M-Branes: Lessons from M2’s and Hopes for M5’s

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In this talk we will review the construction of M2-brane SCFT’s highlighting some novelties and the role of 3-algebras. Parts of our discussion will closely follow parts of [1]. Next we will discuss M5-branes: the basics, the obstacles as well as various attempts to construct the associated SCFT and potential relations between M2-branes and M5-branes.

1. Introduction

I was asked to give a review talk on the construction of the M2-brane SCFT’s and also to detail the issues and problems associated to the infamous M5-brane SCFT’s. Not wanting to feel left out I also tried to add something of my own recent work which I hoped would be of interest to my colleagues at the conference. Therefore the plan of this talk is split into three themes:

i) M2-branes and 3-algebras
ii) M5-branes and the (2,0) theory
iii) A (2,0) system

The aim of the first theme is to review the construction of the M2-brane Chern–Simons SCFT’s with a view to emphasizing the role of 3-algebras. However I also want to point out that although we have Lagrangian descriptions for arbitrary numbers of M2-branes in many eleven-dimensional backgrounds, these Lagrangians do not have all the symmetries that one expects. Instead these only arise in the quantum theory at strong coupling through non-perturbative operators. In the second theme I will discuss the objections to obtaining a Lagrangian description of M5-branes and review a few attempts to define the associated SCFT using lower dimensional Lagrangian theories. Finally in the remaining theme I would like to present an explicit representation of the (2,0) super-algebra on a set of fields and show that by making certain choices for the solutions to the constraints one recovers various Lagrangian descriptions of M5-branes and M2-branes.

On the other hand there is much work on M5-branes that I will not discuss. Not out of a lack of interest but out of a lack of time and knowledge. Amongst the plethora of work that I will not mention are:

i) Results arising from reduction to 4D and below such as novel non-Lagrangian field theories, dualities, surface operators, AGT etc (e.g. [2], …)
ii) Bootstrap results for M5-branes (e.g. [3], …)
iii) AdS$_7$/CFT$_6$ (e.g. [4,5], …)

The moral of this talk is that although M2-branes are essentially a done deal there are details in the fine print that could hold lessons for M5-branes. And furthermore although an explicit M5-brane construction from some Lagrangian-like system seems unlikely there is still hope that novel techniques and physics can emerge and that we will learn new things about quantum field theory.

2. M2-Branes

The M2-brane SCFT arises as the strong coupling limit of $N$ D2-branes. These are described in the decoupling limit by 3D maximally supersymmetric Yang–Mills (MSYM) with gauge group $U(N)$. The strong coupling limit corresponds to the IR limit. However the lift to M-theory implies that at strong coupling through non-perturbative operators. In the second theme I will discuss the objections to obtaining a Lagrangian description of M5-branes and review a few attempts to define the associated SCFT using lower dimensional Lagrangian theories. Finally in the remaining theme I would like to present an explicit representation of the (2,0) super-algebra on a set of fields and show that by making certain choices for the solutions to the constraints

$$SO(1, 2)_L \times SO(7)_R \to SO(1, 2)_L \times SO(8)_R.$$ (1)

here $L$ stands for Lorentz symmetry and $R$ for R-symmetry. Ultimately these are subgroups of the ten and eleven-dimensional Lorentz groups. Therefore the standard type IIA/M-theory dictionary predicts that there is a 3D SCFT with maximal supersymmetry and $SO(8)$ R-symmetry corresponding to the IR limit of 3D super-Yang–Mills. Thus although we started with string theory and M-theory we have reached a conclusion that is simply about gauge theory and QFT. A prediction so to speak.

The relevant Lagrangians for these theories have now been constructed. The first example with maximal ($\mathcal{N} = 8$) manifest supersymmetry is BLG.[6,7] It is a Chern–Simons-matter theory with gauge group $SU(2) \times SU(2)$ or $(SU(2) \times SU(2))/\mathbb{Z}_2$. However it is limited in that it only describes two or three M2’s on an orbifold.

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For arbitrary number of M2-branes one has $\mathcal{N} = 6$ manifest SUSY and the ABJM or ABJ models. Here one gives up manifest maximal supersymmetry and instead has only 12 supercharges. It is again a Chern–Simons–matter theory but with gauge group $U(M) \times U(N)$ and it describes $N \leq M$ branes in an eleven-dimensional orbifold.

There is now a zoology of Chern–Simons–matter theories with extended SUSY $\mathcal{N} = 4, 5, 6, 8$ corresponding to a motley list of gauge groups. For $\mathcal{N} = 3$ there is no restriction on the gauge group.

2.1. 3-Algebras

A central ingredient to all these theories is a 3-algebra. This is a vector space $\mathcal{Y}$ with a triple product

$$[\cdot, \cdot, \cdot] : \mathcal{Y} \otimes \mathcal{Y} \otimes \mathcal{Y} \to \mathcal{Y},$$

such that the endomorphism $\varphi(\cdot) = [\cdot, U, V] : \mathcal{Y} \to \mathcal{Y}$, for fixed $U, V \in \mathcal{Y}$, is a derivation. This leads to the so-called fundamental identity:

$$\varphi([A, B, C]) = [\varphi(A), B, C] + [A, \varphi(B), C] + [A, B, \varphi(C)].$$

(3)

For physics we require that there is a positive definite inner product on $\mathcal{Y}$:

$$\{\cdot, \cdot\} : \mathcal{Y} \otimes \mathcal{Y} \to \mathbb{R},$$

which induces an invariant inner product on the space of derivations:

$$(T, \varphi) = (T(U), V).$$

(5)

There is also a complex version of a 3-algebra:

$$[\cdot, \cdot, \cdot] : \mathcal{Y} \otimes \mathcal{Y} \otimes \mathcal{Y} \to \mathcal{Y},$$

(6)

with complex positive definite inner product: For physics we require that there is a positive definite inner product on $\mathcal{Y}$:

$$\{\cdot, \cdot\} : \mathcal{Y} \otimes \mathcal{Y} \to \mathbb{C}.$$  

(7)

Again the analogue of adjoint map

$$\varphi_{U, V}(X) = [X, U; V], \quad \varphi_{U, V}(\bar{X}) = -[\bar{X}, \bar{V}; U],$$

(8)

is a derivation

$$\varphi_{U, V}([X, Y; Z]) = [\varphi_{U, V}(X), Y; Z] + [X, \varphi_{U, V}(Y); Z] + [X, Y; \varphi_{U, V}(Z)].$$

(9)

The fundamental identity tells us that the action of $\varphi$ on $\mathcal{Y}$ is that of a Lie algebra $\mathfrak{g}$ generated by $\varphi_{U, V}$ for all $U, V \in \mathcal{Y}$. In other words $\mathcal{Y}$ is representation of $\mathfrak{g}$. Thus a 3-algebra defines a Lie algebra $\mathfrak{g}$ along with a preferred representation.

In fact the reverse is also true: given a Lie algebra and a representation (along with invariant inner products) one can always construct a triple product satisfying the fundamental identity via the so-called Faulkner map. Such 3-algebras, including ones with mixed signature inner products (which also have applications to gauge theory) have been classified, see for example.

One need not think of a 3-algebra and just think of the gauge group and matter representation. However the triple product fixes all the terms in the Lagrangian. Furthermore the amount of manifest supersymmetry fixes the symmetry properties of the triple product which in turn restricts the choice of 3-algebra and hence which gauge algebras and representations arise. This is a rather novel situation as the amount of manifest supersymmetry is determined by the gauge algebra and matter representations, unlike the case of super-Yang–Mills theories where the gauge group is arbitrary. Furthermore even though the gauge fields are related to the matter fields by supersymmetry they do not sit in the same representation of the gauge group. This is possible as the Chern–Simons structure means that the gauge fields do not carry on-shell degrees of freedom.

2.2. Examples

Let us look at some examples.

2.2.1. $\mathcal{N} = 8$ Supersymmetry: BLG

We take $\mathcal{Y}$ real and $[\cdot, \cdot, \cdot]$ totally antisymmetric and

$$\delta X^I = i\epsilon \Gamma^I \Psi$$

(10a)

$$\delta \Psi = D_\nu X^I \Gamma^\nu \Gamma^I \epsilon - \frac{1}{6} [X^I, X^J, X^K] \Gamma^{IJK} \epsilon$$

(10b)

$$\delta A_\mu (\cdot) = i\epsilon \Gamma_\mu \Gamma I [\cdot, X^I, \Psi],$$

(10c)

(11)

But for a positive definite choice of inner product (which we take to be the identity) there is just one finite-dimensional solution:

$$[T^a, T^b, T^c] = \frac{4\pi}{k} \varepsilon^{abc d} T^d \quad a, b, c, d = 1, 2, 3, 4.$$  

(12)

The gauge algebra generated by $\varphi$ is $\mathfrak{so}(4) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$ and

$$([T^+, T^-], (W^+, W^-)) = \frac{k}{4\pi} \text{tr}(T^+ W^+) - \frac{k}{4\pi} \text{tr}(T^- W^-),$$

(13)
so we find an $\text{su}(2)_+ \oplus \text{su}(2)_-$. Chern–Simons Lagrangian with opposite levels:

$$
\mathcal{L}_{CS} = \frac{k}{4\pi} e^{i \alpha} \text{tr} \left( A_i^+ \partial_j A_i^+ + \frac{1}{3} A_i^+ [A_i^+, A_i^+] \right)
$$

(14)

$$
- \frac{k}{4\pi} e^{i \alpha} \text{tr} \left( A_i^- \partial_j A_i^- + \frac{1}{3} A_i^- [A_i^-, A_i^-] \right).
$$

The fields $X^i$, $\Psi$ are in the $4 = 2 + \overline{2}$ = bifundamental. A standard result tells us that $k \in \mathbb{Z}$ - no continuous parameter.

### 2.2.2. $\mathcal{N} = 6$ Supersymmetry: ABJM

We need a little less symmetry and a complex $\mathcal{Y}$. To this end we write $X^i$ as four complex scalar fields $Z^A$, $A = 1, 2, 3, 4$ in $4$ of $\text{SU}(4)$ with $U(1)$ charge $1$. And $\Psi$ is now written as $4$ complex fermions $\psi_{AB}$ in $4$ with $U(1)$ charge $1$. Lastly the $16$ components of $\epsilon$ are reduced to $\epsilon^{AB} = -\epsilon^{BA}$ in $6$ of $\text{SU}(4)$ with $U(1)$ charge $0$.

We can now write down the supersymmetry transformations:

$$
\delta Z^A = i \epsilon^{AB} \psi_B
$$

(15a)

$$
\delta \psi_B = \gamma^\mu D_\mu Z^A \epsilon^{AB} + [Z^C, Z^A] \epsilon_{AB}^{CD} \psi_D
$$

(15b)

$$
\delta A_\mu (\cdot) = i \epsilon^{AB} \gamma_\mu [\cdot, \psi_B] - i \epsilon^{AB} \gamma_\mu [\cdot, \bar{Z}_A] \psi_B,
$$

(15c)

and Lagrangian

$$
\mathcal{L} = -(D^\mu \bar{Z}_A, D_\mu Z^A) - i (\bar{\psi}^A \gamma^\mu, A_\mu) - V
$$

$$
- i (\bar{\psi} \gamma^\mu [\psi_A, Z^B], \bar{Z}_B) + 2i (\bar{\psi} \gamma^\mu [\psi_B, Z^B], \bar{Z}_A)
$$

$$
+ \frac{i}{2} f_{ABCD} (\bar{\psi}^A, [Z^C, Z^D], \psi_B)
$$

(16a)

$$
- \frac{i}{2} f_{ABCD} (\bar{\psi}^A, [\bar{Z}_C, \bar{Z}_D], \psi_B)
$$

$$
+ \delta^{A\lambda} \delta^{B\nu} (A_\mu, A_\nu) + \frac{1}{3} \delta^{A\lambda} \delta^{B\nu} (A_\mu, [A_\nu, A_\lambda]),
$$

where the potential is

$$
V = \frac{2}{3} \Gamma_{\overline{B} \overline{D}} \overline{\gamma}_{\overline{C} \overline{D}}
$$

(16b)

with

$$
\Gamma_{\overline{B} \overline{D}} = [Z^\overline{C}, Z^\overline{D}, \bar{Z}_\overline{E}] - \frac{1}{2} \delta_{\overline{B} \overline{D}} [Z^\overline{E}, Z^\overline{B}, \bar{Z}_\overline{E}] + \frac{1}{2} \delta_{\overline{B} \overline{D}} [Z^\overline{E}, Z^\overline{D}, \bar{Z}_\overline{E}]
$$

(16c)

An infinite class of solutions are given by $M \times N$ complex matrices with $(A, B) = \text{tr}(AB^\dagger)$ and

$$
[A, B; C] = \frac{4\pi}{k} (AC^\dagger B - BC^\dagger A).
$$

(17)

The gauge transformation generated by $\delta Z^A = [Z^A, U, \bar{V}]$ is

$$
\delta Z^A = M Z^A - Z^A N,
$$

(18)

where $M = -V^\dagger U$, $N = MU^\dagger$ are $M \times M$ and $N \times N$ matrices respectively and

$$
(M, M') = \frac{k}{4\pi} \text{tr}(MM') \quad (N, N') = -\frac{k}{4\pi} \text{tr}(NN').
$$

(19)

Thus we find the gauge group $U(M) \times U(N)$ with matter in the bi-fundamental. Cases with $M > N$ are known as the ABJ theories. For $N = M$ one actually just finds $SU(N) \times SU(N)$ but the missing $U(1)$ factors can be added in by hand as they are super-symmetry singlets. In the special case of $SU(2) \times SU(2)$ we recover the BLG theory in complex notation.

The list of examples continues with less supersymmetry depending on the symmetry properties of the structure constants

$$
[T^A, T^V; T_z] = f^{ab\overline{c}d} T^{\overline{d}},
$$

(20)

but the actions are essentially the same.

### 2.3. Novelties

As we have mentioned above these actions break some super-symmetry 'rules'.

i) Gauge fields and matter fields are in the same multiplet but not in the same representation of the gauge group.

ii) The amount of supersymmetry is determined by the gauge group

In particular (for example see [15]):

$$
\Gamma_{abcd} = \Gamma^{abcd} \iff \mathcal{N} = 8 \iff \text{su}(2) \oplus \text{su}(2)
$$

(21a)

$$
\left(f_{abcd} \right) = (f^{abcd} ) \iff \mathcal{N} = 6 \iff \text{su}(N) \oplus \text{su}(M)
$$

(21b)

$$
\left(f_{abcd} \right)^* = (f^{abcd} )^* \iff \mathcal{N} = 5 \iff \text{so}(7) \oplus \text{su}(2) \oplus \text{so}(2)
$$

(21c)

The 3-algebra formalism is a neat way of encoding all this data even though in the end one is always just talking about a Chern–Simons-matter field theory based on a gauge group and choice of representation. This relationship is illustrated in Figure 1.

### 2.4. Essential Dynamics

Having constructed these theories it begs the question as to whether or not they really describe M2-branes. For a start there is no obvious free centre of mass multiplet. I warn you now that his is a rather lengthy and detailed section so please bare with me (or skip to the end). I am mentioning it here to help illustrate
some points later: namely that one has to work hard, and within the quantum theory, to see the correct physics.

The first thing to look at is the vacuum moduli space. This tells us the space of all the zero-energy configurations of the M2-branes. We will just stick to ABJM:

\[ \{Z^A, Z^B, Z_C = 0 \iff Z^A Z_C Z^B = Z^B Z_C Z^A. \quad (22) \] Generically this implies that all the \( Z^A \) commute (c.f. D-branes):

\[ Z^A = \text{diag}(z_1^A, \ldots, z_n^A). \quad (23) \]

To see that this is all requires one to evaluate the mass formula for small fluctuations which one finds is non-zero (generically: there are special points where extra massless modes arise but are expected to be lifted by non-perturbative effects).

We must identify fields that differ by gauge transformations:

\[ Z^A \to g_A Z^A g_R^{-1}. \quad (24) \]

We could set \( g_A = g_R \) so that this is an adjoint action, as with D-branes. This allows us to put \( Z^A \) in diagonal form (as we have already done) and in addition acts as

\[ z_i^A \to z_j^A \quad \text{for any } i \neq j. \quad (25) \]

e.g. for \( i, j = 1, 2 \) these are generated by

\[ g_{i j} = g_R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (26) \]

These generate the action of the symmetric group \( S_n \) on \( z_i^A \).

Unlike D-branes we also have continuous gauge transformations:

\[ z_i^A \to e^{i \theta_i} z_i^A. \quad (27) \]

These arise from taking

\[ g_A = g_R^{-1} = \text{diag}(e^{i \theta_1/2}, \ldots, e^{i \theta_n/2}). \quad (28) \]

To see the effect of this on the vacuum moduli space we must examine the Lagrangian for the moduli \( z_i^A \), including the gauge fields. The Lagrangian on the moduli space is

\[ \mathcal{L} = -\frac{1}{2} \sum_i D_{\mu} z_i^A D^\mu z_{\mu i} + k \frac{e^{i \mu \nu k}}{4\pi} \sum_i \bar{A}_{\mu i} \bar{A}_{\nu i} - k \frac{e^{i \mu \nu k}}{4\pi} \sum_i A_{\mu i} ^A \partial_\nu A_{\nu i}^B. \quad (29a) \]

where

\[ A_{\mu i} ^L = \text{diag}(A_{\mu i} ^L, \ldots, A_{\mu i} ^N), A_{\mu i} ^R = \text{diag}(A_{\mu i} ^R, \ldots, A_{\mu i} ^N), \quad (29b) \]

and

\[ D_{\mu} z_i^A = \partial_\mu z_i^A - i(A_{\mu i} ^L - A_{\mu i} ^R)z_i^A. \quad (29c) \]

Note that \( z_i^A \) only couples to \( B_{\mu i} = A_{\mu i} ^L - A_{\mu i} ^R \) and not to \( Q_{\mu i} = A_{\mu i} ^L + A_{\mu i} ^R \):

\[ \mathcal{L} = -\frac{1}{2} \sum_i D_{\mu} z_i^A D^\mu z_{\mu i} + k \frac{e^{i \mu \nu k}}{4\pi} \sum_i B_{\mu i} \partial_\nu Q_{\mu i}, \quad (30) \]

with \( D_{\mu} z_i^A = \partial_\mu z_i^A - i B_{\mu i} z_i^A \).

It is helpful to dualize \( Q_{\mu i} \) by introducing a Lagrange multiplier \( \sigma_i \):

\[ \mathcal{L} \equiv -\frac{1}{2} \sum_i D_{\nu} z_i^A D^\nu z_{\nu i} + k \frac{e^{i \mu \nu k}}{4\pi} \sum_i B_{\mu i} \partial_\nu H_{\mu i} + \frac{1}{8\pi} e^{i \mu \nu k} \partial_\nu \sigma_i H_{\mu i}, \quad (31) \]

where \( H_{\mu i} = \partial_\nu Q_{\mu i} - \partial_\mu Q_{\nu i} \).

Integrating out \( H_{\mu i} \) tells us \( B_{\mu i} = -k^{-1} \partial_\nu \sigma_i \) and everything is pure gauge:

\[ \mathcal{L} = -\frac{1}{2} \sum_i \partial_\nu w_i^A \partial^\nu \bar{w}_{\nu i}, \quad (32) \]

where \( w_i^A = e^{i \theta_i/2} z_i^A \) is gauge invariant. Next we observe that \( \sigma_i \) is periodic:

\[ \int \mathcal{L}(\sigma_i + 2\pi) - \int \mathcal{L}(\sigma_i) = -\frac{1}{4} \sum_i \int e^{i \mu \nu k} \partial_\nu H_{\mu i} = -\frac{1}{2} \sum_i \int dH = -\frac{1}{2} \sum_i \int dF^L + \int dF^R \in 2\pi \mathbb{Z}, \quad (33) \]

because of the Dirac quantization rule

\[ \int dF \in 2\pi \mathbb{Z}, \quad (34) \]
as well as the fact that \( B_i = -k^{-1}d\sigma_i \) implies \( d\xi = F^k - F^R = 0 \). This means that (recall \( u_i^A = e^{i\varphi_i/k}z_i^A \))

\[
w_i^A \cong e^{2\pi i/k} w_i^A.
\]

Thus there is an extra orbifold action in space-time

\[
\mathbb{R}^4 \to \mathbb{C}^4 / \mathbb{Z}_k,
\]

and the vacuum moduli space is

\[
\mathcal{M} = \text{Sym}^4(\mathbb{C}^4 / \mathbb{Z}_k).
\]

Corresponding to \( N \) M2-branes in an \( \mathbb{C}^4 / \mathbb{Z}_k \) transverse space. And indeed and M2-brane in this orbifold preserves 12 super-symmetries. This explains why there is no translational mode for generic \( k \) (including the classical, large \( k \), limit). But it should be there for \( k = 1 \) and we will find it later.

Let us return to the moduli space. It follows that we can think of

\[
Z^A = \begin{pmatrix}
   z^A \\
   \vdots \\
   z^A
\end{pmatrix},
\]

as describing the positions of \( N \) M2-branes in \( \mathbb{C}^4 / \mathbb{Z}_k \). Furthermore the natural circle for the M-theory direction is the overall phase.

Suppose we wanted to describe \( N \) M2-branes moving along the M-theory circle with different speeds. One might expect that this corresponds to

\[
Z^A = \begin{pmatrix}
   z^A e^{i\omega t/2} \\
   \vdots \\
   z^A e^{i\omega t/2}
\end{pmatrix},
\]

But this is pure gauge! We can un-do it by taking

\[
g_L = g_R^{-1} = \begin{pmatrix}
   e^{-i\omega t/2} \\
   \vdots \\
   e^{-i\omega t/2}
\end{pmatrix}.
\]

So how do the M2-branes ‘explore’ the full transverse space? Let us set the fermions to zero and construct the Hamiltonian

\[
H = \text{tr} \int d^2x \sum_{A=1}^4 D_i Z^A D_i Z^A + V
\]

\[
+ \left( i Z^A \Pi_{Z^A} - i \Pi_{Z^A} Z^A - \frac{k}{2\pi} F^L_{12} \right) A^L_0
\]

\[
+ \left( i Z^A \Pi_{Z^A} - i \Pi_{Z^A} Z^A + \frac{k}{2\pi} F^R_{12} \right) A^R_0.
\]

As usual the time-components of the gauge field give constraints:

\[
\frac{k}{2\pi} F^L_{12} = i Z^A \Pi_{Z^A} - i \Pi_{Z^A} Z^A + \frac{k}{2\pi} F^R_{12}.
\]

Let us consider the vacuum moduli again:

\[
Z^A = \left( \begin{array}{c}
   \frac{1}{\sqrt{2}} R_A^a e^{i\theta_a^A} \\
   \vdots \\
   \frac{1}{\sqrt{2}} R_A^a e^{i\theta_a^A}
\end{array} \right).
\]

The constraint is

\[
\frac{k}{2\pi} F^L_{12} = \frac{k}{2\pi} F^R_{12} = \left( \sum_a (R_A^a)^2 \theta_a^A \right) = \left( \sum_a (R_A^a)^2 \theta_a^A \right).
\]

In other words the momentum around the M-theory circle is given by the magnetic flux. In spirit this is the same as dualization:

\[
\partial_\theta X^{10} = \frac{1}{2} F^{10} \Rightarrow \partial_\theta X^{10} = F_{12}.
\]

This raises the next question: how do we compute quantities with eleven-dimensional momentum. In particular the gauge invariant observables appear to only carry vanishing \( U(1) \) charges:

\[
\mathcal{O} = \text{tr}(Z^A \bar{Z}^A Z^\cdots), \quad \text{OK},
\]

\[
\mathcal{O} = \text{tr}(Z^A \bar{Z}^B Z^\cdots), \quad \text{not OK}
\]

and hence don’t really explore all eleven dimensions.

This brings us to monopole or ‘t Hooft operators: we want to create states that carry magnetic charge. These operators are defined as a prescription for computing correlators in the path integral. They are not constructed as a local expression of the fields. In particular, a monopole operator \( \mathcal{M}(y) \) is defined by modifying the boundary conditions of the fields about the point \( y \) in the path integral

\[
(\mathcal{M}(y) \mathcal{O}(z) \cdots) = \int d^{10}x_\parallel f_\perp F = Q_M D D^2 D \mathcal{A} \mathcal{O}(z) e^{-\mathcal{S}},
\]

in other words we require the fields in the path integral to have a specific singularity

\[
F = \ast \frac{Q_M}{2} \left( \frac{1}{|x - y|} \right) + \text{nonsingular}.
\]

\( Q_M \in u(n) \times u(n) \) is the magnetic flux and is subject to the standard Dirac quantization condition

\[
e^{2\pi i Q_M} = 1.
\]
Next we note that due to the Chern–Simons term monopole operators transform locally under a gauge transformation
\[ \delta A^8_{\mu} = D_{\mu} \omega_{U/\mathbb{R}} \] (with \( \omega \rightarrow 0 \) at infinity) as
\[ \mathcal{M}_{Q_{\mathbb{R}}} (x) \rightarrow e^{i \frac{k}{2} \omega} f (D_{\mu} \omega, F_{\mu\nu}) \mathcal{M}_{Q_{\mathbb{R}}} (x) = e^{i \frac{k}{2} \omega} \mathcal{M}_{Q_{\mathbb{R}}} (x). \]

Note that by construction we have broken the gauge group to \( U(1)^{\mathbb{R}} \times U(1)^{\mathbb{R}} \). This is enough to tell us that under full gauge transformations the monopole operators transform in the representation of \( U(n) \times U(n) \) whose highest weight is
\[ \Lambda = k (Q_{\mathbb{R}} - Q_{\mathbb{R}}). \]

The monopole operators are just
\[ \mathcal{M}(y) = e^{i \sigma_{\mu} y_{\mu}}, \]

since
\[ \langle \mathcal{M}(y) \rangle_{\mathcal{O}(z) \cdots} = \int Dz DB DQ e^{i \sigma_{\mu} y_{\mu}} \mathcal{O}(z) e^{-i \int dx \mathcal{L}_{\mathcal{O}}} \]
\[ = \int Dz DB DQ \mathcal{O}(z) e^{-i \int dx \left[ \mathcal{L}_{\mathcal{O}} + i \sigma_{\mu} y_{\mu} \right]}. \]

This is the same as taking
\[ \frac{1}{8 \pi} \epsilon^{\mu \nu \lambda \sigma_{\mu \nu \lambda}} H_{\lambda \sigma_{\mu \nu \lambda}} \rightarrow \frac{1}{8 \pi} \epsilon^{\mu \nu \lambda \sigma_{\mu \nu \lambda}} H_{\lambda \sigma_{\mu \nu \lambda}} + 8 \pi \delta(x - y). \]

i.e. inserting a magnetic charge at \( x = y \).

Thus our gauge invariant operator on the moduli space is just
\[ u_{\alpha} = e^{i \sigma_{\mu} y_{\mu}} = \langle \mathcal{M} \rangle_{\mathcal{O}^{\alpha}} \]

and indeed \( \mathcal{M} \) has charge \( k \) under \( U(1) \times U(1) \).

Even at \( k = 1 \) translations in the transverse space are not symmetries of the Lagrangian:
\[ P_{\mu}^A = tr (D_{\mu} Z^A), \quad \text{not OK} \]

But now we can construct the conserved current (but only for \( k = 1 \)):
\[ P_{\mu}^A = tr (\mathcal{M}_{z_{\alpha} = 1} D_{\mu} Z^A), \quad \text{OK} \]

as well as the additional two supersymmetries that enhance \( \mathcal{N} = 6 \rightarrow 8 \):
\[ S_{\mu} = tr (\mathcal{M}_{z_{\alpha} = -1} \Psi_{\mathcal{A}} D_{\mu} Z^A). \quad \text{OK} \]

Finally we ask how BLG fits in? To cut a long story short:
\[ \begin{align*}
\text{i)} & \quad \text{BLG } (SU(2) \times SU(2))/\mathbb{Z}_2 \text{ at } k = 1 \text{ is dual to ABJM } U(2) \times U(2) \text{ at } k = 1, \text{i.e. 2 M2's in } \mathbb{R}^4 \\
\text{ii)} & \quad \text{BLG } SU(2) \times SU(2) \text{ at } k = 2 \text{ is dual to ABJM } U(2) \times U(2) \text{ at } k = 2, \text{i.e. 2 M2's in } \mathbb{R}^4/\mathbb{Z}_2 \\
\text{iii)} & \quad \text{BLG } (SU(2) \times SU(2))/\mathbb{Z}_2 \text{ at } k = 4 \text{ is dual to ABJM } U(2) \times U(3) \text{ at } k = 2, \text{i.e. 2 M2's in } \mathbb{R}^4/\mathbb{Z}_2 \\
\text{iv)} & \quad \text{BLG } (SU(2) \times SU(2))/\mathbb{Z}_2 \text{ at } k = 3 \text{ is dual to ABJM } U(3) \times U(3) \text{ at } k = 1 \text{ without the centre of mass multiplet, i.e. the interacting part of 3 M2's in } \mathbb{R}^4 \\
\end{align*} \]

So BLG describes 2 or even 3 M2-branes in \( \mathbb{R}^4 \) or \( \mathbb{R}^4/\mathbb{Z}_2 \) with all symmetries manifest (although not translations in the former case).

### 2.5. Lessons and Questions

So let me close the discussion of M2-branes with some lessons and a question. It is in general too much to ask for all symmetries to be manifest in the classical Lagrangian. In particular the true symmetries and dynamics only arise in the quantum theory using ‘quantum’ operators, i.e. operators that are not constructed directly out of the fields and which do not have a classical analogue. Furthermore the role of the gauge group is very non-trivial and global issues matter.

Lastly my question is: is there a role for the general BLG theories (i.e. for \( k > 4 \))? They exist as maximally supersymmetric field theories which have a weakly coupled limit as \( k \rightarrow \infty \). Due to their moduli space they seem rather non-geometric but perhaps slightly deeper in the sense that one can find the M2-brane theories by taking a \( \mathbb{Z}_k \) quotient of them.[16]

### 3. M5-Branes

The decoupling limit of \( N \) M5-branes leads to an interacting CFT in 5+1 dimensions with an \( SO(5) \) R-symmetry coming from rotations in the transverse 5-plane in eleven dimensions. In the Abelian case \( N = 1 \) the dynamics are known.[19–21]

The field content consists of five scalars \( X^i \) (so now \( i = 6, 7, 8, 9, 10 \) and \( \mu = 0, 1, 2, 3, 4, 5 \)), a 2-form \( B \) with self-dual field strength \( H \) and a 16-component fermion \( \Psi \). At the linearized level we simply have
\[ \partial_{\mu} X^i = 0, \]
\[ H_{\mu \nu \lambda} = 3 \delta_{\mu \nu} B_{\lambda}, \hspace{1cm} H_{\mu \nu \lambda} = \frac{3}{2} \epsilon_{\mu \nu \lambda \rho \sigma} H_{\rho \sigma}, \]
\[ i \Gamma^\mu \partial_{\mu} \Psi = 0. \]

For \( N > 1 \) one finds the interacting \( A_{N-1} (2,0) \) theory.
The dynamics are thought to arise from self-dual strings associated to M2-branes ending on M5-branes, just as D-brane dynamics arise from the end point of open strings (see Figure 2). These are the natural BPS states and Wilson-lines are replaced by surface operators. The Abelian case has been long understood[23] however the non-Abelian case of great interest as a higher gauge theory analogue of the Nahm transform.[23] Finally we mention that AdS/CFT predicts that the number of 'degrees of freedom' of \( N \) M5-branes scales as \( N^3 \).[24]

### 3.1. Reduction on \( S^1 \)

Let us wrap \( N \) M5-branes on \( S^1 \) of radius \( R_5 \). According to the M-theory dictionary this leads to \( N \) D4-branes in type IIA string theory with coupling \( g_s = R_5/l_s \). These are in turn described by \( U(N) \) 5D SYM and coupling \( g^2 = 4\pi^2 R_5 \). So the (2,0) theory is a UV completion of 5D SYM with enhanced Lorentz symmetry[25]

\[
\text{SO}(1, 4)_L \times \text{SO}(5)_R \longrightarrow \text{SO}(1, 5)_L \times \text{SO}(5)_R.
\]  

(61)

This is another 'prediction' of M-theory for quantum field theory: there exists a 6D SCFT that provides a UV completion of 5D SYM.

In this story the Kaluza-Klein momenta are carried by instanton-solitons \( F = \pm \phi \phi^* \);[26]

\[
P_\phi = \frac{n}{R_5}, \quad n = \frac{1}{8\pi^2} \text{tr} \int F \wedge F.
\]  

(62)

These states carry charges of the topological current

\[
J^\mu = \frac{1}{32\pi^2} \text{tr} \int g^{\mu\nu\rho\sigma} F_{\nu\rho} \rho_{\sigma},
\]  

(63)

for which all perturbative states are uncharged.

### 3.2. Reduction on \( T^2 \)

Let us reduce again on another \( S^1 \) with radius \( R_6 \). Here we find 4D \( U(N) \) SYM with coupling \( g^2 = 2\pi R_6/R_5 \). This theory has an S-duality that swaps perturbative modes with monopoles and \( R_4 \leftrightarrow R_6 \). However from the 6D point of view this is a modular transformation of \( T^2 = S^1 \times S^1 \) which is a diffeomorphism and hence is, or should be, a manifest symmetry of the (2,0) theory.\(^3\)

### 3.3. No Action?!

There are several arguments/challenges/opportunities\(^4\) against constructing a 6D action for the (2,0) theory. Let us discuss some.

i) Even without worrying about self-duality there are no ‘good’ interacting Lagrangians in 6D. In particular the Lagrangian must take the form

\[
\mathcal{L}_{6D} \sim H_{ijk} F^{ijk} + D_\mu X^i D^\mu X^i + (X) F_{\mu\nu} F^{\mu\nu} + (X)^3 + \text{non-renormalizable,}
\]  

(64)

which is problematic as the interactions are either non-renormalizable or unbounded or both. So what would the Lagrangian look like? In what space of classical theories does it exist?

ii) How can one obtain the 4D SYM action which takes the form

\[
S_{\text{SYM}} = \frac{R_4}{2\pi R_5} \int d^4 x \mathcal{L}_{4\text{SYM}},
\]  

(65)

from the standard Kaluza-Klein result

\[
S_{\text{DYM}} = \int d^6 x \mathcal{L}_{6\text{DYM}} = 4\pi^2 R_4 R_5 \int d^4 x \mathcal{L}_{4\text{DYM}} \text{zero-modes},
\]  

(66)

since the dependence on \( R_6 \) is inverted between the two\(^2\).\(^28\)

iii) Let us consider dimensional reduction to \( \mathbb{R}^{1,1} \) on some \( \mathcal{M} \). This leads to a 2D theory with \( b^+ (\mathcal{M}) \) chiral bosons and \( b^- (\mathcal{M}) \) anti-chiral bosons. However it is known that there is no modular invariant partition function if \( \sigma (\mathcal{M}) = b^+ (\mathcal{M}) - b^- (\mathcal{M}) \in 8\mathbb{Z} \). On the other hand there is Rohlin’s theorem which states that for compact 4D spin-manifolds \( \sigma (\mathcal{M}) \in 16\mathbb{Z} \). Thus it almost seems as if things should go the other way: the existence of a (2,0) action would imply a weaker version of Rohlin’s theorem (weaker by a factor of 2). However one knows that non-spin manifolds, such as \( \mathbb{C} P^2 \) with \( \sigma (\mathbb{C} P^2) = 1 \), can arise in M-theory and hence the M5-brane on \( \mathbb{C} P^2 \) should make sense. But it cannot have an action, so therefore no diffeomorphism invariant action in 6D.\(^2\)

iv) We have seen that the (2,0) theory exists for ADE gauge groups but it is also known that reduction on \( S^1 \) with a boundary condition that twists by an outer-automorphism gives 5D SYM with \( B, C \) gauge groups. Thus if one had an

---

\(^3\) 4D SYM is only self-dual for ADE gauge groups so the (2,0) theory can only exist for ADE gauge groups. Indeed it was first constructed by a decoupling limit of type IIB on \( \mathbb{K}^3 \) with an ADE singularity.[27]

\(^4\) Delete as appropriate.
action it should be subjected to the Tachikawa Test\footnote{NB: $SO(2n+1)\not\subset SU(2n)$}, given a $SU(2n)$ $(2,0)$ theory action with an $\mathbb{Z}_2$ twist along $S^1$, does it give $SO(2n+1)$ 5D MSYM?\footnote{NB: This relies heavily on the fact that $g^2 \propto R_5$ so we find weakly coupled 5D SYM, which is a unique feature of the (2,0) theory compared with compactifications of Lagrangian theories.}

### 3.4. Constructions

Let us now review some constructions of the (2,0) theory that have been proposed.

#### 3.4.1. DLCQ

Consider null-compactification: $x^\pm = x^0 \pm x^3$, $x^i$, $i = 1, 2, 3, 4$

\[
x^- \cong x^- + 2\pi R_\epsilon \quad \text{and fix} \quad P_\epsilon = K / R_\epsilon.
\]

We should view this as the limit of an infinite boost $v = 1 - \epsilon^2 \to 1$ of a space-like compactification $x^\pm \cong x^3 + 2\pi R_5$ resulting in $R_\epsilon = R_5 / \epsilon$.

To keep $R_\epsilon$ finite one must shrink $R_5 \to 0$ and hence the (2,0) theory on $S^5$ is well described 5D MSYM with fixed $P_\epsilon = K / R_5$. In this limit $K$ is given by the instanton number

\[
K = \frac{1}{8\pi^2} \text{tr} \int F \wedge F.
\]

and we are looking at the sector of 5D MSYM with instanton number $K$. Thus the dynamics are reduced to quantum mechanics on the moduli space of $U(N)$ instantons with instanton number $K$.

#### 3.4.2. Deconstruction

Construct a quiver (moose) diagram arising from the brane diagram in Figure 3 (which was stolen from \cite{32}), where the left and right sides are identified into a periodic direction. The D4-branes are described by $(SU(K))^N$ SYM with $N_f = 2K$ fields in the bifundamental of each $SU(K)$. This gives a 4D $\mathcal{N} = 2$ SCFT.

The next step is to go out on the Higgs’ branch breaking $(SU(K))^N \to SU(K)$. A careful tuning of parameters: scalar vev’s, coupling $g$ and number of nodes $N$ leads to a well-defined limit as $N \to \infty$.

The periodicity leads to a finite but large tower of states which for low energy look like a KK-tower. However there is also an S-duality of the quiver field theory so that the KK-tower is enhanced non-perturbatively to an $SL(2, \mathbb{Z})$ multiplet of two towers. Thus as $N \to \infty$ one reconstructs a 6D theory with $SO(5)$ R-symmetry.\footnote{NB: The roles of $K$ here is the same as $N$ elsewhere in our discussion and the role of $N$ here has no analogue elsewhere.}

3.4.3. 5D MSYM

Here the idea is that maybe 5D MSYM is actually well defined non-perturbatively, despite being perturbatively non-renormalizable, and it is an exact description of the (2,0) theory on $S^5$.\footnote{NB: This relies heavily on the fact that $g^2 \propto R_5$ so we find weakly coupled 5D SYM, which is a unique feature of the (2,0) theory compared with compactifications of Lagrangian theories.} In particular it contains a complete KK tower of soliton states so any UV completion would have to remove these and replace them with Fourier modes of some fields. So why bother? Then one must hope that the perturbative divergences are removed by small instanton-soliton effects.\footnote{NB: One cannot define 5D MSYM without also defining the (2,0) theory on $S^5$.} In addition it seems that for this to work we need to include zero-sized instantons but one can see $N$! behaviour.\footnote{NB: This relies heavily on the fact that $g^2 \propto R_5$ so we find weakly coupled 5D SYM, which is a unique feature of the (2,0) theory compared with compactifications of Lagrangian theories.}

In this scenario the extra momentum can be inserted by ‘instanton’ operators\footnote{NB: This relies heavily on the fact that $g^2 \propto R_5$ so we find weakly coupled 5D SYM, which is a unique feature of the (2,0) theory compared with compactifications of Lagrangian theories.}

\[
\langle J (y_\mu \nu) \theta (z, \cdots) \rangle = \int \frac{D\Psi}{\Psi_1} \frac{DA}{A} \theta (z) e^{-S}
\]

which are analogous to the monopole operators that we saw before for M2-branes.

If so then 5D MSYM does provide an ‘action’ for the (2,0) theory on $S^5$ for any radius. We note that if $\mathcal{M}_4 = S^5 \times \mathcal{M}_4$ then $\hat{b}_4 (\mathcal{M}_4) = \hat{b}_4 (\mathcal{M}_4)$ and hence no-chiral modes, in agreement with the discussion above.

We could consider instead $\mathcal{M}_4$ as multi-Taub-NUT space with $\hat{b}_4 (\mathcal{M}_4) \neq 0$. This is non-compact but has a nontrivial $S^1$ fibre. Reducing to IIA on the fibre leads to D4-branes intersecting with D6-branes. Here there are 2D chiral charged modes that are localised where fibre shrinks to zero size. The 5D MSYM that arises from reduction on a circle fibration has been discussed by \cite{41,42}. In this case one finds that the required chiral modes exist as solitons\footnote{This relies heavily on the fact that $g^2 \propto R_5$ so we find weakly coupled 5D SYM, which is a unique feature of the (2,0) theory compared with compactifications of Lagrangian theories.} and are described by a 5D version of a WZWN model.

There is a related proposal where the (2,0) theory on $\mathbb{R} \times S^5$, is reduced to 5D MSYM on $\mathbb{R} \times \mathbb{C} P^4$ with a Chern–Simons term.\footnote{This relies heavily on the fact that $g^2 \propto R_5$ so we find weakly coupled 5D SYM, which is a unique feature of the (2,0) theory compared with compactifications of Lagrangian theories.} In this case the coupling is related to the Chern–Simons level and so quantised.

3.4.4. Interrelations and Other Proposals

In fact these three descriptions are all related:

i) The DLCQ description of the (2,0) theory must also give the UV completion of 5D MSYM. But it only uses information arising from the classical IR dynamics of instanton-solitons of 5D MSYM. So somehow the IR behaviour of the theory is enough to determine its UV completion. This suggests that 5D MSYM is indeed well-defined without additional degrees of freedom.

ii) The action obtained from deconstruction is a ‘lattice’-like regularization of the 5D MSYM action.\footnote{This relies heavily on the fact that $g^2 \propto R_5$ so we find weakly coupled 5D SYM, which is a unique feature of the (2,0) theory compared with compactifications of Lagrangian theories.}

iii) One cannot define 5D MSYM without also defining the (2,0) theory on $S^5$.\footnote{This relies heavily on the fact that $g^2 \propto R_5$ so we find weakly coupled 5D SYM, which is a unique feature of the (2,0) theory compared with compactifications of Lagrangian theories.}
In addition to these proposals there also exist some action constructions in the literature. Let us list a few here:

i) Twistor-inspired and higher gauge theory action\cite{47,48}
ii) 5D SYM with KK-tower\cite{49}
iii) Mixed 5D/6D action\cite{50}
iv) $G \times G$ action\cite{51}
v) Non-local 6D action\cite{52,53}
vi) (1,0) Lagrangians and tensor hierarchy\cite{54}

3.5. Relations to M2-Branes

There are also a few ways that we expect M5s to arise from M2’s.

3.5.1. ‘T-Duality’

Here we consider a three-torus $T^3$ and reduce to IIA on the first $S^1$, T-dualize to type IIB on the second $S^1$, T-dualize back to IIA on the third $S^1$ and finally lift back up to M-theory, now on a dual $\hat{T}^3$. Using the standard rules we find that

i) M5’s on $T^3$ map to M2’s orthogonal to $\hat{T}^3 \times R^3$
ii) M2’s orthogonal to $T^3$ map M5’s on $\hat{T}^3$

However we must take the decoupling (low energy) limit to isolate the M-brane field theories. This requires that we take $R \rightarrow 0$ in the original $T^3$ (so $\hat{R} \rightarrow \infty$ in the dual $\hat{T}^3$).

The first relation is rather trivial: the M5-brane on $T^3$ gives 3D MSYM and shrinking the torus takes it to strong coupling. Thus we recover the M2-brane SCFT as a strong coupling IR limit of 3D MSYM.

The second relation says that we can construct M5-branes by looking at M2-branes in a shrinking transverse $T^3$. However enacting this is more tricky because translational symmetry is not manifest in the M2-brane Lagrangian. An attempt was tried in [55] and gives a modified version of 5D MSYM.

3.5.2. Flux Background

When M2-branes are placed in a background 3-form flux they expand into M5-branes on $S^1$ by the Myers effect. The resulting M5-brane action was constructed from the M2-brane action in [56] but one finds 5D MSYM on $S^3$. However when the monopole states in ABJM are included one finds that these map to instanton-soliton states 5D MSYM.\cite{57}

3.5.3. M2’s with a Nambu Bracket

There are infinite dimensional 3-algebras that can be used in the BLG theory. In particular the Nambu bracket

$$[X^I, X^J, X^K] = \epsilon^{ijk} \partial_i X^I \partial_j X^J \partial_k X^K,$$  \hspace{1cm} (71)

is an example where $X^I$ are functions of some three-manifold $\Sigma$. It has been observed that substituting this into BLG leads to an Abelian M5-brane wrapped on an auxiliary $\Sigma$.\cite{58,59}

3.6. Open Problems and Wishes

Let me close this discussion of M5-branes with some open problems and wish list of results:

i) Provide a field definition/construction of the (2,0) theory i.e. without recourse to String Theory or M theory
ii) Find the mathematical structures that best capture aspects of the (2,0) theory e.g. Non-Abelian periods of 2-forms. Twisters, Lie-2-Groups etc.
iii) Obtain calculable formulations of the (2,0) theory with 6D Diffeomorphisms and Lorentz!
iv) Construct an action (?), Partition function(s), families of actions or something action-like.
v) Better understand ‘quantum operators’ such monopole and instanton operators.
vii) Explore the relation between M2’s and M5’s more
viii) Make S-duality manifest?
ix) Make the $N^3$ behaviour more apparent

4. A Representation of the (2,0) Superalgebra

Finally in this last section I wanted to indulge myself by reporting on my own recent work that I hope is of interest to the conference.

![Figure 3. The (2,0) quiver.](image-url)
crowd and I welcome any suggestions. In particular in [60,61] my collaborators and I constructed a representation of the (2,0)
superalgebra acting on a set of fields:

\[ \begin{align*}
\delta X &= i \epsilon \Gamma^a \Psi, \\
\delta Y &= \frac{i}{2} \epsilon \Gamma_{\mu \nu} \sigma \Gamma^{\mu \nu} \Psi, \\
\delta H_{\mu \nu \lambda} &= 3i \epsilon \Gamma_{[\mu} \sigma \delta_{\nu \lambda]} \psi + i \epsilon \Gamma^1 \Gamma_{\mu \nu \lambda} \psi \] 
\end{align*} \]

and \( \psi \) is a constant. Lastly we have

\[ \begin{align*}
\Gamma_{012345} \epsilon &= 0, \quad \Gamma_{012345} \psi &= -\psi. \\
(73) 
\end{align*} \]

A standard (but trust me tedious) calculation shows that this system indeed closes on the following equations of motion (in this section we will omit fermions as much as possible for the sake of clarity)

\[ \begin{align*}
0 &= \Gamma^a D_a \psi + \Gamma_{\mu} \Gamma^\mu \psi + \frac{i}{12} \epsilon \sigma_{\mu \nu \rho} \Gamma_{\rho \tau} \Gamma^{\mu \nu \lambda} \psi, \\
0 &= D^2 X^i + [Y^\mu, X^i, X^j, \ldots] \\
&+ \frac{1}{2} \epsilon \sigma_{\mu \nu \rho} \psi, \\
0 &= D_\mu H_{\nu \lambda}, \quad \mu \sigma \nu \lambda, \mu, \nu, \lambda \] 
\end{align*} \]

as well as constraints:

\[ \begin{align*}
F_{\mu \nu} &= [Y^\mu, H_{\nu \lambda}], \quad \ldots \psi + \epsilon \sigma_{\mu \nu \rho} \psi, \\
0 &= D_\mu Y^\mu - \frac{1}{2} \epsilon \sigma_{\mu \nu \rho} \psi, \\
0 &= [Y^\mu, D_\mu, X^i, \ldots], \quad + \frac{1}{2} \epsilon \sigma_{\mu \nu \rho} \psi, \\
0 &= \Gamma^a \sigma_{\mu \nu \rho} \psi, \\
0 &= \epsilon \sigma_{\mu \nu \rho} \psi, \\
0 &= \epsilon \sigma_{\mu \nu \rho} \psi. \\
(75) 
\end{align*} \]

There is a conserved energy-momentum tensor:

\[ \begin{align*}
T_{\mu \nu} &= \frac{\pi}{2} (H_{\mu \nu \rho}, H_{\nu \lambda \rho}) \\
&+ 2\pi (D_\mu X^i, D_\lambda X^i) - \pi \eta_{\mu \nu} (D_\lambda X^i, D^i X^\lambda) \\
- \frac{\pi}{2} \eta_{\mu \nu} ([Y_\mu, x^i], [Y_\lambda, x^i], [Y^\mu, x^i, X^i]) \\
+ \frac{2\pi}{3} \left( \epsilon \sigma_{\mu \nu \rho} \psi, - \frac{1}{6} \epsilon \sigma_{\mu \nu \rho} \psi \right) \\
&\times ([Y^\mu, X^i, X^j], [X^i, X^j, X^\lambda]) \\
+ \frac{\pi}{12} \epsilon \sigma_{\mu \nu \rho} \psi, ([Y^\mu, X^i, X^j], [X^i, X^j, X^\lambda]), \quad + \text{fermions.} \\
(76) 
\end{align*} \]

One can also compute the supercurrent, superalgebra and central charges but let's not list those here.

Even I think this is an unconventional system and cannot decide if it is ugly (probably) or beautiful (possibly). But let us explore it.

4.1. M5-Branes

Let us start with the case \( C_{\mu \nu \lambda} = 0 \). Here \( D_\mu Y^\mu = 0 \) and we can fix

\[ Y^\mu = V^\mu T^4. \]

where \( T^4 \) is some generator of the 3-algebra and \( V^\mu \) a constant vector. All components of the fields along \( T^4 \) become free - the 6D centre of mass (2,0) multiplet. The remaining modes are acted on by an \( su(2) \) gauge algebra. The constraint \([Y^\mu, D_\lambda, \ldots] = 0 \) implies that these modes only depend on the coordinates orthogonal to \( V^\mu \). We also note that we can extend to any gauge group by taking a Lorentzian 3-algebra.

But there are still some choices for \( V^\mu \).

4.1.1. Space-Like \( Y^\mu \)

First we take \( V^\mu = 2\pi R_5 \delta_5^\mu \). The constraints then say that the remaining dynamical fields only depend on \( x^4, \ldots, x^8 \) and

\[ \[ F_{\mu \nu} = 2\pi R_5 H_{\mu \nu \lambda}. \]

The dynamical equations then all arise from the action

\[ S = -\frac{4\pi^2}{R_5} \text{tr} \int d^5 x \left( \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} D_\mu X^i D^i X^\mu \right. \\
- \frac{1}{4} [X^i, X^j]^2 \right) + \text{fermions}, \]

\[ (79) \]
We can view a null choice of $Y^\mu$ as a limit of an infinite boost of a space-like $Y^\mu$ where we saw that the spatial momentum was $n/R_s$. Thus we are looking at an M5-brane with $P_\perp = n/R_s$. This reproduces the DLCQ description of the dynamics of M5-brane.

### 4.2. M2-Branes

#### 4.2.1. Space-Like $C_{\mu \nu \lambda}$

First take $C_{45} = l^3$ non-vanishing. The constraint

$$[Y^\mu, D_\alpha, \cdots] + \frac{1}{3} [D_\alpha Y^\mu, \cdots] = 0 \quad (85)$$

suggests setting $\partial_\alpha = 0$, $\alpha = 3, 4, 5$ and $Y^\nu = 0$, $\alpha = 0, 1, 2$. In this case the constraint

$$C_{\mu \nu \lambda} D_\lambda (\cdot) - [Y^\mu, \nu^\nu, \cdot] = 0 \quad (86)$$

implies

$$A_\alpha (\cdot) = \frac{1}{2 l^3} \epsilon_{\alpha \beta \gamma} [Y^\beta, \nu^\gamma, \cdot]. \quad (87)$$

From this the remaining constraints can be solved leading to

$$H_{\beta \gamma} = -\frac{1}{l^2} [Y^\beta, Y^\gamma, Y^\nu],$$

$$H_{\beta \gamma \delta} = \frac{1}{l^2} \epsilon_{\beta \gamma \delta} D_\nu Y^\nu, \quad (88a)$$

and

$$F_{\alpha \beta} (\cdot) = \frac{1}{l^3} \epsilon_{\alpha \beta \gamma} [Y^\gamma, D_\nu Y^\nu, \cdot], \quad (88b)$$

$$F_{\alpha \beta} (\cdot) = \frac{1}{l^2} [Y^\beta, [Y^\gamma, Y^\nu, Y^\nu], \cdot].$$

Let us write $X^\mu = l^{-3/2} Y^\mu$, then everything is derived from the action (taking $I = 3, 4, 5, \ldots, 10$)

$$S = \int d^4 x \left\{ (D_\mu X^I, D^\mu X^I) + \frac{1}{6} \left[ [X^I, X^J, X^K], [X^I, X^J, X^K] \right] + \epsilon^{\alpha \beta \gamma} \left( A_\alpha, \partial_\beta A_\gamma \right) - \frac{1}{3} \epsilon^{\alpha \beta \gamma} \left( A_\alpha, [A_\beta, A_\gamma] \right) \right\} + \text{fermions}.$$  

i.e. BLG.
This is consistent with a T-duality along the directions of $C_{\mu\nu\lambda}$:

$$M_5 : 0 1 2 3 4 5 \overset{T_{25}}{\cong} M_2 : 0 1 2 \quad (90)$$

### 4.2.2. Time-Like $C_{\mu\nu\lambda}$

We can also take a ‘time-like’ $C_{\mu\nu5} = 1$. This leads to a Euclidean M2-brane theory with $SO(2, 6)$ R-symmetry. The Lagrangian is similar in structure to the normal maximally supersymmetric M2-brane case but with some funny signs so we won’t bother to give it here.

This is consistent with [66] where a time-like T-duality of M-theory leads to $M^{5*}$-theory with signature $(2, 9)$

$$M_5 : 0 1 2 3 4 5 \overset{T_{25}}{\cong} E_3 : 1 2 5 \quad (91)$$

And thus an E3-brane in this theory would indeed have $SO(2, 6)$ R-symmetry.

### 4.2.3. Light-Like $C_{\mu\nu\lambda}$

We can also take a null $C_{\mu\nu\lambda} = 1^{[67]}$ which leads to a rather odd system. In particular the fields depend only on $x^\mu$, $x^\nu$, $x^\lambda$ and now $Y^1$, $Y^4$, $Y^6$ are non-zero. Furthermore $Y^3$ joins up with $X^i$ to form an $SO(6)$ multiplet which we denote by $X^i$. As before $H_{\mu\nu\lambda}$ is largely determined in terms of $Y^1$, $Y^4$, $Y^6$ but self-duality implies $Z = Y^4 + iY^6$ is holomorphic $\tilde{D}Z = 0$ where $z = x^3 + ix^6$.

Lastly $H = H_{453} = iH_{4\lambda\mu}$ is undetermined.

One finds that the dynamics can be obtained from the action[64]

$$S = \int d^2x dx^5 \left\{ \frac{1}{4} (D_\mu Z, D_\mu \tilde{Z}) + (D\tilde{Z}, H) + (\tilde{D}Z, H) ight. 
- \langle DX^i, \tilde{D}X^i \rangle - \frac{i}{4} \left\langle D_\mu , X^i , \left[ Z, \tilde{Z}, X^i \right] \right\rangle 
- \frac{1}{8} \left\langle X^i, X^j, Z \right\rangle , X^i, X^j, Z \right\rangle 
+ \frac{i}{2} \left\langle A_1, F_{23} \right\rangle 
+ \frac{i}{2} \left\langle A_2, F_{34} \right\rangle 
+ \frac{i}{2} \left\langle A_3, F_{45} \right\rangle \right\}_{\text{fermions}},$$

where $\Psi_{\pm} = \frac{1}{i} (1 \pm \Gamma_{03})\Psi$.

This is a novel field theory in 2+1 dimensions invariant under 16 supersymmetries, translations in space and time, spatial $SO(2)$ rotations and an $SO(6)$ R-symmetry, but again no boost symmetry.

Note that $H = H_{i23}$ acts as a Lagrange multiplier imposing

$$\tilde{D}Z = 0. \quad (93)$$

Furthermore there is a Gauss Law constraint arising from the $A_i$ equation of motion:

$$F_{23}(\cdot) = -\frac{1}{4} [X^i, [Z, \tilde{Z}, X^i], \cdot] + \cdots. \quad (94)$$

Thus the motion is constrained to the Hitchin moduli space.

As above we can view $C_{\mu\nu\lambda}$ as the limit of an infinite boost along $x^5$ of the $C_{\mu\nu5}$ case. Indeed the Hitchin-system gives rise to a momentum along $x^5$:

$$\mathcal{P}_5 \sim \oint \{ (Z, \tilde{D}Z) dz + (\tilde{Z}, DZ) dz \}, \quad (95)$$

which appears as a winding of the M2-branes around $x^4, x^5$.

So we are looking at intersecting M2-branes that have been boosted along $x^5$. This is a T-dual relation to momentum modes of the M5-brane (see Figure 4).

### 4.3. Observations and Provocations

So our representation of the $(2,0)$ superalgebra gives various field theories associated to M-branes.

- i) 5D SYM as the M5 on $S^1$
- ii) Maximally supersymmetric M2 branes
- iii) Null M5-branes: QM on instanton moduli space
- iv) Null intersecting M2-branes: QM on Hitchin moduli space

The later two are novel non-Lorentz invariant field theories whose on-shell dynamics reduces to one-dimensional motion on moduli space and breaks $1/2$ the supersymmetry.

The field theories that we obtain from this system are all consistent with the notion of ‘T-duality’ (really a U-duality) in M-theory on $T^3$ along $x^a$, $x^b$, $x^c$ with radii $R_a$, $R_b$, $R_c$ and

$$C_{\mu\nu\lambda} = (2\pi)^3 R_a R_b R_c, \quad (96)$$

but one needs to generalise all this to more than two branes in order to make it more concrete!
This (2,0) system is reminiscent of doubled field theory where $X^i$ is a position coordinate and $Y^i$ is a winding coordinate. Under T-duality along $x^i$ the corresponding $Y^i$'s become position coordinates. Furthermore the $Y^i D_i = 0$ constraint is like a section condition. Although it should be noted that the fields are only functions of ordinary 6D coordinates $x^i$ (i.e. not of the winding coordinates). It would be interesting to see if there is a deeper geometrical significance to the various constraints of the (2,0) system.

Conflict of Interest

The author has declared no conflict of interest.

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