Stability Investigation of Nonlinear Quadratic Discrete Dynamics Systems in the Critical Case

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Abstract. Many important processes (e.g. in industry and biology) are mathematically simulated with systems of discrete equations with quadratic right-hand sides. The paper presents stability results of quadratic discrete systems in the critical case (in the presence of a simple eigenvalue of the matrix of linear terms that is equal to unity and the others are smaller than unity in absolute value). In addition to the stability investigation of a zero solution, we also estimate stability domains.

1. Introduction

The main results of the stability theory of difference equations are presented, e.g., in the books by [1] and [2]. Instability problems are discussed, e.g., in the paper [3]. Notice that stability and instability results have often a local character being usually obtained without any estimation of the stability domain, or without investigating the character of instability.

Many important processes (e.g. in industry and biology) are mathematically simulated by systems of discrete equations with quadratic right-hand sides. In this paper, we give conditions for the stability of a zero solution of difference systems with quadratic nonlinearities in the critical case where there exists a simple eigenvalue $\lambda = 1$ of the matrix of linear terms. In addition to the stability investigation, we derive an estimation of stability domains.

In the sequel, the norms used for vectors and matrices, are defined as $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ for a vector $x = (x_1, \cdots, x_n)^T$ and $\|A\| = \left(\lambda_{\max}(A^T A)\right)^{1/2}$ for any $m \times n$ matrix $A$. Here and in the sequel, $\lambda_{\max}()$ (or $\lambda_{\min}()$) is the maximal (or the minimal) eigenvalue of the corresponding symmetric and positive definite matrix.

Consider a nonlinear difference system with a quadratic right-hand side. As was emphasized, e.g., in [2, 4], such a system can be written in a general matrix form

$$x(k+1) = Ax(k) + X^T(k) B x(k), \quad k = 0, 1, 2, \cdots,$$

where $x = (x_1, x_2, \cdots, x_n)^T$, $A$ is an $n \times n$ constant square matrix, $X^T$ is an $n \times n^2$ matrix, $B$ is an $n^2 \times n$ constant matrix, $X^T = (X_1^T, X_2^T, \cdots, X_n^T)$ and $B^T = (B_1, B_2, \cdots, B_n)$; all the elements of the

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square matrices \( X_i^T, i = 1, \cdots, n \) are equal to zero except the \( i \)-th line, which equals \( x^T \), and matrices \( B_j = \{b_{ij}\}, i, k, j = 1, \cdots, n \) are \( n \times n \) constant and symmetric.

The stability of the zero solution of system (1) depends on the stability of the matrix \( A \). If all the eigenvalues of \( A \) are within the circle with unit radius, i.e., if \( |\lambda_i(A)| < 1, i = 1, \cdots, n \), then the zero solution of system (1) is asymptotically stable for arbitrary matrix \( B \). In this case, matrix \( B \) impacts on the dimension of the stability domain of the equilibrium state only.

If zero solution of (1) is investigated on stability by the second Lyapunov method and Lyapunov function is taken as the quadratic form \( V(x) = x^T H x \) with a suitable \( n \times n \) constant real symmetric matrix \( H \), then the first difference \( \Delta V \) along trajectories of (1) equals

\[
\Delta V(x(k)) = V(x(k+1)) - V(x(k)) = x^T(k) \left[ A^T HA - H + A^T HX^T (k)B + B^T X(k)HA + B^T X(k)HX^T (k)B \right] x(k) = x^T(k) \left[ (A^T HA - H) + 2B^T X(k)HA + B^T X(k)HX^T (k)B \right] x(k). 
\]

Let the matrix \( A \) be asymptotically stable. Then, for arbitrary positive definite symmetric matrix \( C \), the matrix Lyapunov equation \( A^T HA - H = -C \) has a unique solution \( H \) - a positively definite symmetric matrix and, as follows from (2),

\[
\Delta V(x(k)) \leq - \left[ \lambda_{\text{min}}(C) - 2 \|B\| \|HA\| \|x(k)\| - \|B\|^2 \|HA\| \|x(k)\|^2 \right] \|x(k)\|^2. 
\]

Analysing (3), we deduce that the first difference \( \Delta V(x(k)) \) will be negatively definite in a neighborhood of the steady state \( x(k) = 0 \). As a consequence of the computations performed, in the case of stability, a concrete neighbourhood of the zero solution is found for which the fulfillment of the definition of stability is guaranteed. For such a kind of neighbourhoods, the term \textit{guaranteed domain of stability} (GDS) was used previously (see e.g. [5]).

2. Main results
In this section we derive stability results of the zero solution of system (1) in the critical case. More exactly, we consider the critical case if the matrix \( A \) has a simple eigenvalue \( \lambda = 1 \).

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2.1. Instability in One-dimensional Case
We start by discussing a simple scalar equation when the eigenvalue of the matrix \( A \) equals one, i.e., when \( n = 1, a_{11} = 1 \). Then (1) takes the form

\[
x(k+1) = x(k) + bx^2(k), \quad k = 0, 1, 2, \cdots
\]

and it is easy to see that the \textit{trivial solution is unstable for arbitrary} \( b \neq 0 \).

This example shows that the presence of stability in the case of system (1) has an extraordinary significance. Results on stability (for \( n \neq 1 \)) lose their meaning for \( n = 1 \) when we deal with instability. If \( n \neq 1 \) and \( B \) satisfies certain assumptions, the zero solution is stable.

2.2. Stability in a General Two-dimensional Case
Let \( n = 2 \). Then system (1) with the matrix \( A \) having a simple eigenvalue \( \lambda = 1 \) reduces (after a linear transformation of the dependent variables if necessary) to

\[
x_i(k+1) = ax_i(k) + b_{i1} x_{i1}^2(k) + 2b_{i2} x_i(k) x_{i2}(k) + b_{i2} x_{i2}^2(k), \quad i = 1, 2.
\]
In this case
\[
A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} b_{11}^1 & b_{12}^1 \\ b_{12}^1 & b_{22}^1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_{11}^2 & b_{12}^2 \\ b_{12}^2 & b_{22}^2 \end{pmatrix}.
\]

Let \( h_{11} \) be a positive number. We define: \( \alpha = h_{11}(1-a^2), \beta_1 = 2h_{11}ab_{11}^1, \beta_2 = 2h_{11}ab_{12}^1 + b_{11}^1, \gamma_1 = h_{11}(b_{11}^1)^2 + (b_{12}^1)^2, \gamma_2 = 4h_{11}(b_{11}^1)^2, \delta_1 = 2h_{11}b_{11}^1b_{12}^1. \)

**Theorem 1.** Assume \( |a| < 1 \) and \( b_{12}^1 = b_{12}^2 = b_{22}^2 = 0, h_{11}, b_{11}^1 \neq 0 \). Then the zero solution of system (4), (5) is stable in the Lyapunov sense and the GDS is described by an inequality
\[
h_{11}x_1^2 + x_2^2 \leq (r^*)^2
\]
where \( h_{11} \) is an arbitrary positive number and \( r^* \) is defined as the third coordinate of a solution \((x_1, x_2, r)\) with minimal positive \( r \) of the nonlinear system:
\[
\gamma_1 x_1^2 + 2\gamma_2 x_2^2 + 2\beta_2 x_1 + 2\beta_2 x_2 = \alpha,
\]
\[
h_{11}x_1^2 + x_2^2 = r^2,
\]
\[
h_{11}\delta_1 x_1 x_2 + \gamma_2 x_2 + \beta_2) - x_2(\gamma_1 x_1 + \delta_1 x_2 + \beta_1) = 0.
\]

**Proof.** Let \( x_2 = y \) and \( x = (x_1, y) \). We rewrite system (4), (5) as
\[
x_1(k+1) = ax_1(k) + x^T(k)B_1x(k),
\]
\[
y(k+1) = y(k) + x^T(k)B_2x(k).
\]

To investigate the stability of the zero solution we use an appropriate Lyapunov function \( V \). Let \( H \) be a positive definite symmetrix constant matrix. We set
\[
V(x(k)) = V(x_1(k), y(k)) := x^T(k)Hx(k) = (x_1(k), y(k))^T \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} \begin{pmatrix} x_1(k) \\ y(k) \end{pmatrix}.
\]

The first difference of the function \( V \) along the trajectories of system (4), (5) equals
\[
\Delta V(x(k)) = h_{11}(a^2-1)x_1^2(k) + 2h_{11}(a-1)x_1(k)y(k)
\]
\[
+ h_{11} \left\{ 2ax_1(k) \left[ x^T(k)B_1x(k) \right] + \left[ x^T(k)B_1x(k) \right]^2 \right\}
\]
\[
+ 2h_{12} \left\{ ax_1(k) \left[ x^T(k)B_2x(k) \right] + y(k) \left[ x^T(k)B_2x(k) \right] + \left[ x^T(k)B_2x(k) \right]^2 \right\}
\]
\[
+ h_{22} \left\{ 2y(k) \left[ x^T(k)B_2x(k) \right] + \left[ x^T(k)B_2x(k) \right]^2 \right\}.
\]
It is easy to see that \( \Delta V \) does not preserve the sign if \( h_{12} \neq 0 \). Therefore we put \( h_{12} = 0 \) and
\[ \Delta V(x(k)) = h_1(a^2 - 1)x_i^2(k) + h_1 \left\{ 2ax_i(k)\left[ x^T(k)B_1x(k) \right] + \left[ x^T(k)B_1x(k) \right]^2 \right\} + h_{22} \left\{ 2y(k)\left[ x^T(k)B_2x(k) \right] + \left[ x^T(k)B_2x(k) \right]^2 \right\} = h_1(a^2 - 1)x_i^2(k) + F_3(x_i(k), y(k)) + F_4(x_i(k), y(k)) \]

where

\[ F_3(x_i(k), y(k)) = 2h_1ab_i x_i^3(k) + 2\left[ 2h_1ab_i^2 + bh_i^2 \right] x_i^3(k)y(k) + 2\left[ h_1ab_i^2 + 2h_{22}b_i^2 \right] x_i(k)y^2(k) + 2\left[ h_1b_i^2 + bh_i^2 \right] x_i(k)y^3(k) + 4h_1b_i^2 x_i(k)y^4(k). \]

If \(|a| < 1\), then \( \Delta V \) will be non-positive in a small neighborhood of the zero solution if multipliers of the terms \( x_i(k)y^2(k) \), \( y^3(k) \) and \( x_i(k)y^3(k) \) equal zero, and the multiplier of the term \( y^4(k) \) is non-positive, i.e., if \( h_1ab_i^2 + 2h_{22}b_i^2 = 0 \), \( h_1b_i^2 + bh_i^2 = 0 \), \( h_i(b_i^2)^2 + bh_i^2 \). As long as Lyapunov function is positively definite, \( h_1 > 0 \) and \( h_{22} > 0 \).

Therefore above conditions hold if and only if \( b_i^2 = 0 \), \( b_{22}^2 = 0 \). Then the initial system (4), (5) turns into

\[
\begin{align*}
  x_i(k + 1) &= ax_i(k) + \left[ 2h_1b_i^2 + 2h_{22}b_i^2 \right] x_i(k)x_2(k), \\
  x_2(k + 1) &= x_i(k) + bh_i^2 x_i^2(k)
\end{align*}
\]

and \( \Delta V \) (without loss of generality we put \( h_{22} = 1 \), i.e., \( V(x_i, y) = h_1x_i^2 + y^2 \)) into

\[ \Delta V(x(k)) = -\left[ h_1(1 - a^2) - 2h_1ab_i x_i(k) - 2\left[ 2h_1ab_i^2 + bh_i^2 \right] y(k) \right] x_i^2(k) - \left[ h_1(b_i^2)^2 + bh_i^2 \right] x_i^2(k) - 4h_1b_i^2 x_i(k)y(k) - 4h_1(b_i^2)^2 y^2(k) = -\left\{ \alpha - 2\beta_1 x_i(k) - 2\beta_2 y(k) - \gamma_1 x_i^2(k) - 2\delta x_i(k)y(k) - \gamma_2 y^2(k) \right\} x_i^2(k).
\]

Because \( h_1 > 0 \) and \(|a| < 1\), \( \alpha = h_1(1 - a^2) > 0 \) and the first difference of the Lyapunov function is non-positive in a sufficiently small neighborhood of the origin. In other words, the zero solution is stable in the Lyapunov sense. The \( GDS \) can be defined by inequalities

\[ \gamma_1 x_i^2 + 2\gamma_2 x_i x_2 + 2\gamma_2 x_2^2 + 2\beta_1 x_i + 2\beta_2 x_2 \leq \alpha, \quad h_1 x_i^2 + x_2^2 \leq r^2 \]

where \( r > 0 \). Both inequalities geometrically express closed ellipses. For the second inequality this is obvious. For the first one, this follows from inequalities: \( \gamma_1 > 0, \gamma_2 > 0 \)

\[ \gamma_1^2 - \delta_1 = \left[ h_1(b_i^2)^2 + bh_i^2 \right] \left[ 4h_1(b_i^2)^2 \right] - 4h_1(b_i^2)^2 = 4h_1(b_i^2)^2 > 0. \]

Therefore \( GDS \) is, in general, the intersection of two ellipses. Moreover, the second ellipse \( h_1 x_i^2 + x_2^2 \leq r^2 \) shrinks to the origin for \( r \to 0 \), i.e., there exists such \( r = r^* \) that, for \( r \in (0, r^*) \), the second ellipse lies inside the first one and for \( r = r^* \) there exists at least one common boundary point...
of both ellipses. Let us find the value \( r^\ast \). It is characterized by the requirement that the slope coefficients \( k_1 \) and \( k_2 \) of both ellipses are the same at the point of contact. Therefore

\[
k_1 = -\frac{\gamma_1 x_1 + \delta_1 y + \beta_1}{\delta_1 x_1 + \gamma_2 y + \beta_2}, \quad k_2 = -\frac{h_1 x_1}{y}
\]

where we assume (without loss of generality) that the denominators are nonzero. Thus we get a quadratic system of three equations to find the contact points \((x, y)\) and the corresponding values of \( r^\ast \):

\[
\gamma_1 x^2 + 2\delta_1 x y + \gamma_2 y^2 + 2\beta_1 x + 2\beta_2 y = \alpha,
\]

\[
h_1 x^2 + y^2 = r^2,
\]

\[
h_1 x_1 (\delta_1 x_1 + \gamma_2 y + \beta_2) - y(\gamma_1 x_1 + \delta_1 y + \beta_1) = 0.
\]

In accordance with the geometrical meaning of this system, we take a solution \((x^\ast_1, y^\ast, r^\ast)\) where \( r^\ast \) is the minimal positive number, and put \( r = r^\ast \).

### 2.3. Stability in a General n-dimensional Case

Consider an \( n \)-dimensional case. We assume that the matrix \( A \) has one eigenvalue that is equal to unity and the others are smaller than unity in absolute value. Then we can assume (without loss of generality) that the matrix \( A \) has a block form, i.e.,

\[
A = \text{diag}(A_0, 1), \quad A_0 = \{a_{ij}^0\}, \quad i, j = 1, 2, \ldots, n - 1
\]

and all the eigenvalues of the matrix \( A_0 \) lie inside of a unit circle. Let

\[
B_n^0 = \begin{pmatrix}
  b_{11}^0 & b_{12}^0 & \cdots & b_{1,n-1}^0 \\
  b_{12}^0 & b_{22}^0 & \cdots & b_{2,n-1}^0 \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{1,n-1}^0 & b_{2,n-1}^0 & \cdots & b_{n-1,n-1}^0
\end{pmatrix}, \quad \bar{B} = \begin{pmatrix}
  b_{1,1}^0 & \cdots & b_{1,n-1}^0 & b_{2,1}^0 & \cdots & b_{2,n-1}^0 & \cdots & b_{n-1,1}^0 & \cdots & b_{n-1,n-1}^0 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  b_{1,n-1}^0 & b_{2,n-1}^0 & \cdots & b_{n-1,n-1}^0 & b_{n-1,1}^0 & \cdots & b_{n-1,n-1}^0 & \cdots & b_{n-1,n-1}^0 & \cdots & b_{n-1,n-1}^0
\end{pmatrix},
\]

We assume that there exists a symmetric positive definite \((n - 1) \times (n - 1)\) matrix \( H \) such that the matrix \( C = -A_n^0 H - HA_n^0 \) is positively definite and symmetric. Let \( h \) be a fixed positive number. We define \( x^T = (x_1, x_2, \ldots, x_{n-1}) \), \( y = x_n \),

\[
\gamma_1 = \|BH \bar{B}^T + h((B_n^0)^T)^2\|, \quad \delta_1 = 2\|BH \bar{B}^T\|, \quad \gamma_2 = 4\|H \bar{B}\|, \quad \alpha = \lambda_{\min}(C), \quad \beta_1 = \|A_0 H \bar{B}^T\|, \quad \beta_2 = \|BA_n^0 + 3A_n^0 H \bar{B}^T + h(B_n^0)^T\|.
\]

The scheme of the proof of the following theorem is similar to the proof of Theorem 1, but involves a lot of technical difficulties. Therefore we omit the proof.

**Theorem 2.** Assume \( b_{11}^0 = b_{22}^0 = \cdots = b_{nn}^0 = 0 \), \( b_{1n}^0 = b_{2n}^0 = \cdots = b_{n-1,n}^0 = 0 \). Then a zero solution of system (1) is stable by Lyapunov and the GDS is described by inequalities

\[
\gamma_1 \|x\|^2 + 2\delta_1 \|x\| \|y\| + \gamma_2 \|y\|^2 + 2\beta_1 \|y\|^2 + 2\beta_2 \|y\| \leq \alpha,
\]

\[
h \|y\|^2 + H \|x\|^2 \leq r^2
\]

where \( r > 0 \) is so small that the second domain is embedded into the first one.
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