LOGARITHMIC EXTENSIONS OF MINIMAL MODELS: CHARACTERS
AND MODULAR TRANSFORMATIONS

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ABSTRACT. We study logarithmic conformal field models that extend the \((p, q)\) Virasoro minimal models. For coprime positive integers \(p\) and \(q\), the model is defined as the kernel of the two minimal-model screening operators. We identify the field content, construct the \(W\)-algebra \(W_{p,q}\) that is the model symmetry (the maximal local algebra in the kernel), describe its irreducible modules, and find their characters. We then derive the \(SL(2, \mathbb{Z})\)-representation on the space of torus amplitudes and study its properties. From the action of the screenings, we also identify the quantum group that is Kazhdan–Lusztig-dual to the logarithmic model.

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1. Introduction

1.1. Logarithmic conformal field theory can be interesting for two reasons at least. The first is their possible applications in condensed-matter systems: the quantum Hall effect [1, 2, 3], self-organized critical phenomena [4, 5], the two-dimensional percolation problem [6, 7, 8], and others (see, e.g., [9] and the references therein). The second is the general category-theory aspects of conformal field theory involving vertex-operator algebras with nonsemisimple representation categories [10, 11] (also see [12] and the references therein). Unfortunately, there are few well-investigated examples in logarithmic conformal field theory. Presently, only the \((2,1)\) model [13, 14, 15, 16, 17, 18] is formulated with sufficient completeness. A number of results exist on the \((p,1)\) models [19, 20, 21, 22], but no model has been investigated so completely as the Ising model, for example.

An essential new feature of nonsemisimple (logarithmic) conformal field theories, in comparison with semisimple (rational) theories, already occurs in constructing the space of states. In the semisimple case, it suffices to take the sum of all irreducible representations in each chiral sector. But in the nonsemisimple case, there are various indecomposable representations, constructed beginning with first extensions of irreducible representations, then taking their further extensions, and so on, ending up with projective modules (the largest possible indecomposable extensions, and hence the modules with the largest Jordan cells, for the scaling dimension operator \(L_0\), that can be constructed for a given set of irreducible representations). The space of states is therefore given by the sum over all nonisomorphic indecomposable projective modules,

\[
P = \bigoplus_i P_i.
\]

This affects not only theory but also applications: the physically important operators (thermal, magnetic, and so on) in specific models may often be identified with the field corresponding to the “highest-weight” vector in a projective module \(P_i\), not with the primary field (of the same dimension) corresponding to the highest-weight vector of the irreducible quotient of \(P_i\). Also, whenever indecomposable representations are involved, there are more possibilities for constructing modular invariants by combining the chiral and antichiral spaces of states.

But an even more essential point about nonsemisimple theories is that before speaking of the representations, their characters, fusion, etc., one must find the symmetry algebra whose representations, characters, fusion, etc. are to be considered; this algebra is typically not the “naive,” manifest symmetry algebra. This point was expounded in [21]. The symmetry algebras of logarithmic conformal field theory models are typically nonlinear extensions of the naive symmetry algebra (e.g., Virasoro), i.e., are some \(W\)-algebras. The
first examples of $W$-algebras arising in this context were studied in [13, 14, 15], also see [23] and the references therein.

In a given logarithmic model, the chiral $W$-algebra must be big enough to ensure that only a finite number of its irreducible representations are realized in the model. Only then can one expect to have finite-dimensional fusion rules and modular group representation (we recall that a finite-dimensional modular group representation is a good test of the consistency of the model. Also, once finitely many irreducible representations are involved, there are finitely many projective modules, and the sum in (1.1) makes sense.

In this paper, we systematically investigate the logarithmic extensions of the $(p, q)$ models of conformal field theory. This task was already set in [22], in the line of an appropriate extension of the results obtained for the $(p, 1)$ models. The method in [21, 22] is to define and construct logarithmic conformal field models in terms of free fields and screenings. The chiral algebra $W$ that is the symmetry algebra of the model is then derived from the kernel of the screenings in the vacuum representation of a lattice vertex-operator algebra $L$. Irreducible representations of $W$ are identified with the images and cohomology of (certain powers of) the screenings in irreducible representations of $L$. The projective $W$-modules in (1.1) are then to be constructed as the projective covers of the irreducible representations (this construction may be rather involved; a notable exception is provided by the $(2, 1)$ model, see [16, 17, 18]).

In what follows, we set $p = p_+$ and $q = p_-$, a fixed pair of coprime positive integers.

In the rational $(p_+, p_-)$ model, the chiral symmetry algebra is the vertex-operator algebra $M_{p_+, p_-}$ defined as the cohomology of screenings that act in the vacuum representation of the appropriate lattice vertex-operator algebra $L_{p_+, p_-}$; irreducible representations of $M_{p_+, p_-}$ can then be identified with the cohomology of (powers of) the screenings in irreducible representations of $L_{p_+, p_-}$.

In the logarithmic $(p_+, p_-)$ model, the vacuum representation of the chiral algebra $W_{p_+, p_-}$ extends $M_{p_+, p_-}$ such that $M_{p_+, p_-} = W_{p_+, p_-}/R$, where $R$ is the maximal vertex-operator ideal in $W_{p_+, p_-}$. The $W$-algebra $W_{p_+, p_-}$ can be defined as the intersection of the screening kernels in $L_{p_+, p_-}$. (As we see in what follows, $W_{p_+, p_-}$ is generated by the energy–momentum tensor $T(z)$ and two Virasoro primaries $W^+(z)$ and $W^-(z)$ of conformal dimension $(2p_+ - 1)(2p_- - 1)$.) The irreducible representations of $W_{p_+, p_-}$ are of two different kinds. The first are the $\frac{1}{2}(p_+ - 1)(p_- - 1)$ irreducible modules of the Virasoro minimal model or, in other words, the modules annihilated by $R$. The second are the $2p_+ p_-$ modules that admit a nontrivial action of $R$. They can be identified with the images of (powers of) the screenings in the respective irreducible representations of the lattice vertex-operator algebra $L_{p_+, p_-}$ and decompose into infinitely many irreducible Virasoro modules. (The Virasoro embedding structure in the $(3, 2)$ and $(5, 2)$ logarithmic models was recently arduously explored in [24].)
The characters of the $\frac{1}{2}(p_+ - 1)(p_- - 1) + 2p_+ p_-$ irreducible representations of $\mathcal{W}_{p_+, p_-}$ (or, in slightly different terms, the Grothendieck ring\footnote{The free Abelian group generated by symbols $[M]$, where $M$ ranges over all representations subject to relations $[M] = [M'] + [M''].$} $\mathcal{G}$) do not exhaust the space $\mathcal{C}$ of torus amplitudes of the logarithmic $(p_+, p_-)$ model: we only have that $\mathcal{C} \supset \mathcal{G}$. This is a characteristic feature of nonsemisimple (logarithmic) conformal field theories, cf.\cite{25,26,21,27,28,22}. Because $\mathcal{C}$ carries a representation of $SL(2, \mathbb{Z})$, a much better approximation to this space is $\bar{\mathcal{C}}$, the $SL(2, \mathbb{Z})$-representation generated from $\mathcal{G}$:

$$\mathcal{C} \supset \bar{\mathcal{C}} \supset \mathcal{G}$$

(where most probably $\mathcal{C} = \bar{\mathcal{C}}$).

1.2. Theorem. In the logarithmic $(p_+, p_-)$ model,

1. $\dim \mathcal{G} = 2p_+ p_- + \frac{1}{2}(p_+ - 1)(p_- - 1)$;
2. $\dim \bar{\mathcal{C}} = \frac{1}{2}(3p_+ - 1)(3p_- - 1)$;
3. the $SL(2, \mathbb{Z})$-representation $\pi$ on $\bar{\mathcal{C}}$ has the structure

\begin{equation}
\bar{\mathcal{C}} = R_{\text{min}} \oplus R_{\text{proj}} \oplus \mathbb{C}^2 \otimes (R_{\mathbb{Z}} \oplus R_{\mathbb{Z}}) \oplus \mathbb{C}^3 \otimes R_{\text{min}},
\end{equation}

where $\mathbb{C}^2$ is the standard two-dimensional representation, $\mathbb{C}^3 \cong S^3(\mathbb{C}^2)$ is its symmetrized square, $R_{\text{min}}$ is the $\frac{1}{2}(p_+ - 1)(p_- - 1)$-dimensional $SL(2, \mathbb{Z})$-representation on the characters of the $(p_+, p_-)$ Virasoro minimal model, and $R_{\text{proj}}$, $R_{\mathbb{Z}}$, and $R_{\mathbb{Z}}$ are $SL(2, \mathbb{Z})$-representations of the respective dimensions $\frac{1}{2}(p_+ + 1) \cdot (p_- + 1)$, $\frac{1}{2}(p_+ - 1)(p_- + 1)$, and $\frac{1}{2}(p_+ + 1)(p_- - 1)$.

We note that the space $R_{\text{min}} \oplus R_{\text{proj}}$ is spanned by the characters of irreducible modules of the lattice vertex-operator algebra $\mathcal{L}_{p_+, p_-}$ and coincides with the space of theta functions.\footnote{Technically, the extension from $R_{\text{min}} \oplus R_{\text{proj}}$ to $\bar{\mathcal{C}}$ involves derivatives of the theta functions, which gives rise to the explicit occurrences of the modular parameter $\tau$ equalizing the modular weights. The highest theta-function derivative and simultaneously the top power of $\tau$ thus occurring, $n$, is equal to 2 for $(p_+, p_-)$ models and to 1 for $(p, 1)$ models\cite{22}, which can be considered the origin of the corresponding $\mathbb{C}^{n+1}$ in \cite{12} and in a similar formula in \cite{22}.} The space $R_{\text{proj}}$ can be identified with the characters of projective $\mathcal{W}_{p_+, p_-}$-modules.

1.2.1. Remark. The $2p_+ p_- + \frac{1}{2}(p_+ - 1)(p_- - 1)$ irreducible $\mathcal{W}_{p_+, p_-}$-representations are rather naturally arranged into a Kac table as follows. First, the $\frac{1}{2}(p_+ - 1)(p_- - 1)$ rational-model representations occupy the standard positions, with the standard identifications, in the boxes of the standard $(p_+ - 1) \times (p_- - 1)$ Kac table. Next, each box of the extended $p_+ \times p_-$ Kac table contains two, a “plus” and a “minus,” of the $\mathcal{W}_{p_+, p_-}$-representations labeled $(A_{r,s}^{+})_{1 \leq r \leq p_+}^{s \leq p_-}$ in what follows. We note that a logarithmic conformal field theory containing only finitely many irreducible or indecomposable Virasoro representations...
does not seem to exist; in particular, each \((p_+, p_-)\) model considered in this paper contains infinitely many indecomposable Virasoro representations, usually with multiplicity greater than 1, and we use the term “Kac table” exclusively to refer to a finite set of irreducible representations of the \(\mathcal{W}_{p_+, p_-}\) algebra.

Theorem 1.2 also implies that there exist \(SL(2, \mathbb{Z})\)-representations \(\tilde{\pi}\) and \(\pi^*\) on \(\tilde{\mathcal{C}}\) such that
\[
\pi(\gamma) = \pi^*(\gamma)\tilde{\pi}(\gamma), \quad \tilde{\pi}(\gamma)\pi^*(\gamma') = \pi^*(\gamma')\tilde{\pi}(\gamma), \quad \gamma, \gamma' \in SL(2, \mathbb{Z}).
\]
The representation \(\tilde{\pi}\) can be restricted to \(\mathcal{G}\), which then decomposes in terms of \(SL(2, \mathbb{Z})\) representations as
\[
\mathcal{G} = R_{\text{min}} \oplus R_{\text{proj}} \oplus R_{\mathbb{Z}} \oplus R_{\mathbb{Z}} \oplus R_{\text{min}}.
\]
This decomposition can be taken as the starting point for constructing a logarithmic Verlinde formula as in [21].

In view of the fundamental importance of the \(SL(2, \mathbb{Z})\) action, this theorem gives a very strong indication regarding the field content of a consistent conformal field theory model: the \(\frac{1}{2}(3p_+ - 1)(3p_- - 1)\)-dimensional space of \textit{generalized characters} \(\tilde{\mathcal{C}}\) is a strong candidate for the space of torus amplitudes of the logarithmic \((p_+, p_-)\) model. The following conjecture appears to be highly probable.

1.3. Conjecture. The \(SL(2, \mathbb{Z})\)-representation generated from \(\mathcal{G}\) coincides with the space of torus amplitudes:
\[
\tilde{\mathcal{C}} = \mathcal{C}
\]
as \(SL(2, \mathbb{Z})\) representations.

1.4. To move further in constructing the full space of states of the logarithmic theory as in (1.1), one must construct projective covers of all the \(2p_+p_- + \frac{1}{2}(p_+ - 1)(p_- - 1)\) irreducible \(\mathcal{W}_{p_+, p_-}\)-representations. This is a separate, quite interesting task.\(^3\)

Some useful information on the structure of the chiral-algebra projective modules can be obtained from the Kazhdan–Lusztig correspondence. In general, it is a correspondence between a chiral algebra \(\mathcal{W}\) and its representation category \(\mathcal{M}\) realized in conformal field theory, on the one hand, and some “dual” quantum group and its representation category on the other hand. In some “well-behaved” cases, the occurrence of the Kazhdan–Lusztig-dual quantum group can be seen by taking \(\bigoplus d_i P_i\), a direct sum of projective \(\mathcal{W}\)-modules with some multiplicities \(d_i\) chosen such that they are additive with respect to the direct

\(^3\)In particular, the question about logarithmic partners of the energy–momentum tensor \(T(z)\) and other fields takes the form of a well-posed mathematical problem about the structure of projective \(\mathcal{W}_{p_+, p_-}\)-modules. Logarithmic partners have been discussed, e.g., in [7, 29, 30, 31, 32] in the case \(c = 0\), where the differential polynomial in \(T(z)\) whose logarithmic partner is sought coincides with \(T(z)\) itself.
sum\textsuperscript{4} and multiplicative with respect to the quasitensor product in $\mathcal{W}$. If such a choice of the $d_i$ is possible, then

\begin{equation}
\mathfrak{g} = \text{End}_\mathcal{W}\left( \bigoplus_i d_i P_i \right)
\end{equation}

can be endowed with the structure of a (Hopf) algebra and $d_i$ are the dimensions of its irreducible representations (constructing the comultiplication requires certain conditions, which we do not discuss here). In this case, the category $\mathcal{W}$ is equivalent (as a quasitensor category) to the category of finite-dimensional representations of $\mathfrak{g}$. Such an extremely well-behaved case is realized in $(p,1)$ logarithmic models [18]: classification of indecomposable representation of the Kazhdan–Lusztig-dual quantum group $\mathfrak{g}$, which is not difficult to obtain using quite standard means, gives the classification of indecomposable $\mathcal{W}_{p,1}$-representations. In particular, the structure of projective $\mathcal{W}_{p,1}$-modules is thus known.

In the $(p_+, p_-)$ logarithmic models, the Kazhdan–Lusztig-dual quantum group $\mathfrak{g}_{p_+, p_-}$ (obtained as a subalgebra in the quotient of the Drinfeld double of the algebra of screenings for the $\mathcal{W}_{p_+, p_-}$ algebra) is not Morita-equivalent to $\mathcal{W}_{p_+, p_-}$, but nevertheless provides important information on the structure of indecomposable $\mathcal{W}_{p_+, p_-}$-modules. On the one hand, the quantum group $\mathfrak{g}_{p_+, p_-}$ and its representation category give only an “approximation” to the structure of the $\mathcal{W}_{p_+, p_-}$-representation category (the representation categories are certainly not equivalent, in contrast to the $(p,1)$ case [18]; in particular, there are $2p_+ p_-$ indecomposable projective $\mathfrak{g}_{p_+, p_-}$-modules but $2p_+ p_- + \frac{1}{2}(p_+ - 1)(p_- - 1)$ indecomposable projective $\mathcal{W}_{p_+, p_-}$-modules). On the other hand, this “approximation” becomes the precise correspondence as regards the modular group representations: naturally associated with $\mathfrak{g}_{p_+, p_-}$ is the $SL(2,\mathbb{Z})$-representation on its center [34], which turns out to be equivalent to the $SL(2,\mathbb{Z})$-representation on the $\mathcal{W}_{p_+, p_-}$-algebra characters and generalized characters in (1.2).

The $\mathfrak{g}_{p_+, p_-}$ quantum group acts in the space $\mathbb{F}$ obtained as a certain extension (“dressing”) of the irreducible $\mathcal{L}_{p_+, p_-}$-modules. Moreover, $\mathbb{F}$ is in fact a $(\mathcal{W}_{p_+, p_-}, \mathfrak{g}_{p_+, p_-})$-bimodule. This bimodule structure plays an essential role in the description of the full conformal field theory on Riemann surfaces of different genera, defect lines, boundary conditions, etc.

\textsuperscript{4}For a given chiral algebra $\mathcal{W}$, such sums are assumed to be finite; from a somewhat more general standpoint, this must follow from a set of fundamental requirements on $\mathcal{W}$ in a given nonsemisimple model.

First, the algebra itself must be generated by a finite number of fields $W_i(z)$. The category $\mathcal{W}$ of $\mathcal{W}$-modules with locally nilpotent action of the positive modes of $W_i(z)$ is then well defined. Second, the $\mathcal{W}$-modules must be finitely generated, such that for any collection of positive integers $(N_i)$, the coinvariants with respect to the subalgebra $\mathcal{W}(N_1, \ldots, N_m) \subset \mathcal{W}$ generated by the modes $W_i[n_i]$ with $n_i \leq N_i$ be finite-dimensional in any module from $\mathcal{W}$. In particular, this means that the category $\mathcal{W}$ contains a finite number of irreducible representations, and hence a finite number of projective modules (cf. [33, 27]).
1.5. From the standpoint of applications in condensed-mater physics, the definition of both the $W$-algebra $W_{p_+,p_-}$ and the quantum group $g_{p_+,p_-}$ refers to the Coulomb-gas picture [35, 36], where the starting point is a two-dimensional scalar field $\varphi$ with the action written in complex coordinates as

$$S_0 = -\frac{1}{8\pi} \int \partial \varphi \bar{\partial} \varphi dz d\bar{z}.$$  

(The normalization is chosen such that the propagator has the form $\langle \varphi(z, \bar{z}) \varphi(0,0) \rangle = \log |z|$ and vertex operators $\exp(\alpha \varphi(z, \bar{z}))$ do not involve an $i$ in the exponent.) Furthermore, the field is taken to be compactified to a circle (of the radius $\sqrt{2p_+ p_-}$), which just means that the fields $\varphi$ and $\varphi + 2i\pi \sqrt{2p_+ p_-}$ are considered equivalent. Then the observables are given by vertex operators $\exp(\frac{n}{\sqrt{2p_+ p_-}} \varphi(z, \bar{z}))$ with $n \in \mathbb{Z}$.

This model with central charge $c = 1$ has a large symmetry algebra, the lattice vertex-operator algebra $L_{p_+,p_-}$ mentioned above. The minimal models can be regarded as conformal points with $c < 1$ [37] to which the system renormalizes after the perturbation

$$S = S_0 + \int \lambda_+ e^{\alpha_+ \varphi(z, \bar{z})} dz d\bar{z} + \int \lambda_- e^{\alpha_- \varphi(z, \bar{z})} dz d\bar{z}$$

with the appropriate $\alpha_+$ and $\alpha_-$ (and some constant $\lambda_\pm$). The symmetry of the model thus obtained is the vertex-operator algebra $M_{p_+,p_-}$, which is “much smaller” than $L_{p_+,p_-}$.

Logarithmic conformal points occur in this approach for quenched random systems [38] whose action is given by

$$S = S_0 + \int \lambda_+ (z, \bar{z}) e^{\alpha_+ \varphi(z, \bar{z})} dz d\bar{z} + \int \lambda_- (z, \bar{z}) e^{\alpha_- \varphi(z, \bar{z})} dz d\bar{z},$$

where $\lambda_\pm (z, \bar{z})$ are quenched random variables with appropriately chosen correlators $\lambda_\pm (z, \bar{z}), \lambda_\pm (z_1, \bar{z}_1) \lambda_\pm (z_2, \bar{z}_2)$, and so on. The parameters involved in these fixed correlators renormalize such that the system occurs in a new infrared fixed point with the same $c < 1$ as in the minimal model, but with the symmetry algebra given by a $W$-algebra (a subalgebra of $L_{p_+,p_-}$). The entire system can thus be regarded as the tensor product of the original Coulomb-gas model and an additional model describing a quenched disorder through the chosen correlators of the $\lambda_\pm (z, \bar{z})$. We do not know how these correlators must be chosen in order to produce just the $(p_+, p_-)$ logarithmic conformal field theory model; instead, we take a more algebraic stand and study the $W$-algebra $W_{p_+,p_-}$ of the expected fixed point. It is then possible to make contact with quenched disorder by studying the projective modules of this $W$-algebra.

This paper is organized as follows. In Sec. 2 we introduce the basic notation and describe some facts in the free-field description of minimal models. We introduce vertex operators in [2, 21] define the free-field and Virasoro modules in [2, 22] and introduce the lattice vertex-operator algebra $L_{p_+,p_-}$ and its modules that are important ingredients in constructing the $W_{p_+,p_-}$-representations [2, 23]. In Sec. 3 we describe the action of screenings
on the vertices in \(3.1\) and introduce the Kazhdan–Lusztig-dual quantum group \(\mathfrak{g}_{p_+, p_-}\) (we reformulate the action of the screenings in terms of a Hopf algebra \(\mathcal{H}\) in \(3.2\)) and then construct its Drinfeld double \(\mathfrak{g}_{p_+, p_-}^\ast\) in \(3.3\). In Sec. 3 we finally construct the \(\mathcal{W}_{p_+, p_-}\) algebra (we introduce it in \(4.1\)) formulate the structural result in \(4.2\) and describe the irreducible and Verma \(\mathcal{W}_{p_+, p_-}\)-modules in \(4.3\) and \(4.4\) a \((\mathcal{W}_{p_+, p_-}, \mathfrak{g}_{p_+, p_-})\)-bimodule structure of the space of states in given \(4.5\). In Sec. 5 we calculate the \(SL(2, \mathbb{Z})\)-representation generated from the \(\mathcal{W}_{p_+, p_-}\)-characters; on the resulting space of generalized characters (as noted above, most probably the torus amplitudes), we then decompose the \(SL(2, \mathbb{Z})\)-action as in \(1.2\). In \(5.1\) we calculate characters of the irreducible \(\mathcal{W}_{p_+, p_-}\)-modules; in \(5.2\) we introduce generalized characters, calculate the \(SL(2, \mathbb{Z})\) action on \(\mathcal{C}\), and give a direct proof of \(1.2\). Several series of modular invariants are considered in \(5.3\). Some implications of the Kazhdan–Lusztig correspondence and several open problems are mentioned in the conclusions.

**Notation.** We fix two coprime positive integers \(p_+\) and \(p_-\) and set

\[
\alpha_- = -\sqrt{\frac{2p_+}{p_-}}, \quad \alpha_+ = \sqrt{\frac{2p_-}{p_+}}, \quad \alpha_0 = \alpha_+ + \alpha_-.
\]

With the pair \((p_+, p_-)\), we associate the sets of indices

\[
J_0 = \left\{ (r, s) \mid 0 \leq r \leq p_+, \quad 0 \leq s \leq p_-, \quad p_-r + p_+s \leq p_+p_-, \quad (r, s) \neq (0, p_-) \right\},
\]

\[
J_1 = \left\{ (r, s) \mid 1 \leq r \leq p_+ - 1, \quad 1 \leq s \leq p_- - 1, \quad p_-r + p_+s \leq p_+p_- \right\},
\]

\[
J_\varnothing = \left\{ (r, s) \mid 0 \leq r \leq p_+ - 1, \quad 1 \leq s \leq p_- - 1, \quad p_-r + p_+s \leq p_+p_- \right\},
\]

\[
J_\varnothing = \left\{ (r, s) \mid 1 \leq r \leq p_+ - 1, \quad 0 \leq s \leq p_- - 1, \quad p_-r + p_+s \leq p_+p_- \right\}.
\]

The numbers of elements in these sets are \(|J_0| = \frac{1}{2}(p_+ + 1)(p_- + 1)\), \(|J_1| = \frac{1}{2}(p_+ - 1) \cdot (p_- - 1)\), \(|J_\varnothing| = \frac{1}{2}(p_+ + 1)(p_- - 1)\), and \(|J_\varnothing| = \frac{1}{2}(p_+ - 1)(p_- + 1)\).

The building blocks for the characters are the so-called theta-constants \(\theta_{s,p}(q)\), \(\theta^{(1)}_{s,p}(q)\), and \(\theta^{(2)}_{s,p}(q)\), where

\[
\theta_{s,p}(q) = \theta_{s,p}(q, 1), \quad \theta^{(m)}_{s,p}(q) = \left( z \frac{\partial}{\partial z} \right)^m \theta_{s,p}(q, z) \bigg|_{z = 1}, \quad m \in \mathbb{N},
\]

and the theta function is defined as

\[
\theta_{s,p}(q, z) = \sum_{j \in \mathbb{Z} + \frac{s}{2p}} q^{p_j^2} z^{p_j}, \quad |q| < 1, \quad z \in \mathbb{C}, \quad p \in \mathbb{N}, \quad s \in \mathbb{Z}.
\]

To further simplify the notation, we resort to the standard abuse by writing \(f(\tau)\) for \(f(e^{2i\pi \tau})\), with \(\tau\) in the upper complex half-plane; it is tacitly assumed that \(q = e^{2i\pi \tau}\).

There are the easily verified properties

\[
\theta^{(m)}_{s+2pa,p}(\tau) = \theta^{(m)}_{s,p}(\tau), \quad \theta^{(m)}_{s-pa,p}(\tau) = (-1)^m \theta^{(m)}_{s,p}(\tau), \quad p \in \mathbb{N}, \quad m \in \mathbb{N}_0, \quad a \in \mathbb{Z}.
and \( \theta'_0, p (\tau) = \theta'_0, p (\tau) = 0 \). We often write
\[
\theta_{s, p}, p_{+}, p_{-} (\tau) \equiv \theta_{s}, \quad \theta_{s, p_{+}, p_{-}} (\tau) \equiv \theta'_{s}, \quad \theta_{s, p_{+}, p_{-}} (\tau) \equiv \theta''_{s}.
\]

Similar abbreviations are used for the characters: we write
\[
\chi_{r, s} (\tau) \equiv \chi_{r, s}, \quad \chi_{\pm, r, s} (\tau) \equiv \chi_{\pm, r, s}.
\]

Finally, we use the \( \eta \)-function
\[
\eta (q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).
\]

2. Free-field preliminaries

We introduce a free bosonic field and describe its energy–momentum tensor, vertex operators, screenings, and representations of the Virasoro algebra.

2.1. Free field and vertices. Let \( \varphi \) denote a free scalar field with the OPE
\[
\partial \varphi (z) \partial \varphi (w) = \frac{1}{(z - w)^2}
\]
and the mode expansion
\[
\partial \varphi (z) = \sum_{n \in \mathbb{Z}} \varphi_n z^{-n-1}.
\]

The energy–momentum tensor is given by
\[
T(z) = \frac{1}{2} \partial \varphi (z) \partial \varphi (z) + \frac{\alpha_0}{2} \partial^2 \varphi (z)
\]
(see (1.4)). The modes of \( \partial \varphi (z) \) span the Heisenberg algebra and the modes of \( T(z) \) span the Virasoro algebra \( \text{Vir}_{p_{+}, p_{-}} \) with the central charge
\[
c = 1 - 6 \left( \frac{p_{+} - p_{-}}{p_{+} + p_{-}} \right)^2.
\]

The vertex operators are given by \( e^{j (r, s) \varphi (z)} \) with \( j(r, s) = \frac{1 - s}{2} \alpha_- + \frac{1 - r}{2} \alpha_+ \), \( r, s \in \mathbb{Z} \). Equivalently, these vertex operators can be parameterized as
\[
V_{r, s; n} (z) = e^{\frac{p_{+} (1 - r) - p_{-} (1 - s) + p_{+} p_{-} n}{\sqrt{p_{+} + p_{-}}} \varphi (z)}, \quad 1 \leq r \leq p_{+}, \quad 1 \leq s \leq p_{-}, \quad n \in \mathbb{Z}.
\]

The conformal dimension of \( V_{r, s; n} (z) \) assigned by the energy–momentum tensor is
\[
\Delta_{r, s; n} = \frac{(p_{+} + p_{-} - r + p_{+} p_{-} n)^2 - (p_{+} - p_{-})^2}{4 p_{+} p_{-}}.
\]

We note that
\[
\Delta_{r, s; n} = \Delta_{r, s; n}, \quad \Delta_{r + kp_{+}, s + kp_{-}; n} = \Delta_{r, s; n}, \quad \Delta_{r, s + kp_{-}; n} = \Delta_{r, s; n + k}.
\]

The vertex operators satisfy the braiding relations
\[ (2.8) \quad V_{r,s;n}(z_1)V_{r',s';n'}(z_2) = \\
= q^{(p_-(1-r')-p_+(1-s') + p_+p_-n')(p_-(1-r')-p_+(1-s') + p_+p_-n')} V_{r',s';n'}(z_2) V_{r,s;n}(z_1), \]

where
\[ (2.9) \quad q = e^{\frac{i\pi}{2p_+p_-}}. \]

(The convention is to take the OPE of the operators in the left-hand side with \(|z_1| > |z_2|\) and make an analytic continuation to \(|z_1| < |z_2|\) by moving \(z_1\) along a contour passing below \(z_2\), i.e., as \(z_1 \to z_2 \to e^{i\pi}(z_1 - z_2)\).)

In what follows, we also write \(V_{r,s;0}(z) \equiv V_{r,s}(z)\) and \(\Delta_{r,s;0} \equiv \Delta_{r,s}\).

2.2. Definition of modules. For \(1 \leq r \leq p_+\), \(1 \leq s \leq p_-\), and \(n \in \mathbb{Z}\), let \(\mathcal{F}_{r,s;n}\) denote the Fock module of the Heisenberg algebra generated from (the state corresponding to) the vertex operator \(V_{r,s;n}(z)\). The zero mode \(\varphi_0 = \frac{1}{2i\pi} \oint dz \varphi(z)\) acts in \(\mathcal{F}_{r,s;n}\) by multiplication with the number
\[ \varphi_0 v = \frac{p_-(1-r) - p_+(1-s) + p_+p_-n}{\sqrt{2p_+p_-}} v, \quad v \in \mathcal{F}_{r,s;n}. \]

We write \(\mathcal{F}_{r,s} \equiv \mathcal{F}_{r,s;0}\). For convenience of notation, we identify \(\mathcal{F}_{0,s;n} \equiv \mathcal{F}_{p_+,s;n+1}\) and \(\mathcal{F}_{r;0,n} \equiv \mathcal{F}_{r,p_-;n-1}\).

Let \(\mathcal{Y}_{r,s;n}\) with \(1 \leq r \leq p_+\), \(1 \leq s \leq p_-\), and \(n \in \mathbb{Z}\) denote the Virasoro module that coincides with \(\mathcal{F}_{r,s;n}\) as a linear space, with the Virasoro algebra action given by (2.3) (see [39]). As with the \(\mathcal{F}_{r,s;n}\), we also write \(\mathcal{Y}_{r,s} \equiv \mathcal{Y}_{r,s;0}\).

2.2.1. Subquotient structure of the modules \(\mathcal{Y}_{r,s;n}\). We recall the subquotient structure of the Virasoro modules \(\mathcal{Y}_{r,s;n}\) [40]. We let \(\mathcal{J}_{r,s;n}\) denote the irreducible Virasoro module with the highest weight \(\Delta_{r,s;n}\) (as before, \(1 \leq r \leq p_+\), \(1 \leq s \leq p_-\), and \(n \in \mathbb{Z}\)). Evidently, \(\mathcal{J}_{r,s;n} \cong \mathcal{J}_{p_+-r,p_-;s;-n}\). The \(\frac{1}{2}(p_+-1)(p_-+1)\) nonisomorphic modules among the \(\mathcal{J}_{r,s;0}\) with \(1 \leq r \leq p_+-1\) and \(1 \leq s \leq p_-+1\) are the irreducible modules from the Virasoro \((p_+, p_-)\) minimal model. We also write \(\mathcal{J}_{r,s} \equiv \mathcal{J}_{r,s;0}\). For convenience of notation, we identify \(\mathcal{J}_{0,s;n} \equiv \mathcal{J}_{p_+,s;n+1}\) and \(\mathcal{J}_{r;0,n} \equiv \mathcal{J}_{r,p_-;n-1}\).

The well-known structure of \(\mathcal{Y}_{r,s}\) for \(1 \leq r \leq p_+-1\) and \(1 \leq s \leq p_-+1\) is recalled in Fig. [1].

2.2.2. The Fock spaces introduced above constitute a free-field module
\[ (2.10) \quad \mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{r=1}^{p_+} \bigoplus_{s=1}^{p_-} \mathcal{F}_{r,s;n}. \]

It can be regarded as (the chiral sector of) the space of states of the Gaussian Coulomb gas model compactified on the circle of radius \(\sqrt{2p_+p_-}\). This model has \(c = 1\) and its symmetry is a lattice vertex operator algebra, which is described in the next subsection.
Figure 1. Embedding structure of the Feigin–Fuchs module $\mathcal{Y}_{r,s}$. The notation is as follows. The cross $\times$ corresponds to the subquotient $\mathcal{J}_{r,s}$, the filled dots $\bullet$ to $\mathcal{J}_{r,p-r-s;2n+1}$ with $n \in \mathbb{N}_0$, the triangles $\triangle$ to $\mathcal{J}_{r,s;2n}$ with $n \in \mathbb{N}$, and the open dots $\circ$ to $\mathcal{J}_{r,s;2n}$ with $n \in \mathbb{N}$. The arrows correspond to embeddings of the Virasoro modules and are directed toward submodules. The notation $[a,b;n]$ in square brackets is for subquotients isomorphic to $\mathcal{J}_{a,b;n}$. The filled dots constitute the socle of $\mathcal{Y}_{r,s}$.

2.3. The lattice vertex-operator algebra. Let $\mathcal{L}_{p+,p-}$ be the lattice vertex-operator algebra (see [41, 42, 43]) generated by the vertex operators

$$V_{1;2n}(z) = e^{n\sqrt{2}p_+ p_-}(z), \quad n \in \mathbb{Z}.$$ \hspace{1cm} (2.11)

The underlying vector space (the vacuum representation) of $\mathcal{L}_{p+,p-}$ is

$$\mathcal{L}_{p+,p-} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{1;2n}.$$ \hspace{1cm} (2.11)

The vertex-operator algebra $\mathcal{L}_{p+,p-}$ has $2p_+ p_-$ irreducible modules, denoted in what follows as $\mathcal{V}_{r,s}^\pm$ with $1 \leq r \leq p_+$ and $1 \leq s \leq p_-$. Their Fock-module decompositions are given by

$$\mathcal{V}_{r,s}^+ = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{p_+ - r - p_-;2n}, \quad 1 \leq r \leq p_+, \quad 1 \leq s \leq p_-.$$ \hspace{1cm} (2.12)

$$\mathcal{V}_{r,s}^- = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{p_+ - r - p_-;2n+1}, \quad 1 \leq r \leq p_+, \quad 1 \leq s \leq p_-.$$

In this numerology, the vacuum representation $\mathcal{L}_{p+,p-}$ coincides not with the usual $\mathcal{V}_{1,1}^+$ but with $\mathcal{V}_{p_+-1, p_-+1}^+$; such a “twist of notation” turns out to be convenient in what follows.

The model with the space of states (2.10) is rational with respect to $\mathcal{L}_{p+,p-}$, i.e., the space $\mathcal{F}$ is a finite sum of irreducible $\mathcal{L}_{p+,p-}$-modules:
corresponding Fock module. Vertex operators the Heisenberg subalgebra acts in each two-strand “braid” (labeled by the last integer in to be described in Sec. 4). The action of distance of the Virasoro algebra. With the same notation as in Fig. 1, we describe the Virasoro structure We need to recall a number of standard facts known in the representation theory of the At the moment, the reader is asked to ignore the dotted lines in the figures (they are Evidently, each \( V_{r,s}^\pm \) is a Virasoro module by virtue of the free-field construction\(^{(2,3)}\). We need to recall a number of standard facts known in the representation theory of the Virasoro algebra. With the same notation as in Fig. 1, we describe the Virasoro structure of the \( V_{r,s}^\pm \) in Fig. 2\((V_{p_+-r,p_-s}^+)\), Fig. 3\((V_{r,-p_-s}^-)\), Fig. 4\((V_{r,p_-s}^+)\), and Fig. 5\((V_{p_+,-p_-}^-)\).

At the moment, the reader is asked to ignore the dotted lines in the figures (they are to be described in Sec. 4). The action of \( L_{p_+p_-} \) in \( V_{r,s}^\pm \) is as follows. In Figs. 2 and 3 the Heisenberg subalgebra acts in each two-strand “braid” (labeled by the last integer in the square brackets at the top of each braid, even in Fig. 2 and odd in Fig. 3) as in the corresponding Fock module. Vertex operators \( V_{1,1;2n}(z) \) map between the braids over the distance \( n \) (to the left for \( n > 0 \)). In Figs. 4 and 5 the Heisenberg subalgebra acts in each vertical strand and \( V_{1,1;2n}(z) \) act between strands similarly.
We next consider the socle (the maximal semisimple submodule) of $\mathcal{V}_{r,s}^\pm$ (the irreducible $\mathcal{L}_{p_+,p_-}$-module defined in (2.12)), viewed as a Virasoro module.

**2.3.1. Definition.** With the $\mathcal{V}_{r,s}^\pm$ modules regarded as Virasoro modules, we set

$\mathcal{X}_{r,s}^\pm = \text{soc } \mathcal{V}_{p_+-r,p_-+s}^\pm, \quad 1 \leq r \leq p_+-1, \quad 1 \leq s \leq p_-1,$

$\mathcal{X}_{r,p_-}^\pm = \text{soc } \mathcal{V}_{p_+-r,p_-}^\pm, \quad 1 \leq r \leq p_+-1,$

$\mathcal{X}_{p_+,s}^\pm = \text{soc } \mathcal{V}_{p_+,p_-+s}^\pm, \quad 1 \leq s \leq p_-1,$

$\mathcal{X}_{p_+,p_-}^\pm = \text{soc } \mathcal{V}_{p_+,p_-}^\pm.$

We note that $\mathcal{X}_{p_+,p_-}^\pm = \mathcal{V}_{p_+,p_-}^\pm$. Therefore, in particular,

\begin{equation}
\text{soc } \mathcal{F} = \bigoplus_{r=1}^{p_+} \bigoplus_{s=1}^{p_-} (\mathcal{X}_{r,s}^+ \oplus \mathcal{X}_{r,s}^-).
\end{equation}

The space $\mathcal{X}_{r,s}^+$ is represented by the collection of filled dots in Figs. 2 and 5, $\mathcal{X}_{p_+-r,s}^-$ by the collection of boxes in Fig. 3, and $\mathcal{X}_{p_+-r,p_-}^-$ by the collection of open dots in Fig. 4. They decompose into direct sums of irreducible Virasoro modules as follows.
Figure 4. Structure of the modules $V_{r,p}^+$. The notation is the same as in Fig. 1. Open dots $\circ$ denote Virasoro representations that are combined into $\mathcal{X}_{p_+ - r, p_-}^-$. The dotted lines show the action of the $W^\pm(z)$ generators of $W_{p_+, p_-}$ in $\mathcal{X}_{p_+ - r, p_-}^-$. 

Figure 5. Structure of the modules $V_{r,p}^-$. The notation is the same as in Fig. 1. Filled dots $\bullet$ denote Virasoro representations that are combined into $\mathcal{X}_{r, p_-}^+$. The dotted lines show the action of the $W^\pm(z)$ generators of $W_{p_+, p_-}$ in $\mathcal{X}_{r, p_-}^+$. 

\[ [r, p_-, 0] \]
\[ [p_+ - r, p_-; 1] \]
\[ W^+ \]
\[ W^0 \]
\[ [r, p_-; 2] \]
\[ [r, p_-; 2] \]
\[ W^- \]
\[ W^- \]
\[ [p_+ - r, p_-; 3] \]
\[ [p_+ - r, p_-; 3] \]
\[ W^+ \]
\[ W^0 \]
\[ W^0 \]
\[ W^- \]
\[ W^- \]
\[ [p_+ - r, p_-; 5] \]
\[ [p_+ - r, p_-; 5] \]
\[ [r, p_-; 4] \]
\[ [r, p_-; 4] \]
\[ [r, p_-; 4] \]
\[ W^+ \]
\[ W^- \]
\[ W^- \]
\[ [r, p_-; 6] \]
\[ [r, p_-; 6] \]
\[ [r, p_-; 6] \]
\[ W^+ \]
\[ W^- \]
\[ W^- \]
\[ [r, p_-; 6] \]
\[ [r, p_-; 6] \]
\[ [r, p_-; 6] \]
2.3.2. Lemma. As a Virasoro module, the space $X^\pm_{r,s}$ for $1 \leq r \leq p_+$ and $1 \leq s \leq p_-$ decomposes as

$$X^+_r \simeq \bigoplus_{a \geq 0} (2a + 1) J_{r,p_+ - s;2a+1}, \quad X^-_r \simeq \bigoplus_{a \geq 1} 2a J_{r,p_+ - s;2a},$$

with the identification $J_{0,s;n} \equiv J_{p_+,s;n+1}$ and $J_{r,0;n} \equiv J_{r,p_-;n-1}$ introduced above.

As we see in what follows, the $X^\pm_{r,s}$ become $W$-algebra representations. Describing the $W$-algebra requires studying the action of screenings, which we consider in the next section.

3. SCREENING OPERATORS AND THE QUANTUM GROUP

In this section, we introduce screening operators and study the quantum group associated with them. Because the screenings do not act in the free-field module $\mathcal{F}$, we have to extend $\mathcal{F}$ to a larger space $\hat{\mathcal{F}}$ in 3.1. In 3.2 we then reformulate the action of the screenings as a representation of a Hopf algebra, with the result formulated in 3.3. A subalgebra in a quotient of the Drinfeld double of this Hopf algebra is the quantum group that is Kazhdan–Lusztig dual to the $W$-algebra. In 3.4 we next describe the spaces $X^\pm_{r,s}$ introduced above in terms of the screenings. We also show in 3.5 that the relevant spaces carry a representation of the $s\ell(2, \mathbb{C})$ algebra commuting with the Virasoro action. All these ingredients are to be used in the next section in the study of the $W_{p_+,p_-}$ algebra.

3.1. Screening operators and dressing. Free-field construction of both the minimal model and its logarithmic extension involves screening operators $e_+$ and $f_-$ that commute with the energy–momentum tensor, $[e_+, T(z)] = [f_-, T(z)] = 0$. They have the standard form

$$e_+ = \oint dz e^{\alpha_+ \varphi(z)}, \quad f_- = \oint dz e^{\alpha_- \varphi(z)}.$$

The operators $e_+$ and $f_-$ do not act in the space $\mathcal{F}$ (see (2.10)), and to complete their definition, we must extend this space appropriately. For this, we introduce the space $\hat{\mathcal{F}}$ spanned by dressed fields (cf. [44, 25])

$$\int_{\Gamma} dz_1 \ldots dz_m dw_1 \ldots dw_j e^{\alpha_+ \varphi(z_1)} \ldots e^{\alpha_+ \varphi(z_j)} e^{\alpha_- \varphi(w_1)} \ldots \times e^{\alpha_- \varphi(w_m)} \mathcal{P}(\partial \varphi) V_{r,s;n}(z), \quad 0 \leq j \leq p_+ - 1, \quad 0 \leq m \leq p_- - 1,$$

where $\Gamma$ is a local system defined as in [44] and $\mathcal{P}(\partial \varphi)$ is a differential polynomial in $\partial \varphi$.

The action of $e_+$ and $f_-$ on each vector (3.2) can then be evaluated by manipulations with contour integrals, as described in [44, 25]. In particular,

$$e^r_+ : \mathcal{F}_{r,s;n} \to \mathcal{F}_{p_+,r,s;n+1};$$
\[ f_s^r : \mathcal{F}_{r,s;n} \rightarrow \mathcal{F}_{r,p-s;n-1}. \]

Evidently, these spaces and the maps between them constitute the Felder complexes \[25\] whose cohomology gives the standard minimal model.

### 3.2. The Hopf algebra of screenings.

The action of screening operators in \( \mathbb{F} \) gives rise to a Hopf algebra representation. The following lemma basically restates some known facts about the screenings \[44, 25\].

#### 3.2.1. Lemma.

The space \( \mathbb{F} \) defined in 3.1 admits the action of the operators

\[ (3.3) \quad e_+, \quad f_-, \quad k = e^{i \pi \sqrt{2p + p} - \varphi_0}, \]

which satisfy the relations

\[ (3.4) \quad e^p_+ = f^p_- = 0, \quad k^{4p + p} = 1, \quad e_+ f_- = f_- e_+, \]

\[ ke_+ k^{-1} = q_+ e_+, \quad kf_- k^{-1} = q_- f_-, \]

where (see (2.9))

\[ q_+ = q^{2p} = e^{i \pi}, \quad q_- = q^{2p} = e^{i \pi}. \]

Let \( \mathcal{H} \) denote the associative algebra generated by \( e_+, f_- \), and \( k \) with relations (3.4). Clearly, the PBW basis in \( \mathcal{H} \) is given by

\[ (3.5) \quad e_{jmn} = k^j e^m_+ f^n_-, \quad 0 \leq j \leq 4p + p - 1, \quad 0 \leq m \leq p_+ - 1, \quad 0 \leq n \leq p_+ - 1. \]

#### 3.2.2. Lemma.

The algebra \( \mathcal{H} \) is a Hopf algebra with the counit

\[ (3.6) \quad \epsilon(e_+) = \epsilon(f_-) = 0, \quad \epsilon(k) = 1, \]

comultiplication

\[ (3.7) \quad \Delta(k) = k \otimes k, \quad \Delta(e_+) = e_+ \otimes 1 + k^{2p} \otimes e_+, \quad \Delta(f_-) = f_- \otimes 1 + k^{2p} \otimes f_-, \]

and antipode

\[ (3.8) \quad S(k) = k^{-1}, \quad S(e_+) = -k^{2p} e_+, \quad S(f_-) = -k^{2p} f_. \]

We note that the counit is given by the trivial representation on the vertex \( V_{1,1} = 1. \) The comultiplication is taken from operator product expansion; for example, the standard manipulations lead to

\[ e_+(e^{j_1 \varphi(w)} e^{j_2 \varphi(u)}) = \oint e^{\alpha + \varphi(z)}(e^{j_1 \varphi(w)} e^{j_2 \varphi(u)}) dz = \]

\[ = (\oint e^{\alpha + \varphi(z)} e^{j_1 \varphi(w)} dz) e^{j_2 \varphi(u)} + e^{i \pi \alpha + j_1} e^{j_1 \varphi(w)} \left( \oint e^{\alpha + \varphi(z)} e^{j_2 \varphi(u)} dz \right) = \]

\[ = e_+ e^{j_1 \varphi(w)} e^{j_2 \varphi(u)} + k^{2p} e^{j_1 \varphi(w)} e_+ e^{j_2 \varphi(u)}, \]
which gives the comultiplication for \( e_+ \), and similarly for \( f_- \) and \( k \). With the counit and comultiplication thus fixed, the antipode \( S \) is uniquely calculated from the comultiplication, with the result in (3.8).

The full quantum group realized in a given conformal field theory model involves not only the screening but also the contour-removal operators. A convenient procedure for introducing the latter is to take Drinfeld’s double (see [45, 46] and B.1) of the algebra \( \mathcal{H} \) generated by the screenings.

3.3. Theorem. The double \( D(\mathcal{H}) \) of \( \mathcal{H} \) is a Hopf algebra generated by \( e_\pm, f_\pm, k, \) and \( \kappa \) with the relations

\[
\begin{align*}
(3.9) & \quad e_{\pm}^p = f_{\pm}^p = 0, \quad k^{4p} = \kappa^{4p} = 1, \\
(3.10) & \quad k_\pm^4 e_\pm^{-1} = q_\pm e_\pm, \quad k_\pm^4 f_\pm^{-1} = q_\pm f_\pm, \quad \kappa e_\pm^{-1} = q_\pm e_\pm, \quad \kappa f_\pm^{-1} = q_\pm f_\pm, \\
(3.11) & \quad k \kappa = \kappa k, \quad e_+ e_- = e_- e_+, \quad f_+ f_- = f_- f_+, \quad e_+ f_- = f_- e_+, \quad e_- f_+ = f_+ e_-,
\end{align*}
\]

\[
(3.12) \quad [e_\pm, f_\pm] = \frac{k^{4p} - \kappa^{4p}}{q_\pm^p - q_\pm^{-1}};
\]

\[
(3.13) \quad \Delta(k) = k \otimes k, \quad \Delta(e_+) = e_+ \otimes 1 + k^{2p} \otimes e_+, \quad \Delta(f_-) = f_- \otimes 1 + k^{-2p} \otimes f_-;
\]

\[
(3.14) \quad \Delta(\kappa) = \kappa \otimes \kappa, \quad \Delta(f_+) = f_+ \otimes \kappa^{2p} + 1 \otimes f_+, \quad \Delta(e_-) = e_- \otimes \kappa^{-2p} + 1 \otimes e_-;
\]

\[
(3.15) \quad S(k) = k^{-1}, \quad S(e_+) = -k^{-2p} e_+, \quad S(f_-) = -k^{2p} f_-;
\]

\[
(3.16) \quad S(\kappa) = \kappa^{-1}, \quad S(f_+) = -f_+ \kappa^{-2p}, \quad S(e_-) = -e_- \kappa^{2p};
\]

\[
(3.17) \quad \epsilon(e_\pm) = \epsilon(f_\pm) = 0, \quad \epsilon(k) = \epsilon(\kappa) = 1.
\]

We prove this in B.2 by a routine application of the standard construction.

The double thus introduces contour-removal operators \( e_- \) and \( f_+ \), dual to \( e_+ \) and \( f_- \) in the sense that is fully clarified in the proof in Appendix B. But the doubling procedure also yields the dual \( \kappa \) to the Cartan element \( k \) in \( \mathcal{H} \), which is to be eliminated by passing to the quotient

\[
(3.18) \quad D(\mathcal{H}) = D(\mathcal{H}) / (k\kappa - 1)
\]

over the Hopf ideal generated by the central element \( k\kappa - 1 \). We next take a subalgebra in \( D(\mathcal{H}) \) (which, unlike \( D(\mathcal{H}) \), is a factorizable ribbon quantum group, see [34]).

3.3.1. Definition. Let \( g_{p_+, p_-} \) be the subalgebra in \( D(\mathcal{H}) \) generated by \( e_+, f_+, e_-, f_- \), and

\[
(3.19) \quad K = k^2.
\]

Explicit relations among the generators are a mere rewriting of the corresponding formulas in the theorem,

\[
e_{\pm}^p = f_{\pm}^p = 0, \quad K^{2p} = 1,
\]
\[ K e_{\pm} K^{-1} = q_{\pm}^2 e_{\pm}, \quad K f_{\pm} K^{-1} = q_{\pm}^{-2} f_{\pm}, \]
\[ e_+ e_- = e_- e_+, \quad f_+ f_- = f_- f_+, \quad e_+ f_- = f_- e_+, \quad e_- f_+ = f_+ e_-, \]
\[ [e_\pm, f_\pm] = \frac{K^{\pm p_{\pm}} - K^{\mp p_{\pm}}}{q_{\pm}^{p_{\pm}} - q_{\pm}^{-p_{\pm}}}, \]
with the comultiplication, antipode, and counit defined in (3.13)–(3.17). The structure of \( g_{p_+, p_-} \) and its relation to the logarithmic \((p_+, p_-)\) model are investigated in [34].

This quantum group has \( 2p_+ p_- \) irreducible representations, which we label as \( X_{r, s}^\pm \) with \( 1 \leq r \leq p_+ \) and \( 1 \leq s \leq p_- \). The \( g_{p_+, p_-} \)-module \( X_{r, s}^\pm \) is generated by \( e_+ \) and \( f_- \) from the eigenvector of \( K \) with the eigenvalue \( \pm q_{r}^{s-1} q_{-s+1}^{-1} \), and we have \( \dim X_{r, s}^\pm = rs \). In what follows, we use the \( g_{p_+, p_-} \)-modules to describe the bimodule structure of \( F \).

### 3.4. Kernels and images of the screenings

Our aim in what follows is to introduce some other module structures in the spaces \( X_{r, s}^\pm \), so far defined as Virasoro modules in section 2.3.1. For this, we first describe them an intersection of the images of screenings.

The socle of the space \( F \) (see (2.13)), still viewed as a Virasoro module, can also be written as

\[ \text{soc } F = \text{im } e_{p_+}^{p_+ - 1} \cap \text{im } f_{p_-}^{p_- - 1} \subset F \]

Equivalently (and somewhat more convenient technically), the spaces \( X_{r, s}^\pm \) that constitute \( \text{soc } F \) are described as intersections of the images of the screenings as

\[
X_{r, s}^\pm = \text{im } e_{p_+}^{p_+ - r} \cap \text{im } f_{p_-}^{p_- - s} \quad \text{in } V_{p_+, p_-}^{\pm} \\
\text{for } 1 \leq r \leq p_+ - 1, \\
\text{and } 1 \leq s \leq p_- - 1,
\]

(3.20)

\[
X_{r, p_-}^\pm = \text{im } e_{p_+}^{p_+ - r} \quad \text{in } V_{p_+, p_-}^{\pm} \\
\text{for } 1 \leq r \leq p_+ - 1,
\]

\[
X_{p_+, s}^\pm = \text{im } f_{p_-}^{p_- - s} \quad \text{in } V_{p_+, p_-}^{\pm} \\
\text{for } 1 \leq s \leq p_- - 1.
\]

In Fig. 2 the image of \( e_{p_+}^{p_+ - r} \) is given by with the collection of filled and open dots and the image of \( f_{p_-}^{p_- - s} \) by the collection of filled dots and boxes.

In addition to the images, we also consider the kernels, namely, \( \ker e_+ \cap \ker f_- \subset F \), which decomposes similarly to (2.13) as

\[
\ker e_+ \cap \ker f_- = \bigoplus_{r=1}^{p_+} \bigoplus_{s=1}^{p_-} (K_{r, s}^+ \oplus K_{r, s}^-),
\]

where the \( K_{r, s}^\pm \) can equivalently be identified in \( V_{p_+, p_-}^{\pm - r, p_- - s} \) as

\[
K_{r, s}^\pm = \ker e_{p_+}^r \cap \ker f_{p_-}^s \quad \text{in } V_{p_+, p_-}^{\pm - r, p_- - s} \\
\text{for } 1 \leq r \leq p_+ - 1, \\
\text{and } 1 \leq s \leq p_- - 1,
\]

(3.21)

\[
K_{r, p_-}^\pm = \ker e_{p_+}^r \quad \text{in } V_{p_+, p_-}^{\pm - r} \\
K_{p_+, s}^\pm = \ker f_{p_-}^s \quad \text{in } V_{p_+, p_-}^{\pm - s} \\
\text{for } 1 \leq s \leq p_- - 1
\]

(and \( K_{p_+, p_-}^\pm = V_{p_+, p_-}^{\pm} \) for uniformity of notation). Clearly, the \( K_{r, s}^\pm \) are Virasoro modules.
In Fig. 2 the kernel of \( e_+^r \) coincides with the collection of filled and open dots and the cross and the kernel of \( f_+^s \) coincides with the collection of filled dots, boxes, and the cross. It is easy to see that

\[
\mathcal{K}_{r,s}^+ \supset \mathcal{X}_{r,s}^+ \quad \text{with} \quad \mathcal{K}_{r,s}^+/\mathcal{X}_{r,s}^+ = J_{r,s} \quad \text{for} \quad 1 \leq r \leq p_+ - 1, \quad 1 \leq s \leq p_- + 1,
\]

\[
\mathcal{K}_{r,s}^- = \mathcal{X}_{r,s}^- \quad \text{for} \quad 1 \leq r \leq p_+, \quad 1 \leq s \leq p_-,
\]

\[
\mathcal{K}_{r,s}^+ = \mathcal{X}_{r,s}^+ \quad \text{whenever} \quad r = p_+ \quad \text{or} \quad s = p_-.
\]

In what follows, we see that the spaces \( \mathcal{X}_{r,s}^\pm \) are in fact irreducible \( \mathcal{W}_{p_+,p_-} \)-representations; on the other hand, the \( \mathcal{K}_{r,s}^\pm \), which are also \( \mathcal{W}_{p_+,p_-} \)-representations, are not all irreducible. However, they play a crucial role in the \( \mathcal{W}_{p_+,p_-} \) representation theory; \( \mathcal{K}_{1,1}^+ \) is the vacuum representation of the \( \mathcal{W}_{p_+,p_-} \) algebra, and the \( \mathcal{K}_{r,s}^\pm \) can be identified with the preferred basis of the \( \mathcal{W}_{p_+,p_-} \) fusion algebra \( \mathcal{G} \).

3.5. **A Lusztig extension of the quantum group.** Virasoro modules \( \mathcal{X}_{r,s}^\pm \) and \( \mathcal{K}_{r,s}^\pm \) admit an action of the \( \mathfrak{sl}(2, \mathbb{C}) \) algebra.

3.5.1. **Lemma.** The spaces \( \mathcal{X}_{r,s}^\pm \) and \( \mathcal{K}_{r,s}^\pm \) admit an \( \mathfrak{sl}(2, \mathbb{C}) \)-action that is a derivation of the operator product expansion. The Virasoro algebra \( \mathfrak{Vir}_{p_+,p_-} \), see (2.3), commutes with this \( \mathfrak{sl}(2, \mathbb{C}) \).

The construction of the \( \mathfrak{sl}(2, \mathbb{C}) \) action is based on Lusztig’s divided powers and can be briefly described as follows [47]. Morally, the \( e \) and \( f \) generators of \( \mathfrak{sl}(2, \mathbb{C}) \) are given by \( e_+^r/[p_+]_+! \) and \( f_+^r/[p_-]_-! \), where we use the standard notation

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [1]_q[2]_q \ldots [n]_q
\]

for \( q \)-integers and \( q \)-factorials and set \([m]_+ = [m]_q \) and \([m]_- = [m]_q \). But if taken literally, both \( e_+^r/[p_+]_+! \) and \( f_+^r/[p_-]_-! \) are given by \( 0/0 \) ambiguities for \( e_+ \) and \( f_- \) acting as described above and for \( q \) in Eq. (2.9) (because \([p_\pm]_\pm = 0\)). To resolve the ambiguities, we consider a deformation of \( q \) in Eq. (2.9) as

\[
q_\epsilon = e^{\alpha \epsilon} \quad \text{or, equivalently, a deformation of the parameter} \quad \alpha_0.
\]

We thus obtain \( e_+(\epsilon), f_-(\epsilon) \), and \( q \)-factorials \([n]_\epsilon \) depending on \( \epsilon \), such that \([p_\pm]_\epsilon \) \( \neq 0 \). The limits

\[
e = \lim_{\epsilon \to 0} \frac{e_+(\epsilon)^{p_+}}{[p_+]_\epsilon!} \quad \text{and} \quad f = \lim_{\epsilon \to 0} \frac{f_-(\epsilon)^{p_-}}{[p_-]_\epsilon!}
\]

act in \( \mathcal{K}_{r,s}^\pm \) and are independent of the deformation details, and are therefore well-defined operators in \( \mathcal{K}_{r,s}^\pm \). Because \( e \) and \( f \) commute with the Virasoro algebra action in \( \mathcal{K}_{r,s}^\pm \), \( \mathcal{X}_{r,s}^\pm \) are also representations of the \( \mathfrak{sl}(2, \mathbb{C}) \) algebra generated by \( e \) and \( f \).

We also note that \( J_{r,s} = \mathcal{K}_{r,s}^+ / \mathcal{X}_{r,s}^+ \) (where \( 1 \leq r \leq p_+ - 1 \) and \( 1 \leq s \leq p_- - 1 \)) are \( \mathfrak{sl}(2, \mathbb{C}) \)-singlets.
3.5.2. Lemma. The spaces $K_{r,s}^+$ and $X_{r,s}^+$ have the structure of $(\text{Vir}_{p+}, s\ell(2, \mathbb{C}))$-bimodules given by

$$K_{r,s}^+ = \bigoplus_{n \in \mathbb{N}} C_n \otimes \ell_{2n-1}, \quad 1 \leq r \leq p_+-1, \quad 1 \leq s \leq p_- - 1$$

and

$$X_{r,s}^+ \simeq \bigoplus_{n \in \mathbb{N}} J_{r,p_--s;2n-1} \otimes \ell_{2n-1},$$

where $\ell_n$ is the $n$-dimensional irreducible $s\ell(2, \mathbb{C})$ representation and

$$C_n = \begin{cases} J_{r,p_--s;2n-1}, & n \geq 2, \\ J_{r,s} \times \rightarrow J_{r,p_--s;1}, & n = 1, \end{cases}$$

where we use the same notation as in Fig. 1 for the extension of the Virasoro module $J_{r,s}$ by the Virasoro module $J_{r,p_--s;1}$.

Each direct summand in the decomposition of $X_{r,s}^+$ in (3.26) corresponds to a horizontal row of filled dots in Figs. 2 and 5 and each direct summand in the decomposition of $X_{p_--r,s}^-$ corresponds to a horizontal row of boxes and open dots in Figs. 3 and 4 respectively, with the $s\ell(2, \mathbb{C})$ algebra acting in each row. We thus have 1, 3, 5, ...-dimensional $s\ell(2, \mathbb{C})$-representations in Figs. 2 and 5 and 2, 4, 6, ...-dimensional representations in Figs. 3 and 4.

We now have all the ingredients needed for constructing the $W$-algebra $W_{p_+,p_-}$ of the $(p_+, p_-)$ logarithmic conformal field theory model.

4. VERTEX-OPERATOR ALGEBRA FOR THE $(p_+, p_-)$-CONFORMAL FIELD THEORY AND ITS REPRESENTATIONS

In this section, we define the chiral symmetry algebra $W_{p_+,p_-}$ of the $(p_+, p_-)$ logarithmic model and describe irreducible and Verma $W_{p_+,p_-}$-modules.

4.1. Definition. The algebra $W_{p_+,p_-}$ is the subalgebra of $\mathcal{L}_{p_+,p_-}$ with the underlying vector space $W_{p_+,p_-} = K_{1,1}^+ \subset \mathcal{L}_{p_+,p_-}$ (see (3.21)).

The $(\text{Vir}_{p_+}, s\ell(2, \mathbb{C}))$-bimodule structure of the vacuum $W_{p_+,p_-}$-representation $W_{p_+,p_-}$ is described in 3.5.2 and can be understood better using Fig. 2 as the part of the figure consisting of the cross and filled dots.\footnote{In the Coulomb-gas picture, $W_{p_+,p_-}$ can be viewed as the symmetry algebra of a fixed-point system with quenched disorder.}
4.2. Theorem.

(1) The algebra \( \mathcal{W}_{p_+, p_-} \) is generated by \( T(z) \) in (2.3) and the two currents \( W^+(z) \) and \( W^-(z) \) given by

\[
W^+(z) = (f_-)^{p_+ - 1}V_{1, p_- - 1; 3}(z), \quad W^-(z) = (e_+)^{p_+ - 1}V_{p_+ - 1, 1; -3}(z),
\]

which are Virasoro primaries of conformal dimension \( (2p_+ - 1)(2p_- - 1) \).

(2) The maximal (and the only nontrivial) vertex-operator ideal \( \mathcal{R} \) of \( \mathcal{W}_{p_+, p_-} \) is generated by \( W^+(z) \) and \( W^-(z) \).

(3) The quotient \( \mathcal{W}_{p_+, p_-} / \mathcal{R} \) is the vertex-operator algebra \( \mathcal{M}_{p_+, p_-} \) of the \((p_+, p_-)\) Virasoro minimal model.

4.2.1. Remark. In accordance with the definition of the screening action in (4.1), the \( W^\pm(z) \) currents can be written as (with normal ordering understood)

\[
W^+(z) = P^+_{3p_+ - 3p_- p_- + 1} \psi^{p_+ - 1}(z), \\
W^-(z) = P^-_{3p_+ - 3p_- p_- + 1} \psi^{p_+ - 1}(z),
\]

where \( P^\pm_j \) are polynomials of degree \( j \) in \( \varphi \), \( n \geq 1 \) (with \( \deg \varphi = n \)). The OPE of \( W^+(z) \) and \( W^-(z) \) is

\[
W^+(z)W^-(w) = \frac{S_{p_+, p_-}(T)}{(z - w)^{p_+ - 3p_- + 1}} + \text{less singular terms},
\]

where \( S_{p_+, p_-}(T) \) is the vacuum singular vector — the polynomial of degree \( \frac{1}{2}(p_+ - 1) \cdot (p_- - 1) \) in \( T \) and \( \varphi^\alpha T \), \( n \geq 1 \), such that \( S_{p_+, p_-}(T) = 0 \) is the polynomial relation for the energy–momentum tensor in the \((p_+, p_-)\) Virasoro minimal model. We note that this OPE is already quite difficult for direct analysis in the simplest case of \((3, 2)\) model, see Appendix A.

4.2.2. Remark. As follows from 4.1 and 3.5.2, \( \mathcal{W}_{p_+, p_-} \) admits an \( \mathfrak{sl}(2, \mathbb{C}) \) action and the \( W^\pm(z) \) generators are highest- and lowest-weight vectors of an \( \mathfrak{sl}(2, \mathbb{C}) \)-triplet. We note that the \( W \) algebra of \((p, 1)\) logarithmic models bears the name \emph{triplet} \([16, 14]\) because it admits such an \( \mathfrak{sl}(2, \mathbb{C}) \) action and the corresponding \( W^\pm(z) \) generators are also the highest- and lowest-weight vectors in a triplet.

Proof of 4.2. With decomposition 3.25 taken into account, it suffices to construct \(( \text{Vir}_{p_+}, \mathfrak{sl}(2, \mathbb{C}) \) highest-weight vectors in each direct summand in order to show part (1). This can be done as follows. The highest-weight vector of each component \( C_n \otimes \ell_{2n-1} \) (corresponding to the rightmost dot in each row of filled dots in Fig.2) is identified with the field \( W^{-, n}(z) \), \( n \geq 1 \), defined as the first nonzero term in the OPE of \( W^-(z) \) with \( W^{-, n-1}(z) \); the recursion base is \( W^{-, 1} \equiv 1 \). The highest-weight vector of \( C_1 \otimes \ell_1 \) is identified with the identity operator 1. This shows part (1).
Next, it follows from 3.5.2 that $X_{1,1}^{+} = \im e_{+}^{p_{1}-1} \cap \im f_{-}^{p_{-}-1}$ is invariant under $W_{p_{+},p_{-}}$. The space $X_{1,1}^{+}$ is the maximal $W_{p_{+},p_{-}}$-submodule in $W_{p_{+},p_{-}} = \mathcal{K}_{1,1}^{+}$, and therefore there exists a map $W_{p_{+},p_{-}} \to M_{p_{+},p_{-}}$ with the kernel $X_{1,1}^{+}$. We set $\mathcal{R} = X_{1,1}^{+}$, which is the maximal vertex-operator ideal in $W_{p_{+},p_{-}}$. Evidently, $W^{\pm}(z)$ generate some subspace $\mathcal{R}'$ in $\mathcal{R}$, but the decomposition into the direct sum of $\mathfrak{sl}(2, \mathbb{C})$-representations allows calculating the character of $\mathcal{R}'$, which coincides with the character of $\im e_{+}^{p_{1}-1} \cap \im f_{-}^{p_{-}-1}$ calculated from the Felder resolution, thus showing parts (2) and (3).

4.3. Irreducible $W_{p_{+},p_{-}}$-modules. We now describe irreducible $W_{p_{+},p_{-}}$-modules. First, the irreducible Virasoro modules $\mathcal{J}_{r,s}$ with $(r,s) \in I_{1}$ are at the same time $W_{p_{+},p_{-}}$-modules, with the ideal $\mathcal{R}$ acting by zero. We write $X_{r,s}$ for the $\mathcal{J}_{r,s}$ regarded as $W_{p_{+},p_{-}}$-modules. Second, there are $2p_{+}p_{-}$ irreducible $W_{p_{+},p_{-}}$-modules where $\mathcal{R}$ acts nontrivially. These are the spaces $X_{r,s}^{\pm}$ introduced in 2.3.1 which are evidently $W_{p_{+},p_{-}}$-modules because of their description in (3.20).

4.3.1. Proposition. $X_{r,s}^{\pm}$ is an irreducible $W_{p_{+},p_{-}}$-module.

Proof. Taking decomposition (3.26) into account, we show the irreducibility of $X_{r,s}^{\pm}$ by literally repeating the construction for the $(\text{Vir}_{p_{+},p_{-}}, \mathfrak{sl}(2, \mathbb{C}))$ highest-weight vectors in the proof of 4.2.

The lowest-dimension Virasoro primary in $X_{r,s}^{+}$ is $V_{r,p_{-}-s;1}$ (of dimension $\Delta_{r,p_{-}-s;1}$) and two lowest-dimension Virasoro primaries in $X_{r,s}^{-}$ are $(e_{+})^{p_{+}-r}V_{p_{+}-r,s;-2}$ and $(f_{-})^{p_{-}-s}V_{r,p_{-}-s;2}$ (of dimension $\Delta_{p_{+}-r,s;-2}$). The irreducible representations can be arranged into a $p_{+} \times p_{-}$ Kac table, with $X_{r,s}^{+}$ and $X_{r,s}^{-}$ in each box (in addition, the standard $(p_{+}-1) \times (p_{-}-1)$ Kac table containing $\frac{1}{2}(p_{+}-1)(p_{-}-1)$ distinct representations is inherited from the minimal model); the $(3,2)$ example is given in Table 1. The $W_{p_{+},p_{-}}$-action in $X_{r,s}^{\pm}$ is shown by dotted lines in Figs. 2–5 for $X_{r,s}^{+}$ with $1 \leq r \leq p_{+}-1$ and $1 \leq s \leq p_{-}-1$ in Fig. 2 for $X_{r,s}^{-}$ with $1 \leq r \leq p_{+}-1$ and $1 \leq s \leq p_{-}-1$ in Fig. 3 for $X_{r,p_{+}-r,p_{-}}^{+}$ with $1 \leq r \leq p_{+}-1$ in Fig. 4 and for $X_{r,p_{+}-r,p_{-}}^{-}$ with $1 \leq r \leq p_{+}-1$ in Fig. 5.

Table 1. The $p_{+} \times p_{-}$ $W$-algebra Kac table for $p_{+} = 3$ and $p_{-} = 2$. Each $(r,s)$ box contains the dimension of the highest-weight vectors of $X_{r,s}^{+}$ and $X_{r,s}^{-}$, in this order. The $(p_{+}-1) \times (p_{-}-1) = 2 \times 1$ subtable also contains the dimensions of $X_{r,s}^{-}$ (which are $X_{1,1}$ in this case), shown in parentheses. The (infinite) Virasoro content follows from the decompositions in 2.3.2.

| $r$ | $s$ | $p_{+}$ | $p_{-}$ |
|-----|-----|--------|--------|
| $5$ | $8$ | $33$   | $8$    |
| $24$ | $24$ | $35$   | $24$   |
| $2, 7$ | $0$ | $1, 5$ | $0$ |
| $1, 3$ | $10$ | $3$    | $3$    |
4.4. “Verma” modules. The \( L_{p+\cdot p-} \)-modules \( V^\pm_{r,s} \) (see 2.12) are \( W_{p+\cdot p-} \)-modules (simply because \( W_{p+\cdot p-} \) is a subalgebra in \( L_{p+\cdot p-} \)). In referring to the modules \( V^\pm_{r,s} \) as \( W_{p+\cdot p-} \)-modules, it is convenient to call them the Verma modules of the \( W_{p+\cdot p-} \) algebra. (Their counterparts for the Kazhdan–Lusztig-dual quantum group 34 are indeed Verma modules, but investigation of the Verma properties of \( V^\pm_{r,s} \) is a separate problem, which we do not consider here and only use the convenient and suggestive name for these modules.) We now describe their subquotients.

4.4.1. Proposition. The subquotient structure of the \( V^\pm_{r,s} \) is as follows.

1. \( V^\pm_{p+\cdot p-} \) is not considered here and only use the convenient and suggestive name for these modules. For \( 1 \leq r \leq p+ - 1 \), \( X^\pm_{p+ - r, p-} \subset V^\pm_{r,p-} \) and \( V^\pm_{r,p-} / X^\pm_{p+ - r, p-} \) is as follows.
2. For \( 1 \leq r \leq p+ - 1 \), \( X^\pm_{p+ - r, p-} \subset V^\pm_{r,s} \) and \( V^\pm_{r,s} / X^\pm_{p+ - r, p-} \) is as follows.
3. For \( 1 \leq r \leq p+ - 1 \) and \( 1 \leq s \leq p- - 1 \), \( V^\pm_{r,s} \) admits a filtration

\[
\mathcal{H}_0 \subset \mathcal{H}_1 \subset V^\pm_{r,s},
\]

where \( \mathcal{H}_0 \simeq X^\pm_{r,s} / X^\pm_{r,s} \) and \( V^\pm_{r,s} / \mathcal{H}_1 \simeq X^\pm_{r,s} \);

and \( V^\pm_{r,s} \) admits a filtration

\[
\mathcal{H}_0 \subset \mathcal{H}_1 \subset V^\pm_{r,s},
\]

where \( \mathcal{H}_0 \simeq X^\pm_{r,s} / X^\pm_{r,s} \) and \( V^\pm_{r,s} / \mathcal{H}_1 \simeq X^\pm_{r,s} \).

The subquotient structure is clear from Figs. 2, 3, 4, and 5. For example, we consider Fig. 2. The filled dots \( \bullet \) constitute the subquotient \( X^+_r \), the cross \( \times \) is \( X^r_s \), the \( \circ \) and \( \square \) are combined into \( X^-_{r,-s} \) and \( X^-_{p+, -s} \), and the \( \bigtriangleup \) are combined into \( X^+_{p+, -s} \).

4.5. Bimodule structure of \( \mathbb{F} \). The quantum group \( g_{p+\cdot p-} \) acts in the space \( \mathbb{F} \), which is in fact a \( (W_{p+\cdot p-} \cdot g_{p+\cdot p-}) \)-bimodule.

The subquotient structure of the bimodule \( \mathbb{F} \) shows the origin of the multiplicities in decomposition 4.4: each indecomposable \( W_{p+\cdot p-} \)-module enters in several copies produced by the action of \( \mathcal{H} \).

4.5.1. Proposition. As a \( (W_{p+\cdot p-} \cdot g_{p+\cdot p-}) \)-bimodule, the space \( \mathbb{F} \) decomposes as

\[
\mathbb{F} = \bigoplus_{(r,s) \in \mathbb{Z}} Q_{r,s},
\]

where \( Q_{r,s} \) are indecomposable \( (W_{p+\cdot p-} \cdot g_{p+\cdot p-}) \)-bimodules with the following structure.

1. If \( (r, s) = (0, 0) \), then \( Q_{0,0} = X^+_0 \cap X^+_0 \);
2. If \( (r, s) = (p+, 0) \), then \( Q_{p+,0} = X^-_{p+,0} \cap X^-_{p+,0} \);
3. If \( 1 \leq r \leq p+ - 1 \) and \( s = 0 \), then \( Q_{r,0} \) has the subquotient structure described in Fig. 6.
(4) if \( r = 0 \) and \( 1 \leq s \leq p_- - 1 \), then \( Q_{0,s} \) has the subquotient structure obtained from Fig. 6 by changing \( \chi^+_{r,p_-} \rightarrow \chi^+_{p_+,s} \), \( \chi^+_{p_+ - r,p_-} \rightarrow \chi^+_{p_+ - p_- - s} \), \( \chi^+_{r,p_-} \rightarrow \chi^+_{p_+,s} \), \( \chi^+_{p_+ - r,p_-} \rightarrow \chi^+_{p_+ - p_- - s} \), and \( e_+ \rightarrow f_- \);

(5) if \( (r, s) \in I_1 \), then \( Q_{r,s} \) has the subquotient structure described in Fig. 7.

![Figure 6](image6.png)

**Figure 6.** Subquotient structure of \( Q_{r,0} \). \( \boxtimes \) denotes the external tensor product. Solid lines denote the \( \mathcal{W}_{p_+,p_-} \)-action and dashed lines denote the \( \mathfrak{g}_{p_+,p_-} \)-action.

![Figure 7](image7.png)

**Figure 7.** Subquotient structure of \( Q_{r,s} \). We use the notation

- \( \bullet = \chi^+_{r,s} \), \( \circ = \chi^+_{r,p_- - s} \), \( \square = \chi^+_{p_+ - r,s} \), \( \blacklozenge = \chi^+_{p_+ - p_- - s} \), and \( \times = \chi^+_{r,s} \) for \( \mathcal{W}_{p_+,p_-} \)-modules, and

- \( \blacklozenge = \chi^+_{r,s} \), \( \blacklozenge = \chi^+_{r,p_- - s} \), \( \spadesuit = \chi^+_{p_+ - r,s} \), \( \spadesuit = \chi^+_{p_+ - p_- - s} \)

for quantum-group modules. Solid lines show the action of \( \mathcal{W}_{p_+,p_-} \)-generators and dashed lines show the action of \( \mathfrak{g}_{p_+,p_-} \)-generators, and \( \boxtimes \) denotes external tensor product.
4.5.2. In particular, forgetting the quantum-group structure, we obtain a decomposition of \( \mathbb{F} \) into a finite sum of \( \mathcal{W}_{p^+,p^-} \) Verma modules with multiplicities (dimensions of irreducible \( \mathfrak{g}_{p^+,p^-} \)-modules) as

\[
\mathbb{F} = \bigoplus_{r=1}^{p^+ - 1} \bigoplus_{s=1}^{p^+ - 1} (p^+ - r)(p^- - s) \left( V^+_{r,s} \oplus V^-_{r,s} \right) + \bigoplus_{r=1}^{p^+ - 1} (p^+ - r)p^- \left( V^+_{r,p^-} \oplus V^-_{r,p^-} \right) + \bigoplus_{s=1}^{p^+ - 1} (p^- - s)p^+ \left( V^+_{p^+,s} \oplus V^-_{p^+,s} \right) + p^+ p^- \left( V^+_{p^+,p^-} \oplus V^-_{p^+,p^-} \right).
\]

The structure of \( \mathbb{F} \) as a \( \mathcal{W}_{p^+,p^-} \)-\( \mathfrak{g}_{p^+,p^-} \)-bimodule can be taken as a starting point for constructing the projective \( \mathcal{W}_{p^+,p^-} \)-modules by the method in [17].

5. THE SPACE OF TORUS AMPLEUTES

In this section, we find the \( SL(2, \mathbb{Z}) \)-representation generated from the space of characters \( \mathcal{S} \) in the logarithmic \( (p^+, p^-) \) model. With Conjecture 1.3 assumed, this gives the space of torus amplitudes. In 5.1, we calculate the characters of irreducible \( \mathcal{W}_{p^+,p^-} \)-modules in terms of theta-constants, see (1.9). It is then straightforward to calculate modular transformations of the characters, which we do in 5.2. This extends the space of characters by \( \mathbb{C}[\tau]/\tau^3 \), giving rise to the space of generalized characters \( \bar{\mathcal{C}} \) (which, as noted above, most probably coincides with the space \( \mathcal{C} \) of torus amplitudes). (Theta-function derivatives enter the characters through the second order; this can be considered a “technical” reason for the explicit occurrences of \( \tau \) and \( \tau^2 \) in the modular transformation properties.) The analysis of the \( SL(2, \mathbb{Z}) \)-representation on \( \bar{\mathcal{C}} \) then yields the results in 1.2. In 5.3, we use the established decomposition of the \( SL(2, \mathbb{Z}) \) action to briefly consider modular invariants.

5.1. \( \mathcal{W}_{p^+,p^-} \)-characters. Let

\[
\chi^\pm_{r,s}(q) = \text{Tr}_{\mathcal{X}^\pm_{r,s}} q^L q^{-\frac{c}{24}}, \quad 1 \leq r \leq p^+, \quad 1 \leq s \leq p^-,
\]

with \( c \) given by (2.4), be the characters of the irreducible \( \mathcal{W}_{p^+,p^-} \)-representations \( \mathcal{X}^\pm_{r,s} \) and \( \mathcal{X}_{r,s} \) (see 4.3).

5.1.1. Proposition. The irreducible \( \mathcal{W}_{p^+,p^-} \)-representation characters are given by

\[
\chi_{r,s} = \frac{1}{\eta} (\theta_{p^+,s-p^-} - \theta_{p^+,s+p^-}), \quad (r, s) \in J_1,
\]

\[
\chi^+_{r,s} = \frac{1}{(p^+ p^-)^2 \eta} \left( \frac{(p^+ s + p^- s)^2}{4} \theta_{p^+,s+p^-}^r + \frac{(p^+ s - p^- s)^2}{4} \theta_{p^+,s-p^-}^r - (p^+ s + p^- s) \theta_{p^+,s+p^-}^r + (p^+ s - p^- s) \theta_{p^+,s-p^-}^r \right), \quad 1 \leq r \leq p^+, \quad 1 \leq s \leq p^-,
\]

\[
\chi^-_{r,s} = \frac{1}{(p^+ p^-)^2 \eta} \left( \frac{(p^+ s + p^- s)^2}{4} \theta_{p^+,s+p^-}^r + \frac{(p^+ s - p^- s)^2}{4} \theta_{p^+,s-p^-}^r - (p^+ s + p^- s) \theta_{p^+,s+p^-}^r + (p^+ s - p^- s) \theta_{p^+,s-p^-}^r \right), \quad 1 \leq r \leq p^+, \quad 1 \leq s \leq p^-.
\]
In particular, for the characters of (5.3)

\[ (p_+ p_- p_+ s - p_- r) \theta_{p_+ p_- p_+ s - p_- r} + (p_- s - p_- r) \theta_{p_+ p_- p_+ s - p_- r} + \frac{(p_- s - p_- r)^2 - (p_+ p_-)^2}{4} \theta_{p_+ p_- p_+ s - p_- r} - \frac{(p_- s - p_- r)^2 - (p_+ p_-)^2}{4} \theta_{p_+ p_- p_+ s - p_- r}, \]

where \(\theta_{\cdot}\) is given by

\[ \text{Setting } \theta_{p_+ p_- p_+ s - p_- r} \text{ which gives (5.1) when rewritten in terms of the theta-constants.} \]

Proof. We first recall the well-known irreducible Virasoro characters \[ \text{(5.4) char } J_{r,s;n}(q) = \frac{q^{1-c}}{\eta(q)} \left( \sum_{m \geq 0} q^{\Delta_{r,s;2m}} + \sum_{m \geq 1} q^{\Delta_{r,s;2m+1}} \right). \]

In particular, for the characters of \(\mathcal{X}_{r,s}\), we immediately have

\[ \text{char } \mathcal{X}_{r,s}(q) = \frac{q^{1-c}}{\eta(q)} \left( \sum_{m \in \mathbb{Z}} (q^{\Delta_{r,s;2m}} - q^{\Delta_{r,s;2m+1}}) \right), \]

which gives (5.1) when rewritten in terms of the theta-constants.

Next, from \[ \text{(2.3.2) we have that the character of each } \mathcal{X}_{r,s}^+ \text{ for } 1 \leq r \leq p_+ \text{ and } 1 \leq s \leq p_- \text{ is given by} \]

\[ \chi_{r,s}^+(q) = \text{Tr}_{\mathcal{X}_{r,s}^+} q^{L_0 - \frac{c}{24}} = \sum_{a \geq 0} (2a + 1) \text{ char } J_{r,s-p_-;2a+1}(q). \]

Substituting (5.4) here, we have

\[ \chi_{r,s}^+(q) = \Sigma_1 + \Sigma_2, \]

where

\[ \Sigma_1 = \sum_{a \geq 0} (2a + 1) q^{\frac{1-c}{24}} \left( \sum_{m \geq 0} q^{\Delta_{r,s-p_-;2m+2}} + \sum_{m \geq 1} q^{\Delta_{r,s-p_-;2m+1}} \right), \]

\[ \Sigma_2 = -\sum_{a \geq 0} (2a + 1) q^{\frac{1-c}{24}} \left( \sum_{m \geq 0} q^{\Delta_{r,s-p_-;2m+2}} + \sum_{m \geq 0} q^{\Delta_{r,s-p_-;2m+1}} \right). \]

Setting \(\bar{\Delta}_{r,s;n} = \Delta_{r,s;n} + \frac{1-c}{24}\) for brevity, we obtain

\[ \Sigma_1 = \sum_{m \geq 0} \sum_{a=0}^{m} (2a + 1) q^{\bar{\Delta}_{r,s-p_-;2m+2}} + \sum_{m \leq -2} \sum_{a=0}^{m-2} (2a + 1) q^{\bar{\Delta}_{r,s-p_-;2m+1}} \]

\[ = \sum_{m \in \mathbb{Z}} (m+1)^2 q^{\bar{\Delta}_{r,s-p_-;2m+1}} = \sum_{m \in \mathbb{Z}} m^2 q^{p_+ p_-(m - \frac{p_+ p_-}{2p_+ p_-})}. \]

\[ \Sigma_2 = \sum_{m \in \mathbb{Z}} (m - \frac{p_+ p_-}{2p_+ p_-})^2 q^{p_+ p_-(m - \frac{p_+ p_-}{2p_+ p_-})}. \]
We see shortly, the space of Verma-module characters coincides with the space of theta-functions (not including their derivatives). We also note that similar expressions for characters involving second derivatives of theta-functions were proposed in [49].

5.1.2. Remark. From the definition of the Verma modules $V^\pm_{r,s}$ in Eqs. (2.12), we calculate their characters as

$$\text{char } V^+_{r,s}(q) = \frac{q^{1+}}{\eta(q)} \sum_{n \in \mathbb{Z}} q^{\Delta_{r,s},2n} = \frac{1}{\eta(q)} \theta_{p+s-p-r,p+p-}(q),$$

$$\text{char } V^-_{r,s}(q) = \frac{q^{1-}}{\eta(q)} \sum_{n \in \mathbb{Z}} q^{\Delta_{r,s},2n+1} = \frac{1}{\eta(q)} \theta_{p+p+s-p-r,p+p-}(q).$$

On the other hand, from the subquotient structure of Verma modules described in 4.4.1 we obtain the identities

$$\text{char } V^+_{r,s}(q) = \chi_{r,s}(q) + \chi_{r,s}(q) + \chi_{r,p_-,q}(q) + \chi_{p_+,r,q}(q),$$

$$\text{char } V^-_{r,s}(q) = \chi_{r,s}(q) + \chi_{r,s}(q) + \chi_{p_+,r,s}(q) + \chi_{p_+,r,s}(q)$$

for $(r, s) \in J_1$, which shows that, as could be expected, the space of Verma-module characters coincides with the space of theta-functions. We also note that similar expressions for characters involving second derivatives of theta-functions were proposed in [49].

5.2. The space $\mathfrak{g}$. We now construct the $SL(2, \mathbb{Z})$-representation $\mathfrak{g}$ generated from the space $\mathfrak{g}$ of the irreducible-representation characters and simultaneously obtain the decomposition in 1.2

5.2.1. First, $\mathfrak{g}$ contains the subspace $R_{\text{min}}$ spanned by $\chi_{r,s}$ with $(r, s) \in J_1$ — the Virasoro minimal-model characters, which evidently carry an $SL(2, \mathbb{Z})$-representation. Second, as we see shortly, the $\frac{1}{2}(p_+ + 1)(p_- + 1)$-dimensional space $R_{\text{proj}} \subset \mathfrak{g}$ linearly spanned by the functions

$$\chi_{r,s} = \chi_{r,s} + 2\chi_{r,s} + 2\chi_{r,p_-,r,s} + 2\chi_{p_+,r,s} + 2\chi_{p_+,r,p_-,s}, \quad (r, s) \in J_1,$$

$$\chi_{0,s} = 2\chi_{p_+,r,p_-,s} + 2\chi_{p_-,s}, \quad 1 \leq s \leq p_--1,$$

$$\chi_{r,0} = 2\chi_{p_+,r,p_-} + 2\chi_{r,p_-}, \quad 1 \leq r \leq p_+-1,$$

$$\chi_{0,0} = 2\chi_{p_+,p_-},$$

(5.5)
\[ \chi_{p_+,0} = 2 \chi_{p_+,p_-}^- \]

(which can be identified with the characters of projective \( \mathcal{W}_{p_+,p_-} \)-modules) also carries an \( SL(2, \mathbb{Z}) \)-representation.

Next, the \((p_+ p_- - 1)\)-dimensional complement of \( R_{\text{min}} \oplus R_{\text{proj}} \) in \( \mathcal{G} \) contains no more subspaces closed under the \( SL(2, \mathbb{Z}) \)-action. It is convenient to choose the functions

\[
\rho_{r,s} = \frac{p_+ s - p_- r}{2} \chi_{r,s} - p_+(p_- - s)(\chi_{r,s}^+ + \chi_{p_+,p_-}^-),
\]

\[
\rho_{0,s} = p_+(s \chi_{p_+,p_-} - (p_- - s) \chi_{p_+,s}),
\]

\[
\rho_{r,0} = p_-(r \chi_{p_+,p_-} - (p_+ - r) \chi_{p_+,s}),
\]

\[
\rho_{r,s} = p_+ p_- \left( (p_+ - r)(p_- - s) \chi_{r,s}^+ + rs \chi_{p_+,p_-}^- - \frac{(p_+ s - p_- r)^2}{4 p_+ p_-} \chi_{r,s} \right),
\]

as a basis in this complement.

We then define

\[
\varphi_{r,s}(\tau) = \tau \rho_{r,s}(\tau),
\]

\[
\varphi_{r,s}(\tau) = \tau \rho_{r,s}(\tau),
\]

\[
\psi_{r,s}(\tau) = 2 \tau \rho_{r,s}(\tau) + i \pi p_- \chi_{r,s}(\tau),
\]

\[
\varphi_{r,s}(\tau) = \tau^2 \rho_{r,s}(\tau) + i \pi p_- \chi_{r,s}(\tau),
\]

(5.6)

5.2. Proposition. The \( SL(2, \mathbb{Z}) \)-representation \( \mathcal{C} \) generated from \( \mathcal{G} \) is the linear span of the \( \frac{1}{2} (3p_+ - 1)(3p_- - 1) \) functions

\[ \chi_{r,s}(\tau), \rho_{r,s}(\tau), \psi_{r,s}(\tau), \text{and} \varphi_{r,s}(\tau) \text{ with } (r, s) \in J_1, \]

\[ \chi_{r,s}(\tau) \text{ with } (r, s) \in J_0, \]

\[ \rho_{p_+,s}(\tau) \text{ and } \varphi_{p_+,s}(\tau) \text{ with } (r, s) \in J_{12}, \text{ and} \]

\[ \rho_{r,s}(\tau) \text{ and } \varphi_{r,s}(\tau) \text{ with } (r, s) \in J_{13}. \]

This is summarized in Table 2, where we also indicate the \( SL(2, \mathbb{Z}) \)-representation and its dimension spanned by each group of functions.

The proof amounts to the following two lemmas, the first of which shows that \( \mathcal{C} \) is closed under the \( S \)-transformation and the second that it is closed under the \( T \)-transformation.
5.2.3. Lemma.

(5.7) \[ \chi_{r,s}(-\frac{1}{\tau}) = -\frac{2\sqrt{2}}{\sqrt{p_+p_-}} \sum_{(r',s') \in I_1} (-1)^{r's'+r's} \sin \frac{\pi p_- r'r'}{p_+} \sin \frac{\pi p_+ s's'}{p_-} \chi_{r',s'}(\tau), \]

\[(r, s) \in I_1,\]

(5.8) \[ \varphi_{r,s}(-\frac{1}{\tau}) = \frac{\sqrt{2}}{\sqrt{p_+p_-}} \left( \sum_{(r',s') \in I_1} 2(-1)^{r's'+r's} \cos \frac{\pi p_- r'r'}{p_+} \cos \frac{\pi p_+ s's'}{p_-} \varphi_{r',s'}(\tau) \right. \]

\[+ \sum_{r'-1}^p (-1)^{r's} \cos \frac{\pi p_- r'r}{p_+} \chi_{r',0}(\tau) + \sum_{s'=1}^{p_+} (-1)^{r's'} \cos \frac{\pi p_+ s's'}{p_-} \varphi_{0,s'}(\tau) \]

\[+ \frac{1}{2} \rho_{0,0}(\tau) + \frac{1}{2} (-1)^{p_-r+p_+s} \varphi_{r,0}(\tau), \]

\[(r, s) \in I_0,\]

(5.9) \[ \rho^{\oplus}_{r,s}(-\frac{1}{\tau}) = -i \frac{\sqrt{2}}{\sqrt{p_+p_-}} \left( \sum_{(r',s') \in I_1} 2(-1)^{r's'+r's} \sin \frac{\pi p_+ s's'}{p_-} \cos \frac{\pi p_- r'r'}{p_+} \varphi^{\oplus}_{r',s'}(\tau) \right. \]

\[+ \sum_{s'=1}^{p_-} (-1)^{r's} \sin \frac{\pi p_+ s's'}{p_-} \varphi^{\oplus}_{0,s'}(\tau), \]

\[(r, s) \in I_1,\]

(5.10) \[ \varphi^{\ominus}_{r,s}(-\frac{1}{\tau}) = i \frac{\sqrt{2}}{\sqrt{p_+p_-}} \left( \sum_{(r',s') \in I_1} 2(-1)^{r's'+r's} \sin \frac{\pi p_+ s's'}{p_-} \cos \frac{\pi p_- r'r'}{p_+} \rho^{\ominus}_{r',s'}(\tau) \right. \]

\[+ \sum_{s'=1}^{p_-} (-1)^{r's} \sin \frac{\pi p_+ s's'}{p_-} \rho^{\ominus}_{0,s'}(\tau), \]

\[(r, s) \in I_1,\]

(5.11) \[ \rho^{\ominus}_{r,s}(-\frac{1}{\tau}) = -i \frac{\sqrt{2}}{\sqrt{p_+p_-}} \left( \sum_{(r',s') \in I_1} 2(-1)^{r's'+r's} \sin \frac{\pi p_+ s's'}{p_-} \cos \frac{\pi p_- r'r'}{p_+} \varphi^{\ominus}_{r',s'}(\tau) \right. \]

\[+ \sum_{r'=1}^{p_+} (-1)^{r's} \sin \frac{\pi p_- r'r}{p_+} \varphi^{\ominus}_{r,0}(\tau), \]

\[(r, s) \in I_2,\]

(5.12) \[ \varphi^{\ominus}_{r,s}(-\frac{1}{\tau}) = i \frac{\sqrt{2}}{\sqrt{p_+p_-}} \left( \sum_{(r',s') \in I_1} 2(-1)^{r's'+r's} \sin \frac{\pi p_+ s's'}{p_-} \cos \frac{\pi p_- r'r'}{p_+} \rho^{\ominus}_{r',s'}(\tau) \right. \]
there, we find the transformation of the theta-constants as

\[ \rho_{r,s}(\frac{-1}{\tau}) = -\frac{2\sqrt{2}}{\sqrt{p_+p_-}} \sum_{(r',s') \in \mathcal{J}_1} (-1)^{r'+s'} \sin \frac{\pi p_- r'}{p_+} \varphi_{r',s'}(\tau), \]

\[ (r,s) \in \mathcal{J}_1, \]

\[ \varphi_{r,s}(\frac{-1}{\tau}) = -\frac{2\sqrt{2}}{\sqrt{p_+p_-}} \sum_{(r',s') \in \mathcal{J}_1} (-1)^{r'+s'} \sin \frac{\pi p_- r'}{p_+} \rho_{r',s'}(\tau), \]

\[ (r,s) \in \mathcal{J}_1. \]

Proof. The standard modular transformation formulas for the theta-constants are

\[ \theta_{s,p}(\frac{-1}{\tau}) = \sqrt{\frac{-i\tau}{2p}} \sum_{r=0}^{2p-1} e^{-i\pi \frac{rs}{p}} \theta_{r,p}(\tau), \]

\[ \theta'_{s,p}(\frac{-1}{\tau}) = \tau \sqrt{\frac{-i\tau}{2p}} \sum_{r=0}^{2p-1} e^{-i\pi \frac{rs}{p}} \theta'_{r,p}(\tau), \]

\[ \theta''_{s,p}(\frac{-1}{\tau}) = \sqrt{\frac{-i\tau}{2p}} \sum_{r=0}^{2p-1} e^{-i\pi \frac{rs}{p}} (\tau^2 \theta''_{r,p}(\tau) + i\pi p \tau \theta_{r,p}(\tau)) \]

and the eta function transforms as

\[ \eta(\tau + 1) = e^{i\pi \eta(\tau)}, \quad \eta(-\frac{1}{\tau}) = \sqrt{-i\tau} \eta(\tau). \]

We next use the resummation formula (C.2). Substituting

\[ g(r) = e^{-i\pi \frac{rs}{p_+p_-}}, \quad h_+(r) = \frac{1}{\sqrt{2p_+p_-}} \theta_{\tau}(\tau), \quad u_+(r,s) = \kappa_{r,s}, \quad u_-(r,s) = \chi_{r,s} \]

there, we find the transformation of the theta-constants as

\[ \frac{\theta_s(\frac{-1}{\tau})}{\eta(\frac{-1}{\tau})} = \frac{1}{\sqrt{2p_+p_-}} \sum_{r=0}^{2p-1} e^{-i\pi \frac{rs}{p_+p_-}} \theta_{r}(\tau), \]

\[ = \frac{1}{\sqrt{2p_+p_-}} \left( \frac{1}{2} \kappa_{0,0}(\tau) + (-1)^s \kappa_{0,0}(\tau) + \sum_{r'=1}^{p_-} \cos \frac{\pi r'}{p_+} \kappa_{r',0}(\tau) + \sum_{s'=1}^{p_-} \cos \frac{\pi s'}{p_-} \kappa_{0,s'}(\tau) \right) \]

\[ + 2 \sum_{(r',s') \in \mathcal{J}_1} \left( \cos \frac{\pi r'}{p_+} \cos \frac{\pi s'}{p_-} \kappa_{r',s'}(\tau) - \sin \frac{\pi r'}{p_+} \sin \frac{\pi s'}{p_-} \chi_{r',s'}(\tau) \right) \].

Taking the sum and the difference of these formulas with \( s \mapsto p_-r - p_+s \) and \( s \mapsto p_-r + p_+s \), we obtain (5.7) and (5.8).
Substituting
\[ g(r) = e^{-i\pi \frac{r^s}{p_+ p_-}}, \quad h_+(r) = \frac{\tau}{\sqrt{2 p_+ p_-}} \theta'_\tau(\tau), \quad u^+_{\tau}(r, s) = \rho_{r,s}^\Theta, \quad u^-_{\tau}(r, s) = \rho_{r,s}^\Sigma \]
in (C.2), we find the transformation of the theta-constant derivatives as
\[ \frac{\theta'_s\left(-\frac{1}{\tau}\right)}{\eta\left(-\frac{1}{\tau}\right)} = -\frac{2i\pi}{\sqrt{2 p_+ p_-}} \left( \sum_{(r', s') \in \mathcal{J}_1} \left( \begin{array}{c} \pi r' s' \\ p_+ \cos \frac{\pi s' s}{p_+} + \cos \frac{\pi s' s}{p_-} \rho_{r', s'}^\Theta(\tau) + \sin \frac{\pi s' s}{p_+} \rho_{r', s'}^\Sigma(\tau) \\ + \frac{1}{2} \sum_{r' = 1}^{p_+ - 1} \sin \frac{\pi r' s'}{p_+} \rho_{r', 0}^\Theta(\tau) + \frac{1}{2} \sum_{s' = 1}^{p_- - 1} \sin \frac{\pi s' s}{p_-} \rho_{0, s'}^\Sigma(\tau) \end{array} \right) \right). \]

Taking the sum and the difference of these formulas with \( s \mapsto p_- r - p_+ s \) and \( s \mapsto p_- r + p_+ s \), we obtain (5.6), (5.10) and (5.11), (5.12).

For the second derivatives of the theta-constants, we have
\[ \frac{1}{\eta\left(-\frac{1}{\tau}\right)} \left( \theta''_{p_+ s + p_- r}\left(-\frac{1}{\tau}\right) - \theta''_{p_+ s - p_- r}\left(-\frac{1}{\tau}\right) \right) = \sqrt{2} \frac{1}{\eta\left(-\frac{1}{\tau}\right)} \sum_{(r', s') \in \mathcal{J}_1} (-1)^{r' s' + s' r' + 1} \sin \frac{\pi r' s'}{p_+} \sin \frac{\pi s' s}{p_-} \]
\[ \times \left( \tau^2 (\theta''_{p_+ s' + p_- r'}(\tau) - \theta''_{p_+ s' - p_- r'}(\tau)) + i\pi p_+ p_- (\theta_{p_+ s' + p_- r'}(\tau) - \theta_{p_+ s' - p_- r'}(\tau)) \right), \]
whence (5.13), (5.14), and (5.15) immediately follow.

**5.2.4. Lemma.**
\[ \chi_{r,s}(\tau + 1) = \lambda_{r,s} \chi_{r,s}(\tau), \quad \chi_{r,s}(\tau + 1) = \chi_{r,s}(\tau), \]
\[ \rho_{r,s}^\Theta(\tau + 1) = \lambda_{r,s} \rho_{r,s}^\Theta(\tau), \quad \varphi_{r,s}^\Theta(\tau + 1) = \lambda_{r,s} \left( \varphi_{r,s}^\Theta(\tau) + \rho_{r,s}^\Theta(\tau) \right), \]
\[ \rho_{r,s}^\Sigma(\tau + 1) = \lambda_{r,s} \rho_{r,s}^\Sigma(\tau), \quad \psi_{r,s}^\Sigma(\tau + 1) = \lambda_{r,s} \left( \psi_{r,s}^\Sigma(\tau) + \rho_{r,s}^\Sigma(\tau) \right), \]
\[ \rho_{r,s}(\tau + 1) = \lambda_{r,s} \rho_{r,s}(\tau), \quad \psi_{r,s}(\tau + 1) = \lambda_{r,s} \left( \psi_{r,s}(\tau) + 2 \rho_{r,s}(\tau) \right), \]
\[ \varphi_{r,s}(\tau + 1) = \lambda_{r,s} \left( \varphi_{r,s}(\tau) + \psi_{r,s}(\tau) + \rho_{r,s}(\tau) \right), \]
where
\[ \lambda_{r,s} = e^{2i\pi (\Delta_{r,s} - \frac{1}{12})} = (-1)^{r s} e^{i\pi (\frac{p_+}{2 p_+ p_-} r^2 + \frac{p_-}{2 p_-} s^2 - \frac{1}{12})}. \]

**Proof.** Elementary calculation.

**5.2.5.** It may be useful to give the formulas inverse to those in [5.2.1] Let \( \mathcal{J} \) denote the set of indices
\[ \mathcal{J} = \{(r, s) \mid 1 \leq r \leq p_+, 1 \leq s \leq p_-\} \]
(actually labeling the Kac-table boxes) and let \( \mathcal{J}_1 = \mathcal{J} \setminus \mathcal{J}_1 \). For \( (r, s) \in \mathcal{J}_1 \), it is convenient to set
\[ \rho_{r,s} = \rho_{p_+ - r, p_- - s}, \quad \rho_{r,s}^\Theta = -\rho_{p_+ - r, p_- - s}^\Theta, \quad \rho_{r,s}^\Sigma = -\rho_{p_+ - r, p_- - s}^\Sigma, \]
\( \kappa_{r,s} = \kappa_{p_+ - r, p_- - s}, \quad \chi_{r,s} = \chi_{p_+ - r, p_- - s}. \)

Then the formulas that invert those in 5.2.1 are

\[
\kappa^+_{r,s} = \frac{1}{(p_+ + p_-)^2} \left( \rho_{r,s} - p_- p_{r,s} - p_+ s p_{r,s} \right)
+ \frac{p_+ + p_- r}{2} \kappa_{r,s} - \frac{p^2 s^2 + p_+^2 r^2}{4} \chi_{r,s}, \quad (r, s) \in \mathcal{I},
\]

\[
\chi^+_{p_+, s} = \frac{1}{p_+ + p_-} \left( \rho_{0_{p_+ - s} - s} \right), \quad 1 \leq s \leq p_- - 1,
\]

\[
\chi^+_{r, p_-} = \frac{1}{p_+ + p_-} \left( p_+ + p_- r_0 \right) \chi_{p_+ - r_0, 0}, \quad 1 \leq r \leq p_+ - 1,
\]

\[
\chi^+_{p_+, p_-} = \frac{1}{2} \kappa_{0, 0}.
\]

\[
\chi^-_{r, s} = \frac{1}{(p_+ + p_-)^2} \left( -\rho_{p_+ - r, s} - p_- r p_{p_+ - r, s} + p_+ s p_{p_+ - r, s} \right)
+ \frac{p_+ + p_- r}{2} \kappa_{p_+ - r, s} + \frac{p^2 s^2 + p_+^2 r^2 - (p_+ + p_-)^2}{4} \chi_{p_+ - r, s}, \quad (r, s) \in \mathcal{I},
\]

\[
\chi^-_{p_+, s} = \frac{1}{p_+ + p_-} \left( p_+ + p_- s_0 \right), \quad 1 \leq s \leq p_- - 1,
\]

\[
\chi^-_{r, p_-} = \frac{1}{p_+ + p_-} \left( p_+ - r \right) \chi_{0, r_0}, \quad 1 \leq r \leq p_+ - 1,
\]

\[
\chi^-_{p_+, p_-} = \frac{1}{2} \kappa_{0, 0}.
\]

### 5.3. Modular invariants.

The decomposition of the \( SL(2, \mathbb{Z}) \) action established in 1.2 considerably simplifies finding sesquilinear modular invariants. We illustrate this by giving several easily constructed series.

First, in the \( \rho_0 \), \( \rho_1 \), and \( \rho \) sectors (see Table 2), modular invariants necessarily involve \( \tau \) explicitly: they are given by

\[
\rho_0 \left( \tau, \bar{\tau} \right) = \prod_{r=1}^{p_+ - 1} \text{im} \left( \rho_{r, 0} \left( \tau \right) \right)^2 + 2 \sum_{(r, s) \in \mathcal{I}} \text{im} \left( \rho_{r, s} \left( \tau \right) \right)^2,
\]

a “symmetric” expression with the \( \rho_{r, s} \), and

\[
\rho \left( \tau, \bar{\tau} \right) = \sum_{(r, s) \in \mathcal{I}} \bar{\rho}_{r, s} \left( \bar{\tau} \right) \left( 8 \text{im} \left( \tau \right)^2 \rho_{r, s} \left( \tau \right) + 4 p_+ p_- \text{im} \left( \tau \right) \pi \chi_{r, s} \left( \tau \right) \right)
+ \bar{\chi}_{r, s} \left( \bar{\tau} \right) \left( 4 p_+ p_- \text{im} \left( \tau \right) \pi \rho_{r, s} \left( \tau \right) + \left( \pi p_+ p_- \right) \chi_{r, s} \left( \tau \right) \right).
\]

All these are expressed through the characters \( \chi_{r, s}^\pm \) and \( \chi_{r, s} \) in accordance with the formulas in 5.2.1.

Next, in the \( \kappa \) sector, we have the A-series invariants

\[
\kappa_{A} \left( \tau, \bar{\tau} \right) =
\]
and additional \( D \)-series invariants in the case where \( p_+ \equiv 0 \mod 4 \):

\[
\chi_{[D]}(\tau, \bar{\tau}) = |\chi_{0,0}(\tau) + \chi_{p_+,0}(\tau)|^2 + \sum_{r=1}^{p_+-1} |\chi_{r,0}(\tau) + \chi_{p_+-r,0}(\tau)|^2 \\
+ \sum_{2 \leq s \leq p_- - 1 \atop s \text{ even}} |\chi_{0,s}(\tau) + \chi_{0,p_- - s}(\tau)|^2 + \sum_{(r,s) \in I_1 \atop s \text{ even}} 2|\chi_{r,s}(\tau) + \chi_{r,p_- - s}(\tau)|^2.
\]

Of course, “symmetric” \( D \)-invariants exist whenever \( p_+ \equiv 0 \mod 4 \). Again, all these invariants are expressed through the characters via the formulas in 5.2.1.

We also note an \( E_6 \)-like invariant for \((p_+, p_-) = (5, 12)\):

\[
\chi_{[E_6]}(\tau, \bar{\tau}) = |\chi_{0,1}(\tau) - \chi_{0,7}(\tau)|^2 + |\chi_{0,2}(\tau) - \chi_{0,10}(\tau)|^2 + |\chi_{0,5}(\tau) - \chi_{0,11}(\tau)|^2 \\
+ 2|\chi_{1,1}(\tau) - \chi_{1,7}(\tau)|^2 + 2|\chi_{2,1}(\tau) - \chi_{2,7}(\tau)|^2 + 2|\chi_{2,5}(\tau) - \chi_{3,1}(\tau)|^2 \\
+ 2|\chi_{2,2}(\tau) - \chi_{3,2}(\tau)|^2 + 2|\chi_{1,5}(\tau) - \chi_{4,1}(\tau)|^2 + 2|\chi_{1,2}(\tau) - \chi_{4,2}(\tau)|^2.
\]

It would be quite interesting to systematically obtain all modular invariants as quantum-group invariants.

6. Conclusions

In the logarithmically extended \((p_+, p_-)\) minimal models, we have constructed the chiral vertex-operator algebra \( \mathcal{W}_{p_+, p_-} \), its irreducible representations and Verma modules, and calculated the \( SL(2, \mathbb{Z}) \)-representation generated by the characters. The space of generalized characters carrying that representation most probably coincides with the space of torus amplitudes. A great deal of work remains to be done, however.

6.1. First and foremost, constructing the chiral sector of the space of states requires building projective modules of \( \mathcal{W}_{p_+, p_-} \). Unfortunately, little is known about their structure (some indirect but quite useful information has recently become available in [24]). One of the clues is the known structure [34] of projective modules of the Kazhdan–Lusztig-dual quantum group \( \mathfrak{g}_{p_+, p_-} \), which must be a piece of the structure of projective \( \mathcal{W}_{p_+, p_-} \)-modules. That is, taking an irreducible \( \mathcal{W}_{p_+, p_-} \)-module \( \mathcal{X}_{r,s}^+ \) (with \( 1 \leq r \leq p_+ - 1 \) and \( 1 \leq s \leq p_- - 1 \)), replacing it with the \( \mathfrak{g}_{p_+, p_-} \)-module \( \mathfrak{X}_{r,s}^+ \), taking the universal projective cover \( \mathcal{P}_{r,s}^+ \) of the latter, and translating the result back into the \( \mathcal{W}_{p_+, p_-} \)-language, we obtain the structure of sixteen subquotients in Fig. 8 where solid and dotted lines denote the elements of \( \text{Ext}^1 \) and the symbolic notation for the modules is as in Fig. 7, i.e., \( \blacklozenge = \mathcal{X}_{r,s}^+ \), \( \Box = \mathcal{X}_{p_+, p_- - s}^- \), \( \circ = \mathcal{X}_{r,p_- - s}^- \), and \( \blacktriangle = \mathcal{X}_{p_+, p_- - s}^+ \). However, in addition to these subquotients and embeddings, the projective \( \mathcal{W}_{p_+, p_-} \)-module \( \mathcal{P}_{r,s}^+ \) (the universal projective
cover of $\mathcal{X}_{r,s}^+$) must also contain subquotients isomorphic to the minimal-model representations $\mathcal{X}_{r,s}$. Furthermore, the $\mathcal{W}_{p_+,p_-}$-representation category contains the projective covers $P_{r,s}$ of the irreducible representations $\mathcal{X}_{r,s}$, which have no projective-module counterparts in the $\mathfrak{g}_{p_+,p_-}$-representation category (the $P_{r,s}$ may be interesting because of their relation to the boundary-condition-changing operator whose 4-point correlation function gives the Cardy formula [6] for the crossing probability).

6.2. The foregoing is directly relevant to establishing the relation between our construction, specialized to $(p_+ = 3, p_- = 2)$, and the analysis in [38, 50, 29, 9], where a family of $c = 0$ models was considered, labeled by a parameter $b$. The approach in [9], in principle, allows adding various primary fields to the theory and, in this sense, is not aimed at fixing a particular chiral algebra. The logarithmic minimal models in this paper, on the other hand, are minimal extensions of the rational $(p_+, p_-)$ models, minimal in the sense that consistent “intermediate” models — with a subalgebra of $\mathcal{W}_{p_+,p_-}$ but with a finite number of fields — do not seem to exist.\(^6\) There are various ways to construct “larger” logarithmic extensions of minimal models, for example, by taking the kernel not of two but of one screening (cf. [17]). In our “minimal” setting, in particular, the $\mathcal{W}_{3,2}$-primary fields are just those whose dimensions are in Table 1 and similarly for all the $(p_+, p_-)$ models, as stated in 1.2 (We note once again that the Virasoro field content, which appears to have been the subject of some discussion in other approaches, then follows uniquely from the decompositions in 2.3.2 — in fact, from the structure of the relevant complexes. In particular, the number of Virasoro primary fields is not limited to an integer multiple of either the standard $(p_+ - 1) \times (p_- - 1)$ or the extended $p_+ \times p_-$ Kac table.) In the setting in this paper, fixing the value of $b$ or another similar parameter requires constructing projective

---

\(\text{Figure 8. Sixteen subquotients, which do not suffice to build a projective } \mathcal{W}_{p_+,p_-}\text{-module.}\)
W-modules, which will be addressed elsewhere (among other things, it will completely settle the “logarithmic partner” issues).

6.3. The $SL(2, \mathbb{Z})$-representation on the generalized characters (presumably, torus amplitudes) coincides with the $SL(2, \mathbb{Z})$-representation on the center of the Kazhdan–Lusztig dual quantum group $\mathfrak{g}_{p_+, p_-}$. This remarkable correspondence deserves further study, as do other aspects of the Kazhdan–Lusztig correspondence.

The Kazhdan–Lusztig correspondence suggests that in some generality, the space of torus amplitudes $W$ and the conformal field theory center $Z_{\text{cft}}$ are related by conformal-field-theory analogues of the Radford and Drinfeld maps known in the theory of quantum groups. Then, under the identification of $Z_{\text{cft}}$ with the space of boundary conditions preserved by $W$, the images of irreducible characters under the “$W$-Radford map” are the Ishibashi states and the images under the “$W$-Drinfeld map” are the Cardy states.

We also give a fusion algebra suggested by the Kazhdan–Lusztig correspondence. This fusion is the $\mathfrak{g}_{p_+ p_-}$ Grothendieck ring $\mathfrak{g}$, with the preferred basis of $2p_+ p_-$ irreducible representations. In terms of the $W_{p_+ p_-}$ algebra, we identify the preferred basis elements with the $2p_+ p_-$ representations $K_{r,s}^+ = K_{r,s}^-$ defined in (3.21). For all $1 \leq r, r' \leq p_+$, $1 \leq s, s' \leq p_-$, and $\alpha, \beta = \pm$, we thus have the algebra

$$K_{r,s}^\alpha K_{r',s'}^\beta = \sum_{u=|r-r'|+1} \sum_{v=|s-s'|+1} \mathcal{K}_{u,v}^\alpha \mathcal{K}_{r',s'}^\beta$$

where

$$\mathcal{K}_{r,s}^\alpha = \begin{cases} K_{r,s}^\alpha, & 1 \leq r \leq p_+, 1 \leq s \leq p_-, \\ K_{2p_+ - r, s}^\alpha + 2K_{r-p_+, s}^\alpha, & p_+ + 1 \leq r \leq 2p_+ - 1, 1 \leq s \leq p_-, \\ K_{r,2p_- - s}^\alpha + 2K_{r,s-p_-}^\alpha, & 1 \leq r \leq p_-, p_- + 1 \leq s \leq 2p_- - 1, \\ K_{2p_+ - r, 2p_- - s}^\alpha + 2K_{2p_+ - r, s-p_-}^\alpha + 2K_{r-p_+, 2p_- - s}^\alpha + 4K_{r-p_+, s-p_-}^\alpha, & p_+ + 1 \leq r \leq 2p_+ - 1, p_- + 1 \leq s \leq 2p_- - 1. \end{cases}$$

This algebra has several noteworthy properties:

1. it is generated by two elements $K_{1,2}^+$ and $K_{2,1}^+$;
2. its radical is generated by the algebra action on $K_{p_+, p_-}^+$; the quotient over the radical coincides with the fusion of the $(p_+, p_-)$ Virasoro minimal models;
3. $K_{1,1}^+$ is the identity;
4. $K_{1,1}^-$ acts as a simple current, $K_{1,1}^- K_{r,s}^\alpha = K_{r,s}^{-\alpha}$. 
A functor between the representation categories of $\mathcal{W}_{p_+,p_-}$ and $\mathfrak{g}_{p_+,p_-}$ is not yet known, and the identification of the basis of irreducible $\mathfrak{g}_{p_+,p_-}$-representation with the $\mathcal{K}_{r,s}^\pm$ is based on indirect arguments. First, it is clear that the minimal-model representations $\mathcal{X}_{r,s}$ act by zero on the other basis elements in the fusion algebra, simply because of the vanishing of three-point functions involving two minimal-model representations and one representation on which the ideal $\mathcal{R}$ acts nontrivially. We next recall that the $2p_+,p_-$ representations $\mathcal{K}_{r,s}^\pm$ are defined as kernels of the screenings. Some of them are reducible, see (3.22), “inasmuch as” the Felder complexes have a nonzero cohomology (the minimal-model representations $J_{r,s} \equiv \mathcal{X}_{r,s}$). Their role as counterparts of the irreducible $\mathfrak{g}_{p_+,p_-}$-representations under the Kazhdan–Lusztig correspondence is supported by the numerical evidence in [24]: decomposing $\mathcal{K}_{r,s}^\pm$ into the Virasoro modules (see 3.5.2) shows that the $(3,2)$ and $(5,2)$ results in [24] agree with (6.1). Needless to say, it would be quite interesting to properly define the $\mathcal{W}_{p_+,p_-}$-fusion and establish (6.1) by some “relatively direct” (or just numerical, as the first step) calculation of the $\mathcal{W}_{p_+,p_-}$ (not just Virasoro) coinvariants.

We also note that the quantum group $\mathfrak{g}_{p_+,p_-}$ can be quite useful in constructing the full (holomorphic + antiholomorphic) space of states, by taking $\mathfrak{g}_{p_+,p_-}$-invariants in the product of $(\mathcal{W}_{p_+,p_-}, \mathfrak{g}_{p_+,p_-})$-bimodules.

Acknowledgments. We are grateful to A. Belavin, J. Fuchs, A. Isaev, S. Parkhomenko, and P. Pyatov for the useful discussions and comments. This paper was supported in part by the RFBR Grant 04-01-00303 and the RFBR–JSPS Grant 05-01-02934YaF. The work of AMG, AMS and IYuT is supported in part by the LSS-4401.2006.2 grant. The work of IYuT was supported in part by the RFBR Grant 05-02-17217 and the “Dynasty” foundation.

APPENDIX A. (3,2)-MODEL EXAMPLES

In the simplest case of the (3, 2) model, the minimal-model character is trivial, $\chi_{1,1}(q) = 1$, and 12 nontrivial characters are expressed through theta-constants in accordance with Eqs. (5.2)–(5.3). Explicitly, for example, the characters of $\mathcal{X}_{1,1}^\pm$ are

$$q^{-2}\chi_{1,1}^+(q) = 1 + q + 2q^2 + 2q^3 + 4q^4 + 4q^5 + 7q^6 + 8q^7 + 12q^8 + 14q^9 + 21q^{10} + 24q^{11} + 34q^{12} + 44q^{13} + 58q^{14} + 72q^{15} + \ldots$$

(with the character of $\mathcal{K}_{1,1}^+$ given by $1 + \chi_{1,1}^+(q)$), and

$$q^{-7}\chi_{1,1}^-(q) = 2 + 2q + 4q^2 + 6q^3 + 10q^4 + 12q^5 + 20q^6 + 26q^7 + 36q^8 + 48q^9$$

The fusion algebra in (6.1) also follows from the above $SL(2,\mathbb{Z})$ action on the characters via a procedure generalizing the Verlinde formula, similar to that in [21]; the (somewhat bulky) details will be given elsewhere.
(the dimensions of the respective highest-weight vectors are \(\Delta_{1,1} = 2\) and \(\Delta_{2,1} = 7\). The modular transformation properties of the characters follow by expressing them through the basis in Table 2 (where now \(|J_1| = 1\), \(|J_0| = 6\), \(|J_2| = 2\), and \(|J_3| = 3\) in accordance with the formulas in \(5.2.5\) and using \(5.2.3\). Under the \(SL(2,\mathbb{Z})\) action, the 13 characters (including \(\chi_{1,1}\)) give rise to the dimension-20 space of generalized characters. To give examples of modular transformations, we use Eqs. \(5.6\) and express the \(S\)-transformed characters through the characters with \(\tau\)-dependent coefficients:

\[
\chi_{1,1}^+(\tau) = (-\frac{13}{144} + \frac{i(\sqrt{3} + 18\pi)}{108})\chi_{1,1}(\tau) + (\frac{\tau^2}{144} - \frac{\tau^2}{3})\chi_{1,1}(\tau)
\]

\[
+ (\frac{1}{12\sqrt{3}} + \frac{i\tau}{6})\chi_{1,2}(\tau) - (\frac{i\tau}{6\sqrt{3}} + \frac{\tau^2}{6})\chi_{1,1}(\tau) - \frac{\tau}{12\sqrt{3}}\chi_{3,1}(\tau)
\]

\[
\chi_{1,1}^-(-\frac{1}{\tau}) = (-\frac{23}{144} - \frac{i(\sqrt{3} + 18\pi)}{108} + \frac{\tau^2}{144} + \frac{\tau^2}{3})\chi_{1,1}(\tau) + (\frac{\tau^2}{6\sqrt{3}} + \frac{\tau^2}{6})\chi_{1,1}(\tau)
\]

\[
+ (\frac{1}{12\sqrt{3}} + \frac{i\tau}{6})\chi_{1,2}(\tau) + (\frac{i\tau}{6\sqrt{3}} + \frac{\tau^2}{6})\chi_{1,1}(\tau) - \frac{\tau}{12\sqrt{3}}\chi_{3,1}(\tau)
\]

We next consider fusion relations \(6.1\). To write them explicitly for \((p_+, p_-) = (3, 2)\), we recall that \(K_{1,1}^-\) acts as \(K_{r,s}^-K_{1,1}^- = K_{r,s}^-\), and therefore the entire 12 \times 12 multiplication table essentially reduces to its 6 \times 6 block, where (recalling that \(K_{1,1}^+\) acts as identity) the \(\frac{1}{2} \cdot 5 \cdot 6\) independent relations are

\[
K_{1,2}^+K_{1,2}^+ = 2K_{1,1}^++2K_{1,1}^+, \quad K_{1,2}^+K_{2,1}^+ = K_{1,2}^+, \quad K_{1,2}^+K_{2,2}^+ = 2K_{2,1}^++2K_{2,2}^+
\]

\[
K_{1,2}^+K_{3,1}^+ = K_{3,2}^+, \quad K_{1,2}^+K_{3,2}^+ = 2K_{3,1}^-+2K_{3,1}^+
\]

\[
K_{2,1}^+K_{2,1}^- = K_{1,1}^++K_{3,1}^+, \quad K_{2,1}^+K_{2,2}^- = K_{1,2}^-+K_{3,2}^+, \quad K_{2,1}^+K_{3,1}^- = 2K_{1,1}^-+2K_{2,1}^+
\]

\[
K_{2,1}^+K_{3,2}^- = 2K_{1,2}^-+2K_{2,2}^+, \quad K_{2,2}^+K_{2,1}^- = 2K_{1,1}^-+2K_{1,1}^-+2K_{3,1}^++2K_{3,1}^+, \quad K_{2,2}^+K_{2,2}^- = 2K_{1,2}^-+2K_{2,2}^+
\]

\[
K_{2,2}^+K_{3,1}^- = 2K_{2,1}^-+2K_{1,1}^-+2K_{3,1}^++2K_{3,1}^+, \quad K_{3,1}^+K_{3,2}^- = 2K_{2,1}^-+2K_{1,1}^-+4K_{1,1}^-+4K_{1,1}^++4K_{1,1}^++4K_{2,1}^-+2K_{2,2}^+
\]

\[
K_{3,2}^+K_{3,2}^- = 4K_{1,1}^-+4K_{2,1}^-+4K_{1,1}^-+4K_{1,1}^++4K_{1,1}^++4K_{2,1}^-+2K_{2,2}^+.\n\]
Finally, more as a curiosity than for any practical purposes, we give explicit free-field expressions for the $W_{3,2}$-algebra generators in (4.1) (arbitrarily normalized):

$$W^+ = \left( \frac{35}{27} \partial^4 \varphi \right)^2 + \frac{56}{27} \partial^3 \varphi \partial^2 \varphi + \frac{28}{27} \partial^2 \varphi \partial \varphi + \frac{8}{27} \partial \varphi \partial^2 \varphi - \frac{280}{9 \sqrt{3}} \partial^2 \varphi \partial^2 \varphi$$

$$- \frac{70}{3 \sqrt{3}} \partial^4 \varphi \partial^2 \varphi - \frac{280}{9 \sqrt{3}} \partial^2 \varphi \partial^2 \varphi - \frac{56}{3 \sqrt{3}} \partial^2 \varphi \partial^2 \varphi - \frac{28}{9 \sqrt{3}} \partial^2 \varphi \partial^2 \varphi$$

$$+ \frac{35}{3} (\partial^2 \varphi)^4 + \frac{280}{9} \partial^3 \varphi (\partial^2 \varphi)^2 \partial \varphi + \frac{280}{9} (\partial^2 \varphi)^2 (\partial^2 \varphi)^2 + \frac{140}{9} \partial^4 \varphi \partial^2 \varphi (\partial^2 \varphi)^2$$

$$+ \frac{56}{9} \partial^5 \varphi (\partial^2 \varphi)^3 - \frac{140}{3 \sqrt{3}} (\partial^2 \varphi)^3 (\partial^2 \varphi)^2 - \frac{560}{3 \sqrt{3}} \partial^2 \varphi (\partial^2 \varphi)^2 - \frac{70}{3 \sqrt{3}} \partial^4 \varphi (\partial^2 \varphi)^4$$

$$+ 70 (\partial^2 \varphi)^4 (\partial^2 \varphi)^2 + \frac{56}{3} \partial^3 \varphi (\partial^2 \varphi)^5 - \frac{28}{\sqrt{3}} \partial^2 \varphi (\partial^2 \varphi)^6 + (\partial^2 \varphi)^8 - \frac{1}{27 \sqrt{3}} \partial^2 \varphi)^4 \right),$$

$$W^- = \left( \frac{217}{192} (\partial^2 \varphi)^2 - \frac{2653}{3456} \partial^2 \varphi \partial^2 \varphi - \frac{23}{384} \partial^3 \varphi \partial^2 \varphi - \frac{11}{1152} \partial^2 \varphi \partial^2 \varphi - \frac{1}{768} \partial^2 \varphi \partial^2 \varphi$$

$$- \frac{1225}{64 \sqrt{3}} \partial^4 \varphi \partial^2 \varphi + \frac{13475}{576 \sqrt{3}} (\partial^4 \varphi)^2 \partial^2 \varphi + \frac{2695}{64 \sqrt{3}} \partial^5 \varphi \partial^3 \varphi \partial^2 \varphi + \frac{2555}{192 \sqrt{3}} \partial^5 \varphi \partial^3 \varphi \partial^2 \varphi$$

$$+ \frac{2891}{576 \sqrt{3}} \partial^6 \varphi (\partial^2 \varphi)^2 - \frac{1351}{192 \sqrt{3}} \partial^6 \varphi \partial^3 \varphi \partial^2 \varphi - \frac{103}{192 \sqrt{3}} \partial^7 \varphi \partial^2 \varphi \partial^2 \varphi - \frac{13}{384 \sqrt{3}} \partial^8 \varphi (\partial^2 \varphi)^2$$

$$+ \frac{3535}{32} (\partial^2 \varphi)^2 (\partial^2 \varphi)^2 - \frac{735}{16} (\partial^2 \varphi)^3 \partial \varphi - \frac{3395}{54} \partial^4 \varphi (\partial^2 \varphi)^3 + \frac{245}{24} \partial^4 \varphi \partial^2 \varphi \partial^2 \varphi$$

$$+ \frac{12695}{128} (\partial^4 \varphi)^2 (\partial^2 \varphi)^2 - \frac{2443}{288} \partial^4 \varphi \partial^2 \varphi (\partial^2 \varphi)^2 - \frac{19}{96} \partial^7 \varphi (\partial^2 \varphi)^3 - \frac{13405}{144 \sqrt{3}} (\partial^2 \varphi)^5 + \frac{8225}{24} \partial^3 \varphi (\partial^2 \varphi)^3 \partial \varphi$$

$$- \frac{105 \sqrt{3}}{4} \partial^4 \varphi \partial^2 \varphi (\partial^2 \varphi)^2 + \frac{665}{24 \sqrt{3}} \partial^4 \varphi (\partial^2 \varphi)^2 (\partial^2 \varphi)^2$$

$$- \frac{245}{8 \sqrt{3}} \partial^5 \varphi \partial^2 \varphi (\partial^2 \varphi)^3 - \frac{91}{24 \sqrt{3}} \partial^6 \varphi (\partial^2 \varphi)^4 + \frac{1605}{144} (\partial^2 \varphi)^4 (\partial^2 \varphi)^2 + \frac{385}{4} \partial^4 \varphi (\partial^2 \varphi)^2 (\partial^2 \varphi)^3$$

$$+ \frac{525}{8} (\partial^3 \varphi)^2 (\partial^2 \varphi)^4 + \frac{35}{3} \partial^4 \varphi \partial^2 \varphi (\partial^2 \varphi)^4 - 7 \partial^5 \varphi (\partial^2 \varphi)^5 + \frac{665}{3 \sqrt{3}} (\partial^2 \varphi)^3 (\partial^2 \varphi)^2$$

$$+ \frac{105 \sqrt{3}}{2} \partial^4 \varphi \partial^2 \varphi (\partial^2 \varphi)^5 - \frac{35}{3 \sqrt{3}} \partial^4 \varphi (\partial^2 \varphi)^6 + \frac{455}{6} (\partial^2 \varphi)^2 (\partial^2 \varphi)^6 + 5 \partial^3 \varphi (\partial^2 \varphi)^7$$

$$+ \frac{25}{3 \sqrt{3}} \partial^2 \varphi (\partial^2 \varphi)^8 + (\partial^2 \varphi)^8 - \frac{1}{13824 \sqrt{3}} \partial^2 \varphi)^4 \right) e^{-2 \sqrt{3} \varphi},$$

where, despite the brackets introduced for the compactness of notation, the nested normal ordering is from right to left, e.g., \( \partial^4 \varphi (\partial^2 \varphi (\partial^2 \varphi e^{2 \sqrt{3} \varphi})) \). These dimension-15 operators have the OPE

$$W^+(z) W^-(w) = 2^7 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 17 \frac{T(w)}{(z - w)^{28}} + \ldots,$$

with the energy–momentum tensor given by (2.3), that is,

$$T(z) = \frac{1}{2} \partial \varphi(z) \partial \varphi(z) - \frac{1}{2 \sqrt{3}} \partial^2 \varphi(z).$$

In the (3, 2) model, the minimal-model vertex-operator algebra \( M_{p+, p-} \) is trivial, or in other words, \( T(z) \) is in the ideal \( \mathcal{R} \).
B.1. Drinfeld double of a quantum group. We recall \cite{45,46} that the space $H^*$ of linear functions on a Hopf algebra $H$ is a Hopf algebra with the multiplication, comultiplication, unit, counit, and antipode given by

$$
\langle \beta \gamma, x \rangle = \sum_{(x)} \langle \beta, x' \rangle \langle \gamma, x'' \rangle, \quad \langle \Delta(\beta), x \otimes y \rangle = \langle \beta, yx \rangle,
$$

(B.1)

$$
\langle 1, x \rangle = \epsilon(x), \quad \epsilon(\beta) = \langle \beta, 1 \rangle, \quad \langle S(\beta), x \rangle = \langle \beta, S^{-1}(x) \rangle
$$
for any $\beta, \gamma \in H^*$ and $x, y \in H$. The Drinfeld double \cite{45,46} $D(H)$ is a Hopf algebra with the multiplication, comultiplication, unit, counit, and antipode given (in addition to the formulas for $H$ and $H^*$) by

$$
x \beta = \sum_{(x)} \beta(S^{-1}(x')x')x'', \quad x \in H, \quad \beta \in H^*.
$$

(B.2)

B.2. Proof of 3.3. By induction, it is easy to see that the comultiplication in the PBW basis in $\mathcal{H}$ is given by

$$
\Delta(e_{jmn}) = \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} q_+^{p_r(r-m)} q_-^{p_s(s-n)} e_{j+2p_r(m-r)-2p_s(n-s),r,s} \otimes e_{j-m-r,n-s}.
$$

(B.3)

With (B.5), we define $\kappa, f_+, e_+ \in \mathcal{H}^*$ by the relations

$$
\langle \kappa, e_{jmn} \rangle = \delta_{m,0} \delta_{n,0} q^j,
$$

$$
\langle f_+, e_{jmn} \rangle = -\delta_{m,1} \delta_{n,0} \frac{q_+^j}{q_+^p - q_-^p}, \quad \langle e_-, e_{jmn} \rangle = -\delta_{m,0} \delta_{n,1} \frac{q_-^{-j}}{q_+^p - q_-^p}
$$
and then follow the standard step-by-step construction of a Drinfeld double, based on Eqs. (B.1) and (B.2), which now become

$$
k \beta = \beta(k^{-1} \beta) k, \quad e \beta = \beta(\beta e) 1 + \beta(\beta k p) e - \beta(e k 2 p - k 2 p^2) k 2 p,
$$

(B.5)

$$
f \beta = \beta(\beta f) 1 + \beta(\beta k 2 p) f - \beta(f k 2 p - k^2 2 p) k 2 p^2.
$$

We here use (B.3). The following formulas are then obtained by direct calculation:

$$
\kappa(k^{-1} k) = \kappa, \quad \kappa(e e) = 0, \quad \kappa(k 2 p) = q^{\kappa} \kappa,
$$

$$
\kappa(e k 2 p - k^2 2 p) = 0, \quad \kappa(f 2 p - k 2 p) = 0,
$$

$$
f_+(k^{-1} k) = q^{f_+} f_+, \quad f_+(e e) = -\frac{\kappa 2 p}{q_+^p - q_-^p}, \quad f_+(k 2 p) = f_+, \quad f_+(f 2 p - k 2 p) = q^{f_+} f_+
$$

$$
f_+(e k 2 p - k^2 2 p) = -\frac{1}{q_+^p - q_-^p}, \quad f_+(f f) = 0, \quad f_+(k 2 p) = f_+,
$$

$$
f_+(f k 2 p + k^2 2 p) = 0,
$$
\[ e_-(k^{-1} ? k) = q_- e_-, \quad e_-(?e_+) = 0, \quad e_-(?k^{2p-}) = e_- , \]
\[ e_-(e_k^{-2p} ? k^{2p-}) = 0, \quad e_-(?f_+) = \frac{-\kappa^{-2p+}}{q^{p_+} - q^{-p_+}}; \quad e_-(?k^{2p+}) = e_- , \]
\[ e_-(f^{-k^{2p+}} ? k^{2p+}) = \frac{-1}{q^{p_+} - q^{-p_+}}. \]

Applying the first relation in (B.1) to (B.3), we subsequently obtain
\[ \langle \kappa^a, e_{jmn} \rangle = q^{aj} \delta_{m0} \delta_{n0} , \]
\[ \langle f^a_+, e_{jmn} \rangle = (-1)^a \delta_{ma} \delta_{n0} \frac{[a]_+!}{(q^{p_+} - q^{-p_-})^a} q_+^{aj + p_- \frac{a(a-1)}{2}} , \]
\[ \langle e^a_-, e_{jmn} \rangle = (-1)^a \delta_{m0} \delta_{na} \frac{[a]_-!}{(q^{p_+} - q^{-p_-})^a} q_-^{aj + p_+ \frac{a(a-1)}{2}} . \]
\[ \langle f^a_+ e^b_- \kappa^c, e_{jmn} \rangle = \delta_{ma} \delta_{nb} \frac{(-1)^{a+b}[a]_+! [b]_-!}{(q^{p_+} - q^{-p_-})^a (q^{p_+} - q^{-p_-})^b} q_+^{aj + p_- \frac{a(a-1)}{2}} q_-^{bj + p_+ \frac{b(b-1)}{2}} q^c_j. \]

It is now straightforward to prove that the \( 4p_+^{-2}p_-^2 \) elements \( \{ f^a_+ e^b_- \kappa^c \} \) with \( 0 \leq a \leq p_+ - 1 \), \( 0 \leq b \leq p_- - 1 \), and \( 0 \leq c \leq 4p_+p_- - 1 \) are linearly independent, cf. [22], and that the relations claimed in the theorem are indeed satisfied.

**APPENDIX C. SUMMATION OVER 2p_+p_- CONSECUTIVE VALUES**

Here, we isolate elementary but bulky formulas needed in the derivation of modular transformations. For any \( f \) satisfying \( f(r + 2p_+p_-) = f(r) \), we have the obvious identity
\[
\sum_{r=0}^{2p_+p_-} f(r) = \sum_{r'=0}^{p_+ - 1} \sum_{s'=0}^{p_- - 1} f(p_-r' + p_+s') + \sum_{r'=0}^{p_- - 1} \sum_{s'=1}^{p_+ - 1} f(p_-r' - p_+s')
\]
\[ = f(0) + f(-p_+p_-) + \sum_{r'=1}^{p_- - 1} \sum_{s'=1}^{p_+ - 1} (f(p_-r' + p_+s') + f(p_-r' - p_+s'))
\]
\[ + \sum_{r'=1}^{p_- - 1} (f(p_-r') + f(-p_-r')) + \sum_{s'=1}^{p_+ - 1} (f(p_+s') + f(-p_+s')). \]

Next, for \( f(r) = g(r) h_\pm(r) \), where \( g(r + 2p_+p_-) = g(r) \), \( h_\pm(r + 2p_+p_-) = h_\pm(r) \), and in addition \( h_\pm(-r) = \pm h_\pm(r) \), we have
\[ (C.1) \sum_{r=0}^{2p_+p_- - 1} g(r) h_\pm(r) = g(0) h_\pm(0) + g(-p_+p_-) h_\pm(-p_+p_-)
\]
\[ + \sum_{(r',s') \in \mathbb{F}_1} \left( (g(p_-r' + p_+s') \pm g(-p_-r' - p_+s')) h_\pm(p_-r' + p_+s')
\]
\[ + (g(p_-r' - p_+s') \pm g(-p_-r' + p_+s')) h_\pm(p_-r' - p_+s') \right) . \]
In terms of the combinations 

\[ u_+^+(r, s) = h_+(p_r + p_s) + h_+(p_r - p_s), \]

\[ u_-(r, s) = h_+(p_r + p_s) - h_+(p_r - p_s), \quad (r, s) \in \mathcal{J}_1, \]

Eq. (C.1) is written as

\[
\sum_{r=0}^{2p_+p_- - 1} g(r)h_+(r) = g(0)h_+(0) + g(-p_+p_-)h_+(-p_+p_-) \\
+ \frac{1}{2} \sum_{(r', s') \in \mathcal{J}_1} \left( (g(p_r' + p_s') \pm g(-p_r' - p_s'))u_+^+(r', s') \right. \\
\left. + g(p_r' - p_s') \pm g(-p_r' + p_s'))u_-^+(r', s') \right) \\
+ \sum_{r'=1}^{p_+ - 1} (g(p_r') \pm g(-p_r'))h_+(p_r') + \sum_{s'=1}^{p_- - 1} (g(p_s') \pm g(-p_s'))h_+(p_s').
\]

This rearrangement of the sum of \(2p_+p_-\)-periodic functions over \(2p_+p_-\) consecutive values is an efficient way to find modular transformations of the \((p_+, p_-)\)-model characters.

REFERENCES

[1] V. Gurarie, M. Flohr, and C. Nayak, The Haldane–Rezayi quantum Hall state and conformal field theory, Nucl. Phys. B498 (1997) 513–538 [cond-mat/9701212].

[2] F.H.L. Essler, H. Frahm, and H. Saleur, Continuum limit of the integrable sl(2/1) 3-3 superspin chain, Nucl. Phys. B712 (2005) 513–572 [cond-mat/0501197].

[3] H. Saleur, Lectures on non perturbative field theory and quantum impurity problems, cond-mat/9812110.

[4] G. Piroux and P. Ruelle, Pre-logarithmic and logarithmic fields in a sandpile model, J. Stat. Mech. 0401 (2004) P005 [hep-th/0407143].

[5] M. Jeng, Conformal field theory correlations in the Abelian sandpile model, Phys. Rev. E 71 (2005) 016140 [cond-mat/0407115].

[6] J. Cardy, Critical percolation in finite geometries, J. Phys. A25 (1992) L201–L206, hep-th/9111026.

[7] G.A.M. Watts, A crossing probability for critical percolation in two dimensions, J. Phys. A29 (1996) L363 [cond-mat/9603167].

[8] H. Saleur and B. Duplantier, Exact determination of the percolation hull exponent in two dimensions, Phys. Rev. Lett. 58, 22 (1987) 2325–2328.

[9] V. Gurarie and A.W.W. Ludwig, Conformal field theory at central charge \(c = 0\) and two-dimensional critical systems with quenched disorder, hep-th/0409105.
[10] Y.-Z. Huang, J. Lepowsky, and L. Zhang, *A logarithmic generalization of tensor product theory for modules for a vertex operator algebra*, math.QA/0311235.

[11] M. Miyamoto, *Modular invariance of vertex operator algebras satisfying C\(_2\)-cofiniteness*, math.QA/0209101.

[12] J. Fuchs, *On non-semisimple fusion rules and tensor categories*, hep-th/0602051.

[13] H.G. Kausch, *Extended conformal algebras generated by a multiplet of primary fields*, Phys. Lett. B 259 (1991) 448.

[14] M.R. Gaberdiel and H.G. Kausch, *A rational logarithmic conformal field theory*, Phys. Lett. B386 (1996) 131–137 [hep-th/9606050].

[15] M.R. Gaberdiel and H.G. Kausch, *A local logarithmic conformal field theory*, Nucl. Phys. B538 (1999) 631–658 [hep-th/9807091].

[16] H.G. Kausch, *Symplectic fermions*, Nucl. Phys. B583 (2000) 513–541 [hep-th/0003029].

[17] J. Fjelstad, J. Fuchs, S. Hwang, A.M. Semikhatov, and I.Yu. Tipunin, *Logarithmic conformal field theories via logarithmic deformations*, Nucl. Phys. B633 (2002) 379–413 [hep-th/0201091].

[18] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Kazhdan–Lusztig correspondence for the representation category of the triplet W-algebra in logarithmic CFT*, math.QA/0512621.

[19] M. Flohr, *Bits and pieces in logarithmic conformal field theory*, Int. J. Mod. Phys. A18 (2003) 4497–4592 [hep-th/0111228].

[20] M.R. Gaberdiel and H.G. Kausch, *Indecomposable fusion products*, Nucl. Phys. B477 (1996) 293–318 [hep-th/9604026].

[21] J. Fuchs, S. Hwang, A.M. Semikhatov, and I.Yu. Tipunin, *Nonsemisimple fusion algebras and the Verlinde formula*, Commun. Math. Phys. 247 (2004) 713–742 [hep-th/0306274].

[22] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center*, 265 (2006) 47–93 [hep-th/0504093].

[23] N. Carqueville and M. Flohr, *Nonmeromorphic operator product expansion and C\(_2\)-cofiniteness for a family of W-algebras*, J. Phys. A39 (2006) 951–966 [math-ph/0508015].

[24] H. Eberle and M. Flohr, *Virasoro representations and fusion for general augmented minimal models*, hep-th/0604097.

[25] G. Felder, *BRST approach to minimal models*, Nucl. Phys. B317 (1989) 215–236.

[26] T. Kerler and V.V. Lyubashenko, *Non-semisimple Topological Quantum Field Theories for 3-Manifolds with Corners*, Springer Lecture Notes in Mathematics 1765, Springer Verlag (2001).

[27] M. Miyamoto, *A theory of tensor products for vertex operator algebra satsifying C\(_2\)-cofiniteness*, math.QA/0309350.

[28] M. Flohr and M.R. Gaberdiel, *Logarithmic torus amplitudes*, hep-th/0509075.

[29] V. Gurarie and A.W.W. Ludwig, *Conformal algebras of 2d disordered systems*, J. Phys. A: Math. Gen. 35 (2002) L377–L384 [cond-mat/9911392].

[30] I. I. Kogan, A. Nichols, *Stress Energy tensor in LCFT and the Logarithmic Sugawara construction*, Int.J.Mod.Phys. A18 (2003) 4771–4788 [hep-th/0112008].

[31] I. I. Kogan, A. Nichols, *Stress Energy Tensor in c=0 Logarithmic Conformal Field Theory*, [hep-th/0203207].

[32] M. Flohr and A. Müller-Lohmann, *Notes on non-trivial and logarithmic CFTs with c = 0*, J. Stat. Mech. 0604 (2006) P002 [hep-th/0510096].

[33] C. Dong, H. Li, and G. Mason, *Twisted representations of vertex operator algebras*, Math. Ann. 310 (1998) 571-600.
[34] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I.Yu. Tipunin, *Kazhdan–Lusztig-dual quantum group for logarithmic extensions of Virasoro minimal models*, math.QA/0606506.

[35] B. Nienhuis, *Critical behavior of two-dimensional spin models and charge asymmetry in the Coulomb gas*, J. Stat. Phys. 34 (1984) 731–761.

[36] V.I.S. Dotsenko and V.A. Fateev, *Conformal algebra and multipoint correlation functions in 2D statistical models*, Nucl. Phys. B240 [FS 12] (1984) 312–348.

[37] A.B. Zamolodchikov, *‘Irreversibility’ of the flux of the renormalization group in a 2D field theory*, JETP. Lett. 43 (1986) 730–732.

[38] J. Cardy, *The stress tensor in quenched random systems*, cond-mat/0111031.

[39] B.L. Feigin and D.B. Fuchs, *Representations of Infinite-Dimensional Lie Groups and Lie Algebras*, Gordon and Breach, New York (1989).

[40] B.L. Feigin and D.B. Fuks, *Verma modules over the Virasoro algebra*, Funct. Anal. Appl. 17 (1983) 241.

[41] V. Kac, *Vertex Operator Algebras for Begginers*, University Lecture Series, Vol. 10, (AMS, Boston, 1998).

[42] I.B. Frenkel, Y.-Z. Huang, and J. Lepowsky, *On axiomatic approach to vertex operator algebras and modules*, Memoirs Amer. Math. Soc. 104 (1989).

[43] J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Mathematics, Birkhäuser, 2004.

[44] P. Bouwknegt, J. McCarthy, and K. Pilch, *Fock space resolutions of the Virasoro highest weight modules with $c \leq 1$*, Lett. Math. Phys. 23 (1991) 193–204 [hep-th/9108023].

[45] C. Kassel, *Quantum Groups*, Springer Verlag, New York, 1995.

[46] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press (1994).

[47] B.L. Feigin and I.Yu. Tipunin, unpublished.

[48] A. Rocha–Caridi, *Vacuum vector representations of the Virasoro algebra*, in: *Vertex Operators in Mathematics and Physics*, MSRI Publications 3 (Springer, Heidelberg, 1984) 451–473.

[49] A. Nichols, *Extended chiral algebras and the emergence of SU(2) quantum numbers in the Coulomb gas*, hep-th/0302075.

[50] J. Cardy, *Logarithmic correlations in quenched random magnets and polymers*, cond-mat/9911024.