GORENSTEIN HOMOLOGICAL DIMENSIONS AND AUSLANDER CATEGORIES

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Abstract. In this paper, we study Gorenstein injective, projective, and flat modules over a Noetherian ring $R$. For an $R$-module $M$, we denote by $\text{Gpd}_R M$ and $\text{Gfd}_R M$ the Gorenstein projective and flat dimensions of $M$, respectively. We show that $\text{Gpd}_R M < \infty$ if and only if $\text{Gfd}_R M < \infty$ provided the Krull dimension of $R$ is finite. Moreover, in the case that $R$ is local, we correspond to a dualizing complex $D$ of $\hat{R}$, the classes $A'(R)$ and $B'(R)$ of $R$-modules. For a module $M$ over a local ring $R$, we show that $M \in A'(R)$ if and only if $\text{Gpd}_R M < \infty$ or equivalently $\text{Gfd}_R M < \infty$. In dual situation by using the class $B'(R)$, we provide a characterization of Gorenstein injective modules.

1. Introduction

Throughout this paper, $R$ will denote a commutative ring with nonzero identity and $\hat{R}$ will denote the completion of a local ring $(R, \mathfrak{m})$. When discussing the completion of a local ring $(R, \mathfrak{m})$, we will mean the $\mathfrak{m}$-adic completion.

Auslander and Bridger [3] introduced the G-dimension, $G - \dim_R M$, for every finitely generated $R$-module $M$ (see also [2]). They proved the inequality $G - \dim_R M \leq \text{pd}_R M$, with equality $G - \dim_R M = \text{pd}_R M$ when $\text{pd}_R M$ is finite. The G-dimension has strong parallels to the projective dimension. For instance, over a local Noetherian ring $(R, \mathfrak{m})$, the following conditions are equivalent:

(i) $R$ is Gorenstein.
(ii) $G - \dim_R R/\mathfrak{m} < \infty$.
(iii) All finitely generated $R$-modules have finite $G$-dimension.

This characterization of Gorenstein rings is parallel to Auslander-Buchsbaum-Serre characterization of regular rings. G-dimension also differs from projective dimension in that it is defined only for finitely generated modules. Enochs and Jenda defined in [9] Gorenstein projective modules (i.e. modules of G-dimension 0) whether the modules are finitely generated or not. Also, they defined a homological dimension, namely the Gorenstein projective dimension, $\text{Gpd}_R (-)$, for arbitrary (non-finitely generated) modules. It is known that for finitely generated modules, the Gorenstein projective dimension agrees with the G-dimension. Along the same lines, Gorenstein flat and Gorenstein injective modules were introduced in [9,10].

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Let $R$ be a Cohen-Macaulay local ring admitting a dualizing module $D$. Foxby [12] defined the class $G_0(R)$ to be those $R$-modules $M$ such that $\text{Tor}_i^R(D, M) = \text{Ext}_i^R(D, D \otimes_R M) = 0$ for all $i \geq 1$ and such that the natural map $M \to \text{Hom}_R(D, D \otimes_R M)$ is an isomorphism, and $I_0(R)$ to be those $R$-modules $N$ such that $\text{Ext}_i^R(D, N) = \text{Tor}_i^R(D, \text{Hom}_R(D, N)) = 0$ for all $i \geq 1$ and such that the natural map $D \otimes_R \text{Hom}_R(D, N) \to N$ is an isomorphism. In [11] Enochs, Jenda and Xu characterize Gorenstein injective, projective and flat dimensions in terms of $G_0(R)$ and $I_0(R)$.

Let $R$ be a Noetherian ring with dualizing complex $D$. The Auslander categories $A(R)$ and $B(R)$ with respect to $D$ are defined in [4, 3.1]. In [5], it is shown that the modules in $A(R)$ are precisely those of finite Gorenstein projective dimension (Gorenstein flat dimension), see [5, Theorem 4.1], and the modules in $B(R)$ are those of finite Gorenstein injective dimension, see [5, Theorem 4.4]. This may be viewed as an extension of the results of [11]. Note that, by [4, Proposition 3.4], if $R$ is a Cohen-Macaulay local ring with a dualizing module, then an $R$-module $M$ is in $A(R)$ if and only if $M \in G_0(R)$ (resp. an $R$-module $M$ is in $B(R)$ if and only if $M \in I_0(R)$).

The main aim of this paper is to extend the characterization of finiteness of Gorenstein dimensions in [5] to arbitrary local Noetherian rings.

Let $R$ be a local Noetherian ring probably without dualizing complex, and let $D$ denote the dualizing complex of $R$. We define $A'(R)$ to be those $R$-modules $M$ such that $R \otimes_R M \in A(R)$ and $B'(R)$ to be those $R$-modules $N$ such that $\text{Hom}_R(R, N) \in B(R)$. In sections 2, 3, and 4, we characterize Gorenstein injective, projective, and flat modules in terms of the classes $A'(R)$ and $B'(R)$. To be more precise, we show the following results.

**Theorem 1.1.** Let $R$ be a local Noetherian ring and $M$ an $R$-module.
(i) (See Theorem 2.5) $M$ is Gorenstein flat if and only if $M$ belongs to $A'(R)$ and $\text{Tor}_i^R(L, M) = 0$ for all injective $R$-modules $L$ and all $i > 0$.
(ii) (See Corollary 3.3) $M$ is Gorenstein projective if and only if $M$ belongs to $A'(R)$ and $\text{Ext}_i^R(M, P) = 0$ for all projective $R$-modules $P$ and all $i > 0$.
(iii) (See Theorem 4.8) $M$ is Gorenstein injective if and only if $M$ belongs to $B'(R)$, $M$ is cotorsion and $\text{Ext}_i^R(E, M) = 0$ for all injective $R$-modules $E$ and all $i > 0$.

Even more generally, by using the classes $A'(R)$ and $B'(R)$, we characterize modules of finite Gorenstein injective, projective and flat dimensions. Namely, we prove the following two results.

**Theorem 1.2.** (See Theorems 3.4 and 3.5) Let $R$ be a Noetherian ring of finite Krull dimension and $M$ an $R$-module. Then the following conditions are equivalent:
(i) $\text{Gfd}_R M < \infty$.
(ii) $\text{Pfd}_R M < \infty$.
(More precisely, if $\text{Pfd}_R M < \infty$ or $\text{Gfd}_R M < \infty$, then $\text{Max}\{\text{Gfd}_R M, \text{Pfd}_R M\} \leq \text{dim} R$).
Moreover, if $R$ is local, then the above conditions are equivalent to the following
(iii) $M \in A'(R)$.

**Theorem 1.3.** (See Theorem 4.10) Let $(R, m)$ be a local Noetherian ring of dimension $d$ and $\text{Ext}_i^R(R, M) = 0$ for all $i > 0$. Then Gorenstein injective dimension of $M$ is finite if and only if $M$ belongs to $B'(R)$. In particular, if $M \in B'(R)$ then $\text{Gid}_R(M) \leq d$. 


Setup and notation If $M$ is any $R$-module, we use $\text{pd}_R M$, $\text{id}_R M$ and $\text{id}_R M$ to denote the usual projective, flat and injective dimension of $M$, respectively. Furthermore, we write $\text{Gpd}_R M$, $\text{Gfd}_R M$ and $\text{Gid}_R M$ for the Gorenstein projective, Gorenstein flat and Gorenstein injective dimension of $M$, respectively. Let $\mathcal{X}$ be any class of $R$-modules and let $M$ be an $R$-module. An $\mathcal{X}$-precover of $M$ is an $R$-homomorphism $\varphi : X \rightarrow M$, where $X \in \mathcal{X}$ and such that the sequence,

$$\Hom_R(X', X) \xrightarrow{\Hom_R(X', \varphi)} \Hom_R(X', M) \rightarrow 0$$

is exact for every $X' \in \mathcal{X}$. If, moreover, $f \varphi = \varphi$ for $f \in \Hom_R(X, M)$ implies $f$ is an automorphism of $M$, then $\varphi$ is called an $\mathcal{X}$-cover of $M$. Also, an $\mathcal{X}$-preenvelope and $\mathcal{X}$-envelope of $M$ are defined “dually”. By $P(R)$, $F(R)$ and $I(R)$ we denote the classes of all projective, flat and injective $R$-modules, respectively. Furthermore, we let $\overline{P(R)}$, $\overline{F(R)}$ and $\overline{I(R)}$ denote the classes of all $R$-modules with finite projective, flat and injective dimension, respectively.

We may use the following facts without comment. If $R$ is Noetherian of finite Krull dimension, then $\overline{P(R)} = P(R)$ (see [16, Theorem 4.2.8]). Also, if $R$ is Noetherian then for any $M \in \overline{P(R)}$, we have $\text{pd}_R(M) \leq \dim R$ (see [15, p. 84]).

2. Gorenstein flat dimension

Let $R$ be a local Noetherian ring and let $\mathbf{D}$ denote the dualizing complex of $\hat{R}$. Let $A(\hat{R})$ denote the full subcategory of $\mathbf{D}_b(\hat{R})$, consisting of those complexes $X$ for which $\mathbf{D} \otimes^L_R X \in \mathbf{D}_b(\hat{R})$ and the canonical morphism

$$\gamma_X : X \rightarrow R \text{Hom}_R(\mathbf{D}, \mathbf{D} \otimes^L_R X),$$

is an isomorphism. Here, $\mathbf{D}_b(\hat{R})$ denote the full subcategory of $\mathbf{D}(\hat{R})$ (the derived category of $\hat{R}$-modules) consisting of complexes $X$ with $H_n(X) = 0$ for $|n| >> 0$, see [4].

Now, we define $A'(R)$ to be the class of all $R$-modules $M$ such that $\hat{R} \otimes_R M \in A(\hat{R})$.

**Lemma 2.1.** Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of modules over a local Noetherian ring $R$. Then if any two of $M', M, M''$ are in $A'(R)$, so is the third.

**Proof.** The exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ yields, the exact sequence $0 \rightarrow \hat{R} \otimes_R M' \rightarrow \hat{R} \otimes_R M \rightarrow \hat{R} \otimes_R M'' \rightarrow 0$. Now, the conclusion follows by using [5, Theorem 4.1] and [13, Theorem 2.24]. □

**Proposition 2.2.** Let $R$ be a local Noetherian ring and let $M$ be an $R$-module. If $\text{Gfd}_R M < \infty$, then $M \in A'(R)$.

**Proof.** By [13, Proposition 3.10], we have $\text{Gfd}_R(\hat{R} \otimes_R M) < \infty$. Using [5, Theorem 4.1], we conclude that $\hat{R} \otimes_R M$ belongs to $A(\hat{R})$. So, the assertion follows by the definition. □

In the proof of the following lemma we use the method of the proof of [11, Lemma 3.1].

**Lemma 2.3.** Suppose $K$ is cotorsion of finite flat dimension and suppose $M$ is an $R$-module. If $\text{Tor}_i^R(E, M) = 0$ for all $i > 0$ and all injective $R$-modules $E$, then $\text{Ext}^i_R(M, K) = 0$ for all $i > 0$. 

Proof. We prove by induction on $\text{fd}_R K$. First, let $K$ be flat and cotorsion. Then $K$ is a summand of a module of the form $\text{Hom}_R(E, E')$ where $E$ and $E'$ are injective ([8, Lemma 2.3]). It is enough to show that $\text{Ext}^i_R(M, \text{Hom}_R(E, E')) = 0$ for all $i > 0$. We have

$$\text{Ext}^i_R(M, \text{Hom}_R(E, E')) \cong \text{Hom}_R(\text{Tor}^R_i(M, E), E')$$

for all $i \geq 0$. Thus $\text{Ext}^i_R(M, K) = 0$ for all $i > 0$. Now, let $K$ be cotorsion and of finite flat dimension. Let $F_0 \to K$ be a flat cover of $K$ with kernel $L$. Then $L$ is cotorsion, see [8, Lemma 2.2]. Also, we have the exact sequence

$$\text{Ext}^i_R(M, F_0) \to \text{Ext}^i_R(M, K) \to \text{Ext}^{i+1}_R(M, L).$$

Since $K$ and $L$ are cotorsion, then so is $F_0$. Hence, by inductive hypothesis $\text{Ext}^i_R(M, K) = 0$ for all $i > 0$. □

Lemma 2.4. Let $R$ be a Noetherian ring and $M$ an $R$-module.

(i) If $R$ be a local ring and $M \in A^i(R)$, then there exists a monomorphism $M \to L$ with $\text{fd}_R L < \infty$.

(ii) Assume $\psi : M \to L$ is a monomorphism such that $\text{fd}_R L < \infty$ and that $\text{Tor}^i_R(N, M) = 0$ for all injective $R$-modules $N$ and all $i > 0$. Then $M$ possesses a monic $\overline{F(R)}$-preenvelope $M \to F$, in which $F$ is flat.

(iii) Let $R$-homomorphism $f : M \to L'$ be an $\overline{F(R)}$-preenvelope. Assume $\varphi : M \to L$ is a monomorphism such that $\text{pd}_R L < \infty$ and that $\text{Ext}^i_R(M, N) = 0$ for all projective $R$-modules $N$ and all $i > 0$. Then there exists a monic $\overline{F(R)}$-preenvelope $M \to P$, in which $P$ is projective.

Proof. (i) Since $M$ belongs to $A^i(R)$, $\text{Gfd}_R(M \otimes_R \hat{R})$ is finite by the definition and [5, Theorem 4.1]. Therefore, by [5, lemma 2.19], we have an exact sequence of $\hat{R}$-modules and $\hat{R}$-homomorphisms $0 \to M \otimes_R \hat{R} \to L$, where flat dimension of $L$ is finite as an $\hat{R}$-module. So, we obtain an exact sequence $0 \to M \to L$ of $R$-modules and $R$-homomorphism, where flat dimension of $L$ is finite as an $R$-module. Not that every flat $\hat{R}$-module is also flat as an $R$-module.

(ii) Using [7, Proposition 5.1], there exists a flat preenvelope $f : M \to F$. We show that $f$ is $\overline{F(R)}$-preenvelope. To this end, let $\psi' : M \to L'$ be an $R$-homomorphism such that $\text{fd}_R L' < \infty$ and let $0 \to K \to F' \overset{\pi} \to L' \to 0$ be an exact sequence such that $\pi : F' \to L'$ is a flat cover. Then $K$ is of finite flat dimension and also by [8, lemma 2.2], it is cotorsion. Lemma 2.3 implies that $\text{Ext}^i_R(M, K) = 0$ for all $i > 0$. So, we have the exact sequence

$$0 \to \text{Hom}_R(M, K) \to \text{Hom}_R(M, F') \to \text{Hom}_R(M, L') \to \text{Ext}^1_R(M, K) = 0.$$ 

Therefore, there exists an $R$-homomorphism $h : M \to F'$ such that $\pi h = \psi'$. Since $f : M \to F$ is flat preenvelope, there exists an $R$-homomorphism $g : F \to F'$ such that $h = gf$. Hence, there exists the $R$-homomorphism $\pi g : F \to L'$ such that $\pi gf = \psi'$. Thus $f$ is $\overline{F(R)}$-preenvelope. Consequently, $f$ is monic, because $\psi$ is monic.

(iii) Since $\varphi : M \to L$ is monic, it turns out that $f : M \to L'$ is also monic. Now, let $0 \to K \to P \overset{\pi} \to L' \to 0$ be an exact sequence such that $P$ is projective $R$-module. It is easy to see that $K \in \overline{P(R)}$. On the other hand, by hypothesis and induction on projective dimension,
By the exact sequence be an exact sequence of $R$ modules and let Hom$_R$ be an injective $R$-module. Therefore, Ext$_R^i(M, K) = 0$ for all $i > 0$. Hence $f : M \rightarrow L'$ has a lifting $M \rightarrow P$ which is monic and still an $\overline{P(R)}$-preenvelope. □

**Theorem 2.5.** Let $(R, m)$ be a local Noetherian ring and $C$ an $R$-module. Then the following conditions are equivalent:

(i) $C$ is Gorenstein flat.

(ii) $C$ belongs to $A'(R)$ and Tor$_i^R(L, C) = 0$ for all injective $R$-modules $L$ and all $i > 0$.

**Proof.** (i) $\Rightarrow$ (ii) By Proposition 2.2, $C$ belongs to $A'(R)$. Also, [13, Theorem 3.6], implies the last assertion in (i).

(ii) $\Rightarrow$ (i) By [13, Theorem 3.6], it is enough to show that $C$ admits a right flat resolution

$$X = 0 \rightarrow C \rightarrow F^0 \rightarrow F^1 \rightarrow F^2 \rightarrow \ldots$$

such that Hom$_R(X, Y)$ is exact for all flat $R$-modules $Y$ (i.e. $C$ admits a co-proper right flat resolution). Lemma 2.4 (i) implies that there exists an exact sequence $0 \rightarrow C \rightarrow L$ of $R$-modules and $R$-homomorphisms such that fd$_R L < \infty$. Using Lemma 2.4 (ii), there exists a monomorphism $f : C \rightarrow K$ which is a flat preenvelope. We obtain the short exact sequence $0 \rightarrow C \xrightarrow{f} K \rightarrow B \rightarrow 0$ and so for every flat $R$-module $F'$ we have the short exact sequence

$$0 \rightarrow \text{Hom}_R(B, F') \rightarrow \text{Hom}_R(K, F') \rightarrow \text{Hom}_R(C, F') \rightarrow 0.$$

Let $E$ be an injective $R$-module. Since Hom$_R(E, E_R(R/m))$ is a flat $R$-module, we conclude that

$$0 \rightarrow C \otimes_R E \rightarrow K \otimes_R E \rightarrow B \otimes_R E \rightarrow 0$$

is an exact sequence. So, Tor$_i^R(E, B) = 0$ for all $i > 0$ and all injective $R$-modules $E$, because $K$ is a flat $R$-module. Also, by Lemma 2.1 and Proposition 2.2, we obtain $B \in A'(R)$. Then proceeding in this manner, we get the desired co-proper right flat resolution of $C$. □

**Corollary 2.6.** Let $(R, m)$ be a local Noetherian ring of dimension $d$ and let $M \in A'(R)$. Then Gfd$_R(M) = \hat{\text{Gfd}}_R(\hat{R} \otimes_R M)$. In particular, if $M \in A'(R)$ then Gfd$_R M \leq \dim R$.

**Proof.** By [13, Proposition 3.10], Gfd$_R(\hat{R} \otimes_R M) \leq \hat{\text{Gfd}}_R(M)$. We show that Gfd$_R(M) \leq \hat{\text{Gfd}}_R(\hat{R} \otimes_R M)$ and so [13, Theorem 3.24] completes the proof. As $M$ belongs to $A'(R)$, we get that $\hat{R} \otimes_R M$ belongs to $A(\hat{R})$. So, by [5, Theorem 4.1] Gfd$_R(\hat{R} \otimes_R M)$ is finite. Set Gfd$_R(\hat{R} \otimes_R M) = t$ and let

$$0 \rightarrow C \rightarrow P_{t-1} \rightarrow \ldots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

be an exact sequence of $R$-modules and $R$-homomorphisms such that $P_i$’s are projective. We obtain the exact sequence

$$0 \rightarrow \hat{R} \otimes_R C \rightarrow \hat{R} \otimes_R P_{t-1} \rightarrow \ldots \rightarrow \hat{R} \otimes_R P_1 \rightarrow \hat{R} \otimes_R P_0 \rightarrow \hat{R} \otimes_R M \rightarrow 0.$$

By [13, Theorems 3.14], $\hat{R} \otimes_R C$ is a Gorenstein flat $\hat{R}$-module. Also, Lemma 2.1 and the above exact sequence, imply that $C$ belongs to $A'(\hat{R})$. In view of Theorem 2.5, it is enough to show that Tor$_i^\hat{R}(C, E) = 0$ for all injective $R$-modules $E$ and all $i > 0$. Let $E$ be an injective $R$-module and let Hom$_R(\cdot, E_R(R/m))$ denote by $(\cdot)^\vee$. From the natural monomorphism $E \rightarrow (E^\vee)^\vee$, we
conclude that $E$ is a direct summand of $(E^\vee)^\vee$. So, it is enough to show that \( \text{Tor}^R_i((E^\vee)^\vee, C) = 0 \) for all \( i > 0 \). By the next result, \( \text{id}_R((E^\vee)^\vee) \) is finite. It therefore follows from \([13, \text{Theorem 3.14}]\) that \( \text{Tor}^R_i(C \otimes_R \hat{R}, (E^\vee)^\vee) = 0 \) for all \( i > 0 \). Suppose \( F \to C \) is a flat resolution of \( C \). For every \( i > 0 \), we have

\[
\text{Tor}^R_i((E^\vee)^\vee) \cong H_i((F \otimes_R \hat{R}) \otimes_R (E^\vee)^\vee)
\cong H_i(F \otimes_R \hat{R}, (E^\vee)^\vee)
\cong \text{Tor}^R_i(C \otimes_R \hat{R}, (E^\vee)^\vee)
\]

The last isomorphism comes from the fact that \( F \otimes_R \hat{R} \) is a flat resolution of \( C \), considered as an \( \hat{R} \)-module. Thus, \( \text{Tor}^R_i((E^\vee)^\vee) = 0 \) for all \( i > 0 \).

**Lemma 2.7.** Let \( (R, \mathfrak{m}) \) be a local Noetherian ring and let \( K \) be an \( R \)-module such that \( \text{id}_R(K) \) is finite. Let \( \text{Hom}_R(-, E_R(R/\mathfrak{m})) \) denote by \( (-)^\vee \). The \( R \)-module \( (K^\vee)^\vee \) considered with the \( \hat{R} \)-module structure coming from \( E_R(R/\mathfrak{m}) \), that is, \( (\hat{r} f)(x) = \hat{r}(f(x)) \), for all \( \hat{r} \in \hat{R} \), \( f \in \text{Hom}_R(K^\vee, E_R(R/\mathfrak{m})) \) and \( x \in K^\vee \). Then \( \text{id}_R((K^\vee)^\vee) \) is finite.

**Proof.** We deduce that \( \text{fd}_R(K^\vee) \) is finite. It is easy to see that \( \text{fd}_R(K^\vee \otimes_R \hat{R}) \) is finite. By the adjoint isomorphism, we have the following isomorphism

\[
\text{Hom}_R((K^\vee \otimes_R \hat{R}, E_R(R/\mathfrak{m})) \cong \text{Hom}_R(K^\vee, E_R(R/\mathfrak{m})),
\]

as an \( \hat{R} \)-modules. This ends the proof, because the injective dimension of \( \text{Hom}_R(K^\vee \otimes_R \hat{R}, E_R(R/\mathfrak{m})) \) is finite as an \( \hat{R} \)-module.

3. Gorenstein projective dimension

In this section, we show that Gorenstein projective dimension of an \( R \)-module is finite if and only if its Gorenstein flat dimension is finite.

**Proposition 3.1.** Let \( R \) be a Noetherian ring with finite Krull dimension and \( C \) be an \( R \)-module. Then \( \text{Gpd}_R(C) \leq \text{Gfd}_R(C) \).

**Proof.** See \([13, \text{Remark 3.3 and Proposition 3.4}]\). \( \square \)

**Theorem 3.2.** Let \( R \) be a Noetherian ring of finite Krull dimension and \( M \) an \( R \)-module. Then the following conditions are equivalent:

(i) \( M \) is Gorenstein projective.

(ii) \( \text{Gfd}_R M < \infty \) and \( \text{Ext}_R^i(M, P) = 0 \) for all projective \( R \)-modules \( P \) and all \( i > 0 \).

**Proof.** Assume that \( M \) is Gorenstein projective. Then \( \text{Gfd}_R M < \infty \), by Proposition 3.1. Also, \([13, \text{Proposition 2.3}]\), implies that \( \text{Ext}_R^i(M, P) = 0 \) for all projective \( R \)-modules \( P \) and all \( i > 0 \).

Next, we show that (ii) \( \Rightarrow \) (i). By \([13, \text{Proposition 2.3}]\), it is enough to show that \( M \) admits a right projective resolution

\[
X = 0 \to M \to P^0 \to P^1 \to P^2 \to \cdots
\]
such that \( \text{Hom}_R(X, Y) \) is exact for every projective \( R \)-module \( Y \) (i.e. \( M \) admits a co-proper right projective resolution).

Since \( \text{Gfd}_R M < \infty \), it follows from [5, Lemma 2.19] that there exists a monomorphism \( M \rightarrow L \) with \( \text{fd}_R L < \infty \).

Let \( i > 0 \). By assumption and induction on projective dimension, \( \text{Ext}^i_R(M, Q) = 0 \) for all \( Q \in P(R) \). On the other hand, we have

\[
\text{Ext}^i_R(M, \text{Hom}_R(E, E')) \cong \text{Hom}_R(\text{Tor}_i^R(M, E), E')
\]

for all injective \( R \)-modules \( E \) and \( E' \). Therefore, \( \text{Tor}_i^R(M, E) = 0 \) for all injective \( R \)-modules \( E \). Note that, for each nonzero \( R \)-module \( N \), there exists an injective \( R \)-module \( E' \) such that \( \text{Hom}_R(N, E') \neq 0 \).

Now, using parts (ii) and (iii) of Lemma 2.4, there exists a monomorphism \( \psi : M \rightarrow Q \) which is a projective preenvelope. We consider the exact sequence

\[
0 \rightarrow M \xrightarrow{\psi} Q \rightarrow B \rightarrow 0,
\]

where \( B = \text{Coker} \psi \). Let \( P \) be a projective \( R \)-module. Applying the functor \( \text{Hom}_R(\cdot, P) \) to the above exact sequence, we see that \( \text{Ext}^i_R(B, P) = 0 \) for all \( i > 0 \) because \( \psi : M \rightarrow Q \) is a projective preenvelope. Also, \( \text{Gfd}_R B < \infty \), by [13, Theorem 3.15]. Then proceeding in this manner, we get the desired co-proper right projective resolution for \( M \).

We can deduce from Proposition 2.2, Corollary 2.6 and Theorem 3.2 the following result.

**Corollary 3.3.** Let \( R \) be a local Noetherian ring and \( M \) an \( R \)-module. Then the following conditions are equivalent:

(i) \( M \) is Gorenstein projective.

(ii) \( M \in A'(R) \) and \( \text{Ext}^i_R(M, P) = 0 \) for all projective \( R \)-modules \( P \) and all \( i > 0 \).

**Theorem 3.4.** Let \( R \) be a Noetherian ring of finite dimension \( d \) and \( M \) be an \( R \)-module. Then the following conditions are equivalent:

(i) \( \text{Gfd}_R M < \infty \).

(ii) \( \text{Gpd}_R M < \infty \).

Moreover, if one of the above conditions holds, then \( \text{Gpd}_R M \leq d \).

**Proof.** (i) \( \Rightarrow \) (ii) We prove the claim by induction on \( \text{Gfd}_R M \). First, let \( M \) be a Gorenstein flat \( R \)-module. Let \( F \) be a flat \( R \)-module. Consider the minimal pure injective resolution

\[
0 \rightarrow F \rightarrow PE^0(F) \rightarrow PE^1(F) \rightarrow \cdots
\]

(see [16, pages 39 and 92]). Note that, by [16, Lemma 3.1.6], \( PE^n(F) \) is flat for all \( n \geq 0 \) and also, by [16, Corollary 4.2.7], \( PE^n(F) = 0 \) for all \( n > d \). Since, every pure injective module is cotorsion, by [13, Proposition 3.22], \( \text{Ext}^j_R(M, PE^i(F)) = 0 \) for all \( i \geq 0 \) and all \( j \geq 1 \). Therefore, \( \text{Ext}^{d+i}_R(M, F) \cong \text{Ext}^i_R(M, PE^d(F)) \) for all \( i \geq 1 \), and so \( \text{Ext}^{d+i}_R(M, F) = 0 \) for all \( i \geq 1 \). Next, let

\[
0 \rightarrow C \rightarrow P_{d-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0
\]
be an exact sequence such that $P_i$’s are projective. We have $\text{Ext}_R^{d+i}(M,F) \cong \text{Ext}_R^{i}(C,F)$ for all $i \geq 1$, and so $\text{Ext}_R^i(C,F) = 0$ for all $i \geq 1$. On the other hand, using [13, Theorem 3.15], we conclude that $\text{Gfd}_R C < \infty$. Therefore, by Theorem 3.2, $C$ is Gorenstein projective, and hence $\text{Gpd}_R M \leq d$.

Now, let $\text{Gfd}_R M = t > 0$ and let $0 \to K \to P \to M \to 0$ be an exact sequence such that $P$ is projective. By [13, Proposition 3.12], $\text{Gfd}_R K = t - 1$. Hence, induction hypothesis implies that $\text{Gpd}_R M < \infty$.

(ii) $\Rightarrow$ (i) This follows from Proposition 3.1.

Now, if either $\text{Gpd}_R M < \infty$ or equivalently $\text{Gfd}_R M < \infty$, then, by [5, Lemma 2.17], $\text{Gpd}_R M = \text{pd}_R H$, where $H$ is an $R$-module. This completes the proof. □

Now, we are ready to deduce the main result of this section by using Proposition 2.2, Corollary 2.6 and Theorem 3.4.

**Theorem 3.5.** Let $R$ be a local Noetherian ring and $M$ an $R$-module. Then the following conditions are equivalent:

(i) $\text{Gfd}_R M < \infty$.

(ii) $\text{Gpd}_R M < \infty$.

(iii) $M \in A'(R)$.

Moreover, if one of the above conditions holds, then $\text{Gpd}_R M \leq \dim R$.

4. Gorenstein injective dimension

Let $R$ be a local Noetherian ring and let $D$ denote the dualizing complex of $\hat{R}$. Let $B(\hat{R})$ denote the full subcategory of $D_b(\hat{R})$, consisting of those complexes $X$ for which $R\text{Hom}_\hat{R}(D,X) \in D_b(\hat{R})$ and the canonical morphism

$$\tau_X : D \otimes^L_R R\text{Hom}_\hat{R}(D,X) \to X,$$

is an isomorphism, see [4, 3.1].

Now, we define $B'(R)$ to be the class of all $R$-modules $M$ such that $\text{Hom}_R(\hat{R},M) \in B(\hat{R})$.

In the Theorem 4.8, we want to characterize Gorenstein injective modules in terms of the class $B'(R)$. To prove Theorem 4.8, we need the following results.

**Definition 4.1.** (See [6, Definition 5.10]) For every $R$-module $M$, we show the large restricted injective dimension by $\text{Ed}_R M$ and define

$$\text{Ed}_R M = \sup \{ i \in \mathbb{N}_0 \mid \exists L \in F(\hat{R}) \mid \text{Ext}_R^i(L,M) \neq 0 \}.$$

**Theorem 4.2.** (Dimension inequality) Let $R$ be a Noetherian ring of finite Krull dimension. For every $R$-module $M$, we have the following inequality:

$$\text{Ed}_R M \leq \text{Gid}_R M \leq \text{id}_R M.$$
Hence, we obtain the exact sequence
\[ 0 \rightarrow M \rightarrow T \rightarrow K \rightarrow 0 \]
such that \( T \) is injective \( R \)-module and \( \text{Gid}_R K = n - 1 \). By induction, we have \( \text{Ed}_R K \leq \text{Gid}_R K = n - 1 \), and so \( \text{Ext}_R^1(L, K) = 0 \) for all \( L \in \overline{F}(R) \) and all \( j > n - 1 \). For each \( i > n \) and each \( L \in \overline{F}(R) \), we have the following exact sequence
\[ 0 = \text{Ext}_R^{i-1}(L, K) \rightarrow \text{Ext}_R^i(L, M) \rightarrow \text{Ext}_R^i(L, T) = 0. \]
So \( \text{Ed}_R M \leq n = \text{Gid}_R M \). This ends the proof. \( \square \)

By Theorem 4.2, every Gorenstein injective \( R \)-module over a Noetherian ring of finite Krull dimension is strongly cotorsion (see [16, Definition 5.4.1]). The following example shows that there exists an \( R \)-module with finite Gorenstein injective dimension over a regular local ring which is not cotorsion.

**Example 4.3.** Let \( R \) be a regular local ring of Krull dimension one which is not complete. By [1, Lemma 3.3], \( \text{Hom}_R(\hat{R}, R) = 0 \). So, \( \hat{R} \) is not a projective \( R \)-module. Therefore, \( \text{pd}_R(\hat{R}) = 1 \) and consequently there exists an \( R \)-module \( M \) such that \( \text{Ext}_R^1(\hat{R}, M) \neq 0 \). On the other hand, \( \text{id}_R M \leq 1 \). So, \( M \) is an \( R \)-module with finite Gorenstein injective dimension which is not cotorsion.

**Proposition 4.4.** Let \( R \) be a local Noetherian ring and \( M \) an \( R \)-module.

(i) If \( M \) is a Gorenstein injective \( R \)-module, then \( \text{Hom}_R(\hat{R}, M) \) is Gorenstein injective as an \( \hat{R} \)-module.

(ii) If \( M \) is a Gorenstein injective \( R \)-module, then \( M \in B'(R) \).

**Proof.** (i) Let
\[ X = \ldots \rightarrow E_2 \rightarrow E_1 \rightarrow E_0 \xrightarrow{\rho^0} G^0 \rightarrow G^1 \rightarrow \ldots \]
be an exact sequence of injective \( R \)-modules such that \( \text{Hom}_R(I, X) \) is exact for every injective \( R \)-modules \( I \) with \( \ker \rho^0 = M \). If
\[ 0 \rightarrow G'' \rightarrow E \rightarrow G' \rightarrow 0 \]
is an exact sequence such that \( G' \), \( G'' \) are Gorenstein injective and \( E \) is injective, then Theorem 4.2 yields the short exact sequence,
\[ 0 \rightarrow \text{Hom}_R(\hat{R}, G'') \rightarrow \text{Hom}_R(\hat{R}, E) \rightarrow \text{Hom}_R(\hat{R}, G') \rightarrow 0. \]
Hence, we obtain the exact sequence
\[ Y = \ldots \rightarrow \text{Hom}_R(\hat{R}, E_1) \rightarrow \text{Hom}_R(\hat{R}, E_0) \xrightarrow{\text{Hom}_R(\hat{R}, \rho^0)} \text{Hom}_R(\hat{R}, G^0) \rightarrow \ldots \]
of \( \hat{R} \)-modules and \( \hat{R} \)-homomorphisms in which \( \ker(\text{Hom}_R(\hat{R}, \rho^0)) \cong \text{Hom}_R(\hat{R}, M) \). On the other hand, if \( E \) is an injective \( R \)-module, we can conclude that \( \text{Hom}_R(\hat{R}, E) \) is injective as an \( \hat{R} \)-module,
because $\text{Hom}_R(-, \text{Hom}_R(\hat{R}, E)) \cong \text{Hom}_R(- \otimes_R \hat{R}, E)$. It is enough to show that $\text{Hom}_R(E', Y)$ is exact, for all injective $\hat{R}$-modules $E'$. This follows from the following isomorphisms of complexes

$$\text{Hom}_R(E', Y) \cong \text{Hom}_R(E', \text{Hom}_R(\hat{R}, X)) \cong \text{Hom}_R(E', X)$$

and the fact that every injective $\hat{R}$-module is also injective as an $R$-module.

(ii) Let $M$ be Gorenstein injective. By (i), $\text{Hom}_R(\hat{R}, M)$ is Gorenstein injective $\hat{R}$-module. Hence, by [5, Theorem 4.4], $\text{Hom}_R(\hat{R}, M) \in B(\hat{R})$, and so $M \in B'(R)$, by the definition. □

**Proposition 4.5.** An $R$-module $M$ is Gorenstein injective if and only if $\text{Ext}_R^i(M, E) = 0$ for all injective $R$-modules $E$ and for all $i > 0$ and there exists an exact sequence

$$X = \ldots \rightarrow E_2 \rightarrow E_1 \rightarrow E_0 \rightarrow M \rightarrow 0$$

of $R$-modules and $R$-homomorphisms with $E_i$ is injective $R$-module for all $i \geq 0$, such that $\text{Hom}_R(E, X)$ is exact for all injective $R$-modules $E$ (i.e. $M$ admits a proper left injective resolution).

**Proof.** It is the dual version of [13, Proposition 2.3] and we leave the proof to the reader. □

**Lemma 4.6.** (i) Let $R$ be a local Noetherian ring and $M$ a cotorsion $R$-module such that $M$ belongs to $B'(R)$. Then there exists an epimorphism $L \rightarrow M$ with $\text{id}_R(L) < \infty$.

(ii) Let $R$ be a Noetherian ring and $\varphi : L \rightarrow M$ an $R$-epimorphism with $\text{id}_R(L) < \infty$ and $\text{Ext}_R^i(N, M) = 0$ for all injective $R$-modules $N$ and all $i > 0$. Then there exists an epic $\overline{I(R)}$-precover $E \rightarrow M$, in which $E$ is injective.

**Proof.** (i) Since $M$ belongs to $B'(R)$, then $\text{Hom}_R(\hat{R}, M))$ belongs to $B(\hat{R})$. So, $\text{Hom}_R(\hat{R}, M))$ has finite Gorenstein injective dimension as an $\hat{R}$-module by [5, Theorem 4.4]. By [5, Lemma 2.18], there are an $\hat{R}$-module $L$ and an $\hat{R}$-epimorphism $L \rightarrow \text{Hom}_R(\hat{R}, M)$ such that injective dimension of $L$ as an $\hat{R}$-module is finite. Since every injective $\hat{R}$-module is injective as an $R$-module, injective dimension of $L$ as an $R$-module is finite. Consider the following exact sequence

$$0 \rightarrow R \rightarrow \hat{R} \rightarrow \hat{R}/R \rightarrow 0,$$

that yields the following exact sequence

$$\text{Hom}_R(\hat{R}, M)) \rightarrow \text{Hom}_R(R, M)) \rightarrow \text{Ext}_R^1(\hat{R}/R, M).$$

On the other hand, since $\hat{R}/R$ is a flat $R$-module and $M$ is a cotorsion $R$-module, $\text{Ext}_R^1(\hat{R}/R, M) = 0$. So, the natural $R$-homomorphism $\text{Hom}_R(\hat{R}, M)) \rightarrow M$ is epic. The result follows.

(ii) By [16, Theorem 2.4.3], there exists an $I(R)$-precover $f : E \rightarrow M$. We claim that $f$ is an $\overline{I(R)}$-precover. Let $\varphi' : L' \rightarrow M$ be an $R$-homomorphism such that $\text{id}_R(L') < \infty$. Consider an exact sequence

$$0 \rightarrow L' \rightarrow E' \rightarrow K \rightarrow 0$$

such that $E'$ is an injective $R$-module. It is clear that injective dimension of $K$ is finite. By induction on injective dimension, we can deduce from assumption that $\text{Ext}_R^1(K, M)$ is zero. We
obtain the following exact sequence

\[ 0 \to \text{Hom}_R(K, M) \to \text{Hom}_R(E', M) \to \text{Hom}_R(L', M) \to \text{Ext}^1_R(K, M) = 0. \]

Hence, we conclude that there exists an \( R \)-homomorphism \( \psi : E' \to M \) such that \( \varphi' = \psi g \). On the other hand, since \( f \) is an \( I(R) \)-precover, there exists an \( R \)-homomorphism \( h : E' \to E \) such that \( \psi = fh \). Hence, there exists an \( R \)-homomorphism \( h'g : L' \to E \) such that \( f(hg) = \varphi' \). It therefore follows that \( f \) is an \( \overline{I(R)} \)-precover. Consequently \( f \) is epic, because \( \varphi \) is epic. \( \Box \)

**Lemma 4.7.** Let \((R, \mathfrak{m})\) be a local Noetherian ring, \( M \) a cotorsion \( R \)-module, and \( K \) a cotorsion \( \hat{R} \)-module. Then

(i) \( \text{Ext}^i_R(F, M) = 0 \) for all flat \( R \)-modules \( F \) and all \( i > 0 \).

(ii) \( K \) is cotorsion as an \( R \)-module.

(iii) For all \( j > 0 \), \( \text{Ext}^j_R(E, M) = 0 \) for all injective \( R \)-modules \( E \) if and only if \( \text{Ext}^j_R(I, \text{Hom}_R(\hat{R}, M)) = 0 \) for all injective \( \hat{R} \)-modules \( I \).

**Proof.** (i) See the proof of [16, Proposition 3.1.2].

(ii) Suppose \( F \) is a flat \( R \)-module and \( \text{P}_\bullet \to F \) a projective resolution of \( F \). For all \( i > 0 \), we have

\[
\text{Ext}^i_R(F, K) \cong H^i(\text{Hom}_R(\text{P}_\bullet, K)) \\
\cong H^i(\text{Hom}_R(\text{P}_\bullet \otimes R \hat{R}, K)) \\
\cong \text{Ext}^i_R(F \otimes R \hat{R}, K).
\]

The last isomorphism comes from the fact that \( K \) is a cotorsion \( \hat{R} \)-module and \( F \otimes R \hat{R} \) is flat as an \( \hat{R} \)-module for all flat \( R \)-modules \( F \). This ends the proof of (ii).

(iii) Suppose \( L \) is an \( \hat{R} \)-module and \( \text{F}_\bullet \to L \) is a free resolution of \( L \), considered as an \( \hat{R} \)-module. For every \( j > 0 \), we have

\[
\text{Ext}^j_R(L, \text{Hom}_R(\hat{R}, M)) \cong H^j(\text{Hom}_R(\text{F}_\bullet, \text{Hom}_R(\hat{R}, M))) \\
\cong H^j(\text{Hom}_R(\text{F}_\bullet \otimes R \hat{R}, M)) \\
\cong H^j(\text{Hom}_R(\text{F}_\bullet, M)) \\
\cong \text{Ext}^j_R(L, M)
\]

The last isomorphism follows from the fact that \( M \) is cotorsion and every flat \( \hat{R} \)-module is flat as an \( R \)-module.

\( \Rightarrow \) We know that every injective \( \hat{R} \)-module is injective as an \( R \)-module. So, the result follows from the above isomorphism.

\( \Leftarrow \) By assumption, it is easy to see that

\[
\text{Ext}^i_R(N, \text{Hom}_R(\hat{R}, M)) = 0,
\]

for all \( \hat{R} \)-modules \( N \) of finite injective dimension and all \( i > 0 \). Let \( E \) be an injective \( R \)-module and let \( \text{Hom}_R(-, E_R(R/\mathfrak{m})) \) denote by \( (-)^{\vee} \). From the natural monomorphism \( E \to (E^{\vee})^{\vee} \), we conclude that \( E \) is a direct summand of \( (E^{\vee})^{\vee} \). So, it is enough to show that \( \text{Ext}^i_R((E^{\vee})^{\vee}, M) = 0 \).
for all $i > 0$. Since, by Lemma 2.7, $\text{id}_{\hat{R}}((E^\vee)^\vee) < \infty$, the result follows from the above isomorphism.

\[\square\]

**Theorem 4.8.** Let $R$ be a local Noetherian ring and $M$ an $R$-module. Then the following conditions are equivalent:

(i) $M$ is Gorenstein injective.

(ii) $M$ is cotorsion and $\text{Hom}_R(\hat{R}, M)$ is Gorenstein injective as an $\hat{R}$-module.

(iii) $M \in B'(R)$, $M$ is cotorsion and $\text{Ext}^i_{\hat{R}}(E, M) = 0$ for all injective $R$-modules $E$ and all $i > 0$.

**Proof.** (i) $\Rightarrow$ (ii) This follows from Theorem 4.2 and Proposition 4.4.

(ii) $\Rightarrow$ (iii) By [5, Theorem 4.4], $\text{Hom}_R(\hat{R}, M)$ belongs to $B(\hat{R})$, and so $M$ belongs to $B'(R)$.

Also, Proposition 4.5 implies that

$$\text{Ext}_R^i(I, \text{Hom}_R(\hat{R}, M)) = 0$$

for all injective $\hat{R}$-modules $I$ and all $i > 0$. The result follows from Lemma 4.7 (iii).

(iii) $\Rightarrow$ (i) In view of Proposition 4.5, it is enough to show that $M$ admits a proper left injective resolution. It follows from Lemma 4.6 (i) and (ii) that there exists an exact sequence

$$0 \rightarrow B \rightarrow E \rightarrow f \rightarrow M \rightarrow 0$$

such that $f$ is an $I(R)$-precover and $E$ an injective $R$-module. It is enough to show that $B$ satisfies the given assumptions on $M$.

Let $I$ be an injective $R$-module. It is easy to deduce from the above exact sequence that $\text{Ext}_{\hat{R}}(I, B) = 0$ for all $i \geq 2$. Also, we have the following exact sequence

$$\text{Hom}_R(I, E) \rightarrow \text{Hom}_R(I, M) \rightarrow \text{Ext}_{\hat{R}}^1(I, B) \rightarrow \text{Ext}_{\hat{R}}^1(I, E) = 0.$$  

On the other hand, $\text{Hom}_R(I, f)$ is epimorphism. So $\text{Ext}_{\hat{R}}^1(I, B) = 0$.

Now, we prove that $B$ is a cotorsion $R$-module. In view of assumption and Lemma 4.7, we conclude that

$$\text{Ext}_{\hat{R}}^i(I, \text{Hom}_R(\hat{R}, M)) = 0$$

for all injective $\hat{R}$-modules $I$ and all $i > 0$. On the other hand, $M \in B'(R)$ implies that $\text{Hom}_R(\hat{R}, M) \in B(\hat{R})$. Therefore, by [5, Lemma 4.7], $\text{Hom}_R(\hat{R}, M)$ is Gorenstein injective as an $\hat{R}$-module. Hence, we have an exact sequence

$$0 \rightarrow K \rightarrow E' \rightarrow \text{Hom}_R(\hat{R}, M) \rightarrow 0,$$

of $\hat{R}$-modules and $\hat{R}$-homomorphism such that $E'$ is an injective and $K$ is a Gorenstein injective $\hat{R}$-module. By Theorem 4.2, $K$ is a cotorsion $\hat{R}$-module. Lemma 4.7 implies that $K$ is cotorsion as an $R$-module. Now, let $\varphi : \text{Hom}_R(\hat{R}, M) \rightarrow M$ be the natural $R$-homomorphism. Consider the following diagram

$$\begin{array}{cccccc}
0 & \rightarrow & K & \rightarrow & E' & \rightarrow & \text{Hom}_R(\hat{R}, M) & \rightarrow & 0 \\
& & & \downarrow{\varphi} & & & \\
0 & \rightarrow & B & \rightarrow & E & \rightarrow & M & \rightarrow & 0.
\end{array}$$
Since $E'$ is an injective $R$-module and $f : E \to M$ is an $\hat{R}(R)$-precover, there exists an $R$-homomorphism $\psi : E' \to E$ such that the following diagram is commutative.\[
\begin{array}{cccccc}
0 & \to & K & \to & E' & \to & \text{Hom}_R(\hat{R}, M) & \to & 0 \\
& & \downarrow\psi & & \downarrow\varphi & & & \\
0 & \to & B & \to & E & \to & M & \to & 0.
\end{array}
\]
It is easy to see that there exists an $R$-homomorphism $\theta : K \to B$ such that the following diagram is commutative.\[
\begin{array}{cccccc}
0 & \to & K & \to & E' & \to & \text{Hom}_R(\hat{R}, M) & \to & 0 \\
& & \downarrow\theta & & \downarrow\psi & & \downarrow\varphi & & \\
0 & \to & B & \to & E & \to & M & \to & 0.
\end{array}
\]
Suppose $F$ is a flat $R$-module. Then we obtain the following commutative diagram
\[
\begin{array}{cccc}
\text{Hom}_R(F, \text{Hom}_R(\hat{R}, M)) & \xrightarrow{\beta} & \text{Ext}^1_R(F, K) & \to & 0 \\
\downarrow\text{Hom}_R(F, \varphi) & & \downarrow\theta_1 & & (\ast) \\
\text{Hom}_R(F, M) & \xrightarrow{\delta} & \text{Ext}^1_R(F, B) & \to & 0.
\end{array}
\]
The natural exact sequence\[
0 \to R \to \hat{R} \to \hat{R}/R \to 0,
\]
yields the exact sequence\[
0 \to \text{Hom}_R(\hat{R}/R, M) \to \text{Hom}_R(\hat{R}, M) \xrightarrow{\varphi} M \to 0,
\]
because $M$ is a cotorsion $R$-module and $\hat{R}/R$ is a flat $R$-module. Thus, we obtain the following exact sequence\[
0 \to \text{Hom}_R(F, \text{Hom}_R(\hat{R}/R, M)) \to \text{Hom}_R(F, \text{Hom}_R(\hat{R}, M)) \xrightarrow{\text{Hom}_R(F, \varphi)} \text{Hom}_R(F, M) \to \\
\to \text{Ext}^1_R(F, \text{Hom}_R(\hat{R}/R, M)).
\]
Since $M$ is a cotorsion and $\hat{R}/R$ is a flat $R$-module,
\[
\text{Ext}^1_R(F, \text{Hom}_R(\hat{R}/R, M)) \cong \text{Ext}^1_R(F \otimes_R \hat{R}/R, M).
\]
On the other hand, $F \otimes_R \hat{R}/R$ is a flat $R$-module, so $\text{Ext}^1_R(F \otimes_R \hat{R}/R, M)$ is zero $R$-module. Therefore $\text{Hom}_R(F, \varphi)$ is an epimorphism. By $\ast$, $\theta_1\beta$ is epic and so $\theta_1$ is epic. Thus, since $K$ is a cotorsion $R$-module, $\text{Ext}^1_R(F, B)$ is the zero module. This means that $B$ is cotorsion.

Now, we apply the functor $\text{Hom}_R(\hat{R}, -)$ on the following exact sequence\[
0 \to B \to E \to M \to 0,
\]
and obtain the exact sequence\[
0 \to \text{Hom}_R(\hat{R}, B) \to \text{Hom}_R(\hat{R}, E) \to \text{Hom}_R(\hat{R}, M) \to 0.
\]
It is easy to see that $\text{Hom}_R(\hat{R}, E)$ is an injective $\hat{R}$-module. Since $\text{Hom}_R(\hat{R}, M)$ is Gorenstein injective as an $\hat{R}$-module, by [13, theorem 2.25], $\text{Hom}_R(\hat{R}, B)$ has finite Gorenstein injective
dimension. So, it follows from [5, Theorem 4.4] that \( B \in B'(R) \). This ends the proof. \( \square \)

The following example shows that the dual version of Theorem 3.4 is not true.

**Example 4.9.** Let \( R \) be a non-complete local Noetherian domain which is not Gorenstein. By [14, Theorem 2.1], \( \text{Gid}_R(R) = \infty \). On the other hand, by [1, Lemma 3.3], \( \text{Hom}_R(\hat{R}, R) = 0 \). So \( R \) has infinite Gorenstein injective dimension as an \( R \)-module but \( R \notin B'(R) \).

**Theorem 4.10.** Let \((R, m)\) be a local Noetherian ring of dimension \( d \) and \( \text{Ext}_R^i(\hat{R}, M) = 0 \) for all \( i > 0 \). Then the Gorenstein injective dimension of \( M \) is finite if and only if \( M \) belongs to \( B'(R) \). In particular, if \( M \in B'(R) \) then \( \text{Gid}_R(M) \leq d \).

**Proof.** \( \Rightarrow \) Let \( \text{Gid}_R M = t \) and

\[
0 \to M \to G^0 \to G^1 \to G^2 \to \ldots \to G^t \to 0
\]

be an exact sequence such that \( G^i \) is Gorenstein injective for all \( 0 \leq i \leq t \). Using hypothesis, we obtain the following exact sequence

\[
0 \to \text{Hom}_R(\hat{R}, M) \to \text{Hom}_R(\hat{R}, G^0) \to \ldots \to \text{Hom}_R(\hat{R}, G^t) \to 0.
\]

By Proposition 4.4 (i), \( \text{Gid}_R(\text{Hom}_R(\hat{R}, M)) \) is finite as an \( \hat{R} \)-module and so by [5, Theorem 4.4], \( \text{Hom}_R(\hat{R}, M) \) belongs to \( B(\hat{R}) \). The assertion follows from the definition.

\( \Leftarrow \) Since \( M \) belongs to \( B'(R) \), \( \text{Hom}_R(\hat{R}, M) \) belongs to \( B(\hat{R}) \). Now, by using [5, Theorem 4.4], the Gorenstein injective dimension of \( \text{Hom}_R(\hat{R}, M) \) is finite as an \( \hat{R} \)-module. By [13, Theorem 2.29], \( \text{Gid}_R(\text{Hom}_R(\hat{R}, M)) \leq \text{FID}(R) \), where \( \text{FID}(R) = \sup \{ \text{id}_R(M) \mid M \text{ is an } R \text{-module of finite injective dimension} \} \). It is known that \( \text{id}_R(N) = \text{id}_R(\text{Hom}_R(N, E_R(R/m))) \), for all \( R \)-modules \( N \). So, we have \( \text{Gid}_R(\text{Hom}_R(\hat{R}, M)) \leq d \).

Consider the following exact sequence

\[
0 \to M \to E^0 \to E^1 \to \ldots \to E^{d-1} \to L \to 0,
\]

of \( R \)-modules and \( R \)-homomorphisms such that \( E^i \) is injective \( R \)-module for all \( 0 \leq i \leq d - 1 \). We have the following exact sequence,

\[
0 \to \text{Hom}_R(\hat{R}, M) \to \ldots \to \text{Hom}_R(\hat{R}, E^{d-1}) \to \text{Hom}_R(\hat{R}, L) \to 0.
\]

So, by [13, Theorem 2.22], \( \text{Hom}_R(\hat{R}, L) \) is a Gorenstein injective \( \hat{R} \)-module. On the other hand, for any flat \( R \)-module \( F \) and any \( i > 0 \), we have

\[
\text{Ext}_R^i(F, L) \cong \text{Ext}_R^{i+d}(F, M).
\]

Therefore, \( \text{Ext}_R^i(F, L) \) is zero for all \( i > 0 \), because the projective dimension of \( F \) is less than \( d + 1 \). So, \( L \) is cotorsion. It therefore follows from Theorem 4.8 that \( L \) is a Gorenstein injective \( R \)-module. Thus, \( \text{Gid}_R(M) \leq d \). \( \square \)
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