Gravitational collapse of null fluid on the brane

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Abstract

We first obtain the analogue of Vaidya’s solution on the brane for studying the
collapse of null fluid onto a flat Minkowski cavity on the brane. Since the back-
reaction of the bulk onto the brane is supposed to strengthen gravity on the brane,
it would favour formation of black hole as against naked singularity. That is the
parameter window in the initial data set giving rise to naked singularity in the 4D
Vaidya case would now get partially covered.

KEY WORDS: Gravitational collapse, naked singularity, cosmic censorship, brane
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1 Introduction

General relativity (GR) is a successful theory of gravitation in the energy range accessible
to observation. However, at very high energies where all fields are expected to move
towards unification, considerations of modification to GR would be quite in order. In
this context, string theory proposes a grand framework for unification of all forces and it
requires dimensions larger than the usual four. That is, at ultra high energy, there should
be a new theory of gravitation, of which GR should turn out to be the low energy limit.
Of course we are far from that theory, yet it has inspired considerable interest in higher
dimensional gravity.

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The canonical scenario in the higher dimensional models is that the matter fields are confined to a $1 + 3$-dimensional 3-brane in $1 + 3 + d$ dimensional spacetime, while gravitational field propagates in the extra $d$ dimensions as well [1, 2]. Earlier the extra dimensions were taken to be very compact, but recently this condition has also been relaxed by Randall and Sundrum [3]. They have shown for $d = 1$ gravitational field could be confined in the neighbourhood of the 3-brane without any restriction of finiteness of the fifth extra dimension (see also [4]). Shiromizu, Maeda and Sasaki [5] have worked out an elegant geometric framework for the R-S brane model. The 5-dimensional spacetime is called bulk while the Universe we live in brane, in which the standard models and theories of fields and matter enjoy the observational support. In the R-S model, it is envisaged that the bulk is a solution of the Einstein equation with negative cosmological constant, $\Lambda$. Gravitational field gets “reflected” from the bulk onto the brane through the projection of the bulk Weyl curvature on the brane. This projection would appear as trace free matter field on the brane. By ”reflection” from bulk the effective Einstein equation on the brane would in addition to the usual stresses include quadratic terms in them plus the trace free part which arises from free gravitational field [3]. For seeking solution of a problem in the brane world model is first to find the solution with negative $\Lambda$ in the bulk and then solution of the effective equation on the brane which has to be matched with the bulk solution satisfying the proper boundary conditions [5].

In view of the involved process of finding solution, it would be understandably quite difficult to find the complete solution. A solution for a black hole on the brane has recently been proposed [3], which takes into account the trace free matter field induced on the brane from the bulk. It is given by the Reissner-Nordström metric for a charged black hole. Here the induced charge is the tidal gravitational charge, and not the electric charge, and it is the measure of ”reflected” gravitational field energy as a matter field on the brane. The solution was obtained by solving for the trace free condition and null energy condition on the brane. It has not been matched with the $\Lambda$ vacuum in the bulk. The null energy condition is required to hold good on the horizon and not necessarily everywhere off the horizon. It is however an approximate solution which is good in the neighbourhood of the horizon [3]. It has been argued that induced energy density on the brane must be negative so as to strengthen hole’s gravity and as well as not to radically change the singularity structure of spacetime. When an isolated object is sitting in an energy distribution which vanishes asymptotically, it would contribute in line with object’s gravity only when its energy density is negative [8]. The modification ensuing from bulk’s back-reaction would thus strengthen black hole’s gravitational field. It would therefore be pertinent to study gravitational collapse in this model. Would the parameter window in the initial data set giving rise to naked singularity shrink owing to effective strengthening of gravity? This is
the question we wish to address in this letter.

The cosmic censorship conjecture (CCC) is one of the most important unresolved question in classical general relativity [9]. It is of course well-known that gravitational collapse under very general conditions in GR ultimately leads to a singularity [10]. The question arises whether singularity so formed is visible to asymptotic observer or hidden behind an event horizon. The CCC states that the latter is the case. In its strong form, it demands spacetime to be globally hyperbolic (and hence its invisibility to any observer), while the weak form demands formation of a horizon and its invisibility only to an asymptotic observer (see [11], for reviews on CCC). In view of its profound importance, the question also attains pertinence in situations that may call for modification of GR. We shall therefore like to address this question for the brane world model. That is, what effect would the back-reaction of the bulk cause on gravitational collapse on the brane?

We shall consider collapse of the null fluid described by the Vaidya solution onto a flat Minkowski cavity. The exterior to the Vaidya region would be the Schwarzschild region. That is the brane-generalized Vaidya would be matched to the brane-generalized Schwarzschild solution. For this we shall first find the analogue of the Vaidya solution on the brane taking into account the bulk induced trace free matter field. In finding the end product of collapse, we need only to know the generalized Vaidya solution on the brane. In the next section, we write the effective Einstein equation on the brane and the generalized Vaidya solution on the brane, which would be followed by discussion of occurrence of naked singularity. We conclude with a discussion.

2 Vaidya solution on the brane

The Einstein equation for the 5-dimensional bulk spacetime with a negative $\Lambda$ and brane energy-momentum as source [1, 3]:

$$\tilde{G}_{AB} = \tilde{\kappa}^2 [\tilde{\Lambda} \tilde{g}_{AB} + \delta(\xi)(-\lambda g_{AB} + T_{AB})]$$

The tildes indicate the 5-dimensional bulk analogous of the standard GR quantities, and $\tilde{\kappa}^2 = 8\pi / \tilde{M}_p^3$, where $\tilde{M}_p$ is the fundamental 5-dimensional Planck mass, which is much less than the brane Planck mass, $M_p = 1.2 \times 10^{19}$ GeV. The brane is located at $\xi = 0$ which suggests the coordinates $(x^\mu, \xi)$ where $x^\mu = (t, x^i)$ are the spacetime coordinates on the brane. $\lambda$ is the brane tension and induced metric on the brane is given by $g_{AB} = \tilde{g}_{AB} - n_A n_B$ where $n_A$ is the space-like unit normal to the brane. The brane stress tensor $T_{AB}$ refers to the brane confined matter fields with $T_{AB}n^A = 0$. 
The effective equation on the brane is obtained by Shiromizu et al. \[5\] by using the Gauss-Codacci equations, the matching conditions and the $Z_2$ symmetry of the spacetime. It reads as follows:

$$ G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \tilde{\kappa}^4 S_{\mu\nu} - \mathcal{E}_{\mu\nu}, \quad (2) $$

where

$$ S_{\mu\nu} = \frac{1}{12} T_{\alpha\beta} T^{\alpha\beta} - \frac{1}{4} T^{\mu\alpha} T_{\nu\alpha} + \frac{1}{24} g_{\mu\nu} \left[ 3 T_{\alpha\beta} T^{\alpha\beta} - (T_\alpha^\alpha)^2 \right] \quad (3) $$

and

$$ \mathcal{E}_{AC} = \tilde{C}_{ABCD} n^B n^D. \quad (4) $$

Here $\tilde{C}_{ABCD}$ is the bulk Weyl curvature and hence $\mathcal{E}_{AB} = \mathcal{E}_{BA}$, $\mathcal{E}_A^A = 0$, $\mathcal{E}_{AB} n^B = 0$. The induced modifications from the bulk are of two kinds, one the quadratic stress corrections and the other nonlocal effects of the free gravitational filed in the bulk transmitted through the projection of the bulk Weyl curvature. Thus $\mathcal{E}_{\mu\nu}$ represents reflected nonlocal gravitational degrees of freedom from bulk to the brane and it includes tidal (Coulomb), gravito-magnetic as well as transverse traceless (gravitational wave) effects.

The energy scales are related by

$$ \lambda = 6 \frac{\kappa^2}{\tilde{\kappa}^4}, \quad \Lambda = \frac{4\pi}{M_p} \left[ \tilde{\Lambda} + \left( \frac{4\pi}{3M_p^3} \right) \lambda^2 \right] \quad (5) $$

where $\kappa^2 = 8\pi/M_p^2$.

For obtaining the analogue of the Vaidya solution \[12\] on the brane, we note that $T_{\mu\nu} = \sigma k_\mu k_\nu$, $k_\mu k^\mu = 0$ where $\sigma$ is the density of null fluid. We set $\Lambda = 0$, and the equation we need to solve is

$$ G_{\mu\nu} = \kappa^2 \sigma k_\mu k_\nu - \mathcal{E}_{\mu\nu} \quad (6) $$

because for the null fluid $S_{\mu\nu} = 0$. The solution in $(v, r, \theta, \phi)$ coordinates is given by

$$ ds^2 = -(1 - \frac{2m(v)}{r} - \frac{e^2(v)}{r^2}) dv^2 + 2 dv dr + r^2 d\Omega^2 \quad (7) $$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, $v$ represents advanced Eddington time, in which $r$ is decreasing towards the future along a ray $v = \text{const.}$, the two arbitrary functions $m(v) = M(v)/M_p^2$ and $e(v)^2 = M^2(v)/M_p^2$ refer respectively to the mass and reflected tidal charge at advanced
time \( v \). They would be only restricted by the proper energy conditions \([8]\). In contrast to the positive energy density of electric field for the charged black hole, the energy density of reflected gravitational field is negative which accounts for the negative sign before \( e^2/r^2 \) term in the metric.

We have thus obtained an analogue of the Vaidya solution on the brane in the same manner as the Schwarzschild’s analogue \([6]\). It would also attract the same criticism that it would be valid only close to the horizon. In our analysis the behaviour of solution near the horizon is most relevant. Thus even though this solution may not be valid globally, it would be good enough for our purpose here to probe occurrence of naked singularity or black hole.

3 Collapse on the brane

In this section, we employ the above metric for investigation of formation of black hole or naked singularity in collapse of null fluid in the context of CCC. The physical situation is that of a radial influx of null fluid in an initially empty region of the Minkowski spacetime. The first shell arrives at \( r = 0 \) at time \( v = 0 \) and the final at \( v = T \). A central singularity of growing mass is developed at \( r = 0 \). For \( v < 0 \) we have \( m(v) = e(v) = 0 \), i.e., an empty Minkowski metric, and for \( v > T \), \( \dot{m}(v) = \dot{e}(v) = 0 \), \( m(v) \) and \( e^2(v) \) are positive definite. The metric for \( v = 0 \) to \( v = T \) is the brane-generalised Vaidya, and for \( v > T \) we have the brane-generalised Schwarzschild solution.

In order to proceed further we would require the specific forms of the functions \( m(v) \) and \( e^2(v) \) which we choose as follows:

\[
m(v) = \begin{cases} 
0, & v < 0, \\
\lambda v (\lambda > 0), & 0 \leq v \leq T, \\
\mu v (\mu > 0), & v > T.
\end{cases}
\]  

(8)

and

\[
e^2(v) = \begin{cases} 
0, & v < 0, \\
\mu^2 v^2 (\mu^2 > 0), & 0 \leq v \leq T, \\
\mu^2 v^2 (\mu^2 > 0), & v > T.
\end{cases}
\]  

(9)

Then the space-time is self-similar \([13]\), admitting a homothetic Killing vector

\[
\xi^a = r \frac{\partial}{\partial r} + v \frac{\partial}{\partial v}
\]  

(10)

which is given by the Lie derivative

\[
L_{\xi} g_{ab} = \xi_{a,b} + \xi_{b,a} = 2 g_{ab}
\]  

(11)
L denotes the Lie derivative. Let \( K^a = dx^a/dk \) be the tangent vector to the null geodesics, where \( k \) is an affine parameter. Then

\[
g_{ab} K^a K^b = 0. \tag{12}\]

It follows that along null geodesics, we have

\[
\xi^a K_a = r K_r + v K_v = C \tag{13}\]

Following [14], we introduce

\[
K^v = \frac{P}{r} \tag{14}\]

and, from the null condition, we obtain

\[
K^r = \left[ 1 - \frac{2m(v)}{r} - \frac{e^2(v)}{r^2} \right] \frac{P}{2r}. \tag{15}\]

To study the singularity we employ the method developed by Dwivedi and Joshi [14]. Consider the equation for radial, outgoing null geodesics

\[
\frac{dr}{dv} = \frac{1}{2} \left[ 1 - \frac{2m(v)}{r} - \frac{e^2(v)}{r^2} \right]. \tag{16}\]

This is an ordinary differential equation with a singular point \( v = 0, r = 0 \). This singularity is (at least locally) naked if there are geodesics starting at it with a definite tangent. If no geodesic exists, then singularity is not naked and strong CCC holds. To investigate the behaviour near singular point, define

\[
y = \frac{v}{r}. \tag{17}\]

Eq. (16), upon using eqs. (8), (9) and (17), turns out to be

\[
\frac{dr}{dv} = \frac{1}{2} \left[ 1 - 2\lambda y - \mu^2 y^2 \right]. \tag{18}\]

If singularity is naked, there exists some value of \( \lambda \) and \( \mu \) such that at least one positive finite value \( y_0 \) exists which solves the algebraic equation

\[
y_0 = \lim_{r \to 0} \lim_{v \to 0} y = \lim_{r \to 0} \lim_{v \to 0} \frac{v}{r}. \tag{19}\]
Using (18) and L’Hôpital’s rule we get
\[ y_0 = \lim_{r \to 0} \lim_{v \to 0} y = \lim_{r \to 0} \lim_{v \to 0} \frac{v}{r} = \lim_{r \to 0} \frac{dv}{dr} = \frac{2}{1 - 2\lambda y_0 - \mu^2 y_0^2} \] (20)
which can be written in explicit form as,
\[ \mu^2 y_0^3 + 2\lambda y_0^2 - y_0 + 2 = 0. \] (21)
The central shell focusing singularity is naked, if eq. (21) admits one or more positive real roots. Hence in the absence of positive real roots, the collapse will always lead to a black hole. It can be shown that real roots of eq. (21) exist only when \( \Delta = \lambda^2 - 18\lambda\mu^2 - 16\lambda^3 + \mu^2 - 27\mu^4 \geq 0. \) It is easy to check that two of three real roots of eq. (21) are always positive. We have summarised the results in the following table. Hence we can say that the nature of the singularity (naked or covered) depends on the behaviour of \( \Delta. \) The behaviour of \( \Delta \) in \((\lambda, \mu)\) space is shown in Figs. I-III.

| \( \lambda \) | \( \mu \) | Roots \((X_0)\) |
|---|---|---|
| 0.06 | 0.03 | 3.57311, 4.40124 |
| 0.05 | 0.05 | 2.90354, 5.66840 |
| 1/16 | 0.00 | 4, 4 |
| 0.0005 | 0.19 | 2.80212, 3.25127 |
| 0.0 | 0.10 | 2.09149, 8.78885 |

It can be seen that collapse always leads to naked singularity if \( \lambda \leq 0.06 \) and \( \mu \leq 0.03 \) (Fig. I). Whereas for \( \mu > 0.19 \) it is always a black hole for all values of \( \lambda \) (Fig. II). When \( \mu = 0 \), we come back to the Vaidya metric, and eq. (21) admit positive roots for \( 0 < \lambda \leq 1/16 \) and hence singularities are naked for this range of \( \lambda \). [11]

**Strength of Naked Singularities:** A singularity is termed gravitationally strong or simply strong, if it destroys by crushing or stretching any object to zero volume which falls into it. A sufficient condition [15] for a strong singularity as given by Tipler [16] is that for at least one non-spacelike geodesic with affine parameter \( k \), in limiting approach to singularity, we must have
\[ \lim_{k \to 0} k^2 \psi = \lim_{k \to 0} k^2 R_{ab} K^a K^b > 0 \] (22)
where \( R_{ab} \) is the Ricci tensor. Eq. (22) can be expressed as
\[ \lim_{k \to 0} k^2 \psi = \lim_{k \to 0} 2 \left[ m(v) + \frac{e(v) \dot{e}(v)}{r} \right] \left[ \frac{k P}{r^2} \right]^2 \] (23)
Figure 1: NS at all points of this \((\lambda \leq .06, \mu \leq .03)\) space

Figure 2: BH occurs at all points of the \((\lambda, \mu)\) space for which \(\mu > .19\)
Our purpose here is to investigate the above condition along future directed null geodesics coming out from the singularity. For this, solution $P$ is required. Eq. (13), because of eqs. (8), (14) and (15), yields

$$P = \frac{2C}{2 - y + 2\lambda y^2 + \mu y^3}$$

(24)

and the geodesics are then completely known. Further, we note that

$$\frac{dy}{dk} = \frac{1}{r}K^v - \frac{X}{r}K^r$$

(25)

which, on inserting the expressions for $K^v$ and $K^r$, become

$$\frac{dy}{dk} = (2 - y + \lambda y^2 + \mu y^3) \frac{P}{2r^2} = \frac{C}{r^2}.$$  

(26)

Using the fact that as singularity is approached, $k \to 0$, $r \to 0$ and $X \to a_+$ (a root of (21)) and using L’Hôpital’s rule, we observe

$$\lim_{k \to 0} \frac{kP}{r^2} = \frac{2}{1 + \mu y_0^2}$$

(27)

and hence eq. (23) gives

$$\lim_{k \to 0} k^2 \psi = \frac{8\lambda}{(1 + \mu y_0^2)^2} > 0.$$
Thus along radial null geodesics strong curvature condition is satisfied. Therefore, one may say that generically, the naked singularity is gravitationally strong [10]. Having seen that the naked singularity in our model is a strong curvature singularity, we also examine its scalar polynomial character. The Kretschmann scalar ($K = R^a_{bced} R^{bced}$, $R^{aecd}$ is the Riemann tensor) with the help of eqs. (8) and (9), takes the form

$$K = \frac{48}{r^4} \left[ \lambda^2 y^2 + 2 \lambda \mu^2 y^3 + \frac{7}{6} \mu^4 y^4 \right]$$

which diverges at the naked singularity and hence the singularity is a scalar polynomial singularity [10].

Finally, we compute the Ricci scalar ($R = R_a^a$) for the metric (8) is

$$R = 4 \left[ \frac{\mu^4 y^4}{r^4} \right]$$

The Ricci scalar, which vanishes in 4D Vaidya case, has the same divergence rate as the Kretschmann scalar.

4 Discussion

We have generalized the Vaidya solution in the same spirit as the Schwarzschild solution done in [8] to describe a null fluid on the brane. The reflected energy density of free gravitational field from the bulk is negative and hence it would contribute positively to gravity of the collapsing null radiation [8]. This should cover part of the parameter window in the initial data set for naked singularity. This is what has been demonstrated. That is, the parameter set which gave rise to naked singularity in GR collapse may now lead to black hole on the brane. The window gets covered with the increased strength of the reflected tidal charge. There exists a threshold value for $\mu$, as shown in Fig. II, the parameter window gets fully covered ensuring formation of black hole for all values of $\lambda$. That is when $\mu > 0.19$ the CCC is always respected. On the other hand the CCC is always violated for $\lambda \leq 0.06$ and $\mu \leq 0.03$ (Fig. I). The singularity is always Tipler strong curvature singularity. Of course the tidal charge goes as $M^2/\tilde{M}_p^2$ which cannot take arbitrarily large value. The important point is that collapse on the brane would favour black hole in comparison to naked singularity. This is what was however expected because of strengthening of gravity resulting from the back-reaction of the bulk.

One of the important issues about existence of naked singularities is their stability, and there exists no well formulated criterion for it. In such a situation, it is prudent to examine the question under various conditions. For instance very recently, the effect of immersing
the Vaidya null fluid collapsing system into a constant potential bath was considered and was shown that naked singularity does persist. The present consideration could as well be considered from this viewpoint. That is, how does it fare with respect to the possible high energy modifications to GR in the brane world model? It seems to be stable with partially covered parameter window in the initial data set for naked singularity.

It would thus be fair to say that the brane inspired modification of GR at high energies seems to have tendency towards the CCC. However, the parameter window allowing violation of the CCC gets fully covered only for very large “reflected” charge, which may not be realistically sustainable. This is the issue for further probing in future.

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