Decompositions of Reflexive Bimodules over Maximal Abelian Selfadjoint Algebras

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Abstract
We generalize the notion of ‘diagonal’ from the class of CSL algebras to masa bimodules. We prove that a reflexive masa bimodule decomposes as a sum of two bimodules, the diagonal and a module generalizing the w*-closure of the Jacobson radical of a CSL algebra. The latter module turns out to be reflexive, a result which is new even for CSL algebras. We show that the projection onto the direct summand contained in the diagonal is contractive and preserves compactness and reduces rank of operators. Stronger results are obtained when the module is the reflexive hull of its rank-one subspace.

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1 Introduction
In this paper we attempt a generalisation of the concept of the diagonal of a CSL algebra to reflexive spaces of operators which are modules over maximal abelian selfadjoint algebras (masas).

Recall [2] that a CSL algebra is an algebra \( \mathcal{A} \) of operators on a Hilbert space \( H \) which can be written in the form
\[
\mathcal{A} = \{ A \in B(H) : AP = PAP \text{ for all } P \in \mathcal{S} \}
\]
where \( \mathcal{S} \) is a commuting family of projections. Note that \( \mathcal{A} \) contains any masa containing \( \mathcal{S}'' \).

More generally, a reflexive masa bimodule \( \mathcal{U} \) of operators from \( H \) to another Hilbert space \( K \) can be written in the form
\[
\mathcal{U} = \{ T \in B(H, K) : TP = \phi(P)TP \text{ for all } P \in \mathcal{S} \}
\]
where $S$ is a commuting family of projections on $H$ and $\phi$ maps them to commuting projections on $K$ (see below for details).

The diagonal $\mathcal{A} \cap \mathcal{A}^*$ of a CSL algebra $\mathcal{A}$ is a von Neumann algebra, which equals the commutant

$$S' = \{ A \in B(H) : AP = PA \text{ for all } P \in S \}$$

of the corresponding invariant projection family. The natural corresponding object for a reflexive masa bimodule $\mathcal{U}$ is a ternary ring of operators (TRO)

$$\Delta(\mathcal{U}) = \{ T \in B(H, K) : TP = \phi(P)T \text{ for all } P \in S \}$$

which is also a reflexive masa bimodule.

This ‘diagonal’ $\Delta(\mathcal{U})$ is the primary object of study of the present paper. We decompose $\mathcal{U}$ as a sum $\mathcal{U}_0 + \Delta(\mathcal{U})$, where $\mathcal{U}_0$ also turns out to be reflexive (Theorem 5.2). This is new even for the case of CSL algebras; note, however, that for nest algebras reflexivity of $\ast$-closed bimodules is automatic \[7\]. An analogous decomposition for the case of nest subalgebras of von Neumann algebras is in \[11\].

We also prove (Corollary 5.3) that the bimodule $\mathcal{U}_0$ has in our context the role corresponding to the $\ast$ closure of the Jacobson radical of a CSL algebra.

The diagonal $\Delta(\mathcal{U})$ is proved to be generated by a partial isometry and natural von Neumann algebras associated to $\mathcal{U}$ (Theorem 4.1).

The above decomposition may be further refined to a direct sum: $\mathcal{U} = \mathcal{U}_0 \oplus \mathcal{M}$ where $\mathcal{M}$ is a TRO ideal of the diagonal $\Delta(\mathcal{U})$ (Theorem 3.3), containing the compact operators of the diagonal (Proposition 6.3).

In case $\mathcal{U}$ is strongly reflexive (that is, coincides with the reflexive hull of the rank one operators it contains) we show (Theorem 7.4) that $\mathcal{M}$ coincides with the $\ast$-closed linear span of the finite rank operators of the diagonal, an equality which fails in general.

As in the case of von Neumann algebras, we show that every TRO decomposes in an ‘atomic’ and a ‘nonatomic’ part. The ‘atomic’ part of the diagonal $\Delta(\mathcal{U})$ is contained (properly in general) in $\mathcal{M}$ (Proposition 6.3).

We also study the projection $\theta : \mathcal{U} \longrightarrow \mathcal{M}$ defined by the above direct sum decomposition. We prove that it is contractive and maps compact operators to compact operators and finite rank operators to operators of at most the same rank.

In case $\mathcal{U}$ is strongly reflexive, we show that $\theta = D|\mathcal{U}$, where $D$ is the natural projection onto the ‘atomic’ part of the diagonal $\Delta(\mathcal{U})$. 

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A main tool used to obtain these results is an appropriate sequence of projections \((U_n)\) on \(B(H,K)\) which depend on \(U\). This sequence behaves analogously to the net of ‘diagonal sums’ used in nest algebras (see for example [2]).

In nest algebra theory, the net of diagonal sums of a compact operator converges in norm to a compact operator in the ‘atomic’ part of the diagonal. This has been generalised to CSL algebras by Katsoulis [10]. Here we show (Proposition 6.10) that for every compact operator \(K\), the sequence \((U_n(K))\) converges in norm to \(D(K)\).

We present some definitions and concepts we use in this work. All Hilbert spaces will be assumed separable.

If \(S\) is a set of operators then \(R_1(S)\) denotes the subset of \(S\) which contains the rank 1 operators and the zero operator. If \(H\) is a Hilbert space and \(S \subset B(H)\), the set of orthogonal projections of \(S\) is denoted by \(\mathcal{P}(S)\).

If \(H_1, H_2\) are Hilbert spaces, \(C_1(H_1, H_2)\) are the trace class operators and \(\mathcal{R}\) a subset of \(C_1(H_1, H_2)\), we denote by \(\mathcal{R}^0\) the set of operators which are annihilated by \(\mathcal{R}\):

\[
\mathcal{R}^0 = \{ T \in B(H_2, H_1) : tr(TS) = 0 \text{ for all } S \in \mathcal{R} \}.
\]

Let \(H_1, H_2\) be Hilbert spaces and \(U\) a subset of \(B(H_1, H_2)\). Then the reflexive hull of \(U\) is defined [12] to be the space

\[\operatorname{Ref}(U) = \{ T \in B(H_1, H_2) : Tx \in \overline{Ux} \text{ for each } x \in H_1 \}.\]

Simple arguments show that

\[\operatorname{Ref}(U) = \{ T \in B(H_1, H_2) : \text{for all projections } E, F : EFU = 0 \Rightarrow ETF = 0 \}\]

A subspace \(U\) is called reflexive if \(U = \operatorname{Ref}(U)\). It is called strongly reflexive if there exists a set \(L \subset B(H_1, H_2)\) of rank 1 operators such that \(U = \operatorname{Ref}(L)\).

Now we present some concepts introduced by Erdos [5].

Let \(\mathcal{P}_i = \mathcal{P}(B(H_i)), i = 1, 2\). Define \(\phi = \operatorname{Map}(U)\) to be the map \(\phi : \mathcal{P}_1 \to \mathcal{P}_2\) which associates to every \(P \in \mathcal{P}_1\) the projection onto the subspace
\[ TPy \colon T \in \mathcal{U}, y \in H_1 \]. The map \( \phi \) is \( \vee - \) continuous (that is, it preserves arbitrary suprema) and 0 preserving.

Let \( \phi^* = \text{Map}(\mathcal{U}^*) \), \( S_{1,\phi} = \{ \phi^*(P)^\perp : P \in \mathcal{P}_2 \} \), \( S_{2,\phi} = \{ \phi(P) : P \in \mathcal{P}_1 \} \).

Erdos has proved that \( S_{1,\phi} \) is meet complete and contains the identity projection, \( S_{2,\phi} \) is join complete and contains the zero projection, while \( \phi|_{S_{1,\phi}} : S_{1,\phi} \to S_{2,\phi} \) is a bijection. In fact

\[
(\phi|_{S_{1,\phi}})^{-1}(Q) = \phi^*(Q^\perp)
\]

for all \( Q \in S_{2,\phi} \) and

\[
\text{Ref}(\mathcal{U}) = \{ T \in \mathcal{B}(H_1, H_2) : \phi(P)^\perp TP = 0 \text{ for each } P \in S_{1,\phi} \}.
\]

We call the families \( S_{1,\phi}, S_{2,\phi} \) the semilattices of \( \mathcal{U} \).

A C.S.L. is a complete abelian lattice of projections which contains the identity and the zero projection.

If \( \mathcal{A}_1 \subset \mathcal{B}(H_1) \), and \( \mathcal{A}_2 \subset \mathcal{B}(H_2) \) are algebras, a subspace \( \mathcal{U} \subset \mathcal{B}(H_1, H_2) \) is called an \( \mathcal{A}_1, \mathcal{A}_2 - \) bimodule if \( \mathcal{A}_2 \mathcal{U} \mathcal{A}_1 \subset \mathcal{U} \).

A subspace \( \mathcal{M} \) of \( \mathcal{B}(H_1, H_2) \) is called a ternary ring of operators (TRO) if \( \mathcal{M} \mathcal{M}^* \mathcal{M} \subset \mathcal{M} \). Katavolos and Todorov [9] have proved that a TRO \( \mathcal{M} \) is \( w^* \) closed if and only if it is \( wot \) closed if and only if it is reflexive. In this case, if \( \chi = \text{Map}(\mathcal{M}) \), then

\[
\mathcal{M} = \{ T \in \mathcal{B}(H_1, H_2) : TP = \chi(P)T \text{ for all } P \in S_{1,\chi} \}.
\]

They also proved that if \( \mathcal{M} \) is a strongly reflexive TRO, then there exist families of mutually orthogonal projections \( (F_n), (E_n) \) such that \( \mathcal{M} = \sum_{n=1}^{\infty} \oplus E_n \mathcal{B}(H_1, H_2) F_n \). We present a new proof of this result in Corollary 6.9.

The following proposition is easily proved.

**Proposition 1.1** Let \( H_1, H_2 \) be Hilbert spaces, \( \mathcal{A}_1 \subset \mathcal{B}(H_1), \mathcal{A}_2 \subset \mathcal{B}(H_2) \) masas and \( \mathcal{U} \) a \( \mathcal{A}_1, \mathcal{A}_2 - \) bimodule. Then

\[
\text{Ref}(\mathcal{U}) = \{ T \in \mathcal{B}(H_1, H_2) : E \in \mathcal{P}(\mathcal{A}_2), F \in \mathcal{P}(\mathcal{A}_1), EUF = 0 \implies ETF = 0 \}.
\]

The next section contains some preliminary results.
2 Decomposition of a reflexive TRO.

In this section we show that a \( w^* \)-closed TRO decomposes into a ‘nonatomic’ and a ‘totaly atomic’ part.

Let \( H_1, H_2 \) be Hilbert spaces, \( \mathcal{M} \subset B(H_1, H_2) \) be a \( w^* \)-closed TRO and \( \mathcal{B}_1 = (\mathcal{M}^*\mathcal{M})'', \mathcal{B}_2 = (\mathcal{M}\mathcal{M}^*)''. \)

**Remark 2.1** We suppose that \( \mathcal{M}_0 \) is a \( w^* \)-closed TRO ideal of \( \mathcal{M} \); namely, \( \mathcal{M}_0 \) is a linear subspace of \( \mathcal{M} \) and \( \mathcal{M}_0\mathcal{M} \subset \mathcal{M} \) and \( \mathcal{M}\mathcal{M}_0 \subset \mathcal{M}_0 \).

It follows that \( \mathcal{M}\mathcal{M}_0'\mathcal{M} \subset \mathcal{M}_0 \) \([4]\).

Now, we observe that there exist projections \( Q_i \) in the centre of \( \mathcal{B}_i \), \( i = 1, 2 \) such that \( \mathcal{M}_0 = \mathcal{M}Q_1 = Q_2\mathcal{M} \). Hence \( \mathcal{M}_0 \) is a \( \mathcal{B}_1, \mathcal{B}_2 \)-bimodule.

**Proof**

Let \( \mathcal{J}_1 = [\mathcal{M}_0'\mathcal{M}_0]^{-w*} \) and \( \mathcal{J}_2 = [\mathcal{M}_0\mathcal{M}_0']^{-w*} \).

We can easily verify that \( \mathcal{J}_i \) is an ideal of \( \mathcal{B}_i \), \( i = 1, 2 \). Hence there is a projection \( Q_i \) in the centre of \( \mathcal{B}_i \) so that \( \mathcal{J}_i = \mathcal{B}_iQ_i \), \( i = 1, 2 \).

One easily checks that

\[
\mathcal{M}\mathcal{B}_1 \subset \mathcal{M}, \quad \mathcal{B}_2\mathcal{M} \subset \mathcal{M},
\]

\[
\mathcal{M}\mathcal{J}_1 \subset \mathcal{M}_0, \quad \mathcal{J}_2\mathcal{M} \subset \mathcal{M}_0
\]

We observe that \( \mathcal{M}Q_1 \subset \mathcal{M}\mathcal{J}_1 \subset \mathcal{M}_0 \).

Let \( T \in \mathcal{M}_0 \) then \( T^*T \in \mathcal{J}_1 \), so \( T^*T = T^*TQ_1 \) and thus \( T = TQ_1 \). Hence \( T \in \mathcal{M}Q_1 \). We conclude that \( \mathcal{M}_0 \subset \mathcal{M}Q_1 \) and hence equality holds.

Similarly one shows that \( \mathcal{M}_0 = Q_2\mathcal{M} \). \( \square \)

Since \( [R_1(\mathcal{M})]^{-w*} \) is a strongly reflexive TRO, by Proposition 3.5 in \([9]\) there exist mutually orthogonal projections \( (F_n) \) in the centre of \( \mathcal{B}_1 \) and \( (E_n) \) in the centre of \( \mathcal{B}_2 \) such that \( [R_1(\mathcal{M})]^{-w*} = \sum_{n=1}^{\infty} \oplus E_n B(H_1, H_2)F_n \). We write \( E = \vee_n E_n \), \( F = \vee_n F_n \).

**Theorem 2.2** The space \( \mathcal{M} \) decomposes in the following direct sum

\[
\mathcal{M} = (\mathcal{M} \cap (R_1(\mathcal{M}))^0) \oplus [R_1(\mathcal{M})]^{-w*}.
\]
The spaces \( M \cap (R_1(M)^*)^0 \) and \([R_1(M)]^{-w^*}\) are TRO ideals of \( M \). Moreover
\[
[R_1(M)]^{-w^*} = MF = EM = EMF
\]
\[
M \cap (R_1(M)^*)^0 = MF^\perp = E^\perp M = E^\perp MF^\perp.
\]

Proof

We observe that \([R_1(M)]^{-w^*}\) is a TRO ideal of \( M \).

By Remark 2.1 there exists projection \( Q \) in the centre of \( B_1 \) such that
\[
[R_1(M)]^{-w^*} = MQ.
\]

For every \( m \in \mathbb{N} \), we have \( E_m B(H_1, H_2) F_m \subset MQ \).
It follows that \( E_m B(H_1, H_2) F_m = E_m B(H_1, H_2) F_m Q \), so \( F_m = F_m Q \). We conclude that \( \vee_m F_m = F \leq Q \).

Since \( F \in B_1 \) we get \( MF \subset M \), therefore \( MF = MFQ \subset MQ \).

It follows that
\[
[R_1(M)]^{-w^*} = MQ \supset MF \supset [R_1(M)]^{-w^*} F = [R_1(M)]^{-w^*}.
\]

We proved that \([R_1(M)]^{-w^*} = MF\).

If \( M \in \mathcal{M} \) and \( R \in R_1(M) \), then \( R = RF \) so \( tr(MF^\perp R^*)\)
\[
= tr(M(RF^\perp)^*) = tr(M0) = 0.
\]

We conclude that
\[
MF^\perp \subset \mathcal{M} \cap (R_1(M)^*)^0
\]

Hence \( \mathcal{M} = MF^\perp + MF \subset \mathcal{M} \cap (R_1(M)^*)^0 + [R_1(M)]^{-w^*} \subset \mathcal{M} \).

It follows that
\[
\mathcal{M} = (\mathcal{M} \cap (R_1(M)^*)^0) + [R_1(M)]^{-w^*}.
\]

We shall prove that this sum is direct.

If \( T \in [R_1(M)]^{-w^*} \cap (R_1(M)^*)^0 \) then \( T = \sum_{n=1}^{\infty} E_n TF_n \). If \( R \) is a rank 1 operator then \( tr(TR) = \sum_{n=1}^{\infty} tr(E_n TF_n R) = \sum_{n=1}^{\infty} tr(TF_n RE_n) \).

But for every \( n \in \mathbb{N} \), \( tr(TF_n RE_n) = tr(T(E_n R^* F_n)^*) = 0 \) since \( E_n R^* F_n \in R_1(M) \) and \( T \in (R_1(M)^*)^0 \).

Thus \( tr(TR) = 0 \) for every rank 1 operator \( R \), hence \( T = 0 \). This shows that \([R_1(M)]^{-w^*} \cap (R_1(M)^\perp)^* = 0\).
We have shown that \( M = (M \cap (R_1(M))^*)^0 \oplus [R_1(M)]^{-w^*}. \)

Since \( M = MF^\perp \oplus MF, [R_1(M)]^{-w^*} = MF \) and \( MF^\perp \subset M \cap (R_1(M))^* \) we conclude that
\[
MF^\perp = M \cap (R_1(M))^*.
\]

The equalities \( E^\perp M = M \cap (R_1(M))^* \), \( E M = [R_1(M)]^{-w^*} \) are proved similarly. □

**Proposition 2.3** Let \( \theta : M \to M \) be the projection onto \([R_1(M)]^{-w^*}\) defined by the decomposition in Theorem 2.2. Then \( \theta(T) = \sum_{n=1}^\infty E_n T F_n \) for every \( T \in M \).

**Proof**

Since \( M \) decomposes as the direct sum of the \( B_1, B_2 \)-bimodules \( M \cap (R_1(M))^* \) and \([R_1(M)]^{-w^*}\), \( \theta \) is a \( B_1, B_2 \)-bimodule map:
\[
\theta(B_2 T B_1) = B_2 \theta(T) B_1
\]
for every \( T \in M, B_1 \in B_1, B_2 \in B_2 \).

Since \( (E_n) \subset B_1, (F_n) \subset B_2 \) we have that:
\[
\theta(T) = \sum_{n=1}^\infty E_n \theta(T) F_n = \sum_{n=1}^\infty \theta(E_n T F_n) = \sum_{n=1}^\infty E_n T F_n. \quad \square
\]

**3 Decomposition of a reflexive masa bimodule**

Let \( H_1, H_2 \) be Hilbert spaces, \( P_i = \mathcal{P}(B(H_i)), i = 1, 2, \mathcal{D}_i \subset B(H_i), i = 1, 2 \) be masas, \( \mathcal{U} \subset B(H_1, H_2) \) be a reflexive \( \mathcal{D}_1, \mathcal{D}_2 \)-bimodule. Write
\[
\phi = \text{Map}(\mathcal{U}), \quad \phi^* = \text{Map}(\mathcal{U}^*),
\]
\[
S_2,\phi = \phi(P_1), \quad S_{1,\phi} = \{P^\perp : P \in \phi^*(P_2)\}
\]
\[
A_2 = (S_2,\phi)', \quad A_1 = (S_{1,\phi})'.
\]

Observe that \( S_{i,\phi} \subset \mathcal{D}_i \) hence \( \mathcal{D}_i \subset A_i, i = 1, 2. \) We define
\[
\mathcal{U}_0 = [\phi(P)TP^\perp : T \in \mathcal{U}, P \in S_{i,\phi}]^{-w^*}.
\]
\[ \Delta(U) = \{ T : TP = \phi(P)T \text{ for all } P \in \mathcal{S}_{1,\phi} \}. \]

We remark that \( U_0 \) and \( \Delta(U) \) are \( \mathcal{D}_1, \mathcal{D}_2 \)-bimodules contained in \( \mathcal{U} \) and \( \Delta(U) \) is a reflexive TRO. We call \( \Delta(U) \) the \textit{diagonal} of \( \mathcal{U} \).

**Theorem 3.1** \( \mathcal{U} = U_0 + \Delta(U) \).

**Proof**

As noted in the introduction

\[ \mathcal{U} = \{ T \in B(H_1, H_2) : \phi(P)^\perp TP = 0 \text{ for all } P \in \mathcal{S}_{1,\phi} \}. \]

Since the Hilbert spaces \( H_1, H_2 \) are separable we can choose a sequence \( (P_n) \subset \mathcal{S}_{1,\phi} \) such that

\[ \mathcal{U} = \{ T \in B(H_1, H_2) : \phi(P_n)^\perp TP_n = 0 \text{ for all } n \in \mathbb{N} \}. \]

We define

\[ V_n : B(H_1, H_2) \to B(H_1, H_2) : V_n(T) = \phi(P_n)TP_n + \phi(P_n)^\perp TP_n^\perp, n \in \mathbb{N}. \]

One easily checks that \( V_n \) is idempotent and a norm contraction.

We also define \( U_n = V_n \circ V_{n-1} \circ \ldots \circ V_1, n \in \mathbb{N} \).

Let \( T \in \mathcal{U} \), then

\[ T = U_1(T) + \phi(P_1)TP_1^\perp \]

\[ U_1(T) = U_2(T) + \phi(P_2)U_1(T)P_2^\perp \]

by induction

\[ U_{n-1}(T) = U_n(T) + \phi(P_n)U_{n-1}(T)P_n^\perp \]

for all \( n \in \mathbb{N} \).

Adding the previous equalities we obtain

\[ T = U_n(T) + M_n \]

where

\[ M_n = \phi(P_1)TP_1^\perp + \phi(P_2)U_1(T)P_2^\perp + \ldots + \phi(P_n)U_{n-1}(T)P_n^\perp \in \mathcal{U}_0 \]
for all $n \in \mathbb{N}$.

We observe that $\phi(P_i)^\perp U_n(T)P_i = \phi(P_i)U_n(T)P_i^\perp = 0$ for $i = 1, 2, \ldots, n$ and $\|U_n(T)\| \leq \|U_{n-1}(T)\| \leq \ldots \leq \|T\|$ for all $n \in \mathbb{N}$.

The sequence $(U_n(T))$ is bounded, so there exists a subsequence $(U_{n_m}(T))$ that converges in the weak-$*$ topology to an operator $L$.

Then $M_{n_m} = T - U_{n_m}(T) \overset{w^*}{\to} T - L = M \in U_0$.

Since $\phi(P_i)^\perp LP_i = \phi(P_i)LP_i^\perp = 0$ for all $i \in \mathbb{N}$ we have $L \in \Delta(U)$ and $T = M + L \in U_0 + \Delta(U)$. □

Remark 3.2 The following are equivalent:

i) $U$ is a TRO
ii) $U = \Delta(U)$
iii) $U_0 = 0$.

Theorem 3.3 There exist projections $Q_i \in D_i, i = 1, 2$ such that:

$U = U_0 \oplus (I - Q_2)\Delta(U)(I - Q_1) = U_0 \oplus (I - Q_2)\Delta(U) = U_0 \oplus \Delta(U)(I - Q_1)$.

Proof

We make the following observations:

i) $U\Delta(U)^*\Delta(U) \subset U, \ \Delta(U)^*\Delta(U)U \subset U$.

Proof

Let $T \in U, M, N \in \Delta(U)$. Then for every $P \in S_{1,\phi}$ we have

$\phi(P)^\perp T M^* N P = \phi(P)^\perp T M^* \phi(P) N = \phi(P)^\perp T P M^* N = 0 M^* N = 0$.

Thus $T M^* N \in U$. Similarly we have that $M N^* T \in U$.

ii) $U_0\Delta(U)^*\Delta(U) \subset U_0, \ \Delta(U)\Delta(U)^*U_0 \subset U_0$.

Proof

Let $T \in U, M, N \in \Delta(U)$. Then for every $P \in S_{1,\phi}$ we have

$\phi(P)TP^\perp M^* N = \phi(P)TP M^* \phi(P)^\perp N = \phi(P)TP M^* N P^\perp$.

It follows by (i) that $T M^* N \in U$ so $\phi(P)TP^\perp M^* N \in U_0$.

Taking the $w^*$ closed linear span we get $S M^* N \in U_0$ for all $S \in U_0, M, N \in \Delta(U)$. Similarly we have that $\Delta(U)\Delta(U)^*U_0 \subset U_0$.

iii) The space $U_0 \cap \Delta(U)$ is a TRO ideal of $\Delta(U)$.

Proof

Since $\Delta(U)$ is a TRO $(U_0 \cap \Delta(U))\Delta(U)^*\Delta(U) \subset \Delta(U)$. □
Using observation \((ii)\) we have that \((U_0 \cap \Delta(U))\Delta(U)^* \Delta(U) \subset U_0\).

It follows that \((U_0 \cap \Delta(U))\Delta(U)^* \Delta(U) \subset U_0 \cap \Delta(U)\).

Analogously we get \(\Delta(U)\Delta(U)^*(U_0 \cap \Delta(U)) \subset U_0 \cap \Delta(U)\).

We conclude that the space \(U_0 \cap \Delta(U)\) is a TRO ideal of \(\Delta(U)\).

So there exist projections \(Q_i \in D_i, i = 1, 2\), such that \(U_0 \cap \Delta(U) = \Delta(U)Q_1 = Q_2\Delta(U)\) (Remark 2.1).

By Theorem 3.1 we have

\[ U = U_0 + \Delta(U) = U_0 + \Delta(U)Q_1 + \Delta(U)(I - Q_1) = U_0 + \Delta(U)(I - Q_1). \]

Clearly \(U_0 \cap \Delta(U)(I - Q_1) = 0\).

Similarly one shows that \(U = U_0 \oplus (I - Q_2)\Delta(U)\) and it therefore follows that \(U = U_0 \oplus (I - Q_2)\Delta(U)(I - Q_1)\). □

**Remark 3.4** The projection \(\theta : U \to U\) onto \((I - Q_2)\Delta(U)(I - Q_1)\) defined by the decomposition in Theorem 3.1 is a contraction.

Indeed, if \(T \in U\), as in Theorem 3.1 we have \(T = M + S\) where \(M \in \Delta(U), S \in U_0\) and \(\|M\| \leq \|T\|\) (see the proof).

Since \(\theta(T) = (I - Q_2)M(I - Q_1)\), we obtain \(\|\theta(T)\| \leq \|T\|\).

Let \(N_i = \text{Alg}(S_i, \phi) = \{T : P^\perp TP = 0\text{ for all } P \in S_{i, \phi}\}, i = 1, 2, \) and \(L_i = [P^\perp TP^\perp : T \in N_i, P \in S_{i, \phi}]^{-w^*}, i = 1, 2.\)

**Lemma 3.5** i) \(A_2\Delta(U)A_1 \subset \Delta(U)\).

ii) \(\Delta(U)^*A_2\Delta(U) \subset A_1, \Delta(U)A_1\Delta(U)^* \subset A_2.\)

iii) \(U = N_2U N_1.\)

iv) \(U_0 = N_2 U_0 N_1.\)

v) \(U L_1 \subset U_0, L_2U \subset U_0.\)

vi) \(\Delta(U)^*U \subset N_1, U \Delta(U)^* \subset N_2.\)

vii) \(\Delta(U)^*U_0 \subset L_1, U_0 \Delta(U)^* \subset L_2.\)

**Proof**

Claims \(i, (ii)\) are obvious and \(iii)\) is Lemma 1.1 in [9].

iv) If \(N_1 \in N_1, N_2 \in N_2, T \in U\) and \(P \in S_{i, \phi}\) then

\[ N_2\phi(P)TP^\perp N_1 = \phi(P)N_2\phi(P)TP^\perp N_1P^\perp \in U_0. \]
since $N_2\phi(P)TP_\perp N_1 \in U$ by (iii). Taking the $w^*$ closed linear span we get $N_2U_0\phi N_1 \subset U_0$.  

v) If $N_1 \in N_1\setminus T \in U$ and $P \in S_{1,\phi}$ then 
\[ TPN_1 P_\perp = \phi(P)TP_\perp N_1 P_\perp \in U_0 \]
since $TPN_1 \in UN_1 \subset U_0$. Taking the $w^*$ closed linear span we get $TK \in U_0$ for every $K \in L_1$. The second inclusion follows by symmetry.

vi) Let $M \in \Delta(U), T \in U, P \in S_{1,\phi}$. Then $PM^*TP = M^*\phi(P)TP = M^*TP$ so $M^*T \in N_1$.

Similarly one shows that $TM^* \in N_2$.

vii) Let $M \in \Delta(U), T \in U, P \in S_{1,\phi}$ then $M^*\phi(P)TP_\perp = PM^*TP_\perp \in L_1$
since $M^*T \in \Delta(U)^*U \subset N_1$. Taking the $w^*$ closed linear span we get $M^*S \in L_1$ for every $S \in U_0$.

Similarly one shows that $U_0\Delta(U)^* \subset L_2$.  

□

**Proposition 3.6** The following are equivalent:

i) $U = U_0$.

ii) $\Delta(U)^*\Delta(U) \subset L_1 \cap A_1$.

iii) $\Delta(U)\Delta(U)^* \subset L_2 \cap A_2$.

**Proof**

If $U = U_0$ then $\Delta(U) \subset U_0$, hence $\Delta(U)^*\Delta(U) \subset \Delta(U)^*U_0 \subset L_1$ by the previous lemma.

Since $\Delta(U)^*\Delta(U) \subset A_1$ we get $\Delta(U)^*\Delta(U) \subset L_1 \cap A_1$.

If conversely $\Delta(U)^*\Delta(U) \subset L_1 \cap A_1$, then $\Delta(U)^*\Delta(U)(I - Q_1) \subset L_1 \cap A_1$, so by the previous lemma $\Delta(U)^*\Delta(U)(I - Q_1) \subset UC_1 \subset U_0$.

(Q_1 is the projection in Theorem 3.3).

Since $U_0 \cap \Delta(U)$ is a TRO ideal of $\Delta(U)$ (Theorem 3.3) we have that $\Delta(U)^*\Delta(U)^*\Delta(U)Q_1 = \Delta(U)^*\Delta(U)^*(\Delta(U) \cap U_0) \subset \Delta(U) \cap U_0 \subset U_0$.

We conclude that $\Delta(U)^*\Delta(U)^* \subset U_0$.

Since $\Delta(U)$ is a TRO its subspace $\Delta(U)^*\Delta(U)$ is norm-dense [4]. Therefore $\Delta(U) \subset U_0$ and so $U = U_0$.

The equivalence $(i) \Leftrightarrow (iii)$ is proved similarly.  

□

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Proposition 3.7 The following are equivalent:

i) \( U = U_0 \oplus \Delta(U) \).

ii) \( \Delta(U) (L_1 \cap A_1) = 0 \).

iii) \( (L_2 \cap A_2) \Delta(U) = 0 \).

Proof

Note by Lemma 3.5 that \( \Delta(U)(L_1 \cap A_1) \subset \Delta(U) A_1 \subset \Delta(U) \) and \( \Delta(U)(L_1 \cap A_1) \subset U L_1 \subset U_0 \).

Thus if the sum \( U = U_0 + \Delta(U) \) is direct then \( \Delta(U) (L_1 \cap A_1) = 0 \).

Suppose conversely that \( \Delta(U)(L_1 \cap A_1) = 0 \).

Using again Lemma 3.5 we have that \( (U_0 \cap \Delta(U))^* (U_0 \cap \Delta(U)) \subset \Delta(U)^* \Delta(U) \subset A_1 \) and \( (U_0 \cap \Delta(U))^* (U_0 \cap \Delta(U)) \subset \Delta(U) \Delta(U) \subset U_0 \).

But since \( U_0 \cap \Delta(U) \) is a TRO (Theorem 3.3), its subspace \( (U_0 \cap \Delta(U))^* (U_0 \cap \Delta(U)) \) is norm-dense \([4]\). Therefore \( U_0 \cap \Delta(U) = 0 \).

This shows that (i) and (ii) are equivalent.

The proof of the equivalence of (i) and (iii) is analogous. \( \square \)

4 The diagonal

Let \( U, U_0, \Delta(U), \phi \) be as in section 3 and \( \chi = \text{Map}(\Delta(U)) \).

Theorem 4.1 There exists a partial isometry \( V \in \Delta(U) \) such that \( \Delta(U) = [A_2 V A_1]^{-w^*} \) (recall that \( A_i = (S_i, \phi)' \)).

Proof

If \( T \in \Delta(U) \) and \( T = U|T| \) is the polar decomposition of \( T \), then \( U \in \Delta(U) \) and \( |T| \in A_i \) : Proposition 2.6 in \([3]\).

By Zorn’s lemma there exists a maximal family of partial isometries \( (V_n) \subset \Delta(U) \) such that: \( V_n^* V_n \perp V_m^* V_m, V_n V_m^* \perp V_m V_n^* \) for \( n \neq m \).

Let \( V = \sum_{n=1}^{\infty} V_n \). Then \( V \) is a partial isometry in \( \Delta(U) \).
First we show that
\[ \Delta(\mathcal{U}) = \{ T \in B(H_1, H_2) : E \in \mathcal{P}(\mathcal{A}'_1), F \in \mathcal{P}(\mathcal{A}'_2), FVE = 0 \Rightarrow FTE = 0 \} \] (4.1)

Let \( T \) be such that, if \( FVE = 0 \) for \( E \in \mathcal{P}(\mathcal{A}'_1) \) and \( F \in \mathcal{P}(\mathcal{A}'_2) \), then \( FTE = 0 \). Since \( \phi(P) \perp VP = \phi(P) \perp = 0 \) for every \( P \in S_{1,\phi} \) and \( S_{i,\phi} \subset \mathcal{A}'_i, i = 1, 2 \) we have \( \phi(P)^\perp TP = \phi(P)^\perp = 0 \) for every \( P \in S_{1,\phi} \) so \( T \in \Delta(\mathcal{U}) \).

For the converse let \( T \in \Delta(\mathcal{U}) \) and \( T = U|T| \) be the polar decomposition of \( T \).

If \( E \in \mathcal{P}(\mathcal{A}'_1), F \in \mathcal{P}(\mathcal{A}'_2) \) are such that \( FVE = 0 \), since \( |T| \in \mathcal{A}_1 \), we have \( FTE = FU|T|E = FUE|T| \).

Hence it suffices to show that \( FUE = 0 \).

We observe that:
\[
V^*V(FUE)^*FUE = (V^*V)EU^*FUE = E(V^*V)U^*FUE \quad (V^*V \in \mathcal{A}_1)
\]
\[
= EV^*(VU^*)FUE = EV^*F(VU^*)UE \quad (VU^* \in \mathcal{A}_2)
\]
\[
= 0VU^*UE = 0
\]

hence
\[
(FUE)^*FUE \leq I - V^*V. \tag{4.2}
\]

Similarly, one shows that
\[
FUE(FUE)^* \leq I - VV^*. \tag{4.3}
\]

Since \( FUE \) is a partial isometry in \( \Delta(\mathcal{U}) \), the maximality of \( V \) and (4.2),(4.3) imply that \( FUE = 0 \).

Thus claim (4.1) holds.

Let \( \mathcal{M} = \mathcal{A}_2V\mathcal{A}_1 \) \( w^* \).

We observe that \( \mathcal{M} \) is a TRO which is contained in \( \Delta(\mathcal{U}) \). Since \( \mathcal{M} \) is \( w^* \) closed, it is reflexive.

If \( \zeta = \text{Map}(\mathcal{M}) \) then for every projection \( P \),
\[
\zeta(P) = [A_2V\mathcal{A}_1Py : A_i \in \mathcal{A}_i, i = 1, 2, y \in H_1]^{-}. \]
We observe that $\zeta(P) \in \mathcal{A}_2'$ for every projection $P$ so $S_{2,\zeta} \subset \mathcal{A}_2'$. Similarly if $\zeta^* = \text{Map}(\mathcal{M}^*)$ then $S_{2,\zeta^*} \subset \mathcal{A}_1'$ but $S_{1,\zeta} = \{P^\perp : P \in S_{2,\zeta^*}\}$ so we have that $S_{1,\zeta} \subset \mathcal{A}_1'$.

Now since $V \in \mathcal{M}$ we conclude that $\zeta(P)^\perp VP = 0$ for every $P \in S_{1,\zeta}$.

From claim (4.1) we obtain $\zeta(P)^\perp \Delta(U)P = 0$ for every $P \in S_{1,\zeta}$, so since $\mathcal{M}$ is reflexive $\Delta(U) \subset \mathcal{M}$. □

By the previous theorem it follows that if $\mathcal{M}$ is a $w^*$-closed TRO masa bimodule and $\zeta = \text{Map}(\mathcal{M})$ then there exists a partial isometry $V \in \mathcal{M}$ so that $\mathcal{M} = [(S_{2,\zeta})/V(S_{1,\zeta})]'^{-w^*}$.

But we shall prove a stronger result:

**Theorem 4.2** Let $\mathcal{M}$ a $w^*$-closed TRO masa bimodule and $\mathcal{B}_1 = [\mathcal{M}^*,\mathcal{M}]^{-w^*} \mathcal{B}_2 = [\mathcal{M}\mathcal{M}^*]^{-w^*}$. Then there exists a partial isometry $V$ such that $\mathcal{M} = [\mathcal{B}_2V\mathcal{B}_1]^{-w^*}$.

**Proof**

Let $\mathcal{D}_i \subset B(H_i), i = 1, 2$ be masas such that $\mathcal{D}_2\mathcal{M}\mathcal{D}_1 \subset \mathcal{M}$ and put $\zeta = \text{Map}(\mathcal{M})$.

We shall prove that $\mathcal{B}_2'\mathcal{M}\mathcal{B}_1' \subset \mathcal{M}$.

In [9], Theorem 2.10 it is shown that

$$\mathcal{B}_2' = (\mathcal{M}\mathcal{M}^*)' \subset \mathcal{D}_2|_{\zeta(I)} \oplus B(\zeta(I)^\perp(H_2))$$

and

$$\mathcal{B}_1' = (\mathcal{M}^*\mathcal{M})' \subset \mathcal{D}_1|_{\zeta^*(I)} \oplus B(\zeta^*(I)^\perp(H_1)).$$

So it suffices to show that

$$(\mathcal{D}_2|_{\zeta(I)} \oplus B(\zeta(I)^\perp(H_2))) \mathcal{M} (\mathcal{D}_1|_{\zeta^*(I)} \oplus B(\zeta^*(I)^\perp(H_1))) \subset \mathcal{M}.$$ 

But this is true because $\mathcal{D}_2\mathcal{M}\mathcal{D}_1 \subset \mathcal{M}, \zeta(I) \in \mathcal{D}_2, \zeta^*(I) \in \mathcal{D}_1$ and $\mathcal{M} = \zeta(I)\mathcal{M}\zeta^*(I)$.

Now, we shall follow the proof of the previous theorem:

By Zorn’s lemma there exists a maximal family of partial isometries $(V_n) \subset \mathcal{M}$ such that: $V_n^*V_n \perp V_m^*V_m, V_nV_m^* \perp V_mV_n^*$ for $n \neq m$.

Let $V = \sum_{n=1}^\infty V_n$. Then $V$ is a partial isometry in $\mathcal{M}$.

We shall show that

$$\mathcal{M} \subset \{T \in B(H_1, H_2) : E \in \mathcal{P}(\mathcal{B}_1'), F \in \mathcal{P}(\mathcal{B}_2'), FVE = 0 \Rightarrow FTE = 0\}$$

(4.4)
Let $T \in \mathcal{M}$ and $T = U|T|$ be the polar decomposition of $T$. Then $|T| \in (\mathcal{M}^*\mathcal{M})''$ and $U \in \mathcal{M}$, (Proposition 2.6 in [9]). If $E \in \mathcal{P}(\mathcal{B}_1')$, $F \in \mathcal{P}(\mathcal{B}_2')$ are such that $FVE = 0$, since $|T| \in (\mathcal{M}^*\mathcal{M})''$ and $E \in \mathcal{B}_1' = (\mathcal{M}^*\mathcal{M})'$, we have $FTE = FU|T|E = FUE|T|$. Hence it suffices to show that $FUE = 0$.

As in the proof of the previous theorem we have that $V^*V \perp (FUE)^*(FUE)$ and $VV^* \perp (FUE)(FUE)^*$. But $FUE \in \mathcal{B}_2'M\mathcal{B}_1' \subset \mathcal{M}$, so by the maximality of $V$ we have that $FUE = 0$.

Let $W = [\mathcal{B}_2VM\mathcal{B}_1]^{-w}$. We observe that $W \subset \mathcal{M}$. For the converse, we follow the proof of the previous theorem and we use the relation (4.4) \( \Box \)

An alternative proof of the previous theorem was communicated to us by I. Todorov, based on his paper [14].

**Theorem 4.3** The semilattices of $\Delta(U)$ are the following:

\[ S_{1,\chi} = \chi^*(I)^\perp \oplus \chi^*(I)\mathcal{P}((S_{1,\phi})'') \]

\[ S_{2,\chi} = \chi(I)\mathcal{P}((S_{2,\phi})'') \]

The map $\chi : S_{1,\chi} \longrightarrow S_{2,\chi}$ is such that

\[ \chi(\chi^*(I)^\perp \oplus \chi^*(I)Q) = \chi(I)\phi(Q) \quad \text{for every } Q \in S_{1,\phi}. \quad (4.5) \]

**Proof**

i) In Theorem 4.1 we showed that there exists a partial isometry $V$ in $\Delta(U)$ such that $\Delta(U) = [(S_{2,\phi})'V(S_{1,\phi})]'^{-w}$. So if $P \in S_{1,\chi}$ then $\chi(P)$ is the projection onto $[(S_{2,\phi})'V(S_{1,\phi})'P(H_1)]^-$. We conclude that $\chi(P) \in (S_{2,\phi})''$. Hence $S_{2,\chi} \subset (S_{2,\phi})''$.

If $H$ is a Hilbert space, $\mathcal{B}$ is a subset of $\mathcal{B}(H)$ and $Q$ a projection in $\mathcal{B}'$ the set $\{T|_{Q(H)} : T \in \mathcal{B}\}$ is denoted by $\mathcal{B}|_{Q}$.

We have shown that $(S_{2,\chi})''|_{\chi(I)} \subset (S_{2,\phi})''|_{\chi(I)}$.

Let $P \in S_{1,\phi}$ then $\Delta(U)P = \phi(P)\Delta(U)$. Hence $\chi(P) = \phi(P)\chi(I)$. So $\chi(I)S_{2,\phi} \subset S_{2,\chi}$ hence, $(S_{2,\phi})''|_{\chi(I)} \subset (S_{2,\chi})''|_{\chi(I)}$.
We proved that

\[(S_2,\phi)|_{\chi(I)} = (S_2,\chi)|_{\chi(I)}.\]

Since \(\Delta(U)\) is a TRO, using Theorem 2.10 in [9] (see the proof) we have that

\[S_2,\chi|_{\chi(I)} = \mathcal{P}((S_2,\phi)|_{\chi(I)}).\]

It follows that

\[S_2,\chi = \chi(I)\mathcal{P}((S_2,\phi)).\]

Applying this to \(\Delta(U)^* = \Delta(U^*)\),

\[S_2,\chi^* = \chi^*(I)\mathcal{P}((S_2,\phi)).\]

Since \(S_{1,\phi} = \{Q^\perp : Q \in S_{2,\phi}\}\), see the introduction, we have that

\[S_2,\chi^* = \chi^*(I)\mathcal{P}((S_{1,\phi})).\]

But

\[S_{1,\chi} = \{Q^\perp : Q \in S_{2,\phi}\} = \{(\chi^*(I)Q)^\perp : Q \in \mathcal{P}((S_{1,\phi}))\}\]

\[= \{\chi^*(I)^\perp \oplus \chi^*(I)Q : Q \in \mathcal{P}((S_{1,\phi}))\}.\]

ii) If \(Q \in S_{1,\phi}\) then

\[\chi(\chi^*(I)^\perp \oplus \chi^*(I)Q) = \chi(\chi^*(I)Q) \quad (\chi(\chi^*(I)^\perp) = 0)\]

\[= \chi(Q) \quad (\Delta(U)\chi^*(I) = \Delta(U))\]

\[= \phi(Q)\chi(I) \quad (\Delta(U)Q = \phi(Q)\Delta(U))\]

\[\square\]

**Remark 4.4** The smallest ortholattice containing the commutative family \(\chi(I)S_{2,\phi}\) is easily seen to be \(\chi(I)\mathcal{P}((S_{2,\phi}))\), which equals \(S_2,\chi\); similarly the family \(\chi^*(I)^\perp \oplus \chi^*(I)S_{1,\phi}\) generates the complete ortho-lattice \(S_{1,\chi}\).

Therefore, since \(\chi|_{S_{1,\chi}}\) is a complete ortho-lattice isomorphism (Theorem 2.10 in [9]) equality (4.5) determines the map \(\chi\).
Proposition 4.5  The families $\chi^*(I)S_{1,\phi}$ and $\chi(I)S_{2,\phi}$ are complete lattices and the map
\[
\vartheta : \chi^*(I)S_{1,\phi} \to \chi(I)S_{2,\phi} : \vartheta(\chi^*(I)P) = \chi(I)\phi(P)
\]
is a complete lattice isomorphism.

Proof

We use Theorem 4.3 and the fact 9 that the map $\chi|_{S_{1,\chi}}$ is a complete ortholattice isomorphism.

Let $(P_i)_{i \in I} \subset S_{1,\phi}$. We claim that
\[
\wedge_{i \in I} \chi(I)\phi(P_i) = \chi(I)\phi(\wedge_{i \in I} P_i).
\] (4.6)

Indeed, by (4.5),
\[
\wedge_{i \in I} \chi(I)\phi(P_i) = \wedge_{i \in I} \chi(\chi^*(I) \downarrow \chi^*(I)P_i)
= \chi(\wedge_{i \in I} (\chi^*(I) \downarrow \chi^*(I)P_i)) = \chi(\chi^*(I) \downarrow \chi^*(I)(\wedge_{i \in I} P_i)).
\]

Since $\wedge_{i \in I} P_i \in S_{1,\phi}$ we get that $\chi(\chi^*(I) \downarrow \chi^*(I)(\wedge_{i \in I} P_i)) = \chi(I)\phi(\wedge_{i \in I} P_i)$ again using (4.5).

By (1.1), there exist $(Q_i)_{i \in I} \subset S_{1,\phi^*}$ such that $\phi^*(Q_i) = P_i$ for every $i \in I$.

We shall prove that
\[
\vee_{i \in I} \chi^*(I)P_i = \chi^*(I)(\phi^*(\wedge_{i \in I} Q_i)) \downarrow.
\] (4.7)

Since $\Delta(U^*) = \Delta(U)^*$ we have that $\chi^* = Map(\Delta(U^*))$ and so applying equation (4.6) to $\chi^*$ we have that
\[
\wedge_{i \in I} \chi^*(I)\phi^*(Q_i) = \chi^*(I)\phi^*(\wedge_{i \in I} Q_i) \Rightarrow
\vee_{i \in I} (\chi^*(I)\phi^*(Q_i)) \downarrow = (\chi^*(I)\phi^*(\wedge_{i \in I} Q_i)) \downarrow \Rightarrow
\vee_{i \in I} (\chi^*(I) \downarrow \chi^*(I)(\phi^*(\wedge_{i \in I} Q_i)) \downarrow = \chi^*(I) \downarrow \chi^*(I)(\phi^*(\wedge_{i \in I} Q_i)) \downarrow \Rightarrow
\vee_{i \in I} \chi^*(I)P_i = \chi^*(I)(\phi^*(\wedge_{i \in I} Q_i)) \downarrow.
\]

From equalities (4.6) and (4.7) we conclude that the families $\chi^*(I)S_{1,\phi^*}$, $\chi(I)S_{2,\phi}$ are complete lattices.
Since \( \chi(\chi^*(I)^\perp \oplus \chi^*(I)Q) = \chi(I)\phi(Q) \) for every \( Q \in \mathcal{S}_{1,\phi} \) and \( \chi|_{\mathcal{S}_{1,\chi}} \) is 1−1 the map \( \vartheta \) is a bijection.

It remains to show that \( \vartheta \) is sup and inf continuous.

Let \((P_i)_{i \in I} \subseteq \mathcal{S}_{1,\phi}\) and \((Q_i)_{i \in I} \subseteq \mathcal{S}_{1,\phi}^\perp\) be such that \( \phi^*(Q_i)^\perp = P_i \), equivalently by equation (1.1) \( \phi(P_i)^\perp = Q_i \) for every \( i \in I \).

Then, since \( \land_{i \in I} P_i \subseteq \mathcal{S}_{1,\phi} \), by the definition of \( \vartheta \) we have

\[
\vartheta(\land_{i \in I} \chi^*(I)P_i) = \vartheta(\chi^*(I)(\land_{i \in I} P_i)) = \chi(I)\phi(\land_{i \in I} P_i) = \land_{i \in I} \chi(I)\phi(P_i) = \land_{i \in I} \vartheta(\chi^*(I)P_i).
\]

Using equations (4.7) and (1.1) we have that

\[
\vartheta(\lor_{i \in I} \chi^*(I)P_i) = \vartheta(\chi^*(I)(\phi^*(\land_{i \in I} Q_i))^\perp) = \chi(I)\phi((\phi^*(\land_{i \in I} Q_i))^\perp) = \chi(I)(\land_{i \in I} Q_i)^\perp = \lor_{i \in I} \chi(I)\phi(P_i) = \lor_{i \in I} \vartheta(\chi^*(I)P_i). \quad \square
\]

5 The space \( \mathcal{U}_0 \) is reflexive.

Let \( \mathcal{U}, \mathcal{U}_0, \Delta(\mathcal{U}), \phi \) be as in section 3 and \( \chi = \text{Map}(\Delta(\mathcal{U})), \psi = \text{Map}(\mathcal{U}_0) \).

**Lemma 5.1** If \( \Delta(\mathcal{U}) \) is essential, i.e. \( \chi(I) = I, \chi^*(I) = I \), then \( \mathcal{S}_{1,\psi} \subseteq \mathcal{S}_{1,\phi} \) and \( \mathcal{S}_{2,\psi} \subseteq \mathcal{S}_{2,\phi} \).

**Proof**

Since \( \chi(I) = I \) we have \( \phi(I) = I \), so by Proposition 4.5, \( \mathcal{S}_{2,\phi} \) is a C.S.L. Since \( \chi^*(I) = I \), \( \mathcal{S}_{2,\phi}^\perp \) is a C.S.L. and so \( \mathcal{S}_{1,\phi} \) is a C.S.L.

If \( E \) is a projection, then \( \text{Alg}(\mathcal{S}_{2,\phi})\mathcal{U}_0 E \subseteq \mathcal{U}_0 E \) (Lemma 3.5).

It follows that \( \psi(E)^\perp \text{Alg}(\mathcal{S}_{2,\phi})\psi(E) = 0 \). Hence \( \psi(E) \in \text{Lat}(\text{Alg}(\mathcal{S}_{2,\phi})) \).

Since commutative subspace lattices are reflexive \([\mathbb{I}]\), it follows that \( \psi(E) \in \mathcal{S}_{2,\psi} \). We get that \( \mathcal{S}_{2,\psi} \subseteq \mathcal{S}_{2,\phi} \).

Analogously \( \mathcal{U}_0 \text{Alg}(\mathcal{S}_{1,\phi}) \subseteq \text{Alg}({\mathcal{S}_{1,\phi}}) \) so \( \text{Alg}(\mathcal{S}_{1,\phi}^\perp)\mathcal{U}_0 \subseteq \mathcal{U}_0^\perp \). As above we obtain \( \mathcal{S}_{2,\phi}^\perp \subseteq \mathcal{S}_{1,\phi}^\perp \) hence \( \mathcal{S}_{1,\psi} \subseteq \mathcal{S}_{1,\phi} \). \( \square \)

**Theorem 5.2** The space \( \mathcal{U}_0 \) is reflexive

**Proof**
Firstly, we suppose that $\Delta(\mathcal{U})$ is essential ($\chi(I) = I, \chi^*(I) = I$). Now, by Theorem 4.3 we have that $\mathcal{S}_{1,\chi} = \mathcal{P}((\mathcal{S}_{1,\phi})^\prime\prime), \mathcal{S}_{2,\chi} = \mathcal{P}((\mathcal{S}_{2,\phi})^\prime\prime)$ and $\chi|_{\mathcal{S}_{1,\phi}} = \phi$.

If $E \in \mathcal{S}_{1,\phi}$, then $\phi(E), \psi(E) \in \mathcal{P}((\mathcal{S}_{2,\phi})^\prime\prime)$ so there exists a unique $F \in \mathcal{P}((\mathcal{S}_{1,\phi})^\prime\prime)$ such that $\chi(F) = \phi(E) - \psi(E)$.

We observe that $\chi(F) \leq \phi(E) = \chi(E)$. Since $\chi$ is a lattice isomorphism $F \leq E$ and so $\psi(F) \leq \psi(E)$; therefore $\chi(F) \perp \psi(F)$.

Since $\chi = \text{Map}(\Delta(\mathcal{U}))$ and $\psi = \text{Map}(\mathcal{U}_0)$ we obtain that $\Delta(\mathcal{U})F(H_1) \perp \text{Ref}(\mathcal{U}_0)F(H_1)$ and so $\Delta(\mathcal{U})F \cap \text{Ref}(\mathcal{U}_0)F = 0$.

By Theorem 3.1 $\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U})$, hence $\mathcal{U}F = \text{Ref}(\mathcal{U}_0)F \oplus \Delta(\mathcal{U})F$ and $\mathcal{U}F = \mathcal{U}_0F \oplus \Delta(\mathcal{U})F$.

It follows that $\mathcal{U}_0F = \text{Ref}(\mathcal{U}_0)F$ and so $\mathcal{U}_0F$ is reflexive.

Let

$$P = \vee\{F \in \mathcal{P}((\mathcal{S}_{1,\phi})^\prime\prime) : \chi(F) = \phi(E) - \psi(E), E \in \mathcal{S}_{1,\phi}\}.$$ 

By the previous arguments the space $\mathcal{U}_0P$ is reflexive.

Since $\chi$ is $\vee$-continuous we have that

$$\chi(P) = \vee\{\phi(E) - \psi(E), E \in \mathcal{S}_{1,\phi}\}.$$

Let $Q = \chi(P)^\perp$ then $Q\phi(E) = Q\psi(E)$ for all $E \in \mathcal{S}_{1,\phi}$. Therefore, it follows that

$$\mathcal{QU} = \{T : Q\phi(E)^\perp TE = 0 \text{ for all } E \in \mathcal{S}_{1,\phi}\} =$$

$$= \{T : Q\psi(E)^\perp TE = 0 \text{ for all } E \in \mathcal{S}_{1,\phi}\}.$$ 

Using the previous lemma ($\mathcal{S}_{1,\psi} \subset \mathcal{S}_{1,\phi}$) we obtain that $\mathcal{QU}$ is contained in the space:

$$\{T : Q\psi(E)^\perp TE = 0 \text{ for all } E \in \mathcal{S}_{1,\psi}\} =$$

$$= Q = \text{Ref}(\mathcal{U}_0) = \text{Ref}(\mathcal{QU}_0) \subset \mathcal{QU}.$$ 

We proved that $\mathcal{QU} = \text{Ref}(\mathcal{QU}_0)$.

Katavolos and Todorov [9] have proved that $\Delta(\mathcal{U}) \subset (\mathcal{U})_{\min}$ where $(\mathcal{U})_{\min}$ is the smallest $w^*$-closed masa bimodule such that $\text{Ref}((\mathcal{U})_{\min}) = \mathcal{U}$.

So $Q\Delta(\mathcal{U}) \subset Q(\mathcal{U})_{\min} = (\mathcal{QU})_{\min}$. But since $\mathcal{U}_0$ is a $w^*$-closed masa bimodule such that $\text{Ref}(\mathcal{QU}_0) = \mathcal{QU}$ it follows that $Q\Delta(\mathcal{U}) \subset \mathcal{QU}_0$.

Now $Q\Delta(\mathcal{U}) = \chi(P)^\perp \Delta(\mathcal{U}) = \Delta(\mathcal{U})P^\perp$, hence $\Delta(\mathcal{U})P^\perp \subset \mathcal{U}_0$. 

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So \( \mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U})P^\perp + \Delta(\mathcal{U})P = \mathcal{U}_0 + \Delta(\mathcal{U})P \) and so \( \mathcal{U}P^\perp = \mathcal{U}_0 P^\perp \).

We conclude that \( \mathcal{U}_0 P^\perp \) is reflexive. Since \( \mathcal{U}_0 P \) is reflexive too, \( \mathcal{U}_0 \) is reflexive.

Now, relax the assumption that \( \Delta(\mathcal{U}) \) is essential.

Let \( \mathcal{W} = \chi(I)\mathcal{U}|_{\chi^*(I)} \). This is a masa bimodule in \( B(\chi^*(I)(H_1), \chi(I)(H_2)) \).

We have that
\[
\mathcal{W} = \{ T : \chi(I)\phi(L)\perp TL|_{\chi^*(I)} = 0 \text{ for all } L \in S_{1,\phi} \}.
\]

By Proposition 4.5 the families \( S_{1,\phi}|_{\chi^*(I)}, S_{2,\phi}|_{\chi(I)} \) are complete lattices and the map \( S_{1,\phi}|_{\chi^*(I)} \to S_{2,\phi}|_{\chi(I)} : P|_{\chi^*(I)} \to \phi(P)|_{\chi(I)} \) is a complete lattice isomorphism.

By the Lifting theorem of J.Erdos \[5\] it follows that the (semi)lattices of \( \mathcal{W} \) are the families \( S_{1,\phi}|_{\chi^*(I)}, S_{2,\phi}|_{\chi(I)} \).

Therefore, \( \mathcal{W}_0 = [\chi(I)\phi(L)\perp TL|_{\chi^*(I)} : T \in \mathcal{W}, L \in S_{1,\phi}]^{-w^*} = \chi(I)\mathcal{U}_0|_{\chi^*(I)} \).

By the proof in the essential case we have that the space \( \chi(I)\mathcal{U}_0\chi^*(I) \) is reflexive.

But \( \chi(I)^\perp \mathcal{U} = \chi(I)^\perp \mathcal{U}_0 \) and \( \mathcal{U}\chi^*(I)^\perp = \mathcal{U}_0\chi^*(I)^\perp \) so the spaces \( \chi(I)^\perp \mathcal{U}_0 \) and \( \mathcal{U}_0\chi^*(I)^\perp \) are reflexive.

Finally the space \( \mathcal{U}_0 \) is reflexive. \( \square \)

For the rest of this section let \( \mathcal{S} \) be a C.S.L. \( \mathcal{U} = Alg(\mathcal{S}), \mathcal{J} = [PTP^\perp : T \in \mathcal{U}, P \in \mathcal{S}]^{-\|\cdot\|}, \text{Rad}(\mathcal{U}) \) be the radical of \( \mathcal{U}, \mathcal{U}_0 = \mathcal{J}^{-w^*}, \psi = \text{Map}(\mathcal{U}_0) \).

It is known that \( \mathcal{J} \subset \text{Rad}(\mathcal{U}) \). The equality \( \mathcal{J} = \text{Rad}(\mathcal{U}) \) is an open problem (Hopenwasser’s conjecture), \[8,3\].

I.Todorov \[13\] has proved that \( \mathcal{J} \) and \( \text{Rad}(\mathcal{U}) \) have the same reflexive hull.

We improve this by showing the next corollary.

**Corollary 5.3** *The spaces \( \mathcal{J} \) and \( \text{Rad}(\mathcal{U}) \) have the same \( w^* \)-closure.*

**Proof**

\[
\mathcal{U}_0 = \mathcal{J}^{-w^*} \subset \text{Rad}(\mathcal{U})^{-w^*} \subset \text{Ref}(\text{Rad}(\mathcal{U})) = \text{Ref}(\mathcal{J}) = \mathcal{U}_0.
\]

\( \square \)

**Corollary 5.4** *\( \text{Rad}(\mathcal{U})^{-w^*} = \{ T : \psi(E)^\perp TE = 0 \text{ for all } E \in \mathcal{S} \} \).*

**Proof**
Rad(\mathcal{U})^{-w^*} = \mathcal{U}_0 = \{T : \psi(E)^\perp T E = 0 \text{ for every projection } E \} \subset \{T : \psi(E)^\perp T E = 0 \text{ for all } E \in \mathcal{S}\}.

Using Lemma 5.1 the last space is contained in the space:
\{T : \psi(E)^\perp T E = 0 \text{ for all } E \in \mathcal{S}_1, \psi\} = \mathcal{U}_0 = \text{Rad}(\mathcal{U})^{-w^*}.

□

Now we are ready to give the form of the decomposition of \mathcal{U} in the case that \mathcal{U} is a C.S.L. algebra:

**Proposition 5.5** Let \( Q = \vee \{E - \psi(E) : E \in \mathcal{S}\} \) then
\[ \mathcal{U} = \text{Rad}(\mathcal{U})^{-w^*} \oplus QS'. \]

**Proof**
We observe that \( Q^\perp E = Q^\perp \psi(E) \) for all \( E \in \mathcal{S} \), so we have:
\[ Q^\perp \mathcal{U} = \{T : Q^\perp E^\perp T E = 0 \text{ for all } E \in \mathcal{S}\} = \{T : Q^\perp \psi(E)^\perp T E = 0 \text{ for all } E \in \mathcal{S}\}. \]

By the previous corollary the last space is the space \( Q^\perp \text{Rad}(\mathcal{U})^{-w^*} \).
So we have that \( Q^\perp \mathcal{S}' \subset Q^\perp \text{Rad}(\mathcal{U})^{-w^*} \subset \text{Rad}(\mathcal{U})^{-w^*} \).
Since \( \mathcal{U} = \text{Rad}(\mathcal{U})^{-w^*} + \mathcal{S}' \) we have \( \mathcal{U} = \text{Rad}(\mathcal{U})^{-w^*} + QS' \).

It suffices to show that \( \text{Rad}(\mathcal{U})^{-w^*} \cap QS' = 0 \).
Let \( E \in \mathcal{S} \) and \( T \in \mathcal{U}_0 \cap (E - \psi(E))\mathcal{S}' \) then \( T = (E - \psi(E))T = \psi(E)^\perp ET = \psi(E)^\perp T E = 0 \), because \( T \in \mathcal{U}_0 \).
If \( T \in \mathcal{U}_0 \cap QS' \Rightarrow (E - \psi(E))T \in \mathcal{U}_0 \cap (E - \psi(E))\mathcal{S}' = 0. \)
So \( (E - \psi(E))T = 0 \) for all \( E \in \mathcal{S} \).
But \( T = (\vee \{E - \psi(E) : E \in \mathcal{S}\})T \). It follows that \( T = 0 \) □

6 Decomposition of compact operators in reflexive masa bimodules

Let \( \mathcal{U}, \mathcal{U}_0, \Delta(\mathcal{U}), \phi, \mathcal{D}_1, \mathcal{D}_2, \mathcal{Q}_1 \) be as in section 3 and \( \chi = \text{Map}(\Delta(\mathcal{U})) \).

We denote by \( \mathcal{K} \) the set of compact operators and by \( C_p \) the set of p-Schatten class operators in \( B(H_1, H_2) \).

**Proposition 6.1** If \( T \in R_1(\mathcal{U}) \), there exist \( L \in R_1(\Delta(\mathcal{U})) \) and \( S \in [R_1(\mathcal{U}_0)]^{-\|\cdot\|_1} \) such that \( T = L + S. \)
Proof

Write $U = \{ X : \phi(P_n)^\perp XP_n = 0 \text{ for all } n \in \mathbb{N} \}$ for an appropriate sequence $(P_n) \subset S_{1,\phi}$ and let $T \in R_1(U)$.

As in the proof of Theorem 3.1

$$T = L_1 + \phi(P_1)TP_1^\perp,$$

where $L_1 = \phi(P_1)TP_1 + \phi(P_1)^\perp TP_1^\perp$.

Since $\phi(P_1)^\perp TP_1 = 0$ and $T$ has rank 1 either $\phi(P_1)^\perp T = 0$ or $TP_1 = 0$, hence either $L_1 = \phi(P_1)TP_1$ or $L_1 = \phi(P_1)^\perp TP_1^\perp$.

$$L_1 = L_2 + \phi(P_2)L_1P_2^\perp,$$

where $L_2 = \phi(P_2)L_1P_2 + \phi(P_2)^\perp L_1P_2^\perp$.

Since $\phi(P_2)^\perp L_1P_2 = 0$, either $L_2 = \phi(P_2)L_1P_2$ or $L_2 = \phi(P_2)^\perp L_1P_2^\perp$. Similarly

$$L_{n-1} = L_n + \phi(P_n)L_{n-1}P_n^\perp,$$

where $L_n = \phi(P_n)L_{n-1}P_n + \phi(P_n)^\perp L_{n-1}P_n^\perp$.

As before, either $L_n = \phi(P_n)L_{n-1}P_n$ or $L_n = \phi(P_n)^\perp L_{n-1}P_n^\perp$ for all $n \in \mathbb{N}$.

We conclude that there exist projections $(Q_n) \subset D_2, (R_n) \subset D_1$ such that

$$L_n = (\bigwedge_{i=1}^n Q_i)T(\bigwedge_{i=1}^n R_i), n \in \mathbb{N}.$$  

We observe that $T = L_n + M_n$ where $M_n = \phi(P_1)TP_1^\perp + \phi(P_2)L_2P_2^\perp + ... + \phi(P_n)L_nP_n^\perp$, $n \in \mathbb{N}$.

Since $\bigwedge_{i=1}^n Q_i \Rightarrow \bigwedge_{i=1}^{\infty} Q_i, \bigwedge_{i=1}^n R_i \Rightarrow \bigwedge_{i=1}^{\infty} R_i$ and $T$ has rank 1

$$L_n \parallel_{\parallel_1} (\bigwedge_{i=1}^{\infty} Q_i)T(\bigwedge_{i=1}^{\infty} R_i) = L, \text{say.}$$

Now $\phi(P_i)^\perp L_n P_i = \phi(P_i)L_n P_i^\perp = 0, i = 1, 2, ...n$ for all $n \in \mathbb{N}$, therefore $\phi(P_i)^\perp LP_i = \phi(P_i)LP_i^\perp = 0$ for all $i \in \mathbb{N}$.

Thus $L \in R_1(\Delta(U))$.

We have $M_n = T - L_n \parallel_2 T - L = S \in [R_1(U_0)]^{\parallel_1}$.

Proposition 6.2 $U_0 \subset (R_1(\Delta(U))^*0$.

Proof

Let $T \in U, P \in S_{1,\phi}, R \in R_1(\Delta(U))$. Then

$$tr(\phi(P)TP^\perp R^*) = tr(T(\phi(P)RP^\perp)^*) = tr(T0) = 0.$$  

Taking the $w^*$ closed linear span we get $tr(SR^*) = 0$ for every $S \in U_0$.  \hfill $\square$
Proposition 6.3  

i) \( R_1(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U})(I - Q_1) \).

ii) \( \Delta(\mathcal{U}) \cap K = [R_1(\Delta(\mathcal{U}))]^{-\| \cdot \|} \subset \Delta(\mathcal{U})(I - Q_1) \).

Proof

Let \( R \in R_1(\Delta(\mathcal{U})) \) then as in Theorem 3.3 \( RQ_1 \in \Delta(\mathcal{U})Q_1 = \mathcal{U}_0 \cap \Delta(\mathcal{U}) \subset \mathcal{U}_0 \).

By the previous proposition we have: \( tr(RQ_1R^*) = 0 \) \( \Rightarrow tr(R^*RQ_1) = 0 \) \( \Rightarrow RQ_1 = 0 \) \( \Rightarrow R = R(I - Q_1) \).

We conclude that \( R_1(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U})(I - Q_1) \).

For part (ii), observe that if \( K \in \Delta(\mathcal{U}) \cap K \) then \( K \) can be approximated in the norm topology by sums of rank 1 operators in \( \Delta(\mathcal{U}) \) : Proposition 3.4 in [9]. \( \square \)

Remark 6.4 We will see below that if \( \mathcal{U} \) is a strongly reflexive masa bimodule then \( [R_1(\Delta(\mathcal{U}))]^{-w^*} = \Delta(\mathcal{U})(I - Q_1) \). This is not true in general.

For example take \( \mathcal{U} \) to be a TRO which is not strongly reflexive. Then \( [R_1(\Delta(\mathcal{U}))]^{-w^*} \) is strictly contained in \( \Delta(\mathcal{U})(I - Q_1) = \mathcal{U} \).

Proposition 6.5 \( \Delta(\mathcal{U}) \subset (R_1(\mathcal{U}_0)^*)^0 \).

Proof

Let \( T \in R_1(\mathcal{U}_0) \). Then as in Proposition 6.1 we have \( T = L + M \) where

\[
L \in R_1(\Delta(\mathcal{U})) \text{ and } M \in [R_1(\phi(P_n)\mathcal{U}P_n^\perp) : n \in \mathbb{N}]^{-\| \cdot \|} \subset \mathcal{U}_0.
\]

So \( L = T - M \in \mathcal{U}_0 \cap R_1(\Delta(\mathcal{U})) \).

Using Proposition 6.3, \( \mathcal{U}_0 \cap R_1(\Delta(\mathcal{U})) \subset \mathcal{U}_0 \cap \Delta(\mathcal{U})(I - Q_1) \) which vanishes by Theorem 3.3 so \( L = 0 \) and hence \( T = M \).

We conclude that

\[
R_1(\mathcal{U}_0) \subset [R_1(\phi(P_n)\mathcal{U}P_n^\perp) : n \in \mathbb{N}]^{-\| \cdot \|}.
\]  \( (6.1) \)

Let \( A \in \Delta(\mathcal{U}) \). We want to show that \( tr(A^*R) = 0 \) for every \( R \in R_1(\mathcal{U}_0) \).

Using (6.1) it suffices to show that \( tr(A^*R) = 0 \) for every \( R \in R_1(\phi(P_n)\mathcal{U}P_n^\perp) \), and \( n \in \mathbb{N} \).
Let $R$ a rank 1 operator such that $R = \phi(P_n)RP_n^\perp$ then
\[
tr(A^*R) = tr(A^*\phi(P_n)RP_n^\perp) = tr(P_n^\perp A^*\phi(P_n)R) = tr((\phi(P_n)AP_n^\perp)^*R) = tr(0R) = 0. \tag*{□}
\]

Let $P \in S_{1,\phi}$. We suppose that $\forall \{\phi(L) : L \in S_{1,\phi}, \phi(L) < \phi(P)\} < \phi(P)$.
Since $S_{2,\phi}$ is join complete there exists $P_0 \in S_{1,\phi}$ such that
\[
\phi(P_0) = \forall \{\phi(L) : L \in S_{1,\phi}, \phi(L) < \phi(P)\}.
\]

We call the projection $P - P_0$ an atom of $\mathcal{U}$ and we denote the projection $\phi(P) - \phi(P_0)$ by $\delta(P - P_0)$.

**Proposition 6.6** Let $F$ be an atom of $\mathcal{U}$.
i) The projection $F$ is minimal in the algebra $(S_{1,\phi})''$.
ii) The projection $\chi(I)\delta(F)$ is minimal in the algebra $\chi(I)(S_{2,\phi})''$.
iii) $\chi(I)\delta(F)B(H_1, H_2)F \subset \Delta(\mathcal{U})$.
iv) $\chi(I)\delta(F)B(H_1, H_2)F \subset \mathcal{U}_0$.

**Proof**
i) Let $P, P_0 \in S_{1,\phi}$ be such that $\phi(P_0) = \forall \{\phi(L) : L \in S_{1,\phi}, \phi(L) < \phi(P)\} < \phi(P)$ and $F = P - P_0$.
If $Q \in S_{1,\phi}$ either $P \leq Q$ or $QP < P$.
If $P \leq Q$ then $QF = F$.
If $QP < P$ then (since $QP \in S_{1,\phi}$ and $\phi$ is 1-1 on $S_{1,\phi}$) $\phi(QP) < \phi(P) \Rightarrow \phi(QP) \leq \phi(P_0) \Rightarrow QP \leq P_0$, so $QF = 0$.

We conclude that $QFB(H_1)F = FFB(H_1)QF$ for all $Q \in S_{1,\phi}$, therefore $F\mathcal{B}(H_1)F \subset (S_{1,\phi})''$, hence $F$ is a minimal projection in $(S_{1,\phi})''$.

ii) Since $P, P_0 \in S_{1,\phi}$ we have that $\phi(P)\Delta(\mathcal{U}) = \Delta(\mathcal{U})P$ and $\phi(P_0)\Delta(\mathcal{U}) = \Delta(\mathcal{U})P_0$ hence
\[
\delta(F)\Delta(\mathcal{U}) = \Delta(\mathcal{U})F \text{ and so } \chi(I)\delta(F) = \chi(F).
\]

Let $Q \in S_{1,\phi}$.
\[
If \quad QF = 0 \text{ then } \chi(I)\delta(F)\phi(Q) = 0. \quad (6.2)
\]
Indeed, \( \delta(F)\Delta(\mathcal{U}) = \Delta(\mathcal{U})F \) so \( \delta(F)\Delta(\mathcal{U})Q = 0 \) so \( \delta(F)\chi(Q) = 0 \) so \( \chi(I)\delta(F)\phi(Q) = 0 \).

If \( FQ = F \) then \( \chi(I)\delta(F)\phi(Q) = \chi(I)\delta(F) \).

(6.3)

Indeed, \( \delta(F)\Delta(\mathcal{U}) = \Delta(\mathcal{U})F \) so \( \delta(F)\Delta(\mathcal{U})Q = \Delta(\mathcal{U})F \) so \( \delta(F)\chi(Q) = \chi(F) \) so \( \chi(I)\delta(F)\phi(Q) = \chi(I)\delta(F) \).

Using equations (6.2), (6.3) as in (i) we have that \( \chi(I)\delta(F) \) is a minimal projection in \( \chi(I)(\mathcal{S}_{2,\phi})'' \).

iii) Let \( T \in B(H_1, H_2) \) and \( Q \in \mathcal{S}_{1,\phi} \).

From equations (6.2), (6.3) it follows that \( \phi(Q)\chi(I)\delta(F)TF = \chi(I)\delta(F)TFQ \),

so \( \chi(I)\delta(F)TF \in \Delta(\mathcal{U}) \).

iv) If \( T \in \mathcal{U} \) then \( \chi(I)^+T \in \mathcal{U}_0 \). Indeed, by Theorem 3.1 there exist \( T_1 \in \mathcal{U}_0 \), \( T_2 \in \Delta(\mathcal{U}) \) so that \( T = T_1 + T_2 \).

But \( T_2 = \chi(I)T_2 \) so \( \chi(I)^+T = \chi(I)^+T_1 \in \mathcal{U}_0 \).

Now it suffices to show that \( \delta(F)B(H_1, H_2)F \subset \mathcal{U} \).

Let \( T \in B(H_1, H_2) \) and \( Q \in \mathcal{S}_{1,\phi} \).

If \( FQ = 0 \) then \( \phi(Q)^+\delta(F)TFQ = 0 \).

If \( FQ = F \) then \( P - P_0 \leq Q \) hence \( \delta(F) = \phi(P) - \phi(P_0) \leq \phi(P - P_0) \leq \phi(Q) \) so \( \phi(Q)^+\delta(F)TFQ = 0 \).

We conclude that \( \delta(F)TF \in \mathcal{U} \). \( \square \)

Remark 6.7 There exists a simple example of a reflexive masa bimodule \( \mathcal{U} \) so that \( \delta(F)B(H_1, H_2)F \subset \mathcal{U}_0 \) for any atom \( F \) in \( \mathcal{U} \).

(Take \( \mathcal{U} \) to be the set of \( 3 \times 3 \) matrixes with zero diagonal.)

This is an example of the different behaviour of algebras and bimodules: it is known that if \( \mathcal{U} \) is a CSL algebra in a Hilbert space \( H \) and \( F \) is an atom in \( \mathcal{U} \) then \( FB(H)F \subset \Delta(\mathcal{U}) \).

We thank Dr. I. Todorov for suggesting the ‘atomic decomposition’ in the theorem below.

Theorem 6.8 Let \( \{F_n : n \in \mathbb{N}\} = \{F : F \text{ atom of } \mathcal{U}\} \). Then

\[
[R_1(\Delta(\mathcal{U}))]^{-w^*} = \sum_{n=1}^{\infty} \oplus \chi(I)\delta(F_n)B(H_1, H_2)F_n.
\]
Proof

By the previous proposition it follows that

\[ [R_1(\Delta(U))]^{-w^*} \supset \sum_{n=1}^{\infty} \oplus \chi(I)\delta(F_n)B(H_1, H_2)F_n. \]

Let \( R = x \otimes y^* \in \Delta(U). \)

For every \( Q \in S_{1,\phi} \) we have that \( x \otimes (Qy)^* = (\phi(Q)x) \otimes y^* \) so \( \phi(Q)x \neq 0 \iff Qy \neq 0 \iff \phi(Q)x = x \iff Qy = y. \)

Let \( P = \wedge\{Q \in S_{1,\phi} : Qy = y\}, \) then \( P \in S_{1,\phi}. \)

If \( Q \in S_{1,\phi} \) so that \( \phi(Q) < \phi(P) \) then \( \phi(Q)x = 0. \)

(If \( \phi(Q)x = x \) then \( Qy = y \) so \( Q \geq P). \)

Let \( P_0 \in S_{1,\phi} \) with \( \phi(P_0) = \vee\{\phi(L) : L \in S_{1,\phi}, \phi(L) < \phi(P)\}. \)

We observe that \( \phi(P_0)x = 0 \) and \( \phi(P)x = x \), hence \( \phi(P_0) < \phi(P). \)

We conclude that \( F = P - P_0 \) is an atom of \( \mathcal{U}. \)

The equalities \( (P - P_0)y = y \) and \( (\phi(P) - \phi(P_0))x = x \) imply that \( R = \delta(F)RF. \) But \( R = \chi(I)R \) so \( R = \chi(I)\delta(F)RF. \)

The proof is complete. \( \square \)

Every strongly reflexive TRO is a masa bimodule \[9\]. So using the previous theorem we have a new proof of the following result in \[9\].

Corollary 6.9 If \( \mathcal{M} \) is a strongly reflexive TRO, \( \zeta = \text{Map}(\mathcal{M}) \) and \( \{A_n : n \in \mathbb{N}\} = \{A : A \text{ atom of } \mathcal{M}\}, \) then

\[ \mathcal{M} = \sum_{n=1}^{\infty} \oplus \zeta(A_n)B(H_1, H_2)A_n. \]

Let \( (P_n) \subset S_{1,\phi} \) be a sequence such that

\[ \mathcal{U} = \{T \in B(H_1, H_2) : (P_n)^{-1}TP_n = 0 \text{ for all } n \in \mathbb{N}\}. \]

Let \( V_n, U_n : B(H_1, H_2) \longrightarrow B(H_1, H_2), n \in \mathbb{N} \) be as in theorem 3.1.

By Theorem 6.8

\[ [R_1(\Delta(\mathcal{U}))]^{-w^*} = \sum_{n=1}^{\infty} \oplus E_nB(H_1, H_2)F_n, \]
where $F_n$ atom of $U$ and $E_n = \chi(I)\delta(F_n)$ for all $n \in \mathbb{N}$.

Thus $[R_1(\Delta(U))]^{-w^*}$ is the range of the contractive projection $D$ defined by

$$D : B(H_1, H_2) \to B(H_1, H_2) : D(T) = \sum_{n=1}^{\infty} E_n TF_n.$$ 

**Proposition 6.10** Let $K \in K$, then the sequence $(U_n(K))$ converges to $D(K)$ in norm.

**Proof**

We observe that $(V_n|_{C_2})$ is a commuting sequence of orthogonal projections in the Hilbert space $C_2$.

Hence $(U_n|_{C_2})$ is a decreasing sequence of orthogonal projections. Therefore if $T \in C_2$ the sequence $(U_n(T))$ converges in the Hilbert-Schmidt norm $\| \cdot \|_2$.

Let $K \in K$. Then for $\varepsilon > 0$ there exist $K_\varepsilon \in C_2$ such that $\|K - K_\varepsilon\| < \frac{\varepsilon}{3}$ and $n_0 \in \mathbb{N}$ such that $\|U_n(K_\varepsilon) - U_m(K_\varepsilon)\|_2 < \frac{\varepsilon}{3}$ for every $n, m \geq n_0$.

Then

$$\|U_n(K) - U_m(K)\|$$

$$\leq \|U_n(K) - U_n(K_\varepsilon)\| + \|U_n(K_\varepsilon) - U_m(K_\varepsilon)\| + \|U_m(K_\varepsilon) - U_m(K)\|$$

$$\leq \|K - K_\varepsilon\| + \|U_n(K_\varepsilon) - U_m(K_\varepsilon)\|_2 + \|K - K_\varepsilon\| < \varepsilon$$

for every $n, m \geq n_0$.

Thus $(U_n(K))$ converges in norm. Let $D_1(K) = \| \cdot \| - \lim U_n(K)$.

Since $\phi(P_i)^*U_n(K)P_i = \phi(P_i)U_n(K)P_i^* = 0$ for every $i = 1, 2, \ldots n$, the limit $D_1(K)$ belongs to the diagonal $\Delta(U)$.

Since $\|U_n(K)\| \leq \|K\|$ for all $n \in \mathbb{N}$, $D_1$ is a contraction.

We observe that if $K \in \Delta(U) \cap K$ then $U_n(K) = K$ for all $n \in \mathbb{N}$ hence $D_1$ projects onto $\Delta(U) \cap K$.

Now $D_1|_{C_2}$ is the orthogonal projection onto $\Delta(U) \cap C_2$, being the infimum of the sequence $(U_n|_{C_2})$.

We can also observe that $D|_{C_2}$ is an orthogonal projection in the Hilbert space $C_2$. 

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If $T \in \Delta(U) \cap C_2$ then by Proposition 6.3 $T = \sum_{n=1}^{\infty} E_n T F_n = D(T)$.

We conclude that $D|_{C_2}$ and $D|_{C_2}$ are both orthogonal projections onto $\Delta(U) \cap C_2$, hence $D|_{C_2} = D_1|_{C_2}$.

Since $C_2$ is norm dense in $\mathcal{K}$ and $D|_{\mathcal{K}}, D_1$ are norm continuous, $D|_{\mathcal{K}} = D_1$. □

**Proposition 6.11** Suppose that $\forall_n F_n = F$. Then the sequence $(U_n(T)F)$ converges strongly to the operator $D(T)$ for every $T \in B(H_1, H_2)$.

**Proof**

First we observe that if $x \in F_m(H_1), m \in \mathbb{N}$, then the operator $x \otimes x^*$ is in $(S_{1, \phi})'$.

Indeed, let $y \in E_m(H_2)$, then $R = y \otimes x^* \in \Delta(U)$. It follows that $R^* R = \|y\|^2 x \otimes x^* \in \Delta(U)^* \Delta(U) \subset (S_{1, \phi})'$.

Let $T \in B(H_1, H_2)$ and $x \in F_m(H_1), m \in \mathbb{N}, \|x\| = 1$.

By Proposition 6.10

$$U_i(T x \otimes x^*) \xrightarrow{\|\|} D(T x \otimes x^*), i \rightarrow \infty,$$

hence

$$U_i(T x \otimes x^*) (x) \xrightarrow{\|\|} D(T x \otimes x^*) (x), i \rightarrow \infty \quad (6.4)$$

$$D(T x \otimes x^*) (x) = \sum_{n=1}^{\infty} E_n(T x \otimes x^*) F_n(x) = E_m T(x) \quad (6.5)$$

$$D(T)(x) = \sum_{n=1}^{\infty} E_n T F_n(x) = E_m T(x) \quad (6.6)$$

We have that

$$V_i(T x \otimes x^*) = \phi(P_i) (T x \otimes x^*) P_i + \phi(P_i)^\perp (T x \otimes x^*) P_i^\perp, i \in \mathbb{N}$$

since $x \otimes x^* \in (S_{1, \phi})'$,

$$V_i(T x \otimes x^*) = (\phi(P_i) T P_i) (x \otimes x^*) + (\phi(P_i)^\perp T P_i^\perp) (x \otimes x^*), i \in \mathbb{N},$$

hence

$$U_i(T x \otimes x^*) = U_i(T) x \otimes x^* \Rightarrow U_i(T x \otimes x^*) (x) = U_i(T)(x), i \in \mathbb{N} \quad (6.7)$$
Using (6.4), (6.5), (6.6), (6.7)

\[ U_i(T)(x) \xrightarrow{\mathcal{H}} D(T)(x), \quad i \to \infty \text{ for all } x \in \bigcup_{m=1}^{\infty} F_m(H_1). \]

Since the \( U_i \) are contractions \( U_i(T)(x) \xrightarrow{\mathcal{H}} D(T)(x), \) for all \( x \in F(H_1) \)

Observe that \( D(T)F = D(T). \)

\[ \square \]

\textbf{Remark 6.12} The sequence \( (U_n(T)) \) has similar properties to the net of finite diagonal sums in the case of nest algebras. (Propositions 6.10, 6.11 are analogous to Propositions 4.3, 4.4 in [2].)

\textbf{Theorem 6.13} Let \( K \in \mathcal{U} \) be compact. Then there exist unique compact operators \( K_1 \in \mathcal{U}_0, K_2 \in \Delta(\mathcal{U}) \) such that \( K = K_1 + K_2. \) Moreover \( K_2 = D(K). \)

\textbf{Proof}\n
Let \( K_2 = D(K) \) and \( K_1 = K - K_2 \) then \( K_1 = \lim(K - U_n(K)) \) (Proposition 6.10).

As in Theorem 3.1 \( K - U_n(K) \in \mathcal{U}_0 \) for all \( n \in \mathbb{N}. \) Hence \( K_1 \in \mathcal{U}_0. \)

The decomposition \( K = K_1 + K_2 \) in \( \mathcal{U}_0 + \Delta(\mathcal{U}) \cap K \) is unique because by Proposition 6.3, \( \Delta(\mathcal{U}) \cap K \subset \Delta(\mathcal{U})(I - Q_1), \) while by Theorem 3.3, \( \mathcal{U} = \mathcal{U}_0 \oplus \Delta(\mathcal{U})(I - Q_1). \)

\[ \square \]

\textbf{Corollary 6.14} Let \( F \in \mathcal{U} \) be a finite rank operator. Then there exist unique finite rank operators \( F_1 \in \mathcal{U}_0, F_2 \in \Delta(\mathcal{U}) \) such that \( F = F_1 + F_2. \) Moreover \( \text{rank} F_2 \leq \text{rank} F \) and \( F_2 = D(F). \)

\textbf{Proof}\n
It can be shown that for each \( n \in \mathbb{N} \) we have \( \text{rank}(U_n(F)) \leq \text{rank}(F). \)

Therefore if \( F_2 = \| \cdot \| -\lim U_n(F) \) then \( \text{rank}(F_2) \leq \text{rank}(F) \) and \( F_2 = D(F). \)

Setting \( F_1 = F - F_2 \) we obtain the desired decomposition.

\[ \square \]

\textbf{Corollary 6.15} Let \( K \in \mathcal{U} \cap C_p, 1 \leq p < \infty. \) Then there exist unique operators \( K_1 \in \mathcal{U}_0 \cap C_p, K_2 \in \Delta(\mathcal{U}) \cap C_p \) such that \( K = K_1 + K_2. \) Moreover \( \| K_2 \|_p \leq \| K \|_p. \)
Proof
As in Theorem 6.13 $K = K_1 + D(K)$ where $K_1 \in U_0$.
We observe that $D(K) \in C_p$ and $\|D(K)\|_p \leq \|K\|_p$. □

7 Decomposition of a strongly reflexive masa bimodule

Let $U$, $U_0$, $\Delta(U)$, $\phi$, $D_1$, $D_2$ be as in section 3 and $\chi = \text{Map}(\Delta(U))$.

We now assume that $U$ is a strongly reflexive masa bimodule.

Proposition 7.1 The space $U_0$ is strongly reflexive.

Proof
Let $T \in U$, $P \in S_1$. Since $U$ is a strongly reflexive masa bimodule there exists a net $(R_i) \subset [R_1(U)]$ such that $R_i \wot T$ : Corollary 2.5 in [3].
So we have that $\phi(P)R_iP^\perp \wot \phi(P)TP^\perp$. Since $(\phi(P)R_iP^\perp) \subset [R_1(U_0)]$ we conclude that $\phi(P)TP^\perp \in [R_1(U_0)]^{-\wot}$.
We proved that $\phi(P)UP^\perp \subset [R_1(U_0)]^{-\wot}$ for all $P \in S_1, \phi$. Hence $U_0 = [R_1(U_0)]^{-\wot}$. □

Remark 7.2 The diagonal of a strongly reflexive masa bimodule is not necessarily strongly reflexive. For example if $U$ is a nonatomic nest algebra, then $\Delta(U)$ does not contain rank 1 operators.

Proposition 7.3 i) $U_0 = U \cap (R_1(\Delta(U))^*)^0$.
   ii) $U_0 \cap \Delta(U) = \Delta(U) \cap (R_1(\Delta(U))^*)^0$.

Proof
By Proposition 6.2 we have $U_0 \subset (R_1(\Delta(U))^*)^0$.
It suffices to show that $U \cap (R_1(\Delta(U))^*)^0 \subset U_0$.

Since $U \cap (R_1(\Delta(U))^*)^0$ is masa bimodule, as in Theorem 3.1 we can decompose it in the next sum:

$$U \cap (R_1(\Delta(U))^*)^0 = U_0 \cap (R_1(\Delta(U))^*)^0 + \Delta(U) \cap (R_1(\Delta(U))^*)^0.$$ 

Now we must prove that $\Delta(U) \cap (R_1(\Delta(U))^*)^0 \subset U_0$. 30
Using Theorem 2.2, there exist projections $P_1 \in D_1, P_2 \in D_2$ such that 

$$[R_1(\Delta(U))]^{-w^*} = P_2 \Delta(U) P_1$$

and 

$$\Delta(U) \cap (R_1(\Delta(U))^*)^0 = P_2^\perp \Delta(U) P_1^\perp.$$

Let $T \in \Delta(U) \cap (R_1(\Delta(U))^*)^0$.

Since $U$ is a strongly reflexive masa bimodule there exists a net $(R_i) \subset [R_1(U)]$ such that $R_i \stackrel{wot}{\to} T$.

By Proposition 6.1 there exist $M_i \in [R_1(\Delta(U))], L_i \in U_0$ such that $R_i = M_i + L_i$.

Thus $M_i + L_i \stackrel{wot}{\to} T$ so $P_2^\perp M_i P_1^\perp + P_2^\perp L_i P_1^\perp \stackrel{wot}{\to} P_2^\perp T P_1^\perp$ and thus $P_2^\perp L_i P_1^\perp \stackrel{wot}{\to} T$. It follows that $T \in U_0$. □

**Theorem 7.4** $U = U_0 \oplus [R_1(\Delta(U))]^{-w^*}$.

**Proof**

By Theorem 2.2,

$$\Delta(U) = \Delta(U) \cap (R_1(\Delta(U))^*)^0 \oplus [R_1(\Delta(U))]^{-w^*}$$

so by Proposition 7.3 $\Delta(U) = U_0 \cap \Delta(U) + [R_1(\Delta(U))]^{-w^*}$.

Since $U = U_0 + \Delta(U)$ we have that $U = U_0 + [R_1(\Delta(U))]^{-w^*}$.

By Proposition 6.3 and Theorem 3.3 the previous sum is direct. □

Propositions 3.6 and 3.7 have the following consequences:

**Corollary 7.5** i) The following are equivalent:

a) $R_1(\Delta(U)) = 0$.

b) $\Delta(U)^* \Delta(U) \subset L_1 \cap A_1$.

c) $\Delta(U) \Delta(U)^* \subset L_2 \cap A_2$.

ii) The following are equivalent:

a) $\Delta(U)$ is strongly reflexive.

b) $\Delta(U) (L_1 \cap A_1) = 0$.

c) $(L_2 \cap A_2) \Delta(U) = 0$.

Theorems 6.8, 7.4 and Corollary 5.3 give the following form of the decomposition of $U$ when it is a strongly reflexive C.S.L. algebra.

**Corollary 7.6** If $S$ is a completely distributive CSL in a Hilbert space $H$ and \{ $A_n : n \in \mathbb{N}$ \} = \{ $A : A$ atom of $S$ \} then:

$$\text{Alg}(S) = \text{Rad}(\text{Alg}(S))^{-w^*} \oplus \sum_{n=1}^{\infty} \oplus A_n B(H) A_n.$$
Recall the notation \( [R_1(\Delta(U))]^{-w^*} = \sum_{n=1}^{\infty} \oplus \chi(I)\delta(F_n)B(H_1, H_2)F_n \), where \( \{F_n : n \in \mathbb{N}\} = \{F : F \text{ atom of } U\} \) and
\[
D : B(H_1, H_2) \rightarrow B(H_1, H_2) : D(T) = \sum_{n=1}^{\infty} \chi(I)\delta(F_n)TF_n.
\]

**Proposition 7.7** Let \( \theta : U \rightarrow U \) be the projection onto \( [R_1(\Delta(U))]^{-w^*} \) defined by the decomposition in Theorem 7.4. Then \( \theta = D|_U \).

**Proof**

Since \( U \) decomposes as the direct sum of the masa bimodules \( U_0 \) and \( [R_1(\Delta(U))]^{-w^*} \), the map \( \theta \) is a masa bimodule map:
\[
\theta(D_2TD_1) = D_2\theta(T)D_1
\]
for every \( T \in U, D_1 \in D_1, D_2 \in D_2 \).

Hence if \( T \in U \):
\[
\theta(T) = \sum_{n=1}^{\infty} \chi(I)\delta(F_n)\theta(T)F_n = \sum_{n=1}^{\infty} \theta(\chi(I)\delta(F_n)TF_n)
\]
\[
= \sum_{n=1}^{\infty} \chi(I)\delta(F_n)TF_n = D(T). \quad \square
\]

**Proposition 7.8** \( U_0 = \{T \in U : \chi(I)\delta(F)TF = 0 \text{ for every atom } F \text{ of } U\} \).

**Proof**

Let \( F \) be an atom of \( U \).
If \( P \in S_{1,\phi} \), as in Proposition 6.6 either \( PF = F \Rightarrow P^\perp F = 0 \) or \( PF = 0 \Rightarrow \chi(I)\delta(F)\phi(P) = 0 \).
So \( \chi(I)\delta(F)\phi(P)TP^\perp F = 0 \) for all \( P \in S_{1,\phi} \) and \( T \in U \), thus \( \chi(I)\delta(F)U_0F = 0 \) for every atom \( F \).
It follows that \( U_0 \subset \{T \in U : \chi(I)\delta(F)TF = 0, \text{ for every atom } F \in U\} \).

For the converse, let \( T \in U : \chi(I)\delta(F)TF = 0 \) for every atom \( F \) in \( U \).
By the previous proposition \( D(T) = 0 \), hence \( T \in U_0 \). \square

It is known that the linear span of the rank 1 operators in a strongly reflexive masa bimodule is wot dense in the module.
This is not true generally for the ultraweak topology \([6]\).

For the previous problem we have the next equivalence in proposition 7.10.

Firstly, we need the following lemma.

**Lemma 7.9** If \(U\) is a reflexive masa bimodule (not necessarily strongly reflexive) then:

\[
[R_1(U)]^{-w^*} = [R_1(U_0)]^{-w^*} \oplus [R_1(\Delta(U))]^{-w^*}.
\]

**Proof**

Since \(R_1(\Delta(U)) \subset \Delta(U)(I - Q_1)\) (Proposition 6.3), by Theorem 3.3 the previous sum is direct.

Clearly

\[
[R_1(U)]^{-w^*} \supset [R_1(U_0)]^{-w^*} \oplus [R_1(\Delta(U))]^{-w^*}.
\]

For the converse, let \(T \in [R_1(U)]^{-w^*}\).

There is a net \((R_i) \subset [R_1(U)]\) with \(R_i \overset{w^*}{\to} T\).

As in Proposition 6.1, we may decompose \(R_i = L_i + M_i\) where \(L_i \in \left[R_1(U_0)\right]^{-\|\cdot\|_1}\) and \(M_i \in \left[R_1(\Delta(U))\right]^{-w^*}\) for all \(i\).

Since \(M_i = D(R_i)\) (Corollary 6.14) and \(D\) is \(w^*\)-continuous, we have \(M_i \overset{w^*}{\to} M \in \left[R_1(\Delta(U))\right]^{-w^*}\).

So \(L_i = R_i - M_i \overset{w^*}{\to} T - M = L \in \left[R_1(U_0)\right]^{-w^*}\).

Thus \(T = L + M \in \left[R_1(U_0)\right]^{-w^*} \oplus \left[R_1(\Delta(U))\right]^{-w^*}.\)

\(\square\)

**Proposition 7.10** If \(U\) is a strongly reflexive masa bimodule, then:

\[U = [R_1(U)]^{-w^*} \iff U_0 = [R_1(U_0)]^{-w^*}.\]

**Proof**

Suppose \(U = [R_1(U)]^{-w^*}\). Then by Theorem 7.4 we have

\[U = U_0 \oplus [R_1(\Delta(U))]^{-w^*}.
\]

It follows from the previous lemma that \(U_0 = [R_1(U_0)]^{-w^*}\).

If conversely \(U_0 = [R_1(U_0)]^{-w^*}\) then again by Theorem 7.4

\[U = U_0 \oplus [R_1(\Delta(U))]^{-w^*} = [R_1(U_0)]^{-w^*} \oplus [R_1(\Delta(U))]^{-w^*} = [R_1(U)]^{-w^*}\]

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by Lemma 7.9. □

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