Event-triggered control of singularly perturbed linear system with DoS attacks

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Funding information
National Nature Science Foundation of China, Grant/Award Number: 61573036

1 | INTRODUCTION

In networked control systems (NCSs), the traditional periodic sampling control (or called time-triggered control) may execute some control tasks unnecessarily from the viewpoint of stability. Therefore, in recent years, more and more studies pay attention to the aperiodic sampling control. As a method of aperiodic control, event-triggered control is proposed (see [1–4], and the references therein). Different from the periodic sampling control, in event-triggered control systems, sampling behaviour obeys some exact conditions (named as event-triggering conditions) depending on real-time information of the systems (see [5–7]). Specifically, event-triggering mechanism (ETM) verifies continuously whether the event-triggering condition is violated. Once it is violated, the state or output signals are sampled for updating the controller inputs. A lot of works showed that event-triggered control can significantly mitigate communication traffic over the networks while guaranteeing a desirable performance (see [8–13]).

Although NCS offers several benefits, such as low cost and convenience to maintain, the communication networks pose several challenges. One of the most challengeable issues in NCSs is the cyber security at the cyber layer. In NCSs, packet loss, time delay, network attacks and quantisation have strong influence on the system performance (see [14, 15]). Compared with these adverse influences, cyberattacks are more destructive and get a lot of studies in recent years. Generally, network attacks can be classified as deception attacks (see [5, 16]) and DoS attacks (see [6, 17, 18]). The deception attacks mainly affect the trustworthiness of the communication data. Unlike the deception attacks, DoS attacks mainly affect the timeliness of the communications among different network units. Through the limited average duration and frequency of attacks in any finite time, De Persis and Tesi [19] studied the input-to-state stability (ISS) of the linear system with bounded disturbance in the presence of DoS attacks. Considering the existence of valid and invalid DoS attacks, Cui et al. [20] proposed a method to improve the robustness of the closed-loop systems with DoS attacks. Moreover, using a predictor-based controller, Feng and Tesi [21] studied the resilient control of NCSs under DoS attacks and maximised the tolerable frequency and duration of attacks. In addition, some studies used the probabilistic hypotheses for the DoS attack strategy, such as [22, 23]. Many other studies considered the optimal attack strategy from the consideration of attackers, such as [24, 25]. Considering attackers and defenders simultaneously, Yan et al. [26] studied the game between attackers and defenders for figuring out their influences.

However, the above-mentioned research did not relate to the cyberattacks in the singularly perturbed systems. Also, singularly
perturbed system is an important class of systems. The states in singularly perturbed systems can be grouped as slowly varying states and fast varying states (see [27–29]). Parameter $\varepsilon \in (0,1)$ in the state derivative is a symbol of singularly perturbed systems. Because of the influence of parameter $\varepsilon$, the main properties of singularly perturbed systems cannot be fully described by some traditional methods. Considering the existence of fast varying states, the approximation method can be used in its analysis when the fast varying states are convergent in a short time. The fast–slow decomposition method dominates the studies of singularly perturbed systems (see [30–32]). In the study of event-triggered control on singularly perturbed systems, Ren and Hao [33] investigated the robust stabilisation of model-based event-triggered control in linear cases, while the effects of input saturation were considered in [34]. In [35], the event-triggered control based on the slow dynamics of non-linear singularly perturbed systems was considered. Ma et al. [36] considered the event-triggered state estimation of the singularly perturbed system with time delay. However, the stability and performance, guaranteed by these studies, required a reliable network circumstance; otherwise, it could be completely destroyed by DoS attacks. But, the network security in the control of singularly perturbed system is rarely considered in the literature.

Based on the observations above, we consider the event-triggered control of singularly perturbed systems under DoS attacks. The main contributions of this paper are summarised as follows.

DoS attack is first considered in the event-triggered control of singularly perturbed systems. The ultimately bounded stability of the closed-loop singularly perturbed systems is obtained. From the fact that the cost of reconstructing slow dynamics is lower than that of the computing and communication in the network (see [33]), the slow subsystem model of the singularly perturbed systems is used in ETM1 and ETM2 in Figure 1. Second, by designing the parameters in two model-based event-triggering conditions, Zeno behaviour in ETM1 and ETM2 is eliminated, which is difficult since the two event-triggering conditions are coupled and the detection signals in ETM2 are incommensurate.

The remainder of this paper is organised as follows. The problem description is given in Section 2. The main results are provided in Section 3. Section 4 provides the numerical simulations to illustrate the feasibility and efficiency of the results. Section 5 summarises the conclusions of this paper.

Notations. The set of real numbers is denoted by $\mathbb{R}$. The set of non-negative real numbers is denoted by $\mathbb{R}_{\geq 0}$. $\mathbb{N}_{\geq 0}$ denotes the set of non-negative integers. $\| \cdot \|$ denotes the Euclidean norm of real vectors. The Euclidean-induced matrix norm of $A \in \mathbb{R}^{n \times n}$ is denoted by $\| A \| = \sqrt{\lambda_{\max}(A^T A)}$, where $A^T$ is the transpose of matrix $A$, and $\lambda_{\max}(A^T A)$ denotes the maximal eigenvalues of matrix $A^T A$. For a symmetric matrix $P \in \mathbb{R}^{n \times n}$, $P > 0 (P < 0)$ denotes positive (negative) definiteness. $0 (I)$ represents zero (identity) matrix of appropriate dimensions. $E_{m \times n}$ denotes an $m \times n$ matrix with all the elements equal to 1, and $\text{diag}\{ \cdots \}$ denotes the block-diagonal matrix with its entries listed. A scalar function $f(\varepsilon)$ with $\varepsilon \in (0,1)$ is said to be $\mathcal{O}(\varepsilon)$ if there exists a positive constant $D > 0$ such that $|f(\varepsilon)| \leq D\varepsilon$ for $\varepsilon \in (0,1)$. For matrix $A$, $\mu_\gamma = \max|\lambda|\in\text{spectrum}(A^T A^\gamma)$ is its logarithmic norm. For function $f(t)$, define by $f(t^\ast)$ the limit when $t \to t^\ast$ and $f(t^-)$ are the limits when $t \to t^-$.

## 2 | PROBLEM STATEMENT

Consider the following singularly perturbed system:

$$
\begin{align*}
\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t), \\
\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t),
\end{align*}
$$

(1)

where $x_1(t) \in \mathbb{R}^n$, $x_2(t) \in \mathbb{R}^n$, $x_1(\theta_0) \in \mathbb{R}^n$ and $x_2(\theta_0) \in \mathbb{R}^n$ are the state vectors and the corresponding initial states. $0 < \varepsilon < 1$ is the singularly perturbed parameter. $u(t) \in \mathbb{R}^p$ is the control input vector. $A_{ij}, (i,j \in \{1,2\})$ and $B$ are matrices with appropriate dimensions, where $A_{22}$ is Hurwitz. The pair $(A_{11} + A_{12}A_{22}^{-1}A_{21}, B)$ is assumed to be stabilisable.

For system (1) without DoS attacks as in Figure 1, define by $\{\tau_i\}, (r \in \mathbb{N}_{\geq 0})$ the set of the triggering time instants of ETM1 in the sensor node, and define by $\{\tau_j\}, (k \in \mathbb{N}_{\geq 0})$ the set of triggering time instants of ETM2 in the controller node. The models in Figure 1 are given as follows:

$$
\dot{x}_m(t) = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_m(t) + Bu_m(t),
$$

(2)

for $t \in [\tau_i, \tau_{i+1})$ with $x_m(\tau_i) = x_1(\tau_i)$ and $x_m(0) = x_m(0)$. The control input in the models is $u_m(t) = Kx_m(t)$, where $K$ is the matrix such that $A_{11} - A_{12}A_{22}^{-1}A_{21} + BK$ is Hurwitz. Based on plant (1) and model (2), the input $u(t)$ is given by

$$
\begin{align*}
u(t) &= \begin{cases} 
Kx_m(T^+), & t \in [T_k, T_{k+1}), k \in \mathbb{N}_{\geq 0}; \\
0, & t \in [0, T_0),
\end{cases}
\end{align*}
$$

(3)
2.1 DoS attacks

DoS attacks prevent the communication between different system units. This paper considers the case of DoS attacks in the measurement channel only and the case of DoS attacks simultaneously blocking the measurement and control channels. Define by \( \{b_i\}_{i \in \mathbb{N}_0} \) the set of DoS on/off transitions with \( b_i \geq 0 \). Let

\[
H_n := \{b_i\} \cup \{b_{i+1} b_i + \tau_s\}
\]

be the \( n \)th DoS attack interval with length \( \tau_s \geq 0 \). For \( t \geq \tau \geq 0 \), define

\[
\Xi(t, \tau) := \bigcup_{n \in \mathbb{N}_0} [t, t + \tau] \land \Theta(t, \tau) := [\tau, \tau] \setminus \Xi(t, \tau).
\]  

(4)

Accordingly, \( \Xi(t, \tau) \) and \( \Theta(t, \tau) \) in (4) represent the set of time instants at which the communications are denied and allowed, respectively. Given \( t, \tau \in \mathbb{R}_\geq 0 \) with \( t \geq \tau \geq 0 \), let \( n(t, \tau) \) be the number of DoS on/off transitions in the interval \([t, t + \tau] \). The DoS attacks considered here are restricted in terms of average dwell time and frequency as in [19], that is, DoS attacks satisfy the following assumptions.

Assumption 2.1 (DoS frequency). There exist \( \eta \in \mathbb{R}_{\geq 1} \) and \( \tau_f \in \mathbb{R}_{>0} \) such that

\[
|n(t, \tau)| \leq \eta + \frac{t - \tau}{\tau_f}
\]  

for \( t, \tau \in \mathbb{R}_\geq 0 \) with \( t \geq \tau \).

Assumption 2.2 (DoS duration). There exist \( \pi > 0 \) and \( T \in \mathbb{R}_{\geq 1} \) such that

\[
|\Xi(t, \tau)| \leq \pi + \frac{t - \tau}{T}
\]  

for \( t, \tau \in \mathbb{R}_\geq 0 \) with \( t \geq \tau \).

Remark 1. In Assumptions 2.1 and 2.2, the positive parameters \( \eta \) and \( \pi \) are necessary to guarantee inequalities (5) and (6). For the pulsing DoS attack interval \( H_n \), it is obvious that \( |H_n| = 0 \) and \( n(b_{i+1}, b_i) = 1 \). Then \( |n(b_{i+1}, b_i)| = 1 > \eta + \frac{b_{i+1} - b_i}{\tau_f} = 0 \) with \( \eta = 0 \), which is contradictory with (5). For the interval \( H_n \) and \( T \in \mathbb{R}_{\geq 1} \), we have \( |\Xi(b_{i+1}, b_i + \tau_s)| = \tau_s \) and \( |\Xi(b_{i+1}, b_i + \tau_s)| \geq \pi + \frac{T}{\tau_s} \) with \( \pi = 0 \), which is contradictory with (6). Hence, \( \eta \in \mathbb{R}_{\geq 1} \) and \( \pi \in \mathbb{R}_{>0} \) are necessary for (6) and (5). The meaning of Assumptions 2.1 and 2.2 is that the average frequency and duration of DoS attacks are restricted in any interval \([t, \tau] \) with \( t \geq \tau \geq 0 \).

For system (1) under DoS attacks in the measurement channel, define by \( \{\mathfrak{X}_r\}_{r=0}^\infty \) the set of the successful transmission time instants of ETM1 and define by \( \{\mathfrak{T}_r\}_{r=0}^\infty \) the set of the successful transmission time instants of ETM2. As in Figure 2, the model in the controller node sends back an acknowledge (ACK) signal to ETM1 once it receives a data packet. In the model at the sensor node, the failure in the reception of ACK signals at triggering time instants indicates that the system is under DoS attacks. Once the failure happens, the ETM1 detects the event-triggering condition by \( D_1 \), period with \( D_1 > 0 \) until the reception of the ACK signals from the controller node. These ACK signals are introduced to detect the happening of the DoS attacks and then to ensure the model states to be identical for the two models in the controller and sensor nodes with and without DoS attacks. Then, in the presence of DoS attacks in the measurement channel, the model can be given by

\[
\dot{x}_m(t) = (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_m(t) + B u_m(t),
\]  

(7)

for \( t \in [\mathfrak{X}_r, \mathfrak{X}_{r+1}] \) and \( x_m(\mathfrak{X}_r) = x_1(\mathfrak{X}_r) \), and the controller can be given as

\[
u(t) = \begin{cases} 
K x_m(\mathfrak{T}_k^+), & t \in [\mathfrak{T}_k, \mathfrak{T}_{k+1}), k \in \mathbb{N}_0; \\
0, & t \in [0, \mathfrak{T}_0). 
\end{cases}
\]  

(8)

For the DoS attacks in the measurement and control channels as in Figure 3, define by \( \{\mathfrak{X}_k\}_{k=0}^\infty \) the set of the successful transmission time instants of ETM1 and define by \( \{\mathfrak{T}_k\}_{k=0}^\infty \) the set of the successful transmission time instants of ETM2. Based on the same principle as the ACK signals in Figure 2, the actuator sends back an ACK signal to ETM2 once it receives a data packet. The failure in the reception of ACK signals at triggering instants of ETM2 indicates that the system is under DoS attacks. Once the DoS attacks are detected, the ETM2 detects the event-triggering condition by \( D_2 \), period with \( D_2 > 0 \) until the reception of the ACK signals from the actuator. In Figure 3, the mechanism of ACK signals from controller to sensor nodes is similar to that in Figure 2. Then, in the presence of the DoS attacks in the measurement and control channels, the model is similar to (7) with \( t \in [\mathfrak{X}_r, \mathfrak{X}_{r+1}] \) and \( x_m(\mathfrak{T}_k^+) = x_1(\mathfrak{X}_r) \), and the

FIGURE 2 Configuration of the event-triggered control systems under denial-of-service (DoS) attacks in the measurement channel (sensor to controller)
controller can be written as

$$u(t) = \begin{cases} K\nu_0(T_k^+), & t \in [T_k, T_{k+1}], k \in \mathbb{N}_0; \\ 0, & t \in [0, T_0). \end{cases}$$  \hspace{1cm} (9)

In the absence of DoS attacks, define $e_1(t) := x_m(t) - x_1(t)$ for $t \in [\hat{T}_k, \hat{T}_{k+1})$, and $e_2(t) := x_m(T_k^+) - x_m(t)$ for $t \in [T_k, T_{k+1})$. For the DoS attacks in the measurement channel, define $e_1(t) := x_m(t) - x_1(t)$ for $t \in [\hat{\hat{T}}_k, \hat{\hat{T}}_{k+1})$, and $e_2(t) := x_m(T_k^+) - x_m(t)$ for $t \in [T_k, T_{k+1})$. For the DoS attacks in the measurement and control channels, define $e_1(t) := x_m(t) - x_1(t)$ for $t \in [\hat{T}_k, \hat{T}_{k+1})$, and $e_2(t) := x_m(T_k^+) - x_m(t)$ for $t \in [T_k, T_{k+1})$.

Then, the event-triggering conditions in ETM1 and ETM2 are given as follows:

$$\|e_1(t)\|^2 \leq \sigma_1,$$  \hspace{1cm} (10)

$$\|e_2(t)\|^2 \leq \sigma_2,$$  \hspace{1cm} (11)

for $t \geq 0$, where $\sigma_1$ and $\sigma_2$ satisfying $\sigma_2 > \sigma_1 > 0$ are parameters to be designed.

System (1) with the controller (3) in the absence of DoS attacks and the controller (8) in the presence of DoS attacks can be written as

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} A_{11} + BK & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} (e_2(t) + e_1(t) + x_1(t))$$

$$= \begin{pmatrix} A_{11} + BK & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} BK \\ 0 \end{pmatrix} \begin{pmatrix} e_2(t) + e_1(t) \end{pmatrix},$$  \hspace{1cm} (12)

for $t \in [0, \infty)$.

Before the problem statement, the following definition is given.

**Definition 2.3** ([37]). A system is said to be ultimately bounded if for every initial state $x(\theta_0)$, there exist $T(x(\theta_0))$ and a bounded set $S$ such that for $t > T(x(\theta_0))$, we have $x(t) \in S$.

Then, the interest of this paper is given as follows.

(I) For system (12) resulting from (1) under controller (3) without DoS attacks, design parameters $\sigma_1$ and $\sigma_2$ in event-triggering conditions (10) and (11) such that closed-loop system (12) is ultimately bounded.

(II) For system (12) resulting from (1) under controller (8) with DoS attacks in the measurement channel, based on the results obtained in (I), the conditions are given for parameters $\eta, \tau_D, \pi, T$ in Assumptions 2.1 and 2.2 and $D_1$ under which the closed-loop system remains ultimately bounded. Moreover, for system (12) resulting from (1) under controller (9), the conditions for parameters $\eta, \tau_D, \pi, T, D_1$ and $D_2$ are given such that the system is ultimately bounded.

(III) Positive lower bounds of the inter-event time sequences of ETM1 and ETM2 are given to eliminate Zeno behaviour from the closed-loop system without or with different forms of DoS attacks.

Before ending this section, the following lemma is provided, which is used in the analysis of the main results.

**Lemma 1** ([39]). If matrix $N$ is Hurwitz and $N_0$ is a constant matrix with the same dimensions, then there exists a sufficiently small constant $\rho$ such that matrix $N + \rho N_0$ is also Hurwitz.

## 3 \hspace{1cm} THE MAIN RESULTS

In this section, we first study the stability and sampling performance of closed-loop system (12) without DoS attacks. Then we present the stability results for closed-loop system (12) under different forms of DoS attacks.

### 3.1 \hspace{1cm} The stability and sampling performance without DoS attacks

For the states $x_1(t)$ and $x_2(t)$ in closed-loop system (12), we introduce the following transformation:

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \begin{pmatrix} I - \varepsilon H & -\varepsilon H \\ L & I \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$  \hspace{1cm} (13)

where $H$ and $L$ are parameter matrices to be given later. Then from (13), one has

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} I & \varepsilon H \\ -L & I - \varepsilon L H \end{pmatrix} \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix},$$  \hspace{1cm} (14)
and

\[
\begin{pmatrix}
\xi(t) \\
\eta(t)
\end{pmatrix} =
\begin{pmatrix}
A_{011} & A_{012} \\
A_{021} & A_{022}
\end{pmatrix}
\begin{pmatrix}
\xi(t) \\
\eta(t)
\end{pmatrix} +
\begin{pmatrix}
B_1 \\
B_2
\end{pmatrix}
\begin{pmatrix}
e_2(t) + e_1(t)
\end{pmatrix},
\]

(15)

where

\[
A_{011} = (I - \varepsilon HL)A_{11} + BK - HA_{21}
+ (I - \varepsilon HL)A_{12}L - HA_{22}L,
\]

\[
A_{012} = \varepsilon(I - \varepsilon HL)A_{11}H - \varepsilon HL A_{21}H + (I - \varepsilon HL)A_{12}
- HA_{22}(I - \varepsilon LH),
\]

\[
A_{021} = \varepsilon L(A_{11} + BK) + A_{21} - (\varepsilon L A_{12} + A_{22}) L,
\]

\[
A_{022} = \varepsilon^2(A_{11} + BK)H + \varepsilon A_{21}H + (\varepsilon L A_{12} + A_{22})(I - \varepsilon LH),
\]

\[
B_1 = (I - \varepsilon HL)BK, \quad B_2 = \varepsilon LBK.
\]

Based on the method in [32], using \(A_{012} = A_{021} = 0\), we can obtain the following two equations with respect to matrices \(H\) and \(L\):

\[
\varepsilon(I - \varepsilon HL)A_{11}H - \varepsilon HL A_{21}H + (I - \varepsilon HL)A_{12}
- HA_{22}(I - \varepsilon LH) = 0,
\]

\[
\varepsilon L(A_{11} + BK) + A_{21} - (\varepsilon L A_{12} + A_{22}) L = 0.
\]

(16)

For the equations in (16), using the same process as in [33], there exists an \(\varepsilon_1 > 0\) such that the solutions of matrices \(H\) and \(L\) are \(H = A_{12}A_{22}^{-1} + \mathcal{O}(\varepsilon)E_{\Delta x}\) and \(L = A_{22}^{-1}A_{21} + \mathcal{O}(\varepsilon)E_{\Delta x}\), for any \(\varepsilon \in (0, \varepsilon_1]\), respectively. With the solutions of \(H\) and \(L\), we can obtain \(A_{011} = A_{11} + BK - A_{12}A_{22}^{-1}A_{21} + \mathcal{O}(\varepsilon)E_{\Delta x}\), \(A_{022} = A_{22} + \mathcal{O}(\varepsilon)E_{\Delta x}\), \(B_1 = BK + \mathcal{O}(\varepsilon)E_{\Delta x}\), and \(B_2 = \mathcal{O}(\varepsilon)E_{\Delta x}\). By these matrices, (15) can be written as

\[
\begin{pmatrix}
\xi(t) \\
\eta(t)
\end{pmatrix} =
\begin{pmatrix}
A_{011} & 0 \\
A_{022}
\end{pmatrix}
\begin{pmatrix}
\xi(t) \\
\eta(t)
\end{pmatrix} +
\begin{pmatrix}
B_1 \\
B_2
\end{pmatrix}
\begin{pmatrix}
e_2(t) + e_1(t)
\end{pmatrix},
\]

for \(t \in [0, \infty)\).

For system (17), because of Lemma 1 and the Hurwitzness of \(A_{11} - A_{12}A_{22}^{-1}A_{21} + BK\) and \(A_{22}\), there exists an \(\varepsilon > 0\) such that matrices \(A_{011}\) and \(A_{022}\) are Hurwitz for \(0 < \varepsilon < \varepsilon_0\). Then for \(\varepsilon \in (0, \min(\varepsilon_1, \varepsilon_2)]\), one has

\[
\frac{\xi(t)}{\eta(t)} = \frac{A_{011}}{A_{022}}(\xi(t) + \frac{B_1}{\varepsilon} \eta(t) + \frac{B_2}{\varepsilon} (e_2(t) + e_1(t))),
\]

where \(\frac{A_{011}}{A_{022}}\) and \(\frac{B_1}{\varepsilon}\) and \(\frac{B_2}{\varepsilon}\) are Hurwitz because matrix \(A_{022}\) is Hurwitz. Based on the analysis above, we present the following lemma.

**Lemma 2.** System (17) with the event-triggering conditions (10) and (11) is globally bounded for sufficiently small \(\varepsilon\).

**Proof.** Based on the analysis above, system (17) can be written as

\[
\begin{pmatrix}
\xi(t) \\
\eta(t)
\end{pmatrix} =
\begin{pmatrix}
A_{011} & 0 \\
A_{022}
\end{pmatrix}
\begin{pmatrix}
\xi(t) \\
\eta(t)
\end{pmatrix} +
\begin{pmatrix}
B_1 \\
B_2
\end{pmatrix}
\begin{pmatrix}
e_2(t) + e_1(t)
\end{pmatrix}
\]

(18)

with matrices \(A_{011}\) and \(A_{022}\) being Hurwitz for sufficiently small \(\varepsilon\). From the event-triggering conditions (10) and (11), \(\|e_1(t)\|\) and \(\|e_2(t)\|\) are bounded. Therefore, system (18) is globally bounded for sufficiently small \(\varepsilon\).

Then, the analysis of the inter-event times is necessary to give a positive lower bound from the implementability of ETM1 and ETM2. Based on Lemma 2, we present the results of the sampling performance of ETM1 and ETM2 in the following theorem.

**Theorem 3.1.** For the event-triggering conditions (10) and (11), based on Lemma 2, there exist positive lower bounds for the triggering time sequences of ETM1 and ETM2, that is, Zeno behaviour can be eliminated.

**Proof.** For the event-triggering condition (10), the lower bound of the inter-event times can be obtained by the time that it takes for \(\|x_n(t) - x_0(t)\|\) to evolve from 0 to \(\sqrt{\Delta t}\). We consider the dynamic of \(\|x_n(t) - x_0(t)\|\). For \(x_n(t)\) and \(x_0(t)\), \(n<\gamma\) satisfies \(\dot{x}_n(t) = A_0 x_n(t) + B_n u(t)\) for \(t \in [\gamma_n, \gamma_{n+1}]\) and \(\dot{x}_0(t) = \xi(t) + \varepsilon H \eta(t)\). Then one has

\[
\dot{x}_0(t) = A_{011} \xi(t) + B_1 (e_2(t) + e_1(t))
+ H (A_{022} \eta(t) + B_2 (e_2(t) + e_1(t)))
= A_{011} \xi(t) + H A_{022} \eta(t) + (B_1
+ H B_2) e_2(t) + (B_1 + H B_2) e_1(t)
\]

and

\[
\dot{x}_n(t) - \dot{x}_0(t) = (A_0 + BK) x_n(t) - A_{011} \xi(t) - HA_{022} \eta(t)
- (B_1 + H B_2) e_2(t)
- (B_1 + H B_2) e_1(t).
\]

From Lemma 2 and model (2), \(\|x_n(t)\|, \|\xi(t)\|, \|\eta(t)\|, \|e_1(t)\|, \|e_2(t)\|\) are bounded for \(t \in [0, \infty)\). Then, there exists a bounded constant \(G > 0\) such that \(\|A_0 + BK\| x_n(t) - A_{011} \xi(t) - HA_{022} \eta(t) - (B_1 + H B_2) e_2(t)\| < G\) for \(t \in [0, \infty)\). From (19), one has \(\|e_1(t)\| \leq \|e_1(t)\| < G\|B_1 + H B_2\| \|e_1(t)\| + G,\) with \(\|e_1(t)\| = 0\) and hence

\[
\|e_1(t)\| \leq \int_{\gamma_n}^{\gamma_{n+1}} e^{\int_{\tau}^{\gamma_n+\gamma} d\tau} G = \int_0^{\gamma_{n+1}} e^{\int_0^{\gamma_n+\gamma} d\tau} G,
\]"
where \( B := B_1 + H B_2 \). Using \( \int_0^\Delta \| e(t) \| d t G = \sqrt{\sigma_1}, \) we can obtain \( \Delta_1 = \frac{1}{\| B \|} \ln \left( \frac{\sqrt{\sigma_2}}{G} + 1 \right) \). Thus the lower bound for inter-event times of ETM1 can be given by \( \Delta_1 > 0 \).

For the event-triggering condition (11), using the same process as it for the event-triggering condition (10), one has

\[
\frac{d}{dt} \| e_2(t) \| \leq \| e_2(t) \| \\
\leq \| (A_0 + BK) \| \| x_2(t) \| \\
\leq \| A_0 + BK \| \| G \| (20)
\]

for \( t \in [T_{k}, T_{k+1}) \) if there is no triggering time instant \( S_2 \) in the interval \( [T_{k}, T_{k+1}) \), where \( \| G \| > 0 \) is a constant satisfying \( \| x_2(t) \| < \| G \| \) for \( t \in \mathbb{R}_{\geq 0} \). And the lower bound of the inter-event times for the event-triggering condition (11) can be given as \( \frac{\sqrt{\sigma_2}}{\| \| A_0 + BK \| \| G \|} > 0 \). Otherwise, if there exists a triggering time instant \( S_2 \) in the interval \( [T_{k}, T_{k+1}) \), we have \( \| x_2(T_k) - x_2(T_{k+1}) \| = \sqrt{\sigma_1} < \sqrt{\sigma_2} \) from ETM2. Then, the lower bound of the inter-event times for the event-triggering condition (11) can be obtained by the time it takes for \( \| e_2(t) \| \) to evolve from 0 to \( \| x_2(T_k) - x_2(T_{k+1}) \| \) and from \( \| x_2(T_k) - x_2(T_{k+1}) \| \) to \( \sqrt{\sigma_2} \).

Hence, one has \( \Delta_2 = \frac{\sqrt{\sigma_2} - \sqrt{\sigma_1}}{\| \| A_0 + BK \| \| G \|} > 0 \) if there are two or more triggering time instants \( S_2 \) \(( \in \mathbb{N}_{\geq 0} \) in the interval \([T_k, T_{k+1})\), the lower bound of the inter-event times of ETM2 can be given by \( \Delta_2 \). \( \square \)

Based on Theorem 3.1, we present the following theorem on the stability of system (17).

**Theorem 3.2.** For system (17) and the given positive definite matrices \( Q_1 > 0 \) and \( Q_2 > 0 \), let \( P_1 \) and \( P_2 \) be the unique solutions of the Lyapunov equations \( A_0^T P_1 + P_1 A_0 = -Q_1 \) and \( A_{12}^T P_2 + P_2 A_{12} = -Q_2 \), respectively. Define

\[
\lambda := \min \left\{ \lambda_{\text{min}}(Q_1) - \mu_1 - \mu_2, \lambda_{\text{min}}(Q_2) - \mu_3 - \mu_4 \right\}
\times \min \left\{ \frac{1}{\lambda_{\text{max}}(Q_1)}, \frac{1}{\lambda_{\text{max}}(Q_2)} \right\} > 0
\]

with \( \mu_i \in \mathbb{R}_{\geq 0} \) \(( i \in \{1, 2, 3, 4\}) \) such that \( \lambda_{\text{min}}(Q_1) - \mu_1 - \mu_2 > 0 \) and \( \lambda_{\text{min}}(Q_2) - \mu_3 - \mu_4 > 0 \). Then there exists an \( \varepsilon^* > 0 \) such that system (17) is ultimately bounded and the ultimate bound \( \frac{1}{\lambda} J(\sigma_1, \sigma_2) \) is related to parameters \( \sigma_1 \) and \( \sigma_2 \) with

\[
\tilde{f}(\sigma_1, \sigma_2) := \left( \frac{1}{\mu_1} \| B_1^T P_1 \|^2 + \frac{1}{\mu_3} \| B_2^T P_2 \|^2 \right) \sigma_2 \\
+ \left( \frac{1}{\mu_2} \| B_1^T P_1 \|^2 + \frac{1}{\mu_4} \| B_2^T P_2 \|^2 \right) \sigma_1
\]

when \( \varepsilon \in (0, \varepsilon^*) \).

Proof. Define \( V'(t) := V_1(\xi(t)) + V_2(\eta(t)) \), where \( V_1(\xi(t)) = \xi(t)^T P_1 \xi(t) \) and \( V_2(\eta(t)) := \eta(t)^T P_2 \eta(t) \) with matrices \( P_1 \) and \( P_2 \) defined in the theorem. Then we have

\[
\frac{d}{dt} V_1(\xi(t)) = \frac{d}{dt} \| e_2(t) \| \leq \| (A_0 + BK) \| \| x_2(t) \| \\
\leq \| A_0 + BK \| \| G \| (20)
\]

and

\[
\frac{d}{dt} V_2(\eta(t)) = \frac{d}{dt} \| e_2(t) \| \leq \| (A_0 + BK) \| \| G \| (20)
\]

where \( \mu_i > 0 \) for \( i \in \{1, 2, 3, 4\} \), and matrices \( Q_1, Q_2 \) and parameters \( \mu_1, \mu_2, \mu_3, \mu_4 \) satisfy \( \lambda_{\text{min}}(Q_1) - \mu_1 - \mu_2 > 0 \) and \( \lambda_{\text{min}}(Q_2) - \mu_3 - \mu_4 > 0 \). Using the definition of \( V'(t) \) and (21), (22), one has

\[
\frac{d}{dt} V'(t) \leq -\lambda_{\text{min}}(Q_1) - \mu_1 - \mu_2 \xi^T(t) \xi(t) \\
- \lambda_{\text{min}}(Q_2) - \mu_3 - \mu_4 \eta^T(t) \eta(t) \\
+ \left( \frac{1}{\mu_1} \| B_1^T P_1 \|^2 + \frac{1}{\mu_3} \| B_2^T P_2 \|^2 \right) \| e_2(t) \|^2 \\
+ \left( \frac{1}{\mu_2} \| B_1^T P_1 \|^2 + \frac{1}{\mu_4} \| B_2^T P_2 \|^2 \right) \| e_1(t) \|^2. (23)
\]
From the event-triggering conditions (10) and (11), we have \( \| \xi (t) \| \leq \sigma_1 \) and \( \| \chi (t) \| \leq \sigma_2 \). Combining this with (23), one has

\[
\frac{d}{dt} V(t) \leq -\min \{ \lambda_{\min}(Q_1) - \mu_1 - \mu_2, \lambda_{\min}(Q_2) - \mu_3 - \mu_4 \} \times \left( \| \xi (t) \|^2 + \| \chi (t) \|^2 \right)
+ \left( \frac{1}{\mu_1} \| B_1^T P_1 \|^2 + \frac{1}{\mu_2} \| B_2^T P_2 \|^2 \right) \sigma_2
+ \left( \frac{1}{\mu_2} \| B_1^T P_1 \|^2 + \frac{1}{\mu_4} \| B_2^T P_2 \|^2 \right) \sigma_1.
\]

Using the definition of \( \lambda \), (24) can be written as

\[
\frac{d}{dt} V(t) \leq -\lambda V(t) + \left( \frac{1}{\mu_1} \| B_1^T P_1 \|^2 + \frac{1}{\mu_2} \| B_2^T P_2 \|^2 \right) \sigma_2
+ \left( \frac{1}{\mu_2} \| B_1^T P_1 \|^2 + \frac{1}{\mu_4} \| B_2^T P_2 \|^2 \right) \sigma_1.
\]

Then one has

\[
V(t) \leq -\lambda V(t) + \int \frac{d}{dt} \int \xi (t) + \frac{1}{\lambda} (1 - e^{-\lambda t}) \int \sigma_1, \sigma_2) + \frac{1}{\lambda} (1 - e^{-\lambda t}) \sigma_2.
\]

Defining \( \varepsilon^* := \min \{ \varepsilon_1, \varepsilon_2 \} \), from (26), closed-loop system (17) is ultimately bounded and the ultimate bound \( \frac{1}{\lambda} \int \sigma_1, \sigma_2 \) is related to parameters \( \sigma_1 \) and \( \sigma_2 \) when \( \varepsilon \in (0, \varepsilon^*) \).

From (14) and the results in Theorem 3.2, system (12) is also ultimately bounded and the ultimate bound is related to parameters \( \sigma_1 \) and \( \sigma_2 \). Summarising this result, we have the following corollary.

**Corollary 3.3.** For system (12), if the conditions in Theorem 3.2 are satisfied, then there exists an \( \varepsilon^* \in \mathbb{R}_{>0} \) such that system (12) is ultimately bounded and the ultimate bound is related to parameters \( \sigma_1 \) and \( \sigma_2 \) when \( \varepsilon \in (0, \varepsilon^*) \).

**Remark 2.** From Theorem 3.1, there exist positive lower bounds for the triggering intervals of ETM1 and ETM2. Therefore, hybrid system (17) has the maximum solution (see [38]) and the ultimately bounded stability of (17) in Theorem 3.2 can be obtained. From the results in Theorem 3.2 and Corollary 3.3, the ultimate bounds of \( \| \xi (t) \| \) and \( \| \chi (t) \| \) depend on parameters \( \sigma_1 \) and \( \sigma_2 \). From the results in Theorem 3.1, parameters \( \sigma_1 \) and \( \sigma_2 \) also affect the lower bounds of inter-event times. Larger \( \sigma_1 \) and \( \sigma_2 \) would result in a larger ultimate bound but a lower triggering frequency of ETM1 and ETM2, respectively. Hence, parameters \( \sigma_1 \) and \( \sigma_2 \) give a trade-off between the sampling performance and stability of the closed-loop systems.

### 3.2 The stability under DoS attacks in the measurement channel

In this subsection, we consider closed-loop system (12) with DoS attacks in the measurement channel. Define \( \{ \ell_k \}_{k \in \mathbb{N}} \) as the set of the attempted triggering time instants in the DoS attack intervals and \( \mathcal{L}_e := \{ k | \ell_k \in \mathcal{H}_e, k \in \mathbb{N} \} \) as the set of the index related to the attempted triggering time instants in the \( n \)th DoS attack interval. Define

\[
\varepsilon_a := \begin{cases} 0, & \text{if } \mathcal{L}_a = \emptyset; \\ t_{\max[\mathcal{L}_a]} - t_{\min[\mathcal{L}_a]}, & \text{otherwise}, \\ \end{cases}
\]

and \( \mathcal{H}_e := [t_{\min[\mathcal{L}_e]}, t_{\max[\mathcal{L}_e]} + \varepsilon_a + \sigma_a] \). Define a set \( \mathcal{P}_a \) as follows:

\[
\mathcal{P}_a := \{ t_{\min[\mathcal{L}_a]} \} \cup \{ t_{\max[\mathcal{L}_a]} \} + \varepsilon_a + \sigma_a + \mathcal{D}_1.
\]

Define \( \mathcal{Z}_m := \sum_{\mathcal{P}_a \leq t_{\min[\mathcal{L}_a]} < \mathcal{P}_{a+1}} | \tilde{H}_e \setminus \tilde{H}_{e+1} | \). Then

\[
\tilde{\mathcal{X}}(\tau, t) := \bigcup_{\mathcal{P}_a \in \mathcal{N}} \mathcal{X}_m \cap \{ \tau, t \}
\]

and

\[
\Theta(\tau, t) := \bigcup_{\mathcal{P}_a \in \mathcal{N}} \psi_{a-1} \cap \{ \tau, t \},
\]

where \( \mathcal{X}_m := \{ \mathcal{P}_m, \mathcal{P}_a + \mathcal{Z}_m \}, \), \( \psi := \{ \mathcal{P}_m, \mathcal{Z}_m, \mathcal{P}_{a+1} \} \) and \( \mathcal{P}_{a+1} = \mathcal{Z}_{a+1} = 0 \) define the intervals over which the event-triggering condition (10) holds and does not hold, respectively, in the interval \( [\tau, t] \). Based on the parameters in Theorem 3.2, define

\[
\tilde{X}_1 := \frac{1}{\mu_1} \| B_1^T P_1 \|^2 + \frac{1}{\mu_3} \| B_2^T P_2 \|^2,
\]

\[
\tilde{X}_2 := \frac{1}{\mu_2} \| B_1^T P_1 \|^2 + \frac{1}{\mu_4} \| B_2^T P_2 \|^2
\]

and

\[
Z_1 := \max \{ 1, \varepsilon^2 \| H \| \}, \ Z_2 := \max \left\{ \frac{1}{\lambda_{\min}(P_1)}, \frac{1}{\lambda_{\min}(P_2)} \right\},
\]
For matrix $Q_1 > 0$, noting the Hurwitzness of $A_0 + BK$, let $P > 0$ be the solution of the Lyapunov equation $(A_0 + BK)^TP + P(A_0 + BK) + Q_1 = 0$. Define

$$
\hat{G} = \sqrt{\frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)}}, \quad \beta := 4\hat{\alpha}_1 Z_2 + 8\hat{\alpha}_2 \hat{G}^2 Z_2.
$$

Then, based on Theorem 3.2, we present the following theorem.

**Theorem 3.4.** For closed-loop system (12) with DoS attacks in the measurement channel, if the parameters of DoS attacks satisfy Assumptions 2.1 and 2.2 with

$$
1 + \frac{D_1}{\tau_D} \leq \frac{\lambda}{\beta + \lambda},
$$

then there exists an $\varepsilon^* > 0$ such that closed-loop system (12) is ultimately bounded under DoS attacks when $\varepsilon \in (0, \varepsilon^*)$.

**Proof.** For closed-loop system (12), if there exist attacks in the measurement channel, the event-triggering condition (10) may be violated. By the dynamic of $\dot{x}_m(t) = (A_0 + BK)x_m(t)$ and the definition of $\hat{G}$, one has $\|x_m(t)\| \leq \hat{G}\|x_m(P_m)\|$ for $t \in [P_m, P_m + Z_m]$. In addition, the event-triggering condition (10) can be used at $t = P_m$ and we have $\|x_m(P_m) - x_t(P_m)\| = \|x_t(P_m)\| \leq \sqrt{\varepsilon_1}$. Combining the analysis above, one has

$$
\|x_m(t)\|^2 \leq 2\hat{G}^2(\|x_t(P_m)\|^2 + \sigma_1)
$$

$$
\leq 4\hat{G}^2[\|\xi(P_m)\|^2 + \varepsilon^2H^2\|\eta(P_m)\|^2] + 2\hat{G}^2\sigma_2(t) \tag{30}
$$

when $t \in [P_m, P_m + Z_m]$, where the second inequality uses the relation of $x_t(t)$ and $\xi(t)$ as well as $\eta(t)$ in (14). From (30) and the definition of $e_1(t)$, one has

$$
\|e_1(t)\|^2 = \|x_m(t) - x_t(t)\|^2
$$

$$
\leq 2\|x_m(t)\|^2 + 2\|x_t(t)\|^2
$$

$$
= 2\|x_m(t)\|^2 + 2\|\xi(t) + \varepsilon H\eta(t)\|^2
$$

$$
\leq 2\|x_m(t)\|^2 + 4\|\xi(t)\|^2 + 4\|\xi(t)\|^2 + \varepsilon^2H^2\|\eta(t)\|^2
$$

$$
\leq 8\hat{G}^2[\|\xi(P_m)\|^2 + \varepsilon^2H^2\|\eta(P_m)\|^2]
$$

$$
+ 4\hat{G}^2\sigma_1(t) + 4\|\xi(t)\|^2 + \varepsilon^2H^2\|\eta(t)\|^2, \tag{31}
$$

for $t \in \chi_m$. Then from (31), we have

$$
\|e_1(t)\|^2 \leq 8\hat{G}^2 Z_2(\|\xi(P_m)\|^2 + \|\eta(P_m)\|^2)
$$

$$
+ 4\hat{G}^2\sigma_1(t) + 4\|\xi(t)\|^2 + \|\eta(t)\|^2
$$

$$
\leq 8\hat{G}^2 Z_2 V(P_m) + 4\hat{G}^2\sigma_1(t) + 4Z_2 V'(t), \tag{32}
$$

for $t \in \chi_m$. Combining (25) and (32), one has

$$
V'(t) \leq -\lambda V(t) + \hat{\alpha}_1\sigma_2 + \hat{\alpha}_2\|e_1(t)\|^2
$$

$$
\leq -\lambda V(t) + \hat{\alpha}_1\sigma_2 + \hat{\alpha}_2[8\hat{G}^2 Z_2 V'(P_m) + 4\hat{G}^2\sigma_1(t) + 4Z_2 V'(t)]
$$

$$
\leq [4\hat{\alpha}_1 Z_2 + 8\hat{\alpha}_2 \hat{G}^2 Z_2] \max\{V(t), V(P_m)\}
$$

$$
+ \hat{\alpha}_1\sigma_2 + 4\hat{G}^2\hat{\alpha}_1\sigma_1 \tag{33}
$$

and

$$
V'(t) \leq e^{\beta(t-P_m)}V(P_m) + \frac{1}{\beta}(e^{\beta(t-P_m)} - 1)(\hat{\alpha}_1\sigma_2 + 4\hat{G}^2\hat{\alpha}_1\sigma_1)
$$

$$
\leq e^{\beta(t-P_m)}V(P_m) + \frac{1}{\beta}(e^{\beta(t-P_m)} - 1)(\hat{\alpha}_1\sigma_2 + 4\hat{G}^2\hat{\alpha}_1\sigma_1) \tag{34}
$$

for $t \in \chi_m$. On the other hand, from the results in Theorem 3.2, one has

$$
V(t) \leq e^{-\lambda(t-P_m)} Z_m V(P_m + Z_m) + \frac{1}{\lambda} \hat{\gamma}(\sigma_1, \sigma_2), \tag{35}
$$

for $t \in \psi_m$. By (34) and (35), using the induction method as Lemma 3 in [19], one has

$$
V(t) \leq e^{\beta[\varepsilon(0)](1-\lambda)} V(0)
$$

$$
\cdot \left[ 1 + 2 \sum_{P \leq \tau \in \mathbb{N}_{\geq 0}} e^{\beta[\varepsilon(0)](1-\lambda) [\eta(P, t)]} \right] g(\sigma_1, \sigma_2) \tag{36}
$$

for $t \in \mathbb{R}_{\geq 0}$, where

$$
g(\sigma_1, \sigma_2) = \frac{1}{\lambda} \hat{\gamma}(\sigma_1, \sigma_2) + \frac{1}{\beta}(\hat{\alpha}_1\sigma_2 + 4\hat{G}^2\hat{\alpha}_1\sigma_1).
$$

By Assumption 2.2 and the definition of $\hat{\Xi}(t, \theta)$, we have $\hat{\Xi}(t, \theta) \leq \hat{\Xi}(\tau, t) + \eta(\tau, t) + 1)D_1 \leq \tau + \frac{\theta - \tau}{\tau_D} + (\theta + 1 + \frac{\tau - \tau}{\tau_D})D_1$ and $\hat{\Theta}(t, \theta) = t - \tau - \hat{\Xi}(\tau, t) \geq t - \tau - (\hat{\Xi}(\tau, t) + \eta(\tau, t) + 1)D_1 \geq t - \tau - (\tau + \frac{\theta - \tau}{\tau_D} + (\theta + 1 + \frac{\tau - \tau}{\tau_D})D_1$. Hence, one has

$$
\sum_{P \leq \tau \in \mathbb{N}_{\geq 0}} e^{\beta[\varepsilon(0)](1-\lambda) [\eta(P, t)]} \leq \sum_{P \leq \tau \in \mathbb{N}_{\geq 0}} e^{\beta[\varepsilon(0)](1-\lambda) [\eta(\tau, t) + 1)D_1]}
$$

$$
= e^{\beta[\varepsilon(0)](1-\lambda) [\eta(\tau, t) + 1)D_1] \sum_{P \leq \tau \in \mathbb{N}_{\geq 0}} e^{\beta[\varepsilon(0)](1-\lambda) [\eta(\tau, t) + 1)D_1]}. \tag{37}
$$
Combining Assumption 2.2 and condition (29), one has \( \lambda - \left( \frac{1}{T} + \frac{\lambda}{T_D} \right) (\alpha \dot{\beta}) > 0 \). From Assumption 2.1, we have \( I - \tau = (\eta I, \eta) \tau_D \) and

\[
\sum_{P_i \leq \tau, \in \mathbb{N}_{\geq 0}} e^{-\lambda I P_{i, \tau}} \leq \sum_{P_i \leq \tau, \in \mathbb{N}_{\geq 0}} e^{\lambda P_{i, \tau}} \sum_{n=0}^{t_P} e^{-\lambda I D_{i, T_D} n}
\]

\[
\leq e^{\lambda P_{i, \tau}} \sum_{n=0}^{t_P} e^{-\lambda I D_{i, T_D} n}
\]

\[
\leq e^{\lambda P_{i, \tau}} \left[ \frac{1}{1 - e^{-\lambda I D_{i, T_D}}} \right],
\]

(38)

where \( \tilde{\lambda} := \lambda - \left( \frac{1}{T} + \frac{\lambda}{T_D} \right) (\alpha \dot{\beta}) > 0 \). Combining (36)–(38), one has

\[
V'(t) \leq e^{-\tilde{\lambda} I} [\pi (\eta I + 1/\tau_D) (\alpha \dot{\beta})] V(0)
\]

\[
+ \left[ 1 + 2 e^{\tilde{\lambda} I} (\pi (\eta I + 1/\tau_D) (\alpha \dot{\beta}) \sum_{n=0}^{t_P} \frac{1}{1 - e^{-\tilde{\lambda} I D_{i, T_D}}} \right] g(\sigma_1, \sigma_2).
\]

(39)

Then from (39), it is obvious that system (12) is ultimately bounded and the ultimate bound

\[
\left[ 1 + 2 e^{\tilde{\lambda} I} (\pi (\eta I + 1/\tau_D) (\alpha \dot{\beta}) \sum_{n=0}^{t_P} \frac{1}{1 - e^{-\tilde{\lambda} I D_{i, T_D}}} \right] g(\sigma_1, \sigma_2)
\]

is related to parameters \( \sigma_1 \) and \( \sigma_2 \) under the DoS attacks in the measurement channel satisfying (29) and Assumptions 2.1 and 2.2.

\[
\square
\]

3.3 The stability under DoS attacks in the measurement control channels

In this section, we consider synchronous DoS attacks in the measurement and control channels as in Figure 3. In this case, the communications of measurement and control channels are interrupted simultaneously in the presence of DoS attacks. According to chronological arrangement of time, define by \( \{\tilde{t}, \tilde{i} \in \mathbb{N}_{\geq 0}\} \) the set of attempted triggering time instants of ETM1 and ETM2, in the presence of synchronous DoS attacks. Define \( \tilde{I}_s := \{\tilde{t}, \tilde{i} \in \mathbb{I}_{s, i} \in \mathbb{N}_{\geq 0}\} \),

\[
\tilde{t}_s := \begin{cases} 0, & \text{if } I_s = \emptyset; \\ \max[I_s] - \min[I_s], & \text{otherwise,} \end{cases}
\]

\[
\tilde{t}_s := \begin{cases} 0, & \text{if } I_s = \emptyset; \\ \max[D_1, D_2], & \text{otherwise.} \end{cases}
\]

Define \( \tilde{H}_s := [\min[I_s] + \tilde{t}_s + \tilde{\eta}] \). Then \( \tilde{H}_s \) is the interval in which at least one of the event-triggering conditions (10) and (11) is violated. Define a set \( \{\tilde{P}_{m, \tilde{m}} \in \mathbb{N}_{\geq 0}\} \) recursively as follows:

\[
\tilde{P}_0 := \min[I_s],
\]

\[
\tilde{P}_{m+1} := \min\{\min[I_s] > \tilde{P}_m, \tilde{P}_{m+1} > \min[I_s] + \tilde{\eta} \},
\]

Define \( \tilde{H}_s := \sum_{\tilde{P}_m \leq \min[I_s], \tilde{P}_{m+1} > \min[I_s]} V(0), \)

\[
\tilde{H}_s := \sum_{\tilde{P}_m \leq \min[I_s], \tilde{P}_{m+1} > \min[I_s]} |\tilde{H}_s| \cap [\tau, t],
\]

(40)

and

\[
\tilde{H}_s := \sum_{\tilde{P}_m \leq \min[I_s], \tilde{P}_{m+1} > \min[I_s]} V(0), \]

\[
\tilde{H}_s := \sum_{\tilde{P}_m \leq \min[I_s], \tilde{P}_{m+1} > \min[I_s]} V(0),
\]

(41)

with \( \tilde{t}_s := [\tilde{P}_m + \tilde{t}_s + \tilde{\eta}], \tilde{t}_s := [\tilde{P}_m + \tilde{t}_s + \tilde{\eta}], \tilde{P}_{m+1} = \tilde{\eta} = 0. \) Then \( \tilde{t}_s \) describes the combination of intervals over which the event-triggering conditions (10) and (11) hold in the interval \( [\tau, t] \) and \( \tilde{t}_s \) represents the intervals over which either of the event-triggering conditions (10) and (11) is violated.

Recalling the definitions of \( \tilde{e}_2(t), Z_1 \) and \( Z_2 \), one has

\[
\|e_2(t)\|^2 \leq 2\|e_2(t)\|^2 + 2\|e_2(t)\|^2
\]

\[
\leq 2(\tilde{G}^2 + 1)\|e_2(t)\|^2
\]

\[
\leq 4(\tilde{G}^2 + 1)\|e_2(t)\|^2 + \sigma_1
\]

\[
\leq 8(\tilde{G}^2 + 1)Z_2 V'(\tilde{P}_m) + 4(\tilde{G}^2 + 1)\sigma_1
\]

(42)

for \( t \in \tilde{t}_s \). Following the same process as obtaining (32), we also have

\[
\|e_1(t)\|^2 \leq 8\tilde{G}^2 Z_2 V'(\tilde{P}_m) + 4\tilde{G}^2 \sigma_1 + 4Z_2 V'(t),
\]

(43)

for \( t \in \tilde{t}_s \). Combining (25), (42) and (43), one has

\[
V'(t) \leq -\lambda V'(t) + \tilde{\lambda}_1 [8(\tilde{G}^2 + 1)Z_2 V'(\tilde{P}_m) + 4(\tilde{G}^2 + 1)\sigma_1]
\]

\[
+ \tilde{\lambda}_2 [8\tilde{G}^2 Z_2 V'(\tilde{P}_m) + 4\tilde{G}^2 \sigma_1 + 4Z_2 V'(t)]
\]

\[
= (4\tilde{\lambda}_2 Z_2 - \lambda) V'(t) + [8(\tilde{G}^2 + 1)\tilde{\lambda}_1 Z_2 + 8\tilde{G}^2 Z_2 \tilde{\lambda}_2] V'(\tilde{P}_m)
\]

\[
+ 4(\tilde{G}^2 + 1)\tilde{\lambda}_1 \sigma_1 + 4\tilde{G}^2 \tilde{\lambda}_2 \sigma_1
\]

\[
\leq [8(\tilde{G}^2 + 1)\tilde{\lambda}_1 Z_2 + 8\tilde{G}^2 Z_2 \tilde{\lambda}_2]
\]

\[
+ 4(\tilde{G}^2 + 1)\tilde{\lambda}_1 \sigma_1 + 4\tilde{G}^2 \tilde{\lambda}_2 \sigma_1,
\]

(44)
for \( t \in  \mathcal{X}_i \).

Compared to (33), (44) has the same form as (33). Replacing \( \beta \) by \( \hat{\beta} \):

\[
g(\sigma_1, \sigma_2) = \frac{1}{\hat{\lambda}} \left( \sigma_1 \right) + \frac{1}{\beta} \left( 4(\hat{G}^2 + 1) \lambda_1 \sigma_1 + 4 \hat{G}^2 \lambda_2 \sigma_1 \right)
\]

and \( D_1 \) by \( \hat{D}_1 = \max\{D_1, D_2\} \), the results similar to Theorem 3.4 can be obtained using the replaced parameters above.

**Theorem 3.5.** For closed-loop system (12) with DoS attacks in the measurement and control channels as in Figure 3, if the parameters of DoS attacks satisfy Assumptions 2.1 and 2.2 with \( 1 + \frac{\hat{D}_1}{\tau_D} < \hat{\lambda} < \lambda + \hat{\beta} \), then there exists an \( \varepsilon^* \in (0, 1) \) such that closed-loop system (12) is ultimately bounded with the ultimate bound

\[
\left[ 1 + 2e^{(\lambda + \hat{\beta})(\sigma + \mu + 1)\hat{D}_1 + \lambda \tau_D} \right] \frac{1}{1 - e^{-\lambda \tau_D}} \hat{g}(\sigma_1, \sigma_2)
\]

under DoS attacks when \( \varepsilon \in (0, \varepsilon^*) \).

**Remark 3.** From Theorem 3.1, Zeno behaviour is eliminated in system (12) without DoS attacks. In the presence of DoS attacks, it is obvious that the sampling interval is \( D_1 \) for ETM1 in the interval of DoS attacks in the measurement channel. Similarly, there exist positive lower bounds for inter-event times of ETM1 and ETM2 in the presence of DoS attacks in the measurement and control channels. Therefore, Zeno behaviour can be eliminated for system (12) under two forms of DoS attacks.

### 4 | SIMULATIONS

In this section, two examples are given to illustrate the efficiency and feasibility of the obtained results.

**Example 4.1.** Consider parameters \( A_{11} = 1, A_{12} = 3, A_{21} = \frac{1}{3}, A_{22} = \frac{1}{2}, B = 2, K = -2, Q_1 = 6, Q_2 = 5, \theta_1 = 0.5, \sigma_1 = 0.08, \sigma_2 = 0.4, D_1 = 0.02, D_2 = 0.03 \) and \( \mu_i = 0.5, \forall i \in \{1, 2, 3, 4\} \). For the given parameters, one has \( A_{11} - A_{12} A_{22} A_{21} = 3 > 0 \), that is, the open-loop system is not stable. By these parameters and the initial value \( x_1(0) = 10, x_2(0) = -9, x_m = 10 \), the simulation results without DoS attacks are given in Figures 4 and 5 with the simulation step size \( 1 \times 10^{-4} \).

From Figure 4, system (12) is ultimately bounded.

For the DoS attacks in the measurement channel, the simulation results are given in Figures 6 and 7 with a random attack sequence. By the given parameters, the upper bound of \( \frac{1}{T + \frac{D_1}{\tau_D}} \) is 0.06. However, from Figure 6, under the DoS attacks with the attack proportion being 30% for \( t \in [0, 10] \), system (12) is ultimately bounded. This reflects the theoretical conservatism in determining parameters \( T \) and \( \tau_D \). Matrices \( Q_1, Q_2 \) and the control gain \( K \) have the main effect in determining the bounds of parameters \( T \) and \( \tau_D \). For the attacks in the measurement channel.
and control channels, the simulation results are given in Figures 8 and 9. From Figure 8, closed-loop system (12) is ultimately bounded.

From Figure 6, the event-triggering condition of ETM1 cannot be triggered in the DoS attack intervals but some events of ETM2 can occur in the DoS attack intervals. Moreover, the updates times of the control signals are synchronous with the triggering instants of ETM2. This is consistent with the fact of the DoS attacks in the measurement channel. Moreover, from Figure 6, the delay of the latest triggering time of ETM1 after the DoS attacks are smaller than $D_1 + D_2$ with $D_1 + D_2 = 0.05$ if the attacks are valid.

Table 1 gives the maximal, minimum and average inter-event times of ETM1 and ETM2 in Figures 5, 7 and 9, respectively. Combining the simulation step size $1 \times 10^{-4}$ and Table 1, it is obvious that Zeno behaviour is eliminated in system (12) in the presence and absence of DoS attacks.

**Example 4.2.** We consider the active suspension system in [40]. The system can be written as

$$
\begin{bmatrix}
\dot{X}_1(t) \\
\dot{X}_2(t)
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
X_1(t) \\
X_2(t)
\end{bmatrix}
+ 
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u(t),
$$

(45)

where $A_{11} = \begin{bmatrix} 0 & 1 \\ 1 & \beta \end{bmatrix}$, $A_{12} = \begin{bmatrix} 0 & -1 \\ -\beta & 0 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0 & 0 \\ \alpha & \alpha \beta \end{bmatrix}$, $A_{22} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha \beta \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ -\alpha \end{bmatrix}$, $X_1(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $X_2(t) = \begin{bmatrix} x_3(t) \\ x_4(t) \end{bmatrix}$. For the given parameters $\alpha = 1$ and $\beta = 0.6$, we have
Therefore, the slow subsystem of the open-loop system is not stable and matrix $A_{22}$ is Hurwitz. By selecting $K = [2, 1.6]$, matrix $(A_{11} - A_{12}A_{22}^{-1}A_{21}) + (B_1 - A_{12}A_{22}^{-1}B_2)K = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$ is stable. For $\epsilon = 0.01$, $\sigma_1 = 0.08$, $\sigma_2 = 0.4$ and the initial value $x_1(0) = 10$, $x_2(0) = 5$, $x_3(0) = 6$, $x_4(0) = 8$, the simulations of the state trajectories and the triggering intervals are given in Figures 10 and 11. For the DoS attacks in the measurement channel and the synchronous DoS attacks in measurement and control channels with the same parameters as in Example 4.1, the simulation results are given in Figures 12, 13, and Figures 14, 15, respectively.

From Figures 10, 12 and 14, it is obvious that the closed-loop system is ultimately bounded. From Figure 12, the events of ETM1 cannot occur in the DoS attack intervals but the events of ETM2 can occur in some of the DoS attack intervals. This is consistent with the DoS attacks in the measurement chan-
From Figure 14, all the events of ETM1 and ETM2 cannot occur in the DoS attack intervals, which is consistent with the synchronous DoS attacks in the measurement and control channels. Moreover, from Figure 12, the delays between the latest events of ETM1 and the finished time of the valid DoS attacks are small than $D_1 = 0.02$. From Figure 14, the delays between the latest events of ETM1 or ETM2 and the finished time of the valid DoS attacks are small than $D_1 + D_2 = 0.05$. This is caused by the detection periods $D_1$ of ETM1 and $D_2$ of ETM2 in the presence of DoS attacks.

For the inter-event times in Figures 11, 13 and 15, the maximum, minimum and average triggering intervals of ETM1 and ETM2 are given in Table 2. Table 2 and the simulation step size $1 \times 10^{-4}$, Zeno behaviour is eliminated in system (45) in the presence and absence of DoS attacks.

### Table 2

| Figure | ETM 1       | ETM 2       |
|--------|-------------|-------------|
| Min    | 0.0130      | 0.0414      |
| Max    | 1.4285      | 0.9533      |
| Ave    | 0.1924      | 0.1886      |
| Figure 13 | ETM1 | ETM2 |
| Min    | 0.0130      | 0.0414      |
| Max    | 1.8978      | 0.6445      |
| Ave    | 1.3328      | 0.1429      |
| Figure 15 | ETM1 | ETM2 |
| Min    | 0.0130      | 0.0414      |
| Max    | 1.1341      | 0.9615      |
| Ave    | 0.2365      | 0.1951      |

### 5 | CONCLUSIONS

This paper has studied the event-triggered control stabilisation problem of linear singularly perturbed systems in the presence of DoS attacks. We used two event-triggering conditions in measurement and control channels including the model of the slow subsystem of the singularly perturbed systems. Especially, the detection signals in ETM2 were inconsecutive. We proved the positive lower bounds of the inter-event times through the parameter design of ETM1 and ETM2. The ultimate bounded stability of the closed-loop plant was obtained under different forms of DoS attacks. Finally, the feasibility and efficiency of the theoretical results were verified through numerical examples.

### ACKNOWLEDGEMENTS

This work was supported by National Nature Science Foundation of China under grant 61573036 and the Academic Excellence Foundation of BUAA for Ph.D. students.

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How to cite this article: Ren X, Hao F. Event-triggered control of singularly perturbed linear system with DoS attacks. IET Control Theory Appl. 2021;15:1028–1041. https://doi.org/10.1049/cth2.12100