On the Electromagnetic Response of Charged Bosons Coupled to a Chern-Simons Gauge Field: A Path Integral Approach

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Abstract

We analyze the electromagnetic response of a system of charged bosons coupled to a Chern-Simons gauge field. Path integral techniques are used to obtain an effective action for the particle density of the system dressed with quantum fluctuations of the CS gauge field. From the action thus obtained we compute the $U(1)$ current of the theory for an arbitrary electromagnetic external field. For the particular case of a homogeneous external magnetic field, we show that the quantization of the transverse conductivity is exact, even in the presence of an arbitrary impurity distribution. The relevance of edge states in this context is analyzed. The propagator of density fluctuations is computed, and an effective action for the matter density in the presence of a vortex excitation is suggested.

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1 Introduction

Quantum Field Theory (QFT) techniques applied to Condensed Matter systems have been extensively used in the recent past [1] [2]. Specially, path integral methods have been applied successfully over the years in many problems of Condensed Matter physics such as superconductivity, superfluidity, plasmas, crystalization [3] and quantum Hall effect (QHE) [4]. The essence of this approach is the evaluation of an effective partition function, obtained from the original one by integration over some degrees of freedom. In general, this technique leads to the problem of solving a fermion or boson functional determinant, to be evaluated by some approximation scheme. An application to the study of two-body interactions in the QHE can be found in ref. [5]. Also, an interesting application to quasi one-dimensional interfaces was developed in ref. [6].

In the last years, models involving matter fields coupled to a $U(1)$ Chern-Simons gauge field were analyzed, not only as QFT models [4], but also as phenomenological Condensed Matter models [8]. The most interesting feature of these models is that the Chern-Simons field attaches quantum fluxes to the particle positions (or to the particle density distribution) thus changing the statistics of the original fields. In particular, if we attach an even number of quantum fluxes to a matter field, its statistics is preserved (bosonic or fermionic). On the other hand, if we attach an odd number of fluxes, the statistic of the relevant degrees of freedom changes from bosonic to fermionic and vice-versa. This means that a system of fermions can be regarded as either a system of bosons with an odd number of attached fluxes or a system of (another kind of) fermions plus an even number of fluxes attached to them. The first possibility has been explored in the construction of the Chern-Simons-Landau-Ginzburg (CSLG) theory for the Fractional Quantum Hall Effect (FQHE) [9]; the second one has led to the “composite fermion” model for the FQHE [10] [11].

The electromagnetic response of the CSLG theory was calculated in ref. [9]. In that paper, the Hall conductivity is evaluated in the saddle point approximation plus gaussian fluctuations. The essence of this approximation scheme can be described as follows: first, the path integral of charged bosons in the presence of an external gauge field is evaluated in the gaussian approximation, yielding a quadratic effective action for the external gauge field; then, the Chern-Simons coupling is turned on, and the integral over the statistical gauge field is computed. In this way, an interesting relation between bosonic superconductivity and incompressibility of fermionic degrees of freedom is established.

Several authors argue that the quantization of the transverse conductivity should be exact due to the topological character of the Chern-Simons term. Nevertheless, to
the best of our knowledge, a complete explicit calculation, involving random impurities, is still lacking. In the case of the Integer Quantum Hall Effect (IQHE) (where electron-electron interaction is not considered), it was shown that the Hall conductivity is not affected by a delta-type impurity \[12\] \[13\]. However, the study of a general type of impurities combined with a Chern-Simons gauge coupling is a very difficult task and no exact results are available.

Another aspect of these models is the study of density fluctuations around the ground state. At the classical level, topological \[14\] \[15\] \[16\] and non-topological \[17\] solutions to the equations of motion were found. However, it would be interesting to study if the quantum fluctuations of the gauge field do really modify the density profiles.

Motivated by these ideas, we will consider in this paper the electromagnetic response of non-relativistic bosons coupled to a Chern-Simons Gauge Field. To this aim, the path integral technique is employed, so that all degrees of freedom of the model but the particle density are integrated out; hence, the resulting partition function — a functional integral over the charge density alone — takes into account the quantum fluctuations of the gauge field. (The charge density is, perhaps, the simplest of gauge invariant objects.) In particular, the Chern-Simons constraint (that attaches fluxes to particles) is implemented exactly, rather than being artificially imposed by means of some approximation scheme.

This technique allows us to clarify some aspects of this system. First, we can write an explicit expression for the \(U(1)\) current of the model submitted to an arbitrary electromagnetic field. Specializing this general expression for the current to the case of a static and homogeneous external magnetic field, we can explicitly show that the quantization of the transverse conductivity is exact, even in the presence of any type of impurities. Thus, we generalize Prange’s result \[12\] \[13\] to the FQHE and to any kind of impurities in the context of the CSLG theory.

This paper is organized as follows: In \(\S2\) we deduce the effective action for the bosons coupled to a Chern-Simons field plus an arbitrary electromagnetic field, in the density representation. In \(\S3\) we evaluate the \(U(1)\) current of the model. We show the exact quantization of the Hall conductivity and the role played by the edge states. In \(\S4\) we analyze the dynamics of density fluctuations around the ground state. Finally we discuss our results and present our conclusions in \(\S5\).
2 The Effective Action for Bosons Coupled to a Chern-Simons Gauge Field in the Density Representation.

Let us begin by considering the Euclidean action for non-relativistic bosons coupled to a $U(1)$ Chern-Simons gauge field $a_\mu(x)$ and an arbitrary external electromagnetic field $A_\mu(x)$,

$$S = \int d^2x d\tau \left\{ \phi^\ast \left( \partial_\tau + i[A_0 + a_0] - i\mu \right) \phi - \frac{1}{m} \left| \left( \frac{1}{i} \vec{\nabla} - e(\vec{A} + \vec{a}) \right) \phi \right|^2 \right\} +$$

$$- \frac{1}{2} \int d^2x d^2y d\tau \left( \phi^\ast(x) \phi(x) - \vec{\rho} \right) V(x - y) \left( \phi^\ast(y) \phi(y) - \vec{\rho} \right) +$$

$$- \frac{i}{2} \left( \frac{\pi}{\theta} \right) \int d^2x d\tau \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho .$$

(1)

where $V(x - y)$ is an arbitrary two-body potential, and the last term is the well-known Chern-Simons term. Due to this term, the classical equation of motion related to $a_0$

$$\rho(x) = \phi^\ast \phi = \frac{\pi}{\theta} \vec{\nabla} \times a$$

(2)

is a pure constraint that attaches $\frac{\theta}{\pi}$ fluxes to the particle density. In order to represent fermionic degrees of freedom from the bare bosonic fields, we must choose $\theta = (2n+1)\pi$, with integer $n$.

The partition function is

$$Z(A_0, A_i) = \int D\phi^* D\phi D\phi D\phi D\phi \quad G_F(\phi, a_\mu) \quad e^{-S(\phi, \phi^*, a_\mu, A_\mu)}$$

(3)

where $G_F(\phi, a_\mu)$ is a gauge fixing functional, necessary to avoid overcounting of physical states in the functional integral.

The presence of the Chern-Simons field allows us to extract the external electromagnetic field $A_\mu(x)$ from the kinetic operator of the action. Making the following change of variables:

$$a_\mu \rightarrow a_\mu - A_\mu$$

$$D a_\mu \rightarrow D a_\mu$$

(4)
we obtain

\[ S = \int d^2 x d\tau \left\{ \phi^* (\partial_\tau + i a_0 - i \mu) \phi - \frac{1}{m} \left( \frac{1}{i} \tilde{\nabla} + \tilde{a} \right) \phi \right\} \]

\[ - \frac{1}{2} \int d^2 x d^2 y d\tau \ (\phi^*(x)\phi(x) - \bar{\rho}) V(x - y) (\phi^*(y)\phi(y) - \bar{\rho}) \]

\[ - \frac{i \pi}{2 \theta} \int d^2 x d\tau \ e^{i\theta} a_i \partial_\theta a_j - \frac{i \pi}{\theta} \int d^2 x d\tau \ a_0 (\tilde{\nabla} \times \tilde{a} + B) \]

\[ - \frac{i \pi}{\theta} \int d^2 x d\tau \ e^{i\theta} a_i E_j - \frac{i \pi}{2 \theta} \int d^2 x d\tau \ e^{i\theta} A_\mu \partial_\nu A_\rho \]  

(5)

where the matter field couples to the dynamical gauge field only. Here we use \( \vec{E} \) and \( B \) to denote the external electric and magnetic field, respectively. (In what follows, greek indices run from 0 to 2 and latin indices run from 1 to 2.)

Going over the density representation, it is useful to perform another change of variables,

\[ \phi(x) = \rho^{1/2}(x) e^{i\theta(x)} \]

\[ \phi^*(x) = \rho^{1/2}(x) e^{-i\theta(x)} \]  

(6)

with trivial Jacobian

\[ \mathcal{D}\phi \mathcal{D}\phi^* = \mathcal{D}\rho \mathcal{D}\theta \]  

(7)

Replacing (6) in (5) we have

\[ S = \int d^2 x d\tau \left\{ \frac{1}{2} \partial_\tau \rho + i \rho (\partial_\tau \theta + a_0) \right\} - \frac{1}{2m} \left\{ \frac{1}{4\rho} \tilde{\nabla} \rho \cdot \tilde{\nabla} \rho + \rho \left| \tilde{\nabla} \theta - \tilde{a} \right|^2 \right\} \]

\[ - \frac{1}{2} \int d^2 x d^2 y d\tau \ (\rho(x) - \bar{\rho}) V(x - y) (\rho(y) - \bar{\rho}) + \mu \int d^2 x d\tau \rho(x) \]

\[ - \frac{i \pi}{2 \theta} \int d^2 x d\tau \ e^{i\theta} a_i \partial_\theta a_j - \frac{i \pi}{\theta} \int d^2 x d\tau \ a_0 (\tilde{\nabla} \times \tilde{a} + B) \]

\[ - \frac{i \pi}{\theta} \int d^2 x d\tau \ e^{i\theta} a_i E_j - \frac{1}{2 \theta} \int d^2 x d\tau \ e^{i\theta} A_\mu \partial_\nu A_\rho \]  

(8)

The idea is to integrate out the gauge fields \( a_0(x), a_i(x) \) and the phase \( \theta(x) \), in order to obtain an effective action in terms of the density \( \rho(x) \) only. At this point, it is useful to discuss the gauge fixing functional \( G_F(\phi, a_\mu) \).

There is great freedom in choosing \( G_F(\phi, a_\mu) \), and the partition function \( Z(A_0, A_i) \) must be gauge invariant, i.e., it must not depend on any particular choice. Notice that in the “polar coordinates” we are using, a gauge transformation reads
\[ \theta(x) \rightarrow \theta(x) + \Lambda(x) \quad (9) \]
\[ a_\mu(x) \rightarrow a_\mu(x) + \partial_\mu \Lambda(x) \quad . \quad (10) \]

So, we can always choose \( \Lambda(x) = -\theta(x) \) to have \( \theta(x) = 0 \) in the action. This corresponds to choosing the fixing functional to be

\[ G_F(\theta, a_\mu) = \delta(\theta(x)) \quad . \quad (11) \]

We must prove that this is an accessible gauge, and that it does completely fix the gauge. It is not difficult to show \[2\] that this gauge is equivalent to Coulomb’s gauge

\[ G_F(\theta, a_\mu) = \delta(\vec{\nabla} \cdot \vec{a}) \quad . \quad (12) \]

Demonstrating such equivalence consists of showing the existence of a continuous gauge transformation that leads from condition (11) to (12). Decomposing the field \( a_i \) into a transversal \( a_i^\perp \) plus a longitudinal component \( a_i^L \), it is possible to rewrite the integration measure in the Coulomb’s gauge in the following form,

\[ \int D a_i D \rho D \theta \delta(\vec{\nabla} \cdot \vec{a}) \ldots = \int D a_i^\perp D \rho D \theta \ldots \quad (13) \]

as the Coulomb’s gauge sets the longitudinal component of the gauge field to zero. Let us now perform a gauge transformation

\[ a_i(x) \longrightarrow a_i(x) - \partial_i \theta(x) \quad . \quad (14) \]

This transformation removes the phase of \( \phi(x) \) from the action but adds a longitudinal component \( a_i^L(x) = -\partial_i \theta(x) \) to \( a_i^\perp(x) \). Then, we can replace \( D \theta \) with \( D a_i^L(x) \) in (13) (with a trivial Jacobian) obtaining

\[ (13) = \int D a_i^\perp D a_i^L(x) D \rho \ldots = \int D a_i^\perp D \rho D \theta \delta(\theta(x)) \ldots \quad (15) \]

In this way, we show that (11) and (12) are equivalent.

Now, we are ready to begin with the integration. We see that equation (8) has only linear terms in \( a_0(x) \). Integration is trivially carried out, yielding the well-known Chern-Simons constraint

\[ \int D a_0 \quad e^{-i \int d^2 x d \tau \ a_0(\rho - \frac{\pi}{\theta} [\vec{\nabla} \times \vec{a} + B])} = \delta(\rho - \frac{\pi}{\theta} [\vec{\nabla} \times \vec{a} + B]) \quad . \quad (16) \]
The quantization of this model in the framework of constrained systems has been considered by several authors; in particular, we find the gauge invariant approach of reference [14] very instructive. However, in the present paper, we prefer to follow a functional approach, i.e., we will integrate over all configurations of $\rho$ and $a_i$ that satisfy

$$\rho(x) = \frac{\pi}{\theta} [\vec{\nabla} \times \vec{a} + B].$$

Note that the attachment of magnetic fluxes to the particle density $\rho(x)$ is not a classical effect. It remains in all the integration over $\rho$ and $a_i$. A very interesting physical realization of this phenomenon can be found in [18]. Thus, using (16) and (11) we find for the partition function

$$Z(A_0, A_i) = \int \mathcal{D}\rho \mathcal{D}\vec{a} \delta \left( \rho - \frac{\pi}{\theta} [\vec{\nabla} \times \vec{a} + B] \right) e^{-S_F}$$

with

$$S_F = -\frac{1}{2m} \int d^2 x d\tau \left\{ \frac{1}{4\rho} \vec{\nabla} \rho \cdot \vec{\nabla} \rho + \rho |\vec{a}|^2 \right\} + \mu \int d^2 x d\tau \rho(x)$$

$$- \frac{1}{2} \int d^2 x d^2 y d\tau \left( \rho(x) - \bar{\rho} \right) V(x - y) \left( \rho(y) - \bar{\rho} \right)$$

$$- \frac{i}{2} \int d^2 x d\tau \epsilon^{ij} a_i \partial_j \rho - \frac{i}{2} \int d^2 x d\tau \epsilon^{ij} a_i E_j$$

$$- \frac{i}{2} \int d^2 x d\tau \epsilon^{\mu \nu \rho} A_\mu \partial_\nu A_\rho$$

Clearly, $S_F$ is not gauge invariant, as we have fixed the gauge to perform the integrals. Of course, after the integration, this symmetry must be restored. Another observation about $S_F$ is that the density $\rho(x)$ has no independent dynamics; rather, all of its dynamics depends on its coupling to the gauge field. This is an important fact in the study of the dynamic of density fluctuations. We will return to this point in section §4.

The next step of our development is to integrate out the field $a_i$, to obtain an effective action only for $\rho(x)$. In order to do that, it is useful to decompose the bidimensional vector $a_i$ into a longitudinal plus a transversal part

$$a_i = \partial_i \phi + \epsilon_{ij} \partial_j \eta$$

$$7$$
This is a linear transformation. So,

$$D a_1 D a_2 \rightarrow D \eta D \varphi$$  \hspace{1cm} (21)

This decomposition is equivalent to describing a vector field through its rotational and divergence, since

$$\vec{\nabla} \cdot \vec{a} = \nabla^2 \varphi$$  \hspace{1cm} (22)

$$\vec{\nabla} \times \vec{a} = -\nabla^2 \eta$$  \hspace{1cm} (23)

Replacing (20) in (19) we find

$$S_F = -\frac{1}{2m} \int d^2 x d \tau \left\{ \frac{1}{4 \rho} \vec{\nabla} \rho \cdot \vec{\nabla} \rho \right\} + \mu \int d^2 x d \tau \rho(x)$$

$$- \frac{1}{2} \int d^2 x d^2 y d \tau \ (\rho(x) - \bar{\rho}) V(x - y) (\rho(y) - \bar{\rho})$$

$$- \frac{1}{2m} \int d^2 x d \tau \rho(x) \left( \vec{\nabla} \varphi \cdot \vec{\nabla} \varphi + \vec{\nabla} \eta \cdot \vec{\nabla} \eta \right) - i \frac{\pi}{\theta} \int d^2 x d \tau \ \varphi \partial_\theta \nabla^2 \eta$$

$$+ \frac{i \pi}{\theta} \int d^2 x d \tau \ \varphi (\vec{\nabla} \times \vec{E}) + \eta (\vec{\nabla} \cdot \vec{E})$$

$$- \frac{i}{2} \int d^2 x d \tau \ e^{i \mu \nu} A_\mu \partial_\nu A_\rho$$  \hspace{1cm} (24)

This action is quadratic in the field \( \varphi \). Moreover, the \( \delta \) – functional in (18) depends only on \( \eta \). Therefore, we can integrate over \( \varphi \) using the relation

$$\int D \varphi e^{-S_F} \int d^2 x \ \frac{1}{2} \varphi(x) \hat{O} \varphi(x) + J(x) \varphi(x) = D e^{-1/2} (\hat{O}) e^{\frac{1}{2} \int d^2 x d y \ J(x) \hat{O}^{-1}(x-y) J(y)}$$  \hspace{1cm} (25)

thus obtaining

$$Z(A_0, A_i) = \int D \rho D \eta \ \delta(\rho - \frac{\pi}{\theta} [B - \nabla^2 \eta]) \ e^{-S_F^\prime}$$  \hspace{1cm} (26)

with

$$S_F^\prime = -\frac{1}{2m} \int d^2 x d \tau \left\{ \frac{1}{4 \rho} \vec{\nabla} \rho \cdot \vec{\nabla} \rho \right\} + \mu \int d^2 x d \tau \rho(x) - \frac{1}{2} \text{Tr} \ln(\vec{\nabla} \cdot \rho \vec{\nabla})$$

$$- \frac{1}{2} \int d^2 x d^2 y d \tau \ (\rho(x) - \bar{\rho}) V(x - y) (\rho(y) - \bar{\rho})$$

8
\[- \frac{1}{2m} \int d^2x d\tau \, \rho(x) \vec{\nabla} \eta \cdot \vec{\nabla} \eta \]
\[+ \frac{m}{4} \frac{\pi^2}{\theta^2} \int d^2x d^2y d\tau \, (\partial_0 \nabla^2 \eta(x) - \vec{\nabla} \times \vec{E}(x)) \int d^2 \eta \, \frac{1}{\nabla^2} (\partial_0 \nabla^2 \eta(y) - \vec{\nabla} \times \vec{E}(y)) \]
\[+ i \frac{\pi}{\theta} \int d^2x d\tau \, \eta(\vec{\nabla} \cdot \vec{E}) - i \frac{\pi}{2\theta} \int d^2x d\tau \, \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \]

(27)

where we have used the fact that \( \ln \text{Det} \hat{O} = Tr \ln \hat{O} \).

Finally, we must integrate over the transversal component of \( a_i \), say \( \eta(x) \). This is not a difficult task because \( \eta(x) \) is in the argument of a \( \delta \)–functional (see eq. (16)). From (17) we see that the support of the \( \delta \)–functional consists in all the field configurations that satisfy
\[ \nabla^2 \eta = B - \frac{\theta}{\pi} \rho \]

(28)

We can invert this linear differential equation by using
\[ \eta(x) = -\frac{\theta}{\pi} \int d^2y \, G(x - y)(\rho(y) - \frac{\pi}{\theta} B(y)) \]

(29)

where \( G(x - y) \) is the Green function for the Laplacian operator,
\[ \nabla^2 G(x - y) = \delta(x - y) \]

(30)

The well-known property of the \( \delta \)–function
\[ \delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, \quad f(x_i) = 0. \]

(31)

can be extended to the \( \delta \)–functional, yielding
\[ \delta \left( \rho - \frac{\pi}{\theta} (B - \nabla^2 \eta) \right) = \frac{1}{|\nabla^2 \delta|} \delta \left( \eta(x) + \frac{\theta}{\pi} \int d^2y \, G(x - y) \left\{ \rho(y) - \frac{\pi}{\theta} B(y) \right\} \right) \]

(32)

The factor \( |\nabla^2 \delta| \) is field-independent and can be absorbed in the global normalization constant of the partition function. Using (32), it is now simple to functionally integrate out the field \( \eta \), obtaining
\[ Z(A_0, A_i) = e^{\frac{\pi^2}{\theta} \int d^2x d\tau \, \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho} \int \mathcal{D} \rho \, e^{-S_{\text{eff}}(\rho, \vec{E}, B)} \]

(33)
\[ S_{\text{eff}} = \frac{m}{2} \int d^3x d^3y \partial_0 \rho \left\{ \frac{1}{\nabla \cdot (\rho \nabla)} \right\} \partial_0 \rho - \frac{1}{2m} \int d^2x d^2r \left\{ \frac{1}{4\rho} \vec{\nabla} \rho \cdot \vec{\nabla} \rho \right\} 
\]

\[- \frac{1}{2} \int d^2x d^2y d^2r \left( \rho(x) - \rho \right) V(x-y) \left( \rho(y) - \rho \right) - \frac{1}{2} Tr \ln \left( \vec{\nabla} \cdot \frac{\rho}{m} \vec{\nabla} \right) \]

\[- \frac{1}{2} \int d^2x d^2y d^2r \left( \rho(x) - \frac{\pi}{\theta} B(x) \right) F(x-y) \left( \rho(y) - \frac{\pi}{\theta} B(y) \right) \]

\[- \frac{1}{2m} \frac{\theta^2}{\pi^2} \int d^2x d^2y d^2zd^2r \vec{\nabla} G(x-y) \cdot \vec{\nabla} G(x-z) \times \]

\[ \left( \rho(x) - \frac{\pi}{\theta} B(x) \right) \left( \rho(y) - \frac{\pi}{\theta} B(y) \right) \left( \rho(z) - \frac{\pi}{\theta} B(z) \right) \]

\[- \int d^2x d^2y d^2r \left( \rho(x) - \frac{\pi}{\theta} B(x) \right) G(x-y) \vec{\nabla} \cdot \vec{E}(y) \]  (34)

where

\[ F(x-y) = \frac{\theta}{\pi} \frac{1}{m} \int d^2z B(z) \vec{\nabla} G(z-x) \cdot \vec{\nabla} G(z-y) \]  (35)

Equations (33) and (34) are the main results of this section. \( S_{\text{eff}} \) is the effective action for the Chern-Simons Bosons in the density representation, interacting with an arbitrary electromagnetic field. This is a gauge invariant action, as it only depends on \( \rho(x) \), \( \vec{E} \) and \( B \). The coupling of the charge density to the electromagnetic field is very peculiar. The magnetic field acts as a “background density”, entering the action only through terms of the form \( (\rho(x) - \frac{\pi}{\theta} B(x)) \). The electric field couples to the density only through its divergence \( \vec{\nabla} \cdot \vec{E} \), which is obviously proportional to the external charges (impurities, for example). So the last term of (34) is simply the two-dimensional Coulomb energy between the external charges and the charge fluctuations over the magnetic background. In this term we absorbed the chemical potential as it can be simulated by a uniform background charge density. Note that, in particular, a homogeneous electric field do not couple to the matter field.

The coupling of the gauge field with the bosonic matter field had two main consequences in the particle density of the system. The integration over the longitudinal gauge field produced a non-local dynamical term for \( \rho(x) \) (see first term of (14)) and a non-local interaction term given by \( Tr \ln (\vec{\nabla} \cdot \frac{1}{m} \rho \vec{\nabla}) \). The integration over the transversal gauge field induced a two-body interaction (third term of (34)) and a three-body interaction also. The induced two-body interaction is the basic characteristic of the fluctuation of the transversal degrees of freedom of Chern-Simons field and, for the
particular case of $B(z) \equiv B = \text{const.}$,

$$F(x - y) = -\frac{\theta B}{\pi m} G(x - y),$$  \hspace{1cm} (36)

we obtain the well-known logarithmic two-body interaction, extensively explored in the past. This interaction is responsible for opening a gap in the spectrum of excitations over the ground state of the system. Moreover, the longitudinal fluctuations are non-trivial. In particular, the non-local dynamical term will modify the propagator of the density fluctuations. Also, we note that a Chern-Simons term for the external electromagnetic field has been factored out from the action. In the next section we will take advantage of these properties to analyze the electromagnetic response of this system.

### 3 Electromagnetic Response

This section is devoted to the calculation of the electromagnetic response of the theory described in the last section.

Due to gauge symmetry, the conserved current is given by

$$< J_\mu(x) > = \frac{\delta}{\delta A_\mu(x)} \ln(Z) .$$  \hspace{1cm} (37)

Explicitly evaluating the functional derivatives, we find using (33) and (34):

$$< J_0(x) > = i\pi \theta B + \partial_i \left< \frac{\delta S_{\text{eff}}(\vec{E}, B)}{\delta E_i} \right>$$  \hspace{1cm} (38)

$$< J_i(x) > = -i\frac{\pi}{\theta} \epsilon_{ij} E_j + \partial_0 \left< \frac{\delta S_{\text{eff}}(\vec{E}, B)}{\delta E_i} \right> + \epsilon_{ij} \partial_j \left< \frac{\delta S_{\text{eff}}(\vec{E}, B)}{\delta B} \right>$$  \hspace{1cm} (39)

As $S_{\text{eff}}$ is gauge invariant, it depends on $A_i$ only through $\vec{E}$ and $B$. This implies the existence of a topological current, automatically conserved (last term of (39)). We will see that this topological current is responsible for the exact quantization of the Hall conductivity, even in the presence of impurities, a fact closely related to the gauge invariance and edge state excitations.

We can easily evaluate expression (38), obtaining the obvious result

$$< iJ_0 > = < \rho(x, \tau) >$$  \hspace{1cm} (40)
The calculation of (39) is direct, but the result is less trivial:

\[
< J_i > = -i \pi \theta \epsilon_{ij} \left( E_j - \partial_j \int d^2 y \, G(x - y) \nabla \cdot \vec{E} \right) 
+ i \int d^2 y \, \partial_i G(x - y) \partial_0 < \rho(y) - \frac{\pi}{\theta} B(y) > 
+ J_i^T(x)
\]

where

\[
J_i^T(x) = \frac{1}{m \pi} \epsilon_{ij} \partial_j \Delta(x)
\]

and

\[
\Delta(x) = \int d^2 z d^2 y \, \nabla G(x - y) \cdot \nabla y G(y - z) < (\rho(y) - \frac{\pi}{\theta} B(y))(\rho(z) - \frac{\pi}{\theta} B(z)) >
\]

This is the first result of this section. The current (41) is the response of the system to an arbitrary electromagnetic field. Before we go over to the calculation, it is necessary at this point to check the conservation of the current (41). So, let us evaluate the divergence of \( < J_i > \). The last and the second terms of the first line of (41) are automatically conserved because both of them are topological terms. That means, they are of the form \( \epsilon_{ij} \partial_j f(x) \), with arbitrary \( f \). The other part of the current needs some attention. From (41) we have

\[
\partial_i < J_i > = -i \pi \theta \epsilon_{ij} \partial_j E_j + i \int d^2 y \, \nabla^2 G(x - y) \partial_0 < \rho(y) - \frac{\pi}{\theta} B(y) >
\]

Using (30) we can integrate the last term obtaining

\[
\partial_i < J_i > = i \partial_0 < \rho(y) > -i \pi \theta \left( \nabla \times \vec{E} + \partial_0 B \right)
\]

From (40) (remembering that the external electromagnetic field satisfies Faraday’s Law), we obtain the current conservation law (in Euclidean space)

\[
\partial_i < J_i > + \partial_0 < J_0 > = 0
\]

For an arbitrary electromagnetic field, expression (41) can turn out to be quite a complicated one. However, it is useful for the study of the structure of the current.
for specific configurations of the electromagnetic field. For example, let us analyze the case of a static and uniform magnetic field, and a static but arbitrary electric field. This case is the relevant one in the study of the Quantum Hall Effect. In this case, equation (41) reduces to

$$< J_i > = -i \frac{\pi}{\theta} \epsilon_{ij} \left( E_j - \partial_j \int d^2 y \ G(x-y) \vec{\nabla} \cdot \vec{E} \right) + J_i^T(x) \quad (47)$$

The first line of (47) does not depend on any specific detail of the system. Clearly, the dynamical properties of the system (two-body interactions, etc.) are all contained in the topological current $J_i^T(x)$. So, let us analyze this current more carefully. From (43), we have

$$\Delta(x) = \int d^2 z d^2 y \ \vec{\nabla} G(x-y) \cdot \vec{\nabla} y G(y-z) < \delta \rho(y) \delta \rho(z) > \quad (48)$$

where the expectation value is evaluated with the partition function $Z_\sigma$, and $\delta \rho$ is the density fluctuation around the magnetic background. The action $S_{eff}$ only depends on $\vec{E}$ through its divergence. So, in a clean sample (without external charges), $\Delta(x)$ does not depend on the electric field. Moreover, we have that $< \rho >$ is constant due to translation invariance and $< \rho > = \bar{\rho}$ fixes charge neutrality. Also, we saw that the effect of the quantum fluctuations of the Chern-Simons field was that of inducing a new two-body and three-body interactions with neutralizing background $\pi \theta B$. In order to define the density fluctuations $\delta \rho$ of equation (48), correctly, we must have

$$\bar{\rho} = \frac{\pi}{\theta} B \quad (49)$$

thus implying a Landau filling factor $\nu = \frac{\pi}{\theta}$. In this case the two-point correlation function of density fluctuations $< \delta \rho(x) \delta \rho(y) >$ depends on $|x-y|$. In the case of a sample with impurities, we must calculate the average of the correlation function with some weight factor (gaussian, for example). Thus, upon averaging the random impurities we restore the translation invariance and

$$< \delta \rho(x) \delta \rho(y) > = \sigma(x-y) \quad (50)$$

(the bar means “average over random impurities”). In any case, we can write for $\Delta$,

$$\Delta(x) = \int d^2 z d^2 y \ \vec{\nabla} G(x-y) \cdot \vec{\nabla} y G(y-z) \sigma(y-z)$$

$$= \int d^2 y \ \vec{\nabla} G(x-y) \cdot \vec{\nabla} y \int d^2 z G(y-z) \sigma(y-z) \quad (51)$$
In an infinite plane, the last integral of (51) is constant and therefore $\Delta(x) = 0$, for $\nu = \pi/\theta$. So, in an infinite system the topological current $j^T_i$ given by (42) is zero.

In a finite plane, $j^T_i$ is no longer zero, but we will show that it does not contribute to the total current. To see this, let us write the total current due to the topological density current as

$$I^T_i = \int_D j^T_i \, dS = \frac{1}{m} \frac{\theta}{\pi} \int_D \epsilon_{ij} \partial_j \Delta(x) \, dS$$  \hspace{1cm} (52)$$

The domain of integration $D$ is simply the surface of the sample. Suppose that $D$ is limited by a border $\partial D$. In this case we can represent the function $\Delta$ as $\Delta(x) \Theta(D)$, where $\Theta(D)$ is one in $D$ and zero otherwise. So,

$$\partial_j (\Delta(x) \Theta(D)) = \partial_j \Delta(x) \Theta(D) + \Delta(x) \partial_j \Theta(D) = \partial_j \Delta(x) \Theta(D) - \hat{n}_j \Delta(x) \delta(\partial D)$$ \hspace{1cm} (53)$$

where $\delta(\partial D)$ is the Dirac delta-functional with support in $\partial D$, and $\hat{n}_j$ is the $j$-th component of the versor contained in the plane, normal to $\partial D$ and external to $D$. Hence,

$$I^T_i = \frac{1}{m} \frac{\theta}{\pi} \left\{ \int_D \epsilon_{ij} \partial_j \Delta(x) \Theta(D) \, dS - \int \epsilon_{ij} \hat{n}_j \Delta(x) \delta(\partial D) \right\}$$

$$= \frac{1}{m} \frac{\theta}{\pi} \left\{ \oint_{\partial D} d\vec{l} \, \hat{t}_i \Delta(x) - \int \epsilon_{ij} \hat{n}_j \Delta(x) \delta(\partial D) \, ds \right\}$$ \hspace{1cm} (54)$$

where $t_i = \epsilon_{ij} \hat{n}_j$ is a tangent versor to $\partial D$. But,

$$\int \epsilon_{ij} \hat{n}_j \Delta(x) \delta(\partial D) \, ds = \oint_{\partial D} d\vec{l} \, \hat{t}_i \Delta(x)$$ \hspace{1cm} (55)$$

which implies that $I^T_i = 0$.

The r.h.s of (55) represents an edge current, responsible for the cancellation of the topological total current. It means that, due to the edge currents, the main contribution to the total current comes only from the first line of (41). The relevance of edge states in the QHE was first stressed in [19] and then, its theory was developed in refs. [20] [21] [22].

It is interesting to note that, for $\nu \neq \pi/\theta$, there are two different “uniform backgrounds”, say, $\tilde{\rho}$ and $(\pi/\theta)B$. In this case, it is not possible for the mean value of the density being constant. The translation invariance is broken and the density fluctuations no longer depend simply on $|x - y|$, but show a non-trivial dependence on $x$ and $y$. This shows that for $\nu \neq \pi/\theta$, the current will not be given by the first line of (41), but the topological density current $J^T_i$ will contribute in a non-trivial way.
Let us analyze now the main part of (41), namely

\[ < J_i > = -i \frac{\pi}{\theta} \epsilon_{ij} \left( E_j - \partial_j \int d^2 y \ G(x - y) \nabla \cdot \vec{E} \right) \] (56)

Its second term has support only in the region \( \rho(x) \neq 0 \) and depends on the divergence of \( \vec{E} \), only. Thus, in a clean sample, \( \nabla \cdot \vec{E} = 0 \) and the current is the well-known Hall current (turning back to usual units)

\[ < J_i > = \frac{1}{2n + 1} \frac{e^2}{h} \epsilon_{ij} E_j \] (57)

In a real sample, where impurities are present, \( \nabla \cdot \vec{E} \neq 0 \). In this case we can decompose the electric field \( \vec{E} \) in two parts

\[ \vec{E} = \vec{E}' + \vec{E}^{imp} \] (58)

where \( \nabla \cdot \vec{E}' = 0 \) and \( \nabla \cdot \vec{E}^{imp} = \rho_e \), with \( \rho_e \) the external charge due to impurities. With these definitions we can rewrite (59) in the following form:

\[ < J_i > = -i \frac{\pi}{\theta} \epsilon_{ij} E_j' - i \frac{\pi}{\theta} \epsilon_{ij} \left( E_j^{imp} - \partial_j \int d^2 y \ G(x - y) \nabla \cdot \vec{E}^{imp} \right) \] (59)

It is simple to show that

\[ E_j^{imp} - \partial_j \int d^2 y \ G(x - y) \nabla \cdot \vec{E}^{imp} = 0. \] (60)

The second term of the l.h.s of the previous equation can be interpreted as the electric bidimensional field created by a density charge \( \rho_e = \nabla \cdot \vec{E}^{imp} \).

Thus, from (59) we have (in usual units)

\[ < J_i > = \frac{1}{2n + 1} \frac{e^2}{h} \epsilon_{ij} E_j' \] (61)

So, the quantization of the Hall conductivity for \( \nu = \pi/\theta \) is exact in this model and does not depend on any impurity distribution. In other words, the transverse current only “sees” divergenceless fields. System dynamics and random impurities only affect the edge states of the system, and this the reason for the exact quantization of the Hall conductance. This result generalize Prange’s result \([\text{12}]\) for a \( \delta \) – type impurity, to the case of a general impurity distribution in the context of the CSLG theory for the FQHE.

15
4 Dynamics of Density Fluctuations

In the last section we calculated the electromagnetic current and, from the resulting expression, we were able to understand the role of the Chern-Simons term, of the edge states — that are responsible for the exact quantization of the conductivity — and the role of impurities, too. However, for a complete understanding of the electromagnetic response, it is necessary to evaluate the polarization tensor

$$\Pi_{\mu\nu}(x, y) = \frac{\delta^2 \ln Z(A)}{\delta A^\mu(y) \delta A^\nu(x)}$$ (62)

The calculation of the functional derivatives is straightforward and the result is formally expressed in terms of mean values of the density fluctuation $\langle \delta \rho(x) \delta \rho(y) \rangle$. In particular, the expression for $\Pi_{00}$ is very simple

$$\Pi_{00}(x, y) = \langle \delta \rho(x) \delta \rho(y) \rangle$$ (63)

were $\delta \rho(x) = \rho(x) - \langle \rho(x) \rangle$

In order to explicitly calculate this type of objects we need to evaluate the effective action for the density fluctuation $\delta \rho$. The action (34) is the appropriate one for the study of the dynamics of density fluctuations around the ground state of the model. The structure of the ground state depends on the specific configuration of the external electromagnetic field. We will specialize our present analysis to the case of a static and homogeneous field, the relevant case to the study of the QHE. The ground state for $\nu = \pi/\theta$ is homogeneous, and is given by

$$\langle \rho(x) \rangle = \bar{\rho} = \frac{\pi}{\theta} B$$ (64)

so that $\rho(x) = \bar{\rho} + \delta \rho(x)$. The aim of the present section is to obtain from (34) an effective action for $\delta \rho(x)$ and to extract from it the Feynman’s rules to calculate mean values. We need to perform a Taylor expansion of the action around the configuration $\rho(x) = \bar{\rho}$. This expansion is straightforward, except for two terms originated from integrating over the longitudinal component of the gauge field. These terms are the kinetic term $\partial_0 \rho \left\{ \frac{\psi_i(\rho \nabla \bar{\rho})}{\psi_i(\rho \nabla)} \right\} \partial_0 \rho$, and the determinant obtained when integrating over $\varphi$, namely, $\text{Tr} \ln(\nabla \cdot \frac{\psi_i(\rho \nabla)}{\psi_i(\rho \nabla)})$. Let us analyze these two terms carefully. We can write the kinetic term of (34) in the following way:

$$I_K = \int d^3z_1 d^3z_2 \delta \dot{\rho}(z_1) \mathcal{K}(z_1, z_2) \delta \dot{\rho}(z_2)$$ (65)
where
\[ \vec{\nabla}_{z_1} \cdot \left( \rho(z_1) \vec{\nabla}_{z_1} K(z_1, z_2) \right) = \delta(z_1 - z_2) \tag{66} \]
(\(\delta \dot{\rho}\) means the derivative of \(\delta \rho\) with respect to time). We have also used that the ground state density is static. Note that \(K\) depends on \(\rho\) through the implicit equation (66). Thus, we can expand the kernel \(K\) in powers of \(\delta \rho\) in the following form,
\[ K(z_1, z_2) = K(z_1, z_2) \big|_{\rho = \bar{\rho}} + \int d^3 x \left( \frac{\delta K(z_1, z_2)}{\delta \rho(x)} \right) \big|_{\rho = \bar{\rho}} \delta \rho(x) + \ldots \tag{67} \]
From (66) we see that
\[ K(z_1, z_2) \big|_{\rho = \bar{\rho}} = \frac{1}{\bar{\rho}} G(z_1, z_2) \tag{68} \]
where \(G(z_1, z_2)\) is the Green’s function of the Laplacian (see eq. (30)). Also, we can show (see Appendix) that
\[ \frac{\delta K(z_1, z_2)}{\delta \rho(x)} \big|_{\rho = \bar{\rho}} = \frac{1}{\bar{\rho}^2} \vec{\nabla} G(z_1, x) \cdot \vec{\nabla} G(x, z_2) \tag{69} \]
Introducing (68) and (69) into (65) we find
\[ I_K = \frac{1}{\bar{\rho}} \int d^3 z_1 d^3 z_2 \delta \dot{\rho}(z_1) G(z_1, z_2) \delta \dot{\rho}(z_2) \]
\[ + \frac{1}{\bar{\rho}^2} \int d^3 z_1 d^3 z_2 d^3 x \vec{\nabla} G(z_1, x) \cdot \vec{\nabla} G(x, z_2) \delta \dot{\rho}(z_1) \delta \dot{\rho}(z_2) \delta \rho(x) \]
\[ + \ldots \tag{70} \]
The next interesting term to be considered is
\[ I_{\text{det}} = Tr \ln \left( \vec{\nabla} \cdot \frac{\rho}{m} \vec{\nabla} \right), \tag{71} \]
which can be rewritten as
\[ I_{\text{det}} = Tr \ln \left( \frac{\bar{\rho}}{m} \nabla^2 + \frac{\delta \rho}{m} \vec{\nabla} \right) + Tr \ln(1 + \frac{\delta \rho}{\bar{\rho}}) \tag{72} \]
Observe that this term is due to the interaction of the density field with the longitudinal component of the gauge field. This process is governed by low-momentum transfer, so the logarithmic term with two derivatives can be neglected, and we obtain
\[ I_{\text{det}} \approx Tr \ln \left( \frac{\bar{\rho}}{m} \nabla^2 \right) + Tr \ln(1 + \frac{\delta \rho}{\bar{\rho}}) \tag{73} \]
The first term in this expression is a constant and can be absorbed into the global renormalization factor of the partition function. The other term can be expanded in powers of $\delta \rho$, yielding

$$I_{det} \approx -\frac{1}{2} \int d^3x \frac{\delta \rho^2}{\bar{\rho}^2} + \frac{1}{3} \int d^3x \frac{\delta \rho^3}{\bar{\rho}^3}$$  \hspace{1cm} (74)$$

The linear term can be absorbed by a suitable renormalization of the chemical potential; the only effect of the quadratic term is redefining the two-body potential $V(x, y)$ in the low-momentum limit. In the case of local potentials, this term may be used to redefine the coupling constant, but it is irrelevant in the case of long-distance interactions.

Collecting all the terms, we obtain the effective action for the density fluctuations,

$$S = \frac{m}{2\bar{\rho}} \int d^3x d^3y \delta \rho(x)G(x - y)\delta \rho(y)$$
$$+ \frac{1}{2} \int d^3x d^3y \delta \rho(x) \left\{ \frac{1}{4m\bar{\rho}} \nabla^2 \delta(x - y) - \left( V(x - y) - \frac{\bar{\rho}}{m} \frac{\theta}{\pi}^2 G(x - y) \right) \right\} \delta \rho(y)$$
$$+ \frac{1}{8m\bar{\rho}^2} \int d^3x \vec{\nabla} \delta \rho(x) \cdot \vec{\nabla} \delta \rho(x) \delta \rho(x) - \frac{1}{3\bar{\rho}^{3/2}} \int d^3x \delta \rho^3 + \ldots$$
$$- \frac{1}{2m\pi^2} \int d^3x d^3y d^3z \vec{\nabla} G(x - y) \cdot \vec{\nabla} G(x - z) \delta \rho(x) \delta \rho(y) \delta \rho(z)$$
$$+ \frac{m}{2} \frac{1}{\bar{\rho}^2} \int d^3x d^3y d^3z \vec{\nabla} G(x - z) \cdot \vec{\nabla} G(z - y) \delta \rho(x) \delta \rho(y) \delta \rho(z)$$
$$- \int d^2x d^2y \delta \rho(x)G(x - y)\vec{\nabla} \cdot \vec{E}$$  \hspace{1cm} (75)$$

As the fluctuation $\delta \rho$ is small, the quadratic term of the effective action may be regarded as defining the dynamics of the fluctuations, whereas the cubic terms standing for perturbations to such dynamics. In this sense, the propagator of the density fluctuation in Fourier space may be written as

$$< \delta \rho(\omega, \mathbf{k}) \delta \rho(-\omega, -\mathbf{k}) > = \frac{2\pi k^2}{\omega^2 - \left\{ \left( \frac{k^2}{2m} \right)^2 + \frac{\bar{\rho}^2}{m} V(-\mathbf{k}^2) + \frac{\rho^2}{m^2} \frac{\theta}{\pi}^2 \right\}}$$  \hspace{1cm} (76)$$

This propagator has very interesting information encoded. For example, the compressibility of the system is defined as:

$$\kappa = \lim_{k \to 0} \Pi_{00}(0, \mathbf{k})$$  \hspace{1cm} (77)$$
From (63) and the propagator (77) we have for all the potentials with the general form
\[ V(-k^2) \propto k^\alpha, \]
\[ \kappa = \lim_{k \to 0} - \frac{2\bar{\rho} k^2}{(k^2/2m)^2 + \frac{\bar{\rho} k^2 V(-k^2)}{m^2(\theta/\pi)^2}} = 0 \]
(78)
showing that the system is in an incompressible state.

The relation of this incompressibility to boson superconductivity was first shown in ref. [9]. In the context of the present work, incompressibility depends on two facts. The first one is the existence of a gap in the spectrum of excitations. However, the gap existence is a necessary but not sufficient condition to determine the incompressibility of the ground state. The second fact is the presence of a \(k^2\) factor in the numerator of the propagator. The gap in the spectrum is opened by the transverse gauge field fluctuations and its value \(\delta \Delta = \frac{\bar{\rho}^2}{m^2(\theta/\pi)^2}\) coincides with those found in refs. [9], [11] and [14]. This gapfull excitation corresponds to the cyclotron mode excitation, and was identified as an inter-Landau level excitation. The \(k^2\) factor in the numerator is induced by the longitudinal gauge field fluctuations that affect the dynamics of the density fluctuations. Remember that these fluctuations are completely equivalent to the phase fluctuations of the matter field.

It is useful to write the propagator (76) in the following form:
\[ < \delta \rho(\omega, k)\delta \rho(-\omega, -k) > = \frac{\pi}{\theta} k^2 \left( \frac{1}{\omega - \omega_k} - \frac{1}{\omega + \omega_k} \right) + O(k^3) \]
(79)
with
\[ \omega_k = \sqrt{\left(\frac{k^2}{2m}\right)^2 + \frac{\bar{\rho} \omega}{m^2(\theta/\pi)^2}}. \]
(80)
This shows that our approach is consistent with Kohon’s theorem [23]. This theorem asserts that in a planar translation-invariant system submitted to a perpendicular magnetic field. The density-density correlation function up to order \(k^2\) must have a gap equal to \(\omega_c = \frac{\bar{\rho} B}{m}\). This is an exact result, independent of the microscopic details of the system. Thus, a consistent calculus of radiative corrections must take into account that constraint at each order of the perturbation expansion, in addition to the vertices read from (74). Note that this result is valid only for two-body potentials that vanish as \(r \to \infty\). For example, a logarithmic potential \(V(-k^2) \propto 1/k^2\) modifies the gap, as can be confirmed from (80).

The dispersion relation (80) coincides with that calculated in [9] and [11] in the low-momentum limit. For the Coulomb potential \(V(-k^2) \propto 1/|k|\) the dispersion is
linear, as it was first suggested in ref.\[24\]. Moreover, equation (80) coincides with the dispersion relation of quasiparticles in an anyon superfluid\[14\]. It is interesting to note that, in ref. \[14\] the model is written in terms of gauge invariant operators

\[
\Phi(x) \equiv \phi(x)e^{i\varphi(x)} \quad \Phi(x)^* \equiv e^{-i\varphi(x)}\phi(x)
\]

where \(\varphi(x)\) is the longitudinal component of the gauge field given by \(24\). The theory is canonically quantized and then, the excitations of quasiparticles are calculated in the Bogoliubov approximation. In our functional approach, the gauge invariant degree of freedom is chosen to be the particle density \(\rho(x)\) and integration over all the remaining fields is carried out. Studying density fluctuations, we find that the dispersion relation of the cyclotron excitation is exactly the same as the former. This should mean that the term \((k^2/2m)^2\) of (80) is closely related to fluctuations of gauge invariant objects.

It is well known that this model presents another type of excitations called topological vortices. The typical asymptotic behavior of a vortex is (for \(|x| \to \infty\))

\[
\phi(x) = \rho^{1/2}e^{i\gamma(x)} \quad a_i(x) = \partial_i\gamma(x) = \epsilon_{ij}\frac{x_j}{|x|^2}
\]

where \(\gamma\) is the azimuthal angle. Finite energy requirements conduce to flux quantization

\[
\oint a \cdot ds = 2\pi
\]

and due to the Chern-Simons constraint the charge of the vortex is also quantized. Explicit solutions to the classical equations of motion were found in several references \[14\] \[15\] \[16\]. Here, we would like to find an effective action for the particle density containing the vortex dressed with quantum fluctuations of the gauge field. The effective action \(73\) has lost its topological information and contains no vortex. So far, we worked out all the functional integrations by implicitly fixing trivial boundary conditions on the fields. In order to obtain a density representation for the effective action in the presence of a vortex, though, we must fix non-trivial boundary conditions when performing the functional integrals.

In \(11\) we fixed the gauge by imposing \(\delta(\theta(x))\), and we have shown that this is equivalent to choosing Coulomb’s gauge. However, this is not the only possible choice. Suppose we allow a singular gauge transformation,

\[
\begin{align*}
\theta(x) & \longrightarrow \theta(x) + \Lambda(x) + \gamma(x) \\
a_\mu(x) & \longrightarrow a_\mu(x) + \partial_\mu\Lambda(x) + \partial_\mu\gamma(x)
\end{align*}
\]

20
(γ is the azimutal angle) and choose Λ = −θ. This transformation fixes the phase to the value θ = γ in the action, and induces a longitudinal gauge component \( a^L_i = \partial_i \theta + \partial_i \gamma \). Thus, we can change the integration variables from \( \theta \) to \( a^L_i \). Retracing the steps that lead from (11) to (15), we conclude that we can also choose the gauge fixing functional to be

\[
G_F(\theta, a_i) = \delta(\theta - \gamma)
\]

This gauge fixing automatically establishes topological boundary conditions for the phase or, equivalently, for \( a^L_i \). A similar technique was also used in [25], in the context of the abelian Higgs model.

With these boundary conditions, the action in terms of density fluctuations takes the form

\[
S_V(\rho) = S_{\text{eff}}(\rho) + S_\gamma(\rho)
\]

where \( S_{\text{eff}} \) is the effective action with trivial boundary conditions given by (34) and

\[
S_\gamma(\rho) = \frac{1}{m} \int d^3 x \left( -\frac{1}{2} \rho |\vec{\nabla} \gamma|^2 + \rho \vec{\alpha} \cdot \vec{\nabla} \gamma \right).
\]

Integrating over the gauge field, we obtain

\[
S_\gamma(\rho) = -\frac{1}{4m} \int d^3 x \rho |\vec{\nabla} \gamma|^2 + \frac{\pi}{m} \epsilon_{ij} \int d^3 x d^3 y \partial_i \rho(x) G(x - y) \gamma(x) \partial_j \rho(y)
\]

The first term of this equation comes from the integration of the longitudinal fluctuation of the gauge field, and can be interpreted as the interaction with an external charge density, as it can be rewritten in the form

\[
\frac{1}{4m} \int d^3 x \rho |\vec{\nabla} \gamma|^2 = \frac{1}{m} \int d^3 x d^3 y \rho(x) G(x - y) \rho_e(y)
\]

where

\[
\rho_e(x) = \nabla^2 |\vec{\nabla} \gamma|^2 = 1/|x|^4.
\]

The second term of (90) comes from the integration of the transverse gauge field fluctuation; such fluctuation induces a new two-body interaction relating density gradients in orthogonal directions. Note that \( S_\gamma \) is not translationally invariant because \( \gamma(x) \) is not well defined at the origin (the nucleus of the vortex). Thus equation (88), together with (34) and (90), is the effective action for the density charge of bosons coupled to Chern-Simons gauge field in the presence of a vortex centered at the origin of coordinates. It would be possible, upon minimizing this action, to obtain a vortex profile dressed with the quantum fluctuation of the gauge fields.
5 Discussions and Conclusions

We have considered in this paper a system of non-relativistic bosons coupled to a Chern-Simons gauge field and an arbitrary external electromagnetic field. We have used path integral techniques to deduce an effective action in terms of the matter field density only. We found the coupling of the charge density with the electromagnetic field very interesting. The magnetic field acts as a “background density”, as it only enters the action through terms of the form \((\rho(x) - \frac{\pi}{\theta} B(x))\). The electric field only couples to the density through its divergence \(\vec{\nabla} \cdot \vec{E}\), which in its turn is proportional to the external charges (impurities, for example). So, in the absence of impurities, the external electric field does not couple with the matter field. This peculiar fact is due to the Chern-Simons structure of the dynamical gauge field and it is the main reason for the exact quantization of the transverse conductivity.

The coupling of the gauge field with bosonic matter had two main consequences on the density of the system. The integration over the longitudinal gauge field produced a non-local dynamical term for \(\rho(x)\) and also a non-local interaction given by \(Tr \ln(\vec{\nabla} \cdot \frac{1}{m} \rho \vec{\nabla})\). The integration over the transversal gauge field induced a two and a three-body interaction terms. The induced two-body interaction is the basic characteristic of a fluctuation of the transversal degrees of freedom of the Chern-Simons field, and it is responsible for opening a gap in the spectrum of excitations. Moreover, the longitudinal fluctuations are also non-trivial. In particular the non-local dynamical term modifies the propagator of density fluctuations, leading, in addition to the appearance of a gap, to an incompressible ground state. Another interesting aspect of this action is that, apart from the interaction terms, a pure Chern-Simons term in the external electromagnetic field is factored out. This fact is important in the quantization of the transversal conductivity.

Using these properties of the action, we were able to build an exact expression for the conserved \(U(1)\) current of the model. In particular, we have explicitly shown that, in the case of a homogeneous and static magnetic field, the quantization of the transverse conductivity is exact, even in the presence of any type of impurities. Thus, we generalized Prange’s result [2] for the FQHE to any kind of impurity distribution, in the context of the CSLG theory. This development is based on the fact that all the microscopic dynamics of the model and the coupling with impurities are entirely contained in an automatically conserved current (topological current) that does not contribute to the total current. In the case of an infinite system, this topological density current is zero. In a finite system the edge states exactly cancel out the contribution of
this density to the total current. Moreover, we have shown that the transverse current only “sees” divergenceless field, thus canceling any contribution from external charges.

We have computed the propagator of density fluctuations, using our effective action. The propagator shows a gap in the excitation spectrum that coincides with the inter-Landau level excitations calculated previously using semiclassical approximations [9]. The dispersion relation of this excitation coincides with that calculated in ref. [14] by a gauge-invariant canonical formalism in the framework of anyon superconductivity. This observation suggests that the form of the dispersion relation is related to fluctuations of gauge invariant objects. Of course, in the low-momentum (long-distance) limit, the dispersion relation is linear (as noted in [24]) and coincides with all the other approaches in this limit.

We have also analyzed the structure of the effective action in the presence of a vortex excitation. The main difference in this case is that the integration over the longitudinal gauge field produces an interaction with an induced external density charge of the form \( \rho_e = 1/|x|^4 \). The integration over the transversal gauge field induces a new two-body interaction between the gradients of the density in orthogonal directions. Upon minimizing this action we should obtain the density profile of the vortex dressed with the quantum fluctuation of the gauge field. In order to actually minimize this action, we must face the problem of solving a system of a non-local and non-linear integro-differential equations. This work is in progress and will be presented elsewhere.

The action \( S_V \) should not be confused with the dual actions developed in references [26] and [27]. Those dual actions represent vortex densities, whereas the action in the present work represents the particle density in the presence of a vortex. In order to obtain an action for vortex excitation with our formalism, we would have to introduce singularities into configuration space in a way similar to that of ref. [28]. This study is under development and it is certainly beyond the scope of this paper.

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A Appendix
A.1 Expansion of the Kinetic term

In order to deduce the dynamics of density fluctuations, we must expand the kinetic term of (34) by performing a functional Taylor expansion around the function \( \rho(x) = \bar{\rho} \). Here, we would like to sketch the principal steps in developing such expansion. As indicated in (65) and (66), the kinetic term of (34) can be written as

\[
I_K = \int d^3z_1 d^3z_2 \delta \dot{\rho}(z_1) \mathcal{K}(z_1, z_2) \delta \dot{\rho}(z_2)
\]  

(92)

where

\[
\vec{\nabla}_{z_1} \cdot \left( \rho(z_1) \vec{\nabla}_{z_1} \mathcal{K}(z_1, z_2) \right) = \delta(z_1 - z_2)
\]

(93)

The kernel \( \mathcal{K} \) is a functional of \( \rho(x) \) given implicitly by eq. (93). Putting \( \rho(x) = \bar{\rho} + \delta \rho(x) \) and expanding up to the first order in \( \delta \rho \), we find

\[
\mathcal{K}(z_1, z_2)[\delta \rho] = \mathcal{K}(z_1, z_2)|_{\rho = \bar{\rho}} + \int d^3x \delta \mathcal{K}(z_1, z_2) \left|_{\rho = \bar{\rho}} \right. \frac{\delta \rho(x)}{\rho(x)} + \ldots
\]

(94)

Calculation of the first term is straightforward. From (93), we have

\[
\vec{\nabla}_{z_1} \cdot \left( \bar{\rho} \vec{\nabla}_{z_1} \mathcal{K}(z_1, z_2) \right) = \bar{\rho} \nabla^2_{z_1} \mathcal{K}(z_1, z_2)|_{\rho = \bar{\rho}} = \delta(z_1 - z_2)
\]

(95)

Thus,

\[
\mathcal{K}(z_1, z_2)|_{\rho = \bar{\rho}} = \frac{1}{\bar{\rho}} G(z_1, z_2)
\]

(96)

where \( G(z_1, z_2) \) is the Green’s function of the Laplacian operator.

The second term of (94) is more involved. Functionally differentiating equation (93) with respect to \( \rho \) we find

\[
\frac{\delta}{\delta \rho(x)} \vec{\nabla}_{z_1} \cdot \left( \rho(z_1) \vec{\nabla}_{z_1} \mathcal{K}(z_1, z_2) \right) = \vec{\nabla}_{z_1} \cdot \left( \delta(z_1 - x) \vec{\nabla}_{z_1} \mathcal{K}(z_1, z_2) \right) + \vec{\nabla}_{z_1} \cdot \left( \rho(z_1) \vec{\nabla}_{z_1} \frac{\delta}{\delta \rho(x)} \mathcal{K}(z_1, z_2) \right) = 0
\]

(97)

Observing that the operator \( \vec{\nabla}_z \cdot \left( \rho(z) \vec{\nabla}_z \ldots \right) = \int dy \mathcal{K}^{-1}(z, y) \ldots \) we can rewrite equation (97) as

\[
\int dy \mathcal{K}^{-1}(z_1, y) \frac{\delta}{\delta \rho(x)} \mathcal{K}(y, z_2) = -\vec{\nabla}_{z_1} \cdot \left( \delta(z_1 - x) \vec{\nabla}_{z_1} \mathcal{K}(z_1, z_2) \right)
\]

(98)
Inverting this equation we find

\[
\frac{\delta}{\delta \rho(x)} \mathcal{K}(z_1, z_2) = - \int dy \mathcal{K}(z_1, y) \{ \nabla_y \delta(y - x) \cdot \nabla_y \mathcal{K}(y, z_2) + \delta(y - x) \nabla_y^2 \mathcal{K}(y, z_2) \} 
\] (99)

and integrating the \( \delta \) – functions we finally obtain

\[
\frac{\delta}{\delta \rho(x)} \mathcal{K}(z_1, z_2) = \nabla_x \mathcal{K}(z_1, x) \cdot \nabla_x \mathcal{K}(x, z_2) 
\] (100)

This is a functional differential equation for \( \mathcal{K} \). Finding its solutions for arbitrary \( \rho(x) \) is a very difficult task, but for our purposes only the special case \( \rho = \bar{\rho} \) needs to be considered. Using (99) and (100) we have

\[
\frac{\delta \mathcal{K}(z_1, z_2)}{\delta \rho(x)} \bigg|_{\rho = \bar{\rho}} = \frac{1}{\bar{\rho}^2} \nabla \mathcal{G}(z_1, x) \cdot \nabla \mathcal{G}(x, z_2) 
\] (101)

that, of course, coincides with equation (69).
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