BIHARMONIC REEB CURVES IN SASAKIAN MANIFOLDS

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Abstract. Sasakian manifolds provide explicit formulae of some Jacobi operators which describe the biharmonic equation of curves in Riemannian manifolds. In this paper we characterize non-geodesic biharmonic curves in Sasakian manifolds which are either tangent or normal to the Reeb vector field. In the three-dimensional case, we prove that such curves are some helices whose geodesic curvature and geodesic torsion satisfy a given relation.

1. Introduction

The notions of harmonic and biharmonic maps between Riemannian manifolds have been introduced by J. Eells and J.H. Sampson (see [8]). For a map \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds the energy functional \( E_1 \) is defined by

\[
E_1(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g.
\]

Critical points of \( E_1 \) are called harmonic maps and are then solutions of the corresponding Euler-Lagrange equation

\[
\tau_1(\phi) = \text{trace} \nabla^\phi d\phi.
\]

Here \( \nabla^\phi \) denotes the induced connection on the pull-back bundle \( \phi^{-1}(TN) \) and \( \tau_1(\phi) \) is called the tension field of \( \phi \).

Biharmonic maps are the critical points of the functional bienergy

\[
E_2(\phi) = \frac{1}{2} \int_M |\tau_1(\phi)|^2 v_g,
\]

whose Euler-Lagrange equation is given by the vanishing of the bitension field (cf. [12]) defined by

\[
\tau_2(\phi) = -\Delta^\phi \tau_1(\phi) - \text{trace} R^N(d\phi, \tau_1(\phi)) d\phi,
\]

where \( \Delta^\phi = -\text{trace}_g(\nabla^\phi \nabla^\phi - \nabla^\phi_\phi) \) is the Laplacian on the sections of \( \phi^{-1}(TN) \), and \( R^N \) is the Riemannian curvature operator of \( (N, h) \). Note that

\[
\tau_2(\phi) = J_\phi(\tau_1(\phi))
\]

where \( J_\phi \) is the Jacobi operator along \( \phi \) defined by

\[
J_\phi(X) = -\Delta^\phi X - \text{trace} R^N(d\phi, X) d\phi, \quad \forall X \in \phi^{-1}(TN).
\]

Harmonic maps are obviously biharmonic and are absolute minimum of the bienergy.

2000 Mathematics Subject Classification. 53D10, 31A30.

Key words and phrases. biharmonic curves, Reeb vector fields, contact manifolds.
Nonminimal biharmonic submanifolds of the pseudo-euclidean space and of the spheres have been studied in [3] and [4].
Biharmonic curves have been investigated on many special Riemannian manifolds like Heisenberg groups [6], [9], invariant surfaces [11], Damek-Ricci spaces [7], Sasakian manifolds [10], etc.
As in the general theory of metric contact manifolds an important role is played by the Reeb vector field whose dynamics can be used to study the structure of the contact manifold or even the underlying manifold using techniques of Floer homology such as symplectic field theory and embedded contact homology.
The main purpose of this work is to study non-geodesic biharmonic curves in a $(2n+1)$-dimensional Sasakian manifold, which are either tangent or normal to the Reeb vector field.

2. Sasakian manifolds

A contact manifold is a $(2n+1)$-dimensional manifold $M$ equipped with a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M$. It has an underlying almost contact structure $(\eta, \varphi, \xi)$ where $\xi$ is a global vector field (called the characteristic vector field) and $\varphi$ a global tensor of type $(1, 1)$ such that
\begin{equation}
\eta(\xi) = 1, \ \varphi \xi = 0, \ \eta \varphi = 0, \ \varphi^2 = -I + \eta \otimes \xi.
\end{equation}
A Riemannian metric $g$ can be found such that
\begin{equation}
\eta = g(\xi, .), \ d\eta = g(., \varphi.), \ g(., \varphi.) = -g(\varphi., .).
\end{equation}
$(M, \eta, g)$ or $(M, \eta, g, \xi, \varphi)$ is called a contact metric manifold. If the almost complex structure $J$ on $M \times \mathbb{R}$ defined by
\begin{equation}
J(X, f \frac{d}{dt}) = (\varphi X - f \xi, \eta(X) \frac{d}{dt}),
\end{equation}
is integrable, $(M, \eta, g)$ is said to be Sasakian.
The following relations play an important role in the present work:

**Lemma 2.1.** [2] *On a Sasakian manifold $(M, \eta, g)$ we have*
\begin{equation}
R(X, \xi)X = -\xi
\end{equation}
and
\begin{equation}
R(\xi, X)\xi = -X
\end{equation}
*for any unit vector field $X$ orthogonal to the Reeb vector field $\xi$, where $R$ denotes the Riemannian curvature of $(M, g)$.*

3. Biharmonic curves in Sasakian manifolds

Let $\gamma : I \rightarrow (M^{2n+1}, \eta, g)$ be a regular curve parametrized by its arc length in a $(2n+1)$-dimensional Sasakian manifold and $\{T, N_1, ..., N_{2n}\}$ be the Frenet frame in $M^{2n+1}$, defined along $\gamma$, where $T = \gamma'$ is the unit tangent vector field of $\gamma$.
It holds:
Lemma 3.1. [10] The Frenet equations of $\gamma$ are given by

\[
\begin{align*}
\nabla_T T &= \chi_1 N_1 \\
\nabla_T N_1 &= -\chi_1 T + \chi_2 N_2 \\
& \vdots \\
\nabla_T N_k &= -\chi_k N_{k-1} + \chi_{k+1} N_{k+1}, \quad k = 2, \ldots, 2n - 1, \\
& \vdots \\
\nabla_T N_{2n} &= -\chi_{2n} N_{2n-1},
\end{align*}
\]

where $\chi_1 = |\nabla_T T|$, $\chi_2 = \chi_2(s), \ldots, \chi_{2n} = \chi_{2n}(s)$ are real valued functions, where $s$ is the arc length of $\gamma$.

Definition 3.2. If the functions $\chi_k$, $k = 1, \ldots, 2n$ are all constant, then $\gamma$ is said to be a helix.

The tension field $\tau_1(\gamma)$ and the bitension field $\tau_2(\gamma)$ of the curve $\gamma$ are given in the Frenet frame $(T, N_1, \ldots, N_{2n})$ by:

Proposition 3.3. [10]

(9) \[ \tau_1(\gamma) = \chi_1 N, \]

and

(10) \[
\tau_2(\gamma) = -3\chi_1 \chi_1' T + (\chi''_1 - \chi^3_1 - \chi_1 \chi^2_2) N_1 \\
\quad - (2\chi'_1 \chi_2 + \chi_1 \chi'_2) N_2 + \chi_1 \chi_2 \chi_3 N_3 - \chi_1 R(T, N_1) T.
\]

From Proposition 3.3 we get:

Proposition 3.4. If $\gamma$ is either tangent or normal to the Reeb vector field, then

(11) \[ \tau_2(\gamma) = -3\chi_1 \chi_1' T + (\chi''_1 - \chi^3_1 - \chi_1 \chi^2_2) N_1 \\
\quad - (2\chi'_1 \chi_2 + \chi_1 \chi'_2) N_2 + \chi_1 \chi_2 \chi_3 N_3.
\]

Proof

Let $\gamma$ be non-geodesic biharmonic curve in a Sasakian manifold $(M, \eta, g)$. Assume that $\gamma$ is tangent to the Reeb vector field $\xi$; that is $T = \xi$.

The relation (10) in Proposition 3.3 becomes

\[
\tau_2(\gamma) = -3\chi_1 \chi_1' T + (\chi''_1 - \chi^3_1 - \chi_1 \chi^2_2) N_1 \\
\quad - (2\chi'_1 \chi_2 + \chi_1 \chi'_2) N_2 + \chi_1 \chi_2 \chi_3 N_3 - \chi_1 R(\xi, N_1) \xi \\
\quad -3\chi_1 \chi_1' T + (\chi''_1 - \chi^3_1 - \chi_1 \chi^2_2) N_1 \\
\quad - (2\chi'_1 \chi_2 + \chi_1 \chi'_2) N_2 + \chi_1 \chi_2 \chi_3 N_3 + \chi_1 N_1,
\]

according to lemma 2.1.

It follows that

(12) \[
\tau_2(\gamma) = -3\chi_1 \chi_1' T + (\chi''_1 - \chi^3_1 - \chi_1 \chi^2_2 + \chi_1) N_1 \\
\quad - (2\chi'_1 \chi_2 + \chi_1 \chi'_2) N_2 + \chi_1 \chi_2 \chi_3 N_3
\]
We assume now that $\gamma$ is normal to the Reeb vector field; that is $N_1 = \xi$.

The relation (10) in Proposition 3.3 becomes then

$$
\tau_2(\gamma) = -3\chi_1\chi' T + (\chi'' - \chi_1^3 - \chi_1\chi_2^2)N_1 \\
- (2\chi_1\chi_2 + \chi_1\chi_3^2)N_2 + \chi_1\chi_2\chi_3 N_3 - \chi_1 R(T, \xi) T \\
= -3\chi_1\chi' T + (\chi'' - \chi_1^3 - \chi_1\chi_2^2)N_1 \\
- (2\chi_1\chi_2 + \chi_1\chi_3^2)N_2 + \chi_1\chi_2\chi_3 N_3 - \chi_1\xi 
$$

We obtain then again

$$
\tau_2(\gamma) = -3\chi_1\chi' T + (\chi'' - \chi_1^3 - \chi_1\chi_2^2 + \chi_1)N_1 \\
- (2\chi_1\chi_2 + \chi_1\chi_3^2)N_2 + \chi_1\chi_2\chi_3 N_3
$$

Thus we get the relation (11) in both cases.

From Proposition 3.4, we get the following result.

**Theorem 3.5.** Non-geodesic biharmonic curves in Sasakian manifolds which are either tangent or normal to the Reeb vector field are characterized by:

$$
\begin{align*}
\chi_1 &= \text{constant} \in [-1, 0[U[0, 1], \\
\chi_2 &= \pm \sqrt{1 - \chi_1^2}, \\
\chi_2\chi_3 &= 0.
\end{align*}
$$

where $\chi_1$, $\chi_2$ and $\chi_3$ are functions defined in lemma 3.1.

**Proof**

$\tau_2(\gamma) = 0$ with $\chi_1 \neq 0$ implies $\chi' = 0$ according to the first component in (11). So $\chi_1$ is constant. Then the second component gives $\chi_2^2 + \chi_2^2 = 1$. Thus (14) is satisfied.

**Corollary 3.6.** If $\chi_1 = \pm 1$ then $\chi_2 = 0$. And if $\chi_1 \neq \pm 1$ then $\chi_3 = 0$.

**Remark 3.7.** From the conditions given in (14), it is clear that in general non-geodesic biharmonic curves in Sasakian manifolds are not helixes since only the functions $\chi_1$ and $\chi_2$ have to be constant and maybe $\chi_3$.

In three-dimensional Sasakian manifolds the Frenet frame is given by

$$
\begin{align*}
\nabla_T T &= \chi_1 N_1, \\
\nabla_T N_1 &= -\chi_1 T + \chi_2 N_2, \\
\nabla_T N_2 &= -\chi_2 N_1,
\end{align*}
$$

where $\chi_1 = |\nabla T|$, $\chi_2 = \chi_2(s)$ is a real valued function, where $s$ is the arc length of $\gamma$.

And the equation characterizing the non-geodesic biharmonic curves which are either tangent or normal to Reeb vector field is reduced to

$$
\tau_2(\gamma) = -3\chi_1\chi' T + (\chi'' - \chi_1^3 - \chi_1\chi_2^2 + \chi_1)N_1 \\
- (2\chi_1\chi_2 + \chi_1\chi_2^2)N_2.
$$

So we have the following result.
Theorem 3.8. Non-geodesic biharmonic curves which are either tangent or normal to the Reeb vector field in three-dimensional Sasakian manifolds are helixes whose geodesic curvature $\chi_1$ and geodesic torsion $\chi_2$ are related by:

$$\chi_1^2 + \chi_2^2 = 1, \text{ with } \chi_1 \neq 0.$$ 

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