NON-COMPACT NEWTON BOUNDARY AND WHITNEY EQUISINGULARITY FOR NON-ISOLATED SINGULARITIES

CHRISTOPHE EYRAL AND MUTSUO OKA

Abstract. In an unpublished lecture note, J. Briancón observed that if \( \{f_t\} \) is a family of isolated complex hypersurface singularities such that the Newton boundary of \( f_t \) is independent of \( t \) and \( f_t \) is non-degenerate, then the corresponding family of hypersurfaces \( \{f_t^{-1}(0)\} \) is Whitney equisingular (and hence topologically equisingular). A first generalization of this assertion to families with non-isolated singularities was given by the second author under a rather technical condition. In the present paper, we give a new generalization under a simpler condition.

1. Introduction

Let \((t, z) := (t, z_1, \ldots, z_n)\) be coordinates for \( \mathbb{C} \times \mathbb{C}^n \), let \( U \) be an open neighbourhood of \( 0 \in \mathbb{C}^n \) and \( D \) be an open disc centered at \( 0 \in \mathbb{C} \); finally, let
\[
f: (D \times U, D \times \{0\}) \to (\mathbb{C}, 0), (t, z) \mapsto f(t, z),
\]
be a polynomial function. As usual, we write \( f_i(z) := f(t, z) \) and we denote by \( V(f_i) \) the hypersurface in \( U \subseteq \mathbb{C}^n \) defined by \( f_i \). We are interested in the local structure of the singular loci of the hypersurfaces \( V(f_i) \) at the origin \( 0 \in \mathbb{C}^n \) as the parameter \( t \) varies from a “small” non-zero value \( t_0 \neq 0 \) to \( t = 0 \). More precisely, we are looking for easy-to-check conditions on the members \( f_t \) of the family \( \{f_t\} \) that guarantee equisingularity (in a sense to be specified) for the corresponding family of hypersurfaces \( \{V(f_t)\} \).

In an unpublished lecture note \[1\], J. Briançon made the following observation.

Assertion 1.1 (Briançon). Suppose that for all \( t \) sufficiently small, the following three conditions are satisfied:

1. \( f_t \) has an isolated singularity at the origin \( 0 \in \mathbb{C}^n \);
2. the Newton boundary \( \Gamma(f_t; z) \) of \( f_t \) at \( 0 \) with respect to the coordinates \( z \) is independent of \( t \);
3. \( f_t \) is non-degenerate (in the sense of the Newton boundary as in \[3, 6\]).

Then the family of hypersurfaces \( \{V(f_t)\} \) is Whitney equisingular.

We say that a family \( \{V(f_t)\} \) of (possibly non-isolated) hypersurface singularities is Whitney equisingular if there exist a Whitney stratification of the hypersurface \( V(f) := f^{-1}(0) \) in an open neighbourhood \( \mathcal{U} \) of the origin \( (0, 0) \in \mathbb{C} \times \mathbb{C}^n \) such that the \( t \)-axis \( \mathcal{U} \cap (D \times \{0\}) \) is a stratum. By “Whitney stratification” we mean a Whitney stratification in the sense of \[2\]—that is, we do not require that the frontier condition holds. However, note that if \( \mathcal{S} \) is Whitney stratification of

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\( \mathcal{U} \cap V(f) \) with the \( t \)-axis as a stratum, then so is the partition \( \mathcal{S}^c \) consisting of the connected components of the strata of \( \mathcal{S} \); moreover, \( \mathcal{S}^c \) satisfies the frontier condition (see [2] for details). Whitney equisingularity is quite a strong form of equisingularity. Combined with the Thom-Mather first isotopy theorem (cf. [2][4][12]), it implies topological equisingularity. Here, we say that the family \( \{V(f_t)\} \) is topologically equisingular if for all sufficiently small \( t \), there is an open neighbourhood \( U_t \subseteq U \) of \( 0 \in \mathbb{C}^n \) together with a homeomorphism \( \varphi_t: (U_t, 0) \to (\varphi_t(U_t), 0) \) such that \( \varphi_t(V(f_0) \cap U_t) = V(f_t) \cap \varphi(U_t) \). Note that a family of isolated hypersurface singularities (as in Assertion [11]) is Whitney equisingular if and only if \( V(f) \setminus (D \times \{0\}) \) is smooth and Whitney (b)-regular over \( D \times \{0\} \) in an open neighbourhood of the origin in \( \mathbb{C} \times \mathbb{C}^n \). Here, it is worth to observe that, in general, even if the smooth part of \( V(f) \) is Whitney (b)-regular along the \( t \)-axis, the family of hypersurfaces \( \{V(f_t)\} \) may fail to be topologically equisingular. The simplest example illustrating this phenomenon is due to O. Zariski [13] and is as follows. Consider the family defined by the polynomial function \( f(t, z_1, z_2) := t^2z_1^2 - z_2^2 \). The singular locus of \( V(f) \) consists of two lines, namely the \( t \)-axis and the \( z_1 \)-axis. Clearly, the smooth part of \( V(f) \) is Whitney (b)-regular along the \( t \)-axis. However, there is no local ambient homeomorphism sending \( V(f_0), 0 \) onto \( (V(f_t), 0) \).

To conclude with Assertion [11] let us mention that its proof is based on a famous theorem due to A. G. Kouchnirenko [3] and the second author [6]. This theorem says that if \( h(z) \) is a non-degenerate polynomial function with an isolated singularity at \( 0 \), then its Milnor fibration and its Milnor number at \( 0 \), as well as the local ambient topological type of the corresponding hypersurface \( V(h) \) at \( 0 \), are determined by the Newton boundary \( \Gamma(h; z) \) of \( h \) with respect to the coordinates \( z \).

In [8], the second author gave a generalization of Briançon’s assertion to families of non-isolated singularities under a relatively technical condition (so-called “simultaneous IND-condition”). Essentially, the technical nature of this condition comes the fact that, in [8], the question of Whitney equisingularity is treated in the general framework of complete intersection varieties. In the present paper, we restrict ourselves to the case of hypersurfaces, and we give a new generalization under a rather simple condition. As in [9], when dealing with non-isolated singularities, we need to consider not only the compact faces of the Newton polygon (i.e., the faces involved in the usual Newton boundary) but also “essential” non-compact faces. The union of the compact and essential non-compact faces forms the non-compact Newton boundary, which was considered not only in [8] but also in [11] to study the Milnor fibration and some geometric properties such as Thom’s condition or the transversality of the nearby fibres of a polynomial function \( h(z) \) with non-isolated singularities. (Actually, in [11], the second author investigated the Milnor fibration and the Thom condition in the more general case of a “mixed” polynomial function \( h(z, \overline{z}) \).) A crucial ingredient introduced in [11], to handle the essential non-compact faces is the “local tameness.” We shall see below that the local tameness—more precisely, a uniform version in \( t \) of the local tameness—also plays a key role in our generalization of Briançon’s assertion.

2. Non-compact Newton boundary and local tameness

In this section, we recall important definitions—due to A. G. Kouchnirenko [3] and the second author [6][11]—that will be used in this paper. A special emphasis
will be given to the “local tameness” introduced by the second author in [11] and which will be crucial for our purpose.

Let \( z := (z_1, \ldots, z_n) \) be coordinates for \( \mathbb{C}^n \), let \( U \) be an open neighbourhood of the origin \( 0 \in \mathbb{C}^n \), and let

\[
h: (U, 0) \to (\mathbb{C}, 0), \quad z \mapsto h(z) = \sum \alpha c_\alpha z^\alpha
\]

be a polynomial function, where \( \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), \( c_\alpha \in \mathbb{C} \) and \( z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n} \).

As usual, we write \( V(h) \) for the hypersurface in \( U \subseteq \mathbb{C}^n \) defined by \( h \). For any subset \( I \subseteq \{1, \ldots, n\} \), we set

\[
\mathbb{C}^I := \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : z_i = 0 \text{ if } i \not\in I \},
\]

\[
\mathbb{C}^{*I} := \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : z_i = 0 \text{ if and only if } i \not\in I \}.
\]

In particular, \( \mathbb{C}^0 = \mathbb{C}^*0 = \{ 0 \} \) and \( \mathbb{C}^{*\{1,\ldots,n\}} = (\mathbb{C}^*)^n \). (As usual, \( \mathbb{C}^* := \mathbb{C} \setminus \{ 0 \} \).

The Newton polygon of \( h \) at \( 0 \) with respect to the coordinates \( z \) (denoted by \( \Gamma_+(h; z) \)) is the convex hull in \( \mathbb{R}_+^n \) of the set

\[
\bigcup_{\alpha \neq 0} (\alpha + \mathbb{R}_+^n),
\]

where \( \mathbb{R}_+^n = \{ x := (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for } 1 \leq i \leq n \} \). The Newton boundary of \( h \) at \( 0 \) with respect to \( z \) (denoted by \( \Gamma(h; z) \)) is the union of the compact faces of \( \Gamma_+(h; z) \). For any system of weights \( w := (w_1, \ldots, w_n) \in \mathbb{N}^n \setminus \{ 0 \} \), there is a linear map \( \mathbb{R}^n \to \mathbb{R} \) given by

\[
x := (x_1, \ldots, x_n) \mapsto \sum_{1 \leq i \leq n} x_i w_i.
\]

Let \( l_w \) be the restriction of this map to \( \Gamma_+(h; z) \), let \( d_w \) be the minimal value of \( l_w \), and let \( \Delta_w \) be the face of \( \Gamma_+(h; z) \) defined by the locus where \( l_w \) takes this minimal value. It is easy to see that

\[
d_w := \inf_{\alpha \neq 0} \deg_w(z^\alpha),
\]

where \( \deg_w(z^\alpha) \) is the \( w \)-degree of the monomial \( z^\alpha \), which is defined by

\[
\deg_w(z^\alpha) := \sum_{1 \leq i \leq n} \alpha_i w_i = l_w(\alpha).
\]

Note that \( \Delta_w = \{ x \in \Gamma_+(h; z) : l_w(x) = d_w \} \), and if \( w_i > 0 \) for each \( 1 \leq i \leq n \), then \( \Delta_w \) is a (compact) face of the Newton boundary \( \Gamma(h; z) \).

**Definition 2.1.** The non-compact Newton boundary of \( h \) at \( 0 \) with respect to \( z \) (denoted by \( \Gamma_{nc}(h; z) \)) is obtained from the usual Newton boundary \( \Gamma(h; z) \) by adding the “essential” non-compact faces. Here, a non-compact face \( \Delta \) is said to be essential if there are weights \( w := (w_1, \ldots, w_n) \in \mathbb{N}^n \setminus \{ 0 \} \) such that the following two conditions hold:

(i) \( \Delta = \Delta_w \) (i.e., \( \Delta \) is the face defined by the locus where \( l_w \) takes its minimal value) and \( h|_{\mathbb{C}w} = 0 \), where \( I_w := \{ i \in \{1, \ldots, n\} : w_i = 0 \} \);

(ii) for any \( i \in I_w \) and any point \( \alpha \in \Delta \), the half-line \( \alpha + \mathbb{R}_+ e_1 \) is contained in \( \Delta \), where \( e_1 \) is the unit vector in the direction of the \( x_1 \)-axis.

The set \( I_w \) does not depend on the choice of the weights \( w \). It is called the non-compact direction of \( \Delta \) and is denoted by \( I_\Delta \).
Definition 2.5. Suppose that subspaces $C$ and $\Delta$. Eyral and M. Oka complex numbers $h$ does not identically vanish neither on $C$, nor on $\Delta$. Indeed, we can take $w = (1, 3, 0)$. Then $\Delta = \Delta_w$, $I_\Delta = \{3\}$ and $h(0, 0, z_3) = 0$ for any $z_3$. On the other hand, the non-compact face containing the edge $AB$ (respectively, the edge $BC$), where $B = (0, 3, 0)$, is not essential. Indeed, $h$ does not identically vanish neither on $\mathbb{C}^{[1,2]}$ nor on $\mathbb{C}^{[2,3]}$. See Figure 1.

Remark 2.3. Note that an essential non-compact face is not necessarily a maximal face of the Newton polygon. Indeed, in the above example, the 1-dimensional non-compact face $\Xi := C + \mathbb{R}_+ e_3$ is also essential. Indeed, we can take $w' = (1, 2, 0)$. Then $\Xi = \Delta_{w'}$, $I_{\Xi} = \{3\}$ and $h(0, 0, z_3) = 0$ for any $z_3$.

Definition 2.4. The function $h$ is said to be non-degenerate if for any (compact) face $\Delta \subseteq \Gamma(h; \textbf{z})$, the face function

$$h_\Delta(\textbf{z}) := \sum_{\alpha \in \Delta} c_\alpha \textbf{z}^\alpha$$

has no critical point on $(\mathbb{C}^*)^n$.

Let $\mathcal{J}_w(h)$ (respectively, $\mathcal{J}_v(h)$) be the set of all subsets $I \subseteq \{1, \ldots, n\}$ such that $h_{|\mathbb{C}^I} \neq 0$ (respectively, $h_{|\mathbb{C}^I} = 0$). For any $I \in \mathcal{J}_w(h)$ (respectively, any $I \in \mathcal{J}_v(h)$), the subspace $\mathbb{C}^I$ is called a non-vanishing (respectively, a vanishing) coordinates subspace. For any $u_{i_1}, \ldots, u_{i_m} \in \mathbb{C}^*$ ($m \leq n$), we set

$$\mathbb{C}^{*\{i_1, \ldots, i_m\}} := \{(z_1, \ldots, z_n) \in (\mathbb{C}^*)^n : z_{i_j} = u_{i_j} \text{ for } 1 \leq j \leq m\}.$$  

Definition 2.5. Let $\Delta \subseteq \Gamma_w(h; \textbf{z})$ be an essential non-compact face, and let $w = (w_1, \ldots, w_n)$ be a system of weights satisfying the conditions (i) and (ii) of Definition 2.1. Suppose that $I_\Delta := I_w = \{i_1, \ldots, i_m\}$ (i.e., $w_i = 0$ if and only if $i \in \{i_1, \ldots, i_m\}$). We say that the face function

$$h_\Delta(\textbf{z}) := \sum_{\alpha \in \Delta} c_\alpha \textbf{z}^\alpha$$

is locally tame if there exists a positive number $r(h_\Delta) > 0$ such that for any non-zero complex numbers $u_{i_1}, \ldots, u_{i_m} \in \mathbb{C}^*$ with

$$|u_{i_1}|^2 + \cdots + |u_{i_m}|^2 < r(h_\Delta)^2,$$
$h_\Delta$ has no critical point in $\mathbb{C}^\ast_{w_1,\ldots,w_m}$ as a function of the $n - m$ variables $z_{i_{n+1}},\ldots,z_{i_n}$.

(Here, $\{i_{n+1},\ldots,i_n\} = \{1,\ldots,n\} \backslash \{i_1,\ldots,i_m\}$.) We say that $h$ is locally tame along a vanishing coordinates subspace $C^I$ if for any essential non-compact face $\Delta \subseteq \Gamma_{nc}(h;\mathbf{z})$ with $h_\Delta = I$, the face function $h_\Delta$ is locally tame. Finally, we say that $h$ is locally tame along the vanishing coordinates subspaces if it is locally tame along $C^I$ for any $I \in \mathcal{S}_c(h)$.

**Notation 2.6.** For any $I \in \mathcal{S}_c(h)$, let

$$r_I(h) := \inf_{I_0=I} r(h_{I_0}).$$

Also, let

$$r_{nc}(h) := \inf_{I \in \mathcal{S}_c(h)} r_I(h).$$

**Example 2.7.** If $h(z_1,z_2) = z_1^2 + z_1^2 + 2z_1z_2^2$, then $\Gamma_{nc}(h;\mathbf{z})$ has two essential non-compact faces $\Delta_1 := A + \mathbb{R}_+e_2$ and $\Delta_2 := B + \mathbb{R}_+e_1$, where $A = (2,3)$ and $B = (3,2)$. Here, $I_{\Delta_1} = \{2\}$ and $I_{\Delta_2} = \{1\}$. For any $u_2 \in \mathbb{C}$ with $|u_2| < 1/2$, the function

$$z_1 \mapsto h_{\Delta_1}(z_1,u_2) = z_1^2u_2^3 + 2z_1u_2^4$$

of the variable $z_1$ has no critical point on $\mathbb{C}^\ast_{u_2}$. Thus the face function $h_{\Delta_1}$ is locally tame (we can take $r(h_{\Delta_1}) = 1/2$). Similarly, for any $u_1 \in \mathbb{C}$, the function

$$z_2 \mapsto h_{\Delta_2}(u_1,z_2) = u_1^2z_2^3$$

of the variable $z_2$ has no critical point on $\mathbb{C}^\ast_{u_1}$, and hence the face function $h_{\Delta_2}$ is locally tame. Altogether, $h$ is locally tame along the vanishing coordinates subspaces.

**Example 2.8.** If $h(z_1,z_2,z_3) = z_1^2z_3^2 - z_2^2z_3 + z_3$, then $\Delta := \overline{AB} + \mathbb{R}_+e_1 + \mathbb{R}_+e_2$ is an essential non-compact face, where $\overline{AB}$ is the edge with endpoints $A = (2,0,2)$ and $B = (0,3,2)$. Here, $I_\Delta = \{1,2\}$. For any positive number $r > 0$, there exist $u_1,u_2 \in \mathbb{C}$ such that $|u_1|^2 + |u_2|^2 < r^2$ but the function

$$z_3 \mapsto h_{\Delta}(u_1,u_2,z_3) = u_1^2z_3^2 - u_2^2z_3^3$$

of the variable $z_3$ has critical points on $\mathbb{C}^\ast_{(u_1,u_2)}$. Indeed, the derivative of $z_3 \mapsto h_{\Delta}(u_1,u_2,z_3)$ is zero along the curve $u_1^2 - u_2^3 = 0$. It follows that $h_{\Delta}$ (and therefore $h$) is not locally tame.

3. Admissible Families and Whitney Equisingularity

Let $(t,\mathbf{z}) := (t,z_1,\ldots,z_n)$ be coordinates for $\mathbb{C} \times \mathbb{C}^n$, let $U$ be an open neighbourhood of $0 \in \mathbb{C}^n$, let $D$ be an open disc centered at $0 \in \mathbb{C}$, and let

$$f: (D \times U, D \times \{0\}) \to (\mathbb{C},0), \quad (t,\mathbf{z}) \mapsto f(t,\mathbf{z}),$$

be a polynomial function. (This notation implies that $f(D \times \{0\}) = \{0\}$.) As above, we write $f_t(\mathbf{z}) := f(t,\mathbf{z})$ and we denote by $V(f_t)$ the hypersurface in $U \subseteq \mathbb{C}^n$ defined by $f_t$. In terms of the present section, we introduce a condition (admissibility condition) that will guarantee Whitney equisingularity for families of non-isolated singularities. Roughly, a family $\{f_t\}$ is admissible if for all $t$ small enough, the non-compact Newton boundary $\Gamma_{nc}(f_t;\mathbf{z})$ is independent of $t$, the polynomial function $f_t$ is non-degenerate, and the radius $r_{nc}(f_t)$ which appear in Definition 2.5 and...
Notation $\| \cdot \|$ is greater than or equal to a fixed positive number $\rho > 0$. But before to go into further details, we first need to show that if the Newton boundary $\Gamma(f_t; z)$ is independent of $t$ and $f_t$ is non-degenerate for all small $t$, then, in a neighbourhood of the origin $0 \in \mathbb{C}^n$, the hypersurface $V(f_t)$ is smooth along $\mathbb{C}^{nd}$ for any $I \in \mathcal{I}_m(f_t)$ and any $t$ small enough (cf. [3.1]).

3.1. **Smoothness along the non-vanishing coordinates subspaces.** The following proposition is a uniform version of [9, Chapter III, Lemma (2.2)] and [10, Theorem 19].

**Proposition 3.1.** Suppose that for all $t$ sufficiently small, the following two conditions are satisfied:

1. the Newton boundary $\Gamma(f_t; z)$ of $f_t$ at $0$ with respect to the coordinates $z$ is independent of $t$ (in particular, $\mathcal{I}(f_t)$ is independent of $t$);
2. the polynomial function $f_t$ is non-degenerate.

Then there exists a positive number $R > 0$ such that for any $I \in \mathcal{I}(f_0)$ and any $t$ sufficiently small, the set $V(f_t) \cap \mathbb{C}^{nd} \cap B_R$ is non-singular and intersects transversely with $S_r$, for any $r < R$, where $B_R$ (respectively, $S_r$) is the open ball (respectively, the sphere) with centre the origin $0 \in \mathbb{C}^n$ and radius $R$ (respectively, $r$).

**Proof.** As there are only finitely many subsets $I \in \mathcal{I}(f_0)$, it suffices to show that for a fixed $I \in \mathcal{I}(f_0)$, there is $R > 0$ such that for any $t$ small enough, $V(f_t) \cap \mathbb{C}^{nd} \cap B_R$ is non-singular and intersects transversely with the sphere $S_r$ for any $r \leq R$. To simplify, we may assume that $I = \{1, \ldots, m\}$.

We start with the “smoothness” assertion. We argue by contradiction. Suppose that there exists a sequence $\{(t_N, z_N)\}$ of points in $V(f) \cap (D \times \mathbb{C}^{nd})$ converging to $(0, 0)$ and such that $z_N$ is a critical point of the restriction of $f_{t_N}$ to $\mathbb{C}^I$. Then $(0, 0)$ is in the closure of the set

$$W := \left\{ (t, z) \in D \times \mathbb{C}^{nd}; f_t(z) = 0 \quad \text{and} \quad \frac{\partial (f_t(z))}{\partial z_i} = 0 \quad \text{for} \quad 1 \leq i \leq m \right\}.$$ 

Therefore, by the curve selection lemma [5], there is a real analytic curve

$$(\tau(s), z(s)) = (t(s), z_1(s), \ldots, z_m(s), 0, \ldots, 0)$$ 

such that $(\tau(0), z(0)) = (0, 0)$ and $(\tau(s), z(s)) \in W$ for any $s \neq 0$. For $1 \leq i \leq m$, consider the Taylor expansions

$$t(s) = t_0 s^v + \ldots,$$

$$z_i(s) = a_i s^m + \ldots,$$

where $t_0, a_i \neq 0$ and $v, w_i > 0$. Here, the dots stand for the higher order terms. Let $a := (a_1, \ldots, a_m, 0, \ldots, 0) \in \mathbb{C}^{nd}$ and $w := (w_1, \ldots, w_m, 0, \ldots, 0) \in \mathbb{N}^n \setminus \{0\}$, and let $\Delta$ be the face of $\Gamma(f_{t(s)}; z) = \Gamma(f_0; z)$ defined by the locus where the map

$$x := (x_1, \ldots, x_m, 0, \ldots, 0) \in \Gamma(f_{t(s)}; z) \mapsto \sum_{1 \leq i \leq m} x_i w_i$$

takes its minimal value $d$. For any $1 \leq i \leq m$ and any $s \neq 0$,

$$0 = \frac{\partial (f_{t(s)}(z))}{\partial z_i} (z(s)) = \frac{\partial (f_{t(s)}(z))}{\partial z_i}(a) s^{w_i} + \ldots.$$ 


where \( \left( f_{(i)} \right)_{\Delta} \) is the face function associated with \( f_{(i)} \) and \( \Delta \). It follows that
\[
\frac{\partial \left( f_{0|C^l} \right)_{\Delta}}{\partial z_i} (\mathbf{a}) = 0
\]
for any \( 1 \leq i \leq m \). Therefore, \( \mathbf{a} \in C^l \) is a critical point of \( \left( f_0 \right)_{\Delta} : C^l \to C \). In particular, this implies that \( f_0 \) is not non-degenerate as a function of the variables \( z_1, \ldots, z_m \). This contradicts Proposition 7 of [10] which says that if a polynomial function \( f_0 \) is non-degenerate and if \( f_0 \neq 0 \), then \( f_0 \) must be non-degenerate as well.

To prove the “transversality” assertion, we also argue by contradiction. Suppose that there exists a sequence \( \{ (t_N, z_N) \} \) of points in \( V(f) \cap (D \times C^l) \) converging to \( (0, 0) \) and such that \( V(f_N) \cap C^l \) does not intersect the sphere \( S_{\|z_N\|} \) transversely at \( z_N \). Then \( (0, 0) \) is in the closure of the set consisting of points \( (t, z) \in D \times C^l \) such that
\[
f_{(i)}(z) = 0 \quad \text{and} \quad \text{grad} f_{(i)}(z) = \lambda z \text{ for } \lambda \in C^\ast.
\]

Here, \( \text{grad} f_{(i)}(z) \) is the gradient vector of \( f_{(i)} \) at \( z \), that is,
\[
\text{grad} f_{(i)}(z) := \left( \frac{\partial f_{(i)}}{\partial z_1}(z), \ldots, \frac{\partial f_{(i)}}{\partial z_m}(z), 0, \ldots, 0 \right),
\]
where the bar stands for the complex conjugation. Thus, by the curve selection lemma, we can find a real analytic curve
\[
(t(s), z(s)) = (t(s), z_1(s), \ldots, z_m(s), 0, \ldots, 0)
\]
and a Laurent series \( \lambda(s) \) such that:

(i) \( (t(0), z(0)) = (0, 0) \);
(ii) \( (t(s), z(s)) \in D \times C^l \) for \( s \neq 0 \);
(iii) \( f_{(i)}(z(s)) = 0 \);
(iv) \( \text{grad} f_{(i)}(z(s)) = \lambda(s) z(s) \).

Consider the Taylor expansions
\[
t(s) = t_0 s^v + \cdots,
\]
\[
z_i(s) = a_i s^w + \cdots \quad (1 \leq i \leq m),
\]
where \( t_0, a_i \neq 0 \) and \( v, w_i > 0 \), and the Laurent expansion
\[
\lambda(s) = \lambda_0 s^w + \cdots,
\]
where \( \lambda_0 \neq 0 \). Then define \( \mathbf{a}, \mathbf{w}, d \) and \( \Delta \) as above. By reordering, we may assume that \( w_1 = \cdots = w_k < w_j \) \( (k < j \leq m) \). Then, by (iv), we have \( d - w_1 = \omega + w_1 \) and
\[
(3.1) \quad \frac{\partial \left( f_{0|C^l} \right)_{\Delta}}{\partial z_i} (\mathbf{a}) = \left\{ \begin{array}{ll} \lambda_0 a_i & \text{for } 1 \leq i \leq k, \\ 0 & \text{for } k < i \leq m. \end{array} \right.
\]

Since the polynomial \( \left( f_0 \right)_{\Delta} \) is weighted homogeneous with respect to the weights \( \mathbf{w} \) and has weighted degree \( d \), it follows from the Euler identity that
\[
(3.2) \quad d \cdot \left( f_0 \right)_{\Delta}(\mathbf{a}) = \sum_{1 \leq i \leq m} w_i a_i \frac{\partial \left( f_{0|C^l} \right)_{\Delta}}{\partial z_i} (\mathbf{a}).
\]
As \( f(t(s),z(s)) = 0 \) for any \( s \), we have \( (f_0)_\text{v}_I(a) = 0 \). Therefore, by combining (3.1) and (3.2), we get a contradiction:

\[
0 = \sum_{1 \leq i \leq m} w_i a_{i} \frac{\partial (f_0)_\text{v}_I}{\partial z_i}(a) = \lambda_0 \sum_{1 \leq i \leq k} w_i |a_i|^2 \neq 0.
\]

This completes the proof of Proposition 3.1. \( \square \)

**Remark 3.2.** Under the same assumption as in Proposition 3.1, the second author showed in [7] that there also exists a positive number \( R' > 0 \) such that for any \( 0 < R' \leq R'' \), the sphere \( S = \{ |z| = R'' \} \). Under the same assumption as in Proposition 3.1, the second author showed that if an element \( f \) of the family (3.1) and (3.2), we get a contradiction:

\[
\Gamma_{nc}(f;\zeta) = \Gamma_{nc}(f_0;\zeta) \neq \Gamma_{nc}(f_0;\zeta) \neq \Gamma_{nc}(f_0;\zeta).
\]

**Remark 3.3.** \( \Gamma_{nc}(f;\zeta) = \Gamma_{nc}(f_0;\zeta) \neq \Gamma_{nc}(f_0;\zeta) \neq \Gamma_{nc}(f_0;\zeta) \).

By Proposition 3.1, we know that there exists a positive number \( R > 0 \) such that for any \( I \in \mathcal{I}_m(f_0) = \mathcal{I}_m(f_0) \) and any \( t \) small enough, \( V(f_0) \cap \mathbb{C}^d \cap B_R \) is non-singular. It follows immediately that in a sufficiently small open neighbourhood \( \mathcal{U} \subseteq D \times U \) of the origin of \( \mathbb{C} \times \mathbb{C}^n \), the set \( V(f) \cap (\mathbb{C} \times \mathbb{C}^d) \) is non-singular for any \( I \in \mathcal{I}_m(f_1) \). Therefore, in such a neighbourhood, we can stratify \( \mathbb{C} \times \mathbb{C}^n \) in such a way that the hypersurface \( V(f) := f^{-1}(0) \) is a union of strata. More precisely, we consider the following three types of strata:

- \( A_I := \mathcal{U} \cap (V(f) \cap (\mathbb{C} \times \mathbb{C}^d)) \) for \( I \in \mathcal{I}_m(f_0) \);
- \( B_I := \mathcal{U} \cap ((\mathbb{C} \times \mathbb{C}^d) \setminus (V(f) \cap (\mathbb{C} \times \mathbb{C}^d))) \) for \( I \in \mathcal{I}_m(f_0) \);
- \( C_I := \mathcal{U} \cap (\mathbb{C} \times \mathbb{C}^d) \) for \( I \in \mathcal{I}_m(f_0) \).

The (finite) collection

\[
\mathcal{I} := \{A_I, B_I; I \in \mathcal{I}_m(f_0)\} \cup \{C_I; I \in \mathcal{I}_m(f_0)\}
\]

is a stratification (i.e., a partition into complex analytic submanifolds) of the set \( \mathcal{U} \cap (\mathbb{C} \times \mathbb{C}^n) \) for which \( \mathcal{U} \cap V(f) \) is a union of strata. Note that for \( I = \emptyset \), which is an element of \( \mathcal{I}_m(f_0) \), the stratum \( C_0 := \mathcal{U} \cap (\mathbb{C} \times \mathbb{C}^n) \) of \( \mathcal{I} \) is nothing but the \( t \)-axis \( \mathcal{U} \cap (\mathbb{C} \times \{0\}) \).

**Remark 3.4.** A similar stratification but for a single polynomial function \( h(\zeta) \) (not for a family) is already considered in [11] where the second author shows that if \( h \) is non-degenerate and locally tame along the vanishing coordinates subspaces, then it satisfies Thom’s \( a_h \) condition with respect to this stratification.

**Definition 3.5.** We say that the family \( \{f_i\} \) is admissible (at \( t = 0 \)) if it satisfies the conditions (I) and (II) above and if, furthermore, there exists a positive number \( \rho > 0 \) such that for any sufficiently small \( t \),

\[
\inf\{R, r_{nc}(f_i)\} \geq \rho.
\]
where \( R \) is given by Proposition 3.1.

In particular, if the family \( \{ f_t \} \) is admissible, then it is uniformly locally tame along the vanishing coordinates subspaces—that is, \( f_t \) is locally tame along \( \mathbb{C}^I \) for any \( I \in \mathcal{I}(f_0) \) and \( r_{nc}(f_t) \geq \rho \) for all small \( t \).

Here is our main result.

**Theorem 3.6.** If the family of polynomial functions \( \{ f_t \} \) is admissible, then the canonical stratification \( \mathcal{S} \) of \( U \cap (\mathbb{C} \times \mathbb{C}^n) \) described above is a Whitney stratification, and hence, the corresponding family of hypersurfaces \( \{ V(f_t) \} \) is Whitney equisingular.

We recall that a stratification of a subset of \( \mathbb{C}^N \) is a Whitney stratification if the closure \( \overline{S} \) of each stratum \( S \) and the complement \( \overline{S} \setminus S \) are both analytic sets, and if for any pair of strata \( (S_2, S_1) \) and any point \( p \in S_1 \cap \overline{S}_2 \), the stratum \( S_2 \) is Whitney \((b)\)-regular over the stratum \( S_1 \) at the point \( p \). The latter condition means that for any sequences of points \( \{ p_k \} \) in \( S_1 \), \( \{ q_k \} \) in \( S_2 \) and \( \{ a_k \} \) in \( \mathbb{C} \) satisfying:

(i) \( p_k \to p \) and \( q_k \to p \);
(ii) \( T_{q_k}S_2 \to T \);
(iii) \( a_k(p_k - q_k) \to v \);

we have \( v \in T \). (As usual, \( T_{q_k}S_2 \) is the tangent space to \( S_2 \) at \( q_k \).) For details, we refer the reader to [2].

**Remark 3.7.** Observe that if \( M \) is a smooth manifold and \( N \subseteq M \) is a closed smooth submanifold, then \( M \setminus N \) is Whitney \((b)\)-regular over \( N \) at any point.

Theorem 3.6 will be proved in Section 4. The proof will show that for isolated singularities, if \( \Gamma(f_t;z) \) is independent of \( t \) and \( f_t \) is non-degenerate for all small \( t \), then the conclusions of Theorem 3.6 still holds true without assuming the uniform local tameness (cf. Remark 4.2). In other words, Theorem 3.6 includes Assertion 1.1 as a special case.

**Remark 3.8.** In [8, §8], another Whitney stratification of \( \mathcal{U} \cap (\mathbb{C} \times \mathbb{C}^n) \), with the \( t \)-axis as a stratum and such that \( \mathcal{U} \cap V(f) \) is a union of strata, is constructed under a different assumption (so-called “simultaneous IND-condition”). However this stratification is different from our. Especially, it has a larger number of strata.

Combined with the Thom-Mather first isotopy theorem (cf. [2,4,12]), Theorem 3.6 implies the following result.

**Corollary 3.9.** If the family of polynomial functions \( \{ f_t \} \) is admissible, then the corresponding family of hypersurfaces \( \{ V(f_t) \} \) is topologically equisingular.

**Remark 3.10.** Topological equisingularity is also proved in [8, Theorem (8.2)] under the simultaneous IND-condition.

**Proof of Corollary 3.9.** By Theorem 3.6, \( \mathcal{U} \cap (\mathbb{C} \times \mathbb{C}^n) \) is a Whitney stratified set. Hence, by the Thom-Mather first isotopy theorem, it is topologically locally trivial (see, e.g., Theorem (5.2) and Corollary (5.5) of [2]). As the stratum \( C_b \) is nothing but the \( t \)-axis, Corollary 3.9 follows. \( \square \)
4. Proof of Theorem 3.6

First of all, observe that if $I \subseteq J$, then $C^I$ is contained in the closure $\overline{C^J}$ of $C^J$. Moreover, if $I \subseteq J$ and $J \in \mathcal{I}_s(f_0)$, then $I \in \mathcal{I}_s(f_0)$ too. Therefore, to prove the theorem, it suffices to check that the Whitney ($b$)-regularity condition holds for all the pairs of strata satisfying one of the following three conditions:

1. $C_I \cap \overline{C_J} \neq \emptyset$ with $I \subseteq J$ and $J \in \mathcal{I}_s(f_0)$;
2. $C_I \cap \overline{A_J} \neq \emptyset$ or $C_I \cap \overline{B_J} \neq \emptyset$ with $I \subseteq J$ and $I \in \mathcal{I}_s(f_0)$, $J \in \mathcal{I}_m(f_0)$;
3. $A_I \cap \overline{A_J} \neq \emptyset$, $A_I \cap \overline{B_J} \neq \emptyset$ or $B_I \cap \overline{B_J} \neq \emptyset$ with $I \subseteq J$ and $I, J \in \mathcal{I}_m(f_0)$.

Except for the case of a pair of strata of the form $(A_J, C_I)$, the Whitney ($b$)-regularity condition immediately follows from Remark 3.7. Thus, to prove our result, it suffices to show that for any $J \in \mathcal{I}_m(f_0)$ and any $I \in \mathcal{I}_s(f_0)$, with $I \subseteq J$, $A_I := \mathcal{V} \cap (V(f) \cap (C \times C^I))$ is Whitney ($b$)-regular over $C_I := \mathcal{V} \cap (C \times C^I)$ at any point $(\tau, q) = (\tau, q_1, \ldots, q_n) \in C_I \cap \overline{A_I}$ sufficiently close to the origin of $C \times C^n$. To simplify, without loss of generality, we may assume that $J = \{1, \ldots, n\}$ and $I = \{1, \ldots, m\}$ with $1 \leq m \leq n-1$. (For $I = \emptyset$, see Remark 4.1.) In particular, $q_i \neq 0$ if and only if $1 \leq i \leq m$. By the curve selection lemma [5], it is enough to show that the Whitney ($b$)-regularity condition holds along arbitrary real analytic paths

\[
\gamma(s) := (t(s), z(s)) := (t(s), z_1(s), \ldots, z_n(s))
\]

\[
\mathfrak{c}(s) := (\bar{t}(s), \bar{z}(s)) := (\bar{t}(s), \bar{z}_1(s), \ldots, \bar{z}_n(s))
\]

such that $\gamma(0) = \mathfrak{c}(0) = (\tau, q)$, and $\gamma(s) \in C_I$ and $\mathfrak{c}(s) \in A_I$ for $s \neq 0$. Consider the Taylor expansions (where, as above, the dots stand for the higher order terms):

\[
t(s) = \tau + b_0 s + \cdots, \quad z_i(s) = a_i s^w_i + b_i s^{w_i+1} + \cdots,
\]

\[
\bar{t}(s) = \tau + \bar{b}_0 s + \cdots, \quad \bar{z}_i(s) = q_i + \bar{b}_i s + \cdots,
\]

where $w_i = 0$ and $a_i = q_i$ for $1 \leq i \leq m$ while $w_i > 0$ and $a_i \neq 0$ for $i > m$. Note that, for any $s$, we have $\bar{z}_i(s) = 0$ if $i > m$. Let

\[
\ell(s) := \mathfrak{c}(s) \gamma(s) = (\ell_0(s), \ell_1(s), \ldots, \ell_n(s))
\]

where

\[
\ell_i(s) := \begin{cases} 
(b_0 - \bar{b}_0)s + \cdots & \text{for } i = 0, \\
(b_i - \bar{b}_i)s + \cdots & \text{for } 1 \leq i \leq m, \\
{a_i s^w_i} + \cdots & \text{for } m + 1 \leq i \leq n.
\end{cases}
\]

By reordering, we may suppose that

\[
w_{m+1} = \cdots = w_{m+m_1} < w_{m+m_1+1} = \cdots = w_{m+m_1+m_2} < \cdots < w_{m+m_1+\cdots+m_{k-1}+1} = \cdots = w_{m+m_1+\cdots+m_k} = w_n,
\]

for some non-negative integers $m_1, \ldots, m_k \in \mathbb{N}$ with $m + m_1 + \cdots + m_k = n$. To show that the pair of strata $(A_J, C_I)$ satisfies the Whitney ($b$)-regularity condition at the point $(\tau, q)$, we have to prove that

\[
\lim_{s \to 0} \frac{\langle \ell(s), \nabla f(\gamma(s)) \rangle}{\|\ell(s)\| \cdot \|\nabla f(\gamma(s))\|} = 0.
\]
where \( \langle \cdot, \cdot \rangle \) is the standard Hermitian inner product on \( \mathbb{C} \times \mathbb{C}^n \) and \( \nabla f(\gamma(s)) \) is the gradient vector of \( f \) at \( \gamma(s) \), that is,

\[
\nabla f(\gamma(s)) := \left( \frac{\partial f}{\partial t}(\gamma(s)), \frac{\partial f}{\partial z_1}(\gamma(s)), \ldots, \frac{\partial f}{\partial z_n}(\gamma(s)) \right),
\]

where the bar stands for the complex conjugation. Let

\[
\text{ord } \ell(s) := \inf_{0 \leq i \leq n} \text{ord } \ell_i(s)
\]

where \( \text{ord } \ell_i(s) \) is the order (in \( s \)) of the \( i \)-th component \( \ell_i(s) \) of \( \ell(s) \). Clearly, \( \text{ord } \ell(s) \leq w_{m+1} \). Note that if \( \text{ord } \ell(s) < w_{m+1} \), then

\[
\lim_{s \to 0} \text{ord } \ell(s) = (*, \ldots, *, 0, \ldots, 0),
\]

where each term marked with a star “*” represents a complex number which may be zero or not. On the other hand, if \( \text{ord } \ell(s) = w_{m+1} \), then

\[
\lim_{s \to 0} \text{ord } \ell(s) = (*, \ldots, *, a_{m+1}, \ldots, a_{m+m_1}, 0, \ldots, 0).
\]

Let \( w := (w_1, \ldots, w_n) = (0, \ldots, 0, w_{m+1}, \ldots, w_n) \), and let \( l_w : \Gamma \to \mathbb{R} \) be the restriction of the linear map

\[
(x_1, \ldots, x_n) \in \mathbb{R}^n \mapsto \sum_{1 \leq i \leq n} x_i w_i \in \mathbb{R},
\]

where \( \Gamma \) is the non-compact Newton boundary \( \Gamma_{\text{nc}}(f; z) \), which is independent of \( t \). Denote by \( d_w \) the minimal value of \( l_w \), and write \( \Delta_w \) for the face of \( \Gamma \) defined by the locus where \( l_w \) takes this minimal value. Clearly, \( \Delta_w \) is an essential non-compact face, and \( \Delta_{w_0} = I \). Finally, let \( a := (a_1, \ldots, a_n) \). As \( \Gamma \) does not depend on \( t \),

\[
\frac{\partial f}{\partial z_i}(\gamma(s)) = \frac{\partial (f_t)_{\Delta_w}}{\partial z_i}(a) s^{d_w - w_i} + \ldots
\]

for any \( 1 \leq i \leq n \), while

\[
\lim_{s \to 0} \left( \frac{1}{|s|^{d_w - 1}} \frac{\partial f}{\partial t}(\gamma(s)) \right) = 0.
\]

(As usual, \( (f_t)_{\Delta_w} \) is the face function associated with \( f_t \) and \( \Delta_w \).) Also, note that, by (4.1),

\[
d_w - w_{m+1} = \cdots = d_w - w_{m+m_1} > d_w - w_{m+m_1+1} = \cdots = d_w - w_{m+m_1+m_2} > \cdots > d_w - w_n.
\]

Let \( o(s) := \text{ord } (\nabla f(\gamma(s))) \), that is,

\[
o(s) := \inf \left\{ \text{ord } \left( \frac{\partial f}{\partial t}(\gamma(s)) \right), \inf_{1 \leq i \leq n} \text{ord } \left( \frac{\partial f}{\partial z_i}(\gamma(s)) \right) \right\}.
\]

By the uniform local tameness (i.e., the condition \( r_{\rho}(f_t) \geq \rho \) for all small \( r \)), if \( (\tau, \mathbf{q}) \) is close enough to \((0, \mathbf{0}) \in \mathbb{C} \times \mathbb{C}^n \), then there exists an integer \( t_0 \in \{m + 1, \ldots, n\} \) such that

\[
\frac{\partial (f_t)_{\Delta_w}}{\partial z_{t_0}}(a) \neq 0.
\]
Combined with (4.5), the relation (4.8) shows that
\[ o(s) \leq d_w - w_i \leq d_w - w_{m+1} \leq d_w - 1. \]

Then, by (4.6),
\[ \lim_{s \to 0} \left( \frac{1}{|s|^{o(s)}} \frac{\partial f}{\partial t}(\gamma(s)) \right) = 0, \]
and since \( w_i = 0 \) for \( 1 \leq i \leq m \),

(4.9)
\[ \lim_{s \to 0} \frac{\text{grad} f(\gamma(s))}{|s|^{o(s)}} = \begin{cases} (0, 0, \ldots, 0, \underbrace{* \ldots *}_{m+m_1 \text{ zeros}, \ n-m-m_1 \text{ terms}}) & \text{if } o(s) < d_w - w_{m+1}, \\ (0, 0, \ldots, 0, \underbrace{\partial (f_\tau)_{\Delta w}(a), \ldots, \partial (f_\tau)_{\Delta w}(a), \underbrace{* \ldots *}_{m \text{ zeros}, \ n-m-m_1 \text{ terms}}}_{\text{if } o(s) = d_w - w_{m+1}.} \end{cases} \]

Since \( \|\ell(s)\| \sim c_1 |s|^{\text{ord}\ell(s)} \) and \( \|\text{grad} f(\gamma(s))\| \sim c_2 |s|^{o(s)} \) as \( s \to 0 \) (\( c_1, c_2 \) constants), it follows immediately from (4.3), (4.4) and (4.9) that the relation (4.2) is satisfied if \( o(s) < d_w - w_{m+1} \) or if \( o(s) = d_w - w_{m+1} \) and \( \text{ord} \ell(s) < w_{m+1} \). In order to show that (4.2) also holds when \( o(s) = d_w - w_{m+1} \) and \( \text{ord} \ell(s) = w_{m+1} \), we must prove that

(4.10)
\[ \sum_{i=m+1}^{m+m_1} a_i \frac{\partial (f_\tau)_{\Delta w}}{\partial z_i} = 0. \]

To prove (4.10), we proceed as follows. The polynomial \( (f_\tau)_{\Delta w} \) is weighted homogeneous with respect to the weights \( w \) and has weighted degree \( d_w \). Then, by the Euler identity, for any \( z = (z_1, \ldots, z_n) \) we have:

(4.11)
\[ \sum_{1 \leq i \leq n} w_i z_i \frac{\partial (f_\tau)_{\Delta w}}{\partial z_i}(z) = d_w \cdot (f_\tau)_{\Delta w}(z). \]

As \( f(\gamma(s)) = 0 \) for any \( s \), we have \( (f_\tau)_{\Delta w}(a) = 0 \). Therefore, by (4.11),

(4.12)
\[ \sum_{1 \leq i \leq n} w_i a_i \frac{\partial (f_\tau)_{\Delta w}}{\partial z_i}(a) = 0. \]

Combined with (4.5) and (4.7), the equality \( o(s) = d_w - w_{m+1} \) implies that for any \( i > m + m_1 \),

(4.13)
\[ \frac{\partial (f_\tau)_{\Delta w}}{\partial z_i}(a) = 0. \]

Indeed, if there were \( i_1 > m + m_1 \) such that (4.13) does not hold, then, by (4.5),
\[ o(s) \leq d_w - w_{i_1} \]
which is a contradiction. As \( w_i = 0 \) for \( 1 \leq i \leq m \), it follows from (4.12) and (4.13) that
\[ \sum_{i=m+1}^{m+m_1} w_i a_i \frac{\partial (f_\tau)_{\Delta w}}{\partial z_i}(a) = w_{m+1} \sum_{i=m+1}^{m+m_1} a_i \frac{\partial (f_\tau)_{\Delta w}}{\partial z_i}(a) = 0. \]

As \( w_{m+1} > 0 \), the equality (4.10) follows. This completes the proof.
Remark 4.1. In the above proof, we have assumed \( l \neq \emptyset \) (i.e., \( m \geq 1 \)). However, a straightforward modification shows that the argument still works when \( l = \emptyset \). We just observe that, in this case, the face \( \Delta_w \) is compact, and hence, in (4.5), instead of the uniform local tameness, it suffices to invoke the non-degeneracy condition.

Remark 4.2. Note that in the special case where the functions \( f_t \) have an isolated singularity at the origin, we recover the Briançon assertion mentioned in the introduction (cf. Assertion 1.1). Indeed, in this case, as observed by Briançon in [1], by adding monomials of the form \( z_i^{N_i} \) for large values of \( N_i \), we may assume that \( f_t \) is convenient, that is, the intersection of the Newton boundary \( \Gamma(f_t; z) \) with each coordinates subspace is non-empty. (In other words, the family of hypersurfaces \( \{ V(f_t) \} \) defined by the polynomial functions \( f_t(z) := f_t(z) + c_1 z_1^{N_1} + \cdots + c_n z_n^{N_n} \) is Whitney equisingular if and only if the original family \( \{ V(f_t) \} \) is.) But for convenient polynomials, the only vanishing coordinates subspace is \( \mathbb{C}^0 = \{ 0 \} \), and by Remark 4.1, the partition
\[
\{ A_I : I \in \mathcal{I}_m(f_0) \} \cup \{ C_0 \}
\]
of \( \mathcal{V} \cap \mathcal{V}(f_t) \) is a Whitney stratification with the \( t \)-axis as a stratum if \( \Gamma(f_t; z) \) is independent of \( t \) and \( f_t \) is non-degenerate.

5. EXAMPLES OF ADMISSIBLE FAMILIES

In this section, we give some examples of admissible (and therefore Whitney equisingular) families with non-isolated singularities.

5.1. A family of curves with non-isolated singularities. Consider the family given by the polynomial function
\[
f(t, z_1, z_2) := z_1^2 z_2^3 + z_1^4 z_2 + t z_1^2 z_2^2.
\]
Let \( A = (2, 3) \) and \( B = (3, 2) \). For \( t \) small enough, the singular locus of \( f_t \) in a sufficiently small open neighbourhood of the origin in \( \mathbb{C}^2 \) consists of the coordinates axes. The non-compact Newton boundary \( \Gamma_{nc}(f_t; z) \), which is clearly independent of \( t \), has one compact 1-dimensional face (namely, the face \( \overline{AB} \)), two 0-dimensional faces (\( A \) and \( B \)), and two essential non-compact faces: \( \Xi_1 := B + \mathbb{R}_+ e_1 \) and \( \Xi_2 := A + \mathbb{R}_+ e_2 \). We easily check that for each compact face \( \Delta \) and each \( t \), the face function \( (f_t)_{\Delta} \) has no critical point on \( (\mathbb{C}^*)^2 \) (i.e., \( f_t \) is non-degenerate). We claim that for any \( l \in \mathcal{I}_t(f_0) = \mathcal{I}_t(f_t) \), the family \( \{ f_t \} \) is uniformly locally tame along \( \mathbb{C}^l \). Indeed, a trivial calculation shows that for any fixed \( u_1 \in \mathbb{C}^* \), the function
\[
z_2 \mapsto (f_t)_{\Xi_1}(u_1, z_2) := u_1 z_2^2
\]
of the variable \( z_2 \) has no critical point on \( \mathbb{C}^{*(1,2)} \). Similarly, for any fixed \( u_2 \in \mathbb{C}^* \) with \( |u_2| < 1/|t| \) (if \( t \neq 0 \)), the function
\[
z_1 \mapsto (f_t)_{\Xi_2}(z_1, u_2) := z_1^2 u_2^3 + t z_1^2 u_2^4
\]
of the variable \( z_1 \) has no critical point on \( \mathbb{C}^{*(1,2)}_{u_2} \). So we can take
\[
r_{nc}(f_t) = \begin{cases} 
\frac{1}{|t|} & \text{for } t \neq 0, \\
\infty & \text{for } t = 0,
\end{cases}
\]
and we have \( r_{nc}(f_t) > \rho := 1 \) for all \( t \) with \( |t| < 1 \). It follows that the family \( \{ f_t \} \) is admissible.
5.2. Families with “big” exponents for \( t \)-dependent monomials. Let \( h(z) \) be a polynomial function on \( \mathbb{C}^n \) and let \( g(t, z) \) be a polynomial function on \( \mathbb{C} \times \mathbb{C}^n \). As usual, we write \( g_t(z) := g(t, z) \). Suppose that for all small \( t \), \( \Gamma^+(g_t; z) \subseteq \Gamma^+(h; z) \) and \( \Gamma^+_{nc}(g_t; z) \cap \Gamma^+_{nc}(h; z) = \emptyset \). Under this assumption, if \( h \) is non-degenerate and locally tame along the vanishing coordinates subspaces, then the family \( \{f_t\} \) defined by \( f_t(z) := h(z) + g_t(z) \) is admissible. For example, the family given by \( f(t, z_1, z_2) := z_1^2 + z_2^2 + t z_1 z_2 \) is admissible.

5.3. Admissible families and branched coverings. Take a positive integer \( p \in \mathbb{N}^* \), and consider the branched covering
\[
\varphi^p : \mathbb{C}^n \to \mathbb{C}^n, (z_1, \ldots, z_n) \mapsto (z_1^p, \ldots, z_n^p),
\]
whose ramification locus is given by the coordinates hyperplanes \( z_i = 0 \) \((1 \leq i \leq n)\). In [11 Proposition 22], the second author showed that if \( h(z) \) is a non-degenerate polynomial function which is locally tame along \( \mathbb{C}^l \) for any \( l \in \mathcal{I}_v(h) \), then \( h^p(z) := h \circ \varphi^p(z) = h(z_1^p, \ldots, z_n^p) \) is non-degenerate, \( \mathcal{I}_v(h^p) = \mathcal{I}_v(h) \), and \( h^p \) is locally tame along \( \mathbb{C}^l \) for any \( l \in \mathcal{I}_v(h^p) \). Actually, the proof shows that if \( \{f_t\} \) is an admissible family of polynomial functions, then so is the family \( \{f_t^p\} \), where \( f_t^p(z) := f_t \circ \varphi^p(z) \). Indeed, it is not difficult to see that the independence of \( \Gamma_{nc}(f_t; z) \) with respect to \( t \) implies that of \( \Gamma_{nc}(f_t^p; z) \). Also, it is easy to check that \( \mathcal{I}_v(f_t^p) \supseteq \mathcal{I}_v(f_t) \) and \( \mathcal{I}_{nc}(f_t^p) \supseteq \mathcal{I}_{nc}(f_t) \). To see that \( f_t^p \) is non-degenerate, we argue by contradiction. Take any compact face \( \Xi \subseteq \Gamma(f_t^p; z) \), and suppose there exists \( a := (a_1, \ldots, a_n) \in (\mathbb{C}^*)^n \) such that
\[
\frac{\partial (f_t^p)}{\partial z_i}(a) = 0
\]
for all \( 1 \leq i \leq n \). Clearly, \( (f_t^p)_\Delta(z) = (f_t)_\Delta \circ \varphi^p(z) = (f_t)_\Delta(z_1^p, \ldots, z_n^p) \), where \( \Delta \) is the compact face of \( \Gamma(f_t; z) \) corresponding to \( \Xi \)—that is, if \( \Delta \cap \mathbb{N}^2 = \{(\alpha_1, \alpha_2), (\beta_1, \beta_2), \ldots\} \), then \( \Xi \cap \mathbb{N}^2 = \{(a_1^p, a_2^p), (\beta_1^p, \beta_2^p), \ldots\} \) (cf. Figure 2). Therefore, by (5.1),
\[
\frac{\partial (f_t)_\Delta}{\partial z_i}(a_1^{\rho}, \ldots, a_n^{\rho}) \cdot pa_i^{\rho - 1} = 0,
\]
and hence,
\[
\frac{\partial (f_t)_\Delta}{\partial z_i}(a_1^{\rho}, \ldots, a_n^{\rho}) = 0
\]
for all \( 1 \leq i \leq n \). This contradicts the non-degeneracy of \( f_t \).

Claim 5.1. The family \( \{f_t^p\} \) is uniformly locally tame along the vanishing coordinates subspaces.

Proof. By hypothesis, we know that the family \( \{f_t\} \) is uniformly locally tame along the vanishing coordinates subspaces—that is, \( r_{nc}(f_t) \geq \rho \) for all small \( t \). Without loss of generality, we may assume that \( \rho < 1 \). Consider a subset \( I \in \mathcal{I}_v(f_t^p) \). For simplicity, let us assume that \( I = \{1, \ldots, m\} \). Let \( u_1, \ldots, u_m \) be non-zero complex numbers such that
\[
|u_1|^2 + \cdots + |u_m|^2 \leq \rho^2.
\]
Take any essential non-compact face \( \Xi \in \Gamma_{nc}(f_t^p; z) \) such that \( I_{\Xi} = I \), and consider the corresponding face \( \Delta \in \Gamma_{nc}(f_t; z) \). (Note that \( I_{\Delta} = I \) too.) We want to
Figure 2. A face $\Delta \subseteq \Gamma_{nc}(f_1; z)$ and its corresponding face $\Xi \subseteq \Gamma_{nc}(f^p_1; z)$

show that the face function $(f^p_1)_{\Xi}$ has no critical point on $\C^{n+1}_{u_1, \ldots, u_m}$ as a function of the variables $z_{m+1}, \ldots, z_n$. Again we argue by contradiction. Suppose $(u_1, \ldots, u_m, a_{m+1}, \ldots, a_n)$ is a critical point. Then, for $m+1 \leq i \leq n$,

$$0 = \frac{\partial (f^p_1)_{\Xi}}{\partial z_i}(u_1, \ldots, u_m, a_{m+1}, \ldots, a_n)$$

$$= \frac{\partial (f_1)_{\Delta}}{\partial z_i}(u^p_1, \ldots, u^p_m, a^p_{m+1}, \ldots, a^p_n) \cdot p\alpha_i^{p-1},$$

and therefore,

$$\frac{\partial (f_1)_{\Delta}}{\partial z_i}(u^p_1, \ldots, u^p_m, a^p_{m+1}, \ldots, a^p_n) = 0.$$  \hspace{1cm} (5.2)

As $\rho < 1$,

$$|u^p_1|^2 + \cdots + |u^p_m|^2 \leq \rho^2,$$

and hence (5.2) contradicts the uniform local tameness of the family $\{f_1\}$. \hspace{1cm} $\square$

Altogether, the family $\{f^p_1\}$ is admissible.

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C. EYRAL, INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, ŚNIADECKICH 8, 00-656 WARSAW, POLAND
E-mail address: eyralchr@yahoo.com

M. OKA, DEPARTMENT OF MATHEMATICS, TOKYO UNIVERSITY OF SCIENCE, 1-3 KAGURAZAKA, SHINJUKU-KU, TOKYO 162-8601, JAPAN
E-mail address: oka@rs.kagu.tus.ac.jp