Hidden Supersymmetry of Electrostatic Fields

Juan D García-Muñoz and A Raya

1Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Ciudad Universitaria, Francisco J. Mújica S/N Col. Felicitas del Río, 58040 Morelia, Michoacán, México.
2Centro de Ciencias Exactas, Universidad del Bío-Bío. Avda. Andrés Bello 720, Casilla 447, 3800708, Chillán, Chile.

Email: juan.domingo.garcia@umich.mx and alfredo.raya@umich.mx

Abstract
A hidden supersymmetry of electrostatic fields is evidenced. Proposing an ansatz for the electrostatic potential as the natural logarithm of a nodeless function, it is demonstrated that the electrostatic fields fulfill the Bernoulli equation associated to a second-order confluent supersymmetric transformation. By using the so-called confluent algorithm, it is possible, given a charge density, to find the corresponding electrostatic field as well as the supersymmetric potentials. Furthermore, the associated charge density and the electrostatic field profile of Schrödinger-like solvable potentials can be determined.

Keywords: Electrostatic Fields, Supersymmetric Quantum Mechanics, Schrödinger-like potentials

I Introduction
Supersymmetric Quantum Mechanics (SUSY-QM) is an algebraic method allowing to intertwine two Schrödinger-like Hamiltonians dubbed as SUSY partners [1–6]. It has a direct connection with the factorization method [7–10] and the Darboux transformation [11]. By means of the so-called intertwining operator, which is a $k$-th order differential operator, one obtains the operational intertwining between the two SUSY partner Hamiltonians, thus defining a $k$-th order transformation [12]. In literature, the most common transformation is of the first-order, which has applications as generation of solvable quantum potentials [13], in particular shape-invariant potentials [14]; determination of the generalized Heisenberg algebras for the SUSY partner potentials [15,16], as well their coherent states [17–19]; and more recently, in calculation of exact solutions for matrix Hamiltonians (analogous to the Dirac Hamiltonian) describing 2D materials, such as graphene [20–22]; and the electron propagator in non-trivial magnetic backgrounds [23]. Moreover, nowadays the second-order transformation has been proven useful in the above quoted applications and more, including spectral manipulation [24, 25]; and more specifically, in the description of the bilayer graphene [26, 27].

On the other hand, from freshman courses in physics, electromagnetism is known to be a fundamental phenomenon for understanding the nature of our universe. However, the mathematical description of electric and magnetic fields, summarised in Maxwell’s equations, turns out be a hard nut to crack, generally speaking. A physical situation where it is more often possible to find the...
corresponding solutions is the static case. It is interesting that electrostatic systems as described by Maxwell’s equations show a supersymmetric nature defining a first-order SUSY transformation fulfilling the standard superalgebra given by Witten [28], in which the supersymmetric Hamiltonian is similar to the one-dimensional Dirac Hamiltonian [29]. As a natural generalization, in this paper we show that electrostatic fields also develop a second-order hidden supersymmetry with a quadratic superalgebra. For a self-contained exposition of ideas, the organisation of the remaining of this work is as follows: In Section II we describe the second-order transformation corresponding to electrostatic fields in a linear medium; Section III shows some particular cases corresponding to known charge densities. These examples are used to illustrate the SUSY transformation finding the associated intertwining Hamiltonians and the corresponding electrostatic fields. We further consider the quantum harmonic oscillator potential and obtain its corresponding charge density via supersymmetry. Finally, in Section IV we present our conclusions.

II Supersymmetry of Electrostatic Fields

In electrostatics, the electric field $E$ obeys the Maxwell’s equations in an linear medium,

$$\begin{align*}
\nabla \cdot D &= \rho, \\
\nabla \times E &= 0,
\end{align*}$$

(1)

where $\rho$ is the charge density and $D$ is the electric displacement field given by

$$D = \varepsilon_0 E + P,$$

(2)

with $P$ being the electric polarization of the medium and $\varepsilon_0$ the vacuum permittivity [30].

Let us consider an electrostatic field pointing out and changing only along a fixed direction, i.e., $E(x) = E(x) \hat{x}$. Due to the Maxwell-Faraday eq. (1), the electrostatic field $E(x)$ must be the gradient of a scalar function $\varphi(x)$, called electrostatic potential. In other words,

$$E(x) = -\nabla \varphi(x) \Rightarrow E(x) = -\frac{d\varphi(x)}{dx}.$$  

(3)

Taking into account that such electrostatic potential $\varphi(x)$ can be expressed as the natural logarithm of a nodeless function $w(x)$, namely,

$$\varphi(x) = \varphi_0 \ln \left[ \frac{w(x)}{A} \right],$$

(4)

with $\varphi_0$ and $A$ being constants with the appropriate units to ensure the correct dimensionality. Then, substituting the above expression into eq. (3), it turns out that†

$$E(x) = -\varphi_0 \frac{w'(x)}{w(x)}.$$  

(5)

Thus, upon taking the derivative of the previous expression, we arrive at

$$E'(x) = \frac{E^2(x)}{\varphi_0} - \varphi_0 \frac{w''(x)}{w(x)}.$$  

(6)

†We use the notation $f'(x) = \frac{df}{dx}$. 

2
To further continue, we consider that the function $w(x)$ has the following form

$$w(x) = w_0 - \int_{x_0}^{x} u^2(y)dy,$$

(7)

where $w_0$ is a parameter that together with the function $u(x)$ guarantee that $w(x)$ remains nodeless. Here, $x_0$ is a point in the appropriate $x$-domain. Calculating the derivatives of the function $w(x)$ and substituting them back in eq. (6), we obtain

$$\frac{E'(x)}{\varphi_0} = \left(\frac{E(x)}{\varphi_0}\right)^2 + 2\beta(x)\left(\frac{E(x)}{\varphi_0}\right),$$

(8)

with $\beta(x) = u'(x)/u(x)$. Equation (8) has the well-known form of a Bernoulli equation which describes a second-order confluent supersymmetric transformation, where the function $u(x)$ is referred to as seed solution [12]. By sticking to the supersymmetric quantum mechanics algorithm, we can define two one-dimensional Schrödinger-like supersymmetric partner Hamiltonians $H^+$ fulfilling the following intertwining relation

$$H^+L^- = L^-H^-,$$

(9)

where $L^-$ is a second-order differential operator known as the intertwining operator, which is explicitly given by

$$L^- = \frac{d^2}{dx^2} + \eta(x) \frac{d}{dx} + \gamma(x), \quad \eta(x) = \frac{u'(x)}{u(x)},$$

$$\gamma(x) = \left(\frac{\eta(x)}{2}\right)^2 + \frac{\eta'(x)}{2} + \left(\frac{\eta'(x)}{2\eta(x)}\right)^2 - \frac{\eta''(x)}{2\eta(x)},$$

(10)

with $w(x)$ defined as in eq. (7). The corresponding SUSY partner potentials $V^\pm(x)$ can be written as follows

$$V^-(x) = \beta'(x) + \beta^2(x) + \varepsilon, \quad \beta(x) = \frac{u'(x)}{u(x)},$$

$$V^+(x) = V^-(x) + 2\eta'(x),$$

(11)

and $\varepsilon$ denoting the so-called factorization energy associated to the seed solution $u(x)$, which is solution of the Schrödinger-like equation

$$-u''(x) + V^-(x)u(x) = \varepsilon u(x).$$

(12)

In other words, $u(x)$ is an eigenfunction of the Hamiltonian $H^-$ (not necessarily square-integrable) with energy eigenvalue $\varepsilon$. Now, suppose the solutions of the Hamiltonian $H^-$ are known, i.e., its eigenvalues $\varepsilon_n$ and eigenfunctions $\psi_n^-(x)$ are given in advance. Thus, the eigenfunctions $\psi_n^+(x)$ of the Hamiltonian $H^+$ are related with the functions $\psi_n^-(x)$ by means of the following expressions

$$\psi_n^+(x) = \frac{L^-\psi_n^-(x)}{|\varepsilon_n|}, \quad \psi_n^-(x) = \frac{L^+\psi_n^+(x)}{|\varepsilon_n^-|},$$

(13)

where $L^+$ is the hermitian conjugate of $L^-$ in eq. (10). Furthermore, the eigenfunction associated to the factorization energy of the Hamiltonian $H^+$ is inversely proportional to the seed solution, i.e., $\psi_\varepsilon^+(x) \propto 1/u(x)$.
In order to determine the confluent transformation defined in eq. (8), notice that eq. (4) allows us to know \( w(x) \) as a function of the electrostatic potential \( \varphi(x) \) as

\[
w(x) = Ae^{\varphi(x)/\varphi_0} \Rightarrow \eta(x) = \frac{E(x)}{\varphi_0},
\]

whereas the seed solution can be expressed as

\[
u(x) = \sqrt{AE(x)/\varphi_0}e^{\varphi(x)/\varphi_0} \propto e^{\int \beta(x)dx}.
\]

If we substitute \( u(x) \) in eq. (12), we obtain that the corresponding factorization energy \( \varepsilon = 0 \). Moreover, given the form of \( u(x) \) in eq. (15), it is reasonable to think that the seed solution is asymptotically zero, since, away from the charge distributions, electrostatic potentials and fields tend to vanish. Thus, the seed solution should be square-integrable and then, the constant \( A \) can be chosen so that the integral term in eq. (7) equals 1. Because the function \( w(x) \) is nodeless, the constant \( w_0 \) does not lie in the range \((0, 1)\). Then, comparing with eq. (14), we get that \( w_0 = 0 \). Consequently, the factorization energy level does not belong to the spectrum of the Hamiltonian \( H^+ \), while the corresponding function \( \psi^+_\varepsilon(x) \propto 1/u(x) \) is not square-integrable. However, it could happen the seed solution in eq (15) is not square-integrable, though the eigenfunction \( \psi^+_\varepsilon(x) \) is. Despite of this, since the integral term in eq. (7) must be convergent, it can also be concluded that \( w_0 = 0 \) in this case. Hence, the spectra of the Hamiltonians \( H^\pm \) are the same except for the factorization energy level, see Figure 1.

It is worthwhile to mention that the supersymmetric transformation described in eq. (8) fulfils the quadratic supersymmetric algebra

\[
\{Q_i, Q_j\} = \delta_{ij}H_{SS}, \quad i, j = \pm,
\]

Figure 1: Spectral scheme of the of \( H^\pm \) and the intertwining liking them, when the confluent SUSY transformation is carried out by using the ground state eigenfunction of \( H^- \) as seed solution.
with the supersymmetric charges defined as follows
\[
Q_- = \begin{pmatrix} 0 & L^- \\ 0 & 0 \end{pmatrix}, \quad Q_+ = \begin{pmatrix} 0 & 0 \\ L^+ & 0 \end{pmatrix},
\]
\[
(17)
\]
and the supersymmetric Hamiltonian \( H_{SS} \) given by
\[
H_{SS} = \begin{pmatrix} L^-L^+ & 0 \\ 0 & L^+L^- \end{pmatrix} = \begin{pmatrix} (H^+)^2 & 0 \\ 0 & (H^-)^2 \end{pmatrix}.
\]
\[
(18)
\]
Such Hamiltonian cannot be associated to a Dirac-like Hamiltonian, in contrast to what is shown in Ref. [29], where first-order SUSY QM is applied. From eq. (13), one can obtain that the eigenvectors \( \Psi_n(x) \) of \( H_{SS} \) are given in terms of the eigenfunctions \( \psi_n^\pm(x) \) of the form
\[
\Psi_n(x) \propto \begin{pmatrix} \psi_n^+(x) \\ \psi_n^-(x) \end{pmatrix},
\]
\[
(19)
\]
whose corresponding eigenvalues are \((\varepsilon_n^\pm)^2\). It is important to mention the existence of the zero energy eigenvector \( \Psi_0(x) \propto \begin{pmatrix} 0 \\ u(x) \end{pmatrix} \) defining the supersymmetric transformation and being straightly related to the electrostatic potential through eq. (15).

Note that in the previous analysis, we have only worked the Maxwell-Faraday equation. However, the electric field \( E \) must also satisfy the Gauss law. Then, Maxwell’s equations (1) are transformed in the following system of equations
\[
\eta'(x) = \frac{\rho(x)}{\epsilon \varphi_0}, \quad \eta'(x) = \eta^2(x) + 2\beta(x)\eta(x),
\]
\[
(20)
\]
where \( \eta(x) \) and \( \beta(x) \) are the supersymmetric functions to be determined from the charge density \( \rho(x) \). By substituting the first equation in the second one, we have that
\[
\beta(x) = \frac{1}{2} \left( \frac{\rho(x)}{\int \rho dx} - \frac{1}{\epsilon \varphi_0} \int \rho dx \right).
\]
\[
(21)
\]
Thereby, in order to find \( \eta(x) \), it is enough to use eq. (21) in eq. (20). By returning to the original variables, it turns out that we have two solutions for the electric field, namely,
\[
E_+(x) = \frac{1}{\epsilon} \int \rho(x) dx, \quad E_-(x) = -\varphi_0 \frac{\rho(x)}{\int \rho(x) dx}.
\]
\[
(22)
\]
Furthermore, the SUSY potential \( V^-(x) \) can be directly calculated by means of eq. (11). Thus, if we are given the charge density \( \rho(x) \), by using the confluent algorithm, we obtain two SUSY partner potentials as well as the corresponding electrostatic field. It is worth mentioning \( E_\pm(x) \) are solutions of the Maxwell’s equations (1) in an electrostatic situation calculated from the electric field flux, which is implicit in eq. (22). A complementary analysis is necessary to give a right interpretation of these solutions. One of the advantages of the SUSY-QM formalism is that it allows to approach in another different way the connection between electrostatic fields and quantum one-dimensional Hamiltonians. Specifically, assuming that we can obtain the eigenfunctions
\footnote{The notation \((f g)^T\) symbolizes the transpose matrix operation.}
and energy eigenvalues of the Schrödinger-like Hamiltonian $H^-$, we can take an eigenfunction as seed solution. Then, upon performing the confluent supersymmetric transformation, we can obtain the associated electrostatic field. In the next section we develop some particular examples, which help to illustrate the connection between electrostatic fields fulfilling Maxwell's eqs. (1) and the supersymmetric partner Schrödinger-like Hamiltonians intertwined by means of the relation (9).

### III Particular examples

#### 1 Infinite charged sheet

As a first example, let us consider a situation where the charge distribution is known and the electrostatic field is derived straightforwardly. We then look for the transformed quantum mechanical problem corresponding to a couple of SUSY partner potentials. We take a uniform surface charge distribution localized on an infinite sheet separating into two regions the space filled by a dielectric material with permittivity constant $\epsilon$. In this case, $\rho(x) = \sigma \delta(x)$, $\sigma > 0$. Calculating the integral of this charge density and substituting in eq. (21), it turns out that

$$\beta(x) = \frac{1}{2} \left( \delta(x) - \frac{\sigma}{\epsilon \varphi_0} \right).$$

Consequently, using the previous form of $\beta(x)$ in eq. (22), the electrostatic field solutions are given by

$$E_+(x) = \frac{\sigma}{\epsilon}, \quad E_-(x) = -\varphi_0 \delta(x).$$

As we mentioned in the previous section, this solutions are derived, implicitly, from the electric field flux. Nevertheless, we associate the solution $E_-(x)$ with the field in the place where the charge lies, while the solution $E_+(x)$ is the field in the remaining of space. Evidently, it considers the flux contribution of the two infinite sheet surfaces, been half of it the correct magnitude of the electric field in each of the regions of space. In Fig. 2(a) we display a graph of the electrostatic field and the charge density. Substituting the function $\beta(x)$ (23) into eq. (11), we arrive at the following supersymmetric partner potentials

$$V^-(x) = V^+(x) = \frac{\delta'(x)}{2} + \frac{1}{4} \left( \delta(x) - \frac{\sigma}{\epsilon \varphi_0} \right)^2.$$  

In Fig. 2(b) we show a plot of these potentials and $u(x)$, which can be obtained from eq. (15), using eq. (23), the properly normalized seed solution has the form

$$u(x) = \sqrt{\frac{\sigma}{2\epsilon \varphi_0}} e^{-\frac{x}{2\epsilon \varphi_0}} e^{\frac{x}{4} + \frac{1}{2}}, \quad x > 0.$$  

It is worth mentioning that since the infinite charged sheet in the electrostatic problem appears as an infinite potential barrier, eq. (25) in the SUSY-QM problem, the $x$-domain splits into two regions and it is enough to analyze one of them, for the result in the other region being analogous. The results obtained in eqs. (25) and (26) are similar to those obtained in Ref. [29], where by means of first-order supersymmetric quantum mechanics, the case of an infinite charged sheet is also addressed. However, the confluent supersymmetric algorithm allows us to find an extra solution $E_-(x)$ of the Maxwell’s equations, which is associated to the region where $E_+(x)$ is not physically appropriate. This solution cannot be obtained by means of the first-order SUSY QM.
2 Constant charge density

As a second example, we consider a positive constant charge density $\rho = \rho_0$ uniformly distributed in an infinite dielectric box with finite width and a permittivity constant $\epsilon$. The function $\beta(x)$ is given by

$$\beta(x) = \frac{1}{2} \left( \frac{1}{x} - \frac{\rho_0}{\epsilon \varphi_0} \right).$$

A straightforward calculation gives the electrostatic field solutions

$$E_+(x) = \frac{\rho_0}{\epsilon} x, \quad E_-(x) = -\frac{\varphi_0}{x}.$$  

(28)

We can observe the solution $E_+(x)$ correspond to the electric field inside the dielectric box and the solution $E_-(x)$ is the electric field outside the box. In this case, $\varphi_0$ is a potential difference between a point $x \rightarrow \pm \infty$ and the corresponding surface of the box, which, since the electric field must be continuous, equals $\varphi_0 = -\rho_0 d^2/\epsilon$. The electric field and the charge density are displayed in Fig. 3(a). Performing the confluent transformation, we directly obtain the pair of SUSY partner potentials

$$V^-(x) = \omega^2 x^2 - \frac{1}{4x^2} - \omega,$$

$$V^+(x) = \omega^2 x^2 - \frac{1}{4x^2} + \omega,$$

(29)

with $\omega = \rho_0/\epsilon \varphi_0$. Figure 3(b) shows the SUSY partner potentials in eq. (29) as well as the seed solution, which can be written as

$$u(x) = \sqrt{\omega} x e^{-\frac{\omega}{4} x^2}, \quad x > 0.$$  

(30)
For the constant charge density: (a) Plot of the charge density divided by $\epsilon$ and the electrostatic field $E(x)/d$. (b) The SUSY partner potentials $V^\pm(x)$ as well a representation of the seed solution $u(x)$. The scale of the graphs is fixed by the parameters $\omega = \varphi_0 = -1$ and $\epsilon = \rho_0 = 1$.

We must mention that this seed solution is not square-integrable, while $\psi_0^+$ is. On the other hand, the potentials in eq. (29) have a “centrifugal” term and consequently, the appropriate $x$-domain is the range $[0, \infty)$. Analogous conclusions can be drawn for the range $(-\infty, 0]$. Then, we can assume that the box lies in the range $[-d, d]$. Finally, $\psi_0^+(x)$ is in agreement with the results obtained in Ref. [31], where the so-called isotonic potential $V(x) = \omega^2 x^2/2 + g/2x^2$ is solved assuming $g$ constant. The potential $V^+(x)$ in eq. (29) is the limit case with $g = -1/4$, where there exists bound states.

### 3 Harmonic oscillator potential

As a final example, we address the case where a quantum mechanical potential is known and look for the associated electrostatic field and the charge distribution originating it. For this purpose, we choose the potential $V(x) = \omega^2 x^2$, $\omega > 0$, $x \in \mathbb{R}$. Its well-known that the eigenfunctions $\psi_n^-(x)$ for this problem are given in terms of the Hermite polynomials, namely,

$$\psi_n^-(x) = C_n e^{-\frac{x^2}{2}} H_n(\sqrt{\omega}x), \quad (31)$$

with eigenvalues $E_n = \omega(2n + 1)$. We can observe none of the bound states has a zero energy eigenvalue. Then, we need to subtract from the potential the energy of the bound state which will be used as seed solution. In other words, we consider a harmonic oscillator with an energy shift guaranteeing a zero energy bound state level. Taking the ground state eigenfunction as the seed solution and substituting in eq. (7), the function $w(x)$ turns out to be

$$w(x) = -\frac{1}{2} \left(1 + \text{Erf}(\sqrt{\omega}x)\right), \quad (32)$$
For the harmonic oscillator potential: (a) The corresponding charge density, which is asymptotically zero for $x \to \infty$, while for $x \to -\infty$ it tends to 2. (b) Plot of the SUSY partner potentials $V^\pm(x)$ and a representation of the seed solution $u(x)$ used to perform the confluent transformation. The scale of the graphs is set by the parameters $\omega = \varphi_0 = \epsilon = 1$.

where $\text{Erf}(x)$ is the error function. Furthermore, using eq. (32) in eq. (10), we obtain the function $\eta(x)$, and taking its derivative, the charge density has the following intricated form

$$\rho(x) = 4\epsilon \varphi_0 \sqrt{\frac{\omega}{\pi}} \frac{e^{-\omega x^2}}{1 + \text{Erf}(\sqrt{\omega}x)} \left(\omega x + \sqrt{\frac{\omega}{\pi}} \frac{e^{-\omega x^2}}{1 + \text{Erf}(\sqrt{\omega}x)}\right), \quad (33)$$

Note that this density permeates the full space. Thus, we have an infinite charge distributed in it. Therefore, we should consider the electric field per unit of length rather than the field itself. Such linear electric field density behaves similarly to $\rho(x)/\epsilon$, see Fig. 4(a). Hence, this example reveals a highly non-trivial charge distribution which would be really tough to realize in the laboratory. However, from a theoretical viewpoint, it is interesting that by means of the supersymmetric transformation, it is associated to a simpler quantum mechanical problem. The SUSY partner potentials are given by

$$V^{-}(x) = \omega^2 x^2 - \omega, \quad V^{+}(x) = V^{-}(x) + 2\frac{\rho(x)}{\epsilon \varphi_0}. \quad (34)$$

These potentials and the seed solution are shown in Fig. 4(b).

IV Conclusions

The connection between electric fields fulfilling the electrostatic Maxwell’s equations and Schrödinger-like second-order confluent supersymmetric partner Hamiltonians has been evidenced. It is
worth noticing that the confluent supersymmetric transformation defined in eq. (8) is not arbitrary, since it is carried out by means of an eigenfunction of $H^{-}$, associated to the zero eigenvalue, as seed solution and choosing the parameter $w_0 = 0$. Moreover, the associated quadratic superalgebra in eq. (16) defines the matrix Hamiltonian $H_{SS}$, which is similar to the Hamiltonians describing 2D materials. It is important to observe that given the charge density, the confluent second-order SUSY-QM allows us to solve Maxwell’s equations and determine the electrostatic field in all the $x$-domain, a remarkable difference as compared with the case in which a first-order supersymmetric transformation is used, as can be seen in the first two examples of Section III. Furthermore, for such particular profiles of charge density, the charge discontinuity is translated as infinite barrier or well in the quantum potential. On the other hand, by considering a solvable potential as $V^-(x)$, the associated charge density and the electric field can be obtained, as seen in the third example in previous Section, wherein, since there is a infinite charge, it is appropriate consider an linear electric field density rather than the field itself. Finally, we must mention that the charge density profile in eq. (33), associated to the harmonic oscillator potential, could be pretty difficult to realize in the laboratory. Nevertheless, from a theoretical point of view, it is interesting to determine charge densities and electrostatic fields associated to solvable quantum potentials.

Acknowledgments

We acknowledge financial support from CONACYT Project FORDECYT-PRONACES/61533/2020.

References

[1] A. Gangopadhyaya, J. Mallow, and C. Rasinariu. Supersymmetric Quantum Mechanics. World Scientific, Singapore, second edition, 2018.

[2] G. Junker. Supersymmetric Methods in Quantum, Statistical and Solid State Physics. IOP Publishing Ltd, Bristol, second edition, 2019.

[3] B.K. Bagchi. Supersymmetry In Quantum and Classical Mechanics. Monographs and Surveys in Pure and Applied Mathematics. CRC Press, 2000.

[4] Fred Cooper, Avinash Khare, and Uday Sukhatme. Supersymmetry and quantum mechanics. Physics Reports, 251(5):267–385, 1995.

[5] J. David and C. Fernández. Supersymmetric quantum mechanics. AIP Conference Proceedings, 1287(1):3–36, 2010.

[6] D. J. Fernández. Trends in supersymmetric quantum mechanics. In Şengül Kuru, Javier Negro, and Luis M. Nieto, editors, Integrability, Supersymmetry and Coherent States: A Volume in Honour of Professor Véronique Hussin, pages 37–68, Cham, 2019. Springer International Publishing.

[7] L. Infeld and T. E. Hull. The factorization method. Rev. Mod. Phys., 23:21–68, Jan 1951.
[8] Bogdan Mielnik. Factorization method and new potentials with the oscillator spectrum. *Journal of Mathematical Physics*, 25(12):3387–3389, 1984.

[9] E. Schrödinger. Further studies on solving eigenvalue problems by factorization. *Proc. R. Irish Acad. A: Math. Phys. Sc.*, 46(00358975):183–206, 1940.

[10] E. Schrödinger. The factorization of the hypergeometric equation. *Proc. R. Irish Acad. A: Math. Phys. Sc.*, 47:53–54, 1941.

[11] Vladimir B Matveev and M A Salle. *Darboux transformations and solitons*. Springer-Verlag, 1991.

[12] David J. Fernández C. and Nicolás Fernández-García. Higher-order supersymmetric quantum mechanics. *AIP Conference Proceedings*, 744(1):236–273, 2004.

[13] J I Díaz, J Negro, L M Nieto, and O Rosas-Ortiz. The supersymmetric modified pöschl-teller and delta well potentials. *Journal of Physics A: Mathematical and General*, 32(48):8447–8460, nov 1999.

[14] Ranabir Dutt, Avinash Khare, and Uday P. Sukhatme. Supersymmetry, shape invariance, and exactly solvable potentials. *American Journal of Physics*, 56(2):163–168, 1988.

[15] Juan M Carballo, David J Fernández C, Javier Negro, and Luis M Nieto. Polynomial heisenberg algebras. *Journal of Physics A: Mathematical and General*, 37(43):10349–10362, oct 2004.

[16] David J Fernández C and Véronique Hussin. Higher-order SUSY, linearized nonlinear heisenberg algebras and coherent states. *Journal of Physics A: Mathematical and General*, 32(19):3603–3619, jan 1999.

[17] David J Fernández, Véronique Hussin, and Oscar Rosas-Ortiz. Coherent states for hamiltonians generated by supersymmetry. *Journal of Physics A: Mathematical and Theoretical*, 40(24):6491–6511, may 2007.

[18] E Díaz-Bautista, Y Concha-Sánchez, and A Raya. Barut–girardello coherent states for anisotropic 2d-dirac materials. *Journal of Physics: Condensed Matter*, 31(43):435702, jul 2019.

[19] E Díaz-Bautista, M Oliva-Leyva, Y Concha-Sánchez, and A Raya. Coherent states in magnetized anisotropic 2d dirac materials. *Journal of Physics A: Mathematical and Theoretical*, 53(10):105301, feb 2020.

[20] S Hernández-Ortíz, G Murguía, and A Raya. Hard and soft supersymmetry breaking for ‘graphinos’ in uniform magnetic fields. *Journal of Physics: Condensed Matter*, 24(1):015304, dec 2011.

[21] S. Kuru, J. Negro, and L. M. Nieto. Exact analytic solutions for a Dirac electron moving in graphene under magnetic fields. *J. Phys.: Condens. Matter*, 21:455305, 2009.
[22] Y Concha, A Huet, A Raya, and D Valenzuela. Supersymmetric quantum electronic states in graphene under uniaxial strain. *Materials Research Express*, 5(6):065607, jun 2018.

[23] Y Concha-Sánchez, E Díaz-Bautista, and A Raya. Ritus functions for graphene-like systems with magnetic fields generated by first-order intertwining operators. *Physica Scripta*, 97(9):095203, aug 2022.

[24] Alonso Contreras-Astorga and David J Fernández C. Supersymmetric partners of the trigonometric pöschl–teller potentials. *Journal of Physics A: Mathematical and Theoretical*, 41(47):475303, oct 2008.

[25] David J Fernández C and Barnana Roy. Confluent second-order supersymmetric quantum mechanics and spectral design. *Physica Scripta*, 95(5):055210, feb 2020.

[26] David J. Fernández C., Juan D. García M., and Daniel O-Campa. Electron in bilayer graphene with magnetic fields leading to shape invariant potentials. *J. Phys. A: Math. Theor.*, 53(43):435202, oct 2020.

[27] David J Fernandez, Juan Domingo García, and Daniel Ortiz Campa. Bilayer graphene in magnetic fields generated by supersymmetry. *Journal of Physics A: Mathematical and Theoretical*, apr 2021.

[28] Edward Witten. Dynamical breaking of supersymmetry. *Nucl. Phys. B*, 188(3):513–554, 1981.

[29] G González, J Méndez, R Díaz, and F Javier González. Electrostatic simulation of the jackiw-rebbi zero energy state. *Revista Mexicana de Física E*, 65(1):30–33, 2019.

[30] J.D. Jackson. *Classical Electrodynamics*. John Wiley & Sons, Limited, 2021.

[31] Sameer M. Ikhdair and Ramazan Sever. Relativistic and nonrelativistic bound states of the isotonic oscillator by nikiforov-uvarov method. *Journal of Mathematical Physics*, 52(12):122108, 2011.