CROSS-CONSTRAINED VARIATIONAL METHOD AND NONLINEAR SCHRÖDINGER EQUATION WITH PARTIAL CONFINEMENT

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Abstract. In this paper, we study the nonlinear Schrödinger equation with a partial confinement. By applying the cross-constrained variational arguments and invariant manifolds of the evolution flow, the sharp condition for global existence and blowup of the solution is derived.

1. Introduction. This paper considers the nonlinear Schrödinger equation with a one-dimensional harmonic potential

\[ i\psi_t + \Delta \psi - x_N^2 \psi + |\psi|^{p-1} \psi = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \tag{1} \]

where \( \psi = \psi(t, x) \) is a complex-valued wave function of \( (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N \), \( N \geq 2 \), \( i = \sqrt{-1} \), \( x_N \) is the \( N \)-th component of \( x = (x_1, ..., x_N) \in \mathbb{R}^N \), \( \Delta \) is the Laplacian in \( x \), and \( 1 + \frac{4}{N-1} \leq p < 1 + \frac{4}{(N-2)^{+}} \). Here \( 1 + \frac{4}{(N-2)^{+}} = \infty \) when \( N = 2 \). (1) arises in various branches of physics, such as the Bose-Einstein condensate with a partial confined potential trap and the propagation of mutually incoherent wave packets in nonlinear optics. For more details we refer to [2,3,6,12,16]. We impose the initial data of (1) as follows

\[ \psi(0, x) = \psi_0(x), \quad x \in \mathbb{R}^N. \tag{2} \]

The aim of this paper is to study the sharp conditions of global existence in time and blowup in finite time for the Cauchy problem (1)-(2). For this topic, many known results have been achieved.

Firstly, we recall the classical nonlinear Schrödinger equation

\[ i\psi_t + \Delta \psi + |\psi|^{p-1} \psi = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N. \tag{3} \]

In [18], Weinstein provided a sharp sufficient condition of global existence and blowup for (3) in \( L^2 \)-critical nonlinearity case \( p = 1 + \frac{4}{N} \) by a variational argument. In [17], Stubbe treated the sharp global existence for (3) in terms of the ground state for \( L^2 \)-supercritical nonlinearity case \( 1 + \frac{4}{N} < p < 1 + \frac{4}{(N-2)^{+}} \). In [21], Zhang proposed the cross-constrained variational method and studied the similar
problems for (3) in the case of \(1 + \frac{4}{N} \leq p < 1 + \frac{4}{(N-2)^+}\). It is worth noting that the sharp global existence problem goes back to the work of Payne and Sattinger on the Klein-Gordon equation [14]. Furthermore, Kenig and Merle [11] gave a sharp condition for scattering of the energy-critical nonlinear Schrödinger equation. Then Duyckaerts, Holmer and Roudenko [7], Holmer and Roudenko [9] treated this problem for (3) with \(p = 3, N = 3\) by following Kenig and Merle’s ideas [11].

Then we recall the nonlinear Schrödinger equation with a harmonic potential

\[
  i\psi_t + \Delta \psi - |x|^2 \psi + |\psi|^{p-1} \psi = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N,
\]

which is a standard model for magnetic traps in the context of Bose-Einstein condensation (see e.g. [10, 16]). Based on the method of Weinstein [18], Zhang [20] showed a similar sharp condition for global existence of (4) in \(L^2\)-critical nonlinearity case \(p = 1 + \frac{4}{N}\). In [22], Zhang gave the sharp conditions for global existence and blowup for (4) via cross-constrained variational method proposed in [21]. This method is also used to study the sharp conditions for global existence and blowup in [8, 19].

In this paper, we further exploit this method to study the sharp conditions of global existence and blowup for (1). We note that both (3) and (4) have variance identity for the variance \(V = \int_{\mathbb{R}^N} |\psi|^2 dx\), by which one can construct the sharp blowup solutions for (3) and (4) under the cross-constrained variational scheme (see [21, 22]). But for (1), this variance identity may not be suitable because partial confinement \(x_N^2 \psi\) prevents complete dispersion. Thanks to Ohta [13], who provided an irregular variance identity by \(J(t) = \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} x_j^2 |\psi|^2 dx\), we can apply the cross-constrained variational argument and show the sharp conditions of global existence for (1). Indeed, such a sharp condition obtained by the cross-constrained variational argument can be used to further study the instability of the standing waves for (1) (see e.g. [4, 15, 23, 24]).

The rest of this paper is organized as follows. In section 2, we state some preliminaries. In section 3, we construct cross-constrained variational problem and establish the invariant manifolds for the Cauchy problem (1)-(2). In Section 4, we prove the sharp conditions of global existence and blowup for the Cauchy problem (1)-(2).

2. Preliminaries. For (1), we define a function space

\[
  H := \{ \varphi \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} x_N^2 |\varphi|^2 dx < \infty \} \tag{5}
\]

with inner product

\[
  \langle \varphi, \phi \rangle = \int_{\mathbb{R}^N} \nabla \varphi \nabla \bar{\phi} + \varphi \bar{\phi} + x_N^2 \varphi \bar{\phi} dx
\]

and induced norm \(\| \cdot \|_H\). Then \(H\) is a Hilbert space and can be continuously embedded in \(H^1(\mathbb{R}^N)\).

We define the energy functional in \(H\) as follows:

\[
  E(\varphi) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \varphi|^2 + x_N^2 |\varphi|^2 - \frac{2}{p+1} |\varphi|^{p+1} dx.
\]

The local well-posedness of the Cauchy problem (1)-(2) in \(H\) is stated as follows (see Cazenave [5] and Antonelli, Carles, Silva [1]).

**Lemma 2.1.** Assume that \(1 < p < 1 + \frac{4}{(N-2)^+}\). For \(\psi_0 \in H\), there exist \(T \in (0, \infty]\) and a unique solution \(\psi(t) \in C([0, T); H)\) of the Cauchy problem (1)-(2). If \(T = \infty\),
then \( \psi(t) \) exists globally; if \( T < \infty \), then \( \lim_{t \to T} ||\psi||_H = \infty \) (finite time blowup). In addition, \( \psi(t) \) satisfies

\[
\int_{\mathbb{R}^N} |\psi|^2 \, dx = \int_{\mathbb{R}^N} |\psi_0|^2 \, dx, \quad E(\psi) = E(\psi_0)
\]

for all \( t \in [0,T) \).

Now, we define another function space

\[
\Sigma := \{ \varphi \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |x|^2 |\varphi|^2 \, dx < \infty \}.
\]

Let

\[
J(t) = \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} x_j^2 |\psi_j|^2 \, dx.
\]

From Ohta [13] and Cazenave [5], we have the following virial identity.

**Lemma 2.2.** Assume that \( 1 < p < 1 + \frac{4}{(N-2)^+} \). Let \( \psi_0 \in \Sigma \) and \( \psi(t,x) \) be a solution of the Cauchy problem (1)-(2). Then one has

\[
J''(t) = 8 \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j \psi|^2 \, dx - \frac{8\alpha}{p+1} \int_{\mathbb{R}^N} |\psi|^{p+1} \, dx
\]

for all \( t \in [0,T) \), where \( \alpha = \frac{(N-1)(p-1)}{2} \).

We state the following Heisenberg’s inequality (also see Ohta [13]).

**Lemma 2.3.** Let \( \varphi \in \Sigma \). Then

\[
\int_{\mathbb{R}^N} |\varphi|^2 \, dx \leq \frac{2}{N-1} \left( \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} x_j^2 |\varphi|^2 \, dx \right)^{\frac{1}{2}} \left( \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j \varphi|^2 \, dx \right)^{\frac{1}{2}}.
\]

3. The cross-constrained variational problem. For \( u \in H \), \( 1 < p < 1 + \frac{4}{(N-2)^+} \) and \( \alpha = \frac{(N-1)(p-1)}{2} \), we define the following functionals:

\[
I(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 + x_N^2 |u|^2 - \frac{2}{p+1} |u|^{p+1} \, dx,
\]

\[
S(u) := \int_{\mathbb{R}^N} |\nabla u|^2 + x_N^2 |u|^2 + |u|^2 - |u|^{p+1} \, dx,
\]

\[
P(u) := \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j u|^2 \, dx - \frac{\alpha}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx.
\]

We define a set in \( H \) as follows:

\[
M := \{ u \in H, P(u) = 0, S(u) < 0 \}.
\]

Then we have the following result.

**Lemma 3.1.** Assume that \( 1 + \frac{4}{N-1} \leq p < 1 + \frac{4}{(N-2)^+} \). Then \( M \) is not empty.
Proof. From [13], we see that there exists a solution $u \in H \setminus \{0\}$ of the stationary equation
\[-\Delta u + u + x_N^2 u - |u|^{p-1} u = 0, \quad x \in \mathbb{R}^N.\] (12)
For (12), multiplying by $u$ and integrating with respect to $x$ on $\mathbb{R}^N$, we obtain $S(u) = 0$. In addition, multiplying by $\sum_{j=1}^{N-1} x_j \cdot \partial_j u$ and integrating with respect to $x$ on $\mathbb{R}^N$, we get the Pohozaev identity
\[
\sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j u|^2 dx - \frac{N-1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 + x_N^2 |u|^2 dx + \frac{2(N-1)}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx = 0.
\]
Note that $S(u) = 0$, then we derive $P(u) = 0$. Let
\[v = \lambda \frac{x}{|x|^2} u(\lambda x_1, ..., \lambda x_{N-1}, x_N) \text{ for all } \lambda > 0.
\]
Then by a direct calculation, it follows that
\[S(v) = \lambda^{\theta+2} \left( \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j u|^2 dx - \int_{\mathbb{R}^N} |u|^{p+1} dx \right) + \lambda^\theta \int_{\mathbb{R}^N} |u|^2 + x_N^2 |u|^2 + |\partial_N u|^2 dx,
\]
\[P(v) = \lambda^{\theta+2} \left( \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j u|^2 dx - \frac{\alpha}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx \right),
\]
where $\theta = \frac{4-(N-1)(p-1)}{p-1} \leq 0$. Thus $S(u) = 0$ implies that there exists $\lambda_0 > 1$ such that $S(v) < 0$. On the other hand, for all $\lambda_0 > 0$ and $P(u) = 0$, we still have $P(v) = 0$. So $v \in M$. \qed

Now we define a cross-constrained variational problem
\[d_M := \inf_{u \in M} I(u).
\]
Then the following result is true.

**Lemma 3.2.** Assume that $1 + \frac{4}{N-1} \leq p < 1 + \frac{4}{(N-2)^+}$. Then $d_M > 0$.

**Proof.** Let $u \in M$. Then $S(u) < 0$ yields that $u \neq 0$. From $P(u) = 0$, we have
\[I(u) = \frac{\alpha - 2}{2\alpha} \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |\partial_N u|^2 + |u|^2 + x_N^2 |u|^2 dx.
\]
(14)

Since $1 + \frac{4}{N-1} \leq p < 1 + \frac{4}{(N-2)^+}$, we have $\alpha \geq 2$. It follows from $u \neq 0$ and (14) that $I(u) > 0$ for all $u \in M$. Hence (13) implies that $d_M > 0$. In the following we have to divide the proof into two cases: $\alpha = 2$ and $\alpha > 2$.

We first treat the case $\alpha = 2$. From $S(u) < 0$ and $P(u) = 0$, then one can get
\[
\int_{\mathbb{R}^N} |\partial_N u|^2 + |u|^2 + x_N^2 |u|^2 dx < \frac{p-1}{2} \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j u|^2 dx.
\]
(15)

By $S(u) < 0$ and the Sobolev’s embedding inequality, it follows that
\[
\int_{\mathbb{R}^N} |\nabla u|^2 + x_N^2 |u|^2 + |u|^2 dx \leq c(\int_{\mathbb{R}^N} |\nabla u|^2 + x_N^2 |u|^2 + |u|^2 dx)^{\frac{p+1}{p}}.
\]

Here and hereafter, $c$ denotes the various positive constants. From $p > 1$, we deduce that
\[\int_{\mathbb{R}^N} |\nabla u|^2 + x_N^2 |u|^2 + |u|^2 dx \geq c > 0.
\]
(16)
Combine (15) with (16), we have
\[ \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j u|^2 dx \geq c > 0. \] (17)

From (14), (17) and \( \alpha = 2 \), one has
\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 + x_N^2 |u|^2 dx \geq c > 0. \] (18)

Thus, (18) yields that \( d_M > 0 \) for \( \alpha = 2 \).

Next we treat the case \( \alpha > 2 \). In this case, (14) can be rewritten as
\[ I(u) = \frac{\alpha - 2}{2\alpha} \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 + x_N^2 |u|^2 dx + \frac{1}{\alpha} \int_{\mathbb{R}^N} |\partial_N u|^2 + |u|^2 + x_N^2 |u|^2 dx. \] (19)

From \( S(u) < 0 \) and the Sobolev’s embedding inequality, we have
\[ \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 + x_N^2 |u|^2 dx \geq c > 0. \] (20)

Since \( \alpha > 2 \), then it follows from (19) and (20) that \( I(u) \geq c > 0 \) for all \( u \in M \). Hence \( d_M > 0 \) for \( \alpha > 2 \).

Therefore, we proved \( d_M > 0 \) for \( 1 + \frac{4}{N-1} \leq p < 1 + \frac{4}{(N-2)^+}. \) \( \square \)

Now we define another constrained variational problem
\[ d_B := \inf_{u \in B} I(u), \] (21)

where
\[ B := \{ u \in H \setminus \{0\}, S(u) = 0 \}. \]

Then we get the following result.

**Lemma 3.3.** \( d_B > 0 \).

**Proof.** It is clear that \( B \) is nonempty. Let \( u \in B \), then \( S(u) = 0 \) and \( u \neq 0 \). From \( S(u) = 0 \), by the Sobolev’s embedding inequality, one can derive
\[ \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 + x_N^2 |u|^2 dx \geq c > 0. \] (22)

On the other hand, by \( S(u) = 0 \) again, it follows that
\[ I(u) = \frac{p-1}{2(p+1)} \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 + x_N^2 |u|^2 dx. \] (23)

Since \( p > 1 \), we see from (22) and (23) that \( I(u) \geq c > 0 \). Thus (21) yields \( d_B > 0 \) for all \( u \in B \). \( \square \)

We define
\[ d := \min \{ d_M, d_B \}. \] (24)

From (13), (21) and (24), we have the following theorem.

**Theorem 3.4.** Assume that \( 1 + \frac{4}{N-1} \leq p < 1 + \frac{4}{(N-2)^+} \). Then \( d > 0 \).

**Proof.** From Lemma 3.2 and Lemma 3.3, it is clear that \( d > 0 \) provided \( 1 + \frac{4}{N-1} \leq p < 1 + \frac{4}{(N-2)^+} \). \( \square \)
Now, we define the following manifolds:
\[ K_1 := \{ u \in H, I(u) < d, P(u) < 0, S(u) < 0 \}, \]
\[ K_2 := \{ u \in H, I(u) < d, P(u) > 0, S(u) < 0 \}, \]
\[ K_3 := \{ u \in H, I(u) < d, S(u) < 0 \}, \]
\[ K_4 := \{ u \in H, I(u) < d, S(u) > 0 \}. \]

Then the following theorem is obtained.

**Theorem 3.5.** Assume that \( 1 + \frac{4}{N-2} \leq p < 1 + \frac{4}{N-2} \). Then for \( j = 1, 2, 3, 4 \), \( K_j \) is the invariant manifold of the Cauchy problem (1)-(2). That is, if \( \psi_0 \in K_j, j = 1, 2, 3, 4 \), then for any \( t \in [0, T) \), the solution \( \psi(t, x) \) of the Cauchy problem (1)-(2) also satisfies \( \psi(t, x) \in K_j \).

**Proof.** Let \( \psi_0 \in K_1 \). Then from Lemma 2.1, the Cauchy problem (1)-(2) has a unique solution \( \psi(t) \in C([0, T]; H) \) with \( T \leq \infty \). For any \( t \in [0, T) \), we see from (6) that
\[ I(\psi(t)) = I(\psi_0). \] (25)
Therefore, \( I(\psi_0) < d \) yields that \( I(\psi(t)) < d \) for any \( t \in [0, T) \).

Suppose that there exists \( t_1 \in [0, T) \) such that \( S(\psi(t_1)) \geq 0 \). Then, by the continuity of the function \( t \mapsto S(u(t)) \), one has \( t_0 \in (0, t_1) \) such that \( S(\psi(t_0)) = 0 \). From (25), we have \( \psi(t_0) \neq 0 \). Thus \( \psi(t_0) \in B \). By (21) and (24), we get \( I(\psi(t_0)) \geq d \), which contradicts with \( I(\psi(t_0)) < d \) for \( t \in [0, T) \). Therefore, for all \( t \in [0, T) \), we have \( S(\psi(t)) < 0 \).

Suppose that there exists \( t_2 \in [0, T) \) such that \( P(\psi(t_2)) \geq 0 \). Then, by the continuity of the function \( t \mapsto P(u(t)) \), one has \( t_3 \in (0, t_1) \) such that \( P(\psi(t_3)) = 0 \). From (25) and \( S(\psi(t_3)) < 0 \), we have \( \psi(t_3) \in M \). Then it follows from (13) and (24) that \( I(\psi(t_3)) \geq d \), which contradicts with \( I(\psi(t_3)) < d \) for \( t \in [0, T) \). Therefore, for all \( t \in [0, T) \), we have \( P(\psi(t)) < 0 \).

By the similar arguments as above, we can prove that \( K_2, K_3, K_4 \) are also invariant manifolds of the Cauchy problem (1)-(2).

4. Sharp condition for global existence and blowup.

**Theorem 4.1.** Assume that \( 1 + \frac{4}{N-2} \leq p < 1 + \frac{4}{N-2} \). If \( \psi_0 \in K_2 \cup K_4 \), then the solution \( \psi(t, x) \) of the Cauchy problem (1)-(2) exists globally in \( t \in (0, \infty) \).

**Proof.** If \( \psi_0 \in K_2 \cup K_4 \), then it follows from Lemma 2.1 and Theorem 3.5 that the Cauchy problem (1)-(2) has a unique solution \( \psi(t, x) \in C([0, T]; H) \) and satisfies \( \psi(t, x) \in K_2 \cup K_4 \) for \( t \in [0, T) \). For fixed \( t \in [0, T) \), we denote \( \psi(t, x) = \psi \).

We first treat the case \( \psi \in K_2 \). From \( P(\psi) > 0 \) and \( I(\psi) < d \), one has
\[ \frac{\alpha - 2}{2\alpha} \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j \psi|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |\partial_N \psi|^2 + |\psi|^2 + x_\alpha^2 |\psi|^2 dx < d. \] (26)
In the following we have to divide the proof into two cases: \( \alpha = 2 \) and \( \alpha > 2 \).

For the case \( \alpha = 2 \), from (26), it follows that
\[ \int_{\mathbb{R}^N} |\partial_N \psi|^2 + |\psi|^2 + x_\alpha^2 |\psi|^2 dx < 2d. \] (27)
For \( \mu > 0 \), let
\[ \psi^\mu = \mu^{\frac{N-1}{2\alpha}} \psi(\mu x_1, \ldots, \mu x_{N-1}, x_N). \]
By (11), we get
\[ P(\psi^\mu) = \mu^{\frac{2(p-1)}{p+1}} \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j \psi|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} |\psi|^{p+1} dx. \]
Since \( P(\psi) > 0 \), then there exists \( 0 < \mu_* < 1 \) such that \( P(\psi^\mu) = 0 \). It follows from (9) and (11) that
\[ I(\psi^\mu) = \frac{1}{2} \mu_*^{-\frac{4}{p+1}} \int_{\mathbb{R}^N} |\partial_N \psi|^2 + |\psi|^2 + x_N^2 |\psi|^2 dx. \] (28)
Combine (27) with (28), one has
\[ I(\psi^\mu) < \mu_*^{-\frac{4}{p+1}} d. \] (29)

Now we see \( S(\psi^\mu) \), which has two possibilities. One is \( S(\psi^\mu) < 0 \), the other is \( S(\psi^\mu) \geq 0 \). For the case \( S(\psi^\mu) < 0 \), note that \( P(\psi^\mu) = 0 \), then (13) and (24) yield that
\[ I(\psi^\mu) \geq d_M \geq d > I(\psi). \]
It follows that
\[ I(\psi) - I(\psi^\mu) < 0. \]

That is
\[ \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j \psi|^2 dx < \frac{\mu_*^{-\frac{4}{p+1}} - 1}{1 - \mu_*^{\frac{4}{p+1}}} \int_{\mathbb{R}^N} |\partial_N \psi|^2 + |\psi|^2 + x_N^2 |\psi|^2 dx. \] (30)
Combine (27) and (30), we get that
\[ \int_{\mathbb{R}^N} |\nabla \psi|^2 + |\psi|^2 + x_N^2 |\psi|^2 dx < c. \] (31)

For the case \( S(\psi^\mu) \geq 0 \), it follows from (29) that
\[ I(\psi^\mu) - \frac{1}{p+1} S(\psi^\mu) < \mu_*^{-\frac{4}{p+1}} d. \]

More precisely,
\[ \mu_*^2 \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j \psi|^2 dx + \int_{\mathbb{R}^N} |\partial_N \psi|^2 + |\psi|^2 + x_N^2 |\psi|^2 dx < \frac{2(p+1)}{p-1} d. \] (32)
It follows that
\[ \int_{\mathbb{R}^N} |\nabla \psi|^2 + |\psi|^2 + x_N^2 |\psi|^2 dx < c. \] (33)
By (31) and (33), we always get \( \|\psi\|_H \) is bounded for any \( t \in [0, T) \). Thus, from Lemma 2.1, we get that \( \psi(t, x) \) exists globally in \( t \in [0, \infty) \).

For the case \( \alpha > 2 \), from (26), we always get
\[ \int_{\mathbb{R}^N} |\nabla \psi|^2 + |\psi|^2 + x_N^2 |\psi|^2 dx < c. \] (34)
Therefore, Lemma 2.1 implies that \( \psi(t, x) \) exists globally in \( t \in [0, \infty) \).

Thus for \( \psi_0 \in K_2 \), we proved that the solution \( \psi(t, x) \) of the Cauchy problem (1)-(2) exists globally in \( t \in [0, \infty) \).

Next, we treat the case \( \psi \in K_4 \). From \( I(\psi) < d \) and \( S(\psi) > 0 \), one has
\[ \int_{\mathbb{R}^N} |\nabla \psi|^2 + |\psi|^2 + x_N^2 |\psi|^2 dx < \frac{2(p+1)}{p-1} d. \] (35)
Since $1 + \frac{4}{N - 1} \leq p < 1 + \frac{4}{(N - 2)^2}$, then $\|\psi\|_H$ is bounded for any $t \in [0, \infty)$. Thus Lemma 2.1 implies that $\psi(t, x)$ exists globally in $t \in [0, \infty)$.

Moreover, by (1) and (2) with $1 + \frac{4}{N - 1} \leq p < 1 + \frac{4}{(N - 2)^2}$ and $\psi_0 \in K_2 \cup K_4$, then the solution $\psi(t, x)$ of the Cauchy problem (1)-(2) exists globally in $t \in (0, \infty)$.

**Theorem 4.2.** Assume that $1 + \frac{4}{N - 1} \leq p < 1 + \frac{4}{(N - 2)^2}$. If $\psi_0 \in K_1 \cap \Sigma$, then the solution $\psi(t, x)$ of the Cauchy problem (1)-(2) blows up in a finite time.

**Proof.** If $\psi_0 \in K_1 \cap \Sigma$, then it follows from Lemma 2.1 and Theorem 3.5 that the Cauchy problem (1)-(2) has a unique solution $\psi(t, x) \in C([0, T); H)$ and satisfies $\psi(t) \in K_1 \cap \Sigma$. For fixed $t \in [0, T)$, we denote $\psi(t, x) = \psi$.

For $\lambda > 0$, let

$$\psi^\lambda = \lambda^{\frac{N+1}{4}} \psi(\lambda x_1, ..., \lambda x_{N-1}, x_N).$$

From (9) and (10), one has that

$$I(\psi^\lambda) = \frac{\lambda^2}{2} \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j \psi|^2 dx - \frac{\lambda^\alpha}{p + 1} \int_{\mathbb{R}^N} |\psi|^{p+1} dx + \frac{1}{2} \int_{\mathbb{R}^N} |\partial_N \psi|^2 + x_N^2 |\psi|^2 + |\psi|^2 dx,$$

$$S(\psi^\lambda) = \lambda^\alpha \sum_{j=1}^{N-1} \int_{\mathbb{R}^N} |\partial_j \psi|^2 dx - \lambda^\alpha \int_{\mathbb{R}^N} |\psi|^{p+1} dx + \int_{\mathbb{R}^N} |\partial_N \psi|^2 + x_N^2 |\psi|^2 + |\psi|^2 dx.$$

Since $P(\psi) < 0$, then

$$S(\psi^\lambda) < \left( \frac{\alpha \lambda^2}{p + 1} - \lambda^\alpha \right) \int_{\mathbb{R}^N} |\psi|^{p+1} dx + \int_{\mathbb{R}^N} |\partial_N \psi|^2 + x_N^2 |\psi|^2 + |\psi|^2 dx.$$

By $\alpha \geq 2$ and $S(\psi) < 0$, then there exists $\lambda_0 \in (0, \infty)$ such that $S(\psi^\lambda_0) = 0$.

We consider the function

$$f(\lambda) = I(\psi^\lambda) - \frac{1}{2} \lambda^2 P(\psi)$$

$$= \frac{1}{2} \int_{\mathbb{R}^N} |\partial_N \psi|^2 + x_N^2 |\psi|^2 + |\psi|^2 dx + \frac{\alpha \lambda^2 - 2 \lambda^\alpha}{2(p + 1)} \int_{\mathbb{R}^N} |\psi|^{p+1} dx.$$

It is clear that $f(\lambda)$ attains its maximum at $\lambda = 1$.

From (21), (24) and $P(\psi) < 0$, it follows that

$$d \leq d_B \leq I(\psi^{\lambda_0}) \leq I(\psi^{\lambda_0}) - \frac{1}{2} \lambda_0^2 P(\psi) \leq I(\psi) - \frac{1}{2} P(\psi).$$

Moreover, by $I(\psi) < d$, (36) and the Virial identity (7), we have

$$J'(t) = 8P(\psi) \leq 16[I(\psi) - d] < 0$$

for all $t \in [0, T)$. Thus, one has that $T < \infty$ and $\lim_{t \to T^-} J(t) = 0$. It follows that $\lim_{t \to T^-} \int_{\mathbb{R}^N} |\nabla \psi|^2 dx = \infty$. By Lemma 2.3, that is $\lim_{t \to T^-} \|\psi\|_H = \infty$.

**Remark 1.** Assume that $1 + \frac{4}{N - 1} \leq p < 1 + \frac{4}{(N - 2)^2}$. Then

$$\{ \psi \in H \setminus \{0\}, I(\psi) < d \} = K_1 \cup K_2 \cup K_4.$$

Therefore, Theorem 4.1 and Theorem 4.2 provide a sharp condition for global existence and blowup of the Cauchy problem (1)-(2) for $1 + \frac{4}{N - 1} \leq p < 1 + \frac{4}{(N - 2)^2}$. Since our results strongly depend on Ohta’s virial identity (see [13]), we cannot give the sharp conditions of global existence and blowup for the Cauchy problem (1)-(2) with $1 + \frac{4}{N} < p < 1 + \frac{4}{(N - 2)^2}$, which is still open.
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