Progress in manufacturing and a deeper understanding of electronic devices have caused an increased interest in the field of spintronics [1]. For example, the spin-orbit coupling occurring in various materials such as semiconductors provides a handle for the electron spin. In fact, the spin-orbit coupling is a main ingredient for both the spin Hall effect [2] and its companion, the anomalous Hall effect (AHE) [3]. This makes these effects attractive from a practical point of view, and not only as phenomena on their own right. However, despite its long history, the underlying physics of especially the latter, the AHE, has remained a controversial subject [4].

Already E. Hall discovered that the Hall effect is greatly enhanced in ferromagnetic compared to non-magnetic materials [5]. This became known as the AHE. It is only recently, with knowledge of geometrical phases, that a more complete picture of the AHE has been obtained [6]. The processes contributing to the AHE can be divided into three categories: (i) intrinsic (deriving from the band structure, and consequently from the spin-orbit coupling), (ii) skew scattering (originating from distorted scattering of impurities), and (iii) side-jump (often identified as the contribution making up for the difference between the total AHE conductivity and the intrinsic plus skew conductivities). While the first has a topological origin, the last two contributions are due to impurities in the material. Numerous measurements of the AHE conductivity have been presented [8], but identifying, and even controlling, the different mechanisms underlying the AHE current remains an extremely non-trivial task. In the realm of spintronics, it would therefore be desirable to come up with alternative systems free from impurities such that the AHE current derives solely from the spin-orbit coupling, which could, for example, lead to accomplishing effective Datta-Das transistors [2]. Overcoming the obstacle of impurities with solid state devices seems very difficult, and one should preferably look in other directions.

Two possible candidates, both known to exhibit Hall characteristics, are either rotating Bose-Einstein condensates [8] or cold atoms in optical lattices [9]. Due to the purity and high controlled versatility of these systems, great interest has been paid to them during the recent past [10]. The AHE has been predicted to occur on the p-band of cold fermions in an optical lattice [11]. An optional system considers cold atoms coupled to spatially varying light fields [12], in which the spin Hall effect has been predicted [13]. Regardless of the development of novel experimental techniques in cooling and trapping cold atom gases, presently none of the different Hall effects nor non-Abelian properties have been realized utilizing cold atoms.

Yet another system endowed from impurities and possessing long coherence time scales is cavity quantum electrodynamics (QED); a single atom interacting with an isolated set of cavity modes [14]. During the last two decades, cavity QED has successfully demonstrated among others, entanglement generation [15], the quantum measurement problem and the quantum-classical transition [16], and verification of the graininess of the quantized electromagnetic field [17]. Moreover, recent experiments have realized coherent coupling of single quantum dots [18] or Bose-Einstein condensates [19] to a cavity mode, paving the way for the possibility of reaching a super-strong coupling regime of cavity QED.

Notwithstanding the many advantages provided by cavity QED models, relatively few schemes employing them as quantum simulators have been suggested. In Ref. [20] it was presented how effective non-Abelian gauge potentials arise in bimodal cavity QED models. Similar cavity QED systems had also been shown to mimic Jahn-Teller models frequently appearing in molecular and condensed matter physics [21]. A proposal how to realize magnetic monopoles by the means of cavity QED systems was outlined in [22]. A model similar in structure, the cold trapped ion system, was analyzed.
in [23] and it was put forward how it may simulate relativistic particles (see also [24]). In this paper we demonstrate the appearance of transverse phase space currents, evocative of intrinsic AHEs, in a bimodal cavity QED system.

We consider a high-$Q$ cavity containing a two-level atom (quantum-dot) dipole-interacting with two degenerate but orthogonal cavity modes. The system Hamiltonian reads

$$\hat{H} = \hat{H}_f + \hat{H}_a + \hat{H}_I$$

where

$$\hat{H}_f = \hbar \omega \left( \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \right), \quad \hat{H}_a = \sum_{j=1,2} E_j |j\rangle \langle j|,$$

$$\hat{H}_I = \vec{d} \cdot \vec{E}(\mathbf{x}).$$

Here, $\hat{a}^\dagger$ and $\hat{b}^\dagger$ ($\hat{a}$ and $\hat{b}$) represent the creation (annihilation) operators for the two modes, $\omega$ is their common mode frequency, $E_j$ the internal atomic energy of state $|j\rangle$, $\vec{d}$ is the dipole transition moment, and $\vec{E}(\mathbf{x})$ is the electric field. In terms of the internal atomic states, the components of the dipole moment operator becomes $d_a = -e|1\rangle\langle 2| - e|2\rangle\langle 1|$, where $e$ is the electron charge and $\alpha = x, y, z$. In the dipole approximation we set $\mathbf{x} = 0$, and the field is written as

$$\vec{E} = \varepsilon_1 \mathbf{E}_1 i \left( \hat{a} - \hat{a}^\dagger \right) + \varepsilon_2 \mathbf{E}_2 i \left( \hat{b} - \hat{b}^\dagger \right),$$

where $\varepsilon_j$ is the polarization vector for mode $j$ and $\mathbf{E}_j$ its corresponding field amplitude. A deeper insight of the system characteristics is obtained by expressing the fields in their quadrature operators

$$\hat{X}_1 = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger), \quad \hat{P}_1 = \frac{i}{\sqrt{2}} (\hat{a} - \hat{a}^\dagger),$$

$$\hat{X}_2 = -\frac{i}{\sqrt{2}} (\hat{b} + \hat{b}^\dagger), \quad \hat{P}_2 = \frac{\hbar}{\sqrt{2}} (\hat{b} - \hat{b}^\dagger)$$

obeying the canonical commutation relations; $[\hat{X}_k, \hat{P}_{k'}] = i\hbar \delta_{kk'}$. For brevity we introduce the Pauli operators $\hat{\sigma}_x = |1\rangle\langle 2| + |2\rangle\langle 1|$, $\hat{\sigma}_y = -i|1\rangle\langle 2| + i|2\rangle\langle 1|$, $\hat{\sigma}_z = |1\rangle\langle 1| - |2\rangle\langle 2|$, and set the zero energy such that $E_1 = \hbar \Omega / 2$ and $E_2 = -\hbar \Omega / 2$. Furthermore, we choose $\hat{g}_1 \equiv \vec{d} \cdot \varepsilon_1 \mathbf{E}_1 \sqrt{2} = -g \hat{\sigma}_x$, and assume $\hat{g}_2 \equiv \vec{d} \cdot \varepsilon_2 \mathbf{E}_2 \sqrt{2} = -g \hat{\sigma}_y$ with $g$ real. The resulting Hamiltonian is the $E \times \varepsilon$ Jahn-Teller one [21]

$$\hat{H}_{E \times \varepsilon} = \hbar \omega \sum_{k=1,2} \left( \frac{\hat{P}_k^2}{2} + \frac{\hat{X}_k^2}{2} \right) + \hbar \Omega / 2 \hat{\sigma}_z - \hbar g \left( \hat{\sigma}_x \hat{P}_1 + \hat{\sigma}_y \hat{P}_2 \right),$$

first introduced in the molecular physics community [23]. Via a simple $\pi$-rotation around the $\hat{\sigma}_x$ axis, the atom-field interaction term attains a Rashba form [24]. In the absence of $\hat{X}_1, \hat{X}_2$-terms, the same type of Hamiltonian is frequently used to describe a two dimensional gas of free spin-orbit coupled electrons. Adding the parabolic potential renders, on the other hand, a stereotype model representing spin-orbit coupled quantum dots [24]. Such confinement naturally restricts the particle motion, but nonetheless, as will be shown spin-orbit induced transverse currents still play a crucial role for the system dynamics.

The adiabatic potential surfaces (APS) of $\hat{H}_{E \times \varepsilon}$, defined as $V_{\text{APS}}(P_1, P_2) = \hbar \omega (P_1^2 + P_2^2) / 2 \pm \hbar \sqrt{\Omega^2 / 4 + g^2 (P_1^2 + P_2^2)}$, are envisaged in Fig. 1. These have a polar symmetry and possess a conical intersection at the origin [24]. The $\hat{\sigma}_z$-term split the degeneracy at the origin, and whenever $g < \sqrt{\omega \Omega}$ the sombrero shape of the lower APS is lost and instead a global minimum at the origin is attained. Conical intersections appearing in momentum space are frequently referred to as Dirac points due to their linear dispersions.

![Figure 1: The APSs of the $E \times \varepsilon$ Hamiltonian (5). The conical intersection is located at the origin, $P_1 = P_2 = 0$.](image-url)

The Hamiltonian is conveniently written on the form

$$\hat{H}_{E \times \varepsilon} = \hbar \omega \sum_{j=1,2} \left( \frac{\hat{P}_j - \hat{A}_j}{2} \right)^2 + \frac{\hbar \Omega}{2} \hat{\sigma}_z + \hat{\Phi}$$

where

$$(\hat{A}_1, \hat{A}_2) = \frac{g}{\hbar} (\hat{\sigma}_x, \hat{\sigma}_y), \quad \hat{\Phi} = -\hbar \frac{g^2}{\omega}.$$

The quantities $\hat{A}_k$ and $\hat{\Phi}$ are effective vector and scalar potentials respectively. The gauge invariance follows directly from considering their response to unitary transformations;

$$\hat{A} \rightarrow U^\dagger (\hat{X}, t) \hat{A} U (\hat{X}, t) - U^\dagger (\hat{X}, t) \frac{\partial}{\partial \hat{X}} U (\hat{X}, t),$$

$$\hat{\Phi} \rightarrow U^\dagger (\hat{X}, t) \hat{\Phi} U (\hat{X}, t) - i \hbar U^\dagger (\hat{X}, t) \frac{\partial}{\partial t} U (\hat{X}, t).$$

Pointed out in Ref. [28] the intrinsic AHE is a direct result of a non-zero Berry phase originating from a vector potential (also called Mead-Berry curvature). One way to illustrate the appearance of the Hall effect is by the generalized spin-dependent Lorenz force analogous to the
work by Wong on color-charged classical particles moving in non-Abelian fields. As an outcome of the gauge potentials, there is an associating magnetic field

\[ \hat{B}_i = \frac{1}{2} \varepsilon_{ijk} \hat{F}_{kl}, \quad \hat{F}_{kl} = \partial_k \hat{A}_l - \partial_l \hat{A}_k - i[\hat{A}_k, \hat{A}_l]. \]  

The first two terms of \( \hat{F}_{kl} \) are identically zero in our model. On the other hand, the last term, deriving from the non-Abelian structure of the system, is indeed non-zero. The effective magnetic field, being proportional to \( \hat{A}_z \), induces a state (spin) dependent force; the magnetic field is either in the positive or negative \( z \)-direction depending on the internal atomic state \( |1\rangle \) or \( |2\rangle \).

Due to the opposite direction of the force on the internal states, the system may act as a spin-filter (Datta-Das transistor) which explains its interest for spintronics. Here, however, the intrinsic transverse AHE current does not take place in real space as in solid state devices, but in a phase space representation of the field states.

For typical atom cavity QED parameters it is legitimate to apply the rotating wave approximation, which would cast the Hamiltonian into a bimodal Jaynes-Cummings one. On the other hand, for circuit cavity QED employing solid state quantum dots/SQUIDs instead of true atoms, imposing the rotating wave approximation is usually not justified. In fact, in the parameter regimes where this approximation is applicable one finds that the transverse current rendered by the spin-orbit coupling is rather weak, and spontaneous emission of the atom as well as cavity decay will thereby become significant within the time needed to build up a noticeable transverse current.

For example, the transverse AHE current is driven by an electric field having the effect of tilting the 2D \( X_1 X_2 \)-plane. In this cavity QED setting, a comparable tilt is generated by external driving of the cavity. Pumping with an amplitude \( \eta \) of say mode 1, implies a term \( \hat{H}_{\text{pump}} = \hbar \eta \left( \hat{a}_1 + \hat{a}_1^\dagger \right) = \hbar \sqrt{2} \eta \hat{X}_1 \) in the Hamiltonian. This induces a position shift of oscillator 1. An alternative way of picturing the pumping is to displace the initial field state of mode 1 by the operator \( \tilde{D}(\alpha) = e^{\alpha \hat{a}_1^\dagger - \alpha^* \hat{a}_1} \), with \( \alpha = \eta \sqrt{2}/\omega \). In our numerical simulations we will consider this alternative viewpoint by assuming the initial state of mode 1 to be displaced by \( \alpha \), whereby we omit any pumping term \( \hat{H}_{\text{pump}} \) in our numerical calculations.

In a solid, impurities prevents the electrons to accelerate to infinity causing an equilibrium situation. It is known that a constantly accelerating situation results in no net transverse current. The present model lacks impurities, but on the other hand the confining potential of Eq. (10) restricts the motion in the \( X_1, X_2 \)-plane and the spin-orbit induced AHE is thereby still apparent in the phase space motion.

To demonstrate the parallel of an intrinsic AHE in the present system we solve the Schrödinger equation employing the split-operator wave packet method. In particular we utilize the experimental parameters of the circuit QED experiment presented in Ref. [18] and moreover assume the atom (quantum-dot) to be initially in its ground state \(|1\rangle \), mode 2 in vacuum \(|0\rangle_2\), and mode 1 in a coherent state \( |\alpha_{10}\rangle_1 \) (displaced vacuum);

\[ \psi_1(X_1, 0) = \left( \frac{1}{\pi} \right)^{1/4} e^{-i P_{10} X_1} e^{-\left(X_1 - X_{10}\right)^2/2} \]  

Here, \( X_{10} \) and \( P_{10} \) are respectively the initial position and momentum of the coherent state, related to the amplitude as \( \alpha_{10} = (X_{10} - i P_{10})/\sqrt{2} \). By choosing \( P_{10} = 0 \) and \( X_{10} > 0 \), the combined field wave packet starts out with a zero momentum at \( (X_1, X_2) = (X_{10}, 0) \). Apart from the potential force causing the wave packet to slide down the harmonic potential, in addition it experiences the effective transverse Lorentz force originating from the magnetic field. For an Abelian gauge potential \( (\hat{g}_1 \propto \hat{g}_2) \), the Lorentz force vanishes and the field wave packet would remain localized along the \( X_1 \)-axis throughout the propagation. In the present model, however, the oscillatory motion will depart from the \( X_1 \)-axis inducing a non-zero field in mode 2. As time progresses, population is transferred from mode 1 to mode 2 until finally mode 1 is in vacuum. Consequently, in phase space, mode 1 will follow an inward spiral trajectory while mode 2 follows an outward trajectory until the population is completely swapped between the two modes after some time \( t_\varphi \). This is analogous to the intrinsic AHE; considering polar coordinates \( r \) and \( \varphi \) instead, one has that the radial motion along \( r \) induces a transverse rotational current in the \( \varphi \)-direction. For longer time periods, the process starts over again with the population of the two modes interchanged. The spiral motion is depicted by the black

![Figure 2: (Color online) The upper plot (a) displays the average radius of the phase space distributions as function of time. Black curves shows the ideal situation without losses, while for the light blue curve losses have been taken into account. The two lower plots present the same time evolution but of \((X_1, X_2)\) vs. \(\langle X_2 \rangle\) without (b) and with (c) losses. The system parameters are \( P_{10} = 0 \), \( X_{10} = 0 \), \( X_{20} = 0 \), \( \Omega/2\pi = 6.9 \) GHz, \( \omega/2\pi = 5.7 \) GHz, \( g/2\pi = 105 \) MHz, \( \gamma/2\pi = 1.9 \) MHz, and \( \kappa/2\pi = 250 \) kHz.](image)
curve in Fig. 2 (a) showing the averages \( \langle X_i^2 \rangle + \langle \dot{P}_i \rangle ^2 \) for initial values \( X_{20} = P_{10} = P_{20} = 0, X_{10} = 10 \), and the atom initially in \( |1\rangle \). Figure 2 (b) displays the corresponding time evolution of \( \langle X_1 \rangle \) versus \( \langle X_2 \rangle \). At about 80 ns (for the utilized parameters), the oscillations are almost entirely along the \( X_2 \)-axis indicating the swapping of population between the two modes.

So far we considered a closed system neglecting system losses. From Fig. 2 it is clear that the two modes perform a very large number of individual oscillations within the time \( t_s \). Thus, decoherence of both the cavity fields and the atom may become important over such time periods. To investigate the effect of losses, we consider the effective non-hermitian Hamiltonian \( H_{\text{eff}} = H_{E \times E} - i \kappa \left( \hat{a} \hat{\alpha} + \hat{b} \hat{\beta} \right) - i \gamma |2\rangle \langle 2| \). Here \( \kappa \) and \( \gamma \) are the photon decay rate and the atomic spontaneous emission rate respectively. The results corresponding to the black curves of Fig. 2 but where experimental values of \( \kappa \) and \( \gamma \) have been considered, are shown as light blue curves in Fig. 2. The AHE clearly survives despite the relatively long interaction time, only a slight drop in field intensity, reminiscent of momentum relaxation, is found. It is understood that this analysis of the effect of an environment does not correctly capture decay of coherence of the system state. This, however, should not be a problem in the present situation as we are here only interested in the field amplitude of the two modes and not coherent superpositions nor the precise values of the quadrature operators.

Summarizing, in this work we have shown how transverse phase space currents, reminiscent of AHEs, naturally occur in bimodal cavity QED systems. The phenomenon has been explained in terms of an effective non-Abelian gauge field and its corresponding state-dependent Lorentz force acting upon the combined phase space distributions. In particular, the effective gauge potentials induces phase space trajectories having spiral shapes; starting in coherent states, one field will follow an inward spiral trajectory and the other an outward spiral trajectory causing in general a swapping of population between the two modes. Utilizing experimental parameters, we showed that the intrinsic AHE should be visible in current experiments even in the presence of losses. Finally, we point out that bimodal cavity experiments have been performed \[33\], and moreover that, apart from field intensities, even the field quadrature operators \[4\] are easily measured. Indeed, the ENS group of S. Haroche recently presented experimental results were the full phase space distribution of a cavity mode was assessed \[34\]. It should also be mentioned that the general idea presented in this work is not restricted to a two-level atom. It equally well applies to other atom-field configurations, such as for example a three-level Λ-atom coupled to two cavity modes.

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