A NOTE ON ERDŐS-KO-RADO SETS OF GENERATORS IN HERMITIAN POLAR SPACES

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Abstract. The size of the largest Erdős-Ko-Rado set of generators in a finite classical polar space is known for all polar spaces except for $H(2d-1, q^2)$ when $d \geq 5$ is odd. We improve the known upper bound in this remaining case by using a variant of the famous Hoffman’s bound.

1. INTRODUCTION

An Erdős-Ko-Rado set of generators in a finite classical polar space is a set of generators of the polar space that have mutually non-trivial intersection. The largest size of an Erdős-Ko-Rado set of generators in a finite classical polar space was determined in [7] for all finite classical polar spaces except for the hermitian polar space $H(2d-1, q^2)$ of odd rank $d \geq 5$. Here rank means vector space rank and not projective dimension. The best known upper bound for this remaining case was proven in [4]. The idea of the proof was to formulate a linear optimization problem whose solution gives an upper bound. This idea goes back to Delsarte and uses the primitive idempotents of the association scheme related to the set of generators of a polar space, see [1]. In [4] we were however not able to determine the optimal solution of the optimization problem. Using a slightly different approach, the previous bound can be improved as follows.

Theorem 1. If $S$ is an Erdős-Ko-Rado set of generators of $H(2d-1, q^2)$, $d \geq 5$ odd, then

\begin{equation}
|S| \leq ((q^2 + q + 1)q^{2d-3} + 1) \prod_{2 \leq d \leq 1} (q^{2^d-1} + 1).
\end{equation}

Remark. 1 The set consisting of all generators of $H(2d-1, q^2)$, $d$ odd, on a point has size roughly $q^{d^2}$ whereas the bound given in the theorem has size roughly $q^{d^2+1}$.

2 It can be shown that equality can not occur in the theorem. At the end of Section 2 we sketch a proof of this fact.

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3. The proof of Theorem 1 does not use linear optimization. For small \( d \) it can be checked by computer that the bound (1) is in fact the solution of the optimization problem mentioned above. I am convinced this is true for all \( d \).

2. Proof of the theorem

Consider the graph whose vertices are the generators of the hermitian polar space \( H(2d - 1, q^2) \) of odd rank \( d \geq 3 \). Let \( N \) be the number of generators and number them as \( G_1, \ldots, G_N \). For \( 0 \leq i \leq d \), let \( A_i \) be the real symmetric \((N \times N)\)-matrix whose \((r, s)\)-entry is 1, if \( G_r \cap G_s \) has rank \( d - i \), and 0 otherwise. These real matrices are symmetric and commute pairwise, so they can simultaneously be diagonalized.

It is known that there are exactly \( N \) common eigenspaces \( V_0, \ldots, V_d \) of these matrices. Also one of the eigenspaces is \( \langle j \rangle \) where \( j \) is the all one vector of length \( N \). We choose notation so that \( V_0 = \langle j \rangle \). If \( P_{t,j} \) denotes the eigenvalue of \( A_j \) on \( V_t \), then with a suitable ordering of the eigenspaces we have, see [8], Theorem 4.3.6

\[
P_{t,d} = (-1)^i q^{(d-i)^2 + i(i-1)},
\]

\[
P_{t,d-2} = \frac{2}{d} (-1)^{i+u} \left[ \begin{array}{c} d - i \\ 2 - u \end{array} \right] q^{(d-2+u-i)^2 + (i-u)(i-u-1)}. \]

We want to apply Hoffman’s bound (see below) to the generalized adjacency matrix \( A := A_d - fA_{d-2} \) where we use for \( f \) the value for which the smallest eigenvalue of \( A \) is as large as possible. The eigenvalues of \( A \) are of course \( P_{t,d} - fP_{t,d-2}, i = 0, \ldots, d \). An investigation shows that the best choice for \( f \) is obtained when \( P_{d,d} - fP_{d,d-2} = P_{1,d} - fP_{1,d-2} \), which results in the following definition.

\[
f := \frac{(q^{d-1} - 1)q^{4(d-2)}}{\left[ \begin{array}{c} d-1 \\ 2 \end{array} \right] q^{2d-5} - \left[ \begin{array}{c} d-1 \\ 2 \end{array} \right] q^{d-3}}.
\]

A direct calculation shows that

\[
f = \frac{(q^{2d} - 1)(q^{d-1} - 1)q^{4(d-2)}}{\left[ \begin{array}{c} d \\ 2 \end{array} \right] (q^{d-2} - 1)(q^{d-1} - 1)q^{4(d-2)}}.
\]

\[
= \frac{(q^2 - 1)(q^4 - 1)(q^{d-1} - 1)q^{4(d-2)}}{\left[ \begin{array}{c} d \\ 2 \end{array} \right] (q^{d-2} - 1)(q^{d-1} - 1)q^{4(d-2)}}.
\]

\[
< q^2 - 1.
\]

Lemma 1. The matrix \( A \) has constant row sum

\[
K = q^2 - f \left[ \begin{array}{c} d \\ 2 \end{array} \right] q^{4(d-2)^2} > 0.
\]

Proof. The row sum of \( A \) is the eigenvalue of \( A \) on the eigenspace \( \langle j \rangle \). Using \( f < q^2 - 1 \), it follows that \( K > 0 \).

Lemma 2. The smallest eigenvalue of \( A \) is

\[
\lambda := -q^{d(d-1)} + f \left[ \begin{array}{c} d \\ 2 \end{array} \right] q^{4(d-2)(d-3)}
\]

and we have \( \lambda < -q^{d^2 - 2d + 2} \).
Lemma 3. It follows from the list of eigenvalues that \( \lambda \) is the eigenvalue of \( A = A_d - fA_{d-2} \) on the eigenspace \( V_d \). Also, the way we determined \( f \), namely in such a way that

\[
P_{d,d} - fP_{d,d-2} = P_{1,d} - fP_{1,d-2}
\]

shows that the eigenvalue of \( A \) on \( V_1 \) is also \( \lambda \). A straightforward calculation shows that

\[
\lambda = -\frac{(q + 1)(q^{2d} - q^{2d-3} + q - 1)q^{d^2-d-2}}{(q^{d-2} + 1)(q^{2d-1} - q^{d-2} - q^{d-3} + 1)}
\]

and, using this expression, it is easy to see that \( \lambda < -q^{d^2-2d+2} \).

The eigenvalue of \( A \) on \( V_0 \) is \( K \) and the previous lemma shows that \( K > 0 \). For \( 2 \leq i \leq d - 1 \), the eigenvalue of \( A \) on \( V_i \) is \( P_{i,d} - fP_{i,d-2} \), and we show in the remaining part of the proof that this eigenvalue is larger than \( \lambda \).

First consider the case when \( i \) is odd. Then in the above formula for \( P_{i,d-2} \) as a sum over \( u \in \{0, 1, 2\} \), only the term corresponding to \( u = 1 \) is positive. Hence

\[
P_{i,d-2} \leq \left[ \frac{d-i}{1} \right] q^{(d-1-i)^2+(i-1)(i-2)} \leq \frac{q^{(d-i)^2+i^2-i+3}}{(q^2-1)^2}.
\]

Using \( f < q^2 - 1 \), we obtain the following bound for the eigenvalue of \( A \) on \( V_i \),

\[
P_{i,d} - fP_{i,d-2} \geq P_{i,d} - f \cdot \frac{q^{(d-i)^2+i^2-i+3}}{(q^2-1)^2} \geq -q^{(d-i)^2+i^2-i} - \frac{q^{(d-i)^2+i^2-i+3}}{q^2-1} \geq -q^{d^2-2d+2} > \lambda.
\]

Here we have used that \( 3 \leq i \leq d - 2 \) (since \( i \) is odd).

If \( i \) is even, then \( P_{i,d} > 0 \) and it is not difficult to see that \( P_{i,d-2} < 0 \). In this case the eigenvalue \( P_{i,d} - fP_{i,d-2} \) of \( A \) on \( V_i \) is positive.

Lemma 3. If \( N \) is the number of generators of \( H(2d-1, q^2) \), then

\[
\frac{-\lambda N}{K - \lambda} = ((q^2 + q + 1)q^{2d-3} + 1) \prod_{\substack{r \geq 1 \\text{odd} \\text{and} \ r \leq d - 1}} (q^{2r-1} + 1).
\]

Proof. We denote by \( f_1 \) and \( f_2 \) the nominator and denominator in the definition of \( f \). We have

\[
\frac{-\lambda}{K - \lambda} = \frac{q^{d(d-1)}f_2 - (q^{d-1} - 1)q^{d(d-2)}[d]_2}{q^{d^2} + q^d(d-1)}f_2 - (q^{d-1} - 1)q^{d(d-2)}[d]_2(q^{d-2} + q(d-2)(d-3))
\]

\[
= \frac{q^2f_2 - (q^{d-1} - 1)[d]_2}{q^2(q^d + 1)f_2 - (q^{d-1} - 1)[d]_2(q^{d-2} + 1)}
\]

An easy calculation gives

\[
f_2(q^{2d} - 1) = \left[ \frac{d}{2} \right] g
\]

where

\[
g = (q^4 - 1)q^{2d-5} - (q^{2d-4} - 1) + (q^{2d} - 1)q^{d-3}.
\]
Hence
\[
\frac{-\lambda}{K - \lambda} = \frac{q^2 g - (q^{d-1} - 1)(q^d - 1)}{q^2 (q^d + 1) g - (q^{d-1} - 1)(q^{2d} - 1)(q^{d-2} + 1)}.
\]
Thus
\[
\frac{-\lambda(q^d + 1)}{K - \lambda} = \frac{q^2 g - (q^{d-1} - 1)(q^d - 1)}{q^2 g - (q^{d-1} - 1)(q^d - 1)(q^{d-2} + 1)} = 1 + \frac{(q^{d-1} - 1)(q^d - 1)q^{d-2}(1 - q^2)}{(q^2 - 1)(q^{2d-1} + 1)(q^{d-2} + 1)} = 1 - \frac{(q^{d-1} - 1)(q^d - 1)q^{d-2}}{(q^{2d-1} + 1)(q^{d-2} + 1)} = \frac{q^2 + q + 1}{(q^{d-2} + 1)}.
\]
As \( N = \prod_{i=1}^{d}(q^{2i-1} + 1) \), the assertion follows. \( \square \)

Let \( G \) be a simple and non-empty graph with \( N \) vertices \( v_1, \ldots, v_N \). A real symmetric \( N \times N \) matrix \( A \) with diagonal entries zero is called an extended weight matrix of \( G \) if \( A_{rs} \leq 0 \) whenever \( r \neq s \) and \( \{v_r, v_s\} \) is not an edge of the graph \( G \), and if \( A_{rs} \neq 0 \) for at least one edge \( \{v_r, v_s\} \) of the graph. It is called a \( K \)-regular extended weight matrix, if it has in addition constant row sum \( K \). The following result appeared in various forms in the literature, we present it in the form of Corollary 3.3 in [2] but it already appeared in Lemma 6.1 of [3] when applied to the matrix \( A - \lambda I \), and it was also mentioned in [5].

**Theorem 2 ([2]).** Let \( \Gamma \) be a finite simple and non-empty graph with \( N \) vertices and suppose that \( A \) is a \( K \)-regular generalized weight matrix of \( G \) with least eigenvalue \( \lambda \). Then every independent set \( S \) of \( G \) satisfies
\[
|S| \cdot (K + |\lambda|) \leq |\lambda| \cdot N.
\]

From Theorem 2 applied to \( A = A_d - \frac{1}{2} A_{d-2} \) and from Lemma 3, we find that an Erdős-Ko-Rado set \( S \) of \( H(2d - 1, q^2) \) satisfies the inequality (1) of Theorem 1. This completes the proof of Theorem 1.

We remark that equality in (1) is impossible and sketch a proof. Suppose that \( |S| \) satisfies the bound (1) with equality. Then the standard proof of Theorem 2 gives information on the characteristic vector \( \chi \) of \( S \), in fact, it must lie in the span of the all-one-vector \( j \) and the eigenspace of \( A \) for the eigenvalue \( \lambda \). Our arguments show that this eigenspace of \( A \) is \( V_1 + V_d \), hence \( \chi = \frac{|S|}{N} j + v_1 + v_d \) with \( v_1 \in V_1 \) and \( v_d \in V_d \) (here we use \( j^T v_1 = j^T v_d = 0 \), \( j^T j = N \) and \( j^T \chi = |S| \)). The vectors \( v_1 \) and \( v_d \) are also eigenvectors of \( A_1, \ldots, A_d \) and the eigenvalues are known. Since \( S \) is an Erdős-Ko-Rado set, the entries of \( A_d \chi \) corresponding to elements of \( S \) are zero. This gives a linear equation for the entries \( a_1 \) and \( a_d \) of \( v_1 \) and \( v_d \) corresponding to some element of \( S \). A second linear equation comes from \( \chi = \frac{|S|}{N} j + v_1 + v_d \), namely \( 1 = \frac{|S|}{N} + a_1 + a_d \). The two equations are linearly independent, so \( a_1 \) and \( a_d \) can be calculated and are of course independent of the element of \( S \). With this information the entries of \( A_i \chi \) corresponding to elements of \( S \) can be calculated for all \( i \), which gives the number of elements of \( S \) that meet a given element of \( S \) in
dimension $d - i$. It turns out that not all these numbers are integers, which is the desired contradiction. The same argument was used in the last section of [6].

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