A SHARP ESTIMATE OF WEIGHTED DYADIC SHIFTS OF COMPLEXITY 0 AND 1

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Abstract. A simple shortcut to proving sharp weighted estimates for the Martingale Transform and for the Hilbert transform is presented. It is a unified proof for these both transforms.

1. Introduction

Let \( \sigma := w^{-1} \).

Notations. We call a shift by \( n \) generations, or \( SH_n \) any sub-bilinear operator of the following form

\[
(SH_n f_1, f_2) = \sum_{J \subset I, |J| = 2^{-n}|I|} 2^{-\frac{n}{2}} c_{IJ} (f_1, h_I)(f_2, h_J),
\]

where \( |c_{IJ}| \leq 1 \).

Theorem 1.1. \( (SH_1 f_1, f_2) \leq C [w]_{A_2} \|f_1\|_w \|f_2\|_\sigma \).

Proof. Let \( Q := [w]_{A_2} \).

We know from [26], [40] that

Theorem 1.2. There exists a function \( B_Q \) of 6 variables \((X, Y, x, y, u, v)\) defined in \( \Omega_Q := \{(X, Y, x, y, u, v) : x^2 \leq Xv, y^2 \leq Yu, 1 \leq uv \leq Q\} \) such that

\[
B_Q(X, Y, x, y, u, v) \leq C Q (X + Y),
\]

and

\[
d^2 B_Q(X, Y, x, y, u, v) \geq |dx||dy|.
\]

In particular, one can conclude that having two points \( a = (a_1, ..., a_6), b = (b_1, ..., b_6) \) in \( \Omega_Q \) connected by segment \([a, b] \) lying entirely inside \( \Omega_Q \) one can introduce the parametrization \( c(t) = at + b(1 - t) \), consider

\[
q(t) = B_Q(c(t))
\]

and claim, using (1.2) that

\[
- q''(t) \geq |a_3 - b_3||a_4 - b_4|.
\]

We will need the same thing for some other segments \([a, b] \) not lying entirely inside \( \Omega_Q \) (but with \( a, b \in \Omega_Q \)).
The problem is of course that \( \Omega_Q \) is not convex.

Now let us apply \( B_{40Q} \). We choose \( I \) and put

\[
b := (f_1^2 w_I, f_2^2 \sigma_I, (f_1)_I, (f_2)_I, \langle w \rangle_I, \langle \sigma \rangle_I),
\]

\[
b_+ = (f_1^2 w_{I+}, f_2^2 \sigma_{I+}, (f_1)_+, (f_2)_+, \langle w \rangle_+, \langle \sigma \rangle_+),
\]

\[
b_- = (f_1^2 w_{I-}, f_2^2 \sigma_{I-}, (f_1)_-, (f_2)_-, \langle w \rangle_-, \langle \sigma \rangle_-),
\]

\[
b_{ij} = (f_1^2 w_{I_{ij}}, f_2^2 \sigma_{I_{ij}}, (f_1)_{I_{ij}}, (f_2)_{I_{ij}}, \langle w \rangle_{I_{ij}}, \langle \sigma \rangle_{I_{ij}}),
\]

where \( i, j = \pm \).

We want to estimate from below

\[
D := B_{40Q}(b) - \frac{1}{4} \sum_{i, j = \pm} B_{40Q}(b_{ij}) = A + B + C,
\]

where

\[
A := B_{40Q}(b) - \frac{1}{2} (B_{40Q}(b_+) + B_{40Q}(b_-)),
\]

\[
B := \frac{1}{2} (B_{40Q}(b_+) - \frac{1}{2} (B_{40Q}(b_{++}) + B_{40Q}(b_{+-}))),
\]

\[
C := \frac{1}{2} (B_{40Q}(b_-) - \frac{1}{2} (B_{40Q}(b_{-+}) + B_{40Q}(b_{--}))).
\]

Let \( b = (\cdot, \cdot, x, y, \cdot, \cdot) \), \( b_+ = (\cdot, \cdot, x + \alpha, y + \lambda, \cdot, \cdot) \), \( b_- = (\cdot, \cdot, x - \alpha, y - \lambda, \cdot, \cdot) \),

\[
b_{++} = (\cdot, \cdot, x + \alpha + \beta_1, y + \lambda + \delta_1, \cdot, \cdot),
\]

\[
b_{+-} = (\cdot, \cdot, x + \alpha - \beta_1, y + \lambda - \delta_1, \cdot, \cdot),
\]

\[
b_{-+} = (\cdot, \cdot, x - \alpha + \beta_2, y - \lambda + \delta_2, \cdot, \cdot),
\]

\[
b_{--} = (\cdot, \cdot, x - \alpha - \beta_2, y - \lambda - \delta_2, \cdot, \cdot).
\]

We do not know the signs of \( \alpha, \lambda, \beta_1, \beta_2, \delta_1, \delta_2 \).

We want to show that there exists an absolute positive constant \( c \) such that

\[
(1.4) \quad D \geq c |\alpha| (|\delta_1| + |\delta_2|).
\]

Consider several cases. First of all notice that not only all \( b, b_-, b_+, b_{ij} \) are in \( \Omega_Q \) but the segments \([b, b_{ij}]\) are in \( \Omega_{40Q} \). This follows from the following geometric lemma.

Lemma 1.3. Let three point \( A, B, C \) be in \( \Omega_Q \) and let \( M = \frac{A + B}{2} \). Assume \( [A, B] \subset \Omega_Q \) and \( [C, M] \subset \Omega_Q \). Then \( [C, A], [C, B] \subset \Omega_{40Q} \).

Proof. Let’s prove the statement for \([C, A]\).

Case 1: \( C_1 \leq A_1, C_2 \leq A_2 \). Then there is nothing to prove, since if we have a line segment with positive slope, who’s endpoints are in \( \Omega_Q \), then the whole segment lies in \( \Omega_Q \).

Case 2: \( C_1 \geq A_1 \) or \( C_2 \geq A_2 \). Without loss of generality, assume \( C_1 \geq A_1 \).

Denote \( S = \frac{A + C}{2} \) the middle of \([A, C]\). Denote also \( O = [C, M] \cap [B, S] \). Since \([C, M]\) and \([B, S]\) are two medians of the triangle \( ABC \), we have that \( O \) is the center of \( ABC \).

Therefore,

\[
(1.5) \quad O = \frac{1}{3} B + \frac{2}{3} S,
\]

\[
(1.6) \quad O = \frac{1}{3} C + \frac{2}{3} M.
\]
Therefore, for $k \in \{1, 2\}$ we have

(1.7) \hspace{1cm} O_k \geq \frac{2}{3} M_k,

(1.8) \hspace{1cm} O_k \geq \frac{1}{3} C_k.

On the other hand,

\[ S_1 = \frac{A_1 + C_1}{2} \leq C_1 \leq 3O_1. \]

Therefore,

\[ S_1S_2 \leq 3O_1 \cdot \frac{3}{2}O_2 = \frac{9}{2}O_1O_2. \]

But $O \in [C, M] \subset \Omega_Q$, so

\[ S_1S_2 \leq \frac{9}{2}Q. \]

Therefore, $S \in \Omega_{\frac{9}{2}Q}$, and so are $A$ and $C$. Thus, $[A, C] \in \Omega_{40Q}$, which finishes the proof. \qed
The statement for segments \([b, b_{ij}]\) follows from this lemma. Indeed, we have a triangle \(bb_+ b_−\) such that \([b_+, b_−] \subset \Omega_Q\) and, moreover, since endpoints and the middle of the line segment \(b−bb_+\) are in \(\Omega_Q\), we conclude that \([b, b_+] \in \Omega_{2Q}\). Therefore, the median of mentioned triangle is in \(\Omega_{2Q}\), thus, all sides are in \(\Omega_{40Q}\).

Lemma 1.4. Let points \(P_i, i = 1, 2, 3, 4\) be in \(\Omega_Q\) and \(P\) be a baricenter of \(P_i\). Then all segments \([P, P_i]\) are in \(\Omega_{40Q}\).

Now fix \(i, j\), say, \(i = +, j = −\). Consider function

\[
f_{+-}(t) = B_{40Q}(tb_+ + (1−t)b)
\]

and write

\[
f_{+-}(0)−f_{+-}(1) = −f′(0)−\frac{1}{2}f″(ξ) = −\nabla B_{40Q}(b)⋅(b_+−b)+\frac{1}{2}|x+α−β_1−x||y+λ−δ_1−y|.
\]

This is because of Theorem 1.2.

We do this for all \(f_{ij}, i = ±, j = ±\), add and divide by 4. Then we get the first estimate on \(D\):

\[
D ≥ −\nabla B_{40Q}(b)⋅(\frac{1}{4}(b_+ + b++ + b− + b−−) − b)
\]

\[\text{(1.9)}\]

\[
+ \frac{1}{2}((|α−β_1||λ−δ_1| + |α+β_1||λ+δ_1|) + (|α−β_2||λ−δ_2| + |α+β_2||λ+δ_2|)).
\]

The first term is zero. If we have the case that \(|β_1| ≤ \frac{1}{2}|α|\) and \(|β_2| ≤ \frac{1}{2}|α|\), then we get from the first bracket of the second term at least \(|α||δ_1|\), and from the second bracket at least \(|α||δ_1|\). In this case \((1.9)\) is proved.

Suppose now that \(|β_1| ≥ \frac{1}{2}|α|\) and \(|β_2| ≥ \frac{1}{2}|α|\). Then we notice that \(D = A + B + C\). Moreover, \(A ≥ 0\) as \(B_{40Q}\) is concave, and \([b−, b_+] \subset \Omega_{40Q}\) (see Lemma 1.4), point \(b\) being the center of this segment. On the other hand, by Theorem 1.2

\[
2B ≥ B_{40Q}(b_+ − \frac{1}{2}(B_{40Q}(b_+ + B_{40Q}(b−−)) ≥ c|β_1||δ_1| ≥ \frac{c}{2}|α||δ_1|
\]

by our assumption. Symmetrically we will have
Combining the last two inequalities we also have
\[ D = A + B + C \geq c' \|\alpha\| \|\delta_1\|, \]
which is \((1.4)\) we want.

Now suppose \(|\beta_1| \leq \frac{1}{2}|\alpha|\) and \(|\beta_2| \geq \frac{1}{2}|\alpha|\). Then we write \(2D = D + A + B + C\). We estimate \(D\) by \((1.3)\), omitting the second (positive) bracket of the second term, and writing for the first bracket of the second term the following estimate:
\[ D \geq \|\alpha\| \|\delta_1\|. \]
We again use that \(A \geq 0\) and also use that \(B \geq 0\) by the same concavity and the fact that \([b_{+}, b_{+}]) \subset \Omega_{40Q}\).

On the other hand, by Theorem 1.2
\[ 2C \geq B_{40Q}(b_{-}) - \frac{1}{2}(B_{40Q}(b_{+}) + B_{40Q}(b_{-})) \geq c |\beta_2| \|\delta_2\| \geq \frac{c}{2} |\alpha| \|\delta_2\| \]
by our assumption \(|\beta_2| \geq \frac{1}{2}|\alpha|\). Now combining \(2D = D + A + B + C\) and the last two inequalities we get \((1.4)\).

We are left with the fourth case: \(|\beta_1| \geq \frac{1}{2}|\alpha|\) and \(|\beta_2| \leq \frac{1}{2}|\alpha|\). But it is totally symmetric to the previous case. So \((1.4)\) is always proved.

Now we repeat the usual Bellman function summation over dyadic tree (we have above the inequality for the node \(I\), we repeat it for nodes \(I_{+}, I_{-}\) et cetera). In other words we use integration of discrete Laplacian and discrete Green's formula to get (we use also \((1.1)\) of course):
\[ \sum_{I} \frac{1}{|I|} \sum_{J \subset I} |L||\Delta_J f_1||(|\Delta_{J_{-}} f_2| + \Delta_{J_{+}} f_2)| \leq C_{40Q} (\langle f^2 w \rangle_I + \langle g^2 \sigma \rangle_I). \]
Our Theorem 1.1 is completely proved.

2. Points over i’s

We gave a simple proof of linear estimate of any shift of complexity 1. So, for example, it gives the way to deduce Stefanie’s result from [26]. Below we give a very simple proof of [40]. This is up to the existence of \(B_Q\). In the next section we give a proof of such an existence.

3. The heart of the matter: a reduction to bilinear embedding estimate

To prove Theorem 1.2 we need a key inequality. It is an inequality established by Wittwer [40] (see also [26] on which [citeWit is based]).
\[ \sum_{I} |(\phi w, h_I)||(|\psi \sigma, h_I)| \leq C[w]_{A_2} \|\phi\|_w \|\psi\|_{\sigma}. \]

In fact, if \((3.1)\) is proved we just put
\[ B_Q(X, Y, x, y, u, v) := \sup \left\{ \frac{1}{|J|} \sum_{I \in J} |(\phi w, h_I)||(|\psi \sigma, h_I)| : \langle \phi^2 w \rangle_I = X, \langle \psi^2 \sigma \rangle_I = Y, \langle \phi \rangle_I = x, \langle \psi \rangle_I = y, \langle w \rangle_I = u, \langle \psi \rangle_I = v \right\}. \]
All properties (1.1)–(1.3) can be easily checked as soon as (3.1) is proved. We give here an easy proof of (3.1)–considerably easier than in [40].

**Lemma 3.1.** Below $I$’s are dyadic intervals. We have the following decomposition:

$$h_I = \alpha_I h^w_I + \beta_I \chi_I \sqrt{I},$$

where

1) $|\alpha_I| \leq \sqrt{\langle w \rangle_I}$,
2) $|\beta_I| \leq |\Delta_I w| \langle w \rangle_I$,
3) \{h^w_I\}_I is an orthonormal basis in $L^2(w)$,
4) $h^w_I$ assumes on $I$ two constant values, one on $I_+$ and another on $I_-.$

We write

$$\sum_I |\langle \phi w, h_I \rangle| |\langle \psi \sigma, h_I \rangle| \leq$$

$$\sum_I |\langle \phi w, h^w_I \rangle| \sqrt{\langle w \rangle_I} |\langle \psi \sigma, h^w_I \rangle| \sqrt{\langle \sigma \rangle_I} +$$

$$\sum_I |\langle \psi \sigma \rangle_I | \frac{\Delta_I w}{\langle w \rangle_I} |\langle \psi \sigma, h^w_I \rangle| \sqrt{\langle \sigma \rangle_I} \sqrt{I} +$$

$$\sum_I |\langle \phi \psi \rangle_I | \frac{\Delta_I \sigma}{\langle \sigma \rangle_I} |\langle \psi \sigma, h^w_I \rangle| \sqrt{\langle w \rangle_I} \sqrt{I} +$$

$$\sum_I |\langle \phi \psi \rangle_I | \frac{\Delta_I w}{\langle w \rangle_I} \frac{\Delta_I \sigma}{\langle \sigma \rangle_I} \sqrt{I} \sqrt{I} =: I + II + III + IV.$$

Obviously

(3.2) \hspace{1cm} I \leq C[w]^{1/2}_{A_2} \|\phi\|_w \|\psi\|_{\sigma}.

Terms $II, III$ are symmetric, so consider $II$. Using Bellman function one can prove now that

(3.3) \hspace{1cm} II \leq C[w]_{A_2} \|\phi\|_w \|\psi\|_{\sigma}.

(3.4) \hspace{1cm} III \leq C[w]_{A_2} \|\phi\|_w \|\psi\|_{\sigma}.

If we do the same in $IV$ by using Cauchy’s inequality, we would get

$$IV \leq C[w]^{3/2}_{A_2} \|\phi\|_w \|\psi\|_{\sigma},$$

which is not our coveted linear estimate. So $I, II, III$ are fine and linear estimate of exterior sum $\sigma_{11e}$ is equivalent to the linear estimate of $IV$. 


4. Carleson measures built on \( w \in A_2 \) and their estimates

Let us introduce bi-sublinear sum

\[
B(\phi w, \psi \sigma) := \sum_{I} \frac{|\langle \phi w \rangle_I| |\langle \psi \sigma \rangle_I| |\Delta_I w| |\Delta_I \sigma|}{\langle w \rangle_I \langle \sigma \rangle_I} |I| .
\]

Everything is reduced to the estimate of this bi-sublinear sum.

We can rewrite it as

\[
\sum_{I} \frac{|\langle \phi w \rangle_I| |\langle \psi \sigma \rangle_I| |\Delta_I w| |\Delta_I \sigma|}{\langle w \rangle_I \langle \sigma \rangle_I} |I| \leq [w]_{A_2} \| \phi \|_{L^2(L,\sigma)} \| \psi \|_{L^2(L,\sigma)} .
\]

This is immediately reductive to Carleson measure estimate. In fact, the LHS of (4.1) can be rewritten as

\[
\sum_{I} \frac{|\langle \phi w \rangle_I| |\langle \psi \sigma \rangle_I| |\Delta_I w| |\Delta_I \sigma|}{\langle w \rangle_I \langle \sigma \rangle_I} |I| \leq B \int L M_w \phi(x) M_\sigma \psi(x) dx ,
\]

where \( B \) is the Carleson norm of the measure given by the formula

\[
\alpha_I = |\Delta_I w| |\Delta_I \sigma| |I| .
\]

In fact, (4.2) is a simple geometric argument: exercise!

But the RHS of (4.2) is estimated by Cauchy inequality independently of \([w]_{A_2}\) (we learnt this other trick from [4]):

\[
\int L M_w \phi(x) M_\sigma \psi(x) dx = \int L M_w \phi(x) M_\sigma \psi(x) \sqrt{w(x)} \sqrt{\sigma(x)} dx \leq
\]

\[
\| M_w \phi \|_w \| M_\sigma \chi_L \|_\sigma \leq A \| \phi \|_{L^2(w)} \| \psi \|_{L^2(\sigma)} .
\]

Combining this with (4.2) we obtain that everything follows from

**Theorem 4.1.**

\[\| \{ \alpha_I \}_I \|_{\text{Carl}} \leq A [w]_{A_2} .\]

**Proof.** In the paper [35] it is shown that if for all \( I \in D \) we have that two positive functions \( u, v \) satisfy

\[ \langle u \rangle_I \langle v \rangle_I \leq 1 \]

then for any \( L \in D \) we also have

\[
\frac{1}{|L|} \sum_{I \in D, I \subseteq L} |\Delta_I u| |\Delta_I v| |I| \leq A \sqrt{\langle u \rangle_L \langle v \rangle_L} .
\]

Take our \( w \in A_2 \) and put \( u = w/[w]_{A_2}, v = \sigma \). Then the assumption is satisfied, and we immediately get

\[
\sum_{I \in D, I \subseteq L} |\Delta_I w| |\Delta_I \sigma| |I| \leq A [w]_{A_2}^{1/2} \sqrt{\langle w \rangle_L \langle \sigma \rangle_L} |L| .
\]

In particular, we obtain

\[
\sum_{I \in D, I \subseteq L} |\Delta_I w| |\Delta_I \sigma| |I| \leq A [w]_{A_2} |L| .
\]

This is exactly (4.1) for measure \( \{ \alpha_I \}_I \)!
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