RTT relations, a modified braid equation and noncommutative planes

A.Chakrabarti

Centre de Physique Théorique, Ecole Polytechnique, 91128 Palaiseau Cedex, France.

Abstract

With the known group relations for the elements \((a, b, c, d)\) of a quantum matrix \(T\) as input a general solution of the RTT relations is sought without imposing the Yang-Baxter constraint for \(R\) or the braid equation for \(\hat{R} = P R\). For three biparametric deformations, \(GL(p,q)(2), GL(g,h)(2)\) and \(GL(q,h)(1/1)\), the standard, the nonstandard and the hybrid one respectively, \(R\) or \(\hat{R}\) is found to depend, apart from the two parameters defining the deformation in question, on an extra free parameter \(K\), such that

\[
\hat{R}_{(12)}\hat{R}_{(23)} - \hat{R}_{(23)}\hat{R}_{(12)}\hat{R}_{(23)} = \left(\frac{K}{K_1} - 1\right)\left(\frac{K}{K_2} - 1\right)\left(\hat{R}_{(23)} - \hat{R}_{(12)}\right)
\]

with \((K_1, K_2) = (1, 1), (1, 1)\) and \((1, 1)\) respectively. Only for \(K = K_1\) or \(K = K_2\) one has the braid equation. Arbitray \(K\) corresponds to a class (conserving the group relations independent of \(K\)) of the MQYBE or modified quantum YB equations studied by Gerstenhaber, Giaquinto and Schak. Various properties of the triparametric \(\hat{R}(K; p, q), \hat{R}(K; g, h)\) and \(\hat{R}(K; q, h)\) are studied. In the larger space of the modified braid equation (MBE) even \(\hat{R}(K; p, q)\) can satisfy \(\hat{R}^2 = 1\) outside braid equation (BE) subspace. A generalized, \(K\)-dependent, Hecke condition is satisfied by each 3-parameter \(\hat{R}\). The role of \(K\) in noncommutative geometries of the \((K; p, q), (K; g, h)\) and \((K; q, h)\) deformed planes is studied. \(K\) is found to introduce a “soft symmetry breaking”, preserving most interesting properties and leading to new interesting ones. Further aspects to be explored are indicated.
1 Introduction

Our starting point will be the group relations of the elements of the quantum matrix

\[ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]  

(1.1)

Three known cases will be considered:

1. The biparametric \((p, q)\) or standard deformation of \(GL(2)\)

2. The biparametric \((g, h)\) or nonstandard deformation of \(Gl(2)\)

3. The \((q, h)\) or ”hybrid” deformation of\(GL(1/1)\).

Each set will be presented explicitly below. These three have been studied in [1] where many original sources can be found cited. We start directly with the biparametric cases since the one - parameter deformations can then be systematically obtained through suitable constraints ( \(p = q^{-1}, g = h\) and so on ).

For the given group relations we construct for each case the matrix \(R\) satisfying

\[ RT_1T_2 = T_2T_1R \]  

(1.2)

where

\[ T_1 = T \otimes I_2, T_2 = I_2 \otimes T \]

To start with, we do not require \(R\) to satisfy the Yang - Baxter equation \((YB)\). It will be found that, apart from the parameters concerned ( \((p, q)\), \((g, h)\)or \((q, h)\) ) the solution for \(R\) satisfying (2) contains a supplementary arbitrary parameter \(K\). Two particular values of \(K\) ( say \(K_1\) and \(K_2\) ) will give the two solutions of \((YB)\) related through

\[ ((21)R(K_1))^{-1} = R(K_2) \]  

(1.3)

both satisfying

\[ R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} = 0 \]

The existence of such a pair of solutions is assured by the fact that (1.2) can be written as

\[ T_1T_2R^{-1} = R^{-1}T_2T_1 \]

The germ of our paper is in the question: what structure is obtained when \(K\) is not restricted to the values \(K_1\) and \(K_2\).

For arbitrary \(K\) the situation is best expressed in terms of

\[ \hat{R} = PR \]

where \(P\) is the permutation matrix and for our 4 × 4 case it permutes the second and the third rows.
One obtains, for the normalizations we will choose,

\[
\hat{R}_{(12)} \hat{R}_{(23)} \hat{R}_{(12)} - \hat{R}_{(23)} \hat{R}_{(12)} \hat{R}_{(23)} = \left( \frac{K}{K_1} - 1 \right) \left( \frac{K}{K_2} - 1 \right) \left( \hat{R}_{(23)} - \hat{R}_{(12)} \right)
\]

(1.4)

This is our modified braid equation (MBE). (See Discussion for further comments.)

In terms of \( R \) one obtains

\[
R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} = \left( \frac{K}{K_1} - 1 \right) \left( \frac{K}{K_2} - 1 \right) \left( (123)R_{(12)} - (213)R_{(23)} \right)
\]

(1.5)

where \((123)\) and \((213)\) denote corresponding permutations of the tensor factors of \( V^\otimes 3 \) (\( R \) acting on \( V \otimes V \)). [Having pointed out the structure (1.5) we will use throughout (1.4) as our fundamental relation.]

Thus (1.2) by itself is seen to lead to a particular class of solutions of the ”modified quantum Yang - Baxter equations” (MQYBE) introduced by Gerstenhaber et al. [2]. Our (1.4) has the same structure as the equation (2.4) of [3] for ”quantum transpositions” (\( \sigma_{12}, \sigma_{23} \)) defined by the authors, though we do not impose in general their ”unitarity” leading to

\[
\hat{R}^2 = I
\]

An example of a solution of (1.2) with an arbitrary \( K \) can be found in [1].

We present below some particularly interesting explicit examples. Their properties will reveal that the existence of such a class of more general solutions of (MBE) is more than an accident and can play a significant role in various domains, such as noncommutative geometry.

2 Explicit solutions

2.1 Standard \((p, q)\) deformation of \( GL(2) \)

The elements \((a, b, c, d)\) of \( T \) satisfy

\[
ab = qba, \quad pac = ca,
\]

\[
ad = da + (q - p)bc, \quad pqbc = cb, \quad (2.6)
\]

\[
pbd = db, \quad cd = qdc.
\]

Apart from a possible normalizing factor, the solution of (1.2) turns out to be (writing directly \( \hat{R} = PR \) and assuming \( p \) to be nonzero)

\[
\hat{R}(K; p, q) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & (1 - K) & \frac{K}{p} & 0 \\
0 & Kq & (1 - \frac{Kq}{p}) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(2.7)
This is found to satisfy
\[ \hat{R}_{(12)} \hat{R}_{(23)} \hat{R}_{(12)} - \hat{R}_{(23)} \hat{R}_{(12)} \hat{R}_{(23)} = \left( K - 1 \right) \left( \frac{Kq}{p} - 1 \right) \left( \hat{R}_{(23)} - \hat{R}_{(12)} \right) \] (2.8)
with
\[ K_1 = 1, \quad K_2 = \frac{p}{q} \]

2.2 Nonstandard \((g, h)\) deformation of GL(2)

The group relations are,
\[ ca = ac - gc^2, \quad cb = bc - gdc - hac + ghc^2, \]
\[ cd = dc - hc^2, \quad da = ad - gdc + hac, \]
\[ db = bd + g(ad - bc + hac - d^2) \]
\[ ba = ab - h(ad - bc + hac - a^2) \]

From (1.2) one obtains
\[ \hat{R}(K; g, h) = \begin{pmatrix} 1 & -hK & hK & ghK \\ 0 & (1 - K) & K & gK \\ 0 & K & (1 - K) & -gK \\ 0 & 0 & 0 & 1 \end{pmatrix}. \] (2.10)

This is found to satisfy
\[ \hat{R}_{(12)} \hat{R}_{(23)} \hat{R}_{(12)} - \hat{R}_{(23)} \hat{R}_{(12)} \hat{R}_{(23)} = \left( K - 1 \right)^2 \left( \hat{R}_{(23)} - \hat{R}_{(12)} \right) \] (2.11)
with, independently of \((g, h),\)
\[ K_1 = K_2 = 1 \]

2.3 Hybrid \((q, h)\) deformation of G(1/1)

The group relations are
\[ ba = ab + hcd, \quad ac = qca, \quad bc = qcb, \]
\[ dc + qcd = 0, \quad ad - da + (1 - q)cb = 0, \]
\[ bd + db = hca, \]
\[ ha^2 = hd^2 + (q + 1)b^2, \quad c^2 = 0 \]
One obtains from (1.2)

\[
\hat{R}(K; q, h) = \begin{pmatrix}
1 & 0 & 0 & K h \\
0 & (1 - K) & K q & 0 \\
0 & K & (1 - K q) & 0 \\
0 & 0 & 0 & (1 - K(q + 1))
\end{pmatrix}.
\]  

(2.13)

This satisfies

\[
\hat{R}_{(12)} \hat{R}_{(23)} \hat{R}_{(12)} - \hat{R}_{(23)} \hat{R}_{(12)} \hat{R}_{(23)} = \left( K - 1 \right) \left( K q - 1 \right) \left( \hat{R}_{(23)} - \hat{R}_{(12)} \right)
\]  

(2.14)

Here \( K_1 = 1, K_2 = q^{-1} \), both being independent of \( h \).

3 Properties

3.1 \( K \) and triangularity

The matrix \( R \) is called "triangular" if

\[ (21) R = R^{-1} \]

when

\[ \hat{R}^2 = (PR)^2 = I \]

In [2] the term "unitary" is used in this context. For an \( R \)-matrix satisfying the Yang-Baxter constraint \((YB)\) the following features are well known.

(1): For standard \((q \text{ or } \sqrt{\text{or}}(p, q))\) deformations the \( R \) satisfying \((YB)\) is "quasitriangular" and

\[ \hat{R}^2 \neq I \]

(2): For nonstandard \(( h \text{ or } (g, h))\) deformations for \( R \) satisfying \((YB)\) one has "triangularity" or

\[ \hat{R}^2 = I \]

( It is in this sense that we use the term triangular, without \( R \) being necessarily strictly upper or lower triangular.)

In presence of an arbitrary \( K \) the modified braid equation \((MBE)\) breaks this dichotomy. Specifically in the preceding three cases one has the following situation:

\[ (21) R(K; p, q) = (R(K'; p, q))^{-1} \]  

(3.15)

where

\[ K' = K(K(1 + qp^{-1}) - 1)^{-1} \]
\begin{align*}
(21) R(K; g, h) &= (R(K'; g, h))^{-1} \\
\text{where} \\
K' &= K(2K - 1)^{-1}
\end{align*}

\begin{align*}
(21) R(K; q, h) &= (R(K'; q, h))^{-1} \\
\text{where} \\
K' &= K(K(1 + q) - 1)^{-1}
\end{align*}

Thus in each case one obtains

\begin{align*}
K' &= K\left(\frac{K}{K_1} + \frac{K}{K_2} - 1\right)^{-1}
\end{align*}

In general none is triangular (or unitary). On the other hand \textit{in each case one can have triangularity} by choosing

\begin{align*}
K' &= K
\end{align*}

or

\begin{align*}
K &= 2K_1K_2(K_1 + K_2)^{-1}
\end{align*}

For the three previous cases this gives respectively

\begin{align*}
K &= 2p(p + q)^{-1}, \quad 1, \quad 2(1 + q)^{-1}
\end{align*}

Thus for the nonstandard case trianguarity coincides with the \((YB)\) property. In contrast, for the other two cases triangularity implies a nonzero right hand side in (1.4). In particular for the \((p, q)\) case one obtains (permuting the second and the third rows of \(\hat{R}\)) for

\begin{align*}
K &= 2p(p + q)^{-1}
\end{align*}

\begin{align*}
R &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{2pq}{p + q} & \frac{p - q}{p + q} & 0 \\
0 & \frac{q - p}{p + q} & \frac{2}{p + q} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\end{align*}

Now one has,

\begin{align*}
R(K; p, q) &= ((21)M)^{-1}M
\end{align*}

where one can set, choosing an upper triangular form,
\begin{equation}
M = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{(2pq)^{\frac{1}{2}}}{p+q} & \frac{p-q}{(2pq(p+q))^\frac{1}{2}} & 0 \\
0 & 0 & \frac{(p+q)^{\frac{1}{2}}}{2pq} & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\end{equation}

\(R\) is invariant under

\[M \rightarrow VM\]

where

\[(21)V = V\]

\[.\]

### 3.2 Projectors

For each case \((K; p, q), (K; g, h), (K; q, h)\) one obtains , \(I\) being the 4 \(\times\) 4 unit matrix,

\[\hat{R}^2 = X\hat{R} + (1 - X)I\]  \hspace{1cm} (3.23)

\[X = 2 - \left( \frac{K}{K_1} + \frac{K}{K_2} \right)\]

Thus for the three cases \(((p, q), (g, h), q, h)\) one has respectively

\[X = 2 - K(1 + qp^{-1})\]

\[X = 2(1 - K)\]

\[X = 2 - K(1 + q)\]

Two special cases are

\[X = 0, \quad (\hat{R})^2 = I\]

and

\[X = 2, \quad (\hat{R} - I)^2 = 0\]

For \(X \neq 2\) one obtains for each deformation considered two projectors \((P^2 = P)\) as follows

\[P_1 = \frac{(\hat{R} - I)}{(X - 2)}\]  \hspace{1cm} (3.25)

\[P_2 = \frac{(\hat{R} - (X - 1)I)}{(2 - X)}\]  \hspace{1cm} (3.26)

Finally one has
\[
\hat{R} = (X - 1)P_1 + P_2
\]  
(3.27)

with
\[
P_1 + P_2 = I, \quad P_1P_2 = 0
\]

Note that one obtains a canonical formalism valid for all the deformations considered.

It follows from the preceding results that if \( C \) is a column vector (with 4 rows) a constraint (\( n \) being a constant)
\[
C = n\hat{R}C
\]
is only consistent with the actions of the projectors for
\[
n = 1; \quad (P_1C = 0, C = P_2C)
\]
or
\[
n = (X - 1)^{-1}; \quad (P_2C = 0, C = P_1C)
\]
This fact should be kept in mind for what follows.

4 \textit{K and noncommutative planes}

Detailed study of noncommutativity implemented via \( \hat{R} \) in the plane and higher dimensional spaces can be found in [4, 5, 6] where numerous sources are cited. Here we limit our considerations to the two dimensional plane. But we let our \( \hat{R} \) be more general by letting it depend on an extra arbitrary parameter \( K \) permitted by our \((MBE)\). Our \( \hat{R} \) will depend on 3 parameters. The biparametric nonstandard deformation with differential calculus was first presented (for \( K = 1, g = h, h = h' \)) in [7]. The original formalism is due to Wess and Zumino [8].

We use the following notations:
\[
x^i = (x^1, x^2) = (x, y)
\]
\[
dx^i = \xi^i = (\xi^1, \xi^2) = (\xi, \eta)
\]
\[
(\alpha, \beta) = (p, q), (g, h), (q, h)
\]

We postulate
\[
x^i x^j = (\hat{R}(K; \alpha, \beta))^{ij}_{i'j'} x^{i'} x^{j'}
\]
(4.28)
i.e.,
\[
(P_1)^{(ij)}_{i'j'} x^{i'} x^{j'} = 0
\]
\[
\xi^i \xi^j = - \frac{1}{(1 - X)} (\hat{R}(K; \alpha, \beta))^{(ij)} \xi^i \xi^j 
\]  
(4.29)

i.e.,

\[
(P_2)^{(ij)} \xi^i \xi^j = 0
\]

\[
x^i \xi^j = \frac{1}{(1 - X)} (\hat{R}(K; \alpha, \beta))^{(ij)} \xi^i x^j 
\]  
(4.30)

where

\[
(1 - X) = \left( \frac{K}{K_1} + \frac{K}{K_2} - 1 \right)
\]

The bilinear constraints (4.28),(4.29),(4.30), related through derivations, are required to satisfy suitable consistency relations. (See, for example, Sec. 4 of [4] and [7].) Following the usual procedure the required consistency for our case can be shown to be assured precisely by our generalized Hecke condition, namely,

\[
\left( \hat{R}(K; \alpha, \beta) - I \right) \left( \frac{\hat{R}(K; \alpha, \beta)}{(1 - X)} + I \right) = 0 
\]  
(4.31)

or

\[
P_1 P_2 = 0
\]

This generalizes some well known results. Thus, for example, setting

\[
p = q^{-1}, \quad K_1 = 1, \quad K = K_2 = q^{-2}
\]

and changing the normalization of \( R \) by a factor \( q \) one obtains the result (4.4.15) of [4]. One obtains analogous generalizations for the other cases. Note that the consistency is obtained for our case by \textit{by implementing} \( K \) \textit{nontrivially} through the factor \((1 - X)\) for the \((\xi, \eta)\) constraints. But once this is done the final consequences of (4.28) and (4.29) turn out to be \textit{systematically independent} of \( K \). (Those of (4.30) do involve \( K \) but, as will be shown below, in a particularly simple fashion.) We recapitulate for completeness the first two sets of results which are the same as one would obtain with \( K = (K_1, K_2) \).

One obtains for \((\alpha, \beta) = (p, q)\)

\[
pxy = yx 
\]  
(4.32)

\[
\xi^2 = 0, \quad \eta^2 = 0, \quad \eta \xi + q \xi \eta = 0
\]  
(4.33)

For \((\alpha, \beta) = (g, h)\)

\[
xy - yx = gy^2 
\]  
(4.34)

\[
\xi^2 = h \xi \eta, \quad \eta^2 = 0, \quad \eta \xi + \xi \eta = 0
\]  
(4.35)
The results above are for $GL(2)$. For $GL(1/1)$, namely for

$$(\alpha, \beta) = (q, h)$$

one obtains

$$xy = qyx, \quad y^2 = 0 \quad (4.36)$$

$$(1 + q)\xi^2 + h\eta^2 = 0, \quad \eta\xi + \xi\eta = 0 \quad (4.37)$$

For the deformed $GL(1/1)$ $y$ becomes fermionic. (After exhibiting as above how the three cases can be treated uniformly in our formalism, in what follows we will consider only the deformations $(K; p, q)$ and $(K; g, h)$ of $GL(2)$. Those for $GL(1/1)$ can easily be added.)

In contrast to the foregoing results, the consequences of (4.30) involve $K$ nontrivially. For $\hat{R}(K; p, q)$ one obtains

$$x\xi = \frac{1}{(1 - X)}\xi x, \quad x\eta = \frac{1}{(1 - X)}(\xi y + \frac{K}{p}\Phi_1)$$

$$y\xi = \frac{1}{(1 - X)}(\eta x - \frac{Kq}{p}\Phi_1), \quad y\eta = \frac{1}{(1 - X)}\eta y \quad (4.38)$$

with

$$\Phi_1 = (\eta x - p\xi y) \quad (4.39)$$

For $p = q^{-1}, K = q^{-2}$ and again suitably choosing the normalization of $\hat{R}$ these results reduce to (4.1.8) of [4]. In order to compare with $\kappa$ of (4.1.10) of [4] one can show by reordering terms

$$\Phi_1^2 = \frac{1}{(1 - X)}(-qp + Kqp + p^2 - Kqp + pq - p^2)(\xi x y) = 0 \quad (4.40)$$

Note that $K$ reappears on reordering but the coefficient of $K$ in the numerator vanishes separately. Thus, apart from the overall factor, $K$ appears as a factor of the nilpotent $\Phi_1$. Moreover one can show that

$$px\Phi_1 = \frac{1}{(1 - X)}K\Phi_1 x, \quad y\Phi_1 = \frac{1}{(1 - X)}Kq\Phi_1 y$$

$$\frac{1}{(1 - X)}(p + q - Kq)\xi\Phi_1 = -\Phi_1\xi, \quad \frac{1}{(1 - X)}(p + q - Kq)\eta\Phi_1 = -pq\Phi_1\eta \quad (4.41)$$

For the prescriptions indicated before ($K = q^{-2}$ and so on) one finds back the corresponding results of Sec.4.1.13 and Sec.4.1.17 of [4].

For $\hat{R}(K; g, h)$ one obtains (compare (3.1.6) of [5] where $K = 1$ and $g = h$)

$$x\xi = \frac{1}{(1 - X)}(\xi x + Kh\Phi_2), \quad x\eta = \frac{1}{(1 - X)}(\xi y + K\Phi_2)$$
\[ y \xi = \frac{1}{(1 - X)} (\eta x - K \Phi_2), \quad y \eta = \frac{1}{(1 - X)} \eta y \] (4.42)

with
\[ \Phi_2 = (\eta x - \xi y + g \eta y) \] (4.43)

and
\[ \Phi_2^2 = 0 \] (4.44)

Moreover,
\[ x \Phi_2 = \frac{1}{(1 - X)} (K \Phi_2 x + K (g - h) \Phi_2 y), \quad y \Phi_2 = \frac{1}{(1 - X)} K q \Phi_2 y \]
\[ \frac{1}{(1 - X)} (2 - K) \xi \Phi_2 = -(\Phi_2 \xi + (h - g) \Phi_2 \eta), \quad \frac{1}{(1 - X)} (2 - K) \eta \Phi_2 = -\Phi_2 \eta \] (4.45)

The results for the \((g, h)\) case can of course be obtained independently. But they are obtained more efficiently and with a deeper understanding by starting from the corresponding ones for \((p, q)\) and using the "contraction" studied in the following section. It is instructive to see, in particular, how the \((g - h)\) factors in the results above arise (end of the next section). These terms are present even for the \((YB)\) subspace \((K = 1)\) unless \(g = h\). Finally, for \(K = 1\) and \(g = h\) one obtains the simple results of Sec.4.1.17 of [4].

5 \(((K; p, q) \rightarrow (K; g, h)) : \text{singular limit of a transformation}\)

In Sec.4 of [1] such a passage was presented for the case where \(R(p, q)\) and \(R(g, h)\) both satisfied \((YBE)\). Here we generalize it to include an arbitrary \(K\). In fact the same transformation will work again, leading to a well defined \(R(K; g, h)\). We want to emphasize this fact. It underlines again the "soft symmetry breaking" role of \(K\). Moreover we will display here how the corresponding features of the two noncommutative plains are related systematically through this "contraction" procedure. The \((K; g, h)\) -deformed plane emerges in full detail from the \((K; p, q)\)-deformed one. Some previous sources are cited in [1], which in turn lead to some original ones.

Setting
\[ G = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \] (5.46)

and with \(R = \hat{P} \hat{R}\) one obtains
\[(G^{-1} \otimes G^{-1})R(K; p, q)(G \otimes G) = \begin{pmatrix}
1 & -K(q-1)\omega & \frac{K}{p}(q-1)\omega & -\frac{K}{p}(p-1)(q-1)\omega^2 \\
0 & Kq & (1 - K) & \frac{K}{p} \\
0 & (1 - K) & \frac{K}{p} & \frac{K}{p}(p-1)\omega \\
0 & \frac{K}{p} & \frac{K}{p}(p-1)\omega & 1 
\end{pmatrix}. \quad (5.47)\]

Now, as in [1], let \(p \to 1, q \to 1\) in such a way that \((p - 1)(q - 1)^{-1}\) remains constant.
And \(\omega_0\) being a constant, define
\[\omega = \omega_0((p - 1)(q - 1))^{-\frac{1}{2}}\]

Now one can define finite constants \((g, h)\) such that as \(p \to 1\) and \(q \to 1\)
\[((1-p)\omega) \to g, \quad ((q-1)\omega) \to h \quad (5.48)\]

Now from (2.10) and (5.47) (with \(R = \hat{P}\bar{R}\)), one obtains
\[
(G^{-1} \otimes G^{-1})R(K; p, q)(G \otimes G) \to R(K; g, h) \quad (5.49)
\]

The same procedure works for \((a, b, c, d)\), \((x, y)\) and \((\xi, \eta)\). In this section, to distinguish the cases \((p, q)\) and \((g, h)\), we will use for the latter the notations
\[(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}); (\tilde{x}, \tilde{y}); (\tilde{\xi}, \tilde{\eta})\]

Consistently with transformation of \(R\) one defines (with the previous definitions of \(G\) and \(T\))
\[G^{-1}TG = \begin{pmatrix}
\tilde{a} & \tilde{b} \\
\tilde{c} & \tilde{d}
\end{pmatrix}.\]
\[G^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}.\]
\[G^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix}.\]

Let us now consider some examples to appreciate how the technique works. From the preceding definitions one obtains
\[\tilde{a} = a - \omega c, \quad \tilde{b} = (b - \omega d) + \omega(a - \omega c)\]
\[\tilde{c} = c, \quad \tilde{d} = d + \omega c \quad (5.50)\]
The inverse relations are easily obtained. Using them and the group relations for \((a, b, c, d)\) one obtains
\[
\tilde{c}a = c(a - \omega c) = pac - \omega c^2 = p\tilde{a} + \omega \tilde{c}c - \omega \tilde{c}^2 = p\tilde{a} - (1 - p)\omega \tilde{c}^2
\]
Using the definition of \(g\) now one obtains, in the limit,
\[
\tilde{c}a = \tilde{a}c - gc^2 \quad (5.51)
\]
Again,
\[
\tilde{c}b = c(\omega a + b - \omega^2 c - \omega d) = pqbc - \omega qdc + \omega a c - \omega^2 c^2
\]
\[
= pq\tilde{b}c + q(p - 1)\omega \tilde{d}c - p(q - 1)\omega \tilde{a}c + (1 - p)(q - 1)\omega^2 \tilde{c}^2
\]
giving in the limit
\[
\tilde{c}b = \tilde{b}c - gc\tilde{c} - h\tilde{a}c + gh\tilde{c}^2 \quad (5.52)
\]
We have thus obtained the first two group relations (with tildes added to avoid confusion) for the nonstandard case (2.9). The others can be obtained analogously. Let us now look at the \((K; g, h)\)-deformed plane. One obtains from the definitions introduced
\[
\tilde{x} = x - \omega y, \quad \tilde{y} = y
\]
Hence, using the constraints for \((x, y)\)
\[
\tilde{x}\tilde{y} - \tilde{y}\tilde{x} = (x - \omega y)y - y(x - \omega y) = xy - yx = (1 - p)xy = (1 - p)(\tilde{x}\tilde{y} + \omega y^2)
\]
Now taking limit and using the definition of \(g\),
\[
\tilde{x}\tilde{y} - \tilde{y}\tilde{x} = q\tilde{y}^2 \quad (5.54)
\]
This is the nonstandard version (with tildes added)
Similarly starting with
\[
\tilde{\xi} = \xi - \omega \eta, \tilde{\eta} = \eta
\]
and using
\[
\xi^2 = 0, \quad \eta^2 = 0, \quad (\xi\eta + q\eta\xi) = 0
\]
one obtains in the limit the expected results
\[
\tilde{\xi}^2 = h\tilde{\xi}\eta, \quad \tilde{\eta}^2 = 0, \quad (\tilde{\xi}\tilde{\eta} + \tilde{\eta}\tilde{\xi}) = 0
\]
These simple cases have been presented to give a feeling for the limiting process at work. But they have further usefulness. For the important nilpotent operators of the preceding section one easily obtains, taking our limits,
\[
\Phi_1 = (\eta x - p\xi y) \rightarrow (\tilde{\eta}\tilde{x} - \tilde{\xi}\tilde{y} + g\tilde{\eta}\tilde{y}) = \Phi_2
\]
Hence avoiding a lengthy reordering process one obtains directly from

\[ \Phi_1^2 = 0 \]

\[ (\Phi_1^2) \rightarrow \Phi_2^2 = 0 \]

The commutators of \( \Phi_2 \) can again be derived simply from those of \( \Phi_1 \). The terms involving \((g - h)\) in the latter set can be seen to emerge as follows:

\[
(p^{-1} - q)\omega = (p^{-1}(1 - p)\omega - (q - 1)\omega) \rightarrow (g - h)
\]

\[
(pq - 1)\omega = (p(q - 1)\omega - (1 - p)\omega) \rightarrow (h - g)
\]

### 6 Discussion

The following points are worth noting.

(1): If \( \hat{R}(K;\alpha,\beta) \) depend linearly on \( K \) and satisfy the braid equation for \( K = (K_1, K_2) \), then the right hand side of (1.4) becomes almost evident as follows. One can set

\[
\hat{R}_{(12)}\hat{R}_{(23)}\hat{R}_{(12)} - \hat{R}_{(23)}\hat{R}_{(12)}\hat{R}_{(23)} = \left(\frac{K}{K_1} - 1\right)\left(\frac{K}{K_2} - 1\right)Z \quad (6.56)
\]

The first two factors on the right assure the braid property for \( K = (K_1, K_2) \). Next one notes the following points:

The left hand side is trilinear in \( K \). Hence \( Z \), coming after the first two factors, should be linear in \( K \).

The left side is antisymmetric under the exchange

\[ \hat{R}_{(12)} \leftrightarrow \hat{R}_{(23)} \]

Hence \( Z \) should have the same property. Thus the evident ansatz is

\[
Z = \left(\hat{R}_{(23)} - \hat{R}_{(12)}\right) \quad (6.57)
\]

This is indeed found to be correct. A possible \( K \)-independent constant factor can be normalized to unity, as we have done.

(The following two properties have been pointed out to the author by Daniel Arnaudon.)

(2): For the \((p, q)\) and the \((q, h)\) cases one can write

\[
\hat{R}(K;\alpha,\beta) = c_1 \hat{R}(K_1;\alpha,\beta) + c_2 \hat{R}(K_2;\alpha,\beta) \quad (6.58)
\]
where
\[ c_1 + c_2 = 1; \quad c_1 K_1 + c_2 K_2 = K \]

However, for the \((g, h)\) case (since \(K_1 = K_2 = 1\)) such a relation does not hold for \(K \neq 1\).

(3): For
\[ \hat{R}_{(12)}\hat{R}_{(23)}\hat{R}_{(12)} - \hat{R}_{(23)}\hat{R}_{(12)}\hat{R}_{(23)} = \lambda \left( \hat{R}_{(23)} - \hat{R}_{(12)} \right) \]
and
\[ \hat{R}^2 = X \hat{R} + (1 - X) I \]

(\(\lambda\) and \(X\) not being necessarily restricted to the values considered previously) defining
\[ \hat{S} = (\hat{R} - \mu I) \]

one can verify that
\[ \hat{S}_{(12)}\hat{S}_{(23)}\hat{S}_{(12)} - \hat{S}_{(23)}\hat{S}_{(12)}\hat{S}_{(23)} = (\lambda + X \mu - \mu^2) \left( \hat{S}_{(23)} - \hat{S}_{(12)} \right) \]

This generalizes an analogous result of [2] since we do not restrict \(R\) to be “unitary”. One can choose \(\mu\) so that \(\hat{S}\) satisfies the braid equation. Directly connected with the last two equations is the following, canonical relation valid for all the cases considered before,
\[ \hat{R}(K; \alpha, \beta) = \left( \frac{K}{K_i} \right) \hat{R}(K_i; \alpha, \beta) - \left( \frac{K}{K_i} - 1 \right) I \]

Here \(K_i\) denotes either one of the “braid” (or YB) values of \(K\). This is of basic importance. The parameter \(K/(K_i - K)\) can be shown to provide the prescription for Baxterization. In fact, (6.63) can be recognized to correspond to the usual ansatz for Baxterization [9].

(4): The works of Gerstenhaber, Giaquinto and Schak [2, 3] assure that our \((MBE)\) encodes deformations satisfying basic criteria but removing certain restrictive features of the standard \((BE)\) (or \((YB)\)). For our class the factor \(\lambda\) on the right in (6.59) is neither zero nor entirely arbitrary. It has the specific form given by (1.4) arising out of our basic condition: \(K\)-independence of the group relations. Our parametrization of this factor carries information. The YB or the braid solutions are obtained effortlessly as byproducts. This leads also to agreeable properties designated here as ”soft symmetry breaking ” role of \(K\) in the noncommutative geometries studied. For all \(K\) (and all the case studied) one obtains the crucial, canonical Hecke condition we have emphasized. It permits us to introduce consistently and uniformly the noncommutativity constraints. Let us recapitulate the remarkable consequences.

(a): The bilinear constraints for the coordinates and those for the differentials remain independent of \(K\).

(b): In the constraints involving both coordinates and differentials \(K\) does appear but in a “minimal” fashion. Apart from a simple overall factor, in the linear combinations on the right \(K\) appears as a factor of a nilpotent combination(\(\Phi_1\) or \(\Phi_2\)). Along with the commutation
relations satisfied by these nilpotents, this has the consequence that reordering any higher order product one obtains, apart from an overall factor, finally linearity in $K$. The operator $\Phi$, crucial for constructing covariant derivatives [4], remains nilpotent for arbitrary $K$ (see (4.40)). The main point is that conserving the $(x, y)$ and the $(\xi, \eta)$ commutators and without violating the constraints imposed by the postulated actions of exterior derivations one can implement the parameter $K$ in the mixed commutators $((x, \xi), \text{etc.})$, even there conserving good properties.

(c): The "contraction" procedure leading from standard $(p, q)$ to nonstandard $(g, h)$ deformations is not perturbed by $K$. Even the titles of previous papers [10, 11] give an idea of the scope of this approach. It is reassuring to note that one can continue to implement it in presence of $K$.

Having noted some interesting features of the results obtained let us now look at further developments they suggest. One naturally thinks of the following aspects:

(1): Extension of our results to higher dimensional (MBE). Firstly by going beyond the $4 \times 4$ cases for deformations of $GL(2)$. Secondly by starting from group relations for deformed $SL(N)$ and $SO(N)$. Higher dimensional cases have already been studied in [2, 3]. Our aim would be to obtain explicit structures corresponding to conserved group relations for such cases. Then one can see if our soft symmetry breaking still gets implemented and in what fashion.

(2): For $K = (K_1, K_2)$ the $R$ matrix flips the tensor components of coproducts. Having obtained more general modified $R$ matrices it would be important to examine the consequences for coproducts as $K$ moves away from the $(Y B)$ values.

(3): A more complete study of the role of $K$ in noncommutative geometries induced by 3-parameter deformations $(K; \alpha, \beta)$. Even for the 2-plane we have stopped at a certain point leaving much to be done. After constructing higher dimensional matrices $\hat{R}(K; \alpha, \beta)$ one can implement them in higher dimensional spaces.

(4): Study of twists in the context of "modified" $R$ matrices. In particular, the fact that one can implement triangularity for all types of deformations by suitably choosing $K$ suggests intriguing possibilities. Various aspects studied in [12, 13, 14] can be reexamined in this broader context.

(5): Our MBE (or MYBQE of Gerstenhaber et al.) and Baxterization can be seen to be (see (6.63)) two facets of the same underlying construction, namely the general solution of the $RTT$ relations. In the first case the parameter $K$ is kept fixed in each term and the right hand side of the braid equation is allowed to be nonzero. In the second one the right is held fixed at zero and to permit this the parameter is suitably varied from term to term. The two procedures are complementary! This links MBE with integrable models.

(6): What are the consequences for knot invariants associated to an $\hat{R}$ as $K$ moves away from the "braid values"? Can a conceptually consistent generalization (parametrized third Reidemeister move) be implemented fruitfully?

Presumably this list is not exhaustive. Some of these objectives should be directly accessible. Elsewhere one may encounter obstructions. We hope to explore different directions.
in future studies.

This work owes much to sustained and reassuring help from Daniel Arnaudon. It goes beyond results explicitly attributed to him. Our treatment of noncommutative planes took shape from successive discussions with John Madore.

References

[1] B.L.Aneva, D.Arnaudon, A.Chakrabarti, V.K.Dobrev and S.G.Mihov, On combined standard-nonstandard or hybrid \((q, h)\)-deformations (QA/0006206)

[2] M.Gerstenhaber and A.Giaquinto, Boundary solutions of the quantum Yang-Baxter equation and solutions in three dimensions (q-alg/9710033)

[3] M.Gerstenhaber, A.Giaquinto and S.D.Schak, Construction of quantum groups from Belavin-Drinfeld infinitesimals, (Israel Mathematical Conference Proceedings, Vol.7, p.45 - 64, 1993)

[4] J.Madore, An Introduction to Noncommutative Differential Geometry and its Physical Applications (C.U.P., Second edition, 1999)

[5] S.Cho, J.Madore and K.S.Park, Noncommutative geometry of the \(h\)-deformed quantum plane, J.Phys., A 31, 2639 (1998)

[6] G.Fiore and J.Madore, The geometry of the quantum Euclidean space (QA/9904027)

[7] A.Aghamohammadi, The two-parametric extension of \(h\) deformation of \(GL(2)\) and the differential calculus on its quantum plane, Mod.Phys.Lett. A8, 2607 (1993)

[8] J.Wess and B.Zumino, Covariant differential calculus on the quantum hyperplane, Nucl.Phys.(Proc.Suppl) 18B, 302 (1990)

[9] A.Klimyk and K.Schmudgen, Quantum groups and their representations, p.296 (Springer, 1997).

[10] B.Abdesselam, A.Chakrabarti and R.Chakrabarti, Towards a general construction of nonstandard \(R_h\)-matrices as contraction limits of \(R_q\)-matrices: \(U_h(SL(N))\) algebra case, Mod.Phys.Lett. A13, 779 (1998)

[11] A.Chakrabarti and R.Chakrabarti, The Gervais-Neveu-Felder equation for the Jordanian quasi-Hopf \(U_{h,y}(SL(2))\) algebra, J. Phys. A: Math. Gen.33,1 (2000) (math.QA/0001015)
[12] B. Abdesselam, A. Chakrabarti, R. Chakrabarti and J. Segar,
Maps and twists relating $U(SL(2))$ and the nonstandard $U_{h}(SL(2))$: unified construction, Mod. Phys. Lett. A14, 765 (1999)

[13] P. P. Kulish, Symmetries related to Yang-Baxter equation and reflection equation, Int. J. Mod. Phys. B13, 2943 (1999)

[14] P. P. Kulish, V. D. Lyakhovsky and A. I. Mudrov, Extended Jordanian twists for Lie algebras, (math.QA/9806014)