Anchored Lagrangian submanifolds and their Floer theory

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Abstract. We introduce the notion of (graded) anchored Lagrangian submanifolds and use it to study the filtration of Floer's chain complex. We then obtain an anchored version of Lagrangian Floer homology and its (higher) product structures. They are somewhat different from the more standard non-anchored version. The anchored version discussed in this paper is more naturally related to the variational picture of Lagrangian Floer theory and so to the likes of spectral invariants. We also discuss rationality of Lagrangian submanifold and reduction of the coefficient ring of Lagrangian Floer cohomology of thereof.

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1. Introduction

Lagrangian Floer theory associates to each given pair of Lagrangian submanifolds \( L_0, L_1 \subset M \) a group \( HF(L_1, L_0) \), called the Floer cohomology group. Floer cohomology group can be regarded as a \((\infty/2)\)-dimensional) homology group of the space of paths \( \Omega(L_0, L_1) \) joining \( L_0 \) to \( L_1 \):

\[
\Omega(L_0, L_1) = \{ \ell : [0, 1] \to M \mid \ell(0) \in L_0, \ell(1) \in L_1 \}.
\]

Floer used Morse theory to rigorously define this cohomology group. The exterior derivative of the ‘Morse function’ Floer used is the action one-form \( \alpha \) defined by

\[
\alpha(\ell)(\xi) = \int_0^1 \omega(\dot{\ell}(t), \xi(t)) \, dt
\]

for each tangent vector \( \xi \in T_\ell \Omega(L_0, L_1) \).

In general the one form \( \alpha \) is closed but not necessarily exact. So one needs to use Novikov’s Morse theory \([N]\) of closed one forms. In order to take care of non-compactness of the moduli space of connecting orbits which occurs from non-exactness of the closed one form involved, Novikov uses a kind of formal power series ring, the so called Novikov ring for his Morse theory of closed one forms.

Floer, and later Hofer-Salamon \([HS]\) and the fourth named author \([On]\), used a similar Novikov ring for Floer homology of periodic Hamiltonian system. The present authors also used a Novikov ring to study Lagrangian Floer homology in \([FOOO00]\). They however introduced a slightly different ring which they call universal Novikov ring. The same universal Novikov ring was used in \([Fu2]\) to associate a filtered \( A_\infty \) category (Fukaya category) to a symplectic manifold, which combine Floer cohomologies of various pairs of Lagrangian submanifolds, together with their (higher) product structures.

In Section 5.1 \([FOOO08]\), the relationship between the Floer cohomology over a (traditional) Novikov ring and the one over the universal Novikov ring is discussed, which concerns pairs of Lagrangian submanifolds. The discussion thereof involves a systematic choice of associating base points on the connected components of \( \Omega(L_0, L_1) \) when the pair \( (L_0, L_1) \) varies. In this paper we extend this to the cases of three or more Lagrangian submanifolds, which enter in the product structure of Floer cohomology.

We remark that the closed one form \( \alpha \) above, determines a single-valued function on an appropriate covering space of \( \Omega(L_0, L_1) \) up to addition of a constant. The choice of this additive constant, which is closely related to the choice of a base point, determines the filtration of Floer cohomology. When more than two Lagrangian submanifolds are involved, to equip filtrations of the Floer cohomologies ‘in a consistent way’ for all pairs \((L_0, L_1)\) is a somewhat nontrivial problem. The problem of finding a systematic choice of the base point for the filtration shares some similarity with the corresponding problems for the degree (dimension) and for the orientation of the moduli space of pseudo-holomorphic strips or polygons. (See Definition 3.7.)

For the purpose of systematically finding the base points of the path spaces, we use the notion of anchored Lagrangian submanifold.

DEFINITION 1.1 (Anchored Lagrangian submanifolds). Fix a base point \( y \in M \). An anchor of a Lagrangian submanifold \( L \subset M \) to \( y \) is a path \( \gamma : [0, 1] \to M \).
such that $\gamma(0) = y, \gamma(1) \in L$. A pair $(L, \gamma)$ is called an anchored Lagrangian submanifold.

Roughly speaking, Floer cohomology group is a cohomology group of a chain complex $CF(L_1, L_0)$ which is generated by the set of intersections $L_0 \cap L_1$ and whose boundary operator $\partial$ is defined by ‘counting’ the number of solutions $u : \mathbb{R} \times [0, 1] \to M$ of the (nonlinear) Cauchy-Riemann equation

\[
\begin{cases}
\frac{\partial u}{\partial \tau} + J \frac{\partial u}{\partial \imath} = 0 \\
u(\tau, 0) \in L_0, \quad u(\tau, 1) \in L_1.
\end{cases}
\]

The moduli space of pseudo-holomorphic strips entering in this counting problem is an appropriate compactification of the solution space.

To properly defined the Floer cohomology group, we need to study:

1. (Filtration): a filtration of the Floer’s chain complex $CF(L_1, L_0)$
2. (Z-Grading): a $\mathbb{Z}$-grading with respect to which $\partial$ has degree 1
3. (Sign): a sign on the generators which induces a $\mathbb{Z}$-module (or at least a $\mathbb{Q}$ vector space) structure on $CF(L_1, L_0)$ with respect to which $\partial$ is a $\mathbb{Z}$-module (resp. a $\mathbb{Q}$ vector space) homomorphism.

In all of these structures, the relative version, i.e., the ‘difference’ between two generators $q, p \in L_0 \cap L_1$ is canonically defined : for (1) it is the symplectic area, for (2) it is nothing but the so called Maslov-Viterbo index [Y] [F12] and for (3) it is based on the choice of orientation of the determinant bundle $\det D_u \overline{\partial} \to \mathcal{M}(p, q; L_0, L_1)$ at a solution $u$ of [L3]. More precisely, the gluing formula for the indices shares a similar behavior with the problems on (1) and (2). (See Remarks 6.6 and 6.8.)

Denote any of these invariants associated to the Floer trajectory $u$ by $I(q, p; u)$. The main problem to solve to provide these structures then is to see if there exists some family of functions $I = I(q)$ independent of the choice of $u \in \mathcal{M}(p, q; L_0, L_1)$ such that

\[
I(q, p; u) = I(p) - I(q).
\]

This is not possible in general unless one puts various restrictions on the triple $(L_0, L_1; M)$: for (1) exactness of $(M, \omega)$ and of $(L_0, L_1)$, for (2) vanishing of $c_1$ of $(M, \omega)$ and of the associated Maslov indices of $L_0, L_1$ and for (3) spinness of the pair $L_0, L_1$ or (more generally relative spinness of the pair $(L_0, L_1)$). Under these restrictions respectively, it has been well understood by now that such a choice is always possible. See [F11], [Se2] for (1), [F12] and [FOOO00] for (2) and [FOOO00] for (3) respectively. (See also [Fu2] where (1), (2) and (3) are described in the setting of Fukaya category. There are some technical errors and/or inconsistency with [FOOO08], in the description of [Fu2], which are corrected in this paper.)

In this paper we define a filtered $A_\infty$ category on each symplectic manifold, which is an anchored version of Fukaya category. (See Theorem 8.14.) Its objects are anchored Lagrangian submanifolds $(L, \gamma)$ equipped with some extra data: (bounding cochain, spin structure and grading.) The morphism is an (anchored version of) Floer’s chain complex. We however emphasize that the cohomology group $HF((L_1, \gamma_1), (L_0, \gamma_0))$ is different from usual Floer cohomology group $HF(L_1, L_0)$ which is defined in [FOOO00], [FOOO08]. Namely the former is a component of the latter where only one of the connected components of $\Omega(L_0, L_1)$ is used for the construction. The anchored versions of (higher) compositions $m_k$ are also different.
from the usual one. The precise relationship between the anchored version and the non-anchored one is rather complicate to describe.

Necessity of studying this non-canonicality of filtration appears in several situations: one is in the construction of Seidel’s long exact sequence as studied in [Oh3] and the other is in the study of Galois symmetry in Floer homology [Fu3].

Leaving the first problem to [Oh3], we will discuss the latter problem in Section 10 of this paper. This involves the detailed discussion of the universal Novikov ring. An element of the universal Novikov ring has the form

$$
\sum_i a_i e^{\mu_i} T^{\lambda_i}
$$

which is either a finite sum or an infinite sum with \( \lambda_i \leq \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = +\infty \).

Here \( a_i \) is an element of a ground ring \( R \) (for example \( R = \mathbb{Q} \)) and \( \lambda_i \) are real numbers. We consider the subring consisting of elements \( (1.5) \) such that \( \lambda_i \in \mathbb{Q} \) in addition. We denote it by \( \Lambda^{rat}_{0,nov} \). We say that a Lagrangian submanifold \( L \) of a symplectic manifold \( M \) is rational if the subgroup

$$
\Gamma_\omega(M, L) = \{ \omega(\alpha) \mid \alpha \in \pi_2(M, L) \} \subset \mathbb{R}
$$

is discrete.

Now we assume \([\omega] \in H^2(M; \mathbb{Q})\). Then there exists \( m_{amb} \in \mathbb{Z}_+ \) and a complex line bundle \( \mathcal{P} \) with connection \( \nabla \) such that the curvature form of \((\mathcal{P}, \nabla)\) is \( 2\pi \sqrt{-1} m_{amb}\omega \). We call \((\mathcal{P}, \nabla)\) the pre-quantum bundle. A Lagrangian submanifold \( L \) is called Bohr-Sommerfeld rational or simply BS-rational if holonomy group of the restriction of \((\mathcal{P}, \nabla)\) to \( L \) is of finite order. (Such a Lagrangian submanifold is called ‘cyclic’ in [Oh1] and just ‘rational’ in [Fu3]. Since a Lagrangian submanifold \( L \) is called a Bohr-Sommerfeld orbit when the holonomy group is trivial, the name ‘BS-rational’ seems to be a more reasonable choice.)

**Theorem 1.2.** To each \((M, \omega)\) with \([\omega] \in H^2(M; \mathbb{Q})\) with \( c_1(M) = 0 \), we can associate a filtered \( A_\infty \) category with \( \Lambda^{rat}_{nov} \) coefficients. Its object consists of a system \((L, \mathcal{L}, sp, b, s, S_L)\) where \( L \) is a BS-rational Lagrangian submanifold of \( M \) with its Maslov class \( \mu_L = 0 \), \( \mathcal{L} \) is a flat \( U(1) \) bundle on \( L \) with finite holonomy group, \( sp \) is a spin structure of \( L \), \( b \) is a bounding cochain, \( s \) is a \( \mathbb{Z} \)-grading, and \( S_L \) is a rationalization of \( L \). We denote this category by \( \mathcal{Fuk}^{rat}(M, \omega) \).

If \( m_{amb}[\omega] \in H^2(M; \mathbb{Z}) \), then there exists a \( m_{amb}\hat{Z} \) action on this category which is compatible with the \( \hat{Z} \) action of \( \Lambda^{rat}_{0,nov} \) as continuous Galois group.

We will explain the notions appearing in the theorem in Section 10. In fact our attempt to further reduce to a smaller ring leads us to considering a collection of Lagrangian submanifolds for which one can associate an \( A_\infty \) category over a Novikov ring like \( \mathbb{Q}[[T^{1/m}]][T^{-1}][e, e^{-1}] \).

**Theorem 1.3.** Let \((M, \omega)\) be rational and \((\mathcal{P}, \nabla)\) be the pre-quantum line bundle of \( m_{amb}\omega \). Then for each fixed \( N \in \mathbb{Z}_+ \), there exists a filtered \( A_\infty \) category \( \mathcal{Fuk}_N(M, \omega) \) over the ring \( \mathbb{C}[[T^{1/N}]][T^{-1}][e, e^{-1}] \):

1. its objects are \((L, \mathcal{L}, sp, b, S_L)\) where \( L \) is a \( N \) BS-rational Lagrangian submanifold, \( \mathcal{L} \) is a flat complex line bundle with its holonomy group \( G(L, \nabla) \) in \( \{ \exp(2\pi k\sqrt{-1}/N) \mid k \in \mathbb{Z} \} \).
2. The set of morphisms between two such objects is \( CF(L_1, L_1, b_1, sp_1, S_{L_1}), (L_0, L_0, b_0, sp_0, S_{L_0}); \mathbb{C}[[T^{1/N}]][T^{-1}][e, e^{-1}] \).
There are also the anchored versions of Theorems 1.2, 1.3. See Subsection 10.3.

The category could be empty for some \( N \). For example, one necessary condition for \( \mathcal{F}uk_N(M,\omega) \) to be non-empty is that \( N \) should be divided by \( m_{amb} \). This leads us to the notions of \( N \)-rational Lagrangian submanifolds: \( L \) is called \( N \)-rational if \((\mathcal{P}^{\otimes N/m_{amb}},\nabla^{\otimes N/m_{amb}})|_L\) is trivial. Then \( \mathcal{F}uk_N(M,\omega) \) is generated by \( N \)-rational Lagrangian submanifolds for each fixed \( N \). The following question seems to be interesting to study

**Question 1.4.** Is \( \mathcal{F}uk_N(M,\omega) \) generated by a finite number of objects? More specifically, is the number of the Hamiltonian isotopy class of compact BS \( N \)-rational Lagrangian submanifolds finite?

It is shown in Section 10 that the system \((\mathcal{F}uk_N(M,\omega); <)\) with respect to the partial order ‘\( N < N' \)' if and only if \( N|N' \) forms an inductive system. By definition, \( \mathcal{F}uk_{rat}(M,\omega) \) will be the corresponding inductive limit.

2. Novikov rings

The following ring was introduced in [FOOO00] which plays an important role in the rigorous formulation of Lagrangian Floer theory.

**Definition 2.1 (Universal Novikov ring).** Let \( R \) be a commutative ring with unit. (In many cases, we take \( R = \mathbb{Q} \).) We define

\[
\Lambda_{nov} = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{\mu_i/2} \mid a_i \in R, \lambda_i \in \mathbb{R}, \lambda_i \leq \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = \infty \right\}
\]

\[
\Lambda_{0,nov} = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{\mu_i/2} \in \Lambda_{nov} \mid \lambda_i \geq 0 \right\}.
\]

There is a natural filtration on these rings provided by the multiplicative non-Archimedean valuation defined by

\[
v \left( \sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{\mu_i/2} \right) := \inf \{ \lambda_i \mid a_i \neq 0 \}.
\]

Here we assume \((\lambda_i, \mu_i) \neq (\lambda_j, \mu_j)\) for \( i \neq j \).

This is well-defined by the definition of the Novikov ring and induces a filtration \( E^v \Lambda_{nov} := v^{-1}([\lambda, \infty)) \) on \( \Lambda_{nov} \). The function \( e^{-v} : \Lambda_{nov} \to \mathbb{R}_+ \) also provides a natural non-Archimedean norm on \( \Lambda_{nov} \).

Let \( \mathcal{C} \) be a free \( R \) module. We consider \( \mathcal{C} \otimes_R \Lambda_{nov} \) or \( \mathcal{C} \otimes_R \Lambda_{0,nov} \). We define valuation \( v \) on it by \( v(\sum x_i e_i) = \inf v(x_i) \), where \( e_i \) is the basis of \( \mathcal{C} \). It defines a metric. We take the completion and denote it by \( \hat{\mathcal{C}} \otimes_R \Lambda_{nov} \) or \( \hat{\mathcal{C}} \otimes_R \Lambda_{0,nov} \), respectively.

In the point of view of Novikov’s Morse theory of closed one forms, it is natural to use the version of Novikov ring that is a completion of the group ring of an appropriate quotient group of the fundamental group of \( \Omega(L_0, L_1) \). This is the point of view taken in many classical references of various Floer theories. The universal Novikov ring introduced above is slightly different from this Novikov ring. In this paper we also use this more traditional Novikov ring, whose definition is now in order.
We consider the space of paths \([1.1]\), on which we are given the action one-form \(\alpha \ [1.2]\). By definition
\[
\text{Zero}(\alpha) = \{ \tilde{\rho} : [0,1] \to M \mid p \in L_0 \cap L_1, \ \tilde{\rho} \equiv p \}.
\]
Note that \(\Omega(L_0, L_1)\) is not connected but has countably many connected components. We pick up a based path \(\ell_{01} \in \Omega(L_0, L_1)\) and consider the corresponding component \(\Omega(L_0, L_1; \ell_{01})\). We now review the definition of Novikov covering we used in Section \([2]\). Let \(g : \tilde{\Omega}(L_0, L_1; \ell_{01}) \to \tilde{\Omega}(L_0, L_1; \ell_{01})\) be an element of deck transformation group of the universal cover \(\tilde{\Omega}(L_0, L_1; \ell_{01})\) of \(\Omega(L_0, L_1; \ell_{01})\). It induces a map \(w : [0,1]^2 \to M\) with \(w(0,t) = \ell_{01}(t) = w(1,t), w(s,0) \in L_0, w(s,1) \in L_1\). (Namely \(s \mapsto w(s,\cdot)\) represent the path corresponding to \(g\)). We put
\[
E(g) = \int_{[0,1]^2} w^* \omega.
\]
We also obtain a Lagrangian loop \(\alpha_{01;\lambda_{01}}\) defined on \(\partial[0,1]^2\) by
\[
\begin{align*}
\alpha_{01;\lambda_{01}}(0,t) &= \alpha_{01;\lambda_{01}}(1,t) = \lambda_{01}(t), \\
\alpha_{01;\lambda_{01}}(s,0) &= T_{w(s,1)}L_0, \quad \alpha_{01;\lambda_{01}}(s,1) = T_{w(s,1)}L_1.
\end{align*}
\]
Here \(\lambda_{01}\) is any path of Lagrangian subspaces along \(\ell_{01}\) with
\[
\lambda_{01}(0) = T_{\ell_{01}(0)}L_0, \quad \lambda_{01}(1) = T_{\ell_{01}(1)}L_1.
\]
We denote by \(\mu(g)\) be the Maslov index of this Lagrangian loop. See Section \([5.1]\) for the definition of Maslov index of this Lagrangian loop. We remark that this index does not depend on the choice of \(\lambda_{01}\) and can be expressed as the index of a bundle pair over the annulus independently of this choice. (See [FOOO00].)

**Definition 2.2.** The Novikov covering is the covering space of \(\Omega(L_0, L_1; \ell_{01})\) which corresponds to the kernel of the homomorphism
\[
(E, \mu) : \pi_1(\Omega(L_0, L_1; \ell_{01})) \to \mathbb{R} \times \mathbb{Z}.
\]

Since \(\Pi(L_0, L_1; \ell_{01})\) is the deck transformation group of Novikov covering it follows that there exists an (injective) group homomorphism
\[
(E, \mu) : \Pi(L_0, L_1; \ell_{01}) \to \mathbb{R} \times \mathbb{Z}.
\]
Let \(\Lambda_{\text{nov}}\) be the field of fraction of \(\Lambda_{0,\text{nov}}\). \((E, \mu)\) induces a ring homomorphism
\[
\Lambda(L_0, L_1; \ell_{01}) \to \Lambda_{\text{nov}}
\]
by
\[
\sum g c_g[g] \mapsto \sum g e^{\mu(g)/2} T^{E(g)}.
\]
On \(\tilde{\Omega}(L_0, L_1; \ell_{01})\) we have a unique single valued action functional \(\mathcal{A}\) such that
\[
d\mathcal{A} = \pi^* \alpha, \quad \mathcal{A}(\ell_{01}) = 0
\]
where \(\ell_{01}\) is a base point of \(\tilde{\Omega}(L_0, L_1; \ell_{01})\).

We then denote by \(\Pi(L_0, L_1; \ell_{01})\) the group of deck transformations. We define the associated Novikov ring \(\Lambda(L_0, L_1; \ell_{01})\) as a completion of the group ring \(\mathbb{Q}[\Pi(L_0, L_1; \ell_{01})]\).
such that \( a_g \in \mathbb{Q} \) and that for each \( C \in \mathbb{R} \), the set

\[
\# \{ g \in \Pi(L_0, L_1; \ell_{01}) \mid E(g) \leq C, \ a_g \neq 0 \} < \infty.
\]

We put \( \Lambda(L_0, L_1; \ell_{01}) = \bigoplus_k \Lambda_k(L_0, L_1; \ell_{01}) \).

We call this graded ring the Novikov ring of the pair \((L_0, L_1)\) relative to the path \( \ell_{01} \). Note that this ring depends on the connected component of \( \ell_{01} \).

### 3. Anchors and abstract index

In this paper we always assume that \( L_0 \) intersects \( L_1 \) transversely.

Let \( p, q \in L_0 \cap L_1 \). We denote by \( \pi_2(p, q) = \pi_2(p, q; L_0, L_1) \) the set of homotopy classes of smooth maps \( u : [0, 1] \times [0, 1] \to M \) relative to the boundary

\[
u(0, t) \equiv p, \ u(1, t) = q; \ u(s, 0) \in L_0, \ u(s, 1) \in L_1
\]

and by \([u] \in \pi_2(p, q)\) the homotopy class of \( u \) and by \( B \) a general element in \( \pi_2(p, q) \). For given \( B \in \pi_2(p, q) \), we denote by \( \text{Map}(p, q; B) \) the set of such \( w \)'s in class \( B \). Each element \( B \in \pi_2(p, q) \) induces a map given by the obvious gluing map \([p, w] \to [q, w\#u]\) for \( u \in \text{Map}(p, q; B) \). There is also the natural gluing map

\[
\pi_2(p, q) \times \pi_2(q, r) \to \pi_2(p, r)
\]

induced by the concatenation \((u_1, u_2) \to u_1 \# u_2\). These 'relative' homotopy classes are canonically defined.

On the other hand, if we have chosen a base path \( \ell_{01} \in \Omega(L_0, L_1) \), then we can define the set of path homotopy classes of the maps \( w : [0, 1]^2 \to M \) satisfying the boundary condition

\[
w(0, t) = \ell_{01}(t), \ w(1, t) \equiv p, \ w(s, 0) \in L_0, \ w(s, 1) \in L_1.
\]

We denote the corresponding set of homotopy classes of the maps by \( \pi_2(\ell_{01}; p) \). Then we have the obvious gluing map

\[
\pi_2(\ell_{01}; p) \times \pi_2(p, q) \to \pi_2(\ell_{01}; q); (\alpha, B) \mapsto \alpha \# B.
\]

Now we would like to generalize this construction for a chain \( \mathcal{E} = (L_0, \cdots, L_k) \) of more than two Lagrangian submanifolds, i.e., with \( k \geq 2 \). (We call such \( \mathcal{E} \) the Lagrangian chain and \( k + 1 \) the length of \( \mathcal{E} \).)

To realize this purpose, we use the notion of anchors of Lagrangian submanifolds in this paper.

**Definition 3.1.** Fix a base point \( y \) of ambient symplectic manifold \((M, \omega)\).

Let \( L \) be a Lagrangian submanifold of \((M, \omega)\). We define an anchor of \( L \) to \( y \) is a path \( \gamma : [0, 1] \to M \) such that \( \gamma(0) = y, \ \gamma(1) \in L \). We call a pair \((L, \gamma)\) an anchored Lagrangian submanifold.

A chain \( \mathcal{E} = ((L_0, \gamma_0), \cdots, (L_k, \gamma_k)) \) is called an anchored Lagrangian chain. \( \mathcal{E} = (L_0, \cdots, L_k) \) is called its underlying Lagrangian chain.
It is easy to see that any homotopy class of path in $\Omega(L, L')$ can be realized by a path that passes through the given point $y$. Motivated by this observation, when we are given a Lagrangian chain $(L_0, L_1, \cdots, L_k)$ we also consider a chain of anchors $\gamma_i : [0, 1] \to M$ of $L_i$ to $y$ for $i = 0, \cdots, k$. These anchors give a systematic choice of a base path $\ell_{ij} \in \Omega(L_i, L_j)$ by concatenating $\gamma_i$ and $\gamma_j$:

\[
\ell_{ij}(t) = \begin{cases} 
\gamma_i(1 - 2t) & t \leq 1/2 \\
\gamma_j(2t - 1) & t \geq 1/2.
\end{cases}
\]

The upshot of this construction is the following overlapping property

\[
\ell_{ij}(t) = \ell_{i-1,j-1}(t) \quad \text{for } 0 \leq t \leq \frac{1}{2}, \quad \ell_{ij}(t) = \ell_{ij}(t) \quad \text{for } \frac{1}{2} \leq t \leq 1
\]

for all $j, \ell$.

Let $(L_0, \cdots, L_k)$ be a Lagrangian chain and $p_{(i+1)i} \in L_i \cap L_{i+1}$. (Let $p_{(k+1)k} = p_{0k}$ and $L_{k+1} = L_0$ as convention.) We write $\vec{p} = (p_{01}, \cdots, p_{(k+1)k})$. Let $\chi_i = \exp(-2\pi i \sqrt{-1}/k)$. We consider the set of homotopy class of maps $v : D^2 \to M$ such that $v(\chi_i \chi_j) \subset L_i$ and $v(\chi_i) = p_{(i+1)i}$. We denote it by $\pi_2(\mathcal{E}, \vec{p})$. If $\mathcal{E}$ is an anchored Lagrangian chain and $\mathcal{L}$ be its underlying Lagrangian chain we write $\pi_2(\mathcal{E}; \vec{p})$ some times by abuse of notation.

**Definition 3.2.** Let $\mathcal{E} = \{(L_i, \gamma_i)\}_{0 \leq i \leq k}$ be a chain of anchored Lagrangian submanifolds. A homotopy class $B \in \pi_2(\mathcal{E}; \vec{p})$ is called admissible to $\mathcal{E}$ if it can be obtained by a polygon that is a gluing of $k$ bounding strips $w^-(i+1) : [0, 1] \times [0, 1] \to M$ satisfying

\[
\begin{align*}
& w^{-}_{i(i+1)}(s, 0) \in L_i, \quad w^{-}_{i(i+1)}(s, 1) \in L_{i+1} \\
& w^{-}_{i(i+1)}(0, t) = p_{(i+1)i}, \\
& w^{-}_{i(i+1)}(1, t) = \begin{cases} 
\gamma_i(1 - 2t) & 0 \leq t \leq \frac{1}{2} \\
\gamma_{i+1}(2t - 1) & \frac{1}{2} \leq t \leq 1
\end{cases}
\end{align*}
\]

When this is the case, we denote the homotopy class $B$ as

\[
B = [w^{-}_{01}]# [w^{-}_{12}]# \cdots # [w^{-}_{k0}]
\]

and the set of admissible homotopy classes by $\pi_2^{ad}(\mathcal{E}; \vec{p})$.

We note that not all homotopy classes in $\pi_2(\mathcal{E}; \vec{p})$ is admissible for a given anchored Lagrangian chain. (See however Lemma 3.6)

**Definition 3.3.** Let $(L_i, \gamma_i)$, $i = 0, 1$ be anchored Lagrangian submanifolds. We say $p \in L_0 \cap L_1$ is admissible (with respect to the pair $((L_0, \gamma_0), (L_1, \gamma_1))$ if there exists $w = w_{01}$ satisfying $\mathbf{3.6a}$ for $i = 0$ and $\mathbf{3.6b}$ for $i = 0, p_{10} = p$.

Note $p$ is admissible if and only if $\pi_2(\ell_{01}; p)$ is nonempty. (Here $\ell_{01}$ is as in $\mathbf{3.3}$.) Let us go back to the case $k = 1$. First note that we have:

\[
\pi_2(p, q) = \pi_2(\mathcal{L}; \vec{p})
\]

where $\mathcal{L} = (L_0, L_1)$, $\vec{p} = (p, q)$ and the left hand side is as in the beginning of this section.

**Lemma 3.4.** Let $k = 1$ and $\mathcal{E} = ((L_0, \gamma_0), (L_1, \gamma_1))$. Then $\pi_2^{ad}(\mathcal{E}, (p, q)) = \pi_2(\mathcal{E}, (p, q))$ if $p, q$ are admissible. Otherwise $\pi_2^{ad}(\mathcal{E}, (p, q))$ is empty.
The proof is easy and omitted.

**Lemma 3.5.** Let \( L_0, L_1 \) be a pair of Lagrangian submanifold and \( p \in L_0 \cap L_1 \). Then for each given anchor \( \gamma_0 \) of \( L_0 \) there exists an anchor \( \gamma_1 \) of \( L_1 \) such that \( p \) is admissible with respect to the pair \((L_0, \gamma_0), (L_1, \gamma_1)\).

The proof is easy and omitted.

The proof of the following two lemmas are also easy and so is omitted.

**Lemma 3.6.** Let \( L \) be a Lagrangian chain and \( B \in \pi_2(L; \rho) \). Then there exist anchors \( \gamma_i \) of \( L_i \) (\( i = 0, \cdots, k \)) such that \( B \) is admissible with respect to \( E \), where

\[ E = ((L_0, \gamma_0), \cdots, (L_k, \gamma_k)). \]

The anchors in Lemmas 3.5, 3.6 are not necessarily unique (even up to homotopy). It is rather complicated to describe how many there are. (See Section 9 for some illustration.) The following definition can be used to study the gluing formulas of symplectic areas and Maslov indices of pseudo-holomorphic polygons that enter in the construction of the anchored version of Fukaya category.

**Definition 3.7.** Let \( R \) be a module. We say a collection of maps

\[ I = \{ I_k : \pi_2^{ad}(E; \rho) \to R \}_{k=1}^\infty \]

an abstract index over the collection of anchored Lagrangian chains \( E \), if they satisfy the following gluing rule: whenever the gluing is defined, we have

\[ I_{k+1}([w_0^-] \# \cdots \# [w_{k-1}^-] \# [w_k^0]) = \sum_{i=0}^{k} I_1([w_i^-_{i(i+1)})]. \]

In subsection 10.3 we will use another abstract index, a normalized symplectic area over the class of BS-rational Lagrangian submanifolds with \( R = \mathbb{Q} \) or with \( R = \frac{1}{N} \cdot \mathbb{Z} \) for integers \( N \).

### 4. Anchors, action functional and action spectrum

For given two anchors \( \gamma, \gamma' \) homotopic to each other, we denote by \( \pi_2(\gamma, \gamma'; L) \) the set of homotopy classes of the maps \( w : [0, 1]^2 \to M \) satisfying

\[ w(0, t) = \gamma(t), w(1, t) = \gamma'(t), w(s, 0) \equiv y, \text{ and } w(s, 1) \in L. \]

For any such map \( w \), we define

\[ a_{(\gamma, \gamma'; L)}(w) = \int w^* \omega. \]

(4.1)

It is immediate to check that this function pushes down to \( \pi_2(\gamma, \gamma': L) \) which we again denote by \( a_{(\gamma, \gamma'; L)} \).

We denote by \( G(\gamma, \gamma' : L) \subset \mathbb{R} \) the image of \( a_{(\gamma, \gamma'; L)} \). The following is easy to check whose proof we omit. Let \( \gamma + \gamma' \) be an element \( \Omega(L, L; M) \) obtained by concatenating \( \gamma \) (where \( \gamma(t) = \gamma(1-t) \)) and \( \gamma' \) in the same way as \( \Omega_{\gamma + \gamma'}(L, L; M) \) the connected component of \( \Omega(L, L; M) \) containing it.

**Lemma 4.1.** \( \pi_2(\gamma, \gamma' : L) \) is a principal homogeneous space of \( \pi_1(\Omega_{\gamma + \gamma'}(L, L; M)) \). and so \( G(\gamma, \gamma' : L) \) is a principal homogeneous space of the group

\[ \{ \omega(C) \mid C \in \pi_1(\Omega_{\gamma + \gamma'}(L, L; M)) \}. \]
The action functional $A = A_{(\gamma_0, \gamma_1; L)} : \bar{\Omega}(L_0, L_1; \ell_{01}) \to \mathbb{R}$ is defined by

$$A([\ell, w]) = \int w^* w.$$  

Note an element of $\bar{\Omega}(L_0, L_1; \ell_{01})$ is identified with a pair $[\ell, w]$ where $\ell \in \Omega(L_0, L_1; \ell_{01})$ and $w : [0, 1]^2 \to M$ satisfies

\begin{align}
(4.2) \quad w(0, t) = \ell_{01}(t), \quad w(1, t) = \ell(t), \quad w(s, 0) \in L_0, \quad w(s, 1) \in L_1.
\end{align}

We identify $[\ell, w]$ with $[\ell, w']$ if

\begin{align}
(4.3) \quad \int (\mp \# w)^* \omega = 0, \quad \mu(\mp \# w) = 0.
\end{align}

Here $\mp(s, t) = w'(1 - s, t)$ and $\mu$ is an appropriate Maslov index. (See \cite{2,5} and Definition \ref{def:maslov}.)

We now study dependence of the action functional $A_{(L_0, \gamma_0), (L_1, \gamma_1)}$ on their anchors. Let $\gamma_0, \gamma_0'$ and $\gamma_1, \gamma_1'$ be two anchors of $L_0$ and $L_1$ respectively. It defines $\ell_{01}$ and $\ell'_{01}$ by (4.3). We assume that there exist paths $w_0, w_1$ connecting them respectively. Then $\mp \# w_1$ induces a diffeomorphism

$$\Phi_{\mp \# w_1} : \bar{\Omega}(L_0, L_1; \ell_{01}) \to \bar{\Omega}(L_0, L_1; \ell'_{01})$$

defined by

\begin{align}
(4.4) \quad \Phi_{\mp \# w_1}([\ell, u]) = [\ell, (\mp \# w_1) \# u].
\end{align}

For the clarity of notations, we will use $\#$ for two dimensional concatenations and by $\ast$ for one dimensional ones.

**Proposition 4.2.** Let $\gamma_i, \gamma_i'$ and $w_i$ for $i = 0, 1$ be as above. Consider $[\ell, u] \in \bar{\Omega}(L_0, L_1; \ell_{01})$. Then we have

$$A_{(L_0, \gamma_0), (L_1, \gamma_1)} - \Phi_{\mp \# w_1} A_{(L_0, \gamma_0'), (L_1, \gamma_1')} \equiv \omega(\mp \# w_1).$$

**Proof.** Obvious from the definition. \hfill $\square$

Now we define:

**Definition 4.3 (Action spectrum).** Denote by $\text{Spec}((L_0, \gamma_0), (L_1, \gamma_1))$ the set of critical values of $A_{(L_0, \gamma_0), (L_1, \gamma_1)}$ and call the action spectrum of the pair $(L_0, \gamma_0), (L_1, \gamma_1)$.

An immediate corollary of Proposition 4.2 and this definition is the following

**Corollary 4.4.** We assume that $\gamma_i$ is homotopic to $\gamma_i'$ for $i = 0, 1$. Then there exists a real constant $c = c((L_0, \gamma_0), (L_1, \gamma_1); (L_0, \gamma_0'), (L_1, \gamma_1'))$ depending on the pair $(L_0, \gamma_0), (L_1, \gamma_1)$ such that

$$\text{Spec}((L_0, \gamma_0), (L_1, \gamma_1)) = \text{Spec}((L_0, \gamma_0'), (L_1, \gamma_1')) + c$$

as a subset of $\mathbb{R}$.

**Proof.** Let $\gamma_i, \gamma_i'$ and $w_i$ for $i = 0, 1$ be as above. By Proposition 4.2 we have

$$\text{Crit} A_{(L_0, \gamma_0), (L_1, \gamma_1')} + \omega(\mp \# w_1) = \text{Crit} A_{(L_0, \gamma_0'), (L_1, \gamma_1)}$$

for any choice of $w_0, w_1$ joining $\gamma_0, \gamma_0'$ and $\gamma_1, \gamma_1'$ respectively.

Just take $c = \omega(\mp \# w_1)$. This finishes the proof. \hfill $\square$
Next we consider the Lagrangian chains with 3 or more elements in them. When we are given an anchored Lagrangian chain
\[ E = ((L_0, \gamma_0), (L_1, \gamma_1), \cdots, (L_k, \gamma_k)) \]
these anchors give a systematic choice of a base path \( \ell_{ij} \in \Omega(L_i, L_j) \) by concatenating \( \gamma_i \) and \( \gamma_j \) as in (3.4). Inside the collection of anchored Lagrangian submanifolds \((L, \gamma)\) we are given a coherent system of single valued action functionals \( A : \tilde{\Omega}_0(L_i, L_j; \ell_{ij}) \to \mathbb{R} \).

We will use the action functional associated to \( \ell_{ij} \) to define an energy level on the critical point set \( A : \text{Crit} A \to \mathbb{R} \). By the overlapping property (3.5), the following proposition is immediate whose proof we omit.

**Proposition 4.5.** Denote by \( E \) an anchored Lagrangian chain. Consider the map \( I_{\omega,k} : \pi_2(\vec{p}; E) \to \mathbb{R} \) defined by the symplectic area \( I_{\omega,k}(\alpha) = \omega(\alpha) \) for \( k = 1, \cdots \). Then the collection denoted by \( I_{\omega} = \{I_{\omega,k}\}_{k=1}^\infty \) defines an abstract index of anchored Lagrangian chains.

### 5. Grading and filtration

A familiar description of generators of Floer chain module as the set of equivalence classes \([p, w]\) in the Novikov covering space is useful as far as the study of filtration on the Floer complex is concerned. However for the study of grading and signs on the Floer complex, we have to have additional structures on the Floer chain module which requires some geometric condition on the Lagrangian side, e.g., spin structure or graded structure. There has been a few different approach to how one incorporates these additional structures. In this section, we describe them by using anchors.

#### 5.1. Maslov index in Lagrangian Grassmannian

In this subsection, we review the definition of Maslov index in Lagrangian Grassmannian. The Lagrangian Grassmannian \( \text{Lag}(S, \omega) \) of a symplectic vector space \((S, \omega)\) is defined to be
\[ \text{Lag}(S, \omega) = \{ V \mid V \text{ is a Lagrangian subspace of } (S, \omega) \} . \]

When we equip \( S \) a compatible complex structure \( J \) and define \( U(S) \) to be the group of unitary transformations of \( S \), any \( V_0, V_1 \subset \text{Lag}(S, \omega) \) can be written as \( V_1 = A \cdot V_0 \) for some \( A \in U(S) \). In [A], this fact is used to show that \( H^1(\text{Lag}(S, \omega), \mathbb{Z}) \cong \mathbb{Z} \). It generator \( \mu \in H^1(\text{Lag}(S, \omega), \mathbb{Z}) \) is the Maslov class \( [A] \) and two loops \( \gamma_1, \gamma_2 \) are homotopic if and only if \( \mu(\gamma_1) = \mu(\gamma_2) \).

We give an elementary description of the Maslov class below. We fix \( V_0 \in \text{Lag}(S, \omega) \) and put
\[ \text{Lag}_1(S, \omega) = \{ V \in \text{Lag}(S, \omega) \mid \dim(V \cap V_0) \geq 1 \} . \]

It is proven in [A] that \( \text{Lag}_1(S, \omega) \) is co-oriented and so defines a cycle whose Poincaré dual is precisely the Maslov class \( \mu \in H^1(\text{Lag}_1(S, \omega), \mathbb{Z}) \).

The tangent space \( T_{V_0} \text{Lag}(S, \omega) \) is canonically isomorphic to the set of quadratic forms on \( V_0 \).

**Definition 5.1.** We say any tangent vector pointing the chamber of nondegenerate positive-definite quadratic forms is positively directed.

The following is also proved in [A].
Lemma 5.2. There exists a neighborhood $U$ of $V_0 \in \text{Lag}(S, \omega)$, the set
\[ U \setminus \text{Lag}_1(S, \omega; V_0) \]
has exactly $n + 1$ connected components each of which contains $V_0$ in its closure.

We refer readers to [A] or see Proposition 3.3 of [FOOO06] for the proof of the following proposition.

Proposition 5.3. Let $(S, \omega)$ be a symplectic vector space and $V_0 \in (S, \omega)$ be a
given Lagrangian subspace. Let $V_1 \in \text{Lag}(S, \omega) \setminus \text{Lag}_1(S, \omega; V_0)$ i.e., be a Lagrangian
subspace with $V_0 \cap V_1 = \{0\}$. Consider smooth paths $\alpha : [0, 1] \to \text{Lag}(S, \omega)$ satisfying
\begin{enumerate}
\item $\alpha(0) = V_0$, $\alpha(1) = V_1$.
\item $\alpha(t) \in \text{Lag}(S, \omega) \setminus \text{Lag}_1(S, \omega; V_0)$ for all $0 < t \leq 1$.
\item $\alpha'(0)$ is positively directed.
\end{enumerate}
Then any two such paths $\alpha_1$, $\alpha_2$ are homotopic to each other via a homotopy $s \in [0, 1] \to \alpha_s$ such that each $\alpha_s$ also satisfies the 3 conditions above.

Let $\text{Lag}^+(S, \omega)$ be the double cover of $\text{Lag}(S, \omega)$. Its element is regarded as an
element $V$ of $\text{Lag}(S, \omega)$ equipped with an orientation of $V$.

5.2. Anchors and grading. To use the anchor in the definition of a grading
in the Floer complex, we need to equip each anchor with an additional decoration.

Let $y \in M$ be the base point. We fix an oriented Lagrangian subspace $V_y \in \text{Lag}^+(T_y M)$.

Definition 5.4. Consider an anchored Lagrangian $(L, \gamma)$. We denote by $\lambda$ a
section of $\gamma^* \text{Lag}^+(M, \omega)$ such that
\[ \lambda(0) = V_y, \quad \lambda(1) = T_{\gamma(1)} L. \]
We call such a pair $(\gamma, \lambda)$ a graded anchor of $L$ (relative to $(y, V_y)$) and a triple
$(L, \gamma, \lambda)$ a graded anchored Lagrangian submanifold.

Remark 5.5. We remark that a notion similar to the graded anchor also ap-
ppears in Welchinger’s recent work [W].

Let $(L_0, \gamma_0, \lambda_0)$ and $(L_1, \gamma_1, \lambda_1)$ be graded anchored Lagrangian submanifolds
relative to $(y, V_y)$. Assume that $L_0$ and $L_1$ intersect transversely. We define $\lambda_{01}(t) \in \text{Lag}^+(T_{\gamma_0(t)} M)$ by concatenating $\lambda_0$ and $\lambda_1$ as follows:
\[ \lambda_{01}(t) = \begin{cases} 
\lambda_0(1 - 2t) & t \leq 1/2 \\
\lambda_1(2t - 1) & t \geq 1/2.
\end{cases} \]
We consider a pair $[p, w]$ where $p \in L_0 \cap L_1$, and $w : [0, 1]^2 \to M$ as in (3.2). To put a grading at $[p, w]$, we recall the definition of Maslov-Morse index introduced in [FOOO06]. For given $w$, we associate a Lagrangian loop $\alpha_{[p, w]; \lambda_{01}}$ defined on $\partial[0, 1]^2$ by
\[ \alpha_{[p, w]; \lambda_{01}}(0, t) = \lambda_{01}(t), \quad \alpha_{[p, w]; \lambda_{01}}(s, 0) = T_{w(s, 0)} L_0, \]
\[ \alpha_{[p, w]; \lambda_{01}}(s, 1) = T_{w(s, 1)} L_1, \quad \alpha_{[p, w]; \lambda_{01}}(1, t) = \alpha^+_y(t) \]
where $\alpha^+_y : [0, 1] \to T_y M$ is a path connecting from $T_p L_0$ to $T_p L_1$ in $\text{Lag}(T_p M, \omega_p)$
whose homotopy class is the unique one as described in Proposition 5.3.

Let $p \in L_0 \cap L_1$ and $w : [0, 1]^2 \to M$ satisfy (3.2). Choose a symplectic
trivialization $\Phi = (\pi, \phi) : w^* TM \to [0, 1]^2 \times T_p M \cong [0, 1]^2 \times \mathbb{R}^{2n}$ where $\pi$ :
$w^*TM \to [0, 1]^2$ and $\phi : w^*TM \to T_pM$ are the corresponding projections to $[0, 1]^2$ and $T_pM$ respectively. $\Phi$ is homotopically unique. Now we denote by $\alpha^{\Phi}_{[p, w]; \lambda_{01}}$ the Lagrangian loop

$$\alpha^{\Phi}_{[p, w]; \lambda_{01}} = \phi((\alpha_{[p, w]; \lambda_{01}} \circ c)).$$

Here we fix a piecewise smooth parametrization $c : S^1 \cong \mathbb{R}/\mathbb{Z} \to \partial[0, 1]^2$ of $\partial[0, 1]^2$ with positive orientation with $c(0) = (1, 0)$.

**Definition 5.6.** We define the Maslov-Morse index, denoted by $\mu([p, w]; \lambda_{01})$, to be the Maslov index of this Lagrangian loop $\alpha^{\Phi}_{[p, w]; \lambda_{01}}$ in $(T_pM, \omega)$.

This definition does not depend on the trivialization $\Phi$ or on the (positive) parametrization $c$ of $\partial[0, 1]^2$ and so well-defined.

**Remark 5.7.** Here and hereafter we uses the symbol $\lambda$ for a path in $\text{Lag}^+$, the oriented Lagrangian Grassmannian and $\alpha$ for a path in $\text{Lag}$, the un-oriented Lagrangian Grassmannian.

**Lemma 5.8.** Let $p, w, \lambda_{01}$ be as in Definition 5.6. We put

$$w^-(s, t) = w(s, 1 - t), \quad \lambda_{10}(t) = \lambda_{01}(1 - t).$$

Then

$$\mu([p, w]; \lambda_{01}) + \mu([p, w^-]; \lambda_{10}) = n.$$

**Proof.** Let $\Phi$ and $\alpha^{\Phi}_{[0, 1]; \lambda_{01}}$ be as above. If we denote $\iota : [0, 1]^2 \to [0, 1]^2$ to be the map $\iota(s, t) = (s, 1 - t)$, we have $w^- = w \circ \iota$. Therefore we can trivialize $(w^-)^*TM = \iota^*w^*TM$ by the map $\Phi^- : (w^-)^*TM \to [0, 1]^2 \times \mathbb{R}^{2n}$ defined by $\Phi^- = \Phi \circ \iota^*$. Then

$$\alpha^\sim_{[p, w]; \lambda_{01}} = \phi((\alpha_{[p, w]; \lambda_{01}} \circ \overline{c})$$

where $(\overline{c})$ denotes the inverse path, e.g., $\overline{c}(\theta) = c(-\theta)$. By definition of $\alpha^{\Phi^-}_{[p, w^-]; \lambda_{10}}$, the path $\phi((\alpha_{[p, w]; \lambda_{01}} \circ c))$ coincides with $\alpha^{\Phi^-}_{[p, w^-]; \lambda_{10}}$ (up to parametrization) except on the segment $c^{-1}(\{1\} \times [0, 1])$.

Therefore the composition $\alpha^{\Phi}_{[p, w]; \lambda_{01}} * \alpha^{\Phi^-}_{[p, w^-]; \lambda_{10}}$ of $\alpha^{\Phi^-}_{[p, w^-]; \lambda_{10}}$ and $\alpha^{\Phi}_{[p, w]; \lambda_{01}}$ is homotopic to a path $\alpha = \alpha^- \cup \alpha^+ : S^1 \cong I^- \cup I^+ \to \text{Lag}(\mathbb{R}^{2n}, \omega_0)$ with $I^\pm = \{1\} \times [0, 1]$ such that both $\alpha^\pm : I^\pm \to \text{Lag}(\mathbb{R}^{2n}, \omega_0)$ are the paths positively directed at $t = 0$ provided in Proposition 5.3 and satisfy

$$\alpha^+(0) = \alpha^-(1) = \phi(T_pL_0), \quad \alpha^+(1) = \alpha^-(0) = \phi(T_pL_1).$$

It is easy to see that the Maslov index of such $\alpha$ is $n$ and hence we obtain

$$n = \mu(\alpha^{\Phi}_{[p, w]; \lambda_{01}} * \alpha^{\Phi^-}_{[p, w^-]; \lambda_{10}}) = \mu(\alpha^{\Phi}_{[p, w]; \lambda_{01}}) + \mu(\alpha^{\Phi^-}_{[p, w^-]; \lambda_{10}}).$$

By definition, the last sum is nothing but $\mu([p, w]; \lambda_{01}) + \mu([p, w^-]; \lambda_{10})$. This finishes the proof of (5.3). \hfill $\square$

**5.3. Polygonal Maslov index.** Consider a chain of Lagrangian submanifolds $\mathcal{L} = \{L_0, \cdots, L_k\}$ and a chain of intersection points $\{(p_{00}, p_{k(k-1)}, \cdots, p_{10})\}$ with $p_{i(i-1)} \in L_{i-1} \cap L_i$ for $i = 0, \cdots, k$. We consider the disc with marked points $\{z_{00}, z_{k(k-1)}, \cdots, z_{10}\}$ and denote $D^2 = D^2 \setminus \{z_{00}, z_{k(k-1)}, \cdots, z_{10}\}$. We assume $z_{00}, z_{k(k-1)}, \cdots, z_{10}$ respects counter clock-wise cyclic order of $\partial D^2$. 
Remark 5.9. Here and hereafter the suffix $j$ is regarded as modulo $k+1$. Namely $p_{(j+1)j}$ in case $j=k$ means $p_0$, for example. We also put $p_{jj}=p_{jj}$.

For the following discussion, we will consider the cases $k \geq 1$, i.e., the cases with length $\mathcal{L} \geq 2$.

For each given such chains, we define the set of maps

$$C^\infty(D^2, \mathcal{L}; \vec{p})$$

to be the set of all $w : D^2 \to M$ such that

$$(5.5) \quad w(z_{(j+1)j}) \subset L_j, \quad w(z_{j(j-1)}) = p_{jj} \in L_j \cap L_{j-1},$$

and that it is continuous on $D^2$ and smooth on $\partial D^2$. We will define a topological index, which is associated to each homotopy class $B \in \pi_2(\mathcal{L}, \vec{p})$. We denote it by $\mu(\mathcal{L}, \vec{p}; B)$. Let $w \in C^\infty(D^2, \mathcal{L}; \vec{p})$ be a map such that $[w] = B$. We denote by

$$\mathcal{F}(\mathcal{L}, \vec{p}; B) \subset C^\infty(D^2, \mathcal{L}; \vec{p})$$

the set of such maps.

We identify $[0, 2\pi]/(0 \sim 2\pi) \cong S^1$ by $t \mapsto e^{\sqrt{-1}t}$. (The direction $t$ increase then becomes counter-clockwise order of $S^1$.)

Under a symplectic trivialization of the bundle $w^*TM$, the map

$$\alpha_w : S^1 = \partial D^2 \to \text{Lag}(\mathbb{R}^{2n}, \omega_0); \quad t \mapsto T_w(t)L_i \quad \text{if} \quad t \in \pi_{i+1}^{i+1}$$

defines a piecewise smooth path with discontinuities at $(k+1)$ points $z_{i(i-1)} \in \partial D^2$ for $i = 0, 1, \cdots, k$, at which we have

$$(5.6) \quad \lim_{t \to z_{i(i-1)}^-} \alpha_w(t) = T_{p_{ii-1}} L_i, \quad \lim_{t \to z_{i(i-1)}^+} \alpha_w(t) = T_{p_{ii-1}} L_{i-1}.$$

By the transversality hypothesis, $T_{p_{ii-1}} L_i$ and $T_{p_{ii-1}} L_{i-1}$ are Lagrangian subspaces in $(\mathbb{R}^{2n}, \omega_0)$ with $T_{p_{ii-1}} L_i \cap T_{p_{ii-1}} L_{i-1} = \{0\}$. We fix a smooth path $\alpha_{i(i-1)}^- : [0, 1] \to \text{Lag}(\mathbb{R}^{2n}, \omega_0)$ for each $i = 0, 1, \cdots, k$ so that

$$(5.7) \quad \alpha_{i(i-1)}^- (0) = T_{p_{ii-1}} L_i, \quad \alpha_{i(i-1)}^- (1) = T_{p_{ii-1}} L_{i-1}$$

and $-(\alpha_{i(i-1)}^-)'(1)$ is positively directed in the sense of Definition 5.1.

In other words $\alpha_{i(i-1)}^- (t) = \alpha_{i(i-1)}^+ (1-t)$, where the right hand side is as in $(5.2)$. By Proposition 5.3 such a choice is unique up to homotopy relative to the end points $t = 0, 1$. Inserting $\alpha_{i(i-1)}^-$ into the map $\alpha_w$ at each $z_{i(i-1)}$, we obtain a continuous loop $\widetilde{\alpha}_w$ in $\text{Lag}(\mathbb{R}^{2n}, \omega_0)$.

Definition 5.10. Let $\mathcal{L} = (L_0, \cdots, L_k)$ be a Lagrangian chain. We define the topological index, denoted by $\mu(\mathcal{L}, \vec{p}; B)$, to be the Maslov index of the loop $\widetilde{\alpha}_w$, i.e.,

$$\mu(\mathcal{L}, \vec{p}; B) = \mu(\widetilde{\alpha}_w).$$

This definition is essentially reduced to the Maslov-Viterbo index [V] for the pairs $(L_0, L_1)$ when $k = 1$ and reduces to the one given in Section A3 [KS] for the case where $L_i$ are all affine.

Remark 5.11. We remark that $L_0, L_1, \cdots, L_k$ are put on the boundary of the disc $D^2$ in a clockwise order. On the other hand, $z_{0k}, z_{k(k-1)}, \cdots, z_{21}, z_{10}$ in the counter clockwise order.

This is consistent with the case $k = 1$ discussed in [FOOO08]. See Remark 3.7.23 (1) [FOOO08].
Now consider a chain of graded anchored Lagrangian submanifolds $\mathcal{E} = (L_0, \ldots, L_k)$, $L_i = (L_i, \gamma_i, \lambda_i)$. It induces a grading $\lambda_{ij}$ along $\ell_i$ as in (5.1), we remark that $\lambda_{ij}$ also satisfy the overlapping property

\[(5.8)\] 
\[\lambda_{ij} | [0, \frac{1}{2}] = \lambda_{ij} | [\frac{1}{2}, 1] = \lambda_{ij} | [\frac{1}{2}, 1].\]

Let $p_{(i+1)i} = p_{(i+1)i} \in L_i \cap L_{i+1}$. We put

\[(5.9)\] 
\[w_{ij}^{+} - (s, t) = w_{ij}^{-} + (1 - s, t)\]

where the right hand side is as in Definition 3.2.

**Lemma 5.12.** Let $\mathcal{E}$ be a graded anchored Lagrangian chain. Suppose $B \in \pi_{2}^{ad}(\mathcal{E}, \tilde{p})$ given as Lemma 3.2. Then we have

\[(5.10)\] 
\[\mu(\mathcal{E}, \tilde{p}; B) + \sum_{i=0}^{k} \mu([p_{(i+1)i}; w_{ij}^{+}); \lambda_{ij-1}) = 0.\]

**Proof.** Since $\mu([p_{(i+1)i}; w_{ij}^{+}); \lambda_{ij-1})$ is defined as the Maslov index of the loop $\alpha_{[p_{(i+1)i}; w_{ij}^{+}); \lambda_{ij-1}]$ (Definition 5.6), the equality (5.10) follows from

\[\sum_{i=0}^{k} \alpha_{[p_{(i+1)i}; w_{ij}^{+}); \lambda_{ij-1}] + \tilde{\alpha}_{w} \sim 0,\]

where $\tilde{\alpha}_{w}$ is as in Definition 5.10 with $B = [w_{01}^{-}]# \cdots # [w_{(k-1)k}^{-}] # [w_{k0}^{-}]$ and $\sim$ means homologous.

When the length of $\mathcal{E}$ is $k+1$, we define $\mu_k(B) = \mu(\mathcal{E}, \tilde{v}; B)$ where $B \in \pi_{2}^{ad}(\mathcal{E}, \tilde{p})$.

**Corollary 5.13.** Define $\mu_1 : \pi_{2}(\ell_0, p) \to \mathbb{Z}$ by setting $\mu_1(\alpha) := -\mu([p, w]; \lambda_{01})$ for a representative $[p, w]$ of the class $\alpha \in \pi_{2}(\ell_0; p)$. Then the sequence of maps $\mu = \{\mu_k\}_{k=1}^{\infty}$ with $\mu_k : \pi_{2}^{ad}(\mathcal{E}; \tilde{p}) \to \mathbb{Z}, k \geq 1$ defines an abstract index.

## 6. Orientation

To be able to define various operators in Floer theory, we need to provide a compatible system of orientations on the Floer moduli spaces and other moduli spaces of pseudo-holomorphic polygons. We here explain the way to give orientations, which is basically the same as [FOOO06].

**Definition 6.1.** A submanifold $L \subset M$ is called relatively spin if it is orientable and there exists a class $st \in H^2(M, \mathbb{Z}_2)$ such that $st|L = w_2(TL)$ for the Stiefel-Whitney class $w_2(TL)$ of $TL$.

A chain $(L_0, L_1, \ldots, L_k)$ or a pair $(L_0, L_1)$ of Lagrangian submanifolds is said to be relatively spin if there exists a class $st \in H^2(M, \mathbb{Z}_2)$ satisfying $st|L_i = w_2(TL_i)$ for each $i = 0, 1, \ldots, k$.

We fix such a class $st \in H^2(M, \mathbb{Z}_2)$ and a triangulation of $M$. Denote by $M^{(k)}$ its $k$-skeleton. There exists a real vector bundle $V(st)$ on $M^{(3)}$ with $w_1(V(st)) = 0, w_2(V(st)) = st$. Now suppose that $L$ is relatively spin and $L^{(2)}$ be the 2-skeleton of $L$. Then $V \oplus TL$ is trivial on the 2-skeleton of $L$. We define

**Definition 6.2.** We define a $(M, st)$-relative spin structure of $L$ to be a choice of $V$ and a spin structure of the restriction of the vector bundle $V \oplus TL$ to $L^{(2)}$.

The relative spin structure of a chain of Lagrangian submanifolds $(L_0, \ldots, L_k)$ is defined in the same way by using the same $V$ for all $L_i$. 
Let \( p, q \in L_0 \cap L_1 \) and \( B \in \pi_2(p, q) \). We consider \( u : \mathbb{R} \times [0,1] \to M \) such that

\[
(6.1a) \quad \frac{du}{d\tau} + J \frac{du}{d\tau} = 0 \\
(6.1b) \quad u(\tau, 0) \in L_0, \quad u(\tau, 1) \in L_1, \quad \int u^* \omega < \infty \\
(6.1c) \quad u(-\infty, \cdot) \equiv p, \quad u(\infty, \cdot) \equiv q.
\]

It induces a continuous map \( \overline{u} : [0,1]^2 \to M \) with \( \overline{u}(0, t) \equiv p, \ u(1, t) \equiv q \) in an obvious way. With an abuse of notation, we denote by \( [u] \) the homotopy class of the map \( \overline{u} \) in \( \pi_2(p, q) \). We denote by \( \mathcal{M}^\circ(p, q; B) \) the moduli space consisting of the maps \( u \) satisfying (6.1) and compactify \( \mathcal{M}^\circ(p, q; B)/\mathcal{R} \) its quotient by the \( \tau \)-translations by using an appropriate notion of stable maps as in Section 3 \cite{FOOO00}. We denote the compactification by \( \mathcal{M}(p, q; B) \). We call this the Floer moduli space. It carries the structure of a space with Kuranishi structure.

If \( (L_0, L_1) \) is a relatively spin pair, then \( \mathcal{M}(p, q; B) \) is orientable. Furthermore a choice of relative spin structures gives rise to a compatible system of orientations for \( \mathcal{M}(p, q; B) \) for all pair \( p, q \in L_0 \cap L_1 \) and \( B \in \pi_2(p, q) \). For completeness’ sake, we now recall from \cite{FOOO00} how the relative spin structure gives rise to a system of coherent orientations.

Let \( p \in L_0 \cap L_1 \) and \( w \) satisfies (6.2) . We denote by \( \text{Map}(\ell_{01}; p; L_0, L_1; \alpha) \) the set of such maps \([0,1]^2 \to M\) its homotopy class \([w] = \alpha \) in \( \pi_2(\ell_{01}; p) \). Let \( w \in \text{Map}(\ell_{01}; p; L_0, L_1; \alpha) \). Let \( \Phi : w^*TM \to [0,1]^2 \times T_pM \) be a (homotopically unique) symplectic trivialization as before. The trivialization \( \Phi \), together with the boundary condition, \( w(0, t) = \ell_{01}(t) \) and the Lagrangian path \( \lambda_{01} \) along \( \ell_{01} \), defines a Lagrangian path

\[
\lambda^\Phi = \lambda^\Phi_{([p,w];\lambda_{01})} : [0,1] \to T_pM
\]

satisfying \( \lambda^\Phi(0) = T_pL_0, \ \lambda^\Phi(1) = T_pL_1 \). The homotopy class of this path does not depend on the trivialization \( \Phi \) but depends only on \([p,w]\) and the homotopy class of \( \lambda_{01} \). Hereafter we omit \( \Phi \) from notation.

We remark that relative spin structure determines a trivialization of \( V_{\lambda_{01}(0)} \oplus T_{\lambda_{01}(0)}L_0 = V_{\lambda_{01}(0)} \oplus \lambda_{01}(0) \) and \( V_{\lambda_{01}(1)} \oplus T_{\lambda_{01}(1)}L_1 = V_{\lambda_{01}(1)} \oplus \lambda_{01}(1) \). We take and fix away to extend this trivialization to the family \( \ell_{01}^*V \oplus \lambda_{01} \) on \([0,1]\).

We consider the following boundary valued problem for the section \( \xi \) of \( w^*TM \) on \( \mathbb{R}_{\geq 0} \times [0,1] \) of \( W^{1,p} \) class such that:

\[
(6.2a) \quad D_w \overline{\partial}(\xi) = 0 \\
(6.2b) \quad \xi(0, t) \in \lambda_{01}(t), \quad \xi(\tau, 0) \in T_pL_0, \quad \xi(\tau, 1) \in T_pL_1.
\]

Here \( D_w \overline{\partial} \) is the linearization operator of the Cauchy-Riemann equation.

We define \( W^{1,p}(\mathbb{R}_{\geq 0} \times [0,1], T_pM; \lambda_{01}) \) to be the set of sections \( \xi \) of \( w^*TM \) on \( \mathbb{R}_{\geq 0} \times [0,1] \) of \( W^{1,p} \) class satisfying (6.2b). Then (6.2a) induces an operator

\[
D_w \overline{\partial} : W^{1,p}(\mathbb{R}_{\geq 0} \times [0,1], T_pM; \lambda_{01}) \to L^p(\mathbb{R}_{\geq 0} \times [0,1], T_pM \otimes \Lambda^{0,1}),
\]

which we denote by \( \overline{\partial}_{([p,w];\lambda_{01})} \). The following proposition was proved in Lemma 3.7.69 \cite{FOOO00}.

**Proposition 6.3.** We have

\[
(6.3) \quad \text{Index} \overline{\partial}_{([p,w];\lambda_{01})} = \mu([p,w];\lambda_{01}).
\]
We denote its determinant line by
\[ \det \overline{\mathcal{D}}_{(p,w);\lambda_1}. \]
By varying \( w \) in its homotopy class \( \alpha \in \pi_2(\ell_{01};p) = \pi_2(\ell_{01};p;L_0,L_1) \), these lines define a line bundle
\[ (6.4) \quad \det \overline{\mathcal{D}}_{(p,w);\lambda_1} \to \text{Map}(\ell_{01};p;L_0,L_1;\alpha). \]
The bundle (6.4) is trivial if \((L_0,L_1)\) is a relatively spin pair. (See Section 8.1 [FOOO08].)

We need to find a systematic way to orient (6.4) for various \( \alpha \in \pi_2(\ell_{01};p) \) simultaneously. Following Subsection 8.1.3 [FOOO08] we proceed as follows. Let \( \lambda_p : [0,1] \to T_pM \) be a path connecting from \( T_pL_0 \) to \( T_pL_1 \) in \( \text{Lag}^+(T_pM,\omega) \). The relative spin structure determines a trivialization of \( V_p \oplus T_pL_0 = V_p \oplus \lambda_p(0) \) and of \( V_p \oplus T_pL_1 = V_p \oplus \lambda_p(1) \). We fix an extension of this trivialization of the \([0,1]\) parametrized family of vector spaces \( V_p \oplus \lambda_p \). We define
\[ (6.5) \quad Z_+ = \{ (\tau,t) \in \mathbb{R}^2 \mid \tau \leq 0, \ 0 \leq t \leq 1 \} \cup \{ (\tau,t) \mid \tau^2 + (t - 1/2)^2 \leq 1/4 \} \]

**Remark 6.4.** We would like to remark that attaching the semi-disc to the side of the semi-strip \( t = 0 \) is not necessary for the definition of \( Z_+ \). However for the consistency with the notation of Subsection 8.1.3 [FOOO08], we keep using \( Z_+ \) instead of the simpler \( (-\infty,0] \times [0,1] \).

We consider maps \( \xi : Z_+ \to T_pM \) of \( W^{1,p} \) class and study the linear differential equation
\[ (6.6a) \quad \overline{\partial} \xi = 0 \]
\[ (6.6b) \quad \xi(\pi(t-1/2)/2 + i/2) \in \lambda_p(t), \ \xi(\tau,0) \in T_pL_0, \ \xi(\tau,1) \in T_pL_1. \]

It defines an operator
\[ W^{1,p}(Z_+,T_pM;\lambda_p) \to L^p(Z_+,T_pM \otimes \Lambda^{0,1}), \]
which we denote by \( \overline{\partial}_{\lambda_p} \). Let \( \text{Index} \overline{\partial}_{\lambda_p} \) be its index, which is a virtual vector space. The following theorem is proved in the same way Chapter 8 [FOOO06].

**Theorem 6.5.** Let \((L_0,L_1)\) be a relatively spin pair of oriented Lagrangian submanifolds. Then for each fixed \( \alpha \) the bundle \((6.4)\) is trivial.

If we fix a choice of system of orientations \( \alpha_p \) on Index \( \overline{\partial}_{\lambda_p} \) for each \( p \), then it determines orientations on (6.4), which we denote by \( \alpha_{(p,w)} \).

Moreover \( \alpha_p, \alpha_{(p,w)} \) determine an orientation of \( \mathcal{M}(p,q;B) \) denoted by \( o(p,q;B) \) by the gluing rule
\[ (6.7) \quad o_{[q,w\#B]} = o_{(p,w)} \# o(p,q;B) \]
for all \( p,q \in L_0 \cap L_1 \) and \( B \in \pi_2(p,q) \) so that they satisfy the gluing formulae
\[ \partial o(p,r;B) = o(p,q;B_1) \# o(q,r;B_2) \]
whenever the virtual dimension of \( \mathcal{M}(p,r;B) \) is 1. Here \( \partial o(p,r;B) \) is the induced boundary orientation of the boundary \( \partial \mathcal{M}(p,r;B) \) and \( B = B_1 \# B_2 \) and \( \mathcal{M}(p,q;B_1) \# \mathcal{M}(q,r;B_2) \) appears as a component of the boundary \( \partial \mathcal{M}(p,r;B) \).
Remark 6.6. In the last statement in Theorem 6.5, we assumed that $\mathcal{M}(p, r; B)$ is one-dimensional. In general, we have
\[ \partial (p, r; B) = (-1)^r o(p, q; B_1) \# o(q, r; B_2), \]
where $\epsilon = \dim \mathcal{M}(q, r; B_2)$, which is presented in the proof of Proposition 8.7.3 in [FOOO08]. For the definition of the orientation of the moduli spaces for the filtered bimodule structure, see sections 8.7 and 8.8 (Definition 8.8.11) in [FOOO08].

Proof. The first paragraph follows from Section 8.1 [FOOO08]. We glue the end $(-\infty, t)$ of $Z_+$ with $(+\infty, t)$ of $\mathbb{R} \times [0, 1]$ to obtain $(\mathbb{R} \times [0, 1]) \# Z_+$. We ‘glue’ operators (6.2) and (6.6) in an obvious way to obtain an operator $D_w \overline{\partial}((\mathbb{R} \times [0, 1]) \# Z_+)$ on $(\mathbb{R} \times [0, 1]) \# Z_+$. We have an isomorphism of (family of) virtual vector spaces:
\[ \text{Index} \left( D_w \overline{\partial}((\mathbb{R} \times [0, 1]) \# Z_+) \right) \cong \text{Index} \left( \overline{\partial}_{(p, w; \lambda_0)} \right) \oplus \text{Index} \left( \overline{\partial}_{\lambda p} \right) \]
We fixed a trivialization of the family of vector spaces $V_p \oplus \lambda_p$ and $\ell_{01} \oplus \lambda_0$, which extends a trivialization of $V \oplus TL_0$, $V \oplus TL_1$ on the two skeletons of $L_0$ and $L_1$ respectively, which is given by the relative spin structure. It induces a canonical orientation of the index bundle $\text{Index} \left( D_w \overline{\partial}((\mathbb{R} \times [0, 1]) \# Z_+) \right)$ by Lemma 3.7.69 [FOOO08]. Therefore the orientation of $\text{Index} \overline{\partial}_{\lambda p}$ induces an orientation of $\text{Index} \overline{\partial}_{(p, w; \lambda_0)}$ in a canonical way. This implies the second paragraph.

The third paragraph is a consequence of (6.7), which is similar to the proof of Theorem 8.1.14 [FOOO08].

One can generalize the above discussion to the moduli space of pseudo-holomorphic polygons in a straightforward way, which we describe below.

Consider a disc $D^2$ with $k + 1$ marked points $z_{0i}, z_{k(i-1)}, \cdots, z_{10} \in \partial D^2$ respecting the counter clockwise cyclic order of $\partial D^2$. We take a neighborhood $U_i$ of $z_{i(i-1)}$ and a conformal diffeomorphism $\varphi_i : U_i \setminus \{ z_{i(i-1)} \} \subset D^2 \cong (-\infty, 0) \times [0, 1]$ of each $z_{i(i-1)}$. For any smooth map $w : D^2 \rightarrow M; w(z_{i(i-1)}) = p_t(i-1), w(z_{i(i-1)}) \subset L_i$ we deform $w$ so that it becomes constant on $\varphi_i^{-1}((-\infty, -1) \times [0, 1]) \subset U_i$, i.e., $w(z) \equiv p_t(i-1)$ for all $z \in \varphi_i^{-1}((-\infty, -1) \times [0, 1])$. So assume this holds for $w$ from now on. We now consider the Cauchy-Riemann equation
\begin{align}
(6.9a) \quad & D_w \overline{\mathcal{J}}(\xi) = 0 \\
(6.9b) \quad & \xi(\theta) \in T_{w(\theta)}L_i \quad \text{for} \quad \theta \in \varphi_i(z_{i(i-1)}) \subset \partial D^2.
\end{align}
We remark that on $U_i = (-\infty, 0) \times [0, 1]$ the boundary condition (6.9b) becomes
\[ \xi(s, 0) \in L_{i-1}, \quad \xi(s, 1) \in L_i. \]
(6.9b) induces a Fredholm operator, which we denote by
\[ \overline{\mathcal{J}}_{w; \xi} : W^{1,p}(D^2; w^*TM; \xi) \rightarrow L^p(D^2; w^* \Lambda^{0,1}). \]
Moving $w$ we obtain a family of Fredholm operators $\overline{\mathcal{J}}_{(\xi, \tilde{\xi}, B)}$ parametrized by a suitable completion of $\mathcal{F}(\tilde{\xi}, \xi; B)$ for $B \in \pi_2(\tilde{\xi}; \xi)$. Therefore we have a well-defined determinant line bundle
\[ \det \overline{\mathcal{J}}_{(\xi, \tilde{\xi}, B)} \rightarrow \mathcal{F}(\xi; \tilde{\xi}; B). \]
The following theorem is an extension of the above Theorem 6.5.
Theorem 6.7. Suppose $\Sigma = (L_0, \cdots, L_k)$ is a relatively spin Lagrangian chain. Then each $\det J_{(\varepsilon; p; B)}$ is trivial.

Moreover we have the following: If we fix orientations $\alpha_{p_{ij}}$ on Index $\overline{\mathcal{J}}_{p_{ij}}$ as in Theorem 6.3 for all $p_{ij} \in L_i \cap L_j$, with $L_i$ transversal to $L_j$, then we have a system of orientations, denoted by $\alpha_{k+1}(\overline{\mathcal{P}}; \Sigma; B)$, on the bundles (6.12) so that it is compatible with gluing map in an obvious sense.

Proof. Let $w^+_{(i+1)i} \in \pi_2(\ell_{i(i+1)}; p_{(i+1)i})$ be as in (5.9) and we consider the operator $\overline{\mathcal{J}}_{([p_{i(i+1)i}^+; \lambda_i])}$. We glue it with $\overline{\mathcal{J}}_{([\varepsilon; p; B])}$ at $U_{i+1}$. (6.10) implies that the boundary condition can be glued.) After gluing all of $\overline{\mathcal{J}}_{([p_{i(i+1)i}^+; \lambda_i])}$ we have an index bundle of a Fredholm operator

\begin{equation}
\overline{\mathcal{J}}_{([\varepsilon; p; B])} \# \sum_{i=0}^k \overline{\mathcal{J}}_{([p_{i(i+1)i}^+; \lambda_i])},
\end{equation}

By Lemma 3.7.69 [FOOO08], the index bundle of (6.13) has canonical orientation. On the other hand, index virtual vector spaces of $\overline{\mathcal{J}}_{([p_{i(i+1)i}^+; \lambda_i])}$ are oriented by Theorem 6.5. Theorem 6.7 follows. \end{proof}

We can prove that the orientation of $\overline{\mathcal{J}}_{([\varepsilon; p; B])}$ depends on the choice of $\alpha_{p_{i(i+1)i}}$ (and so on $\alpha_p$) with $i = 0, \cdots, k$ but is independent of the choice of $w^+_{(i+1)i}$ etc. This is a consequence of the proof of Theorem 6.5. (We omit the detail of this point. See Remark 8.1.15 (3) [FOOO08].) Therefore the orientation in Theorem 6.7 is independent of the choice of anchors.

Remark 6.8. In order to give an orientation of $\mathcal{M}(\Sigma; \varepsilon; p; B)$, we have to take the moduli parameters of marked points and the action of the automorphism group into account. We also treat the intersection point $p_{i(i-1)}$ as if it is a chain of codimension $\mu([p_{i(i-1)}; w^+_{i(i-1)i}]; \lambda_i)$ in a similar way to Chapter 8, section 8.5 in [FOOO08].

7. Floer chain complex

In this subsection, we will describe construction of the boundary map. We also mention some (minor) modification needed in its construction in the context with anchored Lagrangian submanifolds.

Let $(L_i, \gamma_i) i = 0, 1$ be anchored Lagrangian submanifolds. We write $\Sigma = ((L_0, \gamma_0), (L_1, \gamma_1))$. Let $p, q \in L_0 \cap L_1$ be admissible intersection points. We defined the set $\pi_2(p, q) = \pi_2((L_0, L_1), (p, q))$ in Section 6. We also defined $\pi_2(\ell_{01}; p)$ there. We now define:

**Definition 7.1.** $\operatorname{CF}((L_1, \gamma_1), (L_0, \gamma_0))$ is a free $R$ module over the basis $[p, w]$ where $p \in L_0 \cap L_1$ is an admissible intersection points and $[w] \in \pi_2(\ell_{01}; p)$.

Here $R$ is a ground ring such as $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{C}$ or $\mathbb{R}$. (The choice $\mathbb{Z}$ or $\mathbb{Z}_2$ requires some additional conditions.)

**Remark 7.2.** We remark that the set of $[p, w]$ where $p$ is the admissible intersection point is identified with the set of the critical point of the action functional $\mathcal{A}$ defined on the Novikov covering space of $\Omega(L_0, L_1; \ell_{01})$. The group $\Pi(L_0, L_1; \ell_{01})$ defined in Section 6 acts freely on it so that the quotient space is the set of admissible intersection points.
We next take a grading \( \lambda_i \) to \((L_i, \gamma_i)\) as in Subsection 7.2. It induces a grading of \([p, w] \) given by \( \mu([p, w]; \lambda_0) \), which gives the graded structure on \( CF(L_1, L_0; \ell_{01}) \)
\[
CF(L_1, L_0; \ell_{01}) = \bigoplus_k CF^k(L_1, L_0; \lambda_{01})
\]
where \( CF^k(L_1, L_0; \lambda_{01}) = \text{span}_R \{[p, w] \mid \mu([p, w]; \lambda_{01}) = k \} \).

For given \( B \in \pi_2(p, q) \), we denote by \( \text{Map}(p, q; B) \) the set of such \( w \)'s in class \( B \).

We summarize the extra structures added in the discussion of Floer homology for the anchored Lagrangian submanifolds in the following

**Situation 7.3.** We assume that \((L_0, L_1)\) is a relatively spin pair. We consider a pair \((L_0, \gamma_0), (L_1, \gamma_1)\) of anchored Lagrangian submanifolds and the base path \( \ell_{01} = \tau_0 \ast \gamma_1 \). We fix a grading \( \lambda_i \) of \( \gamma_i \) for \( i = 0, 1 \), which in turn induces a grading of \( \ell_{01}, \lambda_{01} = \lambda_0 \ast \lambda_1 \). We also fix an orientation \( o_p \) of Index \( \partial_p \) for each \( p \in L_0 \cap L_1 \) as in Theorem 6.5.

We sometimes do not explicitly write these extra data in our notations below as long as there is no danger of confusion.

Let us consider Situation 7.3. Orientations of the Floer moduli space \( \mathcal{M}(p, q; B) \) is induced by Theorem 6.5. Using virtual fundamental chain technique we can take a system of multisecions and obtain a system of rational numbers \( n(p, q; B) = \#(\mathcal{M}(p, q; B)) \) whenever the virtual dimension of \( \mathcal{M}(p, q; B) \) is zero. Finally we define the Floer ‘boundary’ map \( \partial : CF(L_1, L_0; \ell_{01}) \rightarrow CF(L_1, L_0; \ell_{01}) \) by the sum
\[
\partial([p, w]) = \sum_{q \in L_0 \cap L_1} \sum_{B \in \pi_2(p, q)} n(p, q; B)[q, w#B].
\]

By Remark 7.2, \( CF(L_1, L_0; \ell_{01}) \) carries a natural \( \Lambda(L_0, L_1; \ell_{01}) \)-module structure and \( CF^k(L_1, L_0; \lambda_{01}) \) a \( \Lambda^{(0)}(L_0, L_1; \ell_{01}) \)-module structure where
\[
\Lambda^{(0)}(L_0, L_1; \ell_{01}) = \left\{ \sum a_g [g] \in \Lambda(L_0, L_1; \ell_{01}) \mid \mu([g]) = 0 \right\}.
\]

We define
\[
C(L_1, L_0; \ell_{01}) = CF(L_1, L_0; \ell_{01}) \otimes_{\Lambda(L_0, L_1; \ell_{01})} \Lambda_{\text{nov}}
\]
where we use the embedding 7.2.

We write the \( \Lambda_{\text{nov}} \) module 7.2 also as
\[
C((L_1, \gamma_1), (L_0, \gamma_0); \Lambda_{\text{nov}}).
\]

**Definition 7.4.** We define the energy filtration \( F^\lambda CF((L_1, \gamma_1), (L_0, \gamma_0)) \) of the Floer chain complex \( CF(L_1, \gamma_1), (L_0, \gamma_0)) \) (here \( \lambda \in \mathbb{R} \)) such that \([p, w]\) is in \( F^\lambda CF((L_1, \gamma_1), (L_0, \gamma_0)) \) if and only if \( A([p, w]) \geq \lambda \).

This filtration also induces a filtration on 7.2.

**Remark 7.5.** We remark that this filtration depends (not only of the homotopy class of) but also of \( \gamma_i \) itself.

It is easy to see the following from the definition of \( \partial \) above:

**Lemma 7.6.**
\[
\partial (F^\lambda CF((L_1, \gamma_1), (L_0, \gamma_0)) \subseteq F^{\lambda + \epsilon} CF((L_1, \gamma_1), (L_0, \gamma_0))).
\]
8. Obstruction and $A_{\infty}$ structure

Let $(L_0, L_1)$ be a relatively spin pair with $L_0$ intersecting $L_1$ transversely and fix a $(M, st)$-relatively spin structure for the pair $(L_0, L_1)$.

According to the definition \([7,1]\) of the map $\partial$, we have the formula for its matrix coefficients

\[
(8.1) \quad \langle \partial \partial [p, w], [r, w \# B] \rangle = \sum_{q \in L_0 \cap L_1} \sum_{B = B_1 \# B_2 \in \pi_2(p, r)} n(p, q; B_1) n(q, r; B_2) T^{\omega(B)}
\]

where $B_1 \in \pi_2(p, q)$ and $B_2 \in \pi_2(q, r)$.

To prove, $\partial \partial = 0$, one needs to prove $\langle \partial \partial [p, w], [r, w \# B] \rangle = 0$ for all pairs $[p, w]$, $[r, w \# B]$. On the other hand it follows from definition that each summand

\[
n(p, q; B_1) n(q, r; B_2) T^{\omega(B)} = n(p, q; B_1) T^{\omega(B_1)} n(q, r; B_2) T^{\omega(B_2)}
\]

and the coefficient $n(p, q; B_1) n(q, r; B_2)$ is nothing but the number of broken trajectories lying in $\mathcal{M}(p, q; B_1) \# \mathcal{M}(q, r; B_2)$. This number is nonzero in the general situation we work with.

To handle the problem of obstruction to $\partial \circ \partial = 0$ and of bubbling-off discs in general, a structure of filtered $A_{\infty}$ algebra $(\mathfrak{C}, \mathfrak{m})$ with non-zero $\mathfrak{m}_0$-term is associated to each Lagrangian submanifold $L$.

8.1. $A_{\infty}$ algebra. In this subsection, we review the notion and construction of filtered $A_{\infty}$ algebra associated to a Lagrangian submanifold. In order to make the construction consistent to one in the last section, where $\Lambda(L_0, L_1; t_{01})$ is used for the coefficient ring rather than the universal Novikov ring, we rewrite them using smaller Novikov ring $\Lambda(L)$ which we define below. Let $L$ be a relatively spin Lagrangian submanifold. We have a homomorphism

\[
(E, \mu) : H_2(M, L; \mathbb{Z}) \to \mathbb{R} \times \mathbb{Z}
\]

where $E(\beta) = \beta \cap [\omega]$ and $\mu$ is the Maslov index homomorphism. We put $g \sim g'$ for $g, g' \in H_2(M, L; \mathbb{Z})$ if $E(g) = E(g')$ and $\mu(g) = \mu(g')$. We write $\Pi(L)$ the quotient with respect to this equivalence relation. It is a subgroup of $\mathbb{R} \times \mathbb{Z}$. We define

\[
\Lambda(L) = \left\{ \sum c_g[g] \mid g \in \Pi(L), c_g \in R, E(g) \geq 0, \forall E_0 \# \{ g \mid c_g \neq 0, E(g) \leq E_0 \} < \infty \right\}
\]

There exists an embedding $\Lambda(L) \to \Lambda_{0, nov}$, defined by $[g] \mapsto e^{\mu(g)/2} T^{E(g)}$.

Let $\mathcal{C}$ be a graded $R$-module and $\mathcal{C} = \mathcal{C} \otimes_R \Lambda(L)$. Here and hereafter we use symbol $\mathcal{C}F$ for the modules over $\Lambda(L)$ or $\Lambda(L_0, L_1)$ and $\mathcal{C}$ for the modules over the universal Novikov ring.

We denote by $\mathcal{C}F[1]$ its suspension defined by $\mathcal{C}F[1]^k = \mathcal{C}F^{k+1}$. We denote by $\deg(x) = [x]$ the degree of $x \in C$ before the shift and $\deg'(x) = |x| - 1$. Define the bar complex $B(\mathcal{C}F[1])$ by

\[
B_k(\mathcal{C}F[1]) = (\mathcal{C}F[1])^{k \otimes}, \quad B(\mathcal{C}F[1]) = \bigoplus_{k=0}^{\infty} B_k(\mathcal{C}F[1]).
\]
Here $B_0(CF[1]) = R$ by definition. The tensor product is taken over $\Lambda(L)$. We provide the degree of elements of $B(CF[1])$ by the rule

$$\langle x_1 \otimes \cdots \otimes x_k \rangle' := \sum_{i=1}^{k} |x_i|' = \sum_{i=1}^{k} |x_i| - k$$

where $| \cdot |'$ is the shifted degree. The ring $B(CF[1])$ has the structure of graded coalgebra.

**Definition 8.1.** The structure of strict filtered $A_\infty$ algebra over $\Lambda(L)$ is a sequence of $\Lambda(L)$ module homomorphisms

$$m_k : B_k(CF[1]) \to CF[1], \quad k = 1, 2, \cdots,$$

of degree +1 such that the coderivation $d = \sum_{k=1}^{\infty} m_k$ satisfies $dd = 0$, which is called the $A_\infty$-relation. Here we denote by $\hat{m}_k : B(CF[1]) \to B(CF[1])$ the unique extension of $m_k$ as a coderivation on $B(CF[1])$. A filtered $A_\infty$ algebra is an $A_\infty$ algebra with a filtration for which $m_k$ are continuous with respect to the induce non-Archimedean topology.

In particular, we have $m_1m_1 = 0$ and so it defines a complex $(CF, m_1)$. We define the $m_1$-cohomology by

$$H(CF, m_1) = \ker m_1 / \im m_1.$$

A filtered $A_\infty$ algebra is defined in the same way, except that it also includes

$$m_0 : R \to B(CF[1]).$$

The first two terms of the $A_\infty$ relation for a $A_\infty$ algebra are given as

$$m_1(m_0(1)) = 0 \quad \text{(8.4)}$$

$$m_1m_1(x) + (-1)^{|x|'} m_2(x, m_0(1)) + m_2(m_0(1), x) = 0. \quad \text{(8.5)}$$

In particular, for the case $m_0(1)$ is nonzero, $m_1$ will not necessarily satisfy the boundary property, i.e., $m_1m_1 \neq 0$ in general.

**Remark 8.2.** Here we use the Novikov ring $\Lambda(L)$. In [FOOO06] we defined a filtered $A_\infty$ algebra over the universal Novikov ring $\Lambda_{0,\text{nov}}$. A filtered $A_\infty$ algebra over $\Lambda(L)$ induces one over $\Lambda_{0,\text{nov}}$ in an obvious way. On the other hand, an appropriate gap condition is needed for a filtered $A_\infty$ algebra over $\Lambda_{0,\text{nov}}$ to induce one over $\Lambda(L)$.

We now describe the $A_\infty$ operators $m_k$ in the context of $A_\infty$ algebra of Lagrangian submanifolds. For a given compatible almost complex structure $J$, consider the moduli space of stable maps of genus zero

$$\mathcal{M}_{k+1}(\beta; L) = \{(w, (z_0, z_1, \cdots, z_k)) \mid \partial w = 0, z_i \in \partial D^2, [w] = \beta \text{ in } \pi_2(M, L)\} / \sim$$

where $\sim$ is the conformal reparameterization of the disc $D^2$. We require that $z_0, \cdots, z_k$ respects counter clockwise cyclic order of $S^1$. (We wrote this moduli space $\mathcal{M}_{k+1}^{\text{main}}(\beta; L)$ in [FOOO08]. The symbol ‘main’ indicates the compatibility of $z_0, \cdots, z_k$, with counter clockwise cyclic order. We omit this symbol in this paper since we always assume it.)

$\mathcal{M}_{k+1}(\beta; L)$ has a Kuranishi structure and its dimension is given by

$$n + \mu(\beta) - 3 + (k + 1) = n + \mu(\beta) + k - 2. \quad \text{(8.6)}$$
Finally we take the sum and continuous with respect to non-Archimedean topology. We extend \( \partial \) and consider the fiber product

\[
e_{\nu_0} : \mathcal{M}_{k+1}(\beta; L) \times_{(\nu_1, \ldots, \nu_k)} (P_1 \times \cdots \times P_k) \rightarrow L.
\]

A simple calculation shows that the expected dimension of this chain is given by

\[
deg [\mathcal{M}_{k+1}(\beta; L) \times_{(\nu_1, \ldots, \nu_k)} (P_1 \times \cdots \times P_k), e_{\nu_0}] = \sum_{j=1}^{n} (\deg P_j - 1) + 2 - \mu(\beta).
\]

For each given \( \beta \in \pi_2(M, L) \) and \( k = 0, \ldots, \), we define \( m_{1,0}(P) = \pm \partial P \) and

\[
m_{k,\beta}(P_1, \ldots, P_k) = [\mathcal{M}_{k+1}(\beta; L) \times_{(\nu_1, \ldots, \nu_k)} (P_1 \times \cdots \times P_k), e_{\nu_0}] \in C(L; \mathbb{Q})
\]

(8.7)

(More precisely we regard the right hand side of (8.7) as a smooth singular chain by taking appropriate multi-valued perturbation (multisection) and choosing a simplicial decomposition of its zero set.)

We put

\[
CF(L) = C(L; \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda(L).
\]

We define \( m_k : B_k CF(L)[1] \rightarrow B_k CF[1] \) by

\[
m_k = \sum_{\beta \in \pi_2(M, L)} m_{k,\beta} \otimes [\beta].
\]

Then it follows that the map \( m_k : B_k CF(L)[1] \rightarrow CF(L)[1] \) is well-defined, has degree 1 and continuous with respect to non-Archimedean topology. We extend \( m_k \) as a coderivation \( \hat{m}_k : BCF[1] \rightarrow BCF[1] \) where \( BCF(L)[1] \) is the completion of the direct sum \( \oplus_{n=0}^{\infty} B_k CF(L)[1] \) where \( B_k CF(L)[1] \) itself is the completion of \( CF(L)[1]^{\otimes k} \). \( BCF(L)[1] \) has a natural filtration defined similarly as Definition [let].

Finally we take the sum

\[
\hat{d} = \sum_{k=0}^{\infty} \hat{m}_k : BCF(L)[1] \rightarrow BCF(L)[1].
\]

We then have the following coboundary property:

**Theorem 8.3.** Let \( L \) be an arbitrary compact relatively spin Lagrangian submanifold of an arbitrary tame symplectic manifold \((M, \omega)\). The coderivation \( \hat{d} \) is a continuous map that satisfies the \( A_\infty \) relation \( \hat{d} \hat{d} = 0 \), and so \( CF(L), m \) is a filtered \( A_\infty \) algebra over \( \Lambda(L) \).

We put

\[
C(L; \Lambda_{0, nov}) = CF(L) \otimes_{\Lambda(L)} \Lambda_{0, nov}
\]

on which a filtered \( A_\infty \) structure on \( C(L; \Lambda_{0, nov}) \) (over the ring \( \Lambda_{0, nov} \)) is induced. This is the filtered \( A_\infty \) structure given in Theorem A [FOOO06]. The proof is the same as that of Theorem A [FOOO06].

In the presence of \( \hat{m}_0, \hat{m}_1 \hat{m}_1 = 0 \) no longer holds in general. This leads to consider deforming Floer’s original definition by a bounding cochain of the obstruction cycle arising from bubbling-off discs. One can always deform the given (filtered) \( A_\infty \)
algebra \((CF(L),\mathfrak{m})\) by an element \(b \in CF(L)[1]^0\) by re-defining the \(A_{\infty}\) operators as

\[
m_k^b(x_1, \ldots, x_k) = \mathfrak{m}(e^b, x_1, e^b, x_2, e^b, x_3, \ldots, x_k, e^b)
\]

and taking the sum \(d^b = \sum_{k=0}^{\infty} \widehat{m}_k^b\). This defines a new filtered \(A_{\infty}\) algebra in general. Here we simplify notations by writing

\[
e^b = 1 + b + b \otimes b + \cdots + b \otimes \cdots \otimes b + \cdots.
\]

Note that each summand in this infinite sum has degree 0 in \(CF(L)[1]\) and converges in the non-Archimedean topology if \(b\) has positive valuation, i.e., \(v(b) > 0\). (See Section 2 for the definition of \(v\).)

**Proposition 8.4.** For the \(A_{\infty}\) algebra \((CF(L),\mathfrak{m}_k^b), \mathfrak{m}_0^b = 0\) if and only if \(b\) satisfies

\[
\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \ldots, b) = 0.
\]

This equation is a version of Maurer-Cartan equation for the filtered \(A_{\infty}\) algebra.

**Definition 8.5.** Let \((CF(L),\mathfrak{m})\) be a filtered \(A_{\infty}\) algebra in general and \(BCF(L)[1]\) be its bar complex. An element \(b \in CF(L)[1]^0 = CF(L)^1\) is called a bounding cochain if it satisfies the equation (8.8) and \(v(b) > 0\). We denote by \(\mathcal{M}(L; \Lambda(L))\) the set of bounding cochains.

In general a given \(A_{\infty}\) algebra may or may not have a solution to (8.8). In our case we define:

**Definition 8.6.** A filtered \(A_{\infty}\) algebra \((CF(L),\mathfrak{m})\) is called unobstructed over \(\Lambda(L)\) if the equation (8.8) has a solution \(b \in CF(L)[1]^0 = CF(L)^1\) with \(v(b) > 0\).

One can define the notion of homotopy equivalence between two bounding cochains and et al as described in Chapter 4 of \(\text{FOOO06}\). We denote by \(\mathcal{M}(L; \Lambda(L))\) the set of equivalence classes of bounding cochains of \(L\).

**Remark 8.7.** In Definition 8.5 above we consider bounding cochain contained in \(CF(L) \subset C(L; \Lambda_0)\) only. This is the reason why we write \(\mathcal{M}(L; \Lambda(L))\) in place of \(\mathcal{M}(L)\). (The latter is used in \(\text{FOOO06}\).)

### 8.2. \(A_{\infty}\) bimodule.

Suppose we are in Situation 7.3. Once the \(A_{\infty}\) algebra is attached to each Lagrangian submanifold \(L\), we then construct a structure of filtered \(A_{\infty}\) bimodule on the module \(CF((L_1, \gamma_1), (L_0, \gamma_0))\), which was introduced in Section 7 as follows. This filtered \(A_{\infty}\) bimodule structure is by definition a family of operators

\[
n_{k_1, k_0} : B_{k_1}(CF(L_1)[1]) \otimes_{\Lambda(L_1)} CF((L_1, \gamma_1), (L_0, \gamma_0)) \otimes_{\Lambda(L_0)} B_{k_0}(CF(L')[1]) \to CF((L_1, \gamma_1), (L_0, \gamma_0))
\]

for \(k_0, k_1 \geq 0\). Here the left hand side is defined as follows: It is easy to see that there are embeddings \(\Lambda(L_0) \to \Lambda(L_0, L_1; \ell_{01}), \Lambda(L_1) \to \Lambda(L_0, L_1; \ell_{01})\). Therefore a \(\Lambda(L_0, L_1; \ell_{01})\) module \(CF((L_1, \gamma_1), (L_0, \gamma_0))\) can be regarded both as \(\Lambda(L_0)\) module and \(\Lambda(L_1)\) module. Hence we can take tensor product in the left hand side. (\(\otimes_{\Lambda(L_0)}\) is the completion of this algebraic tensor product.) The left hand side then becomes a \(\Lambda(L_0, L_1; \ell_{01})\) module, since the rings involved are all commutative.
We briefly describe the definition of $n_{k_1,k_0}$. A typical element of the tensor product

$$B_{k_1}(CF(L_1)[1]) \hat{\otimes}_{\Lambda(L_1)} \text{CF}((L_1, \gamma_1), (L_0, \gamma_0)) \hat{\otimes}_{\Lambda(L_0)} B_{k_0}(CF(L_0)[1])$$

has the form

$$P_{1,1} \otimes \cdots \otimes P_{1,k_1} \otimes [p, w] \otimes P_{0,1} \otimes \cdots \otimes P_{0,k_0}$$

with $p \in L_0 \cap L_1$ being an admissible intersection point. Then the image $n_{k_0,k_1}$ thereof is given by

$$\sum_{q,B} T^{w(B)e_B(B)/2} \# (\mathcal{M}(p, q; B; P_{1,1}, \cdots, P_{1,k_1}; P_{0,0}, \cdots, P_{0,k_0})) [q, B#w].$$

Here $B$ denotes homotopy class of Floer trajectories connecting $p$ and $q$, the summation is taken over all $[q, B]$ with

$$\dim \mathcal{M}(p, q; B; P_{1,1}, \cdots, P_{1,k_1}; P_{0,1}, \cdots, P_{0,k_0}) = 0,$$

and $\# (\mathcal{M}(p, q; B; P_{1,1}, \cdots, P_{1,k_1}; P_{0,1}, \cdots, P_{0,k_0}))$ is the ‘number’ of elements in the ‘zero’ dimensional moduli space $\mathcal{M}(p, q; B; P_{1,1}, \cdots, P_{1,k_1}; P_{0,1}, \cdots, P_{0,k_0})$. Here the moduli space $\mathcal{M}(p, q; B; P_{1,1}, \cdots, P_{1,k_1}; P_{0,1}, \cdots, P_{0,k_0})$ is the Floer moduli space $\mathcal{M}(p, q; B)$ cut-down by intersecting with the given chains $P_{1,i} \subset L_1$ and $P_{0,j} \subset L_0$. (See Section 3.7 [FOOO08].) An orientation on this moduli space is induced by $o_{[p,w]}$, $o_{[q,w]}$, which we obtained by Theorem 6.5.

**Theorem 8.8.** Let $((L_0, \gamma_0), (L_1, \gamma_1))$ be a pair of anchored Lagrangian submanifolds. Then the family $\{n_{k_1,k_0}\}$ defines a left $(\text{CF}(L_1), m)$ and right $(\text{CF}(L_0), m)$ filtered $A_\infty$-bimodule structure on $\text{CF}((L_1, \gamma_1), (L_0, \gamma_0))$.

See Section 3.7 [FOOO08] for the definition of filtered $A_\infty$ bimodules. (In [FOOO08] the case of universal Novikov ring as a coefficient is considered. It is easy to modify to our case of $\Lambda(L_0, L_1)$ coefficient.) The proof of Theorem 8.8 is the same as that of Theorem 3.7.21 [FOOO08].

In the case where both $L_0$, $L_1$ are unobstructed, we can carry out this deformation of $n$ using bounding cochains $b_0$ and $b_1$ of $\text{CF}(L_0)$ and $\text{CF}(L_1)$ respectively, in a way similar to $m^b$. Namely we define $\delta_{b_1,b_0} : \text{CF}((L_1, \gamma_1), (L_0, \gamma_0)) \rightarrow \text{CF}((L_1, \gamma_1), (L_0, \gamma_0))$ by

$$\delta_{b_1,b_0}(x) = \sum_{k_1,k_0} n_{k_1,k_0} (b_1^\otimes k_1 \otimes x \otimes b_0^\otimes k_0) = \mathfrak{n}(e^{b_1}, x, e^{b_0}).$$

We can generalize the story to the case where $L_0$ has clean intersection with $L_1$, especially to the case $L_0 = L_1$. In the case $L_0 = L_1$ we have $n_{k_1,k_0} = m_{k_0+k_1+1}$. So in this case, we have $\delta_{b_1,b_0}(x) = m(e^{b_1}, x, e^{b_0})$.

We define Floer cohomology of the pair $(L_0, \gamma_0, \lambda_0), (L_1, \gamma_1, \lambda_1)$ by

$$HF((L_1, \gamma_1, b_1), (L_0, \gamma_0, b_0)) = \text{Ker} \delta_{b_1,b_0} / \text{Im} \delta_{b_1,b_0}.$$  

This is a module over $\Lambda(L_0, L_1; \ell_{01})$.

**Theorem 8.9.** $HF((L_1, \gamma_1, b_1), (L_0, \gamma_0, b_0)) \otimes_{\Lambda(L_0, L_1)} \Lambda_{nov}$ is invariant under the Hamiltonian isotopies of $L_0$ and $L_1$ and under the homotopy of bounding cochains $b_0, b_1$.

The proof is the same as the proof of Theorem 4.1.5 [FOOO08] and so omitted.
8.3. Products. Let $\mathcal{L} = (L_0, L_1, \cdots, L_k)$ be a chain of compact Lagrangian submanifolds in $(M, \omega)$ that intersect pairwise transversely without triple intersections.

Let $\vec{z} = (z_{0k}, z_{k(k-1)}, \cdots, z_{10})$ be a set of distinct points on $\partial D^2 = \{ z \in \mathbb{C} \mid |z| = 1 \}$. We assume that they respect the counter-clockwise cyclic order of $\partial D^2$.

The group $PSL(2; \mathbb{R}) \cong \text{Aut}(D^2)$ acts on the set in an obvious way. We denote by $M_{k+1,\text{main}}$ be the set of $PSL(2; \mathbb{R})$-orbits of $(D^2, \vec{z})$.

In this subsection, we consider only the case $k \geq 2$ since the case $k = 1$ is already discussed in the last subsection. In this case there is no automorphism on the domain $(D^2, \vec{z})$, i.e., $PSL(2; \mathbb{R})$ acts freely on the set of such $(D^2, \vec{z})$’s. Let $p_j(j-1) \in L_j \cap L_{j-1}$ ($j = 0, \cdots, k$), be a set of intersection points.

We consider the pair $(w, \vec{z})$ where $w : D^2 \to M$ is a pseudo-holomorphic map that satisfies the boundary condition

\[
\begin{align*}
(8.9a) & \quad w(z_{(j-1)i}z_{(j+1)j}) \subset L_j, \\
(8.9b) & \quad w(z_{(j+1)j}) = p_{(j+1)j} \in L_j \cap L_{j+1}.
\end{align*}
\]

We denote by $\tilde{M}(\mathcal{L}, \vec{p})$ the set of such $((D^2, \vec{z}), w)$.

We identify two elements $((D^2, \vec{z}), w), ((D^2, \vec{z}'), w')$ if there exists $\psi \in PSL(2; \mathbb{R})$ such that $w \circ \psi = w'$ and $\psi(z_{(j-1)i}) = z_{(j-1)i}$. Let $M(\mathcal{L}, \vec{p})$ be the set of equivalence classes. We compactify it by including the configurations with disc or sphere bubbles attached, and denote it by $M(\mathcal{L}, \vec{p})$. Its element is denoted by $((\Sigma, \vec{z}), w)$ where $\Sigma$ is a genus zero bordered Riemann surface with one boundary components, $\vec{z}$ are boundary marked points, and $w : (\Sigma, \partial \Sigma) \to (M, L)$ is a bordered stable map.

We can decompose $M(\mathcal{L}, \vec{p})$ according to the homotopy class $B \in \pi_2(\mathcal{L}, \vec{p})$ of continuous maps satisfying $w : \mathcal{L} \to M$, into the union

\[
M(\mathcal{L}, \vec{p}) = \bigcup_{B \in \pi_2(\mathcal{L}, \vec{p})} M(\mathcal{L}, \vec{p}; B).
\]

In the case we fix an anchor $\gamma_i$ to each of $L_i$ and put $E = ((L_0, \gamma_0), \cdots, (L_k, \gamma_k))$, we consider only admissible classes $B$ and put

\[
M(\mathcal{E}, \vec{p}) = \bigcup_{B \in \pi_2(\mathcal{E}, \vec{p})} M(\mathcal{E}, \vec{p}; B).
\]

Theorem 8.10. Let $\mathcal{L} = (L_0, \cdots, L_k)$ be a chain of Lagrangian submanifolds and $B \in \pi_2(\mathcal{L}; \vec{p})$. Then $M(\mathcal{L}, \vec{p}; B)$ has an oriented Kuranishi structure (with boundary and corners). Its (virtual) dimension satisfies

\[
\dim M(\mathcal{L}, \vec{p}; B) = \mu(\mathcal{L}, \vec{p}; B) + n + k - 2,
\]

where $\mu(\mathcal{L}, \vec{p}; B)$ is the polygonal Maslov index of $B$.

Proof. We consider the operator $\bar{\partial}_{w; \mathcal{L}}$ in (6.11). It is easy to see that

\[
\text{Index } \bar{\partial}_{w; \mathcal{L}} + k - 2 = \dim M(\mathcal{L}, \vec{p}; B).
\]

In fact $k - 2$ in the left hand side is the dimension of $M_{k+1,\text{main}}/PSL(2; \mathbb{R})$.

We next consider the Fredholm operator $[0,13]$. By (8.11), Lemma 5.12 and index sum formula, we have

\[
\text{Index } \bar{\partial}_{w; \mathcal{L}} - \mu(\mathcal{L}, \vec{p}; B) = \text{Index of } [0,13].
\]
We remark that the operator (6.13) is a Cauchy-Riemann operator of the trivial \( \mathbb{C}^n \) bundle on \( D^2 \) with boundary condition determined by a certain loop in \( \text{Lag}(\mathbb{C}^n, \omega) \). By construction it is easy to see that this loop is homotopic to a constant loop. Therefore, the index of (6.13) is \( n \). Theorem 8.10 follows.

We next take graded anchors \((\gamma_i, \lambda_i)\) to each \( L_i \) and fix the data as in Situation 8.3. We assume that \( B \) is admissible and write \( B = [w_0^+][w_{12}^-]# \cdots #[w_{k0}^-] \) as in Definition 5.2. We put \( w^+_{i(i+1)}(s,t) = w^-_{i(i+1)}(1-s, t) \) as in (5.9). We also put \( w^+_{k0}(s,t) = w^+_{0k}(s,1-t), ([w^+_{0k}] \in \pi_1(\ell_{k0}; p_{k0}). \) We also put \( \lambda_{k0}(t) = \lambda_{0k}(1-t) \).

**Lemma 8.11.** If \( \dim \mathcal{M}(\mathcal{L}, \vec{p}^i; B) = 0 \), we have

\[
(8.13) \quad \mu([p_{k0}, w^+_{0k}]; \lambda_{0k}) - 1 = 1 + \sum_{i=1}^{k} \mu([p_{i(i-1)}, w^+_{i(i-1)}]; \lambda_{i(i-1)} - 1).
\]

**Proof.** Lemma 5.12 and Theorem 8.10 implies

\[
\sum_{i=0}^{k} \mu([p_{(i+1)i}, w^+_{(i+1)i}]; \lambda_{i(i+1)}) = n + k - 2
\]

in the case \( \dim \mathcal{M}(\mathcal{L}, \vec{p}^i; B) = 0 \). By Lemma 5.8 we have

\[
\mu([p_{k0}, w^+_{0k}]; \lambda_{0k}) = -\mu([p_{k0}, w^+_{0k}]; \lambda_{0k}) + n.
\]

Substituting this into the above identity and rearranging the identity, we obtain the lemma.

Using the case \( \dim \mathcal{M}(\mathcal{L}, \vec{p}^i; B) = 0 \), we define the \( k \)-linear operator

\[ m_k : \text{CF}((L_k, \gamma_k), (L_{k-1}, \gamma_{k-1})) \otimes \cdots \otimes \text{CF}((L_1, \gamma_1), (L_0, \gamma_0)) \rightarrow \text{CF}((L_k, \gamma_k), (L_0, \gamma_0)) \]

as follows:

\[
(8.14) \quad m_k([p_{k(k-1)}, w^+_{k(k-1)}], [p_{(k-1)(k-2)}, w^+_{(k-1)(k-2)}], \cdots, [p_{10}, w^+_{10}]) = \sum \#(\mathcal{M}_{k+1}(\mathcal{L}, \vec{p}^i; B); p_{k0}, w^+_{0k}).
\]

Here the sum is over the basis \([p_{k0}, w^+_{0k}]\) of \( \text{CF}((L_k, \gamma_k), (L_0, \gamma_0)) \), where \( \vec{p} = (p_{k0}, p_{k(k-1)}, \cdots, p_{10}) \), \( B \) as in Definition 3.2 and \( w^+_{i(i+1)}(s,t) = w^-_{i(i+1)}(1-s, t) \).

The formula (8.13) implies that \( m_k \) above has degree one.

In general the operator \( m_k \) above does not satisfy the \( A_\infty \) relation by the same reason as that of the case of boundary operators (see Section 7). We need to use bounding cochains \( b_i \) of \( L_i \) to deform \( m_k \) in the same way as the case of \( A_\infty \)-bimodules (Subsection 8.2), whose explanation is now in order.

Let \( m_0, \cdots, m_k \in \mathbb{Z}_{\geq 0} \) and \( \mathcal{M}_{m_0, \cdots, m_k}(\mathcal{L}, \vec{p}^i; B) \) be the moduli space obtained from the set of \(( (D^2, \vec{z}),(\vec{z}^{(0)}, \cdots, \vec{z}^{(k)}), w) \) by taking the quotient by \( PSL(2, \mathbb{R}) \)-action and then by taking the stable map compactification as before. Here \( \vec{z} = (z_1^{(i)}, \cdots, z_k^{(i)}) \) and \( z_i^{(i)} \in \mathbb{P}^1 \) such that \( z_{i(i+1)i}^{(i)}, z_{i(i+1)i}^{(i)} \) respects the counter clockwise cyclic ordering.

\[
(\vec{D}^2, \vec{z}, (\vec{z}^{(0)}, \cdots, \vec{z}^{(k)}), w) \mapsto (w(z_1^{(0)}), \cdots, w(z_k^{(m_k)}))
\]

induces an evaluation map:

\[
ev = (ev^{(0)}, \cdots, ev^{(k)}) : \mathcal{M}_{m_0, \cdots, m_k}(\mathcal{L}, \vec{p}^i; B) \rightarrow \prod_{i=0}^{k} L^{m_i}_i.
\]
Let $P_j^{(i)}$ be smooth singular chains of $L_i$ and put

$$\tilde{P}^{(i)} = (P_1^{(i)}, \cdots, P_m^{(i)}), \quad \tilde{P} = (\tilde{P}^{(0)}, \cdots, \tilde{P}^{(k)})$$

We then take the fiber product to obtain:

$$\mathcal{M}_{m_0, \cdots, m_k}(\xi, \tilde{p}; \tilde{P}; B) = \mathcal{M}_{m_0, \cdots, m_k}(\xi, \tilde{p}; B) \times_{ev} \tilde{P}.$$ 

We use this to define

$$m_{k;m_0, \cdots, m_k} : B_{mk}(CF(L_k)) \otimes CF((L_{k-1}, \gamma_{k-1})) \otimes \cdots \otimes CF((L_1, \gamma_1), (L_0, \gamma_0)) \rightarrow CF((L_k, \gamma_k), (L_0, \gamma_0))$$

by

$$m_{k;m_0, \cdots, m_k}(\tilde{P}^{(k)}; [p_{k+1}, w_{k+1}^0], \cdots, [p_{10}, w_{10}^0], \tilde{P}^{(0)}) = \sum \#(\mathcal{M}_{k+1}(\xi, \tilde{p}^{(i)}; B)) [p_{k0}, w_{k0}].$$

Finally for each given $b_i \in CF(L_i)[1]^0$ ($b_i \equiv 0 \mod \Lambda_+$), $\tilde{b} = (b_0, \cdots, b_k)$, and $x_i \in CF((L_{i-1}, \gamma_{i-1})), (L_{i-2}, \gamma_{i-2}))$, we put

$$(8.15) \quad m_k^\xi(x_k, \cdots, x_1) = \sum m_{k;m_0, \cdots, m_k}(b_0^{m_0} x_k, b_{k-1}^{m_{k-1}} \cdots, x_1, b_0^{m_0}).$$

**Theorem 8.12.** If $b_i$ satisfies the Maurer-Cartan equation (8.8) then $m_k^\xi$ in (8.15) satisfies the $A_\infty$ relation

$$(8.16) \quad \sum_{k_1, k_2} (-1)^* m_{k}(x_k, \cdots, m_{k_2}(x_{k-i-1}, \cdots, x_{k-i-2}), \cdots, x_1) = 0$$

where we take sum over $k_1 + k_2 = k + 1$, $i = -1, \cdots, k - 2$. (We write $m_k$ in place of $m_k^\xi$ in (8.15).) The sign $*$ is $* = i + \deg x_k + \cdots + \deg x_{k-i}$.

The non-anchored version is proved in Theorem 4.17 [Fu2]. In order to translate it to the anchored version we only need to show the following.

**Lemma 8.13.** Let $E = ((L_0, \gamma_0), \cdots, (L_i, \gamma_i), \cdots, (L_j, \gamma_j)), (L_{j-1}, \gamma_{j-1}), (L_k, \gamma_k))$, $\tilde{b} = (b_0^{(k+1)}, \cdots, p_{j-1}^{i-1}, \cdots, p_{10})$ and $B \in \pi^{ad}_2(E, \tilde{p})$ be admissible. Suppose that the sequence $u_i \in \mathcal{M}(\xi, \tilde{p}^{(i)}; B)$ converges to an element in the product $\mathcal{M}(\xi', \tilde{p}'^{(i)}; B_1) \times \mathcal{M}(\xi'', \tilde{p}''^{(i)}; B_2)$ where

$$\xi' = (L_0, \cdots, L_i, E_{j+1}, L_j), \quad \xi'' = (L_i, L_{i+1}, \cdots, L_j)$$

and

$$\tilde{p}' = (p_{0k}, p_{(k+1)}^{(i-1)}, \cdots, p_{j-1}^{i-1}, \cdots, p_{10}), \quad \tilde{p}'' = (p_{ij}, p_{(j-1)}^{i-1}, \cdots, p_{(i+1)}^{j-1})$$

for some $p_{ij} = p_{ji} \in L_i \cap L_j$.

Then $B_1, B_2$ are $E'$, $E''$ admissible, respectively.

**Proof.** For simplicity of notations, we only consider the case $k = 3$, $i = 1$, $j = 3$. Let $u_n \in \mathcal{M}(\xi, \tilde{p}; B)$ which converges to $u_\infty = (u_{\infty, 1}, u_{\infty, 2})$ where

$$u_{\infty, 1} \in \mathcal{M}((L_0, L_1, \ell_1), (p_{03}, p_{31}, p_{10}); B_1), \quad u_{\infty, 2} \in \mathcal{M}((L_1, L_2, L_3), (p_{13}, p_{32}, p_{21}); B_2),$$

and $p_{13} = p_{31}$. By definition of $E$-admissibility of $B$, there exist homotopy classes $[w_{i+1}^0] \in \pi_2(L_{i+1}; \ell_{i+1})$ for $0 \leq i \leq 3$ such that $B = [w_{01}^0, w_{12}^0, w_{23}^0, w_{30}^0]$. 
To prove the required admissibility of $B_1$, $B_2$, we need to prove the existence of homotopy classes $[w_{13}] \in \pi_2(\ell_{13}; p_{13})$ and $[w_{31}] \in \pi_2(\ell_{31}; p_{31})$ such that

$$[w_{01} # w_{13} # w_{30}] = B_1, \quad [w_{12} # w_{23} # w_{31}] = B_2$$

and $[w_{13} # w_{31}] = [\hat{p}_{13}, p_{13}]$ where $\hat{p}_{13}$ is the constant map to $p_{13}$. In fact, we will select both homotopy classes to be that of $\hat{p}_{13} = \hat{p}_{31}$.

Exploiting $\mathcal{E}$-admissibility of $B$, we can take the sequences of points $x_a \in u_a(D^2) \subset M$, of paths $\gamma_a : [0, 1] \to M$ and $\gamma_{a,i} : [0, 1] \to u_a(D^2)$ such that

$$\gamma_a(0) = y, \quad \gamma_a(1) = x_a,$$

$$\gamma_{a,i}(0) = x_a, \quad \gamma_{a,i}(1) \in L_i$$

and that $\gamma_a \ast \gamma_{a,i}$ is homotopic to the given anchor $\gamma_i$.

Deforming these choices further, we may use the convergence hypothesis to achieve the following additional properties of $x_a$, $\gamma_a$ and $\gamma_{a,i}$'s:

1. $\lim_{a \to \infty} x_a = p_{13}$,
2. $\lim_{a \to \infty} \gamma_{a,i}(t) \equiv p_{13}$ for $i = 1, 3$,
3. $\gamma_{a,i}$ converges to a path $\gamma_{\infty,i}$ as $a \to \infty$,
4. $\gamma_a$ converges to a path $\gamma_{\infty}$ as $a \to \infty$.

From this and by construction of $\gamma_i$, $\gamma_{a,i}$, it is easy to see that $B_1$ and $B_2$ are $((L_0, \gamma_{\infty,0}), (L_1, \gamma_{\infty}), (L_3, \gamma_{\infty}))$ admissible and $((L_1, \gamma_{\infty}), (L_2, \gamma_{2,\infty}), (L_3, \gamma_{\infty}))$ admissible respectively. In fact, since $\gamma_{\infty}$ is homotopic to $\gamma_i$ for $i = 1, 3$, $\gamma_{\infty,j}$ is homotopic to $\gamma_j$ for $j = 0, 2$, we can express

$$B_1 = [w_{01}^{-} \hat{p}_{13}^{+} w_{30}^{-}], \quad B_2 = [w_{12}^{-} w_{23}^{-} \hat{p}_{31}^{+}]$$

This finishes the proof.

We summarize the above discussion as follows:

**Theorem 8.14.** We can associate an filtered $A_\infty$ category to a symplectic manifold $(M, \omega)$ such that:

1. Its object is $((L, \gamma, \lambda), b, sp)$ where $(L, \gamma, \lambda)$ is a graded anchored Lagrangian submanifold, $[b] \in \mathcal{M}(CF(L))$ is a bounding cochain and $sp$ is a spin structure of $L$,
2. The set of morphisms is $CF((L_1, \gamma_1), (L_0, \gamma_0))$.
3. $m_k^b$ are the operations defined in [8.12].

**Remark 8.15.** In Situation 8.13 beside the choices spelled out in $((L, \gamma, \lambda), b, sp)$, the choice of orientations $\sigma_p$ of $\partial \nu_{\lambda_p}$ is included. This choice in fact does not affect the module structure $CF((L_1, \gamma_1), (L_0, \gamma_0))$ up to isomorphism: if we take an alternative choice $\sigma'_p$ at $p$, then all the signs appearing in the operations $m_k^b$ that involves $[p, w]$ for some $w$ will be reversed. Therefore $[p, w] \mapsto -[p, w]$ gives the required isomorphism.

**Remark 8.16.** In [Fu2], the filtered $A_\infty$ category is defined over $\Lambda_{0, nov}$. The situation of Theorem 8.14 is slightly different in that $\Lambda(L_0, L_1; \ell_0)$ or $\Lambda(L)$ are used as the coefficient rings and hence the coefficient rings vary depending on the objects involved. It is easy to see that the notion of filtered $A_\infty$ category can be generalized to this context.
We can also change the coordinate ring to $\Lambda_{nov}$ by using the map $[p, w] \mapsto T \omega_{e^{(p, w)/2}}$. The resulting filtered $A_\infty$ category is still different from the non-anchored version in the case $M$ is not simply connected.

**Remark 8.17.** In Theorem 8.14, we assume that our Lagrangian submanifold $L$ is spin. We can slightly modify the construction to accommodate the relatively spin case as follows: We will construct the filtered $A_\infty$ category of $((M, \omega), st)$ for each choice of $st \in H^2(M; \mathbb{Z}_2)$. Its objects consist of $((L, \gamma, \lambda), b, sp)$ where $L$ satisfies $w_2(L) = i^*(st)$ and $(L, \gamma, \lambda, b)$ are as before. Finally $sp$ is the stable conjugacy class of relative spin structure of $L$. (See Definition 8.1.5 [FOOO08] for its definition.) In this way we obtain a filtered $A_\infty$ category. The same remark applies to the non-anchored case.

The operations $m_k$ are compatible with the filtration. Namely we have

**Proposition 8.18.** If $x_i \in F^\lambda CF((L_i, \gamma_i), (L_{i-1}, \gamma_{i-1}))$, then

$$m_k^\lambda(x_1, \cdots, x_1) \in F^\lambda CF((L_k, \gamma_k), (L_0, \gamma_0))$$

where $\lambda = \sum_{i=1}^k \lambda_i$.

Studying the behavior of filtration under the $A_\infty$ operations, one can define (higher-order) spectral invariants of Lagrangian Floer theory in a way similar to the one carried out in [Oh2]. Then Proposition 8.18 implies a similar estimate as Theorem I(4) [Oh2]. This is a subject of future study.

9. Comparison between anchored and non-anchored versions

The anchored Lagrangian Floer theory presented in this paper is somewhat different from the one developed in [FOOO00] [FOOO06] (for one and two Lagrangian submanifolds) [Fu2] (for 3 or more Lagrangian submanifolds) in several points. In this section we examine their relationship and make some comments on some aspects of their applications.

We first point out the following obvious fact:

**Proposition 9.1.** If $M$ is simply connected the anchored version of Floer homology is isomorphic to non-anchored version together with all of its multiplicative structures.

We also remark that the way how we treat the orientation for the anchored version in Section 4 is actually the same as the one used for the non-anchored version in [FOOO06] and [Fu2].

9.1. Examples. We start with simple examples that illustrate some difference between the two.

We consider the symplectic manifold $(T^2, dx \wedge dy)$, where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $(x, y)$ is the standard coordinate of $\mathbb{R}^2$. Let $L_0 = \{[x, 0] \mid x \in \mathbb{R}\}, L_1 = \{[x, 3x] \mid x \in \mathbb{R}\}$.

$$L_0 \cap L_1 = \{(0, 0), [1/3, 0], [2/3, 0]\}.$$  

Let $[1/2, 0]$ be the base point and take anchors 

$$\gamma_0(t) = [(1 - t)/2, 0], \quad \gamma_1^0(t) = [(1 - t)/2, 0],$$

of $L_0$ and $L_1$ respectively. It is easy to see that $[0, 0]$ is $((L_0, \gamma_0), (L_1, \gamma_1^0))$ admissible. It is also easy to see by drawing pictures that $[1/3, 0], [2/3, 0]$ are not
((L₀, γ₀), (L₁, γ₁)) admissible. The set of homotopy classes of anchors of L₁ is identified with \( \mathbb{Z} \) and \([k/3, 0]\) is \(((L₀, γ₀), (L₁, γ₁))\) admissible if \( k \equiv \ell \mod 3 \).

Here anchor \( γ_1^i \) is a concatenation of \( γ_0^i \) and \( t \mapsto [it/3, 0] \). We also remark that \( Π(L₀), Π(L₁), Π(L₀, L₁) \) are trivial. Therefore \( Λ(L₀) = Λ(L₁) = Λ(L₀, L₁) = \mathbb{Q} \). Thus we have

\[
HF((L₁, γ₁^i), (L₀, γ₀)) \cong \mathbb{Q}
\]

for any choice of anchor \( γ₁^i \).

**Remark 9.2.** In this subsection, we always take 0 as the bonding cochain \( b \) and we omit it from the notation of Floer cohomology.

On the other hand we have

\[
HF(L₁, L₀; Λ_{nov}) \cong Λ_{nov}^{3y}.
\]

It is easy to see that \( π₀(Ω(L₀, L₁)) \) consists of 3 elements, which we denote \( ℓ_{i₀} \) \((i = 0, 1, 2)\). Moreover \([Σ₀ * γ₁^i] = ℓ_{i₀} \) with \( i \equiv j \mod 3 \).

Hence we have the decomposition

\[
HF(L₁, L₀; Λ_{nov}) \cong \bigoplus_{i=0}^{2} HF(L₁, L₀; ℓ_{i₀}^i; Λ_{nov}).
\]

This is the decomposition given in Remark 3.7.46 [FOOO08]. (We note that we have the isomorphism

\[
HF(L₁, L₀; ℓ_{i₀}^i; Λ_{nov}) \cong HF((L₁, γ₁^i), (L₀, γ₀)) \otimes Λ_{nov}.
\]

We next consider the same \( T² \) and

\[
L₀ = \{[0, y] \mid y ∈ \mathbb{R}\}, \quad L₁ = \{[x, 0] \mid x ∈ \mathbb{R}\}.
\]

Then \( L₀ \cap L₁ \) consists of one point \([0, 0]\). Therefore

\[
HF((L₁, γ₁), (L₀, γ₀)) \cong \mathbb{Q}
\]

for any anchor \( γ₀ \) and \( γ₁ \). In fact \( Ω(L₀, L₁) \) is connected in this case. We next consider the third Lagrangian submanifold \( L₂ = \{[x, −x] \mid x ∈ \mathbb{R}\} \). It is also easy to see that

\[
HF(L₂, L_i; Λ_{nov}) \cong HF((L₂, γ₂), (L_i, γ_i)) \otimes Λ_{nov} \cong Λ_{nov},
\]

for \( i = 0, 1 \) and any anchor \( γ_j \) of \( L_j \).

We take \([0, 0]\) as base point and take anchors

\[
γ₀^k(t) = [kt, 0], \quad γ₁^k(t) = [0, kt], \quad γ₁^k(t) = [kt/2, kt/2].
\]

of \( L_i \) for \( i = 0, 1, 2 \). For each \( k, ℓ ∈ \mathbb{Z} \) and \( i, j \) \((i, j ∈ \{0, 1, 2\}, i ≠ j)\) we have \( HF((L_i, γ_i^k), (L_j, γ_j^ℓ)) \cong \mathbb{Q} \). Let \( x_{i,j}^{k,ℓ} = [0, 0], w_{i,j,ℓ} \) be its canonical generator. Here \([w_{i,j,ℓ}]\) represents the unique element of \( π₂(γ_i^k * γ_j^ℓ, [0, 0]) \).

Let \( x_{i,i} = ([0, 0], \mathbb{R}) \) be also the canonical generator of the (non-anchored) Floer homology \( HF(L_i, L_i) \). (We refer readers to Section 10.2 of present paper and Subsection 5.1.3 [FOOO08] for the definition of \([0, 0]\).)

Now the product \( m_2 \) is described as follows:
Proposition 9.3. In the case of non-anchored version we have

\[ m_2(x_{21}, x_{10}) = \left( \sum_{k \in \mathbb{Z}} T^{k^2/2} \right) x_{20}. \]  

In the anchored version we have

\[ m_2(x_{m^k}, x_{\ell^k}) = x_{m^k}. \]

Proof. We first remark that \( \pi_2((L_0, L_1, L_2), (p_{02}, p_{21}, p_{10})) \cong \mathbb{Z} \). Moreover each of the homotopy class is realized by holomorphic disc uniquely. This implies (9.1).

To prove (9.2), it suffices to see that for each \( \gamma_0^k, \gamma_2^\ell \) the set

\[ \pi_2^{adm}((L_0, \gamma_0^k), (L_1, \gamma_1^\ell), (L_2, \gamma_2^m), (p_{02}, p_{21}, p_{10})) \]

of admissible class consists of one element. We will prove it below.

Let \( B \) be an element of (9.3). We write \( B = [w_{01}] \# [w_{12}] \# [w_{20}] \) as in Definition 9.2. Let \( \mathbb{R}^2 \to T^2 \) be the universal covering. We lift anchors \( \gamma_0^k, \gamma_1^\ell, \gamma_2^m \) to \( \tilde{\gamma}_0^k, \tilde{\gamma}_1^\ell, \tilde{\gamma}_2^m \) such that \( \tilde{\gamma}_0^k(0) = \tilde{\gamma}_1^\ell(0) = \tilde{\gamma}_2^m(0) = 0 \).

We then lift \( w_{01} \) such that (a part of) its boundary is \( \tilde{\gamma}_0^k \) and \( \tilde{\gamma}_1^\ell \). We lift \( w_{12} \) and \( w_{20} \) in a similar way. We thus obtain a lift \( \tilde{w} \) of \( w \). It is easy to see that the boundary of \( \tilde{w}(D^2) \) is contained in \( \tilde{L}_0^k \cup \tilde{L}_1^\ell \cup \tilde{L}_2^m \) where

\[ \tilde{L}_0^k = \{(k, y) \mid y \in \mathbb{R}\}, \quad \tilde{L}_1^\ell = \{(x, \ell) \mid x \in \mathbb{R}\}, \quad \tilde{L}_2^m = \{(x, m-x) \mid x \in \mathbb{R}\}. \]

Thus the admissible homotopy class of \( B \) is unique.

Remark 9.4. We remark that (9.1) is the formula appearing in Kontsevich [Ko] where the homological mirror symmetry proposal first appeared. So it seems that the anchored version is not suitable for the application to mirror symmetry, when \( M \) is not simply connected. On the other hand, the anchored version is more closely related to the variational theoretical origin of Floer homology and so seems more suitable to study spectral invariant for example.

The above proof also implies the following:

Lemma 9.5. If \( B \) is \( (L_0, \gamma_0^k), (L_1, \gamma_1^\ell), (L_2, \gamma_2^m) \) admissible then

\[ B \cap \omega = \frac{(m - k - \ell)^2}{2}. \]

Remark 9.6. In the case of Lemma 9.3 we obtain the non-anchored version by summing up anchored versions appropriately. In general the non-anchored version is an appropriate sum of anchored versions. However the way of summing up anchored versions to obtain the non-anchored one does not look so simple to describe.

9.2. Relationship with the grading of Lagrangian submanifolds. In [Fu2] the first named author followed the method of Seidel [Se1] (and Kontsevich) to define a grading of Floer cohomology. In this section we discuss its relation to the formulation of Section 5.

We first briefly recall the notion of gradings in the sense of [Se1]. Consider the tangent space \( T_p M \) and let \( Lag^+(T_p M) \) be the set of oriented Lagrangian subspaces. The union \( Lag^+(M) := \cup_{p \in M} Lag^+(T_p M) \) forms a fiber bundle over \( M \). If \( L \) is an oriented Lagrangian submanifold, the Gauss map \( p \mapsto T_p L \) provides a canonical section of the restriction \( Lag^+(M)|_L \to L \). We denote the canonical section by \( \pi_L \).
We first consider the case \((M, \omega)\) with \(c^1(M) = 0\).

The fundamental group of \(\text{Lag}^+(T_p M)\) is \(\mathbb{Z}\). The condition \(c^1(M) = 0\) is equivalent to the condition that there exists a (global) \(\mathbb{Z}\) fold covering \(\text{Lag}(M)\) of \(\text{Lag}^+(M)\), which restricts to the universal covering on each fiber \(\text{Lag}^+(T_p M)\). (See Lemma 2.6 [Fu2].)

The section \(s_L\) lifts to a section \(\tilde{s}\) of \(\text{Lag}(M)|_L\) if and only if the Maslov class \(\mu_L \in H^1(L; \mathbb{Z})\) of \(L\) is zero. (Recall if \(c^1(M) = 0\), then the Maslov class \(\mu_L\) is well-defined.) For each Lagrangian submanifold \(L\) with \(\mu_L = 0\) a lift \(\tilde{s}\) of \(s_L\) is said to be a grading of \(L\). The pair \((L, \tilde{s})\) of Lagrangian submanifold \(L\) and grading \(\tilde{s}\) is called a graded Lagrangian submanifold.

Let \((\lambda, \tilde{s}_{\lambda})\) be a graded Lagrangian submanifold. Then for \(p \in L_0 \cap L_1\) we consider any path \(\lambda\) from \(\tilde{s}_0(p)\) to \(\tilde{s}_1(p)\) in \(\text{Lag}(T_p M)\) and denote its projection to \(\text{Lag}(T_p M)\) by \(\lambda\). Then we compute the intersection number of \(\lambda\) with the Maslov cycle \(\text{Lag}(T_p M; T_p L_0)\) (relative to \(T_p L_0\)) to define a degree \(\text{deg} p \in \mathbb{Z}\) for each \(p \in L_0 \cap L_1\). This definition is independent of the choice of \(\lambda\) with \(\tilde{\lambda}(0) = \tilde{s}_0(p)\), \(\tilde{\lambda}(1) = \tilde{s}_1(p)\). (See [Sc1], [Fu2] for the details.)

Now we explain how the grading \(\lambda\) of \((L, \gamma)\) and the grading \(\tilde{s}\) of \(L\) are related to each other. For this purpose, we fix, once and for all, an element \(V_\gamma\) of \(\text{Lag}(T_y M)\) which projects to \(V_\gamma \in \text{Lag}(T_y M)\) at the base point \(y\) in Definition 5.4.

First, we go from \(\tilde{s}\) to \(\lambda\). We consider any anchored Lagrangian submanifold \((L, \gamma)\) with \(\mu_L = 0\). Let \(\tilde{s}\) be a grading of \(L\). We take a section of \(\tilde{\lambda}_{\gamma}\) of the pull-back \(\gamma^* \text{Lag}(TM) \to [0, 1]\) such that

\[
\tilde{\lambda}_{\gamma}(0) = V_\gamma, \quad \tilde{\lambda}_{\gamma}(1) = s_i(\lambda_i(1)).
\]

Such path is unique up to homotopy because \([0, 1]\) is contractible and so \(\gamma^* \text{Lag}(TM)\) is simply connected. We push it out and obtain a section \(\lambda\) in \(\gamma^* \text{Lag}(TM)\). In this way, a graded Lagrangian submanifold \((L, \tilde{s})\) canonically determines a grading \(\lambda\) of an anchored Lagrangian submanifold \((L, \gamma)\). Namely \((\gamma, \lambda)\) becomes a graded anchor of \(L\) in the sense of Definition 5.4.

We remark that the path \(\lambda_{01}\) induced by these graded anchors lifts to \(\tilde{\lambda}_{01}\) joining \(s_0(\ell_{01}(0))\) to \(s_1(\ell_{01}(1))\).

We then define \(\mu([p, w])\) using this path \(\lambda_{01}\) as in Section 3.

**Lemma 9.7.** \(\mu([p, w])\) is independent of \(w\). Moreover we have

\[
(9.5) \quad \mu([p, w]) = \text{deg}(p).
\]

**Proof.** Independence of the degree of \(w\) is a consequence of our assumption that Maslov index of \(L_0, L_1\) are zero. Then the equality \((9.5)\) follows easily by comparing the definitions. We omit the detail. \(\square\)

Thus the degree of [Fu2] and of this paper coincides under the assumption that Maslov index is 0.

Now we go from \(\lambda\) to \(\tilde{\lambda}\). For any given grading \(\lambda\) of \((L, \gamma)\), we lift \(\lambda\) to a section of \(\gamma^* \text{Lag}(TM)\) so that \(\tilde{\lambda}(0) = V_\gamma\). Then \(\tilde{\lambda}(1)\) is a lifting of \(\lambda(1)\) in \(\text{Lag}(T_1 M)\). Since the lifting of \(\tilde{\lambda}\) of \(\lambda\) is homotopically unique, \(\tilde{\lambda}(1)\) depends only on \((L, \gamma)\) and the fixed \(V_\gamma\). Therefore if \(\mu_L = 0\), then this determines a unique grading \(\tilde{s}\) of \(L\) with \(\tilde{s}(\gamma(1)) = \tilde{\lambda}(1)\).
The above discussion can be generalized to the case of $\mathbb{Z}_{2N}$-grading where the Maslov index is divisible by $2N$ for some positive integer $N$ rather than being zero. We leave this discussion to the readers.

### 10. Reduction of the coefficient ring and Galois symmetry

In this section, we study a reduction of the coefficient ring $\Lambda_{nov}$ to the subring $\Lambda_{rat,nov}$ or to the ring $\mathbb{Q}[[T^{1/N}]][[T^{-1}]]$.

**Definition 10.1.** We put

$$\Lambda_{nov}^{rat} = \left\{ \sum_{i=1}^{\infty} T^{\lambda_i} e^{\mu_i/2} a_i \in \Lambda_{nov} \mid \lambda_i \in \mathbb{Q} \right\}$$

We also define $\Lambda_{0,nov}^{rat}$ in a similar way.

This problem was studied by the first named author in [Fu3] in relation to the Galois symmetry of Floer cohomology over rational symplectic manifolds. Theorem 2.4 in [Fu3] is Theorem 1.2 of the present paper. Its proof was given in [Fu3] as far as $m_k (k = 0, 1)$ concerns. The case for $k \geq 2$ was left to the reader in [Fu3]. In this section we give the detail of the discussion for the case $k \geq 2$.

### 10.1. Rational versus BS-rational Lagrangian submanifolds

In this subsection, we first clarify somewhat confusing usages of the terminology ‘rational’ Lagrangian submanifolds in the literature (e.g. in [Oh1], [Fu3] etc.).

First we assume that there exists an integer $m_{amb}$ with $m_{amb}\omega \in H^2(M; \mathbb{Z})$, i.e., $(M, m_{amb}\omega)$ is integral or pre-quantizable. Then we choose a complex line bundle $\mathcal{P}$ with a unitary connection $\nabla$ such that its curvature $F_\nabla$ satisfies

$$F_\nabla = 2\pi \sqrt{-1} m_{amb}\omega.$$

The pair $(\mathcal{P}, \nabla)$ is called a pre-quantum bundle of $(M, m_{amb}\omega)$. We note that the connection is flat on any Lagrangian submanifold by (10.1).

**Definition 10.2.** We say that a Lagrangian submanifold $L$ is Bohr-Sommerfeld $m$-rational or simply BS-$m$-rational if the image of the holonomy group $(\mathcal{P}|_L, \nabla|_L)$ is contained in $\{ \exp(2\pi \sqrt{-1} km_{amb}/m) \mid k \in \mathbb{Z} \}$. We say $L$ is Bohr-Sommerfeld rational or simply BS-rational if it is BS-rational for some $m$. We denote the smallest such integer by $m_L$.

When $(\mathcal{P}, \nabla)$ is trivial on $L$ and $m_{amb} = 1$, we call $L$ a Bohr-Sommerfeld orbit.

By definition, it is easy to see $m_{amb}|m_L$ for any BS-rational Lagrangian submanifold $L$.

**Remark 10.3.** In [Oh1] and [Fu3], the corresponding notion is called cyclic and just rational respectively. In this paper, we adopt the name Bohr-Sommerfeld rational which properly reflects the kind of rationality of holonomy group of the quantum line bundle $(\mathcal{P}, \nabla)$ relative to the Lagrangian submanifold.

A Lagrangian submanifold is often called (spherically) rational in literature when $\{ \omega(\pi_2(M,L)) \} \subset \mathbb{R}$ is discrete. This is related to but not exactly the same as the BS-rationality in the above definition.

**Definition 10.4.** Let $(M, \omega)$ be rational. We say that a Lagrangian submanifold is rational if $\Gamma_\omega(L) := \{ \omega(\alpha) \mid \alpha \in \pi_2(M,L) \} \subset \mathbb{R}$ is discrete.
The following lemma shows the relationship between the BS rationality and the rationality.

**Lemma 10.5.** If $L$ is BS m-rational, then $L$ is rational. Moreover $\Gamma_\omega(L) \subseteq \{\exp(2\pi\sqrt{-1}km_{amb}/m) \mid k \in \mathbb{Z}\}$.

The converse does not hold in general when $L$ is not simply connected. In fact for the case $M = (T^2, dx \wedge dy)$ and $L_t = \{(t, y) \mid y \in \mathbb{R}\}$, (Here we regard $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.) every $L_t$ is rational but only countably many of $L_t$’s are BS-rational. (The question on which $T^2$ is rational depends on the choice of pre-quantum bundle $(\mathcal{P}, \nabla)$.) It is easy to see that we may choose the pre-quantum bundle so that $L_t$ is BS-rational if and only if $t \in \mathbb{Q}$.

Using Lemma 10.5 it is easy to show that the coefficient ring of the filtered $A_\infty$ algebra $C(L; \Lambda_{0, nov})$ can be reduced to the ring $\mathbb{Q}[[T^{1/N}]](\epsilon, \epsilon^{-1}) \subseteq \Lambda_{0,nov}^{rat}$. In particular, if we define

\begin{equation}
C(L; \Lambda_{0,nov}^{rat}) = C(L; \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_{0,nov}^{rat} \subset C(L; \Lambda_{0,nov})
\end{equation}

the operations $m_k$ induce a filtered $A_\infty$ structure on $C(L; \Lambda_{0,nov}^{rat})$.

**10.2. Reduction of the coefficient ring: non-anchored version.** In this subsection, we explain the way to reduce the coefficient ring of the filtered $A_\infty$ category associated to a symplectic manifold to the ring $\Lambda_{0,nov}^{rat}$. Let $(M, \omega)$ be a symplectic manifold with $m_{amb} \omega \in H^2(M; \mathbb{Z})$. We fix a pre-quantum bundle $(\mathcal{P}, \nabla)$ of $(M, m_{amb} \omega)$.

We fix any integer $N \in \mathbb{Z}_{+}$ and consider the set of BS $N$-rational Lagrangian submanifolds.

**Definition 10.6.** The $N$-rationalization of $L$ (in $(M, \omega)$, $(\mathcal{P}, \nabla)$) is a global section $S_L$ of $\mathcal{P}^\otimes N/m_{amb}$ such that

$$\|S_L\| \equiv 1, \quad \nabla^\otimes N/m_{amb}S_L = 0.$$ 

The following lemma is easy to show.

**Lemma 10.7.** $L$ is BS $N$-rational if and only if it has an $N$-rationalization.

Let $L_i$, $i = 0, 1$ be a pair of $N$ BS-rational Lagrangian submanifolds. Let $S_{L_i}$ be $N$-rationalizations $L_i$. For each $p \in L_0 \cap L_1$, we define $c(p)$ to be the smallest nonnegative real number such that

\begin{equation}
\exp(2\pi Nc(p)\sqrt{-1}/m_{amb})S_{L_0}(p) = S_{L_1}(p).
\end{equation}

**Proposition 10.8.** Let length($\mathcal{L}$) = $k + 1 \geq 2$. Define

\begin{equation}
E'(B) := \int_B \omega - \sum_{i=0}^{k} c(p_{i+1}i)
\end{equation}

for $B \in \pi_2(\mathcal{L}; \tilde{p})$. Then $E'$ has values in $\mathbb{Z}[1/N]$ and satisfies the gluing rule $E'(B) = E'(B_1) + E'(B_2)$ whenever $B = B_1 \# B_2$ in the sense of Lemma 8.13.

**Proof.** Let $w \in C^\infty(\partial D^2; \mathcal{L}; \tilde{p})$ be a map such that $[w] = B$. We consider the pull back bundle $w^*\mathcal{P}^\otimes N/m_{amb}$. Let $\gamma : [0, 1] \rightarrow \partial D^2$ be the map $t \mapsto e^{2\pi\sqrt{-1}t}$. Using $S_L$, we can construct a section $s$ on $\gamma^*w^*\mathcal{P}^\otimes N/m_{amb}$ such that

$$\nabla s = 0, \quad s(1) = \exp(2\pi Nc(p)\sqrt{-1}/m_{amb})s(0).$$
Using the fact that the curvature of $\mathcal{P}^{\otimes N/m_{\text{amb}}}$ is $2N\pi \omega$, we conclude $E'(B) \in \mathbb{Z}[1/N]$. The gluing rule for $E'$ is obvious from its definition (10.10).

We put
\[
[p]\ell_0 = T^{-f} w^\omega + c(p)[p,\omega]
\in CF(L_1, L_0; \ell_0) \otimes \Lambda_{L_1, L_0; \ell_0} \Lambda_{\text{nov}} \subset C(L_1, L_0; \Lambda_{\text{nov}}).
\]

**Lemma 10.9.** If we change the choice of the base point $\ell_0$, then there exists $k \in \mathbb{Z}$ such that $\langle [p]\ell_0 \rangle = e^k \langle [p]\ell'_0 \rangle$ where $e$ is the formal parameter encoding the degree.

The proof of the lemma is easy and left to the reader.

This lemma together with the discussion on the degree in the last section shows that the role of the choice of the base point $\ell_0$ is to fix a connected component of $\Omega(L_0, L_1)$ and does not play an essential role in Lagrangian Floer theory.

**Definition 10.10.** We consider the $\Lambda^{\text{rat}}_{\text{nov}}$-submodule of $C(L_1, L_0; \Lambda_{\text{nov}})$ generated by $\langle [p]\rangle$ ($p \in L_0 \cap L_1$) and denote it by $C(L_1, L_0; \Lambda^{\text{rat}}_{\text{nov}})$. We define the module $C(L_1, L_0; \mathbb{Q}[T^{1/N}][T^{-1}][e, e^{-1}])$ in the same way.

**Remark 10.11.** (1) We note that the $\Lambda^{\text{rat}}_{\text{nov}}$-submodule $C(L_1, L_0; \Lambda^{\text{rat}}_{\text{nov}})$ of $C(L_1, L_0; \Lambda_{\text{nov}})$ depends on the choice of the rationalizations $S_L$. We omit them from notation however.

(2) In Subsection 5.1.3 [FOOO08] we put
\[
\langle p \rangle = T^{-f} w^\omega [p, w].
\]

Then $C(L_1, L_0; \Lambda_{\text{nov}})$ is a free $\Lambda_{\text{nov}}$ module over the basis $\{ \langle p \rangle \mid p \in L_1 \cap L_0 \}$. The difference of $\langle p \rangle$ from the present basis $\langle [p]\rangle$ is $T^{-c(p)}$. Namely
\[
\langle p \rangle = T^{-c(p)} \langle [p] \rangle
\]

$\langle p \rangle$ coincides with the identity in $\text{Hom}(L_p, L_p)$ which appeared in (2.30) [Fr2] and is used there to construct a filtered $A_\infty$ category.

We now consider the operator
\[
m_{k_{m_0, \ldots, m_k}} : B_{m_k}(C(L_k; \Lambda_{0, \text{nov}})) \otimes C(L_k, L_{k-1}; \Lambda_{\text{nov}}) \otimes \cdots
\]
\[
\otimes C(L_1, L_0; \Lambda_{\text{nov}}) \otimes B_{m_0}(C(L_0; \Lambda_{0, \text{nov}})) \to C(L_k, L_0; \Lambda_{\text{nov}})
\]

**Proposition 10.12.** The image of
\[
B_{m_k}(C(L_k; \Lambda^{\text{rat}}_{0, \text{nov}})) \otimes C(L_k, L_{k-1}; \Lambda^{\text{rat}}_{\text{nov}}) \otimes \cdots
\]
\[
\otimes C(L_1, L_0; \Lambda^{\text{rat}}_{0, \text{nov}}) \otimes B_{m_0}(C(L_0; \Lambda^{\text{rat}}_{0, \text{nov}}))
\]

by $m_{k_{m_0, \ldots, m_k}}$ is in $C(L_k, L_0; \Lambda^{\text{rat}}_{\text{nov}})$. The same conclusion holds for $\mathbb{Q}[T^{1/N}][T^{-1}][e, e^{-1}]$.

**Proof.** Let $B \in \pi_2(\mathcal{L}, \vec{p})$ and $M(\mathcal{L}, \vec{p}; B)$ be as in Section 7.

For simplicity, we will prove the proposition for the case $m_0 = \cdots = m_k = 0$. Let $\langle [p] \rangle = T^{-f} w^\omega [p, w]$. By (8.14), we have
\[
\langle m_k(\langle [p_{k-1}] \rangle, \cdots, \langle [p_0] \rangle) \rangle = \sum_{B \in \pi_2(\mathcal{L}, \vec{p})} T^{B \cdot \omega} e^{\mu(B)/2} \# M(\mathcal{L}, \vec{p}; B).
\]
Here the left hand side denotes the \( \langle p_{k0} \rangle \)-coefficient of \( m_k(\langle p_{k(k-1)} \rangle, \cdots , \langle p_{10} \rangle) \). Therefore by (10.5) we have the matrix coefficients

\[
\langle m_k([p_{k(k-1)}]), \cdots , [p_{10}]), [p_{k0}] \rangle
\]

\[= \sum_{B \in \pi_2(\mathcal{L}, 0)} T^{B \cap \omega - \sum_{i=0}^{k} c(p_{i+1}), \epsilon(B)/2} \# M(\mathcal{L}, \mathfrak{p}, B)
\]

for \( k \geq 1 \). Since \( B \cap \omega - \sum_{i=0}^{k} c(p_{i+1}) = E'(B) \) is rational by Proposition 10.8 the right hand side of (10.7) lies in \( C(\mathcal{L}_k, \mathcal{L}_0; A^\text{rat}_{nov}) \) as required.

Now we are ready to wrap up the proof of Theorem 1.2. By the assumption \( c_1(M) = 0 \) and vanishing of Maslov indices of Lagrangian submanifolds, all Lagrangian submanifolds in the discussion below carry a grading \( \tilde{s} \). We just denote \( s \) for \( \tilde{s} \) below to simplify the notation.

For each given \( N \), with \( m_{amb}/N \), we construct a filtered \( A_\infty \) category over \( \mathbb{Q}[[T^{1/N}]]/[T^{-1}] \). Its object is \( (L, sp, b, s, S_L) \) where \( L \) is a BS \( N \)-rational Lagrangian submanifold \( sp \) its spin structure, \( s \) a grading, \( b \) is a bounding cochain, and \( S_L \) is \( N \)-rationalization. We assume that \( b \in C^1(L; \mathbb{Q}[[T^{1/N}]]). \)

For two such objects we obtain a \( \mathbb{Q}[[T^{1/N}]]/[T^{-1}] \) module

\[ C(L_1, L_0; \mathbb{Q}[[T^{1/N}]]/[T^{-1}]). \]

By Proposition 10.12 the operation \( m^L \) is defined over this \( \mathbb{Q}[[T^{1/N}]]/[T^{-1}] \).

We thus obtained a filtered \( A_\infty \) category over \( \mathbb{Q}[[T^{1/N}]]/[T^{-1}] \), which we denote by

\[ \mathcal{F}uk_N(M, \omega). \]

To include all the BS-rational Lagrangian submanifolds and obtain a filtered \( A_\infty \) category over \( A^\text{rat}_{nov}(0) \) we proceed as follows. Let \( L_0, L_1 \) be Lagrangian submanifolds which are \( m_0 \)-BS rational and \( m_1 \)-BS rational, respectively. We take there \( m_0 \) (resp. \( m_1 \)) rationalization \( S_{L_0} \) (resp. \( S_{L_0} \)). Take any \( N \) such that \( m_0, m_1 \mid N. \) \( S_{L_0} \) (resp. \( S_{L_0} \)) induce an N rationalization \( S^N_{L_0} \) (resp. \( S^N_{L_0} \)) in an obvious way.

(Namely \( S^N_{L_0} = (S_{L_0})^\otimes N/m_0. \))

For \( p \in L_0 \cap L_1 \), we use (10.3) to obtain \( c(p) \). To make \( N \)-dependence of \( c(p) \) explicit, we write \( c_N(p) \) for \( c(p) \). Then for each given \( (N, N') \) with \( |N|/N' \), we have \( N' c_N(p) - c_{N'}(p) =: \Delta(p) \in \mathbb{Z}_{\geq 0}. \) We put

\[ c_{N'}(p) = c_N(p) - \frac{\Delta(p)}{N'}. \]

We write \([p]_N \) and \([p]_{N'} \) to distinguish the generators of Floer chain complex over \( \mathbb{Q}[[T^{1/N}]]/[T^{-1}] \) and over \( \mathbb{Q}[[T^{1/N'}]]/[T^{-1}] \). We consider the map

\[ [p]_N \mapsto T^{-1/\Delta(p)} [p]_{N'}, \]

that induces an isomorphism

\[ C(L_1, L_0; \mathbb{Q}[[T^{1/N}]]/[T^{-1}]) \otimes_{\mathbb{Q}[[T^{1/N}]]} \mathbb{Q}[[T^{1/N'}]]/[T^{-1}] \]

\[ \rightarrow C(L_1, L_0; \mathbb{Q}[[T^{1/N'}]]/[T^{-1}]) \]

which respect all the \( A_\infty \) operations.

Therefore the system \( (\mathcal{F}uk_N(M, \omega); <) \) with respect to the partial order ‘\( N < N' \) if and only if \( N \mid N' \) forms an inductive system. We define the \( A_\infty \)-category \( \mathcal{F}uk^\text{rat}(M, \omega) \) to be the associated inductive limit.
Now let \( \widehat{\mathbb{Z}} \) be the profinite completion of \( \mathbb{Z} \). As in \cite{Fu3}, we will define an action of \( \widehat{\mathbb{Z}} \) on \( \mathcal{F}_{\text{Fuk}}(M, \omega) \). To define a \( \widehat{\mathbb{Z}} \) action we need to include a flat line bundle \( \mathcal{L} \) over \( L \) and take \( R = \mathbb{C} \) in place of \( R = \mathbb{Q} \). Namely we take \((L, \mathcal{L}, sp, b, s, S_L)\) where \((L, sp, b, s, S_L)\) is as before and \( \mathcal{L} \) is a flat \( U(1) \) bundle over \( L \). We say \( \mathcal{L} \) is \( N \)-rational if the image of the holonomy representation \( \pi_1(L) \to U(1) \) is contained in \( \{\exp(2\pi \sqrt{-1}k/N) \mid k \in \mathbb{Z}\} \).

Now let \((L_i, \mathcal{L}_i, sp_i, b_i, s_i, S_{L_i})\) be as above such that \( \mathcal{L}_i \) are \( N \)-rational. We put

\[
C((L_1, \mathcal{L}_1), (L_0, \mathcal{L}_0); \mathbb{C}[[T^{1/N}]][[T^{-1}]]) = \bigoplus_{p \in \mathbb{Q} \cap \mathbb{Z}} \mathbb{Q}[[T^{1/N}]][[T^{-1}]][[p]] \otimes_{\mathbb{Q}} \text{Hom}_\mathbb{C}((\mathcal{L}_1)_p, (\mathcal{L}_0)_p).
\]

(10.9)

We then modify operations \( m_k \) by using the holonomy of \( \mathcal{L}_i \): Namely we incorporate the holonomy weight in \( U(1) \) as defined in (3.28) \cite{Fu2} into the right hand side of (10.7). Taking an inductive limit in the same way, we obtain a filtered \( \mathbb{A}_\infty \) category over \( \Lambda_{\text{nov}}^{(0)} \mathbb{C} = \widehat{\Lambda}_{\text{nov}}^{(0)} \otimes_{\mathbb{Q}} \mathbb{C} \).

The \( m_{\text{amb}} \widehat{\mathbb{Z}} \) action on it is defined as follows: Let \( m_{\text{amb}} \in m_{\text{amb}} \mathbb{Z}/(NZ) \) be the standard generator. We define an action on the set of objects by

\[
m_{\text{amb}} \cdot (L, \mathcal{L}, sp, b, s, S_L) = (L, \mathcal{L} \otimes \mathcal{P}_L, sp, b, s, S_L).
\]

Since \( \mathcal{P}^\otimes N/m_{\text{amb}} \) is a trivial bundle on \( L \), this defines an action of \( m_{\text{amb}} \mathbb{Z}/NZ \) on the category over \( \mathbb{C}[[T^{1/N}]][[T^{-1}] \).

**Remark 10.13.** Note the (Galois) action of 1 on \( \mathbb{C}[[T^{1/N}]][[T^{-1}]] \) is

\[T^{1/N} \mapsto \exp(2\pi \sqrt{-1}/N)T^{1/N}.
\]

This action is consistent with the above action, as was shown in \cite{Fu3}.

We then take the inductive limit and obtain an action of \( m_{\text{amb}} \widehat{\mathbb{Z}} \) on the category over \( \Lambda_{\text{nov}}^{(0)} \mathbb{C} \). The proof of Theorem \cite{Fu1} is complete.

**Remark 10.14.** We may take the maximal abelian extension of \( \mathbb{Q} \), that is the field adding all the roots of unity to \( \mathbb{Q} \), in place of \( \mathbb{C} \).

**Remark 10.15.** We use the coefficient ring \( \Lambda_{\text{nov}}^{(0)} \mathbb{C} \) which is the subring of \( \widehat{\Lambda}_{\text{nov}}^{(0)} \mathbb{C} \) consisting of the series not involving the grading parameter \( e \). This is because we include grading \( s \) in the object of our category and so the Floer cohomology has absolute \( \mathbb{Z} \) grading such that all the operations \( m_k \) is of degree 1 (after shifted). (We also choose the bounding cochain \( b \) so that it is degree 1.)

We may consider the \( \mathbb{Z}_N \)-grading instead, then the category is defined over \( \Lambda_{\text{nov}}^{(0)} \mathbb{C} [e^N, e^{-N}] \).

If we take \((L, \mathcal{L}, sp, b, S_L)\) as an object (that is we do not include grading at all) then the category is defined over \( \Lambda_{\text{nov}}^{(0)} \mathbb{C} \).

We note that the \( \widehat{\mathbb{Z}} \) action exists in all of these versions.

**10.3. The reduction of coefficient ring: anchored version.** To see the relation between the construction of the last subsection to the critical value, it is useful to consider anchor. Let \( y \) be the base point of \( M \) we also fix \( V_y \in \text{Log}(T_y M, \omega) \).

We take and fix an element \( S_y \in V_y \) such that \( \|S_y\| = 1 \). Let \((L_i, \gamma_i)\) be anchored
Lagrangian submanifolds. We assume $L_i$ are $N$-rational. Then it is easy to see that there exists a unique $N$-rationalization $S_N$ such that $S_N(\gamma_i(1))$ is a parallel transport of $S_p$ along $\gamma_i$. Using this rationalization we discuss in the same way as the last subsection to obtain a non-anchored version: More specifically, we have the following anchored version of Proposition 10.8.

**Proposition 10.16.** Let $E$ be an anchored Lagrangian chain of length $\geq 2$. Define a map $E'_k : \pi_2^d(E; p) \to \mathbb{R}$

\begin{equation}
E'_k(B) := \int_B \omega - \sum_{i=0}^k c(p_{(i+1)1})
\end{equation}

for $k \geq 2$, and

$$E'_1(\alpha) = \int_\alpha \omega - c(p_{10})$$

for $k = 1$ and $\alpha \in \pi_2(\ell_{01}; p_{01})$. Then $E'_k$ have their values in $\mathbb{Z}[1/N]$ for all $k = 1, \cdots$ and the collection $E' = \{E'_k\}$ defines an abstract index on the collection of $N$-rational anchored Lagrangian submanifolds.

The module of morphisms is

$$CF((L_1, \gamma_1), sp, b_1, \lambda_1), ((L_0, \gamma_0), sp, b_0, \lambda_0)) \otimes_{\Lambda(L_0, L_1; ell_{01})} \Lambda_{rat}$$

and higher products are defined in the same way as before. To define a $\hat{\mathbb{Z}}$ action on the corresponding $A_\infty$ category, we include flat $U(1)$ bundles with finite holonomy on $L_i$ as before.

We remark that we obtain the same filtered $A_\infty$ category as the non-anchored version when $M$ is simply connected. However in general the two are different.

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