RANDOM ATTRACTOR FOR THE 2D STOCHASTIC NEMATIC LIQUID CRYSTALS FLOWS

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Abstract. We consider the long-time behavior for stochastic 2D nematic liquid crystals flows with the velocity field perturbed by an additive noise. The presence of the noises destroys the basic balance law of the nematic liquid crystals flows, so we can not follow the standard argument to obtain uniform a priori estimates for the stochastic flow even in the weak solution space under non-periodic boundary conditions. To overcome the difficulty we use a new technique some kind of logarithmic energy estimates to obtain the uniform estimates which improve the previous result for the orientation field that grows exponentially w.r.t.time t. Considering the existence of random attractor, the common method is to derive uniform a priori estimates in functional space which is more regular than the solution space. We can follow the common method to prove the existence of random attractor in the weak solution space. However, if we consider the existence of random attractor in the strong solution space, it is very difficult and very complicated for such highly nonlinear stochastic system with no basic balance law and non-periodic boundary conditions. Here, we use a compactness arguments of the stochastic flow and regularity of the solutions to the stochastic model to obtain the existence of the random attractor in the strong solution space, which implies the support of the invariant measure lies in a more regular space. As far as we know, it is the first article to attack the long-time behavior of stochastic nematic liquid crystals.

1. Introduction. The paper is concerned with the following stochastic hydrodynamical model for the flow of nematic liquid crystals in $D \times \mathbb{R}^+$, where $D \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\Gamma$.

$$\begin{align*}
v_t + [(v \cdot \nabla)v - \mu \Delta v + \nabla p] + \lambda \nabla \cdot (\nabla d \circ \nabla d) &= \dot{W}, \\
\nabla \cdot v(t) &= 0,
\end{align*}$$

(1) (2)

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\[ d_t + [(\mathbf{v} \cdot \nabla) \mathbf{d} - \gamma (\Delta \mathbf{d} - f(\mathbf{d})) = 0. \] (3)

The unknowns for the 2D stochastic hydrodynamical model are the fluid velocity field \( \mathbf{v} = (v_1, v_2) \in \mathbb{R}^2 \), the averaged macroscopic/continuum molecular orientations \( \mathbf{d} = (d_1, d_2, d_3) \in \mathbb{R}^3 \) and the scalar function \( p(x, t) \) representing the pressure (including both the hydrostatic and the induced elastic part from the orientation field). The positive constants \( \nu, \lambda \) and \( \gamma \) stand for viscosity, the competition between kinetic energy and potential energy, and macroscopic elastic relaxation time (Debroah number) for the molecular orientation field. Without loss of generality, we assume \( \mu = \lambda = \gamma = 1 \). \( W \) is a standard Wiener process in \( \mathbf{H} \) defined below and has the form of

\[ W(t) := \sum_{i=1}^{\infty} \lambda_i^{\frac{1}{2}} e_i B_i(t), \]

where \( ((B_i(t))_{t \in \mathbb{R}})_{i \in \mathbb{N}} \) is a sequence of one-dimensional, independent, identically distributed Brownian motions defined on the complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), \( (e_i)_{i \in \mathbb{N}} \) is an orthonormal basis in \( \mathbf{H} \), \( (\lambda_i)_{i \in \mathbb{N}} \) is a convergent sequence of positive numbers which ensure \( W \) is a standard Wiener process in \( \mathbf{H} \). Here \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is a general polynomial function whose details will be given later. The symbol \( \nabla \mathbf{d} \odot \nabla \mathbf{d} \) denote the \( 2 \times 2 \) matrix whose \( (i, j) \)-th entry is given by

\[ [\nabla \mathbf{d} \odot \nabla \mathbf{d}]_{i,j} = \sum_{k=1}^{3} \partial_x d^{(k)} \partial_x d^{(k)}, \quad i, j = 1, 2. \]

In this paper, we consider the following initial boundary conditions for the stochastic nematic liquid crystals equations. The boundary conditions are

\[ \mathbf{v}(t, x) = 0, \quad \mathbf{d}(x, t) = \mathbf{d}_0(x), \quad \text{for } (x, t) \in \Gamma \times \mathbb{R}^+. \] (4)

The initial conditions are

\[ \mathbf{v}|_{t=0} = \mathbf{v}_0(x) \text{ with } \nabla \cdot \mathbf{v}_0 = 0, \quad \mathbf{d}|_{t=0} = \mathbf{d}_0(x), \quad \text{for } x \in \mathbf{D}, \]

where \( \mathbf{n} \) is the outward unite normal vector to \( \Gamma \). In [21], F.H. Lin proposed a corresponding deterministic model of (1)-(3) as a simplified system of the original Ericksen-Leslie system (see [12, 20]). By Ericksen-Leslie’s hydrodynamical theory of the liquid crystals, the simplified system describing the orientation as well as the macroscopic motion reads as follows

\[ d_v + [(\mathbf{v} \cdot \nabla) \mathbf{v} - \mu \Delta \mathbf{v} + \nabla p]dt = \lambda \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}) dt, \]

\[ \nabla \cdot \mathbf{v}(t) = 0, \]

\[ \partial_t \mathbf{d} + [(\mathbf{v} \cdot \nabla) \mathbf{d} = \gamma (\Delta \mathbf{d}(t) + |\nabla \mathbf{d}|^2 \mathbf{d}), \quad |\mathbf{d}| = 1. \] (8)

In order to avoid the nonlinear gradient in (8), usually one uses the Ginzburg-Landau approximation to relax the constraint \( \mathbf{d} = 1 \). The corresponding approximate energy is

\[ \int_{\mathbf{D}} \left[ \frac{1}{2} |\nabla \mathbf{d}|^2 + \frac{1}{4\eta^2} (|\mathbf{d}|^2 - 1)^2 \right] dx \]

where \( \eta \) is a positive constant. Then one arrives at the approximation system (6)-(8) with \( f(\mathbf{d}) \) and \( F(\mathbf{d}) \) given by

\[ f(\mathbf{d}) = \frac{1}{\eta^2} (|\mathbf{d}|^2 - 1) \mathbf{d} \quad \text{and} \quad F(\mathbf{d}) = \frac{1}{4\eta^2} (|\mathbf{d}|^2 - 1)^2. \] (10)
In this work, we consider a more general polynomial function $f(d)$ which contains as a special case the (10). We define a function $\tilde{f}: [0, \infty) \to \mathbb{R}$ by

$$\tilde{f}(x) = \sum_{k=0}^{N} a_k x^k, \quad x \in \mathbb{R}_+,$$

where $a_N > 0$ and $a_k \in \mathbb{R}, k = 0, 1, 2, ..., N - 1$. Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(d) = \tilde{f}(|d|^2)d.$$

Denote by $F: \mathbb{R}^3 \to \mathbb{R}$ the Fréchet differentiable map such that for any $d \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$

$$F'(d)[\xi] = f(d) \cdot \xi.$$

Set $\tilde{F}$ to be an antiderivative of $\tilde{f}$ such that $\tilde{F}(0) = 0$. Then

$$\tilde{F}(x) = \sum_{k=0}^{N} \frac{a_k}{k+1} x^{k+1}, \quad x \in \mathbb{R}_+.$$

The Ericksen–Leslie system is well suited for describing many special flows for the materials, especially for those with small molecules, and is widely accepted in the engineering and mathematical communities studying liquid crystals. System (1)–(3) with $f(d)$ given by (10) can be possibly viewed as the simplest mathematical model, which keeps the most important mathematical structure as well as most of the essential difficulties of the original Ericksen–Leslie system (see [22]). This deterministic system with Dirichlet boundary conditions has been studied in a series of work not only theoretically (see [22, 23]) but also numerically (see [27, 28]).

The introduction of stochastic processes in nematic liquid crystals flows is aimed at accounting for a number of uncertainties and errors. The state of the nematic liquid crystals is strongly dependent on the state of the environment. In natural systems external noise is often quite large. At an instability point the system is sensitive even to infinitesimally small perturbations and the role of external noise has to be investigated in the vicinity of a transition point. And experimental investigation also showed that, in the case of electrohydrodynamic instabilities, the average value of the voltage necessary for the transition is shifted to higher and higher values as the intensity of the external noise is increased i.e., the amplitude of the voltage fluctuations, is increased. Further study showed that the average voltage necessary to induce the transition to turbulent behavior, increases with the variance of the voltage fluctuations (see [16]). For more details one can also refer to [17, 33, 34].

Rheological predictions of the behavior of complex fluids like these, often start with the derivation of macroscopic, approximate equations for quantities of interest using various closure approximations. The difficulty in obtaining accurate closures has motivated the extensive, in recent years, use of direct simulations, either of the PDE governing the orientation distribution function, or of the equivalent stochastic differential equation, via Brownian dynamics simulations. The latter have the advantage that they are amenable to use with models with many internal degrees of freedom (as opposed to the PDE approach in which the curse of dimensionality precludes realistic computation). For more details one can see [13, 18, 26, 31, 32].

Despite the developments in the deterministic case, the theory for the stochastic nematic liquid crystals remains underdeveloped. To the best of our knowledge, there are few works on the stochastic nematic liquid crystals. In the papers [5, 6],...
Z. Brzezniak, E. Hausenblas and P. Razafimandimby have considered the model perturbed by multiplicative Gaussian noise and have proved the global well-posedness for the weak solution and strong solution in 2-D case. When the noise is jump and the dimension is two, Z. Brzezniak, U. Manna and A.A. Panda in [7] obtained the same result as the case of Gaussian noise. A weak martingale solution is also established for the three dimensional stochastic nematic liquid crystals with jump noise in [7].

One natural problem arising from this global existence result is the dynamical behavior of the 2D stochastic system.

As it is seen in the system (1)-(5), only the velocity field is disturbed by the noise. Why don’t we consider the problem that both velocity field and orientation field are perturbed? This is because of the particular geometric structure of the nematic liquid crystals equations (see [22] for the basic balance law). Specifically, to obtain the energy estimates of velocity field in $H^1$ we should estimate orientation field in the more regular space $H^2$ at the same time. As we see, if (3) is also perturbed by the additive noise, by introducing two known Ornstein-Uhlenbeck processes $z$ and $z_1$ with enough regularity and letting $u = v - z$ and $d - z_1 = \theta$, (1)-(5) is equivalent to the following system,

$$
\begin{align*}
\mathbf{u}_t + ((u + z) \cdot \nabla (u + z) - \mu \Delta u + \nabla p) + \lambda \nabla \cdot ((\nabla \theta + \nabla z_1) \otimes (\nabla \theta + \nabla z_1)) &= 0, \\
\nabla \cdot u &= 0, \\
\theta_t + v \cdot \nabla (\theta + z_1) - \gamma \Delta \theta + \gamma f(\theta + z_1) &= 0,
\end{align*}
$$

where $u$ and $\theta$ satisfy the following initial boundary conditions

$$
\begin{align*}
\mathbf{u}(x, t) &= 0, \quad \theta(x, t) = \mathbf{d}_0(x), \quad \forall (x, t) \in \Gamma \times \mathbb{R}_+, \\
\mathbf{u}|_{t=0} &= \mathbf{v}_0(x) \text{ with } \nabla \cdot \mathbf{v}_0 = 0, \quad \theta(x)|_{t=0} = \mathbf{d}_0(x), \quad \forall x \in \mathcal{D}.
\end{align*}
$$

For simplicity, we assume $f$ is given by (10). Then we will arrive at some kind of the following estimates

$$
\begin{align*}
\frac{1}{2} \left(\frac{\partial u^2}{\partial t} + |\nabla \theta|^2 + |u|^2\right) + \left(\frac{\partial \theta^2}{\partial t} + |\Delta \theta|^2 + |\nabla u|^2 + |\theta|^2\right) &\leq c|\theta|_6^6 + \cdots, \\
\frac{1}{2} \left(\frac{\partial \theta^2}{\partial t} + \int_{\mathcal{D}} (|\theta|^2 - 1)^2 dx + |u|^2\right) + \left(\frac{\partial u^2}{\partial t} + |\Delta \theta - f(\theta)|^2\right) &\leq c \int_{\mathcal{D}} (|\theta|^2 - 1)^2 |\theta|^2 dx + \cdots.
\end{align*}
$$

where $|\cdot|_{L^m}$ is the norm of the usual Lebesgue space $L^m, m > 1$ and $c$ is a positive constant which is bigger than one. From the above inequalities, we see that it is difficult to obtain the energy estimates of $(u, \theta)$ or $(v, d)$ if both velocity field and orientation field are perturbed by the noises. Therefore, in this article, we only consider the long-time behavior when the velocity field is perturbed.

The understanding of asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. For the deterministic nematic liquid crystals equations, the existence of global attractor for the strong solution is established (see [15, 30]). For the corresponding stochastic model, there is no result about the existence of random attractor. One reason is the absence of the basic balance law which results in the failure of applying the method of deterministic model to the stochastic model.
In this article, we firstly improve the bounds for the solutions to (1)-(5). These bounds are uniform with respect to present time and initial time (see Lemma 3.2). These estimates of the uniform boundedness improve previous bounds obtained in [5, 6, 7], in which the bounds of $d$ grow exponentially with respect to present time or initial time. In obtaining these time-uniform a priori estimates for orientation field $d$ (for example in space $(L^2(D))^3$), the power of $d$ from the nonlinear $f(d)$ will be much bigger (see (3)) than two. Using the Gronwall inequality (a standard argument) we will obtain that the solutions have exponential growth which is not sufficient to ensure the existence of random attractor. To overcome the difficulty, our ideal is that taking advantage of the property of the logarithmic function we reduce the power from nonlinear term. Roughly speaking, for a positive $f(t), t \in \mathbb{R}_+$, if we want to consider it’s uniform boundedness with respect to time $t$, we just need to estimate $\ln(1 + f(t))$. If $\ln(1 + f(t))$ is uniformly bounded with respect to $t$, so is $f(t)$. Using this new technique we obtain the uniform estimates for orientation field $d$ (see Lemma 3.1) which opens a way to study the long-time behavior of stochastic nematic liquid crystals.

Our main goal of this article is to show the existence of random attractor in the strong solution space $V \times H^2$. Although, we also obtain the existence of random attractor in the weak solution space $H \times H^1$. As we know, the sufficient condition for ensuring the existence of random attractor in $V \times H^2$ is to obtain an absorbing ball which is compact in $V \times H^2$. One method is to derive uniform a priori estimates in the functional space $H^2 \times H^3$. However, it is very difficult and very complicated for such highly non-linear stochastic system with non-periodic boundary conditions. Here, we use a compactness arguments of the stochastic flow and regularity of the solutions to construct a compact absorbing ball in the strong solution space $V \times H^2$. For simplicity, we will outline the method of proving the existence of random attractor in the weak solution space in detail. The method of proving the existence of random attractor in the strong solution space is the same as the case of weak solution space, but with more complicated calculus. We complete the proof of the existence of random attractor in the weak solution space by four steps. Firstly, using the Lemma 3.2, we obtain the absorbing ball in the weak solution $H \times H^1$ (see Proposition 3). Secondly, we will verify the two a priori estimates of Aubin-Lions compact lemma to obtain a convergent subsequence of $(v, d)$ which converges almost everywhere with respect to time $t \in [s, T], -\infty < s < T < \infty$ (see Proposition 3 and Proposition 4). Thirdly, by showing the continuity of the weak solution in the space $H \times H^1$ with respect to time $t$ and with respect to the initial data $(v_0, d_0)$, we prove that the weak solution operators $S(t, s; \omega)$ are indeed a stochastic dynamic system and are almost surely compact in $H \times H^1$ for all $(t, s)$ satisfying $-\infty < s < t < \infty$ (see Proposition 5). Finally, in Proposition 6 using the compact solution operator to act on the the absorbing ball yields a new set which is a compact and absorbing in $H \times H^1$. The existence of the random attractor in the weak solution space $H \times H^1$ follows directly from Proposition 6.

**Remark 1.** Concerning the regularity of weak solution stated in the third step above, we show the following regularity in Corollary 1 for the weak solution to (1)-(5).

$$v \in C([0, T]; H) \text{ and } d \in C([0, T]; H^1),$$

which is not available before. To show the existence of random attractor in the strong solution space $V \times H^2$, the regularity for the strong solutions to (1)-(5) in [5, 6]
is not enough for our purpose. In \cite{5, 6}, the strong solutions are Lipschitz continuous in the space $H \times H^1$ with respect to initial data. To improve the regularity of the strong solutions, in Proposition 4, we obtain the continuity of the strong solution in $V \times H^2$ with respect to initial data. The new difficulty involved here in proving the continuity in space $V \times H^2$ with respect to initial data is the high non-linearity of the stochastic system. We overcome this difficulty by a careful and delicate a priori estimates.

From the proof of the existence of random attractor in the strong solution space $V \times H^2$, one will see some advantage of our method over the method in \cite{9, 10}. As we will see if the uniform a priori estimates for the stochastic flow to (1)-(5) in the strong solution space $V \times H^2$ hold, our method not only shows the existence of random attractor in the weak solution space $H \times H^1$ but also the existence of random attractor in the strong solution space $V \times H^2$. This improves the regularity of the support of the invariant measure which not only lies in $H \times H^1$ but also in $V \times H^2$ (see Remark 2). We should point out that support of the invariant measure in $V \times H^2$ may be not easily proved using the Krylov-Bogoliubov method. It is a new result for this stochastic model and may open a way to study the long-time behavior of stochastic nematic liquid crystals.

Recently, we note these nice works \cite{3, 4, 19} which study the existence random attractor for the stochastic hydrodynamic equations including stochastic Navier-Stokes equations defined on unbounded domains with an irregular noise. As we know, the study of long-term behavior for stochastic partial differential equations (SPDEs) defined on unbounded domains is difficult. Fortunately, a general frame which can be applied to prove the existence of random attractor for SPDEs defined on both the bounded case and the unbounded case is established in \cite{3}. Furthermore, in \cite{4}, as an application of the beautiful notation 'asymptotic compactness random dynamical system' given in \cite{4}, the existence of invariant measure is obtained for stochastic Navier-Stokes equations defined on unbounded domains. This is the first result for stochastic hydrodynamic equations defined on unbounded domains. Finally, a natural question arises, could the method in this article be applied to SPDEs on unbounded domains? We think we need some work and we will extend the method to the unbounded domains in the next step.

The remaining of this paper is organized as follows. In section 2, we state some preliminaries and recall some results. The existence of random attractor in weak solution space is presented in section 3, and section 4 is for the existence of random attractor in the strong solution space. As usual, constants $c$ may change from one line to the next, unless, we give a special declaration ; we denote by $c(a)$ a constant which depends on some parameter $a$.

2. Preliminaries. For $1 \leq p \leq \infty$, let $L^p(D)$ be the usual Lebesgue spaces with the norm $\|\cdot\|_p$. For a positive integer $m$, we denote by $(H^{m,p}(D), \|\cdot\|_{m,p})$ the usual Sobolev spaces, see\cite{1}). When $p = 2$, we denote by $(H^m(D), \|\cdot\|_m)$ with inner product $\langle \cdot, \cdot \rangle_{H^m}$. Let

$$\mathcal{V} = \{v \in (C_0^\infty(\bar{D}))^2 : \nabla \cdot v = 0\}.$$ 

We denote by $H, V,$ and $H^m$ be the closure spaces of $\mathcal{V}$ in $(L^2(D))^2, (H^1(D))^2,$ and $(H^m(D))^2$ respectively for positive integer $m \geq 2$. And set $\|\cdot\|_2$ and $\langle \cdot, \cdot \rangle$ to be the norm and inner product of $H$ respectively. The notation $\langle \cdot, \cdot \rangle$ is also used to denote the inner product in $(L^2(D))^2$. By the Poincaré inequality, there exists a
constant \( c \) such that for any \( \mathbf{v} \in \mathbf{V} \) we have \( \| \mathbf{v} \|_1 \leq c |\nabla \mathbf{v}|_2 \). Without confusion, we let \( \| \cdot \|_1 \) and \( \langle \cdot, \cdot \rangle_\mathbf{V} \) stand for the norm and the inner product in \( \mathbf{V} \) respectively, where \( \langle \cdot, \cdot \rangle_\mathbf{V} \) is defined by

\[
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle_\mathbf{V} := \int_\mathcal{D} \nabla \mathbf{v}_1 \cdot \nabla \mathbf{v}_2 \, dx, \quad \text{for } \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}.
\]

Denote by \( \mathbf{V}' \) the dual space of \( \mathbf{V} \). And define the linear operator \( A_1 : \mathbf{V} \mapsto \mathbf{V}' \), as the following:

\[
\langle A_1 \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_\mathbf{V}, \quad \text{for } \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}.
\]

Since the operator \( A_1 \) is positive selfadjoint with compact resolvent, by the classical spectral theorems there exists a sequence \( \{ \alpha_j \}_{j \in \mathbb{N}} \) of eigenvalues of \( A_1 \) such that

\[
0 < \alpha_1 \leq \alpha_2 \leq \cdots, \quad \alpha_j \to \infty
\]

corresponding to the eigenvectors \( e_j \) which consist of an orthonormal basis of \( \mathbf{H} \).

Assume

\[
\sum_{i=1}^{\infty} \lambda_i \alpha_i^2 < \infty. \tag{14}
\]

For arbitrary constant \( T > 0 \) and \( j \in \mathbb{N} \), we define

\[
z^j(t) = \sum_{n=1}^{j} \sqrt{\lambda_n} \int_0^t e^{-A_1(t-s)} e_n d\mathbf{B}_n(s), \quad t \in [0, T]
\]

and

\[
z(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \int_0^t e^{-A_1(t-s)} e_n d\mathbf{B}_n(s), \quad t \in [0, T].
\]

Obviously,

\[
z^j(w) \in C([0, T]; \mathbf{H}^2), \quad \mathbb{P} - a.e. \ \omega \in \Omega.
\]

For \( k \in \mathbb{N} \) and \( k > j \), in view of an infinite dimensional version of Burkholder-Davis-Gundy type of inequality for stochastic convolutions (see [11, 25] and references therein), we have

\[
E \sup_{t \in [0, T]} \| A_1(z^j - z^k) \|_2^2 \leq CT \sum_{n=j+1}^{k} \lambda_n \alpha_n^2 \to 0, \quad \text{as } j \to \infty.
\]

Therefore

\[
z(w) \in C([0, T]; \mathbf{H}^2), \quad \mathbb{P} - a.e. \ \omega \in \Omega. \tag{15}
\]

Obviously \( z \) is the unique solution to the problem below

\[
dz - \Delta z = dW_1, \tag{16}
\]

\[
z(x, t) = 0, \quad \forall (x, t) \in \Gamma \times \mathbb{R}_+ \tag{17},
\]

\[
z(x)|_{t=0} = 0, \quad \forall x \in \mathcal{D}. \tag{18}
\]

Let \( \mathbb{H}^m = (H^m(\mathcal{D}))^3, m = 0, 1, 2, \ldots \). When \( m = 0 \), set \( \mathbb{H} = \mathbb{H}^0 = (L^2(\mathcal{D}))^3 \) for simplicity. Denote the dual space of \( \mathbb{H}^m \) by \( \mathbb{H}^{-m} \). Then, similarly, we define the linear operator \( A_2 : \mathbb{H}^1 \mapsto \mathbb{H}^{-1} \) as

\[
\langle A_2 \mathbf{d}_1, \mathbf{d}_2 \rangle = \langle \mathbf{d}_1, \mathbf{d}_2 \rangle_{\mathbb{H}^1}, \quad \text{for } \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{H}^1.
\]
Similarly, if we assume
\[ \sum_{i=1}^{\infty} \lambda_i \alpha_i^3 < \infty, \]  
we have that \( z(\omega) \in C([0, T]; H^3), P - a.e. \omega \in \Omega. \)

Let \( D(A_1) := \{ \eta \in H, A_1 \eta \in H \} \) and \( D(A_2) := \{ \theta \in H, A_2 \theta \in H \}. \) Because \( A_1^{-1} \) and \( A_2^{-1} \) are self-adjoint compact operators in \( H \) and \( H \) respectively, thanks to the classic spectral theory, we can define the power \( A_t^s \) for any \( s \in \mathbb{R} \). Then \( D(A_1)' = D(A_1^{-1}) \) is the dual space of \( D(A_1) \). Furthermore, we have the compact embedding relationship
\[ D(A_1) \subset V \subset H \subset V' \subset D(A_1)', \]
and
\[ \langle \cdot, \cdot \rangle_V = \langle A_1^{\frac{1}{2}} \cdot, A_1^{\frac{1}{2}} \cdot \rangle. \]

Similarly,
\[ D(A_2) \subset H^1 \subset H \subset H^{-1} \subset D(A_2)', \]
and
\[ \langle \cdot, \cdot \rangle_{H^1} = \langle A_2^{\frac{1}{2}} \cdot, A_2^{\frac{1}{2}} \cdot \rangle. \]

**Definition 2.1.** We say a continuous \( H \times H^1 \) valued \((F_t) = (\sigma(W(s), s, 0, t))\) adapted random field \((v(., t), B(., t))_{t \in [0, T]}\) defined on \((\Omega, F, P)\) is a weak solution to problem (1)-(5) if for \((v_0, d_0) \in H \times H^1\) the following conditions hold:
\[ v \in C([0, T]; H) \cap L^2([0, T]; V), \]
\[ d \in C([0, T]; H^1) \cap L^2([0, T]; H^2), \]
and the integral relation
\[ \langle v(t), v \rangle + \int_0^t \langle A_1 v(s), v \rangle ds + \int_0^t \langle v(s), \nabla v(s), v \rangle ds \]
\[ + \int_0^t \langle v(s), \nabla (d(s) \ominus d(s)), v \rangle ds = \langle v_0, v \rangle + \langle W(t), v \rangle, \]
\[ \langle d(t), d \rangle + \int_0^t \langle A_2 d(s), d \rangle ds + \int_0^t \langle v(s), \nabla d(s), d \rangle ds \]
\[ = \langle d_0, d \rangle - \int_0^t \langle f(d(s)), d \rangle ds, \]
hold a.s. for all \( t \in [0, T] \) and \((v, d) \in V \times H.\)

**Definition 2.2.** We say a continuous \( V \times H^2 \) valued \((F_t) = (\sigma(W(s), s, 0, t))\) adapted random field \((v(., t), B(., t))_{t \in [0, T]}\) defined on \((\Omega, F, P)\) is a strong solution to problem (1)-(5) if for \((v_0, d_0) \in V \times H^2\) the following conditions hold:
\[ v \in C([0, T]; V) \cap L^2([0, T]; H^2), \]
\[ d \in C([0, T]; H^2) \cap L^2([0, T]; H^3), \]
and the integral relation
\[ v(t) + \int_0^t A_1 v(s) ds + \int_0^t v(s) \cdot \nabla v(s) ds \]
In Corollary 1, we improve the regularity and obtain that \( V(1)-(5) \) is continuous with respect to initial data in strong solution space. We will show in Proposition 4 that the strong solution to initial data is not enough for us to prove the existence of random attractor in the \( H \times H \) tem. And using this continuity, we can prove that , the solution operator is compact which ensures the solution operator is a time continuous stochastic dynamical sys-

\( \omega \) parametrized by \( 0 \) at two -sided Wiener space of \( X \) closed set valued measurable map \( K \) said to be measurable if for each \( x \) hold

Let \( \Theta \) be a polish space and ( \( ˜\Omega \), \( \tilde{\mathcal{F}} \), \( \tilde{\mathbb{P}} \)) be a probability space, where \( ˜\Omega \) is the two -sided Wiener space \( C_0(\mathbb{R}; X) \) of continuous functions with values in \( X \), equal to \( 0 \) at \( t = 0 \). We consider a family of mappings \( S(t, s; \omega) : X \rightarrow X \), \( -\infty < s \leq t < \infty \), parametrized by \( \omega \in ˜\Omega \), satisfying for \( \tilde{\mathbb{P}}\text{-a.e. } \omega \) the following properties (i)-(iv):

\( i \) \( S(t, r; \omega)S(r, s; \omega)x = S(t, s; \omega)x \) for all \( s \leq r \leq t \) and \( x \in X \);

\( ii \) \( S(t, s; \omega) \) is continuous in \( X \), for all \( s \leq t \);

\( iii \) \( \omega \mapsto S(t, s; \omega)x \) is measurable from \( (\tilde{\Omega}, \tilde{\mathcal{F}}) \) to \( (X, \mathcal{B}(X)) \) where \( \mathcal{B}(X) \) is the Borel-\( \sigma \)- algebra of \( X \);

\( iv \) for all \( t, x \in X \), the mapping \( \omega \mapsto S(t, s; \omega) \) is right continuous at any point.

A set valued map \( K : \tilde{\Omega} \rightarrow 2^X \) taking values in the closed subsets of \( X \) is said to be measurable if for each \( x \in X \) the map \( \omega \mapsto d(x, K(\omega)) \) is measurable, where \( d(A, B) = \sup \{ \inf \{ \inf (x, \{ y \} : y \in B) \} : x \in A \} \) for \( A, B \in 2^X \), \( A, B \neq \emptyset \); and \( d(x, B) = d(\{ x \}, B) \). Since \( d(A, B) = 0 \) if and only if \( A \subset B, d \) is not a metric. A closed set valued measurable map \( K : \tilde{\Omega} \rightarrow 2^X \) is named a random closed set.
Given $t \in \mathbb{R}$ and $\omega \in \tilde{\Omega}$, $K(t, \omega) \subset X$ is called an attracting set at time $t$ if, for all bounded sets $B \subset X$,

$$d(S(t, s; \omega)B, K(t, \omega)) \to 0, \quad \text{provided } s \to -\infty.$$ 

Moreover, if for all bounded sets $B \subset X$, there exists $t_B(\omega)$ such that for all $s \leq t_B(\omega)$

$$S(t, s; \omega)B \subset K(t, \omega),$$

we say $K(t, \omega)$ is an absorbing set at time $t$.

Let $\{\vartheta_t : \tilde{\Omega} \to \tilde{\Omega}\}, t \in T, T = \mathbb{R}$, be a family of measure preserving transformations of the probability space $(\tilde{\Omega}, \tilde{F}, \tilde{P})$ such that for all $s < t$ and $\omega \in \tilde{\Omega}$

(a) $(t, \omega) \to \vartheta_t \omega$ is measurable;
(b) $\vartheta_t(\omega)(s) = \omega(t + s) - \omega(t)$;
(c) $S(t, s; \omega)x = S(t - s, 0; \vartheta_s \omega)x$.

Thus $(\vartheta_t)_{t \in T}$ is a flow, and $(\tilde{\Omega}, \tilde{F}, \tilde{P}, (\vartheta_t)_{t \in T})$ is a measurable dynamical system.

**Definition 2.4.** Given a bounded set $B \subset X$, the set

$$A(B, t, \omega) = \bigcap_{t \leq T} \bigcup_{s \leq T} S(t, s; \omega)B$$

is said to be the $\Omega$-limit set of $B$ at time $t$. Obviously, if we denote $A(B, 0, \omega) = A(B, \omega)$, we have $A(B, t, \omega) = A(B, \vartheta_t \omega)$.

We may identify

$$A(B, t, \omega) = \{x \in X : \text{there exists } s_n \to -\infty \text{ and } x_n \in B \text{ such that } \lim_{n \to \infty} S(t, s_n; \omega)x_n = x\}.$$

Furthermore, if there exists a compact attracting set $K(t, \omega)$ at time $t$, it is not difficult to check that $A(B, t, \omega)$ is a nonempty compact subset of $X$ and $A(B, t, \omega) \subset K(t, \omega)$.

**Definition 2.5.** If, for all $t \in \mathbb{R}$ and $\omega \in \tilde{\Omega}$, the random closed set $\omega \to A(t, \omega)$ satisfying the following properties:

1. $A(t, \omega)$ is a nonempty compact subset of $X$,
2. $A(t, \omega)$ is the minimal closed attracting set, i.e., if $\hat{A}(t, \omega)$ is another closed attracting set, then $A(t, \omega) \subset \hat{A}(t, \omega)$,
3. it is invariant, in the sense that, for all $s \leq t$,

$$S(t, s; \omega)A(s, \omega) = A(t, \omega),$$

$A(t, \omega)$ is called the random attractor.

Let

$$A(\omega) = A(0, \omega).$$

Then the invariance property writes

$$S(t, s; \omega)A(\vartheta_s \omega) = A(\vartheta_t \omega).$$

To prove the existence of the random attractor, we will use the following sufficient condition given in [9]. For the convenience of reference, we cite it here.
Theorem 2.6. Let \((S(t,s;\omega))_{t\geq s,\omega\in\Omega}\) be a stochastic dynamical system satisfying (i), (ii), (iii) and (iv). Assume that there exists a group \(\vartheta_t, t \in \mathbb{R}\), of measure preserving mappings such that condition (c) holds and that, for \(\tilde{\mathbb{P}}\)-a.e. \(\omega\), there exists a compact attracting set \(K(\omega)\) at time 0. For \(\tilde{\mathbb{P}}\)-a.e. \(\omega\), we set
\[
A(\omega) = \bigcup_{B \subseteq X} A(B, \omega)
\]
where the union is taken over all the bounded subsets of \(X\). Then we have for \(\tilde{\mathbb{P}}\)-a.e. \(\omega \in \bar{\Omega}\).

1. \(A(\omega)\) is a nonempty compact subset of \(X\), and if \(X\) is connected, it is a connected subset of \(K(\omega)\).
2. The family \(A(\omega), \omega \in \bar{\Omega}\), is measurable.
3. \(A(\omega)\) is invariant in the sense that
\[
S(t,s;\omega)A(\vartheta_t\omega) = A(\vartheta_t\omega), \quad s \leq t.
\]
4. It attracts all bounded sets from \(-\infty\): for bounded \(B \subseteq X\) and \(\omega \in \bar{\Omega}\)
\[
d(S(t,s;\omega)B, A(\vartheta_t\omega)) \to 0, \quad \text{when } s \to -\infty.
\]
Moreover, it is the minimal closed set with this property: if \(\tilde{A}(\vartheta_t\omega)\) is a closed attracting set, then \(A(\vartheta_t\omega) \subseteq \tilde{A}(\vartheta_t\omega)\).

5. For any bounded set \(B \subseteq X\), \(d(S(t,s;\omega)B, A(\vartheta_t\omega)) \to 0\) in probability when \(t \to \infty\).

And if the time shift \(\vartheta_t, t \in \mathbb{R}\) is ergodic.

6. there exists a bounded set \(B \subseteq X\) such that
\[
A(\omega) = A(B, \omega).
\]

7. \(A(\omega)\) is the largest compact measurable set which is invariant in sense of Definition 2.4.

In this article, we will use the two lemmas below to prove our main results. The first lemma is Aubin-Lions Lemma whose proof can be found in [2, 24].

Lemma 2.7. Let \(B_0, B, B_1\) be Banach spaces such that \(B_0, B_1\) are reflexive and \(B_0 \subseteq B \subseteq B_1\). Define, for \(0 < T < \infty\),
\[
X := \left\{ h \left| h \in L^2([0,T]; B_0), \frac{dh}{dt} \in L^2([0,T]; B_1) \right. \right\}.
\]
Then \(X\) is a Banach space equipped with the norm \(|h|_{L^2([0,T]; B_0)} + |h'|_{L^2([0,T]; B_1)}\).
Moreover,
\[
X \subseteq L^2([0,T]; B).
\]

The following lemma, a special case of a general result of Lions and Magenes [29], will help us to show the continuity of the solution to stochastic nematic liquid crystals with respect to time.

Lemma 2.8. Let \(V, H, V'\) be three Hilbert spaces such that \(V \subset H = H \subset V'\), where \(H'\) and \(V'\) are the dual spaces of \(H\) and \(V\) respectively. Suppose \(u \in L^2([0,T]; V)\) and \(u' \in L^2([0,T]; V')\). Then \(u\) is almost everywhere equal to a function continuous from \([0,T]\) into \(H\).
3. Existence of random attractor in \( H \times H^1 \). One of our main results in this article is to prove:

**Theorem 3.1.** Let \( v_0 \in H, d_0 \in H^1 \) in (4) and \( f(d) \) is given by (12). Assume (14) hold. Then the solution operator \( (S(t, s; \omega))_{t \geq s, \omega \in \Omega} \) of (1)–(5) : \( S(t, s; \omega)(v_s, d_s) = (v(t), d(t)) \) has properties (i)–(iv) of Theorem 2.2 and possesses a compact absorbing ball \( B(0, \omega) \) in \( H \times H^1 \) at time 0. Furthermore, for \( \mathbb{P} \)-a.e. \( \omega \), the set

\[
\mathcal{A}(\omega) = \bigcup_{B \subset H \times H^1} \mathcal{A}(B, \omega)
\]

where the union is taken over all the bounded subsets of \( H \times H^1 \) is the random attractor of (1)-(5) and possesses the properties (1) – (7) of Theorem 2.6 with space \( H \times H^1 \).

**Proof.** The results of this theorem follows directly from Proposition 6 and Theorem 2.6.

The rest of this section is to find a compact absorbing ball for (1)-(5) in \( H \times H^1 \). We will achieve our goal by six steps. In subsection 3.1, we use a new technique logarithmic energy estimates to obtain the uniform a priori estimates in \( L^{4N+2}(D) \) which is very important to study the long-time behavior of the stochastic nematic liquid system (see Lemma 3.2 and the proof of Proposition 1). Then in subsection 3.2, we obtain the absorbing ball for the solution to (1)-(5) in the space \( H \times H^1 \) in Proposition 1. As the third step, we prove the solution operator is a stochastic dynamical system in subsection 3.3. In the next subsection, by Proposition 3 and Proposition 4 we verified two a priori estimates of the Aubin-Lions lemma which is used to obtain a convergent subsequence of the solutions to (1)-(5). In subsection 3.5, taking advantage of the convergent subsequence and the continuity of the solutions with respect to initial data in \( H \times H^1 \), we prove the solution operator \( S(t, s; \omega) \) is almost surely compact from \( H \times H^1 \) to \( H \times H^1 \) for all \( s, t \in \mathbb{R}, s < t \) (see Proposition 5). Finally, using the existence of absorbing ball and compactness of the solution operator, we construct a compact absorbing ball in Proposition 6.

To study the long time behavior of (1)-(5), we introduce a modified stochastic convolution. Let \( t \in \mathbb{R} \) and \( \beta \in \mathbb{R}_+ \). For simplicity, we still define

\[
z(t) := \int_{-\infty}^{t} e^{-(t-s)(A_1+\beta)}dW(s).
\] (20)

Then by (15), we have \( z(\omega) \in C([0, T]; H^2), \mathbb{P} \)-a.e. and satisfies the linear equation

\[
dz = (-A_1z - \beta z)dt + dW,
\]

\[
z(x, t) = 0, \quad \forall (x, t) \in \Gamma \times \mathbb{R},
\]

\[
z(t_0) = z_{t_0},
\]

where \( z_{t_0} = \int_{-\infty}^{t_0} e^{s(A_1+\beta)}dW(s) \). Let \( (v_{t_0}, d_{t_0}) \in H \times H^1 \), then in view of Theorem 2.1, \((v, d)\) is the unique global weak solution to (1)-(5) on \([t_0, \infty)\) with \( v(t_0) = v_{t_0} \) and \( d(t_0) = d_{t_0}(x) \). Making the classic change \((v, d) = (u+z, d)\), then \((u, d)\) satisfies the following system (21)-(25).

\[
u_t + ((u+z) \cdot \nabla (u+z) + \nabla p - \Delta u) + \nabla \cdot (\nabla d \odot \nabla d) = \beta z,
\] (21)

\[
\nabla \cdot u = 0,
\] (22)
Lemma 3.2. Denote by $\text{Absorbing ball of } d$ then it is almost surely uniformly bounded w.r.t. time $t$.

Furthermore, a way for finding the random attractor in the weak solution space $H$ obtain the absorbing balls for the solution in various function spaces (see (38)). The estimates of [5] and bounds in Proposition 5.4 of [7]. These uniform bounds also allow us to improve bounds obtained in Proposition C.1.

Taking inner product with (23) in $H$

Proof. Taking inner product with (23) in $H$ with $|d|^{4N}d$ we have

$$|d(t)|^{4N+2} = |d(t_0)|^{4N+2} - (4N + 2) \int_{t_0}^{t} \langle |d|^{4N}d, -\Delta d + (u + z) \cdot \nabla d + f(d) \rangle ds$$

$$= |d(t_0)|^{4N+2} - (4N + 2)(4N + 1) \int_{t_0}^{t} \int_{\Omega} |d|^{4N} |\nabla d|^2 dx ds$$

$$- (4N + 2) \int_{t_0}^{t} \langle |d|^{4N}d, f(d) \rangle ds,$$

which implies that

$$d|d(t)|^{4N+2} + (4N + 2)(4N + 1) \int_{\Omega} |d|^{4N} |\nabla d|^2 dx + (4N + 2) \langle |d|^{4N}d, f(d) \rangle = 0. \quad (27)$$

By Young’s inequality, for small positive constant $\varepsilon$ there exists a positive constant $c$ such that

$$|a_k| |d|^{2k+4N+2} \leq \frac{\varepsilon}{N} |d|^{6N+2} + \frac{c}{N} |d|^{4N+2}, \quad k = 0, 1, \ldots, N - 1.$$

Therefore,

$$\langle |d|^{4N}d, f(d) \rangle = \sum_{k=0}^{N} a_k |d|^{2k+4N+2}$$

$$\geq - \varepsilon |d|^{6N+2} - c|d|^{4N+2} + a_N |d|^{6N+2}. \quad (28)$$

Combining (27) and (28) yields

$$d|d(t)|^{4N+2} + c|d(t)|^{6N+2} dt \leq c|d(t)|^{4N+2} dt,$$
which implies
\[ d(|\mathbf{d}(t)|_{4N+2}^{4N+2} + 1) + c(|\mathbf{d}(t)|_{6N+2}^{6N+2} + 1) dt \leq c(|\mathbf{d}(t)|_{4N+2}^{4N+2} + 1) dt. \] (29)

Diving \(|\mathbf{d}(t)|_{4N+2}^{4N+2} + 1\) on both sides of (29) yields,
\[ \frac{d}{dt} \ln(|\mathbf{d}(t)|_{4N+2}^{4N+2} + 1) + c\ln(|\mathbf{d}(t)|_{4N+2}^{4N+2} + 1) \leq c. \] (30)

Since
\[ \ln(1 + |x|) \leq 1 + |x| \leq (1 + |x|)^{\frac{2N+1}{N}} \], for all \( x \in \mathbb{R} \), (31)

By (30)-(31) we have
\[ \frac{d}{dt} \ln(|\mathbf{d}(t)|_{4N+2}^{4N+2} + 1) + c \ln(|\mathbf{d}(t)|_{4N+2}^{4N+2} + 1) \leq c. \] (32)

Let \( y(t) = \ln(|\mathbf{d}(t)|_{4N+2}^{4N+2} + 1) \). Then multiplying \( e^{ct} \) on both sides yields
\[ \frac{d}{dt} (y(t) e^{ct}) \leq c e^{ct}, \]
which implies
\[ y(t) \leq y(t_0) e^{-c(t-t_0)} + \int_{t_0}^{t} e^{-c(t-s)} ds. \] (33)

By (33), \( y(t) \) in uniformly bounded with respect to time \( t \) and initial time \( t_0 \) provided the initial data \( y(t_0) \) is bounded. Therefore, this in turn implies the uniform boundedness of \( \mathbf{d} \) in \((L^{4N+2}(\mathbb{D}))^3\) with respect to time \( t \) and initial time \( t_0 \). (26) follows by the uniform boundedness of \( \mathbf{d} \) with respect to time and initial time in \((L^{4N+2}(\mathbb{D}))^3\). \( \square \)

3.2. Absorbing ball of \( \mathbf{d} \) in \( \mathbf{H} \times \mathbf{H}^1 \).

**Proposition 1.** There exists an absorbing ball for the weak solution \((\mathbf{v}, \mathbf{d})\) to (1)-(5) at any time \( t(\in \mathbb{R}) \) in \( \mathbf{H} \times \mathbf{H}^1 \).

**Proof.** Taking inner product of (23) with \( \Delta \mathbf{d} - f(\mathbf{d}) \) yields
\[ \frac{1}{2} \frac{d(|\nabla \mathbf{d}|^2_{L^2} + \int_{\mathbb{D}} F(|\mathbf{d}|^2) \, dx)}{dt} + |\Delta \mathbf{d} - f(\mathbf{d})|^2_{L^2} = ((\mathbf{u} + \mathbf{z}) \cdot \nabla \mathbf{d}, \Delta \mathbf{d} - f(\mathbf{d})). \] (34)

Taking inner product of (21) with \( \mathbf{u} \) in \( \mathbf{H} \) yields,
\[ \frac{1}{2} \frac{d(|\nabla \mathbf{u}|^2_{L^2})}{dt} + |\nabla \mathbf{u}|^2_{L^2} = \langle \mathbf{u} \cdot \nabla \mathbf{d} + \mathbf{z} \cdot \nabla \mathbf{z}, \mathbf{u} \rangle - \langle \nabla \cdot (\nabla \mathbf{d} \otimes \nabla \mathbf{d}), \mathbf{u} \rangle + \beta \langle \mathbf{z}, \mathbf{u} \rangle. \] (35)

Since by integration by parts and boundary conditions (24),
\[ \langle \mathbf{u} \cdot \nabla \mathbf{d}, \Delta \mathbf{d} \rangle = \int_{\mathbb{D}} \mathbf{u}^i \partial_{x_i} \mathbf{d}^k \partial_{x_j} \mathbf{d}^k \]
\[ = - \int_{\mathbb{D}} \partial_{x_j} \mathbf{u}^i \partial_{x_i} \mathbf{d}^k \partial_{x_j} \mathbf{d}^k \, d\mathbb{D} - \int_{\mathbb{D}} \mathbf{u}^i \partial_{x_i} \mathbf{d}^k \partial_{x_j} \mathbf{d}^k \, d\mathbb{D} \]
\[ = - \int_{\mathbb{D}} \partial_{x_j} \mathbf{u}^i \partial_{x_i} \mathbf{d}^k \partial_{x_j} \mathbf{d}^k \, d\mathbb{D}, \]

and
\[ -\langle \nabla \cdot (\nabla \mathbf{d} \otimes \nabla \mathbf{d}), \mathbf{u} \rangle = - \int_{\mathbb{D}} \partial_{x_i} (\partial_{x_i} \mathbf{d}^k \partial_{x_j} \mathbf{d}^k) \mathbf{u}^j \, d\mathbb{D} \]
\[ = \int_{\mathbb{D}} \partial_{x_i} \mathbf{d}^k \partial_{x_j} \mathbf{d}^k \partial_{x_i} \mathbf{u}^j \, d\mathbb{D}, \]
combing (34) and (35) together yields
\[ \frac{1}{2} \frac{d}{dt} (\|u\|_2^2 + \|\nabla d\|_2^2 + \int_D \tilde{F}(|d|^2) dx) + \|\nabla u\|_2^2 + \|\Delta d - f(d)\|_2^2 = (u \cdot \nabla z + z \cdot \nabla z, u) + (z \cdot \nabla d, \Delta d) \]
\[ \leq c(\|z\|_{\infty}^2 \|u\|_2^2 + c|z|_{\infty}^2 \|u\|_2^2 + c|\Delta d|_2^2 + c|z|_{\infty}^2 |\nabla d|_2^2. \quad (36) \]

Since by the Poincaré inequality and the Minkowski inequality
\[ c|\nabla d|_2^2 \leq \frac{1}{2} \|\Delta d|_2^2 \leq \|\Delta d - f(d)\|_2^2 + |f(d)|_2^2, \quad (37) \]
then in view of the Poincaré inequality, the Hölder inequality, the Sobolev imbedding theorem, (36)-(37) and Lemma 3.1, we have
\[ \frac{1}{2} \frac{d}{dt} (\|u\|_2^2 + \|\nabla d\|_2^2 + \int_D \tilde{F}(|d|^2) dx) + c|\nabla u|_2^2 + c|\Delta d|_2^2 + c(1 - \|z\|_1^2) \int_D \tilde{F}(|d|^2) dx \]
\[ \leq c|f(d)|_2^2 + c(1 - \|z\|_1^2) \int_D \tilde{F}(|d|^2) dx + c|z|_{\infty}^2 |\nabla d|_2^2 \]
\[ + |\nabla z|_{\infty} \|u\|_2^2 + |z|_{\infty}^2 |u|_2^2 + |\nabla z|_{\infty}^2 \]
\[ \leq c(\|z\|_1 + \|z\|_{\infty}^2) (|u|_2^2 + |\nabla d|_2^2) + c + c\|z\|_1^2. \quad (38) \]

Let \( g(t) = |u(t)|_2^2 + |\nabla d(t)|_2^2 + \int_D \tilde{F}(|d(t)|^2) dx, \) with \( t \in \mathbb{R}. \) Then by (38) we have
\[ \frac{d}{dt} g(t) + c(1 - \|z\|_1^2) g(t) \leq c + c\|z\|_1^2. \]

Therefore,
\[ \frac{d}{dt} \left( g(t) e^{\int_0^t c(1 - \|z(s)\|_1^2) ds} \right) \leq c(1 + \|z\|_1^2) e^{\int_0^t c(1 - \|z(s)\|_1^2) ds}, \]
which implies
\[ g(t) \leq g(t_0) e^{-\int_0^t c(1 - \|z(s)\|_1^2) ds} + c \int_{t_0}^t (\|z\|_1^2 + 1) e^{-\int_0^s c(1 - \|z(u)\|_1^2) ds} ds. \quad (39) \]

Since, the process \( z(t) \) (see (20)) is stationary and ergodic and \( \|z(t)\|_1 \) has polynomial growth when \( t \to -\infty, \) following the classic arguments (see [9, 11] and other references), (39) gives us the desired uniform estimate which yields an absorbing ball for \((u, d)\) in \( \mathbb{H} \times \mathbb{H}^1. \) Following the standard arguments, from (38) we can show that for any constant \( r(>0), \) \( t(\in \mathbb{R}) \) and given bounded initial data \((v_0, d_0)\)
\[ \int_{t-r}^t (|\nabla u(s)|_2^2 + |\Delta d(s)|_2^2) ds \text{ is uniformly bounded} \]
\[ \text{w.r.t. initial time } t_0(\leq t - r). \quad (40) \]

\[ \square \]

Remark 4. As we see from (38) that the uniform bounds for \( \tilde{F}(d) \) obtained in Lemma 3.2 play a vital role to obtain the absorbing ball for \((v, d)\) in \( \mathbb{H} \times \mathbb{H}^1. \)
3.3. The solution operator $S(t, s; \omega)$ to (1)-(5) is a stochastic dynamical system. In this part, we need to show the solution operator to (1)-(5) is indeed a stochastic dynamical system. For this purpose, we will introduce some notations below. Define $\tilde{e}_j = (e_j, 0, 0, 0)^T, j \in \mathbb{N}$, where the $(e_j)_{j \in \mathbb{N}}$ is an orthonormal basis of $H$ (see section 2). Let

$$W := \sum_{i=1}^{\infty} \lambda_i^{\frac{1}{2}} \tilde{e}_j B_i(t), \ t \in \mathbb{R},$$

where $\lambda_i$ and $B_i$ are illustrated in the introduction. Then $W$ is the two-sided $H \times H^1$-valued Wiener process, which has a version $\omega$ in $C_0(\mathbb{R}, H \times H^1) := \Omega$, the space of continuous functions which are zero at zero. In what follows we consider a canonical version of $W$ given by the probability space $(C_0(\mathbb{R}, H \times H^1), B(C_0(\mathbb{R}, H \times H^1)), \mathbb{P})$ where $\mathbb{P}$ is the Wiener-measure generated by $W$. Then

$$W(t, \omega) = \omega(t), \ t \in \mathbb{R}, \ \omega \in \Omega.$$ (41)

On this probability space we can also introduce the shift

$$\vartheta_{s,t}(\omega) = \omega(t + s) - \omega(s), \ s, t \in \mathbb{R}.$$ By Theorem 2.3, for $s \in \mathbb{R}$ and $(v_s, d_s) \in H \times H^1$ a.s., there exists a weak unique solution defined on $[s, \infty)$ to (1)−(5), $(v(t, \omega), d(t, \omega))$ such that

$$(v(s, \omega), d(s, \omega)) = (v_s, d_s), \ \mathbb{P} - a.s.$$ (42)

Define the solution operator $(S(t, s; \omega))_{t \geq s, \omega \in \Omega}$ to (1)-(5) by

$$S(t, s; \omega)(v_s, d_s) = (u(t, \omega) + z(t, \omega), d(t, \omega))$$ (43)

where $(u, d)$ is the unique weak solution to (21)-(25).

**Proposition 2.** The solution operator $(S(t, s; \omega))_{t \geq s, \omega \in \Omega}$ to (1)-(5) and $\vartheta_t, t \in \mathbb{R}$, satisfy properties (i)−(iv) and (a)−(c) in the preliminaries, respectively.

**Proof.** Denote

$$H := \left\{(v, d) \in (L^2(D))^2 \times H^1 : \nabla \cdot v = 0 \text{ in } D, v(t, x) = 0, d(t, x) = d_0(x), \right\}.$$ (44)

Define $P_H$ to be the Leray type projection operator from $(L^2(D))^2 \times H^1$ onto $H$. The principal linear portion of (1)-(5) is defined by

$$AU = P_H(\lambda_1 v, \lambda_2 d)^T,$$ (44)

where for a matrix $C$, $C^T$ stands for the transpose of $C$ and $U = (v, d) \in D(A) := D(A_1) \times D(A_2)$. Take, for $U \in D(A_1) \times H^1$, we define

$$B_1(U) = P_H(v \cdot \nabla v, f)^T, \ B_2(U) = P_H(\nabla \cdot (\nabla d \odot \nabla d), v \cdot \nabla d)^T.$$ (45)

Let $B(U) := B_1(U) + B_2(U)$. Collecting the operators defined above we reformulate (1)-(5) as the following abstract evolution system

$$dU + (AU + BU)dt = dW, \ U(0) = U_0.$$ (46)

Obviously, for $U_0 \in H \times H^1$ or $U_0 \in V \times H^1$, in view Theorem 2.3 there exists a unique weak solution or a unique strong solution to (46). The definitions of weak solution and strong solution to (46) are similar to Definition 2.1 and Definition 2.2, respectively. Moreover, for $U_0 \in H \times H^1$ or $U_0 \in V \times H^2$, the solution $U$ is Lipschitz
continuous with respect to \( U_0 \) in \( H \times \mathbb{H}^1 \) or \( V \times \mathbb{H}^2 \), respectively. For the later case we will give a proof in Proposition 4.

Then the conclusions of this proposition follow from the global existence, uniqueness and regularity of the solutions to (46).

\[ \square \]

3.4. Two a priori estaimes for the Aubin-Lions Lemma. To establish that the solution operator to (1)-(5) is compact in \( H \times \mathbb{H}^1 \), the first step is to use the Aubin-Lions lemma to obtain a convergent subsequence of \( (v, d) \) or equivalently a convergent subsequence of \( (u, d) \) which converges almost everywhere with respect to time \( t \in [s, T], s, T \in \mathbb{R} \) and \( s < T \), in \( H \times \mathbb{H}^1 \). Then we have to verify the two a priori estimates of Aubin-Lions lemma. The first one is to obtain a priori estimates about \( (u, A_{\mathbb{H}}^2 d) \) in \( L^2([s, T]; V \times \mathbb{H}^1) \), which is proved in the following Proposition 3. The other one is to obtain a priori estimates of \( (\frac{d}{dt} u,\frac{d}{dt} A_{\mathbb{H}}^2 d) \) in \( L^2([s, T]; H^{-1} \times \mathbb{H}^1) \), which is obtained in the Proposition 4.

**Proposition 3.** Let \( B = \{ (v_0, d_0) \in H \times \mathbb{H}^1 ||v_0||_2 + ||d_0||_1 \leq M \} \) for some positive constant \( M \), then \( \mathbb{P} \text{-a.s.} \)

\[
\sup_{(v_0, d_0) \in B} \left( \int_0^T ||u(t)||_2^2 dt + \int_0^T ||d(t)||_2^2 dt \right) < \infty. \tag{47}
\]

**Proof.** Proposition 3 follows directly from Lemma 3.2 and (38). \[ \square \]

**Proposition 4.** Let \( B = \{ (v_0, d_0) \in H \times \mathbb{H}^1 ||v_0||_2 + ||d_0||_1 \leq M \} \) for some positive constant \( M \), then \( \mathbb{P} \text{-a.s.} \)

\[
\sup_{(v_0, d_0) \in B} \left( \int_0^T \frac{d}{dt} u(t)||_2^2 dt + \int_0^T \frac{d}{dt} A_{\mathbb{H}}^2 d(t)||_2^2 dt \right) < \infty. \tag{48}
\]

**Proof.** For \( \eta \in \mathbb{H}^1 \), taking inner product of (23) with \( A_{\mathbb{H}}^2 \eta \) in \( H \) yields,

\[
\langle \frac{d}{dt} A_{\mathbb{H}}^2 d, \eta \rangle = \langle \frac{d}{dt} d, A_{\mathbb{H}}^2 \eta \rangle = \langle \frac{d}{dt} A_{\mathbb{H}}^2 \eta \rangle
\]

\[
= \langle dA_{\mathbb{H}}^2 d, A_{\mathbb{H}}^2 \eta \rangle - \langle u \cdot \nabla d, A_{\mathbb{H}}^2 \eta \rangle - \langle f(d), A_{\mathbb{H}}^2 \eta \rangle
\]

\[
\leq ||dA_{\mathbb{H}}^2 d||_2 ||\eta||_1 + ||u||_4 ||\nabla d||_4 ||\eta||_1 + ||f(d)||_2 ||\eta||_1
\]

\[
\leq ||dA_{\mathbb{H}}^2 d||_2 ||\eta||_1 + c||u||_2 ||\nabla d||_2 ||\eta||_1 + ||f(d)||_2 ||\eta||_1. \tag{49}
\]

By (49) and the Hölder inequality we have

\[
\left| \frac{d}{dt} A_{\mathbb{H}}^2 d \right|_2 \leq c||dA_{\mathbb{H}}^2 d||_2^2 + c||u||_2^2 ||\nabla d||_2^2 + c||f(d)||_2^2. \tag{50}
\]

where \( | \cdot |_{\mathbb{H}^{-m}} \) is the norm of Sobolev space \( \mathbb{H}^{-m} \) which is the dual space of \( \mathbb{H}^m \). Therefore, from Theorem 2.3 and (49) we conclude

\[
\frac{d}{dt} A_{\mathbb{H}}^2 d \text{ is bounded in } L^2([0, T]; H^{-1}). \tag{51}
\]

For \( \xi \in H^1 \), taking inner product of (21) with \( \xi \) in \( H \) yields,

\[
\langle \frac{d}{dt} u, \xi \rangle = \langle u \cdot \nabla u, \xi \rangle - \langle u \cdot \nabla z, \xi \rangle - \langle z \cdot \nabla u, \xi \rangle - \langle z \cdot \nabla z, \xi \rangle - \langle \Delta u, \xi \rangle + \int_D \partial_{x^i} d^k \partial_{x^j} d^k \partial_{x^i} \xi^j. \tag{52}
\]
By the incompressible property of the fluid (see (2)), the boundary condition (4) and integration by parts we obtain
\[-(u \cdot \nabla u_\xi) = \langle u \cdot \nabla \xi, u \rangle \leq |\nabla \xi|_2 |u|^2 \leq c||\xi||_1 |u_2||u|_1 \tag{53}\]

In view of (52) and (53) we have
\[\frac{d}{dt} \xi \leq c|u_2||u||\xi||_1 + |u_2|\nabla z|_4 |\xi|_4 + |\nabla u_2|z|_4 |\xi|_4
+ |u_2||u|\xi||_1 + c|u_2||z||\xi||_1 + c|u_1||\xi||_1
+ c||z||_1^1 + c||u_1||\xi||_1 + c||d_1||d_2||\xi||_1,\]

which implies
\[\frac{d}{dt} |H^{-1}| \leq c|u_2||u||^2 + ||u||^2||z||^2 + c||u||^2 + c||d_1||d_2||^2. \tag{54}\]

where $| \cdot |_{H^{-1}}$ is the norm of the Sobolev space $H^{-1}$ which is the dual space of $V$. Then Theorem 2.3, (15) and (54) imply
\[\frac{d}{dt} \text{ is bounded in } L^2([0,T];H^{-1}). \tag{55}\]

Therefore, (48) follows by (51) and (55).

3.5. The solution $S(t,s;\omega)$ operator to (1)-(5) is impact in $H \times H^1$. By virtue of Proposition 3, Proposition 4 and Lemma 2.8 with $V = V$ or $H^1$, $H = H$ or $H$ and $V' = H^{-1}$ or $H^{-1}$ we infer that

**Corollary 1.**
\[v \in C([0,T];H), \text{ and } d \in C([0,T];H^1), \text{ for arbitrary } T > 0, \text{ a.s..}\]

Then we will use Aubin-Lions lemma and Corollary 1 to show the following compactness result for the solution operators to (1)-(5).

**Proposition 5.** For $\omega \in \Omega$, $S(t,s;\omega)$ is compact from $H \times H^1$ to $H \times H^1$, for all $s,t \in \mathbb{R}$ and $s \leq t$.

**Proof.** In Proposition 1, we have obtained absorbing ball for $(S(t,s;\omega))_{t \geq s, \omega \in \Omega}$ at any time $t \in \mathbb{R}$. We denote by $B(s,r(\omega))$, the absorbing ball at time $s$ with center $0 \in H \times H^1$ and radius $r(\omega)$. Denote by $B$ a bounded subset $H \times H^1$ and set $\mathcal{C}_T$ as a subset of the function space:
\[\mathcal{C}_T := \left\{(v, A^1_d d) | (v(s), d(s)) \in B, (v(t), d(t)) = S(t,s;\omega)(v(s), d(s)), \right\},
\]

$t \in [s,T], s \leq T$.\]

Since both embedding $H^1 \subset H$ and embedding $H^1 \subset H$ are compact, then the embedding $H^1 \times H^1 \subset H \times H$ is also compact. For arbitrary $(v(s), d(s)) \in B$, by Proposition 2 and Proposition 3 we infer that
\[(u, A^1_d d) \text{ is bounded in } L^2([s,T];H^1 \times H^1)\]
and
\[(\partial_t u, \partial_t A^1_d d) \text{ is bounded in } L^2([s,T];H^{-1} \times H^{-1}).\]
Therefore, by Lemma 2.7 with
\[ B_0 = H^1 \times H^1, \quad B = H \times H, \quad B_1 = H^{-1} \times H^{-1}, \]
\( \mathcal{C}_T \) is compact in \( L^2([s, T]; H \times H) \). In order to show that for any fixed \( t \in (s, T], \omega \in \Omega, S(t, s; \omega) \) is a compact operator in \( H \times H^1 \), we take any bounded sequences \( \{(v_{0, n}, d_{0, n})\}_{n \in \mathbb{N}} \subset \mathcal{B} \) and we want to extract, for any fixed \( t \in (s, T] \) and \( \omega \in \Omega \), a convergent subsequence from \( \{S(t, s; \omega)(v_{0, n}, d_{0, n})\} \). Since \( \{(v, A_T^1d)\} \subset \mathcal{C}_T, \) by Lemma 2.7, there is a function \((v_*, d_*)\):
\[ (v_*, d_*) \in L^2([s, T]; H \times H^1), \]
and a subsequence of \( \{S(t, s; \omega)(v_{0, n}, d_{0, n})\}_{n \in \mathbb{N}}, \) still denoted by \( \{S(t, s; \omega)(v_{0, n}, d_{0, n})\}_{n \in \mathbb{N}} \) for simplicity, such that
\[ \lim_{n \to \infty} \int_s^T \|S(t, s; \omega)(v_{0, n}, d_{0, n}) - (v_*(t), d_*(t))\|_{H \times H^1}^2 dt = 0, \quad (56) \]
where \( \| \cdot \|_{H \times H^1} \) denotes the norm of the product space \( H \times H^1 \). By measure theory, convergence in mean square implies almost sure convergence of a subsequence. Therefore, it follows from (56) that there exists a subsequence of \( \{S(t, s; \omega)(v_{0, n}, d_{0, n})\}_{n \in \mathbb{N}}, \) still denoted by \( \{S(t, s; \omega)(v_{0, n}, d_{0, n})\}_{n \in \mathbb{N}} \) for simplicity, such that
\[ \lim_{n \to \infty} \|S(t, s; \omega)(v_{0, n}, d_{0, n}) - (v_*(t), d_*(t))\|_{H \times H^1} = 0, \quad a.e. \ t \in (s, T]. \quad (57) \]
Fix any \( t \in (s, T], \) by (57), we can select a \( t_2 \in (s, t) \) such that
\[ \lim_{n \to \infty} \|S(t_2, s; \omega)(v_{0, n}, d_{0, n}) - (v_*(t_2), d_*(t_2))\|_{H \times H^1} = 0. \]
Then by the continuity of the map \( S(t, t_2; \omega) \) in \( H \times H^1 \) with respect to initial value, we have
\[ S(t, s; (v_{0, n}, d_{0,n})) = S(t, s_0; \omega)S(t_2, s; \omega)(v_{0, n}, d_{0, n}) \rightarrow S(t, t_2; \omega)(v_*(t_2), d_*(t_2)), \quad \text{in } H \times H^1. \]
Hence for any \( t \in (s, T], \) \( \{S(t, s; \omega)(v_{0, n}, d_{0, n})\}_{n \in \mathbb{N}} \) contains a subsequence which is convergent in \( H \times H^1 \), which implies that for any fixed \( t \in (s, T], \omega \in \Omega, S(t, s; \omega) \) is a compact operator in \( H \times H^1 \).

3.6. The existence of compact absorbing ball in \( H \times H^1 \).

**Proposition 6.** There exists a compact absorbing ball at any time \( t \in \mathbb{R} \) for the stochastic dynamical system (1)-(5) in \( H \times H^1 \).

**Proof.** Using the notations given in Proposition 5, for \( s < t, \) let 
\[ \mathcal{B}(t, \omega) = S(t, s; \omega)B(s, r(\omega)) \]
be the closed set of \( S(t, s; \omega)B(s, r(\omega)) \) in \( H \times H^1 \). Then, by the above arguments, we infer \( \mathcal{B}(t, \omega) \) is a random compact set in \( H \times H^1 \) for each \( \omega \). More precisely, \( \mathcal{B}(t, \omega) \) is a compact absorbing set in \( H \times H^1 \) at time \( t \in \mathbb{R} \). Indeed, for \( (v_{0, n}, d_{0, n}) \in \mathcal{B}, \) there exists \( s_0(\mathcal{B}) \in \mathbb{R} \) such that if \( s_0 \leq s(\mathcal{B}) \), we have 
\[ S(t, s_0; \omega)(v_{0, n}, d_{0, n}) = S(t, s_0; \omega)(v_{0, n}, d_{0, n}) \subset S(t, s; \omega)B(s, r(\omega)) \subset \mathcal{B}(t, \omega). \]
Proposition 7. There exists an absorbing ball in $\text{Absorbing ball of the strong solution to (1)-(5)}$.

4.1. where the union is taken over all the bounded subsets of $\mathbb{V}$

Remark 5. By Proposition 4.5 in [10], the existence of a random attractor as constructed in the proof of Theorem 3.1 implies the existence of an invariant Markov measure $\mu \in \mathcal{P}_1(\mathbb{H} \times \mathbb{H}^1)$ for $\varphi$ (see Definition 4.1 in [10]). In view of [8] there exists an invariant measure $\mu$ for the Markovian semigroup $\mathbb{P}[f(S(t,0,x))]$ satisfying

$$\mu(B) = \int_{\Omega} \mu_\omega(B) \mathbb{P}(d\omega) \text{ and } \mu(A(\omega)) = 1, \; \mathbb{P} - \text{a.e.},$$

where $B \in \mathcal{B}(\mathbb{H} \times \mathbb{H}^1)$ which is a Borel $\sigma$-algebra on $\mathbb{H} \times \mathbb{H}^1$. If the invariant measure for $\mathbb{P}_t$ is unique, then the invariant Markov measure $\mu$ for $\varphi$ is unique and given by

$$\mu_\omega = \lim_{t \to \infty} \varphi(t, \vartheta_{-t}\omega) \mu.$$

4. Existence of random attractor in $\mathbb{V} \times \mathbb{H}^2$. The main goal of this section is to show the existence of random attractor in $\mathbb{V} \times \mathbb{H}^2$ which will be proved in four steps. Firstly, in subsection 4.1 we obtain the absorbing ball in $\mathbb{V} \times \mathbb{H}^2$ for the system $(1)-(5)$ in Proposition 3. Then, by careful and delicate a priori estimates we verify the two a priori estimates of Aubin-Lions Lemma in Proposition 3 and Proposition 4 of subsection 4.2. Thirdly, in the next subsection we establish the continuity of the strong solution in $\mathbb{V} \times \mathbb{H}^2$ with respect to time and initial data improving the corresponding continuity of the strong solutions in [5]. Finally, in the last section, with Aubin-Lions Lemma and regularity of strong solutions to $(1)-(5)$, following the same line of Proposition 5 and Proposition 6, the compact property of the solution operators in $\mathbb{V} \times \mathbb{H}^2$ and the existence of a compact absorbing ball in $\mathbb{V} \times \mathbb{H}^2$ hold, see Proposition 5 and Proposition 6 respectively. Theorem 4.1 one of our main result of this article is the direct result of Theorem 2.6 and Proposition 6.

Theorem 4.1. Let $v_0 \in \mathbb{V}, d_0 \in \mathbb{H}^2$ in (4) and $f(d)$ is given by (12). Assume (15) hold. Then the solution operator $(S(t,s,\omega))_{t \geq s, \omega \in \Omega}$ of $(1)-(5)$ : $S(t,s,\omega)(v_\epsilon d_\epsilon) = (v(t), d(t))$ has properties (i) - (iv) of Theorem 2.6 and possesses a compact absorbing ball $B(0,\omega)$ in $\mathbb{V} \times \mathbb{H}^2$ at time 0. Furthermore, for $\mathbb{P}$-a.e. $\omega$, the set

$$A(\omega) = \bigcup_{B \subset \mathbb{V} \times \mathbb{H}^2} A(B,\omega)$$

where the union is taken over all the bounded subsets of $\mathbb{V} \times \mathbb{H}^2$ is the random attractor of $(1)-(5)$ and possesses the properties $(1)-(7)$ of Theorem 2.6 with space $X$ replaced by space $\mathbb{V} \times \mathbb{H}^2$.

4.1. Absorbing ball of the strong solution to (1)-(5).

Proposition 7. There exists an absorbing ball in $\mathbb{V} \times \mathbb{H}^2$ at time $-1$ for the strong solutions $(v, d)$ to $(1)-(5)$.

Remark 6. In fact, we can prove that there exists an absorbing ball in $\mathbb{V} \times \mathbb{H}^2$ at any time. But here, to prove the Theorem 4.1, we only need the existence of an absorbing ball in $\mathbb{V} \times \mathbb{H}^2$ at some time $t$, i.e. $t = -1$.

Proof. Taking inner product of (14) with $\Delta u$ in $\mathbb{H}$ yields

$$\frac{d}{dt} |\nabla u|^2 + |\Delta u|^2 - \langle (u + z) \cdot \nabla (u + z), \Delta u \rangle = \langle \nabla \cdot (\nabla d \odot \nabla d), \Delta u \rangle + \beta(z, \Delta u).$$  (58)
Since it is given in [22] that
\[ \langle \nabla \cdot (\nabla d \odot \nabla d), \Delta u \rangle = \langle \nabla d \Delta d, \Delta u \rangle + \langle \nabla (\frac{|\nabla d|^2}{2}), \Delta u \rangle = \langle \nabla d \Delta d, \Delta u \rangle \]
\[ = \langle \nabla d (\Delta d - f(d)), \Delta u \rangle + \langle \nabla d f, \Delta u \rangle, \]  \hspace{1cm} (59)
where the second equality follows by (22). Taking Fréchet derivative with respect to \(|\Delta d - f(d)|^2\) we have
\[ \frac{d}{dt} |\Delta d - f(d)|^2 = 2 \langle \Delta d - f(d), \Delta d_t - f'(d) d_t \rangle \]
\[ = -2 |\nabla (\Delta d - f(d))|^2 + \int_D (\Delta d - f(d)) (-\nabla (\Delta d - f(d)) (\Delta d - f(d) - (u + z) \cdot \nabla d) dx \]
\[ + 2 \int_D (\Delta d - f(d)) (-\Delta (u + z) \nabla d - (u + z) \nabla (\Delta d) - 2 \nabla (u + z) (\nabla^2 d)) dx. \]  \hspace{1cm} (60)
By integration by parts, we obtain
\[ \int_D (\Delta d - f(d)) (u + z) \nabla (\Delta d) dx \]
\[ = \int_D (\Delta d - f(d)) (u + z) \nabla (\Delta d - f(d)) dx + \int_D (\Delta d - f(d)) (u + z) \nabla f(d) dx \]
\[ = \int_D (\Delta d - f(d)) (u + z) \nabla f(d) dx \]
\[ = \int_D (\Delta d - f(d)) (u + z) f'(d) \nabla d dx \]  \hspace{1cm} (61)
and
\[ \int_D (\Delta d - f(d)) \nabla (u + z) \nabla^2 d dx = - \int_D (\Delta d - f(d))_{x_j} (u_{x_k}^j + z_{x_k}^j) d_{x_k} dx. \]  \hspace{1cm} (62)
In view of (48)-(62) we have
\[ \frac{1}{2} \frac{d}{dt} (|\nabla u|^2 + |\Delta d - f(d)|^2) + |\Delta u|^2 + \langle \nabla (\Delta d - f(d)), \Delta d \rangle \]
\[ = \langle (u + z) \cdot \nabla (u + z), \Delta u \rangle + \langle \nabla d, f(d) \Delta u \rangle \]
\[ + \langle \nabla d (\Delta d - f(d)), \Delta u \rangle + \beta (z, \Delta u) - \int_D (\Delta d - f(d)) f'(d) (\Delta d - f(d)) dx \]
\[ + \int_D (\Delta d - f(d)) f'(d) (u + z) \cdot \nabla d dx - \int_D (\Delta d - f(d)) (u + z) \nabla d dx \]
\[ - \int_D (\Delta d - f(d)) f'(d) (u + z) \cdot \nabla d dx - 2 \int_D (\Delta d - f(d))_{x_j} (u + z)_{x_k}^j d_{x_k} dx \]
\[ = \langle (u + z) \cdot \nabla (u + z), \Delta u \rangle + \beta (z, \Delta u) + (\nabla d, f(d) \Delta u) \]
\[ - \int_D (\Delta d - f(d)) f'(d) (\Delta d - f(d)) dx - \int_D (\Delta d - f(d)) (\Delta u) \]
\[ - 2 \int_D (\Delta d - f(d))_{x_j} (u_{x_k}^j + z_{x_k}^j) d_{x_k} dx. \]  \hspace{1cm} (63)
By the Hölder inequality, the interpolation inequality and Young’s inequality we have

\[ \langle (u + z) \cdot \nabla (u + z), \Delta u \rangle \leq c|\Delta u|_2 |\Delta(u + z)|_2^{\frac{1}{2}} |\nabla(u + z)|_2^{\frac{1}{2}} |u + z|_2^{\frac{1}{2}} |\nabla(u + z)|_2^{\frac{1}{2}} \leq c|\Delta u|_2 + c|\Delta z|_2^{\frac{1}{2}} + c|u + z|_2^{\frac{1}{2}} |\nabla(u + z)|_2^{\frac{1}{2}} \leq c|\Delta u|_2^{\frac{1}{2}} + c|\Delta z|_2^{\frac{1}{2}} + c|u + z|_2^{\frac{1}{2}}(|\nabla u|_2^2 + |\nabla z|_2^2). \]  

(64)

In view of Young’s inequality we have

\[ \beta(z, \Delta u) \leq c|\Delta u|_2^2 + c|z|_2^2. \]  

(65)

By integration by parts,

\[ \langle (\nabla d)f(d), \Delta u \rangle = 0. \]  

(66)

By the Hölder inequality and the Sobolev inequality,

\[ \int_D (\Delta d - f(d))f'(d)(\Delta d - f(d))dx \leq c|f'(d)|_\infty |\Delta d - f(d)|_2^2 \leq c|d|_1^2(N + 1)|\Delta d - f(d)|_2^2. \]  

(67)

Taking a similar argument as in (64) yields,

\[ -\int_D (\Delta d - f(d))\Delta z \nabla d dx \leq |\Delta d - f(d)|_2 |\Delta z|_2 |\nabla d|_2 \leq |\Delta d - f(d)|_2 |\Delta z|_2^{\frac{1}{2}} |\nabla d|_2^{\frac{1}{2}} |\Delta d|_2^{\frac{1}{2}} \leq c|\Delta d - f(d)|_2^2 + c|\Delta d|_2^2 + c|d|_3^2 |\nabla d|_2^2 \leq c|\Delta d - f(d)|_2^2 + c|d|_3^2 |\nabla d|_2^2. \]  

(68)

Similarly,

\[ -\int_D (\Delta d - f(d))_x (u_x^k + z_{x_k}) dx \leq |\nabla(\Delta d - f(d))_x (u + z)|_2 |\nabla d|_2 \leq c|\nabla(\Delta d - f(d))_x (u + z)|_2 + c|\nabla u|_2^2 + c|\nabla z|_2^2 + c(u + z)_x^2 |\nabla d|_2^2 |\Delta d - f(d)|_2^2 \leq c|\nabla u|_2^2 + c|\nabla z|_2^2 + c|d|_3^2 |\nabla d|_2^2 + c(u + z)_x^2 |\nabla d|_2^2 |\Delta d - f(d)|_2^2. \]  

(69)

By virtue of (63)-(69), we reach

\[ \frac{d}{dt}(|\nabla u|_2^2 + |\Delta d - f(d)|_2^2) \leq c|\Delta z|_2^2 + c|u + z|_2^2 |\nabla(u + z)|_2^2 + c|d|_4^2(N + 1)|\Delta d - f(d)|_2^2 \leq c|\nabla u|_2^2 + c(u + z)_x^2 |\nabla d|_2^2 + c(u + z)_x^2 |\nabla d|_2^2 + c(u + z)_x^2 |\nabla d|_2^2 + c(u + z)_x^2 |\nabla d|_2^2 + c(u + z)_x^2 |\nabla d|_2^2. \]  

(70)

Integrating (70) on interval \([t_0, 1]\) yields,

\[ |\nabla u(-1)|_2^2 + |\Delta d(-1) - f(d)(-1)|_2^2 \leq |\nabla u(t_0)|_2^2 + |\Delta d(t_0) - f(d)(t_0)|_2^2 + c \int_{t_0}^1 (|\Delta z|_2^2 + c|u + z|_2^2 (|\nabla u|_2^2 + |\nabla z|_2^2) + c|d|_4^2(N + 1)|\Delta d - f(d)|_2^2)ds \]

\[ + c \int_{t_0}^1 |z|_3^2 |\nabla d|_2^2 + c(u + z)_x^2 |\nabla d|_2^2 + c(u + z)_x^2 |\nabla d|_2^2 ds. \]  

(71)
By Gronwall inequality, we obtain
\[ |\nabla u(-1)|^2 + |\Delta d(-1) - f(d)(-1)|^2 \leq \left( |\nabla u(t_0)|^2 + |\Delta d(t_0) - f(d)(t_0)|^2 \right. \]
\[ + \int_{t_0}^{-1} \left( |\Delta z|^2 + c|u + z||\nabla z|^2 + |z| |\nabla d|^2 + (|u + z||\nabla d|^2 + 1)|f(d)|^2 ds \right) \]
\[ \times e^{c_0^{-1} \int_{t_0}^{-1} |d|^2 dt + |u + z||\nabla u|^2 ds}. \quad (72) \]

Taking integration of (72) over \([-2, -1]\) yields,
\[ |\nabla u(-1)|^2 + |\Delta d(-1) - f(d)(-1)|^2 \leq \left( \int_{-1}^{-2} \left( |\nabla u(t_0)|^2 + |\Delta d(t_0) - f(d)(t_0)|^2 \right) dt_0 \right. \]
\[ + \int_{-1}^{-2} \left( |\Delta z|^2 + c|u + z||\nabla z|^2 + |z| |\nabla d|^2 + (|u + z||\nabla d|^2 + 1)|f(d)|^2 ds dt_0 \right) \]
\[ \times e^{c_0^{-1} \int_{-1}^{-2} |d|^2 dt + |u + z||\nabla u|^2 ds}. \quad (73) \]

Since we have shown in section 3 that \((u, d)\) is uniformly bounded in \(H \times \mathbb{H}^1\), with respect to initial time, then (40) and (73) imply the uniform boundedness of \((\nabla u(-1), \Delta d(-1))\) in \(H \times \mathbb{H}^1\). Therefore, we obtain the absorbing ball for \((u, d)\) in \(V \times \mathbb{H}^2\) at time \(t = -1\).

4.2. Two a priori estimates for the Aubin-Lions Lemma.

**Proposition 8.** Let \(B = \{(v_0, d_0) \in V \times \mathbb{H}^2 | \|v_0\|_1 + \|d_0\|_2 \leq M\}\) for some positive constant \(M\), then \(\mathbb{P}\)-a.s.

\[ \sup_{(v_0, d_0) \in B} \left( \int_0^T \|u(t)\|_2^2 dt + \int_0^T \|d(t)\|_3^2 dt \right) < \infty. \quad (74) \]

**Proof.** Proposition 8 follows directly from Theorem 2.3 and (70). \(\square\)

**Proposition 9.** Let \(B = \{(v_0, d_0) \in H \times \mathbb{H}^1 | \|v_0\|_1 + \|d_0\|_2 \leq M\}\) for some positive constant \(M\), then \(\mathbb{P}\)-a.s.

\[ \sup_{(v_0, d_0) \in B} \left( \int_0^T \frac{d}{dt} u(t) \|_{\mathbb{H}^1}^2 dt + \int_0^T \frac{d}{dt} A_2 d(t) \|_{\mathbb{H}^1}^2 dt \right) < \infty. \quad (75) \]

**Proof.** For \(\eta \in \mathbb{H}^2\), taking inner product of (23) with \(A_2 \eta\) in \(\mathbb{H}\) yields,
\[ \langle \frac{d}{dt} A_2 d, \eta \rangle = \langle \frac{d}{dt} d, A_2 \eta \rangle = \langle \Delta d, A_2 \eta \rangle - \langle (u + z) \cdot \nabla d, A_2 \eta \rangle - \langle f(d), A_2 \eta \rangle \]
\[ \leq \|\Delta d\|_1 \|\eta\|_1 + |\nabla (u + z)|_4 \|\nabla d\|_4 \|\eta\|_1 \]
\[ + |(u + z)|_4 \|\nabla d\|_4 \|\eta\|_1 + |\nabla f(d)|_2 \|\eta\|_1 \]
\[ \leq \|d\|_3 \|\eta\|_1 + c|u + z|_2 \|d\|_2 \|\eta\|_1 \]
\[ + |u + z|_3 \|d\|_3 \|\eta\|_1 + (1 + \|d\|_2^{2N+1}) \|\eta\|_1. \quad (76) \]

Given the bounded initial data \((v_0, d_0) \in H \times \mathbb{H}^2\), by (74) we obtain that
\[ \frac{d}{dt} A_2 d \text{ is bounded in } L^2([0, T]; \mathbb{H}^{-1}). \quad (77) \]
For $\xi \in H^1$, taking inner product of (21) with $\xi$ in $H$ yields,

$$\langle \frac{d}{dt}A^\frac{1}{2}u, \xi \rangle = \langle \frac{du}{dt}, A^\frac{1}{2}\xi \rangle$$

$$= - (u \cdot \nabla u, A^\frac{1}{2}\xi) - (u \cdot \nabla z, A^\frac{1}{2}\xi) - (z \cdot \nabla u, A^\frac{1}{2}\xi)$$

$$- (z \cdot \nabla z, A^\frac{1}{2}\xi) - \langle \Delta u, A^\frac{1}{2}\xi \rangle + \int_{\Gamma} \partial_x d^3 \partial_x d^3 \partial_x A^\frac{1}{2} \xi^j$$

$$\leq |\nabla u|_4 |u|_4 \xi + |u|_4 |\nabla z|_4 \xi + |\nabla u|_4 |z|_4 \xi + |\Delta u|_2 \xi + c|d||\xi|_1$$

$$
\leq c|u|_1 |u|_2 \xi + |u|_1 \xi + c|d|_2 \xi.$$  \hspace{1cm} (78)

Given the bounded initial data $(v_0, d_0) \in H^1 \times H^2$, by (76) we obtain that

$$\frac{d}{dt}A^\frac{1}{2}u \text{ is bounded in } L^2([0, T]; H^{-1}).$$  \hspace{1cm} (79)

4.3. Continuity of the strong solutions in $V \times H^2$ w. r. t. time and initial data. By virtue of Proposition 8, Proposition 9 and Lemma 2.8 with $V = V$ or $V' = H^{-1}$ or $H$ we infer that

**Corollary 2.**

$v \in C([0, T]; V)$, and $d \in C([0, T]; H^2)$, for arbitrary $T > 0$, a.s.

**Proposition 10.** Given $(v_0, d_0) \in V \times H^2$, the unique strong solution $(v, d)$ to (1)-(5) is Lipschitz continuous in $V \times H^2$ with respect to the initial data $(v_0, d_0)$.

**Proof.** Let $(v_1, p_1, d_1)$ and $(v_2, p_2, d_2)$ be two strong solutions to (1)-(5) with the initial data $(v_{0,1}, d_{0,1})$ and $(v_{0,2}, d_{0,2})$ in $V \times H^2$ respectively. Denote $v := v_1 - v_2$, $d := d_1 - d_2$ and $p := p_1 - p_2$. Then $(v, p, d)$ satisfies the following equations

$$v + v \cdot \nabla v + v_2 \cdot \nabla v - \nabla v + \nabla p$$

$$+ \nabla \cdot (\nabla d_1 \circ \nabla d_1) - \nabla \cdot (\nabla d_2 \circ \nabla d_2) = 0,$$  \hspace{1cm} (80)

$$\nabla \cdot v(t) = 0,$$  \hspace{1cm} (81)

$$\nabla \cdot d_1 + v_2 \cdot \nabla d_1 - \nabla d_1 + f(d_1) - f(d_2) = 0,$$  \hspace{1cm} (82)

$$v(x, t) = 0, \quad d(x, t) = d_0(x) := d_{1,0} - d_{2,0} (x), \quad (x, t) \in \Gamma \times [t_0, \infty),$$  \hspace{1cm} (83)

$$v|_{t=0} = v_0(x) = v_{0,1}(x) - v_{0,2}(x) \text{ with } \nabla \cdot v_0 = 0, \quad d|_{t=0} = d_0(x), x \in D.$$  \hspace{1cm} (84)

Taking inner product of (80) with $-\Delta v$ in $H$ yields,

$$\frac{d}{dt} |\nabla v|^2 + |\Delta v|^2 = (v \cdot \nabla v_1, \Delta v) + (v_2 \cdot \nabla v, \Delta v)$$

$$+ \langle \nabla \cdot (\nabla d_1 \circ \nabla d_1) - \nabla \cdot (\nabla d_2 \circ \nabla d_1), \Delta v \rangle.$$  \hspace{1cm} (85)

Taking a similar argument as in (59) yields,

$$\langle \nabla \cdot (\nabla d_1 \circ \nabla d_1) - \nabla \cdot (\nabla d_1 \circ \nabla d_1), \Delta v \rangle = (\nabla d_1 \Delta d_1 - \nabla d_2 \Delta d_2, \Delta v).$$  \hspace{1cm} (86)

By the Hölder inequality, the interpolation inequality and Young’s inequality we have

$$\frac{d}{dt} |\nabla v|^2 + |\Delta v|^2 \leq |\Delta v|_2 |\nabla v|_4 |v|_4 + |\Delta v|_2 |\nabla v|_4 |v_2|_4$$
Taking a similar argument as in (60) yields,
\[
\frac{1}{2} \frac{d|\Delta d|^2}{dt} = \langle \Delta d, \Delta d \rangle
\]
\[
= \langle \Delta d, \Delta^2 d \rangle + \langle \Delta v \nabla d_1, \Delta d \rangle + \langle v \nabla \Delta d_1, \Delta d \rangle + 2 \int_D \nabla v \nabla^2 d_1 \Delta d dx
\]
\[
+ \langle \Delta v_2 \nabla d, \Delta d \rangle + \langle v_2 \nabla \Delta d, \Delta d \rangle + 2 \int_D \nabla v_2 \nabla^2 d \Delta d dx
\]
\[
+ \langle \Delta(f(d_1) - f(d_2)), \Delta d \rangle. \tag{88}
\]
Since by integration by parts, the Hölder inequality and the interpolation inequality we have
\[
\langle \Delta(f(d_1) - f(d_2)), \Delta d \rangle
\]
\[
= \langle \nabla(f(d_1) - f(d_2)), \nabla \Delta d \rangle
\]
\[
\leq \varepsilon |\nabla \Delta d|_2 (|\nabla f(d_1)|_2^2 + |\nabla f(d_2)|_2^2)
\]
\[
\leq \varepsilon |\nabla \Delta d|^2_2 + c \left( (|d_1|_1^N + 1)|d_1|_2^2 + (|d_2|_1^N + 1)|d_2|_2^2 \right)
\]
\[
\leq \varepsilon |\nabla \Delta d|^2_2 + c (|d_1|^N_2 + |d_1|^2_2 + |d_2|^N_2 + |d_2|^2_2). \tag{89}
\]
By the Sobolev imbedding theorem and Young’s inequality we have
\[
2 \int_D \nabla v \nabla^2 d_1 \Delta d dx + 2 \int_D \nabla v_2 \nabla^2 d \Delta d dx
\]
\[
\leq \varepsilon |\nabla v|_\infty |d_1|_2 |\Delta d|_2 + c |\nabla v_2|_\infty |\Delta d|_2^2
\]
\[
\leq \varepsilon (|\Delta v|^2_2 + |\nabla v|^2_2) + c (|d_1|^2_2 + |d_2|_2^2) |\Delta d|^2_2. \tag{90}
\]
In view of (88)-(90), the Hölder inequality, the interpolation inequality and Young’s inequality we obtain
\[
\frac{1}{2} \frac{d|\Delta d|^2}{dt} + |\nabla \Delta d|^2_2 \leq |\Delta v|_2 |\nabla d_1|_4 |\Delta d|_4 + |\nabla \Delta d_1|_2 |\nabla d_1|_4 |
\]
\[
\quad + |\Delta v_2|_2 |\nabla d_1|_4 |\Delta d|_4 + |\nabla \Delta d_2|_2 |\nabla d_2|_4 |
\]
\[
\quad + |\Delta d_2|_2 |\nabla d_2|_4 |
\]
\[
\quad + c (|d_1|^N_2 + |d_1|^2_2 + |d_2|^N_2 + |d_2|^2_2)
\]
\[
\quad + \varepsilon (|\Delta v|^2_2 + |\nabla v|^2_2) + c (|d_1|^2_2 + |d_2|_2^2) |\Delta d|^2_2
\]
\[
\leq \varepsilon (|\Delta v|^2_2 + |\nabla v|^2_2) + c (|d_1|^2_2 + |d_2|_2^2) |\Delta d|^2_2.
\]
\[
\leq \varepsilon (|\Delta v|^2_2 + |\nabla v|^2_2) + c (|d_1|^2_2 + |d_2|_2^2) |\Delta d|^2_2.
\]
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For minor revisions, we have the following conclusions.

Remark 7. By Proposition 4.5 in [10], the existence of a random attractor as constructed in the proof of Theorem 4.1 implies the existence of an invariant Markov measure $\mu \in \mathcal{P}(V \times \mathbb{H}^2)$ for $\varphi$ (see Definition 4.1 in [10]). In view of [8] there exists an invariant measure $\mu$ for the Markovian semigroup $P_t[f(S(t,0,x))]$ satisfying

$$\mu(B) = \int_B \mu_\omega(B) P_t(d\omega), \text{ and } \mu(A(\omega)) = 1, P - a.e.,$$

where $A$ is given by Theorem 4.1 and $B \in \mathcal{B}(V \times \mathbb{H}^2)$ which is a Borel $\sigma$-algebra on $V \times \mathbb{H}^2$. If the invariant measure for $P_t$ is unique, then the invariant Markov measure $\mu$ for $\varphi$ is unique and given by

$$\mu_\omega = \lim_{t \to \infty} \varphi(t, \vartheta_{-t}\omega)\mu.$$

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