Abstract. We show that if two lattice 3-polytopes \( P \) and \( P' \) have the same Ehrhart function then they are \( \text{GL}_3(\mathbb{Z}) \)-equidecomposable; that is, they can be partitioned into relatively open simplices \( U_1, \ldots, U_k \) and \( U'_1, \ldots, U'_k \) such that \( U_i \) and \( U'_i \) are unimodularly equivalent, for each \( i \).

1. Introduction

1.1. Motivation. Consider the rational polygons \( P \) with vertices \( \left\{ \left( -\frac{4}{0}, \left( -\frac{1}{0}, \left( -\frac{3}{2} \right) \right) \right) \right\} \) and \( P' \) with vertices \( \left\{ \left( \frac{1}{0}, \left( \frac{3}{0}, \left( \frac{1}{1} \right) \right) \right) \right\} \), depicted here:

We claim that for every \( k \in \mathbb{Z}_{>0} \), the dilations \( kP \) and \( kP' \) contain the same number of integer points: \( |kP \cap \mathbb{Z}^2| = |kP' \cap \mathbb{Z}^2| \). One way to see this is to compute this number for every \( k \), which can be done for example using Ehrhart theory. A more insightful argument is to decompose \( P \) and \( P' \) as follows:

These decompositions yield a bijection between the rational points in \( P \) and \( P' \) that preserves denominators, hence a bijection \( kP \cap \mathbb{Z}^2 \leftrightarrow kP' \cap \mathbb{Z}^2 \) for every \( k \). In this paper we address the question of whether every Ehrhart-equivalence admits an explanation via equipartitions, as happens in the example. We show the answer to be positive for 3-dimensional lattice polytopes.

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This question is reminiscent of Hilbert’s third problem about the equidecomposability of polytopes of the same volume into pieces that can be congruently bijected. Instead of the volume, the valuation under consideration is the Ehrhart polynomial; instead of rigid motions we consider unimodular transformations.

1.2. Framework. We study decompositions of a lattice polytope $P$ into rational polytopes. That is, $P \subset \mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{Z}^d$, and each piece $Q$ in the decomposition will be the convex hull of finitely many points in $\mathbb{Q}^d$.

The denominator of a rational polytope $Q$ is the minimum positive integer $D \in \mathbb{Z}_{\geq 1}$ such that the dilation $DQ$ is a lattice polytope. The dimension $\dim Q$ of $Q$ is the dimension of the affine subspace $\text{aff} Q \subset \mathbb{R}^d$ spanned by $Q$. The relative interior $\text{relint} Q$ of $Q$ is the interior of $Q$ inside $\text{aff} Q$. Similarly, any subset of $\mathbb{R}^d$ is relatively open if it is open inside its affine span.

Ehrhart’s Theorem [4, Théorème 38] states that the counting function $k \mapsto |kQ \cap \mathbb{Z}^d|$ agrees for $k \in \mathbb{Z}_{\geq 1}$ with a quasipolynomial $\text{ehr}(Q; k)$ of degree $\dim Q$ with period $D$. That is, for every $k$ we have $\text{ehr}(Q; k) = c_{\dim Q}(k) k^{\dim Q} + \ldots + c_1(k) k + c_0(k)$ with periodic functions $c_0, \ldots, c_{\dim Q}: \mathbb{Z} \to \mathbb{Q}$ such that $c_{\dim Q}$ is not identically zero and $c_i(k + D) = c_i(k)$ for all $i$ and $k$. In particular, if $P$ is a lattice polytope then $\text{ehr}(P; k)$ is an honest polynomial.

Definition 1. Two rational polytopes $Q_1, Q_2 \subset \mathbb{R}^d$ are called Ehrhart-equivalent if they have the same Ehrhart quasipolynomial. That is, if $|kQ_1 \cap \mathbb{Z}^d| = |kQ_2 \cap \mathbb{Z}^d|$ for all $k \in \mathbb{Z}_{\geq 1}$.

Example 2. Consider the 1-dimensional polytopes $Q = [0, 1]$, $Q' = [1/5, 6/5]$ and $Q'' = [2/5, 7/5]$. Then $\text{ehr}(Q; k) = k + 1$ while $\text{ehr}(Q'; k) = \text{ehr}(Q''; k) = k + c_0(k)$ where $c_0(k) = 1$ if $k \in 5\mathbb{Z}$ and $c_0(k) = 0$ else. So $Q'$ and $Q''$ are Ehrhart-equivalent to each other but not to $Q$.

The Ehrhart quasipolynomial is invariant under the group $\text{Aff}_d(\mathbb{Z}) := \text{GL}_d(\mathbb{Z}) \ltimes \mathbb{Z}^d$ of lattice preserving affine maps. We call such maps unimodular transformations. Now we can define what kind of “nice explanation” for Ehrhart-equivalence we seek.

Definition 3. We say that two polytopes $P, Q \subset \mathbb{R}^d$ are $\text{GL}(\mathbb{Z})$-equidecomposable if there are relatively open simplices $T_1, \ldots, T_r$ and unimodular transformations $U_1, \ldots, U_r \in \text{GL}_d(\mathbb{Z}) \ltimes \mathbb{Z}^d$ such that

$$P = \bigsqcup_{i=1}^r T_i \quad \text{and} \quad Q = \bigsqcup_{i=1}^r U_i(T_i).$$

(Here, $\bigsqcup$ indicates disjoint union.)

It would be nice if the converse was true.

Main Question ([6, Question 4.1]). Is it true that every pair of Ehrhart-equivalent polytopes are $\text{GL}(\mathbb{Z})$-equidecomposable?

One case where this is true is when both $P$ and $P'$ admit a unimodular triangulation, that is, a triangulation into simplices that are $\text{GL}(\mathbb{Z})$-equivalent to the standard simplex $\text{conv}(0, e_1, \ldots, e_k)$. In this case the Ehrhart quasipolynomial contains the same information as the $f$-vector of such triangulations [2], and unimodular triangulations with the same $f$-vector clearly yield a $\text{GL}(\mathbb{Z})$-equidecomposition.
This in particular implies a positive answer to the Main Question in dimension two, since all lattice polygons have unimodular triangulations. Peter Greenberg proved an even stronger statement [5, Theorem 2.4]: Ehrhart-equivalent lattice polygons can be related to one another by a sequence of GL($\mathbb{Z}$)-equidecompositions of a particular type that he calls 1-triangulated homeomorphisms. Imre Bárány and Jean-Michel Kantor ask a similar question under the stronger hypothesis that $|P \cap \Lambda| = |P' \cap \Lambda|$ for every super lattice $\Lambda \supseteq \mathbb{Z}^d$ [1].

In dimension 3 existence of unimodular triangulations does not hold for every lattice polytope. Even more, the following two polytopes $P$ and $P'$ have the same Ehrhart polynomial but $P'$ admits a unimodular triangulation while $P$ does not:

$$P = \text{conv} \begin{bmatrix} 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{bmatrix} \quad \text{and} \quad P' = \text{conv} \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}.$$  

Jean-Michel Kantor conjectures that, in general dimension, one cannot even find a piece-wise unimodular homeomorphism [8, p. 212] between every pair of Ehrhart equivalent lattice polytopes.

It is worth pointing out that the answer to the Main Question turns out to be negative if we extend it to rational polytopes, even in dimension 1.

**Example 4.** The polytopes $Q' = [1/5, 6/5]$ and $Q'' = [2/5, 7/5]$ from Example 2 are Ehrhart equivalent but they cannot be GL($\mathbb{Z}$)-equidecomposable: $Q'$ contains three points from the Aff$_1(\mathbb{Z})$-orbit of 1/5 (namely, 1/5, 4/5 and 6/5) while $Q''$ contains only two (4/5 and 6/5). See [10, 11] for more examples in dimension 2.

**Remark 5.** What happens in the above example is that all points of the form $a/5$ with $a \in \mathbb{Z} \setminus 5\mathbb{Z}$ have the same Ehrhart function, but are not in the same Aff$_1(\mathbb{Z})$-orbit. This can be formalized as follows:

Let $\Lambda = \frac{1}{9} \mathbb{Z}^d$ be a super lattice of $\mathbb{Z}^d$. The group Aff$_d(\mathbb{Z})$ of unimodular transformations acts on the cosets $\Lambda/\mathbb{Z}^d$. Denote by $O = O(\Lambda) = \text{Aff}_d(\mathbb{Z}) \setminus \Lambda/\mathbb{Z}^d$ the set of orbits of this action, so that $[\lambda] \in O$ is the orbit of $\lambda \in \Lambda$. We identify an orbit $o \in O$ with the corresponding set of $\Lambda$-points $\{\lambda \in \Lambda : [\lambda] = o \}$. Then GL($\mathbb{Z}$)-equidecomposable polytopes $P$, $P'$ satisfy $|P \cap o| = |P' \cap o|$ for all $\Lambda$ and all $o$. (This approach is different from [8, §1.3].)

For $\Lambda = \frac{1}{5} \mathbb{Z} \supseteq \mathbb{Z}$, we get three orbits $O = \{\mathbb{Z}, \{\frac{1}{5}, \frac{2}{5}\} + \mathbb{Z}, \{\frac{3}{5}, \frac{4}{5}\} + \mathbb{Z}\}$. The 0-dimensional polytopes $\{1/5\}$ and $\{2/5\}$ are Ehrhart-equivalent but not GL($\mathbb{Z}$)-equidecomposable.

On the other hand, a weakened version of the Main Question does hold for arbitrary rational polytopes. If we allow transformations in GL$_d(\mathbb{Z}) \ltimes \mathbb{Q}^d$, that is, if we allow rational translations, then any two Ehrhart-equivalent polytopes are equidecomposable [6, Prop. 4.3]. Observe, however, that for this group of motions the converse implication fails: the polytopes $Q$ and $Q'$ from Example 2 are equivalent under GL$_1(\mathbb{Z}) \ltimes \mathbb{Q}^1$, but they have different Ehrhart quasipolynomials. This sublety goes away if we insist on integral vertices [6, Cor. 4.4].

**1.3. Result and structure of proof.** The main result of the present paper is that Ehrhart-equivalence and GL$_3(\mathbb{Z})$-equidecomposability are the same for 3-dimensional lattice polytopes.

**Theorem 6.** Ehrhart-equivalent lattice 3-polytopes are GL$_3(\mathbb{Z})$-equidecomposable into half-unimodular simplices.
Here a half-unimodular simplex is a simplex whose second dilation is a unimodular simplex (Definition 7).

The two ingredients in the proof of Theorem 6 are a classification of half-unimodular simplices in dimension three (Section 2) together with the fact that all empty tetrahedra (hence all lattice simplices in \( \mathbb{R}^3 \)) admit a decomposition into relatively open half-unimodular simplices. The latter is well-known [9, 12] but in Section 3 we show that the decomposition uses using only half-unimodular simplices of certain types. In Section 4 we show that these types have Ehrhart quasipolynomials that are linearly independent in the vector space of all quasipolynomials, which implies that the decompositions constructed in Section 3 for Ehrhart-equivalent polytopes \( P_1 \) and \( P_2 \) use exactly the same number of half-unimodular simplices of each type, hence providing a \( \text{GL}(\mathbb{Z}) \)-equidecomposition.

2. Classification of half-unimodular simplices in \( \mathbb{R}^3 \)

In this section we will give a full classification of half-unimodular simplices under \( \mathbb{Z}^3 \)-equivalence, with the following definition.

**Definition 7.** An \( i \)-simplex \( \Delta \) in \( \mathbb{R}^n \) is called **half-unimodular** if \( 2\Delta \) is a unimodular lattice simplex. That is,

\[
2\Delta \cong \text{conv}(0, e_1, \ldots, e_i),
\]

where \( e_1, \ldots, e_n \in \mathbb{Z}^n \) are the standard basis in \( \mathbb{R}^n \).

For any \( n \), we have at least the following \( 2^n - 1 \) half-unimodular simplices:

\[
\begin{align*}
\Delta_i^1 & := \frac{1}{2} \text{conv}(0, e_1, \ldots, e_i), & i & = \{0, \ldots, n\}, \\
\Delta_i^0 & := \frac{1}{2} \text{conv}(e_1, \ldots, e_{i+1}), & i & = \{0, \ldots, n-1\}.
\end{align*}
\]

Observe that the subindex denotes dimension and the superindex 0 or 1 denotes the number of lattice points, so these simplices are indeed not Ehrhart-equivalent to one another.

For \( n = 3 \) there are additionally the following triangle and tetrahedron:

\[
\begin{align*}
\Delta'_2 & := \left( \frac{1}{2}, \frac{1}{2}, 0 \right) + \frac{1}{2} \text{conv}(0, e_1, e_2), \\
\Delta'_3 & := \left( \frac{1}{2}, \frac{1}{2}, 0 \right) + \frac{1}{2} \text{conv}(0, e_1, e_2, e_3).
\end{align*}
\]

None of \( \Delta'_2 \) and \( \Delta'_3 \) contain lattice points (which shows they are not equivalent to \( \Delta_2^1 \) and \( \Delta_1^3 \)), and \( \Delta'_2 \) is distinguished from \( \Delta_0^0 \) by the fact that its affine span contains lattice points. (In particular, \( \text{ehr}(\Delta_0^0, 2k+1) = 0 \) for all \( k \) while \( \text{ehr}(\Delta'_2, 2k+1) > 0 \) for sufficiently big \( k \)).

**Lemma 8.** Every half-unimodular simplex in \( \mathbb{R}^3 \) is equivalent to one of the nine defined above.

**Proof.** We have already justified that the nine simplices above are non-equivalent. Consider now an arbitrary half-unimodular simplex \( \Delta \) of dimension \( i \) and let’s see that it is equivalent to one of the nine.

Note that every half-unimodular simplex contains at most one integer vertex. If \( \Delta \) contains an integer point, without loss of generality assume it to be the origin.
Then, the unimodular transformation sending $2\Delta$ to $2\Delta^1$ can be chosen to fix the origin, which implies $\Delta$ is equivalent to $\Delta^1$.

For the rest let $H$ be the affine span of $\Delta$. If $H$ does not contain integer points (which implies $i \leq 2$), let $a \in \mathbb{Z}^3$ be such that $\text{conv}\{(a) \cup H\} \cap \mathbb{Z}^3 = \{a\}$. Then, $\Delta$ can be characterized as being the facet opposite to the unique integer point in $\Delta_a = \text{conv}\{(a) \cup \Delta\}$. Since $\Delta_a$ is, by the previous paragraph, equivalent to $\Delta^1_{i+1}$, this gives an equivalence between $\Delta$ and $\Delta^0_1$.

Then the only case left is when $H$ contains lattice points but $\Delta$ does not, which can only happen if $i \geq 2$. Observe that, by definition of half-unimodular simplex, $\Delta$ is equivalent to $p + \Delta^1_i$, for some $p \in \{0, \frac{1}{2}\}^3$. If $i = 2$ there is no loss of generality in assuming that $H = \mathbb{R}^2 \times \{0\}$, so that the only choice of $p$ that makes $\Delta$ not have integer points is indeed $p = (\frac{1}{2}, \frac{1}{2}, 0)$.

If $i = 3$ we have four possibilities for $p$, namely

$$p \in \left\{ \left( \frac{1}{2}, \frac{1}{2}, 0 \right), \left( \frac{1}{2}, 0, \frac{1}{2} \right), \left( 0, \frac{1}{2}, \frac{1}{2} \right), \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right\}. $$

It is left to the reader to check that these four possibilities give equivalent simplices. \qed

3. Decomposing empty tetrahedra into half-unimodular simplices

The main result in this section is that every lattice 3-polytope admits a partition into (relatively open) half-unimodular simplices using only seven of the nine possible types described in Section 2.

Since every lattice polytope can be triangulated into empty simplices, to prove such a statement we can restrict ourselves to empty tetrahedra:

**Definition 9.** An empty simplex is a lattice simplex with no other lattice points apart of its vertices.

The classification of empty tetrahedra is classical and relatively simple:

**Theorem 10** (White 1964 [13]). Every empty tetrahedron of determinant $q$ is unimodularly equivalent to

$$T(p, q) := \text{conv}\{(0, 0, 0), (1, 0, 0), (0, 0, 1), (p, q, 1)\},$$

for some $p \in \mathbb{Z}$ with $\gcd(p, q) = 1$. Moreover, $T(p, q)$ is $\mathbb{Z}$-equivalent to $T(p', q)$ if and only if $p' = \pm p^{\pm 1} \pmod{q}$.

The most important feature of this classification is the fact that all empty tetrahedra have width one: they are the convex hull of two edges lying in consecutive parallel lattice planes. In the coordinatization of Theorem 10 (and in the rest of this section) those planes are $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$. In particular, the half-integer points in $T(p, q)$ are:

(1) Its four vertices,
(2) The six mid-points of edges, and
(3) The following $q - 1$ additional points in the parallelepiped $Q(p, q)$ with vertices $\frac{1}{2}(0, 0, 1), \frac{1}{2}(p, q, 1), \frac{1}{2}(1, 0, 1), \text{and} \ \frac{1}{2}(1 + p, q, 1)$:

$$a_i := \frac{1}{2}\left( \left\lfloor \frac{ip}{q} \right\rfloor, i, 1 \right) = \frac{1}{2}\left( \frac{ip + q - (ip \pmod{q})}{q}, i, 1 \right), \quad i = 1, \ldots, q - 1.$$

See a picture of $Q(7, 12)$ in Figure 1.
Observe that the second coordinate makes these $a_i$'s form a naturally ordered sequence. We extend this sequence by setting $a_0 = \frac{1}{2}(0, 0, 1)$ and $a_q = \frac{1}{2}(p, q, 1)$ (the choice $a_0 = \frac{1}{2}(1, 0, 1)$ and $a_q = \frac{1}{2}(p, q, 1)$ would be equally valid for what follows, but we do need a choice).

Apart of ordering these $q - 1$ points "vertically" according to the second coordinate, we can equally order them "horizontally" according to the functional $f(x_1, x_2, x_3) = qx_1 - px_2$ that takes the value 0 on the edge $\text{conv} \{ (\frac{1}{2}(0, 0, 1), \frac{1}{2}(p, q, 1)) \}$ of $Q(p, q)$ and value $q$ on the opposite edge $\text{conv} \{ (\frac{1}{2}(1, 0, 1), \frac{1}{2}(1 + p, q, 1)) \}$. This gives a new sequence $\{ b_j \}_{j=0}^{q}$ which coincides as a set with $\{ a_i \}_{i=0}^{q}$ but where now $f(b_j) = j$. More explicitly, calling $p' \in \{1, \ldots, q - 1\}$ the inverse of $-p$ modulo $q$ we have that

$$b_j := \frac{1}{2} \left( j + (jp' \mod q) p, j p' \mod q, 1 \right), \quad j = 1, \ldots, q - 1.$$  

As before, we extend the sequence with $b_0 = \frac{1}{2}(0, 0, 1)$ and $b_q = \frac{1}{2}(p + 1, q, 1)$. See Figure 1 for an illustration for $q = 12$ and $p = 7$, where an affine transformation has been made so that $Q$ appears as a square in the picture.

With this we can now prove the main result in this section:

**Theorem 11.** Let $T$ be an empty $3$-simplex, and without loss of generality assume $T \subset \mathbb{R}^2 \times [0, 1]$. Let $T^- = T \cap (\mathbb{R}^2 \times [0, \frac{1}{2}])$ and $T^+ = T \cap (\mathbb{R}^2 \times [\frac{1}{2}, 1])$ be the two halves of it. Then, both $T^-$ and $T^+$ have half-unimodular triangulations in which all tetrahedra contain an integer vertex.

**Proof.** We triangulate both parts separately. In both cases, observe that we can consider the quadrilateral $Q = T^- \cap T^+$ as a lattice parallelogram with $q - 1$ interior lattice points and only its vertices as boundary lattice points, with respect to the lattice $\Lambda = \frac{1}{2}\mathbb{Z}^3$.

To triangulate $T^-$, consider the path of vertices $a_0a_1\ldots a_q$, which is monotone with respect to the coordinate $x_3$ and divides $Q$ into two (non-convex) parts $Q_l$ and $Q_r$. Triangulate $Q_l$ and $Q_r$ arbitrarily but using all lattice points as vertices.
Figure 2. A triangulation of the fundamental parallelogram $Q(7, 12)$ using the monotone path $a_0 a_1 \ldots a_q$. It is shown both in the quadrilateral $Q(7, 12)$ and in the bottom half $T^-$ of $T(7, 12)$.

which gives exactly $2q$ triangles in total (and, although this is less important, $q$ in each of $Q_l$ and $Q_u$). These two triangulations, call them $T_l$ and $T_r$, are unimodular with respect to $\Lambda$. Figure 2 shows them for $q = 12$ and $p = 7$.

The triangulation $T^-$ of $T^-$ consists of:

- The $q$ tetrahedra obtained joining $T_l$ to $(0, 0, 0)$.
- The $q$ tetrahedra obtained joining $T_r$ to $(1, 0, 0)$.
- The following $2q$ tetrahedra, two for each $i = 1, \ldots, q$:

$$\text{conv} \left( (0, 0, 0), \left( \frac{1}{2}, 0, 0 \right), a_{i-1}, a_i \right), \quad \text{conv} \left( \left( \frac{1}{2}, 0, 0 \right), (1, 0, 0), a_{i-1}, a_i \right).$$

All these simplices are half-unimodular: for the first two groups it follows from the fact that they are the join of a half-unimodular triangle and a point at distance $\frac{1}{2}$ from the hyperplane containing it; for the last group it is an easy calculation to verify it. It is also clear by construction that each of these tetrahedra contains one (and only one) of the two integer points in $T^-$. We omit to proof that $T^-$ is a triangulation, but see Remark 12 below.

For $T^+$ we use the same idea, except now we use the path $b_0 b_1 \ldots b_q$, which is monotone with respect to the functional $f$ constant on the second pair of edges of $Q$. Again, this path divides $Q$ into two (non-convex) parts $Q_u$ and $Q_d$ that we triangulate as before, producing $T_u$ and $T_d$, and we take as triangulation $T^+$ of $T^+$:

- The $q$ simplices obtained joining $T_d$ to $(0, 0, 1)$.
- The $q$ simplices obtained joining $T_u$ to $(p, q, 1)$.
- The following $2q$ tetrahedra, two for each $i = 1, \ldots, q$:

$$\text{conv} \left( (0, 0, 1), \left( \frac{p}{2}, \frac{q}{2}, 1 \right), b_{j-1}, b_j \right), \quad \text{conv} \left( \left( \frac{p}{2}, \frac{q}{2}, 1 \right), (p, q, 1), b_{j-1}, b_j \right).$$

$$\square$$

Remark 12. In this proof we skipped some details, in particular the proof that the sets of tetrahedra $T^+$ and $T^-$ so defined indeed triangulate $T^-$ and $T^+$. But these triangulations we construct are nothing but (scaled down versions of) the lattice
triangulations of the upper and lower halves of $2T(p,q)$ that appear in [12, Sect. 4] and are implicit in [9, Sect. 2] (see also [3, Sect. 9.3.2] or [7, Sect. 4.1]).

**Corollary 13.** Every lattice 3-polytope admits a decomposition into relatively open half-unimodular simplices taken from the following seven classes:

$$\Delta_0^1, \Delta_1^1, \Delta_2^1, \Delta_3^1, \Delta_0^0, \Delta_1^0, \Delta_2^0.$$

**Proof.** Let $P$ be a lattice polytope. First decompose $P$ into relatively open empty simplices. Those of dimensions 0, 1, and 2 can then trivially be decomposed into half-unimodular tetrahedra of the types

$$\Delta_0^1, \Delta_1^1, \Delta_2^1.$$

For the ones of dimension three, use the decomposition coming from the triangulations $T^+$ and $T^-$ from Theorem 13. (To make this a decomposition, consider only the simplices lying in the interior of $T^+$ and $T^-$, plus those from the triangulation of, say, $T^+$ and lying in the relative interior of $Q = T^+ \cap T^-$).

The fact that all tetrahedra in the triangulations $T^+$ and $T^-$ have a lattice point implies they are all equivalent to $\Delta_3^1$, whose boundary consists of the following types and numbers of relatively open simplices:

$$3\Delta_2^1 + \Delta_0^0 + 3\Delta_1^1 + 3\Delta_0^1 + \Delta_1^0 + 3\Delta_0^0.$$

□

4. Putting the pieces together

For each of the half-unimodular simplices $\Delta_i^j$, $i \in \mathbb{Z}_{\geq 1}$, $j = 0, 1$ from Section 2 let

$$E_i^j(k) := \#(k \text{relint}(\Delta_i^j) \cap \mathbb{Z}^3)$$

denote the Ehrhart function of the relative interior of $\Delta_i^j$, which is a quasipolynomial of period two.

**Proposition 14.** The $2n + 1$ quasipolynomials

$$\{E_i^1 : i = 0, \ldots, n\} \cup \{E_i^0 : i = 0, \ldots, n - 1\}$$

form a basis for the linear span of all Ehrhart quasipolynomials of half-lattice polytopes in $\mathbb{R}^n$.

**Proof.** Ehrhart quasipolynomials of half-lattice polytopes of dimension at most $n$ have period two and degree at most $n$, so they can be written as linear combinations of the following $2n$ quasi-monomials:

$$1, k, k, \ldots, k^n, (-1)^k, (-1)^k k, \ldots, (-1)^k k^n.$$

But the quasi-monomial $(-1)^k k^n$ is not used by polytopes in $\mathbb{R}^n$, because the coefficient of degree $n$ in $\text{ehr}(P, k)$ for a $P \subset \mathbb{R}^n$ equals the $n$-dimensional volume of $P$. This implies that Ehrhart quasipolynomials of half-lattice polytopes in $\mathbb{R}^n$ generate a vector space of dimension at most $2n - 1$. The $E_i^0$’s and $E_i^1$’s are independent, since there are two of each degree $i$ and their leading terms are different (see, e.g., the remark below). Hence they form a basis for this vector space. □

**Proof of Theorem 6.** Given a lattice 3-polytope $P$, we can decompose it into relatively open half-unimodular simplices of the form $\Delta_0^0$ or $\Delta_1^1$ by Corollary 13. Denoting by $f_i^j$ the number of simplices of type $\Delta_i^j$ in this decomposition for each
For $i \in \{0, 1, 2, 3\}$ and $j \in \{0, 1\}$, we can write the Ehrhart quasipolynomial of $P$ as

$$\text{ehr}(P; k) = \sum_{i,j} f_{ij} E_{ij}(k).$$

Conversely, by Proposition 14, if $P$ and $P'$ have the same Ehrhart quasipolynomials then the coefficients in the above expression for $P$ and $P'$ are the same, so that the decompositions of Corollary 13 for $P$ and $P'$ are in fact $\text{GL}(\mathbb{Z})$-equidecompositions of them.

**Remark 15.** The Ehrhart quasipolynomials $E^0_i$ and $E^1_i$ admit simple closed formulas. For the even values of $k$, being a half-unimodular simplex implies that $E^0_i(2k) = \binom{k}{i-1}$. For the odd values of $k$:

- For $E^0_i$, the fact that the affine span of $\Delta^0_i$ contains no lattice points implies $E^0_i(2k - 1) = 0$. Together with the previous fact, this gives:

$$E^0_i(k) = 1 + \frac{(-1)^k}{2} \left( \left\lfloor \frac{k}{2} \right\rfloor - 1 \right). \tag{1}$$

- For $E^1_i$ we have $E_i(2k) = E^1_i(2k - 1)$, which implies

$$E^1_i(k) = \binom{\left\lfloor \frac{k}{2} \right\rfloor - 1}{i}. \tag{2}$$

The reason is that the facet of $(2k - 1)\Delta^1_i$ opposite to its unique lattice vertex does not contain lattice points, so enlarging it to $2k\Delta^1_i$ does not add interior lattice points. (See Figure 3 for an illustration of $7\Delta^1_3$).
These closed forms readily show that they are independent, since their evaluations start with:

\[ E_0^0(k) = 1 \ldots \]
\[ E_0^1(k) = 0 1 \ldots \]
\[ E_1^0(k) = 0 0 1 \ldots \]
\[ E_1^1(k) = 0 0 0 1 \ldots \]
\[ E_0^2(k) = 0 0 0 0 1 \ldots \]
\[ E_1^2(k) = 0 0 0 0 0 1 \ldots \]

Observe that all the above values can also be checked in appropriate faces of \(k \Delta^1\), \(k = 1, \ldots, 7\) (compare Figure 3). Incidentally, these evaluations (which generalize naturally to higher dimensions) imply that the quasipolynomials of Proposition 14 form an integer basis (and not only a linear basis as stated in the proposition) for the lattice of integer-valued period-two quasipolynomials, which includes all Ehrhart quasipolynomials of half-lattice polytopes.

**Remark 16.** Formulas (1) and (2), together with direct calculations for \(E'_2(2k+1)\) and \(E'_3(2k+1)\), give us the following formulas for the nine Ehrhart quasipolynomials of half-unimodular simplicies in \(\mathbb{R}^3\) in terms of the quasi-monomials mentioned in the proof of Proposition 14:

\[
E_0^1(k) = 1 \\
E_1^1(k) = \frac{1}{4}(2k + (-1)^k - 1) \\
E_2^1(k) = \frac{1}{16} \left(2k^2 - 2((-1)^k + 5)k + 5(-1)^k + 11\right) \\
E_3^1(k) = \frac{1}{96} \left(2k^3 - 3((-1)^k + 7)k^2 + (21(-1)^k + 67)k - 33(-1)^k - 63\right) \\
E_0^2(k) = \frac{1+(-1)^k}{2^2} \\
E_1^2(k) = \frac{1+(-1)^k}{4}(k - 2) \\
E_2^2(k) = \frac{1+(-1)^k}{16}(k^2 - 6k + 8) \\
E'_2(k) = \frac{1}{16} \left(2k^2 - 6((-1)^k + 1)k + 9(-1)^k + 7\right) \\
E'_3(k) = \frac{1}{96} \left(2k^3 - (15(-1)^k + 9)k^2 + (45(-1)^k + 43)k - 51(-1)^k - 45\right)
\]

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