Critical Scaling at Zero Virtuality in QCD

Romuald A. Janik$^1$, Maciej A. Nowak$^1$, Gábor Papp$^2$ and Ismail Zahed$^3$

1 Department of Physics, Jagellonian University, 30-059 Krakow, Poland.
2 ITP, Univ. Heidelberg, Philosophenweg 19, D-69120 Heidelberg, Germany
3 Institute for Theoretical Physics, Eötvös University, Budapest, Hungary

Department of Physics and Astronomy, SUNY, Stony Brook, New York 11794, USA.

(14 September 14, 2018)

We show that at the critical point of chiral random matrix models, novel scaling laws for the inverse moments of the eigenvalues are expected. We evaluate explicitly the pertinent microscopic spectral density, and find it in agreement with numerical calculations. We suggest that similar sum rules are of relevance to QCD at the critical temperature, and even above if the transition is amenable to a Ginzburg-Landau description.

PACS numbers : 11.30.Rd, 11.38.Aw, 64.60.Fr

1. A large number of physical phenomena can be modeled using random matrix models $^4$. An important aspect of these models is their ability to capture the generic form of spectral correlations in the ergodic regime of quantum systems. This regime is reached by electrons traveling a long time in disordered metallic grains $^2$ or virtual quarks moving a long proper time in a small Euclidean volume $^3$.

In QCD, the ergodic regime is characterized by a huge accumulation of quarks eigenvalues near zero virtuality. This is best captured by the Banks-Casher $^5$ relation $\langle |\vec{q}q| \rangle \equiv \Sigma = \pi \rho(0)$, where the nonvanishing of the chiral condensate in the vacuum signals a finite quark density $\rho(\lambda = 0) \neq 0$ at zero virtuality. This behavior is at the origin of spectral sum rules $^6$, which are reproduced by chiral random matrix models $^4$. These sum rules reflect on the distribution of quark eigenvalues and correlations $^6$.

If QCD is to undergo a second or higher order chiral transition, then at the critical point there is a dramatic reorganization of the light quark states near zero virtuality as the quark condensate vanishes. In section 2, we suggest that such a reorganization is followed by new scaling laws, which are captured by a novel microscopic limit. In section 3, we use a chiral random matrix model with a mean-field transition to illustrate our point. In section 4, we explicitly construct the pertinent microscopic spectral distribution in the quenched case and compare it to numerical calculations. In sections 5,6 we argue that the sum rules following from the new scaling, may be of relevance to QCD at the chiral critical point, provided that the transition follows the general lore of universality, and suggest that spectral correlations persist even above the transition temperature.

2. The nonvanishing of $\Sigma$ in the QCD vacuum implies that the number of quark states in a volume $V$, $N(E) = V \int d\lambda \rho(\lambda)$ in the virtuality band $E$ around 0 grows linearly with $E$, that is $N(E) \sim EV$. As a result, the level spacing $\Delta = dE/dN \sim 1/V$ for $N \sim 1$, and the eigenvalues of the quark operator obey spectral sum rules $^3$. During a second or higher order phase transition $\Sigma$ vanishes in the chiral limit. Scaling arguments give $\Sigma \sim m^{1/\delta}$ at the critical point, where $m$ is the current quark mass $^3$. It follows again from the Banks-Casher relation $^5$, that for small virtualities $\Sigma, \rho(\lambda) \sim |\lambda|^{1/\delta}$ to leading order in the current quark mass $^3$. Hence, $N(E) \sim VE^{1+1/\delta}$, and the level spacing is now $\Delta_\ast \sim 1/V^{3/\delta+1}$ at $N \sim 1$.

For a mean-field exponent $\delta = 3$, and we have $\Delta_\ast = V^{-3/4}$, which is intermediate between $V^{-1}$ in the spontaneously broken phase and $V^{-1/4}$ in free space. At the critical point there are still level correlations in the quark spectrum except for the free limit, corresponding formally to $\delta = 1/3$. We now conjecture that at the critical point, the rescaling of the quark eigenvalues through $\lambda \to \lambda/\Delta_\ast$ yields new spectral sum rules much like the rescaling with $\Delta = 1/V$ in the vacuum $^5$. The master formula for the diagonal sum rules is given by the (dimensionless) microscopic density of states

$$\nu_\ast(s) = \lim_{V \to \infty} (V\Delta_\ast) \rho(s\Delta_\ast)$$

and similarly for the off-diagonal sum rules in terms of the microscopic multi-level correlators. Since we lack an accurate effective action formulation of QCD at $T = T_c$ (a possibility based on universality is discussed below), the nature and character of these sum rules is not a priori known, but could easily be established using lattice simulations in QCD.

Could these sum rules be shared by random matrix models? We will postpone the answer to this question till section 5, and instead show in what follows that the present scaling laws hold at zero virtuality for chiral random matrix models with mean-field exponents.

3. Consider the set of chiral random plus deterministic matrices

$$\mathcal{M} = \begin{pmatrix} im & t + A^\dagger \\ t + A & im \end{pmatrix}$$

where $A$ is an $N \times N$ complex matrix with Gaussian weight, $m$ a ‘mass’ parameter, and $t$ a ‘temperature’ parameter. Such matrices or variant thereof have been investigated by a number of authors in the recent past $^4$. Their associated density of states is
\[ \rho(\lambda) = \frac{1}{2N} (\text{Tr} \delta(\lambda - \mathcal{M})) \]

where the averaging is carried using the weight associated to the following partition function

\[ Z[m, t] = \int dA \delta \mathcal{M}^{N_f} e^{-N \text{Tr} \mathcal{A} \mathcal{A}^+}. \]

For \( t = m = 0 \) the density of states is \( \rho(0) = 1/\pi \), while zero for \( t \geq 1 \) and \( m = 0 \). At \( t = 1 \), \( \rho(\lambda) = |\lambda|^{1/3} \), which indicates that the \( t \)-driven transition is mean-field with \( \rho(0) \) as an order parameter. In particular, at \( t = 1 \) the level spacing is \( \Delta_\ast = N^{-1/3} \) near zero.

Standard bosonization of the partition function \( Z \) yields

\[ Z[m, t] = \int dP dP^+ e^{N \log \det \left[ (m + P)(m + P^+) + i^2 \right]} - N \text{Tr} PP^+ \]

where \( P \) is an \( N_f \times N_f \) complex matrix. We may shift \( P \) by the mass matrix \( Q = P + m \) and get

\[ Z[m, t] = \int dQ dQ^+ e^{N \log \det \left[ (Q + m)(Q^+ + m^2) + i^2 \right]} - N \text{Tr} QQ^+ \]

dropping the irrelevant normalization factor. We will now specialize to the critical temperature \( t = 1 \), and denote the rescaled mass by \( x = im/\Delta_\ast \), and eigenvalue by \( s = \lambda/\Delta_\ast \). This suggests the rescaling \( Q \rightarrow \tilde{Q} = N^{1/4}Q \), so that

\[ Z[x] = \int d\tilde{Q} d\tilde{Q}^+ e^{-4 \text{Tr}(\tilde{Q} \tilde{Q}^+)^2} e^{ix \text{Tr}(\tilde{Q} + \tilde{Q}^+)} \]

reducing to

\[ Z[x] = \int_0^{\infty} r dr e^{-4 x^2} J_0(2xr) \]

for one flavor. Expanding this integral in powers of \( x \),

\[ Z[m] = Z[0] \left[ 1 + \sum_{k=1}^{\infty} \frac{2^k (m/\Delta_\ast)^{2k}}{(k!)^2 \sqrt{\pi}} \left( \frac{k + 1}{2} \right) \right] \]

gives rise to spectral sum rules for the moments of the reciprocals of the eigenvalues of \( \mathcal{M} \) by matching the mass power in the spectral representation of the partition function,

\[ Z[m]/Z[0] = \left\langle \prod_{\lambda_k > 0} \left( 1 + \frac{m^2}{\lambda_k^2} \right) \right\rangle_0 \]

where the averaging is done over the Gaussian randomness with the additional measure \( \prod_{\lambda_k > 0} \lambda_k^2 \). Matching the terms of order \( m^2 \) yields

\[ \left\langle \sum_{\lambda_k > 0} \frac{1}{\lambda_k^2} \right\rangle_0 = \frac{2}{\sqrt{\pi} \Delta_\ast^2}, \]

and matching the terms of order \( m^4 \) gives

\[ \left\langle \left( \sum_{\lambda_k > 0} \frac{1}{\lambda_k^3} \right)^2 \right\rangle_0 - \left\langle \sum_{\lambda_k > 0} \frac{1}{\lambda_k^2} \right\rangle_0 = \frac{1}{\Delta_\ast^4}. \]

The relations \( 11 \) are examples of microscopic sum rules at the critical point \( t = 1 \).

One should note here that the preceding calculation has been performed for the gaussian matrix model. It turns out that if we were to add terms of the form

\[ e^{-N g_0 \text{Tr}(PP^+)^2 - N g_1 (\text{Tr} PP^+)^2} \]

to the measure in \( 3 \), then the spectral sum rules are in general unchanged. Indeed, for \( N_f = 1 \) the addition amounts to a global shift \( x \rightarrow x/(1+2g_0+2g_1) \). Higher powers of \( PP^+ \) are subleading after rescaling and do not affect the sum rules in large \( N \).

4. The diagonal moments of the reciprocals of the rescaled eigenvalues are generated by the microscopic density \( \rho(x) \), in the limit \( N \rightarrow \infty \) and \( \lambda \rightarrow 0 \) but \( s = \lambda/\Delta_\ast \) fixed. The microscopic density \( \rho(x) \) for \( N_f \) flavours in a fixed topological sector \( n \) : \( \nu_n, N_f, n \) can be evaluated using supersymmetric methods \( 10 \). For example,

\[ \nu_n(0, s) = -\frac{48}{\pi^2} (k_2(s) j_0(s) + k_0(s) j_2(s) - k_1(s) j_1(s)) \]

where \( k_n(s) \) and \( j_n(s) \) are given by

\[ j_n(s) = \int_0^{\infty} dz \ z^{2n+1} J_0(2z s) e^{-z^2/4} \]

\[ k_n(s) = \int_0^{\infty} dz \ z^{2n+2} K_1(2z s) e^{-z^2/4} \]

and where the integration contour \( C \) is the sum of two lines: \( C = (-1+i)\infty, 0 \cup [0, (1-i)\infty). \) In this spirit, the first sum rule \( 11 \) reads

\[ \int_0^{\infty} \frac{1}{s^2} \nu_{1,0}(s) ds = \frac{2}{\sqrt{\pi}}. \]

The second sum rule \( 12 \) involves the 2-level microscopic correlator for \( N_f = 1 \) and \( n = 0 \) which can be obtained using a similar reasoning.

In Fig. 2 we compare the quenched \( N_f = 1 \) (left) and unquenched \( N_f = 1 \) (right) results to the numerically generated microscopic spectral density at \( t = 1 \) using \( N = 100 \) size matrices. The agreement suggests that the present method of finding the scaling properties of microscopic spectral distributions can be used to accurately determine the value of the critical exponent \( \delta \) in lattice simulations.
5. In QCD the finite temperature transition is still debated. For very light quark masses, it is suspected to be second order, although a cross over is not ruled out. The character of the transition depends crucially on the number of flavors $N_f$ and the fate of the $U_A(1)$ quantum breaking. For two light flavors, we may assume with Pisarski and Wilczek [12] that the transition is from an $SU(2)$ spontaneously broken phase to $Z_2 \times SU(2) \times SU(2) \sim O(4)$ [3]. Choosing the order parameter $\Phi = \sigma + i \bar{\tau} \cdot \vec{\pi}$ (vacuum analogous to a ferromagnet), implies for the Ginzburg-Landau potential

$$V(\Phi) = +m \text{Tr}(\Phi^4 + \Phi) + g_0(T) \text{Tr}(\Phi^4) + g_1(T) \left(\text{Tr}(\Phi^4)\right)^2 + \ldots$$ (18)

at zero vacuum angle $\theta$. The dots refer to marginal or irrelevant terms. For a second order transition, $g_0(T) \sim T - T_c$, which is negative below $T_c$ and positive above. Note, that in this section we analyze this hamiltonian just from the mean-field point of view by truncating it to the space of constant modes.

For $T = T_c$ the potential (18) is reminiscent of the one for the chiral random matrix model discussed above [3]. If we note that the measure on the manifold with restored symmetry is $e^{-\beta VV^\dagger} \equiv e^{-VV^\dagger}$, we conclude that (13) is enough to accommodate for the level spacing $\Delta_c = 1/\sqrt{\delta/(\delta+1)}$ with $\delta = 3$ (mean-field). Indeed after the rescaling $x = im/\Delta_c$ we recover (13) with $g_1 = 1/2$ and the proper identification of the manifold.

For $T > T_c$ we can use (13) to define new sum rules for the quark eigenvalues in the sector with zero winding number [3]. In particular ($N_f = 2$)

$$\frac{1}{V^2} \left( \sum_{\lambda>0} \hat{\lambda}^2 \right)_0 \sim \frac{1}{2N_f} \frac{2\pi}{2\pi} \left( \text{Tr}(e^{-\sigma T} \Phi^4 + \Phi e^{-\sigma T}) \right)^2 .$$ (19)

The rhs measures the variance in the scalar direction on an invariant $O(4)$ manifold with $e^{-VV^\dagger}$ as a measure. As $T \to T_c$, from above, the scalar susceptibility averaged over ‘$\theta$-states’ (rhs of (19)) diverges since $\sigma$ and $\pi$ become degenerate. These modes are the analogue of the ones originally discussed by Hatsuda and Kunihiro [14] using an effective model of QCD. For the present case, the calculation is done readily by rescaling $\Phi \to \sqrt{\Phi}$ and noting that the quartic contribution in (18) becomes subleading in large $V$. Hence,

$$\frac{1}{V} \left( \sum_{\lambda>0} \frac{1}{\lambda^2} \right)_0 = \frac{1}{2g_0(T)} .$$ (20)

Near $T_c$ from above it is seen to diverge as $1/(T - T_c)$ with the critical exponent $\gamma = 1$ (mean-field). We note that above the critical temperature, the level spacing is $V^{-1/2}$, which is intermediate between $V^{-1}$ in the vacuum and $V^{-1/4}$ in free space. Other sum rules are of course possible.

The possibility of microscopic sum rules above the critical temperature reflects on the assumption that the symmetry restoration in two-flavor QCD is driven by universality. The level correlations in the Dirac spectrum attests to the correlations still present on the $O(4)$ manifold despite the gap developing in the eigenvalue density. The gap is due to the fact that near zero virtuality the accumulation of eigenvalues is not commensurate with the volume $V$. At high temperature, the quark eigenvalues are typically of order $T$, and both $\pi$ and $\sigma$ correlations are dissolved in the ‘plasma’ upsetting the present universality arguments. In QCD we expect this to take place at a temperature $T \sim 3T_c$ [13].

6. The idea of microscopic sum rules above the critical temperature may be readily checked using an (unquenched) chiral random matrix model with one flavor. Again we may use the bosonized form of the partition function [3], but now instead introduce the variable $y = \sqrt{\lambda} \sim \sqrt{N} \lambda$, and rescale $Q$ by $Q \to Q = Q\sqrt{N}$. This leads to the following expression for the partition function

$$Z(y) = t^{2N} \int_0^\infty dr e^{-(1+\frac{1}{\lambda})r^2} J_0(2yr)e^{y^2} = \frac{\pi t^{2N}}{N} \frac{t^2}{t^2 - 1} e^{-\frac{t^2}{t^2 - 1} y^2} .$$ (21)

Expanding the partition function in powers of $y$ leads again to modified sum rules, the simplest example being

$$\left( \sum_{\lambda>0} \frac{1}{\lambda^2} \right)_0 = \frac{1}{t^2 - 1} .$$ (22)

which is seen to diverge at $t = 1$ as expected.

The comparison to numerical simulation is shown in Fig. 2 using a random set of matrices $M$ distributed with a Gaussian measure. For high temperatures or large matrix sizes the agreement is good. At the critical temperature, the finite size effects are important. Amusingly, we note the drop by two orders of magnitude at $t \sim 3 = 3t_c$. 

FIG. 1. left: $\nu_{0,0,0}(s)$ at $t = 1$ and $N_f = 0$ for matrices of size $N=100$ (dots) and the theoretical prediction (13) (solid line). right: $\nu_{1,0,0}(s)$ at $t = 1$ and $N_f = 1$ for matrices of size $N = 20$ (dotted), $N = 50$ (dashed) and $N = 100$ (solid).
The Green’s function both for quenched (the mass playing the role of an external parameter) and one flavor chiral random matrix model in the rescaled variables may be readily obtained,

\[ G(y) = \frac{1}{\sqrt{N}} \sum_i \frac{1}{y - \sqrt{N} \lambda_i} = -\frac{1}{\sqrt{N}} \frac{y}{t^2 - 1}. \]  

(23)

Since for \( T > T_c \) the eigenvalue spectrum develops a gap, there are no eigenvalues for small values of \( y \), hence the resolvent is purely real. For finite sizes however, \( y \) may get out of the gap and our result breaks down as well as the scaling arguments. This is seen in Fig. 3, where for temperatures slightly above the critical one we are entering the nonzero eigenvalue density part of the spectra. This effect is shifted for higher values with increasing temperature and matrix size.

**FIG. 3.** Scaled and quenched resolvent (23) (solid line) in comparison to a numerical simulation with \( N = 20 \) (long dashes), \( N = 50 \) (short dashes) and \( N = 100 \) (dotted line) random matrices.

7. Using arguments based on universality we have suggested that the QCD Dirac spectrum may exhibit universal spectral correlations at \( T = T_c \) that reflect on the nature of the chirally restored phase. To illustrate our points, we have used a chiral random matrix model. Although the matrix model is based on a Gaussian weight, we provided physical arguments for why the results are insensitive to the choice of the weight at the critical point. In QCD this can be readily checked using our recent arguments at finite temperature, since the closest singularity to zero in the virtuality plane is persistently ‘pionic’ for \( T \leq T_c \).

The existence of a microscopic spectral density at \( T = T_c \) for QCD opens up the interesting possibility of measuring both the critical temperature \( T_c \) and the critical exponent \( \delta \) by simply monitoring the pertinently rescaled distribution \( \nu_\epsilon(s) \) of eigenvalues in lattice QCD simulation. We have also suggested that these correlations may persist even above \( T_c \).

**Acknowledgements**

After completing this paper we noticed the paper by Brezin and Hikami [cond-mat/9804023] were similar issues are discussed using different arguments. This work was supported in part by the US DOE grant DE-FG-88ER40388, by the Polish Government grant Project (KBN) grants 2P03B04412 and 2PB03B00814 and by the Hungarian grants FKFP-0126/1997 and OTKA-T022931. RAJ is supported by the Foundation for Polish Science (FNP).

[1] K. Efetov, ‘Supersymmetry in Disorder and Chaos’, Camb. UP, NY (1997), and references therein.
[2] D.J. Thouless, Phys. Rep. 13 (1974) 93.
[3] R.A. Janik, M.A. Nowak, G. Papp and I. Zahed, hep-ph/9803288.
[4] T. Banks and A. Casher, Nucl. Phys. B169 (1980) 103.
[5] H. Leutwyler and A. Smilga, Phys. Rev. D46 (1992) 5607.
[6] E. Shuryak and J. Verbaarschot, Nucl. Phys. A560 (1993) 306.
[7] J. Verbaarschot and I. Zahed, Phys. Rev. Lett. 70 (1993) 3852.
[8] A. Kocic, J. Kogut and M. Lombardo, Nucl. Phys. B398 (1993) 376.
[9] R.A. Janik, M.A. Nowak and I. Zahed, Phys. Lett. B392 (1997) 155.
[10] E. Brezin, S. Hikami and A. Zee, Phys. Rev. E51 (1995) 5442; A. Jackson and J. Verbaarschot, Phys. Rev. D53 (1996) 7223; T. Wettig, A. Schafer and H. Weidenmüller, Phys. Lett. B367 (1996) 28; M.A. Nowak, G. Papp and I. Zahed, Phys. Lett. B389 (1996) 137.
[11] K. Efetov, Adv. Phys. 32 (1983) 53.
[12] R. Pisarski and F. Wilczek, Phys. Rev. D29 (1984) 338.
[13] The \( Z_2 \) factor is from the persistent \( U_A(1) \) breaking effects.
[14] T. Hatsuda and T. Kunihiro, Phys. Rev. Lett. 55 (1985) 158.
[15] I. Zahed, Act. Phys. Pol. B25 (1994) 99.