Global wellposedness to the $n$-dimensional compressible Oldroyd-B model without damping mechanism

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Abstract

We are concern with the Cauchy problem of the compressible Oldroyd-B model without damping mechanism in $\mathbb{R}^n$ with $n \geq 2$. By exploiting the intrinsic structure of the equations and introducing several new quantities between density, velocity and stress tensor to overcome the lack of dissipation in density and stress tensor, we prove the global solutions to this system with initial data restricted in the critical $L^p$ Besov spaces, which implies large highly oscillating velocity fields are allowed. As a byproduct, we obtain the optimal time decay rates of the solutions by using the pure energy argument. Our result still be valid for the compressible viscoelastic system without “div-curl” structure assumption, and thus can be regarded as an improvement of [21], [33], [35].

Keywords: Global solutions; Compressible Oldroyd-B model; Time decay

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1. Introduction and the main results

In the last several decades, non-Newtonian fluids which do not satisfy a linear relationship between the stress tensor and the deformation tensor have been widely applied in engineering and industry. One particular subclass of non-Newtonian fluids is the Oldroyd-B fluid, which has been found to approximate the response to many dilute polymeric liquids. Fluids of this type has a memory and can describe the motion of some viscoelastic flows, for example, the system coupling fluids and polymers. Formulations about viscoelastic flows of Oldroyd-B type are first introduced by Oldroyd [31] and are extensively discussed in [3].

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In this paper, we mainly consider the wellposedness of the compressible Oldroyd-B model which has the following form [31], [38]:

\[
\begin{aligned}
&\partial_t \rho + \text{div} (\rho u) = 0, \\
&\rho (\partial_t u + (u \cdot \nabla) u) - \nu (\Delta u + \nabla \text{div} u) + \nabla P = \mu_1 \text{div} \tau, \\
&\partial_t \tau + (u \cdot \nabla) \tau + g(\tau, \nabla u) + \beta \tau = \mu_2 D(u), \quad x \in \mathbb{R}^n, \quad t > 0,
\end{aligned}
\]  

(1.1)

where \(\rho, u, \tau\) are the density, velocity and symmetric tensor of constrains, respectively. The smooth function \(P = P(\rho)\) is the pressure.

\[
\begin{aligned}
g(\tau, \nabla u) &\overset{\text{def}}{=} \tau \Omega(u) - \Omega(u) \tau - b (D(u) \tau + \tau D(u)), \quad \beta \text{ is a parameter in } [-1, 1], \\
D(u) &\text{ is the symmetric part of } \nabla u, \quad \Omega(u) \text{ is the skew-symmetric part of } \nabla u, \quad \text{namely}
\end{aligned}
\]

\[
\begin{aligned}
D(u) &= \frac{1}{2} (\nabla u + (\nabla u)^T), \\
\Omega(u) &= \frac{1}{2} (\nabla u - (\nabla u)^T),
\end{aligned}
\]  

(1.2)

The parameters \(\nu, \mu_1, \mu_2, \beta\) are such that \(\nu > 0, \mu_1 > 0, \mu_2 > 0, \beta \geq 0\). For more explanations on the modeling, see [4], [28], [31], [37] and references therein.

The theory of Oldroyd-B fluids attracts continuous attentions of mathematicians. Most of the results on Oldroyd-B fluids in the literature are about the incompressible model, i.e., \(\rho = \text{Constant in (1.1)}\). For the incompressible Oldroyd-B model, the local-in-time well-posedness, as well as the global-in-time well-posedness with small data, in various spaces is known due to the contribution of Chemin and Masmoudi [5], Elgindi and Liu [12], Elgindi and Rousset [13], Guillopè and Saut [17], [18], Lin [26], Renardy [36], Zi et al. [45]. We also mention that Lions and Masmoudi [27] obtained the global-in-time existence of weak solutions, under the corotational derivative setting. However, it’s still open for the global-in-time existence of weak solutions in the general situation.

Compared with the incompressible case, there has few known results concerning compressible Oldroyd-B models. Lei [23] and Guillopè et al. [16] investigated the incompressible limit problem of the compressible Oldroyd-B model in a torus and bounded domain \(\mathbb{R}^3\), respectively. Fang and Zi [14] further studied the incompressible limit problem in \(\mathbb{R}^n, n \geq 2\), with ill prepared initial data in the Besov spaces. Recently, Zi [44] obtained the global small solutions of (1.1) in the critical \(L^2\) Besov spaces. Barrett et al. [2] and Lu and Zhang in [29]
studied the existence of global in- time weak solutions in $\mathbb{R}^2$ and $\mathbb{R}^3$ for the compressible Oldroyd-B model, respectively. For the compressible Oldroyd type model based on the deformation tensor, see the results [21], [22], [32], [33], [35] and references therein.

In the present paper, we main concern with the global wellposedness of (1.1) without damping mechanism ($\beta = 0$ in (1.1)) in $\mathbb{R}^n (n \geq 2)$. To make the whole paper seems to be net, we shall set all the parameters appeared in (1.1) equal to 1 for convenience, i.e., we consider the following system:

$$\begin{cases}
\partial_t \rho + \text{div} (\rho u) = 0, \\
\rho (\partial_t u + (u \cdot \nabla) u) - (\Delta u + \nabla \text{div} u) + \nabla P = \text{div} \tau, \\
\partial_t \tau + (u \cdot \nabla) \tau + g(\tau, \nabla u) = D(u), \quad x \in \mathbb{R}^n, \quad t > 0, \\
(\rho, u, \tau)|_{t=0} = (\rho_0, u_0, \tau_0).
\end{cases} \tag{1.3}$$

The system (1.3) is supplemented with the following initial conditions:

$$\rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x), \quad \tau|_{t=0} = \tau_0(x), \quad x \in \mathbb{R}^n,$$

and with far field behaviors

$$\rho \to \bar{\rho}, \quad u \to 0, \quad \tau \to 0 \quad \text{as} \quad |x| \to \infty.$$

Setting $P'(1) = 1$ and

$$a = \rho - \bar{\rho} \overset{\text{def}}{=} \rho - 1, \quad \varphi = P(1 + a) - P(1), \tag{1.4}$$

then we can rewrite (1.3) into the following new system:

$$\begin{cases}
\partial_t \varphi + u \cdot \nabla \varphi + \text{div} u = -k(a) \text{div} u, \\
\partial_t \tau + u \cdot \nabla \tau + g(\tau, \nabla u) - D(u) = 0, \\
\partial_t u + u \cdot \nabla u - \Delta u - \nabla \text{div} u + \nabla \varphi - \text{div} \tau = I(a) (\nabla \varphi - \text{div} \tau - \Delta u - \nabla \text{div} u),
\end{cases} \tag{1.5}$$

with

$$k(a) \overset{\text{def}}{=} (1 + a) P'(1 + a) - 1, \quad I(a) \overset{\text{def}}{=} \frac{a}{1 + a}.$$
out that the most of the global solutions constructed for the incompressible or compressible Oldroyd-B model depend heavily on the damping term $\tau$ in the third equation of (1.1). The damping term can help us get $L^1$ or $L^2$ integration about time, this point is very important for us to deal with the linear term $\text{div} \, \tau$ in the momentum equation when construct global small solutions. For the incompressible Oldroyd-B model without damping mechanism on the stress tensor, Zhu in [42] obtained the global small solutions in $\mathbb{R}^3$, by constructing two special time-weighted energies. This result was further generalized for a more general dimension by Chen and Hao [6] and Zhai [40], in the critical $L^2$ Besov spaces and critical $L^p$ Besov spaces, respectively. Recently, Zhu [43] and Pan et al. [33] obtained the global small solutions to compressible viscoelastic flows without structure assumption, in $\mathbb{R}^3$ with Sobolev initial data and in $\mathbb{R}^n$ with Besov initial data, respectively.

To the author’s knowledge, it’s still an open problem to construct the global solutions with a class of large highly oscillating initial velocity, for the compressible Oldroyd-B model without damping mechanism, even for the damping case. The main barrier lies in that we cannot get the damping effect of the density and the stress tensor in the high frequencies part in the $L^p$ type Besov spaces, which is different from the compressible Navier-Stokes equations treated in [7], [8], [10], [20]. The aim of the present paper is to break this barrier and solve this open problem by exploiting the intrinsic structure of the equations and introducing several new quantities between density, velocity and stress tensor.

Let $\mathcal{S}(\mathbb{R}^n)$ be the space of rapidly decreasing functions over $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ its dual space. For any $z \in \mathcal{S}'(\mathbb{R}^n)$, the lower and higher frequency parts are expressed as

$$z^\ell \overset{\text{def}}{=} \sum_{j \leq j_0} \hat{\Delta} j z \quad \text{and} \quad z^h \overset{\text{def}}{=} \sum_{j > j_0} \hat{\Delta} j z$$

for some fixed integer $j_0 \geq 1$ (the value of which follows from the proof of the main theorems). The corresponding truncated semi-norms are defined as follows:

$$\|z\|^\ell_{\dot{B}^s_{p,1}} \overset{\text{def}}{=} \|z^\ell\|_{\dot{B}^s_{p,1}} \quad \text{and} \quad \|z\|^h_{\dot{B}^s_{p,1}} \overset{\text{def}}{=} \|z^h\|_{\dot{B}^s_{p,1}}.$$

Denote

$$\Lambda \overset{\text{def}}{=} \sqrt{-\Delta}, \quad \text{and} \quad \mathcal{P} = \mathcal{I} - \mathcal{Q} := \mathcal{I} - \nabla\Delta^{-1}\text{div}.$$

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Now, we can state the first theorem of the paper:

**Theorem 1.1.** Let \( n \geq 2 \) and

\[
2 \leq p \leq \min(4, 2n/(n-2)) \quad \text{and, additionally,} \quad p \neq 4 \quad \text{if} \quad n = 2.
\]

For any \( (a^\ell_0, u^\ell_0, \tau^\ell_0) \in \dot{B}^{\frac{n}{2}-1} \_2(\mathbb{R}^n) \), \( u^h_0 \in \dot{B}^{\frac{n}{p}-1} \_p(\mathbb{R}^n) \), \( (a^h_0, \tau^h_0) \in \dot{B}^{\frac{n}{p}} \_p(\mathbb{R}^n) \). If there exists a positive constant \( c_0 \) depending on \( n, \omega, \Re, \mathcal{W} \), We such that,

\[
\| (a^\ell_0, u^\ell_0, \tau^\ell_0) \|_{\dot{B}^{\frac{n}{2}-1} \_2} + \| u^h_0 \|_{\dot{B}^{\frac{n}{p}-1} \_p} + \| (a^h_0, \tau^h_0) \|_{\dot{B}^{\frac{n}{p}} \_p} \leq c_0,
\]

then the system (1.5) has a unique global solution \( (a, u, \tau) \) so that for any \( T > 0 \)

\[
a^\ell \in C_b([0, T]; \dot{B}^{\frac{n}{2}-1} \_2(\mathbb{R}^n)), \quad a^h \in C_b([0, T]; \dot{B}^{\frac{n}{p}} \_p(\mathbb{R}^n)),
\]

\[
\tau^\ell \in C_b([0, T]; \dot{B}^{\frac{n}{2}-1} \_2(\mathbb{R}^n)), \quad \tau^h \in C_b([0, T]; \dot{B}^{\frac{n}{p}} \_p(\mathbb{R}^n)),
\]

\[
(\Lambda^{-1} \mathcal{P} \div \tau)^\ell \in L^1([0, T]; \dot{B}^{\frac{n}{2}+1} \_2(\mathbb{R}^n)), \quad (\Lambda^{-1} \mathcal{P} \div \tau)^h \in L^1([0, T]; \dot{B}^{\frac{n}{p}+1} \_p(\mathbb{R}^n)),
\]

\[
u \in C_b([0, T]; \dot{B}^{\frac{n}{2}-1} \_2 \cap L^1([0, T]; \dot{B}^{\frac{n}{p}+1} \_p(\mathbb{R}^n)), u^h \in C_b([0, T]; \dot{B}^{\frac{n}{p}-1} \_p \cap L^1([0, T]; \dot{B}^{\frac{n}{p}+1} \_p(\mathbb{R}^n)).
\]

Moreover, there exists some constant \( C = C(p, n, \omega, \Re, \mathcal{W}) \) such that

\[
\| (a, u, \tau) \|_{L^\infty_t(\dot{B}^{\frac{n}{2}-1})} + \| u \|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1})} + \| (a, \tau) \|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1})} + \| u \|_{L^1_t(\dot{B}^{\frac{n}{p}+1})} + \| u \|_{L^1_t(\dot{B}^{\frac{n}{p}+1})}
\]

\[
+ \| (u, (\Lambda^{-1} \mathcal{P} \div \tau)) \|_{L^1_t(\dot{B}^{\frac{n}{p}+1})} + \| \Lambda^{-1} \mathcal{P} \div \tau \|_{L^1_t(\dot{B}^{\frac{n}{p}+1})} \leq C_0.
\]

**Remark 1.2.** Like the classical compressible Navier-Stokes equations, one may construct the unique global solution for a class of large highly oscillating initial velocity. A typical example is

\[
u_0(x) = \sin \left( \frac{x_1}{\varepsilon} \right) \phi(x), \quad \phi(x) \in \mathcal{S}(\mathbb{R}^n), \quad p > n
\]

which satisfies for any \( \varepsilon > 0 \)

\[
\| u_0^\ell \|_{\dot{B}^{\frac{n}{2}-1}} + \| u_0^h \|_{\dot{B}^{\frac{n}{p}-1}} \leq C \varepsilon^{1-\frac{n}{p}},
\]

here \( C \) is a constant independent of \( \varepsilon \). (see [17], Proposition 2.9).

**Remark 1.3.** By using a similar argument, one can get the global small solutions for the compressible viscoelastic system without any "div-curl" structure assumption. Thus, our theorem improves considerably the recent results in [32], [33], [35].
Remark 1.4. Let \( \rho \) be a constant, (1.1) reduces to the incompressible Oldroyd-B model without damping mechanism, the above theorem generalizes the Theorem 1.2 in [6] and coincides with the Theorem 1.2 in [40].

With the global solutions constructed above, next, a natural problem is what is the large time asymptotic behavior of this solutions. The study of the large-time behavior of solutions to the partial different equations is also an old subject. Here, we only set the compressible Navier-Stokes equations as an example. Starting with the pioneering work by Matsumura and Nishida [30], in which the authors proved that if the initial data are a small perturbation in \( H^3(\mathbb{R}^3) \times L^1(\mathbb{R}^3) \) of \((\bar{\rho}, 0)\) then
\[
\| \nabla^k(\rho - \bar{\rho}, u)(t) \|_{L^2} \leq C(1 + t)^{-\frac{3}{4} - \frac{k}{2}} \quad \text{for} \quad k = 0, 1.
\]

Subsequently, Ponce [34] obtained more general \( L^p \) decay rates
\[
\| \nabla^k(\rho - \bar{\rho}, u)(t) \|_{L^p} \leq C(1 + t)^{-\frac{2}{p} (1 - \frac{1}{p}) - \frac{k}{2}}, \quad 2 \leq p \leq \infty \quad 0 \leq k \leq 2 \quad n = 2, 3.
\]

One can also refer to [11] for the decay rate of the compressible Navier-Stokes equations by small perturbation in Besov spaces. Recently, Xin and Xu [39] obtained the decay rate of the compressible Navier-Stokes equations without the smallness of low frequencies of initial data. However, there is almost no result for the decay rate of the compressible Oldroyd-B model even with the damping mechanism. Motivated by [11], [19], [39], we give the time-decay estimates of the global solutions constructed in Theorem 1.1. Due to some technical reasons, here we only consider the pressure \( P(\rho) \) is a linear function of \( \rho \) in (1.5). For the general pressure, there may need some more complicated argument in Besov spaces which is not what we pursue in this paper. Now, we can state the second theorem of the paper:

**Theorem 1.5.** Let \((a, u, \tau)\) be the global small solutions addressed by Theorem 1.1 with \( p = 2 \). If in addition \((a, u_0, \tau_0) \in \dot{B}_{2,1}^\sigma(\mathbb{R}^n) \) with \(-\frac{n}{2} < \sigma < \frac{n}{2} - 1\). For any \( 2 \leq q \leq \infty \) and \( \frac{n}{q} - \frac{n}{2} + \sigma < \alpha \leq \frac{n}{q} - 1 \), there holds
\[
\| \Lambda^\alpha(u, \Lambda^{-1}v) \|_{L^q} \leq C(1 + t)^{-\frac{n}{q} - \frac{(\alpha - \sigma)n - n}{2q}},
\]
with \( v \) def \( \nabla a - \text{div} \tau \).
Structure of the proof of Theorem 1.1 and Theorem 1.5

Let us now briefly describe the main difficulties that arise when we try to construct the global solutions and give the basic ideas used in solving them.

We start explaining the main ingredients of the proof of Theorem 1.1. To make the thought here more clear, we only analysis the linearized equations of (1.3) with \( \rho = 1 + a \):

\[
\begin{align*}
\partial_t a + \text{div} \, u &= 0, \\
\partial_t \tau - D(u) &= 0, \\
\partial_t u - \left( \Delta u + \nabla \text{div} \, u \right) + \nabla a - \text{div} \, \tau &= 0.
\end{align*}
\] (1.8)

Neglect the effect of the stress tensor \( \tau \), the above system (1.8) goes back to the linearized compressible Navier-Stokes equations. From [7], [8], [10], [20], we have known that the density and velocity have the smoothing effect in the low frequencies, in the \( L^2 \) type Besov spaces. Moreover, in the high frequencies part, the velocity field has the smoothing effect and the density has the damping effect in the \( L^p \) type Besov spaces.

Let \( a \) be a constant, (1.8) is the same as the linearized incompressible Oldroyd-B model. From [6], [40], one can get the smoothing effect of \((\mathcal{P}u, \Lambda^{-1} \mathcal{P} \text{div} \, \tau)\) in the low frequencies, the smoothing effect of \( \mathcal{P}u \) and the damping effect of \( \Lambda^{-1} \mathcal{P} \text{div} \, \tau \) in the high frequencies.

Let us go back to the couple system \((a, u, \tau)\) in (1.8), one can follow the method for compressible Navier-Stokes equations to get the smoothing effect \((a, \mathcal{Q}u, \Lambda^{-1} \mathcal{Q} \text{div} \, \tau)\) in the low frequencies. However, we cannot get the damping effect of \( a \) and \( \tau \) in the high frequencies part. Indeed, to find the damping effect of \( a, \Lambda^{-1} \mathcal{P} \text{div} \, \tau \), we rewrite the third equation of (1.8) into the following form:

\[
\partial_t \mathcal{Q}u - 2\Delta (\mathcal{Q}u - \Delta^{-1} \nabla a + \Delta^{-1} \mathcal{Q} \text{div} \, \tau) = 0.
\]

Just like the compressible Navier-Stoke equations, we shall introduce “effective” velocity field

\[
\Gamma = \mathcal{Q}u - \Delta^{-1} \nabla a + \Delta^{-1} \mathcal{Q} \text{div} \, \tau,
\]

from which

\[
\text{div} \, \mathcal{Q}u = \text{div} \, \Gamma + a - \Delta^{-1} \text{div} \, (\mathcal{Q} \text{div} \, \tau), \quad \Lambda \mathcal{Q}u = \Lambda \Gamma + \Lambda^{-1} \nabla a - \Lambda^{-1} (\mathcal{Q} \text{div} \, \tau).
\]
Substitute the above equality into the first two equations in (1.8), one can get
\[
\begin{aligned}
\partial_t a + a &= -\text{div} \Gamma - \Lambda^{-1} \text{div} (\Lambda^{-1} Q \text{div} \tau), \\
\partial_t \Lambda^{-1} (\text{div} \tau) + \Lambda^{-1} (\text{div} \tau) &= -\Lambda \Gamma - \Lambda^{-1} \nabla a.
\end{aligned}
\] (1.9)

Due to the smoothing effect of $\Gamma$, the terms $\text{div} \Gamma, \Lambda \Gamma$ can be controlled easily. Yet, the rest terms $\Lambda^{-1} \text{div} (\Lambda^{-1} Q \text{div} \tau)$ and $\Lambda^{-1} \nabla a$ cannot be controlled mutually, since they have the same regularity in the high frequencies part. The difficult comes from that there have three variables in (1.8), which is different from the two variables in compressible Navier-Stokes equations. That is to say, the damping effect of $a$ and $\tau$ in (1.9) is not self-governed. Thus, we cannot expect the damping effect of $a, \tau$ in the high frequencies part, which brings the seriously difficulty to bound the nonlinear term $\frac{1}{1+a} \nabla P(1+a)$ in the momentum equations. We need some trick to overcome this difficulty. In fact, from the above analysis of $a, \tau$ in high frequencies part, we find that a new combination of $\Lambda^{-1}(\nabla a - \text{div} \tau)$ has the damping effect, although the single $a$ or $\tau$ don’t have. Based on this observation, we will set $\Lambda^{-1}(\nabla a - \text{div} \tau)$ as a new variable to close our energy estimates, see more details in the following Section 3.

From the proof of the global solutions in Theorem 1.1, we have known that only the velocity field and the couple of $\nabla a - \text{div} \tau$ have the smoothing effect, hence, we only expect to get the time decay of $(u, \nabla a - \text{div} \tau)$. The decay rate will be obtained by solving a Lyapunov-type inequality which depends on the interpolation inequality in the low frequencies and high frequencies, see Section 4 for more details.

The rest of the paper unfolds as follows. In the second section, we shall collect some basic facts on Littlewood-Paley analysis and various product laws in Besov spaces. In Section 3, we will use four subsections to prove the main Theorem 1.1. Finally in the last Section, we present the proof of Theorem 1.5.

Notations : Let $A, B$ be two operators, we denote $[A, B] = AB - BA$, the commutator between $A$ and $B$. For $a \lesssim b$, we mean that there is a uniform constant $C$, which may be different on different lines, such that $a \leq C b$. Given a Banach space $X$, we shall denote $\|(a, b)\|_X = \|a\|_X + \|b\|_X$. 

8
2. Preliminaries

For readers’ convenience, in this section, we list some basic knowledge on Littlewood-Paley theory.

**Definition 2.1.** Let us consider a smooth function $\phi$ on $\mathbb{R}$, the support of which is included in $[\frac{3}{4}, \frac{8}{3}]$ such that

$$\forall \tau > 0, \sum_{j \in \mathbb{Z}} \phi(2^{-j} \tau) = 1, \quad \text{and} \quad \chi(\tau) \overset{\text{def}}{=} 1 - \sum_{j \geq 0} \phi(2^{-j} \tau) \in \mathcal{D}([0, 4/3]).$$

Then one can define

$$\dot{\Delta}_j u = \mathcal{F}^{-1}(\phi(2^{-j} |\xi|) \hat{u}), \quad \text{and} \quad \dot{S}_j u = \mathcal{F}^{-1}(\chi(2^{-j} |\xi|) \hat{u}).$$

Let us remark that, for any homogeneous function $A$ of order 0 smooth outside 0, we have

$$\forall p \in [1, \infty], \quad \|\dot{\Delta}_j (A(D)f)\|_{L^p} \leq C\|\dot{\Delta}_j f\|_{L^p}.$$ 

Let $p, r$ be in $[1, +\infty]$ and $s$ in $\mathbb{R}$, $u \in S'(\mathbb{R}^n)$. We define the Besov norm by

$$\|u\|_{\dot{B}^s_{p,r}} \overset{\text{def}}{=} \left\|(2^{js}\|\dot{\Delta}_j u\|_{L^p})_j\right\|_{\ell^r(\mathbb{Z})}.$$ 

We then define the spaces $\dot{B}^s_{p,r} \overset{\text{def}}{=} \{u \in S'_h(\mathbb{R}^n), \|u\|_{\dot{B}^s_{p,r}} < \infty\}$, where $u \in S'_h(\mathbb{R}^n)$ means that $u \in S'(\mathbb{R}^n)$ and $\lim_{j \to -\infty} \|\dot{S}_j u\|_{L^\infty} = 0$ (see Definition 1.26 of [1]).

**Lemma 2.2.** Let $B$ be a ball and $C$ a ring of $\mathbb{R}^n$. A constant $C$ exists so that for any positive real number $\lambda$, any non-negative integer $k$, any smooth homogeneous function $\sigma$ of degree $m$, and any couple of real numbers $(p, q)$ with $1 \leq p \leq q \leq \infty$, there hold

$$\text{Supp} \ \hat{u} \subset \lambda B \Rightarrow \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^k + n + m \left(\frac{1}{p} - \frac{1}{q}\right) \|u\|_{L^p},$$

$$\text{Supp} \ \hat{u} \subset \lambda C \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leq \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p},$$

$$\text{Supp} \ \hat{u} \subset \lambda C \Rightarrow \|\sigma(D) u\|_{L^q} \leq C_{\sigma,m} \lambda^m + n \left(\frac{1}{p} - \frac{1}{q}\right) \|u\|_{L^p}.$$ 

Let us now state some classical properties for the Besov spaces.
Lemma 2.3. Let $1 \leq p \leq \infty$ and $s_1, s_2 \in \mathbb{R}$ with $s_1 > s_2$, for any $u \in \dot{B}_{p,1}^{s_1} \cap \dot{B}_{p,1}^{s_2}(\mathbb{R}^n)$, there holds
\[
\|u^\ell\|_{\dot{B}_{p,1}^{s_1}} \leq C\|u^\ell\|_{\dot{B}_{p,1}^{s_2}}, \quad \|u^h\|_{\dot{B}_{p,1}^{s_1}} \leq C\|u^h\|_{\dot{B}_{p,1}^{s_2}}.
\]

If $s_1 \neq s_2$ and $\theta \in (0, 1)$, $\left[\dot{B}_{p,1}^{s_1}, \dot{B}_{p,1}^{s_2}\right]_\theta = \dot{B}_{p,1}^{\theta s_1 + (1-\theta)s_2}$.

For any smooth homogeneous of degree $m \in \mathbb{Z}$ function $A$ on $\mathbb{R}^n \setminus \{0\}$, the operator $A(D)$ maps $B^s_{p,1}$ in $\dot{B}^{s-m}_{p,1}$.

In this paper, we frequently use the so-called "time-space" Besov spaces or Chemin-Lerner space first introduced by Chemin and Lerner [1].

Definition 2.4. Let $s \in \mathbb{R}$ and $0 < T \leq +\infty$. We define
\[
\|u\|_{\dot{L}_T^q(\dot{B}^s_{p,1})} \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} 2^{js} \left( \int_0^T \|\Delta_j u(t)\|_{L^p}^q dt \right)^{\frac{1}{q}}
\]
for $q, p \in [1, \infty)$ and with the standard modification for $p, q = \infty$.

By Minkowski’s inequality, we have the following inclusions between the Chemin-Lerner space $\dot{L}_T^\lambda(\dot{B}^s_{p,1})$ and the Bochner space $L_0^T(\dot{B}^s_{p,1})$:
\[
\|u\|_{\dot{L}_T^\lambda(\dot{B}^s_{p,1})} \leq \|u\|_{L_0^T(\dot{B}^s_{p,1})} \quad \text{if } \lambda \leq r, \quad \|u\|_{\dot{L}_T^\lambda(\dot{B}^s_{p,1})} \leq \|u\|_{L_0^T(\dot{B}^s_{p,1})} \quad \text{if } \lambda \geq r.
\]

To study product laws between distributions, we need para-differential decomposition of Bony in the homogeneous context:
\[
uv = T_u v + T_v u + \bar{R}(u,v), \quad (2.1)
\]

where
\[
T_u v = \sum_{j \in \mathbb{Z}} \delta_{j-1} u \tilde{\Delta}_j v, \quad \bar{R}(u,v) = \sum_{j \in \mathbb{Z}} \Delta_j u \tilde{\Delta}_j v, \quad \tilde{\Delta}_j v = \sum_{|j-j'| \leq 1} \Delta_j v.
\]

The paraproduct $\bar{T}$ and the remainder $\bar{R}$ operators satisfy the following continuous properties.

Lemma 2.5 (1). For all $s \in \mathbb{R}$, $\sigma \geq 0$, and $1 \leq p, p_1, p_2 \leq \infty$, the paraproduct $\bar{T}$ is a bilinear, continuous operator from $\dot{B}_{p,1}^{-\sigma} \times \dot{B}_{p,1}^{s}$ to $\dot{B}_{p,1}^{s-\sigma}$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. The remainder $\bar{R}$ is bilinear continuous from $\dot{B}_{p,1}^{s_1} \times \dot{B}_{p,1}^{s_2}$ to $\dot{B}_{p,1}^{s_1+s_2}$ with $s_1 + s_2 > 0$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. 

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In view of (2.1), Lemmas 2.2, 2.5 one easily deduces the following product laws:

**Lemma 2.6.** (see [11], Proposition A.1) Let $1 \leq p, q \leq \infty$, $s_1 \leq \frac{n}{q'}$, $s_2 \leq n \min\{\frac{1}{p}, \frac{1}{q}\}$ and $s_1 + s_2 > n \max\{0, \frac{1}{p'} + \frac{1}{q'} - 1\}$. For $\forall (u, v) \in B_{q,1}^{s_1}(\mathbb{R}^n) \times B_{p,1}^{s_2}(\mathbb{R}^n)$, we have

$$\|uv\|_{B_{q,1}^{s_1+s_2}-\frac{n}{q}} \lesssim \|u\|_{B_{q,1}^{s_1}} \|v\|_{B_{p,1}^{s_2}}.$$

**Lemma 2.7.** Let $n \geq 2$ and $2 \leq p \leq \min(4, 2n/(n - 2))$ and, additionally, $p \neq 4$ if $n = 2$. Let $u \in B_{p,1}^{\frac{n}{q}}(\mathbb{R}^n), v^\ell \in B_{2,1}^{\frac{n}{q}}(\mathbb{R}^n)$ and $v^h \in B_{p,1}^{\frac{n}{q}-1}(\mathbb{R}^n)$, then we have

$$\|(uv)^\ell\|_{B_{2,1}^{\frac{n}{q}-1}} \lesssim (\|v^\ell\|_{B_{2,1}^{\frac{n}{q}}-1} + \|v^h\|_{B_{p,1}^{\frac{n}{q}-1}}) \|u\|_{B_{p,1}^{\frac{n}{q}}}.$$\hspace{1cm} (2.2)

**Proof.** Thanks to Bony’s decomposition, one can write

$$\hat{S}_{j_0+1}(uv) = \hat{S}_{j_0+1}(Tu + \hat{R}(v, u)) + Tu \hat{S}_{j_0+1}v + [\hat{S}_{j_0+1}, Tu]v.$$

By Lemma 2.5 and the embedding relation $\dot{B}_{p,1}^{\frac{n}{q}}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$, we obtain

$$\|Tu \hat{S}_{j_0+1}v\|_{B_{2,1}^{\frac{n}{q}-1}} \lesssim \|u\|_{L^\infty} \|v^\ell\|_{B_{2,1}^{\frac{n}{q}}-1} \lesssim \|v^\ell\|_{B_{2,1}^{\frac{n}{q}}-1} \|u\|_{B_{p,1}^{\frac{n}{q}}}.$$

Denote $\frac{1}{p^*} = \frac{1}{2} - \frac{1}{p}$, one can infer that $p \leq p^*$, hence

$$\|\hat{S}_{j_0+1}(Tu + \hat{R}(v, u))\|_{B_{2,1}^{\frac{n}{q}-1}} \lesssim \|v^\ell\|_{B_{2,1}^{\frac{n}{q}}-1} \|u\|_{B_{p^*,1}^{\frac{n}{q}}} \lesssim \|v\|_{B_{p^*,1}^{\frac{n}{q}}} \|u\|_{B_{p,1}^{\frac{n}{q}}}.$$

By Lemma 6.1 in [10], we can further get

$$\|\hat{S}_{j_0+1}, Tu]v\|_{B_{2,1}^{\frac{n}{q}-1}} \lesssim \|\nabla u\|_{B_{p^*,1}^{\frac{n}{q}}-1} \|v\|_{B_{2,1}^{\frac{n}{q}}-1} \lesssim \|v\|_{B_{p,1}^{\frac{n}{q}-1}} \|u\|_{B_{p,1}^{\frac{n}{q}}}.$$

Thus, combining above estimates and using the fact

$$\|v\|_{B_{p,1}^{\frac{n}{q}-1}} \leq C(\|v^\ell\|_{B_{2,1}^{\frac{n}{q}}-1} + \|v^h\|_{B_{p,1}^{\frac{n}{q}-1}}),$$

we can arrive at the desired estimate (2.2).

---

**Lemma 2.8.** Let $2 \leq p \leq 2n$, for any $u^\ell \in B_{2,1}^{\frac{n}{q}}(\mathbb{R}^n), v^\ell \in B_{2,1}^{\frac{n}{q}}(\mathbb{R}^n), u^h \in \dot{B}_{p,1}^{\frac{n}{q}}(\mathbb{R}^n), v^h \in \dot{B}_{p,1}^{\frac{n}{q}-1}(\mathbb{R}^n)$, there holds

$$\|(uv)^\ell\|_{B_{2,1}^{\frac{n}{q}-1}} \lesssim \|u^\ell\|_{B_{2,1}^{\frac{n}{q}}-1} \|v^\ell\|_{B_{2,1}^{\frac{n}{q}}-1} + \|u^h\|_{B_{p,1}^{\frac{n}{q}}} \|v^h\|_{B_{p,1}^{\frac{n}{q}-1}}.$$\hspace{1cm} (2.3)
Proof. We use again decomposition (2.1)

\[ uv = \hat{T}_u v + \hat{T}_v u + \hat{R}(u,v). \]

By the Hölder inequality and Bernstein’s Lemma 2.2

\[ \| (\hat{T}_u v)^\ell \|_{B^n_{2,1}} \lesssim \sum_{j \leq j_0} 2^{j(n-1)} \sum_{|j-k| \leq 4} \| \Delta_j (\hat{S}_{k-1} u \Delta_k v) \|_{L^2} \]
\[ \lesssim \sum_{j \leq j_0} 2^{j(n-1)} \sum_{|j-k| \leq 4} ( \sum_{k' \leq k-2} \| \Delta_{k'} u \|_{L^\infty} ) \| \Delta_k v \|_{L^2} \]
\[ \lesssim \sum_{j \leq j_0} 2^{j(n-1)} \sum_{|j-k| \leq 4} \| u^\ell \|_{B^{n-1}_{2,1}} 2^k \| \Delta_k v \|_{L^2} \]
\[ \lesssim \| u^\ell \|_{B^{n-1}_{2,1}} \| v^\ell \|_{B^{n}_{2,1}}. \] \hspace{1cm} (2.4)

Similarly,

\[ \| (\hat{T}_v u)^\ell \|_{B^n_{2,1}} \lesssim \| u^\ell \|_{B^{n-1}_{2,1}} \| v^\ell \|_{B^{n}_{2,1}}. \] \hspace{1cm} (2.5)

For the remainder term,

\[ \| \hat{R}(u,v)^\ell \|_{B^n_{2,1}} \lesssim \sum_{j \leq j_0} 2^{j(n-1)} 2^{j(n-1)n_j} \| \Delta_j \hat{R}(u,v) \|_{L^{p/2}} \]
\[ \lesssim \sum_{j \leq j_0} 2^{j(n-1)j} \left( \sum_{j-3 \leq k \leq j} \| \Delta_{k} u \|_{L^p} \| \Delta_{k} v \|_{L^q} + \sum_{k > j} \| \Delta_{k} u \|_{L^p} \| \Delta_{k} v \|_{L^q} \right) \]
\[ \lesssim \| u^\ell \|_{B^{n-1}_{2,1}} \| v^\ell \|_{B^{n}_{2,1}} + \| u^h \|_{B^p_{p,1}} \| v^h \|_{B^p_{p,1}}. \] \hspace{1cm} (2.6)

Hence, (2.3) is followed by (2.4), (2.5) and (2.6) directly. \hspace{1cm} \Box

The following classical commutator’s estimate is often used:

Lemma 2.9. (see [1, Lemma 2.100]) Let \( n \geq 2, 1 \leq p, q \leq \infty, v \in \dot{B}^s_{q,1}(\mathbb{R}^n) \) and \( \nabla u \in \dot{B}^n_{p,1}(\mathbb{R}^n) \).

Assume that

\[ -n \min \left\{ \frac{1}{p}, 1 - \frac{1}{q} \right\} < s \leq 1 + n \min \left\{ \frac{1}{p}, \frac{1}{q} \right\} \]

Then there holds the commutator estimate

\[ \| [\Delta_j, u \cdot \nabla] v \|_{L^q} \lesssim d_j 2^{-js} \| \nabla u \|_{B^p_{p,1}} \| v \|_{\dot{B}^s_{q,1}}. \]

The following estimates are implied from [41, Lemma 2.16].
Lemma 2.10. Let $2 \leq p \leq \min\{4, 2n/(n - 2)\}$ for $n > 2$ and $2 \leq p < 4$ for $n = 2$. Assume $A(D)$ a zero-order Fourier multiplier. For $v^\ell \in B_{2,1}^{n-1}(\mathbb{R}^n)$, $v^h \in B_{p,1}^{\frac{n}{p}-1}(\mathbb{R}^n)$ and $\nabla u \in B_{p,1}^{\frac{n}{p}}(\mathbb{R}^n)$, we have

$$\sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j} \| \Delta_j A(D), u \cdot \nabla \|_{L^2} \leq C(\| \nabla u^\ell \|_{B_{2,1}^{\frac{n}{2}}} + \| \nabla u^h \|_{B_{p,1}^{\frac{n}{p}}})(\| v^\ell \|_{B_{2,1}^{n-1}} + \| v^h \|_{B_{p,1}^{\frac{n}{p}-1}}),$$

$$\sum_{j \leq j_0} 2^{(\frac{n}{2}-1)j} \| \Delta_j A(D), u \cdot \nabla \|_{L^2} \leq C(\| \nabla u \|_{B_{p,1}^{\frac{n}{p}}}(\| v^\ell \|_{B_{2,1}^{n-1}} + \| v^h \|_{B_{p,1}^{\frac{n}{p}-1}}), \quad \text{if} \ \text{div} \ u = 0,$$

for a constant dependent on $j_0$.

Remark 2.11. This lemma has been proved in [41] in the case of $n = 3$. Following a similar processes, we can generalized it to more general dimensional $n \geq 2$ under the condition of the above lemma. Here, we omit the details for convenience.

System (1.5) also involves compositions of functions (through $I(a)$ and $k(a)$) that are bounded thanks to the following classical result:

Proposition 2.12. (III) Let $F : \mathbb{R} \to \mathbb{R}$ be smooth with $F(0) = 0$. For all $1 \leq p, r \leq \infty$ and $s > 0$, it holds that $F(u) \in \dot{B}^{s}_{p,r} \cap L^{\infty}$ for $u \in \dot{B}^{s}_{p,r} \cap L^{\infty}$, and

$$\| F(u) \|_{\dot{B}^{s}_{p,r}} \leq C \| u \|_{\dot{B}^{s}_{p,r}}$$

with $C$ depending only on $\| u \|_{L^{\infty}}$, $F'$ (and higher derivatives), $s$, $p$ and $n$.

In the case $s > - \min\left(\frac{n}{p}, \frac{n}{p'}\right)$ then $u \in \dot{B}^{s}_{p,r} \cap \dot{B}^{s}_{p,1}$ implies that $F(u) \in \dot{B}^{s}_{p,r} \cap \dot{B}^{s}_{p,1}$ and

$$\| F(u) \|_{\dot{B}^{s}_{p,r}} \leq C(1 + \| u \|_{\dot{B}^{s}_{p,1}}) \| u \|_{\dot{B}^{s}_{p,r}}.$$

3. The proof of Theorem 1.1

In this section, the global solutions with small initial data to the Cauchy problem (1.5) will be proved, we usually employ the standard continuity argument, namely, we obtain the local solutions first and then extend it to a global-in-time solutions by establishing the \textit{a priori} estimates of the solutions. But the local solutions can be obtained by some usual method, so we omit the process of the local solutions in this paper, just focus on the \textit{a priori} estimates of the solutions.
We divide the proof into four subsections to get the a priori estimates. In the first subsection, we get the $L^\infty$ estimates of $(\varphi, u, \tau)$ in the low frequencies. Then, we obtain the smoothing effect of $u$ and the hidden dissipation of $\nabla \varphi - \text{div} \tau \overset{\text{def}}{=} v$ in the low frequencies by introducing two new quantities in the second subsection. In the third subsection, we follow the approach in [20] and introduce so-called “effective” velocity field to find the smoothing effect of $u$ and the damping effect of $v$ in the high frequencies. We complete the proof of Theorem 1.1 by a standard continuity arguments in the last subsections.

Note that $\varphi = P(1 + a) - P(1), P'(1) = 1$, then there exists a small $\epsilon_0$, if $\|a\|_{L^\infty([0,T];\mathbb{R}^n)} \leq \epsilon_0$, one can express $a$ by a smooth function of $\varphi$, for convenience, we set $a = \psi(\varphi)$. Throughout we make the assumption that

$$\sup_{t \in \mathbb{R}^+, x \in \mathbb{R}^n} |a(t, x)| \leq \frac{1}{2}$$

(3.1)

which will enable us to use freely the composition estimate stated in Proposition 2.12. Note that as $\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$, condition (3.1) will be ensured by the fact that the constructed solution has small norm in (1.7).

3.1. The $L^\infty$ estimates of $(\varphi, u, \tau)$ in the low frequencies

Apply the operator $\Delta_j$ to (1.5). By using the standard energy argument, we arrive at the following three equalities:

$$\frac{1}{2} \frac{d}{dt} \|\Delta_j \varphi\|_{L^2}^2 + \int \Delta_j \text{div} u \cdot \Delta_j \varphi \, dx$$
$$= - \int u \cdot \nabla \cdot \Delta_j \varphi \cdot \Delta_j \varphi \, dx - \int [\Delta_j, u \cdot \nabla] \varphi \cdot \Delta_j \varphi \, dx - \int \Delta_j(k(a) \text{div} u) \cdot \Delta_j \varphi \, dx,$$

(3.2)

$$\frac{1}{2} \frac{d}{dt} \|\Delta_j \tau\|_{L^2}^2 - \int \Delta_j D(u) \cdot \Delta_j \tau \, dx$$
$$= - \int u \cdot \nabla \cdot \Delta_j \tau \cdot \Delta_j \tau \, dx - \int [\Delta_j, u \cdot \nabla] \tau \cdot \Delta_j \tau \, dx - \int \Delta_j(g(\tau, \nabla u)) \cdot \Delta_j \tau \, dx,$$

(3.3)

and

$$\frac{1}{2} \frac{d}{dt} \|\Delta_j u\|_{L^2}^2 + \int |\Delta_j \nabla u|^2 \, dx + \int |\Delta_j \text{div} u|^2 \, dx + \int \Delta_j \nabla \varphi \cdot \Delta_j u \, dx - \int \Delta_j \text{div} \tau \cdot \Delta_j u \, dx$$
$$= - \int u \cdot \nabla \cdot \Delta_j u \cdot \Delta_j u \, dx - \int [\Delta_j, u \cdot \nabla] u \cdot \Delta_j u \, dx$$
$$+ \int \Delta_j(I(a)(v - \Delta u - \nabla \text{div} u)) \cdot \Delta_j u \, dx.$$

(3.4)
Combining with (3.2)–(3.4), integrating from 0 to \( t \), then multiplying the resultant equations by \( 2^{(j-1)j} \), we get by summing up for any \( j \leq j_0 \) that

\[
\|(\varphi, u, \tau)\|^2_{L_t^\infty(B_{2,1}^{\frac{n}{2}})} \lesssim \|(\varphi_0, u_0, \tau_0)\|^2_{B_{2,1}^{\frac{n}{2}}} + \int_0^t \|\nabla u\|_{L^\infty} \|(\varphi, u, \tau)\|^2_{B_{2,1}^{\frac{n}{2}}} ds
\]
\[
+ \int_0^t \|(k(a)\div u)^\ell\|^2_{B_{2,1}^{\frac{n}{2}}} ds + \int_0^t \|(I(a)\nu)^\ell\|^2_{B_{2,1}^{\frac{n}{2}}} ds
\]
\[
+ \int_0^t \|(g(\tau, \nabla u))^\ell\|^2_{B_{2,1}^{\frac{n}{2}}} ds + \int_0^t \|(I(a)(\Delta u + \nabla \div u))^\ell\|^2_{B_{2,1}^{\frac{n}{2}}} ds
\]
\[
+ \int_0^t \sum_{j\leq j_0} 2^{(j-1)j} \|(\Delta_j, u \cdot \nabla)(\varphi, u, \tau)\|_{L^2_{x,t}} ds
\]

(3.5)
in which we have used the following cancellations:

\[
\int \Delta_j \div u \cdot \Delta_j \varphi dx + \int \Delta_j \nabla \varphi \cdot \Delta_j u dx = 0, \quad \int \Delta_j D(u) \cdot \Delta_j \tau dx + \int \Delta_j \div \tau \cdot \Delta_j u dx = 0.
\]

By virtue of embedding relation \( \dot{B}_{p,1}^\frac{n}{2}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \), the second term on the right hand side of (3.5) can be bounded by

\[
\|\nabla u\|_{L^\infty} \|(\varphi, u, \tau)\|^\ell_{B_{2,1}^{\frac{n}{2}}} \lesssim \|\nabla u\|_{B_{p,1}^\frac{n}{2}} \|(\varphi, u, \tau)\|^\ell_{B_{2,1}^{\frac{n}{2}}}
\]
\[
\lesssim (\|u^\ell\|_{B_{2,1}^{\frac{n}{2}+1}} + \|u^h\|_{B_{p,1}^\frac{n}{2}+1}) \|(\varphi, u, \tau)\|^\ell_{B_{2,1}^{\frac{n}{2}}}. \tag{3.6}
\]

Thanks to Lemma 2.8, one can infer that

\[
\|(k(a)\div u)^\ell\|^2_{B_{2,1}^{\frac{n}{2}}} \lesssim \|(k(a))^\ell\|_{B_{2,1}^{\frac{n}{2}}+1} \|(\div u)^\ell\|^2_{B_{2,1}^{\frac{n}{2}}} + \|(k(a))^h\|_{B_{p,1}^\frac{n}{2}} \|(\div u)^h\|^2_{B_{p,1}^{\frac{n}{2}}}
\]
\[
\lesssim (\|k(a)\|^\ell_{B_{2,1}^{\frac{n}{2}}+1} + \|k(a)\|^h_{B_{p,1}^\frac{n}{2}}) \|u^h\|_{B_{p,1}^\frac{n}{2}+1}. \tag{3.7}
\]

For bounding \( k(a) \), we first write

\[
k(a) = k'(0) a + \tilde{k}(a) \quad \text{with } \tilde{k}(0) = 0.
\]

Then, by Lemma 2.7, Proposition 2.12 and (3.1), we can get

\[
\|(k(a))^h\|_{B_{p,1}^\frac{n}{2}} \lesssim (1 + \|a\|_{L^\infty})^2 \|a\|_{B_{p,1}^\frac{n}{2}+1} \lesssim \|a^\ell\|_{B_{2,1}^{\frac{n}{2}}+1} + \|a^h\|_{B_{p,1}^\frac{n}{2}},
\]
\[
\|(k(a))^\ell\|_{B_{2,1}^{\frac{n}{2}}} \lesssim k'(0) \|a^\ell\|_{B_{2,1}^{\frac{n}{2}}+1} + \|a\tilde{k}(a)\|_{B_{2,1}^{\frac{n}{2}}}
\]
\[
\lesssim k'(0) \|a^\ell\|_{B_{2,1}^{\frac{n}{2}}+1} + \|a^\ell\|_{B_{2,1}^{\frac{n}{2}}+1} + \|a^h\|_{B_{p,1}^\frac{n}{2}+1}) \|\tilde{k}(a)\|_{B_{2,1}^{\frac{n}{2}}},
\]
\[
\lesssim k'(0) \|a^\ell\|_{B_{2,1}^{\frac{n}{2}}+1} + \|a^\ell\|_{B_{2,1}^{\frac{n}{2}}+1} + \|a^h\|_{B_{p,1}^\frac{n}{2}+1}) (1 + \|a\|_{L^\infty})^2 \|a\|_{B_{p,1}^\frac{n}{2}}
\]
\[
\lesssim (\|a^\ell\|_{B_{2,1}^{\frac{n}{2}}+1} + 1) \|a^\ell\|_{B_{2,1}^{\frac{n}{2}}+1} + \|a^h\|_{B_{p,1}^\frac{n}{2}}^2. \tag{3.8}
\]

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The above estimate (3.8) still be valid if we substitute \( k(a) \) for \( I(a) \).

From (3.8), the previous estimation (3.7) can be further bounded by

\[
\| (k(a) \text{div } u)^\ell \|_{B^{n}_{2,1}} \lesssim (\| \phi^h \|_{B^{n}_{2,1}} + 1) \| \phi^h \|_{B^{n}_{2,1}} \| u^h \|_{B^{n}_{2,1}} + (\| \phi^h \|_{B^{n}_{2,1}} + 1) \| \phi^h \|_{B^{n}_{2,1}} \| u^h \|_{B^{n}_{2,1}}. \quad (3.9)
\]

Repeating the same argument as above gives

\[
\| (I(a) v)^\ell \|_{B^{n}_{2,1}} \lesssim (\| I(a)^\ell \|_{B^{n}_{2,1}} \| v^\ell \|_{B^{n}_{2,1}} + (\| I(a)^h \|_{B^{n}_{2,1}} \| v^h \|_{B^{n}_{2,1}} + (\| I(a)^h \|_{B^{n}_{2,1}} + 1) \| \phi^h \|_{B^{n}_{2,1}} \| v^h \|_{B^{n}_{2,1}}). \quad (3.10)
\]

With the aid of Lemmas 2.7, 2.8 one has

\[
\| (g(\tau, \nabla u))^\ell \|_{B^{n}_{2,1}} \lesssim (\| I(a)^\ell \|_{B^{n}_{2,1}} \| v^\ell \|_{B^{n}_{2,1}} + (\| I(a)^h \|_{B^{n}_{2,1}} \| v^h \|_{B^{n}_{2,1}} + (\| I(a)^h \|_{B^{n}_{2,1}} + 1) \| \phi^h \|_{B^{n}_{2,1}} \| v^h \|_{B^{n}_{2,1}}). \quad (3.11)
\]

Finally the commutator term may be handled according to Lemma 2.10 which ensures that

\[
\sum_{j \leq 0} 2^{(\frac{n}{2} - 1) j} \| [\hat{\Delta}_j, u \cdot \nabla] (\varphi, u, \tau) \|_{L^2} \lesssim (\| I(a)^\ell \|_{B^{n}_{2,1}} + (\| I(a)^h \|_{B^{n}_{2,1}} + (\| I(a)^h \|_{B^{n}_{2,1}} + 1) \| \phi^h \|_{B^{n}_{2,1}} + \| u^h \|_{B^{n}_{2,1}}). \quad (3.12)
\]

Inserting (3.6), (3.9)+(3.12) into (3.5) gives

\[
\| (\varphi, u, \tau)^\ell \|_{L^\infty_t(B^{n}_{2,1})} \lesssim (\| \varphi_0, u_0, \tau_0 \|_{B^{n}_{2,1}} + \int_0^t (\| \phi^h \|_{B^{n}_{2,1}} + 1) \| \phi^h \|_{B^{n}_{2,1}} (\| u^h \|_{B^{n}_{2,1}} + \| v^h \|_{B^{n}_{2,1}}) \, ds + \int_0^t (\| \phi^h \|_{B^{n}_{2,1}} + 1) \| \phi^h \|_{B^{n}_{2,1}} (\| u^h \|_{B^{n}_{2,1}} + \| v^h \|_{B^{n}_{2,1}}) \, ds + \int_0^t (\| \phi^h \|_{B^{n}_{2,1}} + 1) \| \phi^h \|_{B^{n}_{2,1}} (\| u^h \|_{B^{n}_{2,1}} + \| v^h \|_{B^{n}_{2,1}}) \, ds. \quad (3.13)
\]
3.2. The smoothing effect of \((u, \nabla \varphi - \text{div} \tau)\) in the low frequencies

In this subsection, we shall derive the smoothing effect of \((u, \nabla \varphi - \text{div} \tau)\) in the low frequencies, to do so, we introduce a new quantity \(v \overset{\text{def}}{=} \nabla \varphi - \text{div} \tau\), then one can deduce from (1.5) that \((u, v)\) satisfies the following equations:

\[
\begin{cases}
\partial_t v + u \cdot \nabla v + \Delta u + \nabla \text{div} u = f_1, \\
\partial_t u + u \cdot \nabla u - (\Delta u + \nabla \text{div} u) + v = f_2,
\end{cases}
\tag{3.14}
\]

with

\[
f_1 \overset{\text{def}}{=} \nabla u^T \nabla \tau - \nabla u^T \nabla \varphi - \nabla (k(a) \text{div} u) + \text{div} g(\tau, \nabla u), \quad f_2 \overset{\text{def}}{=} I(a)(v - \Delta u - \nabla \text{div} u).
\]

With the aid of operators \(P\) and \(Q\), we further separate the system (3.14) from compressible part and incompressible part:

\[
\begin{cases}
\partial_t Qv + 2\Delta Qu = Qf_1 - Q(u \cdot \nabla v), \\
\partial_t Qu - 2\Delta Qu + Qv = Qf_2 - Q(u \cdot \nabla u),
\end{cases}
\tag{3.15}
\]

and

\[
\begin{cases}
\partial_t Pv + u \cdot \nabla P v + \Delta Pu = Pf_1 - [P, u \cdot \nabla]v, \\
\partial_t Pu + u \cdot \nabla Pu - \Delta Pu + P v = Pf_2 - [P, u \cdot \nabla]u.
\end{cases}
\tag{3.16}
\]

It’s obvious that the above two systems (3.15) and (3.16) have the same linear structure. Thus, we only pick out the one, for example (3.16), to present the details of finding the hidden dissipation of \(v\). Let

\[
\Lambda \overset{\text{def}}{=} \sqrt{-\Delta}, \quad w \overset{\text{def}}{=} Pu + Pv,
\]

we deduce from (3.16) that

\[
\begin{cases}
\partial_t \Lambda^{-1} P v - \Lambda Pu = f_3, \\
\partial_t w + u \cdot \nabla w + P v = f_4, \\
\partial_t Pu + u \cdot \nabla Pu - \Delta Pu + P v = f_5,
\end{cases}
\tag{3.17}
\]

with

\[
f_3 \overset{\text{def}}{=} \Lambda^{-1} Pf_1 - \Lambda^{-1} P(u \cdot \nabla v), \quad f_4 \overset{\text{def}}{=} Pf_1 + Pf_2 - [P, u \cdot \nabla]v - [P, u \cdot \nabla]u, \quad f_5 \overset{\text{def}}{=} Pf_2 - [P, u \cdot \nabla]u.
\]
Apply the operator $\hat{A}_j$ to (3.17), then testing the first equation by $\hat{A}_j \Lambda^{-1} \mathcal{P} v$, the second equation by $\hat{A}_j \mathcal{P} w$ and the third equation by $\hat{A}_j \mathcal{P} u$, using integrating by parts and a standard commutator’s argument, respectively, we can get the following three inequalities:

\[
\frac{1}{2} \frac{d}{dt} \|\hat{A}_j \Lambda^{-1} \mathcal{P} v\|_{L^2}^2 - \int \hat{A}_j \mathcal{P} v \cdot \hat{A}_j \mathcal{P} u \, dx \lesssim \int |\hat{A}_j f_3 \cdot \hat{A}_j \Lambda^{-1} \mathcal{P} v| \, dx, \tag{3.18}
\]

\[
\frac{1}{2} \frac{d}{dt} \|\hat{A}_j \mathcal{P} w\|_{L^2}^2 + \|\hat{A}_j \mathcal{P} v\|_{L^2}^2 + \int \hat{A}_j \mathcal{P} v \cdot \hat{A}_j \mathcal{P} u \, dx \\
\lesssim \int |\div \mathcal{Q} u| |\hat{A}_j \mathcal{P} w|^2 \, dx + \int |[\hat{A}_j, u \cdot \nabla] w \cdot \hat{A}_j \mathcal{P} w| \, dx + \int |\hat{A}_j f_4 \cdot \hat{A}_j \mathcal{P} w| \, dx, \tag{3.19}
\]

\[
\frac{1}{2} \frac{d}{dt} \|\hat{A}_j \mathcal{P} u\|_{L^2}^2 + \|\hat{A}_j \nabla \mathcal{P} u\|_{L^2}^2 + \int \hat{A}_j \mathcal{P} v \cdot \hat{A}_j \mathcal{P} u \, dx \\
\lesssim \int |\div \mathcal{Q} u| |\hat{A}_j \mathcal{P} u|^2 \, dx + \int |[\hat{A}_j, u \cdot \nabla] \mathcal{P} u \cdot \hat{A}_j \mathcal{P} u| \, dx + \int |\hat{A}_j f_5 \cdot \hat{A}_j \mathcal{P} u| \, dx. \tag{3.20}
\]

Let $\eta \in (0, 1)$ be a small enough constant which be determined later, summing up $3.19 \times \eta$, $3.20 \times (1 - \eta)$ and $3.18$ leads to

\[
\frac{1}{2} \frac{d}{dt} (\|\hat{A}_j \Lambda^{-1} \mathcal{P} v\|_{L^2}^2 + \eta \|\hat{A}_j \mathcal{P} w\|_{L^2}^2 + (1 - \eta) \|\hat{A}_j \mathcal{P} u\|_{L^2}^2) + (1 - \eta) \|\hat{A}_j \nabla \mathcal{P} u\|_{L^2}^2 + \eta \|\hat{A}_j \mathcal{P} v\|_{L^2}^2 \\
\lesssim \int |\div \mathcal{Q} u| (|\hat{A}_j \mathcal{P} w|^2, |\hat{A}_j \mathcal{P} u|^2) \, dx + \int |[\hat{A}_j, u \cdot \nabla] (\mathcal{P} u, \hat{A}_j \mathcal{P} u)| \, dx \\
+ \int |\hat{A}_j f_4 \cdot \hat{A}_j \mathcal{P} w| \, dx + \int |\hat{A}_j f_5 \cdot \hat{A}_j \mathcal{P} u| \, dx + \int |\hat{A}_j f_3 \cdot \hat{A}_j \Lambda^{-1} \mathcal{P} v| \, dx. \tag{3.21}
\]

When $j \leq j_0$ with the large integer $j_0$, it holds that

\[
\|\hat{A}_j \mathcal{P} w\|_{L^2} \lesssim \|\hat{A}_j \mathcal{P} u\|_{L^2} + \|\hat{A}_j \mathcal{P} v\|_{L^2} \lesssim \|\hat{A}_j \mathcal{P} u\|_{L^2} + 2^{j_0} \|\hat{A}_j \Lambda^{-1} \mathcal{P} v\|_{L^2}.
\]

Thus, we can find an $\eta > 0$ small enough such that

\[
\|\hat{A}_j \Lambda^{-1} \mathcal{P} v\|_{L^2}^2 + \eta \|\hat{A}_j \mathcal{P} w\|_{L^2}^2 + (1 - \eta) \|\hat{A}_j \mathcal{P} u\|_{L^2}^2 \geq \frac{1}{6} (\|\hat{A}_j \mathcal{P} u\|_{L^2}^2 + \|\hat{A}_j \Lambda^{-1} \mathcal{P} v\|_{L^2}^2).
\]

Integrating (3.21) over $[0, t]$, multiplying the resultant inequality by $2^{(j_0 - 1)}$ and summing up about $j$ for $j \leq j_0$, we get that

\[
\|(\mathcal{P} u, \Lambda^{-1} \mathcal{P} v)^j\|_{L_t^q(B_{2,1}^{\frac{d}{2}})} + \int_0^t \|(\mathcal{P} u, \Lambda^{-1} \mathcal{P} v)^j\|_{B_{2,1}^{\frac{d}{2}}} \, ds \\
\lesssim \|(\mathcal{P} u_0, \Lambda^{-1} \mathcal{P} v_0)^j\|_{B_{2,1}^{\frac{d}{2}}} + \int_0^t \sum_{j_0 \leq j} 2^{(j_0 - 1)} \|(\hat{A}_j, u \cdot \nabla] (\mathcal{P} u, \mathcal{P} v)\|_{L^2}) \, ds \\
+ \int_0^t \|\nabla u\|_{L^\infty} \|(\mathcal{P} u, \mathcal{P} v)^j\|_{B_{2,1}^{d-1}} \, ds + \int_0^t \|(f_4, f_5, f_3)^j\|_{B_{2,1}^{d-1}} \, ds. \tag{3.22}
\]
from which and $\|([P, u \cdot \nabla]v)^{\ell}\|_{B^{\frac{n}{2},1}} \lesssim \|(\Lambda^{-1}(u \cdot \nabla v))^{\ell}\|_{B^{\frac{n}{2},1}}$, we have
\[
\|(P\Lambda^{-1}P)\|_{L^\infty(B^{\frac{n}{2},1})} + \int_0^t \|(P\Lambda^{-1}P)\|_{B^{\frac{n}{2},1}} \, ds 
\lesssim \|(P\Lambda^{-1}P)\|_{B^{\frac{n}{2},1}} + \int_0^t \sum_{j < l_0} 2^{j/2-1}(\|[(\Delta_j, u \cdot \nabla)](P\Lambda^{-1}P)\|_{L^2}) \, ds 
+ \int_0^t \|\nabla u\|_{L^\infty}(P\Lambda^{-1}P)\|_{B^{\frac{n}{2},1}} \, ds + \int_0^t \|\nabla u\|_{L^2} \, ds 
+ \int_0^t \|\nabla u\|_{L^1} \, ds + \int_0^t \|(\Lambda^{-1}(u \cdot \nabla v))\|_{B^{\frac{n}{2},1}} \, ds 
+ \int_0^t \|(\Lambda^{-1}(u \cdot \nabla v))\|_{B^{\frac{n}{2},1}} \, ds + \int_0^t \|(\Lambda^{-1}(u \cdot \nabla v))\|_{B^{\frac{n}{2},1}} \, ds.
\]

The nonlinear terms appeared on the right hand side of the above inequality (3.23) are the same as (3.5) up to the last four terms. Hence, we only estimate the last four terms in (3.26).

Firstly, due to $\Lambda_j([P, u \cdot \nabla]u) = [\Lambda_jP, u \cdot \nabla]u - [\Lambda_j, u \cdot \nabla]P\Lambda^{-1}P$, we can get from Lemma 2.10 that
\[
\|(P\Lambda^{-1}P)\|_{B^{\frac{n}{2},1}} \lesssim (\|u^\ell\|_{B^{\frac{n}{2},1}} + \|u^\ell\|_{B^{\frac{n}{2},1}} + \|u^\ell\|_{B^{\frac{n}{2},1}} + \|u^\ell\|_{B^{\frac{n}{2},1}}).
\]

Secondly, we express $u\nabla v$ using Bony’s decomposition:
\[
u\nabla v = T_u \nabla v + T_{\nabla v} u + R(u, \nabla v).
\]

Thanks to Bernstein’s inequality and the Hölder inequality, we have
\[
\|(\Lambda^{-1}(T_u \nabla v))\|_{B^{\frac{n}{2},1}} \lesssim \sum_{j \leq j_0} 2^{j/2-2} \sum_{|j-k| \leq 4} \|\Delta_j (S_{k-1}u \Delta_k \nabla v)\|_{L^2} 
\lesssim \sum_{j \leq j_0} 2^{j/2-2} \sum_{|j-k| \leq 4} (\sum_{k' \geq k-2} \|\Delta_k \nabla v\|_{L^\infty} \|\Delta_k \nabla v\|_{L^2} 
\lesssim \sum_{j \leq j_0} 2^{j/2-2} \sum_{|j-k| \leq 4} \|u^\ell\|_{B^{\frac{n}{2},1}} 2^k \|\Delta_k \nabla v\|_{L^2} 
\lesssim \|u^\ell\|_{B^{\frac{n}{2},1}} \|v^\ell\|_{B^{\frac{n}{2},1}}.
\]

The above estimate still holds for the second term in (3.25):
\[
\|(\Lambda^{-1}(T_{\nabla v} u))\|_{B^{\frac{n}{2},1}} \lesssim \|u^\ell\|_{B^{\frac{n}{2},1}} \|v^\ell\|_{B^{\frac{n}{2},1}}.
\]

(3.27)
Finally, we control the rest term by

\[
\| (\Lambda^{-1}\bar{R}(u, \nabla v))^\ell \|_{L^2_{B_{2,1}^u}} \lesssim \sum_{j \leq j_0} 2^{j-2} \sum_{j-3 \leq k \leq j} \| \Delta_k u \|_{L^\infty} \| \tilde{\Delta}_k \nabla v \|_{L^2} \\
+ \sum_{j \leq j_0} 2^{(2n-1) j} \sum_{k > j} (\| \Delta_k \Lambda^{-1} u \|_{L^p} \| \tilde{\Delta}_k \nabla v \|_{L^p} + \| \Delta_k u \|_{L^p} \| \tilde{\Delta}_k v \|_{L^p})
\]

\[
\lesssim \| u^\ell \|_{L^2_{B_{2,1}^u}} \| v^\ell \|_{L^2_{B_{2,1}^u}} + \| \Lambda^{-1} u^h \|_{L^\infty_{B_{p,1}^u}} \| \nabla v^h \|_{L^2_{B_{p,1}^u}} + \| u^h \|_{L^\infty_{B_{p,1}^u}} \| v^h \|_{L^2_{B_{p,1}^u}}
\]

\[
\lesssim \| u^\ell \|_{L^2_{B_{2,1}^u}} \| v^\ell \|_{L^2_{B_{2,1}^u}} + \| u^h \|_{L^\infty_{B_{p,1}^u}} \| v^h \|_{L^2_{B_{p,1}^u}}
\]

\[
\lesssim \| u^\ell \|_{L^2_{B_{2,1}^u}} \| v^\ell \|_{L^2_{B_{2,1}^u}} + \| u^h \|_{L^\infty_{B_{p,1}^u}} \| v^h \|_{L^2_{B_{p,1}^u}} + \| v^\ell \|_{L^2_{B_{p,1}^u}} \| u^h \|_{L^\infty_{B_{p,1}^u}} + \| v^h \|_{L^2_{B_{p,1}^u}} \| u^h \|_{L^\infty_{B_{p,1}^u}}.
\]

The inequalities (3.26), (3.27) and (3.28) yield

\[
\| (\Lambda^{-1}(u \cdot \nabla v))^\ell \|_{L^2_{B_{2,1}^u}} \lesssim \| u^\ell \|_{L^2_{B_{2,1}^u}} \| v^\ell \|_{L^2_{B_{2,1}^u}} + \| u^h \|_{L^\infty_{B_{p,1}^u}} \| v^h \|_{L^2_{B_{p,1}^u}} + \| v^\ell \|_{L^2_{B_{p,1}^u}} \| u^h \|_{L^\infty_{B_{p,1}^u}} + \| v^h \|_{L^2_{B_{p,1}^u}} \| u^h \|_{L^\infty_{B_{p,1}^u}}.
\]

To estimate \( \nabla u^T \nabla \varphi \), we first write

\[
\nabla u^T \nabla \varphi = \hat{T} \nabla u \nabla \varphi + \hat{T} \nabla \varphi \nabla u + \hat{R}(\nabla u, \nabla \varphi).
\]

(3.30)

Now, the same procedure as (3.26) gives

\[
\| (\Lambda^{-1}(\hat{T} \nabla u \nabla \varphi))^\ell \|_{L^2_{B_{2,1}^u}} \lesssim \sum_{j \leq j_0} 2^{j-2} \sum_{|j-k| \leq 4} \| \Delta_j \hat{S}_{k-1} \nabla u \Delta_k \nabla \varphi \|_{L^2}
\]

\[
\lesssim \sum_{j \leq j_0} 2^{j-2} \sum_{|j-k| \leq 4} \sum_{k' \leq 2} \| \Delta_k' \nabla u \|_{L^\infty} \| \Delta_k \nabla \varphi \|_{L^2}
\]

\[
\lesssim \sum_{j \leq j_0} 2^{j-2} \sum_{|j-k| \leq 4} \| (\nabla u)^\ell \|_{L^2_{B_{2,1}^u}} \| \hat{\Delta}_k \nabla \varphi \|_{L^2}
\]

\[
\lesssim \| (\nabla \varphi)^\ell \|_{L^2_{B_{2,1}^u}} \| \hat{\nabla} u \|_{L^2_{B_{2,1}^u}} \| \hat{\Delta}_k \nabla \varphi \|_{L^2}
\]

\[
\lesssim \| \varphi^\ell \|_{L^2_{B_{2,1}^u}} \| u^\ell \|_{L^2_{B_{2,1}^u}} \| \varphi^\ell \|_{L^2_{B_{2,1}^u}}
\]

Similarly,

\[
\| (\Lambda^{-1}(\hat{T} \nabla \varphi \nabla u))^\ell \|_{L^2_{B_{2,1}^u}} \lesssim \| u^\ell \|_{L^2_{B_{2,1}^u}} \| \varphi^\ell \|_{L^2_{B_{2,1}^u}} \| \varphi^\ell \|_{L^2_{B_{2,1}^u}}
\]

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and

\[ \|(\Lambda^{-1}\hat{R}(\nabla u, \nabla \varphi))_\ell\|_{B^{\frac{n}{2}}_2,1} \lesssim \sum_{j \leq 0} 2^{j\left(\frac{n}{2}-2\right)} \sum_{j-3 \leq k \leq j} \|	ilde{\Delta}_k \nabla u\|_{L^\infty} \|	ilde{\Delta}_k \nabla \varphi\|_{L^2} \]
\[ + \sum_{j \leq 0} 2\left(\frac{n}{2}-1\right)j \sum_{k > j} (\|	ilde{\Delta}_k u\|_{L^p} \|	ilde{\Delta}_k \nabla \varphi\|_{L^p} + \|	ilde{\Delta}_k p\|_{L^p} \|	ilde{\Delta}_k \nabla u\|_{L^p}) \]
\[ \lesssim \|\varphi^\ell\|_{B^{\frac{n}{2}}_2,1} \|u^\ell\|_{B^{\frac{n}{2}}_2,1} + \|\varphi^h\|_{B^\frac{n}{p},1} \|u^h\|_{B^\frac{n}{p},1}. \quad (3.31) \]

Summing up the above three estimates, we have

\[ \|\Lambda^{-1}(\nabla u^T \nabla \varphi)^\ell\|_{B^{\frac{n}{2}}_2,1} \lesssim (\|\varphi^\ell\|_{B^{\frac{n}{2}}_2,1} + \|\varphi^h\|_{B^\frac{n}{p},1}) (\|u^\ell\|_{B^{\frac{n}{2}}_2,1} + \|u^h\|_{B^\frac{n}{p},1}). \quad (3.32) \]

Similarly,

\[ \|\Lambda^{-1}(\nabla u^T \nabla \tau)^\ell\|_{B^{\frac{n}{2}}_2,1} \lesssim (\|\varphi^\ell\|_{B^{\frac{n}{2}}_2,1} + \|\varphi^h\|_{B^\frac{n}{p},1}) (\|u^\ell\|_{B^{\frac{n}{2}}_2,1} + \|u^h\|_{B^\frac{n}{p},1}). \quad (3.33) \]

Inserting (3.6)-(3.12), (3.24), (3.29), (3.32), (3.33) into (3.23) and combining with (3.13) give

\[ \|\varphi, u, \tau\|_{L^\infty_1(B^{\frac{n}{2}}_2,1)} + \int_0^t \|\mathcal{P} u, \Lambda^{-1}\mathcal{P} v\|_{B^{\frac{n}{2}}_2,1} \|u^\ell\|_{B^{\frac{n}{2}}_2,1} ds \]
\[ \lesssim \|\varphi_0, u_0, \tau_0\|_{B^{\frac{n}{2}}_2,1} + \int_0^t (\|\varphi^\ell\|_{B^{\frac{n}{2}}_2,1} + 1) \|\varphi^\ell\|_{B^{\frac{n}{2}}_2,1} (\|u^\ell\|_{B^{\frac{n}{2}}_2,1} + \|v^\ell\|_{B^{\frac{n}{2}}_2,1}) ds \]
\[ + \int_0^t (\|\varphi^h\|_{B^\frac{n}{p},1} + 1) \|\varphi^h\|_{B^\frac{n}{p},1} (\|u^h\|_{B^\frac{n}{p},1} + \|v^h\|_{B^\frac{n}{p},1}) ds \]
\[ + \int_0^t (\|\varphi, u, \tau\|_{B^{\frac{n}{2}}_2,1} + \|u^h\|_{B^\frac{n}{p},1} + \|(\varphi, \tau)^h\|_{B^\frac{n}{p},1}) (\|u^\ell\|_{B^{\frac{n}{2}}_2,1} + \|u^h\|_{B^\frac{n}{p},1}) ds. \quad (3.34) \]

As the linear structure of the compressible part (3.15) is the same as (3.16), thus, the above inequality is still valid if we instead of \(\|\mathcal{P} u, \Lambda^{-1}\mathcal{P} v\|_{B^{\frac{n}{2}}_2,1}\) by \(\|(Q u, \Lambda^{-1} Q v)\|_{B^{\frac{n}{2}}_2,1}\).

Thus, we can finally obtain

\[ \|\varphi, u, \tau\|_{L^\infty_1(B^{\frac{n}{2}}_2,1)} + \int_0^t \|u, \Lambda^{-1}\mathcal{P} v\|_{B^{\frac{n}{2}}_2,1} \|u^\ell\|_{B^{\frac{n}{2}}_2,1} ds \]
\[ \lesssim \|\varphi_0, u_0, \tau_0\|_{B^{\frac{n}{2}}_2,1} + \int_0^t (\|\varphi^\ell\|_{B^{\frac{n}{2}}_2,1} + 1) \|\varphi^\ell\|_{B^{\frac{n}{2}}_2,1} (\|u^\ell\|_{B^{\frac{n}{2}}_2,1} + \|v^\ell\|_{B^{\frac{n}{2}}_2,1}) ds \]
\[ + \int_0^t (\|\varphi^h\|_{B^\frac{n}{p},1} + 1) \|\varphi^h\|_{B^\frac{n}{p},1} (\|u^h\|_{B^\frac{n}{p},1} + \|v^h\|_{B^\frac{n}{p},1}) ds \]
\[ + \int_0^t (\|\varphi, u, \tau\|_{B^{\frac{n}{2}}_2,1} + \|u^h\|_{B^\frac{n}{p},1} + \|(\varphi, \tau)^h\|_{B^\frac{n}{p},1}) (\|u^\ell\|_{B^{\frac{n}{2}}_2,1} + \|u^h\|_{B^\frac{n}{p},1}) ds. \quad (3.35) \]
3.3. The estimates of \((\varphi, u, \tau)\) in the high frequencies

In this section, we shall get the smoothing effect of \(u\) and the damping effect of \(v\) in the high frequencies, by introducing the so-called “effective” velocity field in [20].

As we have done in the Section 3, we find the smoothing effect of \(v\) and the damping effect of \(v\) in the compressible part and the incompressible part, respectively. Different from the previous section, here we set the compressible part for example to get the smoothing effect of \(u\) and the damping effect of \(v\) in the high frequencies.

Define
\[
\Gamma_1 \overset{\text{def}}{=} Qu - \frac{1}{2} \Delta^{-1} Qv, \quad \Gamma_2 \overset{\text{def}}{=} Pu - \Delta^{-1} Pv,
\]

one can deduce from (3.15) that
\[
\partial_t \Gamma_1 - 2\Delta \Gamma_1 = Qu + Qf_2 - Q(u \cdot \nabla u) - \Delta^{-1}(Qf_1 - Q(u \cdot \nabla v))
\]
\[
= \Gamma_1 + \frac{1}{2} \Delta^{-1} Qv + Qf_2 - Q(u \cdot \nabla u) - \Delta^{-1}(Qf_1 - Q(u \cdot \nabla v)). \tag{3.36}
\]

We get by a standard energy argument that
\[
\|\Gamma_1^h\|_{L_{t}^{\infty}(B^{#}_{p,1})} + \|\Gamma_1^h\|_{L_{t}^{1}(B^{#}_{p,1})} \lesssim \|(\Gamma_1)^h\|_{B^{#}_{p,1}} + \int_0^t \|\Gamma_1^h\|_{B^{#}_{p,1}} + \frac{1}{2} \|Qv\|_{B^{#}_{p,1}} ds + \int_0^t \|Qf_2, \Delta^{-1} Qf_1\|_{B^{#}_{p,1}} ds
\]
\[
+ \int_0^t \|Q(u \cdot \nabla u)\|_{B^{#}_{p,1}} + \|\Delta^{-1} Q(u \cdot \nabla v)\|_{B^{#}_{p,1}} ds. \tag{3.37}
\]

Plugging \(Qu = \Gamma_1 + \frac{1}{2} \Delta^{-1} Qv\) into the first equation in (3.15) gives
\[
\partial_t Qv + u \cdot \nabla Qv + Qv = -2\Delta \Gamma_1 + Qf_1 - [Q, u \cdot \nabla]v. \tag{3.38}
\]

Applying the operator \(\hat{\Delta}_j\) on (3.38) and multiplying by \(|\Delta_j Qv|^{p-2} \Delta_j Qv\) to the resultant equation, we get by summing up for the high frequencies \(\Delta_j Qv\) only that
\[
\|(Qv)^h\|_{L_{t}^{\infty}(B^{#}_{p,1})} + \|(Qv)^h\|_{L_{t}^{1}(B^{#}_{p,1})} \lesssim \|(Qv)^h\|_{B^{#}_{p,1}} + \int_0^t \|\Gamma_1^h\|_{B^{#}_{p,1}} + \|(Qf_1)^h\|_{B^{#}_{p,1}} + \|([Q, u \cdot \nabla]v)^h\|_{B^{#}_{p,1}} ds
\]
\[
+ \int_0^t \|\text{div } u\|_{L^\infty} \|(Qv)^h\|_{B^{#}_{p,1}} ds + \int_0^t \sum_{j \geq j_0} 2^{(\frac{p}{2^j} - 1)j} \|\hat{\Delta}_j u \cdot \nabla]Qv\|_{L^p} ds. \tag{3.39}
\]
Owing to the high frequency cut-off at $|\xi| \sim 2^j$, hence,

$$\|\Gamma_1\|_{L^1(B_{p,1}^{q-1})}^h \lesssim 2^{-2j_0} \|\Gamma_1\|_{L^1(B_{p,1}^{q-1})}^h \quad \text{and} \quad \|Qv\|_{L^1(B_{p,1}^{q-1})}^h \lesssim 2^{-2j_0} \|Qv\|_{L^1(B_{p,1}^{q-1})}^h.$$ 

Multiplying by a suitable large constant on both hand side of (3.37) and then plugging (3.39), we get

$$\|\Gamma_1\|_{L^1(B_{p,1}^{q-1})}^h + \|Qv\|_{L^1(B_{p,1}^{q-1})}^h \lesssim \|\Gamma_1\|_{L^1(B_{p,1}^{q-1})}^h + \|Qv\|_{L^1(B_{p,1}^{q-1})}^h + \int_0^t \|Qf_1 + Qf_2\|_{L^1(B_{p,1}^{q-1})}^h + \|\text{div} u\|_{L^\infty} \|Qv\|_{L^1(B_{p,1}^{q-1})}^h \, ds$$

$$+ \int_0^t \|Q(u \cdot \nabla u)\|_{L^1(B_{p,1}^{q-1})}^h + \|\Delta^{-1}Q(u \cdot \nabla u)\|_{L^1(B_{p,1}^{q-1})}^h \, ds.$$  

(3.40)

By Lemma 2.6, we have

$$\|Qf_1\|_{B_{p,1}^{q-1}} \lesssim \|\nabla u^T \nabla \varphi\|_{B_{p,1}^{q-1}} + \|\nabla \varphi\|_{B_{p,1}^{q-1}} + \|\text{div} (k(a) \text{div} u)\|_{B_{p,1}^{q-1}} + \|\text{div} g(\tau, \nabla u)\|_{B_{p,1}^{q-1}}$$

$$\lesssim \|\nabla \varphi\|_{B_{p,1}^{q-1}} + \|k(a)\|_{B_{p,1}^{q-1}} + \|\text{div} \tau\|_{B_{p,1}^{q-1}} + \|\text{div} g(\tau, \nabla u)\|_{B_{p,1}^{q-1}}$$

(3.41)

To estimate the first term in $Qf_2$, we decompose it into low frequencies and high frequencies

$$I(a)v = I(a)v^\ell + I(a)v^h.$$  

(3.42)

By Lemma 2.6, we have

$$\|I(a)v^\ell\|_{B_{p,1}^{q-1}} \lesssim \|I(a)v^\ell\|_{B_{p,1}^{q-1}} \lesssim \|a - aI(a)\|_{B_{p,1}^{q-1}} \|v^\ell\|_{B_{p,1}^{q-1}}$$

$$\lesssim \|a\|_{B_{p,1}^{q-1}} (1 + \|I(a)\|_{B_{p,1}^{q-1}}) \|v^\ell\|_{B_{p,1}^{q-1}} \lesssim \|a\|_{B_{p,1}^{q-1}} (1 + \|a\|_{B_{p,1}^{q-1}}) \|v^\ell\|_{B_{p,1}^{q-1}}$$

$$\lesssim (\|a^\ell\|_{B_{p,1}^{q-1}} + 1) \|a^\ell\|_{B_{p,1}^{q-1}} \|v^\ell\|_{B_{p,1}^{q-1}} + (\|a^h\|_{B_{p,1}^{q-1}} + 1) \|a^h\|_{B_{p,1}^{q-1}} \|v^h\|_{B_{p,1}^{q-1}}$$

(3.43)
Similarly, \[
\| I(a) v^h \|_B^{\frac{n}{p},1} \lesssim \| I(a) \|_B^{\frac{n}{p},1} \| v^h \|_B^{\frac{n}{p},-1} \lesssim \left( \| \varphi^\ell \|_B^{\frac{n}{2},1} + \| \varphi^h \|_B^{\frac{n}{p},1} \right) \| v^h \|_B^{\frac{n}{p},-1}. \tag{3.44}
\]

The combination of (3.43) and (3.44) gives \[
\| I(a) v \|_B^{\frac{n}{p},1} \lesssim (\| \varphi^\ell \|_B^{\frac{n}{2},1} + \| \varphi^h \|_B^{\frac{n}{p},1}) \| v^h \|_B^{\frac{n}{p},-1} + \left( (\| \varphi^\ell \|_B^{\frac{n}{2},1} + 1) \| \varphi^\ell \|_B^{\frac{n}{p},1} + (\| \varphi^h \|_B^{\frac{n}{p},1} + 1) \| \varphi^h \|_B^{\frac{n}{p},1} \right) \| v^h \|_B^{\frac{n}{p},-1}. \tag{3.45}
\]

Similarly, from Lemma 2.6, we have \[
\| I(a) (\Delta u + \nabla \text{div } u) \|_B^{\frac{n}{p},1} \lesssim \| I(a) \|_B^{\frac{n}{p},1} \| \Delta u + \nabla \text{div } u \|_B^{\frac{n}{p},-1} \lesssim (\| \varphi^\ell \|_B^{\frac{n}{2},1} + \| \varphi^h \|_B^{\frac{n}{p},1}) (\| u^\ell \|_B^{\frac{n}{2},1} + \| u^h \|_B^{\frac{n}{p},+1} + \| v^h \|_B^{\frac{n}{p},-1}). \tag{3.46}
\]

Combining (3.45) with (3.46) gives \[
\| (Qf_2) \|_B^{\frac{n}{p},1} \lesssim (\| \varphi^\ell \|_B^{\frac{n}{2},1} + \| \varphi^h \|_B^{\frac{n}{p},1}) (\| u^\ell \|_B^{\frac{n}{2},1} + \| u^h \|_B^{\frac{n}{p},+1} + \| v^h \|_B^{\frac{n}{p},-1}) + \left( (\| \varphi^\ell \|_B^{\frac{n}{2},1} + 1) \| \varphi^\ell \|_B^{\frac{n}{p},1} + (\| \varphi^h \|_B^{\frac{n}{p},1} + 1) \| \varphi^h \|_B^{\frac{n}{p},1} \right) \| v^h \|_B^{\frac{n}{p},-1}. \tag{3.47}
\]

By embedding relation $B^{\frac{n}{p},1}_p(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ and $\| v^h \|_B^{\frac{n}{p},-1} \lesssim \| (\varphi, \tau) \|_B^{\frac{n}{p},-1}$, we can get \[
\| \text{div } u \|_{L^\infty} (\| Qv \|_B^{\frac{n}{p},1}) \lesssim \| v^h \|_B^{\frac{n}{p},-1} \| \nabla u \|_B^{\frac{n}{p},1} \lesssim \| v^h \|_B^{\frac{n}{p},-1} (\| u^\ell \|_B^{\frac{n}{p},1} + \| u^h \|_B^{\frac{n}{p},1}) \lesssim \| (\varphi, \tau) \|_B^{\frac{n}{p},-1} (\| u^\ell \|_B^{\frac{n}{p},1} + \| u^h \|_B^{\frac{n}{p},1}). \tag{3.48}
\]

Similarly, \[
\| (Q(u \cdot \nabla u)) \|_B^{\frac{n}{p},1} \lesssim (\| u^\ell \|_B^{\frac{n}{p},1} + \| u^h \|_B^{\frac{n}{p},1}) (\| u^\ell \|_B^{\frac{n}{p},1} + \| u^h \|_B^{\frac{n}{p},1}). \tag{3.49}
\]

From $\Delta_j([Q, u \cdot \nabla]) = [\Delta_j Q, u \cdot \nabla] v - [\Delta_j, u \cdot \nabla] Qv$ and Lemma 2.9, we have \[
\sum_{j \geq j_0} 2^{\frac{j(n-1)}{p}} \| [\Delta_j, u \cdot \nabla] Qv \|_{L^p} + \| ([Q, u \cdot \nabla] v)^h \|_B^{\frac{n}{p},-1} \lesssim \| v \|_B^{\frac{n}{p},-1} \| \nabla u \|_B^{\frac{n}{p},1} \lesssim (\| v^\ell \|_B^{\frac{n}{2},1} + \| v^h \|_B^{\frac{n}{p},1}) (\| u^\ell \|_B^{\frac{n}{p},1} + \| u^h \|_B^{\frac{n}{p},1}) \lesssim (\| (\varphi, \tau) \|_B^{\frac{n}{p},1} + \| (\varphi, \tau) \|_B^{\frac{n}{p},1} (\| u^\ell \|_B^{\frac{n}{p},1} + \| u^h \|_B^{\frac{n}{p},1}). \tag{3.50}
\]
Using \( u \cdot \nabla v = \text{div} (u \otimes v) - \nu \text{div} u \) and the embedding relation in the high frequencies, we can get

\[
\|((\Delta^{-1} \mathcal{Q}(u \cdot \nabla v))^h\|_{L^\infty_t(B^\|u\|_0^p,1)} \lesssim \|uv\|_{L^\infty_t(B^\|u\|_0^p,1)} + \|\nu \text{div} u\|_{L^\infty_t(B^\|u\|_0^p,1)}
\]

\[
\lesssim \|u\|_{L^2_t(B^\|u\|_0^p,1)}^2 + \|v\|_{L^\infty_t(B^\|v\|_0^p,1)}^2 + \|\nu \|_{L^\infty_t(B^\|\nu\|_0^p,1)} \|\text{div} u\|_{L^\infty_t(B^\|u\|_0^p,1)}
\]

\[
\lesssim \|u\|_{L^2_t(B^\|u\|_0^p,1)}^2 + \|v\|_{L^\infty_t(B^\|v\|_0^p,1)}^2 + \|\nu \|_{L^\infty_t(B^\|\nu\|_0^p,1)} \|\text{div} u\|_{L^\infty_t(B^\|u\|_0^p,1)}
\]

\[
\lesssim \|u^\ell\|_{B^\|u\|_1^2} \|u^\ell\|_{B^\|u\|_1^2} + \|u^h\|_{B^\|u\|_1^h} \|u^h\|_{B^\|u\|_1^h} + \|v^\ell\|_{B^\|v\|_1^2} \|v^\ell\|_{B^\|v\|_1^2} + \|v^h\|_{B^\|v\|_1^h} \|v^h\|_{B^\|v\|_1^h}.
\]

(3.51)

Inserting (3.41), (3.47)–(3.51) into (3.40) gives

\[
\|\Gamma^h_t\|_{L^\infty_t(B^\|\Gamma^h\|_0^p,1)} + \|\mathcal{Q}v\|_{L^\infty_t(B^\|v\|_0^p,1)} + \|\Gamma^h_t\|_{L^1_t(B^\|\Gamma^h\|_0^p,1)} + \|(\mathcal{Q}v)^h\|_{L^1_t(B^\|\nu\|_0^p,1)}
\]

\[
\lesssim \|(\Gamma_1)_t\|_{B^\|\Gamma\|_0^p,1} + \|(\phi, \tau)_t\|_{B^\|\phi\|_0^p,1} + \|\mathcal{Q}v\|_{L^1_t(B^\|\nu\|_0^p,1)} + \|\mathcal{Q}v\|_{L^1_t(B^\|\nu\|_0^p,1)}
\]

\[
\lesssim \|(\Gamma_1)_t\|_{B^\|\Gamma\|_0^p,1} + \|(\phi, \tau)_t\|_{B^\|\phi\|_0^p,1} + \|\mathcal{Q}v\|_{L^1_t(B^\|\nu\|_0^p,1)} + \|\mathcal{Q}v\|_{L^1_t(B^\|\nu\|_0^p,1)}
\]

\[
+ \int_0^t \left( (\|\phi^\ell\|_{B^\|\phi\|_1^2} + 1) \|\phi^\ell\|_{B^\|\phi\|_1^2} + (\|\phi^h\|_{B^\|\phi\|_1^h} + 1) \|\phi^h\|_{B^\|\phi\|_1^h} \right) \|\mathcal{Q}v\|_{B^\|\nu\|_1^2} ds.
\]

(3.52)

The above estimate is still valid for \( \Gamma_2 \) and \( \mathcal{P}v \):

\[
\|\Gamma^h_2\|_{L^\infty_t(B^\|\Gamma^h\|_0^p,1)} + \|\mathcal{P}v\|_{L^\infty_t(B^\|\nu\|_0^p,1)} + \|\Gamma^h_2\|_{L^1_t(B^\|\Gamma^h\|_0^p,1)} + \|(\mathcal{P}v)^h\|_{L^1_t(B^\|\nu\|_0^p,1)}
\]

\[
\lesssim \|(\Gamma_2)_t\|_{B^\|\Gamma\|_0^p,1} + \|(\phi_0, \tau_0)_t\|_{B^\|\phi\|_0^p,1} + \|\mathcal{P}v\|_{L^1_t(B^\|\nu\|_0^p,1)} + \|\mathcal{P}v\|_{L^1_t(B^\|\nu\|_0^p,1)}
\]

\[
+ \int_0^t \left( (\|\phi^\ell\|_{B^\|\phi\|_1^2} + 1) \|\phi^\ell\|_{B^\|\phi\|_1^2} + (\|\phi^h\|_{B^\|\phi\|_1^h} + 1) \|\phi^h\|_{B^\|\phi\|_1^h} \right) \|\mathcal{P}v\|_{B^\|\nu\|_1^2} ds.
\]

(3.53)

A simple computation implies

\[
\|u^h\|_{L^\infty_t(B^\|u\|_0^p,1)} \lesssim \|(\Gamma_1)_t + \Delta^{-1} (\mathcal{P}v)^h + (\Gamma_2)_t + \Delta^{-1} (\mathcal{Q}v)^h\|_{L^\infty_t(B^\|\nu\|_0^p,1)}
\]

\[
\lesssim \|(\Gamma_1, \Gamma_2)_t\|_{L^\infty_t(B^\|\nu\|_0^p,1)} + \|\nu^h\|_{L^\infty_t(B^\|\nu\|_0^p,1)}.
\]

(3.54)

\[
\|u^h\|_{L^1_t(B^\|u\|_0^p,1)} \lesssim \|(\Gamma_1, \Gamma_2)_t\|_{L^1_t(B^\|\nu\|_0^p,1)} + \|\nu^h\|_{L^1_t(B^\|\nu\|_0^p,1)}.
\]

(3.55)
Combining (3.52) with (3.53) and using (3.54), (3.55) imply that

\[ \| u^h \|_{L_t^\infty(B^+_{p,1})} + \| v^h \|_{L_t^\infty(B^+_{p,1})} + \| u^h \|_{L_t^1(B^+_{p,1})} + \| v^h \|_{L_t^1(B^+_{p,1})} \]
\[ \lesssim \| u^h_0 \|_{B^+_{p,1}} + \| \varphi_0, \tau_0 \|_{B^+_{p,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}}. \] (3.56)

With the estimate of \( \| u^h \|_{L_t^1(B^+_{p,1})} \) in hand, we can find the \( \| \varphi, \tau \|_{L_t^\infty(B^+_{p,1})} \) by exploiting the original equations. Indeed, from the first equation in (1.5), one infers that

\[ \| \phi^h \|_{L_t^\infty(B^+_{p,1})} \lesssim \| \phi_0^h \|_{B^+_{p,1}} + \int_0^t \| \text{div } u^h \|_{B^+_{p,1}} + \| (k(a) \text{div } u^h) \|_{B^+_{p,1}} \] \[ + \int_0^t \| \text{div } u \|_{L_t^\infty} \| \phi^h \|_{B^+_{p,1}} \] \[ + \int_0^t \| \phi^h \|_{B^{2,1}_{2,1}} + \| u^h \|_{B^{2,1}_{2,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}}. \] (3.57)

Similarly,

\[ \| \tau^h \|_{L_t^\infty(B^+_{p,1})} \lesssim \| \tau_0^h \|_{B^+_{p,1}} + \int_0^t \| u^h \|_{B^{2,1}_{2,1}} \] \[ + \int_0^t \| u^h \|_{B^{2,1}_{2,1}} + \| u^h \|_{B^{2,1}_{2,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}}. \] (3.58)

Multiplying by a suitable large constant on both hand side of (3.40) and then plugging (3.57) and (3.58), we get

\[ \| (u^h, v^h) \|_{L_t^\infty(B^+_{p,1})} + \| (\phi^h, \tau^h) \|_{L_t^\infty(B^+_{p,1})} + \| u^h \|_{L_t^1(B^+_{p,1})} + \| v^h \|_{L_t^1(B^+_{p,1})} \]
\[ \lesssim \| u^h_0 \|_{B^+_{p,1}} + \| \varphi_0, \tau_0 \|_{B^+_{p,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \phi^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}} + \| \tau^h \|_{B^{2,1}_{2,1}}. \] (3.59)
Summing up (3.35) and (3.59) results in the following complete estimate

$$
\| (\varphi, u, \tau)^\ell \|_{L^\infty_t(B^\frac{n}{2} \mathcal{L}^1)} + \| u^h \|_{L^\infty_t(B^\frac{n-1}{2} \mathcal{L}^1)} + \| (\varphi^h, \tau^h) \|_{L^\infty_t(B^\frac{n}{2} \mathcal{L}^1)}
+ \| (u, \Lambda^{-1} \varphi)^\ell \|_{L^1_t(B^\frac{n}{2} \mathcal{L}^1)} + \| u^h \|_{L^1_t(B^\frac{n+1}{2} \mathcal{L}^1)} + \| v^h \|_{L^1_t(B^\frac{n}{2} \mathcal{L}^1)}
\leq \| (\varphi_0, u_0, \tau_0)^\ell \|_{B^\frac{n}{2} \mathcal{L}^1} + \| u_0^h \|_{B^\frac{n-1}{2} \mathcal{L}^1} + \| (\varphi_0^h, \tau_0^h) \|_{B^\frac{n}{2} \mathcal{L}^1}
+ \int_0^t \left( \| (\varphi, u, \tau)^\ell \|_{B^\frac{n}{2} \mathcal{L}^1} + \| (\varphi, \tau)^h \|_{B^\frac{n}{2} \mathcal{L}^1} \right) \| v^h \|_{B^\frac{n}{2} \mathcal{L}^1} ds
+ \int_0^t \left( \| \varphi^h \|_{B^\frac{n}{2} \mathcal{L}^1} + \| u^h \|_{B^\frac{n+1}{2} \mathcal{L}^1} + \| v^h \|_{B^\frac{n}{2} \mathcal{L}^1} \right) ds
+ \int_0^t \left( \| (\varphi, u, \tau)^\ell \|_{B^\frac{n}{2} \mathcal{L}^1} + \| u^h \|_{B^\frac{n}{2} \mathcal{L}^1} + \| (\varphi, \tau)^h \|_{B^\frac{n}{2} \mathcal{L}^1} \right) (\| u^h \|_{B^\frac{n+1}{2} \mathcal{L}^1} + \| u^h \|_{B^\frac{n}{2} \mathcal{L}^1}) ds. \quad (3.60)
$$

3.4. Complete the proof of Theorem 1.1 by the continuity arguments

Now, we can complete the proof of Theorem 1.1 by the continuity arguments. As we have stated in the beginning of this section, we omit the main details to prove the local wellposedness of (1.5). The interested readers can refer to [6, 14, 35] for similar arguments.

Thus, we can deduce that there exists a positive time $T$ so that the system (1.5) has a uniqueness local solution $(a, u, \tau)$ on $[0, T^*)$, where $T^*$ is the maximal existence time of $(a, u, \tau)$. Then, the proof of Theorem 1.1 is reduced to show that $T^* = \infty$ under the assumption of (1.6). Denote

$$
X(t) \overset{\text{def}}{=} \| (\varphi, u, \tau)^\ell \|_{L^\infty_t(B^\frac{n}{2} \mathcal{L}^1)} + \| u^h \|_{L^\infty_t(B^\frac{n-1}{2} \mathcal{L}^1)} + \| (\varphi^h, \tau^h) \|_{L^\infty_t(B^\frac{n}{2} \mathcal{L}^1)}
+ \| (u, \Lambda^{-1} \varphi)^\ell \|_{L^1_t(B^\frac{n}{2} \mathcal{L}^1)} + \| u^h \|_{L^1_t(B^\frac{n+1}{2} \mathcal{L}^1)} + \| v^h \|_{L^1_t(B^\frac{n}{2} \mathcal{L}^1)}.
$$

From estimate (3.60) and the Gronwall inequality, we have

$$
X(t) \leq C e^{CX(t)} \| (\varphi_0, u_0, \tau_0)^\ell \|_{B^\frac{n}{2} \mathcal{L}^1} + \| u_0^h \|_{B^\frac{n-1}{2} \mathcal{L}^1} + \| (\varphi_0^h, \tau_0^h) \|_{B^\frac{n}{2} \mathcal{L}^1}. \quad (3.61)
$$

Now let $\varepsilon$ be a positive constant, which will be determined later on. For any $T^* \in [0, T^*)$, we define

$$
T^{**} \overset{\text{def}}{=} \sup \left\{ t \in [0, T^*) : X(t) \leq \varepsilon \right\}.
$$
From (3.61), we have for any \( t \in [0, T^*] \) there holds

\[
X(t) \leq C_1e^{C_1\varepsilon} \| (\varphi_0, u_0, \tau_0) \|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \| u^h_0 \|_{\dot{B}^{\frac{5}{2}}_{p,1}} + \| (\varphi^h, \tau^h_0) \|_{\dot{B}^{\frac{5}{2}}_{p,1}}.
\] (3.62)

Choosing \( \varepsilon < \frac{1}{4C_1} \) fixed and then letting

\[
\| (\varphi_0, u_0, \tau_0) \|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \| u^h_0 \|_{\dot{B}^{\frac{5}{2}}_{p,1}} + \| (\varphi^h, \tau^h_0) \|_{\dot{B}^{\frac{5}{2}}_{p,1}} < \frac{1}{8C_1},
\]

we can get from (3.62) that

\[
X(t) \leq \frac{\varepsilon}{2}, \quad \forall t \in [0, T^*],
\]

this contradicts with the definition of \( T^* \), thus we conclude that \( T^* = T \).

Moreover, from the above argument, we have

\[
\| a^\ell \|_{\dot{B}^{\frac{5}{2}}_{2,1}} \lesssim \| (\varphi(\varphi)) \|_{\dot{B}^{\frac{5}{2}}_{2,1}}, \quad \| a^h \|_{\dot{B}^{4\ell}_{p,1}} \lesssim \| (\varphi(\varphi)) \|_{\dot{B}^{\frac{5}{2}}_{p,1}} \lesssim \| \varphi^0 \|_{\dot{B}^{\frac{5}{2}}_{p,1}}.
\]

As \( v = \nabla \varphi - \text{div} \tau \), we can get \( \mathcal{P}v = -\mathcal{P}\text{div} \tau \), thus, we can get only the smoothing effect of incompressible part of \( \text{div} \tau \), for any \( T > 0 \)

\[
(\Lambda^{-1}\mathcal{P}\text{div} \tau)^\ell \in L^1([0, T]; \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R}^n)), \quad (\Lambda^{-1}\mathcal{P}\text{div} \tau)^h \in L^1([0, T]; \dot{B}^{\frac{5}{2}}_{p,1}(\mathbb{R}^n)),
\]

Consequently, we complete the proof of Theorem 1.1 by standard continuation argument.

4. The proof of Theorem 1.5

In this section, we shall follow the method used in [19] and [39] to get the decay rate of the solutions constructed in the previous section. From the proof of Theorem 1.1 we can get the following inequality (see the derivation of (3.60) for more details):

\[
\frac{d}{dt} \left( \| (u, \Lambda^{-1}v)^\ell \|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \| u^h \|_{\dot{B}^{\frac{5}{2}}_{p,1}} + \| (\Lambda^{-1}v)^h \|_{\dot{B}^{\frac{5}{2}}_{p,1}} \right)
+ C \left( \| (u, \Lambda^{-1}v)^\ell \|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \| u^h \|_{\dot{B}^{\frac{5}{2}}_{p,1}} + \| (\Lambda^{-1}v)^h \|_{\dot{B}^{\frac{5}{2}}_{p,1}} \right)
\leq C \left( \| (u, \Lambda^{-1}v)^\ell \|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \| u^h \|_{\dot{B}^{\frac{5}{2}}_{p,1}} + \| (\Lambda^{-1}v)^h \|_{\dot{B}^{\frac{5}{2}}_{p,1}} \right)
\times \left( \| (\varphi, u, \tau)^\ell \|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \| \varphi^\ell \|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \| u^h \|_{\dot{B}^{\frac{5}{2}}_{p,1}} + \| (\varphi, \tau)^h \|_{\dot{B}^{\frac{5}{2}}_{p,1}} + \| \varphi^h \|_{\dot{B}^{\frac{5}{2}}_{p,1}} \right). \] (4.1)
Noticing that (1.7) in Theorem 1.1 it is easy to see that

\[
\|(\varphi, u, \tau)^{\ell}\|_{L_t^\infty(B_{2,1}^u)} + \|u^h\|_{L_t^\infty(B_{2,1}^u)} + \|(\varphi^h, \tau^h)\|_{L_t^\infty(B_{2,1}^u)} \leq C_0 \ll 1, \quad \text{for all } t \geq 0,
\]

thus, absorbing all the terms on the right hand side of (4.1) to left gives

\[
\frac{d}{dt}(\|u, \Lambda^{-1}v\|_{B_{2,1}^u} + \|u^h\|_{B_{2,1}^u} + \|\Lambda^{-1}v^h\|_{B_{2,1}^u}) + c(\|u, \Lambda^{-1}v\|_{B_{2,1}^u} + \|u^h\|_{B_{2,1}^u} + \|\Lambda^{-1}v^h\|_{B_{2,1}^u}) \leq 0.
\]  (4.2)

In order to derive the decay estimate of the solutions given in Theorem 1.1, we need to get a Lyapunov-type differential inequality from (4.2). This inequality can be obtained from interpolation inequality which heavily relies on the viability of the bound

\[
\|(u, \Lambda^{-1}v)\|_{B_{2,1}^u} \leq C, \quad -\frac{\nu}{2} < \sigma < \frac{\nu}{2} - 1.
\]  (4.3)

From the definition of \(v = \nabla \varphi - \text{div} \tau\), we only need to control the bound of \(\|(\varphi, u, \tau)\|_{B^{\nu}_{2,1}}\). Due to some technical reasons and as we have assumed before the Theorem 1.5 we only consider the pressure function \(P(\rho)\) is a linear function of \(\rho\) in (1.5) which implies \(\varphi = \hat{c}a\) for some fixed positive constant \(\hat{c}\). Hence, in the following argument, we only need to control the bound of

\[
\|(a, u, \tau)\|_{B^{\nu}_{2,1}} \leq C, \quad -\frac{\nu}{2} < \sigma < \frac{\nu}{2} - 1.
\]

From (1.5), we can get by a similar derivation of (3.5) that

\[
\|(a, u, \tau)^{\ell}\|_{L_t^\infty(B_{2,1}^{\nu})} \lesssim \|(a_0, u_0, \tau_0)^{\ell}\|_{B_{2,1}^{\nu}} + \int_0^t \|\nabla u\|_{L^{\infty}} \|(a, u, \tau)^{\ell}\|_{B^{\nu}_{2,1}} \ ds
\]

\[
+ \int_0^t \|a(\text{div} u)^{\ell}\|_{B^{\nu}_{2,1}} \ ds + \int_0^t \|g(\tau, \nabla u)^{\ell}\|_{B^{\nu}_{2,1}} \ ds
\]

\[
+ \int_0^t \|I(\Delta u + \nabla \text{div} u)^{\ell}\|_{B^{\nu}_{2,1}} \ ds + \sum_{j \leq j_0} 2^{\sigma j} \|\nabla_j (a, u, \tau)\|_{L^2} \ ds.
\]  (4.4)

By virtue of embedding relation \(B^{\nu}_{2,1}(\mathbb{R}^n) \hookrightarrow L^{\infty}(\mathbb{R}^n)\), we have

\[
\|\nabla u\|_{L^{\infty}} \|(a, u, \tau)^{\ell}\|_{B^{\nu}_{2,1}} \lesssim \|\nabla u\|_{B^{\nu}_{2,1}} \|(a, u, \tau)^{\ell}\|_{B^{\nu}_{2,1}} \lesssim (\|u^{\ell}\|_{B^{\nu+1}_{2,1}} + \|u^h\|_{B^{\nu+1}_{2,1}}) \|(a, u, \tau)^{\ell}\|_{B^{\nu}_{2,1}}.
\]  (4.5)
In the further argument, we shall use repeatedly the following product law:

\[ \| (bc^h)^{\ell} \|_{B^{\sigma}_{2,1}} \lesssim \| b \|_{B^{\frac{n}{2}-1}_2} \| c^h \|_{B^{n-1}_{2,1}} \quad \text{for any } -1 < \sigma < \frac{n}{2} - 1. \quad (4.6) \]

Indeed, by Bony’s decomposition:

\[ bc^h = T_{ch}b + R(c^h, b) + T_bc^h. \]

The first term can be bounded from Lemma 2.5 that

\[ \| T_{ch}b \|_{B^{\sigma}_{2,1}} \lesssim \| T_{ch}b \|_{B^{\frac{n}{2}-1}_2} \lesssim \| c^h \|_{B^{n-1}_{\infty, \infty}} \| b \|_{B^{\frac{n}{2}-1}_2} \]

\[ \lesssim \| c^h \|_{B^{1-n}_{2,1}} \| b \|_{B^{\frac{n}{2}-1}_2} \lesssim \| c^h \|_{B^{n-1}_{2,1}} \| b \|_{B^{\frac{n}{2}-1}_2} \quad (4.7) \]

where we have used the high frequency property of \( c^h \) and the fact \(-1 < \sigma < \frac{n}{2} - 1\) in the last inequality. Similarly,

\[ \| T_bc^h \|_{B^{\sigma}_{2,1}} \lesssim \| T_bc^h \|_{B^{\frac{n}{2}-1}_2} \lesssim \| b \|_{B^{1-n}_{\infty, \infty}} \| c^h \|_{B^{n-1}_{2,1}} \]

\[ \lesssim \| b \|_{B^{\frac{n}{2}-1}_2} \| c^h \|_{B^{1-n}_{2,1}} \| b \|_{B^{\frac{n}{2}-1}_2} \| c^h \|_{B^{n-1}_{2,1}}. \quad (4.8) \]

By using the low frequency property and the condition \(-1 < \sigma < \frac{n}{2} - 1\), the rest term can be estimated from Lemma 2.5 that

\[ \| R(c^h, b) \|_{B^{\sigma}_{2,1}} \lesssim \| R(c^h, b) \|_{B^{\frac{n}{2}-1}_2} \lesssim \| R(c^h, b) \|_{B^{0}_{1, \infty}} \]

\[ \lesssim \| c^h \|_{B^{1-n}_{2,1}} \| b \|_{B^{\frac{n}{2}-1}_2} \lesssim \| c^h \|_{B^{n-1}_{2,1}} \| b \|_{B^{\frac{n}{2}-1}_2} \quad (4.9) \]

which together with (4.7), (4.8) gives the desired estimate (4.6).

Thanks to Lemma 2.6, one can infer that

\[ \| (a \text{div } u)^{\ell} \|_{B^{\sigma}_{2,1}} \lesssim \| a \|_{B^{\sigma}_{2,1}} \| \text{div } u \|_{B^{\frac{n}{2}-1}_2} \]

\[ \lesssim (\| a^\ell \|_{B^{\sigma}_{2,1}} + \| a^h \|_{B^{\sigma}_{2,1}}) \| u \|_{B^{\frac{n}{2}-1}_2} \]

\[ \lesssim (\| a^\ell \|_{B^{\sigma}_{2,1}} + \| a^h \|_{B^{\sigma}_{2,1}}) (\| u^\ell \|_{B^{\frac{n}{2}-1}_2} + \| u^h \|_{B^{\frac{n}{2}-1}_2}). \quad (4.10) \]

Here and in after, we shall use repeatedly the following fact:

\[ \| a^h \|_{B^{\sigma}_{2,1}} \lesssim \| a^h \|_{B^{\frac{n}{2}-1}_2}, \quad -1 < \sigma < \frac{n}{2} - 1. \quad (4.11) \]
We get by a similar derivation of (4.10) that
\[ \| (g(\tau, \nabla u)^\ell \|^{B_{2,1}^\ell} \leq \| \tau \|^{B_{2,1}} + \| u^\ell \|^{B_{2,1}} \| u^h \|^{B_{2,1}}. \] (4.12)

Regarding the nonlinear term \( \|(I(a)v)^\ell\|^{B_{2,1}^\ell} \), the calculation is a little bit careful. We first get from Lemma 2.6 and (4.6) that
\[ \| (I(a)v)^\ell \|^{B_{2,1}^\ell} \leq \| I(a)\|^{B_{2,1}^\ell} + \| I(a)^h \|^{B_{2,1}^\ell} \]
\[ \leq \| I(a)\|^{B_{2,1}^\ell} + \| I(a)^h \|^{B_{2,1}^\ell}. \] (4.13)

Recall that \( I(a) = \frac{a}{1+a} = a - aI(a) \), hence,
\[ \| I(a)\|^{B_{2,1}^\ell} \leq \| a - aI(a)\|^{B_{2,1}^\ell} \leq (1 + \| I(a)\|^{B_{2,1}^\ell}) \| a\|^{B_{2,1}^\ell} \]
\[ \leq (1 + \| a\|^{B_{2,1}^\ell}) (\| a^\ell \|^{B_{2,1}^\ell} + \| a^h \|^{B_{2,1}^\ell}) \]
\[ \leq (1 + \| a^\ell \|^{B_{2,1}^\ell} + \| a^h \|^{B_{2,1}^\ell}) (\| a\|^{B_{2,1}^\ell} + \| I(a)\|^{B_{2,1}^\ell}). \] (4.14)

similarly,
\[ \| I(a)^h \|^{B_{2,1}^\ell} \leq (1 + \| a^\ell \|^{B_{2,1}^\ell} + \| a^h \|^{B_{2,1}^\ell}) (\| a\|^{B_{2,1}^\ell} + \| I(a)\|^{B_{2,1}^\ell}). \] (4.15)

Taking (4.14) and (4.15) into (4.13) gives
\[ \| (I(a)v)^\ell \|^{B_{2,1}^\ell} \leq (1 + \| a^\ell \|^{B_{2,1}^\ell} + \| a^h \|^{B_{2,1}^\ell}) (\| v^\ell \|^{B_{2,1}^\ell} + \| a^\ell \|^{B_{2,1}^\ell} + \| a^h \|^{B_{2,1}^\ell}) \]
\[ + (1 + \| a^\ell \|^{B_{2,1}^\ell} + \| a^h \|^{B_{2,1}^\ell}) (\| v^\ell \|^{B_{2,1}^\ell} + \| a^\ell \|^{B_{2,1}^\ell} + \| a^h \|^{B_{2,1}^\ell}) \]
\[ + (\| a^\ell \|^{B_{2,1}^\ell} + \| a^h \|^{B_{2,1}^\ell}) (1 + \| a^\ell \|^{B_{2,1}^\ell} + \| a^h \|^{B_{2,1}^\ell}) (\| v^\ell \|^{B_{2,1}^\ell} + \| a^\ell \|^{B_{2,1}^\ell} + \| a^h \|^{B_{2,1}^\ell}). \] (4.16)

The term \( \| (I(a)(\Delta u + \nabla \text{div } u))^\ell \|^{B_{2,1}^\ell} \) can be dealt with the same processes:
\[ \| (I(a)(\Delta u + \nabla \text{div } u))^\ell \|^{B_{2,1}^\ell} \leq \]
With the aid of Lemma 2.10 one has

\[
\sum_{j \leq j_0} 2^{\sigma j} \| (\Delta_j, u \cdot \nabla) (a, u, \tau) \|_{L^2} \lesssim \| \nabla u \|_{B^{\sigma}_{2,1}} \| (a, u, \tau) \|_{B^{\sigma}_{2,1}}
\]

\[
\lesssim (\| u^\ell \|_{B^{\sigma}_{2,1}} + \| u^h \|_{B^{\sigma}_{2,1}})((\| u^h \|_{B^{\sigma}_{2,1}} + \| (a, \tau)^h \|_{B^{\sigma}_{2,1}})
\]

\[+ (\| u^\ell \|_{B^{\sigma}_{2,1}} + \| u^h \|_{B^{\sigma}_{2,1}})((a, u, \tau)^\ell \|_{B^{\sigma}_{2,1}}). \tag{4.18}
\]

Inserting (4.15), (4.10), (4.12), (4.16) – (4.18) into (4.4) gives

\[
\| (a, u, \tau)^\ell \|_{L^\infty_t(B^{\sigma}_{2,1})} \lesssim (a_0, u_0, \tau_0)^\ell + \int_0^t G_2(s) \, ds + \int_0^t G_1(s) \| (a, u, \tau)^\ell \|_{B^{\sigma}_{2,1}} \, ds \tag{4.19}
\]

in which

\[
G_1(t) = (\| a^\ell \|_{B^{\sigma}_{2,1}} + \| a^h \|_{B^{\sigma}_{2,1}} + 1)((\| u^\ell \|_{B^{\sigma}_{2,1}} + \| u^h \|_{B^{\sigma}_{2,1}} + \| v^\ell \|_{B^{\sigma}_{2,1}}),
\]

\[
G_2(t) = (\| a^\ell \|_{B^{\sigma}_{2,1}} + \| a^h \|_{B^{\sigma}_{2,1}} + 1)(\| (a, \tau)^h \|_{B^{\sigma}_{2,1}} + \| u^\ell \|_{B^{\sigma}_{2,1}} + \| v^\ell \|_{B^{\sigma}_{2,1}}
\]

\[+ (\| a^\ell \|_{B^{\sigma}_{2,1}} + \| (a, \tau)^h \|_{B^{\sigma}_{2,1}})((\| a^\ell \|_{B^{\sigma}_{2,1}} + \| a^h \|_{B^{\sigma}_{2,1}} + 1)((\| u^\ell \|_{B^{\sigma}_{2,1}} + \| v^\ell \|_{B^{\sigma}_{2,1}}).
\]

As \((a, u, \tau)\) is the global solution constructed in Theorem 1.1 thus, we can get from (1.7) that

\[
\int_0^t G_1(s) \, ds + \int_0^t G_2(s) \, ds \leq C \tag{4.20}
\]

from which and the Gronwall inequality applied to (4.19), we have

\[
\| (a, u, \tau)^\ell \|_{B^{\sigma}_{2,1}} \leq C, \quad -\frac{n}{2} < \sigma < \frac{n}{2} - 1, \tag{4.21}
\]

with \(C\) is a constant depending on \(n\) and \(a_0, u_0, \tau_0\).

Moreover, from \(v = \nabla a + \text{div } \tau\), we can get

\[
\| (\Lambda^{-1} v)^\ell \|_{B^{\sigma}_{2,1}} \leq C \| (a, \tau)^\ell \|_{B^{\sigma}_{2,1}} \leq C, \quad -\frac{n}{2} < \sigma < \frac{n}{2} - 1. \tag{4.22}
\]

Now, for any \(-\frac{n}{2} < \sigma < \frac{n}{2} - 1\), it follows from interpolation inequality in Lemma 2.3 that

\[
\| (u, \Lambda^{-1} v)^\ell \|_{B^{\sigma}_{2,1}} \leq C \| (u, \Lambda^{-1} v)^\ell \|_{B^{\sigma}_{2,1}}^{\theta_1} \| (u, \Lambda^{-1} v)^\ell \|_{B^{\sigma+1}_{2,1}}^{1-\theta_1}
\]

\[
\leq C \| (u, \Lambda^{-1} v)^\ell \|_{B^{\sigma}_{2,1}}^{1-\theta_1}, \quad \theta_1 = \frac{4}{n-2\sigma+2} \in (0, 1),
\]

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this implies that
\[
\| (u, \Lambda^{-1}v) \|_{B^{\sigma+1}_{2,1}}^\ell \geq C \left( \| (u, \Lambda^{-1}v) \|_{B^\sigma_{2,1}}^\ell \right)^{\frac{1}{1-\sigma_1}}. \tag{4.23}
\]

Due to the embedding relations in the high frequencies part, one can deduce from (1.7) that
\[
\| u \|_{B^\sigma_{2,1}}^h \geq C \left( \| u \|_{B^\sigma_{2,1}}^h \right)^{\frac{1}{1-\sigma_1}}, \quad \| \Lambda^{-1}v \|_{B^\sigma_{2,1}}^h \geq C \left( \| \Lambda^{-1}v \|_{B^\sigma_{2,1}}^h \right)^{\frac{1}{1-\sigma_1}}. \tag{4.24}
\]

Thus, substituting (4.23) and (4.24) into (4.2) yields
\[
\frac{d}{dt} \left( \| (u, \Lambda^{-1}v) \|_{B^{\sigma+1}_{2,1}}^\ell + \| u \|_{B^\sigma_{2,1}}^h + \| \Lambda^{-1}v \|_{B^\sigma_{2,1}}^h \right) + \tilde{c} \left( \| (u, \Lambda^{-1}v) \|_{B^{\sigma+1}_{2,1}}^\ell + \| u \|_{B^\sigma_{2,1}}^h + \| \Lambda^{-1}v \|_{B^\sigma_{2,1}}^h \right)^{\frac{n-2\sigma+2}{n-2\sigma-2}} \leq 0.
\]

Solving this differential inequality directly, we obtain
\[
\| (u, \Lambda^{-1}v) \|_{B^{\sigma+1}_{2,1}}^\ell + \| u \|_{B^\sigma_{2,1}}^h + \| \Lambda^{-1}v \|_{B^\sigma_{2,1}}^h \leq C \left( \gamma_0^{-\frac{4}{n-2\sigma-2}} + \frac{4\tilde{c}}{n-2\sigma-2} \right)^{-\frac{n-2\sigma-2}{4}} \leq C(1 + t)^{-\frac{n-2\sigma-2}{4}}.
\]

Moreover, we further get
\[
\| (u, \Lambda^{-1}v) \|_{B^\sigma_{2,1}}^h \leq C \left( \| (u, \Lambda^{-1}v) \|_{B^\sigma_{2,1}}^h \right)^{\frac{1}{1-\sigma}} \leq C(1 + t)^{-\frac{n-2\sigma-2}{4}}. \tag{4.25}
\]

For any \( \sigma < \gamma < \frac{n}{2} - 1 \), on the one hand, one can get from the embedding relations in the high frequencies that
\[
\| (u, \Lambda^{-1}v) \|_{B^\gamma_{2,1}}^h \leq C \left( \| u \|_{B^\gamma_{2,1}}^h + \| \Lambda^{-1}v \|_{B^\gamma_{2,1}}^h \right) \leq C(1 + t)^{-\frac{n-2\sigma-2}{4}}, \tag{4.26}
\]

On the other hand, by the interpolation inequality we have
\[
\| (u, \Lambda^{-1}v) \|_{B^\gamma_{2,1}}^\ell \leq C \left( \| (u, \Lambda^{-1}v) \|_{B^\gamma_{2,1}}^\ell \right)^{\theta_2} \left( \| (u, \Lambda^{-1}v) \|_{B^\gamma_{2,1}}^\ell \right)^{1-\theta_2}, \quad \theta_2 = \frac{n}{2} - 1 - \gamma \in (0, 1),
\]

which combines (4.21), (4.22) with (4.25) gives
\[
\| (u, \Lambda^{-1}v) \|_{B^\gamma_{2,1}}^\ell \leq C(1 + t)^{-\frac{(\gamma - \sigma - 1)\theta_2}{2}} = C(1 + t)^{-\frac{\gamma - \sigma}{2}}. \tag{4.27}
\]

The combination of (4.26) and (4.27) gives
\[
\| (u, \Lambda^{-1}v) \|_{B^\gamma_{2,1}}^h \leq C \left( \| (u, \Lambda^{-1}v) \|_{B^\gamma_{2,1}}^h \right) \leq C(1 + t)^{-\frac{\gamma - \sigma}{2}}.
\]

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Thanks to the embedding relation $B^0_{2,1}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$, one infer that
\[ \| \Lambda^\gamma(u, \Lambda^{-1}v) \|_{L^2} \leq C(1 + t)^{-\frac{\gamma}{2-\sigma}}. \]

For any $2 \leq q \leq \infty$ and $\frac{n}{q} - \frac{n}{2} + \sigma < \alpha \leq \frac{n}{q} - 1$, by the Gagliardo-Nirenberg type interpolation inequality, which can be found in the Chap. 2 of [1], taking
\[ \kappa \theta_3 + m(1 - \theta_3) = \alpha + n\left(\frac{1}{2} - \frac{1}{q}\right), \quad m = \frac{n}{2} - 1, \]
we get
\[
\| \Lambda^\alpha(u, \Lambda^{-1}v) \|_{L^q} \leq C \| \Lambda^m(u, \Lambda^{-1}v) \|_{L^2}^{1-\theta_3} \| \Lambda^k(u, \Lambda^{-1}v) \|_{L^2}^{\theta_3}
\leq C \left\{ (1 + t)^{-\frac{m-\sigma}{2-\sigma}} \right\}^{1-\theta_3} \left\{ (1 + t)^{-\frac{k-\sigma}{2-\sigma}} \right\}^{\theta_3}
\leq C (1 + t)^{-\frac{n}{2} \left(\frac{1}{2} - \frac{1}{q}\right) - \frac{\alpha - \sigma}{2-\sigma}}.
\]

Consequently, we have completed the proof of our theorem. \(\square\)

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