QUASILOCAL ENERGY FOR SPIN-NET GRAVITY

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Abstract. The Hamiltonian of the gravitational field defined in a bounded region is quantized. The classical Hamiltonian, and starting point for the regularization, is a boundary term required by functional differentiability of the Hamiltonian constraint. It is the quasilocal energy of the system and becomes the ADM mass in asymptopia. The quantization is carried out within the framework of canonical quantization using spin networks. The result is a gauge invariant, well-defined operator on the Hilbert space induced by the state space on the whole spatial manifold. The spectrum is computed. An alternate form of the operator, with the correct naive classical limit, but requiring a restriction on the Hilbert space, is also defined. Comparison with earlier work and several consequences are briefly explored.

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1. Introduction

Fundamental physical observables of the gravitational field, such as energy and angular momentum, are notoriously hard to define satisfactorily. In asymptopia, where there exist the well-defined Bondi [1, 2] and Arnowitt-Deser-Misner (ADM) [3] masses, the quantities are non-local. The inherent non-locality is usually attributed to the equivalence principle; one needs at least two observers to distinguish geodesic deviation from acceleration-dependent quantities. Nevertheless, given the existence of the Bondi and ADM energy definitions on 2-surfaces at null and spatial infinities, it is reasonable to expect a degree of localization is possible. One possibility given by Penrose [4] is to associate quasilocal quantities to compact, oriented spatial 2-surfaces. Given the difficulty in defining gravitational observables, it is perhaps not surprising that there exists a plethora of proposed quasilocal quantities [5] - [15]. The definitions of quasilocal energy may be distinguished by the ability to satisfy various criteria. These include vanishing in flat space, taking reasonable values for spherically symmetric solutions, and approaching the ADM and Bondi values in appropriate limits. It is by no means simple to satisfy these criteria [10], [13].

With the complications involved in defining energy even in classical general relativity, it may seem perilous to define quantum quasilocal energy. However, in the context of the canonical (3 + 1) theory there is a clear selection criteria for the correct physical quantity. When defined within a bounded region, functional differentiability of the constraints requires that, generically, surface terms must be added to make the theory well-defined. The surface term required by the Hamiltonian constraint is the Hamiltonian for the system on shell. In general, the method of
functional differentiability generates boundary conditions on the phase space variables, gauge parameters, and surface terms (as is explored in some depth in Ref. [16] for gravity and BF theory).

Originally, it was noticed that a surface term had to be added to the action so as to match the asymptotic expression. However, Regge and Teitelboim found that gravitational theory was simply inconsistent without this boundary term [17]. The gravitational action has derivatives of the phase space functions, so the variation generates surface terms. It is only when these terms vanish (or are canceled) that the theory has a well-defined variational principle. The boundary Hamiltonian or quasilocal energy is of this form; it cancels a term arising in the variation of the Hamiltonian constraint.

This paper provides a quantization of the surface observable

$$H_{\partial \Sigma}(N) = \frac{1}{4\pi G} \int_{\partial \Sigma} d^2x \epsilon^{ijk} N n_a A^a_i E^b_j F^{ak}.$$  

This reduces to the quasilocal energy of Brown and York [14], to the ADM energy in asymptopia, and to the Misner-Sharp mass in spherical symmetry [16].

The quantization is carried out in the context of canonical quantum gravity in the real new variables [18, 19] or, succinctly, “spin-net gravity” [20] in which the state space is built from functions of holonomies based on graphs. (See Ref. [20] for a recent review.) Perhaps the most remarkable result of this study is the discreteness in geometric operators. Length [21], area [22, 24, 29], volume [22, 25–28, 33], and angle [30] have been found to have fully discrete spectra. In this approach to quantum gravity space is discrete.

There have been two previous quantizations of Eq. (1). The first, by Baez, Muniain, and Piriz, was completed before many of the spin-net techniques were developed [31]. The action of the operator was qualitatively described as a “shift.” In the second, two ADM energy operators were introduced in the framework of Thiemann’s quantum spin dynamics [32 – 37]. The classical expression used in that quantization was different than the one used here. In fact, the expression is only weakly equal to Eq. (1) [38]. Nevertheless, the resulting quantum operator is very similar and reduces to this ADM expression under the same restrictions. A more extensive comparison between the operators is given in Section 4. The present work completes the quantization of the boundary Hamiltonian, even in a non-asymptotically flat context.

The quantum definition is to satisfy modest criteria. I only ask that the operator be well-defined on the Hilbert space induced from the full gauge invariant space. Important criteria, including the proper behavior on semiclassical states and the appropriate algebra of boundary observables, is left for further investigation. Definitions of simple semiclassical states or weaves [39] are explored in [40].

The remainder of this paper is organized as follows: Spin-net gravity is briefly reviewed in Section 2. This serves to base the definition of the quasilocal energy in

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1A word on the choice of name: “Spin-net gravity” is chosen to reflect the current state of affairs. It is widely recognized that the kinematic state space of the canonical quantization of gravity in terms of real connection variables has a basis in terms of spin networks. The descriptive name, spin-net gravity, emphasizes the fundamental importance of spin networks to the theory both as a kinematic state space (in which case one might say that one has “quantum geometry”) and the very critical assumption that spin networks also serve to describe the dynamics. There are a number of results which suggest that spin networks fill this role – as this paper does – but the issue is not resolved.
the framework of the classical theory and provides an opportunity to fix notation and units. The main developments are contained in Section 3 where the operator is regularized in 3.1 and given final form in 3.2. An alternate quantization is given in 3.3 and the full spectra of the resulting operators is presented in 3.4. This paper concludes with a comparison with Thiemann’s $E_{ADM}$ operator and some discussion on wider implications of the quantum quasilocal energy operator.

2. The setting: Canonical Quantum Gravity

This section sets the classical framework for the operator definitions and provides the basic elements of spin-net quantization. Meanwhile, I use the opportunity to fix signs, factors, and units. For an introduction, rather than a brief review, the reader is encouraged to read Refs. [20, 23, 29, 41–44].

For four-dimensional spacetimes $M$ with $M = \Sigma \times \mathbb{R}$, where $\Sigma$ is compact, the $(3 + 1)$-action for vacuum, Riemannian general relativity is

$$S[E^{ai}, A^i_a; \Lambda^i, N^a, N] = \frac{1}{8\pi G} \int_{t_1}^{t_2} dt \int_{\Sigma} d^3x \left[ E^{a_i} \dot{A}^i_a - NH - N^a D_a - \Lambda^i F^i \right].$$

The action is written in terms of a real, $su(2)$-valued connection one-form $A^i_a$ and its momenta, a densitized inverse triad $E^{ai}$. I have included the overall factor depending on Newton’s constant $G$ ($c = 1$ throughout the paper). The phase space variables satisfy

$$\{A^i_a(x), E^{b_j}(y)\} = 8\pi G \delta^i_x \delta^j_y \delta(x, y).$$

The Lagrange multipliers $N$, $N^a$, and $\Lambda^i$ are, respectively, the lapse, shift, and $SU(2)$ gauge rotation parameters. Varying the action with respect to the last two functions gives the constraints

$$G^i \equiv D_a E^{a_i} \approx 0$$

$$\mathcal{D}^i_a \equiv E^{b_i} \partial_a A^i_b - \partial_b (E^{b_i} A^i_a) \approx 0$$

where $D_a \lambda^i = \partial_a \lambda^i + \epsilon^{ijk} A^j_a \lambda^k$. Varying the lapse gives the Hamiltonian constraint. Defining $F_{ab}^i = \partial_a A^i_b + \epsilon^{ijk} A^j_b A^k_a$, one may express the constraint in integrated form as

$$H(N) = \frac{1}{8\pi G} \int_{\Sigma} d^3x \left( \epsilon^{ijk} E^{a_i} E^{b_j} F_{ab}^k - 4E^{a_i}_l E^{b_j}_l (A^i_a - \Gamma^i_a) (A^j_b - \Gamma^j_b) \right) \approx 0.$$ 

Due to the choice of using real variables, the additional term has been added to the constraint. The triad-compatible connection $\Gamma^i_a$ satisfies $\mathcal{D}_a E^{b_i} = \partial_a E^{b_i} + \epsilon^{ijk} \Gamma^i_a E^{bk} + \Gamma^b_a E^{ci} - \Gamma^c_a E^{bi} = 0$. This constraint generates time evolution. A key observation which affects the definition of the quasilocal energy is that the Hamiltonian constraint has density weight +2 so the lapse function has density weight -1.

As bounded spatial regions are the subject of this work, it is best to start by fixing notation. The calculations are in a spacetime of the form $M = \Sigma \times \mathbb{R}$, although the space has at least one compact subset $I \subset \Sigma$ such that the boundary of the interior $I$, $\partial I$, is homeomorphic to a 2-sphere. I usually denote the boundary $\partial I$ as the

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2As Immirzi has emphasized, in the canonical transformation used to define the connection there is a family of choices generated by one non-zero, real parameter $\gamma$, $\gamma A^i_a = \Gamma^i_a - \gamma K_a^i$. $\gamma E^{ai} = (1/\gamma) E^{ai}$. I take $\gamma = 1$ until the final section, when it is included in the spectra of the quasilocal energy.
surface $S$. The topology of $\Sigma$ is not specified but two examples are worth keeping in mind. One is a compact spatial slice $\Sigma$ with one boundary and two “interiors” $I$ and its complement $I^\ast$. The other is the topology of the asymptotically flat spacetimes in which $\Sigma$ is homeomorphic to $\mathbb{R}^3$ with a compact ball cut out.

When the theory of Eq. (4) is applied to a bounded spatial region $I$, it is no longer well-defined \[16\]. The problem arises because the phase space of the compact theory does not contain the physical solutions of the bounded one \[17\]; the variational problem has no solutions. This comes about as boundary terms arise in the variation, making Hamilton’s equations ill defined; dynamics takes the system outside of the phase space.

The $(3+1)$-action, or the constraints, must be functionally differentiable. When the theory is defined in a finite region, this requires the addition of surface terms and/or the imposition of boundary conditions. These surface terms, without which the theory would be inconsistent, are the fundamental observables and necessarily satisfy the same algebra as the constraints \[16\].

Under the variation of the connection, the Hamiltonian constraint generates a surface term which must either vanish through the imposition of boundary conditions or be canceled by another surface term. If the lapse is non-vanishing on $\partial I$ then the surface term of Eq. (4) must be added to the constraint. It is this term which is the quasilocal energy and reduces to the ADM energy in asymptopia. To ensure that the theory is dynamically well-defined one must impose a complete set of conditions on the boundary $\partial I$. To be concrete, I consider the specific boundary conditions:

\[ \begin{align*}
\delta E^{\alpha i}\big|_{\partial I} & = 0 \text{ which includes fixing the “area density” } n_a\delta E^{\alpha i}\big|_{\partial I} = 0 \\
\delta N^a\big|_{\partial I} & = 0; \ \delta N\big|_{\partial I} = 0
\end{align*} \]

See Ref. \[16\] (or, for a more general setting, Ref. \[45\]) for details.

This work is devoted to quantizing the observable of Eq. (1). Before beginning this, I review the quantization program in which the quasilocal energy operation is defined. The mathematically precise formulation is developed in Refs. \[46, 47, 49–53\].

The natural framework for a diffeomorphism invariant gauge theory is based on Wilson loops \[54, 55\]. However, to build three-dimensional geometry it is necessary to use a more general structure based on graphs. The quantum configuration space is most appropriately constructed from holonomies along the edges of a graph. I denote a graph embedded in $\Sigma$ by $G$. It contains a set of $N$ edges $e$ and a set of vertices $v$. The key idea is that every smooth connection $A$ associates a group element to an edge $e$ of $e$ via the holonomy,

\[ U_e(A) := \mathcal{P} \exp \int_e dt \dot{e}^\alpha A_\alpha(e(t)). \]

Here, $A_\alpha := A^i_\alpha \tau^i$ with $\tau^i$ proportional to the Pauli matrices via $\tau^i = -\frac{i}{2} \sigma^i$. Given an embedded graph, holonomies along the edges, and a complex (Haar-integrable) function on $SU(2)^N$, one may define a “cylindrical function”

\[ \Psi_{G,f}(A) = f(U_{e_1}(A), \ldots, U_{e_N}(A)) \]

(The function $\Psi$ is “cylindrical” since $f$ only depends on a finite number of directions in the space of connections; it is constant on all other directions.) With the appropriate norm, the completion of this function space gives the Hilbert space $\mathcal{H}$. 
A basis on $\mathcal{H}$ is given by spin network states $|s\rangle$. I denote a spin network $\mathcal{N}$ by the triple $(G, i, n)$ of an oriented graph $G$, labels on the vertices (or “intertwiners”), $i$, and integer edge labels, $n$, indexing the representation carried by the edge. The corresponding spin net state $|s\rangle$ in $\mathcal{H}$ is defined in the connection representation as

$$(A | s) \equiv \langle A | G \rangle := \prod_{v \in V(G)} i_v \otimes \otimes_{e \in E(G)} U_{e}^{\langle n_e \rangle}[A]$$

where the holonomy $U_{e}^{\langle n_e \rangle}$ along edge $e$ is in the $n_e/2$ irreducible representation of $SU(2)$. When the intertwiners “tie up” all the incident edges – when they are invariant tensors on the group – these states are gauge invariant.

As the triads are dual to pseudo two-forms, they most comfortably live on 2-surfaces, denoted by $S$.

$$E^{i}_S = \int_S d^2 \sigma n_a(\sigma) E^{ai}(x(\sigma))$$

in which $\sigma$ are coordinates on the surface and $n_a = \epsilon_{abc} \frac{ds^b}{d\sigma} \frac{ds^c}{d\sigma}$ is the normal. The action of the triads on a function of holonomies may be computed from the Poisson brackets, Eq. (3). A short calculation shows that the bracket with a single edge $U_e$ is given by

$$\{E^{ai}(x), U_e(A)\} = -8\pi G \int_d dt \dot{e}^a(t) \delta(x, e(t)) J^i_{(e)} \cdot U_e(A)$$

in which $J^i_{(e)} \cdot U_e$ denotes the action of the left (or right) invariant vector fields on the group element $U_e$. The bracket will generally have a sum of terms. When a cylindrical function based on a graph $G$, say $c_G(A)$, is used, the triad defined on a surface, $E^{i}_S$, gives a sum over all intersections, $v$, of the surface and the graph $G$

$$\{E^{i}_S, c_G(A)\} = -\frac{8\pi G}{2} \sum_{v \in S \cap G I \vdash v} \chi^S_I J^i_{I} \cdot c_G.$$ 

The sum is over all edges $I$ incident to $v$ (denoted $I \vdash v$). The geometric factor $\chi^S_I$ is defined by

$$\chi^S_I = \begin{cases} +1 & \text{when the orientation of } e_I \text{ is aligned with } n_a \\ 0 & \text{when the edge is tangent} \\ -1 & \text{when the orientation of } e_I \text{ is anti-aligned with } n_a. \end{cases}$$

This geometrical factor ties the tangent space of the edge (through $\dot{e}_I$) to the orientation of the surface. Due to the behavior of $J^i_{(e)}$ under edge orientation reversal, this factor maintains edge orientation independence and surface orientation dependence.

There are two further remarks to make. First, the result is non-vanishing only when there is at least one intersection between the graph $G$ and the surface $S$. Second, the overall factor of $\frac{1}{2}$ can be seen to arise from a “thickened surface” regularization \[23\].

\[3\]While $J^i_{(e)}$ does satisfy the algebra of Eq. (11), it has an additional property: under orientation reversal of the edge $e$, $J^i_{(e)}$ changes sign \[29\]. This may be seen directly from Eq. (10).
With this preparation, one may define the quantum triad operator on a spin network state \( |s\rangle = |Gin\rangle \)

\[
\hat{E}^i_S |s\rangle := -(4\pi G)^2 \sum_{v \in S \cap G} \sum_{I \vdash v} \chi^S_I \hat{J}^i_I |s\rangle
\]

where the angular momentum-like operator \( \hat{J}^i_e \equiv i\hbar J^i_e \) satisfies the usual algebra

\[
\left[ \hat{J}^i_e, \hat{J}^j_{e'} \right] = i\hbar \epsilon^{ijk} \delta_{e,e'} \hat{J}^k_e.
\]

(The \( \delta \)-function restricts the relation to one edge; \( \hat{J}^i \) on distinct edges commute.)

The triad operator of Eq. (10) is essentially self-adjoint in the Hilbert space of quantum gravity \([23, 50]\). The diagrammatic form of this operator is the “one-handed” \([44]\)

\[
\hat{E}^i_S |s\rangle = -il^2 \sum_{v \in S \cap G} \sum_{I,J} \sqrt{\chi^S_I \chi^S_J} \hat{J}^i_I \cdot \hat{J}^j_J |s\rangle
\]

in which the index \( i \) is the internal space index. The length scale of the theory, \( l \), is defined by \( l^2 = 4\pi \hbar G \). The grasping is chosen such that, in the plane of the diagram, the 2-line is on the left when the orientation on the edge points up (vice versa for downward orientations) \([22–24]\). Though the overall action is orientation independent, such a diagrammatic representation of grasping involves the choice of a sign. The geometric factor \( \chi^S_I \) in the expression for the “unclasped hand” in Eq. (12) ensures that the operator is independent of edge orientation \([43]\).

The quantum configuration space based on graphs and triads which act as signed angular momentum operators are the basic elements of spin-net gravity. The definition of the quasilocal energy makes use of two further results, the geometric operators of area and volume.

The area of a surface \( S \) may be, for simplicity, specified by \( z = 0 \) in an adapted coordinate system. Expressed in terms of the triad \( E^a_i \), the area of the surface only depends on the 3rd vector component via \([22–24]\)

\[
A_S = \int_S d^2x \sqrt{E^3_i E^3_i}.
\]

The quantum operator is defined using the operators of Eq. (10) and by partitioning the surface \( S \) so that only one edge or vertex threads through each cell in the partition. The integral of Eq. (13) becomes a sum over operators that act only at intersections of the surface with the spin network

\[
\hat{A}_S |s\rangle = (4\pi G)^2 \sum_{v \in S \cap G} \sum_{I,J} \sqrt{\chi^S_I \chi^S_J} \hat{J}^i_I \cdot \hat{J}^j_J |s\rangle.
\]

The spectrum may be computed with recoupling theory \([24]\) or operator methods \([28]\). In both cases it is found by considering all the intersections of the spin network with the surface \( S \), including vertices which lie in the surface. The edges incident to a vertex in the surface may be divided into three categories: those which have tangents aligned with the surface normal \( j_p \), anti-aligned \( j_n \), and tangent to the surface \( j_z \). Summing over all contributions, Eq. (14) becomes \([23, 24]\)

\[
\hat{A}_S |s\rangle = \frac{l^2}{2} \sum_{v \in S \cap G} \left[ 2j_{p_v}(j_{p_v} + 1) + 2j_{n_v}(j_{n_v} + 1) - j_{z_v}(j_{z_v} + 1) \right]^{1/2} |s\rangle.
\]
This result suggests that space is discrete; measurements of area can only take quantized values. Other geometric operators such as length [23], angle [30], and volume share this property.

The construction of the volume operator is more complicated than the area construction. Nevertheless, it is possible, starting from the classical expression

\[ V_R = \int_R d^3x \sqrt{q} = \int_R d^3x \sqrt{\frac{1}{3!} \epsilon_{abc} \epsilon^{ijk} E^a_i E^b_j E^c_k}, \]

to regularize the quantity and define the operator [22], [29] - [28]

\[ \hat{V}_R |s\rangle = (4\pi G)^{\frac{3}{2}} \sum_{v \in G} \frac{1}{8 \cdot 3!} \sum_{I,J,K} \chi_{IJK} \epsilon_{ijk} \hat{J}^I_i \hat{J}^J_j \hat{J}^K_k |s\rangle \]

where the sign factor is given by \( \chi_{IJK} = \text{sgn}(\det(\dot{e}_I(0), \dot{e}_J(0), \dot{e}_K(0))) \). (Incident edges are oriented outward for ease of writing.) The spectra of such an operator can be worked out using recoupling theory as in Refs. [42], [28], and [27]. Introductions to diagrammatic recoupling theory may be found in Refs. [44, 60].

3. THE QUASILOCAL ENERGY OPERATOR

To begin, I give a regularization of the Hamiltonian \( H_S(N) \) with a scalar lapse. This term may be made into a well-defined quantum operator on the appropriate state space. However, it does not have the dimensions of energy. Two normalizations to fix this problem are explored, one using area and the other using volume. In both cases the full spectrum may be computed. There are differences. The operator which, in the naive classical limit, gives the familiar expression for the ADM energy and quasilocal energy requires a restriction on the Hilbert space. The alternate operator is defined on the full Hilbert space but does not have the correct naive classical limit. This is discussed in Sections 3.3 and 4. The next section provides a regularization of \( H_S(N) \). A recoupling identity studied in 3.2 suggests the final form of the operators. The spectra are given in Section 3.4.

3.1. Regularization of the classical observable. The classical quantity to be promoted to a quantum operator is given by

\[ H_S(N) = \frac{1}{4\pi G} \int_S d^2x N n_a(x) \epsilon^{ijk} A^i_a \tilde{E}^{bj} \tilde{E}^{ck} \]

(For the remainder of this paper I explicitly give the density weights using tildes. This notation also serves to distinguish the two forms of the quasilocal energy operator.) The regularization is based on the observation that the boundary Hamiltonian has two parts. One part, \( n_a \tilde{E}^a_k \), is the familiar triad operator integrated over a 2-surface as in Eq. (7). The second part, \( A^a_i \tilde{E}^{aj} \), may be roughly described as a projection of the connection along an edge of a spin network. These two parts are regularized by point splitting.

The regularization of the classical Hamiltonian is in several steps. The surface is thickened, as in the definition of the area operator in Ref. [24]. This is done by introducing a smooth coordinate \( r \) over a finite neighborhood of \( S \) with “thickness”
The boundary itself is located at $r = 0$. The thickened surface has a natural foliation in terms of $r$ and gives a way to regulate the operator. This portion of the regularization handles the tangent space factor (the sign $\chi_S$). The boundary integration is transformed to an integration over the thickened surface, a region $\mathcal{R}$:

$$
\int_S d^2 \sigma \rightarrow \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dr \int_S d^2 \sigma = \frac{1}{\tau} \int_{\mathcal{R}} d^3 x.
$$

The region $\mathcal{R}$ is further partitioned into cells $\mathcal{R}_c$ which have the property that the coordinate width, denoted by $\epsilon$, is tied to the coordinate height so that $\tau = \epsilon^k$. The power $k$ is restricted to lie between 1 and 2. By construction, the number of cells $n_r$ is tied to the same limit. The fully regularized operator is averaged over the leaves of the foliation of $\mathcal{R}$ and point split. The classical expression is defined by

$$
[H_S(N)]_\epsilon := \frac{1}{4\pi G \tau} \sum_{c=1}^{n_r} \int d^3 x d^3 y \epsilon^{ijk} (A_a^i \tilde{E}^{aj}(y) N(x) n_b(x) \tilde{E}^{bj}(x)).
$$

The regulated Hamiltonian is well-defined and goes to Eq. (1) in the limit as $\epsilon$ vanishes.

In the next steps I re-express the classical variables $A_a^i$ and $\tilde{E}^{aj}$ in a suitable form for quantization. Making use of the identity $\epsilon^{ijk} = (-2) \text{Tr} \left[ [\tau^i, \tau^j] \tau^k \right]$, $[H_S(N)]_\epsilon$ is transformed into the convenient expression

$$
[H_S(N)]_\epsilon = \frac{1}{2\pi G} \sum_{c=1}^{n_r} \int ds N(e_I(t)) n_b(e_I(t)) \tilde{e}^b_I(t) \times \text{Tr} \left[ [A_a(e_J(s)) \tilde{e}^a_J(s), J_J] J_I \right].
$$

Next, the “edge-projected connection” $A_a \tilde{e}^a_J$ is expressed in terms of holonomies. This can be done in a straightforward manner as smooth connections satisfy

$$
U_a(s, s + \delta s)^{\pm 1} = 1 \pm \delta s \tilde{e}^a(s) A_a(e(s)) + O(\delta s^2).
$$

The partitioning of the thickened surface splits the integration along the edges in Eq. (17) into segments of maximum coordinate length $\epsilon$. The integral $\int ds$ is approximated in the limit, by $\sum \epsilon$.

Meanwhile, the integral along the edge $e_I$ in $t$ contains the geometrical factor

$$
\int_{e \in \mathcal{R}_c} dt \ n_a(e(t)) \dot{e}^a(t) = \begin{cases} 
+\tau & \text{when } \dot{e}^a \text{ is aligned with } n_a \\
0 & \text{when the edge is tangent} \\
-\tau & \text{when } \dot{e}^a \text{ is anti-aligned with } n_a 
\end{cases} \equiv \chi_e^S \tau.
$$

\footnote{As will be clear in the following discussion the reason for this restriction is identical to the one used in Ref. [4].}
This result is a consequence of the construction of the cells. As is clear from this expression, the action of the regularized expression is, in the limit, only non-vanishing when an edge or a vertex of the cylindrical function’s graph $G$ contains a transverse intersection with the bounding surface. For a finite graph, the sum over cells becomes a finite sum over these intersections.

As $\epsilon$ tends to zero $[H_S(N)\epsilon$ of Eq. (17) becomes

$$-8\pi G \sum_{v \in S \cap G} N_v \chi_f^2 \text{Tr} \left[ U_{e_j} J_j U_{e_j^{-1}} J_1 \right] + O(\epsilon^2)$$

in which $N_v$ is the value of the lapse at the vertex $v$. The quantum expression may then be defined as

$$\hat{H}_S(N) \ket{s} := -8\pi G \sum_{v \in S \cap G} \sum_{I,J \vdash v} N_v \chi_f^2 \text{Tr} \left[ U_{e_j} \hat{J}_j U_{e_j^{-1}} \hat{J}_1 \right] \ket{s}$$

on a spin network state $\ket{s}$. The regularization in which the classical Hamiltonian is averaged over a thickened surface and point-split produces a gauge invariant, seemingly well-defined boundary Hamiltonian. It is a new operator on the kinematic state space. Its action has an intriguing “self-measuring” property: reading from left to right, the operator grasps an edge then alters the state by adding the edge, $e_j^{-1}$. The operator measures the angular momentum of the edge which it modified. Before exploring the details of the action, a more obvious problem must be addressed.

This operator does not have the correct dimension. The source of this trouble is the (neglected) density weight tucked into the lapse; the expression is missing a factor of dimension inverse volume. There are two ways to fix this problem. First, as the energy is defined on a 2-surface, the natural unit of length is provided by the area operator of Eq. (14) which may be used to normalize the surface term. The quantum quasilocal energy operator on a surface $S$ is defined by

$$\hat{E}_S(N) \ket{s} := -8\pi G \sum_{v \in S \cap G} \sum_{I,J \vdash v} N_v \chi_f^2 \text{Tr} \left[ U_{e_j} \hat{J}_j U_{e_j^{-1}} \hat{J}_1 \right] \ket{s}.$$  

The operator $\hat{A}_v$ acts only on the vertex $v$. As the area operator is a well-defined operator on the state space, it can be replaced with its spectral resolution. The second solution to the correct normalization is to include the volume factor in the regularization. A regularization of the latter form is given in Section 3.3.

3.2. A recoupling identity. This section is devoted to a recoupling identity which provides a more tractable form of the quasilocal energy operator defined in the last section. The definition simplifies dramatically. The techniques employed are those of diagrammatic recoupling theory introduced to canonical quantum gravity in Ref.

\footnote{This is the reason for the restriction on the relation between $\tau$ and $\epsilon$.}$^{[24]}$ $^{7}$ $J$ is or, shortly is promoted to, an operator. This means that the leading order contribution of the factor $U_{e_j} J_j U_{e_j^{-1}}$ is linear in $\epsilon$. If desired one can include an explicit subtraction term $-J_j J_j$. The resulting operator is identical in action, but the spectrum differs from the one considered here by a factor of 2.

\footnote{The quantum “Hamiltonian” has dimensions $\hbar G$ – the $\hbar$ is in the definition of the operator $J$ – so has the units of area rather than the dimensions $\sqrt{\hbar/G}$ of energy.}$^{[25]}$ $^{8}$
and further developed in Refs. [28, 42–44]. A complete development of the techniques in terms of “framed spin networks” or Temperley-Lieb recoupling theory is in Ref. [60].

Starting from the definition of the quasilocal energy operator Eq. (20) a direct, but moderately lengthy, recoupling calculation yields the full spectrum. However, there is a shorter method. The idea is to use recoupling to investigate and re-express the action of

$$\text{Tr} \left[ U_{e_J} \hat{J}_J U_{e_J}^{-1} \hat{J}_J \right].$$

This factor, acting on the edge $e_J$, can be reduced to a simple form. To see this, note that the two hands, $\hat{J}_J$ and $\hat{J}_I$, grasp edges independently. The unusual action is entirely contained in $U_{e_J} \hat{J}_J U_{(e_J)^{-1}}$ which has the effect of overlaying a segment of the edge $e_J$, grasping the altered edge, and then retracing the segment. The overlaying can easily be accounted for using the “edge addition formula” [45, 59, 60]

$$\begin{align*}
\frac{1}{n} & = \frac{1}{n+1} - \frac{n}{n+1}.
\end{align*}$$

Thus, when the holonomy $U_{(e_J)^{-1}}$ acts on the edge $e_J$ (labeled by $n$), the result is the linear combination

$$\frac{1}{n} - \frac{n}{n+1}.$$

The operator $\hat{J}_J$ acts next. It is tied to the same edge and so measures the angular momentum on the edge that was modified. The edge $e_J$ is grasped, yielding a 2-line and an overall factor of $n \pm 1$

$$(n+1) - \frac{n}{n+1}.$$

Finally, the “loop” ($U_{e_J} U_{(e_J)^{-1}}$) is closed and the second hand is included to give the result

$$(n+1) - \frac{n(n-1)}{(n+1)}.$$

This is the diagrammatic action of the factor $U_{e_J} \hat{J}_J U_{(e_J)^{-1}} \hat{J}_J$. By the Wigner-Eckart theorem each of these diagrams is equivalent to a single trivalent vertex [44, 45, 61]. Diagrammatically, the reduction is accomplished in two stages, each of which removes a triangular loop. The first is given by

$$\frac{1}{2(n+1)}.$$

While $\hat{J}_J$ must act on the edge $e_J$, there is some ambiguity in where, diagrammatically, it grasps the edge. Not surprisingly, the results are the same, even when the operator grasps the edge above or below the modified edge.
The needed recoupling coefficients are given in the appendix (See Eqs. (38) and (39)). Inserting these results in Eq. (21), one finds

\[
(n + 1) \frac{1}{2}\sum_{n}^{\infty} - n \frac{(n - 1)}{2(n + 1)} \frac{1}{2}\sum_{n}^{\infty} = \left[- n \frac{(n + 3)}{2(n + 1)} - n \frac{(n - 1)}{2(n + 1)}\right] \frac{1}{2}\sum_{n}^{\infty} = -n \frac{1}{2}\sum_{n}^{\infty}.
\]

(22)

Therefore, we learn that the numerator of the quasilocal energy is most simply expressed in terms of a signed, “spin-orbit coupling” term:

\[
\text{Tr} \left[U e^{-J e} J J U^{-1} e^{-J e} J J \right] \equiv \frac{1}{2}\sum_{n}^{\infty} n \chi S I \hat{J} J \cdot \hat{J} J \cdot (\sqrt{A \tilde{E}}) 3.
\]

(23)

The properties of the operator are explored in Section 3.4 after the “volume normalized” operator is in place. At this stage it is worth commenting on the operator ordering in Eq. (20) which now can be phrased in terms of the recoupling identity. The natural orderings of the expression are given by the cyclic permutations of the operators in the trace. A short investigation shows that there is one ordering which is distinct from the action given in the last section: The order \(\text{Tr}[U e^{-J e} J J U^{-1} e^{-J e} J J] \) differs in that, diagrammatically, \(\hat{J} J\) grasps the edge \(e J\) before the operator overlays an edge. This produces a diagram distinct from those above. Nevertheless, the two terms created with the edge addition formula trivially re-sum to give a loop in the 2-line of the operator. Such a loop in a 2-line is equivalent to a 2-line. Therefore, this is again proportional to \(\text{Tr}[\hat{J} J J J]\).

We have seen that the regularization of the boundary Hamiltonian with scalar lapse led almost directly to this result, Eq. (23). The key element of choice is the “normalization.” In the next section I regulate the true boundary Hamiltonian. As we will see, this regularization is successful but the resulting operator requires a strong restriction on the state space.

3.3. Alternate regularization and operator. There is an alternate regularization of the density in which the number is no longer absorbed by the lapse. In this case the classical quantity to be quantized is

\[
H_S(N) = \frac{1}{4\pi G} \int_S d^2 x N e^{i j k} m_a A^i_{a} E^{a j} E^{b k} / \sqrt{q}.
\]

The density factor is also promoted to an operator. To regularize this expression one may follow Thiemann [37] and simultaneously soften the divergence of \(\sqrt{q}\) and...
point-split with
\[ V_{\epsilon}(x) := \int_{\Sigma} d^3y f_{\epsilon}(x, y) \sqrt{q}(y) \]
in which the smoothed characteristic function \( f_{\epsilon} \) satisfies
\[ \lim_{\epsilon \to 0} \frac{f_{\epsilon}(x, y)}{\epsilon^3} = \delta^{(3)}(x, y). \]

Note that \( \lim_{\epsilon \to 0} V_{\epsilon}(x) / \epsilon^3 = \sqrt{q}(x) \). Proceeding in a manner similar to the regularization in Section 3.1, the two parts are point-split to give the regularized quasilocal energy
\[ \left[ H_{S}(N) \right]_{\epsilon} := \frac{1}{2\pi G} \int_{S} d^2x n_{b}(x) \frac{N(x)}{V_{\epsilon}(x)} \int d^3y \frac{f_{\epsilon}(x, y)}{\epsilon^3} Tr \left[ [A_{\alpha}, \tilde{E}^{\alpha}](y)\tilde{E}^{b}(x) \right]. \]

To reach the quantum operator, the triads and connection are expressed in forms suitable for quantization. Replacing the triads with the form in Eq. (8) and the factor \( \epsilon^3 \sqrt{q}(x) \) with \( V_{\epsilon}(x) \), one finds that
\[
\left[ H_{S}(N) \right]_{\epsilon} = -(8\pi G) \sum_{I,J} \int_{S} d^2x n_{b}(x) \frac{N(x)}{V_{\epsilon}(x)} \int ds \hat{\epsilon}_{I}(s) \delta(x, e_{I}(s)) \times \int d^3y \int dt \hat{\delta}(y, e_{J}(t)) f_{\epsilon}(x, y) Tr \left[ [A_{\alpha}(y)\hat{\epsilon}^{b}_{J}(t), J_{J}]J_{I} \right] \\
= -(8\pi G) \sum_{I,J} \int_{S} d^2x n_{b}(x) \frac{N(x)}{V_{\epsilon}(x)} \times \int dt \hat{\epsilon}^{b}_{J}(t) \delta(x, e_{J}(t)) \int ds f_{\epsilon}(x, e_{J}(s)) Tr \left[ [A_{\alpha}(e_{J}(s))\hat{\epsilon}^{b}_{J}(s), J_{J}]J_{I} \right]
\]
with the delta function eliminating the \( y \) integration in the second step.

The connection may be replaced by a holonomy. This is possible as the factors inside the commutator are all evaluated on the edge \( e_{J} \). It is also convenient to partition the edge \( e_{J} \) using the definition of holonomy
\[
\left[ H_{S}(N) \right]_{\epsilon} = \lim_{n \to \infty} -(8\pi G) \sum_{I,J} \int_{S} d^2x n_{b}(x) \frac{N(x)}{V_{\epsilon}(x)} \int dt \hat{\epsilon}^{b}_{J}(t) \delta(x, e_{I}(t)) \times \sum_{k=1}^{n} f_{\epsilon}(x, e_{J}(s_{k-1})) Tr \left[ U_{I}(s_{k-1}, s_{k})J_{J}U_{I}(e_{J})^{-1}(s_{k-1}, s_{k})J_{J} \right] \]
\[
= \lim_{n \to \infty} -(8\pi G) \sum_{v \in S} \sum_{I,J} \chi^{S}_{I} \frac{N_{v}}{V_{\epsilon}(v)} \sum_{k=1}^{n} f_{\epsilon}(v, e_{J}(s_{k-1})) \times Tr \left[ U_{I}(t_{k-1}, t_{k})J_{J}U_{I}(e_{J})^{-1}(t_{k-1}, t_{k})J_{J} \right]
\]
where the usual definition for \( \chi^{S}_{I} \) [Eq. (9)] is used. I denote the intersection between the edge \( e_{I} \) and the surface \( S \) by \( v \). The volume \( V_{\epsilon}(v) \) acts on the vertex \( v \).

In the limit as \( \epsilon \) vanishes, \( f_{\epsilon}(v, e_{J}) \) goes to 1 if, and only if, the edge \( e_{J} \) is incident to \( v \). Thus, only one term in the sum over partitions of the edge \( e_{J} \) survives and the sum over of \( J \) is tied to the vertex \( v \). Also in the limit, the volume goes to
$V(v)$, the volume at the vertex $v$. As the regulator is removed,
\[
\lim_{\epsilon \to 0} [H_S(N)]_e = -(8\pi G) \sum_{v \in \partial S} \sum_{I,J=v} \chi^S_I N_e \frac{\text{Tr}[U_e J_I(U_{e,J})^{-1} J_I]}{V(v)}
\]
in which the holonomy is defined to start at the incident end of $e_I$ (the “germ” of Ref. [29]).

The quantization is now immediate. Let me pull the units out of the geometric operator so that
\[
\hat{l}^3 \hat{V}_e := \lim_{\epsilon \to 0} V_e(v).
\]

Using the recoupling identity of Eq. (22), quantum boundary Hamiltonian is defined as
\[
\hat{H}_S(N) = -\frac{1}{(\sqrt{\pi} \hbar^3 G)} \sum_v \sum_{I,J=v} \chi^S_I N_e \frac{\text{Tr}[\hat{J}_I \hat{J}_J]}{V_v}.
\]

The reciprocal of the volume operator is evaluated using its spectral resolution. Though the regulation of the boundary Hamiltonian is possible, the result suffers from the same difficulty as the $\hat{E}_{ADM}$ operator has in Ref. [37]: The signed angular momentum operator $\chi_I^S \hat{J}_I$ does not commute with the volume. To prevent the whole operator from diverging, one can restrict the state space so that the graph has no edges tangent to the surface. This is the tangle property of Ref. [37]. With this modified state space, denoted by $|s_i\rangle$, the quasilocal energy becomes
\[
\hat{E}_S(N) |s_i\rangle = \frac{1}{\sqrt{4\pi\hbar^3 G}} \sum_v \sum_{I,J=v} N_e \chi^S_I \frac{\hat{J}_I \cdot \hat{J}_J}{V_v} |s_i\rangle
\]
The spin network state $|s_i\rangle$ satisfies the tangle property on $S$. This is nearly identical to the $\hat{E}_{ADM}$ operator of Ref. [37]. Aside from an overall factor, this operator differs from $\hat{E}_{ADM}$ in the sign $\chi^S_I$ and the lapse $N_v$. Indeed, under the same asymptotically flat conditions as in Ref. [37] (unit lapse, tangle property, and all edges outgoing), this is $\hat{E}_{ADM}$.

A detailed comparison between $\hat{E}_S(N)$ and $\hat{E}_{ADM}$ is in Section 4, but it is worth offering one observation here. In the context of a quasilocal operator the tangle property seems to be overly restrictive. Since the quasilocal quantity may be applied to any surface in $\Sigma$, restricting the domain of this operator, and thus the Hilbert space, implies that there are no surfaces with edges inside them. Since every edge lies within some surface, the property effectively eliminates the entire state space. The tangle property is too strong for the quasilocal energy operator.

3.4. The spectrum of quasilocal energy. The operators of Eqs. (19), (23), and (27) are quite similar to the geometric operators of spin-net gravity and can be treated with the same methods. The spectra may be computed using the recoupling methods of Refs. [42, 44, 59] or the operator methods of Refs. [23, 29, 41]. I first give the spectrum of the numerator, the operator “$\chi_I^S \hat{J}_I \cdot \hat{J}_J$.” For completeness, the calculation is in the space of gauge non-invariant spin networks, the “extended spin networks” of Ref. [29].

It is convenient to order the edges of each vertex into categories according to the geometric factor $\chi^S_I$. For a vertex of valence $d$, the $a$ edges for which $\chi^S$ is positive are labeled $e_1$ to $e_a$. The $b$ edges for which $\chi^S$ vanishes are labeled $e_{a+1}$ to $e_{a+b}$.
Finally, the $c$ edges for which $\chi^S$ is negative are labeled $e_{a+b+1}$ to $e_d$. There is no restriction on the order of edges within these partitions. Diagrammatically, this ordering is equivalent to selecting three intertwiner trees which grow from internal edges labeled by $p$, $z$, and $n$, according to the value of $\chi^S$. The angular momentum operators are similarly partitioned so that $\hat{J}_{(p)} := \hat{J}_{(e_1)} + \hat{J}_{(e_2)} + \cdots + \hat{J}_{(e_a)}$ for edges with $\chi^S = 1$; $\hat{J}_{(z)} := \hat{J}_{(e_{a+1})} + \cdots + \hat{J}_{(e_{a+b})}$ for edges with $\chi^S = 0$; and $\hat{J}_{(n)} := \hat{J}_{(e_{a+b+1})} + \cdots + \hat{J}_{(e_d)}$ for edges with $\chi^S = -1$. Using the methods from the quantum mechanics of angular momentum, one finds

$$\sum_{I,J} \chi^I_S \hat{J}_I \cdot \hat{J}_J \equiv \left( \hat{J}_{(p)} - \hat{J}_{(n)} \right) \cdot \left( \hat{J}_{(p)} + \hat{J}_{(z)} + \hat{J}_{(n)} \right)$$

(26)

$$= \hat{J}_{(p)}^2 - \hat{J}_{(n)}^2 + \hat{J}_{(p)} \cdot \hat{J}_{(z)} - \hat{J}_{(n)} \cdot \hat{J}_{(z)}.$$

Two properties of this operator are immediately clear. First, the operator simplifies when evaluated on gauge invariant spin networks. In fact, the spectrum collapses and contains only the value zero. This is obvious from Eq. (26) as this operator contains

$$\hat{G}^2 := \sum_j \hat{J}_j^2$$

which is the quantum version of the Gauss constraint, Eq. (4) of Ref. [10]. Of course, one may simply compute the result as well. For general vertices a recoupling calculation of the terms in Eq. (26) shows that the operator vanishes. The recoupling terms are of two forms. The first is diagrammatically

$$\hat{J}_{(n)}^2 \Phi = \frac{\hbar^2}{2} p^2 n^2 \Phi = \hbar^2 \frac{n(n+2)}{4} \Phi,$$

(28)

making use of identity Eq. (40). The second is

$$\hat{J}_{(p)} \cdot \hat{J}_{(z)} \Phi = - \frac{\hbar^2}{2} p z \Phi = \hbar^2 \frac{p(p+2) + z(z+2) - n(n+2)}{4} \Phi,$$

(29)

making use of identity Eq. (41). Thus, as a consequence of summing up all the components of the connection along the incident edges (Eq. (18)), the quasilocal energy operator vanishes on gauge invariant states.

The second property seems equally serious. Since the operator of Eq. (26) contains two terms which do not commute, $\hat{J}_{(p)} \cdot \hat{J}_{(z)}$ and $\hat{J}_{(n)} \cdot \hat{J}_{(z)}$, the spectrum is not well-defined. These terms cannot be simultaneously diagonalized. This result may be seen in the diagrammatic picture as well. In the gauge non-invariant case, vertices of this type have an intertwiner with four edges, one of which is not grasped by either operator. (This extra edge can be thought of as leaving the 3-dimensional manifold.) The operator of Eq. (26) requires that all three edges labeled by $p$, $n$, and $z$ are part of a single trivalent vertex. This is not possible with a four-valent intertwiner. These properties suggest that the state space be carefully examined.

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10The Gauss constraint, although it is expressed in terms of $\hat{J}$ operators, is edge orientation invariant. In the current work, the invariance comes from the “edge-projected connection” in Eq. (17). When the orientation of the edge $J$ is reversed, both the operator $\hat{J}_{J}$ and the component along the connection change sign.
The space on which this operator acts is not characterized by general, gauge invariant vertices. When the spatial manifold is cut into regions with boundary, the Hilbert space $\mathcal{H}^I$ of a bounded region $I$ is based on “open” graphs (graphs with edges ending in vertices of valence 1). Given a graph in $\Sigma$, I define the open subgraph $G^I$ to be the portion of $G$ within the interior $I$ and within the boundary $\partial I$, i.e. $G^I = \mathcal{T} \cap G$. Thus, all the open edges are incident to the boundary $S$ and these gauge non-invariant edges are confined to the intersection of the graph with the boundary; the “extended spin networks” of Ref. [23] are restricted to lie on the boundary. Only edges that are incident to $v$ and in $G^I$ – including tangent edges – contribute to the energy. The Hilbert space is defined as before, only now the graph used is $G^I$. With $\mathcal{H}^I$ it is possible to give the spectrum.

The clearest way to express spectrum of “$\chi^S \hat{J}_I \cdot \hat{J}_J$” on $\mathcal{H}^I$ is to choose the orientations of the edges incident to the boundary to be outward pointing. This is suggests a form for the intertwiner “core”\textsuperscript{11}. It is a trivalent vertex labeled by $p, n$ and $z$ for the values of $\chi^S$. With the orientations outward pointing, then the labels $p, n, z$ take the meanings outside the surface, inside the surface, and tangent to the surface, respectively. As the $p$-edges are not in $\mathcal{H}^I$, they are not seen by the operator. Therefore the operator of Eq. (20) reduces to

$$\hat{J}_{(n)} \cdot \hat{J}_{(p)}$$

using $-\hat{J}_{(z)} = \hat{J}_{(n)} + \hat{J}_{(p)}$. For $| s_i \rangle \in \mathcal{H}^I$ based on a single vertex $v$ the spectrum is

$$\sum_{I,J \leftrightarrow v} \chi^S_{IJ} \hat{J}_I \cdot \hat{J}_J | s_i \rangle = \frac{\hbar^2}{2} \left[ \frac{p(p+2) + n(n+2) - z(z+2)}{4} \right] | s_i \rangle.$$

It is easy to generalize to more than one vertex; simply sum over all contributing vertices. The spectrum is not positive definite and, acting on a single vertex, is explicitly bounded (for finite spin). There are two cases worth identifying. When the vertex has no tangent edges, the spectrum, which I call type (i), is always positive definite and is proportional to $n(n+2)$. The general case when there are tangent edges, called type (ii), is given in Eq. (31).

The form of the operator given in Eq. (30) also provides a simple way to check two critical properties of the quasilocal energy, gauge invariance and compatibility with the area operator. Gauge invariance is clear from Eq. (24), may be directly checked\textsuperscript{11} and can also be made manifest with

$$2\hat{J}_{(p)} \cdot \hat{J}_{(n)} = \hat{J}^2_{(p)} + \hat{J}^2_{(n)} - \hat{J}^2_{(z)}.$$

This expression is also useful to show that the two operators $\chi^S \hat{J}_I \cdot \hat{J}_J$ and $\hat{A}_v$ are compatible on the Hilbert space $\mathcal{H}^I$. Thus, the terms in the numerator and denominator may be diagonalized simultaneously.

\textsuperscript{11}This is by no means necessary. The spectrum can be computed without assigning orientations although the computation is considerably longer.

\textsuperscript{12}When the edges are partitioned into three categories, as is often convenient in quantum geometry, the external edges are connected in trees which end in one principle, internal edge. The core of the intertwiner is the trivalent vertex which connects these three internal edges. It is the only part of the intertwiner which must be specified before completing the diagrammatic calculation of the spectrum.

\textsuperscript{13}The crux of the matter is that

$$[\hat{J}_{(p)} \cdot \hat{J}_{(n)}, G^I] = -i\hbar \epsilon^{ijk} \left( \hat{J}^j_{(p)} \hat{J}^k_{(n)} + \hat{J}^k_{(p)} \hat{J}^j_{(n)} \right) = 0.$$
It is now possible to assemble all the factors for the full spectrum of the quasilocal energy operator on $\mathcal{H}^I$. For the operator defined in Eq. (23), using the spectrum of the area operator, Eq. (15), one has for type (i)

$$\hat{E}_S(N) \mid s \rangle = m \sum_{v \in S \cap G^I} N_v \sqrt{4 n_v (n_v + 2)} \mid s \rangle.$$

(32)

For type (ii):

$$\hat{E}_S(N) \mid s \rangle = m \sum_{v \in S \cap G^I} N_v \frac{p_v (p_v + 2) + n_v (n_v + 2) - z_v (z_v + 2)}{\left[2 p_v (p_v + 2) + 2 n_v (n_v + 2) - z_v (z_v + 2)\right]^2} \mid s \rangle.$$

The fundamental mass scale is defined as

$$m := \sqrt{\frac{\hbar}{4 \pi G}}.$$

The simple transverse vertices of type (i) yield a positive definite spectrum while the others do not. The results in Eq. (32) are the full spectrum of the quasilocal energy operator with area normalization. The operator $\hat{E}_S(N)$ does not require any restrictions on the Hilbert space. It is well-defined on the Hilbert space $\mathcal{H}^I$; all the needed properties of are naturally induced from the full gauge invariant Hilbert space in $\Sigma$.

There is one example of a spin-net state which is particularly interesting. When the intersections between the graph of the spin network state and the surface are entirely transversal (type (i)), the quasilocal operator becomes

$$\hat{E}_S(N) \mid s_i \rangle = m \sqrt{2} \sum_{v \in S \cap G^I} N_v \sqrt{4 j_v (j_v + 1)} \mid s_i \rangle \equiv m \sqrt{2} \sum_{v} N_v \sqrt{a_v} \mid s \rangle,$$

in which $a_v$ is the eigenvalue of the area operator for the vertex $v$. The operator is the “square root of the area”! This could have been anticipated with a little dimensional analysis. Nevertheless, it is remarkable that energy is so closely related to area. (The close connection is also present in the volume normalization.) For large spins the energy scales as $\sqrt{j_v}$.

The next and final section begins with a summary of the operator definitions and continues with comparisons to earlier work and a discussion of some wider implications.

4. Discussion

To be well-defined, the action of a theory must be functionally differentiable. This simple observation provides a key to the form of all surface observables and boundary conditions of a given action [15, 16]. In the case of the (3+1) gravitational theory defined in a bounded region, the variation generates a surface observable associated to the Hamiltonian constraint. This term is precisely the negative of the Hamiltonian of the system. In this paper, this boundary Hamiltonian is quantized within the framework of spin-net gravity.

Spin-net gravity is a background-metric independent, canonical quantization of gravity using real connection variables and the methods of spin networks. Under this rubric is also included the assumption that the kinematic state space is rich enough to describe the full physical state space, including dynamics.

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14 Note that the spectrum can not be dependent only on the tangent edges; the operator vanishes unless there is at least one transverse edge.
There are two different, inequivalent expressions for the boundary Hamiltonian corresponding to two separate normalizations. Both operators are well-defined quantum operators (but on different spaces). The area normalized operator has the following spectrum: For type (i) vertices:

$$\hat{E}_S(N) \mid s \rangle = m \sqrt{2\gamma} \sum_v N_v \sqrt{j_v(j_v + 1)} \mid s \rangle$$

and for type (ii) vertices:

$$\hat{E}_S(N) \mid s \rangle = m \sqrt{2\gamma} \sum_v N_v \left[ j_v(j_v + 1) + j_{(n_v)}(j_{(n_v)} + 1) - j_{(z_v)}(j_{(z_v)} + 1) \right]^{-3/4} \mid s \rangle.$$  

All the dimensionful constants and the Immirzi parameter are included; $m = \sqrt{\hbar / 4\pi G} \sim 40\text{GeV}/c^2$. This operator is defined on the full Hilbert space $H_I$. This comes about through an interplay between the operator and the gauge invariant Hilbert space which allows the operator to be defined on the induced Hilbert space. As can be easily seen from the spectrum, only transversal edges contribute energy. Tangential edges remove energy.

The alternate form of the operator, with volume normalization and defined only on states which are of type (i), is

$$\hat{E}_S(N) \mid s_i \rangle = m \sqrt{2\gamma} \sum_v N_v \left[ j_{(n_v)}(j_{(n_v)} + 1) - j_{(z_v)}(j_{(z_v)} + 1) \right]^{-3/4} \mid s_i \rangle$$

where $j_{(n_v)} = n_v/2$ and $\lambda_v$ is the eigenvalue of the volume. On account of the state restriction, the numerator always has the simple form of the area operator; the operator is proportional to $\hat{A}_v^2 / \hat{V}_v$. The volume operator does not have as simple a form as the area operator, so the full spectrum cannot be presented, as in Eq. (32). The eigenvalues $\lambda_v$ can be computed using recoupling theory as in Refs. [27, 42]. This form of the quasilocal energy has the correct naive classical limit in that it is a direct quantization of the classical surface Hamiltonian. This operator requires tight restrictions on the Hilbert space; there can be no edges tangent to the surface $S$. As mentioned in Section 3.3, since every edge is tangent to some surface, this restriction introduces a contradiction in the construction of this quasilocal energy operator.

The operators share some general features. They are defined on bounding 2-surfaces. Although they have a strange inclination to be positive, both operators share the property that the spectrum is fully discrete and bounded (for arbitrarily large but finite graphs and spins). The area normalized operator does not have a positive spectrum. The reason may be traced to the geometric factor in the new operator $\chi_S \hat{J}_I \cdot \hat{J}_J$. (This in turn is a result of the Hamiltonian’s two factors, the edge-projected connection and the $\hat{E}_S$ piece.) On account of this factor, the quasilocal energy depends on the orientation of the surface. It does not, however, depend on the orientation of the edges.

The volume normalized operator has a similar form to the $\hat{E}_{ADM}$ energy of Ref. [37], in which a different, but weakly equivalent, classical expression was quantized. The present operator generalizes the ADM operator in one important way. It is a quasilocal operator defined on all bounding surfaces in the spatial manifold. It also
shares a key property: to be defined at all, the tangle property of Ref. [37] must be imposed. In the language of Section 3.4, all vertices are of type (i). In fact, when the tangle property is satisfied and when the lapse is fixed at unity, the expression for \( \hat{E}_{S}(N) \) is simply identical to the \( \hat{E}_{ADM} \) of Eq. (3.18) in Ref. [37] (up to an overall numerical factor). The volume normalized quasilocal energy reduces to the ADM energy. It is remarkable that such different classical expressions yield nearly the same form of the operator.

Clearly, only one quasilocal energy operator gives physically correct values. While \( \hat{E}_{S}(N) \) has the correct naive limit, this criterion is not the only physical condition which must be met. More importantly, the expectation value of the quantum energy operator in an appropriate semiclassical state must approximate the classical energy, up to small quantum corrections. In addition, the algebra of boundary observables ought to be anomaly free. Until such states and such operators are investigated in full detail, it is hard to definitively select a quasilocal energy operator. To complete this investigation would require quantization of the boundary rotation and boosts – a project which will be left to further work (there is preliminary work already in Ref. [37]). Other approaches may also help fix the correct quantization, perhaps through the matching of quantum to semiclassical results.

In advance of a more complete investigation, there is one feature which does differentiate the two operators. The volume normalized operator requires a restriction on the induced Hilbert space. While this may be acceptable for the asymptotically flat setting where the property was introduced, this is a too severe a priori restriction on the whole Hilbert space. Therefore, for the remainder of this section I restrict the majority of my comments to the area normalized \( \hat{E}_{S}(N) \).

The previous quantization of the boundary Hamiltonian has a qualitatively different action of “shifting” edges incident to the boundary [31]. The quasilocal energy operator does not share this qualitative behavior. From the perspective of the spin-net framework, it is clear that the shift is an artifact of the partial quantization. In fact, there are two terms which shift “up and back” as may be seen in Eq. (17) or in the gauge invariant form of the operator. The “shift” is incorporated into the gauge invariant operator, which leads to the “self-interaction” measurement of edge addition as shown in Section 3.2.

There are a few immediate results which follow from the definition of the quasilocal energy operator. Gravitational energy takes on quantized values with the smallest gap in energy of

\[
e_{o} = \frac{\sqrt{3}}{m}
\]

due the the addition or removal of a spin-1/2 edge. Suppose that a system’s bounding surface only intersected spin-1/2 edges of the underlying spin network. Then, transitions due to adding or removing a spin-1/2 edge would result in radiation with the rather energetic fundamental frequency of \( \omega_{o} = \frac{\sqrt{3}}{m} \). This mechanism is similar to the Bekenstein-Mukhanov quantization [12], in which the area is given by integer multiples of a fundamental area. In this case, however, the quasilocal energy is quantized in integer units, \( E_{S} = n e_{o} \), so area scales as \( n^{2} \). Of course, before we tune our radio receivers to listen to black holes, there is much more to understand. In particular, we need a characterization of semiclassical states and the dynamics. To see that this strongly affects the radiation, we need only study
The energy gap narrows, as would be expected from the naive semiclassical limit.

There are also some wider implications of the quasilocal energy. The number of directions is enormous and I confine my comments to four brief remarks: On a first glance at the quasilocal energy operator of Eq. (20) it appears that the quasilocal energy operator depends on the orientation of the edges. Since the kinematic state space of $SU(2)$ spin networks is independent of orientation this would be odd. In fact, the quasilocal energy operator does not depend on orientation of the edges. There are two aspects of this property. This is perhaps easiest to see by noting that when the orientation of an edge is changed, both the sign factor $\chi_{II}$ and the angular momentum operator $\hat{J}_I$ change sign \[29\]. In an abuse of notation the same $\hat{J}_I$ are used in the Gauss operator $\sum_j \hat{J}_I$, although the Gauss operator is independent of edge orientation. On the other hand, the operator definitely does depend on the orientation of the surface. When the orientation of the surface changes, only the geometric factor changes sign. Orienting the edges to be outgoing from the vertices simply places the edges in three, simple categories: inside the region, outside the region, and tangent to the surface. One tantalizing aspect to this third category is the interpretation of edges which leave the region $I$. They take energy with them. It remains to be seen whether these could provide structure for the 4-dimensional space – “pillars of time” in a 4-dimensional spin network model – or more exotic manifolds.

Given the definition of the quasilocal energy operator and the induced Hilbert space $\mathcal{H}_I$, the question arises whether the operator is consistent. If the spatial manifold $\Sigma$ is compact, is it true that $E_S + E_{S^*}$ vanishes? The natural language in which to consider such questions is topological quantum field theory (TQFT) \[64\], in which one studies diffeomorphism invariant theories on manifolds with boundary. While the manifolds are cut and sewn together, the theories associate maps to interiors and vector spaces to boundaries. A more careful study will be left to further work. But the question may addressed directly. On a particular spin network state, say $| s_{\Sigma} \rangle$, the two operators act on the two “halves” of the state. One is based on $\bar{T}$; the other is based on $\bar{T}^\dagger$. As $| s_{\Sigma} \rangle$ is a general, gauge invariant state, there are two types of vertices to consider, types (i) and (ii). A short calculation shows that the contributions to the energies are numerically identical \[15\].

As this quasilocal energy is the boundary Hamiltonian, one may describe the time development of the Lorentzian theory on $S \times \mathbb{R}$,

$$\hat{U}(t) = e^{i\hat{E}_{S}(N_t)t/\hbar}.$$  

(Recall that the boundary conditions fix the lapse and metric on the boundary. A more complete treatment would also include terms with non-vanishing shift $N^a$, giving rotations and boosts.) The energy operator is diagonal in the spin network basis, so the unitary evolution operator has a simple, well-defined form (at least for type (i) vertices). It describes the system in terms of an observer on the surface $S$.

Likewise, the quasilocal energy operator gives the partition function $e^{\beta \hat{E}_S}$. As the quasilocal energy is the true Hamiltonian, this function is the partition function for the statistical mechanics of spin-net gravity. This observation is the starting point for a vast range of physical questions. For instance, is the energy quanta $e_o$

\[15\] Naturally, one needs a consistent orientation on $\Sigma$ to perform the sum.
statistically favored? What is the selection criteria for the ground state(s)? What is the entropy of a bounded system in spin-net gravity?

What is particularly striking is the delightful number of physical questions that can be addressed with this operator in the current framework on spin-net gravity. This is but one result of the techniques which have been developed since 1995 and which offer methods for detailed study of these questions. In fact, it appears that these techniques are powerful and rich enough to bring us from theoretical modeling to physical predictions.

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Appendix A. Recoupling quantities

This appendix contains recoupling formulae needed in the calculations of the spectrum. The conventions are those of Kauffman and Lins [60] (for $A = -1$). Short introductions and definitions of the basic recoupling quantities can be found in [42, 44].

The diagrammatics for the “$\mathcal{J} \cdot \mathcal{J}$” operator require careful use of the “$\lambda$-move”

$$
\chi^{ab} = \lambda^{ab}_{c} \chi^{c}
$$

where $\lambda^{ab}_{c}$ is

$$
\lambda^{ab}_{c} = (-1)^{a(a+3)+b(b+3)-c(c+3)/2}
$$

The function $\theta(m, n, l)$ is given by

$$
\theta(m, n, l) = (-1)^{a+b+c} \frac{(a + b + c + 1)!a!b!c!}{(a+b)!(b+c)!(a+c)!}
$$

where $a = (l + m - n)/2$, $b = (m + n - l)/2$, and $c = (n + l - m)/2$.

The tetrahedral symbol is given by

$$
\text{Tet} \begin{bmatrix} a & b & c & d & e \end{bmatrix} = N \sum_{m \leq s \leq S} (-1)^{s} \frac{(s + 1)!}{\prod_{i} (s - a_{i})! \prod_{j} (b_{j} - s)!} \\
N = \frac{\prod_{i,j} [b_{j} - a_{i}]!}{a!b!c!d!e!f!}
$$

in which

$$
a_{1} = \frac{1}{4}(a + d + e) \quad b_{1} = \frac{1}{4}(b + d + e + f) \\
a_{2} = \frac{1}{4}(b + c + e) \quad b_{2} = \frac{1}{4}(a + c + e + f) \\
a_{3} = \frac{1}{4}(a + b + f) \quad b_{3} = \frac{1}{4}(a + b + c + d) \\
a_{4} = \frac{1}{4}(c + d + f) \quad m = \max \{a_{i}\} \quad M = \min \{b_{j}\}
$$
The quantities needed in the calculation for the simplification of the quasilocal energy operator are:

\[
\begin{align*}
\text{Tet} \left[ \begin{array}{ccc} 1 & 1 & n \\ n+1 & n+1 & 2 \end{array} \right] & = \frac{(n+3)}{2(n+1)}; \\
\theta(n+1, n, 1) & = -\frac{n}{2(n+1)}; \\
\text{Tet} \left[ \begin{array}{ccc} 1 & 1 & n \\ n-1 & n-1 & 2 \end{array} \right] & = \frac{1}{2}; \\
\theta(n, n, 2) & = 1.
\end{align*}
\]

For instance,

\[
\begin{align*}
\text{Tet} \left[ \begin{array}{ccc} 1 & 1 & n \\ n+1 & n+1 & 2 \end{array} \right] & = \frac{(n+3)}{2(n+1)}; \\
\theta(n+1, n, 1) & = -\frac{n}{2(n+1)};
\end{align*}
\]

This is used in the recoupling calculation of Eq. (22).

A “bubble” diagram is proportional to a single edge. In particular,

\[
\begin{align*}
\text{Tet} \left[ \begin{array}{ccc} 1 & 1 & n \\ n+1 & n+1 & 2 \end{array} \right] & = \frac{(n+3)}{2(n+1)}; \\
\theta(n+1, n, 1) & = -\frac{n}{2(n+1)}; \\
\text{Tet} \left[ \begin{array}{ccc} 1 & 1 & n \\ n-1 & n-1 & 2 \end{array} \right] & = \frac{1}{2}; \\
\theta(n, n, 2) & = 1.
\end{align*}
\]

Such a 2-line spanning a vertex is

\[
\begin{align*}
\text{Tet} \left[ \begin{array}{ccc} 1 & 1 & n \\ p & p & z \end{array} \right] & = \frac{(n+3)}{2(n+1)}; \\
\theta(p, n, 2) & = -\frac{n}{2(n+1)};
\end{align*}
\]

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