Optimal Sample Complexity for Matrix Completion and Related Problems via $\ell_2$-Regularization

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Abstract

We study the strong duality of non-convex matrix factorization: we show under certain dual conditions, non-convex matrix factorization and its dual have the same optimum. This has been well understood for convex optimization, but little was known for non-convex matrix factorization. We formalize the strong duality of non-convex matrix factorization through a novel analytical framework, and show that the duality gap is zero for a wide class of matrix factorization problems. Although matrix factorization problems are hard to solve in full generality, under certain conditions the optimal solution of the non-convex program is the same as its bi-dual, and we can achieve global optimality of the non-convex program by solving its bi-dual. This analytical framework might be of independent interest to non-convex optimization more broadly.

We apply our framework to matrix completion and robust Principal Component Analysis (PCA), which are examples of efficiently recovering a hidden matrix given limited reliable observations of it. While a long line of work has studied these problems, for basic problems in this area such as matrix completion, the information-theoretically optimal sample complexity was not known, and the sample complexity bounds if one also requires computational efficiency are even larger. In this work, we show that exact recoverability and strong duality hold with nearly-optimal sample complexity guarantees for matrix completion and robust PCA. For matrix completion, under the standard incoherence assumption that the underlying rank-$r$ matrix $X^* \in \mathbb{R}^{n \times n}$ with skinny SVD $U \Sigma V^T$ has $\max\{\|U^T e_i\|_2^2, \|V^T e_i\|_2^2\} \leq \frac{\mu r}{n}$ for all $i$, to the best of our knowledge we give (1) the first non-efficient algorithm achieving the optimal $O(\mu r \log n)$ sample complexity, and (2) an efficient algorithm in our framework achieving $O(\kappa^2 \mu r \log(n) \log_2(n))$ sample complexity.

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1 Introduction

In this work, we develop a novel framework to analyze a class of non-convex matrix factorization problems with strong duality, which lead to exact recoverability for matrix completion and robust Principal Component Analysis (PCA) via the solution to a convex problem. Non-convex matrix factorization problems have been an emerging object of study in theoretical computer science [JNS13, Har14, SL15, RSW16], optimization [WYZ12, SWZ14], machine learning [BNS16, GLM16, GHJY15, JMD10, LLR16, WX12], and many other domains. In theoretical computer science and optimization, the study of such models has led to significant advances in provable algorithms that converge to local minima in linear time [JNS13, Har14, SL15, AAZB+16, AZ16]. In machine learning, matrix factorization serves as a building block for large-scale prediction and recommendation systems, e.g., the winning submission for the Netflix prize [KBV09].

The focus of this work is on a wide class of matrix factorization problems. The problems can be stated as factorizing an unknown target matrix $X^*$ in the form of $X^* = AB$, by minimizing an $\ell_2$-regularized non-convex function $H(AB) + \frac{1}{2}\|AB\|^2_F$ over factor matrices $A \in \mathbb{R}^{n_1 \times r}$ and $B \in \mathbb{R}^{r \times n_2}$ with a known value of $r \ll \min\{n_1, n_2\}$.

Our work is motivated by several promising areas where our analytical framework for non-convex matrix factorizations is applicable. In low-rank matrix completion, an incoherent matrix can be exactly recovered by finding a solution of the form $AB$ that is consistent with the observed entries [JNS13, SL15, GLM16]. This problem has received a tremendous amount of attention due to its important role in optimization and its wide applicability in many areas such as quantum information theory and collaborative filtering [Har14, ZLZ16, BZ16]. In robust PCA, a fundamental problem of interest is recovering both the low-rank and the sparse components exactly from their superposition [CLMW11, NNS+14, GWL16, ZLZC15, ZLZ16, YPCC16]. The low-rank component corresponds to the product of $A$ and $B$ while the sparse component is captured by a proper choice of function $H(\cdot)$, e.g., the $\ell_1$ norm $\|\cdot\|_1$ [CMW11, ABHZ16]. We believe our analytical framework can be potentially applied to other non-convex problems more broadly, e.g., matrix sensing [TBR15], dictionary learning [SQW17a], weighted low-rank approximation [RSW16, LLR16], and deep linear neural network [Kaw16], which might be of independent interest.

Despite the large amount of work on matrix factorization problems, many fundamental issues remain unresolved. One of the long-standing questions is to achieve information-theoretically nearly-optimal sample complexity in a computationally efficient way. Without assumptions on the structure of the objective function, direct formulations of matrix factorization problems are NP-hard to optimize in general [HMRW14, ZLZ13]. With standard assumptions on the structure of the problem and with sufficiently many samples, these optimization problems can be solved efficiently given an initialization close enough to the global solution in the basin of attraction by running local search algorithms [JNS13, Har14, SL15, GHJY15, JGN+17]. However, there is a significant gap between the information-theoretic lower bound on the sample complexity and the sample complexity required of these methods; see Table 1 for a comparison. The problem becomes more challenging when the number of samples is small enough that the sample-based initialization is far from the desired solution, in which case the algorithm can run into a local minimum or a saddle point.

Another line of work has focused on studying the loss surface of matrix factorization problems, providing positive results for approximately achieving global optimality. One nice property in this line of research is the absence of spurious local minima for specific applications such as matrix completion [GLM16], matrix sensing [BNS16], dictionary learning [SQW17a], phase retrieval [SQW16], linear deep neural networks [Kaw16], etc. However, these results are based on concrete forms of objective functions. For many problems, such as those in linear deep neural networks, even when any local minimum is guaranteed to be globally optimal, in general it remains NP-hard to escape high-order saddle points [AG16]. Most importantly, all existing results rely on strong assumptions on the sample size.
1.1 Our Results

Our work studies the exact recoverability problem for a variety of non-convex matrix factorization problems. The goal is to provide a unified framework to analyze a large class of matrix factorization problems, and to achieve efficient algorithms. Our main results show that although matrix factorization problems are hard to optimize in general, under certain dual conditions the duality gap is zero, and thus the problem can be converted to an equivalent convex program. One of our main theorems is the following.

Theorems 4.3 (Strong Duality. Informal). Under certain dual conditions, strong duality holds for the non-convex optimization problem

$$(\tilde{A}, \tilde{B}) = \arg\min_{A \in \mathbb{R}^{n_1 \times r}, B \in \mathbb{R}^{r \times n_2}} F(A, B) = H(AB) + \frac{1}{2} \|AB\|_F^2, \quad H(\cdot) \text{ is convex and closed.} \quad (1)$$

In other words, problem (1) and its bi-dual problem

$$\tilde{X} = \arg\min_{X \in \mathbb{R}^{n_1 \times n_2}} H(X) + \|X\|_{r^*}, \quad (2)$$

have exactly the same optimal solutions in the sense that $\tilde{AB} = \tilde{X}$, where $\|X\|_{r^*}$ is a convex function defined by $\|X\|_{r^*} = \max_M \langle M, X \rangle - \frac{1}{2} \|M\|_F^2$ and $\|M\|_r^2 = \sum_{i=1}^r \sigma_i^2(M)$ is the sum of the first $r$ largest squared singular value.

Theorem 4.3 connects the non-convex program (1) to its convex counterpart via strong duality; see Figure 1. We mention that the strong duality condition rarely happens in the non-convex optimization literature: low-rank matrix approximation [OW92] and quadratic optimization with two quadratic constraints [BE06] are among the few paradigms that enjoy such a nice guarantee of strong duality. Other than these, little was known about the duality gap of non-convex problems. Furthermore, we also state a complementary lower bound to formalize the hardness of the above problem in general, which, assuming that the random 4-SAT problem is hard (see Conjecture 1) [RSW16], gives a strong negative result for deterministic algorithms. If also $\text{BPP} = \text{P}$ (see Section 7 for a discussion), then the same conclusion holds for randomized algorithms succeeding with probability at least 2/3. The negative results demonstrate the hardness of matrix factorization problems in the extreme cases.

Theorem 7.1 (Informal Hardness Statement). Assuming that random 4-SAT is hard on average, there is a problem in the form of (1) such that any deterministic algorithm achieving $(1 + \epsilon)OPT$ in the objective function value with $\epsilon \leq \epsilon_0$ requires $2^{\Omega(n_1 + n_2)}$ time, where $\text{OPT}$ is the optimum and $\epsilon_0 > 0$ is an absolute constant. If $\text{BPP} = \text{P}$, then the same conclusion holds for randomized algorithms succeeding with probability at least 2/3.

Given strong duality, the computational issues of the original problem can be overcome by solving the convex bi-dual problem (2), and thus only the dual condition in Theorem 4.3 need to be verified. We will show prototypical applications which obey the conditions. These are linear inverse problems of form (1) with a proper choice of function $H$. In these problems, a fundamental question of interest is to exactly recover a hidden matrix $X^*$ with $\text{rank}(X) \leq r$ given a limited number of linear observations of it. Matrix completion and robust PCA are two special cases of linear inverse problems.
Table 1: Comparison of matrix completion methods. Here $\kappa = \sigma_1(X^*) / \sigma_r(X^*)$ is the condition number of $X^* \in \mathbb{R}^{n_1 \times n_2}$, $\epsilon$ is the accuracy such that the output $\hat{X}$ obeys $\|X - \hat{X}\|_F \leq \epsilon$, $n(1) = \max\{n_1, n_2\}$ and $n(2) = \min\{n_1, n_2\}$. The first line of ours is an information-theoretic upper bound and the second line is an efficient approach.

| Work | Sample Complexity | $\mu$-Incoherence |
|------|-------------------|-------------------|
| [JNS13] | $O(\kappa^2 \mu^2 \sigma^4 r n(1) \log n(1) \log (\frac{\|X\|_F}{\epsilon}))$ | Condition (3) |
| [Har14] | $O(\mu r n(1) (r + \log \frac{n(1) \|X\|_F}{\epsilon} \|X\|_F^2 \sigma^2))$ | Condition (3) |
| [GLM16] | $O(\max\{\mu^3 r^2, \mu^4 r^4\} n(1) \log^2 n(1))$ | $\|X^*\|_2 \leq \frac{n}{\sqrt{n(2)}} \|X^*\|_F$ |
| [SL15] | $O(r n(1) \kappa^2 \max\{\mu \log n(2), \sqrt{\frac{n(1)}{n(2)} \mu^2 r^6 \kappa^4}\})$ | Condition (3) |
| [ZL16] | $O(\mu^4 n(1) \kappa^4 \max(\mu, \log n(1)))$ | Condition (3) |
| [GLZ17] | $O(\mu^2 r^4 \kappa^2 + \mu r \log (\frac{\|X\|_F}{\epsilon}) n(1) \log (\frac{\|X\|_F}{\epsilon}))$ | Condition (3) |
| [ZWL15] | $O(\mu^4 n(1) \log (\frac{\epsilon}{\mu}))$ | Condition (3) |
| [KMO10a] | $O(n(2)^2 r \sqrt{\frac{n(1)}{n(2)} \kappa^2} \max\{\mu \log n(2), \mu^2 r \sqrt{\frac{n(1)}{n(2)} \kappa^4}\})$ | Similar to (3) and (14) |
| [Che15] | $O(\mu r n(1) \log^2 n(1))$ | Condition (3) |
| Ours | $O(\mu r n(1) \log n(1))$ | Condition (3) |
| Ours | $O(\kappa^2 \mu r n(1) \log_2 n(1))$ | Condition (3) |

In the matrix completion problem, the linear measurements are of the form $\{X_{ij}^* : (i, j) \in \Omega\}$, where $\Omega$ is the support set which is uniformly distributed among all subsets of $[n_1] \times [n_2]$ of cardinality $m$. With strong duality, we can either study the exact recoverability of a non-convex primal problem (1), or investigate the validity of its convex dual (or bi-dual) problem (2). For matrix completion, we consider the former where we apply tools from geometric functional analysis to study the exact recoverability problem.

In the analysis of matrix completion, one typically requires an $\mu$-incoherence condition for a given rank-$r$ matrix $X$ with skinny SVD $U \Sigma V^T$ [Rec11, CT10]:

$$\|U^T e_i\|_2 \leq \sqrt{\frac{\mu r}{n_1}} \quad \text{and} \quad \|V^T e_i\|_2 \leq \sqrt{\frac{\mu r}{n_2}},$$

(3)

where $e_i$’s are vectors with $i$-th entry equal to 1 and other entries equal to 0. The incoherence condition claims that information spreads evenly throughout the left and right singular vectors and is quite standard in the matrix completion literature. Under this standard condition, we have the following results.

**Theorems 5.1, 5.3, and 5.2 (Matrix Completion: Informal).** $X^* \in \mathbb{R}^{n_1 \times n_2}$ is the unique matrix of rank at most $r$ that is consistent with the $m$ measurements with high probability, provided $m = O((\mu n_1 + n_2) r \log (n_1 + n_2))$ and $X^*$ satisfies incoherence (3). In addition, there exists a convex optimization for matrix completion in the form of (2) that exactly recovers $X^*$ with high probability, provided that $m = O(\kappa^2 \mu (n_1 + n_2) r \log (n_1 + n_2) \log_2 n(1) n(2))$, where $\kappa$ is the condition number of $X^*$.

To the best of our knowledge, our result is the first to connect convex matrix completion to non-convex matrix completion, two parallel lines of research that have received significant attention in the past few years. Table 1 compares our result with prior results. Note that Table 1 compares the best known sample complexity of solving matrix completion with any method and the same assumptions, not necessarily an existing method, and indeed our method differs from previous ones.

For the problem of matrix completion, we study exact recoverability of a non-convex optimization problem (1) and apply strong duality to build the validity of its convex counterpart (2). An alternative way is

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1The method of [GLM16] works only for the positive semi-definite matrices $X^*$.
to investigate problem (2) directly. In particular, this is how we analyze the robust PCA problem. The robust PCA problem is to decompose a given matrix $D = X^* + S^*$ into the sum of a low-rank component $X^*$ and a sparse component $S^*$ [ANW12]. We obtain the following theorem for robust PCA.

**Theorems 6.1 (Robust PCA. Informal).** There exists a convex optimization formulation for robust PCA in the form of problem (2) that exactly recovers the incoherent matrix $X^* \in \mathbb{R}^{n_1 \times n_2}$ and $S^* \in \mathbb{R}^{n_1 \times n_2}$ with high probability, even if $\text{rank}(X^*) = \Theta \left( \frac{\min(n_1, n_2)}{\mu \log^2 \max(n_1, n_2)} \right)$ and the size of the support of $S^*$ is $m = \Theta(n_1 n_2)$, where the support set of $S^*$ is uniformly distributed among all sets of cardinality $m$, and the incoherence parameter $\mu$ satisfies constraints (3) and (14).

The bounds in Theorem 6.1 match the best known results in the robust PCA literature when the supports of $S^*$ are uniformly sampled [CLMW11], under an arguably more intuitive assumption; see Section 6. Note that the results hold even when $X^*$ is close to full rank and a constant fraction of the entries have noise.

Independently of our work, Ge et al. [GJY17] developed a framework to analyze the loss surface of matrix completion and robust PCA. Their bounds are: for matrix completion, the sample complexity is $O(...)$; for robust PCA, the outlier entries are deterministic and the number that the method can tolerate is $O(...)$.

Zhang et al. [ZWG17] also studied the robust PCA problem using non-convex optimization, where the outlier entries are deterministic and the number of outliers that their algorithm can tolerate is $O(...)$.

The strong duality approach is unique to our work.

**1.2 Our Techniques**

**Reduction to Low-Rank Approximation.** Our results are inspired by the low-rank approximation problem:

$$
\min_{A \in \mathbb{R}^{n_1 \times r}, B \in \mathbb{R}^{r \times n_2}} \frac{1}{2} \| -\tilde{\Lambda} - AB \|_F^2.
$$

We know that all local solutions of (4) are globally optimal (see Lemma 4.1) and that strong duality holds for any given matrix $-\tilde{\Lambda} \in \mathbb{R}^{n_1 \times n_2}$ [GRG16]. To extend this property to our more general problem (1), our main insight is to reduce problem (1) to the form of (4) using the $\ell_2$-regularization term. While some prior work attempted to apply a similar reduction, their conclusions either depended on unrealistic conditions on local solutions, e.g., all local solutions are rank-deficient [HYV14, GRG16], or their conclusions relied on strong assumptions on the objective functions, e.g., that the objective functions are twice-differentiable [HV15]. Instead, our general results formulate strong duality via the existence of a dual certificate $\tilde{\Lambda}$. For concrete applications, the existence of a dual certificate is then converted to mild assumptions, e.g., that the measurements are random.

More specifically, denote by $(\tilde{A}, \tilde{B})$ the optimal solution to (1). Define $T = \{ \tilde{A}L + M\tilde{B} : L \in \mathbb{R}^{r \times n_2}, M \in \mathbb{R}^{n_1 \times r} \}$, $T^\perp$ the complement of $T$, and $P_T$ the orthogonal projection onto subspace $T$. Let $\partial H(X) = \{ \Lambda \in \mathbb{R}^{n_1 \times n_2} : H(Y) \geq H(X) + \langle \Lambda, Y - X \rangle \text{ for any } Y \}$ be the sub-differential of function $H$ evaluated at $X$. To perform the reduction from problem (1) to (4), we study the Lagrangian $L(A, B, \Lambda)$ of (1), which is equivalent to problem (4) if we fix $\Lambda = \tilde{\Lambda}$. We show that, for a fixed Lagrangian multiplier $\Lambda \in \partial H(\tilde{A}\tilde{B})$, minimizing the primal problem (1) reduces to minimizing the Lagrangian function $L(A, B, \tilde{\Lambda})$, i.e., problem (4), thus strong duality holds, if $(\tilde{A}, \tilde{B})$ remains globally optimal to $L(A, B, \tilde{\Lambda})$ (i.e., problem (4)). This can be translated to: a) $\forall \Lambda \in \partial H(\tilde{A}\tilde{B})$, b) $(\tilde{A}, \tilde{B})$ is a stationary point of the Lagrangian $L(A, B, \tilde{\Lambda})$ so that c) $\tilde{A}\tilde{B} = \text{svd}_r(-\tilde{\Lambda})$, where $\text{svd}_r(\Lambda) = U_{:, 1:r} \Sigma_{1:r, 1:r} V_{1:r, :}^T$ if $U \Sigma V^T$ is the SVD of $-\tilde{\Lambda}$, and $U_{:, 1:r}$ and $\Sigma_{1:r, 1:r}$ are the first $r$ columns of $U$ and top left $r \times r$ submatrix of $\Sigma$, respectively. We note that conditions b) and c) can be rephrased as $P_T(-\tilde{\Lambda}) = \tilde{A}\tilde{B}$ and $\sigma_1(P_T \perp \Lambda) \leq \sigma_r(\tilde{A}\tilde{B})$, respectively. To satisfy conditions a), b) and c) simultaneously, one may want to find a certificate $\tilde{\Lambda}$ such that among all matrices $\Lambda \in \partial H(\tilde{A}\tilde{B})$ (i.e., condition a)) with $P_T(-\tilde{\Lambda}) = \tilde{A}\tilde{B}$ (i.e.,...
condition b), \( \tilde{A} \) is the one with minimum Frobenius norm, so that condition c) is easier to satisfy. Following this principle, we build our dual certificate \( \tilde{A} \) by \(-P_{\partial H} P_T (P_T P_{\partial H} P_T)^{-1} (A B)\). It can be easily checked that conditions a) and b) hold for our construction. Thus the remainder is to prove condition c) for specific applications. We observe that \( \partial H = \Omega \) for matrix completion, where \( \Omega \) is the linear space that characterizes the sample support. This nice property serves as a bridge, connecting our analytical framework to the concrete application of matrix completion. We will then state how to prove condition c) by randomness as follows.

**The Bless of Randomness.** Unfortunately, the desired dual certificate \( \tilde{A} \) obeying conditions a), b), and c) may not exist in the deterministic world: the hardness result above shows that for the problem of weighted low-rank approximation, which can be cast in the form of (1), without some randomization in the measurements made on the underlying low rank matrix, it is NP-hard to achieve a good objective value, not to mention to achieve strong duality. A similar phenomenon was observed for deterministic matrix completion [HM12]. Thus we should utilize such randomness to analyze condition c). For matrix completion, where we aim at recovering a low-rank matrix \( X^* \) given limited measurements, we make the usual assumption that the measurements are random. With randomness, the angle between spaces \( \Omega \) and \( T \) is small with high probability, namely, \( X^* \) is almost the unique matrix in the space \( T \) that is consistent with the measurements. Thus, our dual certificate can be represented as another form of a convergent Neumann series concerning the projection operators on the spaces \( \Omega \) and \( T \). The remainder of the proof is to show that such a construction obeys our dual condition.

To prove the dual condition for matrix completion, we use the fact that the subspace \( \Omega \) and the complement space \( T^\perp \) are almost orthogonal when the sample size is sufficiently large. This implies the projection of our dual certificate on the space \( T^\perp \) has a very small norm, which exactly matches our dual condition.

**Non-Convex Geometric Analysis.** Strong duality implies that the primal problem (1) and its bi-dual problem (2) have exactly the same solutions in the sense that \( \tilde{A} B = \tilde{X} \). Thus, to show exact recoverability of linear inverse problems such as matrix completion and robust PCA, it suffices to study either the non-convex primal problem (1) or its convex counterpart (2). Here we do the former analysis for matrix completion. We mention that traditional techniques [CT10, Rec11, CRPW12] for convex optimization break down for our non-convex problem, since the subgradient of a non-convex objective function may not even exist [BV04]. Instead, we apply tools from geometric functional analysis [Ver09] to analyze the geometry of problem (1). Our non-convex geometric analysis is in stark contrast to prior techniques of convex geometric analysis [Ver15] where convex combinations of non-convex constraints were used to define the Minkowski functional (e.g., in the definition of atomic norm) while our method uses the non-convex constraint itself. For matrix completion, problem (1) has two hard constraints: a) the rank of the output matrix should be no larger than \( r \), as implied by the form of \( A B \); b) the output matrix should be consistent with the sampled measurements, i.e., \( P_\Omega (A B) = P_\Omega (X^*) \). We study the feasibility condition of problem (1) from a geometric perspective: \( \tilde{A} B = X^* \) is the unique feasible solution to problem (1) if and only if starting from \( X^* \), the rank of \( X^* + D \) increases for all directions \( D \)'s in the constraint set \( \Omega^\perp = \{ D \in \mathbb{R}^{n_1 \times n_2} : P_\Omega (X^* + D) = P_\Omega (X^*) \} \) (a.k.a. the feasibility condition). This can be geometrically interpreted as the requirement that the descent cone \( D_S (X^*) = \{ t (X - X^*) \in \mathbb{R}^{n_1 \times n_2} : \text{rank}(X) \leq r, t \geq 0 \} \) and the constraint set \( \Omega^\perp \) must intersect uniquely at \( 0 \) (see Figure 2), which means \( X^* \) is the unique matrix that satisfies the constraints a) and b).

Then the following tangent cone argument shows exact recoverability for matrix completion.

Let \( S \) be the set of all matrices with rank at most \( r \) around the underlying matrix \( X^* \). In the tangent cone argument, by definition, \( D_S (X^*) \) is a subset of the tangent cone of \( S \) at \( X^* \). The latter cone of interest has a very nice form, namely, it is just the space \( T \) mentioned above. Our information-theoretic upper bound for matrix completion can now leverage results from prior work which imply \( T \cap \Omega^\perp = \{ 0 \} \) with a large

![Figure 2: Feasibility.](image-url)
enough sample size. In other words, among all matrices of the form \( X^* + D \), \( D = 0 \) is the only matrix such that \( \text{rank}(X^* + D) \leq \tau \) and \( X^* + D \) is consistent with the observations.

Our positive results are complemented by known sample complexity lower bounds, up to a constant factor.

**Putting Things Together.** We summarize our new framework with the following figure.

![Diagram showing the relationship between Non-Convex Problem (1) and Convex Problem (2).]

**Other Techniques.** An alternative method is to investigate the exact recoverability of problem (2) via standard convex analysis. We find that the sub-differential of our induced function \( \| \cdot \|_{r^*} \) is very similar to that of the nuclear norm. With this observation, we prove the validity of robust PCA in the form of (2) by combining this property of \( \| \cdot \|_{r^*} \) with standard techniques from [CLMW11].

## 2 Related Work

Non-convex matrix factorization is a popular topic studied in theoretical computer science [JNS13, Har14, SL15, RSW16], machine learning [BNS16, GLM16, GHJY15, JMD10, LLR16], and optimization [WYZ12, SWZ14]. We review several lines of research on studying the global optimality of such optimization problems.

**Global Optimality of Matrix Factorization.** While lots of matrix factorization problems have been shown to have no spurious local minima, they either require additional conditions on the local minima, or are based on particular forms of the objective function. Specifically, Burer and Monteiro [BM05] showed that one can minimize \( F(AA^T) \) for any convex function \( F \) by solving for \( A \) directly without introducing any local minima, provided that the rank of the output \( A \) is larger than the rank of the true minimizer \( X_{\text{true}} \). However, such a condition is often impossible to check as \( \text{rank}(X_{\text{true}}) \) is typically unknown a priori. To resolve the issue, Bach et al. [BMP08] and Journée et al. [JBAS10] proved that \( X = AA^T \) is a global minimizer of \( F(X) \), if \( A \) is a rank-deficient local minimizer of \( F(AA^T) \) and \( F(X) \) is a twice differentiable convex function. Haeffele and Vidal [HV15] further extended this result by allowing a more general form of objective function \( F(X) = G(X) + H(X) \), where \( G \) is a twice differentiable convex function with compact level set and \( H \) is a proper convex function such that \( F \) is lower semi-continuous. However, a major drawback of this line of research is that these results fail when the local minimizer is of full rank.

**Matrix Completion.** Matrix completion is a prototypical example of matrix factorization. One line of work on matrix completion builds on convex relaxation (e.g., [SS05, CR09, CT10, Rec11, CRPW12, NW12]), which does not achieve the optimal sample complexity. Recently, Ge et al. [GLM16] showed that matrix completion has no spurious local optimum, when \( |\Omega| \) is sufficiently large and the matrix \( Y \) is incoherent. The result is only for positive semi-definite matrices and their sample complexity is not optimal.

Another line of work is built upon good initialization for global convergence. Recent attempts showed that one can first compute some form of initialization (e.g., by singular value decomposition) that is close to the global minimizer and then use non-convex approaches to reach global optimality, such as alternating minimization, block coordinate descent, and gradient descent [KMO10b, KMO10a, JNS13, Kes12, Har14, ...]
BKS16, ZL15, ZWL15, TBSR15, CW15, SL15]. In our result, in contrast, we can reformulate non-convex matrix completion problems as equivalent convex programs, which guarantees global convergence from any initialization.

**Robust PCA.** Robust PCA is also a prototypical example of matrix factorization. The goal is to recover both the low-rank and the sparse components exactly from their superposition [CLMW11, NNS+14, GWL16, ZLZC15, ZLZ16, YPCC16]. It has been widely applied to various tasks, such as video denoising, background modeling, image alignment, photometric stereo, texture representation, subspace clustering, and spectral clustering.

There are typically two settings in the robust PCA literature: a) the support set of the sparse matrix is uniformly sampled [CLMW11, ZLZ16]; b) the support set of the sparse matrix is deterministic, but the non-zero entries in each row or column of the matrix cannot be too large [YPCC16, GJY17]. In this work, we discuss the first case. Our framework provides results that match the best known work in setting (b) [CLMW11].

**Other Matrix Factorization Problems.** Matrix sensing is another typical matrix factorization problem [CRPW12, JNS13, ZWL15]. Bhojanapalli et al. [BNS16] and Tu et al. [TBSR15] showed that the matrix recovery model \( \min_{A, B} \frac{1}{2} \| A (AB - Y) \|_{F}^{2} \), achieves optimality for every local minimum, if the operator \( A \) satisfies the restricted isometry property. They further gave a lower bound and showed that the unstructured operator \( A \) may easily lead to a local minimum which is not globally optimal.

Some other matrix factorization problems are also shown to have nice geometric properties such as the property that all local minima are global minima. Examples include dictionary learning [SQW17a], phase retrieval [SQW16], and linear deep neural networks [Kaw16]. In multi-layer linear neural networks where the goal is to learn a multi-linear projection \( X^{*} = \prod_{i} W_{i} \), each \( W_{i} \) represents the weight matrix that connects the hidden units in the \( i \)-th and \( (i + 1) \)-th layers. The study of such linear models is central to the theoretical understanding of the loss surface of deep neural networks with non-linear activation functions [Kaw16, CHM+15]. In dictionary learning, we aim to recover a complete (i.e., square and invertible) dictionary matrix \( A \) from a given signal \( X \) in the form of \( X = AB \), provided that the representation coefficient \( B \) is sufficiently sparse. This problem centers around solving a non-convex matrix factorization problem with a sparsity constraint on the representation coefficient \( B \) [BMP08, SQW17a, SQW17b, ABGM14]. Other high-impact examples of matrix factorization models range from the classic unsupervised learning problems like PCA, independent component analysis, and clustering, to the more recent problems such as non-negative matrix factorization, weighted low-rank matrix approximation, sparse coding, tensor decomposition [BCMV14, AGH+14], subspace clustering [ZLZG15, ZLZG14], etc. Applying our framework to these other problems is left for future work.

**Atomic Norms.** The atomic norm is a recently proposed function for linear inverse problems [CRPW12]. Many well-known norms, e.g., the \( \ell_{1} \) norm and the nuclear norm, serve as special cases of atomic norms. It has been widely applied to the problems of compressed sensing [TBSR13], low-rank matrix recovery [CR13], blind deconvolution [ARR14], etc. The norm is defined by the Minkowski functional associated with the convex hull of a set \( A \): \( \| X \|_{A} = \inf \{ t > 0 : X \in tA \} \). In particular, if we set \( A \) to be the convex hull of the infinite set of unit-\( \ell_{2} \)-norm rank-one matrices, then \( \| \cdot \|_{A} \) equals to the nuclear norm. We mention that our objective term \( \| AB \|_{F} \) in problem (1) is similar to the atomic norm, but with slight differences: unlike the atomic norm, we set \( A \) to be the infinite set of unit-\( \ell_{2} \)-norm rank-\( r \) matrices for \( \text{rank}(X) \leq r \). With this, we achieve better sample complexity guarantees than the atomic-norm based methods.
3 Preliminaries

We will use calligraphy to represent a set, bold capital letters to represent a matrix, bold lower-case letters to represent scalars. Specifically, we denote by $X^* \in \mathbb{R}^{n_1 \times n_2}$ the underlying matrix. We use $X_{i:} \in \mathbb{R}^{n_1 \times 1}$ ($X_{t:} \in \mathbb{R}^{1 \times n_2}$) to indicate the $t$-th column (row) of $X$. The entry in the $i$-th row, $j$-th column of $X$ is represented by $X_{ij}$. The condition number of $X$ is $\kappa = \sigma_1(X)/\sigma_r(X)$. We let $n_1 = \max\{n_1, n_2\}$ and $n_2 = \min\{n_1, n_2\}$. For a function $H(M)$ on an input matrix $M$, its conjugate function $H^*$ is defined by $H^*(A) = \max_{M} \langle A, M \rangle - H(M)$. Furthermore, let $H^{**}$ denote the conjugate function of $H^*$.

We will frequently use $\text{rank}(X) \leq r$ to constrain the rank of $X$. This can be equivalently represented as $X = AB$, by restricting the number of columns of $A$ and rows of $B$ to be $r$. For norms, we denote by $\|X\|_F = \sqrt{\sum_{ij}X_{ij}^2}$ the Frobenius norm of matrix $X$. Let $\sigma_1(X) \geq \sigma_2(X) \geq \ldots \geq \sigma_r(X)$ be the non-zero singular values of $X$. The nuclear norm (a.k.a. trace norm) of $X$ is defined by $\|X\|_* = \sum_{i=1}^r \sigma_i(X)$, and the operator norm of $X$ is $\|X\|_\infty = \max_{ij} |X_{ij}|$. For two matrices $A$ and $B$ of equal dimensions, we denote by $\langle A, B \rangle = \sum_{ij} A_{ij}B_{ij}$. We denote by $\partial H(X) = \{A \in \mathbb{R}^{n_1 \times n_2} : H(Y) \geq H(X) + \langle A, Y - X \rangle$ for any $Y\}$ the sub-differential of function $H$ evaluated at $X$. We define the indicator function of convex set $C$ by $I_C(X) = \begin{cases} 0, & \text{if } X \in C; \\ +\infty, & \text{otherwise}. \end{cases}$ For any non-empty set $C$, denote by $\text{cone}(C) = \{ tX : X \in C, t \geq 0 \}$.

We denote by $\Omega$ the set of indices of observed entries, and $\Omega^\perp$ its complement. Without confusion, $\Omega$ also indicates the linear subspace formed by matrices with entries in $\Omega^\perp$ being 0. We denote by $P_\Omega : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{n_1 \times n_2}$ the orthogonal projector of subspace $\Omega$. We will consider a single norm for these operators, namely, the operator norm denoted by $\|A\|$ and defined by $\|A\| = \sup_{\|X\|_F = 1} \|A(X)\|_F$. For any orthogonal projection operator $P_T$ to any subspace $T$, we know that $\|P_T\| = 1$ whenever $\dim(T) \neq 0$. For distributions, denote by $N(0, 1)$ a standard Gaussian random variable, $\text{Uniform}(m)$ the uniform distribution of cardinality $m$, and $\text{Ber}(p)$ the Bernoulli distribution with success probability $p$.

4 $\ell_2$-Regularized Matrix Factorizations: A Unified Framework

In this section, we develop a unified framework to analyze a general class of $\ell_2$-regularized matrix factorization problems. Our framework can be widely applied to many specific problems and leads to nearly optimal sample complexity guarantees. In particular, we study the $\ell_2$-regularized matrix factorization problem

\[
\begin{array}{ll}
\text{(P)} & \min_{A \in \mathbb{R}^{n_1 \times r}, B \in \mathbb{R}^{r \times n_2}} F(A, B) = H(AB) + \frac{1}{2} \|AB\|_F^2, \\
& H(\cdot) \text{ is convex and closed.}
\end{array}
\]

We show that the duality gap between (P) and its dual (bi-dual) problem is zero, so problem (P) can be converted to an equivalent convex problem.

4.1 Strong Duality

We first consider an easy case where $H(AB) = \frac{1}{2} \|\hat{Y}\|_F^2 - \langle \hat{Y}, AB \rangle$ for a fixed $\hat{Y}$, leading to the objective function $\frac{1}{2} \|\hat{Y} - AB\|_F^2$. For this case, we establish the following lemma.

Lemma 4.1. For any given matrix $\hat{Y}$, any local minimum of $f(A, B) = \frac{1}{2} \|\hat{Y} - AB\|_F^2$ is globally optimal, given by $\text{svd}_r(\hat{Y})$. The objective function $f(A, B)$ around any saddle point has a negative second-order directional curvature. Moreover, $f(A, B)$ has no local maximum.\footnote{Prior work studying the loss surface of low-rank matrix approximation assumes that matrix $\hat{A}$ is of full rank and does not have the same singular values [BH89]. In this work, we generalize this result by removing these two assumptions.}
Given this lemma, we can reduce $F(A, B)$ to the form $\frac{1}{2} \| \hat{Y} - AB \|_F^2$ for some $\hat{Y}$ plus an extra term:

\[
F(A, B) = \frac{1}{2} \| AB \|_F^2 + H(AB) = \frac{1}{2} \| AB \|_F^2 + H^*(AB) = \max_{\Lambda} \frac{1}{2} \| AB \|_F^2 + \langle \Lambda, AB \rangle - H^*(\Lambda)
\]

where we define $L(A, B, \Lambda) \triangleq \frac{1}{2} \| - \Lambda - AB \|_F^2 - \frac{1}{2} \| \Lambda \|_F^2 - H^*(\Lambda)$ the Lagrangian of problem (P), and the second equality holds because $H$ is closed and convex w.r.t. the argument $AB$. For any fixed value of $\Lambda$, by Lemma 4.1, any local minimum of $L(A, B, \Lambda)$ is globally optimal, because minimizing $L(A, B, \Lambda)$ is equivalent to minimizing $\frac{1}{2} \| - \Lambda - AB \|_F^2$ for a fixed $\Lambda$.

The remaining part of our analysis is to choose a proper $\tilde{\Lambda}$ such that $(\tilde{A}, \tilde{B}, \tilde{\Lambda})$ is a primal-dual saddle point of $L(A, B, \Lambda)$, so that $\min_{A, B} L(A, B, \tilde{\Lambda})$ and problem (P) have the same optimal solution $(\tilde{A}, \tilde{B})$. For this, we introduce the following condition, and later we will show that the condition holds with high probability.

**Condition 1.** For a solution $(\tilde{A}, \tilde{B})$ to problem (P), there exists an $\tilde{\Lambda} \in \partial_X H(X)|_{X=\tilde{A}B}$ such that

\[
-\tilde{A} \tilde{B} \tilde{B}^T = \tilde{\Lambda} \tilde{B}^T \quad \text{and} \quad \tilde{A}^T (-\tilde{A} \tilde{B}) = \tilde{A}^T \tilde{\Lambda}.
\]

We note that $\nabla_A L(A, B, \Lambda) = ABB^T + AB^T$ and $\nabla_B L(A, B, \Lambda) = A^T AB + A^T \Lambda$ for a fixed $\Lambda$. In particular, if we set $\Lambda$ to be the $\tilde{\Lambda}$ in (6), then $\nabla_A L(A, B, \tilde{\Lambda})|_{A=\tilde{A}} = 0$ and $\nabla_B L(\tilde{A}, \tilde{B}, \tilde{\Lambda})|_{B=\tilde{B}} = 0$. So $(\tilde{A}, \tilde{B})$ is either a saddle point or a local minimizer of $L(A, B, \Lambda)$ as a function of $(A, B)$ for the fixed $\tilde{\Lambda}$. The following lemma states that if it is a local minimizer, then strong duality holds.

**Lemma 4.2 (Dual Certificate).** Let $(\tilde{A}, \tilde{B})$ be a global minimizer of $F(A, B)$. If there exists a dual certificate $\tilde{\Lambda}$ satisfying Condition 1 and the pair $(\tilde{A}, \tilde{B})$ is a local minimizer of $L(A, B, \tilde{\Lambda})$ for the fixed $\tilde{\Lambda}$, then strong duality holds. Moreover, we have the relation $\tilde{A} \tilde{B} = \text{svd}_r(-\tilde{\Lambda})$.

**Proof Sketch.** By the assumption of the lemma, we can show that $(\tilde{A}, \tilde{B}, \tilde{\Lambda})$ is a primal-dual saddle point to the Lagrangian $L(A, B, \Lambda)$ (see Appendix B). To show strong duality, by the fact that $F(A, B) = \max_A L(A, B, \Lambda)$ and that $\tilde{\Lambda} = \arg \max_{\Lambda} L(\tilde{A}, \tilde{B}, \Lambda)$, we have $F(\tilde{A}, \tilde{B}) = L(\tilde{A}, \tilde{B}, \tilde{\Lambda}) \leq L(\tilde{A}, \tilde{B}, \tilde{A})$, for any $\tilde{A}, \tilde{B}$, where the inequality holds because $(\tilde{A}, \tilde{B}, \tilde{\Lambda})$ is a primal-dual saddle point of $L$. So on the one hand, $\min_{A,B} \max_{\Lambda} L(A, B, \Lambda) = F(\tilde{A}, \tilde{B}) \leq \min_{A,B} L(A, B, \tilde{A}) \leq \max_{\Lambda} \min_{A,B} L(A, B, \Lambda)$. On the other hand, by weak duality, $\min_{A,B} \max_{\Lambda} L(A, B, \Lambda) = \max_{\Lambda} \min_{A,B} L(A, B, \Lambda)$, i.e., strong duality holds. Therefore, $\tilde{A} \tilde{B} = \arg \min_{A,B} L(A, B, \tilde{\Lambda}) = \arg \min_{A,B,\tilde{\Lambda}} \frac{1}{2} \| AB \|_F^2 + \langle \tilde{\Lambda}, AB \rangle - H^*(\tilde{\Lambda}) = \arg \min_{A,B} \| - \tilde{\Lambda} - AB \|_F^2 = \text{svd}_r(-\tilde{\Lambda})$, as desired.

This lemma then leads to the following theorem.

**Theorem 4.3.** Denote by $(\tilde{A}, \tilde{B})$ the optimal solution of problem (P). Define a matrix space

\[
\mathcal{T} \triangleq \{ \tilde{A}X^T + Y\tilde{B}, \ X \in \mathbb{R}^{n_2 \times r}, \ Y \in \mathbb{R}^{n_1 \times r} \}.
\]

Then strong duality holds for problem (P), provided that

\[
(1) \quad \tilde{\Lambda} \in \partial_X H(\tilde{A} \tilde{B}) \triangleq \Psi,
\]

\[
(2) \quad \mathcal{P}_T(-\tilde{\Lambda}) = \tilde{A} \tilde{B},
\]

\[
(3) \quad \| \mathcal{P}_{T^\perp}(-\tilde{\Lambda}) \| < \sigma_r(\tilde{A} \tilde{B}).
\]

---

3One can easily check that $L(A, B, \Lambda) = \min_M L'(A, B, M, \Lambda)$, where $L'(A, B, M, \Lambda)$ is the Lagrangian of the constraint optimization problem $\min_{A,B,M} \frac{1}{2} \| AB \|_F^2 + H(M)$, s.t. $M = AB$. With a little abuse of notation, we call $L(A, B, \Lambda)$ the Lagrangian of the unconstrained problem (P) as well.
The proof is deferred to Appendix C.

Therefore, the dual conditions in (8) are equivalent to (1) incoherence condition (3), \( \mu \) holds for many random matrices with incoherence parameter (3) to the low-rank matrix \( n \times n \) intuition is captured by the incoherence conditions. Formally, denote by unified framework in Section 4 to matrix completion, by setting \( M = (I - \tilde{A}A^\dagger)M(I - \tilde{B}B^\dagger) \) and so \( \|P_{T^\perp}M\| \leq \|M\| \), a fact that we will frequently use in the sequel. Denote by \( \|E\| < \sigma_r(\tilde{A}\tilde{B}) \) and \( \|E\| < \sigma_r(\tilde{A}\tilde{B}) \) and condition (b) imply \( \tilde{A}\tilde{B} = \text{svd}_r(-\tilde{\Lambda}) \). Conversely, \( \|E\| < \sigma_r(\tilde{A}\tilde{B}) \) and condition (b) imply \( \tilde{A}\tilde{B} = \text{svd}_r(-\tilde{\Lambda}) \). Therefore, the dual conditions in (8) are equivalent to (1) \( \tilde{\Lambda} \in \partial H(\tilde{A}\tilde{B}) \triangleq \Psi \); (2) \( P_T(-\tilde{\Lambda}) = \tilde{A}\tilde{B} \); (3) \( \|P_{T^\perp}\tilde{\Lambda}\| < \sigma_r(\tilde{A}\tilde{B}) \).

To show the dual condition in Theorem 4.3, intuitively, we need to show that the angle \( \theta \) between subspace \( T \) and \( \Psi \) is small (see Figure 3) for a specific function \( H(\cdot) \). In the following (see Section 5.1), we will demonstrate applications that, with randomness, obey this dual condition with high probability.

5 Matrix Completion

In matrix completion, there is a hidden matrix \( X^* \in \mathbb{R}^{n_1 \times n_2} \) with rank \( r \). We are given measurements \( \{X^*_{ij} : (i,j) \in \Omega\} \), where \( \Omega \sim \text{Uniform}(m) \), i.e., \( \Omega \) is sampled uniformly at random from all subsets of \( [n_1] \times [n_2] \) of cardinality \( m \). The goal is to exactly recover \( X^* \) with high probability. Here we apply our unified framework in Section 4 to matrix completion, by setting \( H(\cdot) = \mathbf{1}_{\{M : P_M(M) = P_M(X^*)\}}(\cdot) \).

A quantity governing the difficulties of matrix completion is the incoherence parameter \( \mu \). Intuitively, matrix completion is possible only if the information spreads evenly throughout the low-rank matrix. This intuition is captured by the incoherence conditions. Formally, denote by \( U\Sigma V^T \) the skinny SVD of a fixed \( n_1 \times n_2 \) matrix \( X \) of rank \( r \). Candès et al. [CLMW11, CR09, Rec11, ZLZ16] introduced the \( \mu \)-incoherence condition (3) to the low-rank matrix \( X \). For condition (3), it can be shown that \( 1 \leq \mu \leq \frac{n(1)}{r} \). The condition holds for many random matrices with incoherence parameter \( \mu \) about \( \sqrt{r \log n(1)} \) [KMO10a].

We have two positive results. The first result is an information-theoretic upper bound: with the standard incoherence condition (3), \( X^* \) is the unique matrix of rank at most \( r \) that is consistent with the observations. The proof is deferred to Appendix C.
Theorem 5.1 (Information-Theoretic Upper Bound). Let $\Omega \sim \text{Uniform}(m)$ be the support set uniformly distributed among all sets of cardinality $m$. Suppose that $m \geq c \mu n(1)^r \log n(1)$ for an absolute constant $c$. Then $X^*$ is the unique $n_1 \times n_2$ matrix of rank at most $r$ with $\mu$-incoherence condition (3) such that $P_{\Omega}(X) = P_{\Omega}(X^*)$, with probability at least $1 - n(1)^{-10}$.

We describe a simple finite-time inefficient algorithm given Theorem 5.1 in Section G. Our positive results match a lower bound from prior work. The lower bound claims that the sample complexity in Theorem 5.1 is optimal.

Theorem 5.2 (Information-Theoretic Lower Bound. [CT10], Theorem 1.7). Let $\Omega \sim \text{Uniform}(m)$ be the support set uniformly distributed among all sets of cardinality $m$. Suppose that $m \leq c \mu n(1)^r \log n(1)$ for an absolute constant $c$. Then there exist infinitely many $n_1 \times n_2$ matrices $X'$ of rank at most $r$ obeying $\mu$-incoherence (3) such that $P_{\Omega}(X') = P_{\Omega}(X^*)$, with probability at least $1 - n(1)^{-10}$.

We note that Theorem 5.2 is information-theoretic, meaning that any algorithm (computationally inefficient or efficient) cannot exactly recover a matrix if the sample size is smaller than $c \mu n(1)^r \log n(1)$ for an absolute constant $c$. Therefore, our information-theoretic upper bound in Theorem 5.1 matches this lower bound.

Our second positive result converts the feasibility problem in Theorem 5.1 to a convex optimization problem, which can be efficiently solved.

Theorem 5.3 (Efficient Matrix Completion). Let $\Omega \sim \text{Uniform}(m)$ be the support set uniformly distributed among all sets of cardinality $m$. Suppose $X^*$ has condition number $\kappa = \sigma_1(X^*)/\sigma_r(X^*)$. Then there are absolute constants $c$ and $c_0$ such that with probability at least $1 - c_0 n(1)^{-10}$, the output of the convex problem

$$\tilde{X} = \arg\min_X \|X\|_{r^*}, \quad \text{s.t.} \quad P_{\Omega}(X) = P_{\Omega}(X^*),$$

is unique and exact, i.e., $\tilde{X} = X^*$, provided that $m \geq c \kappa^2 \mu r \log_2(n(1)) \log(n(1))$ and $X^*$ obeys $\mu$-incoherence (3).

5.1 Proof of Theorem 5.3: Strong Duality for Matrix Completion

We have shown in Theorem 5.1 that the optimization problem $(\tilde{A}, \tilde{B}) = \arg\min_{A, B} \frac{1}{2} \|AB\|_2^2$, s.t. $P_{\Omega}(AB) = P_{\Omega}(X^*)$, exactly recovers $X^*$, i.e., $\tilde{A}B = X^*$, with the optimal sample complexity. So if strong duality holds, this non-convex optimization problem can be equivalently converted to the convex program (9). Then Theorem 5.3 is straightforward from strong duality.

It now suffices to apply our unified framework in Section 4 to prove the strong duality. We show that the dual condition in Theorem 4.3 holds with high probability. Let $(\hat{A}, \hat{B})$ be a global solution to problem (9). For $H(\tilde{X}) = I_{\{M \in \mathbb{R}^{n_1 \times n_2}; P_{\Omega}M=P_{\Omega}X^*\}}(\tilde{X})$, we have

$$\Psi = \partial H(\tilde{A}\tilde{B}) = \{G \in \mathbb{R}^{n_1 \times n_2} : \langle G, \tilde{A}\tilde{B} \rangle \geq \langle G, Y \rangle, \text{ for any } Y \in \mathbb{R}^{n_1 \times n_2} \text{ s.t. } P_{\Omega}Y = P_{\Omega}X^*\}$$

$$= \{G \in \mathbb{R}^{n_1 \times n_2} : \langle G, X^* \rangle \geq \langle G, Y \rangle, \text{ for any } Y \in \mathbb{R}^{n_1 \times n_2} \text{ s.t. } P_{\Omega}Y = P_{\Omega}X^*\} = \Omega,$$

where the third equality holds since $\tilde{A}\tilde{B} = X^*$. Then we only need to show

$$\tilde{A} \in \Omega,$$

$$P_{\Omega}(-\tilde{A}) = \tilde{A}\tilde{B},$$

$$\|P_{\Omega}(\tilde{A})\| < \frac{2}{3} \sigma_r(\tilde{A}\tilde{B}).$$

We have the following lemma.
Lemma 5.4. If we can construct an $\Lambda$ such that

(a) $\Lambda \in \Omega$,

(b) $\|P_T(-\Lambda) - \tilde{A}\tilde{B}\|_F \leq \sqrt{\frac{r}{3n^2(1)}} \sigma_r(\tilde{A}\tilde{B})$,

(c) $\|P_T\perp\Lambda\| < \frac{1}{3} \sigma_r(\tilde{A}\tilde{B})$,

then we can construct an $\tilde{\Lambda}$ such that Eqn. (10) holds with probability at least $1 - n^{-10}$.

Proof. To prove the lemma, we first claim the following theorem.

Theorem 5.5 ([CR09], Theorem 4.1). Assume that $\Omega$ is sampled according to the Bernoulli model with success probability $p = \Theta\left(\frac{m}{n_1 n_2}\right)$, and incoherence condition (3) holds. Then there is an absolute constant $C_R$ such that for $\beta > 1$, we have

$$\|p^{-1}P_T P_\Omega P_T - P_T\| \leq C_R \sqrt{\frac{\beta \mu n(1)]^{r \log n(1)}}{m}} \triangleq \epsilon,$$

with probability at least $1 - 3n^{-\beta}$ provided that $C_R \sqrt{\frac{\beta \mu n(1)]^{r \log n(1)}}{m}} < 1$.

Suppose that Condition (11) holds. Let $Y = \tilde{\Lambda} - \Lambda \in \Omega$ be the perturbation matrix between $\Lambda$ and $\tilde{\Lambda}$ such that $P_T(-\Lambda) = \tilde{A}\tilde{B}$. Such a $Y$ exists by setting $Y = P_\Omega P_T (P_T P_\Omega P_T)^{-1} (P_T(-\Lambda) - \tilde{A}\tilde{B})$. So $\|P_T Y\|_F \leq \sqrt{\frac{1}{3n^2(1)}} \sigma_r(\tilde{A}\tilde{B})$. We now prove Condition (3) in Eqn. (10). Observe that

$$\|P_T \perp \tilde{\Lambda}\| \leq \|P_T \perp \Lambda\| + \|P_T \perp Y\|
\leq \frac{1}{3} \sigma_r(\tilde{A}\tilde{B}) + \|P_T \perp Y\|.
$$

(12)

So we only need to show $\|P_T \perp Y\| \leq \frac{1}{3} \sigma_r(\tilde{A}\tilde{B})$.

Before proceeding, we begin by introducing a normalized version $Q_\Omega : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}^{n_1 \times n_2}$ of $P_\Omega$:

$$Q_\Omega = p^{-1}P_\Omega - I.$$

With this, we have

$$P_T P_\Omega P_T = pP_T(I + Q_\Omega)P_T.$$

Note that for any operator $P : \mathcal{T} \rightarrow \mathcal{T}$, we have

$$P^{-1} = \sum_{k \geq 0} (P - P)^k \text{ whenever } \|P - P\| < 1.$$

So according to Theorem 5.5, the operator $p(P_T P_\Omega P_T)^{-1}$ can be represented as a convergent Neumann series

$$p(P_T P_\Omega P_T)^{-1} = \sum_{k \geq 0} (-1)^k (P_T Q_\Omega P_T)^k,$$

because $\|P_T Q_\Omega P_T\| \leq \epsilon < \frac{1}{2}$ once $m \geq C\mu n(1)]^{r \log n(1)}$ for a sufficiently large absolute constant $C$. We also note that

$$p(P_T \perp Q_\Omega P_T) = P_T \perp P_\Omega P_T,$$
because \(P_T \perp P_T = 0\). Thus
\[
\|P_T \perp Y\| = \|P_T \perp \Omega P_T (P_T P_O P_T)^{-1}(P_T (-\Lambda) - \tilde{A} \tilde{B})\|
\]
\[
= \|P_T \perp Q_T \perp p \perp (P_T P_O P_T)^{-1}((P_T (-\Lambda) - \tilde{A} \tilde{B}))\|
\]
\[
= \|\sum_{k \geq 0} (-1)^k P_T \perp \Omega (P_T Q_T \Omega P_T)^k((P_T (-\Lambda)) - \tilde{A} \tilde{B}))\|
\]
\[
\leq \sum_{k \geq 0} \|(-1)^k P_T \perp \Omega (P_T Q_T \Omega P_T)^k((P_T (-\Lambda)) - \tilde{A} \tilde{B})\|_F
\]
\[
\leq \|Q_T\| \sum_{k \geq 0} \|P_T Q_T P_T^k \| \|P_T (-\Lambda) - \tilde{A} \tilde{B})\|_F
\]
\[
\leq \frac{4}{p} \|P_T (-\Lambda) - \tilde{A} \tilde{B})\|_F
\]
\[
\leq \Theta \left( \frac{n_1 n_2}{m} \right) \sqrt{\frac{r}{3 n_2^2} \sigma_r (\tilde{A} \tilde{B})}
\]
\[
\leq \frac{1}{3} \sigma_r (\tilde{A} \tilde{B})
\]
with high probability. The proof is completed. \(\Box\)

It thus suffices to construct a dual certificate \(\Lambda\) such that all conditions in (11) hold. To this end, partition \(\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_b\) into \(b\) partitions of size \(q\). By assumption, we may choose
\[
q \geq \frac{128}{3} C \beta \kappa^2 \mu r n(1) \log n(1) \quad \text{and} \quad b \geq \frac{1}{2} \log 2 \kappa \left( 24^2 n^2 \kappa^2 \right)
\]
for a sufficiently large constant \(C\). Let \(\Omega_j \sim \text{Ber}(q)\) denote the set of indices corresponding to the \(j\)-th partitions. Define \(W_0 = \tilde{A} \tilde{B}\) and set \(\Lambda_k = \frac{n_1 n_2}{q} \sum_{j=1}^k P_{\Omega_j}(W_{j-1})\), \(W_k = \tilde{A} \tilde{B} - P_T(\Lambda_k)\) for \(k = 1, 2, ..., b\). Then by Theorem 5.5,
\[
\|W_k\|_F = \left\| W_{k-1} - \frac{n_1 n_2}{q} P_T P_{\Omega_k}(W_{k-1}) \right\|_F = \left\| \left( P_T - \frac{n_1 n_2}{q} P_T P_{\Omega_k} P_T \right) (W_{k-1}) \right\|_F \leq \frac{1}{2 \kappa} \|W_{k-1}\|_F.
\]
So it follows that \(\|W_b\|_F \leq (2 \kappa)^{-b} \|W_0\|_F \leq (2 \kappa)^{-b} \sqrt{r} \sigma_1(\tilde{A} \tilde{B}) \leq \sqrt{\frac{r}{24^2 n^2 (1)}} \sigma_r (\tilde{A} \tilde{B}).
\]

The following lemma together implies the strong duality of (9) straightforwardly.

Lemma 5.6. Under the assumptions of Theorem 5.3, the dual certification \(W_b\) obeys the dual condition (11) with probability at least \(1 - n^{-10}_1\).

6 Robust Principal Component Analysis

In this section, we develop our theory for robust PCA based on our framework. In the problem of robust PCA, we are given an observed matrix of the form \(D = X^* + S^*\), where \(X^*\) is the ground-truth matrix and \(S^*\) is the corruption matrix which is sparse. The goal is to recover the hidden matrices \(X^*\) and \(S^*\) from the observation \(D\). We set \(H(X) = \lambda \|D - X\|_1\).

To make the information spreads evenly throughout the matrix, the matrix cannot have one entry whose absolute value is significantly larger than other entries. For the robust PCA problem, Candès et al. [CLMW11] introduced an extra incoherence condition
\[
\|UV_T^T\|_\infty \leq \sqrt{\frac{\mu r}{n_1 n_2}}.
\]
Theorem 6.1 (Robust PCA). Suppose $X^*$ is an $n_1 \times n_2$ matrix of rank $r$, and obeys incoherence (3) and (14). Assume that the support set $\Omega$ of $S^*$ is uniformly distributed among all sets of cardinality $m$. Then with probability at least $1 - cn_{(1)}^{-10}$, the output of the optimization problem

$$\tilde{X}, \tilde{S} = \arg\min_{X, S} \|X\|_{\text{r*}} + \lambda\|S\|_1, \text{ s.t. } D = X + S,$$

with $\lambda = \frac{\sigma_r(X^*)}{\sqrt{n_{(1)}}}$ is exact, namely, $\tilde{X} = X^*$ and $\tilde{S} = S^*$, provided that $\text{rank}(X^*) \leq \rho_r \frac{n_{(2)}}{\log^* n_{(1)}}$ and $m \leq \rho_s n_1 n_2$, where $c$, $\rho_r$, and $\rho_s$ are all positive absolute constants, and function $\| \cdot \|_{\text{r*}}$ is given by (15).

The bounds on the rank of $X^*$ and the sparsity of $S^*$ in Theorem 6.1 match the best known results for robust PCA in prior work when we assume the support set of $S^*$ is sampled uniformly [CLMW11].

7 Computational Aspects

Computational Efficiency. We discuss our computational efficiency given that we have strong duality. We note that the dual and bi-dual of primal problem (P) are given by (see Appendix H)

$$(\text{Dual, D1}) \max_{\Lambda \in \mathbb{R}^{n_1 \times n_2}} -H^*(\Lambda) - \frac{1}{2}\|\Lambda\|^2_r, \text{ where } \|\Lambda\|^2_r = \sum_{i=1}^r \sigma^2_i(\Lambda),$$

$$(\text{Bi-Dual, D2}) \min_{M \in \mathbb{R}^{n_1 \times n_2}} H(M) + \|M\|_{\text{r*}}, \text{ where } \|M\|_{\text{r*}} = \max_{X} \langle M, X \rangle - \frac{1}{2}\|X\|^2_r.$$ 

Problems (D1) and (D2) can be solved efficiently due to their convexity. In particular, Grussler et al. [GRG16] provided a computationally efficient algorithm to compute the proximal operators of functions $\frac{1}{2}\| \cdot \|_r^2$ and $\| \cdot \|_{\text{r*}}$. Hence, the Douglas-Rachford algorithm can find global minimum up to an $\epsilon$ error in function value in time $\text{poly}(1/\epsilon)$ [HY12].

Computational Lower Bounds. Unfortunately, strong duality does not always hold for general non-convex problems (P). Here we present a very strong lower bound based on the random 4-SAT hypothesis. This is by now a fairly standard conjecture in complexity theory [Fei02] and gives us constant factor inapproximability of problem (P) for deterministic algorithms, even those running in exponential time.

If we additionally assume that $\text{BPP} = \text{P}$, where $\text{BPP}$ is the class of problems which can be solved in probabilistic polynomial time, and $\text{P}$ is the class of problems which can be solved in deterministic polynomial time, then the same conclusion holds for randomized algorithms. This is also a standard conjecture in complexity theory, as it is implied by the existence of certain strong pseudorandom generators or if any problem in deterministic exponential time has exponential size circuits [IW97]. Therefore, any subexponential time algorithm achieving a sufficiently small constant factor approximation to problem (P) in general would imply a major breakthrough in complexity theory.

The lower bound is proved by a reduction from the Maximum Edge Biclique problem [AMS11], similar to the reduction done in [RSW16]. The details are presented in Appendix F.
Theorem 7.1 (Computational Lower Bound). Assume the hardness of Random 4-SAT (see Conjecture 1). Then there exists an absolute constant $\epsilon_0 > 0$ for which any deterministic algorithm achieving $(1 + \epsilon)OPT$ in the objective function value for problem $(P)$ with $\epsilon \leq \epsilon_0$, requires $2^{\Omega(n_1 + n_2)}$ time, where $OPT$ is the optimum. If in addition, $BPP = P$, then the same conclusion holds for randomized algorithms succeeding with probability at least $2/3$.

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A Proof of Lemma 4.1

**Lemma 4.1 (Restated).** For any given matrix \( \hat{Y} \), any local minimum of \( f(A, B) = \frac{1}{2} \| \hat{Y} - AB \|_F^2 \) is globally optimal. The objective function \( f(A, B) \) around any saddle point has a negative second-order directional curvature. Moreover, \( f(A, B) \) has no local maximum.

**Proof.** \((A,B)\) is a critical point of \( f(A, B) \) if and only if \( \nabla_A f(A, B) = 0 \) and \( \nabla_B f(A, B) = 0 \), or equivalently,

\[
AB^T = B^TY \quad \text{and} \quad A^TAB = A^T\hat{Y}.
\]

(16)

Note that for any fixed matrix \( A \) (resp. \( B \)), the function \( f(A, B) \) is convex in the coefficients of \( B \) (resp. \( A \)).

To prove the desired lemma, we have the following claim.

**Claim 1.** If two matrices \( A \) and \( B \) define a critical point of \( f(A, B) \), then the global mapping \( M = AB \) is of the form

\[
M = \mathcal{P}_A \hat{Y},
\]

with \( A \) satisfying

\[
AA^\dagger \hat{Y}Y^T = AA^\dagger \hat{Y}Y^TA^\dagger = \hat{Y}Y^TA^\dagger. \tag{17}
\]

**Proof.** If \( A \) and \( B \) define a critical point of \( f(A, B) \), then (16) holds and the general solution to (16) satisfies

\[
B = (A^TA)^\dagger A^TY + (I - A^\dagger A)L, \tag{18}
\]

for any matrix \( L \). So \( M = AB = A(A^TA)^\dagger A^TY = AA^\dagger \hat{Y} = \mathcal{P}_A \hat{Y} \) by the property of the Moore-Penrose pseudo-inverse: \( A^\dagger = (A^TA)^\dagger A^T \).

By (16), we also have

\[
AB^T A^\dagger = \hat{Y}B^T A^\dagger \quad \text{or equivalently} \quad MM^T = \hat{Y}M^T.
\]

Plugging in the relation \( M = AA^\dagger \hat{Y} \), (A) can be rewritten as

\[
AA^\dagger \hat{Y}Y^TA^\dagger = \hat{Y}Y^TA^\dagger.
\]

Note that the matrix \( AA^\dagger \hat{Y}Y^TA^\dagger \) is symmetric. Thus

\[
AA^\dagger \hat{Y}Y^T = AA^\dagger \hat{Y}Y^T A^\dagger,
\]

as desired. \( \square \)

To prove Lemma 4.1, we also need the following claim.

**Claim 2.** Denote by \( \mathcal{I} = \{i_1, i_2, ..., i_r\} \) any ordered \( r \)-index set (ordered by \( \lambda_{i_1}, j \in [r] \) from the largest to the smallest) and \( \lambda_i, i \in [n_1], \) the eigenvalues of \( \hat{Y}Y^T \in \mathbb{R}^{n_1 \times n_1} \) with \( p \) distinct values. Let \( U \in \mathbb{R}^{n_1 \times n_1} \) denote the matrix formed by the orthonormal eigenvectors \( U = [u_{i_1}, u_{i_2}, ..., u_{i_r}] \) of \( \hat{Y}Y^T \) associated with the ordered \( r \)-index set \( \mathcal{I} \), whose multiplicities are \( m_1, m_2, ..., m_p \) \( (m_1 + m_2 + ... + m_p = n_1) \). Then two matrices \( A \) and \( B \) define a critical point of \( f(A, B) \) if and only if there exists an ordered \( r \)-index set \( \mathcal{I} \), an invertible matrix \( C \), and an \( r \times n \) matrix \( L \) such that

\[
A = (UD)_{\mathcal{I}}C \quad \text{and} \quad B = A^\dagger \hat{Y} + (I - A^\dagger A)L, \tag{19}
\]

where \( D \) is a \( p \)-block-diagonal matrix with each block equal to the orthogonal projector of dimension \( m_i \). For such a critical point, we have

\[
AB = \mathcal{P}_A \hat{Y},
\]
\[ f(A, B) = \frac{1}{2} \left( \text{tr}(\hat{Y}Y^T) - \sum_{i \in I} \lambda_i \right) = \frac{1}{2} \sum_{i \notin I} \lambda_i. \]  

(20)

**Proof.** Note that \( \hat{Y}Y^T \) is a real symmetric covariance matrix. So it can always be represented as \( U\Lambda U^T \), where \( U \in \mathbb{R}^{n_1 \times n_1} \) is an orthonormal matrix consisting of eigenvectors of \( \hat{Y}Y^T \) and \( \Lambda \in \mathbb{R}^{n_1 \times n_1} \) is a diagonal matrix with non-increasing eigenvalues of \( \hat{Y}Y^T \).

If \( A \) and \( B \) satisfy (19) for some \( C, L, \) and \( \mathcal{I} \), then
\[
AB^T = \hat{Y}B^T \quad \text{and} \quad A^T AB = A^T \hat{Y},
\]
which is (16). So \( A \) and \( B \) define a critical point of \( f(A, B) \).

For the converse, notice that
\[
\mathcal{P}_{U^TA} = U^T A (U^T A)^\dagger = U^T A A^\dagger U = U^T \mathcal{P}_A U,
\]
or equivalently, \( \mathcal{P}_A = U \mathcal{P}_{U^TA} U^T \). Thus (17) yields
\[
U \mathcal{P}_{U^TA} U^T U A U^T = U A U^T U \mathcal{P}_{U^TA} U^T,
\]
or equivalently, \( \mathcal{P}_{U^TA} A = \Lambda \mathcal{P}_{U^TA} \). Notice that \( \Lambda \in \mathbb{R}^{n_1 \times n_1} \) is a diagonal matrix with \( p \) distinct eigenvalues of \( \hat{Y}Y^T \). So \( \mathcal{P}_{U^TA} \) is a block-diagonal matrix with \( p \) blocks, each of which is an orthogonal projector of dimension \( m_i \), corresponding to the eigenvalues \( \lambda_i \), \( i \in [p] \). Therefore, there exists an index set \( \mathcal{I} \) such that \( \mathcal{P}_{U^TA} = D_{\mathcal{I}} D_{\mathcal{I}}^T \), where \( D \) is a block-diagonal matrix. It follows that
\[
\mathcal{P}_A = U \mathcal{P}_{U^TA} U^T = UD_{\mathcal{I}} D_{\mathcal{I}}^T U^T = (UD_{\mathcal{I}} (UD_{\mathcal{I}})^T.
\]
Since the column space of \( A \) coincides with the column space of \( (UD)_{\mathcal{I}} \), \( A \) is of the form \( A = (UD)_{\mathcal{I}} C \), and \( B \) is given by (18). Thus \( AB = A (A^T A)^\dagger A^T \hat{Y} + A (I - A^\dagger A) L = \mathcal{P}_A \hat{Y} \) and
\[
\frac{1}{2} \| \hat{Y} - AB \|^2_F = \frac{1}{2} \| \hat{Y} - \mathcal{P}_A \hat{Y} \|^2_F = \frac{1}{2} \| \mathcal{P}_A \hat{Y} \|^2_F = \frac{1}{2} \sum_{i \notin \mathcal{I}} \lambda_i.
\]

\( \Box \)

So the local minimizer of \( f(A, B) \) is given by (19) with \( \mathcal{I} = \Phi \), which is globally optimal according to (20), where \( \Phi \) is the index set corresponding to the \( r \) largest eigenvalues of \( \hat{Y}Y^T \). We then show that when \( \mathcal{I} \) consists of other combinations of indices of eigenvalues, i.e., \( \mathcal{I} \neq \Phi \), the corresponding pair \( (A, B) \) given by (19) is a strict saddle point.

**Claim 3.** If \( \mathcal{I} \neq \Phi \), then the pair \( (A, B) \) given by (19) is a strict saddle point.

**Proof.** Let \( i \in \Phi \) but \( i \notin \mathcal{I} \), and denote by \( UD = R \). It is enough to slightly perturb the column space of \( A \) towards the direction of an eigenvector of \( \lambda_i \). More precisely, fix two indices \( i \) and \( j \) such that \( i \in \Phi \),
where the inequality holds because

Thus

Therefore,

as desired.

On the other hand, by weak duality,

argmax

in

0

∈

\tilde{A}\|^2_F + \lambda \tilde{A} \overline{w.r.t.}\text{ variable } \tilde{A}

\implies \tilde{A}B \in \partial \lambda^* H^*(\tilde{A})|_{\Lambda = \tilde{A}} \text{ by the convexity of function } H, \text{ meaning that } \tilde{A} \in \partial \lambda^* \min L(\tilde{A}, \tilde{B}, \tilde{A}) \text{ due to the concavity of } L(\tilde{A}, \tilde{B}, \Lambda) \text{ w.r.t. variable } \Lambda.

Thus \((\tilde{A}, \tilde{B}, \tilde{A})\) is a primal-dual saddle point of \(L(\tilde{A}, \tilde{B}, \Lambda)\).

We now prove the strong duality. By the fact that \(F(\tilde{A}, \tilde{B}) = \max_{\Lambda} L(\tilde{A}, \tilde{B}, \Lambda)\) and that \(\tilde{A} = \arg\max_{\Lambda} L(\tilde{A}, \tilde{B}, \Lambda)\), we have

\[
F(\tilde{A}, \tilde{B}) = L(\tilde{A}, \tilde{B}, \tilde{A}) \leq L(\tilde{A}, \tilde{B}, \tilde{A}), \quad \forall \tilde{A}, \tilde{B},
\]

where the inequality holds because \((\tilde{A}, \tilde{B}, \tilde{A})\) is a primal-dual saddle point of \(L\). So on the one hand, we have

\[
\min_{\tilde{A}, \tilde{B}} \max_{\Lambda} L(\tilde{A}, \tilde{B}, \Lambda) = F(\tilde{A}, \tilde{B}) \leq \min_{\tilde{A}, \tilde{B}} L(\tilde{A}, \tilde{B}, \tilde{A}) \leq \max_{\Lambda} \min_{\tilde{A}, \tilde{B}} L(\tilde{A}, \tilde{B}, \Lambda).
\]

On the other hand, by weak duality,

\[
\min_{\tilde{A}, \tilde{B}} \max_{\Lambda} L(\tilde{A}, \tilde{B}, \Lambda) \geq \max_{\Lambda} \min_{\tilde{A}, \tilde{B}} L(\tilde{A}, \tilde{B}, \Lambda).
\]

Therefore, \(\min_{\tilde{A}, \tilde{B}} \max_{\Lambda} L(\tilde{A}, \tilde{B}, \Lambda) = \max_{\Lambda} \min_{\tilde{A}, \tilde{B}} L(\tilde{A}, \tilde{B}, \Lambda)\), i.e., strong duality holds. Hence,

\[
\tilde{A}\tilde{B} = \arg\min_{\tilde{A}, \tilde{B}} L(\tilde{A}, \tilde{B}, \tilde{A})
\]

\[
= \arg\min_{\tilde{A}, \tilde{B}} \frac{1}{2} \| - \tilde{A} - \tilde{A}\tilde{B} \|_F^2 - \frac{1}{2} \| \tilde{A} \|_F^2 - H^*(\tilde{A})
\]

\[
= \arg\min_{\tilde{A}, \tilde{B}} \frac{1}{2} \| - \tilde{A} - \tilde{A}\tilde{B} \|_F^2
\]

\[
= \text{svd}_r(-\tilde{A}),
\]

as desired.
C Proof of Theorem 5.1

Theorem 5.1 (Information-Theoretic Upper Bound. Restated). Let \( \Omega \sim \text{Uniform}(m) \) be the support set, which is uniformly distributed among all sets of cardinality \( m \). Suppose that \( m \geq c \mu n_1^r \log n_1 \) for an absolute constant \( c \). Then \( X^* \) is the unique \( n_1 \times n_2 \) matrix of rank at most \( r \) with \( \mu \)-incoherence (3) such that \( P_\Omega(X) = P_\Omega(X^*) \), with probability at least \( 1 - n_1^{-10} \).

Proof. We note that the sampling model Uniform\((m)\) is equivalent to the sampling model Ber\((p\) with \( p = \Theta\left(\frac{m}{n_1 n_2}\right)\), which we will frequently use in the sequel (see Appendix I). We consider the feasibility of the matrix completion problem:

\[
\text{Find a matrix } X \in \mathbb{R}^{n_1 \times n_2} \text{ such that } P_\Omega(X) = P_\Omega(X^*), \quad \text{rank}(X) \leq r.
\] (21)

Our proof first identifies a feasibility condition for problem (21), and then shows that \( X^* \) obeys this feasibility condition when the sample size is large enough. We denote by

\[
S = \{X \in \mathbb{R}^{n_1 \times n_2} : \text{rank}(X) \leq r\},
\]

and define

\[
D_S(X^*) = \{t(X - X^*) \in \mathbb{R}^{n_1 \times n_2} : \text{rank}(X) \leq r, \ t \geq 0\}.
\]

We have the following proposition for the feasibility of problem (21).

Proposition C.1 (Feasibility Condition). \( X^* \) is the unique feasible solution to problem (21) if \( D_S(X^*) \cap \Omega^\perp = \{0\} \).

Proof. Notice that problem (21) is equivalent to another feasibility problem

\[
\text{Find a matrix } D \in \mathbb{R}^{n_1 \times n_2} \text{ such that } \text{rank}(X^* + D) \leq r, \ D \in \Omega^\perp.
\]

Suppose that \( D_S(X^*) \cap \Omega^\perp = \{0\} \). Since \( \text{rank}(X^* + D) \leq r \) implies \( D \in D_S(X^*) \), and note that \( D \in \Omega^\perp \), we have \( D = 0 \), which means \( X^* \) is the unique feasible solution to problem (21).

The remainder of the proof is to show \( D_S(X^*) \cap \Omega^\perp = \{0\} \). To proceed, we note that the “escaping through a mesh” techniques for matrix sensing do not work for matrix completion since \( \Omega \) is not drawn from the Grassmanian according to the Haar measure. To address this issue, we instead need the following lemmas. The first lemma claims that the tangent cone of the set \( S \) evaluated at \( X^* \) is slightly larger than the cone \( \text{cone}(S - \{X^*\}) \).

Lemma C.2 ([Jah07], Theorem 4.8). Let \( S \) be a non-empty subset of a real normed space. If \( S \) is star-shaped w.r.t. some \( X^* \in S \), i.e., \( t(S - \{X^*\}) \subseteq S - \{X^*\} \) for all \( t \in [0, 1] \), then it follows

\[
\text{cone}(S - \{X^*\}) \subseteq T(S, X^*),
\]

where \( T(S, X^*) \) is the tangent cone of the set \( S \) at point \( X^* \) defined by

\[
T(S, X^*) = \{\Xi \in \mathbb{R}^{n_1 \times n_2} : \exists X_n \subseteq S, \ (a_n) \subseteq \mathbb{R}^+ \text{ s.t. } X_n \rightarrow X^*, \ a_n(X_n - X^*) \rightarrow \Xi\}.
\]

The second lemma states that the tangent cone of the set \( S \) evaluated at \( X^* \) can be represented in a closed form.

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Lemma C.3 ([SU15], Theorem 3.2). Let $X^* = U\Sigma V^T$ be the skinny SVD of matrix $X^*$. The tangent cone $T(S, X^*)$ of the set $S = \{X \in \mathbb{R}^{n_1 \times n_2} : \text{rank}(X) \leq r\}$ at $X^*$ is a linear subspace given by

$$T(S, X^*) = \{UL^T + MV^T : L \in \mathbb{R}^{n_2 \times r}, M \in \mathbb{R}^{n_1 \times r}\} \triangleq \mathcal{T}.$$ 

Now we are ready to prove Theorem 5.1. By Lemma C.2 and C.3, we have

$$\mathcal{D}_S(X^*) = \text{cone}(S - \{X^*\}) \subseteq T(S, X^*) = \mathcal{T},$$

where the first equality holds by the definition of $\mathcal{D}_S(X^*)$. So if $\mathcal{T} \cap \Omega^\perp = \{0\}$, then $\mathcal{D}_S(X^*) \cap \Omega^\perp = \{0\}$, meaning that $X^*$ is the unique feasible solution to the problem (21). Thus the rest of proof is to find a sufficient condition for $\mathcal{T} \cap \Omega^\perp = \{0\}$. We have the following lemma.

Lemma C.4. Assume that $\Omega \sim \text{Ber}(p)$ and the incoherence condition (3) holds. Then with probability at least $1 - n_{(1)}^{-10}$, we have $\|P_{\Omega^\perp} P_T\| \leq \sqrt{1 - p} + \epsilon p$, provided that $p \geq C_0 e^{-2(\mu r \log n_{(1)})/n_{(2)}}$, where $C_0$ is an absolute constant.

Proof. If $\Omega \sim \text{Ber}(p)$, we have, by Theorem 5.5, that with high probability

$$\|P_T - p^{-1} P_T P_{\Omega^\perp} P_T\| \leq \epsilon,$$

provided that $p \geq C_0 e^{-2(\mu r \log n_{(1)})/n_{(2)}}$. Note, however, that since $I = P_{\Omega^\perp} + P_{\Omega^\perp}$,

$$P_T - p^{-1} P_T P_{\Omega^\perp} P_T = p^{-1} (P_T P_{\Omega^\perp} P_T - (1 - p) P_T)$$

and, therefore, by the triangle inequality

$$\|P_T P_{\Omega^\perp} P_T\| \leq \epsilon p + (1 - p).$$

Since $\|P_{\Omega^\perp} P_T\|^2 \leq \|P_T P_{\Omega^\perp} P_T\|$, the proof is completed. \hfill \Box

We note that $\|P_{\Omega^\perp} P_T\| < 1$ implies $\Omega^\perp \cap \mathcal{T} = \{0\}$. The proof is completed. \hfill \Box

D Proof of Lemma 5.6

Lemma 5.6 (Restated). Under the assumptions of Theorem 5.3, the dual certification $W_b$ obeys the dual condition (11) with probability at least $1 - n_{(1)}^{-10}$.

Proof. It is well known that for matrix completion, the Uniform model $\Omega \sim \text{Uniform}(m)$ is equivalent to the Bernoulli model $\Omega \sim \text{Ber}(p)$, where each element in $[n_1] \times [n_2]$ is included with probability $p = \Theta(m/(n_1 n_2))$ independently; see Section I for a brief justification. By the equivalence, we can suppose $\Omega \sim \text{Ber}(p)$.

To prove Lemma 5.6, as a preliminary, we need the following lemmas.

Lemma D.1 ([Che15], Lemma 2). Suppose $Z$ is a fixed matrix. Suppose $\Omega \sim \text{Ber}(p)$. Then with high probability,

$$\|(I - p^{-1} P_{\Omega}) Z\| \leq C_0' \left( \frac{\log n_{(1)}}{p} \|Z\|_\infty + \sqrt{\frac{\log n_{(1)}}{p} \|Z\|_{\infty,2}} \right),$$

where $C_0' > 0$ is an absolute constant and

$$\|Z\|_{\infty,2} = \max \left\{ \max_i \left( \sum_b Z_{ib}^2 \right), \max_j \left( \sum_a Z_{aj}^2 \right) \right\}. $$

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Lemma D.2 ([CLMW11], Lemma 3.1). Suppose $\Omega \sim \text{Ber}(p)$ and $Z$ is a fixed matrix. Then with high probability,
\[
\|Z - p^{-1} \mathcal{P}_\Omega Z\|_\infty \leq \epsilon \|Z\|_\infty,
\]
provided that $p \geq C_0 \epsilon^{-2} (\mu \log n(1))/n(2)$ for some absolute constant $C_0 > 0$.

Lemma D.3 ([Che15], Lemma 3). Suppose that $Z$ is a fixed matrix and $\Omega \sim \text{Ber}(p)$. If $p \geq c_0 \mu \log n(1)/n(2)$ for some $c_0$ sufficiently large, then with high probability,
\[
\|(p^{-1} \mathcal{P}_\Omega - \mathcal{P}_T)Z\|_{\infty,2} \leq \frac{1}{2} \sqrt{\frac{n(1)}{\mu r}} \|Z\|_\infty + \frac{1}{2} \|Z\|_{\infty,2}.
\]

Observe that by Lemma D.2,
\[
\|W_j\|_\infty \leq \left(\frac{1}{2}\right)^j \|\tilde{A}\|_\infty,
\]
and by Lemma D.3,
\[
\|W_j\|_{\infty,2} \leq \frac{1}{2} \sqrt{\frac{n(1)}{\mu r}} \|W_{j-1}\|_\infty + \frac{1}{2} \|W_{j-1}\|_{\infty,2}.
\]

So
\[
\|W_j\|_{\infty,2} \leq \left(\frac{1}{2}\right)^j \sqrt{\frac{n(1)}{\mu r}} \|\tilde{A}\|_\infty + \frac{1}{2} \|W_{j-1}\|_{\infty,2} \leq j \left(\frac{1}{2}\right)^j \sqrt{\frac{n(1)}{\mu r}} \|\tilde{A}\|_\infty + \left(\frac{1}{2}\right)^j \|\tilde{A}\|_{\infty,2}.
\]

Therefore,
\[
\|\mathcal{P}_{\mathcal{T}_\perp} \Lambda_b\|
\leq \sum_{j=1}^b \|n(1) n(2)/q \mathcal{P}_{\mathcal{T}_\perp} \mathcal{P}_\Omega, W_{j-1}\|
= \sum_{j=1}^b \|\mathcal{P}_{\mathcal{T}_\perp}(n(1) n(2)/q \mathcal{P}_\Omega, W_{j-1} - W_{j-1})\|
\leq \sum_{j=1}^b \left\|n(1) n(2)/q \mathcal{P}_\Omega, (W_{j-1})\right\|
\leq C_0' \frac{\log n(1)}{p} \sum_{j=1}^b \|W_{j-1}\|_\infty + C_0' \frac{\log n(1)}{p} \sum_{j=1}^b \|W_{j-1}\|_{\infty,2}.
\]

Let $p$ denote $\Theta\left(\frac{q}{n(1)n(2)}\right)$. By Lemma D.1,
\[
\|\mathcal{P}_{\mathcal{T}_\perp} \Lambda_b\|
\leq C_0' \frac{\log n(1)}{p} \sum_{j=1}^b \|W_{j-1}\|_\infty + C_0' \frac{\log n(1)}{p} \sum_{j=1}^b \|W_{j-1}\|_{\infty,2}
\leq C_0' \frac{\log n(1)}{p} \sum_{j=1}^b \left(\frac{1}{2}\right)^j \|\tilde{A}\|_\infty + C_0' \frac{\log n(1)}{p} \sum_{j=1}^b \left[j \left(\frac{1}{2}\right)^j \sqrt{\frac{n(1)}{\mu r}} \|\tilde{A}\|_\infty + \left(\frac{1}{2}\right)^j \|\tilde{A}\|_{\infty,2}\right]\]
\leq C_0' \frac{\log n(1)}{p} \|\tilde{A}\|_\infty + 2C_0' \sqrt{\frac{\log n(1)}{p}} \sqrt{\frac{n(1)}{\mu r}} \|\tilde{A}\|_{\infty} + C_0' \sqrt{\frac{\log n(1)}{p}} \|\tilde{A}\|_{\infty,2}.
\]
Setting $\widetilde{A}\widetilde{B} = X^*$, we note the facts that (we assume WLOG $n_2 \geq n_1$)

$$
\|X^*\|_{\infty,2} = \max_i \|e_i^T U \Sigma V^T\|_2 \leq \max_i \|e_i^T U\| \sigma_1(X^*) \leq \sqrt{\frac{\mu r}{n_1}} \sigma_1(X^*) \leq \sqrt{\frac{\mu r}{n_1}} \kappa \sigma_r(X^*),
$$

and that

$$
\|X^*\|_{\infty} = \max_{ij} \langle X^*, e_i e_j^T \rangle = \max_{ij} \langle U \Sigma V^T, e_i e_j^T \rangle = \max_{ij} \langle e_i^T U \Sigma, e_j^T V \rangle \leq \max_{ij} \|e_i^T U \Sigma V^T\|_2 \|e_j^T V\|_2 \leq \max_j \|X^*\|_{\infty,2} \|e_j^T V\|_2 \leq \sqrt{\frac{\mu r}{n_1}} \kappa \sigma_r(X^*).
$$

Substituting $p = \Theta \left( \frac{\kappa^2 \mu r n_1 \log(n_1)}{n_1 n_2} \right)$, we obtain $\|P_{T\perp} \bar{A}\| < \frac{1}{3} \sigma_r(X^*)$. The proof is completed. \qed

## E Proof of Theorem 6.1

**Theorem 6.1 (Robust PCA. Restated).** Suppose $X^*$ is $n_1 \times n_2$, obeys incoherence (3) and (14). Assume that the support set $\Omega$ of $S^*$ is uniformly distributed among all sets of cardinality $m$. Then with probability at least $1 - c n_1^{-10}$, the output of the optimization problem

$$
(\bar{X}, \bar{S}) = \arg\min_{X,S} \|X\|_{r*} + \lambda \|S\|_1, \quad \text{s.t.} \quad D = X + S,
$$

with $\lambda = \frac{\sigma_2(X^*)}{\sqrt{m(n_1)}}$ is exact, namely, $\bar{X} = X^*$ and $\bar{S} = S^*$, provided that $\operatorname{rank}(X^*) \leq \rho_r n_1^{(2)}$ and $m \leq \rho_s n_1 n_2$, where $c$, $\rho_r$, and $\rho_s$ are all positive absolute constants, and function $\| \cdot \|_{r*}$ is given by (15).

### E.1 Subgradient of the $r*$ Function

**Lemma E.1.** Let $U \Sigma V^T$ be the skinny SVD of matrix $X^*$ of rank $r$. The subdifferential of $\| \cdot \|_{r*}$ evaluated at $X^*$ is given by

$$
\partial \|X^*\|_{r*} = \{X^* + W : U^T W = 0, W V = 0, \|W\| \leq \sigma_r(X^*)\}.
$$

**Proof.** Note that for any fixed function $f(\cdot)$, the set of all optimal solutions of the problem

$$
f^*(X^*) = \max_Y \langle X^*, Y \rangle - f(Y)
$$

form the subdifferential of the conjugate function $f^*(\cdot)$ evaluated at $X^*$. Set $f(\cdot)$ to be $\frac{1}{2} \| \cdot \|^2$ and notice that the function $\frac{1}{2} \| \cdot \|^2$ is unitarily invariant. By Von Neumann’s trace inequality, the optimal solutions to problem (23) are given by $[U, U^{\perp}] \operatorname{Diag}([\sigma_1(Y), \ldots, \sigma_r(Y), \sigma_{r+1}(Y), \ldots, \sigma_{n_2}(Y))][V, V^{\perp}]^T$, where $\{\sigma_i(Y)\}_{i=r+1}^{n_2}$ can be any value no larger than $\sigma_r(Y)$ and $\{\sigma_i(Y)\}_{i=1}^r$ are given by the optimal solution to the problem

$$
\max_{\{\sigma_i(Y)\}_{i=1}^r} \sum_{i=1}^r \sigma_i(X^*) \sigma_i(Y) - \frac{1}{2} \sum_{i=1}^r \sigma_i^2(Y).
$$

The solution is unique such that $\sigma_i(Y) = \sigma_i(X^*)$, $i = 1, 2, \ldots, r$. The proof is complete. \qed
E.2 Dual Certificates

**Lemma E.2.** Assume that \( \|P_\Omega P_T\| \leq 1/2 \) and \( \lambda < \sigma_r(X^*) \). Then \((X^*, S^*)\) is the unique solution to problem (6.1) if there exists a pair \((W, F)\) for which

\[
X^* + W = \lambda (\text{sign}(S^*) + F + P_\Omega K),
\]

where \( W \in T^\perp, \|W\| \leq \frac{\sigma_r(X^*)}{2}, F \in \Omega^\perp, \|F\|_\infty \leq \frac{1}{2}, \) and \( \|P_\Omega K\|_F \leq \frac{1}{4} \).

**Proof.** Let \((X^* + H, S^* - H)\) be any optimal solution to problem (22). By the definition of the subgradient, the inequality follows

\[
\|X^* + H\|_{r*} + \lambda \|S^* - H\|_1 \geq \|X^*\|_{r*} + \lambda \|S^*\|_1 + (X^* + W^*, H) - \lambda \langle \text{sign}(S^*) + F^*, H \rangle
\]

\[
= \|X^*\|_{r*} + \lambda \|S^*\|_1 + (X^* - \lambda \text{sign}(S^*), H) + (W^*, H) - \lambda \langle F^*, H \rangle
\]

\[
= \|X^*\|_{r*} + \lambda \|S^*\|_1 + (\lambda F + \lambda P_\Omega K - W, H) + \lambda \langle X^* \rangle \|P_{T^\perp} H\|_\ast + \lambda \|P_{\Omega^\perp} H\|_1
\]

\[
= \|X^*\|_{r*} + \lambda \|S^*\|_1 + \frac{\sigma_r(X^*)}{2} \|P_{T^\perp} H\|_\ast + \frac{\lambda}{2} \|P_{\Omega^\perp} H\|_1 - \frac{\lambda}{4} \|P_{\Omega} H\|_F.
\]

We note that

\[
\|P_{\Omega} H\|_F \leq \|P_{\Omega} P_T H\|_F + \|P_{\Omega} P_{T^\perp} H\|_F
\]

\[
\leq \frac{1}{2} \|H\|_F + \|P_{T^\perp} H\|_F
\]

\[
\leq \frac{1}{2} \|P_{\Omega} H\|_F + \frac{1}{2} \|P_{\Omega^\perp} H\|_F + \|P_{T^\perp} H\|_F,
\]

which implies that \( \frac{1}{4} \|P_{\Omega} H\|_F \leq \frac{1}{4} \|P_{\Omega^\perp} H\|_F + \frac{1}{2} \|P_{T^\perp} H\|_F \leq \frac{1}{4} \|P_{\Omega^\perp} H\|_1 + \frac{1}{2} \|P_{T^\perp} H\|_\ast. \) Therefore,

\[
\|X^* + H\|_{r*} + \lambda \|S^* - H\|_1 \geq \|X^*\|_{r*} + \lambda \|S^*\|_1 + \frac{\sigma_r(X^*)}{2} \|P_{T^\perp} H\|_\ast + \frac{\lambda}{4} \|P_{\Omega^\perp} H\|_1
\]

\[
\geq \|X^* + H\|_{r*} + \lambda \|S^* - H\|_1 + \frac{\sigma_r(X^*)}{2} \|P_{T^\perp} H\|_\ast + \frac{\lambda}{4} \|P_{\Omega^\perp} H\|_1,
\]

where the second inequality holds because \((X^* + H, S^* - H)\) is optimal. Thus \( H \in T \cap \Omega. \) Note that \( \|P_{\Omega} P_T\| < 1 \) implies \( T \cap \Omega = \{0\}. \) This completes the proof. \( \square \)

According to Lemma E.2, to show the exact recoverability of problem (22), it is sufficient to find an appropriate \( W \) for which

\[
W \in T^\perp, \quad \|W\| \leq \frac{\sigma_r(X^*)}{2}, \quad \|P_{\Omega^\perp} (X^* + W - \lambda \text{sign}(S^*))\|_F \leq \frac{1}{4},
\]

\[
\|P_{\Omega^\perp} (X^* + W)\|_\infty \leq \frac{\lambda}{4}.
\]

E.3 Dual Certification by Least Squares and the Golfing Scheme

The remainder of the proof is to construct \( W \) such that the dual condition (24) holds true. Before introducing our construction, we assume \( \Omega \sim \text{Ber}(p) \), or equivalently \( \Omega^\perp \sim \text{Ber}(1 - p) \), where \( p \) is allowed be as large as an absolute constant. Note that \( \Omega^\perp \) has the same distribution as that of \( \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_{j_0} \), where the \( \Omega_j \)'s are drawn independently with replacement from \( \text{Ber}(q) \), \( j_0 = \lceil \log n(1) \rceil \), and \( q \) obeys \( p = (1 - q)^{j_0} \) \( (q = \Omega(1/ \log n(1)) \) implies \( p = O(1) \)). We construct \( W \) based on such a distribution.
Our construction separates \( W \) into two terms: \( W = W^L + W^S \). To construct \( W^L \), we apply the golfing scheme introduced by [Gro11, Rec11]. Specifically, \( W^L \) is constructed by an inductive procedure:

\[
Y_j = Y_{j-1} + q^{-1}P_\Omega P_T (X^* - Y_{j-1}), \; Y_0 = 0, \quad W^L = P_{T \perp} Y_{j_0}.
\]

To construct \( W^S \), we apply the method of least squares by [CLMW11], which is

\[
W^S = \lambda P_{T \perp} \sum_{k \geq 0} (P_\Omega P_T P_\Omega)^k \text{sign}(S^*).
\]

Note that \( \|P_\Omega P_T\| \leq 1/2 \). Thus \( \|P_\Omega P_T P_\Omega\| \leq 1/4 \) and the Neumann series in (26) is well-defined. Observe that \( P_\Omega W^S = \lambda (P_\Omega - P_\Omega P_T P_\Omega) (P_\Omega - P_\Omega P_T P_\Omega)^{-1} \text{sign}(S^*) = \lambda \text{sign}(S^*) \). So to prove the dual condition (24), it suffices to show that

\[
\begin{align*}
(a) \quad & \|W^L\| \leq \frac{\sigma_T(X^*)}{4}, \\
(b) \quad & \|P_\Omega (X^* + W^L)\|_F \leq \frac{\lambda}{4}, \\
(c) \quad & \|P_\Omega (X^* + W^L)\|_\infty \leq \frac{\lambda}{4}, \\
(d) \quad & \|W^S\| \leq \frac{\sigma_T(X^*)}{4}, \\
(e) \quad & \|P_\Omega \cdot W^S\|_\infty \leq \frac{\lambda}{4}.
\end{align*}
\]

E.4 Proof of Dual Conditions

Since we have constructed the dual certificate \( W \), the remainder is to show that \( W \) obeys dual conditions (27) and (28) with high probability. We have the following.

Lemma E.3. Assume \( \Omega_j \sim \text{Ber}(q) \), \( j = 1, 2, \ldots, j_0 \), and \( j_0 = 2[\log n(1)] \). Then under the other assumptions of Theorem 6.1, \( W^L \) given by (25) obeys dual condition (27).

Proof. Let \( Z_j = P_T (X^* - Y_j) \in T \). Then we have

\[
Z_j = P_T Z_{j-1} - q^{-1} P_T P_\Omega P_T Z_{j-1} = (P_T - q^{-1} P_T P_\Omega P_T) Z_{j-1},
\]

and \( Y_j = \sum_{k=1}^j q^{-1} P_\Omega_k Z_{k-1} \in \Omega^\perp \). We set \( q = \Omega(e^{-2\mu r \log n(1)/n(2)}) \).

Proof of (a). It holds that

\[
\begin{align*}
\|W^L\| &= \|P_{T \perp} Y_{j_0}\| \leq \sum_{k=1}^{j_0} \|q^{-1} P_{T \perp} P_\Omega Z_{k-1}\| \\
&= \sum_{k=1}^{j_0} \|P_{T \perp} (q^{-1} P_\Omega Z_{k-1} - Z_{k-1})\| \\
&\leq \sum_{k=1}^{j_0} \|q^{-1} P_\Omega Z_{k-1} - Z_{k-1}\| \\
&\leq C_0' \left( \frac{\log n(1)}{q} \sum_{k=1}^{j_0} \|Z_{k-1}\|_\infty + \sqrt{\frac{\log n(1)}{q} \sum_{k=1}^{j_0} \|Z_{k-1}\|_\infty^2} \right). \quad \text{(by Lemma D.1)}
\end{align*}
\]
We note that by Lemma D.2,
\[ \|Z_{k-1}\|_\infty \leq \left( \frac{1}{2} \right)^{k-1} \|Z_0\|_\infty, \]
and by Lemma D.3,
\[ \|Z_{k-1}\|_{\infty,2} \leq \frac{1}{2} \sqrt{\frac{n(1)}{\mu r}} \|Z_{k-2}\|_\infty + \frac{1}{2} \|Z_{k-2}\|_{\infty,2}. \]
Therefore,
\[ \|Z_{k-1}\|_{\infty,2} \leq \left( \frac{1}{2} \right)^{k-1} \sqrt{\frac{n(1)}{\mu r}} \|Z_0\|_\infty + \frac{1}{2} \|Z_{k-2}\|_{\infty,2} \]
and so we have
\[ \|W^L\| \leq C_0' \left[ \log_q n(1) \sum_{k=1}^{j_0} \left( \frac{1}{2} \right)^{k-1} \|Z_0\|_\infty + \sqrt{\log_q n(1)} \sum_{k=1}^{j_0} \left( \frac{1}{2} \right)^{k-1} \sqrt{\frac{n(1)}{\mu r}} \|Z_0\|_\infty + \left( \frac{1}{2} \right)^{k-1} \|Z_0\|_{\infty,2} \right] \]
\[ \leq 2C_0' \left[ \frac{n(2)}{\mu r} \|X^*\|_\infty + \sqrt{\frac{n(1)n(2)}{\mu r}} \|X^*\|_\infty + \sqrt{\frac{n(1)n(2)}{\mu r}} \|X^*\|_{\infty,2} \right] \leq \frac{\sigma_r(X^*)}{4}, \quad \text{(by incoherence (14))} \]
where we have used the fact that
\[ \|X^*\|_{\infty,2} \leq \sqrt{n(1)}\|X^*\|_\infty \leq \sqrt{\frac{\mu r}{n(2)}} \sigma_r(X^*). \]

Proof of (b). Because \( Y_{j_0} \in \Omega^\perp \), we have \( P_\Omega(X^* + P_\perp Y_{j_0}) = P_\Omega(X^* - P_\perp Y_{j_0}) = P_\Omega Z_{j_0}. \) It then follows from Theorem 5.5 that
\[ \|Z_{j_0}\|_F \leq \epsilon_0 \|X^*\|_F \leq \epsilon_0 \sqrt{n_1n_2} \|X^*\|_\infty \leq \epsilon_0 \sqrt{n_1n_2} \sigma_r(X^*) \leq \frac{\lambda}{8}. \]

Proof of (c). By definition, we know that \( X^* + W^L = Z_{j_0} + Y_{j_0}. \) Since we have shown \( \|Z_{j_0}\|_F \leq \lambda/8 \), it
suffices to prove $\|Y_{j0}\|_\infty \leq \lambda/8$. We have

$$\|Y_{j0}\|_\infty \leq q^{-1} \sum_{k=1}^{j0} \|P_{\Omega k}Z_{k-1}\|_\infty$$

$$\leq q^{-1} \sum_{k=1}^{j0} e^{k-1} \|X^*\|_\infty$$ (by Lemma D.2)

$$\leq \frac{n(2)^2}{C_0 \mu r \log n(1)} \sqrt{\frac{\mu r}{n_1 n_2}} \sigma_r(X^*)$$ (by incoherence (14))

$$\leq \frac{\lambda}{8},$$

if we choose $\epsilon = C \left( \frac{\mu r (\log n(1))^2}{n(2)^2} \right)^{1/4}$ for an absolute constant $C$. This can be true once the constant $\rho_r$ is sufficiently small.

We now prove that $W^S$ given by (26) obeys dual condition (28). We have the following.

**Lemma E.4.** Assume $\Omega \sim \text{Ber}(p)$. Then under the other assumptions of Theorem 6.1, $W^S$ given by (26) obeys dual condition (28).

**Proof.** According to the standard de-randomization argument [CLMW11], it is equivalent to studying the case when the signs $\delta_{ij}$ of $S_{ij}^*$ are independently distributed as

$$\delta_{ij} = \begin{cases} 
1, & \text{w.p. } p/2, \\
0, & \text{w.p. } 1 - p, \\
-1, & \text{w.p. } p/2.
\end{cases}$$

**Proof of (d).** Recall that

$$W^S = \lambda P_{\bot} \sum_{k \geq 1} (P_{\Omega} P_{\Omega} P_{\Omega})^k \text{sign}(S^*)$$

$$= \lambda P_{\bot} \text{sign}(S^*) + \lambda P_{\bot} \sum_{k \geq 1} (P_{\Omega} P_{\Omega} P_{\Omega})^k \text{sign}(S^*).$$

To bound the first term, we have $\|\text{sign}(S^*)\| \leq 4 \sqrt{\mu r(n)}$ [Ver10]. So $\|\lambda P_{\bot} \text{sign}(S^*)\| \leq \lambda \|\text{sign}(S^*)\| \leq 4\sqrt{\mu r}(X^*) \leq \sigma_r(X^*)/8$.

We now bound the second term. Let $G = \sum_{k \geq 1} (P_{\Omega} P_{\Omega} P_{\Omega})^k$, which is self-adjoint, and denote by $N_{n1}$ and $N_{n2}$ the $1/2$-nets of $S^{n-1}$ and $S^{n-1}$ of sizes at most $6^{n1}$ and $6^{n2}$, respectively [Led05]. We know that [Ver10]

$$\|G(\text{sign}(S^*))\| = \sup_{x \in S^{n-1}, y \in S^{n-1}} \langle G(yx^T), \text{sign}(S^*) \rangle$$

$$\leq 4 \sup_{x \in N_{n2}, y \in N_{n1}} \langle G(yx^T), \text{sign}(S^*) \rangle.$$ 

Consider the random variable $X(x, y) = \langle G(yx^T), \text{sign}(S^*) \rangle$ which has zero expectation. By Hoeffding’s inequality, we have

$$\Pr(|X(x, y)| > t |\Omega) \leq 2 \exp \left( -\frac{2t^2}{\|G(yx^T)\|_{F}^2} \right) \leq 2 \exp \left( -\frac{2t^2}{\|G\|_{F}^2} \right).$$
Therefore, by a union bound,
\[
\Pr(\|G(\text{sign}(S^\ast))\| > t|\Omega) \leq 2 \times 6^{n_1 + n_2} \exp\left(-\frac{t^2}{8\|G\|^2}\right).
\]

Note that conditioned on the event \(\{\|P_\Omega P_T\| \leq \sigma\}\), we have \(\|G\| = \left\|\sum_{k \geq 1} (P_\Omega P_T P_\Omega)^k\right\| \leq \frac{\sigma^2}{1-\sigma^2}\). So
\[
\Pr(\lambda\|G(\text{sign}(S^\ast))\| > t) \leq 2 \times 6^{n_1 + n_2} \exp\left(-\frac{t^2}{8\lambda^2} \left(\frac{1-\sigma^2}{\sigma^2}\right)^2\right) \Pr(\|P_\Omega P_T\| \leq \sigma) + \Pr(\|P_\Omega P_T\| > \sigma).
\]

The following lemma guarantees that event \(\{\|P_\Omega P_T\| \leq \sigma\}\) holds with high probability for a very small absolute constant \(\sigma\).

**Lemma E.5** ([CLMW11], Cor 2.7). Suppose that \(\Omega \sim \text{Ber}(p)\) and incoherence (3) holds. Then with probability at least \(1 - n_{10}^{-10}\), \(\|P_\Omega P_T\|^2 \leq p + \epsilon\), provided that \(1 - p \geq C_0 e^{-2\mu r \log n_{(1)} / n_{(2)}}\) for an absolute constant \(C_0\).

Setting \(t = \frac{\sigma(X^\ast)}{8}\), this completes the proof of (d).

**Proof of (e).** Recall that \(W^S = \lambda P_T \sum_{k \geq 0} (P_\Omega P_T P_\Omega)^k \text{sign}(S^\ast)\) and so
\[
P_{\Omega^\perp} W^S = \lambda P_{\Omega^\perp} (I - P_T) \sum_{k \geq 0} (P_\Omega P_T P_\Omega)^k \text{sign}(S^\ast)
\]
\[
= -\lambda P_{\Omega^\perp} P_T \sum_{k \geq 0} (P_\Omega P_T P_\Omega)^k \text{sign}(S^\ast).
\]

Then for any \((i, j) \in \Omega^\perp\), we have
\[
W_{ij}^S = \langle W^S, e_i e_j^T \rangle = \left(\lambda \text{sign}(S^\ast), -\sum_{k \geq 0} (P_\Omega P_T P_\Omega)^k P_\Omega P_T (e_i e_j^T)\right).
\]

Let \(X(i, j) = -\sum_{k \geq 0} (P_\Omega P_T P_\Omega)^k P_\Omega P_T (e_i e_j^T)\). By Hoeffding’s inequality and a union bound,
\[
\Pr\left(\sup_{ij} |W_{ij}^S| > t|\Omega\right) \leq 2 \sum_{ij} \exp\left(-\frac{2t^2}{\lambda^2 \|X(i, j)\|_F^2}\right).
\]

We note that conditioned on the event \(\{\|P_\Omega P_T\| \leq \sigma\}\), for any \((i, j) \in \Omega^\perp\),
\[
\|X(i, j)\|_F \leq \frac{1}{1 - \sigma^2} \|P_T (e_i e_j^T)\|_F
\]
\[
\leq \frac{1}{1 - \sigma^2} \sigma \sqrt{1 - \|P_T^\perp (e_i e_j^T)\|_F^2}
\]
\[
= \frac{1}{1 - \sigma^2} \sigma \sqrt{1 - \|((I - UU^T) e_i 2 \|_2^2 (I - VV^T) e_j 2 \|_2^2}
\]
\[
\leq \frac{1}{1 - \sigma^2} \sqrt{1 - \left(1 - \frac{\mu r}{n_1}\right) \left(1 - \frac{\mu r}{n_2}\right)}
\]
\[
\leq \frac{1}{1 - \sigma^2} \sqrt{\frac{\mu r}{n_1} + \frac{\mu r}{n_2}}.
\]
Then unconditionally,
\[
\Pr\left( \sup_{ij} |W_{ij}^\mathcal{S}| > t \right) \leq 2n_1n_2 \exp \left( -\frac{2t^2}{\lambda^2} \frac{(1 - \sigma^2)^2n_1n_2}{\sigma^2 \mu \mathcal{R}(n_1 + n_2)} \right) \Pr(\|P_{\Omega}P_T\| \leq \sigma) + \Pr(\|P_{\Omega}P_T\| > \sigma).
\]

By Lemma E.5 and setting \( t = \lambda/4 \), the proof of (e) is completed.

**F Proof of Theorem 7.1**

Our computational lower bound for problem (P) assumes the hardness of random 4-SAT.

**Conjecture 1** (Random 4-SAT). Let \( c > \ln 2 \) be a constant. Consider a random 4-SAT formula on \( n \) variables in which each clause has 4 literals, and in which each of the \( 16n^4 \) clauses is picked independently with probability \( c/n^3 \). Then any algorithm which always outputs 1 when the random formula is satisfiable, and outputs 0 with probability at least 1/2 when the random formula is unsatisfiable, must run in \( 2^{c'n} \) time on some input, where \( c' > 0 \) is an absolute constant.

Based on Conjecture 1, we have the following computational lower bound for problem (P). We show that problem (P) is in general hard for deterministic algorithms. If we additionally assume \( \text{BPP} = \text{P} \), then the same conclusion holds for randomized algorithms with high probability.

**Theorem 7.1** (Computational Lower Bound. Restated). Assume Conjecture 1. Then there exists an absolute constant \( \epsilon_0 > 0 \) for which any algorithm that achieves \( (1 + \epsilon)\text{OPT} \) in objective function value for problem (P) with \( \epsilon \leq \epsilon_0 \), and with constant probability, requires \( 2^{\Omega(n_1 + n_2)} \) time, where OPT is the optimum. If in addition, \( \text{BPP} = \text{P} \), then the same conclusion holds for randomized algorithms succeeding with probability at least 2/3.

**Proof.** Theorem 7.1 is proved by using the hypothesis that random 4-SAT is hard to show hardness of the Maximum Edge Biclique problem for deterministic algorithms, similar to [RSW16].

**Definition 1** (Maximum Edge Biclique). The problem is

**Input:** An \( n \)-by-\( n \) bipartite graph \( G \).

**Output:** A \( k_1 \)-by-\( k_2 \) complete bipartite subgraph of \( G \), such that \( k_1 \cdot k_2 \) is maximized.

[GL04] showed that under the random 4-SAT assumption there exist two constants \( \epsilon_1 > \epsilon_2 > 0 \) such that no efficient deterministic algorithm is able to distinguish between bipartite graphs \( G(U, V, E) \) with \( |U| = |V| = n \) which have a clique of size \( (n/16)^2(1 + \epsilon_1) \) and those in which all bipartite cliques are of size \( \leq (n/16)^2(1 + \epsilon_2) \). The reduction uses a bipartite graph \( G \) with at least \( tn^2 \) edges with large probability, for a constant \( t \).

Given a given bipartite graph \( G(U, V, E) \), define \( H(\cdot) \) as follows. Define the matrix \( Y \) and \( W : Y_{ij} = 1 \) if edge \( (U_i, V_j) \in E \), \( Y_{ij} = 0 \) if edge \( (U_i, V_j) \not\in E \); \( W_{ij} = 1 \) if edge \( (U_i, V_j) \in E \), and \( W_{ij} = \text{poly}(n) \) if edge \( (U_i, V_j) \not\in E \). Choose a large enough constant \( \beta > 0 \) and let \( H(AB) = \beta \sum_{ij} W_{ij}^2 (Y_{ij} - (AB)_{ij})^2 \). Now, if there exists a biclique in \( G \) with at least \( (n/16)^2(1 + \epsilon_2) \) edges, then the number of remaining edges is at most \( tn^2 - (n/16)^2(1 + \epsilon_1) \), and so the solution to \( \min H(AB) + \frac{\beta}{2} \|AB\|^2_F \) has cost at most \( \beta [tn^2 - (n/16)^2(1 + \epsilon_1)] + n^2 \). On the other hand, if there does not exist a biclique that has more than \( (n/16)^2(1 + \epsilon_2) \) edges, then the number of remaining edges is at least \( (n/16)^2(1 + \epsilon_2) \), and so any solution to \( \min H(AB) + \frac{\beta}{2} \|AB\|^2_F \) has cost at least \( \beta [tn^2 - (n/16)^2(1 + \epsilon_2)] \). Choose \( \beta \) large enough so that \( \beta [tn^2 - (n/16)^2(1 + \epsilon_2)] > \beta [tn^2 - (n/16)^2(1 + \epsilon_1)] + n^2 \). This combined with the result in [GL04] completes the proof for deterministic algorithms.
To rule out randomized algorithms running in time \(2^\alpha(n_1+n_2)\) for some function \(\alpha\) of \(n_1, n_2\) for which \(\alpha = o(1)\), observe that we can define a new problem which is the same as problem \((P)\) except the input description of \(H\) is padded with a string of 1’s of length \(2^{(\alpha/2)(n_1+n_2)}\). This string is irrelevant for solving problem \((P)\) but changes the input size to \(N = \text{poly}(n_1, n_2) + 2^{(\alpha/2)(n_1+n_2)}\). By the argument in the previous paragraph, any deterministic algorithm still requires \(2^\Omega(n) = N^{o(1)}\) time to solve this problem, which is super-polynomial in the new input size \(N\). However, if a randomized algorithm can solve it in \(2^\alpha(n_1+n_2)\) time, then it runs in \(\text{poly}(N)\) time. This contradicts the assumption that \(\text{BPP} = \text{P}\). This completes the proof. \(\square\)

G Matrix Completion by Information-Theoretic Upper Bound

Theorem 5.1 formulates matrix completion as a feasibility problem. However, it is a priori unclear if there is an algorithm for finding \(X^*\) with \(\mathcal{O}(\mu n(1)r \log n(1))\) sample complexity and incoherence (3) via solving the feasibility problem. To answer this question, we mention that matrix completion can be solved in finite time under these minimum assumptions, namely, we note that the feasibility problem is equivalent to finding a zero of the polynomial

\[
\sum_{(i,j) \in \Omega} (e_i^T AB e_j - X_{ij}^*)^2 = 0
\]

w.r.t. the \((n_1 + n_2)r\) unknowns \(A\) and \(B\). Since \(A\) can be assumed to be orthogonal, if the entries of \(X^*\) can be written down with \(\text{poly}(n)\) bits, then \(\|B\|_F \leq \text{exp}(\text{poly}(n))\), which means if one rounds each of the entries of \(B\) to the nearest additive grid multiple of \(1/\text{exp}(\text{poly}(n))\), then we will get a rank-\(k\) matrix \(B\) where each entry represents the true entry of the optimal \(B\) up to additive \(1/\text{exp}(\text{poly}(n))\) error (of course one cannot write down \(B\) in some cases if the entries are irrational). Such an \(A\) and \(B\) can be found in \(\text{exp}((n_1 + n_2)r)\) time [Ren92a, Ren92b, BPR96]. This gives an exponential time algorithm to solve the feasibility problem in Theorem 5.1 for matrix completion.

H Dual and Bi-Dual Problems

In this section, we derive the dual and bi-dual problems of non-convex program \((P)\). According to (5), the primal problem \((P)\) is equivalent to

\[
\min_{\Lambda, B} \max_{\Lambda} \frac{1}{2} \| \Lambda - \Lambda AB \|_F^2 - \frac{1}{2} \| \Lambda \|_F^2 - H^*(\Lambda).
\]

Therefore, the dual problem is given by

\[
\max_{\Lambda, B} \min_{\Lambda} \frac{1}{2} \| \Lambda - \Lambda AB \|_F^2 - \frac{1}{2} \| \Lambda \|_F^2 - H^*(\Lambda)
\]

\[
= \max_{\Lambda} \frac{1}{2} \sum_{i=r+1}^{n(2)} \sigma_i^2(\Lambda) - \frac{1}{2} \| \Lambda \|_F^2 - H^*(\Lambda)
\]

\[
= \max_{\Lambda} -\frac{1}{2} \| \Lambda \|_F^2 - H^*(\Lambda), \tag{D1}
\]

where \(\| \Lambda \|_F^2 = \sum_{i=1}^r \sigma_i^2(\Lambda)\). The bi-dual problem is derived by

\[
\min_{\Lambda, A', \Lambda'} \max_{\Lambda} \left[ \frac{1}{2} \| \Lambda \|_F^2 - H^*(\Lambda') + \langle M, \Lambda' - \Lambda \rangle \right]
\]

\[
= \min_{\Lambda, A'} \left[ \min_{\Lambda} \left( \langle M, -\Lambda \rangle - \frac{1}{2} \| -\Lambda \|_F^2 \right) + \max_{\Lambda'} \left( \langle M, \Lambda' \rangle - H^*(\Lambda') \right) \right]
\]

\[
= \min_M \| M \|_{r+} + H(M), \tag{D2}
\]
where \( \|M\|_{r^*} = \max_X \langle M, X \rangle - \frac{1}{2} \|X\|_2^2 \) is a convex function, and \( H(M) = \max_{\Lambda'} [\langle M, \Lambda' \rangle - H^*(\Lambda')] \) holds by the definition of conjugate function.

Problems (D1) and (D2) can be solved efficiently due to their convexity. In particular, Grussler et al. [GRG16] provided a computationally efficient algorithm to compute the proximal operators of functions \( \frac{1}{2} \|\cdot\|_2^2 \) and \( \|\cdot\|_{r^*} \). Hence, the Douglas-Rachford algorithm can find the global minimum up to an \( \epsilon \) error in function value in time \( \text{poly}(1/\epsilon) \) [HY12].

I Equivalence of Bernoulli and Uniform Models

**Lemma I.1.** Let \( n \) be the number of Bernoulli trials and suppose that \( \Omega \sim \text{Ber}(m/n) \). Then with probability at least \( 1 - n^{-10} \), \( |\Omega| = \Theta(m) \), provided that \( m \geq c \log n \) for an absolute constant \( c \).

**Proof.** By the scalar Chernoff bound, with \( \epsilon > 0 \) we have
\[
\Pr(|\Omega| \leq m - n\epsilon) \leq \exp\left(-\frac{\epsilon^2 n^2}{2m}\right),
\]
and
\[
\Pr(|\Omega| \geq m + n\epsilon) \leq \exp\left(-\frac{\epsilon^2 n^2}{3m}\right).
\]
Taking \( \epsilon = m/(2n) \) and \( m \geq c_1 \log n \) in (29) for an appropriate absolute constant \( c_1 \), we have
\[
\Pr(|\Omega| \leq m/2) \leq \exp(-m/4) \leq \frac{n^{-10}}{2}.
\]
Taking \( \epsilon = m/n \) and \( m \geq c_2 \log n \) in (30) for an appropriate absolute constant \( c_2 \), we have
\[
\Pr(|\Omega| \geq 2m) \leq \exp(-m/3) \leq \frac{n^{-10}}{2}.
\]
Given (31) and (32), we conclude that \( m/2 < |\Omega| < 2m \) with probability at least \( 1 - n^{-10} \), provided that \( m \geq c \log n \) for an absolute constant \( c \). \( \square \)

**References**

[AAZB+16] Naman Agarwal, Zeyuan Allen-Zhu, Brian Bullins, Elad Hazan, and Tengyu Ma. Finding approximate local minima for nonconvex optimization in linear time. *arXiv preprint arXiv:1611.01146*, 2016.

[ABGM14] Sanjeev Arora, Aditya Bhaskara, Rong Ge, and Tengyu Ma. More algorithms for provable dictionary learning. *arXiv preprint arXiv:1401.0579*, 2014.

[ABHZ16] Pranjal Awasthi, Maria-Florina Balcan, Nika Haghtalab, and Hongyang Zhang. Learning and 1-bit compressed sensing under asymmetric noise. In *Annual Conference on Learning Theory*, pages 152–192, 2016.

[AG16] Anima Anandkumar and Rong Ge. Efficient approaches for escaping higher order saddle points in non-convex optimization. *arXiv preprint arXiv:1602.05908*, 2016.

[AGH+14] Animashree Anandkumar, Rong Ge, Daniel J Hsu, Sham M Kakade, and Matus Telgarsky. Tensor decompositions for learning latent variable models. *Journal of Machine Learning Research*, 15(1):2773–2832, 2014.
[AMS11] Christoph Ambühl, Monaldo Mastrolilli, and Ola Svensson. Inapproximability results for maximum edge biclique, minimum linear arrangement, and sparsest cut. *SIAM Journal on Computing*, 40(2):567–596, 2011.

[ANW12] Alekh Agarwal, Sahand Negahban, and Martin J Wainwright. Noisy matrix decomposition via convex relaxation: Optimal rates in high dimensions. *The Annals of Statistics*, pages 1117–1197, 2012.

[ARR14] Ali Ahmed, Benjamin Recht, and Justin Romberg. Blind deconvolution using convex programming. *IEEE Transactions on Information Theory*, 60(3):1711–1732, 2014.

[AZ16] Zeyuan Allen-Zhu. Katyusha: The first direct acceleration of stochastic gradient methods. *arXiv preprint arXiv:1603.05953*, 2016.

[BCMV14] Aditya Bhaskara, Moses Charikar, Ankur Moitra, and Aravindan Vijayaraghavan. Smoothed analysis of tensor decompositions. In *ACM Symposium on Theory of Computing*, pages 594–603, 2014.

[BE06] Amir Beck and Yonina C Eldar. Strong duality in nonconvex quadratic optimization with two quadratic constraints. *SIAM Journal on Optimization*, 17(3):844–860, 2006.

[BH89] Pierre Baldi and Kurt Hornik. Neural networks and principal component analysis: Learning from examples without local minima. *Neural Networks*, 2(1):53–58, 1989.

[BKS16] Srinadh Bhojanapalli, Anastasios Kyrillidis, and Sujay Sanghavi. Dropping convexity for faster semi-definite optimization. In *Annual Conference on Learning Theory*, pages 530–582, 2016.

[BM05] Samuel Burer and Renato DC Monteiro. Local minima and convergence in low-rank semidefinite programming. *Mathematical Programming*, 103(3):427–444, 2005.

[BMP08] Francis Bach, Julien Mairal, and Jean Ponce. Convex sparse matrix factorizations. *arXiv preprint arXiv:0812.1869*, 2008.

[BNS16] Srinadh Bhojanapalli, Behnam Neyshabur, and Nati Srebro. Global optimality of local search for low rank matrix recovery. In *Advances in Neural Information Processing Systems*, pages 3873–3881, 2016.

[BPR96] Saugata Basu, Richard Pollack, and Marie Francoise Roy. On the combinatorial and algebraic complexity of quantifier elimination. *J. ACM*, 43(6):1002–1045, 1996.

[BV04] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[BZ16] Maria-Florina Balcan and Hongyang Zhang. Noise-tolerant life-long matrix completion via adaptive sampling. In *Advances in Neural Information Processing Systems*, pages 2955–2963, 2016.

[Che15] Yudong Chen. Incoherence-optimal matrix completion. *IEEE Transactions on Information Theory*, 61(5):2909–2923, 2015.

[CHM+15] Anna Choromanska, Mikael Henaff, Michael Mathieu, Gérard Ben Arous, and Yann LeCun. The loss surfaces of multilayer networks. In *International Conference on Artificial Intelligence and Statistics*, 2015.
[CLMW11] Emmanuel J. Candès, Xiaodong Li, Yi Ma, and John Wright. Robust principal component analysis? *Journal of the ACM*, 58(3):11, 2011.

[CR09] Emmanuel J. Candès and Ben Recht. Exact matrix completion via convex optimization. *Foundations of Computational Mathematics*, 9(6):717–772, 2009.

[CR13] Emmanuel J. Candès and Benjamin Recht. Simple bounds for recovering low-complexity models. *Mathematical Programming*, pages 1–13, 2013.

[CRPW12] Venkat Chandrasekaran, Benjamin Recht, Pablo A Parrilo, and Alan S Willsky. The convex geometry of linear inverse problems. *Foundations of Computational Mathematics*, 12(6):805–849, 2012.

[CT10] Emmanuel J. Candès and Terence Tao. The power of convex relaxation: Near-optimal matrix completion. *IEEE Transactions on Information Theory*, 56(5):2053–2080, 2010.

[CW15] Yudong Chen and Martin J. Wainwright. Fast low-rank estimation by projected gradient descent: General statistical and algorithmic guarantees. *arXiv preprint arXiv:1509.03025*, 2015.

[Fei02] Uriel Feige. Relations between average case complexity and approximation complexity. In *Proceedings of the 17th Annual IEEE Conference on Computational Complexity, Montréal, Québec, Canada, May 21-24, 2002*, page 5, 2002.

[GHJY15] Rong Ge, Furong Huang, Chi Jin, and Yang Yuan. Escaping from saddle points – online stochastic gradient for tensor decomposition. In *Annual Conference on Learning Theory*, pages 797–842, 2015.

[GIY17] Rong Ge, Chi Jin, and Zheng Yi. No spurious local minima in nonconvex low rank problems: A unified geometric analysis. *arXiv preprint: 1704.00708*, 2017.

[GL04] Andreas Goerdt and André Lanka. An approximation hardness result for bipartite clique. In *Electronic Colloquium on Computational Complexity, Report*, volume 48, 2004.

[GLM16] Rong Ge, Jason D Lee, and Tengyu Ma. Matrix completion has no spurious local minimum. In *Advances in Neural Information Processing Systems*, pages 2973–2981, 2016.

[GLZ17] David Gamarnik, Quan Li, and Hongyi Zhang. Matrix completion from $O(n)$ samples in linear time. *arXiv preprint arXiv:1702.02267*, 2017.

[GRG16] Christian Grussler, Anders Rantzer, and Pontus Giselsson. Low-rank optimization with convex constraints. *arXiv preprint arXiv:1606.01793*, 2016.

[Gro11] D. Gross. Recovering low-rank matrices from few coefficients in any basis. *IEEE Transactions on Information Theory*, 57(3):1548–1566, 2011.

[GWL16] Quanquan Gu, Zhaoran Wang, and Han Liu. Low-rank and sparse structure pursuit via alternating minimization. In *International Conference on Artificial Intelligence and Statistics*, pages 600–609, 2016.

[Har14] Moritz Hardt. Understanding alternating minimization for matrix completion. In *IEEE Symposium on Foundations of Computer Science*, pages 651–660, 2014.

[HM12] Moritz Hardt and Ankur Moitra. Algorithms and hardness for robust subspace recovery. *arXiv preprint: 1211.1041*, 2012.
[HMRW14] Moritz Hardt, Raghu Meka, Prasad Raghavendra, and Benjamin Weitz. Computational limits for matrix completion. In Annual Conference on Learning Theory, pages 703–725, 2014.

[HV15] Benjamin D Haeffele and René Vidal. Global optimality in tensor factorization, deep learning, and beyond. arXiv preprint arXiv:1506.07540, 2015.

[HY12] Bingsheng He and Xiaoming Yuan. On the $O(1/n)$ convergence rate of the douglas–rachford alternating direction method. SIAM Journal on Numerical Analysis, 50(2):700–709, 2012.

[HYV14] Benjamin Haeffele, Eric Young, and Rene Vidal. Structured low-rank matrix factorization: Optimality, algorithm, and applications to image processing. In International Conference on Machine Learning, pages 2007–2015, 2014.

[IW97] Russell Impagliazzo and Avi Wigderson. $P = BPP$ if $E$ requires exponential circuits: Derandomizing the XOR lemma. In ACM Symposium on the Theory of Computing, pages 220–229, 1997.

[Jah07] Johannes Jahn. Introduction to the theory of nonlinear optimization. Springer Berlin Heidelberg, 2007.

[JBAS10] Michel Journée, Francis Bach, P-A Absil, and Rodolphe Sepulchre. Low-rank optimization on the cone of positive semidefinite matrices. SIAM Journal on Optimization, 20(5):2327–2351, 2010.

[JGN+17] Chi Jin, Rong Ge, Praneeth Netrapalli, Sham M Kakade, and Michael I Jordan. How to escape saddle points efficiently. arXiv preprint arXiv:1703.00887, 2017.

[JMD10] Prateek Jain, Raghu Meka, and Inderjit S Dhillon. Guaranteed rank minimization via singular value projection. In Advances in Neural Information Processing Systems, pages 937–945, 2010.

[JNS13] Prateek Jain, Praneeth Netrapalli, and Sujay Sanghavi. Low-rank matrix completion using alternating minimization. In ACM Symposium on Theory of Computing, pages 665–674, 2013.

[Kaw16] Kenji Kawaguchi. Deep learning without poor local minima. arXiv preprint arXiv:1605.07110, 2016.

[KBV09] Yehuda Koren, Robert Bell, and Chris Volinsky. Matrix factorization techniques for recommender systems. IEEE Computer, 42(8):30–37, 2009.

[Kes12] Raghunandan Hulikal Keshavan. Efficient algorithms for collaborative filtering. PhD thesis, Stanford University, 2012.

[KMO10a] Raghunandan H Keshavan, Andrea Montanari, and Sewoong Oh. Matrix completion from a few entries. IEEE Transactions on Information Theory, 56(6):2980–2998, 2010.

[KMO10b] Raghunandan H Keshavan, Andrea Montanari, and Sewoong Oh. Matrix completion from noisy entries. Journal of Machine Learning Research, 11:2057–2078, 2010.

[Led05] Michel Ledoux. The concentration of measure phenomenon. Number 89. American Mathematical Society, 2005.

[LLR16] Yuanzhi Li, Yingyu Liang, and Andrej Risteski. Recovery guarantee of weighted low-rank approximation via alternating minimization. In International Conference on Machine Learning, pages 2358–2367, 2016.
[NNS+14] Praneeth Netrapalli, UN Niranjan, Sujay Sanghavi, Animashree Anandkumar, and Prateek Jain. Non-convex robust PCA. In Advances in Neural Information Processing Systems, pages 1107–1115, 2014.

[NW12] Sahand Negahban and Martin J Wainwright. Restricted strong convexity and weighted matrix completion: Optimal bounds with noise. Journal of Machine Learning Research, 13:1665–1697, 2012.

[OW92] Michael L Overton and Robert S Womersley. On the sum of the largest eigenvalues of a symmetric matrix. SIAM Journal on Matrix Analysis and Applications, 13(1):41–45, 1992.

[Rec11] Benjamin Recht. A simpler approach to matrix completion. Journal of Machine Learning Research, 12:3413–3430, 2011.

[Ren92a] James Renegar. On the computational complexity and geometry of the first-order theory of the reals, part I: introduction. preliminaries. the geometry of semi-algebraic sets. the decision problem for the existential theory of the reals. J. Symb. Comput., 13(3):255–300, 1992.

[Ren92b] James Renegar. On the computational complexity and geometry of the first-order theory of the reals, part II: the general decision problem. preliminaries for quantifier elimination. J. Symb. Comput., 13(3):301–328, 1992.

[RSW16] Ilya Razenshteyn, Zhao Song, and David P. Woodruff. Weighted low rank approximations with provable guarantees. In ACM Symposium on Theory of Computing, pages 250–263, 2016.

[SL15] Ruoyu Sun and Zhi-Quan Luo. Guaranteed matrix completion via nonconvex factorization. In IEEE Symposium on Foundations of Computer Science, pages 270–289, 2015.

[SQW16] Ju Sun, Qing Qu, and John Wright. A geometric analysis of phase retrieval. In IEEE International Symposium on Information Theory, pages 2379–2383, 2016.

[SQW17a] Ju Sun, Qing Qu, and John Wright. Complete dictionary recovery over the sphere I: Overview and the geometric picture. IEEE Transactions on Information Theory, 63(2):853–884, 2017.

[SQW17b] Ju Sun, Qing Qu, and John Wright. Complete dictionary recovery over the sphere II: Recovery by Riemannian trust-region method. IEEE Transactions on Information Theory, 63(2):885–914, 2017.

[SS05] Nathan Srebro and Adi Shraibman. Rank, trace-norm and max-norm. In International Conference on Computational Learning Theory, pages 545–560. Springer, 2005.

[SU15] Reinhold Schneider and André Uschmajew. Convergence results for projected line-search methods on varieties of low-rank matrices via Lojasiewicz inequality. SIAM Journal on Optimization, 25(1):622–646, 2015.

[SWZ14] Yuan Shen, Zaiwen Wen, and Yin Zhang. Augmented lagrangian alternating direction method for matrix separation based on low-rank factorization. Optimization Methods and Software, 29(2):239–263, 2014.

[TBSR13] Gongguo Tang, Badri Narayan Bhaskar, Parikshit Shah, and Benjamin Recht. Compressed sensing off the grid. IEEE Transactions on Information Theory, 59(11):7465–7490, 2013.

[TBSR15] Stephen Tu, Ross Boczar, Mahdi Soltanolkotabi, and Benjamin Recht. Low-rank solutions of linear matrix equations via procrustes flow. arXiv preprint arXiv:1507.03566, 2015.
[Ver09] Roman Vershynin. Lectures in geometric functional analysis. pages 1–76, 2009.

[Ver10] Roman Vershynin. Introduction to the non-asymptotic analysis of random matrices. arXiv preprint: 1011.3027, 2010.

[Ver15] Roman Vershynin. Estimation in high dimensions: A geometric perspective. In Sampling theory, a renaissance, pages 3–66. Springer, 2015.

[WX12] Yu-Xiang Wang and Huan Xu. Stability of matrix factorization for collaborative filtering. In International Conference on Machine Learning, pages 417–424, 2012.

[WYZ12] Zaiwen Wen, Wotao Yin, and Yin Zhang. Solving a low-rank factorization model for matrix completion by a nonlinear successive over-relaxation algorithm. Mathematical Programming Computation, 4(4):333–361, 2012.

[YPCC16] Xinyang Yi, Dohyung Park, Yudong Chen, and Constantine Caramanis. Fast algorithms for robust PCA via gradient descent. In Advances in neural information processing systems, pages 4152–4160, 2016.

[ZL15] Qinqing Zheng and John Lafferty. A convergent gradient descent algorithm for rank minimization and semidefinite programming from random linear measurements. In Advances in Neural Information Processing Systems, pages 109–117, 2015.

[ZL16] Qinqing Zheng and John Lafferty. Convergence analysis for rectangular matrix completion using Burer-Monteiro factorization and gradient descent. arXiv preprint arXiv:1605.07051, 2016.

[ZLZ13] Hongyang Zhang, Zhouchen Lin, and Chao Zhang. A counterexample for the validity of using nuclear norm as a convex surrogate of rank. In European Conference on Machine Learning and Principles and Practice of Knowledge Discovery in Databases, volume 8189, pages 226–241, 2013.

[ZLZ16] Hongyang Zhang, Zhouchen Lin, and Chao Zhang. Completing low-rank matrices with corrupted samples from few coefficients in general basis. IEEE Transactions on Information Theory, 62(8):4748–4768, 2016.

[ZLZC15] Hongyang Zhang, Zhouchen Lin, Chao Zhang, and Edward Chang. Exact recoverability of robust PCA via outlier pursuit with tight recovery bounds. In AAAI Conference on Artificial Intelligence, pages 3143–3149, 2015.

[ZLZG14] Hongyang Zhang, Zhouchen Lin, Chao Zhang, and Junbin Gao. Robust latent low rank representation for subspace clustering. Neurocomputing, 145:369–373, 2014.

[ZLZG15] Hongyang Zhang, Zhouchen Lin, Chao Zhang, and Junbin Gao. Relations among some low rank subspace recovery models. Neural Computation, 27:1915–1950, 2015.

[ZWG17] Xiao Zhang, Lingxiao Wang, and Quanquan Gu. A nonconvex free lunch for low-rank plus sparse matrix recovery. arXiv preprint arXiv:1702.06525, 2017.

[ZWL15] Tuo Zhao, Zhaoran Wang, and Han Liu. A nonconvex optimization framework for low rank matrix estimation. In Advances in Neural Information Processing Systems, pages 559–567, 2015.