Cross-Analyzing Radio and $\gamma$-Ray Time Series Data: Fermi Marries Jansky

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Abstract. A key goal of radio and $\gamma$-ray observations of active galactic nuclei is to characterize their time variability in order to elucidate physical processes responsible for the radiation. I describe algorithms for relevant time series analysis tools – correlation functions, Fourier and wavelet amplitude and phase spectra, structure functions, and time-frequency distributions, all for arbitrary data modes and sampling schemes. For example radio measurements can be cross-analyzed with data streams consisting of time-tagged gamma-ray photons. Underlying these methods is the Bayesian block scheme, useful in its own right to characterize local structure in the light curves, and also prepare raw data for input to the other analysis algorithms. One goal of this presentation is to stimulate discussion of these methods during the workshop.

1. Introduction

Active galactic nuclei (AGN) are highly variable at all wavelengths. A major fraction of their total luminosity fluctuates over time scales ranging from the shortest for which statistically significant signals can be obtained, to the longest time intervals over which data are available. Characterizing this variability has yielded growing insight into the physical processes powering the large AGN luminosities – a trend that will accelerate as observations, data analysis, and theory proliferate.

This paper outlines time series methods for analysis of the disparate data modes of radio, $\gamma$-ray, and other astronomical observations. The next section introduces a data structure that generalizes data modes traditionally used for time-sequential observations. This abstraction yields methods for estimating, from arbitrary time series data, including heterogeneous mixtures of data modes, all of the standard analysis functions:

- light curves
- autocorrelations
- Fourier power and phase spectra
- wavelet representations
- structure functions
- time-frequency distributions

As indicated in Table 1, for essentially arbitrary data modes these methods yield amplitude and phase information for single or multiple time series (auto- and cross-analysis, respectively) – if desired, conditional on auxiliary variables.

Table 1. Time Series Analysis for All Data Modes

|                | Auto | Cross | Amp | Phase | Condit. |
|----------------|------|-------|-----|-------|---------|
| Correlation    | yes  | yes   | -   | -     | yes     |
| Fourier        | yes  | yes   | yes | yes   | yes     |
| Wavelet        | yes  | yes   | yes | location | ?       |
| Struct Fcn     | yes  | yes   | -   | -     | yes     |
| Time-Freq      | yes  | yes   | yes | yes   | yes     |

There are significant difficulties in the astrophysical interpretation of these quantities. The methods described here are of use in some of these, such as separation of observational errors from stochastic source variability (both of which, unfortunately, are often called noise). But I do not discuss other more difficult problems, which are probably beyond the scope of time series analysis methods, such as assessing the importance of cosmic variance, identifying causal or otherwise physically connected relationships in multi-wavelength time series data, etc.

Subsequent sections discuss each of the above-listed functions and give sample applications.

2. Abstract Data Cells

The time series algorithms to be described below can be applied to almost any type of time-sequential astronomical data. This generality is facilitated by identifying those features of the data modes that are necessary for analysis algorithms.

Each individual act of measurement may yield a large set of data values relevant to estimation of the signal amplitude, and its uncertainty, as a function of time. Of these, two pieces of information, related to the independent variable (time of the measurement) and the dependent variable (amplitude of the signal at that time), are necessary

\[ \text{In practice we always use a discrete time representation, such as a micro-second scale computer clock tick, or the finite time interval of signal averaging.} \]
for any time series algorithm. In radio astronomy the typical example is the measurement of the flux of a source averaged over a short interval of time. In γ-ray astronomy the typical example is the detection of individual photons. The arrival time of the photon is obviously the timing quantity, but what about the signal? One scheme is to represent an individual photon with a delta-function in time. But more information can be extracted by incorporating the time interval between photons. Specifically, for each photon consider the interval starting half way back to the previous photon and ending half way forward to the subsequent photon. This interval, namely

\[ \Delta t_n = \frac{t_{n+1} - t_{n-1}}{2} \tag{1} \]

is the set of times closer to \( t_n \) than to any other time and has length equal to the average of the two intervals connected by photon \( n \), namely

\[ \Delta t_n = \frac{t_{n+1} - t_{n-1}}{2} \tag{2} \]

Then the reciprocal

\[ x_n = \frac{1}{\Delta t_n} \tag{3} \]

is taken as an estimate of the signal amplitude corresponding to observation \( n \). When the photon rate is large, the corresponding intervals are small. Figure 1 demonstrates the data cell concept, including the simple modifications to account for variable exposure and for weighting by photon energy.

Consider gaps in the data. By this we mean that there are portions of the total observation interval during which the detection system is completely off (exposure zero). This situation is readily handled by defining the start of the data cell for the first photon detected after the gap at the end time of the gap. Correspondingly the data cell for the last photon before the gap is set at the start time of the gap. The statistical nature of independent events assures that this procedure rigorously estimates the true photon rate at the edge of the gap. Of course, no information is available about the signal during such a gap, and the various algorithms deal with gaps accordingly.

Now we consider data modes generally. Three common examples are: (a) measurements of a quasi-continuous physical variable (e.g. radio astronomy flux measurements) (b) the time of occurrence of discrete events (e.g. photons) and (c) counts of events in bins. The signal of interest is the time dependence of the measured quantity in case (a), or of the event rate in case (b). Case (c) is actually very similar to (b), but is often described as density estimation or determination of the event distribution function. In all cases it is useful to introduce the concept of *cells* to represent the measurements. Letting \( x_n \) be the estimate of the signal amplitude for a cell at time \( t_n \), a data set of \( N \) sequential observations is denoted

\[ C_n \equiv \{ x_n, t_n \}, \quad n = 1, 2, \ldots, N. \tag{4} \]

The specific meaning of the quantities \( x_n \) depends on the type of data. For example, in the three cases mentioned above the array \( x \) contains (a) the sum or average over the measurement interval of an extensive or intensive quantity, respectively, plus one or more quantifiers of measurement uncertainty, (b) coordinates of events, such as photon arrival times, and (c) sizes and locations of the bins, and the count of events in them.

A major reason for constructing this abstract data representation is that it unifies all data modes into a common format that makes construction of universal algorithms easy. As we will see in the next sections, even mixtures of data types – either in the sense of cross-analyzing two very different data types, or mixing data within a single time series – can be handled.

### 3. Light Curve Analysis: Bayesian Blocks

The simplest and most direct way to study variability is to construct a representation of the intensity of
the source as a function of time. More can be done than just plotting the intensity measurements as a function of the time of the measurement. Smoothing, interpolation, gap filling, etc. are all techniques meant to enhance one’s understanding of the variability. Here we discuss a different procedure, namely construction of a simple, generic, non-parametric model of the data that as much as possible shows the actual variability of the source, and minimizes the effect of observation errors. The model adopted is the simplest possible non-parametric representation of time series data, namely a piece-wise constant model. Details of this approach are given in (Scargle, Norris, Jackson and Chiang 2010); the improved algorithm given there replaces the approximate one described in (Scargle 1998).

The Bayesian Blocks algorithm finds the best partition of the observation interval into blocks, such that the source intensity is modeled as varying from block to block, but constant within each block. This is just a step-function representation of the data. The meaning of the “best” model is the one that maximizes a measure of goodness-of-fit function described in detail in (Scargle, Norris, Jackson and Chiang 2010). Another change since the earlier reference is the use of a very simple maximum likelihood fitness function, preferable to the Bayesian posterior previously used because it is invariant to a scale change in the time variable, thus eliminating a parameter from the analysis.

Figure 2 shows the Bayesian blocks analysis of two AGN in the OVRO/Fermi joint program. The data shown are from somewhat earlier in the program, where the overlap between the two instruments was not huge. Also these were just the first and third objects in the long list of observed sources, and were not particularly selected for being highly variable cases.

4. Correlation Functions

A rather underutilized technique for studying correlated variability of two observables (such as time series for different wavelengths) is to construct a scatter plot of one against the other. If done carefully, this approach allows study of joint probability distributions for the two variables; these contain more statistical information than correlation functions or any of the other functions discussed here. The challenges of this approach include the difficulty of depicting the all-important time-sequence connecting the points in the scatter plot, and the need to consider plotting lagged versions of the variables, for a number of values of the lag. The understanding that comes from careful study of scatter plots most often makes it worthwhile to conquer these difficulties.

But probably the most used tool for studying statistical variability properties of a single time series is the auto-correlation function (ACF) or, for studying relations between the variability in two or more sets of simultaneous time series, the cross-correlation function (CCF). The meaning of the latter can be understood by modeling one

Fig. 2. Bayesian Block representations of the lightcurves of 3C273 and 3C279, two AGN in the OVRO/Fermi project. The co-aligned times are in days, relative to an arbitrary zero point; amplitudes are on a common relative scale. Binned LAT data is shown for comparison, but the BB representation is based on the photon data only.

Fig. 3. Summation schemes for autocorrelation functions. The points represent data cells, derived from measured values (as in radio astronomy) or time-tagged events (as in Fermi photon data). Top: Summation over data with arbitrary spacing in the Edelson and Krolik algorithm. From each point average over all points within a bin $d\tau$ distant by $\tau$; $\tau$ is binned, but $t$ is not. Bottom: Standard lag summation over evenly spaced data. From each point (except near the ends) there is another point distant by exactly $\tau = \text{an integer multiple of } \Delta t.$
time series as a lagged version of the other, and evaluating the posterior distribution of the lag \( \tau \), yielding

\[
P(\tau) \sim e^{\frac{R_{X,Y}(\tau)}{K}},
\]

where \( K \) is a constant and \( R_{X,Y}(\tau) \) is the cross-correlation function defined below [Scargle 2001b].

Concentrating on the CCF, of which the ACF is really a special case, and following the notation and definitions of [Papoulis 1965] and [Papoulis 1977], we have this definition of the cross-correlation function of two real processes \( x(t) \) and \( y(t) \)

\[
R_{xy}(t_1, t_2) = \langle x(t_1) y(t_2) \rangle
\]

Assuming the processes are stationary, the time dependence is on only the difference \( \tau = t_2 - t_1 \) and we have

\[
R_{xy}(\tau) = \langle x(t) y(t + \tau) \rangle
\]

The symbol \( \langle \rangle \) means the expected value, informally to be thought of as an average over realizations of the underlying random process \( X \). In data analysis this theoretical quantity is typically not known, and must be therefore be estimated from the data at hand, e.g.

\[
E[X(t)Y(t + \tau)] \equiv \frac{1}{N(\tau)} \sum_{n} x_{n} y_{n+\tau}
\]

where \( x_{n} \) and \( y_{n} \) are the samples of the variable \( X,Y \), and \( N(\tau) \) is the number of terms for which the sum can be taken.

Figure 3 is a cartoon of the lag relationships for correlation functions of evenly spaced data (bottom), as well as a solution to the difficulty posed by unevenly spaced time samples in general, and event data in particular. For a given sample or event at \( t_{n} \) there will in general not be a corresponding one at \( t_{n} + \tau \), no matter what restriction is placed on \( \tau \).

For this problem an ingenious if straightforward algorithm [Edelson and Krolik 1988] is in wide use. The basic idea is to pre-define a set of bins in the variable \( \tau \) in order to construct a histogram of the corresponding time separations \( \tau = t_{m} - t_{n} \), weighted by the corresponding \( x_{n}y_{m} \) product. To be more specific, and modifying slightly Edelson and Krolik’s formulas for our case (including not subtracting the process means), define for all measured pairs \( (x_{n}, y_{m}) \) the quantity

\[
UDCF_{nm} = \frac{x_{n}y_{m}}{\sqrt{(\sigma_{x}^{2} - \epsilon_{x}^{2})(\sigma_{y}^{2} - \epsilon_{y}^{2})}},
\]

\[\text{Caution: It is common to center the processes about their means, to yield the cross-covariance and auto-covariance functions. Such mean-removal can have unfortunate consequences, such as distortion of the low-frequency power spectrum. In addition, the nomenclature is not completely standard. Various terms are used for the cases where the means of the processes have been subtracted off, and/or the resulting function normalized to unity at } \tau = 0.\]
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5. Fourier Power and Phase Spectra

Perhaps the most used time series analysis technique in astronomy is estimation of the Fourier power spectrum, mainly with the goal of detecting and then characterizing periodic signals hidden in noisy data, but also for analyzing non-periodic signals such as quasi periodic oscillations and colored, or \( \frac{1}{f} \), noise. There are methods for direct estimation of Fourier power (Scargle 1982) and phase (Scargle 1989) spectra from time series data. However, it is often more convenient to make use of the well-known result that the power spectrum is the Fourier transform of the ACF computed as described above in §4. The sliding window power spectra depicted in Fig. 2 were computed in this way.

6. Wavelet Representations

It is relatively straightforward to compute the wavelet transform for any time series that can be put into the standard data cell representation. The wavelet shape (in this case the piecewise constant Haar wavelet) is integrated against the empirical signal amplitude assigned by the data cells. Figure 6 shows the scalegrams, or wavelet power spectra (Scargle et al. 1993), for the same AGN data as in Figure 4. There is not enough data to yield much detail in these spectral representations, but the rough power law characteristic of \( \frac{1}{f} \) processes can be seen, as well as the noise floor for the LAT data.

7. Structure Functions

Another concept in wide usage is the structure function. For the most part its auto- and cross-versions are a repackaging of the same information contained in the corresponding correlations. This point has recently been emphasized by (Emmanoulopoulos, McHardy and Uttley 2010). In addition to summarizing some of the caveats and problems associated with structure functions, these authors give a formal proof of the exact relation between structure functions and the corresponding auto- and cross-correlation functions. In addition, the literature contains a number of claims for the superiority of the structure function that seem unwarranted, especially in view of the relation just mentioned. An example is the misconception that structure functions are somehow immune from sampling effects, including aliasing. Finally, some analysts believe that at short timescales the structure function always becomes flat; the actual generic behavior can be derived from eq. (A10) of (Emmanoulopoulos, McHardy and Uttley 2010); the normalized structure function satisfies

\[
NSF(\tau) = 2[1 - ACF(\tau)] \to C\tau^2
\]

for \( \tau \to 0 \), since autocorrelation functions are even in \( \tau \). In practice this dependence may seem flat compared to
steeper behavior at intermediate time scales, transitioning to the typical asymptotic loss of correlation at large time scale expressed as $NSF(\tau) \to 2$, correctly assessed as flat.

A few other points perhaps favor the use of structure functions (beyond the fact that they have been widely used in the past, and therefore arguably should be computed if only for comparison with previous work). When the structure and correlation functions are estimated from actual data, this equivalence result quoted above does not hold exactly. There can in fact be significant departures from the theoretical relations in Appendix A of [Emmanoulopoulos, McHardy and Uttley 2010], due to end effects always present for finite data streams. In addition, when measuring slope of powerlaw relationships it can be slightly more convenient to fit polynomials to the typical shape of a structure function than to the corresponding correlation function or power spectrum.

8. Time-Frequency Distributions

The term time-frequency distribution refers to techniques for studying the time-evolution of the power spectrum of time series. This concept must deal with the fact that the spectrum is a property of the entire time interval, so that estimating it locally in time results in the need for trading off time resolution against frequency resolution. See [Flandrin 1999] for a complete exposition of these issues.

There are many algorithms for computing time-frequency distributions, but little has been done for the case of event data, one exception being the approach described in [Galleani, Cohen, Nelson, and Scargle 2001]. Although there are advanced techniques based on the Wigner-Ville distribution, Cohen’s class of distribution, and others, in many applications the sliding window power spectrum is of considerable use. The idea is simple: compute the power spectrum of a subsample of the data within a restricted time-interval, small compared to the total interval. Information on the time dependence results from the fact that the window is slid along the observation interval. Information on frequency dependence is contained in the power spectrum. The tradeoff of time- and frequency resolution is mediated by the length of the time window: a short window yields high time resolution and low spectral resolution, and vice versa for a long window. Implementation of this approach is straightforward through use of the techniques in [1] and [5].

Fig. 8. Sampling histograms: the distributions of the time intervals between the samples for the data in Figure 7.

9. Conclusions

Rather than regurgitating the discussion above, I end with a few practical suggestions. They may seem obvious or trivial, but I have found them surprisingly useful in practice.

When addressing time series data in the form of eq. [4], the first step should be to study the time intervals $t_{n+1} - t_n$; in particular compute, plot, and study the their distribution with suitably constructed histograms. (Even if the provider of the data swears the times are evenly spaced, check it!) This often reveals many defects in the data, such as duplicate entries and observations out of order. The outliers of the distribution signal peculiarities, perhaps expected (such as known sampling irregularities, regularities, or semi-regularities) but often unexpected surprises. Figure 8 shows examples from the data for which time-frequency distributions were shown above. The reader is invited to see what conclusions can be deduced from these distributions.

Don’t subtract the mean value! Or at least do so with attention to its effects. Too often time series data are detrended without careful consideration of the resulting effects on the estimated functions. Mean removal is a special case of detrending.

While there are some cases where the distinction between stationary and non-stationary processes is important, with limited data it is difficult or impossible to make this distinction in practice. For different reasons, the distinction between linearity and non-linearity is best left to the realm of physical models rather than data analysis. Linearity is a property of physical processes, and mathematical definitions [Priestly 1988, Tong 1990] may or may not connect meaningfully to physical concepts.
Finally, in thinking about AGN variability in general it is useful to think in terms of the mathematical concept of doubly stochastic (or Cox) processes. Essentially, this is a picture in which there are two distinct random processes: the intrinsic variability of the source (truly random, periodic, quasi-periodic, etc.) and the observation process. The latter is random due to observational errors from photon counting, detector noise, background variability, etc. It is a major data analysis challenge to cleanly separate out the observational process to reveal the true variability of the astronomical source.

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