Feynman quadrics-motive of the massive sunset graph

The MIT Faculty has made this article openly available. Please share how this access benefits you. Your story matters.

**Citation**
Marcolli, Matilde and Gonçalo Tabuada. “Feynman quadrics-motive of the massive sunset graph.” Journal of number theory, vol. 195, 2019, pp. 159-183 © 2019 The Author(s)

**As Published**
10.1016/J.JNT.2018.06.001

**Publisher**
Elsevier BV

**Version**
Original manuscript

**Citable link**
https://hdl.handle.net/1721.1/126182

**Terms of Use**
Creative Commons Attribution-NonCommercial-NoDerivs License

**Detailed Terms**
http://creativecommons.org/licenses/by-nc-nd/4.0/
We prove that the Feynman quadrics-motive of the massive sunset graph is “generically” not mixed-Tate. Moreover, we explicitly describe its “extra” complexity in terms of a Prym variety.

1. Introduction

Feynman motive. After the seminal work of Bloch-Esnault-Kreimer [11], there has been a lot of research concerned with the construction of motives associated to Feynman graphs $\Gamma$; consult [3, 4, 13, 14, 16, 18, 19, 20, 21, 28, 29, 30, 32, 33]. Several different approaches have been developed for the construction of such “Feynman motives”. In all the cases the main problem is the construction of a motive $M_\Gamma$ such that the (renormalized) Feynman integral of $\Gamma$ is a period of $M_\Gamma$. There are different possible ways of writing the Feynman integral (e.g. Feynman parametric form, momentum space and configuration space) and each one of these ways leads to a different Feynman motive. The most commonly studied approach is the Feynman parametric form. In this case, given a base field $F$, the Feynman motive $M_\Gamma$ is defined as the Voevodsky’s mixed motive $M(\mathbb{P}^{n-1}\setminus X_\Gamma)_\mathbb{Q} \in \text{DM}_{gm}(F)_\mathbb{Q}$, where $n$ stands for the number of internal edges of $\Gamma$ and $X_\Gamma$ for the graph hypersurface defined by the vanishing of the Kirchhoff–Symanzik polynomial of $\Gamma$; consult [7, 11, 18, 32] for details. When renormalization is taken into account, $\mathbb{P}^{n-1}\setminus X_\Gamma$ needs to be replaced by a certain blow-up of itself; see [11, 13]. Currently, one of the most important open questions concerning Feynman graphs is the following:

Question: Given a Feynman graph $\Gamma$, is the associated Feynman motive $M_\Gamma$ mixed-Tate? If not, how to describe its “extra” complexity?

On the “positive side”, $M_\Gamma$ is known to be mixed-Tate whenever $\Gamma$ has less than 14 edges; see [34, 35]. Consult also [5] for infinite families of mixed-Tate Feynman motives. On the “negative side”, there exist examples of Feynman graphs $\Gamma$ with 14 edges for which the Feynman motive $M_\Gamma$ is not mixed-Tate; see [25, 32].

All the above can be generalized to the case where the Feynman graph $\Gamma$ is equipped with a mass parameter $m$. In this generality, the computation of the associated Feynman motive $M_{(\Gamma,m)}$ becomes much more difficult and only a few computations are currently known; consult [2, 12, 15, 37].

Feynman quadrics-motive. As observed by Bloch-Esnault-Kreimer in [11, §5] (see also [28, §1]), the Feynman parametric form is not the only way of writing the
Feynman integral of \( \Gamma \). An alternative approach is to write the Feynman integral of \( \Gamma \) in terms of edge propagators. This alternative approach is developed in \( \S 2.2 \), where we also consider the case where \( \Gamma \) is equipped with a mass parameter \( m \). As explained in \textit{loc. cit.}, given a spacetime dimension \( D \), this approach leads to the Feynman quadrics-motive \( M^Q_{(\Gamma, m)} := M((\mathbb{P}^{b_1(\Gamma)}D)\bigcap Q_{(\Gamma, m)})_\mathbb{Q} \in \text{DM}_{\text{gm}}(F)_\mathbb{Q} \), where \( b_1(\Gamma) \) stands for the first Betti number of \( \Gamma \) and \( Q_{(\Gamma, m)} \) for the union \( \bigcup_{i=1}^{n} Q_{i,\epsilon} \) of certain “deformed” quadric hypersurfaces; consult Definition 2.16. Intuitively speaking, the Feynman motive and the Feynman quadrics-motive are two different “motivic incarnations” of the same period (= Feynman integral of \( (\Gamma, m) \)); see \( \S 2.3 \).

\textbf{Statement of results.} Assume that the base field \( F \subseteq \mathbb{C} \) is algebraically closed. Consider the \textit{massive sunset graph}\(^1\) with mass parameter \( m = (m_1, m_2, m_3) \in \mathbb{Q}^3 \):

\begin{center}
\begin{tikzpicture}
\draw (-1,0) -- (1,0);
\draw (0,-1) -- (0,1);
\draw (-1,0) circle (1);
\draw[fill=white] (0,0) circle (0.1);
\node at (0,1) {$m_1$};
\node at (0,-1) {$m_3$};
\node at (1,0) {$m_2$};
\node at (-1,0) {$m_1$};
\node at (0,0) {$l_1$};
\node at (0,1) {$l_2$};
\end{tikzpicture}
\end{center}

In this case, the “deformed” quadric hypersurfaces \( Q_{1,\epsilon}, Q_{2,\epsilon}, \) and \( Q_{3,\epsilon} \), corresponding to the 3 internal edges, are odd-dimensional. Hence, following Beauville [6, \S 6.2], whenever \( Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon} \) is a complete intersection, we have an associated (abelian) Prym variety \( \text{Prym}(\tilde{C}/C) \), where \( C \) stands for the discriminant divisor of the quadric fibration associated to the triple intersection and \( \tilde{C} \) for the \( \acute{e} \)tale double cover of the curve \( C \).

Our main result answers the above important question, with the Feynman motive replaced by the Feynman quadrics-motive, in the case of the massive sunset graph:

\textbf{Theorem 1.1.} Let \( (\Gamma, m) \) be the massive sunset graph. Assume that the spacetime dimension \( D \) is \( \geq 2 \) and that the mass parameter \( m = (m_1, m_2, m_3) \) satisfies \( m_3^2 \neq m_1^2 + m_2^2 \). Under these assumptions, the following holds:

(i) The Feynman quadrics-motive \( M^Q_{(\Gamma, m)} \) is not mixed-Tate.

(ii) The Feynman quadrics-motive \( M^Q_{(\Gamma, m)} \) belongs to the smallest subcategory of \( \text{DM}_{\text{gm}}(F)_\mathbb{Q} \) which can be obtained from the following set of motives \( \{ M(\text{Prym}(\tilde{C}/C))_\mathbb{Q}, \mathbb{Q}(-1), \mathbb{Q}(1) \} \) by taking direct sums, shifts, summands, tensor products, and at most 5 cones.

Roughly speaking, Theorem 1.1 shows that the Feynman quadrics-motive associated to the massive sunset graph is not mixed-Tate for a “generic” choice of the mass parameter \( m \). Moreover, Theorem 1.1 provides an explicit “upper bound” for the complexity of the Feynman quadrics-motive. In particular, the “obstruction” to the mixed-Tate property is explicitly realized by the Prym variety \( \text{Prym}(\tilde{C}/C) \).

\(^1\) \( l_1 \) and \( l_2 \) are called the associated “loop variables”; consult \( \S 2.2 \) for details.
Remark 1.2 (Related work). Let \((\Gamma, m)\) be the massive sunset graph. In the particular case of equal masses, Bloch-Vanhove proved in [15, Lem. 6.1] that the Feynman motive \(M_{(\Gamma, m)}\) is also not mixed-Tate. Their approach is based on an explicit description of the parametric Feynman integral in terms of elliptic curves. In contrast, our approach to prove that the Feynman quadrics-motive \(M_{(\Gamma, m)}^Q \neq M_{(\Gamma, m)}\) is not mixed-Tate is based on an explicit description of the Chow motives of complete intersections of two and three quadric hypersurfaces; consult §3 for details.

2. Feynman quadrics-motive

Let \(D > 0\) be the spacetime dimension and \((\Gamma, m, \kappa)\) a Feynman graph equipped with a mass parameter \(m\) and with external momentum \(\kappa\). Recall that \(\Gamma\) is a finite, connected, and direct graph. In what follows, we will write \(E(\Gamma), E_{\text{ext}}(\Gamma)\) and \(E_{\text{int}}(\Gamma)\) for the set of edges, external edges, and internal edges, respectively. Similarly, we will write \(V(\Gamma)\) and \(V_{\text{int}}(\Gamma)\) for the set of vertices and internal vertices, respectively. Given an edge \(e \in E(\Gamma)\), we will denote by \(s(e), t(e)\) and \(\partial(e)\) its source, target, and boundary, respectively. The mass parameter \(m = (m_i)\) consists of a rational number \(m_i \in \mathbb{Q}\) indexed by the internal edges \(e_i \in E_{\text{int}}(\Gamma)\). In the same vein, the external momentum \(\kappa = (\kappa_j)\) consists of a vector \(\kappa_j = (\kappa_{j,r}) \in \mathbb{Q}^D\) indexed by the external edges \(e_j \in E_{\text{ext}}(\Gamma)\).

2.1. Feynman integral. To every internal edge \(e_i \in E_{\text{int}}(\Gamma)\) associate a “momentum variable” \(k_i = (k_{i,r}) \in \mathbb{R}^D\) and the following edge propagator:

\[
q_i(k_i) = \sum_{r=1}^D k_{i,r}^2 + m_i^2.
\]

Under these notations, recall from [28, §3.1] that the (unrenormalized) Feynman integral \(I_{(\Gamma, m, \kappa)}\) associated to the triple \((\Gamma, m, \kappa)\) is defined as follows:

\[
C \int \frac{\prod_{v \in V(\Gamma)} \delta(\sum_{e_i \in E_{\text{int}}(\Gamma)} \epsilon_{v,i} k_i + \sum_{e_j \in E_{\text{ext}}(\Gamma)} \epsilon_{v,j} \kappa_j)}{\prod_{v \in E_{\text{int}}(\Gamma)} q_i(k_i)} \prod_{e_i \in E_{\text{int}}(\Gamma)} \frac{d^D k_i}{(2\pi)^D}. \tag{2.2}
\]

Some explanations are in order: \(C\) stands for the product \(\prod_v \lambda_v (2\pi)^{-D}\) with \(\lambda_v\) the coupling constant at the vertex \(v\); \(\epsilon_{v,i}\) for the incidence matrix with entries 1, -1, 0, according to whether \(v = s(e), v = t(e), v \notin \partial(e)\), respectively (similarly for \(\epsilon_{v,j}\)); \(\prod_{e_i} d^D k_i\) for the standard volume form in \(\mathbb{R}^D\) with \(n := \#E_{\text{int}}(\Gamma)\); the number of internal edges of \(\Gamma\); and finally \(\delta\) stands for the delta function.

2.2. Quadrics-motive. Let us denote by \(n := \#E_{\text{int}}(\Gamma)\) the number of internal edges of \(\Gamma\). In what follows, we will always assume that the mass parameter \(m\) is positive, i.e. that \(m_i > 0\) for every internal edge \(e_i \in E_{\text{int}}(\Gamma)\).

Notation 2.3. Given any two vectors \(v = (v_{i,r}) \in \mathbb{R}^D\) and \(v' = (v'_{i,r}) \in \mathbb{R}^D\), let \(\langle v, v' \rangle := \sum_{i=1}^n \sum_{r=1}^D v_{i,r} v'_{i,r}\) and \(v^2 := \langle v, v \rangle = \sum_{i,r} v_{i,r}^2\).

Note that due to the presence of the mass parameter \(m\), the polynomial (2.1) in \(D\) variables is non-homogeneous. By (formally) adding an homogeneous coordinate \(x\), we can consider the associated homogeneous polynomial in \(D + 1\) variables:

\[
q_i'(k_i, x) := \sum_{r=1}^D k_{i,r}^2 + m_i^2 x^2. \tag{2.4}
\]
Moreover, under the identification of \( k_i = (k_{i,r}) \in A^D \) with the vector \( v = (v_{j,r}) \) of \( A^{nD} \), if \( i = j \) and 0 otherwise, the polynomial (2.4) can be considered as an homogeneous polynomial in \( nD + 1 \) variables (where \( k = (k_i) \in A^{nD} \)):  

\[
q^{(1)}_i(k, x) := k_i^2 + m_i^2 x^2.
\]

Let us denote by \( Q'_i \subset \mathbb{P}^{nD} \) the associated quadric hypersurfaces. The delta function \( \delta \) in the numerator of (2.2) imposes linear relations between the “momentum variables” \( k_i = (k_{i,r}) \in A^D \). Concretely, every internal vertex \( v \in V_{\text{int}}(\Gamma) \) yields the following linear relations:

\[
\sum_{e_i \in E_{\text{int}}(\Gamma) \atop s(e_i) = v} k_i + \sum_{e_j \in E_{\text{ext}}(\Gamma) \atop t(e_j) = v} \kappa_j = \sum_{e_i \in E_{\text{int}}(\Gamma) \atop t(e_i) = v} k_i + \sum_{e_j \in E_{\text{ext}}(\Gamma) \atop t(j) = v} \kappa_j.
\]

Let us write \( N \) for the number of independent linear relations imposed by (2.6), and choose \( n - N \) independent variables \( l_i \) among \( \{k_1, \ldots, k_n\} \). One usually refers to the variables \( l_i \) as the “loop variables”. Indeed, it is known that \( N = \# V_{\text{int}}(\Gamma) - 1 \). Therefore, the difference \( n - N = \# E_{\text{int}}(\Gamma) - \# V_{\text{int}}(\Gamma) + 1 \) is equal to the first Betti number \( b_1(\Gamma) \) of the graph \( \Gamma \); see [24, §5.2][27, §8]. In what follows, we will write \( L := b_1(\Gamma) \) for the “loop number” of \( \Gamma \).

**Lemma 2.7.** Assume that \( \kappa = 0 \). Under this assumption, the intersection of the quadric hypersurfaces \( Q'_i \subset \mathbb{P}^{nD} \) with the following linear subspace

\[
H_{\Gamma} := \bigcap_{v \in V_{\text{int}}(\Gamma)} \{ \sum_{e_i \in E_{\text{int}}(\Gamma) \atop s(e_i) = v} k_i - \sum_{e_j \in E_{\text{ext}}(\Gamma) \atop t(e_j) = v} k_i = 0 \} \subset \mathbb{P}^{nD}
\]

determines quadric hypersurfaces \( Q_i \subset \mathbb{P}^{LD} \), \( i = 1, \ldots, n \).

**Proof.** When \( \kappa = 0 \), the relations (2.6) become homogeneous in the variables \( k_i \) and determine the linear subspace \( H_{\Gamma} \). Note that the loop variables \( l_i \) and the auxiliary variable \( x \) are homogeneous coordinates of \( H_{\Gamma} \). Let us then write \( q_\ell(\ell, x) \) for the restriction of (2.5) to \( H_{\Gamma} \). Since \( H_{\Gamma} \simeq \mathbb{P}^{LD} \), the intersections \( Q_i := Q'_i \cap H_{\Gamma} \) agree with the quadric hypersurfaces in \( \mathbb{P}^{LD} \) defined by the equations \( \{ q_\ell(\ell, x) = 0 \} \).

**Notation 2.8.** Let \( u = (u_0 : \cdots : u_{LD}) \) be projective coordinates on the projective space \( \mathbb{P}^{LD} \) corresponding to the following variables:

\[
(2.9) \quad u_0 := x \quad u_1, \ldots, u_D := \ell_1 \quad \cdots \quad u_{(L-1)D}, \ldots, u_{LD} := \ell_L.
\]

Note that the parameterizing space of all quadric hypersurfaces in \( \mathbb{P}^{LD} \) is the projective space \( \mathbb{P}^{(LD+2) - 1} \) of symmetric \((LD+1) \times (LD+1)\)-matrices up to scalar multiples. Inside this parameterizing space we have the discriminant hypersurface \( D \) consisting of all those quadratic forms with non-trivial kernel. Recall that a net of \( n \) quadric hypersurfaces in \( \mathbb{P}^{LD} \) consists of an embedding \( \rho : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{(LD+2) - 1} \).

Consider the following net of quadrics

\[
(2.10) \quad \rho : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{(LD+2) - 1} \quad (0 : \cdots : 0 : 1 : 0 : \cdots : 0) \mapsto Q_i,
\]

where \( Q_i \) stands for the quadric hypersurface of the above Lemma 2.7.

**Lemma 2.11.** The above net of quadrics (2.10) has the following properties:

(i) The quadric hypersurfaces \( Q_i \) belong to \( \mathbb{P}^{(LD+2) - 1}(\mathbb{R}) \), i.e. the defining quadratic form \( q_i \) of the quadric \( Q_i \) is real.
(ii) The symmetric matrices \( A_i \), defined by the equalities \( q_i(u) = \langle u, A_i u \rangle \), can be written as \( A_i = T_i^T T_i \), where \( T_i^T \) stands for the adjoint of the matrix \( T_i \) with respect to the bilinear form of Notation 2.3.

(iii) Let \( P \) be the projection \((u_0, \ldots, u_{LD}) \mapsto (u_1, \ldots, u_{LD})\) and \( \bar{T}_i \) the matrix \( P T_i P \). Under these notations, the matrices \( \bar{T}_i \) satisfy the following momentum conservation condition: \( \sum_{s(e_i)=v} \bar{T}_i = \sum_{l(e_i)=u} \bar{T}_i \).

Proof. Item (i) follows from the combination of Lemma 2.7 with the description (2.5) of the quadratic form \( q_i' \). The description (2.5) of the quadratic forms \( q_i' \) also implies that the matrix \( A_i \) can be written as \( A_i = T_i^T T_i \) with \( q_i(u) = \langle T_i u, T_i u \rangle \). This shows item (ii). In what concerns item (iii), note that thanks to Lemma 2.7 we can order the internal edges of the Feynman graph \( \Gamma \) in such a way that the first \( L \) quadratic forms \( q_i \) do not depend on the variables \( l_j \in \mathbb{A}^D \) with \( j \neq i \). The dependence of the remaining quadratic forms \( q_i \) on the \( l_i \) variables is dictated by the momentum conservation condition (2.6) (with \( \kappa_j = 0 \)). Note that the matrices \( \bar{T}_i \) are diagonal and that its entries are either 1 or 0. This corresponds to which variables \( u_j, j \geq 1 \), occur in \( q_i \) or not. Therefore, the condition (2.6) (with \( \kappa_j = 0 \)) can be written as \( \sum_{s(e_i)=v} \bar{T}_i u = \sum_{l(e_i)=u} \bar{T}_i u \). Finally, since the latter equality holds for all the variables \( u_j \), it can be re-written as the following momentum conservation condition: \( \sum_{s(e_i)=v} \bar{T}_i = \sum_{l(e_i)=u} \bar{T}_i \). This proves item (iii).

Definition 2.12. A one-parameter deformation of a net of \( n \) quadric hypersurfaces \( \rho : \mathbb{P}^{n-1} \to \mathbb{P}^{(LD+2)-1} \) is a morphism \( \tilde{\rho} : \mathbb{P}^{n-1} \times \mathbb{A}^1 \to \mathbb{P}^{(LD+2)-1} \) such that \( \rho = \tilde{\rho}|_{\mathbb{P}^{n-1} \times \{0\}} \). Given a point \( \epsilon \in \mathbb{A}^1(\mathbb{Q}) \), with \( \epsilon \neq 0 \), we will write \( \rho_\epsilon \) for the associated net of quadrics \( \tilde{\rho}|_{\mathbb{P}^{n-1} \times \{\epsilon\}} \) and call it the \( \epsilon \)-deformation of \( \rho \).

Proposition 2.13. There exists a one-parameter deformation \( \tilde{\rho} \) of the net of quadrics (2.10) such that for sufficiently small points \( \epsilon \in \mathbb{A}^1(\mathbb{Q}) \) the \( \epsilon \)-deformations

\[
(2.14) \quad \rho_\epsilon : \mathbb{P}^{n-1} \to \mathbb{P}^{(LD+2)-1} \quad (0 : \cdots : 0 : 1 : 0 : \cdots : 0) \mapsto Q_{i,\epsilon}
\]

have the following properties:

(i) The quadrics \( Q_{i,\epsilon} \) belong to \( \mathbb{P}^{(LD+2)-1}\setminus\mathcal{D} \), i.e. they are smooth.

(ii) The quadrics \( Q_{i,\epsilon} \) belong to \( \mathbb{P}^{(LD+2)-1}(\mathbb{R}) \).

(iii) The symmetric matrices \( A_{i,\epsilon} \) can be written as \( A_{i,\epsilon} = T_{i,\epsilon}^T T_{i,\epsilon} \).

(iv) Let \( P \) be the projection \((u_0, \ldots, u_{LD}) \mapsto (u_1, \ldots, u_{LD}) \) and \( T_{i,\epsilon} \) the matrix \( P T_{i,\epsilon} P \). Under these notations, the matrices \( T_{i,\epsilon} \) satisfy the following momentum conservation condition: \( \sum_{s(e_i)=v} T_{i,\epsilon} = \sum_{l(e_i)=u} T_{i,\epsilon} \).

Corollary 2.15. The quadrics \( Q_{i,\epsilon} \) don’t have real points.

Proof. Items (ii)-(iii) of Proposition 2.13 imply that the symmetric matrices \( A_{i,\epsilon} \) have real non-negative eigenvalues \( \lambda_{i,\epsilon} \). Moreover, item (i) implies that these eigenvalues \( \lambda_{i,\epsilon} \) are strictly positive. Therefore, we have \( q_{i,\epsilon}(u) = \sum_{i=0}^{LD} \lambda_{i,\epsilon} u_i^2 \) with \( \lambda_{i,\epsilon} > 0 \). This shows that the associated quadrics \( Q_{i,\epsilon} \) don’t have real points.

Proof. (of Proposition 2.13) Thanks to Lemma 2.11, the quadrics \( Q_i \) of Lemma 2.7 belong to \( \mathbb{P}^{(LD+2)-1}(\mathbb{R}) \). However, they are singular in general, i.e. they belong to the discriminant divisor \( \mathcal{D} \). Nevertheless, since the complement \( \mathbb{P}^{(LD+2)-1}\setminus\mathcal{D} \) is a Zariski open set, there exists then a one-parameter deformation \( \tilde{\rho} \) of the net of quadrics (2.10) such that for a generic point \( \epsilon \in \mathbb{A}^1(\mathbb{Q}) \) the associated \( \epsilon \)-deformation
(2.14) satisfies conditions (i)-(ii). The corresponding quadratic forms can then be diagonalized $q_{i,\epsilon}(u) = \sum \lambda_i u_i^2$. Moreover, the eigenvalues $\lambda_i$ are real, non-zero, and converge to the eigenvalues $\lambda_i$ of the quadratic forms $q_i$ when $\epsilon \to 0$. If $\lambda_i > 0$, then for a sufficiently small point $\epsilon \in A^1(Q)$ we also have $\lambda_i > 0$. If $\lambda_i = 0$, then for a sufficiently small point $\epsilon \in A^1(Q)$, we have $\lambda_i > 0$ or $\lambda_i < 0$. In the latter case, we can always change the sign in order to make all the eigenvalues positive. This yields a new $\epsilon$-deformation which not only satisfies conditions (i)-(ii) but also condition (iii).

Let us now prove condition (iv). Recall that a spanning tree $\tau$ of a connected graph $\Gamma$ is a connected subgraph of $\Gamma$ which is a tree and which contains all the vertices of $\Gamma$. The Euler characteristic formula implies immediately that the complement $\Gamma \setminus \tau$ consists of $L = b_1(\Gamma)$ edges. We need to show that if the original matrices $T_i$ satisfy the momentum conservation condition $\sum_{e \in \tau} t_i = \sum_{\ell \in \ell_i} t_i$, then there is a non-empty set of $\epsilon$-deformations $\rho_{q_i}$ such that the associated matrices $T_i,\epsilon$ also satisfy the momentum conservation condition $\sum_{e \in \tau} T_i,\epsilon = \sum_{\ell \in \ell_i} T_i,\epsilon$.

We will show that this is possible by first choosing a spanning tree $\tau$ for the Feynman graph $\Gamma$, then by constructing a $\epsilon$-deformation $q_{i,\epsilon}$ of the quadratic forms $q_i$ associated to the $L$ edges in the complement of the spanning tree, and finally by showing that there is a unique way to extend the deformation to the remaining quadratic forms $q_i$ associated to the edges of the spanning tree so that momentum conservation condition (as well as the above conditions (i)-(iii)) holds.

If $\tau$ is a spanning tree of $\Gamma$, then contracting $\tau$ to a single vertex gives a bouquet of $L$ circles. Hence, the complement $\Gamma \setminus \tau$ provides a choice of $L$ edges all belonging to different loops (=different generators of the first homology) of $\Gamma$. This implies that, in the loop variables $\ell_i$, we can write the quadratic forms $q_i$ with $i = 1, \ldots, L$, associated to the edges $e_i$ in the complement $\Gamma \setminus \tau$ as functions of a single loop variable $\ell_i = (\ell_i,1, \ldots, \ell_i,D)$ and of the variable $x$, independently of the remaining variables $\ell_j$ with $j \neq i$; see Lemma 2.7. Now, we can deform $q_i$ into $q_{i,\epsilon}$ just by adding terms of the form $\lambda_{j,k,\epsilon} \ell_{j,k}^2$ for the remaining variables that do not appear in $q_i$. Concretely, we have $q_{i,\epsilon}(x, \ell_1, \ldots, \ell_L) := q_i(\ell_i) + \sum_{j \neq i} \lambda_{j,k,\epsilon} \ell_{j,k}^2$, where the terms $\lambda_{j,k,\epsilon}$ are chosen so that the above conditions (i)-(ii)-(iii) are satisfied. It remains to show that it is possible to compatibly choose the deformations $q_{i,\epsilon}$ of the remaining $q_i$, with $i = L + 1, \ldots, n$, so that the momentum conservation condition $\sum_{e \in \tau} T_i,\epsilon = \sum_{\ell \in \ell_i} T_i,\epsilon$ as well as the above conditions (i)-(iii) also holds. These remaining edges are the edges of the spanning tree $\tau$.

The strategy is to extend the deformation to the edges of the spanning tree, by imposing the momentum conservation condition, starting with the ends of the tree and proceed inward. Consider first the vertices that are incident to only one edge in the spanning tree. This means that, for all other edges incident to the same vertex, the deformation $q_{i,\epsilon}$ has already been assigned. Thus, the matrices $T_i,\epsilon$ for all but one of the edges are known, and the momentum conservation equation at the vertex fixes what the matrix $T_i,\epsilon$ for the last remaining edge (the one in the spanning tree) should be.

Since the coefficient of $x^2$ is fixed to be the mass $m_i^2 > 0$, which we can leave undeformed, determining $T_i,\epsilon$ suffices to determine the full $T_i,\epsilon$ for this remaining edge, hence the quadratic form $q_{i,\epsilon}$ is also determined.

To proceed to the next step, observe that, after this step there must be vertices in the spanning tree for which the $q_{i,\epsilon}$ for all but one of the adjacent edges have
already been determined. Indeed, we can just remove from the graph all the vertices for which all adjacent edges have \( q_{i,\epsilon} \) already determined. The intersection of the original spanning tree with the remaining graph is a spanning tree for this smaller graph and we can repeat the first step.

This implies that we can again uniquely determine the \( \tilde{T}_{i,\epsilon} \) (hence the \( T_{i,\epsilon} \) and the \( q_{i,\epsilon} \)) for the remaining edge by imposing the momentum conservation condition. Iterating this procedure exhausts all the edges of the spanning tree. Each time, in diagonal form the \( T_{j,\epsilon} \) have non-zero eigenvalues with either positive or negative sign, hence the corresponding \( A_{j,\epsilon} = T_{j,\epsilon}^\dagger T_{j,\epsilon} \) has strictly positive \( \lambda_{j,\epsilon} > 0 \) and satisfies (i)-(ii)-(iii) in addition to satisfying (iv) by construction. □

**Definition 2.16.** Let \((\Gamma, m)\) be a Feynman graph equipped with a mass parameter \( m \) (and with trivial external momentum \( \kappa \)). The associated Feynman quadrics-motive \( M^Q_{\Gamma, m} \) is defined as the Voevodsky’s mixed motive \( M(\mathbb{P}^{LD} \backslash Q_{(\Gamma, m)})_Q \in \text{DM}_{gm}(F)_Q \), where \( Q_{(\Gamma, m)} := \bigcup_{i=1}^n Q_{i,\epsilon} \) stands for the union of the quadric hypersurfaces \( Q_{i,\epsilon} \subset \mathbb{P}^D \) introduced in Proposition 2.13.

**Remark 2.17.** Let \((\Gamma, m, \kappa)\) be a Feynman graph equipped with a mass parameter and with external momentum. Note that Definition 2.16 holds similarly in the case where the ingoing momentum (in the left-hand side of (2.6)) equals the outgoing momentum (in the right-hand side of (2.6)). In particular, we can also consider the massive sunset graph equipped with (ingoing=going) momentum.

### 2.3. Regularization and Renormalization.

In this subsection we express the divergent Feynman integral (2.2) as a period of the Feynman quadrics-motive \( M^Q_{\Gamma, m} \). Recall from [27, §8] that the superficial degree of divergence of the Feynman graph \( \Gamma \) is defined as \( \delta(\Gamma) := LD - 2n \). When \( \delta(\Gamma) \geq 0 \), the Feynman integral (2.2) is divergent at infinity (=UV divergence). In the particular case where \( \delta(\Gamma) = 0 \) the Feynman integral (2.2) is moreover logarithmically divergent.

Following Notation 2.8, consider the following differential form

\[
(2.18) \quad \omega := \sum_{i=0}^{LD} (-1)^i u_1 \wedge \cdots \wedge \widehat{du_i} \wedge \cdots \wedge du_{LD}
\]

as well as the algebraic differential forms

\[
(2.19) \quad \eta_\alpha := \frac{\omega}{\prod_{i=1}^n q_i^\alpha}, \quad \eta_{\alpha,\epsilon} := \frac{\omega}{\prod_{i=1}^n q_i^{\alpha,\epsilon}},
\]

where \( \alpha \in \mathbb{N} \) and \( q_i \), resp. \( q_{i,\epsilon} \), stands for the quadratic form corresponding to the quadric hypersurface \( Q_i \), resp. \( Q_{i,\epsilon} \), of Lemma 2.7, resp. Proposition 2.13.

**Lemma 2.20.** The algebraic differential forms (2.19) have the following properties:

(i) When \( \alpha = 1 \), the Feynman integral (2.2) is equal to

\[
(2.21) \quad C \frac{1}{(2\pi)^D} \int_{\mathbb{A}^{LD}(\mathbb{R})} \eta_1,
\]

where \( \mathbb{A}^{LD} \subset \mathbb{P}^{LD} \) stands for the affine chart of coordinates \((1, u_1, \ldots, u_{LD})\).

(ii) When \( \alpha > \frac{LD}{2\pi} \), the following integral

\[
(2.22) \quad \int_{\mathbb{A}^{LD}(\mathbb{R})} \eta_{\alpha,\epsilon} = \int_{\mathbb{P}^{2D}(\mathbb{R})} \eta_{\alpha,\epsilon}
\]

is convergent and a period of \( \mathbb{P}^{LD} \backslash Q_{(\Gamma, m)} \).
Proof. The restriction of the differential form $\omega$ to the affine chart $\mathbb{A}^{LD}$ of coordinates $(1,u_1,\ldots,u_{LD})$ is the standard affine volume form $du_1 \wedge \cdots \wedge du_{LD}$. In other words, it is the volume form $d^D \ell_1 \cdots d^D \ell_L = \delta(H_F) d^D k_1 \cdots d^D k_n$ in the integrand of (2.2), with $\delta(H_F)$ the delta function. The denominator in the integrand of (2.2) is the product of the quadratic forms $q_j$ (as in $\eta_{\alpha=1}$) and the integration in (2.2) is over real variables. Hence, the locus of integration is the set of real points $\mathbb{A}^{LD}(\mathbb{R})$ of the affine chart $\mathbb{A}^{LD}$. This shows item (i).

The presence of the exponent $\alpha$ in the denominator of (2.19) changes the superficial degree of convergence of the integral from $\delta(\Gamma) = LD - 2n$ to $\delta(\Gamma) = DL - 2n\alpha$. When $DL - 2n\alpha < 0$, the integral on the left-hand-side of (2.22) is convergent at infinity. Since we always assume that $m_i > 0$ for every edge $e_i$, there are no further divergences in the domain of integration $\mathbb{A}^{LN}(\mathbb{R})$. The choice of the $\epsilon$-deformations $Q_{1,\epsilon}$ of Proposition 2.13 ensures that the differential form $\eta_{\alpha,\epsilon}$ also has no poles on the hyperplane at infinity $\mathbb{P}^{LN}(\mathbb{R}) \backslash \mathbb{A}^{LN}(\mathbb{R}) = \mathbb{P}^{LN-1}(\mathbb{R})$; otherwise, $q_{i,\epsilon}$ would have zeros on the real locus $\mathbb{P}^{LN}(\mathbb{R})$. This yields the equality (2.22). The proof of item (ii) follows now from the fact that (2.22) is manifestly a period of $\mathbb{P}^{LD}(Q_{1,m})$. □

Recall that the process of extraction of finite values from divergent Feynman integrals consists of two main steps:

(i) **Regularization**: the replacement of divergent Feynman integrals by meromorphic functions with poles at the exponents of divergence.

(ii) **Renormalization**: a pole subtraction procedure on these meromorphic functions performed consistently with the combinatorics of subgraphs and quotient graphs (nested subdivergences).

A general procedure for carrying out these steps is provided by the Connes–Kreimer formalism of algebraic renormalization [23]; consult also [24, §1][28, §5]. In our setting, regularization is obtained by combination the $\epsilon$-deformation of quadric hypersurfaces with an Igusa zeta function. On the other hand, renormalization is as in the Connes–Kreimer setting. Consider the following Igusa zeta function

\[
\mathcal{I}(s) = \int_{\mathbb{P}^{LD}(\mathbb{R})} \eta_{s,\epsilon},
\]

where the integer $\alpha \in \mathbb{N}$ has been replaced by a complex variable $s$.

**Proposition 2.24.** The Igusa zeta function $\mathcal{I}(s)$ has a Laurent series expansion

\[
\mathcal{I}(s) = \sum_{k \geq N} \gamma_k \, (s - \alpha)^k \quad \alpha \in \mathbb{Z}
\]

for some $N \in \mathbb{Z}$, where the coefficients $\gamma_k$ are periods of $(\mathbb{P}^{LD}(Q_{1,m})) \times \mathbb{A}^k$.

**Proof.** Similarly to the proof of Lemma 2.20, we observe that the integral defining $\mathcal{I}(s)$ is convergent for $\Re(s) > \frac{LD}{2n}$. By writing it as in the left-hand-side of (2.22), we can then use [10, Cor. 4.7] in order to extend $\mathcal{I}(s)$ to a meromorphic function $\mathcal{I}_F(s)$ on the entire complex plane. This extension satisfies the functional equation

\[
\mathcal{I}_F(s) = a_1(s) \mathcal{I}(s + 1) + \cdots + a_k(s) \mathcal{I}(s + k)
\]

for some $k \in \mathbb{N}$, where the $a_i(s)$ are rational functions. For $\alpha > \frac{LD}{2n}$, we write the Laurent expansion at $\alpha$ as follows:

\[
\mathcal{I}_F(s) = \sum_{k \geq 0} \frac{(-1)^k}{k!} \int_{\mathbb{P}^{LD}(\mathbb{R})} \eta_{0,\epsilon} \cdot \log^k(\prod_{j=1}^{n} q_{j,\epsilon}) \cdot (s - \alpha)^k.
\]
We then argue as in [8], using the identity
\[
\log(f(u)) = \int_0^1 \theta(u, t) \quad \text{with} \quad \theta(u, t) = \frac{f(u) - 1}{(f(u) - 1)t + 1} \, dt,
\]
to show that \( \gamma_k \) is a period
\[
\gamma_k = \frac{(-1)^k}{k!} \int_{\mathbb{P}^D \times [0,1]^k} \eta_{\alpha,\epsilon} \wedge \theta(u,t_1) \wedge \cdots \wedge \theta(u,t_k)
\]
of \((\mathbb{P}^D \setminus \mathbb{P}(\Gamma,m)) \times \mathbb{A}^k\). Using the functional equation (2.26), it is then possible to argue inductively as in [8]: the same property continues to hold for integers \( \alpha \leq \frac{kD}{m} \), using the Laurent expansion of the terms on the right-hand-side of the functional equation to define that of the left-hand-side.

**Remark 2.27.** The coefficients \( \gamma_k \) of the Laurent series expansion (2.25) are periods of \((\mathbb{P}^D \setminus \mathbb{P}(\Gamma,m)) \times \mathbb{A}^k\). Therefore, thanks to \( \mathbb{A}^1 \)-homotopy invariance, it suffices to understand the nature of the Feynman quads-motive \( M^Q_{(\Gamma,m)} \).

We now briefly recall how the formalism of algebraic renormalization of [23] can be used in order to obtain a renormalized value from the regularized Feynman integral \( \mathcal{I}_T(s) \). Let \( \mathcal{H}_{CK} \) be the Hopf algebra of Feynman graphs. As a commutative algebra, \( \mathcal{H}_{CK} \) is the commutative polynomial algebra in the connected and 1-edge connected (1PI in the physics terminology) Feynman graphs. The coproduct \( \Delta \) is non-cocommutative and is defined by a certain sum over certain subgraphs (consult [24, §5.3 and §6.2] for details):
\[
\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma} (\gamma \otimes \Gamma) / \gamma.
\]
The Hopf algebra \( \mathcal{H}_{CK} \) is graded by the loop number \( L \) (or by the number of internal edges \( n \)) and is connected. The antipode is defined inductively by the formula \( S(\Gamma) = -\Gamma + \sum S(\Gamma') X'' \) for \( \Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum X' \otimes X'' \) with \( X' \) and \( X'' \) of lower degree. Let \( \mathcal{R} \) be the algebra of Laurent series centered at \( s = 1 \) and let \( \mathcal{T} \) be the projection onto its polar part. This is a Rota–Baxter operator of degree \(-1\), i.e. the following equality holds \( \mathcal{T}(f_1) \mathcal{T}(f_2) = \mathcal{T}(f_1 \mathcal{T}(f_2)) + \mathcal{T}(\mathcal{T}(f_1) f_2) - \mathcal{T}(f_1 f_2) \). The operator \( \mathcal{T} \) determines a splitting \( \mathcal{R}_+ = (1 - \mathcal{T}) \mathcal{R} \) and \( \mathcal{R}_- = \mathcal{T} \mathcal{R} \), called the unitization of \( \mathcal{T} \). Given any morphism of commutative algebras \( \phi : \mathcal{H}_{CK} \to \mathcal{R} \), the coproduct on \( \mathcal{H} \) and the Rota–Baxter operator on \( \mathcal{R} \) determine a Birkhoff factorization of \( \phi \) into algebra homomorphisms \( \phi_\pm : \mathcal{H}_{CK} \to \mathcal{R}_\pm \). These algebra homomorphisms are determined inductively by the following formulas
\[
\phi_- (\Gamma) = -\mathcal{T}(\phi(\Gamma)) + \sum_{\Gamma' \subset \Gamma} \phi_- (\Gamma') \phi(\Gamma''),
\]
\[
\phi_+ (\Gamma) = (1 - \mathcal{T})(\phi(\Gamma)) + \sum_{\Gamma' \subset \Gamma} \phi_- (\Gamma') \phi(\Gamma''),
\]
for \( \Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum (X' \otimes X'') \) in such a way that \( \phi = (\phi_- \circ S) \circ \phi_+ \), with \( \phi_1 \circ \phi_2 (\Gamma) := \langle \phi_1 \otimes \phi_2, \Delta(\Gamma) \rangle \). Given a Feynman graph \( \Gamma \), the Laurent series \( \phi_+(\Gamma)(s) \) is regular at \( s = 1 \) and the value \( \phi_+(\Gamma)(1) \) is the renormalized value. More explicitly, we have the following equality
\[
\phi_+(\Gamma)(s) = (1 - \mathcal{T})(\mathcal{I}_T(s)) + \sum_{\gamma \subset \Gamma} \phi_- (\gamma)(s) \cdot \mathcal{I}_{\Gamma/\gamma}(s),
\]
where \( \phi_- \) is defined inductively as above.
Remark 2.28. In this article, we focus solely on the leading term \((1 - T)I_T(s)|_{s = 1}\) of the renormalized \(\phi_+(\Gamma)\), which is a period of \((\mathbb{P}^2D \setminus Q(\Gamma,m)) \times \mathbb{A}^1\).

3. Proof of Theorem 1.1

By construction, the massive sunset graph \((\Gamma, m)\) has 2 vertices and 3 internal edges \(e_1, e_2, e_3\). Following §2, let us write \((k_1, k_2, k_3) \in \mathbb{A}^{3D}\) for the associated “momentum variables”. Under these notations, the two linear relations \((2.6)\) reduce to the single relation \(k_1 + k_2 + k_3 = 0\). Therefore, \(N = 1\) and \(n - N = 2\). Let us now choose \(\ell_1 := k_1\) and \(\ell_2 := k_2\) as the “loop variables”. Equivalently, let us use the variables \(u = (u_0, u_1, \ldots, u_{2D})\) with \(u_0 = x, \ell_1 = (\ell_{1,1}, \ldots, \ell_{1,D}) = (u_1, \ldots, u_D)\), and \(\ell_2 = (\ell_{2,1}, \ldots, \ell_{2,D}) = (u_{D+1}, \ldots, u_{2D})\). Under these choices, the quadric hypersurfaces \(Q_1, Q_2, Q_3 \subset \mathbb{P}^{2D}\) of Lemma 2.7 can be written as follows:

\[
\begin{align*}
Q_1 &= \{q_1(u) = (u, A_1 u) = 0\} \quad \text{with} \quad A_1 = \text{diag}(m_1^2, 1, \ldots, 1, 0, \ldots, 0) \\
Q_2 &= \{q_2(u) = (u, A_2 u) = 0\} \quad \text{with} \quad A_2 = \text{diag}(m_2^2, 0, \ldots, 0, 1, \ldots, 1) \\
Q_3 &= \{q_3(u) = (u, A_3 u) = 0\} \quad \text{with} \quad A_3 = \text{diag}(m_3^2, 1, \ldots, 1, 1, \ldots, 1).
\end{align*}
\]

Let us write \(Q_1, Q_2, Q_3 \subset \mathbb{P}^{2D}\) for the associated \(\epsilon\)-deformations of Proposition 2.13. An explicit choice for these \(\epsilon\)-deformations is the following:

\[
\begin{align*}
Q_{1,\epsilon} &= \{q_{1,\epsilon}(u) = (u, A_{1,\epsilon} u) = 0\} \quad \text{with} \quad A_{1,\epsilon} = \text{diag}(m_1^2, 1, \ldots, 1, \epsilon^2, \ldots, \epsilon^{2D}) \\
Q_{2,\epsilon} &= \{q_{2,\epsilon}(u) = (u, A_{2,\epsilon} u) = 0\} \quad \text{with} \quad A_{2,\epsilon} = \text{diag}(m_2^2, \epsilon^2, \ldots, \epsilon^{2D}, 1, \ldots, 1) \\
Q_{3,\epsilon} &= \{q_{3,\epsilon}(u) = (u, A_{3,\epsilon} u) = 0\},
\end{align*}
\]

where \(A_{3,\epsilon}\) stands for the following diagonal matrix:

\[
A_{3,\epsilon} = \text{diag}(m_3^2, (1 + \epsilon)^2, \ldots, (1 + \epsilon^D)^2, (1 + \epsilon)^2, \ldots, (1 + \epsilon^D)^2).
\]

Proposition 3.2. There exists a Zariski open subset \(W(m) \subset \mathbb{A}^1\) (which depends on the mass parameter \(m = (m_i)\)) such that for every \(\epsilon \in W(m)\) the above \(\epsilon\)-deformations \(Q_{1,\epsilon}, Q_{2,\epsilon}, Q_{3,\epsilon} \subset \mathbb{P}^{2D}\) have not only the properties (i)-(iv) of Proposition 2.13 but also the following additional properties:

(v) The intersection \(Q_{1,i,\epsilon} \cap Q_{3,j,\epsilon}\) with \(i \neq j \in \{1, 2, 3\}\), is a complete intersection.

(vi) The intersection \(Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}\) is a complete intersection.

Proof. As explained in [26, Prop. 17.18], an intersection of \(r\) transversally intersecting hypersurfaces in \(\mathbb{P}^s\), with \(r < s\), is always a complete intersection. Therefore, the proof will consist on showing that the quadric hypersurfaces \(Q_{1,\epsilon}, Q_{2,\epsilon}, Q_{3,\epsilon} \subset \mathbb{P}^{2D}\) intersect transversely. This condition can be checked in an affine chart. Concretely, the tangent space at a point \(\bar{u}\) of the quadric hypersurface \(Q_{i,\epsilon} = \{q_{i,\epsilon}(u) = 0\}\) is defined by the equation \(\langle \nabla q_{i,\epsilon}(\bar{u}), (u - \bar{u}) \rangle = 0\) (see Notation 2.3), where \(\nabla q_{i,\epsilon}\) stands for the gradient vector. The gradient vectors \(\nabla q_{1,\epsilon}(u), \nabla q_{2,\epsilon}(u),\) and \(\nabla q_{3,\epsilon}(u),\) are
given, respectively, by the following expressions:

\[
\begin{align*}
(2m_1^2u_0, 2(u_1, \ldots, u_D), 2(\epsilon^2 u_{D+1}, \ldots, \epsilon^{2D} u_{2D})) \\
(2m_2^2u_0, 2(\epsilon^2 u_1, \ldots, \epsilon^{2D} u_D), 2(u_{D+1}, \ldots, u_{2D}) \\
(2m_3^2u_0, 2((1 + \epsilon)^2 u_1, \ldots, (1 + \epsilon)^2 u_D), 2((1 + \epsilon)^2 u_{D+1}, \ldots, (1 + \epsilon)^2 u_{2D}).
\end{align*}
\]

Hence, in order to prove item (v) it suffices to show that for every point \(u \in \bigcup_{i \neq j}(Q_{i,\epsilon} \cap Q_{j,\epsilon}) \subset \mathbb{P}^{2D}\) any two of the three gradient vectors are linearly independent. Note that the points of \(\mathbb{P}^{2D}\) which lie in at least one of the quadric hypersurfaces \(Q_{i,\epsilon}\) have at least two nonzero coordinates \(u_i\) (if all but one of the \(u_i\) are zero, then by the equation \(g_{i,\epsilon}(\epsilon) = 0\) the last coordinate must also be zero, which would not be a point in projective space). Thus, it is enough to check that at all points of \(\mathbb{P}^{2D}\) with at least two non-zero coordinates, the vectors are linearly independent. This is equivalent to checking that the following \(2 \times 2\) matrices have non-zero determinant:

\[
\begin{pmatrix}
1 & \epsilon^j \\
\epsilon^j & m_1^2 \\
\end{pmatrix}
\begin{pmatrix}
\epsilon^{2k} & m_1^2 \\
m_1 & m_2 \\
\end{pmatrix}
\begin{pmatrix}
j \neq k \\
1 & \epsilon^{2k} \\
k \neq j \\
1 & \epsilon^j \\
\end{pmatrix}
\begin{pmatrix}
1 & \epsilon^j \\
(1 + \epsilon)^2 & m_1^2 \\
\end{pmatrix}
\begin{pmatrix}
\epsilon^{2k} & m_1^2 \\
(1 + \epsilon)^2 & m_2 \\
\end{pmatrix}
\begin{pmatrix}
j \neq k \\
1 & \epsilon^{2k} \\
(1 + \epsilon)^2 & m_2 \\
\end{pmatrix}
\begin{pmatrix}
1 & \epsilon^j \\
(1 + \epsilon)^2 & m_1^2 \\
\end{pmatrix}
\begin{pmatrix}
\epsilon^{2k} & m_1^2 \\
(1 + \epsilon)^2 & m_2 \\
\end{pmatrix}
\begin{pmatrix}
1 & \epsilon^j \\
(1 + \epsilon)^2 & m_2 \\
\end{pmatrix}.
\]

The locus where at least one of these determinants is equal to zero defines a polynomial equation in \(\epsilon\) that depends on the value of the masses \(m = (m_i)\). Thus, the set of \(\epsilon\)'s where all the determinants are nonzero is the complement of the solutions of these polynomial equations, hence a Zariski open set \(U = U(m)\). The intersection of this open set \(U(m)\) with the open set of sufficiently small \(\epsilon\)'s for which conditions (i)-(iv) of Proposition 2.13 hold is also a Zariski open set.

In the same vein, in order to prove (vi), it suffices to show that at every point \(u \in Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon} \subset \mathbb{P}^{2D}\) the three gradient vectors are linearly independent. This means that, at every such point, at least one \(3 \times 3\)-minor of the matrix formed by the three gradient vectors is non-zero. It is therefore sufficient to show that the following \(3 \times 3\) matrices have non-zero determinant:

\[
\begin{pmatrix}
1 & \epsilon^{2j} & m_1^2 \\
\epsilon^{2j} & 1 & m_2 \\
(1 + \epsilon)^2 & (1 + \epsilon)^2 & m_3 \\
\end{pmatrix}
\begin{pmatrix}
1 & \epsilon^{2k} & m_1^2 \\
\epsilon^{2k} & 1 & m_2 \\
(1 + \epsilon)^2 & (1 + \epsilon)^2 & m_3 \\
\end{pmatrix}.
\]
Notation 3.3. (i) Let $X$ be a $F$-scheme of finite type. Following Voevodsky [38], we will write $M(X)_{\mathbb{Q}}$, resp. $M^c(X)_{\mathbb{Q}}$, for the mixed motive, resp. mixed motive with compact support, associated to $X$. Recall from loc. cit. that whenever $X$ is proper, we have a canonical isomorphism $M^c(X)_{\mathbb{Q}} \cong M(X)_{\mathbb{Q}}$. (ii) Let $\text{Chow}(F)_{\mathbb{Q}}$ be the Grothendieck’s (additive) category of Chow motives; see [1, §4]. Given a smooth projective $k$-scheme $X$, we will write $h(X)_{\mathbb{Q}}$ for the associated Chow motive; when $X = \text{Spec}(k)$, we will write $1_{\mathbb{Q}}$ instead.

Remark 3.4. As proved by Voevodsky in [38, Prop. 2.1.4] (consult also [1, §18.3]), there exists a fully-faithful (contravariant) functor $\Phi: \text{Chow}(F)_{\mathbb{Q}} \to \text{DM}_{gm}(F)_{\mathbb{Q}}$ such that $\Phi(h(X)_{\mathbb{Q}}) \cong M(X)_{\mathbb{Q}}$ for every smooth projective $F$-scheme $X$. Moreover, the functor $\Phi$ sends the Lefschetz motive $\mathbb{L}$ to the Tate motive $\mathbb{Q}(1)/[2]$.

Proof of item (i).

Lemma 3.5. For every smooth $F$-scheme $X$, the motive $M(X)_{\mathbb{Q}}$ is mixed-Tate if and only if the motive $M^c(X)_{\mathbb{Q}}$ is mixed-Tate.

Proof. Without loss of generality, we can assume that $X$ is equidimensional; let $d$ be its dimension. As proved by Voevodsky in [38, Thm. 4.3.7], the dual $M(X)^\vee$ of $M(X)$ is isomorphic to $M^c(X)_{\mathbb{Q}}(-d)[-2d]$. Hence, the proof follows from the fact that the category of mixed-Tate motives is stable under duals and Tate-twists. □

Recall from Definition 2.16 that the Feynman quadrics-motive $M^Q_{\Gamma,m}$ is defined as $M(\mathbb{P}^{2D}\setminus Q_{\gamma,m})_{\mathbb{Q}} \in \text{DM}_{gm}(F)_{\mathbb{Q}}$, where $Q_{\gamma,m} = Q_{1,\varepsilon} \cap Q_{2,\varepsilon} \cap Q_{3,\varepsilon}$. Since $\mathbb{P}^{2D}\setminus Q_{\gamma,m}$ is smooth, we hence conclude from Lemma 3.5 that

\begin{equation}
M^Q_{\Gamma,m} \text{ mixed-Tate} \iff M^c(\mathbb{P}^{2D}\setminus Q_{\gamma,m})_{\mathbb{Q}} \text{ mixed-Tate}.
\end{equation}

Let $X$ be a $F$-scheme of finite type. Recall from [31, Ex. 16.18] that given a Zariski open cover $X = U \cup V$, we have an induced Mayer-Vietoris distinguished triangle:

\begin{equation}
M^c(X)_{\mathbb{Q}} \to M^c(U)_{\mathbb{Q}} \oplus M^c(V)_{\mathbb{Q}} \to M^c(U \cap V)_{\mathbb{Q}} \to M^c(X)_{\mathbb{Q}}[1].
\end{equation}

In the same vein, given a Zariski closed subscheme $Z \subset X$ with open complement $U$, recall from [38, Prop. 4.1.5] that we have an induced Gysin distinguished triangle:

\begin{equation}
M^c(Z)_{\mathbb{Q}} \to M^c(X)_{\mathbb{Q}} \to M^c(U)_{\mathbb{Q}} \to M^c(Z)_{\mathbb{Q}}[1].
\end{equation}
Lemma 3.9. The motive $M^c(\mathbb{P}^{2D}\setminus Q_{i,\epsilon})_Q$ is mixed-Tate for every $i \in \{1, 2, 3\}$.

Proof. Since $2D$ is even, the hypersurface quadric $Q_{i,\epsilon} \subset \mathbb{P}^{2D}$ is odd-dimensional. Consequently, we have the following motivic decomposition (see [17, Rk. 2.1]):

$$b(Q_{i,\epsilon})_Q \simeq 1_Q \oplus L \oplus L^\otimes 2 \oplus \cdots \oplus L^\otimes (2D-1).$$

This implies that the motive $M^c(Q_{i,\epsilon})_Q \simeq M(Q_{i,\epsilon})_Q$ is mixed-Tate; see Remark 3.4. Using the fact that the motive $M^c(\mathbb{P}^{2D})_Q \simeq M(\mathbb{P}^{2D})_Q$ is mixed-Tate, we hence conclude from the general Gysin triangle (3.8) (with $X := \mathbb{P}^{2D}$ and $Z := Q_{i,\epsilon}$) that the motive $M^c(\mathbb{P}^{2D}\setminus Q_{i,\epsilon})_Q$ is also mixed-Tate. \qed

Proposition 3.11. Assume that the motive $M^c(\mathbb{P}^{2D}\setminus (Q_{i,\epsilon} \cap Q_{j,\epsilon}))_Q$ is mixed-Tate for every $i \neq j \in \{1, 2, 3\}$. Under this assumption, the motive $M^c(\mathbb{P}^{2D}\setminus Q_{(i,m)})_Q$ is mixed-Tate if and only if the motive $M^c(\mathbb{P}^{2D}\setminus (Q_{i,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}))_Q$ is mixed-Tate.

Proof. Let $U := \mathbb{P}^{2D}\setminus (Q_{1,\epsilon} \cup Q_{2,\epsilon})$ and $V := \mathbb{P}^{2D}\setminus Q_{3,\epsilon}$. Note that $U \cap V = \mathbb{P}^{2D}\setminus Q_{(i,m)}$. Thanks to Lemma 3.9, the motive $M^c(V)_Q$ is mixed-Tate. Moreover, since by assumption the motive $M^c(\mathbb{P}^{2D}\setminus (Q_{i,\epsilon} \cap Q_{2,\epsilon}))_Q$ is mixed-Tate, Lemma 3.14 below implies that the motive $M^c(U)_Q$ is also mixed-Tate. Therefore, we conclude from the general Mayer-Vietoris triangle (3.7) (with $X := U \cup V$) that

$$M^c(\mathbb{P}^{2D}\setminus Q_{(i,m)})_Q \text{ mixed-Tate } \iff M^c(U \cup V)_Q \text{ mixed-Tate}.$$\hspace{2cm} (3.12)

Now, let $U_{13} := \mathbb{P}^{2D}\setminus (Q_{1,\epsilon} \cap Q_{3,\epsilon})$ and $U_{23} := \mathbb{P}^{2D}\setminus (Q_{2,\epsilon} \cap Q_{3,\epsilon})$. Note that $U_{13} \cap U_{23} = \mathbb{P}^{2D}\setminus ((Q_{1,\epsilon} \cap Q_{3,\epsilon}) \cup (Q_{2,\epsilon} \cap Q_{3,\epsilon})) = \mathbb{P}^{2D}\setminus (((Q_{1,\epsilon} \cup Q_{2,\epsilon}) \cap Q_{3,\epsilon}) = U \cup V$ and that $U_{13} \cup U_{23} = \mathbb{P}^{2D}\setminus (Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon})$. Therefore, since by assumption the motives $M^c(\mathbb{P}^{2D}\setminus (Q_{1,\epsilon} \cap Q_{3,\epsilon}))_Q$ and $M^c(\mathbb{P}^{2D}\setminus (Q_{2,\epsilon} \cap Q_{3,\epsilon}))_Q$ are mixed-Tate, we conclude from the general Mayer-Vietoris triangle (3.7) (with $X := U_{13} \cup U_{23}$) that

$$M^c(U \cup V)_Q \text{ mixed-Tate } \iff M^c(\mathbb{P}^{2D}\setminus (Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}))_Q \text{ mixed-Tate}.\hspace{2cm} (3.13)$$

The proof follows now from the combination of (3.12) with (3.13). \qed

Lemma 3.14. The motive $M^c(\mathbb{P}^{2D}\setminus (Q_{i,\epsilon} \cap Q_{j,\epsilon}))_Q$ is mixed-Tate if and only if the motive $M^c(\mathbb{P}^{2D}\setminus (Q_{i,\epsilon} \cap Q_{j,\epsilon}))_Q$ is mixed-Tate.

Proof. Thanks to Lemma 3.9, the proof follows from the general Mayer-Vietoris triangle (3.7) (with $X := \mathbb{P}^{2D}\setminus (Q_{i,\epsilon} \cap Q_{j,\epsilon})$, $U := \mathbb{P}^{2D}\setminus Q_{i,\epsilon}$ and $V := \mathbb{P}^{2D}\setminus Q_{j,\epsilon}$); note that under these choices we have $U \cap V = \mathbb{P}^{2D}\setminus (Q_{i,\epsilon} \cap Q_{j,\epsilon})$. \qed

Thanks to Proposition 3.2, the intersection $Q_{i,\epsilon} \cap Q_{j,\epsilon}$, with $i \neq j \in \{1, 2, 3\}$, is a complete intersection of two odd-dimensional quadrics. Therefore, as proved in [9, Cor. 2.1], the Chow motive $b(Q_{i,\epsilon} \cap Q_{j,\epsilon})_Q$ admits the motivic decomposition:

$$1_Q \oplus L \oplus L^\otimes 2 \oplus \cdots \oplus L^\otimes (2D-2) \oplus (L^\otimes (2D-1)) \oplus (L^\otimes (2D+2)) \oplus L^\otimes D \oplus \cdots \oplus L^\otimes (2D-2).$$\hspace{2cm} (3.15)

This implies that the motive $M^c(Q_{i,\epsilon} \cap Q_{j,\epsilon})_Q \simeq M(Q_{i,\epsilon} \cap Q_{j,\epsilon})_Q$ is mixed-Tate; see Remark 3.4. Using the fact that the motive $M^c(\mathbb{P}^{2D})_Q \simeq M(\mathbb{P}^{2D})_Q$ is mixed-Tate, we hence conclude from the general Gysin triangle (3.8) (with $X := \mathbb{P}^{2D}$ and $Z := Q_{i,\epsilon} \cap Q_{j,\epsilon}$) that the motive $M^c(\mathbb{P}^{2D}\setminus (Q_{i,\epsilon} \cap Q_{j,\epsilon}))_Q$ is also mixed-Tate. Consequently, thanks to Proposition 3.11, we obtain the (unconditional) equivalence:

$$M^c(\mathbb{P}^{2D}\setminus Q_{(i,m)})_Q \text{ mixed-Tate } \iff M^c(\mathbb{P}^{2D}\setminus (Q_{1,\epsilon} \cap Q_{2,\epsilon} \cap Q_{3,\epsilon}))_Q \text{ mixed-Tate}.\hspace{2cm} (3.16)$$
Using the fact that the motive $M^c(\mathbb{P}^{2D})_\mathbb{Q} \simeq M(\mathbb{P}^{2D})_\mathbb{Q}$ is mixed-Tate, we conclude from the Gysin triangle (3.8) (with $X := \mathbb{P}^{2D}$ and $Z := Q_{1,c} \cap Q_{2,c} \cap Q_{3,c}$) that $M^c(\mathbb{P}^{2D} \setminus (Q_{1,c} \cap Q_{2,c} \cap Q_{3,c}))_\mathbb{Q}$ is mixed-Tate. Therefore, as proved in [9, Cor. 2.1], the Chow motive $b(\mathbb{P}^{2D} \setminus (Q_{1,c} \cap Q_{2,c} \cap Q_{3,c}))_\mathbb{Q}$ admits the following motivic decomposition

$$M^c(\mathbb{P}^{2D} \setminus (Q_{1,c} \cap Q_{2,c} \cap Q_{3,c}))_\mathbb{Q} = \bigoplus_{n=0}^{D-2} (J^{n-2}(Q_{1,c} \cap Q_{2,c} \cap Q_{3,c}))_\mathbb{Q} \otimes \mathbb{Q}(n) \otimes \mathbb{Q}(D-1),$$

where $J^{n-2}(Q_{1,c} \cap Q_{2,c} \cap Q_{3,c})$ stands for the $(2n-2)$th intermediate algebraic Jacobian of $Q_{1,c} \cap Q_{2,c} \cap Q_{3,c}$. Moreover, as proved by Beauville in [6, Thm. 6.3], the Abel-Jacobi variety $J^{n-2}(Q_{1,c} \cap Q_{2,c} \cap Q_{3,c})$ is isomorphic, as a principally polarized abelian variety, to the Prym variety $\text{Prym}(\bar{C}/C)$ mentioned in §1.

We now claim that the motive $M^c(Q_{1,c} \cap Q_{2,c} \cap Q_{3,c})_\mathbb{Q}$ is not mixed-Tate. Recall from Totaro [36, Cor. 7.3] that $M^c(Q_{1,c} \cap Q_{2,c} \cap Q_{3,c})_\mathbb{Q}$ is mixed-Tate if and only if the Chow motive $b(\mathbb{P}^{2D} \setminus (Q_{1,c} \cap Q_{2,c} \cap Q_{3,c}))_\mathbb{Q}$ is a direct summand of a finite direct sum of Lefschetz motives. Given a Weil cohomology group $H^*(Q_{1,c} \cap Q_{2,c} \cap Q_{3,c})_\mathbb{Q}$ is isomorphic, as a principally polarized abelian variety, to the Prym variety $\text{Prym}(\bar{C}/C)$ mentioned in §1.

The combination of equivalence (3.6) with the preceding two equivalences leads then to the following equivalence:

$$H^1(Q_{1,c} \cap Q_{2,c} \cap Q_{3,c})_\mathbb{Q} = H^1(\text{Prym}(\bar{C}/C))_\mathbb{Q}$$

This implies the preceding claim. Consequently, the proof of item (i) follows now automatically from the above equivalence (3.16).

**Proof of item (ii).** Recall from above that we have the Mayer-Vietoris triangles

$$M^c(U \cup V)_\mathbb{Q} \rightarrow M^c(U)_\mathbb{Q} \oplus M^c(V)_\mathbb{Q} \rightarrow M^c(U \cup V)_\mathbb{Q} \rightarrow M^c(U \cup V)_\mathbb{Q}[1]$$

$$M^c(\mathbb{P} \setminus Q_{123})_\mathbb{Q} \rightarrow M^c(U_{13})_\mathbb{Q} \oplus M^c(U_{23})_\mathbb{Q} \rightarrow M^c(U \cup V)_\mathbb{Q} \rightarrow M^c((\mathbb{P} \setminus Q_{123})_\mathbb{Q})_\mathbb{Q} [1],$$

where $P := \mathbb{P}^{2D}$, $U := \mathbb{P}^{2D} \setminus (Q_{1,c} \cap Q_{2,c} \cap Q_{3,c})$, $V := \mathbb{P}^{2D} \setminus Q_{3,c}$, $U_{13} := \mathbb{P}^{2D} \setminus (Q_{1,c} \cap Q_{3,c})$, $U_{23} := \mathbb{P}^{2D} \setminus (Q_{2,c} \cap Q_{3,c})$, and $Q_{123} := Q_{1,c} \cap Q_{2,c} \cap Q_{3,c}$.

Recall also from above that we have the following Gysin triangles:

$$M^c(Q_{1,c} \cup Q_{2,c})_\mathbb{Q} \rightarrow M^c(U)_\mathbb{Q} \rightarrow M^c(Q_{1,c} \cup Q_{2,c})_\mathbb{Q}[1]$$

$$M^c(Q_{3,c})_\mathbb{Q} \rightarrow M^c(U)_\mathbb{Q} \rightarrow M^c(Q_{3,c})_\mathbb{Q}[1]$$

$$M^c(Q_{1,c} \cap Q_{3,c})_\mathbb{Q} \rightarrow M^c(U_{13})_\mathbb{Q} \rightarrow M^c(Q_{1,c} \cap Q_{3,c})_\mathbb{Q}[1]$$

$$M^c(Q_{2,c} \cap Q_{3,c})_\mathbb{Q} \rightarrow M^c(U_{23})_\mathbb{Q} \rightarrow M^c(Q_{2,c} \cap Q_{3,c})_\mathbb{Q}[1]$$

$$M^c(Q_{123})_\mathbb{Q} \rightarrow M^c(\mathbb{P})_\mathbb{Q} \rightarrow M^c(Q_{123})_\mathbb{Q}[1].$$

In what follows, we make use of the direct sum of the 1st and 2nd Gysin triangles as well as of the direct sum of the 3rd and 4th Gysin triangles. Since $M^c(\mathbb{P})_\mathbb{Q} \simeq M(\mathbb{P})_\mathbb{Q} \simeq \bigoplus_{i=0}^{2D} \mathbb{Q}(i)[2i]$, we can then conclude from the above motivic computations (3.10), (3.15) and (3.17) (see Remark 3.4) and from the isomorphism $J^{2D-2}(Q_{123}) = \text{Prym}(\bar{C}/C)$ that the motive $M^c(\mathbb{P} \setminus Q_{123})_\mathbb{Q}$ belongs to the smallest subcategory of $\text{DM}_{gm}(\mathbb{F})_\mathbb{Q}$ which can be obtained from the set of motives $\{M(\text{Prym}(\bar{C}/C))_\mathbb{Q}, \mathbb{Q}(0), \mathbb{Q}(1)\}$ by taking direct sums, shifts, summands,
tensor products, and at most 5 cones. Now, recall from [38, Thm. 4.3.7] that since $\mathbb{P}^{2D} \setminus Q(\Gamma, m)$ is smooth and $2D$-dimensional, we have a canonical isomorphism between the dual $M(\mathbb{P}^{2D} \setminus Q(\Gamma, m))^\vee$ of $M(\mathbb{P}^{2D} \setminus Q(\Gamma, m))$ and $M(\mathbb{P}^{2D} \setminus Q(\Gamma, m)) Q(−2D)[−4D]$. Using the fact that the duality functor $(−)^\vee$ preserves direct sums, shifts, summands, tensor products, and cones, we hence conclude that the Feynman quadrics-motive $M(\mathbb{P}^{2D} \setminus Q(\Gamma, m))$ belongs to the smallest subcategory of $\text{DM}_{gm}(F)$ which can be obtained from the set of motives $\{M(\text{Prym}(\tilde{C}/C))^\vee, Q(−1)^\vee, Q(1)^\vee\}$ by taking direct sums, shifts, summands, tensor products, and at most 5 cones. The proof follows now from the isomorphisms $Q(i)^\vee ≃ Q(−i)$ and $M(\text{Prym}(\tilde{C}/C))^\vee ≃ M(\text{Prym}(\tilde{C}/C)) Q(−d)[−2d]$, where $d$ stands for the dimension of the Prym variety.

References

[1] Y. Andrè, Une introduction aux motifs (motifs purs, motifs mixtes, périodes). Panoramas et Synthèses, vol. 17, Société Mathématique de France, Paris, 2004.
[2] P. Aluffi and M. Marcolli, Feynman motives of banana graphs. Commun. Number Theory Phys. 3 (2009) no. 1, 1–57.
[3] , Parametric Feynman integrals and determinant hypersurfaces. Adv. Theor. Math. Phys. 14 (2010), no. 3, 911–963.
[4] , Graph hypersurfaces and a dichotomy in the Grothendieck ring. Lett. Math. Phys. 95 (2011), no. 3, 223–232.
[5] , Feynman motives and deletion-contraction relations. Topology of algebraic varieties and singularities, 21–64, Contemp. Math., 538, Amer. Math. Soc., 2011.
[6] A. Beauville, Variétés de Prym et Jacobiennes intermédiaires. Ann. Sci. de l’ENS 10 (1977) 309–391.
[7] P. Belkale and P. Brosnan, Matroids, motives, and a conjecture of Kontsevich. Duke Math. J. 116 (2003), 147–188.
[8] P. Belkale and P. Brosnan, Periods and Igusa local zeta functions. Int. Math. Res. Not. 49 (2003), 2655–2670.
[9] M. Bernardara and G. Tabuada, Chow groups of intersections of quadrics via homological projective duality and (Jacobians of) non-commutative motives. Izv. Math. 80 (2016) no. 3, 463–480.
[10] I. N. Bernstein, Analytic continuation of generalized functions with respect to a parameter. Functional Anal. Appl. 6 (1972), 273–285 (1973).
[11] S. Bloch, H. Esnault and D. Kreimer, On motives associated to graph polynomials. Comm. Math. Phys. 267 (2006), no. 1, 181–225.
[12] S. Bloch, M. Kerr and P. Vanhove, A Feynman integral via higher normal functions. Compos. Math. 151 (2015) no. 12, 2329–2375.
[13] S. Bloch and D. Kreimer, Mixed Hodge structures and renormalization in physics. Comm. Number Theory Phys. 2 (2008), no. 4, 637–718.
[14] , Feynman amplitudes and Landau singularities for one-loop graphs. Comm. Number Theory Phys. 4 (2010), no. 4, 709–753.
[15] S. Bloch and P. Vanhove, The elliptic dilogarithm for the sunset graph. J. Number Theory 148 (2015), 328–364.
[16] C. Bogner and F. Brown, Feynman integrals and iterated integrals on moduli spaces of curves of genus zero. Comm. Number Theory Phys. 9 (2015), no. 1, 189–238.
[17] P. Brosnan, On motivic decompositions arising from the method of Białynicki-Birula. Invent. Math. 161 (2005), no. 1, 91–111.
[18] F. Brown and O. Schnetz, A K3 in $φ^4$. Duke Math. J. 161 (2012), no. 10, 1817–1862.
[19] , Modular forms in quantum field theory. Comm. Number Theory Phys. 7 (2013), no. 2, 293–325.
[20] F. Brown, O. Schnetz and K. Yeats, Properties of $c_2$ invariants of Feynman graphs. Adv. Theor. Math. Phys. 18 (2014), no. 2, 323–362.
[21] O. Ceyhan and M. Marcolli, *Feynman integrals and motives of configuration spaces*. Comm. Math. Phys. 313 (2012), no. 1, 35–70.

[22] ______, *Algebraic renormalization and Feynman integrals in configuration spaces*. Adv. Theor. Math. Phys. 18 (2014), no. 2, 469–511.

[23] A. Connes and D. Kreimer, *Renormalization in quantum field theory and the Riemann–Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem*. Comm. Math. Phys. 210 (2000) no. 1, 249–273.

[24] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*. Colloquium Publications, Vol. 55, American Mathematical Society, 2008.

[25] D. Doryn, *On one example and one counterexample in counting rational points on graph hypersurfaces*. Lett. Math. Phys. 97 (2011), no. 3, 303–315.

[26] J. Harris, *Algebraic geometry. A first course*. Graduate Texts in Mathematics, 133. Springer-Verlag, New York, 1992

[27] C. Itzykson and J.B. Zuber, *Quantum Field Theory*. McGraw-Hill, 1980.

[28] M. Marcolli, *Feynman motives*. World Scientific Publishing, NJ, 2010.

[29] ______, *Feynman integrals and motives*. European Congress of Mathematics, 293–332, Eur. Math. Soc., Zürich, 2010.

[30] M. Marcolli and X. Ni, *Rota-Baxter algebras, singular hypersurfaces, and renormalization on Kausz compactifications*. J. Singul. 15 (2016), 80–117.

[31] C. Mazza, V. Voevodsky and C. Weibel, *Lecture notes on motivic cohomology*. Clay Mathematics Monographs, 2. American Mathematical Society, 2006.

[32] O. Schnetz, *Quantum field theory over $\mathbb{F}_q$*. Electron. J. Combin. 18 (2011), no. 1, paper 102.

[33] ______, *Quantum periods: a census of $\phi^4$-transcendentals*. Comm. Number Theory Phys. 4 (2010), no. 1, 1–47.

[34] R. Stanley, *Spanning trees and a conjecture of Kontsevich*. Ann. Comb. 2 (1998), no. 4, 351–363.

[35] J. Stembridge, *Counting points on varieties over finite fields related to a conjecture of Kontsevich*. Ann. Comb. 2 (1998), no. 4, 365–385.

[36] B. Totaro, *The motive of a classifying space*. Geom. Topol. 20 (2016), no. 4, 2079–2133.

[37] P. Vanhove, *The physics and the mixed Hodge structure of Feynman integrals*. Proc. Symp. Pure Math. 88 (2014) 161–194.

[38] V. Voevodsky, *Triangulated categories of motives over a field*. Cycles, transfers, and motivic homology theories, 188238, Ann. of Math. Stud., 143, Princeton Univ. Press, NJ, 2000.