ČECH COHOMOLOGY, HOMOCLINIC Trajectories AND ROBUSTNESS OF NON-SADDLE SETS

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Abstract. In this paper we study flows having an isolated non-saddle set. We see that the global structure of a flow having an isolated non-saddle set $K$ depends on the way $K$ sits in the phase space at the cohomological level. We construct flows on surfaces having isolated non-saddle sets with a prescribed global structure. We also study smooth parametrized families of flows and continuations of isolated non-saddle sets.

1. Introduction and preliminaries. In this paper we study global properties of flows that have a special kind of isolated invariant sets known as non-saddle. Roughly speaking, a non-saddle set is an invariant compactum such that the trajectories of nearby points remain close either in positive or negative time. For instance, stable attractors, repellers and certain attractors with mild forms of instability are non-saddle sets [26]. The theory of non-saddle sets was first studied by Bhatia [7] and Ura [28], although, according to Ura, it was introduced by Seibert in an oral communication. We are interested in the case when the non-saddle set is isolated since this allows the use topological techniques as has been done in [12]. Locally, isolated non-saddle sets present a simple structure divided into pieces that are either attracted or repelled. The main differences with the aforementioned theories of stable attractors, repellers and attractors with mild forms of instability appear when looking at the global structure of the flows. More specifically, the complexity of a flow having an isolated non-saddle set is related to the structure of the region of influence of this invariant set. This is especially rich in presence of the so-called dissonant points, a phenomenon that does not appear in the basin of attraction of an attractor neither stable nor unstable. The region of influence of an isolated non-saddle set has been deeply studied in [5, 6].

In spite of the similarities between the local dynamics of isolated non-saddle sets and attractors, while attractors are robust objects in both the topological and

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dynamical sense, it is well-known that isolated non-saddle sets are not. More specifically, small perturbations of a flow preserve attractors and some of their basic topological properties [25] while small perturbations may transform isolated non-saddle sets into saddle ones with distinct topological structures [13]. However, as established in [3, 6] there are some situations in which the robustness of some topological properties is equivalent to the robustness of non-saddleness.

The aim of this paper is to study some connections between the topological structure of the region of influence of an isolated non-saddle set and the way in which the non-saddle set lies in the phase space at the cohomological level. In particular, we see that phase spaces with little cohomology do not support isolated non-saddle sets with overly complicated regions of influence. This is motivated by some results from [23, 24] about certain unstable attractors. In addition, motivated by the results in [6] and [3] about the continuation properties of isolated non-saddle sets we find necessary and sufficient conditions for the property of being non-saddle to be robust for families of smooth flows defined on smooth manifolds without further assumptions about the dimension or cohomology of the phase space.

In order to make the paper more readable we recall some basic concepts about topology and dynamical systems. Throughout the paper we shall use the letter $M$ to denote the phase spaces of the flows considered. The general assumption about $M$ is that it is a locally compact ANR. Sometimes we shall need to impose that $M$ is a manifold. Whenever we impose this condition to the phase space we will mention it explicitly.

**Manifolds.** We recall that an $n$-dimensional manifold is a second countable Hausdorff topological space such that each point has a neighborhood homeomorphic to $\mathbb{R}^n$. A second countable Hausdorff space is said to be an $n$-manifold with boundary if each point has either a neighborhood homeomorphic to $\mathbb{R}^n$ or to the upper half-space $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. A smooth manifold is a manifold endowed with an open covering $\{U_i\}_{i \in I}$ together with homeomorphisms $\psi_i : U_i \to \mathbb{R}^n$ such that if $U_i \cap U_j \neq \emptyset$ the transition maps $\psi_{i,j} = \psi_j^{-1} \circ \psi_i : U_i \cap U_j \to \psi_j^{-1}(\psi_i(U_i \cap U_j))$ are $C^\infty$. Smooth manifolds with boundary are defined in an analogous way. Throughout this paper we shall use the word surface to refer to a connected 2-manifold.

**ANRs.** A metric space $X$ is said to be an Absolute Neighborhood Retract or, shortly, an ANR if it satisfies the following: whenever there exists an embedding $f : X \to Y$ of $X$ into a metric space $Y$ such that $f(X)$ is closed in $Y$, there exists a neighborhood $U$ of $f(X)$ such that $f(U)$ is a retract of $U$. Some examples of ANRs are manifolds, CW-complexes and polyhedra. An open subset of an ANR is an ANR and a retract of ANR is also an ANR. More information about ANRs can be found in [15].

**Algebraic Topology.** In this paper we use singular homology and cohomology, Čech cohomology and Alexander’s duality theorem. The notation $H_* (\cdot; G)$ and $H^* (\cdot; G)$ is used for singular homology and cohomology respectively and $\tilde{H}^* (\cdot; G)$ is used for Čech cohomology. The coefficients group $G$ is always assumed to be either $\mathbb{Z}$ or $\mathbb{Z}_2$. Since Čech and singular cohomology theories agree on ANRs [17, Theorem 1] we sometimes use both interchangeably. Some good references for this material are the book of Spanier [27] and Hatcher [14].

**Shape theory.** There is a form of homotopy which is very convenient to study the global topological properties of the invariant spaces involved in dynamics, namely the shape theory introduced and studied by Karol Borsuk. The books
are standard references in this topic. The papers [16, 21, 25, 26] show some applications of shape theory to dynamical systems. We shall use the fact that Čech
[9, 18] are standard references in this topic. The papers [16, 21, 25, 26] show some
x
point cohomology is a shape invariant.

Limit sets. We recall that the omega-limit and the negative omega-limit of a
point \( x \in M \) are the sets
\[
\omega(x) = \bigcap_{t>0} x[t, \infty), \quad \omega^*(x) = \bigcap_{t>0} x[-\infty, -t].
\]

Sections and parallelizable flows. Given a flow \( \varphi : M \times \mathbb{R} \to M \), by a section
\( S \), we mean a set which intersects each trajectory exactly in a point.

The flow \( \varphi \) is said to be parallelizable if it admits a section \( S \) such that the map
\( \sigma : M \to \mathbb{R} \) defined by the property \( x\sigma(x) \in \sigma \) is continuous. Notice that, if one
section satisfies this condition, all of them do.

If a flow is parallelizable and \( S \) is a section, the map \( h : S \times \mathbb{R} \to M \) defined by
\( (x, t) \mapsto xt \) is a homeomorphism. A direct consequence of these considerations
is that a section \( S \) of a parallelizable flow is a strong deformation retract of \( M \) and
the deformation retraction is provided by the flow.

Invariant manifolds, stability, attractors and repellers. The stable and
unstable manifolds of an invariant compactum \( K \) are defined respectively as the sets
\[
W^s(K) = \{ x \in M \mid \emptyset \neq \omega(x) \subset K \}, \quad W^u(K) = \{ x \in M \mid \emptyset \neq \omega^*(x) \subset K \}.
\]

Throughout this paper an attractor will be an asymptotically stable set while a
repeller will be a negatively asymptotically stable set. More precisely, an invariant
compactum \( K \) is said to be an attractor if it possesses a neighborhood \( U \) such
that \( \emptyset = \omega(x) \subset K \) for every \( x \in U \) and, in addition, every neighborhood \( V \) of \( K \)
contains a neighborhood \( W \) of \( K \) such that \( W[0, \infty) \subset V \). The latter condition is
known as stability. In this case the stable manifold of \( K \) turns out to be an open
set and is called the basin of attraction of \( K \) and denoted by \( \mathcal{A}(K) \). A repeller is
an attractor for the reverse flow. In an analogous way, the unstable manifold of a
repeller \( K \) is an open set called the basin of repulsion of \( K \) and denoted by \( \mathcal{R}(K) \).

Since we are dealing with locally compact metric phase spaces, if \( K \) is an attractor
(resp. repeller), the restriction flow \( \varphi|_{\mathcal{A}(K) \setminus K} \) (resp. \( \varphi|_{\mathcal{R}(K) \setminus K} \)) is parallelizable and
its sections are compact [8, p. 83].

Although throughout this paper we require attractors to be stable, sometimes
stability is dropped from the definition to consider a more general kind of attractors.
We shall refer to those as unstable attractors. For the reader interested in a detailed
study of unstable attractors we recommend the papers [1, 2, 19, 23].

Isolated invariant sets and isolating blocks. A compact invariant set \( K \) is
said to be an isolated invariant set if it possesses a so-called isolating neighborhood,
that is, a compact neighborhood \( N \) such that \( K \) is the maximal invariant set in \( N \),
or setting
\[
N^+ = \{ x \in N \mid x[0, +\infty) \subset N \}, \quad N^- = \{ x \in N \mid x(-\infty, 0] \subset N \};
\]
such that \( K = N^+ \cap N^- \). Notice that \( N^+ \) and \( N^- \) are compact and, respectively, positively and negatively invariant. For instance, attractors and repellers are
isolated invariant sets.

To avoid trivial cases when considering an isolated invariant set we shall assume
without further mention that it is a non-empty proper subset of the phase space. We
make an exception to this rule in Section 4 where, for technical reasons regarding
continuations, we have to deal with the empty set as an isolated invariant set. Hence
in Section 4 we shall explicity mention whenever the isolated invariant sets involved
in the statements are necessarily non-empty.

We shall make use of a special type of isolating neighborhoods, the so-called
isolating blocks, which have good topological properties. More precisely, an isolating
block $N$ is an isolating neighborhood such that there are compact sets $N^i, N^o \subset \partial N$,
called the entrance and exit sets, satisfying

1. $\partial N = N^i \cup N^o$;
2. for every $x \in N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset M \setminus N$
   and for every $x \in N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset M \setminus N$,
3. for every $x \in \partial N \setminus N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset \bar{N}$
   and for every $x \in \partial N \setminus N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset \bar{N}$.

Notice that by $\partial N$ and $\bar{N}$ we refer to the topological frontier and interior of $N$.
If the phase space is a smooth manifold and the flow is $C^\infty$, the proof of [11, The-
orem 1.5] ensures that isolated invariant sets possess bases of neighborhoods com-
prised of isolating blocks that are smooth manifolds with boundary which contain
$N^i$ and $N^o$ as submanifolds of their boundary and such that $\partial N^i = \partial N^o = N^i \cap N^o$.
Here $\partial N^i$ and $\partial N^o$ are the boundary of $N^i$ and $N^o$ as manifolds. This kind of iso-
lating blocks will be called isolating block manifolds. For more information about
isolated invariant sets and isolating blocks see [10, 11].

**Structure of the paper.** The paper is structured as follows: in Section 2 we
recall the basic notions of the theory of isolated non-saddle sets, including that of
the region of influence, and their fundamental properties. In Section 3 we intro-
duce the so-called complexity of the region of influence of an isolated non-saddle
set. This complexity is a number which encapsulates how complicated the region
of influence of the isolated non-saddle set is. We see that this complexity has a
strong relationship with the cohomology of the phase space (Theorem 3.6) and, in
particular, with the homomorphisms induced in Čech cohomology by the inclusion
of the isolated non-saddle set in the phase space. As a consequence, we can infer
interesting dynamical features only by looking at topological relationships. For in-
fact, we see in Proposition 2 that the complexity of the region of influence of a
connected isolated non-saddle set in the $n$-dimensional torus is at most 1. We also
see in Proposition 4 that the region of influence of a connected isolated non-saddle
sets in a closed orientable surface of genus $g$ has complexity at most $g$. In addition,
Proposition 5 establishes the existence of flows on closed orientable surfaces which
have connected isolated non-saddle sets whose regions of influence have complexity
$g$ and satisfy some additional conditions. Finally, in Section 4 we study the robust-
ness of isolated non-saddle sets from the point of view of continuation theory. The
main results of this section are Proposition 6 and Theorem 4.3. Both results estab-
lish necessary and sufficient conditions for a connected isolated non-saddle set to be
locally continued to a family of isolated non-saddle sets. In particular, Theorem 4.3
establishes the equivalence between the continuation of non-saddleness and the con-
tinuation of certain cohomological relations, i.e., the continuation of the dynamical
property of non-saddleness turns out to be equivalent to the continuation of some
properties of topological nature.
2. Isolated non-saddle sets and their region of influence. Let $M$ be a locally compact ANR and $\varphi : M \times \mathbb{R} \to M$ a flow on $M$. We recall that an invariant compactum $K$ is said to be saddle whenever there exists a neighborhood $U$ of $K$ such that for every neighborhood $V$ of $K$ there exists $x \in V$ such that the trajectory of $x$ leaves $U$ in the past and in the future, i.e., such that $x(0, +\infty) \notin U$ and $x(-\infty, 0] \notin U$. Otherwise $K$ is said to be non-saddle.

In this paper we are interested in those non-saddle sets which are isolated as invariant sets. Isolated non-saddle sets of flows defined on locally compact ANRs have the shape of finite polyhedra and, hence, finitely generated Čech homology in all dimensions and nonzero only for a finite number of them [12, Theorem 4]. These sets are characterized by the property of admitting isolating blocks of the form $N = N^+ \cup N^-$ [6, Proposition 3]. Moreover, if $K$ is connected and $N$ is a connected isolating block, then $N$ is of the form $N = N^+ \cup N^-$ [6, Remark 4]. If $K$ is an isolated non-saddle set and $N = N^+ \cup N^-$ is an isolating block the inclusion $i : K \hookrightarrow N$ is a shape equivalence [26, Proof of Theorem 3.5] and hence induces isomorphisms in Čech cohomology groups. As a consequence, an isolated non-saddle set on a locally compact ANR has a finite number of components and all of them are isolated non-saddle.

The local dynamics near an isolated non-saddle set is rather simple. Each component of $N \setminus K$ is either attracted or repelled by $K$. In addition, the flow provides a deformation retraction from $N \setminus K$ onto $\partial N$. Notice that $\partial N$ is an ANR [6, Proof of Proposition 18], so it has a finite number of components. In spite of the simplicity of the local structure of these flows, their global structure may be far more complicated than the structure of a flow having either an attractor or an isolated unstable attractor without external explosions. This complexity is exhibited in the structure of the region of influence of the isolated non-saddle set $K$. The region of influence of an isolated non-saddle set $K$ is defined as the set

$$I(K) = W^s(K) \cup W^u(K).$$

This set is an open subset of the phase space and its topological and dynamical structures have been extensively studied in [6]. Although these structures share many features with those of the basin of attraction of an isolated attractor without external explosions, the global structure of $I(K)$ may be much richer. As a matter of fact, while the flow restricted to the complement of an isolated stable or unstable attractor without external explosions in its basin of attraction is always parallelizable, this is not generally the case for the flow restricted to $I(K) \setminus K$ (see Figure 2).

The region of influence $I(K)$ is composed of three different kinds of points.

1. Purely attracted points, that is, points $x \in I(K)$ with $\omega(x) \subset K$ and $\omega^*(x) \notin K$.
2. Purely repelled points, that is, points $x \in I(K)$ with $\omega^*(x) \subset K$ and $\omega(x) \notin K$.
3. Homoclinic points, that is, points $x \in I(K)$ with $\omega^*(x) \subset K$ and $\omega(x) \subset K$.

We denote by $A^*(K)$, $R^*(K)$ and $H(K)$ the sets of purely attracted, purely repelled and homoclinic points respectively. The three sets are invariant subsets of $M$ and they satisfy that $A^*(K) \cup K$ and $R^*(K) \cup K$ are closed in $I(K)$ and $H(K) \setminus K$ is open (see [6, Proposition 12]). The situation one would expect at this point would be that $I(K) \setminus K$ was decomposed as a finite union of components comprised entirely of purely attracted points, components comprised entirely of purely repelled points and components comprised entirely of homoclinic points.
However, this happens if and only if \( \mathcal{H}(K) \) is a closed set which is not the case in general (see Figure 2). What actually happens is that \( \mathcal{I}(K) \setminus K \) decomposes as a finite union of components comprised entirely of purely attracted points, components comprised entirely of purely repelled points, components comprised entirely of homoclinic points and components that contain points of the three kinds [6, Proposition 20]. Moreover, the flow restricted to the components comprised entirely of purely attracted points, purely repelled points and homoclinic points is parallelizable while the flow restricted to the components that contain the three kinds of points is not. Notice that the components that contain the three kinds of points are exactly those that contain boundary points of \( \mathcal{H}(K) \).

We call dissonant those points in \( \partial \mathcal{H}(K) \) which are not in \( K \). The previous discussion illustrates that all the interesting dynamical features in \( \mathcal{I}(K) \setminus K \) occur in the components containing dissonant points. In fact, in the absence of dissonant points it has been seen in [5, Theorem 19] the dynamics in \( \mathcal{I}(K) \setminus K \) is qualitatively the same as the dynamics in the basin attraction of an isolated attractor without external explosions studied in [23].

3. **Dynamical complexity of the region of influence of isolated non-saddle sets and the cohomology of the phase space.** In this section we study to what extent the topology of the phase space and the way in which the isolated non-saddle continuum \( K \) sits in it at the cohomological level affects the structure of the region of influence of \( K \). Some results in this spirit were obtained by Sánchez-Gabites [23, 24] in the case of isolated attractors without external explosions and by Barge and Sanjurjo [6] for isolated non-saddle continua in closed manifolds. Although the results we present here can be regarded as generalizations of the aforementioned results, they stress again that the structure of the region of influence of an isolated non-saddle set is much more subtle than the region of attraction of an isolated attractor without external explosions.

The following result, which generalizes [6, Theorem 25] establishes cohomological obstructions to the existence of homoclinic trajectories and dissonant points in the region of influence of an isolated non-saddle continuum for a flow defined on a locally compact ANR.

**Theorem 3.1.** Let \( M \) be a connected locally compact ANR and \( K \) a connected isolated non-saddle set of a flow on \( M \). Suppose that \( H^1(M; G) = 0 \) or, more generally, that the homomorphism \( i^* : \check{H}^1(M; G) \to \check{H}^1(K; G) \) induced in 1-dimensional Čech cohomology by the inclusion \( i : K \hookrightarrow M \) is a monomorphism. Then \( K \) does not have dissonant points. Moreover, if \( U \) is a component of \( M \setminus K \), then the flow restricted to \( U \) is either locally attracted by \( K \) (i.e. all points lying in \( U \) near \( K \) are attracted by \( K \)) or locally repelled by \( K \). Furthermore, if \( N \) is an isolating block of \( K \) of the form \( N = N^+ \cup N^- \) then each component of \( M \setminus K \) contains exactly one component of \( \partial N \).

**Proof.** Consider an isolating block \( N \) of \( K \) such that \( N = N^+ \cup N^- \). Then the homomorphism \( j^* : \check{H}^1(M; G) \to \check{H}^1(N; G) \) induced in 1-dimensional Čech cohomology by the inclusion \( j : N \hookrightarrow M \) is a monomorphism. This follows from the equality \( i^* = k^* \circ j^* \), where \( k^* \) is the isomorphism induced in 1-dimensional Čech cohomology by the inclusion \( k : K \hookrightarrow N \).

Consider the initial part of the long exact sequence of Čech cohomology for the pair \( (M, N) \)
0 \to \hat{H}^0(M, N; G) \to \hat{H}^0(M; G) \to \hat{H}^0(N; G) \to \hat{H}^1(M, N; G)
\to \hat{H}^1(M; G) \xrightarrow{j^*} \hat{H}^1(N; G) \to \ldots

Since \(M\) and \(N\) are connected, the homomorphism \(\hat{H}^0(M; G) \to \hat{H}^0(N; G)\) is an isomorphism and, since \(j^*\) is a monomorphism, the exactness of the sequence ensures that \(\hat{H}^i(M, N; G) = 0\) for \(i = 0, 1\).

On the other hand, by excision we get
\[
\hat{H}^i(M, N; G) \cong \hat{H}^i(M \setminus K, N \setminus K; G)
\]
and, as a consequence, \(\hat{H}^i(M \setminus K, N \setminus K; G) = 0\) for \(i = 0, 1\). Taking this into account in the long exact sequence of Čech cohomology of the pair \((M \setminus K, N \setminus K)\) we get that the inclusion \(N \setminus K \hookrightarrow M \setminus K\) induces an isomorphism between \(\hat{H}^0(M \setminus K; G)\) and \(\hat{H}^0(N \setminus K; G)\). This proves that each component of \(M \setminus K\) contains exactly one component of \(N \setminus K\). Besides, since \(N^+ \cap N^- = K\) it easily follows that every component of \(N \setminus K\) must be either contained in \(N^+ \setminus N^-\) or \(N^- \setminus N^+\). This shows that each component of \(M \setminus K\) is either locally attracted or locally repelled by \(K\), which prevents \(K\) from having homoclinic trajectories in \(\mathcal{I}(K) \setminus K\) and, hence, dissonant points. The remaining part of the statement follows easily from the fact that the flow provides a deformation retraction from \(N \setminus K\) onto \(\partial N\). 

\textbf{Corollary 1.} Let \(M\) be a connected locally compact ANR and \(K\) a connected isolated non-saddle set of a flow on \(M\). Suppose that the homomorphism \(i^*: \hat{H}^1(M; G) \to \hat{H}^1(K; G)\) induced in 1-dimensional Čech cohomology by the inclusion \(i: K \hookrightarrow M\) is a monomorphism and that \(K\) does not separate \(M\). Then \(K\) is an attractor or a repeller.

\textbf{Corollary 2.} Let \(M\) be a connected locally compact ANR and \(K \subset M\) is an isolated non-saddle continuum. Suppose that \(K\) does not separate \(\mathcal{I}(K)\). Then \(K\) is an attractor or a repeller if and only if the homomorphism \(i^*: \hat{H}^1(\mathcal{I}(K); G) \to \hat{H}^1(K; G)\) induced in 1-dimensional Čech cohomology by the inclusion \(i: K \hookrightarrow M\) is a monomorphism.

\textbf{Proof.} For the if part observe that if \(K\) is an attractor, the region of influence \(\mathcal{I}(K)\) is just the basin of attraction of \(K\). Then \([16, \text{Theorem 3.6}]\) ensures that the inclusion \(i: K \hookrightarrow \mathcal{I}(K)\) is a shape equivalence and, hence, \(i^*: \hat{H}^1(\mathcal{I}(K); G) \to \hat{H}^1(K; G)\) is a monomorphism. If \(K\) is a repeller the proof is analogous working with the reverse flow.

The only if part follows directly from Corollary 1 taking into account that \(\mathcal{I}(K)\) is an open neighborhood of \(K\) in \(M\). 

Isolated non-saddle sets for flows defined on locally compact ANRs have a finite number of components and all them are isolated non-saddle. For this reason it may seem that the assumption about connectedness in Theorem 3.1 and the subsequent results is only placed to simplify the statements without the general case being genuinely more complicated. The following example illustrates that an analog of Theorem 3.1 does not hold in the general situation.

\textbf{Example 3.2.} Let \(S^2\) be the 2-dimensional sphere, \(N\) and \(S\) the north and south poles of \(S^2\) and \(m \subset S^2\) a meridian, i.e., a semicircle of a great circle whose endpoints are \(N\) and \(S\). We consider a flow \(\varphi: S^2 \times \mathbb{R} \to S^2\) with the following properties:

- The pair \((S, N)\) is an attractor-repeller decomposition.
• The trajectory of any point not contained in \( m \) flows downwards along its meridian.
• The meridian \( m \) splits into five trajectories: the fixed points \( N \) and \( S \), a fixed point \( p \) situated in the equator, the oriented open segment of \( m \) that connects \( N \) with \( p \) and the oriented open segment of \( m \) that connects \( p \) with \( S \).

The compactum \( K = \{ N, S \} \) is a non-separating isolated non-saddle set that is neither an attractor nor a repeller. In addition, every point in \( m \setminus K \) is dissonant.

The situation described in Theorem 3.1, where \( K \) acts either as an attractor or a repeller in each component of \( M \setminus K \), is the simplest one that appears when dealing with isolated non-saddle sets. In this case, given an isolating block of the form \( N = N^+ \cup N^- \), each component of \( I(K) \setminus K \) contains exactly one component of \( N \setminus K \). This suggests that a possible way to measure how complicated is the flow in a component \( C \) of \( I(K) \setminus K \) may be to count the number of components of \( N \setminus K \) contained in \( C \) and motivates the following definition.

**Definition 3.3.** Let \( M \) be a locally compact ANR and suppose that \( K \) is an isolated non-saddle set of a flow defined on \( M \). Let \( C \) be a component of \( I(K) \setminus K \) and consider an isolating block for \( K \) of the form \( N = N^+ \cup N^- \). We define the local complexity of \( C \) as the difference \( k - 1 \) where \( k \) denotes the number of components of \( N \setminus K \) contained in the component \( C \).

The complexity \( c \) of \( I(K) \) is defined as the sum of the local complexities of its components.

**Remark 1.**
• The concept of local complexity is well-defined. That is, it does not depend on the choice of an isolating block of the form \( N = N^+ \cup N^- \). To see this it is enough to observe that given two isolating blocks \( N_1 = N_1^+ \cup N_1^- \) and \( N_2 = N_2^+ \cup N_2^- \) of an isolated non-saddle set \( K \) and a component \( C \) of \( I(K) \setminus K \), the flow provides a bijection between the components of \( N_1 \setminus K \) and \( N_2 \setminus K \) that are contained in \( C \). In fact, the local complexity encapsulates topological information about the different ways \( K \) can be approached from \( C \).
• The complexity of \( I(K) \) can also be defined as the difference \( k - m \) where \( k \) denotes the number of components of \( N \setminus K \) for any isolating block of the form \( N = N^+ \cup N^- \) and \( m \) is the number of components of \( I(K) \setminus K \).

**Proposition 1.** Suppose that \( K \) is an isolated non-saddle set of a flow defined on a locally compact ANR. Let \( C \) be a component of \( I(K) \setminus K \). Then

(i) \( C \) does not contain homoclinic points if and only if \( C \) has local complexity zero.
(ii) If every point in \( C \) is homoclinic then \( C \) has local complexity one.
(iii) If the local complexity of \( C \) is greater than one then \( C \) contains dissonant points.

**Proof.** The key point in this proof is that [6, Proposition 18] ensures that if \( C \) does not contain dissonant points the flow on \( C \) is parallelizable with section any component of \( \partial N \) contained in \( C \). Hence \( C \) contains at most one component of \( N^+ \setminus K \) and one component of \( N^- \setminus K \). Moreover, \( C \) contains both if and only if \( C \) contains a homoclinic point, in which case it is comprised entirely of homoclinic points. This shows that if \( C \) does not contain dissonant points its local complexity is at most one and is one if and only if \( C \) is comprised entirely of homoclinic points.
Since whenever $C$ contains dissonant points it also contains homoclinic points the result follows.

Remark 2. The following remarks are direct consequences of Proposition 1.

- $\mathcal{I}(K) \setminus K$ does not contain homoclinic points if and only if the complexity of $\mathcal{I}(K)$ is zero.
- In the absence of dissonant points the complexity of $\mathcal{I}(K)$ is exactly the number of components of $\mathcal{I}(K) \setminus K$ comprised entirely of homoclinic points.
- In the general case, the complexity is an upper bound for the number of components of $\mathcal{I}(K) \setminus K$ which contain homoclinic points.
- Theorem 3.1 ensures that the region of influence of a connected isolated non-saddle set $K$ of a flow on an ANR $M$ has complexity zero whenever $H^1(M; G) = 0$ or, more generally, the homomorphism induced in 1-dimensional Čech cohomology by the inclusion $i: K \hookrightarrow M$ is a monomorphism.

The following two examples illustrate the fact that the local complexities carry more information than the complexity of $\mathcal{I}(K)$.

Example 3.4. Let $M$ be a closed orientable surface of genus 2. We consider $M$ endowed with the flow $\varphi : M \times \mathbb{R} \to M$ depicted in Figure 1. This flow has an isolated non-saddle continuum $K$ which is homeomorphic to a sphere with the interiors of four disjoint closed topological disks removed. The flow in $K$ is stationary and $\mathcal{I}(K) \setminus K$ has two connected components $C_1$ and $C_2$ both of them comprised entirely of homoclinic trajectories. As a consequence, both components $C_1$ and $C_2$ have local complexity 1 and, then, the complexity of $\mathcal{I}(K)$ is 2. Notice that $K$ does not have dissonant points.

Example 3.5. Consider the flow $\tilde{\varphi} : M \times \mathbb{R} \to M$ defined on a closed orientable surface of genus 2, depicted in Figure 2. In this case, there is an isolated non-saddle continuum $K'$ which is homeomorphic to a sphere with the interiors of four disjoint
closed topological disks removed. The flow \( \hat{\phi} \) is stationary on \( K' \) and \( \mathcal{I}(K') \setminus K' \) has two connected components \( C'_1 \) and \( C'_2 \). \( C'_1 \) is homeomorphic to an open annulus, every point in it is purely repelled by \( K' \) and, hence, the local complexity of \( C'_1 \) is zero. On the other hand, \( C'_2 \) is homeomorphic to a 2-dimensional sphere with four punctures. The local complexity of \( C'_2 \) is 2 and, as a consequence, it contains dissonant points. Observe that the dissonant points are those which lie in the stable and unstable manifolds of the fixed point \( p_2 \in \overline{C'_2} \). It follows that the complexity of \( \mathcal{I}(K) \) is 2.

Examples 3.4 and 3.5 illustrate two flows defined on a closed orientable surface of genus 2. Both of them have isolated non-saddle continua whose regions of influence have complexity 2. In addition, both \( \mathcal{I}(K) \setminus K \) and \( \mathcal{I}(K') \setminus K' \) have two connected components. However, \( K \) does not have dissonant points while \( K' \) does. This stresses that the complexity does not predict in general the existence of dissonant points. On the other hand, if we look at the local complexities, we see that \( \mathcal{I}(K') \setminus K' \) contains a component with local complexity 2 and, hence, it must contain dissonant points. This discussion points out that the complexity can be seen as an upper bound for how complicated the dynamics in the region of influence is, while the local complexities actually record the structure of the flow in \( \mathcal{I}(K) \setminus K \). In the 2-dimensional case, the existence of dissonant points can be detected using the Euler characteristic [6, Theorem 32].

The next result makes precise the relationship between the complexity of the region of influence of a connected isolated non-saddle set \( K \) and the way in which \( K \) sits in the phase space at the cohomological level. Before stating the result we recall that a manifold \( M \) is said to be \( G \)-orientable, where \( G \) is either \( \mathbb{Z} \) or \( \mathbb{Z}_2 \), whenever is possible to choose a generator of \( H_n(M,M \setminus \{x\};G) \cong G \) for every \( x \in M \) in a consistent way (see [14, p. 235]). We also recall that orientable manifolds are those that are \( \mathbb{Z} \)-orientable and that all manifolds are \( \mathbb{Z}_2 \)-orientable. This rather technical assumption is placed in order to avoid issues with orientability when applying Alexander duality.

**Theorem 3.6.** Let \( K \) be an isolated non-saddle continuum of a flow defined on a connected locally compact ANR \( M \) and \( i^*: \check{H}^k(M;G) \rightarrow \check{H}^k(K;G) \) the homomorphism induced in \( k \)-dimensional Čech cohomology by the inclusion \( i : K \hookrightarrow M \). Suppose that the complexity of the region of influence of \( K \) is \( c \). Then there exist

\[ \alpha_1, \ldots, \alpha_c \in \check{H}^1(M;G) \]

which are independent non-torsion elements satisfying that \( i^*(\alpha_j) = 0 \) for every \( j = 1, \ldots, c \). Moreover, if \( M \) is a closed, connected and \( G \)-orientable \( n \)-manifold, then there exist

\[ \beta_1, \ldots, \beta_c \in \check{H}^{n-1}(M;G) \quad \text{and} \quad \gamma_1, \ldots, \gamma_c \in \check{H}^{n-1}(K;G) \]

which are independent non-torsion elements such that \( i^{n-1}(\beta_j) = \gamma_j \) for each \( j = 1, \ldots, c \).

**Proof.** Let \( N = N^+ \cup N^- \) be an isolating block of \( K \). Reasoning as in the proof of Theorem 3.1 it follows that the homomorphism \( j^*: \check{H}^1(M;G) \rightarrow \check{H}^1(N;G) \) induced in 1-dimensional Čech cohomology by the inclusion \( j : N \hookrightarrow M \) satisfies that \( \ker j^* = \ker i^* \). Consider the initial part of the long exact sequence of Čech
Figure 2. Flow on a double torus having an isolated non-saddle set $K'$ comprised of stationary points that is a sphere with the interiors of four closed topological disks removed. The region of influence of $K'$ is the double torus with the fixed points $p_1$ and $p_2$ removed. $\mathcal{I}(K') \setminus K'$ has two connected components $C_1'$ and $C_2'$ with local complexities 0 and 2 respectively.

Cohomology of the pair $(M, N)$,

$$0 \to \tilde{H}^0(M, N; G) \to \tilde{H}^0(M; G) \to \tilde{H}^0(N; G) \to \tilde{H}^1(M, N; G)$$

$$\quad \to \tilde{H}^1(M; G) \xrightarrow{j_*} \tilde{H}^1(N; G) \to \cdots$$

Since $M$ and $N$ are connected, the second homomorphism is an isomorphism and, hence, $\tilde{H}^0(M, N; G) = 0$ and $\tilde{H}^1(M, N; G) \cong \ker i^*$. Then, by excising $K$, we obtain that $\tilde{H}^0(M \setminus K, N \setminus K; G) = 0$ and $\tilde{H}^1(M \setminus K, N \setminus K; G) \cong \ker i^*$. Therefore, the initial part of the long exact sequence of Čech cohomology of the pair $(M \setminus K, N \setminus K)$ takes the form

$$0 \to \tilde{H}^0(M \setminus K; G) \to \tilde{H}^0(N \setminus K; G) \to \ker i^* \to \cdots$$

Let $C_1, \ldots, C_k$ be the components of $M \setminus K$. The exactness of the latter sequence ensures that $\ker i^*$ has a subgroup isomorphic to $H_1 \oplus \cdots \oplus H_k$ where

$$H_i = G \oplus m_i^{-1} \oplus G$$

and $m_i$ is the number of components of $N \setminus K$ contained in $C_i$. Then, the first part of the result follows by observing that given a component $C_i$ of $M \setminus K$, each component $U_j$ of $\mathcal{I}(K) \setminus K$ contained in $C_i$ contributes with at least $l_j$ summands $G$ to $H_i$, where $l_j$ is the local complexity of $U_j$.

Let us prove the second part of the statement. Observe that by Alexander duality

$$\tilde{H}^n(K; G) \cong H_0(M, M \setminus K; G) = 0.$$
Consider the long exact sequence of Čech cohomology of the pair \((M, K)\),
\[
\cdots \to 
\check{H}^{n-1}(M; G) \to \check{H}^{n-1}(K; G) \to \check{H}^n(M, K; G) \\
\to \check{H}^n(M; G) \to \check{H}^n(K; G) = 0.
\]
This sequence breaks into the short exact sequence
\[
0 \to \text{coker } i^* \to \check{H}^n(M, K; G) \to \check{H}^n(M; G) \to 0,
\]
and, hence, \(\check{H}^n(M, K; G) \cong \text{coker } i^* \oplus G\). Another application of Alexander duality theorem ensures that \(H_0(M \setminus K; G) \cong \text{coker } i^* \oplus G\). Then
\[
\text{rk } \check{H}^{n-1}(K; G) = \text{rk } \text{coker } i^* + \text{rk } \text{im } i^* = \text{rk } \check{H}_0(M \setminus K; G) + \text{rk } \text{im } i^*.
\]
On the other hand, Alexander duality ensures that
\[
H_1(\check{N}, \check{N} \setminus K; G) \cong \check{H}^{n-1}(K; G).
\]
Taking this into account in the long exact sequence of reduced singular homology of the pair of ANRs \((\check{N}, \check{N} \setminus K)\) it follows that
\[
\text{rk } \check{H}_0(\check{N} \setminus K; G) \leq \text{rk } \check{H}_0(M \setminus K; G) + \text{rk } \text{im } i^*.
\]
From Remark 1 it follows that the complexity of \(\varepsilon\) of \(\mathcal{I}(K)\) is the difference between the number of components of \(N \setminus K\) and the number of components of \(\mathcal{I}(K) \setminus K\). Hence,
\[
\varepsilon = \text{rk } \check{H}_0(N \setminus K; G) - \text{rk } \check{H}_0(\mathcal{I}(K) \setminus K; G).
\]
Since \(N = N^+ \cup N^-\) we have that \(N \setminus K\) is homeomorphic to \(\partial N \times [0, +\infty)\) by a homeomorphism that carries \(\partial N\) to \(\partial N \times \{0\}\). This follows from [4, Theorem 3] and [4, Theorem 6]. Therefore, \(\text{rk } \check{H}_0(N \setminus K; G) = \text{rk } \check{H}_0(\check{N} \setminus K; G)\) and
\[
\varepsilon = \text{rk } \check{H}_0(\check{N} \setminus K; G) - \text{rk } \check{H}_0(\mathcal{I}(K) \setminus K; G) \leq \text{rk } \check{H}_0(\check{N} \setminus K; G) - \text{rk } \check{H}_0(M \setminus K; G).
\]
The last inequality follows from the fact that \(\mathcal{I}(K)\) is an open neighborhood of \(K\) in \(M\) and, as a consequence, each component of \(M \setminus K\) contains at least one component of \(\mathcal{I}(K) \setminus K\). Then, there must be \(\gamma_1, \ldots, \gamma_\varepsilon\) independent non-torsion cohomology classes in \(\text{im } i^*\) and the result follows.

A direct consequence of Theorem 3.6 is the following result that generalizes [24, Theorem 4.6].

**Corollary 3.** Suppose that \(M\) is a connected locally compact ANR and \(K \subset M\) is an isolated non-saddle continuum. Then the complexity of \(\mathcal{I}(K)\) is zero if and only if the homomorphism \(i^* : \check{H}^1(\mathcal{I}(K); G) \to \check{H}^1(K; G)\) induced in 1-dimensional Čech cohomology by the inclusion \(i : K \hookrightarrow M\) is a monomorphism.

**Proof.** Suppose that the complexity of \(\mathcal{I}(K)\) is zero. Then, each component \(C\) of \(\mathcal{I}(K) \setminus K\) has local complexity zero and Proposition 1 ensures that either all the points in \(C\) are purely attracted or they are purely repelled by \(K\). Reasoning as in [16, Theorem 3.6] it is possible to construct a shape deformation retraction from \(\mathcal{I}(K)\) to \(K\) using the flow. Therefore, the inclusion \(i : K \hookrightarrow \mathcal{I}(K)\) is a shape equivalence and, hence, \(i^* : \check{H}^1(\mathcal{I}(K); G) \to \check{H}^1(K; G)\) is a monomorphism.

The only if part follows directly from Theorem 3.6.

The following result extends [23, Example 24] to the case of isolated non-saddle sets.
Proposition 2. Suppose $K$ is an isolated non-saddle continuum in the $n$-dimensional torus $T^n$. Then the complexity of $\mathcal{I}(K)$ is at most 1.

Proof. We shall assume that $n > 1$ since for $n = 1$ the result follows directly from Theorem 3.6. Let $i^*: \check{H}^k(T^n;\mathbb{Z}_2) \to \check{H}^k(K;\mathbb{Z}_2)$ be the homomorphism induced in $k$-dimensional Čech cohomology by the inclusion $i : K \hookrightarrow T^n$. To prove the result we use the fact that the cohomology ring $\check{H}^*(T^n;\mathbb{Z}_2)$ is the exterior algebra over $\check{H}^1(T^n;\mathbb{Z}_2) \cong \mathbb{Z}_2^n$. In particular, we use this ring structure to show that if $	ext{rk}(\ker i^*) > 1$, then $i^{*n-1} = 0$. Suppose that there exist $\alpha_1, \alpha_2 \in \ker i^*$, with $\alpha_1$ and $\alpha_2$ linearly independent. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of $\check{H}^1(T^n;\mathbb{Z}_2)$ containing $\alpha_1, \alpha_2$. Since the cohomology ring $\check{H}^*(T^n;\mathbb{Z}_2)$ is the exterior algebra over $\check{H}^1(T^n,\mathbb{Z}_2)$ it follows that any element $\beta \in \check{H}^{n-1}(T^n;\mathbb{Z}_2)$ is of the form $\sum_{i=1}^n m_i(\alpha_1 \sim \ldots \sim \hat{\alpha}_i \sim \ldots \sim \alpha_n)$, where the hat symbol $\hat{}$ over $\alpha_i$ denotes that this cohomology class is removed from the cup product. Then, since

$$i^{*n-1}(\beta) = \sum_{i=1}^n m_i(i^{*1}(\alpha_1) \sim \ldots \sim \hat{i^{*1}(\alpha_i)} \sim \ldots \sim i^{*1}(\alpha_n))$$

and each summand must contain either $i^{*1}(\alpha_1)$ or $i^{*1}(\alpha_2)$, it follows that $i^{*n-1}(\beta) = 0$. Therefore $i^{*n-1}$ is the zero homomorphism and the result follows from Theorem 3.6.

The last part of this section deals with the case of isolated non-saddle continua in closed surfaces. In this context, Theorem 3.6 allows us to get sharp estimates on the complexity of the region of influence of $K$. In addition, Corollary 1 allows us to ensure that some isolated non-saddle continua must be either attractors or repellers.

Proposition 3. Let $K$ be an isolated non-saddle continuum of a flow on a closed surface $M$. If $\text{rk} \check{H}^1(K;\mathbb{Z}_2) = \text{rk} \check{H}^1(M;\mathbb{Z}_2)$ and $K$ does not separate $M$, then the complexity of $\mathcal{I}(K)$ must be zero. Moreover, $K$ is an attractor or a repeller.

Proof. Since $K$ is a non-separating continuum, Alexander duality ensures that $\check{H}^2(M, K;\mathbb{Z}_2) \cong H_0(M \setminus K;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Let us consider the long exact sequence of reduced Čech cohomology of the pair $(M, K)$,

$$0 \to \check{H}^1(M, K;\mathbb{Z}_2) \to \check{H}^1(M;\mathbb{Z}_2) \to \check{H}^1(K;\mathbb{Z}_2) \to \check{H}^2(M, K;\mathbb{Z}_2) \to \check{H}^2(M;\mathbb{Z}_2) \to 0.$$

The previous observation guarantees that the last homomorphism must be an isomorphism. As a consequence, the homomorphism $i^*: \check{H}^1(M;\mathbb{Z}_2) \to \check{H}^1(K;\mathbb{Z}_2)$ induced in 1-dimensional Čech cohomology by the inclusion $i : K \hookrightarrow M$ is surjective and, since $\text{rk} \check{H}^1(K;\mathbb{Z}_2) = \text{rk} \check{H}^1(M;\mathbb{Z}_2)$, it must be an isomorphism. The result follows from Theorem 3.6 and Corollary 1.

Proposition 4. Let $K$ be a connected isolated non-saddle set of a flow defined on a closed, orientable surface $M$ of genus $g$. Then the complexity of $\mathcal{I}(K)$ is at most $g$.

Proof. Since $K$ is a proper subcontinuum of a surface and $\check{H}^1(K;\mathbb{Z}_2)$ is finitely generated [22, Corolario B.9] ensures that $K$ has the shape of a wedge of $r = \text{rk} \check{H}^1(K;\mathbb{Z}_2)$ circumferences. Hence $\check{H}^1(K;G) \cong G \oplus \cdots \oplus G$. Let $i^*: \check{H}^1(M;G)$
The homomorphism induced in 1-dimensional Čech cohomology by the inclusion \( i : K \to M \) is denoted by \( \tilde{H}^1(K;G) \). Then \( \tilde{H}^1(M;G) \cong \ker i^* \oplus \text{im} i^* \) and using the fact that \( \text{rk} \tilde{H}^1(M;G) = 2g \) we get that either the rank of \( \ker i^* \) or the rank of \( \text{im} i^* \) is at most \( g \). The result follows from Theorem 3.6.

The last result of this section shows that the upper bound from Proposition 4 is sharp. Before stating it we introduce some definitions.

**Definition 3.7.** We shall say that a fixed point \( p \) of a flow \( \phi : M \times \mathbb{R} \to M \) defined on a surface is **topologically hyperbolic** if it possesses a neighborhood \( U_p \) such that the flow in \( U_p \) is topologically equivalent to the flow on \( \mathbb{R}^2 \) induced by the vector field \( X(x,y) = (\lambda x, \mu y) \), where \( \lambda \) and \( \mu \) are either +1 or −1. We shall say that \( p \) is a **topologically hyperbolic saddle** whenever \( \lambda = 1 \) and \( \mu = -1 \) or vice versa.

**Definition 3.8.** Under the same assumptions, we shall say that \( p \) is a **degenerate saddle** if it possesses a neighborhood \( U_p \) such that the flow in \( U_p \) is topologically equivalent to a flow in \( \mathbb{R}^2 \) generated by a vector field of the form \( X(x,y) = (\rho(x,y),0) \) where \( \rho : \mathbb{R}^2 \to [0, +\infty) \) is a non-negative smooth function which takes the value 0 only at \((0,0)\) (see Figure 3).

**Proposition 5.** Assume that \( K \) is a connected isolated non-saddle set of a flow defined on a closed, orientable surface \( M \) of genus \( g \). Let \( k_1, \ldots, k_n \) be non-negative integers such that \( g = k_1 + \ldots + k_n \). Then there exists a flow on \( M \) having an isolated non-saddle continuum \( K \) whose region of influence \( I(K) \) has complexity \( g \) and satisfying that \( I(K) \setminus K \) has \( n \) components \( C_1, \ldots, C_n \) with local complexities \( k_1, \ldots, k_n \) respectively. Moreover, the flow can be constructed with the following properties:

1. Every component \( C_i \) with non-zero local complexity has dissonant points.
2. If we denote by \( k, m \) and \( l \) the number of components \( C_i \) with complexities greater, equal, and less than 1 respectively, \( M \setminus I(K) \) is comprised of \( g - k + l \) isolated fixed points where \( g - k - m \) are topologically hyperbolic saddles, \( m \) are degenerate saddle fixed points and \( l \) are attracting fixed points.

**Proof.** Consider non-negative integers \( k_1, \ldots, k_n \) such that \( k_1 + \ldots + k_n = g \) and suppose that \( k_i = 0 \) for every \( i \leq l \) and \( k_i \geq 1 \) if \( i \geq l + 1 \). We first observe that a closed orientable surface of genus \( g \) can be constructed by glueing together the 2-manifolds with boundary \( K, D_1, \ldots, D_l \) and \( H_{l+1}, \ldots, H_n \) where:

1. \( K \) is a sphere with the interiors of \( g + n \) disjoint closed topological disks removed.
2. Each \( D_i \) is a closed topological disk.
Figure 4. Flow on $S^1 \times [0,1]$ which has $S^1 \times \{0\}$ as a repelling circle of fixed points, $S^1 \times \{1\}$ as an attracting circle of fixed points and the point $\{z\} \times \{1/2\}$ as a degenerate saddle fixed point.

3. Each $H_i$ is a sphere with the interiors of $k_i + 1$ disjoint closed topological disks removed.

We obtain a closed surface $M$ by attaching to $K$ the $D_i$ and the $H_i$ in such a way that each $D_i$ caps a hole of $K$ and each $H_i$ connects $k_i + 1$ holes of $K$. This can be done so that the closed surface $M$ obtained is orientable. The Euler characteristic of this surface is

$$
\chi(M) = \chi(K) + \sum_{i=1}^{l} \chi(D_i) + \sum_{i=l+1}^{n} \chi(H_i) = (2 - g - n) + l + \sum_{i=l+1}^{n} (1 - k_i)
$$

$$
= (2 - g - n) + l + (n - l) - g = 2 - 2g
$$

and, as a consequence, $M$ is a closed orientable surface of genus $g$.

We shall define a flow $\varphi : M \times \mathbb{R} \to M$ with the desired properties. We assume that $\varphi$ is stationary on $K$. As a consequence, the $D_i$ and the $H_i$ are also invariant and the flow is stationary on their boundaries.

Each disk $D_i$ comes equipped with a homeomorphism $h_i : D_i \to D$ to the closed unit disk $D \subset \mathbb{R}^2$. To define the flow on each $D_i$ we pull back using $h_i^{-1}$ any flow on $D$ that is stationary on $\partial D$ and has the pair $\{(0), \partial D\}$ as an attractor-repeller decomposition.

To define the flow on the $H_i$ we have to distinguish two different situations: (A) $k_i = 1$ and (B) $k_i > 1$. To construct the flow in case (A) observe that if $k_i = 1$ there exists a homeomorphism $\tilde{h}_i : H_i \to S^1 \times [0,1]$. We fix a point $z \in S^1$ and consider in $S^1 \times [0,1]$ a flow which is stationary on the boundary $S^1 \times \{0,1\}$ and such that the trajectories of points in $(S^1 \times \{0,1\}) \setminus (z \times (0,1))$ move from $S^1 \times \{0\}$ to $S^1 \times \{1\}$ along the fibers while the fiber $\{z\} \times (0,1)$ is broken into three orbits, covering $\{z\} \times (0,1/2)$, $\{z\} \times \{1/2\}$ and $\{z\} \times (1/2,1)$ respectively (see Figure 4). We pull back this construction to $H_i$ using $\tilde{h}_i^{-1}$. 
Figure 5. Flow defined on a sphere with the interior of four closed topological disks removed. This flow has a Morse decomposition \( \{M_1, M_2, M_3\} \) where \( M_1 \) is the attracting outer circle of fixed points, \( M_2 \) is the union of two topologically hyperbolic saddle fixed points and \( M_3 \) is the union of the three repelling inner circles of fixed points.

In case (B) we consider a flow on \( H_i \) that is stationary on \( \partial H_i \) and has a Morse decomposition \( \{M_1, M_2, M_3\} \) where \( M_1 \) is an attracting boundary component of \( H_i \), \( M_2 \) is the union of \( k_i - 1 \) topologically hyperbolic saddle fixed points and \( M_3 \) is a repeller comprised of the remaining boundary components of \( H_i \) (see Figure 5). This flow can be constructed by modifying the gradient flow induced by the height function on a sphere \( S \subset \mathbb{R}^3 \) embedded in such a way that it has \( k_i \) maxima, all of them contained in the same level set, \( k_i - 1 \) saddle critical points, all of them in the same level set, and one minimum (see Figure 6).

It is clear from the construction that \( K \) is an isolated non-saddle set and that if we denote by \( L \) the set comprised of the isolated fixed points, \( \mathcal{I}(K) = M \setminus L \). In addition, \( \mathcal{I}(K) \setminus K \) is the disjoint union of \( \hat{D}_1, \ldots, \hat{D}_l, \hat{H}_{l+1}, \ldots, \hat{H}_n \), where the symbol \( \hat{\ } \) indicates that we are removing the fixed points. By construction each component \( \hat{D}_i \) has local complexity \( k_i = 0 \) and each component \( \hat{H}_i \) has local complexity \( k_i \geq 1 \). As a consequence, \( \mathcal{I}(K) \) has complexity \( g \). Moreover, each \( \hat{H}_i \) contains the sets \( W^u(p_i) \setminus \{p_i\} \) and \( W^s(p_i) \setminus \{p_i\} \) for some isolated saddle fixed point \( p_i \), which are comprised of dissonant points for \( K \). The number of attracting, topologically hyperbolic saddle and degenerate saddle fixed points is clear from the construction.

As it happened with Theorem 3.1, the connectedness assumption on the isolated non-saddle set \( K \) plays a crucial role in Theorem 3.6. In fact, the construction carried out in the proof of Proposition 5 can be adapted to find flows on \( S^2 \) that have a non connected isolated non-saddle \( K \) whose region of influence has arbitrarily
large complexity. More specifically, it is possible to achieve that \( I(K) \setminus K \) has as many components as desired with prescribed local complexities.

4. Dynamical and homological robustness of isolated non-saddle sets. In this section we study necessary and sufficient conditions for the preservation of the property of being non-saddle by continuations. We start by recalling the basic notions of continuation theory.

Let \( M \) be an \( n \)-dimensional manifold. We say that the family of flows \( \varphi_{\lambda} : M \times \mathbb{R} \to M \), with \( \lambda \) in the unit interval \( I \), is a \emph{continuous parametrized family of flows} if the map \( \varphi : M \times \mathbb{R} \times I \to M \) given by \( \varphi(x, t, \lambda) = \varphi_{\lambda}(x, t) \) is continuous. In this context, the family \( (K_\lambda)_{\lambda \in J} \), where \( J \subset [0, 1] \) is a closed (non-degenerate) subinterval and, for each \( \lambda \in J \), \( K_\lambda \) is an isolated invariant set for \( \varphi_{\lambda} \) is said to be a \emph{continuation} if any of the following two equivalent conditions are satisfied:

(i) For each \( \lambda_0 \in J \) and each \( N_{\lambda_0} \) isolating neighborhood for \( K_{\lambda_0} \), there exists \( \delta > 0 \) such that \( N_{\lambda_0} \) is an isolating neighborhood for \( K_\lambda \) for every \( \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap J \).

(ii) For each \( \lambda_0 \in J \) there exist an isolating neighborhood \( N_{\lambda_0} \) for \( K_{\lambda_0} \) and a \( \delta > 0 \) such that \( N_{\lambda_0} \) is an isolating neighborhood for \( K_\lambda \), for every \( \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap J \).

If the family \( (K_\lambda)_{\lambda \in J} \) satisfies any of the equivalent conditions (i) and (ii) we say that it is a continuation of \( K_{\lambda_0} \) for each \( \lambda_0 \in J \).
[20, Lemma 6.1] ensures that if \( K_{\lambda_0} \) is an isolated invariant set for \( \varphi_{\lambda_0} \), there always exists a continuation \( (K_\lambda)_{\lambda \in J_{\lambda_0}} \) of \( K_{\lambda_0} \) for some closed (non-degenerate) subinterval \( \lambda_0 \in J_{\lambda_0} \subset [0, 1] \). Notice that the possibility of some (or even all) of the \( K_\lambda \) to be empty is not excluded from the definition. The equivalence between (i) and (ii) follows from [20, Lemma 6.1] and [20, Lemma 6.2].

It is well known (see [25, Theorem 4]) that attractors are robust from the dynamical and topological points of view since an attractor \( K \) locally continues to a family of attractors with the shape of \( K \). This is not the case for isolated non-saddle sets as shown in [13] (see Figure 7). In fact, neither the property of being non-saddle nor the topological properties of the original isolated non-saddle set are preserved by local continuation. However, it turns out that there exist some relations between the preservation of certain topological properties by continuation and the preservation of the dynamical property of non-saddleness. For instance, if the phase space is a surface [3, Theorem 26] or a closed orientable smooth manifold with trivial integral first cohomology group [6, Theorem 39], the property of being non-saddle is preserved by continuation if and only if the shape is preserved.

From now on we are going to consider smooth parametrized families of flows defined on smooth manifolds. That is \( M \) is a smooth manifold and the map \( \varphi : M \times \mathbb{R} \times I \to M \) given by \( \varphi(x, t, \lambda) = \varphi_\lambda(x, t) \) is smooth. It is quite likely that the following results hold without the smoothness assumptions but so far we do not know how to overcome this difficulty.
The next result can be extracted from the proof of \cite[Theorem 5]{13} and is a consequence of the robustness properties of transversality.

**Lemma 4.1.** Let $\varphi : M \times \mathbb{R} \to M$ be a smooth parametrized family of flows (parametrized by \( \lambda \in I \), the unit interval) defined on an \( n \)-dimensional smooth manifold. Suppose that $K_0$ is a connected isolated non-saddle set for $\varphi_0$ and that $(K_\lambda)_{\lambda \in [0, \delta]}$ is a continuation of $K_0$ for some $\delta > 0$ such that each $K_\lambda$ is non-empty. Suppose that $N = N^+ \cup N^-$ is an isolating block manifold for $K_0$. Then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ the compactum $N$ is an isolating block for $K_\lambda$ with the same entrance and exit sets as for $\varphi_0$ and is of the form $N = N^+ \cup N^-$ whenever $K_\lambda$ is non-saddle.

The following proposition gives a necessary and sufficient condition for the robustness of non-saddleness.

**Proposition 6.** Let $\varphi_\lambda$, with $\lambda \in [0, 1]$, be a smooth parametrized family of flows on a smooth \( n \)-manifold $M$ and let $K_0$ be a connected isolated non-saddle set of $\varphi_0$. Suppose $K_0$ continues to a family $(K_\lambda)_{\lambda \in [0, \delta]}$ of non-empty isolated invariant compacta for some $\delta > 0$. Then $K_\lambda$ is non-saddle for $\lambda > 0$ sufficiently small if and only if there exists a connected isolating block manifold $N$ of $K_0$ such that $N$ isolates $K_\lambda$ and each component of $N \setminus K_\lambda$ contains exactly one component of $\partial N$.

**Proof.** Suppose that $K_\lambda$ is non-saddle for $\lambda$ sufficiently small. By \cite[Theorem 1.5]{11} we can find a connected isolating block manifold $N$ that isolates $K_0$. Then, Lemma 4.1 ensures that there exists $\lambda_0 > 0$ such that, for $\lambda \in [0, \lambda_0)$, $N$ is an isolating block of the form $N = N^+ \cup N^-$ for $K_\lambda$. The necessity follows from the fact that $\partial N$ is a deformation retract of $N \setminus K_\lambda$.

Conversely, suppose that there exists a connected isolating block manifold $N$ of $K_0$ and some $\lambda_1 > 0$ such that, for every $\lambda \in [0, \lambda_1)$, $N$ isolates $K_\lambda$ and each component of $N \setminus K_\lambda$ contains exactly one component of $\partial N$. Lemma 4.1 guarantees that there exists $\lambda'_1 > 0$ such that $N$ is an isolating block of $K_\lambda$ for $\lambda \in (0, \lambda'_1)$ satisfying that the entrance and exit sets for $\varphi_\lambda$ are disjoint since they agree with those for $\varphi_0$. We may assume that in fact $\lambda'_1 = \lambda_1$. Suppose that $K_\lambda$ is saddle for some $\lambda \in (0, \lambda_1)$. Then there exists $x_0 \in N \setminus (N^+ \cup N^-)$. The trajectory segment of $x_0$ in $N$ is a path in $N \setminus K_\lambda$ joining a component of $N^1$ with a component of $N^0$ which, by the previous discussion must be different components of $\partial N$. This contradiction proves the converse statement.

**Corollary 4.** Suppose $\varphi_\lambda : M \times \mathbb{R} \to M$ is a smooth parametrized family of flows defined on a $n$-dimensional smooth manifold $M$. Let $K_0$ be a connected isolated non-saddle set for $\varphi_0$ and $(K_\lambda)_{\lambda \in [0, \delta]}$ a continuation of $K_0$. Suppose that there exists $\lambda_0 > 0$ such that, for $\lambda \in [0, \lambda_0)$, $K_0 \subset K_\lambda$. Then $K_\lambda$ is non-saddle for $\lambda > 0$ sufficiently small.

The following result is a reformulation in terms of Čech cohomology of \cite[Theorem 39]{6}.

**Theorem 4.2.** Let $\varphi_\lambda$, with $\lambda \in [0, 1]$, be a smooth parametrized family of flows on a $G$-orientable smooth $n$-manifold $M$ with $H^1(M; G) = 0$ and $K_0$ be a connected isolated non-saddle set for $\varphi_0$ that continues to a family $(K_\lambda)_{\lambda \in [0, \delta]}$ of non-empty isolated invariant compacta. Then $K_\lambda$ is non-saddle for $\lambda > 0$ sufficiently small if and only if $\tilde{H}^k(K_0; G) \cong \tilde{H}^k(K_\lambda; G)$ for every $k \geq 0$. 
Observe that in this reformulation we have removed the assumption about the phase space being compact and connected and we replaced the orientability by $G$-orientability. The proof is almost the same so we do not reproduce it here. The only differences are that to prove Theorem 4.2 one has to work with the connected component that contains $K_0$ and use Theorem 3.1 instead of [6, Theorem 39].

Theorem 4.2 establishes that the robustness of a topological property, namely Čech cohomology, is equivalent to the robustness of the dynamical property of non-saddleness if the phase space is a smooth manifold with trivial first cohomology group. So far we are unable to establish an equivalence between the robustness of non-saddleness and the robustness of the Čech cohomology for isolated non-saddle continua without further assumptions. However, we can prove the equivalence between the robustness of non-saddleness with a strong form of the robustness of Čech cohomology.

**Theorem 4.3.** Suppose $\varphi_\lambda : M \times \mathbb{R} \to M$ is a smooth parametrized family of flows defined on a connected smooth $n$-dimensional manifold $M$. Let $K_0$ be a connected isolated non-saddle set for $\varphi_0$ and suppose that $K_0$ continues to a family $(K_\lambda)_{\lambda \in [0,\delta)}$ of non-empty isolated invariant compacta. Then $K_\lambda$ is non-saddle for $\lambda > 0$ sufficiently small if and only if there exists a connected isolating block $N$ of $K_0$ such that $N$ isolates $K_\lambda$ and the inclusion $i_\lambda : K_\lambda \hookrightarrow N$ induces isomorphisms in $k$-dimensional Čech cohomology with $\mathbb{Z}_2$ coefficients for every $k \geq 0$.

**Proof.** Suppose that $K_\lambda$ is non-saddle for $\lambda > 0$ sufficiently small. Consider an isolating block manifold $N$ of $K_0$ that exists because of [11, Theorem 1.5]. Notice that the connectedness of $N$ ensures that $N = N^+ \cup N^-$. Since $K_\lambda$ is non-saddle, Lemma 4.1 ensures that $N$ is an isolating block for $K_\lambda$ of the form $N = N^+ \cup N^-$ and, hence, the inclusion $i_\lambda : K_\lambda \hookrightarrow N$ induces isomorphisms in $k$-dimensional Čech cohomology with $\mathbb{Z}_2$ coefficients for every $k \geq 0$ as desired.

Conversely, suppose that there exists a connected isolating block $N$ of $K_0$ such that $N$ isolates $K_\lambda$ for every $\lambda$ smaller that $\lambda_0 > 0$ and that the inclusion $i_\lambda : K_\lambda \hookrightarrow N$ induces isomorphisms in $k$-dimensional Čech cohomology with $\mathbb{Z}_2$ coefficients for every $k \geq 0$. Let $N'$ be a connected isolating block manifold for $K_0$ contained in $N$ that exists because of the proof of [11, Theorem 1.5]. Notice that $N$ and $N'$ are of the form $N^+ \cup N^-$ and $N'^+ \cup N'^-$ respectively. Hence, the flow $\varphi_0$ provides a deformation retraction from $N$ onto $N'$. We may assume that $N'$ isolates $K_\lambda$ for $\lambda \in [0,\lambda_0)$ since, otherwise, we only have to choose a smaller $\lambda_0$. Since the inclusion $i_\lambda : K_\lambda \hookrightarrow N$ is the composition of the inclusions $i'_\lambda : K_\lambda \hookrightarrow N'$ and $j : N' \hookrightarrow N$ and $i_\lambda$ and $j$ induce isomorphisms in $k$-dimensional Čech cohomology for every $k \geq 0$, it follows that $i'_\lambda$ also induces isomorphisms in $k$-dimensional Čech cohomology for every $k \geq 0$. As a consequence, $H^k(N', K_\lambda; \mathbb{Z}_2) = 0$ for every $k \geq 0$ and $\lambda \in [0,\lambda_0)$. Then, Alexander duality ensures that $H_{n-k}(N' \setminus K_\lambda, \partial N'; \mathbb{Z}_2) = 0$ for each $k$ and from the terminal part of the long exact sequence of singular homology of the pair $(N' \setminus K_\lambda, \partial N')$, we deduce that the homomorphism $i'_{\lambda*} : H_0(\partial N'; \mathbb{Z}_2) \to H_0(N' \setminus K_\lambda; \mathbb{Z}_2)$, induced in 0-dimensional singular homology by the inclusion $i'_\lambda : \partial N' \hookrightarrow N' \setminus K_\lambda$, is an isomorphism for each $\lambda \in [0,\lambda_0)$. Therefore, each component of $N' \setminus K_\lambda$ contains exactly one component of $\partial N'$ and the result follows from Proposition 6.

**Remark 3.** Notice that both Theorem 4.2 and Theorem 4.3 can be established in shape theoretical terms by changing the isomorphisms by shape equivalences in both statements.
Figure 8. Flow on a double torus having an isolated non-saddle set $K''$ whose region of influence has complexity zero but after an arbitrarily small perturbation becomes topologically equivalent to the flow depicted in Figure 2.

Although we shall not discuss it thoroughly in the present paper, it would be interesting to study how the structure of $I(K)$ and the quantities introduced in Section 3 behave under small perturbations. Notice that the global character of these features makes it more difficult to handle since some control over the phenomena that may appear “far” from the isolated non-saddle set is needed. The following example shows that arbitrarily small perturbations may reduce the number of components of $I(K) \setminus K$ increasing the complexity of $I(K)$ after arbitrarily small perturbations.

**Example 4.4.** Consider the flow $\psi : M \times \mathbb{R} \rightarrow M$ defined on a closed orientable surface of genus 2, depicted in Figure 8. In this case, as it happens in Example 3.5, there is an isolated non-saddle continuum $K''$ comprised of fixed points which is homeomorphic to a sphere with the interiors of four disjoint closed topological disks removed. In addition, there are two isolated non-saddle circles $K_1$ and $K_2$ comprised of fixed points. In this case $K''$ and $I(K'') \setminus K''$ has four connected components $C_1''$, $C_2''$, $C_3''$ and $C_4''$. The components $C_1''$, $C_2''$ and $C_3''$ are purely repelled by $K''$ while the component $C_4''$ is purely attracted by $K''$. Hence, Proposition 1 ensures that the local complexity of each $C_i''$ is zero. As a consequence, the complexity of $I(K'')$ is zero. However, the isolated non-saddle circles $K_1$ and $K_2$ may be removed after an arbitrarily small perturbation in such a way that the obtained flow is topologically equivalent to the flow $\hat{\psi}$ from Example 3.5.

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