ADDENDUM: “THE PROBLEM OF DEFICIENCY INDICES FOR DISCRETE SCHRÖDINGER OPERATORS ON LOCALLY FINITE GRAPHS” [J. MATH. PHYS. (52), 063512 (2011)]

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Abstract. In this note we answer negatively to our conjecture concerning the deficiency indices. More precisely, given any non-negative integer \( n \), there is locally finite graph on which the adjacency matrix has deficiency indices \((n, n)\).

Given a closable and densely defined symmetric operator \( T \) acting on a complex Hilbert space, the deficiency indices of \( T \) are defined by \( \eta_{\pm}(T) := \dim \ker(T^* \mp i) \in \mathbb{N} \cup \{ +\infty \} \). The operator \( T \) possesses a self-adjoint extension if and only if \( \eta_{\pm}(T) = \eta_{\mp}(T) \). If this is the case, we denote the common value by \( \eta(T) \) and the self-adjoint extensions of \( T \) are parametrized by the unitary group \( U(\eta(T)) \), e.g., [RS78, Section X.1]. The operator \( T \) is essentially self-adjoint, i.e., its closure is self-adjoint, if and only if \( \eta(T) = 0 \). In this note we discuss the possible values of \( \eta(T) \), when \( T \) is the adjacency matrix acting on a locally finite and simple graph.

We recall some standard definitions of graph theory to fix notation. A (simple, undirected) graph is a pair \((E, V)\), where \( V \) is a countable set and \( E : V \times V \to \{ 0, 1 \} \) is a symmetric function with \( E(v, v) = 0 \) for all \( v \in V \). The elements of \( V \) are called vertices. Two vertices \( v, w \in V \) with \( E(v, w) = 1 \) form an edge \((v, w)\), are neighbours, and we write \( v \sim w \). The set of neighbours of \( v \in V \) is \( \mathcal{N}_{G}(v) := \{ w \in V \mid v \sim w \} \). The number of neighbours of \( v \) is the degree \( d_{G}(v) := |\mathcal{N}_{G}(v)| \) of \( v \). The graph \( G = (V, E) \) is locally finite, if \( d_{G}(v) < \infty \) for all \( v \in V \). In this note, all graphs are simple, undirected and locally finite.

A path of length \( n \in \mathbb{N} \) in \( G \) is a tuple \((v_0, v_1, \ldots, v_n) \in V^{n+1} \) such that \( v_{j-1} \sim v_j \) for all \( j \in \{1, \ldots, n\} \). Such a path connects \( v_0 \) and \( v_n \) and is called \( v_0 \)-\( v_n \)-path. Being connected by a path is an equivalence relation on \( V \), and the equivalence classes are called connected components of the graph. A graph is connected, if all its vertices belong to the same connected component. The vertex set \( V \) of a connected graph is equipped with the graph metric \( \rho_{G} : V \times V \to \mathbb{R} \), \( \rho_{G}(v, w) := \inf \{ n \in \mathbb{N} \mid \text{there exists a } v \)-\( w \)-path of length \( n \} \). Note that we use the convention \( 0 \in \mathbb{N} \), so that each vertex is connected to itself with a path of length 0.

We now define trees. An edge \( e \in V \times V \), \( E(e) = 1 \), in a connected graph \( G = (E, V) \) is pivotal, if the graph \( G \) with the edge \( e \) removed, i.e. \((\tilde{E}, V)\) with \( \tilde{E}(e) = 0 \) and \( \tilde{E}(e') = E(e') \) for all \( e' \in V \setminus \{ e \} \), is disconnected. A tree is a connected graph, which has only pivotal edges.

We associate to a graph \( G \) the complex Hilbert space \( \ell^{2}(V) \). We denote by \( \langle \cdot, \cdot \rangle \) and by \( \| \cdot \| \) the scalar product and the associated norm, respectively. The set of complex functions with finite support in \( V \) is denoted by \( C_{c}(G) \). One may define different discret operators acting on \( \ell^{2}(V) \). For instance, the (physical) Laplacian is defined by

\[
(\Delta_{G,c}f)(x) := \sum_{y \in \mathcal{N}_{G}(x)} (f(x) - f(y)), \text{ with } f \in C_{c}(G)
\]

It is well known that it is symmetric and essentially self-adjoint on \( C_{c}(G) \), see [Woj07].

In this note we focus on the study of the adjacency matrix of \( G \), which is defined by:

\[
(A_{G,c}f)(x) := \sum_{y \in \mathcal{N}_{G}(x)} f(y), \text{ with } f \in C_{c}(G).
\]

Date: May 2, 2014.
Key words and phrases. adjacency matrix, deficiency indices, locally finite graphs.
This operator is symmetric and thus closable. We denote the closure by $A_G$. We denote the domain by $D(A_G)$, and its adjoint by $(A_G)^*$. Unlike the Laplacian, $A$ may have several self-adjoint extensions. We investigate its deficiency indices. Since the operator $A_G$ commutes with complex conjugation, its deficiency indices are equal, see [RS78] Theorem X.3. This means that $A_G$ possesses a self-adjoint extension. Note that $\eta(A_G) = 0$ if and only if $A_G$ is essentially self-adjoint on $C_c(G)$.

In [MOSB, Mil87], one constructs adjacency matrices for simple trees with positive deficiency indices. In fact, it follows from their proofs that the deficiency indices are infinite in both references. As a general result, a special case of [GST11] Theorem 1.1] gives that, given a locally finite simple tree $G$, one has the following alternative:

\[(3)\quad \eta(A_G) \in \{0, +\infty\}.
\]

The value of $\eta(A_G)$ is discussed in [GST11] and linked with the growth of the tree.

In [MW89] Section 3], one finds:

**Theorem 1.** For all $n \in \mathbb{N} \cup \{\infty\}$, there is a simple graph $G$, such that $\eta(A_G) = n$.

Their proof is unfortunately incomplete. However, the statement is correct, this is aim of this note. In [MW89], they provided simple and locally finite graph $G$ such that $\eta(A_G) \geq 1$ but did not check that $\eta(A_G) = 1$. The problem comes from the fact that they considered a tree. More precisely, they refered to the works of [MOSB, Mil87]. Therefore, [3] gives $\eta(A_G) = \infty$ in their case. Keeping that in mind and strongly motivated by some other examples, we had proposed a drastically different scenario and had conjectured in [GST11] that for any simple graph $G$, one has [3].

We now turn to the proof of Theorem [1] and therefore disprove our conjecture. First, we show that the validity of Theorem t:main in particular states the existence of a simple graph $G$ with

\[(4)\quad \eta(A_G) = 1.
\]

Of course, Theorem t:main in particular states the existence of $G$. We focus on the other implication. We denote the positive integers with $\mathbb{N}^*$.

**Lemma 2.** Let $n \in \mathbb{N}^*$ and $G$ be a locally finite and connected graph. Then there exists a locally finite and connected graph $G$ such that

\[\eta(A_G) = n \times \eta(A_G)\]

**Proof.** Let $\hat{G} := (\hat{E}, \hat{V})$ be the disjoint union of $n$ copies. We have: $\hat{G} := (\hat{E}, \hat{V})$ with $\hat{V} := \{1, \ldots, n\} \times V$ and $\hat{E}((i, v), (j, w)) := \delta_{i,j} E(v, w)$. Note that $\eta(A_{\hat{G}}) = n \times \eta(A_G)$ since we have a direct sum. Take now $v_0 \in V$ and connect the copies of $G$ by adding an edge between $(i, v_0)$ and $(i + 1, v_0)$, for all $i = 1, \ldots, n - 1$, and denote the resulting graph by $\tilde{G}$. Note that $A_{\tilde{G}}$ is bounded perturbation of $A_{\hat{G}}$. Therefore, by Proposition p:stab in Appendix [A] we have $\eta(A_{\tilde{G}}) = n \times \eta(A_G)$.

Our example of a graph $G$ with [4] is an antitree, a class of graphs which we define next. See also [BK]. The sphere of radius $n \in \mathbb{N}$ around a vertex $v \in V$ is the set $S_n(v) := \{w \in V \mid d_G(v, w) = n\}$. A graph is an antitree, if there exists a vertex $v \in V$ such that for all other vertices $w \in V \setminus \{v\}$

\[\mathcal{N}_G(w) = S_{n-1}(v) \cup S_{n+1}(v),\]

where $n = d_G(v, w) \geq 1$. See Figure [1] for an example. The distinguished vertex $v$ is the root of the antitree. Antitrees are bipartite and enjoy radial symmetry, which means that each permutation of $V$, which fixes the spheres around the root, induces a graph isomorphism on $G$.

We denote the root by $v$, the spheres by $S_n := S_n(v)$, and their sizes by $s_n := |S_n|$. Further, $\rho_G(v, x)$ is the distance of $x \in V$ from the root. The operator $P : \ell^2(V) \to \ell^2(V)$, given by

\[Pf(x) := \frac{1}{\rho_G(v, x)} \sum_{y \in S_{\rho_G(v, x)}} f(y), \text{ for all } f \in \ell^2(V) \text{ and } x \in V,\]

averages a function over the spheres. Thereby, $P = P^2 = P^*$ is the orthogonal projection onto the space of radially symmetric functions in $\ell^2(V)$. A function $f : V \to \mathbb{C}$ is radially symmetric, if
it is constant on spheres, i.e., for all nodes \(x, y \in V\) with \(|x| = |y|\), we have \(f(x) = f(y)\). For all radially symmetric \(f\), we define \(\bar{f} : \mathbb{N} \to \mathbb{C}, \bar{f}(|x|) := f(x)\), for all \(x \in V\). Note that

\[
Pf^2(V) = \{ f : V \to \mathbb{C}, f \text{ radially symmetric}, \sum_{n \in \mathbb{N}} s_n|f(n)|^2 < \infty \} \simeq \ell^2(\mathbb{N}, (s_n)_{n \in \mathbb{N}}),
\]

where \((s_n)_{n \in \mathbb{N}}\) is now a sequence of weights. The key observation of [BK] Theorem 4.1 is that

\[
\mathcal{A}_G = PA_GP \text{ and } \tilde{\mathcal{A}}_G Pf(|x|) = s_{|x|}^{-1} \tilde{P} f(|x| - 1) + s_{|x|+1} \tilde{P} f(|x| + 1),
\]

for all \(f \in \mathcal{C}_c(V)\), with the convention \(s_{-1} = 0\). Using the unitary transformation \(U : \ell^2(\mathbb{N}, (s_n)_{n \in \mathbb{N}}) \to \ell^2(\mathbb{N})\), \(U f(n) = \sqrt{s_n} f(n)\), we see that \(\tilde{\mathcal{A}}_G\) is unitarily equivalent to the direct sum of 0 on \((Pf^2(V))^\perp\) and a Jacobi matrix acting on \(\ell^2(\mathbb{N})\) with 0 on the diagonal and the sequence \((\sqrt{s_n}/\sqrt{s_{n+1}})_{n \in \mathbb{N}}\) on the off-diagonal.

**Proposition 3.** Set \(\alpha > 0\). Let \(G\) be the antitree with sphere sizes \(s_n\), where \(s_0 := 1, s_n := \lfloor n^\alpha \rfloor, n \geq 1\). Then,

\[
\eta(\mathcal{A}_G) = \begin{cases} 
0, & \text{if } \alpha \in (0, 1], \\
1, & \text{if } \alpha > 1.
\end{cases}
\]

**Proof.** Using Proposition 5 from Appendix A, we have \(\eta(\mathcal{A}_G) = \eta(J)\), where \(J\) is the Jacobi matrix given by \(a_n = \sqrt{s_{n}s_{n+1}}\) on the off-diagonal and \(b_n = 0\) on the diagonal. Let \(\tilde{J}\) be the Jacobi matrix given by \(\tilde{a}_n = \sqrt{n^\alpha(n+1)^\alpha}\) and \(\tilde{b}_n = 0\). Now note that

\[
0 \leq \check{a}_n - a_n \leq \sqrt{n^\alpha(n+1)^\alpha} - \sqrt{(n^\alpha - 1)((n+1)^\alpha - 1)} = \frac{(n+1)^\alpha + n^\alpha - 1}{\sqrt{n^\alpha(n+1)^\alpha} + \sqrt{(n^\alpha - 1)((n+1)^\alpha - 1)}} \xrightarrow{n \to \infty} 1,
\]

therefore \(\check{a}_n - a_n\) is bounded. Hence, \(\tilde{J} - J\) is a bounded operator, and by Proposition 5, cf. Appendix A, we have \(\eta(J) = \eta(\tilde{J})\).

Now note \(\sum_{n \in \mathbb{N}} \check{a}_n = \infty\), iff \(\alpha \leq 1\), and

\[
\check{a}_{n-1} \check{a}_{n+1} = \sqrt{(n-1)^\alpha n^\alpha(n+1)^\alpha(n+2)^\alpha} = \sqrt{(n^2-1)^\alpha(n+1)^2-1} \leq n^\alpha(n+1)^\alpha = \check{a}_n^2.
\]

By Theorem 4, see Appendix A, applied to \(\tilde{J}\) we get the result.

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**APPENDIX A. USEFUL FACTS**

The theory of Jacobi matrices, as developed in [Ber68, Chapter VII], provides the following general theorem.

**Theorem 4.** Let \(J\) be the Jacobi matrix with off-diagonal entries \(a_n > 0\) and diagonal entries \(b_n \in \mathbb{R}, n \in \mathbb{N}\), acting on \(\ell^2(\mathbb{N})\).
(1) If \( \sum_{n \in \mathbb{N}} a_n^{-1} = \infty \), then \( J \) is essentially self-adjoint on \( C_c(\mathbb{N}) \).

(2) If \( \sum_{n \in \mathbb{N}} a_n^{-1} < \infty \), then \( J \) is not essentially self-adjoint on \( C_c(\mathbb{N}) \) and has deficiency index 1.

We also recall that the deficiency indices are stable under the Kato-Rellich class of perturbation and refer to [GS11, Proposition A.1] for a proof.

**Proposition 5.** Given two closed and densely defined symmetric operators \( S, \ T \) acting on a complex Hilbert space and such that \( D(S) \subset D(T) \). Suppose there are \( a \in [0, 1) \) and \( b \geq 0 \) such that
\[
\| Tf \| \leq a \| Sf \| + b \| f \|, \text{ for all } f \in D(S).
\]
Then, the closure of \( (S + T)|_{D(S)} \) is a symmetric operator that we denote by \( S + T \). Moreover, one obtains that \( D(S) = D(S + T) \) and that \( \eta_{\pm}(S) = \eta_{\pm}(S + T) \). In particular, \( S + T \) is self-adjoint if and only if \( S \) is self-adjoint.

**Acknowledgments:** We would like to thank Matthias Keller for helpful discussions and Thierry Jecko for precious remarks.

**References**

[Ber68] Ju. M. Berezanskii, *Expansions in eigenfunctions of selfadjoint operators*, American Mathematical Society, Providence, Rhode Island, 1968.

[BK] J. Breuer and M. Keller, *Spectral analysis of certain spherically homogeneous graphs*, to be published.

[GS11] S. Golénia and C. Schumacher, *The problem of deficiency indices for discrete schrödinger operators on locally finite graphs*, J. Math. Phys. 52 (2011), no. 6, 17, 063512.

[MO85] B. Mohar and M. Omladič, *The spectrum of infinite graphs with bounded vertex degrees*, Teubner, Leipzig, 1985.

[Müü87] V. Müller, *On the spectrum of an infinite graph*, Linear Algebra Appl. 93 (1987), 187–189.

[MW89] B. Mohar and W. Woess, *A survey on spectra of infinite graphs*, J. Bull. Lond. Math. Soc. 21 (1989), no. 3, 209–234.

[RS78] M. Reed and B. Simon, *Methods of modern mathematical physics, tome i–iv: Analysis of operators*, Academic Press, 1978.

[Woj07] R. Wojciechowski, *Stochastic completeness of graphs*, Ph.D. thesis, City University of New York, 2007, p. 72.

[Woj11] __________, *Stochastically incomplete manifolds and graphs*, Progress in Probability 64 (2011), 163–179.

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