LONG TIME EXISTENCE OF THE 
(n − 1)-PLURISUBHARMONIC FLOW

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Abstract. We consider the (n−1)-plurisubharmonic flow, suggested by Tosatti-Weinkove, and prove a formula for its maximal time of existence. This includes estimates that will be useful in further investigating the flow.

1. Introduction

Let \( M \) be a compact complex manifold of dimension \( n > 2 \) with \( g \) and \( g_0 \) Hermitian metrics on \( M \). We define the associated real \((1,1)\)-form

\[
\omega = \sqrt{-1} g_{\overline{j}j} dz^i \wedge d\overline{z}^j
\]

which will will also refer to as a metric. The \((n−1)\)-plurisubharmonic flow is the equation

\[
\frac{\partial}{\partial t} \omega_{n-1} = -(n-1) \text{Ric}^C(\omega_t) \wedge \omega^{n-2}, \quad \omega_t|_{t=0} = \omega_0.
\]

where \( \text{Ric}^C(\omega_t) = -\sqrt{-1} \partial \overline{\partial} \log \omega^n_t \) is the Chern-Ricci form of \( \omega_t \). In the case of \( n = 2 \), (1.1) becomes the Chern-Ricci flow (see [8, 9, 10, 13, 17, 20, 23, 24, 27]). This flow was originally suggested by Tosatti-Weinkove in their work on the elliptic Monge-Ampère equation for \((n−1)\)-plurisubharmonic forms [25, 26].

We say that a metric \( \omega_0 \) is balanced [16] if

\[
d\omega_0^{n-1} = 0
\]

Gauduchon [5] if

\[
\partial \overline{\partial} \omega^{n-1} = 0
\]

and strongly Gauduchon (recently introduced by Popovici in [18]) if \( \overline{\partial} \omega_0^{n-1} \) is \( \partial \)-exact.

When \( \omega \) is a Kähler metric

\[
d\omega = 0
\]

then the \((n−1)\)-plurisubharmonic flow preserves all three of the above conditions imposed on \( \omega_0 \). If instead \( \omega \) is an Astheno-Kähler metric (see [12])

\[
\partial \overline{\partial} \omega^{n-2} = 0
\]

Supported by NSF RTG grant DMS-0838703.
the flow preserves the Gauduchon and strongly Gauduchon conditions, but not necessarily the balanced condition. Indeed, the flow is equivalent to
\[ \frac{\partial}{\partial t} \omega^{n-1}_t = -(n-1) \text{Ric}^C(\omega_t) \wedge \omega^{n-2} + \sqrt{-1} \partial \bar{\partial} \theta(t) \wedge \omega^{n-2} \]
where
\[ \theta(t) = \log \frac{\det(g_t)_{n-1}}{\det g^{n-1}}. \]

Defining
\[ \Phi_t = \omega^{n-1}_0 - t(n-1) \text{Ric}^C(\omega_t) \wedge \omega^{n-2} \]
we see that a solution to (1.1) is of the form
\[ \omega^{n-1}_t = \Phi_t + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} \]
for some real valued function \( u \) on \( M \). One can check that if \( \omega \) is Kähler and \( \omega_0 \) is balanced (respectively Gauduchon, strongly Gauduchon), then the family of metrics \( \omega_t \) is balanced (respectively Gauduchon, strongly Gauduchon) for all \( t \) along the flow. Similarly for \( \omega \) Asthen-Kähler and \( \omega_0 \) Gauduchon or strongly Gauduchon.

We prove the following formula for the maximal time of existence of the flow assuming \( \omega_0 \) and \( \omega \) are Hermitian metrics.

**Theorem 1.1.** Let \( M \) be a compact complex manifold of dimension \( n \geq 3 \) and let \( \omega_0 \) and \( \omega \) be Hermitian metrics on \( M \). Then there exists a unique solution of the \((n-1)\)-plurisubharmonic flow (1.1) on the maximal time interval \([0, T)\) where
\[ T = \sup \{ t > 0 \mid \exists \psi \in C^\infty(M) \text{ such that } \Phi_t + \sqrt{-1} \partial \bar{\partial} \psi \wedge \omega^{n-2} > 0 \} . \]

Note that if we define an equivalence relation of real \((n-1,n-1)\)-forms by
\[ \Psi \sim \Psi' \iff \Psi = \Psi' + \sqrt{-1} \partial \bar{\partial} \psi \wedge \omega^{n-2} \text{ for some } \psi \in C^\infty(M) \]
then \( T \) depends only on \( \omega \) and the equivalence class of \( \omega_0^{n-1} \). This is analogous to the result of Tian-Zhang for the Kähler-Ricci flow [22] and of Tosatti-Weinkove for the Chern-Ricci flow [23]. Much like these related results, this theorem suggests that the \((n-1)\)-plurisubharmonic flow is a natural object of study that reflects the geometry of the manifolds.

Every Hermitian metric is conformal to a Gauduchon metric [5] on a compact complex manifold. However if \( \omega \) is only assumed to be Gauduchon then the \((n-1)\)-plurisubharmonic flow (1.1) does not preserve the Gauduchon condition of \( \omega_0 \). To alleviate this problem we consider the new flow
\[ \frac{\partial}{\partial t} \omega^{n-1}_t = -(n-1) \text{Ric}^C(\omega_t) \wedge \omega^{n-2} - (n-1) \text{Re} \left( \sqrt{-1} \partial (\log \omega^n_t) \wedge \bar{\partial} (\omega^{n-2}) \right) . \]

If the fixed metric \( \omega \) is Gauduchon and the initial metric \( \omega_0 \) is Gauduchon or strongly Gauduchon, so is the solution to (1.2) for as long as it exists. To
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see this, we compute as above. A solution to this new flow \((1.2)\) is of the form

\[
\omega_t^{n-1} = \hat{\Phi}_t + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} + \text{Re} \left( \sqrt{-1} \partial u \wedge \bar{\partial} (\omega^{n-2}) \right)
\]

where

\[
\hat{\Phi}_t = \omega_0^{n-1} - t(n-1) \left( \text{Ric}^C(\omega) \wedge \omega^{n-2} + \text{Re} \left( \sqrt{-1} \partial (\log \omega) \wedge \bar{\partial} (\omega^{n-2}) \right) \right).
\]

We conjecture that this flow has a similar theorem for its maximal existence time, but we are currently unable to prove the estimates that would give this result.

**Conjecture 1.2.** Let \(M\) be a compact complex manifold of dimension \(n \geq 3\), \(\omega\) a Gauduchon metric, and \(\omega_0\) a Hermitian metric on \(M\). Then there exists a unique solution of \((1.2)\) on the maximal time interval \([0, T)\) where

\[
T = \sup \left\{ t > 0 \mid \exists \psi \in C^\infty(M) \text{ such that } \hat{\Phi}_t + \sqrt{-1} \partial \bar{\partial} \psi \wedge \omega^{n-2} + \text{Re} \left( \sqrt{-1} \partial \psi \wedge \bar{\partial} (\omega^{n-2}) \right) > 0 \right\}.
\]

The estimates required to prove the above conjecture are the same as those needed to prove Gauduchon’s conjecture:

**Conjecture 1.3.** (Gauduchon, 1977 [6]) Let \(M\) be a compact complex manifold and let \(\psi\) be a closed real \((1,1)\)-form on \(M\) with \([\psi] = \epsilon_1^BC(M)\). Then there exists a Gauduchon metric \(\tilde{\omega}\) on \(M\) with

\[
\text{Ric}^C(\tilde{\omega}) = \psi.
\]

This is a generalization of the famous Calabi-Yau theorem in Kähler geometry [28]. Popovici [19] and Tosatti-Weinkove [26] have both recently shown that proving Gauduchon’s conjecture is equivalent to solving

\[
(1.3) \quad \det (\Phi_u) = e^{F+b} \det (\omega^{n-1})
\]

with

\[
\Phi_u = \omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} + \text{Re} \left( \sqrt{-1} \partial u \wedge \bar{\partial} (\omega^{n-2}) \right) > 0
\]

with \(\sup_M u = 0\) and \(\omega\) Gauduchon. The missing ingredient for the solution is a second order estimate for \(u\) solving \((1.3)\). Consider \((1.3)\) where we remove the last term in the definition of \(\Phi_u\):

\[
(1.4) \quad \det (\omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2}) = e^{F+b} \det (\omega^{n-1})
\]

with

\[
\omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} > 0, \quad \sup u = 0
\]

Fu-Wang-Wu [3] proved that \((1.4)\) has a smooth solution when \(\omega\) is Kähler and has nonnegative orthogonal bisecional curvature and Tosatti-Weinkove have proven this result with no assumptions on \(\omega\) other than being a Hermitian metric [25, 26]. The estimates of [26] are crucial in the proof of the main theorem which we now summarize.
The general strategy is similar to that of the analogous results for the Kähler-Ricci flow [22] (see also [21]) and Chern-Ricci flow [23]. Note that the flow (1.1) cannot exist beyond $T$ as defined in the main theorem, so we assume that the flow has a maximal time of existence $S < T$. The $(n - 1)$-plurisubharmonic flow is reduced to the parabolic scalar flow
\[
\frac{\partial}{\partial t} u = \log \left( \frac{\hat{\omega}_t + \frac{1}{n-1} (\Delta u) \omega - \sqrt{-1} \partial \bar{\partial} u) )}{\Omega} \right)^n, \quad u|_{t=0} = 0
\]
with
\[
\hat{\omega}_t + \frac{1}{n-1} (\Delta u) \omega - \sqrt{-1} \partial \bar{\partial} u) > 0
\]
on $[0, S)$. The maximum principle gives uniform bounds for $u$, $\dot{u}$, and the volume form $\hat{\omega}_t^n$ where
\[
\hat{\omega}_t = \hat{\omega}_t + \frac{1}{n-1} (\Delta u) \omega - \sqrt{-1} \partial \bar{\partial} u).
\]
We then apply the maximum principle to obtain the estimate
\[
\text{tr}_\omega \hat{\omega}_t \leq C \left( \sup_{M \times [0, S]} |\nabla u|^2 g + 1 \right)
\]
which is the parabolic version of the estimate from [26] and the proof uses many similar elements. Following [25], we use a Liouville theorem and blow-up argument to uniformly bound $|\nabla u|^2 g$. Applying the Evans-Kyrlov method (see [7, 15] and [8] in the complex setting for parabolic equations) gives the $C^{2+\alpha} (M, g)$ estimate and then from standard parabolic theory we produce higher order estimates. This allows us to extend the flow beyond the time $S$ contradicting the maximality of $S$.

2. Reduction to Monge-Ampère and notation

We define the Christoffel symbols of the Hermitian metric $g$ in local holomorphic coordinates $(z^1, \ldots, z^n)$ by
\[
\Gamma^k_{ij} = g^{kl} \partial_i g_{jl}
\]
and the covariant derivative with respect to $g$ by
\[
\nabla_i a_l = \partial_i a_l - \Gamma^p_{il} a_p.
\]
The torsion of $g$ is the tensor
\[
T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}.
\]
Note that if $g$ is a Kähler metric, then $T^k_{ij} = 0$. The Chern curvature of $g$ is
\[
R_{k\ell i}^p = -\partial_i \Gamma^p_{\ell i}
\]
and it obeys the usual commutation identities for curvature. For example,
\[
[\nabla_i, \nabla_j] a_l = -R_{ijl}^p a_p, \quad [\nabla_i, \nabla_j] \omega_m = R_{ij}^p \omega_m a_p.
\]
We will make use of the commutation formulas
\[ u_{ji} = u_{ij} - u_{pj} R_{ji}^p, \quad u_{pqr} = u_{qrp} - T_{mq}^p u_{pqr}, \quad u_{ql} - T_{ili}^p u_{pql}. \] (2.6)

The Chern-Ricci form \( \text{Ric}^C(\omega) \) is given by
\[ \text{Ric}^C(\omega) = \sqrt{-1} R_{ij} dz^i \wedge d\bar{z}^j \]
where
\[ R_{ij} = g^{km} R_{ijkm} = -\partial_i \partial_j \log \det g. \]

A real \((n-1,n-1)\)-form \( \Psi \) is defined to be positive definite if for every nonzero \((1,0)\)-form \( \gamma \),
\[ \Psi \wedge \sqrt{-1} \gamma \wedge \bar{\gamma} \geq 0 \]
with equality if and only if \( \gamma = 0 \). The determinant of \( \Psi \) is given by the determinant of the matrix \( (\Psi_{ij}) \) where
\[ \Psi = (\sqrt{-1})^{n-1} (n-1)! \sum_{i,j} (\text{sgn}(i,j)) \Psi_{ij} dz^i \wedge d\bar{z}^j \wedge \ldots \wedge d\bar{z}^l \wedge \ldots \wedge d\bar{z}^n. \]

Using this formula,
\[ \det (\omega^{n-1}) = (\det g)^{n-1}. \]

We say that a constant \( C > 0 \) is uniform if it only depends on the initial data for the \((n-1)\)-plurisubharmonic flow. In our calculations a uniform constant \( C \) may change from line to line.

Now we set up the proof of the main theorem. Suppose that \( S \) is such that \( 0 < S < T \). Then there exists a smooth function \( \psi \) such that
\[ \Psi_S := \Phi_S + \sqrt{-1} \partial \bar{\partial} \psi \wedge \omega^{n-2} > 0. \] (2.7)

We define \( \Psi_t \) to be the straight line path from \( \omega_0^{n-1} \) to \( \Psi_S \) on \([0,S] \)
\[ \Psi_t = \frac{1}{S} ((S-t)\omega_0^{n-1} + t (\Phi_S + \sqrt{-1} \partial \bar{\partial} \psi \wedge \omega^{n-2})) \]
\[ = \omega_0^{n-1} + t \chi \wedge \omega^{n-2} \]
where \( \chi = \frac{1}{S} \sqrt{-1} \partial \bar{\partial} \psi - (n-1) \text{Ric}^C(\omega) \). From its definition, note that \( \Psi_t \) is uniformly bounded in the sense that there exists a uniform constant \( C \) such that
\[ \frac{1}{C} \omega^{n-1} \leq \Psi_t \leq C \omega^{n-1} \] (2.9)
on \( M \times [0,S] \). Define a family of Hermitian metrics \( \hat{\omega}_t \) by
\[ \hat{\omega}_t = \frac{1}{(n-1)!} * \Psi_t = \hat{\omega}_0 + \frac{t}{n-1} ((\text{tr} \omega \chi) \omega - \chi). \]
where $*$ is the Hodge star operator with respect to $g$ and

$$\tilde{\omega}_0 = \frac{1}{(n-1)!} * \omega_0^{n-1}. $$

From (2.9) we also have

$$(2.10) \quad \frac{1}{C} \omega \leq \tilde{\omega}_t \leq C \omega. $$

on $M \times [0, S]$ for some uniform $C$.

Suppose that $u$ satisfies (1.5)

$$\frac{\partial}{\partial t} u = \log \left( \frac{\tilde{\omega}_t + \frac{1}{n-1}((\Delta u)\omega - \sqrt{-1} \partial \bar{\partial} u)}{\Omega} \right), \quad u|_{t=0} = 0$$

with $\tilde{\omega}_t + \frac{1}{n-1}((\Delta u)\omega - \sqrt{-1} \partial \bar{\partial} u) > 0$ and $\Omega := e^{\psi/S} \omega^n$. Note that

$$\frac{\partial}{\partial t} u = \log \left( \frac{\det * \left( \tilde{\omega}_t + \frac{1}{n-1}((\Delta u)\omega - \sqrt{-1} \partial \bar{\partial} u) \right)}{\det \omega^{n-1}} \right) - \frac{1}{S} \psi$$

Then if we define

$$\omega_t^{n-1} := \Psi_t + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2},$$

equations (2.8) and (2.11) show that

$$\frac{\partial}{\partial t} \omega_t^{n-1} = \chi \wedge \omega^{n-2} + \sqrt{-1} \partial \bar{\partial} \left( \frac{\partial}{\partial t} u \right) \wedge \omega^{n-2}$$

$$= -(n-1) \text{Ric}^C(\omega_t) \wedge \omega^{n-2}.$$  

Conversely, suppose that $\omega_t^{n-1}$ as defined in (2.12) satisfies (1.1), then

$$\sqrt{-1} \partial \bar{\partial} \left( \frac{\partial}{\partial t} u \right) \wedge \omega^{n-2} = \frac{\partial}{\partial t} \left( \omega_t^{n-1} - \Psi_t \right)$$

$$= \left( \sqrt{-1} \partial \bar{\partial} \left( \frac{\partial}{\partial t} u \right) \right) \wedge \omega^{n-2}.$$  

Using the equalities in (2.11), we see that $\omega_t^{n-1}$ satisfies (1.1) if and only if $u$ satisfies (1.5).

We define the Hermitian metric $\check{\omega}$ by

$$\check{\omega}_t := \tilde{\omega}_t + \frac{1}{n-1}((\Delta u)\omega - \sqrt{-1} \partial \bar{\partial} u).$$

To simplify notation we drop the $t$ subscripts on the metrics and use $\check{\omega}$ and $\tilde{\omega}$ to denote $\check{\omega}_t$ and $\tilde{\omega}_t$. However, $\omega$ will still denote the fixed Hermitian metric $\omega$ and we will not refer to the family of metrics $\omega_t$ solving (1.1) for the remainder of this paper.
3. Preliminary estimates

We prove uniform bounds for $u$, $\dot{u}$, and the volume form $\tilde{\omega}^n$. The estimate for $u$ is actually simpler than in the elliptic case \cite{25, 26} since we can apply the parabolic maximum principle to (1.5).

Lemma 3.1. Suppose $u$ satisfies (1.5) on $M \times [0, S)$. Then there exists a uniform $C > 0$ such that

1. $|u| \leq C$
2. $|\dot{u}| \leq C$
3. $\frac{1}{C} \Omega \leq \tilde{\omega}^n \leq C \Omega$

on $M \times [0, S)$.

To prove this, we need a maximum principle that will work in this context.

Lemma 3.2. Let $v$ be a smooth real-valued function on a compact complex manifold $M$ with Hermitian metric $\omega$. Then at a point $x_0$ where $v$ achieves a maximum,

$$(\Delta v) \omega - \sqrt{-1} \partial \bar{\partial} v \leq 0.$$ 

Proof. Choose coordinates at $x_0$ so that $g_{\overline{i}j} = \delta_{\overline{i}j}$ and $v_{\overline{i}j} := \partial_{\overline{i}} \partial_j v = \lambda_i \delta_{\overline{i}j}$. Since $x_0$ is where $v$ attains a maximum $\lambda_i \leq 0$ for all $i = 0, \ldots, n$. Then at $x_0$,

$$(\Delta v) g_{\overline{i}j} = \left( \sum_{i=1}^{n} \lambda_i \right) \delta_{\overline{i}j} \leq \lambda_i \delta_{\overline{i}j} = v_{\overline{i}j}.$$ 

We will also make use of the tensor

$$\Theta^\overline{i}j = \frac{1}{n-1} \left( (\text{tr}_g g)^\overline{i}j - g^\overline{i}j \right) > 0$$

and the operator $L$ acting on smooth functions $v$ on $M$ defined by

$$Lv = \Theta^\overline{i}j \partial_{\overline{i}} \partial_j v.$$ 

Taking trace of (2.14), we have the useful relation

$$n = \text{tr}_g \tilde{\omega} + Lu.$$ 

Using this, we can prove Lemma 3.1 via maximum principle similar to the analogous estimates for the Kähler-Ricci flow (see \cite{21} for example).

Proof. For (1), define a quantity $Q = u - At$ where $A$ is a constant to be determined later and fix $0 < t' < S$. Then suppose that a maximum of $Q$ on $M \times [0, t']$ occurs at a point $(x_0, t_0)$ with $t_0 > 0$. Applying the previous
lemma and the usual maximum principle at \((x_0,t_0)\),

\[
0 \leq \frac{\partial}{\partial t}Q = \log \left( \frac{\hat{\omega} + \frac{1}{n-1}((\Delta u)\omega - \sqrt{-1}\partial \bar{\partial} u)^n}{\Omega} \right) - A \leq \log \frac{\hat{\omega}^n}{\Omega} - A \leq C - A
\]

where on the last line we used (2.10). Choosing \(A = C + 1\), we get a contradiction. Since \(t'\) is arbitrary, we conclude that \(Q\) achieves its maximum at \(t_0 = 0\) and so we have a uniform upper bound for \(u\). The lower bound follows similarly.

For (2), we compute the evolution equation for \(\dot{u}\). Using (1.5),

\[
(3.16) \quad \frac{\partial}{\partial t} \dot{u} = \text{tr}_{\bar{\omega}} \left( \frac{\partial}{\partial \bar{\omega}} \hat{\omega} \right) = \frac{1}{n-1} \text{tr}_{\bar{\omega}} \left( (\text{tr}_{\omega} \chi)\omega - \chi + (\Delta \dot{u})\omega - \sqrt{-1}\partial \bar{\partial} \dot{u} \right)
\]

Then we have

\[
(3.17) \quad L\dot{u} = \frac{1}{n-1} \left( (\text{tr}_{\bar{\omega}} g^{ij} - \bar{\omega}^{ij}) \partial_i \partial_j \dot{u} \right) = \frac{1}{n-1} \left( (\Delta \dot{u})(\text{tr}_{\omega} \chi) - \text{tr}_{\omega} \sqrt{-1}\partial \bar{\partial} \dot{u} \right).
\]

Now consider the quantity \(Q = (n-1)\dot{u} - Au\) where \(A\) is a constant to be determined. Combining (3.15), (3.16), and (3.17),

\[
\left( \frac{\partial}{\partial t} - L \right) Q = (\text{tr}_{\bar{\omega}} \omega)(\text{tr}_{\omega} \chi) - \text{tr}_{\omega} \chi - A\dot{u} + An - A\text{tr}_{\bar{\omega}} \hat{\omega}.
\]

Using (2.10), we can choose \(A\) large enough so that

\[
A\hat{\omega} \geq (\text{tr}_{\omega} \chi)\omega - \chi
\]

which gives

\[
\left( \frac{\partial}{\partial t} - L \right) Q \leq -A\dot{u} + An.
\]

Hence at a point \((x_0,t_0)\) at which \(Q\) achieves a maximum, \(\dot{u}(x_0,t_0) \leq n\). Then since \(Q\) is bounded above by its value at \((x_0,t_0)\),

\[
\dot{u} \leq \frac{1}{n-1} \left( A \sup_{M \times [0,S]} u + n(n-1) - Au(x_0,t_0) \right) \leq C
\]

where for the last inequality we used the above uniform bound for \(u\).

To prove the lower bound, consider the quantity

\[
Q = (n-1)(S - t + \varepsilon)\dot{u} + u + nt
\]
where $\epsilon > 0$ is a constant to be determined. Again applying (3.15), (3.16), and (3.17),

$$
\left( \frac{\partial}{\partial t} - L \right) Q = -\dot{u} + (S - t + \epsilon) ((\text{tr}\tilde{\omega})(\text{tr}\omega\chi) - \text{tr}\tilde{\omega}\chi) + \dot{u} - n + \text{tr}\tilde{\omega}\dot{\omega} + n
$$

$$
= \text{tr}\tilde{\omega}(\dot{\omega}s + \epsilon((\text{tr}\omega\chi)\omega - \chi))
$$

$$
> 0
$$

provided we choose $\epsilon > 0$ small enough. If $Q$ achieves a minimum at a point $(x_0, t_0)$ with $t_0 > 0$, we have a contradiction. Hence $Q$ must be bounded from below by its infimum over $M$ at time $t = 0$. When combined with the uniform bound for $u$, this gives the lower bound for $\dot{u}$.

To finish the lemma, (3) follows immediately from (2) since we have

$$
\dot{u} = \log \frac{\tilde{\omega}^n}{\Omega}.
$$

4. Second order estimate

We obtain a second order estimate for $u$ in terms of $\text{tr}\tilde{\omega}\dot{\omega}$. This estimate is the parabolic version of the estimates from Hou-Ma-Wu [11] and Tosatti-Weinkove [25, 26] and the proof follows a similar method.

Lemma 4.1. There exists a uniform $C > 0$ such that

$$
\text{tr}\omega\tilde{\omega} \leq C \left( \sup_{M \times [0, S)} |\nabla u|^2_g + 1 \right)
$$

on $M \times [0, S)$.

Proof. As in [26] we consider the tensor

$$
\eta_{ij} = u_{ij} + (\text{tr}\tilde{g})g_{ij} - (n - 1)\tilde{g}_{ij} = (\text{tr}\tilde{g})g_{ij} - (n - 1)\tilde{g}_{ij}.
$$

Fix a $t'$ such that $0 < t' < S$. Define the quantity

$$
H(x, \xi, t) = \log(\eta_{ij}^{\xi_i} \xi_j) + c \log \left( g^{\pi\rho}_{\eta_{ij}^{\xi_i} \xi_j} \right) + \varphi \left( |\nabla u|^2_g \right) + \nu(u)
$$

for $x \in M$, $\xi \in T^{1,0}_xM$ a $g$-unit vector, $t \in [0, t']$, and $c > 0$ a small constant to be determined. The above functions are

$$
\varphi(s) = -\frac{1}{2} \log \left( 1 - \frac{s}{2K} \right), \quad 0 \leq s \leq K - 1
$$

$$
\nu(s) = -A \log \left( 1 + \frac{s}{2L} \right), \quad -L + 1 \leq s \leq L - 1,
$$

where

$$
K = \sup_{M \times [0, t']} |\nabla u|^2_g + 1, L = \sup_{M \times [0, S)} |u| + 1, A = 3L(C_1 + 1)
$$

with $C_1$ a uniform constant to be determined during the proof. Note that $L$ is uniformly bounded by Lemma 3.1. This setup is similar to [11, 25, 26],
the difference being that we have a time dependence. Evaluating at $|\nabla u|^2$, we have the bounds

\begin{equation}
0 \leq \varphi \leq C, \quad 0 < \frac{1}{4K} \leq \varphi' \leq \frac{1}{2K}, \quad \varphi'' = 2(\varphi')^2 > 0
\end{equation}

and evaluating at $u$,

\begin{equation}
|\nu| \leq C, \quad C_1 + 1 = \frac{A}{3L} \leq -\nu' \leq \frac{A}{L}, \quad \frac{2\varepsilon}{1 - \varepsilon} (\nu')^2 \leq \nu'', \quad \text{for all } \varepsilon \leq \frac{1}{2A + 1}
\end{equation}
on $M \times [0, t']$ for uniform $C > 0$.

Similar to [11], we define the set

$$W = \{(x, \xi, t) \mid \eta(x, t) \xi^i \xi^j \geq 0, \xi \in T_x^{1,0} M \text{ a } g\text{-unit vector, } t \in [0, t']\}.$$ 

Then $W$ is compact, $H = -\infty$ on the boundary of a cross section $W_t$ for fixed time $t$, and $H$ is upper semi-continuous on $W_t$. Thus if $H$ has a maximum at a point $(x_0, \xi_0, t_0)$ in $W$, $(x_0, \xi_0)$ is in the interior of $W_{t_0}$. We assume without loss of generality that $t_0 > 0$.

Choose holomorphic coordinates $(z^1, \ldots, z^n)$ centered at $x_0$ such that at $(x_0, \xi_0, t_0)$

$$g_{ij} = \delta_{ij}, \quad \eta_{ij} = \delta_{ij} \eta_{ii}, \quad \eta_{11} \geq \eta_{22} \geq \ldots \geq \eta_{nn}.$$ 

From the definition of $\eta_{ij}$

$$\tilde{g}_{ij} = \frac{1}{n - 1} \left(-\eta_{ij} + (\text{tr}_g \tilde{g}) g_{ij}\right)$$

so that $\tilde{g}_{ij}$ is also diagonal at $(x_0, t_0)$ and we may define $\lambda_i$ by

$$\tilde{g}_{ij} = \lambda_i \delta_{ij}.$$ 

at $(x_0, t_0)$. Using (4.19),

\begin{equation}
\eta_{ii} = \sum_{j=1}^{n} \lambda_j - (n - 1)\lambda_i
\end{equation}

which gives

$$0 < \lambda_1 \leq \ldots \leq \lambda_n$$

and

\begin{equation}
\frac{1}{n} \text{tr}_{\omega} \tilde{\omega} \leq \lambda_n \leq \eta_{11} \leq (n - 1)\lambda_n \leq (n - 1)\text{tr}_{\omega} \tilde{\omega}.
\end{equation}

Following [26], choosing $c < 1/(n - 3)$ when $n > 3$ or $c$ any positive real number when $n = 3$, the quantity

$$\log(\eta_{ij} \xi^i \xi^j) + c \log \left(g^{pq} \eta_{pq} \eta_{ij} \xi^i \xi^j\right)$$

is maximized at $(x_0, t_0)$ by $\xi_0 = \partial/\partial z^1$ since $\eta_{11}$ is the largest eigenvalue of $\eta_{ij}$. We extend $\xi_0$ over our coordinate patch to the unit vector field

$$\xi_0 = g_{11}^{1/2} \frac{\partial}{\partial z^1}.$$
Now we consider the quantity

\[
Q(x, t) = H(x, \xi_0, t) = \log \left( g^{-1}_{ij} \eta_{ij} \right) + \varphi \left( |\nabla u|^2_g \right) + \nu(u)
\]

deﬁned in a neighborhood of \((x_0, t_0)\) chosen small enough so that \(Q\) attains its maximum at \((x_0, t_0)\). The proof of the estimate follows from applying the maximum principle to this quantity to obtain the bound

\[
\eta_{ij}(x_0, t_0) \leq CK = C \left( \sup_{M \times [0, t']} |\nabla u|^2_g + 1 \right).
\]

which will complete the proof: at any point \((x, t) \in M \times [0, t']\) using (4.23),

\[
\text{tr} \tilde{\omega} (x, t) \leq n \eta_{ij}(x, t)
\]

\[
\leq n \sup_{M \times [0, t']} \left( \left( g^{ij} \eta_{ij} \eta_{kl} \xi^i \xi^j \right)^{1/2} \left( \left( g^{jk} \eta_{jk} \eta_{pq} \xi^j \xi^k \right)^{c/(1+2c)} \right) \right)
\]

\[
\leq Ce^Q(x_0, t_0)
\]

\[
\leq C \left( \sup_{M \times [0, t']} |\nabla u|^2_g + 1 \right).
\]

Since \(C > 0\) is uniform we get the desired estimate (4.18).

We begin the proof of the estimate (4.25). First, we collect some useful facts. At the point \((x_0, t_0)\),

\[
\sum_i \Theta_i^\pi = \text{tr} \tilde{g}
\]

and we may assume that at this point

\[
|\tilde{u}| \leq 2|\eta_{ij}|
\]

since our goal is to prove a uniform bound for \(\eta_{ij}(x_0, t_0)\). As in [26] we have at \((x_0, t_0)\)

\[
L(Q) \geq (1 + 2c) \sum_i \frac{\Theta_i^\pi \eta_{ij}}{\eta_{ij}} + \frac{c}{2} \sum_i \sum_{p \neq 1} \frac{\Theta_i^\pi |\eta_{ij}|^2}{(\eta_{ij})^2} + \frac{c}{2} \sum_i \sum_{p \neq 1} \frac{\Theta_i^\pi |\eta_{ij}|^2}{(\eta_{ij})^2}
\]

\[
- (1 + 2c) \sum_i \frac{\Theta_i^\pi |\eta_{ij}|^2}{(\eta_{ij})^2} + \nu' \sum_i \Theta_i^\pi \tilde{u}^2_i + \nu'' \sum_i \Theta_i^\pi |u_i|^2
\]

\[
+ \varphi'' \sum_i \Theta_i^\pi \left( \sum_p u_{ip} u_{pi} + \sum_p u_{ip}^2 \right)^2 + \varphi' \sum_{i,p} \Theta_i^\pi \left( |u_{pi}|^2 + |u_{ip}|^2 \right)
\]

\[
+ \varphi' \sum_{i,p} \Theta_i^\pi \left( u_{pi}^2 u_{pi} + u_{pi} u_{ip} \right) - C \text{tr} \tilde{g}
\]

for a uniform \(C > 0\) where the subscripts denote covariant derivatives with respect to the fixed Hermitian metric \(g\).
Computing the time evolution of $Q$ at $(x_0, t_0)$,

$$
\frac{\partial}{\partial t}Q = (1 + 2c) \frac{\dot{\eta}}{\eta} + \varphi' \left( \sum p \dot{u}_p u_p + \sum_p \ddot{u}_p u_p \right) + \nu' \dot{u}.
$$

Using the definition of $\eta$ (4.19),

$$
\dot{\eta}_{ij} = \dot{u}_{ij} + \left( \text{tr}_g \frac{\partial}{\partial t} \hat{g} \right) g_{ij} - (n - 1) \frac{\partial}{\partial t} \hat{g}_{ij} = \dot{u}_{ij} + (\text{tr}_g \chi) g_{ij} - (\text{tr}_g \chi) g_{ij} - \chi_{ij}.
$$

Evaluating at $(x_0, t_0)$,

$$
\dot{\eta}_{11} = \dot{u}_{11} + \chi_{11}.
$$

Covariantly differentiating the flow (1.5) with respect to $g$,

$$
\dot{u}_l = g^{ij} \nabla_i \hat{g}_{ij} - \frac{1}{S} \psi_l
$$

and

$$
\dot{u}_{lm} = g^{ij} \nabla_i \hat{g}_{ij} - g^{ij} g^{pq} \nabla_i \hat{g}_{pq} \nabla_l \hat{g}_{ij} - \frac{1}{S} \psi_{lm}.
$$

Using the definition of $\hat{g}$ (2.14),

$$
\dot{u}_l = \Theta^{ij} u_{ijl} + \frac{1}{S} \psi_l
$$

and letting $\hat{h}_{ij} = (n - 1) \hat{g}_{ij}$,

$$
\dot{u}_{lm} = \Theta^{ij} u_{ijlm} + \frac{1}{S} \psi_{lm}
$$

At $(x_0, t_0)$, these become

$$
\dot{u}_p = \sum_i \Theta^{ip} u_{ip} + \frac{1}{S} \psi_p
$$

and

$$
\dot{u}_{1T} = \sum_i \Theta^{1T} u_{iT} + \frac{1}{S} \psi_{1T} - H
$$

where

$$
H = \frac{\sum_{i,j} \hat{g}^{ij} \hat{g}^{ij} \left( g_{ij} \sum_a u_{ia} - u_{ij} + \hat{h}_{11} \right) \left( g_{ij} \sum_b u_{ib} - u_{ij} + \hat{h}_{11} \right)}{(n - 1)^2}.
$$
Applying the commutation rule \( (2.6) \), \( (4.35) \) becomes

\[
\dot{u}_i = - H + \sum \Theta \hat{u}_i \eta \eta + \sum \hat{u}_i \hat{g}_i \hat{g}_i - \frac{1}{S} \psi \eta \\
+ \sum \Theta \hat{u}_i \left( u_{i1} - u_{i1} \right) \\
- \sum \Theta \hat{u}_i \left( T_1 + u_{i1} + T_1 \right).
\]

Combining \( (4.30), (4.31), (4.37) \), and the fact that

\[
u_1 = \eta_1 + \hat{h}_1 - (\text{tr}_g \hat{g})_1
\]

we have the evolution equation

\[
\begin{align*}
\frac{\partial}{\partial t} Q &= -(1 + 2c) \frac{H}{\eta_1} + (1 + 2c) \sum \Theta \eta_1 \\
&\quad + \frac{1 + 2c}{\eta_1} \left( \chi_1 - \frac{1}{S} \psi \eta \eta + \sum \Theta \left( u_{i1} - u_{i1} \right) \right) \\
&\quad + \sum \hat{g}_i \hat{g}_i + \sum \Theta \left( \hat{h}_1 - (\text{tr}_g \hat{g}) \right) \\
&\quad - \frac{2(1 + 2c)}{\eta_1} \sum \Theta \left( T_1 + u_{i1} + T_1 \right) - \frac{1 + 2c}{\eta_1} \sum \Theta \left( T_1 + T_1 \right) \\
&\quad + \varphi \left( \sum \hat{u}_p + u_p + \sum \hat{u}_p u_p \right) + \nu \hat{u}.
\end{align*}
\]
Subtracting \((4.38)\) and \((4.29)\) we obtain the evolution equation bound at \((x_0, t_0)\),

\[
0 \leq \left( \frac{\partial}{\partial t} - L \right) Q
\]

\[
\leq -(1 + 2c) \frac{H}{\eta_{1T}}
\]

\[
- \frac{c}{2} \sum_i \sum_{p \neq 1} \frac{\Theta^{\varpi|\eta_p T_1|}}{(\eta_{1T})^2} - \frac{c}{2} \sum_i \sum_{p \neq 1} \frac{\Theta^{\varpi|\eta_p \varpi_i|}}{(\eta_{1T})^2} + (1 + 2c) \sum_i \frac{\Theta^{\varpi|\eta_{1T1}|}}{(\eta_{1T})^2}
\]

\[
C tr g + \frac{1 + 2c}{\eta_{1T}} \left( \chi_{1T} - \frac{1}{2} \psi_{1T} + \sum_i \Theta^{\varpi} \left( u_p R_{1T}^p - u_p T_{1T}^p \right) \right)
\]

\[
+ \sum_i \eta^{\varpi} \hat{g}_{\varpi T_1} + \sum_i \Theta^{\varpi} \left( \hat{h}_{1T1} - (tr g)_{1T} \right)
\]

\[
- \frac{2(1 + 2c)}{\eta_{1T}} \sum_{i,p} \Theta^{\varpi} Re \left( \overline{T_{1T}^p u_{1pT}} \right) - \frac{1 + 2c}{\eta_{1T}} \sum_{i,p} \Theta^{\varpi} \overline{T_{1T}^p T_{1T}^q u_{1pT}}
\]

\[
+ \nu' \left( \frac{\partial}{\partial t} - L \right) u
\]

\[
- \nu'' \sum_i \Theta^{\varpi} |u_i|^2 - \varphi'' \sum_i \Theta^{\varpi} \left| \sum_p u_p u_{1p} + \sum_p u_p u_{1pT} \right|^2
\]

\[
- \varphi' \sum_{i,p} \Theta^{\varpi} \left( |u_{1p}|^2 + |u_{1pT}|^2 \right)
\]

\[
+ \varphi' \sum_p \left( \left( \hat{u}_p - \sum_i \Theta^{\varpi} u_{1pT} \right) u_{1T} + u_{1T} \left( i u_p - \sum_i \Theta^{\varpi} u_{1pT} \right) \right)
\]

\[
= (1) + (2) + (3) + (4) + (5) + (6) + (7) + (8) + (9)
\]

where (1) through (9) correspond to the lines in the last inequality. We now bound each of the lines of \((4.39)\) from above.

Lines (3) and (4): Using \((4.27)\) and \((4.28)\) we have the upper bound

\[
(3) + (4) \leq C tr g + C.
\]

Line (5): As in \([26]\), using the second term from line (2) we can bound line (5). Covariantly differentiating \((4.19)\),

\[
u_{1pT} = \eta_{1pT} - (tr g)_{1T} + \hat{h}_{1T1}
\]
and so
\[
-\frac{2(1 + 2c)}{\eta_{1T}} \sum_{i,p} \Theta \Re \left( \overline{T_{1i}^p} u_{\eta_{1T}} \right) \leq -\frac{2(1 + 2c)}{\eta_{1T}} \sum_{i,p} \Theta \Re \left( \overline{T_{1i}^p} \eta_{1T} \right) + \text{tr}_g g.
\]

Since \( T_{11}^1 = 0 \), the term from the sum with \( p = 1 \) is
\[
-\frac{2(1 + 2c)}{\eta_{1T}} \sum_i \Theta \Re \left( \overline{T_{1i}^1} \eta_{1T} \right) = -\frac{2(1 + 2c)}{\eta_{1T}} \sum_{i \neq 1} \Theta \Re \left( \overline{T_{1i}^1} \eta_{1T} \right).
\]
The remaining summands can be bounded by
\[
-\frac{2(1 + 2c)}{\eta_{1T}} \sum_i \sum_{p \neq 1} \Theta \Re \left( \overline{T_{1i}^p} \eta_{1T} \right) \leq \frac{c}{4} \sum_i \sum_{p \neq 1} \Theta \Re \left| \eta_{1i}^p \right|^2 (\eta_{1T})^2 + \text{tr}_g g.
\]

Putting together (4.40), (4.41), (4.42) and controlling the second term in (6) using (4.28) we have the bound
\[
(6) \leq -\frac{2(1 + 2c)}{\eta_{1T}} \sum_{i \neq 1} \Theta \Re \left( \overline{T_{1i}^1} \eta_{1T} \right) + \frac{c}{4} \sum_i \sum_{p \neq 1} \Theta \Re \left| \eta_{1i}^p \right|^2 (\eta_{1T})^2 + \text{tr}_g g.
\]

Line (6): Applying (3.15), the uniform bound for \( \dot{u} \), and (4.21),
\[
(6) = \nu' \dot{u} - \nu v + \nu' \text{tr}_g \dot{g}
\leq 3C(C_1 + 1) + 3(C_1 + 1)n - (C_1 + 1) \text{tr}_g \dot{g}
\leq C - (C_1 + 1) \text{tr}_g \dot{g}
\]
remembering that \( C_1 > 0 \) is to be determined.

Lines (8) and (9): For line (9), commuting covariant derivatives and recalling
\[
(9) = \varphi' \sum_p \left( \left( \dot{u}_p - \sum_i \Theta \Re (u_{\eta_{1i}^p}) \right) u_{\eta_{1i}^p} + u_{\eta_{1i}^p} \left( \dot{u}_{\eta_{1i}^p} - \sum_i \Theta \Re (u_{\eta_{1i}^p}) \right) \right)
- \varphi' \sum_{i,p} \Theta \Re u_{\eta_{1i}^p} R_{i \eta_{1i}^p} + 2 \Re \varphi' \sum_{i,p,q} \Theta \Re u_{\eta_{1i}^p} T_{i \eta_{1i}^p}^q
= \varphi' \sum_{i,p} \Theta \Re u_{\eta_{1i}^p} R_{i \eta_{1i}^p} + 2 \Re \varphi' \sum_{i,p,q} \Theta \Re u_{\eta_{1i}^p} T_{i \eta_{1i}^p}^q.
\]

Thankfully, \( \varphi' \) can be used to control the single derivatives of \( u \) via (4.20). Combining this and (4.27),
\[
(9) \leq C + C \text{tr}_g \dot{g} + \frac{1}{10} \varphi' \sum_{i,p} \Theta \Re \left( |u_{\eta_{1i}^p}|^2 + |u_{\eta_{1i}^p}|^2 \right).
\]
Together with (8) we have the upper bound
\[(8) + (9) \leq C + C \text{tr} \tilde{g} - \frac{9}{10} \phi' \sum_{i, p} \Theta_i \left( |u_{pi}|^2 + |u_{pi}|^2 \right).\]

Combining the above estimates for the lines in (4.39), we have
\[0 \leq - (1 + 2c) \frac{H}{\eta_{1T}} \sum_{i, p \neq 1} \frac{\Theta_i |\eta_{pi}|^2}{(\eta_{1T})^2} - \frac{c}{4} \sum_{i, p \neq 1} \frac{\Theta_i |\eta_{pi}|^2}{(\eta_{1T})^2} + (1 + 2c) \sum_i \frac{\Theta_i |\eta_{1T}|^2}{(\eta_{1T})^2} \]
\[+ \nu'' \sum_i \Theta_i |u_i|^2 - \varphi'' \sum_i \Theta_i \left| \sum_p u_{pi} u_{pi} + \sum_p |u_{pi}^2| \right|^2 \]
\[+ C + C_0 \text{tr} \tilde{g} - \frac{9}{10} \phi' \sum_{i, p} \Theta_i \left( |u_{pi}|^2 + |u_{pi}|^2 \right) \]
\[- \frac{2(1 + 2c)}{\eta_{1T}} \sum_{i \neq 1} \Theta_i \text{Re} \left( T_{11}^i \eta_{1T} \right) - (C_1 + 1) \text{tr} \tilde{g}.\]

This is the same inequality as part way through the second order estimate in [26]. Since we are fixed at the point \((x_0, t_0)\), \(\tilde{g}\) is a fixed Hermitian metric. This lets us choose \(C_1 > 0\) uniform and large such that
\[(C_0 + 2) \text{tr} \tilde{g} \leq (C_1 + 1) \text{tr} \tilde{g}.\]

The remainder of the estimate goes through exactly as in [26] and we will not reproduce it here. This gives the bound
\[\eta_{1T}(x_0, t_0) \leq CK\]
for uniform \(C > 0\) which completes the proof as discussed above. □

5. First order estimate

Given the form of our second order estimate we require a first order estimate for \(u\). For the proof we modify the argument of [25] to apply in this parabolic setting.

Lemma 5.1. There exists a uniform \(C > 0\) such that
\[\sup_{M \times [0, S]} |\nabla u|^2 \leq C.\] (5.43)

The proof of this lemma requires a bit of machinery which we will recall from [25]. Let \(\beta\) be the Euclidean Kähler form on \(\mathbb{C}^n\) and \(\Delta\) the Laplacian with respect to \(\beta\). Let \(\Omega \subset \mathbb{C}^n\) be a domain. We say that an upper semi-continuous function
\[u : \Omega \to \mathbb{R} \cup \{-\infty\}\]
in $L^1_{loc}(\Omega)$ is $(n-1)$-PSH if

$$P(u) := \frac{1}{n-1} \left( (\Delta u)\beta - \sqrt{-1} \partial \bar{\partial} u \right) \geq 0$$

as a $(1,1)$-current. A continuous $(n-1)$-PSH function $u$ is maximal if for any relatively compact open set $\Omega' \Subset \Omega$ and any continuous $(n-1)$-PSH function $v$ on a domain $\Omega' \Subset \Omega'' \Subset \Omega$ and with $v \leq u$ on $\partial \Omega'$, then $v \leq u$ on $\Omega'$.

We need the following Liouville-type theorem from [25].

**Theorem 5.2.** (Tosatti-Weinkove) If $u : \mathbb{C}^n \to \mathbb{R}$ is an $(n-1)$-PSH function in $\mathbb{C}^n$ which is Lipschitz continuous, maximal, and satisfies

$$\sup_{\mathbb{C}^n} (|u| + |\nabla u|) < \infty$$

then $u$ is constant.

The proof of this result uses an idea of Dinew-Kołodziej [1]. With these definitions and the Liouville-type theorem, we now begin the proof of Lemma 5.1.

**Proof.** Suppose for contradiction that (5.43) does not hold. Then there exists a sequence $(x_j, t_j)$ in $M \times [0, S)$ with $t_j \to S$ such that

$$\lim_{j \to \infty} |\nabla u(x_j, t_j)|^2_g = \infty.$$  

Without loss of generality we assume our $t_j$ are such that

$$\sup_{x \in M} |\nabla u(x, t_j)|^2_g = \sup_{M \times [0, t_j]} |\nabla u|_g^2.$$  

Additionally, we choose our $x_j$ to be a point at which $|\nabla u(\cdot, t_j)|_g$ attains its maximum. We define

$$C_j := |\nabla u(x_j, t_j)|^2_g = \sup_{M \times [0, t_j]} |\nabla u|_g^2$$

which has the property $C_j \to \infty$ as $j \to \infty$.

With this setup, we are ready to apply the blow-up argument and the Liouville-type theorem from [25] to obtain a contradiction. After passing to a subsequence, there exists an $x$ in $M$ such that $x_j \to x$ as $j \to \infty$. Fix holomorphic coordinates $(z^1, \ldots, z^n)$ centered at $x$ with $\omega(x) = \beta$ and identifying with the ball $B_2(0) \subset \mathbb{C}^n$. Also assume that $j$ is sufficiently large so that $x_j \in B_1(0)$. We define

$$u_j(x) = u(x, t_j)$$

$$\Phi_j(z) = C_j^{-1} z + x_j$$

and

$$\hat{u}_j(z) := (u_j \circ \Phi_j(z)) = u_j \left( C_j^{-1} z + x_j \right) \text{ for } z \in B_{C_j}(0).$$

Note that by construction $\hat{u}_j$ achieves its maximum at $z = 0$ and

$$|\nabla \hat{u}_j|_\beta(0) = C_j^{-1} |\nabla u(x_j)|_g = 1.$$  

(5.44)
We also have the uniform bounds
\[ \sup_{B_{C_j}(0)} |\hat{u}_j|_\beta \leq C, \quad \sup_{B_{C_j}(0)} |\nabla \hat{u}_j|_\beta \leq 1. \]

Using Lemma 4.1 on \([0, t_j]\) (see (4.26))
\[ \sup_{y \in M} \left| \sqrt{-1} \partial \bar{\partial} u(y, t_j) \right| g \leq C' \left( \sup_{M \times [0, t_j]} |\nabla u|_g^2 + 1 \right) = C' C_j^2 + C' \]
which gives the estimate
\[ \sup_{B_{C_j}(0)} \left| \sqrt{-1} \partial \bar{\partial} \hat{u}_j \right|_{\beta} \leq \frac{C}{C_j^2} \sup_{y \in M} \left| \sqrt{-1} \partial \bar{\partial} u(y, t_j) \right| g \leq C''. \]

For every compact \( K \subset \mathbb{C}^n \), every \( 0 < \alpha < 1 \), and every \( p > 1 \) there exists uniform \( C > 0 \) such that
\[ ||\hat{u}_j||_{C^{1,\alpha}(K)} + ||\hat{u}_j||_{W^{2,p}(K)} \leq C \]
using the Sobolev embedding theorem. From this we have a function \( u \in W^{2,p}_{loc}(\mathbb{C}^n) \) such that a subsequence \( \hat{u}_j \) converges strongly in \( C^{1,\alpha}_{loc}(\mathbb{C}^n) \) and weakly in \( W^{2,p}_{loc}(\mathbb{C}^n) \) to \( u \). Thus from the estimates for \( \hat{u}_j \) we have the uniform bounds
\[ \sup_{\mathbb{C}^n} (|u| + |\nabla u|) \leq C \]
and from (5.44) \( u \) is nonconstant. Following the remainder of the argument for the elliptic case in [25] shows that \( u \) is maximal and is hence constant by the Liouville-type theorem, a contradiction. \( \square \)

6. Higher order estimates and proof of the main theorem

To finish the proof of the main theorem, it sufficed to prove the uniform higher order estimates
\[ ||u||_{C^k(M, g)} \leq C_k \]
for \( k = 0, 1, 2, \ldots \). With these estimates the flow converges smoothly as \( t \to S \) to a metric \( \omega_S \). We extend the flow to \([0, S]\) with \( \omega_t|_{t=S} = \omega_S \) allowing us to begin the flow once more. This contradicts the fact that \( S \) is maximal so we must have \( S = T \) since the flow cannot exist beyond \( T \). We now prove the higher order estimates.

Summarizing our current estimates for \( u \), we have
\[ \sup_{M \times [0, S]} |u| + \sup_{M \times [0, S]} |\nabla u|_g + \sup_{M \times [0, S]} |\sqrt{-1} \partial \bar{\partial} u|_g + \sup_{M \times [0, S]} |\hat{u}|_g \leq C \]
for a uniform \( C > 0 \). Note that from the volume form bound in Lemma 3.1 and the trace bound in Lemma 4.1 we have that \( \tilde{g} \) is uniformly equivalent to \( g \):
\[ \frac{1}{C} g \leq \tilde{g} \leq C g. \]
Using standard parabolic theory, the higher order estimates follow from a uniform parabolic \( C^{2+\alpha}(M, g) \) bound for \( u \) for some \( \alpha > 0 \). This can be done via the parabolic Evans-Krylov method as in \[8\] with some modification (also see \[7, 15\]).

Let \( B_R \) be a small ball in \( \mathbb{C}^n \) of radius \( R > 0 \) centered at the origin. Let \( \varepsilon > 0 \) and fix \( t_0 \in [\varepsilon, T) \). We work in the parabolic cylinder

\[
Q(R, t_0) = \{(x, t) \in B_R \times [0, S) \mid t_0 - R^2 \leq t \leq t_0 \}.
\]

Let \( \{\gamma_i\} \) be a basis for \( \mathbb{C}^n \). For the \( C^{2+\alpha}(M, g) \) estimate it suffices to prove the bound

\[
\sum_{i=1}^{n} \text{osc}_{Q(R, t_0)}(u_{\gamma_i}) + \text{osc}_{Q(R, t_0)}(\dot{u}) \leq CR^\delta
\]

for any \( t_0 \in [\varepsilon, S) \), for some uniform \( C > 0 \), some \( R > 0 \) sufficiently small, and some \( \delta > 0 \).

We first rewrite the flow (1.5) as

\[
(6.46) - \frac{\partial}{\partial t} u + \log \det \tilde{g} = \tilde{F}
\]

where \( \tilde{F} = \psi/S + \log \Omega \). Let \( \gamma \) be an arbitrary unit vector in \( \mathbb{C}^n \). We differentiate the flow covariantly and commute derivatives as in (4.35) and (4.37) to obtain

\[
- \frac{\partial}{\partial t} u_{\gamma \gamma} + \Theta^{\gamma \gamma} u_{\gamma \gamma} \geq G + \frac{H}{\eta_{\gamma \gamma}} - C \sum_{p,q} |u_{p \gamma \gamma}| + H \eta_{\gamma \gamma} - C C' \sum_{p,q} |u_{p \gamma \gamma}| - C' \geq C \sum_{p,q} |u_{p \gamma \gamma}| - C'
\]

as in (4.36). Converting the covariant derivatives to partial derivatives,

\[
- \frac{\partial}{\partial t} u_{\gamma \gamma} + \Theta^{\gamma \gamma} \partial_\gamma \partial_\gamma u_{\gamma \gamma} \geq G + H - C \sum_{p,q} |u_{p \gamma \gamma}|
\]

for a larger \( C > 0 \). The latter two terms cancel because we have the estimate

\[
\frac{H}{\eta_{\gamma \gamma}} \geq \frac{1}{C'} \left( g^{\gamma \gamma} g_{p \gamma} g_{p \gamma} (g_{p \gamma} g^{\alpha \gamma} u_{p \gamma} - u_{p \gamma}) - C' \right)
\]

\[
\geq \frac{1}{C'} \left( (n-2) \left| g^{\gamma \gamma} u_{p \gamma} \right|^2 + g^{\gamma \gamma} g_{p \gamma} u_{p \gamma} u_{p \gamma} \right) - C'
\]

\[
\geq C' \sum_{p,q} |u_{p \gamma \gamma}| - C'
\]

for a uniform constant \( C' > 0 \), giving the bound

\[
(6.47) - \frac{\partial}{\partial t} u_{\gamma \gamma} + \Theta^{\gamma \gamma} \partial_\gamma \partial_\gamma u_{\gamma \gamma} \geq G.
\]
We also have
\begin{equation}
-\frac{\partial}{\partial t} \tilde{\omega} + \Theta \tilde{\omega} \partial_i \partial_j \tilde{\omega} = \frac{(\text{tr} \tilde{g} g)(\text{tr} \tilde{g} \chi) - \text{tr} \tilde{g} \chi}{n - 1} \leq C
\end{equation}
using (3.16), (3.17), Lemma 4.1, and Lemma 5.1 for a uniform $C > 0$.

As in [25, 26] we define a metric $g'_{ij} = g_{ij}(x_0)$ on $B_R$. This fixed metric allows us to contract tensors that would otherwise be at different points in space and time. We will also use the tensor
$$\hat{\Theta} = \frac{1}{n-1} \left( (\text{tr} \tilde{g} g') g'_{ij} - \tilde{g}_{ij} \right)$$
and the operator
$$\Delta' = g'_{ij} \partial_i \partial_j.$$

By the mean value inequality, for all $x$ in $B_R$,
\begin{equation}
\sum_{i,j} \frac{\partial \Phi}{\partial \tilde{g}(y,s)} \left( \tilde{g}(y,s) - \tilde{g}(x,t) \right) \geq \Phi(\tilde{g}(x,t)) - \Phi(\tilde{g}(y,s)).
\end{equation}

Using (6.49), equation (6.50) becomes
\begin{equation}
\dot{u}(x,t) - \dot{u}(y,s) + \sum_{i,j} \tilde{g}_{ij}(y,s) \left( \tilde{g}_{ij}(y,s) - \tilde{g}_{ij}(x,t) \right) \leq CR
\end{equation}
after applying the mean value inequality to $\tilde{F}$. We need to further bound the last term on the left hand side. Computing from the definition of $\tilde{g}$
\begin{equation}
\sum_{i,j} \tilde{g}_{ij}(y,s) \left( \tilde{g}(y,s) - \tilde{g}(x,t) \right) = \sum_{i,j} \tilde{g}_{ij}(y,s) \left( \tilde{g}(y,s) - \tilde{g}(x,t) \right) + \frac{1}{n-1} \sum_{i,j} \tilde{g}_{ij}(y,s) \left( ((\Delta u)g_{ij} - u_{ij})(y,s) - ((\Delta u)g_{ij} - u_{ij})(x,t) \right).
\end{equation}

The mean value inequality in $Q(R,t_0)$ along with the uniform bounds for $\tilde{g}$ and $\tilde{g}$ gives
\begin{equation}
\sum_{i,j} \tilde{g}_{ij}(y,s) \left( \tilde{g}(y,s) - \tilde{g}(x,t) \right) \leq CR.
\end{equation}

Then with (6.52), (6.53), and the uniform bounds for $u_{ij}$ and $\tilde{g}_{ij}$ equation (6.51) becomes
\begin{equation}
\frac{1}{n-1} \sum_{i,j} \tilde{g}_{ij}(y,s) \left( ((\Delta u)g_{ij} - u_{ij})(y,s) - ((\Delta u)^{\prime} g_{ij} - u_{ij})(x,t) \right) + \dot{u}(x,t) - \dot{u}(y,s) \leq CR.
\end{equation}
Here is where we use the fixed metric $g'$. Since
\[
\sum_{i,j} \hat{\Theta}^i_j(y,s) u_{ij}(z,r) = \sum_{i,j} \hat{\Theta}^i_j(y,s) \left( (\Delta' u) g'_{ij} - u_{ij} \right)(z,r),
\]
for any $(z,r) \in B_R \times [0,S)$ we have the estimate
\[
(6.55) \quad \dot{u}(x,t) - \dot{u}(y,s) + \sum_{i,j} \hat{\Theta}^i_j(y,s) \left( u_{ij}(y,s) - u_{ij}(x,t) \right) \leq CR.
\]
Following [8] (or [7, 15]) we find finitely many unit vectors $\gamma_1, \ldots, \gamma_n$ in $\mathbb{C}^n$ and real valued functions $\beta_\nu$ on $B_R \times [0,S)$ with
\[
0 < C^{-1} < \beta_\nu < C
\]
for $\nu = 1, \ldots, N$ such that
\[
\hat{\Theta}^i_j(y,s) = \sum_{\nu=1}^{N} \beta_\nu(y,s) (\gamma_\nu)^i (\gamma_\nu)^j.
\]
For $\nu = 1, \ldots, N$ define
\[
w_\nu = u_{\gamma_\nu \overline{\gamma_\nu}}
\]
and for $\nu = 0$,
\[
w_0 = -\dot{u}, \text{ and } \beta_0 = 1.
\]
From (6.55),
\[
(6.56) \quad \sum_{\nu=0}^{N} \beta_\nu(y,s) (w_\nu(y,s) - w_\nu(x,t)) \leq CR
\]
and for all $\nu = 0, 1, \ldots, N$,
\[
(6.57) \quad -\frac{\partial}{\partial t} w_\nu + \Theta^i_j \partial_i \partial_j w_\nu \geq G
\]
where $G$ is a uniformly bounded function using (6.47) and (6.48). With the key estimates (6.56) and (6.57) we can complete the $C^{2+\alpha}(M,g)$ estimate exactly as in [8] for the parabolic complex Monge-Ampère equation. This finishes the proof of the main theorem.

7. Acknowledgments

The author thanks Valentino Tosatti for several helpful discussions and suggestions and especially thanks Ben Weinkove for his encouragement, support, and advice.
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