RECURSION OPERATOR AND
RATIONAL LAX REPRESENTATION

K. Zheltukhin
Department of Mathematics, Faculty of Sciences
Bilkent University, 06533 Ankara, Turkey
Phone: 90 312 2901938 (office)
Fax : 90 312 2664579
e-mail: zhelt@fen.bilkent.edu.tr

Abstract

We consider equations arising from rational Lax representations. A general method to construct recursion operators for such equations is given. Several examples are given, including a degenerate bi-Hamiltonian system with a recursion operator.

PACS: 0230.Ik; 0230.Sr

Keywords: integrable system, recursion operator.
I. Introduction

Recently a new method of constructing a recursion operator from Lax representation was introduced in [1]. This construction depends on Lax representation of a given system of PDEs. Let

\[ L_t = [A, L] \]  

be Lax representation of an integrable nonlinear system of PDEs. Then a hierarchy of symmetries can be given by

\[ L_{tn} = [A_n, L], \quad n = 0, 1, 2 \ldots \]  

where \( t_0 = t, A_0 = A \) and \( A_n, \quad n = 0, 1, 2 \ldots \) are Gel’fand-Dikii operators given in terms of \( L \). The recursion relation between symmetries can be written as

\[ L_{tn+1} = LL_{tn} + [R_n, L], \quad n = 0, 1, 2 \ldots \]  

where \( R_n \) is an operator such that \( ordR_n = ordL \).

This symmetry relation allows to find \( R_n \), hence \( L_{tn+1} \), in terms of \( L \) and \( L_{tn} \).

In [1], [2] this method was applied to construct recursion operators for Lax equations with different classes of scalar and shift operators, corresponding to field and lattice systems respectively. In [3] the method was applied to Lax equations on a Poisson algebra of Laurent series

\[ \Lambda = \left\{ \sum_{-\infty}^{+\infty} u_i p^i : u_i \text{ - smooth functions} \right\} \]
with the polynomial Lax function. Such equations give systems of hydrodynamic type. They were also discussed in [4]–[7]. The Hamiltonian structure of the Lax equation on a Poisson algebra was studied in [8].

Here we consider the Lax equation on the Poisson algebra $\Lambda$ with a rational Lax function

$$L = \frac{\Delta_1}{\Delta_2},$$

(5)

where $\Delta_1, \Delta_2$ are polynomials of degree $N$ and $M$, respectively, and $N > M$. The Lax equation is

$$\frac{\partial L}{\partial t_n} = \{(L)_{n}^{N-M+n}, L\},$$

(6)

where the Poisson bracket is given by

$$\{f, g\} = p \left( \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} \right).$$

First we study the symmetry relation (3) for the rational Lax function. Then we give some examples.

In particular, we find a recursion operator $R$ for equation (3) with the Lax function

$$L = p + S + \frac{P}{p+Q},$$

(7)

which leads to the system (4)

$$S_t = P_x,$$
$$P_t = PS_x - QP_x - PQ_x,$$
$$Q_t = QS_x - QQ_x.$$

(8)
The recursion operator is given by

\[
R = \begin{pmatrix}
S & 1 & PQ^{-1} + P_x D_x^{-1} \cdot Q \\
2P & S - Q & -2P + (PS_x - (PQ)_x) D_x^{-1} \cdot Q \\
Q & 1 & PQ^{-1} + S - Q + (QS_x - QQ_x) D_x^{-1} \cdot Q
\end{pmatrix}
\]  

(9)

In [4] bi-Hamiltonian representation of this equation was constructed with Hamiltonian operators

\[
D_1 = \begin{pmatrix} 0 & P & Q \\ P & -2PQ & -Q^2 \\ Q & -Q^2 & 0 \end{pmatrix} D_x + \begin{pmatrix} 0 & P_x & Q_x \\ 0 & -(PQ)_x & -QQ_x \\ 0 & -QQ_x & 0 \end{pmatrix}
\]  

(10)

and

\[
D_2 = \begin{pmatrix} 2P & P(S - 3Q) & Q(S - Q) \\ P(S - 3Q) & P(2P - 2SQ + 4Q^2) & Q(2P - SQ + Q^2) \\ Q(S - Q) & Q(2P - SQ + Q^2) & 2Q^2 \end{pmatrix} D_x +
\]

(11)

These Hamiltonian operators are degenerate, so, one can not use them to find a recursion operator. But it turns out that they are related to the recursion operator \(R\). One can easily check that the following equality holds

\[RD_1 = D_2.\]
We observe that the degeneracy in the bi-Hamiltonian operators is due to the following fact. Let \( p' = p + F \) then the Lax function becomes

\[
L = p' + G + \frac{P}{p'}.
\]

(12)

This means that we have two independent variables \( P \) and \( G \), where \( G = S - F \). The equation corresponding to the Lax function (12) has been studied in [3].

To remove degeneracy one can take the Lax function as

\[
L = p + S + \frac{P}{p} + \sum_{i=1}^{m} \frac{Q_i}{p + F_i}.
\]

(13)

As an example we shall consider the equation (6) with the Lax function

\[
L = p + S + \frac{P}{p} + \frac{Q}{p + F}.
\]

(14)

II. Symmetry Relation for Rational Lax Representation.

Following [1] we consider the hierarchy of symmetries for the Lax equation (6) with the Lax function (5)

\[
\partial L \partial t_n = \{(L^{n+1+n} + n)_{\geq 0}, L\}.
\]

(15)

Lemma 1. For any \( n = 0, 1, 2, \ldots \),

\[
\partial L \partial t_n = L \frac{\partial L}{\partial t_{n-1}} + \{R_n, L\}.
\]

(16)
Function $R_n$ has a form

$$R_n = A + \frac{B}{\Delta_2}$$  \hspace{1cm} (17)

where $A$ is a polynomial of degree $(N - M)$ and $B$ is a polynomial of degree $(M - 1)$.

**Proof.** We have

$$(L^{\frac{1}{N-M} + n})_{\geq 0} = [L(L^{\frac{1}{N-M} + (n-1)})_{\geq 0} + L(L^{\frac{1}{N-M} + (n-1)})_{< 0}]_{\geq 0}$$

So,

$$(L^{\frac{1}{N-M} + n})_{\geq 0} = L(L^{\frac{1}{N-M} + (n-1)})_{\geq 0} + (L(L^{\frac{1}{N-M} + (n-1)})_{< 0})_{\geq 0} - (L(L^{\frac{1}{N-M} + (n-1)})_{\geq 0})_{< 0}.$$ 

If we take

$$R_n = (L(L^{\frac{1}{N-M} + (n-1)})_{< 0})_{\geq 0} - (L(L^{\frac{1}{N-M} + (n-1)})_{\geq 0})_{< 0} ,$$  \hspace{1cm} (18)

then

$$(L^{\frac{1}{N-M} + n})_{\geq 0} = L(L^{\frac{1}{N-M} + (n-1)})_{\geq 0} + R_n.$$ 

Hence,

$$\frac{\partial L}{\partial t_n} = \left\{ (L^{\frac{1}{N-M} + n})_{\geq 0}; L \right\} = \left\{ L(L^{\frac{1}{N-M} + (n-1)})_{\geq 0} + R_n; L \right\} = L \frac{\partial L}{\partial t_n} + \{ R_n; L \},$$

and (16) is satisfied. The remainder $R_n$ has form (17). Indeed in (18) we set

$$A = (L(L^{\frac{1}{N-M} + (n-1)})_{< 0})_{\geq 0}$$

and

$$B = \Delta_2 \cdot (L(L^{\frac{1}{N-M} + (n-1)})_{\geq 0})_{< 0}.$$
Then $A$ is a polynomial of degree $(N - M - 1)$ and $B$ is a polynomial of degree $(M - 1)$. □

Now we can apply the Lemma 1 to find recursion operators.

**III. Examples.**

**Example 2.** Let us consider the equation (8) given in introduction.

**Lemma 3.** A recursion operator for (8) is given by (9).

**Proof.** Using (17) for $R_n$, we have $R_n = A + \frac{B}{p + Q}$. So, the symmetry relation (16) is

$$
\frac{\partial S}{\partial t_n} + \frac{\partial P}{\partial t_n} \cdot \frac{1}{p + Q} + \frac{\partial Q}{\partial t_n} \cdot \frac{P}{(p + Q)^2} = 
$$

$$
\left( p + S + \frac{P}{p + Q} \right) \left( \frac{\partial S}{\partial t_{n-1}} + \frac{\partial P}{\partial t_{n-1}} \cdot \frac{1}{p + Q} + \frac{\partial Q}{\partial t_{n-1}} \cdot \frac{P}{(p + Q)^2} \right) + 
$$

$$
p \left( A_x + \frac{B_x}{p + Q} + \frac{-BQ_x}{(p + Q)^2} \right) \left( 1 + \frac{-P}{(p + Q)^2} \right) 
$$

$$
- \frac{pB}{(p + Q)^2} \left( S_x + \frac{P_x}{p + Q} + \frac{-PQ_x}{(p + Q)^2} \right) 
$$

To have the equality the coefficients of $p$ and $(p + Q)^{-3}$ must be zero. It gives the recursion relations to find $A$ and $B$. Then the coefficients of $p^0$, $(p + Q)^{-1}$, $(p + Q)^{-2}$ give expressions for $\frac{\partial S}{\partial t_n}$, $\frac{\partial P}{\partial t_n}$, $\frac{\partial Q}{\partial t_n}$. □

**Example 4.** The Lax equation (8) with the Lax function (14), for $n = 1$, gives the following system
\[ S_t = P_x + Q_x, \]
\[ P_t = P S_x, \]
\[ Q_t = Q S_x - F Q_x - Q F_x, \]
\[ F_t = F S_x - F F_x. \]  

(19)

**Lemma 5.** A recursion operator for (19) is given by

\[
\begin{pmatrix}
S & 2 + P_x D_x^{-1} \cdot P^{-1} & 1 & Q F^{-1} + Q_x D_x^{-1} \cdot F^{-1} \\
2P & S + Q F^{-1} + P S_x D_x^{-1} \cdot P^{-1} & P F^{-1} & -2PQF^{-2} \\
2Q & -Q F^{-1} & S - F & -2PQF^{-2} - 2Q \\
F & 1 + (P_x - P F^{-1} F_x) D_x^{-1} \cdot P^{-1} & -1 & P F^{-1} - F + (F S_x - F F_x) D_x^{-1} \cdot F^{-1} \\
\end{pmatrix}
\]

(20)

**Proof.** Using (17) for \( R_n \), we have \( R_n = C + \frac{A}{p} + \frac{B}{p + F} \). So, the symmetry relation (10) is

\[
\frac{\partial S}{\partial t_n} + \frac{\partial P}{\partial t_n} \cdot \frac{1}{p} + \frac{\partial Q}{\partial t_n} \cdot \frac{1}{(p + F)} + \frac{\partial F}{\partial t_n} \cdot \frac{-Q}{(p + F)^2} =
\]

\[
\left( p + S + \frac{P}{p} + \frac{Q}{p + F} \right) \left( \frac{\partial S}{\partial t_{n-1}} + \frac{\partial P}{\partial t_{n-1}} \cdot \frac{1}{p} + \frac{\partial Q}{\partial t_{n-1}} \cdot \frac{1}{(p + F)} + \frac{\partial F}{\partial t_{n-1}} \cdot \frac{-Q}{(p + F)^2} \right) +
\]

7
\[ p \left( \frac{-B}{p^2} + \frac{-C}{(p+F)^2} \right) \left( S_x + \frac{P_x}{p} + \frac{Q_x}{(p+F)} + \frac{-QF_x}{(p+F)^2} \right) - \\
p \left( A_x + \frac{B_x}{p} + \frac{C_x}{(p+F) + \frac{-CF_x}{(p+F)^2}} \right) \left( 1 + \frac{P}{p} + \frac{-Q}{(p+F)^2} \right) \]

Therefore, the coefficients of \( p \), \( p^{-2} \), and \((p+F)^{-3}\) must be zero, it gives recursion relations to find \( A \), \( B \) and \( C \). Then the coefficients of \( p^0 \), \( p^{-1} \), \((p+F)^{-1}\) and \((p+F)^{-2}\), give expressions for \( \frac{\partial S}{\partial t_n} \), \( \frac{\partial P}{\partial t_n} \), \( \frac{\partial Q}{\partial t_n} \) and \( \frac{\partial F}{\partial t_n} \).

\[ \square \]

Acknowledgments

I thank Professors Metin Gürses, Atalay Karasu and Maxim Pavlov for several discussions. This work is partially supported by the Scientific and Technical Research Council of Turkey.

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