The Complete KLT-Map Between Gravity and Gauge Theories

Poul H. Damgaard, Rijun Huang, Thomas Søndergaard, Yang Zhang

Niels Bohr International Academy and Discovery Center,
The Niels Bohr Institute, Blegdamsvej 17,
DK-2100 Copenhagen Ø, Denmark
emails: {phdamg, huang, tsonderg, zhang}@nbi.dk

Abstract: We present the complete map of any pair of super Yang-Mills theories to supergravity theories as dictated by the KLT relations in four dimensions. Symmetries and the full set of associated vanishing identities are derived. A graphical method is introduced which simplifies counting of states, and helps in identifying the relevant set of symmetries.

Keywords: Gravity, Yang-Mills theory, Supersymmetry, Amplitudes
1. Introduction

While general relativity has a beautiful and compact formulation in terms of the Einstein-Hilbert action, it is striking that all simplicity is lost once attempts to view that same action from the perturbative (graviton) point of view. The simple and almost trivial problem is that there is no way to treat both the metric $g_{\mu \nu}$ and its inverse as a background with a small single-term perturbative correction. As a consequence, the Einstein-Hilbert action explodes into an infinite series of terms when expanded about a fixed background: There is an infinite series of vertices of increasing order in the number of graviton fields. Beyond four-point scattering amplitudes this makes it essentially impossible to compute tree-level graviton scattering amplitudes on the basis of conventional Feynman-diagram techniques.

A most surprising and powerful way to circumvent this obstacle to perturbative calculations in gravity (or theories coupled to gravity) is provided by the Kawai-Lewellen-Tye (KLT) relation [1]. Based on the factorization of closed-string amplitudes into a product of two open-string amplitudes, it yields, in the field theory limit, a remarkable connection between gravity and Yang-Mills theory. This is totally obscure at the level of the actions of gravity and Yang-Mills theory. Symbolically, it tells us that gravity, as far as tree-level $n$-point amplitudes are concerned, is a very particular kind of 'square' of Yang-Mills theory,

$$Gravity \sim (Gauge \text{ theory}) \times (Gauge \text{ theory}).$$
This is just the field theory limit of the KLT-relation, and because of its existence in the full superstring case, it really, more generally, relates supergravity theories to super Yang-Mills theories. The precise map in field theory has recently been derived by means of on-shell recursion for all \(n\)-point functions [2],

\[
M_n = \sum_{\gamma, \beta \in S_{n-3}} \tilde{A}_n(n-1, n, \gamma, 1) \mathcal{S}[\gamma|\beta]_{p_1} A_n(1, \beta, n - 1, n)
\]  

(1.1)

Here, \(M_n\) indicates an \(n\)-point gravity amplitude, and on the right hand side we have Yang-Mills \(n\)-point amplitudes (the possibility of combining two different kinds of Yang-Mills amplitudes \(A_n\) and \(\tilde{A}_n\) will be discussed in details below). The object \(\mathcal{S}\) that glues the two Yang-Mills amplitudes together, the S-kernel [2], will be defined below. It serves in a very precise manner to remove the double poles of the amplitude product and, simultaneously, to ensure full permutation symmetry of the gravity amplitude. Note that in the sum above only permutation symmetry of \(n-3\) of the \(n\) legs is manifest. With a very simple modification, the S-kernel of field theory generalizes to the full string theory case [3].

In string theory, the factorization of closed-string amplitudes into two open-string amplitudes is not unique [1]. Indeed, the S-kernel that ties the two Yang-Mills amplitudes together is also not unique: there is a whole family of kernels that all do the factorization correctly. Conversely, this must imply amplitude relations on the Yang-Mills side. This intuition turns out to be correct: the Bern-Carrasco-Johansson (BCJ) relations that were first conjectured in field theory [4] and later shown to follow from monodromy in string theory [5, 6], are intimately tied to the KLT-relations\(^1\). The BCJ-relations have also, like the KLT-relations, been derived in field theory by means of on-shell recursion [8, 9], and the S-kernel of the KLT-map can be seen as a generator of BCJ-relations [2, 3]. A variant of BCJ-relations that builds on numerators of given amplitude representations [4] constructs amplitudes for gravity by squaring numerators. This has been proven to give a remarkable alternative representation of gravity amplitudes by means of products of two Yang-Mills amplitudes [10]. Perhaps the most exciting aspect is that it points directly towards applications of KLT squaring relations also at loop level [11]. There are also applications of KLT relations directly at one-loop order [12].

A common theme in these recent developments is the appearance of surprising traces of Yang-Mills theory in perturbative gravity. From scattering amplitudes it is as if the color group is replaced by a kinematical group. This is an idea that originates in the way Jacobi relations of the color group in Yang-Mills theory have (generalized) mirrors in the kinematical factors that appears in numerators [4, 13–16]. These notions have become beautifully synthesized in recent work of Monteiro and O’Connell [17], where, conversely, the diffeomorphism invariance of gravity re-surfaces in the kinematics of Yang-Mills theory. The connection between BCJ-relations and the weak-weak duality between gauge theory and gravity is more and more leading us towards an algebra of amplitudes [18]. This serves as additional motivation to establish the precise relationship between two (super) Yang-Mills theories and the associated (super) gravity theory as dictated by the KLT-map.

\(^1\)See also [7].
Several aspects of this gauge theory to gravity map can already be found, scattered through the scientific literature, in terms of specific examples. Here, we provide the comprehensive catalog of the maps between four-dimensional gauge theories with supersymmetries $\mathcal{N} = 4, 3, 2, 1$ and 0 to the corresponding supergravity theories. The $\mathcal{N} = 0$ theory is nothing but pure Yang-Mills theory. Yet, as is well known, it maps not to pure Einstein gravity, but to gravity coupled to two real scalars: in string theory language an axion-dilaton pair.

Indeed, the recent field-theory proof of KLT-relations [2] hinged in an essential manner on a corresponding set of “vanishing relations” [19], quadratic identities among Yang-Mills amplitudes that were proven independently. These vanishing identities correspond to gravity amplitudes with an odd number of (complex) scalars, which vanish. As observed in [20], such identities indeed follow from string theory. Alternatively, if seen as embedded in maximally supersymmetric theories, they correspond to KLT-relations that violate $R$-symmetry on the gravity side [21, 22]. It is interesting that even if one considers only the conventional KLT-map of vector bosons to gravitons with like helicities, the scalars never appear. Yet, as we have mentioned, they are essential for the proof [2] of these KLT-relations even in the complete absence of scalars as external states. Much can be learned already from this simplest example, since it shows that by gluing two pure Yang-Mills theories together through the S-kernel of the KLT-map, we do not get just pure gravity, and that the additional states on the gravity side are crucial for understanding the gauge theory to gravity map. Let us explain this in slightly greater detail.

We start by considering gravity as the low-energy limit of string theory. It consists of the following states and their corresponding polarization tensors:

- **Graviton.** $e^{\pm
u}_{\mu
u}(k) = e_{\mu}^{\pm}(k)\epsilon_{\nu}(k), e_{\mu
u}^{-+}(k) = e_{\mu}^{-}(k)\epsilon_{\nu}^{+}(k)$.
- **Axion.** $e_{\mu
u}(k) = e_{\mu}^{+}(k)\epsilon_{\nu}^{-}(k) - e_{\mu}^{-}(k)\epsilon_{\nu}^{+}(k)$.
- **Dilaton.** $e_{\mu
u}(k) = e_{\mu}^{+}(k)\epsilon_{\nu}^{-}(k) + e_{\mu}^{-}(k)\epsilon_{\nu}^{+}(k)$.

Gravity amplitudes with axions and dilations have conserved quantum numbers, and these constraints lead to vanishing identities when combined with KLT-relations. To see the origin of these identities, let us present the action of this theory [23]. The four-dimensional coupled axion-dilaton gravity action reads as follows in the Einstein frame:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-G}(R - 2\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{12}e^{-4\phi}H_{\mu\nu\rho}H^{\mu\nu\rho}). \quad (1.2)$$

The Poincare dual of $H_{\mu\nu\rho}$ is the axion,

$$\partial_{\mu}b = \frac{1}{6}e^{-4\phi}e_{\mu\nu\rho\sigma}H^{\mu\nu\rho}. \quad (1.3)$$

We can now combine the axion and the dilaton into the complex combination $z = b + ie^{-2\phi}$, leading to

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-G} \left( R - \frac{1}{2} \frac{1}{(\text{Im}z)^2} \right). \quad (1.4)$$
Here, $z$ takes value in the upper complex plane, as can be seen from its definition. This upper part of the complex plane is the moduli space of this theory, which has an $SL(2, \mathbb{R})$ global symmetry:

$$g_{\mu\nu} \mapsto g_{\mu\nu}, \quad z \mapsto \frac{az + b}{cz + d}, \quad ad - bc = 1.$$  \hspace{1cm} (1.5)

We may choose the vacuum expectation value as $\langle z \rangle = i$. Then the remaining manifest symmetry is $U(1)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$  \hspace{1cm} (1.6)

We can also perform a redefinition of the scalar field in order to make the symmetry linearly realized. This is achieved by

$$z = \frac{2\kappa w}{1 + i\kappa w} + i,$$  \hspace{1cm} (1.7)

which is a Möbius transformation which transforms the upper half plane to the Poincare disc, $|w| < 1/\kappa$. The action then becomes

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-G} R - \int d^4x \sqrt{-G} \frac{\partial_\mu w \partial^\mu \bar{w}}{(1 - \kappa^2 |w|^2)^2},$$  \hspace{1cm} (1.8)

and the $U(1)$ symmetry acts on $w$ as $w \mapsto e^{2i\theta} w$. This charges the states $\epsilon^+_\mu(k)\epsilon^-_\nu(k)$ and $\epsilon^-_\mu(k)\epsilon^+_\nu(k)$. The origin of the vanishing identities as arrived from KLT-relations is now clear. The action of the remaining two generators of $SL(2, \mathbb{R})$ changes the vacuum expectation value of $z$. The axion and the dilaton can thus be regarded as the two Goldstone bosons associated with the global symmetry breaking $SL(2, \mathbb{R}) \sim SU(1, 1) \rightarrow U(1)$. We will see analogs of these two Goldstone bosons in all the cases to be discussed below. In the maximally supersymmetric case, the scalars live in the well-known coset of $E_{7(7)}/SU(8)$.

The pure gauge theory case discussed above gives a very direct proof of the vanishing identities. However, as stressed in ref. [22], it can also be profitable to understand these identities from violation of $R$-symmetry in the maximally supersymmetric version of the KLT-map. This may appear puzzling, since supersymmetry plays no direct role in the identities themselves. However, in taking such a top-down approach, one sees every single four-dimensional KLT-map between gauge theory and gravity as a sub-map of the maximal map between two $\mathcal{N} = 4$ super Yang-Mills theories to $\mathcal{N}_G = 8$ supergravity. Crucial in this connection is a most powerful extension of the proof of the KLT-formula in terms of superfields by Feng and He [21]. We shall here use this extended KLT-formula to project out more and more fields from the maximally supersymmetric case, in this way generating a complete catalog of KLT-maps between theories with less and less supersymmetry in four dimensions. The $\mathcal{N} = 4$ superfield of Nair [24] and the associated definition of a superamplitude out of given helicity amplitudes, is essential in this context. For each entry in the KLT-table we identify global symmetries including $R$-symmetries, if applicable, and these symmetries determine the set of vanishing identities.

Our paper is organized as follows. In section 2 we briefly review the KLT-relation and its generalization to maximal supersymmetry in its superfield formulation. In section 3 we discuss the KLT-relations in superfields for $\mathcal{N} < 4$ super Yang-Mills theories, and describe
the full set of supergravity theories that can be obtained from KLT products. As an aid in
the construction, we introduce a graphical tool ("diamond diagrams") for KLT-relations of
non-maximal theories. In section 4 we study the invariant symmetry groups that emerge
naturally from KLT products for the various supergravity theories. Conclusions and an
outlook are given in the final section.

2. A brief overview of KLT-relations

The full, stringy, KLT-relation \[1\] has recently been shown to take the following explicit
form \[3\]:

\[M_{n}^{\text{closed}} = \sum_{\gamma,\beta} \tilde{A}_{n}^{\text{open}}(n-1,n,\gamma,1)S_{\alpha'}[\gamma|\beta]_{p_{1}}A_{n}^{\text{open}}(1,\beta,n-1,n),\] (2.1)

where we sum over two sets of \((n-3)\) permutations \(\beta\) and \(\gamma\). The momentum kernel \(S_{\alpha'}\)
glues two open string-theory amplitudes \(A_{n}^{\text{open}}, \tilde{A}_{n}^{\text{open}}\) together to form the closed string-amplitude \(M_{n}^{\text{closed}}\). The explicit form of \(S_{\alpha'}\), given in ref. \[3\], resembles very much its field
theory analog \(S\), which will be defined below. The two open string amplitudes may be
bosonic or supersymmetric depending on whether the closed string amplitude is bosonic or
supersymmetric. Alternatively, one of the open string amplitudes may be bosonic and the
other supersymmetric if one wishes to construct a heterotic string amplitude\(^{3}\). This implies
that in the field theory limit, we can either have KLT-relations between gravity and Yang-
Mills theory, or we can have relations between supergravity and super Yang-Mills theories.
The \(\alpha'-\)dependent momentum kernel \(S_{\alpha'}\) is totally independent of the types of closed and
open strings considered, and thus the relation (2.1) is completely general and universal.
In the field theory limit of \(\alpha' \to 0\), the momentum kernel \(S_{\alpha'}\) reduces directly to the field
theory S-kernel. Therefore also in field theory the S-kernel is universal, and independent of
whether the involved Yang-Mills and gravity amplitudes arise in supersymmetric theories
or not.

It is clear, then, that in the field theory limit, the KLT-relations express a given gravity
amplitude as a particular sum of gauge field amplitudes squared. An explicit expression of
this relation with manifest \((n-3)!\) permutation symmetry has been proven to be \[2\]

\[M_{n} = \sum_{\gamma,\beta \in S_{n-3}} \tilde{A}_{n}(n-1,n,\gamma,1)S[\gamma|\beta]_{p_{1}}A_{n}(1,\beta,n-1,n),\] (2.2)

where \(\gamma\) and \(\beta\) are permutations over the legs \(2, \ldots, n-2\) and the S-kernel is defined by

\[S[i_{1}, \ldots, i_{m}|j_{1}, \ldots, j_{m}]_{p_{1}} = \prod_{t=1}^{m}(s_{i_{1}} + \sum_{q>t}^{m} \theta(i_{t}, i_{q})s_{i_{t}i_{q}}),\] (2.3)

with

\[\theta(i_{a}, i_{b}) = \begin{cases} 1 & \text{if } i_{a} \text{ appears after } i_{b} \text{ in the sequence } \{j_{1}, \ldots, j_{m}\} \\ 0 & \text{if } i_{a} \text{ appears before } i_{b} \text{ in the sequence } \{j_{1}, \ldots, j_{m}\} \end{cases}.\]

\(^{2}\)The relation that is derived from the low-energy limit of string theory comes with an overall irrelevant
sign factor \[3\] which we ignore in this paper.

\(^{3}\)Some explicit examples can be found in refs. \[25\].
The shown expression is not unique: there is a whole class of different but equivalent KLT-relations written in terms of S-kernels \[2,3\]. One of these, the one containing fewest terms, coincides with an explicit representation conjectured already in Appendix A of ref. \[26\].

As discussed in the introduction, the KLT-formula \(2.2\) is quite surprising since the gravity amplitude on the left hand side receives contributions from Feynman diagrams with, depending on the number of external legs \(n\), graviton vertices that continue up to infinite order while the Yang-Mills amplitudes on the right hand side of course receive contributions of only three- and four-point vertices. Another surprising aspect of the KLT-relation is that Yang-Mills amplitudes are colored objects, but gravity should know nothing about color. Magically, it is only the color-stripped Yang-Mills amplitudes that enter. But these are not symmetric under permutations of the external legs, while of course the gravity amplitude is totally symmetric under such permutations. Though not obvious, the right hand side of \(2.2\) is indeed symmetric under permutations of all \(n\) legs, rather than just the manifest symmetry under \((n-3)!\) of such permutations. Finally, there is magic with respect to locality. A simple Feynman-diagram analysis quickly shows that the product of two gauge theory amplitudes has double poles that cannot be allowed in a gravity amplitude. The S-kernel cleverly manages to precisely cancel these unwanted double poles, rendering the correct behavior required for the gravity amplitude on the left hand side. A crucial series of very finely tuned cancellations clearly occur in the KLT-relation. It is from this point of view even more striking that the S-kernel is not unique, and that a whole series of such kernels can do the job. This is precisely the origin of the BCJ-relations, which can be viewed as a consequence of the equivalence between these different parametrization of the KLT-relations.\(^4\)

\(^4\)A first example of this, at the six-point level, was already implicit in the original KLT-paper \[1\].
where the $\eta_i$’s are Grassmann variables labeled by $SU(4)_R$ symmetry indices. In the on-shell formalism the supercharges are given by

$$\tilde{Q}_a = \sum_{i=1}^{n} |i\rangle \eta_a, \quad Q^a = \sum_{i=1}^{n} |i\rangle \frac{\partial}{\partial \eta_a},$$

which relates all the 16 states in one supermultiplet.

It is useful to think of the $\Phi$’s as super-states and introduce a superamplitude

$$\mathcal{A}^{N=4}_N(\Phi_1, \Phi_2, \ldots, \Phi_n),$$

which represents a sum of amplitudes of all different helicity assignments and choices of external states. The expansion coefficients, which uniquely identify a given component helicity amplitude, are precisely the $\eta_i$’s, one set for each external line. Because the amplitudes must be invariant under $SU(4)_R$ symmetry, this puts constraints on the combinations of indices that can occur for non-vanishing amplitudes. Hence many of the amplitudes in the direct expansion will vanish since they are $SU(4)_R$ symmetry violating. Schematically, what is left is thus an expansion of the form

$$\mathcal{A}^{N=4}_N = \sum A^{N_{\text{MHV}}}^N(\eta)^8 + \sum A^{N_{\text{NMHV}}}^N(\eta)^{12} + \ldots + \sum A^{N_{\text{MHV}}}^N(\eta)^{4n-8},$$

where each $SU(4)_R$ symmetry index ($a = 1, 2, 3, 4$) appears the same number of times in each monomial of the $\eta_i$’s. Here $A^{N_{\text{MHV}}}^N$ denotes the actual component helicity amplitudes. They can be extracted from the superamplitudes by acting with the corresponding differential operators (or integrals) that single out the desired components.

The $\mathcal{N}_G = 8$ supergravity theory has an on-shell formalism that is completely analogous to $\mathcal{N} = 4$ super Yang-Mills theory. The superfield of $\mathcal{N}_G = 8$ supergravity contains one graviton $h_{\pm}$, 8 gravitinos $\psi_{\pm}$, 28 graviphotons $v_{\pm}$, 56 graviphotinos $\chi_{\pm}$ and 70 real scalars $\phi$. It can be represented as

$$\Phi^{\mathcal{N}_G=8} = h_+ + \eta_A \psi_+ + \frac{1}{3!} \eta_A \eta_B \psi_{AB} + \frac{1}{4!} \eta_A \eta_B \eta_C \chi_{ABC} + \frac{1}{5!} \eta_A \eta_B \eta_C \eta_D \phi^{ABCD} + \frac{1}{6!} \eta_A \eta_B \eta_C \eta_D \eta_E \psi_{ABCD} + \frac{1}{7!} \eta_A \eta_B \eta_C \eta_D \eta_E \eta_F \psi_{ABCD} + \frac{1}{8!} \eta_A \eta_B \eta_C \eta_D \eta_E \eta_F \eta_G \psi_{ABCD} + \eta_A \eta_B \eta_C \eta_D \eta_E \eta_F \eta_G \psi_{ABCD} + \eta_A \eta_B \eta_C \eta_D \eta_E \eta_F \eta_G \psi_{ABCD} + \eta_A \eta_B \eta_C \eta_D \eta_E \eta_F \eta_G \psi_{ABCD} + \eta_A \eta_B \eta_C \eta_D \eta_E \eta_F \eta_G \psi_{ABCD},$$

and we can likewise introduce superamplitudes for $\mathcal{N}_G = 8$ supergravity amplitudes,

$$\mathcal{M}^{\mathcal{N}_G=8}_n(\Phi_1, \Phi_2, \ldots, \Phi_n).$$

When analogously expanded out in terms of the $\eta_{i,A}$’s, this gives a sum of all possible component amplitudes $M^{N_{\text{MHV}}}^N$ dressed with strings of $\eta_{i,A}$’s, where the capital letters run from 1 to 8. The $SU(8)_R$ invariance dictates that only amplitudes dressed with a string of $\eta_A$’s where each $A$ index appears an equal amount of times can be non-vanishing.

In this on-shell formalism, the KLT-relations between maximally supersymmetric supergravity and super Yang-Mills theories can be formulated in an extremely compact way. As is perhaps now almost evident, the $n$-point KLT-relations between $\mathcal{N}_G = 8$ supergravity
superamplitudes $\mathcal{M}^{N_G=8}_n$ and the product of two $\mathcal{N} = 4$ super Yang-Mills superamplitudes $\tilde{A}_{n}^{\mathcal{N}=4}$ and $A_{n}^{\mathcal{N}=4}$ can be wrapped into
\[
\mathcal{M}^{N_G=8}_n = \sum_{\gamma, \beta \in S_{n-3}} \tilde{A}_{n}^{\mathcal{N}=4}(n-1, n, \gamma, 1) S[\gamma|\beta] p_{1} A_{n}^{\mathcal{N}=4}(1, \beta, n-1, n) .
\] (2.8)

Here $\gamma$ and $\beta$ are again just permutations over the legs 2, $\ldots$, $n-2$ and the S-kernel is the same as defined in (2.3). $\mathcal{M}^{N_G=8}_n$, $\tilde{A}_{n}^{\mathcal{N}=4}$ and $A_{n}^{\mathcal{N}=4}$ are the superamplitudes of on-shell superfields as defined above. This maximally supersymmetric KLT-relation has been proven by BCFW on-shell recursion relations by Feng and He [21]. The superfield expansions on each side automatically yield all the correct component relations when the $\eta$'s on the supergravity side are correctly identified as the union of $\tilde{\eta}$'s of the two super Yang-Mills theories. The explicit operator map of states between the two $\mathcal{N} = 4$ super Yang-Mills theories and $\mathcal{N}_G = 8$ supergravity has been worked out in ref. [28].

Interestingly, the superamplitude formulation of the maximally supersymmetric KLT-relation of eq. (2.8) contains all information required to construct KLT-relations for theories with less supersymmetry as well. Superamplitudes for supersymmetric Yang-Mills theories with less than maximal supersymmetry were introduced in [29]. Since $SU(8)_R \supset SU(4)_R \otimes SU(4)_R$, there is perfect matching between $SU(4)_R$ indices 1, 2, 3, 4 of one $\mathcal{N} = 4$ super Yang-Mills amplitude $\tilde{A}_{n}^{\mathcal{N}=4}$ and $SU(4)_R$ indices 5, 6, 7, 8 of the other amplitude $A_{n}^{\mathcal{N}=4}$. The product will label all states in the $\mathcal{N}_G = 8$ supergravity theory with $SU(8)_R$ indices 1, 2, $\ldots$, 8. In order to get the component KLT-relations, we need only to expand superamplitudes of eq. (2.8) into their component amplitudes dressed with their strings of $\eta_{i,A}$ and $\eta_{i,a}, \eta_{i,b}$, where $i = 1, \ldots, n$, $A = 1, \ldots, 8$, $a = 1, 2, 3, 4$ and $b = 5, 6, 7, 8$. By picking up the appropriate coefficients of $\eta$-strings on the left and right sides of eq. (2.8), and taking care of signs when exchanging Grassmann variables, we get the KLT-relations for component amplitudes. For example, if we wish to get the KLT-relation for a pure graviton MHV amplitude $M_{n}(h^-, h^-, h^+, \ldots, h^+)$, from the $\mathcal{N}_G = 8$ superfield expansion, we know that this amplitude is just the coefficient of the string $\prod_{A=1}^{5} \eta_{i,A} \prod_{A=1}^{8} \eta_{i,A} \prod_{b=5}^{8} \eta_{i,b} \prod_{b=5}^{8} \eta_{i,b}$ on the left hand side of eq. (2.8). This string of $\eta$'s decomposes into $\prod_{A=1}^{5} \eta_{i,A} \prod_{A=1}^{4} \eta_{i,a} \prod_{b=5}^{8} \eta_{i,b} \prod_{b=5}^{8} \eta_{i,b}$, whose coefficient on the right hand side is nothing but two pure-gluon MHV amplitudes of proper helicities. Thus the component KLT-relation for the pure graviton MHV amplitude is a sum of a product of two pure gluon MHV amplitudes with the S-kernel in-between and the corresponding permutations of the external legs.

The superamplitude version of KLT-relations is actually far more powerful. In this maximally supersymmetric KLT-formulation we can consider the violation of $SU(8)_R$ symmetry on the supergravity side, while each $SU(4)_R$ symmetry of the super Yang-Mills side is kept intact. $\tilde{A}_{n}^{\mathcal{N}=4}$ should be invariant under $SU(4)_R$, and thus the power $k$ of each $\eta_{a,a} = 1, 2, 3, 4$ should be the same. Similarly, also $A_{n}^{\mathcal{N}=4}$ should be invariant under $SU(4)_R$ and the power $k'$ of each $\eta_{b,b} = 5, 6, 7, 8$ should be the same. Seen from the super Yang-Mills side of the KLT-relation, the power $k$ is not necessarily equal to $k'$, but the product of these two super Yang-Mills theory amplitudes should produce a supergravity amplitude and hence possess $SU(8)_R$ symmetry. Thus if $k \neq k'$, the resulting supergravity amplitude will obviously violate $SU(8)_R$ symmetry and vanish [21, 29]. We thereby get
vanishing identities of superamplitudes such as

\[ 0 = \sum_{\gamma, \beta \in S_{n-3}} \tilde{A}_{n}^{N_k \text{MHV}}(n-1, n, \gamma, 1) S[\gamma/\beta]_{j_1} A_{n}^{N_{k'} \text{MHV}}(1, \beta, n-1, n), \]  

(2.9)

where \( k \neq k' \). As we describe in a little more detail below, this explains all the vanishing identities originally found in [19], and gives the general prescription for how to obtain these identities systematically. For the pure gauge theory case, the analogous argument was given in ref. [20].

We will present the non-maximal supersymmetric KLT-relations in the next section. These supersymmetric KLT-relations will also make the vanishing identities manifest in these cases.

3. KLT-relations with less supersymmetry: the full map

In this section, we derive the supersymmetric KLT-relations for \( \mathcal{N}_G < 8 \) supergravity theories, by removing and integrating out components in the \( \mathcal{N}_G = 8 \) formalism.

We begin with a brief review of the \( \Phi-\Psi \) on-shell superfield formalism for super Yang-Mills theory. As is well-known, all states in a maximally supersymmetric theory are related under the action of supercharge generators \( \tilde{Q}_a \) and \( Q^a \). This means that starting from the highest helicity state \(+h\) we can lower the helicity by \( 1/2 \) each time when acting with \( \tilde{Q}_a \) all the way down to the lowest helicity state \(-h\). Thus we can pack all states into one superfield (2.5), and since \( \Phi^{N=4} \) is CPT self-conjugate, this superfield is already complete. But for super Yang-Mills theories with \( \mathcal{N} < 4 \), this is not true. For example, for the \( \mathcal{N} = 2 \) super Yang-Mills theory we have \( \tilde{Q}^a, a = 1, 2 \). Starting from the \(+1\) helicity state we can at most lower the helicity to \( 0 \), and hence we cannot reach the CPT-conjugate state of helicity \(-1\).

Following [29] we can always get one of the \( \mathcal{N} < 4 \) superfields \( \Phi^\mathcal{N} \) from a truncation of the \( \mathcal{N} = 4 \) superfield by simply setting the unwanted \( \eta \)'s to zero, i.e.

\[ \Phi^\mathcal{N}<4 = \Phi^{N=4}|_{\eta_{N+1},... \eta_{4} \to 0}. \]  

(3.1)

For example, by setting \( \eta_4 \to 0 \) in (2.5), we get

\[ \Phi^{N=3} = g_+ + \eta_0 f_+ + \frac{1}{2!} \eta_0 \eta_2 s^{ab} + \eta_1 \eta_2 \eta_3 f_{123}, \]  

(3.2)

where \( a, b = 1, 2, 3 \). This superfield contains one plus-helicity gluon, three plus-helicity fermions, three real scalars and one minus-helicity fermion. It is therefore not complete and we should add the CPT-conjugate superfield. This additional superfield can be obtained by integrating out the unwanted \( \eta \)'s in eq. (2.5). We therefore introduce another \( \mathcal{N} < 4 \) superfield \( \Psi^\mathcal{N} \) by

\[ \Psi^\mathcal{N}<4 = \int \prod_{a=N+1}^{4} \, d\eta_a \Phi^{N=4}. \]  

(3.3)
As an example, for \( \mathcal{N} = 3 \), we have

\[
\Psi^{\mathcal{N}=3} = f_+^{(4)} - \eta_a s^{a(4)} + \frac{1}{2} \eta_a \eta_b f^{ab(4)} - \eta_1 \eta_2 \eta_3 g^{123(4)} ,
\]

where \( a, b = 1, 2, 3 \). The index 4 is placed in parenthesis because the corresponding Grassmann parameter \( \eta_4 \) has been integrated out. We have only indices 1, 2, 3 for the \( SU(3)_R \) symmetry in \( \mathcal{N} = 3 \) super Yang-Mills theory, and the index 4 is what we will call a hidden index from \( SU(4)_R \). Although it has nothing to do with the \( SU(3)_R \) symmetry, we keep it for reasons that will be explained in the next section. Using the combined \( \Phi-\Psi \) formalism, all states now have their CPT-conjugate partners, and it is thus complete. The \( \mathcal{N} < 4 \) superamplitudes can now directly be obtained from the \( \mathcal{N} = 4 \) superamplitudes. Suppose the \( i_1 < i_2 < \ldots < i_m \) external legs are in the \( \Psi \) superfield representation, while the \( j_1 < j_2 < \ldots < j_l \) external legs are in the \( \Phi \) representation and \( m + l = n \). The \( \mathcal{N} < 4 \) superamplitude is then

\[
A_{\mathcal{N}<4}^{\mathcal{N}=4} = \left[ \int d\eta_{i_1 \ldots i_m} d\eta_{j_1 \ldots j_l} A_{\mathcal{N}=4}^{\mathcal{N}=4} (\Phi_1, \Phi_2, \ldots, \Phi_n) \right]_{\eta_{N+1}, \ldots, \eta_4 \to 0} .
\]

In order to keep track of fields in a systematic and graphical manner, we introduce a diagrammatic notation which we have found useful. The idea is to express the superfields in terms of (generalized) “diamond diagrams”, which keep track of helicities and \( R \)-symmetry quantum numbers. A diamond is a set of on-shell component fields which are all related by supersymmetric transformations. A diamond may not be a full supermultiplet since it may not be CPT complete. We illustrate this in figure 1 for all four super Yang-Mills cases. We use solid lines to connect all states that are related under operation by supercharges. This means that we can reach all states within one diamond by applying \( \tilde{Q}^a \) or \( Q^a \) a sufficient number of times on any arbitrary initial state. In contrast, there is no way to reach states in a different diamond by a similar procedure. The diamonds without hidden indices represent \( \Phi \) superfields while those with hidden indices represent \( \Psi \) superfields, the

---

**Figure 1:** Diamond diagrams for superfields of super Yang-Mills theories with increasing amount of supersymmetry. The \( SU(N)_R \) indices \( a, b, c \) are labeled as superscripts, where \( a < b < c \) with \( a, b, c = 1, 2, \ldots, N \). The hidden indices, which refer to where they originated from in the maximally supersymmetric multiplet are indicated in parentheses. The numbers inside the diamonds show the number of corresponding states on each horizontal line.
CPT conjugates of Φ superfields. Superamplitudes of these superfields should be invariant under their $SU(N)_R$ transformation. This puts constraints on the corresponding $SU(N)_R$ indices $a, b, c$, but clearly not on the hidden indices inside parentheses.

The convenience of using this diamond representation is that we straightforwardly can obtain all states on the supergravity side from the product of two super Yang-Mills theories while we explicitly keep track of both the $SU(N)_R$ and the hidden indices. It is also easy to count the number of states on the supergravity side. We can construct diamonds for superfields of all the different supergravity theories. For maximal $N_G = 8$ supergravity we evidently need only one diamond, which contains all states from helicity $+2$ to helicity $-2$. Analogously to the situation on the gauge theory side, for $N_G < 8$ we need more than one diamond to express a complete set of states in the supergravity theory.

3.1 The equivalence of $N = 3$ and $N = 4$ super Yang-Mills theory

Before proceeding to the $N_G < 8$ super-KLT relations, we want to briefly dwell on the equivalence of $N = 3$ and $N = 4$ super Yang-Mills theories. This is textbook material, but it is instructive to see it in the light of our diamond representation. We find it directly from the diamonds in figure 1. For the $N = 3$ superfields, the two diamonds combined will contribute with one gluon, four fermions and six real scalars, which is exactly the same as the field content of $N = 4$ super Yang-Mills theory. We can recover $\eta_4$ of $\Psi^{N=3}$ and combine the two superfields $\Phi^{N=3}$ and $\Psi^{N=3}$ in the following way

$$\Phi^{N=3} + \eta_4 \Psi^{N=3}.$$ (3.6)

Using results of eqs. (3.2) and (3.4) we immediately see that

$$\Phi^{N=3} + \eta_4 \Psi^{N=3} = \Phi^{N=4},$$ (3.7)

which means that the two $N = 3$ superfields are nothing but a rewriting of the $N = 4$ superfield. For the $N \neq 3$ superfields there is no way of combining the Φ and Ψ superfields into one of larger $N$. In the following discussion, we will simply treat the $N = 3$ theory as equivalent to the $N = 4$ theory. In supergravity there is a completely equivalent phenomenon associated with the $N_G = 7$ superfields which can be combined into that of $N_G = 8$,

$$\Phi^{N_G=7} + \eta_8 \Psi^{N_G=7} = \Phi^{N_G=8},$$ (3.8)

so that also the $N_G = 7$ and $N_G = 8$ supergravity theories are equivalent. Actually, since we intend to construct the supergravity theories from gauge theories through super-KLT relations, we will only assume the equivalence of the $N = 3$ and $N = 4$ gauge theories. The equivalence of $N_G = 7$ and $N_G = 8$ supergravity theories is then an obvious consequence of the former equivalence.

3.2 Diamond diagrams and the $N_G < 8$ KLT-relations

In order to obtain $N < 4$ superfields from the $N = 4$ superfield, we set to zero or integrate out the unwanted $\eta_a$’s, as prescribed in eq. (3.1) and (3.3). Their resulting superfields can
be expressed graphically in terms of our diamond diagrams, as shown in figure [1]. Because of the super-KLT relation between two \( \mathcal{N} = 4 \) gauge theories to \( \mathcal{N}_G = 8 \) supergravity, it is possible to write not only the superfields of supergravity theories with smaller \( \mathcal{N}_G \) in terms of combinations of two gauge superfields, but also the supergravity superamplitude in those variables.

The identification of the expressions for the \( \mathcal{M}^{\mathcal{N}_G \leq 8}_n \) superamplitudes will be done in terms of its embedding in \( \mathcal{N}_G = 8 \) supergravity and the use of the canonical splitting of the \( SU(8)_R \) indices into the two subsets 1, 2, 3, 4 and 5, 6, 7, 8, for the two super Yang-Mills superamplitudes \( \tilde{A}^N_n \) and \( A^N_n \), respectively. So in general, we classify the external states of the supergravity superamplitudes into one of four representations, \( (\Phi, \Phi), (\bar{\Phi}, \Phi), (\Phi, \bar{\Phi}), (\bar{\Phi}, \bar{\Phi}) \), which will be explained in greater details below (here \( \Phi \) and \( \bar{\Phi} \) is just shorthand for \( \Phi^N \) and \( \bar{\Phi}^N \), respectively). In particular, if \( \tilde{N} = 4 \), then \( \Phi = \bar{\Phi} \) and the number of representations is reduced by a factor of 2.

By taking the KLT-product between two arbitrary superamplitudes we thereby get

\[
\begin{align*}
&\sum_{\gamma, \beta \in S_{n-3}} \tilde{A}^{\tilde{N} \leq 4}_{n, i_1, \ldots, i_m} (n - 1, n, \gamma, 1) S[|\beta|_{p_1}] A^{N \leq 4}_{n, i_1, \ldots, i_m} (1, \beta, n - 1, n) \\
&\quad = \sum_{\gamma, \beta \in S_{n-3}} \left[ \int \prod_{\tilde{a}_1 = \tilde{N} + 1}^4 d\eta_{\tilde{a}_1} \prod_{\tilde{a}_m = \tilde{N} + 1}^4 d\eta_{\tilde{a}_m} \tilde{A}^{\tilde{N} = 4}_n (n - 1, n, \gamma, 1) \right] \times S[|\gamma|_{p_1}] \left[ \int \prod_{a_1 = N + 5}^8 d\eta_{a_1} \prod_{a_m = N + 5}^8 d\eta_{a_m} A^{N = 4}_n (1, \beta, n - 1, n) \right] \\
&\quad = \left[ \int \prod_{\tilde{a}_1 = \tilde{N} + 1}^4 d\eta_{\tilde{a}_1} \prod_{\tilde{a}_m = \tilde{N} + 1}^4 d\eta_{\tilde{a}_m} \tilde{A}^{\tilde{N} = 4}_n (n - 1, n, \gamma, 1) S[|\beta|_{p_1}] A^{N = 4}_n (1, \beta, n - 1, n) \right] \\
&\quad = \mathcal{M}^{\mathcal{N}_G \leq 8}_{n, (i_1, \ldots, \tilde{i}_m); (i_1, \ldots, \tilde{i}_m)} \tag{3.9}
\end{align*}
\]

where the subscripts \((\tilde{i}_1, \ldots, \tilde{i}_m)\) and \((i_1, \ldots, \tilde{i}_m)\) label the external legs given by \( \tilde{\Psi} \) and \( \Psi \) fields, respectively. \( \tilde{m} \leq n \) and \( m \leq n \). In the second to last step, we used the \( \mathcal{N}_G = 8 \) super KLT-relation. \( \mathcal{M}^{\mathcal{N}_G \leq 8}_n \) in the last line is the superamplitude for \( \mathcal{N}_G \leq 8 \) supergravity, obtained from \( \mathcal{N}_G = 8 \) by setting to zero or integrating out unwanted \( \eta \)'s. To see this explicitly, we present the four possible superfields for an external leg \( k \).
\documentclass{article}

\usepackage{amsmath}

\begin{document}

\begin{itemize}
  \item \((\tilde{\Phi}, \Phi)\): if \(k \not\in (\tilde{i}_1, \ldots, \tilde{i}_m)\) and \(k \not\in (i_1, \ldots, i_m)\), we set all \(\eta_k, \tilde{N}+1, \ldots, \eta_k, 4\) and \(\eta_k, \tilde{N}+5, \ldots, \eta_k, 8\) to zero, and the resulting superfield is
  \[
  \Phi_k^{N_G=\tilde{N}+N} = \Phi_k^{N_G=8}|_{\eta_k, \tilde{N}+1, \ldots, \eta_k, 4, \eta_k, \tilde{N}+5, \ldots, \eta_k, 8 \to 0}. \tag{3.10}
  \]

  \item \((\bar{\Psi}, \Psi)\): if \(k \in (\tilde{i}_1, \ldots, \tilde{i}_m)\) and \(k \not\in (i_1, \ldots, i_m)\), we get another superfield
  \[
  \Psi_k^{N_G=\tilde{N}+N} = \int \prod_{a=\tilde{N}+1}^{4} d\eta_{k,a} \prod_{b=\tilde{N}+5}^{8} d\eta_{k,b} \Phi_k^{N_G=8}. \tag{3.11}
  \]

  These two superfields combine to form a full \(SU(N_G)\) supergravity multiplet.

  \item \((\bar{\Phi}, \Phi)\): if \(k \in (\tilde{i}_1, \ldots, \tilde{i}_m)\) and \(k \not\in (i_1, \ldots, i_m)\), we have
  \[
  \Theta_k^{N_G=\tilde{N}+N} = \int \prod_{a=\tilde{N}+1}^{4} d\eta_{k,a} \Phi_k^{N_G=8}|_{\eta_k, \tilde{N}+1, \ldots, \eta_k, 4, \eta_k, \tilde{N}+5, \ldots, \eta_k, 8 \to 0}. \tag{3.12}
  \]

  \item \(\Phi, \Psi)\): \(k \not\in (\tilde{i}_1, \ldots, \tilde{i}_m)\) and \(k \in (i_1, \ldots, i_m)\),
  \[
  \Gamma_k^{N_G=\tilde{N}+N} = \int \prod_{b=\tilde{N}+5}^{8} d\eta_{k,b} \Phi_k^{N_G=8}|_{\eta_k, \tilde{N}+1, \ldots, \eta_k, 4, \eta_k, \tilde{N}+5, \ldots, \eta_k, 8 \to 0}. \tag{3.13}
  \]

  The latter two superfields combine to form an \(SU(N_G)\) matter supermultiplet, if \(\tilde{N} < 3\) and \(N < 3\).

Thus \(\mathcal{M}_{N_G \leq 8}^{N_G \leq 8}((i_1, \ldots, i_m);(\tilde{i}_1, \ldots, \tilde{i}_m))\) in eq. (3.9) is the \(N_G \leq 8\) supergravity amplitude based on all four superfields \(\Phi_k^{N_G}, \Psi_k^{N_G}, \Theta_k^{N_G}\) and \(\Gamma_k^{N_G}\). This completes the formal construction of all \(N_G < 8\) super KLT-relations.

Since the states of the supergravity theories arise from the tensor product between two super Yang-Mills states, we can also label the superfield of the pertinent supergravity theory by the tensor product of two gauge theory superfields through our diamond diagrams. As an example, we illustrate this by the diamond diagram of the \(N_G = 8\) super KLT-relation. This is shown in figure 2. From this we can very easily establish how all states of the \(N_G = 8\) superfield arise from the tensor product; the one graviton state \(h^\pm\) comes from
  \[
  (+1) \otimes (+1)\quad \text{and} \quad (-1) \otimes (-1),
  \]
the \(8 \times 2\) gravitino states \(\psi^\pm\) from
  \[
  (+1/2)^4 \otimes (+1),\quad (+1) \otimes (+1/2)^4 \quad \text{and} \quad (-1/2)^4 \otimes (-1),\quad (-1) \otimes (-1/2)^4,
  \]
the \(28 \times 2\) vector states \(v^\pm\) arise through
  \[
  (0)^6 \otimes (+1),\quad (+1/2)^4 \otimes (+1/2)^4,\quad (+1) \otimes (0)^6 \quad \text{and} \quad (0)^6 \otimes (-1),\quad (-1/2)^4 \otimes (-1/2)^4,\quad (-1) \otimes (0)^6,
  \]

\end{itemize}

\end{document}
The 56 × 2 spin-1/2 fermions $\chi^\pm$ are built out of

$$(-1/2)^4 \otimes (+1), (0)^6 \otimes (+1/2)^4, (+1/2)^4 \otimes (0)^6, (1) \otimes (-1/2)^4,$$

$$(-1/2)^4 \otimes (-1), (0)^6 \otimes (-1/2)^4, (-1/2^4) \otimes (0)^6, (-1) \otimes (+1/2)^4,$$

and finally the 70 scalars

$$(-1) \otimes (+1), (-1/2)^4 \otimes (+1/2)^4, (0)^4 \otimes (0)^4, (+1/2)^4 \otimes (-1/2)^4, (+1) \otimes (-1),$$

where the superscripts denote the degeneracies of the states. This is exactly the field content of the $\mathcal{N}_G = 8$ supergravity theory. The $SU(8)_R$ indices can be recovered by combining two sets of $SU(4)_R$ indices. This is shown in figure 1.

In a similar manner we will now obtain all possible $\mathcal{N}_G < 8$ supergravity theories that can be constructed from KLT products. We will explicitly work out their field content and the corresponding diamond diagrams. This gives us the complete table of supergravity theories obtained from tensor products of super Yang-Mills theories in four dimensions. We have for convenience summarized our results in table 1.

We can classify all of these theories into three categories. Category I consists of maximal $\mathcal{N}_G = 8$ supergravity, its equivalent $\mathcal{N}_G = 7$ supergravity theory, and the $(\tilde{\mathcal{N}} = 3) \otimes (\mathcal{N} = 3)$ theory. Since their field contents are all the same, one must consider them as describing the same theory just encoded in slightly different ways. The super KLT-relations for these theories are equivalent to eq. 2.8.

Category II contains all minimal supergravity theories with $4 \leq \mathcal{N}_G < 8$. These theories contain only the minimal supergravity multiplet (the one-graviton supermultiplet). They are given in terms of two diamonds (which arise from the two superfields $\Phi$ and $\Psi$) to describe the complete set of states. These theories all arise from the KLT product $(\tilde{\mathcal{N}} = 4) \otimes (\mathcal{N} \leq 2)$ (or $(\tilde{\mathcal{N}} = 3) \otimes (\mathcal{N} \leq 2)$ due to the equivalence between $\tilde{\mathcal{N}} = 3$ and

**Figure 2:** Diamond diagrams that demonstrates the matching of states in (supergravity)$_{\mathcal{N}_G=8} = (\text{super Yang-Mills})_{\tilde{\mathcal{N}}=4} \otimes (\text{super Yang-Mills})_{\mathcal{N}=4}$. The numbers inside the diamonds indicate the number of states on each line, and the number next to the dots indicate the helicities. Only the highest, lowest and zero helicities have been labeled explicitly.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$\mathcal{N}_G$ & $\mathcal{N} \otimes \mathcal{N}$ & Description \\
\hline
8 & $4 \otimes 4$ & Maximal $\mathcal{N}_G = 8$ Supergravity \\
7 & $4 \otimes 3$ & Maximal $\mathcal{N}_G = 8$ Supergravity \\
6 & $4 \otimes 2$ & Minimal $\mathcal{N}_G = 6$ Supergravity with $SU(6)$ supergravity multiplet \\
6 & $3 \otimes 3$ & Maximal $\mathcal{N}_G = 8$ Supergravity \\
5 & $4 \otimes 1$ & Minimal $\mathcal{N}_G = 5$ Supergravity with $SU(5)$ supergravity multiplet \\
5 & $3 \otimes 2$ & Minimal $\mathcal{N}_G = 6$ Supergravity with $SU(6)$ supergravity multiplet \\
4 & $4 \otimes 0$ & Minimal $\mathcal{N}_G = 4$ Supergravity with $SU(4)$ supergravity multiplet \\
4 & $3 \otimes 1$ & Minimal $\mathcal{N}_G = 5$ Supergravity with $SU(5)$ supergravity multiplet \\
4 & $2 \otimes 2$ & $\mathcal{N}_G = 4$ Supergravity multiplet coupled to vector multiplet \\
3 & $3 \otimes 0$ & Minimal $\mathcal{N}_G = 4$ Supergravity with $SU(4)$ supergravity multiplet \\
3 & $2 \otimes 1$ & $\mathcal{N}_G = 3$ Supergravity multiplet coupled to vector multiplet \\
2 & $2 \otimes 0$ & $\mathcal{N}_G = 2$ Supergravity multiplet coupled to vector multiplet \\
2 & $1 \otimes 1$ & $\mathcal{N}_G = 2$ Supergravity multiplet coupled to hypermultiplet \\
1 & $1 \otimes 0$ & $\mathcal{N}_G = 1$ Supergravity multiplet coupled to chiral multiplet \\
0 & $0 \otimes 0$ & Einstein gravity coupled to two scalars \\
\hline
\end{tabular}
\caption{Full list of all possible supergravity theories that can be constructed from KLT-relations of minimal super Yang-Mills theories with varying degree of supersymmetry.}
\end{table}

$\tilde{N} = 4$). The corresponding super KLT-relations can be expressed as

$$
\mathcal{M}_n^{\mathcal{N}_G=4+\mathcal{N}}(\Phi_{i_1,...,i_{m_1}}^{\mathcal{N}_G}, \Psi_{j_1,...,j_{m_2}}^{\mathcal{N}_G}) = \\
\sum_{\gamma,\beta \in S_{n-3}} \mathcal{A}_n^{\mathcal{N} = 4}(\Phi_{1,...,n}^{\mathcal{N} = 4}) \times S[\gamma|\beta]_{p_1} \times \mathcal{A}_n^{\mathcal{N} \leq 2}(\Phi_{i_1,...,i_{m_1}}, \Psi_{j_1,...,j_{m_2}}^{\mathcal{N} \leq 2}),
$$

(3.14)

where the indices $(i_1,...,i_{m_1})$ and $(j_1,...,j_{m_2})$ denote the legs of the corresponding superfields and $m_1 + m_2 = n$.

Category III includes the remaining theories: they describe minimal supergravity coupled to a variety of matter multiplets, and requires four diamonds to describe the full CPT-complete state space. This means that we have four kinds of superfields: $\Phi$, $\Psi$, and $\Theta$. The super KLT-relation can be compactly expressed as

$$
\mathcal{M}_n^{\mathcal{N}_G=\tilde{N}+\mathcal{N}}(\Phi_{i_1,...,i_{m_1}}^{\mathcal{N}_G}, \Psi_{j_1,...,j_{m_2}}^{\mathcal{N}_G}, \Theta_{k_1,...,k_{m_3}}^{\mathcal{N}_G}, \Gamma_{l_1,...,l_{m_3}}^{\mathcal{N}_G}) = \\
\sum_{\gamma,\beta \in S_{n-3}} \mathcal{A}_n^{\tilde{N} \leq 2}(\Phi_{i_1,...,i_{m_1}}, \Psi_{j_1,...,j_{m_2},k_1,...,k_{m_3}}^{\tilde{N} \leq 2}) \\
\times S[\gamma|\beta]_{p_1} \times \mathcal{A}_n^{\mathcal{N} \leq 2}(\Phi_{i_1,...,i_{m_1},k_1,...,k_{m_3}}, \Psi_{j_1,...,j_{m_2},l_1,...,l_{m_3}}^{\mathcal{N} \leq 2}),
$$

(3.15)

where again $(i_1,...,i_{m_1}), (j_1,...,j_{m_2}), (k_1,...,k_{m_3})$ and $(l_1,...,l_{m_3})$ label legs of the corresponding superfields and $m_1 + m_2 + 2m_3 = n$. Note that the number of $\Theta$ superfields must match the number of $\Gamma$ superfields, otherwise the $SU(\mathcal{N}_G)_R$ symmetry will be violated.

It is interesting to note that there are no KLT-maps (from minimal super Yang-Mills theories) that provide just minimal supergravity multiplets for $\mathcal{N}_G < 4$: in all cases we get...
supergravity coupled to matter multiplets. However, if we want to project out the subset of KLT-relations that would appear in these minimal supergravity theories we need to restrict the sum over indices on the right hand side of the KLT-relations. We can achieve such a projection by making the restriction $m = \tilde{m}$ and $i_{j} = i_{j}$ for $j = 1, \ldots, m$ in eq. (3.3).

There are also vanishing identities for the $\mathcal{N} < 4$ super Yang-Mills amplitudes that follow from these $\mathcal{N}_{G} < 8$ super KLT-relations much like the ones in eq. (2.9) for the maximally supersymmetric case. We will later see how to explain the vanishing identities from the point of view of less than maximal supersymmetry.

We will now go through all the different cases summarised in table A using the diamond diagrams. The explicit expressions for the superfields can be found in Appendix A.

### 3.2.1 Diamond diagrams for the $\mathcal{N}_{G} = 7$ theory

![Diamond diagram](image)

**Figure 3:** Diamond diagram for (supergravity)$\mathcal{N}_{G}=7$ = (super Yang-Mills)$\mathcal{N}=4 \otimes$ (super Yang-Mills)$\mathcal{N}=3$. The two diamonds represent the $\Phi^{\mathcal{N}_{G}=7}$ and $\Psi^{\mathcal{N}_{G}=7}$ superfields, respectively. Note that there is a hidden $SU(8)_{R}$ index in the $\Psi$ field, for instance, the negative helicity graviton state is $-2^{1234567(8)}$.

The KLT-product between $\mathcal{N} = 4$ and $\mathcal{N} = 3$ super Yang-Mills superamplitudes maps exactly to $\mathcal{N}_{G} = 7$ supergravity. This is illustrated by the tensor product between the superfields represented by diamonds in figure 3. The $SU(\mathcal{N})_{R}$ indices for component fields of the super Yang-Mills theories have already been shown in figure 2. The $SU(\mathcal{N}_{G})_{R}$ indices come from combining these two sets. The index “8” is now a hidden index in the $\mathcal{N} = 3$ diamond, and thus it is also a hidden index for the $\Psi^{\mathcal{N}_{G}=7}$ superfield, which is shown as the second diamond on the right hand side of figure 3. From this it is easy to see that

$$\Phi^{\mathcal{N}_{G}=8} = \Phi^{\mathcal{N}_{G}=7} + \eta_{8} \Psi^{\mathcal{N}_{G}=7},$$  

which again displays the fact that $\mathcal{N}_{G} = 7$ supergravity is just a rewriting of $\mathcal{N}_{G} = 8$ supergravity. Likewise, the content of the super KLT-relation for $\mathcal{N}_{G} = 7$ is the same as the $\mathcal{N}_{G} = 8$ version in figure 2, just encoded in a slightly different way.

### 3.2.2 Diamond diagrams for the $\mathcal{N}_{G} = 6$ theories

The case of $(\mathcal{N} = 4) \otimes (\mathcal{N} = 2)$ is rather straightforward. There is an exact correspondence between minimal $\mathcal{N}_{G} = 6$ supergravity superamplitudes and the KLT-product between
Figure 4: Diamond diagram for \( (\text{supergravity})_{\mathcal{N}G=6} = (\text{super Yang-Mills})_{\mathcal{N}=4} \otimes (\text{super Yang-Mills})_{\mathcal{N}=2} \). The two diamonds represent the \( \Phi_{\mathcal{N}G=6} \) and \( \Psi_{\mathcal{N}G=6} \) superfields, respectively. There are two hidden indices (78) for the \( \Psi \) field.

\( \mathcal{N} = 4 \) and \( \mathcal{N} = 2 \) super Yang-Mills superamplitudes. This gravity theory consists of 1 graviton \( h_{\pm} \), 6 gravitinos \( \psi_{\pm} \), 16 vectors \( v_{\pm} \), 26 spin-1/2 fermions \( \chi_{\pm} \) and 30 scalars. This is illustrated in figure 4.

\[ \Phi_{\mathcal{N}G=8} = \Phi_{\mathcal{N}G=6} + \eta_4 \Theta_{\mathcal{N}G=6} + \eta_8 \Gamma_{\mathcal{N}G=6} + \eta_4 \eta_8 \Psi_{\mathcal{N}G=6} . \]  (3.17)

This is just a rewriting of maximal \( \mathcal{N}_G = 8 \) supergravity. This is expected since \( \mathcal{N} = 3 \) super Yang-Mills is just a rewriting of \( \mathcal{N} = 4 \) super Yang-Mills. However, using the aforementioned restriction \( m = \bar{m} \) and \( \bar{i}_j = i_j \) for \( j = 1, \ldots, m \) in eq. (3.9), we can project out the \( \mathcal{N}_G = 6 \) superamplitudes from the \( (\mathcal{N} = 3) \otimes (\mathcal{N} = 3) \) product if desired.

3.2.3 Diamond diagrams for the \( \mathcal{N}_G = 5 \) theories

There is again two cases to consider. The first one is from the product between \( \mathcal{N} = 4 \) and \( \mathcal{N} = 1 \) super Yang-Mills theory, see figure 6. There is one \( (+2, +\frac{5}{2}, +\frac{1}{2}, 0, -\frac{11}{2} \) ) supergravity diamond plus its CPT conjugate, and a total of 1 graviton \( h_{\pm} \), 5 gravitinos
Figure 6: Diamond diagram for (supergravity)\(\mathcal{N}_G=5\) = (super Yang-Mills)\(\mathcal{N}=4\) \(\otimes\) (super Yang-Mills)\(\mathcal{N}=1\). The two diamonds represent the \(\Phi^{\mathcal{N}_G=5}\) and \(\Psi^{\mathcal{N}_G=5}\) superfields. There are hidden indices (678) for the \(\Psi\) field.

\(\psi_{\pm}\), 10 vectors \(v_{\pm}\), 11 spin-1/2 fermions \(\chi_{\pm}\) and 10 scalars. This is minimal \(\mathcal{N}_G = 5\) supergravity.

Figure 7: Diamond diagram for (supergravity)\(\mathcal{N}_G=5\) = (super Yang-Mills)\(\mathcal{N}=3\) \(\otimes\) (super Yang-Mills)\(\mathcal{N}=2\). The four diamonds represent the four superfields \(\Phi^{\mathcal{N}_G=5}\), \(\Theta^{\mathcal{N}_G=5}\), \(\Gamma^{\mathcal{N}_G=5}\) and \(\Psi^{\mathcal{N}_G=5}\). The hidden indices are (4) for \(\Theta\), (78) for \(\Gamma\) and (478) for \(\Psi\).

The second case is obtained by tensoring \(\tilde{\mathcal{N}} = 3\) with \(\mathcal{N} = 2\), as shown in figure 7. There are four diamonds, decreasing in helicity by half a unit. By recovering the hidden indices from the diagrams, one sees immediately that we can write

\[
\Phi^{\mathcal{N}_G=6} = \Phi^{\mathcal{N}_G=5} + \eta_4 \Theta^{\mathcal{N}_G=5}, \quad \Psi^{\mathcal{N}_G=6} = \Gamma^{\mathcal{N}_G=5} + \eta_4 \Psi^{\mathcal{N}_G=5}.
\]

(3.18)

So the four superfields of this theory are just a rewriting of the two superfields \(\Phi^{\mathcal{N}_G=6}\) and \(\Psi^{\mathcal{N}_G=6}\) of minimal \(\mathcal{N}_G = 6\) supergravity.

3.2.4 Diamond diagrams for the \(\mathcal{N}_G = 4\) theories

In this case we have three KLT constructions given by \((\tilde{\mathcal{N}} = 4) \otimes (\mathcal{N} = 0)\), \((\tilde{\mathcal{N}} = 3) \otimes (\mathcal{N} = 1)\) and \((\tilde{\mathcal{N}} = 2) \otimes (\mathcal{N} = 2)\). The first one is shown in figure 8. It contains only a supergravity multiplet and its CPT conjugate, with a total of 1 graviton \(h_{\pm}\), 4 gravitinos \(\psi_{\pm}\), 6 vectors
\( v_{\pm} \), 4 spin-1/2 fermions \( \chi_{\pm} \) and two scalars. This is minimal \( \mathcal{N}_G = 4 \) supergravity. It was this particular KLT-map that was recently used by Bern et al. to study finiteness properties of \( \mathcal{N}_G = 4 \) supergravity [30].

The second construction is illustrated in figure 8. As there is just half a unit of helicity between both the \( \Phi^{\mathcal{N}_G=4} \) and \( \Theta^{\mathcal{N}_G=4} \) superfields and the \( \Gamma^{\mathcal{N}_G=4} \) and \( \Psi^{\mathcal{N}_G=4} \) superfields, we can readily recover the hidden indices, and reassemble these four superfields into

\[
\Phi^{\mathcal{N}_G=5} = \Phi^{\mathcal{N}_G=4} + \eta_4 \Theta^{\mathcal{N}_G=4}, \quad \Psi^{\mathcal{N}_G=5} = \Gamma^{\mathcal{N}_G=4} + \eta_4 \Psi^{\mathcal{N}_G=4}.
\]

Indeed, this theory is nothing but a rewriting of minimal \( \mathcal{N}_G = 5 \) supergravity.

The third construction is shown in figure 9. In contrast to the result of the previous two cases, this theory contains two extra diamonds besides the minimal supergravity multiplet. These two diamonds of additional states cannot be reassembled into new superfields.
with higher $\mathcal{N}_G$. Therefore, for the first time in our systematic approach we here have a case where in addition to a simple supergravity the KLT-map also provides us with a matter multiplet. In detail, besides the field content of the $(\tilde{\mathcal{N}} = 4) \otimes (\mathcal{N} = 0)$ theory, the additional matter fields combine to form a vector multiplet. It consists of 2 vector fields with helicity $\pm 1$, 8 fermion fields of helicity $\pm 1/2$ and 12 scalars. The resulting theory is $\mathcal{N}_G = 4$ supergravity coupled to these matter fields.

**Figure 10:** Diamond diagram for $(\text{supergravity})_{\mathcal{N}_G=4} = (\text{super Yang-Mills})_{\tilde{\mathcal{N}}=2} \otimes (\text{super Yang-Mills})_{\mathcal{N}=2}$. The four diamonds represent the $\Phi_{\mathcal{N}_G=4}$, $\Theta_{\mathcal{N}_G=4}$ vector, $\Gamma_{\mathcal{N}_G=4}$ vector and $\Psi_{\mathcal{N}_G=4}$ superfields. $\Theta$ and $\Gamma$ correspond to CPT self-conjugate vector multiplets, but they have different sets of $SU(4)_R$ indices. The hidden indices are $(34)$ for $\Theta$, $(78)$ for $\Gamma$ and $(3478)$ for $\Psi$.

### 3.2.5 Diamond diagrams for the $\mathcal{N}_G = 3$ theories

There are two cases to consider. The first comes from the product $(\tilde{\mathcal{N}} = 3) \otimes (\mathcal{N} = 0)$, which is illustrated in figure 11. Superficially this appears to be a supergravity multiplet coupled to a gravitino supermultiplet. However, we notice that the first two diamonds have a helicity difference of half a unit, likewise for the last two diamonds. Thus we can

**Figure 11:** Diamond diagram for $(\text{supergravity})_{\mathcal{N}_G=3} = (\text{super Yang-Mills})_{\tilde{\mathcal{N}}=3} \otimes (\text{Yang-Mills})_{\mathcal{N}=0}$. The four diamonds represent the four superfields $\Phi_{\mathcal{N}_G=3}$, $\Theta_{\mathcal{N}_G=3}$, $\Gamma_{\mathcal{N}_G=3}$ and $\Psi_{\mathcal{N}=3}$. The hidden indices are $(3)$ for $\Theta$, $(5678)$ for $\Gamma$ and $(45678)$ for $\Psi$. 
reconstruct each of them into a bigger superfield by recovering the hidden index, i.e.

$$\Phi_{\mathcal{N}_G=4} = \Phi_{\mathcal{N}_G=3} + \eta_4 \Theta_{\mathcal{N}_G=3}, \quad \Psi_{\mathcal{N}_G=4} = \Gamma_{\mathcal{N}_G=3} + \eta_4 \Psi_{\mathcal{N}_G=3}. \quad (3.20)$$

The resulting theory is simply minimal $\mathcal{N}_G = 4$ supergravity encoded in a slightly different way. This is as expected since $\mathcal{N} = 3$ super Yang-Mills can be identified with $\mathcal{N} = 4$ super Yang-Mills theory.

The second case is $(\tilde{\mathcal{N}} = 2) \otimes (\mathcal{N} = 1)$, see figure 12. Besides the usual field content of minimal $\mathcal{N}_G = 3$ supergravity, this theory has additional fields from the matter supermultiplet, which includes 1 vector field $\pm 1$, 4 fermion fields $\pm 1/2$ and 6 scalars. This is a theory of minimal $\mathcal{N}_G = 3$ supergravity coupled to a vector multiplet.

### 3.2.6 Diamond diagrams for the $\mathcal{N}_G = 2$ theories

There are again two constructions to consider. The first one comes from the product $(\tilde{\mathcal{N}} = 2) \otimes (\mathcal{N} = 0)$, as shown in figure 13. This gives a theory containing a minimal $\mathcal{N}_G = 2$ supergravity multiplet coupled to a vector multiplet. The supergravity multiplet contains 1 graviton $h_{\pm}$, 2 gravitinos $\psi_{\pm}$ and 1 vector $\chi_{\pm}$, while the vector multiplet contains 1 vector field of helicity $\pm 1$, 2 fermion fields of helicity $\pm 1/2$ and two scalars.

The second construction is given by the product of $(\tilde{\mathcal{N}} = 1) \otimes (\mathcal{N} = 1)$, as illustrated in figure 14. This provides also a theory of minimal $\mathcal{N}_G = 2$ supergravity coupled to matter, but now to a hypermultiplet $\Theta_{\text{hyper}}$, $\Gamma_{\text{hyper}}$, which is different from the vector multiplet of the first construction. Besides the usual field content of the minimal $\mathcal{N}_G = 2$ supergravity multiplet, there are now also 2 fermion fields of helicity $\pm 1/2$ and 4 scalars from the hypermultiplet.

### 3.2.7 Diamond diagrams for the $\mathcal{N}_G = 1$ theory

There is only one KLT-construction for this theory, namely $(\tilde{\mathcal{N}} = 1) \otimes (\mathcal{N} = 0)$, and it is shown in figure 15. This is a theory of minimal $\mathcal{N}_G = 1$ supergravity coupled to a
Figure 13: Diamond diagram for \((\text{supergravity})_{\mathcal{N}_G=2} = (\text{super Yang-Mills})_{\mathcal{N}=2} \otimes (\text{Yang-Mills})_{\mathcal{N}=0}\). The four diamonds represent \(\Phi^{N_G=2}\), \(\Theta^{N_G=2}\), \(\Gamma^{N_G=2}\) vector and \(\Psi^{N_G=2}\). The hidden indices are \((34)\) for \(\Theta\), \((5678)\) for \(\Gamma\) and \((345678)\) for \(\Psi\).

Figure 14: Diamond diagram for \((\text{supergravity})_{\mathcal{N}_G=2} = (\text{super Yang-Mills})_{\mathcal{N}=1} \otimes (\text{super Yang-Mills})_{\mathcal{N}=1}\). The four diamonds represent the superfields \(\Phi^{N_G=2}\), \(\Theta^{N_G=2}\), \(\Gamma^{N_G=2}\) hyper and \(\Psi^{N_G=2}\). Hidden indices are \((234)\) for \(\Theta\), \((678)\) for \(\Gamma\) and \((234678)\) for \(\Psi\).

hypermultiplet. There is 1 graviton \(h_\pm\) and 1 gravitino \(\psi_\pm\) from the supergravity multiplet, and 1 helicity-1/2 fermion field, as well as 2 scalars from the hypermultiplet. We can work out all the states from tensor product of the (super) Yang-Mills theories, while keeping track of both \(R\)-indices and hidden \(R\)-indices. We have

\[\Phi^{N_G=1}: \quad (+1) \otimes (+1) = +2, \quad (+1) \otimes (+1) = +\frac{3}{2},\]

\[\Theta_{\text{chiral}}^{N_G=1}: \quad \left(-\frac{1}{2}\right)^{(234)} \otimes (+1) = +\frac{1}{2}, \quad (-1)^{(234)} \otimes (+1) = 0^{(234)};\]

\[\Gamma_{\text{chiral}}^{N_G=1}: \quad (+1) \otimes (-1)^{(5678)} = 0^{(5678)}, \quad (-1)^{\frac{1}{2}} \otimes (-1)^{(5678)} = -\frac{1}{2}^{(5678)};\]

\[\Psi^{N_G=1}: \quad \left(-\frac{1}{2}\right)^{(234)} \otimes (-1)^{(5678)} = -\frac{3}{2}^{(2345678)}, \quad (-1)^{(234)} \otimes (-1)^{(5678)} = -2^{(2345678)};\]
and all states are thus completely identified.

### 3.2.8 Diamond diagrams for the \( \mathcal{N}_G = 0 \) theory

![Diamond diagram for \( \mathcal{N}_G = 0 \)](image)

**Figure 16:** “Diamond” diagram for \((\text{gravity})_{\mathcal{N}_G=0} = (\text{Yang-Mills})_{\mathcal{N}=0} \otimes (\text{Yang-Mills})_{\mathcal{N}=0}\). This is just the classic case of gravity as the square of two Yang-Mills theories. The origin of the two additional scalars is clear in the language of hidden indices.

For the \((\tilde{\mathcal{N}} = 0) \otimes (\mathcal{N} = 0)\) product the diamonds are reduced to simple dots, *i.e.* single states. We can still draw the “diamond diagram” shown in figure [16]. This is Einstein gravity coupled to two scalars. The two scalars come from tensor product of \((+1) \otimes (-1_{(5678)}), (-1_{(1234)}) \otimes (+1)\). Although the scalars do not have explicit \(R\)-indices, they do carry different hidden \(R\)-indices.

### 4. Symmetry groups of supergravity theories from super KLT-relations

In this section, we will discuss the symmetry groups of the different supergravity theories
that can be constructed from KLT-products. Much of this may be found in the supergravity literature, but it is quite instructive to see the emergence of supergravity symmetries from the tensoring of two super Yang-Mills theories.

4.1 Maximal supersymmetric theories

For supersymmetric theories with $\mathcal{N}$ supercharge generators $Q^A, \tilde{Q}_A, A = 1, 2, \ldots, \mathcal{N}$, it is possible to have an $U(\mathcal{N})_R$ group that rotates $Q^A$ or $\tilde{Q}_A$. It is customary to decompose the $U(\mathcal{N})_R$ into $U(\mathcal{N})_R = SU(\mathcal{N})_R \otimes U(1)_R$. The Grassmann variables $\eta_A$ transform under $SU(\mathcal{N})_R$, and the superamplitudes should be invariant under such a transformation. This gives constraints on the $SU(\mathcal{N})_R$ indices of a given superamplitude. For both supergravity and super Yang-Mills theories, the $SU(\mathcal{N})_R$ group will surely be an invariant.

Let us now first briefly review the lack of an additional non-trivial $U(1)_R$ group in maximally supersymmetric theories. If there would be a non-trivial $U(1)_R$ group for the $\mathcal{N} = 4$ super Yang-Mills theory, then at least some of the component fields should carry non-vanishing $U(1)_R$ charges. It is obvious that we cannot assign non-trivial $U(1)_R$ charges to the gauge bosons (and such charges would be in contradiction with the existence of non-vanishing pure-gluon MHV amplitudes). We could also try to assign a $U(1)_R$ charge $\beta$ for the scalars, but recall that the superfield for $\mathcal{N} = 4$ super Yang-Mills theory is CPT self-conjugate. In our terminology, all states are inside one single diamond, as shown in figure [1]. The scalars in this diamond satisfy a self-duality relation under complex conjugation. As a consequence, if we assign, say, $U(1)_R$ charge $\beta$ to the scalar $\phi^{12}$, then the scalar $\phi^{34} = (\phi^{12})^\dagger$ will carry a $U(1)_R$ charge of $-\beta$. Being states inside the same diamond, they should carry the same charge. This gives $\beta = -\beta$, and forces all potential $U(1)_R$ charges for the scalars to vanish. Finally, could there be non-trivial $U(1)_R$ charges for the fermions? The Yukawa couplings to the scalars forbid these, since the scalars are neutral. From this argument one concludes that even if there is potentially a $U(1)_R$ symmetry group in $\mathcal{N} = 4$ super Yang-Mills theory, all component fields have vanishing charges under this symmetry and it therefore plays no role. For a review of global symmetries in $\mathcal{N} = 4$ super Yang-Mills theory, see [31].

A similar argument carries through for the maximally supersymmetric supergravity theory. Also here a crucial ingredient is that the superfield of $\mathcal{N}_G = 8$ supergravity is CPT self-conjugate, so all states belong to one single diamond as shown in figure [2]. The $U(1)_R$ charge for the graviton must be zero. The $U(1)_R$ charges for the scalars should also be zero, because scalars and their complex-conjugate partners are all inside the same diamond; they must hence carry the same $U(1)_R$ charges, which clearly forces all these charges to vanish. For the remaining fields we need to look at their interactions. In the present setting, this is actually most easily done by evaluating on-shell three-point amplitudes, which directly correspond to interaction vertices in the supergravity Lagrangian. In maximally supersymmetric supergravity, we find non-vanishing three-point amplitudes such as

$$\mathcal{M}(v^-, \chi_-, \chi^-) = \frac{\langle 12 \rangle^3 \langle 13 \rangle^3 \langle 23 \rangle^2}{\langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2} = \langle 12 \rangle \langle 13 \rangle,$$  \hspace{1cm} (4.1)
which represents two Weyl spinors coupled to a graviphoton. Indeed, this is what in the Lagrangian corresponds to a helicity-changing three-vertex through a gauge invariant Pauli term.

There is also

\[
\mathcal{M}(\phi, \chi^-, \psi^-) = \frac{\langle 12 \rangle \langle 13 \rangle^3 \langle 23 \rangle^4}{(12)^2 (23)^2 (31)^2} = \frac{\langle 13 \rangle \langle 23 \rangle^2}{\langle 12 \rangle},
\]

(4.2)

which is the derivative coupling for a scalar field to a gravitino and a spin-1/2 fermion. And another three-vertex corresponds to

\[
\mathcal{M}(\phi, v^-, v^-) = \frac{\langle 12 \rangle^2 \langle 13 \rangle \langle 23 \rangle^4}{(12)^2 (23)^2 (31)^2} = \langle 23 \rangle^2,
\]

(4.3)

which is a scalar coupled to two graviphotons. Since all of these amplitudes (and their corresponding vertices) are non-vanishing, their total \(U(1)_R\) charges must be zero. From the vanishing charge of the scalar and three-vertex \(\mathcal{M}(\phi, v^-, v^-)\) we know that the \(U(1)_R\) charge for graviphoton \(v\) with helicity \(-1\) should be zero. By further taking into account the three-vertex corresponding to the amplitudes \(\mathcal{M}(v^-, \chi^-, \chi^-)\) we assure that the \(U(1)_R\) charge for graviphotino \(\chi\) with helicity \(-1/2\) should be zero. Finally, from the three-vertex corresponding to \(\mathcal{M}(\phi, \chi^-, \psi^-)\) we conclude that \(U(1)_R\) charge for the gravitino \(\psi\) of helicity \(-3/2\) should be zero. The same vanishing \(U(1)_R\) charge must clearly be assigned to their complex conjugate partners. From this simple argument, we conclude that even if there is a potential \(U(1)_R\) symmetry group for maximally supersymmetric supergravity, all the component fields will have zero \(U(1)_R\) charge. So such a group can play no role in this theory either.

The global linear symmetry groups of \(N_G = 8\) supergravity and \(N = 4\) super Yang-Mills theory are therefore \(SU(8)\) and \(SU(4)\), respectively. For \(N_G = 8\) supergravity, there are also non-linear global symmetries, and the total global symmetry group is \(E_7(7)\) [32]. However, the non-linear symmetries do not generate vanishing identities and they only manifest themselves in the soft limit of scattering amplitudes [33].

### 4.2 \(4 \leq N_G < 8\) minimal supergravity

For the minimal \(N_G = 4, 5, 6\) supergravity theories, there are only supergravity multiplets, and therefore, in our notation, only the two diamonds representing the \(\Phi\) and \(\Psi\) fields. Again there is a \(U(N_G)_R\) group that rotates \(Q^A\) and \(\tilde{Q}_A\), where \(A = 1, 2, \ldots, N_G\). The \(SU(N_G)_R\) group will for sure be an invariant group for the amplitudes, but what about the \(U(1)_R\) group? Let us take the \(N_G = 6\) minimal supergravity theory as an example, and discuss the role of \(U(1)_R\) symmetry in such minimal supergravity theories.

The diamond diagram for \(N_G = 6\) minimal supergravity is shown in figure 4. We can read off the explicit \(U(1)_R\) symmetry of \(N_G = 6\) by truncating the \(N_G = 8\) theory. We remind ourselves that the component fields not only carry \(SU(6)_R\) indices but also the hidden (78) indices. These hidden indices will appear in the component fields of the \(\Psi\) superfield, and they always come out neatly joined. The generators of the symmetry group
acting on the $R$-indices in the $\mathcal{N}_G = 6$ theory are $6 \times 6$ matrices. They are embedded in
the Lie-algebra of $SU(8)$. Consider the following $8 \times 8$ traceless Hermitian matrix
\[
\begin{pmatrix}
\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\
\end{pmatrix}, \tag{4.4}
\]
which is divided into two blocks. The first $6 \times 6$ block acts on the $R$-indices $1, 2, \ldots, 6$ and
the second $2 \times 2$ block $B_{ij}$ acts on the hidden indices $(78)$. Since we are considering the role
of the $U(1)_R$ symmetry, we take the first block to commute with all $SU(6)_R$ generators. It
must therefore be a diagonal matrix proportional to the identity, and we write it as $\alpha I_{6 \times 6}$
in the above matrix. This will make each $R$-index $1, 2, \ldots, 6$ correspond to a charge $\alpha$.
What is the effect of $B_{ij}$ acting on the hidden indices $(78)$?

For a more general discussion, we will consider the effect of a $k \times k$ matrix $B_{x_i x_j}$ acting
on the indices $(x_1, x_2, \ldots, x_k)$. Since Grassmann variables $\eta$ are totally anti-commuting,
this actually means a $k \times k$ matrix $B_{ij}$ acting on $\eta_{x_1} \wedge \eta_{x_2} \wedge \cdots \wedge \eta_{x_k}$, where we consider
the $k$ Grassmann variables $\eta$ as spanning a superspace, with each $\eta_{x_i}$ being a basis vector
$\eta_{x_i} = (0, 0, 1, 0, \ldots, 0)^T$, and the $1$ is in the $i$-th position. Then the matrix $B_{ij}$ acting
on the basis vector $\eta_{x_i}$ gives
\[
\begin{pmatrix}
B_{11} & \cdots & \cdots & B_{1k} \\
\vdots & \ddots & & \vdots \\
\vdots & & \ddots & \vdots \\
B_{k1} & \cdots & \cdots & B_{kk}
\end{pmatrix}_{k \times k}
\begin{pmatrix}
0 \\
\vdots \\
1 \\
\vdots \\
0
\end{pmatrix}
= \begin{pmatrix}
B_{11} \\
\vdots \\
B_{2i} \\
\vdots \\
B_{ki}
\end{pmatrix}_{k \times i} = \sum_{j=1}^{k} B_{ji} \eta_{x_j}, \tag{4.5}
\]
so
\[
B(\eta_{x_1} \wedge \eta_{x_2} \wedge \cdots \wedge \eta_{x_k}) = \sum_{i=1}^{k} \eta_{x_1} \wedge \cdots \wedge (B_{\eta_{x_i}}) \wedge \cdots \wedge \eta_{x_k}
= \sum_{i=1}^{k} \eta_{x_1} \wedge \cdots \wedge (\sum_{j=1}^{k} B_{ji} \eta_{x_j}) \wedge \cdots \wedge \eta_{x_k}
= \sum_{i=1}^{k} \eta_{x_1} \wedge \cdots \wedge (B_{ii} \eta_{x_i}) \wedge \cdots \wedge \eta_{x_k}
= (\sum_{i=1}^{k} B_{ii}) (\eta_{x_1} \wedge \eta_{x_2} \wedge \cdots \wedge \eta_{x_k}),
\]
which means that only the trace part of the matrix $B_{ij}$ is important when acting on the
hidden indices. Note that this is true only when the hidden indices appear together, so
that they can be expressed as \((\eta_{x_1} \wedge \eta_{x_2} \wedge \cdots \wedge \eta_{x_k})\). If we denote \(\beta = Tr(B_{ij})\), we can then assign a charge of \(\beta\) to the hidden indices \((x_1, x_2, \ldots, x_k)\).

Let us now return to the \(\mathcal{N}_G = 6\) minimal supergravity theory. For each \(R\)-index \(1, 2, \ldots, 6\) we assigned a charge \(\alpha\), and for the hidden indices \((78)\) we assigned a charge of \(\beta\). Because of the traceless condition on \(SU(8)\) generators we have \(\beta = -6\alpha\). In this setup we can hence assign charges for all component fields through their \(SU(6)_R\) and/or hidden indices. For the \(\Phi\)-superfield diamond we have

| Helicity | KLT Products | Charge |
|----------|--------------|--------|
| +2       | \((1) \otimes (+1)\) | 0      |
| \(\frac{3}{2}\) | \((+\frac{1}{2} a_1) \otimes (+1), \ (+\frac{1}{2} a_1) \otimes (+\frac{1}{2} b_1)\) | \(\alpha\) |
| +1       | \((0^a \cdot a_2) \otimes (+1), \ (+\frac{1}{2} a_1) \otimes (+\frac{1}{2} b_1)\) | \(2\alpha\) |
| \(\frac{1}{2}\) | \((-\frac{1}{2} a_1 \cdot a_2 a_3) \otimes (+1), \ (0^a \cdot a_2) \otimes (+\frac{1}{2} b_1), \ (+\frac{1}{2} a_1) \otimes (0^{56})\) | \(3\alpha\) |
| 0        | \((-\frac{1}{2} 1234) \otimes (+1), \ (-\frac{1}{2} a_1 a_2 a_3) \otimes (+\frac{1}{2} b_1), \ (0^a \cdot a_2) \otimes (0^{56})\) | \(4\alpha\) |
| \(-\frac{1}{2}\) | \((-\frac{1}{2} 1234) \otimes (+\frac{1}{2} b_1), \ (-\frac{1}{2} a_1 a_2 a_3) \otimes (0^{56})\) | \(5\alpha\) |
| -1       | \((-\frac{1}{2} 1234) \otimes (0^{56})\) | \(6\alpha\) |

where \(a_i = 1, 2, 3, 4,\) and \(b_i = 5, 6\). For the \(\Psi\) superfield we have

| Helicity | KLT Products | Charge |
|----------|--------------|--------|
| +1       | \((1) \otimes (0^{78})\) | \(-6\alpha\) |
| \(\frac{1}{2}\) | \((1) \otimes (-\frac{1}{2} b_1^{(78)}), \ (+\frac{1}{2} a_1) \otimes (0^{78})\) | \(-5\alpha\) |
| 0        | \((1) \otimes (-\frac{1}{2} 56^{(78)}), \ (+\frac{1}{2} a_1) \otimes (-\frac{1}{2} b_1^{(78)}), \ (0^a \cdot a_2) \otimes (0^{78})\) | \(-4\alpha\) |
| \(-\frac{1}{2}\) | \((+\frac{1}{2} a_1) \otimes (-\frac{1}{2} 56^{(78)}), \ (0^a \cdot a_2) \otimes (-\frac{1}{2} b_1^{(78)}), \ (-\frac{1}{2} a_1 a_2 a_3) \otimes (0^{78})\) | \(-3\alpha\) |
| -1       | \((0^a \cdot a_2) \otimes (-\frac{1}{2} 56^{(78)}), \ (-\frac{1}{2} a_1 a_2 a_3) \otimes (-\frac{1}{2} b_1^{(78)}), \ (-\frac{1}{2} 1234) \otimes (0^{78})\) | \(-2\alpha\) |
| \(-\frac{3}{2}\) | \((-\frac{1}{2} a_1 a_2 a_3) \otimes (-\frac{1}{2} 56^{(78)}), \ (-\frac{1}{2} 1234) \otimes (-\frac{1}{2} b_1^{(78)})\) | \(-\alpha\) |
| -2       | \((-\frac{1}{2} 1234) \otimes (-\frac{1}{2} 56^{(78)})\) | 0 |

Since the \(\Psi\) superfield is the CPT-conjugate of the \(\Phi\) superfield, the charges of its component fields are opposite to the charges of their CPT-conjugated component partners in the \(\Phi\) superfield. This is exactly as one can read off from the above two tables in our \(\mathcal{N}_G = 6\) example. The charges of component fields within each diamond are different, which just means that this symmetry does not commute with supercharges. The freedom of choice of \(\alpha\) corresponds to an additional \(U(1)_R\) invariance group for superamplitudes. Combined with the \(SU(6)_R\) invariant group, we infer that superamplitudes of \(\mathcal{N}_G = 6\) minimal supergravity must be invariant under the larger \(U(6)_R = SU(6)_R \otimes U(1)_R\) group.

Similarly, for minimal supergravity theories of \(4 \leq \mathcal{N}_G < 8\), there is always a freedom of assigning an abelian charge \(\alpha\) to the component fields of these theories, and the superamplitudes are therefore invariant under the larger \(U(\mathcal{N}_G)_R = SU(\mathcal{N}_G)_R \otimes U(1)_R\) group.
4.3 $0 \leq \mathcal{N}_G \leq 4$ minimal gravity coupled to matter multiplets

For $0 \leq \mathcal{N}_G \leq 4$ supergravity theories, there will, besides the usual supergravity multiplets, also emerge matter supermultiplets from the KLT product. In detail, we have $\Phi$-$\Psi$ superfields for the supergravity multiplets, and $\Theta$-$\Gamma$ superfields for the matter multiplets. The $U(\mathcal{N}_G)_R$ group that rotates $Q^A$ and $\bar{Q}_A$ is still there, and the full $SU(\mathcal{N}_G)_R$ group is an invariant group for the superamplitudes. In order to study other possible invariant groups for these kind of theories, let us consider $(\mathcal{N}_G = 4) = (\tilde{\mathcal{N}} = 2) \otimes (\mathcal{N} = 2)$ supergravity from the KLT-construction. The associated diamond diagrams are shown in figure 10, and there are hidden indices (34) from the $\tilde{\mathcal{N}} = 2$ super Yang-Mills theory and (78) from the other. The $SU(4)_R$ indices for this $\mathcal{N}_G = 4$ supergravity theory are then 1, 2, 5, 6. Consider now the following $8 \times 8$ matrix in the Lie-algebra of $SU(8)$,

\[
\begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 \\
B_{33} & B_{34} \\
B_{43} & B_{44}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \alpha \\
0 & 0 & \alpha \\
C_{77} & C_{78} \\
C_{87} & C_{88}
\end{pmatrix}.
\]

(4.6)

Since we are again looking for the effects of $U(1)$ subgroups, we take the $4 \times 4$ matrix of $\mathcal{N}_G = 4$ supergravity, i.e. the matrix constructed from row 1, 2, 5, 6 and column 1, 2, 5, 6 of the above $8 \times 8$ matrix, to commute with all generators of $SU(4)_R$. It will thus be a diagonal matrix proportional to the identity $\alpha I_{4\times 4}$, as already written above. $B_{ij}$ is a $2 \times 2$ matrix acting on the hidden indices (34) while $C_{ij}$ is another $2 \times 2$ matrix acting on the hidden indices (78). From the previous discussion we know that only the trace part of $B_{ij}$ and $C_{ij}$ acts on the hidden indices. The indices (34) and (78) are not paired, so we should treat the two matrices separately. If we denote $\beta = Tr(B_{ij})$ and $\gamma = Tr(C_{ij})$, then for each $SU(4)_R$ R-index 1, 2, 5, 6 we can assign a charge $\alpha$, while for (34) we assign a charge $\beta$ and for (78) a charge $\gamma$. The condition of tracelessness of $SU(8)$ generators implies the constraint $\gamma = -\beta - 4\alpha$.

With this, we can now assign corresponding $U(1)$ charges for all component fields in the $\Phi$, $\Theta$, $\Gamma$ and $\Psi$ superfields. More explicitly, for the $\Phi$ superfield we have

| Helicity | KLT Product | Charge |
|----------|-------------|--------|
| +2       | $(+1) \otimes (+1)$ | 0      |
| $+\frac{3}{2}$ | $(+\frac{a_1}{2}) \otimes (+1)$ , $(+1) \otimes (+\frac{b_1}{2})$ | $\alpha$ |
| +1       | $(0^{12}) \otimes (+1)$ , $(+\frac{a_1}{2}) \otimes (+\frac{b_1}{2})$ , $(+1) \otimes (0^{56})$ | $2\alpha$ |
| $+\frac{1}{2}$ | $(0^{12}) \otimes (+\frac{b_1}{2})$ , $(+\frac{a_1}{2}) \otimes (0^{56})$ | $3\alpha$ |
| 0        | $(0^{12}) \otimes (0^{56})$ | $4\alpha$ |

where $a_i = 1, 2$ and $b_i = 5, 6$. For the $\Theta$ superfield we have
| Helicity | KLT Product | Charge |
|---------|-------------|--------|
| +1      | $(0^{(34)}) \otimes (1)$ | 0 + $\beta$ |
| $+\frac{1}{2}$ | $(0^{(34)}) \otimes (+\frac{1}{2})$, $(-\frac{1}{2}a_{1(34)}) \otimes (1)$ | $\alpha + \beta$ |
| 0       | $(0^{(34)}) \otimes (0^{56})$, $(-\frac{1}{2}a_{1(34)}) \otimes (+\frac{1}{2})$, $(-1^{12(34)}) \otimes (1)$ | $2\alpha + \beta$ |
| $-\frac{1}{2}$ | $(-\frac{1}{2}a_{1(34)}) \otimes (0^{56})$, $(-1^{12(34)}) \otimes (+\frac{1}{2})$ | $3\alpha + \beta$ |
| -1      | $(-1^{12(34)}) \otimes (0^{56})$, $(+1^{b_{1(34)}})$ | $4\alpha + \beta$ |

For the $\Gamma$-superfield

| Helicity | KLT Product | Charge |
|---------|-------------|--------|
| +1      | $(+1) \otimes (0^{(78)})$ | $-4\alpha - \beta$ |
| $+\frac{1}{2}$ | $(+1) \otimes (-\frac{1}{2}b_{1(78)})$, $(+1)^{a_{1}} \otimes (0^{(78)})$ | $-3\alpha - \beta$ |
| 0       | $(0^{12}) \otimes (0^{(78)})$, $(+1)^{a_{1}} \otimes (-\frac{1}{2}b_{1(78)})$, $(+1) \otimes (-1^{56(78)})$ | $-2\alpha - \beta$ |
| $-\frac{1}{2}$ | $(0^{12}) \otimes (-\frac{1}{2}b_{1(78)})$, $(+1)^{a_{1}} \otimes (-1^{56(78)})$ | $-\alpha - \beta$ |
| -1      | $(0^{12}) \otimes (-1^{56(78)})$ | $0 - \beta$ |

and finally for the $\Psi$-superfield

| Helicity | KLT Product | Charge |
|---------|-------------|--------|
| 0       | $(0^{(34)}) \otimes (0^{(78)})$ | $-4\alpha$ |
| $-\frac{1}{2}$ | $(0^{(34)}) \otimes (-\frac{1}{2}b_{1(78)})$, $(-\frac{1}{2}a_{1(34)}) \otimes (0^{(78)})$ | $-3\alpha$ |
| -1      | $(0^{(34)}) \otimes (-1^{56(78)})$, $(-\frac{1}{2}a_{1(34)}) \otimes (-\frac{1}{2}b_{1(78)})$, $(-1^{12(34)}) \otimes (0^{(78)})$ | $-2\alpha$ |
| $-\frac{3}{2}$ | $(-\frac{1}{2}a_{1(34)}) \otimes (-1^{56(78)})$, $(-1^{12(34)}) \otimes (-\frac{1}{2}b_{1(78)})$ | $-\alpha$ |
| -2      | $(-1^{12(34)}) \otimes (-1^{56(78)})$ | 0 |

From the charges of the component fields in these four superfields, we see that (1) The charge of the graviton $h$ is indeed zero as it had to be, (2) The $\Psi$ and $\Gamma$ superfields are CPT conjugate of the $\Phi$ and $\Theta$ superfields, respectively, and thus the charge of a given component field is opposite to its CPT-conjugate partner in the corresponding superfield, (3) If we only consider the $\alpha$ charge and set $\beta = 0$, we see that component fields in each diamond have different charges corresponding to the usual $U(1)^{R}$ charge, (4) If we only consider the $\beta$ charge and set $\alpha = 0$, then there is an extra charge for the matter supermultiplet, and every component field inside the same diamond has the same charge $\beta$ (or $-\beta$ in the $\Gamma$ diamond). This indicates that there is an extra $U(1)$ invariant group for the matter multiplet that commutes with all supercharge generators $Q^{A}, \tilde{Q}_{A}$. This $U(1)$ group is different from the $U(1)^{R}$ group that comes from $U(N_{G})^{R}$ rotations.

From the discussion above we learn that there is an $U(1)^{R}$ invariant group from the freedom of assigning a charge $\alpha$ to the component fields of all supermultiplets, and another $U(1)$ invariant group from the freedom of assigning a charge $\beta$ to component fields in the matter multiplets. We conclude that the superamplitudes of the $N_{G} = 4$ supergravity theory considered here is invariant under the group $SU(4)^{R} \otimes U(1)^{R} \otimes U(1)$. Similarly, for $0 \leq N_{G} < 4$ supergravity theories that describe minimal supergravity multiplets coupled
to matter supermultiplets, the superamplitudes are invariant under the group \( U(N_G)_R \otimes U(1) = SU(N_G)_R \otimes U(1)_R \otimes U(1) \).

4.3.1 **Examples of** \( SU(N_G)_R \otimes U(1)_R \otimes U(1) \) **symmetry**

In order to illustrate the impact of \( SU(N_G)_R \) and the additional \( U(1)_R \otimes U(1) \) group on gravity amplitudes, let us start by considering some amplitudes of gravitons coupled to two scalars in the \( N_G = 4 \) supergravity theory constructed from two \( N = 2 \) super Yang-Mills amplitudes, see figure 10. The hidden indices are \((34)\) and \((78)\), and \( SU(4)_R \)-indices are \(1, 2, 5, 6\). If we assign a charge \( \alpha \) to each of the four \( SU(4)_R \)-indices, and \( \beta \) for the hidden indices \((34)\), then the hidden indices \((78)\) will carry a charge of \(-4\alpha - \beta\) because of the traceless condition of the generators of \( SU(8)_R \). Let us take the following two scalars from the \( \Theta^{N_G=4}_{\text{vector}} \) diamond

\[
\phi_1 = (0^{(34)}) \otimes (0^{56}), \quad \phi_2 = (-1^{12(34)}) \otimes (1), \tag{4.7}
\]

which carry the \( U(1)_R \) charge \( 2\alpha \) and \( U(1) \) charge \( \beta \). From the \( \Gamma^{N_G=4}_{\text{vector}} \) superfield we pick

\[
\phi_3 = (0^{12}) \otimes (0^{78}), \quad \phi_4 = (+1) \otimes (-1^{56(78)}), \tag{4.8}
\]

with \( U(1)_R \) charge \(-2\alpha\) and \( U(1) \) charge \(-\beta\). Consider now the following two-scalar MHV amplitude

\[
M^{N_G=4}_{n}(\phi_i, \phi_j, h^-, h^+, \ldots, h^+), \tag{4.9}
\]

where the two scalars \( \phi_i, \phi_j \), with \( i, j = 1, \ldots, 4 \), can be any one of the above four kinds of scalars in (4.7) and (4.8). This amplitude can be readily calculated from the KLT-relations, and the only two non-vanishing amplitudes are

\[
M^{N_G=4}_{n}(\phi_1, \phi_3, h^-, h^+, \ldots, h^+), \quad M^{N_G=4}_{n}(\phi_2, \phi_4, h^-, h^+, \ldots, h^+), \tag{4.10}
\]

which neither violates \( SU(4)_R \) symmetry or the conservation of \( U(1)_R \otimes U(1) \) charge. The first is constructed from products of two \( N = 2 \) super Yang-Mills MHV amplitudes with a scalar and its CPT-conjugated partner and all other legs being gluons. The second arises from two pure-gluon MHV amplitudes. All other of these two-scalar amplitudes vanishes, as can be easily inferred from the gauge theory part. The super Yang-Mills amplitudes either contain only one scalar or two of the same scalars.

These vanishing two-scalar supergravity amplitudes can be explained from the violation of \( SU(4)_R \) and/or \( U(1)_R \otimes U(1) \) symmetry. For the following amplitudes

\[
M^{N_G=4}_{n}(\phi_1, \phi_1, h^-, h^+, \ldots, h^+), \quad M^{N_G=4}_{n}(\phi_2, \phi_2, h^-, h^+, \ldots, h^+), \quad M^{N_G=4}_{n}(\phi_3, \phi_3, h^-, h^+, \ldots, h^+), \quad M^{N_G=4}_{n}(\phi_4, \phi_4, h^-, h^+, \ldots, h^+), \tag{4.11}
\]

\( SU(4)_R \) is violated as well as \( U(1)_R \otimes U(1) \). For the amplitudes

\[
M^{N_G=4}_{n}(\phi_1, \phi_2, h^-, h^+, \ldots, h^+), \quad M^{N_G=4}_{n}(\phi_3, \phi_4, h^-, h^+, \ldots, h^+), \tag{4.12}
\]
the SU(4)\(_R\) is not violated, but the total U(1)\(_R\) \(\otimes\) U(1) charge is non-zero, and thus they vanish. For the amplitudes

\[ M^n_{N_G=4}(\phi_1,\phi_4,h^-,h^+,\ldots,h^+), \quad M^n_{N_G=4}(\phi_2,\phi_3,h^-,h^+,\ldots,h^+) \], \quad (4.13)\]

the U(1)\(_R\) \(\otimes\) U(1) charge is zero, but SU(4)\(_R\) is violated.

The U(1)\(_R\) and U(1) groups can also be violated individually. To illustrate this, let us have a look at some two-graviphoton coupled to graviton amplitudes. In the \(\Theta^{NG=4}_{vector}\) superfield, we have the graviphoton \(v^+_1\) with charge \((0 + \beta)\) and \(v^-_2\) with charge \((4\alpha + \beta)\), while in the \(\Gamma^{NG=4}_{vector}\) superfield we have \(v^+_2\) with charge \((-4\alpha - \beta)\) and \(v^-_1\) with charge \((0 - \beta)\).

With this we can have non-vanishing amplitudes like

\[ M^n_{N_G=4}(v^+_1,v^-_1,h^-,h^+,\ldots,h^+), \quad M^n_{N_G=4}(v^+_2,v^-_2,h^-,h^+,\ldots,h^+) \], \quad (4.14)\]

which satisfy both the SU(4)\(_R\) symmetry and conserves both the U(1)\(_R\) and the U(1) charge. However, it is easy to see that, for example, the amplitude

\[ M^n_{N_G=4}(v^-_1,v^-_1,h^-,h^+,\ldots,h^+) \], \quad (4.15)\]

does not violate the SU(4)\(_R\) symmetry and conserve the U(1)\(_R\) charge, but not the U(1) charge. Thus the violation of U(1) ensures the vanishing of this amplitude. Similarly, the amplitude

\[ M^n_{N_G=4}(v^-_1,v^-_2,h^-,h^+,\ldots,h^+) \], \quad (4.16)\]

does not violate the SU(4)\(_R\) symmetry, have zero U(1) charge but the U(1)\(_R\) charge is not conserved, thus the violation of U(1)\(_R\) implies the vanishing of this amplitude.

### 4.4 Summary of the symmetry groups from the KLT-construction

As shown in the examples discussed above, we see that the invariant symmetry groups for supergravity theories constructed out of KLT-products are directly linked to the type of diamonds for such a theory. In general there is \(U(N_G)_R \sim SU(N_G)_R\) and U(1)\(_R\) symmetry. For theories of maximal supersymmetry, or theories with exactly the same field content as maximal supersymmetry, all component fields are forced to have vanishing U(1)\(_R\) charge, and thus U(1)\(_R\) plays no role in these theories. For theories with only supergravity multiplets which are not CPT self-conjugate, component fields in the same diamond will have different U(1)\(_R\) charges, and opposite U(1)\(_R\) charges for the complex-conjugate partners in the CPT-conjugate diamond. For theories that contains matter supermultiplets, an extra U(1) group appears naturally. All component fields in the same matter multiplet diamond have the same additional U(1) charges, opposite to those of the component fields in the CPT-conjugate diamond. We list all possible supergravity theories constructed from KLT-products of super Yang-Mills theories in table 2, along with the invariant symmetry group that can be inferred from the KLT-product.

Superamplitudes of the different supergravity theories are invariant under their symmetry groups, and both SU(N\(_G\))\(_R\) and U(1)\(_R\) \(\otimes\) U(1) symmetries induce vanishing identities.
Table 2: Field content of the supergravity theories that can be constructed from super KLT-relations, and their invariant groups as inferred from our diamond diagrams. The total number of states for specific component fields is obtained by adding states in the different diamonds of the given theory. The linear global symmetry groups for minimal $4 \leq N_G \leq 8$ supergravities are also listed in [34].

5. Conclusions

In this paper we have identified all possible four-dimensional KLT-maps between gauge theories with decreasing supersymmetry to associated gravity theories with correspondingly decreasing degree of supersymmetry. For the case of $\mathcal{N} = 0$, we recover the well-known map between pure Yang-Mills theory and Einstein gravity. That map is actually slightly larger in that also two scalars couple on the gravity side. These two scalars are the only remnants of string theory in that simplest case: the axion and the dilaton. The existence of this additional set of scalars and their corresponding conserved $U(1)$ quantum number is the closest we get to a conserved $R$-charge in that case. Conservation of this charge is what ensures the existence of “vanishing identities” among the pure Yang-Mills amplitudes when helicities do not match in the product of amplitudes.

For theories with maximal supersymmetry to theories without supersymmetry, we have derived the full catalog of equivalences between gravity and gauge theories, and explored the linear symmetries that can be inferred from the KLT-map. The pattern is very interesting, and in particular for supergravity theories with $N_G \leq 4$, there are additional matter multiplets (the two scalars coupled to gravity in the $N_G = 0$ case can be viewed as an example of this phenomenon). We have established the precise maps, with the aid of diamond diagrams that graphically illustrates the combination of states on the gauge
theory side into supergravity states on the gravity side. Again, the complete set of linear global symmetries builds up the full set of vanishing identities.

Although the supergravity theories that follow from the gauge theory map may contain additional matter multiplets, one can readily project out those by fixing appropriate $R$-symmetry indices on the supergravity side. In this sense one can for, instance, construct minimal $\mathcal{N}_G = 4$ supergravity from both $(\tilde{\mathcal{N}} = 4) \otimes (\mathcal{N} = 0)$ directly or from $(\tilde{\mathcal{N}} = 2) \otimes (\mathcal{N} = 2)$ together with the projection discussed after eq. (3.15).

We have here restricted ourselves to KLT-maps between pure Yang-Mills theories with varying degrees of supersymmetry. It could be interesting to explore corresponding maps based on Yang-Mills theories with matter fields as well.

Acknowledgment

Numerous discussions with Emil Bjerrum-Bohr, Donal O’Connell, Henry Tye and Yi Yin are gratefully acknowledged.

A. Explicit expressions for superfields of $\mathcal{N}_G < 8$ supergravity

In this appendix we present the explicit expressions for the superfields we have represented by diamond diagrams throughout our paper. The superfields of supergravity multiplets will be denoted by $\Phi^{\mathcal{N}_G}$ and their CPT-conjugates by $\Psi^{\mathcal{N}_G}$. The needed matter multiplets will be denoted by $\Theta^{\mathcal{N}_G}$ and their CPT-conjugates by $\Gamma^{\mathcal{N}_G}$.

For the $\mathcal{N}_G = 7$ theory, we have the $(+2, +\frac{7}{2}, +1^{21}, +\frac{1}{2}^{35}, 0^{35}, -\frac{1}{2}^{21}, -1^{7}, -\frac{3}{2})$ supergravity multiplet diamond

\[
\Phi^{\mathcal{N}_G=7} = \Phi^{\mathcal{N}_G=8}|_{\eta_8 \to 0} = h^+_+ + \sum_{i=1}^7 \eta_i \eta_i^+ + \sum_{i<j}^7 \eta_i \eta_j v^+ + \sum_{i<j<k}^7 \eta_i \eta_j \eta_k \chi_i^+ + \sum_{i<j<k<l}^7 \eta_i \eta_j \eta_k \eta_l \chi_{ijkl} + \sum_{i<j<k<l<m}^7 \eta_i \eta_j \eta_k \eta_l \eta_m \chi_{ijklm} + \sum_{i<j<k<l<m<p}^7 \eta_i \eta_j \eta_k \eta_l \eta_m \eta_p v^+ + \eta_1 \eta_2 \eta_3 \eta_4 \eta_5 \eta_6 \eta_7 \psi^+_{1234567},
\]

(A.1)

where the superscripts denote the the degeneracies of states.
We also have the \((+\frac{3}{2}, +1^7, +\frac{1}{2}^{21}, 0^{35}, -\frac{1}{2}^{35}, -1^{21}, -\frac{3}{2}^7, -2)\) supergravity multiplet diamond

\[
\psi^{N_G=7} = \int d\eta_8 \Phi^{N_G=8} = \psi_{\varphi}^{(8)} - \sum_{i=1}^{7} \eta_i v_{\varphi}^{(8)} + \sum_{i<j=1}^{7} \eta_i \eta_j \chi_{\varphi}^{(8)} - \sum_{i<j<k=1}^{7} \eta_i \eta_j \eta_k \phi_{ijk}^{(8)} + \sum_{i<j<k=l=1}^{7} \eta_i \eta_j \eta_k \eta_l \chi_{ijkl}^{(8)} - \sum_{i<j<k<l=m=1}^{7} \eta_i \eta_j \eta_k \eta_l \eta_m \chi_{ijklm}^{(8)}
\]

\(+(\text{terms involving}\ \eta_i \eta_j \eta_k \eta_l \eta_m \eta_p \psi_{\varphi}^{(8)} - \eta_1 \eta_2 \eta_3 \eta_4 \eta_5 \eta_6 \eta_7 \psi_{\varphi}^{1234567(8)})\).

(A.2)

For the \(N_G = 6\) theory, we have the \((+2, +\frac{3}{2}, +1^{15}, +\frac{1}{2}^{20}, 0^{15}, -\frac{1}{2}^6, -1)\)-supergravity multiplet diamond

\[
\Phi^{N_G=6} = \Phi^{N_G=8}|_{\eta_7, \eta_8 \to 0} = h_{\varphi} + \sum_{i=1,2,3,4,5,6} \eta_i v_{\varphi} + \sum_{i<j=1,2,3,4,5,6} \eta_i \eta_j \chi_{\varphi} + \sum_{i<j<k=1,2,3,4,5,6} \eta_i \eta_j \eta_k \phi_{ijk} + \sum_{i<j<k<l=1,2,3,4,5,6} \eta_i \eta_j \eta_k \eta_l \chi_{ijkl} + \sum_{i<j<k<l=m=1,2,3,4,5,6} \eta_i \eta_j \eta_k \eta_l \eta_m \chi_{ijklm} + \eta_1 \eta_2 \eta_3 \eta_4 \eta_5 \eta_6 \psi_{\varphi}^{123456},
\]

(A.3)

and the \((+1, +\frac{1}{2}^6, 0^{15}, -\frac{1}{2}^{20}, -1^{15}, -\frac{3}{2}^7, -2)\) supergravity multiplet diamond

\[
\psi^{N_G=6} = \int d\eta_7 d\eta_8 \Phi^{N_G=8} = -v_{\varphi}^{(78)} - \sum_{i=1,2,3,4,5,6} \eta_i \chi_{\varphi}^{(78)} - \sum_{i<j=1,2,3,4,5,6} \eta_i \eta_j \phi_{ij}^{(78)} - \sum_{i<j<k=1,2,3,4,5,6} \eta_i \eta_j \eta_k \chi_{\varphi}^{(78)} - \sum_{i<j<k<l=1,2,3,4,5,6} \eta_i \eta_j \eta_k \eta_l \chi_{ijkl}^{(78)} - \sum_{i<j<k<l=m=1,2,3,4,5,6} \eta_i \eta_j \eta_k \eta_l \eta_m \chi_{ijklm}^{(78)} - \eta_1 \eta_2 \eta_3 \eta_4 \eta_5 \eta_6 \psi_{\varphi}^{1234567(78)}.
\]

(A.4)

We also have the \((+\frac{3}{2}, +1^6, +\frac{1}{2}^{15}, 0^{20}, -\frac{1}{2}^{15}, -1^6, -\frac{3}{2}^{2})\) gravitino supermultiplet diamond

\[
\Theta^{N_G=6} \equiv \int d\eta_4 \Phi^{N_G=8}|_{\eta_8 \to 0} = \psi_{\varphi}^{(4)} - \sum_{i=1,2,3,4,5,6,7} \eta_i v_{\varphi}^{(4)} + \sum_{i<j=1,2,3,4,5,6,7} \eta_i \eta_j \chi_{\varphi}^{(4)} - \sum_{i<j=k=1,2,3,4,5,6,7} \eta_i \eta_j \eta_k \phi_{ijk}^{(4)} + \sum_{i<j<k=l=1,2,3,4,5,6,7} \eta_i \eta_j \eta_k \eta_l \chi_{ijkl}^{(4)} - \sum_{i<j<k<l=m=1,2,3,4,5,6,7} \eta_i \eta_j \eta_k \eta_l \eta_m \chi_{ijklm}^{(4)} - \eta_1 \eta_2 \eta_3 \eta_4 \eta_5 \eta_6 \eta_7 \psi_{\varphi}^{1234567(4)}.
\]

(A.5)
and its CPT-conjugate gravitino supermultiplet diamond

$$\Gamma^{\mathcal{N}_G=6} \equiv \int d\eta_8 \Phi^{\mathcal{N}_G=8}|_{\eta_4 \to 0} = \psi^+(8) - \sum_{i=1,2,3,5,6,7} \eta_i v^i(8) + \sum_{i<j=1,2,3,5,6,7} \eta_i \eta_j \chi^+(8) - \sum_{i<j<k=1,2,3,5,6,7} \eta_i \eta_j \eta_k \phi^{ijk}(8) + \sum_{i<j<k<l=1,2,3,5,6,7} \eta_i \eta_j \eta_k \eta_l \chi^{ijkl}(8) - \sum_{i<j<k<l<m=1,2,3,5,6,7} \eta_i \eta_j \eta_k \eta_l \eta_m v^m(8) + \eta_1 \eta_2 \eta_3 \eta_5 \eta_6 \psi^-^{123567}(8).$$

(A.6)

For the $\mathcal{N}_G = 5$ theory, we have the $(+2, +\frac{3}{10}, +1^{10}, +\frac{1}{2}, 0^{5}, -\frac{1}{2})$ supergravity multiplet diamond

$$\Phi^{\mathcal{N}_G=5} = \Phi^{\mathcal{N}_G=8}|_{\eta_6, \eta_7, \eta_8 \to 0} = h_+ + \sum_{i=1,2,3,4,5} \eta_i \psi^i + \sum_{i<j=1,2,3,4,5} \eta_i \eta_j v^i + \sum_{i<j<k=1,2,3,4,5} \eta_i \eta_j \eta_k \chi^+$$

$$+ \sum_{i<j<k<l=1,2,3,4,5} \eta_i \eta_j \eta_k \eta_l \phi^{ijkl} + \eta_1 \eta_2 \eta_3 \eta_4 \eta_5 \chi_+^{12345},$$

(A.7)

and the $(+ \frac{1}{2}, 0^5, -\frac{1}{2}, -1^{10}, -\frac{3}{10}, -\frac{5}{2}, -2)$ supergravity multiplet diamond

$$\Psi^{\mathcal{N}_G=5} = \int \prod_{A=6}^{8} d\eta_A \Phi^{\mathcal{N}_G=8} = - \chi^{(678)} + \sum_{i=1,2,3,4,5} \eta_i \phi^i(678) - \sum_{i<j=1,2,3,4,5} \eta_i \eta_j \chi^-(678)$$

$$+ \sum_{i<j<k=1,2,3,4,5} \eta_i \eta_j \eta_k v^- (678) - \sum_{i<j<k<l=1,2,3,4,5} \eta_i \eta_j \eta_k \eta_l \chi^- (678)$$

$$+ \eta_1 \eta_2 \eta_3 \eta_4 \eta_5 h^- (678),$$

(A.8)

There is also a $(+ \frac{3}{2}, +1^{5}, +\frac{1}{2}, 0^{10}, -\frac{5}{2}, -1)$ gravitino supermultiplet diamond

$$\Theta^{\mathcal{N}_G=5} \equiv \int d\eta_4 \Phi^{\mathcal{N}_G=8}|_{\eta_7, \eta_8 \to 0} = \psi^+(4) - \sum_{i=1,2,3,5,6} \eta_i v^i(4) + \sum_{i<j=1,2,3,5,6} \eta_i \eta_j \chi^+(4)$$

$$- \sum_{i<j<k=1,2,3,5,6} \eta_i \eta_j \eta_k \phi^{ijk}(4) + \sum_{i<j<k<l=1,2,3,5,6} \eta_i \eta_j \eta_k \eta_l \chi^{ijkl}(4)$$

$$- \eta_1 \eta_2 \eta_3 \eta_5 \eta_6 \psi^- (4),$$

(A.9)

as well as the $(+1, +\frac{3}{2}, 0^{10}, -\frac{1}{2}, -1^{5}, -\frac{3}{2})$ gravitino supermultiplet diamond

$$\Gamma^{\mathcal{N}_G=5} \equiv \int d\eta_7 d\eta_8 \Phi^{\mathcal{N}_G=8}|_{\eta_4 \to 0} = - v^+(78) - \sum_{i=1,2,3,5,6} \eta_i \chi^+(78) - \sum_{i<j=1,2,3,5,6} \eta_i \eta_j \phi^{ij}(78)$$

$$- \sum_{i<j<k=1,2,3,5,6} \eta_i \eta_j \eta_k \chi^- (78) - \sum_{i<j<k<l=1,2,3,5,6} \eta_i \eta_j \eta_k \eta_l \phi^{ijkl}(78)$$

$$- \eta_1 \eta_2 \eta_3 \eta_5 \eta_6 \psi^- (78),$$

(A.10)
For the $N_G = 4$ theory, we have the $(+2,+\frac{3}{2},+1^6,+\frac{1}{2},0)$ supergravity multiplet diamond
\[
\Phi^{N_G=4} = \Phi^{N_G=8}_{|\eta_5,\eta_6,\eta_7,\eta_8 \rightarrow 0} = h_+ + \sum_{i=1,2,3,4} \eta_i \psi^i_+ + \sum_{i<j=1,2,3,4} \eta_i \eta_j \phi^{ij}_+ + \sum_{i<j<k=1,2,3,4} \eta_i \eta_j \eta_k \chi^{ijk}_+ + \eta_1 \eta_2 \eta_3 \eta_4 \phi^{1234}_+ ,
\] (A.11)
and the $(0,-\frac{1}{2},-1^6,-\frac{3}{2},-2)$ supergravity multiplet diamond
\[
\Psi^{N_G=4} = \int \prod_{A=5}^{8} d\eta_A \Phi^{N_G=8}_{|\eta_6,\eta_7,\eta_8 \rightarrow 0} = \phi^{(5678)} + \sum_{i=1,2,3,4} \eta_i \chi^i_{(5678)} + \sum_{i<j=1,2,3,4} \eta_i \eta_j \phi^{ij}_{(5678)} + \sum_{i<j<k=1,2,3,4} \eta_i \eta_j \eta_k \chi^{ijk}_{(5678)} + \eta_1 \eta_2 \eta_3 \eta_4 \phi^{1234}_{(5678)} .
\] (A.12)
There is also a $(+\frac{3}{2},+1^4,+\frac{1}{2},0^4,-\frac{1}{2})$ gravitino supermultiplet diamond
\[
\Theta^{N_G=4} \equiv \int d\eta_4 \Phi^{N_G=8}_{|\eta_6,\eta_7,\eta_8 \rightarrow 0} = \psi^i_{(4)} + \sum_{i=1,2,3,5} \eta_i \psi^i_{(4)} + \sum_{i<j=1,2,3,5} \eta_i \eta_j \chi^{ij}_{(4)} - \sum_{i<j<k=1,2,3,5} \eta_i \eta_j \eta_k \phi^{ijk}_{(4)} + \eta_1 \eta_2 \eta_3 \eta_5 \chi^{1235}_{(4)} ,
\] (A.13)
and a $(+\frac{1}{2},0^4,-\frac{1}{2},-1^4,-\frac{3}{2})$ gravitino supermultiplet diamond
\[
\Gamma^{N_G=4} \equiv \int d\eta_6 d\eta_7 d\eta_8 \Phi^{N_G=8}_{|\eta_4 \rightarrow 0} = -\chi^i_{(678)} + \sum_{i=1,2,3,5} \eta_i \phi^i_{(678)} - \sum_{i<j=1,2,3,5} \eta_i \eta_j \chi^{ij}_{(678)} + \sum_{i<j<k=1,2,3,5} \eta_i \eta_j \eta_k \phi^{ijk}_{(678)} - \eta_1 \eta_2 \eta_3 \eta_5 \chi^{1235}_{(678)} ,
\] (A.14)
as well as a $(+1,+\frac{3}{2},0^6,-\frac{1}{2},-1)$ vector multiplet diamond
\[
\Theta^{N_G=4}_{\text{vector}} \equiv \int d\eta_3 d\eta_4 \Phi^{N_G=8}_{|\eta_7,\eta_8 \rightarrow 0} = -\psi^i_{(34)} - \sum_{i=1,2,5,6} \eta_i \psi^i_{(34)} - \sum_{i<j=1,2,5,6} \eta_i \eta_j \phi^{ij}_{(34)} - \sum_{i<j<k=1,2,5,6} \eta_i \eta_j \eta_k \chi^{ijk}_{(34)} - \eta_1 \eta_2 \eta_5 \eta_6 \psi^{1256}_{(34)} ,
\] (A.15)
with its CPT-conjugate
\[
\Gamma^{N_G=4}_{\text{vector}} \equiv \int d\eta_7 d\eta_8 \Phi^{N_G=8}_{|\eta_3,\eta_4 \rightarrow 0} = -\psi^i_{(78)} - \sum_{i=1,2,5,6} \eta_i \psi^i_{(78)} - \sum_{i<j=1,2,5,6} \eta_i \eta_j \phi^{ij}_{(78)} - \sum_{i<j<k=1,2,5,6} \eta_i \eta_j \eta_k \chi^{ijk}_{(78)} - \eta_1 \eta_2 \eta_5 \eta_6 \psi^{1256}_{(78)} .
\] (A.16)
For the $\mathcal{N}_G = 3$ theory, we have the $(+2, +\frac{3}{2}, +1^3, +\frac{4}{2})$ supergravity multiplet diamond

$$\Phi_{\mathcal{N}_G = 3} = \Phi_{\mathcal{N}_G = 8}|_{\eta_4, \eta_6, \eta_7, \eta_8 \to 0} = h_+ + \sum_{i=1,2,3} \eta_i \psi^i_+ + \sum_{i<j=1,2,3} \eta_i \eta_j \psi^{ij}_+ + \eta_1 \eta_2 \eta_3 \chi^{123}_+ ,$$

(A.17)

and the $(-\frac{1}{2}, -1^3, -\frac{3}{2}, -2)$ supergravity multiplet diamond

$$\Psi_{\mathcal{N}_G = 3} = \int \prod_{A=4}^8 d\eta_A \Phi_{\mathcal{N}_G = 8}$$

$$= \chi^{(45678)}_+ - \sum_{i=1,2,3} \eta_i \psi^{(45678)}_+ + \sum_{i<j=1,2,3} \eta_i \eta_j \psi^{ij(45678)}_+ - \eta_1 \eta_2 \eta_3 h^{123(45678)}_+ .$$

(A.18)

Again there is also a $(+\frac{3}{2}, +1^3, +\frac{4}{2}, 0)$ gravitino supermultiplet diamond

$$\Theta_{\mathcal{N}_G = 3} = \int d\eta_4 \Phi_{\mathcal{N}_G = 8}|_{\eta_5, \eta_6, \eta_7, \eta_8 \to 0}$$

$$= \psi^{(4)}_+ - \sum_{i=1,2,3} \eta_i \psi^{i(4)}_+ + \sum_{i<j=1,2,3} \eta_i \eta_j \psi^{ij(4)}_+ - \eta_1 \eta_2 \eta_3 \phi^{123(4)} .$$

(A.19)

and a $(0, -1^3, -\frac{3}{2}, -\frac{1}{2})$ gravitino supermultiplet diamond

$$\Gamma_{\mathcal{N}_G = 3} = \int d\eta_5 d\eta_6 d\eta_7 d\eta_8 \Phi_{\mathcal{N}_G = 8}|_{\eta_4 \to 0}$$

$$= \phi^{(5678)}_+ + \sum_{i=1,2,3} \eta_i \chi^{i(5678)}_+ + \sum_{i<j=1,2,3} \eta_i \eta_j \psi^{ij(5678)}_+ + \eta_1 \eta_2 \eta_3 \psi^{123(5678)}_+ .$$

(A.20)

And there is a $(+1, +\frac{3}{2}, 0^3, -\frac{1}{2})$ vector multiplet diamond

$$\Theta_{\text{vector}}^{\mathcal{N}_G = 3} = \int d\eta_3 d\eta_4 \Phi_{\mathcal{N}_G = 8}|_{\eta_5, \eta_7, \eta_8 \to 0}$$

$$= -\psi^{(34)}_+ - \sum_{i=1,2,5} \eta_i \chi^{i(34)}_+ - \sum_{i<j=1,2,5} \eta_i \eta_j \phi^{ij(34)}_+ - \eta_1 \eta_2 \eta_5 \chi^{125(34)}_+ ,$$

(A.21)

and a $(+\frac{1}{2}, 0^3, -\frac{3}{2}, -1)$ vector multiplet diamond

$$\Gamma_{\text{vector}}^{\mathcal{N}_G = 3} = \int d\eta_6 d\eta_7 d\eta_8 \Phi_{\mathcal{N}_G = 8}|_{\eta_3, \eta_4 \to 0}$$

$$= -\chi^{(678)}_+ + \sum_{i=1,2,5} \eta_i \psi^{i(678)}_+ - \sum_{i<j=1,2,5} \eta_i \eta_j \chi^{ij(678)}_+ + \eta_1 \eta_2 \eta_5 \psi^{125(678)}_+ .$$

(A.22)

For the $\mathcal{N}_G = 2$ theory, we have the $(+2, +\frac{3}{2}, +1)$ supergravity multiplet diamond

$$\Phi_{\mathcal{N}_G = 2} = \Phi_{\mathcal{N}_G = 8}|_{\eta_2, \eta_3, \eta_4, \eta_6, \eta_7, \eta_8 \to 0} = h_+ + \eta_1 \psi^1_+ + \eta_5 \psi^5_+ + \eta_6 \psi^{15}_+ ,$$

(A.23)
and the \((-1, -\frac{3}{2}, -2)\) supergravity multiplet diamond

\[
\Psi^{N_G=2} = \int \prod_{A=2}^{4} d\eta_A \prod_{A=6}^{8} d\eta_A \Phi^{N_G=8} = -v_{-}^{(234678)} - \eta_1 \psi_{1}^{1(234678)} - \eta_5 \psi_{5}^{5(234678)} - \eta_1 \eta_5 h_{-}^{15(234678)}.
\] (A.24)

There is also a \((+1, +\frac{1}{2}, 0)\) vector multiplet diamond

\[
\Theta^{N_G=2}_{vector} \equiv \int d\eta_3 d\eta_4 \Phi^{N_G=8}_{|\eta_5, \ldots, \eta_8 \to 0} = -\phi^{(5678)} + \eta_1 \chi^{1(5678)} + \eta_2 \chi^{2(5678)} + \eta_1 \eta_2 \phi^{12(5678)},
\] (A.25)

and a \((0, -\frac{1}{2}, -1)\) vector multiplet diamond

\[
\Gamma^{N_G=2}_{vector} \equiv \int \prod_{a=5}^{8} d\eta_a \Phi^{N_G=8}_{|\eta_2, \ldots, \eta_4 \to 0} = -\chi^{(5678)} + \eta_1 \phi^{1(5678)} + \eta_5 \phi^{5(5678)} - \eta_1 \eta_5 \chi^{15(5678)}.
\] (A.26)

Besides this vector supermultiplet, there is also a \((+\frac{1}{2}, 0^2, -\frac{1}{2})\) hypermultiplet diamond

\[
\Theta^{N_G=2}_{hyper} \equiv \int \prod_{a=2}^{4} d\eta_a \Phi^{N_G=8}_{|\eta_6, \ldots, \eta_8 \to 0} = -\chi^{(234)} + \eta_1 \phi^{1(234)} + \eta_5 \phi^{5(234)} - \eta_1 \eta_5 \chi^{15(234)},
\] (A.27)

and its CPT-conjugate partner

\[
\Gamma^{N_G=2}_{hyper} \equiv \int \prod_{a=6}^{8} d\eta_a \Phi^{N_G=8}_{|\eta_2, \ldots, \eta_4 \to 0} = -\chi^{(678)} + \eta_1 \phi^{1(678)} + \eta_5 \phi^{5(678)} - \eta_1 \eta_5 \chi^{15(678)}.
\] (A.28)

For the \(N_G = 1\) theory, we have the \((+2, +\frac{3}{2})\) and \((-\frac{3}{2}, -2)\) supergravity multiplet diamond

\[
\Phi^{N_G=1} = \Phi^{N_G=8}_{|\eta_2, \ldots, \eta_8 \to 0} = h_+ + \eta_1 \psi_+^1,
\] (A.29)

\[
\Psi^{N_G=1} = \int \prod_{A=2}^{8} d\eta_A \Phi^{N_G=8} = -\psi_{-}^{(2345678)} + \eta_1 h_+^{1(2345678)},
\] (A.30)

as well as the \((+\frac{1}{2}, 0)\) and \((0, -\frac{1}{2})\) chiral supermultiplet diamond

\[
\Theta^{N_G=1}_{chiral} \equiv \int \prod_{a=2}^{4} d\eta_a \Phi^{N_G=8}_{|\eta_5, \ldots, \eta_8 \to 0} = -\chi^{(234)} + \eta_1 \phi^{1(234)},
\] (A.31)

\[
\Gamma^{N_G=1}_{chiral} \equiv \int \prod_{a=5}^{8} d\eta_a \Phi^{N_G=8}_{|\eta_2, \ldots, \eta_4 \to 0} = \phi^{(5678)} + \eta_1 \chi^{1(5678)}.
\] (A.32)
Finally, for the theory without any supersymmetry (the “$\mathcal{N}_G = 0$” theory), the “superfield” contains only one single state. This is the graviton

$$
\Phi^{\mathcal{N}_G=0} = \Phi^{\mathcal{N}_G=8}|_{\eta_1, \ldots, \eta_8 \to 0} = h_+, \quad (A.33)
$$

$$
\Psi^{\mathcal{N}_G=0} = \int \prod_{A=1}^{8} d\eta_A \Phi^{\mathcal{N}_G=8} = h_{(12345678)}^- , \quad (A.34)
$$

as well as the two scalars discussed at length in the text,

$$
\Theta^{\mathcal{N}_G=0} = \int \prod_{a=1}^{4} d\eta_a \Phi^{\mathcal{N}_G=8}|_{\eta_5, \ldots, \eta_8 \to 0} = \phi^{(1234)} , \quad (A.35)
$$

$$
\Gamma^{\mathcal{N}_G=0} = \int \prod_{a=5}^{8} d\eta_a \Phi^{\mathcal{N}_G=8}|_{\eta_1, \ldots, \eta_4 \to 0} = \phi^{(5678)} . \quad (A.36)
$$

References

[1] H. Kawai, D. C. Lewellen and S. H. H. Tye, Nucl. Phys. B 269 (1986) 1.

[2] N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, Phys. Rev. D 82 (2010) 107702 [arXiv:1005.4367 [hep-th]]; JHEP 1009, 067 (2010) [arXiv:1007.3111 [hep-th]].

[3] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, JHEP 1101 (2011) 001 [arXiv:1010.3933 [hep-th]].

[4] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D 78 (2008) 085011 [arXiv:0805.3993 [hep-ph]].

[5] N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, Phys. Rev. Lett. 103 (2009) 161602 [arXiv:0907.1425 [hep-th]].

[6] S. Stieberger, arXiv:0907.2211 [hep-th].

[7] Q. Ma, Y. -J. Du and Y. -X. Chen, JHEP 1202 (2012) 061 [arXiv:1109.0685 [hep-th]].

[8] B. Feng, R. Huang and Y. Jia, Phys. Lett. B 695 (2011) 350 [arXiv:1004.3417 [hep-th]].

[9] Y. -X. Chen, Y. -J. Du and B. Feng, JHEP 1102 (2011) 112 [arXiv:1101.0009 [hep-th]].

[10] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. Lett. 105 (2010) 061602 [arXiv:1004.0476 [hep-th]].

Z. Bern, T. Dennen, Y. -t. Huang and M. Kiermaier, Phys. Rev. D 82 (2010) 065003 [arXiv:1004.0693 [hep-th]].

[11] Z. Bern, C. Boucher-Veronneau and H. Johansson, Phys. Rev. D 84 (2011) 105035 [arXiv:1107.1935 [hep-th]]; C. Boucher-Veronneau and L. J. Dixon, JHEP 1112 (2011) 046 [arXiv:1110.1132 [hep-th]]; Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, arXiv:1201.5366 [hep-th].

[12] S. G. Naculich, H. Nastase and H. J. Schnitzer, JHEP 1201 (2012) 041 [arXiv:1111.1675 [hep-th]].

[13] S. H. Henry Tye and Y. Zhang, JHEP 1006 (2010) 071 [Erratum-ibid. 1104 (2011) 114] [arXiv:1003.1732 [hep-th]].
[14] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, JHEP 1006 (2010) 003 [arXiv:1003.2403 [hep-th]].

[15] Z. Bern and T. Dennen, Phys. Rev. Lett. 107 (2011) 081601 [arXiv:1103.0312 [hep-th]].

[16] C. R. Mafra, O. Schlotterer and S. Stieberger, JHEP 1107 (2011) 092 [arXiv:1104.5224 [hep-th]].

[17] R. Monteiro and D. O’Connell, JHEP 1107 (2011) 007 [arXiv:1105.2565 [hep-th]].

[18] N. E. J. Bjerrum-Bohr, P. H. Damgaard, R. Monteiro and D. O’Connell, JHEP 1206 (2012) 061 [arXiv:1203.0944 [hep-th]].

[19] N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, Phys. Lett. B 691 (2010) 268 [arXiv:1006.3214 [hep-th]]; J. Phys. Conf. Ser. 287 (2011) 012030 [arXiv:1101.5555 [hep-ph]].

[20] H. Tye and Y. Zhang, Phys. Rev. D 82 (2010) 087702 [arXiv:1007.0597 [hep-th]].

[21] B. Feng and S. He, JHEP 1009 (2010) 043 [arXiv:1007.0055 [hep-th]].

[22] H. Elvang and M. Kiermaier, JHEP 1010 (2010) 108 [arXiv:1007.4813 [hep-th]].

[23] J. H. Schwarz, hep-th/9209125.

[24] V. P. Nair, Phys. Lett. B 214 (1988) 215.

[25] Z. Bern, A. De Freitas and H. L. Wong, Phys. Rev. Lett. 84 (2000) 3531 [hep-th/9912033]; N. E. J. Bjerrum-Bohr and O. T. Engelund, Phys. Rev. D 81 (2010) 105009 [arXiv:1002.2279 [hep-th]].

[26] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, Nucl. Phys. B 546 (1999) 423 [hep-th/9811140].

[27] T. Sondergaard, Adv. High Energy Phys. 2012 (2012) 726030 [arXiv:1106.0033 [hep-th]].

[28] M. Bianchi, H. Elvang and D. Z. Freedman, JHEP 0809 (2008) 063 [arXiv:0805.0757 [hep-th]].

[29] H. Elvang, Y. -t. Huang, C. Peng, Y. -t. Huang and C. Peng, JHEP 1109 (2011) 031 [arXiv:1102.4843 [hep-th]].

[30] Z. Bern, S. Davies, T. Dennen and Y. -t. Huang, arXiv:1202.3423 [hep-th].

[31] M. F. Sohnius, Phys. Rept. 128, 39 (1985).

[32] E. Cremmer and B. Julia, Phys. Lett. B 80, 48 (1978).

[33] N. Arkani-Hamed, F. Cachazo and J. Kaplan, JHEP 1009, 016 (2010) [arXiv:0808.1446 [hep-th]].

[34] E. Cremmer and B. Julia, Nucl. Phys. B 159, 141 (1979).