On The Dependence Structure of Wavelet Coefficients for Spherical Random Fields

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Abstract

We consider the correlation structure of the random coefficients for a class of wavelet systems on the sphere (labelled Mexican needlets) which was recently introduced in the literature by [15]. We provide necessary and sufficient conditions for these coefficients to be asymptotic uncorrelated in the real and in the frequency domain. Here, the asymptotic theory is developed in the high frequency sense. Statistical applications are also discussed, in particular with reference to the analysis of Cosmological data.

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• AMS 2000 Classification: Primary 60G60; Secondary 62M40, 42C40, 42C10

1 Introduction

There is currently a rapidly growing literature on the construction of wavelets systems on the sphere, see for instance [14], [2], [7], [19], [30], [35] and the references therein. Some of these attempts have been explicitly motivated by extremely interesting applications, for instance in the framework of Astronomy

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and/or Cosmology; concerning the latter, special emphasis has been devoted to wavelet techniques for the statistical study of the Cosmic Microwave Background (CMB) radiation [25]. CMB can be viewed as providing observations on the Universe in the immediate adjacency of the Big Bang, and as such it has been the object of immense theoretical and applied interest over the last decade [9].

Among spherical wavelets, particular attention has been devoted to so-called needlets, which were introduced into the Functional Analysis literature by [26, 27]; their statistical properties were first considered by [3], [4], [13], and [21]. In particular, it has been shown in [3] that random needlet coefficients enjoy a capital uncorrelation property: namely, for any fixed angular distance, random needlets coefficients are asymptotically uncorrelated as the frequency parameter grows larger and larger. The meaning of this uncorrelation property must be carefully understood, given the specific setting of statistical inference in Cosmology. Indeed, CMB can be viewed as a single realization of an isotropic random field on a sphere of a finite radius [9]. The asymptotic theory is then entertained in the high frequency sense, i.e. it is considered that observations at higher and higher frequencies (smaller and smaller scales) become available with the growing sophistication of CMB satellite experiments. Of course, uncorrelation entails independence in the Gaussian case: as a consequence, from the above-mentioned property it follows that an increasing array of asymptotically i.i.d. coefficients can be derived out of a single realization of a spherical random field, making thus possible the introduction of a variety of statistical procedures for testing non-Gaussianity, estimating the angular power spectrum, testing for asymmetries, implementing bootstrap techniques, testing for cross-correlation among CMB and Large Scale Structure data, and many others, see for instance [3], [4], [17], [21], [13], [29], [23], [8], [6], [20], [31]. We note that the relevance of high frequency asymptotics is not specific to Cosmology (cf. e.g. financial data).

Given such a widespread array of techniques which are made feasible by means of the uncorrelation property, it is natural to investigate to what extent this property should be considered unique for the construction in [26, 27], or else whether it is actually shared by other proposals. In particular, we shall focus here on the approach which has been very recently advocated by [15], see also [14] for a related setting. This approach (which we shall discuss in Section 2) can be labelled Mexican needlets, for reasons to be made clear later. Its analysis is made particularly interesting by the fact that, as we shall discuss below, the Mexican needlets can be considered asymptotically equivalent to the Spherical Mexican Hat Wavelet (SMHW), which is currently the most popular wavelet procedure in the Cosmological literature (see again [25]). As such, the investigation of their properties will fill a theoretical gap which is certainly of interest for CMB data analysis.

Our aim in this paper is then to investigate the correlation properties of the Mexican needlets coefficients. The stochastic properties of wavelets have been very extensively studied in the mathematical statistics literature, starting from the seminal papers [10, 11]. We must stress, however, that our framework here is very different: indeed, as explained earlier we shall be concerned with
circumstances where observations are made at higher and higher frequencies on a single realization of a spherical random field. As such, no form of mixing or related properties can be assumed on the data and the proofs will rely more directly on harmonic methods, rather than on standard probabilistic arguments.

We shall provide both a positive and a negative result: namely, we will provide necessary and sufficient conditions for the Mexican needlets coefficients to be uncorrelated, depending on the behaviour of the angular power spectrum of the underlying (mean square continuous and isotropic) random fields. In particular, on the contrary of what happens for the needlets in [26, 27], we shall show that there is indeed correlation of the random coefficients when the angular power spectrum is decaying faster than a certain limit. However, higher order versions (already considered in [15]) of the Mexican needlets can indeed provide uncorrelated coefficients, depending on a parameter which is related to the decay of the angular power spectrum. In some sense, a heuristic rationale under these results can be explained as follows: the correlation among coefficients is introduced basically by the presence in each of these terms of random elements which are fixed (with respect to growing frequencies) in a given realization of the random field, because they depend only on very large scale behaviour (this is known in the Physical literature as a Cosmic Variance effect). Because of the compact support in frequency in the needlets as developed by [26, 27], these low-frequency components are always dropped and uncorrelation is ensured. On the other hand, the same components can be dominant for Mexican needlets, in which cases it becomes necessary to introduce suitably modified versions which are better localized in the frequency domain (i.e., they allow less weight on very low frequency components).

As well-known, there is usually a trade-off between localization properties in the frequency and real domains, as a consequence of the Uncertainty Principle (“It is impossible for a non-zero function and its Fourier transform to be simultaneously very small”, see for instance [18]). In view of this, an interesting consequence of our results can be loosely suggested as follows: the better the localization in real domain, the worst the correlation properties. This is clearly a paradox, and we do not try to formulate it more rigorously from the mathematical point of view - some numerical evidence will be collected in an ongoing, more applied work. However, we hope that the previous discussion will help to shed some light within the class of needlets; in particular, it should clarify that the uncorrelation property of wavelets coefficients does not follow at all by their localization properties in real domain. Indeed, given the fixed-domain asymptotics we are considering, perfect localization in real space does not ensure any form of uncorrelation (all random values at different locations on the sphere have in general a non-zero correlation).

The plan of the paper is as follows: in Section 2 we shall review some basic results on isotropic random fields on the sphere and the (Mexican and standard) needlets constructions. In Section 3 and 4 we establish our main results, providing necessary and sufficient conditions for the uncorrelation properties to hold; in Section 5 we review some statistical applications.
2 Isotropic Random Fields and Spherical Needlets

2.1 Spherical Random Fields

In this paper, we shall always be concerned with zero-mean, finite variance and isotropic random fields on the sphere, for which the following spectral representation holds, in the mean square sense:

\[ T(x) = \sum_{lm} a_{lm} Y_{lm}(x), \quad x \in S^2, \]  

(1)

where \( \{a_{lm}\}_{l,m} \) is a triangular array of zero-mean, orthogonal, complex-valued (for \( m \neq 0 \)) random variables with variance \( \mathbb{E}|a_{lm}|^2 = C_l \), the angular power spectrum of the random field. The functions \( \{Y_{lm}(x)\} \) are the so-called spherical harmonics, i.e. the eigenvectors of the Laplacian operator on the sphere \([12], [34], [32] \)

\[ \Delta_{S^2} Y_{lm}(\theta, \varphi) = \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] Y_{lm}(\theta, \varphi) \]

\[ = -l(l+1)Y_{lm}(\theta, \varphi) \]

where we have moved to spherical coordinates \( x = (\theta, \varphi) \), \( 0 \leq \theta \leq \pi \) and \( 0 \leq \varphi < 2\pi \). It is a well-known result that the spherical harmonics provide a complete orthonormal systems for \( L^2(S^2) \). There are many routes for establishing (1), usually by means of Karhunen-Loève arguments, the Spectral Representation Theorem, or the Stochastic Peter-Weyl theorem, see for instance [1]. The spherical harmonic coefficients \( a_{lm} \) can be recovered by means of the Fourier inversion formula

\[ a_{lm} = \int_{S^2} T(x) Y_{lm}(x) dx. \]  

(2)

If the random field is mean-square continuous, the angular power spectrum \( \{C_l\} \) must satisfy the summability condition

\[ \sum_l (2l+1)C_l < \infty. \]

We shall introduce a slightly stronger condition, as follows (see [3, 4, 13, 21]).

**Condition 1** For all \( B > 1 \), there exist \( \alpha > 2 \), and \( \{g_j(.)\}_{j=1,2,...} \) a sequence of functions such that

\[ C_l = l^{-\alpha}g_j \left( \frac{1}{B^j} \right) > 0, \text{ for } B^{j-1} < l < B^j, \quad j = 1, 2, ... \]  

(3)

where

\[ c_0^{-1} \leq g_j \leq c_0 \text{ for all } j \in \mathbb{N}, \text{ and } \sup_{j} \sup_{B^{-1} \leq u \leq B} \left| \frac{d^r}{du^r}g_j(u) \right| \leq c_r, \]

some \( c_0, c_1, ..., c_M > 0 \), \( M \in \mathbb{N} \).
In practice, random fields such as CMB are not fully observed, i.e. there are some missing observations in some regions of \(S^2\); (2) is thus unfeasible in its exact form, and this motivates the introduction of spherical wavelets such as needlets.

### 2.2 NPW Needlets

The construction of the standard needlet system is detailed in [26, 27]; we can label this system **NPW needlets** and we sketch here a few details for completeness. Let \(\phi\) be a \(C^\infty\) function supported in \(|\xi| \leq 1\), such that \(0 \leq \phi(\xi) \leq 1\) and \(\phi(\xi) = 1\) if \(|\xi| \leq 1/B\), \(B > 1\). Define

\[
b^2(\xi) = \phi\left(\frac{\xi}{B}\right) - \phi(\xi) \geq 0 \text{ so that } \forall|\xi| \geq 1, \sum_{j=0}^\infty b^2\left(\frac{\xi}{B^j}\right) = 1. \tag{4}
\]

It is immediate to verify that \(b(\xi) \neq 0\) only if \(\frac{1}{B} \leq |\xi| \leq B\). The needlets frame \(\{\varphi_{jk}(x)\}\) is then constructed as

\[
\varphi_{jk}(x) := \sqrt{\lambda_{jk}} \sum_{l} b\left(\frac{l}{B^j}\right) \sum_{m=-l}^{l} Y_{lm}(\xi_{jk}) Y_{lm}(x). \tag{5}
\]

Here, \(\{\lambda_{jk}\}\) is a set of cubature weights corresponding to the cubature points \(\{\xi_{jk}\}\); they are such to ensure that, for all polynomials \(Q_l(x)\) of degree smaller than \(B^{j+1}\)

\[
\sum_k Q_l(\xi_{jk}) \lambda_{jk} = \int_{S^2} Q_l(x) dx.
\]

The main localization property of \(\{\varphi_{jk}(x)\}\) is established in [26], where it is shown that for any \(M \in \mathbb{N}\) there exists a constant \(c_M > 0\) s.t., for every \(\xi \in S^2:\)

\[
|\varphi_{jk}(\xi)| \leq \frac{c_M B^j}{(1 + B^j \arccos(\xi_{jk}, \xi))^M} \text{ uniformly in } (j, k).
\]

More explicitly, needlets are almost exponentially localized around any cubature point, which motivates their name. In the stochastic case, the (random) spherical needlet coefficients are then defined as

\[
\beta_{jk} = \int_{S^2} T(x) \varphi_{jk}(x) dx = \sqrt{\lambda_{jk}} \sum_{l} b\left(\frac{l}{B^j}\right) \sum_{m=-l}^{l} a_{lm} Y_{lm}(\xi_{jk}). \tag{6}
\]

It is then immediate to derive the correlation coefficient

\[
\text{Corr}(\beta_{jk}, \beta_{jk'}) = \frac{E\beta_{jk}\beta_{jk'}}{\sqrt{E\beta_{jk}^2}E\beta_{jk'}} = \frac{\sqrt{\lambda_{jk}\lambda_{jk'}} \sum_{l \geq 1} b^2\left(\frac{l}{B^j}\right) \frac{2l+1}{4\pi} C_l \left((\xi_{jk}, \xi_{jk'})\right)}{\sqrt{\lambda_{jk}\lambda_{jk'}} \sum_{l \geq 1} b^2\left(\frac{l}{B^j}\right) \frac{2l+1}{4\pi} C_l}. \tag{5}
\]
where $P_l$ is the ultraspherical (or Legendre) polynomial of order $\frac{1}{2}$ and degree $l$; the last step follows from the well-known identity [34].

$$L_l(⟨\xi, \eta⟩) := \sum_{m=-l}^{l} Y_{lm}(\xi) Y_{lm}(\eta) = \frac{2l+1}{4\pi} P_l(⟨\xi, \eta⟩).$$

The capital stochastic property for random needlet coefficients is provided by [3], where it is shown that under Condition 1 the following inequality holds

$$|\text{Corr}(\beta_{jk}, \beta_{jk'})| \leq \frac{C_M}{(1 + B_jd(\xi_{jk}, \xi_{jk'}))^M}, \text{ some } C_M > 0,$$

where $d(\xi_{jk}, \xi_{jk'}) = \arccos(⟨\xi_{jk}, \xi_{jk'}⟩)$ is the standard distance on the sphere.

### 2.3 Mexican Needlets

The construction in [15] is in a sense similar to NPW needlets in [26, 27] (see also [14]), insofar as a combination of Legendre polynomials with a smooth function is proposed; the main difference is that for NPW needlets the kernel is taken to be compactly supported, which allows on one hand for an exact reconstruction function (needlets make up a tight frame), at the same time granting exact localization in the frequency-domain. It should be added, however, that the approach by [15] enjoys some undeniable strong points: firstly, it covers general oriented manifolds and not simply the sphere; moreover it yields Gaussian localization properties in the real domain. A further nice benefit is that it can be formulated in terms of an explicit recipe in real space, a feature which is certainly valuable for practitioners. In particular, as we report below in the high-frequency limit the Mexican needlets are asymptotically close to the Spherical Mexican Hat Wavelets, which have been exploited in several Cosmological papers but still lack a sound stochastic investigation.

More precisely, [15] propose to replace $b(l/B^j)$ in (5) by $f(l(l+1)/B^{2j})$, where $f(.)$ is some Schwartz function vanishing at zero (not necessarily of bounded support) and the sequence $\{-l(l+1)\}_{l=1,2,...}$ represents the eigenvalues of the Laplacian operator $\Delta_{S^2}$. In particular, Mexican needlets can be obtained by taking $f(s) = \exp(-s)$, so to obtain

$$\psi_{jk;1}(x) = \sqrt{\lambda_{jk}} \sum_{l \geq 1} \frac{l(l+1)}{B^{2j}} e^{-\frac{(l+1)}{B^{2j}}} L_l(⟨x, \xi_{jk}⟩).$$

The resulting functions make up a frame which is not tight, but very close to, in a sense which is made rigorous in [15]. Exact cubature formulae cannot hold (in particular, $\{\lambda_{jk}\}$ are not exactly cubature weights in this case), because polynomials of infinitely large order are involved in the construction, but again this entails very minor approximations in practical terms. More generally, it is possible to consider higher order Mexican needlets by focussing on $f(s) =$
\( s^p \exp(-s) \), so to obtain

\[
\psi_{jk,p}(x) := \sqrt{\lambda_{jk}} \sum_{l \geq 1} \left( \frac{(l+1)}{B^{2j}} \right) e^{-l(l+1)/B^{2j}} L_l \left( \langle x, \xi_{jk} \rangle \right).
\]

The random spherical Mexican needlet coefficients are immediately seen to be given by

\[
\beta_{jk,p} = \int_{S^2} T(x) \psi_{jk}(x) \, dx = \int_{S^2} \sum_{l \geq 0} a_{lm} Y_{lm}(x) \psi_{jk,p}(x) \, dx
\]
\[
= \sqrt{\lambda_{jk}} \sum_{l \geq 1} \left( \frac{l(l+1)}{B^{2j}} \right) e^{-l(l+1)/B^{2j}} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\xi_{jk}),
\]

whence their covariance is

\[
E[\beta_{jk,p} \beta_{jk',p}] = \sqrt{\lambda_{jk} \lambda_{jk'}} \sum_{l \geq 1} \left( \frac{(l+1)}{B^{2j}} \right) e^{-2l(l+1)/B^{2j}} \frac{2l+1}{4\pi} C_l P_l \left( \langle \xi_{jk}, \xi_{jk'} \rangle \right).
\]

Throughout this paper, we shall only consider weight functions of the form \( f(s) = s^p \exp(-s) \). It is certainly possible to consider more general constructions; however this specific shape lends itself to very neat results, allowing us to produce both upper and lower bounds for the coefficients’ correlation. Also, it makes possible a clear interpretation of the final results, i.e. the effect of varying \( p \) on the structure of dependence is immediately understood; this is, we believe, a valuable asset for practitioners. In the sequel, we shall drop the subscript \( p \), whenever possible without risk of confusion.

**Remark 2** As mentioned earlier, it is suggested from results in \([15]\) that Mexican needlets provide asymptotically a very good approximation to the widely popular Spherical Mexican Hat Wavelets (SMHW), which have been used in many physical papers; the asymptotic analysis of the stochastic properties of SMHW coefficients is still completely open for research. The discretized form of the SMHW can be written as

\[
\Psi_{jk}(\theta; B^{-j}) = \frac{1}{(2\pi)^{1/2}} \sqrt{2 B^{-j}} (1 + B^{-2j} + B^{-4j})^{1/2} \left[ 1 + \left( \frac{y^2}{2} \right)^2 \right]^2 \left[ 2 - \frac{y^2}{2(2j)^2} \right] e^{-y^2/4B^{-j^2}},
\]

where the coordinates \( y = 2 \tan \frac{\theta}{2} \) follows from the stereographic projection on the tangent plane in each point of the sphere; here we take \( \theta = \theta_{jk}(x) := d(x, \xi_{jk}) \).

Now write

\[
\psi_{jk,p}(\theta_{jk}(x)) = \psi_{jk,p}(\theta);
\]

by following the arguments in \([15]\) and developing their bounds further, it can be argued that

\[
|\Psi_{jk}(\theta; B^{-j}) - K_{jk} \psi_{jk,p}(\theta)| = B^{-j} \mathcal{O} \left( \min \{\theta^1 B^4, 1\} \right), \quad (8)
\]
for some suitable normalization constant $K_{jk} > 0$. Equation (8) suggests that our results below can be used as a guidance for the asymptotic theory of random SMHW coefficients. The validity of this approximation over relevant cosmological models and its implications for statistical procedures of CMB data analysis are currently being investigated.

### 3 Stochastic properties of Mexican needlet coefficients, I: upper bounds

As mentioned in the Introduction, having established (7) opened the way to several developments for the statistical analysis of spherical random fields. It is therefore a very important question to establish under what circumstances these results can be extended to other constructions, such as Mexican needlets. In this and the following Section, we provide a full characterization with positive and negative results. We start by writing the expression for the correlation coefficients, which is given by

$$\text{Corr}(\beta_{j_1k_1}; \beta_{j_2k_2}) = \sum_{l \geq 1} \frac{\ell(l+1)B(l+1)}{B^2} p e^{-\frac{2(l+1)}{B^2}} \frac{e^{-\frac{2l}{B^2}}}{\sqrt{B^2}} (2l+1)C_l P_l(\langle \xi_{j_1k_1}, \xi_{j_2k_2} \rangle)$$

We now provide upper bounds on the correlation of random coefficients, as follows.

**Theorem 3** Assume Condition (4) holds with $\alpha < 4p + 2$ and $M \geq 4p + 2 - \alpha$; then there exist some constant $C_M > 0$

$$|\text{Corr}(\beta_{j_1k_1}; \beta_{j_2k_2})| \leq \frac{C_M}{1 + B(j_1+j_2)/2 - \log_B (j_1+j_2)/2d(\xi_{j_1k_1}, \xi_{j_2k_2})(4p+2-\alpha)}.$$  

(9)

**Proof.** We prove (9) following some ideas in [26]. For notational simplicity, we focus first on the case where $j_1 = j_2$; we have

$$\text{Corr}(\beta_{jk}; \beta_{jk'; p}) = \sum_{l \geq 1} \frac{\ell(l+1)}{B^2} p e^{-2l(l+1)/B^2} \frac{e^{-2l^2/B^2}}{4\pi} \frac{2l+1}{4\pi} C_l P_l(\langle \xi_{jk}, \xi_{jk'} \rangle)$$

We now provide upper bounds on the correlation of random coefficients, as follows.

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1While finishing this paper, we learned by personal communication that working independently and at the same time as us, A. Mayeli has obtained a result similar to Theorem 3 for the case $j_1 = j_2$, see [24]. The statements and the assumptions in the two approaches are not equivalent and the methods of proofs are entirely different; we believe both are of independent interest.
Now replace \( C_l = l^{-\alpha} g_j \left( \frac{d}{dx} \right) \) in the denominator of the above representation, for which we get
\[
C_l = \frac{1}{4\pi} \left( \frac{l(l+1)}{B^2_j} \right)^{2p} \frac{e^{-2(l(l+1))/B^2_j}}{4\pi} \frac{2l+1}{4\pi} C_l \leq \frac{1}{4\pi} \frac{B^{-(4p-2-\alpha)}j}{4p+2-\alpha}.
\]

Denoting \( \theta = \arccos \left( \xi_{j,k}, \xi_{j,k'} \right) \), the numerator can be written as
\[
\sum_{l \geq 1} \left( \frac{l(l+1)}{B^2_j} \right)^{2p} e^{-2(l(l+1))/B^2_j} \frac{2l+1}{4\pi} \sum_{\mu} \sum_{\pm} \int_\theta \frac{\sin \left( \frac{l+\frac{1}{2}}{B^2_j} \varphi \right)}{\cos \theta - \cos \varphi} \frac{1}{2\pi} d\varphi
\]
where we have used the Dirichlet-Mehler integral representation for the Legendre polynomials \([16]\).

The following steps and notations are very close to \([26]\). We write
\[
C_{B,g_j} = \sum_{l \geq 1} \left( \frac{l(l+1)}{B^2_j} \right)^{2p} \frac{e^{-2(l(l+1))/B^2_j}}{4\pi} \frac{2l+1}{4\pi} l^{-\alpha} g_j \left( \frac{l}{B^2_j} \right) \sin \left( l + \frac{1}{2} \right) \varphi_{12}
\]
where
\[
h_{j \pm}(u) = \frac{u(u+1)}{B^2_j} \sum_{\mu} \sum_{\pm} \left( 
\begin{array}{c}
\sum_{j \in \mathbb{Z}} h_{j \pm}(l) = \frac{1}{2} \sum_{l \in \mathbb{Z}} h_{j \pm}(l) = \frac{1}{2} \sum_{\mu \in \mathbb{Z}} \hat{h}_{j \pm}(2\pi\mu)
\end{array}
\right.
\]

Denote
\[
G_{\alpha,j} (t) := t^{2p-\alpha} g_j (t) e^{-2(t+2^{-j})} I_{(t,\infty)}.
\]

Let us now recall the following standard property of Fourier transforms:
\[
(i \omega)^k \frac{d^m}{d\omega^m} \hat{f}(\omega) = (\frac{d^k}{dx^k} \{ x^m f(x) \}) \hat{f}(\omega).
\]

Some simple computations yield
\[
\hat{h}_{\pm}(2\pi\mu) = \frac{B^{(2-\alpha)}_{j} \pm i\varphi}{4\pi} \left\{ \frac{2p}{m} \left[ \left( \frac{2p}{m} \right) + \left( \frac{2p}{m} \right) \right] B^{-(2p+1-m)j} \frac{d^m}{d\omega^m} + B^{-(2p+1)j} + 2 \frac{d^{2p+1}}{d\omega^{2p+1}} \right\}
\]
\[
\times \int_{-\infty}^{\infty} G_{\alpha,j} (t/B^j) e^{i(t+\frac{1}{2})\varphi - i\omega t} dt \big|_{\omega=2\pi\mu}
\]
\[
= \frac{B^{(2-\alpha)}_{j} \pm i\varphi}{4\pi} \left\{ \frac{2p}{m} \left[ \left( \frac{2p}{m} \right) + \left( \frac{2p}{m} \right) \right] B^{-(2p+1-m)j} \frac{d^m}{d\omega^m} \right\} \hat{G}_{\alpha,j} (\omega) \big|_{\omega=B^j(2\pi\mu \mp \varphi)}
\]
\[
+ \frac{B^{(2-\alpha)}_{j} \pm i\varphi}{4\pi} \left\{ B^{-(2p+1)j} + 2 \frac{d^{2p+1}}{d\omega^{2p+1}} \right\} \hat{G}_{\alpha,j} (\omega) \big|_{\omega=B^j(2\pi\mu \mp \varphi)}.
\]
where
\[ \widehat{G}_{\alpha,j}(\omega) = \int_{\mathbb{R}} G_{\alpha,j}(t) e^{-it\omega} dt. \]

For all positive integers \( k \leq M \), we can obtain
\[
\left| \int_{B^{-j}} \frac{d^k}{dt^k} \{ i^m G_{\alpha,j}(t) \} dt \right| \leq \begin{cases} \frac{k! \Gamma(m+2p-\alpha)C_g B^{-j(2p+1+m-\alpha-k)}}{k! \Gamma(m+2p-\alpha)C_g(j \log B)} , & \text{for } 2p+1-\alpha+m \neq k \\ k \Gamma(m+2p-\alpha)C_g(j \log B) , & \text{for } 2p+1-\alpha+m = k \end{cases}
\]
where \( C_g = \max \{ c_0, ..., c_M \} \) and
\[
L(p,m,\alpha,k) := \begin{cases} \Gamma(2p+1+m-\alpha-k) \text{ when } (2p+1+m-\alpha-k) > 0 , \\ (\Gamma(k-2p-1-m+\alpha))^{-1} \text{ when } (2p+1+m-\alpha-k) < 0 . \end{cases}
\]

It should be noticed that our argument here differs from the one in [26], because we cannot assume the integrand function on the left-hand side to be in \( L^1 \) for all \( k \leq M \). Let us now focus on the case where \( k = 4p+2-\alpha \), with \( m = 2p+1 \); we obtain
\[
\left| \frac{d^{2p+1}}{d\omega^{2p+1}} \widehat{G}_{\alpha,j}(\omega) \right| \leq \left| B^{j(2\pi\mu-\varphi)} \right|^4 \Gamma(4p+1-\alpha) C_g(j \log B);
\]
therefore
\[
\left| \frac{d^{2p+1}}{d\omega^{2p+1}} \widehat{G}_{\alpha,j}(\omega) \right| \leq \frac{C_{2p+1,\alpha,g} B^{-j(4p+2-\alpha)+\log B} j}{(2\pi\mu-\varphi)^{4p+2-\alpha}},
\]
where
\[
C_{2p+1,\alpha,g} = (4p+2-\alpha) \Gamma(4p+1-\alpha) C_g \log B.
\]
The modifications needed for other cases are obvious and we obtain
\[
B^{-(2p+1-m)j} \left| \frac{d^m}{d\omega^m} \widehat{G}_{\alpha,j}(\omega) \right| \leq \frac{C_{m,\alpha,g} B^{-j(4p+2-\alpha)}}{(2\pi\mu-\varphi)^{4p+2-\alpha}},
\]
where
\[
C_{m,\alpha,g} = \frac{(4p+2-\alpha) \Gamma(m+2p-\alpha) C_g}{L(p,m,\alpha,k)}.
\]
Now let \( C_{\alpha,g} = \max \{ C_{m,\alpha,g}, m = 0, ..., 2p+1 \} \); we have
\[
\left| \widehat{h}_\pm(2\pi\mu) \right| \leq \frac{B^{(2-\alpha)j}}{4\pi} C_{\alpha,g} \left\{ \sum_{m=0}^{2p} \binom{2p}{m} B^{-j(4p+2-\alpha)} + B^{-j(4p+2-\alpha)+\log B j} \right\}
\leq \frac{2^{2p} C_{\alpha,g} B^{-4pj+\log B j}}{(2\pi\mu-\varphi)^{4p+2-\alpha}}, \quad \mu = 1, 2, ...
\]
\[
(15)
\]
Therefore
\[
\|C_{B,j}\| \leq 2^{2p} C_{\alpha,g} \left( \frac{1}{2\varphi^{4p+2-\alpha}} + \sum_{\mu \in N} \frac{1}{|2\pi \mu \pm \varphi|^{4p+2-\alpha}} \right) B^{-4pj+\log B} j
\]
\[
\leq 2^{2p} \left( \frac{1}{2\varphi^{4p+2-\alpha}} + |4p + 1 - \alpha|^{\pi \alpha - 4p - 1} \right) C_{\alpha,g} B^{-4pj+\log B} j.
\]

Hence, for the numerator of the correlation we have the bound
\[
\|C_{B,j}\| \leq CB^{-4pj+\log B} j \int_0^\pi \frac{\pi^{\alpha - 4p - 1}}{(\cos \theta - \cos \varphi)^{1/2}} d\varphi.
\]

As in (10), when \(0 \leq \theta \leq \pi/2\), we can get the following inequality
\[
\|C_{B,j}\| \leq C_{\alpha,k,g} B^{-4pj+\log B} j \int_0^\pi \frac{\pi^{\alpha - 4p - 1}}{0.27 \varphi^{4p+2-\alpha} (\varphi^2 - \theta^2)^{1/2}} d\varphi \leq C_1 B^{-4pj+\log B} j \theta^{\alpha - 4p - 2}.
\]

If \(\pi/2 \leq \theta \leq \pi\), letting \(\bar{\theta} = \pi - \theta, \bar{\varphi} = \pi - \varphi\), we can obtain the same bound.

Going back to (11), we obtain
\[
\text{Corr}(\beta_{j,k}, \beta_{j',k'}) \leq \frac{C g^{\alpha - 4p - 2} B^{-4pj+\log B} j}{B^{(2-\alpha)j}} \leq C \theta^{\alpha - 4p - 2} B^{-(j-\log B)(4p+2-\alpha)} \to 0, \text{ as } j \to \infty.
\]

We thus get inequality (9) for \(j_1 = j_2\).

Now let us consider \(j_1 \neq j_2\). As the proof is very similar to the arguments above, we omit many details. In the sequel, for any two sequences \(a_l, b_l\), we write \(a_l \approx b_l\) if and only if \(a_l = O(b_l)\) and \(b_l = O(a_l)\).

First, we consider the variance of the random coefficients, which can be represented by:
\[
\sum_{l \geq 1} \left( \frac{l(l+1)}{B^{2j_l}} \right) 4^p e^{-2l(l+1)/B^{2j_l}} (2l+1) C_l \approx B^{(2-\alpha)j_1} \left\{ \left( \int_0^1 + \int_1^\infty \right) 4^p e^{-2l(l+1)/B^{2j_l}} g_{j_1}(t) dt \right\}
\]
\[
= B^{(2-\alpha)j_1} \left\{ \frac{c_0}{4p + 2 - \alpha} B^{-j_1(4p+2-\alpha)} + O(1) \right\};
\]

therefore
\[
\left\{ \sum_{l \geq 1} \left( \frac{l(l+1)}{B^{2j_l}} \right) 4^p e^{-2l(l+1)/B^{2j_l}} (2l+1) C_l \right\}^{1/2}
\]
\[
= B^{(1-\alpha/2)(j_1+j_2)} \left\{ \frac{c_0}{4p + 2 - \alpha} B^{-j_1(4p+2-\alpha)} + O(1) \right\}^{1/2}
\]
\[
= O(1) B^{(1-\alpha/2)(j_1+j_2)}.\]
Thus (9) is established.

Without loss of generality, we can always assume $j_1 < j_2$. We can implement the same argument as before, provided we replace $C_{B,g_j}$ in (12) by

$$C_{B,g,j_1,j_2} = \sum_{l \geq 1} \left( \frac{l(l+1)}{Bj_1+j_2} \right)^2 p e^{-l(l+1)(B^{-2j_1}+B^{-2j_2}) \frac{2l+1}{4\pi} l^{-\alpha} g_{j_1,1} \left( \frac{l}{B^2} \right) \sin \left( \frac{l+1}{2} \right) \varphi}$$

$$= : \frac{1}{2} \sum_{l \geq 1} (h_{j_1,j_2,+}(l) - h_{j_1,j_2,-}(l))$$

where

$$h_{j_1,j_2,\pm}(u) = \left( \frac{u(u+1)}{Bj_1+j_2} \right)^2 p \frac{2u+1}{4\pi} u^{-\alpha} g_{j_1} \left( \frac{u}{B^2} \right) e^{-u(u+1)(B^{-2j_1}+B^{-2j_2}) \pm (u+\frac{1}{2}) \varphi}.$$  

Again, by Poisson summation formula, we have

$$\sum_{l \geq 1} h_{j_1,j_2,\pm}(l) = \frac{1}{2} \sum_{l \in \mathbb{Z}} h_{j_1,j_2,\pm}(l) = \frac{1}{2} \sum_{\mu \in \mathbb{Z}} \hat{h}_{j_1,j_2,\pm}(2\pi \mu).$$

Denote

$$G_{\alpha,j_1,j_2}(t) := t^{2p-\alpha} g_{j_1}(t) e^{-t(1+B^{-j_1})(1+B^{2j_1-j_2})} I_{(B^{-j_2},\infty)};$$

by the same argument and notation as in (13), we have

$$\left| \hat{h}_{j_1,j_2,\pm}(2\pi \mu) \right| = \frac{B^{2p(j_1-j_2)+(2-\alpha)j_1}}{4\pi} \left\{ \sum_{m=1}^{2p} \frac{2p}{m-1} B^{-2p(2p+1-m)j_1} \frac{d^m}{d\omega^m} G_{\alpha,j_1,j_2}(\omega) \bigg| \omega = Bj_1(2\pi \mu \mp \varphi) \right. $$

$$+ \frac{B^{2p(j_1-j_2)+(2-\alpha)j_1}}{4\pi} \frac{d^{2p+1}}{d\omega^{2p+1}} \left\{ B^{-2p(2p+1)j_1} + 2 \frac{d^{2p+1}}{d\omega^{2p+1}} \right\} \hat{G}_{\alpha,j_1,j_2}(\omega) \bigg| \omega = Bj_1(2\pi \mu \mp \varphi) \right\} \frac{\varepsilon}{(2\pi \mu \mp \varphi)^{4p+2-\alpha}}, \mu = 1, 2, ...$$

Therefore

$$\left| C_{B,g,j_1,j_2} \right| \leq 2^{2p} C_{\alpha,g} \frac{1}{2\varphi^{4p+2-\alpha}} + |4p+1-\alpha| \pi^{\alpha-4p-1} B^{-2p(j_1+j_2)+\log_B j_1}.$$  

It is then straightforward to conclude as in the case where $j_1 = j_2$, to obtain

$$\text{Corr} \left( \beta_{j_1,k}, \beta_{j_2,k} \right) \leq C \frac{\theta^{\alpha-4p-2} B^{-2p(j_1+j_2)+\log_B j_1}}{B^{(1-\alpha/2)(j_1+j_2)}} \leq C \theta^{\alpha-4p-2} B^{-\left[j_1+j_2\right]/2-\log_B (j_1+j_2)/2(4p+2-\alpha)} \to 0, \quad \text{as } j_2 \to \infty.$$  

Thus (9) is established. ■
Remark 4 By careful manipulation, we can obtain an explicit expression for the constant $C_M$ in (9), i.e.

$$C_M = 2^{2p} r^{M+1} (4p + 2 - \alpha)^2 \Gamma (4p + 1 - \alpha) c_0 C_g \log B.$$

The previous result shows that Mexican needlets can enjoy the same uncorrelation properties as standard needlets, in the circumstances where the angular power spectrum is decaying “slowly enough”. The extra log term in (9) is a consequence of a standard technical difficulty when dealing with a boundary case in the integral in (14).

Remark 5 To obtain central limit results for finite-dimensional statistics based on nonlinear transformations of the Mexican needlets coefficients, it would be sufficient to consider the case where $j = j'$. The asymptotic uncorrelation we established in Theorem 3 is stronger; indeed, for many applications it is useful to consider different scales $\{j\}$ at the same time. Because of this, it is also important to focus on the correlation of Mexican needlets coefficients at different $j, j'$. We stress that the need for such analysis was much more limited for standard needlets; indeed in the latter case, given the compactly supported kernel $b(\cdot)$, the frequency support of the various coefficients is automatically disjoint when $|j - j'| \geq 2$, whence (for completely observed random fields) standard needlet coefficients can be correlated only for $|j - j'| = 1$.

4 Stochastic properties of Mexican needlet coefficients, II: lower bounds

In this Section, we complete the previous analysis, establishing indeed that the random Mexican needlets coefficients are necessarily correlated at some angular distance in the presence of faster memory decay. This is clearly different from needlets, which are always uncorrelated. As mentioned in the Introduction, the heuristic rationale behind this duality can be explained as follows: it should be stressed that we are focussing on high-resolution asymptotics, i.e. the asymptotic behaviour of random coefficients at smaller and smaller scales in the same random realization. For such asymptotics, a crucial role can be played by terms which remain constant across different scales. In the case of usual needlets, which have bounded support over the multipoles, terms like these are simply dropped by construction. This is not so for Mexican needlets, which in any case include components at the lowest scales. These components are dominant when the angular power spectrum decays fast, and as such they prevent the possibility of asymptotic uncorrelation. In particular, we have correlation when the angular power spectrum is such that $\alpha > 4p + 2$. 

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Theorem 6 Under condition $\mathbf{1}$ for $\alpha > 4p + 2$, $\forall \varepsilon \in (0, 1)$, there exists a positive $\delta \leq \varepsilon \left(1 + c_0^2\right)^{-1/(\alpha - 4p - 2)}$ such that

$$\liminf_{j \to \infty} \text{Corr} \left( \beta_{jk}; \beta_{j'k'} \right) > 1 - \varepsilon ,$$

for all $\{\xi_{jk}, \xi_{j'k'}\}$ such that $d(\xi_{jk}, \xi_{j'k'}) \leq \delta$.

Proof. We first divide the variance of the coefficients into three parts, as follows

$$
\left( \sum_{1 \leq l < \ell_1, B_j} \frac{\ell(l + 1)}{B^{2j}} + \sum_{\ell_1, B_j \leq l < \ell_2 B_j} \frac{\ell(l + 1)}{B^{2j}} \right) \frac{2p}{4\pi} e^{-2l(l+1)/B^{2j}} \frac{2l + 1}{C_l} \leq 2 \sum_{1 \leq l < \ell_1, B_j} \frac{l}{B^{4p+2}} \frac{l}{\pi} \gamma_j(l) \frac{l}{B^{2j}} e^{-2l(l+1)/B^{2j}} \frac{2l + 1}{C_l} \leq 2 \sum_{1 \leq l < \ell_1, B_j} \frac{l}{B^{4p+2}} \frac{l}{\pi} \gamma_j(l) \frac{l}{B^{2j}} e^{-2l(l+1)/B^{2j}} \frac{2l + 1}{C_l} =: A_{1j} + A_{3j} + A_{3j}.
$$

Intuitively, our idea is to show that the first sum is of exact order $O(B^{(2-\alpha)j} \times B^{(\alpha - 4p - 2)j}) = O(B^{-4pj})$, while the second two are smaller ($O(B^{(2-\alpha)j})$ and $o(B^{(2-\alpha)j})$, respectively). Indeed, for the first part we obtain easily

$$A_{1j} = \sum_{1 \leq l < \ell_1, B_j} \frac{\ell(l + 1)}{B^{2j}} \frac{2p}{4\pi} e^{-2l(l-1)/B^{2j}} \frac{2l + 1}{C_l} \leq 2 \int_{B-j}^{B} x^{4p+1-\alpha} dx = 2 \frac{c_0 (B^{(\alpha - 4p - 2)j} - B^{(4p+2)j})}{\pi (\alpha - 4p - 2)} B^{(2-\alpha)j},$$

and

$$\sum_{1 \leq l < \ell_1, B_j} \frac{\ell(l + 1)}{B^{2j}} \frac{2p}{4\pi} e^{-2l(l+1)/B^{2j}} \frac{2l + 1}{C_l} \geq \frac{c_0 (B^{(\alpha - 4p - 2)j} - B^{(4p+2)j})}{\pi (\alpha - 4p - 2)} e^{-2l(l+1)/B^{2j}} \frac{2l + 1}{C_l}.$$

Similarly, for the second part

$$\frac{c_2 (B^{(\alpha - 4p - 2)j} - B^{(4p+2)j})}{\pi (\alpha - 4p - 2)} e^{-2l(l+1)/B^{2j}} \frac{2l + 1}{C_l} \leq 2 \frac{c_0 (B^{(\alpha - 4p - 2)j} - B^{(4p+2)j})}{\pi (\alpha - 4p - 2)} e^{-2l(l+1)/B^{2j}} \frac{2l + 1}{C_l}.$$

and for the third part

$$\sum_{l \geq \ell_2 B_j} \frac{l}{B^{2j}} \frac{2p}{4\pi} e^{-2l(l+1)/B^{2j}} \frac{2l + 1}{C_l} \leq \frac{3c_0}{4\pi} B^{(2-\alpha)j} \int_{\ell_2}^{\infty} x^{4p+1-\alpha} e^{-x^2} dx \leq \frac{3c_0}{4\pi} B^{(2-\alpha)j} \int_{\ell_2}^{\infty} x^2 e^{-x^2} dx \leq \frac{3c_0}{4\pi} B^{(2-\alpha)j} \int_{\ell_2}^{\infty} x^{4p+1-\alpha} e^{-x^2} dx , \quad (\ell_2 > 1).$$
The last inequality follows from the asymptotic formula
\[
\int_{y}^{\infty} e^{-x^2/2} dx \sim \frac{1}{y} e^{-y^2/2}, \quad y \to \infty.
\]
We have then that
\[
\frac{A_{3j}}{A_{1j}} = \left\{ \sum_{1 \leq l < \epsilon_1 B^j} \left( \frac{l(l+1)}{B^{2j}} \right)^{2p} e^{-2l(l+1)/B^{2j}} \frac{2l+1}{4\pi} C_l \right\}^{-1}
\times \left\{ \sum_{l \geq \epsilon_2 B^j} \left( \frac{l(l+1)}{B^{2j}} \right)^{2p} e^{-2l(l+1)/B^{2j}} \frac{2l+1}{4\pi} C_l \right\}
= O\left(B^j(4p+2-\alpha)\right) = o(1), \text{ as } j \to \infty.
\]
On the other hand, for any positive \( \epsilon < 1 \), if we choose \( \epsilon_1 = NB^{-j} \), where \( N \) is sufficiently large that
\[
N^{4p+2-\alpha} \left( 1 + \frac{2\epsilon^2}{\epsilon_0^2} \right) < 1,
\]
whence
\[
\frac{A_{2j}}{A_{1j}} < \frac{\epsilon}{2} \text{ as } j \to \infty.
\]
Thus we obtain that
\[
\left\{ \sum_{l \geq 1} \left( \frac{l(l+1)}{B^{2j}} \right)^{2p} e^{-2l(l+1)/B^{2j}} \frac{2l+1}{4\pi} C_l \right\}^{-1} = \frac{1}{A_{1j}} \left\{ 1 + \frac{A_{2j}}{A_{1j}} + \frac{A_{3j}}{A_{1j}} \right\}^{-1}
\geq \frac{1}{A_{1j}} \left\{ 1 + \frac{\epsilon}{2} + o(1) \right\}^{-1}
\geq cB^{-4pj}, \text{ some } c > 0.
\]
More explicitly, the variance at the denominator has the same order as for the summation restricted to the elements in the range \( 1 \leq l < \epsilon_1 B^j \).

To analyze the numerator, we start by recalling that
\[
\sup_{\theta \in [0, \pi]} P_l(\cos \theta) = P_l(\cos 0) = 1, \text{ and } \sup_{\theta \in [0, \pi]} \left| \frac{d}{d\theta} P_l(\cos \theta) \right| \leq 3l.
\]
As a consequence, for any \( \epsilon > 0 \) there exists a \( \delta > 0 \), s.t. if \( 0 < \theta \leq \delta \leq \epsilon/6N \), then,
\[
|P_l(\cos \theta) - P_l(\cos 0)| \leq 3l \theta \leq \epsilon,
\]
for all \( l > N \), where the above inequalities follow from
\[
0 \leq \cos(\theta + \theta) - \cos \theta = 2 \sin^2 \frac{\theta}{2} \leq \theta, \quad \forall \theta \in [0, 2\pi).
\]
Therefore, for any \( \xi_{jk}, \xi_{jk'} \in S^2 \) s.t. \( \arccos (\xi_{jk}, \xi_{jk'}) \leq \delta \), we have

\[
\text{Corr} \left( \beta_j, \beta_{j'} \right) = \sum_{l \geq 1} \frac{\left( \frac{l(l+1)}{B^2} \right)^{2p} e^{-2l(l+1)/B^2} 2l+1}{4\pi} C_l \left( \langle \xi_{jk}, \xi_{jk'} \rangle \right)
\]

\[
\geq \sum_{1 \leq l \leq N} \frac{\left( \frac{l(l+1)}{B^2} \right)^{2p} e^{-2l(l+1)/B^2} 2l+1}{4\pi} C_l P_l \left( \langle \xi_{jk}, \xi_{jk'} \rangle \right) - P_l (\cos 0)
\]

\[
\geq \sum_{1 \leq l \leq N} \frac{\left( \frac{l(l+1)}{B^2} \right)^{2p} e^{-2l(l+1)/B^2} 2l+1}{4\pi} C_l P_l \left( \langle \xi_{jk}, \xi_{jk'} \rangle \right)
\]

\[
+ \sum_{l > N} \frac{\left( \frac{l(l+1)}{B^2} \right)^{2p} e^{-2l(l+1)/B^2} 2l+1}{4\pi} C_l
\]

\[
\geq \sum_{1 \leq l \leq N} \frac{\left( \frac{l(l+1)}{B^2} \right)^{2p} e^{-2l(l+1)/B^2} 2l+1}{4\pi} C_l \times (1 - \varepsilon)
\]

\[
+ O(B^{j(4p+2-\alpha)})
\]

\[
= \frac{(1 - \varepsilon)}{1 + \varepsilon/2} + o(1) = (1 - \varepsilon') + o(1), \text{ as } j \to \infty, \varepsilon' = \frac{3\varepsilon}{2 + \varepsilon}.
\]

Thus the proof is completed. \( \blacksquare \)

As mentioned earlier, the results in the previous two Theorems illustrate an interesting trade-off between the localization and correlation properties of spherical needlets. In particular, we can always achieve uncorrelation by choosing \( p > (\alpha - 2)/4 \); of course \( \alpha \) is generally unknown and must be estimated from the data (in this sense standard needlets have better robustness properties). Introducing higher order terms implies lowering the weight of the lowest multipoles, i.e., improving the localization properties in frequency space. On the other hand, it may be expected that such an improvement of the localization properties in the frequency domain will lead to a worsening of the localization in pixel space, as a consequence of the Uncertainty Principle we mentioned in the Introduction (see for instance [18]). We do not investigate this issue here, but we shall provide some numerical evidence on this phenomenon in an ongoing work.
Remark 7 In Theorem 6, we decided to keep the assumptions as close as possible to Theorem 3, in order to ease comparisons and highlight the symmetry between the two results. However, it is simple to show that the correlation result holds in much greater generality, for angular power spectra that have a decay which is faster than polynomial. In particular, assume that

$$C_l = H(l) \exp(-lp) \ , \ l = 1, 2, ...$$

where $H(l)$ is any kind of polynomial such that $H(l) > c > 0$ and $p > 0$. Then it is simple to establish the same result as in Theorem 6, by means of a simplified version of the same argument. The underlying rationale should be easy to get: for exponentially decaying power spectra the dominating components are at the lowest frequencies, and they introduce correlations among all random coefficients which cannot be neglected.

5 Statistical Applications

The previous results lend themselves to several applications for the statistical analysis of spherical random fields, in particular with a view to CMB data analysis. Similarly to [3], let us consider polynomials functions of the normalized Mexican needlets coefficients, as follows

$$h_{u,N_j} := \frac{1}{\sqrt{N_j}} \sum_{k=1}^{N_j} \sum_{q=1}^{Q} w_{uq} H_q(\tilde{\beta}_{jk;p}) \ , \ \tilde{\beta}_{jk;p} := \frac{\beta_{jk;p}}{\sqrt{E\beta_{jk;p}^2}} \ , \ u = 1, ..., U ,$$

where $H_q(.)$ denotes the $q$-th order Hermite polynomials (see [33]), $N_j$ is the cardinality of coefficients corresponding to frequency $j$ (where we take $\{\xi_{jk}\}$ to form a $B^{-j}$-mesh, see [4], so that $N_j \approx B^{2j}$), and $\{w_{uq}\}$ is a set of deterministic weights that must ensure these statistics are asymptotically nondegenerate, i.e.

Condition 8 There exist $j_0$ such that for all $j > j_0$

$$\text{rank}(\Omega_j) = U \ , \ \Omega_j := Eh_{N_j} h'_{N_j} \ , \ h_{N_j} := (h_{1,N_j}, ..., h_{U,N_j})'.$$

Condition 8 is a standard invertibility assumption which will ensure our statistics are asymptotically nondegenerate (for instance, it rules out multicollinearity). Several examples of relevant polynomials are given in [3]; for instance, given a theoretical model for the angular power spectrum $\{C_l\}$, it is
suggested in that reference that a goodness-of-fit statistic might be based upon

\[
\frac{1}{\sqrt{N_j}} \sum_{k=1}^{N_j} H_2(\hat{\beta}_{jk;p}) = \frac{1}{\sqrt{N_j}} \sum_{k=1}^{N_j} (\hat{\beta}_{jk;p}^2 - 1) = \frac{1}{\sqrt{N_j}} \sum_{k=1}^{N_j} (\lambda_{jk} \sum_{l \geq 1} b^2 \left( l^2 \frac{H_1(\hat{\beta}_{jk;p})}{\pi^2} \right) \frac{2l+1}{4\pi} C_l - 1). 
\]

The statistic

\[
\hat{\Gamma}_j = \frac{1}{N_j} \sum_{k=1}^{N_j} \frac{\beta_{jk;p}^2}{\lambda_{jk}}
\]

can then be viewed as an unbiased estimator for

\[
\Gamma_j = E\hat{\Gamma}_j = \sum_{l \geq 1} b^2 \left( l^2 \frac{H_1(\hat{\beta}_{jk;p})}{\pi^2} \right) \frac{2l+1}{4\pi} C_l.
\]

We refer to [13], [8] for the analysis of this estimator in the presence of missing observations and noise, and for its application to CMB data in the standard needlets case. Our results below can be viewed as providing consistency and asymptotic Gaussianity (for fully observed maps and without noise) in the Mexican needlets approach. As always in this framework, consistency has a non-standard meaning, as we do not have convergence to a fixed parameter, but rather convergence to unity of the ratio \(\hat{\Gamma}_j/\Gamma_j\).

Likewise, tests of Gaussianity could be implemented by focussing on the skewness and kurtosis of the wavelets coefficients (see for instance [25]), i.e. by focussing on

\[
\frac{1}{\sqrt{N_j}} \sum_{k=1}^{N_j} \left\{ H_3(\hat{\beta}_{jk;p}) + H_1(\hat{\beta}_{jk;p}) \right\} = \frac{1}{\sqrt{N_j}} \sum_{k=1}^{N_j} \beta_{jk;p}^3 \text{ and }
\]

\[
\frac{1}{\sqrt{N_j}} \sum_{k=1}^{N_j} \left\{ H_4(\hat{\beta}_{jk;p}) + 6H_2(\hat{\beta}_{jk;p}) \right\} = \frac{1}{\sqrt{N_j}} \sum_{k=1}^{N_j} \left\{ \hat{\beta}_{jk;p}^4 - 3 \right\}.
\]

The joint distribution for these statistics is provided by the following results:

**Theorem 9** Assume \(T\) is a Gaussian mean square continuous and isotropic random field; assume also that Conditions 1, 8 are satisfied and choose \(p > (\alpha + \delta)/4\), some \(\delta > 0\). Then as \(N_j \to \infty\)

\[
\Omega_j^{-1/2} h_{N_j} \to_d N(0, I_U).
\]

**Proof.** The asymptotic behaviour of our polynomial statistics can be established by means of the method of moments. In particular, it is possible to
exploit the diagram formula for higher order moments of Hermite polynomial, as explained for instance in [28, 33]. The details are the same as in [3], and thus they are omitted for brevity’s sake. We only note that, in order to be able to use Lemma 6 in that reference, we need to ensure that

$$\left|\text{Corr} (\beta_{jk';p}; \beta_{jk;p})\right| \leq \frac{C}{\left(1 + Bd(\xi_{jk}, \xi_{jk'})\right)^{2+\delta}}$$

some $C > 0$.

In view of (9), this motivates the tighter limit we need to impose on the value of $p$. ■

It may be noted that the covariance matrix $\Omega_j$ can itself be consistently estimated from the data at any level $j$, for instance by means of the bootstrap/subsampling techniques that are detailed in [4]. Again, the arguments of that paper lend themselves to straightforward extensions to the present circumstances, as they rely uniquely upon the covariance inequalities for the random wavelet coefficients. Likewise, the previous results may be extended to cover statistical functionals over different frequencies, for instance the bispectrum ([21, 22]). A much more challenging issue relates to the relaxation of the Gaussianity assumptions, which is still under investigation.

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