Complex analysis/Harmonic analysis

A characterization of Möbius transformations

Une caractérisation des transformations de Möbius

Konstantin M. Dyakonov

ICREA and Universitat de Barcelona, Departament de Matemàtica Aplicada i Anàlisi, Gran Via 585, E-08007 Barcelona, Spain

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ABSTRACT

We prove that the derivative $\theta'$ of an inner function $\theta$ is outer if and only if $\theta$ is a Möbius transformation. An alternative characterization involving a reverse Schwarz–Pick type estimate is also given.

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RÉSUMÉ

Étant donnée une fonction intérieure $\theta$, on démontre que sa dérivée $\theta'$ est extérieure si et seulement si $\theta$ est une transformation de Möbius.

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1. Introduction and main result

Let $H^\infty$ stand for the algebra of bounded holomorphic functions on the disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$. A function $\theta \in H^\infty$ is called inner if $\lim_{r \to 1^-} |\theta(r \xi)| = 1$ at almost every point $\xi$ of the circle $T := \partial D$. Among the nonconstant inner functions, the simplest ones are undoubtedly the conformal automorphisms of the disk, also known as Möbius transformations; these are of the form

$$\theta_{\lambda, a}(z) := \frac{\lambda z - a}{1 - \bar{a}z}$$

for some $\lambda \in \mathbb{T}$ and $a \in \mathbb{D}$. A calculation shows that

$$\theta'_{\lambda, a}(z) = \lambda \frac{1 - |a|^2}{(1 - \bar{a}z)\bar{z}} ,$$

which happens to be an outer function. (A nonvanishing holomorphic function $f$ on $\mathbb{D}$ is said to be outer if $\log |f|$ agrees with the harmonic extension of an integrable function on $\mathbb{T}$.)

In this note, we prove that the property of $\theta'$ being outer actually characterizes the Möbius transformations among all inner functions $\theta$.

E-mail address: konstantin.dyakonov@icrea.cat.

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Before stating the result rigorously, we need to recall that the Nevanlinna class \( \mathcal{N} \) (resp., the Smirnov class \( \mathcal{N}^+ \)) is formed by the functions that can be written as \( u/v \), where \( u, v \in H^\infty \) and \( v \) is zero-free (resp., outer) on \( \mathbb{D} \). The reader is referred to [5, Chapter II] for further information on \( \mathcal{N} \) and \( \mathcal{N}^+ \), including the canonical factorization theorem for functions from these spaces. We also mention the fact that, for \( \theta \) inner, one has \( \theta' \in \mathcal{N} \) if and only if \( \theta' \in \mathcal{N}^+ \); see [1] for a proof. In what follows, we are forced to require that \( \theta' \) be in \( \mathcal{N} \) (or \( \mathcal{N}^+ \)), since this is apparently the weakest natural assumption that allows us to speak of the inner–outer factorization for \( \theta' \).

**Theorem 1.1.** Let \( \theta \) be a nonconstant inner function with \( \theta' \in \mathcal{N} \). Then \( \theta' \) is outer if and only if \( \theta \) is a Möbius transformation.

In some special cases, the fact that the derivative of a non-Möbius inner function will have a nontrivial inner part may be obvious or due to known results. First of all, \( \theta' \) will certainly vanish at the multiple zeros of \( \theta \), if any. Secondly, a result of Ahern and Clark (see [1, Corollary 4]) tells us that the singular factor of \( \theta \), if existent, gets inherited by \( \theta' \), provided that the latter function is in \( \mathcal{N} \). Thus, in a sense, singular factors can be thought of as responsible for the (boundary) zeros of infinite multiplicity. Thirdly, if \( \theta \) is a finite Blaschke product with at least two zeros, then \( \theta' \) is sure to have zeros in \( \mathbb{D} \) (see [6] for a more precise information on the location of these), so \( \theta' \) will again be non-outer. The remaining case, where \( \theta \) is an infinite Blaschke product with simple zeros, seems however to be new.

**2. Proof of Theorem 1.1**

To prove the nontrivial part of the theorem, assume that \( \theta \) is inner and \( \theta' \) is an outer function in \( \mathcal{N} \).

For all \( z \in \mathbb{D} \) and almost all \( \zeta \in \mathbb{T} \), Julia’s lemma (see [2] or [5, p. 41]) yields

\[
\frac{|\theta(\zeta) - \theta(z)|^2}{1 - |\theta(z)|^2} \leq |\theta'(\zeta)| \cdot \frac{|\zeta - z|^2}{1 - |z|^2},
\]

or equivalently,

\[
\frac{1 - |z|^2}{1 - |\theta(z)|^2} \cdot \frac{1 - \theta(z)\overline{\theta'(\zeta)}}{1 - 2\overline{\zeta}} \leq |\theta'(\zeta)|.
\]

Further, we associate with every (fixed) \( z \in \mathbb{D} \) the \( H^\infty \)-function

\[
\Phi_z(w) := \frac{1 - |z|^2}{1 - |\theta(z)|^2} \cdot \left(\frac{1 - \theta(z)\overline{\theta'(w)}}{1 - 2w}\right)^2
\]

and rewrite (2.2) in the form

\[
\left|\Phi_z(\zeta)\right| \leq |\theta'(\zeta)|, \quad \zeta \in \mathbb{T}.
\]

Since \( \Phi_z \in H^\infty \) and \( \theta' \) is outer, the ratio \( \psi_z := \Phi_z/\theta' \) will be in \( \mathcal{N}^+ \); and since, by (2.4), \( |\psi_z| \leq 1 \) a.e. on \( \mathbb{T} \), it follows that \( \psi_z \) is in \( H^\infty \) and has norm at most 1. In other words, the estimate (2.4) extends into the disk, so that

\[
\left|\Phi_z(w)\right| \leq |\theta'(w)|, \quad w \in \mathbb{D}.
\]

In particular, putting \( w = z \), we obtain

\[
\left|\Phi_z(z)\right| \leq |\theta'(z)|.
\]

A glance at (2.3) reveals that

\[
\Phi_z(z) = \frac{1 - |\theta(z)|^2}{1 - |z|^2},
\]

and plugging this into (2.5) gives

\[
\frac{1 - |\theta(z)|^2}{1 - |z|^2} \leq |\theta'(z)|.
\]

In conjunction with the Schwarz–Pick estimate

\[
|\theta'(z)| \leq \frac{1 - |\theta(z)|^2}{1 - |z|^2}
\]

(see [5, Chapter I, Section 1]), this means that we actually have equality in (2.6). This last fact is known to imply that \( \theta \) is a Möbius transformation (see ibid), and the proof is complete.
3. An alternative characterization and open questions

The primary purpose of this note, essentially accomplished by now, can be described as giving a short and self-contained proof of a result from [4]. In that paper, our main concern was a certain reverse Schwarz–Pick type inequality for unit-norm $H^\infty$ functions (see also [3] for an earlier version); the above characterization of Möbius transformations was then deduced as a corollary. In addition, it was shown in [4, Section 2] that, among the nonconstant inner functions $\theta$ with $\theta^0 \in \mathcal{N}$, the Möbius transformations are also characterized by the property that

$$\eta \left( \frac{1 - |\theta(z)|^2}{1 - |z|^2} \right) \leq |\theta'(z)|, \quad z \in \mathbb{D},$$

(3.1)

for some nondecreasing function $\eta : (0, \infty) \to (0, \infty)$. We now improve this last result by relaxing the a priori assumptions on $\theta$. In fact, it turns out that we need not restrict our attention to inner functions from the start. Instead, we shall verify that $\theta$ will have to be inner (and with derivative in $\mathcal{N}$) automatically, under the milder hypotheses below.

Proposition 3.1. Let $\theta \in H^\infty$ be a nonconstant function with $\|\theta\|_\infty \leq 1$. The following conditions are equivalent:

(i) $\theta$ is a Möbius transformation,

(ii) there is a nondecreasing function $\eta : (0, \infty) \to (0, \infty)$ with $\lim_{t \to \infty} \eta(t) = \infty$ making (3.1) true.

Proof. Of course, (i) implies (ii) with $\eta(t) = t$. To prove the nontrivial implication (ii) $\Rightarrow$ (i), observe that

$$\inf \left\{ \frac{1 - |\theta(z)|^2}{1 - |z|^2} : z \in \mathbb{D} \right\} > 0$$

(by Schwarz’s lemma), and so (3.1) yields $\inf_{z \in \mathbb{D}} |\theta'(z)| > 0$. Hence $1/\theta^0 \in H^\infty$ and $\theta^0 \in \mathcal{N}$; in particular, $\theta^0$ has radial limits almost everywhere on $\mathbb{T}$.

Now, if $\zeta \in \mathbb{T}$ is a point at which $\lim_{r \to 1^-} |\theta(r\zeta)| < 1$, then (3.1) shows that $\lim_{r \to 1^-} |\theta'(r\zeta)| = \infty$. Consequently, the set of such $\zeta$’s has zero measure on $\mathbb{T}$. It follows that $\theta$ has radial limits of modulus 1 almost everywhere, and is therefore an inner function. To complete the proof, it remains to invoke the above-mentioned result from [4]. □

We conclude by mentioning two open questions that puzzle us. First, we would like to know which inner functions $I$ can be written as $I = \text{inn}(\theta^0)$ (where “inn” stands for “the inner factor of”), as $\theta$ ranges over the nonconstant inner functions with $\theta^0 \in \mathcal{N}$. Does every inner $I$ arise in this way?

To pose the other question, let us introduce the notation $\sigma(I)$ for the boundary spectrum of an inner function $I$. Thus, $\sigma(I)$ is the smallest closed set $E \subset \mathbb{T}$ such that $I$ is analytic across $\mathbb{T} \setminus E$. Now, if $\theta$ is inner (and nonconstant) with $\theta^0 \in \mathcal{N}$, and if $I = \text{inn}(\theta^0)$, then it is easy to see that $\sigma(I) \subset \sigma(\theta)$. Do we actually have $\sigma(I) = \sigma(\theta)$? An affirmative answer seems plausible to us, but so far, we have only succeeded in verifying it under an additional hypothesis.

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