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Exponentiated transformation of Gumbel Type-II distribution for modeling COVID-19 data

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Received 8 August 2020; revised 30 September 2020; accepted 30 September 2020
Available online 21 October 2020

KEYWORDS
Gumbel model; COVID-19; Entropies; Stochastic order; Stress-strength analysis; Reliability analysis

Abstract The aim of this study is to analyze the number of deaths due to COVID-19 for Europe and China. For this purpose, we proposed a novel three parametric model named as Exponentiated transformation of Gumbel Type-II (ETGT-II) for modeling the two data sets of death cases due to COVID-19. Specific statistical attributes are derived and analyzed along with moments and associated measures, moments generating functions, uncertainty measures, complete/incomplete moments, survival function, quantile function and hazard function, etc. Additionally, model parameters are estimated by utilizing maximum likelihood method and Bayesian paradigm. To examine efficiency of the ETGT-II model a simulation analysis is performed. Finally, using the data sets of death cases of COVID-19 of Europe and China to show adaptability of suggested model. The results reveal that it may fit better than other well-known models.

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1. Introduction

The literature includes a variety of models for analyzing the lifetime data. In extreme value (e.v.) study of extreme events, Gumbel model, also recognized as type-1 e.v. model, has gained considerable research attention, in particular over years. Pinheiro and Ferrari [1] investigation can be considered for an overview of the recent innovations and implementations of Gumbel model. There is no doubt that Gumbel type-II model has not been commonly used in statistical modeling until now, and reason might not be far from its lack of data modeling fits. Traditional probability models are usually criticized for their lack of fit in complex data simulation. Through this context, users of such model in various areas in general, and in particular mathematics and statistics, have been fantastically inspired to establish sophisticated models of probability from basic models. Exponentiated models have been implemented to address issue of lack of fitting typically experienced when using the regular models of probability to model complex data. Findings from this development have always proved to be more accurate than one based on standard models. Exponentiating models are also a strong statistical modelling technique which offers an efficient way to incorporate extra shape parameter to regular model to obtain versatility and robustness. This approach of generalizing models of

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Peer review under responsibility of Faculty of Engineering, Alexandria University.

https://doi.org/10.1016/j.aej.2020.09.060
1110-0168 © 2020 The Authors. Published by Elsevier B.V. on behalf of Faculty of Engineering, Alexandria University.
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Probability is identifiable to study of Gupta et al. [2]. He presented EE (exponentiated exponential) model via simplified version of traditional exponential model by simply increasing cdf to a power of constant. Since development of EE model, exponentiated models have gained rational features in modelling data from different complicated phenomena. A large number of regular models of probabilities have its respective exponentiated forms. Gupta et al. [2] proposed Weibull exponentiated (WE) model via generalization of regular Weibull model. Nadarajah and Kotz [3] revised the approach of Gupta et al. [2] and described EF (Exponentiated Frechet) model by generalizing model of Frechet. Nadarajah [4] developed Exponentiated Gumbel (EG) model as a generalization of regular Gumbel model using same approach (Nadarajah and Kotz [3]). Exponentiated Weibull family model being a generalization of Weibull family model was developed by Mudholkar and Srivastava [5]. Ashour and Eltehiwy [6] established EPL (exponentiated power Lindley) model by generalizing model of power Lindley and many more. Here it should be observed that almost all extensions added extra parameters to base model. Such outcomes manages to complexities in future assumptions. On other hands, additional parameters offer more versatility but at same time introduce uncertainty in estimation of parameter(s). Possibly, taking that into consideration, Kumar et al. [7] suggested a DUS transformation, in order to achieve new model. If \( G(y) \) be baseline cdf, above mentioned transformation offers a new cdf \( F(y) \) as shown below:

\[
F(y) = \frac{e^{G(y)} - 1}{e - 1},
\]

and consider exponential model as base model and they named it exponential DUS model. It had observed that it provides a model containing non-constant hazard rates. The benefit of using above transformation is that new model retains characteristic of being parsimonious in parameter, since it does not include any extra parameters.

Thus, current study is aimed to propose a new model by using DUS transformation by introducing a new extra parameter. And for this purpose we use a traditional Gumbel type-II model [8–12] as a basic model to a wider distribution class, in order to enhance its efficiency and promote its suitability, in modelling varieties of complex datas. The proposed model is known as the Exponentiated transformation of Gumbel Type-II (ETGT-II).

The ETGT-II model can be obtained as follows: Let \( Y \) be a r.v. (random variable) with \( G(y) \) cdf and let \( g(y) \) be respective pdf to be consider as basic model.

\[
F(y) = \frac{e^{g(y)} - 1}{e - 1},
\]

and corresponding pdf is

\[
f(y) = \frac{gg(y)e^{g(y)}}{e^2 - 1}; \xi > 0.
\]

ETGT-II model is more flexible, since it includes extra shape parameter \( \xi \) to regulate the transformation.

In December, Wuhan City, China confirmed the first case of respiratory disease, pneumonia, with symptoms close to the serious acute 34 respiratory SARS-CoV [13]. COVID-19 symptoms involved fever, shortness of breath, cough and occasional watery diarrhea [14]. In February 2020, 17,238 infections cases of COVID-19 and 361 deaths in China were reported [15]. The objective of current study is to analyze the number of daily deaths due to COVID-19 in Europe (1st – 30th Mrch) and China (23 January to 28 March). For this purpose, we introduced a novel three parametric model known as Exponentiated transformation of Gumbel Type-II (ETGT-II) for modeling said data sets. Specific statistical and reliable attributes, along with survival and hazard functions, reliability function, cumulative hazard rate function, Quantile function (qf), measure of skewness and kurtosis, central and non-central moments, means and variance moments generating function, order statistics (OS), stochastic ordering (SO), characteristic function, factorial generating function, incomplete non central moments, conditional moments and mean deviation are derived and studied. Moreover, uncertainty measures including information generating function and entropy measures namely Renyi, Verma, Tsallis etc are also derived and represented graphically. The Log-likelihood, density and trace plots are also plotted for both data sets. Additionally, model parameter is estimated via maximum likelihood and Bayesian paradigm. To examine the efficiency of the ETGT-II model a simulation analysis is performed. In the end, using COVID-19 data sets of Europe and China to show adaptability of suggested model. The results reveal that it may fit better than other well-known models.

### 2. Proposed distribution (ETGT-II)

For illustration, the baseline distribution is assumed to be a Gumbel type-II distribution with parameters \( \gamma, \delta \) having pdf as

\[
g(y|\gamma, \delta) = \gamma \delta y^{-1} \exp \{-\delta y^{-\gamma}\}; \gamma, \delta, y > 0.
\]

and corresponding cdf as

\[
G(y|\gamma, \delta) = \exp \{-\delta y^{-\gamma}\}; \gamma, \delta, y > 0.
\]

Using transformation given in Eq. (1), the pdf and cdf of proposed model, subsequently mentioned as Exponentiated transformation of Gumbel Type-II (ETGT-II) can conveniently be obtained as:

\[
f(y|\gamma, \delta, \xi) = \frac{\gamma \delta y^{-1} e^{-\delta y^{-\gamma}} \exp \{\xi e^{-\delta y^{-\gamma}}\}}{e^2 - 1}; \gamma, \delta, \xi > 0.
\]

\[
F(y|\gamma, \delta, \xi) = \frac{e^{-\delta y^{-\gamma}} - 1}{e^2 - 1}; \gamma, \delta, \xi > 0.
\]

It is simple and clear to notice that \( F(y|\gamma, \delta, \xi) \) differentiable and increases in \( 0 \to \infty \) and \( \lim_{y \to 0} F(y) = 0 \) and \( \lim_{y \to \infty} F(y) = 1 \).

#### Reliability Function

The \( R(y) = P(Y > y) = 1 - F(y) \) determines reliability or survival function of a r.v. \( Y \). It could be defined as the probability that a device does not fail within a certain prescribed time \( t \). The reliability function of ETGT-II is defined as

\[
R(y|\gamma, \delta, \xi) = P(Y > y) = \frac{e^2 - e^{-\delta y^{-\gamma}}}{e^2 - 1}.
\]

#### Hazard rate function

The \( h(y|\gamma, \delta, \xi) = f(y|\gamma, \delta, \xi)/[1 - F(y|\gamma, \delta, \xi)] \) is a very useful tool in lifetime analysis. The immediate failure rate of a r.v. \( Y \) is probability that a mechanism collapses as it has persisted to the present time \( t \) is shown by
The odd ratio is defined as
\[ Y(y; \gamma, \delta, \xi) = \frac{R_y(y)}{h_y(y)} = \left( \frac{\exp\left(\xi e^{\delta-y}\right)}{\exp\left(-\delta^y\right)} \right), \]
where \( R_y(y) \) and \( h_y(y) \) is defined in (7) and (8).

Cumulative Hazard rate function
The cumulative hrf is defined as
\[ H(y) = \int_0^y h(t; \gamma, \delta, \xi) dt, \]
Therefore,
\[ H(y) = -\log\left\{ \exp(\xi) - \exp\left(\xi \exp\{-\delta^y\}\right) \right\}. \]
Using computing packages like MATLAB, Maple, Minitab, Mathematica, and R, equations (1–10) can be readily assessed numerically. The plots of (5) (6) and (8) are shown in Figs. 1–3 for different choices of parameters. Fig. 1 demonstrates how parameter \( \xi \) influence the density of ETGT-II and shows flexibility of pdf (6) from where skewness, little symmetry, high tails and modality can be directly measured. These curves demonstrate ETGT-II distribution's versatility. Fig. 2 shows plots of cdf of ETGT-II distribution. Fig. 3 shows the decreasing and upside-down bathtub pattern of hrfs of ETGT-II distribution.

2.1. Shape of model and hazard rates
The shape of model is a significant attribute as it provides an idea about nature of the model. For analysis of shapes of hazard rate we have implemented Glaser [16]. He established the term \( \zeta(y) = -\frac{f(y)}{f_0(y)}, \) where \( f(y) \) defines density function and \( f_0(y) \) defines first derivative of \( f(y) \) in relation to \( y, \) then following theorem has been stated.

**Theorem 1.** (a) If \( \zeta'(y) < 0 \) for all \( y > 0, \) then model has DFR (decreasing failure rate). (b) Assume there exists \( y_0 > 0 \) such that \( \zeta'(y) > 0 \) \( \forall \ y \in (0, y_0), \) \( \zeta(y_0) = 0, \) and \( \zeta'(y) < 0 \) for all \( y > y_0, \) then \( h(y) \) is uni-modal (called upside bathtub (UBT)).

**Proof.** For the suggested distribution, we have:
\[ \zeta(y) = \frac{1 + \gamma - y}{y^{\gamma+1}}, \]
and
\[ \zeta'(y) = -\frac{e^{\gamma y}(1 + \gamma)(\gamma - y\delta) + \gamma\delta[\gamma(1 + \gamma) - \gamma\delta]}{\gamma^2[1 + \gamma e^{-\beta}]} \]
It can easily be verified that the given two situations may arise:
From (12), we can easily noted that \( \zeta'(y) < 0 \) for all \( y > 0. \) Hence the proposed model has decreasing failure rate and has observed from Fig. 3.

![Fig. 1](image-url)  
**Fig. 1** Graphs of density curves of ETGT-II at different parameter values.
Also, \( \psi'(y) > 0 \) for all \( y \in (0, y_0) \), \( \psi'(y_0) = 0 \), and \( \psi'(y) < 0 \) for \( y \in (y_0, \infty) \), where, \( y_0 \) can be attained by solving the below equation

\[
-y_0^\gamma e^{y_0^\gamma}(1 + \gamma)y_0^\gamma - \gamma \delta y_0^\gamma(1 + \gamma) - \gamma \delta_0^\gamma y_0^\gamma = 0. \tag{13}
\]

We have drawn hazard function (8) with different combinations of parameters in Fig. 3, which shows that \( h(.) \), for given value of \( (\gamma, \delta, \zeta) > 0 \), is uni-modal in \( y \). The turning point of hazard function, say \( y_0 \) is a solution of following equation

\[
e^{\gamma \delta + \zeta e^{\gamma \delta}}[\psi'(1 + \gamma) - \gamma \delta] - e^{y_0^\gamma} \gamma \delta \zeta e^{\gamma \delta} + \gamma \delta_0^\gamma e^{\gamma \delta} = 0.
\tag{14}
\]

### 2.2. Limiting models of sample extremes

Let \( Y_1, Y_2, \ldots, Y_n \) be a r.s. of size \( n \) from an absolutely continuous model with \( f(y|\gamma, \delta, \zeta) \) pdf and \( F(y|\gamma, \delta, \zeta) \) cdf. Limiting model of sample maxima \( Y_{n,n} = \max(Y_1, Y_2, \ldots, Y_n) \) is a long stand area in usages of probability and statistics. First, we con-
sider the below asymptotical outcomes for $Y_{n,\alpha}$ (see Arnold et al. [17]).

For the maximum order statistic $Y_{n,\alpha}$, here

$$
\lim_{n \to \infty} P(Y_{n,\alpha} \leq c_n + d_n t) = e^{-e^{-t}}, \quad -\infty < t < \infty,
$$

where $c_n = F^{-1}(1 - \frac{1}{n})$ and $d_n = \frac{1}{\eta(\alpha)}$, if

$$
\lim_{y \to \infty} \frac{1}{d_n} \frac{1}{d_n(y)} = 0.
$$

The next result provides the limiting models of the highest order statistics from ETGT-II model.

**Proposition 1.** Let $Y_{n,\alpha}$ be the highest order statistics from ETGT – II(γ, δ, ζ) model. Then

$$
\lim_{n \to \infty} \frac{d}{d_n} \frac{1}{d_n(y)} = e^{-e^{-t}}, \quad -\infty < t < \infty,
$$

where $c_n = F^{-1}(1 - \frac{1}{n})$ and $d_n = \frac{1}{\eta(\alpha)}$, if and f(.), $F^{-1}(\cdot)$, respectively are given by Equations, (5) and (39).

**Proof.** For ETGT-II model, we have

$$
\lim_{y \to \infty} \frac{d}{d_n} \frac{1}{d_n(y)} = \lim_{y \to \infty} \frac{1}{d_n} \frac{1}{d_n(y)} \left( \frac{1}{y^{\gamma - 1}} \right) = 0.
$$

Hence, the statement follows from (15) and (16). □

### 2.3. Order Statistics (OS)

Let $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(n)}$ be OS of a r.s. of size $n$ from distribution $F(y)$. Then, for $m = 1, 2, \ldots, n$, the pdf of $m^{th}$ OS, $Y_{(m)}$ is as below

$$
f_{m}(y|\gamma, \delta, \xi) = \Psi \{1 - F(y|\gamma, \delta, \xi)\}^{m-1} F(y|\gamma, \delta, \xi)^{-n},
$$

where $\Psi = \frac{n!}{(m-1)!n!}$. Thus from (5), (6) and (19) the pdf of $Y_{(m)}$ results as

$$
f_{m}(y|\gamma, \delta, \xi) = \frac{\xi e_{\delta}^{-\delta e_{\delta}^{-\delta e_{\delta}}}}{2\nu e_{\delta}^{-\nu}} \sum_{j=0}^{n-m-1} \binom{n}{j} \binom{n-m}{j} (1 - \frac{\xi e_{\delta}^{-\delta e_{\delta}}}{\nu e_{\delta}^{-\nu}})^{j}.
$$

The cdf of $Y_{(m)}$ is

$$
F_{m}(y|\gamma, \delta, \xi) = \sum_{j=m}^{n} \binom{n}{j} F(y|\gamma, \delta, \xi)^{j} (1 - F(y|\gamma, \delta, \xi))^{n-j},
$$

then cdf of $m^{th}$ order statistic, $Y_{(m)}$ of ETGT-II model is given by

$$
F_{m}(y|\gamma, \delta, \xi) = \sum_{j=m}^{n} \binom{n}{j} \binom{n-m}{j} (1 - \frac{\xi e_{\delta}^{-\delta e_{\delta}}}{\nu e_{\delta}^{-\nu}})^{j}.
$$

In general, cdfs of $Y_{(n)}$ and $Y_{(i)}$ are given as

$$
F_{(n)}(y) = (1 - F(y))^{n}, \quad F_{(i)}(y) = F^{i}(y),
$$

$$
F_{(i)}(y|\gamma, \delta, \xi) = \frac{\exp \{\xi e_{\delta}^{-\delta e_{\delta}} - (\xi e_{\delta}^{-\delta e_{\delta}})\}}{\exp(\xi) - 1},
$$

$$
F_{(i)}(y|\gamma, \delta, \xi) = 1 - \left[ \frac{\exp(\xi) - \exp(\xi e_{\delta}^{-\delta e_{\delta}})}{\exp(\xi) - 1} \right]^{n}.
$$

Let $Q_{n}(\cdot)$ be (for $0 < q < 1$) qf of $Y_{(n)}$. Then we obtain from (24) and (25)

$$
Q_{n}(q) = Q\left\{1 - \left[1 - q\right]^{n}\right\},
$$

where $Q(\cdot)$ is qf of $Y$. Hence, from (26) and (39), we cannot define qfs of $Y_{(n)}$ and $Y_{(i)}$ in closed-form. In the case of i.i.d. random variables, it is possible to attain an expression for $\rho$ ordinary moment of OS, when $\rho < \infty$. So, as Silva et al. [18], we can represent $m^{th}$ moment of $m^{th}$ OS $Y_{(m)}$ as

$$
\rho_{m}' = \sum_{j=m}^{n} \binom{n}{j} \binom{n-m}{j} (1 - \frac{\xi e_{\delta}^{-\delta e_{\delta}}}{\nu e_{\delta}^{-\nu}})^{j} I_{j}\{r\},
$$

where $I_{j}\{r\} = \int_{0}^{\infty} r^{j-1} [1 - F(y)] dy$. Specifically, for ETGT-II model, we get

**Proposition 2.** Let $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(n)}$ be OS of a random sample of $n$ size from ETGT-II model.

The next outcome shows the $r^{th}$ moment of $m^{th}$ OS $Y_{(m)}$ of pdf (5) can be described as

$$
\rho_{m}' = \sum_{j=m}^{n} \binom{n}{j} \binom{n-m}{j} (1 - \frac{\xi e_{\delta}^{-\delta e_{\delta}}}{\nu e_{\delta}^{-\nu}})^{j} I_{j}\{r\},
$$

where $I_{j}\{r\} = \sum_{j=m}^{n} \binom{n}{j} \binom{n-m}{j} (1 - \frac{\xi e_{\delta}^{-\delta e_{\delta}}}{\nu e_{\delta}^{-\nu}})^{j} I_{j}\{r\}.$

**Proof.**
2.4. Stochastic ordering

It is of interest to define, for practical purposes, the intrinsic stochastic ordering (SO) of such members according to the parameters when we engage with a general family of models. In this respect, some distributional functions can be used as function of the cdf, hrf, likelihood ratio function. Now, we’re concentrating on the order of likelihood ratios defined as below. For r.v. X and Y, we state, $X \leq_{st} Y$, if ratio of two respective pdfs a decreasing function in $y$.

Stochastic ordering of continuous positive r.v. is a significant mechanism for evaluating relative behavior. We must remember certain basic definitions. It is assumed that r.v. X is smaller than Y in the

(i) stochastic order $X \leq_{st} Y$ if $F_X(y) \leq F_Y(y)$ for all $y$; (ii) hazard rate order $X \leq_{h} Y$ if $h_X(x) \geq h_Y(x)$ for all $y$; (iii) likelihood ratio order $X \leq_{lr} Y$, if $\frac{f_X(y)}{f_Y(y)}$ decreases in $y$. The consequences below are well known, see Ross [19] chapter 9:

$$X \leq_{st} Y \Rightarrow X \leq_{lr} Y \Rightarrow X \leq_{st} Y.$$ 

(31)

The ETGT-II models are ordered with strongest “likelihood ratio” ordering as given below.

**Theorem 2.** Let $X \sim \text{ETGT-II}(\gamma_1, \delta_1, \xi_1)$, and $Y \sim \text{ETGT-II}(\gamma_2, \delta_2, \xi_2)$. If (i) $\gamma_1 = \gamma_2 = \gamma$, $\delta_1 = \delta_2 = \delta$ and $\xi_1 \leq \xi_2$, and (ii) $\gamma_1 = \gamma_2 = \gamma$, $\xi_1 = \xi_2 = \xi$ and $\delta_1 \leq \delta_2$, then $X \leq_{st} Y$ ($X \leq_{sr} Y, X \leq_{lr} Y$).

**Proof.** The likelihood ratio (LR) is

$$\frac{f_X(y)}{f_Y(y)} = \frac{\exp\left(-\gamma_1^2y^{\gamma_1-1}+\gamma_2^2y^{\gamma_2-1}+2\gamma_1\gamma_2y^{\gamma_1+\gamma_2-2}\right)}{\exp(-\gamma y)}$$

(32)

now differentiate above equation w.r.t $y$, we obtain

$$d\left[\frac{f_X(y)}{f_Y(y)}\right] = \frac{y^{\gamma_1-1}e^{-\gamma}y^{\gamma_1-2}+\gamma_2\gamma_1\gamma_2y^{\gamma_1+\gamma_2-3}}{(\exp(-\gamma y))}$$

(33)

Hence it shows that $X \leq_{st} Y$, and according to above Eq. (31) these both are $X \leq_{st} Y, X \leq_{st} Y$ also hold.

Case (ii) if $\gamma_1 = \gamma_2 = \gamma$, $\xi_1 = \xi_2 = \xi$ and $\delta_1 \leq \delta_2$, then $\frac{f_X(y)}{f_Y(y)}$ becomes

$$\frac{f_X(y)}{f_Y(y)} = \frac{\exp(-\gamma_1^2y^{\gamma_1-1}+\gamma_2^2y^{\gamma_2-1}+2\gamma_1\gamma_2y^{\gamma_1+\gamma_2-2})}{\exp(-\gamma y)}$$

(34)

now differentiate above equation w.r.t $y$, we obtain

$$d\left[\frac{f_X(y)}{f_Y(y)}\right] \leq 0.$$
get with Table 1 that there are considerable effects of $\zeta$ on these measures. It is observed that different combination of the parameters provide different values for $\mu_1^*, \mu_2^*, \mu_3^*, Q_{1.1}^{1.1}, Q_{3.1}^{1.1}, \text{Median}^{1.1}, \text{C.V.},$ and $K_{3.1}^{1.1}$. This Table shows that for the fixed value of $\delta$ and $\zeta$, as the values of $\gamma$ increases, all measures of the proposed model decrease except $\mu_4^*$ and $\mu_5^*$. On the other side, for enhancement in $\zeta$, while $\gamma$ and $\delta$ are fixed, the first three non-central moments, $Q_{1.1}^{1.1}$, $\text{Median}^{1.1}$, and Variance increase.

4. Reliability in multicomponent stress–strength model

The stress strength model in the theory of reliability gives the scheme of the existence of a device or component. If
The purpose of this section is to determine the expression of the non-central moment \( \mu_r \) of \( Y \) in terms of gamma function.

**Theorem 3.** For \( \gamma, \delta, \xi > 0 \), the \( r \)-th non-central moment of \( Y \) can be expressed as

\[
\mu_r(Y) = \int_0^\infty y^r dF(y|\gamma, \delta, \xi); \quad r = 1, 2, \ldots
\]

where, \( \Gamma(\cdot) \) is usual gamma function.

**Proof.** Consider the following integral

\[
\mu_r(Y|\gamma, \delta, \xi) = \int_0^\infty y^{r-1} \exp\left\{-\delta y^{-1}\right\} \exp\left\{\xi \exp\left\{-\delta y^{-1}\right\}\right\} dy.
\]

After using Taylor expansion of the function \( \exp\left\{-\delta y^{-1}\right\} \), we have

\[
\mu_r(Y|\gamma, \delta, \xi) = \sum_{l=0}^\infty \frac{\xi^{l+1} \delta^l}{l!} \int_0^\infty y^{r-1} \exp\left\{-\delta y^{-1}\right\} \left\{\exp\left\{\xi \exp\left\{-\delta y^{-1}\right\}\right\}\right\} \exp\left\{-\delta(l+1) y^{-1}\right\} dy.
\]

Let \( t = (1+l)y^{-1} \) then \( -\delta y^{-1} dy = -\frac{dt}{t^2} \), we have after some algebraic manipulation

\[
\mu_r(Y|\gamma, \delta, \xi) = \sum_{l=0}^\infty \frac{\xi^{l+1} \delta^l}{l!} \int_0^\infty t^{r-1} \exp(-\delta t)(dt).
\]

After integrating, we obtain the final result. That completes the proof. \( \square \)

In particular, the mean of \( Y \):

\[
E(Y) = \int_0^\infty y \mu_r(Y|\gamma, \delta, \xi) dy = \int_0^\infty y \left\{\int_0^\infty y^{r-1} \exp\left\{-\delta y^{-1}\right\} \exp\left\{\xi \exp\left\{-\delta y^{-1}\right\}\right\} dy\right\}_{r=1} \Gamma(1 - 1/\gamma).
\]
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\[ Var(Y) = \sum_{j=0}^{\infty} \frac{\xi^{1+j}}{\beta^{1+j}} \Gamma(1-2/\gamma) 
- \left\{ \sum_{j=0}^{\infty} \frac{\xi^{1+j}}{\beta^{1+j}} \Gamma(1-1/\gamma) \right\}^2. \]

The \( r^{th} \) central moment of the class can be obtained by using Equation (46), as:

\[ \mu_r(y; \delta, \xi) = \sum_{i=0}^{\infty} \frac{\xi^{i+j}}{\beta^{i+j}} \delta \Gamma(1-(2/\gamma)) \int_0^\infty y^{r-1+j} \exp\{-\delta(1+t)y^{-\gamma}\} dy. \]

Again using the above substitution, we have

\[ \mu_r(y; \delta, \xi) = \sum_{i=0}^{\infty} \frac{\xi^{i+j}}{\beta^{i+j}} \delta \Gamma(1-(2/\gamma)) \int_0^\infty y^{r-1+j} \exp\{-\delta(1+t)^{1-1/\gamma}\}. \]

The mgf (moment generating function) gives the base for an alternative path to analytical outcomes compared to working specifically with pdf and cdf and is broadly used in characterization of distribution. The mgf of ETGT-II distribution may be indicated as

\[ M(y|\gamma, \delta, \xi) = \sum_{i=0}^{\infty} \frac{\xi^{i+j}}{\beta^{i+j}} \delta \Gamma(1-1/\gamma) \int_0^\infty y^{r} \exp\{-\delta(1+t)y^{-\gamma}\} dy. \]

5.1. Characteristic function (cf)

The characteristic function of \( Y \) can be evaluated as

\[ \Phi(y|\gamma, \delta, \xi) = \int_0^\infty e^{iyt} dF(y|\gamma, \delta, \xi). \]

After using Taylor expansion, we have

\[ \Phi(y|\gamma, \delta, \xi) = \sum_{i=0}^{\infty} \frac{(iy)^n}{n!} \int_0^\infty y^{r} dF(y|\gamma, \delta, \xi). \]

Hence, we obtain

\[ \Phi(y|\gamma, \delta, \xi) = \sum_{i=0}^{\infty} \frac{(iy)^n}{n!} \delta \Gamma(1-(2/\gamma)) \int_0^\infty y^{r-1+j} \exp\{-\delta(1+t)^{1-1/\gamma}\}. \]

That concludes the objective evidence.

5.2. Factorial generating function (fgf)

The fgf of ETGT-II model is extracted as

\[ F_y(y|\gamma, \delta, \xi) = \int_0^\infty e^{\log(1+y/t)} dF(y|\gamma, \delta, \xi), \]

so, we can compose the integral component above as

\[ F_y(y|\gamma, \delta, \xi) = \sum_{\psi=0}^{\infty} \log(1+y/t)^{\psi} \frac{\xi^{1+j}}{\beta^{1+j}} \delta \Gamma(1-(2/\gamma)). \]

Which establishes objective result.

5.3. Incomplete non-central moments

The incomplete non-central moments of the distribution play a major role in evaluating inequality, including Lorenz and Bonferroni’s income quantiles and curves, that are centered on the incomplete distribution moments.

Proposition 3. The \( r^{th} \) incomplete moment \( \mu_y(z) \) of \( Y \) is given by

\[ \mu_y(z) = \int_0^\infty y^{r} dF(y|\gamma, \delta, \xi), r = 1, 2, \ldots \]

Proof. By definition

\[ \mu_y(z) = \int_0^\infty y^{r} dF(y|\gamma, \delta, \xi), \]

After using Taylor expansion of the function \( e^{\delta(1+t)^{1-1/\gamma}} \), and substitute \( t = \delta(1+m)z^{-\gamma} \), we have

\[ \mu_y(z) = \int_0^\infty y^{r} \delta \Gamma(1-(2/\gamma)) \int_0^\infty y^{r-1+j} \exp\{-\delta(1+t)^{1-1/\gamma}\} \exp\{-\delta(1+t)\} dt. \]

where, \( \Gamma(a, x) = \int_x^\infty e^{-t} e^{-t} dt \) is upper incomplete gamma function.

Lemma 1. Suppose

\[ \mathcal{L}(y|\gamma, \delta, \xi) = \sum_{i=0}^{\infty} \frac{\xi^{1+j}}{\beta^{1+j}} \delta \Gamma(1-(2/\gamma)) \int_0^\infty y^{r-1} \exp\{-\delta(1+m)z^{-\gamma}\} dt, \]

where \( \Gamma(a, x) = \int_x^\infty e^{-t} e^{-t} dt \) is lower incomplete gamma function.
5.4. Conditional moments and mean deviations

In predictive inference, it is advantageous in interaction with lifetime distributions to evaluate the conditional moments $E(Y|Y > t)_{t=1,2,...}$. The $r$th conditional moment of $Y$ is provided as

$$E(Y^r|Y > t) = \frac{1}{\exp(t)} \left[ E(Y^r) - \int_0^t y^r f(y)dy \right].$$

After a little simplification we have

$$H = \exp(-1) \left( \sum_{n=0}^{\infty} \frac{(\exp(t)-1)^n}{\exp(t) - \exp(n \cdot \exp(t))} \right).$$  \hspace{1cm} (66)

The mean deviations of $Y$ about the mean $\mu = E(Y)$ and the median $\mu$ can be stated as

$$\Theta(Y) = \int_0^\infty |y - \mu| dF(y)_{\gamma, \delta, \zeta},$$ \hspace{1cm} (67)

and

$$\Theta_2(Y) = \int_0^\infty |y - \mu| dF(y)_{\gamma, \delta, \zeta},$$ \hspace{1cm} (68)

respectively. The quantity $\Theta_1(Y)$ and $\Theta_2(Y)$ can be calculated by the following results

$$\Theta_1(Y) = 2 \int_0^\infty yf(y)dy + 2\mu F(\mu) - 2\mu,$$ \hspace{1cm} (69)

$$\Theta_2(Y) = 2 \int_0^\infty yF(y)dy - \mu,$$ \hspace{1cm} (70)

using Lemma 1, we have

$$\Theta_1(Y) = 2\mu \gamma + 2\mu F(\mu) - 2\mu,$$ \hspace{1cm} (71)

$$\Theta_2(Y) = 2\mu \gamma - \mu,$$ \hspace{1cm} (72)

where $F(\mu)$ is specified in (6).

6. Uncertainty measures

Information generating function, Renyi entropy, Verma, Tsallis and other entropies for the distribution of ETGT-II model are being investigated in this section.

6.1. Information generating function (IGf)

For the ETGT-II model the information generating function for $Y$ is estimated as

$$I_0(f) = \int E[f(Y|\gamma, \delta, \zeta)]dy = \int_0^\infty f^0(Y|\gamma, \delta, \zeta)dy,$$ \hspace{1cm} (73)

$$I_0(f) = \int_0^\infty \left( \frac{\exp(t-1)\times e^{-t\times e^{-y}}}{(e^t-1)} \right) dy,$$ \hspace{1cm} (74)

$$= \sum_{q=0}^\infty \left( \frac{\exp(t-1)\times e^{-t\times e^{-y}}}{(e^t-1)} \right) dy,$$ \hspace{1cm} (75)

Now making the transformation $(\omega + q)y^{-1} = t$ in (74) and after a little simplification we have

$$I_0(f) = \sum_{q=0}^\infty \left( \frac{\exp(t-1)\times e^{-t\times e^{-y}}}{(e^t-1)} \right) \Gamma \left( 1 + \frac{q}{\gamma} (\gamma + 1) \right).$$ \hspace{1cm} (76)

6.2. Entropy measures

Entropy is a significant idea in several relevant fields communications, thermodynamics, information theory, statistical mechanics, topological dynamics, measure-preserving dynamical systems, etc., as a calculation of various characteristics like disorder, energy that cannot produce work, randomness, uncertainty, complexity, etc. There are several concepts of entropy and they are not necessarily perfect for all applications.

6.2.1. Renyi entropy

The Renyi entropy $	ilde{H}_\alpha(Y)$ for $Y$ with ETGT-II model is

$$\tilde{H}_\alpha(Y) = \frac{1}{1-\alpha} \log \int_0^\infty f^\alpha(Y|\gamma, \delta, \zeta)dy, \alpha \neq 1, \alpha > 0,$$ \hspace{1cm} (77)

where

$$f^\alpha(Y|\gamma, \delta, \zeta) = \left( \frac{\exp(t-1)\times e^{-t\times e^{-y}}}{(e^t-1)} \right)^\alpha,$$ \hspace{1cm} (78)

$$= \sum_{q=0}^\infty \left( \frac{\exp(t-1)\times e^{-t\times e^{-y}}}{(e^t-1)} \right)^\alpha dy.$$ \hspace{1cm} (79)

Now substituting $(\omega + q)y^{-1} = t$, the above integral becomes

$$\int_0^\infty f^\alpha(Y|\gamma, \delta, \zeta)dy = \int_0^\infty \sum_{q=0}^\infty \left( \frac{\exp(t-1)\times e^{-t\times e^{-y}}}{(e^t-1)} \right)^\alpha dy.$$ \hspace{1cm} (80)

After simplification, we have

$$\tilde{H}_\alpha(Y) = \sum_{q=0}^\infty \left( \frac{\exp(t-1)\times e^{-t\times e^{-y}}}{(e^t-1)} \right)^\alpha dy,$$ \hspace{1cm} (81)

Finally, Renyi entropy (RE) becomes

$$\tilde{H}_\alpha(Y) = \frac{1}{1-\alpha} \log \left\{ \sum_{q=0}^\infty \left( \frac{\exp(t-1)\times e^{-t\times e^{-y}}}{(e^t-1)} \right)^\alpha dy \right\}.$$ \hspace{1cm} (82)

It is important to mention that Shannon entropy (SE) of a r.v. $Y$ is obtained as a special case of RE for $\alpha \to 1$.

6.2.2. Verma entropy

The Verma entropy $\mathcal{V}_{\alpha, \beta}(Y)$ for $Y$ with ETGT-II model is

$$\mathcal{V}_{\alpha, \beta}(Y) = \frac{1}{\alpha - \beta} \log \int_0^\infty f^{\alpha-\beta-1}(Y|\gamma, \delta, \zeta)dy, \hspace{1cm} \alpha - 1 < \beta < \alpha, \alpha \geq 1, \alpha \neq \beta,$$ \hspace{1cm} (83)

where
Exponentiated transformation of Gumbel Type-II distribution for modeling COVID-19 data

\[ f^{g(x)}(y|\gamma, \delta, \xi) = \left( \frac{\delta^g}{(e^b - 1)^g - 1} \right)^{a - g} - \frac{\delta^g}{(e^b - 1)^g - 1} y^{-\delta^g - 1}(x + \beta - 1) e^{\delta^g(x + \beta - 1) y^{-\delta^g - 1}}. \]  

(83)

It is important to notice that, when \( z \to 1 \), in (82), it reduces to the RE. On the other hand, if \( \beta \to 1 \) and \( x \to 1 \), in (82), then it approaches to the Shannon entropy. By using the above information, we have

\[ H(\mu) = \int_0^\infty f^{g(x)}(y|\gamma, \delta, \xi)dy. \]

(84)

Now substituting, \((x + \beta - 1) y^{-\delta^g} = t\), the above integral becomes

\[ H(\mu) = \int_0^\infty f^{g(x)}(y|\gamma, \delta, \xi)dy - \sum_{q=0}^{\infty} \frac{\delta^g}{q!} \left( \frac{\delta^g}{(e^b - 1)^g - 1} \right)^{a - g} \left( e^b - 1 \right)^{e^b - q} \left( a - \theta - 1 \right)^{\Delta} \sum_{q=0}^{\infty} \frac{\delta^g}{q!} \left( \frac{\delta^g}{(e^b - 1)^g - 1} \right)^{a - g} \left( e^b - 1 \right)^{e^b - q} \left( a - \theta - 1 \right)^{\Delta}. \]

(85)

After simplification, we have

\[ H(\mu) = \int_0^\infty f^{g(x)}(y|\gamma, \delta, \xi)dy - \sum_{q=0}^{\infty} \frac{\delta^g}{q!} \left( \frac{\delta^g}{(e^b - 1)^g - 1} \right)^{a - g} \left( e^b - 1 \right)^{e^b - q} \left( a - \theta - 1 \right)^{\Delta} \sum_{q=0}^{\infty} \frac{\delta^g}{q!} \left( \frac{\delta^g}{(e^b - 1)^g - 1} \right)^{a - g} \left( e^b - 1 \right)^{e^b - q} \left( a - \theta - 1 \right)^{\Delta}. \]

(86)

Finally, \( V_{\theta}(Y) \) becomes

\[ V_{\theta}(Y) = \frac{1}{\alpha - \beta} \times \log \left[ \frac{\delta^g}{(e^b - 1)^g - 1} \left( e^b - 1 \right)^{e^b - q} \left( a - \theta - 1 \right)^{\Delta} \right]. \]

(87)

6.2.3. Tsallis entropy

Tsallis entropy \( T_\omega(Y) \) of \( Y \) with ETGT-II model is defined as

\[ T_\omega(Y) = \frac{1}{\omega - 1} \left( 1 - \int_0^\infty f^{g(x)}(y|\gamma, \delta, \xi)dy \right), \quad \omega \neq 1. \]

(88)

As, \( f^{g(x)}(y|\gamma, \delta, \xi) \) and \( \int_0^\infty f^{g(x)}(y|\gamma, \delta, \xi)dy \) are calculated in (77)-(80) respectively. Therefore, by using this information, \( T_\omega(Y) \) takes the following form

\[ T_\omega(Y) = \frac{1}{\omega - 1} \left( 1 - \sum_{q=0}^{\infty} \frac{\delta^g}{(e^b - 1)^g - 1} \left( e^b - 1 \right)^{e^b - q} \left( a - \theta - 1 \right)^{\Delta} \right). \]

(89)

6.2.4. Mathai-Houbold entropy

Classical Shannon Entropy has been generalized in many directions one of them is \( \omega \) generalized entropy introduced by Mathai and Houbold [20] and is defined by

\[ \tilde{I}_{\omega}(Y) = \frac{1}{\omega_1 - 1} \left( \int_0^\infty f^{\omega_1}(y|\gamma, \delta, \xi)dy - 1 \right). \]

(90)

Similar arguments to \( f^{\omega_1}(y|\gamma, \delta, \xi) \) gives

\[ f^{\omega_1}(y|\gamma, \delta, \xi) = \left( \frac{\delta^g}{(e^b - 1)^g - 1} \right)^{a - g} \left( e^b - 1 \right)^{e^b - q} \left( a - \theta - 1 \right)^{\Delta}. \]

(91)

by using the above information, we get

\[ \tilde{I}_{\omega}(Y) = \frac{1}{\omega_1 - 1} \left( \int_0^\infty f^{\omega_1}(y|\gamma, \delta, \xi)dy - 1 \right). \]

(92)

Therefore, the final form of the above integral is

\[ \tilde{I}_{\omega}(Y) = \frac{1}{\omega_1 - 1} \left( \sum_{q=0}^{\infty} \frac{\delta^g}{(e^b - 1)^g - 1} \left( e^b - 1 \right)^{e^b - q} \left( a - \theta - 1 \right)^{\Delta} \right). \]

(93)

6.2.5. Kapur entropy

Kapur entropy \( I_{K,\beta}(Y) \) of \( Y \) with ETGT-II model is defined as

\[ I_{K,\beta}(Y) = \frac{1}{\beta - \alpha} \left( \int_0^\infty f^{\omega}(y) dy \right) \log \left( \int_0^\infty f^{\omega}(y) dy \right). \]

(94)

\[ I_{K,\beta}(Y) = \frac{1}{\beta - \alpha} \left( \int_0^\infty f^{\omega}(y) dy \right) - \log \left( \int_0^\infty f^{\omega}(y) dy \right). \]

(95)

Similar arguments to \( f^{\omega} \) incorporating in Eq. (94), we get the following results

\[ I_{K,\beta}(Y) = \frac{1}{\beta - \alpha} \left( \int_0^\infty f^{\omega}(y) dy \right) \log \left( \int_0^\infty f^{\omega}(y) dy \right). \]

(96)

6.2.6. \( \omega \)-Entropy

\( \omega \)-entropy \( H_\omega(Y) \) of \( Y \) with ETGT-II model is defined as

\[ H_\omega(Y) = \frac{1}{\omega - 1} \left( 1 - \int_0^\infty f^{\omega}(y|\gamma, \delta, \xi)dy \right), \quad \omega \neq 1. \]

(97)

As, \( f^{\omega}(y|\gamma, \delta, \xi) \) and \( \int_0^\infty f^{\omega}(y|\gamma, \delta, \xi)dy \) are calculated in (77)-(80) respectively. Therefore, by using this information, \( H_\omega(Y) \) takes the following form

\[ H_\omega(Y) = \frac{1}{\omega - 1} \left( 1 - \sum_{q=0}^{\infty} \frac{\delta^g}{(e^b - 1)^g - 1} \left( e^b - 1 \right)^{e^b - q} \left( a - \theta - 1 \right)^{\Delta} \right). \]

(98)

Three dimensional behavior of the entropies of the ETGT-II model are plotted in Fig. 7.

7. Estimation

We continue by showing estimates of the parameters of the proposed distribution through various techniques. We use
maximum likelihood (ML) estimation methodology and the Bayesian method for estimation objective. Working the Matlab (log lik), R (optimum and MaxLik features), the Ox program (subroutine MaxBFGS), or SAS (PROC NLMIXED), the parameters of the ETGT-II distribution can be assessed from the log-likelihood depending on the sample. In addition, some goodness-of-fit statistics are provided for comparing estimates of density and choice of models.

7.1. Maximum likelihood estimation

The maximum likelihood estimates (MLEs) are provided by optimizing this equation according to \( \gamma, \delta \) and \( \xi \). They are also characterized as the maximum of the log-likelihood function defined by

\[
L(y|\gamma, \delta, \xi) = \prod_{i=1}^{n} \frac{\gamma \delta y_i^{-\gamma-1} \exp\{-\delta y_i^{-\gamma}\} \exp\{\xi \exp\{\delta y_i^{-\gamma}\}\}}{\exp(\xi) - 1}.
\]

(99)

The log-likelihood function for the ETGT-II model is provided by the data set \( y_1, \ldots, y_n \),

\[
b_{\gamma, \delta, \xi} = -n \log(e^{\gamma} - 1) + n \log(y_1) + n \log(\delta) + n \log(\xi) - (\gamma + 1) \sum_{i=1}^{n} \log y_i - \delta \sum_{i=1}^{n} y_i^{-\gamma} + \xi \sum_{i=1}^{n} e^{-\delta y_i^{-\gamma}}.
\]

(100)

By solving the non-linear likelihood equation, we obtain the MLEs of the parameters \( \gamma, \delta \) and \( \xi \) obtained by differentiating (100). We obtain the components of score vector \( \lambda_{\gamma, \delta, \xi} = (\lambda_{\gamma}, \lambda_{\delta}, \lambda_{\xi})^T \)

\[
\lambda_{\gamma} = \frac{\partial b_{\gamma, \delta, \xi}}{\partial y} = \frac{n}{\gamma} - \sum_{i=1}^{n} \log(y_i) + \delta \sum_{i=1}^{n} y_i^{-\gamma} \log(y_i) + \xi \sum_{i=1}^{n} e^{-\delta y_i^{-\gamma}} \log(y_i).
\]

(101)

\[
\lambda_{\delta} = \frac{\partial b_{\gamma, \delta, \xi}}{\partial \delta} = \frac{n}{\delta} - \sum_{i=1}^{n} y_i^{-\gamma} - \delta \sum_{i=1}^{n} y_i^{-\gamma} e^{-\delta y_i^{-\gamma}},
\]

(102)

\[
\lambda_{\xi} = \frac{\partial b_{\gamma, \delta, \xi}}{\partial \xi} = -\frac{ne^{\xi}}{(e^{\xi} - 1)} + \frac{n}{\xi} + \sum_{i=1}^{n} e^{-\delta y_i^{-\gamma}}.
\]

(103)

Setting \( \Lambda_{\gamma}, \Lambda_{\delta}, \Lambda_{\xi} = 0 \) and solving ultimately provides the MLEs for the parameters of the ETGT-II model. To solve them numerically an iterative process such as Newton-Raphson approach is required. Now, using simulation, we scrutinize the performance of the MLEs regarding sample size \( n \). We follow the following steps to perform simulation study: stimulate \( 5000 \) samples of size \( n = 25, 150, 250, 350 \) from ETGT-II \((0.2, 0.3, 0.7, (2, 3, 4))\); obtain the MLEs for \( 5000 \) samples, say \( \hat{\gamma}, \hat{\delta}, \text{ and } \hat{\xi} \), for \( m = 1, 2, \ldots, 5000 \); quantify MLEs, estimated biases and mean squared errors (MSEs); where \( \text{MLE} = \frac{1}{5000} \sum_{m=1}^{5000} \hat{\phi} \), absolute bias = \( \frac{1}{5000} \sum_{m=1}^{5000} |\hat{\phi} - \phi| \) and MSE = \( \frac{1}{5000} \sum_{m=1}^{5000} (\hat{\phi} - \phi)^2 \). Table 2 lists the ml, bias and mSE for the different estimates. The mean square errors are calculated for each set of parameters and sample size. We directly observed that the MSEs decrease by \( n \). It is noted from Table 2, choosing large (small) values of the parameters, larger (smaller) values of MSE are observed.

Fig. 7 Graphical behaviour of Reny, Verma, Tsallis Mathai, Kapur and \( \omega \) entropy of the ETGT-II model.
7.2. Bayesian analysis

In this section, we continue by presenting estimation of suggested structure parameters by a Bayesian mechanism. We assume the parameters $\gamma$, $\delta$, and $\xi$ are random variables. Here, the following independent priors are assumed as $\gamma \sim \text{gamma} (v_1, \sigma_1)$, $\delta \sim \text{gamma} (v_2, \sigma_2)$, and $\xi \sim \text{gamma} (v_3, \sigma_3)$ where $v_i$, and $\sigma_i$ $\in \mathbb{R}^+$. $i = 1, 2, 3$. The joint posterior density of $\gamma$, $\delta$, and $\xi$ has the following form

$$
    g(\gamma, \delta, \xi | x) = \frac{L(y | \gamma, \delta, \xi) p(\gamma)p(\delta)p(\xi)}{\int \int \int L(y | \gamma, \delta, \xi) p(\gamma)p(\delta)p(\xi)}
$$

The Bayes estimator (BE) of the parameter $\gamma$ is provided under squared error loss function (SELF) as follows.

$$
    \hat{\gamma}_{BE} = E(\gamma | x) = \frac{\int \int \int \gamma L(y | \gamma, \delta, \xi) p(\gamma)p(\delta)p(\xi)}{\int \int \int L(y | \gamma, \delta, \xi) p(\gamma)p(\delta)p(\xi)}
$$

in a similar fashion $\hat{\delta}_{BE} = E(\delta | x)$, and $\hat{\xi}_{BE} = E(\xi | x)$. In addition, the Bayes risk is determined by the relation, $\text{Var}(\gamma | x) = E(\gamma^2 | x) - (E(\gamma | x))^2$, where

$$
    E(\gamma^2 | x) = \int \int \int \gamma^2 L(y | \gamma, \delta, \xi) p(\gamma)p(\delta)p(\xi)
$$

One may realize that estimates are not analytically tractable, so we practise Adaptive Metropolis Hasting algorithm of Monte Carlo Markov Chain (MCMC) to obtain estimates. The trace plots and estimated posterior densities for the MCMC estimates of the model parameters for simulated sets have been presented in Fig. 8. These plots depicted good convergence of the estimates. The density plots show the shape of the marginal distributions. One can notice that, density plots of $\gamma$, $\delta$ and $\xi$ are slightly symmetrical. In addition, Bayes estimates, 95% Bayesian intervals, posterior variance are also reported in Table 3. Furthermore, all Bayes estimates are within the 95% Bayesian intervals. Same behaviour is obtained for the implementation of real life data sets.

8. Real data implementations

We present two examples in this section to explain the efficiency of the proposed model. We use R software to demonstrate improved efficiency of the ETGT-II model and numerical calculations. We consider the following distributions, for comparative purposes: (i) Generalized Inverse Weibull distribution [21], (ii) Frechet Distribution (Frechet), (iii) Additive Gumbel Type-II (AGT-II) (iv) Gumbel Type-II (v) Exponentiated Burr XII (EB-XII) model. Different techniques of segregation based on the log-likelihood function assessed at the MLEs were also considered: Akaike Information Criterion (AIC) computed through AIC: $2\left[ \log(\bar{y}) + \frac{1}{2}\log(n) \right]$, Corrected Akaike information Criterion AICC = AIC + $\frac{2(2+c+1)}{n-c-2}$, Hannan Quinn Information Criterion HQIC = $-2L\left( \hat{\gamma}, \hat{\delta}, \hat{\xi}; y \right) + 2k\log\log(n)$, where $\gamma$ represents the number of parameters to be estimated and $\hat{\gamma}, \hat{\delta}, \hat{\xi}$ the estimates of $\gamma, \delta, \xi$. The best model is the one that gives the minimum values of those benchmarks. The results revealed that ETGT-II model is a better model than the above mentioned models in all cases. To check whether the data fits ETGT-II ($\gamma, \delta, \xi$) distribution. We use here, Kolmogorov-Smirnov (K-S) distances between the empirical distribution function and the fitted distribution function. In Tables 6,7, the negative log likelihood value, the K-S test value and the corresponding critical value are given. The small K-S distance and the large critical-value for the test indicate that this data matches perfectly almost well with ETGT-II ($\gamma, \delta, \xi$) distribution.

The standardized goodness-of-fit measures are now implemented to validate which distribution fits these data best. The Anderson–Darling and Cramér–von Mises tests statistics are considered. In general, the smaller the Anderson – Darling ($A^2$) and Cramér – von Mises ($W^2$), values are, the better the data fit. The values of the statistics Anderson–Darling and Cramér–von Mises are given in Tables 6,7. In Figs. 10,11,
the estimated densities of the ETGT-II ($\gamma, \delta, \xi$) and the competitor distributions were graphed for more visual comparison.

Data 1: Number of deaths due to COVID-19 in Europe
This data is reported in (https://covid19.who.int/) which represents daily deaths due to COVID-19 in Europe from 1st March to 30 March.

Data 2: Number of deaths due to COVID-19 in China
This data is reported in (https://www.worldometers.info/coronavirus/country/china/) which represents daily

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**Table 3** Posterior summaries of the ETGT-II model for simulated data.

| $n$ | Parameter | 2.5%  | 97.5%  | Posterior Mean | Posterior Variance |
|-----|-----------|-------|--------|----------------|-------------------|
| 20  | $\gamma$  | 0.32758 | 0.70016 | 0.49691 | 0.00882 |
|     | $\delta$  | 0.38223 | 0.93174 | 0.62422 | 0.01932 |
|     | $\xi$     | 0.38829 | 1.73208 | 0.94961 | 0.12227 |
| 40  | $\gamma$  | 0.51876 | 0.88215 | 0.69003 | 0.00849 |
|     | $\delta$  | 0.57567 | 1.20878 | 0.86062 | 0.02606 |
|     | $\xi$     | 0.46949 | 2.02583 | 1.12829 | 0.16287 |
| 70  | $\gamma$  | 0.76522 | 1.13714 | 0.94397 | 0.00891 |
|     | $\delta$  | 0.69718 | 1.35226 | 1.00023 | 0.02771 |
|     | $\xi$     | 0.496294 | 2.11858 | 1.18145 | 0.17183 |

**Table 4** Descriptive measures for Data Sets.

| Data | $Q_1$ | $Q_3$ | Median | Mean | Trimmed Mean | Min | Max | SD | Range | Skew | Kurtosis |
|------|------|------|--------|------|--------------|-----|-----|----|-------|------|----------|
| I    | 120.8 | 1414.0 | 385.0  | 841.4 | 703.96       | 6   | 2824 | 938.69 | 2818  | 0.92 | −0.63    |
| II   | 13.00 | 82.75  | 33     | 49.74 | 44.8         | 3   | 150  | 43.87 | 147   | 0.82 | −0.62    |
deaths due to COVID-19 in China from 23 January to 28 March. The data are: 8, 16, 15, 24, 26, 28, 38, 43, 46, 45, 57, 64, 65, 73, 71, 86, 89, 97, 108, 97, 146, 121, 143, 142, 105, 98, 136, 114, 118, 109, 97, 150, 71, 52, 44, 47, 35, 42, 31, 38, 31, 28, 27, 22, 17, 22, 11, 7, 13, 10, 14, 13, 11, 8, 3, 7, 6, 9, 7, 4, 6, 5, 3, 5.

Fig. 9 shows the summary statistics through box plot for both data sets. The shaded lines in graphs show the quantile point 25%, 50% and 75% respectively. Box plot is a tool for graphically illustrating the data. It gives a good indication of how the values in the data are spread out. Figs. 12, 13 list the PP-plots of the fitted distributions. The proposed model clo-

Table 5: Estimates of the parameters with Standard errors (in parentheses) for Data Set I and Data Set II.

| Model  | Parameter | Data I       | Error     | Data II      | Error     |
|--------|-----------|--------------|-----------|--------------|-----------|
| ETGT-II| \( \hat{\gamma} \) | 0.69454      | 0.09256   | 1.08612      | 0.10218   |
|        | \( \hat{\beta} \) | 11.6824      | 8.28909   | 10.6887      | 4.54291   |
|        | \( \hat{\lambda} \) | 3.17107      | 2.20390   | 2.43135      | 1.34383   |
| GIWD   | \( \hat{\alpha} \) | 2.70212      | 334.481   | 9.10197      | 811.843   |
|        | \( \hat{\beta} \) | 0.57051      | 0.07536   | 0.91594      | 0.08369   |
|        | \( \hat{\lambda} \) | 9.18399      | 648.661   | 1.78997      | 46.234    |
| Frechet| \( \hat{\alpha} \) | 99.7713      | 29.7155   | 17.1868      | 2.45283   |
|        | \( \hat{\lambda} \) | 0.59186      | 0.07787   | 0.91595      | 0.08368   |
| AGT-II | \( \hat{\alpha} \) | 9.79414      | 803.73    | 7.47910      | 116.202   |
|        | \( \hat{\lambda} \) | 6.40543      | 803.75    | 13.4320      | 3.46155   |
|        | \( \hat{\lambda} \) | 0.57062      | 0.11674   | 4.48651      | 6.66264   |
|        | \( \hat{\lambda} \) | 0.57071      | 0.21937   | 0.91372      | 0.09052   |
| GT-II  | \( \hat{\alpha} \) | 0.57057      | 0.07536   | 0.91595      | 0.08369   |
|        | \( \hat{\lambda} \) | 16.1933      | 5.71448   | 13.5324      | 3.10943   |
| EB-XII | \( \hat{\alpha} \) | 0.29162      | 0.13471   | 0.42854      | 0.16904   |
|        | \( \hat{\lambda} \) | 52.0316      | 54.4001   | 59.0695      | 58.8060   |

Table 6: Values of the considered measures for Data Set I.

| Distribution | ETGT-II | GIWD | Frechet | AGT-II | GT-II | EB-XII |
|--------------|---------|------|---------|--------|-------|--------|
| \( W^5 \)    | 0.12316 | 0.15402 | 0.15425 | 0.15404 | 0.15402 | 0.12913 |
| \( A^5 \)    | 0.81477 | 0.98445 | 0.98601 | 0.98453 | 0.98444 | 0.84641 |
| \( K - S \)  | 0.14104 | 0.15964 | 0.17716 | 0.15960 | 0.15964 | 0.14954 |
| p-value      | 0.54270 | 0.38800 | 0.26980 | 0.38820 | 0.38790 | 0.46870 |
| \( AIC \)    | 472.771 | 475.394 | 474.074 | 477.394 | 473.394 | 473.437 |
| \( CAIC \)   | 473.694 | 476.317 | 474.518 | 478.994 | 473.838 | 474.360 |
| \( BIC \)    | 476.974 | 479.597 | 476.876 | 482.998 | 476.196 | 477.641 |
| \( HQIC \)   | 474.115 | 476.738 | 474.970 | 479.187 | 474.290 | 474.782 |
| \( -\hat{\gamma} \) | 233.385 | 234.697 | 235.037 | 234.697 | 234.697 | 233.719 |

Table 7: Values of the considered measures for Data Set II.

| Distribution | ETGT-II | GIWD | Frechet | AGT-II | GT-II | EB-XII |
|--------------|---------|------|---------|--------|-------|--------|
| \( W^5 \)    | 0.22812 | 0.28434 | 0.28433 | 0.28444 | 0.28434 | 0.23364 |
| \( A^5 \)    | 1.45916 | 1.75835 | 1.75829 | 1.75861 | 1.75835 | 1.48713 |
| \( K - S \)  | 0.11417 | 0.12843 | 0.12845 | 0.12888 | 0.12843 | 0.11457 |
| p-value      | 0.35580 | 0.22640 | 0.22620 | 0.22290 | 0.22640 | 0.35170 |
| \( AIC \)    | 664.316 | 668.203 | 666.203 | 670.162 | 666.203 | 664.695 |
| \( CAIC \)   | 664.703 | 668.590 | 666.394 | 670.818 | 666.397 | 665.082 |
| \( BIC \)    | 670.885 | 674.772 | 670.583 | 678.921 | 670.583 | 671.264 |
| \( HQIC \)   | 666.912 | 670.799 | 667.934 | 673.623 | 667.934 | 667.291 |
| \( -\hat{\gamma} \) | 329.158 | 331.102 | 331.102 | 331.081 | 331.102 | 329.348 |
sely followed the box plot and PP-plots. Log-likelihood plots 
are given in Figs. 14 and 15 (see Table 4).

We estimate the unknown parameters by maximum likeli-
hood for each model. Table 5 lists the MLEs of the above 
models with their respective standard errors (evaluated by 
inverting the information matrices). The calculations were 
made using the R programming language. Posterior sum-
maries and densities plots of the ETGT-II model for data set 
I and II are given in Table 8 and plotted in Fig. 16.

9. Concluding remarks

A novel model called Exponentiated transformation of 
Gumbel Type-II (ETGT-II) is proposed in this study, and its 
mathematical properties have been analyzed in detail. Also, 
graphical representation of some statistical functions are 
included. The general non-central complete and incomplete 
moments are also discussed. Entropies for uncertainty mea-
sures (such as Renyi, Verma Tsallis, Mathai-Houbold and 
Kumar entropy) are derived and represented graphically. 
The estimation of the parameters is derived using the maxi-
mum probability method and the Bayesian paradigm. 
Through using the classical goodness of fit indicators and P-
P graphs, we evaluate the efficiency of the ETGT-II model 
with its five significant counterparts. The log-likelihood, den-
sity and trace plots are also sketched for both data sets. These 
findings are in line with the fact that suggested model is quite 
suitable for real-life applications.
Exponentiated transformation of Gumbel Type-II distribution for modeling COVID-19 data.

Fig. 12  PP-graphs for daily deaths due to COVID-19 in Europe.

Fig. 13  PP-graphs for daily deaths due to COVID-19 in China.
Fig. 14  The curves log-likelihood function of \((\gamma, \delta, \xi)\) for data set I.

Fig. 15  The curves log-likelihood function of \((\gamma, \delta, \xi)\) for data set II.

Table 8  Posterior summaries of the ETGT-II model for data sets I and II.

| Data | \(n\) | Parameter | 2.5\% | 97.5\% | Posterior Mean | Posterior Variance |
|------|-------|-----------|-------|--------|----------------|-------------------|
| I    | 30    | \(\gamma\) | 0.294248 | 0.45244 | 0.36953 | 0.00163 |
|      |       | \(\delta\) | 1.550709 | 2.72582 | 2.09653 | 0.09207 |
|      |       | \(\xi\) | 0.196083 | 0.85450 | 0.47634 | 0.02858 |
| II   | 40    | \(\gamma\) | 0.359340 | 0.57046 | 0.46023 | 0.00294 |
|      |       | \(\delta\) | 0.161395 | 0.46259 | 0.28618 | 0.00590 |
|      |       | \(\xi\) | 6.773682 | 17.09584 | 11.20747 | 6.90395 |

Fig. 16  Density plots and Trace plots of parameters \((\gamma, \delta, \xi)\) for data set I (left) data set II (right).
Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors would like to thanks the reviewers and the editors for their comments that helped to improve the article substantially.

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