Single-photon observables and preparation uncertainty relations

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Abstract
We propose a procedure to define all single-photon observables in a consistent and unified picture based on operational approach to quantum mechanics. We identify the suppression of zero-helicity states as a projection from an extended Hilbert space onto the physical single-photon Hilbert space. We show that all single-photon observables are in general described by positive-operator valued measures (POVMs), obtained by applying this projection to opportune projection-valued measures (PVMs) defined on the extended Hilbert space. The POVMs associated to momentum and helicity reduce to PVMs, unlike those associated to position and spin. This fact reflects the intrinsic unsharpness of these observables. We apply this formalism to study the preparation uncertainty relations for position and momentum and to compute the probability distribution of spin, for a broad class of Gaussian states. Results show quantitatively the enhancement of the statistical character of the theory.

Keywords: single-photon formalism, quantum Measurements, photon spin, relativistic quantum mechanics, POVMs, irreducible representations

(Some figures may appear in colour only in the online journal)

1. Introduction

The investigation of single-photon properties has experienced an increasing interest over the last years [1–6]. One of the reasons has to be sought in the escalating request, in many quantum information and cryptography protocols [7, 8], for highly accurate manipulations of spatial and polarization single-photon properties.
The most appropriate description of single-photon observables is the subject of an ongoing debate. Photon position, in particular, has been considered a controversial concept since Newton and Wigner first stated [9] that no position operator can be defined in the usual sense for particles with mass \( m = 0 \). Later, Wightman [10] extended the search for a notion of photon localization based on a projection-valued measure (PVM), again obtaining a negative result. This evidence suggested Kraus to define the single-photon position observable as a positive operator valued measure (POVM) [11]. After the appearance of Kraus’ seminal work, several authors proposed alternative definitions of the photon position POVM relying on the theory of quantum estimate [12, 13] or on explicit models describing actual measurements performed in photocounting experiments [1, 14, 15].

Despite such remarkable achievements, the most appropriate single-photon description remains controversial, since the above approaches were conceived to solve the specific problem of photon localization, and do not appear amenable of an immediate generalization to all other single-photon observables. In particular, the photon spin is another notoriously delicate topic [16–19], often ignored in favour of the more familiar notions of helicity and polarization. A renewed interest, due to recent experimental developments especially concerning quantum cryptography protocols [2], for the spin of single photons nevertheless calls for an appropriate and manageable description of such observable, in a common picture with that of position.

In the present work, we generalize Kraus’ construction of the single-photon position observable, given within Ludwig’s axiomatic formulation of quantum mechanics [20], to a formalism in which all the fundamental single-photon observables are given in a unified way in terms of POVMs [20, 21].

Hence, the present work provides a concrete example of a situation where the conventional formulation of quantum observables as self-adjoint operators cannot be sufficient. Other physically relevant examples are the phase observables of a single-mode electromagnetic field [21] and the momentum observable of a confined particle [22, 23].

Following the construction of the single-photon Hilbert space given by Kraus and Moses [11, 24, 25], based on the representation theory of the Poincaré group for mass \( m = 0 \) and spin \( s = 1 \) particles, given by Wigner [26], we interpret the notorious suppression of zero-helicity photon states as a projection from an extended Hilbert space onto the single-photon Hilbert space.

Extending Kraus’ construction of the single-photon position observable, we show that all single-photon observables are described by POVMs obtained by applying this projection to PVMs defined on the extended Hilbert space and mutated from the well-established quantum description of relativistic massive particles.

We provide explicit expressions for the POVMs describing the joint measurement of spin and momentum, and of spin and position. The results show that momentum and helicity are described by PVMs, while spin and position by POVMs. Such difference naturally reflects the well-known [21, 27–29] circumstance that POVMs describe unsharp observables reflecting either practical limits in the precision of measurements (in which case POVMs typically correspond to coarse-grained version of PVMs) or inherent difficulties in realizing a preparation in which the value of an observable is perfectly defined [29–31]. In particular, the intrinsic unsharpness of position and spin results from the coupling between momentum and spin introduced by the suppression of zero-helicity states, a specific consequence of the mass \( m = 0 \) and spin \( s = 1 \) of the photon.

We finally apply this formalism to assess the increase of the statistical character of single-photon observables naturally brought along by the intrinsic unsharpness of POVMs [20, 28]. For this purpose we investigate preparation uncertainty relations for position and momentum,
as well as the spin probability distribution. These quantities are analytically calculated for a broad class of physically meaningful single-photon states, namely Gaussian states with definite polarization and projections of Gaussian states with definite spin. The reasons behind the choice of Gaussian states range from their great theoretical and experimental relevance to the fact that, in the non-relativistic context, they saturate the notorious inequality $\Delta X \Delta P \geq \frac{\hbar}{2}$, identifying themselves as the most suitable candidate to investigate the increment of the statistical character of quantum theory brought into stage by the POVMs.

Our results show that the emergence of POVMs systematically increases the randomness of the outcomes [32]. In particular, the inequality $\Delta X \Delta P \geq \frac{\hbar}{2}$ is saturated only in the limiting case of infinitely sharp states in the momentum space; for any finite Gaussian width, we observe instead an increase in the product $\Delta X \Delta P$, of which we give a fully analytic estimate. We observe a similar increase of randomness in the spin probability distribution.

Such increment appears to be a manifestation of the unsharpness of position and spin, and of the inherent impossibility of sharply localizing a single photon in a bounded space region [3, 33], and of preparing it with definite spin along a spatial direction independent on its momentum [30].

The paper is organized as follows: in section 2 the quantum mechanical description of a single free photon is reviewed; in section 3 the procedure for constructing single-photon observables is delineated and the POVMs and probability densities of such observables are explicitly given. Finally, in section 4, a detailed study of the preparation uncertainty relations for position and momentum and of the spin probability distributions of Gaussian states is presented, and conclusions are drawn in section 5.

2. Single-photon states

In the present section, a detailed review of the single-photon formalism formulated by Kraus in [11] will be given. Particular attention will be devoted to the representation theory of the Poincaré group [24–26] and to the introduction of an isomorphism based on the representation of the $SU(2)$ group for spin $s = 1$ particles [24, 25, 34]. All these elements will provide a framework for the discussion in section 3, where all the single-photon observables observables, including spin, will be defined as POVMs in a unified picture generalizing Kraus’ treatment of the position observable [11]. It is worth reminding that the term photon does not correspond to a unique notion in literature [27]: photons are either treated as spin $s = 1$ and mass $m = 0$ irreducible representations of the Poincaré group [26, 35], or as occupations of electromagnetic field modes. In the present work we will rely on the first approach, which naturally brings to the introduction of POVMs. Wigner’s seminal work [26] on the representation theory of the Poincaré group, revealed the existence of a deep connection between the symmetries underlying special relativity and the measurable quantities of an elementary particle. Concretely, the mathematical description of a relativistic particle existing in space-time should reflect its Poincaré invariance, and consequently the state space of such particle should carry an irreducible representation, characterized by its spin and mass, of the Poincaré group [24–27, 36].

We construct the above mentioned $s = 1, m = 0$ irreducible representation of the Poincaré group by a suitable modification of the well-known [36, 37] $s = 1$ irreducible projective representation of the Galilei group. This representation acts on $L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$ and, for roto-translations, reads:
\[
\left( \hat{U}(\mathbf{a}, R)\phi \right)(\mathbf{p}) = e^{-\frac{i}{\hbar}\mathbf{p} \cdot \mathbf{a}} \sum_{k=1}^{3} \left( e^{-\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{S}_k} \right)^k \phi^k \left( R^{-1}\mathbf{p} \right),
\]

(1)

where \( R \) is the matrix associated to the rotation of an angle \( \varphi \) around the axis \( \mathbf{n} \), \( \mathbf{a} \) is the vector associated to a spatial translation, and \( \mathbf{S} \) denotes the vector of \( 3 \times 3 \) matrices:

\[
S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = -\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0 \end{pmatrix}.
\]

(2)

The key observation is that the matrices \( R \) and \( \mathbf{S} \) are related to each other through

\[
R = V^\dagger e^{-\frac{i}{\hbar} \mathbf{v} \cdot \mathbf{S} V},
\]

(3)

where the unitary matrix \( V \) reads [34]

\[
V = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\
0 & 0 & -1 \\
-\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}.
\]

(4)

This is a peculiar prerogative of spin \( s = 1 \) particles. In such case, the matrices \( \mathbf{S} \) and \( \mathbf{i} \hbar \mathbf{A} \), with \( \mathbf{A} \) denoting the generators of the rotations in \( \mathbb{R}^3 \), satisfy the same commutation relations. Then, the irreducibility of the \( s = 1 \) representation implies the existence of a unitary matrix \( V \) such that \( \mathbf{S} V = V \mathbf{i} \hbar \mathbf{A} V \).

Equation (3) suggests therefore to introduce the following unitary transformation:

\[
\phi(\mathbf{p}) \mapsto \psi(\mathbf{p}) \equiv V^\dagger \phi(\mathbf{p})
\]

(5)

on \( L^2(\mathbb{R}^3) \otimes \mathbb{C}^3 \). It is immediately noticed that wavefunctions \( \psi(\mathbf{p}) \) in \( L^2(\mathbb{R}^3) \otimes V^\dagger \mathbb{C}^3 \) transform as vector fields under roto-translations \( \hat{U}(\mathbf{a}, R) \equiv V^\dagger \hat{U}(\mathbf{a}, R) V \)

\[
\left( \hat{U}(\mathbf{a}, R)\psi \right)(\mathbf{p}) = e^{-\frac{i}{\hbar}\mathbf{p} \cdot \mathbf{a}} \sum_{k=1}^{3} R_k \psi^k \left( R^{-1}\mathbf{p} \right).
\]

(6)

It is worth reminding that the difference between (1) and (6) consists in the replacements \( \phi \mapsto \psi \), \( \hat{U} \mapsto \hat{U} \) and correspondingly \( e^{-\frac{i}{\hbar} \mathbf{v} \cdot \mathbf{S}} \mapsto R \). In the relativistic case rotations are replaced by transformations in the proper Lorentz group. This suggests a heuristic, but very natural, procedure to move to the relativistic case:

- promoting \( \psi(\mathbf{p}) \) to four-component functions:

\[
\psi(\mathbf{p}) = \begin{pmatrix}
\psi^0(p) \\
\psi^1(p) \\
\psi^2(p) \\
\psi^3(p)
\end{pmatrix}
\]

(7)

of the four-momentum \( p = (p^0, \mathbf{p}) \) satisfying the mass-shell condition \( p^\mu g_{\mu\nu} p^\nu = 0 \), where \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) and Einstein’s convention is assumed. Therefore \( p^0 = |\mathbf{p}| \). The functions (7) lie in the vector space:

\[
\mathcal{H} = L^2 \left( \mathbb{R}^3, \frac{d^3\mathbf{p}}{|\mathbf{p}|} \right) \otimes \mathbb{C}^4
\]

(8)
• equipping $\mathcal{H}$ with the following sesquilinear form:

$$\langle \psi_A | \psi_B \rangle = -\int \frac{d^3p}{|p|} (\psi_A^* p) g_{\mu\nu} \psi_B^*(p)$$  \hspace{1cm} (9)

in which the Poincaré invariant measure $\frac{d^3p}{|p|}$ replaces the Galilei invariant measure $d^3p$ and $^*$ denotes complex conjugation.

• generalizing (6) by the introduction of the following linear representation of the Poincaré group [26]:

$$\hat{U}(a, \Lambda)\psi^\mu(p) = e^{i\int d^3x \cdot a \cdot \Lambda} \psi^\mu\Lambda^{-1}p$$  \hspace{1cm} (10)

with $a$ being a four-vector and $\Lambda$ a Lorentz matrix. Notice that $\hat{U}(a, \Lambda)$ is an isomorphism on $\mathcal{H}$, and $\psi(p)$ transforms as a four-vector under Poincaré transformations.

Since the Minkowski inner product $g_{\mu\nu}$, and thus the scalar product (9), are non-positive definite, $\mathcal{H}$ is a pseudo-Hilbert space. This prevents the possibility of giving a probabilistic interpretation to this formalism. Nevertheless, such obstacle can be overcome in the special case of massless particles, such as photons. In fact, the massless condition identifies a subspace $S \subset \mathcal{H}$

$$S = \{ \psi(p) : p^\mu g_{\mu\nu} \psi^\nu(p) = 0 \text{ for almost all } p \}$$  \hspace{1cm} (11)

with two desirable properties. Firstly, $S$ is invariant under Poincaré transformations, since:

$$p^\mu g_{\mu\nu} \left( \hat{U}(a, \Lambda)\psi^\nu(p) \right) = \psi^\nu(p) g_{\mu\nu} (a + \Lambda^{-1}p) = \left( \Lambda^{-1}p \right)^\mu g_{\mu\nu} \psi^\nu(p) = 0$$  \hspace{1cm} (12)

and therefore $\psi(p) \in S \iff \hat{U}(a, \Lambda)\psi(p) \in S$ for all Poincaré transformations. Second, functions in $S$ have non-negative norm. To prove this statement, it is most convenient to expand the spatial part of $\psi(p) \in S$ on the intrinsic frame $\{ \hat{e}_i(p) \}_{i=1}^3$ given by:

$$\hat{e}_1(p) = \frac{p \times (m \times p)}{|p||m \times p|}, \hspace{1cm} \hat{e}_2(p) = \frac{m \times p}{|m \times p|}, \hspace{1cm} \hat{e}_3(p) = \frac{p}{|p|}$$  \hspace{1cm} (13)

where $m$ is an arbitrary unit vector [14, 38] independent on $p$. It is worth of notice that the first elements of the intrinsic frame (13) are related to the circular polarization vectors $\hat{e}_l(p)$, solutions of the eigenvalue equation $p \times \hat{e}_l(p) = \mp |p| \hat{e}_l(p)$, by the following relation:

$$\hat{e}_l(p) = \frac{\hat{e}_1(p) \mp i\hat{e}_2(p)}{\sqrt{2}}.$$  \hspace{1cm} (14)

On this basis, functions in $S$ are expressed as:

$$\psi^\mu(p) = \begin{pmatrix} \psi^0(p) \\ \psi(p) \end{pmatrix}$$  \hspace{1cm} (15)

where:

$$\psi(p) = \sum_{i=1}^3 \hat{e}_i(p) \bar{\psi}(p).$$  \hspace{1cm} (16)

The condition $\psi(p) \in S$ translates into $\bar{\psi}^0(p) = \bar{\psi}^3(p)$, and the restriction of the sesquilinear form (9) onto $S$ reads:
Equation (17) is manifestly non-negative, and does not involve the components \( \tilde{\psi}^0(p), \tilde{\psi}^3(p) \). Following [11, 24, 25] we interpret these components as irrelevant degrees of freedom, this fact being precursive of the gauge symmetry of the electromagnetic theory [39, 40]. This interpretation is confirmed if the action of Poincaré transformations on functions in \( S \) is taken in consideration. In fact, it is immediate to show that \( \hat{\chi}^i(p) \equiv \left( \hat{U}(a, \Lambda) \right)^i_0 (p) \) has components \( \hat{\chi}^{1,2}(p) \) which depend only on \( \tilde{\psi}^{1,2}(p) \) through the relation:

\[
\hat{\chi}^i(p) = \sum_{j=1}^{2} \hat{e}_j(p) \cdot \Lambda \hat{e}_j(A^{-1}p) \tilde{\psi}^i(A^{-1}p),
\]

(18)

where \( \Lambda \) indicates the spatial part of \( A^\mu_\nu \). These arguments lead immediately to the construction of the single-photon Hilbert space \( \mathcal{H}_S \). Equation (17) defines a seminorm on \( S \), and we can construct a normed space out of \( S \) taking the quotient \( S/\sim \) of \( S \) by the equivalence relation:

\[
\phi \sim \psi \iff \| \phi - \psi \| = 0,
\]

(19)

where \( \| \cdot \| \) denotes the seminorm induced by (9) on \( S \) [11]. \( S/\sim \) is equipped with the scalar product (9), which is now manifestly positive-definite, and carries the irreducible representation of the Poincaré group (18). Each element of \( S/\sim \) is an equivalence class of functions in \( S \), parametrized by a pair:

\[
\begin{pmatrix}
\tilde{\psi}^1(p) \\
\tilde{\psi}^2(p)
\end{pmatrix}
\]

(20)

of complex-valued square-integrable functions, thus making \( S/\sim \) isomorphic to \( L^2(\mathbb{R}^3, \frac{dp}{|p|}) \otimes \mathbb{C}^2 \) [11, 41]. Each equivalence class in \( S/\sim \) has a representative with the following transversal form:

\[
\psi(p) = \begin{pmatrix} 0 \\ \tilde{\psi}(p) \end{pmatrix}, \quad p \cdot \tilde{\psi}(p) = 0 \rightarrow \tilde{\psi}(p) = \sum_{j=1}^{2} \tilde{\psi}^j(p) \hat{e}_j(p).
\]

(21)

All other elements of the equivalence class are related to the transversal representative by the addition of an unphysical component \( (\tilde{\psi}^3(p), 0, 0, \tilde{\psi}^1(p))^T \). This fact shows that the single-photon Hilbert space \( S/\sim \) is also isomorphic to the space of square-integrable transverse wavefunctions:

\[
\mathcal{H}_S = \left\{ \tilde{\psi}(p) : \tilde{\psi}(p) = \sum_{j=1}^{2} \tilde{\psi}^j(p) \hat{e}_j(p), \tilde{\psi}(p) \in L^2(\mathbb{R}^3, \frac{dp}{|p|}) \right\}
\]

(22)

retrieving the result given by Kraus in equations (2), (3) of [11].

The construction outlined above has led in a very natural way to the introduction of the single-photon state space \( \mathcal{H}_S \) by just considering the conditions of mass \( m = 0 \) and spin \( s = 1 \), and requiring that \( \mathcal{H}_S \) carries an irreducible representation of the Poincaré group. The concrete realization of the single-photon Hilbert space \( \mathcal{H}_S \) has the advantage of providing a one-to-one correspondence between states and transverse vector functions.

The physically relevant states with linear and circular polarization are elements of \( \mathcal{H}_S \) of the form:
Finally, one might wish to relate such construction of the single-photon state space to the familiar one based on the requirement of helicity $\pm 1$ [42]. The equivalence of the two approaches is readily proved by recalling the isomorphism $V$, introduced to relate the spin matrices and the generators $A$ of rotations in (3). In fact, the images of the intrinsic frame vectors (13) under the action of the matrix $V$ are closely related to the eigenstates of the helicity operator:

$$
\psi_1 = \pm \frac{\mathbf{p} \cdot \mathbf{v}}{\sqrt{2}} \psi_0
$$

(25)

In particular $V\psi_3(p)$ is the eigenstate relative to the eigenvalue $0$, while $V\psi_{\pm 1}(p)$ are the eigenstates $\psi_{\pm 1}$ relative to eigenvalues $\pm 1$ respectively.

3. Single-photon observables

3.1. The extended Hilbert space

In the present section single-photon observables will be introduced as POVMs, generalizing the treatment of position given by Kraus in [11]. With such purpose, the single-photon Hilbert space $\mathcal{H}_S$ introduced in the previous section 2 has to be regarded as a subspace of the extended Hilbert space $\mathcal{H}_{A} = \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^3) \otimes \mathbb{C}^3$ whose elements have the form:

$$
f(p) = \sum_{i=1}^{3} \psi_i(p) \psi_i(p)
$$

(26)

and thus differ from elements (22) of $\mathcal{H}_S$ by the addition of a longitudinal component $\psi_3(p) \psi_3(p)$. In equation (26), and in the remainder of the present work, we express states as functions of $p$ rather than $\hat{p}$ with harmless abuse of notation.

$\mathcal{H}_S$ is obtained from $\mathcal{H}_{A}$ by means of the projection operator:

$$
\pi : \mathcal{H}_A \rightarrow \mathcal{H}_S, \quad (\pi f)(p) = \sum_{k} \pi^k(p) f^{\pm}(p) \quad \forall p \in \mathbb{R}^3
$$

(27)

with

$$
\pi^k(p) = \delta^k - \frac{p^k p^k}{|p|^2} \quad \forall p \in \mathbb{R}^3.
$$

(28)

The projector (28) eliminates the longitudinal component of the triple $\{\psi_i(p)\}_{i=1}^{3}$ and can thus be interpreted as an analogue of the Helmholtz projection used for decomposing the electric
and magnetic field of classical electrodynamics into a longitudinal and a transverse component. Under roto-translations, states in $\mathcal{H}$ must transform as vector fields:

$$U(a,R)f(p) = e^{-i\pi^R p R f}\left(R^{-1}p\right)$$

(29)

in order to retrieve (18). The introduction of the spin and helicity observables, nevertheless, requires a transformation law under roto-translations involving the vector of Pauli matrices, as in (1), instead of the rotation matrix, as in (6). This can be achieved using the matrix $V$ due to (3). Throughout the remainder of the present work, we will denote using the symbol $\mathcal{H}_{S}$ the isomorphic image of $\mathcal{H}$ through the isomorphism $V$, and make use of the fact that $\mathcal{H}_{S}$ corresponds to the image of $\mathcal{H}$ through the action of the operator $\pi V^\dagger$, see figure 1.

The role of the matrix $V$ elucidates the equivalence between the conditions of transversality and non-zero helicity. Moreover it leads to a unitary representation of the roto-translation group that, as in the case of non-relativistic particles, involves the vector $S$ of spin matrices. In fact, for a state $\tilde{f}(p)$ in $\mathcal{H}$, equations (29) and (3) imply:

$$U^*(a,R)\tilde{f}(p) = e^{-i\pi^S \frac{p}{2} s} f(R^{-1}p).$$

(30)

Equation (30) closely resembles (1) which holds in non-relativistic context.

3.2. Observables as POVMs on $\mathcal{H}_{S}$

Having constructed the single-photon state space, we will now define single-photon observables. To begin, let us consider the case of massive relativistic particles with spin $s$, where each physical observable $\mathcal{O}$ is described in terms of a self-adjoint operator $\hat{O}$ on $L\left(\mathbb{R}^3; \frac{1}{2} p^2\right) \otimes \mathbb{C}^{2s+1}$. Let $\mathcal{O} \subseteq \mathbb{R}$ be the measurable space where $\mathcal{O}$ takes values and $\mathcal{G}$ a suitable $\sigma-$algebra on $\mathcal{O}$. It is well-known that, by making use of the spectral theorem [43, 44], the probability that $\mathcal{O}$ takes values in a measurable set $\mathcal{M} \subseteq \mathcal{G}$ is given by the expectation value of a projector $\hat{E}_\mathcal{O}(\mathcal{M})$. The function $\mathcal{M} \mapsto \hat{E}_\mathcal{O}(\mathcal{M})$ is notoriously a PVM [43, 44].

This construction can be generalized to a set $\mathcal{O}_1, \ldots, \mathcal{O}_n$ of $n$ compatible observables, each taking values in the sample space $\mathcal{O}_i \subseteq \mathbb{R}$ equipped with a suitable sigma-algebra $\mathcal{G}_i$. The compatible observables $\mathcal{O}_1, \ldots, \mathcal{O}_n$, in fact, take value in the sample space $\mathcal{O}_1 \times \ldots \times \mathcal{O}_n \subseteq \mathbb{R}^n$, equipped with the product sigma-algebra $\mathcal{G}_1 \times \ldots \times \mathcal{G}_n$. Moreover, they are described by a set $\hat{O}_1, \ldots, \hat{O}_n$ of $n$ self-adjoint commuting operators on $L\left(\mathbb{R}^3; \frac{1}{2} p^2\right) \otimes \mathbb{C}^{2s+1}$. Also in such situation, the spectral theorem allows to express the probability that $\mathcal{O}_1, \ldots, \mathcal{O}_n$ take values in a measurable set $\mathcal{M} \subseteq \mathcal{G}_1 \times \ldots \times \mathcal{G}_n$ as the expectation value of a projector $\hat{E}_{\mathcal{O}_1, \ldots, \mathcal{O}_n}(\mathcal{M})$. The

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**Figure 1.** Diagram illustrating the action of the matrix $V$ and projection $\pi$. Wavefunctions $\psi$ in $\mathcal{H}_S$ are represented by couples of complex-valued functions. Functions $f$ in $\mathcal{H}_A$ transform according to (29) under roto-translations, while elements $\tilde{f}$ in $\mathcal{H}_A$ according to (30).
map $\mathcal{M} \mapsto \hat{E}_{\mathcal{O}_1 \ldots \mathcal{O}_n} (\mathcal{M})$ is referred to as the joint PVM of the compatible observables $\mathcal{O}_1 \ldots \mathcal{O}_n$, and the PVMs associated to the observables $\mathcal{O}_i$ are readily obtained as marginals of the joint PVM through the operation:

$$\hat{E}_{\mathcal{O}_i} (\mathcal{M}_i) = \hat{E}_{\mathcal{O}_1 \ldots \mathcal{O}_n} (c_{\mathcal{M}_1} \times \ldots \times c_{\mathcal{M}_i} \times \ldots \times c_{\mathcal{M}_n}), \quad \mathcal{M}_i \in \mathcal{G}_i. \quad (31)$$

For massive particles, the fundamental observables of spin, momentum and position are described by the familiar self-adjoint operators $\hat{S}, \hat{P}, \hat{X}^{NW}$, where the latter is the well-known Newton–Wigner position operator [9].

Motivated by the seminal work by Kraus [11, 45], we define, for each single-photon observable $\mathcal{O}$, a self-adjoint operator $\hat{O}$ on the Hilbert space $\mathcal{H}$ with the same structure as in the case of massive particles, e.g.

$$\left( \hat{P}_f \right)_j (p) = p f_j (p)$$

$$\left( \hat{S}_f \right)_j (p) = \sum_k s_{jk} f_k (p)$$

$$\left( \hat{X}^{NW}_f \right)_j (p) = i \hbar \frac{\partial f_j (p)}{\partial p} - \frac{i \hbar}{2 |p|^2} f_j (p). \quad (32)$$

In the light of equation (30), these observables are covariant under roto-translations [24, 25].

Nevertheless, since the physical space of a single photon is $\mathcal{H}_\mathcal{S}$, the image of $\mathcal{H}$ through the operator $\pi V$, the probability that $\mathcal{O}$ takes values in a measurable set $\mathcal{M} \in \mathcal{G}$ is the expectation value of the positive operator:

$$\hat{F}_\mathcal{O} (\mathcal{M}) = \left( \pi V \right)^\dagger \hat{E}_\mathcal{O} (\mathcal{M}) (V \pi) = \hat{\Omega}_\mathcal{O} (\mathcal{M}) \hat{\Omega}_\mathcal{O} (\mathcal{M}), \quad (33)$$

where $\hat{\Omega}_\mathcal{O} (\mathcal{M}) = \hat{E}_\mathcal{O} (\mathcal{M}) V \pi$ and the equality holds by virtue of the idempotence of the PVM $\hat{E}_\mathcal{O} (\mathcal{M})$. The PVM $\mathcal{M} \mapsto \hat{E}_\mathcal{O} (\mathcal{M})$ is turned into the POVM $\mathcal{M} \mapsto \hat{F}_\mathcal{O} (\mathcal{M})$ (33) on $\mathcal{H}_\mathcal{S}$. In fact, the operators $\hat{F}_\mathcal{O} (\mathcal{M})$, referred to as *effect operators* in published literature, are still positive and bounded by the identity operator, but in general are not projectors. The idempotence property characterizing PVMs is recovered if only if the single-photon Hilbert space $\mathcal{H}_\mathcal{S}$ is invariant under the action of the projectors $\hat{E}_\mathcal{O} (\mathcal{M})$.

In the remainder of the present section, we will show that relevant examples of such projective or sharp observables are momentum, polarization and helicity. On the other hand, position and spin are described by POVMs.

In published literature [27, 29, 31, 46], observables described by POVMs are referred to as unsharp, since the emergence of POVMs reflects either practical limits in the precision of measurements performed on the system (in which case POVMs appear as coarse-grained versions of PVMs) [14, 15] or the inherent impossibility of realizing a preparation in which the value of an observable is perfectly defined [21, 27, 31].

In the case of photons, after explicitly constructing the PVMs and POVMs associated to the fundamental single-photon observables, we will interpret the need of describing position and spin as POVMs rather than PVMs as a consequence of the elimination of zero-helicity states. In the context of open quantum systems’ theory, POVMs are obtained as projections of opportune PVMs defined on larger Hilbert spaces, describing the system plus a suitable environment. From a mathematical point of view, both these situations are described in terms of the well-known Naimark’s dilation theorem [20, 28, 45, 47].
3.3. Fundamental POVMs and probability distributions

In the remainder of the present section, we apply the general procedure (33) to explicitly show fundamental examples of POVMs.

3.3.1. Joint probability distribution of $S_z$ and $P$. As first application, we consider the momentum and spin-$z$ observables, which admit the representation (32) on $\mathcal{F}_A$, giving rise to the familiar joint PVM:

$$ (\mathcal{M}, hm_z) \mapsto \left( \hat{E}_{\mathcal{P}, S_z}(\mathcal{M}, hm_z) \right)_s(p) = 1_{\mathcal{M}}(p) \delta_{s,s'} f_s(p), \quad (34) $$

where $1_{\mathcal{M}}(p)$ is the indicator function of the Borel set $\mathcal{M} \in \mathbb{R}^3$ and $m_z = 2 - s$ with $s = 1, 2, 3$. The corresponding POVM on $\mathcal{F}_S$, obtained by applying (33) to such PVM, reads:

$$ (\mathcal{M}, hm_z) \mapsto \left( \hat{F}_{\mathcal{P}, S_z}(\mathcal{M}, hm_z) \psi \right)_s(p) = 1_{\mathcal{M}}(p) \delta_{s,s'} \sum_{i=1}^2 \psi_i^*(p) [V \psi_i(p)], $$

and gives rise to the following joint probability distribution:

$$ p\left( P \in \mathcal{M}, S_z = hm_z \right) = \langle \psi | \hat{F}_{\mathcal{P}, S_z}(\mathcal{M}, hm_z) | \psi \rangle = \left\| \hat{\Delta}_{\mathcal{P}, S_z}(\mathcal{M}, hm_z) \psi \right\|^2 = \int_{\mathcal{M}} \frac{d^3p}{|p|} \left[ \sum_{i=1}^2 \psi_i^*(p) [V \psi_i(p)] \right]^2. \quad (35) $$

3.3.2. Joint probability distribution of $S_z$ and $X$. The joint eigenfunctions of the Newton–Wigner position $\hat{X}^{\text{NW}}$ and spin-$z$ operators (32) are the following elements of $\mathcal{F}_A$:

$$ u_{x,s}(p) = \sqrt{|p|} \frac{e^{-\frac{i}{\hbar} x \cdot p}}{(2\pi \hbar)^{3/2}} e_s, \quad (37) $$

and the associated PVM is:

$$ (\mathcal{M}, hm_z) \mapsto \left( \hat{E}_{X^{\text{NW}}, S_z}(\mathcal{M}, hm_z) \right)_s(p) = \int_{\mathcal{M}} d^3x \left[ \int \frac{d^3p}{|p|} u_{x,s}(p) \cdot \mathbf{T}(p) [u_{x,s}(p)]^* \right], \quad (38) $$

where $\mathcal{M} \in \mathbb{R}^3$ denotes again a Borel set. The position observable $\hat{X}$ on $\mathcal{F}_S$ is the POVM obtained applying (33) to the PVM (38), yielding the following joint probability distribution:

$$ p\left( X \in \mathcal{M}, S_z = hm_z \right) = \langle \psi | \hat{F}_{X^{\text{NW}}, S_z}(\mathcal{M}, hm_z) | \psi \rangle = \left\| \hat{\Delta}_{X^{\text{NW}}, S_z}(\mathcal{M}, hm_z) \psi \right\|^2 = \int_{\mathcal{M}} d^3x \left| \psi(x) \right|^2, \quad (39) $$

where the probability amplitude $\psi(x)$ reads:

$$ \psi(x) = \int \frac{d^3p}{|p|} \frac{e^{\frac{i}{\hbar} x \cdot p}}{(2\pi \hbar)^{3/2}} \sum_{i=1}^2 \psi_i^*(p) [V \psi_i(p)], \quad (40) $$
and can be therefore regarded to as the wave-function for the photon in the configuration space \([48–51]\). It can be finally noticed that, by virtue of \((6)\), the amplitude \((40)\) is covariant under roto-translations:

\[
\psi'(x) = e^{-\frac{i}{\hbar} n \cdot \mathbf{S}} \psi\left(R^{-1}(x - a)\right)
\]

reproducing the expected transformation law for the joint probability density of spin and position. We stress again that the position observable \(X\) in \((39)\) is radically different from the Newton–Wigner position operator \(X^{NW}\) in \((32)\) defined on \(\mathcal{H}_A\). This fact has a prominent impact on the evaluation of the variance, as will be explored in more detail in section 3.3.5 and afterwards. The position POVM obtained as marginal of \((38)\) coincides with the photon position observable constructed by Kraus’ in \([11]\).

3.3.3. Probability distribution of helicity. On the single-photon Hilbert space \(\mathcal{H}_S\), states with definite circular polarization take the form \((24)\). It is easy to verify that such states are eigenstates of the helicity operator \(\epsilon\) with eigenvalues \(\pm 1\). Consequently, the joint POVM for momentum and helicity on \(\mathcal{H}_S\) takes the form:

\[
(M, \pm 1) \mapsto \left(\hat{E}_{P,\epsilon}(M, \pm 1) \psi\right)(p) = 1_M(p) \left(\psi^\dagger(p) \pm i \hat{p}^2(p)\right)^{\frac{1}{2}}
\]

where \(M \in \mathbb{R}^3\) is a Borel set. The joint probability distribution of momentum and helicity reads:

\[
p(\mathcal{M} \in M, \epsilon = \pm 1) = \int_M \frac{d^3p}{|p|} \left|\psi^\dagger(p) \pm i \hat{p}^2(p)\right|^2
\]

from which the probability distribution of helicity is obtained choosing \(\mathcal{M} = \mathbb{R}^3\). \(\mathcal{H}_S\) is invariant under the action of \((42)\). Therefore \((42)\) is a PVM alongside with its marginals, describing momentum and helicity.

3.3.4. Probability distribution of \(S_n\). The probability distribution of the spin projection \(S_n = S \cdot n\) along an arbitrary spatial direction \(n\) is readily found recalling that the matrix \(S \cdot n\) admits in \(\mathbb{C}^3\) the spectral decomposition \(S \cdot n = \sum_s \hbar m_s |\phi_{n,s}\rangle \langle \phi_{n,s}|\), with \(s = 1, 2, 3\) and \(m_s = 2 - s\). The joint PVM of \(S_n\) and momentum on \(\mathcal{H}_A\) reads:

\[
(M, \hbar m_s) \mapsto \left(\hat{E}_{P,S_n}(M, m_s) \Gamma\right)(p) = 1_M(p) \left[|\phi_{n,s}\rangle \langle \phi_{n,s}'| \right] \phi_{n,s}^* \cdot \Gamma(p)
\]

where \(M \in \mathbb{R}^3\) is a Borel set. The corresponding POVM on \(\mathcal{H}_S\), obtained by applying \((33)\) to such PVM, leads to the joint probability distribution:

\[
p(\mathcal{M} \in M, S_n = \hbar m_s) = \int_M \frac{d^3p}{|p|} \left|\sum_{i=1}^2 \psi^\dagger(p) \phi_{n,s,i}^* \cdot \hat{V}_e(p)\right|^2
\]

from which \(p(S_n = \hbar m_s)\) is readily obtained choosing \(\mathcal{M} = \mathbb{R}^3\).

3.3.5. Uncertainty relations for position and momentum. In the light of \((36)\) and \((39)\), it becomes interesting to investigate the preparation uncertainty relations for position and momentum observables of a single photon. On \(\mathcal{H}_A\), both these observables are defined in terms of self-adjoint operators \((32)\) with usual commutator \([\hat{X}^{NW}_i, \hat{P}_j] = i\hbar \delta_{ij}\). Therefore the familiar inequality:
\[ \Delta X_k^\text{NW} \Delta R \geq \frac{\hbar}{2} \]  

holds, and is saturated by Gaussian states with definite spin along an arbitrary spatial direction.

On \( \mathcal{R}_S \), \( \Delta X_k^\text{NW} \) must be replaced with the variance \( \Delta X_k \) of the position observable. \( \Delta X_k \Delta R \geq \frac{\hbar}{2} \) still holds but is no longer saturated by the same minimum uncertainty states due to the projection \( \pi \).

Mean values and variances have to be computed taking the marginals of the joint probability distributions (36) and (39) over the spin degrees of freedom. In the case of position, we are left with the following expression:

\[ p(X \in \mathcal{M}) = \int_{\mathcal{M}} d^3x \left| \tilde{\psi}(x) \right|^2, \quad \mathcal{M} \in \mathbb{R}^3, \]  

where the configurational wave-function \( \tilde{\psi}(x) \) is known to exhibit polynomial [14, 33, 52] or even exponential [3] decay, ensuring the existence of the first momenta of (47).

The latter are readily obtained inserting (40) in (47):

\[
\begin{align*}
\langle X_k \rangle &= i\hbar \sum_{s=1}^{3} \int d^3p \frac{\tilde{\psi}_s(p)}{\sqrt{|p|}} \frac{\partial}{\partial p_k} \left( \frac{\tilde{\psi}_s(p)}{\sqrt{|p|}} \right) \\
\langle X_k^2 \rangle &= -\hbar^2 \sum_{s=1}^{3} \int d^3p \frac{\tilde{\psi}_s(p)}{\sqrt{|p|}} \frac{\partial^2}{\partial p_k^2} \left( \frac{\tilde{\psi}_s(p)}{\sqrt{|p|}} \right),
\end{align*}
\]

and \( (\Delta X_k)^2 \equiv \langle X_k^2 \rangle - \langle X_k \rangle^2 \).

In the remainder of the work, the quantity \( \Delta X_k \Delta R \) will be referred to as uncertainty product.

### 3.4. Interpretation and generalization

The formalism outlined in section 3.2 is a very direct application of Naimark’s dilation theorem to the single-photon case, where the choice of the extended Hilbert space is a very natural consequence of the \( m = 0 \) and \( s = 1 \) conditions. In section 3.3 we have considered several relevant single-photon observables and derived their POVMs, according to (33), as projections on \( \mathcal{R}_S \) of PVMs defined on \( \mathcal{R}_A \), whose expression is borrowed from the quantum theory of relativistic massive particles. The application of (33) to these new concrete situations represents the main contribution of the present work.

It is worth of notice that position and spin are described by POVMs, while momentum and helicity by PVMs, a circumstance which is amenable of a clear physical interpretations.

Observables \( \mathcal{O} \) represented by PVMs are commonly understood to correspond to measurements with perfect accuracy, and are therefore called \textit{sharp observables}. In fact, the generic projector \( \tilde{E}_{\mathcal{O}}(\mathcal{M}) \), \( \mathcal{M} \in G \), identifies the subspace:

\[
\mathcal{S}_{\mathcal{O}}(\mathcal{M}) = \left\{ |\psi\rangle : \tilde{E}_{\mathcal{O}}(\mathcal{M})|\psi\rangle = |\psi\rangle \right\},
\]

where \( \mathcal{O} \) takes values in \( \mathcal{M} \) with probability (1) This property is notoriously not shared by generic POVMs, which are therefore associated to \textit{unsharp observables} [21, 46]. In this sense, we understand that the emergence of POVMs enhances the statistical character of quantum theory: concretely, the use of POVMs typically reflects either imperfect measurements or inherent difficulties in realizing a preparation in which the event \( \mathcal{O} \in \mathcal{M} \) certainly.
In the case of photons, the need of treating position and spin as unsharp observables reflects the properties of single-photon preparations rather than those of measurement procedures.

In particular, the suppression of zero-helicity states determines a coupling between momentum and spin, expressed by the key role of the helicity. This limits the possibility to prepare a photon with definite spin along a fixed spatial direction independent on its momentum. Moreover, this coupling prevents the possibility of localizing photons in the configuration space. In fact single-photon wavefunctions are linear combinations of $\hat{e}_1(p)$ and $\hat{e}_2(p)$, which are not analytic functions of $p$ [33]. Then the Paley–Wiener theorem [52] excludes the possibility of preparing a photon in a bounded region of space. This non-localizability meets the causality conditions discussed by Hegerfeldt in [53–56].

Poses severe limitations on the possibility of localizing photons in the configuration space [3, 33]. These inherent difficulties find a direct correspondence in our construction, which highlights the need of treating spin and position as unsharp observables.

Finally, it is worth of notice that (33) can be further generalized to the case of a set of compatible unsharp observables $\mathcal{O}_1\ldots\mathcal{O}_n$ on $\mathcal{A}$. In this case the probability that $\mathcal{O}_1\ldots\mathcal{O}_n$ take values in a measurable set $\mathcal{M} \in \mathcal{G}$ is given in terms of the expectation value of an effect operator $\hat{F}_{\mathcal{O}_1\ldots\mathcal{O}_n}(\mathcal{M})$. The family of such operators is referred to as the POVM associated to the unsharp joint measurement of the observables $\mathcal{O}_1\ldots\mathcal{O}_n$ [21, 28]. The projection onto the physical single-photon space $\mathcal{H}$ leads therefore to another POVM which is related to that defined on $\mathcal{A}$ by means of the equation:

$$\hat{F}_{\mathcal{O}_1\ldots\mathcal{O}_n}(\mathcal{M}) = (\pi V^*) \hat{F}_{\mathcal{O}_1\ldots\mathcal{O}_n}(\mathcal{M})(V\pi).$$

This expression, which clearly extends (33), can be used for example to describe the joint measurement of position and momentum.

4. Results

In the present section we will consider two classes of physically relevant single-photon states, namely Gaussian states with definite polarization and projections of Gaussian states with definite spin, and extensively investigate: (i) the preparation uncertainty products $\Delta X_k \Delta P_k$ and (ii) the probability distribution of spin along a certain direction.

The study of both these properties will highlight and quantitatively estimate the increase of the statistical character naturally brought along by the intrinsic unsharpness of POVMs [20, 21, 28, 32, 47]. The results show both an increment in the product $\Delta X_k \Delta P_k$ for Gaussian states, which in non-relativistic framework saturate the inequality $\Delta X_k \Delta P_k \geq \hbar/2$, and the increase of randomness in the probability distribution of spin $S_z$.

4.1. Gaussian states with definite polarization

Let us consider Gaussian states with definite polarization in $\mathcal{H} \approx \mathcal{L}^2(\mathbb{R}^3, \frac{dp}{|p|^2}) \otimes \mathbb{C}^2$, i.e. states of the form:

$$\begin{pmatrix} \tilde{\psi}^1(p) \\ \tilde{\psi}^2(p) \end{pmatrix} = \sqrt{|p|} g(p) e^{\frac{i}{\hbar} p \cdot x_0} \begin{pmatrix} \gamma^1 \\ \gamma^2 \end{pmatrix}, \quad g(p) \equiv e^{\frac{|p-p_0|^2}{4|\delta p_0|^2}} \left( 4\pi |\delta p_0|^2 \right)^{-3/2}. \quad (51)$$
where \( \mathbf{p}_0 = \langle \hat{\mathbf{P}} \rangle, \mathbf{x}_0 = \langle \hat{\mathbf{X}} \rangle, \sum_{i=1}^{2} |y_i|^2 = 1 \) and:

\[
a = \frac{(\Delta p)^2}{2p_0^2}, \quad p_0 = |\mathbf{p}_0|
\]

is a positive, dimensionless parameter which takes into account for the wavefunction’s spread in momentum space. In the following, without any loss of generality, we choose \( p_0 = p_0 \mathbf{e}_z \).

4.1.1. Preparation uncertainty relations for position and momentum. The uncertainty products both for \( |y_1|^2 = 1 \) and \( |y_2|^2 = 1 \) read:

\[
\Delta Z \Delta P_z = \frac{\hbar}{2} \sqrt{1 - 4a + 4 \sqrt{a} \left( 1 + 2a \right) D \left( \frac{1}{2,\sqrt{a}} \right)}
\]

\[
\Delta X \Delta P_x = \Delta Y \Delta P_y = \frac{\hbar}{2} \sqrt{1 + 8a - 16a^{3/2} D \left( \frac{1}{2,\sqrt{a}} \right)}
\]

where \( D(x) \) denotes the Dawson’s function (sometimes also referred to as Dawson’s Integral) of argument \( x \) [57]. These results are derived in appendix A.1 and illustrated in figure 2. These are monotonically increasing functions of the parameter \( a \) and remarkably \( \lim_{a \to 0} \Delta X_i \Delta P_i = \frac{\hbar}{2} \) is obtained for every component. This indicates that, for an infinitely sharp wavefunction in the momentum space, the familiar Heisenberg limit is retrieved.

Figure 2 shows that all the uncertainty products converge to \( \lim_{a \to \infty} \Delta X_i \Delta P_i = \frac{\hbar}{\sqrt{\pi}} \hbar \).

This increase in the uncertainty products is a direct consequence of the unsharpness of position observable, and gives a direct example of the circumstance, well known in the quantum theory of measurement [20, 21, 27, 28, 47], that POVMs enhance the randomness of measurement outcomes and the statistical character of the underlying quantum theory.

4.1.2. Probability distribution of \( S_z \). The probability distribution of \( S_z \) for states of the form (51) reads:

\[
p(S_z = \hbar m_z) = \sum_{\xi} \langle y_{\xi}| \Sigma (m_z) | y_{\xi} \rangle y_{\xi}^z.
\]
where \( m_s = 2 - s \) and
\[
\left[ \Sigma (m_s) \right]_{ij} = \int d^3p \left[ \hat{e}_i(p) \right]^* \left[ \hat{e}_j(p) \right] g^2(p).
\] (55)

The concrete expression for the three matrices \( \left[ \Sigma (m_s) \right]_{ij} \) will be given in appendix A.2.

Figure 3 shows the probability distribution of \( S_z \) for a Gaussian state with circular polarization \( \gamma_\pm \) (24). In the limit \( a \to 0 \), the probability distribution of \( S_z \) is concentrated in \(-\hbar\) while in the opposite limit \( a \to \infty \) \( p(S_z = \hbar m_s) = \frac{1}{2} \) for \( m_s = 0, \pm 1 \), this meaning that all the information about the spin is lost.

4.2. Gaussian states with definite spin

Gaussian states with definite spin along an arbitrary spatial axis will now be considered, which are defined in \( \mathcal{F}_A \simeq \mathcal{L}^2 \left( \mathbb{R}^3, \frac{d^3p}{|p|} \right) \otimes \mathbb{C}^3 \) as:
\[
\hat{f}(p) = \sqrt{|p|} \ g(p) \ h \quad h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \quad \sum_{i=1}^3 |h_i|^2 = 1.
\] (56)

The corresponding physical states are elements in \( \mathcal{F}_S \) given by:
\[
\hat{\psi}(p) = \frac{\pi V^* \hat{f}(p)}{K},
\] (57)

the quantity
\[
K^2 = \langle \pi V^* \hat{f} | \pi V^* \hat{f} \rangle = \langle V \pi V^* \hat{f} | V \pi V^* \hat{f} \rangle
\] (58)

being a normalization constant.

4.2.1. Preparation uncertainty relations for position and momentum. For states of the form (57) the uncertainty products on the \( z \) and \( x(y) \) axis, respectively parallel and perpendicular to \( p_0 \), are given by:
• if \( \mathbf{h} = (1, 0, 0)^T \), or \( \mathbf{h} = (0, 0, 1)^T \):

\[
\Delta Z \cdot \Delta P_z = \hbar \sqrt{1 + 16a^2 u_1(a) - \left(\frac{1 - 2a u_1(a)}{1 - 2a u_2(a)}\right)^2 \left[\frac{1}{8a} + \left(\frac{1}{2} - \frac{1}{8a}\right)u_2(a) - u_1(a)\right]} \quad (59)
\]

\[
\Delta X \cdot \Delta P_x = \Delta Y \cdot \Delta P_y = \hbar \sqrt{2a \left(1 - 4a u_2(a)\right) \left(\frac{1}{2} u_1(a) - u_2(a)\right) \left(\frac{1}{2} + \frac{1}{4a}\right) + \frac{3}{8a}} \quad (60)
\]
Expressions (59) and (61) are presented in figures 4 and 5 respectively.

Again the uncertainty products show a non-trivial dependence on the width parameter \(a\). For \(h = (1, 0, 0)^T\), the uncertainty products are monotonically increasing functions of \(a\), converging to \(\hbar\) in the \(a \to 0\) limit and to finite values in the opposite \(a \to \infty\) limit. The asymptotic expansions of (59) and (61) are listed in table A3.

For \(h = (0,1,0)^T\), on the other hand, this behaviour is exhibited only by the uncertainty product along the direction of \(p_0\), while the remaining uncertainty products are monotonically decreasing functions of \(a\). This apparently anomalous behaviour is due to the fact that the state (57) has, for decreasing \(a\), vanishing projection on the physically relevant space \(\mathcal{H}_s\).

Finally, for a generic vector \(h \in \mathbb{C}^3\) the uncertainty products are quite involved expressions that will be included in the appendix A.3, together with a discussion about their maximization and minimization.

### 4.2.2. Probability distribution of \(S_z\)

The probability distribution of \(S_z\) for a spin state (57) reads:

\[
p(S_z = h m_s) = \frac{\sum_{ij} h_i^* \left[ \sum (m_s) \right]_{ij} h_j}{K},
\]

where \(m_s = 2 - s\),

\[
\left[ \sum (m_s) \right]_{ij} = \int \frac{d^3 p}{p^3} |\Pi(p)\rangle_{ss} \langle \Pi(p) | g^2(p)
\]

and \(K\) is given by equation (58). Concrete expression for the three matrices \(\left[ \sum (m_s) \right]_{ij}\) will be given in appendix A.4. Figure 6 shows the cases \(h = (1,0,0)^T\), \((0, 1, 0)^T\), \((0,0,1)^T\).

Due to the projection onto the physical Hilbert space, the spin of the photon ceases to be a definite quantity and is characterized by a probability distribution resulting by integration (36) over the momentum variable. This result is consequence of the fact that states with definite spin exist only in the extended Hilbert space and puts under an even more clear evidence the enhancement of the statistical character of the quantum theory stemming from the unsharpness of POVMs.

### 5. Conclusions

In the present work, we have constructed a unified procedure for treating all single-photon observables as POVMs, including the notoriously delicate position and spin observables. This also allows to compute all their joint probability distributions.

Starting from the representation theory of the Poincaré group for spin \(s = 1\) and mass \(m = 0\) particles, we have demonstrated how the suppression of the zero-helicity component of
the single-photon wavefunction, which corresponds through the isomorphism $V$ to the suppression of the longitudinal component, can be viewed as the projection from an extended Hilbert space onto the physical Hilbert space.

We have then shown how this theoretical construction naturally brings along the notion of POVMs, since any observable can be introduced on the extended Hilbert space as a PVM, whose form is very naturally suggested by the well-established quantum theory of relativistic particles, and then turned, by means of the projection, into a directly manageable POVM on $\mathcal{H}_S$.

The suppression of zero-helicity states determines a coupling between momentum and spin, which reflects in the sharpness of helicity observable. The intrinsic unsharpness of spin

**Figure 6.** Probability distribution of $S_z$ (red dashed line for the eigenvalue $m_z = 1$, green solid line for $m_z = -1$, blue dotted-dashed line for $m_z = 0$) on the spin states (57) with $\mathbf{h} = (1, 0, 0)^T$ (a), $(0, 1, 0)^T$ (b) and $(0, 0, 1)^T$ (c).
and position, on the other hand, reflects the impossibility of preparing a photon with definite spin along a fixed arbitrary direction or localizing it in a bounded region of the configuration space, according to the Paley–Wiener theorem.

We have explicitly derived the joint probability distributions for spin and momentum, and for spin and position, along with their marginals and the probability distribution of helicity.

Finally, we have extensively investigated the consequences of the introduction of the POVMs, by studying the preparation uncertainty relations for position and momentum and the probability distribution of $S_z$. We have considered different classes of Gaussian states, which in the non-relativistic context saturate the inequality $\Delta X^\text{NW}_x \Delta R_\perp \geq \frac{\hbar}{2}$, namely those with definite polarization and with definite spin along a spatial axis. The results we have found show a clear enhancement of the statistical character of position and spin observables brought into stage by the appearance of POVMs. This ranges from the increment of the uncertainty products for any value of the Gaussian width $a$, to the impossibility of preparing a state with definite spin along an axis independent on momentum. Only in the limiting case of infinitely sharp wavefunctions in the momentum space (which corresponds to $a \to 0^+$), a single-photon state can have a defined value of the spin and the uncertainty products return to the value $\frac{\hbar}{2}$. Such behaviour is observed both for Gaussian states with definite polarization and spin, confirming the coherence of the underlying formalism, which can be immediately applied to all single-photon observables and to all single-photon states and density matrices.

We believe there exist favourable prospects for this formalism to be applied in a high-precision description of interference phenomena at the basis of experiments with single photons [2, 6, 58, 59].

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Appendix A. Calculation details

This appendix is devoted to present in detail the intermediate calculations leading to the results (53), (59) and (61).

A.1. Intermediate results for subsection 4.1.1

Application of (36) readily shows that the mean values and the variances of momentum are:

\[
\langle P^j \rangle = p^j_0
\]

\[
\left\langle (P^j)^2 \right\rangle = \left( p^j_0 \right)^2 + 2a p^j_0.
\]

(65)

Application of (48) shows that the mean values of $X_j$ and $X_j^2$ are bilinear functions of the coefficients $\gamma^i$, defined by the matrices:
Table A1. Limiting behaviour of the uncertainty product (46), in units of $\hbar$, for states with definite polarization along directions parallel ($z$ axis) and perpendicular ($x$ or $y$ axis) to $p_0$.

| Axis | Small $a$ | Large $a$ |
|------|-----------|-----------|
| $z$  | $\frac{1}{2} + 4a^2$ | $\sqrt{\frac{T}{12}} - \frac{1}{5} \sqrt{21} a + \frac{11}{700} \sqrt{21} a^2$ |
| $x$ ($y$) | $\frac{1}{2} + 2a - 16a^2$ | $\sqrt{\frac{T}{12}} - \frac{1}{5} \sqrt{21} a + \frac{1}{175} \sqrt{21} a^2$ |

\[
\left[ X^1 \right]_{kl} = i\hbar \int d^3 p \ g^2(p) \ \hat{e}_k(p) \cdot \partial_{p_l} \hat{e}_l(p)
\]
\[
\left[ (X^1)^2 \right]_{kl} = -\hbar^2 \int d^3 p \ g^2(p) \left[ \hat{e}_k(p) \cdot \partial^2_{p_l,p_j} \hat{e}_j(p) - \frac{p_j - p_{lj}}{4ap_0^2} \hat{e}_k(p) \cdot \partial_{p_l} \hat{e}_l(p) \right] + \hbar^2 \left( x_i \right)^2 \delta_{kl} \tag{66}
\]

with $g^2(p) = \frac{e^{-m^2/4a}}{(4\pi a p_0)^2}$. Integrating (66) in shifted spherical coordinates gives:

\[
\left[ X^1 \right]_{kl} = 0 \quad \forall j = 1, 2, 3
\]
\[
\left[ (X^1)^2 \right]_{kl} = h^2 \delta_{kl} - \frac{1 - 4a + 4\sqrt{a} (1 + 2a) D \left( \frac{1}{2,\sqrt{a}} \right)}{8ap_0^2}
\]
\[
\left[ (X^1)^2 \right]_{kl} = h^2 \delta_{kl} - \frac{1 + 8a - 16a^{3/2} D \left( \frac{1}{2,\sqrt{a}} \right)}{8ap_0^2}
\]
\[
\left[ (X^2)^2 \right]_{kl} = h^2 \delta_{kl} - \frac{1 + 8a - 16a^{3/2} D \left( \frac{1}{2,\sqrt{a}} \right)}{8ap_0^2}
\]

where $D(x)$ denotes the Dawson’s function of argument $x$ [57]. These results immediately lead to (53), whose asymptotic expansions are given in table A1.

A.2. Intermediate results for 4.1.2

Equation (54) is a quadratic function of $\left( \frac{x^1}{y^2} \right)$ and can be expressed in terms of the functions:

\[
u_1(a) = 1 - 6a + 12a^{3/2} D \left( \frac{1}{2,\sqrt{a}} \right)
\]
\[
u_2(a) = 1 - 2\sqrt{a} D \left( \frac{1}{2,\sqrt{a}} \right)
\]
\[
u_3(a) = \frac{1}{2} \left( \frac{2\sqrt{a}}{\sqrt{\pi}} e^{-\frac{1}{4a}} + (1 - 2a) \operatorname{erf} \left( \frac{1}{2\sqrt{a}} \right) \right)
\]

(68)
The matrices $\Sigma(m_s)$ are given by

$$\Sigma(1) = \begin{pmatrix}
\frac{i}{3} + \frac{u_1(a)}{6} & i u_3(a) \\
-i u_3(a) & \frac{i}{3} + \frac{u_1(a)}{6}
\end{pmatrix},
$$

$$\Sigma(0) = \begin{pmatrix}
2au_2(a) & 0 \\
0 & 2au_2(a)
\end{pmatrix},$$

$$\Sigma(-1) = \begin{pmatrix}
\frac{i}{3} + \frac{u_1(a)}{6} & -iu_3(a) \\
iu_3(a) & \frac{i}{3} + \frac{u_1(a)}{6}
\end{pmatrix}$$

from which it can be immediately seen that:

(a) the eigenvectors of (69) are the states $\gamma_\pm$ with definite circular polarization, see equation (24).

(b) $p(S_z = 0)$ for $\gamma_+$ and $p(S_z = 0)$ for $\gamma_-$ are equal to each other.

(c) $p(S_z = h)$ for $\gamma_+$ and $p(S_z = -h)$ for $\gamma_-$ are equal to each other.

The asymptotic behaviour of $p(S_z = h m_s)$ for Gaussian states with definite polarization $\gamma_\pm$ is listed in table A2.

### A.3. Intermediate results for 4.2.1

The square norm $K^2$ and the first momenta of the position and momentum probability distributions are quadratic functions of $h$ defined by the matrices:

$$[K]_{kl} = \int d^3p \, g^2(p) [\Pi(p)]_{kl},$$

$$[p^j]_{kl} = \int d^3p \, g^2(p) \, p^j \, [\Pi(p)]_{kl},$$

$$\left[p^2\right]_{kl} = \int d^3p \, g^2(p) \, \left[p^j\right]^2 \, [\Pi(p)]_{kl},$$

$$\left[X^j\right]_{kl} = i h \int d^3p \, g^2(p) \left[\frac{\partial}{\partial p^j} [\Pi(p)]_{kl} - \frac{p^j - p_0^j}{4ap_0^2} [\Pi(p)]_{kl}\right]$$

| $S_z/\hbar$ | Polarization | Small $a$ | Large $a$ |
|------------|-------------|----------|----------|
| 1          | $\gamma_+$  | $2a^2$   | $\frac{1}{3} - \frac{1}{2\hbar} + \frac{1}{60a}$ |
| 0          | $\gamma_+$  | $2a - 4a^2$ | $\frac{1}{3} - \frac{1}{30a}$ |
| -1         | $\gamma_+$  | $1 - 2a + 2a^2$ | $\frac{1}{3} + \frac{1}{2\hbar} + \frac{1}{60a}$ |
| 1          | $\gamma_-$  | $1 - 2a + 2a^2$ | $\frac{1}{3} + \frac{1}{2\hbar} + \frac{1}{60a}$ |
| 0          | $\gamma_-$  | $2a - 4a^2$ | $\frac{1}{3} - \frac{1}{30a}$ |
| -1         | $\gamma_-$  | $2a^2$ | $\frac{1}{3} - \frac{1}{2\hbar} + \frac{1}{60a}$ |
\[
\left[ (X^2)^{1/2} \right]_{kl} = -\hbar^2 \int d^3 p \, g^2(p) \left[ \frac{\partial^2}{\partial p_j^2} [\Pi(p)]_{kl} - 2 \frac{p^j - p^j_0}{4ap_0^2} \frac{\partial}{\partial p_j} [\Pi(p)]_{kl} \right]
+ \left[ \frac{(p^j - p^j_0)^2}{4ap_0^2} \right] \frac{1}{4ap_0^2} [\Pi(p)]_{kl} \right].
\]

where \( \Pi(p) \equiv V_\pi V^\dagger \). We give below their analytic expression, resulting from integrations in shifted spherical coordinates. All the results, which can be expressed solely in terms of the functions \( u_1(a) \) and \( u_2(a) \) in (68), read:

\[
K = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} + 2a \, u_2(a) \begin{pmatrix}
-1 & 0 & 0 \\
0 & 2 & 2 \\
0 & 0 & -1
\end{pmatrix}; \quad (70)
\]

\[
P^1 \equiv P_1 = 2p_0 \, a \, u_1(a) \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & -\frac{1}{\sqrt{2}} & 0
\end{pmatrix}; \quad (71)
\]

\[
P^2 \equiv P_2 = 2p_0 \, a \, u_1(a) \begin{pmatrix}
0 & -\frac{i}{\sqrt{2}} & 0 \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
0 & \frac{i}{\sqrt{2}} & 0
\end{pmatrix}; \quad (72)
\]

\[
P^3 \equiv P_3 = p_0 \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} + 2p_0 \, a \, u_1(a) \begin{pmatrix}
-1 & 0 & 0 \\
0 & 2 & 2 \\
0 & 0 & -1
\end{pmatrix}; \quad (73)
\]

\[
\left( P^1 \right)^2 = \left( 2p_0 \right)^2 \frac{a}{2} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} + \left( 2p_0 \right)^2 a^2 \, u_1(a) \begin{pmatrix}
-2 & 0 & 1 \\
0 & 4 & 0 \\
1 & 0 & -2
\end{pmatrix}; \quad (74)
\]

\[
\left( P^2 \right)^2 = \left( 2p_0 \right)^2 \frac{a}{2} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} + \left( 2p_0 \right)^2 a^2 \, u_1(a) \begin{pmatrix}
-2 & 0 & -1 \\
0 & 4 & 0 \\
-1 & 0 & -2
\end{pmatrix}; \quad (75)
\]

\[
\left( P^3 \right)^2 = p_0^2 \begin{pmatrix}
1 & 0 & 0 \\
0 & 4a & 0 \\
0 & 0 & 1
\end{pmatrix} + \left( 2p_0 \right)^2 4a^2 \, u_1(a) \begin{pmatrix}
1 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & 1
\end{pmatrix}; \quad (76)
\]

and

\[
X^1 \equiv X = 0 \\
X^2 \equiv Y = 0 \\
X^3 \equiv Z = 0 \quad (77)
\]
\[
\left( x^1 \right)^2 = \frac{h^2}{\rho_0^2} \begin{pmatrix}
\frac{1}{4}u_1(a) - u_2(a) \left( \frac{1}{2} + \frac{1}{4a} \right) + \frac{3}{8a} & \frac{-u_1(a) + 2u_2(a)}{4} & 0 \\
0 & 2u_2(a) - u_1(a) & 0 \\
\frac{u_1(a) + 2u_2(a)}{4} & 0 & \frac{1}{2}u_1(a) - u_2(a) \left( \frac{1}{2} + \frac{1}{4a} \right) + \frac{3}{8a}
\end{pmatrix}
\]

\[
\left( x^2 \right)^2 = \frac{h^2}{\rho_0^2} \begin{pmatrix}
\frac{1}{4}u_1(a) - u_2(a) \left( \frac{1}{2} + \frac{1}{4a} \right) + \frac{3}{8a} & 0 & \frac{-u_1(a) + 2u_2(a)}{4} \\
0 & 2u_2(a) - u_1(a) & 0 \\
\frac{u_1(a) + 2u_2(a)}{4} & 0 & \frac{1}{2}u_1(a) - u_2(a) \left( \frac{1}{2} + \frac{1}{4a} \right) + \frac{3}{8a}
\end{pmatrix}
\]

\[
\left( x^3 \right)^2 = \frac{h^2}{\rho_0^2} \begin{pmatrix}
\frac{1}{4a} + \frac{4a-1}{8a}u_2(a) - u_1(a) & 0 & 0 \\
0 & \frac{3}{4a} - \frac{12a + 3}{4a}u_2(a) + 2u_1(a) & 0 \\
0 & 0 & \frac{1}{4a} + \frac{4a-1}{8a}u_2(a) - u_1(a)
\end{pmatrix}
\]

We remark that the matrices associated to the components of momentum and position along the \( x \) and \( y \) axes share the same spectrum. These results immediately lead to (59) and (61), whose asymptotic expansions are given in table A3. Finally we identify the quantities \( h_{\text{max}} \) and \( h_{\text{min}} \) which respectively maximize and minimize the Heisenberg Uncertainty Relations (59) and (61). For the purpose of simplifying the forthcoming calculations, let us parametrize \( h \in \mathbb{C}^3 \) as follows:

\[
h = \begin{pmatrix}
\tilde{h}_1 \\
\tilde{h}_2 \\
\tilde{h}_3
\end{pmatrix}
\]

where \( \tilde{h}_1 = \rho_0 \left( \frac{1}{\sqrt{1 - 2a\rho}} \right) \), \( \tilde{h}_2 = \rho_0 \left( \frac{1}{\sqrt{2a\rho}} \right) \), and \( \tilde{h}_3 = \rho_0 \left( \frac{1}{\sqrt{2a\rho}} \right) \).

It can be easily verified that \( h \cdot K h = 1 \) if and only if \( \tilde{h}_1 \tilde{h}_2 + \tilde{h}_2 \tilde{h}_3 + \tilde{h}_3 \tilde{h}_1 = 1 \). This choice leads to the following expressions:

\[
\begin{align*}
\langle P \rangle &= \rho_0 \left[ \frac{1 - 2au_1(a)}{1 - 2au_2(a)} (1 - \rho) + \frac{u_1(a)}{u_2(a)} \rho \right] \\
\langle (P)^2 \rangle &= \rho_0^2 \left[ \frac{1 + 16a^2u_1(a)}{1 - 2au_2(a)} (1 - \rho) + \frac{1 - 8au_1(a)}{u_2(a)} \rho \right] \\
\langle Z \rangle &= 0 \\
\langle (Z)^2 \rangle &= \frac{h^2}{\rho_0^2} \left[ \frac{1 + \frac{u_2(a)}{2} + \frac{u_2(a)}{8a}}{1 - 2au_2(a)} - u_1(a) (1 - \rho) + \frac{3}{4a} - \left( \frac{3 + \frac{3}{4a}}{4au_2(a)} \right) u_2(a) + 2u_1(a) \rho \right].
\end{align*}
\]

where \( \tilde{h}_1^2 = \rho, \tilde{h}_2^2 = 1 - \rho, \tilde{h}_3^2 = 1 - \rho \). The square of the Heisenberg product constructed from (80) is a polynomial of third degree in the variable \( \rho \) possessing, for each fixed value of \( a \), a global maximum at a value \( \rho_{\text{max}}(a) \) in the interval \((0, 1)\) and a global minimum at

\[23\]
\[ \rho_{\text{min}}(a) = \Theta(a - a^q), \text{ where } a^q \sim 6.13116. \] We do not detail the rather intricate analytic form of the minimum and maximum values of the Heisenberg product, but limit ourselves to show, in figure A1, their values against those in figures 4 and 5 relative to the choices \( h = (1, 0, 0)^T \) and \( h = (0, 1, 0)^T \).

On the \( x \) axis, it is convenient to parametrize also \( \hat{h} \) as follows:

\[
\hat{h} = \begin{pmatrix}
\sqrt{\lambda} e^{\phi_1} \\
\sqrt{1 - \lambda} \sqrt{1 - \xi} e^{\phi_2}
\end{pmatrix}
\]

obtaining:

\[
\left\langle \hat{p}_i \right\rangle = -\left(2p_0\right) \frac{2au_i(a)}{\sqrt{(1 - 2au_z(a))(4au_z(a))}} (1-\lambda) \sqrt{\xi - \xi^2} \cos(\phi_2)
\]

\[
\left\langle \left(\hat{p}_i\right)^2 \right\rangle = \left(2p_0\right)^2 a \left[ \frac{1 - 2au_z(a)}{2(1 - 2au_z(a))} (1 - \lambda) \sqrt{1 - \xi} + \frac{1 - 6au_1(a)}{2(1 - 2au_z(a))} (1 - \xi) + \frac{u_1(a)}{u_2(a)} (1 - \lambda) \xi \right]
\]

\[
\langle \hat{x} \rangle = 0
\]

\[
\left\langle \hat{p}_z \right\rangle = \frac{\hbar^2}{(2p_0)^2} \left[ \frac{u_1(a) - \frac{u_z(a)}{a} - 4u_z(a) + \frac{3}{2a}}{1 - 2au_z(a)} \right]
\]

\[
+ \left( \frac{3au_1(a) - u_z(a) + \frac{3}{2a}}{1 - 2au_z(a)} (1 - \lambda)(1 - \xi) + \frac{2u_z(a) - u_1(a)}{au_z(a)} (1 - \lambda) \xi \right]
\]

The square of the Heisenberg uncertainty product is a linear and monotonically decreasing function of \( \cos^2(\phi_2) \). The minimum is attained at \( \cos^2(\phi_2) = 1, \xi = 0 \) and \( \lambda = \Theta(a^q - a^q), a^q \sim 2.6095, \) and the maximum at \( \cos^2(\phi_2) = 0, \xi = \xi_{\text{max}}(a) \) and \( \lambda = 0 \). \( \xi_{\text{max}}(a) \) results from a straightforward but quite lengthy maximization procedure. The minimum and maximum values of the Heisenberg uncertainty product are shown in figure A1 against the corresponding values (59) for \( h = (1, 0, 0)^T \) and \( h = (0, 1, 0)^T \).

Finally the asymptotic expansions of the uncertainty products for small and large values of the parameter \( a \) are listed in table A4 in units of \( \hbar \).

**Table A3.** Limiting behaviour of the uncertainty product (46), in units of \( \hbar \), for several choice of the vector \( h \) (first column), along directions parallel (\( z \) axis) and perpendicular (\( x \) axis) to \( p_0 \) (second column). The expansions relative to the vector \( \hat{h} = (0, 0, 1)^T \) are not listed being equal to the case \( h = (1, 0, 0)^T \).

| \( h \) | Axis | Small \( a \) | Large \( a \) |
|---|---|---|---|
| \( (1, 0, 0)^T \) | \( z \) | \( \frac{1}{2} + 4a^2 \) | \( \frac{221}{5} - \frac{59\sqrt{2}}{350\sqrt{a}} \) |
| \( (1, 0, 0)^T \) | \( x \) | \( \frac{1}{2} + a + 40a^3 \) | \( \frac{3\sqrt{41}}{2} - \frac{1361}{2800\sqrt{41a}} \) |
| \( (0, 1, 0)^T \) | \( z \) | \( \frac{1}{2} + 16a^2 + 272a^3 \) | \( \frac{9}{10} - \frac{1}{175a} \) |
| \( (0, 1, 0)^T \) | \( x \) | \( 1 - 8a^2 + 32a^3 \) | \( \frac{2\sqrt{21}}{5} + \frac{2\sqrt{5}}{175\sqrt{2a}} \) |
A.4. Intermediate results for 4.2.2

Equation (63) is the ratio between two quadratic functions of \( h \). The denominator is explicitly given by (71), while the matrix \( m(s) \Sigma \) in the numerator is most conveniently expressed in terms of the function \( u_{a}(a) = 1 - 2au_{2}(a) \) and reads:

\[
\Sigma(1) = \begin{pmatrix}
4a + (1 - 6a)u_{4}(a) & 0 & 0 \\
0 & 2(1 + 12a)u_{4}(a) - (1 + 8a) & 0 \\
0 & 0 & 1 + 4a - (1 + 6a)u_{4}(a)
\end{pmatrix}
\]

\[
\Sigma(0) = \begin{pmatrix}
4a - (2a + 24a^{2})u_{2}(a) & 0 & 0 \\
0 & (1 + 6a)8u_{2}(a) - 4a & 0 \\
0 & 0 & 4a - (2a + 24a^{2})u_{2}(a)
\end{pmatrix}
\]

**Figure A1.** Minimum and maximum (blue lines) of the Heisenberg uncertainty product along directions parallel (left panel) and perpendicular (right panel) to \( p_{0} \), in comparison with \( h = (1, 0, 0)^{T} \) (red dotted line) and \( h = (0, 1, 0)^{T} \) (green dotted–dashed line).

A.4. Intermediate results for 4.2.2

Equation (63) is the ratio between two quadratic functions of \( h \). The denominator is explicitly given by (71), while the matrix \( \Sigma(m) \) in the numerator is most conveniently expressed in terms of the function \( u_{a}(a) = 1 - 2au_{2}(a) \) and reads:

\[
\Sigma(1) = \begin{pmatrix}
4a + (1 - 6a)u_{4}(a) & 0 & 0 \\
0 & 2(1 + 12a)u_{4}(a) - (1 + 8a) & 0 \\
0 & 0 & 1 + 4a - (1 + 6a)u_{4}(a)
\end{pmatrix}
\]

\[
\Sigma(0) = \begin{pmatrix}
4a - (2a + 24a^{2})u_{2}(a) & 0 & 0 \\
0 & (1 + 6a)8u_{2}(a) - 4a & 0 \\
0 & 0 & 4a - (2a + 24a^{2})u_{2}(a)
\end{pmatrix}
\]
\begin{table}
\centering
\caption{Limiting behaviour of the Heisenberg uncertainty relations (46), in units of $\hbar$, for $h_{\text{min}}$ and $h_{\text{max}}$ (first column), along directions parallel ($z$ axis) and perpendicular ($x$ axis) to $p_0$ (second column).}
\begin{tabular}{cccc}
\hline
$h$ & Axis & Small $a$ & Large $a$ \\
\hline
$h_{\text{min}} (a)$ & $z$ & $\frac{1}{2} + 4a^2$ & $\frac{9}{\sqrt{13}}$ \\
$h_{\text{max}} (a)$ & $z$ & $\frac{1}{2} + a + 12a^2$ & $\frac{2}{\sqrt{13}} \left( \frac{317}{364a} \right)^{\frac{1}{2}}$ \\
$h_{\text{min}} (a)$ & $x$ & $\frac{1}{2} + 4a^2$ & $\frac{9}{23}$ \\
$h_{\text{max}} (a)$ & $x$ & $1 - 8a^2 + 32a^3$ & $2\sqrt{\frac{2}{13}}$ \\
\hline
\end{tabular}
\end{table}

Table A5. Asymptotic behaviour of the probability distribution of $S_z$ for the choices of $h$ reported in figure 6.

| $S_z/h$ | $h$ | Small $a$ | Large $a$ |
|---------|-----|-----------|-----------|
| 1       | $101$ | $1 - 2a + 8a^2$ | $\frac{7}{10}$ + $\frac{5}{180a}$ |
| 0       | $101$ | $2a - 16a^2$ | $\frac{1}{2}$ + $\frac{1}{3} 1400a$ |
| $-1$    | $101$ | $8a^2$ | $\frac{20}{3}$ - $\frac{4}{3} 54a$ |
| 1       | $101$ | $1 - 4a + 8a^2$ | $\frac{1}{2}$ + $\frac{1}{6} 175a$ |
| 0       | $101$ | $8a - 16a^2$ | $\frac{8}{10}$ - $\frac{13}{15} 175a$ |

\[ \Sigma (-1) = \begin{pmatrix}
1 + 4a - (1 + 6a)u_4(a) & 0 & 0 \\
0 & 2(1 + 12a)u_4(a) - (1 + 8a) & 0 \\
0 & 0 & 4a + (1 - 6a)u_4(a)
\end{pmatrix} \]  

(82)

from which it is evident that:

(a) $p(S_z = 1) = p(S_z = -1)$ for the spin state relative to the choice $h = (0, 1, 0)^T$

(b) $p(S_z = 1)$ for the spin state with $h = (1, 0, 0)^T$ and $p(S_z = -1)$ for the spin state with $h = (0, 0, 1)^T$ are equal to each other.

The asymptotic expansions of the probability distributions for $S_z$ are listed in table A5.

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