Non-Equilibrium Steady State generated by a moving defect: the supersonic threshold

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We consider the dynamics of a system of free fermions on a 1D lattice in the presence of a defect moving at constant velocity. The defect has the form of a localized time-dependent variation of the chemical potential and induces at long times a Non-Equilibrium Steady State (NESS), which spreads around the defect. We present a general formulation which allows recasting the time-dependent protocol in a scattering problem on a static potential. We obtain a complete characterization of the NESS. In particular, we show a strong dependence on the defect velocity and the existence of a sharp threshold when such velocity exceeds the speed of sound. Beyond this value, the NESS is not produced and remarkably the defect travels without significantly perturbing the system. We present an exact solution for a $\delta$–like defect traveling with an arbitrary velocity and we develop a semiclassical approximation which provides accurate results for smooth defects.

Recent experimental advances in the context of cold atoms [1–16] converted the study of out-of-equilibrium closed quantum systems from an academic debate to a concrete and extremely active research topic. In this context, the simplest protocol is known as \textit{quantum quench} [17], where the system is brought out-of-equilibrium by a sudden change of a coupling constant. Despite the unitary time evolution, in the thermodynamic limit, local observables reach a time-independent expectation value and the system locally equilibrates [18]. One dimensional systems have played a special role because of the presence of special techniques, such as \textit{conformal field theory} [19] and \textit{integrability} [20, 21]. In particular, \textit{homogeneous quenches} a global parameter is modified and several studies with a number of integrable models and initial conditions [22–31] (see also [32] for a review) have confirmed the validity of the Generalized Gibbs Ensemble (GGE) [33]. The GGE hypothesis prescribes that the steady state assumes a thermal form, but with an extended set of temperatures, conjugated to each (quasi) local conserved quantity present in the model [33, 34]. The value of such temperatures is then fixed by the initial expectation value of conserved quantities [35–49].

A more complex scenario emerges for \textit{inhomogeneous quenches}, where, because of an asymmetry in the initial state or in the final Hamiltonian, translational invariance is explicitly broken. The simplest case is the one in which a localized defect perturbs an otherwise homogeneous system. As the spreading of correlations is bounded by a maximal speed of sound $v_s$, this defect cannot affect immediately the whole (thermodynamically large) system [50, 51]. In the presence of \textit{ballistic} dynamics, at late times $t$ and large distance $x$ from the defect, the system reaches a \textit{Locally Quasi-Stationary State} (LQSS) [52], whose properties depend only on the ray $\zeta = x/t$ inside the lightcone $|\zeta| < v_s$. The infinite time limit at finite distance (i.e. $\zeta = 0$) corresponds instead to the Non-Equilibrium Steady State (NESS). Our present understanding of LQSS is based on numerical studies [53–55], free models [56–68], CFT [69–74] and only recently on truly interacting integrable models [52, 75–79]. In particular, the hydrodynamic description [52, 78–80] has led to exact results with possible applications to several contexts [81–86].

In this Letter, we consider instead a \textit{moving defect}. This setting offers an additional parameter to control the stationary state, particularly interesting in those systems whose excitation possess a maximum velocity. Moving impurities have already been experimentally probed, in particular in [15], the case of a Tonks Girardeau gas [87] (intimately linked to the free fermion case here analyzed) was considered. While several works have considered moving impurities in several contexts [88–98], the long-time out-of-equilibrium dynamics has never been addressed so far. In particular, does the system still reacts forming a LQSS? How the LQSS changes for different velocities of the defect? We explore these questions in the prototypical case of hopping fermions on a lattice, which is amenable to a full analytical treatment still retaining a rich phenomenology. At time $t > 0$, the dynamics is governed by the following time dependent Hamiltonian:

$$H = \sum_j -\frac{1}{2} \left( d_j^\dagger d_{j+1} + d_{j+1}^\dagger d_j \right) + V(j - vt)d_j^\dagger d_j .$$

where $d_j$ are the fermionic operators $\{d_j, d_j^\dagger\} = \delta_{j,i}$. With the current choice of couplings, the unperturbed system has a sound velocity $v_s = 1$. This lattice model can be mapped in the XX spin chain [99–101], where the defect plays the role of a traveling magnetic impurity. The system is assumed to be initially in the unperturbed ground state ($V = 0$) at fixed particle number (the finite temperature case being a trivial generalization). Here, the impurity is suddenly created and put in motion; other similar settings (e.g. the motion of a pre-existing defect)
would lead to the same late time physics. We show the emergence of a moving LQSS, whose rays refer to the instantaneous position of the defect. The amplitude of the LQSS is showed to be suppressed when the velocity of the defect is increased and the formation of a LQSS becomes impossible for a supersonic impurity. We fully determine the exact LQSS generated by a \( \delta \)-like perturbation and provide a semiclassical expression of the LQSS, valid for smooth defects. Even though the model we are considering is free, the emergent LQSS displays absolutely non-trivial features understood thanks to its exact description, while approximated methods commonly used could lead to misleading conclusions (see [102] for the discussion of the Luttinger-Liquid approximation [103, 104]).

**The LQSS as a scattering problem.** — The effect of the moving impurity is best understood in a pictorial representation where the initial state can be regarded as a gas of excitations. The excitations move freely in the space until the defect is met, then a scattering event takes place and the excitation continues in a free motion, with a different momentum. The non-trivial LQSS is due to the scattered particles, spreading ballistically from the defect. Thereafter, we make this argument rigorous. The initial state is a filled Fermi sea, with gaussian correlations. As the post-quench Hamiltonian is quadratic, all the local observables at any time are fixed by the two-point correlators via the Wick theorem. Thereafter, we focus only on the case \( v > 0 \). Changing the reference frame to set the defect at rest would remove the explicit time-dependence of the problem, but such a program is foiled by the discreteness of the lattice. This difficulty can be circumvented through a map to a continuous fermionic model \( \{ c_x, c_x^\dagger \} = \delta(x - y) \) with Hamiltonian

\[
H_c = \int dx - \frac{1}{2} \left( c_x^\dagger c_{x+1} + c_{x+1}^\dagger c_x \right) + V(x - vt) c_x^\dagger c_x . \tag{2}
\]

From this model all the discrete correlation functions are exactly recovered. Indeed, \( H_c \) only couples a coordinate \( x \) with \( x + n, n \in \mathbb{Z} \). On this sublattice, continuous and discrete (normal ordered) correlation functions satisfy the same equation of motion, thus their solution is the same provided consistent initial conditions \( \langle c_i^\dagger c_i \rangle_{t=0} = \langle d_i^\dagger d_i \rangle_{t=0} \) have been chosen. We can thus employ (2) to study the dynamics of the system and later restrict ourselves to integer positions. This approach leaves us the freedom of arbitrarily choosing the correlator at non-integer values; a convenient choice is to assume the initial state in the momentum space is described by the same Fermi sea of the discrete model. As \( x \) is a continuous coordinate, we can now introduce a reference frame \( c_x = c_{x-vt} \) where the defect is at rest. In terms of this new field, the equation of motion can be derived from the time-independent Hamiltonian

\[
\tilde{H}_c = \int dx iv c_x^\dagger \partial_x c_x - \frac{1}{2} \left( c_x^\dagger \partial_x c_{x+1} + c_{x+1}^\dagger \partial_x c_x \right) + V(x) c_x^\dagger c_x . \tag{3}
\]

Clearly, the dynamics induced by (3) depends on the details of \( V(x) \). However, being the defect localized, we can use scattering theory. We introduce the mode operators \( \eta_k = \int dx \psi_k(x) c_x \) where the \( \psi_k(x) \) is the normalized wavefunction satisfying the Lippmann-Schwinger equation [105]. In other words, far away from the defect, it assumes the simple form of a scattering problem [123]

\[
\psi_k(x) = \theta(-vx(k)) \frac{e^{ikx}}{\sqrt{2\pi}} + \sum_{k_n} S_{k-k_n} \theta(xv(k_n)) \frac{e^{ik_nx}}{\sqrt{2\pi}}, \tag{4}
\]

where the incoming wave is expanded into outgoing waves weighted with the scattering amplitudes \( S_{k-k_n} \). The wavevectors \( k_n \) are obtained via energy conservation \( E(k) = E(k_n) \), with \( E(k) = -\cos(k) - vk \) the single-particle energy in the defect reference frame and \( v(k) = dE(k)/dk \) its group velocity. Corrections to (4) vanish exponentially in the distance and the scattering amplitude takes the form

\[
S_{k-k_n} = \delta_{k,k_n} - 2i\pi |v(k_n)|^{-1} \langle k_n | \tilde{V} | \psi_k \rangle \tag{5}
\]

where we introduced a bra/ket notation with \( \langle x|k \rangle = e^{ikx}/\sqrt{2\pi} \) and \( \tilde{V}(x')|x \rangle = V(x') \delta(x - x')|x \rangle \). The unitarity of the Lippmann-Schwinger equation permits to derive an exact sum rule whose explicit derivation is left to [102]. The initial two point correlator is diagonal in the momentum space, thus we are ultimately led to consider the time-evolution of plane waves \( f_k(x,t) = \langle x|e^{-iHt}|k \rangle \). In terms of the eigenbasis \( |\psi_q \rangle \) is

\[
f_k(x,t) = \sqrt{2\pi} \int dq e^{-iE(q)t} \langle \psi_q(x)|\psi_k \rangle . \tag{6}
\]

The large time behavior of \( f_k \) is readily extracted using (4) together with the aforementioned sum rule for \( S \), as the corrections to the asymptotic approximation (4)
are ineffective in the LQSS scaling limit [102]. We can then obtain the LQSS two-point correlator in the form of a ray-dependent GGE with an excitation density \( \rho_\zeta(k) \), being the ensemble gaussian and (locally) diagonal in the momentum space [106–109]

\[
\langle \hat{c}_x^\dagger \hat{c}_y \rangle_t = \int \frac{dk}{2\pi} \rho_\zeta(k) e^{i k (y-x)} = \int_{-k_f}^{k_f} \frac{dk}{2\pi} e^{i k (y-x)} \left[ 1 - \theta(\zeta v(k))\theta(|v(k)| - |\zeta|) \right] + \sum_{k_n} |S_{k \to k_n}|^2 \theta(v(k_n))\theta(|v(k_n)| - |\zeta|) e^{ik_n(y-x)}.
\]

(7)

As the two point correlator is a decaying function in the relative distance, in the large time limit, one must set \( x/t \sim y/t \sim \zeta \). Eq. (7) is easily interpreted: when a particle of momentum \( k \) collides with the defect, it can scatter in several channels \( k_n \) with amplitudes \( S_{k \to k_n} \). This produces a hole in the old momentum and a flux of particles with the new momenta: these spread ballistically at their own velocity \( v(k_n) \) leading to the ray dependence on \( \zeta \). In Fig. 1, we plot the space-time profile of the fermion density for a \( \delta - \) like defect, whose detailed analysis will be considered later. With no further information on \( S \), we can discuss the behavior of the LQSS in terms of \( v \) analyzing the scattering channels, identified by \( E(k_n) = E(k) \). For generic \( v \), several channels are open (a divergent number letting \( v \to 0 \)) and they diminish increasing \( v \) until, for a supersonic defect, the excitations are unavoidably purely transmitted. In this case, the sum rule obeyed by \( S \) forces \( |S_{k \to k}| = 1 \) [102] (simply interpreted as particle conservation). Therefore, in the supersonic regime, the LQSS can never be produced (see also Fig. 1). Note that the LQSS (7) emerges at late times and large distances from the defect, when Eq. (4) holds: despite the absence of LQSS, a supersonic defect still gives non-trivial effects localized on the impurity. We will further analyze this behavior within the semiclassical approximation.

An exactly solvable case. — We now consider an example of defect for which the LQSS can be exactly determined, i.e. the limit of an extremely narrow defect \( V(x) = c \delta(x) \). In the discrete model (1) the Dirac-\( \delta \) is ill-defined when \( v = 0 \). However, this is not the case for a defect in motion \( v \neq 0 \); the \( \delta - \) defect represents an impulsive kick traveling along the lattice and leads to well-defined equations of motion. The detailed calculations can be found in [102], here we simply report the result. Referring to eq. (5), we have

\[
2\pi \langle k_n|V|\psi_k \rangle = -\left[ \sum_{k_m} \frac{1}{2|v(k_m)|} - \frac{1}{2v} \cot \left( \frac{c}{2v} \right) + \mathcal{I}(k) \right]^{-1}.
\]

(8)

Above, \( \mathcal{I}(k) = \mathcal{P} \int \frac{dq}{2\pi} (E(k) - E(q))^{-1} \), where the principal value prescription in integrating the singular points is assumed. In Fig. 2 the exact solution for the fermion density is tested against numerical simulations. Numerical data show persistent oscillations due to the interference of the various scattering channels. These oscillations decay far away from the defect and are therefore inessential in the LQSS scaling limit, but are nevertheless captured by scattering theory (see [102]). The non-trivial density profile is only one of the manifestations of the LQSS, being the complicated structure of the underlying scattering best appreciated in the excitation density profile propagating from the defect (Fig. 3).

The semiclassical approximation. — For general potential, determining the scattering amplitudes requires some approximation schemes. Here, we develop a semiclassical analysis in which the scattering interpretation is most easily displayed. Our derivation is based on the Wigner distribution [110–112]. Semiclassical approaches are commonly found in literature [113–117] (see also [118–121] for quantum corrections) even though, at the best of our knowledge, the problem at hand has never been addressed. Consider the two-point correlator \( \mathcal{C}(x,s) = \int \frac{dk}{2\pi} e^{i k (x-s)} \rho_\zeta(k) \)
The terms of the Wigner function approximation are verified for a smooth potential $V(x)$. Being the initial state homogeneous, this is the classical Hamiltonian. In the semiclassical approximation, the density distribution of particles subjected to the classical equations of motion $\dot{x} = v(k)$, $k = -\partial_x V(x)$. The precise solution $x(t), k(t)$ of these equations depends on the details of $V(x)$, but the trajectories $k(x)$ of the particles can be easily computed combining the conservation of the classical energy and the determination of the turning points, i.e., those points where $dx/dt = 0$ (see [102]). In the semiclassical language, computing the LQSS is reduced to simple classical scattering processes: whether a particle is reflected or transmitted by the defect is simply determined reconstructing the trajectories. The case of a supersonic defect is remarkably simple, since the equation of motion does not have turning points. In this case we readily obtain

$$\lim_{t \to \infty} C_t(x, s) = \int_{-k_f}^{k_f} \frac{dk}{2\pi} \frac{v(k)}{v(q_{k,x})} e^{-iq_{k,x} s},$$

valid also in proximity of the defect (within the semiclassical approximation). Above, $q_{k,x}$ is determined by energy conservation

$$-\cos(q_{k,x}) - \nu q_{k,x} + V(x) = -\cos(k) - \nu k.$$

which has a unique solution for a supersonic $\nu$. The semiclassical results and the numerics are compared in Fig. 4, further details about the semiclassical approximation are left to [102].

Conclusions and outlook. We analyzed a local quench where the defect is moving at constant velocity in a system of hopping fermions on a lattice. In particular, we focused on the emergence of a Locally-Quasi-Stationary State and we studied the dependence of the latter in terms of the velocity of the defect. With general arguments, we showed the impossibility of a LQSS formation for supersonic defects. We provided exact results for a $\delta-$like defect and a semiclassical analysis for general shapes. We are confident that our framework can be studied in forthcoming cold-atoms experiments, as in Ref. [15], a very similar setting was realized. The moving defect would then be a intriguing way to induce nonequilibrium dynamics and to probe the scattering properties of quasi-particle excitations.

Being the hopping fermions a free model, its simplicity allowed for exact computations: an intriguing question concerns the same problem in a truly interacting model like integrable spin chains. We expect that a promising investigation line could be the recently introduced generalized hydrodynamics [52, 78–86], which for a free model, reduces precisely to the semiclassical approach we used. This will be the subject of a forthcoming publication. Another natural approach would be the use of the recently introduced curved-CFT formalism [122].

Acknowledgment. We are grateful to Mario Collura, Pasquale Calabrese, Fabian Essler and Giuseppe Mussardo for useful discussions. This work was supported by

FIG. 4: The numerical fermionic density is tested against the semiclassical LQSS for a gaussian shaped repulsive potential $V(x) = e^{-x^2}$ with $\sigma = 0.04$, $k_f = \pi/2$. Top: subsonic defect ($\nu = 0.3$) and $t = 636$. The defect is placed where the discontinuity occurs and a LQSS is produced. Bottom: supersonic defect ($\nu = 1.5$, $t = 388$) and placed in correspondence with the rightmost peak: the LQSS is indeed absent and the semiclassical prediction captures the density profile on the defect (inset). The corrections to the analytic prediction are a combined effect of $i)$ the particles initially sat on the defect that have not yet managed to spread $ii)$ the delay time experienced by the particles scattering on the defect. Both these effects are negligible at late time and in the scaling limit. Quantum effects can be recognized in the propagating ripples [118–120], more evident in the supersonic case.
the EPSRC Quantum Matter in and out of Equilibrium Ref. EP/N01930X/1 (A.D.L.).

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Supplementary Material

Non Equilibrium Steady State generated by a moving defect: 
the supersonic threshold

Here we report the technical details of the results presented in the Letter. The Supplementary Material is organized as follows

1. In Section we briefly discuss the moving defect in the Luttinger Liquid’s approximation, showing the incapability of the latter of describing any LQSS in the present context.

2. In Section we apply the general framework of the Lippmann-Schwinger equation to our scattering problem. In particular, in Section we derive the sum rules obeyed by the scattering amplitudes and in Section we derive the LQSS.

3. In Section we exactly compute the scattering data for a δ−like defect.

4. In Section we present a detailed explanation of the semiclassical approximation.

THE LUTTINGER LIQUID APPROXIMATION

The Luttinger Liquid approach [103, 104] provides a well established approximated description of the low energy sectors of several gapless one dimensional models, among those the hopping fermions system at hand. However, it is not limited to free models and it widely succeeds in exploring non-perturbative effects in interacting models. It is then natural to investigate the moving defect in this approximation scheme, aiming at identifying the universal features. However, when applied to the moving impurity, the Luttinger Liquid is unable to capture the out-of-equilibrium steady state that is observed on the lattice. The Luttinger Liquid maps the low energy sector of interacting fermions in a free massless bosonic field with Lagrangian

\[ \mathcal{L}_\text{LL} = \int dx \left( \frac{1}{K v_s} \partial_\mu \phi \partial^\mu \phi \right). \tag{S1} \]

Where \( K \) is the adimensional Luttinger parameter, whose value depends on the interaction (\( K = 1 \) for free fermions), \( v_s \) is the sound velocity that it appears also in the covariant derivative \( \partial_\mu = (\partial_t, v_s \partial_x) \). Correlation values of the fields are recovered from the vertex operators and the descendants of the field \( \phi \). For example, it holds the correspondence

\[ c_j^\dagger c_j \simeq n_0 - \frac{1}{\pi} \partial_x \phi, \tag{S2} \]

where \( n_0 \) is the background constant fermionic density. In this perspective, the moving defect in the Luttinger Liquid approach would be described by a Lagrangian

\[ \mathcal{L}_\text{LL} = \int dx \frac{1}{K v_s} \partial_\mu \phi \partial^\mu \phi + \frac{1}{\pi} V(x - vt) \partial_x \phi. \tag{S3} \]

The Luttinger Liquid approximation holds as long as we remain in the low energy sector, i.e. for weak and smooth defects. This simple Luttinger description is further enriched in literature through the so-called depletion model [92, 93, 95, 97, 98], where an impurity of finite mass is phenomenologically coupled to a Luttinger Liquid mimicking the low energy excitations of the surrounding environment, but such an analysis goes beyond our current purposes. We discuss now the solution of the out-of-equilibrium protocol: from the quantum equation of motion

\[ \partial_\mu \partial^\mu \phi = - \frac{K v_s}{\pi} \partial_x V(x - vt) \tag{S4} \]

we immediately see that the potential term can be absorbed through a shift of the bosonic field (if \( v \neq v_s \))

\[ \phi(x, t) = \tilde{\phi}(x, t) + J(x - vt) \tag{S5} \]

where

\[ J(x) = - \int x \frac{K v_s V(x')}{\pi(v^2 - v_s^2)}. \tag{S6} \]
Above, the integration constant of the indefinite integral will not affect any observable and can be left unspecified. The field \( \phi \) evolves with the homogeneous equation of motion \( \partial_t \phi \partial^\mu \phi = 0 \) and thus it can be split in left/right movers and the solution easily follows using that, at time \( t \) in the kinetic and potential term we report the \( t > 0 \) evolution of the density fluctuations, being similar conclusions valid for other observables

\[
\langle \partial_x \phi(x, t) \rangle = \frac{K v_s V(x + v_s t)}{2\pi(v^2 - v_s^2)} \left( 1 - \frac{v}{v_s} \right) + \frac{K v_s V(x - v_s t)}{2\pi(v^2 - v_s^2)} \left( 1 + \frac{v}{v_s} \right) - \frac{K v_s V(x - vt)}{\pi(v^2 - v_s^2)} . \tag{S7}
\]

We can recognize three contributions. Two of them spread at sounds velocity \( v_s \) from the initial position of the defect: these are simply the signal of having activated the defect that spreads in the homogeneous system. The third contribution instead moves with the defect, thus at velocity \( v \). For any velocity, the LQSS is absent and not peculiar behavior emerges passing from a subsonic to a supersonic regime. The Luttinger Liquid result is consistent with our scattering theory, in fact an incoming particle at wavevector \( k \) can scatter in an outgoing particle at vector \( q \) only if it respects the energy conservation

\[-vk - \cos(k) = -vq - \cos(q) . \tag{S8}\]

In the Luttinger Liquid paradigm only small wavevectors around the edges of the Fermi sea are considered, thus if we linearize around the right edge (the left case is analogous) \( k \to k_F + \delta k \) and \( q \to k_F + \delta q \) we obtain

\[-v\delta k - v_s \delta k = -v\delta q - v_s \delta q , \tag{S9}\]

that has always the unique solution (if \( v \neq v_s \) \( \delta k = \delta q \). This means that the particles do not scatter and are purely transmitted by the defect without producing an LQSS. As we observed in the main text, this conclusion is only true for a supersonic defect; indeed, the linearization of the dispersion relation at the basis of the LL derivation does not capture the several scattering channels emerging in the subsonic case.

**EIGENSTATES IN THE ASYMPTOTIC REGION AND LIPPMANN-SCHWINGER EQUATION**

The modes of the continuous Hamiltonian (Eq. (3) in the main text) in the defect reference frame are solutions of the time-independent version of the Schrödinger equation

\[ E\psi(x) = iv\partial_x \psi(x) - \frac{\psi(x + 1) + \psi(x - 1)}{2} + V(x)\psi(x) . \tag{S10} \]

Assuming a localized potential around \( x = 0 \), we can use the scattering theory to describe the eigenfunctions. In particular, the latter can be labeled in terms of the asymptotic in-states (plane waves) and must satisfy the Lippmann-Schwinger equation \([105]\]

\[ |\psi_k\rangle = |k\rangle + \frac{1}{E(k) - T + iv0^+} V|\psi_k\rangle , \tag{S11} \]

where we adopted a bra/ket notation. Above, \( \langle x|k\rangle = e^{ikx}/\sqrt{2\pi} \), \( E(k) = -\cos(k) - vk \) and we split the Hamiltonian in the kinetic and potential term

\[ \langle x|T|\psi_k\rangle = iv\partial_x \psi_k(x) - \frac{\psi_k(x + 1) + \psi_k(x - 1)}{2} , \quad \langle x|V|\psi_k\rangle = V(x)\psi_k(x) . \tag{S12} \]

The kinetic term is clearly diagonal in the momentum basis \( \langle k|T|q\rangle = \delta(k - q)E(k) \): inserting a decomposition of the identity in the momentum basis in Eq. (S11) and contracting it with \( \langle x| \), we immediately recover the Lippmann-Schwinger equation in the coordinate space

\[ \psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} + \int dx' K_k(x - x')V(x')\psi_k(x') , \tag{S13} \]

where the kernel \( K_k(x) \) is defined as

\[ K_k(x) = \int \frac{dq}{2\pi} \frac{e^{iqx}}{E(k) - E(q) + i0^+} . \tag{S14} \]
Aiming to describe a scattering theory, we are interested in the behavior of $\psi_k(x)$ in the asymptotic region $|x| \to \infty$. In this limit the integral in (S14) can be closed in the upper (lower) complex half-plane for $x > 0$ ($x < 0$). Then, all the poles with a non-vanishing imaginary part only give exponential corrections to the contribution of singularities on the real axis. This leads to

$$K_k(x) = \sum_{k_n} \frac{\theta(v(k_n)x)}{i|v(k_n)|} e^{ik_n x} + \ldots \quad |x| \to \infty,$$  
(S15)

where $\theta$ is the Heaviside Theta function and $v(k)$ is the classical velocity $v(k) = \sin k - v$. The scattered wavevectors $k_n$ are all and only the solutions of $E(k) = E(k_n)$, i.e. the energy conservation condition. The wavefunction in the asymptotic region is thus immediately derived

$$\psi_k(x) = \theta(-xv(k)) \frac{e^{ikx}}{\sqrt{2\pi}} + \sum_{k_n} S_{k \to k_n} \theta(xv(k_n)) \frac{e^{ik_n x}}{\sqrt{2\pi}}.$$  
(S16)

Where

$$S_{k \to k_n} = \begin{cases} \frac{\sqrt{2\pi}}{\pi|v(k)|} \int dk e^{-ik_n x'} V(x') \psi_k(x') = \frac{2\pi}{\pi|v(k)|} \langle k_n | V | \psi_k \rangle & k_n \neq k \\ 1 + \frac{\sqrt{2\pi}}{\pi|v(k)|} \int dk e^{-ikx'} V(x') \psi_k(x') = 1 + \frac{2\pi}{\pi|v(k)|} \langle k | V | \psi_k \rangle & k_n = k \end{cases}$$  
(S17)

Of course the above expressions are only an implicit solution for the scattering amplitudes that cannot be determined without the exact knowledge of $\psi_k$. This issue will be consider in Section and Section, thereafter we instead derive some conclusions valid for generic scattering amplitudes.

### Sum rules for the scattering amplitudes

The scattering amplitudes that appear in (S16) must obey proper sum rules induced by the unitarity of the Lippmann-Schwinger equation. Here we provide a derivation of these sum rules for the scattering problem at hand. The Lippmann-Schwinger equation is equivalent to unitarily evolve the asymptotic in-state with an adiabatic turning

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on of the defect [[105]], thus it preserves the scalar products and we must have $\langle \psi_k | \psi_q \rangle = \langle k | q \rangle = \delta(k - q)$. Computing the scalar product $\langle \psi_k | \psi_q \rangle$, the singular behavior is due only to the wavefunction in the asymptotic region: thus we can compute $\langle \psi_k | \psi_q \rangle$ using directly (S16). We now impose the conservation of the scalar product

$$\delta(k - q) = \int dx \psi_k^*(x) \psi_q(x) = \sum_{q_n, k_m} S_{q \to q_n}^* S_{k \to k_m} \theta(v(k_m)v(q_n)) \int dx \frac{2\pi}{\pi|v(k)|} \theta(xv(k_m)) e^{i(k_m - q_n)x} +$$

$$+ \sum_{q_n} S_{q \to q_n}^* \theta(-v(k)v(q_n)) \int dx \frac{2\pi}{\pi|v(k)|} \theta(-xv(k)) e^{i(k_m - q_n)x} + \sum_{n} S_{k \to k_n} \theta(-v(k_m)v(q)) \int dx \frac{2\pi}{\pi|v(k)|} \theta(-xv(k)) e^{i(k_m - q)x} +$$

$$+ \theta(v(k)v(q)) \int dx \frac{2\pi}{\pi|v(k)|} \theta(-xv(k)) e^{i(k_m - q)x} + \ldots$$  
(S18)

The neglected terms are the corrections due to the replacement of the true eigenfunctions with their approximation in the asymptotic region. Integrating the $\theta$ functions according to

$$\int dx \theta(x) e^{i\omega x} = iP \left( \frac{1}{\omega} \right) + \pi \delta(\omega),$$  
(S19)

where $P(\ldots)$ is the principal value, we obtain the sought singular terms. Equating the coefficients $\delta-$singular parts in the left and right-hand side of (S18) the following equality is obtained

$$\sum_{q_n, k_m} S_{q \to q_n}^* S_{k \to k_m} \delta(k_m - q_n) = \delta(k - q),$$  
(S20)

where all the $q_n$ and $k_m$ are respectively functions of $k$ and $q$, given by the energy conservation $E(k_m) = E(k)$ and $E(q_n) = E(q)$. In order to compare the non singular pre-factors of the two members, the Jacobian $dk_m/dk$ (and $dq_n/dq$) must be taken into account. Using the fact that $E(k) = E(k_n)$ we readily have

$$\frac{dk_m}{dk} = \frac{dE(k_m)}{dk_m} \left( \frac{dE(k)}{dk} \right)^{-1} = \frac{v(k)}{v(k_m)}.$$  
(S21)
With this last piece of information (and a slight change of notation in order to have a compact expression) we obtain the orthogonality condition

\[ \sum_{k_m} S_{k_m \rightarrow k_m}^* S_{k_m' \rightarrow k_m} |v(k_m)| = |v(k_n)| \delta_{n, n'}. \tag{S22} \]

The physical interpretation of the above is especially simple when \( k_n = k_{n'} \). In this case it is nothing else than the familiar normalization for the scattering matrix that sums to unity. As usually the incoming particles and outgoing particles have the same velocity (apart from signs) the extra factors \( |v(k_n)| \) do not appear. We can interpret the above as a conservation of flux: the flux of outgoing particles is given by the probability of having such a particle (i.e. the modulus square of the scattering amplitude) times its velocity. Another sum rule can be obtained similarly: being \( |\psi_k\rangle \) a complete set we must have

\[ \int dk \langle q | \psi_k \rangle \langle \psi_k | k' \rangle = \delta(q - k'). \tag{S23} \]

As in the previous calculations, the singular behavior is due to the asymptotic wavefunction (S16). The derivation of the forthcoming orthogonality relations is slightly more cumbersome compared with the previous case. In this approximation the scalar product is:

\[ \langle \psi_k | k' \rangle = \int dx \psi_k^*(x) e^{ik'x} \sqrt{2\pi} = \frac{1}{2\pi} \left( \frac{i}{\text{sgn}(v(k))(k - k') + i\epsilon} + \sum_{k_n} \frac{iS_{k \rightarrow k_n}^*}{-\text{sgn}(v(k_n))(k_n - k') + i\epsilon} \right) + \ldots, \tag{S24} \]

where the limit \( \epsilon \to 0^+ \) must be taken. Thus

\[ \int dk \langle q | \psi_k \rangle \langle \psi_k | k' \rangle = \int \frac{dk}{(2\pi)^2} \left( \frac{i}{\text{sgn}(v(k))(k - k') + i\epsilon_1} + \sum_{k_n} \frac{iS_{k \rightarrow k_n}^*}{-\text{sgn}(v(k_n))(k_n - k') + i\epsilon_1} \right) \times \left( \frac{i}{-\text{sgn}(v(k))(k - q) + i\epsilon_2} + \sum_{k_{n'}} \frac{iS_{k \rightarrow k_{n'}}}{\text{sgn}(v(k_{n'}))(k_{n'} - q) + i\epsilon_2} \right) + \ldots. \tag{S25} \]

Above we must take the limits \( \epsilon_1 \to 0^+ \) and \( \epsilon_2 \to 0^+ \) independently. Expanding the brackets we obtain several terms

\[ \int dk \langle q | \psi_k \rangle \langle \psi_k | k' \rangle = \int \frac{dk}{(2\pi)^2} \left( \frac{i}{\text{sgn}(v(k))(k - k') + i\epsilon_1} \frac{i}{-\text{sgn}(v(k))(k - q) + i\epsilon_2} + \right. \]

\[ + \sum_{k_n} \frac{iS_{k \rightarrow k_n}^*}{-\text{sgn}(v(k_n))(k_n - k') + i\epsilon_1} \frac{i}{-\text{sgn}(v(k_n))(k_n - k) + i\epsilon_2} + \]

\[ + \sum_{k_{n'}} \frac{iS_{k \rightarrow k_{n'}}}{\text{sgn}(v(k_{n'}))(k_{n'} - q) + i\epsilon_2} \frac{i}{\text{sgn}(v(k))(k - k') + i\epsilon_1} + \]

\[ + \sum_{k_{n'}, k_n} \frac{iS_{k \rightarrow k_{n'}}}{\text{sgn}(v(k_{n'}))(k_{n'} - q) + i\epsilon_2} \frac{iS_{k \rightarrow k_n}^*}{-\text{sgn}(v(k_n))(k_n - k') + i\epsilon_1} \right). \tag{S26} \]

The singular part is associated with the singularities on real axis when \( \epsilon_{1, 2} \to 0^+ \), however the singular terms are produced only when the two poles in each of the above terms pinch the real axis in the limit \( \epsilon_{1, 2} \to 0^+ \). This is verified in the first and last line. For example, consider the first integral

\[ \int \frac{dk}{(2\pi)^2} \frac{i}{\text{sgn}(v(k))(k - k') + i\epsilon_1} \frac{i}{-\text{sgn}(v(k))(k - q) + i\epsilon_2} \approx \int \frac{dk}{(2\pi)^2} \frac{1}{(k - k') + i\epsilon_1} \frac{1}{(k - q) - i\epsilon_2} = \]

\[ \approx \frac{2\pi i}{(2\pi)^2} \frac{i}{(k - q) - i\epsilon_2} = \frac{1}{2} \delta(k' - q) + \ldots. \tag{S27} \]

where we neglected non singular terms at each passage. Thus, the singular part of (S26) is:

\[ \delta(k' - q) = \int dk \langle q | \psi_k \rangle \langle \psi_k | k' \rangle = \frac{1}{2} \delta(k' - q) + \frac{1}{2} \sum_{k_{n'}, k_n} \int dk S_{k \rightarrow k_{n'}} S_{k \rightarrow k_n}^* \delta(k_{n'} - q) \delta(k_n - k'). \tag{S28} \]
Using Eq. (S21) as before to compare the non singular amplitudes of the above equation, we get a second orthogonality condition

$$\sum_{k_n} \frac{S_{k_n \rightarrow k_{m'} S_{k_n \rightarrow k_{m'}}^*}}{|v(k_n)|} = \delta_{m,m'}. \quad (S29)$$

**Evolution of the plane wave and LQSS**

As we reported in the Letter, the key point in the derivation of the LQSS is computing the evolution of the plane wave $|k\rangle$, at late times and in the asymptotic region

$$f_k(x,t) = \sqrt{2\pi} \int dq e^{-iE(q)t} \langle \psi_q | x \rangle \langle \psi_q | k \rangle . \quad (S30)$$

Being interested in the asymptotic region, we replace $\psi_q(x)$ with (S16) at the price of neglecting contributions exponentially vanishing away from the defect. This leads to

$$f_k(x,t) = \int dq e^{-iE(q)t} \left( \theta(-x\nu(q)) e^{iqx} + \sum_{q_n} S_{q \rightarrow q_n} \theta(x\nu(q_n)) e^{iq_n x} \right) \langle \psi_q | k \rangle . \quad (S31)$$

Even though we do not know exactly the overlap $\langle \psi_q | k \rangle$, being interested in the large time limit we can replace $\psi_q(x)$ with its expression in the asymptotic region (S16), i.e. Eq. (S24). In fact, only the singularities of the integrand contribute to the late time behavior and any non singular contribution decays because of the fast oscillating phases $e^{iq_n x - iE(q)t}$. At fixed ray $\zeta = x/t$, the neglected contribution will vanish as $\sim t^{-1/2}$ as one sees via the saddle point approximation (a slower decay $\sim t^{-1/3}$ is obtained if $\zeta$ is at the edges of the lightcone). Thus in this approximation we have

$$f_k(x,t) = \int dq \frac{e^{-iE(q)t}}{2\pi} \left( \theta(-x\nu(q)) e^{iqx} + \sum_{q_n} S_{q \rightarrow q_n} \theta(x\nu(q_n)) e^{iq_n x} \right) \times$$

$$\left( \frac{i}{\nu(q)(q-k)} + \sum_{q_n'} -\frac{iS_{q \rightarrow q_n'}^*}{\nu(q_n')(q_n' - k) + i\epsilon} \right). \quad (S32)$$

Again, only the singular contributions survive at late times. Using the identity (in the distribution sense)

$$\lim_{t \rightarrow \infty, \zeta \text{ fixed}} \frac{i e^{-iE(k)t} + i k_{x_n x}}{-\nu(q_n') (q_n' - k) + i \epsilon} = 2\pi e^{-iE(k)t} + i k_{x_n x} \theta(\nu(q_n')) (\zeta - \nu(q_n)) \delta(q_n' - k), \quad (S33)$$

we can compute the asymptotic evolved plane waves. Again, correction to Eq. (S33) can be estimated through the saddle point method and can be proved to vanish at least as $\sim t^{-1/3}$. Using (S33) in (S32) and (S21) to carry on the explicit integration over the $\delta-$functions we readily get

$$e^{iE(k)t} f_k(x,t) = \theta(-\zeta \nu(k)) e^{ikx} + \sum_{k_n} e^{ik_{x_n x}} \theta(\zeta \nu(k_n)) \theta(\nu(k_n) - |\zeta|) S_{k \rightarrow k_n} +$$

$$+ \sum_{q,q_n,q_n'} \left[ e^{iq_{x_n x}} \theta(\zeta \nu(q_n)) \theta(\nu(q_n) - |\zeta|) S_{q \rightarrow q_n} S_{q \rightarrow q_n'}^* \frac{\nu(q_n')}{\nu(q)} \right]_{q_n' = k}. \quad (S34)$$

The summation of the last line is greatly simplified by the orthogonality condition (S29), leading to the final result we were looking for

$$\lim_{t \rightarrow \infty} e^{iE(k)t} f_k(x,t) = e^{ikx} \theta(-\zeta \nu(k)) + e^{ik_{x_n x}} \theta(\zeta \nu(k_n)) \theta(\nu(k_n) - |\zeta|) + \sum_{k_n} e^{ik_{x_n x}} \theta(\zeta \nu(k_n)) \theta(\nu(k_n) - |\zeta|) S_{k \rightarrow k_n}. \quad (S35)$$

With this last result, we can analyze the two point correlator

$$\langle \tilde{c}_x^\dagger \tilde{c}_y \rangle_t = \int_{-k_f}^{k_f} \frac{dk}{2\pi} f_k^*(x,t) f_k(y,t) \quad (S36)$$
in the large time limit. Inserting (S35) in the above we get a sum of several oscillating phases: being interested in
the LQSS scaling limit we can rely on the further assumption of large distances from the defect and thus discard several
contributions, finally leading to the desired expression

\( \langle c_x \bar{c}_y \rangle_t = \int_{-k_f}^{k_f} \frac{dk}{2\pi} e^{ik(y-x)} [1 - \theta(v(k))\theta(|v(k)| - |\zeta|)] + \sum_{\kappa_n} |S_{k \rightarrow k_n}|^2 \theta(v(k_n))\theta(|v(k_n)| - |\zeta|) e^{i\kappa_n(y-x)}. \) (S37)

Discarding the oscillating corrections to (S37) requires some additional care. Despite being vanishing corrections,
the simple saddle point arguments predict a slow algebraic (\( \sim 1/\sqrt{|x|} \approx 1/\sqrt{|y|} \)) decay that can affect the LQSS on a
practical purpose. In particular, such oscillations are relevant for the NESS description, when only time is sent to
infinity while the distance from the defect is kept finite. Persistent oscillations are found in the case of a \( \delta \)-like defect,
that we are going to analyze.

**AN EXACT RESULT: THE \( \delta \)-LIKE DEFECT**

In this section we determine the exact eigenfunctions of the time-independent Schröedinger equation (S10) in the
case of a \( \delta \)-like defect, i.e. \( V(x) = \delta(x) \). Obviously, the exact eigenfunctions permit an exact determination of the
scattering amplitudes \( S_{k \rightarrow k_n} \) and thus of the LQSS generated by such a defect. Attempting to solve directly the
Schröedinger equation is complicated because of the presence of the hopping term, on the other hand the Lippmann-
Schwinger equation leads to some ambiguities for the limiting case \( V(x) = \delta(x) \): the best way to solve the problem
is combining the two approaches. Consider the Lippman-Swinger equation in the coordinate space (S13), specializing
to a \( \delta \)-like potential we have

\[ \psi_k(x) = \frac{e^{ikx} + \mathcal{K}_k(x)\alpha_k}{\sqrt{2\pi}}, \] (S38)

where \( \alpha_k \) is a constant formally defined as

\[ \alpha_k = c\sqrt{2\pi} \int dx' \delta(x')\psi_k(x') \] (S39)

We could be tempted of explicitly performing the integral by the \( \delta \)-function. However, \( \psi_k \) is not continuous in \( x = 0 \)
(this simply because \( \mathcal{K}_k(x) \) is discontinuous in \( x = 0 \) and therefore

\[ \int dx' \delta(x')\psi_k(x') \neq \psi_k(0) \] (S40)

The arbitrariness in the regularization of (S40) is resolved looking directly at the Schröedinger equation (S10), that
we report below for convenience

\[ E(k)\psi_k(x) = iv\partial_x\psi_k(x) - \frac{\psi_k(x+1) + \psi_k(x-1)}{2} + \delta(x)\psi(x). \] (S41)

We consider now what happens at the singularity in \( x = 0 \), matching the singular terms in the above: obviously
\( \delta(x)\psi_k(x) \) is singular. Then, \( \psi_k \) cannot be singular on its own and this leaves out only \( iv\partial_x\psi_k \) as a possible singular term.
Thus, retaining in the above only the singular terms, we formally get

\[ iv\partial_x\psi_k + \delta(x)\psi_k(x) = 0 \quad \Rightarrow \quad \psi_k(x) = e^{i\frac{x}{\hbar}} \int_{x_0}^{x} dx' \delta(x')\psi_k(x_0) . \] (S42)

Thus, the discontinuity of the wavefunction in \( x = 0 \) is determined as

\[ \psi_k(0^+) = e^{i\frac{\pi}{\hbar}} \psi_k(0^-) . \] (S43)

We can supplement the explicit form of the wavefunction (S38) with this last information and determine the constant
\( \alpha_k \)

\[
\begin{cases}
\psi_k(0^+) = \frac{1}{\sqrt{2\pi}} (1 + \mathcal{K}_k(0^+)\alpha_k) \\
\psi_k(0^-) = \frac{1}{\sqrt{2\pi}} (1 + \mathcal{K}_k(0^-)\alpha_k) \quad \Rightarrow \quad \alpha_k = \frac{1 - e^{i\frac{\pi}{\hbar}}}{e^{i\frac{\pi}{\hbar}\mathcal{K}_k(0^-) - \mathcal{K}_k(0^+)}}. \quad (S44)
\end{cases}
\]
subtracting the poles we can thus write (S15) has been extracted from the poles along the real axis in the integral (S14). Adding and produced and the system is completely unaware of the traveling defect. We now proceed in computing K waves, i.e. the eigenvectors of the Hamiltonian in absence of the defect: in this case neither LQSS or any transient is periodicity is fulfilled by the exact eigenvectors. In particular, for \( c/v = 2\pi\nu \) the eigenvectors simply become the plane waves, i.e. the eigenvectors of the Hamiltonian in absence of the defect: in this case neither LQSS or any transient is produced and the system is completely unaware of the traveling defect. We now proceed in computing \( K_k(x) \). The asymptotic expansion (S15) has been extracted from the poles along the real axis in the integral (S14). Adding and subtracting the poles we can thus write

\[
K_k(x) = \sum_{k_n} \frac{\theta(v(k_n)x)}{i|v(k_n)|} e^{ik_n x} + \int \frac{dq}{2\pi} \left( \frac{e^{iqx}}{E(k) - E(q) + i\epsilon} - \sum_{k_n} \frac{e^{iqx}}{-v(k_n)(q - k_n) + i\epsilon} \right). \tag{S45}
\]

Now, the term in brackets is no longer singular when \( \epsilon \to 0 \), thus we can directly take the limit. Consider now the integral

\[
\tau(k, x) = \int \frac{dq}{2\pi} \left( \frac{e^{iqx}}{E(k) - E(q)} + \sum_{k_n} \frac{e^{iqx}}{v(k_n)(q - k_n)} \right). \tag{S46}
\]

\( \tau \) has a continuous and a discontinuous contribution when \( x \to 0^\pm \). In order to extract it, we split the integral into two regions with a cut off \( \Delta \)

\[
\tau(k, x) = \int_{|q|<\Delta} dq \left( \frac{e^{iqx}}{E(k) - E(q)} + \sum_{k_n} \frac{e^{iqx}}{v(k_n)(q - k_n)} \right) + \int_{|q|>\Delta} dq \left( \frac{e^{iqx}}{E(k) - E(q)} + \sum_{k_n} \frac{e^{iqx}}{v(k_n)(q - k_n)} \right). \tag{S47}
\]

The first integral is obviously continuous when \( x \to 0 \), since the integration domain is finite and there are no singularities. For the second term, by choosing \( \Delta \) to be fixed but large, we can replace \( E(k) - E(q) \approx -vq \). We have therefore

\[
\int_{|q|>\Delta} dq \frac{e^{iqx}}{2\pi q} = 2i \int_{\Delta} dq \frac{\sin(qx)}{2\pi q} = \text{sgn}(x) \int_{\Delta} dq \frac{\sin(qx)}{2\pi q} = \frac{1}{2} \text{sgn}(x). \tag{S48}
\]

Thus, the limit \( x \to 0^\pm \) of \( \tau \) is

\[
\tau(k, 0^\pm) = \lim_{\Delta \to \infty} \left[ \mathcal{P} \int_{|q|<\Delta} dq \frac{1}{2\pi E(k) - E(q)} \right] \mp \frac{i}{2} \left( \frac{1}{v} - \sum_{k_n} \frac{1}{v(k_n)} \right). \tag{S49}
\]
Above, it must be used the Principal Value prescription in the integration of singular points. Coming back to $\mathcal{K}_k$, we finally have

$$\mathcal{K}(k,0^\pm) = \pm\frac{1}{2iv} + \sum_{k_n} \frac{1}{2|v(k_n)|} + \mathcal{P} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{E(k) - E(q)},$$

that leads to the following expression for $\alpha_k$

$$\alpha_k = \left[ \sum_{k_n} \frac{1}{2|v(k_n)|} - \frac{1}{2v} \cot \left( \frac{c}{2v} \right) + \mathcal{P} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{E(k) - E(q)} \right]^{-1}.$$

This explicit expression permits an exact determination of the wavefunction $\psi_k$ (S38) and therefore of the scattering amplitudes, leading to the result (8) in the main text. In the Letter we compared the analytic LQSS with the numerics for a $\delta-$like defect in Fig. 2 of the main text. In that figure, we focused on the $t \to \infty$ limit which leads to the LQSS. Therefore, slow vanishing oscillations were smeared out with (local) spatial averages. If no averaging procedure is performed, the density profile looks like Fig. S6 (left panel): at fixed coordinate $x$, the oscillations do not display any damping in time, but decay (though slowly) away from the defect, thus disappearing in the LQSS limit. Nevertheless, recalling that the asymptotic wavefunction (S16) is exponentially accurate away from the defect, such oscillations can be captured by the scattering theory. Indeed, inserting the full expression of (S35) in (S36), one obtains oscillating contribution to (S37), which are in agreement with the numerical simulations (right panel of Fig. S6). It is interesting to inspect more in detail the origin of this oscillations. For the parameters chosen in the Fig. S6 ($v = 0.3$ and $k_f = \pi/3$), each incoming wavevector $k$ in the Fermi sea has exactly three possible scattering channels $k_1, k_2, k_3$. These scattering channels are plotted in Fig. S7 (left panel) together with their velocities (right panel): in the two point correlator, passing from (S36) to (S37), the leading corrections given by the oscillating phases $\sim e^{i(k_n - k_{n'})\pi}$ are given by the saddle point approximation ruled by the second derivative of the phase $\partial^2 \Phi_k(k_n - k_{n'})$. As it is clear from the left panel of Fig. S7, $k_n$ have a small curvature as functions of $k$ and this explains the slow vanishing oscillations. In order to understand why the oscillations in Fig. S6 appear only on the left of the defect, we should consider the velocities associated with the scattering channels (right panel Fig. S7), together with the expression for the evolved plane wave (S35). For what concerns the scattering channels $k_n$, the Heaviside Theta functions in (S35) guarantee that the plane-wave associated with $k_n$ has support only on the right (left) of the defect if $v(k_n) > 0$ ($v(k_n) < 0$). The right panel of Fig. S7 shows that for any momentum $k$ there is only one scattering channel with positive velocity, therefore in the expression for the two point correlator (S36) no oscillating phases are produced on the right of the defect. The situation is different on the left side: as one can see in Fig. S7, there are always two channels with negative velocity causing interference in (S36).

**FIG. S6:** Density profile generated by a delta-like potential $V(x) = c\delta(x)$ with $c = 0.5$, $v = 0.3$ and $k_f = \pi/3$ at time $t = 620$. Left panel: the numerical result of the whole density profile is displayed, showing slow vanishing oscillations on the left of the defect (placed where the discontinuity occurs). Right panel: the density profile is zoomed in a few-sites region around zero and the numerics is compared with the analytic prediction.
Here we present further details about the semiclassical approximation derived in the main text. Under the assumption of a smooth defect, the Wigner distribution $\rho_t(k,x)$ evolves as the density distribution of classical particles subjected to the classical equation of motion

$$\dot{x} = v(k), \quad \dot{k} = -\partial_x V,$$

(S52)

that are associated with the classical Hamiltonian

$$H_{cl} = -\cos(k) - vk + V(x).$$

(S53)

At initial time, because of the initial state we choose, the Wigner distribution is homogeneous in $x$ and flat in $k$ within the Fermi sea, zero outside of it. Immediately after the defect has been switched on, the Wigner function evolves accordingly to two contributions: i) the particles initially sat on the defect support and ii) those that initially resided outside the defect support. In the perspective of determining the LQSS, the first group of particles does not contribute at all: these particles will be scattered away from the defect at different velocities, thus they will be spread at late times, being irrelevant for local quantities. However, especially for large defects, the spreading of these particles could require long times. The second group of particles is what matters at late times: the constant flux of in-going particles scatters with the defect, creating a flux of out-going particles, i.e. the classical LQSS. In order to describe the classical scattering events, the first step is to describe the \textit{classical trajectories}: while the solution of the equation of motion can be complicated, a rather simple picture based on the classical \textit{turning points} (i.e. where the particles invert their motion, thus $v(k) = 0$) is more easily accessed. The trajectories can be simply constructed following the rules reported in the main text, that we quote hereafter for convenience.

1. Until a turning point is reached, the particle maintains the same direction. When a turning point is reached, the particle inverts its motion.

2. The momentum $k$ changes continuously, moreover from $\dot{k} = -\partial_x V$ it can be seen that $k$ increases if $V$ has a negative slope and decreases otherwise.

Thereafter, we discuss the determination of the trajectory of a classical particle on the bell-shaped potential of Fig. S8. Along the trajectory the classical energy is conserved, so let $T(k)$ be the kinetic energy $T(k) = -\cos(k) - vk$, we must have $T(k(t)) + V(x(t)) = E$. Actually, it is convenient to think about the energy conservation as $E - T(k(t)) = V(x(t))$ and plot, on the same energy axis, the two members (Fig. S8): in this picture, the turning points are nothing else than the stationary points of the kinetic part. The particle will be initially placed at some point in the position/momentum phase space and we can now construct the trajectories using the aforementioned rules. As it should be clear from the explicit example of Fig. S8, no bound states are allowed for a repulsive bell-shaped potential for $v \neq 0$ (differently from the case at $v = 0$ [[117]]). The same method can be used to study the classical scattering processes, that will lead to determining the LQSS in the semiclassical approximation: for a detailed discussion we refer

![FIG. S7: Left panels: scattering channels for a moving defect at $v = 0.3$ and the incoming momentum in the Fermi sea with $k_f = \pi/3$. Right panel, the velocities associated with the scattering channels.](image-url)
FIG. S8: We consider the trajectory of a classical particle initially placed in a bell-shaped potential with a given momentum. On the right the potential energy \( V(x) \) is plotted as a function of position, on the left \( E - T(k) \), where \( E \) is the initial energy. The ordinate axis of the two plots is the same to enforce the energy conservation \( E - T(k) = V(x) \) along the trajectory. The blue dots represent the initial condition of the particles, the red crosses are the turning points: the numeration represents the order in which the turning points are visited. The initial condition in such that \( \partial_k T(k) = v(k) < 0 \) thus the particle starts moving on the left until a turning point is reached. The first turning point to be reached is 1, not 2: even though 2 appears closer than 1 to the initial position of the particle, in the momentum space the point 2 cannot be reached if the particle does not have crossed 1 before. Thus, the particle travels until 1 is reached (as long as \( \partial_x V < 0 \), we must have \( k > 0 \), thus the momentum increases). After 1 has been reached, the particle inverts its motion and the next turning point is 2, then the particle inverts its motion again: there are no longer turning points able to stop the particles before it reaches the other side of potential (i.e. \( \partial_x V > 0 \)). Here the motion in the momentum space is reverted, thus the particle reaches again the same turning point in the momentum space, but differently placed in the space plot (point 3). As the particle’s journey proceeds, the last turning point to be visited is 6: then the particle keeps traveling toward the left forever, its momentum being the solution of \( T(k) = E \) closer to the turning point 6.

directly to Fig. S9 and the relative caption. In general, whether a particle is reflected or transmitted it depends on the height of the potential barrier, but there exists a window of momenta such that a left mover (we assume the defect velocity to be positive \( v > 0 \)) is unavoidably transmitted. This classically purely transmissive window of momenta encloses the corresponding quantum counterpart, but actually is much larger than the latter: this because several channels, while energetically allowed, are classically forbidden because of the absence of tunneling. Consistently with the quantum case, when \( v \) is increased more and more particles are surely transmitted until the sound velocity is reached: beyond that point, all the particles are unavoidably transmitted.

The semiclassical LQSS

Here we derive the semiclassical expression for the LQSS, it is convenient to briefly recap the different contributions

1. The particles initially sat on the defect are ignored, since contribute as a transient vanishing in time.

2. Left movers that are initially placed on the left of the defect of course never meet the latter and thus do not scatter. Symmetrically, right movers that are initially on the right of the defect do not undergo a scattering process.

3. Left movers initially placed on the right of the defect, as well as right movers coming from the left of the defect, undergo a scattering event. Let \( k \) be the momentum of the incoming particle, we denote as \( k^s \) the momentum of the scattered particle, the latter being determined through the rules previously described.

4. Along the time evolution, the phase space measure is conserved (Liouville Theorem) thus

\[
\rho_t(x(t), k(t))dx(t)dk(t) = \rho_{t=0}(x, k)dxdk \text{ where } (x(t), k(t)) \text{ are the time-evolved coordinates with initial}
\]
FIG. S9: In this figure the various possible scattering events on a bell-shaped repulsive potential with positive velocity are illustrated, the blue dots represent the initial position of the particle and the red crosses are the final momenta after the scattering event occurred. We start discussing the richer phenomenology of particles coming from the right, thus left movers. If the kinetic energy of the incoming particle is less than the kinetic energy of the second nearest turning point with momentum larger than the initial momentum (panel (a)), then the particle is transmitted independently on the potential height. If instead it is not the case, the particle is reflected if the height of the defect is larger than the difference between the kinetic energy of the nearest turning point with momentum larger than the initial one and the initial kinetic energy (panel (b)), if instead the height of the barrier is smaller the particle is transmitted (panel (c)). The destiny of the right movers depends on the height of the defect: if the height is larger than the difference in kinetic energy between the nearest turning point smaller than the initial momentum and the initial kinetic energy, then the particle is reflected, otherwise is transmitted (panels (d) and (e) respectively).

conditions \((x, k)\). After the scattering event has occurred, the time evolution is simply

\[
x(t) = v(k^*) \left( t - t_d(k) - \frac{x}{v(k)} \right), \quad k(t) = k^*.
\] (S54)

were \(t_d(k)\) is the delay time experienced by the particle undergoing the scattering event. As it is clear, the Jacobian of the change of coordinate \((x(t), k(t)) \rightarrow (x, k)\) simply amounts to compute \(dk^*/dk\). Using the conservation of energy similarly to Eq. (S21) and the Liouville Theorem

\[
\rho_t(x(t), k(t)) = \rho_{t=0}(x, k) \left| \frac{v(k)}{v(k^*)} \right|
\] (S55)

The corrections to the initial ensemble given by the emergent LQSS are most easily described in terms of holes and occupancies. The holes describe the removal of particles from their unperturbed trajectories due to the scattering process on the defect: the front of propagating holes at a given momentum \(k\) can reach at most a position \(x = v(k)t\). On the other hand, the occupancies describe the phase space populated after the scattering event: the front of particles
that undergoes the scattering $k \rightarrow k^s$ can travel at most up to $x = v(k^s)t$ (here we neglect the time delay, since it is irrelevant in the large time limit and thus for the determination of the LQSS). These considerations, together with the fact that the two point correlator $C_t(x,s) = \langle \hat{c}^\dagger_{x+s/2} \hat{c}_{x-s/2} \rangle_t$ is the Fourier transform of the Wigner function (Eq. (10) of the main text), provide us the two point correlator in the large time limit and outside of the defect support

$$C_t(x,s) \simeq \int_{-k_f}^{k_f} \frac{dk}{2\pi} \left(1 - \theta(v(k))\theta(|v(k)| - \zeta)\right) e^{-iks} + \theta(v(k^s))\theta(|v(k^s)| - \zeta) \left|\frac{v(k)}{v(k^s)}\right| e^{-ik^ss}, \quad (S56)$$

where, as usual, $\zeta = x/t$. As expected, the semiclassical LQSS coincides with the quantum LQSS in the case where only a scattering channel is possible, i.e. when we completely neglect the tunneling through non-classical trajectories.

**The semicclassic correlator on a supersonic defect**

In the semiclassical regime, as well as in the true quantum case, a supersonic defect cannot produce a LQSS: in this case the non-trivial two point correlator that persists at late times must be localized on the defect. Deriving such a correlator in the purely quantum case would require knowing the eigenfunctions $\psi_k(x)$ on the defect: while in principle possible for the exactly solvable case of the $\delta-$defect, it still involves some rather tedious calculations. Instead, a simple and intuitive derivation can be outlined within the semiclassical approximation. In fact, the defect sees a flux of incoming particles (from the far right, assuming a positive velocity $v$): at infinity, the energy of the particles is simply equal to their kinetic energy. While a particle travels across the defect, its momentum is unambiguously determined by energy conservation (in the supersonic case, turning points are absent). Let $q_{k,x}$ be the momentum that the particle has at point $x$ if it had momentum $k$ far away from the defect, thus $q_{k,x}$ is determined by energy conservation

$$-\cos(q_{k,x}) - vq_{k,x} + V(x) = -\cos(k) - vk. \quad (S57)$$

Along the motion the phase space changes as in (S55) with the only difference of using $q_{k,x}$ rather than $k^s$. Thus, we finally obtain our two point correlator for a supersonic defect in the infinite time limit

$$\lim_{t \to \infty} C_t(x,s) = \int_{-k_f}^{k_f} \frac{dk}{2\pi} \left|\frac{v(k)}{v(q_{k,x})}\right| e^{-i q_{k,x}s}. \quad (S58)$$

which is the result presented in the main text. In principle, a similar semiclassical analysis could be pursued even for subsonic defects, however extra complications arise due the presence of turning points and thus multiple solutions to the energy conservation must be considered.