Abstract

We use a $G_2$-structure on a 7-dimensional Riemannian manifold with a fixed metric to define an octonion bundle with a fiberwise non-associative product. We then define a metric-compatible octonionic covariant derivative on this bundle that is compatible with the octonion product. The torsion of the $G_2$-structure is then shown to be an octonionic connection for this covariant derivative with curvature given by the component of the Riemann curvature that lies in the 7-dimensional representation of $G_2$. We also interpret the choice of a particular $G_2$-structure within the same metric class as a choice of gauge and show that under a change of this gauge, the torsion does transform as an octonion-valued connection 1-form. Finally, we also show an explicit relationship between the octonion bundle and the spinor bundle, define an octonionic Dirac operator and explore an energy functional for octonion sections. We then prove that critical points correspond to divergence-free torsion, which is shown to be an octonionic analog of the Coulomb gauge.

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1 Introduction

Seven-dimensional manifolds with a $G_2$-structure have been of great interest in differential geometry and theoretical physics ever since Alfred Gray studied vector cross products on orientable manifolds in 1969 \[18\]. It turns out that a 2-fold vector cross product - that is, one that takes two vectors and outputs another one, exists only in 3 dimensions and in 7 dimensions. The 3-dimensional vector cross product is very well known in $\mathbb{R}^3$ and on a general oriented 3-manifold it comes from the volume 3-form, so it is a special case of a $(n-1)$-fold vector cross product in a $n$-dimensional space, where it also comes from the volume form. In 7 dimensions, however, the vector cross product structure is even more special, since it is not part of an infinite sequence. The 3-dimensional and 7-dimensional vector cross products do however have something in common since they are closely related to the normed division algebras - the quaternions and octonions, which are 4 and 8 dimensional, respectively. In fact, the 3-dimensional vector cross product can be obtained by restricting the quaternion product to the purely imaginary quaternions and then taking the projection to the imaginary part. The 7-dimensional vector cross product is induced from the octonion product in a similar way. Note that the only other normed division algebras are $\mathbb{R}$ and $\mathbb{C}$, so the only non-trivial vector cross products obtained this way are in 3 and 7 dimensions. In 3 dimensions, the group that preserves the vector product is $SO(3)$, therefore, on a 3-dimensional manifold, given an oriented orthonormal frame bundle, i.e. an $SO(3)$-structure, we can always define a vector cross product, and moreover in will be parallel with respect to the Levi-Civita connection. On a 7-manifold, the group that preserves the vector cross product is now $G_2$ - this is the automorphism group of the octonion algebra, which is in particular a 14-dimensional exceptional Lie group. Therefore, in order to be able to define a vector cross product globally on a 7-manifold, we need to introduce a $G_2$-structure, which is now a reduction of the frame bundle to $G_2$. There is now a further topological obstruction for such a reduction - apart from the first Stiefel-Whitney class $w_1$ vanishing (which gives orientability), we also need the second Stiefel-Whitney class $w_2$ to vanish \[15\]. This is the condition for the manifold to admit a spin structure. Once we have a 7-manifold with both $w_1$ and $w_2$ vanishing, any Riemannian metric will give rise to an $SO(7)$-structure, and this could then be reduced to a $G_2$-structure. By specifying a $G_2$-principal bundle, we are effectively also defining a $G_2$-invariant 3-form $\varphi$, which gives rise to the structure constants for the vector cross product. A good review of vector cross product geometries can also be found in \[34\].

In general, the 3-form $\varphi$ will not be parallel with respect to the Levi-Civita connection, and thus the $G_2$-structure will have torsion. The different torsion classes have originally been classified by Fernández and Gray \[15\]. Understanding the existence properties of different torsion classes is of particular interest to theoretical physics, because in a compactification of 11-dimensional M-theory to an observable 4-dimensional space, it is necessary to use a 7-dimensional manifold which will necessarily admit a $G_2$-structure. The torsion of this $G_2$-structure will then affect the physical properties of the theory \[33\]. Of even greater interest, both in mathematics and physics, are torsion-free $G_2$-structures. A torsion-free $G_2$-structure then corresponds to a Riemannian metric with a reduced holonomy group. In particular, the holonomy group would have to be a subgroup of $G_2$. In even dimensions, thanks to Yau’s Theorem \[40\], we have necessary and sufficient conditions for the existence of $SU(n)$ holonomy metrics - the Calabi-Yau metrics. For $G_2$ holonomy manifolds currently there is even no conjecture as to what the conditions could be.

Due to the close relationship between $G_2$ and octonions, it is natural to introduce an octonionic structure on a 7-manifold with a $G_2$-structure. The aim of this paper is to develop the properties of an octonion bundle on a 7-manifold with $G_2$-structure. A number of properties of $G_2$-structures are re-expressed in a very natural form using the octonion formalism, and we believe that further progress in the study of $G_2$-structures could be made using this approach.

In Section 2 we give a brief introduction to $G_2$-structures and some of their basic properties. More detailed accounts of properties of $G_2$-structures can be found in \[10\] \[19\] \[21\] \[30\] \[31\]. In Section 3 we then introduce the octonion bundle - which is a rank 8 bundle with fibers $\mathbb{R} \oplus T_pM$. The scalar part corresponds to the real part of an octonion and the vector part corresponds to the imaginary part. The $G_2$-structure is then used to define a fiberwise nonassociative normed division algebra. We then also give some properties of the associator that will be used later. The subbundle of unit octonion sections then has a fiberwise Moufang loop structure on it - this is a nonassociative analog of a group, but with associativity replaced by weaker
properties.

It is well-known that given a Riemannian metric $g$ on a 7-manifold $M$ that admits $G_2$-structures, there is a family of compatible $G_2$-structures. Pointwise, such a family is parametrized by $SO(7)/G_2 \cong \mathbb{R}P^7$. In particular, given a $G_2$-structure 3-form $\varphi$, any unit section $A$ of the octonion bundle can be used to define a new $G_2$-structure $\sigma_A(\varphi)$ where $\sigma_A : \Omega^3(M) \to \Omega^3(M)$ is a map of 3-forms that is quadratic in $A$. The original expression for $\sigma_A(\varphi)$ is due to Bryant [10]. All the $G_2$-structures that correspond to $g$ are then of the form $\sigma_A(\varphi)$ for some unit octonion section $A$ and are called isometric $G_2$-structures. In Section 4 we focus on the properties of isometric $G_2$-structures and the map $\sigma_A$. In particular, we show that the action of $\sigma_A$ on $\varphi$ corresponds to the action of the adjoint map $\text{Ad}_A$ on the octonions (Theorem 4.8) and we use that to show that $\sigma$ is compatible with octonion multiplication - that is, $\sigma_A(\sigma_B(\varphi)) = \sigma_{AB}(\varphi)$ where $AB$ is the octonion product of $A$ and $B$ with respect to $\varphi$ (Theorem 4.8). This gives also a representation of the unit octonions on 3-forms.

In Section 5 we give a brief overview of the $G_2$-structure torsion and in Section 6 we use the torsion $T$ and octonion multiplication to introduce an octonionic covariant derivative $D$ given

$$D_X V = \nabla_X V - VT_X$$

where $T_X = X \cdot T$ is interpreted as an imaginary octonion section, and $VT_X$ is the octonion product of $T_X$ and $V$. This is then shown to be partially compatible with octonion multiplication, that is, given two octonion sections $U$ and $V$,

$$D_X (UV) = (\nabla_X U)V + U(D_X V)$$

where $T_X = X \cdot T$ is interpreted as an imaginary octonion section, and $VT_X$ is the octonion product of $T_X$ and $V$. This is then shown to be partially compatible with octonion multiplication, that is, given two octonion sections $U$ and $V$,

$$D_X (UV) = (\nabla_X U)V + U(D_X V)$$

We then prove that it is moreover metric compatible. The $G_2$-structure torsion then is interpreted as an Im $\mathbb{O}$-valued 1-form. In our case, the tangent bundle to the unit octonion Moufang loop is precisely the space of imaginary octonions, so this is the exact analog of a “Lie algebra-valued 1-form” that represents a connection on a principal bundle. Note that while the idea of constructing $G_2$-compatible connections using the torsion has been used in the past, such as in \cite{1, 2, 3}, the idea to interpret the $G_2$ torsion as a connection 1-form on a nonassociative bundle is new and is the key point in this paper. The curvature of this connection has a standard part that comes from Levi-Civita connection and a part that comes from the octonion structure. We prove that the octonion part of the curvature of this octonionic connection is $\frac{1}{4}\pi_7 \text{Riem}$, which is the component of the Riemann curvature that lies in the 7-dimensional representation of $G_2$. It is well known \cite{8, 31} that the vanishing of $\pi_7 \text{Riem}$ is a necessary condition for a torsion-free $G_2$-structure, and now we have a new interpretation of this quantity as an Im $\mathbb{O}$-valued 2-form that is the octonionic exterior covariant derivative of the torsion 1-form.

The octonion covariant derivative $D$ is defined with respect to a fixed $G_2$-structure $\varphi$. However, we know that we have a choice of isometric $G_2$-structures given by $\sigma_V(\varphi)$ for any unit octonion section $V$. In Section 7 we consider how $D$ is affected by a change of the $G_2$-structure within the metric class. To do this, we first prove in Theorem 7.2 how the torsion 1-form $T(V)$ for the $G_2$-structure $\sigma_V(\varphi)$ is related to the original torsion $T$. In turns out that

$$T(V) = VTV^{-1} + V(\nabla V^{-1}) = -(DV)V^{-1}.$$  \hspace{1cm} (1.3)

This relationship further reinforces the idea that the correct way of thinking of $G_2$ torsion is to regard it as an octonionic connection 1-form, since the expression (1.3) is very similar to the transformation of a principal bundle connection 1-form under a change of trivialization, i.e. a change of gauge. Moreover, this shows that the choice of the particular $G_2$-structure 3-form within the metric class corresponds to picking a gauge. Finally, using (1.3), we conclude that $D$ is indeed covariant with respect to a change of $G_2$-structure within the same metric class. In particular, if $D$ is the covariant derivative with respect to $\sigma_V(\varphi)$, then

$$D(AV^{-1}) = (DA)V^{-1}.$$  \hspace{1cm} (1.4)

This is then used to note that the metric class contains a torsion-free $G_2$-structure if and only if there exists a nowhere-vanishing octonion section that is parallel with respect to $D$. This condition is independent of the initial choice of the $G_2$-structure.
In turns out that much of the structure of the octonion bundle mirrors that of the spinor bundle on a 7-manifold. In Section 8 we make this relationship precise. It is well known that a $G_2$-structure may be defined by a unit spinor on the manifold. Under the correspondence between the spinor bundle and the octonion bundle, the fixed spinor is then mapped to 1. A change of the unit spinor then corresponds to a transformation $\sigma_V$ of the $G_2$-structure for some appropriate $V$. As it is well known, the enveloping algebra of the octonions, i.e. the algebra of left multiplication maps by an octonion under composition, is isomorphic to the Clifford algebra on spinors [24]. However, the enveloping algebra is by definition associative, so the correspondence of the octonion bundle with the spinor bundle only captures part of the structure of the octonion bundle. The full non-associative structure of the octonion bundle cannot be seen in the spinor bundle, therefore it is expected that the octonion carries more information than the spinor bundle, although the difference is subtle. In particular, while there is no natural binary operation on the spinor bundle, we can multiply octonions. In fact, Clifford multiplication of a vector and a spinor translates to multiplication of two octonions - therefore we are implicitly using the triality correspondence between vector and spinor representations.

Using the correspondence between spinors and octonions, in Section 9 we define an octonionic Dirac operator. In the torsion-free case, an octonionic Dirac operator had been defined by Karigiannis in [32], however this is the generalization for arbitrary $G_2$-structure torsion. Using octonion algebra we then give a direct proof of the octonionic Lichnerowicz-Weitzenböck formula. It also follows that the Dirac operator of an octonion section $\sigma$ determines the 1-dimensional and 7-dimensional components of the torsion of the corresponding $G_2$-structure $\sigma$ (this has also been recently shown directly for spinors in [2]). It is then a simple observation to note that on compact manifold, a $G_2$-structure that corresponds to a metric with vanishing total scalar curvature is torsion-free if and only if the 1-dimensional and 7-dimensional components of the torsion vanish.

The interpretation of the $G_2$-structure torsion as a connection 1-form and the choice of $G_2$-structure within a fixed metric class as a choice of gauge suggests that there needs to be a way to select the “best” gauge. A natural way of doing this is to consider critical points of a functional. On a compact manifold, a reasonable functional to consider is the $L_2$-norm of the torsion tensor, now regarded as a functional $\mathcal{E}(V)$ on the space of unit octonion sections, so that the metric remains fixed:

$$\mathcal{E}(V) = \frac{1}{2} \int_M |T(V)|^2 \, \text{vol} = \frac{1}{2} \int_M |DV|^2 \, \text{vol}. \quad (1.5)$$

We consider the basic properties of this functional in Section 10. Such a functional, but without the constraint on the metric, has been considered from different points of view by Weiss and Witt [39, 40] - as a functional on 3-forms, and by Ammann, Weiss and Witt [4] - as a functional on spinors. In our case, this reduces to an energy functional on unit octonion sections. The equation for a critical point is then shown to be

$$D^* DV - |DV|^2 V = 0. \quad (1.6)$$

This is very similar to the equation for harmonic unit vector fields [17, 33, 44], which is one of the equations that a vector field must satisfy in order to be a harmonic map from a manifold to the unit tangent bundle. In terms of $G_2$-structures, critical points of (1.5) then are shown to correspond to divergence-free torsion, i.e. a Coulomb gauge. This is not surprising, since in gauge theory, the Coulomb gauge corresponds precisely to critical points of the $L_2$-norm of the connection [14, 35]. This further reinforces the point of view that $G_2$-structure torsion is a connection 1-form for a non-associative gauge theory.

Conventions In this paper we will be using the following convention for Ricci and Riemann curvature:

$$\text{Ric}_{ij} = g^{ij} \text{Riem}_{ijkl} \quad (1.7)$$

Also, the convention for the orientation of a $G_2$-structure will same as the one adopted by Bryant [10] and follows the author’s previous papers. In particular, this causes $\psi = * \varphi$ to have an opposite sign compared to the works of Karigiannis, so many identities and definitions cited from [30, 32, 34] may have differing signs.
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2 $G_2$-structures

The 14-dimensional group $G_2$ is the smallest of the five exceptional Lie groups and is closely related to the octonions, which is the noncommutative, nonassociative, 8-dimensional normed division algebra. In particular, $G_2$ can be defined as the automorphism group of the octonion algebra. Given the octonion algebra $\mathbb{O}$, there exists a unique orthogonal decomposition into a real part, that is isomorphic to $\mathbb{R}$, and an imaginary (or pure) part, that is isomorphic to $\mathbb{R}^7$:

$$\mathbb{O} \cong \mathbb{R} \oplus \mathbb{R}^7$$

(2.1)

Correspondingly, given an octonion $a \in \mathbb{O}$, we can uniquely write

$$a = \text{Re} a + \text{Im} a$$

where $\text{Re} a \in \mathbb{R}$, and $\text{Im} a \in \mathbb{R}^7$. We can now use octonion multiplication to define a vector cross product $\times$ on $\mathbb{R}^7$. Given $u,v \in \mathbb{R}^7$, we regard them as octonions in $\text{Im} \mathbb{O}$, multiply them together using octonion multiplication, and then project the result to $\text{Im} \mathbb{O}$ to obtain a new vector in $\mathbb{R}^7$:

$$u \times v = \text{Im} (uv).$$

(2.2)

The subgroup of $GL(7,\mathbb{R})$ that preserves this vector cross product is then precisely the group $G_2$. A detailed account of the properties of the octonions and their relationship to exceptional Lie groups is given by John Baez in [5]. The structure constants of the vector cross product define a 3-form on $\mathbb{R}^7$, hence $G_2$ is alternatively defined as the subgroup of $GL(7,\mathbb{R})$ that preserves a particular 3-form $\varphi_0$ [29].

**Definition 2.1** Let $(e^1, e^2, ..., e^7)$ be the standard basis for $(\mathbb{R}^7)^*$, and denote $e^i \wedge e^j \wedge e^k$ by $e^{ijk}$. Then define $\varphi_0$ to be the 3-form on $\mathbb{R}^7$ given by

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$  

(2.3)

Then $G_2$ is defined as the subgroup of $GL(7,\mathbb{R})$ that preserves $\varphi_0$.

In general, given a $n$-dimensional manifold $M$, a $G$-structure on $M$ for some Lie subgroup $G$ of $GL(n,\mathbb{R})$ is a reduction of the frame bundle $F$ over $M$ to a principal subbundle $P$ with fibre $G$. A $G_2$-structure is then a reduction of the frame bundle on a 7-dimensional manifold $M$ to a $G_2$-principal subbundle. The obstructions for the existence of a $G_2$-structure are purely topological.

**Theorem 2.2** ([15] [16] [18]) Let $M$ be a 7-dimensional smooth manifold. Then, $M$ admits a $G_2$-structure if and only if the Stiefel-Whitney classes $w_1$ and $w_2$ both vanish.

Thus, given a 7-dimensional smooth manifold that is both orientable ($w_1 = 0$) and admits a spin structure ($w_2 = 0$), there always exists a $G_2$-structure on it.

It turns out that there is a 1-1 correspondence between $G_2$-structures on a 7-manifold and smooth 3-forms $\varphi$ for which the 7-form-valued bilinear form $B_{\varphi}$ as defined by (2.4) is positive definite (for more details, see [9] and the arXiv version of [25]).

$$B_{\varphi} (u, v) = \frac{1}{6} (u \wedge \varphi) \wedge (v \wedge \varphi) \wedge \varphi$$

(2.4)

Here the symbol $\wedge$ denotes contraction of a vector with the differential form:

$$(u \wedge \varphi)_{mn} = u^a \varphi_{amn}.$$ 

(2.5)
parametrized by sections of the induced $\mathbb{R}P^7$-bundle. Thus, also using Theorem 2.3, we have the following holds
\[
g_\varphi(u, v)\text{vol}_\varphi = \frac{1}{6} (u \varphi) \wedge (v \varphi) \wedge \varphi \tag{2.6}
\]
An equivalent way of defining a positive 3-form $\varphi$, is to say that at every point, $\varphi$ is in the $GL(7, \mathbb{R})$-orbit of $\varphi_0$. It can be easily checked that the metric (2.6) for $\varphi = \varphi_0$ is in fact precisely the standard Euclidean metric $g_0$ on $\mathbb{R}^7$. Therefore, every $\varphi$ that is in the $GL(7, \mathbb{R})$-orbit of $\varphi_0$ has an associated Riemannian metric $g$, that is in the $GL(7, \mathbb{R})$-orbit of $g_0$. The only difference is that the stabilizer of $g_0$ (along with orientation) in this orbit is the group $SO(7)$, whereas the stabilizer of $\varphi_0$ is $G_2 \subset SO(7)$. This shows that positive 3-forms forms that correspond to the same metric, i.e., are isometric, are parametrized by $SO(7)/G_2 \cong \mathbb{RP}^7 \cong S^7/\mathbb{Z}_2$. Therefore, on a Riemannian manifold, metric-compatible $G_2$-structures are parametrized by sections of an $\mathbb{RP}^7$-bundle, or alternatively, by sections of an $S^7$-bundle, with antipodal points identified. In fact, the precise parametrization of isometric $G_2$-structures is well-known.

**Theorem 2.3** ([10]) Let $M$ be a 7-dimensional smooth manifold. Suppose $\varphi$ is a positive 3-form on $M$ with associated Riemannian metric $g$. Then, any positive 3-form $\tilde{\varphi}$ for which $g$ is also the associated metric, is given by the following expression:
\[
\tilde{\varphi} = \left( a^2 - |\alpha|^2 \right) \varphi - 2a \alpha \wedge (\ast \varphi) + 2\alpha \wedge (\ast \varphi) \tag{2.7}
\]
where $a$ is a scalar function on $M$ and $\alpha$ is a vector field such that
\[
a^2 + |\alpha|^2 = 1 \tag{2.8}
\]
Note that the relation (2.8) shows that indeed $(a, \alpha) \in S^7$, and moreover, in the expression (2.7), simultaneously changing the sign of $a$ and $\alpha$ keeps $\varphi$ unchanged. The pair $(a, \alpha)$ can in fact be also interpreted as a unit octonion section, where $a$ is the real part, and $\alpha$ is the imaginary part. It turns out that this is a natural formalism in which to study isometric $G_2$-structures, and the main aim of this paper is to develop this point of view.

An alternative way of studying $G_2$-structures is in terms of spinors. A detailed account of the relationship between Spin group and $G_2$ can be found in [5], while explicit descriptions of different $G_2$-structures in terms of spinors can be found in [3] [6], among others. This approach also makes a connection with other types of $G$-structures on manifolds, with $SU(3)$-structures on 6-manifolds being of particular interest. Also, since in this approach, a $G_2$-structure is defined by a single nowhere vanishing spinor, it is also of relevance in theoretical physics, e.g. [34]. Now, given a 7-dimensional manifold $M$ with $w_1 = w_2 = 0$, that is, one that satisfies the conditions for the existence of $G_2$-structures from Theorem 2.2, a Riemannian metric will define a spinor bundle which will necessarily admit a nowhere vanishing section. Any such spinor section will then define a positive 3-form, and hence a compatible $G_2$-structure on $M$. Moreover, spinor sections within the same projective class define the same $G_2$-structure. Therefore, metric-compatible $G_2$-structures are parametrized by sections of the induced $\mathbb{RP}^7$-bundle. Thus, also using Theorem 2.3, we have the following corollary of Theorem 2.2.

**Corollary 2.4** Suppose $M$ is a smooth 7-dimensional manifold that satisfies $w_1 = 0$ and $w_2 = 0$. Given any Riemannian metric $g$ on $M$, there exists a family of $G_2$-structures for which $g$ is the associated metric.

**Definition 2.5** If two $G_2$-structures $\varphi_1$ and $\varphi_2$ on $M$ have the same associated metric $g$, we say that $\varphi_1$ and $\varphi_2$ are in the same metric class.

Using this observation we see that the set of all $G_2$-structures on $M$ is parametrized by the set of Riemannian metrics on $M$, and then within each metric class, by projective classes of unit octonion sections.
The octonion bundle

Let \((M, g)\) be a smooth 7-dimensional Riemannian manifold, with \(w_1 = w_2 = 0\). We know \(M\) admits \(G_2\)-structures. In particular, let \(\varphi\) be a \(G_2\)-structure for \(M\) for which \(g\) is the associated metric. We also use \(g\) to define the Levi-Civita connection \(\nabla\), and the Hodge star \(*\). In particular, \(*\varphi\) is a 4-form dual to \(\varphi\), which we will denote by \(\psi\).

**Definition 3.1** The octonion bundle \(\mathbb{O}M\) on \(M\) is the rank 8 real vector bundle given by

\[
\mathbb{O}M \cong \Lambda^0 \oplus TM
\]

where \(\Lambda^0 \cong M \times \mathbb{R}\) is a trivial line bundle. At each point \(p \in M\), \(\mathbb{O}_p M \cong \mathbb{R} \oplus T_p M\).

The definition \((3.1)\) simply mimics the decomposition of octonions into real and imaginary parts. The bundle \(\mathbb{O}M\) is defined as a real bundle, but which will have additional structure as discussed below. Now let \(A \in \Gamma(\mathbb{O}M)\) be a section of the octonion bundle. We will call \(A\) simply an octonion on \(M\). From \((3.1)\), \(A\) has a scalar component in \(\Gamma(\Lambda^0)\), i.e. just a function on \(M\), as well as a vector component in \(\Gamma(TM)\), i.e. a vector field on \(M\). We then have globally defined projections

\[
\begin{align*}
\text{Re} : \Gamma(\mathbb{O}M) &\rightarrow \Gamma(\Lambda^0) \\
\text{Im} : \Gamma(\mathbb{O}M) &\rightarrow \Gamma(TM).
\end{align*}
\]

Therefore, we write \(A = \text{Re}A + \text{Im}A\). For convenience, we may also write \(A = (\text{Re}A, \text{Im}A)\) or \(A = \begin{pmatrix} \text{Re}A \\ \text{Im}A \end{pmatrix}\). We also have a natural involution on \(\mathbb{O}M\) - conjugation. As for complex or quaternionic sections, define

\[
\bar{A} = (\text{Re}A, -\text{Im}A).
\]

Since \(\mathbb{O}M\) is defined as a tensor bundle, the Riemannian metric \(g\) on \(M\) induces a metric on \(\mathbb{O}M\). Let \(A = (a, \alpha) \in \Gamma(\mathbb{O}M)\). Then,

\[
|A|^2 = \langle A, A \rangle = a^2 + g(\alpha, \alpha) = a^2 + |\alpha|^2
\]

We will be using the same notation for the norm, metric and inner product for sections of \(\mathbb{O}M\) as for standard tensors on \(M\). It will be clear from the context which is being used. If however we need to specify that only the octonion inner product is used, we will use the notation \(\langle \cdot, \cdot \rangle_\mathbb{O}\). The metric \((3.3)\) ensures that the real and imaginary parts are orthogonal to each other.
Definition 3.2 Given the $G_2$-structure $\varphi$ on $M$, we define a vector cross product with respect to $\varphi$ on $M$. Let $\alpha$ and $\beta$ be two vector fields, then define
\[
\langle \alpha \times \varphi \beta, \gamma \rangle = \varphi(\alpha, \beta, \gamma)
\] (3.4)
for any vector field $\gamma$.

In index notation, we can thus write
\[
(\alpha \times \varphi \beta)^a = \varphi_{bc}^a \alpha^b \beta^c
\] (3.5)
Note that $\alpha \times \varphi \beta = -\beta \times \varphi \alpha$. If there is no ambiguity as to which $G_2$-structure is being used to define the cross product, we will simply denote it by $\times$, dropping the subscript.

Using the contraction identity for $\varphi$
\[
\varphi_{abc} \varphi_{mn}^c = g_{am} g_{bn} - g_{an} g_{bm} + \psi_{abmn}
\] (3.6)
we obtain the following identity for the double cross product.

Lemma 3.3 Let $\alpha, \beta, \gamma$ be vector fields, then
\[
\alpha \times (\beta \times \gamma) = \langle \alpha, \gamma \rangle \beta - \langle \alpha, \beta \rangle \gamma + \psi(\sharp, \alpha, \beta, \gamma)
\] (3.7)
where $\sharp$ means that we raise the index using the inverse metric $g^{-1}$.

Using the inner product and the cross product, we can now define the *octonion product* on $\mathcal{O}M$.

Definition 3.4 Let $A, B \in \Gamma(\mathcal{O}M)$. Suppose $A = (a, \alpha)$ and $B = (b, \beta)$. Given the vector cross product (3.4) on $M$, we define the octonion product $A \circ \varphi B$ with respect to $\varphi$ as follows:
\[
A \circ \varphi B = \begin{pmatrix}
ab - \langle \alpha, \beta \rangle \\
\alpha \beta + b \alpha + \alpha \times \varphi \beta
\end{pmatrix}
\] (3.8)
If there is no ambiguity as to which $G_2$-structure is being used to define the octonion product, for convenience, we will simply write $AB$ to denote it. The octonion product behaves as expected with respect to conjugation:

Lemma 3.5 Suppose $A$ and $B$ are sections of $\mathcal{O}M$, then
\[
\overline{AB} = \overline{B} \overline{A} \quad \text{(3.9a)}
\]
\[
\overline{AA} = \overline{A}A = |A|^2 \quad \text{(3.9b)}
\]

Lemma 3.6 Using the octonion product (3.8), the inner product $\langle A, B \rangle$ of two octonions $A = (a, \alpha)$ and $B = (b, \beta)$ is given by
\[
\langle A, B \rangle = \frac{1}{2} (A\overline{B} + B\overline{A}) .
\] (3.10)
The commutator $[A, B]$ of $A$ and $B$ is given by
\[
[A, B] = AB - BA = 2 \alpha \times \beta = 2 \varphi(\sharp, \alpha, \beta)
\] (3.11)
The associator $[A, B, C]$ of three octonions $A, B$ and $C = (c, \gamma)$ is given by
\[
[A, B, C] = A(BC) - (AB) C = 2 \psi(\sharp, \alpha, \beta, \gamma)
\] (3.12)
**Proof.** The identities for the inner product and the commutator follow immediately from (3.8). The identity for (3.12) follows from the double cross product identity (3.7). □

Crucially, (3.12) shows that the associator is skew-symmetric. This property of the octonion algebra is known as *alternativity*. Note that the associator of a non-associative algebra is usually defined with the opposite sign to what we have, however due to the relation with $\psi$ it is more convenient this way. Thus, given a $G_2$-structure on $M$, we can fully transfer the octonion algebra structure to $\mathcal{O}M$. The expressions (3.8)-(3.12) are exactly the same as for the standard octonion algebra, as given, for example, in [5]. Therefore, $\mathcal{O}M$ is a bundle that carries a non-associative division algebra structure on it.

We will need a few identities that the octonions satisfy. The proofs are straightforward computations and are given in Appendix A.

**Lemma 3.7** Let $A = (0, \alpha) \in \Gamma (\text{Im} \mathcal{O}M)$. Then the exponential of $A$, $e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$, is given by

$$e^A = \cos |\alpha| + \alpha \frac{\sin |\alpha|}{|\alpha|}$$

(3.13)

**Corollary 3.8** Suppose $B = (b, \beta) \in \Gamma (\mathcal{O}M)$, then for any $k \in \mathbb{Z}$ (assuming nowhere vanishing $B$ if $k$ is negative)

$$B^k = |B|^k \left( \cos k\theta + \beta \frac{\sin k\theta}{|\beta|} \right)$$

where $\beta = \frac{\alpha}{|B|}$ and $\theta \in \mathbb{R}$ is such that $\cos \theta = \frac{b}{|B|}$ and $\sin \theta = |\beta|$.

The above Lemma 3.7 and Corollary 3.8 are direct analogs of similar well-known results for complex numbers. It is a useful fact that the for any integer power $k$, Im $B^k$ is a real multiple of Im $B$. Using the alternative property of the associator, this also shows that the octonions are *power-associative*, and in fact, any subalgebra generated by two elements and their conjugates is also associative. In Lemma 3.9 we collect some related identities.

**Lemma 3.9** For any $A, B, C \in \Gamma (\mathcal{O}M)$, and $k \in \mathbb{Z}^+$, the following identities hold

1. $[A, B, C] = -[\bar{A}, B, C]$
2. $[A^k, A, C] = 0$
3. $A [A, B, C] = [A, B, C] \bar{A}$
4. $[A, A^k B, C] = \bar{A}^k [A, B, C]$
5. $[A, B A^k, C] = [A, B, C] \bar{A}^k$
6. $[A^{k+1}, B, C] = [A^k, B, C] \bar{A} + [A, B, C] A^k$

For $k = 1$ and $k = 2$, the last identity in Lemma 3.9 gives us important special cases.

**Corollary 3.10** In particular,

1. $[A^2, B, C] = [A, B, C] (A + \bar{A})$
2. $[A^3, B, C] = [A, B, C] (\bar{A}^2 + |A|^2 + A^2)$

For a given octonion $B \in \Gamma (\mathcal{O}M)$ we may define the right translation map $R_B : \Gamma (\mathcal{O}M) \rightarrow \Gamma (\mathcal{O}M)$ and the left translation map $L_B : \Gamma (\mathcal{O}M) \rightarrow \Gamma (\mathcal{O}M)$ by

$$R_B A = AB$$
$$L_B A = BA$$

Whenever $B \neq 0$, these are invertible maps since $R_B R_B^{-1} A = (AB^{-1}) B = A (B^{-1} B) = A$ and similarly for $L$. As expected, the conjugates of $R_B$ and $L_B$ with respect to the octonion metric are given by $\bar{R}_B$ and $\bar{L}_B$ respectively.
Lemma 3.11 For any octonions $A, B, C \in \Gamma(O^M)$, we have
\[
\langle RB A, C \rangle = \langle A, RB C \rangle \quad (3.15)
\]
\[
\langle LB A, C \rangle = \langle A, LB C \rangle \quad (3.16)
\]
From this we can see that
\[
\langle RB A, RB C \rangle = |B|^2 \langle A, C \rangle
\]
and similarly for the left translation map. Therefore, whenever $B \neq 0$, every $RB$ and $LB$ is an element of the conformal group of $\mathbb{R}^8$. Moreover, when $|B| = 1$, $RB$ and $LB$ preserve the octonion metric, and are thus elements of the $O(8)$ group. Due to the nonassociativity of octonion multiplication, neither the left nor the right translation maps form subgroups of $O(8)$. In fact, the left and right maps in general do not commute, and the associator can be thought of as the commutator of the right and left translations:
\[
[L_A, RB] B = A (BC) - (AB) C = [A, B, C] \quad (3.17)
\]
Since multiplication by unit octonions preserves the norm, we can restrict the octonion multiplication to unit octonions. Hence we define the subbundle of unit octonions.

Definition 3.12 Define the unit octonion bundle $S\Omega M$ on $M$ as the unit sphere subbundle of $\Omega M$ where at each point $p \in M$, the fiber is given by $S\Omega p M = \{ A \in \Omega p M : |A| = 1 \}$.

We can restrict octonion multiplication to unit octonions, so the fiber at each point is the 7-sphere $S^7$ with a non-associative binary operation defined on it. In fact, the set of unit octonions form a Moufang loop - an algebraic structure with similar properties to a group, except that it is non-associative. Instead of associativity, we have weaker properties, as given by Lemma 3.9. The bundle $S\Omega M$ can then be regarded as a principal Moufang loop bundle - analogous to a principal bundle.

4 Isometric $G_2$-structures

Since the octonions are power-associative we can unambiguously define the adjoint map.

Definition 4.1 For any nowhere-vanishing $V \in \Gamma(O^M)$, define the map
\[
Ad_V : \Gamma(O^M) \rightarrow \Gamma(O^M)
\]
given by
\[
Ad_V A = VAV^{-1} \quad (4.1)
\]
for any $A \in \Gamma(O^M)$.

The adjoint map satisfies a number of properties. In particular, it is easy to see that $(Ad_V)^{-1} = Ad_{V^{-1}}$, so it is invertible. Also, as we show below, $Ad_V$ preserves the octonion metric. Let $A, B \in \Gamma(O^M)$, then
\[
\langle Ad_V A, Ad_V B \rangle = \langle VAV^{-1}, VBV^{-1} \rangle = \frac{1}{|V|} \langle VAV, VBV \rangle = \langle A, B \rangle
\]
Therefore, $Ad_V \in O(8)$. However, $Ad_V$ preserves the multiplicative identity of $O$, and therefore maps imaginary octonions to imaginary octonions. It also follows trivially that for $\alpha \in \text{Im} \Gamma(O^M)$, $Ad_V \alpha = \overline{V \alpha V^{-1}} = -Ad_V \alpha$. Hence it restricts to pure imaginary octonions, and this restriction $Ad_V|_{\text{Im}O}$ lies in $O(7)$. Note that for brevity we will sometimes use $Ad_V$ to denote the restriction $Ad_V|_{\text{Im}O}$. It will be clear from the context that this is regarded as a map of imaginary octonions.
Note that \( \text{Ad}_kV = \text{Ad}_V \) for any nowhere-vanishing scalar \( k \), so in fact we can always assume that \( V \) is a unit octonion.

Using the octonion multiplication rules in terms of the \( G_2 \)-structure \( \varphi \) we can write out \( \text{Ad}_V |_{\text{Im}O} \) explicitly as an element in \( O(7) \). Suppose \( \beta \) is pure imaginary, and let \( V = (v_0, v) \), then
\[
\text{Ad}_V B = V \beta V^{-1}
\]
\[
= \frac{1}{\|V\|^2} (v_0 + v) \beta (v_0 - v)
\]
\[
= \frac{1}{\|V\|^2} (v_0 + v) ((v, \beta) + v_0 \beta + v \times \beta)
\]
\[
= \frac{1}{\|V\|^2} (v_0^2 \beta + 2v_0 v \times \beta + v \langle v, \beta \rangle + v \times (v \times \beta))
\]
\[
= \frac{1}{\|V\|^2} \left( (v_0^2 - |v|^2) \beta + 2v_0 v \times \beta + 2v \langle v, \beta \rangle \right)
\]
(4.2)

In index notation, this then gives the components of the matrix \( \text{Ad}_V |_{\text{Im}O} \):
\[
\left( \text{Ad}_V |_{\text{Im}O} \right)^a_b = \frac{1}{\|V\|^2} \left( (v_0^2 - |v|^2) \delta^a_b - 2v_0 (v \langle \varphi \rangle)^a_b + 2v^a v_b \right).
\]
(4.3)

Using this explicit description of \( \text{Ad}_V |_{\text{Im}O} \), a computation of the determinant in Maple shows that \( \det \left( \text{Ad}_V |_{\text{Im}O} \right) = +1 \) for any non-zero \( V \). The explicit calculation is somewhat messy. Therefore, in fact, \( \text{Ad}_V |_{\text{Im}O} \in SO(7) \).

Since \( \text{Ad}_V |_{\text{Re}O} = +1 \), we find that \( \text{Ad}_V \in SO(8) \).

Consider also the following identities.

**Lemma 4.2** Given a nowhere-vanishing octonion \( V \), the following identities hold for any \( A, B \in \Gamma (\mathbb{O}M) \)

1. \( (VA) (BV^{-1}) = \text{Ad}_V (AB) + [A, B, V^{-1}] (V + \bar{V}) \)
2. \( (AV^{-1}) (VB) = AB + [A, B, V^{-1}] V \)

The proof of Lemma 4.2 is given in the Appendix. Using these identities we can now see what happens to the octonion product under the action of \( \text{Ad} \):

**Proposition 4.3** Given a nowhere-vanishing octonion \( V \), the octonion product is transformed as follows
\[
(\text{Ad}_V A) (\text{Ad}_V B) = \text{Ad}_V (AB) + [A, B, V^{-1}] \left( V + \bar{V} + \frac{1}{\|V\|^2} V^3 \right)
\]
(4.4)

and in particular,
\[
\text{Ad}_{V^{-1}} [(\text{Ad}_V A) (\text{Ad}_V B)] = AB + [A, B, V^{-3}] V^3
\]
(4.5)
\[
= (AV^{-3}) (V^3 B)
\]
(4.6)

**Proof.** To work out \( (\text{Ad}_V A) (\text{Ad}_V B) \), we first use the identity 2 from Lemma 4.2 and the identities from Lemma 3.9
\[
(\text{Ad}_V A) (\text{Ad}_V B) = (VA) (BV^{-1})
\]
\[
= (VA) (BV^{-1}) + [VA, BV^{-1}, V^{-1}] V
\]
\[
= (VA) (BV^{-1}) + \bar{V} \left[ A, B, V^{-1} \right] \frac{V^2}{|V|^2}
\]
\[
= (VA) (BV^{-1}) + \left[ A, B, V^{-1} \right] \frac{V^3}{|V|^2}
\]
(4.7)
Now we apply identity 1 from Lemma 4.2

\[(\text{Ad}_V A) (\text{Ad}_V B) = \text{Ad}_V (AB) + [A, B, V^{-1}] (\ddot{V} + V) + [A, B, V^{-1}] \frac{V^3}{|V|^2} \]

\[= \text{Ad}_V (AB) + [A, B, V^{-1}] \left(\ddot{V} + V + \frac{V^3}{|V|^2}\right) \tag{4.8}\]

For the second part, we just apply \(\text{Ad}_{V^{-1}}\) to (4.2) and then rewrite using the fact that the subalgebra generated by the two elements \(V\) and \([A, B, V^{-1}]\) is associative.

\[\text{Ad}_{V^{-1}} [(\text{Ad}_V A) (\text{Ad}_V B)] = AB + V^{-1} \left([A, B, V^{-1}] \left(\ddot{V} + V + \frac{V^3}{|V|^2}\right)\right) \]

\[= AB + (V^{-1} [A, B, V^{-1}]) \left(\ddot{V} + V + \frac{V^3}{|V|^2}\right) \tag{4.9}\]

Applying the identities from Lemma 3.9 we get

\[\text{Ad}_{V^{-1}} [(\text{Ad}_V A) (\text{Ad}_V B)] = AB - ([A, B, V] V) \left(\ddot{V} + V + \frac{V^3}{|V|^2}\right) \left(\frac{V}{|V|^2}\right) \]

\[= AB - [A, B, V] \left(\ddot{V} + V + \frac{V^3}{|V|^2}\right) \left(\frac{V^2}{|V|^2}\right) \]

\[= AB - [A, B, V] \left(\ddot{V}^2 + |V|^2 + V^2\right) \frac{V^3}{|V|^6} \tag{4.10}\]

Now we can use Corollary 3.10 to simplify the right-hand side of (4.9) to obtain

\[\text{Ad}_{V^{-1}} [(\text{Ad}_V A) (\text{Ad}_V B)] = AB - |V|^{-6} [A, B, V]^3 \frac{V^3}{V^3}\]

\[= AB + [A, B, V^{-3}] V^3 \tag{4.11}\]

Finally, using the identity 2 from Lemma 4.2 we can rewrite this as

\[\text{Ad}_{V^{-1}} [(\text{Ad}_V A) (\text{Ad}_V B)] = (AV^{-3}) \left(V^3 B\right).\]

Let us use the action of \(\text{Ad}_V\) to define a new octonion product \(A \circ_{V^3} B\) given by

\[A \circ_{V^3} B = \text{Ad}_V [(\text{Ad}_{V^{-1}} A) (\text{Ad}_{V^{-1}} B)] \tag{4.12}\]

The corresponding 3-form \(\varphi_{V^3}\) that defines the product \(A \circ_{V^3} B\) is then given by

\[\varphi_{V^3} (A, B, C) = \varphi (\text{Ad}_{V^{-1}} A, \text{Ad}_{V^{-1}} B, \text{Ad}_{V^{-1}} C) \tag{4.13}\]

We know that \(\text{Ad}_V\) is an invertible map. Therefore, \(\varphi_{V^3}\) is pointwise in the \(GL(7, \mathbb{R})\)-orbit of the original 3-form \(\varphi\), and is therefore another positive 3-form, so it defines a new \(G_2\)-structure. However, since \(\text{Ad}_V\) preserves the metric, \(\varphi_{V^3}\) has the same associated metric \(g\) as \(\varphi\). We will now give an explicit description of (4.13) and show that the descriptions of isometric \(G_2\)-structures (2.7) and (4.13) are equivalent. For convenience, let us define a map of 3-forms that gives (2.7):

**Definition 4.4** Let \(A = (a, \alpha)\) be a nowhere-vanishing octonion. Then, define the map \(\sigma_A : \Omega^3(M) \rightarrow \Omega^3(M)\) given by

\[\sigma_A (\varphi) = \frac{1}{|A|^2} \left(\varphi(a^2 - |\alpha|^2) - 2a \alpha \wedge (*\varphi) + 2a \wedge (\alpha \wedge \varphi)\right) \quad (4.14)\]
In particular, from Theorem 2.3, \( \sigma_A \) is a map of positive 3-forms, and moreover, it preserves the metric class. Note that \( \sigma_A = \sigma_{fA} \) for any nowhere vanishing function \( f \). Therefore, usually it is enough to take \( A \) as a unit octonion.

**Theorem 4.5** Let \((\varphi, g)\) be a \(G_2\)-structure on a smooth 7-dimensional manifold \(M\). Then, for any nowhere-vanishing octonion \(V\),

\[
\sigma_{V^3}(\varphi)(\cdot,\cdot,\cdot) = \varphi(\text{Ad}_{V^{-1}} A, \text{Ad}_{V^{-1}} B, \text{Ad}_{V^{-1}} C)
\]  

(4.15)

**Proof.** Note that for pure imaginary octonions \(A, B, C\),

\[
\varphi(A, B, C) = \langle A \times B, C \rangle = \langle AB, C \rangle
\]

Therefore,

\[
\varphi_{V^3}(A, B, C) = \varphi(\text{Ad}_{V^{-1}} A, \text{Ad}_{V^{-1}} B, \text{Ad}_{V^{-1}} C)
\]

\[
= \langle \text{Ad}_{V^{-1}} A \text{Ad}_{V^{-1}} B, \text{Ad}_{V^{-1}} C \rangle
\]

\[
= \langle \text{Ad}_{V} (\text{Ad}_{V^{-1}} A), \text{Ad}_{V^{-1}} B, C \rangle
\]

where we have used the fact that \(\text{Ad}_{V^{-1}} = \text{Ad}_V\). Now, using (4.5), we have

\[
\varphi_{V^3}(A, B, C) = \langle AB + [A, B, V^3] V^{-3}, C \rangle
\]

\[
= \varphi(A, B, C) + \langle [A, B, V^3] V^{-3}, C \rangle
\]

Suppose now \(V^3 = (u_0, u)\) and for convenience, let \(|V^3|^2 = u_0^2 + |u|^2 = M\), then,

\[
[A, B, V^3] V^{-3} = \frac{u_0}{M} [A, B, u] - \frac{1}{M} [A, B, u] \times u
\]

In index notation, and using (3.12) to express the associator in terms of \(\psi\), we get

\[
(\varphi_{V^3})_{abc} = \varphi_{abc} + \frac{2u_0}{M} \psi_{abcd} u^d - \frac{2}{M} \varphi_{cmn} \psi^m_{\,\,\,\,abd} u^d u^n
\]  

(4.16)

To simplify the last term we need to use a contraction identity between \(\varphi\) and \(\psi\) (see for example [21, 30]).

\[
\varphi_{abc} \psi_{mnp} = -3 \left( g_{a[m} \varphi_{np]b} - g_{b[m} \varphi_{np]a} \right)
\]  

(4.17)

where the square parentheses denote skew-symmetrization. Using this, we get

\[
\varphi_{cmn} \psi^m_{\,\,\,\,abdu^d u^n} = |u|^2 \varphi_{abc} - 3u_{[a} \varphi_{bc]} m u^m
\]

Therefore, (4.16), becomes

\[
(\varphi_{V^3})_{abc} = \left(1 - \frac{2}{M} |u|^2 \right) \varphi_{abc} + \frac{2u_0}{M} \psi_{abcd} u^d + \frac{6}{M} u_{[a} \varphi_{bc]} m u^m
\]

This can now be rewritten in coordinate-free notation as

\[
\varphi_{V^3} = \frac{1}{M} \left( \left( u_0^2 - |u|^2 \right) \varphi - 2u_0 u_{[a} \psi + 2u \wedge (u_{a} \varphi) \right)
\]

Comparing with (4.14), this is precisely equal to \(\sigma_{V^3}(\varphi)\). ■

**Remark 4.6** The expression (4.5) shows that \(\circ_{V^3} = \circ\) if and only if \(V^3\) is real. So if without loss of generality we assume \(V\) is a unit octonion, then the octonion product is preserved by \(\text{Ad}_V\) if and only if \(V^6 = 1\). This beautiful fact was originally discovered by Manogue and Schray in [35].
Example 4.7 Suppose $V = (\cos \frac{\pi}{2}, (\sin \frac{\pi}{2}) v) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}v\right)$ for some unit vector $v$. Then, $V^3 = (-1, 0)$, so $\sigma_{V^3}(\varphi) = \varphi$. However, $\text{Ad}_{V|_{\text{Im}O}} \in SO(7)$ is nontrivial. Using (4.3) we have an explicit expression

$$\left(\text{Ad}_{V|_{\text{Im}O}}\right)^a_b = -\frac{1}{2} g^a_b - \frac{\sqrt{3}}{2} (v \cdot \varphi)^a_b + \frac{3}{2} v^a v_b \quad (4.18)$$

Since $\text{Ad}_{V|_{\text{Im}O}}$ preserves $\varphi$, pointwise, $\text{Ad}_{V|_{\text{Im}O}} \in G_2$. In particular, for any nowhere-vanishing vector field $v$ on $M$, $\text{Ad}_V$ defines an octonion-product preserving map at every point in $M$.

Due to the nonassociativity of octonions, in general $\text{Ad}_U \text{Ad}_V \neq \text{Ad}_{UV}$. However we do have such a composition property for the map $\sigma_V$.

Theorem 4.8 Let $(\varphi, g)$ be a $G_2$-structure on a smooth 7-dimensional manifold $M$. Then, given nowhere-vanishing octonions $U$ and $V$,

$$\sigma_U(\sigma_V \varphi) = \sigma_{UV} (\varphi) \quad (4.19)$$

Before we prove Theorem 4.8 we need a few more properties of octonion products.

Remark 4.9 In Theorem 4.8, the octonion product is defined using $\varphi$. However, we will show that

$$U \circ \varphi V = U \circ_V V.$$  

Suppose that $V = (v_0, v)$. Then,

$$v_0 \sigma_V(\varphi) = \frac{1}{M} v \left[ (v_0^2 - |v|^2) \varphi - 2 v_0 v_0 \varphi + 2 v \wedge (v \cdot \varphi) \right]$$

$$= \frac{1}{M} \left( v_0^2 - |v|^2 \right) v_0 \varphi + 2 |v|^2 v \cdot \varphi$$

Therefore, any product with $V$ using $\circ_V$, defined by $\sigma_V(\varphi)$, will be equal to the product with using $\circ$, that is defined by $\varphi$. Hence, the product $UV$ is unambiguous whether defined using $\varphi$ or using $\sigma_V(\varphi)$.

From Theorem 4.5 and Proposition 4.3 we know that the octonion product defined by $\sigma_V(\varphi)$ is given by

$$A \circ_V B = AB + [A, B, V] V^{-1} = (AV) (V^{-1}B) \quad (4.20)$$

Define $[\cdot, \cdot, \cdot]_V$ to be the associator with respect to the product $\circ_V$. Then using (4.20) as well as the associator identities in Lemma 3.9 we obtain the following expression for $[\cdot, \cdot, \cdot]_V$. The proof is given in the Appendix.

Lemma 4.10 Let $A, B, C \in \Gamma(\mathbb{O}M)$, and define

$$[A, B, C]_V = A \circ_V (B \circ_V C) - (A \circ_V B) \circ_V C \quad (4.21)$$

where the product $\circ_V$ is defined by (4.20) for a nowhere-vanishing octonion $V$. Then,

$$[A, B, C]_V = [A, B, CV] V^{-1} - [A, B, V] (V^{-1}C) \quad (4.22)$$

Using Lemma 4.10 we can now prove Theorem 4.8.

Proof of Theorem 4.8 Let $A, B \in \Gamma(\mathbb{O}M)$. Let $\delta$ be the octonion product defined by $\sigma_U(\sigma_V \varphi)$. Using (4.20), this is then given by

$$A \delta B = A \circ_V B + [A, B, U]_V \circ_V U^{-1} \quad (4.23)$$
since we are now starting with $\phi_V$ that is defined by $\sigma_V(\varphi)$, and are changing it to $\tilde{\sigma}$ that is defined by $\sigma_U(\sigma_V \varphi)$. Therefore, expanding $[1.23]$ using $[1.20]$ and $[1.22]$, we have

$$A\tilde{\sigma}B = AB + [A, B, V]V^{-1} + [A, B, U]V^{-1} + [[A, B, U], V]V^{-1}$$

$$= AB + [A, B, V]V^{-1} + ([A, B, UV]V^{-1} - [A, B, V](V^{-1}U))U^{-1}$$

$$+ [[A, B, UV]V^{-1} - [A, B, V](V^{-1}U), U^{-1}, V]V^{-1}$$

$$= AB + [A, B, V]V^{-1} + ([A, B, UV]V^{-1})U^{-1} + [[A, B, UV]V^{-1}, U^{-1}, V]V^{-1}$$

$$- ([A, B, V](V^{-1}U))U^{-1} - [[A, B, V](V^{-1}U), U^{-1}, V]V^{-1}$$  \hspace{1cm} (4.24)

Note that using Lemma 3.9 and the definition of the associator, we get

$$[[A, B, UV]V^{-1}, U^{-1}, V]V^{-1} = [[A, B, UV], U^{-1}, V] \frac{|V|^2}{|V|^4}$$

$$= - [[A, B, UV], U^{-1}, V^{-1}]$$

$$= [[A, B, UV], V^{-1}, U^{-1}]$$

$$= [A, B, UV]V^{-1} - ([A, B, UV](V^{-1}U))U^{-1}$$  \hspace{1cm} (4.25)

and

$$[[A, B, V](V^{-1}U), U^{-1}, V]V^{-1} = ([[A, B, V](V^{-1}U))(U^{-1}V))V^{-1} - ([A, B, V](V^{-1}U))U^{-1}$$

$$= [A, B, V](V^{-1}U)(U^{-1}V)V^{-1} - ([A, B, V](V^{-1}U))U^{-1}$$

$$= [A, B, V]V^{-1} - ([A, B, V](V^{-1}U))U^{-1}$$  \hspace{1cm} (4.26)

where we have used the fact that $V^{-1}U = (U^{-1}V)^{-1}$ twice - in the second line to conclude that $[A, B, V], V^{-1}U,$ and $U^{-1}V$ associate, and in the third line to simply. Now substituting $[1.25]$ and $[1.26]$ into $[1.24]$, we are left with

$$A\tilde{\sigma}B = AB + [A, B, UV](V^{-1}U^{-1})$$

$$= AB + [A, B, UV](UV)^{-1}$$

Therefore,

$$A\tilde{\sigma}B = A \circ_{UV} B$$

Therefore, $\sigma_U(\sigma_V \varphi) = \sigma_{UV}(\varphi)$.

From Theorem 4.8 we hence see that the action of $\sigma_V$ on positive 3-forms corresponds to octonion multiplication on the left. The map $\sigma_V$ then also gives a representation of the non-zero octonion Moufang loop on 3-forms. Given a fixed “reference” $G_2$-structure, this then allows us to freely work with octonions rather than 3-forms.

A few consequences of Theorem 4.8 are the following. First of all it is clear that $\sigma_{V^{-1}}(\sigma_V \varphi) = \varphi$. Also, note that

$$\sigma_{U^3}(\sigma_{V^3} \varphi) = \sigma_{U^3V^3} \varphi$$  \hspace{1cm} (4.27)

However, for any octonions $A, B, C$

$$(\sigma_{V^3} \varphi)(A, B, C) = \varphi(\text{Ad}_{V^{-1}} A, \text{Ad}_{V^{-1}} B, \text{Ad}_{V^{-1}} C)$$

Hence,

$$\sigma_{U^3}(\sigma_{V^3} \varphi)(A, B, C) = (\sigma_{V^3} \varphi)(\text{Ad}_{U^{-1}V^{-1}}(A), \text{Ad}_{U^{-1}V^{-1}}(B), \text{Ad}_{U^{-1}V^{-1}}(C))$$

$$= \varphi\left(\text{Ad}_{V^{-1}}\left(\text{Ad}_{U^{-1}V^{-1}}(A)\right), \text{Ad}_{V^{-1}}\left(\text{Ad}_{U^{-1}V^{-1}}(B)\right), \text{Ad}_{V^{-1}}\left(\text{Ad}_{U^{-1}V^{-1}}(C)\right)\right)$$  \hspace{1cm} (4.28)
where $\text{Ad}_{U^{-1}}^{V^3}$ means that we are applying the Ad operator with respect to the $G_2$-structure $\sigma_{V^3}\varphi$. Finally,

$$(\sigma_{U^3V^3}\varphi)(A, B, C) = \varphi \left( Ad_{(U^3V^3)^{-\frac{1}{4}}} A, Ad_{(U^3V^3)^{-\frac{1}{4}}} B, Ad_{(U^3V^3)^{-\frac{1}{4}}} C \right)$$

Therefore, we may conclude that

$$\text{Ad}_{(U^3V^3)^{-\frac{1}{4}}}\text{Ad}_{U^{-1}}\text{Ad}_{V^{-1}}^{(V^3)} \in G_2 \quad (4.29)$$

In particular, if $U$ and $V$ are both 6th roots of unity, then $\sigma_{U^3V^3}\varphi = \sigma_{\pm 1}\varphi = \varphi$. Also, $\text{Ad}(V^3) = \text{Ad}$. Therefore, we conclude that in this case

$$\text{Ad}_{V^{-1}}\text{Ad}_{U^{-1}} \in G_2.$$ 

This of course is to be expected, since $\text{Ad}_{U^{-1}} \in G_2$ and $\text{Ad}_{V^{-1}} \in G_2$, so their composition is also in $G_2$ by the group property.

## 5 Torsion of a $G_2$-structure

So far we have only looked at the algebraic properties of $G_2$-structures and octonions. However, given a $G_2$-structure $\varphi$ with an associated metric $g$, we may use the metric to define the Levi-Civita connection $\nabla$. The intrinsic torsion of a $G_2$-structure is then defined by $\nabla\varphi$. Following [21, 31], we can write

$$\nabla_a \varphi_{bcd} = 2T_a^e \psi_{ebcd} \quad (5.1)$$

where $T_{ab}$ is the full torsion tensor. Similarly, we can also write

$$\nabla_a \psi_{bcde} = -8T_a[\psi_{bcde}] \quad (5.2)$$

We can also invert (5.1) to get an explicit expression for $T$

$$T_a^m = \frac{1}{48} (\nabla_a \varphi_{bcd}) \psi^{mbcd}. \quad (5.3)$$

This 2-tensor fully defines $\nabla\varphi$ [21].

**Remark 5.1** The torsion tensor $T$ as defined here is actually corresponds to $\frac{1}{2} T$ in [21] and $-\frac{1}{2} T$ in [31]. Even though this requires extra care when translating various results, it will turn out to be more convenient, because otherwise we would have a factor of $\frac{1}{2}$ everywhere.

In general we can obtain an orthogonal decomposition of $T_{ab}$ according to representations of $G_2$ into torsion components:

$$T = \tau_1 g + \tau_7 \varphi + \tau_{14} + \tau_{27} \quad (5.4)$$

where $\tau_1$ is a function, and gives the $1$ component of $T$. We also have $\tau_7$, which is a 1-form and hence gives the $7$ component, and, $\tau_{14} \in \Lambda_4^2$ gives the $14$ component and $\tau_{27}$ is traceless symmetric, giving the $27$ component. As it was originally shown by Fernández and Gray [15], there are in fact a total of 16 torsion classes of $G_2$-structures that arise depending on which of the components are non-zero. Moreover, as shown in [31], the torsion components $\tau_i$ relate directly to the expression for $d\varphi$ and $d\psi$. It can also be shown [13, 21, 31], that $T$ satisfies a “Bianchi identity”:

**Proposition 5.2** Let $\varphi$ be a $G_2$-structure with an associated metric $g$, and Levi-Civita connection $\nabla$. The torsion tensor then satisfies

$$\nabla_a T_{bc} - \nabla_b T_{ac} + 2T_{am} T_{bn} \varphi^{mn} c - \frac{1}{4} \text{Riem}_{abmn} \varphi^{mn} c = 0 \quad (5.5)$$

where $\text{Riem}$ is the Riemann curvature tensor of the metric $g$. 

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In particular, the term in (5.5) involving $\text{Riem}$, is precisely the component of the curvature 2-form that lies in $\Lambda^2$, namely $\pi_7 \text{Riem}$. Proposition 5.2 then tells us that $\pi_7 \text{Riem}$ is fully determined by the torsion, and in particular, if the torsion vanishes, then $\pi_7 \text{Riem} = 0$. Moreover, if we contract (5.5) with $\phi_{bc}^d$, we obtain an expression for the Ricci curvature $\text{Ric}$ in terms of $T_{13, 21, 31}$:

$$\text{Ric}_{ab} = 2 (\nabla_a T_{nm} - \nabla_n T_{am}) \phi^{nm}_{\ b} - 4 T_{an} T_{b} + 4 \text{Tr} (T) T_{ab}$$

(5.6)

This then shows that if the torsion vanishes, then so does $\text{Ric}$. Of course, Ricci curvature is a function of the metric, so it is invariant over the metric class of $G_2$-structures. In particular, the scalar curvature is given by

$$\frac{1}{4} R = 42 \tau_1^2 + 30 |\tau_7|^2 - |\tau_{14}|^2 - |\tau_{27}|^2 + 6 \text{div} \tau_7$$

(5.7)

When the torsion vanishes, that is $T = 0$, the $G_2$-structure is said to be torsion-free. This is equivalent to $\nabla \varphi = 0$ and also equivalent, by Fernández and Gray, to $d\varphi - d\psi = 0$. Moreover, a $G_2$-structure is torsion-free if and only if the holonomy group $\text{Hol} (g)$ of the corresponding metric $g$ is contained in $G_2$.

The holonomy group is then precisely equal to $G_2$ if and only if the fundamental group $\pi_1$ is finite. As we have seen, for any metric there is a family of compatible $G_2$-structures. Holonomy however is a property of the metric, so we can reformulate the correspondence between $G_2$-structures and holonomy as follows.

**Theorem 5.3** ([15, 29]) Let $(M, g)$ be a smooth 7-dimensional Riemannian manifold with $w_1 = w_2 = 0$. Then, $\text{Hol} (g) \subseteq G_2$ if and only if there exists a torsion-free $G_2$-structure $\varphi$ that is compatible with $g$.

Therefore, in order to understand the holonomy of a given metric we have to understand if the corresponding metric class of $G_2$-structures contains a torsion-free $G_2$-structure. A necessary condition for a metric class to admit a torsion-free $G_2$-structure is $\text{Ric} = 0$. The converse would be true if any metric on a 7-manifold with $\text{Ric} = 0$ has reduced holonomy.

It is possible to explicitly work out the expression for the torsion of a $G_2$-structure $\sigma_V (\varphi)$ in terms of the torsion of $\varphi$, however the expression in index notation is not too illuminating. To get the expression in terms of octonions, we will first define a covariant derivative on sections of the octonion bundle.

## 6 Octonion covariant derivative

Consider the octonion bundle $\mathcal{O} M$ with the octonion algebra defined by the $G_2$-structure $\varphi$ with torsion tensor $T$. Then, we can extend the Levi-Civita connection $\nabla$ to sections of $\mathcal{O} M$. Let $A = (a, \alpha) \in \Gamma (\mathcal{O} M)$, then define the covariant derivative on $\mathcal{O} M$ as

$$\nabla_X A = (\nabla_X a, \nabla_X \alpha)$$

(6.1)

for any $X \in \Gamma (TM)$. Now the question is how does $\nabla$ interact with octonion multiplication.

**Proposition 6.1** Suppose $A, B \in \Gamma (\mathcal{O} M)$. Then, for $X \in \Gamma (TM)$

$$\nabla_X (AB) = (\nabla_X A) B + A (\nabla_X B) - [T_X, A, B]$$

(6.2)

where $T_X = (0, X, T)$.

**Proof.** Suppose $A = (a, \alpha)$ and $B = (b, \beta)$, then, using the definition of octonion multiplication (3.8) we write

$$\nabla_X (AB) = \nabla_X \left( \frac{ab - \langle \alpha, \beta \rangle}{a \beta + b \alpha + \varphi (\alpha, \beta, \sharp)} \right)$$

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Using the Leibniz property and metric compatibility of $\nabla_X$, we then get

$$\nabla_X (AB) = (\nabla_X A) B + B (\nabla_X A) + \begin{pmatrix} 0 \\ (\nabla_X \varphi) (\alpha, \beta, \cdot) \end{pmatrix}$$

However, from (5.1), $\nabla_X \varphi = 2T_X \lrcorner \psi$, and using the relationship between $\psi$ and the associator (3.12), we get

$$(\nabla_X \varphi) (\alpha, \beta, \cdot) = 2\psi (T_X, \alpha, \beta, \cdot) = -[T_X, A, B]$$

Therefore, indeed we obtain (6.2).

Note that if either of $A$ or $B$ in (6.2) is real, then the associator vanishes, and we recover the standard Leibniz rule for $\nabla$.

**Remark 6.2** Proposition 6.1 has two important implications. Firstly, note that $[T_X, A, B]$ vanishes for all $X, A, B$ if and only if $T = 0$. Therefore, the Levi-Civita connection is compatible with octonion multiplication if and only if the $G_2$-structure is torsion-free. Therefore, the torsion $T$ is an obstruction to $\nabla$ being compatible with octonion product. This is of course expected, since the product is defined by $\varphi$, and $T$ is the precisely given by $\nabla \varphi$. Secondly, here we are treating the torsion tensor $T$ as a “pure octonion-valued 1-form” on $M$. That is,

$$T \in \Gamma (T^* M \otimes \text{Im} \, \mathbb{O} M) = \Omega^1 (\text{Im} \, \mathbb{O} M),$$

so that in particular for any vector $X$ on $M$,

$$T_X \in \Gamma (\text{Im} \, \mathbb{O} M).$$

This presents an important shift in perception of what torsion of a $G_2$-structure actually is. Recall that a principal bundle connection can be thought of as a Lie algebra-valued 1-form. A Lie algebra is the tangent space to the identity of a Lie group. In our case, the pure imaginary octonions precisely form the tangent space to 1 in the Moufang loop of unit octonions. Therefore, $T$ is the octonionic analog of a “Lie algebra-valued 1-form”. Alternatively, it can be thought as some kind of a “gauge connection” for a non-associative gauge theory.

For a generic $G_2$-structure, $\nabla$ does not satisfy the derivation property with respect to the octonion product. Let us however define an adapted covariant derivative, using $T$ as a “connection” 1-form.

**Definition 6.3** Define the octonion covariant derivative $D$ such for any $X \in \Gamma (TM)$,

$$D_X : \Gamma (\mathbb{O} M) \rightarrow \Gamma (\mathbb{O} M)$$

given by

$$D_X A = \nabla_X A - AT_X$$

for any $A \in \Gamma (\mathbb{O} M)$.

Using the octonion covariant derivative, we in particular have

$$D_X 1 = -T_X$$

The idea to use the $G_2$-structure torsion to define a new connection is certainly not new (see for example [1, 2, 3], and references therein). However using it to define an octonion covariant derivative is a new concept. This covariant derivative satisfies a partial derivation property with respect to the octonion product.

**Proposition 6.4** Suppose $A, B \in \Gamma (\mathbb{O} M)$ and $X \in \Gamma (TM)$, then

$$D_X (AB) = (\nabla_X A) B + A (D_X B)$$

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Proof. We use the definition of \( D \) to write out \( D_X (AB) \):
\[
D_X (AB) = \nabla_X (AB) - (AB) T_X
\]
Then expanding \( \nabla_X (AB) \) using Proposition 3.11 and applying properties of the associator we get
\[
D_X (AB) = (\nabla_X A) B + A (\nabla_X B) - [T_X , A, B] - (AB) T_X
\]
\[
= (\nabla_X A) B + A (\nabla_X B) - [A, B; T_X] - (AB) T_X
\]
\[
= (\nabla_X A) B + A (\nabla_X B) - A (B T_X) + (AB) T_X - (AB) T_X
\]
\[
= (\nabla_X A) B + A (\nabla_X B - B T_X)
\]
\[
= (\nabla_X A) B + A (D_X B)
\]

Therefore, \( D_X \) satisfies a “one-sided” derivation identity - the derivative on the first term of the right hand side of (6.6) is a standard \( \nabla \), however on the second term we have a \( D \). However, if \( A \) is real in (6.6), then we see that it does give us what we would expect. Moreover, we now show that \( D \) is metric-compatible. Recall from Section 3 that we extend the metric \( g \) on \( M \) to \( \Omega M \) by setting:
\[
g (A, B) = (\text{Re} A) (\text{Re} B) + (\text{Im} A, \text{Im} B)
\]
where the imaginary parts are now regarded as vectors on \( M \).

**Proposition 6.5** Suppose \( A, B \in \Gamma (\Omega M) \) and \( X \in \Gamma (TM) \), then
\[
\nabla_X (g (A, B)) = g (D_X A, B) + g (A, D_X B)
\]

**Proof.** We can rewrite \( g (D_X A, B) \) as
\[
g (\nabla_X A - AT_X, B) = g (\nabla_X A, B) - g (AT_X, B)
\]
\[
= g (\nabla_X A, B) - g (T_X, AB)
\]
where we have used Lemma 3.11 \( L^*_A = L_A \). Similarly,
\[
g (A, D_X B) = g (A, \nabla_X B) - g (A, B T_X)
\]
\[
= g (A, \nabla_X B) - g (T_X, BA)
\]
Combining (6.9) and (6.10), we obtain
\[
g (D_X A, B) + g (A, D_X B) = g (\nabla_X A, B) + g (A, \nabla_X B) - g (T_X, AB + BA)
\]
However, \( T_X \) is pure imaginary, while \( AB + BA \) is real. Therefore, their inner product vanishes. Hence,
\[
g (D_X A, B) + g (A, D_X B) = g (\nabla_X A, B) + g (A, \nabla_X B) = \nabla_X (g (A, B))
\]

Further, given \( \Omega M \)-valued differential forms, we can extend \( D \) to an exterior covariant derivative \( d_D \). Define \( \Omega^p (\Omega M) \) to be sections of the bundle \( \Lambda^p (T^* M) \otimes \Omega M \), that is, octonion-valued \( p \)-forms. Then,
\[
d_D : \Omega^p (\Omega M) \rightarrow \Omega^{p+1} (\Omega M)
\]
is such that
\[
d_D Q = d_{\nabla} Q - (-1)^p Q \overset{\nabla}{\wedge} T
\]
where \( d_{\nabla} \) is the skew-symmetrized \( \nabla \) and \( \overset{\nabla}{\wedge} \) is a combination of exterior product and octonion product. More concretely, in index notation, if we suppose \( Q \) is given by
\[
Q = \frac{1}{p!} Q_{a_1 \ldots a_p} dx^{a_1} \wedge \ldots \wedge dx^{a_p}
\]
where \( Q_{a_1...a_p} \) is an octonion section for any fixed \( \{a_1,...,a_p\} \). Then

\[
(d_D Q)_{b_1...b_{p+1}} = (p+1) \left( \nabla [b_1 Q_{b_2...b_{p+1}}] - (-1)^p Q_{b_1...b_p} \otimes T_{b_{p+1}} \right)
\]  

(6.13)

where each \( T_{b_k} \) is an imaginary octonion for any \( b_k \). Suppose \( A \) is an \( \Omega M \)-valued 0-form, \( B \) is an \( \Omega M \)-value 1-form, and \( C \) is an \( \Omega M \)-value 2-form, we have the following explicit formulas for \( d_D \):

\[
(d_D A)_k = \nabla_k A - AT_k
\]  

(6.14a)

\[
(d_D B)_{kl} = 2 \nabla_{[k} B_{l]} + B_k T_l - B_l T_k
\]  

(6.14b)

\[
(d_D C)_{klm} = 3 \nabla_{[k} C_{lm]} - C_{kl} T_m - C_{km} T_l - C_{ml} T_k
\]  

(6.14c)

Note that even if \( T = 0 \), \( d_D^2 = d_D^2 \) is a function of \( \text{Riem} \) and is in general not equal to zero. Lemma 6.6 below gives the precise statement. This is a standard result for vector-valued differential forms, so the proof is given in Appendix [A]

**Lemma 6.6** The operator \( d^*_D : \Omega^p (\Omega M) \rightarrow \Omega^{p+2} (\Omega M) \) is given by

\[
d^*_D P = \text{Riem} \wedge (\text{Im} P)
\]  

(6.15)

here \( \text{Riem} \) is regarded as a section of \( \Omega^2 (M) \otimes \text{End} (\text{Im} \Omega M) \cong \Omega^2 (M) \otimes \text{End} (TM) \) so in [6.10], \( \text{Riem} \wedge \text{Im} P \) is a wedge product in \( \Omega^* (M) \) and moreover \( \text{Riem} \) acts as an endomorphism on \( \text{Im} P \).

Let us now assume that \( M \) is compact. We can then define the \( L_2 \)-inner product of octonion-valued forms. Suppose \( P \) and \( Q \) are octonion-valued \( p \)-forms, then let

\[
\langle P, Q \rangle_{L^2} = \int_M \langle P, Q \rangle \text{ vol}
\]  

(6.16)

where vol is the standard volume form on \( M \) defined by the metric and the orientation and \( \langle \cdot, \cdot \rangle \) is the canonical extension of \( g \) from \( \Lambda^p (T^* M) \) to \( \Lambda^p (T^* M) \otimes \Omega M \). Using (6.10), we then define the adjoint operator to \( d_D \) - the codifferential \( d^*_D \). Let \( P \) be an \( \Omega M \)-valued \( p \)-form, and \( Q \) an \( \Omega M \)-valued \((p-1)\)-form, then

\[
\langle d^*_D P, Q \rangle_{L^2} = \langle P, d_D Q \rangle_{L^2}
\]  

(6.17)

The codifferential \( d^*_D \) is then a map

\[
d^*_D : \Omega^p (\Omega M) \rightarrow \Omega^{p-1} (\Omega M)
\]  

(6.18)

A direct computation using Stokes' Theorem and Lemma 3.11 gives

\[
\langle P, d_D Q \rangle_{L^2} = \int_M \frac{1}{(p-1)!} \left\langle \nabla_{b_1} P_{b_2...b_p} + (-1)^{p-1} P_{b_1...b_p} T_{b_1}, Q_{b_2...b_p} \right\rangle_\Omega \text{ vol}
\]

\[
= \left( d^*_D P, Q \right)_{L^2}
\]

Therefore,

\[
(d^*_D P)_{b_2...b_p} = - \left( \nabla_{b_1} P_{b_2...b_p} - P_{b_1...b_p} T_{b_1} \right)
\]

(6.19)

In particular,

\[
(d^*_D P)_{b_2...b_p} = -D_{b_1} P_{b_2...b_p}.
\]

We will thus define the divergence of a \( p \)-form \( P \) with respect to \( D \) as the \((p-1)\)-form \( \text{Div} P \), given by

\[
(\text{Div} P)_{b_2...b_p} = -D_{b_1} P_{b_2...b_p}
\]  

(6.20)
and thus,

\[ d^*_D P = - \text{Div} P. \]

This is the complete analog of the standard codifferential \( d^* \) being equal to \(- \text{div}\), where divergence is now with respect to \( \nabla \). Let us now consider the derivatives of the torsion \( T \), which we know is an \( \text{Im} \mathbb{O}M \)-valued 1-form. Applying the definition \((6.4)\) of \( D \), we obtain

\[ D_i T_j = \nabla_i T_j - T_j T_i \]

Expanding the octonion product \( T_j T_i \), we then have

\[ D_i T_j = \nabla_i T_j - T_j T_i \tag{6.21} \]

We then use \((6.21)\) to find \( d_D T \) and \( d^*_D T \).

**Proposition 6.7** Suppose the octonion product on \( \mathbb{O}M \) is defined by the \( G_2 \)-structure \( \varphi \) with torsion \( T \). Then,

\[ d_D T = \frac{1}{4} (\pi_7 \text{Riem}) \]  
\[ d^*_D T = - \text{div} T - |T|^2 \]

where \( \pi_7 \text{Riem} \in \Omega^2 (\text{Im} \mathbb{O}M) \cong \Omega^2 (TM) \) - a vector-valued 2-form. Similarly, \( \text{div} T \in \Omega^0 (\text{Im} \mathbb{O}M) \) and \( |T|^2 \in \Omega^0 (\text{Re} \mathbb{O}M) \).

**Proof.** To obtain the exterior derivative \( d_D T \), we skew-symmetrize \((6.21)\):

\[ (d_D T)_{ij} = 2 (\nabla_{[i} T_{j]} + T_i \times T_j) \tag{6.24} \]

So far we have considered \( T \) as an \( \text{Im} \mathbb{O}M \)-valued 1-form, and have suppressed the octonion index on \( T \). Writing out \((6.24)\) in full, we have

\[ (d_D T)_{ij} = 2 \left( \nabla_{[i} T_{j]} + T_i \beta T_j \gamma \varphi^\alpha_{\beta \gamma} \right) \tag{6.25} \]

However using the Bianchi identity for \( G_2 \) torsion (Proposition \(5.2)\), we see that the right hand side of \((6.25)\) is precisely \( \frac{1}{4} (\pi_7 \text{Riem}) \):

\[ 2 \left( \nabla_{[i} T_{j]} + T_i \beta T_j \gamma \varphi^\alpha_{\beta \gamma} \right) = \frac{1}{4} \text{Riem} \theta_{ij} \gamma \varphi^\alpha_{\beta \gamma} \]

Therefore,

\[ (d_D T)_{ij} = \frac{1}{4} \text{Riem} \theta_{ij} \gamma \varphi^\alpha_{\beta \gamma} = \frac{1}{4} (\pi_7 \text{Riem})_{ij} \alpha. \]

To find \( d^*_D T \), we write

\[ d^*_D T = - D^i T_i = - \text{div} T - |T|^2. \]

In particular, using Proposition \(6.7) and Lemma \(6.6) we can now work out the action of \( d^*_D \) on octonion-valued forms.

**Proposition 6.8** Suppose \( P \in \Omega^p (\mathbb{O}M) \). Then,

\[ d^2_D P = \text{Riem} \wedge (\text{Im} P) - P \wedge d_D T = \text{Riem} \wedge (\text{Im} P) - \frac{1}{4} P \wedge (\pi_7 \text{Riem}) \tag{6.26} \]

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Proof. From the definition of $d_D$ (6.12), we have

$$d_D P = d_V P - (-1)^p P \wedge T$$

Hence,

$$d_D^2 P = d_D (d_V P) - (-1)^p d_D \left( P \wedge T \right)$$

$$= d_V^2 P - (1)^{p+1} d_V P \wedge T - (-1)^p (d_V P) \wedge T - (-1)^{2p} P \wedge d_T T$$

$$= d_V^2 P - P \wedge d_D T$$

The expression (6.20) then follows when we use Lemma 6.6 to rewrite $d_V^2$ and Proposition 6.7 to rewrite $d_D T$. □

In particular, if $P = (p_0, p) \in \Omega^0 (\mathcal{O} M)$, i.e. a section of $\Gamma (\mathcal{O} M)$ then (6.20) gives us:

$$(d_D^2 P) = \text{Riem} (p) - \frac{1}{4} P \circ (\pi_7 \text{Riem}) \quad (6.27)$$

Remark 6.9 In the expression (6.20) we see that $d_D^2 P$ has two components - $\text{Riem} \wedge (\text{Im} P)$, which comes from $d_V^2$ and does not depend on the octonion product, and an “octonionic” part $-P \wedge d_D T$ which is fully determined by the torsion and involves octonion multiplication. This gives $\pi_7 \text{Riem}$ a new interpretation as an octonionic curvature. In particular, $\pi_7 \text{Riem}$ completely determines the real part of $d_D^2$:

$$\text{Re} (d_D^2 P) = \frac{1}{4} \left( -p_0 \wedge \pi_7 \text{Riem} + \langle p \wedge \pi_7 \text{Riem} \rangle \right) \quad (6.28)$$

where in the second term of (6.28), the octonion inner product is combined with the wedge product of differential forms.

7 Change of reference $G_2$-structure

In the previous section we have considered the octonion covariant derivative with respect to a fixed $G_2$-structure $\phi$. However, from Section 4 we know that any nowhere-vanishing octonion section $V$ defines a new $G_2$-structure $\sigma_V (\phi)$ in the same metric class. Recall from (4.20) that the octonion product defined by $\sigma_V (\phi)$ is given by

$$A \circ V B = AB + [A, B, V] V^{-1} = (AV) (V^{-1} B) \quad (7.1)$$

We can work out $\nabla_X (A \circ V B)$ directly.

Lemma 7.1 Let $V$ be a nowhere-vanishing octonion section, and suppose $\circ_V$ is the octonion product defined by the $G_2$-structure $\sigma_V (\phi)$ and $[\cdot, \cdot, V]$ is the corresponding associator. Then, for any $A, B \in \Gamma (\mathcal{O} M)$, and any vector field $X$,

$$\nabla_X (A \circ_V B) = (\nabla_X A) \circ_V B + A \circ_V \nabla_X B - [\text{Ad}_V T_X + V (\nabla_X V^{-1}) , A, B]_V \quad (7.2)$$

The proof of Lemma 7.1 is straightforward, but technical, so we give it in Appendix A. However, from (6.2), the torsion $T (V)$ of $\sigma_V (\phi)$ is given by

$$\nabla_X (A \circ_V B) = (\nabla_X A) \circ_V B + A \circ_V (\nabla_X B) - [T^V_X , A, B]_V \quad (7.3)$$

for any octonion sections $A, B$, and vector field $X$. Comparing (7.2) and (7.3), we therefore have the following result.
Theorem 7.2 Let $M$ be a smooth 7-dimensional manifold with a $G_2$-structure $(\varphi, g)$ with torsion $T \in \Omega^1(\text{Im} \, O_M)$. For a nowhere-vanishing $V \in \Gamma(\text{O}_M)$, consider the $G_2$-structure $\sigma_V (\varphi)$. Then, the torsion $T^{(V)}$ of $\sigma_V (\varphi)$ is given by

$$T^{(V)} = \text{Im} \left( \text{Ad}_V T + V \left( \nabla V^{-1} \right) \right)$$

(7.4)

In particular, if $V$ has constant norm, $T^{(V)}$ is given by

$$T^{(V)} = - (DV) V^{-1}$$

(7.5)

Proof. This is a direct consequence of Lemma 7.1. Since (7.2) is defined for arbitrary $A, B$, by comparing it with (7.3), we find that the imaginary parts of $T^{(V)}$ and $\text{Ad}_V T + V \left( \nabla V^{-1} \right)$ must agree. However, by definition, $T^{(V)} \in \Omega^1(\text{Im} \, O_M)$ is pure imaginary, so (7.4) holds. Note that in general $\text{Ad}_V T + V \left( \nabla V^{-1} \right)$ has a real part:

$$\text{Re} \left( \text{Ad}_V T + V \left( \nabla V^{-1} \right) \right) = \langle \text{Ad}_V T + V \left( \nabla V^{-1} \right), 1 \rangle$$

$$= \langle V \left( \nabla V^{-1} \right), 1 \rangle$$

$$= - \langle (\nabla V) V^{-1}, 1 \rangle$$

$$= - \frac{1}{|V|^2} \langle \nabla V, V \rangle$$

$$= - \frac{1}{2 |V|^2} \nabla |V|^2 = - \nabla \ln |V|$$

(7.6)

In particular, if $|V|$ is constant, then the real part vanishes, and hence

$$T^{(V)} = \text{Ad}_V T + V \left( \nabla V^{-1} \right)$$

$$= VTV^{-1} - (\nabla V) V^{-1}$$

$$= - (\nabla V - VT) V^{-1}$$

$$= - (DV) V^{-1}.$$ 

\[ \blacksquare \]

Remark 7.3 Theorem 7.2 shows that under a transformation of $G_2$-structures given by $\varphi \rightarrow \sigma_V (\varphi)$, the torsion 1-form transforms in a way similar to the transformation of a principal bundle connection 1-form under a change of trivialization. Since in our case $T$ is not the full connection 1-form - it is only part of the connection that also includes the Levi-Civita connection, the transformation rule (7.4) involves $\nabla$ rather than standard partial derivatives as one has on a principal bundle. As Corollary 7.4 shows below, the existence of torsion-free $G_2$-structures in a given metric class reduces to solving the equation $DV = 0$ for some nowhere-zero $V \in \Gamma(\text{O}_M)$.

Corollary 7.4 Let $M$ be a smooth 7-dimensional manifold with a $G_2$-structure $(\varphi, g)$ with torsion $T \in \Omega^1(\text{Im} \, O_M)$. There exists a torsion-free $G_2$-structure $\tilde{\varphi}$ in the same metric class as $\varphi$ if and only if there exists a nowhere-zero octonion section $V$ such that

$$DV = 0$$

(7.7)

where the covariant derivative $D$ is defined by (6.4) using the torsion 1-form $T$. Moreover, $\tilde{\varphi} = \sigma_V (\varphi)$.

Proof. If $DV = 0$, then using the metric compatibility of $D$ from Proposition 6.5, we get $\nabla |V| = 0$. Hence $|V|$ is a non-zero constant. Therefore, from Theorem 7.2, the torsion of $\sigma_V (\varphi)$ is given by

$$T^{(V)} = - (DV) V^{-1} = 0.$$
Conversely, suppose there exists a torsion-free $G_2$-structure $\tilde{\varphi}$ in the same metric class as $\varphi$. Then, by Theorem 2.3, $\tilde{\varphi} = \sigma_U (\varphi)$ for some nowhere-vanishing octonion section $U$. By Theorem 7.2

$$T(U) = \text{Im} \left( \text{Ad}_U T + U \nabla U^{-1} \right) = 0$$

However, let $V = \frac{U}{|U|}$, so that $V$ is a unit octonion. Then,

$$\text{Ad}_U T = \text{Ad}_V T$$

$$U \nabla U^{-1} = - (\nabla U) U^{-1}$$

$$= - \nabla (V |U|) |U|^{-1} V^{-1} = - \frac{1}{|U|} \nabla |U| - (\nabla V) V^{-1}$$

But, $\frac{1}{|U|} \nabla |U|$ is real, so

$$0 = \text{Im} \left( \text{Ad}_U T + U \nabla U^{-1} \right) = \text{Im} \left( \text{Ad}_V T - (\nabla V) V^{-1} \right) = -(DV) V^{-1}$$

Thus $DV = 0$. □

An interesting special case of Corollary 7.4 is when $\varphi$ is already torsion-free. In this case, $D = \nabla$, hence the condition (7.7) becomes simply $\nabla V = 0$. Moreover, now the real and pure imaginary parts of $V$ are differentiated separately, so we just require $\nabla v = 0$ for some vector field $v$ on $M$. Given a unit parallel vector field $v$, any unit octonion $V = (a, bv)$, for constants $a$ and $b$ such that $a^2 + b^2 = 1$, will define a torsion-free $G_2$-structure. This shows that all the other torsion-free $G_2$-structures in the same metric class as $\varphi$ are parametrized by parallel vector fields on $M$ together with a choice of a phase factor.

**Definition 7.5** Let $F_g$ be the space of torsion-free $G_2$-structures that are compatible with the metric $g$.

**Theorem 7.6** Suppose $(M, g)$ is a 7-dimensional Riemannian manifold with $\text{Hol} (g) \subseteq G_2$. If $M$ admits $m$ linearly independent parallel vectors, then

$$F_g \cong \mathbb{R}P^m \quad \text{(7.8)}$$

If moreover, $M$ is compact, then $F_g \cong \mathbb{R}P^{b^1}$, where $b^1$ is the first Betti number. In particular, if $b^1 = 0$, the torsion-free $G_2$-structure in this metric class is unique.

**Proof.** Suppose $\varphi$ is a torsion-free $G_2$-structure on $M$ that is compatible with $g$. From Corollary 7.4, we know that any torsion-free $G_2$-structure $\tilde{\varphi}$ in the same metric class is given by $\tilde{\varphi} = \sigma_V (\varphi)$ for an octonion $V$ with $\nabla V = 0$. This is equivalent to $\nabla v = 0$ where $v = \text{Im} V$ is a vector field on $M$. If $M$ does admit parallel vector fields, then any parallel vector field $v$ defines a torsion-free $G_2$-structure. Suppose now $M$ admits $m > 0$ linearly independent parallel vector fields $v_1, ..., v_m$. We can then define a global $(m + 1)$-subframe on $\Omega M$ spanned by $V_0 = (1, 0)$ and $V_i = (0, v_i)$ for $1 \leq i \leq m$.

Therefore, any octonion in the space spanned by $(V_k)$ for $0 \leq k \leq m$ defines a torsion-free $G_2$-structure. However, as we see from the definition of the map $\sigma_V$ (1.14), any constant multiple of $V_k$ defines the same $G_2$-structure. Therefore, the torsion-free $G_2$-structures are in a 1-1 correspondence with projective lines in $(V_k)$. Hence $F_g \cong \mathbb{R}P^m$.

If $M$ is compact, then it is a standard fact that, since $\text{Ric} = 0$, parallel vector fields are in a 1-1 correspondence with harmonic forms, and thus $m = b^1$. If $b^1 = 0$, then there exist no parallel vectors on $M$, and therefore $\varphi$ is the unique torsion-free $G_2$-structure that is compatible with the metric $g$. □

**Remark 7.7** If a compact Riemannian 7-manifold $(M, g)$ has $\text{Hol} (g) \subseteq G_2$, then the only possible values for $b^1$ are 0, 1, 3, 7 [29] and the number of linearly independent parallel vector fields on $M$ is equal to $b^1$. However even if $M$ is non-compact, the number of parallel vectors $m$ can also be only 0, 1, 3, 7, even if $m \neq b^1$ (in fact $m \leq b^1$). This is easy to see - suppose we have two orthogonal parallel vector fields, $v_1$ and $v_2$. Then, using a torsion-free $G_2$-structure $\varphi$, we can define $v_3 = v_1 \times_\varphi v_2$. This will also be parallel (since $\varphi$ is parallel), and it will be orthogonal to both $v_1$ and $v_2$. Thus once we have at least 2 parallel vector fields, we must actually have 3. Similarly, if we have at least 4 orthogonal parallel vector fields, by considering cross
products, we find that we actually must have 7. Another way of looking at this is that if we have no parallel vector fields, then the octonion bundle globally splits only as \( O \cong \mathbb{R} \oplus \text{Im} O \) where \( \text{Im} O \) corresponds to \( TM \). If we have one parallel vector field, then we in fact have a globally-defined complex plane inside the octonion bundle, so now we get a splitting as \( O \cong \mathbb{C} \oplus V^6 \) where \( V^6 \) now corresponds to the tangent bundle of Calabi-Yau 3-fold. Further, if we have three parallel vector fields, together with the octonion \((1,0)\) these form a globally defined quaternion subspace, where \( O \cong \mathbb{H} \oplus V^4 \), where \( V^4 \) now corresponds to the tangent bundle of a 4 real dimensional Hyperkähler manifold. Finally, if we have a global frame of parallel vector fields, then the octonion bundle just becomes a direct product of \( \mathbb{R}^7 \) and the standard octonion algebra.

The condition (7.7) for existence of a torsion-free \( G_2 \)-structure depends on the initial choice of the \( G_2 \)-structure \( \varphi \). This is the reference \( G_2 \)-structure. However if we chose a different reference \( G_2 \)-structure, the covariant derivative \( D \) would be defined differently. As we have seen from the deformation of the torsion, we can interpret the choice of the reference \( G_2 \)-structure as a choice of trivialization. Therefore we need to understand whether the condition (7.7) is invariant under a change of trivialization.

**Proposition 7.8** Suppose \( (\varphi, g) \) is a \( G_2 \)-structure on a 7-manifold \( M \), with torsion \( T \) and corresponding octonion covariant derivative \( D \). Suppose \( V \) is a unit octonion section, and \( \tilde{\varphi} = \sigma_V (\varphi) \) is the corresponding \( G_2 \)-structure, that has torsion \( \tilde{T} \), given by (7.3), and an octonion covariant derivative \( \tilde{D} \). Then, for any octonion section \( A \), we have

\[
\tilde{D} (AV^{-1}) = (DA)V^{-1}
\]

and equivalently,

\[
\tilde{D}A = (D(AV))V^{-1}
\]

**Proof.** Since \( |V| = 1 \), recall from Theorem 7.2 that \( \tilde{T} \) is given by

\[
\tilde{T} = VTV^{-1} + V(\nabla V^{-1})
\]

Hence,

\[
\tilde{D} (AV^{-1}) = \nabla (AV^{-1}) - (AV^{-1}) \circ_V \tilde{T}
= \nabla (AV^{-1}) - (AV^{-1}V)(V^{-1}\tilde{T})
\]

where we have used the expression (7.20) for \( \circ_V \). Substituting (7.11) into (7.12) and using Proposition 6.1 to expand the first term in (7.12), we obtain

\[
\tilde{D} (AV^{-1}) = \nabla (AV^{-1}) - A (TV^{-1} + \nabla V^{-1})
= (\nabla A)V^{-1} + A(\nabla V^{-1}) - [T, A, V^{-1}] - A (TV^{-1}) - A (\nabla V^{-1})
= (\nabla A)V^{-1} + [A, T, V^{-1}] - A (TV^{-1})
= (\nabla A - AT)V^{-1} = (DA)V^{-1}
\]

The equivalent expression (7.10) follows immediately.

**Remark 7.9** From Proposition 7.8 we also obtain that the octonion covariant exterior derivative (6.11) also transforms in a similar way, and hence, for \( P \in \Omega^0 (\Omega M) \), we see that

\[
\tilde{d}_D P = \tilde{d}_D (d_D (PV)V^{-1}) = (d_D (PV)V^{-1})
\]

Therefore, if \( F \) and \( \tilde{F} \) are the curvature operators for \( G_2 \)-structures \( \varphi \) and \( \sigma_V (\varphi) \), respectively, then we see that

\[
\tilde{F} = R_V^{-1} FR_V
\]

where \( R_V \) is the right multiplication map by \( V \) and the right hand side of (7.14) is a composition of maps. Further study of the algebraic properties of these octonion structures is needed to understand how to interpret (7.14) and whether there are any invariants of the octonion curvature with respect to the transformation (7.14).
From (7.39), we see that $DA = 0$ with respect to the $G_2$-structure $\varphi$ if and only if $\bar{D} (AV^{-1}) = 0$ with respect to the $G_2$-structure $\sigma_V (\varphi)$. Therefore, the solutions of the equation $DA = 0$ are in a 1-1 correspondence with solutions of the equation $\bar{D} A = 0$.

Suppose $\tilde{\varphi} = \sigma_V (\varphi)$, so that $\varphi = \sigma_{V^{-1}} (\tilde{\varphi})$. Then, using Theorem 1.8

$$\sigma_A \varphi = \sigma_A \sigma_{V^{-1}} \tilde{\varphi} = \sigma_{AV^{-1}} \tilde{\varphi}$$

(7.15)

The torsion $T^{(A)}$ of $\sigma_A \varphi$ is then given by $T^{(A)} = -(DA) A^{-1}$ with respect to $G_2$-structure $\varphi$, but with respect to the $G_2$-structure $\tilde{\varphi}$, the expression is

$$T^{(A)} = -\bar{D} (AV^{-1}) \circ_V (AV^{-1})^{-1} = -\bar{D} (AV^{-1}) \circ_V (VA^{-1}) = -(DA) V^{-1} \circ_V (VA^{-1}) = -(DA) A^{-1}$$

where we have used (7.9) and the definition of $\circ_V$ (1.20). Therefore, the two descriptions give the same result and hence the torsion is well-defined.

**Remark 7.10** This shows that the “covariant” derivative $D$ is indeed covariant under change of trivialization. In particular, if we have different choices of the reference $G_2$-structure, say $\varphi$ and $\tilde{\varphi}$, the octonion solutions of $DA = 0$ and $\bar{D} A = 0$ will be different, however they still define the same $G_2$-structures under the map $\sigma$. That is, $\sigma_A (\varphi) = \sigma_{\tilde{A}} (\tilde{\varphi})$.

The equation $DA = 0$ is linear, but it is an overdetermined PDE, since pointwise, we have 56 equations for 8 variables of $A$. If we however restrict to compact manifolds, we can show that this is equivalent to an elliptic equation $D^2 A = 0$ where we define $D^2 = D_i D^i$. From the definition of $D$ this is clearly elliptic, since the leading term is just the ordinary rough Laplacian $\nabla^2 = \nabla_i \nabla^i$.

**Proposition 7.11** Suppose $(\varphi, g)$ is a $G_2$-structure on a compact 7-dimensional manifold $M$. Then, if $A \in C^2 (\Omega M)$ is a twice differentiable octonion section, $DA = 0$ if and only if $D^2 A = 0$.

**Proof.** Clearly, if $DA = 0$, then $D^2 A$ also vanishes. Conversely, suppose $A$ is an octonion section that satisfies $D^2 A = 0$. Using the compatibility of $D$ with the metric (6.3), we have the following identity:

$$\nabla^2 |A|^2 = 2 \nabla^i \left( \langle D_i A, A \rangle_\Omega \right) = 2 \left( \langle D^2 A, A \rangle_\Omega + |DA|^2 \right)$$

(7.16)

However, if $D^2 A = 0$, we get

$$\nabla^2 |A|^2 = 2 |DA|^2$$

(7.17)

Therefore, $\nabla^2 |A|^2 \geq 0$ on $M$, and by the Weak Maximum Principle, $|A|^2$ must be constant, and hence $|DA|^2 = 0$ everywhere on $M$. ■

Combining Corollary 7.4 and Proposition 7.11, we thus have the following important result.

**Theorem 7.12** Let $M$ be a smooth 7-dimensional compact manifold with a $G_2$-structure $(\varphi, g)$ with torsion $T \in \Omega^1 (\text{Im} \Omega M)$. There exists a torsion-free $G_2$-structure $\tilde{\varphi}$ in the same metric class as $\varphi$ if and only if there exists a non-zero $V \in C^2 (\Omega M)$ that satisfies the linear elliptic PDE

$$D^2 V = 0$$

(7.18)

where the covariant derivative $D$ is defined by (6.4) using the torsion 1-form $T$. Moreover, $\tilde{\varphi} = \sigma_V (\varphi)$.
Remark 7.13 Given a fixed metric $g$ on a compact manifold $M$, it is well-known that a $G_2$-structure $\varphi$ in the metric class of $g$ is torsion-free if and only if $\Delta \varphi = 0$. Although this is a linear elliptic PDE for the 3-form $\varphi$, there is an extra non-linear condition to make sure that $\varphi$ is actually compatible with $g$. Equation (7.18) is also linear and elliptic, but it does not require any further conditions. Since any non-zero solution of (7.18) will actually be a parallel octonion (with respect to $D$), it will have constant norm, and will be nowhere-vanishing. The trade-off here is that the equation (7.18) depends on the choice of trivialization, but in a covariant way. This is to be expected, because the existence of solutions to (7.18) is a property of the metric - the dimension of the kernel of $D^2$ tells us the holonomy group of the metric.

8 Relationship to the spinor bundle

It is well-known (e.g. [5, 24]) that quaternions and octonions have a very close relationship with spinors in 3, 4 and 7, 8 dimensions, respectively. In particular, multiplication by imaginary octonions is equivalent to Clifford multiplication on spinors in 7 dimensions. More precisely, the enveloping algebra of the octonion algebra is isomorphic to the Clifford algebra in 7 dimensions. The (left) enveloping algebra of $\mathbb{O}$ consists of left multiplication maps $L_A : V \to AV$ for $A, V \in \mathbb{O}$, under composition [30]. Similarly, a right enveloping algebra may also be defined. Since the binary operation in the enveloping algebra is defined to be composition, it is associative. For octonions $A, B, V$ we thus have

$$L_AL_B(V) + L_BL_A(V) = A(BV) + B(AV)$$

$$= (AB)V + [A,B,V] + (BA)V + [B,A,V]$$

$$= (AB + BA)V$$

Hence, if $A, B$ are pure imaginary, then indeed,

$$L_AL_B + L_BL_A = -2 \langle A, B \rangle \text{Id} \quad (8.1)$$

which is the defining identity for a Clifford algebra. We see that while the octonion algebra does give rise to the Clifford algebra, in the process we lose the nonassociative structure, and hence the octonion algebra has more structure than the corresponding Clifford algebra. Note that also due to non-associativity of the octonions, in general that $L_AL_B \neq L_BL_A$. In fact, we have the following relationship.

Lemma 8.1 Let $A, B, C$ be octonion sections, with multiplication defined by the $G_2$-structure $\varphi$, then

$$A(BC) = (A \circ_C B)C$$

where $\circ_C$ denotes octonion multiplication with respect to the $G_2$-structure $\sigma_C(\varphi)$. In particular,

$$L_AL_B = L_{A\circ_CB}$$

Proof. We can write

$$A(BC) = (AB)C + [A,B,C]$$

$$= [AB + [A,B,C]C^{-1}]C$$

$$= (A \circ_C B)C$$

where we have used [120]. 

Let $\mathcal{S}$ be the spinor bundle on the 7-manifold $M$. It is then well-known that a nowhere-vanishing spinor on $M$ defines a $G_2$-structure via a bilinear expression involving Clifford multiplication. In fact, given a unit norm spinor $\xi \in \Gamma(\mathcal{S})$, we may define

$$\varphi_\xi(\alpha, \beta, \gamma) = -\langle \xi, \alpha \cdot (\beta \cdot (\gamma \cdot \xi)) \rangle_\mathcal{S} \quad (8.2)$$

where $\cdot$ denotes Clifford multiplication, $\alpha, \beta, \gamma$ are arbitrary vector fields and $\langle \cdot, \cdot \rangle_\mathcal{S}$ is the inner product on the spinor bundle. The next lemma shows that we get exactly the same expression if we use the octonion representation of the Clifford algebra.
Lemma 8.2 Let \( \alpha, \beta, \gamma \in \text{Im} \Gamma (\mathbb{O} M) \), and suppose \( V \in \Gamma (\mathbb{O} M) \) is a unit octonion section. Then,

\[
(\sigma V \varphi \alpha, \beta, \gamma) = - \langle V, (\alpha, \beta, \gamma V) \rangle_0 \quad (8.3)
\]

Proof. Using Lemma 8.1 we have

\[
\alpha (\beta (\gamma V)) = \alpha ((\beta \circ V \gamma) V) = (\alpha \circ V (\beta \circ V \gamma)) V
\]

Hence, using the fact that \(|V| = 1|,

\[
\langle V, \alpha (\beta (\gamma V)) \rangle_0 = \langle V, (\alpha \circ V (\beta \circ V \gamma)) V \rangle_0 = (1, \alpha \circ V (\beta \circ V \gamma))_0 = - (\alpha, \beta \circ V \gamma)_0
\]

Therefore,

\[
\langle V, \alpha (\beta (\gamma V)) \rangle_0 = - (\alpha, \beta \circ V \gamma)_0 = - (\sigma V \varphi \alpha, \beta, \gamma)
\]

The main difference between (8.2) and (8.3) is that the right hand side of (8.2) only depends on the Clifford algebra (and hence only on the metric), while the right hand side of (8.3) already assumes a choice of a reference \( G_2 \)-structure. Suppose the reference \( G_2 \)-structure \( \varphi_\xi \) is defined by a unit norm spinor \( \xi \). Using Lemma 8.1, we have

\[
\phi_\xi \text{ is defined by a unit norm spinor } \xi
\]

for any octonion \( V \) and spinor \( \eta \) and where \( \circ \) denotes octonion multiplication with respect to the \( G_2 \)-structure \( \varphi_\xi \), and for \( V = (v_0, v) \), the Clifford product is given by \( V \cdot \eta = v_0 \eta + v \cdot \eta \). Now if \( \eta = A \cdot \xi \) for some octonion section \( A \), then in (8.4b)

\[
\phi_\xi (\eta) = A \quad (8.5)
\]

Note that in 7 dimensions, if we fix a nowhere-zero spinor \( \xi \), then we get a pointwise decomposition of \( \mathcal{S} \) as \( \mathbb{R} \cdot \xi \oplus \{ X \cdot \xi : X \in \mathbb{R}^7 \} \), so given any spinor \( \eta \), we can write it as \( \eta = A \cdot \xi \) for some octonion section \( A \). Therefore, the map \( \phi_\xi \) is in fact pointwise an isomorphism of real vector spaces from spinors to octonions.

Lemma 8.3 The map \( \phi_\xi \) respects the inner product. That is, for spinors \( \eta_1 \) and \( \eta_2 \),

\[
\langle \eta_1, \eta_2 \rangle_S = \langle \phi_\xi (\eta_1), \phi_\xi (\eta_2) \rangle_0 \quad (8.6)
\]

Proof. Suppose \( \eta_1 = V_1 \cdot \xi \) and \( \eta_2 = V_2 \cdot \xi \) where \( V_1 = (a_1, v_1) \) and \( V_2 = (a_2, v_2) \). Then,

\[
\langle \eta_1, \eta_2 \rangle_S = \langle V_1 \cdot \xi, V_2 \cdot \xi \rangle_S = a_1 a_2 |\xi|^2 + \langle v_1 \cdot \xi, v_2 \cdot \xi \rangle_S = a_1 a_2 |\xi|^2 + \langle v_1, v_2 \rangle |\xi|^2 = \langle V_1, V_2 \rangle_0 = \langle \phi_\xi (\eta_1), \phi_\xi (\eta_2) \rangle_0
\]

where we have the property that the Clifford product is skew-adjoint with respect to the spinor inner product [24].
Under the map \( j_\xi \) we then have

\[
\varphi_\xi (\alpha, \beta, \gamma) = - \langle \xi, \alpha \cdot (\beta \cdot (\gamma \cdot \xi)) \rangle_S
\]

\[
= - \langle \xi (\xi), \alpha (\beta (\gamma (\xi))) \rangle_S
\]

\[
= - \langle 1, \alpha (\beta (\gamma)) \rangle
\]

\[
= \langle \alpha, \beta \gamma \rangle
\]

as expected. Then, for \( \eta = A \cdot \xi \), using Lemma 8.3, we get

\[
\varphi_\eta (\alpha, \beta, \gamma) = - \langle \eta, \alpha \cdot (\beta \cdot (\gamma \cdot \eta)) \rangle_S
\]

\[
= - \langle \xi (\eta), \alpha (\beta (\gamma (\xi))) \rangle_S
\]

\[
= - \langle 1, \alpha (\beta (\gamma A)) \rangle
\]

where the octonion multiplication is with respect to \( \varphi_\xi \). Using (8.3) this then shows that

\[
\varphi_A \cdot \xi = \sigma_A (\varphi_\xi)
\]

This shows that our Theorem 4.8 on composition of \( \sigma_U \) and \( \sigma_V \) can be restated in terms of spinors.

**Corollary 8.4** Let \( \xi \) be a spinor of unit norm on a 7-dimensional manifold and let \( \varphi_\xi \) be the \( G_2 \)-structure defined by (8.2). Then, for any unit octonions \( U, V \)

\[
\varphi_U \cdot (V \cdot \xi) = \varphi(UV) \cdot \xi
\]

where the octonion product \( UV \) on the right hand side is defined unambiguously using \( \varphi_\xi \) or \( \varphi_{V \cdot \xi} \) (see Remark 4.9).

**Proof.** Theorem 4.8 tells us that

\[
\sigma_U (\sigma_V \varphi_\xi) = \sigma_{UV} \varphi_\xi
\]

However, from (8.2), we get

\[
\sigma_U (\sigma_V \varphi_\xi) = \sigma_U (\varphi_{V \cdot \xi}) = \varphi_U \cdot (V \cdot \xi)
\]

\[
\sigma_{UV} \varphi_\xi = \varphi(UV) \cdot \xi
\]

which gives us (8.8). \( \square \)

It is also well-known (e.g. [2, Definition 4.2 and Lemma 4.3]) that given the spinorial covariant derivative \( \nabla^S \) on \( S \), which is obtained by lifting the Levi-Civita connection to \( S \), we get

\[
\nabla^S_X \xi = -T^{(\xi)}_X \cdot \xi
\]

where \( T^{(\xi)} \) is the torsion tensor of \( \varphi_\xi \). Note that in [2], the torsion endomorphism is denoted by \( S \), and compared to our conventions \( S = -T \). The negative sign is due to a different sign in the definition (8.2) of \( \varphi_\xi \) in terms of the spinor \( \xi \). Equation (8.9) gives us an important relationship between the spinorial covariant derivative and the octonion covariant derivative.

**Theorem 8.5** Let \( \xi \in \Gamma (S) \) be a unit spinor on a 7-manifold \( M \) and let \( \varphi_\xi \) be the \( G_2 \)-structure defined by \( \xi \) via (8.2). Then, for any \( \eta \in \Gamma (S) \)

\[
j_\xi \cdot (\nabla^S_X \eta) = D^{(\xi)}_X (j_\xi (\eta))
\]

where \( D^{(\xi)} \) is the octonion covariant derivative [6.4] with respect to the \( G_2 \)-structure \( \varphi_\xi \).
Proof. Using (8.9), we have

\[ j_\xi (\nabla_X S^\xi) = -T^{(\xi)}_X = D^{(\xi)}_X j_\xi (\xi) \] (8.11)

(8.12)

Then, for an arbitrary spinor \( \eta = A \cdot \xi \),

\[ \nabla^S_X \eta = (\nabla_X A) \cdot \xi + A \cdot \nabla^S_X \xi \]

and using the properties of \( j_\xi \) (8.4), we conclude

\[ j_\xi (\nabla^S_X \eta) = (\nabla_X A) \cdot j_\xi (\xi) + A \cdot j_\xi (\nabla^S_X \xi) \]

\[ = \nabla_X A - AT^{(\xi)}_X \]

\[ = D^{(\xi)}_X A = D^{(\xi)}_X j_\xi (\eta). \] (8.13)

Remark 8.6 Theorem 8.5 thus shows that with respect to the real vector bundle isomorphism \( S \cong \Omega M \) given by \( j_\xi \), the spin bundle connection \( \nabla^S \) is mapped precisely to the octonion connection \( D \). From the above properties it may seem that given a fixed nowhere vanishing spinor, the octonion bundle is isomorphic to the spinor bundle. The isomorphism however is only at the level of real vector bundles and connections. However, similar to the relationship between the Clifford algebra and the enveloping algebra of the octonions, the two structures are not fully isomorphic, precisely due to the fact that the octonion bundle has a nonassociative product which is not present in the spinor bundle. Therefore, the octonion bundle retains all of the information from the spinor bundle, but has some additional structure. This is also reflected in the fact that there is no natural binary operation on spinors. When transitioning to octonions, we are implicitly applying the triality relationship between spinors and vectors to define the octonion multiplication [5].

Remark 8.7 Theorem 8.5 also shows that our condition \( DV = 0 \) (7.7) for the existence of a torsion-free \( G_2 \)-structure is equivalent to the well-known condition \( \nabla^S \eta \) = 0 for some nowhere-vanishing spinor \( \eta \).

9 Dirac operator

We may define a distinguished \( \text{Im} \mathbb{O} \)-valued 1-form \( \delta \in \Omega^1 (\text{Im} \mathbb{O} M) \) such that for any vector \( X \) on \( M \), \( \delta (X) \in \Gamma (\text{Im} \mathbb{O} M) \), with components given by

\[ \delta (X) = (0, X). \] (9.1)

Therefore in particular, \( \delta \) is the isomorphism that takes vectors to imaginary octonions. In components, its imaginary part is simply represented by the Kronecker delta:

\[ \delta_i = (0, \delta^\alpha_i). \] (9.2)

Note that left multiplication by \( \delta \) gives a representation of the Clifford algebra, so these are precise analogs of the gamma-matrices used in physics. Below are some properties of \( \delta \):

Lemma 9.1 Suppose \( \delta \in \Omega^1 (\text{Im} \mathbb{O} M) \) is defined by (9.2) on a 7-manifold \( M \) with \( G_2 \)-structure \( \varphi \) and metric \( g \). It then satisfies the following properties, where octonion multiplication is with respect to \( \varphi \):

1. \( \nabla \delta = 0 \)
2. \( \delta_i \delta_j = (-g_{ij}, \varphi_{ij}^\alpha) \)
3. \( \delta_i (\delta_j \delta_k) = (-\varphi_{ijk}, \psi_{ijk}^\alpha - \delta_i^\alpha g_{jk} + \delta_j^\alpha g_{ik} - \delta_k^\alpha g_{ij}) \)
4. For any $A = (a_0, \alpha) \in \Gamma (\mathbb{O} M)$, 
\[ \delta_i A = \left( a_0 \delta_i - (\alpha \varphi)_i \right) \]  
(9.3)

**Proof.** It is obvious that $\delta$ is parallel with respect to the Levi-Civita connection. Consider now the octonion product $\delta_i \delta_j$. Writing octonion real and imaginary parts in column notation for clarity, we have
\[ \delta_i \delta_j = \left( \begin{array}{c} 0 \\ \delta_i \end{array} \right) \left( \begin{array}{c} 0 \\ \delta_j \end{array} \right) = \left( \begin{array}{c} -\langle \delta_i, \delta_j \rangle_{\text{Im} \mathbb{O}} \\ \delta_i \times \delta_j \end{array} \right) = \left( \begin{array}{c} -g_{ij} \varphi_{ij} \\ \varphi_{ij} \end{array} \right) \]
and similarly,
\[ \delta_i (\delta_j \delta_k) = \left( \begin{array}{c} 0 \\ \delta_i \end{array} \right) \left( \begin{array}{c} -\varphi_{jk} \\ \varphi_{jk} \end{array} \right) = \left( \begin{array}{c} -\varphi_{ijk} \\ -\delta_i g_{jk} + \varphi_i \gamma_j g_{jk} - \delta_k g_{ij} \end{array} \right) \]
where we have used the contraction identity (3.6) for $\varphi$.

Finally, consider $\delta_i A$. We can write
\[ \delta_i A = a_0 \delta_i + \delta_i (0, \alpha) = a_0 \delta_i + \alpha^j \delta_i \delta_j = a_0 \delta_i + \alpha^j \left( -g_{ij} + \varphi_{ij} \right) = \left( \begin{array}{c} -\alpha_i \\ a_0 \delta_i - (\alpha \varphi)_i \end{array} \right) \]

We can now define the octonion Dirac operator $\mathcal{D}$ using $\delta$ and the octonion covariant derivative $D$ (6.4). Let $A \in \Gamma (\mathbb{O} M)$, then define $\mathcal{D} A$ as
\[ \mathcal{D} A = \delta \circ (D A) \]  
(9.4)
where $\circ$ is a combination of contraction and octonion multiplication. In coordinates, (9.4) is given by
\[ \mathcal{D} A = \delta^{i} \circ (D_{i} A). \]  
(9.5)
This operator is precisely what we obtain by applying the map $j_{\xi}$ to the standard Dirac operator on the spinor bundle. We can use this definition to work some properties of the operator. First of all, let us prove that $\mathcal{D}$ is covariant under a change of the reference $G_2$-structure:

**Proposition 9.2** Suppose $(\varphi, g)$ is a $G_2$-structure on a 7-manifold $M$, with torsion $T$ and corresponding octonion covariant derivative $D$ and Dirac operator $\mathcal{D}$. Suppose $V$ is a unit octonion section, and $\tilde{\varphi} = \sigma_V (\varphi)$ is the corresponding $G_2$-structure, that has torsion $\tilde{T}$, given by (7.7), an octonion covariant derivative $\tilde{D}$ and Dirac operator $\tilde{\mathcal{D}}$. Then, for any octonion section $A$, we have
\[ \tilde{\mathcal{D}} (AV^{-1}) = (\mathcal{D} A) V^{-1} \]  
(9.6)

**Proof.** This follows from the covariant nature of $D$. Suppose $\tilde{\circ}$ denotes octonion product with respect to $\tilde{\varphi}$.
By definition of $\tilde{\mathcal{D}}$, we then have
\[ \tilde{\mathcal{D}} (AV^{-1}) = \delta \circ (\tilde{D} (AV^{-1})) \]
Using Proposition 7.8 and (4.20), we rewrite this as

\[ \tilde{\mathcal{D}} (AV^{-1}) = \delta_{\tilde{\mathcal{D}}} ((DA) V^{-1}) \]

\[ = \delta_{\tilde{\mathcal{D}}} ((DA) V^{-1}) + [\delta_{\tilde{\mathcal{D}}} (DA) V^{-1}, AV^{-1}] \]

\[ = (\delta_{\tilde{\mathcal{D}}} (DA)) V^{-1} + [\delta_{\tilde{\mathcal{D}}} (DA), V] + [\delta (DA), V] \]

\[ = (DA) V^{-1} \]

where we have also used Lemma 3.9.

**Theorem 9.3** Suppose \( V \) is a unit octonion section, and suppose \( \tilde{\varphi} = \sigma_{V} (\varphi) \) has torsion tensor \( \tilde{T} \). Then,

\[ \tilde{\mathcal{D}} V = (7 \tilde{\tau}_{1} - 6 \tilde{\tau}_{7}) V \]

where \( \tilde{T} \) and \( \tilde{T} \) are the 1-dimensional and 7-dimensional components of \( \tilde{T} \), respectively. In particular,

\[ \tilde{\mathcal{D}} 1 = \begin{pmatrix} 7 \tau_{1} \\ -6 \tau_{7} \end{pmatrix} \]

where \( \tau_{1} \) and \( \tau_{7} \) are the corresponding components of \( T \) - the torsion tensor of the \( G_{2} \)-structure \( \varphi \).

**Proof.** Let us first verify (9.8). Indeed, since from (6.3), \( D1 = -T \) we have

\[ \tilde{\mathcal{D}} 1 = \delta^{i} D_{i} 1 = -\delta^{i} T_{i} \]

\[ = -T_{j}^{i} (\delta^{i} \delta_{j}) \]

\[ = -T_{j}^{i} (\frac{g^{ij}}{\varphi^{aij}}) \]

where have also used Lemma 9.1. Thus,

\[ \tilde{\mathcal{D}} 1 = \begin{pmatrix} \text{Tr} T \\ -T_{j} \varphi \end{pmatrix} \]

and (9.8) follows.

Now let \( \tilde{\mathcal{D}} \) be the Dirac operator with respect to the \( G_{2} \)-structure \( \tilde{\varphi} = \sigma_{V} (\varphi) \). To get (9.7), we note that

\[ \tilde{\mathcal{D}} 1 = (7 \tilde{\tau}_{1} - 6 \tilde{\tau}_{7}) \]

however from Proposition 9.2

\[ \tilde{\mathcal{D}} 1 = \tilde{\mathcal{D}} (VV^{-1}) = (\tilde{\mathcal{D}} V) V^{-1} \]

Hence,

\[ (\tilde{\mathcal{D}} V) = (\tilde{\mathcal{D}} 1) V \]

and thus we get (9.7).

Equivalently we can translate the results of Theorem 9.3 into the language of spinors.

**Corollary 9.4** Let \( \xi \) be a unit spinor and suppose \( \tau_{1} \) and \( \tau_{7} \) are the 1- and 7-dimensional components of the torsion of the \( G_{2} \)-structure \( \varphi_{\xi} \). Then, \( \xi \) satisfies

\[ \nabla \xi = 7 \tau_{1} \xi - 6 \tau_{7} \cdot \xi \]

where \( \nabla \) is the Dirac operator on the spinor bundle.
Corollary 9.5 Let $\xi$ be a unit spinor. Then the corresponding $G_2$-structure $\varphi_\xi$ has only 14- and 27-dimensional torsion components if and only if $\nabla_\xi = 0$, that is, $\xi$ is a harmonic spinor.

The result in Corollary 9.5 has also been proved recently using a different method in [2].

The octonionic Dirac operator also satisfies the Lichnerowicz-Weitzenböck formula. Of course given the spinorial Lichnerowicz-Weitzenböck formula, we immediately obtain the octonionic analog using the map $j_\xi$, however we can also prove it using octonionic techniques.

Theorem 9.6 (Lichnerowicz-Weitzenböck) For any smooth octonion section $V$, we have

$$\mathcal{D}^2 V = d^* d D V + \frac{1}{4} R V$$  \hspace{1cm} (9.11)

where $R$ is the scalar curvature.

Proof. If $V$ is identically zero at a point $p$ and a neighborhood around it, then (9.11) is trivially true, since all the operators are local. Suppose $V \neq 0$ at $p$, then at least locally we can change the reference $G_2$-structure to $\sigma_\xi (\varphi)$. Then, if $\tilde{\mathcal{D}}$ is the Dirac operator corresponding to $\sigma_\xi (\varphi)$, Proposition 9.2 shows that

$$\mathcal{D}^2 V = (\tilde{\mathcal{D}}_1 V, \mathcal{D}^2 (\tilde{\mathcal{D}}_1 V)) \hspace{1cm} (9.12)$$

So in this case, it is enough to verify (9.11) for $V = 1$. We then have

$$\mathcal{D}^2 1 = (\delta^i \delta^j) (D_i D_j 1) + [\delta^i, \delta^j, D_i D_j 1] \hspace{1cm} (9.13)$$

From Lemma 9.1 $\delta^i \delta^j = \left( - \frac{g^{ij}}{\varphi^{\alpha ij}} \right)$, hence the first term in (9.13) becomes

$$\begin{align*}
(\delta^i \delta^j) (D_i D_j 1) &= \left( - \frac{g^{ij}}{\varphi^{\alpha ij}} \right) \\
&= \delta^i (D_i D_j) \\
&= d^* d D 1 + \left( \frac{0}{\varphi^{\alpha ij}} \right) \\
&= d^* d D 1 + \left( \frac{\Re (D_i D_j 1)}{\Im (D_i D_j 1)} \right) \\
&= \frac{1}{2} d^2 D 1 - \frac{1}{8} \pi_7 \text{Riem} \hspace{1cm} (9.14)
\end{align*}$$

where we have used the formula (6.19) for $d^2 D$ in coordinates. Note that now in the second term of (9.13) as well as (9.14), the indices $i$ and $j$ are skew-symmetrized, but from Proposition 6.8

$$D_{ij} D_{ij} 1 = \frac{1}{2} d^2 D 1 = - \frac{1}{8} \pi_7 \text{Riem} \hspace{1cm} (9.15)$$

Hence, $\Re (D_i D_j 1) = 0$. Using (9.14) and (9.12), equation (9.13) becomes

$$\mathcal{D}^2 1 = d^* d D 1 + \frac{1}{8} \varphi^{kij} (\pi_7 \text{Riem})_{ij} \left( \delta_k \delta_l \right) - \frac{1}{4} \psi^{kijl} (\pi_7 \text{Riem})_{ijl} \delta_k \hspace{1cm} (9.16)$$

Using Lemma 9.1 $\delta_k \delta_l = - g_{kl} + \varphi_m \delta_m$, thus

$$\begin{align*}
\mathcal{D}^2 1 &= d^* d D 1 + \frac{1}{8} \varphi^{kij} (\pi_7 \text{Riem})_{ij} \delta_k - \frac{1}{4} \varphi^{kij} (\pi_7 \text{Riem}) \varphi_m \delta_m - \frac{1}{4} \psi^{kijl} (\pi_7 \text{Riem})_{ijl} \delta_k \hspace{1cm} (9.16)
\end{align*}$$
However, using the identity (3.6), as well as the Riemannian Bianchi identity,
\[ \varphi_{kij} (\pi_7 \text{Riem})_{ijk} = \varphi_{kij} \text{Riem}_{ijmn} \varphi^{mn} \]
\[ = \text{Riem}_{ijmn} (\psi_{ijmn} + g^{im} g^{jn} - g^{in} g^{jm}) \]
\[ = 2 R \]
where we have used (1.7). Similarly, using the identity (3.6) and the Bianchi identity,
\[ \varphi_{kij} (\pi_7 \text{Riem})_{ijl} \varphi_{mkl} = (\pi_7 \text{Riem})_{ijl} \psi_{ijlm} \]
\[ = (\pi_7 \text{Riem})_{ijl} \psi_{ijlm} \]
Now we are left with
\[ \mathcal{D}^2 1 = d_D^* d_D 1 + \frac{1}{4} R + \frac{1}{8} (\pi_7 \text{Riem})_{ijl} \psi_{ijlm} \delta_m. \] (9.17)
Consider
\[ (\pi_7 \text{Riem})_{ijl} \psi_{ijlm} = - \text{Riem}_{ijpq} \varphi_{i}^{pq} \psi_{ijlm} \]
Using the identity (4.17),
\[ \text{Riem}_{ijpq} \varphi_{i}^{pq} \psi_{ijlm} = -3 \text{Riem}_{ijpq} (g^{[i} \varphi^{j]q} - g^{[i} \varphi^{[j}p) \]
\[ = -6 \text{Riem}_{ijpq} g^{[i} \varphi^{j]q} \]
\[ = 6 \text{Riem}_{ijpq} \varphi^{[i} \varphi^{j]q} \]
\[ = -2 (\text{Ric})_{jq} \varphi^{jq} + 2 (\text{Ric})_{iq} \varphi^{iq} + 2 \text{Riem}_{ijpq} \varphi^{[i} \varphi^{j]q} \]
\[ = 0 \]
Thus, we are left with
\[ \mathcal{D}^2 1 = d_D^* d_D 1 + \frac{1}{4} R. \]
For an arbitrary octonion section \( V \) that is nonzero at a point \( p \), using (9.12) we obtain (9.11).

Now suppose \( V(p) = 0 \) and is not identically zero in any neighborhood of \( p \). Let \( L = \mathcal{D}^2 V - d_D^* d_D V - \frac{1}{4} RV \). Since \( V \) is fixed, this is in particular a continuous map from a neighborhood of \( p \) to the octonions. We can then find a sequence of points \( \{ p_n \} \) in a neighborhood of \( p \) such that \( p_n \to p \) and such that for each \( n \), either \( V(p_n) \neq 0 \) or \( V(p_n) = 0 \) and \( V \equiv 0 \) in a neighborhood of \( p_n \). Using the previous cases in the proof, we find that \( L(p_n) = 0 \) for all \( n \). Therefore, by continuity, \( L(p) = 0 \), and thus the identity is satisfied. We thus conclude that (9.11) is true for any (smooth) octonion section \( V \). □

**Remark 9.7** On a compact manifold \( M \), for a unit octonion \( V \), we then find that
\[ \int_M |\mathcal{D}V|^2 \text{vol} = \int_M |DV|^2 \text{vol} + \frac{1}{4} \int_M R \text{vol} \] (9.18)
This shows that if \( \int_M R \text{vol} = 0 \), i.e. the total scalar curvature is zero, then \( \mathcal{D}V = 0 \) if and only if \( DV = 0 \). This then implies that on a compact manifold, a \( G_2 \)-structure that is compatible with a metric that has vanishing total scalar curvature, is torsion-free if and only if its \( \tau_1 \) and \( \tau_7 \) torsion components both vanish. This can also be obtained directly by integrating the expression (5.7) for the scalar curvature in terms of torsion components. Note that if the scalar curvature vanishes pointwise, then also from (5.7), we see that even without requiring compactness we find that a compatible \( G_2 \)-structure is torsion-free if and only if \( \tau_1 \) and \( \tau_7 \) both vanish.

We can also write out the octonionic Dirac operator explicitly in terms of the \( G_2 \)-structure torsion.
Theorem 9.8 Suppose $A = (a_0, \alpha) \in \Gamma (\mathcal{O} M)$. Then,

$$\slashed{D} A = \begin{pmatrix} - \text{div}_T A \\ \text{grad}_T A + \text{curl}_T A \end{pmatrix}$$

(9.19)

where $\text{div}_T$, $\text{grad}_T$ and $\text{curl}_T$ are given by

\begin{align*}
\text{div}_T A &= \text{div} \alpha - a_0 \text{Tr} T + \langle \alpha, T \varphi \rangle = \text{div} \alpha - 7a_0 \tau_1 - 6 \langle \alpha, \tau_7 \rangle \quad (9.20a) \\
\text{grad}_T A &= \text{grad} a_0 + T (v) \quad (9.20b) \\
\text{curl}_T A &= \text{curl} \alpha + a_0 (T \varphi) - \alpha \text{Tr} T + T_\alpha - \alpha \cdot (T \psi) \quad (9.20c)
\end{align*}

where $(T_\alpha)^k = \alpha^i T_i^k$ and

$$(\text{curl} \alpha)^i = \varphi^{ab} \nabla_b \alpha_a.$$  

(9.21)

Remark 9.9 In [22], Karigiannis gave an expression for an octonionic Dirac operator in the torsion-free case. In the case $T = 0$, (9.19) reduces to

$$\slashed{D} A = (- \text{div} \alpha, \text{grad} a_0 + \text{curl} \alpha)$$

(9.22)

which is precisely the expression given by Karigiannis.

Proof of Theorem 9.8 By definition (9.4), we have

$$\slashed{D} A = \delta^i (\nabla_i A - AT_i)$$

(9.23)

The first term is then

\begin{align*}
\delta^i \nabla_i A &= \begin{pmatrix} 0 \\ g_{ij} \end{pmatrix} \begin{pmatrix} \nabla_i a_0 \\ \nabla_i \alpha^k \end{pmatrix} \\
&= \begin{pmatrix} -g_{ij} \nabla_i \alpha_j \\ \nabla^l a_0 + \varphi^l_{jk} g_{ij} \nabla_l \alpha^k \end{pmatrix} \\
&= (- \text{div} \alpha, \text{grad} a_0 + \text{curl} \alpha)
\end{align*}

which is precisely (9.22) - the term which is independent of the torsion. Now let us look at the second term of (9.23). We have

$$\delta^i (AT_i) = a_0 \delta^i T_i + \alpha^j \delta^i (\delta_j T_i).$$

(9.24)

Now again look at each term separately. The first term in (9.24) is then just

\begin{align*}
a_0 \delta^i T_i &= -a_0 \slashed{D} 1 \\
&= -a_0 \begin{pmatrix} \text{Tr} T \\ -T \varphi \end{pmatrix}
\end{align*}

where we have used (9.9) and the computation before (9.9). Since the components $T_i^k$ are real, the second term in (9.24) can be re-written in the following way

\begin{align*}
\alpha^j \delta^i (\delta_j T_i) &= \alpha^j T_i^k \delta^i (\delta_j \delta_k) \\
&= \alpha^j T_i^k \left( \psi^{mi}_{jk} - g_{mi} g_{jk} + \delta^m_j \delta^i_k - \delta^m_k \delta^i_j \right) \\
&= \left( \alpha \cdot (T \varphi) - T (\alpha) + \alpha \text{Tr} T - T_\alpha \right)
\end{align*}
where we have also used Lemma 9.1. Overall,

$$\mathcal{P}A = \left( - \text{div} \alpha + a_0 \text{Tr} T - \langle \alpha, T \varphi \rangle \right)$$

Applying the definitions (9.20) of $\text{div}_T$, $\text{grad}_T$ and $\text{curl}_T$, we obtain (9.19). The alternative expressions using components of $T$ are then derived using the identities

$$\text{Tr} T = 7 \tau_1$$
$$T \varphi = 6 \tau_7$$
$$T \psi = 4 \tau_7 \varphi - 2 \tau_{14}$$

Remark 9.10 The motivation for the definitions (9.20) is the following. The standard $\text{div}$, $\text{grad}$ and $\text{curl}$ are obtained by computing $\delta^i \nabla_i A$. In particular,

$$\text{div} \alpha = \langle \delta^i, \text{Im} \nabla_i A \rangle$$
$$\text{grad} a_0 = \delta^i (\text{Re} \nabla_i A)$$
$$\text{curl} \alpha = \delta^i \times (\text{Im} \nabla_i A).$$

The expressions (9.20) are then similarly obtained by replacing $\nabla A$ in (9.25) by $DA$:

$$\text{div}_T A = \langle \delta^i, \text{Im} D_i A \rangle$$
$$\text{grad}_T A = \delta^i (\text{Re} D_i A)$$
$$\text{curl}_T A = \delta^i \times (\text{Im} D_i A).$$

Note that since now there is some mixing of real and imaginary parts of $A$, it only makes sense to apply these operators to $A$ as a whole. It is then clear that

$$\mathcal{P}A = \delta^i D_i A = ( - \text{div}_T A, \text{grad}_T A + \text{curl}_T A)$$

It then takes a routine calculation to actually obtain the expressions (9.20) from (9.26).

10 Energy functional

Given a 7-dimensional Riemannian manifold that admit $G_2$-structures, we have a choice of $G_2$-structures that correspond to the given Riemannian metric $g$. After fixing an arbitrary $G_2$-structure $\varphi$ in this metric class, all the other $G_2$-structures that are compatible with $g$ are parametrized by unit octonion sections, up to a sign. Given a unit octonion section $V$, the corresponding $G_2$-structure $\sigma_V(\varphi)$ will have torsion $T(V)$ given by $T(V) = -(DV) V^{-1}$, where $D$ is the octonion covariant derivative with respect to $\varphi$. The question is how to pick the "best" representative of this metric class. The choice of a particular $G_2$-structure in a fixed metric class is akin to choosing a gauge in gauge theory. Obviously, if the metric has holonomy contained in $G_2$, then the "best" representative should be a torsion-free $G_2$-structure that corresponds to that metric. In general however, one approach, at least on compact manifolds, would be to pick a gauge that minimizes some functional. The obvious choice is the $L_2$-norm of the torsion. Suppose $M$ is now compact, and define a functional $\mathcal{E} : \Gamma (SOM) \rightarrow \mathbb{R}$, where $SOM$ is the unit sphere subbundle, by

$$\mathcal{E} (V) = \frac{1}{2} \int_M |T(V)|^2 \text{vol}$$
$$= \frac{1}{2} \int_M |(DV) V^{-1}|^2 \text{vol}$$
$$= \frac{1}{2} \int_M |DV|^2 \text{vol}$$
Thus, this is simply the energy functional for unit octonion sections. Note that a similar energy functional for spinors has been recently studied by Ammann, Weiss and Witt [4], however in their case, the metric was unconstrained, and so the functional was both on spinors and metrics. Using the properties of $D$, we obtain the critical points using standard methods:

**Proposition 10.1** The critical points of $E$ satisfy

$$D^*DV - |DV|^2 V = 0. \quad (10.4)$$

**Proof.** To make the restriction to unit octonions explicit, let us introduce a Lagrange multiplier function $\lambda$, so that now the functional is given by

$$E(V) = \frac{1}{2} \int_M \left( |DV|^2 - \lambda \left( |V|^2 - 1 \right) \right) \text{vol}$$

Variations of $\lambda$ give the pointwise constraint

$$|V|^2 = 1 \quad (10.5)$$

Now, suppose $V(t)$ is a 1-parameter family of unit octonion sections, then

$$\frac{d}{dt}E(V(t)) = \frac{1}{2} \int_M \left( \frac{d}{dt} |DV(t)|^2 - \lambda \frac{d}{dt} |V|^2 \right) \text{vol}$$

$$= \int \left\langle D \frac{d}{dt} V(t), DV(t) \right\rangle - \lambda \left\langle V(t), \frac{d}{dt} V(t) \right\rangle \text{vol}$$

$$= \int \left\langle \frac{d}{dt} V(t), D^* DV(t) - \lambda V(t) \right\rangle \text{vol}$$

Therefore, at a critical point, we must also have

$$D^*DV - \lambda V = 0 \quad (10.6)$$

From (10.5), using the metric-compatible property of $D$ (6.8), we find that

$$\langle DV, V \rangle = 0 \quad (10.7)$$

$$\langle D^* DV, V \rangle - |DV|^2 = 0 \quad (10.8)$$

Therefore, by taking the inner product of (10.6) with $V$, we conclude that $\lambda = |DV|^2$, and hence obtain (10.4). □

**Remark 10.2** In the case when the torsion vanishes, and when restricted to imaginary octonions, since $D = \nabla$, the functional (10.3) reduces to the well-known energy functional of a unit vector field. In that case, the critical points of this energy functional are known as harmonic unit vector fields. The energy functional of a unit vector fields (sometimes also known as the total bending) have been studied independently by Gil-Medrano, Wiegmink, and Wood, among others ([17, 43, 44]). In particular, the equation satisfied by a harmonic unit vector field $v$ is very similar to the equation (10.4):

$$\nabla^* \nabla v - |\nabla v|^2 v = 0 \quad (10.9)$$

Such unit vectors are called harmonic, because the equation (10.9) is one of the equations obtained when considering harmonic maps from a manifold into the unit tangent bundle. A unit vector field can be regarded as a harmonic map if and only if it satisfies (10.9) as well as an equation that depends on the curvature. More details on harmonic unit vector fields can be in particular be found in [7, 23, 26].

Given a unit octonion that is a critical point of the functional $E$, we can interpret the equation (10.4) in terms of the torsion of the corresponding $G_2$-structure $\sigma_V(\varphi)$. 

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Corollary 10.3 A unit octonion \( V \) is a critical point of the functional \( E \) if and only if the torsion \( T^{(V)} \) of the \( G_2 \)-structure \( \sigma_V(\varphi) \) satisfies
\[
\text{div} \, T^{(V)} = 0.
\] (10.10)

Proof. For convenience, let \( \tilde{T} = T^{(V)} \). Note that from (7.5),
\[
\tilde{T} = - (DV) V^{-1}.
\]
Hence, the equation (10.4) can be rewritten as
\[
-D^* \left( \tilde{T} V \right) - |\tilde{T}|^2 V = 0
\]
Using the expression (6.19) for \( D^* \), and the property of \( D \) (6.6) we can write
\[
D^a \left( \tilde{T}_a V \right) - |\tilde{T}|^2 V = \left( \nabla^a \tilde{T}_a \right) V + \tilde{T}_a D^a V - |\tilde{T}|^2 V
\]
However, \( DV = -\tilde{T} V \), so
\[
\tilde{T}_a D^a V = -\tilde{T}_a \left( \tilde{T}^a V \right)
\]
\[
= - \left( \tilde{T}_a \tilde{T}^a \right) V - \left[ \tilde{T}_a, \tilde{T}^a, V \right]
\]
Now,
\[
\tilde{T}_a \tilde{T}^a = \left( \begin{array}{cc} 0 & \tilde{T}^a_m \\ \tilde{T}_a^m & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ \tilde{T}^a_n \end{array} \right) = \left( \begin{array}{c} - |\tilde{T}|^2 \\ \tilde{T}_a^m \tilde{T}^a_n \varphi_{mn} \end{array} \right)
\]
however, \( \tilde{T}_a^m \tilde{T}^a_n \) is symmetric in \( m \) and \( n \), so \( \tilde{T}_a \tilde{T}^a = - |\tilde{T}|^2 \). For the same reason, the associator \( \left[ \tilde{T}_a, \tilde{T}^a, V \right] = 0 \). Hence,
\[
\tilde{T}_a D^a V = |\tilde{T}|^2 V
\]
Therefore,
\[
D^* DV - |DV|^2 V = \left( \text{div} \tilde{T} \right) V
\]
Since \( V \) is nowhere-vanishing, \( D^* DV - |DV|^2 V = 0 \) if and only if \( \text{div} \tilde{T} = 0 \). ■

Remark 10.4 From Corollary 10.3 we obtain a different interpretation of the critical points of \( E \) - the critical points correspond to \( G_2 \)-structures that have divergence-free torsion. This description fits very well with the interpretation of the \( G_2 \)-structure torsion as a connection for a non-associative gauge theory. The condition \( \text{div} \, T = 0 \) is then simply the analog of the Coulomb gauge. It is well-known (e.g. [14, 37]) that in gauge theory, the Coulomb gauge \( d^* A = 0 \), for the gauge connection \( A \), corresponds to critical points of the \( L_2 \)-norm of \( A \). In our situation, we have an exactly similar thing happening. This gives an interesting link between the harmonic map point of view and the Coulomb gauge point of view. Given Uhlenbeck’s existence result for the Coulomb gauge [38], there may be a possibility of an existence result for divergence-free torsion.

The characterization of divergence-free torsion as corresponding to critical points of the functional \( E \) shows that \( G_2 \)-structures with such torsion (whenever they exist) are in some sense special. However the significance of divergence-free torsion still needs to be investigated. In particular, we can see that unit norm eigensections of \( \mathcal{D} \) are in fact critical points of \( E \).

Proposition 10.5 If \( V \) is a unit eigensection of the Dirac operator \( \mathcal{D} \), then \( V \) is a critical point of the functional \( E \).
Proof. Suppose $V$ is a unit eigensection of $\mathcal{D}$ with eigenvalue $\lambda$, then

$$\mathcal{D}V = \lambda V$$
$$\mathcal{D}^2 V = \lambda^2 V$$

However, from the Lichnerowicz-Weitzenböck formula (9.11) we have

$$D^*DV = \mathcal{D}^2 V - \frac{1}{4}RV$$
$$= \left( \lambda^2 - \frac{1}{4}R \right) V$$

Since $|V|^2 = 1$, from (10.8), we have

$$|DV|^2 = \langle D^*DV, V \rangle$$
$$= \lambda^2 - \frac{1}{4}R$$

Therefore, indeed,

$$D^*DV - |DV|^2 V = 0$$

and $V$ is hence a critical point of $E$. ■

Remark 10.6 From (9.7) we see that the condition that $\mathcal{D}V = \lambda V$ for a constant $\lambda$, simply means that the torsion $\tilde{T}$ of $\sigma_V(\varphi)$ has a constant 1-dimensional component and a vanishing 7-dimensional component. Using the $G_2$-structure Bianchi identity (5.5), and in particular, the expressions from [21, Proposition 3.3], which are derived from it, we then obtain that this implies both 14- and 27-dimensional components of $\tilde{T}$ are divergence-free, and hence $\text{div} \tilde{T} = 0$. This is an alternative way to prove the above result. In fact, if $\tilde{T}$ has a vanishing 7-dimensional component, it is true that $\text{div} \tilde{T} = 0$ if and only if $V$ is a constant norm eigenvalue of $\mathcal{D}$.

In general, however, we don’t know if the functional $E$ has any critical points for a given metric, or if a critical point does exist, whether it corresponds to a minimum. However, another approach, that has been successful in the study of harmonic maps (for example, [12]) as well as for other functionals of $G_2$-structures [10, 11, 20, 31, 40, 45] would be to consider a gradient flow of $E$. Since we can always redefine the reference $G_2$-structure to correspond to the initial value of the flow, this would give the following initial value problem

$$\left\{ \begin{array}{l}
\frac{\partial V}{\partial t} = -D^*DV + |DV|^2 V \\
V(0) = 1
\end{array} \right. \quad (10.11)$$

The properties of this flow will be the subject of further study.

11 Concluding remarks

The octonion bundle formalism for $G_2$-structures that has been developed in this paper raises multiple directions for further research. The interpretation of the $G_2$-structure torsion as a connection on a non-associative bundle and $\pi_7$ Riem as its curvature leads to natural questions such as, what is the analogue of a Yang-Mills connection in this case, and what is its interpretation in terms of $G_2$-structures? This also ties in with the interpretation and significance of divergence-free torsion which corresponds to the Coulomb gauge. Then there is also the question of existence of divergence-free connections, that is, critical points of the functional $E(V)$. The equation (10.11) for the critical points is very similar to the harmonic unit vector field equation, however in this case we have additional structure - the equation can be split into real and imaginary octonion parts and we also have the octonion product structure, so it is possible that this could be exploited to give some answers regarding existence. Another possible direction is to consider harmonic maps
from $M$ to the unit octonion bundle of $M$. One of the equations would be precisely (10.4) and there would also be an equation involving curvature. This may also have further interpretation in terms of $G_2$-structures.

In this paper we have been using octonion-valued 1-forms and 2-forms, however for further progress a more rigorous theory of octonion-valued bundles is needed. For quaternions, which are non-commutative but associative, a theory of quaternion-valued modules and bundles has been developed by Joyce [27, 28] and Widdows [41, 42]. For octonion-valued modules and bundles the corresponding theory would necessarily be even more subtle due to the added non-associativity.

Due to the relationship with octonions, manifolds with $G_2$-structure have an intrinsic non-associativity, therefore it is likely that the enigmatic nature of $G_2$-structure can only be truly understood by embracing the non-associativity and using it to define new mathematical structures.

A Proofs of identities

Proof of Lemma 3.7  Let $A = (0, \alpha) \in \text{Im} \Gamma (O M)$, then the exponential of $A$ is defined to be $e^A = \sum_{k=1}^{\infty} \frac{1}{k!} A^k$. From the definition (3.8) of octonion multiplication, we have

$$
A = \alpha \\
A^2 = -|\alpha|^2 \\
A^3 = -|\alpha|^2 \alpha \\
A^4 = |\alpha|^4 \\
...$$

Therefore,

$$
e^A = \left(1 - \frac{1}{2} |\alpha|^2 + \frac{1}{4!} |\alpha|^4 + ...\right) \\
+ \left(1 - \frac{1}{3!} |\alpha|^2 + \frac{1}{5!} |\alpha|^4 - ...\right) \alpha
$$

(A.1)

Note that this converges for any $\alpha$. □

Proof of Corollary 3.8  Suppose $B = (b, \beta) \in \Gamma (O M)$. Since we are proving a pointwise identity, without loss of generality, we may assume $\beta$ (and hence $B$ itself) is nowhere vanishing, since whenever $\beta$ does vanish, the identity (3.14) is satisfied trivially. Then,

$$
B = |B| \left(\hat{b} + \hat{\beta}\right)
$$

where $\hat{b} = \frac{b}{|b|}$ and $\hat{\beta} = \frac{\beta}{|\beta|}$. Since now $\hat{b}^2 + |\hat{\beta}|^2 = 1$, we can find a non-negative real number $\theta$ such that $\cos \theta = \hat{b}$ and $\sin \theta = |\hat{\beta}|$. Therefore, we write

$$
B = |B| \left(\cos \theta + \alpha \frac{\sin \theta}{|\alpha|}\right)
$$

where $\alpha = \frac{\hat{\beta}}{|\hat{\beta}|}$, so that $|\alpha| = \theta$. From Lemma 3.7 we can then rewrite $B = |B| e^A$ where $A = (0, \alpha)$. Hence, for $k \in \mathbb{Z}$

$$
B^k = |B|^k (e^A)^k
$$

It is clear from the expansion (A.1) that $(e^A)^k = e^{kA} = \cos k\theta + \alpha \frac{\sin k\theta}{\theta}$. Hence the result. □

Proof of Lemma 3.9  Let $A, B, C \in \Gamma (O M)$. Then
1. From the expression (3.12) of the associator in terms of the 4-form \( \psi \), we see that \( [A, B, C] \) only depends on the imaginary parts of \( A, B, C - \alpha, \beta, \gamma \), respectively. Hence,

\[
[A, B, C] = \{-\alpha, \beta, \gamma\} = -[A, B, C] \tag{A.2}
\]

2. From Corollary 3.8, \( \text{Im} A^k = \tilde{a} \alpha \) for some \( \tilde{a} \in \mathbb{R} \), hence,

\[
[A^k, A, C] = \tilde{a} [\alpha, \alpha, \gamma] = 0.
\]

3. From the expression for the inner product (3.10), we have

\[
\langle A, [A, B, C] \rangle = \frac{1}{2} \left( A[A, B, C] + [A, B, C] \bar{A} \right) = \frac{1}{2} \left( -A[A, B, C] + [A, B, C] A \right) \tag{A.3}
\]

since \([A, B, C]\) is pure imaginary. However, from (3.12),

\[
\langle A, [A, B, C] \rangle = 2\psi(A, A, B, C) = 0
\]

since \( \psi \) is totally skew-symmetric. Therefore (A.3) yields

\[
A[A, B, C] = [A, B, C] \bar{A} \tag{A.4}
\]

4. First let us consider \([A, AB, C]\). Using (A.2), and then the definition of the associator, we can write

\[
[A, AB, C] = -[\bar{A}, AB, C]
\]

\[
= -\bar{A}((AB)C) + (\bar{A}(AB))C
\]

\[
= -\bar{A}(A(BC) - [A, B, C]) + |A|^2(BC)
\]

\[
= \bar{A}[A, B, C] \tag{A.5}
\]

where we have used the fact that \([\bar{A}, A, \cdot] = 0\). Continuing by induction, we conclude that \([A, A^k B, C] = \bar{A}^k[A, B, C]\).

5. Similarly as above, consider \([A, BA, C]\). Then,

\[
[A, BA, C] = -[A, \bar{A}B, C] = [\bar{A}, \bar{A}B, C]
\]

Applying (A.5) and (A.4) we hence get

\[
[A, BA, C] = A[\bar{A}, B, C] = [A, B, C] \bar{A}.
\]

Continuing by induction for arbitrary \( k \), we conclude that \([A, BA^k, C] = [A, B, C] \bar{A}^k\).

6. Consider \([A^k, B, CA]\). Using the definition of the associator,

\[
[A^k, B, CA] = [B, CA, A^k] = B (CA) A^k - (B (CA)) A^k
\]

\[
= B (CA^{k+1}) - ((BC)A + [B, C, A]) A^k
\]

\[
= [B, C, A^{k+1}] - [A, B, C] A^k \tag{A.6}
\]

However, note that \( \text{Im} A^k = \tilde{a} \alpha \) for some \( \tilde{a} \in \mathbb{R} \), hence using the skew-symmetry of the associator and part 5 of this proof, we have

\[
[A^k, B, CA] = [A^k, B, C] \bar{A}
\]

Therefore, we indeed obtain

\[
[A^{k+1}, B, C] = [A^k, B, C] \bar{A} + [A, B, C] A^k \tag{A.7}
\]
Proof of Lemma 3.11.} Let $A, B, C \in \Gamma (\mathbb{O} M)$, and consider

\[
\langle R_B A, C \rangle = \langle AB, C \rangle = \frac{1}{2} \left( (AB) \bar{C} + C (\bar{B} A) \right)
\]

\[
= \frac{1}{2} \left( A (B \bar{C}) - [A, B, \bar{C}] + (C B) \bar{A} + [C, \bar{B}, \bar{A}] \right)
\]

\[
= \frac{1}{2} \left( A (B \bar{C}) + (C B) \bar{A} \right)
\]

\[
= \langle A, C \rangle = \langle A, R_B C \rangle
\]

where we have used the expression for the metric (3.10) and properties of the associator from Lemma 3.9. Similarly we obtain the result for $L_B$.

Proof of Lemma 4.2.} Let $V$ be a nowhere-vanishing octonion, and $A, B$ arbitrary octonion sections.

1. Using the associator,

\[
(VA) (BV^{-1}) = ((VA) B) V^{-1} + [VA, B, V^{-1}]
\]

\[
= (V (AB) - [V, A, B]) V^{-1} - \frac{1}{|V|^2} [VA, B, V]
\]

\[
= \text{Ad}_V (AB) - [A, B, V] V^{-1} - \frac{V}{|V|^2} [A, B, V]
\]

where we have used identity 4 from Lemma 3.9. Now using identity 3 from Lemma 3.9, we conclude that,

\[
(VA) (BV^{-1}) = \text{Ad}_V (AB) - \frac{1}{|V|^2} [A, B, V] (V + \bar{V})
\]

\[
= \text{Ad}_V (AB) + [A, B, V^{-1}] (V + \bar{V}) \quad (A.8)
\]

2. Using the same identities from Lemma 3.9 we get

\[
(AV^{-1}) (VB) = ((AV^{-1}) V) B + [AV^{-1}, V, B]
\]

\[
= AB - \frac{1}{|V|^2} [A, B, V] V
\]

\[
= AB + [A, B, V^{-1}] V \quad (A.9)
\]

Proof of Lemma 4.10.} Let $A, B, C \in \Gamma (\mathbb{O} M)$, then

\[
[A, B, C]_V = A \circ_V (B \circ_V C) - (A \circ_V B) \circ_V C
\]

\[
= A \circ_V (BC + [B, C, V] V^{-1}) - (AB + [A, B, V] V^{-1}) \circ_V C
\]

\[
= A (BC) + [A, BC, V] V^{-1} + A ([B, C, V] V^{-1}) + [A, [B, C, V] V^{-1}, V] V^{-1}
\]

\[
- (AB) C - [AB, C, V] V^{-1} - ([A, B, V] V^{-1}) C - [[A, B, V] V^{-1}, C, V] V^{-1}
\]

Now note that using Lemma 3.9,

\[
[A, [B, C, V] V^{-1}, V] V^{-1} = [A, [B, C, V], V] V^{-1} V^{-1}
\]

\[
= - [A, [B, C, V], V^{-1}]
\]
and similarly,

\[
[[A, B, V] V^{-1}, C, V] V^{-1} = -[[A, B, V], C, V^{-1}]
= [[A, B, V], V^{-1}, C]
\]

However,

\[
A ((B, C, V) V^{-1}) - [A, (B, C, V), V^{-1}] = (A [B, C, V]) V^{-1}
- ([A, B, V] V^{-1}) C - [[A, B, V], V^{-1}, C]
= -[A, B, V] (V^{-1} C)
\]

Therefore,

\[
[A, B, C]_V = [A, B, C, V] V^{-1} + (A [B, C, V]) V^{-1} - [AB, C, V] V^{-1} + [A, B, C] - [A, B, V] (V^{-1} C)
\]

Expanding each of the first three associators, we get

\[
[A, B, C]_V = [A ((BC) V) - (A (BC)) V - (AB) (CV) + ((AB) C) V + A (B (CV)) - A ((BC) V)] V^{-1}
+ [A, B, C] - [A, B, V] (V^{-1} C)
\]

\[
= [A ((BC) V)] V^{-1} - A (BC) - [[AB] (CV)] V^{-1} + (AB) C + [A (B (CV))] V^{-1}
- [A ((BC) V)] - [[AB] (CV)] V^{-1} - [A, B, V] (V^{-1} C)
\]

\[
= [A (B (CV))] V^{-1} - [[AB] (CV)] V^{-1} - [A, B, V] (V^{-1} C)
\]

\[
= [A, B, CV] V^{-1} - [A, B, V] (V^{-1} C)
\]

**Proof of Lemma 6.6** Suppose \( P = (p_0, \rho) \) where \( p_0 \in \Omega^p (\text{Re} \otimes M) \cong \Omega^p (M) \) and \( \rho \in \Omega^p (\text{Im} \otimes M) \cong \Omega^p (TM) \). Then,

\[
d_\nabla P = (dp_0, d\nabla \rho)
\]

where \( d \) is the ordinary exterior derivative on \( \Omega^p (M) \), and \( d \nabla \rho \) is given by

\[
(d \nabla \rho)_{b_1 \ldots b_{p+1}} = (p + 1) \nabla_{[b_1 \rho_{b_2 \ldots b_{p+1}}} \]

Hence,

\[
d_\nabla^2 P = (d^2 p_0, d^2 \nabla \rho) = (0, d_\nabla \nabla \rho)
\]

where

\[
(d_\nabla^2 \rho)_{b_0 \ldots b_{p+1}} = (p + 2) (p + 1) \nabla_{[b_0} \nabla_{b_1} \rho_{b_2 \ldots b_{p+1}]}^{\alpha}
\]

\[
= -\frac{1}{2} (p + 2) (p + 1) \rho \text{(Riem)}^{c}_{[b_0 b_1 b_2} \rho^{[c|b_3 \ldots b_{p+1}]_{\alpha}}
\]

\[
+ \frac{1}{2} (p + 2) (p + 1) \text{(Riem)}^\alpha_{\beta[b_0 b_1 \rho_{b_2 \ldots b_{p+1}]}}
\]

(A.10)

However, from the Bianchi identity for the Riemann tensor, \( \text{Riem}^{c}_{[b_0 b_1 b_2]} = 0 \), therefore the first term in (A.10) vanishes. The remaining term is then a combination of the wedge product between the 2-form Riem and the \( p \)-form \( \rho \), together with the Riem acting as an endomorphism on the \( \text{Im} \otimes \) index of \( \rho \). Therefore, indeed \( d_\nabla^2 \rho \) only has a pure imaginary part which is given by \( \text{Riem} \wedge (\text{Im} P) \).

**Proof of Lemma 7.1** Recall that

\[
A \circ_V B = AB + [A, B, V] V^{-1} = (AV) (V^{-1} B)
\]

(A.11)

Since we want to rewrite \( \nabla_X (A \circ_V B) \) in terms of \( T \), we will first express \( \circ_V \) in terms of the original product using (A.11), and we will evaluate the derivatives using the relation from Proposition 6.4.

\[
\nabla_X (AB) = (\nabla_X A) B + A (\nabla_X B) - [T_X, A, B]
\]

(A.12)
Then, we will use (A.11) to rewrite all the products in terms of $\circ_V$ again.

Consider

$\nabla_X (A \circ_V B) = \nabla_X ((AV) (V^{-1}B))$

$= (\nabla_X (AV)) (V^{-1}B) + (AV) \nabla_X (V^{-1}B) - [T_X, AV, V^{-1}B]$ \hspace{1cm} (A.13)

Now let us expand the first term in (A.13)

$$\nabla_X (AV)) (V^{-1}B) = ((\nabla_X A) V) (V^{-1}B) + (A \nabla_X V) (V^{-1}B) - [T_X, A, V] (V^{-1}B)$$ \hspace{1cm} (A.14)

We can rewrite

$$\nabla_X V = -V (\nabla_X V^{-1}) V$$

and the first term in (A.14) can be rewritten in terms of $\circ_V$. So that,

$$(\nabla_X (AV)) (V^{-1}B) = (\nabla_X A) \circ_V B - [A (V (\nabla_X V^{-1}) V)] [V^{-1}B] - [T_X, A, V] (V^{-1}B)$$

$$= (\nabla_X A) \circ_V B - [(A (V \nabla_X V^{-1})) V] [V^{-1}B] - [A, V \nabla_X V^{-1}, V] (V^{-1}B)$$

$$- [T_X, A, V] (V^{-1}B)$$

$$= (\nabla_X A) \circ_V B - (A (V \nabla_X V^{-1})) \circ_V B - [T_X + V \nabla_X V^{-1}, A, V] (V^{-1}B)$$ \hspace{1cm} (A.15)

where again we keep rewriting products in terms of $\circ_V$. Note that in the second term of (A.15),

$$A (V \nabla_X V^{-1}) = A \circ_V (V \nabla_X V^{-1}) - [A, V \nabla_X V^{-1}, V] V^{-1}$$

and then,

$$\left( A (V \nabla_X V^{-1}) \right) \circ_V B = (A \circ_V (V \nabla_X V^{-1})) \circ_V B + ([A, V \nabla_X V^{-1}, V] V^{-1}) \circ_V B$$

$$= (A \circ_V (V \nabla_X V^{-1})) \circ_V B + [[A, V \nabla_X V^{-1}, V] V^{-1}] V^{-1}B$$

$$= (A \circ_V (V \nabla_X V^{-1})) \circ_V B - [V \nabla_X V^{-1}, A, V] (V^{-1}B)$$

Thus, (A.15) becomes

$$\nabla_X (AV)) (V^{-1}B) = (\nabla_X A) \circ_V B - (A \circ_V (V \nabla_X V^{-1})) \circ_V B - [T_X, A, V] (V^{-1}B)$$ \hspace{1cm} (A.16)

Similarly, let us consider the second term of (A.13):

$$(AV) \nabla_X (V^{-1}B) = (AV) \left( [\nabla_X V^{-1} B + V^{-1} \nabla_X B - [T_X, V^{-1}, B]] \right)$$

$$= (AV) \left( [V^{-1} (V \nabla_X V^{-1}) B + V^{-1} \nabla_X B - [T_X, V^{-1}, B]] \right)$$

$$= (AV) \left[ V^{-1} ((V \nabla_X V^{-1}) B) - [V^{-1}, V \nabla_X V^{-1}, B] + V^{-1} \nabla_X B \right]$$

$$- (AV) [T_X, V^{-1}, B]$$

$$= A \circ_V ((V \nabla_X V^{-1}) B) + A \circ_V \nabla_X B + (AV) [T_X - V \nabla_X V^{-1}, B, V^{-1}]$$ \hspace{1cm} (A.17)

and in the first term in (A.17) we can write

$$(V \nabla_X V^{-1}) B = (V \nabla_X V^{-1}) \circ_V B - [V \nabla_X V^{-1}, B, V] V^{-1}$$

$$A \circ_V ((V \nabla_X V^{-1}) B) = A \circ_V ((V \nabla_X V^{-1}) \circ_V B) - A \circ_V ([V \nabla_X V^{-1}, B, V] V^{-1})$$

$$= A \circ_V ((V \nabla_X V^{-1}) \circ_V B) - (AV) (V^{-1} [V \nabla_X V^{-1}, B, V] V^{-1})$$

$$= A \circ_V ((V \nabla_X V^{-1}) \circ_V B) - (AV) [V \nabla_X V^{-1}, B, V] [V^{-1}]^2$$

$$= A \circ_V ((V \nabla_X V^{-1}) \circ_V B) + (AV) [V \nabla_X V^{-1}, B, V^{-1}]$$ \hspace{1cm} (A.18)

where we have used Lemma 3.3 in the last two lines. Therefore, (A.17) becomes

$$(AV) \nabla_X (V^{-1}B) = A \circ_V ((V \nabla_X V^{-1}) \circ_V B) + A \circ_V \nabla_X B + (AV) [T_X, B, V^{-1}]$$ \hspace{1cm} (A.19)
Thus, replacing the first term in (A.13) with (A.16) and the second term in (A.13) with (A.19), we get

$$\nabla_X (A \circ_V B) = (\nabla_X A) \circ_V B + A \circ_V \nabla_X B - [V \nabla_X V^{-1}, A, B]_V$$  \hspace{1cm} (A.20)

$$-\left[ T_X, A, V \right] (V^{-1}B) + (AV) \left[ T_X, B, V^{-1} \right] - \left[ T_X, AV, V^{-1}B \right]$$

Consider the terms containing $T_X$ in (A.20). We reorder the associators, so that when we expand them, up to parentheses, the order in each term is $AV T_X V^{-1} B$

$$-\left[ T_X, A, V \right] (V^{-1}B) - (AV) \left[ V^{-1}, B, T_X \right] - \left[ T_X, AV, V^{-1} \right]$$

$$= -\left[ A, V, T_X \right] (V^{-1}B) - (AV) \left[ T_X, V^{-1}, B \right] + \left[ AV, T_X, V^{-1} \right]$$

$$= -\left[ A (VT_X) - (AV) T_X \right] (V^{-1}B) - (AV) \left[ T_X (V^{-1}B) - (T_X V^{-1}) B \right]$$

$$+ (AV) \left( T_X (V^{-1}B) \right) - ((AV) T_X) (V^{-1}B)$$

$$= (AV) \left( (T_X V^{-1}) B + (A (VT_X)) (V^{-1}B) \right)$$  \hspace{1cm} (A.21)

Now note that

$$\left( T_X V^{-1} \right) B = \left( V^{-1} (VT_X V^{-1}) \right) B$$

$$= V^{-1} ((Ad_V T_X) B) - [V^{-1}, Ad_V T_X, B]$$  \hspace{1cm} (A.22)

Therefore, the first term in (A.21) becomes

$$\left( AV \right) \left( (T_X V^{-1}) B \right) = \left( AV \right) \left( V^{-1} ((Ad_V T_X) B) \right) - (AV) \left[ V^{-1}, Ad_V T_X, B \right]$$

$$= A \circ_V \left( (Ad_V T_X) B \right) - (AV) \left[ V^{-1}, Ad_V T_X, B \right]$$  \hspace{1cm} (A.23)

Similarly,

$$A (VT_X) = A ((Ad_V T_X) V)$$

$$= (A (Ad_V T_X)) V + [A, Ad_V T_X, V]$$  \hspace{1cm} (A.24)

Hence, the second term in (A.21) becomes

$$\left( A (VT_X) \right) (V^{-1}B) = \left( A (Ad_V T_X) \right) \circ_V B + [A, Ad_V T_X, V] \left( V^{-1}B \right)$$  \hspace{1cm} (A.25)

Using (A.20), we rewrite

$$\left( Ad_V T_X \right) B = \left( Ad_V T_X \right) \circ_V B - [Ad_V T_X, B, V] V^{-1}$$

$$A (Ad_V T_X) = A \circ_V Ad_V T_X - [A, Ad_V T_X, V] V^{-1}$$  \hspace{1cm} (A.26, A.27)

Thus, using (A.26) to rewrite $(Ad_V T_X) B$, the first term in (A.23) is now

$$A \circ_V \left( (Ad_V T_X) B \right) = A \circ_V \left( (Ad_V T_X) \circ_V B \right) - A \circ_V \left( [Ad_V T_X, B, V] V^{-1} \right)$$  \hspace{1cm} (A.28)

But,

$$A \circ_V \left( [Ad_V T_X, B, V] V^{-1} \right) = \left( AV \right) \left( V^{-1} [Ad_V T_X, B, V] V^{-1} \right)$$

$$= \left( AV \right) \left( [Ad_V T_X, B, V] V^{-1} \right)^2$$

$$= \left( AV \right) \left[ Ad_V T_X, B, V^{-1} \right]$$

$$= -\left( AV \right) \left[ Ad_V T_X, B, V^{-1} \right]$$

Therefore, (A.23) becomes

$$\left( AV \right) \left( (T_X V^{-1}) B \right) = A \circ_V \left( (Ad_V T_X) \circ_V B \right)$$  \hspace{1cm} (A.29)
Similarly, using (A.27) to rewrite \( A(\Ad_V T_X) \), the first term in (A.25) is now
\[
(A(\Ad_V T_X)) \circ_V B = (A \circ_V \Ad_V T_X) \circ_V B - ([A, \Ad_V T_X, V] V^{-1}) \circ_V B
\]
\[
= (A \circ_V \Ad_V T_X) \circ_V B - [A, \Ad_V T_X, V] (V^{-1}B)
\]  
(A.30)

Thus, (A.25) becomes:
\[
(A(\VT_X)) (V^{-1}B) = (A \circ_V \Ad_V T_X) \circ_V B
\]  
(A.31)

Using (A.29) and (A.31) in (A.21), we get that
\[
(\nabla_X (A \circ_V B)) = (A \circ_V \Ad_V T_X) \circ_V B + A \circ_V \nabla_X B - [\Ad_V T_X + V \nabla_X V^{-1}, A, B]_V
\]  
(A.33)

which is precisely (7.2) which we were trying to prove. ■

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