A GENERALIZATION OF A $q$-IDENTITY OF DILCHER

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Abstract. We provide a further extension of a $q$-identity due to Dilcher and of its inversion. The result is used to show a $q$-analog of a Wolstenholme type congruence for multiple harmonic sums.

1. Introduction

In [3], Dilcher established the following identity: for any pair of positive integers $n, s$,

$$
\sum_{k=1}^{n} \binom{n}{k}_q \frac{(-1)^k q^{(k^2)/2} + sk}{[k]_q^s} = - \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_s \leq n} \prod_{i=1}^{s} \frac{q_j}{[j_i]_q}.
$$

(1)

where

$$
\binom{n}{k}_q = \prod_{j=1}^{k} \frac{1 - q^{n-k+j}}{1 - q^j}
$$

is the Gaussian $q$-binomial coefficient and

$$
[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}.
$$

Identity (1) is a generalization of the case $s = 1$ due to Van Hamme [15] and by taking the limit as $n$ goes to infinity one obtains a remarkable $q$-series related to overpartitions and divisor generating functions. For example, if $s = 1$ and $|q| < 1$ then

$$
\sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^{(k+1)/2}}{(1 - q)(1 - q^2) \cdots (1 - q^{k-1})(1 - q^k)^2} = \sum_{j=1}^{\infty} \frac{q^j}{1 - q^j} = \sum_{j=1}^{\infty} d(j)q^j.
$$

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$$

where $d(j)$ is the number of divisors of $j$ (see the pioneering paper of Uchimura [14]). With this motivation, several authors have recently investigated (1) and they extended it along several directions; see for example [1, 2, 4, 5, 6, 7, 9, 10, 12, 13, 16, 19]. In [11], Prodinger shows the inversion of (1),

$$
\sum_{k=1}^{n} \binom{n}{k}_q \frac{(-1)^k q^{(k^2)/2} - nk}{[k]_q^s} \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_s = k} \prod_{i=1}^{s} \frac{q_j}{[j_i]_q} = - \sum_{k=1}^{n} \frac{q^{(s-1)k}}{[k]_q^s},
$$

(2)

which is the $q$-analogue of a formula of Hernandez [8].

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In this short note we provide a further extensions of (1) and (2): let \( s_1, s_2, \ldots, s_l \) be positive integers, then

\[
\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k q^{(k)}}{q} \sum_{1 \leq k_1 < k_2 < \cdots < k_l = k} \prod_{i=1}^{l} q^{(s_i - 1) k_i} \prod_{j_i = 1}^{w} \frac{q^{j_i}}{[j_i]_q},
\]

where \( w = \sum_{i=1}^{l} s_i \) and \( I = \{ s_1, s_1 + s_2, \ldots, s_1 + s_2 + \cdots + s_l \} \). Note that Theorem 2 in [2] is the special case \( s_1 = s_2 = \cdots = s_l = 1 \) of (3).

The paper is structured as follows. After some preliminary results, we prove (3) and (4) in the third section. In the final section we give an application of (3) and we prove a \( q \)-analog of a Wolstenholme type congruence for multiple harmonic sums.

2. PRELIMINARY RESULTS

**Lemma 2.1.** Let \( 1 \leq j \leq n \), then

\[
\sum_{k=j}^{n} \binom{k-1}{j-1} q^{k} = \binom{n}{j} q^{j},
\]

\[
\sum_{k=j}^{n} \binom{n}{k} \frac{(-1)^k q^{(k)}}{j_q} = \frac{n-1}{j-1} q^{j}.
\]

**Proof.** Since

\[
\binom{k}{j} = \binom{k-1}{j-1} q^{k-j} + \binom{k-1}{j} q^{j-1},
\]

it follows that

\[
\sum_{k=j}^{n} \binom{k-1}{j-1} q^{k} = q^{j} \sum_{k=j}^{n} \binom{k-1}{j-1} q^{k-j} = q^{j} \sum_{k=j}^{n} \left( \binom{k}{j} - \binom{k-1}{j} \right) q = \binom{n}{j} q^{j}.
\]

As regards (3), since

\[
\binom{n}{k} q = \binom{n-1}{k} q^{k} + \binom{n-1}{k} q,
\]

we have that

\[
\sum_{k=j}^{n} \binom{n}{k} \frac{(-1)^k q^{(k)}}{j_q} = \sum_{k=j}^{n} \left( \binom{n-1}{k} q^{k} + \binom{n-1}{k} q \right) \frac{(-1)^k q^{(k)}}{j_q}
\]

\[
= \sum_{k=j}^{n} \left( \binom{n-1}{k} q^{(k)} - \binom{n-1}{k} q^{(k+1)} \right) q
\]

\[
= \binom{n-1}{j-1} q^{j}. 
\]

□
The following lemma provides some simple properties of the \( q \)-binomial transform of \( \{a_n\}_{n \geq 1} \) given by

\[
b_n = \sum_{k=1}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q (-1)^k q^{k(\frac{k}{2})} a_k,
\]
and its inverse

\[
a_n = \sum_{k=1}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q (-1)^k q^{(n-k)(\frac{n-k}{2})} b_k.
\]

**Lemma 2.2.** Let \( \{a_n\}_{n \geq 1} \) and \( \{b_n\}_{n \geq 1} \) be two sequences which satisfy \((7)\). Then for any positive integer \( r \),

\[
\sum_{k=1}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right]_q (-1)^j q^{j(\frac{j}{2})} a_k = \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_r \leq n} \prod_{i=1}^{r} q^{k_{i-1}} b_{k_1},
\]
and

\[
\sum_{k=1}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right]_q (-1)^{j-k} q^{j(\frac{j}{2})} \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_r = k} \prod_{i=1}^{r} q^{k_{i-1}} b_{k_1} = \sum_{k=1}^{n} q^{(r-1)k} a_k.
\]

**Proof.** For \( i \geq 0 \), let

\[
b_n(i) = \sum_{k=1}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q (-1)^k q^{k(\frac{k}{2})} a_k(i) \quad \text{with} \quad a_k(i) = \frac{q^{ik} a_k}{[k]_q}.
\]

Hence, by \((5)\),

\[
\sum_{k=1}^{n} \frac{q^k b_k(i)}{[k]_q} = \sum_{k=1}^{n} \frac{q^k}{[k]_q} \sum_{j=1}^{k} \left[ \begin{array}{c} k \\ j \end{array} \right]_q (-1)^j q^{j(\frac{j}{2})} a_j(i) = \sum_{j=1}^{n} (-1)^j q^{j(\frac{j}{2})} a_j(i) \sum_{k=1}^{n} \left[ \begin{array}{c} k \\ j \end{array} \right]_q \frac{q^k}{[j]_q}
\]
\[
= \sum_{j=1}^{n} (-1)^j q^{j(\frac{j}{2})} a_j(i) \sum_{k=j}^{n} \left[ \begin{array}{c} k-1 \\ j-1 \end{array} \right]_q q^k = \sum_{j=1}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right]_q (-1)^j q^{j(\frac{j}{2})+\frac{j^2}{2}} a_j(i)
\]
\[
= \sum_{j=1}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right]_q (-1)^j q^{j(\frac{j}{2})} a_j(i+1) = b_n(i+1).
\]

Therefore

\[
b_n(r) = \sum_{k_r=1}^{n} \frac{q^{k_r} b_{k_r}(r-1)}{[k_r]_q} = \sum_{k_r=1}^{n} \frac{q^{k_r}}{[k_r]_q} \sum_{k_{r-1}=1}^{k_r} \frac{q^{k_r-k_{r-1}} b_{k_{r-1}}(r-2)}{[k_{r-1}]_q} = \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_r \leq n} \prod_{i=1}^{r} q^{k_{i-1}} b_{k_1}
\]
and \((9)\) is proved.

Finally, by Lemma 1 in \(\text{[1]}\) (or use the inverse relation \((8)\) ), it follows that

\[
\sum_{k=1}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q (-1)^k q^{k(\frac{k}{2})-nk} (b_k(r) - b_{k-1}(r)) = \sum_{k=1}^{n} q^{-k} a_k(r),
\]
which yields \((10)\). \(\square\)
3. Proof of identities \((3)\) and \((4)\)

For \(s = (s_1, s_2, \ldots, s_l) \in (\mathbb{N}^*)^l\), let \(l(s) := l\) and \(w(s) := \sum_{i=1}^l s_i\) be respectively the length and the weight of the composition \(s\). In order to reduce the use of symbols indicating multiply nested sums, we introduce two definitions: the *\(q\)-multiple harmonic sum*

\[
H^q_k (s) = \sum_{0 < k_1 < k_2 < \cdots < k_l \leq n} \prod_{i=1}^l q^{(s_i-1)k_i} / [k_i]^q
\]

and

\[
T^q_k (s) = \sum_{i=1}^{w(s)} q^{j_i} / [j_i]^q
\]

where the sum is intended to be taken over all integers satisfying the conditions: \(1 \leq j_i \leq n\), \(j_i < j_{i+1}\) for \(i \in \{s_1, s_1 + s_2, \ldots, s_1 + s_2 + \cdots + s_{l-1}\}\) and \(j_l \leq j_{l+1}\) otherwise. Note that the following recursive relations hold

\[
H^q_k (s, r) = \sum_{k=1}^n q^{(r-1)k} H^q_{k-1} (s) / [k]^q,
\]  
(11)

and

\[
T^q_k (s, 1) = \sum_{k=1}^n q^k T^q_{k-1} (s) / [k]^q \quad \text{and} \quad T^q_k (s, r) = \sum_{k=1}^n q^k T^q_{k} (s, r-1) / [k]^q \quad \text{for } r > 1.
\]  
(12)

**Theorem 3.1.** Let \(s \in (\mathbb{N}^*)^l\) and \(r\) be a positive integer. Then

\[
\sum_{k=1}^n \left[ \begin{array}{c} n \\ k \end{array} \right] q^k_k q^k_2 H^q_{k-1} (s, r) = n q^n \sum_{k=1}^n \left[ \begin{array}{c} n \\ k \end{array} \right] q^k_2 \left( -1 \right)^{k} q^k_2 (r-1) H^q_{k-1} (s) / [k]^q
\]

\[
- \sum_{k=1}^n \left[ \begin{array}{c} n \\ k \end{array} \right] q^k_2 \left( -1 \right)^{k} q^k_2 (r+1) H^q_{k-1} (s) / [k]^q.
\]  
(13)

**Proof.** By using \((11)\) and \((3)\), we have that

\[
\sum_{k=1}^n \left[ \begin{array}{c} n \\ k \end{array} \right] q^k_2 H^q_{k-1} (s, r) = \sum_{k=1}^n \left[ \begin{array}{c} n \\ k \end{array} \right] q^k_2 \sum_{j=1}^{k-1} q^{(r-1)j} H^q_{j-1} (s) / [j]^q
\]

\[
= \sum_{j=1}^{k-1} q^{(r-1)j} H^q_{j-1} (s) / [j]^q \sum_{k=j+1}^n \left[ \begin{array}{c} n \\ k \end{array} \right] \left( -1 \right)^{k} q^k_2
\]

\[
= \sum_{j=1}^{k-1} q^{(r-1)j} H^q_{j-1} (s) / [j]^q \sum_{k=j+1}^n \left[ \begin{array}{c} n \\ k \end{array} \right] \left( -1 \right)^{k} q^k_2 (r+1) H^q_{k-1} (s) / [k]^q
\]

\[
= \sum_{j=1}^{k-1} \left( -q^k_2 (n-j) \right) / [n,j]_q \sum_{j=1}^{n} \left( -q^k_2 (n-j) + (r-1)j \right) / [j]_{q}^{r-1} H^q_{j-1} (s),
\]

and the proof is complete as soon as we note that

\[
q^n / [n]_q - q^j / [j]_q = \frac{q^n (1 - q^j) - q^j (1 - q^n)}{[n]_q[j]_q (1 - q)} = \frac{(-q^j) [n-j]_q}{[n]_q[j]_q}.
\]
In the following theorem, the identities (15) and (16) are equivalent to (3) and (4) respectively.

**Theorem 3.2.** Let \( s \in (\mathbb{N}^*)^l \) and let \( r \) be a positive integer. Then

\[
\begin{align*}
\sum_{k=1}^{n} \left[ \binom{n}{k} q^k \right] (-1)^k q^{(k)} H^n_{k-1}(s) &= (-1)^{l+1} T^n_{n-1}(s), \quad (14) \\
\sum_{k=1}^{n} \left[ \binom{n}{k} q^k \right] \frac{(-1)^k q^{(k)} + rk q^k}{[k]^r_q} H^n_{k-1}(s) &= (-1)^{l+1} T^n_{n}(s, r), \quad (15) \\
\sum_{k=1}^{n} \left[ \binom{n}{k} q^k \right] (-1)^k q^{(k)} \cdot \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_r = k} \prod_{i=1}^{r} \frac{q^{k_i}}{[k_i]^r_q} \cdot T^n_{k_i-1}(s) &= (-1)^{l+1} H^n_{n}(s, r). \quad (16)
\end{align*}
\]

**Proof.** We proceed by induction on the length \( l \).

The base case \( l = 0 \) of (14) is true because by (6)

\[
\sum_{k=1}^{n} \left[ \binom{n}{k} q^k \right] (-1)^k q^{(k)} = -1.
\]

The base case \( l = 0 \) of (15) follows from (9) and the previous equation

\[
\sum_{k=1}^{n} \left[ \binom{n}{k} q^k \right] (-1)^k q^{(k)} \cdot \frac{[k]^r_q}{[k]^r_q} = -T^n_{n}(r).
\]

In a similar way, for \( l = 0 \), (16) implies (10).

Now assume that \( l > 0 \) and let \( s = (t, s) \). By (14),

\[
\begin{align*}
\sum_{k=1}^{n} \left[ \binom{n}{k} q^k \right] (-1)^k q^{(k)} H^n_{k-1}(t, s) &= \frac{q^n}{[n]^q_q} \sum_{k=1}^{n} \left[ \binom{n}{k} q^k \right] \frac{(-1)^k q^{(k)} + (s-1)^k H^n_{k-1}(t)}{[k]^r_q} \\
&\quad - \sum_{k=1}^{n} \left[ \binom{n}{k} q^k \right] (-1)^k q^{(k)} H^n_{k-1}(t).
\end{align*}
\]

If \( s = 1 \) then by the inductive step and (12)

\[
\sum_{k=1}^{n} \left[ \binom{n}{k} q^k \right] (-1)^k q^{(k)} H^n_{k-1}(t, 1) = \frac{q^n}{[n]^q_q} (-1)^{(l+1)} T^n_{n-1}(t) - (-1)^{(l+1)} T^n_{n}(t, 1)
\]

\[
= (-1)^{l+1} \left( T^n_{n}(t, 1) - \frac{q^n T^n_{n-1}(t)}{[n]^q_q} \right) = (-1)^{l+1} T^n_{n-1}(s).
\]

On the other hand, if \( s > 1 \) then by the inductive step and (12)

\[
\sum_{k=1}^{n} \left[ \binom{n}{k} q^k \right] (-1)^k q^{(k)} H^n_{k-1}(t, s) = \frac{q^n}{[n]^q_q} (-1)^{(l+1)} T^n_{n}(t, s-1) - (-1)^{(l+1)} T^n_{n}(t, s)
\]

\[
= (-1)^{l+1} \left( T^n_{n}(t, s) - \frac{q^n T^n_{n}(t, s-1)}{[n]^q_q} \right) = (-1)^{l+1} T^n_{n-1}(s),
\]

and therefore (14) holds.
Putting $a_n = H^q_n(s)$ and $b_n = (-1)^{d+1}T^q_{n-1}(s)$ in (9) and using (14), we immediately get

$$
\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k q^k}{q^k} H^q_{k-1}(s) = \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_r \leq n} \prod_{j=1}^{r} \frac{q^{k_j}}{[k_j]_q} \cdot (-1)^{d+1} T^q_{k_1-1}(s)
$$

and the proof of (15) is complete. Similarly, by (14) and (10), we obtain (16). \hfill \Box

4. An application of [3] to a $q$-congruence

In [17], J. Zhao proved that for any integer $d \geq 0$ and for any prime $p > 2d + 3$,

$$
H_{p-1}(\{1\}^d, 2, \{1\}^d) \equiv \left(-1\right)^d \left(\frac{2d + 3}{d + 1} + 1\right) \frac{pB_{p-(2d+3)}}{2(2d + 3)} \pmod{p^2}
$$

where

$$
H_n(s) = \lim_{q \to 1} H^q_n(s) = \sum_{0 < k_1 < k_2 < \cdots < k_i \leq n} \frac{1}{k_i^s},
$$

is the ordinary multiple harmonic sum, $B_n$ is the $n$-th Bernoulli number, and $\{1\}^d$ means that the number 1 is repeated $d$ times. Now we consider the $q$-analog of the above congruence modulo $[p]_q$, which is an irreducible polynomial in $q$ when $p$ is a prime. Then the next statement holds.

**Theorem 4.1.** For any integer $d \geq 0$ and for any prime $p > 2d + 3$,

$$
H^q_{p-1}(\{1\}^d, 2, \{1\}^d) \equiv -\left(\frac{p + 1}{2d + 3}\right) \left(1 - q\right)^{2d+2} \frac{2}{2p} \pmod{[p]_q}.
$$

**Proof.** For $1 \leq j < p$, $[p - j]_q = q^{-j}([p]_q - [j]_q)$, which implies that

$$
\begin{align*}
\left[\frac{p - 1}{k}\right]_q^k &= \prod_{j=1}^{k} \frac{[p - j]_q}{[j]_q} \equiv \prod_{j=1}^{k} (-q^{-j}) = \left(-1\right)^k q^{-\binom{k}{2}} \pmod{[p]_q}.
\end{align*}
$$

By (3), for $n = p - 1$, we obtain the following congruences modulo $[p]_q$,

$$
-H^q_{p-1}(\{1\}^d, 2, \{1\}^d) \equiv \sum_{1 \leq j_1 < \cdots < j_d < j_{d+1} \leq j_{d+2} < \cdots < j_{2d+2} \leq p-1} \frac{2d+2}{[j]_q} q^{j_i} \prod_{i=1}^{2d+2} \frac{q^{j_i}}{[j]_q}.
$$

The right-hand side can be decomposed as

$$
\sum_{1 \leq j_{d+2} < \cdots < j_{2d+2} \leq j_{d+1} < j_d \leq \cdots \leq j_1 \leq p-1} \left(1 - q\right)[j_{d+1}]_q + q^{j_{d+1}} \prod_{i=1}^{2d+2} \frac{1}{[j]_q} + \sum_{1 \leq j_{d+2} < \cdots < j_1 \leq p-1} \prod_{i=1}^{2d+2} \frac{1}{[j]_q}.
$$

Therefore

$$
-H^q_{p-1}(\{1\}^d, 2, \{1\}^d) \equiv (1 - q)H^q_{p-1}(\{1\}^{2d+1}) + H^q_{p-1}(\{1\}^d, 2, \{1\}^d) + H^q_{p-1}(\{1\}^{2d+2}).
$$
By Corollary 2.2 in [18], for any integer $r > 0$ such that $p > r + 1$ we have

$$H_{p-1}^{q}(\{1\}^r) \equiv \left(\frac{p-1}{r}\right) (1-q)^r \pmod{[p]_q}.$$ 

Hence

$$H_{p-1}^{q}(\{1\}^d, 2, \{1\}^d) = -\frac{1}{2} \left((1-q)H_{p-1}^{q}(\{1\}^{2d+1}) + H_{p-1}^{q}(\{1\}^{2d+2})\right)$$

$$= -\frac{1}{2} \left((1-q) \left(\frac{p-1}{2d+1}\right) (1-q)^{2d+1} + \left(\frac{p-1}{2d+2}\right) (1-q)^{2d+2}\right)$$

$$= -\left(\frac{p+1}{2d+3}\right) \left(1-q\right)^{2d+2} \pmod{[p]_q}.$$ 

□

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