The Riemann problem with additional singularities

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Abstract

The Riemann problem is studied in the case when the unknown function has nonisolated singularities, concentrated on the real axis. The problem is used for the factorization of functions, holomorphic outside of the unit circle and the real axis, in the form of product of two functions which have singularities on the given set of the real axis.

The classical Riemann problem with zeroes is well known (see, e.g. [1]). It is required to construct two functions \( \psi_1(z) \), \( \psi_2(z) \), such that \( \psi_1(z) \) is analytic inside the closed contour \( \Gamma \) and has inside the contour \( n \) zeroes \( \lambda_1, \ldots, \lambda_n \), and \( \frac{1}{\psi_2(z)} \) is analytic outside \( \Gamma \) and has outside the contour the zeroes \( \mu_1, \ldots, \mu_n \). In addition, on the contour \( \Gamma \) the following relation is required:

\[
\psi_1(\xi) = G(\xi)\psi_2(\xi), \quad \xi \in \Gamma,
\]

where \( G(\xi) \) is a given complex-valued function on the contour.

In this work the Riemann problem with additional singularities is proposed and solved in the case when the contour \( \Gamma \) is the unit circle and the zeroes and the singularities of the functions \( \psi_1(z) \) and \( \psi_2(z) \), including non isolated singularities, are concentrated on the real axis. A particular case of such a problem is used in [2].

Notation 1. If the function \( f(z) \) is holomorphic everywhere in complex plane outside the unite circle \( T = \{ \xi, |\xi| = 1 \} \) and the real axis \( R \), then the superindices \( ^+ \) and \( ^- \) denote the limit value of the function from inside \( ^+ \) and from the outside \( ^- \) of the unit circle or from above \( ^+ \) and from below \( ^- \) of the real axis (assuming that these limits exist):

\[
f^\pm(\xi) = \lim_{\epsilon \to +0} f((1 \mp \epsilon)\xi), \quad |\xi| = 1,\]
\[
f^\pm(x) = \lim_{\epsilon \to +0} f(x \pm i\epsilon), \quad -\infty < x < \infty.
\]
Now we formulate the Riemann problem with additional singularities. Let the function

\[ G(e^{i\theta}) = \mu(\theta)e^{i\varphi(\theta)}, \quad -\pi < \theta \leq \pi, \]

be given on the unit circle, where \( \ln \mu(\theta) \) and \( \varphi(\theta) \) are summable functions \( (\mu(\theta) > 0, \varphi(\theta) = \varphi(\theta)), \) and \( \varphi(\theta) \) is bounded. Let \( \varphi(t) = \varphi(t) \) be a bounded summable function on the real axis, which vanishes at least on one interval \( \Delta \subset (-1, 1) \) and on one interval \( \Delta'' \subset (-\infty, -1) \cup (1, \infty) \). It is required to construct a holomorphic function \( R(z) \), which do not have zeroes outside the unit circle and the real axis, and such that

\[ \frac{R^+(e^{i\varphi(\theta)})}{R^-(e^{i\varphi(\theta)})} = \mu(\theta)e^{i\varphi(\theta)}, \quad -\pi < \theta \leq \pi, \quad (2) \]

\[ \frac{R^+(t)}{R^-(t)} = e^{i\varphi(t)}, \quad -\infty < t < \infty, \quad (3) \]

assuming that these limits exist. Here we require the exact equality of the arguments in (2) and (3): \( \arg \frac{R^+(e^{i\varphi(\theta)})}{R^-(e^{i\varphi(\theta)})} = \varphi(\theta), \) \( \arg \frac{R^+(t)}{R^-(t)} = \varphi(t), \) where \( \arg R^\pm(\xi), |\xi| = 1, \) and \( \arg R^\pm(t), -\infty < t < \infty, \) are defined in the following way. Let us fix points \( t' \in \Delta', t'' \in \Delta'', \) in which \( R(z) \) is holomorphic and let \( -\pi < \arg R(t') \leq \pi, -\pi < \arg R(t'') \leq \pi. \) According to the assumption, the function \( R(z) \) has four connected components of holomorphy in which it does not vanish. It is the parts of the upper and lower halfplane restricted by the unit circle. Connecting the point \( z \) \( (\text{Im} z \neq 0, |z| \neq 1) \) with the point \( t' \) or \( t'' \) by a continuous curve, lying in one of the four components, we observe the continuous change of the argument of the function \( R(z) \) along this curve. Now the argument \( \arg R(z) \) is defined uniquely. As \( z \) tends to the real axis (resp. to the unit circle), we find \( \arg R^\pm(t), -\infty < t < \infty \) (resp. \( \arg R^\pm(\xi), |\xi| = 1 \)).

It is easy to see that the Riemann problem with additional singularities is a generalization of the classical Riemann problem with zeroes when the zeroes are concentrated on the real axis. In fact, if on some interval \( \Delta \in \mathbb{R}, \pm 1 \not\in \Delta, \varphi(t) = k\pi, k \in \mathbb{Z}, \) then this means that \( R(z) \) is holomorphic on \( \Delta. \) If in the right and left half-neighborhoods of the point \( t_0 \) the function \( \varphi(t) \) is constant and divisible by \( \pi, \) and at the point the function has a jump of the form \( k\pi, \) this means that \( R(z) \) has at the point \( t_0 \) a pole \( (k > 0) \) or a zero \( (k < 0) \) of the order \( |k|. \) More complicated behavior of \( \varphi(t) \) implies more complicated character of the singularities of \( R(z). \)

**Notation 2.** Introduce

\[ P(z, \gamma) = \exp \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \gamma(t) \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) dt \right\}, \quad (4) \]

\[ \hat{P}(z, \hat{\gamma}) = \exp \left\{ -\frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \hat{\gamma}(\theta) d\theta \right\}, \]

where \( \gamma(t) = \overline{\gamma(t)}, -\infty < t < \infty, \) and \( \hat{\gamma}(t) = \overline{\gamma(t)}, -\pi < t < \pi, \) are bounded measurable functions. The function \( P(z, \gamma) \) is defined and holomorphic at least for non real \( z, \) and \( \hat{P}(z, \hat{\gamma}), \) resp., is defined and holomorphic inside and outside the unit disk. The formulae
of Plemelj-Sokhotsky imply the following equalities, connecting the limit values of \( P(z, \gamma) \) and \( \hat{P}(z, \hat{\gamma}) \) on the real axis and the unit circle resp.:

\[
\begin{align*}
\arg P^+(t, \gamma) &= - \arg P^-(t, \gamma) = \gamma(t), \quad |P^+(t, \gamma)| = |P^-(t, \gamma)|, \quad -\infty < t < \infty, \\
\arg \hat{P}^+(e^{i\theta}, \hat{\gamma}) &= - \arg \hat{P}^-(e^{i\theta}, \hat{\gamma}) = \hat{\gamma}(\theta), \quad |\hat{P}^+(e^{i\theta}, \hat{\gamma})| = |\hat{P}^-(e^{i\theta}, \hat{\gamma})|, \quad -\pi < \theta < \pi.
\end{align*}
\]

Let us define the functions

\[
R^{(1)}(z) = P(z, \frac{\phi}{2}), \quad R^{(2)}(z) = \hat{P}(z, \frac{\hat{\phi}}{2}),
\]

\[
R_{\mu}(z) = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}}{e^{i\theta} - z} \ln \mu(\theta) \, d\theta \right\}.
\]

It is easy to obtain from the formulae of Plemelj-Sokhotsky that

\[
\left| \frac{R^+_{\mu}(e^{i\theta})}{R^-_{\mu}(e^{i\theta})} \right| = \mu(\theta), \quad \arg R^+_{\mu}(e^{i\theta}) = \arg R^-_{\mu}(e^{i\theta}), \quad -\pi < \theta < \pi.
\]

Thus, the following theorem is a simple consequence of the equalities (5), (6) and (8):

**Theorem 1.** The function \( R(z) = R^{(1)}(z)R^{(2)}(z)R_{\mu}(z) \) is a solution of the problem (2), (3).

We observe that the summability of the functions \( \ln \mu(\theta), \varphi(\theta), \varphi(t) \) being provided, we have the existence of the limits \( R^\pm(\xi) \) and \( R^\pm(t) \) almost everywhere on the unit circle and on the real axis. and the functions \( \ln R((1 \mp \varepsilon)\xi), \ |\xi| = 1, \) and \( \ln R(t \pm i\varepsilon), \ t \in \mathbb{R}, \) converge respectively to \( \ln R^\pm(\xi) \) and \( \ln R^\pm(t) \) with respect to the metric of \( L^1 \) (see, e.g., [3]). Without paying attention to the problems of convergence, we will apply the Riemann problem to the factorization of functions with singularities on the unit circle and the real axis.

**Notation 3.** We define on the real axis the map \( V \) of the symmetry with respect to the unit circle:

\[
V(t) = t^{-1}, \quad t \in \mathbb{R}\setminus\{0\}.
\]

For a set \( A \subset \mathbb{R}\setminus\{0\} \) and a function \( \rho(t) \), defined on \( \mathbb{R}\setminus\{0\} \) we denote

\[
V(A) = \{ t \mid t^{-1} \in A \}, \quad V(\rho)(t) = \rho(t^{-1}).
\]

The main result of the work is the following

**Theorem 2.** Let the function \( N(z) \) be holomorphic and not vanishing outside the unit circle \( T \) and a certain closed set \( \Sigma \subset \mathbb{R} \) of the real axis and satisfies the following conditions:

1) In the domain of holomorphy of the function \( N(z) \) we have

\[
N(z^{-1}) = N(z), \quad N(\overline{z}) = \overline{N(z)}, \quad z \notin \Sigma \cup T.
\]
2) There exists such positive constant $C > 0$, that in the upper semi-disk

$$|\arg N(z)| < C, \quad |z| < 1, \quad \text{Im} \ z > 0.$$ 

If $\Sigma = \Omega_1 \cup V(\Omega_1) \cup \Omega_2$, where the sets $\Omega_1$, $V(\Omega_1)$, and $\Omega_2 = V(\Omega_2)$ have the mutual positive distances, then there exists the function $R(z)$, holomorphic outside the set $\Omega \equiv \Omega_1 \cup \Omega_2$ and outside the unit circle $T$, such that

$$N(z) = R(z)R(z^{-1}) \quad (10),$$

and

$$\frac{|R^+(e^{i\theta})|}{|R^-(e^{i\theta})|} = \mu(\theta), \quad (11)$$

where $\mu(\theta) = \mu(-\theta) > 0$, $-\pi < \theta < \pi$, is an arbitrary even function with summable logarithm.

We remark that this theorem reduces to the Riemann problem with additional singularities. Essentially, we construct the function $R(z)$ so that the limit values of the arguments of the function $R(z)R(z^{-1})$ equal the limit values of the argument of $N(z)$. We also require that the function $R(z)$ only have singularities onto the unit circle $T$ and onto the set $\Omega$, and satisfy the additional condition (11) with practically arbitrary $\mu(\theta)$. In order to prove the theorem, we will need two simple lemmas demonstrating the properties of the functions $P(z, \gamma)$ and $\hat{P}(z, \hat{\gamma})$.

**Lemma 1.** The functions $P(z, \gamma)$ and $\hat{P}(z, \hat{\gamma})$ in their domain of holomorphy satisfy the properties:

$$P(z, \gamma_1 + \gamma_2) = P(z, \gamma_1)P(z, \gamma_2), \quad (12)$$

$$P(z, \gamma) = \overline{P(z, \gamma)}, \quad (13)$$

$$P(z^{-1}, \gamma) = P(z, -V(\gamma)), \quad (14)$$

$$\hat{P}(z, \hat{\gamma}_1 + \hat{\gamma}_2) = \hat{P}(z, \hat{\gamma}_1)\hat{P}(z, \hat{\gamma}_2), \quad (15)$$

If $\hat{\gamma}(\theta) = -\hat{\gamma}(-\theta)$ is odd, then

$$\hat{P}(z^{-1}, \hat{\gamma}) = \hat{P}(z, \hat{\gamma})). \quad (16)$$

**Lemma 2.** 1. Let $f_1(z)$ be holomorphic function of bounded argument in the halfplane $\{\text{Im} \ z > 0\}$, which do not have zeroes in the halfplane. Then it can be represented in the form

$$f_1(z) = C_1 P(z, \eta_1), \quad \text{Im} \ z > 0, \quad (17)$$

where $C_1$ is a positive constant and

$$\eta_1(t) = \arg f_1^+(t), \quad -\infty < t < \infty, \quad (18)$$

is the limit value of its argument from above on the real axis.

2. Let $f_2(z)$ be holomorphic function in the disk $|z| < 1$ of bounded argument. Then it can be represented in the form.

$$f_2(z) = C_2 \hat{P}(z, \hat{\eta}_2), \quad |z| < 1,$$
where $C_2 > 0$, and
\[
\hat{\eta}_2(\theta) = \arg f_2^+(e^{i\theta}), \quad -\pi < \theta < \pi,
\]
is its limit value from inside the unit circle.

Proof of lemmas. We will only prove the first part of Lemma 2 (equalities (12)–(16) are obtained by direct calculation). The function $f_1(z)$ is holomorphic and does not vanish in the connected domain \{Im $z > 0$\}. Hence, we can define uniquely a logarithm $\ln f_1(z)$, which is holomorphic in the upper halfplane function with bounded imaginary part:
\[
|\text{Im}(\ln f_1(z))| < C_3.
\]
Hence, the function $(\ln f_1(z) + iC_3i)$ is the function of Nevanlinna (i.e. has positive imaginary part in the upper halfplane) and can be represented in the form (see, e.g., [4])
\[
\ln f_1(z) + iC_3 = \alpha + \beta z + \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) \frac{d\rho(t)}{t},
\]
where the measure $d\rho(t)$ is defined by a nondecreasing function $\rho(t)$ with
\[
\rho(t_2) - \rho(t_1) = \lim_{\varepsilon \to 0} \int_{t_1}^{t_2} \text{Im}(\ln f_1(t + i\varepsilon) + iC_3) dt,
\]
\[\alpha \in \mathbb{R}, \quad \beta \geq 0.\]
Since $\text{Im} \ln f_1(z)$ is bounded, then $\beta = 0$, and the measure $d\rho(t)$ is absolutely continuous $d\rho(t) = (\arg f_1^+(t) + C_3) dt$. Besides,
\[
iC_3 = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) C_3 dt,
\]
hence,
\[
\ln f_1(z) = \alpha + \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) \eta_1(t) dt,
\]
where $\eta_1(t)$ is defined by formula (18). This means that for the function $f_1(z)$ the multiplicative representation (17) is obtained. The second part of the lemma is proven analogously.

Proof of Theorem 1. At first, we will present $N(z)$ in the form of the product $P(z, \gamma) \hat{P}(z, \hat{\gamma})$. It follows from (9) that $\arg N(z)$ is bounded in the part of the upper halfplane that lies outside the unit disk. The function
\[
M(\lambda) = |N(z)|_{z+\lambda \cdot z^{-1}} = M(\lambda),
\]
is holomorphic in the upper and lower halfplane, and its argument is bounded. According to lemma 2,
\[
M(\lambda) = CP(\lambda, \eta),
\]
where $\eta(\tau) = \arg M^+(\tau), \quad -\infty < \tau < \infty$, is a bounded function, and $C > 0$ is a positive constant. (Henceforth we will denote by $C$ positive constants, different for each case.) This representation is true in the both halfplanes, because $M(\overline{\tau}) = \overline{M(\tau)}$, and because for $P(\lambda, \eta)$ we have (13). Let $\chi_{[-2,2]}(t)$ and $\chi_{(-\infty,-2) \cup (2,\infty)}(t)$ be the indicators of the sets $[-2,2] \setminus [-2,2]$, respectively. We define the functions
\[
M_0(\lambda) = P(\lambda, \chi_{[-2,2]} \eta_0),
\]

\[ M_1(\lambda) = P(\lambda, \chi(\infty, -2) \cup (2, \infty))(t) \eta_1. \]

According to property (12), in the domain of holomorphy of \( M(\lambda) \)

\[ M(\lambda) = CM_0(\lambda)M_1(\lambda), \quad (19) \]

where, according to (5), the functions \( M_0(\lambda) \) and \( M_1(\lambda) \) are holomorphic on \( \mathbb{R} \setminus [-2, 2] \) and \((-2, 2)\), respectively. Let us define in the plane of the parameter \( z \) the functions

\[ N_0(z) = N_0(z^{-1}) \equiv M_0(z + z^{-1}), \quad |z| \neq 1, \quad N_1(z) = N_1(z^{-1}) \equiv M_1(z + z^{-1}), \quad z \notin \Sigma. \]

It is evident that the functions \( N_0(z) \) and \( N_1(z) \) are holomorphic and positive on the real line and the unit circle, respectively (except possibly the points \( \pm 1 \)). Taking into account (9), (19) and this representation, we have at first outside, then inside the unit disk

\[ N(z) = CN_0(z)N_1(z), \quad z \notin \Sigma \cup T, \quad (20) \]

where the functions \( N_0(z) \) and \( N_1(z) \), according to Lemma 2, can be represented in the multiplicative form

\[ N_0(z) = \hat{P}(z, \hat{\nu}_0), \quad N_1(z) = P(z, \nu), \quad (21) \]

with an odd function on \((\pi, \pi)\)

\[ \hat{\nu}_0(\theta) = -\hat{\nu}_0(-\theta) \equiv \arg N_0^+(e^{i\theta}) = \arg N_1^+(e^{i\theta}) = \begin{cases} -\eta(e^{i\theta} + e^{-i\theta}), & 0 \leq \theta < \pi, \\ \eta(e^{i\theta} + e^{-i\theta}), & -\pi < \theta < 0, \end{cases} \quad (22) \]

and

\[ \nu(t) = -\nu(t^{-1}) \equiv \arg N_1^+(t) = \arg N_1^+(t) = \begin{cases} \eta(t + t^{-1}), & |t| > 1, \\ -\eta(t + t^{-1}), & |t| < 1. \end{cases} \quad (23) \]

We remark that Lemma 2 guarantees representation (21) for \( N_0(z) \) inside the disk (for \( N_1(z) \) in the upper halfplane). However, the same representation is also true outside the disk (resp. in the lower halfplane), because of (16) and because \( N_0(z) = N_0(z^{-1}) \) (because \( N_1(\overline{z}) = N_1(z) \)). Thus, the problem of the factorization of the function \( N(z) = CN_0(z)N_1(z) \) is reduced to the problem of factorization of two functions \( N_0(z) \) and \( N_1(z) \), represented in the form (21). At first we factorize \( N_1(z) \). Let

\[ \Delta = \mathbb{R} \setminus ([-1, 1] \cup \Sigma) = \mathbb{R} \setminus ([-1, 1] \cup \Omega_1 \cup V(\Omega_1) \cup \Omega_2) = \bigcup_k \Delta_k, \]

where \( \Delta_k = (\alpha_k, \beta_k) \) are mutually disjoint intervals and \( |\alpha_k| \geq 1, |\beta_k| \geq 1. \)

It follows from the definition of \( \Delta \) that the endpoints \((\alpha_k, \beta_k)\) of the interval belong to one of the three disjoint sets \( \Omega_1, V(\Omega_1), \Omega_2 \). Moreover, the number of the intervals, whose endpoints belong to different sets, is finite. In fact, if the endpoints of the interval \((\alpha_k, \beta_k)\) belong to different sets, then

\[ \beta_k - \alpha_k \geq \min\{\text{dist}(\Omega_1, V(\Omega_1)), \text{dist}(\Omega_1, \Omega_2), \text{dist}(V(\Omega_1), \Omega_2)\}. \]

If there were infinitely many of such intervals, then (except the case when the intervals are concentrated at the infinity) some of the intervals would have arbitrarily small length, so that one of the distances \( \text{dist}(\Omega_1, V(\Omega_1)), \text{dist}(\Omega_1, \Omega_2) \) and \( \text{dist}(V(\Omega_1), \Omega_2) \) would vanish, which contradicts to the conditions of the theorem. (If these intervals were concentrated in the infinity, then, according to \( \Omega_2 = V(\Omega_2) \) and \( \Omega_1 = V(\Omega_1) \), we would have that the
distance between the different sets and zero vanish, which contradicts to the conditions of the theorem, too.)

Let us choose among the intervals \( \Delta_k \) such intervals, that one of the endpoints belongs to \( V(\Omega_1) \), and the other belongs to \( \Omega_1 \) or \( \Omega_2 \), or is equal to \( \pm 1 \). Let us renumber the intervals \( \Delta_k \) so that \( \Delta_1, \ldots, \Delta_{k_0} \) are the chosen intervals (their number is finite). We divide the rest of the intervals in two groups: the intervals \( \Delta'_k \), whose both endpoints belong to \( V(\Omega_1) \), and \( \Delta''_k \) (all the rest). Let

\[
\alpha^*_k = \begin{cases} 
\alpha_k, & \text{when } \alpha_k \not\in V(\Omega_1), \\
\alpha_k^{-1}, & \text{when } \alpha_k \in V(\Omega_1),
\end{cases} \quad \beta^*_k = \begin{cases} 
\beta_k, & \text{when } \beta_k \not\in V(\Omega_1), \\
\beta_k^{-1}, & \text{when } \beta_k \in V(\Omega_1).
\end{cases} \quad k = 1, \ldots, k_0.
\]

We observe that \( \{\alpha^*_k\}_{k=1}^{k_0}, \{\beta^*_k\}_{k=1}^{k_0} \) and the intervals \( V(\Delta'_k), \Delta''_k \) have positive distance with the set \( V(\Omega_1) \). We denote by \( \chi_1(t), \chi_2(t), \chi_3^{(k_0)}(t), \chi_4(t), \chi_5(t) \) the indicators of the set \( \Omega_1, \Omega_2, \cup_{k=1}^{k_0} \Delta_k, \cup_k \Delta'_k, \cup_k \Delta''_k \). Evidently,

\[
\begin{align*}
\chi_1(t) + V(\chi_1(t)) + \chi_2(t) + \chi_3^{(k_0)}(t) + V(\chi_3^{(k_0)}(t)) + \\
+ \chi_4(t) + V(\chi_4(t)) + \chi_5(t) + V(\chi_5(t)) = 1, \quad t \neq \pm 1.
\end{align*}
\]

**Lemma 3** (factorization of \( N_1(z) \)). The function \( N_1(z) \) can be factored out as follows:

\[
N_1(z) = CR_{012}(z)R_{012}(z^{-1}),
\]

\[
R_{012}(z) = R_0(z)R'_0(z)R''_0(z)R_1(z)R_2(z),
\]

where

\[
R_0(z) = \prod_{k=1}^{k_0} \left( \frac{z - \beta^*_k}{z - \alpha^*_k} \right)^{n(k)},
\]

\[
n(k) = \frac{1}{\pi} \arg N^+(t) \in \mathbb{Z}, \quad t \in \Delta_k, \ k = 1, \ldots, k_0,
\]

\[
R'_0(z) = P(z, V(\chi_3^{(k_0)})), \quad R''_0(z) = P(z, \chi_3^{(k_0)}),
\]

\[
R_1(z) = P(z, \chi_1), \quad R_2(z) = P(z, -\chi_2),
\]

the constant \( C > 0 \), and the numbers \( \alpha^*_k, \beta^*_k, k = 1, \ldots, k_0 \) are defined by equality (24). Here the function \( R_{012}(z) \) is holomorphic outside the set \( \Omega \) and the points \( \pm 1 \). \[\square\]

**Proof.** The holomorphy of \( R_{012}(z) \) outside the set \( \Omega \cup \{1, -1\} \) easily follows from its definition: the function \( R_0(z)R'_0(z)R''_0(z) \) may have singularities only on the boundary of the set \( \Omega_1 \cup \Omega_2 \cup \{-1, 1\} \), the function \( R_1(z) \) may have singularities only on the set \( \Omega_1 \), and \( R_2(z) \) on \( \Omega_2 \). According to (21), (25), (12),

\[
N_1(z) = CP(z, \nu)
\]

\[
= CP(z, (\chi_1 + V(\chi_1) + \chi_2 + \chi_3^{(k_0)} + V(\chi_3^{(k_0)}) + \chi_4 + V(\chi_4) + \chi_5 + V(\chi_5) \nu) =
\]

\[\footnote{Let us remark that formula (27) does not include the case when one of the points \( \alpha^*_k \) or \( \beta^*_k \) in equal to the infinity, i.e. when 0 belongs to the boundary of the set \( V(\Omega_1) \). In this case the respective factor in \( R_0(z) \) should be defined as \( P(z, \chi(\alpha^*_k, \beta^*_k))n(k)\pi \).} \]

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\[
\begin{align*}
&= CP(z, \chi_1 \nu)P(z, V(\chi_1 \nu))P(z, \frac{1}{2} \chi_2 \nu)P(z, \frac{1}{2} \chi_2 \nu) \times \\
&P(z, \chi_\Delta^{(k_0)} \nu)P(z, V(\chi_\Delta^{(k_0)} \nu))P(z, \chi_\Delta \nu)P(z, V(\chi_\Delta') \nu)P(z, \chi_\Delta'' \nu)P(z, V(\chi_\Delta'') \nu).
\end{align*}
\]

Here, according to the properties (14), (12), and the property \(\nu(t) = -\nu(t^{-1})\) (see (23)),

\[
P(z, \chi_1 \nu)P(z, V(\chi_1 \nu)) = P(z, \chi_1 \nu)P(z^{-1}, \chi_1 \nu) = R_1(z)R_1(z^{-1}),
\]

\[
P(z, \frac{1}{2} \chi_2 \nu)P(z, \frac{1}{2} \chi_2 \nu) = P(z, \frac{1}{2} \chi_2 \nu)P(z^{-1}, \frac{1}{2} \chi_2 \nu) = R_2(z)R_2(z^{-1}),
\]

\[
P(z, \chi_\Delta \nu)P(z, V(\chi_\Delta') \nu) = R_0(z)R_0'(z^{-1}),
\]

\[
P(z, \chi_\Delta'' \nu)P(z, V(\chi_\Delta'') \nu) = R_0''(z)R_0''(z^{-1}),
\]

and, from definition (4),

\[
P(z, \chi_\Delta^{(k_0)} \nu)P(z, V(\chi_\Delta^{(k_0)} \nu)) = \exp \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} (\chi^{(k_0)}(t) + V(\chi^{(k_0)})(t)) \nu(t) \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) dt \right\} =
\]

\[
= \prod_{k=1}^{k_0} \left( \frac{z - \beta_k}{z - \alpha_k} \right)^{n_k} \left( \frac{z - \alpha_k^{*-1}}{z - \beta_k^{*-1}} \right)^{-n_k} \exp \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} (\chi^{(k_0)}(t) + V(\chi^{(k_0)})(t)) \nu(t) \left( \frac{-t}{1 + t^2} \right) dt \right\}.
\]

The exponential in the r.h.s is a constant. Further, it is evident that the right-hand side of the latest equality will not change if we replace \(\alpha_k, \varphi_k\) to \(\alpha_k^*, \varphi_k^*\). That is why

\[
P(z, \chi^{(k_0)} \nu)P(z, V(\chi^{(k_0)} \nu)\Delta \nu) = C \prod_{k=1}^{k_0} \left( \frac{z - \beta_k^{*}}{z - \alpha_k^{*}} \right)^{n_k} \left( \frac{z - \alpha_k^{*-1}}{z - \beta_k^{*-1}} \right)^{n_k}.
\]

But

\[
\frac{z - \beta_k^{*}}{z - \alpha_k^{*}} = \left( \frac{z - \beta_k^{*}}{z - \alpha_k^{*-1}} \right) \frac{1}{\beta_k^{*}} \cdot \frac{z - \alpha_k^{*}}{z - \alpha_k^{*-1}},
\]

so

\[
P(z, \chi^{(k_0)} \nu)P(z, V(\chi^{(k_0)} \nu)) = CR_0(z)R_0(z^{-1})
\]

(with another constant \(C > 0\)). Thus, from the equalities (30)--(35) follows (26).

Having factorized \(N_1(z)\), we now factorize \(N_0(z)\). We define

\[
R_3(z) = \hat{P}(z, \frac{i\nu_0}{2}).
\]

For \(|z| \neq 1\), according to (15), (16), taking into account the oddness of \(\nu_0(\theta)\) and representation (21), we have

\[
R_3(z)R_3(z^{-1}) = \hat{P}(z, \frac{i\nu_0}{2})\hat{P}(z^{-1}, \frac{i\nu_0}{2}) = \hat{P}(z, \frac{i\nu_0}{2})\hat{P}(z, \frac{i\nu_0}{2}) = \hat{?}N_0(z).
\]

Let us define \(R_\mu(z)\) by formula (7). It satisfies property (8). So, from the definition of \(R_3(z)\), the property (6) and the holomorph of \(R_{012}(z)\) on the unit circle (possibly, excepting the points \(\pm 1\)), we have for the function

\[
R_{0123\mu}(z) = R_0(z)R'_0(z)R''_0(z)R_1(z)R_2(z)R_3(z)R_\mu(z)
\]
the relation
\[
\frac{|R_{0123\mu}^+(\xi)|}{|R_{0123\mu}^-|} = \frac{|R_{012}^+(\xi)|}{|R_{012}^-|} \cdot \frac{|R_{2}^+(\xi)|}{|R_{2}^-|} \cdot \frac{|R_{\mu}^+(\xi)|}{|R_{\mu}^-|} = \mu(\xi), \quad |\xi| = 1.
\]

But, as it follows directly from the definition of \(R_{\mu}(z)\), with the use of the evenness of \(\mu(\theta)\),
\[
R_{\mu}(z)R_{\mu}(z^{-1}) = \text{const} > 0.
\] (39)
Thus, for the function \(R_{0123\mu}(z)\), defined by (38), we have from (20), (26), (39), (39)
\[
R_{0123\mu}(z)R_{0123\mu}(z^{-1}) = CN(z), \quad C > 0.
\]
Finally, taking
\[
\hat{R}(z) = \sqrt{C}R_{0123\mu}(z),
\]
we see that \(R(z)\) is the solution of our problem.

Certainly, the factorization (10) is not unique. As it will be seen from the next theorem, with some additional restrictions to the function \(N(z)\) we can require additional conditions to the behavior of the function \(R(z)\) near its singularities, for example, we can ask for the existence of the limits in the metric of \(L^p\), \(p \geq 1\), of the functions \(R(t \pm i\varepsilon)\), \(\varepsilon \to +0\).

**Definition.** Let \(A\) be a certain set on the real axis, \(U_\delta(A)\) be its \(\delta\)-neighborhood, and \(f(z)\) be a holomorphic function in \(U_\delta(A)\setminus A\). We say that the function \(f(z)\) locally belongs to the Hardy class \(H^p\) in the neighborhood of the set \(A\), if for some \(\delta > 0\) the functions \(f(t \pm i\varepsilon), \ t \in \mathbb{R} \cap U_\delta(A)\), converge as \(\varepsilon \to +0\) in the metric of \(L^p\).

**Theorem 3.** Let the function \(N(z)\), the sets \(\Sigma, \Omega, \Omega_2, V(\Omega_1)\) and the function \(\mu(\theta)\) satisfy the conditions of the previous theorem. Assume that the set \(\Omega_2\) can be covered with a finite number of mutually disjoint intervals \(\delta_i\), and on each of them
\[
\text{ess sup}_{t \in \delta_i} \arg N^+(t) - \text{ess inf}_{t \in \delta_i} \arg N^+(t) < \pi.
\]

Then the function \(N(z)\) can be factored out so that (10) and (11) hold, and the function \(R(z)\) locally depends to the Hardy class \(H^2\) in the neighborhood of the set \(\Omega_2\).

**Proof.** We will only show what changes should be done in the proof of Theorem 1 to apply it to Theorem 2. The set
\[
\Delta = \mathbb{R} \setminus ([-1,1] \cup \Omega_1 \cup V(\Omega_1)) = \bigcup_k \Delta_k,
\]
is introduced, where \(\Delta_k = (\alpha_k, \beta_k)\) are mutually disjoint intervals. The set \(\Omega_2\) lie inside of these intervals. It follows from the additional condition of the Theorem 3 that on every set \(\Delta_k \setminus \Omega_2\) the function \(n(t) = n(k) = \frac{1}{2} \arg N(t) \in \mathbb{Z}\) is constant. We extend this function to the whole interval \(\Delta_k\) (possibly, including \(\Omega_2\)), and also we extend it to \(V(\Delta_k)\) by the equality
\[
\tilde{n}(t) = \begin{cases} n(k), & t \in \Delta_k, \\ -n(k), & t \in V(\Delta_k). \end{cases}
\]
Further, we divide \(\Delta_k\) to three groups: the intervals \(\Delta_1, \ldots, \Delta_{k_0}\), which have one of the endpoints in \(V(\Omega_1)\) and the other in \(\Omega_1 \cup \{-1,1\}\); the intervals \(\Delta_k\), whose both endpoints
belong to $V(\Omega_1)$, and the intervals $\Delta_k''$ (all the others). We choose numbers $\alpha_k^*$, $\beta_k^*$, $k = 1, \ldots, k_0$, by the same rule as it was done in the Theorem 2 (by formula (24)). The functions $R_0(z), R_1(z), R_3(z), R_\mu(z)$ are defined by formulae (27), (29), (36), (7). We also define the functions

$$R'_0(z) = P(z, V(\chi_\Delta'')n\pi), \quad R''_0(z) = P(z, \chi_\Delta''n\pi), \quad R_2(z) = P(z, \frac{1}{2}\chi_2(\nu - \tilde{n}\pi)).$$

The rest of the proof does not change. We will explain how to prove that the function $R(z)$ locally belongs to the Hardy space in the neighborhood of $\Omega_2$. The function $R_2(z)$ is represented in the multiplicative form $R_2(z) = P(z, \frac{1}{2}\chi_2(\nu - \tilde{n}\pi))$. Here, the oscillation of the function $\frac{1}{2}\chi_2(t)(\nu(t) - \tilde{n}(t)\pi)$ on the interval $\delta_t$ is less than $\frac{\pi}{2}$. This implies (see, e.g., [5]) that $R_2(z)$ locally belongs to the Hardy space in the neighborhood of any compact subset of the interval $\delta_t$. (Such intervals $\delta_t$ cover $\Omega_2$ according to the condition of the theorem). Simultaneously, the functions $R_0(z), R'_0(z), R''_0(z), R_1(z), R_3(z)$ and $R_\mu(z)$ are holomorphic on $\Omega_2$, from where we obtain that $R(z)$ locally belongs to the Hardy class $H^2$ in the neighborhood of the set $\Omega_2$.

We remark that the functions $R'_0(z), R''_0(z)$ can also be presented in the form of a product (27) (generally speaking, this product is infinite). However, if the intervals $\Delta_k''$ do not belong to a finite set of the real axis, then additional factors will occur in the expression for $R'_0(z)$.

**Remark.** This paper is the translation from [6].

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\footnote{We use the following fact from this book

**Theorem (Smirnov).** If $f(z)$ is analytic in the disk $|z| < 1$ and $\text{Re} f(z) \geq 0$, than $f \in H^p$ for every $p > 1$. (Here the Hardy space in the unit disk $|z| < 1$ is denoted by $H^p$).}