Extending generalized spin representations

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Abstract

We revisit the construction of higher spin representations by Kleinschmidt and Nicolai for \( E_{10} \), generalize it to arbitrary simply laced types, and provide a coordinate-free approach to the \( \frac{3}{2} \)-spin and \( \frac{5}{2} \)-spin representations. Moreover, we discuss the relationship between our findings and the representation theory of \( \text{Sym}_3 \) pointed out to us by Levy.

1 Introduction

Generalized spin representations of the maximal compact subalgebra of the split real Kac–Moody algebra of type \( E_{10} \) have been introduced in [1], [2] and generalized to arbitrary symmetrizable types in [3]. The purpose of this note is to revisit some of the higher spin representations of type \( E_{10} \) studied in [5], notably \( \frac{3}{2} \)-spin and \( \frac{5}{2} \)-spin, generalize these to arbitrary simply laced types, and propose a coordinate-free approach which we carry out for \( \frac{3}{2} \)-spin and \( \frac{5}{2} \)-spin.

Our main result is the following coordinate-free extension of generalized spin representations:

Theorem. Let \( g \) be a simply laced split real Kac–Moody algebra, let \( h \) be a Cartan subalgebra of \( g \), let \( \lambda \) be the set consisting of the simple roots of \( g \) and roots that are sums of two distinct simple roots, let \( k \) be the maximal compact subalgebra of \( g \), and let \((\cdot|\cdot)\) denote the induced invariant bilinear form on \( h^* \). A map \( X: \lambda \to \text{End} (V) \) satisfying the following (anti-)commutator relations for all \( \alpha, \beta \in \lambda \)

\[
[X(\alpha),X(\beta)] = 0 \quad \text{if} \quad (\alpha|\beta) = 0
\]

\[
\{X(\alpha),X(\beta)\} = X(\alpha \pm \beta) \quad \text{if} \quad (\alpha|\beta) = \mp 1 \quad \text{and} \quad \alpha \pm \beta \in \lambda
\]

provides a finite-dimensional representation \( \sigma \) of \( k \) via the assignment

\[
\sigma(X_i) := X(\alpha_i) \otimes \Gamma(\alpha_i)
\]

on the Berman generators \( X_1, \ldots, X_n \) of \( k \), where the \( \Gamma(\alpha_i), 1 \leq i \leq n \) are the anti-symmetric real matrices from (3.11) induced by the generalized spin representation of \( k \).

Define \( X_2: \Delta^\vee \to \text{End} (h^*) \) via

\[
\alpha \mapsto X_2(\alpha) := -\alpha (\cdot|\cdot) + \frac{1}{2} \text{id}_{h^*}.
\]

Moreover, for \( \alpha \in \Delta^\vee \) let \( \pi_\alpha := \alpha (\cdot|\cdot) \in \text{End} (h^*) \) and define \( X_2: \Delta^\vee \to \text{End} (\text{Sym}^2(h^*)) \) via

\[
\alpha \mapsto X_2(\alpha) := \pi_\alpha \otimes \pi_\alpha - (\pi_\alpha \otimes \text{id}_{h^*} + \text{id}_{h^*} \otimes \pi_\alpha) + \frac{1}{2} \text{id}_{h^*} \otimes \text{id}_{h^*}.
\]
Then $X_{\frac{1}{2}}$ and $X_{\frac{3}{2}}$ satisfy the above equalities for all real roots $\alpha$, $\beta$ with $(\alpha|\beta) \in \{0, \pm 1\}$ and thus each provides a representation $\sigma$ of $\mathfrak{k}$.

The results for this note have been obtained during and shortly after the first author’s MSc thesis project in mathematics. It would be interesting to understand how these representations decompose into irreducible components. We refer to [5], [6] for some investigations in this direction.

Paul Levy pointed out to us that both assignments $X_{\frac{1}{2}}$ and $X_{\frac{3}{2}}$ are of the form

$$X(\alpha) := \rho(s_\alpha) - \frac{1}{2} \text{id}$$

where $\rho(s_\alpha)$ denotes the natural reflection action of the fundamental generator $s_\alpha$ induced on $\mathfrak{h}^*$, resp. Sym$^2(\mathfrak{h}^*)$. For simple roots $\alpha$, $\beta$ forming a subdiagram of type $A_2$ one obtains the equivalence

$$\{X(\alpha), X(\beta)\} = X(\alpha \pm \beta) \iff \rho(s_\alpha s_\beta s_\alpha) - \rho(s_\alpha s_\beta) - \rho(s_\beta s_\alpha) + \rho(s_\alpha) + \rho(s_\beta) - \text{id} = 0.$$  

Among the irreducible representations of $\text{Sym}_3$, the trivial and the geometric representations satisfy the above identity, whereas the sign representation does not. One in fact arrives at a characterization of those representations $\rho : W \to \text{GL}(V)$ of the Weyl group $W$ of $\mathfrak{g}$ that can be used for extending generalized spin representations via the assignment $X(\alpha) := \rho(s_\alpha) - \frac{1}{2} \text{id}$: exactly those whose restrictions to any standard subgroup $\text{Sym}_3 \cong \langle s_\alpha, s_\beta \rangle \leq W$ (where $\alpha$, $\beta$ are adjacent simple roots of $\mathfrak{g}$) do not contain a sign representation as an irreducible component will do.

Since neither of the given $W$-modules $\mathfrak{h}^*$ and Sym$^2(\mathfrak{h}^*)$ contain a Sym$_3$-sign representation, both can be used for extending generalized spin representations. The module Sym$^3(\mathfrak{h}^*)$ on the other hand does contain a sign representation and so the $\frac{7}{2}$-spin representations discussed in [5], [6] still remain elusive.

Moreover, note that a map $X : \lambda \to \text{End}(V)$ as in the statement of the Theorem naturally extends to the set of all those positive real roots that can be written as iterated sums of simple roots such that each partial sum itself is a positive real root. It is well-known that in the finite-dimensional situation this set equals the set of all positive (real) roots; in the simply-laced affine case it can be shown that this set also equals the set of all positive real roots (cf. [7]). To the best of our knowledge the question what this set looks like in general is open.

Our note contains several redundancies. First, we reproduce the method to obtain extensions of generalized spin representations of $E_{10}$ and its application to $\frac{1}{2}$ and $\frac{3}{2}$-spin representations proposed by Kleinschmidt and Nicolai in order to make their work [5], [6] accessible to a wider mathematical audience and to point out that their approach actually works for any simply-laced Dynkin diagram. Second, we propose and apply our own coordinate-free method. Third, we interpret our findings in terms of Sym$_3$-representation theory based on Levy’s observations. This organization of our note leads to various existence proofs of $\frac{1}{2}$ and $\frac{3}{2}$-spin representations and to a wealth of starting points for further investigation.

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2 Generalized $\frac{1}{2}$-spin representations

Recall the notion of a Kac–Moody algebra from [2]. Let $A$ be a symmetrizable generalized Cartan matrix and $(h, \Pi, \Pi')$ be a realization of $A$ over $\mathbb{R}$ so that for $h_C := h \otimes \mathbb{C}$ the triple $(h_C, \Pi, \Pi')$ is a realization of $A$ over $\mathbb{C}$. Let $\tau : \mathbb{C} \to \mathbb{C}$ be complex conjugation and denote by $\omega_0$ the $\tau$-semilinear involution on the complex Kac–Moody algebra $g_C(A)$ determined by
\[
\omega_0(e_i) = -f_i, \quad \omega_0(f_i) = -e_i, \quad \omega_0(h) = -h \, \forall \, h \in h_R.
\]

Call $\omega_0$ the compact involution of $g_C(A)$ and $\mathcal{C}(A) := \text{Fix} \, \omega_0$ the maximal compact subalgebra of $g_C(A)$.

Let $g(A)$ be the split real form of $g_C(A)$, i.e., the real Kac–Moody algebra obtained as the fixed points of complex conjugation $\tau$ acting naturally on the complex vector space underlying $g_C(A)$. Let $\omega_C$ and $\omega$ denote the Chevalley involutions on these Kac–Moody algebras and let $\omega_0$ be the compact involution on $g_C(A)$. Then one has
\[
g_C(A) \supset \text{Fix} \, \omega_0 \cong \text{Fix} \, \omega \oplus i\omega_{-1},
\]

where $\omega_{-1}$ denotes the $-1$ eigenspace of $\omega$ on $g(A)$. The fixed point subalgebra $\mathfrak{k}(A) = \text{Fix} \, \omega$ is called the maximal compact subalgebra of $g(A)$.

A Kac–Moody algebra $g(A)$ is called simply laced if its generalized Cartan matrix contains only entries which are $0$ or $-1$ on the off-diagonal.

**Theorem 2.1.** Let $g(A)$ be a simply laced real Kac–Moody algebra and $\mathfrak{k}$ its maximal compact subalgebra. Then $\mathfrak{k}$ is isomorphic to the free Lie algebra over $\mathbb{R}$ of generators $X_1, \ldots, X_n$ modulo the ideal generated by the relations
\[
[X_i, [X_i, X_j]] = -X_j \quad \text{if } a_{ij} = -1 \\
[X_i, X_j] = 0 \quad \text{if } a_{ij} = 0
\]

via the isomorphism given by
\[
X_i \mapsto e_i - f_i.
\]

*Proof.* See [3, Theorem 1.3].

**Definition 2.2.** A representation $\rho : \mathfrak{k} \to \text{End} \, (\mathbb{C})$ is called a generalized spin representation if for the generators $X_1, \ldots, X_n$ of $\mathfrak{k}$ one has
\[
\rho(X_i)^2 = -\frac{1}{4} \text{id}, \, \forall \, i = 1, \ldots, n.
\]

**Proposition 2.3.** Let $\rho : \mathfrak{k} \to \text{End} \, (\mathbb{C})$ be a generalized spin representation and denote by $[A, B] := AB - BA$ the commutator and by $\{A, B\} := AB + BA$ the anti-commutator. Then for $1 \leq i \neq j \leq n$ one has
\[
[\rho(X_i), \rho(X_j)] = 0 \quad \text{if } a_{ij} = 0 \quad \iff \quad (i, j) \text{ do not form an edge of the Dynkin diagram}
\]
\[
\{\rho(X_i), \rho(X_j)\} = 0 \quad \text{if } a_{ij} = -1 \quad \iff \quad (i, j) \text{ form an edge of the Dynkin diagram}.
\]

*Proof.* If $(i, j)$ do not form an edge of the Dynkin diagram then $a_{ij} = 0$ and so $[X_i, X_j] = 0$ according to Theorem 2.1 which is carried over to $\text{End} \, (\mathbb{C})$, since $\rho$ is a homomorphism. If $(i, j)$ form an edge, which is to say $a_{ij} = -1$, then by Theorem 2.1 one has
\[
[X_i, [X_i, X_j]] = -X_j
\]
and setting $A = \rho(X_i)$, $B = \rho(X_j)$ one computes in $\text{End}(\mathbb{C}^n)$

$$[A, [A, B]] = -B$$
$$\Leftrightarrow A^2B - ABA - ABA + BA^2 = -B$$
$$\Leftrightarrow -2ABA = -\frac{1}{2}B$$
$$\Leftrightarrow \frac{1}{2}AB = -\frac{1}{2}BA$$
$$\Leftrightarrow AB + BA = 0.$$

Note that multiplication with $A$ preserves equivalence because $A$ is invertible, since $A^{-1} = -4A.$

Corollary 2.4. Given matrices $A_1, \ldots, A_n \in \mathbb{C}^{s \times s}$ with

(i) $A_i^2 = -\frac{1}{4}\text{id}_s$,

(ii) $[A_i, A_j] = 0$, if $(i, j)$ do not form an edge of the Dynkin diagram,

(iii) $\{A_i, A_j\} = 0$, if $(i, j)$ form an edge of the Dynkin diagram,

the extension of the map $X_i \mapsto A_i$ defines a generalized spin representation $\rho$ from $\mathfrak{k}$ on $\mathbb{C}^n$.

Proof. (i) is a necessary condition by the definition of spin representations. Assertion (ii) ensures that the commutation relations between $X_i, X_j$ are respected by $\rho$ if $(i, j)$ do not form an edge, because in this case $[X_i, X_j] = 0$. Finally, (iii) ensures that for $a_{ij} \neq 0$ the relation

$$[X_i, [X_i, X_j]] = -X_j$$

for $i \neq j$ is respected by $\rho$ since according to the proof of Proposition 2.3 the condition $\{A, B\} = 0$ is equivalent to $[A, [A, B]] = -B$ as long as $A^2 = B^2 = -\frac{1}{4}\text{id}_s$.

The existence of generalized spin representations has been established in [3].

Theorem 2.5. For $1 \leq r < n$ let $\mathfrak{k}_{\leq r} := \langle X_1, \ldots, X_r \rangle$ denote the subalgebra of $\mathfrak{k}$ that is generated by the first $r$ generators. Furthermore, let $\rho : \mathfrak{k}_{\leq r} \rightarrow \text{End}(\mathbb{C}^n)$ be a generalized spin representation as in Definition 2.2.

If $X_{r+1}$ centralizes $\mathfrak{k}_{\leq r}$, that is to say $X_{r+1}$ commutes with all generators $X_1, \ldots, X_r$, then there exists a generalized spin representation $\rho' : \mathfrak{k}_{\leq r+1} \rightarrow \text{End}(\mathbb{C}^n)$ with $\rho'|_{\mathfrak{k}_{\leq r}} = \rho$ given by sending $X_{r+1}$ to $\frac{1}{2i} \cdot \text{id}_s$.

If $X_{r+1}$ does not centralize $\mathfrak{k}_{\leq r}$, then $\rho$ can be extended to a generalized spin representation $\rho' : \mathfrak{k}_{\leq r+1} \rightarrow \text{End}(\mathbb{C}^n \oplus \mathbb{C}^n)$.

For this define a sign automorphism $s_0 : \mathfrak{k}_{\leq r} \rightarrow \mathfrak{k}_{\leq r}$ by

$$s_0(X_i) = \begin{cases} X_i, & \text{if } (i, r+1) \text{ do not form an edge of the Dynkin diagram}, \\ -X_i, & \text{if } (i, r+1) \text{ form an edge of the Dynkin diagram}, \end{cases}$$

and define the extension via

$$\rho'|_{\mathfrak{k}_{\leq r}} = \rho \oplus \rho \circ s_0$$

and

$$\rho'(X_{r+1}) = \frac{1}{2} \text{id}_s \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Proof. See [3] Theorem 3.9. 


Corollary 2.6. Given a simply laced Kac–Moody algebra \( g(A) \) and a maximal coclique of size \( r \), then there exists a generalized spin representation \( \rho : \mathfrak{k} \to \text{End}(C^s) \), where \( s = 2^{n-r} \), with compact image.

Proof. See [3, Corollary 3.10 and Theorem 3.14].

3 Extending a generalized \( \frac{1}{2} \)-spin representation — following Kleinschmidt and Nicolai

Throughout this section let \( g \) be a simply laced split real Kac–Moody algebra with maximal compact subalgebra \( \mathfrak{k} \). By Corollary 2.6 there exists a generalized \( \frac{1}{2} \)-spin representation \( \rho : \mathfrak{k} \to \text{End}(C^s) \). In this section we make use of Clifford algebras in order to define higher generalized spin representations as carried out by Kleinschmidt and Nicolai [3] for \( E_{10} \).

Let \( V \otimes S \) be the tensor product of two \( \mathbb{R} \)-vector spaces \( V \) with basis \( \{e_1, \ldots, e_k\} \) and \( S \) with basis \( \{f_1, \ldots, f_l\} \). Then \( \{e^i \otimes f_j \mid 1 \leq i \leq k, 1 \leq j \leq l\} \) is a natural \( \mathbb{R} \)-basis of \( V \otimes S \). Endow \( V \) with a nondegenerate bilinear form \( q_1 \) and \( S \) with a positive definite bilinear form \( q_2 \) such that the basis \( \{e_1, \ldots, f_l\} \) is orthonormal, i.e.,

\[
q_2 (f_a, f_b) = \delta_{a\beta} \quad \text{for} \quad a, \beta \in \{1, \ldots, l\}.
\]

Let \( (G^{ab})_{1 \leq a,b \leq k} \) denote the Gram matrix of \( q_1 \) with respect to the basis \( \{e_1, \ldots, e_k\} \), i.e., the matrix whose components are given by

\[
G^{ab} = q_1 (e_a, e_b) \quad \text{for} \quad a, b \in \{1, \ldots, k\}.
\]

Define \( q := q_1 \otimes q_2 \) as the bilinear extension of \( q_1 \) and \( q_2 \) to \( V \otimes S \) so that on the chosen basis \( \{e^i \otimes f_j \mid 1 \leq i \leq k, 1 \leq j \leq l\} \) one has for \( a, b \in \{1, \ldots, k\} \)

\[
q (e^a \otimes f_a, e^b \otimes f_{\beta}) = q_1 (e^a, e^b) \cdot q_2 (f_a, f_{\beta}) = G^{ab} \delta_{a\beta}.
\]

The bilinear form \( q \) induces a quadratic form

\[
Q : V \otimes S \to \mathbb{R} : w \mapsto q(w, w).
\]

One defines the Clifford algebra \( S = \text{Cl}(V \otimes S, Q) \) as the quotient of \( T (V \otimes S) \) modulo the ideal \( I_Q \) generated by elements of the form

\[
w \otimes w - \frac{1}{2} Q(w) \cdot 1, \quad w \in V \otimes S.
\]

In \( S \) one therefore has \( w^2 = \frac{1}{2} Q(w) \), which via polarization one can restate this as

\[
ww + vw = q(v, w).
\]

(3.1)

On the level of the basis for \( a, \beta \in \{1, \ldots, l\}, a, b \in \{1, \ldots, k\} \) this reads as

\[
\left( e^a \otimes f_a \right) \left( e^b \otimes f_{\beta} \right) + \left( e^b \otimes f_a \right) \left( e^a \otimes f_{\beta} \right) = G^{ab} \delta_{a\beta},
\]

which one may repackage in a compact notation by defining for \( a \in \{1, \ldots, l\}, A \in \{1, \ldots, k\} \)

\[
\phi^A_a := e^A \otimes f_a.
\]

(3.2)
thus yielding the identity for $\alpha, \beta \in \{1, \ldots, l\}, A, B \in \{1, \ldots, k\}$

$$\left\{ \phi^A_\alpha, \phi^B_\beta \right\} := \phi^A_\alpha \phi^B_\beta + \phi^B_\beta \phi^A_\alpha = G^{AB}_{\alpha \beta}.$$  

(3.3)

Since $\{e^i \otimes f_j \mid 1 \leq i \leq k, 1 \leq j \leq l\}$ is a basis of $V \otimes S$, the set $\{\phi^A_\alpha \mid 1 \leq A \leq k, 1 \leq \alpha \leq l\}$ is a generating set of the $\mathbb{R}$-algebra $S$.

**Lemma 3.1.** For $X, Y \in \mathbb{R}^{k \times k}$ and $S, T \in \mathbb{R}^{l \times l}$ consider the following elements of $S$:

$$\hat{A} := \sum_{A, B=1}^{k} \sum_{\alpha, \beta=1}^{l} X_{AB} S^{\alpha \beta} \phi^A_\alpha \phi^B_\beta,$$

$$\hat{B} := \sum_{C, D=1}^{l} \sum_{\gamma, \delta=1}^{l} Y_{CD} T^{\gamma \delta} \phi^C_\gamma \phi^D_\delta.$$  

Under the hypothesis that for all $\alpha, \beta \in \{1, \ldots, l\}$ and for all $A, B \in \{1, \ldots, k\}$

$$X_{AB} S^{\alpha \beta} = -X_{BA} S^{\beta \alpha},$$  

$$Y_{AB} T^{\alpha \beta} = -Y_{BA} T^{\beta \alpha}$$  

(3.4)

the commutator of $\hat{A}$ and $\hat{B}$ is equal to

$$[\hat{A}, \hat{B}] = \sum_{A, B=1}^{k} \sum_{\alpha, \beta=1}^{l} \phi^A_\alpha \left( [X, Y]_{AB} \{S, T\}^{\alpha \beta} + \{X, Y\}_{AB} [S, T]^{\alpha \beta} \right) \phi^B_\beta.$$  

(3.5)

where (anti-)commutators of $X$ and $Y$, resp. $S$ and $T$ are taken with respect to the bilinear forms as follows:

$$[X, Y]_{AB} = \sum_{C, D=1}^{l} \left( X_{AC} G^{CD} Y_{DB} - Y_{AC} G^{CD} X_{DB} \right),$$  

(3.6)

$$\{X, Y\}_{AB} = \sum_{C, D=1}^{l} \left( X_{AC} G^{CD} Y_{DB} + Y_{AC} G^{CD} X_{DB} \right),$$  

(3.7)

$$[S, T]^{\alpha \beta} = \sum_{\gamma, \delta=1}^{l} \left( S^{\alpha \gamma} \delta_{\gamma \delta} T^{\delta \beta} - T^{\alpha \gamma} \delta_{\gamma \delta} S^{\delta \beta} \right),$$  

(3.8)

$$\{S, T\}^{\alpha \beta} = \sum_{\gamma, \delta=1}^{l} \left( S^{\alpha \gamma} \delta_{\gamma \delta} T^{\delta \beta} + T^{\alpha \gamma} \delta_{\gamma \delta} S^{\delta \beta} \right).$$  

(3.9)

**Remark 3.2.**

i. On level of the tensor product matrices, hypothesis (3.4) simply requires anti-symmetry:

$$X^T \otimes S^T = -X \otimes S.$$  

ii. The statement of the lemma can be found as [15] (4.17), p. 13 and footnote 10, p. 14.

**Proof of Lemma 3.1.** One computes

$$[X, Y]_{AB} \{S, T\}^{\alpha \beta} + \{X, Y\}_{AB} [S, T]^{\alpha \beta}$$  

$$= \left( (XY)_{AB} \left( ST \right)^{\alpha \beta} + \left( TS \right)^{\alpha \beta} \right) + \left( (XY)_{AB} + (YX)_{AB} \right) \left( (ST)^{\alpha \beta} - (TS)^{\alpha \beta} \right)$$  

$$= \left( (XY)_{AB} \left( ST \right)^{\alpha \beta} + (XY)_{AB} \left( TS \right)^{\alpha \beta} - (YX)_{AB} \left( ST \right)^{\alpha \beta} - (YX)_{AB} \left( TS \right)^{\alpha \beta} \right)$$  

$$+ \left( (XY)_{AB} \left( ST \right)^{\alpha \beta} - (XY)_{AB} \left( TS \right)^{\alpha \beta} + (YX)_{AB} \left( ST \right)^{\alpha \beta} - (YX)_{AB} \left( TS \right)^{\alpha \beta} \right)$$  

$$= 2 \left( (XY)_{AB} \left( ST \right)^{\alpha \beta} - (YX)_{AB} \left( TS \right)^{\alpha \beta} \right).$$  

(3.10)
where in analogy to the (anti-)commutators one abbreviates

\[(XY)_{AB} = \sum_{c,d=1}^{k} X_{Ac} G^{cD} Y_{DB}, \quad (YX)_{AB} = \sum_{c,d=1}^{k} Y_{Ac} G^{cD} X_{DB}, \]

\[(ST)^{\alpha \beta} = \sum_{\gamma, \delta=1}^{l} S^{\alpha \gamma} \delta_{\gamma \delta} T^{\delta \beta}, \quad (TS)^{\alpha \beta} = \sum_{\gamma, \delta=1}^{l} T^{\alpha \gamma} \delta_{\gamma \delta} S^{\delta \beta}.\]

Several applications of equality (3.3) yield

\[\left[ \hat{A}, \hat{B} \right] = \sum_{A, B, C, D=1}^{k} \sum_{\alpha, \beta, \gamma, \delta=1}^{l} X_{AB} S^{\alpha \beta} Y_{CD} T^{\gamma \delta} \left( \phi_{\alpha}^{A} \phi_{\beta}^{B} \phi_{\gamma}^{C} \phi_{\delta}^{D} - \phi_{\gamma}^{C} \phi_{\delta}^{D} \phi_{\alpha}^{A} \phi_{\beta}^{B} \right)\]

\[= \sum_{A, B, C, D=1}^{k} \sum_{\alpha, \beta, \gamma, \delta=1}^{l} X_{AB} S^{\alpha \beta} Y_{CD} T^{\gamma \delta} \left( \phi_{\alpha}^{A} \phi_{\beta}^{B} \phi_{\gamma}^{C} \phi_{\delta}^{D} + \phi_{\alpha}^{A} \phi_{\gamma}^{C} \phi_{\delta}^{D} \phi_{\beta}^{B} \right)\]

\[= \sum_{A, B, C, D=1}^{k} \sum_{\alpha, \beta, \gamma, \delta=1}^{l} X_{AB} S^{\alpha \beta} Y_{CD} T^{\gamma \delta} \left( \phi_{\alpha}^{A} \phi_{\beta}^{B} \phi_{\gamma}^{C} \phi_{\delta}^{D} + \phi_{\alpha}^{A} \phi_{\gamma}^{C} \phi_{\delta}^{D} \phi_{\beta}^{B} + G^{DA} \delta_{\gamma \alpha} \phi_{\delta}^{C} \phi_{\beta}^{B} - G^{DA} \delta_{\delta \alpha} \phi_{\gamma}^{C} \phi_{\beta}^{B} \right)\]

\[= \sum_{A, B, C, D=1}^{k} \sum_{\alpha, \beta, \gamma, \delta=1}^{l} X_{AB} S^{\alpha \beta} Y_{CD} T^{\gamma \delta} \left( \phi_{\alpha}^{A} \phi_{\beta}^{B} \phi_{\gamma}^{C} \phi_{\delta}^{D} + \phi_{\alpha}^{A} \phi_{\gamma}^{C} \phi_{\delta}^{D} \phi_{\beta}^{B} - G^{DA} \delta_{\gamma \alpha} \phi_{\delta}^{C} \phi_{\beta}^{B} - G^{DA} \delta_{\delta \alpha} \phi_{\gamma}^{C} \phi_{\beta}^{B} \right)\]

Using the symmetry of \(G^{AB}\) and of \(\delta_{\alpha \beta}\) this is rearranged to

\[\left[ \hat{A}, \hat{B} \right] = \sum_{A, B, C, D=1}^{k} \sum_{\alpha, \beta, \gamma, \delta=1}^{l} \phi_{\alpha}^{A} X_{AB} G^{BC} Y_{CD} S^{\alpha \beta} \delta_{\gamma \delta} T^{\gamma \delta} \phi_{\beta}^{C} - \phi_{\gamma}^{C} Y_{CD} G^{DA} X_{AB} S^{\alpha \beta} T^{\gamma \delta} \delta_{\gamma \delta} \phi_{\beta}^{C}\]

\[+ \phi_{\gamma}^{C} Y_{CD} G^{DA} X_{AB} S^{\alpha \beta} T^{\gamma \delta} \delta_{\gamma \delta} \phi_{\beta}^{C} - \phi_{\beta}^{C} Y_{CD} G^{DA} X_{AB} T^{\gamma \delta} \delta_{\gamma \delta} S^{\alpha \beta} \phi_{\beta}^{C}.\]

This can then be transformed by renaming indices and using symmetry of the bilinear forms and
anti-symmetry of the tensor product matrices (cf. (3.4)): 

\[
[\hat{A}, \hat{B}] = \sum_{A,B,C,D=1}^{k} \sum_{A,B,C,D=1}^{l} \phi^A_{\alpha} \left( X_{ABG} R_{CD} S^{\alpha\beta \gamma \delta, \alpha} T^{\gamma \delta} - Y_{DCG} X_{AB} S^{\alpha\beta \gamma \delta, \alpha} T^{\gamma \delta} \right) \phi^B_{\beta} \\
+ \phi^P_{\delta} \left( Y_{DCG} X_{AB} S^{\alpha\beta \gamma \delta, \alpha} T^{\gamma \delta} - Y_{DCG} X_{AB} S^{\alpha\beta \gamma \delta, \alpha} T^{\gamma \delta} \right) \phi^B_{\beta} \\
= \sum_{A,B,C,D=1}^{k} \sum_{A,B,C,D=1}^{l} \phi^A_{\alpha} \left( X_{ABG} R_{CD} S^{\alpha\beta \gamma \delta, \alpha} T^{\gamma \delta} + X_{ABG} R_{CD} S^{\alpha\beta \gamma \delta, \alpha} T^{\gamma \delta} \right) \phi^B_{\beta} \\
+ \phi^P_{\delta} \left( -Y_{DCG} X_{AB} S^{\alpha, \alpha} T^{\gamma} \gamma - Y_{DCG} X_{AB} S^{\alpha, \alpha} T^{\gamma} \gamma \right) \phi^B_{\beta} \\
= 2 \sum_{A,B,C,D=1}^{k} \sum_{A,B,C,D=1}^{l} \phi^A_{\alpha} \left( (XY)_{CD} (ST)^{\alpha \delta} \phi^P_{\delta} - 2 \sum_{B,D=1, \beta, \delta=1}^{k} \phi^P_{\delta} (XY)_{DB} (TS)^{\beta \delta} \phi^B_{\beta} \right) \\
= 2 \sum_{A,B,C,D=1}^{k} \sum_{A,B,C,D=1}^{l} \phi^A_{\alpha} \left( (XY)_{CD} (ST)^{\alpha \delta} - (XY)_{CD} (TS)^{\alpha \delta} \right) \phi^B_{\beta} \\
\]

which in view of (3.11) completes the proof. 

\[ \square \]

**Remark 3.3.** In fact, we never used in the proof of Lemma 3.1 that the form \( q_2 \) is anisotropic. The computations hold in general for arbitrary non-degenerate forms. The definiteness of (3.3) becomes relevant now, when using the preceding lemma in order to construct various representations of \( f \). The generalized spin representation \( \rho : f \to \text{End}(C^n) \) from Corollary 2.6 provides anti-symmetric real matrices, satisfying

\[ \phi^A_{\alpha} \gamma^A_{\alpha} = 2 \rho (X_i) \quad \text{for all simple roots } \alpha_1, \ldots, \alpha_n \text{ of } f. \]

Taking these as the matrix \( S \) in the ansatz

\[ \hat{A} := \sum_{A,B=1}^{k} \sum_{A,B=1}^{l} X_{AB} S^{\alpha\beta} \phi^A_{\alpha} \phi^B_{\beta} \]

of the lemma leaves one with the task of finding suitable symmetric matrices for \( X \).

Note that, since we assumed \( q_2 \) to be anisotropic and conducted our computations with respect to an orthonormal basis for that form, the formulae given in (3.10) and (3.11) actually coincide with the standard definition of commutators and anti-commutators of matrices. In particular, the results from Proposition 2.3 are applicable.

**Definition 3.4.** Now let \( \lambda \) denote the finite set of real roots

\[ \lambda := \{ \alpha_i | 1 \leq i \leq n \} \cup \{ \alpha_i + \alpha_j \in \Phi^{\text{re}} | (i, j) \text{ form an edge of the Dynkin diagram} \} \]

Note that for \( \alpha, \beta \in \lambda \) one has \( (\alpha|\beta) \in \{ \pm 1, 0 \} \).

**Proposition 3.5.** A map \( X : \lambda \to \mathbb{R}^{k \times k} \) that takes values in the set of symmetric matrices which satisfy for all \( \alpha, \beta \in \lambda \)

\[ [X(\alpha), X(\beta)] = 0, \quad \text{if } (\alpha|\beta) = 0, \]

\[ \{X(\alpha), X(\beta)\} = \begin{cases} \frac{1}{2} X(\alpha \pm \beta), & \text{if } (\alpha|\beta) = \mp 1 \text{ and } \alpha \pm \beta \in \lambda, \end{cases} \]

\[ \]
Applying the commutator with (with respect to the commutator and anti-commutator convention from (3.10) and (3.11) together with the anti-symmetric real matrices \( \Gamma (\alpha_1), \ldots, \Gamma (\alpha_n) \) from (3.11) turns the ansatz

\[
\tilde{J}(\alpha_i) = \sum_{A,B=1}^{k} \sum_{\alpha,\beta=1}^{l} X_{AB}(\alpha_i) \Gamma^{\alpha\beta}(\alpha_i) \phi_\alpha^A \phi_\beta^B
\]

into a finite-dimensional representation \( \sigma \) of \( \mathfrak{k} \) by defining \( \sigma \) on the Berman generators \( X_1, \ldots, X_n \) of \( \mathfrak{k} \) as \( \sigma (X_i) := \tilde{J}(\alpha_i) \).

**Remark 3.6.** The observation that (3.13) and (3.14) are the key identities for extending generalized spin representations has been made in [5, (4.23), p. 15; (5.1), p. 18].

**Proof of Proposition 3.5.** By the homomorphism theorem it suffices to establish that the commutator \([\tilde{J}(\alpha_i), \tilde{J}(\alpha_j)]\) satisfies the relations from Theorem 2.1. By Lemma 3.1 one has

\[
[\tilde{J}(\alpha_i), \tilde{J}(\alpha_j)] = \sum_{A,B=1}^{k} \sum_{\alpha,\beta=1}^{l} \phi_\alpha^A \{ X(\alpha_i), X(\alpha_j) \}_A B \{ \Gamma (\alpha_i), \Gamma (\alpha_j) \}^{\alpha\beta} \phi_\beta^B
\]

\[+
\sum_{A,B=1}^{k} \sum_{\alpha,\beta=1}^{l} \phi_\alpha^A \{ X(\alpha_i), X(\alpha_j) \}_A B \{ \Gamma (\alpha_i), \Gamma (\alpha_j) \}^{\alpha\beta} \phi_\beta^B.
\]

In case \((i,j)\) is not an edge of the Dynkin diagram this yields

\[
[\tilde{J}(\alpha_i), \tilde{J}(\alpha_j)] = 0
\]

as desired, because in this case one has \([\Gamma (\alpha_i), \Gamma (\alpha_j)] = 0 \) by Proposition 2.3 and, furthermore, \((\alpha_i|\alpha_j) = 0\), i.e., \([X(\alpha_i), X(\alpha_j)] = 0\) by hypothesis (3.13).

In case \((i,j)\) is an edge of the Dynkin diagram one has \([\Gamma (\alpha_i), \Gamma (\alpha_j)] = 0\) by Proposition 2.3 and so

\[
\{ X(\alpha_i), X(\alpha_j) \} = \frac{1}{2} X(\alpha_i + \alpha_j)
\]

by hypothesis (3.11). Thus,

\[
[\tilde{J}(\alpha_i), \tilde{J}(\alpha_j)] = \sum_{A,B=1}^{k} \sum_{\alpha,\beta=1}^{l} \phi_\alpha^A \cdot \frac{1}{2} X(\alpha_i + \alpha_j) \{ \Gamma (\alpha_i), \Gamma (\alpha_j) \}^{\alpha\beta} \phi_\beta^B.
\]

Applying the commutator with \( \tilde{J}(\alpha_i) \) again according to Lemma 3.1 yields

\[
[\tilde{J}(\alpha_i), [\tilde{J}(\alpha_i), \tilde{J}(\alpha_j)]] = \frac{1}{2} \sum_{A,B=1}^{k} \sum_{\alpha,\beta=1}^{l} \phi_\alpha^A \{ X(\alpha_i), X(\alpha_i + \alpha_j) \}_A B \{ \Gamma (\alpha_i), \Gamma (\alpha_i), \Gamma (\alpha_j) \}^{\alpha\beta} \phi_\beta^B
\]

\[+
\frac{1}{2} \sum_{A,B=1}^{k} \sum_{\alpha,\beta=1}^{l} \phi_\alpha^A \{ X(\alpha_i), X(\alpha_i + \alpha_j) \}_A B \{ \Gamma (\alpha_i), \Gamma (\alpha_i), \Gamma (\alpha_j) \}^{\alpha\beta} \phi_\beta^B.
\]

Since \((\alpha_i|\alpha_i + \alpha_j) = 1\), by hypothesis (3.11) one has

\[
\{ X(\alpha_i), X(\alpha_i + \alpha_j) \} = \frac{1}{2} X(\alpha_j).
\]

Moreover,

\[
[\Gamma (\alpha_i), \Gamma (\alpha_i), \Gamma (\alpha_j)] = -4 \Gamma (\alpha_j)
\]

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because $\rho(X_i) = \frac{1}{2} \Gamma(\alpha_i)$ is a generalized spin representation of $\mathfrak{k}$ (cf. Proposition 3.5 and its proof). Furthermore,

$$\{ \Gamma(\alpha_i), [\Gamma(\alpha_i), \Gamma(\alpha_j)] \} = \Gamma(\alpha_i) \Gamma(\alpha_i) \Gamma(\alpha_j) - \Gamma(\alpha_i) \Gamma(\alpha_j) \Gamma(\alpha_i) + \Gamma(\alpha_i) \Gamma(\alpha_j) \Gamma(\alpha_i) - \Gamma(\alpha_j) \Gamma(\alpha_i) \Gamma(\alpha_i) = 0,$$

because $\Gamma(\alpha_i) \Gamma(\alpha_i)$ commutes with $\Gamma(\alpha_j)$ (cf. Corollary 2.4). Altogether,

$$\left[\hat{J}(\alpha_i), [\hat{J}(\alpha_i), \hat{J}(\alpha_j)]\right] = \frac{1}{2} \sum_{A,B=1}^{k} \sum_{\alpha,\beta=1}^{l} \phi_A^i \frac{1}{2} X(\alpha_j)_{AB} (-4 \Gamma(\alpha_j)^{\alpha\beta} \phi_B^\beta$$

$$= - \sum_{A,B} \phi_A^i X(\alpha_j)_{AB} \Gamma(\alpha_j)^{\alpha\beta} \phi_B^\beta$$

$$= - \hat{J}(\alpha_j),$$

again as desired in view of Theorem 2.1.

One concludes that the assignment

$$\sigma(X_i) := \hat{J}(\alpha_i)$$

defines a finite-dimensional representation of $\mathfrak{k}$.

4 Extending a generalized $\frac{1}{2}$-spin representation
— a coordinate-free approach

In this section we discuss a coordinate-free version of Proposition 3.5. We stress that in this section we make use of the usual definition of (anti-)commutators: For endomorphisms of a real vector space $V$ define the (anti-)commutator as

$$[\cdot, \cdot] : \text{End}(V) \times \text{End}(V) \to \text{End}(V)$$

$$[A, B] \mapsto A \circ B - B \circ A$$

and

$$\{\cdot, \cdot\} : \text{End}(V) \times \text{End}(V) \to \text{End}(V)$$

$$\{A, B\} \mapsto A \circ B + B \circ A$$

where $A, B \in \text{End}(V)$ and $\circ$ denotes concatenation of (linear) maps.

As in Definition 3.4 let

$$\lambda := \{\alpha_i \mid 1 \leq i \leq n\} \cup \{\alpha_i + \alpha_j \in \Phi^e \mid (i, j) \text{ form an edge of the Dynkin diagram}\}.$$

Proposition 4.1. A map $X : \lambda \to \text{End}(V)$ satisfying for all $\alpha, \beta \in \lambda$

$$[X(\alpha), X(\beta)] = 0 \quad \text{if} \ \ (\alpha|\beta) = 0 \quad (4.1)$$

$$\{X(\alpha), X(\beta)\} = X(\alpha \pm \beta) \quad \text{if} \ (\alpha|\beta) = \mp 1 \ \text{and} \ \alpha \pm \beta \in \lambda \quad (4.2)$$

provides a finite-dimensional representation $\sigma$ of $\mathfrak{k}$ via the assignment

$$\sigma(X_i) := X(\alpha_i) \otimes \Gamma(\alpha_i) \in \text{End}(V \otimes S)$$

on the Berman generators $X_1, \ldots, X_n$ of $\mathfrak{k}$, where the $\Gamma(\alpha_i), 1 \leq i \leq n$ are the anti-symmetric real matrices from (3.11).
Remark 4.2.  

i. Defining $X : \lambda \to \text{End}(\mathbb{R}) = \mathbb{R}$ as the constant map $X \equiv \frac{1}{2}$ provides the generalized spin representation from [3], cf. Corollary 2.4. On the other hand, in the approach taken by Kleinschmidt and Nicolai described in Proposition 3.5, one needs to define $X : \lambda \to \mathbb{R}$ as the constant map $X \equiv \frac{1}{2}$ in order to obtain the generalized spin representation from [3]. This difference in normalization stems from the differences in normalizations of the underlying Clifford algebras when comparing [3] Example 3.2 with (3.1) on page 5. Similar differences are visible in the formulae for the $2$-spin representations given in [3] on page 13 and [3] on page 14 below.

ii. Contrary to Proposition 3.5, the above coordinate-free version does not require the map $X : \lambda \to \text{End}(V)$ to take images in the set of self-adjoint/symmetric operators.

iii. Paul Levy pointed out to us the following. Let $W$ be the Weyl group of $g$ and let $\rho : W \to \text{GL}(V)$ be a representation. The ansatz $X(\alpha) := \rho(s_\alpha) - \text{id}$ leads to

$$\rho(\alpha s_\beta) + \rho(\beta s_\alpha) - \rho(\alpha) - \rho(\beta) + \frac{1}{2}\text{id}$$

$$= \left(\rho(\alpha) - \frac{1}{2}\text{id}\right) \left(\rho(\beta) - \frac{1}{2}\text{id}\right) + \left(\rho(\beta) - \frac{1}{2}\text{id}\right) \left(\rho(\alpha) - \frac{1}{2}\text{id}\right)$$

$$= \{X(\alpha), X(\beta)\}$$

$$= X(\alpha + \beta)$$

$$= \rho(s_{\alpha+\beta}) - \frac{1}{2}\text{id}$$

$$= \rho(s_\alpha s_\beta s_\alpha) - \frac{1}{2}\text{id}$$

for each pair $\alpha, \beta$ forming an $A_2$-subdiagram. One concludes that

$$\{X(\alpha), X(\beta)\} = X(\alpha + \beta)$$

in fact is equivalent to

$$\rho(s_\alpha s_\beta s_\alpha) - \rho(s_\alpha s_\beta) - \rho(s_\beta s_\alpha) + \rho(s_\alpha) + \rho(s_\beta) - \text{id} = 0. \quad (4.3)$$

Similar computations imply that in fact any case covered by (4.2) using the ansatz $X(\alpha) := \rho(s_\alpha) - \text{id}$ is equivalent to (4.3). Furthermore, one quickly computes that $[\rho(s_\alpha) - \text{id}, \rho(s_\beta) - \text{id}] = 0$ whenever $\langle \alpha, \beta \rangle = 0$, because this is equivalent to $s_\alpha s_\beta = s_\beta s_\alpha$. We conclude that for the ansatz $X(\alpha) := \rho(s_\alpha) - \text{id}$ it suffices to check (4.3) for each pair $\alpha, \beta$ forming an $A_2$-subdiagram.

iv. Paul Levy also pointed out to us that the identity

$$\rho(s_\alpha s_\beta s_\alpha) - \rho(s_\alpha s_\beta) - \rho(s_\beta s_\alpha) + \rho(s_\alpha) + \rho(s_\beta) - \text{id} = 0$$

holds if and only if the given representation $W \geq \text{Sym} = \langle s_\alpha, s_\beta \rangle \to \text{GL}(V) : w \mapsto \rho(w)$ does not contain a sign representation as an irreducible component. Indeed, among the irreducible representations of $\text{Sym}$, the trivial and the geometric representations satisfy (4.3) whereas the sign representation does not.

Proof of Proposition 4.4. By the homomorphism theorem it suffices to establish that the commutator $[\sigma(X_1), \sigma(X_3)]$ satisfies the relations from Theorem 2.1.
In case \((i, j)\) do not form an edge, one computes the following:

\[
[s(\alpha_i), s(\alpha_j)] = (X(\alpha_i) \otimes \Gamma(\alpha_i)) \circ (X(\alpha_j) \otimes \Gamma(\alpha_j)) \\
- (X(\alpha_i) \otimes \Gamma(\alpha_j)) \circ (X(\alpha_i) \otimes \Gamma(\alpha_i)) \\
= X(\alpha_i) X(\alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) - X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) \\
+ X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) - X(\alpha_i) X(\alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) \\
= [X(\alpha_i), X(\alpha_j)] \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) + X(\alpha_i) X(\alpha_j) \otimes \{\Gamma(\alpha_i), \Gamma(\alpha_j)\} \\
= 0,
\]

because \([X(\alpha_i), X(\alpha_j)] = 0\) by hypothesis \(\text{[4.1]}\) and \([\Gamma(\alpha_i), \Gamma(\alpha_j)] = 0\) by Proposition \(\text{[2.3]}\).

In case \((i, j)\) is an edge, Proposition \(\text{[2.3]}\) and hypothesis \(\text{[4.2]}\) yield

\[
\{\Gamma(\alpha_i), \Gamma(\alpha_j)\} = 0 \quad \text{and} \quad [X(\alpha_i), X(\alpha_j)] = X(\alpha_i + \alpha_j).
\]

Hence

\[
[s(\alpha_i), s(\alpha_j)] = X(\alpha_i) X(\alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) - X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) \\
= X(\alpha_i) X(\alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) + X(\alpha_i) X(\alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) \\
- X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) - X(\alpha_j) X(\alpha_i) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) \\
= \{X(\alpha_i), X(\alpha_j)\} \otimes \Gamma(\alpha_i) \Gamma(\alpha_j) - X(\alpha_i) X(\alpha_j) \otimes \{\Gamma(\alpha_i), \Gamma(\alpha_j)\} \\
= X(\alpha_i + \alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j).
\]

Moreover, since the matrices \(\frac{1}{2} \Gamma(\alpha_i), \ldots, \frac{1}{2} \Gamma(\alpha_n)\) provide a generalized spin representation, by definition one has \(\Gamma(\alpha_i)^2 = 4\rho(X_i)^2 = -\text{id}_\mathfrak{g}\) and by Proposition \(\text{[2.3]}\) the matrices \(\Gamma(\alpha_i)\) and \(\Gamma(\alpha_j)\) anti-commute. Therefore one has the following:

\[
[s(\alpha_i), [s(\alpha_i), s(\alpha_j)]] = (X(\alpha_i) \otimes \Gamma(\alpha_i)) \circ (X(\alpha_i + \alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j)) \\
- (X(\alpha_i + \alpha_j) \otimes \Gamma(\alpha_i) \Gamma(\alpha_j)) \circ (X(\alpha_i) \otimes \Gamma(\alpha_i)) \\
= (X(\alpha_i) X(\alpha_i + \alpha_j) \Gamma(\alpha_i) \Gamma(\alpha_j)) \\
- (X(\alpha_i) X(\alpha_i + \alpha_j) \Gamma(\alpha_i) \Gamma(\alpha_j)) \\
- (X(\alpha_i + \alpha_j) X(\alpha_i) \Gamma(\alpha_i) \Gamma(\alpha_j)) \\
- (X(\alpha_i + \alpha_j) X(\alpha_i) \Gamma(\alpha_i) \Gamma(\alpha_j)) \\
= -[X(\alpha_i), X(\alpha_i + \alpha_j)] \otimes \Gamma(\alpha_j) \\
- X(\alpha_j) \otimes \Gamma(\alpha_j) = -s(\alpha_j).
\]

\[\square\]

5 Towards \(\frac{3}{2}\)-spin representations

Let \(V := \mathfrak{h}^*\). If the generalized Cartan matrix \(A\) is invertible, then \(V = \text{span}_\mathbb{R}\{\alpha_1, \ldots, \alpha_n\}\); otherwise \(V\) is of higher dimension \(k := 2n - \text{rk}(A)\). In both cases the invariant bilinear form on \(\mathfrak{g}\) induces a nondegenerate bilinear form \((\cdot | \cdot)\) on \(V\). Let \(v^1, \ldots, v^k\) be a basis of \(V\) and define

\[
G^{ab} := (v^a, v^b).
\]

That is, \((G^{ab})_{1 \leq a, b \leq k}\) is the Gram matrix of the bilinear form \((\cdot | \cdot)\) on \(V\) with respect to the basis \(v^1, \ldots, v^k\). Moreover, define \((G_{ab})_{1 \leq a, b \leq k} := (G^{ab})_{1 \leq a, b \leq k}^{-1}\), i.e.,

\[
\sum_{b=1}^{k} G^{ab} G_{bc} = \delta_{ac}.
\]

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Note that Cramer's rule implies that also the matrix \((G_{ab})_{1 \leq a, b \leq k}\) is symmetric.

**Proposition 5.1.** Let \(\alpha = \sum_{i=1}^{k} \alpha_i v^i \in V = \mathfrak{h}^*\) be a real root. Then the map
\[
X : \lambda \to \mathbb{R}^{k \times k}
\]
\[
\alpha \mapsto (X(\alpha)_{ab})_{1 \leq a, b \leq k}
\]
defined via
\[
X(\alpha)_{ab} = -\frac{1}{2} \gamma_a \gamma_b + \frac{1}{4} G_{ab}.
\]
(5.1)
yields a set of matrices that satisfy hypotheses (3.13) and (3.14) of Proposition 3.5. In particular, this provides a finite-dimensional representation of \(k\) via \(\sigma(X_i) = X(\alpha_i) \otimes \Gamma(\alpha_i)\).

**Remark 5.2.** Formula (5.1) is [5, (4.21), p. 15].

**Proof of Proposition 5.1.** It suffices to establish the hypotheses of Proposition 3.5. Note first that the matrices \(X(\alpha)\) are symmetric by definition. For \(\alpha, \beta \in \lambda\) with \((\alpha|\beta) = 0\) one computes
\[
[X(\alpha), X(\beta)]_{ad} = \sum_{b, c, d} \left( -\frac{1}{2} \gamma_a \gamma_b + \frac{1}{4} G_{ab} \right) \left( -\frac{1}{2} \gamma_c \gamma_d + \frac{1}{4} G_{cd} \right) \left( -\frac{1}{2} \gamma_a \gamma_b + \frac{1}{4} G_{ab} \right) \left( -\frac{1}{2} \gamma_c \gamma_d + \frac{1}{4} G_{cd} \right) - \frac{1}{8} \gamma_a \gamma_d \gamma_b \gamma_c.
\]

Moreover, for \((\alpha | \beta) = \mp 1\) one computes

\[
\{X(\alpha), X(\beta)\}_{ad} = \sum_{b,c=1}^{k} \left( -\frac{1}{2} \alpha_a \alpha_b + \frac{1}{4} G_{ab} \right) G^{bc} \left( -\frac{1}{2} \beta_c \beta_d + \frac{1}{4} G_{cd} \right)
\]

\[
= \sum_{b,c=1}^{k} \left( -\frac{1}{2} \beta_a \beta_b + \frac{1}{4} G_{ab} \right) G^{bc} \left( -\frac{1}{2} \alpha_c \alpha_d + \frac{1}{4} G_{cd} \right)
\]

\[
= \frac{1}{4} \alpha_a \beta_d (\alpha | \beta) - \frac{1}{8} \alpha_a \alpha_d - \frac{1}{8} \beta_a \beta_d + \frac{1}{16} G_{ab} G^{bc} G_{cd}
\]

\[
= \frac{1}{4} \left( -\alpha_a \alpha_d - \beta_a \beta_d \mp (\alpha_a \beta_d + \beta_a \alpha_d) + \frac{1}{2} G_{ad} \right)
\]

\[
= \frac{1}{2} \left( -\frac{1}{2} (\alpha_a \pm \beta_a) (\alpha_d \pm \beta_d) + \frac{1}{4} G_{ad} \right)
\]

\[
= \frac{1}{2} X(\alpha \pm \beta)_{ad}. \quad \Box
\]

We conclude this section with the following coordinate-free version of Proposition 5.1.

**Proposition 5.3.** For \(V = \mathfrak{h}^*\) let \((\cdot | \cdot)\) denote the induced invariant bilinear form on \(\mathfrak{h}^*\). Define

\[
X : \Delta^o \to \text{End}(\mathfrak{h}^*) \quad \alpha \mapsto \alpha (\alpha | \cdot) + \frac{1}{2} id_{\mathfrak{h}^*}. \quad (5.2)
\]

Then \(X\) satisfies (1.1) and (2.1) for all real roots \(\alpha, \beta\) with \((\alpha | \beta) \in \{0, \pm 1\}\) and thus provides a representation \(\sigma\) of \(\mathfrak{k}\).

**Proof.** First consider \(\alpha, \beta \in \Delta^o\) such that \((\alpha | \beta) = 0\). Then one has

\[
[X(\alpha), X(\beta)] = \left( -\alpha (\alpha | \cdot) + \frac{1}{2} id_{\mathfrak{h}^*} \right) \left( -\beta (\beta | \cdot) + \frac{1}{2} id_{\mathfrak{h}^*} \right)
\]

\[
- \left( -\beta (\beta | \cdot) + \frac{1}{2} id_{\mathfrak{h}^*} \right) \left( -\alpha (\alpha | \cdot) + \frac{1}{2} id_{\mathfrak{h}^*} \right)
\]

\[
= \alpha (\alpha | \beta) (\beta | \cdot) - \frac{1}{2} \alpha (\alpha | \cdot) - \frac{1}{2} \beta (\beta | \cdot) + \frac{1}{4} id_{\mathfrak{h}^*}
\]

\[
- \beta (\beta | \alpha) (\alpha | \cdot) + \frac{1}{2} \beta (\beta | \cdot) + \frac{1}{2} \alpha (\alpha | \cdot) - \frac{1}{4} id_{\mathfrak{h}^*}
\]

\[
= 0.
\]
Moreover, for \((\alpha|\beta) = \mp 1\) one has the following:

\[
\{X(\alpha), X(\beta)\} = \left(-\alpha (\alpha|\cdot) + \frac{1}{2} \text{id}_{\mathfrak{h}^*}\right) \left(-\beta (\beta|\cdot) + \frac{1}{2} \text{id}_{\mathfrak{h}^*}\right) \\
+ \left(-\beta (\beta|\cdot) + \frac{1}{2} \text{id}_{\mathfrak{h}^*}\right) \left(-\alpha (\alpha|\cdot) + \frac{1}{2} \text{id}_{\mathfrak{h}^*}\right) \\
= \alpha (\alpha|\beta) (\beta|\cdot) - \frac{1}{2} \alpha (\alpha|\cdot) - \frac{1}{2} \beta (\beta|\cdot) + \frac{1}{4} \text{id}_{\mathfrak{h}^*} \\
+ \beta (\beta|\alpha) (\alpha|\cdot) - \frac{1}{2} \beta (\beta|\cdot) - \frac{1}{2} \alpha (\alpha|\cdot) + \frac{1}{4} \text{id}_{\mathfrak{h}^*} \\
= \mp \alpha (\beta|\cdot) \mp \beta (\alpha|\cdot) - \alpha (\alpha|\cdot) - \beta (\beta|\cdot) + \frac{1}{2} \text{id}_{\mathfrak{h}^*} \\
= -(\pm \alpha (\beta|\cdot) \pm \beta (\alpha|\cdot) + \alpha (\alpha|\cdot) + \beta (\beta|\cdot)) + \frac{1}{2} \text{id}_{\mathfrak{h}^*} \\
= - (\alpha \pm \beta) (\alpha \pm \beta|\cdot) + \frac{1}{2} \text{id}_{\mathfrak{h}^*}.
\]

\[
\cdot = \alpha (\alpha \pm \beta) (\beta \pm \alpha|\cdot) - \alpha (\alpha|\cdot) - \beta (\beta|\cdot) + \frac{1}{2} \text{id}_{\mathfrak{h}^*}.
\]

\[\square\]

**Remark 5.4.** Note that the canonical (non-reduced geometric) Weyl group representation \(\rho : W \to \text{GL}(\mathfrak{h}^*)\) acts via \(\rho(s_a)(x) = x - (\alpha|x)\alpha\) and so one has \(X(\alpha) = \rho(s_a) - \frac{1}{2} \text{id}\). Therefore Remark 4.2 applies and the statement of Proposition 5.3 in fact follows from the observation that \(\rho\) (restricted to any standard subgroup \(\text{Sym}_3\)) does not contain the sign representation as an irreducible component.

### 6 Towards \(\pm \frac{5}{2}\)-spin representations

**Definition 6.1.** Let \((T_{ab})_{a,b} \in \mathbb{R}^{k \times k}\) and \((U_{ab})_{a,b} \in \mathbb{R}^{k \times l}\). Then

\[
(T_{(ab)})_{a,b} \in \mathbb{R}^{k \times k}
\]

denotes the matrix with components

\[
T_{(ab)} := \frac{1}{2} T_{ab} + \frac{1}{2} T_{ba}.
\]

Moreover, for \(1 \leq a, b, c \leq k\) and \(1 \leq d \leq l\) define

\[
T_{a(dU_{cd})} := \frac{1}{2} T_{ab} U_{cd} + \frac{1}{2} T_{ba} U_{bd}.
\]

This notation is called the *symmetrizer bracket*.

**Lemma 6.2.** As in Section 5 let \(v^1, \ldots, v^k\) be a basis of \(\mathfrak{h}^*\), let \((G_{ab})_{a,b}\) be the Gram matrix of the invariant form with respect to this basis, let \((G_{ab})_{a,b}\) be its inverse, let \(\alpha = \sum_{i=1}^k \alpha_i v^i, \beta = \sum_{i=1}^k \beta_i v^i \in \mathfrak{h}^*\), and let

\[
\alpha' := \sum_{j=1}^k G^{ij} \alpha_j, \quad \beta' := \sum_{j=1}^k G^{ij} \beta_j.
\]
Then the following identities hold:

\[
\sum_{g,h=1}^{k} \alpha^g \alpha^h G_{g(c)G_{d}h} = \alpha_c \alpha_d = \alpha_{c,c_d} \tag{6.1}
\]

\[
\sum_{g,h=1}^{k} \alpha^g \alpha^h \beta_{(gG_{h})(c\beta_d)} = (\alpha|\beta) \alpha_{c\beta_d} \tag{6.2}
\]

\[
\sum_{e,f,g,h=1}^{k} \alpha_{(aG_{b})(cG_{d})} G_{g(bG_{h})} \beta_{(gG_{h})(c\beta_d)} = \frac{1}{2} \alpha_{(a\beta_b)\alpha_{c\beta_d}} + \frac{1}{2} (\alpha|\beta) \alpha_{(aG_{b})(c\beta_d)} \tag{6.3}
\]

\[
\sum_{e,f,g,h=1}^{k} G_{a(cG_{b})} G^{(cG_{d})} G^{f h} G_{g(eG_{f})} \beta_{(gG_{h})(c\beta_d)} = \beta_{(aG_{b})(c\beta_d)} \tag{6.4}
\]

\[
\sum_{e,f,g,h=1}^{k} G_{a(cG_{b})} G^{(cG_{d})} G^{f h} G_{g(eG_{f})} = G_{a(cG_{d})} \tag{6.5}
\]

Proof. Observe first that

\[
\sum_{g=1}^{k} \alpha^g G_{gc} = \sum_{g,i=1}^{k} G^{gi} \alpha_i G_{gc} = \sum_{g,i=1}^{k} G_{cg} G^{gi} \alpha_i = \alpha_c
\]

and

\[
\sum_{g=1}^{k} \alpha^g \beta_g = \sum_{g,i=1}^{k} \alpha^i \beta_g = \sum_{g,i=1}^{k} \alpha^i \beta_g = (\alpha|\beta).
\]

Equality (6.1) can then be established as follows:

\[
\sum_{g,h=1}^{k} \alpha^g \alpha^h G_{g(c)G_{d}h} = \frac{1}{2} \sum_{g,h,i,j=1}^{k} G^{gi} \alpha_i G^{hj} \alpha_j (G_{gc} G_{dh} + G_{gd} G_{ch})
\]

\[
= \frac{1}{2} \sum_{g,h,i,j=1}^{k} G_{cg} G^{gi} \alpha_i G_{dh} G^{hj} \alpha_j + G_{dg} G^{gi} \alpha_i G_{ch} G^{hj} \alpha_j
\]

\[
= \frac{1}{2} (\alpha_c \alpha_d + \alpha_d \alpha_c) = \alpha_{c,c_d} = \alpha_{c,d}.
\]

A similar computation yields equality (6.2):

\[
\sum_{g,h=1}^{k} \alpha^g \alpha^h \beta_{(gG_{h})(c\beta_d)} = \frac{1}{4} \sum_{g,h=1}^{k} \alpha^g \alpha^h (\beta_g G_{hc} \beta_d + \beta_h G_{gc} \beta_d + \beta_g G_{hd} \beta_c + \beta_h G_{gd} \beta_c)
\]

\[
= \frac{1}{4} ((\alpha|\beta) \alpha_c \beta_d + (\alpha|\beta) \alpha_c \beta_d + (\alpha|\beta) \alpha_c \beta_d + (\alpha|\beta) \alpha_c \beta_d)
\]

\[
= \frac{1}{2} (\alpha|\beta) (\alpha_c \beta_d + \alpha_d \beta_c) = (\alpha|\beta) \alpha_{c,\beta_d}.
\]
For equality (6.3) one computes the following:

\[
16 \sum_{e,f,g,h=1}^k \alpha(a G_b)(e \alpha_f) G^{e g} G^{f h} \beta(g G_h)(c \beta_d) \\
= \sum_{e,f,g,h=1}^k \left( \alpha_a G_{bc} \alpha_f + \alpha_b G_{ac} \alpha_c + \alpha_a G_{bf} \alpha_e + \alpha_b G_{af} \alpha_e \right) G^{e g} G^{f h} \\
\left( \beta_g G_{hc} \beta_d + \beta_h G_{gc} \beta_d + \beta_g G_{hd} \beta_c + \beta_h G_{gd} \beta_c \right) \\
= \sum_{g,h=1}^k \left( \alpha_a \delta_{bg} \alpha^h + \alpha_b \delta_{ah} \alpha^g + \alpha_a \delta_{bh} \alpha^g + \alpha_b \delta_{eg} \alpha^h \right) \\
\left( \beta_g G_{hc} \beta_d + \beta_h G_{gc} \beta_d + \beta_g G_{hd} \beta_c + \beta_h G_{gd} \beta_c \right) \\
= \alpha_a \beta_a \alpha_e \beta_d + (\alpha | \beta) \alpha_a G_{bc} \beta_d + \alpha_a \beta_b \beta_c \alpha_d + (\alpha | \beta) \alpha_a G_{bd} \beta_c \\
+ (\alpha | \beta) \alpha_b G_{ac} \beta_d + \alpha_b \alpha_a \alpha_c \beta_d + (\alpha | \beta) \alpha_b G_{ad} \beta_c + \beta_a \beta_b \beta_c \alpha_d + (\alpha | \beta) \alpha_b G_{bc} \beta_d \\
+ (\alpha | \beta) \alpha_a G_{bc} \beta_d + \alpha_a \beta_b \alpha_c \beta_d + (\alpha | \beta) \alpha_a G_{ad} \beta_c + \alpha_b \beta_b \beta_c \alpha_d + (\alpha | \beta) \alpha_b G_{ad} \beta_c \\
= 2 (\alpha_a \beta_a \alpha_e \beta_d + \alpha_a \beta_b \beta_b \alpha_d + \beta_a \alpha_a \alpha_b \beta_d + \beta_a \alpha_b \beta_c \alpha_d) \\
+ 2 (\alpha | \beta) (\alpha_a G_{bc} \beta_d + \alpha_a G_{bd} \beta_c + \alpha_a G_{ac} \beta_d + \alpha_b G_{ad} \beta_c) \\
= 8 \alpha_a \beta_b \beta_c \alpha_d + 8 (\alpha | \beta) \alpha_a G_{bc} \beta_d \\
\text{and, hence,} \\
\sum_{e,f,g,h=1}^k \alpha(a G_b)(e \alpha_f) G^{e g} G^{f h} \beta(g G_h)(c \beta_d) = \frac{1}{2} \alpha(a \beta_b \alpha_c \beta_d) + \frac{1}{2} (\alpha | \beta) \alpha(a G_b)(c \beta_d). \\
\]

Equality (6.4) can be established as follows:

\[
\sum_{e,f,g,h=1}^k G_{a(e \alpha_f)} G^{e g} G^{f h} \beta(g G_h)(c \beta_d) \\
= \frac{1}{8} \sum_{e,f,g,h=1}^k (G_{ac} G_{fb} + G_{af G_{eb}}) G^{e g} G^{f h} \left( \beta_g G_{hc} \beta_d + \beta_h G_{gc} \beta_d + \beta_g G_{hd} \beta_c + \beta_h G_{gd} \beta_c \right) \\
= \frac{1}{8} \sum_{g,h=1}^k \left( \delta_{ag} \delta_{bh} + \delta_{ah} \delta_{bg} \right) \left( \beta_g G_{hc} \beta_d + \beta_h G_{gc} \beta_d + \beta_g G_{hd} \beta_c + \beta_h G_{gd} \beta_c \right) \\
= \frac{1}{8} \left( \beta_a G_{bc} \beta_d + \beta_b G_{ac} \beta_d + \beta_a G_{bd} \beta_c + \beta_b G_{ad} \beta_c \right) \\
+ \frac{1}{8} \left( \beta_a G_{bc} \beta_d + \beta_b G_{ac} \beta_d + \beta_a G_{bd} \beta_c + \beta_b G_{ad} \beta_c \right) \\
= \frac{1}{4} \left( \beta_a G_{bc} \beta_d + \beta_b G_{ac} \beta_d + \beta_a G_{bd} \beta_c + \beta_b G_{ad} \beta_c \right) \\
= \beta(a G_b)(c \beta_d). \\
\]
Finally, equality (6.5) can be shown as follows:

\[
\sum_{c,f,g,h=1}^{k} G_{a(c,G_f)h} G^c G^f h \in G_{a(G_d)h} \\
= \frac{1}{4} \sum_{g,h=1}^{k} (\delta_{ag} \delta_{bh} + \delta_{ah} \delta_{bg}) (G_{g,b} G_{d,h} + G_{g,d} G_{a,b}) \\
= \frac{1}{4} (G_{ac} G_{db} + G_{ad} G_{cb} + G_{bc} G_{da} + G_{bd} G_{ca}) \\
= \frac{1}{2} (G_{ac} G_{db} + G_{ad} G_{cb}) \\
= G_{a(G_d)h}. 
\]

Throughout this section let \( g \) be a simply laced split real Kac–Moody algebra with maximal compact subalgebra \( \mathfrak{k} \). By Corollary 3.1, there exists a generalized \( \frac{1}{2} \)-spin representation \( \rho : \mathfrak{k} \to \text{End}(\mathbb{C}) \). In analogy to Sections 3 and 5 we make use of Clifford algebras in order to define higher spin representations.

Define \( V := \text{Sym}^2(\mathfrak{h}^*) \). Then, given a basis \( v^1, \ldots, v^k \) of \( \mathfrak{h}^* \), the vector space \( V \) admits the natural basis \( \{ v^{i_1} \otimes v^{i_2} | 1 \leq i_1 \leq i_2 \leq k \} \). Given an orthonormal basis \( f_1, \ldots, f_s \) of \( S \) as in Section 5 one arrives at a basis \( \{ v^{i_1} \otimes v^{i_2} \otimes f_j | 1 \leq i_1 \leq i_2 \leq k, 1 \leq j \leq l \} \) of \( V \otimes S \).

In analogy to (3.2) define

\[
\phi^{ab} := v^a \otimes v^b \otimes f_2 = v^b \otimes v^a \otimes f_2 =: \phi^{ba} \quad (6.6)
\]

The invariant symmetric bilinear form \( (\cdot | \cdot) \) on \( \mathfrak{h}^* \) induces a natural symmetric bilinear form on \( \mathfrak{h}^* \otimes \mathfrak{h}^* \) which, by symmetry, factors through a symmetric bilinear form \( q_1 \) on \( V = \text{Sym}^2(\mathfrak{h}^*) \). If \( (G^{ab})_{1 \leq a,b \leq k} \) as in Section 5 denotes the Gram matrix of \( (\cdot | \cdot) \) with respect to the basis \( v^1, \ldots, v^k \), the computation

\[
q_1 \left( v^a \otimes v^b, v^c \otimes v^d \right) = \frac{1}{4} q_1 \left( v^a \otimes v^b + v^b \otimes v^a, v^c \otimes v^d + v^d \otimes v^c \right) \\
= \frac{1}{4} \left[ q_1 \left( v^a \otimes v^b, v^c \otimes v^d \right) + q_1 \left( v^a \otimes v^b, v^d \otimes v^c \right) + q_1 \left( v^b \otimes v^a, v^c \otimes v^d \right) + q_1 \left( v^b \otimes v^a, v^d \otimes v^c \right) \right] \\
= \frac{1}{4} \left( G^{ac} G^{bd} + G^{ad} G^{bc} + G^{bd} G^{ac} + G^{ab} G^{cd} \right) \\
= \frac{1}{2} \left( G^{bc} G^{ad} + G^{bd} G^{ca} \right) = \frac{1}{2} \left( G^{ac} G^{bd} + G^{ab} G^{cd} \right)
\]

shows that the various symmetrizer brackets

\[
G^{ac} G^{bd} = G^{bc} G^{ad} = G^{bd} G^{ca} = G^{ab} G^{cd} = q_1 \left( v^a \otimes v^b, v^c \otimes v^d \right) = G^{ac} G^{bd} \quad \text{for all } a, b, c, d \in \{1, \ldots, k\}
\]

all describe the Gram matrix of \( q_1 \) with respect to the basis \( \{ v^{i_1} \otimes v^{i_2} | 1 \leq i_1 \leq i_2 \leq k \} \). In analogy to Section 5 define a symmetric bilinear form on the tensor product \( V \otimes S \) via

\[
q(\phi_{\alpha \beta}^{ab}, \phi_{\gamma \delta}^{cd}) := G^{ac} G^{bd} \delta_{\alpha \beta} = G^{ac} G^{bd} \delta_{\alpha \beta}.
\]

The above equality between various symmetrizer brackets makes it meaningful to define

\[
\phi^{A} := \frac{1}{2} \phi_{\alpha \beta}^{ab} + \frac{1}{2} \phi_{\beta \alpha}^{ab} \quad \phi^{B} := \frac{1}{2} \phi_{\beta \alpha}^{ab} + \frac{1}{2} \phi_{\beta \alpha}^{ab}
\]

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and

\[ G^{AB} := G^{a(c Gd)b} = G^{ac}G^{bd} = G^{ca}G^{db} = G^{c(a Gd)b} =: G^{BA} \]

in order to make the formalism of Lemma 3.1 and Proposition 3.5 applicable also in the situation of \( V = \text{Sym}^2(\mathfrak{h}^*) \) by interpreting \( A \) and \( B \) as multi-indices whose constituents vary independently between 1 and \( k \). For instance, expanding the commutator

\[ [X, Y]_{AB} = \sum_{C, D = 1}^k \left( X_{AC}G^{CD}Y_{DB} - Y_{AC}G^{CD}X_{DB} \right) \]

from (3.6) into the current setting with

- \( A \) corresponding to \( a, b, \)
- \( B \) corresponding to \( c, d, \)
- \( C \) corresponding to \( e, f, \)
- \( D \) corresponding to \( g, h \)

then yields

\[ [X, Y]_{ab cd} = \sum_{e, f, g, h = 1}^k \left( X_{ab ef}G^{eg}G^{fh}Y_{gh cd} - Y_{ab ef}G^{eg}G^{fh}X_{gh cd} \right) . \]

**Proposition 6.3.** Let \( \alpha = \sum_{i=1}^k \alpha_i v^i \in \mathfrak{h}^* \) be a real root. Then the matrices given by

\[ X(\alpha)_{ab cd} = \frac{1}{2} \alpha_a \alpha_b \alpha_c \alpha_d - \alpha_a \alpha_b + \frac{1}{4} G_a G_b \]  \hspace{1cm} (6.7) \]

satisfy for all \( \alpha, \beta \in \Delta^{re} \) such that \( (\alpha|\beta) = 0 \)

\[ [X(\alpha), X(\beta)]_{ab cd} = \sum_{e, f, g, h = 1}^k \left( X(\alpha)_{ab ef}G^{eg}G^{fh}X(\beta)_{gh cd} - X(\beta)_{ab ef}G^{eg}G^{fh}X(\alpha)_{gh cd} \right) = 0 \]

and for all \( \alpha, \beta \in \Delta^{re} \) such that \( (\alpha|\beta) = \mp 1 \)

\[ \{X(\alpha), X(\beta)\}_{ab cd} = \sum_{e, f, g, h = 1}^k \left( X(\alpha)_{ab ef}G^{eg}G^{fh}X(\beta)_{gh cd} + X(\beta)_{ab ef}G^{eg}G^{fh}X(\alpha)_{gh cd} \right) = \frac{1}{2} X(\alpha \pm \beta) . \]

In particular, the assignment \( X_i \mapsto X(\alpha_i) \) defines a finite-dimensional representation of \( \mathfrak{k} \).

**Remark 6.4.** Formula (6.7) is [5, (5.4), p. 18].

**Proof of Proposition 6.3.** It suffices to establish the hypotheses of Proposition 3.5. By definition, \( X(\alpha) \) is symmetric.

Define

\[ S_{ab cd} := \sum_{e, f, g, h = 1}^k X(\alpha)_{ab ef}G^{eg}G^{fh}X(\beta)_{gh cd} \]
and calculate the following; for the sake of the exposition in the next calculation we use Einstein’s summation convention, i.e., equal indices are summed over if one is upper and one is lower.

\[
S_{ab,cd} = \left( \frac{1}{2} \alpha_a \alpha_b \alpha_c \alpha_f - \alpha_a G_{b} \alpha_f \right) + \frac{1}{4} G_{a} \left( G_{f} \right) G^{eg} G^{fh}
\]

\[
\left( \frac{1}{2} \beta_d \beta_c \beta_b \beta_d - \beta_a G_{b} \beta_d \right) + \frac{1}{4} G_{a} \left( G_{d} \right) \right)
\]

\[
= \frac{1}{4} \alpha_a \alpha_b \alpha_c \alpha_d - \frac{1}{2} \alpha_a \alpha_b \alpha_c \alpha_d + \frac{1}{8} \alpha_a \alpha_b \alpha_c \alpha_d
\]

Next one computes the commutator \( C_{ab,cd} := \left[ X(\alpha), X(\beta) \right]_{ab,cd} \) in the case of \( (\alpha|\beta) = 0 \) to be equal to

\[
C_{ab,cd} = \sum_{e,f,g,h=1}^{k} \left( X(\alpha)_{ab,ef} G^{eg} G^{fh} X(\beta)_{gh,cd} - X(\beta)_{ab,ef} G^{eg} G^{fh} X(\alpha)_{gh,cd} \right)
\]

\[
= \frac{1}{8} \alpha_a \alpha_b \alpha_c \alpha_d + \frac{1}{2} \alpha_a \beta_b \alpha_c \beta_d - \frac{1}{4} \alpha_a G_{b} \alpha_c \alpha_d
\]

since \( \alpha_a \beta_b \alpha_c \beta_d = \beta_a \alpha_b \beta_c \alpha_d \) and the terms with matching colours cancel. (Here the symbol \( (\alpha \leftrightarrow \beta) \) denotes a repetition of all previous terms with the roles of \( \alpha \) and \( \beta \) interchanged.) In a similar fashion one calculates the anti-commutator

\[
A_{ab,cd} := \sum_{e,f,g,h=1}^{k} \left( X(\alpha)_{ab,ef} G^{eg} G^{fh} X(\beta)_{gh,cd} + X(\beta)_{ab,ef} G^{eg} G^{fh} X(\alpha)_{gh,cd} \right)
\]
for \((\alpha|\beta) = \pm 1\) to be
\[
A_{ab\alpha\beta} := \frac{1}{4} a_\alpha a_\beta b_\alpha b_\beta + \frac{1}{2} a_\alpha a_\alpha c_\beta d_\beta + \frac{1}{8} a_\alpha a_\beta a_\beta a_\beta \\
+ \frac{1}{2} a_\alpha (a_\beta b_\beta b_\alpha b_\beta) + \frac{1}{2} a_\alpha (a_\beta b_\alpha b_\beta a_\beta) + \frac{1}{2} (a_\alpha G_b c_\beta d_\beta) \\
- \frac{1}{4} a_\alpha (a_\alpha G_b b_\beta a_\beta) + \frac{1}{8} b_\alpha b_\beta b_\beta b_\beta - \frac{1}{4} b_\alpha (a_\beta b_\alpha c_\beta d_\beta) + \frac{1}{16} G_{a_\alpha (c_\beta d_\beta)} \\
+ \frac{1}{4} b_\alpha b_\beta a_\alpha a_\beta + \frac{1}{2} b_\beta (a_\alpha b_\alpha a_\beta a_\alpha) + \frac{1}{4} (a_\alpha b_\alpha b_\beta b_\alpha) \\
+ \frac{1}{2} b_\alpha b_\beta b_\beta b_\beta + \frac{1}{2} b_\alpha b_\beta b_\beta a_\alpha + \frac{1}{2} (a_\beta b_\alpha b_\beta a_\beta) + \frac{1}{2} (a_\beta b_\alpha b_\beta c_\beta d_\beta) \\
+ \frac{1}{4} b_\alpha (a_\beta b_\alpha c_\beta d_\beta) + \frac{1}{2} b_\alpha (a_\beta b_\alpha c_\beta d_\beta) + \frac{1}{2} (a_\beta b_\alpha b_\beta c_\beta d_\beta) \\
- \frac{1}{2} (a_\alpha G_b c_\beta d_\beta) + \frac{1}{2} (a_\beta b_\alpha b_\beta c_\beta d_\beta) + \frac{1}{2} (a_\beta b_\alpha b_\beta c_\beta d_\beta) + \frac{1}{8} G_{a_\alpha (c_\beta d_\beta)} \\
+ \frac{1}{2} X(\alpha \mp \beta)_{ab\alpha\beta}.
\]

This proves the claim.

Again, we conclude this section with a coordinate-free version of Proposition 6.5

**Proposition 6.5.** For \(V = h^*\) let \((\cdot|\cdot)\) denote the induced invariant bilinear form on \(h^*\). Moreover, for \(\alpha \in \Delta^\text{re}\) let \(\pi_\alpha := \alpha (\cdot|\cdot) \in \text{End} (h^*)\). Define \(X : \Delta^\text{re} \to \text{End} (\text{Sym}^2 (h^*))\) via
\[
\alpha \mapsto X(\alpha) := \pi_\alpha \otimes \pi_\alpha - (\pi_\alpha \otimes \text{id}_{h^*} + \text{id}_{h^*} \otimes \pi_\alpha) + \frac{1}{2} \text{id}_{h^*} \otimes \text{id}_{h^*}.
\]  

(6.8)

Then \(X\) satisfies (4.1) and (4.2) for all real roots \(\alpha, \beta\) with \((\alpha|\beta) \in \{0, \pm 1\}\) and thus provides a representation \(\sigma\) of \(\mathfrak{g}\) by sending
\[
X_i \mapsto \sigma (X_i) := X (\alpha_i) \otimes \Gamma (\alpha_i).
\]

**Proof.** Observe
\[
\pi_\alpha \pi_\beta = \begin{cases} 
0, & \text{if } (\alpha|\beta) = 0, \\
\pm \alpha (\beta|\cdot), & \text{if } (\alpha|\beta) = \pm 1,
\end{cases}
\]

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and abbreviate $\alpha \pi \beta := \alpha (\beta \cdot)$ and $1 \equiv \text{id}_{h^\ast}$. One computes for $\langle \alpha | \beta \rangle = 0$ that

$$[X(\alpha), X(\beta)] = \pi_\alpha \pi_\beta \otimes \pi_\alpha \pi_\beta - \pi_\alpha \pi_\beta \otimes \pi_\alpha - \pi_\alpha \otimes \pi_\alpha \pi_\beta + \frac{1}{2} \pi_\alpha \otimes \pi_\alpha$$

$$- \pi_\alpha \pi_\beta \otimes \pi_\beta + \pi_\alpha \pi_\beta \otimes 1 + \pi_\alpha \otimes \pi_\beta - \frac{1}{2} \pi_\alpha \otimes 1$$

$$- \pi_\beta \otimes \pi_\alpha \pi_\beta + \pi_\beta \otimes \pi_\alpha + 1 \otimes \pi_\alpha \pi_\beta - \frac{1}{2} \cdot 1 \otimes \pi_\alpha$$

$$+ \frac{1}{2} \pi_\beta \otimes \pi_\beta - \frac{1}{2} (\pi_\beta \otimes 1 + 1 \otimes \pi_\beta) + \frac{1}{4} \cdot 1 \otimes 1$$

$$- (\alpha \leftrightarrow \beta)$$

$$= \frac{1}{2} \pi_\alpha \otimes \pi_\alpha + \pi_\alpha \otimes \pi_\beta - \frac{1}{2} \pi_\alpha \otimes 1 + \pi_\beta \otimes \pi_\alpha$$

$$- \frac{1}{2} \cdot 1 \otimes \pi_\alpha + \frac{1}{2} \pi_\beta \otimes \pi_\beta - \frac{1}{2} (\pi_\beta \otimes 1 + 1 \otimes \pi_\beta) + \frac{1}{4} \cdot 1 \otimes 1$$

$$- (\alpha \leftrightarrow \beta)$$

$$= 0$$

because the first part is symmetric in $\alpha$ and $\beta$. (Here the symbol $(\alpha \leftrightarrow \beta)$ again denotes a repetition of all previous terms with the roles of $\alpha$ and $\beta$ interchanged.)

Before evaluating the anti-commutator consider for $\langle \alpha | \beta \rangle = \mp 1$

$$\pi_{\alpha \pm \beta} = (\alpha \pm \beta) (\alpha \pm \beta \cdot)$$

$$= \pi_\alpha \pm \alpha \pi_\beta \pm \beta \pi_\alpha + \pi_\beta$$

$$= \pi_\alpha - \pi_\alpha \pi_\beta - \pi_\beta \pi_\alpha + \pi_\beta$$

and, thus,

$$\pi_{\alpha \pm \beta} \otimes \pi_{\alpha \pm \beta} = (\pi_\alpha \pm \alpha \pi_\beta \pm \beta \pi_\alpha + \pi_\beta) \otimes (\pi_\alpha \pm \alpha \pi_\beta \pm \beta \pi_\alpha + \pi_\beta)$$

$$= (\pi_\alpha - \pi_\alpha \pi_\beta - \pi_\beta \pi_\alpha + \pi_\beta) \otimes (\pi_\alpha - \pi_\alpha \pi_\beta - \pi_\beta \pi_\alpha + \pi_\beta)$$

$$= \pi_\alpha \otimes \pi_\alpha - \pi_\alpha \otimes \pi_\alpha \pi_\beta - \pi_\alpha \otimes \pi_\beta \pi_\alpha + \pi_\alpha \otimes \pi_\beta$$

$$- \pi_\alpha \pi_\beta \otimes \pi_\alpha + \pi_\alpha \pi_\beta \otimes \pi_\alpha \pi_\beta + \pi_\alpha \pi_\beta \otimes \pi_\beta \pi_\alpha - \pi_\alpha \pi_\beta \otimes \pi_\beta$$

$$- \pi_\beta \pi_\alpha \otimes \pi_\alpha + \pi_\beta \pi_\alpha \otimes \pi_\alpha \pi_\beta + \pi_\beta \pi_\alpha \otimes \pi_\beta \pi_\alpha - \pi_\beta \pi_\alpha \otimes \pi_\beta$$

$$+ \pi_\beta \otimes \pi_\alpha - \pi_\beta \otimes \pi_\alpha \pi_\beta - \pi_\beta \otimes \pi_\alpha \pi_\beta + \pi_\beta \otimes \pi_\beta \pi_\alpha + \pi_\beta \otimes \pi_\beta \pi_\alpha.$$

(6.9)
For the anti-commutator one computes

\[
\{X(\alpha), X(\beta)\} = \pi_\alpha \pi_\beta \otimes \pi_\alpha \pi_\beta - \pi_\alpha \pi_\beta \otimes \pi_\alpha - \pi_\alpha \otimes \pi_\beta \pi_\alpha + \frac{1}{2} \pi_\alpha \otimes \pi_\alpha \\
- \pi_\alpha \pi_\beta \otimes \pi_\beta + \pi_\alpha \pi_\beta \otimes 1 + \pi_\alpha \otimes \pi_\beta - \frac{1}{2} \pi_\alpha \otimes 1 \\
- \pi_\beta \otimes \pi_\alpha \pi_\beta + \pi_\beta \otimes \pi_\alpha + 1 \otimes \pi_\alpha \pi_\beta - \frac{1}{2} \cdot 1 \otimes \pi_\alpha \\
+ \frac{1}{2} \pi_\beta \otimes \pi_\beta - \frac{1}{2} (\pi_\beta \otimes 1 + 1 \otimes \pi_\beta) + \frac{1}{4} \cdot 1 \otimes 1 \\
+ (\alpha \leftrightarrow \beta)
\]

\[
= \pi_\alpha \pi_\beta \otimes \pi_\alpha \pi_\beta - \pi_\alpha \pi_\beta \otimes \pi_\alpha - \pi_\alpha \otimes \pi_\beta \pi_\alpha - \pi_\alpha \pi_\beta \otimes \pi_\beta \\
+ \pi_\beta \pi_\alpha \otimes \pi_\beta \pi_\alpha - \pi_\beta \pi_\alpha \otimes \pi_\beta - \pi_\beta \otimes \pi_\beta \pi_\alpha - \pi_\beta \pi_\alpha \otimes \pi_\beta \\
+ \pi_\beta \otimes \pi_\beta - \pi_\alpha \otimes \pi_\beta + \pi_\alpha \otimes \pi_\beta + \frac{1}{2} \pi_\alpha \otimes \pi_\alpha \\
+ \pi_\beta \pi_\alpha \otimes 1 + \frac{1}{2} \pi_\alpha \otimes 1 + 1 \otimes \pi_\alpha \pi_\beta - \frac{1}{2} \cdot 1 \otimes \pi_\alpha \\
- \pi_\alpha \pi_\beta \otimes \pi_\beta - \frac{1}{2} \pi_\alpha \pi_\beta \otimes 1 - \frac{1}{2} \cdot 1 \otimes \pi_\beta \\
+ \frac{1}{2} \pi_\alpha \pi_\beta \otimes 1 - \frac{1}{2} \pi_\alpha \otimes \pi_\beta - \frac{1}{2} \cdot 1 \otimes \pi_\alpha \\
+ \pi_\alpha \otimes \pi_\beta - \pi_\alpha \otimes \pi_\beta + \pi_\alpha \otimes \pi_\beta + \frac{1}{2} \cdot 1 \otimes 1
\]

The two red lines and the consecutive two lines equal the term

\[
\pi_\alpha \otimes \pi_\alpha - \pi_\alpha \otimes \pi_\alpha \pi_\beta - \pi_\alpha \otimes \pi_\beta \pi_\alpha + \pi_\alpha \otimes \pi_\beta \\
- \pi_\beta \pi_\alpha \otimes \pi_\beta \pi_\alpha - \pi_\beta \pi_\alpha \otimes \pi_\beta - \pi_\beta \otimes \pi_\beta \pi_\alpha - \pi_\beta \pi_\alpha \otimes \pi_\beta \\
- \pi_\alpha \pi_\beta \otimes \pi_\alpha + \pi_\alpha \pi_\beta \otimes \pi_\alpha \pi_\beta - \pi_\alpha \pi_\beta \otimes \pi_\beta + \pi_\alpha \otimes \pi_\beta \\
\pi_\beta \otimes \pi_\beta - \pi_\beta \otimes \pi_\beta \pi_\alpha - \pi_\beta \otimes \pi_\alpha \pi_\beta + \pi_\beta \otimes \pi_\alpha \\
\pi_\alpha \pi_\beta \otimes \pi_\beta + \pi_\alpha \pi_\beta \otimes \pi_\alpha + \pi_\alpha \pi_\beta \otimes \pi_\beta + \pi_\alpha \otimes \pi_\beta + \frac{1}{2} \cdot 1 \otimes 1
\]

which is almost identical to the expression for \(\pi_{\alpha \pm \beta} \otimes \pi_{\alpha \pm \beta}\) derived in formula (6.9) if it were not for the pink terms. Nevertheless, for \(h \in \mathfrak{h}^*\) one evaluates

\[
(\pi_\alpha \pi_\beta \otimes \pi_\beta \pi_\alpha + \pi_\beta \pi_\alpha \otimes \pi_\alpha \pi_\beta) (h, h) = (\mp \alpha (\beta | h) \otimes (\mp \beta (\alpha | h)) + (\mp \beta (\alpha | h) \otimes (\mp \alpha (\beta | h))) \\
= (\alpha | h) (\beta | h) \cdot (\alpha \otimes \beta + \beta \otimes \alpha)
\]

whereas

\[
(\pi_\alpha \otimes \pi_\beta + \pi_\beta \otimes \pi_\alpha) (h, h) = \alpha (\alpha | h) \otimes (\beta | h) \beta \\
+ (\beta | h) \beta \otimes \alpha (\alpha | h) \\
= (\alpha | h) (\beta | h) \cdot (\alpha \otimes \beta + \beta \otimes \alpha) \\
= (\pi_\alpha \pi_\beta \otimes \pi_\beta \pi_\alpha + \pi_\beta \pi_\alpha \otimes \pi_\alpha \pi_\beta) (h, h)
\]

for arbitrary \(h \in \mathfrak{h}^*\). Thus, using the diagonalizability of real symmetric tensors of degree two, the first four lines of the anti-commutator are equal to \(\pi_{\alpha \pm \beta} \otimes \pi_{\alpha \pm \beta}\). The green lines and the two
consecutive lines are evaluated to be

\[ +\pi_\alpha \pi_\beta \otimes 1 - \frac{1}{2} \pi_\beta \otimes 1 + 1 \otimes \pi_\alpha \pi_\beta - \frac{1}{2} \cdot 1 \otimes \pi_\alpha \]

\[-\frac{1}{2} \pi_\beta \otimes 1 - \frac{1}{2} \otimes \pi_\beta \]

\[+\pi_\beta \pi_\alpha \otimes 1 - \frac{1}{2} \pi_\beta \otimes 1 + 1 \otimes \pi_\beta \pi_\alpha - \frac{1}{2} l \otimes \pi_\beta \]

\[-\frac{1}{2} \pi_\alpha \otimes 1 - \frac{1}{2} \cdot 1 \otimes \pi_\alpha \]

\[= \left( \pi_\alpha \pi_\beta - \frac{1}{2} \pi_\alpha - \frac{1}{2} \pi_\beta + \pi_\beta \pi_\alpha - \frac{1}{2} \pi_\beta - \frac{1}{2} \pi_\alpha \right) \otimes 1 \]

\[+1 \otimes \left( \pi_\alpha \pi_\beta - \frac{1}{2} \pi_\alpha - \frac{1}{2} \pi_\beta + \pi_\beta \pi_\alpha - \frac{1}{2} \pi_\beta - \frac{1}{2} \pi_\alpha \right) \]

\[= \left(\pi_\alpha - \pi_\alpha \pi_\beta - \pi_\beta \pi_\alpha + \pi_\beta \right) \otimes 1 \]

\[= - \left( \pi_\alpha - \pi_\alpha \pi_\beta - \pi_\beta \pi_\alpha + \pi_\beta \right) \otimes 1 \]

\[= -\pi_\alpha \pm \beta \otimes 1 - 1 \otimes \pi_\alpha \pm \beta . \]

So one finds that for \((\alpha | \beta) = \mp 1\) one has

\[\{X(\alpha), X(\beta)\} = \pi_\alpha \pm \beta \otimes \pi_\alpha \pm \beta - \pi_\alpha \pm \beta \otimes 1 - 1 \otimes \pi_\alpha \pm \beta + \frac{1}{2} \cdot 1 \otimes 1 \]

\[= \pi_\alpha \pm \beta \otimes \pi_\alpha \pm \beta - \pi_\alpha \pm \beta \otimes 1 - 1 \otimes \pi_\alpha \pm \beta + \frac{1}{2} \cdot 1 \otimes 1 \]

\[= X(\alpha \pm \beta) \]

as desired. \(\square\)

Remark 6.6. Again note that the canonical Weyl group representation \(\rho : W \rightarrow \text{GL}(\text{Sym}^2(h^*))\) yields \(X(\alpha) = \rho(s_\alpha) - \frac{1}{2} \text{id}\). Therefore Remark 6.2 applies and the statement of Proposition 6.5 in fact follows from the observation that \(\rho\) (restricted to any standard subgroup \(\text{Sym}_3\)) does not contain the sign representation as an irreducible component.

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