Analyzing the structure of basic quantum knowledge for instruction

Giacomo Zuccarini*

University of Udine, Department of Mathematics, Computer Science and Physics, via delle scienze 206, 33100, Udine, Italy
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Abstract

In order to support students in the development of expertise in quantum mechanics (QM), as well as to provide insight on teaching, we asked which concepts and structures can act as organizing principles in basic QM (RQ1). The research question has been addressed in a multi-step process based on the analysis of categorization studies, on a content analysis of a sample of upper-undergraduate course textbooks and on the results of existing research on learning difficulties in QM. The answer to RQ1 consists in seven concept maps, intended as models of the organizing principles of quantum knowledge needed to account for the results of measurement and time evolution both at a qualitative and quantitative level. By means of these instruments, it is possible to visualize and explore the different facets of the interplay of the vector structure of the quantum states and the operator structure of the observables, and to highlight the educational significance of the relations between observables. The relations provide tools for predicting the results of the processes, and explain how information on measurement and time evolution is encoded in the modulus and in the phase of the probability amplitudes. In addition, basic phenomena and formal structures are traceable to the emergence of a specific relation: incompatibility of observables. At upper-undergraduate level, the instruments developed in this work can be used by instructors as a support for helping students build a well-organized knowledge structure and by researchers as a basis for the design of investigations into student understanding. While this framework may be adapted to different approaches and interpretive stances, it provides indications in favor of a spin-first approach over a waves-first one. At high school level, a simplified version of the framework has been used as a basis for the design of a teaching-learning sequence.
I. INTRODUCTION

Twenty years of educational research show that the development of expertise in basic, non-relativistic quantum mechanics (QM) is a difficult and challenging task for students.\textsuperscript{1,2} They adopt survival strategies for performing reasonably well during their course, becoming proficient at solving algorithmic problems, but they have difficulty mastering concepts and applying formalism in order to make predictions and to obtain information of a qualitative nature.\textsuperscript{3} In several respects, upper-level undergraduate students in QM display patterns of difficulties which are analogous to those reported for introductory physics students, such as lack of a robust knowledge structure and consistency of reasoning strongly depending on the context.\textsuperscript{2} Results of researches conducted at the end of the course\textsuperscript{4} and at the beginning of graduate instruction\textsuperscript{3} suggest that knowledge fragmentation - i.e. the organization of knowledge in small, disconnected pieces, which can be productively applied locally but which lack global consistency - often persists after a whole semester or year of intensive instruction on the topic. The importance of helping students overcome fragmentation and develop expertise in basic QM at undergraduate level is further supported by two additional facts. Firstly, only a fraction of students goes on to graduate physics programs in QM: for those who do not, the intro course will be their last quantum exposure. Secondly, there is a growing body of evidence to suggest that students’ understanding of undergraduate QM is not substantially improved by instruction at graduate level.\textsuperscript{5–8}

Another important issue, specifically concerning the teaching of QM, is related to the selection and organization of the subject-matter content for instruction. As testified by empirical research\textsuperscript{9} and by recent reviews of quantum textbooks,\textsuperscript{10} there is still no agreement among faculty members as to which approach is preferable to introduce QM at undergraduate level: waves-first or spin-first? The proponents of the spin-first approach believe that it provides a simple two-dimensional vector space to teach the foundations of QM whereas those who discuss, for instance, the infinite square well, first believe that spin is too abstract and continuity with the content discussed in the earlier courses is important.\textsuperscript{9} At lower levels, such as sophomore and high school, the lack of consensus among education researchers and instructors is even worse, as it concerns which topics to treat and the entities that might be viewed as elementary features of the theory.\textsuperscript{11}

Research has shown that expert knowledge in physics it is hierarchical in structure (major
principles and concepts at the top, facts and equations occupying the lower levels), highly organized (each node is richly linked to other relevant knowledge within the hierarchy) and conditionalized for efficient use: that is, it cannot be reduced to sets of isolated propositions, because procedures for applying principles and concepts are bundled with them, as are the contexts in which those principles/concepts apply.\textsuperscript{12,13} As a consequence, choices made on the organization of the subject-matter content as well as on its elementary features necessarily influence the expertise development process. However, educational investigations on the content structure of QM are currently lacking, and though some of the textbooks attempt to locally organize the information by giving a summary at the end of each single chapter,\textsuperscript{14} they do not include, as far as we know, comprehensive attempts to organize the information at a global level.

In this paper, we endeavour to analyze the structure of basic quantum knowledge, in order to identify what entities should be at the top of the knowledge pyramid, as well as their interconnections and the related procedures that make possible an efficient use of this knowledge, such as applying the formal tools of the theory in order to make qualitative predictions on quantum systems. In the rest of the paper, we shall call this specific knowledge network “the organizing principles” of the subject-matter. This analysis aims to provide a basis for helping students overcome fragmentation and build a well-organized knowledge structure in QM, as well as for giving indications on which choice and organization of the content is preferable from an educational standpoint. The research question we endeavour to answer is the following:

RQ1: which concepts and structures can act as organizing principles in basic QM?

Based on the answer to this question, we provide indications on what approach is preferable at upper-undergraduate level, and suggestions on the elementary features of the theory at lower levels.

II. INSTRUMENTS AND METHODS

The methods used to conduct the analysis are drawn from categorization studies and the Model of Educational Reconstruction (MER).\textsuperscript{15}
Categorizing or grouping together problems based upon similarity of solution has been employed both in introductory physics research\textsuperscript{16,17} and in upper-level QM\textsuperscript{9} to infer how physics knowledge is structured in memory and to assess physics students’ level of expertise. In classical mechanics, differences between novice and experienced problem solvers have been identified both in the way they organize their knowledge of physics and how they approach problems. Novice problem solvers use literal objects from the surface attributes of the problems as category criteria (such as “spring” problems or “inclined plane” problems), whereas experienced problem solvers consider the physics concept or principle used to solve the problem when deciding on problem categories, such as grouping conservation of energy problems together.\textsuperscript{17} For the purpose of identifying the organizing principles of quantum knowledge at upper-undergraduate level, we referred to the categorization study by S.-Y. Lin and C. Singh,\textsuperscript{9} the only one available on the subject.

Another useful strategy to navigate this search is provided by the MER, a theoretical framework designed for converting domain specific knowledge into knowledge for instruction. According to the MER, a fundamental stage in this process is called \textit{elementarization}, i.e. the identification of the entities within a complex content domain which may be viewed as elementary features (e.g., basic phenomena, basic principles and general laws), the combination of which helps explain the different aspects of the scientific content considered. Research methods mainly draw on qualitative content analysis of leading textbooks and key publications on the topic under inspection, to be conducted from the standpoint of science education. This analysis can benefit significantly from research into student ideas on the specific subject at hand, as student conceptions may provide a new view of science content and hence allows another, deeper, understanding of the subject under examination. After examining the categorization study, we drew on content analysis of a sample of textbooks in the search of the elementary features needed to explain our findings. The results obtained were tested and further processed by means of existing research on student understanding of QM.

The analysis of the structure of the subject-matter content has been conducted at upper-undergraduate level. At the end of the article, we discuss how to adapt it to high school students.
III. THE TWO PROCESSES

In the categorization study by S.-Y. Lin and C. Singh, 9 22 physics juniors and seniors in two undergraduate quantum mechanics courses and six physics faculty members (professors) were asked to categorize 20 QM problems based upon similarity of solution, using one or more categories for each problem. The goal of the study was to investigate differences in categorization between faculty members and students and whether there are major differences in the ways in which individuals in each group categorize QM exercises. Three of the six faculties involved were asked to evaluate the quality of each category, scoring them as ‘good’ (assigned a score of 2), ‘moderate’ (1) or ‘poor’ (0), without using identifiers and with the categorizations by the faculty and students jumbled up. In several respects, this categorization task gave different results from those obtained in studies on introductory physics. However, categorizations by faculty members were rated higher overall than those by students.

For this reason, we assumed that categories with the best score (5-6) reflect a highly organized knowledge of the subject, and can be used as instruments to identify central features of basic QM. Consequently, instead of focusing on the differences in categorization between groups and individuals in each group (as the authors did), we chose to focus on categories that were assigned the best score, grouping them by similarity of topic and counting the number of problems they were placed in. The categories time dependence of expectation value, time evolution of wavefunction, stationary state, are all specific instances of the quantum evolution of systems in the absence of measurement, and therefore were collected under the label Time Evolution. The categories measurement, expectation-value, collapsed wavefunction, expectation-value and uncertainty concern different aspects of the peculiar behavior of systems in measurement, and therefore were collected under the label Measurement. The categories expansion in eigenfunction and Fourier transform refer to mathematical procedures used to represent a state vector with respect to another eigenbasis, and were collected under the label Change of Basis. The remaining categories among the best scoring ones, i.e. symmetry argument and spin were considered separately. Table I displays the number of problems in which each collective category was placed. A pattern emerges: while in introductory physics categorization is typically based on the fundamental principles used in order to solve problems (e.g., conservation principles and Newton’s laws),
in QM it is primarily based on the process involved (either measurement or time evolution in the absence of measurement) and secondarily on a mathematical procedure often needed to extract from the state information on measurement/time evolution (change of basis). As the authors of the study observe: “faculty members noted that the fundamental principles, e.g. conservation laws, are also important in understanding quantum processes but they are not the focus of an upper-level undergraduate quantum mechanics course.”

| Category                  | Instances |
|---------------------------|-----------|
| Measurement               | 10        |
| Time-Evolution            | 9         |
| Change of Basis           | 4         |
| Spin                      | 1         |
| Symmetry Argument         | 1         |

Further evidence supporting the central role of the two processes in education is provided by the most comprehensive review of student difficulties at upper-undergraduate level published up to date.\(^\text{18}\) If we read this review looking for issues concerning these processes, we notice that - while many topics are discussed - difficulties with measurement and time evolution, and related procedures, occupy the bulk of the study’s report: about 8/13 pages dedicated to the presentation of reasoning difficulties, organized into two sections for time evolution \((\text{Difficulties with the time dependence of a wave function}, 1,5 \text{ pages, and Difficulties with the time dependence of expectation values}, 1,5 \text{ pages})\) and two on measurement \((\text{Difficulties with measurements and expectation values}, 4 \text{ pages, and Difficulties involving the uncertainty principle}, 0,5 \text{ pages})\), while the section devoted to difficulties with Dirac notation has mostly to do with measurement \((0,5 \text{ pages})\).

As concerns the nature of measurement and time evolution, we observe that, if we accept the projection postulate, both of them represent fundamental evolution processes. Independently from this choice, measurement remains at the core of quantum theory, as the evolution in time concerns the probability distributions of measurement outcomes.

Differently from categorization studies in Newtonian mechanics, our search for the organizing principles of quantum knowledge does not end here. As a matter of fact, the two
processes under consideration do not represent the concepts needed to solve quantum problems, but rather the processes which need to be explained. In conclusion, in order to identify the nodes at the top of the pyramid of quantum knowledge, their interconnections and the attached procedures, we have to determine what kind of network might account for the possible results of measurement and time evolution, both at a qualitative and quantitative level (RQ1.1). At the same time, as the complexity of knowledge structure in QM is due to both the requisite conceptual and mathematical knowledge, this network should also explain how information regarding the two processes is encoded in the formal representations of quantum systems (RQ1.2). The two sub-questions are a refinement of the tasks required to answer RQ1.

IV. ACCOUNTING FOR MEASUREMENT AND TIME EVOLUTION

A. The organizing principles of basic quantum knowledge

In order to find candidate answers to the sub-questions formulated at the end of the last section, we examined how the two processes are presented and discussed in a sample of textbooks written for upper-undergraduate courses. We selected four textbooks, two of them starting with QM in one spatial dimension, and two adopting a spin-first approach. As to the former, we analyzed Griffits’ QM text which, according to a recent survey on the use of textbooks in upper-level QM courses, currently dominates the market. Alongside Griffths’, we also included Gasiorowicz’s textbook, quite popular among faculty members when they were taught QM as undergraduate students. As concerns spin-first, we selected McIntyre’s text - to our knowledge, the only QM textbook with contributions from physics education researchers - and Townsend’s, one of the first undergraduate textbooks adopting this approach.

Here we do not report a description of the content analysis, but present the general picture we obtain in the light of the content of the textbooks and the results of research on learning difficulties. Our answer to RQ1.1 is summarized in seven concept maps, intended as models of the organizing principles of quantum knowledge needed to account for the results of measurement and time evolution both at a qualitative and quantitative level.

Three maps directly emerge from the content analysis. In the first two maps, the state
of the system is represented as a superposition of simultaneous eigenstates of a complete set of commuting observables (from now on: CSCO), and the assessment tool for predicting the outcome of the processes at a qualitative level (stochastic/determined for measurement, stationary/non stationary for the time evolution of the state) is the analysis of the superposition, after - if need be - an appropriate change of basis. The third map addresses the identification of the constants of motion (invariant/non-invariant distribution), and the assessment tool is the analysis of the commutator of the observable under examination and the Hamiltonian. The maps are displayed in fig. 1, 2, and 3. In these maps - as in all
not involve complex quantitative procedures (e.g., the change of basis). Fig. 4 and 5 show how the knowledge structure represented in the first two maps is applied to find qualitative pathways of solution to survey questions administered in educational studies.

While the pathways displayed in the third map are quite straightforward, the first two general maps present various elements of interest.

Firstly, by looking at these maps, we notice a strong structural similarity as concerns the knowledge needed for finding the results of measurement and of the time evolution of the state. Only three differences emerge in passing from the former to the latter: the main focus is on energy rather than a generic observable $Q$, solving the energy eigenvalue problem becomes a preliminary task, obtaining quantitative results requires a different procedure.

Secondly, knowledge about the specific context of the problem (e.g., harmonic oscillator, hydrogen atom, etc...) comes into play whenever energy is involved and possibly when we perform a change of basis. In the transition from classical mechanics to QM, the context of a problem acquires a greater importance, as a change of context affects the measurable values of energy, its eigenstates, the commutation relations of the Hamiltonian, and may
FIG. 3. Time evolution of the probability distribution of an observable

involves tensor products of different Hilbert spaces. The greater significance of contexts in QM was also stressed by faculty members interviewed by S.-Y. Lin and C. Singh.⁹

Thirdly, in the maps, we considered a superposition of eigenstates of a CSCO with discrete spectra. If this is not the case, the sum is replaced by an integral. However, this change does not affect the structure of the pathways for finding the results of the processes.

Lastly, despite the general validity of the first two maps, from an educational standpoint they are incomplete, inasmuch as they do not include qualitative pathways of solution to very simple tasks, e.g. deciding whether a position eigenstate is a stationary state, which is exactly the kind of knowledge students struggle to acquire.¹⁸ Research has shown that deep and unexpected learning difficulties emerge in the interplay between state vectors and operators in the Hilbert space. Singh and Marshman report that a very common difficulty is assuming that eigenstates of operators corresponding to all physical observables are the same. More in general, their review includes whole subsections discussing Difficulties in distinguishing between eigenstates of operators corresponding to different observables (concerning measurement) and Difficulties in distinguishing between stationary states and
analyze the superposition

DETERMINATE

MEASUREMENT

STOCHASTIC

$|\psi\rangle \neq \text{ES of } E$

$\text{ES} = \text{eigenstate}$

$E = \text{energy}$

| $\psi \rangle = \sqrt{\frac{2}{7}} |E_1\rangle + \sqrt{\frac{5}{7}} |E_2\rangle$

FIG. 4. Question: write down the probability of measuring an energy value and the expectation value$^3$

eigenstates of operators corresponding to observables other than energy. The textbooks offer very few examples on how to use the commutation relations between observables - and therefore knowledge about their systems of eigenstates - in order to find the results of measurement and the time evolution of the state, while our first two maps incorporate this kind of knowledge only implicitly in the procedure of the change of basis.

In order to highlight how the structures encoded in the relations between one observable and another affect the outcome of the processes, we developed two additional maps. Here the state is not represented as a superposition state, but as an eigenstate of one observable. This concept represents the junction between the vector structure of the states and the operator structure of the observables, as it is a state vector, but at the same time an element of an eigenbasis of a Hermitian operator. The assessment tool for predicting the outcome is the analysis of the commutation relations, thus establishing a structural analogy with the map on the constants of motion. As concerns the pathways of solutions on measurement and the time evolution of the state, their similarities and differences are the same as before. Therefore, for reasons of space, we include only the map on measurement (fig. 6). At a
FIG. 5. Question: consider the following wave functions for a particle in a three-dimensional isotropic harmonic oscillator potential. Does the state change as time evolves? Explain.

At the qualitative level, we can illustrate the new pathways of solution by means of two examples: one in the context of spin $1/2$, the other concerning systems in one spatial dimension (see fig. 7). Using these maps requires a knowledge of a formal representation of the eigenstate under examination, while the context emerges - as before - if energy and change of basis are involved. The picture here is much more complex, as the relations between observables, namely compatibility and incompatibility, reveal internal structures and challenges, and the analysis of the commutator may not be sufficient for finding the results. In many cases, we need to assess whether there is degeneracy or incompatible observables with simultaneous eigenstates (the kernel of their commutator is non-empty), and if yes, we may have to resort again to the change of basis and the analysis of the superposition. However, from an educational point of view, these new maps are not less important than those based on a superposition state, as they allow us to explicitly visualize the fundamental issues involved in working with the concept of eigenstate and the commutation relations between observables.

A further, and last, system of maps allows us to complete the picture of the organizing principles developed thanks to this analysis. These maps display how the issues of degeneracy and a non-empty kernel - which, abstract problems aside, may emerge in undergraduate courses only when angular momenta and their addition are involved - can be removed if, instead of focusing on an eigenstate of just one observable, we refer to the simultaneous...
eigenstate of a CSCO. As we can see in fig. 8, now the analysis of the superposition is out of the picture and the change of basis intervenes only for obtaining quantitative results. The general pathways of solutions are uncomplicated, with the exception of a limited number of cases concerning the components of integer-valued angular momenta (orbital or spin) and their addition.

Within this framework, the nodes at the top of the knowledge pyramids are the intertwined concepts of measurement, time evolution, state, observable, eigenstate and their superposition, relations between observables (i.e. compatibility and incompatibility). The assessment tools attached to these concepts are the analysis of superposition and of the commutation relations (which may involve further tools). Ancillary but needed procedures are those represented in the green boxes: the change of basis, the identification of energy eigenstates and eigenvalues for the system under examination, the calculation of probabilities.
FIG. 7. a) Measurement: spin 1/2; b) Time evolution of the state: one spatial dimension.

FIG. 8. Measurement on an eigenstate of a CSCO (Born rule), the time evolution operator, the generalized Ehrenfest theorem.
B. Information encoded in the formal representation of systems

While it is clear what information is encoded in the modulus of the probability amplitude as concerns the CSCO upon which the state is represented, research reveals that interpreting the physical meaning and the measurable effects of the relative phases is a challenge even for graduate students.\textsuperscript{6,23} The relations between observables allow us to fully explain how information is organized in the formal representations of the quantum state (RQ1.2). If we expand a vector or a wavefunction in the common basis of a CSCO, the modulus of the probability amplitude encodes complete information on measurement of the elements of the CSCO. Phases do not have anything to do with the CSCO. On the contrary, they integrate information contained in the modulus for what concerns the measurement of observables which are incompatible with at least one element of the CSCO. Moving on to consider time evolution, we observe that if a superposition of energy eigenstates corresponding to different eigenvalues is represented with respect to the basis of a CSCO including energy, the modulus of the probability amplitude contains information on constants of motion. Consequently, it becomes clear why only phase relations may evolve in time.

These structural features can be explored in the simple case of position and momentum, if we do not restrict the treatment to the uncertainty principle, but move on to examine the expectation values: while the modulus of the wavefunction in position space provides constraints on the corresponding wavefunction in momentum space ($\Delta x \Delta p \geq \hbar/2$), the phase function contained in $\psi(x) = A(x)e^{i\phi(x)}$ is needed to find the expectation value of momentum:

$$
\langle p \rangle = \langle \psi | \hat{p} | \psi \rangle = \int_{-\infty}^{+\infty} dx \langle \psi | x \rangle \langle x | \hat{p} | \psi \rangle = \int_{-\infty}^{+\infty} dx A(x)e^{-i\phi(x)} \left(-i\hbar \frac{dA(x)e^{i\phi(x)}}{dx} \right)
$$

Carrying out the integral, we obtain

$$
\langle p \rangle = \hbar \int_{-\infty}^{+\infty} dx A(x)^2 \phi'(x)
$$

C. Further roles of the relations between observables

While the state and superposition take central stage in traditional teaching practice, the explanatory power of the relations between observables may be underestimated. In order
to highlight their potential role in education, we discuss some features connected to the appearance of incompatibility.

**Incompatibility and time evolution:** incompatibility of observables is precisely the engine that generates the time evolution of quantum systems. In a quantum world without incompatibility, all of the observables would be compatible with energy, and therefore every observable would be a constant of motion. QM would end up being the picture of a static, dead world.

**Incompatibility and the use of Hilbert space constructs:** let us assume that all observables are compatible. A simultaneous eigenstate of a CSCO cannot be a superposition state of any other observable, otherwise this observable would be incompatible with at least an element of the CSCO. Any superposition of these eigenstates is indistinguishable from statistical mixtures of the component states, which correspond to different sets of values of all the observables. Therefore, there is no particular need of vectors for representing the states. Hermitian operators follow the same fate: in the absence of a non-commutative algebra, they can be represented as a set of diagonal matrices. If all of the matrices are diagonal, they behave as n-tuples of real numbers.

**Incompatibility and discrete spectra:** the discrete nature of some quantities was already accounted for in the old quantum theory. In this respect, QM differs inasmuch as it explains the existence of discrete spectra as a consequence of incompatibility. The number of values some observables can take is limited by the constraint between the position operator and the momentum operator ($[\hat{x}, \hat{p}] = i\hbar$). Let us consider, for instance, the case of the angular momentum: if $[\hat{x}, \hat{p}] = 0$, all of the commutators between different components of the angular momentum would be zero, and therefore no ladder operator would arise to impose the construction of a discrete spectrum.

D. Organizing principles versus postulates

In their survey on current teaching practice in upper-level courses, Dubson et al.\textsuperscript{20} report the existence of a split, among faculty and textbooks, over the importance of presenting the formal postulates of QM in teaching. In our selection of textbooks, at the extremes we find Griffiths, p. 11: “There is no general consensus as to what its fundamental principles are” and McIntyre, who states at p. 1: “The mathematical formalism of quantum mechanics is
based upon six postulates that we will introduce as we develop the theoretical framework.”

Obviously, the organizing principles we presented in this work are a quite different thing from the postulates of QM. Nevertheless, we find it useful to briefly examine the postulates, as presented by McIntyre, through the lens of the organizing principles:

1 and 2 “The state of a quantum mechanical system, including all the information you can know about it, is represented mathematically by a normalized ket”; “A physical observable is represented mathematically by an operator A that acts on kets.” In the light of the interplay of the vector structure of the states and the operator structure of observables, these two principles become one. In fact, in order to link a state vector \(|\psi\rangle\) and the superposition of states to empirical observations, it must be represented in an eigenbasis of at least one observable, e.g., position (\(|\psi\rangle = \int_{-\infty}^{+\infty} dx C(x)|x\rangle\)). If we need to assess the possible results of the processes, we should say something like: ‘the state is located in a space that is physically structured by the relations between observables. If we take this structure into account, the state is a complete description of a quantum system;’

3 and 5 “The only possible result of a measurement of an observable is one of the eigenvalues of the corresponding operator A”; Projection postulate. In our maps, we took a minimal approach as concerns the interpretation of QM, limiting ourselves to the procedures needed to calculate probability distributions. What happens in the measurement process is not included in the maps, so they can be used as a support independently from the interpretive stance of the instructor;

4 Born rule. A fundamental procedure for linking the state to measurement results, and therefore included in the maps;

6 “The time evolution of a quantum system is determined by the Hamiltonian or total energy operator \(H(t)\) through the Schrödinger equation.” As C. Singh observes, the fundamental equation of motion of QM is often de-emphasized in undergraduate courses.\(^3\) It is presented at the beginning of the course program, while a considerable portion of the lectures is spent in solving the eigenvalue equation for energy with complicated potentials and boundary conditions, and time evolution is determined by applying the time evolution operator to the initial state, represented in terms of
the energy eigenvectors. The maps correspond to this operationalization of the time evolution, and therefore do not include the Schrödinger equation.

V. IMPLEMENTATION AT UPPER-UNDERGRADUATE LEVEL

The maps have been designed as visual tools available to instructors for helping students develop a global knowledge structure on the processes which can be applied in every context. These tools can be adapted to different approaches and interpretive stances.

However, they also provide indications on the organization of the subject-matter content for instruction. According to education research, a science curriculum for advanced study that aims to promote learning with understanding should link “new knowledge to what is already known by presenting concepts in a conceptually and logically sequenced order that builds upon previous learning within and across grade levels.”24 As concerns the link with the content of previous courses, such as modern physics and electromagnetism, starting with waves could be seen as a natural choice. On the other hand, it can be argued that this could leave students with the impression that doing theoretical physics is just learning more and more clever tricks to solve differential equations.10 But what about building upon previous learning within the QM course? Our analysis offers a framework for examining this issue at a global level. In fact, the above-mentioned guideline can be translated into a gradual construction of the entities and the interconnections displayed in the maps, adding pieces to the puzzle until we get a complete picture.

We can show that a spin-first approach is more suitable to the purpose. In the context of spin-1/2, the existence of an interplay between states and observables is immediately visible, as well as the meaning of the change of basis, and - more importantly - the multifaceted features of this interplay can be discovered in a very gradual way. In steps:

1. The course starts with measurement: every spin state is an eigenstate of a spin component, all spin components are incompatible with each other. In the map (fig. 9, map 1), we see the boxes on “incompatibility” and “change of basis + Born rule”, but no “compatibility”, “degeneracy”, “kernel of the commutator” - which is necessarily empty in a two dimensional Hilbert space. The organizing principles are in their simplest form. We can add the map based on a superposition state, but only in order to discuss the superposition of two vectors when we do not know what eigenstate we are
dealing with;

2. A new observable, energy, comes into play in the context of Larmor precession. For what concerns measurement, we only need to add the box on “compatibility” (but no degeneracy) and possibly a new procedure: “the identification of energy eigenstates and eigenvalues” (fig. 9, map 2). The map concerning time evolution is presented, thus introducing a small number of new concepts, but without altering the structure of the pathways;

3. Next, we could proceed as Townsend, by covering two-particle systems with spin-half. At this step, we complete the construction of the maps based on the relations between two observables, as we must deal with degeneracy and the possibility of a non-empty kernel of the commutator (fig. 9, map 3). At the end, we can introduce the concept of CSCO and add the maps based on it, thus providing a general solution for measurement and time evolution;

4. The last step is the passage to QM in one and more spatial dimensions. In this context, we face many situations where it is not suitable to interpret a superposition as an eigenstate of a given observable. Therefore, we complete the exploration of the maps based on a superposition state (fig. 9, map 4).

In a waves-first approach, we should start almost immediately with fig. 9, map 3, as degeneracy appears already in the context of the free particle, and nothing ensures that the kernel of the commutator of two operators (on an infinite-dimensional space) is empty.

Finally, this framework may serve as a basis for the design of investigations into student understanding: the maps on the relations between two observables have been used to develop a survey on incompatibility and compatibility.6

VI. IMPLEMENTATION AT HIGH SCHOOL LEVEL

A full appreciation of the explanatory power of the interplay between the state and the observables, and in particular of incompatibility and compatibility, requires interpreting the structure of the Hermitian operators in complex Hilbert spaces. At high school level, this formal sophistication is obviously not viable. In order to develop a Teaching-Learning
Sequence (TLS) for high school students, we bypassed the obstacle by retaining only the essence of the structural features of the interplay.

Measurement is initially discussed at a quantitative level in the context of the linear polarization of light and then at a qualitative level on a hydrogen atom, by focusing on the acquisition, the loss, and the retention of definite values of observables in the process. In this way, we are able to introduce “mutual exclusivity”, “incompatibility” and “compatibility” as binary relations between values. Their definition, in very general terms, is the following:

A system is prepared with a definite value $P_a$, belonging to the physical quantity $O$. We measure the physical quantity $Q$, with $P_b$ being one of its values. The two values, $P_a$ and $P_b$, are:

- **mutually exclusive**: if the system retains $P_a$ and cannot acquire $P_b$ in measurement;
- **incompatible**: if the system loses $P_a$ and has a given probability to acquire $P_b$ in measurement;
- **compatible**: if the system retains $P_a$ and has a given probability to acquire $P_b$ in measurement (if it does not possess $P_b$ already).
By knowing which relation exists between values of the observables that are initially definite on the system and those of the measured observable, it is possible to qualitatively assess measurement outcomes (determinate/stochastic) and to determine which of the aforementioned observables are definite after the measurement, as a preliminary step to the introduction of the concept of quantum state. The development of the TLS and its results will be presented in future works.

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4 giacomo.zuccarini@uniud.it
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