't Hooft expansion of multi-boundary correlators in 2D topological gravity

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ABSTRACT: We study multi-boundary correlators of Witten–Kontsevich topological gravity in two dimensions. We present a method of computing an open string like expansion, which we call the 't Hooft expansion, of the \( n \)-boundary correlator for any \( n \) up to any order by directly solving the Korteweg–De Vries equation. We first explain how to compute the 't Hooft expansion of the one-boundary correlator. The algorithm is very similar to that for the genus expansion of the open free energy. We next show that the 't Hooft expansion of correlators with more than one boundary can be computed algebraically from the correlators with a lower number of boundaries. We explicitly compute the 't Hooft expansion of the \( n \)-boundary correlators for \( n = 1, 2, 3 \). Our results reproduce previously obtained results for Jackiw–Teitelboim gravity and also the 't Hooft expansion of the exact result of the three-boundary correlator which we calculate independently in the Airy case.
1 Introduction

In a recent paper [1] it was shown that the path integral of Jackiw–Teitelboim (JT) gravity [2, 3] is equivalent to a certain double-scaled random matrix model. The genus expansion of this random matrix model describes the splitting/joining of the baby universes [1]. In this correspondence we can consider the average \( \langle Z(\beta) \rangle \) of the partition function \( Z(\beta) = \text{Tr} e^{-\beta H} \) where the average is defined by the integral over the random matrix \( H \).

More generally, we can consider the multi-point function \( \langle \prod_{i=1}^{n} Z(\beta_i) \rangle \) of the partition functions \( Z(\beta_i) \) \((i = 1, \ldots, n)\). On the bulk gravity side it corresponds to the multi-boundary correlator, i.e. the gravitational path integral on the spacetime with \( n \) boundaries with fixed lengths \( \beta_i \). As argued in [4, 5], the connected part \( \langle \prod_{i=1}^{n} Z(\beta_i) \rangle_{\text{conn}} \) of this correlator comes from the contribution of the Euclidean wormhole connecting the \( n \) boundaries.

Of particular interest is the two-point function \( \langle Z(\beta_1)Z(\beta_2) \rangle \) or its analytic continuation \( \langle Z(\beta + it)Z(\beta - it) \rangle \), known as the spectral form factor. The spectral form factor is a useful diagnostic of the quantum chaos [6, 7] and exhibits the characteristic behavior called ramp and plateau, as a function of time. The ramp comes from the eigenvalue correlations [8] while the plateau arises from the pair-creation of eigenvalue instantons [9]. The transition from the ramp to plateau occurs at what is called Heisenberg time \( t_H \sim g_s^{-1} \), where \( g_s \) is the genus-counting parameter. Around this time scale, the operator insertion \( Z(\beta \pm it) \) into the matrix integral back-reacts to the eigenvalue distribution and two eigenvalues are pulled out from the dominant support (or cut) of the eigenvalue distribution.
This reminds us of the “giant Wilson loop” in 4d $\mathcal{N} = 4$ SU($N$) super Yang–Mills theory. In that case, the path integral of the expectation value of the 1/2 Bogomol’nyi–Prasad–Sommerfield Wilson loops reduces to the Gaussian matrix integral [10, 11]. For the winding Wilson loop with winding number $k$, when $k$ is of the order of $N^0$, the dual object is a fundamental string on $AdS_5 \times S^5$, but when $k$ becomes of the order of $N$ the bulk dual object morphs into a D3-brane [12]. In the Gaussian matrix model picture, what is happening for $k \sim \mathcal{O}(N)$ is that one eigenvalue is pulled out from the cut due to the insertion of the large Wilson loop operator into the matrix integral [13, 14]. This is exactly the same mechanism as the ramp–plateau transition in the spectral form factor [9]. The different bulk dual pictures of the winding Wilson loop for $k \sim \mathcal{O}(N^0)$ and $k \sim \mathcal{O}(N)$ are reflected in the different forms of the genus expansion: closed string like expansion for $k \sim \mathcal{O}(N^0)$ and open string like expansion for $k \sim \mathcal{O}(N)$.

The above discussion suggests that one can study the open string like expansion of the correlators $\langle \prod_i Z(\beta_i) \rangle$ by taking the following scaling limit

$$g_s \ll 1, \quad \beta_i \gg 1 \quad \text{with} \quad s_i = \frac{g_s \beta_i}{2} \quad \text{fixed},$$

which we call the ’t Hooft limit. Indeed, in our previous papers [15, 16] we studied the ’t Hooft limit of the multi-boundary correlators in the JT gravity matrix model. In [15] we pointed out that JT gravity is a special case of general Witten–Kontsevich topological gravity, where infinitely many couplings $t_k$ are turned on with a specific value $t_k = \gamma_k$ with

$$\gamma_0 = \gamma_1 = 0, \quad \gamma_k = \frac{(-1)^k}{(k-1)!} \quad (k \geq 2).$$

In this paper we consider the ’t Hooft expansion of the correlators $\langle \prod_i Z(\beta_i) \rangle$ for Witten–Kontsevich topological gravity with general couplings $t_k$. We find that the leading term of this expansion is closely related to the open free energy defined via the Laplace transform of the Baker–Akhiezer function [17, 18]. We also show that the higher-order corrections to the ’t Hooft expansion of the correlators $\langle \prod_i Z(\beta_i) \rangle$ can be systematically obtained from the Korteweg–De Vries (KdV) equation (2.10) or (5.15). It turns out that the ’t Hooft expansion of the one-point function is obtained by using a similar algorithm for the computation of open free energy developed in [19], while the ’t Hooft expansion of an $n$-point function with $n \geq 2$ is determined algebraically from the lower point functions.

This paper is organized as follows. In section 2, we briefly review how to compute the genus expansion of the multi-boundary correlators by solving the KdV equation. In section 3, we present a method of computing the ’t Hooft expansion of the one-point function. In section 4, we explain that the ’t Hooft expansion of the two-point function can be computed algebraically. In section 5, we show that the ’t Hooft expansion of a general $n$-point function can also be computed algebraically. In section 6, we use this method to obtain the ’t Hooft expansion of the three-point function. We also calculate the exact result in the Airy case. Finally, we conclude in section 7. In appendix A, we present a proof of the relation (5.15).

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1We would like thank Shota Komatsu for the discussion on the analogy with giant Wilson loops.
Multi-boundary correlators in topological gravity

In Witten–Kontsevich topological gravity [20, 21] (see e.g. [18] for a recent review) observables are made up of the intersection numbers

\[ \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} = \int_{\mathcal{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}, \quad d_1, \ldots, d_n \in \mathbb{Z}_{\geq 0}. \] (2.1)

They are associated with a closed Riemann surface \( \Sigma \) of genus \( g \) with \( n \) marked points \( p_1, \ldots, p_n \). We let \( \mathcal{M}_{g,n} \) denote the moduli space of \( \Sigma \) and \( \overline{\mathcal{M}}_{g,n} \) the Deligne–Mumford compactification of \( \mathcal{M}_{g,n} \). Here \( \tau_{d_i} = \psi_i^{d_i} \) and \( \psi_i \) is the first Chern class of the complex line bundle whose fiber is the cotangent space to \( p_i \). The generating function for the intersection numbers is defined as

\[ F(\{t_k\}) := \sum_{g=0}^\infty g^{2g-2} F_g(\{t_k\}), \quad F_g(\{t_k\}) := \left\langle e^{\sum_{d=0}^\infty t_d \tau_d} \right\rangle_g. \] (2.2)

In this paper we consider the \( n \)-boundary connected correlator (which we also call the \( n \)-point function) \[22\]

\[ Z_n(\{\beta_i\}, \{t_k\}) = \left\langle \prod_{i=1}^n Z(\beta_i) \right\rangle_{\text{conn}} \simeq B(\beta_1) \cdots B(\beta_n) F(\{t_k\}), \] (2.3)

where

\[ B(\beta) := g_s \sqrt{\frac{\beta}{2\pi}} \sum_{d=0}^\infty \beta^d \frac{\partial}{\partial t_d}. \] (2.4)

\( B(\beta) \) can be thought of as the “boundary creation operator.” The symbol “\( \simeq \)” in (2.3) means that the equality holds up to an additional non-universal part \[22\] when \( 3g-3+n<0 \). Such a deviation appears only in the genus-zero part of \( n = 1, 2 \)-boundary correlators and does not affect their higher genus parts nor correlators with \( n \geq 3 \) (see e.g. [16] for a more detailed explanation).

\( Z_n \) as well as \( F \) satisfy a set of simple differential equations, which allows us to compute their genus expansion. To see this, let us first introduce the notation

\[ h := \frac{g_s}{\sqrt{2}}, \quad x := \frac{t_0}{h}, \quad \tau := \frac{t_1}{h}. \] (2.5)

and

\[ \partial_k := \frac{\partial}{\partial t_k}, \quad \partial_z := h\partial_0, \quad \partial_\tau := h\partial_1. \] (2.6)

The differential equations are simply written in terms of the derivatives

\[ W_n := Z_n', \quad W_0 := F', \quad u := g_s^2 \partial_0^2 F = 2F''. \] (2.7)
Recall that \( u \) satisfies the KdV equation \([20, 21]\)
\[
\dot{u} = uu' + \frac{1}{6}u'''.
\] (2.8)
Integrating this equation once in \( x \) we obtain
\[
\dot{W}_0 = (W'_0)^2 + \frac{1}{6}W'''_0.
\] (2.9)
Since \( B(\beta_i) \) commutes with \( \dot{\ } = \partial_\tau \) and \( \dot{\ }' = \partial_x \), we immediately obtain a differential equation for \( W_n \sim B(\beta_1) \cdots B(\beta_n)W_0 \) by simply applying \( B(\beta_1) \cdots B(\beta_n) \) to both sides of the above equation. The result is \([16]\)
\[
\dot{W}_n(\beta_1, \ldots, \beta_n) = \sum_{I \subset N} W'_I W'_{|N-I|} + \frac{1}{6}W'''_n(\beta_1, \ldots, \beta_n).
\] (2.10)
Here \( N = \{1, 2, \ldots, n\} \), \( W'_I = W'_I(\beta_{i_1}, \ldots, \beta_{i_{|I|}}) \) with \( I = \{i_1, i_2, \ldots, i_{|I|}\} \), and the sum is taken for all possible subsets \( I \) of \( N \) including the empty set.

As explained in \([16]\) one can solve this equation and compute the genus expansion of \( W_n \) up to any order. The genus expansion of \( Z_n \) is then obtained by merely integrating \( W_n \) once in \( x \). This can be done without ambiguity. In \([16]\) we demonstrated this computation in the (off-shell) JT gravity case \( t_k = \gamma_k \) \( (k \geq 2) \), but as detailed in \([23]\), all the results are immediately generalized to the case of general \( t_k \) by merely replacing
\[
B_n \rightarrow (-1)^{n+1}I_{n+1} \quad (n \geq 1).
\] (2.11)
Here
\[
I_n = I_n(u_0, \{t_k\}) = \sum_{\ell=0}^\infty t_{n+\ell} \frac{u_0^\ell}{\ell!} \quad (n \geq 0)
\] (2.12)
are Itzykson–Zuber variables \([24]\),
\[
u_0 := \partial_0^2 F_0
\] (2.13)
is the genus-zero part of \( u \), and \( B_n \) are Itzykson–Zuber variables restricted to the JT gravity case
\[
B_n = (-1)^{n+1}I_{n+1}(u_0, \{t_k = \gamma_k \} \ (k \geq 2)) \quad (n \geq 1)
\] (2.14)
\[
= \sum_{k=0}^\infty \frac{(-1)^k u_0^k}{k!(k+n)!}.
\]
The key to solving (2.10) order by order is the change of variables\(^2\)
\[
\partial_0 = \frac{1}{t} (\partial_{u_0} - I_2 \partial_t), \quad \partial_1 = u_0 \partial_0 - \partial_t.
\] (2.15)
\(^2\)This change of variables was originally introduced by Zograf (see e.g. [25]).
That is, instead of \( t_0 \) and \( t_1 \) we take \( u_0 \) and
\[
\begin{align*}
t := (\partial_0 u_0)^{-1} = 1 - I_1
\end{align*}
\tag{2.16}
\]
as independent variables and regard \( t_k \geq 2 \) as parameters. In the new variables the integration constant is trivially fixed at every step of solving the differential equation. This is ensured by \( F_1 = -\frac{1}{24} \log t \) and by the fact that \( F_g (g \geq 2) \) are polynomials in the generators \( I_{n \geq 2} \) and \( t^{-1} \) \cite{24, 26, 27}.

To summarize, we know that one can compute the small \( g_s \) expansion of the \( n \)-boundary correlator \( Z_n \) up to any order. This expansion can be thought of as a closed string like expansion. Interestingly, \( Z_n \) also admits an open string like expansion. This is again a small \( g_s \) expansion, but is performed in the ’t Hooft regime \((1.1)\). In the rest of the paper we will show that one can also compute this expansion up to any order.

3 ’t Hooft expansion of the one-boundary correlator

In the scaling regime \((1.1)\), the one-point function admits the ’t Hooft expansion\(^3\)
\[
\begin{align*}
\mathcal{F} = \log Z_1 &= \sum_{\tilde{g} = 0}^{\infty} g^{\tilde{g} - 1} \mathcal{F}_{\tilde{g}},
\end{align*}
\tag{3.1}
\]
where \( \mathcal{F}_{\tilde{g}} \) is a function of \( s = g_s \beta/2 \) and \( t_k \). In \cite{15} we calculated \( \mathcal{F}_{\tilde{g}} \) with \( \tilde{g} = 0, 1, 2 \) in the JT gravity case \( t_k = \gamma_k (k \geq 0) \) by saddle point method. In \cite{16} we generalized the calculation to the “off-shell” case \( t_k = \gamma_k (k \geq 2) \) with \( t_0, t_1 \) being unfixed. In what follows we will present a method of computing \( \mathcal{F}_{\tilde{g}} \) up to any \( \tilde{g} \) with general \( t_k \) by solving the differential equation \((2.10)\). The method is very similar to that of computing the genus expansion of the open free energy \cite{19} and is much more efficient than the saddle point calculation.

Instead of directly dealing with \((3.1)\), we first compute the genus expansion
\[
\begin{align*}
G = \log W_1 = \sum_{\tilde{g} = 0}^{\infty} g^{\tilde{g} - 1} G_{\tilde{g}}.
\end{align*}
\tag{3.2}
\]
Since \( G \) and \( \mathcal{F} \) are related by
\[
\begin{align*}
G = \mathcal{F} + \log \partial_x \mathcal{F},
\end{align*}
\tag{3.3}
\]
the expansion \((3.1)\) will immediately be obtained once \((3.2)\) is computed. The differential equation \((2.10)\) for \( n = 1 \) is written as
\[
\begin{align*}
\partial_1 W_1 &= u \partial_0 W_1 + \frac{g_s^2}{12} \partial_0^3 W_1.
\end{align*}
\tag{3.4}
\]

\(^3\)\( \mathcal{F}_{\tilde{g}} \) in this paper are related to those in our previous work \cite{15, 16} by \( \mathcal{F}_{\tilde{g}} \leftrightarrow \sqrt{2}^{1-\beta} \mathcal{F}_{\text{there}} \) with the identification \( \lambda = \sqrt{2} \).
This implies
\[ \partial_1 G = u \partial_0 G + \frac{g_2^2}{12} \partial_0^3 G + 3 \partial_0 G \partial_0^2 G + (\partial_0 G)^3, \tag{3.5} \]
which is rewritten as
\[ -\partial_t G = \frac{g_2^2}{12} (\partial_0 G)^3 + \frac{g_2^2}{4} \partial_0 G \partial_0^2 G + \frac{g_2^2}{12} \partial_0^3 G + (u - u_0) \partial_0 G. \tag{3.6} \]
Recall that the genus expansion
\[ u = \sum_{g=0}^{\infty} g_{2g} u_g \tag{3.7} \]
is computed by solving the KdV equation (2.8) (see e.g. [15, 19] for the results in our convention). By plugging (3.2) and (3.7) into (3.6), one obtains
\begin{align*}
-\partial_t G_0 &= \frac{1}{12} (\partial_0 G_0)^3, \\
DG_1 &= \frac{1}{4} \partial_0 G_0 \partial_0^3 G_0 
\end{align*}
for \( \tilde{g} = 0, 1 \) and
\begin{align*}
DG_{\tilde{g}} &= \frac{1}{12} \sum_{0 \leq i, j, k < \tilde{g}} \partial_0 G_i \partial_0 G_j \partial_0 G_k + \frac{1}{4} \sum_{k=0}^{\tilde{g}-1} \partial_0 G_{\tilde{g}-k-1} \partial_0^2 G_k + \frac{1}{12} \partial_0^3 G_{\tilde{g}-2} + \sum_{k=1}^{\left\lfloor \frac{\tilde{g}}{2} \right\rfloor} u_k \partial_0 G_{\tilde{g}-2k} 
\end{align*}
for \( \tilde{g} \geq 2 \). Here we have introduced the differential operator
\[ D := -\partial_t - \frac{1}{4} (\partial_0 G_0)^2 \partial_0. \tag{3.10} \]

In what follows we will solve the above differential equations and compute \( G_{\tilde{g}} \). First of all, the explicit form of \( G_0 \) is obtained as follows. Recall that \( W_1 \) is related to the Baker–Akhiezer function \( \psi(\xi) \) as [15]
\[ W_1(s) = e^{G(s)} = \int_{-\infty}^{\infty} d\xi e^{2s \xi} \psi(\xi)^2 \tag{3.11} \]
and \( \psi(\xi) \) is expanded as [15, 19]
\[ \psi(\xi) = \exp \left( \sum_{\tilde{g}=0}^{\infty} g_{2\tilde{g}-1} A_{\tilde{g}} \right) \tag{3.12} \]
with
\[ A_0 = -\frac{tz^3}{3} + \sum_{n=1}^{\infty} \frac{I_{n+1}}{(2n+3)!!} z^{2n+3}, \quad z = \sqrt{2(\xi - u_0)}. \tag{3.13} \]
The integral (3.11) can be evaluated by the saddle point method. \( G_0 \) is given by

\[
G_0 = 2s\xi_* + 2A_0(\xi_*),
\]

where the saddle point \( \xi_* \) is determined by the condition

\[
\partial_\xi [2s\xi + 2A_0(\xi)] \bigg|_{\xi = \xi_*} = 0.
\]

This is equivalent to

\[
s = -\partial_\xi A_0 \bigg|_{\xi = \xi_*} = tz_* - \sum_{n=1}^{\infty} \frac{I_{n+1}}{(2n + 1)!!} z_*^{2n+1}, \quad z_* = \sqrt{2(\xi_* - u_0)}.
\]

As we showed in [19], this relation is inverted as

\[
z_* = \sum_{j_n \geq 0} \frac{(2n + k)!}{2^{n+k+1}n!} \prod_{a=1}^{\infty} \frac{p_{a+1}^j}{j_a!(2a + 1)!^{j_a}}.
\]

Plugging this back into (3.14), we obtain the explicit form of \( G_0 \). In fact, \( G_0 \) is exactly twice the genus zero part \( F_0^0 \) of the open free energy studied in [19], for which the following simple expression is available:

\[
G_0 = 2F_0^0 = 2u_0s + 2 \sum_{j_n \geq 0} \frac{(2n + k + 1)!}{(2n + 3)!} \frac{s^{2n+3}}{t^{2n+k+2}} \prod_{a=1}^{\infty} \frac{p_{a+1}^j}{j_a!(2a + 1)!^{j_a}}.
\]

It also follows that [19]

\[
\partial_0 G_0 = 2z_*, \quad \partial_s G_0 = 2\xi_*.
\]

In terms of \( z_* \), the differential equations (3.8) are written as

\[
-\partial_t G_0 = \frac{2}{3} z_*^3,
\]

\[
DG_1 = z_* \partial_0 z_*
\]

and \( D \) in (3.10) becomes

\[
D = -\partial_t - z_*^2 \partial_0.
\]

We saw in [19] that \( G_0 = 2F_0^0 \) indeed satisfies the first equation in (3.20).

The operator \( D \) has interesting properties. (This is analogous to \( D \) in [19].) For instance, we see that

\[
Dz_* = -\partial_t z_* - z_*^2 \partial_0 z_* = z_* \partial_s z_* - z_* \partial_0 (\xi_* - u_0) = z_* \partial_0 \xi_* - z_* \partial_0 (\xi_* - u_0)
\]

\[
= \frac{z_*}{t}
\]

\[
D\xi_* = -\partial_t \xi_* - z_*^2 \partial_0 \xi_* = -z_* \partial_0 z_* - z_*^2 \partial_s z_*
\]

\[
= 0,
\]
where we have used \[19\]

\[-\partial_t z_s = z_s \partial_s z_s.\] (3.23)

We can also show that

\[
D \xi_s^{(n)} = D \partial_s \xi_s^{(n-1)}
= -\partial_t \partial_s \xi_s^{(n-1)} - z_s^2 \partial_0 \partial_s \xi_s^{(n-1)}
= -\partial_s \partial_t \xi_s^{(n-1)} - \partial_s (z_s^2 \partial_0 \xi_s^{(n-1)}) + (\partial_s z_s^2) \partial_0 \xi_s^{(n-1)}
= \partial_s D \xi_s^{(n-1)} + 2z_s (\partial_s z_s) \partial_s z_s^{(n-1)} \partial_0 \xi_s
= \partial_s D \xi_s^{(n-1)} - 2(\partial_t z_s) z_s^{(n)},
\] (3.24)

where

\[
\xi_s^{(n)} := \partial_s^n \xi_s, \quad z_s^{(n)} := \partial_s^n z_s \quad (n \geq 1).
\] (3.25)

From this we find

\[
D \xi_s^{(n)} = -2 \sum_{k=0}^{n-1} \partial_s^k (z_s^{(n-k)} \partial_t z_s)
= -2 \sum_{k=0}^{n-1} \sum_{\ell=0}^k \binom{k}{\ell} (\partial_s^{k-\ell} z_s^{(n-k)}) \partial_s^\ell \partial_t z_s
= 2 \sum_{k=0}^{n-1} \sum_{\ell=0}^k \binom{k}{\ell} z_s^{(n-\ell)} \partial_s^\ell (\partial_s \partial_t z_s)
= 2 \sum_{k=0}^{n-1} \sum_{\ell=0}^k \binom{k}{\ell} z_s^{(n-\ell)} \sum_{m=0}^{\ell} \binom{\ell}{m} z_s^{(\ell-m)} z_s^{(m+1)}
= 2 \sum_{k=0}^{n-1} \sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \frac{k!}{(k-\ell)! (\ell-m)! m!} z_s^{(n-\ell)} z_s^{(\ell-m)} z_s^{(m+1)}.
\] (3.26)

As explained in \[19\], \( z_s^{(n \geq 1)} \) can be expressed in terms of \( \xi_s^{(n \geq 1)} \) and \( z_s \):

\[
\begin{align*}
\frac{z_s^{(1)}}{z_s} &= \frac{\xi_s^{(1)}}{z_s}, \\
\frac{z_s^{(2)}}{z_s} &= \frac{\xi_s^{(2)}}{z_s} - \frac{(\xi_s^{(1)})^2}{z_s^3}, \\
\frac{z_s^{(3)}}{z_s} &= \frac{\xi_s^{(3)}}{z_s} - \frac{3 \xi_s^{(1)} \xi_s^{(2)}}{z_s^3} + \frac{3 (\xi_s^{(1)})^3}{z_s^5}.
\end{align*}
\] (3.27)

\[\text{– 8 –}\]
Therefore $D_{ξ}(n≥1)$ can also be expressed in terms of $ξ(n≥1)$ and $z$. For $n = 1, 2, 3$ we have

\[
D_{ξ}(1) = \frac{2(ξ(1)^2)}{z},
\]

\[
D_{ξ}(2) = \frac{6(ξ(1)ξ(2))}{z^3} - \frac{4(ξ(1))^3}{z^3},
\]

\[
D_{ξ}(3) = \frac{8(ξ(1)ξ(3))}{z} + \frac{6(ξ(2))^2}{z^3} - \frac{24(ξ(1))^2ξ(2)}{z^3} + \frac{18(ξ(1))^4}{z^4}.
\]

(3.28)

On the other hand, as in [19] we evaluate the r.h.s. of (3.9) using

\[
\partial_0 z = \frac{z(1)}{z}, \quad \partial_0 ξ(n) = z(n+1) (n ≥ 0).
\]

(3.29)

In this way, one can express both sides of (3.9) as a polynomial in the variables $t^{−1}$, $I_k≥2$, $z^{−1}$, $(ξ(1))^{−1}$ and $ξ(n≥1)$.

Almost in the same way as in [19], we can formulate the following algorithm to solve (3.9) and obtain $G_{\tilde{g}}$ from the data of $\{G_{\tilde{g}'}\}_{\tilde{g}'<\tilde{g}}$:

(i) Compute the r.h.s. of (3.9) and express it as a polynomial in the variables $t^{−1}$, $I_k≥2$, $z^{−1}$, $(ξ(1))^{−1}$ and $ξ(n≥1)$.

(ii) Let $t^{−m}f(I_k, z, ξ(n))$ denote the highest-order part in $t^{−1}$ of the obtained expression. This part can arise only from

\[
D \left( -\frac{f(I_k, z, ξ(n))}{(m-2)t^{m-2}z^2I_2} \right).
\]

(3.30)

Therefore subtract this from the obtained expression.

(iii) Repeat procedure (ii) down to $m = 3$. Then all the terms of order $t^{−2}$ automatically disappear and the remaining terms are of order $t^{−1}$ or $t^0$. Note also that the expression does not contain any $I_k$.

(iv) In the result of (iii), collect all the terms of order $t^{−1}$ and let $t^{−1}z_0\partial_0 g(z, ξ(n))$ denote the sum of them. This part arises from

\[
D g(z, ξ(n)).
\]

(3.31)

Therefore subtract this from the result of (iii). The remainder turns out to be independent of $t$.

(v) In the obtained expression, let

\[
\frac{h(ξ(n≥2))}{z_0(ξ(1))^m}
\]

(3.32)
denote the part which is of the order $z_s^{-1}$ as well as of the lowest order in $(\xi^{(1)}_s)^{-1}$. This part arises from

$$D \left( \frac{h(\xi^{(n\geq2)}_s)}{2(m+1)(\xi^{(1)}_s)^{m+1}} \right).$$

Therefore subtract this from the obtained expression.

(vi) Repeat procedure (v) until the resulting expression vanishes.

(vii) By summing up all the above-obtained primitive functions, we obtain $G_{\tilde{g}}$.

Using this algorithm we can compute $G_{\tilde{g}}$ up to a high order. ($G_1$ is also obtained by solving (3.20).) The first few $G_{\tilde{g}}$ terms are

$$G_1 = \frac{1}{2} \log \xi^{(1)}_s - \log z_s,$$

$$G_2 = - \frac{I_2}{12t^2 z_s^3} - \frac{5}{12t^3 z_s^4} + \frac{3\xi^{(1)}_s}{4z_s^4} - \frac{\xi^{(2)}_s}{4z_s^4 \xi^{(1)}_s} + \frac{\xi^{(3)}_s}{16 (\xi^{(1)}_s)^2} - \frac{(\xi^{(2)}_s)^2}{12 (\xi^{(1)}_s)^3},$$

$$G_3 = \frac{I_2^2}{12t^4 z_s^5} + \frac{I_3}{24t^3 z_s^5} + \frac{I_2}{4t^3 z_s^4} - \frac{5}{8t^2 z_s^5} - \frac{5I_2 \xi^{(1)}_s}{48t^2 z_s^5} + \frac{I_2 \xi^{(2)}_s}{48t^2 z_s^5 \xi^{(1)}_s} + \frac{5 \xi^{(2)}_s}{16t^2 z_s^5 \xi^{(1)}_s} - \frac{35 \xi^{(1)}_s}{16t z_s^7}$$

$$- \frac{11 \xi^{(2)}_s}{8 z_s^6 \xi^{(1)}_s^2} - \frac{\xi^{(4)}_s}{32 z_s^2 \xi^{(1)}_s^2} + \frac{7 \xi^{(3)}_s \xi^{(2)}_s}{48 z_s^2 (\xi^{(1)}_s)^3} - \frac{(\xi^{(2)}_s)^3}{8 z_s^2 (\xi^{(1)}_s)^4} + \frac{3 (\xi^{(1)}_s)^2}{z_s^5} + \frac{3 \xi^{(3)}_s}{16 z_s^4 \xi^{(1)}_s}$$

$$- \frac{(\xi^{(2)}_s)^2}{8 z_s^4 (\xi^{(1)}_s)^2} + \frac{\xi^{(5)}_s}{192 (\xi^{(1)}_s)^3} - \frac{(\xi^{(3)}_s)^2}{32 (\xi^{(1)}_s)^4} - \frac{\xi^{(4)}_s \xi^{(2)}_s}{24 (\xi^{(1)}_s)^4} + \frac{3 \xi^{(3)}_s (\xi^{(2)}_s)^2}{16 (\xi^{(1)}_s)^5} - \frac{(\xi^{(2)}_s)^4}{8 (\xi^{(1)}_s)^6}.$$  

(3.34)
\( \mathcal{F}_0 = G_0 = 2F_0^* \),
\( \mathcal{F}_1 = G_1 - \log(h \partial_0 \mathcal{F}_0) = \frac{1}{2} \log \xi_0^{(1)} - 2 \log z - \log h, \)
\( \mathcal{F}_2 = G_2 - \frac{\partial_0 \mathcal{F}_1}{\partial_0 \mathcal{F}_0} \)
\[\begin{align*}
\mathcal{F}_2 &= - \frac{I_2}{12t^2 z_s} - \frac{17}{12t z_s^3} + \frac{2 \xi_s^{(1)}}{z_s^4} - \frac{\xi_s^{(2)}}{2 z_s^2 \xi_s^{(1)}} + \frac{\xi_s^{(3)}}{16 (\xi_s^{(1)})^2} - \frac{(\xi_s^{(2)})^2}{12 (\xi_s^{(1)})^3}, \\
\mathcal{F}_3 &= G_3 - \frac{\partial_0 \mathcal{F}_2}{\partial_0 \mathcal{F}_0} + \frac{1}{2} \left( \frac{\partial_0 \mathcal{F}_1}{\partial_0 \mathcal{F}_0} \right)^2 \\
&= \frac{I_3^2}{6 t^4 z_s^2} + \frac{I_3}{12 t^3 z_s^2} + \frac{I_2}{12 t^2 z_s^2} + \frac{13}{4 t^2 z_s^6} - \frac{7 I_2 \xi_s^{(1)}}{48 t^2 z_s^5} + \frac{I_2 \xi_s^{(2)}}{48 t^2 z_s^3 \xi_s^{(1)}} + \frac{I_2 \xi_s^{(3)}}{16 t z_s^5 \xi_s^{(1)}} - \frac{17 \xi_s^{(2)}}{16 t z_s^2 \xi_s^{(1)}} - \frac{153 \xi_s^{(1)}}{16 t z_s^3} \\
&\quad + \frac{7 \xi_s^{(3)} \xi_s^{(2)}}{24 z_s^2 (\xi_s^{(1)})^3} - \frac{3 (\xi_s^{(2)})^2}{8 z_s^4 (\xi_s^{(1)})^2} - \frac{(\xi_s^{(2)})^3}{4 z_s^4 (\xi_s^{(1)})^3} + \frac{\xi_s^{(3)}}{2 z_s^2 \xi_s^{(1)}} - \frac{17 \xi_s^{(2)}}{16 t z_s^5 \xi_s^{(1)}} + \frac{3 \xi_s^{(3)} \xi_s^{(2)}}{16 (\xi_s^{(1)})^5} \\
&\quad - \frac{\xi_s^{(4)} \xi_s^{(2)}}{24 (\xi_s^{(1)})^4} + \frac{10 (\xi_s^{(1)})^2}{z_s^2} - \frac{4 \xi_s^{(2)}}{z_s^5} + \frac{\xi_s^{(5)}}{192 (\xi_s^{(1)})^3} - \frac{(\xi_s^{(3)})^2}{32 (\xi_s^{(1)})^4} - \frac{(\xi_s^{(2)})^4}{8 (\xi_s^{(1)})^6}.
\end{align*}\]

We computed \( \mathcal{F}_g \) for \( \tilde{g} \leq 13 \). We have checked that the above \( \mathcal{F}_g \) with \( \tilde{g} = 0, 1, 2 \) are in perfect agreement (up to the constant part of \( \mathcal{F}_1 \)) with \( \sqrt{2^{1-g}} \mathcal{F}_g \) in [16] under the identification in (2.11).

4 ‘t Hooft expansion of the two-boundary correlator

In this section let us consider ‘t Hooft expansion of the two-boundary correlator. While \( Z_2 \) itself admits ‘t Hooft expansion, for many purposes it is convenient to consider instead the ‘t Hooft expansion of

\[\tilde{Z}_2(s_1, s_2) = \text{Tr} \left[ e^{\beta_1 Q} \Pi e^{\beta_2 Q} \Pi \right] = Z_1(s_1 + s_2) - Z_2(s_1, s_2), \]

where
\[Q := \partial_s^2 + u, \quad \Pi := \int_{-\infty}^{x} dx' \langle x' | x \rangle. \]

Correspondingly, let us define the derivative
\[\tilde{W}_2(s_1, s_2) = \partial_s \tilde{Z}_2(s_1, s_2) = W_1(s_1 + s_2) - W_2(s_1, s_2) \]

and “free energies”
\[K^{(2)} = \log \tilde{Z}_2, \quad G^{(2)} = \log \tilde{W}_2. \]
$G^{(2)}$ and $K^{(2)}$ are related by

$$G^{(2)} = K^{(2)} + \log \partial_s K^{(2)}. \quad (4.5)$$

We consider the expansions

$$G^{(2)} = \sum_{\bar{g}=0}^{\infty} g_{\bar{g}}^{-1} G^{(2)}_{\bar{g}}, \quad K^{(2)} = \sum_{\bar{g}=0}^{\infty} g_{\bar{g}}^{-1} K^{(2)}_{\bar{g}}, \quad (4.6)$$

where $G^{(2)}_{\bar{g}}$ and $K^{(2)}_{\bar{g}}$ are functions of $s_i = g_i \beta_i/2$ ($i = 1, 2$) and $t_k$.

The differential equation (2.10) for $n = 2$ is written as

$$\partial_1 W_2 = u \partial_0 W_2 + \frac{g_s^2}{12} \partial_0^3 W_2 + \sqrt{2} g_s \partial_0 W_1(s_1) \partial_0 W_1(s_2). \quad (4.7)$$

Subtracting (4.7) from (3.4) with $s = s_1 + s_2$, we obtain the differential equation for $\tilde{W}_2(s_1, s_2)$:

$$-\partial_1 \tilde{W}_2 = (u - u_0) \partial_0 \tilde{W}_2 + \frac{g_s^2}{12} \partial_0^3 \tilde{W}_2 - \sqrt{2} g_s \partial_0 W_1(s_1) \partial_0 W_1(s_2). \quad (4.8)$$

This implies

$$\sqrt{2} g_s \partial_0 G(s_1) \partial_0 G(s_2) e^{G(s_1) + G(s_2) - G^{(2)}}$$

$$= \partial_t G^{(2)} + (u - u_0) \partial_0 G^{(2)} + \frac{g_s^2}{12} \left( \partial_0^3 G^{(2)} + 3 \partial_0 G^{(2)} \partial_0^2 G^{(2)} + (\partial_0 G^{(2)})^3 \right). \quad (4.9)$$

In [16] we derived that

$$K^{(2)}_{\bar{g}}(s_1, s_2) = F_0(s_1) + F_0(s_2). \quad (4.10)$$

From this and (4.5) we have

$$G^{(2)}_{\bar{g}}(s_1, s_2) = G_0(s_1) + G_0(s_2) \quad (4.11)$$

as the initial condition.

We observe that starting with (4.11) and comparing both sides of (4.9) order-by-order in the small $g_s$ expansion one can algebraically determine $G^{(2)}_{\bar{g}}$ from the data of $\{G^{(2)}_{\bar{g}'}\}_{\bar{g}' \leq \bar{g}}$ and $\{G^{(2)}_{\bar{g}'}\}_{\bar{g}' \leq \bar{g}}$. For instance, for $\bar{g} = 1$ we obtain

$$G^{(2)}_1 = \frac{1}{2} \log \xi^{(1)}_1 + \frac{1}{2} \log \xi^{(1)}_2 - \log z_1 - \log z_2 - \log(z_1 + z_2) + \frac{3}{2} \log 2. \quad (4.12)$$

As in the case of one-point function, $K^{(2)}_{\bar{g}}$ can also be algebraically determined by (4.5) from the data of $\{G^{(2)}_{\bar{g}'}\}_{\bar{g}' \leq \bar{g}}$. Therefore, given the data of $\{G^{(2)}_{\bar{g}'}\}_{\bar{g}' \leq \bar{g}}$ we can compute $K^{(2)}_{\bar{g}}$ without any integration procedure. For instance, we obtain

$$K^{(2)}_1 = G^{(2)}_1 - \log(h \partial_0 K^{(2)}_0)$$

$$= \frac{1}{2} \log \xi^{(1)}_1 + \frac{1}{2} \log \xi^{(1)}_2 - \log z_1 - \log z_2 - 2 \log(z_1 + z_2) + \frac{1}{2} \log 2 - \log h. \quad (4.13)$$
Using the above method we computed \( K^{(2)}_{\tilde{g}} \) for \( \tilde{g} \leq 8 \). We verified that \( K^{(2)}_{\tilde{g}} \) with \( \tilde{g} = 1, 2 \) are in perfect agreement (up to the constant part of \( K^{(2)}_{1} \)) with \( \sqrt{2^{1-\tilde{g}}} K^{(2)}_{\tilde{g}} \) given in (3.52) of [16].

As a further nontrivial check, let us compare the above results with the low-temperature expansion of the two-point function studied in [16]. As we mentioned, the results in [16] are trivially generalized to the case of general \( t_k \) by the replacement (2.11). Recall that the low-temperature expansion of \( e^{K^{(2)}_{\tilde{g}}} \) is written as (see (4.32) of [16] and notations therein)

\[
e^{K^{(2)}_{\tilde{g}}} = \text{Tr}(e^{\beta_1 Q \Pi} e^{\beta_2 Q \Pi})
\]

\[
= \text{Erfc}(\sqrt{D_0}) e^{\frac{K^2}{2\sqrt{\pi} h}} \sum_{\ell=0}^{\infty} \frac{T^\ell}{\ell!} z_{\ell} - B \sum_{\ell=0}^{\infty} \frac{T^{\ell+1}}{\ell!} g_{\ell}
\]

\[
= \frac{1}{2 \sqrt{\pi} h} e^{\frac{K^2}{2\sqrt{\pi} h} - D_0} \left[ e^{D_0} \text{Erfc}(\sqrt{D_0}) \sum_{\ell=0}^{\infty} \frac{T^\ell}{\ell!} z_{\ell} - 2t \sqrt{\frac{D_0}{\pi}} \sum_{\ell=0}^{\infty} \frac{T^{\ell+1}}{\ell!} g_{\ell} \right].
\]

Using the data of \( z_{\ell}, g_{\ell} \) (0 \( \leq \ell \leq 7 \)) we verified that this expression indeed reproduces the above-obtained \( K^{(2)}_{\tilde{g}} \) in the form of small-\( s \) expansion. We performed the expansion of \( K^{(2)}_{\tilde{g}} (2 \leq \tilde{g} \leq 5) \) up to the order of \( s^{3(6-\tilde{g})} \) and observed perfect agreement. Since small-\( s \) expansion of \( K^{(2)}_{\tilde{g}} (\tilde{g} \geq 2) \) starts at the order of \( s^{3(1-\tilde{g})} \), this serves as a rather nontrivial check.

5 General formalism for multi-boundary correlator

In this section let us consider ’t Hooft expansion of multi-boundary correlators. As in the case of the two-boundary correlator, it is convenient to consider the ’t Hooft expansion

\[
K^{(n)} = \log \tilde{Z}_n = \sum_{\tilde{g}=0}^{\infty} g_{\tilde{g}}^{\tilde{g}-1} K^{(n)}_{\tilde{g}}
\]

(5.1)

with

\[
\tilde{Z}_n(\beta_1, \ldots, \beta_n) = \text{Tr}(e^{\beta_1 Q \Pi} \cdots e^{\beta_n Q \Pi}).
\]

(5.2)
$\mathcal{K}^{(n)}_g$ in (5.1) is a function of $s_i = g_n \beta_i / 2$ ($i = 1, \ldots, n$) and $t_k$. As we saw in [16], $Z_n$ and $\tilde{Z}_n$ are related as

$$Z_1(\beta) = \text{Tr} \left[ e^{\beta Q} \Pi \right]$$

$$Z_2(\beta_1, \beta_2) = \text{Tr} \left[ e^{(\beta_1 + \beta_2) Q} \Pi - e^{\beta_1 Q} \Pi e^{\beta_2 Q} \Pi \right]$$

$$Z_3(\beta_1, \beta_2, \beta_3) = \text{Tr} \left[ e^{(\beta_1 + \beta_2 + \beta_3) Q} \Pi + e^{\beta_1 Q} \Pi e^{\beta_2 Q} \Pi e^{\beta_3 Q} \Pi + e^{\beta_1 Q} \Pi e^{\beta_2 Q} \Pi e^{\beta_3 Q} \Pi \right]$$

(5.3)

where

$$n = 1, 2, 3.$$ In general, the relation is given by the formula [28]

$$Z_n(\beta_1, \ldots, \beta_n) = \text{Tr} \log \left( 1 + \left[ -1 + z_i \prod_{i=1}^{n} (1 + z_i e^{\beta_i Q} \Pi) \right] \right) \bigg|_{\mathcal{O}(z_1 \ldots z_n)}$$

$$= \text{Tr} \log \left( 1 + \sum_{k=1}^{n} \sum_{i_1 < \cdots < i_k} z_{i_1} \cdots z_{i_k} e^{(\beta_{i_1} + \cdots + \beta_{i_k}) Q} \Pi \right) \bigg|_{\mathcal{O}(z_1 \ldots z_n)}.$$ (5.4)

Let us next introduce

$$\tilde{W}_n(\beta_1, \ldots, \beta_n) = \partial_x \tilde{Z}_n(\beta_1, \ldots, \beta_n).$$ (5.5)

One may expect that (2.10) leads to a differential equation for $\tilde{W}_n$ similar to (4.8), which enables us to compute the 't Hooft expansion of $\log \tilde{W}_n$ in the same way as in the last section. However, this is not the case. This is because $\tilde{W}_n(\beta_1, \ldots, \beta_n)$ are only cyclically symmetric with respect to the variables $\beta_i$ and for $n \geq 3$ multiple $\tilde{W}_n$ with different orders of $\beta_i$ appear in the single differential equation (2.10). Consequently, the differential equation is not determinative for $\tilde{W}_n_{\geq 3}$.

On the other hand, by taking a different approach it is still possible to compute 't Hooft expansion of $\tilde{W}_n_{\geq 3}$. Our new approach is based on the fact that $\tilde{Z}_n$ is expressed in terms of the Baker–Akhiezer function $\psi_i := \psi(\xi_i, \{ t_k \})$, which satisfies

$$L \psi_i = \xi_i \psi_i, \quad \psi_i = M \psi_i,$$ (5.6)

where

$$L = Q = \partial_x^2 + u, \quad M = \frac{2}{3} \partial_x^3 + u \partial_x + \frac{1}{2} u'.$$ (5.7)

From (5.6) we are able to derive a new differential equation, which is not for $\tilde{W}_n$ itself, but for its constituents, as we will see below.
We first recall that (5.2) is rewritten as \[16\]
\[
\tilde{Z}_n(\beta_1, \beta_2, \ldots, \beta_n) = \int_{-\infty}^{\infty} d\xi_1 \cdots \int_{-\infty}^{\infty} d\xi_n e^{\sum_{j=1}^{n} \beta_j \xi_j} K_{12} K_{23} \cdots K_{n-1,n} K_{n,1},
\]
where $K_{ij}$ is the Darboux–Christoffel kernel. It is written in terms of the Baker–Akhiezer function $\psi_i$ as

\[
K_{ij} = K(\xi_i, \xi_j) = \int_{-\infty}^{\infty} dx \psi_i \psi_j.
\]

This means that

\[
\partial_x K_{ij} = \psi_i \psi_j.
\]

Let us introduce the notation

\[
\langle k, l \rangle := \int_{-\infty}^{\infty} d\xi_k \int_{-\infty}^{\infty} d\xi_{k+1} \cdots \int_{-\infty}^{\infty} d\xi_l e^{\sum_{j=1}^{l} \beta_j \xi_j} \psi_k K_{k,k+1} K_{k+1,k+2} \cdots K_{l-1,l} \psi_l
\]
for $k \leq l$. For $k = l$ it is understood that

\[
\langle k, k \rangle = \int_{-\infty}^{\infty} d\xi_k e^{\beta_k \xi_k} \psi_k = W_1(\beta_k).
\]

Using (5.10) we see that

\[
\tilde{W}_n(\beta_1, \beta_2, \ldots, \beta_n) = \sum_{k=1}^{n} \langle k, k-1 \rangle.
\]

We find that $\langle 1, n \rangle$ satisfies the simple differential equation

\[
\left( \partial_{\tau} - u \partial_x - \frac{1}{6} \partial_x^3 \right) \langle 1, n \rangle = - \sum_{1 < k \leq n} \langle 1, k-1 \rangle' \langle k, n \rangle'.
\]

The proof of this equation is not difficult but rather lengthy, so that we relegate it to Appendix A. One can easily check that (5.15) indeed reproduces the differential equations (3.4) and (4.8) for the $n = 1, 2$ cases by observing that

\[
\langle 1, 1 \rangle = W_1(\beta_1), \quad \langle 1, 2 \rangle = \langle 2, 1 \rangle = \frac{1}{2} \tilde{W}_2(\beta_1, \beta_2).
\]

To compute the 't Hooft expansion of $\log \langle 1, n \rangle$ we need to know the initial condition, i.e. the genus zero part. This is derived as follows. Recall that $K_{ij}$ is written as (see e.g. [16])

\[
K_{ij} = \frac{\psi_i' \psi_j - \psi_i \psi_j'}{\xi_i - \xi_j}.
\]
Since the Baker–Akhiezer function admits the expansion (3.12), \( K_{ij} \) is expanded as
\[
K_{ij} = e^{A_0(\xi_i) + A_0(\xi_j)} + \mathcal{O}(g_0^2).
\] (5.18)

Plugging (3.12) and (5.18) into (5.11) one obtains
\[
\langle k, l \rangle = e^{\frac{1}{2} \sum_{j=k}^l G_0(s_j)} + \mathcal{O}(g_0^2).
\] (5.19)

By solving the differential equation (5.15) with the initial condition (5.19) one can determine the ‘t Hooft expansion of \( \langle 1, n \rangle \) purely algebraically, as in the case of \( \tilde{W}_2 \). Then, using the relation (5.14), one immediately obtains the ‘t Hooft expansion of \( \tilde{W}_n \) and \( \tilde{Z}_n \).

6 ‘t Hooft expansion of the three-boundary correlator

6.1 General results

In this section let us compute the ‘t Hooft expansion of the three-point function using the formalism developed in the last section. We consider the ‘t Hooft expansion
\[
K^{(3)} = \log \tilde{Z}_3 = \sum_{\tilde{g}=0}^{\infty} \tilde{g}^{\tilde{g}-1} K^{(3)}_{\tilde{g}}.
\] (6.1)

As we mentioned, one can algebraically compute the above genus expansion. The results are
\[
K^{(3)}_0 = F_0(s_1) + F_0(s_2) + F_0(s_3),
\]
\[
K^{(3)}_1 = \frac{1}{2} \log \left[ \lambda_1^{(1)} \lambda_2^{(1)} \lambda_3^{(1)} \right] - \log \left[ \xi_1^*(z_1 + z_3)(z_2 + z_3)(z_2 + z_3) \right] + \ln 2,
\]
\[
K^{(3)}_2 = \left( -\frac{5}{12t^{3/2}_1} - \frac{1}{12t^{3/2}_2} \right) \left[ \frac{1}{z_1^* + z_2^* + z_3^*} \right] - \frac{J_2}{12t^{3/2}_1} + \frac{\xi_1^{(3)}_s}{16(\xi_1^{(1)})^2} - \frac{(\xi_1^{(2)})^2}{12(\xi_1^{(1)})^3}
\]
\[
+ \frac{\xi_1^{(1)}}{4z_1^{(1)*}} \left[ 1 + \frac{1}{z_1^*(z_1 + z_2^*)} + \frac{1}{z_1^*(z_1 + z_3^*)} \right]
\]
\[
+ \frac{\xi_1^{(1)}}{4} \left[ 3 + \frac{3}{z_1^{(1)*}(z_1 + z_2^*)} + \frac{3}{z_1^{(1)*}(z_1 + z_3^*)} \right]
\]
\[
+ \frac{2}{z_1^{(1)*}(z_1 + z_2^*)^2 + z_1^{(1)*}(z_1 + z_3^*)^2} \left[ 2 + \frac{2}{z_1^{(1)*}(z_1 + z_2^*)^2 + z_1^{(1)*}(z_1 + z_3^*)^2} \right]
\]
\[
+ \text{cyclic perm.}
\] (6.2)

We checked that the above \( K^{(3)}_1 \) is in agreement (up to a constant) with the saddle-point result calculated in [16]. We also verified that the above \( K^{(3)}_2 \) restricted to the JT gravity case \( t_k = \gamma_k \) (\( k \geq 0 \)) is in perfect agreement with \( 2^{-1/2} K^{(3)}_2 \) \( |_{y=0,t=1} \) of [16]. Note that in
the case of \( t_k = \gamma_k \) \((k \geq 0)\) we have
\[
t = 1, \quad I_2 = 1, \quad \\
\xi^{(1)}_{i*} = \frac{z_{i*}}{\cos (\sqrt{2}z_{i*})}; \quad \xi^{(2)}_{i*} = \frac{\sqrt{2}z_{i*} \sin (\sqrt{2}z_{i*}) + \cos (\sqrt{2}z_{i*})}{\cos^3 (\sqrt{2}z_{i*})}, \quad (6.3) \]
\[
\xi^{(3)}_{i*} = -4z_{i*} \cos^2 (\sqrt{2}z_{i*}) + 3\sqrt{2} \sin (\sqrt{2}z_{i*}) \cos (\sqrt{2}z_{i*}) + 6z_{i*} \cos^3 (\sqrt{2}z_{i*})
\]

Note also that the \( z_{i*} \) terms here are related to those in [16] by \( z_{i*} = \sqrt{2}z_{i*} \).

### 6.2 Airy case

In this subsection we consider the Airy case corresponding to a particular subspace of couplings \( \{t_n\} \)
\[
t_0, t_1 \neq 0, \quad t_k = 0 \quad (k \geq 2). \quad (6.4)
\]
In this case the Itzykson–Zuber variables \( I_n \) in (2.12) become
\[
I_0 = t_0 + u_0 t_1, \quad I_1 = t_1, \quad I_{k \geq 2} = 0, \quad (6.5)
\]
and from the genus-zero string equation \( u_0 = I_0 \) we find
\[
u_0 = \frac{t_0}{t}, \quad t = 1 - t_1. \quad (6.6)
\]

From (6.5) and (6.6), one can see that various quantities introduced in section 3 take very simple form
\[
F_0^0 = u_0 s + \frac{s^3}{6t^2}, \quad z_* = \frac{s}{t}; \quad \xi_* = u_0 + \frac{s^2}{2t^2}, \quad (6.7)
\]
\[
\xi^{(1)}_* = \frac{s}{t^2} = \frac{z_*}{t}, \quad \xi^{(2)}_* = \frac{1}{t^2}; \quad \xi^{(k \geq 3)}_* = 0.
\]

Using these we obtain the ’t Hooft expansion (6.1) of \( K^{(3)} \) in the Airy case. For instance, \( K^{(3)}_2 \) and \( K^{(3)}_3 \) are given by
\[
K^{(3)}_2 = -\frac{z_{1*}^3 + z_{2*}^3 + z_{3*}^3}{2t_1 z_1 z_2 z_3 (z_1 + z_2) (z_2 + z_3) (z_3 + z_1)},
\]
\[
K^{(3)}_3 = \frac{1}{8t^2 [2z_1 z_2 z_3 (z_1 + z_2) (z_2 + z_3) (z_3 + z_1)]^2}
\times [5(z_1^6 + z_2^6 + z_3^6) + 6((z_2 + z_3) z_1^5 + (z_3 + z_1) z_2^5 + (z_1 + z_2) z_3^5) + 14(z_2 z_3 z_1^4 + z_3 z_1 z_2^4 + z_1 z_2 z_3^4) - 2(z_1^3 z_2^3 + z_2^3 z_3^3 + z_3^3 z_1^3)
\quad + 21z_1^2 z_2^2 z_3^2]. \quad (6.8)
\]

In the Airy case, it is known that multi-point functions \( Z_n(\beta_1, \ldots, \beta_n) \) can be written in the integral representation [29]. In particular, we can write down \( Z_n \) for \( n = 1, 2 \) in a
closed form \[29\]

\[ Z_1(\beta) = \frac{1 - t_1}{g_s \sqrt{2\pi \beta}} \exp \left( \frac{\beta t_0}{1 - t_1} + \frac{g_s^2 \beta^3}{24(1 - t_1)^2} \right), \quad (6.9) \]

\[ Z_2(\beta_1, \beta_2) = Z_1(\beta_1 + \beta_2) \text{Erf} \left( \frac{g_s \sqrt{\beta_2 \beta_3 (\beta_2 + \beta_3)}}{2 \sqrt{2(1 - t_1)}} \right), \]

where \( \text{Erf}(z) \) denotes the error function

\[ \text{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z dx e^{-x^2}. \quad (6.10) \]

For \( n = 3 \), the closed form expression of \( Z_3(\beta_1, \beta_2, \beta_3) \) with \( \beta_1 = \beta_2 = \beta_3 \) is obtained in \[30\] in terms of the Owen’s \( T \)-function

\[ T(z, a) = \frac{1}{2\pi} \int_0^a dx e^{-\frac{1}{2}z^2(1+x^2)} \frac{1}{1 + x^2}. \quad (6.11) \]

We can generalize the result of \[30\] for \( Z_3(\beta_1, \beta_2, \beta_3) \) with arbitrary \( \beta_1, \beta_2, \beta_3 \). As discussed in \[31\], one can determine \( Z_3 \) by solving the KdV equation for \( W_3 \) \( (2.10) \) on the subspace \( (6.4) \) using \( Z_{1,2} \) in \( (6.9) \) as inputs. In this way we find

\[ \frac{Z_3(\beta_1, \beta_2, \beta_3)}{Z_1(\beta_1 + \beta_2 + \beta_3)} = 1 - 4T \left( \frac{g_s \sqrt{\beta_1 (\beta_2 + \beta_3) (\beta_1 + \beta_2 + \beta_3)}}{2(1 - t_1)} \right), \sqrt{\beta_2 \beta_3} \frac{\beta_2 \beta_3}{\beta_1 (\beta_1 + \beta_2 + \beta_3)} \right) \]

\[ - 4T \left( \frac{g_s \sqrt{\beta_2 (\beta_3 + \beta_1) (\beta_1 + \beta_2 + \beta_3)}}{2(1 - t_1)} \right), \sqrt{\beta_3 \beta_1} \frac{\beta_3 \beta_1}{\beta_2 (\beta_1 + \beta_2 + \beta_3)} \right) \]

\[ - 4T \left( \frac{g_s \sqrt{\beta_3 (\beta_1 + \beta_2) (\beta_1 + \beta_2 + \beta_3)}}{2(1 - t_1)} \right), \sqrt{\beta_1 \beta_2} \frac{\beta_1 \beta_2}{\beta_3 (\beta_1 + \beta_2 + \beta_3)} \right). \quad (6.12) \]

One can easily see that this reduces to the result of \[30\] when \( \beta_1 = \beta_2 = \beta_3 \).

From this exact result \( (6.12) \) of \( Z_3 \), one can compute the ’t Hooft expansion \( (6.1) \) of \( \mathcal{K}^{(3)} = \log \tilde{Z}_3 \) in the Airy case. Using the relation between \( Z_3 \) and \( \tilde{Z}_3 \) in \( (5.3) \) and the following property of the Owen’s \( T \)-function

\[ T(z, \infty) = \frac{1}{4} - \frac{1}{4} \text{Erf} \left( \frac{z}{\sqrt{2}} \right), \quad (6.13) \]

we find

\[ \frac{\tilde{Z}_3(\beta_1, \beta_2, \beta_3)}{Z_1(\beta_1 + \beta_2 + \beta_3)} = 2 \tilde{T} \left( \frac{g_s \sqrt{\beta_1 (\beta_2 + \beta_3) (\beta_1 + \beta_2 + \beta_3)}}{2(1 - t_1)} \right), \sqrt{\beta_2 \beta_3} \frac{\beta_2 \beta_3}{\beta_1 (\beta_1 + \beta_2 + \beta_3)} \right) \]

\[ + 2 \tilde{T} \left( \frac{g_s \sqrt{\beta_2 (\beta_3 + \beta_1) (\beta_1 + \beta_2 + \beta_3)}}{2(1 - t_1)} \right), \sqrt{\beta_3 \beta_1} \frac{\beta_3 \beta_1}{\beta_2 (\beta_1 + \beta_2 + \beta_3)} \right) \]

\[ + 2 \tilde{T} \left( \frac{g_s \sqrt{\beta_3 (\beta_1 + \beta_2) (\beta_1 + \beta_2 + \beta_3)}}{2(1 - t_1)} \right), \sqrt{\beta_1 \beta_2} \frac{\beta_1 \beta_2}{\beta_3 (\beta_1 + \beta_2 + \beta_3)} \right). \quad (6.14) \]

\[ ^4 \text{Strictly speaking, the closed form of } \tilde{Z}_{1,2} \text{ is obtained in } [29] \text{ when } t_n = 0 (n \geq 0), \text{ but it is straightforward to generalize the result in } [29] \text{ to the case of our interest } (6.4). \]
where we defined
\[
\tilde{T}(z, a) = T(z, \infty) - T(z, a) = \frac{1}{2\pi} \int_a^\infty dx \frac{e^{-\frac{1}{2}x^2(1+x^2)}}{1+x^2}.
\] (6.15)

In the large \( z \) regime with finite \( a \), this is expanded as
\[
\tilde{T}(z, a) = e^{-\frac{1}{2}(1+a^2)z^2} \sum_{k=0}^{\infty} z^{-2(k+1)} \left( \frac{1}{a} \partial_a \right)^k \frac{1}{a^k(1+a^2)^k}.
\] (6.16)

Using this expansion, we checked that the 't Hooft expansion of the exact result of \( \tilde{Z}_3 \) in (6.14) reproduces \( K_2^{(3)} \) and \( K_3^{(3)} \) in (6.8). This serves as a nontrivial consistency check of our formalism.

7 Conclusions and outlook

In this paper we developed a formalism to compute the 't Hooft expansion of the multi-boundary correlators in topological gravity with general couplings \( t_k \). The 't Hooft expansion of the \( n \)-point function can be obtained from the relation (5.15), which is equivalent to (2.10) for \( n = 1, 2 \). For the one-point function, we developed an algorithm for the computation of 't Hooft expansion in section 3, which is almost parallel to the computation of open free energy studied in [19]. We find that the 't Hooft expansion of an \( n \)-point function \((n \geq 2)\) is determined algebraically from the lower point functions by using (5.15). Our computation reproduces the JT gravity case studied in [15, 16] and the 't Hooft expansion of the exact result of correlators in the Airy case, as it should.

There are several interesting open questions. It is suggested in [32] that we can define the multi-point analogue of the spectral form factor in random matrix models and it exhibits a similar behavior as the ramp and plateau. Using our formalism, it would be possible to study the ramp–plateau transition regime of the multi-point version of the spectral form factor. We leave this as an interesting future problem.

Recently, the quenched free energy \( \langle \log Z(\beta) \rangle \) of JT gravity is studied by the replica method [33–35]. It is shown in [31] that the quenched free energy is written as a certain integral transformation of the generating function of multi-boundary correlators. The low-temperature behavior of quenched free energy in JT gravity is quite interesting since it is suggested in [33] that JT gravity exhibits a spin glass phase at low temperature. In [31, 33] the quenched free energy is analyzed in the Airy regime \( \beta \sim g^{-2/3} \). It would be interesting to study the quenched free energy of JT gravity in the 't Hooft regime \( \beta \sim g^{-1} \) using our formalism.

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A Proof of the master differential equation

In this section we prove (5.15) for general $n$. Let us introduce the notation

$$\langle k^{(p)}, l^{(q)} \rangle := \int_{-\infty}^{\infty} d\xi_k \int_{-\infty}^{\infty} d\xi_{k+1} \cdots \int_{-\infty}^{\infty} d\xi_l \sum_{i=1}^{\beta} \xi_i \frac{\partial}{\partial x_i} \psi_k K_{k,k+1} K_{k+1,k+2} \cdots K_{l-1,l} \frac{\partial}{\partial x_l} \psi_l.$$  

(A.1)

Derivatives of $\langle 1, n \rangle$ are calculated as

$$\langle 1, n \rangle' = \sum_{1<k\leq n} \langle 1, k-1 \rangle \langle k, n \rangle + \left( \langle 1', n \rangle + \langle 1, n' \rangle \right),$$

$$\langle 1, n \rangle'' = 2 \sum_{1<k<l\leq n} \langle 1, k-1 \rangle \langle k, l-1 \rangle \langle l, n \rangle$$

$$+ \sum_{1<k\leq n} \left( 2 \langle 1', k-1 \rangle \langle k, n \rangle + \langle 1, k-1' \rangle \langle k, n \rangle 
+ \langle 1, k-1 \rangle \langle k', n \rangle + 2 \langle 1, k-1 \rangle \langle k, n' \rangle \right)$$

$$+ \left( \langle 1'', n \rangle + \langle 1, n'' \rangle \right) + 2 \langle 1', n' \rangle,$$

$$\langle 1, n \rangle''' = 6 \sum_{1<k<l<m\leq n} \langle 1, k-1 \rangle \langle k, l-1 \rangle \langle l, m-1 \rangle \langle m, n \rangle$$

$$+ \sum_{1<k<l\leq n} \left( 6 \langle 1', k-1 \rangle \langle k, l-1 \rangle \langle l, n \rangle + 3 \langle 1, k-1' \rangle \langle k, l-1 \rangle \langle l, n \rangle 
+ 3 \langle 1, k-1 \rangle \langle k', l-1 \rangle \langle l, n \rangle + 3 \langle 1, k-1 \rangle \langle k, l-1' \rangle \langle l, n \rangle 
+ 3 \langle 1, k-1 \rangle \langle k, l-1 \rangle \langle l', n \rangle + 6 \langle 1, k-1 \rangle \langle k, l-1 \rangle \langle l, n' \rangle \right)$$

$$+ \sum_{1<k\leq n} \left( 3 \langle 1'', k-1 \rangle \langle k, n \rangle + \langle 1, k-1'' \rangle \langle k, n \rangle 
+ \langle 1, k-1 \rangle \langle k'', n \rangle + 3 \langle 1, k-1 \rangle \langle k, n'' \rangle 
+ 6 \langle 1', k-1 \rangle \langle k, n' \rangle + 2 \langle 1, k-1' \rangle \langle k', n \rangle 
+ 3 \langle 1', k-1 \rangle \langle k', n \rangle + 3 \langle 1, k-1' \rangle \langle k, n' \rangle 
+ 3 \langle 1', k-1 \rangle \langle k, n \rangle + 3 \langle 1, k-1 \rangle \langle k', n' \rangle \right)$$

$$+ \left( \langle 1''', n \rangle + \langle 1, n''' \rangle \right) + 3 \left( \langle 1'', n' \rangle + \langle 1', n'' \rangle \right).$$

Next, from (5.6) we see that

$$\left( \partial_\tau - u \partial_\xi \right) \psi_i = \frac{2}{3} \psi_i''' + \frac{1}{2} u' \psi_i$$

$$= \frac{1}{6} \psi_i''' + \frac{1}{2} \partial_x (\partial_x^2 + u) \psi_i - \frac{1}{2} u \psi_i'$$

$$= \frac{1}{6} \psi_i''' + \frac{1}{2} (\xi - u) \psi_i'.$$

(A.3)
Using this we obtain
\[ \dot{K}_{ij} = \int_{-\infty}^{\infty} dx (\dot{\psi}_i \psi_j + \psi_i \dot{\psi}_j) \]
\[ = \int_{-\infty}^{\infty} dx \left[ \frac{2}{3} (\psi_i'' \psi_j + \psi_i \psi_j''') + u(\psi_i' \psi_j + \psi_i \psi_j') + u' \psi_i \psi_j \right] \tag{A.4} \]
and thus
\[ (\partial_x - u \partial_x) K_{ij} = \frac{2}{3} (\psi_i'' \psi_j + \psi_i \psi_j''' - \psi_i' \psi_j') + u \psi_i \psi_j \]

It also follows that
\[ (\xi_i - \xi_j) K_{ij} = \psi_i' \psi_j - \psi_i \psi_j' \] \tag{A.5}

Using \( (A.3) \) and \( (A.5) \), we obtain
\[ (\partial_x - u \partial_x) \langle 1, n \rangle = \frac{2}{3} \sum_{1 < k \leq n} \left( (1, k - 1') \langle k, n \rangle + (1, k - 1) \langle k', n \rangle - (1, k - 1') \langle k', n \rangle \right) \]
\[ + \frac{1}{6} \left( (1''', n) + (1, n'''') \right) + \frac{1}{2} \left( (\xi_1 - u) \langle 1', n \rangle + (\xi_n - u) \langle 1, n' \rangle \right). \tag{A.6} \]

Note that \( (5.17) \) is rewritten as
\[ (\xi_i - \xi_j) K_{ij} = \psi_i' \psi_j - \psi_i \psi_j' \] \tag{A.7}

and thus
\[ (\xi_i - u) K_{ij} = (\xi_j - u) K_{ij} + \psi_i' \psi_j - \psi_i \psi_j'. \tag{A.8} \]

Using this we obtain
\[ (\xi_1 - u) \langle 1', n \rangle = (\xi_n - u) \langle 1', n \rangle + \sum_{1 < k \leq n} \left( (1', k - 1') \langle k, n \rangle - (1', k - 1) \langle k', n \rangle \right), \]
\[ = \langle 1', n'' \rangle + \sum_{1 < k \leq n} \left( (1', k - 1') \langle k, n \rangle - (1', k - 1) \langle k', n \rangle \right), \tag{A.9} \]
\[ (\xi_n - u) \langle 1, n' \rangle = \langle 1'', n' \rangle + \sum_{1 < k \leq n} \left( (1, k - 1) \langle k', n' \rangle - (1, k - 1') \langle k, n' \rangle \right). \]

\( (A.6) \) is then written as
\[ (\partial_x - u \partial_x) \langle 1, n \rangle = \frac{2}{3} \sum_{1 < k \leq n} \left( (1, k - 1') \langle k, n \rangle + (1, k - 1) \langle k', n \rangle - (1, k - 1') \langle k', n \rangle \right) \]
\[ + \frac{1}{6} \left( (1''', n) + (1, n'''') \right) + \frac{1}{2} \left( (1', n'') + (1'', n') \right) \]
\[ + \frac{1}{2} \sum_{1 < k \leq n} \left( (1', k - 1') \langle k, n \rangle - (1', k - 1) \langle k', n \rangle \right) \]
\[ + \langle 1, k - 1 \rangle \langle k', n' \rangle - \langle 1, k - 1' \rangle \langle k, n' \rangle). \tag{A.10} \]
Combining (A.2) and (A.10), we obtain

\[
\left( \partial_x - u \partial_x - \frac{1}{6} \partial_x^3 \right) \langle 1, n \rangle \\
= \frac{1}{2} \sum_{1 < k < n} \left( \langle 1, k - 1'' \rangle \langle k, n \rangle + \langle 1, k - 1 \rangle \langle k'', n \rangle \\
- \langle 1'', k - 1 \rangle \langle k, n \rangle - \langle 1, k - 1 \rangle \langle k'', n' \rangle \right) \\
- \sum_{1 < k \leq n} \left( \langle 1', k - 1 \rangle + \langle 1, k - 1' \rangle \right) \left( \langle k', n \rangle + \langle k, n' \rangle \right) \\
+ \sum_{1 < k < l \leq n} \left( -\langle 1', k - 1 \rangle \langle k, l - 1 \rangle \langle l, n \rangle - \frac{1}{2} \langle 1, k - 1' \rangle \langle k, l - 1 \rangle \langle l, n \rangle \\
- \frac{1}{2} \langle 1, k - 1 \rangle \langle k', l - 1 \rangle \langle l, n \rangle - \frac{1}{2} \langle 1, k - 1 \rangle \langle k, l - 1' \rangle \langle l, n \rangle \\
- \frac{1}{2} \langle 1, k - 1 \rangle \langle k, l - 1 \rangle \langle l', n \rangle - \langle 1, k - 1 \rangle \langle k, l - 1 \rangle \langle l', n' \rangle \right) \\
- \sum_{1 < k < l < m \leq n} \langle 1, k - 1 \rangle \langle k, l - 1 \rangle \langle l, m - 1 \rangle \langle m, n \rangle.
\]

(A.11)

The first term of the r.h.s. can be rewritten by using the relations

\[
\langle 1, k - 1'' \rangle - \langle 1'', k - 1 \rangle = (\xi_{k-1} - u) \langle 1, k - 1 \rangle - (\xi_1 - u) \langle 1, k - 1 \rangle \\
= \sum_{1 < j < k} (\xi_j - \xi_{j-1}) \langle 1, k - 1 \rangle \\
= \sum_{1 < j < k} \left( -\langle 1, j - 1' \rangle \langle j, k - 1 \rangle + \langle 1, j - 1 \rangle \langle j', k - 1 \rangle \right),
\]

(A.12)

\[
\langle k'', n \rangle - \langle k, n'' \rangle = (\xi_k - u) \langle 1, k - 1 \rangle - (\xi_n - u) \langle k, n \rangle \\
= \sum_{k < l \leq n} (\xi_{l-1} - \xi_l) \langle k, n \rangle \\
= \sum_{k < l \leq n} \left( \langle k, l - 1' \rangle \langle l, n \rangle - \langle k, l - 1 \rangle \langle l', n \rangle \right).
\]
Hence we obtain
\[
\left( \partial_x - u \partial_x - \frac{1}{6} \partial_x^3 \right) \langle 1, n \rangle = - \sum_{1 < k \leq n} \left( \langle 1', k - 1 \rangle + \langle 1, k - 1' \rangle \right) \left( \langle k', n \rangle + \langle k, n' \rangle \right) + \sum_{1 < k < l \leq n} \left( \langle 1', k - 1 \rangle \langle k, l - 1 \rangle - \langle 1, k - 1' \rangle \langle k, l - 1 \rangle - \langle 1, k - 1 \rangle \langle k, l - 1 \rangle \langle l, n' \rangle - \langle 1, k - 1 \rangle \langle k, l - 1 \rangle \langle l, n' \rangle \right) - \sum_{1 < k < l < m \leq n} \langle 1, k - 1 \rangle \langle k, l - 1 \rangle \langle l, m - 1 \rangle \langle m, n \rangle \]
\[= - \sum_{1 < k \leq n} \left( \langle 1', k - 1 \rangle + \langle 1, k - 1' \rangle + \sum_{1 < j < k} \langle 1, j - 1 \rangle \langle j, k - 1 \rangle \right) \times \left( \langle k', n \rangle + \langle k, n' \rangle + \sum_{k < l \leq n} \langle k, l - 1 \rangle \langle l, n \rangle \right) - \sum_{1 < k \leq n} \langle 1, k - 1 \rangle' \langle k, n \rangle'. \]

(A.13)

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