Center-of-mass effects for hydrogen atoms in a nonuniform electric field: applications to magnetic fusion, radiofrequency discharges, and flare stars

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Abstract
We study whether or not the Center-of-Mass (CM) motion and the relative motion can be separated for hydrogen atoms in a nonuniform electric field. First, we show that in the general problem of two charges in a nonuniform electric field, the CM and relative motions, rigorously speaking, cannot be separated. Second, we use an approximate analytical method of the separation of rapid and slow subsystems to achieve a pseudoseparation of the CM and relative motions for hydrogenic atoms/ions in an arbitrary nonuniform electric field. Third, we further develop these results for the case of a hydrogen atom in the nonuniform electric field, where the field is due to the nearest (to the hydrogen atom) ion in a plasma. Fourth, we apply the results to the ion dynamical Stark broadening of hydrogen lines in plasmas. Fifth, we present specific examples of laboratory plasmas (e.g., magnetic fusion plasmas or radiofrequency discharges) and astrophysical plasmas (e.g., in atmospheres of flare stars) where the allowance for these CM effects leads to a significant increase of the width of hydrogen spectral lines.

1. Introduction
Center-of-Mass (CM) effects for hydrogenic atoms/ions in a uniform magnetic field are well-studied—see, e.g., papers [1–3] and references therein. The CM motion and the relative (internal) motion are coupled in a magnetic field and, rigorously speaking, cannot be separated. For hydrogen atoms it is possible to achieve a pseudoseparation leading to a Hamiltonian for the relative motion that still depends on a CM integral of the motion called pseudomomentum [3].

As for hydrogenic atoms/ions in a uniform electric field, it is well-known that the CM and relative motions can be separated rigorously (exactly)—see, e.g., [4]. As for hydrogenic atoms/ions in a nonuniform electric field, there seem to be nothing about the separation (or non-separation) of the CM and relative motions in the literature, to the best of our knowledge.

In the present paper we study this issue for hydrogenic atoms/ions in a nonuniform electric field. First, we show that in the general problem of two charges in a nonuniform electric field, the CM and relative motions, rigorously speaking, cannot be separated. Second, we use an approximate analytical method of the separation of rapid and slow subsystems to achieve a pseudoseparation of the CM and relative motions for hydrogenic atoms/ions in an arbitrary nonuniform electric field. Third, we further develop these results for the case of a hydrogen atom in the nonuniform electric field, where the field is due to the nearest (to the hydrogen atom) ion in a plasma. Fourth, we apply the results to the ion dynamical Stark broadening of hydrogen lines in plasmas. Fifth, we present specific examples of laboratory and astrophysical plasmas where the allowance for these CM effects leads to a significant increase of the width of hydrogen spectral lines.
2. General case of the pseudoseparation of the center-of-mass and relative motions in a nonuniform electric field

We consider a system of two charges \( e_1 \) and \( e_2 \) of masses \( m_1 \) and \( m_2 \), respectively, in a nonuniform electric field. The Lagrangian of the system is

\[
L = \left[ m_1 (dr_1/dt)^2 + m_2 (dr_2/dt)^2 \right] /2 - e_1 e_2 / |r_2 - r_1| - e_1 \phi (r_1) - e_2 \phi (r_2),
\]

where \( r_1 \) and \( r_2 \) are radii-vectors of charges \( e_1 \) and \( e_2 \), respectively and \( \phi \) is the potential of the nonuniform electric field. After the substitution

\[
R = (m_1 r_1 + m_2 r_2) / (m_1 + m_2), \quad r = r_2 - r_1,
\]

so that \( R \) and \( r \) are the coordinates related to the CM motion and the relative motion, respectively, the Lagrangian takes the form

\[
L(R, r) = L_{CM}(R) - U(R, r) + L_r(r),
\]

where

\[
L_{CM}(R) = (m_1 + m_2) (dR/dt)^2 /2 - (e_1 + e_2) \phi (R)
\]

is the Langrangian of the CM, \( L_r(r) = \mu (dr/dt)^2 /2 - e_2 R / r \)

is the Lagrangian of the relative motion, and

\[
U(R, r) = \mu (e_1 / m_1 - e_2 / m_2) r F(R)
\]

is the coupling of the CM and relative motions. Here

\[
\mu = m_1 m_2 / (m_1 + m_2)
\]

is the reduced mass of the two particles, and

\[
F(R) = -d\phi (R) / dR
\]

is a nonuniform electric field (in the expansion of the electric potential we disregarded terms higher than the dipole one). In equation (6) and below, for any two vectors \( A \) and \( B \), the notation \( AB \) stands for the scalar product (also known as the dot-product) of the two vectors.

The Hamiltonian, corresponding to the Langrangian from equation (3), has the form

\[
H = H_{CM}(R, P) + U(R, r) + H_r(r, p),
\]

where

\[
H_{CM}(R, P) = P^2 / [2(m_1 + m_2)] + (e_1 + e_2) \phi (R)
\]

is the Hamiltonian of the CM, \( P \) being the momentum of the CM motion, and

\[
H_r(r, p) = p^2 / (2\mu ) + e_2 R / r
\]

is the Hamiltonian of the relative motion, \( p \) being the momentum of the relative motion.

Thus, the above equations show that at the presence of a nonuniform electric field, the CM motion and the relative motion are coupled (by \( U(R, r) \) from equation (6)) and therefore, rigorously speaking, cannot be separated. However, in the case where \( m_1 \ll m_2 \), the CM and relative motions can be separated by using the approximate analytical method of separating rapid and slow subsystems: in this case, the characteristic frequency of the relative motion is much greater than the characteristic frequency of the CM motion, so that the former and the latter are the rapid and slow subsystems, respectively. Below are the details of this method that can be found, e.g., in [5].

The first step is to freeze the coordinates \( R \) of the slow subsystem and to solve for the motion of the rapid subsystem characterized by the truncated Hamiltonian

\[
H_r = H_r(r, p) + U(R, r) = p^2 / (2\mu ) + e_2 R / r + \mu (e_1 / m_1 - e_2 / m_2) r F(R),
\]

where \( R \) is treated as a fixed parameter rather than as the dynamical variable. In the situation where the charges \( e_1 \) and \( e_2 \) are of the opposite sign (say, for definiteness \( e_1 < 0 \) and \( e_2 > 0 \)), this becomes the Hamiltonian of a hydrogenic atom/ion in a ‘uniform’ electric field.

By treating the last term in equation (12) in the first order of the perturbation theory, one obtains the following expression for the energy of the relative motion, i.e., the rapid subsystem (see, e.g., the textbook [6])

\[
E(R) = -\mu e_1 e_2^2 / (2n^2 \hbar^2) + \mu (e_1 / m_1 - e_2 / m_2) (r) F(R)
\]

\[
= -\mu e_1 e_2^2 / (2n^2 \hbar^2) - (3n^2 \hbar^2 / 2)[1 / (m_2 e_2) + 1 / (m_1 e_1)] A F(R),
\]

where \( A F(R) \) is a nonuniform electric field.
where there was used the well-known relation between the mean value \( \langle r \rangle \) of the radius-vector and the Runge-Lenz vector \( \mathbf{A} \) (see, e.g., [7, 8]):

\[
\langle r \rangle = -3e_1e_2A/|E_0|, \quad E_0 = -\mu e_1^2e_2^2/(2n^2h^2).
\] (14)

Here and below \( n \) is the principal quantum number.

By choosing the \( z \)-axis along the Runge-Lenz vector \( \mathbf{A} \), we rewrite equation (13) in the form

\[
E(R) = -\mu e_1^2e_2^2/(2n^2h^2) - (3n|q|h^2/2)[1/(m_1e_1) + 1/(m_2e_2)] \times F(R) \cos \theta(R),
\] (15)

where \( \theta(R) \) is the polar angle of the vector \( \mathbf{F}(R) \) and \( q \) is the electric quantum number \( q = n_1 - n_2 \), where \( n_1 \) and \( n_2 \) are the parabolic quantum numbers. Physically, the quantum number \( q \) is intimately connected to the conservation of the Runge-Lenz vector \( \mathbf{A} \) for the unperturbed hydrogen atom: the eigenvalue of the operator \( \mathbf{A} \) is \( q/n \)—see, e.g., the textbook [6].

The second step of the analytical method of separating rapid and slow subsystems is to proceed to the slow subsystem (the CM motion), for which \( E(R) \) from equation (15) will play the role of an effective potential. The effective Hamiltonian \( H_{CM,eff}(\mathbf{R}, \mathbf{P}) \) for the CM motion becomes (the first, \( R \)-independent term in \( E(R) \) has been omitted because it does not affect the CM motion)

\[
H_{CM,eff}(\mathbf{R}, \mathbf{P}) = P^2/[2(m_1 + m_2)] + (e_1 + e_2)\varphi(R) - (3n|q|h^2/2)
\times [1/(m_1e_1) + 1/(m_2e_2)]F(R) \cos \theta(R).
\] (16)

Thus, the application of this analytical method allowed the pseudo-separation of the CM motion and the relative motion for any two oppositely charged particles of significantly different masses in a nonuniform electric field.

It should be emphasized that in our paper, the CM coordinate \( \mathbf{R} \) is considered as the dynamical variable (which generally depends on time) and that the Hamiltonian \( H_{CM,eff}(\mathbf{R}, \mathbf{P}) \) from equation (16) can be used to solve for the CM motion. This is the primary distinction of our work from papers where the CM coordinate of a hydrogenic atom/ion in a nonuniform electric fields was considered to be fixed\(^1\). In section 3 we actually solve for the CM motion in the situation where the nonuniform electric field is due to the plasma ion nearest to the hydrogen atom, and apply the solution to the dynamical Stark broadening of hydrogen lines in plasmas. This would be impossible if the CM coordinate \( \mathbf{R} \) would not have been treated as the dynamical variable.

We also note that higher order terms (quadrupole, octupole etc) in the expansion of the potential \( \varphi(R) \) in equation (1) can be easily taken into account, if necessary, and this analytical method for the pseudo-separation of the CM motion and the relative motion, with \( \mathbf{R} \) considered as the dynamical variable, would still work.

In the particular case of hydrogen atoms one has

\[
e_1 = e, \quad e_2 = -e, \quad \mu = m_ee_p/(m_e + m_p),
\] (17)

where \( e > 0 \) is the electron charge, \( m_e \) and \( m_p \) are the electron and proton masses, respectively. Then equation (16) simplifies to

\[
H_{CM,eff}(\mathbf{R}, \mathbf{P}) = P^2/(2m) - [3n|q|h^2/(2\mu)e]F(R) \cos \theta(R), \quad m = (m_e + m_p).
\] (18)

3. Analytical solution for the center-of-mass motion of a hydrogen atom in the field of the nearest ion and its application to the dynamical Stark broadening of hydrogen lines in laboratory and astrophysical plasmas

Now we consider the situation where the nonuniform electric field is due to the nearest (to the hydrogen atom) ion of the positive charge Ze and mass \( m \) in a plasma located at the distance \( R \) from the hydrogen atom. Then the Hamiltonian from equation (18) can be rewritten as

\[
H_{CM,eff}(\mathbf{R}, \mathbf{P}) = P^2/(2m) - (D/R^2)\cos \theta, \quad D = [3n|q|h^2/(2\mu)]Z, \quad \cos \theta = AR/AR.
\] (19)

This Hamiltonian represents a particle of mass \( m \) in the dipole potential. Since this particle is relatively heavy \((m \gg m_e)\), its motion can be described classically and the corresponding classical solution is well-known—see, e.g., paper [11]. For this physical system, the radial motion can be exactly separated from the angular motion

\(^1\) An example of such papers is Bekenstein and Krieger article [9]. In that article, while considering hydrogen atoms in a nonuniform electric field, the authors did not treat the CM coordinate as the dynamical variable, but rather considered it fixed, and did not provide any way to solve for the CM motion. We note in passing that among such papers on hydrogen atoms in a nonuniform electric field, Bekenstein and Krieger article of 1970 [9] was in no way a pioneering work. Bekenstein and Krieger seemed unaware of Sholin paper of 1969 [10], where he showed that the primary source of the asymmetry of hydrogen spectral line shapes is the nonuniformity of the plasma ion nearest to the hydrogen atom; moreover, Sholin took into account quadrupole and octupole interactions, while Bekenstein and Krieger allowed only for the quadrupole interaction.
resulting in the following radical equation:

\[ m[R(dR/dt) + (dR/dt)^2] = E_{CM}, \]  

(20)

where \( E_{CM} \) is the total energy of the particle. This equation allows the following exact general solution:

\[ R(t) = (2E_{CM}t^2/m + 2R_0v_0t + R_0^2)^{1/2}, \quad R_0 = R(0), \quad v_0 = (dR/dt)_{t=0}. \]  

(21)

It is well-known that in plasmas of relatively low electron densities \( N_e \), the Stark broadening of the most intense hydrogen lines, i.e., the lines corresponding to the radiative transitions between the levels of the low principal quantum numbers (such as, e.g., Ly-alpha, Ly-beta, H-alpha, etc), is dominated by the ion dynamical broadening—see, e.g., publications [12–18]. The corresponding validity condition is presented in appendix A.

In the so-called ‘conventional theory’ of the dynamical Stark broadening (also known as the ‘standard theory’) [19–22], the relative motion within the pair ‘radiator—perturber’ was assumed to occur along a straight line—as for a free motion (in our case the radiator is a hydrogen atom and the perturber is the perturbing ion).

However, from the preceding discussion it follows that in the more advanced approach, the relative motion within the pair ‘radiator-perturber’ should be treated as the motion in the dipole potential—(D/R^2)cos \( \theta \), as seen from equation (19). The relevant setup of the problem is to choose the instant \( t = 0 \) as the instant of the smallest distance (the closest approach) within the pair ‘radiator-perturber’. Then \( v_0 = (dR/dt)_{t=0} = 0 \), so that equation (21) simplifies to

\[ R(t) = (2E_{CM}t^2/m + R_0^2)^{1/2}. \]  

(22)

The energy \( E_{CM} \) can be represented in the form

\[ E_{CM} = P_0^2/(2m) - (D/R_0^2)\cos \theta_0, \quad P_0 = P(0), \quad \theta_0 = \theta(0). \]  

(23)

By considering the motion within the pair ‘radiator—perturber’ in the reference frame where the perturbing ion is at rest, so that \( P_0 = mV_0 \), where \( V_0 \) is the relative velocity within the pair ‘radiator—perturber’ at \( t = 0 \), the energy \( E_{CM} \) can be rewritten as

\[ E_{CM} = mV_0^2/2 - (D/R_0^2)\cos \theta_0. \]  

(24)

Then equation (22) becomes

\[ R(t) = [V_0^2 - 2D \cos \theta_0/(mR_0^2)]t^2 + R_0^2)^{1/2}. \]  

(25)

By introducing the effective velocity

\[ V_{eff}(R_0, \theta_0) = [V_0^2 - 2D \cos \theta_0/(mR_0^2)]^{1/2}, \]  

(26)

we can make equation (25) to be formally equivalent to the usual case of the rectilinear trajectories:

\[ R(t) = [V_{eff}(R_0, \theta_0)]t^2 + R_0^2)^{1/2}. \]  

(27)

Now we consider a radiative transition between hydrogen energy levels \( a \) and \( b \). In the general case, the ion dynamical broadening operator \( \Phi_{ab} \) is defined as follows (by analogy with the electron dynamical broadening operator defined, e.g., in paper [19]):

\[ \Phi_{ab}(t) = - \int dV_0 f(V_0) N_a V_a(\sigma(V_0, \theta_0, t))_{\theta_0}. \]  

(28)

Here \( \langle \ldots \rangle_{\theta_0} \) denotes the averaging over the angle \( \theta_0 \), and the operator \( \sigma(V_0, \theta_0, t) \) has the form:

\[ \sigma(V_0, \theta_0, t) = \int dR_2 \pi R_0[1 - U(R_0, \theta_0, \theta_0) \ast (t, 0) U(R_0, \theta_0, \theta_0) \ast (t, 0)]_{\theta_{av}}. \]  

(29)

Here \( N_a \) is the ion density, \( f(V_0) \) is the distribution of the velocities (usually assumed to be Maxwellian), \( \rho \) is the impact parameter of the perturbing ion, \( U_a \) and \( U_b \) are the corresponding time-evolution operators, the symbols “*” and \( [\ldots]_{\theta_{av}} \) stand for the complex conjugation and the angular average, respectively. If the time \( t \) would be considered as a parameter, then the diagonal elements of the operator \( \sigma(V_0, t) \) would have the physical meaning of cross-sections of so-called optical collisions, i.e., the cross-sections of collisions leading to virtual transitions inside level \( a \) between its sublevels and to virtual transitions inside level \( b \) between its sublevels, resulting in the broadening of Stark components of the hydrogen spectral line.

By using the trajectories from equation (25) and averaging over the polar angle \( \theta_0 \), one can obtain the evolution operators and then the ion dynamical broadening operator with the allowance for the effect of the CM motion. However, in this general case, the results cannot be obtained analytically.

Therefore, for obtaining the final results analytically (which should help getting the message across in the simple form), we now employ the so-called impact approximation and substitute the evolution operators by the corresponding scattering matrices (see, e.g., papers [20, 21] or books [18, 22]):

\[ \Phi_{ab} = - \int dV_0 f(V_0) N_a V_a(\sigma(V_0, \theta_0))_{\theta_0}. \]  

(30)
\[ \sigma(V_0, \theta_0) = \int dR_0 2\pi R_0 d[1 - S_r(R_0, V_0, \theta_0) S_b^*(R_0, V_0, \theta_0)]_{\text{ang. av}} \]  
(31)

In the case where non-diagonal matrix elements of \( \Phi_{ab} \) are relatively small, the lineshape is a sum of Lorentzians, whose width \( \gamma_{ab} \) and shift \( \Delta_{ab} \) are equal (apart from the sign) to the real and imaginary parts of diagonal matrix elements \( \langle \alpha | \beta | \Phi_{ab} | \beta \rangle | \alpha \rangle \), respectively:

\[ \gamma_{\alpha \beta} = -\text{Re} \{ \langle \alpha | \beta | \Phi_{ab} | \beta \rangle | \alpha \rangle \}, \quad \Delta_{\alpha \beta} = -\text{Im} \{ \langle \alpha | \beta | \Phi_{ab} | \beta \rangle | \alpha \rangle \}. \]  
(32)

Here \( \alpha \) and \( \beta \) correspond to upper and lower sublevels of the levels \( a \) and \( b \), respectively. Here and below, for any operator \( \mathbf{G} \), for brevity we denote its matrix elements \( \langle \alpha | \beta | \mathbf{G} | \beta \rangle | \alpha \rangle \) as \( \alpha \mathbf{G} \beta \).

As we calculate the scattering matrices by the standard time-dependent perturbation theory, we obtain the following expression for the operator \( \sigma \):

\[ \sigma(R_0, V_0, \theta_0) = \int dR_0 2\pi R[K^2 Q(R_0, V_0, \theta_0) / R_0^2]. \]  
(33)

Here

\[ Q(R_0, V_0, \theta_0) = 2\hbar^2 / [3 \mu V_{\text{eff}}(R_0, \theta_0)^2] = Q_0 / [1 - 2D \cos \theta_0 / (\mu V_0^2 R_0^2)], \]
\[ Q_0 = 2\pi^2 \mu V_0^2 / (3 \mu V_0^2 R_0^2), \]  
(34)

and

\[ K^2 = K_0^2 + K_{\text{interf}}^2 + K_{\text{int}}^2, \quad K_0^2 = r_a^2 / a_b^2, \quad K_{\text{interf}}^2 = -2 r_a r_b^2 / a_b^2, \]
\[ K_{\text{int}}^2 = r_b^2 / a_b^2, \quad a_b = \hbar^2 / (\mu e^2), \]  
(35)

where \( a_b \) is the Bohr radius, \( K_{\text{interf}} \) represents the so-called interference term. In the conventional theory \(^{20-22}\), in equation (34) instead of \( V_{\text{eff}}(R_0, \theta_0) \), it would be \( V_0^2 \).

The next step is the averaging of \( 1 / V_{\text{eff}}(R_0, \theta_0) \) in equation (34) over the angle \( \theta_0 \):

\[ (1/2) \int_{-1}^{1} d(\cos \theta_0) / [V_0^2 - 2D \cos \theta_0 / (\mu V_0^2 R_0^2)] = [R_0^2 / (2R_0^3 V_0^2)] \times \ln [(R_0^2 + R_D^2) / (R_0^2 - R_D^2)], \]  
(36)

where

\[ R_D = [2D / (\mu V_0^2)]^{1/2}, \]  
(37)

so that the quantity \( Q(R_0, \theta_0) \) after the averaging over \( \theta_0 \) becomes

\[ Q(R_0) = [Q_0 R_0^2 / (2R_D^2)] \ln [(R_0^2 + R_D^2) / (R_0^2 - R_D^2)]. \]  
(38)

with \( Q_0 \) defined in equation (34).

The way the quantity \( D \) (entering equation (37)) was defined in equation (19) as \( D = [3n|q|\hbar^2 / (2\mu)] \) is valid only for the Lyman lines. For all other hydrogen lines one should use the arithmetic average of the values of \( D \) for the upper and lower Stark sublevels—as suggested in the similar case in paper \(^{24}\) and used in paper \(^{25}\). Therefore, in the present paper for all other hydrogen lines we use the following value of \( D \):

\[ D = 3(n|q| + n'|q'|) Ze^2 a_b / 4, \]  
(39)

where the quantum numbers with the prime symbol and without it relate to the lower and upper levels, respectively.

The next step is the averaging over \( R_0 \). The integral over \( R_0 \) in equation (33) has a weak, logarithmic divergence at both small and large impact parameters—just like in the conventional theory \(^{19-22}\). Therefore, as in the conventional theory, we subdivide collisions into ‘weak’ \( (R_0 > R_{\text{min}}) \) and ‘strong’ \( (R_0 < R_{\text{min}}) \), and introduce also the upper cutoff \( R_{\text{max}} \) (just as in the conventional theory) discussed later. Then the diagonal elements of the cross-section of optical collisions can be represented in the form

\[ a_{\alpha \beta}(\sigma)_{\alpha \beta} = \int_{R_{\text{max}}}^{R_{\text{min}}} dR_0 2\pi R_0 d[\alpha_{\beta}(K^2)_{\alpha \beta} Q(R_0) / R_0^2] + \int_0^{R_{\text{min}}} dR_0 2\pi R_0 C, \]  
(40)

where \( R_{\text{min}} \) is defined by the condition:

\[ a_{\beta}(K^2)_{\beta \alpha} Q(R_{\text{min}}) / R_{\text{min}} = \alpha_{\beta}(K^2)_{\alpha \beta} [Q_0 / (2R_D^2)] \ln \]
\[ \times [(R_{\text{min}}^2 + R_D^2) / (R_{\text{min}}^2 - R_D^2)] = C \]  
(41)

(naturally, \( R_{\text{min}} > R_D \)). Here and below the subscript ‘D’ in \( a_{\beta}(\sigma)_{\alpha \beta} \) signifies that this cross-section was obtained with the allowance for the CM motion. The constant \( C \) in equation (41) is called ‘strong collision constant’ in the conventional theory. It arises from the preservation of the unitarity of the S-matrices:

\[ |1 - S_a(R_0, V_0, \theta_0) S_b^*(R_0, V_0, \theta_0)| = C, \quad C \leq 2. \]  
(42)
For example, according to Griem book [22], page 43, his choice was C = 3/2. More details can be found in paper [23].

As for the upper cutoff $R_{\text{max}}$, following the conventional theory we choose it as the Debye radius

$$ R_{\text{max}} = R_{\text{Debye}} = (T/(4\pi e^2 N_0))^1/2, \tag{43} $$

(though more rigorously, it should have been $R_{\text{max}} = \min(R_{\text{Debye}}, V_0/\Delta \omega)$, where $\Delta \omega$ is the detuning from the center of the spectral line; physically, the requirements $R_{\text{max}} < V_0/\Delta \omega$ being the allowance for incomplete collisions).

By integrating analytically over $R_\theta$ in equation (40) and substituting into the result the expression for the strong collision constant $C$ from equation (41) we obtain:

$$ a_0(\sigma)_{\text{st},0} = 2\pi a_0(\beta^2)_{\text{st},0} Q_0 \left[ \ln \left[ \frac{(R_{\text{max}} - R_D)^4}{(R_{\text{min}} - R_D)^4} \right] + \frac{[R_{\text{max}}^2 + R_D^2]}{(4R_D^2)} \ln \left[ \frac{(R_{\text{max}}^2 - R_D^2)}{(R_{\text{max}}^2 - R_D^2)} \right] \right]. \tag{44} $$

The boundary $R_{\text{min}}$ between the weak and strong collisions in equation (44) is the solution of equation (41) with respect to $R_{\text{min}}$:

$$ R_{\text{min}} = R_D \left[ \exp (2CR_D^2/a_0(\beta^2)_{\text{st},0} Q_0) + 1 \right] \left[ \exp (2CR_D^2/a_0(\beta^2)_{\text{st},0} Q_0) - 1 \right]^{1/2}. \tag{45} $$

The next step is the averaging of several quantities from the above equations over Stark sublevels of the upper and lower levels, so that each of these quantities will have the unique value for the particular hydrogen spectral line. First, the square root of the averaged matrix element $\langle |\alpha(\beta(K^2)|\beta(\alpha)| \rangle$ is asserted to be

$$ [\alpha(\beta^2)_{\text{st},av}^{1/2} = [\alpha(K^2_{\alpha})_{\text{av}}^{1/2} - \beta(K^2_{\beta})_{\text{av}}^{1/2}], \tag{46} $$

following the conventional theory justification [22] that in this form it allows for the partial cancellation of terms in $\alpha(\beta^2)_{\text{st},0}$, when the principal quantum number $n'$ of the lower level is close to the principal quantum number $n$ of the upper level. The diagonal elements of the operators $K^2_{\alpha}$ and $K^2_{\beta}$ have the following form in the parabolic coordinates (see, e.g., [7, 26])

\begin{equation}
\alpha(K^2_{\alpha})_{\text{av}} = (9/8)n^2(n^2 + q^2 - m^2 - 1), \quad \beta(K^2_{\beta})_{\text{av}} = (9/8)n'^2(n'^2 + q'^2 - m'^2 - 1). \tag{47}
\end{equation}

The averaging over Stark sublevels (since $(q^2)_{\text{av}} = (m^2)_{\text{av}}$) results in the following leading term in the quantity $[\alpha(\beta^2)_{\text{st},av}^{1/2}$:

$$ [\alpha(\beta^2)_{\text{st},av} = (9/8)(n^2 - n'^2). \tag{48} $$

We mention that the same result (49) can be obtained after the corresponding averaging in the spherical quantization.

We denote

$$ R_{WA}(C) = [\alpha(\beta^2)_{\text{st},av} Q_0/C]^{1/2} = (3/C)^{1/2}(n^2 - n'^2)hZ/(2\mu V_0). \tag{49} $$

This quantity has the meaning of the so-called Weisskopf radius: it is defined here more accurately than in the conventional theory by Griem [22] (which is why here and below the superscript 'A' stands for 'accurate')—see appendix B.

The next quantity to be averaged over Stark sublevels of the upper and lower levels, so that it will have the unique value for the particular hydrogen spectral line, is the quantity $D$ from equation (39). The result reads:

$$ \langle D \rangle_{\text{av}} = (n^2 + n'^2)Ze^2a_B/4. \tag{50} $$

After substituting this into the definition of $R_D$ in equation (37), we obtain:

$$ \langle R_D \rangle_{\text{av}} = [(n^2 + n'^2)Z/2]^{1/2}/(\mu V_0). \tag{51} $$

Thus, from equations (45), (49), and (51), we get the unique value $(R_{\text{min}})_{\text{av}}$ for the entire hydrogen spectral line:

$$ \langle R_{\text{min}} \rangle_{\text{av}} = \langle R_D \rangle_{\text{av}} \left[ \exp (2\langle R_D \rangle_{\text{av}}^2/R_{WA}(C)^2) + 1 \right] / \left[ \exp (2\langle R_D \rangle_{\text{av}}^2/R_{WA}(C)^2) - 1 \right]^{1/2}. \tag{52} $$

As the last step we substitute $R_{\text{min}}$ by $(R_{\text{min}})_{\text{av}}$ and $R_D$ by $(R_D)_{\text{av}}$ in equation (44), and also introduce dimensionless parameters

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1. Griem explicitly chose $3/2$ for the quantity $1 - S_\alpha(R_{\text{av}}, \theta_0) S_\beta(R_{\text{av}}, \theta_0)$ that we denoted as $C$. To avoid any confusion we note that what Griem called 'strong collision term' was $C/2$. The extra factor $1/2$ arises from the following integral for the strong collision term:

$$ (1/\rho_{\text{min}}^2) \int_{0}^{\sqrt{\rho_{\text{max}}}} d\rho C = C/2. \tag{53} $$

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2. On page 43 of book [22], Griem explicitly chose $3/2$ for the quantity $1 - S_\alpha(R_{\text{av}}, \theta_0) S_\beta(R_{\text{av}}, \theta_0)$ that we denoted as $C$. To avoid any confusion we note that what Griem called 'strong collision term' was $C/2$. The extra factor $1/2$ arises from the following integral for the strong collision term:

$$ (1/\rho_{\text{min}}^2) \int_{0}^{\sqrt{\rho_{\text{max}}}} d\rho C = C/2. \tag{53} $$
In the utmost right part of equation (34), the temperature \(T\) is in eV and the electron density \(N_e\) is in cm\(^{-3}\). While deriving equation (35), the quantity \(1/V_0\) in the expression for \(\langle R_{\Omega}\rangle_{av}\) (given by equation (35)) was substituted by its average over the Maxwell distribution \((1/V_0) = [2m_e/(\pi T)]^{1/2}\)—just as in the conventional theory [22]. For the Lyman lines the expression for \(w\) should be modified to

\[
w = \left[\frac{\epsilon n}{(\mu T)}\right](2m_e N_e)^{1/2} = 1.27 \times 10^{-9} n [Z N_e m_e / m_p]^{1/2} / T,
\]

where

\[
m_e = (m_e + m_p) m_e / (m_e + m_p + m).
\]

The matrix element \(\gamma_{\omega_{\alpha\beta}W^2}\) cancels out from this ratio, so that it becomes indeed a universal function of just two dimensionless parameters \(w\) and \(b\) applicable for any set of the plasma parameters. In the most frequent case, where \(R_{\max}\) is equal to the Debye radius \(R_D\) (given in equation (37)), the parameter \(w\) can be expressed as follows

\[
w = \left[2\epsilon n/(\mu T)\right](n_e^2 + n^2) Z N_e m_e / m_p^{1/2} / T,
\]

where

\[
m_e = (m_e + m_p) m_e / (m_e + m_p + m).
\]

The second example is plasmas in the atmospheres of flare stars. They are characterized by practically the same range of plasma parameters as the edge plasmas of magnetic fusion machines—see, e.g., book [28] and paper [29].

For both the edge of magnetic fusion machines and the atmospheres of flare stars, for the \(H_\alpha\) line emitted from a hydrogen plasma at \(N_e = 5 \times 10^{14}\) cm\(^{-3}\) and \(T = 1\) eV, the ratio from equation (59) yields 1.19 for \(C = 2\) and 1.13 for \(C = 3/2\). Figure 1 presents this ratio (for the \(H_\alpha\) line emitted from a hydrogen plasma) versus the electron density \(N_e\) at \(T = 1\) eV for \(C = 2\) (solid line) and for \(C = 3/2\) (dashed line). It is seen that the allowance for the CM motion increases the ion dynamical Stark width of the \(H_\alpha\) line in these kinds of plasmas by up to (15–20)%.
C = 3/2 (dashed line). It is seen that the allowance for the CM motion increases the ion dynamical Stark width of the H\(\alpha\) line in these kinds of plasmas by up to (15–20)\%.

4. Conclusions

We studied the general problem whether the CM motion and the relative motion can be separated for hydrogenic atoms/ions in a nonuniform electric field. We demonstrated that, strictly speaking, they cannot be separated. Then we used the approximate analytical method of the separation of rapid and slow subsystems to achieve the pseudoseparation of the CM and relative motions for hydrogenic atoms/ions in an arbitrary nonuniform electric field. This is a fundamental result in its own right.

Next we further developed these results for the case of a hydrogen atom in the nonuniform electric field, where the field is due to the nearest (to the hydrogen atom) ion in a plasma. We showed that the effect of the CM motion can be formally taken into account via the substitution of the initial relative velocity \(V_0\) in the pair atom—ion by an effective velocity \(V_{\text{eff}}\) that depends on the quantum numbers of the hydrogen atom, as well as on the initial separation \(R_0\) in the pair atom—ion and on the ion charge \(Z\).

Then we applied the results to the ion dynamical Stark broadening of hydrogen lines in plasmas. We obtained analytical results for the cross-sections of the optical collisions that control the corresponding Stark width. We presented specific examples of laboratory plasmas (such as magnetic fusion plasmas or plasmas of radiofrequency discharges) and astrophysical plasmas (such as in atmospheres of flare stars) where the allowance for these CM effects leads to a significant increase of the width of hydrogen spectral lines—by up to (15–20)\%.

Thus, in addition to the fundamental importance, the results of the present paper seem to have also practical importance for spectroscopic diagnostics of laboratory and astrophysical plasmas.
Appendix A. Validity condition for the ion dynamical Stark broadening

The ion dynamical Stark broadening of hydrogen spectral lines is effective when the number $\nu_{WI}$ of perturbing ions in the sphere of the ion Weisskopf radius is smaller than unity—see, e.g., review [7]. (In the opposite case of $\nu_{WI} \gg 1$, the perturbing ions can be treated in the quasistatic approximation.) By using the ion Weisskopf radius $R_{WA}(C)$ defined in equation (49), one arrives to the following validity condition:

$$\nu_{WI}(C) = \left[3^{1/2}/(2C^{3/2})\right](\mu_{r}/T)^{3/2}[(n^2 - n'^2)\hbar/\mu]^3Z^2N_e < 1. \quad (A.1)$$

For $C = 3/2$ (which is the choice of the strong collision constant in the conventional theory by Griem [22]) the numerical coefficient in the first brackets in the right side of equation (A.1) becomes $2^{1/2}/3$. Thus, the ion dynamical Stark broadening can become effective for the most intense hydrogen spectral lines (i.e., for low values of $n$ and $n'$) in plasmas of relatively low electron densities.

Under the condition (A.1), for the overwhelming majority of perturbing ions, the frequency of the variation of the ion field $v_i/R_N$, where $R_N$ is the mean interionic distance, exceeds the instantaneous Stark splitting in the ion field. Therefore the above requirement is called the modulation-type condition.

We note that there is another condition given by equation (82) in Griem book [22]: $v_i/R_N > \gamma_e$, where $\gamma_e$ is the electron impact width. This kind of requirement is called the damping-type condition. While both the modulation-type condition and the damping-type condition are necessary, the modulation-type condition (A.1) is more restrictive: it requires the electron (and ion) density to be by the factor $\sim(m_e/m_i)^{3/4} \sim 300$ smaller than the damping-type condition. Thus, the modulation-type condition overrides the damping-type condition.

Appendix B

In paper [23] it was shown that choices of the strong collision constant $C$ and of the Weisskopf radius $\rho_W$ are interconnected, so that the latter depends on the former (for the example of $Z = 1$):

$$\rho_W(C) = [2/(3C)]^{1/2}[(\mu_{r}/V_0)][(r_a^2)_{av}/2 - (r_b^2)_{av}/2]/a_b. \quad (B.1)$$

Kepple and Griem [21], while choosing $C = 1$, used the following expression for the Weisskopf radius

$$\hbar(n^2 - n'^2)/(\mu_{r}/V_0) = \rho_{WG}(1) \quad (B.2)$$

(see, e.g., [7, 26]). Since $(q^2)_{av} = (m^2)_{av}$, then

$$\langle\alpha|\alpha\rangle (n^2 - 1)/(9/8) = (9/8)n^2(2l + 1) \quad (B.3)$$

(see, e.g., [7, 26]). The same result can be obtained from the well-known expression for the matrix element of $r_a^2$ in the spherical quantization $(n, l, m)$—see, e.g., Landau–Lifshitz’s textbook [6]:

$$\langle nlm|a_b^2|nlm\rangle = (9/4)n^2(2l + 1) \quad (B.5)$$

Indeed,

$$\langle nlm|a_b^2|nlm\rangle_{av} = \sum_{l=0}^{n-1}(n^2 - l^2 - l - 1)(2l + 1) = (9/8)n^2(2l + 1) \quad (B.6)$$

i.e., the same result as obtained above in the parabolic quantization.

Therefore, the leading term in $\langle nlm|a_b^2|nlm\rangle_{av}$ is $9n^4/8$ rather than the quantity $3n^4/2$ used by Griem. Substituting this more accurate result in equation (24) we obtain the following more accurate expression for the Weisskopf radius

$$\rho_{WA}(C) = (3/C)^{1/2}(\hbar/n^2)/(2m_rV_0) \quad (B.7)$$

for $Z = 1$ and $\mu = m_r$, or

$$\rho_{WA}(C) = (3/C)^{1/2}(\hbar/n^2)Z/(2\mu V_0) \quad (B.8)$$

for any $Z$ and $\mu$. 
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