AN EXTENSION OF RIESZ TRANSFORM

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Abstract. In this paper, we consider the following singular integral

\[ T_j f(x) = K_j * f(x), K_j(x) = \frac{x_j}{|x|^{n+1-\beta}}, \]

where \( x \in \mathbb{R}^n, 0 \leq \beta < n, j = 1, 2, \ldots, n \). When \( \beta = 0 \), it corresponds to the Riesz transform. We will make an estimate the \( L^q \) norm of \( T_j f \), which holds uniformly for \( 0 \leq \beta < n \). In particular, when \( \beta = 0 \), the strong \((q, q)\) type estimate of the Riesz transform for \( 1 < q < \infty \) is recovered from the obtained estimate.

1. Introduction and Main Results

Given \( f(x) \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \) with \( 1 \leq q < \infty \), we consider the singular integral

\[ T_j f(x) = K_j * f(x), K_j(x) = \frac{x_j}{|x|^{n+1-\beta}}, 0 < \beta < n, j = 1, 2, \ldots, n. \] (1.1)

Let \( \hat{f} \) be the Fourier transform of \( f \) defined as

\[ \hat{f}(y) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} f(x) dx. \]

Then the Fourier transform of \( T_j f \) is

\[ \widehat{T_j f}(\xi) = \gamma_\beta \frac{\xi_j}{|\xi|^{\beta+1}}, 0 < \beta < n, \gamma_\beta = i\pi^{n/2-\beta} \frac{\Gamma(\beta+1)}{\Gamma(\frac{n+1-\beta}{2})}, \] (1.2)

Formally, when \( \beta = 0 \), \( T_j f \) is the well-known Riesz transform. In fact, it holds (see [6])

\[ \lim_{\beta \to 0^+} \int_{\mathbb{R}^d} \frac{x_j}{|x|^{k+n-\beta}} \hat{\varphi}(x) dx = \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{x_j}{|x|^{k+n}} \hat{\varphi}(x) dx, \]

where \( \varphi(x) \in \mathcal{S}(\mathbb{R}^d) \) which is the Schwartz space. In view of the Riesz potential estimate (see [6]), it is direct to deduce that, for \( 1 < p < q < \infty \),

\[ \|T_j f\|_q \leq \| \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\beta}} dy \|_q \leq C(\beta) \|f\|_p, \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}, 0 < \beta < n. \] (1.3)

However, the constant \( C(\beta) \) on the right side of (1.3) depends on \( \beta \) in general and is unbounded as \( \beta \to 0 \). A natural question is whether one can obtain an uniform \( L^q \)-estimate of \( T_j f \) with respect to \( \beta > 0 \) such that the strong \((q, q)\) type estimate of the Riesz transform can be recovered when \( \beta \to 0 \). We will answer this question in this paper.

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Here and in what follows, $\|f\|_q$ means the $L^q(\mathbb{R}^n)$ norm of $f$. To simplify the presentation, we omit subscript of $T_j$ and $K_j$ and write (1.1) as

$$Tf(x) = K \ast f(x), K(x) = \frac{x_j}{|x|^{n+1-\beta}}, 0 < \beta < n$$ (1.4)

for any $j = 1, 2, \cdots, n$.

Then our main result can be stated as

**Theorem 1.1.** Let $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $1 < q < \infty$. Then there exists a constant $C = C(n)$ independent of $\beta$ such that

$$\|Tf\|_q \leq C(\|f\|_q + \|f\|_p + \beta \frac{n(q-1)}{q} \|f\|_1)$$ (1.5)

for $0 \leq \beta < \frac{n(q-1)}{q}$ and $1 < p \leq q < \infty$ satisfying $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. Consequently, it holds

$$\|Tf\|_q \leq C(\|f\|_q + L(\beta)\|f\|_1)$$ (1.6)

for $0 \leq \beta < \frac{n(q-1)}{q}$, where $L(\beta) = \frac{\beta \frac{n(q-1)}{q}}{(n(q-1)-\beta q)^{\frac{1}{q}}} + \frac{\beta q}{(q-1)n}$.

**Remark 1.1.** It is addressed that the constant $C$ on the right of (1.5) and (1.6) does not depend on $\beta$ and hence the $(p, p)$ type estimate of the Riesz transform can be recovered from (1.5) and (1.6) respectively when $\beta \to 0$.

To prove Theorem 1.1, we split the singular integral (1.4) into two parts: the part near the origin denoted by $T_1 f$ and the one apart from the origin denoted by $T_2 f$. The estimate on $\|T_2 f\|_q$ is easy to obtain (see Lemma 2.1). The key part is to estimate $\|T_1 f\|_q$. We will use the refined Calderon-Zygmund decomposition to overcome new difficulties encountered in the estimate of $\|T_1 b\|_q$ (see proof of Lemma 3.1). Moreover, we have

**Theorem 1.2.** Let $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $1 \leq q < \infty$. Then there exists a constant $C = C(n)$ independent of $\beta$ such that

$$m\{x : |T_1 f| > t\} \leq C\left(\frac{\|f\|_1}{t} + \frac{\|f\|_q^q}{t^q}\right)$$ (1.7)

for any $t > 0$ and $q \geq 1$ satisfying $\frac{1}{q} = 1 - \frac{\beta}{n}$.

Here $m(A)$ means the Lebesgue measure of a set $A \subset \mathbb{R}^n$. When $\beta \to 0$ it concludes that $q = 1$ and the weak $(1, 1)$ estimate of the Riesz transform can be recovered from (1.7).

The kind of singular integral (1.1) or (1.4) appears in the generalized surface quasigeostrophic (SQG) equation which reads as

$$\begin{cases}
\omega_t + u \cdot \nabla \omega = 0, & (x, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\
u = \nabla^\perp (-\Delta)^{-1+\alpha} \omega, \\
\omega(x, 0) = \omega_0.
\end{cases}$$ (1.8)

Here $0 \leq \alpha \leq \frac{1}{2}$ and $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. The unknown functions $\omega = \omega(x, t)$ and $u = u(x, t) = (u_1(x, t), u_2(x, t))$ are related by (1.8) which can be expressed as

$$u(x) = \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^{2+2\alpha}} \omega(y) dy.$$ (1.9)
Here $x^\perp = (-x_2, x_1)$ and the singular integral \((1.9)\) means the principle value one. When $\alpha = 0$, \((1.8)\) corresponds to the two-dimensional incompressible Euler equations. In this case, the unknown functions $\omega = \omega(x, t)$ and $u = u(x, t)$ are the vorticity and the velocity field respectively. When $\alpha = \frac{1}{2}$, \((1.8)\) corresponds to the surface quasi-geostrophic (SQG) equation which describes a famous approximation model of the nonhomogeneous fluid flow in a rapidly rotating 3D half-space (see [1],[5]). When $0 < \alpha < \frac{1}{2}$, it is called the generalized (or modified) SQG equation. In the case $0 < \alpha \leq \frac{1}{2}$, the unknown functions $\omega = \omega(x, t)$ and $u = u(x, t)$ stand for potential temperature and velocity field respectively. It is noted that when $\alpha = \frac{1}{2}$ the relation between $u = u(x, t)$ and $\omega = \omega(x, t)$ in \((1.9)\) corresponds to the Riesz transform. When $0 < \alpha < \frac{1}{2}$, the relation \((1.9)\) is completely similar to the $T$ operator defined in \((1.4)\) with $\beta = 1 - 2\alpha$ and $n = 2$. It is clear that $\beta$ will vanish as $\alpha \to \frac{1}{2}$. In [3], we investigate the approximation of the SQG equation by the generalized SQG equation as $\alpha \to \frac{1}{2}$. What’s more, when $q = p = 2$ in Theorem 1.1, the following result has been established in [3]:

**Proposition 1.3.** For $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, it holds that

$$
\|Tf\|_2 \leq C(\|f\|_2 + \frac{\beta^{\frac{n}{2}}}{\sqrt{n-2\beta}}\|f\|_1), \quad 0 < \beta < \frac{n}{2}
$$

for some constant $C = C(n)$ independent of $\beta$.

Clearly, Proposition 1.3 is a particular case of Theorem 1.1.

The paper is organized as follows. In Section 2, we will present some preliminary estimates which will be needed later. The proof of Theorem 1.1 and Theorem 1.2 will be given in Sections 3.

2. Preliminaries

Let $\chi(s) \in C_0^\infty(R)$ be the usual smooth cutting-off function which is defined as

$$
\chi(s) = \begin{cases} 
1, & |s| \leq 1, \\
0, & |s| \geq 2,
\end{cases}
$$
satisfying $|\chi'(s)| \leq 2$. Let

$$
\chi_\lambda(s) = \chi(\lambda s),
$$
and define

$$
T_1 f(x) = K_1 * f(x), K_1(x) = K(x)\chi_\beta(|x|),
$$

$$
T_2 f(x) = K_2 * f(x), K_2(x) = K(x)(1 - \chi_\beta(|x|)).
$$

Then it is clear that the operator $T$ in \((1.4)\) can be written as

$$
T = T_1 + T_2.
$$

The following is a $L^q$-estimate of $T_2$:

**Lemma 2.1.** There exists an absolute constant $C > 0$ independent of $\beta$ such that for any $1 < q < \infty$,\n
$$
\|T_2 f\|_q \leq C \frac{\beta^{\frac{n(q-1)}{q}}}{(n(q-1) - \beta q)\frac{n}{3}}\|f\|_1, \quad 0 < \beta < \frac{n(q-1)}{q}.
$$
Proof of Lemma 2.1. Note that
\[ T_2 f(x) = \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1-\beta}} (1 - \chi_{\beta}(|x - y|)) f(y) \, dy. \]
Then direct estimates give
\[
\|T_2 f\|_q \leq \int_{|x - y| \geq \frac{1}{\beta}} \frac{1}{|x - y|^{n-\beta}} |f(y)| \, dy \|f\|_q
\leq \|f\|_1 \int_{|x - y| \geq \frac{1}{\beta}} \left( \frac{1}{|x - y|^{q(n-\beta)}} \right)^{\frac{1}{q}}
\leq C \frac{n(q-1)}{(n(q-1) - \beta q)^{\frac{1}{q}}} \|f\|_1
\]
for any \(0 < \beta < \frac{n(q-1)}{q}\). \(\square\)

Concerning the operator \(T_1\), we first prove that it is of type \((2, 2)\), which has been shown in [3]. For completeness, we give a sketch of proof here.

Lemma 2.2. There exists a constant \(C = C(n)\) independent of \(\beta\) such that
\[
\|T_1 f\|_2 \leq C \|f\|_2, \quad 0 < \beta < n.
\tag{2.4}
\]

Proof of Lemma 2.2. To prove (2.4), our main target is to prove that there exists an absolute constant \(C > 0\) independent of \(\beta\) such that
\[
|\hat{K}_1(y)| \leq C, \quad 0 < \beta < n.
\tag{2.5}
\]
Since \(\int_{S^1} K_1(x) \, ds = 0\) (here \(S^1\) is the unit sphere surface in \(\mathbb{R}^n\)) and \(K_1(x)\) is supported on \(|x| \leq \frac{2}{\beta}\), we have
\[
\hat{K}_1(y) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} K_1(x) \, dx = \int_{|x| \leq \frac{2}{\beta}} (e^{2\pi i x \cdot y} - 1) K_1(x) \, dx
\tag{2.6}
\]
Since (2.5) is a pointwise estimate, we will estimate \(\hat{K}_1(y)\) by different values of \(y\). If \(|y| < \frac{\beta}{2}\), it is direct to estimate
\[
|\hat{K}_1(y)| \leq C |y| \int_{|x| \leq \frac{2}{\beta}} \frac{1}{|x|^{n-\beta}} \, dx
\leq \frac{2^\beta}{\beta + 1} \beta^{-\beta}.
\tag{2.7}
\]
Then there exists an absolute constant \(C > 0\) such that
\[
|\hat{K}_1(y)| \leq C, \quad 0 < \beta < n, \quad |y| < \frac{\beta}{2}.
\tag{2.8}
\]
If \(\frac{\beta}{2} \leq |y| \leq \beta\), we rewrite \(\hat{K}_1(y)\) as
\[
\hat{K}_1(y) = \int_{|x| < \frac{1}{|y|}} e^{2\pi i x \cdot y} K_1(x) \, dx + \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx
= \int_{|x| < \frac{1}{|y|}} (e^{2\pi i x \cdot y} - 1) K_1(x) \, dx + \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx
\]
Similar to (2.7), it deduces
\[ \left| \int_{|x|<\frac{1}{|y|}} (e^{2\pi i x \cdot y} - 1) K_1(x) \, dx \right| \leq \frac{2\beta}{\beta + 1} \beta^{-\beta}. \]

Moreover, we have
\[ \left| \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx \right| \leq C \frac{2\beta - 1}{\beta} \beta^{-\beta}. \]

Consequently, there exists an absolute constant \( C > 0 \) such that
\[ |\widehat{K}_1(y)| \leq C \left( \frac{2\beta}{\beta + 1} \beta^{-\beta} + \frac{2\beta - 1}{\beta} \beta^{-\beta} \right) \leq C, \ 0 < \beta < n, \frac{\beta}{2} \leq |y| \leq \beta. \tag{2.9} \]

If \(|y| > \beta\), \( \widehat{K}_1(y) \) can be divided into
\[ \widehat{K}_1(y) = \int_{|x|<\frac{1}{|y|}} e^{2\pi i x \cdot y} K_1(x) \, dx + \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx \]
\[ = \int_{|x|<\frac{1}{|y|}} (e^{2\pi i x \cdot y} - 1) K_1(x) \, dx + \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx. \tag{2.10} \]

For the first term on the right hand of the above equality, we obtain
\[ \left| \int_{|x|<\frac{1}{|y|}} (e^{2\pi i x \cdot y} - 1) K_1(x) \, dx \right| \leq C|y| \int_{|x|<\frac{1}{|y|}} |x| \frac{1}{|x|^{n-\beta}} \, dx \]
\[ \leq \frac{1}{\beta + 1} \beta^{-\beta}. \tag{2.11} \]

For the second term, we choose \( z = \frac{y}{2|y|^2} \) with \(|z| = \frac{1}{2|y|} < \frac{1}{2\pi} \) such that \( e^{2\pi i y \cdot z} = -1 \) and
\[ \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} K_1(x) \, dx = \frac{1}{2} \int_{\mathbb{R}^n} e^{2\pi i x \cdot y}(K_1(x) - K_1(x - z)) \, dx, \]
so
\[ \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx = \frac{1}{2} \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y}(K_1(x) - K_1(x - z)) \, dx \]
\[ - \frac{1}{2} \int_{\frac{1}{|y|} \leq |x+z|, |x| \leq \frac{1}{|y|}} e^{2\pi i x \cdot y} K_1(x) \, dx \]
\[ + \frac{1}{2} \int_{|x+z| \leq \frac{1}{|y|}, |x| \geq \frac{1}{|y|}} e^{2\pi i x \cdot y} K_1(x) \, dx \]
\[ + \frac{1}{2} \int_{|x+z| \geq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx \tag{2.12} \]
\[ \equiv I + J + K + L. \]
To estimate the term $I$, we have

\[
I = \int_{|y| \leq |z| \leq \frac{|x|}{2}} \left( \frac{x}{|x|^{n+1-\beta}} - \frac{x - z}{|x - z|^{n+1-\beta}} \right) e^{2\pi i x \cdot y} \, dx \\
+ \int_{|\frac{1}{2}| \leq |x| \leq \frac{1}{2}, \ |x - z| \leq \frac{|x|}{2}} \left( \frac{x}{|x|^{n+1-\beta}} - \frac{x - z}{|x - z|^{n+1-\beta}} \right) e^{2\pi i x \cdot y} \, dx \\
+ \int_{\frac{1}{2} \leq |x| < \frac{1}{2}, \ |x - z| \leq \frac{|x|}{2}} \left( \frac{x}{|x|^{n+1-\beta}} - \frac{x - z}{|x - z|^{n+1-\beta}} \beta(x) \right) e^{2\pi i x \cdot y} \, dx \\
+ \int_{\frac{1}{2} \leq |x| < \frac{1}{2}, \ |x - z| \geq \frac{|x|}{2}} \left( \frac{x}{|x|^{n+1-\beta}} - \frac{x - z}{|x - z|^{n+1-\beta}} \beta(x - z) \right) e^{2\pi i x \cdot y} \, dx \\
= I_1 + I_2 + I_3 + I_4.
\]

We first estimate $I_2$. Thanks to $|x - z| \geq |x| - |z| \geq \frac{1}{2} - \frac{1}{2|y|} \geq \frac{1}{2|y|}$, one has

\[
|I_2| \leq \int_{\frac{1}{2} \leq |x| \leq \frac{1}{2}} \frac{1}{|x|^{n-\beta}} \, dx + \int_{\frac{1}{2} \leq |x - z| \leq \frac{|x|}{2}} \frac{1}{|x - z|^{n-\beta}} \, dx \\
\leq C \frac{2\beta - 1}{\beta} \beta^{-\beta} + C \frac{1 - 2\beta}{\beta} \beta^{-\beta}.
\]

Then thanks to $|x| = |x - z + z| \geq |x - z| - |z| \geq \frac{1}{2} - \frac{1}{2|y|} \geq \frac{1}{2|y|}$, $I_3$ is estimated as follows.

\[
|I_3| \leq \int_{\frac{1}{2} \leq |x| \leq \frac{1}{2}} \frac{1}{|x|^{n-\beta}} \, dx + \int_{\frac{1}{2} \leq |x - z| \leq \frac{|x|}{2}} \frac{1}{|x - z|^{n-\beta}} \, dx \\
\leq C \frac{1 - 2\beta}{\beta} \beta^{-\beta} + C \frac{2\beta - 1}{\beta} \beta^{-\beta}.
\]

The term $I_4$ is directly estimated as

\[
|I_4| \leq \int_{\frac{1}{2} \leq |x| \leq \frac{2}{3}} \frac{1}{|x|^{n-\beta}} \, dx + \int_{\frac{1}{2} \leq |x - z| \leq \frac{2}{3}} \frac{1}{|x - z|^{n-\beta}} \, dx \\
\leq C \frac{2\beta - 1}{\beta} \beta^{-\beta}.
\]

Now we deal with $I_1$. Note that

\[
\partial_i \left( \frac{x}{|x|^{n+1-\beta}} \right) = \frac{\vec{e}_i}{|x|^{n+1-\beta}} + (-n - 1 + \beta) \frac{x x_i}{|x|^{n+1-\beta}}, \ i = 1, 2, ..., n.
\]

In this case, since $|x - z| \geq |x| - |z| \geq 2|z| - |z| \geq |z|$, by Taylor expansion, one has

\[
\left| \frac{x - z}{|x - z|^{n+1-\beta}} - \frac{x}{|x|^{n+1-\beta}} \right| \\
\leq \sum_{i=1}^{n} \left( \frac{z_i \vec{e}_i}{|x - z|^{n+1-\beta}} + (-n - 1 + \beta) \frac{(x - z)(x_i - z_i) z_i}{|x - z|^{n+1-\beta}} \right) + C \sum_{k=2}^{\infty} \frac{|z|^k}{k! |x - z|^{n+k-\beta}}.
\]

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Consequently,
\[
|I_1| \leq (n + 2 - \beta)|z| \int_{|z| \leq |x - z| < \frac{1}{\beta}} \frac{1}{|x - z|^{n+1-\beta}} \, dx + C \sum_{k=2}^{\infty} \int_{|z| \leq |x - z| < \frac{1}{\beta}} \frac{|z|^k}{k! |x - z|^{n+k-\beta}} \, dx
\]
\[
\leq C \frac{|z|^\beta}{1 - \beta} \leq C \frac{\beta^\beta}{1 - \beta}.
\]
(2.17)

Substituting (2.14)-(2.17) into (2.13) yields
\[
|I| = \left| \int_{\frac{1}{\beta^n} \leq |x| < \frac{1}{\beta^n}, \ |x - z| \leq \frac{1}{\beta}} \left( \frac{x}{|x|^{n+1-\beta}} - \frac{x - z}{|x - z|^{n+1-\beta}} \right) e^{2\pi i x \cdot y} \, dx \right|
\]
\[
\leq C \left( \frac{2^\beta - 1}{\beta} \beta^{-\beta} + \frac{1 - 2^{-\beta}}{\beta} \beta^{-\beta} + \frac{\beta^{-\beta}}{1 - \beta} \right)
\]
for some absolute constant $C > 0$.

Concerning the term $J$, thanks to $|x| \geq |x + z| - |z| \geq 2|z| - |z| \geq |z|$, one has
\[
|J| \leq \int_{|z| \leq |x| \leq 2|z|} \frac{1}{|x|^{n-\beta}} \, dx
\]
\[
\leq \frac{1 - 2^{-\beta}}{\beta} \beta^{-\beta}.
\]
(2.19)

Concerning the term $K$, thanks to $|x| \leq |x + z| + |z| \leq 2|z| + |z| \leq 3|z|$, one has
\[
|K| \leq \int_{2|z| \leq |x| \leq 3|z|} \frac{1}{|x|^{n-\beta}} \, dx
\]
\[
\leq \frac{\left( \frac{3}{2} \right)^\beta - 1}{\beta} \beta^{-\beta}.
\]
(2.20)

Concerning the term $L$, thanks to $\frac{2}{\beta} \geq |x| \geq |x + z| - |z| \geq \frac{2}{\beta} - \frac{1}{2^\beta} = \frac{3}{2^\beta}$, one has
\[
|L| \leq \int_{\frac{3}{2^\beta} \leq |x| \leq \frac{3}{2}} \frac{1}{|x|^{n-\beta}} \, dx
\]
\[
\leq \frac{1}{\beta} \left[ \left( \frac{3}{2^\beta} \right)^\beta - \left( \frac{2}{\beta} \right)^\beta \right].
\]
(2.21)

Substituting (2.18)-(2.21) into (2.12), we obtain that there exists an absolute constant $C > 0$ such that
\[
\left| \int_{\frac{1}{\beta^n} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx \right| \leq C.
\]
(2.22)

In view of (2.11), (2.22) and (2.10), there exists an absolute constant $C > 0$ such that
\[
|\hat{K}_1(y)| \leq C, \quad 0 < \beta < n, |y| > \beta.
\]
(2.23)

Combining (2.8), (2.9) with (2.23), we finish the proof of (2.5). Applying (2.5), one has
\[
\|T_1 f\|_{L^2} = \|\hat{K}_1 \hat{f}\|_{L^2} \leq C \|\hat{f}\|_{L^2} = C \|f\|_{L^2}.
\]
Hence (2.4) is proved and the proof of the lemma is complete. \[\square\]

The following is a Marcinkiewicz interpolation theorem (see [6]).
Lemma 2.3. Suppose that $1 < r \leq \infty$. Suppose that the following holds:

1. $T$ is a sub-additive mapping from $L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n)$ to the space of measurable functions on $\mathbb{R}^n$: 
   \[ |T(f + g)(x)| \leq |Tf(x)| + |Tg(x)|. \]

2. $T$ is of weak-type $(1,1)$: 
   \[ m\{x : |Tf(x)| > t\} \leq \frac{A_1}{t} \|f\|_1, \quad f \in L^1(\mathbb{R}^n). \]

3. $T$ is of weak-type $(r,r)$: 
   \[ m\{x : |Tf(x)| > t\} \leq \left(\frac{Ar}{t}\right)^r \|f\|_r, \quad f \in L^r(\mathbb{R}^n) \]
   if $r < \infty$ or
   \[ \|Tf\|_\infty \leq A_\infty \|f\|_\infty \]
   if $r = \infty$.

Then $T$ is of type $(p,p)$ for all $1 < p < r$, that is, 
\[ \|Tf\|_p \leq A_p \|f\|_p, \quad f \in L^p(\mathbb{R}^n) \]
for all $1 < p < r$, where $A_p$ depends only on $A_1, A_r, p$ and $r$.

The proof of Lemma 2.3 is referred to [6] and we omit it here.

3. Proof of Main Results

In this section, we give the proof of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. In view of Lemma 2.1, to prove Theorem 1.1, it suffices to prove

Lemma 3.1. Let $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. Then there exists a constant $C = C(n)$ independent of $\beta$ such that
\[ \|T_1 f\|_q \leq C(\|f\|_q + \|f\|_p + \frac{\beta^{\frac{n(q-1)}{q}}}{(n(q-1) - \beta q)\frac{1}{q}}} \|f\|_1) \]
for any $1 < p \leq q < \infty$ satisfying $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$.

Now we prove Lemma 3.1. Given $t > 0$, according to the cube decomposition procedure, $\mathbb{R}^n$ can be divided into the union of countable and disjoint cubes satisfying

(1) there exists a sequence of parallel subcubes $\{K_l\}_{l=1}^\infty$ such that 
\[ t < \frac{1}{m(K_l)} \int_{K_l} |f| < 2nt; \quad (3.1) \]

(2) 
\[ |f| \leq t, \quad a.e. \quad on \quad G = \mathbb{R}^n \setminus \bigcup K_l. \quad (3.2) \]

Denote $F = \bigcup K_l$. Then it follows that that $m(F) \leq \frac{\|f\|_1}{t}$. Let $f = g + b$, where $g$ is defined by
\[ g(x) = \begin{cases} f(x), & x \in G, \\ \frac{1}{m(K_l)} \int_{K_l} f, & x \in K_l, l = 1, 2, \ldots \end{cases} \quad (3.3) \]
and \( b = f - g \) satisfies
\[
b(x) = 0, \quad x \in G, \quad \int_{K_l} b = 0, \quad l = 1, 2, \ldots.
\]

It is easy to get
\[
\|g\|_q \leq \|f\|_q, \quad 1 \leq q \leq \infty.
\]

Let \( \delta_l > 0 \) be the diameter of \( K_l \) and \( B_l \supset K_l \) be a ball with radius \( \delta_l \). Denote \( F^* = \cup B_l, G^* = \mathbb{R}^n \setminus F^* \). Then it yields
\[
m(F^*) \leq n^\frac{2}{n} \omega_n m(F) \leq \frac{C(n)\|f\|_1}{t}.
\]

The operator \( T_1 \) can be decomposed into
\[
T_1 f = T_1 g + T_1 b_{I_{G^*}} + T_1 b_{I_{F^*}}
\]
\[
= (T_1 g + T_1 b_{I_{F^*}}) + T_1 b_{I_{G^*}}
\]
\[
= T_{11} f + T_{12} f,
\]
where \( I_A \) is the characteristic function on a set \( A \), that is, \( I_A = 1 \) for \( x \in A \) and \( I_A = 0 \) for \( x \in \mathbb{R}^n \setminus A \).

Thanks to Lemma 2.2, we have
\[
\|T_{11} f\|_2 \leq \|T_{11} g\|_2 + \|T_{11} b_{I_{F^*}}\|_2
\]
\[
\leq C\|g\|_2 + \|b\|_2.
\]

Direct estimates give
\[
\|g\|_2^2 = \int_G |f|^2 dx + \sum_l \int_{K_l} \left( \frac{1}{|K_l|} \int_{K_l} f \right)^2 dx
\]
\[
\leq \int_G |f|^2 dx + \sum_l \frac{1}{|K_l|} \left( \int_{K_l} f dx \right)^2
\]
\[
\leq \int_G |f|^2 dx + \sum_l \int_{K_l} |f|^2 dx = \|f\|_2^2.
\]

Hence \( \|g\|_2 \leq \|f\|_2 \). Combining the fact that \( \|b\|_2 \leq \|f\|_2 + \|g\|_2 \leq 2\|f\|_2 \), we obtain
\[
\|T_{11} f\|_2 \leq C\|f\|_2,
\]
which implies that \( T_{11} \) is of type \((2, 2)\). Note that, for any \( t > 0 \),
\[
m\{x : |T_{11} f| > t\} \leq m\{x : |T_{11} g| > \frac{t}{2}\} + m\{x : |T_{11} b_{I_{F^*}}| > \frac{t}{2}\}.
\]

Since
\[
\|g\|_2^2 = \int_{\mathbb{R}^n} |g(x)|^2 dx = \int_F |g(x)|^2 dx + \int_G |g(x)|^2 dx
\]
\[
\leq 2^n t^2 m(F) + t \int_G |f(x)| dx
\]
\[
\leq C(n) t \|f\|_1,
\]
we obtain
\[
m\{x : |T_{11} g| > \frac{t}{2}\} \leq \frac{C(n)}{t^2} \frac{\|g\|_2^2}{t} \leq \frac{C(n)}{t} \|f\|_1.
\]
It follows from (3.6) that
\[ m\{x : |T_{11}bI_{F\ast}| > \frac{t}{2}\} \leq m(F\ast) \leq \frac{C(n)\|f\|_1}{t}. \]  
(3.13)
Substitute (3.12) and (3.13) into (3.11) to yield
\[ m\{x : |T_{11}f| > t\} \leq \frac{C(n)\|f\|_1}{t}, \]  
(3.14)
which implies that \( T_{11} \) is of weak-type \((1, 1)\). Therefore, due to (3.10) and (3.14), by Lemma 2.3 and duality method, we obtain
\[ \|T_{11}f\|_p \leq C\|f\|_p, \quad 1 < p < \infty. \]  
(3.15)

Now we estimate \( T_{12} \). Define
\[ b_t = \begin{cases} b, & x \in K_t, \\ 0, & x \notin K_t. \end{cases} \]
Then \( b = \sum_{l=1}^{\infty} b_t \). Noticing that, for any \( x \in \mathbb{R}^n \setminus K_t \),
\[ |Tb| = \left| \int_{K_t} \left( \frac{x_j - y_j}{|x - y|^{n+1-\beta}} - \frac{x_j - \bar{y}_j}{|x - \bar{y}|^{n+1-\beta}} \right) b_t(y) \, dy \right| \leq C\delta_l \int_{K_t} \frac{1}{|x - y|^{n+1-\beta}} |b_t(y)| \, dy \]
(3.16)
\[ = C\delta_l \int_{K_t} \frac{|b_t(y)|I_{K_t}(y)}{|x - y|^{n+1-\beta}} \, dy, \]
where \( \bar{y} \) is the center and \( \delta_l \) is the diameter of the cube \( K_t \) respectively, we obtain
\[ \left( \int_{\mathbb{R}^n \setminus B_l} |Tb| \, dx \right)^{\frac{1}{q}} \leq C\delta_l \left( \int_{\mathbb{R}^n \setminus B_l} \int_{\mathbb{R}^n} \frac{|b_t(y)|I_{K_t}(y)}{|x - y|^{n+1-\beta}} \, dy \, dq \right)^{\frac{1}{q}} \]
(3.17)
\[ \leq C\delta_l \left( \int_{\mathbb{R}^n \setminus B_l} \int_{\mathbb{R}^n} \frac{|b_t(y)|I_{K_t}(y)}{|x - y|^{n+1-\beta}} \, dy \, dx \right)^{\frac{1}{q}} \]
\[ \leq C\delta_l \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} I_{\{|x - y| \geq \frac{\delta_l}{2}\}} |b_t(y)| \, dy \, dx \right)^{\frac{1}{q}} \]
\[ \leq C\|b_t\|_{p, K_t}, \]
where \( 1 \leq p, q < \infty \) satisfying \( \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n} \) and the Young inequality has been used in the last inequality.

Moreover, it holds that
\[ \|TbI_{G\ast}\|_q \leq \sum_l \|Tb_I\|_{\frac{q}{p}, \mathbb{R}^n \setminus B_l} \leq C \sum_l \|b_t\|_{q, p, K_t} \leq C \left( \sum_l \|b_t\|_{p, K_t} \right)^{\frac{q}{p}} \]
\[ = C\|b\|_{q, \mathbb{R}^n} \leq C\|f\|_p \]
for any \( 1 < p \leq q < \infty \) satisfying \( \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n} \). It follows that \( \|TbI_{G\ast}\|_q \leq C\|f\|_p \) and
\[ \|T_{12}f\|_q = \|T_1bI_{G\ast}\|_q \leq \|TbI_{G\ast}\|_q + \|T_2bI_{G\ast}\|_q \]
\[ \leq C(\|f\|_p + \frac{\beta^{\frac{n(q-1)}{q}}}{(n(q - 1) - \beta q)^{\frac{1}{q}}} \|f\|_1), \]
where \( 1 < p \leq q < \infty \) satisfying \( \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n} \). Lemma 3.1 is then proved and the proof of Theorem 1.1 is finished. \( \square \)

In the end, we prove Theorem 1.2 as follows.

**Proof of Theorem 1.2.** According to (3.7), the operator \( T_1 \) can be decomposed into
\[
T_1 f = T_1 g + T_1 b I_{G^*} + T_1 b I_{F^*}
= (T_1 g + T_1 b I_{F^*}) + T_1 b I_{G^*}
≡ T_{11} f + T_{12} f,
\]
(3.18)

Thanks to (3.14), one has
\[
m\{ x : |T_{11} f| > t \} \leq C(n) \| f \|_1, \tag{3.19}
\]
Concerning \( T_{12} \), we take \( p = 1 \) in (3.17) to obtain
\[
\| T b I_{G^*} \|_q \leq \sum_l \| T b_l \|_{q; \mathbb{R}^n \setminus B_l} \leq C \sum_l \| b_l \|_{1; K_l}
= C \| b \|_{1; F} \leq C \| f \|_1
\]
for \( q = \frac{n}{n-\beta} \). Then, in view of Lemma 2.1
\[
\| T_1 b I_{G^*} \|_q \leq \| T b I_{G^*} \|_q + \| T_2 b I_{G^*} \|_q \leq C \| f \|_1,
\]
which implies that \( T_{12} \) is of type \((1, q)\), where \( \frac{1}{q} = 1 - \frac{\beta}{n} = \frac{n-\beta}{n} \). It concludes that, for any \( t > 0 \),
\[
m\{ x : |T_1 f| > t \} \leq m\{ x : |T_{11} f| > \frac{t}{2} \} + m\{ x : |T_{12} f| > \frac{t}{2} \}
\leq C \left( \frac{\| f \|^2}{t^2} + \frac{\| f \|}{ct} \right).
\]
The proof of Theorem 1.2 is proved. \( \square \)

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**References**

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