PARTIAL $W^{2,p}$ REGULARITY FOR OPTIMAL TRANSPORT MAPS

SHIBING CHEN, ALESSIO FIGALLI

Abstract. We prove that, in the optimal transportation problem with general costs and positive continuous densities, the potential function is always of class $W^{2,p}_{loc}$ for any $p \geq 1$ outside of a closed singular set of measure zero. We also establish global $W^{2,p}$ estimates when the cost is a small perturbation of the quadratic cost. The latter result is new even when the cost is exactly the quadratic cost.

1. Introduction

Regularity of optimal transport maps is a very important problem that has been studied extensively in the recent years. For the special case when the cost function is given by $c(x, y) = \frac{1}{2}|x - y|^2$ (or equivalently $c(x, y) = -x \cdot y$, see the discussion in [13, Section 3.1]), Caffarelli [1, 2, 3, 4, 5] developed a deep regularity theory. However, for general costs functions the situation was much more complicated. A major breakthrough happened in 2005 when Ma, Trudinger, and Wang [34] introduced a fourth order condition on the cost function (now known as MTW condition) that guarantees the smoothness of optimal transport map under suitable global assumptions on the data. Later, it was shown by Loeper [31] that the MTW condition is actually a necessary condition. Motivated by these results, a lot of efforts have been devoted to understanding the regularity properties of optimal map under the MTW condition, see for instance [20, 28, 37, 38, 21, 32, 33, 29, 30, 22, 27, 24, 23, 18, 19].

Unfortunately, as observed by Loeper in [31] and further noticed in many subsequent works, the MTW condition is extremely restrictive and many interesting costs do not satisfy this condition. Hence, a natural and important question became the following: What can we say about the regularity of optimal transport maps when the MTW condition fails? A first major answer was given by De Philippis and Figalli [12]: there, the authors proved that, without assuming neither the MTW condition nor any convexity on the domains, for the optimal transport problem with positive continuous (resp. positive smooth) densities, the potential function is always $C^{1,\alpha}_{loc}$ (resp. smooth) outside a closed singular set of measure zero. In a related direction, Caffarelli, Gonzáles, and Nguyen [7] obtained an interior $C^{2,\alpha}_{loc}$ regularity result of optimal transport problem when the densities are $C^\alpha$ and the cost function is of the form $c(x, y) = \frac{1}{p}|x - y|^p$ with $2 < p < 2 + \epsilon$ for some $\epsilon \ll 1$ (or, $p > 1$ and the distance between source and target is sufficiently large). This interior regularity result was later extended by us to a global one [10].

Date: June 17, 2016.
The aim of this work is to further develop the techniques introduced in [10, 11, 12] and prove a partial $W^{2,p}$ regularity result. More precisely we show that, for the optimal transport problem with positive continuous densities, there exists a closed singular set of measure zero outside which the potential function is of class $W^{2,p}_{\text{loc}}$ for any $p > 1$ (in particular, the singular set is independent of the exponent $p$). As a corollary of our techniques together with an argument due to Savin [36], we are able to obtain global $W^{2,p}$ estimates when the domains are convex and the cost function is $C^2$-close to $-x \cdot y$.

The paper is organized as follows. In section 2 we introduce some notation and state our main results. Then, in section 3 we prove our key Proposition 2.4, and finally in the last section we prove our main results.

2. Preliminaries and main results

First, we introduce some conditions which should be satisfied by the cost. Let $X$ and $Y$ be two bounded open subsets of $\mathbb{R}^n$.

(C0) The cost function $c : X \times Y \to \mathbb{R}$ is of class $C^3$, with $\|c\|_{C^3(X \times Y)} < \infty$.

(C1) For any $x \in X$, the map $Y \ni y \mapsto D_x c(x, y) \in \mathbb{R}^n$ is injective.

(C2) For any $y \in Y$, the map $X \ni x \mapsto D_y c(x, y) \in \mathbb{R}^n$ is injective.

(C3) $\det(D_{xy} c)(x, y) \neq 0$ for all $(x, y) \in X \times Y$.

A function $u : X \to \mathbb{R}$ is said $c$-convex if it can be written as

\begin{equation}
(2.1) \quad u(x) = \sup_{y \in Y} \{-c(x, y) + \lambda_y\}
\end{equation}

for some family of constants $\{\lambda_y\}_{y \in Y} \subset \mathbb{R}$. Note that (C0) and (C1) imply that a $c$-convex function is semiconvex, namely, there exists a constant $K$ depending only on $\|c\|_{C^2(X \times Y)}$ such that $u + K|x|^2$ is convex. One immediate consequence of the semiconvexity is that $u$ is twice differentiable almost everywhere.

Thanks to (C0) and (C1) it is well known (see for instance [40, Chapter 10]) that there exists a unique optimal transport map. Also, there exists a $c$-convex function $u$ such that the optimal map is a.e. uniquely characterized in terms of $u$ (and for this reason we denote it by $T_u$) via the relation

\begin{equation}
(2.2) \quad -D_x c(x, T_u(x)) = \nabla u(x) \quad \text{for a.e. } x.
\end{equation}

As explained for instance in [12, Section 2] (see also [13]), the transport condition $(T_u)_# f = g$ implies that $u$ solves at almost every point the Monge-Ampère type equation

\begin{equation}
(2.3) \quad \det\left(D^2 u(x) + D_{xx} c(x, \text{c-exp}_x(\nabla u(x)))\right) = \left|\det\left(D_{xy} c(x, \text{c-exp}_x(\nabla u(x)))\right)\right| \frac{f(x)}{g(\text{c-exp}_x(\nabla u(x)))},
\end{equation}

where $\text{c-exp}$ denotes the $c$-exponential map defined as

\begin{equation}
(2.4) \quad \text{c-exp}_x(p) = y \iff p = -D_x c(x, y).
\end{equation}

Notice that, with this notation, $T_u(x) = \text{c-exp}_x(\nabla u(x))$.

For a $c$-convex function, in analogy with the subdifferential for convex functions, we can talk about its $c$-subdifferential: If $u : X \to \mathbb{R}$ is a $c$-convex function, the $c$-subdifferential of
u at x is the (nonempty) set
\[ \partial_c u(x) := \{ y \in \overline{Y} : u(z) \geq -c(z, y) + c(x, y) + u(x) \quad \forall z \in X \}. \]
We also define Frechet subdifferential of u at x as
\[ \partial^- u(x) := \{ p \in \mathbb{R}^n : u(z) \geq u(x) + p \cdot (z - x) + o(|z - x|) \}. \]
It is easy to check that
\[ (2.5) \quad y \in \partial_c u(x) \implies -D_x c(x, y) \in \partial^- u(x). \]
Also, it is a well-known fact (see for instance [40, Chapter 10]) that the transport map \( T_u \) and the c-subdifferential \( \partial_c u \) are related by the inclusion
\[ T_u(x) \in \partial_c u(x). \]
In particular, since \( \partial_c u(x) \) is a singleton at every differentiability point of u (this follows by (2.5)), we deduce that
\[ (2.6) \quad \partial_c u(x) = \{ T_u(x) \} \quad \text{whenever } u \text{ is differentiable at } x. \]
The analogue of sublevels of a convex functions is played by the sections: given \( y_0 \in \partial_c u(x_0) \), we define
\[ S(x_0, y_0, u, h) := \{ x : u(x) \leq -c(x, y_0) + c(x_0, y_0) + c(x, y_0) + u(x_0) + h \}. \]
Note that, whenever u is differentiable at \( x_0 \) then \( y_0 = T_u(x_0) \). To simplicity the notation, we will use \( S_h(x_0) \) to denote \( S(x_0, y_0, u, h) \) when no confusion arises.
Finally, we recall that given u c-convex, its c-transform \( u^c \) is defined as
\[ u^c(y) := \sup_{x \in X} \{-c(x, y) - u(x)\}. \]
With this definition, \( u^c \) plays the role of u for the transportation problem from \( g \) to \( f \).

Our first main result states that, if \( f \) and \( g \) are positive continuous densities, then u is of class \( W^{2,p}_{loc} \) for any \( p \geq 1 \) outside a closed set of measure zero. A crucial fact in our proof is to show that the singular set \( \Sigma \) is independent of \( p \).

**Theorem 2.1.** Let \( u \) be the potential function for the optimal transport problem from \( (X, f) \) to \( (Y, g) \) with cost \( c \) satisfying (C0)-(C3). Suppose \( f : X \to \mathbb{R}^+ \) and \( g : Y \to \mathbb{R}^+ \) are positive continuous densities. Then there exists a closed set \( \Sigma \subset X \) of measure zero such that \( u \in W^{2,p}_{loc}(X \setminus \Sigma) \) for any \( p \geq 1 \).

By a localization argument, the above theorem yields the following:

**Corollary 2.2.** Let \((M, G)\) be a smooth closed Riemannian manifold, and denote by \( d \) the Riemannian distance induced by \( G \). Let \( f \) and \( g \) be two positive continuous densities, and let \( T \) be the optimal transport map for the cost \( c = \frac{d^2}{2} \) sending \( f \) onto \( g \). Then there exist two closed sets \( \Sigma_1, \Sigma_2 \subset M \) of measure zero, such that \( T : M \setminus \Sigma_1 \to M \setminus \Sigma_2 \) is a diffeomorphism of class \( W^{2,p}_{loc} \) for any \( p \geq 1 \).

In the next result we show that if the cost function is sufficiently close to the “quadratic” cost \( -x \cdot y \) (recall that this cost is equivalent to \( \frac{1}{2}|x - y|^2 \)), then the potential is \( W^{2,p} \) up to the boundary. Observe that the smallness parameter \( \delta \) is independent of \( p \), and that this result is new even in the case \( c(x, y) = -x \cdot y \).
Theorem 2.3. Suppose $X$ and $Y$ are two $C^2$ uniformly convex bounded domains in $\mathbb{R}^n$. Assume $f : X \to \mathbb{R}^+$ and $g : Y \to \mathbb{R}^+$ are two continuous positive densities, and let $u$ be the $c$-convex function associated to the optimal transport problem between $f$ and $g$ with cost $c(x,y)$. Suppose $c$ satisfies (C0)-(C3) and

\begin{equation}
\|c + x \cdot y\|_{C^2(X \times Y)} \leq \delta. \tag{2.7}
\end{equation}

Then there exists $\hat{\delta} > 0$, depending only on $n$, the modulus of continuity of $f$ and $g$, and the uniform convexity and $C^2$-smoothness of $X$, and $Y$, such that $u \in W^{2,p}(X)$ for any $p \geq 1$ provided $\delta \leq \hat{\delta}$.

The proof of above results is based on the following proposition.

Proposition 2.4. Let $f$ and $g$ be two densities supported in $B_{1/K} \subset C_1 \subset B_K$ and $B_{1/K} \subset C_2 \subset B_K$, respectively. Suppose that $C_2$ is convex,

\begin{equation}
\|f - 1\|_{L^\infty(C_1)} + \|g - 1\|_{L^\infty(C_2)} \leq \delta, \tag{2.8}
\end{equation}

and

\begin{equation}
\|c(x,y) + x \cdot y\|_{C^2(B_K \times B_K)} \leq \delta. \tag{2.9}
\end{equation}

Then, for any $p \geq 1$ there exists $\bar{\delta} > 0$, depending only on $n$, $K$, and $p$, such that $u \in W^{2,p}(B_{1/K})$ provided $\delta \leq \bar{\delta}$.

Note that, in the result above, the smallness of the parameter $\delta$ depends on $p$. So, for the proof of Theorems 2.1 and 2.3 and Corollary 2.2, it will be crucial to prove that actually $\delta$ can be chosen independently of $p$ (see Lemma 4.1). Also, as explained in Section 3.2 below, to prove Proposition 2.4 we shall first approximate $u$ with smooth solutions and then obtain $W^{2,p}$ a priori estimates that are independent of the regularization. We note that, in this context, such a regularization procedure is nontrivial and require some attention.

Remark 2.5. As we shall also observe later, the condition “$C_2$ is convex” in Proposition 2.4 can be replaced by the assumption

$$\left\| u - \frac{1}{2}|x|^2 \right\|_{L^\infty(B_{\eta_0})} \leq \delta$$

for some fixed $\eta_0 \leq 1/K$. Under this assumption, for any $p \geq 1$ there exists $\bar{\delta} > 0$, depending only on $n$, $K$, $\eta_0$, and $p$, such that $u \in W^{2,p}(B_{1/\eta_0})$ provided $\delta \leq \bar{\delta}$. Moreover, in the above condition, the function $\frac{1}{2}|x|^2$ can be replaced by a $C^2$ convex function $\nu$ such that $\frac{1}{M}Id \leq D^2\nu \leq MId$, in which case $\bar{\delta}$ depends also on $M$ and the modulus of continuity of $D^2\nu$.

3. Proof of Proposition 2.4

We begin by observing that, under our assumptions, it follows by [12] Theorem 4.3 and the argument in the proof of [11] Theorem 2.1 that $u \in C^{1,\alpha}(B_{1/K})$ for some $\alpha > 0$. Hence, up to replace $K$ by $2K$, we can assume that $u \in C^{1,\alpha}(B_{1/2K})$. In particular it follows by (2.6) that $S(x_0, y_0, u, h) = S(x_0, T_u(x_0), u, h)$, and we can use the notation $S_h(x_0) = S(x_0, T_u(x_0), u, h)$. 


3.1. Engulfing property of sections. The first step consists in establish the engulfing property for sections of $u$, which is stated as the following lemma.

Lemma 3.1 (Engulfing property). There exist universal constants $r_0 > 0$ and $C > 1$ such that, for $h \leq r_0$ and $x_0 \in B_{\frac{2}{3}K}$, 
\[ x_1 \in S_h(x_0) \implies S_h(x_0) \subset S_{Ch}(x_1). \]
Proof. Without loss of generality we may assume $x_0 = 0, y_0 = T_u(x_0) = 0$, and $u(0) = 0$. Up to performing the transformations
\[ c(x, y) = -x \cdot y + O(|x|^2|y| + |x||y|^2). \]
Set $\rho := \left(\frac{|C_1|}{|C_2|}\right)^{1/n}$ so that $|\rho C_2| = |C_1|$, and let $v$ be a convex function satisfying $(\nabla v)_t 1_{C_1} = 1_{\rho C_2}$ with $v(0) = 0$ (we note that $\nabla v$ is the optimal transport map from $\frac{1}{|C_1|} 1_{C_1}$ to $\frac{1}{|C_2|} 1_{C_2}$ for the quadratic cost). By a compactness argument similar to the proof of [12, Lemma 4.1] we have
\[ \|u - v\|_{L^\infty(B_{\frac{1}{\rho K})} \leq \omega(\delta), \]
where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\omega(r) \to 0$ as $r \to 0$. Also, since $\rho C_2$ is convex, it follows by [4] that $v$ is smooth and uniformly convex in $B_{\frac{1}{\rho K}}$.

Thanks to (2.8), (2.9), and (3.2), we can follow the proof of [12, Theorem 4.3] to show that, for small $h$, there exists an affine transform $A$ such that
\[ A(B_{\frac{1}{2}K}) \subset S_h \subset A(B_{3\sqrt{\pi}}), \]
\[ A^{-1}(B_{\frac{1}{2}K}) \subset T_u(S_h) \subset A^{-1}(B_{3\sqrt{\pi}}) \]
and
\[ |u(Ax) - \frac{1}{2}|x|^2| \leq \eta h \quad \text{in } B_{3\sqrt{\pi}} \]
with
\[ \|A\|, \|A^{-1}\| \leq h^{-\theta}, \]
where $\eta, \theta > 0$ can be as small as we want, provided $\delta$ is sufficiently small. Note that (3.2) plays the same role as the fact that $u$ is close to a quadratic function, which is used in the proof of [12, Theorem 4.3]. Furthermore, (3.1) and (3.6) imply that $\text{diam}(S_h(x)) \leq C h^{\frac{1}{2} - \theta}$. Hence, if we choose $r_0$ small enough so that $Cr_0^{\frac{1}{2} - \theta} \leq \frac{1}{4K}$, we see that for $h \leq r_0$ and $x \in B_{\frac{1}{4K}}$, we have $S_h(x) \subset B_{\frac{1}{4K}}$.

Now we perform the transformations $c_1(x, y) := c(Ax, A^{-1}y)$ and $u_1(x) := u(Ax)$, and we use the notation $S_h^1 = S(0, 0, u_1, h)$. By (3.3) and (3.4) we have
\[ B_{\frac{1}{2}K} \subset S_h^1 \subset B_{3\sqrt{\pi}} \]
and
\[(3.8)\quad B_{\frac{1}{\sqrt{3\pi}}} \subset T_{u_1}(S_h^1) \subset B_{3\sqrt{\pi}}.\]

Note that
\[(3.9)\quad 0 \leq u_1(x) + c_1(x, 0) - c_1(0, 0) - u_1(0) \leq h \quad \text{for any } x \in S_h^1.
\]

Also, by (3.1) and (3.6) we see that \(\|c_1\|_{C^2(B_{3\sqrt{\pi}} \times B_{3\sqrt{\pi}})} \leq C\) for some universal constant \(C\). Therefore, thanks to (3.8) and (3.10), for any \(x, x_1 \in S_h^1\) and \(y_1 = T_{u_1}(x_1) \in T_{u_1}(S_h^1)\),
\[
|c_1(x, y_1) - c_1(x_1, y_1) + c_1(x_1, 0) - c_1(x, 0)| \leq \|D c_1\|_{C^0(B_{3\sqrt{\pi}} \times B_{3\sqrt{\pi}})} |x - x_1| |y_1| \leq C_1 h
\]
for some universal constant \(C_1\). Hence, by (3.9) applied to both \(x\) and \(x_1\) we get
\[
u_1(x) + c_1(x, y_1) - c_1(x_1, y_1) - u_1(x_1) = u_1(x) + c_1(x, 0) - c_1(0, 0) - u_1(0)
\]
\[
- (u_1(x_1) + c_1(x_1, 0) - c_1(0, 0) - u_1(0))
\]
\[
+ c_1(x, y_1) - c_1(x_1, y_1) + c_1(x_1, 0) - c_1(x, 0)
\]
\[
\leq h + C_1 h.
\]

Since \(x \in S_h^1 = S(0, 0, u_1, h)\) was arbitrary, this proves that \(S(0, 0, u_1, h) \subset S(x_1, y_1, u_1, (1 + C_1)h)\). Recalling the relation between \(u_1\) and \(u\), this proves the desired result. \(\square\)

As a consequence of this result, one gets the following:

**Corollary 3.2.** There exists a small constant \(r_1\) such that for \(h \leq r_1\) and \(x, y \in B_{\frac{1}{\sqrt{3\pi}}}\) the following holds: suppose \(S_h(x) \cap S_t(y) \neq \emptyset\), \(t \leq h\). Then there exists an universal constant \(C'\) such that \(S_t(y) \subset S_{C'h}(x)\).

**Proof.** Fix \(z \in S_h(x) \cap S_t(y)\). By Lemma 3.1 we have that
\[
S_t(y) \subset S_{C'h}(z) \quad \text{and} \quad x \in S_h(x) \subset S_{C'h}(z)
\]
for some universal constant \(C\). Also, by the argument in the proof of Lemma 3.1, \(z \in B_{\frac{2}{\sqrt{3\pi}}}\) for \(h\) small enough. Hence, using Lemma 3.1 again we have \(S_{C'h}(z) \subset S_{C'^{2}h}(x)\), thus \(S_t(y) \subset S_{C'^{2}h}(x)\) with \(C' := C^2\). \(\square\)

It is well known that the property of sections stated in Corollary 3.2 implies the following Vitali covering lemma (see for instance [16, Lemma 4.6.2] for a proof):

**Lemma 3.3** (Vitali covering). Under the assumptions of Proposition 2.4, let \(D\) be a compact subset of \(B_{\frac{1}{\sqrt{3\pi}}}\), and let \(\{S_{h_x}(x)\}_{x \in D}\) be a family of sections with \(h_x \leq r_1\). Then, there exists a finite number of sections \(\{S_{h_{x_i}}(x_i)\}_{i=1,\ldots,m}\) such that
\[
D \subset \bigcup_{i=1}^{m} S_{h_{x_i}}(x_i)
\]
with \(\{S_{\sigma h_{x_i}}(x_i)\}_{i=1,\ldots,m}\) disjoint, where \(\sigma > 0\) is a universal constant.
3.2. Approximation argument. In the next sections we will prove our $W^{2,p}$ estimates by controlling the measure of the super-level sets of the Hessian of $u$. Because we shall need to use the pointwise value of $D^2 u$, we need an approximation argument in order to work with $C^2$ convex functions. Since in this setting this is not a standard procedure, we now provide the details.

Given $u$ as in Proposition 2.3, we set $\rho := \left( \frac{|C_1|}{|C_2|} \right)^{1/n}$ so that $|\rho C_2| = |C_1|$, and let $v$ be a convex function satisfying $(\nabla v)_i 1_{C_1} = 1_{\rho C_2}$ with $v(0) = 0$. Since
\[ \|u-v\|_{L^\infty(B_{\frac{1}{\rho}})} \to 0 \quad \text{as } \delta \to 0 \]
(see (3.2)), as in the proof of Lemma 3.1 we can choose $\delta$ small enough so that, for any $x \in B_{\frac{1}{\rho}}$ and $h > 0$ small but universal, the section $S_h(x)$ satisfies (3.3), (3.4), (3.5), and (3.6).

We now consider $f_\epsilon : C_1 \to \mathbb{R}$ and $g_\epsilon : C_2 \to \mathbb{R}$ as sequence of $C^\infty$ densities that approximate $f$ and $g$ respectively, and denote by $u_\epsilon$ the potential function for the optimal transport problem from $f_\epsilon$ to $g_\epsilon$ with cost $c$. Without loss of generality, we can assume that $u_\epsilon(0) = u(0)$.

Then, by a compactness argument it follows that
\[ \|u_\epsilon - u\|_{L^\infty(B_{\frac{1}{\rho}})} \to 0 \quad \text{as } \epsilon \to 0 \]
Since $u$ is strictly convex, choosing $\epsilon$ sufficiently small we see that the sections $S_h(x) = S(x, T_{u_\epsilon}(x), u_\epsilon, h)$ satisfy (3.3), (3.4), (3.5), and (3.6) with bounds independent of $\epsilon$.

In particular, assuming $\delta$ is small enough, by [12, Theorem 4.3] applied to $\frac{1}{\delta} u_\epsilon(A\sqrt{h}x)$ we deduce that $u_\epsilon$ is of class $C^{1,6/7}$ in $A(B_{\frac{1}{\delta}\sqrt{h}})$. By duality, similarly we also have that its $c$-transform $u_\epsilon^c$ is of class $C^{1,6/7}$ inside $A^{-1}(B_{\frac{1}{\delta}\sqrt{h}})$. Hence, by [11, Theorem 2.3] we deduce that $u_\epsilon$ is of class $C^2$ in a neighborhood of $x$. Since $x \in B_{\frac{1}{\rho}}$ was arbitrary, this proves that $u_\epsilon \in C^2(B_{\frac{1}{\rho}})$ for any $\epsilon > 0$ small enough.

Hence, up to proving our $W^{2,p}$ estimates with $u_\epsilon$ in place of $u$ and then letting $\epsilon \to 0$, in the next sections we shall directly assume that $u \in C^2$.

3.3. Density estimates. The goal here is to show that, given a section $S_h(x) \subset B_{\frac{1}{\delta\sqrt{h}}}$, the density of “bad points” where the Hessian of $u$ is large has measure that goes to zero as $\delta \to 0$.

Fix $x_0 \in B_{\frac{1}{\rho}}$, and let $y_0 = T_\epsilon(x_0)$. Without loss of generality, we may assume $x_0 = y_0 = 0$. Also, as in the proof of Lemma 3.1 we can assume that (3.1) holds. In this way it follows that, for $h$ small, (3.3), (3.4), (3.5), and (3.6) hold.

Perform the transformations
\[ c(x,y) \mapsto \bar{c}(x,y) := \frac{1}{h} c(\sqrt{h}Ax, \sqrt{h}A^{-1}y); \]
\[ u(x) \mapsto \bar{u}(x) := \frac{1}{h} u(\sqrt{h}Ax); \]
\[ f(x) \mapsto \bar{f}(x) := f(\sqrt{h}Ax), \quad g(y) \mapsto \bar{g}(y) = g(\sqrt{h}A^{-1}y). \]
Note that, by (3.1) and (3.6), we have
\[
\|\bar{c} + x \cdot y\|_{C^2(B_h \times B_h)} \leq \delta
\]
provided \( h \) is sufficiently small. Also, it follows by (3.3), (3.4), and (3.5) that
\[
B_{\frac{\delta}{3}} \subset S(0, 0, \bar{u}, 1) \subset B_3,
\]
(3.11)
\[
B_{\frac{1}{3}} \subset S(0, 0, \bar{u}, 1) \subset B_3,
\]
and
\[
\left\| \bar{u}(x) - \frac{1}{2} |x|^2 \right\|_{L^\infty(B_3)} \leq \eta.
\]
We now construct a smooth function \( w \) that well approximates \( \bar{u} \). Denote
\[
X_1 := S(0, 0, \bar{u}, h)
\]
and \( Y_1 := T_{\bar{u}}(S(0, 0, \bar{u}, h)) \).

**Lemma 3.4.** Set \( \rho := \left( \frac{|X_1|}{|Y_1|} \right)^{1/n} \), and let \( w \) be a convex function such that \( (\nabla w)_2 I_{X_1} = I_{\rho Y_1} \) and \( w(0) = u(0) \). Then, for any \( \gamma > 0 \), there exist \( \delta_\gamma, \eta_\gamma > 0 \) such that
\[
\|\bar{u} - w\|_{L^\infty(B_{1/4})} \leq \gamma
\]
and
\[
\|w\|_{C^3(B_{1/6})} \leq C
\]
provided \( \delta \leq \delta_\gamma \) and \( \eta \leq \eta_\gamma \), where \( C \) is a universal constant.

**Proof.** The bound (3.14) follows from a compactness argument similar to the proof of [12, Lemma 4.1]. Also, taking \( \gamma \leq \eta \), (3.13) and (3.14) imply that
\[
\left\| w(x) - \frac{1}{2} |x|^2 \right\|_{L^\infty(B_{1/4})} \leq 2\eta.
\]
Thanks to (3.16), as in the proof of Lemma 3.1 (see also Step 1 in the proof of [12, Theorem 4.3]) we can apply [4] to deduce that \( \|w\|_{C^3(B_{1/6})} \leq C \) for some universal constant \( C \). \( \square \)

Let \( L \) be the operator defined by
\[
L\bar{u}(x) := D^2\bar{u}(x) + D_{xx}\bar{c}(x, T_{\bar{u}}(x)).
\]
By (2.8) and (3.10), we have
\[
\det(L\bar{u}(x)) = \left| \det \left( D_{xy}\bar{c}(x, T_{\bar{u}}(x)) \right) \right| \frac{f(x)}{g(T_{\bar{u}}(x))} = 1 + O(\delta).
\]
We now follow the argument in [2] to establish the density estimate. Since the argument is rather standard, we shall just emphasize the main points, referring to [2] or [16, Chapter 4.7] for more details.

**Lemma 3.5.** Let \( \bar{u}, w \) be as above, and denote by \( \Gamma \left( \bar{u} - \frac{w}{2} \right) \) the convex envelope of \( \bar{u} - \frac{w}{2} \). Then, for any Borel set \( E \subset B_{1/6} \), we have
\[
\left| \nabla \Gamma \left( \bar{u} - \frac{w}{2} \right)(E) \right| \leq \left( \frac{1}{2^n} + O(\delta) \right) \left| E \cap \left\{ \Gamma \left( \bar{u} - \frac{w}{2} \right) = \bar{u} - \frac{w}{2} \right\} \right|
\]
Proof. Noticing that \( \det D^2 w = 1 \), \( \det D^2 \bar{u} = 1 + O(\delta) \), and \( D_{xx} \bar{c} = O(\delta) \), since \( w \) is uniformly convex and \( \det D^2 \Gamma(\bar{u} - \frac{w}{2}) \) is a measure supported on \( \{ \Gamma(\bar{u} - \frac{w}{2}) = \bar{u} - \frac{w}{2} \} \), it follows by the Area Formula (see for instance [16, Proposition A.4.19]) that

\[
\left| \nabla \Gamma\left( \bar{u} - \frac{w}{2} \right) (E) \right| = \int_{E \cap \{ \Gamma(\bar{u} - \frac{w}{2}) = \bar{u} - \frac{w}{2} \}} \det D^2 \left( \bar{u} - \frac{w}{2} \right) \det \left[ L\bar{u} - \left( D^2 \frac{w}{2} + D_{xx} \bar{c}(x,T_u x) \right) \right] \\
\leq \int_{E \cap \{ \Gamma(\bar{u} - \frac{w}{2}) = \bar{u} - \frac{w}{2} \}} \left( \det(L\bar{u})^{1/n} - \det \left( \left( D^2 \frac{w}{2} + O(\delta) \right)^{1/n} \right) \right)^n \\
\leq \left( \frac{1}{2} + O(\delta) \right)^n \left| E \cap \left\{ \Gamma\left( \bar{u} - \frac{w}{2} \right) = \bar{u} - \frac{w}{2} \right\} \right|
\]

where we used the inequality

\[ \left[ \det(A + B) \right]^{1/n} \geq \left( \det A \right)^{1/n} + \left( \det B \right)^{1/n} \quad \forall A, B \text{ symmetric, nonnegative definite} \]

(see for instance [16, Lemma A.1.3] for a proof).

By using Lemma 3.5, we can follow the lines of proof of [2, Lemma 6] (see also the proof of [16, Lemma 4.7.1]) to establish the estimate

\[
\frac{\left| \{ \Gamma(\bar{u} - \frac{w}{2}) = \bar{u} - \frac{w}{2} \} \cap B_{1/8} \right|}{|B_{1/8}|} \geq 1 - C\delta^{1/2},
\]

from which one immediately obtain the following bound (see [2, Corollary 1] or Step 6 in the proof of [16, Lemma 4.7.1]):

**Lemma 3.6** (Density estimate). Let \( \bar{u} \) be as above. Then there exist universal constants \( N > 1, \eta > 0 \) such that

\[
(3.19) \quad \left| \left\{ x \in S^u_\eta(0) : \| D^2 \bar{u}(x) \| \geq N \right\} \right| \leq N\delta^{1/2} \left| S^u_\eta(0) \right|.
\]

3.4. \( W^{2,p} \) estimate. We now prove our \( W^{2,p} \) interior estimates. Recall that we are assuming that \( u \in C^2 \).

As in the proof of Lemma 3.1, for any \( x \in B_{\frac{1}{2\sqrt{n}}} \) and \( h > 0 \) small enough, there exists an affine transformation \( A \) with \( \det A = 1 \) such that

\[
(3.20) \quad A(B_{\frac{1}{3\sqrt{n}}} \cap S_h(x)) \subset S_h(x) \subset A(B_{\frac{3}{\sqrt{n}}} \cap S_h(x)).
\]

We define the normalized size of the section \( S_h(x) \) as

\[
(3.21) \quad a(S_h(x)) := \| A^{-1} \|^2.
\]

Although \( A \) is not unique, if \( A_1 \) and \( A_2 \) are two affine transformations that satisfy (3.20) then both \( \| A_1^{-1} A_2 \| \) and \( \| A_2^{-1} A_1 \| \) are universally bounded, thus the normalized size is well defined up to universal constants.
With the notation from the previous section, we see that the estimate \(3.19\) can be rewritten in terms of \(u\) and becomes

\[
|S_h(x) \cap \{\|D^2 u(x)\| \geq N a(S_h(x))\}| \leq C \delta^{1/2} |S_h(x)|
\]

(3.22)

for any \(h\) small enough. Also, since \(\det D^2 \bar{u} = 1 + O(\delta)\), it follows that

\[
\|D^2 u\| \leq N \implies D^2 u \geq \frac{1}{2N^{n-1}} \text{Id}.
\]

Thus, up to enlarging \(N\) and using Lemma \(3.6\) again, we deduce that

\[
|S_h(u)| \leq C \left| S_{e\delta h} \cap \left\{ \frac{a(S_h(x))}{N} \leq \|D^2 u\| \leq N a(S_h(x)) \right\} \right|
\]

that combined with (3.22) yields

\[
|S_h(x) \cap \{\|D^2 u\| \geq N a(S_h(x))\}| \leq C \delta^{1/2} \left| S_{e\delta h} \cap \left\{ \frac{a(S_h(x))}{N} \leq \|D^2 u\| \leq N a(S_h(x)) \right\} \right|
\]

(3.23)

Also, by (3.6) we have

\[
\text{diam}(S_h(x)) \leq C h^{1/2} \|A\| \leq C h^{1/2} e^{-\theta} \leq \hat{C} a(S_h(x))^{-\beta},
\]

(3.24)

where \(\beta := \frac{1}{4\delta} - \frac{1}{2}\).

Let \(M \gg 1\) to be fixed later, set \(\rho_0 := \frac{1}{2K}\), and for \(m \geq 1\) we define \(\rho_m\) inductively by

\[
\rho_m := \rho_{m-1} - \hat{C} M^{-m\beta},
\]

(3.25)

where the constants \(\hat{C}, \beta\) are as those in (3.24). Note that, by taking \(M\) large enough so that

\[
\sum_{m=1}^\infty \hat{C} M^{-m\beta} < \frac{1}{4K},
\]

we can ensure that \(\rho_m \geq \frac{1}{4K}\) for all \(m \geq 1\).

Now, for \(k \geq 0\) we set \(D_k := \{x \in B_{\rho_k} : \|D^2 u\| \geq M^k\}\). We shall prove the following lemma.

**Lemma 3.7.** \(|D_{k+1}| \leq N \delta^{1/2} |D_k|\).

**Proof.** Let \(M \gg N\) to be chosen later, and for any \(x \in D_{k+1}\) choose a section \(S_h(x)\) such that

\[
a(S_h(x)) = N M^k.
\]

(3.26)

Such a section always exists because \(a(S_h) \approx 1 < N M^k\) when \(h = h_0\) is a small but fixed universal constant, while

\[
a(S_h) \approx \|D^2 u(x)\| \geq M^{k+1} > N M^k \quad \text{as } h \to 0
\]

(the estimate \(a(S_h) \approx \|D^2 u(x)\|\) follows by a simple Taylor expansion, see for instance [16, Remark 4.7.5]). Hence, by continuity there exists \(h_x \in (0, h_0)\) such that (3.26) holds.

Now, by Lemma 3.3 we can find a finite number of sections \(\{S_{h_x}(x_i)\}_{i=1,\ldots,m}\) covering \(D_{k+1}\) such that \(\{S_{\sigma h_{x_i}}(x_i)\}_{i=1,\ldots,m}\) are disjoint. Then, it follows by (3.23) that

\[
|S_{h_i}(x_i) \cap \{\|D^2 u\| \geq N^2 M^k\}| \leq N \delta^{1/2} |S_{\sigma h_i}(x_i) \cap \{M^k \leq \|D^2 u\| \leq N^2 M^k\}|.
\]

(3.27)
Hence, recalling (3.24) and (3.25), we obtain

\[ |D_{k+1}| \leq \sum_{i=1}^{m} |S_{h_i}(x_i) \cap \{ \|D^2u\| \geq N^2M^k \}| \]
\[ \leq N\delta^{1/2} \sum_{i=1}^{m} |S_{\delta h_i}(x_i) \cap \{ M^k \leq \|D^2u\| \leq N^2M^k \}| \]
\[ \leq N\delta^{1/2}|D_k| \]

provided \( M \geq N^2 \).

\[ \square \]

**Proof of Proposition 2.4.** Thanks to Lemma 3.7, we have

\[ |D_k| \leq (N\delta^{1/2})^k|D_0| \leq \frac{1}{M^{k(p+1)}}|B_{\frac{1}{2K}}| \]

provided \( \delta \leq \frac{1}{N^2M^{2(p+1)}} \). Therefore

\[ \int_{B_{\frac{1}{2K}}} \|D^2u\|^p = p \int_{B_{\frac{1}{2K}}} t^{p-1}|B_{\frac{1}{4K}} \cap \{ \|D^2u\| \geq t \}| \leq C \sum_{k=1}^{\infty} M^{kp}|D_k| \leq C, \]

as desired.

\[ \square \]

4. **Proof of Theorem 2.1 and Corollary 2.2**

4.1. **Proof of Theorem 2.1.** By the argument in [12, Section 3], we only need to establish the following result, which is a strengthened version of Proposition 2.4 for continuous densities. Indeed, the lemma shows that the exponent \( p \) in the \( W^{2,p} \) estimate is independent of the parameter \( \delta \). This is crucial in showing that the singular set \( \Sigma \) can be chosen independently of \( p \).

**Lemma 4.1.** Let \( f, g \) be two continuous densities supported in \( B_{1/K} \subset X_1 \subset B_K \) and \( B_{1/K} \subset Y_1 \subset B_K \) respectively. Suppose that

\[ \|f - 1\|_{L^\infty(X_1)} + \|g - 1\|_{L^\infty(Y_1)} \leq \delta, \]

\[ \|u - \frac{1}{2}|x|^2\|_{L^\infty(B_K)} \leq \delta \]

and

\[ \|c(x,y) + x \cdot y\|_{C^2(B_K \times B_K)} \leq \delta. \]

Then there exists \( \tilde{\delta} > 0 \), depending only on \( n \) and \( K \), such that \( u \in W^{2,p}(B_{\frac{1}{2K}}) \) for any \( p \geq 1 \) provided \( \delta \leq \tilde{\delta} \).
Proof. Fix \( x_0 \in B_{\frac{1}{2\nu}} \), and without loss of generality assume \( x_0 = 0 \), \( T_u(x_0) = 0 \), and \( u(x_0) = 0 \). For small \( h \), similarly to the proof of Lemma 3.1 there exists an affine transformation \( A \) with \( \det A = 1 \), \( \|A\|, \| A^{-1} \| \leq h^{-\theta} \), such that (3.3), (3.4), and (3.6) hold, where \( \theta \) can be as small as we want provided \( \delta \) is sufficiently small. Also, we may assume (3.1) holds.

Given a set \( E \), let \( [E] \) denote its convex hull. By [10, Lemma 3.2] we have that
\[
(4.4) \quad \text{dist}(S_h, [S_h]) \leq C h^{1-6\theta}.
\]
Also, by \( C^{1,\alpha} \) regularity of \( u \) (hence, \( C^{0,\alpha} \) regularity of \( T_u \)), we have
\[
(4.5) \quad \text{dist}(T_u(S_h), T_u([S_h])) \leq C h^{(1-6\theta)\alpha}.
\]
Perform the transformations
\[
u(x) \mapsto \frac{1}{h}u(\sqrt{h}A^{-1}x) := u_1(x); \]
\[
c(x, y) \mapsto \frac{1}{h}c(\sqrt{h}A^{-1}x, \sqrt{h}A' y) := c_1(x, y); \]
\[
f(x) \mapsto f_1(x) := f(\sqrt{h}A^{-1}x), \quad g(y) \mapsto g_1(y) := g(\sqrt{h}A' y); \]
\[
S_h \mapsto \tilde{S} := \frac{1}{\sqrt{h}}A(S_h).
\]
Also, set \( C_1 := [\tilde{S}], C_2 := T_u([\tilde{S}]), \tilde{f} := f_1 1_{C_1}, \) and \( \bar{g} := g_1 1_{C_2}. \)
By (3.3), (3.4), (4.4), (4.5) we have
\[
B_{\frac{1}{4}} \subset C_1 \subset B_4; \]
\[
B_{\frac{1}{4}} \subset C_2 \subset B_4; \]
\[
\|\tilde{f} - 1_{C_1}\|_{L^{\infty}(B_4)} = o(1), \quad \|\bar{g} - 1_{C_2}\|_{L^{\infty}(B_4)} = o(1) \quad \text{as } h \to 0.
\]
It is also easy to check that
\[
\|c_1 + x \cdot y\|_{C^2(B_{\frac{1}{4}} \times B_4)} = o(1) \quad \text{as } h \to 0,
\]
Since \( C_1 \) is convex, we can apply Proposition 2.4 (switch the role of \( x \) and \( y \)) to deduce that, given any \( p \geq 1 \), we can choose \( h \) small enough so that \( u_1' \), the \( c \)-transform of \( u_1 \), belongs to \( W^{2,p}(B_4) \) provided \( h \) is sufficiently small. By a symmetric argument (or using that \( D^2 u \) and \( D^2 u c \) are related), one gets that, given any \( p \geq 1 \), \( u_1 \in W^{2,p}(B_{\frac{1}{4}}) \) provided \( h \) is sufficiently small. Rescaling back to \( u \) this proves that, given \( p \geq 1 \), \( u \in W^{2,p}(B_r) \) provided \( r \) is small enough (the smallness depending on \( h \)). Thanks to this fact, Lemma 4.1 follows from a standard covering argument.

Proof of Theorem 2.1. Theorem 2.1 is an easy consequence of Lemma 4.1, following the argument in [12, Section 3].

Proof of Corollary 2.2. Corollary 2.2 follows by the same reasoning as the proof of [12, Theorem 1.4].
5. Proof of Theorem 2.3

Since interior $W^{2,p}$ estimates follows from Lemma 4.1 and [11] Lemma 3.11, we focus on the estimate near the boundary.

Under the assumptions of Theorem 2.3, it is proved in [9] that, for any $\alpha < 1$, there exists $\delta > 0$ such that $u \in C^{1,\alpha}(\bar{X})$ provided $\delta \leq \bar{\delta}$. Let

$$\bar{h}(x) := \max\{h > 0 : S_h(x) \subset X\},$$

and set $S_x := S_{\bar{h}(x)}(x)$. As in the proof of Lemma 3.1, there exists an affine transformation $A$ with $\det A = 1, \|A\|, \|A^{-1}\| \leq \bar{h}(x)^{-\theta}$, such that

$$\frac{B_{\frac{1}{\theta}}}{\sqrt{\bar{h}(x)}} \subset A(S_x) \subset B_{\frac{3}{\theta}}\sqrt{\bar{h}(x)}.$$

Hence, since $\|A\|, \|A^{-1}\| \leq \bar{h}(x)^{-\theta}$, it follows by (5.1) and the definition of $\bar{h}(x)$ that

$$\text{dist}(x, \partial X) \leq C\bar{h}(x)^{\frac{1}{\theta} - \theta},$$

which proves that

$$S_x \subset X_{\bar{h}(x)^{\frac{1}{\theta} - \theta}} := \left\{z \in X : \text{dist}(z, \partial X) \leq C\bar{h}(x)^{\frac{1}{\theta} - \theta}\right\}.$$

Fix $h_0 > 0$ small but universal. Similarly to the proof of Lemma 3.1, we can find a Vitali covering of $X_{h_0}$, denoted by $\{S_{\bar{h}(x)}(x_i)\}$, such that the sections $\{S_{\sigma\bar{h}(x)}(x_i)\}$ are disjoint.

Now, fix $x_0 \in X_{h_0}$ a point close to $\partial X$. Without loss of generality we may assume $x_0 = 0$, $T_u(x_0) = 0$, and $u(x_0) = 0$. Consider the section $S_{\bar{h}} := S(0, 0, u, \bar{h}(0))$. As in the proof Lemma 4.1, we perform the transformations (3.20), (3.21), (3.22), and (3.23), and we set $C_1 := [\bar{S}], C_2 := T_{u_1}([\bar{S}]), \bar{f} := f_11_{C_1}$, and $\bar{g} := g_11_{C_2}$, so that

$$B_{\frac{1}{\theta}} \subset C_1 \subset C_4;$$

$$B_{\frac{1}{\theta}} \subset C_2 \subset C_4;$$

$$\|\bar{f} - 1_{C_1}\|_{L^\infty(B_1)} = o(1), \quad \|\bar{g} - 1_{C_2}\|_{L^\infty(B_1)} = o(1) \to 0 \text{ as } \bar{h}, \delta \to 0;$$

$$\|c_1 + x \cdot y\|_{C^2(B_4 \times B_4)} = o(1) \to 0 \text{ as } \bar{h}, \delta \to 0.$$

Note that, by (3.5), $u_1$ is arbitrarily close to the function $\frac{1}{2}|x|^2$. Let $v$ be the convex function solving $(\nabla v)_z 1_{C_1} = 1_{\rho C_2}$ with $v(0) = u(0)$ and $\rho := \left(\frac{|C_1|}{|C_2|}\right)^{1/n}$. By a compactness argument we have that

$$\|u_1 - v\|_{L^\infty(B_{\frac{1}{\theta}})} \leq \omega(\delta),$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\omega(r) \to 0$ as $r \to 0$. This implies that also $v$ is uniformly close to the function $\frac{1}{2}|x|^2$ inside $B_{\frac{1}{\theta}}$, hence (3.4) yields that $\|v\|_{C^3(B_{\frac{1}{\theta}})} \leq C$ for some universal constant $C$, and that $v$ is uniformly convex in $B_{\frac{1}{\theta}}$. Thus, if we set $S_t := S(0, 0, u_1, t)$, we can apply Proposition 2.4 to deduce that $\|u_1\|_{W^{2,p}(S_t)} \leq C$ for some universal constants $t, C$.

Rescaling back to $u$, this proves that

$$\int_{S_{\bar{h}(x_0)}} \|D^2u\|^p \leq C\bar{h}(x_0)^{-2p\theta}\|S_{\sigma\bar{h}(x_0)}\|. $$
Now, consider the family of sections $F_k := \{ S_{\bar{h}(x_i)}(x_i) : h_0 2^{-k-1} \leq \bar{h}(x_i) \leq h_0 2^{-k} \}$. Then, since $|X_r| \approx r$ for $r$ small and the sections $\{ S_{\sigma h}(x_i) \}$ are disjoint, it follows by (5.3) and (5.2) that
\[
\sum_{S_{\bar{h}(x_i)}(x_i) \in F_k} \int_{S_{\bar{h}(x_i)}(x_i)} \|D^2 u\|^p \leq C \sum_{S_{\bar{h}(x_i)}(x_i) \in F_k} \bar{h}(x_i)^{-2p\theta} |S_{\sigma \bar{h}(x_i)}| 
\leq C 2^{2k\theta} |X_{C(h_0 2^{-k})^{\frac{1}{2}-\theta}}| 
\leq C 2^{2k\theta} (h_0 2^{-k})^{\frac{1}{2}-\theta} 
\leq C 2^{-k(\frac{1}{4}-3p\theta)}
\]
Choosing $\theta$ small enough so that $3p\theta \leq \frac{1}{4}$, we can sum the above estimate with respect to $k$ to get
\[
\int_{X_{h_0}} \|D^2 u\|^p \leq C.
\]
Since $\int_{X \setminus X_{h_0}} \|D^2 u\|^p \leq C$ by interior regularity, this concludes the proof. \( \square \)

References
1. L. A. Caffarelli. A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. *Ann. of Math.* (2) 131 (1990), no. 1, 129-134.
2. L. A. Caffarelli. Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation. *Ann. of Math.* (2) 131 (1990), no. 1, 135-150.
3. L. A. Caffarelli. Some regularity properties of solutions of Monge Ampère equation. *Comm. Pure Appl. Math.* 44 (1991), no. 8-9, 965-969.
4. L. A. Caffarelli. The regularity of mappings with a convex potential. *J. Amer. Math. Soc.* 5 (1992), no. 1, 99-104.
5. L. A. Caffarelli. Boundary regularity of maps with convex potentials. II, *Ann. of Math.* 144 (1996), 453-496.
6. L. A. Caffarelli. Allocation maps with general cost functions. In Partial differential equations and applications, volume 177 of Lecture Notes in Pure and Appl. Math., pages 29-35. Dekker, New York, 1996
7. L. A. Caffarelli, M. M. González, T. Nguyen. A perturbation argument for a Monge-Ampère type equation arising in optimal transportation. *Arch. Ration Mech. Anal.* 212 (2014), no. 2, 359-414.
8. L. A. Caffarelli, Y. Y. Li. A Liouville theorem for solutions of the Monge-Ampère equation with periodic data. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 21 (2004), no. 1, 97-120.
9. Boundary $C^{1,\alpha}$ regularity of an optimal transport problem with cost close to $-x \cdot y$ *SIAM J. Math. Anal.*, 47(4), 2689-2698.
10. S. Chen, A. Figalli. Boundary $\varepsilon$-regularity in optimal transportation. *Adv. Math.* 273 (2015), 540-567.
11. S. Chen, A. Figalli. Stability results on the smoothness of optimal transport maps with general costs, *J. Math. Pures Appl.*, to appear.
12. G. De Philippis, A. Figalli. Partial regularity for optimal transport maps. *Publ. Math. Inst. Hautes Études Sci.* 121 (2015), 81-112.
13. G. De Philippis, A. Figalli. The Monge-Ampère equation and its link to optimal transportation. *Bull. Amer. Math. Soc. (N.S.)* 51 (2014), no. 4, 527-580.
14. A. Figalli. Regularity of optimal transport maps [after Ma-Trudinger-Wang and Loeper]. (English summary) Séminaire Bourbaki. Volume 2008/2009. Exposés 997-1011. *Astérisque* No. 332 (2010), Exp. No. 1009, ix, 341-368.
15. A. Figalli. Regularity properties of optimal maps between nonconvex domains in the plane. *Comm. Partial Differential Equations* 35 (2010), no. 3, 465-479.
16. A. Figalli. The Monge-Ampère equation and its applications. Zürich Lectures in Advanced Mathematics, to appear.
17. A. Figalli, Y. H. Kim. Partial regularity of Brenier solutions of the Monge-Ampère equation, *Discrete Contin. Dyn. Syst.* 28 (2010), no. 2, 559-565.
18. A. Figalli, Y. H. Kim, R. J. McCann. Hölder continuity and injectivity of optimal maps, *Arch. Ration. Mech. Anal.* 209 (2013), no.3, 1812-1824.
19. A. Figalli, Y.-H. Kim, R.J. McCann. Regularity of optimal transport maps on multiple products of spheres. *J. Eur. Math. Soc. (JEMS)* 15 (2013), no. 4, 1131–1166.
20. A. Figalli, G. Loeper. $C^1$ regularity of solutions of the Monge-Ampère equation for optimal transport in dimension two. *Calc. Var. Partial Differential Equations* 35 (2009), no. 4, 537-550.
21. A. Figalli, L. Rifford. Continuity of optimal transport maps and convexity of injectivity domains on small deformations of $S^2$. *Comm. Pure Appl. Math.*, 62 (2009), no. 12, 1670-1706.
22. A. Figalli, L. Rifford, C. Villani. On the Ma-Trudinger-Wang curvature on surfaces. *Calc. Var. Partial Differential Equations*, 39 (2010), no. 3-4, 307-332.
23. A. Figalli, L. Rifford, C. Villani. Nearly round spheres look convex. *Amer. J. Math.*, 134 (2012), no. 1, 109-139.
24. A. Figalli, L. Rifford, C. Villani. Necessary and sufficient conditions for continuity of optimal transport maps on Riemannian manifolds. *Tohoku Math. J. (2)*, 63 (2011), no. 4, 855-876.
25. D. Gilbarg, N. S. Trudinger. Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.
26. H.-Y. Jian, X.-J. Wang. Continuity estimates for the Monge-Ampère equation. *SIAM J. Math. Anal.* 39 (2007), no. 2, 608-626.
27. Y.-H. Kim, R.J. McCann. Continuity, curvature, and the general covariance of optimal transportation. *J. Eur. Math. Soc.*, 12 (2010), no. 4, 1009-1040.
28. J. Liu, Hölder regularity of optimal mappings in optimal transportation. *Calc. Var. Partial Differential Equations* 34 (2009), no. 4, 435-451.
29. J. Liu, N.S. Trudinger, X.-J. Wang. Interior $C^{2,\alpha}$ regularity for potential functions in optimal transportation. *Comm. Partial Differential Equations* 35 (2010), no. 1, 165-184.
30. Liu, J., Trudinger, N. S. and Wang, X.-J., On asymptotic behaviour and $W^{2,p}$ regularity of potentials in optimal transportation, preprint. *Arch. Rational Mech. Anal.*, 215 (2015) 867-905.
31. G. Loeper. On the regularity of solutions of optimal transportation problems. *Acta Math.* 202 (2009), no. 2, 241-283.
32. G. Loeper. Regularity of optimal maps on the sphere: The quadratic cost and the reflector antenna. *Arch. Ration. Mech. Anal.*, 199 (2011), no. 1, 269-289
33. G. Loeper, C. Villani. Regularity of optimal transport in curved geometry: the nonfocal case. *Duke Math. J.*, 151 (2010), no. 3, 431-485.
34. X. N. Ma, N. S. Trudinger, X. J. Wang. Regularity of potential functions of the optimal transportation problem. *Arch. Ration. Mech. Anal.* 177 (2005), no. 2, 151-183.
35. E. Milakis, L. E. Silvestre. Regularity for fully nonlinear elliptic equations with neumann boundary data. *Comm. Partial Differential Equations* 31 (2006), no. 8, 1227-1252.
36. O. Savin. Global $W^{2,p}$ estimates for the Monge-Ampère equations, *Proc. Amer. Math. Soc.* 141 (2013), 3573-3578
37. N. S. Trudinger, X.-J. Wang. On the second boundary value problem for Monge-Ampère type equations and optimal transportation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 8* (2009), no. 1, 143-174.
38. N. S. Trudinger, X.-J. Wang. On strict convexity and continuous differentiability of potential functions in optimal transportation. *Arch. Ration. Mech. Anal.* 192 (2009), no. 3, 403-418.
39. J. Urbas. On the second boundary value problem for equations of Monge-Ampère type. *J. Reine Angew. Math.*, 487(1997), 115-124.
40. C. Villani. *Optimal Transport. Old and New.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 338. Springer-Verlag, Berlin, 2009.

MATHEMATICAL SCIENCE INSTITUTE, THE AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA ACT 2601
E-mail address: chenshibing1982@hotmail.com
