Discrete Boltzmann model of compressible flows with spherical or
cylindrical symmetry

Aiguo Xu\textsuperscript{1,2,3}, Guangcai Zhang\textsuperscript{1,3}, Chuandong Lin\textsuperscript{4}

\textsuperscript{1} National Key Laboratory of Computational Physics,
Institute of Applied Physics and Computational Mathematics,
P. O. Box 8009-26, Beijing 100088, P.R.China

\textsuperscript{2} Center for Applied Physics and Technology,
MOE Key Center for High Energy Density Physics Simulations,
College of Engineering, Peking University, Beijing 100871, China

\textsuperscript{3} State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics,
Chinese Academy of Sciences, Beijing 100190, China

\textsuperscript{4} State Key Laboratory for GeoMechanics and Deep Underground Engineering,
China University of Mining and Technology, Beijing 100083, P.R.China

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Abstract

To study simultaneously the hydrodynamic and thermodynamic behaviors in compressible flow systems with spherical or cylindrical symmetry, we present a theoretical framework for constructing Discrete Boltzmann Model (DBM) with spherical or cylindrical symmetry in spherical or cylindrical coordinates. To this aim, a key technique is to use local Cartesian coordinates to describe the particle velocity in the kinetic model. Thus, the geometric effects, like the divergence and convergence, are described as a “force term”. Even though the hydrodynamic models are one- or two-dimensional, the DBM needs a Discrete Velocity Model (DVM) with 3 dimensions. We use a DVM with 26 velocities to formulate the DBM which recovers the Navier-Stokes equations with spherical or cylindrical symmetry in the hydrodynamic limit. For the system with global cylindrical symmetry, we formulated also a DBM based on a DVM with 2 dimensions and 16 velocities. In terms of the nonconserved moments, we define two sets of measures for the deviations of the system from its thermodynamic equilibrium state. The extension of current model to the multiple-relaxation-time version is straightforward.

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I. INTRODUCTION

During recent decades the lattice Boltzmann (LB) modeling and simulation have achieved great success in various complex flows [1, 2]. According to the discretization scheme, the LB methods can be roughly classified as the standard LB [1], Finite-Difference (FD) LB [3–5], Finite-Volume (FV) LB [6], and Finite-Element (FE) LB [7, 8], etc. According to the purpose and/or function, the LB methods can be roughly classified as Partial Differential Equations (PDE) solvers, mesoscopic kinetic models, etc. The LB methods working as mesoscopic kinetic models are generally designed to access some nonequilibrium behaviors of the complex system which cannot be or are not convenient to be described by the traditional hydrodynamic models. When constructing LB mesoscopic kinetic models, one has to ensure that the main behaviors under consideration to be included [1, 2]. The formulating of a LB PDE solver can be more flexible. To distinguish from the PDE solvers, the LB mesoscopic kinetic models are referred to as the Discrete Boltzmann Model (DBM) in this work. The appropriately designed DBM should inherit some functions of the Boltzmann equation [2, 9].

Given the great importance of shock waves in many fields of physics and engineering, constructing LB models for high speed compressible flows has attracted considerable interest since the early days of LB research [1]. The LB model for high speed compressible flows has seen significant progress in recent years [9]. In 1992 Alexander et al. [10] formulated a compressible LB model for flows at high Mach number via introducing a flexible sound speed. This model works only for nearly isothermal compressible systems. In 1999 Yan et al. [11] proposed a LB scheme for compressible Euler equations. In the years of 1998 and 2003 Sun and his coworker [12, 13] presented an adaptive LB scheme for the two- and three-dimensional systems, respectively. In this model the particle velocities vary with the Mach number and internal energy, so that the particle velocities are no longer constrained to fixed values. All of those models belong to the standard LB solvers of PDE.

In 1997 Cao et al. [4] proposed to use the FD scheme to calculate the spatial and temporal derivatives in the LB equation and apply nonuniform grids so that the numerical stability can be improved a little. In the past decade, Tsutahara, Watari and Kataoka [14–18] proposed several nice FD LB models for the Euler and Navier-Stokes equations. In 2005 Xu [19, 20] extended the idea to handle binary fluids. However, similar to the case of standard LB models, these FD LB schemes still belong to LB solvers of corresponding PDE since no
function beyond the traditional hydrodynamic models are pointed out in corresponding publications. These FD LB schemes work also only for subsonic flows. To simulate high speed compressible flows, especially those with shocks, many attempts and considerable progress have been achieved in recent years [21–33].

Up to now, most of LB models for compressible fluids are in Cartesian coordinates. In many cases the flows show divergent, convergent, and/or rotational behaviors, for example, in cylindrical or spherical devices. For such flow systems, LB models in polar, spherical, cylindrical or rotational coordinates are more convenient and are less exposed to numerical errors. There have been a number of LB methods in curvilinear coordinates or axisymmetric cylindrical coordinates. Early in 1992, Nannelli and Succi [6] presented a general framework to extend the LB equation to arbitrary lattice geometries. In this work a FV LB was formulated. Then some other versions of FV LB were proposed for irregular meshes [34–38]. In 1997 He and Doolen [39] extended the LB to general curvilinear coordinate systems. In the following year Mei and Shyy [40] developed a FD LB in body-fitted curvilinear coordinates with non-uniform grids. Later, Halliday et al. [41] proposed a Polar Coordinate Lattice Boltzmann (PCLB) for hydrodynamics. In 2005 Premnath and Abraham [42] presented a LB for axisymmetric multiphase flows. In this work source terms were added to a two-dimensional standard LB equation for multiphase flows such that the emergent dynamics can be transformed into the axisymmetric cylindrical coordinate system. But all those LB methods in non-Cartesian coordinates work only for isothermal and nearly incompressible flows. In 2010 Asinari et al. [43] formulated a LB to analyze the radiative heat transfer problems in a participation medium, but did not take into account the effects of fluid flow. In 2011 Watari [44] formulated a FD PCLB to investigate the rotational flow problems in coaxial cylinders. This work presents valuable information on the LB application to the cylindrical system. However, this model works also only for subsonic flow systems. Based on the same discrete velocity model, Lin et al. [45] formulated a PCLB for high-speed compressible flows. Within this model [45], a hybrid scheme being similar to, but different from, the operator-splitting is proposed. The temporal evolution is calculated analytically and the convection term is solved via a Modified Warming-Beam (MWB) scheme. Within the MWB scheme a suitable switch function is introduced.

Besides recovering the macroscopic hydrodynamic equations, designing DBM to access the nonequilibrium behaviors is attracting more attention with time [2, 9]. The idea of
DBM has been further specified and applied in various compressible flow systems via several models [45–49]. Examples of LBGK for compressible flow systems are referred to Refs. [45, 47] where preliminary studies on shocking behaviors are shown. An example of MRT-LB for compressible flow systems is referred to Ref. [49]. In Refs. [46, 48] the Thermodynamic NonEquilibrium (TNE) behaviors in combustion systems are initially investigated via LBGK models. The kinetic nature of DBM for the non-ideal gas systems, particularly, the liquid-vapor system or single-component two phase flows was considered in Ref. [50].

In traditional modeling the implosion and explosion processes, one-dimensional or two-dimensional hydrodynamic equations are frequently used. The one-dimensional hydrodynamic equations are generally used to describe system with spherical symmetry and systems with cylindrical symmetry with translational symmetry. The two-dimensional equations are generally used to describe systems with rotational or cylindrical symmetry. Since the DBM has been proved to be more convenient for measuring the macroscopic behaviors of the system due to its deviating from thermodynamic equilibrium state, in this work we aim to construct the DBM for compressible flow systems with spherical or cylindrical symmetry.

This paper is organized as below. In section II we briefly review the kinetic and hydrodynamic models of the fluid system. In terms of their correlations, we formulate two set of measures for the deviation of the system from its thermodynamic equilibrium. The discrete Boltzmann models are formulated in section III. Section IV presents the conclusion and discussions.

II. BRIEF REVIEW OF FLUID MODELS

A. Kinetic model

The Boltzmann BGK model [51] reads

$$\partial_t f + \mathbf{v} \cdot \nabla f = -\frac{1}{\tau} (f - f^{eq}),$$  \hspace{1cm} (1)

where $f = f(R, \mathbf{v}, t) = f(x, y, z, v_x, v_y, v_z, t)$, $R = x\hat{x} + y\hat{y} + z\hat{z}$ and $\mathbf{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ in Cartesian coordinates.
1. **In cylindrical coordinates**

In cylindrical coordinates, the position \( \mathbf{R} = r \hat{r} + z \hat{z} \), where \( r \hat{r} \) is the projection of the position \( \mathbf{R} \) in the \((x, y)\) plane. The included angle between the \(x\)-axis and the vector \( r \hat{r} \) is \( \theta \). The unit vectors \( \hat{r}, \hat{\theta} \) and \( \hat{z} \) are the changing directions of the position \( \mathbf{R} \) along the three parameters, \( r \), \( \theta \) and \( z \), i.e.,

\[
\mathbf{dR} = \hat{r} dr + r \hat{\theta} d\theta + \hat{z} dz.
\]

Obviously, \( \hat{r}, \hat{\theta} \) and \( \hat{z} \) are orthogonal to each other and satisfy the following relationships,

\[
\begin{align*}
\hat{z} & = \hat{r} \times \hat{\theta}, \\
\hat{\theta} & = \hat{z} \times \hat{r}, \\
\hat{r} & = \hat{\theta} \times \hat{z}.
\end{align*}
\]

It is easy to find that

\[
\begin{align*}
\mathbf{dR} & = (\mathbf{zd\theta}) \times \hat{r} = \hat{\theta} d\theta, \\
\mathbf{dR} & = (\mathbf{zd\theta}) \times \hat{\theta} = -\hat{r} d\theta, \\
\mathbf{dR} & = 0.
\end{align*}
\]

We consider two kinds of cylindrical symmetries here. The system is referred to have *local* cylindrical symmetry if the behavior of the system is independent of the azimuthal coordinate \( \theta \) but maybe dependent of the height \( z \), and is referred to have *global* cylindrical symmetry if the behavior of the system is independent of both the azimuthal coordinate \( \theta \) and the height \( z \).

The description of the particle velocity \( \mathbf{v} \) can use the global Cartesian coordinates \((\hat{x}, \hat{y}, \hat{z})\),

\[
\mathbf{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}.
\]

It can also use the local Cartesian coordinates \( (\hat{r}, \hat{\theta}, \hat{z}) \),

\[
\begin{align*}
\mathbf{v} & = v_r \hat{r} + v_\theta \hat{\theta} + v_z \hat{z} \\
& = \mathbf{v} \cdot \hat{r} \hat{r} + \mathbf{v} \cdot \hat{\theta} \hat{\theta} + \mathbf{v} \cdot \hat{z} \hat{z}.
\end{align*}
\]

Consequently, the distribution function \( f \) can also be described in two different forms,

\[
f = f (r, \theta, z, v_x, v_y, v_z, t)
\]
or

\[ f = f (r, \theta, z, v_r, v_\theta, v_z, t) . \]  

(5)

In this work we will use the latter form, Eq. (5). This is a key technique in this work for constructing the DBM. Under the definition (5), it should be stressed that the particle velocity \( v \) is \textit{locally} fixed, while its two components, \( v_r, v_\theta \), varies with the position \( \mathbf{R} \), when calculating the spatial derivative, \( \nabla f \). We use the symbol “\( \nabla|_v \)” to replace “\( \nabla \)” in Eq. (1) to stress that \( v \) is fixed when calculating the spatial derivatives. Thus, Eq. (1) is rewritten as

\[ \partial_t f + \mathbf{v} \cdot \nabla|_v f = -\frac{1}{\tau} (f - f_{\text{eq}}) . \]  

(6)

According to the definition (5),

\[ \nabla|_v f = \nabla|_{v_r,v_\theta} f + \nabla v_r \frac{\partial f}{\partial v_r} + \nabla v_\theta \frac{\partial f}{\partial v_\theta} , \]  

(7)

where

\[ \nabla v_r = \nabla (\hat{\mathbf{r}} \cdot \mathbf{v}) \]

\[ = (\nabla \hat{\mathbf{r}}) \cdot \mathbf{v} \]

\[ = \left( \hat{\mathbf{r}} \frac{\partial \hat{\mathbf{r}}}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial \hat{\mathbf{r}}}{\partial \theta} + \hat{\mathbf{z}} \frac{\partial \hat{\mathbf{r}}}{\partial z} \right) \cdot \mathbf{v} \]

\[ = \hat{\theta} \frac{1}{r} \cdot \mathbf{v} \]

\[ = \frac{v_\theta}{r} \hat{\theta} , \]  

(8)

similarly,

\[ \nabla v_\theta = -\frac{v_r}{r} \hat{\theta} . \]  

(9)

Substituting Eqs. (7)-(9) into Eq. (6) gives the Boltzmann equation in cylindrical coordinates,

\[ \partial_t f + \mathbf{v} \cdot \nabla|_{v_r,v_\theta} f + \frac{v_\theta^2}{r} \frac{\partial f}{\partial v_r} - \frac{v_r v_\theta}{r} \frac{\partial f}{\partial v_\theta} = -\frac{1}{\tau} (f - f_{\text{eq}}) . \]  

(10)

For macroscopic system with cylindrical symmetry, the distribution function \( f \) does not depend explicitly on the angle \( \theta \). i.e.,

\[ f = f (r, z, v_r, v_\theta, v_z, t) . \]  

(11)
So, the Boltzmann equation becomes

\[
\partial_t f + \left( v_r \frac{\partial f}{\partial r} + v_z \frac{\partial f}{\partial z} \right) + \left( \frac{v_{\theta}^2}{r} \frac{\partial f}{\partial v_r} - \frac{v_r v_{\theta}}{r} \frac{\partial f}{\partial v_{\theta}} \right) = -\frac{1}{\tau} (f - f^{eq}).
\] (12)

It is clear that the term

\[
\left( \frac{v_{\theta}^2}{r} \frac{\partial f}{\partial v_r} - \frac{v_r v_{\theta}}{r} \frac{\partial f}{\partial v_{\theta}} \right)
\]

plays the role of the force term in the Boltzmann equation in the Cartesian coordinates. It creates the divergence or convergence effects in the flow system. When the system is not far from its thermodynamic equilibrium, we can use the approximation, \( f = f^{eq} \), when calculating the force term. If further use the macroscopic condition, \( u_{\theta} = 0 \), the final Boltzmann model for the flow system with cylindrical becomes,

\[
\partial_t f + \left( v_r \frac{\partial f}{\partial r} + v_z \frac{\partial f}{\partial z} \right) + \left[ \frac{v_r v_{\theta}}{rT} - \frac{v_{\theta}^2 (v_r - u_r)}{rT} \right] f^{eq} = -\frac{1}{\tau} (f - f^{eq}).
\] (13)

2. In spherical coordinates

In spherical coordinates, the position \( \mathbf{R} = r \hat{\mathbf{r}} \). The three parameters \( r, \theta \) and \( \varphi \) are the radial, azimuth and zenith angle, respectively. The unit vectors, \( \hat{\mathbf{r}}, \hat{\theta} \) and \( \hat{\varphi} \), are the changing directions of the position vector \( \mathbf{R} \) along the three parameters, \( r, \theta \) and \( \varphi \), respectively, i.e.,

\[
dr = \hat{\mathbf{r}} dr + r \hat{\theta} d\theta + r \sin \theta \hat{\varphi} d\varphi.
\]

Obviously, \( \hat{\mathbf{r}}, \hat{\theta} \) and \( \hat{\varphi} \) are orthogonal to each other and satisfy the following relationships,

\[
\hat{\varphi} = \hat{\mathbf{r}} \times \hat{\theta}, \quad \hat{\theta} = \hat{\varphi} \times \hat{\mathbf{r}}, \quad \hat{\mathbf{r}} = \hat{\theta} \times \hat{\varphi}.
\]

It is easy to find that

\[
d\hat{\mathbf{r}} = (\hat{\varphi} d\theta + \hat{\varphi} d\varphi) \times \hat{\mathbf{r}} = \hat{\theta} d\theta + \hat{\varphi} \sin \theta d\varphi,
\]

\[
d\hat{\theta} = (\hat{\varphi} d\theta + \hat{\varphi} d\varphi) \times \hat{\theta} = -\hat{\mathbf{r}} d\theta + \hat{\varphi} \cos \theta d\varphi,
\]

\[
d\hat{\varphi} = (\hat{\varphi} d\theta + \hat{\varphi} d\varphi) \times \hat{\varphi} = -\left( \hat{\mathbf{r}} \sin \theta + \hat{\theta} \cos \theta \right) d\varphi.
\]
The description of the particle velocity \( \mathbf{v} \) can use the global Cartesian coordinates \( (\mathbf{\hat{x}}, \mathbf{\hat{y}}, \mathbf{\hat{z}}) \). It can also use the local Cartesian coordinates \( (\mathbf{\hat{r}}, \mathbf{\hat{\theta}}, \mathbf{\hat{\phi}}) \).

\[
\mathbf{v} = v_x \mathbf{\hat{x}} + v_y \mathbf{\hat{y}} + v_z \mathbf{\hat{z}} \\
= v_r \mathbf{\hat{r}} + v_\theta \mathbf{\hat{\theta}} + v_\phi \mathbf{\hat{\phi}} \\
= \mathbf{v} \cdot \mathbf{\hat{r}} + \mathbf{v} \cdot \mathbf{\hat{\theta}} + \mathbf{v} \cdot \mathbf{\hat{\phi}}.
\] (14)

The distribution function \( f \) can also be described in two different forms,

\[
f = f(r, \theta, \varphi, v_x, v_y, v_z, t)
\] (15)

or

\[
f = f(r, \theta, \varphi, v_r, v_\theta, v_\phi, t).
\] (16)

In this work we will use the latter form, Eq. (16). Because it is more convenient when describing problems with cylindrical symmetry. Under the definition (16), it should be stressed that the particle velocity \( \mathbf{v} \) is fixed, while its three components, \( v_r, v_\theta, v_\phi \), varies with the position \( \mathbf{R} \), when calculating the spatial derivative, \( \nabla f \). We use the symbol “\( \nabla|_\mathbf{v} \)” to replace “\( \nabla \)” in Eq. (1) to stress that \( \mathbf{v} \) is fixed when calculating the spatial derivatives. Thus, Eq. (1) is rewritten as

\[
\partial_t f + \mathbf{v} \cdot \nabla|_\mathbf{v} f = -\frac{1}{\tau} (f - f^{eq}).
\] (17)

According to the definition (16),

\[
\nabla|_\mathbf{v} f = \nabla|_{v_r, v_\theta, v_\phi} f + \nabla v_r \frac{\partial f}{\partial v_r} + \nabla v_\theta \frac{\partial f}{\partial v_\theta} + \nabla v_\phi \frac{\partial f}{\partial v_\phi},
\] (18)

where

\[
\nabla v_r = \nabla (\mathbf{\hat{r}} \cdot \mathbf{v}) \\
= (\nabla \mathbf{\hat{r}}) \cdot \mathbf{v} \\
= \left( \mathbf{\hat{r}} \frac{\partial \mathbf{\hat{r}}}{\partial r} + \theta \frac{1}{r} \frac{\partial \mathbf{\hat{r}}}{\partial \theta} + \varphi \frac{1}{r \sin \theta} \frac{\partial \mathbf{\hat{r}}}{\partial \varphi} \right) \cdot \mathbf{v} \\
= \left( \frac{\partial \mathbf{\hat{\theta}}}{r} + \frac{\hat{\varphi} \mathbf{\hat{\theta}}}{r} \right) \cdot \mathbf{v} \\
= \frac{v_\theta}{r} \mathbf{\hat{\theta}} + \frac{v_\phi}{r} \mathbf{\hat{\varphi}},
\] (19)
and similarly,
\[ \nabla v_\theta = -\frac{v_r}{r} \hat{\theta} + \frac{v_\varphi \cos \theta}{r \sin \theta} \hat{\varphi}, \]
\[ \nabla v_\varphi = -\left( \frac{v_r}{r} + \frac{v_\theta \cos \theta}{r \sin \theta} \right) \hat{\varphi}. \]

Substituting Eqs. (18)-(21) into Eq. (17) gives the Boltzmann equation in spherical coordinates,
\[ \partial_t f + \mathbf{v} \cdot \nabla f = \frac{v_\theta^2 + v_\varphi^2}{r} \frac{\partial f}{\partial v_r} + \left( \frac{v_\theta^2 \cos \theta}{r \sin \theta} - \frac{v_r v_\theta}{r} \right) \frac{\partial f}{\partial v_\theta} - \left( \frac{v_r v_\varphi}{r} + \frac{v_\varphi v_\theta \cos \theta}{r \sin \theta} \right) \frac{\partial f}{\partial v_\varphi} \]
\[ = -\frac{1}{\tau} (f - f^{eq}). \] (22)

For macroscopic system with spherical symmetry, the distribution function \( f \) does not depend explicitly on the angles \( \theta \) and \( \varphi \), i.e.,
\[ f = f (r, v_r, v_\theta, v_\varphi, t); \] \hspace{1cm} (23)
and \( f \) is invariant under the rotation in the subspace of \( (v_\theta, v_\varphi) \), i.e.,
\[ f (r, v_r, v_\theta, v_\varphi, t) = f (r, v_r, v'_\theta, v'_\varphi, t) \] \hspace{1cm} (24)
only when
\[ v_\theta^2 + v_\varphi^2 = v'_\theta^2 + v'_\varphi^2. \] \hspace{1cm} (25)
So,
\[ \left( v_\theta \frac{\partial f}{\partial v_\varphi} - v_\varphi \frac{\partial f}{\partial v_\theta} \right) f = 0. \] \hspace{1cm} (26)

Using the conditions, (23) and (26), in Eq. (22) gives
\[ \partial_t f + \mathbf{v} \cdot \nabla f = \frac{v_\theta^2 + v_\varphi^2}{r} \frac{\partial f}{\partial v_r} + \left( \frac{v_\theta^2 \cos \theta}{r \sin \theta} - \frac{v_r v_\theta}{r} \right) \frac{\partial f}{\partial v_\theta} - \left( \frac{v_r v_\varphi}{r} + \frac{v_\varphi v_\theta \cos \theta}{r \sin \theta} \right) \frac{\partial f}{\partial v_\varphi} \]
\[ = -\frac{1}{\tau} (f - f^{eq}). \] (27)
It is clear that the term
\[ \left( \frac{v_\theta^2 + v_\varphi^2}{r} \frac{\partial f}{\partial v_r} - \frac{v_r v_\theta}{r} \frac{\partial f}{\partial v_\theta} - \frac{v_r v_\varphi}{r} \frac{\partial f}{\partial v_\varphi} \right) \]
plays the role of the force term in the Boltzmann equation in Cartesian coordinates. It creates the divergence or convergence effects in the flow system. If the system is not far from its thermodynamic equilibrium, we can use the approximation, \( f = f^{eq} \) when calculating the force term. If further use the macroscopic condition, \( u_\theta = u_\varphi = 0 \), the final Boltzmann model for the flow system with spherical symmetry becomes,
\[ \partial_t f + v_r \frac{\partial f}{\partial r} + \left[ \frac{v_r v_\theta^2}{r T} + \frac{v_\varphi^2}{r T} - \frac{(v_\theta^2 + v_\varphi^2) (v_r - u_r)}{r T} \right] f^{eq} = -\frac{1}{\tau} (f - f^{eq}). \] (28)
B. Hydrodynamic models and their correlations to kinetic models

The Navier-Stokes equations in Cartesian coordinates read

\[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_\alpha)}{\partial x_\alpha} = 0, \]  

\[ \frac{\partial (\rho u_\alpha)}{\partial t} + \frac{\partial (\rho u_\alpha u_\beta)}{\partial x_\beta} + \frac{\partial P \delta_{\alpha\beta}}{\partial x_\beta} = \frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta}, \]  

\[ \frac{\partial}{\partial t}(\rho E + \frac{1}{2} \rho u^2) + \frac{\partial}{\partial x_\alpha}[u_\alpha(\rho E + \frac{1}{2} \rho u^2 + P)] = \frac{\partial}{\partial x_\alpha}[-Q_\alpha + u_\beta \sigma_{\alpha\beta}]. \]

In the left-hand side of Eqs. (29a)-(29c), \( \rho, \ u, \ P = \rho RT, \ E = (D + n)RT/2, \ T, \ \gamma = (D + n + 2)/(D + n) \) are the hydrodynamic density, flow velocity, pressure, internal energy, temperature and specific-heat ratio, respectively. \( D \) is the space dimension and \( n \) is the number of extra degrees of freedom whose energy level is \( \eta^2/2 \). \( u^2 = u \cdot u \). In the right-hand side of Eqs. (29b)-(29c),

\[ \sigma_{\alpha\beta} = \mu \left[ \frac{\partial u_\beta}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_\beta} - (1 - \lambda) \frac{\partial u_\gamma}{\partial x_\gamma} \delta_{\alpha\beta} \right] \]

is the viscous stress and

\[ Q_\alpha = -k \frac{\partial E}{\partial x_\alpha} \]

is the heat flux. The two parameters,

\[ \mu = \frac{2}{D} \rho e \tau, \]

\[ \lambda = \frac{D + n - 2}{D + n}, \]

are viscosities and the parameter,

\[ k = \frac{2 (D + n + 2)}{D (D + n)} \rho e \tau, \]

is the heat conductivity. \( e = DRT/2 \) is the translational internal energy. It is clear that \( P = 2\rho e / D \). When the viscosities and heat conductivity vanish, the hydrodynamic equations, (29a) - (29c), become the Euler equations.

The Chapman-Enskog multiscale analysis shows that, in the process of recovering the Navier-Stokes equations, (29a) - (29c), from the Boltzmann BGK equation, (1), the following seven moment relations,

\[ \int \int f_{eq} d\mathbf{v} d\eta = \rho, \]

(32a)
\[ \int \int f^{eq} v_\alpha d\bm{v} d\eta = \rho u_\alpha, \quad (32b) \]
\[ \int \int f^{eq} (v^2 + \eta^2) d\bm{v} d\eta = 2\rho \left( \frac{D + n + 2}{D} e + \frac{u^2}{2} \right), \quad (32c) \]
\[ \int \int f^{eq} v_\alpha v_\beta d\bm{v} d\eta = P \delta_{\alpha\beta} + \rho u_\alpha u_\beta, \quad (32d) \]
\[ \int \int f^{eq} \left( v^2 + \eta^2 \right) v_\alpha d\bm{v} d\eta = 2\rho \left( \frac{D + n + 2}{D} e + \frac{u^2}{2} \right) u_\alpha, \quad (32e) \]
\[ \int \int f^{eq} v_\alpha v_\beta v_\gamma d\bm{v} d\eta = \frac{4}{D} \rho e \left( \frac{D + n + 2}{D} e + \frac{u^2}{2} \right) \delta_{\alpha\beta} + 2\rho u_\alpha u_\beta \left( \frac{D + n + 4}{D} e + \frac{u^2}{2} \right), \quad (32f) \]
\[ \int \int f^{eq} \left( v^2 + \eta^2 \right) v_\alpha v_\beta d\bm{v} d\eta = 4 \rho \left( \frac{D + n + 2}{D} e + \frac{u^2}{2} \right) \delta_{\alpha\beta} + 2\rho u_\alpha u_\beta \left( \frac{D + n + 4}{D} e + \frac{u^2}{2} \right), \quad (32g) \]
are used, where \( \eta \) is the velocity in the \( n \) extra degrees of freedom, \( \eta^2 = \eta \cdot \eta \), and

\[ f^{eq} = g^{eq} (\bm{v}) h^{eq} (\eta) \quad (33) \]

with

\[ g^{eq} (\bm{v}) = \rho \left( \frac{D}{4\pi e} \right)^{D/2} \exp \left[ -\frac{D}{4e} (\bm{v} - \bm{u})^2 \right], \quad (34a) \]
\[ h^{eq} (\eta) = \left( \frac{D}{4\pi ne} \right)^{D/2} \exp \left( -\frac{D}{4ne} \eta^2 \right). \quad (34b) \]

We require \( \int d\eta \ h^{eq} (\eta) = 1 \), and \( n \to 0 \) when \( \eta \to 0 \).

Converting the form of equations, (29a) - (29c), from the Cartesian coordinates to the cylindrical or spherical coordinates gives the hydrodynamic model which can be recovered from the kinetic model (10) or (22) in the continuum limit. The Navier-Stokes equations with spherical symmetry are only one dimensional and read

\[ \partial_t \rho + (\partial_r + \frac{2}{r}) (\rho u) = 0, \quad (35a) \]
\[ \partial_r u + u \partial_r u + \frac{1}{\rho} \partial_r p = \frac{4}{\rho r} \mu \left( \partial_r u_r - \frac{u_r}{r} \right) - \frac{1}{\rho} \partial_r \left[ \mu (1 - \lambda) \left( \partial_r + \frac{2}{r} \right) u_r - 2\mu \partial_r u_r \right], \quad (35b) \]
\[ \partial_t e + u \partial_r e + \frac{p}{\rho} \left[ (\partial_r + \frac{2}{r}) u \right] = \frac{1}{\rho} (\partial_r + \frac{2}{r}) (k \partial_r T) + \frac{2\mu}{\rho} \left[ (\partial_r u)^2 + 2 \left( \frac{u_r}{r} \right)^2 \right] - \frac{1}{\rho} \mu (1 - \lambda) \left[ (\partial_r + \frac{2}{r}) u \right]^2, \quad (35c) \]
which can be recovered from the kinetic model \(^{(28)}\). The Navier-Stokes equations with local cylindrical symmetry are two dimensional and read

\[
\frac{\partial}{\partial t} \rho + (\frac{\partial}{\partial r} + 1/r) (\rho u_r) + \frac{\partial}{\partial z} (\rho u_z) = 0
\]  
(36a)

\[
\frac{\partial}{\partial t} u_r + u_r \frac{\partial}{\partial r} u_r + u_z \frac{\partial}{\partial z} u_r + \frac{1}{\rho} \frac{\partial}{\partial r} p
\]

\[
= \frac{1}{\rho} \frac{\partial}{\partial r} \left[ 2 \mu \frac{\partial}{\partial r} u_r - \mu (1 - \lambda) (\frac{\partial}{\partial r} u_r + \frac{u_r}{r} + \partial_z u_z) \right]
\]

\[
+ \frac{\mu}{\rho} \frac{2 \partial_r u_r - 2 u_r / r}{r} + \partial_z (\partial_z u_r + \partial_r u_z)
\]

(36b)

\[
\frac{\partial}{\partial t} u_z + u_r \frac{\partial}{\partial r} u_z + u_z \frac{\partial}{\partial z} u_z + \frac{1}{\rho} \frac{\partial}{\partial z} p
\]

\[
= \frac{\mu}{\rho} \frac{\partial}{\partial r} (\partial_z u_r + \partial_r u_z) - \frac{\mu (\partial_z u_r + \partial_r u_z)}{\rho} \frac{u_r}{r}
\]

\[
+ \frac{1}{\rho} \partial_z \left[ 2 \mu \frac{\partial_z u_z}{r} - \mu (1 - \lambda) (\frac{\partial}{\partial r} u_r + \frac{u_r}{r} + \partial_z u_z) \right]
\]

(36c)

\[
\frac{\partial}{\partial t} e + u_r \frac{\partial}{\partial r} e + u_z \frac{\partial}{\partial z} e + \frac{1}{\rho} \frac{\partial}{\partial r} (\rho u_r + \frac{u_r}{r} + \partial_z u_z)
\]

\[
= -\frac{1}{\rho} \mu (1 - \lambda) (\frac{\partial}{\partial r} u_r + \frac{u_r}{r} + \partial_z u_z)^2 + \frac{2}{\rho} \mu \frac{\partial}{\partial r} u_r \frac{\partial}{\partial r} u_r
\]

\[
+ \frac{2}{\rho} \mu \frac{u_r}{r} \frac{u_r}{r} + \frac{1}{\rho} \mu (\frac{\partial}{\partial z} u_r + \frac{\partial}{\partial r} u_z)^2
\]

\[
+ \frac{1}{\rho} \mu \frac{\partial_z u_z}{r} \frac{\partial_z u_z}{r} + \frac{1}{\rho} \left[ (\frac{\partial}{\partial r} + \frac{1}{r}) k \frac{\partial}{\partial r} T + \partial_z k \partial_z T \right]
\]

(36d)

which can be recovered from the kinetic model \(^{(13)}\).

C. Measurements of nonequilibrium effects

The Chapman-Enskog multiscale analysis tells that, as the simplest hydrodynamic model of fluid system, the Euler equations ignore completely the Thermodynamic NonEquilibrium (TNE) behavior. The Navier-Stokes equations describe the TNE behavior via the terms in viscosity and heat conductivity. The Euler model works successfully when we consider the fluid system in a time scale \(t_0\) which is large enough compared with thermodynamic relaxation time \(\tau\). Besides the normal high speed compressible flows, the Euler model works also for solid materials under strong shock. From the mechanical side, compared with the
shocking strength, the material strength, for example, the yield, and viscous stress are negligible. Consequently, the Euler model works better with increasing the shock strength. From the side of time scales, when study the shocking procedure, the used time scale \( t_0 \) is generally small enough compared with the time scale \( t_h \) for heat conduction and large enough compared with the thermodynamic relaxation time \( \tau \). In other words, during the time interval under investigation, the heat conduction does not have time to occur significantly and consequently its effects are negligible. For the objective system where the thermodynamic relaxation time \( \tau \) is fixed, if we decreases the observing time scale \( t_0 \), we find more TNE effects. The Boltzmann kinetic model can be used to investigate both the hydrodynamic and thermodynamic behaviors.

Following the seven moment relations, (32a)-(32g), used in recovering the Navier-Stokes equations, we define the following moments,

\[
M_0^*(f,v) = \int \int f \, dv d\eta, \quad (37a)
\]

\[
M_1^*(f,v) = \int \int f \, v dv d\eta, \quad (37b)
\]

\[
M_{2,0}^*(f,v) = \int \int f \, (v \cdot v + \eta^2) \, dv d\eta, \quad (37c)
\]

\[
M_2^*(f,v) = \int \int f \, vv \, dv d\eta, \quad (37d)
\]

\[
M_{3,1}^*(f,v) = \int \int f \, (v \cdot v + \eta^2) \, v \, dv d\eta, \quad (37e)
\]

\[
M_3^*(f,v) = \int \int f \, vvv \, dv d\eta, \quad (37f)
\]

\[
M_{4,2}^*(f,v) = \int \int f \, (v \cdot v + \eta^2) \, vvv \, dv d\eta, \quad (37g)
\]

where \( M_n^* \) means a \( n \)-th order tensor and \( M_{m,n}^* \) means a \( n \)-th-order tensor contracted from a \( m \)-th order tensor. For the case of central moments, the variable \( v \) is replaced with \( v^* = (v - u) \). It is clear \( M_0^* \) and \( M_{2,0}^* \) are scalars. Each of them has only 1 component. \( M_1^* \) and \( M_{3,1}^* \) are vectors. Each of them has 2 independent components in 2-dimensional case or 3 independent components in 3-dimensional case. \( M_2^* \) and \( M_{4,2}^* \) are 2nd order tensors. Each of them has 3 independent components in 2-dimensional case or 6 independent components in 3-dimensional case. \( M_3^* \) is 3rd tensor and has 4 independent components in 2-dimensional case or 10 independent components in 3-dimensional case. Therefore, the constraints, (32a)
are in fact 16 linear equations in \( f^{eq} \) in 2-dimensional case and 30 linear equations in \( f^{eq} \) in 3-dimensional case. We further define

\[
\Delta_{m,n}^*(v) = M_{m,n}^* (f, v) - M_{m,n}^* (f^{eq}, v).
\] (38)

It is clear that \( \Delta_0^*(v) = 0 \), \( \Delta_1^*(v) = 0 \) and \( \Delta_{2,0}^*(v) = 0 \), which is due to the mass, momentum and energy conservations. Except for the three, the quantity \( \Delta_{m,n}^*(v) \) works as a measure for the deviation of the system from its thermodynamic equilibrium. The information of flow velocity \( u \) is taken into account in the definition (38). Similarly,

\[
\Delta_{m,n}^*(v^*) = M_{m,n}^* (f, v^*) - M_{m,n}^* (f^{eq}, v^*).
\] (39)

Except for \( \Delta_0^*(v^*) \), \( \Delta_1^*(v^*) \) and \( \Delta_{2,0}^*(v^*) \), the quantity \( \Delta_{m,n}^*(v^*) \) works as a measure for the deviation of the system from its thermodynamic equilibrium, where only the thermal fluctuations of the molecules are considered.

## III. DISCRETE BOLTZMANN MODELS

There are two key techniques in constructing DBM with force terms. The first is to approximate \( f \) by \( f^{eq} \) in the force term so that the velocity derivative of \( f \), \( \partial f / \partial v \), can be analytically calculated before introducing the Discrete Velocity Model (DVM). The second is that the DVM can be fixed before the main interaction in the code. In other words, the discrete velocities \( v_i \) do not need to be recalculated in each iteration step.

For constructing the DBM for systems with cylindrical symmetry, we use Eq. (13). We have

\[
\partial_t f_i + \left( v_{ir} \frac{\partial f_i}{\partial r} + v_{iz} \frac{\partial f_i}{\partial z} \right) + \left[ \frac{v_{ir} v_{i\theta}^2}{r T} - \frac{v_{i\theta}^2 (v_{ir} - u_r)}{r T} \right] f_i^{eq} = -\frac{1}{\tau} (f_i - f_i^{eq}).
\] (40)

where \( f_i \) (\( f_i^{eq} \)) is the discrete (equilibrium) distribution function; \( v_i \) is the \( i \)-th discrete velocity, \( i = 1, ..., N \); \( N \) is the total number of the discrete velocity. For constructing the DBM for systems with spherical symmetry, we use Eq. (28). We have

\[
\partial_t f_i + v_{ir} \frac{\partial f_i}{\partial r} + \left[ \frac{v_{ir} v_{i\theta}^2}{r T} + \frac{v_{ir} v_{i\phi}^2}{r T} - \frac{(v_{i\theta}^2 + v_{i\phi}^2) (v_{ir} - u_r)}{r T} \right] f_i^{eq} = -\frac{1}{\tau} (f_i - f_i^{eq}).
\] (41)

The fundamental requirement for a DBM is that it should recover the same set of hydrodynamic equations as those given by the original continuous Boltzmann equation. The
Chapman-Enskog multiscale analysis shows that, to formulate a DBM, we need only require that the used moment relations in continuous form can be rewritten in discrete form. To formulate a DBM which recovers the Euler equations, the following five constraints are needed,

\[ \rho = \sum_{i=1}^{N} f_i^{eq} = \sum_{i=1}^{N} f_i , \]  
\[ \rho u_\alpha = \sum_{i=1}^{N} f_i^{eq} v_{i\alpha} = \sum_{i=1}^{N} f_i v_{i\alpha} , \]  
\[ 2\rho \left( \frac{D + n}{D} e + \frac{u^2}{2} \right) = \sum_{i=1}^{N} f_i^{eq} (v_i^2 + \eta_i^2) = \sum_{i=1}^{N} f_i (v_i^2 + \eta_i^2) , \]  
\[ P \delta_{\alpha\beta} + \rho u_\alpha u_\beta = \sum_{i=1}^{N} f_i^{eq} v_{i\alpha} v_{i\beta} , \]  
\[ 2\rho \left( \frac{D + n + 2}{D} e + \frac{u^2}{2} \right) u_\alpha = \sum_{i=1}^{N} f_i^{eq} (v_i^2 + \eta_i^2) v_{i\alpha} , \]

To formulate a DBM which recovers the Navier-Stokes equations, two more constraints,

\[ \rho RT (u_\alpha \delta_{\beta\chi} + u_\beta \delta_{\alpha\chi} + u_\chi \delta_{\alpha\beta} + \rho u_\alpha u_\beta u_\chi = \sum_{i=1}^{N} f_i^{eq} v_{i\alpha} v_{i\beta} v_{i\chi} , \]  
\[ \frac{4}{D} \rho e \left( \frac{D + n + 2}{D} e + \frac{u^2}{2} \right) \delta_{\alpha\beta} + 2\rho u_\alpha u_\beta \left( \frac{D + n + 4}{D} e + \frac{u^2}{2} \right) = \sum_{i=1}^{N} f_i^{eq} (v_i^2 + \eta_i^2) v_{i\alpha} v_{i\beta} , \]

are needed. Following the same idea as in the definitions, (37a) - (37g), we define the following moments of the discrete distribution function \( f_i \),

\[ M_0(f_i, v_i) = \sum_{i=1}^{N} f_i , \]  
\[ M_1(f_i, v_i) = \sum_{i=1}^{N} f_i v_i , \]  
\[ M_{2,0}(f_i, v_i) = \sum_{i=1}^{N} f_i (v_i \cdot v_i + \eta_i^2) , \]  
\[ M_2(f_i, v_i) = \sum_{i=1}^{N} f_i v_i v_i , \]  
\[ M_{3,1}(f_i, v_i) = \sum_{i=1}^{N} f_i (v_i \cdot v_i + \eta_i^2) v_i , \]
\[ M_3(f_i, v_i) = \sum f_i v_i v_i, \quad (43f) \]
\[ M_{4,2}(f_i, v_i) = \sum f_i (v_i \cdot v_i + \eta_i^2) v_i v_i, \quad (43g) \]
where \( M_n \) means a \( n \)-th order tensor and \( M_{m,n} \) means a \( n \)-th-order tensor contracted from a \( m \)-th order tensor. For the case of central moments, the variable \( v \) is replaced with \( v^* = (v - u) \). The constraints, (42a) - (42g), are in fact 16 linear equations in \( f_i^{eq} \) in 2-dimensional case and 30 linear equations in \( f_i^{eq} \) in 3-dimensional case. Following the same idea as in the definitions, (38) - (39), we further define
\[ \Delta_{m,n}(v_i) = M_{m,n}(f_i, v_i) - M_{m,n}(f_i^{eq}, v_i). \quad (44) \]
\[ \Delta_{m,n}(v_i^*) = M_{m,n}(f_i, v_i^*) - M_{m,n}(f_i^{eq}, v_i^*). \quad (45) \]
Except for \( \Delta_0, \Delta_1 \) and \( \Delta_2,0 \), the quantity \( \Delta_{m,n} \) works as a measure for the deviation of the system from its thermodynamic equilibrium.

A key step in formulating a DBM is to find a solution for the discrete equilibrium distribution function \( f_i^{eq} \). The constraints, (42a) - (42g) can also be rewritten as
\[ \tilde{f}^{eq} = Cf^{eq} \quad (46) \]
where \( \tilde{f}^{eq} = [f_k^{eq}]^T \) and \( f^{eq} = [f_k^{eq}]^T \) are column vectors with \( k = 1, 2, \cdots, N \), \( C \) is \( N \times N \) matrix whose components are determined by \( v_i \) if the parameter \( \eta_i \) is fixed. It is clear that
\[ f^{eq} = C^{-1} \tilde{f}^{eq}. \quad (47) \]
Obviously, the choosing of the DVM must ensure the existence of \( C^{-1} \). We work in the frame where the particle mass \( m = 1 \) and the constant \( R = 1 \).

A. DBM for systems with spherical symmetry

If we require the DBM to recover the Navier-Stokes equations in the continuum limit, the DBM needs a DVM with 3 dimensions.

1. Case with \( \gamma = 5/3 \)

We first consider the simple case where ratio of specific rates is fixed, \( \gamma = 5/3 \). We set \( \eta_i = 0 \) and \( n = 0 \) in constraint (42g). Among the seven moment constraints, (42a) -
only five are independent. We do not use the constraints (42d) and (42e). The five independent constraints can be rewritten as 26 independent linear equations in \( f^e_i \). Now, we fix the components \( \hat{f}^e_k \) of \( \hat{f}^e \). Here \( N = 26 \).

From the constraint (42a), we have \( \hat{f}^e_1 = \rho \). From the constraint (42b), we have \( \hat{f}^e_2 = \rho u_r \), \( \hat{f}^e_3 = \rho u_\theta \), \( \hat{f}^e_4 = \rho u_\phi \). From the constraint (42c), we have \( \hat{f}^e_5 = P + \rho u_r^2 \), \( \hat{f}^e_6 = \rho u_r u_\theta \), \( \hat{f}^e_7 = \rho u_r u_\phi \), \( \hat{f}^e_8 = P + \rho u_\theta^2 \). From the constraint (42d), we have \( \hat{f}^e_9 = \rho (T u_\theta + u_r u_\phi) \), \( \hat{f}^e_{10} = \rho (T u_\phi + u_r u_\theta) \), \( \hat{f}^e_{11} = \rho [T (3u_r) + u_\theta^3] \), \( \hat{f}^e_{12} = \rho (T u_\theta + u_r^2 u_\phi) \), \( \hat{f}^e_{13} = \rho (T u_\phi + u_r^2 u_\theta) \), \( \hat{f}^e_{14} = \rho (T u_\phi + u_\theta^2 u_\phi) \), \( \hat{f}^e_{15} = \rho (u_r u_\theta u_\phi) \), \( \hat{f}^e_{16} = \rho (T u_\theta + u_r u_\phi^2) \), \( \hat{f}^e_{17} = \rho [T (3u_\theta) + u_\phi^3] \), \( \hat{f}^e_{18} = \rho (T u_\phi + u_\theta^2 u_\phi) \), \( \hat{f}^e_{19} = \rho [T u_\phi + u_\theta^2 u_\theta] \), \( \hat{f}^e_{20} = \rho [T (3u_\phi) + u_\theta^3] \). From the constraint (42g), we have \( \hat{f}^e_{21} = \rho T (5T + u^2) + \rho u_r^2 (7T + u^2) \), \( \hat{f}^e_{22} = \rho u_r u_\theta (7T + u^2) \), \( \hat{f}^e_{23} = \rho u_r u_\phi (7T + u^2) \), \( \hat{f}^e_{24} = \rho T (5T + u^2) + \rho u_\theta^2 (7T + u^2) \), \( \hat{f}^e_{25} = \rho u_\theta u_\phi (7T + u^2) \), \( \hat{f}^e_{26} = \rho T (5T + u^2) + 2\rho u_\phi^2 (7T + u^2) \).

Since the system is spherically symmetric in macroscopic scale, \( u_\theta = u_\phi = 0 \) and \( u^2 = u_r^2 \) in the above expressions for \( \hat{f}^e = \left[ \hat{f}^e_1, \hat{f}^e_2, \ldots, \hat{f}^e_N \right]^T \).

The components of the matrix \( C = [C_{ki}] = [C_{ki}] \) should be fixed in the same sequence, where \( k = 1, 2, \ldots, 26 \) and \( i = 1, 2, \ldots, 26 \). From the constraint (42a), we have \( C_{1i} = 1 \). From the constraint (42b), we have \( C_{2i} = v_{ir} \), \( C_{3i} = v_{i\theta} \), \( C_{4i} = v_{i\phi} \). From the constraint (42c), we have \( C_{5i} = v_{ir}^2 \), \( C_{6i} = v_{ir} v_{i\theta} \), \( C_{7i} = v_{ir} v_{i\phi} \), \( C_{8i} = v_{i\theta}^2 \), \( C_{9i} = v_{i\theta} v_{i\phi} \), \( C_{10i} = v_{i\phi}^2 \). From the constraint (42d), we have \( C_{11i} = v_{ir}^3 \), \( C_{12i} = v_{ir}^2 v_{i\theta} \), \( C_{13i} = v_{ir}^2 v_{i\phi} \), \( C_{14i} = v_{ir} v_{i\theta}^2 \), \( C_{15i} = v_{ir} v_{i\theta} v_{i\phi} \), \( C_{16i} = v_{ir} v_{i\theta}^2 \), \( C_{17i} = v_{ir} v_{i\theta}^3 \), \( C_{18i} = v_{i\theta}^2 v_{i\phi} \), \( C_{19i} = v_{i\theta} v_{i\phi}^2 \), \( C_{20i} = v_{i\phi}^3 \). From the constraint (42g), we have \( C_{21i} = \left( v_{ir}^2 + v_{i\theta}^2 + v_{i\phi}^2 \right) v_{ir} \), \( C_{22i} = \left( v_{ir}^2 + v_{i\theta}^2 + v_{i\phi}^2 \right) v_{i\theta} \), \( C_{23i} = \left( v_{ir}^2 + v_{i\theta}^2 + v_{i\phi}^2 \right) v_{i\phi} \), \( C_{24i} = \left( v_{ir}^2 + v_{i\theta}^2 + v_{i\phi}^2 \right) v_{ir} v_{i\theta} \), \( C_{25i} = \left( v_{ir}^2 + v_{i\theta}^2 + v_{i\phi}^2 \right) v_{ir} v_{i\phi} \), \( C_{26i} = \left( v_{ir}^2 + v_{i\theta}^2 + v_{i\phi}^2 \right) v_{i\theta} v_{i\phi} \).

An example for the 3-Dimensional 26-Velocity(D3V26) DVM is as below,

\[
\mathbf{v}_i = \begin{cases} 
(0, \pm 1, \pm 1) & c_1 \quad i = 1, \ldots, 4 \\
(\pm 1, 0, \pm 1) & c_1 \quad i = 5, \ldots, 8 \\
(\pm 1, \pm 1, 0) & c_1 \quad i = 9, \ldots, 12 \\
(\pm 1, \pm 1, \pm 1) & c_2 \quad i = 13, \ldots, 20 \\
(\pm 1, 0, 0) & c_3 \quad i = 21, 22 \\
(0, \pm 1, 0) & c_3 \quad i = 23, 24 \\
(0, 0, \pm 1) & c_3 \quad i = 25, 26
\end{cases}
\]

A specific example of the DVM and the expressions for the inverse of the matrix \( C \),
\( C^{-1} = [C_k^{-1}] \), are shown in the appendix A. Up to this step, a the special discretization of the velocity space has been performed. Consequently, the DBM for system with spherical symmetry and \( \gamma = 5/3 \) has been constructed. The spatial and temporal derivatives of the distribution function in the kinetic model can be calculated in the normal way. If we are not interested in the extra degrees of freedom other than the translational, the formulated DBM can be used to study the hydrodynamic and the thermodynamic behaviors of the compressible flow system. In this case, \( E = e \) in the hydrodynamic energy equation.

2. Case with flexible \( \gamma \)

For the case with flexible \( \gamma \), if we are interested only in the hydrodynamic behaviors, we can use the simulation results of the DBM formulated in last subsection. Just get the number \( n \) of the extra degree of freedom using its relation to \( \gamma \), then obtain the total internal energy \( E \) using its definition. If we are interested also in the thermodynamic nonequilibrium behavior, we need continue the formulation of the DBM.

To model the case with flexible ratio of specific heats, we resort to the parameter \( \eta_i \) to describe the contribution of extra degrees of freedom. From the constraints (42d) and (42c) we have

\[
    n\rho RT = \sum_{i=1}^{N} f_i^{eq} \eta_i^2,
\]

(49a)

From the constraints (42e) and (42f), we have

\[
    n\rho RT u_\alpha = \sum_{i=1}^{N} f_i^{eq} v_{i\alpha} \eta_i^2,
\]

(49b)

From the constraint (42g) we have

\[
    n\rho RT (RT \delta_{\alpha\beta} + u_\alpha u_\beta) = \sum_{i=1}^{f_i^{eq}} f_{i\alpha} v_{i\beta} \eta_i^2.
\]

(49c)

The constraints (49a) - (49c) compose 10 linear equations for the variable \( \eta_i^2 \). It is clear that we can set \( \eta_i = 0 \) for \( i = 11, \cdots, 26 \). Therefore, once the DBM for the system with \( \gamma = 5/3 \) are fixed, we can further fixed \( \eta_i^2 \). Via replacing the translational kinetic energy \( e \) with the total internal kinetic energy \( E \), the D2V26-DBM can be used to model system with flexible ratio of specific heats.

The constraints (49a) - (49c) can be rewritten as

\[
    \hat{g} = Dg
\]

(50)
where

\[ \hat{g} = [\hat{g}_k]^T \]
\[ \begin{align*}
\hat{g} &= n\rho RT [1, u_r, u_\theta, u_\varphi, RT + u_r^2, u_r, u_\theta, u_r, RT + u_\varphi^2] 
\end{align*} \]  

(51)

and

\[ g = [g_k]^T \]
\[ g = [\eta_i^2, i = 1, \ldots, 10] \]  

(52)

are column vectors with \( k = 1, 2, \ldots, 10 \), \( D = [D_k] = [D_{ki}] \) is \( 10 \times 10 \) matrix whose components are determined by \( f_{eq}^i \) and \( v_{i\alpha} \). Specifically, \( D_1 = [f_{eq}^i] \), \( D_2 = [f_{eq}^i v_{ir}] \), \( D_3 = [f_{eq}^i v_{i\varphi}] \), \( D_4 = [f_{eq}^2 v_{i\varphi}] \), \( D_5 = [f_{eq} v_{ir}^2] \), \( D_6 = [f_{eq} v_{ir} v_{i\varphi}] \), \( D_7 = [f_{eq} v_{ir} v_{i\varphi}] \), \( D_8 = [f_{eq} v_{i\varphi}^2] \), \( D_9 = [f_{eq} v_{i\varphi} v_{i\varphi}] \), \( D_{10} = [f_{eq}^2 v_{i\varphi}] \).

Since \( f_{eq}^i \) has been determined via

\[ f_{eq} = C^{-1} \hat{f}_{eq} \]  

(53)

finally,

\[ g = D^{-1} \hat{g} \]  

(54)

Via substituting the results \( \eta_i^2 \) to Eqs. (43d) - (43g), then using the Eqs. (44) and (45), the thermodynamic nonequilibrium behavior can be investigated. Up to this step, a DBM with D2V26 for system with flexible specific heat ratio \( \gamma \) has been formulated.

**B. DBM for systems with cylindrical symmetry**

When the system depends also on the value of \( z \), the DBM needs a 3-dimensional DVM with \( N = 26 \). The DVM can be similar to the case with spherical symmetry. The only difference is that the subscript “\( \varphi \)” should be replaced with “\( z \)”.

If the system is also independent of the height \( z \), The discrete Boltzmann equation (40) is simplified as

\[ \partial_t f_i + v_{ir} \frac{\partial f_i}{\partial r} + \left( \frac{v_{ir} v_{i\varphi}^2}{rT} - \frac{v_{i\varphi}^2 (v_{ir} - u_r)}{rT} \right) f_{eq}^i = -\frac{1}{\tau} \left( f_i - f_{eq}^i \right) . \]  

(55)

A complete DBM needs also a DVM with 3 dimensions. But when the degree of freedom in \( z \) can be ignored, to recover the Navier-Stokes equations the continuum limit, the DVM
needs only 2 Dimensions and 13 Velocities(D2V13). In this formulation scheme, we need
the minimum number of discrete velocities, but have to add extra calculations, as shown by
Eq. (54), to fix \( \eta \).

In this section we consider a slightly different scheme. We use a DVM with 2 Dimensions
and 16 Velocities (D2V16) to formulate the DBM which recovers the Navier-Stokes equations
in the continuum limit. From the constraints (42a) - (42g), we choose
\[
\hat{f}_{eq} = \left[ \hat{f}_{eq}^1, \hat{f}_{eq}^2, \ldots, \hat{f}_{eq}^{16} \right]^T,
\]
where \( \hat{f}_{eq}^1 = \rho, \hat{f}_{eq}^2 = \rho u_r, \hat{f}_{eq}^3 = \rho u_\theta, \hat{f}_{eq}^4 = \rho \left[ (n + 2) RT + (u_r^2 + u_\theta^2) \right], \hat{f}_{eq}^5 = P + \rho u_r^2, \hat{f}_{eq}^6 = 
\rho u_r u_\theta, \hat{f}_{eq}^7 = P + \rho u_\theta^2, \hat{f}_{eq}^8 = \rho \left[ (n + 4) RT + (u_r^2 + u_\theta^2) \right] u_r, \hat{f}_{eq}^9 = \rho \left[ (n + 4) RT + (u_r^2 + u_\theta^2) \right] u_\theta, \hat{f}_{eq}^{10} = \rho \left( RT (3u_r + u_r^3) \right), \hat{f}_{eq}^{11} = \rho \left( RT u_\theta + u_\theta^2 u_\theta \right), \hat{f}_{eq}^{12} = \rho \left( RT u_r + u_\theta^2 u_r \right), \hat{f}_{eq}^{13} = \rho \left( RT (3u_\theta + u_\theta^3) \right), \hat{f}_{eq}^{14} = \rho RT \left[ (n + 4) RT + u_r^2 \right] + \rho u_r^2 \left[ (n + 6) RT + u^2 \right], \hat{f}_{eq}^{15} = \rho u_r u_\theta \left[ (n + 6) RT + u^2 \right], \hat{f}_{eq}^{16} = \rho RT \left[ (n + 4) RT + u^2 \right] + \rho u_\theta^2 \left[ (n + 6) RT + u^2 \right].
\]
Since the system is cylindrically symmetric in macroscopic scale and independent of the value of
\( z \), \( u_\theta = u_z = 0 \) in the above expressions for \( \hat{f}_{eq}^i \).

In the same sequence, we fix the matrix \( C = [C_k] = [C_{ki}] \) below. \( C_{1i} = 1, \ C_{2i} = v_{ir}, \ C_{3i} = v_{i\theta}, \ C_{4i} = (v_{ir}^2 + v_{i\theta}^2 + \eta_i^2), \ C_{5i} = v_{ir}^2, \ C_{6i} = v_{ir} v_{i\theta}, \ C_{7i} = v_{i\theta}^2, \ C_{8i} = (v_{ir}^2 + v_{i\theta}^2 + \eta_i^2) v_{ir}, \ C_{9i} = (v_{ir}^2 + v_{i\theta}^2 + \eta_i^2) v_{i\theta}, \ C_{10i} = v_{ir}^3, \ C_{11i} = v_{ir}^2 v_{i\theta}, \ C_{12i} = v_{ir} v_{i\theta}^2, \ C_{13i} = v_{ir}^3, \ C_{14i} = (v_{ir}^2 + v_{i\theta}^2 + \eta_i^2) v_{ir}, \ C_{15i} = (v_{ir}^2 + v_{i\theta}^2 + \eta_i^2) v_{i\theta}, \ C_{16i} = (v_{ir}^2 + v_{i\theta}^2 + \eta_i^2) v_{ir} v_{i\theta} \).

It should be pointed out that, in this formulation scheme, the values of \( \eta_i \) have to be
chosen in such a way that the inverse of the matrix \( C \) exists.

An example of the D2V16 is as below,
\[
v_i = \begin{cases} (\pm1,0) c_1 \\ (0,\pm1) c_1 & , i = 1, \ldots, 8 \\ (\pm1,\pm1) c_2 \end{cases}
\]
\[
v_i = \begin{cases} (\pm1,0) c_3 \\ (0,\pm1) c_3 & , i = 9, \ldots, 16. \\ (\pm1,\pm1) c_4 \end{cases}
\]
A specific example of the DVM with specific \( \eta_i \) and the corresponding components of \( C^{-1} = [C_{ki}^{-1}] \) are shown in appendix B.
IV. CONCLUSION AND DISCUSSIONS

We present a theoretical framework for constructing discrete Boltzmann model in spherical or cylindrical coordinates for the compressible flow systems with spherical or cylindrical symmetry. A common property of the two kinds symmetric systems is that the behavior of the system is independent of the azimuthal coordinate. A key technique here is to use local Cartesian coordinates to describe the particle velocity. Thus, in the Boltzmann equation of such a system the geometric effects, like the divergence and convergence, are described as a “force term”.

For a system with spherical or global cylindrical symmetry, the hydrodynamic equations depend only on the radial coordinate. The hydrodynamic equations for system with local cylindrical symmetry may depend also on the height $z$. For such a system, even though the hydrodynamic models are one- or two-dimensional, the discrete Boltzmann model needs a DVM with 3 dimensions. We use a DVM with 26 velocities to formulate the DBM which recovers the Navier-Stokes equations with spherical or cylindrical symmetry in the hydrodynamic limit.

For the system with global cylindrical symmetry, we formulate also a DBM based on a DVM with 2 dimensions and 16 velocities. But it should be pointed out that, the system described by the DBM based on D2V16 is not the same as that described by the DBM based on D3V26. For example, the equation of state, $P = 2\rho e/D$, is different for the two kinds of systems.

Besides recovering the hydrodynamic equations in the continuum limit, the more important point for a DBM is that, in terms of the nonconserved moments, we can define two sets of measures for the deviation of the system from its thermodynamic equilibrium state. The measurements are clarified. With the DBM we can study simultaneously both the hydrodynamic and thermodynamic behaviors. Since the inverse of the transformation matrix $C$ connecting the discrete equilibrium distribution function $f^{eq}$ and corresponding moments $\hat{f}^{eq}$ has been fixed, the extension to multiple-relaxation-time DBM[27, 49] is straightforward. It should also be pointed out that, fixing the transformation matrix $C$ and its inverse is only one of the possible schemes to get a solution for the discrete equilibrium distribution function $f^{eq}$. A second way to find a solution for the discrete equilibrium distribution function $f^{eq}$ is to follow the ideas used in Refs. [16, 18]. A difference is that the scheme introduced in this
work needs the minimum number of discrete velocities.

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Appendix A: $C^{-1}$ for the D3V26

Specifically, we can choose

$$v_{ir} = c_1 [0, 0, 0, 0, 1, 1, -1, -1, 1, 1, -1, -1]$$
$$v_{i\theta} = c_1 [1, 1, -1, -1, 0, 0, 0, 1, -1, -1, 1, 1]$$
$$v_{i\phi} = c_1 [1, -1, -1, 1, 1, -1, -1, 1, 0, 0, 0]$$

for $i = 1, \ldots, 12$, \hspace{1cm} (58a)

$$v_{ir} = c_2 [1, 1, 1, 1, -1, -1, -1, 1, -1, -1]$$
$$v_{i\theta} = c_2 [1, 1, -1, -1, 1, 1, -1, -1, 1, 1]$$
$$v_{i\phi} = c_2 [1, -1, -1, 1, 1, -1, -1, 1, 0, 0]$$

for $i = 13, \ldots, 20$, \hspace{1cm} (58b)

$$v_{ir} = c_3 [1, -1, 0, 0, 0, 0]$$
$$v_{i\theta} = c_3 [0, 0, 1, -1, 0, 0]$$
$$v_{i\phi} = c_3 [0, 0, 0, 1, -1]$$

for $i = 21, \ldots, 26$, \hspace{1cm} (58c)
The components of the inverse of \( C, \ C^{-1} = [C_k^{-1}] \), are shown below.

\[
C_1^{-1} = [1/4 \frac{c_3^2 c_2^2}{(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)}, 0, -1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)},
-1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)}, -1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)},
0, -1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)}, -3/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - 3c_2^2)},
-1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)}, 0, -1/4 \frac{c_3^2 c_2^2}{c_1^3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)},
-1/4 \frac{c_3^2 c_2^2}{c_1^3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)}, 0, 0, 0, 1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)},
1/4 \frac{c_3^2 c_2^2}{c_1^3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)}, 1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)},
1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)}, -1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)}, 0, 1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)},
0, 0, 1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)}, 1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)},
0, 0, 1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)}, 1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - 3c_2^2)} \] (59a)

\[
C_2^{-1} = [1/4 \frac{c_3^2 c_2^2}{(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)}, 0, -1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)},
1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)}, -1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)},
0, -1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)}, -3/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - 3c_2^2)},
-1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)}, 0, -1/4 \frac{c_3^2 c_2^2}{c_1^3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)},
-1/4 \frac{c_3^2 c_2^2}{c_1^3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)}, 0, 0, 0, 1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)},
1/4 \frac{c_3^2 c_2^2}{c_1^3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)}, 1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)},
1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)}, -1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)}, 0, 1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)},
0, 0, 1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)}, 1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)},
0, 0, 1/4 \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)}, 1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - 3c_2^2)} \] (59b)
\[ C_1^{-1} = \frac{1}{14} \begin{pmatrix} \frac{c_3^2c_2^2}{(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} & 0 & 0 & 1/4 \frac{c_3^2c_2^2}{c_1(c_1^2c_2^2 + c_1^2c_3^2 - 2c_3^2c_2^2)} \\
-1/4 \frac{c_2^2}{c_1(c_1^2c_2^2 + c_1^2c_3^2 - 2c_3^2c_2^2)} & -1/4 \frac{c_2^2}{c_1^2(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} & -3/4 \frac{c_2^2}{c_1^3(c_1^2c_2^2 + c_1^2c_3^2 - 2c_3^2c_2^2)} \\
-1/4 \frac{c_2^2}{c_1^3(c_1^2c_2^2 + c_1^2c_3^2 - 2c_3^2c_2^2)} & -1/4 \frac{c_2^2}{c_1^2 - 2c_2^2} & -1/4 \frac{c_2^2}{c_1^2(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} & 1/4 \frac{1}{c_1^2(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \end{pmatrix} \]  

\[ C_4^{-1} = \frac{1}{14} \begin{pmatrix} \frac{c_3^2c_2^2}{(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} & 0 & 0 & 1/4 \frac{c_3^2c_2^2}{c_1(c_1^2c_2^2 + c_1^2c_3^2 - 2c_3^2c_2^2)} \\
-1/4 \frac{c_2^2}{c_1(c_1^2c_2^2 + c_1^2c_3^2 - 2c_3^2c_2^2)} & -1/4 \frac{c_2^2}{c_1^2(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} & -3/4 \frac{c_2^2}{c_1^3(c_1^2c_2^2 + c_1^2c_3^2 - 2c_3^2c_2^2)} \\
-1/4 \frac{c_2^2}{c_1^3(c_1^2c_2^2 + c_1^2c_3^2 - 2c_3^2c_2^2)} & -1/4 \frac{c_2^2}{c_1^2 - 2c_2^2} & -1/4 \frac{c_2^2}{c_1^2(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} & 1/4 \frac{1}{c_1^2(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \end{pmatrix} \]
\[ C_5^{-1} = \begin{bmatrix} 
\frac{1}{4} \frac{c_3^2 c_2^2}{(2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} - \frac{1}{4} \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & 0, \\
-\frac{1}{4} \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - \frac{1}{4} \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & 0, \\
-\frac{3}{4} \frac{c_1^2 (2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} - \frac{1}{4} \frac{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} & 0, \\
-\frac{1}{4} \frac{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} - \frac{1}{4} \frac{c_1^2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}{c_1^2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \\
0, -\frac{3}{4} \frac{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} & 0, \\
0, - \frac{1}{4} \frac{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} & 0, \\
0, - \frac{1}{4} \frac{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} & 0, \\
\end{bmatrix} \] (59e)

\[ C_6^{-1} = \begin{bmatrix} 
\frac{1}{4} \frac{c_3^2 c_2^2}{(2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} - \frac{1}{4} \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & 0, \\
\frac{1}{4} \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - \frac{1}{4} \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & 0, \\
0, - \frac{3}{4} \frac{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} - \frac{1}{4} \frac{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} & 0, \\
-\frac{1}{4} \frac{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} - \frac{1}{4} \frac{c_1^2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}{c_1^2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \\
0, -\frac{3}{4} \frac{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} & 0, \\
0, - \frac{1}{4} \frac{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} & 0, \\
0, - \frac{1}{4} \frac{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} & 0, \\
\frac{1}{4} \frac{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}{c_1^2 (2 c_1^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} & 0 \\
\end{bmatrix} \] (59f)
\[ C_{7}^{-1} = \left[ \frac{1}{4} \frac{c_2^2 c_3^2}{(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \right] \left[ \frac{1}{4} \frac{c_1^2 c_2^2}{(c_1^2 - c_3^2)(2c_1^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)} \right] \]

\[ = \left[ \frac{1}{4} \frac{c_2^2 c_3^2}{c_1 (c_1^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)} - \frac{1}{4} \frac{c_1^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \right] \]

\[ = 0 - 3/4 \frac{c_2^2}{(c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)}, \quad - \frac{1}{4} \frac{c_1^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \]

\[ - 1/4 \frac{c_2^2}{(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)}, \quad - \frac{1}{4} \frac{c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \]

\[ = 0, \quad -1/4 \frac{c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)}, \quad - \frac{1}{4} \frac{c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \]

\[ = 1/4 \frac{1}{(c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \]

\[ = 1/4 \frac{c_2^2}{(c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \]

\[ C_{8}^{-1} = \left[ \frac{1}{4} \frac{c_3^2 c_2^2}{(2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \right] \left[ \frac{1}{4} \frac{c_3^2 c_2^2}{(c_1^2 c_2^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)} \right] \]

\[ = \left[ \frac{1}{4} \frac{c_3^2 c_2^2}{c_1 (c_1^2 + c_1^2 c_3^2 - 2c_3^2 c_2^2)} - \frac{1}{4} \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \right] \]

\[ = 0, \quad -1/4 \frac{c_3^2 c_2^2}{(c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)}, \quad - \frac{1}{4} \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \]

\[ = 1/4 \frac{c_3^2 c_2^2}{c_1^2 (2c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \]

\[ = 1/4 \frac{c_3^2 c_2^2}{(c_1^2 - c_3^2)(2c_1^2 - 3c_2^2)} \]

(59g)

(59h)
\[ \mathbf{C}_9^{-1} = \begin{bmatrix} 
1/4 \cdot \frac{c_3^2 c_2^2}{(2 c_1^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} & -1/4 \cdot \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \\
-1/4 \cdot \frac{c_3^2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & 0 & -1/4 \cdot \frac{c_3^2 c_2^2}{c_1 (2 c_1^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} \\
-3/4 \cdot \frac{c_2^2}{c_1^2 (2 c_1^2 - 3 c_2^2)} & -1/4 \cdot \frac{c_2 c_2^2 + c_1^2 c_3^2 - c_3^2 c_2^2}{c_1^2 (2 c_1^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} & 0 \\
-1/4 \cdot \frac{c_1^2 c_2^2 - c_1^2 c_3^2 + 2 c_3^2 c_2^2}{c_1 (2 c_1^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} & -1/4 \cdot \frac{c_2 c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & 0 \\
1/4 \cdot \frac{(c_1^2 - c_2^2) c_3^2}{c_1^3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & 0, 1/4 & \frac{c_2^2}{c_1^3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \\
0, -1/4 \cdot \frac{(c_1^2 - c_2^2) c_3^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & 0 & -1/4 \cdot \frac{(c_1^2 - c_3^2) c_2^2}{c_1^2 (2 c_1^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} \\
0, -1/4 \cdot \frac{(c_1^2 - c_3^2) c_2^2}{c_1^3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & 0, 1/4 & \frac{(c_1^2 - c_3^2) c_2^2}{c_1^2 (2 c_1^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} \\
1/4 \cdot \frac{1}{c_1^2 (2 c_1^2 - 3 c_2^2)} & 0, 1/4 & \frac{c_1^2 - c_2^2}{c_1^2 (2 c_1^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} \\
-1/4 \cdot \frac{c_1^2 - 2 c_2^2}{c_1^2 (2 c_1^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} & 0 \\
\end{bmatrix} \]
\[ C_{11}^{-1} = \left[ \begin{array}{c}
\frac{c_3^2 c_2^2}{(2 c_1^2 - c_3^2) (2 c_1^2 - 3 c_2^2)}, \\
\frac{1}{4} c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2), \\
\frac{c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \\
0, \\
-4 c_2^2, \\
-3/4 c_2^2 (2 c_1^2 - 3 c_2^2), \\
0, \\
-1/4 c_2^2 (2 c_1^2 - c_3^2) (2 c_1^2 - 3 c_2^2),
\end{array} \right],
\]
\[ C_{12}^{-1} = \left[ \begin{array}{c}
\frac{c_3^2 c_2^2}{(2 c_1^2 - c_3^2) (2 c_1^2 - 3 c_2^2)}, \\
\frac{1}{4} c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2), \\
\frac{c_2^2}{c_1 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \\
0, \\
-4 c_2^2, \\
-3/4 c_2^2 (2 c_1^2 - 3 c_2^2), \\
0, \\
-1/4 c_2^2 (2 c_1^2 - c_3^2) (2 c_1^2 - 3 c_2^2),
\end{array} \right],
\]

(59k)
18 \left(3a_1^2 - 3a_2^2\right) - 18 \left(3a_2^2 - 3a_1^2\right) - \frac{1}{18} \left(2a_1^2 + a_2^2 + a_1^2 + 2a_2^2\right) + \frac{1}{18} \left(2a_2^2 + a_1^2 + 2a_1^2 + 2a_2^2\right)

\left(\begin{array}{c}
\frac{1}{2} - \frac{1}{3}c_1^2 - \frac{1}{2}c_2^2 \\
\frac{1}{2} - \frac{1}{3}c_2^2 - \frac{1}{2}c_1^2
\end{array}\right)

C_{\text{eq}} = \left[-\frac{1}{4} \left(3a_1^2 - 3a_2^2\right) + 1 \left(2a_1^2 + a_2^2\right) + 1 \left(2a_2^2 + a_1^2\right)\right] 

\left(\begin{array}{c}
\frac{1}{2} - \frac{1}{3}c_1^2 - \frac{1}{2}c_2^2 \\
\frac{1}{2} - \frac{1}{3}c_2^2 - \frac{1}{2}c_1^2
\end{array}\right)
\[ C_{14}^{-1} = \left[ -\frac{1}{4} \frac{c_1^2 c_3^2}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} \right]^1 \frac{1}{8} \frac{c_1^2 c_3^2}{c_2^2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \]

\[
\begin{align*}
1/8 & \frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/8 \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \cdot \\
1/8 & \frac{2 c_1^2 + c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/4 \frac{c_1^2}{c_2 (2 c_1^2 - 3 c_2^2)} \cdot \\
1/8 & \frac{2 c_1^2 + c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/4 \frac{c_1^2}{c_2 (2 c_1^2 - 3 c_2^2)} \cdot \\
-1/8 & \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/8 \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \cdot \\
-1/8 & \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/8 \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \cdot \\
-1/8 & \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/8 \frac{1}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} \cdot \\
-1/8 & \frac{1}{c_2^2 (2 c_1^2 - 3 c_2^2)} - 1/8 \frac{1}{c_2^2 (2 c_1^2 - 3 c_2^2)} - 1/8 \frac{1}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} \cdot \\
1/8 & \frac{1}{c_2^2 (2 c_1^2 - 3 c_2^2)} - 1/8 \frac{1}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} \cdot (59n)
\end{align*}
\]
\[ C_{15}^{-1} = [-1/4 \frac{c_1^2 c_3^2}{(3 c_2^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} - 1/8 \frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} ] \\
-1/8 \frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/8 \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \\
1/8 \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/4 \frac{c_1^2}{c_2 (2 c_1^2 - 3 c_2^2)} - 1/4 \frac{c_1^2}{c_2 (2 c_1^2 - 3 c_2^2)} \\
1/8 \frac{2 c_1^2 + c_3^2}{(3 c_2^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} - 1/4 \frac{2 c_1^2 + c_3^2}{c_2 (2 c_1^2 - 3 c_2^2)} - 1/8 \frac{2 c_1^2 + c_3^2}{(3 c_2^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} \\
1/8 \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \\
-1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/8 c_2^{-3} \\
1/8 \frac{1}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/8 \frac{1}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \\
1/8 \frac{1}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/8 \frac{1}{(3 c_2^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} \\
1/8 \frac{1}{c_2 (2 c_1^2 - 3 c_2^2)} - 1/8 \frac{1}{(3 c_2^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} \\
-1/8 \frac{1}{c_2 (2 c_1^2 - 3 c_2^2)} - 1/8 \frac{1}{(3 c_2^2 - c_3^2) (2 c_1^2 - 3 c_2^2)}] \quad (59o)
\[ C_{16}^{-1} = [-1/4 \frac{c_1^2 c_3^2}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}/] 1/8 \frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \]

\[-1/8 \frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 1/8 \frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \]

\[-1/8 \frac{2 c_1^2 + c_3^2}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} - 1/4 \frac{c_1^2}{c_2 (2 c_1^2 - 3 c_2^2)}, 1/4 \frac{c_1^2}{c_2 (2 c_1^2 - 3 c_2^2)}, \]

\[-1/8 \frac{2 c_1^2 + c_3^2}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} - 1/4 \frac{c_1^2}{c_2 (2 c_1^2 - 3 c_2^2)}, 1/8 \frac{2 c_1^2 + c_3^2}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}], \]

\[-1/8 \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, -1/8 \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}], \]

\[-1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, -1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}], \]

\[-1/8 c_2^{-3}, 1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}], \]

\[-1/8 \frac{1}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, -1/8 \frac{1}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}], \]

\[-1/8 \frac{1}{c_2 (2 c_1^2 - 3 c_2^2)}, -1/8 \frac{1}{c_2 (2 c_1^2 - 3 c_2^2)}, -1/8 \frac{1}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}], \]

\[-1/8 \frac{1}{c_2^2 (2 c_1^2 - 3 c_2^2)}, -1/8 \frac{1}{c_2^2 (2 c_1^2 - 3 c_2^2)}, -1/8 \frac{1}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}]. \]

(59p)
\[
C_{17}^{-1} = [-1/4 \frac{c_1^2 c_3^2}{(3 c_2^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} - 1/8 \frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/8 \frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - 1/4 \frac{c_1^2}{c_2^2 (2 c_1^2 - 3 c_2^2)} - 1/4 \frac{c_1^2}{c_2^2 (2 c_1^2 - 3 c_2^2)} - 1/4 \frac{c_1^2}{c_2^2 (2 c_1^2 - 3 c_2^2)} - 1/8 \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}]
\] (59q)
\[ C_{18}^{-1} = \left[-\frac{1}{4} \frac{c_1^2 c_3^2}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}\right] - \frac{1}{8} \frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \]
\[ \frac{1}{8} \frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \]
\[ \frac{1}{8} \frac{2 c_1^2 + c_3^2}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}, - \frac{1}{4} \frac{c_1^2}{c_2 (2 c_1^2 - 3 c_2^2)}, - \frac{1}{4} \frac{c_1^2}{c_2 (2 c_1^2 - 3 c_2^2)}, \]
\[ \frac{1}{8} \frac{2 c_1^2 + c_3^2}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} - \frac{1}{4} \frac{c_1^2}{c_2 (2 c_1^2 - 3 c_2^2)} - \frac{1}{8} \frac{2 c_1^2 + c_3^2}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}, \]
\[ \frac{1}{8} \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - \frac{1}{8} \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \]
\[ - \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, - \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \]
\[ - \frac{1}{8} \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - \frac{1}{8} \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \]
\[ \frac{1}{8} \frac{1}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} - \frac{1}{8} \frac{1}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)}, \]
\[ \frac{1}{8} \frac{1}{c_2 (2 c_1^2 - 3 c_2^2)}, - \frac{1}{8} \frac{1}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} \]
\[ C_{19}^{-1} = \left( \begin{array}{ccc} \frac{c_1^2 c_3^2}{(3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2)} & -\frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & -\frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \\ -\frac{c_2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & -\frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & -\frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \\ -\frac{1}{8} (3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2) & \frac{1}{4} c_2^2 (2 c_1^2 - 3 c_2^2) & \frac{1}{8} (3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2) \\ \frac{1}{8} & \frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & -\frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \\ \frac{1}{8} & -\frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & -\frac{c_1^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \\ \frac{1}{8} & -\frac{c_2 c_1^2 - c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & -\frac{c_2 c_1^2 - c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} \\ \frac{1}{8} & -\frac{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)} & -\frac{1}{8} (3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2) \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} (3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2) \\ -\frac{1}{8} & -\frac{1}{8} (3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2) & -\frac{1}{8} (3 c_2^2 - c_3^2)(2 c_1^2 - 3 c_2^2) \end{array} \right) \]
\[ C_{20}^{-1} = \left[ -\frac{1}{4} \frac{c_1^2 c_3^2}{(3 c_2^2 - c_3^2) (2 c_1^2 - 3 c_2^2)}, \quad -\frac{1}{8} \frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \right. \\
-\frac{1}{8} \frac{c_1^2 c_3^2}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \quad -\frac{1}{8} \frac{c_1^2}{c_2 (c_1^2 c_3^2)}, \quad -\frac{1}{4} \frac{c_1^2}{c_2 (c_1^2 c_3^2)}, \quad -\frac{1}{4} \frac{c_1^2}{c_2 (c_1^2 c_3^2)}, \quad 1/4 \frac{c_1^2}{c_2 (c_1^2 c_3^2)}, \quad 1/4 \frac{c_1^2}{c_2 (c_1^2 c_3^2)} \right] \\
1/4 \frac{c_1^2}{c_2 (c_1^2 c_3^2)}, \quad -\frac{1}{4} \frac{c_1^2}{c_2 (c_1^2 c_3^2)}, \quad 1/8 \frac{2 c_1^2 + c_3^2}{(3 c_2^2 - c_3^2) (2 c_1^2 - 3 c_2^2)}, \quad 1/8 \frac{(3 c_2^2 - c_3^2) (2 c_1^2 - 3 c_2^2)}{c_2 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \quad 1/8 \frac{c_1^2}{c_2 (c_1^2 c_3^2)}, \quad 1/8 \frac{c_1^2}{c_2 (c_1^2 c_3^2)}, \quad 1/8 \frac{c_1^2}{c_2 (c_1^2 c_3^2)} \right] \\
1/8 \frac{c_1^2}{c_2 (c_1^2 c_3^2)}, -1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_3^2)}, -1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_3^2)}, -1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_3^2)}, -1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_3^2)}, -1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_3^2)}, -1/8 \frac{c_1^2 - c_3^2}{c_2 (c_1^2 c_3^2)} \right] \\
1/8 \frac{c_1^2}{c_2 (c_1^2 c_3^2)}, -1/8 \frac{c_1^2}{c_2 (c_1^2 c_3^2)}, -1/8 \frac{1}{c_2 (2 c_1^2 - 3 c_2^2)}, -1/8 \frac{1}{(3 c_2^2 - c_3^2) (2 c_1^2 - 3 c_2^2)}, -1/8 \frac{1}{(3 c_2^2 - c_3^2) (2 c_1^2 - 3 c_2^2)} \right] \\
\text{(59t)} \]

\[ C_{21}^{-1} = \left[ -\frac{1}{4} \frac{c_1^2 c_2^2}{(3 c_2^2 - c_3^2) (2 c_1^2 - c_2^2)}, \quad 1/2 \frac{c_1^2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \quad 0,0,0,0,1/2 \right. \\
1/2 \frac{4 c_1^2 c_2^2 - 2 c_1^2 c_3^2 - c_3^2 c_2^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, 0,0, -1/2 \frac{(2 c_1^2 + c_3^2) c_2^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)} \right] \\
0, -1/2 \frac{(2 c_1^2 + c_3^2) c_2^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, 1/2 \frac{c_1^2 - c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \quad 0,0, -1/2 \frac{c_1^2 - c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \quad 0,0,0,0,1/2 \frac{c_1^2 - c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \quad 0,0,0,0,1/2 \frac{2 c_2^2 - c_3^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, \quad 0,0,1/2 \frac{2 c_2^2 - c_3^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)} \right] \\
\text{(59u)} \]
\[ C_{22}^{-1} = \left[ \frac{c_1^2 c_2^2}{(3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, -1/2 \frac{c_1^2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 0, 0, 0, 0, 0 \right] \]
\[
1/2 \frac{4 c_1^2 c_2^2 - 2 c_1^2 c_3^2 - c_2^2 c_3^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, 0, 0, -1/2 \frac{(2 c_1^2 + c_3^2) c_2^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)} , c_1^2 - 2 c_2^2, 0, 1/2 \frac{(2 c_1^2 + c_3^2) c_2^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, -1/2 \frac{c_1^2 - 2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 0, 1/2 \frac{c_1^2 - c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 0, 0, 0, 0, -1/2 \frac{2 c_2^2 - c_3^2}{c_3 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, 0, 0, 1/2 \frac{c_2^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)} \]  
(59v)

\[ C_{23}^{-1} = \left[ \frac{c_1^2 c_2^2}{(3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, 0, 1/2 \frac{c_1^2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 0, 1/2 \frac{c_1^2 c_2^2}{c_3 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, 0, 0, 0, 0, 0, 1/2 \frac{c_1^2 - 2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 0, 1/2 \frac{c_1^2 - 2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 0, 1/2 \frac{c_2^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, 0, 0, 1/2 \frac{c_2^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, 0, 0, 1/2 \frac{c_2^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)} \]  
(59w)

\[ C_{24}^{-1} = \left[ \frac{c_1^2 c_2^2}{(3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, 0, -1/2 \frac{c_1^2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 0, 1/2 \frac{c_1^2 - 2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 0, 1/2 \frac{c_1^2 - 2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 0, 1/2 \frac{c_1^2 - 2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 0, 1/2 \frac{c_1^2 - 2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 0, 1/2 \frac{c_1^2 - 2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, 0, 0, 0, 0, -1/2 \frac{2 c_2^2 - c_3^2}{c_3 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, 0, 0, -1/2 \frac{2 c_2^2 - c_3^2}{c_3 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)}, 0, 1/2 \frac{c_2^2}{c_3^2 (3 c_2^2 - c_3^2) (2 c_1^2 - c_3^2)} \]  
(59x)
The DVM with 2 dimensions and 16 velocities can have 4 energy levels.

\[
\mathbf{C}^{-1}_{25} = \begin{bmatrix}
\frac{c_1^2 c_2^2}{(3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)}, & 0, 0, 1/2 & \frac{c_1^2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \\
-1/2 & \frac{(2 c_1^2 + c_3^2) c_2^2}{c_3 (3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)}, & 0, -1/2 & \frac{(2 c_1^2 + c_3^2) c_2^2}{c_3 (3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)}, \\
0, 1/2 & \frac{4 c_1^2 c_2^2 - 2 c_1^2 c_3^2 - c_3^2 c_2^2}{c_3 (3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)}, & 0, 0, 0, 0 & \frac{c_1^2 - c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \\
0, 0, 0, 0 & \frac{c_2^2}{c_3 (3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)}, & 0, 1/2 & \frac{c_2^2}{c_3 (3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)}, \\
-1/2 & \frac{2 c_2^2 - c_3^2}{c_3 (3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)} & \end{bmatrix}
\]

\[
\mathbf{C}^{-1}_{26} = \begin{bmatrix}
\frac{c_1^2 c_2^2}{(3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)}, & 0, 0, 1/2 & \frac{c_1^2 c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \\
-1/2 & \frac{(2 c_1^2 + c_3^2) c_2^2}{c_3 (3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)}, & 0, -1/2 & \frac{(2 c_1^2 + c_3^2) c_2^2}{c_3 (3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)}, \\
0, 1/2 & \frac{4 c_1^2 c_2^2 - 2 c_1^2 c_3^2 - c_3^2 c_2^2}{c_3 (3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)}, & 0, 0, 0, 0 & \frac{c_1^2 - c_2^2}{c_3 (c_1^2 c_2^2 + c_1^2 c_3^2 - 2 c_3^2 c_2^2)}, \\
0, 0, 0, 0 & \frac{c_2^2}{c_3 (3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)}, & 0, 1/2 & \frac{c_2^2}{c_3 (3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)}, \\
-1/2 & \frac{2 c_2^2 - c_3^2}{c_3 (3 c_2^2 - c_3^2)(2 c_1^2 - c_3^2)} & \end{bmatrix}
\]

II. C\textsuperscript{-1} FOR THE D2V16

The DVM with 2 dimensions and 16 velocities can have 4 energy levels.

\[
v_{ir} = [1, 0, -1, 0] c_i \quad i = 1, \ldots, 4,
\]

\[
v_{i\theta} = [0, 1, 0, -1] c_i \quad i = 1, \ldots, 4,
\]

\[
v_{ir} = [1, -1, -1, 1] c_i \quad i = 5, \ldots, 8,
\]

\[
v_{i\theta} = [1, 1, -1, -1] c_i \quad i = 5, \ldots, 8,
\]

\[
v_{ir} = [1, 0, -1, 0] 2 c_i \quad i = 9, \ldots, 12,
\]

\[
v_{i\theta} = [0, 1, 0, -1] 2 c_i \quad i = 9, \ldots, 12,
\]
\[ v_{ir} = [1, -1, -1, 1] \sqrt{2} c_4 \quad \text{and} \quad v_{i\theta} = [1, 1, -1, -1] \sqrt{2} c_4 \quad \text{for} \quad i = 13, \ldots, 16. \quad (60d) \]

If we choose \( \eta_i = \eta_0 \) for \( i = 5, \ldots, 8 \), and \( \eta_i = 0 \) for others, then there are only five variables in the components of the matrix \( C \). The components of the inverse of \( C \), \( C^{-1} = [C_k^{-1}] \), are shown below.

\[
C_1^{-1} = \left[ \begin{array}{c} \frac{c_3^2 c_4^2}{(c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)} - \frac{c_3^2}{c_1(c_1^2 - c_3^2)}, \\
-\frac{1}{2} \frac{c_3^2}{c_1(c_1^2 - c_3^2)}, \\
-\frac{1}{2} \frac{2 c_2^4 - c_2^2 c_3^2 + c_2^2 \eta_0^2 - 2 c_2^2 c_4^2 + c_3^2 c_4^2}{\eta_0^2 (c_1^2 - c_3^2)(c_1^2 - c_3^2)}, \end{array} \right] C_{1,5}^{-1}, 0, C_{1,7}^{-1}, -\frac{1}{2} \frac{(c_2^2 - c_4^2) c_3^2}{c_1 c_4^2 \eta_0^2 (c_1^2 - c_3^2)}, 0, 1/2 \frac{c_2^2 c_3^2 - c_3^2 c_4^2 + c_4^2 \eta_0^2}{c_1 c_4^2 \eta_0^2 (c_1^2 - c_3^2)}, \right] \left[ \begin{array}{c} \frac{2 c_3^2 c_4^2}{(c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)}, \\
0, 1/2 \frac{c_3^2}{c_1(c_1^2 - c_3^2)}, \\
0, 1/2 \frac{c_3^2}{c_1^2 (c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)} \end{array} \right] \quad (61a) \]

\[
C_2^{-1} = \left[ \begin{array}{c} \frac{c_3^2 c_4^2}{(c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)}, \\
0, 1/2 \frac{c_3^2}{c_1(c_1^2 - c_3^2)}, \\
-\frac{1}{2} \frac{2 c_2^4 - c_2^2 c_3^2 + c_2^2 \eta_0^2 - 2 c_2^2 c_4^2 + c_3^2 c_4^2}{\eta_0^2 (c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)}, C_{2,5}^{-1}, 0, C_{2,7}^{-1}, 0, \end{array} \right] \left[ \begin{array}{c} \frac{c_2^2 c_3^2}{c_1 c_4^2 \eta_0^2 (c_1^2 - c_3^2)}, \\
0, 1/2 \frac{c_2^2 c_3^2 + \eta_0^2 c_3^2 - c_3^2 c_4^2 + c_4^2 \eta_0^2}{c_1 c_4^2 \eta_0^2 (c_1^2 - c_3^2)}, \end{array} \right] \left[ \begin{array}{c} \frac{c_2^2 c_3^2}{c_1 c_4^2 \eta_0^2 (c_1^2 - c_3^2)}, \\
1/2 \frac{c_2^2 c_3^2 - c_3^2 c_4^2 + c_4^2 \eta_0^2}{c_1^2 (c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)}, \\
0, 1/2 \frac{c_2^2 c_3^2}{c_1^2 (c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)} \end{array} \right] \quad (61b) \]

\[
C_3^{-1} = \left[ \begin{array}{c} \frac{c_3^2 c_4^2}{(c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)}, \\
0, 1/2 \frac{c_3^2}{c_1(c_1^2 - c_3^2)}, \\
-\frac{1}{2} \frac{2 c_2^4 - c_2^2 c_3^2 + c_2^2 \eta_0^2 - 2 c_2^2 c_4^2 + c_3^2 c_4^2}{\eta_0^2 (c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)}, C_{3,5}^{-1}, 0, C_{3,7}^{-1}, 1/2 \frac{(c_2^2 - c_4^2) c_3^2}{c_1 c_4^2 \eta_0^2 (c_1^2 - c_3^2)}, \end{array} \right] \left[ \begin{array}{c} \frac{c_2^2 c_3^2}{c_1 c_4^2 \eta_0^2 (c_1^2 - c_3^2)}, \\
0, 1/2 \frac{c_2^2 c_3^2 + \eta_0^2 c_3^2 - c_3^2 c_4^2 + c_4^2 \eta_0^2}{c_1 c_4^2 \eta_0^2 (c_1^2 - c_3^2)}, \end{array} \right] \left[ \begin{array}{c} \frac{c_2^2 c_3^2}{c_1 c_4^2 \eta_0^2 (c_1^2 - c_3^2)}, \\
0, 1/2 \frac{c_2^2 c_3^2}{c_1^2 (c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)}, \\
0, 1/2 \frac{c_2^2 c_3^2}{c_1^2 (c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)} \end{array} \right] \quad (61c) \]
\[ C_{4^{-1}} = \left[ \frac{1}{2} \frac{c_3^2 c_4^2}{(c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)}, 0, \frac{1}{2} \frac{c_3^2}{c_1 (c_1^2 - c_3^2)}, -1/2 \frac{2 c_2 c_4^2 - c_2^2 c_1^2 + c_2^2 \eta_0^2 - 2 c_2^2 c_4^2 + c_2^2 c_1^2}{\eta_0^2 (c_1^2 - 2 c_4^2)(c_1^2 - c_3^2)} \right] \]

\[ C_{5^{-1}} = \left[ 0, 0, 0, 1/4 \eta_0^{-2}, -1/4 \eta_0^{-2}, -1/2 \frac{c_4^2}{c_2^2 (\eta_0^2 + 2 c_2^2 - 2 c_4^2)}, -1/4 \eta_0^{-2}, 1/4 \frac{1}{c_2 \eta_0^2}, 1/4 \frac{1}{c_2 \eta_0^2}, -1/4 \frac{1}{c_2 \eta_0^2}, -1/4 \frac{1}{c_2 \eta_0^2}, -1/4 \frac{1}{c_2 \eta_0^2}, -1/4 \frac{1}{c_2^2 (\eta_0^2 + 2 c_2^2 - 2 c_4^2)}, 0 \right] \]

\[ C_{6^{-1}} = \left[ 0, 0, 0, 1/4 \eta_0^{-2}, -1/4 \eta_0^{-2}, 1/2 \frac{c_4^2}{c_2^2 (\eta_0^2 + 2 c_2^2 - 2 c_4^2)}, -1/4 \eta_0^{-2}, 1/4 \frac{1}{c_2 \eta_0^2}, 1/4 \frac{1}{c_2 \eta_0^2}, -1/4 \frac{1}{c_2 \eta_0^2}, -1/4 \frac{1}{c_2 \eta_0^2}, 0 \right] \]

\[ C_{7^{-1}} = \left[ 0, 0, 0, 1/4 \eta_0^{-2}, -1/4 \eta_0^{-2}, -1/2 \frac{c_4^2}{c_2^2 (\eta_0^2 + 2 c_2^2 - 2 c_4^2)}, -1/4 \eta_0^{-2}, -1/4 \frac{1}{c_2 \eta_0^2}, -1/4 \frac{1}{c_2 \eta_0^2}, 1/4 \frac{1}{c_2 \eta_0^2}, 1/4 \frac{1}{c_2 \eta_0^2}, 1/4 \frac{1}{c_2 \eta_0^2}, 1/4 \frac{1}{c_2 \eta_0^2}, 0 \right] \]

\[ C_{8^{-1}} = \left[ 0, 0, 0, 1/4 \eta_0^{-2}, -1/4 \eta_0^{-2}, 1/2 \frac{c_4^2}{c_2^2 (\eta_0^2 + 2 c_2^2 - 2 c_4^2)}, -1/4 \eta_0^{-2}, 1/4 \frac{1}{c_2 \eta_0^2}, -1/4 \frac{1}{c_2 \eta_0^2}, -1/4 \frac{1}{c_2 \eta_0^2}, -1/4 \frac{1}{c_2 \eta_0^2}, 1/4 \frac{1}{c_2 \eta_0^2}, 1/4 \frac{1}{c_2 \eta_0^2}, 0 \right] \]
\[
\begin{align*}
\mathbf{C}_9^{-1} &= \begin{bmatrix}
-1/2 & c_4^2 c_1^2 \\
\frac{c_4^2 c_1^2}{(c_3^2 - 2 c_4^2)(c_1^2 - c_3^2)} & 1/2 & c_1^2 \\
\frac{-1/2 c_4^2 c_1^2 - c_3^2}{c_3 (c_1^2 - c_3^2)} & 0, & 0, \\
\frac{-1/2 c_4^2 c_1^2}{c_3 (c_1^2 - c_3^2)} & 0, & 0,
\end{bmatrix},
\mathbf{C}_9^{-1}, 0, C_{9,7}, 0, \mathbf{C}_{9,5}, 0, 0, \mathbf{C}_{9,1}\end{align*}
\]

\[
\begin{align*}
\mathbf{C}_{10}^{-1} &= \begin{bmatrix}
-1/2 & c_4^2 c_1^2 \\
\frac{c_4^2 c_1^2}{(c_3^2 - 2 c_4^2)(c_1^2 - c_3^2)} & 0, & 1/2 & c_1^2 \\
\frac{-1/2 c_3^2 c_1^2}{c_3 (c_1^2 - c_3^2)} & 0, & 0, & 0, \\
\frac{1/2 c_3^2 c_1^2 - c_2^2}{c_3 (c_1^2 - c_3^2)} & 0, & -1/2 & c_3^2 + c_2^2 \\
\frac{-1/2 c_3^2}{c_3 (c_1^2 - c_3^2)} & 0, & -1/2 & c_3^2 + c_2^2 \\
\frac{-1/2 c_3^2}{c_3 (c_1^2 - c_3^2)} & 0, & -1/2 & c_3^2 + c_2^2
\end{bmatrix},
\mathbf{C}_{10,5}, 0, \mathbf{C}_{10,7}, 0, \mathbf{C}_{10,1}, 0, \mathbf{C}_{10,3}, 0, \mathbf{C}_{10,1}\end{align*}
\]

\[
\begin{align*}
\mathbf{C}_{11}^{-1} &= \begin{bmatrix}
-1/2 & c_4^2 c_1^2 \\
\frac{c_4^2 c_1^2}{(c_3^2 - 2 c_4^2)(c_1^2 - c_3^2)} & 0, & 1/2 & c_1^2 \\
\frac{-1/2 c_3^2 c_1^2}{c_3 (c_1^2 - c_3^2)} & 0, & 0, & 0, \\
\frac{1/2 c_3^2 c_1^2 - c_2^2}{c_3 (c_1^2 - c_3^2)} & 0, & -1/2 & c_3^2 + c_2^2 \\
\frac{-1/2 c_3^2}{c_3 (c_1^2 - c_3^2)} & 0, & -1/2 & c_3^2 + c_2^2
\end{bmatrix},
\mathbf{C}_{11,5}, 0, \mathbf{C}_{11,7}, 0, \mathbf{C}_{11,1}, 0, \mathbf{C}_{11,3}, 0, \mathbf{C}_{11,1}\end{align*}
\]
\[ C_{12}^{-1} = \left[ -\frac{1}{2} \frac{c_4^2 c_1^2}{(c_3^2 - 2c_4^2) (c_1^2 - c_3^2)}, 0, -\frac{1}{2} \frac{c_1^2}{c_3 (c_1^2 - c_3^2)} \right], -\frac{1}{2} \frac{c_1^2 c_2^2 - c_4^2 c_1^2 - 2c_2^4 - c_2^2 \eta_0^2 + 2c_3^2 c_4^2}{\eta_0^2 (c_3^2 - 2c_4^2) (c_1^2 - c_3^2)}, \right], C_{12}^{-1}, 0, C_{12}^{-1}, 0, \]

\[ C_{13}^{-1} = \left[ \frac{1}{4} \frac{c_3^2 c_1^2}{(c_3^2 - 2c_4^2) (c_1^2 - 2c_4^2)}, 0, 0, \frac{1}{4} \frac{2c_1^2 c_2^2 - c_3^2 c_1^2 - 4c_2^4 - 2c_2^2 \eta_0^2 + 2c_3^2 c_4^2}{\eta_0^2 (c_3^2 - 2c_4^2) (c_1^2 - 2c_4^2)}, \right], \frac{1}{4} \frac{2c_1^2 c_2^2 + c_1^2 \eta_0^2 - c_3^2 c_1^2 - 4c_2^4 - 2c_2^2 \eta_0^2 + 2c_3^2 c_4^2 + \eta_0^2 c_3^2}{\eta_0^2 (c_3^2 - 2c_4^2) (c_1^2 - 2c_4^2)}, \right], -\frac{1}{4} \frac{2c_2^2 + \eta_0^2}{c_4^2 (\eta_0^2 + 2c_2^2 - 2c_4^2)}, \right], \frac{1}{4} \frac{2c_1^2 c_2^2 + c_1^2 \eta_0^2 - c_3^2 c_1^2 - 4c_2^4 - 2c_2^2 \eta_0^2 + 2c_3^2 c_4^2 + \eta_0^2 c_3^2}{\eta_0^2 (c_3^2 - 2c_4^2) (c_1^2 - 2c_4^2)}, \right], \frac{1}{4} \frac{c_2^2}{\eta_0^2 c_4^3}, \right], -\frac{1}{4} \frac{c_2^2}{\eta_0^2 c_4^3}, \right], 1/4 \frac{c_2^2 + \eta_0^2}{\eta_0^2 c_4^3}, \right], 1/4 \frac{c_2^2 + \eta_0^2}{\eta_0^2 c_4^3}, \right], 1/4 \frac{1}{\eta_0^2 c_4^3}, \right], -\frac{1}{4} \frac{1}{c_4^2 (\eta_0^2 + 2c_2^2 - 2c_4^2)}, \right], 1/4 \frac{1}{(c_3^2 - 2c_4^2) (c_1^2 - 2c_4^2)} \right] \]
\[ C_{14}^{-1} = \left[ \frac{1}{4} \frac{c_3^2 c_1^2}{(c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)} \right] 0, 0, \]
\[ = \frac{1}{4} \frac{2 c_1^2 c_2^2 - c_3^2 c_1^2 - 4 c_2^4 - 2 c_2^2 \eta_0^2 + 2 c_2^2 c_3^2}{\eta_0^2 (c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)}, \]
\[ - \frac{1}{4} \frac{2 c_1^2 c_2^2 + c_1^2 \eta_0^2 - c_3^2 c_1^2 - 4 c_2^4 - 2 c_2^2 \eta_0^2 + 2 c_2^2 c_3^2 + \eta_0^2 c_3^2}{\eta_0^2 (c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)}, \]
\[ \left( \begin{array}{cc} \frac{2 c_2^2 + \eta_0^2}{c_4^2 (\eta_0^2 + 2 c_2^2 - 2 c_4^2)} & \frac{2 c_2^2 + \eta_0^2}{c_4^2 (\eta_0^2 + 2 c_2^2 - 2 c_4^2)} \\
\frac{2 c_2^2 + \eta_0^2}{\eta_0^2 c_4^3} & \frac{1}{1} \frac{1}{(c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)} \end{array} \right) \]  

\[ (61n) \]

\[ C_{15}^{-1} = \left[ \frac{1}{4} \frac{c_3^2 c_1^2}{(c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)} \right] 0, 0, \]
\[ = \frac{1}{4} \frac{2 c_1^2 c_2^2 - c_3^2 c_1^2 - 4 c_2^4 - 2 c_2^2 \eta_0^2 + 2 c_2^2 c_3^2}{\eta_0^2 (c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)}, \]
\[ - \frac{1}{4} \frac{2 c_1^2 c_2^2 + c_1^2 \eta_0^2 - c_3^2 c_1^2 - 4 c_2^4 - 2 c_2^2 \eta_0^2 + 2 c_2^2 c_3^2 + \eta_0^2 c_3^2}{\eta_0^2 (c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)}, \]
\[ \left( \begin{array}{cc} \frac{2 c_2^2 + \eta_0^2}{c_4^2 (\eta_0^2 + 2 c_2^2 - 2 c_4^2)} & \frac{2 c_2^2 + \eta_0^2}{c_4^2 (\eta_0^2 + 2 c_2^2 - 2 c_4^2)} \\
\frac{2 c_2^2 + \eta_0^2}{\eta_0^2 c_4^3} & \frac{1}{1} \frac{1}{(c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)} \end{array} \right) \]  

\[ (61o) \]
$$C_{16}^{-1} = \left[ \frac{1/4}{\eta_0^2 (c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)}, 0, 0, \right.  \\
1/4 \frac{2 c_1^2 c_2^2 - c_3^2 c_1^2 - 4 c_2^4 - 2 c_2^2 \eta_0^2 + 2 c_2^2 c_3^2}{\eta_0^2 (c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)},  \\
-1/4 \frac{2 c_1^2 c_2^2 + c_1^2 \eta_0^2 - c_3^2 c_1^2 - 4 c_2^4 - 2 c_2^2 \eta_0^2 + 2 c_2^2 c_3^2 + \eta_0^2 c_3^2}{\eta_0^2 (c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)},  \\
-1/4 \frac{2 c_2^2 + \eta_0^2}{c_4^2 (\eta_0^2 + 2 c_2^2 - 2 c_4^2)},  \\
-1/4 \frac{2 c_1^2 c_2^2 + c_1^2 \eta_0^2 - c_3^2 c_1^2 - 4 c_2^4 - 2 c_2^2 \eta_0^2 + 2 c_2^2 c_3^2 + \eta_0^2 c_3^2}{\eta_0^2 (c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)},  \\
-1/4 \frac{2 c_2^2}{\eta_0^2 c_3^4}; 1/4 \frac{1/4 \frac{1}{\eta_0^2 c_3^4}, 1/4 \frac{2 c_2^2}{\eta_0^2 c_4^4}, 1/4 \frac{1/4 \frac{2 c_2^2 + \eta_0^2}{\eta_0^2 c_4^4}}{1/4 \frac{2 c_2^2 + \eta_0^2}{\eta_0^2 c_4^4}},}{1/4 \frac{1}{\eta_0^2 c_4^4}, 1/4 \frac{1}{(c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)},  \\
1/4 \frac{1}{c_4^2 (\eta_0^2 + 2 c_2^2 - 2 c_4^2)}, 1/4 \frac{1}{(c_3^2 - 2 c_4^2) (c_1^2 - 2 c_4^2)} \right] \quad (61p)$$

with

$$C_{m,n}^{-1} = \frac{p_{m,n}}{q_{m,n}}$$

where

$$p_{1,5} = 2 c_1^2 c_2^4 + c_1^2 c_2^2 \eta_0^2 - c_1^2 c_2^2 c_3^2 - 2 c_2^2 c_4^2 c_1^2 - c_1^2 \eta_0^2 c_3^2  \\
+ \eta_0^2 c_3^2 c_4^2 + c_1^2 c_3^2 c_4^2 - c_1^2 c_4^2 \eta_0^2 ;  \\
q_{1,5} = 2 \left( c_1^2 - 2 c_4^2 \right) c_1^2 \eta_0^2 \left( c_1^2 - c_3^2 \right);$$

$$p_{1,7} = 2 c_1^2 c_2^4 - c_1^2 c_2^2 c_3^2 + c_1^2 c_2^2 \eta_0^2 - 2 c_2^2 c_4^2 c_1^2  \\
+ c_1^2 c_3^2 c_4^2 - c_1^2 c_4^2 \eta_0^2 - \eta_0^2 c_3^2 c_4^2 ;  \\
q_{1,7} = 2 \left( c_1^2 - 2 c_4^2 \right) c_1^2 \eta_0^2 \left( c_1^2 - c_3^2 \right);$$

$$p_{2,5} = 2 c_1^2 c_2^4 - c_1^2 c_2^2 c_3^2 + c_1^2 c_2^2 \eta_0^2 - 2 c_2^2 c_4^2 c_1^2  \\
+ c_1^2 c_3^2 c_4^2 - c_1^2 c_4^2 \eta_0^2 - \eta_0^2 c_3^2 c_4^2 ;  \\
q_{2,5} = 2 \left( c_1^2 - 2 c_4^2 \right) c_1^2 \eta_0^2 \left( c_1^2 - c_3^2 \right);$$

$$p_{2,7} = 2 c_1^2 c_2^4 + c_1^2 c_2^2 \eta_0^2 - c_1^2 c_2^2 c_3^2 - 2 c_2^2 c_4^2 c_1^2 - c_1^2 \eta_0^2 c_3^2  \\
+ \eta_0^2 c_3^2 c_4^2 + c_1^2 c_3^2 c_4^2 - c_1^2 c_4^2 \eta_0^2 ;$$
\[ q_{2,7} = 2 \left( c_1^2 - 2 c_4^2 \right) \eta_0^2 \left( c_1^2 - c_3^2 \right), \]

\[ p_{3,5} = 2 c_1^2 c_2^4 + c_1^2 c_2^2 \eta_0^2 - c_1^2 c_2^2 \eta_0 - 2 c_2^2 c_4^2 c_1^2 - c_1^2 \eta_0^2 c_3^2 \]
\[ + \eta_0^2 c_3^2 c_4^2 + c_1^2 c_3^2 c_4^2 - c_1^2 c_4^2 \eta_0^2, \]

\[ q_{3,5} = 2 \left( c_1^2 - 2 c_4^2 \right) \eta_0^2 \left( c_1^2 - c_3^2 \right), \]

\[ p_{3,7} = 2 c_1^2 c_2^4 - c_1^2 c_2^2 c_3^2 + c_1^2 c_2^2 \eta_0^2 - 2 c_2^2 c_4^2 c_1^2 + c_1^2 c_3^2 c_4^2 \]
\[ - c_1^2 c_4^2 \eta_0^2 - \eta_0^2 c_3^2 c_4^2, \]

\[ q_{3,7} = 2 \left( c_1^2 - 2 c_4^2 \right) \eta_0^2 \left( c_1^2 - c_3^2 \right), \]

\[ p_{4,5} = 2 c_1^2 c_2^4 - c_1^2 c_2^2 c_3^2 + c_1^2 c_2^2 \eta_0^2 - 2 c_2^2 c_4^2 c_1^2 \]
\[ + c_1^2 c_3^2 c_4^2 - c_1^2 c_4^2 \eta_0^2 - \eta_0^2 c_3^2 c_4^2, \]

\[ q_{4,5} = 2 \left( c_1^2 - 2 c_4^2 \right) \eta_0^2 \left( c_1^2 - c_3^2 \right), \]

\[ p_{4,7} = 2 c_1^2 c_2^4 + c_1^2 c_2^2 \eta_0^2 - c_1^2 c_2^2 c_3^2 - 2 c_2^2 c_4^2 c_1^2 - c_1^2 \eta_0^2 c_3^2 \]
\[ + \eta_0^2 c_3^2 c_4^2 + c_1^2 c_3^2 c_4^2 - c_1^2 c_4^2 \eta_0^2, \]

\[ q_{4,7} = 2 \left( c_1^2 - 2 c_4^2 \right) \eta_0^2 \left( c_1^2 - c_3^2 \right), \]

\[ p_{9,5} = c_1^2 c_2^2 c_3^2 + c_1^2 \eta_0^2 c_3^2 - c_1^2 c_2^2 c_4^2 - c_1^2 c_4^2 \eta_0^2 - 2 c_3^2 c_2^4 \]
\[ - c_3^2 c_2^2 \eta_0^2 + 2 c_2^2 c_4^2 c_3^2 + \eta_0^2 c_3^2 c_4^2, \]

\[ q_{9,5} = 2 \eta_0^2 c_3^2 \left( c_3^2 - 2 c_4^2 \right) \left( c_1^2 - c_3^2 \right), \]

\[ p_{9,7} = c_1^2 c_2^2 c_3^2 - c_1^2 c_3^2 c_4^2 + c_1^2 c_4^2 \eta_0^2 - 2 c_3^2 c_2^4 - c_3^2 c_2^2 \eta_0^2 \]
\[ + 2 c_2^2 c_4^2 c_3^2 + \eta_0^2 c_3^2 c_4^2, \]

\[ q_{9,7} = 2 \eta_0^2 c_3^2 \left( c_3^2 - 2 c_4^2 \right) \left( c_1^2 - c_3^2 \right), \]

\[ p_{10,5} = c_1^2 c_2^2 c_3^2 - c_1^2 c_3^2 c_4^2 + c_1^2 c_4^2 \eta_0^2 - 2 c_3^2 c_2^4 - c_3^2 c_2^2 \eta_0^2 \]
\[ + 2 c_2^2 c_4^2 c_3^2 + \eta_0^2 c_3^2 c_4^2, \]

\[ q_{10,5} = 2 \eta_0^2 c_3^2 \left( c_3^2 - 2 c_4^2 \right) \left( c_1^2 - c_3^2 \right), \]
\[ p_{10,7} = c_1^2 c_2^2 c_3^2 + c_1^2 \eta_0^2 c_3^2 - c_1^2 c_3^2 c_4^2 - c_1^2 c_4^2 \eta_0^2 - 2 c_3^2 c_2^4 \\
- c_3^2 c_2^2 \eta_0^2 + 2 c_2^2 c_4^2 c_3^2 + \eta_0^2 c_3^2 c_4^2, \\
q_{10,7} = 2\eta_0^2 c_3^2 \left( c_3^2 - 2 c_4^2 \right) \left( c_1^2 - c_3^2 \right), \]

\[ p_{11,5} = c_1^2 c_2^2 c_3^2 + c_1^2 \eta_0^2 c_3^2 - c_1^2 c_3^2 c_4^2 - c_1^2 c_4^2 \eta_0^2 - 2 c_3^2 c_2^4 \\
- c_3^2 c_2^2 \eta_0^2 + 2 c_2^2 c_4^2 c_3^2 + \eta_0^2 c_3^2 c_4^2, \\
q_{11,5} = 2\eta_0^2 c_3^2 \left( c_3^2 - 2 c_4^2 \right) \left( c_1^2 - c_3^2 \right), \]

\[ p_{11,7} = c_1^2 c_2^2 c_3^2 - c_1^2 c_3^2 c_4^2 + c_1^2 c_4^2 \eta_0^2 - 2 c_3^2 c_2^4 - c_3^2 c_2^2 \eta_0^2 \\
+ 2 c_2^2 c_4^2 c_3^2 + \eta_0^2 c_3^2 c_4^2, \\
q_{11,7} = 2\eta_0^2 c_3^2 \left( c_3^2 - 2 c_4^2 \right) \left( c_1^2 - c_3^2 \right), \]

\[ p_{12,5} = c_1^2 c_2^2 c_3^2 - c_1^2 c_3^2 c_4^2 + c_1^2 c_4^2 \eta_0^2 - 2 c_3^2 c_2^4 - c_3^2 c_2^2 \eta_0^2 \\
+ 2 c_2^2 c_4^2 c_3^2 + \eta_0^2 c_3^2 c_4^2, \\
q_{12,5} = 2\eta_0^2 c_3^2 \left( c_3^2 - 2 c_4^2 \right) \left( c_1^2 - c_3^2 \right), \]

\[ p_{12,7} = c_1^2 c_2^2 c_3^2 + c_1^2 \eta_0^2 c_3^2 - c_1^2 c_3^2 c_4^2 - c_1^2 c_4^2 \eta_0^2 - 2 c_3^2 c_2^4 \\
- c_3^2 c_2^2 \eta_0^2 + 2 c_2^2 c_4^2 c_3^2 + \eta_0^2 c_3^2 c_4^2, \\
q_{12,7} = 2\eta_0^2 c_3^2 \left( c_3^2 - 2 c_4^2 \right) \left( c_1^2 - c_3^2 \right), \]

[1] S. Succi, *The Lattice Boltzmann Equation for Fluid Dynamics and Beyond*, Oxford University Press, New York, (2001).

[2] A. Xu, G. Zhang, Y. Li, and H. Li, Progress in Physics (2014, in press).

[3] X. He, and L. S. Luo, Phys. Rev. E 56, 6811 (1997).

[4] N. Cao, S. Chen, S. Jin, and D. Martinez, Phys. Rev. E 55, R21 (1997).

[5] V. Sofonea, R. F. Sekerka, J. Comput. Phys. 184, 422 (2003).

[6] F. Nannelli, S. Succi, J. Stat. Phys. 68, 401 (1992).
[7] Y. Li, E. J. LeBoeuf, and P. K. Basu, Phys. Rev. E 69, 065701(R) (2004).
[8] Y. Li, E. J. LeBoeuf, and P. K. Basu, Phys. Rev. E 72, 046711 (2005).
[9] A. Xu, G. Zhang, Y. Gan, F. Chen, and X. Yu, Front. Phys. 7(5), 582 (2012).
[10] F. J. Alexander, H. Chen, S. Chen and G. D. Doolen, Phys. Rev. A 46, 1967 (1992).
[11] G. Yan, Y. Chen, S. Hu, Phys. Rev. E 59, 454 (1999).
[12] C. H. Sun, Phys. Rev. E 58, 7283 (1998).
[13] C. Sun and A. T. Hsu, Phys. Rev. E 68, 016303 (2003).
[14] T. Kataoka and M. Tsutahara, Phys. Rev. E 69, 056702 (2004).
[15] T. Kataoka and M. Tsutahara, Phys. Rev. E 69, 035701(R)(2004).
[16] M. Watari and M. Tsutahara, Phys. Rev. E 67, 036306 (2003).
[17] M. Watari and M. Tsutahara, Phys. Rev. E 70, 016703 (2004).
[18] M. Watari, Physica A 382, 502 (2007).
[19] A. Xu, Phys. Rev. E 71, 066706 (2005).
[20] A. Xu, Europhys. Lett. 69, 214 (2005).
[21] X. Pan, A. Xu, G. Zhang, and S. Jiang, Int. J. Mod. Phys. C 18, 1747 (2007).
[22] Y. Gan, A. Xu, G. Zhang, and Y. Li, Commun. Theor. Phys. 50, 201 (2008).
[23] Y. Gan, A. Xu, G. Zhang, X. Yu, and Y. Li, Physica A 387, 1721 (2008).
[24] Y. Gan, A. Xu, G. Zhang, and Y. Li, Commun. Theor. Phys. 56, 490 (2011).
[25] Y. Gan, A. Xu, G. Zhang, and Y. Li, Phys. Rev. E 83, 056704 (2011).
[26] F. Chen, A. Xu, G. Zhang, Y. Gan, C. Tao, and Y. Li, Commun. Theor. Phys. 52, 681 (2009).
[27] F. Chen, A. Xu, G. Zhang, Y. Li, S. Succi, EuroPhys. Lett. 90, 54003 (2010).
[28] F. Chen, A. Xu, G. Zhang, Y. Li, Commun. Theor. Phys. 54, 1121, (2010).
[29] F. Chen, A. Xu, G. Zhang, Y. Li, Commun. Theor. Phys. 55, 325 (2011).
[30] F. Chen, A. Xu, G. Zhang, Y. Li, Phys. Lett. A 375, 2129 (2011).
[31] F. Chen, A. Xu, G. Zhang, Y. Li, Commun. Theor. Phys. 56, 333, (2011).
[32] F. Chen, A. Xu, G. Zhang, Y. Li, Theroe. & Appl. Mech. Lett. 1, 052004 (2011).
[33] A. Xu, G. Zhang, Y. Gan, F. Chen, and X. Yu, Front. Phys. 7(5), 582 (2012).
[34] S. Succi, G. Amati, and R. Benzi, J. Stat. Phys. 81, 5 (1995).
[35] G. Amati, S. Succi and R. Benzi, Fluid Dyn. Res. 19, 289 (1997).
[36] G. Peng, H. Xi, C. Duncan and SH. Chou, Phys. Rev. E 58, R4125 (1998).
[37] G. Peng, H. Xi, C. Duncan and SH. Chou, Phys. Rev. E 59, 4676 (1999).
[38] S. Ubertini, G. Bella and S. Succi, Phys. Rev. E 68, 016701 (2003)
[39] X. He, G. Doolen, J. Comput. Phys. 134, 306 (1997).
[40] R. Mei, W. Shyy, J. Comput. Phys. 143, 426 (1998).
[41] I. Halliday, L. A. Hammond, C. M. Care, K. Good, and A. Stevens, Phys. Rev. E 64, 011208 (2001).
[42] K. N. Premnath and J. Abraham, Phys. Rev. E 71, 056706 (2005).
[43] P. Asinari, S. C. Mishra and R. Borchiellini, Numerical Heat Transfer B 57, 126 (2010).
[44] M. Watari, Commun. Comput. Phys. 9, 1293 (2011).
[45] C. Lin, A. Xu, G. Zhang, Y. Li and S. Succi, Phys. Rev. E 89, 013307 (2014).
[46] B. Yan, A. Xu, G. Zhang, Y. Ying, and H. Li, Front. Phys., 8, 94 (2013).
[47] Y. Gan, A. Xu, G. Zhang, and Y. Yang, EPL 103, 24003 (2013).
[48] C. Lin, A. Xu, G. Zhang, Y. Li, e-print arXiv:1308.0653.
[49] F. Chen, A. Xu, G. Zhang, Y. Wang, Front. Phys. 9, 246 (2014).
[50] A. Xu, G. Zhang, Y. Gan, e-print arXiv:1403.3744.
[51] P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. 94, 511 (1954).