A solution to the non-linear equations of $D=10$ super Yang–Mills theory

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In this letter we demonstrate that these off-shell modifications (OPEs) can be resummed to capture the non-linearities in the superfields on-shell, and Bianchi identities lead to the non-linear equations of motion [8],

$$\{\nabla_\alpha, \nabla_\beta\} - \gamma_m^{\alpha\beta} \nabla_m, \quad \mathbb{F}_{mn} \equiv -[\nabla_m, \nabla_n]. \quad (3)$$

One can show that the constraint equation $\mathbb{F}_{\alpha\beta} = 0$ puts the superfields on-shell, and Bianchi identities lead to the non-linear equations of motion [8],

$$\{\nabla_\alpha, \nabla_\beta\} = \gamma_m^{\alpha\beta} \nabla_m$$

$$\left[\nabla_\alpha, \nabla_m\right] = - (\gamma_m \mathbb{W})_\alpha$$

$$\left[\nabla_\alpha, \mathbb{W}^m\right] = \frac{1}{4} (\gamma_m^{\alpha\beta}) \mathbb{F}_{mn}$$

$$\left[\nabla_\alpha, \mathbb{F}^{mn}\right] = \left[\nabla^m, (\gamma^n) \mathbb{W}\right]. \quad (4)$$

In the subsequent, we will construct an explicit solution for the superfields $A_\alpha, \mathbb{W}^\alpha$ and $\mathbb{F}^{mn}$ in (4).

**LINEARIZED MULTIPARTICLE SUPERFIELDS**

In perturbation theory, it is conventional to study solutions $A_\alpha, A_m, \ldots$ of the linearized equations of motion

$$\left\{D_\alpha, A_\beta\right\} = \gamma_m^{\alpha\beta} A_m$$

$$[D_\alpha, A_m] = k_m A_\alpha + (\gamma_m W)_\alpha$$

$$\left\{D_\alpha, W_\beta\right\} = \frac{1}{4} (\gamma_m^{\alpha\beta}) F_{mn}$$

$$[D_\alpha, F^{mn}] = k^m (\gamma^n) W_\alpha. \quad (5)$$

Their dependence on the bosonic coordinates $x$ is described by plane waves $e^{ikx}$ with on-shell momentum $k^2 = 0$. The $\theta$ dependence is known in terms of fermionic power series expansions from [9, 10] whose coefficients contain gluon polarizations and gluino wave functions.
As an efficient tool to determine and compactly represent scattering amplitudes in SYM and string theory, multiparticle versions of the linearized superfields have been constructed in [4]. They satisfy systematic modifications of the linearized equations of motion (5), and their significance for BRST invariance was pointed out in [11]. For example, their two-particle version

\[ A^{12}_\alpha = -\frac{1}{2} [A^{\alpha}_m (k^1 \cdot A^2) + A^{\alpha}_m (\gamma^m W^2) - (1 \leftrightarrow 2)] \]

\[ A^{12}_m = \frac{1}{2} [A^{\alpha}_m F_{mn} + A^{\alpha}_m (k^1 \cdot A^2) + (W_1 \gamma_m W^2) - (1 \leftrightarrow 2)] \]

\[ W^{12}_\alpha = \frac{1}{2} (\gamma^m W^2) \alpha F_{mn}^{12} + W_\alpha^2 (k^2 \cdot A^1) - (1 \leftrightarrow 2) \]  

(6)

\[ F^{12}_{mn} = F^{22}_{mn} (k^2 \cdot A^1) + \frac{1}{2} F^{22}_{mn} F^{12}_{np} \]

\[ + k^1 (W_1 \gamma_n W^2) - (1 \leftrightarrow 2) \]

can be checked via (5) to satisfy

\[ D_{(\alpha A^{12}_\beta)} = \gamma^m_{\alpha \beta} A^{12}_m + (k^1 \cdot k^2) (A^1_\alpha A^2_\beta + A^2_\alpha A^1_\beta) \]

\[ D_{\alpha A^{12}_m} = (\gamma^m W^{12})_\alpha + k^{12} A^{12}_m + (k^1 \cdot k^2) (A^1_\alpha A^2_m - A^2_\alpha A^1_m) \]

\[ D_{\alpha W^{12}_\beta} = \frac{1}{2} (\gamma^m W^{12})_\beta F_{mn}^{12} + (k^1 \cdot k^2) (A^1_\alpha W^{22}_m - A^2_\alpha W^{12}_m) \]

\[ D_{\alpha F_{mn}^{12}} = k^{12}_\alpha (\gamma^m W^{12})_\alpha - k^{12} (\gamma^m W^{12})_\alpha \]

\[ + (k^1 \cdot k^2) (A^1_\alpha F_{mn}^{12} + A^2_\alpha (\gamma^m W^2) - (1 \leftrightarrow 2)) \]  

The modifications as compared to the single-particle equations of motion (5) involve the overall momentum \( k_{12} \equiv k_1 + k_2 \) whose inverse propagator is generically off-shell, \( k^2_{12} = 2(k_1 \cdot k_2) \neq 0 \).

The construction of the two-particle superfields in (6) is guided by the OPEs among integrated vertex operators of the gluon multiplet in the pure spinor formalism [5].

\[ U^i \equiv \partial \theta^\alpha A^{i\alpha}_m + \Pi^{m} A^{i\alpha}_m + d_\sigma W^{\alpha}_i + \frac{1}{2} N^{mn} F^{mi} \]  

(8)

Worldsheet fields \([\partial \theta^\alpha, \Pi^m, d_\sigma, N^{mn}]\) with conformal weight one and well-known OPEs are combined with linearized superfields associated with particle label \( i \). The multiplicity-two superfields in (6) are obtained from the coefficients of the conformal fields in the OPE [4]

\[ U^{12} \equiv -\theta^\alpha A^{1\alpha}_m + \Pi^m A^{1\alpha}_m + d_\sigma W^{\alpha}_i + \frac{1}{2} N^{mn} F^{12}_{mi} \]

where \( \alpha' \) denotes the inverse string tension, and total derivatives in the worldsheet variables \( z_1, z_2 \) have been discarded in the second line. The CIFT-inspired two-particle prescription (6) can be promoted to a recursion leading to superfields of arbitrary multiplicity whose equations of motion generalize along the lines of

\[ \{ D_{(\alpha A^1_{123})} \} = \gamma^m_{\alpha \beta} A^{123}_m + (k^1 \cdot k^2) [A^{1\alpha} A^{2\beta} A^{3\gamma} - (12 \leftrightarrow 3)] \]

\[ + (k^1 \cdot k^2) [A^{1\alpha} A^{2\beta} A^{3\gamma} - (1 \leftrightarrow 2)] \]  

(10)

for suitable definitions of \( A^{123} \) and \( A^{123}_{m} \) [4]. The BCJ symmetries [12, 13] of the multiparticle superfields as well as the momenta \( k_{12}, j \equiv k_1 + k_2 + \ldots + k_j \) in their equations of motion suggest to associate them with tree-level subdiagrams shown in the subsequent figure [4]:

**Berends–Giele currents:** As a convenient basis of multiparticle fields \( K_B \in \{ A^{B\alpha}_m, A^{B\alpha}_{m}, W^{m}_{B}, F^{m}_{B} \} \) with multiparticle label \( B = 12 \ldots p \), we define Berends–Giele currents \( K_B \in \{ A^{B\alpha}_m, A^{B\alpha}_{m}, W^{m}_{B}, F^{m}_{B} \} \), e.g. \( K_{1} \equiv K_{123} \) and [4]

\[ K_{12} \equiv \frac{K_{12}}{s_{12}}, \quad K_{123} \equiv \frac{K_{123}}{s_{123}} + \frac{K_{121}}{s_{23}s_{123}} \]  

(11)

with generalized Mandelstam invariants \( s_{12}, \ldots \equiv \frac{1}{2} k^2_{12}, \ldots \). Berends–Giele currents \( K_B \) are defined to encompass all propagator-dressed tree subdiagrams compatible with the ordering of the external legs in \( B \). As shown in the following figure, the three-particle current in (11) is assembled from the s- and t-channels of a four-point amplitude with an off-shell leg (represented by \( \cdot \)):

\[ K_{123} = \sum_{\rho \in S_{p-1}} S^{-1}[\sigma | \rho] K_{1 \rho (23-p)} \]  

(12)

with permutation \( \sigma \in S_{p-1} \).

The combination of color-ordered trees as in (11) and (12) simplifies their equations of motion [4]

\[ \{ D_{(\alpha A^B_{\beta})} \} = \gamma^m_{\alpha \beta} A^B_m + \sum_{XY=B} (A^{X\alpha} A^{Y\beta} - A^{Y\alpha} A^{X\beta}) \]  

(13)

\[ [D_{\alpha A^B_m} = k^m B_{\alpha B} + (\gamma^m W_{B})_{\alpha} + \sum_{XY=B} (A^{X\alpha} A^{Y\beta} - A^{Y\alpha} A^{X\beta}) \]  

\[ [D_{\alpha W^B_{\beta}} = \frac{1}{2} (\gamma^m W^{B})_{\beta} F_{mn} - \sum_{XY=B} (A^{X\alpha} W^Y_{B} - A^{Y\alpha} W^X_{B}) \]  

\[ [D_{\alpha F^{mn}_{B}} = k^m (\gamma^m (\gamma^m W_{B})_{\alpha} - A^{[m} A^{n]} (\gamma^m W_{B})_{\alpha}) \]

Momenta \( k_B \equiv k_1 + k_2 + \ldots + k_p \) are associated with multiparticle labels \( B = 12 \ldots p \), and \( \sum_{XY=B} \) instructs to sum over all their deconcatenations into \( X = 12 \ldots j \) and \( Y = j + 1 \ldots p \) with \( 1 \leq j \leq p - 1 \). For example, the three-particle equation of motion of \( A^{123}_\alpha \) reads

\[ \{ D_{(\alpha A^{123}_{\beta})} \} = \gamma^m_{\alpha \beta} A^{123}_m \]

\[ + A^{1\alpha} A^{23}_\beta + A^{1\alpha} A^{32}_\beta - A^{23} A^{1\beta} - A^{32} A^{1\beta} \]  

(14)

and a comparison with (10) highlights the advantages of the diagram expansions in (11).
The symmetry properties of the $K_B$ can be inferred from their cubic-graph expansion and summarized as
\[
K_{A,U,B} = 0, \quad \forall A, B \neq \emptyset,
\]
where $\cup$ denotes the shuffle product [16]. For example,
\[
0 = K_{12} + K_{21} = K_{123} - K_{321} = K_{123} + K_{231} + K_{312}.
\]

**GENERATING SERIES OF SYM SUPERFIELDS**

In order to connect multiparticle fields and Berends-Giele currents with the non-linear field equations (4), we define generating series $K \in \{\hat{\alpha}_i, \hat{\alpha}_m, \Omega^a, F^{mn}\}$
\[
K = \sum_i K_i t^i + \sum_{i,j} K_{ij} t^i t^j + \sum_{i,j,k} K_{ijk} t^i t^j t^k + \ldots
\]
(17)
where $t^i$ denote generators in the Lie algebra of the non-abelian gauge group. The second line follows from the symmetry (15), which guarantees that $K$ is a Lie element [16]. As a key virtue of the generating series (17), they allow to rewrite (13) as non-linear equations of motion (where $[\partial^m, K]$ translates into components $k^m_B K_B$)
\[
\{D_{(a}, K_{b)}\} = \gamma^{mn}_a \hat{\alpha}_m + \{\hat{\alpha}_a, \hat{\alpha}_b\}
\]
\[
[D_{a}, \hat{\alpha}_m] = [\partial_m, \hat{\alpha}_a] + (\gamma_m \Omega)_a + [\hat{\alpha}_a, \hat{\alpha}_m]
\]
\[
[D_{a}, \Omega^m] = \frac{1}{2} (\gamma^{mn}, \Omega^m) + \{\hat{\alpha}_a, \Omega^m\}
\]
\[
[D_{a}, F^{mn}] = [\partial^m, \Omega^a] + [\hat{\alpha}_a, F^{mn}]
\]
(18)
They are equivalent to the SYM field equations (4) if the connection in (1) is defined through the representatives $\hat{\alpha}_a$ and $\hat{\alpha}_m$ of the generating series in (17).

Given that the multiparticle superfields satisfy [4]
\[
F^{mn} = k^m_B A^{mn}_B - \sum_{XY=1} (A^X A^Y - A^Y A^X)
\]
(19)
\[
k^m_B (\gamma^m \Omega_B) = \sum_{XY=1} \left[ A^X (\gamma^m \Omega_B) - A^Y (\gamma^m \Omega_B) \right]
\]
\[
k^m_B F^{mn} = \sum_{XY=1} \left[ 2 (W_X \gamma^m \Omega_B) + A^X F^{mn} - A^Y F^{mn} \right],
\]
the above definitions are compatible with (3) and
\[
(\nabla_m, \gamma^m \Omega) = 0, \quad (\nabla_m, F^{mn}) = \gamma_{\alpha \beta} (W^\alpha, \Omega^\beta).
\]
(20)
A linearized gauge transformation in particle one,
\[
\delta_1 A^a_m = D_{a} \Omega, \quad \delta_1 A^m_j = k^m_j \Omega,
\]
(21)
with scalar superfields $\Omega$ and $\delta_1 W^a = \delta_1 F^{mn} = 0$ propagates to multiparticle cases with $B = 12 \ldots p$ via [17]
\[
\delta_1 A^B_m = [D_{a} \Omega_B] + \sum_{XY=1} (\Omega_X A^Y_B) - \sum_{XY=1} (\Omega_X W^X_B)
\]
\[
\delta_1 F^{mn} = \sum_{XY=1} (\Omega_X F^{mn}_Y)
\]
(22)
The multiparticle gauge scalars $\Omega_{12 \ldots p}$ are exemplified in appendix B of [17] and gathered in the generating series
\[
L_1 = \Omega_1 t^1 + \sum \Omega_{jk} [t^j, t^k] + \ldots
\]
(23)
This allows to cast (22) in the standard form of non-linear gauge transformations:
\[
\delta_1 A_a = [\nabla_a, L_1], \quad \delta_1 W^a = [\nabla^a, L_1]
\]
\[
\delta_1 \Omega_m = [\partial_m, L_1], \quad \delta_1 F^{mn} = [\partial^{mn}, L_1].
\]
(24)

**HIGHER MASS DIMENSION SUPERFIELDS**

The introduction of the Lie elements $K$ and their associated supercovariant derivatives allow the recursive definition of superfields with higher mass dimensions,
\[
\Omega^m_{12 \ldots p} = [\Omega^m_1, \Omega^m_{2 \ldots p}],
\]
(25)
\[
F^{mn}_{12 \ldots p} = [\Omega^{mn}_1, F^{mn}_{2 \ldots p}].
\]
Their component fields are defined by
\[
W^m_{12 \ldots p} = \sum_{XY=1} (W^m_X A^Y_1 - W^m_Y A^X_1),
\]
(27)
\[
F^{mn}_{12 \ldots p} = \sum_{XY=1} (F^{mn}_X A^Y_1 - F^{mn}_Y A^X_1).
\]
Note from (27) that the non-linearities in the definition of higher mass superfields do not contribute in the single-particle context where $W^m_{1 \ldots p} = \sum_{i=1}^p W^m_{1 \ldots p}$. The equations of motion at higher mass dimension:

The equations of motion for the superfields of higher mass dimension (25) follow from $[\nabla_a, \nabla^a] = - (\gamma^m \Omega)_a$ and $[\nabla_m, \nabla_n] = - F^{mn}_{mn}$ together with Jacobi identities among iterated brackets. The simplest examples are given by
\[
\{\nabla_a, W^{mn}_B\} = \frac{1}{2} (\gamma^{pq}_{a \alpha}) F^{pq}_{mn},
\]
(28)
\[
\{\nabla^a, W^{mn}_B\} = (W^{pq}_{mn})_{\alpha \beta} F^{pq}_a - (W^{mn}_{pq})_{\alpha \beta} F^{pq}_a,
\]
which translate to
\[
D_a W^{mn}_B = \frac{1}{2} (\gamma^{pq}_{a \alpha} F^{pq}_{mn} + (A^X W^{mn}_Y - A^Y W^{mn}_X)
\]
In general, one can prove by induction that
\[
\begin{align*}
\{ \nabla_\alpha, W^{N\beta} \} &= \frac{1}{4} \delta(pq)_\alpha^\beta F^{N\beta} - \sum_{M \in P(N), M \neq \emptyset} \{ \{ W^{N\alpha} \}, W^{(N\setminus M)\beta} \}, \\
\{ \nabla_\alpha, F^{N[qp]} \} &= \{ W^{N[\gamma q]} \}_\alpha^\gamma - \sum_{M \in P(N), M \neq \emptyset} \{ \{ W^{N\alpha} \}, F^{(N\setminus M)\beta} \}.
\end{align*}
\]
(30)

The vector indices have been gathered to a multi-index \( N \equiv n_1 n_2 \ldots n_k \). Its power set \( P(N) \) consists of the \( 2^k \) ordered subsets, and \( (W^\gamma)_N \equiv (W^\gamma_{1 \ldots n_k \ldots 1}) \).

The higher-mass-dimension superfields obey relations which can be derived from Jacobi identities of nested (anti)commutators. For example, (3) determines their antisymmetrized components
\[
\begin{align*}
W^{[n_1 n_2]n_3 \ldots n_k \gamma} &= [W^{n_3 \ldots n_k \gamma}, F^{n_1 n_2}], \\
F^{[n_1 n_2]n_3 \ldots n_k \gamma} &= [F^{n_3 \ldots n_k \gamma}, F^{n_1 n_2}].
\end{align*}
\]
(31)

Moreover, the definition (25) via iterated commutators together with \( \delta^1 \nabla_m = -[L_1, \nabla_m] \) implies that
\[
\begin{align*}
\nabla^{m[qp]} &= 0, \\
\nabla^{m[qp]} + \nabla^{[pq]}m = 0, \\
\delta^1 W^N &= [L_1, W^N], \\
\delta^1 F^{N[qp]} &= [L_1, F^{N[qp]}],
\end{align*}
\]
and manifold generalizations of (20), (31) and (32) can be generated using these same manipulations.

**OUTLOOK AND APPLICATIONS**

The representation of the non-linear superfields of ten-dimensional SYM theory described in this letter was motivated by the computation of scattering amplitudes in the pure spinor formalism. Accordingly, they give rise to generating functions for amplitudes. For example, color-dressed tree-level amplitudes \( M(1, 2, \ldots, n) \) involving particles 1, 2, \ldots, \( n \) are generated by
\[
\frac{1}{3} \text{Tr}(WVV) = \sum_{n=3}^\infty (n-2) \sum_{i_1 < i_2 < \ldots < i_n} M(i_1, i_2, \ldots, i_n).
\]
(33)

As firstly pointed out in the appendix of [18], the generating series \( V \equiv \lambda^\alpha A^\alpha \) involving the pure spinor \( \lambda^\alpha \) satisfies the field equations \( QV = VV \) of the action \( \text{Tr} \int d^{10}x \left( \frac{1}{3} QVV - \frac{1}{3} VVV \right) \) [19] with BRST operator \( Q \equiv \lambda^\alpha D^\alpha \). The zero mode prescription of schematic form \( \langle \lambda^i \lambda^j \rangle = 1 \) is explained in [5], and the pure spinor representation of SYM amplitudes on the left hand side of (33) is described in [2]. Further details and generalizations to supersymmetrized operators \( F^4 \) and \( D^2 F^4 \) at higher mass dimension will be given elsewhere [20].

The multiparticle superfields of higher mass dimensions can be used to obtain simpler expressions for higher-order kinematic factors of superstring amplitudes, e.g.
\[
T_{12, 3, 4} \equiv \langle \langle \lambda^m \gamma^i \gamma^j W^m_2 \rangle \langle \lambda^r \gamma^s \gamma^t \gamma^u W^m_4 \rangle \rangle \quad (34)
\]
\[
T_{1234} \equiv \langle A^{m_1} T_2, 3, 4 \rangle + \langle \lambda^m \gamma^i \gamma^j W^m_1 \rangle \langle \lambda^r \gamma^s \gamma^t \gamma^u W^{m_2} \rangle \langle \lambda^u \gamma^v W^{m_3} F^{m_4} \rangle
\]

is an equivalent representation for the complicated three-loop kinematic factors generating the operator \( D^6 R^4 \) [6].

Finally, it would be interesting to construct formal solutions to supergravity field equations along similar lines.

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