Abstract

We argue that eleven dimensional supergravity can be described by a non-linear real-

isation based on the group $E_{11}$. This requires a formulation of eleven dimensional super-

gravity in which the gravitational degrees of freedom are described by two fields which are 

related by duality. We show the existence of such a description of gravity.
0. Introduction

There is a widespread belief that all superstring theories are manifestations of a single theory that has been called M theory. However, little is known for sure about M theory. The conjectured existence of M theory has largely relied on the properties of the supergravity theories. In particular, the unique eleven dimensional supergravity [1], the maximally supersymmetric ten dimensional supergravity theory which are the IIA [2,3,4] and IIB [5] supergravity theories as well as the type I supergravity coupled to the Yang-Mills theory [6]. These theories are essentially determined by the type of supersymmetry they possess and hence they are the low energy effective actions of the string theories with the corresponding space-time supersymmetry. One intriguing feature of supergravity theories is the occurrence of coset space symmetries that control the way the scalars occur in these theories. The four dimensional $N = 4$ supergravity theory possess a SL(2,R)/ U(1) symmetry [9], the IIA theory a SO(1,1) symmetry [2], the IIB theory a SL(2,R)/ U(1) [5] and the further reductions of the eleven dimensional supergravity theory possess cosets based on the exceptional groups [10,11,25]. These symmetries have also played an important role in string dualities in recent years [12,13]. The coset construction was extended [14] to include the gauge fields of supergravity theories. This method used generators that were inert under Lorentz transformations and, as such, it is difficult to extend to include either gravity or the fermions. However, this construction did include the gauge and scalar fields as well as their duals and as a consequence the equations of motion for these fields could be expressed as a generalised self-duality condition.

Recently [15], it was shown that the entire bosonic section of eleven dimensional supergravity and the ten dimensional IIA supergravity theory could be formulated as a non-linear realisation. This work made use of two old papers [16,17] which formulated gravity as a non-linear realisation. In this way of proceeding gravity and the gauge fields appeared on an equal footing. It is very likely that this construction can be extended to the bosonic sector of all supergravity theories. Although the extension to include the fermions was not given in reference [15] it was realised that this step implied that eleven dimensional supergravity must be invariant under a group that included Osp(1/64) as a subgroup.

The appearance of the above mentioned coset space symmetries in dimensional reductions of eleven dimensional supergravity on a torus can also be viewed as a consequence of supersymmetry in the dimensionally reduced theory. However, despite their importance, there is no really simple explanation of why these symmetries occur. It has been shown [18,19] that eleven dimensional supergravity does possess an SO(1,2)×SO(16) symmetry, although the SO(1,10) tangent space symmetry is no longer apparent in this formulation. However, it has been thought that the exceptional groups found in the dimensional reductions can not be symmetries of eleven dimensional supergravity. Despite this, it has been noticed that some of the objects which appear in the reductions do appear naturally in the unreduced theory [20]. A similar phenomenon occurs in the reduction of string theories on a torus. For example, if one takes the closed bosonic string to live on the unique self-dual thirty six dimensional Lorentzian lattice one finds that it is invariant under the fake monster algebra [21].

It is indeed peculiar that symmetries should appear in a theory when they are not
present in the theory before it is restricted by the dimensional reduction procedure. In this paper we will examine the possibility that eleven dimensional supergravity when formulated as a coset realisation, as in reference [15], does possess a large symmetry algebra that includes all the symmetries that occur when it is dimensional reduced. Indeed, we conjecture that eleven dimensional supergravity can be formulated as a non-linear realisation based on the group $E_{11}$. It is crucial that the theory is formulated as a non-linear realisation as this allows one to treat all the fields of the theory in the same way. In particular, gravity and the gauge fields are all Goldstone bosons.

We will show that the formulation of eleven dimensional supergravity as a non-linear realisation given in reference [15] is invariant under the Borel subgroup of $E_7$. We will then argue that one can extend this formulation in such a way that the full theory is invariant under a Kac-Moody algebra and show that if this were the case then this algebra must be $E_{11}$. We suggest that this extension can be achieved in two ways. We may take the local subgroup in the coset space construction to be much larger than the choice of the Lorentz group in reference [15]. We may also use an alternative description of eleven dimensional supergravity. In particular, we use a description of gravity that possess two fields which are related by duality. As a result, we propose an enlarged algebra that underlies eleven dimensional supergravity. We show that this algebra contains the Borel subgroup of $E_8$ and leads to the algebra of the ten dimensional IIA supergravity theory.

We will also argue that the effective action for the closed bosonic string in twenty six dimensions is invariant under a Kac-Moody algebra and propose such an algebra which has rank twenty seven.

It is likely that the proper formulation of M theory will involve radically new concepts. However, it is possible that these concepts may be very difficult to guess with our current knowledge. There are, for example, a number of indications that space-time may not be a fundamental concept. Although non-commutative algebras have been suggested as a replacement of space-time there does not exist any definitive and concrete way to implement this suggestion in the framework of M theory. The motivation for the present paper is that M theory may be rather algebraic in character and that the known supergravity theories have hidden within them information on what are the symmetries of M theory.

1. A Review of Kac-Moody Algebras

In this paper we will need some basic facts about Kac-Moody algebras which we now summarise [22]. Associated with any Kac-Moody algebra is a generalised Cartan matrix $A_{ab}$ which satisfies the following properties:

$$A_{aa} = 2,$$  \hspace{1cm} (1.1)

$$A_{ab} \text{ for } a \neq b \text{ are negative integers or zero},$$  \hspace{1cm} (1.2)

and

$$A_{ab} = 0 \text{ implies } A_{ba} = 0.$$  \hspace{1cm} (1.3)

A Kac-Moody algebra can be formulated in terms of its Chevalley generators which consist of the generators of the commuting Cartan subalgebra, denoted by $H_a$, as well as the
generators of the positive and negative simple roots, denoted by $E_a$ and $F_a$ respectively. The Chevalley generators are taken to obey the Serre relations;

$$[H_a, H_b] = 0, \quad (1.4)$$

$$[H_a, E_b] = A_{ab}E_b, \quad (1.5)$$

$$[H_a, F_b] = -A_{ab}F_b, \quad (1.6)$$

$$[E_a, F_b] = \delta_{ab}H_a, \quad (1.6)$$

and

$$[E_a, \ldots [E_a, E_b] \ldots] = 0, \quad [F_a, \ldots [F_a, F_b] \ldots] = 0 \quad (1.7)$$

In equation (1.7) there are $1 - A_{ab}$ number of $E_a$’s in the first equation and the same number of $F_a$’s in the second equation. Given the generalised Cartan matrix $A_{ab}$, one can uniquely reconstruct the entire Kac-Moody algebra from the above Serre relations by taking the multiple commutators of the simple root generators. Hence a Kac-Moody algebra is uniquely specified by its generalised Cartan matrix.

Any Kac-Moody algebra is invariant under the Cartan involution which acts on the generators as

$$E_\alpha \to -F_\alpha, \quad F_\alpha \to -E_\alpha, \quad H_a \to -H_a \quad (1.8)$$

where $\alpha$ is any positive root. It is straightforward to verify that this involution when applied to the Chevalley generators leaves invariant the above Serre relations. We may divide the generators of the Kac-Moody algebra into those that are even and those that are odd under the involution. The even generators are given by $E_\alpha - F_\alpha$. Being invariant under the Cartan involution they must form a subgroup of the original Kac-Moody algebra. The remaining odd generators are of the form $E_\alpha + F_\alpha$ and $H_a$. We note that the subgroup which is invariant under the Cartan involution contains none of the generators of the Cartan sub-algebra of the original Kac-Moody algebra.

2. Identification of $E_{11}$

It has been known for many years that when eleven dimensional supergravity is dimensionally reduced on a torus to $11-n$ dimensions, for $n = 1, \ldots 8$, the resulting scalars can be formulated as a non-linear realisations $[10,11]$. Let us denote the groups so obtained by $E_n$ and the corresponding local subgroups $F_n$. These are given the table below

| Table 1 Coset Spaces of the Maximal Supergravities |

4
The local subgroups $F_n$ are the maximal compact subgroups of $E_n$ and also have the same rank as $E_n$, namely $n$. The local subgroup $F_n$ can be used to choose the coset representatives of $E_n/F_n$ to belong to the Borel sub-group of $E_n$. We recall that the Borel subalgebra consists of the positive root and Cartan generators of the original algebra and hence the group elements of the Borel group can be written as a product of exponentials of these generators. Thus the number of scalar fields is just the number of generators in the Borel subalgebra. It has been shown that not only the scalar sector, but the entire dimensionally reduced theory is invariant under $E_n$.

For $n \leq 8$ the corresponding algebras found in the dimensional reduced theory are finite semi-simple Lie algebras. Dimensional reduction of eleven dimensional supergravity to two dimensions and one dimension is thought to result in theories that are invariant under the affine extension of $E_8$, which is called $E_9$ or $E_8(1)$ and the hyperbolic group $E_{10}$ respectively [23]. Some evidence for this assertion has been given for the two dimensional case in reference [19].

It has not previously been proposed that eleven dimensional supergravity itself should be invariant under any of the groups $E_n$, however, it has been found that a number of the $SO(1,2) \times SO(16)$ covariant objects found in the dimensional reduced theories can be lifted to eleven dimensions. As a result, it has been proposed [20,26] that eleven dimensional supergravity has some kind of exceptional geometry.

We note that the groups that occur in the dimensionally reduced theories, namely $E_n$ and $F_n$, are Kac-Moody Lie groups. Furthermore, the local subgroups $F_n$ for $n = 1, \ldots, 8$ are the subgroups which are invariant under the Cartan involution and were discussed in section one. For the cases of $n = 9, 10$ an involution has been introduced and used to define possible local subgroups [24]. However, in previous discussions of the dimensionally reduced theories, gravity plays a quite different role to that of the gauge fields and scalars under the appropriate $E_n$ group. Indeed, the gauge fields and scalars transform non-trivially, while gravity is inert under $E_n$.

When carrying out the dimensional reduction of eleven dimensional supergravity one has the choice whether to dualise certain fields or leave them as they appear in the usual dimensional reduction procedure. The simplest example of this is provided by a rank two gauge fields in four dimensions that arise from the rank three gauge field in eleven dimensions. These rank two gauge fields can be dualised to become scalars. In fact, the above

| $D$ | $E_n$       | $F_n$       |
|-----|-------------|-------------|
| 11  | $SL(2)$    | 1           |
| 10, IIB | $SO(1,1)/Z_2$ | $SO(2)$    |
| 10, IIA | $GL(2)$    | 1           |
| 9   | $E_3 \sim SL(3) \times SL(2)$ | $U(2)$     |
| 8   | $E_4 \sim SL(5)$ | $USp(4)$   |
| 7   | $E_5 \sim SO(5,5)$ | $USp(4) \times USp(4)$ |
| 6   | $E_6$      | $USp(8)$   |
| 5   | $E_7$      | $SU(8)$    |
| 4   | $E_8$      | $SO(16)$   |
| 3   |            |             |
$E_n$ symmetries only emerge if one does carry out the relevant duality transformations. The relationship between dualisations and the symmetries that emerge in dimensionally reduced theories has been extensively studied in reference [30]. One also has the option of introducing dual fields as well as the original fields for all the scalars and gauge fields and this was the path chosen in the coset formulation of this sector of the theory in reference [14]. This suggestion allows the equations of motion for the scalar and gauge fields to be written as a generalised self duality condition. This approach was further discussed in the papers of reference [31] where the analogy with the Ehlers symmetry of general relativity was explained. The first of these papers also includes a detailed discussion of how the fields that occur in dimensional reduction procedure give rise to the Dynkin diagram of the corresponding $E_n$ symmetries. It was noted that if one did this for all the fields then one would arrive at a rank eleven algebra associated with a dimensionally reduced theory.

Let us now recall the formulation [15] of the bosonic sector of eleven dimensional supergravity as a non-linear realisation. This was based on the group $G_{11}$ with a Lie algebra, also denoted, $G_{11}$ whose commutators are given by

\begin{align}
[K^a_b, K^c_d] &= \delta^c_b K^a_d - \delta^a_d K^c_b, \quad [K^a_b, P_c] = -\delta^a_c P_b, \quad [P_a, P_b] = 0 \quad (2.1)
\end{align}

\begin{align}
[K^a_b, R^{c_1...c_6}] &= \delta^a_b R^{a,c_2...c_6} + \ldots, \quad [K^a_b, R^{c_1...c_3}] = \delta^a_b R^{dc_2c_3} + \ldots, \quad (2.2)
\end{align}

\begin{align}
[R^{c_1...c_3}, R^{c_4...c_6}] &= 2R^{c_1...c_6}, \quad [R^{c_1...c_6}, R^{d_1...d_6}] = 0, \quad (2.3)
\end{align}

\begin{align}
[R^{c_1...c_3}, R^{a_1...a_6}] &= 0 \quad (2.4)
\end{align}

where $+\ldots$ denote the appropriate anti-symmetrisations. The generators $K^a_b$ and $P_c$ generate the affine group $IGL(11)$ while the generators $R^{c_1...c_3}$ and $R^{c_1...c_6}$ form a subalgebra.

The non-linear realisation is built from group elements $g \in G_{11}$ such that it is invariant under

\begin{align}
g \rightarrow g_0 gh^{-1} \quad (2.5)
\end{align}

where $g_0$ is a rigid element of the group $G_{11}$ generated by the above Lie algebra and $h$ is a local element of the Lorentz group. We may take the group element $g$ to be of the form

\begin{align}
g = e^{x^\mu P_\mu} e^{h^a_b K^a_b} e^{A_{c_1...c_3} R^{c_1...c_3}} + A_{c_1...c_6} R^{c_1...c_6} \quad (2.6)
\end{align}

where the fields $h^a_b$, $A_{c_1...c_3}$ and $A_{c_1...c_6}$ depend on $x^\mu$. Invariance under the rigid transformations is gained by considering the $g_0$ invariant forms given by

\begin{align}
\mathcal{V} = g^{-1}dg - w \quad (2.7)
\end{align}

where $w \equiv \frac{1}{2} dx^\mu w_{\mu b} J^b_c$ is the Lorentz connection and so transforms as

\begin{align}
w \rightarrow hw h^{-1} + hdh^{-1} \quad (2.8)
\end{align}

As a result

\begin{align}
\mathcal{V} \rightarrow h\mathcal{V} h^{-1} \quad (2.9)
\end{align}
We observe the important fact that the Cartan forms are inert under the rigid transformations and only transform under the local subgroup. the

Evaluating $\mathcal{V}$ we find that it is given by

$$
\mathcal{V} = dx^\mu (e_\mu \, Pa + \Omega_a \, b K_a \, b + \frac{1}{3!} \tilde{D}_\mu A_{c_1...c_3} R^{c_1...c_3} + \frac{1}{6!} \tilde{D}_\mu A_{c_1...c_6} R^{c_1...c_6}) \tag{2.10}
$$

where

$$
e_\mu \equiv (e^h)_\mu, \quad \tilde{D}_\mu A_{c_1...c_3} \equiv \partial_\mu A_{c_1c_2c_3} + ((e^{-1} \partial_\mu e)_c b A_{bc_2c_3} + ...),
$$

$$
\tilde{D}_\mu A_{c_1...c_6} \equiv \partial_\mu A_{c_1...c_6} + ((e^{-1} \partial_\mu e)_c b A_{bc_2...c_6} + ... - (A_{c_1...c_3} \tilde{D}_\mu A_{c_4...c_6}))
$$

$$
\Omega_{\mu b}^c \equiv (e^{-1} \partial_\mu e)_b^c - w_{\mu b}^c, \tag{2.11}
$$

where $+...$ denotes the action of $(e^{-1} \partial_\mu e)$ on the other indices of $A_{c_1...c_3}$ and $A_{c_1...c_6}$.

Eleven dimensional supergravity is invariant under not just the group $G_{11}$, but also under the infinite dimensional group which is the closure of $G_{11}$ and the eleven dimensional conformal group. This infinite dimensional group is realised by also calculating the Cartan forms for the conformal group and then taking only those $G_{11}$ Cartan forms that can be rewritten in terms of the Cartan forms, or other appropriate forms, of the conformal group. The forms so obtained are then simultaneously covariant under both groups and hence under the infinite dimensional group that is their closure. In fact, the Cartan forms just transform under composite Lorentz transformations. The result of this procedure for the rank three and six forms is that one should only use the simultaneously covariant forms

$$
\tilde{F}_{c_1...c_4} \equiv 4(e_{[c_1} \mu \partial_\mu A_{c_2...c_4]_b} + e_{[c_1} \mu (e^{-1} \partial_\mu e)_c b A_{bc_3c_4]} + ...)
$$

and

$$
\tilde{F}_{c_1...c_7} \equiv 7(e_{[c_1} \mu (\partial_\mu A_{c_2...c_7}] + e_{[c_1} \mu (e^{-1} \partial_\mu e)_c b A_{bc_3...c_7]} + ... + 5 \tilde{F}_{[c_1...c_4} \tilde{F}_{c_5...c_7]}) \tag{2.12}
$$

The invariant equation of motion is then given by

$$
\tilde{F}^{c_1...c_4} = \frac{1}{7!} \epsilon_{c_1...c_11} \tilde{F}^{c_5...c_11} \tag{2.13}
$$

For the Goldstone field $h_a^b$, that is for gravity, the process of finding the simultaneously covariant forms is more complicated and for this we refer the reader to reference [15]. Although the role of the conformal group is crucial, we will for the most part of this paper be concerned with the realisation of the group $G_{11}$. It was also shown in reference [15] that the IIA supergravity theory can also be expressed as a non-linear realisation and it is very likely that all supergravity theories allow such a description.

The scalars that occur in the dimensional reductions of eleven dimensional supergravity have their origin in the graviton and the rank three gauge field. Hence, if the symmetries that occur in the dimensional reductions have their origin in eleven dimensional supergravity, it must be in a formulation of this theory in which all the bosonic fields of the theory, namely the graviton and the rank three gauge field, occur in a similar
manner. One advantage of the coset formulation of eleven dimensional supergravity is that it does treats gravity in the same way as the third rank gauge field; they are both Goldstone bosons. Indeed, general coordinate transformations and the gauge transformations of the third rank gauge field both arise in the coset formulation. Hence, in this paper we will search for hidden symmetries in the coset formulation of eleven dimensional supergravity.

The non-linear realisation used in reference [15] to construct the eleven dimensional supergravity is based on a finite dimensional Lie algebra, namely $G_{11}$ which is not a Kac-Moody algebra. Furthermore, the local subgroup was chosen to be just the Lorentz group and, consequently, the coset representatives given in equation (2.6) are not members of a Borel subgroup of some larger group. Hence, the groups used in the coset formulation appear to be unlike those found in the dimensional reductions of eleven dimensional supergravity. As noted above, the latter are Kac-Moody algebras with local subgroups such that the coset representatives can be written as elements of the Borel subgroups. From this viewpoint, the coset construction of eleven dimensional supergravity would not at first sight appear to contain the Kac-Moody groups found in the dimensional reductions of this theory.

This strongly suggests that, although the coset formulation of reference [15] does lead to the correct equations of motion, it can be extended to involve a larger group than $G_{11}$. We will seek to achieve this in two ways. It is possible to introduce a larger group by introducing a larger local subgroup than the Lorentz group. This step will not introduce extra Goldstone bosons and only implies that the Cartan forms transform under the larger local subgroup (see equation (2.9)). For example, if we adopted a larger local subgroup for the formulation of eleven dimensional supergravity used in reference [15] it would lead to the same field content as the coset formulation of reference [15], the same group element as in equation (2.6) and the same Cartan forms as in equation (2.10). However, one must then show that the field equations are invariant under the larger local subgroup in order to ensure the invariance of the theory under the full group. A second possibility is to use a coset formulation that corresponds to an alternative form of eleven dimensional supergravity. This formulation would describe the same on-shell degrees of freedom, but the field content would be different. As we will explain later in this paper, gravity will be described by two fields in a dual formulation that is analogous to the way two fields $A_{a_1...a_3}$ and $A_{a_1...a_6}$ describe the non-gravitational bosonic degrees of freedom. In fact we can carry out both possibilities, we can first adopt a new formulation of eleven dimensional supergravity and then enlarge the group by introducing a larger local subgroup.

Having carried these steps, we must take the simultaneous non-linear realisation with the conformal group. It is natural to suppose that the resulting non-linear realisation, which is formed from two infinite dimensional groups, is based on a Kac-Moody algebra, $G$ and that the local subalgebra $H$ is the subgroup invariant under the Cartan involution, or some modification of it. The translation generator $P_n$ will play no role in these considerations. It would seem reasonable to suppose that the groups $E_n$ and $F_n$ found in the dimensional reductions of eleven dimensional supergravity are subgroups of the Kac-Moody Lie algebras $G$ and $H$ respectively.

We now consider how to implement this strategy and, in particular, how to identify
the Kac-Moody algebra $G$. Let us decompose the generators of $G_{11}$ into

$$G_{11}^+ = (K^a_b, a < b, a, b = 1, \ldots, 11, R^{c_1 \cdots c_6}, R^{c_1 \cdots c_6}),$$

$$G_{11}^0 = (H_a = K^a_a - K^{a+1}_{a+1}, a = 1, \ldots, 10, D = \sum_a K^a_a)$$

and $K^a_b, a > b$. The generators $K^a_b, a < b$ and $H_a$ are the positive root generators and Cartan sub-algebra of SL(11) respectively. The only other generators of SL(11) are the negative root generators which are the generators $K^a_b, a > b$. As explained above, we should demand that the new Kac-Moody Lie algebra $G$ should contain, among its positive root generators, those in $G_{11}^+$ and that its Cartan sub-algebra should include the generators in $G_{11}^0$. We define the Lie algebra which consists of the generators of $G_{11}^+$ and $G_{11}^0$ by $G_{11}^{0+}$.

As a first step in identifying the Kac-Moody Lie algebra $G$ we now show that $G_{11}^{0+}$ contains the Borel sub-algebra of $E_7$. We identify the positive root generators of $E_7$ to be just the generators of $G_{11}^+$ whose indices are restricted to take values from 5 to 11. Letting this index range be denoted by $i, j, \ldots$, these generators are $K^i_j$ for $i < j$, $R^{i_1 \cdots i_3}$ and $R^{i_1 \cdots i_6}$. Instead of the later generators we may equally well use

$$S_i = \frac{1}{6!} \epsilon_{i_1 \cdots i_6} R^{i_1 \cdots i_6}, \quad \hat{K}^i_j = K^i_j - \frac{1}{7} \delta_i^j \sum_{l=5}^{11} K^l_i.$$

The Cartan subalgebra generators of $E_7$ are identified with $H_i$ and $\hat{D} = \sum_{i=5}^{11} K^i_i$. Using equations (2.1) to (2.4), we find that these generators, together with the generators $\hat{K}^i_j = K^i_j$ for $i > j$, obey the relations

$$[\hat{K}^i_j, \hat{K}^k_l] = \delta^k_j \hat{K}^i_l - \delta^l_i \hat{K}^k_j,$$

$$[\hat{K}^i_j, R^{k_1 \cdots k_3}] = 3 \delta^i_j R^{k_1 \cdots k_3} - \frac{3}{7} \delta^i_j R^{k_1 \cdots k_3},$$

$$[R^{i_1 \cdots i_3}, R^{i_4 \cdots i_6}] = 2 \epsilon^{i_1 \cdots i_6} S_j$$

$$[\hat{D}, \hat{K}^i_j] = 0, \quad [\hat{D}, R^{k_1 \cdots k_3}] = 3 R^{k_1 \cdots k_3},$$

$$[\hat{D}, S_k] = 6 S_k,$$

$$[R^{i_1 \cdots i_3}, S_k] = 0$$

and

$$[\hat{K}^i_j, S_k] = -\delta^i_k S_j + \frac{1}{7} \delta^i_j S_k.$$  

In equations (2.18) to (2.24) we do indeed recognise the correct commutators of the Borel sub-algebra of $E_7$ when written with respect to its SL(7) subgroup. As proposed above, the positive roots generators of $E_7$ are $\hat{K}^i_j$ for $i < j$, $R^{i_1 \cdots i_3}$ and $S_i$ while the Cartan sub-algebra generators of $E_7$ are $H_i = K^i_i - K^{i+1}_{i+1}, i = 5, \ldots, 10$ and $\hat{D}$. We
note that \( G_{11} \) also contains all the generators of the \( \text{SL}(7) \) subgroup of \( E_7 \) in the correct way.

Since \( G_{11} \) is a symmetry of eleven dimensional supergravity it follows that the Borel subalgebra of \( E_7 \) is also a symmetry as is the \( \text{SL}(7) \) subgroup of \( E_7 \).

Under the decomposition of the adjoint (133) of \( E_7 \) into \( \text{SL}(7) \) representations we find that

\[
133 = 48(K^i_j) + 1(\hat{D}) + 35(R_{i_1\ldots i_3}) + 7(S_i) + 35(R_{i_1\ldots i_3}) + 7(S^i) \tag{2.25}
\]

Thus the full \( E_7 \) algebra is found by adding the remaining negative root generators \( S^i \) and \( R_{i_1\ldots i_3} \). To gain a theory invariant under the full \( E_7 \) we would extend the group by adding these generators to the local subgroup and then hope to show that the equations of motion were invariant under the now local \( \text{SU}(8) \) subgroup. In fact, the 63 of the \( \text{SU}(8) \) subgroup decomposes under \( \text{SO}(7) \) as

\[
63 = 21(K_{(ij)}) + 35(R_{i_1\ldots i_3}) + 7(S^i)\tag{2.25}
\]

The simple positive root generators of \( E_7 \) are given by

\[
E^i = K^i_{i+1}, \quad i = 5, \ldots, 10, \quad E_{11} = R^{91011} \tag{2.26}
\]

and an appropriate basis for the generators of the Cartan sub-algebra is given by

\[
H_i = K^i_{i} - K^i_{i+1}, \quad i = 5, \ldots, 10, \quad H_{11} = K^9_{9} + K^{10}_{10} + K^{11}_{11} - \frac{1}{3} \hat{D} \tag{2.27}
\]

It is straightforward to verify that these simple root and Cartan sub-algebra generators do indeed lead, using equations (2.18) to (2.24), and (1.5), to the Cartan matrix of \( E_7 \) that corresponds to its well known Dynkin diagram.

Let us now return to the identification of the Kac-Moody algebra \( G \) associated with the coset formulation of eleven dimensional supergravity. As noted in section one, it is sufficient to determine its Cartan matrix. As a result, we must seek in \( G_{11} \) the Cartan subalgebra and the simple root generators of \( G \). The simple positive root generators are those from which all positive root generators can be constructed by multiple commutators. It is clear that all the generators of \( G^+_{11} \) can be constructed, by taking repeated commutators, from

\[
E_a = K^a_{a+1}, \quad a = 1, \ldots, 10, \quad \text{and} \quad E_{11} = R^{91011}. \tag{2.28}
\]

We therefore identify these as the appropriate simple root generators of the Kac-Moody algebra \( G \) we seek. The \( E_a, a = 1, \ldots, 10 \) are the simple root generators of \( \text{SL}(11) \) and \( E_a, a = 5, \ldots, 11 \) are the simple root generators of \( E_7 \).

The Cartan sub-algebra generators of \( G \) are those contained in \( G^0_{11} \) in equation (2.16). Using equation (1.5), we may read off the generalised Cartan matrix \( A_{ij} \) of the Kac-Moody algebra Lie algebra. However, in order to find an acceptable generalised Cartan matrix \( A_{ij} \), that is one that satisfies equation (1.1) to (1.5), we must adopt an appropriate basis for the Cartan sub-algebra.

We observe that we have the same number of simple roots generators as we have generators in the Cartan subalgebra, namely eleven. We therefore conclude that we are searching for a Kac-Moody algebra \( G \) of rank eleven which must contain \( \text{SL}(11) \), or \( A_{10} \) and \( E_7 \) as a Lie sub-algebras. Examining their Dynkin diagrams we may suspect that the
Kac-Moody Lie algebra we are searching for is $E_{11}$. The Dynkin diagram of $E_{11}$ is given in figure one and we observe that by deleting points in this diagram we readily find the Dynkin diagrams of $A_{10}$ and $E_7$.

Given the embedding of $E_7$ and $A_{10}$ it is straightforward to show that the only allowed basis of the Cartan subalgebra, in the sense of an acceptable Cartan matrix, is given by

$$H_a = K^a_{a} - K^{a+1} a+1, a = 1, \ldots, 10, \quad H_{11} = K^9 9 + K^{10} 10 + K^{11} 11 - \frac{1}{3} D$$

(2.29)

These together with the simple roots $E_a, a = 1, \ldots, 11$ do indeed lead to the Cartan matrix of $E_{11}$. One can also verify that the Serre relations of equation (1.7) for the $E_a$ generators are satisfied, for example

$$[E_8, [E_8, E_{11}]] = 0$$

(2.30)

If we could be sure that there exists a non-linear realisation of eleven dimensional supergravity that is invariant under a Kac-Moody Lie algebra, we could now conclude that it is invariant under $E_{11}$. This follows from the definition of a Kac-Moody algebra given in section one. Since the positive simple roots exist and obey the relevant Serre relations of $E_{11}$, there must by definition exist negative simple root generators which satisfy all the remaining Serre relations. The algebra is then uniquely specified by the occurrence of the Cartan matrix of $E_{11}$ in these relations. However, we can not be sure that eleven dimensional supergravity is invariant under a Kac-Moody algebra. So far we have only shown the Serre relations of equations (1.4), (1.5) and the part of (1.7) which involve the $E_a$ and $H_a$ are satisfied. We have not shown that there exist symmetries corresponding to the generators $F_a$ and that these obey the remaining Serre relations. Indeed, we can not be sure that the $E_a$ and $H_a$ generators have multiple commutators that generate the whole of the Borel subalgebra of the Kac-Moody algebra. The problem is that in the absence of the $F_a$ generators, it can happen that a subset of the positive root generators $E_a$ form an ideal and that this ideal may be trivially realised in eleven dimensional supergravity. As we shall see, with the coset formulation of reference [15] this is indeed the case. However, in the next section, we propose an alternative coset formulation of eleven dimensional supergravity which, we believe does realise a Borel subgroup of a Kac-Moody algebra after the full construction has been carried out.

Since $E_{11}$ is a very large algebra the problem is more tractable if we first consider the smaller algebras that must arise in a restriction of eleven dimensional supergravity. The restriction is obtained by keeping only those generators and fields with indices $i, j, \ldots$ that take values 11 to $11 - n + 1$. The residual invariance group one would expect can be read off from the Dynkin diagram of $E_{11}$ by keeping only the first $n - 1$ nodes on the horizontal line in the Dynkin diagram starting from the right as well as any nodes to which they are attached by vertical lines. One finds that the invariance groups are the groups $E_n$ which are listed for $n = 1, \ldots, 8$ in table one. These are the groups that occur in the dimensional reductions of eleven dimensional supergravity. We would stress that the restriction discussed here is not a dimensional reduction. Indeed, the space-time is not restricted in any way and the residual groups are sub-symmetries of eleven dimensional supergravity.
To completely specify the coset formulation of the theory we must specify the local subgroup corresponding to \( E_{11} \). It is natural to assume that the appropriate subgroup is just that left invariant by the Cartan involution discussed in section one. It is simpler to consider the implications of this suggestion within the context of the restriction discussed above. The local subgroups that occur in the restriction must be the subgroups of \( E_n \) left invariant under the Cartan involution. As such, for \( n = 1, \ldots, 8 \), they must be the subgroups \( F_n \) that occur in table one. Clearly, \( F_n \) must contain the local subgroup in the formulation of reference [15], namely the Lorentz group \( \text{SO}(n) \) for \( n = 1, \ldots, 10 \) and \( \text{SO}(1,10) \) for \( n=11 \). Furthermore, \( F_{n-1} \) must be contained in \( F_n \) which suggests that \( F_{n-1} \) has rank one less than \( F_n \) and so \( F_n \) has rank \( n \). For \( n = 1, \ldots, 8 \) these conditions are satisfied. Although the local subgroups specified by demanding invariance under the Cartan involution are also unique, for \( n \geq 9 \), it is not so obvious what they are in practice. In fact, it could happen that a Cartan invariant subgroup of a Kac-Moody algebra is not itself a Kac-Moody algebra. However, assuming this not to be the case, the local subgroup for \( n = 9 \) is an infinite dimensional Kac-Moody algebra of rank 9. If it can be obtained by adding a node to the Dynkin diagram of \( F_8 \), that is \( D_8 \), and it is an affine Lie algebra there are only two possibilities \( D_8^{(1)} \) and \( B_8^{(1)} \) in the notation of reference [22]. Similarly the possible candidates for the local subgroups for \( n = 10 \) and \( n = 11 \) can be listed.

As explained in section one, the subgroup invariant under the Cartan involution is of the form \( E_{\alpha} - F_{\alpha} \) where \( \alpha \) is a positive root. For a finite dimensional semi-simple Lie algebras these generators are compact in the sense that if the Cartan-Killing metric is used to evaluate the scalar product of these generators it is positive definite. For these groups this definition of compactness coincides with the usual topological definition. Hence, the local subgroups \( F_n \) for \( n = 1, \ldots, 8 \) are compact groups. However, the real form of the local subgroup \( F_{11} \) that occurs in eleven dimensional supergravity must be non-compact as it contains \( \text{SO}(1,10) \) as a subgroup. Thus one might question if the subgroup \( F_{11} \) defined above is really correct.

For an infinite dimensional Kac-Moody Lie algebra one still has an analogue of a Cartan Killing metric which when restricted to the Cartan subalgebra \( H_a \) is equal, as for the finite dimensional case, to the Cartan matrix, that is \( (H_a, H_b) = A_{ab} \) and \( (E_a, F_b) = \delta_{ab} \). These formulae hold for a simply laced algebra, but there also exists a suitable modification for the non-simply laced algebras. For finite dimensional Kac-Moody algebras the eigenvalues of the Cartan matrix are strictly positive and so the Cartan subgroup generators are compact in the above sense. However, for an infinite dimensional Kac-Moody algebra the eigenvalues of the Cartan matrix are not all strictly positive and so the Cartan subalgebra generators are not compact in the above sense. Indeed, in the case of \( E_{11} \) the eigenvalues of the Cartan matrix have the signature \((-,-,\ldots,+)\). In a similar way, the proof of the "compactness" of the Cartan involution invariant subgroups also fails for the infinite dimensional Kac-Moody Lie algebras and there would seem no reason to suppose that \( F_{11} \) could not contain \( \text{SO}(1,10) \).

3. \( E_8 \) Lost

In section two, we observed that if a non-linear realisation of eleven dimensional supergravity was based on a Kac-Moody algebra then this would have to be \( E_{11} \). However, we can not be sure that there does exist a non-linear realisation that is based on a Kac-
Moody algebra. As we noted above, if this is not the case, there can exist non-trivial ideals in the algebra and these can be trivially realised. Nonetheless, we showed in section two that the non-linear realisation of reference [15] did contain the Borel subalgebra of $E_7$. We have also suggested how the non-linear realisation of eleven dimensional supergravity could be modified to possess a full $E_7$ symmetry.

In this section, we consider if the Borel subalgebra of $E_8$ is a symmetry of the non-linear realisation of reference [15] of eleven dimensional supergravity. Let us consider the restriction of the $G_{11}$ algebra such that the indices on the generators take the values $i, j, \ldots = 11, \ldots, 4$. Introducing the generators

\[
\hat{K}^i_j = K^i_j - \frac{1}{8} \delta^i_j \sum_l K^i_l, \quad \hat{D} = \sum_{i=4}^{11} K^i_i, \quad R^{i_1 \ldots i_3}, \quad S_{k_1 k_2} = \frac{1}{6!} \epsilon_{k_1 k_2 i_1 \ldots i_6} R^{i_1 \ldots i_6},
\]

we find, using the $G_{11}$ algebra of equations (2.1) to (2.4), that they obey the algebra

\[
[\hat{K}^i_j, \hat{K}^k_l] = \delta^k_j \hat{K}^i_l - \delta^i_l \hat{K}^k_j, \tag{3.2}
\]

\[
[\hat{K}^i_j, R^{k_1 \ldots k_3}] = 3 \delta^i_j R^{k_1 \ldots k_3} - \frac{3}{8} \delta^i_j R^{k_1 \ldots k_3}, \tag{3.3}
\]

\[
[R^{i_1 \ldots i_3}, R^{i_4 \ldots i_6}] = \epsilon^{i_1 \ldots i_6 j k} S_{j k} \tag{3.4}
\]

\[
[\hat{D}, \hat{K}^i_j] = 0, \quad [\hat{D}, R^{k_1 \ldots k_3}] = 3 R^{k_1 \ldots k_3}, \quad [\hat{D}, S_{j k}] = 6 S_{j k} \tag{3.5}
\]

\[
[\hat{K}^i_j, S_{k_1 k_2}] = -2 \delta^i_j S_{k_1 k_2} + \frac{2}{8} \delta^i_j S_{k_1 k_2}, \tag{3.6}
\]

\[
[S_{k_1 k_2}, S_{j_1 j_2}] = 0 \tag{3.7}
\]

and

\[
[S_{k_1 k_2}, R^{i_1 \ldots i_3}] = 0 \tag{3.8}
\]

Let us now compare this algebra to that of the Borel subalgebra of $E_8$. The 248 adjoint of $E_8$ decomposes into $\text{SL}(8, \mathbb{R})$ representations as $248 = 1 + 63 + (56 + 28 + 8) + (56 + 28 + 8)$. The 63 are the generators of $\text{SL}(8, \mathbb{R})$. The Cartan subalgebra of $E_8$ consists of the Cartan subalgebra of $\text{SL}(8, \mathbb{R})$ and the generator 1 in the above decomposition. The positive root generators of $E_8$ are the positive root generators of $\text{SL}(8, \mathbb{R})$ as well as the $(56 + 28 + 8)$ and the negative root generators of $E_8$ are the negative root generators of $\text{SL}(8, \mathbb{R})$ as well as the $(56 + 28 + 8)$.

We now wish to attempt to precisely identify the generators of the above restriction of the $G_{11}$, that appear in the algebra of equations (4.2) to (4.8), with the Borel subalgebra of $E_8$. In fact, it is just as simple to also include the whole of $\text{SL}(8, \mathbb{R})$ rather than just the Borel subalgebra of $\text{SL}(8, \mathbb{R})$. We can identify $K^i_j$ as the generators of $\text{SL}(8, \mathbb{R})$, $\hat{D}$ as the remaining member of the Cartan subalgebra of $E_8$, the $R^{k_1 \ldots k_3}$ are the 56 and the $S_{k_1 k_2}$ are the 28. Hence the only missing positive root generators in the above decomposition are the 8. These form an ideal in the Borel subalgebra of $E_8$. In fact, the commutation relations of equation (4.2) to (4.8) are those of the Borel subalgebra of $E_8$ except that the
generators of the $\mathfrak{8}$ are trivially realised. Indeed, the commutation relations of equations (4.2) to (4.7) are precisely those of the Borel subalgebra of $E_8$, except that equation (4.8) should have a non-vanishing right-hand side that involves the missing $\mathfrak{8}$ generators.

4. $E_8$ Regain’d

In this section, we propose an extension of the algebra $G_{11}$ that is to be used in a non-linear realisation of eleven dimensional supergravity. Since the algebra has been enlarged the formulation of eleven dimensional supergravity that results will also be modified compared to that used in reference [15]. One advantage of this new algebra is that it realises non-trivially the full Borel subalgebra of $E_8$. This new algebra contains the generators of $G_{11}$ as well as the generators $R^{a_1\ldots a_8,b}$ which is anti-symmetric in $a_1 \ldots a_8$. The new algebra obeys the commutators

$$ [K^a_b, P_c] = -\delta^a_c P_b, \quad [P_a, P_b] = 0 \quad (4.1) $$

$$ [K^a_b, K^c_d] = \delta^c_b K^a_d - \delta^a_b K^c_d, \quad (4.2) $$

$$ [K^a_b, R^{c_1\ldots c_6}] = \delta^{c_1}_b R^{a_2\ldots c_6} + \ldots, \quad [K^a_b, R^{c_1\ldots c_3}] = \delta^{c_1}_b R^{a_2 c_2 c_3} + \ldots, \quad (4.3) $$

$$ [R^{c_1\ldots c_3}, R^{c_4\ldots c_6}] = 2R^{c_1\ldots c_6}, \quad (4.4) $$

$$ [R^{a_1\ldots a_6}, R^{b_1\ldots b_3}] = 3R^{a_1\ldots a_6[b_1 b_2 b_3]}, \quad (4.5) $$

$$ [R^{a_1\ldots a_8,b}, R^{b_1\ldots b_3}] = 0, \quad [R^{a_1\ldots a_8,b}, R^{b_1\ldots b_6}] = 0, \quad [R^{a_1\ldots a_8,b}, R^{c_1\ldots c_8,d}] = 0 \quad (4.6) $$

$$ [K^a_b, R^{c_1\ldots c_8,d}] = (\delta^{c_1}_b R^{a_2\ldots c_8,d} + \ldots) + \delta^d_b R^{c_1\ldots c_8,a}. \quad (4.7) $$

Equations (2.1) to (2.3) are the same as equations (4.1) to (4.4), but equation (2.4) is replaced by equation (4.5) and it is here that the new generator makes its appearance. This algebra satisfies the Jacobi identities provided

$$ R^{[c_1\ldots c_8,d]} = 0. \quad (4.8) $$

The restriction of this new algebra for $n = 8$ gives the generators of equation (4.1) and in addition the generator $R^{i_1\ldots i_8,j} = \epsilon^{i_1\ldots i_8} S^j$. These generators obey the commutators of equations (4.2) to (4.7) as well as the relations

$$ [K^i_j, S^k] = \delta^k_j S^i - \frac{1}{8} \delta^i_j S^k, \quad [S_{k_1 k_2}, R^{j_1 j_2 j_3}] = 3 \delta^{[j_1 j_2} S^{j_3]} \quad (4.9) $$

$$ [R^{i j k}, S^l] = 0, \quad [S_{i j}, S^k] = 0 \quad (4.10) $$

and

$$ [\hat{D}, S^k] = 9 S^k. \quad (4.11) $$

These equations together with those of (4.2) to (4.7) are indeed those of the Borel subalgebra of $E_8$ together with the remaining generators of SL(8,R).

Thus we are led to propose that eleven dimensional supergravity can be described by a non-linear realisation based on the algebra of equations (4.1) to (4.7). To specify the
non-linear realisation we must state the local subgroup. The minimal choice would be to take just the Lorentz group. We would then expect the simultaneous non-linear realisation of this coset with the conformal group to describe eleven dimensional supergravity.

The algebra of equations (4.2) to (4.7) is not a Kac-Moody algebra. However, as we explained in section two, the choice of local subgroup is not unique. We can further enlarge the algebra of equations (4.2) to (4.7) by adding generators that belong to a local subgroup that includes the Lorentz group and in this way hope to arrive at a formulation of eleven dimensional supergravity that is invariant under a Kac-Moody algebra. In effect one treats the generators in equations (4.2) to (4.7), with the exception of $K^a_b$, $a > b$, as part of the Borel sub-algebra of the Kac-Moody algebra and adds the corresponding negative root generators. The final equations of motion are constructed from the Cartan forms which transform non-trivially under the local subgroup. Thus a very strong constraint on the local subgroup is that it leads to the correct equations of motion.

At first sight, one would not appear to arrive at $E_{11}$ through this procedure. However, we are required to take the closure of the resulting algebra with the conformal algebra and construct the simultaneous non-linear realisation. The closure of these two algebras will be an infinite dimensional algebra that contains a number of the important subalgebras of $E_{11}$ and it is hoped that this is actually $E_{11}$. In particular, it would be interesting to examine how the affine nature of $E_9$ arises in this closure.

There is also a puzzle concerning the number of Goldstone bosons that would arise in the construction of such a non-linear realisation. It is well known that a non-linear realisation can result in a theory that has fewer Goldstone bosons than one has generators in the coset. This is due to a mechanism, called the inverse Higgs effect [27], that allows one, under certain circumstances, to solve some Goldstone bosons in terms of some of the others. In fact, this is reason why the infinite dimensional group which is the closure of the conformal group and the group of affine general linear transformations leads only to the Goldstone bosons that belong to gravity. Taking the group $E_{11}$ and the local subgroup that was invariant under the Cartan involution would lead to an infinite number of Goldstone bosons, however, we may expect that most of these can be eliminated by the inverse Higgs effect to leave only a finite number of Goldstone bosons. Indeed these residual Goldstone bosons may be just those correspond to the generators in the algebra of equations (4.1) to (4.7).

We hope to construct this non-linear realisation and examine these conjectures in a future paper. In the rest of this section, we will examine some of the consequences of this suggestion and, as a result, demonstrate a number of detailed checks on the suitability of the algebra of equations (4.1) to (4.7).

In a non-linear realisation based on the algebra of equations (4.1) to (4.7), the corresponding Goldstone fields consist of those of the original formulation of reference [15], that is the fields $h^{ab}$, $A_{a_1...a_3}$, $A_{a_1...a_6}$, but include in addition the fields $A_{a_1...a_8,b}$. The Goldstone bosons $A_{a_1...a_3}$ and $A_{a_1...a_6}$ lead to a first order formulation of the equations of motion of the bosonic non-gravitational degrees of freedom of eleven dimensional supergravity more usually described solely by a rank three anti-symmetric tensor gauge field. The field strengths of the gauge fields $A_{a_1...a_3}$ and $A_{a_1...a_6}$ are related by use of the epsilon symbol. Since the only other on-shell bosonic degrees of freedom are those of the graviton,
the construction implies that there must exist a formulation of gravity involving the fields \( h^a \) and \( h_{a_1...a_8,b} \). We might expect that the field strengths of these fields are related by the epsilon symbol. Indeed, we observe that if we regard the lower index on \( h^a \) as that corresponding to the "gauge field" and the other index as a type of internal index then the field strength would be of the form \( f_{a_1a_2}^b \) and the associated "dual gauge field" would indeed have the index structure of \( h_{a_1...a_8}^b \). Hence in this construction we may expect the graviton equation of motion to have a similar structure to that of the three rank tensor gauge field. It is to be expected that only in such a formulation would the the full symmetry of eleven dimensional supergravity be apparent.

We now show that such a formulation of gravity does exist. We will carry out the construction in a space-time of arbitrary dimension, denoted \( D \). It is well known that Einstein theory of general relativity, whose traditional action is given by

\[
\int d^D x R(e_\mu^a, w_{\mu,a}^b(e))
\]  

(4.12)

can be rewritten in terms of the action

\[
\int d^D x (C_{ca}^a C^{cb} - \frac{1}{2} C_{ab,c} C^{ac,b} - \frac{1}{4} C_{ab,c} C^{ab,c})
\]  

(4.13)

where

\[
C_{\mu\nu}^a = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a.
\]  

(4.14)

It is straightforward to show that this action is equivalent to taking the action

\[
\frac{1}{2} \int d^D x (Y_{ab,c}^a C_{cb}^c + \frac{1}{2} Y_{ab,c} Y^{ac,b} - \frac{1}{2(D-2)} Y_{ca}^a Y^{cb,b})
\]  

(4.15)

Introducing the field \( Y_{c_1...c(D-2),d} \) by

\[
Y_{ab}^d = \frac{1}{(D-2)!} e_{abc_1...c(D-2),d} Y_{c_1...c(D-2)}^d
\]  

(4.16)

The action in terms of these variables is given by

\[
\frac{1}{2} \int d^D x (e^{\mu\nu...\tau_{D-2}} Y_{\tau_1...\tau_{D-2},d}^d C_{\mu\nu,d} + e(-\frac{1}{2} \frac{(D-3)}{(D-2)} Y_{\tau_1...\tau_{D-2},d}^d Y_{\tau_1...\tau_{D-2},d} Y_{\tau_1...\tau_{D-2},d})
\]  

(4.17)

The equations of motion for gravity in terms of the new variables become

\[
\epsilon^{\mu\nu...\tau_{D-1}} \partial_{\tau_1} Y_{\tau_2...\tau_{D-1}}^d = \text{terms of order}(Y_{\tau_1...\tau_{D-2},d})^2
\]  

(4.18)

\[
\epsilon_{\mu\nu...\tau_{D-2},b}^d Y_{\tau_1...\tau_{D-2},b} = -C_{\mu\nu,b} + C_{\nu\mu,b} - C_{\mu\nu,b} + 2(e_{\nu b} C_{\mu c}^c - e_{\mu b} C_{\nu c}^c)
\]  

(4.19)
At lowest order the first of these equations implies that

\[ Y_{\tau_1...\tau_{D-2},b} = \partial_{[\tau_1} h_{\tau_2...\tau_{D-2}],b}. \]  

(4.20)

Consequently, we have found a description of gravity constructed from the field \( e^a_\mu \) and \( h_{\tau_1...\tau_{D-3},b} \). Taking \( D = 11 \), this is precisely of the required type.

Taking a non-linear realisation of eleven dimensional supergravity based on the algebra of equations (4.1) to (4.7) resolves a puzzle concerning the relationship of the coset formulation of IIA supergravity theory to that for the eleven dimensional theory given in reference [15]. Although the coset formulations of these two theories are correct in the sense that they led to equations of motion that do describe the degrees of freedom of both of these theories, the eleven dimensional algebra does not lead in an obvious way to the algebra of the IIA supergravity theory. In particular, the algebra underlying the IIA theory involves the generators, \( K^a_b \) of GL(10) as well as the generators \( R^{a_1...a_p}_{a} \) for \( p = 0, 1, 2, 3, 5, 6, 7, 8 \) and it is very unclear how the generators \( R^{a_1...a_p}_{a} \) for \( p = 7, 8 \) can arise in the eleven dimensional algebra of equations (2.1) to (2.4) since this algebra involves no generators of rank bigger than six. However, we will now show that if we adopt the new eleven dimensional algebra of equations (4.1) to (4.7) then all the generators of the ten dimensional IIA algebra arise naturally and obey in detail the ten dimensional IIA algebra of reference [15].

Treating the eleventh index, denoted by 11, as special, letting \( a, b = 1, \ldots, 10 \) and denoting the resulting generators with \( \tilde{\cdot} \), the generators in the new algebra can be written as

\[ \hat{K}^a_b = K^a_b, \quad \hat{R}^a = K^a_{11}, \quad \hat{R}^{a_1a_2} = R^{a_1a_211}, \quad \hat{R}^{a_1a_2a_3} = R^{a_1a_2a_3} \quad \hat{R}^{a_1...a_5} = R^{a_1...a_511}, \]
\[ \hat{R}^{a_1...a_6} = -R^{a_1...a_6}, \quad \hat{R}^{a_1...a_7} = \frac{1}{2} R^{a_1...a_71111}, \]
\[ \hat{R}^{a_1...a_8} = \frac{3}{8} R^{a_1...a_811}, \quad \hat{R} = \frac{1}{12} (-\sum_{a=1}^{10} K^a_a + 8 K^{11}11). \]  

(4.21)

Evaluating the algebra of equations (4.1) to (4.7) for these generators, dropping the \( \tilde{\cdot} \) one indeed finds the algebra used in the non-linear realisation of IIA supergravity theory of reference [15] which included the relations

\[ [R, R^{a_1...a_p}] = c_p R^{a_1...a_p}, \quad [R^{a_1...a_p}, R^{a_1...a_q}] = c_{p,q} R^{a_1...a_{p+q}} \]  

(4.22)

where

\[ c_1 = -c_7 = -\frac{3}{4}, \quad c_2 = -c_6 = \frac{1}{2}, \quad c_3 = -c_5 = -\frac{1}{4}; \]
\[ c_{1,2} = -c_{2,3} = -c_{3,3} = c_{2,5} = c_{1,5} = 2, \quad c_{1,7} = 3, \quad c_{2,6} = 2, \quad c_{3,5} = 1 \]  

(4.23)

and all other \( c \)'s vanish.

In deriving the IIA algebra, we have required that the generators \( R^{a_1...a_8,b} \) are trivally realised and so do not appear in reference [15]. Use was made of equation (4.8) and, as a
result of this relation, the equation $-8R^{11a_1\cdots a_7,a_8} = R^{a_1\cdots a_7a_8,11}$. Clearly, this calculation provides a detailed check on the proposed eleven dimensional algebra. The result also implies that the IIA algebra contains the Borel subgroup of $E_7$ and that the corresponding Kac-Moody algebra as outlined in section two is $E_{11}$.

5. The Closed Bosonic String and $K_{27}$

The closed bosonic string in 26 dimensions can also be formulated as a non-linear realisation [15]. The underlying group, denoted $G_{26}$, has the generators $K^a_b$, $R$, $R^{a_1a_2}$, $R^{a_1\cdots a_{(D-4)}}$ and $R^{a_1\cdots a_{(D-2)}}$. The generators $K^a_b$ belong to the Lie algebra GL(26) and obey equation (2.7). Their commutation relations with the other generators are the analogues of equation (2.2). The remaining commutation relations are given by

$$[R, R^{a_1\cdots a_p}] = c_p R^{a_1\cdots a_p}, \quad [R^{a_1\cdots a_p}, R^{a_1\cdots a_q}] = c_{p,q} R^{a_1\cdots a_{(p+q)}}$$  \hspace{1cm} (5.1)

where

$$c_2 = -c_{D-4} = \frac{24}{(D-2)}, \quad c_{2,D-4} = 2. \hspace{1cm} (5.2)$$

We take all the other $c$'s to vanish and we have scaled the generator $R \rightarrow \frac{R}{6}$ with respect to reference [15]. The local subgroup is chosen to be the Lorentz group.

The non-linear realisation of $G_D$ is built out of the group element $g = g_hg_A$ where

$$g_h = \exp(h_a^b K_b^a)$$

and

$$g_A = \exp(\frac{A_{a_1\cdots a_{(D-2)}}R^{a_1\cdots a_{(D-2)}}}{(D-2)!}) \exp(\frac{A_{a_1\cdots a_{(D-4)}}R^{a_1\cdots a_{(D-4)}}}{(D-4)!}) \exp(\frac{A_{a_1a_2}R^{a_1a_2}}{(2)!}) \exp(AR)$$ \hspace{1cm} (5.4)

We refer the reader to reference [15] for the derivation of the field equations by taking the simultaneous non-linear realisation with the conformal group.

Much of the discussion of eleven dimensional supergravity also applies to the non-linear realisation of the effective action for the closed bosonic string. We can hope to extend the local subgroup and the formulation of the effective action such that the resulting non-linear realisation is invariant under a Kac-Moody algebra. In this section, using similar arguments as we deployed for eleven dimensional supergravity, we will find what this Kac-Moody algebra is likely to be.

We divide the generators of $G_{26}$ into

$$G^+_{26} = (K^a_b, \ a < b, \ R^{a_1a_2}, R^{a_1\cdots a_{(22)}}, R^{a_1\cdots a_{24}}),$$  \hspace{1cm} (5.5)

$$G^0_{26} = (H_a = K^a_a - K^{a+1}_{a+1}, \ D = \sum_a K^a_a, R)$$  \hspace{1cm} (5.6)

and the remaining generators $K^a_b, \ a > b$ which are the negative root generators of SL(26).
We suppose that \( G_{26}^+ \) are positive root generators contained in the Kac-Moody algebra and its Cartan subalgebra generators contains the generators in \( G_{26}^0 \). To identify the Kac-Moody algebra we must identify the simple roots. It is straightforward to see that the whole of \( G_{26}^+ \) can be found from multiple commutators of

\[
K^a_{a+1}, R^{2526}, R^{5...26}.
\]

Therefore, we identify these as the simple root generators. We observe that we have the same number of generators in the Cartan subalgebra as we have simple roots, namely 27. Thus we are seeking a Kac-Moody algebra of rank 27.

Given these simple roots, we must adopt a basis for the Cartan subalgebra generators such that equation (1.5) leads to an allowed generalised Cartan matrix. It turns out that the basis

\[
H_a, a = 1, \ldots, 25, H_{26} = K_{25}^{25} + K_{26}^{26} - \frac{1}{12} D + \frac{1}{6} R, H_{27} = K_5^{25} + \cdots + K_{26}^{26} - \frac{11}{12} D - \frac{1}{6} R.
\]

satisfies this requirement. We call this algebra \( K_{27} \). The corresponding Dynkin diagram is shown in figure two. We choose the local subgroup to be the one which is invariant under the Cartan involution.

The above process to find the Cartan matrix is not free from ambiguity. However, we can find further evidence for the choice \( K_{27} \) by considering the restriction of the theory obtained by taking the generators and fields to have indices that take values 26 to \( 26 - n + 1 \). The residual invariance group so obtained can be read off from the Dynkin diagram of \( K_{27} \) by keeping only the first \( n - 1 \) nodes on the horizontal line in the Dynkin diagram starting from the right as well as any nodes to which they are attached by vertical lines. The Dynkin diagram so obtained is \( D_n \) provided \( 3 \leq n \leq 22 \), \( D_{24} \) if \( n = 23 \) and affine \( D_{24} \) if \( n = 24 \). It is natural to take the appropriate real form to be \( O(n,n) \) for \( 3 \leq n \leq 22 \) and \( O(24,24) \) if \( n = 23 \). The corresponding local subgroups are taken to be those left invariant under the Cartan involution. For these groups, the local subgroups are then \( O(n) \times O(n) \) for \( 3 \leq n \leq 22 \) and \( O(24) \times O(24) \) if \( n = 23 \). We denote this series of groups by \( G_n \) and their local subgroups by \( H_n \). For \( n < 23 \), the coset spaces agree with that found in reference [29].

We would also expect the scalar fields that occur in the dimensional reduction of the effective action on an \( n \) torus to belong to the coset space \( G_{27}^{\mu} / K_n^{\mu} \). This coset has dimension \( n^2 \) for \( 3 \leq n \leq 22 \) and \((n+1)^2\) if \( n = 23 \). However, we can calculate the number of scalar fields that occur in the dimensional reduction. From the graviton and anti-symmetric tensor we find \( h_i^j \) and \( B_{ij}, i, j = 1, \ldots, n \) scalars and the vectors \( h^{i}_{\mu} \) and \( B_{\mu i} \). If \( 3 \leq n \leq 22 \) we do indeed find \( n^2 \) scalars. However, for \( n = 23 \) the vectors can be dualised to form scalars and so we get \( (n+1)^2 \) scalars. The number of scalars is thus in agreement with that predicted from the Dynkin diagram of \( K_{27} \). The above count does not included the scalar in the original theory for \( n \geq 4 \) and the scalar that results from the anti-symmetric tensor \( B_{\mu \nu} \) by dualisation in four dimensions.

It is interesting to note that the algebra \( K_{27} \) contains the algebra \( E_{11} \) and one might take this as an indication that the closed bosonic string contains the known superstrings.
in ten dimensions and in effect M theory. The closed bosonic string on a torus is invariant under the fake monster Lie algebra and it would be interesting to ask if $K_{27}$ was contained in this algebra.

Conclusion

In this paper we have argued that eleven dimensional supergravity can be formulated as a non-linear realisation based on the group $E_{11}$. We have proposed a new formulation of eleven dimensional supergravity that is an enlargement of the algebra of reference [15]. The simultaneous non-linear realisation of the conformal group with this algebra, when further enlarged by a suitable local subalgebra, is expected to lead to the formulation of eleven dimensional supergravity based on $E_{11}$. Although much remains to be done to verify this conjecture, it has the advantage that it requires calculations, that may be lengthy, but can be carried out as a matter of principle.

Such a non-linear realisation of eleven dimensional supergravity would include new symmetries that mixed the degrees of freedom described by gravity and the rank three tensor gauge field in a very non-trivial manner.

We have shown that some of the necessary conditions for this conjecture to be true are satisfied. In particular, we have shown the presence of the Borel subgroups of $E_7$ and $E_8$. We have also found a formulation of eleven dimensional gravity based on the fields $h_{ab}$ and $h_{a_1...a_8}$ as required by the presence of an $E_8$ symmetry.

The formulation of gravity presented in this paper, and its origin as a non-linear, realisation may be of interest in its own right in that it may lead to the existence of new symmetries in gravity itself. These may explain the presence of the Geroch and related symmetries [28,23] which appear so mysteriously when gravity is constrained to possess a Killing vector.

The motivation for the current work was to identify some of the underlying symmetries of M theory by studying the symmetries of the corresponding supergravity theories. We therefore conjecture that the $E_{11}$ algebra over an appropriate discrete field is a symmetry of M theory and that the closed bosonic string has the $K_{27}$ algebra as a symmetry.

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References

[1] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. 76B (1978) 409.
[2] C. Campbell and P. West, “$N = 2 D = 10$ nonchiral supergravity and its spontaneous compactification.” Nucl. Phys. B243 (1984) 112.
[3] M. Huq and M. Namazie, “Kaluza–Klein supergravity in ten dimensions”, Class. Q. Grav. 2 (1985).
[4] F. Giani and M. Pernici, “$N = 2$ supergravity in ten dimensions”, Phys. Rev. D30 (1984) 325.
[5] J. Schwarz and P. West, “Symmetries and Transformation of Chiral $N = 2 D = 10$ Supergravity”, Phys. Lett. 126B (1983) 301.
[6] P. Howe and P. West, “The Complete N = 2 D = 10 Supergravity”, Nucl. Phys. B238 (1984) 181.
[7] J. Schwarz, “Covariant Field Equations of Chiral N = 2 D = 10 Supergravity”, Nucl. Phys. B226 (1983) 269.
[8] L. Brink, J. Scherk and J.H. Schwarz, “Supersymmetric Yang-Mills Theories”, Nucl. Phys. B121 (1977) 77; F. Gliozzi, J. Scherk and D. Olive, “Supersymmetry, Supergravity Theories and the Dual Spinor Model”, Nucl. Phys. B122 (1977) 253, A.H. Chamseddine, “Interacting supergravity in ten dimensions: the role of the six-index gauge field”, Phys. Rev. D24 (1981) 3065; E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, “Ten-dimensional Maxwell-Einstein supergravity, its currents, and the issue of its auxiliary fields”, Nucl. Phys. B195 (1982) 97; E. Bergshoeff, M. de Roo and B. de Wit, “Conformal supergravity in ten dimensions”, Nucl. Phys. B217 (1983) 143, G. Chapline and N.S. Manton, “Unification of Yang-Mills theory and supergravity in ten dimensions”, Phys. Lett. 120B (1983) 105.
[9] S. Ferrara, J. Scherk and B. Zumino, “Algebraic Properties of Extended Supersymmetry”, Nucl. Phys. B121 (1977) 393; E. Cremmer, J. Scherk and S. Ferrara, “SU(4) Invariant Supergravity Theory”, Phys. Lett. 74B (1978) 61.
[10] E. Cremmer and B. Julia, “The N = 8 supergravity theory. I. The Lagrangian”, Phys. Lett. 80B (1978) 48
[11] B. Julia, “Group Disintegrations”, in Superspace & Supergravity, p. 331, eds. S.W. Hawking and M. Roček, Cambridge University Press (1981).
[12] A. Font, L. Ibanez, D. Lust and F. Quevedo, Phys. Lett. B249 (1990) 35.
[13] C.M. Hull and P.K. Townsend, “Unity of superstring dualities”, Nucl. Phys. B438 (1995) 109, [hep-th/9410167].
[14] E. Cremmer, B. Julia, H. Lu and C. Pope, “Dualisation of dualities II: Twisted self-duality of doubled fields and superdualities”, [hep-th/9806100].
[15] P. West, “Hidden Superconformal Symmetries of M theory”, JHEP., [hep-th/0005270].
[16] V. Ogievetsky, “Infinite-dimensional algebra of general covariance group as the closure of the finite dimensional algebras of conformal and linear groups”. Nuovo. Cimento, 8 (1973) 988.
[17] A. Borisov and V. Ogievetsky, “Theory of dynamical affine and conformal symmetries as the theory of the gravitational field”, Teor. Mat. Fiz. 21 (1974) 329.
[18] B. de Wit and H. Nicolai, Nucl. Phys. B274 (1986) 363; H. Nicolai Phys. Lett. 155B (1985) 47;
[19] H. Nicolai, Phys. Lett. 187B (1987) 316
[20] S. Melosch and H. Nicolai, “New Canonical Variables for D = 11 Supergravity”, [hep-th/9709227]. K. Koespell, H. Nicolai and H. Samtleben, “An Exceptional Geometry for d = 11 Supergravity”, Class. Quantum Grav. 17 (2000) 3689.
[21] G. Moore, “Finite in all directions”, [hep-th/9305139]. P. West, “Physical States and String Symmetries”, [hep-th/9411029]. Int.J.Mod.Phys. A10 (1995) 761. [hep-th/9411029].
[22] For a review see, V. Kac, “Infinite Dimensional Lie Algebras”, Birkhauser, 1983.
[23] B. Julia, in Vertex Operators in Mathematics and Physics, Publications of the Mathematical Sciences Research Institute no 3, Springer Verlag 1984.
[24] H. Nicolai, “Hidden Symmetries in $d = 11$ Supergravity and Beyond”, hep-th/9906106.
[25] E. Cremmer and B. Julia, “The SO(8) Supergravity”, Nucl. Phys. B519 (1979) 141.
[26] B. de Wit and H. Nicolai, “Hidden Symmetries, Central Charges and all That”, hep-th/0011239.
[27] E. A. Ivanov and V. I. Ogievetsky, Teor. Mat. Fiz. 25 (1975) 164.
[28] R. Geroch, J. Math. Phys. 12 (1971) 918; 13 (1972) 394; H. Nicolai, “A Hyperbolic Kac-Moody Algebra from Supergravity”, Phys. Lett. B276 (1992) 333.
[29] J. Maharana and J. Schwarz, Nucl. Phys. B390 (1993) 3.
[30] E. Cremmer, B. Julia, H. Lu and C. Pope, “Dualisation of Dualities. I”, hep-th/9710119.
[31] B. Julia, ”Dualities in the Classical Supergravity Limits”, hep-th/9805083; ”Superdualities: Below and beyond the U-duality”, hep-th/9805083.

Figure 1: The Dynkin Diagram of $E_{11}$.

Figure 2: The Dynkin Diagram of $K_{27}$. 