzero coefficient, and so RFW only needs to compute on average \( d \eta \) gradient coefficients at each iteration, where \( d \) denotes the ambient dimension.

**Stopping criterion.** A side-effect of subsampling the linear oracle is that \( \langle -\nabla f(x_t), s_t - x_t \rangle \), where \( s_t \) is the atom selected by the randomized linear oracle is not, unlike in the non-randomized algorithm, an upper bound on \( f(x_t) - f(x^*) \). This property is a feature of FW algorithms that cannot be retrieved in our variant. As a replacement, the stopping criteria that we propose is to compute a full LMO every \( k \lfloor \frac{t}{h} \rfloor \) iterations, with \( h \in \mathbb{N}^+ \) (\( h = 2 \) is a good default value).

### 2.1. Analysis

In this subsection we prove an \( O(1/t) \) convergence rate for the RFW algorithm. As is often the case for FW-related algorithms, our convergence result will be stated in terms of the curvature constant \( C_f \), which is defined as follows for a
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Algorithm 1: Randomized Frank-Wolfe algorithm

1. **Input:** $x_0 \in \mathcal{M}$, sampling ratio $0 < \eta \leq 1$. for $t = 0, 1, \ldots, T$ do

2. Choose $A_t$ such that $\mathcal{P}(v \in A_t) = \eta$ for all $v \in A$

3. $s_t = \text{LMO}(\nabla f(x_t), A_t)$

4. **Variant 1**
   - $\gamma_t = \arg \max_{\gamma \in [0,1]} f((1-\gamma_t)x_t + \gamma s_t)$
   - $\triangleright$ set $\gamma_t$ by line-search

5. **Variant 2**
   - $\gamma_t = \text{clip}_{[0,1]}(\langle -\nabla f(x_t), s_t - x_t \rangle / C_f)$

6. $x_{t+1} = (1-\gamma_t)x_t + \gamma_t s_t$

convex and differentiable function $f$ and a convex and compact domain $\mathcal{M}$:

$$C_f \overset{\text{def}}{=} \sup_{x, s \in \mathcal{M}, \gamma \in [0,1]} \frac{2}{\gamma^2} \left( f(y) - f(x) - \langle \nabla f(x), y - x \rangle \right).$$

It is worth mentioning that a bounded curvature constant $C_f$ corresponds to a Lipschitz assumption on the gradient of $f$ (Jaggi, 2013).

**Theorem 2.1.** Let $f$ be a function with bounded smoothness constant $C_f$ and subsampling parameter $\eta \in (0, 1]$. Then Algorithm 1 (in both variants) converges towards a solution of (OPT). Furthermore, the following inequality is satisfied:

$$\mathbb{E}[h(x_T)] \leq \frac{2(C_f + f(x_0) - f(x^*))}{\eta T + 2},$$

where $\mathbb{E}$ is a full expectation over all randomness until iteration $T$.

**Proof.** See Appendix A. □

The rate obtained in the previous theorem is similar to known bounds for FW. For example, (Jaggi, 2013, Theorem 1) established for FW a bound of the form

$$h(x_T) \leq \frac{2C_f}{T + 2}.$$  \hspace{1cm} (3)

This is similar to the rate of Theorem 2.1, except for the factor $\eta$ in the denominator. Hence, if our updates are $\eta$ times as costly as the full FW update (as is the case e.g. for the $\ell_1$ ball), then the theoretical convergence rate is the same. This bound is likely tight, as in the worst case one will need to sample the whole atomic set to decrease the objective if there is only one descent direction. This is however a very pessimistic scenario, and in practice good descent directions can often be found without sampling the whole atomic set. As we will see in the experimental section, despite these conservative bounds, the algorithm often exhibits large computational gains with respect to the deterministic algorithm.

3. Randomized Away-steps Frank-Wolfe

A popular variant of the FW algorithm is the Away-steps FW variant of Guélat & Marcotte (1986). This algorithm adds the option to move away from an atom in the current representation of the iterate. In the case of a polytope domain, it was recently shown to have much better convergence properties, such as linear (i.e. exponential) convergence rates for generally-strongly convex objectives (Garber & Hazan, 2013; Beck & Tetruashvili, 2013; Lacoste-Julien & Jaggi, 2015).

In this section we describe the first provably convergent randomized version of the Away-steps FW, which we name Randomized Away-steps FW (RAFW). We will assume throughout this section that the domain is a polytope, i.e. that $\mathcal{M} = \text{conv}(\mathcal{A})$, where $\mathcal{A}$ is a finite set of atoms. We will make use of the following notation.

- **Active set.** We denote by $S_t$ the active set of the current iterate, i.e. $x_t$ decomposes as $x_t = \sum_{v \in S_t} \alpha_v^{(t)} v$, where $\alpha_v^{(t)} > 0$ are positive weights that are iteratively updated.
is performed over and, as in the A way-steps FW, requires computing two linear minimization oracles negligible compared to the cost of an LMO over the full atomic domain. In this regime, and assuming that the atomic domain has a sparse structure that allows gradient coordinate subsampling, RAFW can achieve a per iteration cost that is, like RFW, roughly $|A|/p$ times lower than that of its deterministic counterpart.

**Subsampling parameter.** The method depends on a subsampling parameter $p$. It controls the amount of computation per iteration of the LMO. In this case, the atomic set is finite and $p$ denotes an integer $1 \leq p \leq |A|$. This sampling rate is approximately $[\eta, |A|]$ in the RFW formulation of §2.

The method is described in Algorithm 2 and, as in the Away-steps FW, requires computing two linear minimization oracles at each iteration. Unlike the deterministic version, the first oracle is computed on the subsampled set $S_t \cup A_t$ (Line 3), where $A_t$ is a subset of size $\min \{p, |A \setminus S_t|\}$, sampled uniformly at random from $A \setminus S_t$. The second LMO (Line 5) is computed on the active set, which is also typically much smaller than the atomic domain.

As a result of both oracle calls, we obtain two potential descent directions, the RFW direction $d_{t}^{FW}$ and the Away direction $d_{t}^{A}$. The chosen direction is the one that correlates the most with the negative gradient, and a maximum step size is chosen to guarantee that the iterates remain feasible (Lines 7–10).

**Updating the support.** Line 12 requires updating the support and the associated $\alpha$ coefficients. For a FW step we have $S_{t+1} = \{s_t\}$ if $\gamma_t = 1$ and otherwise $S_{t+1} = S_t \cup \{s_t\}$. The corresponding update of the weights is $\alpha_{v}^{(t+1)} = (1 - \gamma_t)\alpha_{v}^{(t)}$ when $v \in S_t \setminus \{s_t\}$ and $\alpha_{v}^{(t+1)} = (1 - \gamma_t)\alpha_{v}^{(t)} + \gamma_t$ otherwise.

For an away step we instead have the following update rule. When $\gamma_t = \gamma_{\text{max}}$ (which is called a drop step), then $S_{t+1} = S_t \setminus \{v_t\}$. Combined with $\gamma_{\text{max}} < 1$ (or equivalently $\alpha_{v_t} \leq \frac{1}{2}$) we call them bad drop step, as it corresponds to a situation in which we are not able to guarantee a geometrical decrease of the dual gap.

For away steps in which $\gamma_t < \gamma_{\text{max}}$, the away atom is not removed from the current representation of the iterate. Hence $S_{t+1} = S_t$, $\alpha_{v}^{(t+1)} = (1 + \gamma_t)\alpha_{v}^{(t)}$ for $v \in S_t \setminus \{v_t\}$ and $\alpha_{v_t}^{(t+1)} = (1 + \gamma_t)\alpha_{v_t}^{(t)} - \gamma_t$ otherwise.

Note that when choosing Away step in Line 10, it cannot happen that $\alpha_{v_t} = 1$. Indeed this would imply $x_t = v_t$, and so $d_{t}^{A} = 0$. Since we would have $S_t = \{v_t\}$ and the LMO of Line 3 is performed over $S_t \cup A_t$, we necessarily have $\langle -\nabla f(x_t), d_{t}^{FW} \rangle \geq 0$. It thus leads to a choice of FW step, contradiction.

**Per iteration cost.** Establishing the per iteration cost of this algorithm is not as straightforward as for RFW, as the cost of some operations depends on the size of the active set, which varies throughout the iterations. However, for problems with sparse solutions, we have observed empirically that the size of the active set remains small, making the cost of the second LMO and the comparison of Line 7 negligible compared to the cost of an LMO over the full atomic domain. In this regime, and assuming that the atomic domain has a sparse structure that allows gradient coordinate subsampling, RAFW can achieve a per iteration cost that is, like RFW, roughly $|A|/p$ times lower than that of its deterministic counterpart.

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**Algorithm 2: Randomized Away-steps FW (RAFW)**

**Input:** $x_0 \in \mathcal{M}$, $x_0 = \sum_{v \in A} \alpha_v^{(0)} v$ with $|S_0| = s$, a subsampling parameter $1 \leq p \leq |A|$.

**for** $t = 0, 1, \ldots, T$ **do**

1. Get $A_t$ by sampling $\min \{p, |A \setminus S_t|\}$ elements uniformly from $A \setminus S_t$.
2. Compute $s_t = \text{LMO}(\nabla f(x_t), S_t \cup A_t)$
3. Let $d_t^{FW} = s_t - x_t$
4. Compute $v_t = \text{LMO}(-\nabla f(x_t), S_t)$
5. Let $d_t^{A} = x_t - v_t$
6. if $\langle -\nabla f(x_t), d_t^{FW} \rangle \geq \langle -\nabla f(x_t), d_t^{A} \rangle$ then
7. $d_t = d_t^{FW}$ and $\gamma_{\text{max}} = 1$
8. else
9. $d_t = d_t^{A}$ and $\gamma_{\text{max}} = \alpha_{v_t}^{(t)}/(1 - \alpha_{v_t}^{(t)})$
10. **end if**
11. Set $\gamma_t$ by line-search, with $\gamma_t = \text{arg max}_{\gamma \in [0, \gamma_{\text{max}}]} f((1 - \gamma_t)x_t + \gamma_t v_t)$
12. Let $x_{t+1} = x_t + \gamma_t d_t$
13. Let $S_{t+1} = \{v \in A \text{ s.t. } \alpha_{v}^{(t+1)} > 0\}$
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3.1. Analysis

We now provide a convergence analysis of the Randomized Away-steps FW algorithm. These convergence results are stated in terms of the away curvature constant $C_f^A$ and the geometric strong convexity $\mu_f^A$, which are described in Appendix B and in (Lacoste-Julien & Jaggi, 2015). Throughout this section we assume that $f$ has bounded $C_f^A$ (note that the usual assumption of Lipschitz continuity of the gradient over compact domain implies this) and strictly positive geometric strong convexity constant $\mu_f^A$.

Theorem 3.1. Consider the set $\mathcal{M} = \text{conv}(\mathcal{A})$, with $\mathcal{A}$ a finite set of extreme atoms, after $T$ iterations of Algorithm 2 (RAFW) we have the following linear convergence rate

$$\mathbb{E}[h(x_{T+1})] \leq (1 - \eta^2 \rho_f)^{\max\{0,\lfloor(T-s)/2\rfloor\}} h(x_0),$$

with $\rho = \frac{\mu_f^A}{\lambda C_f^A}$, $\eta = \frac{1}{\|\nabla f(x_0)\|}$ and $s = |\mathcal{S}_0|$.

Proof. See Appendix B. ■

Proof sketch. Our proof structure roughly follows that of the deterministic case in (Lacoste-Julien & Jaggi, 2015; Beck & Tetruashvili, 2013) with some key differences due to the LMO randomness, and can be decomposed into three parts.

The first part consists in upper bounding $h_t$ and is no different from the proof of its deterministic counterpart (Lacoste-Julien & Jaggi, 2015; Beck & Tetruashvili, 2013).

The second part consists in lower bounding the progress $h_t - h_{t+1}$. For this algorithm we can guarantee a decrease of the form

$$h_{t+1} \leq h_t (1 - \rho_f (g_t \bar{g}_t)^2) z_t,$$

where $g_t = -\nabla f(x_t)$, $s_t - v_t$ is the partial pair-wise dual gap while $\bar{g}_t$ is the pair-wise dual gap, in which $s_t$ is replaced by the result of a full (and not subsampled) LMO.

We can guarantee a possible geometric decrease on $h_t$ at each iteration, except for bad drop steps, where we can only secure $h_{t+1} \leq h_t$. We mark these by setting $z_t = 0$.

One crucial issue is then to quantify $g_t/\bar{g}_t$. This can be seen as a measure of the quality of the subsampled oracle: if it selects the same atom as the non-subsampled oracle the quotient will be 1, in all other cases it will be $\leq 1$.

To ensure a geometrical decrease we further study the probability of events $z_t = 1$ and $\bar{g}_t = g_t$: first, we produce a simple bound on the number of bad drop steps (where $z_t = 0$). Second, when $z_t = 1$ holds, Lemma 3 provides a lower bound on the probability of $g_t = \bar{g}_t$.

The third and last part of the proof analyzes the expectation of the decrease rate $\prod_{t=0}^{T} (1 - \rho_f (g_t \bar{g}_t)^2)^{z_t}$ given the above discussion. We produce a conservative bound assuming the maximum possible number of bad drop steps. The key element in this part is to make this maximum a function of the size of the support of the initial iterate and of the number of iteration. The convergence bound is then proven by induction. ■

Comparison with deterministic convergence rates. The rate for away Frank-Wolfe in (Lacoste-Julien & Jaggi, 2015, Theorem 8), after $T$ iteration is

$$h(x_{T+1}) \leq (1 - \rho_f)^{\lfloor T/2 \rfloor} h(x_0).$$

Due to the dependency on $\eta^2$ of the convergence rate in Theorem 3.1, our bound does not show that RAFW is computationally more efficient than AFW. Indeed we use a very conservative proof technique in which we measure progress only when the sub-sampling oracle equals the full one. Also, the cost of both LMOs depends on the support of the iterates which is unknown a priori except for a coarse upper bound (e.g. the support cannot be more than the number of iterations). Nevertheless, the numerical results do show speed ups compared to the deterministic method.
Beyond strong convexity. The strongly convex objective assumption may not hold for many problem instances. However, the linear rate easily holds for \( f \) of the form \( g(Ax) \) where \( g \) is strongly convex and \( A \) a linear operator. This type of function is commonly known as a \( \mu \)-generally strongly convex function (Beck & Tetruashvili, 2013; Wang & Lin, 2014) or (Lacoste-Julien & Jaggi, 2015) (see “Away curvature and geometric strong convexity” in Appendix B for definition). The proof simply adapts that of (Lacoste-Julien & Jaggi, 2015, Th. 11) to our setting.

**Theorem 3.2.** Suppose \( f \) has bounded smoothness constant \( C_f^2 \) and is \( \mu \)-generally-strongly convex. Consider the set \( M = \text{conv}(A) \), with \( A \) a finite set of extreme atoms. Then after \( T \) iterations of Algorithm 2, with \( s = |S_0| \) and a \( p \) parameter of sub-sampling, we have

\[
E[h(x_{T+1})] \leq (1 - \eta^2 \tilde{\rho}_f) \max(0, \frac{\rho_f}{\mu^2}) h(x_0),
\]

with \( \tilde{\rho}_f = \frac{\rho_f}{4C_f^2} \) and \( \eta = \frac{\mu}{\|A\|}. \)

**Proof.** See end of Appendix B. ■

4. Applications

In this section we compare the proposed methods with their deterministic versions. We consider two regularized least squares problems: one with \( \ell_1 \) regularization and another one with latent group lasso (LGL) regularization. In the first case, the domain is a polytope and as such the analysis of AFW and RA FW holds.

We will display the FW gap versus number of iterations, and also cumulative number of computed gradient coefficients, which we will label “nbr coefficients of grad”. This allows to better reflect the true complexity of our experiments since sub-sampling the LMO in the problems we consider amounts to computing the gradient on a batch of coordinates.

In the case of latent group lasso, we also compared the performance of RFW against FW in terms of wall-clock time on a large dataset stored in disk and accessed sequentially in chunks (i.e. in streaming model).

4.1. Lasso problem

**Synthetic dataset.** We generate a synthetic dataset following the setting of (Lacoste-Julien & Jaggi, 2015), with a Gaussian design matrix \( A \) of size \((200, 500)\) and noisy measurements \( b = Ax^* + \varepsilon \), with \( \varepsilon \) a random Gaussian vector and \( x^* \) a vector with \( 10\% \) of nonzero coefficients and values in \((-1,+1)\).

In Figures 1 and 2, we consider a problem of the form (OPT), where the domain is an \( \ell_1 \) ball, a problem often referred to as Lasso. We compare FW against RFW, and AFW against RA FW. The \( \ell_1 \) ball radius set to 40, so that the unconstrained optimum lies outside the domain.

**RFW experiments.** Figure 1 compares FW and RFW. Each call to the randomized LMO outputs a direction, likely less aligned with the opposite of the gradient than the direction proposed by FW, which explains why RFW requires more iterations to converge on the upper left graph of Figure 1. Each call of the randomized LMO is cheaper than the LMO in terms of number of computed coefficients of the gradient, and the trade-off is beneficial as can be seen on the bottom left graph, where RFW outperforms its deterministic variant in terms of nbr coefficients of grad.

Finally, the right panels of Figure 1 provide an insight on the evolution of the sparsity of the iterate, depending on the algorithm. FW and RFW perform similarly in terms of the fraction of recovered support (bottom right graph). In terms of the sparsity of the iterate, RFW under-performs FW (upper right graph). This can be explained as follows: because of the sub-sampling, each atom of the randomized LMO provides a direction less aligned with the opposite of the gradient than the one provided by the LMO. Each update in such a direction may result in putting weight on an atom that would better be off the representation of the iterate. It impacts the iterate all along the algorithm as vanilla FW removes past atoms from the representation only by multiplicatively shrinking their weight.

**RA FW experiments.** Unlike RFW, the RA FW method outperforms AFW in terms of number of iterations in the upper left graph in Figure 2. These graphs also show that both have linear rate of convergence. The bottom left graph shows that the gap between RA FW and AFW is even larger when comparing the cumulative number of computed coefficients of the gradient required to reach a certain target precision.
Figure 1. Performance of FW against RFW with subsampling parameter \( \eta = \frac{\hat{\gamma}}{|\mathcal{A}|} = 0.05 \) (chosen arbitrarily) on the lasso problem. 
Upper left: progress in FW dual gap versus number of iterations. Lower left: progress of the FW dual gap versus cumulative number of computed coefficients of gradient per call to LMO, called nbr coefficients of grad here. Lower right: recovered coefficients in support of the ground truth versus number of iterations. Upper right: size of support of iterate versus number of iterations.
Figure 2. Same parameters and setting as in Figure 1 but to compare RAFW and AFW. AFW performed 880 away steps among which 14 were a drop steps while RAFW performed 1242 away steps and 37 drop steps.
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This out-performance of RAFW over AFW in term of number of iteration to converge is not predicted by our convergence analysis. We conjecture that the away mechanism improves the trade-off between the cost of the LMO and the alignment of the descent direction with the opposite of the gradient. Indeed, because of the oracle subsampling, the partial FW gap (e.g. the scalar product of the Randomized FW direction with the opposite of the gradient) in RAFW is smaller than in the non randomized variant, and so there is a higher likelihood of performing an away step.

Finally, the away mechanism enables the support of the RAFW to stay close to that of AFW, which was not the case in the comparison of RFW versus FW. This is illustrated in the right panels of Figure 2.

Real dataset. On figure 3, we test the Lasso problem on the E2006-tf-idf data set (Kogan et al., 2009), which gathers volatility of stock returns from companies with financial reports. Each financial reports is then represented through its TF-IDF embedding. Notation. We write \( |d| \) the set of indices from 1 to \( d \). Consider \( g \subset [d] \) and \( x \in \mathbb{R}^d \), \( x_{(g)} \) represents the projection vector of \( x \) onto its \( g \)-coordinate. We use the notation \( \nabla_{(g)} f(x_t) \) to denote the gradient with respect to the variables in group \( g \). Similarly \( x_{(g)} \in \mathbb{R}^d \) is the vector that equals \( x \) in the coordinates of \( g \) and 0 elsewhere.

Model. As outlined by Jaggi (2013), FW algorithms are particularly useful when the domain is a ball of the latent group norm (Obozinski et al., 2011). Consider a set \( G \) of subset of \([d] \) such that \( \bigcup_{g \in G} g = [d] \) and denote by \( || \cdot ||_g \) any norm on \( \mathbb{R}^{|g|} \). Frank-Wolfe can be tuned to solve (OPT) with \( \mathcal{M} \) being the ball corresponding to the latent group norm

\[
||x||_G \overset{\text{def}}{=} \min_{v_{(g)} \in \mathbb{R}^{|g|}} \sum_{g \in G} ||v_{(g)}||_g \\
\text{s.t. } x = \sum_{v \in G} v_{(g)} .
\]

This formulation matches a constrained version of the regularized (Obozinski et al., 2011, equation (5)) when each \( || \cdot ||_g \) is proportional to the Euclidean norm. From now on we will consider \( || \cdot ||_g \) to be the euclidean norm.

When \( G \) forms a partition of \([d] \) (i.e., there is no overlap between groups), this norm coincides with the group lasso norm.

Sub-sampling. Given an element \( g \) of \( G \), consider the hyper-disk

\[
D_g(\beta) = \{ v \in \mathbb{R}^d \mid v = v_{(g)}, ||v_{(g)}|| \leq \beta \} .
\]

(Obozinski et al., 2011, lemma 8) shows that such constrain set \( M \) is the convex hull of \( \mathcal{A} \overset{\text{def}}{=} \bigcup_{g \in G} D_g \).

At iteration \( t \) of RFW for a random subset \( G_p \) of size \( p \) of \( G \) we then propose to simply run RFW (algorithm 1) with

\[
\mathcal{A}_t \overset{\text{def}}{=} \bigcup_{g \in G_p} D_g .
\]

Denoting by \( g_p = \bigcup_{g \in G_p} g \) the LMO in RFW becomes

\[
\text{LMO}(x_t, \mathcal{A}_t) \in \arg \max_{v \in \mathcal{A}_t} \langle v_{(g_p)}, -\nabla_{(g_p)} f(x_t) \rangle .
\]

This means that we only need to compute the gradient on the \( g_p \) index. Depending on \( G \) and on the sub-sampling rate, this can be a significant computational benefit.

Experiments. We illustrate the convergence speed-up of using RFW over FW for latent group lasso regularized least square regression.

For \( d = 10000 \) we consider a collection \( G \) of groups of size 10 with an overlap of 3 and the associated atomic set \( \mathcal{A} \). We chose the ground truth parameter vector \( w_0 \in \text{conv}(\mathcal{A}) \) with a fraction of 0.01 of nonzero coefficients, where on each active group, the coefficients are generated from a Gaussian distribution. The data is a set of \( n \) pairs \((y_i, w_i) \in \mathbb{R} \times \mathbb{R}^d \) randomly generated from a Gaussian with some additive Gaussian noise. The regularizing parameter is \( \beta = 14 \), set so that the unconstrained optimum lies outside of the constrain set.
Figure 3. Performance of FW and AFW against RFW and RAFW respectively on the lasso problem with TF-IDF 2006 dataset. The subsampling parameter is $\eta = \frac{1}{25} = 0.06$ (again chosen arbitrarily) for RFW and $\eta = 0.25$ for RAFW. Right: Comparison of RAFW against RFW. Left: Comparison of RFW against FW. Upper: progress in FW dual gap versus number of iterations. Lower: progress of the FW dual gap versus cumulative number of computed coefficients in gradient per call to LMO.
Large dataset and Streaming Model. The design matrix is stored in disk. We allow both RFW and FW to access it only through chunks of size $n \times 500$. This streaming model allows a wall clock comparison of the two methods on very large scale problems.

Computing the gradient when the objective is the least squares loss consists in a matrix vector product. Computing it on a batch of coordinates then requires same operation with a smaller matrix. When computing the gradient at each randomized LMO call, the cost of slicing the design matrix can then compensate the gain in doing a smaller matrix vector product.

With data loaded in memory, which is typically the case for large datasets, both the LMO and the randomized LMO have this access data cost. Consider also that RFW allows any scheme of sampling, including one that minimizes the cost of data retrieval.

5. Conclusion and future work

We give theoretical guarantees of convergence of randomized versions of FW that exhibit same order of convergence as their deterministic counter-parts. As far as we know, for the case of RAFW, this is the first contribution of the kind. While the theoretical complexity bounds don’t necessarily imply this, our numerical experiments show that randomized versions often outperform their deterministic ones on $\ell_1$-regularized and latent group lasso regularized least squares. In both cases, randomizing the LMO allows us to compute the gradient only on a subset of its coordinates. We use it to speed up the method in a streaming model where the data is accessed by chunks, but there might be other situations where the structure of the polytope can be leveraged to make subsampling computationally beneficial.

There are also other linearly-convergent FW variants other than AFW, for which it might be possible to derive randomized variants.

Finally many recent results (Goldfarb et al., 2016; 2017; Hazan & Luo, 2016) on FW have combined various improvements of FW (away mechanism, sliding, lazy oracles, stochastic FW, etc.). Randomized oracles add to this toolbox and could further improve its benefits.
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How to Get Away with Subsampling: a Frank-Wolfe Algorithm for Optimizing over Large Atomic Domains

Supplementary material

Appendix notations. We denote by $E_t$ the conditional expectation at iteration $t$, conditioned on all the past and by $E$ a full expectation. We denote by a tilde the values that come from the deterministic analysis of FW. Denote by $r_t = -\nabla f(x_t)$.

For $k \in \mathbb{N}^*$, denote by $[k]$ all integer between 1 and $k$.

Appendix A. Proof of sub-linear convergence for Randomized Frank-Wolfe

In this section we provide a convergence proof for Algorithm 1. The proof is loosely inspired by that of (Locatello et al., 2017, Appendix B.1), with the obvious difference that the result of the LMO is a random variable in our case.

**Theorem 2.1**. Let $f$ be a function with bounded curvature constant $C_f$, Algorithm 1 for $\eta \in (0, 1)$, (with step-size chosen by either variants) converges towards a solution of (OPT), satisfying

$$E(f(x_T)) - f(x^*) \leq \frac{2(C_f + f(x_0) - f(x^*))}{\eta T + 2}. \quad (9)$$

**Proof.** By definition of the curvature constant, at iteration $t$ we have

$$f(x_t + \gamma(s_t - x_t)) \leq f(x_t) + \gamma \langle \nabla f(x_t), s_t - x_t \rangle + \frac{\gamma^2}{2} C_f. \quad (10)$$

By minimizing with respect to $\gamma$ on $[0, 1]$ we obtain

$$\gamma_t = \text{clip}_{[0,1]}(-\nabla f(x_t), s_t - x_t)/C_f, \quad (11)$$

which is the definition of $\gamma_t$ in the algorithm with Variant 2. Hence, we have

$$f(x_{t+1}) \leq f(x_t) + \min_{\gamma \in [0,1]} \left\{ \gamma \langle \nabla f(x_t), s_t - x_t \rangle + \frac{\gamma^2}{2} C_f \right\},$$

an inequality which is also valid for Variant 1 since by the line search procedure the objective function at $x_{t+1}$ is always equal or smaller than that of Variant 1. Denote by $h_t = f(x_t) - f(x^*)$,

$$h_{t+1} \leq h_t + \min_{\gamma \in [0,1]} \left\{ \gamma \langle \nabla f(x_t), s_t - x_t \rangle + \frac{\gamma^2}{2} C_f \right\}. \quad (12)$$

We write $\bar{s}_t$ the FW atom if we had started the FW algorithm at $x_t$, and $E_t$ the expectation conditionned on all the past until $x_t$, we have

$$E_t h_{t+1} \leq h_t + E_t \min_{\gamma \in [0,1]} \left\{ \gamma \langle \nabla f(x_t), s_t - x_t \rangle + \frac{\gamma^2}{2} C_f \right\} \quad (13)$$

$$\leq h_t + P(s_t = \bar{s}_t) \min_{\gamma \in [0,1]} \left\{ \gamma \langle \nabla f(x_t), \bar{s}_t - x_t \rangle + \frac{\gamma^2}{2} C_f \right\} \quad (14)$$

$$\leq h_t + \eta \min_{\gamma \in [0,1]} \left\{ -\gamma h(x_t) + \frac{\gamma^2}{2} C_f \right\} \quad \text{(for any } \gamma \in [0,1], \text{ by definition of min)}, \quad (15)$$
where the second inequality follows from the definition of expectation and the fact that minimum is non-positive since it is zero for $\gamma = 0$. The last inequality is a consequence of uniform sampling as well as it uses that the FW gap is an upper bound on the dual gap, e.g. $\langle -\nabla f(x_t), \bar{s}_t - x_t \rangle \geq h(x_t)$.

**Induction.** From (15) the following is true for any $\gamma \in [0, 1]$

$$E_t(h_{t+1}) \leq h_t(1 - \eta \gamma) + \frac{\gamma^2}{2} \eta C_f.$$  

(16)

Taking unconditional expectation and writing $H_t = \mathbb{E}(h_t)$, we get for any $\gamma \in [0, 1]$

$$H_{t+1} \leq H_t(1 - \eta \gamma) + \frac{\gamma^2}{2} \eta C_f.$$  

(17)

With $\gamma_t = \frac{2}{\eta t + 2} \in [0, 1]$, we get by induction

$$H_t \leq 2 \frac{C_f + \epsilon_0}{\eta t + 2} = \gamma_t(C_f + \epsilon_0),$$  

(18)

where $\epsilon_0 = f(x_0) - f(x^\ast)$. Initialization follows the fact that the curvature constant is positive. For $t > 0$, from (17) and the induction hypothesis

$$H_{t+1} \leq \gamma_t(C_f + \epsilon_0)(1 - \eta \gamma_t) + \frac{\gamma_t^2}{2} \eta C_f$$

$$\leq \gamma_t(C_f + \epsilon_0)(1 - \eta \gamma_t) + \frac{\gamma_t^2}{2} \eta (C_f + \epsilon_0)$$

$$\leq \gamma_t(C_f + \epsilon_0)(1 - \eta \gamma_t + \frac{\gamma_t}{2})$$

$$\leq (C_f + \epsilon_0)(1 - \frac{\gamma_t}{2} \eta)$$

$$\leq (C_f + \epsilon_0) \gamma_{t+1}.$$  

The last inequality comes from the fact that $(1 - \frac{\gamma_t}{2} \eta) \gamma_t \leq \gamma_{t+1}$. Indeed, with $\gamma_t = \frac{2}{\eta t + 2}$, it is equivalent to

$$(1 - \frac{\eta}{\eta t + 2}) \frac{2}{\eta t + 2} \leq \frac{2}{\eta (t+1) + 2}$$

$$\Leftrightarrow (\eta t + 2) - \eta \frac{\eta t + 2}{\eta t + 2} \leq \frac{\eta t + 2}{\eta (t+1) + 2}$$

$$\Leftrightarrow (\eta t + 2 - \eta)(\eta (t+1) + 2) \leq (\eta t + 2)^2$$

$$\Leftrightarrow \eta^2 t^2 + 4\eta t + 4 - \eta^2 \leq \eta^2 t^2 + 4\eta t + 4.$$  

The last being true, it concludes the proof. 

■
Appendix B. Proof of linear convergence for RFW

Away curvature and geometric strong convexity. The away curvature constant is a modification of the curvature constant described in the previous subsection, in which the FW direction $s - x$ is replaced with an arbitrary direction $s - v$:

$$C_f^A \overset{\text{def}}{=} \sup_{x, s, v \in \mathcal{M}} \frac{2}{\gamma^2} \left( f(y) - f(x) - \gamma \langle \nabla f(x), s - v \rangle \right).$$

The geometric strong convexity constant $\mu_f^A$ depends on both the function and the domain (in contrast to the standard strong convexity definition) and is defined as (see “An Affine Invariant Notion of Strong Convexity” in (Lacoste-Julien & Jaggi, 2015) for more details)

$$\mu_f^A = \inf_{x \in \mathcal{M}} \inf_{x^* \in \mathcal{M}} \frac{2}{\gamma^2} \left( B_f(x, x^*) \right)$$

where $B_f(x, x^*) = f(x^*) - f(x) - \langle \nabla f(x), x^* - x \rangle$ and $\gamma^A(x, x^*)$ the positive step-size quantity:

$$\gamma^A(x, x^*) := \frac{\langle -\nabla f(x), x^* - x \rangle}{\langle -\nabla f(x), s_f(x) - v_f(x) \rangle}.$$

In particular $s_f(x)$ is the Frank Wolfe atom starting from $x$. $v_f(x)$ is the away atom when considering all possible expansions of $x$ as a convex combinations of atoms in $\mathcal{A}$. Denote by $S_x := \{ S \mid S \subseteq \mathcal{A} \text{ such that } x \text{ is a proper convex combination of all elements in } S \}$ and by $v_{S(x)} := \arg\max_{v \in \mathcal{S}} \langle \nabla f(x), v \rangle$. $v_f(x)$ is finally defined by

$$v_f(x) \overset{\text{def}}{=} \arg\min_{v = v_{S} \mid S \subseteq S_x} \langle \nabla f(x), v \rangle.$$

Similarly following (Lacoste-Julien & Jaggi, 2015, Lemma 9 in Appendix F), the geometric $\tilde{\mu}$-generally-strongly-convex constant is defined as

$$\tilde{\mu}_f = \inf_{x \in \mathcal{M}} \inf_{x^* \in \mathcal{M}} \frac{1}{\gamma^2} \left( f(x^*) - f(x) - 2\langle \nabla f(x), x^* - x \rangle \right),$$

where $\chi^*$ represents the solution set of (OPT).

Notations. In the context of RFW, $\mathcal{A}$ denotes the finite set of extremes atoms such that $\mathcal{M} = \text{Conv}(\mathcal{A})$. At iteration $t$, $\mathcal{A}_t$ is a random subset of element of $\mathcal{A} \setminus S_t$ where $S_t$ is the current support of the iterate. The Randomized LMO is performed over $\mathcal{V}_t = S_t \cup \mathcal{A}_t$ so that for Algorithm 2, $s_t \overset{\text{def}}{=} \arg\max_{v \in \mathcal{V}_t} \langle -\nabla f(x_t), v \rangle$ is the FW atom at iteration $t$ for RFW.

Note that when $|\mathcal{A} \setminus S_t| \leq p$, Algorithm 2 does exactly the same as AFW. For the sake of simplicity we will consider that this is not the case. Indeed we would otherwise fall back into the deterministic setting and the proof would just be that of (Lacoste-Julien & Jaggi, 2015).

We use tilde notation for quantities that are specific to the deterministic FW setting. For instance, $\tilde{s}_t \overset{\text{def}}{=} \arg\max_{v \in \mathcal{A}} \langle -\nabla f(x_t), v \rangle$ is the FW atom for AFW starting at $x_t$.

Similarly the Away atom is such that $v_t \overset{\text{def}}{=} \arg\min_{v \in \mathcal{S}_t} \langle -\nabla f(x_t), v \rangle$ and it does not depend on the sub-sampling at iteration $t$. Here we do not use any tilde because it is a quantity that appears both in AFW and its Randomized counterpart.

In AFW, $\tilde{g}_t \overset{\text{def}}{=} \langle -\nabla f(x_t), \tilde{s}_t - v_t \rangle = \max_{s \in \mathcal{A}} \langle -\nabla f(x_t), s - v_t \rangle$ is an upper-bound of the dual gap, named the pair-wise dual gap (Lacoste-Julien & Jaggi, 2015). We consider the corresponding partial pair-wise dual gap $\bar{g}_t \overset{\text{def}}{=} \langle -\nabla f(x_t), s_t - v_t \rangle = \max_{s \in \mathcal{V}_t} \langle -\nabla f(x_t), s - v_t \rangle$. It is partial is the sense that the maximum is computed on a subset $\mathcal{V}_t$ of $\mathcal{A}$ which results in the fact that it is not guaranteed anymore to be an upper-bound on the dual-gap.
Structure of the proof. The proof is structured around a main part that uses Lemmas 1 and 3. Lemma 2 is just used to prove Lemma 3.

The main proof follows the scheme of the deterministic one of AFW in (Lacoste-Julien & Jaggi, 2015, Theorem 8). It is divided in three parts. The first part consists in upper bounding \( h_t \triangleq f(x_t) - f(x^*) \) with \( \tilde{g}_t \). It does not depend on the specific construction of the iterates \( x_t \), and thus remains the same as that in (Lacoste-Julien & Jaggi, 2015). The second part provides a lower bound on the progress on the algorithm, namely

\[
h_{t+1} \leq (1 - \rho_f \frac{\tilde{g}_t}{g_t})^2 h_t, \tag{19}
\]

with \( \rho_f = \frac{\mu^A}{2\mu^A} \), when it is not doing a bad drop step (defined above). As a proxy for this event, we use the binary variable \( z_t \) that equals 0 for bad drop steps and 1 otherwise.

The difficulty lies in that we guarantee a geometrical decrease only when \( g_t = \tilde{g}_t \) and \( z_t = 1 \). Because of the sub-sampling and unlike in the deterministic setting, \( z_t \) is a random variable. Lemma 3 provides a lower bound on the probability of interest, \( P(\tilde{g}_t = g_t \mid z_t = 1) \), for the last part of the main proof.

Finally, the last part of the proof constructs a bound on the number of times we can expect both \( z_t = 1 \) and \( g_t = \tilde{g}_t \) subject to the constraint that at least half of the iterates satisfy \( z_t = 1 \). It is done by recurrence.

Appendix B.1. Lemmas

This lemma ensures the chosen direction \( d_t \) in RAFW is a good descent direction, and links it with \( g_t \) which may be equal to \( \tilde{g}_t \).

**Lemma 1.** Let \( s_t, v_t \) and \( d_t \) be as defined in Algorithm 2. Then for \( g_t \triangleq \langle -\nabla f(x_t), s_t - v_t \rangle \), we have

\[
\langle -\nabla f(x_t), d_t \rangle \geq \frac{1}{2} g_t \geq 0. \tag{20}
\]

**Proof.** The first inequality appeared already in the convergence proof of Lacoste-Julien & Jaggi (2015, Eq. (6)), which we repeat here for completeness. By the definition of \( d_t \) we have:

\[
2\langle -\nabla f(x_t), d_t \rangle \geq \langle -\nabla f(x_t), d_t^A \rangle + \langle -\nabla f(x_t), d_t^{FW} \rangle = \langle -\nabla f(x_t), s_t - v_t \rangle = g_t \tag{21}
\]

We only need to prove that \( g_t \) is non-negative. In line 3 of algorithm 2, \( s_t \) is the output of LMO performs of the set of atoms \( S_t \cup A_t \triangleq V_t \),

\[
s_t = \arg \max_{s \in V_t} \langle -\nabla f(x_t), s \rangle,
\]

so that we have \( \langle -\nabla f(x_t), s_t \rangle \geq \langle -\nabla f(x_t), v_t \rangle \). By definition of \( g_t \), it implies \( g_t \geq 0 \). \qed

Lemma 2 is just a simple combinatorial result needed in Lemma 3. Consider a sequence of \( m \) numbers, we lower bound the probability for the maximum of a subset of size greater than \( p \) to be equal to the maximum of the sequence.

**Lemma 2.** Consider any sequence \( (r_i)_{i \in \mathcal{I}} \) in \( \mathbb{R} \) with \( \mathcal{I} = \{1, \ldots, m\} \), and a subset \( \mathcal{I}_p \subseteq \mathcal{I} \) of size \( p \). We have

\[
P(\max_{i \in \mathcal{I}_p} r_i = \max_{i \in \mathcal{I}} r_i) \geq \frac{p}{m}. \tag{22}
\]

**Proof.** Consider \( M = \{i \in \mathcal{I} \mid r_i = \max_{j \in \mathcal{I}} r_j\} \). We have max \( r_i = \max_{i \in \mathcal{I}_p} r_i \) if and only if at least one element of \( \mathcal{I}_p \) belongs to \( M \):

\[
P(\max_{i \in \mathcal{I}_p} r_i = \max_{i \in \mathcal{I}} r_i) = P(|\mathcal{I}_p \cap M| \geq 1). \tag{23}
\]
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By definition $M$ has at least one element $i_0$. Since \{ $i_0 \in I_p$ \} $\subseteq$ \{ $|I_p \cap M| \geq 1$ \}

\[ P(|I_p \cap M| \geq 1) \geq P(\{ i_0 \in I_p \}) . \] (24)

All subsets are taken uniformly at random, we just have to count the number of subset $I_p$ of $I$ of size $p$ with $i_0 \in I_p$

\[ P(\{ i_0 \in I_p \}) = \frac{(p-1)}{p} = \frac{p}{m} \] (25)

\[ P(\max r_i = \max r_i) \geq \frac{p}{m} , \] (26)

In the second part of the main proof we ensure a geometric decrease when both $g_t = \tilde{g}_t$ and $z_t = 1$, i.e. outside of bad drop steps. The following lemma helps quantifying the probability of $g_t = \tilde{g}_t$ holding when $z_t = 1$.

**Lemma 3.** Consider $g_t$ (defined in Lemma 1) to be the partial pairwise (PW) dual gap of RAFW at iteration $t$ with sub-sampling parameter $p$ on the constrained polytope $M = \text{conv}(A)$, where $A$ is a finite set of extremes points of $M$. $\tilde{g}_t = \max_{a \in A} \langle -\nabla f(x_t), a - v_t \rangle$ is the pairwise dual gap of AFW starting at $x_t$ on this same polytope. Denote by $z_t$ the binary random variable that equals 0 when the $t$th iteration of RAFW makes an away step that is a drop step with $\gamma_{\text{max}} < 1$ (a bad drop step), and 1 otherwise. Then we have the following bound

\[ P(g_t = \tilde{g}_t \mid x_t, z_t = 1) \geq \left( \frac{p}{|A|} \right)^2 . \] (PROB)

**Proof.** Recall that $g_t^A \overset{\text{def}}{=} \langle r_t, d_t^A \rangle$. By definition \{ $z_t = 0$ \} $= \{ g_t < g_t^A, \gamma_{\text{max}} < 1, \gamma_t^* = \gamma_{\text{max}} \}$, where $\gamma_t^* \overset{\text{def}}{=} \arg\min_{\gamma \in [0, \gamma_{\text{max}}]} f(x_t + \gamma d_t^A)$. Its complementary \{ $z_t = 1$ \} can thus be expressed as the partition $A_1 \cup A_2 \cup A_3$ where the $A_i$ are defined by

\[ A_1 \overset{\text{def}}{=} \{ g_t \geq g_t^A \} \quad \text{(performs a FW step)} \] (27)

\[ A_2 \overset{\text{def}}{=} \{ g_t < g_t^A, \alpha_v^{(t)}/(1 - \alpha_v^{(t)}) \geq 1 \} \quad \text{(performs away step with } \gamma_{\text{max}} \geq 1) \] (28)

\[ A_3 \overset{\text{def}}{=} \{ g_t < g_t^A, \alpha_v^{(t)}/(1 - \alpha_v^{(t)}) < 1, \gamma_t^* < \alpha_v^{(t)}/(1 - \alpha_v^{(t)}) \} \] (29)

First note that in the case of $A_2$ and $A_3$, $\gamma_{\text{max}} = \alpha_v^{(t)}/(1 - \alpha_v^{(t)})$. Though the right hand side formulation highlights that it is entirely determined by $x_t$, recalling that $\alpha_v^{(t)}$ is the mass along the atom $v_t$ in the decomposition of $x_t$ in §3.

From a higher level perspective, these cases are those for which we can guarantee a geometrical decrease of $h_t = f(x_t) - f(x^*)$ (see second part of main proof). By definition, the $A_i$ are disjoints. $A_1$ represents a choice of a FW step in RAFW contrary to $A_2$ and $A_3$ which stands for an away step choice in RAFW. $A_2$ is an away step for which there is enough potential mass ($\gamma_{\text{max}} > 1$) to move along the away direction and to ensure sufficient objective decreasing. $A_3$ encompasses the situations where there is not a lot of mass along the away direction ($\gamma_{\text{max}} < 1$) but which is not a drop step, e.g. the amount of mass is not a limit to the descent.

Our goal is to lower bound $P = \mathcal{P}(g_t = \tilde{g}_t \mid x_t, z_t = 1)$. The following probabilities will be with respect to the $t$th sub-sampling only. Notice that $g_t^A, \tilde{g}_t$ and $\alpha_v^{(t)}$ are known given $\{ x_t, z_t = 1 \}$. Using Bayes’ rule, and because the $A_i$ are disjoints, we have

\[ P = \mathcal{P}(g_t = \tilde{g}_t \mid x_t, \{ z_t = 1 \}) \]

\[ = \sum_{i=1}^{3} \mathcal{P}(g_t = \tilde{g}_t \mid x_t, A_i) \mathcal{P}(A_i \mid x_t) \]

\[ \sum_{i=1}^{3} \mathcal{P}(A_i \mid x_t) . \] (30)

By definition of $g_t$ and $\tilde{g}_t$, $g_t \leq \tilde{g}_t$, so that measuring the probability of an event like $\{ g_t = \tilde{g}_t \}$ conditionally on $\{ g_t \leq g_t^A \}$ will naturally depend on whether or not, the deterministic condition $\tilde{g}_t \geq g_t^A$ is satisfied. Hence the following case distinction.
Recall $\mathcal{V}_t = S_t \cup A_t$.

**Case $\tilde{g}_t < g_t^A$.**

\[
P = \frac{\sum_{i=1}^{3} P(g_t = \tilde{g}_t \mid x_t, A_t, \tilde{g}_t < g_t^A)P(A_t \mid x_t, \tilde{g}_t < g_t^A)}{\sum_{i=1}^{3} P(A_t \mid x_t, \tilde{g}_t < g_t^A)}. \tag{31}
\]

Recall that $A_1 = \{g_t \geq g_t^A\}$. Since by definition $g_t \leq \tilde{g}_t$, conditionally on $\{\tilde{g}_t < g_t^A\}$, the probability of $A_1$ is zero. Consequently the above reduces to

\[
P = \frac{\sum_{i=2}^{3} P(g_t = \tilde{g}_t \mid x_t, A_t, \tilde{g}_t < g_t^A)P(A_t \mid x_t, \tilde{g}_t < g_t^A)}{\sum_{i=2}^{3} P(A_t \mid x_t, \tilde{g}_t < g_t^A)} \geq \frac{p \sum_{i=2}^{3} P(A_t \mid x_t, \tilde{g}_t < g_t^A)}{|A| \sum_{i=2}^{3} P(A_t \mid x_t, \tilde{g}_t < g_t^A)} = \frac{p}{|A|}. \tag{32}
\]

Where the last inequality is because for $i = 2, 3$ we have $P(g_t = \tilde{g}_t \mid x_t, A_t, \tilde{g}_t < g_t^A) \geq \frac{p}{|A|}$. Indeed for $A_3$ (case $A_2$ is similar) denote

\[
P_1 = P(g_t = \tilde{g}_t \mid x_t, A_3, \tilde{g}_t < g_t^A) = P(\max_{s \in V_t} \langle r_t, s \rangle = \max_{a \in A} \langle r_t, s \rangle \mid x_t, \max_{s \in V_t} \langle r_t, s \rangle < C_0, \max_{s \in V_t} \langle r_t, s \rangle < C_0, \alpha_{v_t}^{(t)}/(1 - \alpha_{v_t}^{(t)}) < 1, \gamma_t^* < \alpha_{v_t}^{(t)}/(1 - \alpha_{v_t}^{(t)})) . \tag{33}
\]

with $C_0 \overset{\text{def}}{=} g_t^A + \langle r_t, v_t \rangle$ and $r_t = -\nabla f(x_t)$ not depending on the $t^{th}$ sub-sampling. Also the event $\{\max_{s \in V_t} \langle r_t, s \rangle < C_0\}$ is a consequence of $\{\max_{s \in A} \langle r_t, s \rangle < C_0\}$ so that $P_1$ simplifies to

\[
P_1 = P(\max_{s \in V_t} \langle r_t, s \rangle = \max_{a \in A} \langle r_t, s \rangle \mid x_t, \max_{s \in V_t} \langle r_t, s \rangle < C_0, \alpha_{v_t}^{(t)}/(1 - \alpha_{v_t}^{(t)}) < 1, \gamma_t^* < \alpha_{v_t}^{(t)}/(1 - \alpha_{v_t}^{(t)})) . \tag{34}
\]

By definition

\[
\gamma_t^* \in \arg\min_{\gamma \in [0, \alpha_{v_t}^{(t)}/(1 - \alpha_{v_t}^{(t)})]} f(x_t + \gamma d_t^A), \tag{35}
\]

so that $\gamma_t^*$ does not depend on the $t^{th}$ sub-sampling. Finally all the conditioning in the probability of (34) do not depend on this $t^{th}$ sub-sampling. Hence we are in the position of using Lemma 2 for the sequence $(\langle r_t, s \rangle)_{s \in A}$. Also by definition of $\mathcal{V}_t = S_t \cup A_t$, we have $|\mathcal{V}_t| \geq p$ so that we finally get

\[
P(g_t = \tilde{g}_t \mid x_t, A_3, \tilde{g}_t < g_t^A) \geq \frac{p}{|A|}. \tag{36}
\]

This was what was needed to conclude (32).

**Case $\tilde{g}_t \geq g_t^A$.** In such a case, $P$ from (30) rewrites as

\[
P = \frac{\sum_{i=1}^{3} P(g_t = \tilde{g}_t \mid x_t, A_t, \tilde{g}_t \geq g_t^A)P(A_t \mid x_t, \tilde{g}_t \geq g_t^A)}{\sum_{i=1}^{3} P(A_t \mid x_t, \tilde{g}_t \geq g_t^A)}. \tag{37}
\]

Here $P(g_t = \tilde{g}_t \mid x_t, A_i, \tilde{g}_t \geq g_t^A) = 0$ for $i = 2, 3$ because $A_i$ implies $g_t < g_t^A$. So that when $\tilde{g}_t \geq g_t^A$ it is then impossible for $g_t$ to equal $\tilde{g}_t$.

\[
P = \frac{P(g_t = \tilde{g}_t \mid x_t, A_1, \tilde{g}_t \geq g_t^A)P(A_1 \mid x_t, \tilde{g}_t \geq g_t^A)}{\sum_{i=1}^{3} P(A_t \mid x_t, \tilde{g}_t \geq g_t^A)}. \tag{38}
\]

Here also we use, and prove later on (see § below the conclusion of the proof of the Lemma), the lower bound

\[
P(g_t = \tilde{g}_t \mid x_t, A_1, \tilde{g}_t \geq g_t^A) \geq \frac{p}{|A|},
\]
that implies
\[ P \geq \frac{p}{|A|} \sum_{i=1}^{3} \mathcal{P}(A_i \mid x_t, \tilde{g}_t \geq g^A_t) \].

Because the \( A_i \) are disjoint, \( \sum_{i=1}^{3} \mathcal{P}(A_i \mid x_t, \tilde{g}_t \geq g^A_t) \leq 1 \) we have
\[ P \geq \frac{p}{|A|} \mathcal{P}(A_1 \mid x_t, \tilde{g}_t \geq g^A_t) \].

Using a similar lower bound as (38), namely
\[ \mathcal{P}(A_1 \mid x_t, \tilde{g}_t \geq g^A_t) \geq \frac{p}{|A|}, \]
we finally get
\[ P \geq \left( \frac{p}{|A|} \right)^2. \]

Since it is hard to precisely count the occurrences of \( \{\tilde{g}_t \geq g^A_t\} \) and \( \{\tilde{g}_t < g^A_t\} \), we use a conservative bound in (40)
\[ \mathcal{P}(g_t = \tilde{g}_t \mid x_t, z_t = 1) \geq \left( \frac{p}{|A|} \right)^2. \]

This will of course make our bound on the rate of convergence very conservative.

**Justification for (38) and (39).**

Let’s denote the left hand side of (38) by \( P_2 \). By definition of \( g_t \) and \( \tilde{g}_t \), with \( r_t = -\nabla f(x_t) \), we have:
\[ P_2 = \mathcal{P}(\max_{s \in V_t} \langle r_t, s - v_t \rangle = \max_{s \in A} \langle r_t, s - v_t \rangle \mid x_t, \max_{s \in V_t} \langle r_t, s - v_t \rangle \geq g^A_t, \max_{s \in A} \langle r_t, s - v_t \rangle \geq g^A_t) \]
\[ = \mathcal{P}(\max_{s \in A} \langle r_t, s \rangle = \max_{s \in V_t} \langle r_t, s \rangle \mid x_t, \max_{s \in A} \langle r_t, s \rangle \geq 0, \max_{s \in A} \langle r_t, s \rangle \geq C_0), \]
where \( C_0 = g^A_t + \langle r_t, v_t \rangle \) and \( r_t \) does not depend on the random sampling at iteration \( t \). Bayes formula leads to
\[ P_2 = \frac{\mathcal{P}(\{\max_{s \in A} \langle r_t, s \rangle = \max_{s \in V_t} \langle r_t, s \rangle \cap \{\max_{s \in V_t} \langle r_t, s \rangle \geq C_0\} \mid x_t, \max_{s \in A} \langle r_t, s \rangle \geq C_0\})}{\mathcal{P}(\max_{s \in V_t} \langle r_t, s \rangle \geq C_0 \mid x_t, \max_{s \in A} \langle r_t, s \rangle \geq C_0)} . \]
Conditionally on \( \{\max_{s \in A} \langle r_t, s \rangle \geq C_0\} \), the event \( \{\max_{s \in V_t} \langle r_t, s \rangle = \max_{s \in A} \langle r_t, s \rangle\} \) implies \( \max_{s \in V_t} \langle r_t, s \rangle \geq C_0 \), which leads to
\[ P_2 = \frac{\mathcal{P}(\max_{s \in A} \langle r_t, s \rangle = \max_{s \in V_t} \langle r_t, s \rangle \mid x_t, \max_{s \in A} \langle r_t, s \rangle \geq C_0)\}}{\mathcal{P}(\max_{s \in V_t} \langle r_t, s \rangle \geq C_0 \mid x_t, \max_{s \in A} \langle r_t, s \rangle \geq C_0)\}} \]
\[ \geq \mathcal{P}(\max_{s \in V_t} \langle r_t, s \rangle = \max_{s \in A} \langle r_t, s \rangle \mid x_t, \max_{s \in A} \langle r_t, s \rangle \geq C_0) \geq \frac{p}{|A|} , \]
where the last inequality is a consequence of applying Lemma 2 on the sequence \( \langle\langle r_t, s\rangle\rangle_{s \in A} \).

Similarly let’s denote the left hand side of (39) by \( P_3 \). The first inequality is justified because conditionally on \( \{\tilde{g}_t \geq g^A_t\} \), \( \{g_t = \tilde{g}_t\} \subset \{g_t \geq g^A_t\} \) and the last by applying, similarly as for (38), Lemma 2 on the sequence \( \langle\langle r_t, s\rangle\rangle_{s \in A} \).
\[ P_3 = \mathcal{P}(g_t \geq g^A_t \mid x_t, \tilde{g}_t \geq g^A_t) \]
\[ \geq \mathcal{P}(g_t = \tilde{g}_t \mid x_t, \tilde{g}_t \geq g^A_t) \]
\[ \geq \mathcal{P}(\max_{s \in V_t} \langle r_t, s \rangle = \max_{s \in A} \langle r_t, s \rangle \mid x_t, \max_{s \in A} \langle r_t, s \rangle \geq C_0) \]
\[ \geq \frac{p}{|A|} . \]
Appendix B.2. Main proof

Theorem 3.1’. Consider the set $\mathcal{M} = \text{conv}(\mathcal{A})$, with $\mathcal{A}$ a finite set of extreme atoms, after $T$ iterations of Algorithm 2 (RAFW) we have the following linear convergence rate

$$
\mathbb{E}[h(x_{T+1})] \leq (1 - \eta^2 \rho_f)^{\max\{0, [(T-s)/2]\}} h(x_0),
$$

with $\rho_f = \mu_f^2 / \mathcal{M}$. $\eta = \frac{p}{|\mathcal{A}|}$ and $s = |\mathcal{S}_0|$.

Proof. The classical curvature constant used in proofs related to non-Away Frank-Wolfe is

$$
C_f := \sup_{\mathcal{M}, \gamma \in [0,1]} \left\{ \frac{2}{\gamma^2} \langle f(y) - f(x) - \nabla f(x), y - x \rangle \right\}.
$$

It is tailored for algorithms in which the update is of the form $x_{t+1} = (1 - \gamma)x_t + \gamma v_t$, but this is not the shape of all updates in away versions of FW. In (Lacoste-Julien & Jaggi, 2015) they introduced a modification of the above curvature constant that we also use to analyze RAFW. It is defined in (Lacoste-Julien & Jaggi, 2015, equation (26)) as

$$
C_f^A := \sup_{\mathcal{M}, \gamma \in [0,1]} \left\{ \frac{2}{\gamma^2} \langle f(y) - f(x) - \gamma \langle \nabla f(x), y - v \rangle, \gamma \rangle \right\}.
$$

It differs from $C_f$ (46) because it allows to move outside of the domain $\mathcal{M}$. We thus require L-lipschitz continuous function on any compact set for that quantity to be upper-bounded. We refer to § curvature constants on (Lacoste-Julien & Jaggi, 2015, Appendix D) for thorough details. The first part of the proof reuses the scheme of (Lacoste-Julien & Jaggi, 2015, Theorem 8).

First part. Upper bounding $h_t$: Considering an iterate $x_t$ that is not optimal (e.g. $x_t \neq x^*$), from (Lacoste-Julien & Jaggi, 2015, Eq. (28)), we have

$$
f(x_t) - f(x^*) = h_t \leq \frac{g^2}{2\mu^2},
$$

where $g_t$ is the pair-wise dual gap defined by $g_t = \langle s_t - v_t, -\nabla f(x_t) \rangle$. $s_t$ and $v_t$ are respectively the FW atom and the away atom in the classical Away step algorithm (conditionally on $x_t$, the away atom of the randomized variant coincides with the away atom of the non-randomized variant). The result is still valid here as it only uses the definition of the affine invariant version of the strong convexity parameter and does not depend on the way $x_t$ are constructed (see upper bounding $h_t$ in (Lacoste-Julien & Jaggi, 2015, Proof for AFW in Theorem 8)).

Second part. Lower bounding progress $h_t - h_{t+1}$. Consider $x_t$ a non-optimal iterate. At step $t$, the update in Algorithm 2 writes $x_{t+1}(\gamma) = x_t + \gamma d_t$. $\gamma$ is optimized by line-search in the segment $[0, \gamma_{\text{max}}]$. Because in either cases $d_t$ is a difference between two elements of $\mathcal{M}$, from the definition of $C_f^A$ and because of the exact line search, we have

$$
f(x_{t+1}) \leq \min_{\gamma \in [0, \gamma_{\text{max}}]} \left( f(x_t) + \gamma \langle \nabla f(x_t), d_t \rangle + \frac{\gamma^2}{2} C_f^A \right),
$$

so that for any $\gamma \in [0; \gamma_{\text{max}}]$

$$
f(x_{t+1}) - f(x_t) \leq \gamma \langle \nabla f(x_t), d_t \rangle + \frac{\gamma^2}{2} C_f^A
$$
or again

$$
\frac{\gamma g_t^2}{\mu^2} - \frac{\gamma^2}{2} C_f^A \leq f(x_t) - f(x_{t+1}),
$$

(49)
where the last inequality is a consequence of Lemma 1. We write \( \gamma_t^B \overset{\text{def}}{=} \frac{g_t}{2C_f} \geq 0 \), the minimizer of the left hand side of (49).

**Case** \( \gamma_{\text{max}} \geq 1 \) and \( \gamma_t^B \leq \gamma_{\text{max}} \). (49) evaluated on \( \gamma = \gamma_t^B \) gives

\[
\frac{g_t^2}{4C_f} - \frac{g_t^2}{8C_f} \leq f(x_t) - f(x_{t+1})
\]

\[
\implies \left( \frac{g_t}{g_t} \right)^2 \frac{g_t^2}{8C_f} \leq h_t - h_{t+1}.
\]

Indeed, \( x_t \) is assumed not to be optimal, so that \( \tilde{g}_t \neq 0 \). Combining (50) with (48) gives

\[
h_{t+1} \leq h_t - \left( \frac{g_t}{g_t} \right)^2 \frac{g_t^2}{8C_f}
\]

\[
\leq h_t - \left( \frac{g_t}{g_t} \right)^2 \frac{\mu_f^2}{4C_f} h_t
\]

\[
= (1 - \rho_f \left( \frac{g_t}{g_t} \right)^2) h_t.
\]

**Case** \( \gamma_{\text{max}} \geq 1 \) and \( \gamma_t^B > \gamma_{\text{max}} \). \( \gamma_t^B = \frac{g_t}{2C_f} \) implies \( g_t \geq 2C_f^A \). (49) transforms into

\[
\frac{g_t^2}{2} (\gamma - \frac{\gamma^2}{2}) \leq f(x_t) - f(x_{t+1})
\]

\[
\frac{g_t}{g_t} \frac{g_t}{g_t} (\gamma - \frac{\gamma^2}{2}) \leq f(x_t) - f(x_{t+1}).
\]

Using \( \tilde{g}_t \geq h_t \) and evaluating at \( \gamma = 1 \), leaves us with

\[
h_{t+1} \leq (1 - \frac{1}{4} \frac{g_t}{g_t}) h_t.
\]

Because \( \mu_f^A \leq C_f^A \) (Lacoste-Julien & Jaggi, 2015, Remark 7.) and \( \rho_f = \frac{\mu_f^A}{4C_f^A} \), the two previous cases resolve in the following inequality

\[
h_{t+1} \leq (1 - \rho_f \left( \frac{g_t}{g_t} \right)^2) h_t.
\]

**Case** \( \gamma_{\text{max}} < 1 \) and \( \gamma_t^* < \gamma_{\text{max}} \). By definition

\[
\gamma_t^* = \arg \min_{\gamma \in [0, \gamma_{\text{max}}]} f(x_t + \gamma d_t) = F(\gamma).
\]

\( f \) is convex and its minimum on \([0; \gamma_{\text{max}}]\) is not reached at \( \gamma_{\text{max}} \). It is then also a minimum on the interval \([0; +\infty]\), and in particular we have

\[
\gamma_t^* = \arg \min_{\gamma \in [0, 1]} f(x_t + \gamma d_t) = F(\gamma).
\]

(49) can then be written with \( \gamma \in [0, 1] \) which leads to the previous case result (55).

**Case** \( \gamma_{\text{max}} < 1 \) and \( \gamma_t^* = \gamma_{\text{max}} \). This corresponds to a particular drop step for which we only guarantee \( h_{t+1} \leq h_t \) (exact line-search). We call this case a bad drop step (indeed \( \gamma_{\text{max}} > 1 \) and \( \gamma_t^* = \gamma_{\text{max}} \) also corresponds to a drop step, but for which we can prove a bound of the form \( h_{t+1} \leq h_t (1 - \rho_f \left( \frac{g_t}{g_t} \right)^2) \)).

We use the binary indicator \( z_t \) to distinguish between the step where (55) is guaranteed or not. Denote by \( z_t = 0 \) when doing a bad drop step and \( z_t = 1 \) otherwise. The second part can be summed-up in

\[
h_{t+1} \leq h_t (1 - \rho_f \left( \frac{g_t}{g_t} \right)^2) z_t.
\]
**Last part.** Consider starting RAFW (Algorithm 2) for \( T \) iterations at \( x_0 \in \text{conv}(V) \), with \( s = |S_0| \geq 0 \). We will now prove there are at most \( \left\lceil \frac{T + s}{2} \right\rceil \) drop steps. Let \( D_T \) be the number of drop steps after iteration \( T \) and \( F_T \) the number of FW step adding a new atom until iteration \( T \). By definition, a FW step is not a drop step so that \( D_T + F_T \leq T \). Also \( |S_T| = |S_0| + |F_T| - |D_T| \), hence \( |S_T| \leq |S_0| - 2|D_T| + T \) so that \( |D_T| \leq \frac{T + s - |S_T|}{2} \). Finally because \( |S_T| \geq 0 \), we have \( |D_T| \leq \left\lceil \frac{T + s}{2} \right\rceil \).

From the first two parts of the main proof, we have that

\[
h_T \leq h_0 \prod_{t=0}^{T-1} (1 - \rho_f \left( \frac{g_t}{g_t} \right)^2)^{z_t},
\]

where \((g_t, z_t)_{t \in [0:T-1]}\) are defined along RAFW starting at \( x_0 \). For \( i \leq j \), we write \( E_{i:j} \) the expectation with respect to all sub-sampling between the \( i^{th} \) iteration and the \( j^{th} \) iteration included. When taking expectation only over sub-sampling \( i \), we write it \( E_i \).

We will now prove by recurrence on \( T \in \mathbb{N}^* \) that

\[
E_{0:T-1} \left( \prod_{t=0}^{T-1} (1 - \rho_f \left( \frac{g_t}{g_t} \right)^2)^{z_t} \right) \leq (1 - \rho_f \eta^2)^{\max\{0, T - \left\lceil \frac{T + s}{2} \right\rceil \}} = F(T, s) \quad \forall s \in \mathbb{N} \ \forall x_0 \in \mathbb{R}^d \quad \text{with} \quad |S_0| = s,
\]

where \( x_0 = \sum_{v \in A} \alpha_v^{(0)} v \) and \( S_0 = \{ v \in A \text{ s.t. } \alpha_v^{(0)} > 0 \} \).

The rate quantity \( \max\{0, T - \left\lceil \frac{T + s}{2} \right\rceil \} \) represents the number of steps (between iteration 0 and \( T - 1 \)) in which \( z_t = 1 \), e.g. the steps in which there is a possibility of having geometrical decrease. Note that the geometrical decrease happens only when \( g_t = \tilde{g}_t \).

The key insight in the global bound is to recall (from section 3) that if the support is a singleton, i.e. \( |S_t| = 1 \), RAFW does a FW step hence \( z_t = 1 \). We consequently distinguish whether or not the first iterate has an initial support of size 1. We then use the recurrence property starting the algorithm at \( x_1 \) and running \( T - 1 \) iterations.

**Initialization.** We will now prove the recurrence property (60) for \( T = 1 \). If \( s \geq 2 \), \( \max\{0, T - \left\lceil \frac{T + s}{2} \right\rceil \} = 0 \) and (60) is true because \( (1 - \rho_f \left( \frac{g_0}{g_0} \right)^2) \leq 1 \). If \( s = 1 \), this implies that the first step needs to be a Frank-Wolfe step. We necessarily have \( z_0 = 1 \) and so

\[
E_0((1 - \rho_f \left( \frac{g_0}{g_0} \right)^2)^z_0) = E_0((1 - \rho_f \left( \frac{g_0}{g_0} \right)^2) \mid z_0 = 1) \leq 1 - \rho_f F(g_0 = \tilde{g}_0 \mid z_0 = 1) \leq 1 - \rho_f \eta^2 \leq 1 \leq F(1, 1),
\]

with \( \eta = \frac{\rho}{\|A\|} \) where \( F \) is defined in (60) and where the last inequality follows from (PROB) in Lemma 3.

**Recurrence.** Consider the property (60) when running \( T - 1 \) iteration. By the tower property of conditional expectations

\[
E_{0:T-1} \left( \prod_{t=0}^{T-1} (1 - \rho_f \left( \frac{g_t}{g_t} \right)^2)^{z_t} \right) = E_{0:T-1} \left[ (1 - \rho_f \left( \frac{g_0}{g_0} \right)^2)^{z_0} E_{1:T-1} \left( \prod_{t=1}^{T-1} (1 - \rho_f \left( \frac{g_t}{g_t} \right)^2)^{z_t} \right) \right].
\]

We can apply the recurrence property with \( T - 1 \) iterations and starting point \( x_1 \) on \( E_{1:T-1} \left( \prod_{t=1}^{T-1} (1 - \rho_f \left( \frac{g_t}{g_t} \right)^2)^{z_t} \right) \) so that

\[
E_{0:T-1} \left( \prod_{t=0}^{T-1} (1 - \rho_f \left( \frac{g_t}{g_t} \right)^2)^{z_t} \right) \leq E_0((1 - \rho_f \left( \frac{g_0}{g_0} \right)^2)^{z_0} F(T - 1, |S_1|), \]

where \( |S_1| \), the support of \( x_1 \), depends on \( z_0 \). Indeed \( z_0 = 0 \) implies a drop step and as such it decreases the support of the iterate. Thus we have to distinguish the case according to the size of the support of \( x_0 \).
Case $|S_0| = 1$. With $x_0 = 0$, RAFW starts with a FW step and as such $z_0 = 1$ as well as $2 \geq |S_1| \geq 1$ so that

$$E_{0:T-1}(\prod_{t=0}^{T-1} (1 - \rho_f (\frac{g_t}{\theta_t})^2)^{z_t}) = E_{0}(1 - \rho_f (\frac{g_0}{\theta_0})^2 | z_0 = 1)F(T-1,|S_1|)$$

by applying (PROB) in Lemma 3. The last equality concludes the heredity in that case.

Case $|S_0| \geq 2$. Here it is possible for $z_0$ to equal 0 or 1. If $z_0 = 1$, then $|S_1| \leq |S_0| + 1$, while if $z_0 = 0$, it implies a drop step, we have $|S_1| = |S_0| - 1$. If we decompose the expectation according to the value of $z_0$ we obtain

$$E_{0:T-1}(\prod_{t=0}^{T-1} (1 - \rho_f (\frac{g_t}{\theta_t})^2)^{z_t}) \leq P(z_0 = 1)\mathcal{P}(z_0 = 1)E_{0}(1 - \rho_f (\frac{g_0}{\theta_0})^2 | z_0 = 1)F(T-1,|S_1|)$$

where the last inequality is just about writing the definition of $F$. It concludes the heredity result.

Conclusion: Starting RAFW at $x_0$, after $T$ iterations, we have

$$h_T \leq h_0 \prod_{t=0}^{T-1} (1 - \rho_f (\frac{g_t}{\theta_t})^2)^{z_t}.$$

Applying (60) we get

$$E_{0:T-1}(h_T) \leq h_0(1 - \rho_f \eta^2)^{\max(0,T-[\frac{T-s}{2}]})$$

where the maximum is on the interval $[0,T]$.

Generalized strongly convex.

**Theorem 3.2.** Suppose $f$ has bounded smoothness constant $C_f^2$ and is $\bar{\mu}$-generally-strongly convex. Consider the set $\mathcal{M} = \text{conv}(A)$, with $A$ a finite set of extreme atoms. Then after $T$ iterations of Algorithm 2, with $s = |S_0|$ and a $p$ parameter of sub-sampling, we have

$$E[h(x_{T+1})] \leq (1 - \eta^2 \bar{\rho}_f)^{\max(0,T-[\frac{T-s}{2}])} E[h(x_0)],$$

with $\bar{\rho}_f = \frac{\bar{\mu}}{4C_f^2}$ and $\eta = \frac{\bar{\mu}}{4C_f^2}$.

**Proof.** The conclusion of proof of (Lacoste-Julien & Jaggi, 2015, Th. 11) is that we have similarly as equation (48) by:

$$f(x_t) - f(x^*) = h_t \leq \frac{\eta^2}{2\bar{\rho}_f},$$

where $\bar{\mu}_f > 0$ is a similar measure of the affine invariant strong convexity constant but for generalized strongly convex function.
We can thus write the twin of equation (58)

\[ h_{t+1} \leq h_t \left( 1 - \tilde{\rho}_f \left( \frac{g_t}{y_t} \right)^2 \right)^z_t, \]

(77)

with \( \tilde{\rho}_f = \frac{\tilde{\rho}_f}{4C_f} \). The rest of the proof follows is the same as that of Theorem 3.1. ■
Appendix C. Technical issues of previous work

In this section we highlight some technical issues present in previous work.

Appendix C.1. Randomized Frank-Wolfe in Frandi et al. (2016)

Frandi et al. (2016) present a Randomized FW algorithm for the case of the $\ell_1$ ball in $\mathbb{R}^d$. Denote by $A = \{ \pm e_i \; \forall i \in [d] \}$, where $e_i$ is the canonical basis (i.e., the vector that is zero everywhere except on the $i$-th coordinate, where it equals one) the extremes atoms of the $\ell_1$ ball. Up to the iterative explicit implementation of the residuals, (Frandi et al., 2016, Algorithm 2) with the sampling size $p \in [n]$ and our RFW (Algorithm 1) are equivalent for the following choice of $A_t$ in RFW

$$A_t = \{ \pm e_i \; \forall i \in \mathcal{I}_p \}, \text{ where } \mathcal{I}_p \text{ is random subset of } [d] \text{ of size } p. \quad (78)$$

Convergence result. In this case, (Frandi et al., 2016, Proposition 2) gives the following convergence bound in expectation after $t$ iterations:

$$\mathbb{E} ( f(x_t)) - f(x^*) \leq \frac{4C_f}{t + 2}. \quad (79)$$

First, it is rather surprising that, unlike in our Theorem 2.1, the sub-sampling size $p$ does not appear in the convergence bound. A closer inspection at their Lemma 2 reveals some errors in their proof. For the remainder of this section we will use the notation in (Frandi et al., 2016).

The point of interest. The proof of their Proposition 2 starts with the following inequality derived from the curvature constant:

$$f(\alpha_{k+1}^\lambda) \leq f(\alpha_k^\lambda) + \lambda (u_k^\lambda - \alpha_k^\lambda)^T \nabla f(\alpha_k^\lambda) + \lambda^2 C_f. \quad (80)$$

Then it is claimed that the following equation, Eq. (24) in their paper, is a direct consequence “after some algebraic manipulations”

$$\mathbb{E}_{S^{(k)}} [ f(\alpha_{k+1}^\lambda) ] \leq f(\alpha_k^\lambda) + \lambda \mathbb{E}_{S^{(k)}} [ (u_k^\lambda - \alpha_k^\lambda)^T \nabla S^{(k)} f(\alpha_k^\lambda) ] + \lambda^2 C_f. \quad (81)$$

which is not clear unless $u_k^\lambda$ is independent of the sampling set, something that is not verified given that it is chosen precisely from the sampling set.

Technical details. $\lambda$ being positive, for Eq. (81) to be true, we should necessarily have the following

$$\mathbb{E}_{S^{(k)}} [ (u_k^\lambda - \alpha_k^\lambda)^T \nabla f(\alpha_k^\lambda) ] \leq \mathbb{E}_{S^{(k)}} [ (u_k^\lambda - \alpha_k^\lambda)^T \nabla S^{(k)} f(\alpha_k^\lambda) ] . \quad (82)$$

$\alpha_k^\lambda$ as well as $\nabla f(\alpha_k^\lambda)$ are deterministic with respect to the $S^{(k)}$ sampling set so the previous equation is equivalent to

$$\mathbb{E}_{S^{(k)}} [ (u_k^\lambda)^T \nabla f(\alpha_k^\lambda) ] - (\alpha_k^\lambda)^T \nabla f(\alpha_k^\lambda) \leq \mathbb{E}_{S^{(k)}} [ (u_k^\lambda)^T \nabla S^{(k)} f(\alpha_k^\lambda) ] - (\alpha_k^\lambda)^T \mathbb{E}_{S^{(k)}} [ \nabla S^{(k)} f(\alpha_k^\lambda) ] \quad (83)$$

Since the sub-sampling of $S^{(k)}$ is uniform and by definition of $\nabla S^{(k)} f(\alpha_k^\lambda)$ in (Frandi et al., 2016, equation (14)) we have

$$\mathbb{E}_{S^{(k)}} [ \nabla S^{(k)} f(\alpha_k^\lambda) ] = \nabla f(\alpha_k^\lambda).$$

Then (82) is equivalent to

$$\mathbb{E}_{S^{(k)}} [ (u_k^\lambda)^T \nabla f(\alpha_k^\lambda) ] \leq \mathbb{E}_{S^{(k)}} [ (u_k^\lambda)^T \nabla S^{(k)} f(\alpha_k^\lambda) ] . \quad (84)$$

Also by definition in (Frandi et al., 2016, equation (22)), $u_k^\lambda$ the FW atom has its support on $S^{(k)}$ as well as from (Frandi et al., 2016, equation (6)) we have that $(u_k^\lambda)^T \nabla f(\alpha_k^\lambda) < 0$. So that $(u_k^\lambda)^T \nabla f(\alpha_k^\lambda) = (u_k^\lambda)^T \nabla S^{(k)} f(\alpha_k^\lambda)$ and finally (82) is equivalent to

$$\frac{|S^{(k)}|}{p} \mathbb{E}_{S^{(k)}} [ (u_k^\lambda)^T \nabla S^{(k)} f(\alpha_k^\lambda) ] \leq \mathbb{E}_{S^{(k)}} [ (u_k^\lambda)^T \nabla S^{(k)} f(\alpha_k^\lambda) ] , \quad (85)$$

this last inequality being false in general because $\frac{|S^{(k)}|}{p} < 1$ and $\mathbb{E}_{S^{(k)}} [ (u_k^\lambda)^T \nabla S^{(k)} f(\alpha_k^\lambda) ] \leq 0$. 

