Research Article

Stabilization for the Stochastic Heat Equation with Boundary Control

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This paper investigates the stabilization of an unstable stochastic heat equation. By the backstepping technique, a boundary feedback control is designed to stabilize the unstable stochastic system, including the 2nd-moment and almost-sure exponential stabilization. We also prove that the system with a disturbance in the control channel possesses good robust stability under our control strategy and suitable conditions. Finally, we also provide numerical simulations which illustrate the effectiveness of the theoretical results.

1. Introduction

From the application point of view, many practical systems are often subjected to stochastic disturbances. A natural and widely acceptable way of describing stochastic factors is using white noises, and such a class of induced systems is called stochastic systems (see [1, 2]). Since the early 1990s, the study of stochastic systems has become a hot topic in control theory. There are many studies on this topic, and we would mention [3–6] for the related work. In particular, Cheng et al. discussed the problem of hidden Markov model-based control for periodic systems subject to singular perturbations in [3]. However, there are very limited studies on the stabilization of stochastic partial differential equations. This paper aims to investigate the stabilization of the stochastic heat equation with boundary control.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a complete filtered probability space, on which a one-dimensional standard Brownian motion \(\{B_t\}_{t \geq 0}\) is defined. (The definition of the complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) can be found in page 201 of [1].) We denote by \(\omega\) a sample point of \(\Omega\), by \(E(\cdot)\) the expectation with respect to the probability measure \(P\), and by \(L^2(\Omega, \mathcal{F}_0, P; L^2(0, 1))\) the space consisting of all \(L^2(0, 1)\)-valued \(\mathcal{F}_0\)-measurable random variable \(X(\cdot)\) such that \(E[X(\cdot)]^2 < +\infty\). Let \(a > 0\) and \(f(\cdot)\) be a real-valued \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted process in \(L^2_\infty(0, +\infty; L^\infty(0, 1))\), which is the space consisting of all \(L^\infty(0, 1)\)-valued \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted bounded process.
This paper investigates the following controlled stochastic heat equation driven by a multiplicative noise:

\[
\begin{cases}
  du(x,t) - u_{xx}(x,t) \, dt - au(x,t) \, dt = f(t)u(x,t) \, dB_t, & (x,t) \in (0,1) \times (0, +\infty), \\
  u(0,t) = 0, & t \in (0, +\infty), \\
  u(1,t) = U(t), & t \in (0, +\infty), \\
  u(x,0) = u_0(x), & x \in (0,1),
\end{cases}
\]

where the initial datum \(u_0(x) \in L^2(\Omega, \mathcal{F}_0, P; L^2(0,1))\) and \(U(\cdot)\) is the control function. Here and in what follows, we omit the variable \(\omega \in \Omega\) in the defined functions if there is no risk of causing any confusion.

For the deterministic case, namely, \(f(\cdot) \equiv 0\), the stabilization of the heat equation has been widely studied (see [7, 8]). However, the study on the stability of stochastic controlled equations has turned out to be a difficult problem, even for linear finite-dimensional stochastic equations. When \(f(\cdot) \neq 0\), it is well known that stochastic heat equation (1) is unstable if the control \(U(\cdot) \equiv 0\) and the number \(a\) is big enough. This point will be discussed in detail in the next section. In this article, we mainly study the stabilization for unstable stochastic heat equation (1). We mention [9] in this direction. The backstepping method is employed to deal with destabilizing terms in system equation (1). The backstepping method is a systematic boundary feedback control approach for partial differential equations (PDEs), which uses an integral transformation to convert an unstable PDE into a stable one. By this technique, the destabilizing terms are eliminated through an invertible integral transformation of the PDEs with boundary feedback. Nowadays, the backstepping method is successfully applied to many types of PDEs. A systematic discussion of this method can be found in the literature [7, 10, 11] and the references therein. The main contribution of the current work is the extension of the backstepping technique to the study of the stabilization of stochastic heat equations, even when there is a disturbance in the control channel.

We organize the paper as follows: In Section 2, some preliminary results are presented. Sections 3 and 4 are devoted to the main results and their proofs. Section 5 shows the numerical simulation results.

### 2. Preliminary Results

We first define that

\[
\mu_k = k^2 \pi^2, \quad \text{and} \quad e_k(x) = \sqrt{2} \sin(k\pi x), \quad (k = 1, 2, 3, \ldots).
\]

Then, \(\{e_k(x)\}_{k=1}^{\infty}\) constitutes an orthonormal basis of \(L^2(0,1)\). Indeed, \(\{\mu_k\}_{k=1}^{\infty}\) is the set of eigenvalues of the operator \(-\Delta\) with homogeneous Dirichlet boundary conditions, and \(\{e_k\}_{k=1}^{\infty}\) is the set of the corresponding eigenfunctions. We denote by \(\| \cdot \|\) and \(\langle \cdot, \cdot \rangle\) the canonical norm and the inner product of \(L^2(0,1)\), respectively, without specific explanation in what follows. It is now useful to provide several definitions of stability that we will use in this paper.

**Definition 1.**

(i) Stochastic equation (1) is said to be second-moment exponentially stable if there exist positive numbers \(C\) and \(\rho\) such that for any initial datum \(u_0(x) \in L^2(\Omega, \mathcal{F}_0, P; L^2(0,1))\), the solution to equation (1) satisfies

\[
E\|u(x,t)\|^2 \leq Ce^{-\rho t}\|u_0\|^2, \quad \text{for} \ t \geq 0.
\]

(ii) Stochastic equation (1) is said to be second-moment asymptotically stable if for any initial datum \(u_0(x) \in L^2(\Omega, \mathcal{F}_0, P; L^2(0,1))\), the solution to equation (1) satisfies

\[
\lim_{t \to \infty} E\|u(x,t)\|^2 = 0.
\]

(iii) Stochastic equation (1) is said to be almost surely exponentially stable if for any initial datum \(u_0(x) \in L^2(\Omega, \mathcal{F}_0, P; L^2(0,1))\), the solution to equation (1) satisfies

\[
\limsup_{t \to \infty} \frac{1}{t} \ln\|u(x,t)\| < 0, \ a.s.
\]

**Remark 1.**

(i) Here and in what follows, the abbreviation a.s. stands for almost surely in the sense of probability \(P\).

(ii) These definitions can be found in [2] (see page 119 and page 127 of [2]).

Now, fixing a real number \(c\), we first discuss the following stochastic heat equation:
Remark 2. By virtue of Lemma 1, we can get that equation (6) satisfied the law of large numbers. □

Lemma 1. For any initial datum $w_0 \in L^2(\Omega, \mathcal{F}_0; P; L^2(0, 1))$, there exists a unique (weak) solution:

\[
w \in C_{\mathcal{F}}\left([0, +\infty); L^2(\Omega; L^2(0, 1))\right),
\]

for equation (6) (see [12, 13]). Moreover, we have the following result for the solution of equation (6).

Lemma 2. Let $X_t$ be a real-valued continuous local martingale vanishing at $t = 0$, and let the process $\langle X_t, X_t \rangle$, be the quadratic variation of $X_t$. Then,

\[
\limsup_{t \to +\infty} \frac{\langle X_t, X_t \rangle}{t} < +\infty, \text{ a.s.} \Rightarrow \lim_{t \to +\infty} \frac{X_t}{t} = 0, \text{ a.s.}
\]

This lemma can be found in [2]. Here, we omit the detailed proof. With the aid of Lemma 2, we can investigate the almost-sure exponentially stability for equation (6).

Lemma 3. For any initial datum $w_0 \in L^2(\Omega, \mathcal{F}_0; P; L^2(0, 1))$, the solution to equation (6) satisfies

\[
\limsup_{t \to +\infty} \frac{1}{t} \ln \|w(x, t)\| \leq -\left(\mu_1 + c\right), \text{ a.s.}
\]

Proof. Write $\lambda_i = \mu_i + c$ ($i = 1, 2, 3, \ldots$), where $\mu_i$ is given in equation (2). By Itô's formula,

\[
d\left[\ln \|w(x, t)\|^2\right] = \left(\frac{-2\|w_x(x, t)\|^2 - 2c\|w(x, t)\|^2}{\|w(x, t)\|^2} - \frac{|f(t)|^2\|w(x, t)\|^4}{\|w(x, t)\|^4}\right) dt + 2f(t) dB_t,
\]

Integrating it over the interval $[0, t]$, we get

\[
\ln \|w(x, t)\|^2 = \ln \|w_0\|^2 + \int_0^t 2f(s) dB_s
\]

\[
+ \int_0^t \left[-2\frac{\|w_x(x, s)\|^2 - 2c\|w(x, s)\|^2}{\|w(x, s)\|^2} - |f(s)|^2\right] ds.
\]

This, together with the Poincaré inequality, yields

\[
\ln \|w(x, t)\|^2 \leq \ln \|w_0\|^2 - 2\lambda_1 t + 2 \int_0^t f(s) dB_s.
\]

Since $f(\cdot) \in L^\infty_{\mathcal{F}}(0, +\infty; L^\infty(\Omega))$, $M(t) = \int_0^t f(s) dB_s$ is a continuous martingale with the quadratic variation:

\[
\langle M(t), M(t) \rangle_t = \int_0^t |f(s)|^2 ds \leq b^2 t,
\]

where $b = \|f(\cdot)\|_{L^\infty_{\mathcal{F}}(0, +\infty; L^\infty(\Omega))}$. It, along with equation (13) and Lemma 2, leads to equation (10). Hence, we complete the proof.

Remark 3. If $\mu_1 + c > 0$, then equation (6) is almost surely exponentially stable.

3. Stability for Stochastic Equation (1)

By Lemmas 1 and 3, we have the following observations: if $a \geq \mu_1 - (b^2/2)$ and $U(t) = 0$ ($b = \|f(\cdot)\|_{L^\infty_{\mathcal{F}}(0, +\infty; L^\infty(\Omega))}$), equation (1) is not second-moment exponentially stable or almost surely exponentially stable.

In this section, we will design a suitable control function $U(\cdot)$ to stabilize stochastic heat equation (1) when $a \geq \mu_1 + (b^2/2)$. For this purpose, we define a linear bounded operator as follows:
where the boundary feedback law is given as follows:

\[ \Lambda: L^2(0,1) \rightarrow L^2(0,1), \]  

(15)

by setting

\[ \Lambda(\phi(x)) = \phi(x) - \int_0^x k(x,y)\phi(y) \, dy, \quad \text{for } \phi(x) \in L^2(0,1), \]  

(16)

where the kernel function \( k(x,y) \) is the solution of the following PDE:

\[
\begin{cases} 
  k_{xx}(x,y) = k_{yy}(x,y) + (c + a)k(x,y), \\
  k(x,0) = 0, \\
  k_x(x,x) + k_y(x,x) + \frac{d}{dx}k(x,x) = -(c + a),
\end{cases}
\]  

(17)

where \( c \) is a fixed positive number. Indeed, equation (17) has a unique and twice continuously differentiable solution \( k(x,y) \) over the following domain (see [7, 8]):

\[ \Sigma = \{(x,y) \in [0,1] \times [0,1] ; y \leq x \}. \]  

(18)

Moreover, we have the following lemma:

**Lemma 4.** The linear operator \( \Lambda \) defined in equation (16) is bounded from \( L^2(0,1) \) to \( L^2(0,1) \). Moreover, \( \Lambda \) has a linear bounded inverse \( \Lambda^{-1} \) from \( L^2(0,1) \) to \( L^2(0,1) \).

The proof can be found in [7]. By the backstepping approach, we now introduce a new variable

\[ w(x,t) = \Lambda(u(x,t)), \quad t \geq 0, \]  

(19)

and then apply transformation equation (16) to equation (1), where the boundary feedback law is given as follows:

\[ \begin{align*} 
    dw(x,t) &= du(x,t) - d\left( \int_0^x k(x,y)u(y,t) \, dy \right) \\
    &= du(x,t) - \left( \int_0^x k(x,y)u_{yy}(y,t) + au(y,t) \, dy \right) dt - f(t) \left( \int_0^x k(x,y)u(y,t) \, dy \right) dB_t, \\
    &= k(x,x)u_x(x,t) - k_y(x,x)u(x,t) + \int_0^x k_{yy}(x,y)u(y,t) \, dy,
\end{align*} \]  

(23)

and

\[ w_{xx}(x,t) = u_{xx}(x,t) - \frac{d}{dx}(k(x,x)u(x,t) - k(x,x)u_x(x,t)) \\
- k_x(x,x)u(x,t) - \int_0^x k_{xx}(x,y)u(y,t) \, dy. \]  

(24)

It follows from the standard method in the stochastic analysis that \( w(x,t) \) is a \( L^2(0,1) \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted process and \( U(t) \) is a real-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted process. According to the well-posedness result for stochastic differential equations (see [12, 13]), system equation (1) admits a unique solution. By this boundary feedback law, we obtain the following result.

**Theorem 1.** Let \( c > 0 \) in equation (17). Then, for any initial datum \( u_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(0,1)) \), the solution to equation (1) with boundary feedback control equation (20) satisfies

\[ \limsup_{t \to +\infty} \frac{1}{t} \ln \|u(x,t)\| < 0, \quad \text{a.s.} \]  

(21)

If we further assume that \( \mu_1 + c > (b^2/2) \), where \( b = \|f(\cdot)\|_{L^\infty([0,1], \mathbb{R}^m)} \) and \( \rho \) such that for any initial datum \( u_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(0,1)) \),

\[ \mathbb{E}[\|u(x,t)\|^2] \leq C e^{-\rho t} \|u_0\|^2, \quad \text{for } t \geq 0. \]  

(22)

**Proof.** The proof will be organized in two steps as follows:

**Step 1.** We prove equation (21) with the help of Lemma 4.

Let \( w(x,t) = \Lambda(u(x,t)) \), where the operator is given in equation (16). By direct computations, we can obtain that

\[ \begin{align*} 
    U(t) &= \int_0^1 k(1,y)u(y,t) \, dx. \\
    \end{align*} \]  

(20)
where the initial datum $\omega_0 \in L^2(\Omega, \mathcal{F}_0; P; L^2(0, 1))$, $a \geq \mu_1 + (b^2/2)$, $U(\cdot)$ is the control function, and $d(\cdot)$ is a real-valued $[\mathcal{F}]_{t \geq 0}$-adapted process, which can be regarded as a disturbance in the control channel. Does the control in equation (20) work well for equation (29) with a disturbance $d(\cdot)$ in the control channel? It should be pointed out that there are many studies on this topic for the deterministic parabolic system (see [15, 16]). We also mention [8, 17–22] for the related work. However, we can only find little work concerned with this problem for stochastic PDEs. Our objective is to stabilize the zero equilibrium of the unstable stochastic heat equation with a disturbance $d(\cdot)$. 

These, along with equation (20), yield that

\[
\begin{align*}
    d \omega(x, t) - \omega_{xx}(x, t) \, dt + c \omega(x, t) \, dt &= f(t) \omega(x, t) \, dB_t, \\
    \omega(0, t) &= 0, \\
    \omega(x, 0) &= \omega_0(x),
\end{align*}
\]

This, together with Lemma 4, equation (19), and $\mu_1 + c > (b^2/2)$, yields equation (22). Hence, we complete the proof. 

4. Robust Stability

In this section, we consider the stability of the perturbed controlled stochastic heat equation:

\[
\begin{align*}
    d u(x, t) - u_{xx}(x, t) \, dt - a u(x, t) \, dt &= f(t) u(x, t) \, dB_t, \\
    u(0, t) &= 0, \\
    u(1, t) &= U(t) + d(t), \\
    u(x, 0) &= u_0(x),
\end{align*}
\]

where $u_0 \in L^2(\Omega, \mathcal{F}_0; P; L^2(0, 1))$, $a \geq \mu_1 + (b^2/2)$, $U(\cdot)$ is the control function, and $d(\cdot)$ is a real-valued $[\mathcal{F}]_{t \geq 0}$-adapted process, which can be regarded as a disturbance in the control channel. Does the control in equation (20) work well for equation (29) with a disturbance $d(\cdot)$ in the control channel? It should be pointed out that there are many studies on this topic for the deterministic parabolic system (see [15, 16]). We also mention [8, 17–22] for the related work. However, we can only find little work concerned with this problem for stochastic PDEs. Our objective is to stabilize the zero equilibrium of the unstable stochastic heat equation with a disturbance $d(\cdot)$. 

\[
\begin{align*}
    \limsup_{t \to \infty} \frac{1}{t} \ln \|w(x, t)\| < -(\mu_1 + c) < 0, \quad \text{a.s.} \\
    \mathbb{E} \|w(x, t)\|^2 \leq e^{-2(\mu_1 + c) t} \mathbb{E} \|w(x, 0)\|^2.
\end{align*}
\]
4.1. An Auxiliary Result. To study the problem of robust stability, we introduce the following nonhomogeneous Dirichlet boundary value problem:

\[
\begin{cases}
\text{d}v(x,t) - \nu_{xx}(x,t) \text{d}t + cv(x,t) \text{d}t = f(t)v(x,t) \text{d}B_t, & (x,t) \in (0,1) \times (0, +\infty), \\
v(0,t) = 0, & t \in (0, +\infty), \\
v(1,t) = g(t), & t \in (0, +\infty), \\
v(x,0) = v_0(x), & x \in (0,1),
\end{cases}
\]  

(30)

where the initial datum \(v_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(0,1))\), and the real-valued \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted process \(g(\cdot)\) satisfies

\[
(H) \quad \begin{cases}
\text{(i)} \text{there exists a positive number } M > 0 \text{ such that } E|g(t)|^2 < M \text{ for } t \in [0, +\infty); \\
\text{(ii)} \lim_{t \to +\infty} E|g(t)|^2 = 0.
\end{cases}
\]

Now, we introduce the following result for equation (30).

**Theorem 2.** Let \(g(\cdot)\) in equation (30) be a real-valued \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted process satisfying (H), and let \(\mu_i + c > (b^2/2)\), where \(b = \|f(\cdot)\|_{L^2(0, +\infty; L^2(\Omega))}\). Then, for any initial datum \(v_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(0,1))\), the solution to equation (30) satisfies

\[
\lim_{t \to +\infty} E\|v(x,t)\|^2 = 0.
\]

(32)

Proof. Let \(v_i(t) = \langle v(x,t), e_i \rangle (i = 1, 2, \ldots)\). Then, we have that for any \(t \in [0, +\infty)\),

\[
v(x,t) = \sum_{i=1}^{\infty} v_i(t)e_i,
\]

(33)

\[
E\|v(x,t)\|^2 = \sum_{i=1}^{\infty} E|v_i(t)|^2.
\]

After some computations, we obtain

\[
\begin{cases}
\text{d}v_i(t) = \left[-\lambda_i v_i(t) + (-1)^{i+1}in g(t)\right] \text{d}t + f(t)v_i(t) \text{d}B_t, & t \in [0, +\infty), \\
v_i(0) = v_i^0,
\end{cases}
\]

(34)

where \(v_i^0 = \langle v_0, e_i \rangle \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})\) and \(\lambda_i = \mu_i + c (i \in \mathbb{N})\).

Let \(\{\theta_i\}_{i=1}^{\infty}\) be a sequence of positive numbers, which will be determined later. It follows from Itô’s formula that

\[
d\left(e^{\theta_i t}|v_i(t)|^2\right) = \theta_i e^{\theta_i t}|v_i(t)|^2 \text{d}t + 2e^{\theta_i t}|v_i(t)|^2 \text{d}v_i + e^{\theta_i t}|f(t)v_i(t)|^2 \text{d}t,
\]

(35)

\[
= \left(\theta_i - 2\lambda_i + b^2\right)e^{\theta_i t}|v_i(t)|^2 \text{d}t + 2e^{\theta_i t}\left[(-1)^{i+1}i\sigma v_i(t)g(t) + |f(t)v_i(t)|^2\right] \text{d}B_t.
\]

Integrating equation (35) over \([0,t]\) and demonstrating the mathematics expectation, we have
\[
E|v_i(t)|^2 \leq e^{-\theta_i t}E|v_i(0)|^2 + (\theta_i - 2\lambda_i + b_i^2)\int_0^t e^{-\theta_j(s)}|v_j(s)|^2 \, ds,
\]
\[
+ 2E\int_0^t e^{-\theta_i(t-s)}(-1)^{i+1}n[v_i(s)] \, ds, \quad i = 1, 2, 3, \ldots,
\]
(36)

where \( b = \|f(.)\|_{L_{\infty}^{1,2}(0,\infty;L_{\infty}^{1,2}(\Omega))} \).

When \( i = 2, 3, \ldots \), we take
\[
\theta_i = 2\lambda_i - b_i^2 - \mu_i \quad (i = 2, 3, \ldots).
\]
(37)

By the definition of \( \lambda_i \) (\( i \in \mathbb{N} \)) and \( b^2 < 2\lambda_1 \), we have the following observations:
\[
\begin{aligned}
&\begin{cases}
(i) 0 < \theta_2 < \theta_3 < \ldots < \theta_n < \ldots;
(ii) \sum_{i=2}^{\infty} \frac{1}{\theta_i} < +\infty.
\end{cases}
\end{aligned}
\]
(38)

Therefore, it follows from the Cauchy–Schwarz inequality that
\[
2E\int_0^t e^{-\theta_i(t-s)}(-1)^{i+1}n[v_i(s)] \, ds,
\]
\[
\leq E\int_0^t e^{-\theta_i(t-s)}|v_i(s)|^2 \, ds + \mu_i E\int_0^t e^{-\theta_i(t-s)}|v_i(s)|^2 \, ds.
\]
(39)

This, together with equations (36) and (37), indicates that
\[
E|v_i(t)|^2 \leq e^{-\theta_i t}E|v_i(0)|^2 + \int_0^t e^{-\theta_i(t-s)}E|g(s)|^2 \, ds.
\]
(40)

It, along with (i) of equation (38), yields
\[
E\sum_{i=2}^{\infty} |v_i(t)|^2 \leq e^{-\theta_i t}E\sum_{i=2}^{\infty} |v_i(0)|^2 + \sum_{i=2}^{\infty} \int_0^t e^{-\theta_i(t-s)}E|g(s)|^2 \, ds.
\]
(41)

Now, we claim that
\[
E\sum_{i=2}^{\infty} |v_i(t)|^2 \longrightarrow 0, \quad as \, t \longrightarrow +\infty.
\]
(42)

It follows from (i) of equation (38) that
\[
\lim_{t \longrightarrow +\infty} e^{-\theta_i t}E\sum_{i=2}^{\infty} |v_i(0)|^2 = 0.
\]
(43)

Now, we are going to prove that
\[
\lim_{t \longrightarrow +\infty} \left( \sum_{i=2}^{\infty} \int_0^t e^{-\theta_i(t-s)}E|g(s)|^2 \, ds \right) = 0.
\]
(44)

By (ii) of (H), we have that for any \( \epsilon > 0 \), there exists a positive number \( N > 0 \), which only depends on \( \epsilon \), such that
\[
E|g(s)|^2 < \epsilon, \quad as \, s > N.
\]
(45)

Therefore,
\[
\theta_i \int_0^t e^{-\theta_i(t-s)}E|g(s)|^2 \, ds \left< \epsilon, \quad as \, t > R, \right.
\]
(46)

By (i) of (H), we obtain
\[
\left[ \theta_i \int_0^t e^{-\theta_i(t-s)}E|g(s)|^2 \, ds \right] \left< \epsilon, \quad as \, t > R, \right.
\]
(47)

Thus, there exists a positive number \( R > N \), which only depends on \( \epsilon \), such that when \( t > R \),
\[
\theta_i \int_0^t e^{-\theta_i(t-s)}E|g(s)|^2 \, ds \left< \epsilon, \quad for \, i = 2, 3, \ldots. \right.
\]
(48)

Together with equations (46) and (48), it shows that when \( t > R \),
\[
\int_0^t e^{-\theta_i(t-s)}E|g(s)|^2 \, ds \left< \frac{\epsilon}{\mu_i}, \quad for \, i = 2, 3, \ldots. \right.
\]
(49)

This, along with (ii) of equation (38), yields equation (44). Thus, equation (42) can be obtained by equations (41), (43), and (44).

Since \( b^2 < 2\lambda_1 = 2\mu_1^2 + 2\lambda_1 > \omega \), we can find a positive number \( \omega \) such that \( 2\lambda_1 - b^2 > \omega \). When \( i = 1 \), we take
\[
\theta_1 = 2\lambda_1 - b^2 - \omega > 0.
\]
(50)

Therefore, it follows from the Cauchy–Schwarz inequality that
\[
2E\int_0^t e^{-\theta_1(t-s)}n[v_1(s)] \, ds,
\]
\[
\leq \mu_1 E\int_0^t e^{-\theta_1(t-s)}|v_1(s)|^2 \, ds + \omega E\int_0^t e^{-\theta_1(t-s)}|g(s)|^2 \, ds.
\]
(51)

This, together with equations (36) and (50), indicates that
\[
E|v_1(t)|^2 \leq e^{-\theta_1 t}E|v_1(0)|^2 + \mu_1 E\int_0^t e^{-\theta_1(t-s)}E|g(s)|^2 \, ds.
\]
(52)

By the same argument in the proof of equation (44), we can also obtain that
\[
\mu_1 E\int_0^t e^{-\theta_1(t-s)}E|g(s)|^2 \, ds \longrightarrow 0, \quad as \, t \longrightarrow +\infty.
\]
(53)

It, along with equation (52), yields
\[
E|v_1(t)|^2 \longrightarrow 0, \quad as \, t \longrightarrow +\infty.
\]
(54)

By equations (33), (42), and (54), we can obtain equation (32). This completes the proof. \( \square \)
Figure 1: A sample path of the solution $u(x, t)$ to equation (1) without control.

Figure 2: The $L^2(0, 1)$ norm of the sample path of the solution $u(x, t)$ to equation (1) without control.

Figure 3: A sample path of the solution $u(x, t)$ to equation (29) with control (20).
4.2. Stability for Stochastic Equation (29). A result on the stability for stochastic equation (29) is given as follows.

**Theorem 3.** Let \( d(\cdot) \) in equation (29) be a real-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted process satisfying (H). Then, for any initial datum \( u_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(0, 1)) \), the solution to equation (29) with boundary feedback control equation (20) satisfies that

\[
\lim_{t \to +\infty} E\|u(x, t)\|^2 = 0. \tag{55}
\]

**Proof.** Let \( w(x, t) = \Lambda(u(x, t)) \), where the operator \( \Lambda \) is given in equation (16). By the same argument in Theorem 1, we have that

\[
\lim_{t \to +\infty} E\|w(x, t)\|^2 = 0. \tag{57}
\]

By Lemma 4,

\[
E\|u(x, t)\|^2 \leq \|\Lambda^{-1}\|^2 E\|w(x, t)\|^2. \tag{58}
\]

This, together with equation (57), shows that equation (55) holds. This completes the proof. \( \square \)

5. Numerical Simulations

In this section, we will carry out a numerical simulation to illustrate the theoretical results. In equation (1), we take \( a = 12 \) and \( f(t) = \sin t \). Let \( U(\cdot) = 0 \), and let the initial datum be

\[
y_0(x) = 10x(1 - x^2). \tag{59}
\]

Now, we discretize the stochastic heat equation using the finite difference method. The time and the space steps are chosen as \( k = 0.00005 \) and \( h = 0.01 \), respectively. We performed the numerical simulation 30 times under the Matlab environment and presented one of them in this section.

Figures 1 and 2 show that equation (1) is unstable without control. In equation (29), we still take \( a = 12 \) and \( b = 1 \). Now, we let \( c = 1 \) in the kernel function and let the disturbance \( d(t) = (1/t + 5) \) in equation (29). It is obvious that \( d(t) \) satisfies (H) given in Section 4. According to [11], we have

\[
k(x, s) = -11s - \frac{I_1\left(\frac{\sqrt{11(x^2 - s^2)}}{\sqrt{11(x^2 - s^2)}}\right)}{I_1\left(\frac{\sqrt{11(x^2 - s^2)}}{\sqrt{11(x^2 - s^2)}}\right)} \tag{60}
\]

where \( I_1 \) is the modified Bessel function of order one. Using this kernel function, we can construct boundary feedback law equation (20).

Figures 3 and 4 confirm that the experiment agrees with the theoretical results in this paper. We performed it 30 times, and these results all show that our method yields satisfactory performance.
Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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