Equivariant Principal Bundles over the 2-Sphere

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Abstract

We classify $\Gamma$-equivariant principal $G$ bundles over $S^2$, where $G$ is a compact connected Lie group and $\Gamma \subset SO(3)$ is finite. Restricted over the 1-skeleton of a carefully constructed $\Gamma$-equivariant CW decomposition, the bundle is determined by its restriction to singular points. The extension over $S^2$ is characterized by a Chern class.

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1 Introduction

Let $\Gamma$ and $G$ be Lie groups. A $\Gamma$-equivariant principal $(\Gamma, G)$-bundle $\xi$ over $X$ is locally trivial, principal $G$-bundle $p : E \to X$ such that $E$ and $X$ are left $\Gamma$-spaces. We denote the bundle $\xi = (E, X, p, G, \Gamma)$. The projection map $p$ is $\Gamma$-equivariant and $\gamma(e \cdot g) = (\gamma e) \cdot g$ where $\gamma \in \Gamma$ and $g \in G$ acting on $e \in E$. Equivariant principal bundles and their generalizations were studied by T. E. Stewart [12], T. tom Dieck [3], R. Lashof [7] together with P. May [9] and G. Segal [8]. These authors use homotopy-theoretic methods. There exists a classifying space $B(\Gamma, G)$ for principal $(\Gamma, G)$-bundles [3], so principal $(\Gamma, G)$-bundles over a $\Gamma$-space $X$ are studied by means of $[X, B(\Gamma, G)]_\Gamma$. If the structure group $G$ of the bundle is abelian, the main result of [8] states that equivariant bundles over a $\Gamma$-space $X$ are classified by ordinary homotopy classes of maps $[X \times_\Gamma ET, BG]$. Recently M. K. Kim [6] has classified equivariant complex bundles over 2-sphere by means of CW decompositions of linear actions of $S^2$. Moreover, J. H. Verrette [13] studies equivariant vector bundles over the 2-sphere with effective actions by the rotational symmetries of the tetrahedron, octahedron and icosahedron for verifying Algebraic Realization Conjecture.

Another approach to equivariant principal bundles is given by Hambleton and Hausmann [5]. This approach proceeds the local invariants arising from isotropy representations at singular points of $(X, \Gamma)$. By an isotropy representation at a $\Gamma$ fixed point $x_0 \in X$, we imply the homomorphism $\alpha_{x_0} : \Gamma_{x_0} \to G$ defined by the formula

$$\gamma \cdot e_0 = \alpha(\gamma) \cdot e_0$$
where $e_0 \in p^{-1}(x_0)$. Denote the collection of isotropy representation of the bundle $\xi$ by $\text{Rep}^G(I)$. The homomorphism $\alpha$ is independent of the choice of $e_0$ up to conjugation in $G$. In general, principal $(\Gamma, G)$-bundles over a $\Gamma$ space $X$ cannot be classified by means of their approach however only on specific base spaces (i.e. split $\Gamma$-space).

In this article, we principally rely on the paper of Hambleton and Hausmann to determine $\Gamma$-equivariant principal $G$ bundles over $S^2$. It is proved that there exists a bijection between the equivalence classes of split $\Gamma$-equivariant $G$-bundles over $X$ and $\text{Rep}^G(I)$, provided that $X$ is a split $\Gamma$-space over $A$ [5].

To begin with, the $\Gamma$-equivariant principal $G$-bundles over split $\Gamma$-space $M$ are determined by its isotropy representation. However, it is obvious that $S^2$ is not a split $\Gamma$-space. On the other hand, 1-skeletons of $S^2$ are split $\Gamma$-space.

After determining the $\Gamma$-equivariant principal $G$-bundles over 1-skeletons lying on $S^2$, homotopy theoretic methods are next tool to separate $\Gamma$-equivariant principal $G$-bundles over $S^2$. Let $A^1 \subset S^2$ be a $\Gamma$-equivariant CW-complex, then

$$A^1 \overset{i}{\to} S^2 \overset{j}{\to} S^2 \cup C(A^1) \overset{k}{\to} \Sigma(A^1) \overset{\Sigma i}{\to} \Sigma(S^2) \overset{\Sigma j}{\to} \Sigma(S^2 \cup C(A^1)) \to \cdots$$

is a cofibration sequence where the cone over $A^1$ is denoted by $C(A^1)$ and suspension of $A^1$ is denoted by $\Sigma(A^1)$.

Compute homotopy classes of maps into space $B(\Gamma, G)$, then the following sequence

$$[\Sigma(S^2), Y] \overset{\Sigma i}{\to} [\Sigma(A^1), Y] \overset{k}{\to} [S^2 \cup C(A^1), Y] \overset{j}{\to} [S^2, Y] \overset{i}{\to} [A^1, Y]$$

is a exact sequence of abelian groups arising from homotopy classes of maps into space $B(\Gamma, G)$ since $B(\Gamma, G)=Y = \Omega Z$ is a loop space [2]. Besides, principal $G$-bundles over the 2-sphere are determined by the Steenrod equivalence theorem [11, Th. 18.5]. The only invariant is the ”Chern class” $c(\xi) \in [S^2, BG] = \pi_2(BG) = \pi_1(G)$. This paper is devoted to proving the following theorem.

**Theorem 1.1.** Let $\xi = (E, S^2, p, G, \Gamma)$ be a $\Gamma$-equivariant principal $G$-bundle over $S^2$ with a compact connected Abelian Lie group $G$ and $\Gamma \subset SO(3)$ be a finite subgroup acting on $S^2$. A $\Gamma$-equivariant principal $G$-bundle over $S^2$ is determined by $\text{Rep}^G(I)$ and $c(\xi) \in \pi_2(BG)$.

**Corollary 1.2.** Let $\xi_1$ and $\xi_2$ be equivariant principal $G$-bundles over $S^2$. If $\text{Rep}^G(I_{\xi_1}) \cong \text{Rep}^G(I_{\xi_2})$ then $c(\xi_1) \equiv c(\xi_2) \mod |\Gamma|$. 

2
2 Preliminaries

In this section, some definitions and theorems are presented from the book of T. tom Dieck \[3\]. In this paper, we consider the group action \( \Gamma \) as a left action. Let \( X \) be a topological space and \( \Gamma \) be a topological group acting on \( X \). We consider the isotropy group (stabilizer group) of each point in \( X \) and the orbit space of the space \( X \) under the group action. For each \( x \in X \), the singular points set of \( X \) is denoted by \( \text{Sing}(X, \Gamma) = \{ x \in X | \Gamma_x \neq 1 \} \) if isotropy subgroup of \( x \) is not identity. An action is called transitive if for every \( x_1, x_2 \in X \) there exists an element \( \gamma \in \Gamma \) such that \( \gamma x_1 = x_2 \) and the action is called free if for every \( x \in X \) the only element of \( \Gamma \) (identity) fixes the point \( x \).

A simplicial \( \Gamma \)-complex is regular if for arbitrary \( \gamma_i \in \Gamma \) and two simplices \((v_1, v_2, ..., v_n)\) and \((\gamma_1 \cdot v_1, \gamma_2 \cdot v_2, ..., \gamma_n \cdot v_n)\), there is an element \( \gamma \in \Gamma \) such that \( \gamma v_i = \gamma_i v_i \) for all \( i \). In general, simplicial complexes may not be regular. However, the simplicial complex that is not regular can be constructed as a regular by means of a barycentric subdivision.

The Riemann-Hurwitz Formula is generally the tool of algebraic geometry. In this context, the analogous result of this formula for the graph is the following;

**Proposition 2.1.** Let \( X \) be a compact connected regular simplicial graph, \( \Gamma \) be a finite group and \( \Gamma \times X \to X \) be a regular left transitive group action with the orbit space \( X/\Gamma \cong A \). Then

\[
\chi(X) = |\Gamma|\chi(A) - \sum_{v_i \in X} (|\Gamma_{v_i}| - 1)
\]

**Definition 2.2 (Split \( \Gamma \)-Space).** Let \( \Gamma \) be a Hausdorff topological group and \( A \) be a topological space. A \( \Gamma \)-space is a topological space equipped with a continuous left action of \( \Gamma \). If the space \( X \) is a \( \Gamma \)-space and \( x \in X \), we denote \( \Gamma_x \) as the stabilizer of \( x \). A split \( \Gamma \)-space over \( A \) is denoted by a triple \((X, \pi, \varphi)\) where

- \( X \) is a \( \Gamma \)-space.
- \( \pi: X \to A \) is a continuous surjective map and, for each \( a \in A \), the preimage \( \pi^{-1}(a) \) is a single orbit.
- \( \varphi: A \to X \) is a continuous section of \( \pi \)

In this definition, we might prefer a split \( \Gamma \)-space over the space \( A \) by omitting the notation of maps \( \pi \) and \( \varphi \). Also, the map \( \pi \) induces \( \tilde{\pi}: X/\Gamma \to A \) which is a homeomorphism since \( \varphi \) provides its continuous inverse and \( \Gamma \) is Hausdorff.

2.1 Split Equivariant Principal Bundles

Let \( X \) be a split \( \Gamma \)-space. The isotropy groupoid and representation of \( X \) can be defined arising from the group \( \Gamma \) and the orbit space.
Definition 2.3 (Isotropy Groupoid). Let \((X, \pi, \varphi)\) be a split \(\Gamma\)-space over the orbit space \(A\). The Isotropy groupoid of \(X\) is denoted by

\[ \mathcal{I}(X) := \{(\lambda, a) \in \Gamma \times A | \lambda \in \Gamma_{\varphi(a)}\} \]

The isotropy groupoid of the space \(X\) is the subspace of \(\Gamma \times A\) such that for each \(a \in A\), the space \(\mathcal{I}_a = \mathcal{I} \cap (\Gamma \times \{a\})\) can be written as the form \(\mathcal{I}_a \times \{a\}\), where \(\mathcal{I}_a\) is a closed subgroup of \(\Gamma\). As a short notation, \(\mathcal{I}\) is used for the isotropy representation of split \(\Gamma\)-space of \(X\) with the orbit space \(A\). A groupoid \(\mathcal{I}\) is called locally maximal if each point \(a \in A\) admits a neighbourhood \(U\) such that \(u\) is a subgroup of \(\mathcal{I}_a\) for all \(\mathcal{I}_a \in U\).

We denote by \(A^{(n)}\) the skelata of \(A\), by \(\Omega = \Omega(A)\) the set of cells of \(A\), by \(d(e)\) the dimension of a cell \(e \in \Omega\), and \(\Omega_n = \{e \in \Omega | d(e) = n\}\) for the CW-complex \(A\). Also, we denote by \(e(a) \in \Omega\) the cell \(e\) of the smallest dimension such that \(a \in A\).

Definition 2.4. \(A (\Gamma, \mathcal{A})\) groupoid \(\mathcal{I}\) is called cellular if it is locally maximal and if \(\tilde{\mathcal{I}}_a = \tilde{\mathcal{I}}_b\) when \(e(a) = e(b)\).

Definition 2.5 (Split Bundle). Let \((X, \pi, \varphi)\) be a split \(\Gamma\)-space over \(A\) with isotropy groupoid \(\mathcal{I}\), and let \(\xi = (E, X, p, G, \Gamma)\) be a \(\Gamma\)-equivariant principal \(G\)-bundle over \(X\). Then, the bundle \(\xi\) is called a split bundle if \(\varphi^* \xi\) is trivial.

\(\Gamma\)-equivariant principal \(G\)-bundle is a split bundle if the orbit space \(A\) is a contractible and paracompact space [5].

Definition 2.6 (Isotropy Representation). Let \(\xi = (E, X, p, G, \Gamma)\) be a \(\Gamma\)-equivariant principal \(G\)-bundle over \(X\) and \((X, \pi, \varphi)\) be a split \(\Gamma\)-space over the orbit space \(A\) and \(G\) be a topological group and \(\mathcal{I}\) be a \((\Gamma, \mathcal{A})\) groupoid of \(X\). A continuous representation of \(\mathcal{I}\) is continuous map

\[ \iota: \mathcal{I} \rightarrow G \]

such that the restriction of \(\iota\) to each point \(a \in A\) is group homomorphism from \(\mathcal{I}_a\) to \(G\) and denoted by \(\iota_a: \mathcal{I}_a \rightarrow G\).

A continuous representation of \(\iota: \mathcal{I} \rightarrow G\) is called locally maximal if for each point \(a \in A\), there exists a neighborhood \(U\) such that \(\mathcal{I}_a\) is subgroup of \(\mathcal{I}_a\) for all \(u \in U\) and cellular if \(\iota_a = \iota_b\) when \(e(a) = e(b)\). Moreover, the isotropy groupoid \(\mathcal{I}\) is called weakly locally maximal if there exists a continuous map \(g : U \rightarrow G\) such that \(\alpha_u(\gamma) = g(u)\alpha_a(\gamma)g(u)^{-1}\) for all \(u \in U\) and \(\gamma \in \mathcal{I}_u\).

The isotropy representation of \(\mathcal{I}\) is the continuous groupoid representation of \(\mathcal{I}\) in \(G\). In fact, it is well-defined up to conjugation by \(\text{Map}(A, G)\). The set of conjugacy classes of locally maximal continuous representations of \(\mathcal{I}\) can be denoted

\[ \text{Rep}^G(\mathcal{I}) = \text{Hom}(\mathcal{I}, G)/\text{Map}(A, G). \]

Let \((X, \pi, \varphi)\) be a split \(\Gamma\)-space with isotropy groupoid \(\mathcal{I}\) and \(A\) be an orbit space of \(X\) by group action \(\Gamma\). Suppose \(\xi\) be a split \(\Gamma\)-equivariant principal \(G\)-bundle over the space
X then there exists a continuous lifting \( \tilde{\varphi}^*(\xi) : A \to E \) of \( \varphi \) since \( \varphi^*(\xi) \) is trivial. The equation

\[
\gamma \tilde{\varphi}(a) = \tilde{\varphi}(a)\alpha_a(\gamma),
\]
(valid for \( a \in A \) and \( \gamma \in I_a \)) determines a continuous representation \( \alpha_{\xi,\tilde{\varphi}} : \mathcal{I} \to G \) which does not depend on the choices \( \tilde{\varphi} \) and depends only on the \( \Gamma \)-equivariant isomorphism class of \( \xi \) [5].

In this case, the class of \( \Gamma \)-equivariant principal \( G \)-bundles over Split \( \Gamma \)-space \( X \) is denoted by \( SBun^G_{\Gamma} \). Then, the following theorem states the bijection between this class and isotropy representation of \( I \).

**Theorem 2.7.** [3] Let \( (X, \pi, \varphi) \) be a split \( \Gamma \)-space over the orbit space \( A \) with the isotropy groupoid \( \mathcal{I} \) of \( X \). Assume that \( A \) is locally compact, the group \( \Gamma \) is a compact Lie group, and \( \mathcal{I} \) is locally maximal. Then for any compact connected Lie group \( G \), the map

\[
\Phi : SBun^G_{\Gamma} \to \text{Rep}^G(\mathcal{I})
\]

is a bijection.

Specifically, if the orbit space is contractible, equivariant bundles turn out split. Except for this case, the following proposition is another result for equivariant bundles provided that the structural group \( G \) of the bundle is abelian.

**Proposition 2.8.** [3] Let \( \Gamma \) be a compact Lie group and let \( (X, \pi, \varphi) \) be a split \( \Gamma \)-space over the orbit space \( A \) with the isotropy groupoid \( \mathcal{I} \). Suppose that \( \mathcal{I} \) is locally maximal and that \( A \) is a locally compact space. If the group \( G \) is a compact connected abelian group, then there exists an isomorphism between abelian groups

\[
(\Phi, \varphi^*) : Bun^G_{\Gamma}(X) \to \text{Rep}^G(\mathcal{I}) \times Bun^G(A).
\]

# 3 Equivariant Principal \( G \)-bundles

## 3.1 1-skeletons on \( S^2 \)

The 2-sphere is not a split \( \Gamma \)-space. On the other hand, 1-skeletons lying on the 2-sphere can be constructed as split \( \Gamma \)-space providing that 1-skeletons are regular simplicial \( \Gamma \)-complex.

**Theorem 3.1.** [10] Let \( \Gamma \) be a finite subgroup of \( SO(3) \). Then \( \Gamma \) is isomorphic to precisely one of the following groups:

i) \( \mathbb{Z}_n \), \( (n \geq 1) \): rotational symmetry group of an \( n \)-pyramid

ii) \( D_n \), \( (n \geq 2) \): rotational symmetry group of an \( n \)-prism

iii) \( A_4 \): rotational symmetry group of a regular tetrahedron
iv) $S_4$: rotational symmetry group of a cube (or a regular octahedron)

v) $A_5$: rotational symmetry group of a regular dodecahedron (or a regular icosahedron).

For each finite subgroup of $SO(3)$, we determine $\Gamma$-equivariant 1-skeleton $A^1 \subset S^2$ and the orbit space $A^1/\Gamma \simeq A$.

**Theorem 3.2.** Let $\Gamma \subset SO(3)$ be a finite subgroup. Then, for each subgroup $\Gamma$, there exists a $\Gamma$-equivariant 1-skeleton $A^1 \subset S^2$ such that the 1-skeleton $A^1$ is a split $\Gamma$-space over the orbit space $A$.

**Proof.** For each case, $\Gamma$-equivariant 1-skeletons can be constructed on the $S^2$ such that the Riemannian-Hurwitz formula is satisfied. 1- skeletons for cyclic and dihedral cases are inductively constructed. By this induction, it can be shown that each group $\Gamma$ acting on 1-skeletons yields fixed orbit spaces. For the other subgroups of the $SO(3)$, it can be shown by direct computation for special $\Gamma$-equivariant CW complex $A^1$. Given any subgroup $\Gamma \subset SO(3)$, 1-skeleton $A^1$ and its orbit space $A^1/\Gamma \cong A$ must satisfy Riemann-Hurwitz formula. Then, the 1-skeleton $A^1$ satisfies the condition of being split $\Gamma$-space.

(i) Cyclic Subgroups

Let $C_n$ be a 1-skeleton lying on $S^2$ such that it contains 2 vertices at north and south poles and $n$ edges longitudinal semicircles through the points

$$(\cos(2\pi k/n), \sin(2\pi k/n))$$

joining the poles for $0 \leq k < n$ and $\mathbb{Z}_n$ be a cyclic group with $n$ elements acting on $C_n$.

Let $\mathbb{Z}_n$ be the cyclic group acting on $C_n$, with the orbit space $E_n \cong C_n/\mathbb{Z}_n$. Hence, the CW-complex $C_n$ becomes a split $\mathbb{Z}_n$-space. Then, the Riemann-Hurwitz Formula must be satisfied for the $E_n \cong C_n/\mathbb{Z}_n$;

$$\chi(C_n) = n\chi(E_n) - \sum_{p \in C_n} (|\mathbb{Z}_{n_p}| - 1)$$

where $n_p$ is the order of the isotropy subgroup of the point $p$. Therefore Euler Characteristic;

$$\chi(C_n) = 2 - n \quad \text{and} \quad \chi(E_n) = 1.$$

We claim that

$$\chi(C_{n+1}) = \chi(C_n) - 1 \quad \text{and} \quad \chi(E_{n+1}) = \chi(E_n).$$

After attaching one new edge to the $C_n$, it turns out to be $C_{n+1}$. Conversely, the orbit space $E_{n+1}$ stays the same as the $E_n$.

Let $n = 2$, $\Gamma = \mathbb{Z}_2$, then $\chi(C_2) = 0$, $\chi(E_2) = 1$, and the Riemann-Hurwitz Formula holds.

Suppose $\chi(C_{n+1}) = 1 - n$ since the CW-complex $C_{n+1}$ has 2 vertices and $n + 1$
edges. Therefore, \( \chi(C_{n+1}) = \chi(C_n) - 1 \), similarly \( C_{n+1}/\mathbb{Z}_{n+1} \cong E_{n+1} \) satisfies the Riemann-Hurwitz Formula.

Let \( I \) denote the isotropy groupoid of the CW-complex \( C_n \). It is cellular since it is defined by \( I_0 = I_1 = \mathbb{Z}_n \) and \( I_{01} = id \). Hence, it can be concluded that there is a continuous section from \( E_n \) to \( C_n \).

(ii) Dihedral Subgroups

Let \( \mathcal{D}_n \) be a 1-skeleton lying on \( S^2 \) with \( 2n + 2 \) vertices and \( 6n \) edges. The vertices of \( \mathcal{D}_n \) are as follows:

- The vertices of the \( n \)-gon on the equator
- The middle points of the edges of the \( n \)-gon
- The south and north poles.

In this case, the 1-skeleton turns out to be a regular simplex. The 1-skeleton contains \( 6n \) edges: longitudinal quarter circles through the points

\[
(\cos(\pi k/n), \sin(\pi k/n))
\]

joining the poles for \( 0 \leq k < n \) and transversal edges on the equator at \( \cos(\pi k/n), \sin(\pi k/n) \).

Let \( D_n \) be the dihedral group acting on \( \mathcal{D}_n \) with the orbit space \( \mathcal{D}_n \cong \mathcal{D}_n/D_n \). Hence, the CW-complex \( \mathcal{D}_n \) becomes a split \( D_n \)-space. Inductively, we show that the orbit space \( \mathcal{D}_n \) and the CW-complex \( \mathcal{D}_n \) satisfy the Riemann-Hurwitz Formula;

\[
\chi(\mathcal{D}_n) = 2n\chi(\mathcal{D}_n) - \sum_{p \in \mathcal{D}_n} (|D_{2n}| - 1)
\]

where \( n_p \) is the order of isotropy subgroup of the point \( p \) and hence Euler Characteristic;

\[
\chi(\mathcal{D}_n) = (2n + 2) - 6n = 2 - 4n \text{ and } \chi(\mathcal{D}_n) = 0.
\]

Attaching a vertex to the CW-complex \( \mathcal{D}_n \) yields the new CW-complex \( \mathcal{D}_{n+1} \). Then, there is a relation between the Euler characteristics of these CW-complexes and their orbit spaces as follows;

\[
\chi(\mathcal{D}_{n+1}) = \chi(\mathcal{D}_n) - 4 \text{ and } \chi(\mathcal{D}_{n+1}) = \chi(\mathcal{D}_n).
\]
Let $n = 4$, $\Gamma = D_4$ and $\mathcal{D}_4$ be a 1-skeleton lying on $S^2$ such that it has a square stating on the equator of $S^2$. Let the vertices of the square be labeled 1, 2, 3, 4.

$$\Gamma = D_4 = \{((), (1234), (1423), (1432), (14)(23), (12)(34), (13), (24))\}$$

then

$$\chi(\mathcal{D}_4) = -6 \quad \chi(D_4) = 0$$

and

$$\chi(\mathcal{D}_n) = 2 - 4n$$

$\chi(\mathcal{D}_{n+1}) = -2 - 4n$ with $4n + 6$ vertices and $8n + 8$ edges. The relation between euler characteristics of $\mathcal{D}_{n+1}$ and $\mathcal{D}_{n+1}$ as follows;

$$\chi(\mathcal{D}_{n+1}) = \chi(\mathcal{D}_n) - 4.$$ 

Hence, $\mathcal{D}_{n+1}$ and $\mathcal{D}_{n+1}$ satisfy the Riemann-Hurwitz Formula.

(iii) Tetrahedral Subgroup

Let $\mathcal{T}$ be an 1-skeleton tetrahedron lying on $S^2$. It is necessary to add 6 vertices in the center of edges and 4 vertices in the center of the faces to obtain a regular CW-complex. Hence, $\mathcal{T}$ is regular with 14 vertices and 24 edges.

$$\chi(\mathcal{T}) = -10$$

Let $A_4$ be the tetrahedral group order 12 acting on $\mathcal{T}$ with the orbit space $T \cong \mathcal{T}/A_4$. Hence, the tetrahedron $\mathcal{T}$ has 4 vertex-rotation with the order 3 and 3 edge-rotation with the order 2. Hence, the tetrahedron $\mathcal{T}$ is split $A_4$-space since the Riemann-Hurwitz Formula holds as follows;

$$-10 = 12\chi(T) - \sum_{p \in \mathcal{T}}(|\mathcal{I}_p| - 1)$$

Figure 2: Orbit space of the dihedral group

The isotropy groupoid $\mathcal{I}$ on the orbit space is calculated as $\mathcal{I}_0 = \mathcal{I}_1 = \mathcal{Z}_2$, $\mathcal{I}_2 = \mathcal{Z}_n$ and $\mathcal{I}_{01} = \mathcal{I}_{12} = \mathcal{I}_{02} = id$. Hence, there exists a continuous section from orbit space to $\mathcal{D}_n$.
\[-10 = 12\chi(T) - (4(2 + 2) + 3(1 + 1))\]

\[\chi(T) = 1\]

The orbit space $T \cong \mathcal{G}/A_4$ is the following:

We can say that the isotropy groupoid $\mathcal{I}$ of $\mathcal{G}$ is cellular and it is given by $\mathcal{I}_0 = \mathbb{Z}_2$ $\mathcal{I}_1 = \mathbb{Z}_3$ and $\mathcal{I}_{01} = \text{id}$. Hence, we can find a continuous section from orbit space to $\mathcal{G}$.

(iv) Octahedral Subgroup

Let $C$ be a 1-skeleton cube lying on $S^2$. Firstly, the cube $C$ has 8 vertices and 12 edges. It is necessary extra 12 vertices in the center of edges and 6 vertices in the center of faces. Hence, the cube $C$ is regular with 26 vertices.

\[\chi(C) = -22\]

Let $S_4$ be an octahedral group order 24 acting on the cube $C$ with the orbit space $O \cong C/S_4$. The cube $C$ has 4 vertex-rotation of order 3, 6 edge-rotation of order 2, and 3 face-rotation of order 4. Therefore, the orbit space $O$ is composed of 3 vertices and 2 edges. The cube $C$ is a split $S_4$-space since the Riemann-Hurwitz Formula holds as follows;

\[\chi(C) = 24\chi(O) - \sum_{p \in C} (|C_p| - 1)\]

\[-22 = 24\chi(O) - (3(3 + 3) + 4(2 + 2) + 6(1 + 1))\]

\[\chi(O) = 1\]
We can say that the isotropy groupoid $\mathcal{I}$ of the cube $\mathcal{C}$ is cellular since it is given by $\mathcal{I}_0 = \mathbb{Z}_2$, $\mathcal{I}_1 = \mathbb{Z}_3$, $\mathcal{I}_2 = \mathbb{Z}_4$, $\mathcal{I}_{01} = \text{id}$ and $\mathcal{I}_{12} = \text{id}$. Hence, we can construct a continuous section from orbit space to $\mathcal{C}$.

(v) Icosahedral Subgroup

Let $\mathcal{O}$ be an 1-skeleton Icosahedron lying on $S^2$. Firstly, the Icosahedron $\mathcal{O}$ has 12 vertices and 30 edges. By adding 30 vertices in the center of edges and 20 vertices in the center of faces, the regular Icosahedron $\mathcal{O}$ has 62 vertices and 120 edges.

$$\chi(\mathcal{O}) = -58$$

Let $A_5$ be an icosahedral group order 60 acting on $\mathcal{O}$ with the orbit space $I \cong \mathcal{O}/A_5$. The Icosahedron $\mathcal{O}$ has 6 vertex-rotation of order 5, 15 edge-rotation of order 2, and 10 face-rotation of order 3. Then, the orbit space $I$ is composed of 3 vertices and 2 edges. Therefore, the Icosahedron $I$ is a split $A_5$-space since it satisfies the Riemann-Hurwitz Formula at proposition 2.1

$$\chi(\mathcal{O}) = 60\chi(I) - \sum_{p \in \mathcal{O}} (|\mathcal{O}_p| - 1)$$

$$-22 = 60\chi(I) - (6(4 + 4) + 15(2 + 2) + 10(1 + 1))$$

$$\chi(I) = 1$$

we can say that the isotropy groupoid $\mathcal{I}$ of the Icosahedron $\mathcal{O}$ is cellular and it is given by $\mathcal{I}_0 = \mathbb{Z}_2$, $\mathcal{I}_1 = \mathbb{Z}_3$, $\mathcal{I}_2 = \mathbb{Z}_5$, $\mathcal{I}_{01} = \text{id}$ and $\mathcal{I}_{12} = \text{id}$. Then we can construct a continuous section from orbit space to $\mathcal{O}$.

Therefore, for each finite subgroup $\Gamma \subset SO(3)$, we construct $\Gamma$-equivariant split 1-skeleton lying on $S^2$.

Now, we show that for each finite subgroup $\Gamma \subset SO(3)$, a $\Gamma$-equivariant 1-skeleton CW-complex $\mathcal{A}^1$ on $S^2$ can be constructed such that the CW-complex $\mathcal{A}^1$ splits over the orbit space $\mathcal{A}$. 

\square
3.2 Classification of G-bundles Over 1-Skeletons On $S^2$

The class of isotropy representations, $Rep^G(\mathcal{I})$, can be quite complex. However, by constructing specific regular 1-skeletons of $S^2$ for each subgroup of $SO(3)$, one can obtain a split $\Gamma$-space. The orbit spaces of cyclic groups, the tetrahedral group, the octahedral group, and the icosahedral group are homeomorphic to the interval $[-1,1]$. Additionally, every $\Gamma$-equivariant principal $G$-bundle can be considered a split $\Gamma$-space, where the orbit space $A$ is both contractible and paracompact. In the case of the dihedral group, the orbit space $A$ is a triangle (i.e. graph), and there exists a bijection between the split bundle space of the CW-complex $A^1$ and the isotropy representation of groupoid $\mathcal{I}$. When the group $G$ is abelian, there is an isomorphism between the bundle spaces of $A^1$ and $Rep^G(\mathcal{I}) \times Bun^G(A)$.

We can deduce that all the orbit spaces are paracompact, as they are all compact. As a result, for any finite subgroup $\Gamma \subset SO(3)$, the orbit spaces can be classified as either contractible or non-contractible.

3.2.1 Contractible Case

All of the equivariant bundles over the space $A^1$ are split bundles, as the orbit space of $A^1$ is both contractible and paracompact. The specific spaces $(\ast, \pi, \varphi)$ can be split over their orbit spaces, as stated in Theorem 2.7. Additionally, all possible orbit spaces are locally compact, and the possible isotropy groupoids are locally maximal. Since the group $\Gamma$ is a compact Lie group, the map

$$\Phi : SBun^G_\Gamma \rightarrow Rep^G(\mathcal{I})$$

is a bijection.

3.2.2 Non-contractible Case

In the same way, the CW-complex $\mathcal{D}_n$ splits over the orbit space $\mathcal{D}_n$ and the map $$ is a bijection. However, it should be noted that not all equivariant principal $G$ bundles are split over the space $A^1$. To handle the non-split bundles over the space $A^1$, we restrict $G$ to be abelian in order to meet the requirement of Proposition 2.8. The space $(\mathcal{D}_n, \pi, \varphi)$ is a split $D_n$-space over $\mathcal{D}_n$ with the isotropy groupoid $\mathcal{I}$, and $\mathcal{D}_n$ is locally compact and $\mathcal{I}$ is locally maximal. Therefore, for any abelian Lie group $G$, the following map is an isomorphism:

$$\Phi, \varphi^* : Bun^G_\Gamma(A^1) \rightarrow Rep^G(\mathcal{I}) \times Bun^G(A).$$

Since the orbit space $\mathcal{D}_n$ is a triangle and is homeomorphic to $S^1$, principal $G$-bundles over $S^1$ can be induced by the following map

$$\Phi, \varphi^* : Bun^G_\Gamma(A^1) \cong Rep^G(\mathcal{I}) \times Bun^G(S^1).$$

In conclusion, for a connected compact Lie group $G$, the following holds:

$$[S^1, BG] \cong \pi_1(BG) \cong \pi_0(G) \cong 0.$$
In the case of a disconnected group \( G \), the analogy is as follows:

\[
[A^1, S^1] \cong \pi_1(BG) \cong \pi_0(G)
\]

provided that group \( G \) is compact.

4 Calculation of \( \text{Rep}^G(\mathcal{I}) \)

Let \( \iota : \mathcal{I} \to G \) be an isotropy representation and let \( \mathcal{I} \) be a \((\Gamma, A)\)-groupoid. For each \( e \in \Omega(A) \), this defines \( \text{Hom}(\mathcal{I}_e, G) \) with face compatibility conditions \( \iota_e = \iota_f \mid \mathcal{I}_e \) whenever \( f \leq e \). The set of conjugacy classes of cellular representations of \( \mathcal{I} \) into \( G \) is denoted by \( \text{Rep}_{\text{cell}}^G(\mathcal{I}) \).

For a cellular representation \( \iota : \mathcal{I} \to G \) and a cell \( e \in \Omega(A) \), the associated conjugacy class of \( \iota \) is denoted by \( [\iota_e] \in \overline{\text{Hom}}(\mathcal{I}, G) \). Hence, the following map \( \beta \) is well-defined:

\[
\beta : \text{Rep}_{\text{cell}}^G(\mathcal{I}) \to \prod_{e \in \Omega(A)} \overline{\text{Hom}}(\mathcal{I}_e, G).
\]

For each \( b_e \in \prod_{e \in \Omega(A)} \overline{\text{Hom}}(\mathcal{I}_e, G) \), faces compatible to each other. Then, we define

\[
\overline{\text{Rep}_{\text{cell}}^G(\mathcal{I})} = \{(b_e) \in \prod_{e \in \Omega(A)} \mid b_e = b_f \mid \mathcal{I}_e \text{ if } f \leq e \}
\]

and we can replace \( \beta \) as a map \( \tilde{\beta} : \text{Rep}_{\text{cell}}^G(\mathcal{I}) \to \overline{\text{Rep}_{\text{cell}}^G(\mathcal{I})} \). Then the diagram is commutative,

\[
\begin{array}{ccc}
\text{Rep}_{\text{cell}}^G(\mathcal{I}) & \xrightarrow{\tau} & \text{Rep}^G(\mathcal{I}) \\
\tilde{\beta} \downarrow & & \downarrow \nu \\
\overline{\text{Rep}_{\text{cell}}^G(\mathcal{I})} & & \text{Rep}_{\text{cell}}^G(\mathcal{I})
\end{array}
\]

when \( \mathcal{I} \) is proper \((\Gamma, A)\)-groupoid.

The map \( \tau \) is obvious since a cellular representation is a representation, which is locally maximal. To define \( \nu(\beta)e \) for \( e \in \Omega(A) \), we choose \( a \in A \) with \( e(a) = e \) and set \( \nu(\beta)e = [\beta a] \). Since cells are connected, \( \nu \) is well defined. On the other hand, none of these maps is either surjective or injective in general.

**Theorem 4.1.** Let \( A^1 \subset S^2 \) and \( G \) be a topological group. Let \( \Gamma \subset SO(3) \), \( A = A^1/\Gamma \) be an orbit space and \( \mathcal{I} \) be a \((\Gamma, A)\)-groupoid. Then \( \tilde{\beta} : \text{Rep}_{\text{cell}}^G(\mathcal{I}) \to \overline{\text{Rep}_{\text{cell}}^G(\mathcal{I})} \) is surjective.

**Proof.** For a finite subgroup \( \Gamma \subset SO(3) \), the orbit spaces of subgroups except dihedral group are tree. Hambleton and Hausmann prove this theorem provided that the orbit space is tree [1]. Now, we only prove the case for the dihedral group. Let \( b \in \overline{\text{Rep}_{\text{cell}}^G(\mathcal{I})} \) and let \( v \) be a vertex of \( A \). We choose \( \iota_v \in \text{Hom}(\mathcal{I}_v, G) \) representing \( b_v \). For an edge
e between v and v’ we define $\iota_e \in \text{Hom}(I_e, G)$ by $\iota_e = \iota_v|I_e$. Since $b \in \text{Rep}_{\text{cell}}^G(\mathcal{I})$, we choose $\iota_{v'} \in \text{Hom}(I_{v'}, G)$ where $\iota_{v'} = \iota_e$. Therefore, we define a cellular representation $\iota_{v,1}$ over the tree $A(v,1)$ of the points of distance smaller than or equal to 1 far from v. We construct $\iota_{v,2}$ over $A(v,2)$ with the same way. Hence, we choose the points of distances smaller than 3 far from v, when $A(v,3)$ is defined. Now, this defines $\iota \in \text{Rep}_{\text{cell}}^G(\mathcal{I})$ with $\tilde{\beta}(\iota) = b$.

Proposition 4.2. [5] Let $\mathcal{I}$ be a $(\Gamma, A)$-groupoid, where $\Gamma \subset SO(3)$ and the orbit space $A$ is a graph. Let $G$ be a path-connected topological group. Then $v : \text{Rep}^G(\mathcal{I}) \to \text{Rep}_{\text{cell}}^G(\mathcal{I})$ is surjective.

Theorem 4.3. Let $\mathcal{I}$ be a proper $(\Gamma, A)$-groupoid with $\Gamma \subset SO(3)$ a finite topological group and let $A$ be an orbit space. Let $G$ be a compact connected Lie group. Then $\tau : \text{Rep}_{\text{cell}}^G(\mathcal{I}) \to \text{Rep}^G(\mathcal{I})$ is bijective.

Proof. Let $\tau(\alpha) = \tau(\alpha')$ with the two cellular representations. One can see $\alpha = \alpha'$ by taking the conjugate of the one of them. The map $\tau$ is surjective. Suppose that $a \in \text{Rep}^G(\mathcal{I})$, it turns out to be a cellular representation. The reason is that isotropy groups are identity except vertices [4].

5 Equivariant Bundles on 1-skeletons

The space of equivariant bundles $\text{Bun}_{\mathbb{Z}_n}^G(\mathcal{A}^1)$ is classified by means of $\text{Rep}^G(\mathcal{I})$ with (1) and (2). $\text{Rep}^G(\mathcal{I})$ is calculated for each finite subgroups of $SO(3)$.

Theorem 5.1. Let $\mathcal{C}_n$ be a $\mathbb{Z}_n$-equivariant 1-skeleton over $S^2$ with 2 vertices and $n$ edges, $\mathbb{Z}_n$ acting on $\mathcal{C}_n$ be a cyclic group with the order $n$. Let $E_n$ be the orbit space of $\mathcal{C}_n$ under the group action of $\mathbb{Z}_n$ with isotropy groupoid $\mathcal{I}_1$. Then, the following map

$$\text{Bun}_{\mathbb{Z}_n}^G(\mathcal{C}_n) \to \text{Rep}^G(\mathcal{I}_1)$$

is a bijection and

$$\text{Rep}^G(\mathcal{I}_1) \cong \overline{\text{Rep}}^G(\mathcal{I}_1) \cong \overline{\text{Hom}}(\mathbb{Z}_n, G) \times \overline{\text{Hom}}(\mathbb{Z}_n, G).$$

Proof. $\mathcal{C}_n$ is a split $\mathbb{Z}_n$-space, all equivariant bundles are split bundles. □

Theorem 5.2. Let $\mathcal{D}_n$ be a $D_n$-equivariant 1-skeleton over $S^2$ with $2n + 2$ vertices, $6n$ edges. Let $D_n$ acting on the CW-complex $\mathcal{D}_n$ be a dihedral group with the order $2n$. Let $\mathcal{D}_n$ be an orbit space under the group action $D_n$ with isotropy groupoid $\mathcal{I}_2$. If $G$ is connected, then there is a bijection

$$\text{Bun}_{D_n}^G(\mathcal{D}_n) \to \text{Rep}^G(\mathcal{I}_2)$$

and

$$\text{Rep}^G(\mathcal{I}_2) \cong \overline{\text{Rep}}^G(\mathcal{I}_2) \cong \overline{\text{Hom}}(\mathbb{Z}_2, G) \times \overline{\text{Hom}}(\mathbb{Z}_2, G) \times \overline{\text{Hom}}(\mathbb{Z}_n, G).$$

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Proof. \( \mathcal{D}_n \) is split \( D_n \)-space and
\[
\text{Bun}^G_1(A^1) \to \text{Rep}^G(I_2) \times \text{Bun}^G(A)
\]
since the group \( G \) is a connected compact Lie group, it follows
\[
[S^1, A^1] \cong \pi_1(BG) \cong \pi_0(G) \cong 0.
\]

**Theorem 5.3.** Let \( \mathcal{T} \) be a \( A_4 \)-equivariant tetrahedron and \( A_4 \) acting on \( \mathcal{T} \) be the tetrahedral group with the order 12. Let \( T \) be an orbit space of \( \mathcal{T} \) under the group action \( A_4 \) with isotropy groupoid \( I_3 \). Then, there is a bijection
\[
\text{Bun}^G_{A_4}(\mathcal{T}) \to \text{Rep}^G(I_3)
\]
and
\[
\text{Rep}^G(I_3) \cong \overline{\text{Rep}}^G(I_3) \cong \overline{\text{Hom}}(\mathbb{Z}_2, G) \times \overline{\text{Hom}}(\mathbb{Z}_3, G).
\]

Proof. \( \mathcal{T} \) is a split \( A_4 \)-space, all equivariant bundles are split bundle.

**Theorem 5.4.** Let \( \mathcal{C} \) be a \( S_4 \)-equivariant cube and \( S_4 \) acting on the cube \( \mathcal{C} \) be the octahedral group with the order 24. Let \( O \) be the orbit space of \( \mathcal{C} \) under the group action \( S_4 \) with isotropy groupoid \( I_4 \). Then, there is a bijection
\[
\text{Bun}^G_{S_4}(\mathcal{C}) \to \text{Rep}^G(I_4)
\]
and
\[
\text{Rep}^G(I_4) \cong \overline{\text{Rep}}^G(I_4) \cong \overline{\text{Hom}}(\mathbb{Z}_2, G) \times \overline{\text{Hom}}(\mathbb{Z}_3, G) \times \overline{\text{Hom}}(\mathbb{Z}_4, G).
\]

Proof. \( \mathcal{C} \) is a split \( S_4 \)-space, all equivariant bundles are split bundle.

**Theorem 5.5.** Let \( \mathcal{O} \) be an \( A_5 \)-equivariant icosahedron and \( A_5 \) acting on \( \mathcal{O} \) be an icosahedral group with the order 60. Let \( I \) be an orbit space of \( \mathcal{O} \) under the group action \( A_5 \) with isotropy groupoid \( I_5 \). Then, there is a bijection
\[
\text{Bun}^G_{A_5}(\mathcal{O}) \to \text{Rep}^G(I_5)
\]
and
\[
\text{Rep}^G(I_5) \cong \overline{\text{Rep}}^G(I_5) \cong \overline{\text{Hom}}(\mathbb{Z}_3, G) \times \overline{\text{Hom}}(\mathbb{Z}_4, G) \times \overline{\text{Hom}}(\mathbb{Z}_5, G).
\]

Proof. \( \mathcal{O} \) is a split \( A_5 \)-space, all equivariant bundles are split bundle.
6 \( \Gamma-G \) Bundles over \( S^2 \)

Let \( \mathcal{A}^1 \subset S^2 \) be a \( \Gamma \)-equivariant 1-skeleton. Hambleton and Hausmann \[4\] provide an isomorphism between \( \Gamma \)-equivariant principal \( G \)-bundles over \( \mathcal{A}^1 \) and the class of isotropy representations. Additionally, it is shown that a \( \Gamma \)-equivariant 1-skeleton \( \mathcal{A}^1 \) is a split \( \Gamma \)-space in Theorem 3.2. On the other hand, we can construct a cofibration sequence derived from the inclusion map \( i : \mathcal{A}^1 \to S^2 \) to determine \( \Gamma \)-equivariant principal \( G \)-bundles over \( S^2 \), since the 2-sphere is not \( \Gamma \)-equivariant.

Then, the following sequence

\[
\mathcal{A}^1 \xrightarrow{i} S^2 \xrightarrow{j} S^2 \cup C(\mathcal{A}^1) \xrightarrow{k} \Sigma(\mathcal{A}^1) \xrightarrow{\Sigma i} \Sigma(S^2) \xrightarrow{\Sigma j} \Sigma(S^2 \cup C(\mathcal{A}^1)) \to \cdots
\]  

(3)

is the cofibration of \( \Gamma \)-equivariant \( CW \)-complexes where a cone

\[
C(\mathcal{A}^1) = (\mathcal{A}^1 \times [0, 1]) / \{(a, 0) \sim \text{single point}\}
\]

and the suspension

\[
\Sigma(\mathcal{A}^1) = (\mathcal{A}^1 \times [-1, 1]) / \{(a, -1) \sim \text{single point}, (a, 1) \sim \text{single point}\}.
\]

The \( \Gamma \)-fixed set of homotopy classes maps into the space \( B(\Gamma, G) \), then the following sequence

\[
[S^2, Y]_{\Gamma} \xrightarrow{\Sigma i_*} [\Sigma(\mathcal{A}^1), Y]_{\Gamma} \xrightarrow{k^*} [S^2 \cup C(\mathcal{A}^1), Y]_{\Gamma} \xrightarrow{j_*} [S^2, Y]_{\Gamma} \xrightarrow{i_*} [\mathcal{A}^1, Y]_{\Gamma}
\]  

(4)

is the exact sequence of abelian groups provided that \( B(\Gamma, G) = Y = \Omega Z \) is a loop space which is defined by Costenoble and Waner \[2\]. Now, \( [S^2, Y] \) is determined by \( j_* \) and \( i_* \).

Topologically, we have homomorphisms such that \( S^2 \cup C(\mathcal{A}^1) \simeq \bigvee S^2 \) (induced from 2-cells) and \( \Sigma(\mathcal{A}^1) \simeq \bigvee S^2 \) (induced from 1-cells). Then the exact sequence at (4) turns out to be the following sequence;

\[
\bigvee_{1-\text{cells}} S^2, Y \xrightarrow{k_*} \bigvee_{2-\text{cells}} S^2, Y \xrightarrow{j_*} [S^2, Y].
\]

\[
[\bigvee_{S^2, Y}]_{\Gamma \neq D_n} \simeq \bigoplus_{1-\chi(\mathcal{A}^1)} \pi_2(BG) \text{ and } \Gamma \text{ acts on product } = I \otimes \pi_1(G)
\]  

(5)

as a \( \Gamma \)-module, or

\[
[\bigvee_{S^2, Y}]_{\Gamma = D_n} \simeq \bigoplus_{1-\chi(\mathcal{A}^1)} \pi_2(BG) \text{ and } \Gamma \text{ acts on product } = (I \oplus \mathbb{Z}_\Gamma) \otimes \pi_1(G)
\]  

(6)

as a \( \Gamma \)-module then, we have

\[
[\bigvee_{2-\text{cells}} S^2, Y]_{\Gamma} \simeq \bigoplus_N \pi_2(BG) \text{ and } \Gamma \text{ acts on product } = \mathbb{Z}_\Gamma \otimes \pi_1(G)
\]  

(7)
as a $\Gamma$-module provided that the ideal $I = \mathbb{Z}\{(\gamma - 1) | \gamma \in \Gamma\} \subset \mathbb{Z}\Gamma$. The number of copy of $\pi_2(BG)$ at the equation (5) is calculated by means of counting rotations and order of groups.

Let $A^1 \subset S^2$ be a $\Gamma$-equivariant 1-skeleton. For the cyclic group $\mathbb{Z}_n$, an equivariant 1-skeleton on $S^2$ is constructed by two vertices and $n$ edges. By collapsing an edge to a point, the other edges turn out to be circles. Therefore, we obtain $(n - 1)$ circles.

For the dihedral group $D_n$, there is an orbit with $2n$ elements. By shrinking an edge to a point, we obtain $(2n - 1)$ circles. The other $2n$ orbits with $2$ elements are reduced to $(2n)$ circles, by collapsing an edge to a point. Therefore, $(4n - 1)$ circles are formed from the group action of the dihedral group.

For the tetrahedral group $A_4$, there are $4$ vertex rotations with order $3$. By collapsing an edge to a point for each rotation, we obtain $8$ circles. There are $3$ edge rotations with order $2$ that are reduced to $3$ circles. In total, we have $11$ circles for the tetrahedral group.

For the octahedral group $S_4$, there are $4$ vertex rotations with order $3$. After collapsing one edge to a point for each rotation, we obtain $8$ circles. There are $6$ edge rotations with order $2$, which yield $6$ circles. There are $3$ face rotations with order $4$, which yield $9$ circles. In total, we have $23$ circles for the octahedral group.

Finally, for the icosahedral group $A_5$, we count rotations and orders using the same method and obtain $59$ circles.

Briefly, we notice that the number of circles for each subgroup $\Gamma$ acting on $A^1$ is $1 - \chi(A^1)$ for each $\Gamma$-equivariant $A^1 \subset S^2$.

Let $N$ denote the number of copies of $\pi_2(BG)$ at (7). It depends on the number of orbits and the order of group $\Gamma$. Except for the dihedral group, finite subgroups of $SO(3)$ have a single orbit. We summarize this in the following table.

| Group             | $1 - \chi(A^1)$ | $N$ |
|-------------------|-----------------|-----|
| Cyclic group      | $n - 1$         | $n$ |
| Dihedral group    | $4n - 1$        | $4n$|
| Tetrahedral group | $11$            | $12$|
| Octahedral group  | $23$            | $24$|
| Icosahedral group | $59$            | $60$|

Table 1: The number of copy of $\pi_2(BG)$.

Since $k^*$ is an injective map, depending on $\Gamma$ the following maps hold;

$0 \to I \otimes \pi_1(G) \xrightarrow{k^*} \mathbb{Z}\Gamma \otimes \pi_1(G) \xrightarrow{j^*} \mathbb{Z} \otimes \pi_1(G)$

or

$0 \to (I \otimes \mathbb{Z}\Gamma) \otimes \pi_1(G) \xrightarrow{k^*} (\mathbb{Z}\Gamma \otimes \mathbb{Z}\Gamma) \otimes \pi_1(G) \xrightarrow{j^*} \mathbb{Z} \otimes \pi_1(G)$

and $\text{Coker}(k^*) \simeq \mathbb{Z} \otimes \pi_1(G) \simeq [S^2, B(\Gamma, G)]\Gamma$.

Now, let $Z$ and $Y$ be two $\Gamma$-space and $f : Z \to Y$ be continuous. Define $f^\gamma(z) = \gamma^{-1}f(\gamma z)$. The map $f \to f^\gamma$ gives an action of $\Gamma$ on $[Z, Y]$. Then $f = f^\gamma \iff \gamma f(z) = f(\gamma z) \iff f$ is a $\Gamma$-map. Therefore, we shall say the following $[Z, Y]_\Gamma = \text{Fix}(\Gamma, [Z, Y])$.
$\text{Fix}(\Gamma, I) = \{x \in I \mid \gamma x = x\} = 0$ and we determine $\text{Fix}(\Gamma, Z\Gamma)$. Let $t \in \Gamma$ be a generator. \(\Gamma\) is acting on \(Z\Gamma = Z \oplus Zt \oplus \cdots \oplus Zt^{n-1}\) where \(|\Gamma| = n\). The fixed set of \(Z\Gamma\) can be determined by \((a_0 + a_1 t + \cdots + a_{n-1}t^{n-1})\gamma = (a_0 + a_1 t + \cdots + a_{n-1}t^{n-1})\) for all \(\gamma \in \Gamma\) implies that fixed elements are \(Z(1 + t + \cdots + t^{n-1})\). Then, \(\text{Fix}(\Gamma, Z\Gamma) = Z(1\sum_{\gamma \in \Gamma} \gamma)\). Therefore,

\[
[\Sigma(\mathcal{A}^1), Y]_\Gamma = \text{Fix}(\Gamma, I \otimes \pi_2 Y) = 0
\]

and

\[
[S^2 \cup C(\mathcal{A}^1), Y]_\Gamma = \text{Fix}(\Gamma, Z\Gamma \otimes \pi_2 Y) = \pi_2(Y),
\]

since we have \([S^2, Y] = \pi_2(Y)\) in the sequence (4).

**Theorem 6.1.** Let \(\xi = (E, S^2, p, G, \Gamma)\) be a \(\Gamma\)-equivariant principal \(G\)-bundle over \(S^2\) with a compact connected Abelian Lie group \(G\) and \(\Gamma \subset SO(3)\) be a finite subgroup acting on \(S^2\). A \(\Gamma\)-equivariant principal \(G\)-bundle over \(S^2\) is determined by \(\text{Rep}^G(I)\) and \(c(\xi) \in \pi_2(BG)\).

**Proof.** Let \([\nu], [\xi] \in [S^2, Y]_\Gamma\), \([S^2, Y]_\Gamma \xrightarrow{\nu} [\mathcal{A}^1, Y]_\Gamma\) and \([\mathcal{A}^1, Y]_\Gamma \cong \text{Rep}^G(I)\). If \(\text{Rep}^G(I_{\nu}) \not\cong \text{Rep}^G(I_{\xi})\) then one concludes that they are non-equivariant to each other. If \(\text{Rep}^G(I_{\nu}) \cong \text{Rep}^G(I_{\xi})\) then

\[
[\Sigma(\mathcal{A}^1), Y]_\Gamma \to [S^2 \cup C(\mathcal{A}^1), Y]_\Gamma \to [S^2, Y]_\Gamma
\]

and by (8) and (9)

\[
0 \to \pi_2(BG) \xrightarrow{[\nu]} [S^2, Y]_\Gamma
\]

then this map is reduced to the following congruence. \(\square\)

**Corollary 6.2.** If \(\text{Rep}^G(I_{\xi_1}) \cong \text{Rep}^G(I_{\xi_2})\) then \(c(\xi_1) \equiv c(\xi_2)\) mod \(|\Gamma|\).

This theorem completes the classification of equivariant principal bundles over the 2-sphere. Future studies will be focusing on how we can apply this theorem to the product space \(S^2 \times S^2\) by these ideas.

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