Non-Adaptive Learning a Hidden Hypergraph

Hasan Abasi\textsuperscript{1}, Nader H. Bshouty\textsuperscript{1}, and Hanna Mazzawi\textsuperscript{2}

\textsuperscript{1} Department of Computer Science
Technion, Haifa, 32000
\textsuperscript{2} Google, London.
76 Buckingham Palace Rd.

Abstract. We give a new deterministic algorithm that non-adaptively learns a hidden hypergraph from edge-detecting queries. All previous non-adaptive algorithms either run in exponential time or have non-optimal query complexity. We give the first polynomial time non-adaptive learning algorithm for learning hypergraph that asks almost optimal number of queries.

1 Introduction

Let $\mathcal{G}_{s,r}$ be a set of all labeled hypergraphs of rank at most $r$ on the set $V = \{1, 2, \ldots, n\}$ with at most $s$ edges. Given a hidden hypergraph $G \in \mathcal{G}_{s,r}$, we need to identify it by asking edge-detecting queries. An edge-detecting query $Q_G(S)$, for $S \subseteq V$ is: does $S$ contain at least one edge of $G$? Our objective is to non-adaptively learn the hypergraph $G$ by asking as few queries as possible.

This problem has many applications in chemical reactions, molecular biology and genome sequencing. In chemical reactions, we are given a set of chemicals, some of which react and some which do not. When multiple chemicals are combined in one test tube, a reaction is detectable if and only if at least one set of the chemicals in the tube reacts. The goal is to identify which sets react using as few experiments as possible. The time needed to compute which experiments to do is a secondary consideration, though it is polynomial for the algorithms we present. See\cite{25,33,13,21,3,28,27,2,15,5,20,30,8,6,17,19,18,4} for more details and many other applications in molecular biology.

In all of the above applications the rank of the hypergraph is much smaller than the number of edges and both are much smaller than the number of vertices $n$. Therefore, throughout the paper, we will assume that $r \leq s$ and $s = o(n)$.

The above hypergraph learning problem is equivalent to the problem of non-adaptively learning a monotone DNF with at most $s$ monomials (monotone term), where each monomial contains at most $r$ variables ($s$-term $r$-MDNF) from membership queries\cite{16}. In this paper we will use the later terminology rather than the hypergraph one.

The adaptive learnability of $s$-term $r$-MDNF was studied in\cite{5,20,6,4}. In\cite{1}, Abasi et. al. gave a polynomial time adaptive learning algorithm for $s$-term $r$-MDNF with almost optimal query complexity. The non-adaptive learnability of $s$-term $r$-MDNF was studied in\cite{33,27,24,20,19,12}.
Torney [33], first introduced the problem and gave some applications in molecular biology. The first explicit non-adaptive learning algorithm for \( s \)-term \( r \)-MDNF was given by Gao et. al. [24]. They show that this class can be learned using \((n, (s, r))\)-cover-free family \(( (n, (s, r))\)-CFF). This family is a set \( A \subseteq \{0, 1\}^n \) of assignments such that for every distinct \( i_1, \ldots, i_s, j_1, \ldots, j_r \in \{1, \ldots, n\} \) there is \( a \in A \) such that \( a_{i_1} = \cdots = a_{i_s} = 0 \) and \( a_{j_1} = \cdots = a_{j_r} = 1 \). Given such a set, the algorithm simply takes all the monomials \((s)\)-term of size \( s \) such that for every distinct \( i \in [n] \) \( \forall a \in A \) \((M(a) = 1 \Rightarrow f(a) = 1)\). It is easy to see that the disjunction of all such monomials is equivalent to the target function. Assuming a set of \((n, (s, r))\)-CFF of size \( N \) can be constructed in time \( T \), this algorithm learns \( s \)-term \( r \)-MDNF with \( N \) queries in time \( O((n)^r + T) \).

In [20,9], it is shown that any set \( A \subset \{0, 1\}^n \) that non-adaptively learns \( s \)-term \( r \)-MDNF is an \((n, (s - 1, r))\)-CFF. Therefore, the minimum size of an \((n, (s - 1, r))\)-CFF is also a lower bound for the number of queries (and therefore also for the time) for non-adaptively learning \( s \)-term \( r \)-MDNF. It is known, [31], that any \((n, (s, r))\)-CFF must have size at least \( \Omega(N(s, r) \log n) \) where

\[
N(s, r) = \frac{s + r}{\log \left(\frac{s + r}{r}\right)}.
\]

Therefore, any non-adaptive algorithm for learning \( s \)-term \( r \)-DNF must ask at least \( N(s - 1, r) \log n = \Omega(N(s, r) \log n) \) queries and runs in at least \( \Omega(N(s, r)n \log n) \) time.

Gao et. al. constructed an \((n, (s, r))\)-CFF of size \( S = (2s \log n / \log(s \log n))^r+1 \) in time \( \tilde{O}(S) \). It follows from [32] that an \((n, (s, r))\)-CFF of size \( O((sr)^{\log^* n} \log n) \) can be constructed in polynomial time. A polynomial time almost optimal constructions of size \( N(s, r)^{1+o(1)} \log n \) for \((n, (s, r))\)-CFF were given in [11,10,12,23] which give better query complexities, but still, the above algorithms have exponential time complexity \( O(n^r) \), when \( r \) is not constant. The latter result implies that there is a non-adaptive algorithm that asks \( Q := N(s, r)^{1+o(1)} \log n \) queries and runs in exponential time \( O(n^r) \). Though, when \( r = O(1) \) is constant, the above algorithms run in polynomial time and are optimal. Therefore, we will assume \( r = \omega(1) \).

Chin et. al. claim in [19] that they have a polynomial time algorithm that constructs an \((n, (s, r))\)-CFF of optimal size. Their analysis is misleading. The size is indeed optimal but the time complexity of the construction is \( O(n^r) \). But even if a \((n, (s, r))\)-CFF can be constructed in polynomial time, the above learning algorithm still takes \( O(n^r) \) time.

Macula et. al., [27,28], gave several randomized non-adaptive algorithms. We first use their ideas combined with the constructions of \((n, (r, s))\)-CFF in [11,10,12,23] to give a new non-adaptive algorithm that asks \( N(s, r)^{1+o(1)} \log^2 n \) queries and runs in \( poly(n, N(s, r)) \) time. This algorithm is almost optimal in \( s \) and \( r \) but quadratic in \( \log n \). We then use a new technique that changes any non-adaptive

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\[3\] Some parts of the construction can indeed be performed in polynomial time, but not the whole construction.
The following table summarizes the results ($r = o(1)$)

| References | Query Complexity | Time Complexity |
|------------|------------------|-----------------|
| 24         | $N(s, r) \cdot (r \log n / \log(s \log n))^{1+\epsilon}$ | $2^n$ |
| 19         | $N(s, r) \log n$ | $2^n$ |
| 111101223  | $N(s, r)^{1+o(1)} \log n$ | $(2^n)$ |
| Ours + 2728 + 12 | $N(s, r)^{1+o(1)} \log^2 n$ | $\text{poly}(n, N(s, r))$ |
| Ours       | $N(s, r)^{1+o(1)} \log n$ | $(n \log n) \cdot \text{poly}(N(s, r))$ |
| Ours, $r = o(s)$ | $N(s, r)^{1+o(1)} \log n$ | $(n \log n) \cdot N(s, r)^{1+o(1)}$ |
| Lower Bound | $N(s, r) \log n$ | $(n \log n) \cdot N(s, r)$ |

This paper is organized as follows. Section 2 gives some definitions and preliminary results that will be used throughout the paper. Section 3 gives the first algorithm that asks $N(s, r)^{1+o(1)} \log^2 n$ membership queries and runs in time $\text{poly}(n, N(s, r))$. Section 4 gives the reduction and shows how to use it to give the second algorithm that asks $N(s, r)^{1+o(1)} \log n$ membership queries and runs in time $(n \log n) \cdot N(s, r)^{1+o(1)}$. All the algorithms in this paper are deterministic. In the full paper we will also consider randomized algorithms that slightly improve (in the $o(1)$ of the exponent) the query and time complexity.

## 2 Definitions

### 2.1 Monotone Boolean Functions

For a vector $w$, we denote by $w_i$ the $i$th entry of $w$. Let $\{e^{(i)} \mid i = 1, \ldots, n\} \subset \{0, 1\}^n$ be the standard basis. That is, $e^{(i)}_i = 1$ if $i = j$ and $e^{(i)}_j = 0$ otherwise. For a positive integer $j$, we denote by $[j]$ the set $\{1, 2, \ldots, j\}$. For two assignments $a, b \in \{0, 1\}^n$ we denote by $(a \land b) \in \{0, 1\}^n$ the bitwise AND assignment. That is, $(a \land b)_i = a_i \land b_i$.

Let $f(x_1, x_2, \ldots, x_n)$ be a boolean function from $\{0, 1\}^n$ to $\{0, 1\}$. For $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $\sigma_1, \ldots, \sigma_k \in \{0, 1\} \cup \{x_1, \ldots, x_n\}$ we denote by $f|_{x_{i_1} \leftarrow \sigma_1, x_{i_2} \leftarrow \sigma_2, \ldots, x_{i_k} \leftarrow \sigma_k}$ the function $f(y_1, \ldots, y_n)$ where $y_{ij} = \sigma_j$ for all $j \in [k]$ and $y_i = x_i$ for all $i \in [n]\{i_1, \ldots, i_k\}$. We say that the variable $x_i$ is relevant in $f$ if $f|_{x_i \leftarrow 0} \neq f|_{x_i \leftarrow 1}$. A variable $x_i$ is irrelevant in $f$ if it is not relevant in $f$. We say that the class is closed under variable projections if for every $f \in C$ and every two variables $x_i$ and $x_j$, $i, j \leq n$, we have $f|_{x_i \leftarrow x_j} \in C$.

For two assignments $a, b \in \{0, 1\}^n$, we write $a \leq b$ if for every $i \in [n]$, $a_i \leq b_i$. A Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is monotone if for every two assignments
a, b ∈ {0, 1}^n, if a ≤ b then f(a) ≤ f(b). Recall that every monotone boolean function f has a unique representation as a reduced monotone DNF, \([1]\). That is, 
\[ f = M_1 \lor M_2 \lor \cdots \lor M_s \]
where each monomial \(M_i\) is an ANDs of input variables, and for every monomial \(M_i\) there is a unique assignment \(a^{(i)} \in \{0, 1\}^n\) such that 
\[ f(a^{(i)}) = 1 \]
and for every \(j \in [n]\) where \(a_j^{(i)} = 1\) we have 
\[ f(a^{(i)}|_{x_j=0}) = 0. \]
We call such assignment a \textit{minterm} of the function f. Notice that every monotone DNF can be uniquely determined by its minterms \([1]\). That is, \(a \in \{0, 1\}^n\) is a minterm of \(f\) iff \(M := \land_{i \in \{j : a_j = 1\}} x_i\) is a monomial in \(f\).

An \textit{s-term} \(r\)-MDNF is a monotone DNF with at most \(s\) monomials, where each monomial contains at most \(r\) variables. It is easy to see that the class \(s\)-term \(r\)-MDNF is closed under variable projections.

\subsection{2.2 Learning from Membership Queries}

Consider a \textit{teacher} that has a \textit{target function} \(f : \{0, 1\}^n \rightarrow \{0, 1\}\) that is \(s\)-term \(r\)-MDNF. The teacher can answer \textit{membership queries}. That is, when receiving \(a \in \{0, 1\}^n\) it returns \(f(a)\). A \textit{learning algorithm} is an algorithm that can ask the teacher membership queries. The goal of the learning algorithm is to \textit{exactly learn} (exactly find) \(f\) with minimum number of membership queries and optimal time complexity.

Let \(c\) and \(H \supset C\) be classes of boolean formulas. We say that \(C\) is \textit{learnable from} \(H\) in time \(T(n)\) with \(Q(n)\) membership queries if there is a learning algorithm that, for a target function \(f \in C\), runs in time \(T(n)\), asks at most \(Q(n)\) membership queries and outputs a function \(h \in H\) that is equivalent to \(C\). When \(H = C\) then we say that \(C\) is \textit{properly learnable} in time \(T(n)\) with \(Q(n)\) membership queries.

In adaptive algorithms the queries can depend on the answers to the previous queries where in non-adaptive algorithms the queries are independent of the answers to the previous queries and therefore all the queries can be asked in parallel, that is, in one step.

\subsection{2.3 Learning a Hypergraph}

Let \(G_{s,r}\) be a set of all labeled hypergraphs on the set of vertices \(V = \{1, 2, \ldots, n\}\) with \(s\) edges of rank (size) at most \(r\). Given a hidden hypergraph \(G \in G_{s,r}\), we need to identify it by asking \textit{edge-detecting queries}. An edge-detecting query \(Q_G(S)\), for \(S \subseteq V\) is: does \(S\) contain at least one edge of \(G\)? Our objective is to learn (identify) the hypergraph \(G\) by asking as few queries as possible.

This problem is equivalent to learning \(s\)-term \(r\)-MDNF \(f\) from membership queries. Each edge \(e\) in the hypergraph corresponds to the monotone term \(\land_{i \in e} x_i\) in \(f\) and the edge-detecting query \(Q_G(S)\) corresponds to asking membership queries of the assignment \(a^{(S)}\) where \(a_i^{(S)} = 1\) if and only if \(i \in S\). Therefore, the class \(G_{s,r}\) can be regarded as the set of \(s\)-term \(r\)-MDNF. The class of \(s\)-term \(r\)-MDNF is denoted by \(G^*_{s,r}\). Now it obvious that any learning algorithm for \(G^*_{s,r}\) is also a learning algorithm for \(G_{s,r}\).
The following example shows that we cannot allow two edges $e_1 \subseteq e_2$. Let $G_1$ be a graph where $V_1 = \{1, 2\}$ and $E_1 = \{\{1\}, \{1, 2\}\}$. This graph corresponds to the function $f = x_1$ that is equivalent to $x_1$ which corresponds to the graph $G_2$ where $V_2 = \{1, 2\}$ and $E_2 = \{\{1\}\}$. Also, no edge-detecting query can distinguish between $G_1$ and $G_2$.

We say that $A \subseteq \{0, 1\}$ is an identity testing set for $G_{s, r}^*$ if for every two distinct $s$-term $r$-MDNF $f_1$ and $f_2$ there is $a \in A$ such that $f_1(a) \neq f_2(a)$. Obviously, every identity testing set for $G_{s, r}^*$ can be used as queries to non-adaptively learns $G_{s, r}^*$.

### 2.4 Cover Free Families

An $(n, (s, r))$-cover free family ($(n, (s, r))$-CFF), [22], is a set $A \subseteq \{0, 1\}^n$ such that for every $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ where $d = s + r$ and every $J \subseteq [d]$ of size $|J| = s$ there is $a \in A$ such that $a_{i_k} = 0$ for all $k \in J$ and $a_{i_j} = 1$ for all $j \in [d] \setminus J$. Denote by $N(n, (s, r))$ the minimum size of such set. Again here we assume that $r \leq s$ and $s = o(n)$. The lower bound in [31, 29] is

$$N(n, (s, r)) \geq \Omega \left( N(s, r) \cdot \log n \right)$$

where $N(s, r)$ is as defined in [11]. It is known that a set of random

$$m = O \left( r^{1.5} \left( \log \left( \frac{s}{r} + 1 \right) \right) \left( N(s, r) \cdot \log n + \frac{N(s, r)}{s + r} \frac{\log 1}{\delta} \right) \right)$$

$$= N(s, r)^{1+o(1)} \log n$$

assignments $a^{(i)} \in \{0, 1\}^n$, where each $a^{(i)}_j$ is 1 with probability $r/(s + r)$, is an $(n, (s, r))$-CFF with probability at least $1 - \delta$.

It follows from [11, 10, 12, 29] that there is a polynomial time (in the size of the CFF) deterministic construction of $(n, (s, r))$-CFF of size

$$N(s, r)^{1+o(1)} \log n$$

where the $o(1)$ is with respect to $r$. When $r = o(s)$ the construction runs in linear time [10, 12].

### 2.5 Perfect Hash Function

Let $H$ be a family of functions $h : [n] \rightarrow [q]$. For $d \leq q$ we say that $H$ is an $(n, q, d)$-perfect hash family ($(n, q, d)$-PHF) [17] if for every subset $S \subseteq [n]$ of size $|S| = d$ there is a hash function $h \in H$ such that $h|_S$ is injective (one-to-one) on $S$, i.e., $|h(S)| = d$.

In [10] Bshouty shows

**Lemma 1.** Let $q \geq 2d^2$. There is a $(n, q, d)$-PHF of size

$$O \left( \frac{d^2 \log n}{\log(q/d^2)} \right)$$

that can be constructed in time $O(qd^2 n \log n / \log(q/d^2))$. 
We now give the following folklore results that will be used for randomized learning algorithms.

**Lemma 2.** Let \( q > d(d-1)/2 \) be any integer. Fix any set \( S \subset [n] \) of \( d \) integers. Consider
\[
N := \frac{\log(1/\delta)}{\log \left( \frac{1}{1-\beta(q,d)} \right)} \leq \frac{\log(1/\delta)}{\log \left( \frac{q}{a(d-1)} \right)}
\]
uniform random hash functions \( h_i : [n] \to [q], i = 1, \ldots, N \) where
\[
g(q,d) := \left(1 - \frac{1}{q}\right) \left(1 - \frac{2}{q}\right) \cdots \left(1 - \frac{d-1}{q}\right)
\]
With probability at least \( 1 - \delta \) one of the hash functions is one-to-one on \( S \).

### 3 The First Algorithm

In this section we give the first algorithm that asks \( N(s,r)^{1+o(1)} \log^2 n \) queries and runs in time \( \text{poly}(n, N(s,r)) \).

The first algorithm is based on the ideas in [27,28] that were used to give a Monte Carlo randomized algorithm.

**Lemma 3.** Let \( A \) be an \((n,(1,r))-\text{CFF}\) and \( B \) be an \((n,(s-1,r))-\text{CFF}\). There is a non-adaptive proper learning algorithm for \( s \)-term \( r \)-MDNF that asks all the queries in \( A \land B := \{a \land b \mid a \in A, b \in B\} \) and finds the target function in time \( |A \land B| \cdot n \).

**Proof.** Let \( f \) be the target function. For every \( b \in B \), let \( A_b = A \land b := \{a \land b \mid a \in A\} \). Let \( I_b \) be the set of all \( i \in [n] \) such that \((a \land b)_i \geq f(a \land b)\) for all \( a \in A \). Let \( T_b := \bigwedge_{i \in I_b} x_i \). We will show that

1. If \( T \) is a term in \( f \) then there is \( b \in B \) such that \( T_b \equiv T \).
2. Either \( T_b = \bigwedge_{i \in [n]} x_i \) or \( T_b \) is a subterm of one of terms of \( f \).

To prove 1, let \( T \) be a term in \( f \) and let \( b \in B \) be an assignment that satisfies \( T \) and does not satisfy the other terms. Such assignment exists because \( B \) is \((n,(s-1,r))-\text{CFF}\). Notice that \( f(x \land b) = T(x) = T(x \land b) \). If \( x_i \) is in \( T \) and \( f(a \land b) = 1 \) then \( T(a \land b) = T(a) = f(a \land b) = 1 \) and \((a \land b)_i = 1 \). Therefore \( i \in I_b \) and \( x_i \) in \( T_b \). If \( x_i \) not in \( T \) then since \( A \) is \((n,(1,r))-\text{CFF}\) there is \( a' \in A \) such that \( T(a') = 1 \) and \( a'_i = 0 \). Then \( (a' \land b)_i = 0 \) where \( f(a' \land b) = 1 \). Therefore \( i \) is not in \( I_b \) and \( x_i \) is not in \( T_b \). Thus, \( T_b \equiv T \).

We now prove 2. We have shown in 1 that if \( b \) satisfies one term \( T \) then \( T_b \equiv T \). If \( b \) does not satisfy any one of the terms in \( f \) then \( f(a \land b) = 0 \) for all \( a \in A \) and then \( T_b = \bigwedge_{i \in [n]} x_i \). Now suppose \( b \) satisfies at least two terms \( T_1 \) and \( T_2 \). Consider any variable \( x_i \). If \( x_i \) not in \( T_1 \) then as before \( x_i \) will not be in \( T_b \). This shows that \( T_b \) is a subterm of \( T_1 \).

This gives the following algorithm

We now have
Learn($G^*_{s,r}$)
1) Construct an $(n, (1, r))$-CFF $A$ and an $(n, (s-1, r))$-CFF $B$.
2) Ask membership queries for all $a \land b$, $a \in A$ and $b \in B$.
3) For every $b \in B$.
4) $T_b \leftarrow 1$.
5) For every $i \in [n]$.
6) If for all $a \in A$, $(a \land b)_i \geq f(a \land b)$
7) then $T_b \leftarrow T_b \land x_i$.
8) $T \leftarrow T \cup \{T_b\}$.
9) Remove from $T$ the term $\land_{i \in [n]} x_i$
and all subterm of a larger term.

Fig. 1. An algorithm for learning $G^*_{s,r}$.

**Theorem 1.** There is a non-adaptive proper learning algorithm for stem $r$-MDNF that asks
\[ N(s, r)^{1+o(1)} \log^2 n \]
queries and runs in time $\text{poly}(n, N(s, r))$.

**Proof.** Constructing a $(n, (1, r))$-CFF of size $|A| = r^2 \log n$ and a $(n, (s-1, r))$-CFF of size $|B| = N(s-1, r)^{1+o(1)} \log n = N(s, r)^{1+o(1)} \log n$ takes $\text{poly}(n, N(s, r))$ time \cite{11101223}. By Lemma 3 the learning takes time $|A \land B| \cdot n = \text{poly}(n, N(s, r))$ time. The number of queries of the algorithm is $|A \land B| \leq |A| \cdot |B| = N(s, r)^{1+o(1)} \log^2 n$. \hfill \qed

## 4 The Second Algorithm

In this section we give the second algorithm.

We first prove the following result

**Lemma 4.** Let $C$ be a class of boolean function that is closed under variable projection. Let $H$ be a class of boolean functions and suppose there is an algorithm that finds the relevant variables of $f \in H$ in time $R(n)$.

If $C$ is non-adaptively learnable from $H$ in time $T(n)$ with $Q(n)$ membership queries then $C$ is non-adaptively learnable from $H$ in time
\[
O \left( qd^2 n \log n + \frac{d^2 \log n}{\log(q/d^2)} (T(q)n + R(q)) \right)
\]
with
\[
O \left( \frac{d^2 Q(q)}{\log(q/d^2)} \log n \right)
\]
membership queries where $d$ is an upper bound on the number of relevant variables in $f \in C$ and $q \geq 2d^2$. 

Proof. Consider the algorithm in Figure 2. Let $A(n)$ be a non-adaptive algorithm that learns $C$ from $H$ in time $T(n)$ with $Q(n)$ membership queries. Let $f \in C_n$ be the target function. Consider the $(n, q, d + 1)$-PHF $P$ that is constructed in Lemma 1 (Step 1 in the algorithm). Since $C$ is closed under variable projection, for every $h \in P$ the function $f_h := f(x_{h(1)}, \ldots, x_{h(n)})$ is in $C_q$. Since the membership queries to $f_h$ can be simulated by membership queries to $f$ there is a set of $|P| \cdot Q(q)$ assignments from $\{0, 1\}^n$ that can be generated from $A(q)$ that non-adaptively learn $f_h$ for all $h \in P$ (Step 2 in the algorithm). The algorithm $A(q)$ learns $f'_h \in H$ that is equivalent to $f_h$.

Then the algorithm finds the relevant variables of each $f'_h \in H$ (Step 3 in the algorithm). Let $V_h$ be the set of relevant variables of $f'_h$ and let $d_{\text{max}} = \max_h |V_h|$. Suppose $x_{i_1}, \ldots, x_{i_d'}, d' \leq d$ are the relevant variables in the target function $f$. There is a map $h' \in P$ such that $h'(i_1), \ldots, h'(i_d')$ are distinct and therefore $f'_{h'}$ depends on $d'$ variables. In particular, $d' = d_{\text{max}}$ (Step 4 in the algorithm).

After finding $d' = d_{\text{max}}$ we have: Every $h$ for which $f'_h$ depends on $d'$ variables necessarily satisfies $h(i_1), \ldots, h(i_{d'})$ are distinct. Consider any other non-relevant variable $x_j \notin \{x_{i_1}, \ldots, x_{i_d'}\}$. Since $P$ is $(n, q, d + 1)$-PHF, there is $h'' \in P$ such that $h''(j), h''(i_1), \ldots, h''(i_{d'})$ are distinct. Then $f'_{h''}$ depends on $x_{h''(i_1)}, \ldots, x_{h''(i_{d'})}$ and not in $x_{h''(j)}$. This way the non-relevant variables can be eliminated. This is Step 6 in the algorithm. Since the above is true for every non-relevant variable, after Step 6 in the algorithm, the set $X$ contains only the relevant variables of $f$. Then in Steps 7 and 8, the target function $f$ can be recovered from any $f'_{h_0}$ that satisfies $|V(h_0)| = d'$.

\begin{algorithm}
\textbf{Algorithm Reduction I}
$A(n)$ is a non-adaptive learning algorithm for $C$ from $H$.
1) Construct an $(n, q, d + 1)$-PHF $P$.
2) For each $h \in P$
   Run $A(q)$ to learn $f_h := f(x_{h(1)}, \ldots, x_{h(n)})$.
   Let $f'_h \in H$ be the output of $A(q)$.
3) For each $h \in P$
   $V_h$ $\leftarrow$ the relevant variables in $f'_h$
4) $d_{\text{max}}$ $\leftarrow$ $\max_h |V_h|$.
5) $X$ $\leftarrow$ \{ $x_1, x_2, \ldots, x_n$ \}.
6) For each $h \in P$
   If $|V_h| = d_{\text{max}}$ then $X$ $\leftarrow$ $X \setminus \{x_i \mid x_{h(i)} \notin V_h\}$
7) Take any $h_0$ with $|V_{h_0}| = d_{\text{max}}$
8) Replace each relevant variable $x_i$ in $f'_{h_0}$ by $x_j \in X$ where $h_0(j) = i$.
9) Output the function resulted in step (8).
\end{algorithm}

Fig. 2. Algorithm Reduction.

We now prove
Theorem 2. There is a non-adaptive proper learning algorithm for $s$-term $r$-MDNF that asks 
\[ N(s, r)^{1+o(1)} \log n \]
queries and runs in time 
\[ (n \log n) \cdot \text{poly}(N(s, r)) \]
time.

Proof. We use Lemma 4. $C = H$ is the class of $s$-term $r$-MDNF. This class is closed under variable projection. Given $f$ that is $s$-term $r$-MDNF, one can find all the relevant variables in $R(n) = \text{poly}(s)$ time. The algorithm in the previous section runs in time $T(n) = \text{poly}(n, N(s, r))$ and asks $Q(n) = N(s, r)^{1+o(1)} \log^2 n$ queries. The number of variables in the target is bounded by $d = rs$. Let $q = 3r^2s^2 \geq 2d^2$. By Lemma 4 there is a non-adaptive algorithm that runs in time

\[ O \left( qd^2n \log n + \frac{d^2 \log n}{\log(q/d^2)} (T(q)n + R(q)) \right) = (n \log n)\text{poly}(N(r, s)) \]

and asks

\[ O \left( \frac{d^2Q(q)}{\log(q/d^2)} \log n \right) = N(s, r)^{1+o(1)} \log n \]
membership queries. \qed

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