EVOLUTION OF SELF-GRAVITATING MAGNETIZED DISKS. II. INTERACTION BETWEEN MAGNETOHYDRODYNAMIC TURBULENCE AND GRAVITATIONAL INSTABILITIES

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ABSTRACT

We present three-dimensional magnetohydrodynamic (MHD) numerical simulations of the evolution of self-gravitating and weakly magnetized disks with an adiabatic equation of state. Such disks are subject to the development of both the magnetorotational and gravitational instabilities, which transport angular momentum outward. As in previous studies, our hydrodynamic simulations show the growth of a strong $m = 2$ spiral structure. This spiral disturbance drives matter toward the central object and disappears when the Toomre parameter, $Q$, has increased well above unity. When a weak magnetic field is present as well, the magnetorotational instability grows and leads to turbulence. In that case, the strength of the gravitational stress tensor is lowered by a factor of $\sim 2$ compared with the hydrodynamic run and oscillates periodically, reaching very small values at its minimum. We attribute this behavior to the presence of a second spiral mode with higher pattern speed than the one that dominates in the hydrodynamic simulations. It is apparently excited by the high-frequency motions associated with MHD turbulence. The nonlinear coupling between these two spiral modes gives rise to a stress tensor that oscillates with a frequency that is a combination of the frequencies of each of the modes. This interaction between MHD turbulence and gravitational instabilities therefore results in a smaller mass accretion rate onto the central object.

Subject headings: accretion, accretion disks — instabilities — MHD — turbulence

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1. INTRODUCTION

In systems such as the disks surrounding low-mass protostars or active galactic nuclei (AGNs), the simultaneous appearance of both gravitational and magnetic instabilities is expected. During the first stages of their evolution, for example, protoplanetary disks are expected to be rather massive because of strong infall from the parent molecular cloud. As the disk builds up in mass as a result of the collapse of an envelope, its surface mass density becomes large enough for gravitational instabilities to develop (e.g., Laughlin & Bodenheimer 1994). These disks are also believed to be sufficiently ionized, at least over some extended regions, to be coupled to a magnetic field (Gammie 1996; Sano et al. 2000; Fromang et al. 2002).

By modeling the outer parts of disks around quasi-stellar objects (QSOs) as steady, viscous, geometrically thin, and optically thick, Goodman (2003) has argued that they are self-gravitating. More precisely, he predicts self-gravitational instabilities to develop beyond $\sim 10^{-2}$ pc from the central object. In addition, it has been suggested by Menou & Quataert (2001) that self-gravitating regions of disks around QSOs are likely to be coupled to a magnetic field.

The stability of a thin, self-gravitating gas disk is controlled by the Toomre $Q$ parameter (Toomre 1964):

$$Q = \frac{c_s \kappa}{\pi G \Sigma},$$

where $c_s$ is the sound speed, $\kappa$ is the epicyclic frequency (see, e.g., Binney & Tremaine 1987), $\Sigma$ is the disk surface mass density, and $G$ is the gravitational constant. Gaseous disks are unstable against axisymmetric perturbations when $Q \leq 1$ and unstable against nonaxisymmetric perturbations when $Q \gtrsim 1$.

Since analytical predictions of the nonlinear evolution of gravitational instabilities are difficult, there have been a large number of numerical simulations of gravitationally unstable disks. Despite the rather daunting technical problems of combining three-dimensional hydrodynamic calculations with rapid and accurate Poisson equation solvers, significant progress has been made. To do so, the energetics must be treated crudely, with the focus squarely on purely dynamical behavior. Using this strategy, the above $Q$ criterion for instability has been confirmed (and shown to still be approximately valid for disks of finite thickness) and the properties of the unstable modes have been studied as a function of the disk parameters (Tohline & Hachisu 1990; Woodward et al. 1994). Several authors have investigated the saturation properties of the instability and have shown that it is capable of transporting a significant amount of mass and angular momentum in a few orbital times (Papaloizou & Savonije 1991; Laughlin & Bodenheimer 1994; Pickett et al. 1996; Laughlin et al. 1997). The first calculations mostly used simple adiabatic equations of state (EOS). More recently, isothermal disks have also been studied (Pickett et al. 1998, 2000; Boss 1998; Mayer et al. 2002). Some new investigations also include a simplified treatment of the disk radiative cooling (Pickett et al. 2003; Rice et al. 2003; Boss 2002).

All these models were purely hydrodynamic and neglected the effect of magnetic fields. However, it is known that the stability of astrophysical disks is extremely sensitive to the presence of weak magnetic fields. In particular, the magnetorotational instability (MRI) completely disrupts laminar Keplerian flow when a subthermal magnetic field of any geometry is present. This was
first understood by Balbus & Hawley (1991). Since then, it has been shown through many numerical simulations that the non-linear outcome of the MRI is MHD turbulence, which, in common with gravitational instabilities, transports angular momentum outward (see Balbus & Hawley 1998 or Balbus 2003 for a review). Since disks around low-mass stars and QSOs may be both magnetized and self-gravitating, the spiral structure transport described above must somehow develop in a medium in the throes of MHD turbulence.

The question naturally arises as to how these two powerful instabilities interact with one another. What is the ultimate effect on the global properties of accretion disks and, in particular, on the critical transport properties of mass and angular momentum? To keep this initial investigation tractable, we must restrict ourselves here to an adiabatic EOS. However, the dynamical behavior of “simple” adiabatic disks is still rich, and contains unanticipated findings. In a companion paper to this one (Fromang et al. 2004, hereafter Paper 1), we carried out two-dimensional axisymmetric numerical simulations of the evolution of massive and magnetized disks. The results show that the MRI behaves in a self-gravitating environment as it does in zero-mass disks. Turbulent transport of angular momentum causes the disk to evolve toward a two-component structure: (1) an inner thin disk in Keplerian rotation fed by Keplerian, strongly influenced by self-gravity. However, an outer thick disk whose rotation profile deviates from flattened at some upper value \( m_{\text{max}} \). We then calculate \( \Phi_\pi \) everywhere on the grid, using a combination of a Fourier transform in \( \phi \) and the two-dimensional successive over-relaxation (SOR) method (Hirsch 1988) in the \((r,z)\)-plane. Although this is an efficient method, the calculation of the self-gravitating potential is still very demanding of computational resources. For the resolution \((N_r, N_\phi, N_z) = (128, 64, 128)\) used in this paper, the time required by the Poisson solver still represents \( \sim 40\% \) of the computation time for \( m_{\text{max}} = 8 \).

2.2. Diagnostics

We introduce and define some key quantities that have been used to analyze the results of the simulations. We denote the ratio of the volume-averaged thermal pressure to the volume-averaged magnetic pressure as \( \langle \beta \rangle \):

\[
\langle \beta \rangle = \frac{P}{\langle B^2/8\pi \rangle}. \tag{10}
\]

This parameter is used primarily as a measure of the initial magnetic field strength.

To solve these equations, we use the GLOBAL code (Hawley & Stone 1995). This uses standard cylindrical coordinates \((r, \phi, z)\) and time-explicit Eulerian finite differences. The magnetic field is evolved using the combined method of characteristics and constrained transport algorithm (MOC-CT), which preserves the divergence of the magnetic field to machine accuracy. Finally, we use outflow boundary conditions in the radial and vertical directions and periodic boundary conditions in \( \phi \).

In its original form, GLOBAL did not include a Poisson solver, and the development of such a routine represents a major technical component of the results we report here. The calculation is done in two steps. The potential, \( \Phi_\pi \), is first computed at the grid boundary, using the spectral decomposition described below, and then calculated on the whole grid using a very rapid method. It is the first step, the boundary calculation, that is computationally expensive.

In the expansion of \( \Phi_\pi \), we have adopted the method of Cohl & Tohline (1999), which uses half-integer Legendre functions in the Green’s function. This method is better suited to cylindrical coordinates than the traditional expansion in spherical harmonics, which are of course tailored to spherical coordinates. Following Cohl & Tohline (1999), \( \Phi_\pi \) may be written

\[
\Phi_\pi(r, \phi, z) = -\frac{G}{\pi \sqrt{r}} \times \int_{\mathcal{V}} d\mathcal{V'} \frac{(r', \phi', z')}{r'} \sum_{m=-1/2}^{\infty} c_m Q_{m-1/2}(\chi) \cos m(\phi - \phi'). \tag{8}
\]

Here \( d\mathcal{V'} = r' dr' d\phi' dz' \) is the elementary volume element, and the integral is taken over the whole computational domain. \( Q_{m-1/2} \) denotes the half-integer order Legendre function of the second type (Abramowitz & Stegun 1965). The argument \( \chi \) is a function of position:

\[
\chi = \frac{r^2 + r'^2 + (z - z')^2}{2rr'}. \tag{9}
\]

The Legendre functions are computed once at the beginning of each simulation and stored in memory. At each time step, we calculate \( \Phi_\pi \) using equation (8), in which the sum over \( m \) is truncated at some upper value \( m_{\text{max}} \). We then calculate \( \Phi_\pi \) everywhere on the grid, using a combination of a Fourier transform in \( \phi \) and the two-dimensional successive over-relaxation (SOR) method (Hirsch 1988) in the \((r,z)\)-plane. Although this is an efficient method, the calculation of the self-gravitating potential is still very demanding of computational resources. For the resolution \((N_r, N_\phi, N_z) = (128, 64, 128)\) used in this paper, the time required by the Poisson solver still represents \( \sim 40\% \) of the computation time for \( m_{\text{max}} = 8 \).
In three-dimensional numerical simulations of magnetized self-gravitating disks, angular momentum is transported by the sum of the Maxwell, Reynolds, and gravitational stress tensors. Following Balbus & Papaloizou (1999) and Hawley (2000), we define the height and azimuthal averages (noted with an overbar) of each of these stresses as

\[ T_{\text{ro}}^{\text{Max}}(r, t) = \frac{\mathcal{E}_e^2}{4\pi} \]

\[ T_{\text{ro}}^{\text{Ren}}(r, t) = \frac{\rho v_r v_\theta}{\rho v_r} \]

\[ T_{\text{ro}}^{\text{grav}}(r, t) = \frac{1}{4\pi G} \frac{\partial \Phi_s}{\partial r} \frac{1}{r} \]  

As in Paper I, volume averages of these quantities will be denoted as \( \langle T_{\text{ro}}^{\text{Max}}(r, t) \rangle \), etc. Note that \( T_{\text{ro}}^{\text{grav}} \) is associated with the gravitational torque resulting from nonaxisymmetric disk structure. This quantity clearly vanishes in an axisymmetric simulation (\( m_{\text{max}} = 0 \)).

In this case, the standard \( \alpha \)-parameter (Shakura & Sunyaev 1973) can be defined as the sum of the Maxwell and Reynolds stress tensors normalized by the gas pressure:

\[ \alpha(r, t) = \frac{T_{\text{ro}}^{\text{Max}}(r, t) + T_{\text{ro}}^{\text{Ren}}(r, t)}{P(r, t)} \]

3. INITIAL MODEL

We start our simulations with a disk model that is as close as possible to hydrostatic equilibrium:

\[-\nabla P - \rho \nabla (\Phi_s + \Phi_c) + \rho \Omega^2 \mathbf{e}_r = 0.\]

Here \( \Omega \) is the angular velocity and \( \mathbf{e}_r \) is the unit vector in the radial direction. The coordinate system has its origin on the disk center. The potential \( \Phi_c \) is due to a central mass \( M_c \). We chose \( M_c = 2M_d \), where \( M_d \) is the disk mass. The initial disk model is gravitationally unstable.

Because of the presence of the disk self-gravity, equation (15) has to be solved iteratively. We use the self-consistent field (SCF) iterative method developed by Hachisu (1986). In this method, the radial profile of the angular velocity, \( \Omega \), or, equivalently, the specific angular momentum, \( j = r^2 \Omega \), is specified. Following Pickett et al. (1996), we fix \( j(r) \):

\[ j = j_0 \left( \frac{M_d + M_c}{2M_d} \right)^q \]

where \( j_0 \) and \( q \) are constant and \( M_d \) is the disk mass within radius \( r \). Setting \( q = 2 \) gives a \( j \)-profile close to that used by Pickett et al. (1996). We begin the iteration with an arbitrary mass density, \( \rho \), from which we can calculate \( \Phi_s \). From \( \rho \) and the above expression for \( j \), we also calculate \( \Omega \) (note that it still depends on the constant \( j_0 \)). The relation (15) is then integrated to give the value of the enthalpy:

\[ h = \frac{5}{2} K \rho^{2/3} = C - (\Phi_s + \Phi_c) + \int r \Omega^2 \, dr \]

where the constant \( C \) and \( j_0 \) are determined from the boundary conditions \( \rho = 0 \) at \( r = R_{\text{in}}, z = 0 \) and at \( r = R_{\text{out}}, z = 0 \). Here \( R_{\text{in}} \) and \( R_{\text{out}} \) are the radial boundaries of the disk. The new density field is then calculated from \( h \) using the normalizing condition that \( \rho_{\text{max}} = 1 \), which determines the polytropic constant \( K \). Upon iterating this procedure, we converge to a model very close to equilibrium.

The resulting disk model (with \( R_{\text{in}} = 0.25 \) and \( R_{\text{out}} = 1 \)) has an \( \Omega \) profile close to Keplerian and the density profile displayed in Figure 1. Note that the disk is rather thin, with an aspect ratio \( H/r \) varying between 0.1 and 0.2.

As noted above, the ratio \( M_c/M_d \) is chosen in such a way that the Toomre \( Q \) parameter is initially close to unity. The radial profile of \( Q \) in the initial disk model is shown in Figure 2. The minimum value of \( Q \) is approximately 1.1, and \( Q \) is close to unity over a large range of radii. We therefore expect strong gravity.
nonaxisymmetric gravitational instabilities to develop in this disk.

4. RESULTS

Table 1 lists the parameters of the different runs we present below. Column (1) gives the label of the model. HD refers to a hydrodynamic run. Models T and P start with a purely toroidal and poloidal magnetic field, respectively. Column (2) gives the computational azimuthal domain, and column (3) gives the highest Fourier component of the gravitational potential. When $m_{\text{max}} = 0$, i.e., when only the $m = 0$ component in the Fourier expansion of $\Phi$, is included, gravitational instabilities cannot develop (recall that $Q > 1$, so that the disk is stable against axisymmetric perturbations). Therefore, models T1 and P1 enable us to study the evolution of MHD turbulence and to compare it with previous work and with the two-dimensional simulations of Paper I. In models T2, T2$_{\text{low}}$, T2*, T3, and P2, both gravitational and magnetic instabilities develop. In model T2*, the nonaxisymmetric part of $\Phi$, is included only after six orbits, i.e., after MHD turbulence has established itself. Column (4) gives the ratio of the volume-averaged thermal and magnetic pressures, and column (5) gives the resolution $(N_r, N_y, N_z)$ of the run.

In all the models, an adiabatic equation of state is used. The computational domain extends radially from 0 to either $0.1$ or $0.2$ and vertically from $0$ to either $\pi/2$ or $\pi$. The smaller range is used in the $m_{\text{max}} = 0$ gravitationally stable cases. Indeed, Hawley (2000) and Papaloizou & Nelson (2003) have shown that an azimuthal domain of $\pi/3$ is generally sufficient to describe the transport properties of MHD turbulence. When we allow for the development of gravitational instabilities, we restrict the azimuthal domain to the half-disk $[0, \pi]$. This saves computational time but of course allows only even modes to develop. The focus of the paper is not on the detailed spectrum of modes that appear in a given disk model, however, but on the interaction between MHD turbulence and the largest scale gravitational modes. This interaction should not be particularly sensitive to whether an integer number of modes exactly fits in the half-disk.

### Table 1

| Model   | $\phi$-Range | $m_{\text{max}}$ | $\langle \beta \rangle$ | Resolution |
|---------|---------------|------------------|--------------------------|------------|
| HD      | $[0, \pi]$    | 8                | $\infty$                | (128, 64, 64) |
| T1      | $[0, \pi/2]$  | 0                | 8                        | (128, 32, 128) |
| T2      | $[0, \pi]$    | 8                | 8                        | (128, 64, 128) |
| T2$_{\text{low}}$ | $[0, \pi]$    | 8                | 8                        | (64, 64, 64) |
| T2*     | $[0, \pi]$    | 0–8*             | 8                        | (128, 64, 128) |
| T3      | $[0, \pi]$    | 16               | 8                        | (128, 64, 128) |
| P1      | $[0, \pi/2]$  | 0                | 300                      | (128, 32, 128) |
| P2      | $[0, \pi]$    | 8                | 300                      | (128, 64, 128) |

**Notes.**—Col. (2) gives the computational azimuthal domain, col. (3) gives the highest Fourier component of the gravitational potential included in the calculation, col. (4) gives the ratio of the volume averaged thermal and magnetic pressures, and col. (5) gives the resolution $(N_r, N_y, N_z)$ of the run. Model HD is hydrodynamic. Models T and P start with a purely toroidal and poloidal magnetic field, respectively. When $m_{\text{max}} = 0$, the disk self-gravitating potential is forced to stay axisymmetric.

* For this run, $m_{\text{max}} = 0$ when $t \in [0, 5.8]$, while $m_{\text{max}} = 8$ for $t > 5.8$.

Time is measured in units of the orbital period at the initial outer edge $R_{\text{out}} = 1$ of the disk model. Typical simulations are carried out for $8$–$10$ orbits at this position. This corresponds to $60$–$80$ orbits at the initial disk inner edge. The simulations are seeded by adding to the mass density at $r > 0.4$ random perturbations with a relative amplitude of $5 \times 10^{-3}$.

We now describe in turn the hydrodynamic run, the simulations with only MHD turbulence, and the runs with both gravitational and magnetic instabilities.

#### 4.1. Control Hydrodynamic Run: Model HD

The time evolution of the Fourier components of the density in the equatorial plane for model HD. The $y$-axis represents the ratio of the amplitude of the $m$th Fourier component of the perturbed density to the unperturbed density. From top to bottom, the different curves correspond to the modes $m = 2$ (solid line), $m = 4$ (dashed line), and $m = 6$ (dot-dashed line).

FIG. 3.—Time evolution of the Fourier components of the density in the equatorial plane for model HD. The $y$-axis represents the ratio of the amplitude of the $m$th Fourier component of the perturbed density to the unperturbed density. From top to bottom, the different curves correspond to the modes $m = 2$ (solid line), $m = 4$ (dashed line), and $m = 6$ (dot-dashed line).
In the presence of a weak magnetic field, we expect our disk model to be unstable to the MRI, regardless of the field geometry. We first perform simulations in which only the MRI develops (models T1 and P1). This allows the properties of the ensuing MHD turbulence to be quantified and compared with previous work. To prevent the growth of nonaxisymmetric gravitational instabilities, we retain only the $m = 0$ component in the Fourier expansion of $\Phi_s$. The resolution is $(N_r, N_\phi, N_z) = (128, 32, 128)$ and the azimuthal domain extends from 0 to $\pi/2$, which would be equivalent to a resolution of $128^3$ over a range of $2\pi$.

### 4.2.1. Initial Toroidal Field: Model T1

We add to the equilibrium toroidal disk model described above a $\langle \beta \rangle = 8$ toroidal magnetic field and run the simulation for $\sim 10$ orbits ($\sim 80$ orbits at the initial disk inner edge).

Figure 5 shows the time evolution of the volume-averaged Maxwell and Reynolds stress tensors and the corresponding $\alpha$-parameter (see eq. [14]). The Maxwell stress increases during the linear phase of the instability. It then saturates after four orbits, at which time the MRI breaks down into turbulence. The presence of turbulence is seen in Figure 6, which shows the density perturbation in the equatorial plane at $t = 6.4$. Turbulent fluctuations are present over the full extent of the disk. It is clear from Figure 5 that the Reynolds stress is significantly smaller than the Maxwell stress over the course of the simulation. This is in agreement with previous non–self-gravitating global simulations of the MRI (Hawley 2000, 2001; Steinacker & Papaloizou 2002). The right panel of Figure 5 shows the radial profile of $\alpha$ at the end of the simulation, i.e., at $t \approx 10$. The typical value of $\alpha$ is a few times $10^{-2}$, similar to what was found in previous simulations starting with a toroidal field with a net flux (Steinacker & Papaloizou 2002).

The Maxwell stress stays roughly constant during our simulation. This indicates that the resolution $(128, 32, 128)$ is
large enough for the turbulence to be sustained over the duration of the run. We therefore adopt it in the following runs (which of necessity are limited in time by the fact that mass is accreted onto the central mass).

4.2.2. Initial Poloidal Field: Model P1

To investigate the sensitivity of our results to the initial field geometry, we run the same calculation as in model T1 but with an initial poloidal magnetic field. We calculate the field from the (toroidal) component of the vector potential in the initial disk model:

$$A_\phi \propto \rho \cos \left( \frac{8\pi r - R_{in}}{R_{out} - R_{in}} \right).$$

This corresponds to four magnetic loops confined inside the disk. The first two-dimensional simulations of a disk permeated by a weak vertical field (Hawley & Balbus 1991) showed the development and growth of “channel” solutions. In three-dimensions, these solutions still exist but they quickly break down into turbulence, as predicted by the analysis of Goodman & Xu (1994). Turbulence is more rapidly established when the field varies on a fairly small scale, which motivates the above choice of $A_\phi$.

The radial and vertical components of the magnetic field are computed from $A_\phi$ and normalized such as to obtain the desired initial value of $\langle \beta \rangle$. Since the linear growth of the vertical field is much more rapid than that of the toroidal field (see below), we chose a much larger initial value of $\langle \beta \rangle = 300$.

The properties of the turbulence are similar to those found when the initial field is toroidal. Figure 7 shows the time evolution of the Maxwell stress for both models T1 and P1. As expected, the linear instability is much more vigorous when a vertical field is present because of the growth of the channel solutions. However, in both cases the stress saturates at a similar value and the level of turbulence is comparable. The evolution of the Maxwell stress in P1 is somewhat similar to what was obtained in Paper I. The important difference is that in three-dimensions, the stress saturates when turbulence is established and does not decay with time as it does in two-dimensions.

4.3. Full MHD Simulations

In this section we report the results of full three-dimensional simulations including the development of both self-gravitational and magnetic instabilities. The resolution is $\left(N_r, N_\phi, N_z\right) = (128, 64, 128)$, and the azimuthal domain extends from 0 to $\pi$.

4.3.1. Initial Toroidal Field: Models T2, T2*, and T3

In this sequence of models, we observe the simultaneous appearance of both MHD turbulence and the $m = 2$ spiral arm familiar from the hydrodynamic calculation. To better understand how angular momentum is transported in the disk, we compare the time evolution of the different stresses with those obtained in the models described in the previous section.

Figure 8 shows the gravitational stress $\langle T_{grav}^{r\phi} \rangle$ as a function of time for both models T2 and HD. Somewhat surprisingly, the presence of both gravitational and MHD instabilities leads to an average $\langle T_{grav}^{r\phi} \rangle$ reduced by a factor of $\sim 2$ compared with the values obtained without a magnetic field. The magnetic torques do not lead to more vigorous gravitational instability. One possible explanation may be that turbulent motions tend to broaden the spiral arms by adding an extra fluctuating component to the thermal pressure, but there is more going on than just this. Figure 8 also shows that the gravitational stress varies nearly periodically with time and can reach very small values. This behavior is in fact associated with the near disappearance of the spiral arms, as can be seen in Figure 9. These snapshots correspond to a maximum and a minimum of the gravitational stress, respectively. The spiral arms are sharp at $t = 4.95$, whereas they lack definition at $t = 5.09$. The arms form, disperse, and reform. This periodic variation is also seen in Figure 10, which shows the mass accretion rate onto the central mass as a function of time. As expected, the accretion rate has a
periodic component with the same frequency as that found in the gravitational stress. The period in both cases is \( \frac{1}{C^2} \approx 0.28 \).

In Figure 11, we compare the evolution of the Maxwell stress in models T2 and T2* (for which only the first six orbits, during which \( m_{\text{max}} = 0 \), are plotted). In contrast to the gravitational stress, the Maxwell stress is significantly larger when the disk is gravitationally unstable. This appears to be due to the systematic compression of the magnetic field lines along the spiral arms, as opposed to, say, an increase of the level of turbulent fluctuations. When the gravitational instability disappears after about seven orbits in model T2, for example, the Maxwell stress decreases to the same value as in model T2*.

As in the hydro model HD, the Toomre \( Q \) parameter rises throughout the body of the disk over the course of the simulation (as mass is transported toward the inner region), until the gravitational instability ceases. But even by \( t \approx 8 \), when there is no longer any gravitational transport, \( Q \) is still larger in model HD than in model T2. This is because the gravitational instability is stronger in the hydrodynamic case, and the disk is depleted more rapidly.

To check the sensitivity of these results to our choice of initial conditions, we conducted the following experiment. In model T2*, the input parameters are the same as in model T2, but the nonaxisymmetric part of \( /C^8_s \) is included only after six orbits, i.e., only after MHD turbulence has been firmly

![Fig. 9.—Logarithm of the density in the equatorial plane for run T2 at \( t = 4.95 \) (left) and \( t = 5.09 \) (right). These snapshots correspond to a maximum and a minimum of the gravitational stress, respectively. The spiral arms are sharp and clear at \( t = 4.95 \), whereas they appear blurred at \( t = 5.09 \). [See the electronic edition of the Journal for color version of these figures.]](image)

![Fig. 10.—Time evolution of the mass accretion rate (mass per unit of time) onto the central mass for run T2. As expected, the accretion rate oscillates with the same period as the gravitational stress.](image)

![Fig. 11.—Time evolution of the volume-averaged Maxwell stress tensor for runs T2* (dashed line), before gravitational transport is turned on, and T2 (dotted line). The Maxwell stress is larger when the disk is gravitationally unstable. This appears to be due to a compression of the magnetic field lines along the spiral arms. When the disk in model T2 becomes gravitationally stable (after \( t \approx 7 \)), the Maxwell stress decreases down to the same value as in model T2*.](image)
established. Figure 12 shows the time evolution of the volume-averaged Maxwell (solid line) and gravitational (dashed line) stress tensors for run T2*. In this run, the nonaxisymmetric part of $\Phi_r$ is included only after six orbits, i.e., only after MHD turbulence has been firmly established. The development of gravitational instabilities coincides with a significant decrease of the Maxwell stress tensor. This may be due to reconnection of the (randomly oriented) field lines in the spiral arms. The gravitational stress tensor has the same amplitude and periodic variations as in run T2.

Figure 12.—Time evolution of the volume-averaged Maxwell (solid line) and gravitational (dashed line) stress tensors for run T2*. In this run, the nonaxisymmetric part of $\Phi_r$ is included only after six orbits, i.e., only after MHD turbulence has been firmly established. The development of gravitational instabilities coincides with a significant decrease of the Maxwell stress tensor. This may be due to reconnection of the (randomly oriented) field lines in the spiral arms. The gravitational stress tensor has the same amplitude and periodic variations as in run T2.

Our next comparison run, model T3, differs from model T2 only in the number $m_{\text{max}}$ of Fourier coefficients in the expansion of $\Phi_r$. (T3 has $m_{\text{max}} = 16$, T2 has $m_{\text{max}} = 8$.) Once again, very similar results emerge, and the choice of $m_{\text{max}}$ does not appear to be critical (cf. § 4.3.3).

To summarize: the evolution of a purely toroidal field in a gravitationally unstable disk leads to a reduction and strong periodicity in the gravitational stress (compared with a purely hydrodynamic model). Compared with gravitationally stable models, the Maxwell stress is larger or smaller depending, respectively, on whether gravitational instabilities develop at the same time as MHD turbulence (magnetic field alignment in spiral arms) or after turbulence is established (reduction in magnetic stress as gravitational stress develops). The behavior of an initial poloidal field is considered next.

4.3.2. Initial Poloidal Field: Model P2

Do our toroidal field findings extend to poloidal field behavior? To answer this question, we begin with an initial poloidal field, calculated as in § 4.2.2 above. Again, we start with $\langle \beta \rangle = 300$. Except for the initial field geometry, model P2 is the same as model T2.

Figure 13 shows the evolution of the gravitational stress tensor for both models P2 and HD (this is the equivalent of Fig. 8). As in the case of a toroidal field, a nonaxisymmetric $m = 2$ spiral grows and becomes nonlinear in model P2. The

Fig. 13.—Time evolution of the volume-averaged gravitational stress tensor for models HD (solid line) and P2 (dotted line). The gravitational stress in model P2 is smaller than in model HD and varies periodically with time.
The fact that gravitational instabilities develop earlier in model P2 than in models T2 and HD (see Figs. 8 and 13) appears to be due to the fact that the strong linear magnetic instability associated with the poloidal field produces large perturbations of the density. Once again, we find that the gravitational stress is smaller than in model HD and varies periodically with time with a period \( \frac{C}{24} \approx 0.38 \), somewhat larger than for model P2.

Figure 14 shows the evolution of the Maxwell stress for models P1 and P2 (this is the equivalent of Fig. 11). Once again, the Maxwell stress is larger when the disk is gravitationally unstable, and gravitational instabilities develop at the beginning of the run (model P2).

Since the state of MHD turbulence in a saturated disk is the same whether an initial poloidal or toroidal field is used, we have not run a case “P2*” with the nonaxisymmetric part of \( \Phi_s \) added later. Such a run is expected to be very similar to T2*, since the initial turbulent states are similar.

The history of run P2 and the above argument together suggest that toroidal and poloidal initial fields behave very similarly.

### 4.3.3. Modal Analysis

The full MHD simulations described above suggest that a general feature of the evolution of gravitationally unstable turbulent disks is a periodic modulation of the gravitational stress. To understand the reason for this modulation, we now analyze in more detail the unstable modes that appear in models HD, T2, and P2. We are especially interested in determining the power spectrum as a function of mode frequency \( \sigma \). Following Papaloizou & Savonije (1991), we Fourier transform in time at each radial zone \( r \) and at a fixed azimuth \( \phi_0 \) the function \( \rho(r, \phi_0, t) \). To get the spectral time evolution, we carry out each Fourier transform over a series of four distinct time intervals. The number of time steps used in each time interval gives a finite frequency resolution \( d\sigma/2\pi = 0.3 \). The contours of constant power are then plotted as a function of frequency and radius for the various time intervals.

Figure 15 shows the contours for model HD. The different panels correspond to the time intervals 1.94–2.91 (upper left), 2.91–3.88 (upper right), 3.88–4.85 (lower left), and 4.85–5.83 (lower right). The contour levels in each of these panels are, in units of \( 10^{-2} \): (0.22, 0.29, 0.44, 0.87, 1.1, 1.6, 3.2), (2.1, 2.6, 3.4, 5.2, 10, 15), (8.4, 11, 14, 21, 42), and (9.4, 12, 16, 23, 47, 62), respectively. One mode (a two-arm spiral) dominates the spectrum in the nonlinear stage (bottom left panel), with a frequency \( \sigma/2\pi = 1 \).
(\(\sigma/2\pi = 2.5\)) mode in the bottom left panel, i.e., at \(t \approx 4.4\). There its amplitude peaks at 0.7. This mode can of course be identified with the two-arm spiral seen in Figure 4. At \(t = 4.4\) the mode is nonlinear. Its amplitude does not increase further, as can be seen in the bottom right panel, i.e., at \(t \approx 5.4\). At this later time, only the frequency of the mode has changed. It is now around 1.5. Both the finite frequency resolution \(d\sigma\) and the increase of the central mass due to accretion may account for this shift in frequency. A third mode with a frequency twice that of the low-frequency mode is seen in the bottom panels. The relationship between the frequencies of these two modes and the fact that their radial structure is very similar suggests that they are harmonics of each other.

Figure 16 shows the contour plots for model T2. In the top left panel, at \(t \approx 2.4\), there is no dominant mode. Instead, there is a large number of high-frequency (with mostly \(\sigma/2\pi = 1.3\)–3.8) perturbations. The amplitude of these fluctuations is a few times \(10^{-2}\). They are associated with the growing MRI. In the top right panel, at \(t \approx 3.4\), two modes emerge, but their amplitude is still rather low. However, these modes subsequently grow and are clearly seen with a larger amplitude in the bottom left panel, at \(t \approx 4.4\). One of these modes has a frequency \(\sigma/2\pi = 1\) and is the same as that seen in the hydrodynamic simulations. Its corotation radius is located at the disk initial outer edge. Its amplitude peaks at a value of about 0.5 in the outer parts of the disk. The other mode has a frequency \(\sigma/2\pi = 2.5\) and an amplitude \(\sim 0.2\) constant over the whole disk. In particular, its amplitude in the disk inner parts is larger than that of the other mode. This mode is probably the same as that seen with a lower amplitude at early times in the hydrodynamic simulations (Fig. 15, top panels). This suggests that this mode is a disk eigenmode that is excited in model T2 by the high-frequency motions associated with the turbulence. We expect nonlinear coupling between these two modes to give rise to beat oscillations, i.e., oscillations whose frequency is a linear combination of the frequencies of the two modes. We have noted above that the gravitational stress tensor in model T2 oscillates with a period \(\sim 0.28\) consistent with the frequency of the oscillations being \(\sigma/2\pi \approx m(\sigma_{HF} - \sigma_H) = 3\), where \(\sigma_{HF}\) and \(\sigma_H\) are the frequencies of the high- and low-frequency modes, respectively. This suggests that the oscillations of the gravitational stress tensor result from a nonlinear coupling between these two modes.

Figure 17 shows the contour plots for model P2 in the time interval 2.72–3.88. The situation is similar to model T2. Here again two modes of comparable amplitude are present. One of these modes has a frequency \(\sigma/2\pi = 1\) and can be identified with the mode that emerges in the hydrodynamic simulations. The other mode has a frequency \(\sigma/2\pi = 2.1\). Given the finite frequency resolution \(d\sigma/2\pi\), this second mode may be the
same as that identified in model T2. However, it may also be a different mode with a lower frequency. In any case, the situation is qualitatively the same here as in model T2. Since the two modes have comparable amplitude, they interact nonlinearly, which results in a periodic modulation of the gravitational stress.

We now vary some of the parameters of model T2 in order to examine the sensitivity of these physical results to the numerical input parameters. Figure 18 shows the contour plots for model T2 \textsubscript{low} in the time interval 2.72–3.88. The resolution for this run in the radial and vertical directions is half that of model T2. The similarity between this plot and the bottom left panel of Figure 16 demonstrates that our results do not depend strongly on the numerical resolution. Figure 19 is the same as Figure 18, but for model T3. Again, it is very similar to the bottom left panel of Figure 16. Since, in model T3, only the parameter \( m_{\text{max}} \) is different from model T2, Figure 19 suggests that the limited number of Fourier components included in the calculation of the self-gravitating potential does not qualitatively affect the main physical results presented in this paper.

In conclusion, models \( P2, T2_{\text{low}}, \) and \( T3, \) taken together, suggest that the physical results described in this paper are insensitive to both the numerical setup of the simulations and the initial magnetic field topology.

5. DISCUSSION

In this paper, we have presented the first three-dimensional numerical simulations of the evolution of self-gravitating and magnetized disks. We have investigated disks in which only gravitational or magnetic instabilities develop and disks in which both types of instabilities occur.

When no magnetic field is present, self-gravitating disks are unstable when the Toomre \( Q \) parameter is close to unity. The spectrum of unstable modes in that case is dominated by a large-scale two-arm spiral whose corotation radius is located near the disk outer edge. The instability is due to the interaction between waves that propagate near the disk outer boundary and inside the inner Lindblad resonance (ILR) (Papaloizou & Savonije 1991).

When a magnetic field is present, two large-scale modes grow in the disk. They both have \( m = 2 \). One of these modes is the same as that seen in the hydrodynamic simulations. The second mode has a higher frequency. The ILR of the low-frequency mode, which is at \( r \approx 0.6 \), is very close to the corotation radius of the high-frequency mode, which is at \( r \approx 0.5 \). Such a pair of modes was seen in the hydrodynamic calculations of self-gravitating disks performed by Papaloizou & Savonije (1991). There both modes were unstable because of the particular vortensity profile. In our simulations, in the absence of magnetic field only the low-frequency mode is unstable. The high-frequency mode seems to be part of the spectrum of normal modes in that case, but it does not grow. The presence of MHD turbulence does not modify the spectrum of large-scale (comparable to the disk radius) modes, since it acts on scales limited by the disk thickness. However, the fact that the high-frequency mode is unstable in the MHD simulations suggests that turbulence acts as a source for high-frequency oscillations.

Nonlinear coupling between these two modes leads to an oscillation of the gravitational stress tensor. Note that such a coupling between two modes with coinciding resonances has been suggested to explain some of the features seen in numerical simulations of galactic disks by Tagger et al. (1987).
These authors argued that the proximity of the resonances made the coupling very efficient. The oscillation of the gravitational stress tensor is accompanied by the periodic disappearance of the spiral arms in the disk. Also, the peak value of this stress is decreased by about half compared to the hydrodynamic simulations.

The results reported here are robust and do not depend on the geometry of the magnetic field. They have important consequences for disks around AGNs and protoplanetary disks. They first show that accretion of a self-gravitating disk onto the central star is slowed down when a magnetic field is present. They also show that the accretion is time-dependent, with a characteristic timescale for the variability being on the order of a fraction of the dynamical timescale at the outer edge of the region where the instabilities develop.

As mentioned in the introduction, protoplanetary disks are probably self-gravitating in the early phases of their evolution. For a disk of ~100 AU, the work presented here suggests variability on a timescale ~10^3 yr. The periodicity in the spatial distribution of knots in jets emanating from such objects is in the range 10–10^3 yr (Reipurth 2000) and is usually thought of as being produced by a time-dependent accretion in the central parts of the disk. The simulations presented in this paper suggest that periodic modulations of the accretion rate might well be the result of the interplay between gravitational instabilities and MHD turbulence, a far from obvious source. Note that the first detection of near-IR variability in a sample of class I protostars was performed recently by Park & Kenyon (2002). However, the poor time coverage of their data prevents a useful measure of the variability timescales from being extracted.

Disks around AGNs display time-dependent phenomena on a large range of timescales (Ulrich et al. 1997). The dynamical timescale for a disk orbiting a 10^8 solar masses black hole at 10^{-2} pc is 9.3 yr, and variations are observed on timescales up to years. This again is consistent with the processes described in this paper.

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