Exact results of the quantum phase transition for the topological order

Jing Yu, Su-Peng Kou, and Xiao-Gang Wen

1Department of Physics, Beijing Normal University, Beijing 100875, China
2Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

In this paper a duality between the $d = 2$ Wen-plaquette model in a transverse field and the $d = 1$ Ising model in a transverse field is used to learn the nature of the quantum phase transition (QPT) between a spin-polarized phase and a topological ordered state with the string-net condensation. The QPT is not induced by spontaneous symmetry breaking and there are no conventional Landau-type local order parameters. Instead the string-like non-local order parameters are introduced to describe the QPT. In particular, the duality character between the open-string and closed string for the QPT are explored.

Keywords: topological order, quantum phase transition, the Wen-plaquette model

PACS numbers: 75.10.Jm, 05.70.Jk

Landau developed a sysmtactical theory for classical statistical systems. Different orders are characterized by different symmetries. The phase transition between different phases are always accompanied with symmetry breaking. However, Landau’s theory cannot describe all the continuous phase transition such as the transitions between quantum ordered phase[1]. Topological ordered state is a special quantum order with full gapped excitations. The first nontrivial example of topological order is the fractional quantum hall (FQH) states which cannot be described by broken symmetries[1]. Recently, people pay attention to the quantum phase transition (QPT) for the topological orders. It is known that there exists a quantum phase transition between a spin-polarized phase and a topological ordered state. The two phases cannot be characterized by a local order parameter. Thus such a quantum phase transition cannot be described by the Landau theory obviously. To find a new approach to learn the nature of such QPT becomes an interesting issue. The quantum phase transition for the Kitaev toric-code model in a transverse magnetic field was studied in Ref.[2]. The numerical simulations demonstrate the condensation of ‘magnetic’ excitations and the confinement of ‘electric’ charges of the phase transition out of the topological phase. And then people find that the “topological entropy” $S_{top}$ serves as an order parameter[3, 4]. At the QPT the “topological entropy” jumps abruptly from a finite number to zero out of the topological phase. In addition, in Ref. [2] it is revealed that the quantum phase transitions between different topological orders can be exactly characterized by a topological order parameter and shows the one dimensional (1D) properties.

In this paper we focus on the Wen-plaquette model on a square lattice[1, 3]. The ground states of the Wen-plaquette model are $Z_2$ topological ordered state which is similar to that for the kitaev toric-code model[3]. After adding the transverse field, a quantum phase transition occurs by raising the strength of the transverse field. However, the model cannot be solved exactly. Fortunately, we find a duality between the $d = 2$ Wen-plaquette model in a transverse field (transverse Wen-plaquette model) and the $d = 1$ Ising model in a transverse field (transverse Ising model). These results indicate a new type of QPT between a topological ordered phase and a non-topological ordered phase, including the non-local order parameters, the 1D properties of the QPT, in particular, the duality between open-string and closed-string.

The Hamiltonian for the $d = 2$ transverse Wen-plaquette model on square lattice[1, 3] is given as:

$$H_{wen} = g \sum_i F_i + h \sum_i \sigma_i^x$$

with $F_i = \sigma_i^x \sigma_{i+\xi_e}^x \sigma_{i+\xi_m}^y \sigma_{i+\xi_m}^y$ and $g < 0$. $\sigma_i^x$, $\sigma_i^y$ are Pauli matrices on sites, $i$.

For the first part of the model, it is an exact solved model[3]. The ground state of it is a topological order described by $Z_2$ projective symmetry groups. And there exists three types of quasiparticles: $Z_2$ charge, $Z_2$ vortex, and fermions, of which three types of string operators $T_1$, $T_2$, $T_3$, are defined, respectively[10]. The fermions can be regarded as bound states of a $Z_2$ charge on even plaquette and a $Z_2$ vortex on odd plaquette. The string operator for $Z_2$ charge has a form $W_c(C) = \prod_{m \in C} \sigma_{l_m}^z$, where $C$ is a string connecting the even-plaquettes of the neighboring links, and $l_m$ are sites on the string. For $Z_2$ vortex, string operator is $W_v(C) = \prod_{m \in C} \sigma_{l_m}^z$ with $C$ as a string connecting the even-plaquettes of the neighboring links. For an open T1 (T2) string, the ends points represent the $Z_2$ charge ($Z_2$ vortex). It is known that the ground state of the topological order has a condensation of the closed-strings as $\langle W_{c,\xi}(C) \rangle \neq 0$. On the contrary, the open strings have mass gap without condensation.

It is known that the Wen-plaquette model ($h = 0$) can be mapped onto nearest neighbor Ising chains[11]. This is because that the Wen-plaquette model ($h = 0$) has an energy spectrum which is identical to that
of the one dimensional Ising chain \[11\]. Furthermore, we find that the transverse Wen-plaquette model on a square lattice in Eq. (1) corresponds to the one dimensional Ising chain in transverse field. In the following part of this paper, we will show the duality and use it to learn the nature of the QPT between a spin-polarized phase and a topological ordered state.

To obtain the duality we define
\[
\sigma^x_i \sigma^y_{i+1} \sigma^z_{i+2} \sigma^y_{i+3} \sigma^y_i = A_i, \quad \sigma^z_i = B_i.
\]

To realize the duality we define
\[
\tau^x_{i+\frac{1}{2}}, \quad \tau^z_{i+\frac{1}{2}} \quad \text{Pauli matrices on sites, } i + \frac{1}{2}.\]

Then the mapping between the two models is given as
\[
\sigma^x_i \sigma^y_{i+1} \sigma^z_{i+2} \sigma^y_{i+3} \sigma^y_i \mapsto \tau^x_{i+\frac{1}{2}}, \quad \sigma^z_i \mapsto \tau^z_{i+\frac{1}{2}} \tau^z_{i+\frac{1}{2}}.
\]

Then we can explicitly denote the original model by the following Hamiltonian, describing the Ising model in a transverse field in \(d = 1\):
\[
\mathcal{H}_{\text{wen}} \rightarrow \mathcal{H}_I = -h \sum_a \sum_i \left( g_I \tau^x_{a,i+\frac{1}{2}} + \tau^z_{a,i+\frac{1}{2}} \tau^z_{a,i+\frac{1}{2}} \right)
\]

where \(a\) is the chain-index which is different for the \(d = 2\) transverse Wen-plaquette model on different lattices with different boundary conditions (see Fig.1). \(h\) is an overall energy scale and \(g_I = \frac{\hbar}{\sqrt{2m} c} > 0\) is a dimensionless coupling constant \(13\). This is a 1D Ising model along diagonal directions. By the mapping above, we find \(g\)-term in wen’s model corresponds to external field term in Ising model, while \(h\)-term corresponds to Ising-term.

In addition, an important feature about the duality is the boundary condition. The \(d = 2\) transverse Wen-plaquette model on \(N \times N\) square lattice with the open boundary condition is dual to \(2N - 1\) decoupled Ising chains (in a transverse field). Here, the chain-index \(a\) is from 1 to \(M + N - 1\). However, on \(N \times N\) square lattice with the periodic boundary condition, the \(d = 2\) transverse Wen-plaquette model 1 corresponds to \(N\) decoupled Ising chains.

**Global phase diagram** — From the duality, one can obtain the global phase diagram for the original model by studying the transverse Ising chain. The model in Eq.\(\text{(1)}\) has the same phase diagram to that for the \(d = 1\) transverse Ising model described by Eq.\(\text{(2)}\). Then the original model in Eq.\(\text{(1)}\) has a \(T = 0\) quantum phase transition at \(g_I = \frac{\hbar}{\sqrt{2m}} = 1\) from topological ordered state with \((g_I > 1)\), to a gapped spin-polarized state with \((g_I < 1)\). For large \(g_I\), \(g_I > 1\), the ground state is topological order which is dual to the spin-polarized state in the 1D Ising model with \(\left\langle \tau^z_{i+\frac{1}{2}} \right\rangle \neq 0\). On the other hand, for small \(g_I\), \(g_I < 1\), the spin-polarized state is in a superposition of \(\sigma^x\) eigenstates, with different sites uncorrected. This state corresponds to the spin ordered state in the dual 1D Ising model with a spontaneous magnetization \(\langle \tau^z_{i+\frac{1}{2}} \rangle \neq 0\).

**The scaling law near the QPT** — To obtain the scaling law near the QPT, the above spin-\(\frac{1}{2}\) transverse Ising model in Eq.\(\text{(2)}\) can be described by the following Hamiltonian
\[
\mathcal{H}_I = \hbar \sum_{a,j} \left[ -(c_{a,j} \dagger c_{a,j}) (c_{a,j+1} + c_{a,j+1}) \right] + \left[ g_I (c_{a,j} \dagger c_{a,j}) (c_{a,j} \dagger c_{a,j}) \right]
\]

after employing Jordan-Wigner transformation of the spin operators to spinless fermions \(14\), \(15\).

In the fermionic representation for the transverse Ising chain, the energy spectrum for fermions
\[
E_k = \pm 2h \sqrt{(g_I - \cos(k\tilde{a}))^2 + \sin^2(k\tilde{a})},
\]

has an energy gap \(E_{k=0} = 2h|g_I - 1|\) which tends to zero at \(g_I \rightarrow 1\). So the energy gap for the fermions must tend to zero at the transition \(m = \frac{\hbar}{2|g_I - 1|}\). For large \(m\), the Majorana fermion which changes sign by tuning the mode across the transition; we have chosen \(m > 0\) to correspond to the topological ordered side \(16\).

**The 1D properties for the QPT** — People have guessed that the energy gap for the elementary excitations will always close at the QPT for topological orders. Our results confirm the conjecture. At the critical point, \(g_I = 1\), the energy for the fermion excitation becomes \(E_k = \text{sech} \cdot k\), where the velocity is \(2\tilde{a}\). Since the result is obtained from the dual Ising chain in Eq.\(\text{(2)}\), the energy gap between the ground state and the first excited
state is \( \delta E_k = c \cdot \delta k = c \cdot \frac{2\pi}{L} \) where \( L \) is the length of the Ising chain. The energy gap turns into zero at QPT in the thermodynamic limit. In Ref. (\cite{15,16}) in the thermodynamic limit, it is predicted that the gap between the ground state and the first excited state closes scales as at the critical point \( \delta E(N) = E_1 - E_0 \sim N^{\nu} \), \( (N = N^2) \). On a \( N \times N \) lattice with periodic boundary condition, the original model is dual to \( N \) decoupled Ising chains, then the gap is determined by the length of each Ising chain which indeed scales as \( \delta E \sim N^{n-1} \sim N^{-\frac{L}{2}} \). The Fig. 2 shows the energy gap \( \delta E_k \) at the critical point on different lattices from above results and those from the exact diagonal numerical results. From the results in Fig. 2, the energy gap doesn’t always scale as \( \delta E(N) \sim N^{-\frac{L}{2}} \).

In addition, the correlation function at the QPT shows the 1D characters. The long-distance limit of the static correlator for \( F_i F_j \) of the 2D transverse Wen-plaquette model can be described by the dual correlator for \( \tau_x^{a,i} \tau_x^{a,j} \) of the 1D Ising model in the transverse field. For a diagonal case, \( F_i F_j = F_i F_i \tau_x^{a,i} \tau_x^{a,j} \), the correlation function at the QPT is given as \( \frac{4}{\pi^2} \).

\[
\langle F_i F_{n-i} \rangle = \frac{4}{\pi^2(4n^2 - 1)}. \tag{4}
\]

Here \( m, n \) are integer numbers.

**The non-local order parameters** — Firstly we try to use the expectation value for \( \sigma_i^z \) and \( F_i \) to be the order parameters to describe the QPT. However, \( \langle \sigma_i^z \rangle \) and \( \langle F_i \rangle \) are finite in both phases but discontinue at the QPT. So one cannot use the expectation values of \( \sigma_i^z \) and \( F_i \) to be the order parameters to describe the QPT.

**Instead, we define two kinds of non-local order parameters** : one is \( \phi_1 = \langle \prod_{i=1}^{i_n} \sigma_i^z \rangle \); the other is \( \phi_2 = \langle \prod_{i=1}^{i_n} F_i \rangle \). Here \( i_n = i - n \hat{e}_x + n \hat{e}_y \) represents the sites along diagonal direction. Let’s explain the physics properties of them. On the one hand, \( \prod_{i=1}^{i_n} \sigma_i^x \) is a T1 or T2 open-string operator with two ends, \( i_n = 1 \) and \( i_n = i \).

On the other hand, \( \prod_{i=1}^{i_n} F_i \) is a T1 or T2 close-string operator around the points \( i_n = 1 \) and \( i_n = i \). Fig. 3 shows the relationship between an open-string and a closed-string. The expectation values of (open or close) string operators \( \prod_{i=1}^{i_n} \sigma_i^x \) and \( \prod_{i=1}^{i_n} F_i \) are given by the Fig. 4.

From it one can see that the QPT can be characterized in terms of the two kinds of non-local order parameters.

**The open-string - closed-string duality** — Another important feature for the QPT between the topological order and the non-topological order is the *open-string - closed-string duality*. From the duality relationship, \( \tau_x^{a,i} \tau_x^{a,j} \rightarrow \tau_x^{a,i} + \tau_x^{a,j} \) and \( \tau_x^{a,i} \rightarrow \tau_x^{a,i} + \tau_x^{a,j} \), one can define the first order parameter as \( \phi_1 = \langle \tau_x^{a,1,i+\frac{1}{2},j+\frac{1}{2}} \rangle \) and the second as \( \phi_2 = \langle \prod_{i=1}^{i_n} \tau_x^{a,i+\frac{1}{2},j+\frac{1}{2}} \rangle \) in the dual model. And \( \phi_1 \) is just the local order parameter in dual model which is dual to \( \phi_2 \), a ‘disorder order parameter’ for the transverse Ising chain. This property leads to the duality between open-string and closed string.
phase, it is known that the open-string operator, 
operators [10] of the topological order has a condensation of the closed-
strings are both condensed while the open-strings of them are not. It is consistent to the fact that the ground state 
that breaks down the topological order. The dis-
genenerate two $Z_2$ vortexes (or $Z_2$ charges) at both ends 
of the string. The nonzero expectation value of it means 
the ‘condensation’ of both $Z_2$ vortex and $Z_2$ charge 
for both $Z_2$ vortex and $Z_2$ charge or the condensation of 
$Z_2$ charge and $Z_2$ charge together. These results indicate a new type of QPT between 
a topological ordered phase and a non-topological 
ordered phase.

The authors acknowledge stimulating discussions with 
S. Chen, N.H. Tong. S. P. Kou acknowledges that this 
research is supported by NFSC Grant no. 10574014.

---

* Electronic address: spkou@bnu.edu.cn

[1] X.-G. Wen, Quantum Field Theory of Many-Body Systems, (Oxford Univ. Press, Oxford, 2004).

[2] S. Trebst, P. Werner, M. Troyer, K. Shtengel, C. Nayak, Phys. Rev. Lett. 98, 070602 (2007).

[3] A. Kitaev and J. Preskill, Phys. Rev. Lett. 96, 110404 (2006).

[4] M. Levin and X.-G. Wen, Phys. Rev. Lett. 96, 110405 (2006).

[5] A. Hamma, D. A. Lidar, quant-ph/0607145; A. Hamma, W. Zhang, S. Haas, D.A. Lidar, quant-ph/0705002.

[6] C. Castelnovo and C. Chamon, cond-mat/0707208.

[7] Xiao-Yong Feng, Guang-Ming Zhang, Tao Xiang, Phys. Rev. Lett. 98, 087204 (2007).

[8] X. G. Wen, Phys. Rev. Lett. 90 (2), 016803 (2003).

[9] A. Kitaev, Annals Phys. 303, 2 (2003).

[10] X. G. Wen, Phys. Rev. D 68, 065003 (2003).

[11] Z. Nussinov, G. Ortiz, cond-mat/0702377.

[12] H. D. Chen and J. P. Hu, preprint, cond-mat/0702366.

[13] A small y-direction transverse field leads to a non-local-interchain interaction for the dual $d = 1$ Ising chains. Such relevant interaction couples isolated Ising chains together. And the effects of it will be explored elsewhere.

[14] E. Lieb, T. Schultz, and D. Mattis, Ann. of Phys. 16, 406 (1961); P. Pfeuty, Ann. of Phys. 57, 79 (1970).

[15] B.M. McCoy et. al., Phys. Rev. A 4, 2331 (1971).

[16] B K Chakrabarti; A Dutta; P Sen, Quantum Ising phases and transitions in transverse Ising models (Lecture Notes in Physics, New Series M, Monographs, M41), (Hardcover - April 1996).

[17] E. Fradkin and L. Susskind, Phys. Rev. D 17, 2637 (1978).