First steps towards \( q \)-deformed Clifford analysis

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Abstract. We consider the extension of the Jackson calculus into higher dimensions and specifically into Clifford analysis.

1 Introduction

In general, \( q \)-deformed stands for quantum-deformed. Essentially, one can see there different directions here. One is the \( q \)-deformed space and the \( q \)-deformed sphere, the fuzzy sphere. The \( q \)-deformed spaces, in turn, lead to non-commutative structures and hence to the \( q \)-analogues of known Lie groups. There are interesting applications in quantum physics and optics. However, this article is not about physics or physical

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mathematics, but about the closely related Jackson calculus. Jackson calculus replaces
the ordinary derivative
\[ f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \]
with the difference operator
\[ D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{qx - x}, \]
where \( \lim_{q \to 1} D_q f(x) = f'(x) \). Jackson calculus is described in the book by Jackson [Jac09] or in the more recent book [KC02]. References to quantum physics and Jackson calculus can be found in [CD96], [Ern12]. The present paper is concerned with the extension of the Jackson calculus into higher dimensions and specifically into Clifford analysis [BDS82, DDS12, GS89, GHS08, GM91]. There is already an extension of the Jackson calculus into Clifford analysis by Coulembier and Sommen [CS10, CS11]. In these papers, a Dirac operator is defined by axioms. In the present work, a Jackson calculus or q-calculus is introduced by defining the partial derivatives by
\[ \partial^q_{x_i} (f(x_1, \ldots, x_n)) = \frac{f(x_1, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{(q - 1)x_i} \]
and thus q-forms. This is then the basis for the q-deformed Dirac operator \( D^q_x = -\sum_{i=1}^m e_i \partial^q_{x_i} \). This Dirac operator does not satisfy the axioms of the Dirac operator of Coulembier and Sommen. Therefore, in this short article, we first consider the q-defined Dirac, Euler and Gamma operators and their symmetry properties. Then the Fisher decomposition and the corresponding Cauchy-Kovalevskya expansion with examples.

2 Preliminaries
2.1 Clifford analysis
We consider the Clifford algebra \( \mathcal{C}l_{0,m} \) over \( \mathbb{R}^m \) with the identity element \( e_0 \) satisfying \( e_0^2 = 1 \) and elements \( e_1, \ldots, e_m \) following the multiplication rules
\[ e_i e_j + e_j e_i = -2\delta_{ij} \]
for \( i, j = 1, \ldots, m \). The \( e_i \) form a basis with \( 2^m \) elements: \( (e_0, e_1, \ldots, e_1 e_2, \ldots, e_1 e_2 e_3, \ldots) \).
Using \( M := \{1, \ldots, m\} \) and \( A := \{h_1, \ldots, h_r \in \mathcal{P}M : 1 \leq h_1 \leq \ldots \leq h_r \leq m\} \). An arbitrary element of the Clifford algebra \( \lambda \in \mathcal{C}l_{0,m} \) is given by
\[ \lambda = \sum_A \lambda_A e_A, \lambda_A \in \mathbb{R} \]
with \( e_A = e_{h_1} \ldots e_{h_r} \). We consider Clifford-valued functions \( f \), i.e.
\[ f(x) = \sum_A f_A(x)e_A = \sum_A f_A(x_1, \ldots, x_m)e_A, f_A : \mathbb{R}^m \to \mathbb{R}. \]
The vector variable is defined as

\[ \mathbf{x} = \sum_{i=1}^{m} x_i e_i. \] (4)

The multiplication rules imply that \( \mathbf{x}^2 \) is scalar valued, \( \mathbf{x}^2 = -\sum_{i=1}^{m} x_i^2 = -|\mathbf{x}|^2 \). Further we can now define the Dirac operator

\[ D_{\mathbf{x}} = -\sum_{i=1}^{m} e_i \partial x_i, \] (5)

where \( (D_{\mathbf{x}})^2 = -\Delta \) is scalar and \( \Delta \) denotes the Laplace operator. A \( C^{l,0}_m \)-valued function \( f = \sum_{A} e_A f_A(\mathbf{x}) \) is called left (right) monogenic if and only if

\[ D_{\mathbf{x}} f = 0 \quad (f D_{\mathbf{x}} = 0). \] (6)

More on Clifford analysis can be found in the following monographs [BDS82], [DSS12].

### 2.2 Spherical harmonics and monogenics

The space of polynomials of degree at most \( k \) is

\[ \Pi_k = \left\{ P(\mathbf{x}) : P(\mathbf{x}) = \sum_{|\alpha| \leq k} c_{\alpha} \mathbf{x}^\alpha \right\} \] (7)

for \( \mathbf{x} \in \mathbb{R}^m, \mathbf{x} = (x_1, ..., x_m) \) and an index vector \( \alpha = [\alpha_1, ..., \alpha_m] \in \mathbb{N}^m \) with \( |\alpha| = \alpha_1 + ... + \alpha_m \) and \( x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m} \).

A polynomial \( P(\mathbf{x}) \) is called homogeneous of degree \( k \) if it satisfies for every \( \mathbf{x} \in \mathbb{R}^m \) and \( \mathbf{x} \neq 0 \) that

\[ P(\mathbf{x}) = |\mathbf{x}|^k P \left( \frac{\mathbf{x}}{|\mathbf{x}|} \right). \] (8)

A homogeneous polynomial of degree \( k \) is an eigenfunction of the Euler operator \( \mathbb{E} \) with eigenvalue \( k \). Using this we can define the space of homogeneous polynomials of degree \( k \)

\[ \mathcal{P}_k = \{ P(\mathbf{x}) : \mathbb{E} P(\mathbf{x}) = k P(\mathbf{x}) \}. \] (9)

Harmonic functions are the null-solutions of the Laplace operator \( \Delta = \sum_{i=1}^{m} \partial^2 x_i \). Therefore the space of \( k \)-homogeneous polynomials that are also harmonic is denoted by

\[ \mathcal{H}_k = \{ P(\mathbf{x}) \in \mathcal{P}_k : \Delta P(\mathbf{x}) = 0 \}. \] (10)

The restriction of a polynomial \( P(\mathbf{x}) \in \mathcal{H}_k \) to the unit sphere \( \mathbb{S}^{m-1} \) is called a spherical harmonic of degree \( k \).
A $\mathcal{C}_{\ell_{0,m}}$-valued homogeneous polynomial is a polynomial

$$P(x) = \sum_A c_A x^A, \ c_A \in \mathcal{C}_{\ell_{0,m}}$$

(11)

$$\iff P(x) = \sum_A e_{AP_A}(x)$$

(12)

with $p_A(x)$ a complex-valued $k$-homogeneous polynomial and $|A| = k$.

The space of monogenic homogeneous polynomials of degree $k$ is denoted by

$$\mathcal{M}_k = \{ M(x) \in \mathcal{P}_k : D_x M = 0 \}.$$  

(13)

That is monogenic polynomials that are also eigenfunctions of the Euler operator. The restriction of a polynomial $M(x) \in \mathcal{M}_k$ to the unit sphere is called a spherical monogenic of degree $k$.

We want to extend Jackson’s calculus to higher dimensions. Therefore we recall some of the one-dimensional Jackson calculus.

2.3 Jackson calculus

For Jackson’s calculus see also [Jac09; Ern12]. For a number $u$ and a deformation parameter $q \in \mathbb{R}^+$ we can define the $q$-deformation of $u$ as

$$[u]_q = \frac{q^u - 1}{q - 1} = 1 + q + q^2 + \ldots + q^{u-1}.$$  

(14)

Taking the limit this satisfies $\lim_{q \to 1} [u]_q = u$. The $q$-derivative or Jackson derivative of a function $f(t)$ is defined as

$$\partial^q t(f(t)) = \frac{f(qt) - f(t)}{(q - 1)t}.$$  

(15)

This yields

$$\partial^q t(t^k) = [k]_q t^{k-1}.$$  

(16)

Further the Leibniz rule is satisfied

$$\partial^q t = qt \partial^q t + 1$$  

(17)

or more in general

$$\partial^q t(f_1(t)f_2(t)) = \partial^q t(f_1(t))f_2(t) + f_1(t)\partial^q t(f_2(t))$$  

$$= \partial^q t(f_1(t))f_2(qt) + f_1(t)\partial^q t(f_2(t)),$$  

(18)

(19)
i.e. there are two versions of a Leibniz rule.
We can also define the $q$-integration. For $q < 1$ the $q$-integral is given by
\begin{equation}
\int_a^0 f(x) d_qx = (1-q)a \sum_{k=0}^{\infty} f(aq^k)q^k. \tag{20}
\end{equation}
Integration on general intervals $[a, b]$ is defined by $\int_a^b = \int_a^0 - \int_0^b$. The $q$-binomial coefficient is
\begin{equation}
\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} \tag{21}
\end{equation}
with the $q$-factorial $[k]_q! = [k]_q[k-1]_q...[1]_q$ and $[0]_q := 1$. Now we can also define the $q$-exponential
\begin{equation}
E_q(t) = \sum_{j=0}^{\infty} \frac{t^j}{[j]_q!} \tag{22}
\end{equation}
and for the inverse we need a second $q$-exponential given by
\begin{equation}
e_q(t) = E_q^{-1}(t). \tag{23}
\end{equation}
These satisfy the following relations:
\begin{align}
E_q(t)e_q(-t) &= 1 \tag{24} \\
\partial_q^k E_q(t) &= E_q(t) \tag{25} \\
\partial_q^k e_q(t) &= e_q(qt). \tag{26}
\end{align}

2.4 $q$-deformed partial derivatives
As we will consider functions dependent on multiple variables we need some sort of $q$-partial derivative. Following Demichev \cite{Dem96} these will be defined by
\begin{equation}
\partial_q^k_x(f(x_1,...,x_n)) = \frac{f(x_1,...,x_{i-1},qx_i,x_{i+1},...x_n) - f(x_1,...,x_n)}{(q-1)x_i}. \tag{27}
\end{equation}
The $q$-partial derivative satisfies the following relations:
\begin{itemize}
\item $\partial_q^{k_i}x_i = x_i\partial_q^{k_i}$ for $i \neq j$
\item $\partial_q^{k_i}x_i = qx_i\partial_q^{k_i} + 1$
\item $\partial_q^{k_i}\partial_q^{k_j}x_j = \partial_q^{k_j}\partial_q^{k_i}x_i$ for $i \neq j$
\item $\partial_q^{k_i}x_i^2 = q^2x_i^2\partial_q^{k_i} + [2]_qx_i$
\item $(\partial_q^{k_i})^2x_i = q^2x_i^2(\partial_q^{k_i})^2 + [2]_q\partial_q^{k_i}$
\item $(\partial_q^{k_i})^2x_i^2 = q^4x_i^4(\partial_q^{k_i})^2 + (q^2+1)[2]_qx_i\partial_q^{k_i} + [2]_q$
\end{itemize}
Now, we can consider an appropriate Clifford analysis by using the $q$-deformed partial derivatives.
3 $q$-deformed Clifford analysis

3.1 $q$-deformed Dirac operator

First we go back to the Dirac operator defined earlier and replace the continuous partial
derivatives by the $q$-deformed partial derivatives. This results in a $q$-Dirac operator

$$D^q_{\mathbb{R}} = - \sum_{i=1}^{m} e_i \partial^q_{x_i}. \quad (28)$$

A generalization of the $q$-deformation and the Jackson calculus was already considered
in Coul./Sommen. There is an approach based on four axioms which are the basis for
their calculation. According to Coulembier and Sommen [CS11, CS10] following axioms
should be satisfied

- (A1) $D^q_{\mathbb{R}}(\mathbf{x}) = [m]_q$
- (A2) $D^q_{\mathbb{R}} \mathbf{x}^2 = q^2 \mathbf{x}^2 D^q_{\mathbb{R}} + (q + 1) \mathbf{x}$
- (A3) $(D^q_{\mathbb{R}})^2$ scalar
- (A4) $D^q_{\mathbb{R}} M_k = 0.$

Our approach is different. By defining the $q$-Dirac operator as seen above (A1) and (A2)
are not satisfied. Instead we have for (A1)

$$D^q_{\mathbb{R}}(\mathbf{x}) = [1]_q \sum_{i=1}^{m} 1 = m \quad (29)$$

and (A2) has to be replaced by

$$D^q_{\mathbb{R}} \mathbf{x}^2 - q^2 \mathbf{x}^2 D^q_{\mathbb{R}} = [2]_q \mathbf{x} + (1 - q^2) \sum_{i=1}^{m} \sum_{j=1}^{m} x_i^2 e_i \partial^q_{x_j}. \quad (30)$$

Axiom (A3) still holds true

$$(D^q_{\mathbb{R}})^2 = - \sum_{i=1}^{m} (\partial^q_{x_i})^2 = -\Delta^q \quad (31)$$

with the $q$-Laplace operator

$$\Delta^q = \sum_{i=1}^{m} (\partial^q_{x_i})^2. \quad (32)$$

Using $\mathbf{x}_i^i = (x_1, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n)$ and $|\mathbf{x}_i^i|^2 = x_1^2 + \ldots + x_{i-1}^2 + q^2 x_i^2 + x_{i+1}^2 + \ldots + x_n^2$
we get

$$D^q_{\mathbb{R}}(\mathbf{x}^2 f(\mathbf{x})) = [2]_q \sum_{i=1}^{m} e_i x_i f(\mathbf{x}_i^i) + |\mathbf{x}_i^i|^2 D^q_{\mathbb{R}}(f(\mathbf{x})) \quad (33)$$
\[
\mathcal{D}_q^2 (x^2 f(x)) = \sum_{i=1}^{m} e_i |x_i|^2 \partial_{x_i}^q (f(x)) + [2]_q x f(x).
\]  

\section*{3.2 \textit{q}-Euler operator and \textit{q}-Gamma operator}

The \textit{q}-partial derivatives satisfy the Weyl relations

\[
\partial_{x_j}^q j - q x_j \partial_{x_j}^q = 1.
\]

Together with the \textit{q}-Dirac operator and the vector variable we can define the \textit{q}-Euler operator \(\mathcal{E}_q\). In the continuous case the Euler operator follows from \(\mathcal{D} X + X \mathcal{D} = 2 \mathcal{E} + m\). Here we get

\[
\mathcal{D}_q^2 x + x \mathcal{D}_q^2 = [2]_q \mathcal{E} + m
\]

with the \textit{q}-Euler operator \(\mathcal{E}^q\)

\[
\mathcal{E}^q = \sum_{i=1}^{m} x_i \partial_{x_i}^q.
\]

Now we can study the symmetry relations between \(x\), \(D_q^2\), \(E^q\) and \(\Delta^q\) using the operations \(\{x, y\} = xy + yx\) and \([x, y] = xy - yx\).

- \(\{x, x\} = -2|x|^2\)
- \(\{D_q^2, D_q^2\} = -2 \Delta^q\)
- \(\{D_q^2, x\} = [2]_q \mathcal{E} + m\)
- \([\mathcal{E}^q, x] = x + (q - 1) \sum_{i=1}^{m} x_i e_i \partial_{x_i}^q\)
- \([\mathcal{E}^q, D_q^2] = D_q^2 + (q - 1) \sum_{i=1}^{m} x_i e_i (\partial_{x_i}^q)^2\)
- \([|x|^2, D_q^2] = [2]_q x + (q^2 - 1) |x|^2 D_q^2\)
- \([\mathcal{E}^q, |x|^2] = [2]_q |x|^2 + (q^2 - 1) \sum_{i=1}^{m} x_i^3 \partial_{x_i}^q\)
- \([\Delta^q, x] = (q^2 - 1) \sum_{i=1}^{m} x_i e_i (\partial_{x_i}^q)^2 - [2]_q D_q^2\)
- \([\mathcal{E}^q, \Delta^q] = (1 - q^2) \sum_{i=1}^{m} x_i (\partial_{x_i}^q)^3 - [2]_q \Delta^q\)
- \([\Delta^q, |x|^2] = [2]_q [2]_q \mathcal{E}^q + [2]_q m + (q^4 - 1) \sum_{i=1}^{m} x_i^2 (\partial_{x_i}^q)^2\)

This results in the usual continuous relations if we take the limit \(q \rightarrow 1\). Similar to the continuous case the \textit{q}-Gamma operator \(\Gamma^q\) results out of

\[
|x, D_q^2| = (1 - q) \mathcal{E} + 2 \Gamma^q - m
\]
with
\[
\Gamma^q = - \sum_{i<j} e_i e_j (x_i \partial_{x_j}^q - x_j \partial_{x_i}^q).
\]

Further the \( q \)-Gamma operator satisfies
\[
\mathbf{x}^T D_x^q = E^q + \Gamma^q.
\]

Following this we now define a \( q \)-deformed monogenic homogeneous polynomial as a homogeneous polynomial of degree \( k \) \( P(\mathbf{x}) = |\mathbf{x}|^k P \left( \frac{\mathbf{x}}{|\mathbf{x}|} \right) \) satisfying \( D_x^q P = 0 \). The space of \( q \)-deformed monogenic homogeneous polynomials of degree \( k \) is then
\[
\mathcal{M}_k^q = \{ M(x) \in P_k : D_x^q M = 0 \}.
\]

In the continuous case homogeneous polynomials are the eigenfunctions of the Euler operator \( EP = kP \). The question remains if they are eigenfunctions of the \( q \)-Euler operator as well. Let \( P(\mathbf{x}) = \sum_{|\alpha|=k} c_\alpha \mathbf{x}^\alpha \) be a homogeneous polynomial of degree \( k \) with \( \alpha \in \mathbb{N}^m \) a multindex. Using \( P(\mathbf{x}) = x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m} \) and \( |\alpha_1| + ... + |\alpha_m| = k \) we get
\[
E^q(P(\mathbf{x})) = \sum_{i=1}^m x_i \partial_{x_i}^q (x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m})
\]
\[
= x_1 \frac{(q^{\alpha_1} - 1)x_1^{\alpha_1-1} x_2^{\alpha_2} ... x_m^{\alpha_m}}{q-1} + x_2 \frac{(q^{\alpha_2} - 1)x_1^{\alpha_1} x_2^{\alpha_2-1} ... x_m^{\alpha_m}}{q-1}
\]
\[
+ ... + x_m \frac{(q^{\alpha_m} - 1)x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m-1}}{q-1}
\]
\[
= [\alpha_1]_q x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m} + [\alpha_2]_q x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m} + ... + [\alpha_m]_q x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m}
\]
\[
= ([\alpha_1]_q + [\alpha_2]_q + ... + [\alpha_m]_q) P(\mathbf{x}).
\]

Therefore homogeneous polynomials are eigenfunctions of the \( q \)-Euler operator to the eigenvalue \( [\alpha_1]_q + ... + [\alpha_m]_q \). As \( \alpha_1 + ... + \alpha_m = k \) this results in \( E^q(P(\mathbf{x})) = kP(\mathbf{x}) \) for \( q \to 1 \). Depending on the order \( k \) and the specific partition of \( k \), we obtain not only the eigenvalue \( k = [1]_q + ... + [1]_q \) because \( [l]_q = 1 + q + ... + q^{l-1}, l \in \mathbb{N} \).

**Example 1** For \( k = 3 \) and \( m = 3 \) we have the following partitions which leads to the corresponding eigenvalues.

| Homogeneous polynomials | Partition | Eigenvalue |
|------------------------|-----------|------------|
| \( x_1 x_2 x_3 \)     | \( 3 + 1 + 1 \) | \([1]_q + [1]_q + [1]_q = 1 + 1 + 1 = 3\) |
| \( x_1^2 x_3, x_1 x_2^2, x_2 x_3^2, x_1^2 x_2, x_2^2 x_3, x_1 x_3^2 \) | \( 3 + 1 + 2 \) | \([1]_q + [2]_q = 1 + 1 + q = 2 + q\) |
| \( x_1^3, x_2^3, x_3^3 \) | \( 3 \) | \([3]_q = 1 + q + q^2\) |

**4 Fischer decomposition**

Using multi-index notation \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m \) we have
We use the standard basis for $k$-homogeneous Clifford-valued polynomials $P_k = \{ x^\alpha : |\alpha| = k \}$. First we define an inner product on the complex vector space $P_k$ as follows.

For $R_1, R_2 \in P_k$ with $R_i(x) = \sum_{|\alpha| = k} x^\alpha a^i_\alpha$, $a^i_\alpha \in \mathbb{C}_{\ell, 0, m}$, $i = 1, 2$ we define

$$\langle R_1, R_2 \rangle_{k, q} = \sum_{|\alpha| = k} [\alpha]_q! (a^1_\alpha a^2_\alpha)_0. \quad (42)$$

This is a scalar product, the so called Fischer inner product.

**Theorem 4.1** For $R_1, R_2 \in P_k$ we get

$$\langle R_1, R_2 \rangle_{k, q} = \langle R_1 (\partial^q x^\alpha) R_2 \rangle_0 \quad (43)$$

where $R_1(\partial^q x^\alpha)$ denotes the operator obtained by replacing the $x_j$ in $R_1$ with the $q$-partial derivative $\partial^q x_j$.

Proof: We use the fact that

$$(\partial^q x^\alpha)^\beta = \begin{cases} [\alpha]_q! , & \alpha = \beta \\ 0 , & \alpha \neq \beta. \end{cases}$$

Using $R_1, R_2 \in P_k$ with coefficients $a^i_\alpha = 1$ we get for the components in (43)

$$R_1 (\partial^q x^\alpha) = (\partial^q x_1)^{\alpha_1} \cdots (\partial^q x_n)^{\alpha_n}, \quad \alpha_1 + \cdots + \alpha_n = k \quad (44)$$

$$R_2(x) = x^{\beta_1} \cdots x^{\beta_n}, \quad \beta_1 + \cdots + \beta_n = k \quad (45)$$

There are three different cases to examine. It is sufficient to verify these with a single variable $x_i$.

The first case is $\alpha_i = \beta_i$ for each $i = 1, \ldots, n$

$$(\partial^q x_i)^{\alpha_i} x_i^{\alpha_i} = (\partial^q x_i)^{\alpha_i-1} \left( \frac{q \alpha_i x_i^{\alpha_i} - x_i^{\alpha_i}}{(q-1)x_i} \right)$$

$$= (\partial^q x_i)^{\alpha_i-1} (\left[\alpha_i\right]_q x_i^{\alpha_i-1})$$

$$= \cdots$$

$$= [\alpha_i]_q!.$$ 

The next case is $\alpha_i > \beta_i$

$$(\partial^q x_i)^{\alpha_i+1} x_i^{\alpha_i} = \partial^q x_i^{[\alpha_i]_q!}$$

$$= \frac{[\alpha_i]_q! - [\alpha_i]_q!}{(q-1)x_i}$$

$$= 0.$$

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Finally we have $\alpha_i < \beta_i$

$$(\partial_{x_i}^q)^{\alpha_i} x_i^{\beta_i}|_{x_i=0} = [\alpha_i]_q! x_i^{\beta_i-\alpha_i}|_{x_i=0}
= 0.$$ 

Together we get for $R_1, R_2 \in \mathcal{P}_k$ with any coefficients $a_n^i \in \mathcal{C}\ell_{0,n}$

$$(\overline{R_1}(\partial_{x_i}^q R_2))_0 = \sum_{|\alpha|=k} [\alpha]_q! (\overline{a_n^i} a_n^2)_0
= \langle R_1, R_2 \rangle_{k,q} \quad \blacksquare$$

**Theorem 4.2** For all $Q \in \mathcal{P}_k$ and $P \in \mathcal{P}_{k+1}$

$$\langle \overline{Q}, P \rangle_{k+1,q} = -\langle Q, D^{\overline{Q}}_{\overline{P}} P \rangle_{k,q}. \quad (46)$$

**Proof:** Due to linearity, it is sufficient only to consider monomials

$$Q(\overline{x}) = e_B x_1^{\beta_1} \ldots x_n^{\beta_n} = x_1^{\beta_1} \ldots x_n^{\beta_n} e_B, \beta_1 + \ldots + \beta_n = k \quad (47)$$

$$P(\overline{x}) = e_A x_1^{\alpha_1} \ldots x_n^{\alpha_n} = x_1^{\alpha_1} \ldots x_n^{\alpha_n} e_A, \alpha_1 + \ldots + \alpha_n = k + 1 \quad (48)$$

Further we only need to consider the variable $x_i$. On the left hand side we get

$$\langle e_i x_i Q, P \rangle_{k+1,q} = (e_i (\partial_{x_i}^q)^{\beta_1} \ldots (\partial_{x_i}^q)^{\beta_n} e_B x_1^{\alpha_1} \ldots x_n^{\alpha_n} e_A)_0
= (e_i (\partial_{x_i}^q)^{\beta_1} \ldots (\partial_{x_i}^q)^{\beta_n} e_B x_1^{\alpha_1} \ldots x_n^{\alpha_n} e_A)_0
= (\overline{Q}_i (\partial_{x_i}^q)^{\beta_1} \ldots (\partial_{x_i}^q)^{\beta_n} e_B x_1^{\alpha_1} \ldots x_n^{\alpha_n} e_A)_0.$$

While the right hand side is

$$\langle Q, e_i \partial_{x_i}^q P \rangle_{k,q} = ((\partial_{x_i}^q)^{\beta_1} \ldots (\partial_{x_i}^q)^{\beta_n} e_B e_i (\partial_{x_i}^q)^{\alpha_1} \ldots x_i^{\alpha_n} e_A)_0
= ((\partial_{x_i}^q)^{\beta_1} \ldots (\partial_{x_i}^q)^{\beta_n} e_B e_i [\alpha_i]_q! x_1^{\alpha_1} \ldots x_i^{\alpha_i-1} \ldots x_n^{\alpha_n} e_A)_0.$$

Because $\overline{x}_i = -e_i$ we get either

$$\langle \overline{Q}, P \rangle_{k+1,q} = -\langle Q, D^{\overline{Q}}_{\overline{P}} P \rangle_{k,q}$$

or both sides equal zero. \quad \blacksquare

This allows the following theorem.

**Theorem 4.3** For $k \in \mathbb{N}$ we have

$$\mathcal{P}_k = \mathcal{M}_k^q \oplus \overline{\mathcal{P}}_{k-1}. \quad (49)$$

Further the subspaces $\mathcal{M}_k^q$ and $\overline{\mathcal{P}}_{k-1}$ of $\mathcal{P}_k$ are orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle_{k,q}$.
Proof: As \( \mathcal{P}_k = x\mathcal{P}_{k-1} \oplus (x\mathcal{P}_{k-1})^\perp \) it suffices to proof \( \mathcal{M}^q_k = (x\mathcal{P}_{k-1})^\perp \). For the first inclusion take any \( R_{k-1} \in \mathcal{P}_{k-1} \) and \( R_k \in \mathcal{P}_k \). Suppose now that 
\[
(\underbrace{xR_{k-1}}, R_k)_{k,q} = 0.
\]
Then we have \( (R_{k-1}, D^q_k R_k)_{k,q} = 0 \) for each \( R_{k-1} \in \mathcal{P}_{k-1} \) (see theorem 2). We put \( R_{k-1} = D^q_k R_k \) and it follows that \( D^q_k R_k = 0 \). Therefore \( R_k \in \mathcal{M}_k \) and finally \( (x\mathcal{P}_{k-1})^\perp \subset \mathcal{M}^q_k \).

For the other inclusion take \( P_k \in \mathcal{M}^q_k \). For each \( R_{k-1} \in \mathcal{P}_{k-1} \)
\[
(\underbrace{xR_{k-1}}, P_k)_{k,q} = -(R_{k-1}, D^q_k P_k)_{k-1,q} = 0.
\]
As \( xR_{k-1} \in x\mathcal{P}_{k-1} \) we have \( P_k \in (x\mathcal{P}_{k-1})^\perp \) and accordingly \( \mathcal{M}^q_k \subset (x\mathcal{P}_{k-1})^\perp \). Therefore \( \mathcal{M}^q_k = (x\mathcal{P}_{k-1})^\perp \).

\[\square\]

Theorem 3 allows a Fischer decomposition of the space of homogeneous polynomials \( \mathcal{P}_k \):
\[
\mathcal{P}_k = \sum_{s=0}^{k} \oplus x^s \mathcal{M}^q_{k-s}.
\]

This follows directly from theorem 3 using
\[
x\mathcal{P}_{k-1} = x\mathcal{M}^q_{k-1} \oplus x^2 \mathcal{P}_{k-2}, x^2 \mathcal{P}_{k-2} = x^2 \mathcal{M}^q_{k-2} \oplus x^3 \mathcal{P}_{k-3}, \ldots.
\]

5 Cauchy-Kovalevskaya extension

To generate monogenic functions, we can use the Cauchy-Kovalevskaya extension theorem.

**Theorem 5.1** Let \( f(x_1, \ldots, x_m) = f(x) \) in \( \mathbb{R}^m \). The \( q \)-deformed Cauchy-Kovalevskaya extension of the function \( f(x) \) is
\[
CK(f(x)) = f^*(x_0, x_1, \ldots, x_m) = \sum_{k=0}^{\infty} \frac{1}{k!q^k} x_0^k (\partial_0^q D^q_x)^k f(x).
\]

\( f^* \) is a monogenic function satisfying \( D^q_x(f^*(x_0, \ldots, x_m)) = 0 \), with \( D^q_x = -\sum_{i=0}^{m} e_i \partial_i^q e_i = -e_0 \partial_0^q + D^q_x \) and \( e_0^q = -1 \). Further \( f^*[x_0=0] = f \).
Example 2 (q-deformed Fueter variables) Applying the CK-extension to the function $x_i$ in $\mathbb{R}^m$ we get

$$f^*(x_0, x_1, ..., x_i, ..., x_m) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} x_0^k (\tau_0 D_{\underline{x}}^q)^k(x_i)$$

$$= \frac{1}{[0]_q!} x_0^0 (\tau_0 D_{\underline{x}}^q)^0(x_i) + \frac{1}{[1]_q!} x_0(\tau_0 D_{\underline{x}}^q)^1(x_i) + 0 + ...$$

$$= x_i - x_0 \tau_0 e_i.$$

For $x_0 = 0$ the CK-extension $f^*$ reduces to the original function $f$. Clearly this is a monogenic function:

$$D_{\underline{x}}^q(x_i - x_0 \tau_0 e_i) = - e_0 \partial^q_{x_0} (x_i - x_0 \tau_0 e_i) - e_i \partial^q_{x_i} (x_i - x_0 \tau_0 e_i)$$

$$= e_0 \tau_0 e_i - e_i = 0.$$

Example 3 We want to compute the Cauchy-Kovalevskaya extension of $x_i x_j$. For $i \neq j$ we obtain

$$(x_i x_j)^* = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} x_0^k (\tau_0 D_{\underline{x}}^q)^k(x_i x_j)$$

$$= \frac{1}{[0]_q!} x_0^0 (\tau_0 D_{\underline{x}}^q)^0(x_i x_j) + \frac{1}{[1]_q!} x_0(\tau_0 D_{\underline{x}}^q)^1(x_i x_j) + \frac{1}{[2]_q!} x_0^2(\tau_0 D_{\underline{x}}^q)^2(x_i x_j) + ...$$

$$= x_i x_j - x_0 \tau_0 (x_j e_i + x_i e_j) + \frac{1}{[2]_q!} x_0^2 (\tau_0 (\tau_0)) (e_j e_i + e_i e_j)$$

$$= x_i x_j - x_0 \tau_0 (x_j e_i + x_i e_j)$$

$$= \frac{1}{2} ((x_i - x_0 \tau_0 e_i)(x_j - x_0 \tau_0 e_j) + (x_j - x_0 \tau_0 e_j)(x_i - x_0 \tau_0 e_i)).$$

But in case of $i = j$ we get

$$(x_i^2)^* = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} x_0^k (\tau_0 D_{\underline{x}}^q)^k(x_i^2)$$

$$= \frac{1}{[0]_q!} x_0^0 (\tau_0 D_{\underline{x}}^q)^0(x_i^2) + \frac{1}{[1]_q!} x_0(\tau_0 D_{\underline{x}}^q)^1(x_i^2) + \frac{1}{[2]_q!} x_0^2(\tau_0 D_{\underline{x}}^q)^2(x_i^2) + ...$$

$$= x_i^2 - x_0 \tau_0 ([2]_q x_i e_i) + \frac{1}{[2]_q!} x_0^2 (\tau_0 (\tau_0)) ([2]_q e_i^2)$$

$$= x_i^2 - [2]_q x_0 x_i e_i - x_0^2 \neq (x_i - x_0 \tau_0 e_i)^2, \text{ because } [2]_q = 1 + q \neq 2 \text{ for } q \neq 1.$$

Therefore, the Cauchy-Kovalevskaya extension of products is not in general the product of $q$-deformed Fueter variables.
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