Oscillator Laser Model

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A laser model is formulated in terms of quantum harmonic oscillators. Emitters in the low lasing states are usual harmonic oscillators, and emitters in the upper states are inverted harmonic oscillators. Diffusion coefficients, consistent with the model and necessary for solving quantum nonlinear laser equations analytically, are found. Photon number fluctuations of the lasing mode and fluctuations of the population of the lasing states are calculated. Collective Rabi splitting peaks are predicted in the intensity fluctuation spectra of the superradiant lasers. Population fluctuation mechanisms in superradiant lasers and lasers without superradiance are discussed and compared with each other.

1. Introduction

Recent technological developments have led to a variety of novel miniature lasers as quantum dot photonic crystal,[1–7] micropillar,[8–10] plasmonic[11] and other kinds of nanolasers.[12] All of them are intensively investigated. The motivation for research of nanolasers is related to fundamental questions, such as general properties of quantum fields with only a few photons in the mode and with practical applications, such as the direct incorporation of lasers into nano-chips.[13–15] Nanolasers often demonstrate exciting and unusual properties, such as the mode-locking with a repetition rate independent of the cavity length.[16,17]

There is a strong interest in superradiant (SR) lasers and nanolasers. Such lasers combine a high gain with a small cavity and operate in the bad-cavity regime.[18–20] The active medium polarization is a dynamical variable in such lasers, where a collective spontaneous emission into the lasing mode is significant. SR lasers have been experimentally realized with cold alkaline earth atoms,[21–24] rubidium atoms,[25] and quantum dots.[26] as the active medium. SR lasers are less sensitive to the cavity-length fluctuations important for atomic clocks.[21,22,25] Superradiance leads to interesting collective effects, such as excitation trapping[23,25] and superthermal photon statistics,[10,26,27] with possible applications in high-visibility optical imaging.[28]

Analytical description of SR lasers meets difficulties. Quantum noise in such lasers is not a perturbation, the laser equations are nonlinear, and the active medium polarization is a dynamical variable not eliminated adiabatically. Strong population fluctuations in SR lasers play a significant role in laser dynamics leading to large relaxation oscillations[29] and the acceleration of spontaneous emission into the lasing mode.[30] High population fluctuations at superradiance were noted previously, for example, at a generation of superradiant pulses.[31]

The first purpose of the present work is to continue calculations and investigation of the population fluctuations in SR lasers at a low excitation using the approach of refs. [29, 30, 32, 33]. The laser at a low excitation does not generate coherent radiation and operates as a LED or, if the collective spontaneous emission (the superradiance) into the lasing mode is significant, as SR LED.

Calculations and investigation of the population fluctuations are necessary for better understanding properties of SR LEDs and lasers, for solving nonlinear laser equations,[10] and for finding quantum characteristics of nanolasers as, for example, the auto-correlation function $g_2$ of the lasing field, widely used in experiments and the theory.[14]

Our previous papers,[32,33] neglected population fluctuations at low laser excitation. In ref. [29], population fluctuations have been taken into account at a high excitation when the laser generates coherent radiation then laser equations can be solved by the usual perturbation procedure. The procedure of ref. [29] cannot account for population fluctuations at a low laser excitation. The paper[30] describes the treatment of nonlinearities and population fluctuations at a low laser excitation. However, the analysis in ref. [30] was simplified and restricted by a very low laser excitation when population fluctuations determined by only the pump and the decay of the upper lasing states but practically do not depend on the lasing field.

The restriction of analysis in ref. [30] appears because of the difficulty in finding diffusion coefficients consistent with the approximate equations. It turns out that the use of well-known diffusion coefficients found from generalized Einstein relations (GER) leads to inconsistencies in the results of the approximate equations, for example, the breaking of Bose-commutation relations for the lasing field operator.[29] The fact that different approximate approaches in quantum optics require different diffusion coefficients is well-known. For example, some diffusion coefficients change in the transition from operator to number Langevin equations – see the discussion in section 12.3 in ref. [35].

Another purpose of this paper, necessary for continuing research,[30] is to find diffusion coefficients consistent with our approximate equations. It will be done with the help of an oscillator laser model (OLM), representing the laser as a combination of the usual and inverted harmonic oscillators. The model of...
Table 1. Definitions of operators and parameters.

| Symbol | Definition | Eq. | Symbol | Definition | Eq. |
|--------|------------|-----|--------|------------|-----|
| \( \hat{a} \) | lasing mode amplitude of \( i \)th emitter | (1) | \( \hat{a}^\dagger \) | rising (lower) operator of \( i \)th emitter | (1) |
| \( \hat{N}_e \) (\( \hat{N}_g \)) | operator of population of excited (ground) state of \( i \)th emitter | (2) | \( \nu \) | active medium polarization operator | (3), (43) |
| \( \hat{F}_a \) | Langlev force operator | (3) | \( N \) | mean population inversion | Section 1 |
| \( \hat{a} (\omega) \) | Fourier-component operator for \( \hat{a}(\tau) \), \( a = \{ a, v, N_e, N_g, \ldots \} \) | (4), (5), etc. | \( \Sigma \) | field-polarization interaction energy operator | (7), (50) |
| \( \hat{\Sigma} \) | photon number operator | (25) | \( \hat{D} \) | dipole–dipole interaction energy operator | (28) |
| \( \hat{b}_i (\xi) \) | Bose-operator for \( i \)th emitter in the excited (ground) state | (36) | \( \hat{b} (\xi) \) | Bose-operator for \( N_e \) (\( N_g \)) emitters in the excited (ground) state | (37) |
| \( \hat{A}_m \) | laser field bath Bose-operator | (39) | \( \hat{b}_{m, \xi_m} \) | polarization baths Bose-operators | (40), (41) |
| \( \hat{D}_j \) | dipole–dipole interaction energy in terms of Bose-operators | (55) | \( \Gamma \) | interaction with bath environments | (1), (36) |
| \( \hat{\sigma}(\omega) \) | Fourier-component operator for fluctuations \( \hat{\sigma}(\tau) \), \( a = \{ n, \Sigma, D, \ldots \} \) | (19) | \( \delta \hat{a}(\omega) \) | Fourier-component operator for fluctuations \( \delta \hat{a}(\xi) \), \( a = \{ n, \Sigma, D, \ldots \} \) | (59) |
| \( \Omega \) | vacuum Rabi frequency | (1) | \( f_i \) | normalized coupling strength of \( i \)th emitter with lasing mode | (1) |
| \( f_{\text{b}} (\xi_{\text{b}}) \) | normalized coupling strength of \( \text{bath} \) emitter in excited (ground) state | (36) | \( f_{\text{b}} (\xi_{\text{b}}) \) | average normalized coupling strength of emitter in excited (ground) state | (37) |
| \( \xi_{\text{b}} \) | average emitter-field coupling strength | (1), (48) | \( \gamma_{\xi} \) | population decay rate | (5) |
| \( N_0 \) | total number of emitters | (1) | \( \kappa \) | lasing mode amplitude decay rate | (4) |
| \( \Gamma_{\text{b}} \) | upper level population decay rate | (6) | \( P \) | upper level dimensionless pump rate, normalized to \( \Gamma_{\text{b}} \) | (6) |
| \( 2D_{e0} \) | diffusion coefficient for Langlev force | (17) | \( \gamma_{\rho} \) | population fluctuation decay rate | (12) |
| \( N_{\text{th}} \) | threshold population inversion in semiclassical laser theory | (21) | \( n(\omega) \) | laser field spectrum | (20) |
| \( \omega_{\text{opt}} \) | optical carrier frequency | (1), (36) | \( \omega \) | deviation from \( \omega_{\text{opt}} \) in \( n(\omega) \); frequency of intensity, etc. fluctuations | (19), (59) |
| \( \omega_{\text{opt}} \) | optical field frequency | after (21) | \( \omega_{\text{opt}} \) | \( \omega_{\text{opt}} \) | (21) |
| \( \delta \hat{N}_e \) | Fourier-component of \( \delta \hat{N}_e \) | (22) | \( S_{\omega_{\text{opt}}} (n) \) | power spectrum of \( \delta \hat{N}_e \) | (22), (23) |
| \( \delta^2 n(\omega) \) | photon number fluctuation spectrum | (60), (A3) | \( \delta^2 n(\omega) \) | photon number fluctuation dispersion | (63) |
| \( \delta^2 N_e (\omega) \) | population fluctuation spectrum | (61) | \( \delta^2 N_e (\omega) \) | population fluctuation dispersion | (64) |
| \( S_{\omega_{\text{opt}}} (n) \) | auxiliary function | (61), (A4) | \( S(\omega) \) | auxiliary function | (61), (A5) |

optical media as a set of oscillators is used widely in nonlinear optics.\(^{36,37}\) Glauber introduced inverted oscillators for the modeling pump bath.\(^{19}\) Inverted oscillators found applications in the quantum theory of linear amplifiers.\(^{39}\) There is a difference between the quantum theory of linear amplifiers and OLM. OLM does not eliminate the active medium polarization adiabatically, quantum theory of linear amplifiers does this.

OLM leads to the same quantum laser equations as in refs.\(^{29, 30, 33}\), finds correct diffusion coefficients in the frame of the input–output theory.\(^{40,41}\) and preserves Bose commutation relations for the field operators. The results of OLM let us find population fluctuations in the first-order approximation.

Using approximate analytical equations with appropriate diffusion coefficients, we find and investigate the photon number and the population fluctuations, their spectra, and variances; describe sources of population fluctuations and compare population fluctuations in the SR LEDs and the LEDs without SR.

Section 2 is introductory. There we describe the exact and approximate two-level laser models. In Section 2.1, we describe the approximate approach and derive Equations (30)–(32) for fluctuations of binary operators.

Section 3 describes the oscillator laser model.

Section 4 presents the calculation of diffusion coefficients for Equations (30)–(32) consistent with the approximation when population fluctuations are neglected. There we find the population fluctuation spectrum in the first-order approximation.

Section 5 contains results about the population fluctuation spectra and variances; describes the influence of population fluctuations on the field intensity fluctuation spectra. This section discusses the mechanisms of population fluctuations and the comparison of population fluctuations in SR LEDs and LEDs without SR.

Concluding Section 6 summarizes the results.

For convenience, Table 1 shows definitions of operators, variables, and parameters with references to equations where the operator or the parameter appears in the main text. Table 1 does not include fluctuation operators, denoted by the symbol \( \delta \), and mean values marked by the same letter as the operator. For
2. Quantum Two-Level Laser Model

We consider a lasing medium with a large number \( N_0 \gg 1 \) of the two-level identical emitters in the optical cavity with the cavity mode resonant to the emitter transitions, shown schematically in Figure 1.

Emitters are fixed in space as in quantum dot lasers.\(^{26,42}\) We investigate the stationary case when, on average, \( N_0 \) emitters are in their ground states and \( N_e \) emitters are in their excited states. The incoherent pump maintains the number of emitters in the excited states and the stationary population inversion \( N = N_e - N_g \), compensating for the energy losses.

We consider a weak excitation of the lasing medium when: a) the number of photons in the cavity is less or of the order of one and b) there is no population inversion in the lasing medium. There is no generation of coherent radiation in such a regime, so the laser operates as a LED. We will call it a weak pump—the pump providing the weak excitation regime.

We write Hamiltonian of the two-level laser, similar to the one used (maybe in different notations) in many papers and books, as refs. \([29, 30, 32, 33, 35, 43, 44]\), written in the interaction picture with the carrier frequency \( \omega_0 \) and the RWA approximation

\[
H = i\hbar \Omega \sum_{i=1}^{N_0} f_i (\hat{a}^\dagger \hat{\sigma}_i - \hat{\sigma}_i^\dagger \hat{a}) + \hat{\Gamma} \tag{1}
\]

Here \( \Omega \) is the vacuum Rabi frequency; \( f_i \) describes the difference in couplings of different emitters with the lasing mode, \( \hat{a} \) is the Bose operator of the lasing mode amplitude, \( \hat{\sigma}_i \) is the lowing operator of \( i \)-th emitter,\(^{43}\) \( \hat{\Gamma} \) describes the interaction of the lasing mode and emitters with the white noise baths of the environment.

Non-zero commutation relations for operators are\(^{29,30,32,33,43}\)

\[
[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{\sigma}_i, \hat{\sigma}_j^\dagger] = (\hat{n}_i^g - \hat{n}_j^g) \delta_{ij}, \quad [\hat{\sigma}_i, \hat{n}_j^g] = (\hat{n}_j^g - 1) \delta_{ij} \quad \delta_{ij} = \delta_i^\dagger \delta_j \tag{2}
\]

where \( \hat{n}_i^g \) and \( \hat{n}_i^e \) are operators of populations of the upper and the low levels of the \( i \)-th emitter, \( \delta_{ij} \) is the Kronecker symbol.

Following, refs. \([29, 30, 33]\) we introduce operators \( \hat{\nu} \) and \( \hat{N}_{c,g} \) of the polarization and populations of all emitters

\[
\hat{\nu} = \sum_{i=1}^{N_0} f_i \hat{\sigma}_i, \quad \hat{N}_{c,g} = \sum_{i=1}^{N_0} \hat{\sigma}_i^{c,g} \tag{3}
\]

Using commutation relations in Equation (2) and Hamiltonian (1), we write Maxwell–Bloch equations for \( \hat{a}, \hat{\nu} \) and \( \hat{N}_g \)

\[
\dot{\hat{a}} = -\kappa \hat{a} + \Omega \hat{\nu} + \hat{F}_a \tag{4}
\]

\[
\dot{\hat{\nu}} = - (\gamma_L / 2) \hat{\nu} + \Omega f a (2 \hat{N}_g - \hat{N}_e) + \hat{F}_\nu \tag{5}
\]

\[
\dot{\hat{N}}_g = -\Omega \hat{\Sigma} + \gamma_1 [P(N_0 - \hat{N}_g) - \hat{N}_e] + \hat{F}_N \tag{6}
\]

where

\[
\hat{\Sigma} = \hat{a}^\dagger \hat{\nu} + \hat{\nu}^\dagger \hat{a} \tag{7}
\]

\( \kappa, \gamma_L \) and \( \gamma_1 \) are decay rates, \( P \gamma_0 \) is the pump rate. We call dimensionless \( P \) a normalized pump; \( \hat{F}_a \) with the index \( a = \{a, \nu, N_e\} \) are Langevin forces of the white noise baths. The total number of emitters is preserved, so \( \hat{N}_e + \hat{N}_g = N_0 \). In Equations (4)–(6) we approximate

\[
f^2 \approx f = N_0 \sum_{i=1}^{N_0} N_i^2 \tag{8}
\]

Physically, \( \hat{\Sigma} \) is the operator of a normalized field-polarization interaction energy analogous to the field-assisted polarization.\(^{26,45}\)

An operator similar to \( \hat{\Sigma} \) has been used previously, for example, in ref. \([46]\).

We rewrite Equations (4)–(6) in the form convenient for our calculations. We separate the operator mean values and fluctuations

\[
\hat{N}_{c,g} = N_{c,g} + \delta \hat{N}_{c,g}, \quad \hat{\Sigma} = \Sigma + \delta \hat{\Sigma} \tag{9}
\]

and write instead of Equations (4)–(6)

\[
\dot{\hat{a}} = -\kappa \hat{a} + \Omega \hat{\nu} + \hat{F}_a \tag{10}
\]

\[
\dot{\hat{\nu}} = - (\gamma_L / 2) \hat{\nu} + \Omega f (\hat{a} N + 2 \hat{a} \delta \hat{N}_g) + \hat{F}_\nu \tag{11}
\]

\[
\delta \hat{N}_g = -\Omega \delta \hat{\Sigma} - \gamma_\rho \delta \hat{N}_g + \hat{F}_N \tag{12}
\]

where \( \gamma_\rho = \gamma_0 (P + 1) \) and

\[
0 = -\Omega \Sigma + \gamma_1 [P(N_0 - \hat{N}_g) - \hat{N}_e] \tag{13}
\]

Diffusion coefficients for correlations of Langevin forces in Equations (10)–(12) are well-known\(^{46,47}\) and derived from generalized Einstein relations.\(^{48}\)
We take the stationary mean photon number \( n = \langle \hat{a}^\dagger \hat{a} \rangle \) and find from Equation (10) \( 0 = -2\kappa n + \Omega \Sigma \). The last equation and Equation (13) lead to the energy conservation law

\[
2\kappa n = \rho \left[ P(N_e - N_h) - N_h \right]
\]

(14)

Nonlinear quantum equations (10)–(12), supplemented by Equations (7), (13), and (14) are our basic set of equations.\(^{[29,30,33]}\) This set is exact (in the frame of the two-level laser model), but it is hard to solve it analytically. We consider the stationary case and solve this set of equations by the approximate analytical approach described in the next subsection.

2.1. Approximate Analytical Approach

One can find an approximate analytical solution of Equations (10)–(12) neglecting population fluctuations. We call this approach a zero-order approximation respectively to population fluctuations or simply a zero-order approximation. Taking \( \delta N_h = 0 \), we reduce the set of Equations (10)–(12) to two linear, on \( \hat{a} \) and \( \hat{v} \), equations

\[
\frac{d}{dt} \hat{a} = -\kappa \hat{a} + \Omega \hat{v} + \hat{F}_a
\]

(15)

\[
\frac{d}{dt} \hat{v} = -\left(\gamma_+ / 2\right) \hat{v} + \Omega N_h \hat{a} + \hat{F}_v
\]

(16)

Non-zero diffusion coefficients \( 2D_{a\beta} \) for Langevin forces in

\[
\langle \hat{F}_a(t)\hat{F}_a(t') \rangle = 2D_{a\beta} \delta(t - t')
\]

(17)

in Equations (15), (16) are\(^{[29]}\)

\[
2D_{a\sigma} = 2\kappa, \quad 2D_{v\sigma} = \gamma_+ N_h, \quad 2D_{v\nu} = \gamma_+ N_e \]

(18)

Diffusion coefficients \( 2D_{a\sigma} \) and \( 2D_{v\nu} \) are different from the “exact” ones found from GER when \( N_{e.g.} \) are operators. We call diffusion coefficients in Equation (18), and the others found for the zero-order approximation equations as zero-order diffusion coefficients. With diffusion coefficients (Equation (18)), Bose commutation relations for \( \hat{a} \) found from Equations (15) and (16) are preserved.

One can solve linear Equations (15) and (16) by the Fourier-transform

\[
\hat{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{a}(\omega)e^{-i\omega t} d\omega
\]

(19)

where \( a = \{ a, v \} \) and calculate the field spectrum \( n(\omega) \)

\[
\langle \hat{a}^\dagger(\omega)\hat{a}(\omega') \rangle = n(\omega)\delta(\omega + \omega')
\]

(20)

It was found in ref.\(^{[29]}\) that

\[
n(\omega) = \frac{(\kappa \gamma_+ / 2) N_e / N_{th}}{[(1 - N_e / N_{th})(\kappa\gamma_+ / 2 - \omega^2)^2 + \omega^2(\kappa + \gamma_+ / 2)^2]}
\]

(21)

where \( N_{th} = 2\Omega^2 / \kappa \gamma_+ \) is the threshold population inversion found in the semiclassical laser theory. The lasing field operator is \( \hat{e} e^{-i\omega t} \), so the frequency \( \omega \) in Equations (19)–(21) is the deviation from the optical carrier frequency \( \omega_0 \). The field spectrum in Equation (21), expressed in terms of the optical frequency \( \omega_{opt} = \omega + \omega_0 \), is \( n(\omega_{opt} - \omega_0) \).

Figure 3 shows examples of the field spectra \( n(\omega_{opt} - \omega_0) \).

With the help of Equation (21), we find the mean photon number \( n = (2\pi)^{-1} \int_{-\infty}^{\infty} n(\omega) d\omega \) as a function of \( N_e \). Then \( N_e \) can be determined from the energy conservation law in Equation (14) as in refs.\(^{[29,30,32,33]}\)

Zero-order approximation leads to interesting results, as the mean photon number \( n(P) \) for SR laser, found beyond the quantum rate equation approach of refs.\(^{[49,50]}\), that is, without adiabatic elimination of polarization; collective Rabi splitting in the laser field spectra.\(^{[29,30,33]}\) Solving Equations (15) and (16) by Fourier transform, one can easily see that the second-order autocorrelation function \( g_2(\omega) \) is due to population fluctuations neglected in Equations (15) and (16).

Recognizing the importance of population fluctuations in SR LEDs and lasers, we come in ref.\(^{[30]}\) to a first-order approximation, which includes population fluctuations.

In the first-order approximation, we take into account the nonlinear term \( \delta \dot{N}_e \) in Equation (11) and calculate \( \delta \dot{N}_e \). We consider the product \( \delta \hat{N}_e \) as a random variable with the Fourier-component operator \( (\delta \hat{N}_e) \) and the power spectrum \( S_{\delta N_e}(\omega) \).

\[
\langle \hat{a}^\dagger(\omega)\delta \hat{N}_e(\omega') \rangle = S_{\delta N_e}(\omega)\delta(\omega + \omega')
\]

(22)

It was shown in ref.\(^{[30]}\) that \( S_{\delta N_e}(\omega) \) is a convolution of the field spectrum \( n(\omega) \) and the population fluctuation spectrum \( \delta^2 N_e(\omega) \)

\[
S_{\delta N_e}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} n(\omega - \omega')\delta^2 N_e(\omega') d\omega'
\]

(23)

A way for analytical calculation of \( S_{\delta N_e}(\omega) \) is that \( n(\omega) \) and \( \delta^2 N_e(\omega) \) in Equation (23) are found in the zero-order approximation. So we calculate approximate \( S_{\delta N_e}(\omega) \) from Equation (23) and then find the first-order approximation solution of Equations (10)–(12) in terms of Fourier-component operators.\(^{[30]}\)

In ref.\(^{[30]}\), we presented a simplified version of the first-order approximation, supposing so weak excitation that the first term on the right in Equation (12) can be neglected, and Equation (12) is approximately replaced by

\[
\delta \dot{N}_e \approx -\gamma_+ \delta \hat{N}_e + \hat{F}_N
\]

(24)

This equation can be easily solved.

In ref.\(^{[30]}\), we found that population fluctuations in SR LEDs significantly enhance spontaneous emission into the lasing mode at certain conditions, increasing the LED radiation. This result confirms the importance of population fluctuations in SR LEDs and lasers. It encourages us to continue working with the first-order approximation, mainly to carry it out without Equation (12) replacement by Equation (24).

To find population fluctuations, in this case, we must know \( \delta \Sigma \). For this, we take Equations (15), (16) and, applying the rule of the differentiation of products, write equations

\[
\hat{\Sigma} = -2\kappa \hat{\Sigma} + \Omega \Sigma + \hat{F}_\Sigma
\]
\[ \dot{\hat{S}} = -(\kappa + \gamma_0/2)\hat{S} + 2\Omega f (\hat{n} \hat{N} + \hat{N} \hat{n}) + \hat{F}_n \]  
(26)

\[ \dot{\hat{D}} = -\gamma_0 \hat{D} + \Omega N \hat{S} + \hat{F}_D \]  
(27)

for the photon number operator \( \hat{n} = \hat{a}^{\dagger} \hat{a}, \hat{S} \), given by Equation (7), and

\[ \dot{\hat{n}} = n + \delta \hat{n}, \quad \dot{\hat{S}} = \Sigma + \delta \hat{S}, \quad \dot{\hat{D}} = D + \delta \hat{D} \]  
(29)

insert Equations (29) into Equations (25)–(27), obtain equations for fluctuations

\[ \delta \hat{n} = -2\kappa \delta \hat{n} + \Omega \delta \hat{S} + \hat{F}_n \]  
(30)

\[ \delta \dot{\hat{S}} = -(\kappa + \gamma_0/2)\delta \hat{S} + 2\Omega f (\delta \hat{n} \hat{N} + \delta \hat{D}) + \hat{F}_n \]  
(31)

\[ \delta \dot{\hat{D}} = -\gamma_0 \delta \hat{D} + \Omega N \delta \hat{S} + \hat{F}_D \]  
(32)

and the set of algebraic equations for the mean values \( n, \Sigma, \) and \( D \)

\[ 0 = -2\kappa n + \Omega \Sigma \]  
(33)

\[ 0 = -(\kappa + \gamma_0/2)\Sigma + 2\Omega f (n \hat{N} + \hat{D} + N_e) \]  
(34)

\[ 0 = -\gamma_0 D + \Omega N \Sigma \]  
(35)

\( n, \Sigma, D \) and \( N_{eg} \) found from Equations (33)–(35) (with the energy conservation law (14) and the relation \( N_e + N_g = N_0 \) are the same as the ones found from Equations (10) and (11).\(^{[29,33]}\)

In the next steps of the procedure, we must solve linear Equations (30)–(32) by the Fourier-transform, find \( \delta \hat{S}, \delta N_e(\omega) \), and \( \delta N_g(\omega) \) from Equation (12) and calculate \( \delta \hat{S}, \delta N_e(\omega) \) in Equation (23). For doing this, we must know diffusion coefficients for Langevin forces in Equations (30)–(32).

An essential part of our procedure is calculating the zero-order diffusion coefficients for Langevin forces in Equations (30)–(32). In these equations (as well as in Equations (15) and (16)), we can not use “exact” diffusion coefficients found from GER; they will be inconsistent, as we will see, with the results of Equations (15) and (16). Calculation of the zero-order diffusion coefficients for Langevin forces in Equations (30)–(32) is carried out below in the frame of the oscillator laser model described in the next section.

3. Laser Equations in Terms of Oscillators

Two zero-order diffusion coefficients \( 2D_{v^*} \) and \( 2D_{v^*} \) for Langevin forces in Equations (15) and (16) have been determined in refs. \([30, 33] \) from GER by setting populations to be constants (i.e., not operators). We found it difficult to determine five zero-order diffusion coefficients for Langevin forces in Equations (25)–(27) or (30)–(32) similar way. It turns out to be easier to use a zero-order approximation Hamiltonian \( H_0 \) leading to Equations (25)–(27) or (30)–(32). The properties of operators in \( H_0 \) help us to find diffusion coefficients for any equations of the zero-order approximation.

Hamiltonian \( H_0 \) can be obtained in an oscillator laser model (OLM). This model describes \( N_e \) emitters in the ground states as conventional (normal) quantum harmonic oscillators with Bose operators \( \hat{c}_i e^{i\omega_0 t}, i = 1 \ldots N_e \). We will describe \( N_g \) emitters in the upper states as inverted harmonic oscillators with Bose-operators \( \hat{b}_j e^{i\omega_0 t}, j = 1 \ldots N_g \). Note the sign “+” in the exponent multiplier \( e^{i\omega_0 t} \) for the inverted oscillator, while the sign “−” in the multiplier \( e^{-i\omega_0 t} \) is for the usual, non-inverted oscillator. \( \hat{b}_j \) and \( \hat{c}_i \) are changed in time much slower than \( e^{i\omega_0 t} \). The normal and the inverted oscillators are shown schematically in Figure 2.

Hamiltonian \( H_0 \) of the OLM, written in the interaction picture and the RWA with the carrier frequency \( \omega_0 \) is

\[ H_0 = \hbar \Omega \left[ \hat{a}^{\dagger} \left( \sum_{i=1}^{N_e} f_{\alpha \beta} \hat{b}_i^{\dagger} + \sum_{j=1}^{N_g} f_{\alpha \gamma} \hat{c}_j \right) - h.c. \right] + \hat{\Gamma} \]  
(36)

Here dimensionless factors \( f_{\alpha \beta} \) describe the difference in couplings of different oscillators with the field; the rest of the notations is the same as for the “exact” two-level laser Hamiltonian (1).
We represent $H_0$ in terms of only three oscillators with Bose operators $\hat{a}$, $\hat{b}$ and $\hat{c}$

$$
\hat{b} = f_{b}^{-1} \sum_{i=1}^{N_b} f_{b,i} \hat{b}_i, \quad \hat{c} = f_{c}^{-1} \sum_{i=1}^{N_c} f_{c,i} \hat{c}_i, \quad f_{b,c} = \left( \sum_{i=1}^{N_{b,c}} f_{b,c,i}^2 \right)^{1/2}
$$

(37)

Bose commutation relations $[\hat{b}, \hat{b}^+] = [\hat{c}, \hat{c}^+] = 1$ follow from Bose commutation relations for $\hat{b}_i$ and $\hat{c}_i$. Hamiltonian (36), in terms of three oscillators, reads

$$
H_0 = i\hbar \Omega \left[ f_{b} (\hat{a}^+ \hat{b} - \hat{b}^+ \hat{a}) + f_{c} (\hat{c}^+ \hat{c} - \hat{c}^+ \hat{c}) \right] + \hat{F}
$$

(38)

Hamiltonian (38) and Bose-commutation relations lead to Heisenberg–Langevin equations for $\hat{a}$, $\hat{b}^+$, and $\hat{c}$

$$
\dot{\hat{a}} = -\kappa \hat{a} + \Omega (\hat{f}_b \hat{b}^+ + \hat{f}_c \hat{c}^+) + \sqrt{2\kappa} \hat{a}_{in}
$$

(39)

$$
\dot{\hat{b}}^+ = -\left(\gamma_v / 2\right) \hat{b}^+ + \Omega \hat{f}_b \hat{a} + \sqrt{\gamma_v} \hat{b}_{in}^+
$$

(40)

$$
\dot{\hat{c}} = -\gamma_v \hat{c} - \Omega \hat{f}_c \hat{a} + \sqrt{\gamma_v} \hat{c}_{in}
$$

(41)

where $\hat{a}_{in}$, $\hat{b}_{in}$, and $\hat{c}_{in}$ are Bose-operators of baths. Non-zero correlations between the bath operators are

$$
\langle \hat{a}_{m}(t) \hat{a}_{m}'(t') \rangle = \delta(t - t')
$$

(42)

where $\hat{a}_{m} = \{\hat{a}_{in}, \hat{b}_{in}, \hat{c}_{in}\}$. Operators of different baths do not correlate with each other. Langevin forces with the bath operators are added in Equations (39)–(41) following the input–output theory.[40,41]

We simplify Equations (39)–(41) by introducing an operator

$$
\hat{v} = \hat{f}_b \hat{b}^+ + \hat{f}_c \hat{c}
$$

of the polarization of the lasing medium (using the same notation $\hat{v}$ as in Equation (3)), and re-write Equations (39)–(41)

$$
\dot{\hat{a}} = -\kappa \hat{a} + \Omega \hat{v} + \sqrt{2\kappa} \hat{a}_{in}
$$

(44)

$$
\dot{\hat{b}}^+ = -\left(\gamma_v / 2\right) \hat{b}^+ + \Omega \hat{f}_b \hat{a} + \sqrt{\gamma_v} \hat{b}_{in}^+
$$

(45)

with the Langevin force

$$
\hat{F}_v = \sqrt{\gamma_v} \left( \hat{f}_b \hat{b}_{in}^+ + \hat{f}_c \hat{c}_{in} \right)
$$

(46)

Langevin forces are delta-correlated $< \hat{F}_v(t) \hat{F}_v(t') > = 2 D_{v,v} \delta(t - t')$ with diffusion coefficients

$$
2 D_{\hat{a},v} = \gamma_v \hat{f}_b^2, \quad 2 D_{\hat{b},v} = \gamma_v \hat{f}_b^2, \quad 2 D_{\hat{c},v} = \gamma_v \hat{f}_c^2, \quad 2 D_{\hat{v},v} = 0
$$

(47)

followed from Equations (42), (43), and (46). We approximate

$$
\hat{f}_{b,c}^2 \approx f N_{b,c}, \quad f = N_b^{-1} \left( \sum_{i=1}^{N_b} f_{b,i}^2 + \sum_{i=1}^{N_c} f_{c,i}^2 \right)
$$

(48)

where $N_{b,c}$ are the mean numbers of emitters in the ground (in the excited) states, and $N_b = N_a + N_b$ is the total number of emitters. With the approximation in Equation (48), Equations (44) and (45) became identical with Equations (15) and (16) and diffusion coefficients in Equation (47) with diffusion coefficients in Equation (18). Such a coincidence confirms the correctness of the OLM.

4. Diffusion Coefficients and Population Fluctuations

Now we will see that the OLM leads to Equations (30)–(32) and obtain diffusion coefficients for these equations. For this, using Equations (44) and (45) and the rule of differentiation of products, we write the equation of motion for $\hat{n} = \hat{a}^+ \hat{a}$

$$
\dot{\hat{n}} = -2 \kappa \hat{n} + \Omega \hat{F} + \hat{F}_n
$$

(49)

where

$$
\hat{F}_n = \sqrt{\gamma_v} \left( \hat{a}_{in} \hat{a}^+ + \hat{a}^+ \hat{a}_{in} \right)
$$

(51)

In Equations (7) and (50), and below, we use the same notation $\Sigma$ for the normalized field-polarization interaction energy. We write, using Equations (44) and (45),

$$
\dot{\Sigma} = -\left( \kappa + \gamma_v / 2 \right) \Sigma + 2 \Omega \left[ \hat{n} (\hat{f}_b^2 - \hat{f}_c^2) + \hat{D}_j + \hat{f}_{c,j}^2 \right] + \hat{F}_\Sigma
$$

(52)

where

$$
\hat{D}_j = \hat{f}_b^2 \hat{b}^+ \hat{b} + \hat{f}_c^2 \hat{c}^+ \hat{c} + \hat{f}_{c,j}^2 \left( \hat{c}^+ \hat{b}^+ + \hat{b} \hat{c}^+ \right)
$$

(53)

and the Langevin force

$$
\hat{F}_\Sigma = \sqrt{2 \gamma_v} \left( \hat{b}_{in} \hat{a}^+ + \hat{b}^+ \hat{a}_{in} \right) + \sqrt{\gamma_v} \left( \hat{b}_{in} \hat{a} + \hat{b}^+ \hat{a}_{in}^+ \right) + \sqrt{2 \gamma_v} \left( \hat{c}_{in} \hat{a}^+ + \hat{c}^+ \hat{a}_{in} \right) + \sqrt{\gamma_v} \left( \hat{c}_{in} \hat{a} + \hat{c}^+ \hat{a}_{in}^+ \right)
$$

(54)

With the derivation of Equation (52), we set the normal ordering of Bose operators replacing $\hat{f}_b \hat{a}^+ \hat{a}$ with $\hat{f}_b \hat{a}^+ \hat{a} + \hat{f}_b$. We obtain the equation for $\hat{D}_j$

$$
\dot{\hat{D}}_j = -\gamma_v \hat{D}_j + \Omega (\hat{f}_b^2 - \hat{f}_c^2) \Sigma + \hat{F}_{\hat{D}_j}
$$

(55)

with the Langevin force

$$
\hat{F}_{\hat{D}_j} = \sqrt{\gamma_v} \left[ \hat{f}_b^2 \left( \hat{b}_{in} \hat{b} + \hat{b}^+ \hat{b}_{in} \right) + \hat{f}_c^2 \left( \hat{c}_{in} \hat{c}^+ + \hat{c}^+ \hat{c}_{in} \right) \right] + \sqrt{\gamma_v} \left( \hat{b}_{in} \hat{a} + \hat{b}^+ \hat{a}_{in} \right) + \sqrt{\gamma_v} \left( \hat{c}_{in} \hat{a} + \hat{c}^+ \hat{a}_{in} \right)
$$

(56)

in the same way as equations for $\hat{n}$ and $\Sigma$. 

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Equations (49), (52), and (55) are the set of linear equations for \( \hat{n} \), \( \Sigma \), and \( \tilde{D} \). Delta-correlations for Langevin forces in these equations have non-zero diffusion coefficients:

\[
\begin{align*}
2D_{n,\alpha} &= 2\kappa n, \\
2D_{\Sigma,\alpha} &= 2\kappa D_{\Omega} + \gamma_{\perp}(f_{\perp}^2 + f_{\parallel}^2) n + (2\kappa + \gamma_{\perp})f_{\parallel}^2 \\
2D_{D,\Omega} &= \gamma_{\perp} [(f_{\perp}^2 + f_{\parallel}^2) D_{\Omega} + 2f_{\parallel}^2 f_{\Omega}]
\end{align*}
\]

(57)

We use approximation in Equation (48), introduce \( D = f D_{\Omega} \), and see Equations (49), (52), and (55) equivalent to Equations (25)–(27). Separating the mean values and fluctuations in operators \( \Sigma \), we come to Equations (30)–(32). Diffusion coefficients for Langevin forces in Equations (25)–(27) or (30)–(32) are

\[
\begin{align*}
2D_{n,\alpha} &= 2\kappa n, \\
2D_{\Sigma,\alpha} &= f(2\kappa D + \gamma_{\perp} N_{\phi} n + (2\kappa + \gamma_{\perp}) N_{\parallel}), \\
2D_{D,\Omega} &= \gamma_{\perp} (N_{\phi} D + 2N_{\parallel} N_{\phi}).
\end{align*}
\]

(58)

Diffusion coefficients followed from GER are all the same as in Equations (58), apart from \( 2D_{D,\Omega} \). The term \( \gamma_{\perp} N_{\phi} N_{\parallel} \) is absent in \( 2D_{D,\Omega} \) found in GER.

We make the Fourier transform

\[
\Delta \hat{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Delta \hat{a}(\omega) e^{-i\omega t} d\omega
\]

(59)
in the linear equations (30)–(32) and obtain algebraic equations for Fourier-component operators. Solving them we find \( \Delta \hat{n}(\omega), \Delta \hat{\Sigma}(\omega) \). From the relation

\[
\langle \Delta \hat{n}(\omega) \Delta \hat{\Sigma}(\omega) \rangle = \Delta \hat{\Sigma}(\omega) \Delta \hat{n}(\omega)
\]

(60)

and the similar one for \( \Delta \hat{\Sigma}(\omega) \), we find the photon number fluctuation spectra \( \Delta \hat{n}(\omega) \) and \( \Delta \hat{\Sigma}(\omega) \) with explicit expressions presented in the Appendix. We use the notation \( \omega \) for the oscillation frequency in Equation (59) and everywhere below. It should not be confused with the same notation \( \omega \) in Equations (19), (20), and in Equation (21) for the field spectrum \( n(\omega) \), where \( \omega \) means the deviation from the optical carrier frequency \( \omega_{\text{c}} \).

We obtain the equation for the Fourier component operator \( \Delta \hat{N}_{\alpha}(\omega) \) from the linear equation (12). We insert there \( \Delta \hat{\Sigma}(\omega) \) found above, find \( \Delta \hat{N}_{\alpha}(\omega) \) and, from the relation \( < \Delta \hat{N}_{\alpha}(\omega) \Delta \hat{N}_{\alpha}(\omega') >= \Delta \hat{N}_{\alpha}(\omega) \Delta \hat{N}_{\alpha}(\omega + \omega') \), obtain the population fluctuation spectrum

\[
\Delta \hat{\Sigma}^2 N_{\alpha}(\omega) = \frac{S_{\Sigma}(\omega)}{\omega^2 + \gamma_{\perp}^2} + \frac{2D_{n,\alpha}N_{\alpha}}{\omega^2 + \gamma_{\parallel}^2}
\]

(61)

Equation (61) is the first-order approximation for \( \Delta \hat{N}_{\alpha}(\omega) \) since \( S_{\Sigma}(\omega) \) and \( \hat{S}(\omega) \), given in Appendix, are calculated in the zero-order approximation and do not depend on population fluctuations. Writing Equation (61), we neglect quantum correlations between the population and the polarization fluctuations of emitters, as we did in refs. [29, 30]. It is a good approximation if the total number of emitters is considerable so that the approximation if the number of photons \( n \ll N_{\text{em}} \).

We take the diffusion coefficient for population fluctuations

\[
2D_{n,\alpha} = \gamma_{\parallel}(PN_{\phi} + N_{\parallel})
\]

(62)

the same as in ref. [29], the rate equation laser theory [49] and calculations from GER [48].

We find that the photon number variance,

\[
\delta^2 n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta^2 n(\omega) d\omega = n(n + 1)
\]

(63)
is the same as for the thermal radiation, consistent with the second-order correlation function \( g_2 = 2 \) found from the solution of Equations (15) and (16). The result in Equation (63) confirms the correctness of the zero-order diffusion coefficients in Equation (58). One can not obtain the result (Equation (63)) with diffusion coefficients from GER, that is, without the term \( \Delta \hat{N}_{\alpha}(\omega) \) in \( 2D_{D,\Omega} \) in Equation (58).

5. Results and Discussion

We calculate zero-order diffusion coefficients (Equation (58)) and found population fluctuations in the first-order approximation from Equations (30)–(32), and (12). Diffusion coefficients (Equation (58)) let us continue the research of ref. [30] and find the approximate analytical solution of Equations (10)–(12) without replacing Equation (12) with Equation (24). We will find this solution in the future.

Below we investigate the photon number and the population fluctuations using their spectra and variances. We compare the results for SR LEDs and LEDs without SR.

In examples we take the values of parameters the same as in refs. [29, 30], \( \Omega/\gamma_{\parallel} = 34 \), \( N_{\phi} = 100 \), \( f = 1/2 \), \( \kappa/\gamma_{\parallel} = 50 \), where \( \gamma_{\parallel} = 10^9 \) rad s\(^{-1} \) and \( \kappa = 50 \times 10^7 \) rad s\(^{-1} \). Similar parameter values can be found, for example, in quantum dot lasers with photonic crystal cavities [30, 51].

We vary the polarization relaxation rate \( \gamma_{\perp} \) such that 1/15 < 2\( \kappa/\gamma_{\perp} < 2 \). The case of 2\( \kappa/\gamma_{\perp} \geq 1 \) corresponds to SR lasers, 2\( \kappa/\gamma_{\perp} \ll 1 \) — to lasers without SR [29]. According to Figures 3 and 4 of ref. [29], the laser at the chosen parameter values has a small mean cavity photon number \(< n > \approx 1 \) and operates in the LED regime for \( P \leq 3 \). So the weak laser excitation is in the interval 0 < \( P < 3 \), and we vary the normalized pump rate \( P \) in this interval. The values of all other laser parameters, apart from \( P \) and \( \gamma_{\perp} \), are fixed.

Figure 3 shows the field spectra \( n(\omega_{\text{opt}} - \omega_0) \) given by Equation (21). We take \( P = 0.7 \) and decrease \( \gamma_{\parallel} \) from curve 1 to curve 5 in Figure 3, so 2\( \kappa/\gamma_{\perp} \leq 1/5 \) (curve 1), 1/3 (2), 1/2 (3), 1 (4), 2 (5), and the contribution of SR in the laser grows from the curve 1 to the curve 5. In Figure 3, we see peaks of the collective Rabi splitting (CRS) [33] on curves 2 - 5 with 2\( \kappa/\gamma_{\perp} \geq 1/3 \). The maxima of the CRS peaks grow with 2\( \kappa/\gamma_{\perp} \), that is, when the laser approaches the superradiant regime.

Figure 4 shows the normalized photon number (or, the same, the field intensity) fluctuation spectra

\[
\sqrt{\delta^2 n(\omega)/n(n + 1)}
\]

for the same parameter values as for Figure 3, \( \delta^2 n(\omega) \) is given by Equation (A1) of Appendix.

When 2\( \kappa/\gamma_{\perp} \) is relatively large, as for curves 3, 4, and 5 in Figure 4, we see the sideband CRS peaks on curves. The maxima of
the binary operator in \( \omega \) (5), with other parameters, fixed and given in the text. Parts of spectra with \( \omega_{\text{opt}} - \omega_0 < 0 \) are symmetric with parts with \( \omega_{\text{opt}} - \omega_0 > 0 \) shown in the figure. All curves, but curve 1, have peaks of CRS near \( \omega_{\text{opt}} / \gamma_\parallel \approx 150 \). CRS peaks grow when the laser approaches the SR regime increasing \( 2 \kappa / \gamma_\perp \).

The second source is fluctuations of the field-polarization interaction energy \( \Sigma \), which is described by the term \( \delta \Sigma \) on the right in Equation (12). As we see in Equation (31), \( \delta \Sigma \) depends on fluctuations \( \delta \hat{D} \) of the dipole-dipole interaction energy of emitters. \( \delta \hat{D} \) is large in SR lasers, so the contribution of \( \delta \Sigma \) to the population fluctuations is more significant in SR lasers than in lasers without SR.

Thus, with the OLM, we found peaks of CRS in the photon number fluctuation spectra of SR LEDs. It supplements the results of ref. [33], where CRS peaks were predicted in the field spectra. CRS peaks in the photon number fluctuation spectra are less visible than in the field spectra (compare Figures 3 and 4) and must be distinguished from the relaxation oscillation (RO) peaks. CRS and RO peaks are present in the field, and the photon number fluctuation spectra of SR lasers, see ref. [29] and Figures 3, 4 here. CRS peaks appear at the SR LED regime. RO peaks appear at high excitation when the laser generates coherent radiation. At the high excitation regime, ROs in the field spectra appear as tiny sidebands of the central peak, as in Figure 7a of ref. [29]. The physical reason for the CRS peaks is the collective Rabi splitting. The physical reason for the RO peaks is the energy exchange between the lasing mode and the active medium.

Now we analyze population fluctuations. We see in Equation (12) two sources of population fluctuations. The first source is fluctuations due to the pump and the energy decay described by the two last terms on the right in Equation (12). This source does not depend on the field and the active medium polarization, so it is the same in lasers with and without SR.

The second source is fluctuations of the field-polarization interaction energy \( \Sigma \), which is described by the term \( \delta \Sigma \) on the right in Equation (12). As we see in Equation (31), \( \delta \Sigma \) depends on fluctuations \( \delta \hat{D} \) of the dipole-dipole interaction energy of emitters. \( \delta \hat{D} \) is large in SR lasers, so the contribution of \( \delta \Sigma \) to the population fluctuations is more significant in SR lasers than in lasers without SR.

Let us see how the contributions of the field-polarization interaction energy and the pump-decay processes to population fluctuations depend on the laser parameters.

Figure 5 shows population fluctuation spectra for the SR laser with \( 2 \kappa / \gamma_\perp = 2 > 1 \) (black curves) and the laser without SR with \( 2 \kappa / \gamma_\perp = 1 / 30 \ll 1 \) (red curves). The normalized pump rate \( P = 1 \) (solid curves), \( P = 2 \) (dashed curves), and \( P = 3 \) (dashed-dotted curves), population fluctuations in the SR laser (black curves) are larger than in the laser without SR (red curves). For small \( P \leq 1 \), population fluctuations depend mostly on the pump and the energy decay, the same for lasers with and without SR, so the population fluctuation spectra for lasers with and without SR for \( P \leq 1 \) (solid red and black curves) practically coincide.

Figure 6 shows population fluctuation spectra for various \( \gamma_\perp \), the same as in Figure 3 for curves 1 – 5 correspondingly, and other parameters are given in the text. Curves 3, 4, and 5 have peaks of CRS at frequencies \( \omega \) approximately two times larger than the values of \( \omega_{\text{opt}} - \omega_0 \) at the maxima of the CRS peaks of the field spectra in Figure 3.
For $P > 1$, population fluctuations in the SR laser with $2\kappa/\gamma_\perp = 2$ became larger than in the laser without SR with small $2\kappa/\gamma_\perp = 1/30$. The difference in population fluctuations in SR lasers and lasers without SR grows with the pump $P$; see in Figure 5 the difference between the dashed and the dotted-dashed curves of different colors. For $P > 1$, population fluctuations depend more on the field and polarization interaction energy fluctuations. So population fluctuations became larger for the SR laser (the black dashed and dashed-dotted curves) than for the laser without SR (the red dashed and dashed-dotted curves).

With a closer look at Figure 5, we see that population fluctuations in lasers without SR (red curves) are progressively (and nonlinearly) reduced with the pump. This effect is caused by well-known “population clamping”\cite{43,52} when the population inver-

tor of $\hat{b}$ atoms, and describes it as an inverted oscillator with the operator $\hat{b}$ in the Hamiltonian (38). This oscillator is hardly saturated (downward the energy levels in Figure 2) by a weak cavity field with the mean number of photons $n \ll N_\gamma$. Similarly, a non-inverted oscillator with the operator $\hat{c}$ combined from the assembly of $N_\delta$ atoms in the ground states can not be saturated by a weak field with $n \ll N_\delta$. So with a weak lasing field and neglecting fluctuations of populations of the lasing states, OLM is a physically reasonable approximation for the exact two-level laser model.

Let us consider some single emitter. In the stationary case, transitions between its levels are population fluctuations, which we neglect. So, in our approximation, the emitter makes only virtual transitions. There are precisely $N_\delta$ emitters in the ground

$$\langle \delta^2 N_e \rangle(P) = \frac{1}{2\pi} \int_{-\infty}^{0} \delta^2 N_e(\omega) d\omega$$

\text{(64)}

for $2\kappa/\gamma_\perp = 2$ (SR laser, solid curves) and for $2\kappa/\gamma_\perp = 1/30$ (laser without SR, dashed curves) and contributions of the pump-decay and the field-polarization interaction energy fluctuations into $\delta^2 N_e(P)$. The red curves are $\delta^2 N_e(P)$, the blue curves are parts of $\delta^2 N_e$ related to the field and the polarization interaction energy fluctuations, and the black curves are parts of $\delta^2 N_e$ related to the pump-decay fluctuations.

For small pump $P < 1$, the pump-decay fluctuations significantly affect population fluctuations. They do not depend on $\gamma_\perp$ and are the same for lasers with and without SR; the red solid and dashed curves in Figure 6 coincide. This effect has been used to simplify population fluctuation calculations at small pumps.\cite{30}

For $P > 1$, the field-polarization interaction energy’s influence on population fluctuations increases, corresponding part of population fluctuations (blue curves in Figure 6) grew. This part is more significant for SR lasers than those without SR because of the dipole–dipole interaction energy $\hat{D}$ contribution to the field-polarization interaction energy $\hat{\Sigma}$, see Equations (26) and (31). $\hat{D}$ and fluctuations $\delta \hat{D}$ are larger for SR lasers than for lasers without SR. So, high dipole–dipole interaction energy fluctuations finally lead to significant population fluctuations in the SR lasers at a large pump.

The contribution of fluctuations of $\hat{D}$ into population fluctuations of SR lasers grows with the pump. It explains the increase of population fluctuations in SR lasers with the pump, as shown by black curves in Figure 5. Such population fluctuation increase may be suppressed by population clamping at the higher pump. We leave the investigation of this for the future.

Let us discuss the OLM. The exact laser model is, of course, not an ensemble of oscillators. However, OLM is a reasonable approximation for the exact model in cases when population fluctuations can be neglected. OLM correctly predicts some mean values, such as the mean photon number, also for SR lasers,\cite{29} and leads to new results, such as the collective Rabi splitting in the field spectra of SR LEDs.\cite{31}

Formal justification of the OLM is the coincidence of equations, followed from OLM, and equations used previously in refs. \cite{29,30,33}. We want to provide also physical arguments in favor of OLM, following the interpretation of quantum inverted oscillators given in ref. \cite{39}. We quote:\cite{39} “A fully excited assembly of two-level atoms displays a ladder of equally spaced levels, which extends downwards for many steps. As long as the process is not saturating these levels behave exactly as an inverted oscillator...”.

The present paper considers two assemblies of $N_e$ excited and $N_\delta$ emitters in the ground states. There is evidence that the number of emitters in nanolasers is at least an order of magnitude larger than the number of photons at the threshold.\cite{10,50,51} So we suppose the mean number $n$ of the cavity photons is small, $n \ll N_{\delta, g}$ at a low laser excitation. In the OLM, we take a ladder of equally spaced levels, combined from the assembly of $N_\delta$ excited atoms, and describe it as an inverted oscillator with the operator $\hat{b}$ in the Hamiltonian (38). This oscillator is hardly saturated (downward the energy levels in Figure 2) by a weak cavity field with the mean number of photons $n \ll N_\gamma$. Similarly, a non-inverted oscillator with the operator $\hat{c}$ combined from the assembly of $N_\delta$ atoms in the ground states can not be saturated by a weak field with $n \ll N_\delta$. So with a weak lasing field and neglecting fluctuations of populations of the lasing states, OLM is a physically reasonable approximation for the exact two-level laser model.
and $N$ — in the excited states with no exchange between manifolds of emitters in that states. When an emitter does not come into the excited (or de-excited) state, it does not matter whether it is a two-level system or an oscillator, so we can replace emitters with oscillators in our approximation. Polarization, non-zero at virtual transitions, can be found separately for manifolds of the normal and the inverted oscillators. We do not eliminate polarization adiabatically in a difference with the quantum theory of normal and the inverted oscillators. We do not eliminate polarizations, can be found separately for manifolds of the excited (or de-excited) state, it does not matter whether it is a two-level system with an active resonant medium.

A harmonic oscillator is a primary system in the oscillation theory used in various areas of physics. Many systems can be modeled as interacting harmonic oscillators in mechanics, optics, chemistry and other physical disciplines. We suggest that OLM will be helpful for the analytical modeling of lasers and other quantum optical systems with an active resonant medium.

6. Conclusion

We present the oscillator laser model—a quantum model of the two-level laser in harmonic oscillators, including inverted harmonic oscillators. OLM can be used when population fluctuations of the lasing transitions can be neglected. OLM is a zero-order approximation in the analytical approach of refs. [29, 30, 32, 33].

With OLM, we calculate zero-order diffusion coefficients in Equation (58) for approximate Equations (30) – (32) for fluctuations in the photon number, the field-polarization, and the dipole–dipole interaction energies of the lasing medium. Diffusion coefficients in Equation (58) differ from the well-known ones obtained from generalized Einstein relations for the exact two-level laser model with Hamiltonian (1). Diffusion coefficients in Equation (58) provide consistent results, in particular, for the photon number fluctuation variance in Equation (63), found from approximate Equations (15), (16), and (30)–(32). Zero-order diffusion coefficients are necessary to complete the approximate analytical procedure of solving laser equations.

Here with OLM, we calculate the photon number (or the intensity) fluctuation spectra of the lasing field and find the collective Rabi splitting peaks in the intensity fluctuation spectra of the superradiant lasers. This calculation supplements the results of ref. [33], where the collective Rabi splitting peaks have been found in the lasing field spectra.

With the OLM, we calculate the population fluctuations in the first-order approximation, investigate and compare them in the SR lasers and the lasers without SR, and suggest the mechanism for the growth of population fluctuations in SR lasers. Population fluctuations at a low laser excitation depend mainly on the pump-decay processes, the same in the lasers with and without SR. When the laser excitation increases, the contribution from the field-polarization interaction energy to the population fluctuations grows and overcomes the contribution of the pump-decay processes. The dipole–dipole interaction between emitters, large in SR lasers, contributes to the field-polarization interaction energy. So, when the laser excitation grows, the population fluctuations become larger in SR lasers than in those without SR.

The authors declare no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Keywords

Heisenberg equations, inverted oscillators, nanolasers

Conflict of Interest

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