Convolution properties of univalent harmonic mappings convex in one direction

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Abstract

Let $*$ and $\bar{*}$ denote the convolution of two analytic maps and that of an analytic map and a harmonic map respectively. Pokhrel \cite{1} proved that if $f = h + \overline{g}$ is a harmonic map convex in the direction of $e^{i\gamma}$ and $\phi$ is an analytic map in the class $DCP$, then $f \bar{*} \phi = h * \phi + \overline{g} * \phi$ is also convex in the direction of $e^{i\gamma}$, provided $f \bar{*} \phi$ is locally univalent and sense-preserving. In the present paper we obtain a general condition under which $f \bar{*} \phi$ is locally univalent and sense-preserving. Some interesting applications of the general result are also presented.

1 Introduction

Let $\mathcal{H}$ denote the class of all complex valued harmonic mappings $f = h + \overline{g}$ defined in the unit disk $E = \{z : |z| < 1\}$. Such harmonic mappings are locally univalent and sense-preserving if and only if $h' \neq 0$ in $E$ and the dilatation function $\omega$, defined by $\omega = g'/h'$, satisfies $|\omega(z)| < 1$ for all $z \in E$. The class of all univalent harmonic and sense-preserving mappings $f = h + \overline{g}$ in $E$, normalized by the conditions $f(0) = 0$ and $f_z(0) = 1$, is denoted by $S_H$. Therefore, a function $f = h + \overline{g}$ in the class $S_H$ has the representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n, \quad z \in E. \quad (1)$$

If the co-analytic part $g(z) \equiv 0$ in $E$, then the class $S_H$ reduces to the usual class $S$ of all normalized univalent analytic functions. We denote by $K_H$ and $C_H$ the subclasses of $S_H$ consisting of those functions which map $E$ onto convex and close-to-convex domains, respectively. $K$ and $C$ will denote corresponding subclasses of $S$. A domain $\Omega$ is said to be convex in a direction $e^{i\gamma}$, $0 \leq \gamma < \pi$, if every line parallel to the line through 0 and $e^{i\gamma}$ has either connected or empty

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intersection with $\Omega$. Let $K_{H(\gamma)}$ and $K_{H}$, $0 \leq \gamma < \pi$ denote the subclass of $S_{H}$ and $S$ respectively, consisting of functions which map the unit disk $E$ on to domains convex in the direction of $e^{i\gamma}$. In particular a domain convex in the horizontal direction will be denoted by $CHD$. Hengartner and Schober \[7\] characterized the mappings in $K_{\pi/2}$ as under:

**Theorem A.** Suppose $f$ is analytic and non constant in $E$. Then

$$\Re[(1 - z^2)f'(z)] \geq 0, \quad z \in E$$

if and only if

(i) $f$ is univalent in $E$;
(ii) $f$ is convex in the direction of the imaginary axis;
(iii) there exist sequences $\{z'_n\}$ and $\{z''_n\}$ converging to $z = 1$ and $z = -1$, respectively, such that

$$\lim_{n \to \infty} \Re(f(z'_n)) = \sup_{|z| < 1} \Re(f(z)) \quad \text{and} \quad \lim_{n \to \infty} \Re(f(z''_n)) = \inf_{|z| < 1} \Re(f(z)).$$

The convolution or the Hadamard product $\phi \ast \psi$ of two analytic mappings $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\psi(z) = \sum_{n=0}^{\infty} A_n z^n$ in $E$ is defined as $(\phi \ast \psi)(z) = \sum_{n=0}^{\infty} a_n A_n z^n$, $z \in E$. The convolution of a harmonic function $f = h + g$ with an analytic function $\phi$, both defined in $E$, is denoted by $f \ast \phi$ and is defined as:

$$f \ast \phi = h \ast \phi + g \ast \phi.$$

Clunie and Sheil-Small \[2\] proved that if $\phi \in K$ and $F \in K_{H}$, then for every $\alpha$, $|\alpha| \leq 1$ $(\alpha \phi + \phi) \ast F \in C_{H}$. They also posed the question: if $F \in K_{H}$, then what is the collection of harmonic functions $f$, such that $F \ast f \in K_{H}$? Ruscheweyh and Salinas \[5\] partially answered the question of Clunie and Sheil-Small by introducing a class $DCP(\text{Direction convexity preserving})$. An analytic function $\phi$ defined in $E$ is said to be in the class $DCP$ if for every $f$ in $K_{H}$, $\phi \ast f \in K_{H}$. Functions in the class $DCP$ are necessarily convex in $E$.

**Theorem B.** (\[5\]) Let $\phi$ be analytic in $E$. Then $F \ast \phi \in K_{H}$ for all $F \in K_{H}$ if and only if $\phi$ is in $DCP$.

Pokhrel \[1\] further investigated convolution properties of functions in the class $DCP$ and proved the following:

**Theorem C.** (\[1\]) Let $f = h + g$ be in $K_{H(\gamma)}$, $0 \leq \gamma < \pi$. Then for an analytic function $\phi$ in $DCP$, $f \ast \phi \in K_{H(\gamma)}$, provided $f \ast \phi$ is locally univalent and sense-preserving in $E$.

In the present paper we find a condition on a univalent harmonic function convex in one direction so that its convolutions with functions of the class $DCP$ are also convex in the same direction. Some interesting applications are also presented.
2 Preliminaries

Following lemmas will be required to prove our main results.

**Lemma 2.1.** Let $\phi$ and $\xi$ be analytic in $E$ with $\phi(0) = \xi(0) = 0$ and $\phi'(0)\xi'(0) \neq 0$. Suppose that for each $\beta(|\beta| = 1)$ and $\sigma(|\sigma| = 1)$ we have

$$\phi(z) * \frac{1 + \beta \sigma z}{1 - \sigma z} \xi(z) \neq 0, \ 0 < |z| < 1.$$  

Then for each function $F$ analytic in $E$, $(\phi * F \xi)/(\phi * \xi)$ takes values in the convex hull of $F(E)$.

**Lemma 2.2.** Let $\chi$ be analytic in $E$ with $\chi(0) = 0$ and suppose that there exist constants $\zeta$ and $\eta$ with $|\zeta| = |\eta| = 1$ such that for each $z$ in $E$

$$\Re \left[ (1 - \zeta z)(1 - \eta z) \frac{\chi(z)}{z} \right] > 0.$$ 

Then for every convex function $\phi$, $\phi(z) * \chi(z) \neq 0$ ($0 < |z| < 1$).

**Lemma 2.3.** A locally univalent harmonic function $f = h + g$ in $E$ is a univalent harmonic mapping of $E$ onto a domain convex in a direction $e^{i\gamma}$ if and only if $h - e^{2i\gamma}g$ is a univalent analytic mapping of $E$ onto a domain convex in the direction $e^{i\gamma}$.

Lemma 2.1 and Lemma 2.2 are due to Ruscheweyh and Sheil-Small [6] whereas Lemma 2.3 is due to Clunie and Sheil-Small [2].

3 Main Results

We begin with the following result.

**Theorem 3.1.** Let $f = h + \overline{g} \in K_{H(\gamma)}$, $0 \leq \gamma < \pi$. If for some complex constants $\eta$ and $\xi$ with $|\xi| = |\eta| = 1$,

$$\Re \left[ (1 - \eta z)(1 - \xi z)(h' - e^{2i\gamma}g') \right] > 0,$$ 

in $E$, (2) then $f * \phi \in K_{H(\gamma)}$ for every analytic function $\phi \in DCP$.

**Proof.** In view of Theorem C, we only need to show that $f * \phi = h * \phi + \overline{g} * \phi$ is locally univalent and sense-preserving in $E$ i.e., the dilatation $\omega = (\phi * g)'/(\phi * h)'$ of $f * \phi$ satisfies $|\omega(z)| < 1$ in
Let \( f = h + \overline{g} \), where \( h(z) + g(z) = z(1 - \alpha z)/(1 - z^2), \alpha \in [-1, 1] \) and \( |g'/h'| < 1 \) in \( E \), be a normalized harmonic mapping in \( E \) and \( \phi \in \text{DCP} \) be an analytic map. Then \( f \circ \phi \in K_{H(z/2)} \).

Proof. Let \( f = h + g \). Since

\[
\Re[(1 - z^2)f'(z)] = \Re \left[ \frac{1 + z^2 - 2\alpha z}{1 - z^2} \right] = \frac{(1 - |z|^2)(1 + |z|^2 - 2\alpha \Re(z))}{|1 - z^2|^2} > 0, \quad z \in E,
\]

we have

\[
\Re(1 + e^{2i\gamma\overline{\omega}})/(1 - e^{2i\gamma\overline{\omega}}) > 0 \quad \text{in} \quad E.
\]

Now

\[
\Re \left[ \frac{1 + e^{2i\gamma\overline{\omega}}}{1 - e^{2i\gamma\overline{\omega}}} \right] = \Re \left[ \frac{(\phi \ast h)' + e^{2i\gamma}(\phi \ast g)'}{(\phi \ast h)' - e^{2i\gamma}(\phi \ast g)'} \right]
\]

\[
= \Re \left[ \frac{\phi \ast z(h' + e^{2i\gamma}g')}{\phi \ast z(h' - e^{2i\gamma}g')} \right]
\]

\[
= \Re \left[ \frac{\phi \ast z(h' - e^{2i\gamma}g')\left[(h' + e^{2i\gamma}g')/(h' - e^{2i\gamma}g')\right]}{\phi \ast z(h' - e^{2i\gamma}g')} \right],
\]

where \( P = (h' + e^{2i\gamma}g')/(h' - e^{2i\gamma}g') = (1 + e^{2i\gamma}g'/h')/(1 - e^{2i\gamma}g'/h') \). Since \( g'/h' \) is the dilatation of \( f \), therefore \( |g'/h'| < 1 \) in \( E \) as \( f \) is univalent and hence locally univalent in \( E \). Thus \( \Re(P) > 0 \) in \( E \) and therefore, in view of Lemma 2.1, we shall get the desired result if for each \( \beta(|\beta| = 1) \) and \( \sigma(|\sigma| = 1) \)

\[
\phi(z) = \frac{1 + \beta \sigma z}{1 - \sigma z} z(h'(z) - e^{2i\gamma}g'(z)) \neq 0, \quad 0 < |z| < 1.
\]

As \( \phi \) is in the class \( \text{DCP} \), so \( \phi \) is convex analytic in \( E \). Therefore, to prove (3) we shall apply Lemma 2.2 and show that for some constants \( \zeta, \eta \) with \( |\zeta| = |\eta| = 1 \),

\[
\Re \left[ (1 - \zeta z)(1 - \eta z) \frac{1 + \beta \sigma z}{1 - \sigma z} z(h'(z) - e^{2i\gamma}g'(z))/z \right] > 0, \quad z \in E.
\]

By setting \( \zeta = \sigma \) and \( \beta = -\frac{\xi}{\sigma} (|\xi| = 1) \) we have

\[
\Re \left[ (1 - \zeta z)(1 - \eta z) \frac{1 + \beta \sigma z}{1 - \sigma z} z(h' - e^{2i\gamma}g')/z \right] = \Re \left[ (1 - \eta z)(1 - \xi z)(h' - e^{2i\gamma}g') \right]
\]

\[
> 0 \quad \text{in} \quad E \quad (\text{in view of (2)}).
\]

Hence \( \Re((1 + e^{2i\gamma\overline{\omega}})/(1 - e^{2i\gamma\overline{\omega}})) > 0 \) and so, \( |\overline{\omega}(z)| < 1 \) for all \( z \in E \). This completes the proof.

As applications of the above theorem we derive the following interesting results.

**Corollary 3.2.** Let \( f_\alpha = h_\alpha + \overline{g}_\alpha \), where \( h_\alpha(z) + g_\alpha(z) = z(1 - \alpha z)/(1 - z^2), \alpha \in [-1, 1] \) and \( |g_\alpha'/h_\alpha'| < 1 \) in \( E \), be a normalized harmonic mapping in \( E \) and \( \phi \in \text{DCP} \) be an analytic map. Then \( f_\alpha \circ \phi \in K_{H(z/2)} \).

Proof. Let \( f_\alpha = h_\alpha + g_\alpha \). Since

\[
\Re[(1 - z^2)f_\alpha'(z)] = \Re \left[ \frac{1 + z^2 - 2\alpha z}{1 - z^2} \right] = \frac{(1 - |z|^2)(1 + |z|^2 - 2\alpha \Re(z))}{|1 - z^2|^2} > 0, \quad z \in E,
\]
Therefore, by Theorem A, the analytic function \( F_\alpha \) is univalent in \( E \) and convex in the direction of the imaginary axis. Consequently, in view of Lemma 2.3, the harmonic function \( f_\alpha = h_\alpha + \overline{g}_\alpha \) is univalent in \( E \) and also convex in the direction of the imaginary axis i.e., \( f_\alpha \in K_{H(\pi/2)}. \) Thus, by keeping (4) in mind, the result immediately follows from Theorem 3.1 by setting \( \eta = -1, \ \xi = 1 \) and \( \gamma = \pi/2. \)

**Corollary 3.3.** Let \( f_\theta = h_\theta + \overline{g}_\theta , \) where \( h_\theta(z) + g_\theta(z) = \frac{1}{2i \sin \theta} \log \left( \frac{1 + ze^{i\theta}}{1 + ze^{-i\theta}} \right), \ \theta \in (0, \pi) \) and \( |g'_\theta/h'_\theta| < 1 \) in \( E. \) Then, for an analytic map \( \phi \) in the class \( DCP, \) \( f_\theta \tilde{\phi} \in K_{H(\pi/2)}. \)

**Proof.** Let \( \rho(z) = (1 - z^2)(h'_\theta + g'_\theta)(z) = \frac{1 - z^2}{(1 + ze^{i\theta})(1 + ze^{-i\theta})}. \) It is easy to verify that \( \rho(0) = 1 \) and for every real number \( \delta, \ \Re \rho(e^{i\delta}) = 0. \) Therefore, by minimum principle for harmonic functions, we have, \( \Re[\rho(z)] = \Re[(1 - z^2)(h'_\theta + g'_\theta)(z)] > 0 \) in \( E. \) Now, the proof follows as in Corollary 3.2.

**Corollary 3.4.** Let \( f = h + \overline{g} \) be a normalized harmonic mapping such that \( h(z) - g(z) = z/(1 - z)^2 \) and \( |g'/h'| < 1 \) in \( E. \) Then \( f \tilde{\phi} \) is CHD for every analytic function \( \phi \in DCP. \)

**Proof.** One can easily verify that \( \Re[(1 - z^2)(h' - g')] > 0 \) in \( E. \) Further, \( h - g = z/(1 - z)^2 \) is univalent and convex in the horizontal direction and \( |g'/h'| < 1 \) in \( E \) implies that \( f \) is locally univalent in \( E. \) Therefore, by Lemma 2.3, \( f = h + \overline{g} \) is univalent and CHD in \( E. \) The desired conclusion now follows immediately from Theorem 3.1 by taking \( \eta = \zeta = 1 \) and \( \gamma = 0. \)

**Corollary 3.5.** Let \( f = h + \overline{g}, \) with \( |g'/h'| < 1 \) in \( E, \) be a normalized \((f(0) = 0, f_z(0) = 1)\) slanted right half-plane mapping in \( E \) given by

\[
\begin{align*}
    h(z) + e^{-2i\alpha}g(z) &= z/(1 - e^{i\alpha}z), \\
    -\pi/2 \leq \alpha \leq \pi/2,
\end{align*}
\]

then \( f \tilde{\phi} \in K_{H(\pi/2-\alpha)} \) for all \( \phi \in DCP. \)

**Proof.** As \( |g'/h'| < 1 \) in \( E, \) so \( f \) is locally univalent in \( E. \) Further for \( -\pi/2 \leq \alpha \leq \pi/2 \)

\[
\frac{z}{1 - e^{i\alpha}z} = h(z) + e^{-2i\alpha}g(z) = h - e^{2i(\pi/2-\alpha)}g
\]

is convex univalent in \( E \) and so, in particular, convex in the direction \( e^{i(\pi/2-\alpha)}, -\pi/2 \leq \alpha \leq \pi/2. \) Hence by Lemma 2.3, \( f = h + \overline{g} \in K_{H(\pi/2-\alpha)}, -\pi/2 \leq \alpha \leq \pi/2. \) Setting \( \eta = \xi = e^{i\alpha} \) and \( \gamma = \pi/2 - \alpha \) in (2), we get

\[
\Re[(1 - ze^{i\alpha})(1 - ze^{i\alpha})(h' - e^{2i(\pi/2-\alpha)}g')] = \Re\left[\frac{(1 - ze^{i\alpha})^2}{(1 - ze^{i\alpha})^2}\right] > 0, z \in E.
\]

The result now follows from Theorem 3.1.
Theorem 3.6. Let \( f = h + \overline{g} \in K_{H(\pi/2)} \) be such that \( F = h + g \) satisfies condition (iii) of Theorem A. Then \( f^*\phi \in K_{H(\pi/2)} \) for every \( \phi \in DCP \).

Proof. As \( f = h + \overline{g} \in K_{H(\pi/2)} \), therefore, by Lemma 2.3, \( F = h + g \) is univalent analytic in \( E \) and convex in the direction of the imaginary axis. As \( F = h + g \) also satisfies condition (iii) of Theorem A, therefore \( \Re[(1 - z^2)(h' + g')] > 0 \) in \( E \) and the desired result follows from Theorem 3.1 (taking \( \eta = -1, \xi = 1 \) and \( \gamma = \pi/2 \)). \( \square \)

An analytic function \( f(z) = z + a_2z^2 + ... \) is said to be typically-real in \( E \) if \( f(z) \) is real if and only if \( z \) is real in \( E \). Rogosinski [8] introduced the class \( T \) of typically-real functions and proved that a function \( f \) is in the class \( T \) if and only if \( f(z) = z/(1 - z^2)P(z) \), where \( P \) has real coefficients, \( P(0) = 1 \) and \( \Re P(z) > 0 \) in \( E \). Functions in the class \( T \) need not be univalent. Let \( TK_{\pi/2} \) be the class of univalent functions in \( T \) which map the unit disk \( E \) onto domains convex in the direction of the imaginary axis. The following results are known.

Theorem D. (a) A function \( H \in TK_{\pi/2} \) if and only if \( zH' \in T \).
(b) If \( H \) is in \( TK_{\pi/2} \) and \( G \) is in \( T \), then \( H * G \in T \).

The result (a) in the above theorem is an observation of Fejer [3] and the result (b) is due to Robertson [4]. We now state and prove our next result.

Theorem 3.7. Let \( f_\alpha = h_\alpha + \overline{g_\alpha} \), where \( h_\alpha(z) + g_\alpha(z) = z(1 - \alpha z)/(1 - z^2), \alpha \in [-1, 1] \) and \( |g'_\alpha/h'_\alpha| < 1 \) in \( E \), be a normalized harmonic mapping defined in \( E \). Then, for every analytic map \( \phi_\beta(z) = z(1 - \beta z)/(1 - z^2), \beta \in [-1, 1] \), defined in \( E \), \( f_\alpha* \phi_\beta \in K_{H(\pi/2)} \), provided \( f_\alpha* \phi_\beta \) is locally univalent and sense-preserving in \( E \).

Proof. If \( f_\alpha* \phi_\beta = h_\alpha * \phi_\beta + g_\alpha * \phi_\beta \) is locally univalent and sense-preserving in \( E \), then in view of Lemma 2.3, \( f_\alpha* \phi_\beta \in K_{H(\pi/2)} \) if and only if \( h_\alpha * \phi_\beta + g_\alpha * \phi_\beta \) is univalent and convex in the direction of the imaginary axis. Let \( F = h_\alpha * \phi_\beta + g_\alpha * \phi_\beta = (h_\alpha + g_\alpha) * \phi_\beta \), so that

\[
zF' = z(h_\alpha + g_\alpha)' * \phi_\beta
\]

Now, if we set \( P(z) = 1 - \beta z \), then \( P(0) = 1 \), coefficients of \( P \) are real and \( \Re P(z) > 0 \) in \( E \) for \( \beta \in [-1, 1] \). Hence

\[
\phi_\beta(z) = \frac{z(1 - \beta z)}{1 - z^2} \in T.
\]

Further, for \( \beta \in [-1, 1] \),

\[
\Re[(1 - z^2)\phi_\beta'(z)] = \Re \left[ \frac{1 + z^2 - 2\beta z}{1 - z^2} \right] = \frac{(1 - |z|^2)(1 + |z|^2 - 2\beta \Re(z))}{|1 - z^2|^2} > 0, \quad z \in E.
\]
Thus, by Theorem A, \( \phi_\beta \) is univalent and convex in the direction of the imaginary axis. Hence \( \phi_\beta \in TK_{\pi/2} \). Again

\[
 z(h_\alpha(z) + g_\alpha(z))' = \left[ \frac{z}{1 - z^2} \frac{1 + z^2 - 2\alpha z}{1 - z^2} \right]
\]

and coefficients of \( P_1(z) = \frac{1 + z^2 - 2\alpha z}{1 - z^2} \) are real, \( P_1(0) = 1 \) and \( \Re P_1(z) = \Re \left[ \frac{1 + z^2 - 2\alpha z}{1 - z^2} \right] > 0 \) in \( E \). Therefore \( z(h_\alpha + g_\alpha)' \in T \). So in view of Theorem D(b), \( zF'' \in T \) and hence \( F \in TK_{\pi/2} \) by Theorem D(a). This completes the proof.

**Remark 3.8.** We observe the functions \( \phi_\beta \) in Theorem 3.7 are not in the class \( DCP \) for \( \beta \in (-1,1) \) as \( \phi_\beta \) are not convex for these values of \( \beta \).

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