Critical behavior of Gauss-Bonnet black holes via an alternative phase space

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Recently, it was argued that charged Anti-de Sitter (AdS) black holes admit critical behavior, without extending phase space, similar to the Van der Waals fluid system in the $Q^2 - \Psi$ plans where $\Psi = 1/v$ (the conjugate of $Q^2$) is the inverse of the specific volume [1]. In this picture, the square of the charge of the black hole, $Q^2$, is treated as a thermodynamic variable and the cosmological constant $\Lambda$ is fixed. In this paper, we would like to examine whether this new approach toward critical behaviour of AdS black holes can work in other gravity such as Gauss-Bonnet (GB) gravity as well as in higher dimensional spacetime. We obtain the equation of state, $Q^2 = Q^2(\Psi, T)$, Gibbs free energy and the critical quantities of the system, and study the effects of the GB coupling $\tilde{\alpha}$ on their behaviour. We find out that the critical quantities have reasonable values, provided the GB coupling constant, $\tilde{\alpha}$, is taken small and the horizon topology is assumed to be $(d-2)$-sphere. Finally, we calculate the critical exponents and show that they are independent of the model parameters and have the same values as the Van der Waals system which is predicted by the mean field theory.

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I. INTRODUCTION

Thermodynamics of black holes has been started around five decades ago in 1970’s by the works of Hawking and Bekenstein. Since the discovery of black holes thermodynamics, a lot of investigations have been carried out to disclose the similarity between the laws of black holes mechanics and the usual thermodynamical systems on the earth. The motivation for this investigation is to understand the microscopic structure of black holes and hence shed light on the quantum theory of gravity as well. Thermodynamics of charged black holes in the background of asymptotically AdS spacetimes is of specific interest, mainly due to the duality between gravity in AdS spacetime and the Conformal Field Theory (CFT) living on its boundary. According to AdS/CFT correspondence [2–4], thermodynamics of black holes in an AdS space can be recognized by that of dual strong coupled CFT on the boundary of the AdS spacetime. Besides, it has been shown that there is a complete analogy between charged black holes in AdS space and the Van der Waals liquid-gas system with their critical exponents coincide with those of the Van der Waals system which is predicted by the mean field theory. In this picture, the phase space of black holes thermodynamics is extended such that the cosmological constant is regarded as the thermodynamic pressure and its conjugate quantity as a thermodynamic volume [5–11]. Interestingly enough, it has been displayed that both systems have extremely similar phase diagrams [11]. This analogy has been generalized to higher dimensional charged black holes [12], rotating black holes [13–15] and dilaton black holes [16]. The studies were also enlarged to the critical behavior of nonlinear black holes [17]. When the gauge field is in the form of Born-Infeld nonlinear electrodynamics, one should extend the phase space and introduce a new thermodynamic quantity conjugate to the Born-Infeld parameter which is necessary for consistency of the first law of thermodynamics as well as the corresponding Smarr relation [18, 19].

It is also of great interest to consider the higher curvature corrections to the Einstein gravity. In these theories the entropy expressions are not proportional to the area of the horizon, and instead are given by a more complicated relation depending on higher-curvature terms [20]. One of the most important and useful batch of these kind of theories are the Lovelock gravity theories [21], which lead to second order differential equations for the metric functions. The second-order Lovelock theory of gravity is well-known as GB gravity, which contains higher curvature terms in the action. The phase structure of GB black holes in AdS spaces has been explored in [22, 23]. Motivated by the idea that the cosmological constant can be regarded as a thermodynamic variable, critical behavior of charged topological GB black holes in $d$-dimensional AdS spacetime has been studied in [24–26]. Thermodynamic analogy between a charged GB-AdS black hole and a Van der Waals liquid gas system has been confirmed and it was shown that the result drastically depend on $\alpha$ and dimensions of the spacetime. It was shown that when one treats the GB coupling constant as a free thermodynamic variable [27], the Van der Waals behavior is occurred, and criticality and reentrant behaviour are observed [28]. Furthermore, the phase structure of asymptotically AdS black holes in Lovelock gravity have also been explored [29].

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In all works mentioned above, the cosmological constant is regarded as thermodynamic pressure which can vary. Although, there are some motivations to consider the cosmological constant as a variable, but it is more reasonable to keep it as a constant parameter. For example, in general relativity the cosmological constant is usually considered as a constant related to the zero point energy of the vacuum. Motivated by the argument given in [1], we want to study the critical behavior of GB black hole via an alternative viewpoint, in which we keep the cosmological constant as a constant parameter and instead treat the charge of the black hole (or more precisely $Q^2$) as an external variable which can vary. The advantages of this approach is that it provides more attractive and straightforward results. Phase structure and critical behavior of BI black holes in an AdS space, where the charge of the system can vary and the cosmological constant (pressure) is fixed have been investigated in [30]. It was shown the system indeed admits a reentrant phase transition. Recently, it was shown that this method also work for investigating the critical behaviour of Lifshitz dilaton black holes [31], which further supports the viability of this new approach.

This paper is organized as follows. In the next section we study the critical behavior of $d$-dimensional charged AdS black hole using an alternative phase space. In section III we review the solution of GB black holes and their thermodynamic features. In section IV, we investigate the $(Q^2 - \Psi)$ phase space of the GB black holes in five dimensions and obtain the critical quantities. Also, we will calculate the critical exponents and Gibbs free energy of the system. In section V we generalize our study to higher dimension by investigating the critical behavior of GB black holes via an alternative method. We use the results of section II to study the accuracy of our calculations in the limit of $\alpha = 0$. The last section is devoted to the summery and conclusions.

II. CRITICAL BEHAVIOUR OF ADS BLACK HOLES IN HIGHER DIMENSIONS

As we mentioned before, an alternative approach towards investigating the critical behaviour of AdS black holes was suggested without extending the phase space [1]. The authors of [1] completed the analogy between charged AdS black holes in four dimensions and Van der Waals fluid system by treating the square of the charge of the black hole, $Q^2$, as the thermodynamic variable and keeping the cosmological constant fixed. Our aim here is to generalize this new approach to higher dimensional charged AdS black holes. The motivations is to check whether this approach does work in higher dimensions or it is only valid in four-dimensions. Besides, this investigation is of great importance since it provides the background for our calculations in the next section, where we would like to investigate the critical behaviour of GB black holes in all higher dimensions.

A. Critical behaviour of black holes in $d$-dimensions

The action of Einstein-Maxwell theory in the background of AdS spacetime in $d$-dimensions is given by

$$S = -\frac{1}{16\pi} \int d^d x \sqrt{-g} (R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}),$$

(1)

where $R$ is the Ricci scalar, $\Lambda = -(d-1)(d-2)/2l^2$ is the cosmological constant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electrodynamic field tensor with $A_\mu$ is the gauge potential [32, 33]

$$A_t = \frac{Q}{(d-3)f^{d-3}}.$$  

(2)

The most general $d$-dimensional static metric with constant curvature boundary may be written as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Sigma_{d-2}^2,$$

(3)

where $d\Sigma_{d-2}^2$ stands for the line elements of a $(d - 2)$-dimensional hypersurface with constant scalar curvature $(d - 2)(d - 3)k$ and volume $\omega_{d-2}$. Here $k$ is a constant and characterizes the curvature of the hypersurface. Without loss of generality, one can take $k = 0, 1, -1$, such that the black hole horizon or cosmological horizon in (3) can be a zero (flat), positive (elliptic) or negative (hyperbolic) constant curvature hypersurface. The function $f(r)$ is given by [32, 33]

$$f(r) = k - \frac{m}{r^{d-3}} + \frac{2Q^2}{(d-2)(d-3)r^{2(d-3)}} + \frac{r^2}{l^2},$$

(4)
where \( m \) and \( Q \) are, respectively, the mass and the charge parameters which are related to the total mass and charge of the black hole via

\[
M = \frac{m(d-2)}{16\pi} \omega_{d-2}, \quad Q = \frac{Q}{4\pi} \omega_{d-2}.
\]

(5)

The horizon radius \( r_+ \) of the black hole is the largest real root of Eq. \( f(r_+) = 0 \). Taking into account the energy formation of the thermodynamic system, it was argued that the mass of AdS black hole, \( M \), is indeed the enthalpy \( H \)\[6\]. It is a matter of calculations to show that in terms of the horizon radius the mass is given by

\[
M = \frac{(d-2)\omega_{d-2}^{d-1}}{16\pi l^2} + \frac{k(d-2)\omega_{d-2}^{r_+^{(d-3)}}}{16\pi} + \frac{\omega_{d-2}Q^2}{8\pi(d-3)r_+^{d-3}}.
\]

(6)

The Hawking temperature of the black hole on the event horizon \( r_+ \) can be calculated as

\[
T = \frac{f'(r_+)}{4\pi} = \frac{(d-1)r_+}{4\pi l^2} + \frac{(d-3)k}{4\pi r_+} - \frac{Q^2}{2\pi(d-2)r_+^{d-3}}.
\]

(7)

The entropy and electric potential \( \Phi \) of the black hole are given by \[18\]

\[
S = \frac{r_+^{d-2}}{4} \omega_{d-2},
\]

(8)

\[
\Phi = \frac{Q}{(d-3)r_+^{d-3}}.
\]

(9)

According to \[18\], in the extended phase space towards study the critical behaviour of AdS black holes, the cosmological constant which is interpreted as a thermodynamic pressure \( P \), and its conjugate quantity, the thermodynamic volume, are given by

\[
P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi l^2}, \quad V = \left( \frac{\partial M}{\partial P} \right)_{Q,S} = \frac{\omega_{d-2}r_+^{d-1}}{d-1}.
\]

(10)

It was shown that all these quantities satisfy the following Smarr formula \[18\]

\[
M = \frac{d-2}{d-3} TS + \Phi Q - \frac{2}{d-3} VP,
\]

(11)

Besides, the first law of thermodynamics with variable \( P \) and fixed \( Q \) is written as

\[
dM = TdS + \Phi dQ + VdP.
\]

(12)

It was shown that charged AdS black holes represent a critical behavior similar to Van der Waals fluid, if one treat the cosmological constant as a thermodynamic variable \[11, 18\]. Although this idea has got a lot of interests in the literatures, it was argued that by keeping the cosmological constant as a fixed parameter and instead considering \( Q \) as a thermodynamic variable, the critical behavior can be seen in \( Q^2 - \Psi \) plane \[1\]. Following \[1\], we replace the term \( \Phi dQ \) in the first law with \( \Psi dQ^2 \),

\[
dM = TdS + \Psi dQ^2 + VdP.
\]

(13)

It is a matter of calculation to show that the Smarr formula takes the form

\[
M = \frac{d-2}{d-3} TS + \Psi Q^2 - \frac{2}{d-3} VP,
\]

(14)

where we have defined

\[
\Psi = \left( \frac{\partial M}{\partial Q^2} \right)_{P,S} = \frac{\omega_{d-2}}{8\pi(d-3)r_+^{d-3}}.
\]

(15)
B. Critical behavior

We start by writing the equation of state in the form \( Q^2(T, \Psi) \) by using Eq. (7). For this purpose, we first write

\[
Q^2(T, r_+) = -2T \pi (d-2) r_+^{d-5} + \frac{(d-1)(d-2) r_+^{2d-4}}{2l^2} + \frac{k(d-3)(d-2) r_+^{2d-6}}{2}.
\] (16)

After replacing \( r_+ \) from Eq. (15), one may rewrite the equation of state as a function of \( \Psi \) and \( T \),

\[
Q^2(T, \Psi) = -2T \pi (d-2) Y^{2d-5} \Psi^{5-2d} + \frac{(d-1)(d-2) Y^{2d-4} \Psi^{4-2d}}{2l^2} + \frac{k(d-3)(d-2) Y^{2d-6}}{2 \Psi^2},
\] (17)

where

\[
Y = \left( \frac{\omega_{d-2}}{8\pi(d-3)} \right)^{\frac{1}{2d-6}}.
\] (18)

In order to investigate the critical behavior of the system and compare with Van der Waals gas, we should plot isotherm diagrams. The isotherm diagrams \( Q^2 - \Psi \) given in Fig. 1 predict a first order phase transition in the system which is in complete analogy with the Van der Waals liquid-gas system. Note that the oscillating part \( \frac{\partial Q^2}{\partial \Psi} > 0 \) of the isotherm diagrams show instable regions. The critical point can be obtained by solving the following equations,

\[
\left. \frac{\partial Q^2}{\partial \Psi} \right|_{T_c} = 0, \quad \left. \frac{\partial^2 Q^2}{\partial \Psi^2} \right|_{T_c} = 0.
\] (19)

It is a matter of calculation to show thermodynamic quantities at the critical point are given by

\[
T_c = \frac{\sqrt{(d-2)(d-1)k}}{\pi l(d-3)}.
\] (20)

\[
\Psi_c = \frac{\omega_{d-2} [(d-1)(d-2)k]^{(d-3)}}{8\pi l (d-3)(d-2)}.
\] (21)

\[
Q_c^2 = \frac{l^{(2d-6)} k^{(d-2)} (d-3)^{(2d-5)}}{2(2d-5)((d-1)(d-2))^{(d-3)}}.
\] (22)

\[
\rho_c = T_c \Psi_c Q_c^2 = \frac{\omega_{d-2} l^{(d-4)} k^{(d-2)}}{16\pi^2 (2d-5)^2 ((d-1)(d-2))^{(d-4)}}.
\] (23)

In the limiting case where \( d = 4 \) and \( k = 1 \), the above equation reduces to \( \rho_c = 1/36\pi \) which is consistent with the result in [1]. Note that \( \rho_c \) is independent of \( l \) only in four dimension. It is clear from the above equation that the critical quantities have reasonable values only in case \( k = 1 \). This implies that the system admits a critical behaviour only with spherical horizon. The critical behaviour of a system is identified by its partition function. Indeed, the thermodynamic potential, which is proportional to the Euclidean action calculating at fixed \( Q \) and \( T_c \), is the Gibbs free energy. To get more information about the phase transition we calculate the Gibbs free energy \( G = M - TS \) as [34]

\[
G(Q^2, T) = \frac{\omega_{d-2}}{8\pi (d-3)} \left( k(d-2)(d-3) r_+^{d-3} - 2\pi (2d-5) T r_+^{d-2} + \frac{(d-2)^2 r_+^{d-1}}{l^2} \right),
\] (24)

where \( r_+ \) is a function of \( T \) and \( Q^2 \) through Eq. (16). The swallowtail behavior of the Gibbs free energy in Fig. 2 represents a first-order phase transition in the system. A first order phase transition occurs when the Gibbs free energy is continuous, but its first derivative is discontinuous. Just like as Van der Waals liquid-gas system.
C. Critical exponents

Our aim here is to calculate the critical exponents by using the alternative approach for higher dimensional charged AdS black holes. The behavior of thermodynamic functions in the vicinity of the critical point are characterized by the critical exponents. To find the critical exponent, let us define the reduced thermodynamic variables,

$$\Psi_r \equiv \frac{\Psi}{\Psi_c}, \quad Q_r^2 \equiv \frac{Q^2}{Q_c^2}, \quad T_r \equiv \frac{T}{T_c}. \quad (25)$$

Since the critical exponents should be studied near the critical point, we write the reduced variables in the form

$$\Psi_r = 1 + \psi, \quad Q_r^2 = 1 + \phi, \quad T_r = 1 + t, \quad (26)$$

where $t$, $\psi$ and $\phi$ indicate the deviation from critical point. One may expand Eq. $(6)$ near the critical point and write

$$\phi = -4(d-2)(d-3)t + 4(d-2)(2d-5)\psi t - \frac{2(d-2)(2d-5)}{3(d-3)^2} \psi^3 + o(t^2, \psi^4). \quad (27)$$

Using the Maxwell’s area law and differentiating Eq. $(27)$ with respect to $\psi$ at a fixed temperature ($t < 0$) leads to

$$\phi = -4(d-2)(d-3)t + 4(d-2)(2d-5)\psi t - 2 \frac{d-2(2d-5)}{3(d-3)^2} \psi^3$$

$$= -4(d-2)(d-3)t + 4(d-2)(2d-5)\psi t - \frac{2(d-2)(2d-5)}{3(d-3)^2} \psi^3. \quad (28)$$

Indeed

$$0 = \Psi_c \int_{\psi_l}^{\psi_s} \psi \left( \frac{\partial Q^2}{\partial \psi} \right) d\psi = \Psi_c \int_{\psi_l}^{\psi_s} \psi \left[ 4(d-2)(2d-5)t - \frac{6(d-2)(2d-5)}{3(d-3)^2} \psi^3 \right] d\psi, \quad (29)$$

where $\psi_l$ and $\psi_s$ denote the event horizon of large and small black hole. The non-trivial solutions of the above equation are given by

$$\psi_l = -\psi_s = \sqrt{6(d-3)}t. \quad (30)$$

From Eq. $(30)$ we conclude that the order parameter is $\beta = 1/2$. To obtain the critical exponent $\gamma$ the isotherm compressibility can be calculated as follows

$$\kappa_T = \left( \frac{\partial \Psi}{\partial Q^2} \right)_T = \frac{\Psi_c}{4(d-2)(2d-5)Q_c^2 t} \implies \gamma = 1. \quad (31)$$

In addition it can be easily seen that

$$\phi|_{t=0} = -\frac{2(d-2)(2d-5)}{3(d-3)^2} \psi^3 \implies \delta = 3. \quad (32)$$
The heat capacity near critical point is $c_Ψ = T \frac{∂S}{∂T} |_{Ψ = 0}$. So, the critical exponent $α = 0$. Therefore, we have shown that the critical exponents of the higher dimensional black holes in the new approach (with fixed $Λ$ and variable $Q^2$) is exactly the same as those presented in [18] (with $Λ$ variable) and coincide with the Van der Waals fluid system. It is worthwhile to mention that the new approach is more realistic from physical and mathematical point of view, because the cosmological constant does not offer a natural variable.

III. THERMODYNAMICS OF GAUSS-BONNET BLACK HOLES IN ADS SPACE

We consider the action of $d$-dimensional Einstein-Gauss-Bonnet-Maxwell theory in the presence of cosmological constant $Λ$ which can be written

$$S = \frac{1}{16\pi} \int d^dx\sqrt{-g} [R - 2Λ - α_{GB} (R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2) - 4\pi F_{\mu\nu}F^{\mu\nu}], \quad (33)$$

where $α_{GB}$ is the GB coefficient with dimension of [length]$^2$ which is proportional to inverse string tension with positive coefficient [35]. The metric function $f(r)$ of charged GB black holes in AdS space is given by

$$f(r) = k + \frac{r^2}{2\tilde{α}} \left[ 1 - \sqrt{1 - \frac{8\tilde{α}Q^2}{(d-2)(d-3)r^{2d-4}} + \frac{64\pi\tilde{α}M}{(d-2)\omega_{d-2}r^{d-1}} - \frac{64\pi\tilde{α}P}{(d-2)(d-1)}} \right]. \quad (34)$$

where $\tilde{α} = (d-3)(d-4)α_{GB}$ and $k$ represents the topology of the horizon, and we have replaced $Λ$ with $P$ by using Eq. (10). The constant $M$ is the mass, while $Q$ is related to the charge of the black hole. The position of the black hole event horizon is determined as a larger root of $f(r_+) = 0$ and hence the mass of black hole which is equivalent with enthalpy is calculated as

$$H = M = \frac{(d-2)\omega_{d-2}r_+^{d-3}}{16\pi} \left( k + \frac{k^2\tilde{α}}{r_+^2} + \frac{16\pi Pr_+^2}{(d-1)(d-2)} + \frac{2Q^2r_+^{d-2d}}{(d-2)(d-3)} \right), \quad (35)$$

where $P$ is pressure and defined in (10). The temperature and entropy of the black hole can be given by

$$T = \frac{f'(r_+)}{4\pi} = \frac{16\pi Pr_+^4}{(d-2)} + (d-3)k\tilde{α}r_+^2 + (d-5)k^2\tilde{α} - \frac{2Q^2}{(d-2)r_+^{d-2d}}, \quad (36)$$

and

$$S = \int_0^{r_+} T^{-1} \left( \frac{∂M}{∂r} \right)_{Q,P} dr = \frac{\omega_{d-2}r_+^{d-2}}{4} \left[ 1 + \frac{2(d-2)\tilde{α}k}{(d-4)r_+^2} \right]. \quad (37)$$
We can also calculate the thermodynamic volume $V$ and the electric potential $\Phi$ as

$$ V = \left( \frac{\partial M}{\partial P} \right)_{Q^2,S} = \frac{\omega_{d-2}r_+^{(d-1)}}{d-1}, $$

$$ \Phi = \frac{Q\omega_{d-2}}{4\pi(d-3)r_+^{d-3}}. $$

These quantities satisfy the first law of black holes thermodynamics [25]

$$ dM = TdS + \Phi dQ + VdP + \Omega d\tilde{\alpha}, $$

where

$$ \Omega = \left( \frac{\partial M}{\partial \tilde{\alpha}} \right)_{Q,S,P}, $$

is the quantity conjugate to the GB coefficient $\tilde{\alpha}$. Since $\tilde{\alpha}$ is a dimensionful parameter, the corresponding term will inevitably appear in the Smarr formula [25]

$$ M = \frac{d-2}{d-3}TS + \Phi Q - \frac{2}{d-3}VP + \frac{2}{d-3}\Omega\tilde{\alpha}. $$

As mentioned before, we want to study the phase structure with a different point of view which $Q^2$ plays the role of thermodynamic variable. We also, define a new variable in an our alternative approach, namely $\Psi$ which is

$$ \Psi = \left( \frac{\partial M}{\partial Q^2} \right)_{P,S} = \frac{\omega_{d-2}r_+^{(3-d)}}{8\pi(d-3)}. $$

The new variable $\Psi$, pressure $P$ and temperature $T$ are intensive parameters conjugate to $Q^2$, $V$ and $S$ respectively. We also replace the term $\Phi dQ$ by $\Psi dQ^2$ in the first law of thermodynamics. By using (43), one may easily show that the Smarr formula and the first law of thermodynamics are now given by

$$ M = \frac{d-2}{d-3}TS + \Psi Q^2 - \frac{2}{d-3}VP + \frac{2}{d-3}\Omega\tilde{\alpha} $$

and

$$ dM = TdS + \Psi dQ^2 + VdP + \Omega d\tilde{\alpha}. $$

IV. CRITICAL BEHAVIOR OF GB BLACK HOLES IN FIVE DIMENSION

In order to study the critical behaviour of GB black hole using the alternative approach, we first consider the case $d = 5$ dimension. In this case, the equation of state is simple enough so that we can solve the equations and obtain critical quantities exactly. Since in this approach the cosmological constant (pressure) is regarded a constant quantity, so we consider the case with $P = 0$ and $P \neq 0$, separately.

A. Critical behavior with $P = 0$

Setting $P = 0$ and $d = 5$, the mass of the GB black hole given in Eq.(35) can be written

$$ M = \frac{3\tilde{\alpha}k^2\omega_3}{16\pi} + \frac{3kr_+^2\omega_3}{16\pi} + \frac{Q^2\omega_3}{64\pi r_+}. $$

Combining the Hawking temperature given in Eq.(36) with the above condition, the equation of state of the black hole can be written as

$$ Q^2(\Psi,T) = \frac{3k\omega_3^2}{256\pi^2\Psi^2} - \frac{3\tilde{\alpha}kT\omega_3^{3/2}}{16\sqrt{\pi}\Psi^{3/2}} - \frac{3T\omega_3^{5/2}}{512\pi^{3/2}\Psi^{5/2}}. $$
Critical points occur at stationary points of inflection in $Q^2 - \Psi$ diagram, where
\[ \frac{\partial Q^2}{\partial \Psi} \bigg|_{T_c} = 0, \quad \frac{\partial^2 Q^2}{\partial \Psi^2} \bigg|_{T_c} = 0. \tag{48} \]

In case of spherical horizon where $k = 1$ and $\omega_3 = 2\pi^2$, one may obtain the critical quantities of GB black hole as
\[ T_c = \frac{1}{\pi \sqrt{30\tilde{\alpha}}}, \quad \Psi_c = \frac{5\pi}{48\tilde{\alpha}}, \quad Q_c^2 = -\frac{36\tilde{\alpha}^2}{125}. \tag{49} \]

We see that in this case $Q_c^2$ is negative, which is physically not acceptable. Therefore, we conclude that in this case there does not exit any phase transition, and therefore $Q^2 - \Psi$ diagram, has no similarity with isotherm diagrams of Van der Waals system. Let us check whether or not there is phase transition or critical behavior for other topology of horizon namely flat ($k = 0$) or hyperbolic ($k = -1$) cases.

In hyperbolic ($k = -1$) case, the equation of state reduces to
\[ Q^2(\Psi, T) = -\frac{3\omega_3^2}{256\pi^2\Psi^2} + \frac{3\tilde{\alpha}T\omega_3^{3/2}}{16\sqrt{\pi}\Psi^{3/2}} - \frac{3T\omega_3^{5/2}}{512\pi^{5/2}\Psi^{5/2}}. \tag{50} \]

In this equation the value of positive term is smaller than the other two terms with negative values because $\tilde{\alpha}$ always has positive small value ($0 < \tilde{\alpha} < 1$). Therefore $Q^2$ is a monotonic function of $\Psi$ and there is no critical point and phase transition. For the flat horizon, the equation of state reads
\[ Q^2(\Psi, T) = -\frac{3\pi^7/2T}{64\sqrt{2}\Psi^{5/2}}. \tag{51} \]

It is clear that this equation is monotonic function which can not cause any phase transition. One may conclude that the existence of $\Lambda$ (pressure) is essential for the critical behavior in this new standpoint. Similarly, the existence of $Q$ is necessary when we treat $\Lambda$ as a thermodynamic variable \cite{11}. It may show a meaningful symmetry between new approach with fixed $\Lambda$ and old one with fixed $Q$.

\section*{B. Critical behavior with $P \neq 0$}

Following the approach taken previously, one may find no critical behavior in the cases with for ($k = 0, -1$). Thus, we just focus on black holes with spherical topology ($k = 1$). In this case Eq.(35) with new conditions, takes the form
\[ M = \frac{3}{8}\pi\tilde{\alpha}k^2 + \frac{3}{8}\pi k r_+^2 + \frac{1}{2} \pi^2 P r_+^4 + \frac{\pi Q^2}{8r_+^4}, \tag{52} \]

where the relation between $r_+$ and new parameter $\Psi$ is $r_+ = \frac{\sqrt{\pi}}{2\sqrt{2\Psi}}$. Using the Hawking temperature in Eq.(36), the equation of state may be obtained as
\[ Q^2(\Psi, T) = \frac{\pi^2}{128\Psi^3} \left( 2\pi^2 P - 48\sqrt{2\pi}\Psi^{3/2}\tilde{\alpha}T - 3\sqrt{2\pi^{3/2}\Psi^{1/2}}T + 6\Psi \right) \tag{53} \]

As isotherm diagram shows in Fig.3, for constant pressure and $T = T_c$, there is an inflection point in $Q^2 - \Psi$ diagrams which is the critical point where the second order phase transition occurs. The critical values reads
\[ T_c = \left[ (3\Gamma + 25)l^2 - 162\tilde{\alpha} \right] \sqrt{-\Gamma - \frac{54\tilde{\alpha}}{l^2}} + 5, \tag{54} \]
\[ \Psi_c = -\frac{\pi \left[ 54\tilde{\alpha} + (\Gamma - 5)l^2 \right]}{96\tilde{\alpha}l^2}, \tag{55} \]
\[ Q_c^2 = -\frac{144\tilde{\alpha}^2 l^4 \left[ 126\tilde{\alpha} + (\Gamma - 5)l^2 \right]}{5 \left[ 54\tilde{\alpha} + (\Gamma - 5)l^2 \right]^3}. \tag{56} \]
where
\[ \Gamma = \sqrt{\frac{36\hat{\alpha} (81\hat{\alpha} - 25l^2)}{l^4}} + 25. \] (57)

We can also find
\[ \rho_c = T_c \Psi_c Q_c^2 = \frac{\sqrt{3\hat{\alpha}} [(3\Gamma + 25)l^2 - 162\hat{\alpha}] \left[\frac{126\hat{\alpha} + (\Gamma - 5)l^2}{54\hat{\alpha} + (\Gamma - 5)l^2}\right]}{1000 \left[\frac{54\hat{\alpha} + (\Gamma - 5)l^2}{l^2}\right]} \sqrt{-\Gamma - \frac{54\hat{\alpha}}{l^2} + 5}. \] (58)

It can be realized that the above equations admit a phase transition with acceptable critical quantities provided the following two constrains are satisfied
\[ \begin{cases} 36\hat{\alpha} (81\hat{\alpha} - 25l^2) & + 25 \geq 0, \\ \Gamma - \frac{54\hat{\alpha}}{l^2} + 5 & \geq 0. \end{cases} \] (59)

As one can see these conditions depend on the amount of \( l \) and \( \hat{\alpha} \). However, they will be satisfied automatically for small \( \hat{\alpha} \) as one may see in the next section. In the limit of small \( \hat{\alpha} \), the series expansions of critical quantities are
\[ \begin{align*}
T_c &= \frac{4\sqrt{3}}{5\pi l} - \frac{72\sqrt{3}\hat{\alpha}}{25\pi l^3}, \\
\Psi_c &= \frac{27\pi\hat{\alpha}}{5l^4} + \frac{3\pi}{8l^2}, \\
Q_c^2 &= \frac{l^4}{45} - \frac{32\hat{\alpha}l^2}{25},
\end{align*} \] (60)

which reduces to the following equation for small \( \hat{\alpha} \)
\[ \rho_c = T_c \Psi_c Q_c^2 = \frac{l}{50\sqrt{3}} - \frac{39}{125} \left( \frac{\sqrt{3}}{l} \right) \hat{\alpha} + O(\hat{\alpha}^2). \] (61)

In the absence GB correction terms (\( \hat{\alpha} = 0 \)) all critical values reduce to the results of section II. The critical behavior of a thermodynamic system can be characterized by the Gibbs free energy, which in our case it can be written as
\[ G(Q^2, T) = \frac{3\pi\hat{\alpha}}{8} + \frac{9\pi r_+^4}{16l^2} - \frac{1}{16} \pi^2 r_+ T \left( \frac{54\hat{\alpha} + 11r_+^2}{32} \right) + \frac{15\pi r_+^2}{32}. \] (62)

The behavior of the Gibbs free energy is depicted in Fig.4 in terms of \( Q^2 \) for various temperature. Evidently, for \( T > T_c \) the Gibbs free energy develops a swallowtail shape which shows first order phase transition. The critical behavior of GB black hole \((k = 1)\) can be characterized by the critical exponent. In order to examine the critical exponents we introduce the following reduced thermodynamic variables
\[ \begin{align*}
\Psi_r &\equiv \frac{\Psi}{\Psi_c} = 1 + \psi, \\
Q_r^2 &\equiv \frac{Q^2}{Q_c^2} = 1 + \phi, \\
T_r &\equiv \frac{T}{T_c} = 1 + t.
\end{align*} \] (63)

The Taylor expansion of Eq.(53) is
\[ \phi = At + B\psi t - C\psi^3 + o(t\psi^2, \psi^4), \] (64)
where \( A = (-24 - \frac{576\hat{\alpha}}{l^2}) \), \( B = (60 + \frac{1296\hat{\alpha}}{l^2}) \) and \( C = (\frac{5}{2} + \frac{18\hat{\alpha}}{l^2}) \). Applying Maxwell’s equal area law and considering the fact that during the phase transition the charge remains constant we have
\[ At + B\psi t - C\psi^3 = At + B\psi t - C\psi_s^3. \] (65)

\[ 0 = \Psi_c \int_{\psi_1}^{\psi_s} \psi \left( \frac{\partial Q^2}{\partial \psi} \right) d\psi = \Psi_c \int_{\psi_1}^{\psi_s} \psi(Bt - 3C\psi^2) d\psi. \] (66)

The nontrivial solution for Eqs.(65) and (66) reads
\[ \psi_t = -\psi_s \longrightarrow |\psi_t - \psi_s| = 2 \sqrt{\frac{-B}{C}} l_s. \] (67)
We conclude that the order parameter $\beta$ which is appropriate with the power of $t$ is \( \beta = \frac{1}{2} \). The next critical exponent is $\gamma$ which can be obtained by the following relation

\[
\chi_T = \left. \frac{\partial \Psi}{\partial Q^2} \right|_T \propto \frac{\Psi_c}{BQ^c_t} t \quad \Rightarrow \quad \gamma = 1
\] (68)

The shape of the critical isotherm at $t = 0$ is given by Eq. (64). We find

\[
\phi|_{t=0} = -C\psi^3 \quad \Rightarrow \quad \delta = 3.
\] (69)

Finally, the heat capacity near the critical point at fixed $\Psi$ reads

\[
c_{\Psi} = T \left. \frac{\partial S}{\partial T} \right|_{\Psi = 0}
\] (70)

Since the entropy is independent of $T$, the critical exponent $\alpha = 0$. Therefore, we have obtained all critical exponents in $Q^2 - \Psi$ plans for GB black holes in five dimensions with spherical horizon. We have treated the charge of the black hole as a thermodynamic variable and kept the pressure as a fixed parameter. We have also confirmed that these critical exponents are similar to those of Van der Waals liquid-gas system.

**V. CRITICAL BEHAVIOR OF GB BLACK HOLES IN ARBITRARY DIMENSIONS**

In this section we are going to extend our investigation on the critical behavior of GB black hole to all higher dimensions. Our approach for calculating the critical quantities and critical exponents in $d > 5$ is exactly the same as in five dimensions. Therefore, for the economic reasons we do not repeat the calculations and only give the results.

Using the Hawking temperature (36), we can write the equation of state in $d$-dimensions as

\[
Q^2(\Psi, T) = \frac{1}{2} Y^{2d-8} \Psi^{\frac{2d-8}{2}} \left( (d-5)(d-2)k^2\tilde{\alpha} + (d-2)k\Psi^{\frac{1}{d-2}} \left( -8\pi T\tilde{\alpha} + Y(d-3)\Psi^{\frac{1}{d-2}} \right) \right.
+ 4Y^3\pi\Psi^{\frac{3}{d-2}} \left. \left( -(d-2)T + 4YP\Psi^{\frac{1}{d-2}} \right) \right).
\] (71)

The behavior of the isotherm diagrams in $d = 6, 7$ are shown in Fig. 5 which show the same behavior as Van der Waals system and predict a first order phase transition in the system. The equation of state is complicated and it is not easy to obtain the critical point analytically, however, we can still calculate the critical quantities for small $\tilde{\alpha}$. As we have explained in the previous section, we do not expect to see critical behavior in the absence of cosmological constant.
(P = 0), or in case with k = 0, −1. Thus, we consider the spherical horizon in the presence of Λ. Using Eq. (48) in case k = 1, one can find

\[ T_c = \frac{d^3 - 6d^2 + 11d - 6}{\pi \sqrt{(d-1)(d-2)(2d-5)}} \cdot \alpha \frac{(d-2)^{3/2}}{(d-1)^{3/2}} \left( -6d^3 + 53d^2 - 155d + 152 \right) + O(\alpha^2), \]  

(72)

\[ \Psi_c = \frac{8\pi}{(d-3)(d-2)^2} \left[ \frac{(d-3)(d-2)^{1/2}}{d-1} \right] \left( 204 + d(-225 + (83 - 10d)d) \right) l^{2d-2} \omega_{d-2} + O(\alpha^2), \]  

(73)

\[ Q_c^2 = \frac{(d-3)^{2d-5}}{2(2d-5)} \left( - \frac{(d-3)^{2d-5}}{2(2d-5)} \right) \left( 2d - 8 \right) \left( d - 2 \right)^{5-d} \left( d - 1 \right)^{4-d} \left( 10d^2 - 51d + 69 \right) l^{2d-8} + O(\alpha^2), \]  

(74)

which leads to

\[ \rho_c = Q_c^2 T_c \Psi_c = \frac{(d-3)^{d-2}}{16\pi^2 (2d-5)^2} \left( \frac{(d-2)(5d-13)}{d^2} \right)^{d-1} \left( 5d^2 - 4rT - 5 + 56rT + 30 \right) + O(\alpha^2). \]  

(75)

It is worthwhile to note that all above equations reduce to the results in section II when \( \alpha \). The Gibbs free energy of the GB black hole \( G = M - TS \) can be calculated as

\[ G(Q^2, T) = \frac{\omega_{d-2}}{64\pi(d-3)} \left( \frac{(d-2)(5d-13)}{d^2} \right)^{d-1} \left( 5d^2 - 4rT - 5 + 56rT + 30 \right) + \alpha \frac{(d-2)r^{d-5}}{(d-4)} \left( 5d^2 + 8r(16 - 5d)rT - 37d + 68 \right) + O(\alpha^2) \]  

(76)

Fig. 6 shows the behaviour of Gibbs free energy versus \( Q^2 \) which shows that the phase transition can occur when the temperature is more than \( T_c \). From these figures we find that GB black holes admit a first order phase transition in higher dimensions. Following the method of the previous section, we can calculate the critical exponents of GB black holes in higher dimensions. The result is

\[ \alpha = 0, \quad \beta = 1/2, \quad \gamma = 1, \quad \delta = 3. \]  

(77)
A close look at the behavior of $f(r)$ in Figs. 7, shows the existence of zero, one or two roots for the metric function depending on the value of $\hat{\alpha}$. It is worthwhile to note that the event horizon disappears with increasing $\hat{\alpha}$. So we have a naked singularity when $\hat{\alpha}$ is larger than specific value. In other words, we did not see any critical behavior for large value of $\hat{\alpha}$ because there is no event horizon in this range of $\hat{\alpha}$.

It is worth studying the behavior of the critical values with respect to $\hat{\alpha}$, which displayed in Figs. 8 and 9. We see from Fig. 8 that $T_c$ and $Q_c^2$ decrease when $\hat{\alpha}$ increases. In addition, the large value of $\hat{\alpha}$ leads to negative values for $T_c$ and $Q_c^2$. This implies that $\hat{\alpha}$ should have an upper bound which depends on the dimension of spacetime. On the
other hand, as one may see from Fig. 9 the values of $\psi_c$ and $\rho_c$ increase with increasing $\tilde{\alpha}$ with no upper bound. We also plot the Gibbs free energy versus $Q^2$ for diffract values of $\tilde{\alpha}$ for both $d = 5$ and $d = 6$ in Fig. 10 which show that all cases have upward trends.

VI. SUMMERY AND CONCLUSION

We have investigated the critical behavior of charged GB black holes in AdS spaces via an alternative phase space. In this approach, one can treat the square of the charge of the black hole as a thermodynamic variable and fix the cosmological constant. It is more reasonable to take the charge of the black hole as an external variable which can vary, instead of the cosmological constant which basically has a constant value. For example, one may think that the charge of the black hole can change by absorbing or emitting the charged particles. The advantages of this new approach is that we do not need to extend the thermodynamical phase space, in order to see the critical behaviour of the system. It was argued that this new approach admits a critical behavior for the black holes similar to the Van der Waals liquid-gas system with the same critical exponents provided one treat the square of the charge of the black hole ($Q^2$) as the thermodynamic variable [1].

In this paper, we first generalized the method developed in [1] to all higher dimensions by investigating the critical behaviour of $d$-dimensional charged AdS black holes and treating $Q^2$ as the thermodynamic variable and keeping $\Lambda$ constant. We found that the critical behaviour of the system in $Q^2 - \Psi$ plane and the critical exponents are similar to the the Van der Waals fluid system. Then, we applied this new approach to string inspired GB gravity. We found out
that the phase transition occurs only for the small values of GB coupling constant ($\tilde{\alpha}$). Besides, the critical quantities are reasonable, provided $\tilde{\alpha}$ to be small and the topology of the horizon is assumed spherical. We found that these black holes may have one or two horizons for small $\tilde{\alpha}$. Therefore, there is neither horizon nor phase transition for larger value of the dilaton coupling constant $\tilde{\alpha}$. We calculated the critical quantities such as $T_c$, $\Psi_c$, $Q_c$ and $\rho_c$ and the critical exponents and observed the critical temperature and critical charge go to zero as $\tilde{\alpha}$ increases. Furthermore, we calculated the Gibbs free energy of the system. The swallowtail shapes of the Gibbs diagrams show the existence of first order phase transition in the system. Also the zero order phase transition are not seen in the diagrams. Finally, we calculated the critical exponents of the GB black holes in all higher dimensions and observed that they are independent of the details of the system and are the same as those of Van der Waals fluid.

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