Function Projective Synchronization of Complex Networks With Distributed Delays via Hybrid Feedback Control

XIULIANG QIU$^1$, WENSHUI LIN$^{2,3}$, (Member, IEEE), AND YIMING ZHENG$^3$

$^1$Chengyi University College, Jimei University, Xiamen 361021, China
$^2$Fujian Key Laboratory of Sensing and Computing for Smart City, Xiamen 361005, China
$^3$School of Informatics, Xiamen University, Xiamen 361005, China

Corresponding author: Wenshui Lin (wslin@xmu.edu.cn)

This work was supported in part by NSFC under Grant 11771362, and in part by the Scientific Research Fund of Fujian Provincial Department of Education under Grant JT180870.

ABSTRACT Due to its applications, some types of synchronization of complex networks have been intensively investigated. In particular, as a more general type of synchronization, function projective synchronization (FPS) has been investigated for complex networks with time delay or with time-varying delay. In this paper, we investigate FPS of complex networks with distributed delays. It is proven that, FPS of such networks can be realized via hybrid feedback control. Finally, two numerical examples are provided to illustrate the effectiveness of the proposed method.

INDEX TERMS Function projective synchronization, complex networks, distributed delays, hybrid feedback control.

I. INTRODUCTION

Complex networks attract a lot of research interests, because they are widely applicable to various fields [1]–[5]. Synchronization is one of the most important and interesting collective behaviors in complex networks. Due to its potential engineering applications (see [6]–[14]), the types of complete synchronization [6]–[8], cluster synchronization [9]–[11], lag synchronization [7], projective synchronization [12], [13], and function projective synchronization (FPS) [14], [15] were intensively investigated in the past few decades.

FPS means synchronizing the driver and response systems up to a scaling function. Hence it is a more general type of chaotic synchronization, and encompasses complete synchronization and projective synchronization. For a complex network, FPS means with a desired scaling function all the nodes are synchronized to an equilibrium point or a periodic orbit. Because the scaling function is hard to predict, FPS has application in secure communication [16]. Hence FPS has attracted a lot of research interests [17]–[20]. In particular, FPS was investigated for complex networks with time delay [19] and those with time-varying delay [20].

Complex networks usually have propagation delays, which have been observed in lasers, neuron models, electronic circuits and so on. Many phenomena in the real world indicate that the current state of a node is affected by those of its neighbors in the previous period. Therefore complex networks with distributed delays were introduced into the model system [21], [22]. In order to serve the practical life better, complex networks with distributed time delays are worthy of serious investigation. Some types of synchronization of such networks have been investigated [23]–[25]. However, to our best knowledge, up to now there are no works concerning FPS for these networks.

In this paper we investigate the problem of FPS for general complex networks with distributed delays. In Section II we introduce the network model, and prove that FPS of such networks can be realized via hybrid feedback control. In Section III, two numerical examples are provided to illustrate the effectiveness of our method.

II. FPS OF COMPLEX NETWORKS WITH DISTRIBUTED DELAYS

Consider a general complex network with $N$ identical nodes $v_1, v_2, \ldots, v_N$, which are linearly coupled. The network topology is represented by the matrix $G = (g_{ij}) \in \mathbb{R}^{N \times N}$,
where for \( i \neq j, g_{ij} \neq 0 \) if \( v_i \) is connected to \( v_j \), and 0 otherwise; And \( g_{ii}, 1 \leq i \leq N \), is defined as \(- \sum_{j=1, j \neq i}^{N} g_{ij} \).

\[ x_i(t) = (x_{i1}, x_{i2}, \ldots, x_{im})^T \in \mathbb{R}^n \] will denote the state of \( v_i \), and the behavior of nodes can be described by a continuously differentiable vector function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \). Let \( u_i(t) \in \mathbb{R}^n \) be the control input. Then a controlled network with distributed delays can be described by the following system of integro-differential equations:

\[
\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} g_{ij} \int_{-\infty}^{t} K(t-s)x_j(s)ds + u_i(t). \tag{1}
\]

**Assumption 1:** The kernel function \( K(\cdot) \) in (1) is a real valued nonnegative continuous function defined on \([0, +\infty)\) and satisfies \( \int_{0}^{+\infty} K(s)ds = 1 \).

**Definition 1 (FPS):** The network (1) is said to achieve function projective synchronization if there exists a continuously differentiable function \( \alpha(t) \) such that

\[
\lim_{t \to +\infty} \|e_i(t)\| = \lim_{t \to +\infty} \|x_i(t) - \alpha(t)s(t)\| = 0,
\]

where \( s(t) \in \mathbb{R}^n \) is an equilibrium point, a periodic orbit, or an orbit of a chaotic attractor, which satisfies \( \dot{s}(t) = f(s(t)) \).

Based on the following result presented in [26], we are able to prove that, FPS of network (1) can be realized via a hybrid feedback control.

**Lemma 1:** For any vectors \( x, y \in \mathbb{R}^n \) and positive definite matrix \( Q \in \mathbb{R}^{n \times n} \), the following matrix inequality holds:

\[
2x^T Q y \leq x^T Q x + y^T Q^{-1} y.
\]

**Theorem 2:** For any given initial conditions \( x_i(0), d_i(0), s(0) \), FPS of network (1) can be realized by the control law:

\[
u_i(t) = u_{i1}(t) + u_{i2}(t), i = 1, 2, \ldots, N,
\]

where

\[
u_{i1}(t) = \alpha(t)f(s(t)) + \dot{\alpha}(t)s(t) - f(x_i(t)), \tag{2}
\]

\[
u_{i2}(t) = -d_i(t)e_i(t), \tag{3}
\]

\[
d_i(t) = k_i e_i^T(t) e_i(t), \tag{4}
\]

and \( k_i > 0 \) is an any constant.

**Proof:** Let \( e_i(t) = x_i(t) - \alpha(t)s(t), i = 1, 2, \ldots, N \). It follows from (1) that

\[
\dot{e}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} g_{ij} \int_{-\infty}^{t} K(t-s)e_j(s)ds + u_i(t) - \dot{\alpha}(t)s(t) - \alpha(t)f(s(t)).
\]

Combining with (2) – (4), we have

\[
\dot{e}_i(t) = \sum_{j=1}^{N} g_{ij} \int_{-\infty}^{t} K(t-s)e_j(s)ds - d_i(t)e_i(t).
\]

Let \( s = t - u \). Then

\[
\dot{e}_i(t) = \sum_{j=1}^{N} g_{ij} \int_{0}^{+\infty} K(u)e_j(t-u)du - d_i(t)e_i(t). \tag{5}
\]

Construct the Lyapunov function

\[
V(t) = \frac{1}{2} \sum_{i=1}^{N} e_i^T(t)e_i(t) + \frac{1}{2} \sum_{i=1}^{N} \int_{t}^{+\infty} K(u)e_i^T(u)du + \frac{1}{2} \int_{0}^{t} \left( \sum_{i=1}^{N} \int_{0}^{+\infty} K(u)e_i^T(u)du \right) dw,
\]

where \( d^* \in \mathbb{R}^{+} \) is to be determined. Consequently, we have

\[
\dot{V}(t) \leq e^T(t)Qh(t) - d^*e^T(t)e(t) + \frac{1}{2} e^T(t)e(t) - \frac{1}{2} h^T(t)h(t)
\]

\[
\leq \frac{1}{2} e^T(t)Qh(t) + \frac{1}{2} h^T(t)h(t) - d^*e^T(t)e(t) + \frac{1}{2} e^T(t)e(t) - \frac{1}{2} h^T(t)h(t)
\]

\[
\leq \rho \left( \frac{QOQ^T}{2} + \frac{1}{2} - d^* \right) e^T(t)e(t),
\]

where \( O^T \) is to be determined.
where $\rho(M)$ is the greatest eigenvalue of a symmetric matrix $M$. Hence, if $d^* = \rho \left( \frac{\partial \phi}{\partial t} \right) + \frac{3}{2}$, then $\dot{V}(t) \leq -e^T(t)e(t)$. Therefore, the error system (5) is asymptotically stable from the Lyapunov stability theory. This completes the proof. ■

### III. COMPUTER SIMULATIONS

In this section, two simulation examples will be employed to illustrate the theoretical result obtained in the previous section.

**Simulation 1:** Consider the following single Lorenz system:

$$
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= (b - x_3)x_1 - x_2 \\
\dot{x}_3 &= x_1x_2 - cx_3
\end{align*}
$$

where $a = 10$, $b = 28$, and $c = \frac{8}{3}$. The chaotic attractor of the system is depicted in Figure 1.

We choose the coupling configuration matrix as

$$
G = \begin{pmatrix}
-1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 1 & -1
\end{pmatrix}.
$$

Then the controlled network with distributed delays is described as

$$
\begin{align*}
\dot{\hat{x}}_{1i}(t) &= 10(\hat{x}_{2i}(t) - x_{1i}(t)) \\
\dot{\hat{x}}_{2i}(t) &= (28 - x_{3i}(t))x_{1i}(t) - x_{2i}(t) \\
\dot{\hat{x}}_{3i}(t) &= x_{1i}(t)x_{2i}(t) - \frac{8}{3}x_{3i}(t) \\
&\quad + \sum_{j=1}^{3} g_{ij} \int_{-\infty}^{t} K(t - s)x_{j}(s)ds + u_i(t),
\end{align*}
$$

where $i = 1, 2, 3$. Choose $K(x) = e^{-x}$ as the kernel. By using Theorem 2 the controllers can be designed as $u_i(t) = u_{i1}(t) + u_{i2}(t)$, $i = 1, 2, 3$, where

$$
\begin{align*}
u_{i1}(t) &= \begin{pmatrix}
10(e_{1i}(t) - e_{2i}(t)) + \dot{\alpha}(t)x_{1i}(t) \\
\frac{1}{2}e_{2i}(t) - x_{1i}(t)x_{2i}(t) + \alpha(t)x_{1i}(t)x_{2i}(t) + \dot{\alpha}(t)x_{1i}(t)
\end{pmatrix}, \\
u_{i2}(t) &= \begin{pmatrix}
-d_{i1}e_{1i}(t) \\
-d_{i2}e_{2i}(t) \\
-d_{i3}e_{3i}(t)
\end{pmatrix},
\end{align*}
$$

The synchronization errors of (6).
can be described as then the controlled complex network with distributed delays is given by, where the attractor of the Chen system is shown in Figure 4. The chaotic attractor of the Chen system is depicted in Figure 4.

Simulation 2: Consider the following single Chen system:

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= (c - a)x_1 - x_1x_3 + cx_2 \\
\dot{x}_3 &= x_1x_2 - bx_3
\end{align*}
\]

where \(a = 35\), \(b = 3\), and \(c = 28\). Figure 4 depicts the chaotic attractor of the Chen system.

The coupling configuration matrix is chosen to be

\[
G = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix}.
\]

Then the controlled complex network with distributed delays can be described as

\[
\begin{align*}
\dot{x}_{1i}(t) &= \frac{35(x_{2i}(t) - x_{1i}(t))}{x_{1i}(t)x_{12}(t) + 28x_{12}(t)} \\
\dot{x}_{2i}(t) &= \frac{-7x_{1i} - x_{1i}x_{2i}(t) + x_{22}(t)}{x_{1i}(t)x_{12}(t) - 3x_{13}(t)} \\
\dot{x}_{3i}(t) &= \sum_{j=1}^{3} g_{ij} \int_{-\infty}^{t} K(t-s)x_{j}(s)ds + u_i(t),
\end{align*}
\]

where \(i = 1, 2, 3\). Choose \(K(x) = e^{-x}\) as the kernel. By using Theorem 2 the controllers can be designed as

\[
\begin{align*}
u_{1i}(t) &= u_{1i}(t) + u_{2i}(t), \quad i = 1, 2, 3, \\
u_{1i}(t) &= \begin{pmatrix} 35(e_{1i}(t) - e_{2i}(t)) + \dot{u}_i(t) \xi_1(t) \\ 7e_{1i}(t) - 28e_{2i}(t) + x_{2i}(t)x_{3i}(t) - 3x_{12}(t)x_{2i}(t) + 28x_{12}(t) \\ 3x_{12}(t)x_{2i}(t) + \alpha(t)x_{1i}(t)x_{2i}(t) + \dot{u}_i(t)x_{3i}(t) \end{pmatrix}, \\
u_{1i}(t) &= \begin{pmatrix} -d_{e1}(t) \\ -d_{e2}(t) \\ -d_{e3}(t) \end{pmatrix}.
\end{align*}
\]

In this numerical simulation, we take the initial states as \(x_1(0) = (17, 14, 13)^T\), \(x_2(0) = (24, 33, 15)^T\), \(x_3(0) = (2, 11, 13)^T\), and \(x(0) = (6, 3, 8)^T\). Set \(k_1 = k_2 = k_3 = 2\), \(d(0) = (10, 1, 13)^T\), and \(\alpha(t) = \cos \frac{\pi t}{6} + 4\). Figures 5 and 6 display the state phases of the Chen system and the synchronization errors, respectively. As well, FPS takes place rapidly in network (7) as expected.

IV. CONCLUSION

FPS schemes for complex networks with distributed delays are investigated in this paper. A hybrid feedback control method is presented to realize FPS in such networks. Finally, two numerical simulations are used to demonstrate the effectiveness of our method.

Recently, quantized techniques were shown to be an effective technique to realize synchronization of complex networks [27]–[32]. On the other hand, finite-time...
control [30]–[32] is more practical for engineering applications. In future works, we will investigate the problem of FPS of complex networks with distributed delays via finite-time quantized control.

ACKNOWLEDGMENT

The authors would like to thank the referees for their valuable comments.

REFERENCES

[1] E. Ott, C. Grebogi, and J. A. Yorke, “Controlling Chaos,” Phys. Rev. Lett., vol. 11, pp. 1196–1199, Mar. 1993.
[2] H. Gang and Z. Qiu, “Controlling spatiotemporal chaos in coupled map lattice systems,” Phys. Rev. Lett., vol. 72, pp. 68–73, Jan. 1994.
[3] D. J. Watts and S. H. Strogatz, “Collective dynamics of ‘small-world’ networks,” Nature, vol. 393, no. 6684, pp. 440–442, 1998.
[4] M. E. J. Newman and D. J. Watts, “Renormalization group analysis of the small-world network model,” Phys. Lett. A, vol. 263, nos. 4–6, pp. 341–346, Dec. 1999.
[5] X. Wang and G. Chen, “Pinning control of scale-free dynamical networks,” Phys. A, Stat. Mech. Appl., vol. 310, nos. 3–4, pp. 521–531, Jul. 2002.
[6] C. Zhou and J. Kurths, “Hierarchical synchronization in complex networks with heterogeneous degrees,” Chaos, Interdiscip. J. Nonlinear Sci., vol. 16, no. 1, Mar. 2006, Art. no. 015104.
[7] Z.-M. Ge and G.-H. Lin, “The complete, lag and anticipated synchronization of a BLDLCM chaotic system,” Chaos, Solitons Fractals, vol. 34, no. 3, pp. 740–764, Nov. 2007.
[8] A. N. Pisarchik and F. R. Ruiz-Oliveras, “Optical chaotic communication using generalized and complete synchronization,” IEEE J. Quantum Electron., vol. 46, no. 3, pp. 279–284, Mar. 2010.
[9] W. Wu, W. Zhou, and T. Chen, “Cluster synchronization of linearly coupled complex networks under pinning control,” IEEE Trans. Circuits Syst. I, Reg. Papers, vol. 56, no. 4, pp. 829–839, Apr. 2009.
[10] J. Cao and L. Li, “Cluster synchronization in an array of hybrid coupled neural networks with delay,” Neural Netw., vol. 22, no. 4, pp. 335–342, May 2009.
[11] W. Lu, B. Liu, and T. Chen, “Cluster synchronization in networks of coupled nonidentical dynamical systems,” Chaos, Interdiscip. J. Nonlinear Sci., vol. 20, no. 1, Mar. 2010, Art. no. 013120.
[12] R. Mainieri and J. Rehacek, “Projective synchronization in three-dimensional chaotic systems,” Phys. Rev. Lett., vol. 82, no. 15, pp. 3042–3045, Apr. 1999.
[13] D. Xu and Z. Li, “Controlled projective synchronization in nonpartially-linear chaotic systems,” Int. J. Bifurcation Chaos, vol. 12, no. 06, pp. 1395–1402, Jun. 2002.
[14] Y. Chen and X. Li, “Function projective synchronization between two identical chaotic systems,” Int. J. Modern Phys. C, vol. 18, no. 5, pp. 883–888, May 2007.
[15] X. Tang, J. Lu, and W. Zhang, “The FPS of chaotic system using backstepping design,” China J. Dynam. Control, vol. 5, pp. 216–219, Jan. 2007.
[16] X.-J. Wu, H. Wang, and H.-T. Lu, “Hyperchaotic secure communication via generalized function projective synchronization,” Nonlinear Anal., Real World Appl., vol. 12, no. 2, pp. 1288–1299, Apr. 2011.
[17] H. Du, Q. Zeng, C. Wang, and M. Ling, “Function projective synchronization in coupled chaotic systems,” Nonlinear Anal., Real World Appl., vol. 11, no. 2, pp. 705–712, Apr. 2010.
[18] H. Du, Q. Zeng, and C. Wang, “A general method for function projective synchronization,” Int. J. Innov. Comput. Inf. Control, vol. 5, no. 8, pp. 2239–2248, Aug. 2009.
[19] H. Du, P. Shi, and N. Lü, “Function projective synchronization in complex dynamical networks with time delay via hybrid feedback control,” Nonlinear Anal., Real World Appl., vol. 14, no. 2, pp. 1182–1190, Apr. 2013.
[20] L. Shi, H. Zhu, X. Zeng, and S. Zhong, “Function projective synchronization of complex dynamical networks with time-varying delay via mixed feedback control,” in Proc. 6th Int. Symp. Comput. Intell. Design, Oct. 2013, pp. 285–288.
[21] C. Wille, J. Lehnert, and E. Scholl, “Synchronization-desynchronization transitions in complex networks: An interplay of distributed time delay and inhibitory nodes,” Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top., vol. 90, no. 3, Sep. 2014.