Computations in the Pre-Bloch group

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Abstract. For compute the five term relations in the pre-Bloch group for specify an infinite-order- element in $K_1(\mathbb{Q}(\sqrt{-m}))$, $m \in \mathbb{N}$ square-free. For the quadratic imaginary number fields $\mathbb{F}$ of discriminant $(-1, -2, -3, -7, -17, -19)$. We use the GAP Programming software to implement our method.

1. Introduction

Let $R$ be an associative ring with unit. The higher algebraic $K$-group of $R$ are defined to be the homotopy groups $K_n(R) := \pi_n(K(R))$ for a space $K(R)$ that is constructed as follows. The infinite general linear group is

$$GL(R) := \bigcup_{n=1}^{\infty} GL_n(R)$$

where the union is formed using the inclusions $GL_n(R) \to GL_{n+1}(R)$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. For a group $G$, the commutator subgroup is $[G, G] := \{ghg^{-1}h^{-1} | g, h \in G\}$, then as an Abelian group, the first $K$-group is $K_1(R) = \frac{GL(R)}{[GL(R), GL(R)]}$.

Any ring map $R \to S$ induces a natural map $GL(R) \to GL(S)$, and hence a map $K_1(R) \to K_1(S)$. Therefore, $K_1$ is a functor from rings to Abelian groups.

**Definition 1.1** If $i \neq j \in \mathbb{N}$ and $r \in \mathbb{R}$, then the elementary matrix $e_{i,j}(r)$ is the matrix in $GL(R)$ which has diagonal entries all 1, $(i, j)$-entry $r$, and 0 elsewhere.

Setting $E_n(R) := \langle e_{i,j}(r) | 1 \leq i, j \leq n, r \in R \rangle$, $E(R) := \bigcup_{n \in \mathbb{N}} E_n(R)$, the subgroup $E(R)$ of elementary matrices in $GL(R)$, equals the commutator : $E(R) = [GL(R), GL(R)]$. Note that for a field $\mathbb{F}$, $E_n(\mathbb{F}) = SL_n(\mathbb{F})$.

1.1. Quillen’s space $BGL(R)^+ = K(R)$

The space $BGL(R)^+ = K(R)$ that we are going to construct is also called Quillen’s “+”-construction of the space $K(R)$ defining algebraic $K$-theory.

For a group $G$, there exist a space $BG$ with $\pi_0(BG) = G$, $\pi_n(BG) = 0$, for all $n \geq 2$. So there is a theoretical construction of $BGL(R)$-the classifying space for group homology.

**Definition 1.2** The notation $BGL(R)^+$ will denote any CW-complex $X$ which has a distinguished map $BGL(R) \to BGL(R)^+$ such that the following are true:
1) $\pi_1BGL(R)^+ \cong K_1(R)$, and the natural map

$$GL(R) = \pi_1BGL(R) \rightarrow \pi_1BGL(R)^+$$

is surjective with kernel $E(R)$.

2) $H_n(BGL(R); M) \cong H_n(BGL(R)^+; M)$ for every $K_1(R)$-module $M$. Such a space $X$ is called a model for $BGL(R)^+$.

**Definition 1.3** An $R$-module $P$ is called projective if there exist an $R$-module $Q$ such that $P \oplus Q$ is free (it has a basis). The set $PR$ of isomorphism classes of finitely generated projective $R$-module, together with direct sum and identity $0$, forms an abelian monoid.

$K_0(R) := (PR)^{-1} PR$ is the Grothendieck group completion.

$K(R)$ is the disjoint union of $|K_0(R)|$ copies of $BGL(R)^+ = K(R) := K_0(R) \times BGL(R)^+ = \sqcup_{P \in K_0(R)} BGL(R)^+$, because $BGL(R)^+$ is a connected space. We recover $K_1(R)$ with the definition $K_n(R) := \pi_n(K(R))$, for all $n \in \mathbb{N}_{\geq 0}$. Note that for all $n \geq 1$, $K_n(R) = \pi_n(BGL(R)^+)$, because in $\sqcup_{P \in K_0(R)} BGL(R)^+$, all connected components are identical, so it does not matter where we place the basepoint.

Now we have a theoretical construction of the higher algebraic $K$-groups, but we do not know yet how any non-trivial element in them looks like.

A theorem of Borel implies that for an imaginary quadratic field $\mathbb{F}$, $K_3(\mathbb{F}) \cong \mathbb{Z} \oplus \mathbb{Z}/\omega_2(\mathbb{F})\mathbb{Z}$ for a natural number $\omega_2(\mathbb{F}) \in \mathbb{N}_{\geq 1}$ which is constructed using Tate twists (we will not go into the details of that construction, because for the present purposes, we are not interested in the torsion).

**Question.** Can we specify an infinite-order element in $K_3(\mathbb{Q}(\sqrt{-m}))$, $m \in \mathbb{N}$ square-free?

For this purpose, we use the Bloch group, and work of de Jen, Gangl, Rahm and Yasaki.

2. **The Bloch group**

For an Abelian group $A$, let $A^2A$ denote the quotient of the group $A \otimes A$ by the subgroup generated by all $a \otimes b + b \otimes a | a, b \in A$.

$$\tilde{A^2} := A \otimes A / \langle a \otimes b + b \otimes a | a, b \in A \rangle .$$

**Definition 2.1** [7] For any field $\mathbb{F}$, the pre-Bloch group $P(\mathbb{F})$ denote the abelian group presented with generator symbols $[x]$ for $x \in \mathbb{F} \setminus \{0\}$ with relations $[1] = [0] = [\infty] = 0$ and for $x \neq y$ in $\mathbb{F} \setminus \{0, 1\}$, the "five-term relations":

$$[x] - [y] + [y/x] - \frac{1 - 1/x}{1 - 1/y} + \frac{1 - x}{1 - y} = 0 .$$

In [4] refer the five term relation is different because of the different definition of the cross-ratio for more details see [3,5]. In addition, in Proposition 2.14 [3] illustrates the equivalence between the two relation of five term relations. If we have $[r] + [r^{-1}] = 0$, $r > 0$ and $[r_1] - [r_2] + [r_1/r_2] - [1 - 1/r_1] + [1 - 1/r_2] = 0$, $s_1 < r_1 < r_2$ and $[r] \neq [\infty, 0, 1]$, $r > 1$, can be translated to defining relation in terms of generators $[s]$, satisfying $[s_1] - [s_2] + [s_1/s_2] - \frac{1 - 1/s_1}{1 - 1/s_2} + \frac{1 - s_1}{1 - s_2} = 0$. Setting $s_1 = \frac{1 - x}{1 - x}$ and $s_2 = \frac{y - xy}{1 - xy}$ in the relation above, we obtain

$$\frac{1 - x}{1 - xy} - \frac{y - xy}{1 - xy} + [y] - [xy] + [x] = 0 .$$

Evidently, $y = \frac{x}{s_1}$ and $x = \frac{1 - x}{1 - s_2}$.

There is a canonical map $\rho(\mathbb{F}) \rightarrow \tilde{A^2} \mathbb{F}$, sending $[1]$ to $0$ and $[x]$ to $x \wedge (1 - x)$ for $x \neq 1$, and Bloch’s group $B(\mathbb{F})$ is define to be its kernel. Thus we have an exact sequence

$$0 \rightarrow B(\mathbb{F}) \rightarrow \rho(\mathbb{F}) \rightarrow \tilde{A^2} \mathbb{F} \rightarrow K_2(\mathbb{F}) \rightarrow 0 .$$
Let $\mathbb{F}(\sqrt{-m})$, $\mu(\mathbb{F}) = \langle$ roots of unity in $\mathbb{F} \rangle$. Then

$$
0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mu(\mathbb{F}) \longrightarrow \mu(\mathbb{F}) \longrightarrow 0
$$

shall be chosen the non-trivial extension. Note that $\mu(\mathbb{Q}(\sqrt{-1})) = \langle i \rangle$, $\mu(\mathbb{Q}(\sqrt{-3})) = \langle \xi \rangle$ and $\mu(\mathbb{F}) = \langle -1 \rangle$ for all other imaginary quadratic fields.

**Theorem 2.2** (Suslin). There is an exact sequence

$$
0 \longrightarrow \tilde{\mu}(\mathbb{F}) \longrightarrow K^\text{ind}_3(\mathbb{F}) \longrightarrow B(\mathbb{F}) \longrightarrow 0
$$

where $K^\text{ind}_3(\mathbb{F}) = K_3(\mathbb{F})/K_3^H(\mathbb{F})$ and the Milnor $K^H_3(\mathbb{F})$ is trivial as $\mathbb{F}$ is totally imaginary.

**Corollary 2.3** All infinite order elements of $K_3(\mathbb{F})$ com from $B(\mathbb{F})$.

We can obtain elements of $B(\mathbb{F})$ from cross-ratios of ideal hyperbolic polyhedra. If their boundary vanishes under the action of $\text{SL}_2(\mathbb{Q}_F)$, with $\mathbb{Q}_F$ the ring of integers in $\mathbb{F}$, technical arguments of de Jeu yield that the element in $B(\mathbb{F})$ has infinite order.

### 2.1 Cross-ratios of ideal hyperbolic tetrahedra

A quadruple $(v_1, \ldots, v_4) \in (\mathbb{P}^1(\mathbb{F}))^4 \subset (\mathbb{P}^1(\mathbb{C}))^4 \approx (\partial \mathbb{H}^3_\mathbb{R})^4$ spans an ideal tetrahedron in $\mathbb{H}^3_\mathbb{R}$, and admits co-ordinates in $(\mathbb{Q}_F^2)^4$. Triangulate an ideal polyhedron $\xi$ into tetrahedra, $\xi = \bigcup_{k=1}^{4} < v_1^k, \ldots, v_4^k >$. Then we get $\beta_\xi := |\tilde{\mu}(\mathbb{F})|, \sum_{k=1}^{4} |\text{Cr}(v_1^k, \ldots, v_4^k)| \in B(\mathbb{F})$.

If $\sum_{k=1}^{4} |\text{Cr}(v_1^k, \ldots, v_4^k)| := \text{vol}(\xi) > 0$, and the triangles in the boundary $\partial(\xi)$ occur in pairs under the action of $\text{SL}_2(\mathbb{Q}_F)$, then de Jeu proves that $\text{ord}(\beta_\xi) = \infty$ if we find $m \in \mathbb{Z}$: $m.\alpha_\xi \equiv \beta_\xi$ in $B(\mathbb{F})$. To prove this congruence, we have to find suitable 5-term relations—a non-deterministic task, since there are infinitely many.

**Definition 2.4** The 6-fold symmetry $[x] = [1 - (1/x)] = [1/(1 - x)] = -[1/x] = -[1 - x] = -[-x/(1 - x)]$ and similar with $[y]$. Also if we have $-x$ it does not mean $-x$.

**Example 2.5** Show $[2] - [1/2] = 0$.

$\bullet \sum_{i=1}^{2} m_i[x_i] = 1[x_1] - 1[x_2] + 1[2] + 1[1/2]$

|   |   |   |   |
|---|---|---|---|
| [2] | [1/2] |
| 1   | 1   |

$\bullet 0 = x_i - x_j + F_3 - F_4 + F_5$

|   |   |   |
|---|---|---|
| [x_i] | [x_j] | F_3 |
| 1   | -1   | 1   |

$\bullet$ Choose $m_1 = 1, m_2 = 1$.

$F_3 = 2 = [1/2] = [1/3], F_4 = [1-1/2] = [1-1/3] = [2], F_5 = [1-1] = [1-2/1] = [-1] = [-2]$. The 6-fold symmetry $[x] = [2] = [1/2] = [-1] = [-1/2] = [1] = [-2], since [2] + [2] = 2[2] = 0$. 

3
2.1 Cross-ratios of ideal hyperbolic tetrahedra

\[
\begin{array}{cccc}
[2] & [1/2] & [1/4] & [-1/2] \\
1 & 1 & 0 & 0 \\
1 & -1 & 1 & -1 \\
\end{array}
\]

- We can merge column with 6-fold symmetry.

Choose \( i=3, j=4 \). 
\( F_3 = \begin{bmatrix} \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \end{bmatrix} \), 
\( F_4 = \begin{bmatrix} \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \end{bmatrix} \) 
\( F_5 = \begin{bmatrix} \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \end{bmatrix} \)

\[
\begin{array}{cccc}
[2]=&[1/2] & [1/4] & [-1/2] =-[2] \\
1 + (-1) & 0 & 0 & +0 \\
-1 + 1 & -1 & 1 & +1 \\
0 & -1 & 2 & \\
\end{array}
\]

- Choose \( i=3, j=4 \). 
\( F_3 = \begin{bmatrix} \frac{\sqrt{2}}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \end{bmatrix} \), 
\( F_4 = \begin{bmatrix} \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \end{bmatrix} \) 
\( F_5 = \begin{bmatrix} \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \end{bmatrix} \)

\[
\begin{array}{cccc}
[-1/2] & [1/4] & [-1/2] & [-1] \\
2 & -1 & 0 & 0 \\
-1 & 1 & -1 & 1 \\
\end{array}
\]

- We can merge column with 6-fold symmetry.

\[
\begin{array}{cccc}
[-1/2]=&[-1/2] & [1/4] & [-1]=[2] \\
2 + 0 & -1 & 0 & +0 \\
-1 + (-1) & 1 & 1 & +(-1) \\
0 & 0 & 0 & \\
\end{array}
\]
Algorithm 2.1 Algorithm for the check \([x] - [y] = 0\) in \(P(F)\)

**Input:** A difference \([x] - [y] = \sum_{i=1}^{k} m_i [x_i]\), where \(x_1, \ldots, x_k \in F\), \(m_1, \ldots, m_k \in \mathbb{Z}\).

**Output:** Either a list of 5-term relations with which \([x] - [y]\) can be seen to be zero in \(P(F)\). Or return "fail" if the algorithm cannot find such 5-term relations.

**Procedure:**

1. Write the vector \([x] - [y]\) in the space \(< x_1, \ldots, x_2 > \approx \mathbb{Z}^k\).
2. check if there are two coefficients \(m_i, m_j\) with the same absolute value.
3. Choose two coefficients with high absolute values \(|m_i|, |m_j|\), \((assume|m_i| \geq |m_j|)\). \(F_3 = \left\lceil \frac{1}{x} \right\rceil\),
   \(F_4 = \left\lceil \frac{1}{1-x} \right\rceil\), \(F_5 = \left\lceil \frac{1}{y} \right\rceil\).
4. Pick \([x1]\) and \([y1]\) with the biggest prime in their denominators: \(x1 = p/q\), \(q = p_1^{m_1} \cdots p_r^{m_r}\) prime factorisation of \(q\), \(p_1, \ldots, p_r\) prime \(m_i \in \mathbb{N}\).
5. Add the 5-term relations \(0 = [x_i] - [x_j] + F_3 - F_4 + F_5\)
6. We just keep the sum: If we instead take \(m_3\) times the row, then we get
   \[
   \begin{bmatrix}
   x_1 & x_2 & \ldots & x_3 & F_3 & F_4 & F_5 \\
   m_1 & m_2 + m_3 & \ldots & m_k & m_3 & -m_3 & m_3
   \end{bmatrix}
   \]
   Here we have to keep track of the sign, so we can enter the coefficient with the correct sign.
7. Merge rows using the 6-fold symmetry.
8. If we arrive at a final row \(\Sigma = 0\), then run the program a second time and print the 5-term relations that have been used.
9. If the number of non-zero columns exceeds a limit that has been defined in advance \(10^m\) then return "fail".

Example 2.6 Show \(2[3] - [-3] = 0\).
To prove the difference class \([3]\) with coefficient 2 and \([-3]\) with coefficient 1, need to find the five terms relations from these classes.
\(\bullet \) \(F(x_i, x_j) = [x_i] - [x_j] + F_3 - F_4 + F_5\)
\(\bullet \) We add \(F(3, -3)\), \(F_3 = \left\lceil \frac{3}{2} \right\rceil = \left\lceil \frac{3}{2} \right\rceil = [1], \ F_4 = \left\lceil \frac{1}{1-1/2} \right\rceil = \left\lceil \frac{1}{1-1/2} \right\rceil = \left\lceil \frac{1}{1/2} \right\rceil = \left\lceil \frac{1}{2} \right\rceil\), \(F_5 = \left\lceil \frac{1}{1-2} \right\rceil = \left\lceil \frac{1}{1-2} \right\rceil = \left\lceil \frac{1}{2} \right\rceil\).

\[
\begin{array}{cccccc}
[3] & [-3] & [-1] & [1/2] & [-1/2] \\
2 & -1 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & -1
\end{array}
\]

Using the 6-fold symmetry we find \([3] = [-1/2] and \([-1] = [1/2]\), so we can merge these columns.

\[
\begin{array}{cccc}
[3] & [-1/2] & [-3] & [-1] & [1/2] \\
2 & -1 & 0 & 0 & 0 \\
-1+(-1) & 1 & -1 & +1
\end{array}
\]

Hence \(+2[3] +1[-3]= 0+1F(3, -3)\), as claimed

Example 2.7 we can implement a GAP function \(\text{CheckEquivalence}(x, y, Cx, Cy)\), which inputs \(x, y\) the coefficient of \([x]\) and the coefficient of \([y]\), and output five term relation.
2.1 Cross-ratios of ideal hyperbolic tetrahedra

GAP session

gap>L:=[2,1/2];;H:=[1,-1];;
gap> CertifyEquivalence(H,L);
We want to show that
+1[2] + -1[1/2] is zero,
in case that this is possible for us. The terms 1[2] and
-1[1/2] are being merged because 1/2 has been found in
the class [ [ 2 ], [ 1/2 ], [ -1 ], [ 1/2 ], [ -1 ], [ 2 ] ]
For j = 2 we get 0
Success:
+1[2] + -1[1/2] = 0
"success"

Example 2.8 For example we 2.6 we can computation by use gap function
CheckEquivalence\((x, y, Cx, Cy)\), which inpute \(x, y\) and coefficient of \(x\), coefficient of \(y\), and the output five
term relation.

GAP session

gap>L:=[3,-3];;H:=[2,-1];;
gap> CheckEquivalence(L,H);
We want to show that
+2[3] + -1[-3] is zero, in case that this is possible for us.
+2[3] + -1[-3]
The terms 1[3] and -1[-1/2] are being merged because -1/2
has been found in the class [ [ 3 ], [ 2/3 ], [ -1/2 ],
[ 1/3 ], [ -2 ], [ 3/2 ] ]
For j = 3 we get 0
The terms -1[-1] and 1[1/2] are being merged because 1/2
has been found in the class [ [ -1 ], [ 2 ], [ 1/2 ],
[ -1 ], [ 2 ], [ 1/2 ] ]
For j = 3 we get 0
Success:
+2[3] + -1[-3] = 0 + -1F(3,-3)
Success:
+2[3] + -1[-3] = 0 + -1F(3,-3)
"success"
2.1 Cross-ratios of ideal hyperbolic tetrahedra

Algorithm 2.2 Algorithm for picking the biggest prime

Input: The list of coefficient and list of classes.

Output: The list of pick two terms as \([C_{x1}, C_{y1}, x_1, y_1]\).

Procedure:

1: \( I = [] \) the list \( I \) is going to contain the maximum of absolute value.
2: for \( j \in 1..N = \text{Length(list of classes)} \) do
3: \( p = \text{Numerator rational(list of classes}[j]) \).
4: \( q = \text{Denominator rational(list of classes}[j]) \).
5: \( \text{absolute values of primes} = [] \);
6: for \( x \in \text{union(prime factors}(p\), prime factors\(q)) \) do
7: \( \text{Add(absolute values of primes,}|x|) \).
8: end for
9: \( \pi_j = \text{Maximum(absolute values of primes)} \).
10: \( \text{Add}(I, \pi_j) \) each element of list of classes produces an element of \( I \), at the same index \( j \).
11: end for
12: for \( i \in I \) do
13: if \( I[i] = \text{maximum}(I) \) then
14: \( j_1 = i \);
15: end if
16: end for
17: \( x_1 = \text{list of classes}[j_1] \).
18: \( C_{x1} = \text{List of coefficient } H[j_1] \).
19: for \( i \in I \) do
20: if \( \text{not } j_1 = i \) then
21: Insert the element \( I[i] \) into reduced list.
22: end if
23: end for
24: for \( i \in I \) do
25: if \( I[i] = \text{maximum(reduced list)} \) and not \( \text{not } j_1 = i \) then
26: \( \text{Add}(L\text{-reduced, list of classes } [i]) \).
27: \( \text{Add}(H\text{-reduced, } H[i]) \).
28: end if
29: end for
30: Apply Algorithm 0.3 to \((H\text{-reduced}, L\text{-reduced})\) and return the output.
31: EndProcedure:

we use the command \texttt{gap PickBiggestPrime(H, L)} which is function input the list of coefficient and list of classes and the output the list of \([C_{x1}, C_{y1}, x_1, y_1]\), where \(C_{x1}, C_{y1}\) the coefficient of \(x1\) and \(y1\) respectively, the algorithm above describe how can pick the biggest prime.

By merging duplication we can computation for the discriminant -3 case with which we prove that the algebraic and geometric elements.

Example 2.9 Let we have the algebraic element \([-3, -1/2*x - 1/2]\) and geometric element \([2, w]\), where \(\text{delta}\equiv-3 \mod 4, d= \text{delta}/4 \) and \(w = Sqrt(d)\). The GAP session below prove the discriminant -3.

We want to show that
\[-2(-\xi_2^3) - 3(\xi_3^2) = 0\]
we can rewrite
\[-2(-\xi_2^3) - 2(\xi_3^2) - 1(\xi_3^2) = 0\]
2.1 Cross-ratios of ideal hyperbolic tetrahedra

By merging duplication
\[ -2[-\zeta_3^2] - 2[\zeta_3^2] = -1[\zeta_3] \]
we have
\[ -1[\zeta_3] - 1[\zeta_3^2] = 0 \]

Applying the 6-fold symmetries has yielded
The terms \(-1[\zeta_3]\) and \(-1[\zeta_3^2]\) are being merged because \(\zeta_3^2\) has been found in the class \([\zeta_3, [-\zeta_3 - 2*\zeta_3^2, -1/3*\zeta_3 - 2/3*\zeta_3^2, [\zeta_3^2], [-2*\zeta_3 - \zeta_3^2, [-2/3*\zeta_3 - 1/3*\zeta_3^2]]]\)

Then we have
\[ +0[\zeta_3] = 0 \]

**GAP session**

We want to show that
\[ +2[-E(3)^2] + -3[E(3)^2] \]
is zero, in case that this is possible for us. Then we have
\[ +2[-E(3)^2] + -3[E(3)^2] \]
Applying the 6-fold symmetries has yielded
\[ +2[-E(3)^2] + -3[E(3)^2] \]
Inserting duplication relations has yielded
\[ +1[E(3)] + -1[E(3)^2] \]
The terms \(-1[E(3)]\) and \(-1[E(3)^2]\) are being merged because \(E(3)^2\) has been found in the class \([\ E(3), [-E(3)-2*E(3)^2], \ [-1/3*E(3)-2/3*E(3)^2], [E(3)^2], [-2*E(3)-E(3)^2], [-2/3*E(3)-1/3*E(3)^2] \] \]
Then we have
\[ +0[E(3)] \]
Applying the 6-fold symmetries has yielded
\[ +2[-E(3)^2] + -3[E(3)^2] \]
Success:
\[ +2[-E(3)^2] + -3[E(3)^2] = 0 \]

In The GAP session below we can computation for the discriminant -7, with algebraic element and geometric element.

**GAP session**

\[ \text{gap> Read("./desktop/Bloch.g.txt");} \]
Over the imaginary quadratic field of discriminant -7, we compare the geometric Bloch group element
\[ +8[-E(7)^3-E(7)^5-E(7)^6] +2[-1/4*E(7)-1/4*E(7)^2-1/2*E(7)^3-1/4*E(7)^4-1/2*E(7)^5-1/4*E(7)^6] +2[-1/2*E(7)-1/2*E(7)^2-1/4*E(7)^3-1/2*E(7)^4-1/4*E(7)^5-1/4*E(7)^6] +2[1/90*E(7)^3+1/90*E(7)^5+1/90*E(7)^6] +2[-1/2] \]
with \(j\) times the algebraic Bloch group element
\[ +2[-3/11*E(7)-3/11*E(7)^2-1/11*E(7)^3-3/11*E(7)^4-1/11*E(7)^5 \]
\[ +2[-3/4*E(7)^2-1/2*E(7)^3-1/4*E(7)^4-1/4*E(7)^5-1/4*E(7)^6] +2[-7/8*E(7)-7/8*E(7)^2-3/4*E(7)^3-7/8*E(7)^4-3/4*E(7)^5-3/4*E(7)^6] +2[1/90*E(7)^3+1/90*E(7)^5+1/90*E(7)^6] +2[-1/2] \]

= 0
2.1 Cross-ratios of ideal hyperbolic tetrahedra

\[ E(7)^{-5} E(7)^{-6} + 2[-1/2 E(7)^{-1} + 2 E(7)^{-2} - 1/2 E(7)^{-3}] + 2 E(7)^{-4} + 2 E(7)^{-5} + 2 E(7)^{-6} \]
\[ + 2[-1/4] + 2[2/11 E(7)^{2} + 2 E(7)^{3} - 1/4 E(7)^{4} - 1/2 E(7)^{5} - 1/4 E(7)^{6}] \]
\[ - E(7)^{-6} + 2[-2/11 E(7)^{2} + 2 E(7)^{3} - 1/4 E(7)^{4} - 1/2 E(7)^{5} - 1/4 E(7)^{6}] \]
\[ - 2 E(7)^{-6} + 2[2/11 E(7)^{2} + 2 E(7)^{3} - 1/4 E(7)^{4} - 1/2 E(7)^{5} - 1/4 E(7)^{6}] \]
\[ - 1/4 E(7)^{-6} + 2[-2/11 E(7)^{2} + 2 E(7)^{3} - 1/4 E(7)^{4} - 1/2 E(7)^{5} - 1/4 E(7)^{6}] \]
\[ - 2 E(7)^{-6} + 2[2/11 E(7)^{2} + 2 E(7)^{3} - 1/4 E(7)^{4} - 1/2 E(7)^{5} - 1/4 E(7)^{6}] \]
\[ - 1/4 E(7)^{-6} + 2[-2/11 E(7)^{2} + 2 E(7)^{3} - 1/4 E(7)^{4} - 1/2 E(7)^{5} - 1/4 E(7)^{6}] \]
\[ - 1/4 E(7)^{-6} + 2[-2/11 E(7)^{2} + 2 E(7)^{3} - 1/4 E(7)^{4} - 1/2 E(7)^{5} - 1/4 E(7)^{6}] \]
\[ + 2 E(7)^{-6} + 2[2/11 E(7)^{2} + 2 E(7)^{3} - 1/4 E(7)^{4} - 1/2 E(7)^{5} - 1/4 E(7)^{6}] \]
\[ + 2[-2/11 E(7)^{2} + 2 E(7)^{3} - 1/4 E(7)^{4} - 1/2 E(7)^{5} - 1/4 E(7)^{6}] \]

\[ j = -3 \text{ yields 22 remaining terms.} \]
\[ j = -2 \text{ yields 1 remaining term.} \]
\[ \text{geobelt} = 2 \text{algbelt} + \]
\[ + 22[-1] \]

where geobelt is the geometric Bloch group element and algbelt the algebraic Bloch group element.

We observe the 6-fold symmetries \[ [ -1 ], [ 2 ], [ 1/2 ], [-1], [ 2 ], [ 1/2 ] \],
which might allow us to indentify the remainder term as torsion.

We want to show that
\[ + 22[-1] \text{ is zero, in case that this is possible for us. Applying the 6-fold symmetries has yielded} \]
\[ + 22[-1] \text{ Inserting duplication relations has yield} \]
\[
\text{Success:} \\
+ 22[-1] \\
= [0] = [1]
\]

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