Modified Friedmann equations in braneworld scenario

Kevin F S Pardede¹, Agus Suroso¹,² and Freddy P Zen¹,²
1Theoretical Physics Division, Institut Teknologi Bandung, Jl. Ganesha 10, Bandung 40132, Indonesia
2Indonesia Center for Theoretical and Mathematical Physics (ICTMP), Jl. Ganesha 10, Bandung 40312, Indonesia
E-mail: kevinfrankly@rocketmail.com

Abstract. Firstly we present a brief discussion about braneworld model and the derivation of Friedmann equations in the braneworld scenario. Next we carry out a dynamical analysis of the modified Friedmann equations. Lastly, we perform a numerical computation using the Metropolis-Hastings algorithm to compute several free parameters of the modified Friedmann equations. Among others, we found that the brane tension is not large enough to support a nucleosynthesis process which correctly reproduce the abundance of present light elements. Also, we found a negative dark-radiation energy density, and thus indicating a possibility of non-singular initial universe.

1. Introduction
Modification of general relativity, has long been one of the main theme of research in theoretical physics. Although the success of general relativity of passing several test for its correctness is indisputable (for example: general relativity predicts the existence of gravitational waves that has only been recently detected), there are still lot of questions remain unanswered. One of those problems is that when we try to apply general relativity to cosmology, it is found that ∼ 95% of our universe is unknown (the dark energy + dark matter). Our past experience regarding the need of inventing of somewhat artificial object (in this case the dark energy/matter) might be a clue that our theory needs to be modified (for example: the existence of aether isn’t necessary after we find a better theory than Newtonian mechanics, which is the theory of special relativity by Einstein).

One of the branch of modified general relativity is to consider a conventional spacetime (3+1 dimension) with extra dimension (which usually is a spatial dimension). In the context of particle physics, the development of braneworld scenario (a model of universe, where a brane which is a hypersurface embedded in the full spacetime/bulk), has its roots from the needs to explain the gauge hierarchy problem (the large discrepancy between the planck mass $M_P$ and electroweak mass scale $M_{EW}$: $M_P/M_{EW} \sim 10^{16}$). One of the earliest model is the large extra dimension developed by Nima, et al [1], which basically “solved” the problem, given there are ≥ 2 extra dimensions compactified within a radius in the millimeter-order. The unusually large extra dimension predicted by the previous model, leads to a generally better model by Randall-Sundrum [2],[3] for the case of two and single brane respectively in a five dimensional warped spacetime, where the radius of the extra dimension is $\sim M_P^{-1}$.

A more general framework to obtain the formulation of general relativity in braneworld scenario is developed by Shiromizu, et al [4]. Assuming that the Einstein equation holds in
five dimension, the projection of the Einstein equation in the usual four dimension giving us the effective Einstein equation. The effective Einstein equation obtained differs from the usual Einstein equation in the sense that it is not closed (contain a terms that depend on the bulk spacetime) and that it contains a term proportional to square of energy momentum tensor.

One way out this problem (non-closed equation) is a method called gradient expansion method [5],[6] developed by Kanno and Soda. This method assume that the energy density of the matter is small relative to the energy scale of the brane (in this case is called brane tension). The basic idea then is to solve the Einstein equation perturbatively with the ratio of energy scale of the matter and brane as the perturbation parameter [7]. Using this method, several generalization of Randall Sundrum braneworld model has been developed, for example the 3 brane model [8],[9] or the braneworld of brane with dimension (4+n) living in a (5+n) dimensional bulk [10],[11].

In this paper, unless otherwise stated, we use the following index convention: $M,N = 0, 1, 2, 3, \ldots, n, \mu, \nu = 0, 1, 2, 3$; and $a, b = 4, 5, 6, \ldots, n$, for $n$-dimensional spacetime. To further distinguish the four or five dimensional tensor, we use the following convention:

$$
({m}) A_{\mu_1 \mu_2 \ldots \nu_1 \nu_2 \ldots} = \text{tensor } A \text{ in } m \text{ dimension spacetime.}
$$

Also as usual, the subscript 0 means that we considered the present epoch value of the concerned cosmological object. For example $H_0$ means the present epoch Hubble constant.

2. Modified Friedmann Equations

Let $\mathcal{M}$ be our five dimensional spacetime whose coordinate is $x^M = (x^\mu, y \equiv x^4)$. Consider a five dimensional braneworld scenario composed of single spacelike brane located at $y = 0$. In the Gaussian normal coordinate, the metric reads

$$
d s^2 = (5) g_{MN} dx^M dx^N = dy^2 + (4) g_{\mu \nu} dx^\mu dx^\nu. \quad (1)
$$

The five and four dimensional metric: $(5) g_{MN}$, $(4) g_{\mu \nu}$, respectively satisfying the following expression

$$(5) g_{MN} = (4) g_{MN} + n_M n_N, \quad (2)$$

where $n_M$ is the unit normal vector with respect to the brane, satisfying $n_M n^M = 1$ and $(4) g_{MN} n^N = 0$.

Assume that the Einstein equation holds in five dimensions

$$(5) G_{MN} = \kappa_5^{25} T_{MN}, \quad (3)$$

Now let us decompose the five dimensional energy-momentum tensor as follows

$$(5) T_{MN} = -\Lambda_5 g_{MN} + (4) T_{MN} \delta(y), \quad (4)$$

where $\Lambda_5$ is the bulk cosmological constant. The four dimensional energy momentum tensor $(4) T_{MN}$ which is localized in the 3-brane ($y = 0$) can be further decomposed as follows

$$(4) T_{MN} = -\lambda q_{MN} + \tau_{MN}, \quad (5)$$

where $\lambda$ is the brane tension and the matter energy momentum tensor $\tau_{MN}$ satisfy $\tau_{MN} n^N = 0$ ($\tau_{MN}$ has no normal component relative to the brane).

Using the Gauss-Codazzi equation the five dimensional Einstein equation can be projected into the four dimensional spacetime, and thus after imposing the $\mathbb{Z}_2$ symmetry on this spacetime (with respect to $y = 0$), resulting the renowned SMS (Shiromizu-Maeda-Sasaki) equation [4],[12]

$$(4) G_{\mu \nu} = -\Lambda_4 q_{\mu \nu} + 8 \pi G_N \tau_{\mu \nu} + \kappa_5^4 \tau_{\mu \nu} - E_{\mu \nu}, \quad (6)$$
where
\[
\Lambda_4 = \frac{1}{2} \kappa_5^2 (\Lambda_5 + \frac{1}{6} \kappa_5^2 \lambda^2),
\]
(7)
\[
G_N = \frac{\kappa_6^3 \lambda}{48\pi},
\]
(8)
\[
\tau_{\mu\nu} = -\frac{1}{4} T_{\mu\alpha} \tau^{\alpha}_{\nu} + \frac{1}{12} \tau_{\mu\nu} + \frac{1}{8} q_{\mu\nu} \tau_{\alpha\beta} \tau^{\alpha\beta} - \frac{1}{24} q_{\mu\nu} \tau^2,
\]
(9)
while \( E_{\mu\nu} \) is the projected five dimensional Weyl tensor \( (5)^A_{BRS} \)
\[
E_{\mu\nu} \equiv (5)^A_{BRS} n_A n_R q^B q^S.
\]
(10)

Now as an application in cosmology, we modeled our four dimensional spacetime as a spatially homogeneous and isotropic space, which mathematically means a spacetime with the Robertson-Walker (RW) metric
\[
(4)^A_{\mu\nu} = \text{diag}(-1, a^2 \gamma_{ij}),
\]
(11)
where \( a(t) \) is the scale factor, and \( \gamma_{ij} \) is the metric for three dimensional maximally symmetric space which is parametrized by \( \kappa = -1, 0, 1 \) (respectively: hyperbolic, flat, and spherical space) Explicitly for \( (x_1, x_2, x_3) = (r, \theta, \phi) \)
\[
d s^2 = \gamma_{ij} d x^i d x^j
\]
\[
= \frac{d r^2}{1 - kr^2} + r^2 d \theta^2 + r^2 \sin^2 \theta d \phi^2,
\]
(12)
where \( k \equiv \kappa / R^2, \) and \( R \) is the radius of the space with the metric \( \gamma_{ij} \) mentioned above. The matter in the brane is modeled as a perfect fluid with the following energy momentum tensor
\[
\tau_{\mu\nu} = \text{diag}(-\rho, p, p, p).
\]
(13)

From the Codazzi equation, our chosen five dimensional energy momentum tensor gives the usual energy momentum conservation law
\[
D_{\nu} \tau_{\mu\nu} = 0.
\]
(14)
Assuming the simplest equation of state
\[
p = w \rho,
\]
(15)
the following relation between scale factor and the energy density could be derived
\[
\rho \propto a^{-v}, \quad v \equiv 3w + 1,
\]
(16)

Using the energy momentum conservation law (14) and assuming the RW metric, the following relation could be inferred \[12\]
\[
E_{00} = 3 \mathcal{E}_0 / a^4,
\]
(17)
where \( 3 \mathcal{E}_0 = E_{00}(t = t_0) \) for \( a(t_0) = 1. \) Because \( E_{00} \propto a^{-4} \) resembles a term from radiation, this term is usually called dark radiation, where the "dark" comes from minus sign in equation (18). From (17) and the energy momentum conservation, we arrived at the modified first and second Friedmann equations
Modified Friedmann equations I

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G_N}{3} \rho - \frac{k}{a^2} + \frac{\Lambda_4}{3} + \frac{\kappa_5^4}{36} \rho^2 - \frac{E_0}{a^4},
\]

where \( \Lambda_4 \equiv \Lambda_5 + \frac{1}{5} \kappa_5^2 \lambda^2 \) is the effective four dimensional cosmological constant.

From the above equations, we see that the correction terms for the conventional Friedmann equations provided by the braneworld model come in two terms: the \( \propto \rho^2 \) term and the dark radiation term. One of the curious feature of the correction terms arises when we consider \( E_0 > 0 \).

In this case, assuming a matterless universe \( \rho = 0 \) and a vanishing cosmological constant \( \Lambda_4 \), the first modified Friedmann equations (18) yields

\[
\frac{H^2}{H_0^2} = 1 + \Omega_E - \frac{\Omega_E}{a^4},
\]

where we have defined

\[
\Omega_E = \frac{E_0}{H_0^2} > 0.
\]

Because \( \frac{H^2}{H_0^2} \geq 0 \), we have

\[
a \geq \sqrt{\frac{\Omega_E}{1 + \Omega_E}} > 0, \text{ for } \Omega_E > 0,
\]

and thus indicating that the universe might has no initial singularity. Although the above arguments is oversimplified, it can be shown that even in a universe that contains matter, almost all cases of \( w \) (parameter for equation of state) leads to a non-singular initial universe (see e.g. [12]).

3. Dynamical Analysis and Benchmark Model

Consider the modified Friedmann equations I (18) for the single matter case

\[
1 = \Omega + \tilde{\Omega} + \Omega_k + \Omega_\epsilon + \Omega_\Lambda,
\]

where

\[
\Omega = \frac{8\pi G_N}{3H^2} \rho, \quad \tilde{\Omega} = \frac{8\pi G_N}{3H^2} \frac{\rho^2}{2\lambda}, \quad \Omega_k = -\frac{k}{a^2H^2},
\]

\[
\Omega_\epsilon = -\frac{E_0}{a^4H^2}, \quad \Omega_\Lambda = \frac{\Lambda_4}{3H^2}.
\]

Next, from (19) and the following relation: \( \frac{\dot{H}}{H^2} = -1 + \frac{\ddot{a}}{aH^2} \), we have

\[
\frac{\dot{H}}{H^2} = -(1 + q);
\]
where the deceleration parameter $q$ is defined as follows

$$ q \equiv \frac{3w + 1}{2} - \Omega + (2 + 3w) \bar{\Omega} + \Omega_{\mathcal{E}} - \Omega_\Lambda. $$

A thorough dynamical analysis for the parameters in (24) has been carried out by Campos and Sopuerta in the case of $\Omega_{\mathcal{E}} = 0$ [13] and for more general case $\Omega_{\mathcal{E}} \neq 0$ [14]. In this section, we only want to analyze the effect of the parameters unique to the braneworld model ($\bar{\Omega}$ and $\Omega_{\mathcal{E}}$). Now, assuming that $\Omega_\Lambda = 0$ and $\Omega_k = 0$ (flat universe), we secure a set of autonomous differential equations for the modified Friedmann equations

$$ \Omega' = (2(1 + q) - 3(1 + w))\Omega, $$

$$ \Omega_{\mathcal{E}}' = 2\Omega_{\mathcal{E}}(q - 1), $$

where $q = -\frac{3}{2} \Omega (w + 1) - (3w + 1) \Omega_{\mathcal{E}} + (2 + 3w)$, $1 = \Omega + \bar{\Omega} + \Omega_{\mathcal{E}}$, and $A' = \dot{A}/H$ for arbitrary $A$.

A fixed point is a point which satisfy $(\Omega', \Omega_{\mathcal{E}}') = (0, 0)$. We found the complete set of fixed points as follows: $(\bar{\Omega}, \Omega, \Omega_{\mathcal{E}}) = (1, 0, 0); (0, 1, 0); (0, 0, 1)$. The complete classification of those fixed points is given in table 1.

| $(\bar{\Omega}, \Omega, \Omega_{\mathcal{E}})$ classification | $(\bar{\Omega}, \Omega, \Omega_{\mathcal{E}}) = (1, 0, 0)$ | $(\bar{\Omega}, \Omega, \Omega_{\mathcal{E}}) = (0, 1, 0)$ | $(\bar{\Omega}, \Omega, \Omega_{\mathcal{E}}) = (0, 0, 1)$ |
|---------------------------------------------------------------|---------------------------------------------------------------|---------------------------------------------------------------|
| A $w < -1$ | $-1 < w < 1/3$ | $w > 1/3$ |
| B $w > -1/3$ | - | $w < -1/3$ |
| C $-1 < w < -1/3$ | $w < -1$, $w > 1/3$ | $-1/3 < w < 1/3$ |
| D - | - | - |
| E - | - | - |

Note: A, B, C, D, E is stable, unstable, saddle, stable spiral, unstable spiral points, respectively.

As for the phase space solution near the fixed points is given in figure 1, 2, 3

![Figure 1](image1.png)

**Figure 1.** Phase space solution $(\Omega, \bar{\Omega})$ near $(0, 0)$ for various value of $w$.

Next, we would like to compare (to fit) the modified Friedmann equations (18) with the Supernova Ia experimental data which has been compiled by Davis, et al. (2007) [15],[16],[17]. From the fitting process, we could determine the present epoch cosmological parameters value.
Now, let us consider the modified Friedmann equations (18) composed of three types of matter: dust, radiation, and cosmological constant $\Lambda$. Thus, we have

$$\rho = \rho_0, r a^{-4} + \rho_{0,m} a^{-3},$$

where the four dimensional cosmological constant energy density $\rho_{0,\Lambda}$ originated from the effective cosmological constant $\Lambda_4$, $\rho_{0,\Lambda} = \Lambda_4/8\pi G_N$.

Let $G \equiv \frac{8\pi G_N}{3c^2}$, and define the following cosmological parameters

$$\Omega_{0,i} = \frac{8\pi G_N \rho_{0,i}}{3c^2 H_0^2}; \quad i = r, m,$$

$$\Omega_{0,\xi} = -\frac{\xi_0}{c^2 H_0^2},$$

where subscript $r, m$ stands for radiation ($w = 1/3$) and dust ($w = 0$) respectively. Assuming that the spatial curvature of our universe is zero, we could calculate the distance modulus as follows

$$\mu = 5 \log \frac{d_L}{10\text{pc}},$$

where the luminosity distance $d_L$ is calculated from

$$d_L = (1 + z) c \int_0^z \frac{dz'}{H(z')},$$
with the following expression for Hubble constant

\[
H(z) = H_0 \left\{ \Omega_{0,\Lambda} + \Omega_{0,m}(1 + z)^3 + (\Omega_{0,r} + \Omega_{0,\phi})(1 + z)^4 + \frac{H_0^2}{2G} \left[ \Omega_{0,m}(1 + z)^3 + \Omega_{0,r}(1 + z)^4 \right]^2 \right\}^{1/2}.
\]

(34)

For simplicity, let us assumed that the contribution of total matter from radiation is negligibly small (\(\Omega_{0,r} \approx 0\)), therefore we conclude that the parameters that we would like to find is the following: \(\Omega_{0,m}, \Omega_{0,\Lambda}, \Omega_{0,\phi}, H_0\), and the brane tension \(\lambda\).

To find the best numerical value of those following parameters, we would like to minimize the following function [18]

\[
\chi^2(\Omega_{0,m}, \Omega_{0,\Lambda}, \Omega_{0,\phi}, H_0, \lambda) = \sum_{i=1}^{N_{\text{exp}}} \frac{[\mu_i - \mu_{\text{exp}}(z_i)]^2}{\sigma_i^2},
\]

(35)

where \(\mu_{\text{exp}}(z_i)\) is the experiment distance modulus for a data with redshift factor \(z = z_i\) and error \(\sigma_i\). Our compiled data, has a total number of experimental data \(N_{\text{exp}} = 192\) [15]. The best fitted parameter has a characteristic of having [18] \(\chi^2/\nu \approx 1\), where \(\nu\) is the difference between of total number of data and the degree of freedoms (number of parameters which we want to determine, which in our case is five).

In this work, we use the Metropolis-Hastings algorithm. Similar cosmological parameter estimation using this algorithm has been performed by Deffayet, et al in the case of DGP (Dvali-Gabadadze-Porrati) brane-induced braneworld model [19]. We found (this method has been tested on the conventional Friedmann equations, which result can be found in the appendix)

\[
\begin{align*}
\Omega_{0,m} &= 0.3453, \\
\Omega_{0,\Lambda} &= 0.6568, \\
\Omega_{0,\phi} &= -0.0328, \\
\lambda &= 10^{35.5669} \text{ kg Mpc}^{-1} \text{ s}^{-2}, \\
H_0 &= 66.1527 \text{ km s}^{-1} \text{ Mpc}^{-1}, \\
\chi^2 &= 197.6811.
\end{align*}
\]

(36)

Firstly, we notice that \(\nu = 192 - 5 = 187\), and thus \(\chi^2/\nu \approx 1.06\). Next, we found that the present epoch Hubble constant \(H_0\) is lower than the Hubble constant obtained from the conventional Friedmann equations [20] where \(H_0 = 71.9 \text{ km s}^{-1} \text{ Mpc}^{-1}\). We also found that the numerical value of brane tension \(\lambda\) doesn’t satisfy the lower bound required for a succesful nucleosynthesis phase [21]: \(\lambda > 1 \text{ (MeV)}^4 = 10^{49} \text{ kg Mpc}^{-1} \text{ s}^{-2}\).

We also found that \(\Omega_{0,\phi} < 0\) which simply means that \(\phi_0 > 0\). It thus can be inferred from \(\phi_0 > 0\), that the evolution of our universe might has no initial singularity: \(a(0) > 0\) [12].

Now because \(\lambda \gg 1\) we can assume that \(\Omega \approx 0\), therefore we found that \(q = -0.5169 < 0\). We thus conclude that the cosmology from braneworld scenario also predicts a universe which expands with positive acceleration.

Using the best fitted parameter we just obtained, the Hubble diagram for the modified Friedmann equation in the braneworld scenario can be compared with the experimental result, and also with the conventional model, as can be seen from Figure 4. As for the conventional cosmology, we use the following best fitted parameter obtained from the latest experiment: \(H_0 = 71.9 \text{ km s}^{-1} \text{ Mpc}^{-1}\) [20], \(\Omega_m = 0.308, \Omega_r \approx 0, \Omega_{\Lambda} = 0.692\) [22].
4. Conclusion and Outlook
We found that the brane tension is not large enough to support a nucleosynthesis process which correctly reproduce the abundance of present light elements. Also, we found a negative dark-radiation energy density, and thus indicating a possibility of evolution without singularity. As for the outlook, it is either we further modify the Randall-Sundrum braneworld model or that we modify our numerical methods. One of the possibility of the modification of Randall-Sundrum braneworld models is to introduce a scalar field in the bulk [23] or in the case of universal extra dimension where the matter is moving freely in the bulk and does not confined in the brane: [24] and [25]. A more general theme of these modifications is to combine the Horndeski model (which it the most general, stable, scalar-tensor theory having second-order equations of motion) and the braneworld model. And for the simplest modification of the numerical methods we have used is to consider a better distribution proposal such as Gaussian distribution, or to use a certain convergence criterion.

Acknowledgments
The work of AS and FPZ is supported by Riset KK ITB 2017 and PUPT-DIKTI 2017.
Appendix

To find the best fitted parameters, we use the Metropolis-Hastings algorithm. In our case, the parameters space is five dimensional: \( x = (\Omega_{0,m}, \Omega_{0,\Lambda}, \Omega_{0,\epsilon}, \lambda, H_0) \). Basically, we varied the parameters randomly to find the best fitted parameters with minimum \( \chi^2 \). We varied parameters using a certain proposal distribution\[26\]. In our work, we use the simplest proposal distributions, which is the uniform distribution with a certain width \( 2\Delta x \) within the following interval \([-\Delta x, \Delta x]\). The proposed parameters will be accepted (the best fitted parameters is updated into the proposed parameters) if the proposed value of \( \chi^2 \) is smaller, but also will be accepted with a certain probability in the case when the value of the proposed \( \chi^2 \) is larger (as to avoid the local minimum of the \( \chi^2 \) function).

In our work, the program will stop after a certain number of loops \( N \). In principle, the results will independent of the initial guess, for a sufficiently large \( N \). To obtain result (36), we set \( N = 10000 \) with the following initial value for the parameters

\[
\begin{align*}
\Omega_{0,m} &= 0.3, \\
\Omega_{0,\Lambda} &= 0.7, \\
\Omega_{0,\epsilon} &= 0, \\
\lambda &= 10^{90}, \\
H_0 &= 67 \text{ km s}^{-1} \text{ Mpc}^{-1}, \\
\Omega_{0,r} &= 0.
\end{align*}
\]

Based on several initial trials, we set \( \Delta x \) as follows

\[
\begin{align*}
\Delta \Omega_{0,m} &= 0.005, \\
\Delta \Omega_{0,\Lambda} &= 0.005, \\
\Delta \lambda &= 1, \\
\Delta \Omega_{0,\epsilon} &= 0.01, \\
\Delta H_0 &= 0.01.
\end{align*}
\]

As we have said in the main section, to ensure the validity of the result, this method has also been tested on the conventional Friedmann equation (where \( \Omega_{0,r} = 0 \) has been assumed)

\[
H(z) = H_0 \left[ \Omega_{0,\Lambda} + \Omega_{0,m}(1 + z)^3 \right]^{1/2}.
\]

For the conventional case we set \( N = 1000 \) with the same \( \Delta x \) as before (A.2). Assuming the spatially flat universe, and by using the following initial parameters value

\[
\begin{align*}
\Omega_{0,m} &= 0.4, \\
\Omega_{0,\Lambda} &= 0.6, \\
H_0 &= 67 \text{ km s}^{-1} \text{ Mpc}^{-1}, \\
\Omega_{0,r} &= 0,
\end{align*}
\]

we found the following parameters value for the conventional case

\[
\begin{align*}
\Omega_{0,m} &= 0.2759, \\
\Omega_{0,\Lambda} &= 0.6880, \\
H_0 &= 67.1397 \text{ km s}^{-1} \text{ Mpc}^{-1}.
\end{align*}
\]
References

[1] N. Arkani Hamed, S. Dimopoulos, and G. Dvali, “The hierarchy problem and new dimensions at a millimeter,” *Physics Letters B* **429** no. 3, (1998).

[2] L. Randall and R. Sundrum, “Large mass hierarchy from a small extra dimension,” *Physical Review Letters* **83** no. 17, (1999).

[3] L. Randall and R. Sundrum, “An alternative to compactification,” *Physical Review Letters* **83** no. 23, (1999).

[4] T. Shiromizu, K. Maeda, and M. Sasaki, “The einstein equations on the 3-brane world,” *Physical Review D* **62** no. 2, (2000).

[5] S. Kanno and J. Soda, “Brane world effective action at low energies and ads/cft correspondence,” *Physical Review D* **66** no. 2, (2002).

[6] S. Kanno and J. Soda, “Radion and holographic brane gravity,” *Physical Review D* **66** no. 8, (2002).

[7] F. P. Zen, B. Gunara, and H. Zainuddin, “The effective equation of motion on the brane world gravity,” *Journal of Mathematical and Fundamental Sciences* **38** no. 1, (2006).

[8] F. P. Zen, B. E. Gunara, and H. Zainuddin, “The low energy effective equations of motion for multibrane world gravity,” *arXiv:0512257*.

[9] L. Cotta-Ramusino and D. Wands, “Low energy effective theory for a randall-sundrum scenario with a moving bulk brane,” *Physical Review D* **75** no. 10, (2007).

[10] S. Feranie and F. P. Zen, “Kaluza-klein two-brane-worlds cosmology at low energy,” *Physical Review D* **81** no. 1, (2010).

[11] Arianto, Freddy. P. Zen, S. Feranie, I. P. Widyatmika, and B. E. Gunara, “Kaluza-klein brane cosmology with a bulk scalar field,” *Physical Review D* **84** no. 4, (2011).

[12] K. Maeda, “The einstein equations on a brane world,” *Progress of Theoretical Physics Supplement* **148** (2002).

[13] A. Campos and C. F. Sopuerta, “Evolution of cosmological models in the brane-world scenario,” *Physical Review D* **63** no. 10, (2001).

[14] A. Campos and C. F. Sopuerta, “Bulk effects in the cosmological dynamics of brane-world scenarios,” *Physical Review D* **64** no. 10, (2001).

[15] T. M. Davis et al., “Scrutinizing exotic cosmological models using essence supernova data combined with other cosmological probes,” *The Astrophysical Journal* **666** no. 2, (2007).

[16] Wood-Vasey, W. Michael, et al., “Observational constraints on the nature of dark energy: first cosmological results from the essence supernova survey,” *The Astrophysical Journal* **666** no. 2, (2007).

[17] A. G. Riess et al., “New hubble space telescope discoveries of type ia supernovae at $z \geq 1$: narrowing constraints on the early behavior of dark energy,” *The Astrophysical Journal* **659** no. 1, (2007).

[18] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical recipes in C: the art of scientific computing*. Cambridge University Press, 1992.

[19] C. Deffayet, S. J. Landau, J. Raux, M. Zaldarriaga, and P. Astier, “Supernovae, cmb, and gravitational leakage into extra dimensions,” *Physical Review D* **66** no. 2, (2002) 024019.

[20] B. V et al., “H0licow v. new cosmograil time delays of he 0435-1223: H0 to 3.8% precision from strong lensing in a flat $\lambda$ cdm model,” *Monthly Notices of the Royal Astronomical Society* **465** no. 4, (2016).

[21] R. Maartens and K. Koyama, “Brane-world gravity,” *Living Reviews in Relativity* **13** no. 1, (2010).

[22] P. A. R. Ade et al., “Planck 2015 results-xiii. cosmological parameters;” *Astronomy & Astrophysics* **594** (2016).

[23] M. Minamitsuji, “Brane worlds with field derivative coupling to the einstein tensor,” *Physical Review D* **89** no. 6, (2014).

[24] A. Suroso and F. P. Zen, “Varying gravitational constant in five dimensional universal extra dimension with nonminimal derivative coupling of scalar field,” *Adv. Stud. Theor. Phys.* **6** no. 27, (2012).

[25] A. Suroso and F. P. Zen, “Cosmological model with nonminimal derivative coupling of scalar fields in five dimensions,” *General Relativity and Gravitation* **45** no. 4, (2013).

[26] A. Heavens, “Statistical techniques in cosmology,” *arXiv:0906.0664*. 
