A CRITERION FOR NORMALITY OF ANALYTIC MAPPINGS

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Abstract In this paper, we give a generalization and improvement of the Pavlović result on the characterization of continuously differentiable functions in the Bloch space on the unit ball in \( \mathbb{R}^m \). Then, we derive a Holland–Walsh type theorem for analytic normal mappings on the unit disk.

Keywords: normal analytic function; Bloch analytic function; hyperbolic distance; spherical distance; Bloch type spaces; Lipschitz type spaces

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1. Introduction

In the eighties, Holland and Walsh [4] published an interesting result on the characterisation of analytic Bloch mappings on the unit disc which involves the expression \( |f(z) - f(w)|/|z - w| \) multiplied with an appropriate weight function depending on the both variables \( z \) and \( w \). More precisely, they obtained that an analytic mapping \( f(z) \) belongs to the Bloch space on the unit disc \( U \) if and only if the expression

\[
\sqrt{1 - |z|^2} \sqrt{1 - |w|^2} \frac{|f(z) - f(w)|}{|z - w|}
\]

is bounded for \( z, w \in U, z \neq w \).

More recently, the same characterisation was given for analytic Bloch functions in the unit ball of \( \mathbb{C}^m \) by Ren and Tu [8]. After that, Ren and Kähler [7] proved this characterisation for harmonic functions on the unit ball in \( \mathbb{R}^m \).

In 2008, Pavlović [6] proved that even continuously differentiable Bloch functions obey the same characterisation. Actually, Pavlović proved more in the following proposition.

**Proposition 1 (Cf. [6]).** A continuously differentiable complex-valued function \( f \) on the unit ball in \( \mathbb{R}^m \) is a Bloch function, i.e.,

\[
\sup_{|x|<1} (1 - |x|^2) \|Df(x)\|
\]
is finite, if and only if the following quantity is finite
\[
\sup_{|x|,|y|<1, x\neq y} \frac{\sqrt{1-|x|^2} \sqrt{1-|y|^2} |f(x) - f(y)|}{|x-y|}.
\]
Moreover, the above numbers are equal.

Therefore, by this proposition, the Bloch semi-norm
\[
\|f\| = \sup_{|x|<1} (1-|x|^2) \|Df(x)\|
\]
of a continuously differentiable (complex-valued) function \(f\) on the unit ball in \(\mathbb{R}^m\) may be expressed in a differential-free way
\[
\|f\| = \sup_{|x|,|y|<1, x\neq y} \frac{\sqrt{1-|x|^2} \sqrt{1-|y|^2} |f(x) - f(y)|}{|x-y|}.
\]

For an analytic mapping \(f\) on \(U\), we have \(|f'(z)| = \|D_f(z)\|\) (for the right side we consider \(D_f(z)\) as a linear mapping from \(\mathbb{R}^2\) into \(\mathbb{R}^2\)), so from the Pavlović result, we can recover the characterization of analytic mappings in the Bloch space on the unit disc obtained by Holland and Walsh [4, Theorem 3] as well as the Ren and Tu results.

Let \(\text{Aut}(U)\) be the group of all conformal transforms of the unit disk onto itself. It is well known that an analytic mapping \(f\) on the unit disk is Bloch if and only if \(\{(f \circ \varphi)(z) - (f \circ \varphi)(0) : \varphi \in \text{Aut}(U)\}\) is a normal family [2].

One says that an analytic mapping \(f\) is normal on \(U\) if \(\{(f \circ \varphi) : \varphi \in \text{Aut}(U)\}\) is a normal family. It is well known that an analytic mapping \(f\) on the unit disc is normal if and only if it satisfies the growth condition
\[
\frac{|f'(z)|}{1 + |f(z)|^2} \leq \frac{C}{1-|z|^2}, \quad z \in U
\]
for a constant \(C > 0\).

The main aim of this article is to obtain a new criterion for normality of analytic mappings on \(U\). This criterion is stated in the proposition which follows. A proof of this proposition follows from the characterisation result (given in the main lemma in the next section) similar to Proposition 1 for continuously differentiable functions that satisfy a certain growth condition.

**Proposition 2.** Let \(f\) be an analytic mapping on the disk \(U\). This mapping is Bloch if and only if
\[
\sqrt{1-|z|^2} \sqrt{1-|w|^2} \frac{|f(z) - f(w)|}{|z-w|}
\]
is bounded as a function of \(z, w \in U\) for \(z \neq w\).

The mapping \(f\) is normal if and only if
\[
\sqrt{1-|z|^2} \sqrt{1-|w|^2} \frac{|f(z) - f(w)|}{\sqrt{1 + |f(z)|^2} \sqrt{1 + |f(w)|^2}} \frac{1}{|z-w|}
\]
is bounded as a function of \(z, w \in U\) for \(z \neq w\).
2. The main lemma

We will introduce here the needed notation and terminology.

Let $\mathbb{B}^m$ be the unit ball in $\mathbb{R}^m$.

For a differentiable mapping $f : D \to \mathbb{R}^n$, where $D \subseteq \mathbb{R}^m$ is a domain, we denote by $D_f(x)$ its differential at $x \in D$, and by

$$\|D_f(x)\| = \sup_{\zeta \in \partial \mathbb{B}^m} |D_f(x)\zeta|$$

the norm of the linear operator $D_f(x) : \mathbb{R}^m \to \mathbb{R}^n$. The class of all continuously differentiable mappings $f : D \to \tilde{D}$ is denoted by $C^1(D, \tilde{D})$.

A weight function is an everywhere positive and continuous function on a domain in $\mathbb{R}^m$. If $\omega$ is a weight function on a domain $D \subseteq \mathbb{R}^m$, the $\omega$-distance between $x \in D$ and $y \in D$ is given by

$$d_\omega(x, y) = \inf_\gamma \int_\gamma \omega(z)|dz|,$$

where $\gamma \subseteq D$ is among all piecewise $C^1$-curves connecting $x$ and $y$.

Let $\omega$ and $\tilde{\omega}$ be weight functions on domains $D \subseteq \mathbb{R}^m$ and $\tilde{D} \subseteq \mathbb{R}^n$, respectively. We will consider mappings $f \in C^1(D, \tilde{D})$ which satisfy the Bloch type growth condition, i.e., the growth condition of the type

$$\tilde{\omega}(f(x))\|D_f(x)\| \leq C\omega(x), \quad x \in D,$$

where $C$ is a positive constant. For such mappings, we introduce

$$\mathcal{B}_f = \sup_{x \in D} \frac{\tilde{\omega}(f(x))}{\omega(x)}\|D_f(x)\|,$$

which will be called the Bloch number of the mapping $f$. We denote by $\mathcal{B}_{\omega, \tilde{\omega}}$ the class of all mappings $f \in C^1(D, \tilde{D})$ for which the Bloch number $\mathcal{B}_f$ is finite.

Note that for $\tilde{D} = \mathbb{R}^n$ and $\tilde{\omega} \equiv 1$ the Bloch number $\mathcal{B}_f$ has the semi-norm properties. Moreover, the class $\mathcal{B}_{\omega, \tilde{\omega}}$ has the linear space structure.

The main aim of this section is to obtain a differential-free description of the class $\mathcal{B}_{\omega, \tilde{\omega}}$ and the differential-free expression for the Bloch number of a continuously differentiable mapping. In order to do that, we will consider mappings $f \in C^1(D, \tilde{D})$ which satisfy the Lipshitz type growth condition, i.e.,

$$|f(x) - f(y)| \leq C_f(x, y)|x - y|,$$

where $C_f(x, y)$ is a positive function.

For given weight functions $\omega$ on $D$ and $\tilde{\omega}$ on $\tilde{D}$, and a mapping $f \in C^1(D, \tilde{D})$, we introduce an everywhere positive function $\Omega_f$ on $D \times D$ such that the following conditions
are satisfied:

\[ \Omega_f(x, y) = \Omega_f(y, x), \quad \Omega_f(x, x) = \tilde{\omega}(f(x)), \quad \liminf_{z \to x} \Omega_f(x, z) \geq \Omega(x, x), \]

and

\[ \Omega_f(x, y) \frac{|f(x) - f(y)|}{|x - y|} \leq \frac{d_\tilde{\omega}(f(x), f(y))}{d_\omega(x, y)}, \quad x, y \in D, \ x \neq y. \]

We say that \( \Omega_f \) is an admissible function for the mapping \( f \) with respect to the given weight functions \( \omega \) and \( \tilde{\omega} \).

Note that if \( \Omega_f \) is not symmetric but satisfies all other conditions stated above, we can replace it by the symmetric function

\[ \tilde{\Omega}_f(x, y) = \max\{\Omega_f(x, y), \Omega_f(y, x)\}, \quad x, y \in D. \]

This new function will be admissible for the same mapping, as it is easy to check.

Also note that if we take \( \tilde{\omega} \equiv 1 \) on the domain \( \tilde{\omega} \), then the distance \( d_\tilde{\omega} \) is equal to the Euclidean distance. In this case, the fourth condition of admissibility is independent of the mapping \( f \), and it reduces on finding an universal admissible function \( \Omega(x, y) \) which satisfies the simplified condition

\[ \Omega(x, y)d_\omega(x, y) \leq |x - y|, \quad x, y \in D. \]

Of course, the admissible function need not be unique, and one may pose the existence question. In the remark given below, we solve the existence question in the general setting.

Introduce now the following quantity

\[ L_f = \sup_{x, y \in D, x \neq y} \Omega_f(x, y) \frac{|f(x) - f(y)|}{|x - y|}, \]

where \( \Omega_f \) is an admissible function for the mapping \( f \) with respect to \( \omega \) and \( \tilde{\omega} \). We call it the Lipschitz number of \( f \). The main lemma stated below says that the Lipschitz number does not depend on the choice of an admissible function (therefore, the definition of \( L_f \) is correct).

The class of all mappings \( f \in \mathcal{C}^1(D, \tilde{D}) \) for which the Lipschitz number \( L_f \) is finite is denoted by \( \mathcal{L}_{\omega, \tilde{\omega}} \). If \( \tilde{D} = \mathbb{R}^n \) and \( \tilde{\omega} = 1 \) then \( L_f \) also has the semi-norm properties, and \( \mathcal{L}_{\omega, \tilde{\omega}} \) is a linear space.

Now we prove our main Lemma 1 which connects the Bloch and Lipschitz number of a continuously differentiable mapping between Euclidean domains. Our main lemma shows that any mapping \( f \in \mathcal{C}^1(D, \tilde{D}) \) satisfies

\[ B_f = L_f. \]

As a consequence, we have that the Lipschitz number is independent of the choice of the admissible function \( \Omega_f \), and the Bloch number may be expressed in the differential-free way

\[ B_f = \sup_{x, y \in D, x \neq y} \Omega_f(x, y) \frac{|f(x) - f(y)|}{|x - y|}, \]

where \( \Omega_f \) is an admissible function for \( f \).
As another consequence, we have the coincidence of the two classes of mappings in $C^1(D, \tilde{D})$, i.e., $B_{\omega, \tilde{\omega}} = L_{\omega, \tilde{\omega}}$. Thus, the Bloch class $B_{\omega, \tilde{\omega}}$ may be described as
\[
\left\{ f \in C^1(D, \tilde{D}) : \sup_{x, y \in D, x \neq y} \Omega_f(x, y) \frac{|f(x) - f(y)|}{|x - y|} < \infty \right\}.
\]

All the results and facts stated above follow from the content of the following lemma.

**Lemma 1.** Let $(D, d_{\omega})$ and $(\tilde{D}, d_{\tilde{\omega}})$ be domains in $\mathbb{R}^m$ and $\mathbb{R}^n$ with distances $d_{\omega}$ and $d_{\tilde{\omega}}$ generated by the weight functions $\omega$ and $\tilde{\omega}$ in $D$ and $\tilde{D}$, respectively. Let $f \in C^1(D, \tilde{D})$, and let $\Omega_f$ be any admissible function for the mapping $f$ with respect to $\omega$ and $\tilde{\omega}$. If one of the numbers $B_f$ and $L_f$ is finite, then both numbers are finite, and these numbers are equal.

**Proof.** For one direction, assume that the Lipschitz number of $f \in C^1(D, \tilde{D})$, i.e., that the quantity
\[
L_f = \sup_{x \neq y} \Omega_f(x, y) \frac{|f(x) - f(y)|}{|x - y|}
\]
is finite, where $\Omega_f$ is an admissible function for $f$ with respect to $\omega$ and $\tilde{\omega}$. We are going to show that $B_f \leq L_f$, which implies that the Bloch number $B_f$ must also be finite.

Let $x \in \Omega$. If we have in mind that
\[
\limsup_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} = \|D_f(x)\|,
\]
we obtain
\[
L_f = \sup_{y \neq z} \Omega_f(y, z) \frac{|f(y) - f(z)|}{|y - z|} \geq \limsup_{z \to x} \Omega_f(x, z) \frac{|f(x) - f(z)|}{|x - z|}
\]
\[
\geq \liminf_{z \to x} \Omega_f(x, z) \limsup_{z \to x} \frac{|f(x) - f(z)|}{|x - z|} \geq \Omega_f(x, x) \|D_f(x)\| = \frac{\tilde{\omega}(f(x))}{\omega(x)} \|D_f(x)\|.
\]

We have used the fact that
\[
\limsup_{y \to x} A(y)B(y) \geq \liminf_{y \to x} A(y) \limsup_{y \to x} B(y)
\]
for non-negative functions $A$ and $B$ on an Euclidean domain.

It follows that
\[
L_f \geq \sup_{x \in D} \frac{\tilde{\omega}(f(x))}{\omega(x)} \|D_f(x)\| = B_f,
\]
which we aimed to prove.

Assume now that the Bloch number $B_f$ of a continuously differentiable mapping $f : D \to \tilde{D}$ is finite. We will prove the reverse inequality $L_f \leq B_f$, which in particular implies that the Lipschitz number $L_f$ is also finite.
Let $\gamma \subseteq D$ be any piecewise $C^1$-curve connecting $x \in D$ and $y \in D$, i.e., such that $\gamma(0) = x$ and $\gamma(1) = y$. Since $f \in C^1(D, \tilde{D})$, the curve $\delta = f \circ \gamma \subseteq \tilde{\Omega}$ (which connects $f(x)$ and $f(y)$), is also piecewise $C^1$ in the domain $\tilde{D}$ and we have

$$d_{\tilde{\omega}}(f(x), f(y)) \leq \int_0^1 \tilde{\omega}(\delta(t))|\delta'(t)|\,dt = \int_0^1 \tilde{\omega}(f \circ \gamma(t))|Df(\gamma(t))\gamma'(t)|\,dt \leq \int_0^1 \tilde{\omega}(f \circ \gamma(t))|Df(\gamma(t))||\gamma'(t)|\,dt \leq B_f \int_0^1 \omega(\gamma(t))|\gamma'(t)|\,dt = B_f d_{\omega}(x, y).$$

If we now take the infimum over all curves $\gamma$, we obtain

$$\frac{d_{\tilde{\omega}}(f(x), f(y))}{d_{\omega}(x, y)} \leq B_f$$

for every $x \in D$ and $y \in D$ such that $x \neq y$. Applying now conditions posed on the admissible function $\Omega_f$, we obtain

$$\Omega_f(x, y) \frac{|f(x) - f(y)|}{|x - y|} \leq \frac{d_{\tilde{\omega}}(f(x), f(y))}{d_{\omega}(x, y)} \leq B_f.$$

It follows that

$$L_f = \sup_{x, y \in D, x \neq y} \Omega_f(x, y) \frac{|f(x) - f(y)|}{|x - y|} \leq B_f,$$

which we aimed to prove.

**Remark 1.** Let us first note that if $D \subseteq \mathbb{R}^m$ is a domain, and $\omega$ a weight function on $D$, then for the $\omega$-distance $d_{\omega}$ on $\Omega$, we have

$$\lim_{y \to x} \frac{d_{\omega}(x, y)}{|x - y|} = \omega(x), \quad x \in \Omega.$$

Indeed, since $\omega$ is continuous, there exists an open ball $B(x, r) \subseteq \Omega$ such that

$$0 < \omega(x) - \varepsilon < \omega(y) < \omega(x) + \varepsilon, \quad y \in B(x, r),$$

where $\varepsilon > 0$ is a sufficiently small number. Now, we have

$$d_{\omega}(x, y) \leq \int_{[x, y]} \omega \leq (\omega(x) + \varepsilon)|x - y| = (\omega(x) + \varepsilon)|x - y|.$$

On the other hand, if $\gamma \subseteq \Omega$ is among curves that connect $x$ and $y$, then

$$d_{\omega}(x, y) = \inf_{\gamma} \int_{[x, y]} \omega \geq (\omega(x) - \varepsilon)|x - y|.$$

Therefore,

$$\omega(x) - \varepsilon \leq \frac{d_{\omega}(x, y)}{|x - y|} \leq \omega(x) + \varepsilon, \quad y \in B(x, r).$$
This means that
\[ \lim_{y \to x} \frac{d_\omega(x, y)}{d(x, y)} = \omega(x). \]

Let us now solve the existence question concerning the admissible function. Let \( f \in C^1(D, \tilde{D}) \) satisfy the condition \( \tilde{\omega}(f(x)) \| D_f(x) \| \leq C \omega(x), \ x \in D \). Then

\[
\Omega_f(x, y) = \begin{cases} 
\frac{d_\omega(f(x), f(y))}{|f(x) - f(y)|} / \frac{d_\omega(x, y)}{|x - y|}, & \text{if } x \neq y, f(x) \neq f(y); \\
\frac{\tilde{\omega}(f(x))}{d_\omega(x, y)} / |x - y|, & \text{if } x \neq y, f(x) = f(y); \\
\frac{\tilde{\omega}(f(x))}{\omega(x)}, & \text{if } x = y, f(x) = f(y).
\end{cases}
\]

is an admissible function for \( f \). Having in mind the preceding remark it follows

\[ \liminf_{y \to x} \Omega_f(x, y) = \lim_{y \to x} \Omega_f(x, y) = \frac{\tilde{\omega}(f(x))}{\omega(x)} = \Omega_f(x, x). \]

Other three admissibility conditions for \( \Omega_f \) are obviously satisfied.

In view of Remark 1, we have the following expected corollary.

**Corollary 1.** Let \( (D, d_\omega) \) and \( (\tilde{D}, d_{\tilde{\omega}}) \) be domains in \( \mathbb{R}^m \) and \( \mathbb{R}^n \) with distances \( d_\omega \) and \( d_{\tilde{\omega}} \) generated by the weight functions \( \omega \) and \( \tilde{\omega} \) in \( D \) and \( \tilde{D} \), respectively. Then a continuously differentiable mapping \( f : D \to \tilde{D} \) satisfies the inequality

\[ \tilde{\omega}(f(x)) \| D_f(\zeta) \| \leq C \omega(x), \]

where \( C \) is a positive constant, if and only if there holds

\[ d_{\tilde{\omega}}(f(x), f(y)) \leq C d_\omega(x, y) \]

for the same constant \( C \).

For example, the result of the last corollary is proved for harmonic mappings of the unit disc into itself by Colonna in [3], where it is also found that the constant \( C \) is less or equal to \( 4/\pi \) for such type of mappings. A variant of this corollary is obtained in [9] (see also Theorem 1 there for analytic functions of several complex variables). A variant is also given in [5].

3. **Characterisations of Bloch and normal mappings**

Based on our main lemma, one may derive Proposition 1. Indeed, if we take \( D = \mathbb{B}^m \), \( \omega(x) = 1/(1 - |x|^2), \ x \in \mathbb{B}^m \), then \( d_\omega \) is the hyperbolic distance on the unit ball which will be denoted by \( d_h \). It is well known that

\[ d_h(x, y) = \text{asinh} \frac{|x - y|}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}, \ x, y \in \mathbb{B}^m. \]

On the other hand, take \( \tilde{D} = \mathbb{R}^n \) and \( \tilde{\omega} \equiv 1 \). Then \( d_{\tilde{\omega}} \) is the Euclidean distance.
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The function \( \Omega(x, y) = \sqrt{1 - |x|^2} \sqrt{1 - |y|^2} \) satisfies the inequality

\[
d_h(x, y) \Omega(x, y) \leq |x - y|, \quad x, \ y \in \mathbb{B}^m,
\]

and therefore it is admissible for any \( f \in C^1(\mathbb{B}^m, \mathbb{R}^n) \) with the growth estimate

\[
(1 - |x|^2) \|Df(x)\| \leq C, \quad x \in \mathbb{B}^m
\]

for a constant \( C \).

Indeed, using the inequality \( \text{asinh} t \leq t \) for \( t \geq 0 \) (to prove it, let \( \phi(t) = \text{asinh} t - t \); then we have \( \phi(0) = 0 \) and \( \phi'(t) = \frac{1}{\sqrt{1 + t^2}} - 1 < 0 \), \( t > 0 \), so the inequality follows from \( \phi(t) \leq \phi(0) = 0 \) one deduces:

\[
\frac{|x - y|}{d_h(x, y)} = |x - y| : \text{asinh} \frac{|x - y|}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}} \geq |x - y| : \frac{|x - y|}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}
\]

\[
= \sqrt{1 - |x|^2} \sqrt{1 - |y|^2} = \Omega(x, y), \quad x, \ y \in \mathbb{B}^m, \ x \neq y.
\]

The Pavlović result in this case now follows. We gave a similar proof in [5].

Applying the following theorem for normal analytic function on \( U \), we immediately obtain the proposition stated in the Introduction.

**Theorem 1.** A continuously differentiable mapping \( f : \mathbb{B}^m \to \mathbb{R}^n \) satisfies the growth condition

\[
\frac{1}{1 - |x|^2} \|Df(x)\| \leq \frac{C}{1 + |f(x)|^2}, \quad x \in \mathbb{B}^m
\]

for a constant \( C \), if and only if there holds

\[
|f(x) - f(y)| \leq C|x - y| \frac{\sqrt{1 + |f(x)|^2} \sqrt{1 + |f(y)|^2}}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}, \quad x, \ y \in \mathbb{B}^m.
\]

**Proof.** In our main lemma, let us take for the domain \( D \) the unit ball \( \mathbb{B}^m \), and for the domain \( \tilde{D} \) the space \( \mathbb{R}^n \). Let, moreover, \( \omega(x) = 1/(1 - |x|^2), \ x \in \mathbb{B}^m \) and \( \tilde{\omega}(y) = 1/(1 + |y|^2), \ y \in \mathbb{R}^n \). As we have already said, \( d_\omega \) is the hyperbolic distance \( d_h \) on the unit ball \( \mathbb{B}^m \). The distance \( d_{\tilde{\omega}} \) is the spherical distance on \( \mathbb{R}^n \) which will be denoted by \( d_s \). For the spherical distance, we have

\[
d_s(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad x, \ y \in \mathbb{R}^m.
\]

Now, we will prove that

\[
\Omega_f(x, y) = \frac{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}{\sqrt{1 + |f(x)|^2} \sqrt{1 + |f(y)|^2}}, \quad x, \ y \in \mathbb{B}^m
\]

is an admissible function for \( f \) with respect to the hyperbolic and spherical weights.
First note that
\[
\Omega_f(x, x) = \frac{1 - |x|^2}{1 + |f(x)|^2} = \frac{1}{1 + |f(x)|^2} : \frac{1}{1 - |x|^2}, \quad x \in \mathbb{B}^m.
\]

Since \(\Omega_f(x, y)\) is symmetric and continuous, it remains only to prove that \(\Omega_f(x, y)\) satisfies the inequality
\[
\Omega_f(x, y) \left| \frac{f(x) - f(y)}{|x - y|} \right| \leq \frac{d_s(f(x), f(y))}{d_h(x, y)}, \quad x, y \in \mathbb{B}^m, x \neq y.
\]

Having in mind the inequality \(\text{asinh} \leq t, t \geq 0\), we obtain
\[
\frac{d_s(f(x), f(y))}{d_h(x, y)} = \sqrt{1 + |f(x)|^2} \sqrt{1 + |f(y)|^2} : \text{asinh} \frac{|x - y|}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}
\]
\[
\geq \frac{|f(x) - f(y)|}{\sqrt{1 + |f(x)|^2} \sqrt{1 + |f(y)|^2}} : \sqrt{1 - |x|^2} \sqrt{1 - |y|^2}
\]
\[
= \frac{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}{\sqrt{1 + |f(x)|^2} \sqrt{1 + |f(y)|^2}} \frac{|f(x) - f(y)|}{|x - y|}
\]
\[
= \Omega_f(x, y) \left| \frac{f(x) - f(y)}{|x - y|} \right|,
\]
which we aimed to prove. \(\square\)

References

1. J. Anderson, J. Clunie and Ch. Pommerenke, On Bloch functions and normal functions, J. die reine und angewandte Math. 270 (1974), 12–37.
2. F. Colonna, Bloch and normal functions and their relation, Rendiconti del Circolo Matematico di Palermo Series 2 38 (1989), 161–180.
3. F. Colonna, The Bloch constant of bounded harmonic mappings, Indiana Univ. Math. J. 38 (1989), 829–840.
4. F. Holland and D. Walsh, Criteria for membership of Bloch space and its subspace, BMOA, Math. Ann. 273 (1986), 317–335.
5. M. Marković, Differential-free characterisation of smooth mappings with given growth, Canad. Math. Bull. 61 (2018), 628–636.
6. M. Pavlović, On the Holland–Walsh characterization of Bloch functions, Proc. Edinburgh Math. Soc. 51 (2008), 439–441.
7. G. Ren and U. Kähler, Weighted Lipschitz continuity and harmonic Bloch and Besov spaces, Proc. Edinb. Math. Soc. 48 (2005), 743–755.
8. G. Ren and C. Tu, Bloch space in the unit ball of \(\mathbb{C}^n\), Proc. Am. Math. Soc. 133 (2005), 719–726.
9. K. Zhu, Distances and Banach spaces of holomorphic functions on complex domains, J. London Math. Soc. 49 (1994), 163–182.