CHARGE CONJUGATION APPROACH TO SCATTERING FOR THE HARTREE TYPE DIRAC EQUATIONS WITH CHIRALITY

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Abstract. We study the Cauchy problems for the Hartree-type nonlinear Dirac equations with Yukawa-type potential derived from pseudoscalar field. We establish scattering for large data but with a relatively small part of initial data associated with charge conjugation by exploiting null structure induced by chiral operator.

1. Introduction

The purpose of this paper is to investigate scattering property for the cubic Dirac equation with the Hartree-type nonlinearity in $\mathbb{R}^{1+3}$ given by

$$
\begin{cases}
-\imath \gamma^\mu \partial_\mu \psi + m\psi = [V_b \ast (\overline{\psi} \gamma^5 \psi)] \gamma^5 \psi & \text{in } \mathbb{R}^{1+3}, \\
\psi|_{t=0} := \psi_0.
\end{cases}
$$

Here $m > 0$, $\ast$ is the convolution in $\mathbb{R}^3$, $\overline{\psi} = \psi^\dagger \gamma^0$, $\psi^\dagger$ is the transpose of complex conjugate of $\psi$, and $\gamma^5 = \imath \gamma^0 \gamma^1 \gamma^2 \gamma^3$. The potential $V_b$ is the Yukawa potential interacting between nucleon and meson and is given by

$$V_b(x) := \frac{g_0^2}{4\pi} \frac{e^{-b|x|}}{|x|}.$$

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where \( g_0 \in \mathbb{R} \) and \( b > 0 \) are given physical parameters. The gamma matrices 
\( \gamma^\mu \in \mathbb{C}^{4 \times 4} \) \((\mu = 0, 1, 2, 3)\) are given by
\[
\gamma^0 = \begin{bmatrix} I_{2 \times 2} & 0 \\ 0 & -I_{2 \times 2} \end{bmatrix}, \quad \gamma^j = \begin{bmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{bmatrix}
\]
with the Pauli matrices \( \sigma^j \in \mathbb{C}^{2 \times 2} \) \((j = 1, 2, 3)\) given by
\[
\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

In this paper we establish global well-posedness and scattering of solutions to 
the system (1) for small charge conjugation in the scaling critical space 
which has extra weighted regularity in the angular variables. To be more precise, 
we let \( \Omega_{ij} = x_i \partial_j - x_j \partial_i \) be the infinitesimal generators of the rotations on \( \mathbb{R}^3 \) 
and \( \Delta_{S^2} = \sum_{1 \leq i < j \leq 3} \Omega_{ij}^2 \) be the Laplace-Beltrami operator on the unit sphere \( S^2 \subset \mathbb{R}^3 \).
Then we can define the fractional power of angular derivative by 
\( \Lambda_{S^2} = (1 - \Delta_{S^2})^{-\frac{1}{2}} \), which will be treated concretely below, 
and define angularly regular space 
\( L^2_x,\sigma \) 
and its norm by
\[
\| f \|_{L^2_x,\sigma} := \| \Lambda_{S^2} f \|_{L^2_x}.
\]
Now we state the main theorem:

**Theorem 1.1.** Let \( \sigma > 0 \) and \( \theta \in \{+, -\} \). Then there exists 
\( a = a(\| \psi_0 \|_{L^2_x,\sigma}(|\mathbb{R}^3|)) > 0 \) such that for all initial data 
\( \psi_0 \in L^2_x,\sigma \) satisfying \( \| \psi_0 + i\gamma^2 \psi_0^* \|_{L^2_x,\sigma} \leq a \), 
the Cauchy problem (1) is globally well-posed and solution \( \psi \) scatters in 
\( L^2_x,\sigma \) to free solutions as \( t \to \pm \infty \).

Here \( \psi_0^* \) denotes the complex conjugate of \( \psi_0 \). The transformed spinor \( i\gamma^2 \psi_0^* \) is referred to as a charge conjugation of \( \psi_0 \), which will be discussed below in detail. 
The condition (2) was observed in [5, 9] for the equations with identity matrix 
in place of \( \gamma^5 \). It is regarded as a perturbation of Majorana condition studied in 
[19, 6, 20].

One may observe that the equation (1) can be derived by decoupling the following 
Dirac-Klein-Gordon system
\[
(-i\gamma^\mu \partial_\mu + m)\psi = g_0 \phi i\gamma^5 \psi, 
\]
\[
(\partial_t^2 - \Delta + M^2)\phi = -g_0 \psi i\gamma^5 \psi. 
\]
The system (3) conserves parity and is Lorentz covariant. The matrix \( \gamma^5 \) was chosen 
for the right-hand side to be a pseudoscalar. For details, see Ch. 10 of [3].

Let us assume that the pseudoscalar field \( \phi \) is a standing wave, i.e., \( \phi(t, x) = e^{i\lambda t} f(x) \) with \( M > |\lambda| \). Then the Klein-Gordon part of (4) becomes
\[
(-\Delta + M^2 - \lambda^2)\phi = -g_0 \psi i\gamma^5 \psi.
\]
Then we put (1) into the Dirac part of (4) and then a spinor field \( \psi \) gives the desired equation.

If \( \gamma^5 \) is replaced by the identity (in this case \( \phi \) is a scalar field), then (1) and 
(4) have been extensively studied [6, 13, 2, 11, 5, 22, 23, 24, 14, 7, 8, 10]. Our work
is motivated from these works and is concerned with the relation between charge 
conjugation and chirality (see (7) below).
Remark 1.2 (Chirality). The gamma matrix $\gamma^5$ represents the chirality of spinors. Let $\psi_R = \frac{1}{2}(1 + \gamma^5)\psi$ and $\psi_L = \frac{1}{2}(1 - \gamma^5)\psi$. Then $\psi = \psi_R + \psi_L$. Using $\gamma^5\gamma^5 = 1$, $\gamma^5\psi_R = \psi_R$ and $\gamma^5\psi_L = -\psi_L$, which lead us to the chirality. Hence $(\psi_R)_R = \psi_R, (\psi_L)_L = \psi_L$ and $(\psi_R)_L = (\psi_L)_R = 0$. Since $\gamma^5\gamma^\mu\gamma^\mu\gamma^5 = 0$ for $\mu = 0, 1, 2, 3$, $\bar{\psi}\gamma^\mu\partial_\mu\psi = 0$ and $\gamma^5(-i\gamma^\mu\partial_\mu)\psi = i\gamma^\mu\partial_\mu\gamma^5\psi$. Hence we conclude that the solution $\psi$ to (1) satisfies
\begin{align}
-i\gamma^\mu\partial_\mu\psi_R &= -M\psi_L + [V_\theta + (\bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L)]\psi_L,
-i\gamma^\mu\partial_\mu\psi_L &= -M\psi_R + [V_\theta + (\bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L)]\psi_R,
\psi_R(0) &= \psi_0,R, \quad \psi_L(0) = \psi_0,L.
\end{align}
(5)

The new system (5) shows that the right(left)-handed chiral field $\psi_R(\psi_L)$ can evolve even though starting off as a completely left(right)-handed chiral state due to the influence of left(right)-handed chiral field. This shows that the chirality is not conserved and a smallness on both data $\psi_0,R$ and $\psi_0,L$ rather than a partial smallness is necessary for the global well-posedness unlike the charge conjugation as stated in Theorem [11].

Remark 1.3. If $b = 0$, then (1) becomes scattering-critical equation due to the log-blowup nature. Hence a non-scattering is plausible in $L^2_x$ space unless the Majorana condition appears. In fact, one may consider scattering states satisfying coercivity condition guaranteeing the full time decay $t^{-\frac{5}{2}}$ (see [12]) as in [10, 8], to which any solution of (1) does not converge in $L^2_x$. Also one may try to remove the log-blowup by modifying phases. It would be very interesting to treat such a modified scattering problem for $b = 0$. See [13] for the Schrödinger case and [21] for the semirelativistic case.

We denote the charge conjugation operator by $C$, which is defined by
\[ C\psi = i\gamma^5\psi^*. \]

As usual, $C\psi$ is interpreted as the wave function of antimatter and a direct calculation shows that $C\psi$ satisfies the equation:
\[ -i\gamma^\mu\partial_\mu C\psi + mC\psi = -[V_\theta + (\overline{C\psi}\gamma^5C\psi)]\gamma^5C\psi. \]

The antimatter field propagates through the interaction of negative Yukawa potential.

To combine the spinor and its charge conjugation we define projection operators $P_\theta^c$ for $\theta \in \{+, -\}$ by
\[ P_\theta^c\psi = \frac{1}{2}(\psi + \theta C\psi). \]

Then we readily get
\[ P_+^c + P_-^c = I, \quad (P_\theta^c)^2 = P_\theta^c, \quad P_\theta^c P_\theta^c = 0. \]
(6)

For this see [9]. Given a spinor field $\psi : \mathbb{R}^{1+3} \rightarrow \mathbb{C}^4$, we further have
\[ P_\theta^c\psi\gamma^5P_\theta^c\psi = 0. \]
(7)

This represents the relation between charge conjugation and chirality. We append the proof in the last section.
Now $P_0^c \psi$ satisfy the system:
\[
-i \gamma^\mu \partial_\mu P_+^c \psi + mP_+^c \psi = V_b \ast \left( \overline{P_+^c \psi} \gamma^5 P_+^c - P_+^c \psi \gamma^5 \overline{P_+^c \psi} \right) \gamma^5 P_+^c \psi,
\]
\[
-i \gamma^\mu \partial_\mu P_-^c \psi + mP_-^c \psi = V_b \ast \left( \overline{P_-^c \psi} \gamma^5 P_-^c - P_-^c \psi \gamma^5 \overline{P_-^c \psi} \right) \gamma^5 P_-^c \psi.
\]

Hence, to solve the equation \((1)\) and to prove Theorem \((1.1)\) we need to consider the system:
\[
-i \gamma^\mu \partial_\mu \varphi + m\varphi = V_b \ast \left( \overline{\varphi} \gamma^5 \chi + \overline{\chi} \gamma^5 \varphi \right) \gamma^5 \varphi,
\]
\[
-i \gamma^\mu \partial_\mu \chi + m\chi = V_b \ast \left( \overline{\varphi} \gamma^5 \chi + \overline{\chi} \gamma^5 \varphi \right) \gamma^5 \chi
\]
with initial data
\[
\varphi(0) = P_+^c \psi_0, \quad \chi(0) = P_-^c \psi_0.
\]

If we set $\psi = P_+^c \varphi + P_-^c \chi$, then by taking $P_+^c \varphi$ to \((8)\) one can turn back to \((1)\). We will discuss the turning-back in the appendix.

Under the above setup we prove the following:

**Theorem 1.4.** Let $\sigma > 0$. Then there exists $0 < \epsilon < 1$ such that for any $A > 0$ and any $0 < a \leq \epsilon A^{-1}$, if the initial data $\psi_0 \in L^{2,\sigma}$ satisfy
\[
\|P_+^c \psi_0\|_{L^{2,\sigma}} \leq a, \quad \|P_-^c \psi_0\|_{L^{2,\sigma}} \leq A,
\]
then the Cauchy problem of \((8)\) is globally well-posed in $C(\mathbb{R}; L^{2,\sigma})$. Furthermore, there exist $\varphi^\ell, \chi^\ell \in L^{2,\sigma}$ such that
\[
-i \gamma^\mu \partial_\mu (\varphi^\ell, \chi^\ell) + m(\varphi^\ell, \chi^\ell) = (0, 0)
\]
\[
\lim_{t \to \pm \infty} \| (\varphi(t), \chi(t)) - (\varphi^\ell, \chi^\ell) \|_{L^{2,\sigma}(\mathbb{R}^3)} = 0.
\]

To show Theorem \((1.4)\) we reveal a null structure in $\Pi_0^c \gamma^0 \gamma^5 \Pi_{\theta'}$. Then we use the adapted function space consisting of $V_0^2$ space equipped with angular regularity, which will be introduced in Section 2. By exploiting the null structure, a contraction argument can be readily carried out by the duality as described in \((4)\) \((5)\) \((8)\). We will sketch the proof in Section 3.

## 2. Preliminaries

This section presents the preliminary setup and notations. In what follows we assume $m = 1$ for simplicity of presentation.

### 2.1. Energy projection

We further decompose the system \((8)\) by energy projection $\Pi_\theta$ for $\theta \in \{+, -\}$, which is defined by
\[
\Pi_\theta (\xi) = \frac{1}{2} \left( I_{4 \times 4} + \sigma \xi^j \gamma^0 \gamma^j + \gamma^0 \right) \Lambda(\xi),
\]
where $\Lambda(\xi) = \sqrt{1 + |\xi|^2}$. Here we used the summation convention. Now we define the Fourier multiplier by the identity $\mathcal{F}_x[\Pi_\theta f](\xi) = \Pi_\theta(\xi) \hat{f}(\xi)$ and $\mathcal{F}_x[\Lambda(D) f](\xi) = \Lambda(\xi) \hat{f}(\xi)$, where $D = -i \nabla$. By an easy computation one easily see the identity $\Pi_+ \Pi_\theta = \Pi_\theta$ and $\Pi_- \Pi_\theta = 0$. Then we have $\psi = \sum_{\theta \in \{+, -\}} \Pi_\theta \psi$. Also we see that $\Lambda(D) (\Pi_+ - \Pi_-) = \gamma^0 \gamma^j (\theta \Lambda(D)) + \gamma^0$ and this leads us to the system: For $\theta, \theta' \in \{+, -\}$
\[
(-i \partial_t + \theta \Lambda(D)) \Pi_\theta \varphi = \Pi_\theta [V_b \ast (\overline{\varphi} \gamma^5 \chi + \overline{\chi} \gamma^5 \varphi) \gamma^0 \gamma^5 \varphi],
\]
\[
(-i \partial_t + \theta' \Lambda(D)) \Pi_{\theta'} \chi = \Pi_{\theta'} [V_b \ast (\overline{\varphi} \gamma^5 \chi + \overline{\chi} \gamma^5 \varphi) \gamma^0 \gamma^5 \chi],
\]
\[
\varphi(0) = \Pi_\theta [P_+^c \psi_0], \quad \chi(0) = \Pi_{\theta'} [P_-^c \psi_0].
\]
Hence solving (5) is equivalent to finding solutions \((\varphi_\theta, \chi_{\theta'})\) to the system of equations:

\[
(-i\partial_t + \theta\Lambda(D))\varphi_\theta = \sum_{\theta_j \in (+,-)} \Pi_\theta [V_0 \ast (\overline{\varphi_{\theta_j}} \gamma^5 \chi_{\theta_j} + \overline{\theta_{\theta_j}} \gamma^5 \varphi_{\theta_j}) \gamma^0 \gamma^5 \varphi_{\theta_k}]
\]

\[
(-i\partial_t + \theta'\Lambda(D))\chi_{\theta'} = \sum_{\theta_j' \in (+,-)} \Pi_{\theta'} [V_0 \ast (\overline{\varphi_{\theta_j'}} \gamma^5 \chi_{\theta_j'} + \overline{\theta'_{\theta_j'}} \gamma^5 \varphi_{\theta_j'}) \gamma^0 \gamma^5 \chi_{\theta_k'}],
\]

(10)

The solutions \(\varphi_\theta, \chi_{\theta'}\) clearly satisfy that \(\Pi_{-\theta} \varphi_\theta = \Pi_{-\theta'} \chi_{\theta'} = 0\).

2.2. Revealing null structure.

\[
\gamma^0 \gamma^5 \Pi_\theta (\xi) = i \gamma^1 \gamma^2 \gamma^3 \Pi_\theta (\xi)
\]

\[
= i \gamma^5 + \frac{\theta}{2} \Lambda (\xi) - i \gamma^1 \gamma^2 \gamma^3 (\xi_1 \gamma^0 \gamma^3 + \gamma^0).
\]

Using \(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 0\) for \(\mu \neq \nu\) and \(\gamma^3 \gamma^0 = -I_{4 \times 4}\), we have

\[
i \gamma^1 \gamma^2 \gamma^3 \xi_0 \gamma^0 = -i \gamma^1 \gamma^2 \gamma^3 \gamma^0 = -\gamma^5 = -\gamma^0 \gamma^5,
\]

\[
i \gamma^1 \gamma^2 \gamma^3 \xi_1 \gamma^0 = \xi_1 \gamma^4 \gamma^3 = -\xi_1 \gamma^0 \gamma^4 \gamma^3,
\]

\[
i \gamma^1 \gamma^2 \gamma^3 \xi_2 \gamma^0 = \xi_2 \gamma^0 = -\xi_2 \gamma^0 \gamma^0 \gamma^3,
\]

and

\[
i \gamma^1 \gamma^2 \gamma^3 \xi_3 \gamma^0 \gamma^3 = \xi_3 \gamma^3 \gamma^5 = -\xi_3 \gamma^0 \gamma^3 \gamma^5.
\]

Hence

\[
\gamma^0 \gamma^5 \Pi_\theta (\xi) = \Pi_{-\theta} \gamma^0 \gamma^5.
\]

By this we deduce that

\[
\Pi_\theta (\xi) \gamma^0 \gamma^5 \Pi_\theta (\eta) = \Pi_\theta (\xi) \Pi_{-\theta} (\eta) \gamma^0 \gamma^5
\]

(11)

from which we expect a spinorial null structure. In fact, one can show

\[
|\Pi_\theta (\xi) \gamma^0 \gamma^5 \Pi_\theta (\eta)| \lesssim \angle (\xi, \eta) + \frac{|\theta| \Lambda (\xi) - \theta' |\eta|}{\Lambda (\xi) \Lambda (\eta)},
\]

(12)

where \(\angle (\xi, \eta)\) is the angle between \(\xi\) and \(\eta\). In the scalar case the sign is positive in the second term. For this see [1, 2, 3].

2.3. Adapted function spaces. Let \(I = \{(t_k)_{k=0}^K : t_k \in \mathbb{R}, t_k < t_{k+1}\}\) be the set of increasing sequences of real numbers. We define the 2-variation of \(v\) to be

\[
|v|_{V^2} = \sup_{(t_k)_{k=0}^K \in I} \left( \sum_{k=0}^K \left| v(t_k) - v(t_{k-1}) \right|^2 \frac{2}{L^2} \right)^{\frac{1}{2}}.
\]

Then the Banach space \(V^2\) can be defined to be all right continuous functions \(v : \mathbb{R} \to L^2_x\) such that the quantity

\[
\|v\|_{V^2} = \|v\|_{L^\infty_t L^2_x} + |v|_{V^2}
\]

is finite. Set \(\|u\|_{V^2} = \|e^{i\theta \Lambda (D)}u\|_{V^2}\). We recall basic properties of \(V^2\) space from [4, 13, 15]. In particular, we use the following lemma to prove the scattering result.

**Lemma 2.1** (Lemma 7.4 of [4]). Let \(u \in V^2_\theta\). Then there exists \(f \in L^2_x\) such that \(\|u(t) - e^{-i\theta \Lambda (D)}f\|_{L^2_x} \to 0\) as \(t \to \pm \infty\).
We refer the readers to \[10, 13\] for more details. We fix a smooth function $\rho \in C_0^\infty(\mathbb{R})$ such that $\rho$ is supported in the interval $(\frac{1}{2}, 2)$ and we let
\[
\sum_{\lambda \in 2^\mathbb{Z}} \rho \left( \frac{t}{\lambda} \right) = 1,
\]
and write $\rho_1(t) = \sum_{\lambda \leq 1} \rho(\frac{t}{\lambda})$ with $\rho_1(0) = 1$. Now we define the standard Littlewood-Paley multipliers, for $\lambda \in 2^\mathbb{N}$, $\lambda > 1$:
\[
P_\lambda = \rho \left( |D| \right), \quad P_1 = \rho_1 \left( |D| \right).
\]
For a dyadic number $N > 1$, we define the spherical dyadic decompositions by
\[
H_N(f)(x) = \sum_{\ell = N}^{2N-1} \sum_{m = -\ell}^{\ell} \langle f(|x|\omega), Y_{\ell m}(\omega) \rangle_{L^2_S(\mathbb{S}^2)} Y_{\ell m}(\frac{x}{|x|}),
\]
\[
H_1(f)(x) = \sum_{\ell = 0, 1} \sum_{m = -\ell}^{\ell} \langle f(|x|\omega), Y_{\ell m}(\omega) \rangle_{L^2_S(\mathbb{S}^2)} Y_{\ell m}(\frac{x}{|x|}),
\]
where $Y_{\ell m}$ is the orthonormal spherical harmonics. Since $-\Delta_{S^2} Y_{\ell m} = \ell (\ell + 1) Y_{\ell m}$, by orthogonality one can readily get
\[
\|\Lambda_{S^2} f\|_{L^2_S(\mathbb{S}^2)} \approx \left\| \sum_{N \in 2^{\mathbb{N}}(0)} N^\sigma H_N f \right\|_{L^2_S(\mathbb{S}^2)}.
\]

3. Trilinear estimates

We define the Banach space associated with the adapted space to be the set
\[
F_\sigma^\theta = \{ \phi \in C(\mathbb{R}; L^2_x): \Pi_{-0 \theta} \phi = 0 \text{ and } \| \phi \|_{F_\sigma^\theta} < \infty \} \quad \text{for } \theta \in \{+, -\},
\]
where the norm is defined by
\[
\| \phi \|_{F_\sigma^\theta} := \left( \sum_{\text{dyadic } \lambda, N \geq 1} N^{2\sigma} \| P_\lambda H_N \phi \|_{L^2_{\sigma^\theta}}^2 \right)^{\frac{1}{2}}.
\]
Note that $\sum_{N \geq 1} \Pi_{-0 \theta} H_N \phi = 0$ for all $\phi \in F_\sigma^\theta$ (see [8]).

A crucial part of the proof of Theorem 1.4 is the following trilinear estimate: Let $\sigma > 0$ and $\theta, \theta_j \in \{+, -\}$. Then for any $\varphi \in F_\sigma^\theta$, $\chi \in F_\sigma^\theta$, and $\psi \in F_\sigma^\theta$, there holds
\[
\| \mathfrak{J}^\theta [\Pi \phi (V_\theta * (\phi^1 \gamma_0^5 \gamma_0^5 \psi) )] \|_{F_\sigma^\theta} \lesssim \| \varphi \|_{F_\sigma^\theta} \| \chi \|_{F_\sigma^\theta} \| \psi \|_{F_\sigma^\theta}.
\]
(13)

Here $\mathfrak{J}^\theta[F]$ is the Duhamel integral which reads
\[
\mathfrak{J}^\theta[F] = \int_0^t e^{-\theta(t-t') \Lambda(D)} F(t') \, dt'.
\]

Now we introduce the following frequency-localised bilinear estimates.
Lemma 3.1. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that the following frequency-localised $L^2$-bilinear estimates hold:

$$\|P_\mu H_N((\Pi_\theta P_\lambda H_{N_1} \varphi) + \gamma_0 \psi_{\lambda_2})\|_{L^2 L^2} \lesssim \mu \left( \frac{\min\{\mu, \lambda_1, \lambda_2\}}{\max\{\mu, \lambda_1, \lambda_2\}} \right)^{\delta} \min\{N_1, N_2\} \|P_\lambda H_{N_1} \varphi\|_{V^2_{\theta_1}} \|P_\lambda H_{N_2} \chi\|_{V^2_{\theta_2}}$$

for any dyadic numbers $\mu, \lambda_1, \lambda_2, N, N_1, N_2 \geq 1$.

In view of the null condition (11) and null bound (12), the bilinear form of Lemma 3.1 has the same type null structure as the bilinear form without $\gamma^5$. Hence one can carry out a similar way of proof for Lemma 3.1 to the one of Proposition 3.1 of [8] by using Lemma 3.1 together with modulation estimates and angular one can carry out a similar way of proof for Lemma 3.1 to the one of Proposition 3.1 of [8] by using Lemma 3.1 together with modulation estimates and angular estimates [4] [8] [7]. Indeed, we have for $\mu \lesssim \lambda_1 \approx \lambda_2$,

$$\|P_\mu (\varphi_{\lambda_1, \gamma, \gamma_0, \psi_{\lambda_2}})\|_{L^2_{\theta, \chi}} \lesssim \mu \|\varphi_{\lambda_1, \gamma_0, \psi_{\lambda_2}}\|_{V^2_{\theta_1}} \|\psi_{\lambda_2}\|_{V^2_{\theta_2}},$$

especially when $\theta_1 = \theta_2$. For the proof, see Proposition 3.7 of [24]. On the other hand, we exploit an extra weight of angular regularity to get the bound as for some $\tilde{\theta} > 0$,

$$\|P_\mu H_N(\varphi_{\lambda_1, N_1, \gamma, \psi_{\lambda_2, N_2}})\|_{L^2_{\theta, \chi}} \lesssim \mu \left( \frac{\mu}{\min\{\lambda_1, \lambda_2\}} \right)^{\delta} \min\{N_1, N_2\} \|\varphi_{\lambda_1, N_1}\|_{V^2_{\theta_1}} \|\psi_{\lambda_2, N_2}\|_{V^2_{\theta_2}}.$$

For this we refer to [17]. Hence we simply combine two bound to get

$$\|P_\mu H_N(\varphi_{\lambda_1, N_1, \gamma, \psi_{\lambda_2, N_2}})\|_{L^2_{\theta, \chi}} \lesssim \mu \left( \frac{\mu}{\min\{\lambda_1, \lambda_2\}} \right)^{\delta} \min\{N_1, N_2\} \|\varphi_{\lambda_1, N_1}\|_{V^2_{\theta_1}} \|\psi_{\lambda_2, N_2}\|_{V^2_{\theta_2}},$$

for an arbitrarily small $\delta \ll 1$, where we write $\psi_{\lambda, N} := P_\lambda H_N \psi$, for brevity.

By Lemma 3.1 and the duality argument for the proof of Proposition 3.1 of [8], we obtain the desired trilinear estimate [13].

4. Proof of Theorem 1.4

For $\theta, \theta' \in \{+, -\}$ we consider the set

$$\mathfrak{M} = \{ \varphi := (\varphi_+, \varphi_-, \chi_+, \chi_-) \in F^{\sigma} := F^\sigma_+ \times F^\sigma_- \times F^\sigma_+ \times F^\sigma_- : \|\varphi_\theta\|_{F^\sigma_\theta} \leq 2\|\varphi_\theta(0)\|_{L^{2, \sigma}}, \|\chi_{\theta'}\|_{F^\sigma_{\theta'}} \leq 2\|\chi_{\theta'}(0)\|_{L^{2, \sigma}} \}$$

and for $A, a > 0$, we define the norm

$$\|\varphi\|_{\mathfrak{M}} := a^{-1} \sum_{\theta \in \{+, -\}} \|\varphi_\theta\|_{F^\sigma_\theta} + a^{-1} \sum_{\theta' \in \{+, -\}} \|\chi_{\theta'}\|_{F^\sigma_{\theta'}}.$$

Then $\mathfrak{M}$ is a complete metric space with the metric derived by the norm. Now we let $\mathcal{M} = (\Phi_+, \Phi_-, X_+, X_-)$ be the inhomogeneous solution map on $\mathfrak{M}$ for (10) given by the Duhamel’s principle. Then the definition of the set $\mathfrak{M}$ and trilinear estimate
(13) give us the estimate: For any \( \phi \in \mathcal{F} \),

\[
\sum_{\theta} \| \Phi_{\theta}(\phi) \|_{F_{\sigma}^{\theta}} \\
\leq \sum_{\theta} \| \phi_{\theta}(0) \|_{L^{2,\sigma}} + C(\sum_{\theta_{1} \in \{+,-\}} \| \phi_{\theta_{1}} \|_{L^{2,\sigma}})^2 (\sum_{\theta_{2} \in \{+,-\}} \| \chi_{\theta_{2}} \|_{F^{\sigma}}) \\
\leq \sum_{\theta} \| \phi_{\theta}(0) \|_{L^{2,\sigma}} + 16C(\sum_{\theta_{1}} \| \phi_{\theta_{1}}(0) \|_{L^{2,\sigma}})^2 (\sum_{\theta_{2}} \| \chi_{\theta_{2}}(0) \|_{L^{2,\sigma}}) \\
\leq (1 + 16CAa) \sum_{\theta} \| \phi(0) \|_{L^{2,\sigma}},
\]

and also

\[
\sum_{\theta'} \| X_{\theta'}(\phi) \|_{F_{\sigma}^{\theta'}} \\
\leq \sum_{\theta} \| \chi_{\theta'}(0) \|_{L^{2,\sigma}} + C(\sum_{\theta_{1} \in \{+,-\}} \| \phi_{\theta_{1}} \|_{L^{2,\sigma}})(\sum_{\theta_{2} \in \{+,-\}} \| \chi_{\theta_{2}} \|_{F^{\sigma}})^2 \\
\leq \sum_{\theta} \| \phi_{\theta}(0) \|_{L^{2,\sigma}} + 16C(\sum_{\theta_{1}} \| \phi_{\theta_{1}}(0) \|_{L^{2,\sigma}})(\sum_{\theta_{2}} \| \chi_{\theta_{2}}(0) \|_{L^{2,\sigma}})^2 \\
\leq (1 + 16CAa) \sum_{\theta'} \| \chi_{\theta'}(0) \|_{L^{2,\sigma}}.
\]

Then we put \( a \leq \frac{1}{16CA} \) and deduce that the map \( \mathcal{M} \) is the flow map from \( \mathcal{F} \) onto \( \mathcal{F} \). The trilinear estimate (13) leads us that the map \( \mathcal{M} \) is a contraction on the set \( \mathcal{F} \). Indeed, suppose that we have \( \phi^{1}, \phi^{2} \in \mathcal{F} \). Then we estimate

\[
\sum_{\theta} \| \Phi_{\theta}(\phi^{1}) - \Phi_{\theta}(\phi^{2}) \|_{F_{\sigma}^{\theta}} \\
\leq 16CAa \sum_{\theta} \| \phi^{1}_{\theta} - \phi^{2}_{\theta} \|_{F_{\sigma}^{\theta}} + 8CA^2 \sum_{\theta'} \| \chi^{1}_{\theta'} - \chi^{2}_{\theta'} \|_{F_{\sigma}^{\theta'}}
\]

and

\[
\sum_{\theta'} \| X_{\theta'}(\phi^{1}) - X_{\theta'}(\phi^{2}) \|_{F_{\sigma}^{\theta'}} \\
\leq 16CAa \sum_{\theta'} \| \chi^{1}_{\theta'} - \chi^{2}_{\theta'} \|_{F_{\sigma}^{\theta'}} + 8CA^2 \sum_{\theta'} \| \phi^{1}_{\theta'} - \phi^{2}_{\theta'} \|_{F_{\sigma}^{\theta'}}.
\]

In consequence we obtain

\[
\| \mathcal{M}(\phi^{1}) - \mathcal{M}(\phi^{2}) \|_{\mathcal{F}} \leq 24CA \sum_{\theta} \| \phi^{1}_{\theta} - \phi^{2}_{\theta} \|_{F_{\sigma}^{\theta}} + 24C \sum_{\theta'} \| \chi^{1}_{\theta'} - \chi^{2}_{\theta'} \|_{F_{\sigma}^{\theta'}} \\
= 24CAa \| \phi^{1} - \phi^{2} \|_{\mathcal{F}}.
\]

Thus by choosing \( \epsilon = \frac{1}{48C} \), the solution map \( \mathcal{M} \) is a contraction on \( \mathcal{F} \) for any \( a \leq \epsilon A^{-1} \). The scattering follows immediately from Lemma 2.1.
5. Appendix

5.1. Proof of (7). We prove (7). By a direct calculation one can readily show that $\bar{P}_\theta \psi \gamma^5 P_\theta \psi$ is purely imaginary. We now write

$$\bar{P}_\theta \psi \gamma^5 P_\theta \psi = \frac{1}{4} (\psi^\dagger + \theta \psi^T (i\gamma^2)) \gamma^0 \gamma^5 (\psi + \theta i\gamma^2 \psi^*),$$

(A1)

$$= \psi^\dagger \gamma^0 \gamma^5 \psi + \psi^\dagger \gamma^0 \gamma^5 (i\gamma^2) \psi^* + \psi^T (i\gamma^2) \gamma^0 \gamma^5 \psi + \psi^T (i\gamma^2) \gamma^0 \gamma^5 (i\gamma^2) \psi^*.$$

Then by the relation $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 0 (\mu \neq \nu)$ and $\gamma^2 \gamma^2 = -I_{4 \times 4}$ we have

$$\psi^T (i\gamma^2) \gamma^0 \gamma^5 (i\gamma^2) \psi^* = \psi^T (i\gamma^2) \gamma^0 \gamma^5 \psi^*.$$

Since $(\psi^\dagger \gamma^0 \gamma^5 \psi)^* = \psi^T \gamma^0 \gamma^5 \psi^*$, the sum of the first and last terms in (A1) are real-valued. By definition of $\gamma^2$, $(\gamma^2)^* = -\gamma^2$ and hence the sum of the second and third terms in (A1) are also real-valued. The LHS of (A1) is purely imaginary, whereas the RHS is purely real. Therefore $\bar{P}_\theta \psi \gamma^5 P_\theta \psi = 0$.

5.2. From (8) to (1). Let $(\varphi, \chi)$ be the solution of (8) satisfying the condition of Theorem 14. Taking $P_c$ to the first equation of (8), we have the equation

$$-i\gamma^\mu \partial_\mu P_c^+ \varphi + m P_c^+ \varphi = V_b \ast (\varphi \gamma^5 \chi + \chi \gamma^5 \varphi) \gamma^5 P_c^+ \varphi.$$

Hence $P_c^+ \varphi$ is the solution with initial data $P_c^+ \varphi(0) = P_c^+ P_c^+ \psi(0) = 0$ and it can be written as

$$\Pi_\theta (P_c^+ \varphi) = i \int_0^t e^{-\theta (t-t')A(D)} \Pi_\theta [V_b \ast (\varphi \gamma^5 \chi + \chi \gamma^5 \varphi) \gamma^5 P_c^+ \varphi] dt'.$$

By trilinear estimates (13) and the choice of $a$ we have

$$\sum_\theta \| \Pi_\theta P_c^+ \varphi \|_{F^\theta} \leq 16 C a A \sum_\theta \| \Pi_\theta P_c^+ \varphi \|_{F^\theta} \leq \frac{1}{2} \sum_\theta \| \Pi_\theta P_c^+ \varphi \|_{F^\theta}.$$

Therefore $P_c^+ \varphi = 0$. In the same way we deduce that $P_c^+ \chi = 0$. We now write the system (8) as

$$-i\gamma^\mu \partial_\mu \varphi + m \varphi = V_b \ast (P_c^+ \gamma^5 P_c^+ \varphi + \gamma^5 P_c^+ \varphi) \gamma^5 \varphi,$$

$$-i\gamma^\mu \partial_\mu \chi + m \chi = V_b \ast (P_c^+ \gamma^5 P_c^+ \chi + \chi \gamma^5 P_c^+ \varphi) \gamma^5 \chi.$$

(A2)

Taking $P_c^+$ and $P_c^-$ to (A2), we finally get

$$-i\gamma^\mu \partial_\mu P_c^+ \varphi + m P_c^+ \varphi = V_b \ast (P_c^+ \varphi \gamma^5 P_c^+ \varphi + \gamma^5 P_c^+ \varphi) \gamma^5 P_c^+ \varphi,$$

$$-i\gamma^\mu \partial_\mu P_c^- \varphi + m P_c^- \varphi = V_b \ast (P_c^+ \varphi \gamma^5 P_c^+ \varphi + \gamma^5 P_c^+ \varphi) \gamma^5 P_c^- \varphi.$$

By setting $\psi = P_c^+ \varphi + P_c^- \chi$ we obtain the original equation (1).

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Data Availability

The data that support the findings of this study are available within the article.
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