Topological Charge and Black Hole Photon Spheres

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Black hole photon spheres or light rings are closely linked to the astronomical phenomena, such as the gravitational waves and the shadow in spherically symmetric or axi-symmetric spacetime. In this paper, we study the photon sphere structures from a novel viewpoint, the topology, for a static, spherically symmetric black hole with asymptotically flat, AdS, and dS behaviors. The topological current and charge for the photon spheres are introduced following Duan’s topological current ϕ-mapping theory. The topological current is nonzero only at the zero point of the vector field determining the location of the photon sphere. So each photon sphere can be assigned a topological charge. Considering the full exterior region, we find the total topological charge always equals -1 for these black hole with different asymptotical behaviors. This result implies that there exists at least one standard photon sphere outside of the black hole. Then we apply the study to the dyonic black holes. We observe that even when more photon spheres are included, the total topological charge still keeps unchanged. Moreover, for a naked singularity, it has a vanishing topological charge, indicating that black hole and naked singularity are in different topological classes. It is expected that this novel topological argument could provide an insightful idea on the study of the black hole photon spheres or light rings, and further cast new light on the black hole astronomical phenomena.

I. INTRODUCTION

The observations of gravitational waves [1] and black hole shadow image [2–4] mark a new era of astrophysical observations. These provide powerful tests to uncover the spacetime structure near a black hole horizon. In the ringdown stage of black hole merge, the radiating gravitational waves can be understood by the quasi-normal modes. It is also well known that in the eikonal limit, the quasinormal modes are linked to the photon spheres (PSs) or light rings (LRs) of the nonrotating or rotating black holes (for recent progress, see Refs. [5–8]). On the other hand, the formation of a black hole shadow mainly depends on the existence of PS or LR rather the horizon. Moreover, the relativistic images of a light source around a compact object are closely dependent of PS and LR. Different relations between these observable phenomena were studied in Refs. [9–13]. All the results confirm that PS and LR play an extremely important role in astronomical observation. Therefore, it is valuable to investigate the characteristic properties of PS or LR for a certain spacetime.

In Ref. [14], Cunha, Berti, and Herdeiro considered the LR stability for ultra-compact objects. Based on the Brouwer degree of a continuous map, they found that the LRs of the compact objects always come in pairs. This introduces a novel scheme to investigate the LRs by using a topological argument, while ignoring the specific field equations. This result is proved to be generally true.

While, there is an important exception when the degenerate light rings present [12,16]. Recently, Cunha and Herdeiro [17] put forward a big step, and generalized the study to a four dimensional stationary, axi-symmetric, asymptotically flat black hole spacetime. They proved that there exists, at least, one standard LR outside the black hole horizon for each rotation sense by calculating the winding number of the vector field defined by an effective potentials on the orthogonal (r, θ)-space. In addition, following their topological argument, a horizonless ultra-compact object, such as the boson star, will show an even number of non-degenerate LRs. An valuable issue concerning the topological argument is to extend the study to the non-rotating black holes. This can help us to understand the topological property of PSs, and to uncover the influence of black hole spin on the PS topology.

We in Ref. [18] also introduced an interesting topological approach to investigate the black hole shadow. Each shadow shape is endowed with a topological covariant quantity. For a black hole shadow, it equals one, while deviates one for a naked singularity. For a multiple disconnected shadow pattern, it can produce the number of the shadow. So through the topological quantity, one can distinguish different spacetimes. This also provides us a distinctive approach to study the black hole shadow from the topological argument.

On the other hand, there is a famous theory known as Duan’s topological current ϕ-mapping theory [10,20], which is a powerful tool to investigate the topological defects in different physical systems, such as gauge theory, superconduction, monopole, magnetic skyrmions, knots, cosmological string, quantum Hall effect, and so
on. Thus, it provides us a natural and effective tool to investigate the topological structure of PSs. In this paper, we will employ this approach to study the topology of PSs in a general static, spherically symmetric black hole with asymptotically flat, AdS, dS behaviors of boundary. Perhaps topology starts to become one powerful tool in the study of black hole physics.

The present paper is organized as follows. In Sec. II we give a brief introduction of PS for a non-rotating black hole. The vector field is also introduced through a regular potential function outside a black hole horizon on the \((r, \theta)\) plane. Then based on the vector field, we in Sec. III present the topological current and charge. In particular, following the topological current \(\phi\)-mapping theory, we express the topological current in term of a \(\phi\) function. The inner structure of the topological charge is also investigated. In Sec. IV we compute the topological charge for a black hole with asymptotically flat, AdS, and dS behaviors of boundary. All these black holes are found to admit a minus one topological charge indicating the existence of at least one standard PS. Next, in Sec. V, as an specific example, we calculate the topological current and charge following Duan’s \(\phi\)-mapping theory. In this paper, we will employ this approach to study the topology of PSs. In this paper, we will employ this approach to study the topology of PSs.

\section{Black Holes and Photon Spheres}

In this paper, we only consider the static, spherically symmetric black hole. The black hole solution is assumed to be in the following form

\[
ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + h(r)(d\theta^2 + \sin^2 \theta d\phi^2).\tag{1}\]

Generally, the radius \(r_h\) of the black hole horizon is the largest root of \(g(r)=0\) or \(f(r)=0\). On the other hand, by solving the null geodesics, one can obtain the radial motion on the equatorial plane

\[\dot{r}^2 + V_{\text{eff}} = 0,\tag{2}\]

where the effective potential is given by

\[V_{\text{eff}} = g(r) \left(\frac{L^2}{h(r)} - \frac{E^2}{f(r)}\right).\tag{3}\]

Here \(E\) and \(L\) are the energy and angular momentum of photon, which are related with the Killing vector fields \(\partial_t\) and \(\partial_\phi\), respectively. Since this solution is spherically symmetric, there exists a PS at \(r_{ps}\) determined by

\[V_{\text{eff}} = 0, \quad \partial_r V_{\text{eff}} = 0.\tag{4}\]

Solving them, we find that the radius of the PS satisfies the following equation

\[
\left(\frac{f(r)}{h(r)}\right)_{r=r_{ps}}' = 0,\tag{5}\]

where the prime indicates the derivative with respect to \(r\). Moreover, \(\partial_r V_{\text{eff}}(r_{ps}) < 0\) indicates the PS is unstable (stable). Carrying out the derivative, (5) reduces to

\[f(r)h(r)' - f(r)'h(r) = 0.\tag{6}\]

It is worth to note that at the horizon where \(f(r_h)=0\), the first term vanishes, while the second term is general nonzero. So the locations of \(r_{ps}\) and \(r_h\) are different. However, when a black hole has more than one horizon, there will be the extremal black hole case, where two horizons coincide. Or for the extremal black hole case, we both have \(f(r_h) = 0\) and \(f(r_h)' = 0\). Significantly, condition (6) is satisfied. So the PS and the extremal black hole horizon naturally coincide for the extremal black hole.

In order to study the topological property of the PS, we introduce the everywhere regular potential function \[14\]

\[H(r, \theta) = \sqrt{-g_{tt}} = \frac{1}{\sin \theta} \left(\frac{f(r)}{h(r)}\right)^{\frac{1}{2}}.\tag{7}\]

Obviously, the radius of the PS locates at the root of \(\partial_r H=0\). Similar to Ref. 17, we can introduce a vector field \(\phi = (\phi^r, \phi^\theta)\) 24

\[
\phi^r = \frac{\partial_r H}{\sqrt{g_{rr}}} = \sqrt{g(r)} \partial_r H, \quad \phi^\theta = \frac{\partial_\theta H}{\sqrt{g_{\theta\theta}}} = \frac{\partial_\theta H}{\sqrt{h(r)}}.\tag{8}\]

Although the circular photon orbit for a spherically symmetric black hole is a PS, which is independent of the coordinate \(\theta\), here we aim to investigate the topological property of the circular photon orbit, so we preserve \(\theta\) in our discussions. Note that the vector can also be reformulated as

\[\phi = ||\phi|| e^{i\phi},\tag{9}\]

where \(||\phi|| = \sqrt{\phi^r \phi^r}\). However, in terms of \(\phi\), a PS occurs at \(\phi=0\). This implies that \(\phi\) in (9) is not well defined for the PS, so we treat the vector as \(\phi = \phi^r + i\phi^\theta\). The normalized vectors are defined as

\[a^a = \frac{\phi^a}{||\phi||}, \quad a = 1, 2,\tag{10}\]

with \(\phi^1 = \phi^r\) and \(\phi^2 = \phi^\theta\).

\section{Topological Current and Charge}

In this section, by treating the PSs as the defects located at the zero points of \(\phi\), we will study their topological current and charge following Duan’s \(\phi\)-mapping topological current theory.

At first we define a superpotential

\[V^{\mu\nu} = \frac{1}{2\pi} e^{\mu\nu\rho} \epsilon_{ab} \partial_\mu a^b, \quad \mu, \nu, \rho = 0, 1, 2.\tag{11}\]
Here \( x^\mu = (t, r, \theta) \). Note that one can reformulate the coordinate \( t \) with other black hole parameters as we will show in what follows. It is clear that the superpotential is an antisymmetric tensor \( V^{\mu \nu} = -V^{\nu \mu} \). Employing the superpotential, we introduce a topological current

\[
j^\mu = \partial_\nu V^{\mu \nu} = \frac{1}{2\pi} \epsilon^{\mu \nu \rho} \epsilon_{ab} \partial_\nu n^a \partial_\rho n^b. \quad (12)
\]

It is easy to find that this topological current satisfies

\[
\partial_\mu j^\mu = 0. \quad (13)
\]

The component \( j^0 \) is the charge density. Integrating it, we will obtain the topological charge at given \( \Sigma \),

\[
Q = \int_\Sigma j^0 d^2 x. \quad (14)
\]

In the next, we aim to uncover the characteristic property of the topological current \( j^\mu \). Inserting (10) into (12), we have

\[
j^\mu = \frac{1}{2\pi} \epsilon^{\mu \nu \rho} \epsilon_{ab} \left( \frac{\partial}{\partial \phi^a} \ln ||\phi|| \right) \partial_\nu \phi^a \partial_\rho \phi^b. \quad (15)
\]

Note that \( \frac{\partial \ln ||\phi||}{\partial \phi^a} = \frac{\delta^a}{||\phi||} \), we can express the topological current as

\[
j^\mu = \frac{1}{2\pi} \epsilon^{\mu \nu \rho} \epsilon_{ab} \left( \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^b} \ln ||\phi|| \right) \partial_\nu \phi^a \partial_\rho \phi^b. \quad (16)
\]

In term of the Jacobi tensor

\[
\epsilon^{ab} J^\mu \left( \frac{\phi}{x} \right) = \epsilon^{\mu \nu \rho} \partial_\nu \phi^a \partial_\rho \phi^b, \quad (17)
\]

we get

\[
j^\mu = \frac{1}{2\pi} \left( \Delta_{\phi^a} \ln ||\phi|| \right) J^\mu \left( \frac{\phi}{x} \right), \quad (18)
\]

where \( \Delta_{\phi^a} = \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^a} \). Using the two-dimensional Laplacian Green function in \( \phi \)-mapping space

\[
\Delta_{\phi^a} \ln ||\phi|| = 2\pi \delta(\phi), \quad (19)
\]

we have the topological current

\[
j^\mu = \delta^2(\phi) J^\mu \left( \frac{\phi}{x} \right). \quad (20)
\]

Multiplying it by \( \epsilon^{\lambda \sigma \rho} \), one can arrive

\[
\frac{dx^\mu}{J^\mu \left( \frac{\phi}{x} \right)} = \frac{dx^\nu}{J^\nu \left( \frac{\phi}{x} \right)}. \quad (22)
\]

After a simple calculation, we have

\[
u^i = \frac{dx^i}{J^0 \left( \frac{\phi}{x} \right)} = \frac{J^i(\phi)}{J^0(\phi)}. \quad (23)
\]

Then from (20), the components of \( j^\mu \) can be expressed in the following form

\[
\begin{align*}
\delta^2(\phi) J^0 \left( \frac{\phi}{x} \right) u^i, & \quad (24) \\
\delta^2(\phi) J^0 \left( \frac{\phi}{x} \right), & \quad (25)
\end{align*}
\]

Therefore, the topological charge reads

\[
Q = \int_\Sigma \delta^2(\phi) J^0 \left( \frac{\phi}{x} \right) d^2 x. \quad (26)
\]

Due to the \( \delta \) function, the charge only does not vanish at the zero point of \( \phi \), where the PS locates at, so we can assign each PS with a topological charge, i.e., the winding number. Significantly, if the boundary \( C = \partial \Sigma \) in the manifold encloses no zero point of \( \phi \) we must have \( Q=0 \). Or if two different closed curves enclose the same zero points, the corresponding topological charges must equal. On the other hand, taking \( \Sigma \) as the manifold of \( x^i \) for certain \( t \), it will give the total topological charge of black hole PSs. So \( Q \) can be used to character different spaces.

Now let examine the inner structure of the topological charge. Considering there are \( N \) zero points of \( \phi \) and the Jacobi determinant \( J^0 \left( \frac{\phi}{x} \right) \neq 0 \), the solution of \( \phi=0 \) can be expressed as

\[
x^i = z^i_n(t), \quad n = 1, 2, \ldots, N. \quad (27)
\]

Near the zero points of \( \phi \), \( \delta^2(\phi) \) can be expressed as

\[
\delta^2(\phi) = \sum_{n=1}^{N} \alpha_n J^0 \left( \frac{\phi}{x} \right) \bigg|_{x=z_n}, \quad (28)
\]

where \( \alpha_n \) is positive expanding coefficients.

According to Duan’s topological current theory [19, 20], the winding number of the \( n \)-th zero point is expressed as \( w_n = w(\phi, z_n) = \alpha_n J^0 \left( \frac{\phi}{x} \right) \bigg|_{x=z_n} \). Considering \( \alpha_n \) is positive, we have

\[
\alpha_n = \frac{|w(\phi, z_n)|}{|J^0 \left( \frac{\phi}{x} \right) \bigg|_{x=z_n}|}. \quad (29)
\]
The \( \phi \)-mapping Hopf index \( \beta_i \) and the Brouwer degree \( \eta_i \) at zero point \( z_n \) are, respectively, given by

\[
\beta_n = |w(\phi, z_n)|, \quad \eta_n = \frac{J^0 \left( \frac{\phi}{x} \right)}{|J^0 \left( \frac{\phi}{x} \right)|}_{x=z_n}.
\]

(30)

Thus,

\[
J^0 \left( \frac{\phi}{x} \right) \delta^2(\phi) = \sum_{n=1}^{N} \beta_n \eta_n \delta^2(x - z_n).
\]

(31)

Finally, the topological charge can be expressed as

\[
Q = \sum_{n=1}^{N} w_n = \sum_{n=1}^{N} \beta_n \eta_n.
\]

(32)

This relation reflects the inner structure of the topological charge.

As shown above, we suppose \( J^0 \left( \frac{\phi}{x} \right) \neq 0 \). However, if this condition violates, there will be the phenomenon, the generation or annihilation. In order to show it, we suppose at least one component of the Jacobi tensor does not vanish, say \( J^1 \left( \frac{\phi}{x} \right) \neq 0 \). Therefore, according to (23), we have

\[
\frac{dx^1}{dt}_{(t_*, z_n)} = \frac{J^1 \left( \frac{\phi}{x} \right)}{J^0 \left( \frac{\phi}{x} \right)}_{(t_*, z_n)} = \infty,
\]

which gives

\[
\frac{dt}{d\phi^1}_{(t_*, z_n)} = 0
\]

(34)

Then at the critical point \( (t_*, z_n) \), we have the following Taylor expansion

\[
t - t_* = \frac{1}{2} \frac{d^2 t}{d(x^1)^2} |_{(t_*, z_n)} (x^1 - z_n^1)^2.
\]

(35)

Because that the topological current is identically conserved, the topological charge of these two defects must be opposite at the critical point. When \( \frac{d^2 t}{d(x^1)^2} |_{(t_*, z_n)} < 0 \) or \( > 0 \), it represents an annihilation or generation [22].

IV. BLACK HOLES AND TOPOLOGICAL CHARGE

As shown above, the topological charge \( Q \) equals the sum of the winding number of each zero point of \( \phi \) for given \( \Sigma \). For a black hole, we takes \( \Sigma \) as a full exterior region outside of the outer horizon. This will give us the total topological charge of the black hole PSs, and which can be used to be characterized different spacetime.

A. Asymptotically flat black holes

Here we first consider an asymptotically flat black hole with the solution described by (21). At \( r \to \infty \), the metric functions have the following asymptotic behaviors

\[
f(r) \sim 1 - \frac{1}{r} + O \left( \frac{1}{r^2} \right),
\]

(36)

\[
g(r) \sim 1 - \frac{1}{r^2} + O \left( \frac{1}{r^3} \right),
\]

(37)

\[
h(r) \sim r^2.
\]

(38)

Considering the \( i \)-th zero point of \( \phi \) is enclosed by a piece-wise smooth and positive oriented closed curve \( C_i \), while other zero points are out of it. The winding number of the vector is

\[
w_i = \frac{1}{2\pi} \oint_{C_i} d\Omega,
\]

(39)

where \( \Omega = \arctan(\phi^2/\phi^1) \). Then the total charge will be

\[
Q = \sum_i w_i.
\]

(40)

On the other hand, due to the \( \delta \) function in the topological current (25), the total charge of the black hole system can be calculated as

\[
Q = \frac{1}{2\pi} \oint_{C} d\Omega,
\]

(41)

where \( C = \sum_i \cup l_i \) or the union of four line segments \( l_1 \sim l_4 : \{ r = \infty, 0 \leq \theta \leq \pi \} \cup \{ \theta = \pi, r_h \leq r < \infty \} \cup \{ r = r_h, 0 \leq \theta \leq \pi \} \cup \{ \theta = 0, r_h \leq r < \infty \} \), see Fig. 1 with the black arrows denoting the direction. In order to calculate \( Q \), we examine the vector field \( \phi^n \) on these line segments. For simplicity, we list \( \phi^n \)

\[
\phi^r = \frac{hf'-fh'}{2h\sin^2 \theta} \sqrt{g},
\]

(42)

\[
\phi^\theta = -\frac{\sqrt{g} \cos \theta}{h\sin^2 \theta}.
\]

(43)

At horizon, one has \( f(r_h) = g(r_h) = 0 \), while \( \sqrt{g}/f \) keeps finite. Therefore, we arrive

\[
\phi^r (r \to r_h^+) > 0, \quad \phi^\theta (r \to r_h^+) \to 0,
\]

(44)

where \( f'(r_h) > 0 \) is used. So as shown in Fig. 1, the vector \( \phi \) is horizontal to the right at the horizon and thus \( \Omega_{l_1} = 0 \). At \( \theta = 0 \) and \( \pi \), we, respectively, have

\[
\phi^r_1 (\theta \to 0^+) \sim \frac{1}{\pi}, \quad \phi^\theta_1 (\theta \to 0^+) \sim -\frac{1}{\pi},
\]

\[
\phi^r_2 (\theta \to \pi^-) \sim \frac{1}{\pi - \theta}, \quad \phi^\theta_2 (\theta \to \pi^-) \sim \frac{1}{(\pi - \theta)^2}.
\]

(45)

(46)

So we have \( \Omega_{l_1} = \frac{\pi}{2} \) and \( \Omega_{l_4} = -\frac{\pi}{2} \). These imply that along line segments \( l_{2,3,4} \), \( \Omega \) does not change, so one has
\[ \Delta \Omega_2 = \Delta \Omega_3 = \Delta \Omega_4 = 0, \text{ and thus } \Omega_3 = \Omega_4 = -\frac{\pi}{2}. \]

Last, let us consider \( \Omega \) along \( l_1 \). When \( r \to \infty \),

\[ \phi_{l_1}^r(r \to \infty) \propto -\frac{1}{r^2 \sin^2 \theta}, \quad \phi_{l_1}^\theta(r \to \infty) = -\frac{\cos \theta}{r^2 \sin \theta} \quad (47) \]

Considering that both \( \phi_{l_1}^r \) and \( \phi_{l_1}^\theta \) are negative, we have \( \Omega_{l_1} = \pi + \arctan(\cot \theta) \). When \( \theta \) varies from 0 to \( \pi \), \( \Omega_{l_1} \) monotonically decreases from \( 3\pi/2 \) to \( \pi/2 \), or from \( -\pi/2 \) to \( \pi/2 \), which means that the vector \( \phi \) changes smoothly at points \( (r, \theta) = (\infty, 0) \) and \( (\infty, \pi) \). So we have \( \Omega_1 = \Omega_2 = 0 \) and

\[ \Delta \Omega_{l_1} = \int_{l_1} d\Omega = -\pi. \quad (48) \]

Therefore, the topological charge is

\[ Q = \frac{1}{2\pi} \left( \Delta \Omega_{l_1} + \Delta \Omega_{l_2} + \Delta \Omega_{l_3} + \Delta \Omega_{l_4} \right) + \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 = -1. \quad (49) \]

### B. Asymptotically AdS black holes

Here we consider the asymptotically AdS black hole case. We assume the metric functions have the following behaviors

\[ f(r) \sim \frac{r^2}{l^2} + 1 - \frac{1}{r} + O \left( \frac{1}{r^2} \right), \quad (50) \]

\[ g(r) \sim \frac{r^2}{l^2} + 1 - \frac{1}{r} + O \left( \frac{1}{r^2} \right), \quad (51) \]

\[ h(r) \sim r^2, \quad (52) \]

where \( l \) is the AdS radius. Note that this asymptotical behaviors only modify the boundary condition at \( r \to \infty \), so the calculation along \( l_2, l_3, \) and \( l_4 \) is the same as the asymptotically flat case. When \( r \to \infty \), we have for a finite \( l \)

\[ \phi_{l_1}^r(r \to \infty) \propto -\frac{1}{r^2 \sin \theta}, \quad \phi_{l_1}^\theta(r \to \infty) = -\frac{\cos \theta}{lr \sin^2 \theta} \quad (53) \]

A detailed analysis shows that \( \Omega \) keeps constant value \( -\frac{\pi}{2} \) for \( \theta \in (0, \pi) \), and constant value \( \frac{\pi}{2} \) for \( \theta \in (\frac{\pi}{2}, \pi) \). Therefore \( \Omega \) only changes at \( \theta = \frac{\pi}{2} \), which gives \( \Delta \Omega_{l_1} = -\pi \). Note that the minus sign comes from that the angle changes in clockwise rotation at \( \theta = \frac{\pi}{2} \). Similarly, we still have \( \Omega_1 = \Omega_2 = 0 \). Then combining with the value of \( \Omega \) along other line segments, we obtain

\[ Q = -1, \quad (54) \]

which is the same as the asymptotically flat black hole case.

### C. Asymptotically dS black holes

Now we turn to the asymptotically dS black hole case. The metric functions will have the following behaviors

\[ f(r) \sim \frac{r^2}{l^2} + 1 - \frac{1}{r} + O \left( \frac{1}{r^2} \right), \quad (55) \]

\[ g(r) \sim \frac{r^2}{l^2} + 1 - \frac{1}{r} + O \left( \frac{1}{r^2} \right), \quad (56) \]

\[ h(r) \sim r^2. \quad (57) \]

Different from the asymptotically flat and AdS cases, besides the black hole horizon, the spacetime admits a cosmological horizon with radius \( r_c > r_h \), which is also a root of \( g(r_c) = f(r_c) = 0 \). So for this case, we only consider the case \( r_h \leq r \leq r_c \). Similarly, we also only need to consider the line segment \( l_1 \). Considering \( g(r_c) = f(r_c) = 0 \), we have

\[ \phi_{l_1}^r(r \to r_c^-) < 0, \quad \phi_{l_1}^\theta(r \to r_c^-) = 0 \quad (58) \]

Therefore, along \( l_1 \), the direction of \( \phi \) is horizontal to the left, which indicates that \( \Omega_1 = \pi \) or \( -\pi \). Therefore \( \Delta \Omega_{l_1} = 0 \) and \( \Omega_1 = \Omega_2 = -\frac{\pi}{2} \). Finally, we achieve

\[ Q = \frac{1}{2\pi} \left( \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 \right) = -1, \quad (59) \]

where \( \Delta \Omega_{l_i} = 0 \ (i=1-4) \) is considered.

In summary, we in this section calculate the topological charge of PS for the asymptotically flat, AdS, and dS black holes in GR. The results show that all them have a total topological charge \( Q=1 \).

Before ending this section, we would like to give several comments on this result. First, since all the black holes with different asymptotical behaviors share the same value of topological charge, they are in the same topological class from the viewpoint of the topology of the black hole PS. Second, there exists at least one unstable PS as suggested in Ref. [17]. However the number
of the PSs are not limited to one. Or one can say that these black holes have an odd number of PSs. While there is always one more unstable photon sphere than the stable photon sphere. Third, as indicating from Ref. [17], the black hole spin does not change the topological charge, so we conjecture Kerr-AdS/dS black holes also have $Q=-1$, and thus they are in the same topological class. Moreover, if other black hole parameters do not affect the asymptotical behaviors, the topological charge keeps unchanged.

V. Dyonic BLACK HOLES

Here we would like to consider a specific case, the Dyonic black hole solution [23], which has rich horizon and PS structures.

The action describing an asymptotically flat solution is

$$S = \frac{1}{16\pi} \int \sqrt{-g} d^4 x \left( R - \alpha_1 F^2 - \alpha_2 \left( (F^2)^{\bar{4}} - 2F(4) \right) \right),$$

where the field strength $F^2 = -F^\mu_\nu F^\nu_\mu$ and $F(4) = F^\mu_\nu F^\nu_\rho F^\rho_\sigma$. Solving the field equation, an exact solution of static and spherically-symmetric dyonic black hole reads [23]

$$ds^2 = -f(r) + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

$$f(r) = 1 - \frac{2M}{r} + \frac{\alpha_1 \mu^2}{r^2} + \frac{q^2}{\alpha_1 \mu^2} \frac{r}{F_1(1,4,1,5; -4p^2 \alpha_2)}.$$  \hspace{1cm} (60)

It is clear that each black hole is characterized by a set of parameters $(M, q, p)$, which is associated with the mass, electric and magnetic charges, respectively. This black hole solution also satisfies the dominant energy condition.

Combining with (1), we see that for this black hole, $g(r) = f(r)$, $h(r) = r^2$, so our discussion in the last section holds. Then the normalized vectors are

$$n^r = \frac{rf' - 2f}{\sqrt{(rf' - 2f)^2 + 4f \cot^2 \theta}},$$

$$n^\theta = \frac{2 \sqrt{f} \cot \theta}{\sqrt{(rf' - 2f)^2 + 4f \cot^2 \theta}}.$$  \hspace{1cm} (61)

Moreover since this black hole is an asymptotically flat solution, we must have the total topological charge $Q=1$. In the following we will perform the detailed study of the topological property of the black hole PS.

A. Case one: $M = \frac{\mu}{343}, p = \sqrt{\frac{396}{343}}, \alpha_1 = 1, \alpha_2 = \frac{196249}{1584}, q = 6.65$

Under this case, the black hole has two horizons located at $r_{h1} = 0.3873$ and $r_{h2} = 7.6249$.

The normalized vector $n$ is shown in Fig. 2. It is clear that there exists a zero point of $n$ near $r = 13.5$ and $\theta = \pi/2$. Solving the equation of PS [4], we find this black hole has only one PS at $r_{ps} = 13.4041$, which exactly coincides with the zero point of $\phi$. Since there is only one sphere photon, the topological charge or the winding number associated with this PS must be $Q = -1$. Any closed curve enclosing this PS will produce $Q = -1$, and other closed curves will give $Q = 0$. For specific example, we would like to calculate the topological charge (59) for the vector along closed circular curves $C_1$ and $C_2$.

Considering that

$$\Omega = \arctan \left( \frac{\phi^2}{\phi^3} \right) = \arctan \left( \frac{n^2}{n^1} \right).$$  \hspace{1cm} (63)

Then, we obtain

$$d\Omega = \frac{n^1 dn^2 - n^2 dn^1}{(n^1)^2 + (n^2)^2} = \epsilon_{ab} n^a dn^b.$$  \hspace{1cm} (64)

Therefore, the topological charge can be rewritten as

$$Q = \frac{\Delta \Omega}{2\pi} = \frac{1}{2\pi} \oint_{C} \epsilon_{ab} n^a \partial_i n^b dx^i.$$  \hspace{1cm} (65)

Here we parameterize the closed curves $C_1$ and $C_2$ by the angle $\vartheta \in (0, 2\pi)$ as

$$\begin{align*}
   r &= a \cos \vartheta + r_0, \\
   \theta &= b \sin \vartheta + \frac{\pi}{2}.
\end{align*}$$  \hspace{1cm} (66)

We choose $(a, b, r_0) = (0.3, 0.3, 13.4041)$ for $C_1$, and $(0.3, 0.3, 14.5)$ for $C_2$. Then we calculate $\Delta \Omega$ along $C_1$ and $C_2$ by using (65). The result is shown in Fig. 3. For $C_1$, see Fig. 3(a), we find $\Delta \Omega$ decreases with $\vartheta$, and approaches $-2\pi$ at $\vartheta = 2\pi$. Thus the topological charge of the vector along $C_1$ is $Q = -1$. While for $C_2$ shown in Fig. 3(b), $\Delta \Omega$ first decreases, then increases, and finally decreases again. After a loop, $\Delta \Omega$ vanishes, and which implies that $Q = 0$ for $C_2$. The difference between them is because that $C_1$ encloses a single PS, while $C_2$ does not. This also reflects the $\delta$ function encoded in the topological current [26].

This result also confirms our conclusion that the topological charge $Q=-1$ for the asymptotically flat black holes. Note that in Ref. [17], the authors dubbed the PSs with $Q=-1$ or 1 as the standard or exotic PSs, respectively. Actually, we can find that the PS with $Q=-1$ is unstable, while the one with $Q=1$ is stable.  

B. Case two: $M = \frac{\mu}{343}, p = \sqrt{\frac{396}{343}}, \alpha_1 = 1, \alpha_2 = \frac{196249}{1584}, q = 6.85$

Generally, a black hole possesses one PS. However, for this kind black hole, in some parameter regions, it can have more than one PS [17]. This provides us a good
FIG. 2: Behavior of the normalized vector $n$ on the $(r, \theta)$ plane for case one. Black dot denotes the zero point of $n$. $C_1$ and $C_2$ are two closed curves.

FIG. 3: $\Delta \Omega$ as a function of $\theta$. (a) for $C_1$, and (b) for $C_2$.

opportunity to study the topological properties of black hole PS when its number is larger than one.

Here we focus on case two: $M = \frac{67}{10}$, $p = \sqrt[3]{\frac{206}{143}}$, $\alpha_1 = 1$, $\alpha_2 = \frac{196249}{22307}$, $q = 6.85$. Solving $f(r_h)=0$, we find that the black hole has two horizons located at $r_h = 0.6302$ and $1.6643$, respectively. Solving the PS equation, we observe three PSs at $r_{ps1} = 2.3066$, $r_{ps2} = 6.3340$, and $r_{ps3} = 12.5075$.

In order to clearly display the behavior of vector $n$, we describe it on the $(r, \theta)$ plane in Fig. 1. We observe that there are three zero points of $n$, which exactly coincide with the locations of PSs. So PSs can be treated as topological defects of the black hole system even when more PSs are included. We show the local behaviors of $n$ near these three zero points in Figs. 4(b), 4(c) and 4(d). Obviously, the vector $n$ has similar behaviors around point $P_1$ and $P_3$. While near $P_2$, it like the electric field of a positive charge. So we conjecture that the topological charges associated with $P_1$ and $P_3$ are the same, while different from that of $P_2$.

To examine the topological charges associated with these three PSs, we construct three closed curves enclosed by $C_{3,4,5}$ shown in Fig. 4(a), respectively. Moreover, we also construct a large closed curve $C_6$, which contains these three PSs. These parameterized forms share the same expression given in Eq. 66 while with different coefficients given in Table I. Then we numerically calculate $\Delta \Omega$ along these curves. The results are given in Fig. 8. With the increases of $\theta$, $\Delta \Omega$ decreases along $C_3$ and $C_5$, and increases along $C_4$. For complete closed curves, we easily observe that the topological charge $Q = -1$ for $P_1$ and $P_3$, and $Q = 1$ for $P_2$. This confirms our conjecture that $P_1$ and $P_3$ has the same topological charge, while different from $P_2$. So we can say that the PSs at $P_1$ and $P_3$ are standard, while the one at $P_2$ is exotic.

On the other hand, since the black hole has three PSs and the total topological charge for its PSs is minus one, any closed curves enclosing these three points must produce $Q = -1$. To check this result, we calculate $\Delta \Omega$ along $C_6$. Nevertheless, $\Delta \Omega$ exhibits a nonmonotonic behavior with $\theta$, it gives $\Delta \Omega = -2\pi$ after a loop, which indicates $Q = 1$. Therefore, it is consistent with our analysis in Sec. IV A for the asymptotically flat black holes.

Interestingly, we can also construct another closed curve, see $C_7$ shown in Fig. 8(a) which only encloses points $P_1$ and $P_3$. The curve $C_7$ is constructed by four segments, $I_1-I_4$, which are also parameterized as (66) with the coefficients given in Table II. Here we show $\Delta \Omega$ as $C_7$’s length parameter rather $\theta$ in Fig. 6(b). Circulating the contour $C_7$ anti-clockwise $\Delta \Omega$ approaches $-4\pi$.

|   | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_9$ | $C_{10}$ |
|---|------|------|------|------|------|------|------|------|------|--------|
| a | 0.3  | 0.3  | 0.5  | 0.5  | 0.5  | 5.7  | 1.0  | 1.0  | 4.0  |
| b | 0.3  | 0.3  | 0.1  | 0.1  | 0.1  | 0.8  | 0.1  | 0.1  | 0.5  |
| $r_0$ | 13.40 | 14.50 | 2.31 | 6.33 | 12.51 | 9.42 | 7.64 | 11.53 | 9.58 |

TABLE I: Parametric coefficients of closed curves $C_i$ ($i=1-6, 8-10$).

|   | $a$ | $b$ | $r_0$ | $\theta$ |
|---|-----|-----|-------|----------|
| $I_1$ | 0.5 | 0.2 | 12.51 | $(\pi, 2\pi)$ |
| $I_2$ | 5.6 | 0.8 | 7.41  | $(0, \pi)$ |
| $I_3$ | 0.5 | 0.2 | 2.31  | $(\pi, 2\pi)$ |
| $I_4$ | 4.6 | 0.6 | 7.41  | $(\pi, 0)$ |

TABLE II: Parametric coefficients of closed curves $C_7$. 
FIG. 4: Behavior of the normalized vector $n$ on the $(r, \theta)$ plane for case two. Black dots denotes the zero points of $n$. (a) Three zero points are completely shown. (b) Behavior of $n$ near zero point $P_1$. (c) Behavior of $n$ near zero point $P_2$. (d) Behavior of $n$ near zero point $P_3$.

FIG. 5: $\Delta \Omega$ as a function of $\vartheta$ for $C_3$, $C_4$, $C_5$, and $C_6$.

FIG. 6: (a) Schematic diagram of closed curve $C_7$. (b) $\Delta \Omega$ as a function of length parameter $\lambda$.

Hence, we have the topological charge $Q=-2$ for $C_7$, which is just the sum of the charges of $C_3$ and $C_5$ as expected.

C. Case three: $M = \frac{67}{10}$, $p = \sqrt{\frac{106}{144}}$, $\alpha_1 = 1$, $\alpha_2 = \frac{106249}{1484}$, $q = 7$.

As shown above, the topological charge $Q=-1$ for the dyonic black holes in case one and two. So these black hole cases are in the same topological class, regardless of the number of PS. Here we wonder whether there exists different structure of PS from the topology. For the
FIG. 7: (a) Behavior of the normalized vector $n$ on the $(r, \theta)$ plane for case three. (b) $\Delta \Omega$ as a function of $\vartheta$ for $C_8$, $C_9$, and $C_{10}$.

Moreover, it is worth pointing out that when $q=6.92$, the naked singularity has four PSs locating at $r_{ps}=1.1358, 1.5794, 6.9090, \text{and} 12.1041$ with the topological charge $1, -1, 1, -1$. And the total topological charge still equals zero.

VI. PHOTON SPHERES ANNIHILATION

From the topology, two defects of opposite topological charge can annihilate, or generate from vacuum with time. For an equilibrium black hole, its PS will be determined and thus no the phenomenon of the generation or annihilation exists. However, when considering the Hawking radiation, the black hole gradually loses its mass and other charges, the PS will be changed. Here we image a black hole system absorbs the charge from its surroundings through a quasistatic process. With the continuous increase of the charge, black hole will turn to a naked singularity. In this case, the electric charge plays the role of evolution time, so we adopt $x^q = q$. In the following, we would like to examine the change of the PS.

For simplicity, we set $\alpha_2=0$. Then the black hole solution in (60) actually describes a charged Reissner-Nordström black hole. The black hole horizons locate at $r_{h1,2} = M \pm \sqrt{M^2 - q^2}$, and the PSs are at $r_{ps} = 1/2(3M \pm \sqrt{9M^2 - 8q^2})$. For black hole case $M > q$, we always have $r_{ps} < r_{h1}$, which means that the inner PS is covered by the outer horizon. Thus only one PS is allowed, which is a standard one and has topological charge $Q=1$. When the black hole is over charged $M < q$, these two horizons disappear. Then the second PS of positive topological charge $Q=1$ at $r_{ps}$ takes action, which leads to $Q = 0$ for the naked singularity. Thus from a black hole to a naked singularity, spacetime undergoes a topological phase transition. We describe the case in Fig. 8.

For naked singularity, with the increase of the charge $q$, these two PSs approach and annihilate at $q = q_*=3M/(2\sqrt{2})$. From the viewpoint of topology, this phenomenon denotes an annihilation of two PSs. Meanwhile the total topological charge does not change.

Further, expanding the charge near $q_*$, we have

$$q - q_* = \frac{1}{2}q''(r - r_{ps}^*)^2,$$  \hfill (67)

$$q'' = \frac{d^2}{dr^2} \bigg|_{(q_*, r=r_{ps}^*)} = -\frac{\sqrt{2}(r - r_{ps}^*)^2}{3M^2}.$$  \hfill (68)

Interestingly, one observes a negative $q''$. So this annihilation process is in accord with that from the topological current $\phi$-mapping theory discussed in [22].

On the other hand, if taking the decrease of $q$ as the evolution direction, this pattern will become the generation of PSs.

VII. DISCUSSIONS AND CONCLUSIONS

In this paper, we studied the topological property of PS for static, spherically symmetric black hole with different
asymptotical behaviors in GR.

Following Duan’s topological current $\phi$-mapping theory, we introduced the superpotential $V^{\mu\nu}$ and topological current $j^\mu$ defined on the $(r, \theta)$ plane outside of a black hole. After some calculations, we expressed the topological charge $Q$ of PSs in terms of a $\delta$ function, see (26), which indicates that black hole PS’s topological property is closely dependent of the zero points of vector field $\phi$ enclosed by the considering parameter region. The inner structure of the topological charge is also investigated.

Then we computed the topological charge for a general black hole solution with asymptotically flat, AdS, and dS boundaries. Here we summarize two universal properties: i) $\Omega_3 = \Omega_4 = -\frac{\pi}{2}$, and $\Delta \Omega_2 = \Delta \Omega_3 = \Delta \Omega_4 = 0$. ii) $\Omega_1 + \Omega_2 + \Delta \Omega_3 = -\pi$. Then after carrying out the detailed analysis, we find that all these black hole systems have the same topological charge $Q=-1$. This suggests that these black holes have at least one standard PS, the same as that for the stationary, axi-symmetric, asymptotically flat black hole [17]. If there are several PSs, the number of standard PS should be one more than the exotic ones. Considering that the black hole spin might not change the total topological charge indicating in Ref. [17], we conjectured that the rotating black holes with asymptotically flat, AdS, and dS boundaries also admit $Q=-1$. This issue is worth for further pursuing.

As a specific example, we computed the topological charge for the dyonic black holes with $M = \frac{\alpha_1}{10}$, $p = \sqrt{\frac{196}{181}}$, $\alpha_1 = 1$, $\alpha_2 = \frac{196249}{1884}$, while with different electric charge $q$. For the case one, $q=6.65$, the black hole exhibits only one standard PS. We clearly showed that for the closed curve enclosing the PS, one has $Q=-1$. While for other closed curves, it produces $Q=0$. This reflects the property of $\delta$ function containing in the topological current (26). For the second case, the electric charge is set to $q=6.85$. The black hole admits more than one PS, and three of them are observed. Two are standard ones associated with $Q=-1$ and one is exotic one with $Q=1$. The total topological charge still keeps $-1$. Therefore case two and case one are in the same topological class. Case three with $q=7$ corresponds a naked singularity rather a black hole. Its topological charge is found to be $Q=0$, which implies that the standard and exotic PSs come in pairs. For example, when $q=7$, we observed a pair of standard and exotic PSs. However two pairs will present when $q=6.92$. The different total topological charges of black holes and naked singularities also indicate that they are in different topological classes.

Finally, we took the charge $q$ as an evolution parameter and investigated the annihilation of the standard and exotic PSs. The result is also in accord with the Duan’s topological current $\phi$-mapping theory.

Our result confirms that static, spherically symmetric black holes with different asymptotically flat, AdS, and dS behaviors have at least one standard PS from the view point of topology. This topological approach provides us a novel insight into the black hole PS structure. Topological phase transition can also be revealed by the charge $Q$. So we expect the topology of black hole PS can play a novel role in investigating the astronomical phenomena related with the black hole PS.

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