A Wave Function approach
to dissipative processes

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Abstract: We present a wave function approach to study the evolution of a small system when it is coupled to a large reservoir. Fluctuations and dissipation originate in this approach from quantum jumps occurring randomly during the time evolution of the system. This approach can be applied to a wide class of relaxation operators in the Markovian regime, and it is equivalent to the standard Master Equation approach.

Published in AIP Conference Proceedings 275, Thirteenth International Conference on Atomic Physics, Munich, Germany 1992; Editors: H. Walther, T. W. Hänsch, and B. Neizert.

The problem of dissipation plays a central role in Atomic Physics and Quantum Optics. The simplest example is the phenomenon of spontaneous emission, where the coupling between an atom and the ensemble of modes of the quantized electromagnetic field gives a finite lifetime to all excited atomic levels. Usually

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the dissipative coupling between a small system and a large reservoir can be treated by a master equation approach \[1, 2, 3, 4\]: one writes a linear equation for the time evolution of the reduced system density matrix \(\rho_S = \text{Tr}_{\text{res}}(\rho)\), trace over the reservoir variables of the total density matrix. If we denote the hamiltonian for the system \(H_S\), this equation can be written:

\[
\dot{\rho}_S = \frac{i}{\hbar}[\rho_S, H_S] + L_{\text{relax}}(\rho_S).
\]

In (1), \(L_{\text{relax}}\) is the relaxation superoperator, acting on the density operator \(\rho_S\). It is assumed here to be local in time, which means that \(\dot{\rho}_S(t)\) depends only on \(\rho_S\) at the same time (Markov approximation). All the system dynamics can be deduced from (1). One can calculate one time average values of a system operator \(A\): 

\[
a(t) = \langle A(t) \rangle = \text{Tr}(\rho_S(t)A),
\]

and also, using the quantum regression theorem \[5\], multi-time correlation functions such as \(\langle A(t+\tau)B(t) \rangle\).

We present here an alternative treatment based on a Monte-Carlo evolution of wave functions of the small system (MCWF) \[6, 7, 8, 9\]. This evolution consists of two elements: evolution with a non hermitian hamiltonian, and randomly decided “quantum jumps”, followed by wave function normalization. This approach, which is equivalent to the master equation treatment, has two main interests. First, if the relevant Hilbert space of the quantum system has a dimension \(N\) large compared to 1, the number of variables involved in a wave function treatment (\(\sim N^2\)) is much smaller than the one required for calculations with density matrices (\(\sim N^2\)). Second, new physical insight may be gained, in particular in the studies of the behavior of a single quantum system.

1 The MCWF procedure

The class of relaxation operators that we consider here is the following:

\[
L_{\text{relax}}(\rho_S) = -\frac{1}{2} \sum_m \left(C_m^\dagger C_m \rho_S + \rho_S C_m^\dagger C_m \right) + \sum_m C_m \rho_S C_m^\dagger.
\]

This type of relaxation operators is very general and it is found in most of the Quantum Optics problems involving dissipation. In \(2\), the \(C_m\)'s are operators acting in the space of the small system. Depending on the nature of the problem there can be one, a few or an infinity of these operators.

For the particular case of spontaneous emission by a two-level system with one stable ground state \(g\) and one excited state \(e\) with a lifetime \(\Gamma^{-1}\), there is just a single operator \(C_1 = \sqrt{\Gamma} |g\rangle \langle e|\) in the relaxation operator \(2\), and one can
check that (2) indeed leads to the well known relaxation part of the optical Bloch equations:

\[
\begin{align*}
\dot{(\rho_s)}_{ee} &= -\Gamma (\rho_s)_{ee} \\
\dot{(\rho_s)}_{gg} &= \Gamma (\rho_s)_{ee} \\
\dot{(\rho_s)}_{eg} &= -\frac{\Gamma}{2} (\rho_s)_{eg} \\
\dot{(\rho_s)}_{ge} &= -\frac{\Gamma}{2} (\rho_s)_{ge} .
\end{align*}
\]

(3)

We now present the procedure for evolving wave functions of the small system. Consider at time \(t\) that the system is in a state with the normalized wave function \(|\phi(t)\rangle\). In order to get the wave function at time \(t + \delta t\), we proceed in two steps:

1. First we calculate the wave function \(|\phi^{(1)}(t + \delta t)\rangle\) obtained by evolving \(|\phi(t)\rangle\) with the non hermitian Hamiltonian:

\[
H = H_S - \frac{i\hbar}{2} \sum_m C_m^\dagger C_m .
\]

(4)

This gives for sufficiently small \(\delta t\):

\[
|\phi^{(1)}(t + \delta t)\rangle = \left(1 - \frac{iH\delta t}{\hbar}\right) |\phi(t)\rangle .
\]

(5)

Since \(H\) is not hermitian, this new wave function is clearly not normalized. The square of its norm is:

\[
\langle \phi^{(1)}(t + \delta t)|\phi^{(1)}(t + \delta t)\rangle = \langle \phi(t)| \left(1 + \frac{iH\delta t}{\hbar}\right) \left(1 - \frac{iH\delta t}{\hbar}\right) |\phi(t)\rangle = 1 - \delta p
\]

(6)

where \(\delta p\) reads:

\[
\delta p = \delta t \frac{i}{\hbar} \langle \phi(t)|H - H^\dagger|\phi(t)\rangle = \sum_m \delta p_m
\]

(7)

\[
\delta p_m = \delta t \langle \phi(t)|C_m^\dagger C_m|\phi(t)\rangle \geq 0 .
\]

(8)

The magnitude of the step \(\delta t\) is adjusted so that this calculation at first order is valid; in particular it requires \(\delta p \ll 1\).

For the particular case of the two-level atom problem, the non hermitian Hamiltonian is

\[
H = H_S - \frac{i\hbar\Gamma}{2} |e\rangle\langle e| .
\]

(9)

This amounts to adding the imaginary term \(-i\hbar\Gamma/2\) to the energy of the unstable excited state, as usual in scattering theory.
Quantum Jumps: probabilities $\delta p_m$

| $\phi(t + \delta t)$ | $= \frac{C_m|\phi(t)\rangle}{\|C_m|\phi(t)\rangle\|}$

Random choice

No Quantum Jump: probability $1 - \sum_m \delta p_m$

| $\phi(t + \delta t)$ | $= \frac{|\phi^{(1)}(t + \delta t)\rangle}{\|\phi^{(1)}(t + \delta t)\rangle\|}$

Figure 1: The possible quantum jumps in the Monte-Carlo evolution

2. The second step of the evolution of $|\phi\rangle$ between $t$ and $t + \delta t$ consists in a possible “quantum jump” (Fig. 1). The various possible “directions” for the jumps are given by the $C_m$ operators, and the probability for making a jump in the “direction” of a particular $C_m$ is given by $\delta p_m$ given in (8). The new normalized wave function after such a jump is given by:

$$|\phi(t + \delta t)\rangle = \frac{C_m|\phi(t)\rangle}{\|C_m|\phi(t)\rangle\|}.$$  \hspace{2cm} (10)

Using (7), we find that the total probability for making a jump is $\delta p$. In the no-jump case, which occurs then with a probability $1 - \delta p$, we take as new normalized wave function at time $t + \delta t$:

$$|\phi(t + \delta t)\rangle = \frac{|\phi^{(1)}(t + \delta t)\rangle}{\|\phi^{(1)}(t + \delta t)\rangle\|}.$$  \hspace{2cm} (11)

Consider again as an example the particular case of the spontaneous emission of a two-level atom. The wave function at time $t$ can be written as:

$$|\phi(t)\rangle = \alpha(t)|e\rangle + \beta(t)|g\rangle.$$  \hspace{2cm} (12)
Since there is a single $C_m$ operator in this case, there is only one possible type of quantum jump. The probability for this quantum jump is:

$$\delta p = \Gamma |\alpha|^2 \delta t$$

and the wave function after the jump is simply $|\phi(t+\delta t)\rangle = |g\rangle$. If no jump occurs, the wave function at time $t+\delta t$ is similar to (12), with the coefficients $\alpha(t+\delta t)$ and $\beta(t+\delta t)$ deduced from $\alpha(t)$ and $\beta(t)$ using the evolution with the non hermitian hamiltonian (9). We see for this particular case that the Monte-Carlo evolution can be understood as the stochastic evolution of the atomic wave function if a continuous detection of the emitted photons is performed. The probability for detecting a photon during a particular time step $\delta t$ is indeed equal to $\delta p$ given in (13), and the new wave function after the detection, according to the standard quantum measurement theory, corresponds to the atom in its ground state $g$.

It is actually quite a general result that the Monte-Carlo evolution outlined above represents a possible history of the system wave function with a suitable continuous detection process taking place [6, 8]. Although this procedure does not make any reference to measurements on the system, it may be useful, in order to get some physical understanding for the result of the simulation, to refer to such a continuous detection process, as if it was really performed. We note in this respect that one might possibly consider several different continuous detection processes for a given quantum system. The various sets of $C_m$’s associated to each of these detection schemes can be deduced from each other by linear combinations, the relaxation equation (2) remaining then of course unchanged [7].

2 Equivalence with the Master Equation

With this set of rules we can propagate a wave function $|\phi(t)\rangle$ in time, and we now show that this procedure is equivalent to the master equation (11). More precisely we consider the quantity $\bar{\sigma}(t)$ obtained by averaging $\sigma(t) = |\phi(t)\rangle\langle\phi(t)|$ over the various possible outcomes at time $t$ of the MCWF evolutions all starting in $|\phi(0)\rangle$, and we prove that $\bar{\sigma}(t)$ coincides with $\rho_S(t)$ at all times $t$, provided they coincide at $t = 0$.

Consider a given MCWF $|\phi(t)\rangle$ at time $t$. At time $t+\delta t$, the average value of $\sigma(t+\delta t)$ is:

$$\overline{\sigma(t+\delta t)} = (1-\delta p) \frac{|\phi^{(1)}(t+\delta t)\rangle \langle \phi^{(1)}(t+\delta t)|}{\|\phi^{(1)}(t+\delta t)\|^{2}}$$
\[ \sum_m \delta p_m \frac{C_m |\phi(t)\rangle}{|C_m |\phi(t)\rangle|} \frac{\langle \phi(t) | C^\dagger_m | \phi(t) \rangle}{\|C_m |\phi(t)\rangle\|} \]  

which gives, using (5), (6) and (10):
\[ \sigma(t + \delta t) = \sigma(t) + \frac{i\delta t}{\hbar} [\sigma(t), H_S] + \delta t \mathcal{L}_{\text{relax}}(\sigma(t)). \]  

We now average this equation over the possible values of \( \sigma(t) \) and we obtain:
\[ \frac{d\bar{\sigma}}{dt} = \frac{i}{\hbar} [\bar{\sigma}, H_S] + \mathcal{L}_{\text{relax}}(\bar{\sigma}). \]  

This equation is identical to the master equation (11). If we assume that \( \rho_S(0) = |\phi(0)\rangle\langle \phi(0)|, \bar{\sigma}(t) \) and \( \rho_S(t) \) coincide at any time, which demonstrates the equivalence between the two points of view. In the case where \( \rho_S(0) \) does not correspond to a pure state, one has first to decompose it as a statistical mixture of pure states, \( \rho(0) = \sum p_i |\chi_i\rangle\langle \chi_i| \) and then randomly choose the initial MCWFs among the \( |\chi_i\rangle \) with the probability law \( p_i \).

As mentioned in the introduction, the master equation approach and the reduced density matrix give access to one time average values \( a(t) = \langle A \rangle(t) = \text{Tr}(\rho_S(t)A) \), which can now also be obtained with the MCWF method. One calculates, for several outcomes \( |\phi^{(i)}(t)\rangle \) of the MCWF evolution, the quantum average \( \langle A |\phi^{(i)}(t)\rangle |A|\phi^{(i)}(t)\rangle \), and one takes the mean value of this quantity over the various outcomes \( |\phi^{(i)}(t)\rangle \):
\[ \langle A \rangle_{(n)}(t) = \frac{1}{n} \sum_{i=1}^{n} \langle A |\phi^{(i)}(t)\rangle |A|\phi^{(i)}(t)\rangle. \]  

For \( n \) sufficiently large, (16) implies that \( \langle A \rangle_{(n)}(t) \sim \langle A \rangle(t) \). The ability to provide expectation values for any system operator makes the MCWF method a computational tool which may be much more efficient than the numerical solution of (1) [7, 9].

As an example of the agreement between the master equation approach and the MCWF approach, we have calculated by both methods the excited state population of a two-level atom coupled to a coherent laser field. The parameters for this Rabi nutation are a zero detuning \( \delta \) between the laser and atomic frequencies, and a Rabi frequency \( \Omega = 3\Gamma \). In Fig. 2a, we show the excited state population for a single “history” for \( |\phi(t)\rangle \). One finds a continuous evolution for this population oscillating between 0 and 1, interrupted by random quantum jumps projecting the atomic wave function into the ground state. In Fig. 2b, we indicate the MCWF result obtained with the average of 100 wave functions. It
shows a damped oscillation as a result of the dephasing of the individual oscillations due to the randomness of the various quantum jumps. The MCWF result is in good agreement with the one derived from the master equation (Optical Bloch Equations). Note that the purpose of this example is to illustrate the convergence of the two methods, and not to provide an efficient way of treating two-level atom problems. For such a small system, there is of course no gain in computing time by using the MCWF method instead of the master equation.

Figure 2: (a) Time evolution of the excited-state population of a two-level atom in the MCWF approach. The dashed lines indicate the projection of the atomic wave function onto the ground state (quantum jump). (b) Excited state population averaged over 100 MCWF starting all in ground state at time 0. The dotted line represents the master equation result.

It appears clearly that the equivalence of the Master Equation and MCWF approaches does not depend on the particular value of the time step $\delta t$. From a practical point of view, the largest possible $\delta t$ is preferable, and one might benefit from using a generalization of (5) to a higher order in $\delta t$, as for example a 4th order Runge-Kutta type calculation. The only requirement on $\delta t$ is that the various $|\eta_i|\delta t$, where the $\hbar\eta_i$ are the eigenvalues of $H$, should be small compared to 1. Of course we assume here that those eigenvalues have been simplified as much as possible in order to eliminate the bare energies of the eigenstates of $H_S$. For instance, for a two-level atom with a transition frequency $\omega_A$ coupled to a laser field with frequency $\omega_L$, one makes the rotating wave approximation in the rotating frame so that the $|\eta_i|$'s are of the order of the natural width $\Gamma$, the Rabi frequency $\Omega$ or the detuning $\delta = \omega_L - \omega_A$; they are consequently much smaller than $\omega_A$.

One might wonder whether there is a minimal size for the time step $\delta t$. In the derivation presented above, it can be chosen arbitrarily small. However one should remember that the derivation of (1) involves a coarse grain average of the real density operator evolution. The time step of this coarse grain average
has to be much larger than the correlation time \( \tau_c \) of the reservoir, which is typically an optical period for the problem of spontaneous emission. Therefore one should be cautious when considering any result derived from this MCWF approach involving details with a time scale of the order of or shorter than \( \tau_c \), and only \( \delta t \) larger than \( \tau_c \) should be applied. This appears clearly if one starts directly from the interaction Hamiltonian between the system and the reservoir in order to generate the stochastic evolution for the system wave function \([6]\). The condition \( \delta t \gg \tau_c \) is then required to prevent quantum Zeno type effects \([10]\). This restriction is discussed in detail in \([11]\) in connection with quantum measurement theory.

### 3 MCWF and other stochastic approaches

The problem of stochastic wave function evolution in connection with the treatment of dissipative systems in quantum optics has recently received a lot of attention. In the context of non-classical field generation, Carmichael \([8]\) has proposed an approach named “quantum trajectories”, inspired by the theory of photoelectron counting sequences \([12]\) and quite similar to the spirit of the present work.

For simple atomic systems (2 or 3 levels) coupled to the electromagnetic field, the dynamics can be interpreted in terms of one or a few delay functions, which give the probability distribution of the time intervals between the emission of two successive photons \([13, 14, 15]\). When these functions are known analytically, they can generate a very efficient Monte-Carlo analysis of the process: just after the emission of the \( n^{th} \) fluorescence photon at time \( t_n \), the atom is in its ground state and the choice of a single random number is sufficient to determine the time \( t_{n+1} \) of emission of the \( n+1^{st} \) photon. This type of Monte-Carlo analysis has been used in \([16]\) to simulate an atomic beam cooling experiment, and in \([14]\) to prove numerically the existence of dark periods in the fluorescence of a 3-level atom (quantum jumps). Very recently, laser cooling of atoms using velocity selective coherent population trapping \([17]\) and lasing without inversion \([18]\) have been analyzed by this type of Monte-Carlo method.

Unfortunately, the delay function cannot be calculated analytically for complex systems involving a large number of levels. Nevertheless, it is possible to generate a Monte-Carlo solution for this problem in which a single random number determines the time of emission of each fluorescence photon \([9]\). The evolution of the system between two quantum jumps has to be integrated step
by step numerically, so that the amount of calculation involved is similar to the one required by the method presented in this paper.

Stochastic approaches have also been introduced in the context of either standard [19, 20] or quantum non demolition [21, 22, 23] measurements of photon number in a given mode of the electromagnetic field. A sequence of random quantum jumps resulting from successive measurements asymptotically leads to a reduction of the field state into a Fock state $|n\rangle$, whose probability distribution is equal to the initial photon number distribution for the case of the non demolition measurement. The main interest of these stochastic approaches, as compared with the usual master equation treatment, is to give explicit individual histories of the quantum field state in a measurement sequence. This is particularly valuable if one wants to optimize the measurement sequence in order to get a complete information on the field state with a minimum number of measurement processes [22]. On the other hand, these stochastic calculations still mostly deal with density matrices and their authors do not seem to consider them as more efficient ways of computing than the master equation.

Another class of stochastic equations for system wave functions, which is also equivalent to the master equation [11], has been introduced by Gisin and Percival [24] (see also the work by Diosi [25]). In this approach, only continuous stochastic equations are considered. The complex Itô stochastic process is given by:

$$|d\phi\rangle = -\frac{i}{\hbar}H_S dt|\phi\rangle + \sum_m \left( \langle C_m^\dagger \rangle C_m - \frac{1}{2} C_m^\dagger C_m - \frac{1}{2} \langle C_m^\dagger \rangle \langle C_m \rangle \right) |\phi\rangle dt$$

$$+ \sum_m \left( C_m - \langle C_m \rangle \right) |\phi\rangle \frac{d\xi_m}{\sqrt{2}}$$  \hspace{1cm} (18)

where $\langle C_m \rangle = \langle \phi | C_m | \phi \rangle$, and where the $d\xi_m$ are independent complex Wiener processes [4]:

$$\overline{d\xi_m} = 0$$

$$\Re(d\xi_m)\Re(d\xi_n) = \Im(d\xi_m)\Im(d\xi_n) = \delta_{m,n} dt$$

$$\Re(d\xi_m)\Im(d\xi_n) = 0$$  \hspace{1cm} (19)

Carmichael has shown that for the particular case of the homodyne detection of the fluorescence light, the Quantum Jump formalism can be transformed into such a continuous stochastic equation [8]. Actually this proof can be extended to the most general case: the first step is to write the relaxation operator $L_{relax}$ as:

$$L_{relax}(\rho_S) = -\frac{1}{2} \sum_{m,\varepsilon} \left( D_{m,\varepsilon}^\dagger D_{m,\varepsilon} \rho_S + \rho_S D_{m,\varepsilon} D_{m,\varepsilon}^\dagger \right) + \sum_{m,\varepsilon} D_{m,\varepsilon} \rho_S D_{m,\varepsilon}^\dagger$$  \hspace{1cm} (20)
where \( \varepsilon = \pm 1 \) and where the \( D_{m,\varepsilon} \) are defined as:

\[
D_{m,\varepsilon} = \frac{\mu \mathbb{1} + \varepsilon C_m}{\sqrt{2}}
\]

One easily shows that \( \mathcal{L}_{\text{relax}} \) in (20) is identical with the one in (2). The coefficient \( \mu \) is arbitrary at this stage; \( \mu^2 \) has the dimension of the inverse of a time, and we just require in the following \( \mu^2 \gg |\eta| \), where \( \hbar \eta \) is a typical eigenvalue for \( H \) (for the two-level atom case, \( \eta \sim \Gamma, \Omega, \delta \)). Using the set of operators \( D_{m,\varepsilon} \), we can now perform a Monte-Carlo evolution of the wave function, equivalent to the master equation (1). Because of the large magnitude of \( \mu^2 \), this simulation with the \( D_{m,\varepsilon} \) operators involves much more quantum jumps in a given time interval \( \Delta t \) than a simulation done with the \( C_m \)'s. But the change of the wave function in a given quantum jump:

\[
|\phi\rangle \rightarrow \frac{D_{m,\varepsilon}|\phi\rangle}{\|D_{m,\varepsilon}|\phi\rangle\|}
\]

is very small since \( D_{m,\varepsilon} \) is nearly proportional to the identity operator \( \mathbb{1} \). In the limit of very large \( \mu \), the Monte-Carlo evolution of the wave function therefore tends towards a continuous stochastic evolution. In Carmichael’s homodyne detection problem, the form (21) for the \( D_{m,\varepsilon} \) has a clear interpretation. These jump operators correspond to the detection of a photon after one has mixed the light emitted by the atomic system with a local oscillator field. The parts in \( \mu \mathbb{1} \) and \( C_m \) correspond respectively to the field originating from the local oscillator and the field emitted by the atom. The condition \( \mu^2 \gg |\eta| \) just states that the intensity of the local oscillator is much larger than the intensity of the light emitted by the atom, as usual in homodyne detection.

To prove the equivalence with (18), we choose a time interval \( \Delta t \) such that

\[
\mu^{-2} \ll \Delta t \ll |\eta|^{-1}.
\]

This implies that the number of jumps \( N_{m,\varepsilon} \) occurring with a given operator \( D_{m,\varepsilon} \) during \( \Delta t \) will be large compared to 1 since \( \mu^2 \Delta t \gg 1 \), but at the same time we expect only a small change in the system wave function since \( |\eta|\Delta t \ll 1 \). The operator \( \mathcal{O} \) describing the action of all those jumps during \( \Delta t \) is a product of the various \( D_{m,\varepsilon} \) and it can be approximated at order 1 in \( \sqrt{\Delta t} |\eta| \) by:

\[
\mathcal{O} \simeq \left( \frac{\mu}{\sqrt{2}} \right)^N \left( \mathbb{1} + \frac{1}{\mu} \sum_m (N_{m,+} - N_{m,-}) C_m \right)
\]

where \( N = \sum_{m,\varepsilon} N_{m,\varepsilon} \) is the total number of jumps occurring during \( \Delta t \). The wave function at time \( t + \Delta t \) can now be written before normalization:

\[
|\phi(t + \Delta t)\rangle = \left( \mathbb{1} - \frac{i}{\hbar} H_S \Delta t - \frac{1}{2} \Delta t \sum_m C_m^\dagger C_m + \sum_m \frac{N_{m,+} - N_{m,-} C_m}{\mu} \right) |\phi(t)\rangle
\]
where we have taken into account both the non-hermitian evolution during $\Delta t$ and the effect of the multiple quantum jumps. The numbers of jumps $N_{m,\varepsilon}$ are poissonian random variables with an average value and a standard deviation given by:

$$N_{m,\varepsilon} \simeq \mu^2 \Delta t / 2 \left( 1 + \frac{\varepsilon}{\mu} \langle C_m + C_m^\dagger \rangle \right)$$

(26)

$$\Delta N_{m,\varepsilon} \simeq \mu \sqrt{2} \sqrt{\Delta t}$$

(27)

where the average value $\langle C_m + C_m^\dagger \rangle$ is taken in $|\phi(t)\rangle$. In the limit of large $N_{m,\varepsilon}$, we can approximate the random variable $N_{m,+} - N_{m,-}$ appearing in (25) by:

$$\frac{N_{m,+} - N_{m,-}}{\mu} = \Delta t \langle C_m + C_m^\dagger \rangle + \Delta \zeta_m$$

(28)

where $\Delta \zeta_m$ is a real gaussian random variable with zero mean and a standard deviation equal to $\sqrt{\Delta t}$. Finally we normalize the wave function (25) and we obtain:

$$|\Delta \phi(t + \Delta t)\rangle = - \frac{i}{\hbar} H_S |\phi(t)\rangle \Delta t$$

$$+ \frac{1}{2} \sum_m \left( \langle C_m + C_m^\dagger \rangle C_m - C_m^\dagger C_m - \frac{1}{4} \langle C_m + C_m^\dagger \rangle^2 \right) |\phi(t)\rangle \Delta t$$

$$+ \frac{1}{2} \sum_m \left( 2C_m - \langle C_m + C_m^\dagger \rangle \right) |\phi(t)\rangle \Delta \zeta_m.$$ 

(29)

In (29), we have kept terms linear in $\Delta \zeta_m$ and $\Delta t$, and we have replaced all the quadratic terms $\Delta \zeta_m \Delta \zeta_{m'}$ by their mean $\Delta t \delta_{m,m'}$. In the limit $\mu \to +\infty$, $\Delta t \to 0$, this equation can be understood as a Itô stochastic equation, corresponding to a real version of (18).

The exact form of (18) can be recovered by taking a slightly more complicated set of $D_{m,\varepsilon}$ operators:

$$D_{m,\varepsilon} = \frac{\mu \mathbb{1} + \varepsilon C_m}{2} \quad \text{with} \quad \varepsilon = \pm 1, \pm i$$

(30)

and by performing an appropriate global phase change of the wave function. The continuous stochastic equation (18) is therefore a limiting case of the quantum jump formalism presented here, and it also has an interpretation in terms of a detection scheme: the information concerning the system is mixed with a “classical field” $\mu \mathbb{1}$, and the sequence of quantum jumps deduced from the whole set of mixed components $D_{m,\varepsilon}$ allows one to determine the subsequent system evolution. Note that on the contrary, the jumps deduced from the action on the system wave function of a single mixed component, such as the $D_{m,+}$’s, are not sufficient to determine this system evolution.
4 Conclusion

We have presented a stochastic evolution for the wave function of a system coupled to a reservoir in the Markovian regime. Each time step in this stochastic evolution consists in two parts: a Hamiltonian but non hermitian evolution and a possible quantum jump. We have proved the equivalence of this Monte-Carlo Wave function approach with the master equation treatment. We have also shown that this simulation with Quantum jumps can be transformed into a continuous stochastic evolution of the wave function, similar to the one of [24].

This approach provides a computational tool which is often more efficient than the standard master equation treatment for systems with a number of states \( N \gg 1 \) (for a detailed discussion see [7]). Indeed a wave function involves only \( N \) components while a density matrix is described by \( N^2 \) terms. This method has already been applied successfully to problems such as the study of the limits of laser cooling in 2 dimensions [26], or the calculation of the spectrum of the light emitted by an assembly of cold atoms [27]. Problems such as the study of collisions between cold atoms, or non linear mixing of quantum fields may also benefit from such an approach.

We have emphasized that this simulation is in many practical cases directly connected to a measurement sequence performed on the system. Each Monte-Carlo trajectory is a possible history for the individual quantum system. In this respect, the noise appearing when one simulates with this method the measurement of a given observable \( A \) is also interesting. The fluctuations in the number of occurrences of a given eigenvalue \( a_i \) of \( A \) correspond to the quantum noise that one would get in a real experiment, performing the relevant detection scheme on an individual quantum system. Since more and more quantum optics and atomic physics experiments are now performed with a single system (single ion or atom, single mode of a cavity), Monte-Carlo wave function methods should therefore have many applications, since they lead to predictions closer to actual experimental signals than the master equation, which rather deals with ensemble averages.

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