Can Error Mitigation Improve Trainability of Noisy Variational Quantum Algorithms?

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Variational Quantum Algorithms (VQAs) are widely viewed as the best hope for near-term quantum advantage. However, recent studies have shown that noise can severely limit the trainability of VQAs, e.g., by exponentially flattening the cost landscape and suppressing the magnitudes of cost gradients. Error Mitigation (EM) shows promise in reducing the impact of noise on near-term devices. Thus, it is natural to ask whether EM can improve the trainability of VQAs. In this work, we first show that, for a broad class of EM strategies, exponential cost concentration cannot be resolved without committing exponential resources elsewhere. This class of strategies includes as special cases Zero Noise Extrapolation, Virtual Distillation, Probabilistic Error Cancellation, and Clifford Data Regression. Second, we perform analytical and numerical analysis of these EM protocols, and we find that some of them (e.g., Virtual Distillation) can make it harder to resolve cost function values compared to running no EM at all. As a positive result, we do find numerical evidence that Clifford Data Regression (CDR) can aid the training process in certain settings where cost concentration is not too severe. Our results show that care should be taken in applying EM protocols as they can either worsen or not improve trainability. On the other hand, our positive results for CDR highlight the possibility of engineering error mitigation methods to improve trainability.

I. Introduction

The prospect of obtaining quantum computational advantage for practical problems, such as simulating systems in chemistry and materials science, has generated much excitement. The past few years have witnessed tremendous progress towards this end, with significant focus on algorithm development for Noisy Intermediate-Scale Quantum (NISQ) computers. In particular, Variational Quantum Algorithms (VQAs) are a leading algorithmic approach because they adapt to the constraints of NISQ devices. Specifically, VQAs minimize a cost function by training a parameterized quantum circuit via a classical-quantum feedback loop [1, 2]. The cost is computed efficiently on a quantum computer whilst the parameter optimization is carried out classically. Different implementations of this versatile framework have been proposed for a broad spectrum of problems from dynamical quantum simulation [3–13] to machine learning [14–20] and beyond [21–40].

A central challenge in the NISQ regime is to combat the effects of noise as full error correction is not possible [41]. Decoherence, gate errors, and measurement noise all conspire to limit the complexity of quantum circuits that can be implemented on NISQ devices. While VQAs themselves offer some strategy to mitigate the impact of noise [1], it is widely viewed that VQAs alone will not be enough, and additional strategies will be needed to obtain quantum advantage in the face of noise. This has spawned the field of error mitigation (EM), and many researchers believe that VQAs combined with EM techniques will be the path forward. Indeed, EM methods like Zero-Noise Extrapolation [10, 42–44], Clifford Data Regression [45], Virtual Distillation [46, 47], Probabilistic Error Cancellation [42, 43] and others [48–52] have been demonstrated to reduce errors of observable expectation values, sometimes by orders of magnitude. Hence, there has been hope that one can simply train the VQA in the presence of noise, and then after training, one can apply an EM method to extract the correct cost value (e.g., the ground state energy in the case of the variational quantum eigensolver [21]).

However, new challenges have recently been discovered for this approach [53, 54]. It is now recognized that noise impacts the trainability of VQAs, that is, the ability of the classical optimizer to find the global cost minimum. For ansatzes (i.e., parameterized quantum circuits) with depth linear or superlinear in the number of qubits and local Pauli noise, the cost function landscape exponentially flattens, leading to an exponentially vanishing cost gradient, a phenomenon known as Noise-Induced Barren Plateaus (NIBPs) [53]. Thus, noise impedes the training process of VQAs, as in such a setting one requires an exponential number of shots per optimization step to resolve the cost landscape against finite sampling noise. As with other barren plateau effects [55, 56], this exponential scaling does not only arise for gradient-based optimizers but also impacts gradient-free methods [57] and optimizers that use higher-order derivatives [58]. NIBPs represent a serious issue for VQA scalability, and could ultimately be a roadblock for near-term quantum advantage. It is therefore crucial to investigate potential methods to mitigate them.

Given the great success of EM methods in suppressing error in observable expectation values, it is natural
to ask whether EM methods could address NIBPs. More
generally, one could simply ask: does it help to use error
mitigation during the training process for VQAs? This
question is precisely the topic of our article. We remark
that error mitigation has been successfully implemented
during the VQA training process for a small-scale prob-
lem [44]. However, it is an open question as to whether
or not EM can resolve large-scale trainability issues
associated with cost concentration. This is due to the fact
that even though EM can reverse the concentration of
cost values, it also increases the stochastic uncertainty in
the mitigated quantities. As summarized in Figure 1, this
is a trade-off that should be carefully considered. Thus,
it is a non-trivial question as to whether or not EM im-
proves the resolvability of cost function values which is
a key factor in determining the trainability of the land-
scape.

In this work, we investigate the effects of error mitiga-
tion on the resolvability of the cost function landscape.
First, we consider a broad class of error mitigation proto-
cols and show that, under the class of local depolarizing
noise that is known to cause NIBPs, in order to reverse
exponential cost concentration any such protocol needs to
spend resources (e.g., shot resources or number of state
copies) scaling at least exponentially in the number of
qubits. This suggests that NIBPs are a serious scaling
issue that cannot be simply resolved with error mitiga-
tion.

Second, we study four specific error mitigation proto-
cols in further detail: Zero Noise Extrapolation, Virtual
Distillation, Probabilistic Error Cancellation, and strate-
gies that implement a linear ansatz which includes Clif-
ford Data Regression. We find that Virtual Distillation
can actually decrease the resolvability of the noisy cost
landscape, and impede trainability. Under more restric-
tive assumptions on the cost landscape, we find a sim-
ilar result for Zero Noise Extrapolation. We also show
that any improvement in the resolvability after applying
Probabilistic Error Cancellation under local depolarizing
noise exponentially degrades with increasing number of
qubits. Finally, for strategies that use a linear ansatz
such as Clifford Data Regression, we show that there is
no change to the resolvability of any pair of cost values
if the same ansatz is used. However, we do observe numeri-
cally that Clifford Data Regression increases trainabil-
ity in some settings. This last observation provides some
hope that a careful choice of error mitigation method can
be useful. It also suggests that researchers could design
and engineer error mitigation methods to enhance VQA
trainability.

The rest of the manuscript is structured as follows.
Section II introduces the framework and notation for our
work. We present our theoretical results in Section III
and our numerical results in Section IV. Finally, our con-
cluding discussions are presented in Section V. The proofs
for our main results are presented in the Appendix.

II. Framework

A. Variational Quantum Algorithms

The main goal of Variational Quantum Algorithms
(VQAs) is to solve an optimization problem by minimiz-
ing a cost function that can be efficiently estimated on
a quantum computer. In this work we consider settings
where the cost function takes the form

$$ C(\theta) = \text{Tr} \left[ U(\theta) \rho_{\text{in}} U(\theta)^\dagger O \right] . $$

(1)
In the above, given some Hilbert space $\mathcal{H}$, we define the set of density operators $\mathcal{S}(\mathcal{H})$ and set of bounded linear operators $\mathcal{B}(\mathcal{H})$. We then denote $\rho_{in} \in \mathcal{S}(\mathcal{H})$ as the input state, $U(\theta) \in \mathcal{B}(\mathcal{H})$ as a unitary that corresponds to a parametrized quantum circuit with trainable parameters $\theta$, and $O \in \mathcal{B}(\mathcal{H})$ is a Hermitian operator. The Variational Quantum Eigensolver [21], variational quantum compiling [33–35, 59], quantum autoencoders [60], and several other VQAs fit under the framework of Eq. (1).

A quantum computer is employed to evaluate the cost function, or gradients thereof, and part of the computational complexity of the algorithm is designated to a classical computer that leverages the power of classical optimizers to solve the problem

\[ \arg \min_{\theta} C(\theta). \]  

(2)

The optimization task defined in Eq. (2) has been shown to be NP-hard [61]. Moreover, on top of the typical difficulties associated with solving classical non-convex optimization problems, there are challenges that arise when training the parameters of a VQA due to the quantum nature of the problem itself.

As quantum mechanics is intrinsically a probabilistic theory, one has to deal with shot noise arising from finite sampling when estimating the cost function (or its gradient). This has led to the development of several quantum-aware optimizers that are frugal in the number of shots [62–64]. Additionally, it has been recently shown that certain properties of the cost function can induce so-called barren plateaus, originating due to highly expressive ansatzes [55, 65, 66], global cost functions [56], high levels of entanglement [67, 68], or the controllability of $U(\theta)$ [69]. When a cost function exhibits a barren plateau, with high probability the cost function partial derivatives are exponentially suppressed across the landscape. This means that an exponentially large number of shots are needed to navigate the flat landscape and determine a cost-minimizing direction [57, 58].

In this work we investigate the effect of noise and error mitigation techniques in solving the optimization task of Eq. (2). For this purpose we investigate the task of resolving two points on the cost function landscape, as presented in Fig. 1. This is a central primitive in the training process that is utilized at each optimization step, regardless of whether one is using gradient-based or gradient-free methods. In gradient-based methods, a common strategy is to use the parameter shift rule, which constructs partial derivatives from two cost function values [70, 71]. Gradient-free methods such as simplex-based methods also compare two or more cost function values at each optimization step [72, 73]. Thus, this task is a key step for both gradient-based and gradient-free optimizers, and it reflects the ability of the optimizer to find a cost-minimizing direction at each step of the optimization. As discussed below, under a finite shot budget this task becomes harder under cost concentration, leading to trainability issues.

B. Effect of noise on the training landscape

Hardware noise can impact the cost function landscape in a variety of ways such as changing the optimal cost function value, shifting the position of minima, and demoting a global minimum to a local minimum. All of the above present further challenges in the training of VQAs. In this section we briefly review some of the literature on the effect of noise on VQAs cost function landscapes. We summarize some of these effects in Fig. 2.

1. Noise resilience

Certain cost functions have been demonstrated to show optimal parameter resilience under particular noise models [34]. This is a phenomenon where the position of the global cost minimum of the cost landscape is invariant under the action of noise. This has important consequences for trainability. There are many VQAs where the goal is to obtain optimal parameters, rather than the optimal cost value, such as when solving combinatorial optimization problems with the Quantum Approximate Optimization Algorithm [24]. If such cost landscapes display optimal parameter resilience, this leaves open the possibility of noisy training even if the cost value of the global minimum is altered by the noise. However, noise can also severely affect the trainability of the landscape in a number of ways, which we summarize below.

2. Noise-induced cost concentration and noise-induced barren plateaus

Here we summarize the phenomenon of noise-induced cost concentration and noise-induced barren plateaux (NIBPs), as well as introduce some notation that we will use throughout the rest of this manuscript. This was formulated in Ref. [53] for a general class of VQAs and a class of Pauli noise that includes as a special case local depolarizing noise. (See also Refs. [54, 74, 75] for other discussions of the impact of noise.) In this section we follow the treatment for local depolarizing noise. Consider a model of noise acting through a depth $L$ circuit with $n$-qubit input state $\rho_{in}$ as

\[ \tilde{\rho} = (\mathcal{N} \circ U_L \circ \cdots \circ \mathcal{N} \circ U_1 \circ \mathcal{N})(\rho_{in}) \]  

(3)

where $\{U_k\}_{k=1}^L$ denote unitary channels that describe collections of gates that act together in a layer, and $\mathcal{N} = \bigotimes_{i=1}^n D_i$ is an instance of local depolarizing channels, with depolarizing probabilities $\{p_i\}_{i=1}^n$. We denote
FIG. 2. Schematic of different effects due to noise on cost landscapes. We present a 1-dimensional slice of a simplified cost landscape corresponding to a single parameter θ. a) Depending on the parameterization strategy, some ansatzes can have degenerate minima. b) Certain types of local Pauli noise can cause the cost landscape to exponentially concentrate on a fixed value. Some can problems display optimal parameter resilience (OPR), where the location of the optimal parameters are invariant under action of the certain noise models. c) Aside from cost concentration, noise can also corrupt the cost landscape by breaking the degeneracy of optimal parameters, and shifting the location of minima.

A noisy cost function as

$$\tilde{C} = \text{Tr}[O\tilde{ρ}] \; ,$$ \hspace{1cm} (4) $$

where $O$ is some Hermitian measurement operator (throughout the article we will use a tilde to denote noisy quantities). In Ref. [53] it was shown that

$$\left| \tilde{C} - \frac{1}{2m}\text{Tr}[O] \right| \leq D(q, n) \; ,$$ \hspace{1cm} (5) $$

where $q = \max\{(1 - p_i)\}_{i=1}^{n} < 1$ and $D(q, n) \in \mathcal{O}(q^\alpha n)$ for some positive constant $\alpha$, if $L \in \Omega(n)$. Thus, in the presence of the class of noise models considered, the noisy cost function exponentially concentrates on a fixed value if the depth scales linearly or superlinearly in the number of qubits.

The gradients across the cost function landscape show similar scaling [53], demonstrating a phenomenon known as NIBPs. This implies that the task of accurately determining gradients or cost function differences during the training process requires an exponential number of shots due to the need to resolve quantities to an exponentially small precision.

3. Cost corruption

In general, a noise model that exhibits cost concentration and NIBPs would not simply uniformly flatten the cost landscape. Instead, we expect noise to additionally alter the cost landscape in many non-trivial ways. We refer to any additional adverse effects on the landscape as cost corruption. For example, it was shown in Ref. [76] that non-unital noise can break the degeneracy of exponentially-occurring global minima, thus proliferating local minima and impacting trainability. In addition, cost functions that do not exhibit optimal parameter resilience [34] limit the quality of noisy optimization, as the optimal parameters of $\tilde{C}(\theta)$ do not correspond to the optimal parameters of $C(\theta)$.

C. Error Mitigation Techniques

We finish the discussion of our framework with a summary of the key features of the error mitigation techniques that we study in this article. For a more detailed review, readers can refer to Refs. [2, 77].

Consider the effects of noise on the cost function in Eq. (1). We suppose the noise can be characterized by a single parameter $\varepsilon$ and we denote the corresponding noisy state and cost function as $\tilde{ρ}(\theta, \varepsilon)$ and $\tilde{C}(\theta, \varepsilon) = \text{Tr}[\tilde{ρ}(\theta, \varepsilon)O]$ respectively. The goal of error mitigation is to construct an experimental protocol which obtains a mitigated cost function estimator $C_m(\theta)$ that approximates the noise-free value $C(\theta)$. The protocol to obtain $C_m(\theta)$ generally consists of running circuits that modify the original circuit of interest by inserting additional gates, preparing multiple copies of a state, changing the measurement operator, and classical post-processing of the expectation values of these circuits. These different utilizations of resources are summarized in a schematic in Fig. 3.

Error mitigation protocols often lead to a larger variance in the statistical outcomes of each experiment, and thus more shots are required to estimate the error-mitigated cost value $C_m(\theta, \varepsilon)$ to a desired precision compared to the unmitigated noisy value $C(\theta, \varepsilon)$. This is often quantified by the error mitigation cost, which is defined below.

**Definition 1** (Error mitigation cost). We define the error mitigation cost as

$$\gamma(\theta, \varepsilon) = \frac{\text{Var}[C_m(\theta, \varepsilon)]}{\text{Var}[C(\theta, \varepsilon)]} \; ,$$ \hspace{1cm} (6) $$

where $\tilde{C}(\theta, \varepsilon)$ denotes the noisy cost function value corresponding to vector of parameters $\theta$ at noise level $\varepsilon$, and $C_m(\theta, \varepsilon)$ denotes the corresponding error-mitigated quantity.
In certain settings we encounter, $\gamma(\theta, \varepsilon)$ is independent of $\theta$. In other cases, where we have to compare $\gamma(\theta, \varepsilon)$ for two different parameters, we will seek parameter-independent bounds.

We now summarize the error mitigation techniques that we study in this article. We note that recently, unified error mitigation techniques have also been proposed that combine two or more of the protocols that we discuss in this section [78–80]. Our results are also applicable to such strategies, however, we will only review the root strategies here.

1. Zero Noise Extrapolation

The goal of Zero Noise Extrapolation is to run a given circuit of interest at $m + 1$ increasing noise levels $\varepsilon < a_1 \varepsilon < \ldots < a_m \varepsilon$, and to use information from the resulting expectation values to obtain an estimate of the zero-noise result. Here we summarize the key features of a protocol using Richardson extrapolation [10, 42], and exponential extrapolation [43].

Richardson Extrapolation. Suppose that $\tilde{C}(\theta_i, \varepsilon)$ admits a Taylor expansion in small noise parameter $\varepsilon$ as

$$\tilde{C}(\theta_i, \varepsilon) = \tilde{C}(\theta_i, 0) + \sum_{k=1}^{m} p_k(\theta_i) \varepsilon^k + \mathcal{O}(\varepsilon^{m+1}),$$

where $p_k$ are unknown parameters and $\tilde{C}(\theta_i, 0) = C(\theta)$ is the zero-noise cost function. By considering the equivalent expansion of $\tilde{C}(\theta_i, a_1 \varepsilon)$ and combining the two equations one obtains

$$C_m(\theta_i) = \frac{a_1 \tilde{C}(\theta_i, a_1 \varepsilon) - \tilde{C}(\theta_i, \varepsilon)}{a_1 - 1} = \tilde{C}(\theta_i, 0) + \mathcal{O}(\varepsilon^2),$$

which is a higher-order approximation of $\tilde{C}(\theta_i, 0)$ compared to simply using $\tilde{C}(\theta_i, \varepsilon)$. This process can be repeated iteratively $m$ times to obtain an estimator which is accurate up to $\mathcal{O}(\varepsilon^{m+1})$ error.

Exponential extrapolation. In some cases the noisy behavior may not be well-depicted by a Taylor expansion. As an alternative one can consider an exponential model

$$\tilde{C}(\theta_i, \varepsilon) = r(\theta_i, \varepsilon)^{-\lambda_i(\theta_i, \varepsilon)} \left( \sum_{k=0}^{m} p_k(\theta_i) \varepsilon^k + \mathcal{O}(\varepsilon^{m+1}) \right),$$

for some $r$ and $t$ which in general can be functions of $\varepsilon$. We can also construct an extrapolation strategy that is tailored towards noisy cost function values that are dominated by NIBP scaling as in Eq. (5), where we model the effects of noise as

$$\tilde{C}(\theta_i, q) = \tilde{C}(\theta_i, 0) + q^k \left( B(\theta_i) + \sum_{k=1}^{m} p_k(1-q)^k \right) + \mathcal{O}((1-q)^{m+1}).$$

Here, $\tilde{C}(\theta_i, 0)$ is the fixed point of the noise and $B(\theta_i) + \tilde{C}(\theta_i, 0)$ is the noise free cost value. For these two strategies we can similarly construct $C_m(\theta_i)$ as linear combinations of $\{\tilde{C}(\theta_i, a_i \varepsilon)\}_{i=0}^{m}$ to achieve $\mathcal{O}(\varepsilon^{m+1})$ approximations of $\tilde{C}(\theta_i, 0)$. We detail these constructions in Section A1a of the Appendix.

2. Virtual Distillation

Virtual Distillation, also known as Error Suppression by Derangement, was proposed concurrently in Refs. [47] and [46]. In this article we consider the two error mitigation protocols in Ref. [47] (denoted "A" and "B") to respectively prepare

$$C_m^{(A)}(\theta_i) = \text{Tr}[\tilde{\rho}_i^M O]/\text{Tr}[\tilde{\rho}_i^M],$$

and

$$C_m^{(B)}(\theta_i) = \text{Tr}[\tilde{\rho}_i^M O]/\lambda_i^M,$$

where $\lambda_i$ is the dominant eigenvalue of $\tilde{\rho}_i \equiv \tilde{\rho}(\theta_i)$. The operator $\tilde{\rho}_i^M$ can be obtained by preparing $M$ copies of $\tilde{\rho}_i$ in a tensor product state $\tilde{\rho}_i^0 \otimes M$ and applying a cyclic shift operator. We note that protocol B presumes access to the dominant eigenvalue beforehand, which could potentially be computed via the techniques of Ref. [37].

3. Probabilistic Error Cancellation

Probabilistic Error Cancellation utilizes many modified circuit runs in order to construct a quasiprobability representation of the noise-free cost function [42, 43]. We assume that the effect of the noise can be described by a quantum channel $\mathcal{N}$ that occurs after a gate that we denote with unitary channel $\mathcal{U}$. Here we make the simplifying assumption that this is the only gate in the circuit, and we treat the general case in Section A1b of the Appendix, as well as provide a more detailed exposition. The goal of this protocol is to simulate the inverse map $\mathcal{N}^{-1}$. Note that, in general, this will not always correspond to a CPTP map. Despite this fact, if one has a basis of (noisy) quantum channels $\{\mathcal{B}_a\}_a$, corresponding to experimentally available channels, one can expand the inverse map in this basis as $\mathcal{N}^{-1} = \sum_a q_a \mathcal{B}_a$, for some set of $q_a \in \mathbb{R}$. By defining a probability distribution
if one has access to the set of CPTP maps \( \mathcal{U}(\theta) \) and \( \mathcal{V}(\theta) \), and

where

\[ p = |q_a|/G_N \] where \( G_N = \sum_\alpha |q_\alpha| \), the noise free-expectation value can then be written as a quasiprobability distribution

\[ C_{\mathcal{U}(\rho)} = G_N \sum_\alpha \text{sgn}(q_\alpha) p_\alpha \text{Tr}[B_\alpha \mathcal{V}(\rho_{in}) O] \] ,

where \( \rho_{in} \) is the input state, \( O \) is the measurement operator, and \( \text{sgn}(q_\alpha) \) denotes the sign of \( q_\alpha \). The idea is that if one has access to the set of CPTP maps \( \{B_\alpha\}_\alpha \) in the noisy native hardware gate set, then one can obtain an estimate of the noise free cost \( C_{\mathcal{U}(\rho)} \) as follows: (1) With probability \( p_\alpha \), prepare the circuit of interest with additional gate \( B_\alpha \) in order to obtain the expectation value \( \text{Tr}[B_\alpha \mathcal{V}(\rho_{in}) O] \). (2) Multiply the result by \( \text{sgn}(q_\alpha) G_N \). (3) Repeat process many times and sum results.

### 4. Clifford Data Regression (CDR) and linear ansatz methods

The main idea of linear ansatz methods is to assume that we can approximately reverse the effects of noise with an affine map, and thus we construct a linear ansatz of the form

\[ C_m(\theta, a) = a_1(\theta) \tilde{C}(\theta) + a_2(\theta) , \]

where \( a(\theta) = (a_1(\theta), a_2(\theta)) \) is a vector of parameters to be determined. In general we expect \( a \) to be highly dependent on \( \theta \). In Ref. [81], the authors use data regression to learn the optimal parameters \( a^*(\theta) \) with training data comprising of pairs of noise-free and corresponding noisy cost function values \( \mathcal{T}_\theta = \{ (C_j, \tilde{C}_j) \} \), where the circuits are predominantly constructed from Clifford gates. The noise-free cost values can be simulated efficiently on a classical computer whilst the noisy cost values can be evaluated directly on the quantum computer. This strategy is known as Clifford Data Regression.

Other methods have been proposed to learn the optimal parameters \( a^*(\theta) \). In Ref. [81] the authors further develop the idea of training-based error mitigation by considering alternative training data comprising of fermionic linear optics circuits. One can also model the noise as global depolarizing noise. Under this assumption, \( a^*(\theta) \) has an exact solution in terms of a single noise parameter. Subsequently, various techniques can be used to estimate the noise parameter [82–86].

### III. Theoretical Results

We present two sets of theoretical results. First, in Section IIIA we show that a broad class of error mitigation techniques cannot undo the exponential resource requirement that exponential cost concentration presents. This has implications for both the trainability of noisy VQAs, as well as the accurate estimation of noise-free cost function values in general. Second, in Section IIIB, we work predominantly in the non-asymptotic regime (in terms of scaling in \( n \)) and investigate to what ex-
tent different error mitigation strategies can improve the resolvability of the noisy cost landscape, assuming that some cost concentration has occurred. For these purposes we introduce a class of quantities which quantify the improvement of the resolvability of the cost function landscape after error mitigation, which we call the relative resolvability (see Defs. 2–4). Using these quantities we study Zero Noise Extrapolation (Sec. III B2), Virtual Distillation (Sec. III B3), Probabilistic Error Cancellation (Sec. III B4) and linear ansatz methods which include Clifford Data Regression (Sec. III B5). In the settings that we consider, we find that in many cases error mitigation impedes the optimizer’s ability to find good optimization steps, and is worse than performing no error mitigation.

A. Asymptotic scaling results (exponential estimator concentration)

In this section we show that full mitigation of exponential cost concentration is not possible for a general class of error mitigation strategies. Specifically, we show that one cannot remove the exponential scaling that local depolarizing noise incurs without investing exponential resources elsewhere in the mitigation protocol.

We start by remarking that, as summarized in Fig. 3, all of the strategies presented in Sec. II C consist of preparing linear combinations of expectation values of the form

$$E_{\sigma,X,M,k} = \text{Tr} \left[ X \left( \sigma^\otimes M \otimes \left| 0 \right\rangle \left\langle 0 \right| \otimes k \right) \right], \quad (16)$$

for some $n$-qubit quantum state $\sigma \in S(\mathcal{H})$ that in general can be prepared by a different circuit to that of the state of interest, for $\left| 0 \right\rangle \left\langle 0 \right| \in S(\mathcal{H}^n)$ and for some $X \in B(\mathcal{H}^\otimes M \otimes \mathcal{H}^\otimes k)$. That is, one can prepare multiple copies of a state, prepare different quantum circuits, and apply general measurement operators. In order to generalize the setting further, we also allow the possibility to utilize multiple clean ancillary qubits at the end of the circuit. By considering linear combinations of such quantities, one also accounts for the ability to post-processing measurement results classically with a linear map, such as is the case with Probabilistic Error Cancellation. In the following theorem we show how quantities of the form (16) concentrate under local depolarizing noise.

**Theorem 1.** Consider an error mitigation strategy that, as a step in its protocol, estimates $E_{\sigma,X,M,k}$ as defined in Eq. (16). Suppose that $\sigma$ is prepared with a depth $L_{\sigma}$ circuit and experiences local depolarizing noise according to Eq. (3). Under these conditions, $E_{\sigma,X,M,k}$ exponentially concentrates with increasing circuit depth on a state-independent fixed point as

$$\left| E_{\sigma,X,M,k} - \text{Tr} \left[ X \left( \frac{1}{2M^n} \otimes \left| 0 \right\rangle \left\langle 0 \right| \otimes k \right) \right] \right| \leq G_{\sigma,X,M}(n), \quad (17)$$

where $\mathbb{1} \in S(\mathcal{H})$ is the $n$-qubit identity operator and

$$G_{\sigma,X,M}(n) = \sqrt{\ln 4 \left\| X \right\|_\infty M^{n^{1/2}} q^{-k+1}}, \quad (18)$$

with noise parameter $q \in [0,1)$.

We remark that a similar result can be obtained for local Pauli noise, provided that each Pauli error occurs with a non-zero probability. Theorem 1 shows that quantities of the form (16) exponentially concentrate in the depth of the circuit. As we summarize in the schematic in Fig. 3, such quantities generalize expectation values that are prepared by many different error mitigation protocols. We now explicitly demonstrate how Theorem 1 affects the mitigated cost values that these protocols output.

**Corollary 1** (Exponential estimator concentration). Consider an error mitigation protocol that approximates the noise-free cost value $C(\theta)$ by estimating the quantity

$$C_m(\theta) = \sum_{(\sigma(\theta),X,M,k) \in T} a_{X,M,k} E_{\sigma(\theta),X,M,k}, \quad (19)$$

where each $E_{\sigma,X,M,k}$ takes the form (16). We denote $M_{\text{max}}$ and $a_{\text{max}}$ as the maximum values of $M$ and $a_{X,M,k}$ respectively accessible from a set $T$ defined by the given protocol. Assuming $\left\| X \right\|_\infty \in O(\text{poly}(n))$, there exists a fixed point $F$ independent of $\theta$ such that

$$\left| C_m(\theta) - F \right| \in O(2^{-\beta n} a_{\text{max}} |T|M_{\text{max}}), \quad (20)$$

for some constant $\beta \geq 1$ if the circuit depths satisfy

$$L_{\sigma(\theta)} \in O(n), \quad (21)$$

for all $\sigma(\theta)$ in the construction (19). That is, if the depths of the circuits scale linearly or superlinearly in $n$ then one requires at least exponential resources to distinguish $C_m$ from its fixed point, for instance by requiring an exponential number of shots, or by requiring an exponential number of state copies $M_{\text{max}}$.

We note that the assumption $\left\| X \right\|_\infty \in O(\text{poly}(n))$ is satisfied in most settings, and in particular is satisfied for all error mitigation protocols discussed in Sec. II C. For instance, in the case of Virtual Distillation, $X$ corresponds to a cyclic shift operator followed by a Pauli observable, and thus $\left\| X \right\|_\infty \in O(1)$. Corollary 1 implies that under conditions that generate a NIBP, in order to obtain an estimate of a noise-free cost value up to some arbitrary additive error, one requires resource consumption that scales exponentially in the number of qubits. In Appendix C we present a more detailed statement that explains how such resources may be consumed.

Whilst the use of clean ancillary qubits as part of an error mitigation protocol, utilized as in Equation (16) and Fig. 3, has not been widely studied, Corollary 1 rules
out the possibility that such resources used at the end of
the circuit would offer advantage in counteracting the exponen-
tial scaling effects due to cost concentration. Indeed,
upon inspecting (17), the ancilla appear explicitly in the
form of the fixed point. This highlights a key difference
between many error mitigation strategies and error cor-
correction, as error correction utilizes resources (such as a
larger Hilbert space) in the middle of the computation,
whilst the error mitigation protocols considered here are
based on processing states obtained at the end of a noisy
computation. Our result leaves open the possibility that
novel error mitigation protocols that move beyond the
framework of (19) and Fig. 3 can have hope of counter-
ting the exponential scaling of exponential cost concentration
and NIBPs.

B. Non-asymptotic protocol-specific results

In this section we present predominantly non-asymptotic results for Zero-Noise Extrapolation
(Sec. III B 2), Virtual Distillation (Sec. III B 3), Probabilistic Error Cancellation (Sec. III B 4), and methods
which use a linear ansatz such as Clifford Data Regression (Sec. III B 5). For each protocol, we investigate
the effect of error mitigation on the resolvability of the
cost landscape, for different classes of noisy states. To
the end, we first define a class of resolvability measures
which quantify how many shots it takes to resolve the
cost landscape at some fixed precision after applying
error mitigation, compared to no mitigation at all.

1. Definitions

Definition 2 (Relative resolvability for two points).
Consider two locations in parameter space \( \theta_1, \theta_2 \) and
their corresponding points on the cost landscape. Denote
the number of shots to resolve these two points up to some
fixed precision with and without error mitigation as \( N_{\text{EM}} \)
and \( N_{\text{noisy}} \), respectively. We define the relative resolvabil-
ity for \( \theta_1 \) and \( \theta_2 \) at error level \( \varepsilon \) as

\[
\chi(\theta_1, \varepsilon) = \frac{N_{\text{noisy}}(\theta_1, \varepsilon)}{N_{\text{EM}}(\theta_1, \varepsilon)}
\]

\[
= \frac{1}{\gamma(\varepsilon)} \left( \frac{\Delta C_m(\theta_1, \varepsilon)}{\Delta C(\theta_1, \varepsilon)} \right)^2
\]

where we have used the shorthand notation \( \chi(\theta_1) = \chi(\theta_1, \theta_2) \), \( \gamma \) is the error mitigation cost as defined in Def-
inition 1, and where we denote

\[
\Delta \tilde{C}(\theta_1, \varepsilon) = \tilde{C}(\theta_1, \varepsilon) - \tilde{C}(\theta_2, \varepsilon)
\]

\[
\Delta C_m(\theta_1, \varepsilon) = C_m(\theta_1, \varepsilon) - C_m(\theta_2, \varepsilon)
\]

We can see that if we have \( \chi(\theta_1, \varepsilon) > 1 \), then error
mitigation has successfully increased the resolvability of
the cost values corresponding to the cost values at \( \theta_1 \) and
\( \theta_2 \). Note that this criterion is a necessary but not su-
fficient condition for error mitigation to reverse the effects
of cost concentration on the cost landscape. Namely, it
does not require the mitigated landscape to accurately
reflect the noise-free landscape, and it does not account
for other trainability issues such as proliferation of min-
ima. If \( \chi(\theta_1, \varepsilon) < 1 \) then \( N_{\text{noisy}}(\theta_1, \varepsilon) < N_{\text{EM}}(\theta_1, \varepsilon) \).
This implies that error mitigation has exacerbated the re-
solvability issues associated with cost concentration and
NIBPs, and it has been counterproductive in fixing these
trainability issues.

For a general cost functions, the relative resolvability
of cost function points after mitigation may vary signif-
icantly across the landscape, or be different for different
choices of ansatzes and noise models. This motivates us
to seek averaged measures of resolvability. We consider
two types of averaging: first, an average over cost func-
tion points generated by a given ansatz, noise and cost;
second, an average over a set of noisy states.

Definition 3 (Average relative resolvability I). Denote
the vector of parameters that corresponds to the global
cost minimum at noise parameter \( \varepsilon \) as \( \theta_\ast \). We then define
the averaged relative resolvability as

\[
\bar{\chi}(\varepsilon) = \frac{1}{\gamma(\varepsilon)} \left( \frac{\Delta C_m(\theta_1, \varepsilon)}{\Delta C(\theta_1, \varepsilon)} \right)^2
\]

where \( \langle \cdot \rangle_i \) denotes the mean over all parameter vectors
\( \theta_i \) accessible with the given ansatz of considera-
tion, and where we denote

\[
\Delta C_m(\theta_i, \varepsilon) = C_m(\theta_i, \varepsilon) - C_m(\theta_\ast, \varepsilon)
\]

\[
\Delta C(\theta_i, \varepsilon) = C(\theta_i, \varepsilon) - C(\theta_\ast, \varepsilon)
\]

Averaging across a given cost landscape gives a result
that is particular to the choice of ansatz, measurement
operator and noise model. In order to evaluate the per-
fomance of error mitigation in a more general setting,
we consider a broader average over noisy states that have
the same spectrum. This choice of class of noisy states is
motivated by the fact that a central mechanism of cost
concentration and NIBPs under unital Pauli noise is the
loss of purity. When we consider the second averaged
relative resolvability for Virtual Distillation in Section
III B 3, it will turn out to be bounded by a function of
the purity for such states.

Definition 4 (Average relative resolvability II). Con-
sider a normalized spectrum \( \lambda \in \mathbb{R}^n \) which corresponds
to the eigenspectrum of some noisy state. We define the
2-design-averaged relative resolvability as

\[
\bar{\chi}_\lambda(\varepsilon) = \frac{1}{\gamma(\lambda)} \left( \frac{\Delta C_m(\rho, U_i)}{\Delta C(\rho, U_i)} \right)^2
\]

where we have used the shorthand notation \( \chi(\theta_1) = \chi(\theta_1, \theta_2) \), \( \gamma \) is the error mitigation cost as defined in Def-
inition 1, and where we denote

\[
\Delta C_m(\theta_i, \varepsilon) = C_m(\theta_i, \varepsilon) - C_m(\theta_\ast, \varepsilon)
\]
2. Zero Noise Extrapolation

In this section we present our results on Zero Noise Extrapolation. First, we consider the simple model of global depolarizing noise.

Proposition 1 (Relative resolvability of Zero Noise Extrapolation with global depolarizing noise, 2 noise levels). Consider a circuit with \( L \) instances of global depolarizing noise of the form Eq. \( \ref{eq:global-depolarizing-noise} \). Consider a Richardson extrapolation strategy based on Eq. \( \ref{eq:richardson-extrapolation} \), an exponential extrapolation strategy based on Eq. \( \ref{eq:exponential-extrapolation} \) and a NIBP extrapolation strategy based on Eq. \( \ref{eq:nibp-extrapolation} \). We presume access to an augmented noisy circuit where the error probability is exactly increased by factor \( a_1 > 1 \) as \( p \rightarrow a_1 p \). Then, we have

\[
\chi_{\text{depol}} \leq \left( \frac{c - (1 - a_1 p)^L}{(1 - p)^L} \right)^2, \tag{33}
\]

where \( \chi_{\text{depol}} \) is the relative resolvability (see Definition 2) for global depolarizing noise, and where

\[
c = \begin{cases} 
    a_1 & \text{for Richardson extrapolation,} \\
    a_1 r(\varepsilon)^{1/\gamma} & \text{for exponential extrapolation,} \\
    a_1^{-L + 1} & \text{for NIBP extrapolation.}
\end{cases}
\tag{34}
\]

Thus, \( \chi_{\text{depol}} \leq 1 \) for all of the above extrapolation strategies with access to 2 noise levels.

We see that for all the above techniques, Zero Noise Extrapolation with access to 2 noise levels decreases the resolvability of the cost function under global depolarizing noise. Further, if one attempts to directly reverse the exponential scaling of NIBPs that global depolarizing noise incurs, one obtains an exponentially worse relative resolvability. We now consider how resolvability behaves under Zero Noise Extrapolation on average across the cost landscape, given a generic noise model.

Proposition 2 (Average relative resolvability of Zero Noise Extrapolation, 2 noise levels). Consider a Richardson extrapolation strategy based on Eq. \( \ref{eq:richardson-extrapolation} \), an exponential extrapolation strategy based on Eq. \( \ref{eq:exponential-extrapolation} \) and a NIBP extrapolation strategy based on Eq. \( \ref{eq:nibp-extrapolation} \). We presume perfect access to an augmented noisy circuit where the noise rate is increased by factor \( a_1 > 1 \). We denote \( \theta_{\varepsilon} \) as the parameter corresponding to the global cost minimum at base noise parameter \( \varepsilon \). Further denote

\[
\rho \xrightarrow{\mathcal{D}} \tilde{\rho} = (1 - p)\rho + p \frac{I}{2n}, \tag{32}
\]

where \( \mathcal{D} \) is the global depolarizing channel and \( p \) is the depolarizing probability. Our justification for studying this noise model is twofold. First, global depolarizing noise provides a clean model of cost concentration with no other cost corrupting effects of the noise. Therefore, if a given error mitigation strategy is to mitigate the effects of cost concentration and NIBPs, we expect it to be able to perform well on this noise model. Second, the structure of many error mitigation strategies is directly motivated by the model of global depolarizing noise \([45, 81–86]\). Indeed, many such strategies have been shown to achieve good or perfect performance with this noise model in mitigating noisy cost function values \([45–47, 82]\). However, we stress that trainability may simultaneously get worse, which is what we will now investigate.
\[
\frac{\langle \Delta \tilde{C}(\theta_{i,\varepsilon}, a_1 \varepsilon) \rangle}{\langle \Delta \tilde{C}(\theta_{i,\varepsilon}, e) \rangle} = z. \text{ Any such noise model has an average relative resolvability}
\]
\[
\bar{\chi} \leq \frac{(z - c)^2}{c^2 + 1}, \tag{35}
\]

where
\[
c = \begin{cases} 
  a_1 & \text{for Richardson extrapolation,} \\
  a_{\alpha}(\epsilon) [\alpha(\epsilon)]^{-1} & \text{for exponential extrapolation,} \\
  r(\alpha(\epsilon))^{-1} & \text{for NIBP extrapolation.}
\end{cases} \tag{36}
\]

Thus, under the assumption that \( z \leq 1 \) and \( \langle \Delta \tilde{C}(\theta_{i,\varepsilon}, a_1 \varepsilon) \rangle_i \geq 0, \bar{\chi} \leq 1 \) for all of the above extrapolation strategies with access to 2 noise levels.

Proposition 2 shows that under mild assumptions of the effect of the noise on the cost landscape, Zero Noise Extrapolation with access to 2 noise levels impairs the resolvability of the cost landscape. These assumptions have physical meaning: \( z \leq 1 \) implies that on average the cost concentrates when the noise parameter is boosted, whilst \( \langle \Delta \tilde{C}(\theta_{i,\varepsilon}, a_1 \varepsilon) \rangle_i \geq 0 \) implies that the landscape is not heavily corrupted after boosting the noise parameter so that the minimum at the base noise level remains below the average cost value. We also see that in the presence of exponential cost concentration and NIBPs, the relative resolvability is exponentially small if one attempts to directly reverse the exponential scaling of NIBPs.

In the Appendix, we study a modification of the averaged resolvability in Definition 4 and find that this is bounded by a function of the purity of the noisy states, such that the resolvability decreases if purity decreases with increasing noise level. This result, along with the proofs of the above propositions, can be found in Appendix D1. Finally, we remark that in the above results we consider a scenario where the Richardson, exponential or NIBP extrapolation strategies utilize expectation values from only two noise levels. In Appendix D1c we show that similar results may be obtained for Richardson extrapolation with access to 3 distinct noise levels.

### 3. Virtual Distillation

Here we present our results on Virtual Distillation. In the following proposition we start again with the simple model of global depolarizing noise.

**Proposition 3** (Relative resolvability of Virtual Distillation with global depolarizing noise). Consider global depolarizing noise of the form in Eq. (32) acting on some \( n \)-qubit pure state \( \rho \) with error probability \( p \). We consider the two error mitigation protocols of Ref. [47] (denoted "A" and "B") to respectively prepare (12) and (13), using \( M \) copies of a quantum state. The relative resolvabilities to resolve any two arbitrary cost function points satisfy
\[
\chi^{(A)}_{\text{depol}} \leq \chi^{(B)}_{\text{depol}} = \Gamma(n, M, p), \tag{37}
\]
for all \( n \geq 1, M \geq 2, p \in [0, 1], \) and where
\[
\Gamma(n, M, p) \leq 1, \tag{38}
\]
is a monotonically decreasing function in \( M \) (with asymptotically exponential decay) for all \( n \geq 1, M \geq 2 \). Within this region the bound is saturated as \( \Gamma(1, 2, p) = 1 \) for all \( p \).

Proposition 3 shows that Virtual Distillation decreases the resolvability of cost landscapes suffering from global depolarizing noise. Moreover, as the number of state copies \( M \) increases, the effect worsens. We find similar results in the following proposition when considering averaged resolvabilities over a class of noisy states.

**Proposition 4** (Average relative resolvability of Virtual Distillation). Consider an error mitigation protocol that prepares estimator \( C_m(\theta_i) = \text{Tr}[\tilde{\rho}_i^m O]/\text{Tr}[\tilde{\rho}_i^m] \) from some noisy parameterized quantum state \( \tilde{\rho}_i \equiv \tilde{\rho}(\theta_i) \). Consider the average relative resolvability \( \bar{\chi}_\lambda \) for noisy states of some spectrum \( \lambda \) with purity \( P \) as defined in Definition 3. We have
\[
\bar{\chi}_\lambda \leq G(n, M, P) \leq 1, \tag{39}
\]
where \( G(n, M, P) \) is a monotonically decreasing function in \( M \) (with asymptotically exponential decay) for all \( n \geq 1, M \geq 2 \). Within this region the bound is saturated as \( G(1, 2, P) = 1 \) for all \( P \), and as \( G(n, M, 1) = 1 \) for all \( n \geq 1, M \geq 2 \).

We present the explicit forms of \( \Gamma(n, M, p) \) and \( G(n, M, P) \) as well as a proof of the above propositions in Sec. D2 of the Appendix. We note that in most cases the bound given by Proposition 4 decreases with decreasing noisy state purity. This indicates that within such settings, the greater the loss of purity due to noise, the worse the impact on resolvability is after error mitigation with Virtual Distillation.

### 4. Probabilistic Error Cancellation

Here we present our results for Probabilistic Error Cancellation. We utilize the optimal quasiprobability decompositions studied in Ref. [87], and the proofs can be found in Section D3 of the Appendix.

**Proposition 5** (Relative resolvability of Probabilistic Error Cancellation under global depolarizing noise). Consider a quasi-probability method that corrects global depolarizing noise of the form (32). For any pair of states
corresponding to points on the cost function landscape, the optimal quasiprobability scheme gives
\[ \chi_{\text{depol}} = \frac{2^nn}{2^nn - p(2 - p)} \geq 1, \]
for all \( n \geq 1, p \in [0, 1] \), which is achieved with access to noisy Pauli gates.

Proposition 5 shows that for the special case of global depolarizing noise, Probabilistic Error Cancellation actually improves the resolvability of the noisy cost landscape. However, this improvement is generally small and is decreasing quickly with the number of qubits \( n \). For instance, for \( n = 1, \chi_{\text{depol}} \) has maximum value \( 4/3 \) (achieved in the limit of maximum depolarization probability). For for \( n = 2, \chi_{\text{depol}} \) has maximum value \( \approx 1.07 \). In the limit of large \( n \), \( \chi_{\text{depol}} \) tends to 1.

We extend this study to local depolarizing noise in Appendix D.3. We find that for a single instance of local depolarizing noise, Probabilistic Error Correction can either improve resolvability or worsen it, depending on the strength of concentration of the cost. In addition, we show in the following proposition that if one wishes to mitigate all the noisy gates in the circuit and one has NIBP scaling, the improvement due to Probabilistic Error Cancellation degrades exponentially, and ultimately for large problem sizes this impairs resolvability.

**Proposition 6** (Scaling of Probabilistic Error Cancellation with local depolarizing noise). Consider local depolarizing noise with depolarizing probability \( p \) to act in \( L \) layers through a depth \( L \) circuit as in Eq. (3). Suppose that the effect of this noise is to cause cost concentration
\[ \langle \Delta \tilde{C}(\theta_i, \ast) \rangle_i = Aq^L \langle \Delta C(\theta_i, \ast) \rangle_i, \]
for some constant \( A \) and noise parameter \( q \in [0, 1] \). The optimal quasiprobability method to mitigate the depolarizing noise in the circuit yields
\[ \chi = \frac{1}{4^{nL}} (Q(p))^n L, \]
where \( Q(p) = 1 - \frac{3p(2 - p)}{4 - p(2 - p)} \in [0, 1] \) for \( p \in (0, 1) \). Thus, the average relative resolvability has unfavourable scaling with system size.

We note that (41) gives the best possible scaling of noisy cost differences allowed by (5) under local depolarizing noise. Thus, Proposition 6 shows that if has NIBP scaling under local depolarizing noise, one still has unfavourable scaling for resolvability.

5. **Linear ansatz methods**

In Proposition 7 we consider a scenario where the same linear ansatz (15) is applied to two points on the noisy cost landscape. For Clifford Data Regression this is a reasonable assumption in scenarios where one is comparing two points that are close in parameter space, for instance, when a simplex-based optimizer is exploring a small local region. However, we remark this is not always true in general settings.

**Proposition 7** (Linear ansatz methods). Consider any error mitigation strategy that mitigates noisy cost function value \( \tilde{C}(\theta) \) by constructing an estimator \( C_m(\theta) \) of the form (15). For any two noisy cost function points to which the same ansatz is applied, we have
\[ \chi = 1, \]
for any noise process.

**Corollary 2** (Linear ansatz methods under global depolarizing noise). Under global depolarizing noise, the optimal linear ansatz gives \( \chi = 1 \) for any pair of cost function points.

Corollary 2 comes simply by noting that the optimal choice of linear ansatz under global depolarizing noise corrects the noise exactly and is state independent [45]. The above results imply that in some settings CDR has a neutral effect on the resolvability of the cost function landscape. This opens up the possibility that in practical settings CDR can improve the trainability of cost landscapes, if it can remedy other cost corrupting effects due to noise outside of cost concentration. This motivates our numerical studies of CDR, which we present in the following section.

IV. **Numerical Results**

As discussed in Sec. III B, in many settings, current state-of-the-art error mitigation methods do not mitigate the effects of cost concentration. Nevertheless, as discussed in Sec. III B 3, trainability of VQAs is also affected by other cost-corrupting effects. We expect that error mitigation can reverse some of the effects due to cost corruption that affect the trainability of VQAs when the effects of cost concentration are not too severe. In this section, we numerically investigate the effects of error mitigation on trainability in such a setting to provide possible evidence towards beneficial effects of error mitigation. To this end, we focus on CDR as in some settings it does not worsen the effects of cost concentration, as shown in Sec. III B 5.

We perform our numerical experiments by simulating the Quantum Approximate Optimization Algorithm (QAOA) [24] for 5-qubit MaxCut problems. We use a realistic noise model of an IBM quantum computer [88], which has been obtained by gate set tomography of IBM’s Ourense quantum device. Further, we assume linear connectivity of the simulated quantum computer.
A MaxCut problem is defined for a graph $G = (V, E)$ of nodes $V$ and edges $E$. The problem is to find a bipartition of the nodes into two sets which maximizes the number of edges connecting the sets. This problem can be reformulated as finding the ground state of a Hamiltonian

$$H_{\text{MaxCut}} = -\frac{1}{2} \sum_{ij \in E} (\mathbb{1} - Z_i Z_j),$$

where $Z_i, Z_j$ are Pauli $Z$ matrices. Here we consider graphs with $n = 5$ vertices, with 36 randomly generated instances according to the Erdős-Rényi model [89], where for each pair of vertices in the graph there is a connecting edge with probability 0.5.

To approximate the ground state of $H_{\text{MaxCut}}$ we simulate the QAOA for number of rounds $p = 1$ to 8. The QAOA ansatz applied to the input state is given as

$$\prod_{j=p,p-1,...,1} e^{i\beta_j H_M} e^{i\gamma_j H_{\text{MaxCut}}} |+\rangle^\otimes n,$$

where $H_M = \sum_j X_j$, $X_j$ are Pauli $X$ matrices, we denote $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, and $\beta_j, \gamma_j$ are variational parameters. We minimize the cost function $\langle H_{\text{MaxCut}} \rangle$ using the Nelder-Mead algorithm [72]. We perform the optimization with shot budgets ranging from $N_{\text{tot}} = 10^7$ to $1.5 \times 10^8$. We define $N_{\text{tot}}$ as total number of shots spent on the optimization. We detail the optimization procedure in Appendix E. In our numerics, the values of $N_{\text{tot}}$ are chosen to enable implementation of the optimization with current quantum computers. To quantify the quality of the solutions of noisy (unmitigated) and CDR-mitigated optimization, we compute approximation ratios of the solutions using the exact expectation value of $\langle H_{\text{MaxCut}} \rangle$. The approximation ratio is defined here as the ratio of a given solution’s energy to the true ground state energy.

We gather our numerical results for CDR in Fig. 5. In the figure we plot the approximation ratio averaged over 36 randomly chosen graphs versus $N_{\text{tot}}$. We compare the quality of the solutions of noisy (unmitigated) and CDR-mitigated optimization and find that CDR-mitigated optimization outperforms noisy optimization for all considered $p$ and $N_{\text{tot}}$ values. We observe that the solutions for $p = 2$ outperform those for $p = 1$ for both CDR-mitigated and noisy optimization. The quality of $p > 2$ solutions decline with increasing $p$ for noisy optimization, while it remains approximately the same for $p = 2$ to 6 for CDR-mitigated optimization. With CDR-mitigated optimization we see a decrease in quality of solution for the largest considered $p = 8$.

The numerical results presented here are obtained for circuits shallow enough to be trainable while using the CDR-mitigated cost function. Therefore they are outside of the NIBP scaling regime. As discussed in Section IIIB, even outside the NIBP regime noise may adversely impact trainability by corrupting the cost function landscape, which error mitigation has a chance to remedy.

Our results give hope that CDR-mitigated optimization may overall offer a trainability advantage for problems with such cost function landscape corruption.

As discussed in Section IIIIB optimizing an error mitigated cost function is not guaranteed to outperform its noisy optimization even outside the NIBP regime. Indeed, we find numerically that for $p = 2, 4$ optimization with Virtual Distillation does not outperform noisy optimization for the considered implementations (see Appendix E2).
V. Discussion

Noise can exponentially degrade the trainability of linear (or superlinear) depth Variational Quantum Algorithms (VQAs) by flattening the cost landscape, thus requiring an exponential precision in system size to resolve its features \cite{53, 54}. This limits the scope for achieving possible quantum advantage with VQAs. At present there are no known strategies to avoid this exponential scaling completely aside from pursuing algorithms with sublinear circuit depth, and current strategies to mitigate this effect consist only of reducing hardware noise rates. Thus, is it a pressing challenge to search for possible solutions to this problem. Error mitigation strategies emerge as a natural candidate to tackle this problem under near-term constraints.

In this work we investigate the effects of error mitigation on the trainability of noisy cost function landscapes in two regimes. First, we work in the asymptotic regime (in terms of scaling with system size) and find that if a VQA is suffering from exponential cost concentration, requiring an exponential number of shots to accurately resolve cost values, then a broad class of error mitigation strategies (including as special cases Zero Noise Extrapolation, Virtual Distillation, Probabilistic Error Cancellation, Clifford Data Regression) cannot remove this exponential scaling. Within the considered paradigm, this exponential scaling implies that at least an exponential number of resources needs to be spent in order to extract accurate information from the cost landscape in order to find a cost-minimizing optimization direction. In Corollary 1 we identify circuit samples (or shots) as well as number of copies of a quantum state as two such resources.

Second, we move out of the asymptotic regime and investigate whether or not particular error mitigation protocols can improve the resolvability of noisy cost landscapes. Should such a landscape be burdened with exponential cost concentration, this would correspond to an improvement in the coefficient in the exponential scaling. Our results indicate that some error mitigation protocols can worsen the resolvability, and ultimately the trainability, of cost landscapes in certain settings. In particular, in Propositions 3 and 4 we show analytically that Virtual Distillation impairs resolvability with worsening resolvability as the number of state copies increases. We obtain similar results for Zero Noise Extrapolation in Propositions 1 and 2 under some assumptions of the cost landscape. Numerical analysis of a particular MaxCut problem indicates that trainability is overall similarly impaired for Virtual Distillation.

Clifford Data Regression (CDR) distinguishes itself from the other error mitigation techniques considered in this article, as in contrast to the other protocols it does not necessarily increase the statistical uncertainty of cost values more than it reverses their concentration. This is reflected in the fact that under a global depolarizing noise model, CDR has neutral impact on resolvability (Corollary 2). However, it is also known that CDR can remedy the effects of more complex noise models. This indicates that CDR could resolve trainability issues arising due to corruptions of the cost function outside of cost concentration, whilst having a neutral effect on cost concentration itself, and thus overall improve trainability. In the numerical example studied, presented in Fig. 5, we observe this to be the case. This points to deeper future work studying the mechanisms that allow error mitigation to improve the trainability of noisy cost landscapes.

Finally, we identify that the broad class of error mitigation protocols we study in our asymptotic analysis all only consist of post-processing expectation values of noisy circuits, as summarized in Fig. 3. This gives intuition as to why they cannot escape the exponential scaling of noise-induced barren plateaus (NIBPs). However, the theory of error correction indicates that with sufficient resources NIBPs can indeed be avoided. This gives hope that there can exist novel error mitigation strategies that move beyond the framework of the protocols considered in this article and thereby avoid the exponential impairment to trainability that NIBPs present.

VI. Acknowledgements

This work was supported by the Quantum Science Center (QSC), a National Quantum Information Science Research Center of the U.S. Department of Energy (DOE). SW was supported by the U.S. DOE, Office of Science, Office of Advanced Scientific Computing Research, under the Quantum Computing Application Teams program. SW was also partially supported by the Samsung GRP grant. Piotr C. and AA were supported by the Laboratory Directed Research and Development (LDRD) program of LANL under project numbers 20190659PRD4 (Piotr C.) and 20210116DR (AA and Piotr. C). MC acknowledge support from the Center for Nonlinear Studies at Los Alamos National Laboratory. MC and LC were also initially supported by the LDRD program of LANL under project number 20190065DR. PJC also acknowledges initial support from the LANL ASC Beyond Moore’s Law project. This research used resources provided by the Los Alamos National Laboratory Institutional Computing Program, which is supported by the U.S. Department of Energy National Nuclear Security Administration under Contract No. 89233218CNA000001.
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Appendices

Road map of appendices. In Appendix A we present some notation and definitions that we need in order to...
prove our main results, as well as provide further details on the error mitigation protocols studied in this article. In Appendix B we derive some useful lemmas that are required for our proofs. In Appendix C we present the proof for our asymptotic results on the exponential concentration of estimators. In Appendix D we present our protocol-specific results on the change in resolvability of the cost landscape under error mitigation. Finally, in Appendix E we discuss details of our numerical implementations for Clifford Data Regression and Virtual Distillation.

Appendix A: Preliminaries

1. Further details on error mitigation techniques

In this section we expand on our discussion in Section IIIC and provide further details on the Zero Noise Extrapolation and Probabilistic Error Cancellation protocols.

a. Zero Noise Extrapolation (ZNE)

For convenience in this section we recall the key points of Zero Noise Extrapolation as summarized in Section (III B2). We also detail the explicit forms of the estimators that can be constructed for exponential extrapolation and an extrapolation strategy tailored towards NIBP effects, which will be required in order to prove our results.

Richardson Extrapolation. We suppose that \( \tilde{C}(\theta, \varepsilon) \) admits a Taylor expansion in small noise parameter \( \varepsilon \) as

\[
\tilde{C}(\theta, \varepsilon) = \tilde{C}(\theta, 0) + \sum_{k=1}^{m} p_k(\theta) \varepsilon^k + \mathcal{O}(\varepsilon^{m+1}) ,
\]

where \( p_k \) are unknown parameters and \( \tilde{C}(\theta, 0) \) is the zero-noise cost function. By considering the equivalent expansion of \( \tilde{C}(\theta_i, a_1 \varepsilon) \) and combining the two equations we can obtain

\[
C_m^{(2)}(\theta_i) = \frac{a_1 \tilde{C}(\theta_i, \varepsilon) - \tilde{C}(\theta_i, a_1 \varepsilon)}{a_1 - 1} = \tilde{C}(\theta_i, 0) + \mathcal{O}(\varepsilon^2) ,
\]

which is a higher-order approximation of \( \tilde{C}(\theta_i, 0) \) compared to simply using \( \tilde{C}(\theta_i, \varepsilon) \). This process can be repeated iteratively \( m \) times to obtain an estimator which is accurate up to \( \mathcal{O}(\varepsilon^{m+1}) \) error. It can be shown that the general form for the estimator that uses \( k \) noise levels can be written as

\[
C_m^{(k)}(\theta_i) = \sum_{j=0}^{k} \beta_j \tilde{C}(\theta_i, a_j \varepsilon) ,
\]

where the coefficients \( \beta_j \) satisfy the linear system of equations \( \sum_{j=0}^{k} \beta_j = 1 \) and \( \sum_{t=0}^{k} \beta_t a_t^j = 0 \) for all \( t \in \{1, ..., k\} \) [90]. For 3 noise levels, (A3) explicitly gives

\[
C_m^{(3)}(\theta_i) = \frac{a_1 a_2 (a_2 - a_1) \tilde{C}(\theta_i, \varepsilon) - a_2 (a_2 - 1) \tilde{C}(\theta_i, a_1 \varepsilon) + a_1 (a_1 - 1) \tilde{C}(\theta_i, a_2 \varepsilon)}{(a_1 - 1) (a_2 - 1) (a_2 - a_1)} = \tilde{C}(\theta_i, 0) + \mathcal{O}(\varepsilon^3) .
\]

Exponential extrapolation. We can also consider an exponential model

\[
\tilde{C}(\theta, \varepsilon) = r(\theta, \varepsilon)^{-t(\theta, \varepsilon)} \left( \sum_{k=0}^{m} p_k(\theta) \varepsilon^k + \mathcal{O}(\varepsilon^{m+1}) \right) ,
\]

for some \( r \) and \( t \), which in general can be functions of \( \varepsilon \). Following in similar steps to Richardson extrapolation we can consider the same expansion at an augmented noise level \( a_1 \varepsilon \). This enables us to construct the estimator

\[
C_m(\theta_i) = \frac{1}{a_1 - 1} \left( a_1 r(\theta_i, \varepsilon)^{t(\theta, \varepsilon)} \tilde{C}(\theta_i, \varepsilon) - r(\theta_i, a_1 \varepsilon)^{t(\theta, a_1 \varepsilon)} \tilde{C}(\theta_i, a_1 \varepsilon) \right) ,
\]

(A6)
which approximates $\tilde{C}(\theta_i, 0)$ to a higher order in $\varepsilon$ compared to $\tilde{C}(\theta_i, \varepsilon)$.

**NIBP extrapolation.** We can construct an alternative Zero Noise Extrapolation strategy that is tailored towards noisy cost function values that are dominated by NIBP scaling as in Eq. (5). We model the effects of noise as

$$\tilde{C}(\theta_i, q) = \tilde{C}(\theta_i, 0) + q^L \left( B(\theta_i) + \sum_{k=1}^{m} p_k (1 - q)^k + \mathcal{O} \left( (1 - q)^{m+1} \right) \right),$$

(A7)

for all noisy cost function points $\tilde{C}(\theta_i, q)$, where $\tilde{C}(\theta_i, 0)$ is the fixed point of the noise and $B(\theta_i) + \tilde{C}(\theta_i, 0)$ is the noise free cost value. (Note that for NIBPs we cannot consider lower-order polynomials of $q$, as else the NIBP condition would be broken for small $q$.) We construct estimators for any given parameter $\theta_i$, as

$$C_m(\theta_i) = \frac{a^{L+1} q^{-L}(\tilde{C}(\theta_i, q/a) - \tilde{C}(\theta_i, 0)) - q^{-L}(\tilde{C}(\theta_i, q) - \tilde{C}(\theta_i, 0))}{a - 1} + K,$$

(A8)

where $\tilde{C}(\theta_i, 0)$ is the parameter-independent fixed value obtained at maximum noise level, and $K = \tilde{C}(\theta_i, 0) - \sum_k p_k$ is an additive constant. As we are only interested in cost function differences for our results, this will cancel out.

### b. Probabilistic Error Cancellation

**General idea.** Probabilistic Error Cancellation utilizes many modified circuit runs in order to construct a quasiprobability representation of the noise-free cost function [42, 43]. We assume that the effect of the noise can be described by a quantum channel $\mathcal{N}$ that occurs after a gate that we denote with unitary channel $\mathcal{U}$. For now we assume this is the only instance of noise in the circuit, however, we will later generalize to many instances of noise. The goal of Probabilistic Error Cancellation is to simulate the inverse map $\mathcal{N}^{-1}$. Note in general this will not always correspond to a CPTP map. Despite this, if we have a basis of (noisy) quantum channels $\{B_\alpha\}_\alpha$, corresponding to experimentally available channels, we can expand the inverse map in this basis as

$$\mathcal{N}^{-1} = \sum_\alpha q_\alpha B_\alpha,$$

(A9)

for some set of $q_\alpha \in \mathbb{R}$. Then, the channel that describes the noiseless gate can be written as

$$\mathcal{U} = \mathcal{N}^{-1} \circ \mathcal{N} \circ \mathcal{U} = \sum_\alpha q_\alpha K_\alpha,$$

(A10)

where we have defined $K_\alpha = B_\alpha \circ \mathcal{N} \circ \mathcal{U}$. Denote the input state to the gate as $\rho_{in}$ and a measurement operator as $O$. The expectation value can be written

$$C_{\mathcal{U}(\rho)} = \text{Tr} [\mathcal{U}(\rho_{in}) O] = \sum_\alpha q_\alpha \tilde{C}_{K_\alpha(\rho)},$$

(A12)

where for simplicity we first assume that $\mathcal{U}$ is the only gate in the circuit, and $\tilde{C}_{K_\alpha(\rho)} \equiv \text{Tr} [K_\alpha(\rho)O]$. Finally, we can explicitly define a probability distribution $p_\alpha = |q_\alpha|/G_N$ where $G_N = \sum_\alpha |q_\alpha|$. This gives us an alternative way to write (A11) and (A12) as

$$\mathcal{U} = G_N \sum_\alpha \text{sgn}(q_\alpha) p_\alpha K_\alpha,$$

(A13)

$$C_{\mathcal{U}(\rho)} = G_N \sum_\alpha \text{sgn}(q_\alpha) p_\alpha \tilde{C}_{K_\alpha(\rho)},$$

(A14)

where $\text{sgn}(q_\alpha)$ denotes the sign of $q_\alpha$. We call this the quasiprobability representation of the gate $\mathcal{U}$. The idea is that if we have access to the set of CPTP maps $\{B_\alpha\}_\alpha$ in our noisy native hardware gate set, then we can obtain an estimate of the noiseless expectation value $C_{\mathcal{U}(\rho)}$ as follows: (1) With probability $p_\alpha$, prepare the circuit corresponding to $K_\alpha(\rho)$ and obtain $\tilde{C}_{K_\alpha(\rho)}$. (2) Multiply the measurement result by $\text{sgn}(q_\alpha)G_N$. (3) Repeat process many times and sum results.
Correcting many gates. So far we have only considered a circuit with a single gate $U$. We can generalize (A13) to a general circuit $\prod_k^{N_g} U_k$ with $N_g$ gates with the quasiprobability representation

$$\prod_k^{N_g} U_k = C_N^{\text{tot}} \sum_i \text{sgn}(q_i) p_i K_i,$$

where $C_N^{\text{tot}} = \prod_k G_k$, $i = (i_1, \ldots, i_N)$, $q_i = \prod_k q_{i_k}$, $p_i = \prod_k p_{i_k}$, $K_i = \prod_k K_{i_k}$. Thus, a similar procedure can be carried out in order to mitigate the noise on each individual gate in the circuit.

Appendix B: Useful Lemmas

1. Noise Induced Cost Concentration

Lemma 1. Consider a parameterized noisy cost function $\tilde{C}(\theta) = \text{Tr} [\tilde{\rho}(\theta) O]$, where $\tilde{\rho}(\theta)$ is an $n$-qubit noisy state given by (3) and $\theta \in \Theta$ is drawn from some set of accessible parameters $\Theta$. Suppose the cost is suffering from exponential cost concentration according to Ref. [53], that is

$$\left| \tilde{C}(\theta) - \frac{\text{Tr}[O]}{2^n} \right| \leq q^L B(n) \|\rho_{in} - \frac{I}{2^n}\|,$$

for all $\theta \in \Theta$, where $\rho_{in}$ is the input state in (3), $0 \leq q < 1$ is some noise parameter, and $B(n) \in \text{poly}(n)$. Then, $\exists A(n) \in \text{poly}(n)$ such that

$$\left| \tilde{C}(\theta_1, q) - \tilde{C}(\theta_2, q) \right| \leq A(n) q^L,$$

for any two sets of parameters $\theta_1, \theta_2 \in \Theta$.

Proof. Starting from Eq. (B1) can simply write

$$\left| \tilde{C}(\theta_1, q) - \tilde{C}(\theta_2, q) \right| \leq 4B(n) q^L,$$

for any two sets of parameters $\theta_1, \theta_2$, where we have used the triangle inequality in 1D and the fact that the trace distance has a maximum value of 1.

For the next lemma we will consider the $n$-qubit channel

$$\mathcal{W} = U_k \circ \mathcal{N} \circ \cdots \circ \mathcal{N} \circ U_2 \circ \mathcal{N} \circ U_1 \circ \mathcal{N},$$

where $\{U_k\}_{k=1}^L$ denote unitary channels that describe collections of gates that act together in a layer, and $\mathcal{N} = \bigotimes_{i=1}^n D_i$ is an instance of local depolarizing channels, with depolarizing probabilities $\{p_i\}_{i=1}^n$. We denote $q = \max\{(1-p_i)\}_{i=1}^n < 1$.

Lemma 2. Consider $\mathcal{W}$ acting on some input state $\rho_{in}$. Then we have

$$\left\| \mathcal{W}(\rho_{in}) - \frac{I}{2^n} \right\|_1 \leq q^k n^{1/2} \sqrt{2\ln 2}.$$

Proof. We have

$$\left\| \mathcal{W}(\rho_{in}) - \frac{I}{2^n} \right\|_1 \leq \sqrt{2\ln 2 D(\mathcal{W}(\rho_{in}) \| \frac{I}{2^n})}$$

$$\leq \sqrt{2\ln 2 q^{2k} D(\rho_{in} \| \frac{I}{2^n})}$$

$$\leq q^k n^{1/2} \sqrt{2\ln 2},$$

where $D(\cdot\|\cdot)$ denotes the relative entropy. The first inequality is Pinsker’s [91], the second inequality is due to $k$ repeated applications of the strong data processing inequality of Ref. [92] and the unitary invariance of the relative entropy, and the third inequality is due to an upper bound on the relative entropy of purity. 

\[\square\]
2. Averages over unitary 2-designs

Lemma 3. Consider the cost function value \( C_\sigma(U) = \text{Tr}[U \sigma U^\dagger O] \) where \( U \) is a \( d \times d \) unitary matrix and \( \sigma \in S(\mathcal{H}) \) is some quantum state. Consider expectation values over \( U_i \in Y \) where \( Y \subset \mathcal{U}(d) \) is a unitary 2-design and \( \mathcal{U}(d) \) is the unitary group of degree \( d \). Denote such expectation values as \( \langle \cdot \rangle_U \). Then, we have

\[
\langle C_\sigma \rangle_U = \frac{1}{d} \text{Tr}[O],
\]

(B9)

\[
\langle C_\sigma C_\rho \rangle_U = \frac{\text{Tr}[O^2] (d \text{Tr}[\rho \sigma] - 1) - \text{Tr}[O]^2 (\text{Tr}[\rho \sigma] - d)}{d(d^2 - 1)},
\]

(B10)

for any two operators \( \sigma, \rho \in B(\mathcal{H}) \), \( \dim(\mathcal{H}) = d \). This implies

\[
\text{Var}[C_\sigma] = \frac{(\text{Tr}[O^2] - \frac{1}{d^2} \text{Tr}[O]^2) (\text{Tr}[\sigma^2] - \frac{1}{d^2})}{d^2 - 1},
\]

(B11)

Proof. We use the following standard expressions for integration with respect to the Haar measure over the unitary group of degree \( d \):

\[
\int_\mathcal{U}(d) d\mu(W) w_{i,j} w_{p,k}^* = \frac{\delta_{i,p} \delta_{j,k}}{d},
\]

(B12)

\[
\int_\mathcal{U}(d) d\mu(W) w_{i_1,j_1} w_{i_2,j_2} w_{i_1',j_1'} w_{i_2',j_2'}^* = \frac{1}{d^2 - 1} \left( \delta_{i_1,i_1'} \delta_{j_2,j_2'} + \delta_{i_2,i_2'} \delta_{j_1,j_1'} + \delta_{i_1,i_2'} \delta_{j_2,j_1'} + \delta_{i_2,i_1'} \delta_{j_1,j_2'} \right) - \frac{1}{d} \frac{1}{(d^2 - 1)} \left( \delta_{i_1,i_1'} \delta_{j_2,j_2'} + \delta_{i_2,i_2'} \delta_{j_1,j_1'} + \delta_{i_1,i_2'} \delta_{j_2,j_1'} + \delta_{i_2,i_1'} \delta_{j_1,j_2'} \right),
\]

(B13)

where \( w_{i,j} \) are the matrix elements of the unitary \( W \in \mathcal{U}(d) \). Then, the expectation values over 2-designs can be evaluated as

\[
\langle C_\sigma \rangle_U = \frac{1}{d} \text{Tr}[O],
\]

(B14)

and

\[
\langle C_\sigma C_\rho \rangle_U = \frac{1}{d^2 - 1} \text{Tr}[O]^2 + \frac{1}{d^2 - 1} \text{Tr}[\rho \sigma] \text{Tr}[O^2] - \frac{1}{d} \frac{1}{(d^2 - 1)} \text{Tr}[\rho \sigma] \text{Tr}[O]^2 - \frac{1}{d^2 - 1} \text{Tr}[O]^2,
\]

(B15)

where we have used \( \text{Tr}[\rho] = \text{Tr}[\sigma] = 1 \). The final statement comes by noting that

\[
\text{Var}[C_\sigma] = \langle C_\sigma^2 \rangle_U - \langle C_\sigma \rangle_U^2
\]

(B16)

\[
= \frac{\text{Tr}[O^2] (d \text{Tr}[\sigma^2] - 1) - \text{Tr}[O]^2 (\text{Tr}[\sigma^2] - d)}{d(d^2 - 1)} - \frac{1}{d^2} \text{Tr}[O]^2
\]

(B17)

\[
= \frac{\text{Tr}[O^2] (d \text{Tr}[\sigma^2] - 1) - \text{Tr}[O]^2 (d \text{Tr}[\sigma^2] - 1)}{d^2(d^2 - 1)},
\]

(B18)

which can be factorized to give the desired result.

\[\square\]

Appendix C: Exponential estimator concentration

We present a proof of Theorem 1, and restate the result here for convenience.

Theorem 1. Consider an error mitigation protocol prepares the quantity

\[
E_{\sigma,X,M,k} = \text{Tr} \left[ X (\sigma^\otimes M \otimes |0\rangle\langle 0|^\otimes k) \right],
\]

(C1)
for some quantum state $\sigma \in S(\mathcal{H})$, for $|0\rangle\langle 0| \in S(\mathcal{H}')$ and for some $X \in B(\mathcal{H}^\otimes M \otimes \mathcal{H}^\otimes k)$. We suppose $\sigma$ is prepared with a depth $L_\sigma$ circuit and experiences noise according to Eq. (3). Under these conditions, $E_{\sigma,X,M,k}$ exponentially concentrates on a state-independent fixed point in the depth of the circuit as

$$|E_{\sigma,X,M,k} - \text{Tr} \left[ X \left( \frac{1}{2Mn} \otimes |0\rangle\langle 0|^{\otimes k} \right) \right]| \leq G_{\sigma,X,M}(n),$$

(C2)

where $\mathbb{1} \in S(\mathcal{H})$ is the $n$-qubit identity operator and

$$G_{\sigma,X,M}(n) = \sqrt{\ln 4} \| X \|_\infty M n^{1/2} q^{L_\sigma + 1},$$

(C3)

with noise parameter $q \in [0, 1)$.

Proof. Consider

$$\left| \text{Tr} \left[ (\sigma^{\otimes M} \otimes |0\rangle\langle 0|^{\otimes k}) X \right] - \text{Tr} \left[ \left( \frac{1}{2n} \otimes \sigma^{\otimes M-1} \otimes |0\rangle\langle 0|^{\otimes k} \right) X \right] \right| \leq \left\| (\sigma - \frac{1}{2n}) \otimes \sigma^{\otimes M-1} \otimes |0\rangle\langle 0|^{\otimes k} \right\|_1 \| X \|_\infty$$

(C4)

$$= \left\| \sigma - \frac{1}{2n} \right\|_1 \text{Tr} \left[ \sigma^{\otimes M-1} \text{Tr} [ |0\rangle\langle 0|^{\otimes k} ] \right] \| X \|_\infty$$

(C5)

$$\leq q^{L_\sigma + 1} n^{1/2} \sqrt{2 \ln 2} \| X \|_\infty.$$

(C6)

The summation of these equations gives

$$\text{Tr} \left[ \sigma^{\otimes M} X \right] - \text{Tr} \left[ \left( \frac{1}{2n} \otimes \sigma^{\otimes M-1} \right) X \right] \leq q^{L_\sigma + 1} n^{1/2} \sqrt{2 \ln 2} \| X \|_\infty,$$

(C8)

$$\text{Tr} \left[ \left( \frac{1}{2n} \otimes \sigma^{\otimes M-1} \right) X \right] - \text{Tr} \left[ \left( \frac{1}{2n} \otimes \frac{1}{2n} \otimes \sigma^{\otimes M-2} \right) X \right] \leq q^{L_\sigma + 1} n^{1/2} \sqrt{2 \ln 2} \| X \|_\infty,$$

(C9)

$$\text{Tr} \left[ \left( \frac{1}{2n} \otimes \frac{1}{2n} \otimes \sigma^{\otimes M-2} \right) X \right] - \text{Tr} \left[ \left( \frac{1}{2n} \otimes \frac{1}{2n} \otimes \frac{1}{2n} \otimes \sigma^{\otimes M-3} \right) X \right] \leq q^{L_\sigma + 1} n^{1/2} \sqrt{2 \ln 2} \| X \|_\infty,$$

(C10)

$$\ldots$$

(C11)

$$\text{Tr} \left[ \left( \frac{1}{2n} \right)^{\otimes M-1} \otimes \sigma \right] X - \text{Tr} \left[ \left( \frac{1}{2n} \right)^{\otimes M} \right] \| X \|_\infty.$$

(C12)

which gives the desired bound.

We now present a more detailed statement of Corollary 1, which explains how one can spend an exponential number of resources in different ways in order to resolve concentrated cost values.

**Corollary 1** (Exponential estimator concentration). Consider an error mitigation protocol that approximates the noise-free cost value $C(\theta)$ by estimating the quantity

$$C_m(\theta) = \sum_{(\sigma(\theta), X, M, k) \in T} a_{X,M,k} E_{\sigma(\theta),X,M,k},$$

(C14)

where each $E_{\sigma,X,M,k}$ takes the form (C1). We denote $M_{max}$ and $a_{max}$ as the maximum values of $M$ and $a_{X,M,k}$ respectively accessible from a set $T$ defined by the given protocol. Assuming $\| X \|_\infty \in O(\text{poly}(n))$, there exists a fixed point $F$ independent of $\theta$ such that

$$|C_m(\theta) - F| \in O(2^{-\beta_n} a_{max} |T| M_{max}),$$

(C15)
for some constant $\beta \geq 1$ if the circuit depths satisfy
\[ L_{\sigma(\theta)} \in \Omega(n), \]  
(C16)
for all $\sigma(\theta)$ in the construction (C15). That is, if the depth of the circuits scale linearly or greater then one requires at least exponential resources to distinguish $C_m$ from its fixed point, for instance in one of the following ways:

- $a_{\max}|T| M_{\max} \in O(\text{poly}(n))$ and an exponentially large number of shots are used to distinguish two quantities with exponentially small separation.
- $a_{\max}|T| \in \Omega(2^{\beta'|n})$ for some constant $\beta' \geq 1$ and an exponentially large number of shots are required to distinguish two quantities with exponentially large statistical uncertainty, as measurement outcomes are multiplied by $a_{\max}|T|$.
- $M_{\max} \in \Omega(2^{\beta''|n})$ for some constant $\beta'' \geq 1$ and an exponentially large number of copies of some quantum state $\sigma$ are required.

Proof. Explicitly applying the results of Theorem (1) to the construction of $C_m(\theta)$ in (C15) we have
\[
\left| C_m(\theta) - \sum_{(\sigma(\theta), X, M, k) \in T} a_{X, M, k} \text{Tr} \left[ X \left( \frac{1}{2^{M_n}} \otimes |0\rangle \langle 0|^\otimes k \right) \right] \right| \leq \sum_{(\sigma(\theta), X, M, k) \in T} a_{X, M, k} G_{\sigma(\theta), X, M}(n) \in O\left( \sum_{(\sigma(\theta), X, M, k) \in T} a_{X, M, k} \|X\|_\infty M_{\max} n^{1/2} q^{L_{\sigma(\theta)}+1} \right)
\]
(C17)
where in the second line we have used (C3). If $L_{\sigma(\theta)} \in \Omega(n)$ then $q^{L_{\sigma(\theta)}+1} \in O(2^{-\beta(\theta)n})$ for some $\beta(\theta) \geq 1$. Thus, we can write
\[
\left| C_m(\theta) - \sum_{(\sigma(\theta), X, M, k) \in T} a_{X, M, k} \text{Tr} \left[ X \left( \frac{1}{2^{M_n}} \otimes |0\rangle \langle 0|^\otimes k \right) \right] \right| \in O(2^{-\beta n a_{\max}|T| M_{\max}})
\]
(C18)
as required, where we can denote $\beta = \min_{\theta} \beta(\theta)$ and the fixed point as $F$, noting that $F$ is indeed parameter independent. From here, we can inspect the three presented cases:

- If $a_{\max}|T| M_{\max} \in O(\text{poly}(n))$ then $C_m$ has exponentially small separation from $F$.
- There exists choice $\beta' \geq 1$ such that if $a_{\max}|T| \in \Omega(2^{\beta'|n})$ such that $C_m$ is not exponentially concentrated on $F$, however, $C_m$ now has an exponentially large statistical uncertainty, as measurement outcomes are multiplied by coefficients of order $a_{\max}|T|$.
- There exists choice of $\beta'' \geq 1$ such that $M_{\max} \in \Omega(2^{\beta''|n})$ and $C_m$ is not exponentially concentrated on $F$.

\[ \square \]

Appendix D: Protocol-specific results

1. Zero Noise Extrapolation

In this section we present our results for Zero Noise Extrapolation. As discussed in Section III B 2 of the main text, we will consider a Richardson extrapolation strategy based on Eq. (A1), an exponential extrapolation strategy based on Eq. (A5) and a NIBP extrapolation strategy based on Eq. (A8). As we deal with two types of noise parameters, throughout this section we will adopt the unifying notation
\[
\tilde{C}(\theta, a) = \begin{cases} 
\hat{C}(\theta, a) & \text{for Richardson/exponential extrapolation} \\
\hat{C}(\theta, q/a) & \text{for NIBP extrapolation,}
\end{cases}
\]
(D1)
for all $a \geq 1$. Thus, $\tilde{C}(\theta, a)$ denotes the noisy cost value at the boosted noise level, and $\tilde{C}(\theta, 1)$ denotes the noisy cost value at the base noise level.

Throughout this section, in order to estimate the sample cost of error mitigation we will make the key assumption that

$$\text{Var}[\tilde{C}(\theta, a)] \geq \text{Var}[\tilde{C}(\theta, 1)],$$

for all $\theta$ and $a \geq 1$ that is, the statistical fluctuations in measurement outcomes at the boosted noise level are no smaller than that at the base noise level. We note that similar assumptions are made in the literature for Zero Noise Extrapolation and Quasi-Probability Methods [43, 77]. Indeed, for noise models with a maximally mixed fixed point, we expect that high noise rates will tend to lead to larger variances. For example, in the simple scenario of a local Pauli measurement, the variance of measurement outcomes takes the form $$(1 - p_0) p_0^N,$$ where $p_0$ is the probability of obtaining a "0" outcome and $N$ is the number of shots. This variance is maximized for $p_0 = \frac{1}{2}$.

### a. Relative resolvability under global depolarizing noise

**Proposition 1** (Relative resolvability of Zero Noise Extrapolation with global depolarizing noise, 2 noise levels). Consider a circuit with $L$ instances of global depolarizing noise of the form (32). Consider a Richardson extrapolation strategy based on Eq. (A1), an exponential extrapolation strategy based on Eq. (A5) and a NIBP extrapolation strategy based on Eq. (A7). We presume access to an augmented noisy circuit where the error probability is exactly increased by factor $a_1 > 1$ as $p \to a_1 p$. Then we have

$$\chi_{\text{depol}} \leq \left( c - \frac{(1 - a_1 p)^L}{(1 - p)^L} \right)^2 c^2 + 1,$$

where $\chi_{\text{depol}}$ is the relative resolvability (see Definition 2) for global depolarizing noise, and where

$$c = \begin{cases} a_1, & \text{for Richardson extrapolation}, \\ a_1 \frac{\varepsilon^{t(\varepsilon)}}{\varepsilon^{t(\varepsilon)}}(a_1 \varepsilon), & \text{for exponential extrapolation}, \\ a_1^{-1}(L + 1) & \text{for NIBP extrapolation}. \end{cases}$$

Thus, $\chi_{\text{depol}} \leq 1$ for all of the above extrapolation strategies with access to 2 noise levels.

**Proof.** Upon inspecting Eqs. (A2), (A6) and (A8), one can verify that the Richardson, exponential and NIBP extrapolation strategies all take the form

$$C_m(\theta) = \frac{A \cdot \tilde{C}(\theta, 1) - B \cdot \tilde{C}(\theta, a)}{D} + E$$

where $A, B \geq 0$ (note that for NIBP extrapolation $E$ contains the state-independent cost value that represents the fixed point of the noise) and where we have adopted the notation defined in (D1). We note that under $L$ instances of global depolarizing noise (of the form (32)) with error probability $p$, noisy cost differences are given by

$$\Delta \tilde{C}(a) = (1 - ap)^L \Delta C,$$

for any pair of cost function points, where $\Delta C$ is the corresponding noise-free cost difference.

The error-mitigated cost function difference $\Delta C_m = C_m(\theta_1) - C_m(\theta_2)$ between two arbitrary points is given by

$$\Delta C_m = \frac{A \cdot \Delta \tilde{C}(1) - B \cdot \Delta \tilde{C}(a)}{D}$$

$$= \frac{A \cdot (1 - p)^L \Delta C - B \cdot (1 - ap)^L \Delta C}{D}.$$
In inspecting (D7), we see that the error mitigation cost can be bounded simply as
\[ \gamma = \frac{A^2 + B^2 \text{Var} \tilde{C}(\theta, a)}{\text{Var} [C(\theta, 1)]} \]

or
\[ \geq \frac{A^2 + B^2}{D^2} \]

where the inequality comes from our core assumption (D2). Inserting \( \gamma, \Delta C_m \) and \( \Delta \tilde{C}(1) \) into Definition 2, we have
\[
\chi_{\text{depol}} = \frac{1}{\gamma} \left( \frac{\Delta C_m}{\Delta \tilde{C}(1)} \right)^2 \leq \frac{(A(1 - p)^L - B(1 - ap)^L)^2}{(A^2 + B^2)(1 - p)^2L} \]
\[
= \frac{(c - (1 - ap)^L)^2}{c^2 + 1},
\]

where we have denoted \( c = A/B \). By inspecting the specific values of \( A \) and \( B \) for the Richardson, exponential and NIBP extrapolation strategies respectively, we obtain the results for each strategy.

\[ \Box \]

b. Average relative resolvability

**Proposition 2** (Average relative resolvability of Zero Noise Extrapolation, 2 noise levels). Consider a Richardson extrapolation strategy based on Eq. (A1), an exponential extrapolation strategy based on Eq. (A5) and a NIBP extrapolation strategy based on Eq. (A7). We presume perfect access to an augmented noisy circuit where the noise rate is increased by factor \( a_1 > 1 \). We denote \( \theta_{\epsilon_*} \) as the parameter corresponding to the global cost minimum at base noise parameter \( \epsilon \). Further denote \( \langle \Delta \tilde{C}(\theta, \epsilon, a_1 \epsilon) \rangle_i = z \). Any such noise model has an average relative resolvability
\[
\chi \leq \frac{(z - c)^2}{c^2 + 1},
\]

where

\[ c = \begin{cases} 
  a_1 & \text{for Richardson extrapolation,} \\
  a_1 r((c)^{(L+1)}) & \text{for exponential extrapolation,} \\
  a_1^{-(L+1)} & \text{for NIBP extrapolation.}
\end{cases} \]

Thus, under the assumption that \( z \leq 1 \) and \( \langle \Delta \tilde{C}(\theta, \epsilon, a_1 \epsilon) \rangle_i \geq 0 \), \( \chi \leq 1 \) for all of the above extrapolation strategies with access to 2 noise levels.

**Proof.** As in the previous proof, we can inspect Eqs. (A2), (A6) and (A8), and see that the Richardson, exponential and NIBP extrapolation strategies all take the form
\[
C_m(\theta) = \frac{A \cdot \tilde{C}(\theta, 1) - B \cdot \tilde{C}(\theta, a)}{D} + E
\]

where \( A, B, D \geq 0 \) (note that \( E \) contains the state-independent cost value that represents the fixed point of the noise) and we have adopted the notation of (D1). The average mitigated cost differences (averaged over accessible parameters \( \{\theta_i\} \)) can be written
\[
\langle \Delta C_m(\theta, \epsilon, \epsilon) \rangle_i = \frac{A \cdot \langle \Delta \tilde{C}(\theta, \epsilon, 1) \rangle_i - B \cdot \langle \Delta \tilde{C}(\theta, \epsilon, a) \rangle_i}{D}. \]

Thus, we have
\[
\frac{\langle \Delta C_m(\theta, \epsilon, \epsilon) \rangle_i}{\langle \Delta \tilde{C}(\theta, \epsilon, \epsilon) \rangle_i} = \frac{A - B \langle \Delta \tilde{C}(\theta, \epsilon, a) \rangle_i}{\langle \Delta \tilde{C}(\theta, \epsilon, 1) \rangle_i} \]
\[
= \frac{A - B z}{D}.
\]

(D17)
finally, by noting once again that the error mitigation cost is simply bounded as $\gamma \geq \frac{A^2 + B^2}{B^2}$ due to (D2), we have

$$\chi \leq \frac{(A - B)^2}{A^2 + B^2},$$

(D19)

where we can obtain the desired form by defining $c = A/B$. Finally, the specific values of $c$ for each extrapolation strategy can be read off by inspecting Eqs. (A2), (A6) and (A8). □

We now introduce a modification of the 2-design-averaged relative resolvability in Definition 4 that we will use to prove an additional result for Zero Noise Extrapolation.

**Definition 5 (Average relative resolvability III).** Consider a spectrum $\lambda \in \mathbb{R}_{>0}^2$ with unit $\ell_1$-norm, which corresponds to a noisy reference state. Then define the unitarily-averaged relative resolvability as

$$\widehat{\chi}_\lambda = \frac{1}{\gamma} \left\langle \left\langle \left( \tilde{C}_m(\rho, U, O) - \text{Tr}[O \lambda]/2^n \right)^2 \right\rangle_{U_i} \right\rangle_{\lambda},$$

(D20)

where $\langle \cdot \rangle_{U_i}$ denotes an average over $U_i$ drawn from a unitary 2-design, and where we denote

$$\tilde{C}(\rho, U_i, O) = \text{Tr}[U_i \rho U_i^\dagger O]$$

(D21)

$$\tilde{C}_m(\rho, U_i, O) = \text{Tr}[U_i \mathcal{M}(\rho) U_i^\dagger O]$$

(D22)

where $\mathcal{M} : S(H) \to B(H)$ is the map that describes the action of the error mitigation protocol.

Note that for Zero Noise Extrapolation we would generally expect to apply the map that describes the error mitigation after conjugation by the unitary $U_i$, counter to the definition presented above. Definition 5 presumes that augmenting the noise level and conjugating the noisy state by a unitary are two processes that approximately commute, and as such its use is not justified for generic cases. However, below we still present a result using this definition as a mathematical curiosity.

**Supplemental Proposition 1 (Average relative resolvability III with Zero Noise Extrapolation).** Consider a Richardson extrapolation strategy based on Eq. (A1), an exponential extrapolation strategy based on Eq. (A5) and a NIBP extrapolation strategy based on Eq. (A7). We presume perfect access to an augmented noisy circuit where the noise rate is increased by factor $a > 1$. Denote the output state at the base and augmented noise levels as $\rho(1)$ and $\rho(a)$ respectively. Then we have

$$\widehat{\chi}_\lambda \leq \frac{c^2 + \frac{P(a) - 1/2^m}{P(1) - 1/2^m}}{c^2 + 1},$$

(D23)

where $\widehat{\chi}_\lambda$ is the averaged relative resolvability defined in Definition 5, $P(1)$ is the purity of $\rho(1)$, $P(a)$ is the purity of $\rho(a)$, and

$$c = \begin{cases} 
\frac{a}{\text{ar}(\varepsilon) \left( \text{ar}(\varepsilon)^{1/(2\gamma)} \right)} & \text{for Richardson extrapolation} \\
\frac{\text{ar}(\varepsilon)^{1/(2\gamma)}}{a^{-L+1}} & \text{for exponential extrapolation} \\
a^{-L+1} & \text{for NIBP extrapolation.}
\end{cases}$$

(D24)

Thus, $\widehat{\chi}_\lambda \leq 1$ when $P(a) \leq P(1)$.

**Proof.** We denote reference states $\tilde{\rho}(\varepsilon)$ and $\tilde{\rho}(a\varepsilon)$ as states with purity $P(\varepsilon)$ and $P(a\varepsilon)$ respectively. Moreover, denote the noisy cost function values $\tilde{C}(U_i, \varepsilon) = \text{Tr}[U_i \tilde{\rho}(\varepsilon) U_i^\dagger O]$ and $\tilde{C}(U_i, a\varepsilon) = \text{Tr}[U_i \tilde{\rho}(a\varepsilon) U_i^\dagger O]$ and further denote $C_m(U_i)$ as the corresponding error mitigated estimator. We start again by noting that in all three Zero Noise Extrapolation strategies the estimator takes the form

$$C_m(U_i) = \frac{A \cdot \tilde{C}(U_i, 1) - B \cdot \tilde{C}(U_i, a)}{D} + E$$

(D25)

where $A, B \geq 0$ (see Eqs. (A2), (A6) and (A8)) and we have adopted the notation of (D1). We first evaluate the relevant expectation values which correspond to integrals over the Haar distribution over the unitary group of degree
We now proceed to derive the result for Richardson/exponential extrapolation, however, we note that the proof follows in a similar way for NIBP extrapolation with the simple substitution \( a \varepsilon \rightarrow q/a \). Utilizing Lemma 3, we have

\[
\langle \tilde{C}(U_i, \varepsilon) \rangle_{U_i} = \frac{1}{2^n} \text{Tr}[\tilde{\rho}(\varepsilon)] \text{Tr}[O] = \frac{1}{2^n} \text{Tr}[O],
\]

(D26)

\[
\langle (\Delta \tilde{C}(U_i, \varepsilon))^2 \rangle_{U_i} = \langle \langle \tilde{C}(U_i, \varepsilon) - \langle \tilde{C}(U_j, \varepsilon) \rangle_{U_j} \rangle_{U_i}^2 \rangle_{U_j} = \frac{\text{Tr}[O^2] - \frac{1}{2^n} \text{Tr}[O]^2}{2^{2n} - 1} \left[ \text{Tr}[\tilde{\rho}(\varepsilon)] \text{Tr}[\tilde{\rho}(\varepsilon)] - \frac{1}{2^n} \text{Tr}[\tilde{\rho}(\varepsilon)] \text{Tr}[\tilde{\rho}(\varepsilon)] \right]
\]

(D27)

\[
\langle (\Delta \tilde{C}(U_i, \varepsilon))(\Delta \tilde{C}(U_i, a \varepsilon)) \rangle_{U_i} = \frac{\text{Tr}[O^2] - \frac{1}{2^n} \text{Tr}[O]^2}{2^{2n} - 1} \left[ \text{Tr}[\tilde{\rho}(\varepsilon)] \text{Tr}[\tilde{\rho}(a \varepsilon)] - \frac{1}{2^n} \text{Tr}[\tilde{\rho}(\varepsilon)] \text{Tr}[\tilde{\rho}(a \varepsilon)] \right]
\]

(D28)

where the inequality comes by observing that \( \text{Tr}[\tilde{\rho}(\varepsilon)] = \text{Tr}[\tilde{\rho}(a \varepsilon)] = 1 \) and further applying Cauchy-Schwarz to \( \text{Tr}[\tilde{\rho}(\varepsilon)] \text{Tr}[\tilde{\rho}(a \varepsilon)] \) and noting that the purity of an \( n \) qubit state is lower bounded by \( 1/2^n \). Inspecting Eq. (D25) we have

\[
\langle C_m(U_i) \rangle_{U_i} = \frac{1}{2^n} \frac{A - B}{D} \text{Tr}[O],
\]

(D31)

\[
\langle (C_m(U_i) - C_m(U_j)) \rangle_{U_i}^2 = \left( \frac{A \cdot \tilde{C}(U_i, \varepsilon) - B \cdot \tilde{C}(U_i, a \varepsilon)}{D} + E - \left( \frac{1}{2^n} \frac{A - B}{D} \text{Tr}[O] + E \right) \right)^2
\]

(D32)

\[
= \left( \frac{A \cdot \Delta \tilde{C}(U_i, \varepsilon) - B \cdot \Delta \tilde{C}(U_i, a \varepsilon)}{D} \right)^2
\]

(D33)

\[
= \frac{A^2 \langle (\Delta \tilde{C}(U_i, \varepsilon))^2 \rangle_{U_i} + B^2 \langle (\Delta \tilde{C}(U_i, a \varepsilon))^2 \rangle_{U_i} - 2AB \langle \Delta \tilde{C}(U_i, \varepsilon) \Delta \tilde{C}(U_i, a \varepsilon) \rangle_{U_i}}{D^2}
\]

(D34)

\[
\leq \frac{\text{Tr}[O^2] - \frac{1}{2^n} \text{Tr}[O]^2}{D^2(2^{2n} - 1)} \left( A^2 \left( P(\varepsilon) - \frac{1}{2^n} \right) + B^2 \left( P(a \varepsilon) - \frac{1}{2^n} \right) \right)
\]

(D35)

The inequality comes by substituting in the expressions for \( \langle (\Delta \tilde{C}(U_i, \varepsilon))^2 \rangle_{U_i} \) and \( \langle (\Delta \tilde{C}(U_i, a \varepsilon))^2 \rangle_{U_i} \) obtained in (D28), and dropping the third term in the numerator, where we have used Eq. (D30). Finally, we note that (D15) gives \( \gamma^{-1} = \frac{B^2}{A^2 + B^2} \). Substituting the obtained expressions for \( \gamma^{-1} \), (D35) and (D28) into Definition 5 we obtain

\[
\chi_{\lambda} \leq \frac{A^2 + B^2 \frac{P(a \varepsilon - 1/2^n)}{P(\varepsilon - 1/2^n)}}{A^2 + B^2},
\]

(D36)

where we can define \( c = A/B \) to obtain the desired result. Further, the explicit form of \( c \) for Richardson, exponential, and NIBP extrapolation can be respectively found by inspecting Eqs. (A2), (A6) and (A8).

\( \square \)

\[ c. \text{ Richardson extrapolation with 3 noise levels} \]

In this section we focus on Richardson extrapolation (see Appendix A 1 a for review) and investigate the change in resolvability under an extrapolation strategy that utilizes 3 distinct noise levels.

**Supplemental Proposition 2** (Relative resolvability of Richardson extrapolation with global depolarizing noise, 3 noise levels). Consider \( L \) instances of global depolarizing noise of the form (32) acting through a circuit. Consider a Richardson extrapolation strategy based on Eq. (A1), an exponential extrapolation strategy based on Eq. (A5) and a NIBP extrapolation strategy based on Eq. (A7) in the appendix. We presume access to two augmented noisy circuits where the error probability is perfectly increased by factors \( a_2 > a_1 > 1 \) as \( p \rightarrow a_1 p \) and \( p \rightarrow a_2 p \) respectively. Then for all three extrapolation strategies and any such choices of \( a_2 \) and \( a_1 \), we have

\[
\chi_{\text{depol}} \leq 1,
\]

(D37)

where \( \chi_{\text{depol}} \) is the relative resolvability (see Definition 2) for global depolarizing noise.
Proof. We start by noting that under \( L \) instances of global depolarizing noise with error probability \( p \) (of the form (32)), noisy cost differences are given by

\[
\Delta \tilde{C}(a) = (1 - ap)^L \Delta C,
\]

for any noise augmentation factor \( a \) and any pair of cost function points, where \( \Delta C \) is the corresponding noise-free cost difference.

The error-mitigated cost function difference \( \Delta C_m(\theta_{1,2}) = C_m(\theta_1) - C_m(\theta_2) \) between two arbitrary points constructed under Richardson extrapolation with 3 noise levels is given by

\[
\Delta C_m = \frac{a_1a_2(a_2 - a_1)\Delta \tilde{C}(p) - a_2(a_2 - 1)\Delta \tilde{C}(a_1p) + a_1(a_1 - 1)\Delta \tilde{C}(a_2p)}{(a_1 - 1)(a_2 - 1)(a_2 - a_1)}
\]

(proof).

\[
\Delta C_m = \frac{a_1a_2(a_2 - a_1)(1-p)^L \Delta C - a_2(a_2 - 1)(1 - a_1p)^L \Delta C + a_1(a_1 - 1)(1 - a_2p)^L \Delta C}{(a_1 - 1)(a_2 - 1)(a_2 - a_1)},
\]

where in order to obtain the first equality we have used \( \langle \rangle \). The second equality comes by substituting in \( \langle D_{38} \rangle \). Inspecting \( (D_{7}) \), we see that the error mitigation cost can be bounded simply as

\[
\gamma = \frac{a_1^2a_2^2(a_2 - a_1)^2 + a_2^2(a_2 - 1)^2 \frac{\text{Var}[\tilde{C}(\theta, a_1p)]}{\text{Var}[\tilde{C}(\theta, p)]} + a_1^2(a_1 - 1)^2 \frac{\text{Var}[\tilde{C}(\theta, a_2p)]}{\text{Var}[\tilde{C}(\theta, p)]}}{(a_1 - 1)^2(a_2 - 1)^2(a_2 - a_1)^2}
\]

for any \( \theta \), where the inequality comes from our core assumption \( (D_2) \). Inserting our expressions for \( \gamma \), \( \Delta C_m \) and \( \Delta \tilde{C}(1) \) into Definition 2, we have

\[
\chi_{\text{depol}} = \frac{1}{\gamma} \left( \frac{\Delta C_m}{\Delta \tilde{C}(1)} \right)^2 \leq \frac{(a_1a_2(a_2 - a_1)(1-p)^L - a_2(a_2 - 1)(1 - a_2p)^L + a_1(a_1 - 1)(1 - a_2p)^L)^2}{(a_1^2a_2^2(a_2 - a_1)^2 + a_2^2(a_2 - 1)^2 + a_1^2(a_1 - 1)^2)(1-p)^2L}
\]

\[
= \frac{a_1a_2(a_2 - a_1) - a_2(a_2 - 1)(1 - a_2p)^L + a_1(a_1 - 1)(1 - a_2p)^L}{a_1^2a_2^2(a_2 - a_1)^2 + a_2^2(a_2 - 1)^2 + a_1^2(a_1 - 1)^2}.
\]

The desired result can be observed by noting that \( a_2(a_2 - 1) > a_1(a_1 - 1) \) and that \( \frac{(1-a_1p)^L}{(1-p)^L} > \frac{(1-a_2p)^L}{(1-p)^L} \).

\[\square\]

Supplemental Proposition 3 (Average resolvability of Richardson extrapolation, 3 noise levels). Consider a Richardson extrapolation strategy based on Eq. (A1), an exponential extrapolation strategy based on Eq. (A5) and a NIBP extrapolation strategy based on Eq. (A7) in the appendix. We presume perfect access to two augmented noisy circuits where the noise rate is increased by factors \( a_2 > a_1 > 1 \). We denote \( \theta_{z*} \) as the parameter corresponding to the global cost minimum at base noise parameter \( \varepsilon \). Further denote \( \langle \Delta \tilde{C}(\theta_{z*, \varepsilon}) \rangle_i = z_1 \) and \( \langle \Delta \tilde{C}(\theta_{z*, \varepsilon}) \rangle_i = z_2 \). Any such noise model has an average relative resolvability

\[
\bar{X} \leq \frac{(a_1a_2(a_2 - a_1) - a_2(a_2 - 1)z_1 + a_1(a_1 - 1)z_2)^2}{a_1^2a_2^2(a_2 - a_1)^2 + a_2^2(a_2 - 1)^2 + a_1^2(a_1 - 1)^2},
\]

where \( \bar{X} \) is the averaged relative resolvability (see Definition 2). Thus, under the assumption that \( z_2 \leq z_1 \leq 1 \) (on average the cost concentrates with increasing noise level) and \( \langle \Delta \tilde{C}(\theta_{z*, \varepsilon}) \rangle_i \geq 0 \) (boosting the noise level does not shift the cost value of the global minimum above the average cost value), then \( \bar{X} \leq 1 \).

Proof. The averaged error-mitigated cost function difference \( \langle \Delta C_m(\theta_{z*, \varepsilon}) \rangle_i = \langle C_m(\theta_1) - C_m(\theta_2) \rangle_i \) between two arbitrary points constructed under Richardson extrapolation with 3 noise levels is given by

\begin{align*}
\langle \Delta C_m(\theta_{z*, \varepsilon}) \rangle_i &= \left( \langle a_1a_2(a_2 - a_1)\Delta \tilde{C}(\theta_{z*, \varepsilon}, p) - a_2(a_2 - 1)\Delta \tilde{C}(\theta_{z*, \varepsilon}, a_1p) + a_1(a_1 - 1)\Delta \tilde{C}(\theta_{z*, \varepsilon}, a_2p) \rangle \right)_{z*, \varepsilon} \\
&= \frac{a_1a_2(a_2 - a_1)(\Delta \tilde{C}(\theta_{z*, \varepsilon}, p))_i - a_2(a_2 - 1)(\Delta \tilde{C}(\theta_{z*, \varepsilon}, a_1p))_i + a_1(a_1 - 1)(\Delta \tilde{C}(\theta_{z*, \varepsilon}, a_2p))_i}{(a_1 - 1)(a_2 - 1)(a_2 - a_1)}.
\end{align*}
As in the previous proof, we can inspect (D7) and we see that the error mitigation cost can be bounded simply as

$$\gamma = \frac{a_1^2 a_2^2 (a_2 - a_1)^2 + a_2^2 (a_2 - 1)^2}{(a_1 - 1)^2 (a_2 - 1)^2} \frac{\text{Var}[\tilde{C}(\theta_i, a_1, p)]}{\text{Var}[\tilde{C}(\theta, p)]} + a_1^2 (a_1 - 1)^2 \frac{\text{Var}[\tilde{C}(\theta_i, a_2, p)]}{\text{Var}[\tilde{C}(\theta, p)]}$$

(D48)

for any $\theta$, where the inequality comes from our core assumption (D2). Inserting our expressions for $\gamma$ and $\Delta C_m$ into Definition 2, we have

$$\chi = \frac{1}{\gamma} \left( \frac{\Delta C_m}{\Delta C(1)} \right)^2 \leq \left( \frac{a_1 a_2 (a_2 - a_1) - a_2 (a_2 - 1)^2}{a_1^2 a_2^2 (a_2 - 1)^2} + a_1 (a_1 - 1)^2 \right) \frac{(\Delta \tilde{C}(\theta_i, a_1, e))_i}{(\Delta \tilde{C}(\theta_i, a_2, e))_i}$$

(D50)

and the desired result comes by denoting $\frac{(\Delta \tilde{C}(\theta_i, a_1, e))_i}{(\Delta \tilde{C}(\theta_i, a_2, e))_i} = z_1$ and $\frac{(\Delta \tilde{C}(\theta_i, a_2, e))_i}{(\Delta \tilde{C}(\theta_i, a_2, e))_i} = z_2$.

As with the results of Proposition 2 we see that $\chi$ decreases with increasing cost concentration.

2. Virtual Distillation

a. Bounds on error mitigation cost

We recall the two error mitigation protocols of Ref. [17], denoted "A" and "B" respectively, to prepare

$$C_m^{(A)}(\theta_i) = \text{Tr}[\tilde{p}_i^A O]/\text{Tr}[\tilde{p}_i^M],$$

(D51)

and

$$C_m^{(B)}(\theta_i) = \text{Tr}[\tilde{p}_i^M O]/\lambda_i^M,$$

(D52)

where $\lambda_i$ is the dominant eigenvalue of $\tilde{p}_i \equiv \tilde{p}(\theta_i)$. The protocols considered explicitly construct these quantities as

$$\text{Tr}[\tilde{p}_i O] = 2\text{prob}_{1,i} - 1,$$

(D53)

$$\text{Tr}[\tilde{p}_i^A O] = 2\text{prob}_{1,i} - 1,$$

(D54)

$$\text{Tr}[\tilde{p}_i^M] = 2\text{prob}_{M,i} - 1,$$

(D55)

where $\text{prob}_{1,i}$, $\text{prob}_{M,i}$, and $\text{prob}_{1,i}'$ are expectation values of a Pauli-Z measurement on a qubit ancillary subsystem.

In order to obtain our results we will herein make the core assumptions

$$\text{Var}[\text{prob}_{1,i}] = \text{Var}[\text{prob}_{1,j}], \quad \forall i \neq j,$$

(D56)

$$\text{Var}[\text{prob}_{M,i}] = \text{Var}[\text{prob}_{M,j}], \quad \forall i \neq j,$$

(D57)

$$\text{Var}[\text{prob}_{M,i}] \geq \text{Var}[\text{prob}_{M,j}], \quad \forall i, M \geq 2,$$

(D58)

that is, the statistical uncertainty of the measurement outcomes of the circuit that prepares $\tilde{p}_i$ and $\tilde{p}_i^M$ can be approximated to be state independent, and the statistical uncertainty of the measurement outcomes of the circuit that prepares $\tilde{p}_i^M$ are at best equal to that of $\tilde{p}_i$. In the case of large $M$ we expect $\text{Var}[\text{prob}_{M,i}]$ to be large, as for any (non-pure) $\tilde{p}$, the quantity $\text{Tr}[- \tilde{p}^M O]$ is close to zero for large $M$. This corresponds to $\text{prob}_{M,i} = \frac{1}{2}$, which maximizes the variance for a binomial distribution.

**Lemma 4** (Bounds on error mitigation cost of virtual distillation). Denote the error mitigation cost (see Definition 1) corresponding to (D51) and (D52) as $\gamma^{(A)}$ and $\gamma^{(B)}$ respectively. We have

$$\gamma^{(A)} \geq \frac{1}{(\text{Tr}[\tilde{p}_i^M])^2}, \quad \gamma^{(B)} = \frac{1}{\lambda_i M^2}.$$

(D59)
Proof. For $\gamma^{(A)}$ and $\gamma^{(B)}$ we need to compute the variances of the estimators of $C_m^{(A)}$, $C_m^{(B)}$ respectively and likewise $\tilde{C} = \text{Tr}[\rho O]$. We have

\[
\text{Var}[\tilde{C}] = \text{Var}[\text{Tr}[\rho O]] = \text{Var}[2\text{prob}_1 - 1],
\]

\[= 4\text{Var}[\text{prob}_1],
\]

\[
\text{Var}[C_m^{(B)}] = \text{Var}\left[\frac{\text{Tr}[\rho^M O]}{\chi^M}\right] = \frac{1}{\chi^M} \text{Var}[2\text{prob}_M - 1],
\]

\[= \frac{4}{\chi^M} \text{Var}[\text{prob}_M],
\]

\[
\text{Var}[C_m^{(A)}] = \text{Var}\left[\frac{\text{Tr}[\rho^M O]}{\text{Tr}[\rho^M]}\right] = 4\text{Var}[\text{prob}_M] \left(\mathbb{E}\left[\frac{1}{2\text{prob}_M - 1}\right]\right)^2 + 4\text{Tr}[\rho^M O]^2 \text{Var}\left[\frac{1}{2\text{prob}_M - 1}\right]
\]

\[+ 4\text{Var}[\text{prob}_M] \text{Var}\left[\frac{1}{2\text{prob}_M - 1}\right],
\]

\[\geq 4\text{Var}[\text{prob}_M] \left(\mathbb{E}\left[\frac{1}{2\text{prob}_M - 1}\right]\right)^2
\]

\[\geq 4\text{Var}[\text{prob}_M] \frac{1}{(\mathbb{E}[2\text{prob}_M - 1])^2}
\]

\[= 4\text{Var}[\text{prob}_M] \frac{1}{(\text{Tr}[\rho^M])^2}.
\]

Equation (D64) comes from the standard formula for the variance of the product of two independent random variables. To obtain the first inequality we simply drop the second and third terms, which are positive. The second inequality is an application of Jensen’s inequality. Recalling the definition of error mitigation cost (Definition 1), the above three equations enable us to write

\[
\gamma^{(A)} \geq \frac{1}{(\text{Tr}[\rho^M])^2}, \quad \gamma^{(B)} = \frac{1}{\chi^M},
\]

where we have used our core assumption that $\text{Var}[\text{prob}_1] \leq \text{Var}[\text{prob}_M]$. 

\[\Box
\]

\subsection*{b. Relative resolvability for global depolarizing noise}

Here we present a proof of Proposition 3, in which we upper bound the relative resolvability for Virtual Distillation, for any two cost function points under global depolarizing noise.

**Proposition 3** (Relative resolvability of Virtual Distillation with global depolarizing noise). Consider $l$ instances of global depolarizing noise $\mathcal{D}$ of the form

\[
\rho \xrightarrow{\mathcal{D}} \tilde{\rho} = q^l \rho + (1 - q^l) \frac{\mathbb{I}}{2^n}
\]

acting on some pure state $\rho$ with some noise parameter $q \in [0, 1]$. We consider the two error mitigation protocols of Ref. [47] (denoted "A" and "B") to respectively prepare (D51) and (D52). The relative resolvability of any pair of arbitrary cost function points satisfies

\[
\chi^{(A)} \leq \chi^{(B)} = \Gamma(n, M, q^l)
\]

for all $n \geq 1$, $M \geq 2$, $q^l \in [0, 1]$, and where

\[
\Gamma(n, M, q^l) \leq 1,
\]

is a monotonically decreasing function in $M$ (with asymptotically exponential decay) in the quadrant $n \geq 1$, $M \geq 2$. The bound is saturated as $\Gamma(1, 2, p) = 1$ for all $p$. 

Proof. In this proof we consider arbitrary cost function differences, that is, given two arbitrarily chosen points in parameter space \( \theta_1 \) and \( \theta_2 \), we consider

\[
\Delta C = C(\theta_1) - C(\theta_2),
\]

and the respective differences for the noisy cost \( \tilde{C}(\theta) \) and the mitigated costs \( C_m^{(A)}(\theta), C_m^{(B)}(\theta) \). In order to evaluate \( \chi_A \) and \( \chi_B \) we need to first evaluate the following quantities:

\[
\Delta \tilde{C}, \Delta C_m^{(A)}, \Delta C_m^{(B)}, \gamma^{(A)}, \gamma^{(B)}
\]

that is, the noisy cost function difference between two points, the difference between the virtual distillation estimators for the same points for both protocols, and the error mitigation rate for both protocols. The noisy cost function difference under global depolarizing noise is simply related the noiseless difference as

\[
\Delta \tilde{C} = q' \Delta C.
\]

To evaluate the other quantities we note that

\[
\tilde{\rho} = \left[ q' + \frac{1}{2n}(1 - q') \right] \rho + \left[ \frac{1}{2n}(1 - q') \right] (I - \rho),
\]

\[
\tilde{\rho}^M = \left[ \left[ q' + \frac{1}{2n}(1 - q') \right]^M - \left[ \frac{1}{2n}(1 - q') \right]^M \right] \rho + \frac{2n}{2nM} (1 - q')^M \frac{1}{2n} (I - \rho),
\]

\[
\text{Tr}[\tilde{\rho}^M] = \left[ \left[ q' + \frac{1}{2n}(1 - q') \right]^M - \left[ \frac{1}{2n}(1 - q') \right]^M \right] \text{Tr}[\rho] + \frac{1}{2nM} (1 - q')^M \text{Tr}[O].
\]

In particular, we highlight that the expression for \( \text{Tr}[\tilde{\rho}^M] \). As the dominant noisy eigenvalue \( \lambda \) and \( \text{Tr}[\tilde{\rho}^M] \) are state independent we have

\[
\Delta C_m^{(A)} = \frac{1}{\text{Tr}[\tilde{\rho}^M]} \left[ \left[ q' + \frac{1}{2n}(1 - q') \right]^M - \left[ \frac{1}{2n}(1 - q') \right]^M \right] \Delta C,
\]

\[
\Delta C_m^{(B)} = \frac{1}{\lambda M} \left[ \left[ q' + \frac{1}{2n}(1 - q') \right]^M - \left[ \frac{1}{2n}(1 - q') \right]^M \right] \Delta C,
\]

where the choice of \( \tilde{\rho} \) is arbitrary.

Now, using Definition 2 and combining (D79), (D80), (D74) along with the result of Lemma 4, we have

\[
\chi^{(A)} \leq \chi^{(B)} = \Gamma(n, M, q'),
\]

where we define the function

\[
\Gamma(n, M, q') = \frac{1}{q'^2} \left[ \left[ q' + \frac{1}{2n}(1 - q') \right]^M - \left[ \frac{1}{2n}(1 - q') \right]^M \right]^2.
\]

First, note that for \( M = 2 \)

\[
\Gamma(q', n, 2) = \frac{1}{q'^2} \left[ \left[ q' + \frac{1}{2n}(1 - q') \right]^2 - \left[ \frac{1}{2n}(1 - q') \right]^2 \right]^2
\]

\[
= \left( q' + \frac{2}{2n}(1 - q') \right)^2.
\]

For \( n = 1 \), \( \Gamma(n, 2, q') = 1 \). For all \( n > 1 \), \( \Gamma(n, 2, q') < 1 \) as \( (1 - q') > 0 \). Thus,

\[
\Gamma(n, 2, q') \leq 1 \quad \forall n \geq 1.
\]
We complete the proof by showing that \( \Gamma(n, M, q^i) \) monotonically decreases with \( M \) for all \( n \geq 1, M \geq 2 \). This can be seen by inspecting the partial derivative (making the decomposition \( \Gamma = (\Gamma^{1/2})^2 \) due to the square in (D82))

\[
\frac{\partial \Gamma}{\partial M} = 2\Gamma^{1/2}\frac{\partial \Gamma^{1/2}}{\partial M}.
\] (D86)

We investigate when this quantity is negative. As \( \Gamma^{1/2} \) is always positive, negativity is determined by the sign of \( \frac{\partial \Gamma^{1/2}}{\partial M} \). Denoting \( \delta = \frac{1}{2^n}(1-q^i) \), we have

\[
\frac{\partial \Gamma^{1/2}}{\partial M} = \frac{1}{q^i} \left[ (q^i + \delta)^M \ln(q^i + \delta) - \delta^M \ln \delta \right]
\] (D87)

\[
= \frac{1}{q^i} \left[ \delta^M \left( \ln(q^i + \delta) - \ln \delta \right) + \ln(q^i + \delta) \left( (q^i + \delta)^M - \delta^M \right) \right]
\] (D88)

\[
\leq \frac{1}{q^i} \left[ q^i \delta^{M-1} + (q^i + \delta - 1) \left( (q^i + \delta)^M - \delta^M \right) \right]
\] (D89)

\[
= \frac{1}{q^i} \left[ q^i \delta^{M-1} - (2^n - 1)\delta \left( (q^i + \delta)^M - \delta^M \right) \right]
\] (D90)

\[
\leq \frac{1}{q^i} \left[ q^i \delta^{M-1} - (2^n - 1)\delta \left( Mq^i \delta^{M-1} + \frac{1}{2} M(M-1)q^{2M-2} \right) \right]
\] (D91)

\[
= \frac{1}{q^i} \left[ q^i \delta^{M-1} - (2^n - 1)\delta \left( \frac{1}{2^n} M(1-q^i)q^i \delta^{M-2} + \frac{1}{2} M(M-1)q^{2M-2} \right) \right]
\] (D92)

\[
= \delta^{M-1} \left[ 1 - \left( 1 - \frac{1}{2^n} \right) M - \frac{1}{2} (2^n - 1)M \left( M - 1 - \frac{2}{2^n} \right) q^i \right]
\] (D93)

\[
\leq \delta^{M-1} \left[ 1 - \frac{1}{2} M - \frac{1}{2} (M - 2) q^i \right] \quad \forall \ n \geq 1,
\] (D94)

where in order to obtain the first inequality we use the inequalities \( \ln(q^i + \delta) - \ln \delta \leq q^i/\delta \) and \( \ln(q^i + \delta) \leq q^i + \delta - 1 \). The second inequality comes from observing that the expansion of \( ((q^i + \delta)^M - \delta^M) \) is a sum of positive terms, and considering only two such terms. The above implies that

\[
\frac{\partial \Gamma}{\partial M} \leq 0 \quad \forall \ n \geq 1, M \geq 2,
\] (D95)

that is, \( \Gamma \) is monotonically decreasing with \( M \) in the quadrant \( n \geq 1, M \geq 2 \). Combined with (D85), we have the proof as required.

\( \square \)

\textit{c. Average relative resolvability}

Here we present a proof of Proposition 4, in which we upper bound the 2-design-averaged relative resolvability for Virtual Distillation.

\textbf{Proposition 4 (Average relative resolvability of Virtual Distillation).} Consider an error mitigation protocol that prepares estimator \( C_m(\theta_i) = \text{Tr}[\tilde{\rho}_i^m O]/\text{Tr}[\tilde{\rho}_i^m] \) from some noisy parameterized quantum state \( \tilde{\rho}_i \equiv \tilde{\rho}(\theta_i) \). Consider the average relative resolvability for noisy states of some spectrum \( \lambda \) with purity \( P_\lambda \) as defined in Definition 3. We have

\[
\bar{\chi}_\lambda \leq G(n, M, P) \leq 1,
\] (D96)

where \( G(n, M, P) \) is a monotonically decreasing function in \( M \) (with asymptotically exponential decay) for all \( n \geq 1, M \geq 2 \). Within this region the bound is saturated as \( G(1, 2, P) = 1 \) for all \( P \) and \( G(n, M, 1) = 1 \) for all \( n \geq 1, M \geq 2 \). Explicitly, we have for \( n = 1 \)

\[
G(P, n = 1, M) = \frac{1}{2^M} \left[ (1 + \sqrt{2P-1})^M - (1 - \sqrt{2P-1})^M \right]^2/2P - 1.
\] (D97)
For $n \geq 2$ and $M = 2$
\[
G(n \geq 2, M = 2, P) = \min \left( \frac{P^2}{P - \frac{1}{2\pi}} \left( 1 - \frac{1}{2n} \right), \frac{4}{2n} + \frac{4}{2n/2}g_2 \sqrt{P - \frac{1}{2n} + 2n g_2 \left( P - \frac{1}{2} \right)} \right),
\]
where we denote $g_k = \left( \frac{2^n - 1}{2^n} \right)^k + \left( \frac{1}{2} \right)^k$. Further, for $n \geq 2$ and $M \geq 3$ we have
\[
G(n \geq 2, M \geq 3, P) = \min \left( \frac{P^M}{P - \frac{1}{2\pi}} \left( 1 - \frac{1}{2n} \right), \frac{2^n}{4} \left( \left( \frac{2}{P - \frac{1}{2\pi}} + \frac{1}{2\pi} \right)^M - \left( \frac{1}{2\pi} \right)^M \right)^2 \right).
\]

Proof. From Definition 4 we have
\[
\bar{\lambda} = \frac{1}{\gamma(\lambda)} \frac{\langle (C_m(\rho_\lambda, U_i) - \text{Tr}[O]/2^n)^2 \rangle_{U_i}}{\langle (\tilde{C}(\rho_\lambda, U_i) - \text{Tr}[O]/2^n)^2 \rangle_{U_i}}.
\]
Let us first evaluate the required averages over unitary 2-designs. The relevant first moments for virtual distillation are given by
\[
\langle \text{Tr}[U\rho_\lambda U^\dagger O] \rangle_U = \text{Tr}[O]/2^n, \quad \langle \text{Tr}[U\rho_\lambda^M U^\dagger O] \rangle_U = \text{Tr}[\rho_\lambda^M]\text{Tr}[O]/2^n,
\]
where we have used Lemma 3. Thus we can see that the numerator and denominator of (D100) correspond to variances which we now evaluate. Again, utilizing Lemma 3, the second moments are given by
\[
\langle (C_m(U_i) - (C_m(U_i))^2 \rangle_{U_i} = \langle (\text{Tr}[U\rho_\lambda U^\dagger O] - \text{Tr}[O]/2^n)^2 \right)_{U_i}
\]
\[
= \frac{(\text{Tr}[O^2] - \frac{1}{2\pi} \text{Tr}[\rho_\lambda^2]) (\text{Tr}[\rho_\lambda^2] - \frac{1}{2\pi} \text{Tr}[\rho_\lambda^2])}{2^{2n} - 1},
\]
\[
\langle (C_m(U_i) - (C_m(U_i))^2 \rangle_{U_i} = \langle \left( \frac{\text{Tr}[U\rho_\lambda^M U^\dagger O]}{\text{Tr}[\rho_\lambda^M]} \right)^2 \rangle_{U_i} - (\text{Tr}[O]/2^n)^2 \rangle_{U_i}
\]
\[
= \frac{(\text{Tr}[O^2] - \frac{1}{2\pi} \text{Tr}[\rho_\lambda^2]) (\frac{\text{Tr}[\rho_\lambda^M]}{\text{Tr}[\rho_\lambda^M]^2} - \frac{1}{\pi})}{2^{2n} - 1},
\]
where in the final equality we have used the fact that $\text{Tr} \left[ \frac{\rho_\lambda^M}{\text{Tr}[\rho_\lambda^M]} \right] = 1$. Using the definition of the averaged relative resolvability II (Definition 4), we can arrive at a bound written explicitly in terms of $\rho_\lambda$ as
\[
\bar{\lambda} = \frac{1}{\gamma} \frac{\langle (C_m(U_i) - (C_m(U_i))^2 \rangle_{U_i}}{\langle (\tilde{C}(U_i) - (\tilde{C}(U_i))^2 \rangle_{U_i}} \leq \frac{\text{Tr}[\rho_\lambda^2] - \frac{1}{\pi} \text{Tr}[\rho_\lambda^2]^2}{\text{Tr}[\rho_\lambda^2] - \frac{1}{\pi} \text{Tr}[\rho_\lambda^2]^2},
\]
where we have used the fact that the error mitigation cost $\gamma \geq 1/(\text{Tr}[\rho_\lambda^M])^2$.

The goal is to now investigate whether or not $f(M) = \text{Tr}[\rho_\lambda^M] - \frac{1}{\pi} \text{Tr}[\rho_\lambda^M]^2$ is monotonically decreasing for $M \in \mathbb{N}_+$. This quantity has two interpretations. First, it can be seen to be a Hilbert Schmidt distance between $\rho_\lambda^M$ and $\text{Tr}[\rho_\lambda^M]\frac{I}{\pi}$. Second, by considering the eigenvalue decomposition of $\rho$, it can be seen to be proportional to the population variance of the distribution $\{\lambda_i^M\}$, where $\lambda_i$ are the eigenvalues of $\rho_\lambda$, that is,
\[
f(M) = 2^n \text{Var}[\lambda_i^M] = \sum_i \lambda_i^2 - \frac{1}{2n} \left( \sum_i \lambda_i^M \right)^2,
\]
where here $\text{Var}(\cdot)$ denotes the population variance of the contained vector. Thus, we can rewrite Eq. (D107) as
\[
\bar{\lambda}(M) \leq \frac{f(M)}{f(1)} = \frac{\text{Var}[\lambda_i^M]}{\text{Var}[\lambda_i^M]}.
\]
Let us first treat the qubit setting of \( n = 1 \). Consider eigenvalue decomposition \( \rho_\lambda = \lambda |\psi\rangle\langle \psi| + (1 - \lambda) |\psi_\perp\rangle\langle \psi_\perp| \), where we have defined \( \lambda_1 = 1 - \lambda \), \( \lambda_2 = \lambda \) and without loss of generality we fix \( 1 - \lambda \geq \lambda \). We define \( G(1, M, P) = f(M)/f(1) \) and will determine \( f(M) \) exactly for single-qubit states. For generic \( M \) we have

\[
f(M) = \lambda^{2M} + (1 - \lambda)^{2M} - \frac{1}{2} (\lambda^M + (1 - \lambda)^M)^2 \quad \text{(D110)}
\]

\[
= \frac{1}{2} ((1 - \lambda)^M - \lambda^M)^2 \quad \text{(D111)}
\]

\[
= \frac{1}{2^{2M+1}} \left[ (1 + \sqrt{2P - 1})^M - (1 - \sqrt{2P - 1})^M \right]^2, \quad \text{(D112)}
\]

where in the final equality we have used the fact that for single-qubit states \( \lambda = \frac{1}{2}(1 - \sqrt{2P - 1}) \). Further, using \( f(1) = P - \frac{1}{2} \) we have the bound as required.

Now let us consider the setting of \( n \geq 2 \). We will construct two bounds, for the respective high purity and low purity limits. We start with the bound for high purity states. We can write the right hand side of Eq. (D109) explicitly as

\[
\text{Var}[\lambda^{(M)}] = \frac{1}{2^n} \sum_i \lambda_i^{2M} - \left( \frac{1}{2^n} \sum_i \lambda_i^M \right)^2
\]

\[
= \frac{1}{2^n} \sum_i \lambda_i^{2M} - \frac{1}{2^n} \sum_i \lambda_i^M - \frac{1}{2^n} \sum_{i \neq j} \lambda_i^M \lambda_j^M
\]

\[
\leq \frac{(2^n - 1)(\sum_i \lambda_i^{2M})}{2^n \sum_i \lambda_i^M - 1}
\]

\[
\leq \frac{(2^n - 1)(\sum_i \lambda_i^M)^2}{2^n \sum_i \lambda_i^M - 1}
\]

\[
= \frac{P^M}{P - \frac{1}{2^n}} \left( 1 - \frac{1}{2^n} \right),
\]

where in order to obtain the first inequality we have dropped the cross terms \( \frac{1}{2^n} \sum_{i \neq j} \lambda_i^M \lambda_j^M \), and in the second inequality we have introduced new cross terms. The final equality comes by substituting in the definition of the purity \( P \). We note this first bound is upper-bounded by \( 1 \) for all \( P \geq \frac{1}{2^n - 1} \).

We can now construct our second bound for strongly mixed states. We will consider bounds on \( \text{Var}[X^M] \) where a random variable \( X \) when it is known that it takes values close to its mean \( \mu \). We consider the decomposition

\[
X^M = ((X - \mu) - \mu)^M
\]

\[
= \mu^M + \sum_{k=1}^M Y_k
\]

where we have defined the random variables \( Y_k = \binom{M}{k} \mu^{M-k} (X - \mu)^k \). Further, we can write

\[
\text{Var}[X^M] = \text{Var} \left[ \sum_{k=1}^M Y_k \right]
\]

\[
= \mathbb{E} \left[ \left( \sum_k Y_k - \mathbb{E} \left[ \sum_k Y_k \right] \right) \left( \sum_j Y_j - \mathbb{E} \left[ \sum_j Y_j \right] \right) \right]
\]

\[
= \sum_{k,j} \mathbb{E} \left[ (Y_k - \mathbb{E}[Y_k]) (Y_j - \mathbb{E}[Y_j]) \right]
\]

\[
= \sum_{k,j} \text{Cov}[Y_k, Y_j]
\]

\[
\leq \sum_{k,j} \sqrt{\text{Var}[Y_k] \text{Var}[Y_j]},
\]

\[
\text{Var}[\lambda^{(M)}] = \frac{1}{2^n} \sum_{i} \lambda_i^{2M} - \left( \frac{1}{2^n} \sum_{i} \lambda_i^M \right)^2
\]
where the inequality is due to Cauchy-Schwarz. We now take $X$ to be the random variable which takes values $\{\lambda_i\}_i$ with uniform probability and mean $\mu = \frac{1}{2^n}$. We will bound $\text{Var}[Y_k]$ under the assumption that $\{\lambda_i\}_i$ are close in value to the maximally mixed value $\frac{1}{2^n}$.

First, we note that each $Y_k$ is a function of $(X-\mu)^k$, and so we must investigate the shifted spectrum which we denote $\hat{\lambda}$ where $\hat{\lambda}_i = \lambda_i - \frac{1}{2^n}$ for all $i$. Using Popoviciu’s inequality, we have the bound

$$\text{Var}[(X-\mu)^k] \leq \frac{1}{4} (\hat{\lambda}_{\max}^k - \hat{\lambda}_{\min}^k)^2. \quad (D125)$$

Now suppose that we have the constraint

$$\lambda_{\max} - \lambda_{\min} = 2b \quad (D126)$$

for some $b \geq 0$. For any $k$, we have

$$\hat{\lambda}_{\max}^k - \hat{\lambda}_{\min}^k \leq |\hat{\lambda}_{\max}|^k + |\hat{\lambda}_{\min}|^k. \quad (D127)$$

Let us now bound the quantity on the right by considering its maximum value over all spectra with constraint (D126). The quantity on the right is maximized by the choice of vector $(|\hat{\lambda}_{\max}|, |\hat{\lambda}_{\min}|)$ that majorizes all others, given some fixed value of $|\hat{\lambda}_{\max}| + |\hat{\lambda}_{\min}|$. Indeed, $|\hat{\lambda}_{\max}| + |\hat{\lambda}_{\min}| = b$ is fixed by our constraint (D126) (\hat{\lambda}_{\min} must be negative in order to preserve trace). Thus the quantity on the right side of (D127) can be bounded by maximizing $\hat{\lambda}_{\max}$ and minimizing $|\hat{\lambda}_{\min}|$. This is achieved by setting all other $\hat{\lambda}_i$ equal to $\hat{\lambda}_{\min}$. We then have pair of constraints

$$\hat{\lambda}_{\max} + (2^n - 1)\hat{\lambda}_{\min} = 0, \quad (D128)$$

$$\hat{\lambda}_{\max} - \hat{\lambda}_{\min} = 2b, \quad (D129)$$

where the first constraint comes from preservation of trace, and the second is our original constraint. This is a linear system of equations with solution

$$\hat{\lambda}_{\max}^* = 2b \frac{2^n - 1}{2^n}, \quad \hat{\lambda}_{\min}^* = -2b \frac{1}{2^n}. \quad (D130)$$

substituting these values into (D127) we have the bound

$$\hat{\lambda}_{\max}^k - \hat{\lambda}_{\min}^k \leq (2b)^k \left( \left( \frac{2^n - 1}{2^n} \right)^k + \left( \frac{1}{2^n} \right)^k \right) \quad (D131)$$

$$\leq (2b)^k. \quad (D132)$$

We will find it necessary to use the tighter bound (D131) in the case of $M = 2$, but the looser bound (D132) will enable us to write a bound with a more compact form for $M \geq 3$.

We now relate $b$ to the purity. We can write a general spectrum that satisfies the constraint in (D126) as $\lambda_{b,c,a} = (\frac{1}{2^n} + b + c, \frac{1}{2^n} - b + c, \frac{1}{2^n} - a_1, ..., \frac{1}{2^n} - a_{d-2})$, for some $c$ and set $\{a_i\}_i$ that satisfy $\sum_{i=1}^{d-2} a_i = 2c$ (in order to preserve trace). The purity that corresponds to this spectrum is given by

$$P(\lambda_{b,c,a}) = \left( \frac{1}{2^n} + b + c \right)^2 + \left( \frac{1}{2^n} - b + c \right)^2 + \sum_{i=1}^{d-2} \left( \frac{1}{2^n} - a_i \right)^2 \quad (D133)$$

$$= \frac{1}{2^n} + 2b^2 + c^2 + \sum_{i=1}^{d-2} a_i^2 + \frac{2}{2^n} \left[ 2c - \sum_{i=1}^{d-2} a_i \right] \quad (D134)$$

$$\geq \frac{1}{2^n} + 2b^2. \quad (D135)$$

Moreover, this purity bound is achievable by the spectrum $\lambda_{b,0,0} = (\frac{1}{2^n} + b, \frac{1}{2^n} - b, \frac{1}{2^n}, ..., \frac{1}{2^n})$ if we have $b \leq \frac{1}{2^n}$. We conclude that for any spectrum $\lambda_b$ that satisfies the constraint (D126), we have

$$b \leq \sqrt{\frac{1}{2} \left( P(\lambda_b) - \frac{1}{2^n} \right)}. \quad (D136)$$
And we now have all the tools to bound \( \text{Var}[Y_k] \) for all \( k \) and subsequently \( \text{Var}[X^M] \)

By combining the bounds (D125) and (D131) we have

\[
\text{Var}[(X - \mu)^k] \leq \frac{1}{4}(2b)^{2k}g_k^2 \tag{D137}
\]

where we have denoted \( g_k = \left(\frac{2^{n-1}}{2^n}\right)^k + \left(\frac{1}{2^n}\right)^k \leq 1 \). This allows us to bound \( \text{Var}[Y_k] \) by writing

\[
\sqrt{\text{Var}[Y_k]} = \left(\binom{M}{k}\right)^k \mu^{M-k} \sqrt{\text{Var}[(X - \mu)^k]}
\leq \frac{1}{2} \left(\binom{M}{k}\right)^{M-k} (2b)^k g_k. \tag{D138}
\]

We first pursue a bound for general \( M \in \mathbb{N} \) and replace each \( g_k \) with 1. We observe that the quantities \( \left(\binom{M}{k}\right)^{M-k} (2b)^k \) are simply the terms in the expansion of \( (2b - \mu)^M - \mu^M \), that is,

\[
\sum_k \sqrt{\text{Var}[Y_k]} \leq \frac{1}{2} ((2b - \mu)^M - \mu^M). \tag{D140}
\]

Returning to (D124), we have

\[
\text{Var}[X^M] \leq \frac{1}{4} ((2b - \mu)^M - \mu^M)^2 \tag{D141}
\]

\[
\leq \frac{1}{4} \left[ \left( 2 \sqrt{\frac{1}{2} \left( P - \frac{1}{2n}\right)} - \mu \right)^M - \mu^M \right]^2 \tag{D142}
\]

where in order to obtain the second inequality we have used (D136) to substitute \( b \) with its bound in terms of the purity. We further note that \( \text{Var}[X] = \frac{1}{2n}(P - \frac{1}{2n}) \), and dividing the two quantities we obtain

\[
\bar{X} \leq \frac{2^n}{4} \left[ \left( \sqrt{2 \left( P - \frac{1}{2n}\right)} + \frac{1}{2n} \right)^M - \left( \frac{1}{2n} \right)^M \right]^2 \tag{D143}
\]

as required. To summarize, combining the two bounds for high purity and low purity, so far we have

\[
G'(n \geq 2, M \geq 2, P) = \min \left( \frac{P^M}{P - \frac{1}{2^n}} \left( 1 - \frac{1}{2^n} \right), \frac{2^n}{4} \left[ \left( \sqrt{2 \left( P - \frac{1}{2n}\right)} + \frac{1}{2n} \right)^M - \left( \frac{1}{2n} \right)^M \right]^2 \right). \tag{D144}
\]

Now we discuss the magnitude of our bound obtained thus far, as well as its monotonicity with respect to \( M \). In particular, we will show that its value can exceed 1 for \( M = 2 \), and so we will pursue a tighter bound for \( M = 2 \). We can evaluate \( G'(n \geq 2, M \geq 2, P) \) explicitly at \( P = \frac{1}{2n-1} \) as

\[
G'(n \geq 2, M \geq 2, P = \frac{1}{2n-1}) = \min \left( \frac{(2^n - 1)^2}{(2n - 1)^M}, \frac{2^n(2^n - 1)}{4} \left[ \left( \sqrt{\frac{2}{2^n(2n - 1)}} + \frac{1}{2^n} \right)^M - \left( \frac{1}{2^n} \right)^M \right]^2 \right). \tag{D145}
\]

Firstly, by inspection this is a decreasing function in \( n \) for all \( M \geq 2 \), so in order to bound its magnitude we can consider \( n = 2 \). At \( M = 2 \) we have

\[
G'(2, 2, \frac{1}{2^2 - 1}) = \min \left( 1, \frac{5 + 2\sqrt{6}}{6} \right) = 1, \tag{D146}
\]
where we note $\frac{\delta + 2\sqrt{\delta}}{6} \geq 1$. As the first function in the minimization of (D144) has negative gradient for $P < \frac{1}{2^{n-1}}$ for $n \geq 2$, $M = 2$, this implies that the exists a set of values $P = \frac{1}{2^{n-1}} - \delta$, where $\delta > 0$ is small, such that the first function has value exceeding 1. The second function also has value exceeding 1 in such a region as it is continuous. Thus, there exist values of $P$ for which the bound $G' > 1$ at $M = 2$. Moving on to $M = 3$, we can numerically verify that $G'\left(2, 3, \frac{1}{2^{n-1}}\right) \leq 1$ with both functions in the minimization having value below 1. As functions of the form $f(x) = a^x - b^x$ where $b \leq a \leq 1$ only have one stationary point which is a maximum, this implies that $G'\left(2, M, \frac{1}{2^{n-1}}\right)$ is decreasing for all $M \geq 3$ and thus $G'\left(2, M \geq 3, \frac{1}{2^{n-1}}\right) \leq 1$.

We will replace $G'\left(n \geq 2, 2, P\right)$ with a tighter bound that is less than 1 for all $n \geq 2$. We return to (D138) and now explicitly consider the $g_k$ terms. Substituting this into (D124) for $M = 2$ we have

$$\text{Var}[X^2] \leq \text{Var}[Y_1] + \text{Var}[Y_2] + 2\sqrt{\text{Var}[Y_1]\text{Var}[Y_2]} \tag{D147}$$

$$= (2\mu)^2\text{Var}[X - \mu] + \text{Var}\left[(X - \mu)^2\right] + 4\mu\sqrt{\text{Var}\left[(X - \mu)^2\right]}\text{Var}[X - \mu] \tag{D148}$$

$$\leq (2\mu)^2\text{Var}[X] + \frac{1}{4}(2b)^4 g_2^2 + 4\mu\sqrt{\frac{1}{4}(2b)^4 g_2^2}\text{Var}[X] \tag{D149}$$

$$\leq \frac{4}{2^n}\text{Var}[X] + \left(P - \frac{1}{2^n}\right)^2 g_2^2 + \frac{4}{2^n}g_2\left(P - \frac{1}{2^n}\right)\sqrt{\text{Var}[X]} \tag{D150}$$

where in the first equality we use the definition of $Y_k$ for $M = 2$, in the first inequality we use (D137) along with the fact that $g_1 = 1$, and in the final inequality we use (D136). Finally, dividing by $\text{Var}[X]$ we have

$$\mathbb{X}(M = 2) \leq \frac{4}{2^n} + \frac{1}{\text{Var}[X]}\left(P - \frac{1}{2^n}\right)^2 g_2^2 + \frac{4}{2^n}g_2\sqrt{P - \frac{1}{2^n}}\frac{1}{\sqrt{\text{Var}[X]}} \tag{D151}$$

$$= \frac{4}{2^n} + 2^n g_2^2\left(P - \frac{1}{2^n}\right) + \frac{4}{2^n}g_2\sqrt{P - \frac{1}{2^n}}, \tag{D152}$$

where we have used $\text{Var}[X] = \frac{1}{2^n}(P - \frac{1}{2^n})$. \hfill \square

We plot our bounds on the 2-design-averaged resolvability in Fig. 6. First, in the left figure we plot the intermediate bound $\frac{\text{Var}[\lambda^{(M)}]}{\text{Var}[\lambda]}$ in (D109) for states with 100 randomly generated spectra for increasing number of qubits $n$ and number of state copies $M$. This visualizes the exponential scaling with $M$ and we observe that broadly, the bound is decreasing with increasing number of qubits. Further, as expected, the bound is always less than or equal to 1. Second, in order to demonstrate the behaviour of our final upper bound (D96) we plot increasing number of state copies $M$ ranging from 2 to 4 for $n = 2$. For each $M$, we randomly generate 10000 states and plot $\frac{\text{Var}[\lambda^{(M)}]}{\text{Var}[\lambda^{(1)}]}$ against the purity of the state as separate points. The final upper bound is then plotted as a line for each value of $M$.

3. Probabilistic Error Cancellation

a. Error mitigation of multiple gates

We first consider the error mitigation cost of mitigating multiple noise channels. Suppose we have two noisy gates which we represent as the channel

$$\mathcal{N}' \circ \mathcal{U}' \circ \mathcal{N} \circ \mathcal{U} \tag{D153}$$

where $\{\mathcal{U}', \mathcal{U}\}$ are channels that represent the ideal gates and $\{\mathcal{N}', \mathcal{N}\}$ are noise channels. Note that this framework also includes as a special case the scenario where two gates act in parallel on different subsystems. Given set of basis gates $\{\mathcal{B}_\alpha\}_\alpha$, we can construct a quasiprobability distribution for the ideal channel as

$$\mathcal{U}' \circ \mathcal{U} = \sum_{\alpha, \beta} k_\alpha k_\beta \mathcal{B}_\alpha \circ \mathcal{N}' \circ \mathcal{U}' \circ \mathcal{B}_\beta \circ \mathcal{N} \circ \mathcal{U}.$$  \hspace{1cm} \tag{D154}
where we have used (A11). From (D154) we see that the error mitigation cost is

\[ \gamma_{\text{tot}} = \sum_{\alpha, \beta} k_\alpha^2 k_\beta^2 = \gamma \gamma' \]  

(D155)

where \( \gamma, \gamma' \) are the individual error mitigation costs for each gate. We can see the above reasoning can be extended inductively to show that the error mitigation cost of a collection of gates with the probabilistic error cancellation is equal to the product of the individual error mitigation costs.

b. Global depolarizing noise

**Proposition 5** (Relative resolvability of Probabilistic Error Cancellation for global depolarizing noise). Consider a quasi-probability method that corrects global depolarizing noise of the form (32). For any pair of states corresponding to points on the cost function landscape, the optimal quasi-probability scheme gives

\[ \chi_{\text{depol}} = \frac{2^{2n}}{2^{2n} - p(2 - p)} \geq 1, \]  

(D156)

for all \( n \geq 1, p \in [0, 1] \), which is achieved with access to noisy Pauli gates.

**Proof.** Ref. [87] gives the optimal quasi-probability decomposition for the inverse noise channel as

\[ \mathcal{D}^{-1} = \left( 1 + \frac{(2^{2n} - 1)p}{2^{2n}(1 - p)} \right) \mathcal{I} - \sum_{i=1}^{2^{2n} - 1} \frac{p}{2^{2n}(1 - p)} \mathcal{P}_i, \]  

(D157)

where \( \mathcal{I} \) is the identity channel and \( \mathcal{P}_i \) is the Pauli channel corresponding to the \( i \)th Pauli tensor product. This has corresponding error mitigation cost

\[ \gamma = \frac{2^{2n} - 2p + p^2}{2^{2n}(1 - p)^2}. \]  

(D158)

Assuming perfect correction we have \( \Delta \tilde{C} = (1 - p) \Delta C \) which implies

\[ \chi_{\text{depol}} = \frac{2^{2n}}{2^{2n} - 2p + p^2}, \]  

(D159)

which is greater than or equal to 1 as \(-2p + p^2 \leq 0\) for all \( 0 \leq p \leq 1 \).
Here we consider a model of cost concentration due to a single instance of local depolarizing noise in a circuit. We presume that the concentration follows a similar form of scaling to global depolarizing noise and a tensor product of local depolarizing noise (see Eq. (5)). We show that, under this assumption, the relative resolvability has regimes of being greater than 1 or less than 1, depending on the strength of the cost concentration.

Supplemental Proposition 4 (Relative resolvability of Probabilistic Error Cancellation with one instance of local depolarizing noise). Consider a single instance of local depolarizing noise occurring with error probability $p$ acting at an arbitrary point in the parameterized circuit. Suppose that due to this noise channel we have

\[
(\Delta \tilde{C}(\theta_{i,*}))_i \geq (1 - b^\alpha p)(\Delta C(\theta_{i,*}))_i
\]

for all $p \in [0, 1]$ where $\langle \cdot \rangle_i$ denotes an average over all available parameters and $b^\alpha$ where is some positive constant.

Then the optimal quasiprobability scheme gives:

- for $b^\alpha \leq \frac{3}{4}$,
  \[ \chi \leq 1, \quad \forall p \in [0, 1], \]

- for $\frac{3}{4} < b^\alpha \leq 1$,
  \[ \chi \leq 1 + \frac{1}{4} p(2 - p) + \mathcal{O}(p^2), \quad \forall p \in [0, 1], \]
  \[ \chi > 1, \quad \forall p \in \left(0, 1 - \frac{1}{\sqrt{3}}(b^{-1} - 1)\right), \]

- for $b^\alpha > 1$,
  \[ \chi > 1 + \frac{p(2 - p)}{4 - p(2 - p)}, \quad \forall p \in (0, 1/b^\alpha]. \]

Proof. From Eq. (D158), we can write the optimal error mitigation cost for one instance of local depolarizing noise acting on one qubit as

\[ \gamma = \frac{4 - 2p + p^2}{4(1 - p)^2}. \]

Now, due to our assumption (D160) and assuming perfect implementation of the basis of noisy gates (leading to perfect correction of the noise) we have

\[ \chi \leq \frac{4(1 - p)^2}{(4 - 2p + p^2)(1 - bp)^2}, \]

and we denote the quantity on the right hand side as $h(p)$. Note that for any value of $b$, $h(p = 0) = 1$ and $h(p = 1) = 0$. The partial derivative can be found to satisfy

\[ \frac{\partial h}{\partial p} \propto (1 - p)(1 - bp) \left(-p^3 + 3p^2 - 3p + \frac{1}{b} - 4\left(\frac{1}{b} - 1\right)\right), \]

where the proportionality factor we omit is positive for all $b \geq 0$ and $p \in [0, 1]$. The third bracket is a cubic form with discriminant

\[ \Delta = \frac{108}{b^2} \left(-8b^2 + 11b - 4\right), \]

which is strictly negative for all $b$. Thus, the cubic form only has one real root and, inspecting its behaviour for large $p$, we can conclude it has negative gradient for all $p$. The cubic form has root at $p = 0$ when $b = 3/4$. More generally, the root can be found to take the form

\[ p' = 1 + \frac{1}{3}(1 - b^{-1}). \]
By evaluating the second derivative of $h(p)$, this root can be seen to correspond to a local maximum of $h(p)$. We now find the maximum value of $h(p)$ over the interval $p \in [0, 1]$ for different regimes of cost concentration strength $b$.

First, we inspect the regime where $b < 3/4$. In this case $p' \leq 0$ and thus $\frac{\partial h}{\partial p} \leq 0$ for $p \in [0, 1]$. Thus the maximum value of $h(p)$ on the interval $p \in [0, 1]$ is $h(0) = 1$. We can then conclude that $\bar{\chi} \leq 1$ with bound saturated in the limit of zero error probability.

Now consider the regime $3/4 < b \leq 1$. In this case $0 < p' \leq 1$ and $\frac{\partial h}{\partial p} > 0$ for small values of $p$. Specifically, it is clear that $\bar{\chi} > 1$ for $0 < p \leq 1 + \sqrt{3(1-b^{-1})}$. The upper limit on $p$ can be raised, however, the exact interval is obtained by solving a quartic equation which we omit here as it is not very insightful. Moreover, the upper limit is tight in the limit $b \to 1$ and we obtain the result that when $b = 1$, $\bar{\chi} > 0$ for all $p \in (0, 1)$.

Finally, consider the regime $b \geq 1$. Now $\frac{\partial h}{\partial p}$ has a different root $p'' = 1/b$ due to the second bracket in (D167). Again, this can be shown to correspond to a maximum of $\bar{\chi}$ and we can write $\bar{\chi} > 1$ for $0 < p \leq 1/b$.

**Proposition 6** (Scaling of Probabilistic Error Cancellation with local depolarizing noise). Consider tensor-product local depolarizing noise with local depolarizing probability $p$ to act in $L$ instances through a depth $L$ circuit as in Eq. 3. Suppose that the effect of this noise is to cause cost concentration

$$\langle \Delta \tilde{C}(\theta_{i,*}) \rangle_i = Aq^L \langle \Delta C(\theta_{i,*}) \rangle_i,$$

for some constant $A$ and noise parameter $q \in [0, 1)$. The optimal quasiprobability method to mitigate the depolarizing noise in the circuit yields

$$\bar{\chi} = \frac{1}{A^2 q^{2L}} (Q(p))^{nL},$$

where $0 \leq Q(p) \leq 1$ for all $p$. Thus, the average relative resolvability has unfavourable scaling with system size.

**Proof.** As shown in Section D3a, error mitigation cost of multiple gates with probabilistic error cancellation is the product of the individual error mitigation costs. Thus, for the collection of gates considered, we have total error mitigation cost

$$\gamma_{\text{tot}} = \left( \frac{4(1-p)^2}{4-2p+p^2} \right)^{nL},$$

where we have used Eq. (D165). We suppose that mitigation perfectly corrects the error, such that $\Delta C_m(\theta_{i,*}) = \Delta C(\theta_{i,*})$. Combining this with our assumption (D170) we obtain the desired result, where we denote

$$Q(p) = \frac{4-2p+p^2}{4(1-p)^2} = 1 - \frac{3p(2-p)}{4-4p(2-p)},$$

which clearly satisfies $0 \leq Q(p) \leq 1$.

4. Linear Ansatz Methods

a. Global depolarizing noise is exactly correctable

Consider the linear ansatz

$$C_m(\alpha) = a_1 \tilde{C} + a_2,$$

where we denote $\alpha = (a_1, a_2)$. As shown in [45], this ansatz is particularly suited to global depolarizing noise and the ansatz can correct the noise exactly. Namely, the $n$-qubit noise channel

$$\rho \xrightarrow{D} (1-p)^L \rho + (1-(1-p)^L) \frac{1}{2^n}$$

(D175)
can be exactly corrected by using

\[ a_1 = \frac{1}{(1-p)L}, \quad a_2 = -\frac{(1-(1-p)L)}{(1-p)L} \text{Tr}[O]/2^n. \]  

(D176)

As correction is exact, \( \Delta C_m = \Delta C \). It can also be seen that \( \Delta \tilde{C} = (1-p)L \Delta C \) and the error mitigation cost is \( \gamma = 1/(1-p)^2L \). This gives \( \chi = 1 \) for any pair of cost function points.

b. Relative resolvability between two points with same ansatz applied

**Proposition 7** (Linear ansatz methods). Consider any error mitigation strategy that mitigates noisy cost function value \( \tilde{C}(\theta) \) by constructing an estimator \( C_m(\theta) \) of the form (15). For any two noisy cost function points to which the same ansatz is applied, we have

\[ \chi = 1, \]  

(D177)

for any noise process.

**Proof.** By applying the same ansatz of the form (15) to two noisy cost function points corresponding to parameter sets \( \theta_1, \theta_2 \), one can write

\[ C_m(\theta_1, a) = a_1 \tilde{C}(\theta_1) + a_2, \]  

(D178)

\[ C_m(\theta_2, a) = a_1 \tilde{C}(\theta_2) + a_2. \]  

(D179)

This gives \( \gamma = a_1 \) and \( \Delta C_m = a_1 \Delta \tilde{C} \). Thus, substituting these quantities into Definition 2 one obtains \( \chi = 1 \) as required.

**Appendix E: Numerical simulations - implementation details**

We perform our optimizations using the MATLAB implementation of the Nelder-Mead algorithm [93]. For each MaxCut graph we perform optimization independently for \( N_i \) random choices of an initial simplex. We evaluate the cost function by performing perfect sampling of the simulated state with \( N_s \) shots. After each iteration of the Nelder-Mead algorithm we compute the total cost of optimization per graph \( N_{\text{tot}} \) by summing the shot budget spent for all \( N_i \) instances of the optimization. To analyze convergence of results with \( N_{\text{tot}} \) as shown in Figs. 5, 7 we take as the result of optimization after \( N_{\text{tot}} \) shots the best of \( N_i \) instances determined according to the optimized cost function. The optimization is terminated when \( N_{\text{tot}} \) exceeds \( 1.5 \times 10^8 \).

1. CDR-mitigated optimization

We perform CDR-mitigated optimization with \( N_i = 30 \) and \( N_s = 1024 \). We use training circuits constructed with a non-Clifford gates projection algorithm of [78]. To construct the training circuits we decompose \( e^{i\gamma_j H_{\text{MaxCut}}} \) to native gates of an IBM quantum computer using a decomposition from [94]. In order to account for linear connectivity of the simulated device we use SWAP gates to implement \( e^{-i\gamma_j Z_k Z_l} \) for non nearest-neighbors terms. The training circuits contain 100 near-Clifford circuits with at most 30 non-Clifford gates. In the case of circuits with fewer than 60 non-Clifford gates we construct training circuits with half of the non-Clifford gates replaced by Clifford gates. We evaluate the cost function for the training circuits using perfect sampling and \( N_s = 1024 \) shots. We perform CDR mitigation for each 2-body term of \( H_{\text{MaxCut}} \) independently. In general, in order to maximize the quality of the mitigation one should construct the training circuits for each of the terms of \( H_{\text{MaxCut}} \) for each new set of QAOA angles. Here, for the sake of shot efficiency, for each new set of parameters we compute the training set from scratch only if the 1-norm distance of its QAOA angles \( (\gamma_1, \beta_1, \gamma_2, \beta_2, \ldots, \gamma_p, \beta_p) \) from the closest point of a simplex is larger than 0.01. Otherwise, we use the CDR linear ansatz for the closest point of the simplex.
FIG. 7. Benchmarking various implementations of the noisy optimization for $p = 4$ 5-qubit MaxCut QAOA. We plot the approximation ratio averaged over 36 Max-Cut graphs chosen randomly from the Erdős-Rényi ensemble as a function of $N_{\text{tot}}$. We compare the results for various numbers $N_i$ of optimization instances initialized randomly and various numbers of shots $N_s$ per cost function evaluation. As a reference we show the results of CDR optimization for $p = 4$. We consider $N_{\text{tot}} = 10^7 - 1.5 \times 10^8$ as in Fig. 5. Additionally, as in Fig. 5 we use the approximation ratio computed with the exact energy to benchmark the optimization, and in the case of $N_i > 1$ we choose as the result of optimization the best instance determined according to the optimized cost function. We consider various values of $N_s = 1024$, $16384$, $262144$ for $N_i = 30$ and various values of $N_i = 30, 150, 3000$ for $N_s = 1024$. We find that $N_i = 3000, N_s = 1024$ yields the best results although differences of quality in between most of the noisy optimization implementations are relatively small in comparison to the CDR mitigated optimization.

For the noisy (no error mitigation) optimization we benchmark various combinations of $N_i$ and $N_s$ values for $p = 4$. In particular we consider increasing $N_s$ for $N_i = 30$ and increasing $N_i$ for $N_s = 1024$. We gather the results in Fig. 7. We find that for $N_{\text{tot}}$ ranging from $10^7$ to $1.5 \times 10^8$ (as considered in Fig. 5) the best results are obtained for $N_i = 3000, N_s = 1024$. We use these values for the noisy optimization presented in Fig. 5.

2. Optimization with Virtual Distillation

In this Appendix we compare 5-qubit MaxCut QAOA optimization of VD-mitigated cost function with optimization of the noisy and CDR mitigated cost function for $p = 2, 4$. We perform the comparison for 36 randomly chosen graphs from the Erdős-Rényi ensemble as in Fig. 5. We perform VD mitigation for each expectation value of a 2-site term of $H_{\text{MaxCut}}$ according to (12). Therefore, a key assumption is that we neglect derangement noise, which would affect realistic VD implementation on hardware [47]. We use the Nelder-Mead algorithm as described in Appendix E. We have $N_i = 30$ as for CDR simulations from Fig. 5 and assign $65536$ shots in order to estimate $\text{Tr}[\hat{\rho}^M Z_i Z_j]$ for each 2-site term of $H_{\text{MaxCut}}$ and $\text{Tr}[\hat{\rho}^M]$. Consequently, the total shot cost of the mitigated cost function estimation is $(n_e + 1) \times 65536$, where $n_e$ is the number of Max-Cut graph edges. We consider $M = 2, 3$ state copies as the shot cost of VD mitigation for given precision grows with increasing $M$ [95] and $M = 2, 3$ was shown to be sufficient for typical applications [96]. We find that $M = 2$ gives better results than $M = 3$ as suggested by our analytical results. Here we allow for $N_{\text{tot}}$ up to $1.5 \times 10^9$, i.e. up to 10 times more shots than considered for CDR mitigated and noisy optimization in Fig. 5.

We gather the results in Fig. 8 comparing them with noisy and CDR mitigated optimization from Fig. 5. We find that even with smaller $N_{\text{tot}}$ the noisy and CDR mitigated optimization outperforms the VD mitigated optimization.
FIG. 8. Virtual Distillation mitigated optimization for $p = 2, 4$, 5-qubit MaxCut QAOA. We plot the approximation ratio averaged over 36 Max-Cut graphs chosen randomly from Erdős-Rényi ensemble as a function of total shot number $N_{\text{tot}}$. The results were obtained with $N_i = 30$ initializations and 65536 shots per $\text{Tr}[\tilde{\rho}^M Z_i Z_j]$ and $\text{Tr}[\tilde{\rho}^M]$ estimation. For reference we also present our results of CDR-mitigated and noisy optimization from Fig. 5. We observe that for this setting the optimization with Virtual Distillation does not outperform the noisy or CDR-mitigated optimization.

This example shows that even for circuits outside of the NIBP regime there is no guarantee that mitigating an error mitigated cost function outperforms noisy cost function optimization. We note that this result does not prohibit VD-mitigated optimization advantage for different choices of $N_{\text{tot}}, M$ or the shot number per cost function evaluation.