Birationally rigid Fano fibre spaces. II

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Abstract. We prove the birational rigidity of large classes of Fano–Mori fibre spaces over a base of arbitrary dimension bounded above by a constant that depends only on the dimension of the fibres. To do this, we first show that if every fibre of a Fano–Mori fibre space satisfies certain natural conditions, then every birational map onto another such space is fibrewise. Then we construct large classes of fibre spaces (whose fibres are either Fano double spaces of index 1 or Fano hypersurfaces of index 1) satisfying these conditions.

Keywords: Fano–Mori fibre space, birational rigidity, maximal singularity, hypertangent divisor, log canonical singularity, linear system, canonical class.

§ 1. Introduction

1.1. Birationally rigid Fano–Mori fibre spaces. In this paper we investigate the problem of the birational rigidity of Fano–Mori fibre spaces $\pi: V \to S$. We assume that the base $S$ is non-singular, $V$ has at most factorial terminal singularities, the anticanonical class $(-K_V)$ is relatively ample and

$$\text{Pic} V = \mathbb{Z}K_V \oplus \pi^* \text{Pic} S.$$ 

Let $\pi': V' \to S'$ be an arbitrary rationally connected fibre space, that is, a morphism of projective algebraic varieties whose base $S'$ and generic fibre $(\pi')^{-1}(s')$, $s' \in S'$, are birationally connected and $\dim V = \dim V'$. Consider a birational map $\chi: V \dasharrow V'$ (if such a map exists). To describe the properties of $\chi$, it is crucially important to know whether $\chi$ is fibrewise (that is, whether $\chi$ transforms the fibres of $\pi$ to fibres of $\pi'$). It is expected (and confirmed by all known examples; see § 1.5) that the answer is affirmative whenever $\pi$ is ‘sufficiently twisted over the base’. This problem may be studied for various classes of Fano–Mori fibre spaces and various interpretations of the property to be ‘twisted over the base’. We shall prove the following fact.

Theorem 1.1. Assume that the Fano–Mori fibre space $\pi: V \to S$ satisfies the following conditions.

(i) Every fibre $F_s = \pi^{-1}(s)$, $s \in S$, is a factorial Fano variety with terminal singularities and Picard group $\text{Pic} F_s = \mathbb{Z}K_{F_s}$. 

AMS 2010 Mathematics Subject Classification. 14E05, 14J45, 14D06, 14M22, 14E30.
(ii) For every effective divisor \( D \in |−nK_{F_s}| \) on an arbitrary fibre \( F_s \), the pair \((F_s, \frac{1}{n} D)\) is log canonical. For every mobile linear system \( \Sigma_s \subset |−nK_{F_s}| \), the pair \((F_s, \frac{1}{n} D)\) is canonical for a general divisor \( D \in \Sigma_s \).

(iii) For every mobile family \( \overline{C} \) of curves on \( S \) that sweeps out \( S \) and for a general curve \( \overline{C} \in \overline{C} \), the class of the algebraic cycle
\[
- N(K_V \cdot \pi^{-1}(\overline{C})) - F
\]
of dimension \( \dim F \) is non-effective for every positive \( N \geq 1 \) (where \( F \) is the fibre of \( \pi \)). In other words, it is not rationally equivalent to an effective cycle of dimension \( \dim F \).

Then every birational map \( \chi : V \rightarrow V' \) onto the total space of a rationally connected fibre space \( V'/S' \) is fibrewise, that is, there is a rational dominant map \( \beta : S \rightarrow S' \) such that the following diagram commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{\chi} & V' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
S & \xrightarrow{\beta} & S'
\end{array}
\]

We now list the standard consequences of Theorem 1.1 and then discuss the issue of how restrictive the conditions (i)–(iii) are.

**Corollary 1.1.** Under the hypotheses of Theorem 1.1, \( V \) admits no structures of a rationally connected fibre space over a base of dimension greater than \( \dim S \). In particular, \( V \) is non-rational. Every birational self-map of \( V \) is fibrewise and induces a birational self-map of \( S \). Hence there is a natural homomorphism of groups \( \rho : \text{Bir} V \rightarrow \text{Bir} S \) whose kernel \( \text{Ker} \rho \) is the group \( \text{Bir} F_\eta = \text{Bir}(V/S) \) of birational self-maps of a generic fibre \( F_\eta \) (over a generic non-closed point \( \eta \) of \( S \)). The group \( \text{Bir} V \) is an extension of the normal subgroup \( \text{Bir} F_\eta \) by the group \( \Gamma = \rho(\text{Bir} V) \subset \text{Bir} S \):

\[
1 \rightarrow \text{Bir} F_\eta \rightarrow \text{Bir} V \rightarrow \Gamma \rightarrow 1.
\]

How restrictive are the conditions (i)–(iii)? Condition (iii) belongs to the same class of conditions as the well-known \( K^2 \)-condition and \( K \)-condition for fibre spaces over \( \mathbb{P}^1 \) (see, for example, [1], Part 4) and Sarkisov’s condition for conic bundles (see [2], [3]). Condition (iii) measures the ‘degree of twistedness’ of the fibre space \( V/S \) over the base \( S \). Below we illustrate this interpretation of condition (iii) by particular examples. We shall see that this condition is not too restrictive. For a fixed method of construction of the fibre space \( V/S \) and a fixed ‘ambient’ fibre space \( X/S \), condition (iii) holds for ‘almost all’ families of fibre spaces \( V/S \).

Condition (iii) may be expressed in terms of the numerical geometry of \( V \) and \( S \) as follows. Let
\[
A^*(V) = \bigoplus_{i=0}^{\dim V} A^i(V)
\]
be the numerical Chow ring of \( V \), graded by codimension. We put
\[
A_i(V) = A^{\dim V - i}(V) \otimes \mathbb{R}.
\]
Let $A_i^{\text{mov}}(V)$ be the closed cone in $A_i(V)$ generated by the classes of mobile cycles whose families sweep out $V$, $A_i^+(V)$ the pseudo-effective cone in $A_i(V)$ generated by the classes of effective cycles, and $A_{i,\leq j}(V)$ the vector subspace of $A_i(V)$ generated by the classes of subvarieties of dimension $i$ whose image on $S$ has dimension not greater than $j$. In the real space $A_{i,\leq j}(V)$ we consider the closed cones $A_{i,\leq j}^{\text{mov}}(V)$ and $A_{i,\leq j}^+(V)$ of mobile and pseudo-effective classes respectively. We similarly define the real vector space $A_i(S)$ and the closed cones $A_i^{\text{mov}}(S)$ and $A_i^+(S)$. If $\delta = \dim F$ is the dimension of the fibre of $\pi$, then the operation of taking the pre-image induces a linear map

$$\pi^*: A_i(S) \to A_{\delta+i,\leq i}(V).$$

We have $\pi^*(A_i^+(S)) \subset A_{\delta+i,\leq i}^+(V)$ and $\pi^*(A_i^{\text{mov}}(S)) \subset A_{\delta+i,\leq i}^{\text{mov}}(V)$. We now define a linear map

$$\gamma: A_1(S) \to A_{\delta,\leq 1}(V)$$

by the formula

$$z \mapsto - (K_V \cdot \pi^* z).$$

Condition (iii) means that the image of the cone $\gamma(A_1^{\text{mov}}(S))$ is contained in the boundary of the pseudo-effective cone $A_{\delta,\leq 1}^+(V)$, that is,

$$\gamma(A_1^{\text{mov}}(S)) \cap \text{Int} A_{\delta,\leq 1}^+(V) = \emptyset.$$

More precisely, for every $z \in A_1^{\text{mov}}(S)$, the intersection of the closed ray

$$\{ \gamma(z) - t[F] \mid t \in \mathbb{R}_+ \},$$

where $[F] \in \text{Int} A_{\delta,\leq 1}^+(V)$ is the class of the fibre of $\pi$, with the cone $A_{\delta,\leq 1}^+(V)$ either is empty or consists of a unique point $\gamma(z)$.

One may suggest that the condition (iii) is close to being a criterion (‘if and only if’), that is, its violation (or the essential deviation from it) means the existence of another structure of a Fano–Mori fibre space on $V$.

The following remark provides an obvious way of checking condition (iii).

Remark 1.1. Assume that $V$ admits a numerically effective divisorial class $L$ such that $(L^\delta \cdot F) > 0$ and the linear function $(\cdot L^\delta)$ is non-positive on the cone $\gamma(A_1^{\text{mov}}(S))$, that is, for every mobile curve $C$ on $S$ we have

$$(L^\delta \cdot K_V \cdot \pi^{-1}(C)) \geq 0.$$  \hfill (1.1)

Then condition (iii) obviously holds.

However, conditions (i) and (ii) are much more restrictive. They mean that all fibres of $\pi$ are varieties of sufficiently general position in their family. It follows that the dimension of the base for a fixed family of fibres is bounded above (by a constant depending on the concrete family to which the fibres belong). In the examples considered in this paper, for sufficiently large dimension $\delta = \dim F$ of the fibre, the dimension of the base is bounded above by a number of order $\frac{1}{2} \delta^2$. The same relation may be stated in the ‘dual’ way: if we fix the dimension $\dim S = w$ of the base, then the dimension of the fibre is bounded below by a constant of order $\sqrt{2w}$. 


In particular, Theorem 1.1 says nothing about fibre spaces with low-dimensional fibres (such as conic bundles or pencils of del Pezzo surfaces) but it is intended to describe the birational geometry of Fano–Mori fibre spaces when the base and the fibre are higher-dimensional. We recall that not a single example was previously known of a fibration into higher-dimensional Fano varieties over a base of dimension two or higher having only one structure of a rationally connected fibre space (see §1.5 for a brief historical survey).

1.2. Fibrations into double spaces of index one. We write \( \mathbb{P} \) for a projective space \( \mathbb{P}^M, M \geq 5 \). Let \( W = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}(2M))) \) be the space of hypersurfaces of degree \( 2M \) in \( \mathbb{P} \).

**Theorem 1.2.** There is a Zariski-open subset \( W_{\text{reg}} \subset W \) such that for every hypersurface \( W \in W_{\text{reg}} \), the double covering \( \sigma: F \to \mathbb{P} \) branched over \( W \) satisfies conditions (i) and (ii) in Theorem 1.1 and we have

\[
\operatorname{codim}(W \setminus W_{\text{reg}}) \subset W \geq \frac{(M-4)(M-1)}{2}.
\]

An explicit description of the set \( W_{\text{reg}} \) and a proof of Theorem 1.2 are given in §2. Fix a number \( M \geq 5 \) and a non-singular rationally connected variety \( S \) of dimension \( \dim S < \frac{1}{2}(M-4)(M-1) \). Let \( \mathcal{L} \) be a locally free sheaf of rank \( M+1 \) on \( S \) and let \( X = \mathbb{P}(\mathcal{L}) = \text{Proj} \bigoplus_{i=0}^{\infty} \mathcal{L}^{\otimes i} \) be the corresponding \( \mathbb{P}^M \)-bundle. We may assume that \( \mathcal{L} \) is generated by its sections. Then the sheaf \( \mathcal{O}_{\mathbb{P}(\mathcal{L})}(1) \) is also generated by its sections. Let \( L \in \text{Pic} X \) be the class of this sheaf. Then

\[
\text{Pic} X = \mathbb{Z}L \oplus \pi_X^* \text{Pic} S,
\]

where \( \pi_X: X \to S \) is the natural projection. Take a general divisor

\[
U \in |2(ML + \pi_X^* R)|,
\]

where \( R \in \text{Pic} S \) is a certain class. If this system is sufficiently mobile, then the assumption about the dimension of \( S \) and Theorem 1.2 enable us to assume that for every point \( s \in S \) the hypersurface \( U_s = U \cap \pi_X^{-1}(s) \) belongs to \( W_{\text{reg}} \). Therefore conditions (i) and (ii) of Theorem 1.1 hold for the double space branched over \( U_s \). Let \( \sigma: V \to X \) be the double covering branched over \( U \). We put \( \pi = \pi_X \circ \sigma: V \to S \). Then \( V \) is a fibration into Fano double spaces of index 1 over \( S \). We recall that the divisor \( U \in |2(ML + \pi_X^* R)| \) is assumed to be sufficiently general.

**Theorem 1.3.** Assume that the divisorial class \((K_S + R)\) is pseudo-effective. Then the conclusions of Theorem 1.1 and Corollary 1.1 hold for the fibre space \( \pi: V \to S \). In particular,

\[
\text{Bir} V = \text{Aut} V = \mathbb{Z}/2\mathbb{Z}
\]

is a cyclic group of order 2.

**Proof.** Since conditions (i) and (ii) of Theorem 1.1 hold by the construction of \( V \), it remains to check condition (iii). We use Remark 1.1. Elementary computations show that the inequality (1.1) takes the following form up to a positive factor:

\[
((K_S + R) \cdot C) \geq 0.
\]
Since the curve $\overline{C}$ belongs to a mobile family that sweeps out $S$, this inequality holds if the class $(K_S + R)$ is pseudo-effective. □

**Example 1.1.** Take $S = \mathbb{P}^m$, where $m < \frac{1}{2}(M-4)(M-1)$, $X = \mathbb{P}^M \times \mathbb{P}^m$ and $W_X$ is a generic hypersurface of bidegree $(2M, 2l)$, where $l \geq m+1$. Then the conclusions of Theorem 1.1 and Corollary 1.1 hold for the double covering $\sigma: V \to X$ branched over $W_X$. Note that when $l \leq m$ the double covering $V$ admits another structure of a Fano fibre space: it is given by the projection $\pi_1: V \to \mathbb{P}^M$. Thus condition (iii) in Theorem 1.1 and its realization in Theorem 1.3 turn out to be precise.

### 1.3. Fibrations into Fano hypersurfaces of index 1

We continue to write $\mathbb{P}$ for a projective space $\mathbb{P}^M$, $M \geq 10$. Fix $M$. Let $F = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(M)))$ be the space of hypersurfaces of degree $M$ in $\mathbb{P}$. Let $S$ be a non-singular rationally connected variety of dimension $\dim S < \frac{1}{2}(M-7)(M-6)-5$. As in §1.2, let $\mathcal{L}$ be a locally free sheaf of rank $M+1$ on $S$, and let $\pi_X: X = \mathbb{P}(\mathcal{L}) = \text{Proj} \bigoplus_{i=0}^{\infty} \mathcal{L}^{\otimes i}$ be the corresponding $\mathbb{P}^M$-bundle in the sense of Grothendieck. We assume that $\mathcal{L}$ is generated by global sections. Let $\pi_X: X \to S$ be the projection, and $L \in \text{Pic} X$ the class of the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{L})}(1)$. Consider a general divisor $V \in |ML + \pi_X^* R|$, where $R \in \text{Pic} S$ is a certain divisor on the base. By the assumption about the dimension of the base and Theorem 1.4 we may assume that conditions (i) and (ii) of Theorem 1.1 hold for the Fano fibre space $\pi: V \to S$, where $\pi = \pi_X|_V$.

**Theorem 1.5.** Assume that the divisorial class $(K_S + (1 - \frac{1}{M})R)$ is pseudo-effective. Then the conclusions of Theorem 1.1 and Corollary 1.1 hold for the Fano fibre space $\pi: V \to S$. In particular, the group $\text{Bir} V = \text{Aut} V$

is trivial.

**Proof.** Conditions (i) and (ii) of Theorem 1.1 hold because the divisor $V$ is general. Up to a positive factor, the inequality (1.1) takes the form $((MK_S + (M - 1)R) \cdot \overline{C}) \geq 0$.

Therefore condition (iii) of Theorem 1.1 also holds by Remark 1.1. □

**Example 1.2.** Take $S = \mathbb{P}^m$, where $m \leq \frac{1}{2}(M-7)(M-6)-6$, $X = \mathbb{P}^M \times \mathbb{P}^m$, and let $V \subset X$ be a sufficiently general hypersurface of bidegree $(M, l)$, where $l$ satisfies the inequality $l \geq \frac{M}{M-1}(m+1)$. 

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Then the hypotheses of Theorem 1.1 (and hence the conclusions of this theorem and Corollary 1.1) hold for the Fano fibre space $V/\mathbb{P}^m$. Note that for $l \leq m$ the variety $V$ admits another structure of a Fano fibre space with projection $V \to \mathbb{P}^M$. Moreover, if we fix the dimension $m$ of the base, then the condition in Theorem 1.5 is close to being optimal for $M \geq m$: it holds for $l \geq m+2$, so that $l = m+1$ is the only integer value of $l$ for which the problem of the birational rigidity of the fibre space $V/\mathbb{P}^m$ remains open. In this case the projection $V \to \mathbb{P}^M$ is a $K$-trivial fibre space.

Remark 1.2. It is natural to expect that Theorem 1.1 is applicable to other classes of Fano–Mori fibre spaces, for example, to spaces whose fibre is a Fano complete intersection of index 1 in a (weighted) projective space. Then analogues of Theorems 1.2, 1.3 or Theorems 1.4, 1.5 also hold for these classes. The main difficulty is in checking conditions (i) and (ii) for Fano complete intersections whose generality can be effectively controlled. In other words, one needs effective estimates for the codimension of the complement to the open set of Fano complete intersections satisfying conditions (i) and (ii). This problem is highly non-trivial. Its difficulty increases when the degree of the complete intersection grows (for a fixed dimension).

1.4. Structure of the paper. This paper is organized in the following way. We prove Theorem 1.1 in §2 and devote §3 to the conditions of general position that must hold for every fibre of $V/S$ to guarantee that conditions (i) and (ii) of Theorem 1.1 hold. The conditions of general position (regularity) are stated for Fano double spaces of index 1 and Fano hypersurfaces of index 1. This enables us to define the sets $\mathcal{W}_{\text{reg}}$ and $\mathcal{F}_{\text{reg}}$, prove Theorem 1.2 and carry out preparatory work for the proof of Theorem 1.4; our main technical result whose obvious corollary is Theorem 1.5 (our geometrically most impressive result).

In §4 we prove Theorem 1.4 or rather establish that condition (ii) of Theorem 1.1 holds for regular Fano hypersurfaces $F \in \mathcal{F}_{\text{reg}}$. The proof combines the technique of hypertangent divisors and the inversion of adjunction. We note that our approach corresponds to the linear method of proving birational rigidity (see [1], Part 7). The technique of the quadratic method (first of all, the technique of counting multiplicities) is not used.

The hypothesis of smoothness of the base $S$ in Theorem 1.1 seems to be unduly strict. It could be replaced by the condition that the singularities of $S$ are at most terminal and $(\mathbb{Q},)$-factorial.

1.5. Historical remarks and acknowledgments. The starting point in the study of the birational geometry of rationally connected fibre spaces seems to be the use of de Jonquiére transformations (see, for example, [4]). The systematic study of such objects in modern algebraic geometry began in the work of Iskovskikh and Gizatullin [5]–[7] on pencils of rational curves over non-closed fields. This work continued Manin’s study of the ‘absolute’ case in [8]–[10]. We also mention Dolgachev’s paper [11] that started the study of $K$-trivial fibrations in the modern era.

After the breakthrough in three-dimensional birational geometry made in the classical paper of Iskovskikh and Manin on the three-dimensional quartic [12],
the problems of ‘relative’ three-dimensional birational geometry were the next to be investigated. The task was to describe birational maps of three-dimensional algebraic varieties fibred into conics over a rational surface or into del Pezzo surfaces over \( \mathbb{P}^1 \). Sarkisov’s famous theorem gave an almost complete solution of the problem of birational rigidity for conic bundles \([2], [3]\). A similar question for pencils of del Pezzo surfaces remained open until 1996 \([13]\). The reason for these difficulties is explained in the introduction to the last paper (the test-class construction turned out to be unsuitable for studying varieties of this type).

The method of proving birational rigidity in \([13]\) can easily be generalized to arbitrary dimension (for varieties fibred into Fano varieties over \( \mathbb{P}^1 \)). In a long series of papers \([14]–[21]\), many classes of Fano fibre spaces over \( \mathbb{P}^1 \) were shown to be birationally rigid. At the same time, the birational geometry of the remaining families of three-dimensional varieties with a pencil of del Pezzo surfaces of degree 1 or 2 was investigated in \([22]–[25]\) with nearly exhaustive results. However, the base of the fibre spaces studied was still one-dimensional, and even Fano fibre spaces over surfaces seemed to be out of reach.

The only exception was a theorem on Fano direct products \([26]\) and the subsequent papers \([27], [28]\) on direct products. Although the Fano fibre spaces in those papers had the base and fibre of arbitrary dimension, they were very special (direct products) and could not pretend to be typical Fano fibre spaces.

The present paper gives, at long last, numerous examples of typical birationally rigid Fano fibre spaces with base and fibre of high dimension (for a fixed dimension \( \delta \) of the fibre, the dimension of the base is bounded by a constant of order \( \sim \frac{1}{2} \delta^2 \)).

Theorem 1.1 may be regarded as a realization of the following well-known principle: if a fibre space is ‘sufficiently twisted’ over the base, then it is birationally rigid. This principle has been confirmed many times in the class of fibre spaces over \( \mathbb{P}^1 \), and now we extend it to the class of fibre spaces over a base of arbitrary dimension.

Since we mainly study fibrations into Fano hypersurfaces of index 1, this paper is a continuation of \([14]\). From the technical viewpoint, its predecessors are \([27], [29]\), where the linear method of proving birational rigidity was developed. However, it is possible that the quadratic technique may also be used in the class of Fano fibre spaces over a base of arbitrary dimension.

Various technical points related to our arguments were discussed in the author’s talks given in 2009–2014 at the Steklov Mathematical Institute. The author is grateful to the members of the divisions of algebraic geometry and algebra and number theory for their interest in his work. The author also thanks his colleagues in the algebraic geometry research group at the University of Liverpool for providing a creative atmosphere and general support.

\section{2. Birationally rigid fibre spaces}

In this section we prove Theorem 1.1. This is done in three steps. First, assuming that the birational map \( \chi: V \rightarrow V' \) is not fibrewise, we prove that \( \chi \) has a maximal singularity that covers the base \( S' \) (\( \S 2.1 \)). Then we construct a sequence of blow-ups \( S^+ \rightarrow S \) such that the image of every maximal singularity on \( S \) is a prime divisor (\( \S 2.2 \)). Finally, using a very mobile family of curves contracted by the projection \( \pi' \),
we obtain a contradiction to condition (iii) of Theorem 1.1 (§2.3). It follows that the map \( \chi : V \to V' \) is fibrewise. This will complete the proof of Theorem 1.1.

We use only standard concepts and definitions (see [1], Parts 2 and 4 for details).

2.1. Maximal singularities of birational maps. In the notation of Theorem 1.1 we fix a birational map \( \chi : V \to V' \) onto the total space \( V' \) of a rationally connected fibre space \( \pi' : V' \to S' \). Consider an arbitrary very ample linear system \( \Sigma' \) on \( S' \). Let \( \Sigma' = (\pi')^* \Sigma \) be its pullback to \( V' \). Then the divisors \( D' \in \Sigma' \) consist of fibres of \( \pi' \) and, therefore, we have \((D' \cdot C) = 0 \) for every curve \( C \subset V' \) contracted by \( \pi' \). Clearly, the linear system \( \Sigma' \) is mobile. Let

\[
\Sigma = (\chi^{-1})_* \Sigma' \subset \left| -nK_V + \pi^*Y \right|
\]

be its strict transform on \( V \), where \( n \in \mathbb{Z}_+ \).

**Lemma 2.1.** For every mobile family of curves \( \mathcal{C} \in \overline{\mathcal{C}} \) (on \( S \)) that sweeps out \( S \), we have \((\overline{\mathcal{C}} \cdot Y) \geq 0 \). In other words, the numerical class of the divisor \( Y \) is non-negative on the cone \( A_{1}^{\text{mov}}(S) \).

**Proof.** This is almost obvious. For a general divisor \( D \in \Sigma \), the cycle \((D \circ \pi^{-1}(\mathcal{C}))\) of the scheme-theoretic intersection of \( D \) and \( \pi^{-1}(\mathcal{C}) \) is effective. The class of this cycle is equal to \(-n(K_V \cdot \pi^{-1}(\mathcal{C})) + (Y \cdot \mathcal{C})F \). Hence the lemma follows from condition (iii) of Theorem 1.1. \( \square \)

Clearly, the map \( \chi \) is fibrewise if and only if \( n = 0 \). Thus, if \( n = 0 \), then the conclusion of Theorem 1.1 holds. We now assume that \( n \geq 1 \) and show that this assumption leads to a contradiction.

The linear system \( \Sigma \) is mobile. We resolve the singularities of \( \chi \): let

\[
\varphi : \tilde{V} \to V
\]

be a birational morphism (a composite of blow-ups with non-singular centres) such that \( \tilde{V} \) is non-singular and the composite \( \chi \circ \varphi : \tilde{V} \to V' \) is regular. Consider the set \( \mathcal{E} \) of prime divisors on \( \tilde{V} \) satisfying the following conditions.

1) Every divisor \( E \in \mathcal{E} \) is \( \varphi \)-exceptional.
2) For every \( E \in \mathcal{E} \), the closed set \( \chi \circ \varphi(E) \subset V' \) is a prime divisor on \( V' \).
3) For every \( E \in \mathcal{E} \), the set \( \chi \circ \varphi(E) \) covers the base: \( \pi'[\chi \circ \varphi(E)] = S' \).

Putting \( \tilde{K} = K_{V} \), we have

\[
\tilde{\Sigma} \subset \left| -n\tilde{K} + \left( \pi^*Y - \sum_{E \in \mathcal{E}} \varepsilon(E)E \right) + \Xi \right|
\]

where, as usual, \( \tilde{\Sigma} \) is the strict transform of the mobile linear system \( \Sigma \) on \( \tilde{V} \), \( \varepsilon(E) \in \mathbb{Z} \) is a certain coefficient, and \( \Xi \) stands for a linear combination of \( \varphi \)-exceptional divisors which do not belong to \( \mathcal{E} \).

**Definition 2.1.** An exceptional divisor \( E \in \mathcal{E} \) is called a maximal singularity of \( \chi \) if \( \varepsilon(E) > 0 \).
Clearly, maximal singularities satisfy the Noether–Fano inequality
\[ \text{ord}_E \varphi^* \Sigma > na(E), \]
where \( a(E) = a(E, V) \) is the discrepancy of the divisor \( E \) with respect to \( V \). In this paper we somewhat modify the standard concept of a maximal singularity: we additionally require that it is realized by a divisor on \( V' \) covering the base. Let \( \mathcal{M} \subset \mathcal{E} \) be the set of all maximal singularities.

**Proposition 2.1.** Maximal singularities exist: \( \mathcal{M} \neq \emptyset \).

*Proof.* Assume the opposite: \( \varepsilon(E) \leq 0 \) for every \( E \in \mathcal{E} \). Let \( C' \) be a family of rational curves on \( V' \) satisfying the following conditions.

1) The curves \( C' \in C' \) are contracted by \( \pi' \).
2) The curves \( C' \in C' \) sweep out a dense open subset of \( V' \).
3) The curves \( C' \in C' \) are disjoint from the indeterminacy locus of the rational map \( (\chi \circ \varphi)^{-1} : V' \rightarrow \widetilde{V} \).

Moreover, we assume that a general curve \( C' \in C' \) intersects every divisor \( \chi \circ \varphi(E), \ E \in \mathcal{E} \), transversally at points of general position. We call such a family of curves *very mobile*. Clearly, very mobile families of rational curves exist.

Let \( \tilde{C} \cong C' \) be the inverse image of a curve \( C' \in C' \) on \( \widetilde{V} \). Since the linear system \( \Sigma' \) is pulled back from the base, we have \( (\tilde{C} \cdot \tilde{D}) = 0 \) for a divisor \( \tilde{D} \in \tilde{\Sigma} \).

On the other hand, \( (\tilde{C} \cdot \tilde{K}) = (C' \cdot K_{V'}) < 0 \) and
\[
(\tilde{C} \cdot (\pi^*Y - \sum_{E \in \mathcal{E}} \varepsilon(E)E)) \geq 0
\]
since \( (\tilde{C} \cdot \pi^*Y) \geq 0 \) by condition (iii) of the theorem and \( -\varepsilon(E) \in \mathbb{Z}_+ \) for all \( E \in \mathcal{E} \) by assumption. Finally, the divisor \( \Xi \) (not necessarily effective) is a linear combination of \( \varphi \)-exceptional divisors \( R \subset \tilde{V} \) such that \( \pi'[\chi \circ \varphi(R)] \) is a proper closed subset of \( S' \). Hence we have \( (\tilde{C} \cdot \Xi) = 0 \). It follows that
\[
(\tilde{C} \cdot \tilde{D}) \geq n > 0,
\]
a contradiction. Hence \( \mathcal{M} \neq \emptyset \). \( \square \)

**Proposition 2.2.** For every maximal singularity \( E \subset \mathcal{M} \) its centre
\[
\text{centre}(E, V) = \varphi(E)
\]
on \( V \) does not cover the base: \( \pi(\text{centre}(E, V)) \subset S \) is a proper closed subset of \( S \).

*Proof.* Assume the opposite: the centre of some maximal singularity \( E \in \mathcal{M} \) covers the base, that is, \( \pi(\text{centre}(E, V)) = S \). Let \( F = \pi^{-1}(s), \ s \in S, \) be a fibre of general position. By assumption, the strict transform \( \tilde{F} \) of the fibre \( F \) on \( \tilde{V} \) has non-empty intersection with \( E \) and, therefore, every irreducible component of the intersection \( \tilde{F} \cap E \) is a maximal singularity of the mobile linear system \( \Sigma_F = \Sigma|_F \subset |-nK_F| \).

However, by condition (ii) of Theorem 1.1, the variety \( F \) admits no mobile linear systems with a maximal singularity, a contradiction. \( \square \)
2.2. Birational modification of the base of the fibre space $V/S$. We now construct a sequence of blow-ups of the base whose composite is a birational morphism $\sigma_S : S^+ \to S$, along with the corresponding sequence of blow-ups of $V$ whose composite is a birational morphism $\sigma : V^+ \to V$, where $V^+ = V \times_S S^+$, in such a way that the following diagram commutes:

$$
\begin{array}{ccc}
V^+ & \xrightarrow{\sigma} & V \\
\downarrow{\pi^+} & & \downarrow{\pi} \\
S^+ & \xrightarrow{\sigma_S} & S
\end{array}
$$

The birational morphism $\sigma_S$ is constructed recursively as a composite of elementary blow-ups $\sigma_i : S_i \to S_{i-1}$, $i = 1, 2, \ldots$, where $S_0 = S$. Assume that $\sigma_i$ have already been constructed for $i \leq k$ (if $k = 0$, then we start with a blow-up of the base $S$). Put $V_k = V \times_S S_k$, and let $\pi_k : V_k \to S_k$ be the projection. Consider the irreducible closed subsets

$$
\pi_k(\text{centre}(E,V_k)) \subset S_k, \quad (2.1)
$$

where $E$ is any element of $\mathcal{M}$. By Proposition 2.2, all of them are proper subsets of $S_k$. If they are all prime divisors on $S_k$, then the construction terminates and we put $S^+ = S_k$ and $V^+ = V_k$. Otherwise we define $\sigma_{k+1}$ as the blow-up of any set (2.1) which is inclusion-minimal over all $E \in \mathcal{M}$.

It is easy to check that the sequence $\overline{\sigma}$ of blow-ups terminates. Indeed, put

$$
\alpha_k = \sum_{E \in \mathcal{M}} a(E,V_k).
$$

Since the birational morphism $\sigma_k : V_k \to V_{k-1}$ is the blow-up of a closed irreducible subset containing the centre of one of the divisors $E \in \mathcal{M}$ on $V_{k-1}$, we have $\alpha_{k+1} < \alpha_k$. By construction, the numbers $\alpha_i$ are non-negative, whence the sequence of blow-ups $\pi_i$ is finite. Thus, for every maximal singularity $E \in \mathcal{M}$, the closed subset $\pi_+(\text{centre}(E,V^+)) \subset S^+$ is a prime divisor.

2.3. The mobile family of curves. We again consider a very mobile family of curves $\mathcal{C}'$ on $V'$ and its strict transform $\mathcal{C}^+$ on $V^+$. Let $C^+ \in \mathcal{C}^+$ be a general curve, and let $C^+ = \pi_+(C^+)$ be the corresponding curve in the family $\mathcal{C}^+$ on $S^+$. Furthermore, let $\Sigma^+$ be the strict transform of the linear system $\Sigma$ on $V^+$. For some class of divisors $Y^+$ on $S^+$ we have

$$
\Sigma^+ \subset |-nK^+ + \pi_+^*Y^+|,
$$

where we write $K^+ = K_{V^+}$ for simplicity of notation. Note that even when $Y$ is an effective or mobile class on $S$, $Y^+$ need not be its strict transform on $S^+$, that is, we are abusing notation. The following observation is crucial.

**Proposition 2.3.** We have $(\overline{C^+} \cdot Y^+) < 0$. In particular, the class $Y^+$ is not pseudo-effective.
Proof. Assume the opposite:

$$(C^+ \cdot \pi^* Y^+) = (\overline{C^+} \cdot Y^+) \geq 0.$$  

We may assume that a resolution of the singularities $\varphi$ of the map $\chi$ factors through the sequence of blow-ups $\sigma : V^+ \to V$. The strict transform $\Sigma$ of the linear system $\Sigma$ on $V$ satisfies

$$\Sigma \subset \left\langle -n\overline{K} + \left(\pi^+_+ Y^+ - \sum_{E \in E} \overline{\varepsilon}(E)E\right) + \Xi \right\rangle,$$

where $\overline{K} = K_{V^+}, \overline{\varepsilon}(E) \in \mathbb{Z}$ and $\Xi$ is a linear combination of exceptional divisors of the birational morphism $\overline{V} \to V^+$ that do not lie in the set $E$. For the strict transform $\overline{C} \in \overline{C}$ of the curve $C^+ \in C^+$ and the divisor $\overline{D} \in \Sigma$ we have $(\overline{C} \cdot \overline{D}) = 0$ as in the proof of Proposition 2.1. By construction of the divisor $\Xi$ we have $(\overline{C} \cdot \Xi) = 0$. Finally, $(\overline{C} \cdot \overline{K}) < 0$, whence we conclude that

$$\left(\overline{C} \cdot \left(\pi^+_+ Y^+ - \sum_{E \in E} \overline{\varepsilon}(E)E\right)\right) < 0.$$

By assumption there is at least one divisor $E \in E$ such that $\overline{\varepsilon}(E) > 0$. This divisor is automatically a maximal singularity: $E \in M$. However, our construction yields more: $E$ is a maximal singularity for the mobile linear system $\Sigma^+$, that is, the pair $(V^+, \frac{1}{n} \Sigma^+)$ is not canonical and $E$ realizes a non-canonical singularity of this pair.

However, $\pi_+(\text{centre}(E, V^+)) = \overline{E} \subset S^+$ is a prime divisor, whence $\pi_+^{-1}(\overline{E}) \subset V^+$ is also a prime divisor. Since the linear system $\Sigma^+$ has no fixed components, the following assertion holds for a general point $s \in \overline{E}$ and the corresponding fibre $F = \pi_+^{-1}(s) \subset V^+$: the linear system $\Sigma_F = \Sigma^+|_F \subset |-nK_F|$ is non-empty and for every $D_F \in \Sigma_F$ the pair $(F, \frac{1}{n} D_F)$ is not log canonical by the inversion of adjunction (see [30]). This contradicts condition (ii) of Theorem 1.1. \qed

We now complete the proof of Theorem 1.1. Let us write down an explicit formula for the divisor $\pi^+_+ Y^+$ in terms of the partial resolution $\sigma$. Let $E^+ = \sigma^*(E) = \sigma^*(E)$ be the set of all exceptional divisors (of the morphism $\sigma$) whose image on $V'$ is a divisor and covers the base $S'$. Hence $E^+$ can be identified with a subset of $E$. In the course of the proof of Proposition 2.3 we established that

$$\mathcal{M}^+ = \mathcal{M} \cap \mathcal{E}^+ \neq \emptyset.$$  

We now write

$$\pi^+_+ Y^+ = \pi^* Y - \sum_{E \in E^+} \varepsilon_+(E)E + \Xi^+.$$

Moreover, we have

$$K^+ = \sigma^* K_V + \sum_{E \in E^+} a_+(E)E + \Xi_K,$$

where all coefficients $a_+(E)$ are positive and the divisor $\Xi_K$ is effective and pulled back from $S^+$, and the image of every irreducible component of $\Xi_K$ on $V'$ has codimension at least 2. Hence the general curve $C^+ \in C^+$ does not intersect the
support of the divisor $\Xi$. Let $C \in \mathcal{C}$ be the image of $C^+$ on the original variety $V$, and let $\mathcal{C} = \pi(C) \in \mathcal{C}$ be the projection of $C$ on the base $S$. For a general divisor $D \in \Sigma$ and its strict transform $D^+ \in \Sigma^+$ on $V^+$, the scheme-theoretic intersection $(D^+ \circ \pi_{+}^{-1}(C^+))$ is well defined. It is an effective cycle of dimension $\delta = \dim F$ on $V^+$ and we have the following representation of its numerical class:

$$(D^+ \circ \pi_{+}^{-1}(C^+)) \sim -n(\sigma^*K_V \cdot \pi_{+}^{-1}(C^+))$$

$$+ \left( \sum_{E \in \mathcal{E}^+} (-na_+(E) - \varepsilon_+(E))E \right) \cdot C^+ \right) F. \quad (2.2)$$

Since $(\mathcal{C} \cdot Y) \geq 0$ and $(C^+ \cdot \pi_+^* Y^+) < 0$, we have

$$\left( -\left[ \sum_{E \in \mathcal{E}^+} \varepsilon_+(E)E \right] \cdot C^+ \right) < 0,$$

whence the intersection of the divisor in square brackets in (2.2) with $C^+$ is negative. Therefore,

$$\sigma_*(D^+ \circ \pi_{+}^{-1}(C^+)) \sim -n(K_V \cdot \pi_{+}^{-1}(C)) + bF,$$

where $b < 0$. Since the left-hand side is an effective cycle of dimension $\delta$ on $V$, this contradicts condition (iii) of Theorem 1.1. □

§ 3. Varieties of general position

In this section we state explicit local conditions of general position for double spaces (§ 3.1) and hypersurfaces (§ 3.2). These conditions determine the sets $\mathcal{W}_{\text{reg}} \subset \mathcal{W}$ and $\mathcal{F}_{\text{reg}} \subset \mathcal{F}$. In § 3.1 we prove Theorem 1.2. In §§ 3.3–3.5 we prove part of Theorem 1.4: the estimate for the codimension of $\mathcal{F} \setminus \mathcal{F}_{\text{reg}}$. Some immediate geometric implications of the conditions of general position are considered in § 3.5.

3.1. Double spaces of general position. The open subset $\mathcal{W}_{\text{reg}} \subset \mathcal{W}$ of hypersurfaces of degree $2M$ in $\mathbb{P} = \mathbb{P}^M$ is determined by local conditions that must hold for every hypersurface $W \in \mathcal{W}_{\text{reg}}$ at every point $o \in W$. These conditions depend on whether the point $o \in W$ is non-singular or singular.

We first consider the condition of general position for a non-singular point $o \in W$. Let $(z_1, \ldots, z_M)$ be a system of affine coordinates with origin at $o$, and let

$$w = q_1 + q_2 + \cdots + q_{2M}$$

be the affine equation of the branch hypersurface $W$, where $q_i(z_*)$ is a homogeneous polynomial of degree $i, \ i = 1, \ldots, 2M$. Then $W$ must satisfy the following condition at the non-singular point $o \in W$ (that is, when $q_1 \neq 0$).

(W1) The rank of the quadratic form $q_2|_{\{q_1=0\}}$ is at least 2.

Proposition 3.1. The violation of condition (W1) imposes $\frac{(M-2)(M-1)}{2}$ independent conditions on the coefficients of the quadratic form $q_2$ (for a fixed linear form $q_1$).

Proof. This is obvious. □
We now define the condition of general position for a singular point \( o \in W \). Let
\[
w = q_2 + q_3 + \cdots + q_{2M}
\]
be the affine equation of the branch hypersurface \( W \) with respect to a system of affine coordinates \((z_1, \ldots, z_M)\) with origin at \( o \). Then \( W \) must satisfy the following condition at the singular point \( o \).

(W2) The rank of the quadratic form \( q_2 \) is at least 4.

**Proposition 3.2.** The violation of the condition (W2) imposes \( \frac{(M-2)(M-1)}{2} \) independent conditions on the coefficients of the quadratic form \( q_2 \).

**Proof.** This is obvious. \( \square \)

We now define the subset \( W_{\text{reg}} \subset W \) by requiring that \( W \in W_{\text{reg}} \) satisfies condition (W1) at every non-singular point and condition (W2) at every singular point. Clearly, \( W_{\text{reg}} \subset W \) is a Zariski-open subset (possibly empty).

**Proposition 3.3.** We have
\[
\text{codim}( (W \setminus W_{\text{reg}}) \subset W ) \geq \frac{(M-4)(M-1)}{2}.
\]

**Proof.** We use standard arguments (see [1], Part 3). Consider the incidence subvariety
\[
I = \{(o, W) \mid o \in W\} \subset \mathbb{P} \times W.
\]

For a fixed point \( o \in \mathbb{P} \), the codimension of the set \( W_{\text{non-reg}}(o) \) of hypersurfaces containing \( o \) and non-regular at \( o \) is given by Propositions 3.1, 3.2. (In the singular case there are \( M \) additional independent conditions since \( q_1 \equiv 0 \).) Then we compute the dimension of the closure of the set
\[
I_{\text{non-reg}} = \bigcup_{o \in \mathbb{P}} \{o\} \times W_{\text{non-reg}}(o)
\]
and consider the projection onto \( W \). \( \square \)

Clearly, for every hypersurface \( W \in W_{\text{reg}} \) the double covering \( F \to \mathbb{P} \) branched over \( W \) is an irreducible algebraic variety. Moreover, by condition (W2), \( F \) belongs to the class of varieties with quadratic singularities of rank at least 5 [31]. We recall that a variety \( \mathcal{X} \) is a variety with quadratic singularities of rank at least \( r \) if, near every point \( o \in \mathcal{X} \), one can realize \( \mathcal{X} \) as a hypersurface in a non-singular variety \( \mathcal{Y} \) in such a way that the local equation of \( \mathcal{X} \) at \( o \) is of the form \( \beta_1(u_o) + \beta_2(u_o) + \cdots = 0 \), where \((u_o)\) is a system of local parameters at \( o \in \mathcal{Y} \) and either \( \beta_1 \not\equiv 0 \), or \( \beta_1 \equiv 0 \) and \( \text{rk} \beta_2 \geq r \). Clearly, \( \text{codim}(\text{Sing} \mathcal{X} \subset \mathcal{X}) \geq r - 1 \). Hence the variety \( F \) is factorial [32].

Furthermore, it is easy to show (see [31]) that the class of quadratic singularities of rank at least \( r \) is stable under blow-ups in the following sense. Let \( B \subset \mathcal{X} \) be an irreducible subvariety. Then there is an open set \( U \subset \mathcal{Y} \) with the following two properties.

1) \( U \cap B \neq \emptyset \) and \( U \cap B \) is a non-singular algebraic variety.
2) Let $\sigma_B : U^+ \to U$ be its blow-up. Then $(\mathcal{X} \cap U)^+ \subset U^+$ is a variety with quadratic singularities of rank at least $r$.

To prove the last property, note the following simple fact. If $Z \ni o$ is a non-singular divisor on $Y$ with $Z \neq \mathcal{X}$ and the scheme-theoretic restriction $\mathcal{X}|_Z$ has a quadratic singularity of rank $l$ at $o$, then $\mathcal{X}$ has a quadratic singularity of rank at least $l$ at $o$. Now, if $B \not\subset \text{Sing} \mathcal{X}$, then the stability assertion is obvious. Therefore we may assume that $B \subset \text{Sing} \mathcal{X}$. One can choose the open set $U \subset Y$ in such a way that $B \cap U$ is a non-singular subvariety and the rank of quadratic points $o \in B \cap U$ is constant and equal to $l \geq r$. But then, in the exceptional divisor $E = \sigma_B^{-1}(B \cap U)$, the divisor $(\mathcal{X} \cap U)^+ \cap E$ is a fibration into quadrics of rank $l$ and, therefore, $(\mathcal{X} \cap U)^+ \cap E$ has at least quadratic singularities of rank at least $l$. Hence $(\mathcal{X} \cap U)^+ \subset U^+$ also has quadratic singularities of rank at least $r$ by the remark above. For an explicit analytic proof, see [31].

Stability under blow-ups implies that the singularities of $F$ are terminal (this is obvious in the particular case of one blow-up: the discrepancy of an irreducible exceptional divisor $(\mathcal{X} \cap U)^+ \cap E$ with respect to $\mathcal{X}$ is positive; every exceptional divisor over $\mathcal{X}$ can be realized by a sequence of blow-ups of the centres). Finally, $F$ satisfies condition (ii) of Theorem 1.1, that is, the condition of divisorial canonicity (see the proof of part (ii) of Theorem 2 in [26] and [29], Theorem 4). This completes the proof of Theorem 1.2.

3.2. Fano hypersurfaces of general position. As in the case of double spaces, the open subset $\mathcal{F}_{\text{reg}} \subset \mathcal{F}$ of hypersurfaces of degree $M$ in $\mathbb{P} = \mathbb{P}^M$ is defined by local conditions that must hold for every hypersurface $F \in \mathcal{F}_{\text{reg}}$ at every point $o \in F$. These conditions are different for singular and non-singular points $o \in F$.

We first consider conditions of general position at a non-singular point $o \in F$. Let $(z_1, \ldots, z_M)$ be a system of affine coordinates with origin at $o$, and let

$$w = q_1 + q_2 + q_3 + \cdots + q_M$$

be the affine equation of the hypersurface $F$, where $q_i(z_*)$ is a homogeneous polynomial of degree $i$, $i = 1, \ldots, M$. Here is the list of conditions of general conditions that must hold for $F$ at the non-singular point $o$.

(R1.1) The sequence $q_1, q_2, \ldots, q_{M-1}$ is regular in the local ring $\mathcal{O}_{o, \mathbb{P}}$, that is, the system of equations

$$q_1 = q_2 = \cdots = q_{M-1} = 0$$

determines a one-dimensional subset: a finite union of lines through $o$ in $\mathbb{P}$. In particular, $q_1 \neq 0$.

The equation $q_1 = 0$ determines the tangent space $T_o F$ (depending on what we need, we regard it either as a vector subspace of $\mathbb{C}^M$, or as its closure: a hyperplane in $\mathbb{P}$). We put $\bar{q}_i = q_i|_{\{q_1=0\}}$ for $i = 2, \ldots, M$. These are polynomials on the vector space $T_o F \cong \mathbb{C}^{M-1}$. Condition (R1.1) means that the sequence $\bar{q}_2, \bar{q}_3, \ldots, \bar{q}_{M-1}$ is regular. This reformulation is more convenient for estimating the codimension of the set of hypersurfaces that do not satisfy the regularity condition.

(R1.2) The quadratic form $\bar{q}_2$ on $T_o F$ is of rank at least 6. The linear span of every irreducible component of the closed algebraic set $\{q_1 = q_2 = q_3 = 0\}$ in $\mathbb{C}^M$ is the hyperplane $\{q_1 = 0\}$, that is, the tangent hyperplane $T_o F$. 
Here is an equivalent restatement of this condition: every irreducible component of the closed set \( \{ q_2 = q_3 = 0 \} \) in \( \mathbb{P}^{M-2} = \mathbb{P}(\{q_1 = 0\}) \) is non-degenerate.

(R1.3) For every hyperplane \( P \subset \mathbb{P} \), \( P \ni o \), other than the tangent hyperplane \( T_o F \subset \mathbb{P} \), the algebraic cycle of scheme-theoretic intersection of the hyperplanes \( P \) and \( T_o F \) and the projective quadric \( \{ q_2 = 0 \} \subset \mathbb{P} \) and \( F \), that is, the cycle

\[
(P \circ \{q_1 = 0\} \circ \{q_2 = 0\} \circ F),
\]

is irreducible and reduced. (Here the vinculum stands for the closure in \( \mathbb{P} \), and \( \circ \) is the operation of taking the cycle of scheme-theoretic intersection on \( \mathbb{P} \).)

We now consider the conditions of general position at a singular point \( o \in F \). Let \( (z_1, \ldots, z_M) \) be a system of affine coordinates with origin at \( o \), and let

\[
f = q_2 + q_3 + \cdots + q_M
\]

be the affine equation of the hypersurface \( F \), where \( q_i(z_*) \) is a homogeneous polynomial of degree \( i \), \( i = 2, \ldots, M \). Here is the list of conditions of general position that must hold for the hypersurface \( F \) at the singular point \( o \).

(R2.1) For every vector subspace \( \Pi \subset \mathbb{C}^M \) of codimension \( c \in \{0, 1, 2\} \), the sequence

\[
q_2|_{\Pi}, \ldots, q_{M-c}|_{\Pi}
\]

is regular in the ring \( \mathcal{O}_{o,\Pi} \), that is, the system of equations

\[
q_2|_{\mathbb{P}(\Pi)} = \cdots = q_{M-c}|_{\mathbb{P}(\Pi)} = 0
\]

determines a finite set of points in \( \mathbb{P}(\Pi) \cong \mathbb{P}^{M-c-1} \).

(R2.2) The quadratic form \( q_2(z_*) \) has rank at least 8.

(R2.3) Here we regard \( (z_1, \ldots, z_M) \) as homogeneous coordinates \( (z_1 : \cdots : z_M) \) on \( \mathbb{P}^{M-1} \). The divisor \( \{q_3|_{q_2=0} = 0\} \) on the quadric \( \{q_2 = 0\} \) is not a sum of three (not necessarily distinct) hyperplane sections of this quadric that belong to the same linear pencil.

Arguing just as in §3.1, we conclude that every hypersurface \( F \in \mathcal{F}_\text{reg} \) is an irreducible projective variety with factorial terminal singularities. Clearly, \( K_F = -H_F \) and \( \text{Pic} F = \mathbb{Z}H_F \), where \( H_F \) is the class of a hyperplane section \( F \subset \mathbb{P} \), that is, \( F \) is a Fano variety of index 1. To prove Theorem 1.4, we must establish two facts: inequality (1.2) and the divisorial log canonicity of the hypersurface \( F \in \mathcal{F}_\text{reg} \), that is, condition (ii) of Theorem 1.1 for the variety \( F \).

These two tasks are dealt with in the remaining part of this section and in §4 respectively.

3.3. Conditions of general position at a non-singular point. Let \( o \in F \) be a non-singular point. We fix an arbitrary non-zero linear form \( q_1 \) and consider the affine space of polynomials

\[
q_1 + \mathcal{P}^{\text{sing}} = \{q_1 + q_2 + \cdots + q_M\},
\]
where $\mathcal{P}^\text{sing}$ is the space of polynomials of the form $f = q_2 + q_3 + \cdots + q_M$. Let $\mathcal{P}_i \subset \{q_1 + \mathcal{P}^\text{sing}\}$ be the closure of the set of all polynomials $f$ that do not satisfy condition (R1.1), $i = 1, 2, 3$. We put

$$c_i = \text{codim}(\mathcal{P}_i \subset \{q_1 + \mathcal{P}^\text{sing}\}).$$

**Proposition 3.4.** For $M \geq 8$ we have

$$\min\{c_1, c_2, c_3\} = c_2 = \frac{(M - 6)(M - 5)}{2}.$$

**Proof.** This can easily be obtained by elementary methods. First, by Lemma 3.1 below (with $M$ replaced by $(M - 1)$) we have

$$c_1 = \frac{(M - 1)(M - 2)}{2} + 2.$$

Furthermore, the violation of the condition $\text{rk} \bar{q}_2 \geq 6$ imposes

$$\frac{(M - 6)(M - 5)}{2} < c_1$$

independent conditions on the coefficients of the quadratic form $q_2$. Assuming that the condition $\text{rk} \bar{q}_2 \geq 6$ holds, we obtain that the quadric $\{\bar{q}_2 = 0\}$ is factorial. It is easily checked that the reducibility or non-reduceness of the divisor $q_3|_{\{\bar{q}_2 = 0\}}$ on this quadric gives

$$\frac{M^3 - 6M^2 - 7M + 54}{6} > \frac{(M - 6)(M - 5)}{2}$$

independent conditions on the coefficients of the cubic form $q_3$.

Finally, consider a hyperplane $P \neq T_0 F$ and the quadric hypersurface

$$q_2|_{P \cap \{q_1 = 0\}} = 0.$$  

The rank of this quadric is at least 5, so it is still factorial. We now give a lower bound for the number of independent conditions imposed on the coefficients of the polynomials $q_3, \ldots, q_M$ by the violation of condition (R1.3). We define numbers $v(\mu), \mu = 0, 1, 2, 3$, by the following table:

| $\mu$ | 0  | 1  | 2  | 3  |
|-------|----|----|----|----|
| $v(\mu)$ | 0  | 1  | $\frac{1}{2}M(M + 1) - 1$ | $\frac{1}{2}M(M + 1) - 1$ |

and put

$$f(j, \mu) = \binom{j + M - 1}{M - 1} - \binom{j + M - 3}{M - 1} - v(\mu) + v(\max(0, \mu - 2)).$$

Now, using the factoriality of the quadric, we obtain a bound

$$c_3 \geq f(M, 3) - (M - 2)$$

$$- \max\left[ \max_{M - 1 \geq j \geq 2} (f(j, 2) + f(M - j, 1)), \max_{M - 1 \geq j \geq 3} (f(j, 3) + f(M - j, 0)) \right].$$

An elementary check shows that the minimum of the right-hand side is strictly larger than $c_2$ (it grows exponentially as $M \to \infty$). □
3.4. Conditions of general position at a singular point. We recall that \( P^{\text{sing}} \) is the space of polynomials of the form
\[
f = q_2 + q_3 + \cdots + q_M
\]
in the variables \( z_*(z_1, \ldots, z_M) \), where \( q_i(z_*) \) is homogeneous of degree \( i \), \( i = 2, \ldots, M \). Let \( P^{\text{sing}}_{\text{reg}} \subset P^{\text{sing}} \) be the subset of polynomials satisfying the conditions (R2.1)–(R2.3).

**Proposition 3.5.** We have
\[
\text{codim}(P^{\text{sing}} \setminus P^{\text{sing}}_{\text{reg}}) \geq \frac{(M-7)(M-6)}{2}.
\]

**Proof.** It suffices to show that the violation of each of the conditions (R2.1)–(R2.3) at the point \( o = (0, \ldots, 0) \) separately imposes at least \( \frac{(M-7)(M-6)}{2} \) independent conditions on the polynomial \( f \). We easily see that violation of (R2.2) imposes precisely \( \frac{(M-7)(M-6)}{2} \) independent conditions on the coefficients of the quadratic form \( q_2(z_*) \). Therefore, considering condition (R2.3), we may assume that (R2.2) holds. Then, in particular, the quadric \( \{q_2 = 0\} \) is factorial, and the number of independent conditions imposed by violation of (R2.3) on the coefficients of the cubic form \( q_3(z_*) \) (for a fixed polynomial \( q_2 \)) is
\[
M \frac{M^2 + 3M - 16}{6} \geq \frac{(M-7)(M-6)}{2}
\]
for \( M \geq 4 \). It remains to consider the violation of (R2.1).

**Lemma 3.1.** The violation of (R2.1) for one value of the parameter \( c = 0 \) imposes
\[
\frac{M(M-1)}{2} + 2 \quad (3.2)
\]
independent conditions on the coefficients of the polynomial \( f \).

**Proof.** We use standard methods (see [1], Part 3). The scheme of argument is as follows. Fix the first moment when the sequence of polynomials \( q_2, \ldots, q_M \) becomes non-regular. Assume that the regularity is first violated for \( q_k \), that is, the closed set \( \{q_2 = \cdots = q_{k-1} = 0 \} \) has the ‘correct’ codimension \( k-2 \) and \( q_k \) vanishes on a component of this set. When \( k \leq M-1 \) we use the method of [33] and obtain that the violation of regularity imposes at least
\[
\binom{M+1}{k} \geq \frac{(M+1)M}{2}
\]
independent conditions on the coefficients of \( f \). The right-hand side is strictly larger than (3.2), as required.

We consider the last possibility:
\[
\{q_2 = \cdots = q_{M-1} = 0\} \subset \mathbb{P}^{M-1}
\]
is a one-dimensional closed set and \( q_M \) vanishes on one of its irreducible components, say, on \( B \). The case when \( B \subset \mathbb{P}^{M-1} \) is a line is a special one: it is easy to check
that the vanishing on a line in $\mathbb{P}^{M-1}$ imposes in total precisely \((3.2)\) independent conditions on the polynomials $q_2, \ldots, q_M$. Hence we may assume that $B$ is not a line, that is, $\dim B < \dim \langle B \rangle = k \geq 2$. We now use the method suggested in [34].

We fix $k$ and the vector subspace $\langle B \rangle$ and consider first the case when $k \leq M - 2$. Then there are subscripts $i_1, \ldots, i_{k-1} \in \{2, \ldots, M - 1\}$ such that the restrictions $q_{i_1}|_{\langle B \rangle}, \ldots, q_{i_{k-1}}|_{\langle B \rangle}$ form a good sequence and $B$ is one of its associated subvarieties (the details of this procedure are described in the proof of Proposition 4 in §3 of [34]). Taking into account that, by construction, $B \subset \langle B \rangle$ is a non-degenerate curve, we see that decomposable polynomials of the form $l_1 \cdots l_a$, where the $l_i$ are linear forms on $\langle B \rangle \cong \mathbb{P}^k$, cannot vanish everywhere on $B$. This gives $jk + 1$ independent conditions for each of the polynomials $q_j$ with $j \not\in \{i_1, \ldots, i_{k-1}\}$. In total we have at least $k(M - k)(M - k + 1)/2 + M - 2k - 1$ independent conditions for these polynomials (the minimum is attained when $i_1 = M - k + 1, \ldots, i_{k-1} = M - 1$). Using the condition $q_M|_{B} \equiv 0$ and the formula for the dimension of the Grassmannian of $k$-dimensional subspaces in $\mathbb{P}^{M-1}$, we obtain at least

$$M^2 - kM + k^2 - M + k + 1$$

independent conditions for $f$. This number is easily seen to be at least $\left(\frac{M-7}{2}\right)(M-6) - 5$.

Finally, if $k = M - 1$ (that is, $B$ is a non-degenerate curve in $\mathbb{P}^{M-1}$), then the condition $q_M|_{B} \equiv 0$ yields at least $M(M - 1) + 1$ independent conditions for $q_M$. $\square$

We now complete the proof of Proposition 3.5.

For a fixed vector subspace $\Pi \subset \mathbb{C}^M$ of codimension $c \in \{0, 1, 2\}$, the violation of regularity of the sequence (3.1) imposes at least $(M - c)(M - c - 1)/2 + 2$ independent conditions on the polynomial $f$. Subtracting the dimension of the Grassmannian of subspaces of codimension $c$ in $\mathbb{C}^M$, we obtain the least value $(M - 3)(M - 6)/2$ for $c = 2$. This completes the proof of Proposition 3.5. $\square$

3.5. Estimating the codimension of the complement of $\mathcal{F}_{\text{reg}}$. We recall that $F \in \mathcal{F}_{\text{reg}}$ if and only if conditions (R1.1)–(R1.3) hold at every non-singular point $o \in F$ and conditions (R2.1)–(R2.3) hold at every singular point $o \in F$. Propositions 3.4, 3.5 yield the following fact.

**Proposition 3.6.** We have

$$\text{codim}( (\mathcal{F} \setminus \mathcal{F}_{\text{reg}}) \subset \mathcal{F} ) \geq \frac{(M - 7)(M - 6)}{2} - 5.$$

**Proof.** The proof is completely analogous to that of Proposition 3.3 and follows from Propositions 3.4, 3.5. $\square$

We now consider some geometric facts that follow immediately from the conditions of general position. These facts will be used in §4 to exclude log maximal
singularities. It was shown in [26] that for every effective divisor $D \sim nH$ on $F$ (where we write $H$ instead of $H_F$ to simplify the notation) the pair $(F, \frac{1}{n}D)$ is canonical at non-singular points $o \in F$. We shall use this fact without special reference. Let $D_2 = \{q_2|_F = 0\}$ be the first hypertangent divisor. Then we have $D_2^+ \in |2H - 3E|$. We recall that $E \subset \mathbb{P}^{M-1}$ is an irreducible quadric of rank at least 8. Clearly, the divisor $D_2 \in |2H|$ satisfies the equality

$$\frac{\text{mult}_o}{\deg} D_2 = \frac{3}{M}.$$  

Here and in what follows the symbol $\text{mult}_o / \deg$ stands for the ratio of the multiplicity at $o$ and the degree.

**Lemma 3.2.** Let $P \subset F$ be a section of the hypersurface $F$ by an arbitrary linear subspace in $\mathbb{P}$ of codimension 2 containing the point $o$. Then the restriction $D_2|_P$ is an irreducible reduced divisor on the hypersurface $P \subset \mathbb{P}^{M-2}$.

**Proof.** The variety $P$ has at most quadratic singularities of rank at least 6. Therefore it is factorial. Hence the reducibility or non-reduceness of $D_2|_P$ means that $D_2|_P = H_1 + H_2$, where $H_1$ and $H_2$ are (possibly coinciding) hyperplane sections of $P$. By condition (R2.2) we have $\text{mult}_oH_i = 2$. However, $\text{mult}_oD_2|_P = 6$. Therefore $D_2|_P$ cannot split into two hyperplane sections. $\Box$

**Proposition 3.7.** The pair $(F, \frac{1}{2}D_2)$ has no non-log-canonical singularities whose centre on $F$ contains the point $o$: $\text{LCS}(F, \frac{1}{2}D_2) \not\ni o$.

**Proof.** Assume the opposite. In any case,

$$\text{codim} \left( \text{LCS} \left( F, \frac{1}{2}D_2 \right) \subset F \right) \geq 6.$$  

Therefore we consider a section $P \subset F$ of the hypersurface $F$ by a general linear subspace of dimension 5 through the point $o$. Then the pair $(P, \frac{1}{2}D_2|_P)$ has $o$ as an isolated centre of a non-log-canonical singularity. Let $\sigma_P : P^+ \rightarrow P$ be the blow-up of the non-degenerate quadratic singularity $o \in P$. Then $E_P = E \cap P^+$ is a non-singular exceptional quadric in $\mathbb{P}^4$. Since

$$\frac{1}{2}(D_2|_P)^+ \sim H_P - \frac{3}{2}E_P, \quad a(E_P, P) = 2 > \frac{3}{2}$$

(while $H_P$ is the class of a hyperplane section of $P \subset \mathbb{P}^5$), the pair $(P^+, \frac{1}{2}(D_2|_P)^+)$ is not log canonical. Consider the union $\text{LCS}(P^+, \frac{1}{2}(D_2|_P)^+)$ of all centres of non-log-canonical singularities of this pair that intersect $E_P$. It is a connected closed subset of the exceptional quadric $E_P$, and any of its irreducible components $S_P$ satisfies the inequality $\text{mult}_{S_P}(D_2|_P)^+ \geq 3$. Reconsidering the original pair $(F, \frac{1}{2}D_2)$, we see that $\text{mult}_{S} D_2^+ \geq 3$ for some irreducible subvariety $S \subset E$, where $S \cap P^+ = S_P$ and, therefore, $\text{codim}(S \subset E) \in \{1, 2, 3\}$.

However, the case $\text{codim}(S \subset E) = 3$ is impossible. Indeed, by the connectedness principle, this equality means that $S_P$ is a point and, therefore, $S \subset E$ is a linear subspace of codimension 3, which is impossible if $\text{rk} q_2 \geq 8$ (a 7-dimensional non-singular quadric contains no linear subspaces of codimension 3).
Consider the case when codim(S ⊂ E) = 2. Let Π ⊂ E be a general linear subspace of maximal dimension. Then $D^+_2|_Π$ is a cubic hypersurface having multiplicity 3 along an irreducible subvariety $S_Π = S ∩ Π$ of codimension 2. Hence $D^+_2|_Π$ is a sum of three (not necessarily distinct) hyperplanes in Π that contain the linear subspace $S_Π ⊂ Π$ of codimension 2. Therefore $D^+_2|_E$ is also a sum of three (not necessarily distinct) hyperplane sections belonging to the same linear pencil, and $S$ is the intersection of the quadric $E$ and a linear subspace of codimension 2. But this is impossible by condition (R2.3).

Finally, if codim(S ⊂ E) = 1, then $D^+_2|_E = 3S$ is a triple hyperplane section of the quadric $E$. This is impossible by condition (R2.3). □

Here is another fact that will be used in what follows.

**Proposition 3.8.** For every hyperplane section $Δ ⊃ o$ of the hypersurface $F$, the pair $(F, Δ)$ is log canonical.

**Proof.** This is a corollary of the following well-known fact (see, for example, [29], [35]). Let $(p ∈ X)$ be a germ of a non-degenerate three-dimensional quadratic singularity, $σ: ˜X → X$ its resolution with exceptional quadric $E_X ≅ ℙ^1 × ℙ^1$, and $D_X$ a germ of an effective divisor with $o ∈ D_X$ and $D_Χ ∼ −βE_X$. Then the pair $(X, 1/βD_X)$ is log canonical at the point $o$. □

§ 4. Exclusion of maximal singularities

In this section we complete the proof of Theorem 1.4. Let $F$ be a fixed hypersurface of degree $M$ in $ℙ$ satisfying the regularity conditions: $F ∈ F_{reg}$. In § 3 we mentioned a result in [26] saying that for every effective divisor $D ∼ nH$, the pair $(F, 1/nD)$ has no maximal singularities whose centre does not lie in the closed set $Sing F$. It was shown in [31] that for every mobile linear system $Σ ⊂ |nH|$, the pair $(F, 1/nD)$ is canonical for a general divisor $D ∈ Σ$, that is, $Σ$ has no maximal singularities. Hence, to complete the proof of Theorem 1.4, it suffices to establish that the pair $(F, 1/nD)$ is log canonical for every effective divisor $D ∼ nH$. Moreover, we may consider only those log-maximal singularities whose centre is contained in $Sing F$.

In § 4.1 we carry out some preparatory work. Using the technique of hypertangent divisors, we obtain bounds for the ratio $\text{mult}_o/\deg$ for some classes of irreducible subvarieties of $F$. Then we fix a pair $(F, 1/nD)$ and assume that it is not log canonical. Our aim is to deduce a contradiction. Let $B^* ⊂ Sing F$ be the centre of a log-maximal singularity of the divisor $D$, $o ∈ B^*$ a point of general position, $F^+ → F$ its blow-up, and $D^+$ the strict transform of $D$. In § 4.2 we study the properties of the pair $(F^+, 1/nD^+)$. We show that this pair has a non-log-canonical singularity whose centre is a subvariety of the exceptional divisor of the blow-up at $o$. Then we show in §§ 4.2, 4.3 that this is impossible. This will complete the proof of Theorem 1.4.

4.1. The method of hypertangent divisors. We fix a singular point $o ∈ F$, a system of coordinates $(z_1, \ldots, z_M)$ on $ℙ$ with origin at $o$, and the equation $f = q_2 + \cdots + q_M$ of the hypersurface $F$.
Proposition 4.1. Assume that the variety $F$ satisfies conditions (R2.1), (R2.2) at the singular point $o$. Then the following assertions hold.

(i) For every irreducible subvariety $Y \subset F$ of codimension 2 we have

$$\frac{\text{mult}_o Y}{\text{deg} Y} \leq \frac{4}{M}.$$

(ii) If $\Delta \ni o$ is an arbitrary hyperplane section of $F$, then for every prime divisor $Y \subset \Delta$ we have

$$\frac{\text{mult}_o Y}{\text{deg} Y} \leq \frac{3}{M}.$$

(iii) If $P \ni o$ is the section of $F$ by an arbitrary linear subspace of codimension 2, then for every prime divisor $Y \subset P$ we have

$$\frac{\text{mult}_o Y}{\text{deg} Y} \leq \frac{4}{M}.$$

Proof. This is proved by the method of hypertangent divisors ([1], Part 3). For $k = 2, \ldots, M - 1$ let $\Lambda_k$ be the linear system

$$\left| \sum_{i=2}^{k} s_{k-i}(q_2 + \cdots + q_i)|_F = 0 \right|$$

(the $k$th hypertangent linear system), where the $s_j(z_*)$ are all possible homogeneous polynomials of degree $j$. Let $\sigma : F^+ \to F$ be the blow-up of the point $o$. Its exceptional divisor $E = \sigma^{-1}(o)$ is naturally realized as a quadric in $\mathbb{P}^{M-1}$. We have

$$\Lambda_k^+ \subset |kH - (k+1)E|,$$

where $\Lambda_k^+$ is the strict transform of the system $\Lambda_k$ on $F^+$. Let $D_k \in \Lambda_k$, $k = 2, \ldots, M - 1$, be general hypertangent divisors.

Let us prove part (i). By condition (R2.1) we have

$$\text{codim}_o (\text{Bs} \Lambda_k \subset F) = k - 1, \quad (4.1)$$

where codim$_o$ means the codimension in a neighbourhood of the point $o$. Hence

$$Y \cap D_4 \cap D_5 \cap \cdots \cap D_{M-1}$$

is a closed one-dimensional set in a neighbourhood of $o$. We construct a sequence of irreducible subvarieties $Y_i \subset F$ of codimension $i$: $Y_2 = Y$ and $Y_{i+1}$ is an irreducible component of the effective cycle $(Y_i \circ D_{i+2})$ with the maximal value of the ratio $\text{mult}_o / \text{deg}$. The cycle $(Y_i \circ D_{i+2})$ is always well defined because inequality (4.1) yields that $Y_i \not\subset D_{i+2}$ for a general hypertangent divisor $D_{i+2}$. By the construction of hypertangent linear systems, at every step of our procedure we have

$$\frac{\text{mult}_o Y_{i+1}}{\deg Y_{i+1}} \geq \frac{i+3}{i+2} \frac{\text{mult}_o Y_i}{\deg Y_i}.$$
Hence the curve $Y_{M-2}$ satisfies the bounds

$$1 \geq \frac{\text{mult}_o Y_{M-2}}{\deg Y_{M-2}} \geq \frac{\text{mult}_o Y}{\deg Y} \cdot \frac{5 \ 6}{4 \ 5} \cdots \frac{M}{M-1},$$

and part (i) follows.

Let us prove part (ii). By Lemma 3.2 the divisor $D_2|\Delta$ is irreducible and reduced. By condition (R2.1) it satisfies the equality

$$\frac{\text{mult}_o D_2|\Delta}{\deg D_2|\Delta} = \frac{3}{M}.$$ 

Hence we can assume that $Y \neq D_2|\Delta$. Then $Y \not\subset D_2$ and the effective cycle $(Y \circ D_2)$ of codimension 2 on $\Delta$ is well defined and satisfies the inequality

$$\frac{\text{mult}_o (Y \circ D_2)}{\deg (Y \circ D_2)} \geq \frac{3}{2} \frac{\text{mult}_o Y}{\deg Y}.$$ 

Let $Y_2$ be an irreducible component of this cycle with the maximal value of the ratio $\text{mult}_o / \deg$. Using the technique of hypertangent divisors for $Y_2$ exactly as in the proof of part (i), we see from condition (R2.1) that the intersection

$$Y_2 \cap D_4|\Delta \cap D_5|\Delta \cap \cdots \cap D_{M-2}|\Delta$$

is a one-dimensional closed set in a neighbourhood of $o$, where $D_4 \in \Lambda_4, \ldots, D_{M-2} \in \Lambda_{M-2}$ are general hypertangent divisors. (Note that in contrast to part (i) the last of them is $D_{M-2}$ and not $D_{M-1}$ because the dimension of $\Delta$ is equal to the dimension of $F$ minus one, and the condition (R2.1) guarantees that the truncated sequence $q_2|\Pi, \ldots, q_{M-1}|\Pi$ is regular, where $\Pi$ is the hyperplane that cuts out $\Delta$ on $F$.) Arguing as in the proof of part (i), we obtain the bound

$$1 \geq \frac{\text{mult}_o Y \cdot 3 \ 5 \ 6}{2 \ 4 \ 5} \cdots \frac{M-1}{M-2}.$$ 

It follows that

$$\frac{\text{mult}_o Y}{\deg Y} \leq \frac{8}{3(M-1)}.$$ 

The right-hand side does not exceed $3/M$ for $M \geq 9$. This proves part (ii).

Let us prove part (iii). We argue as in the proof of (ii) with the only difference being that to estimate the multiplicity of the cycle $(Y \circ D_2|P)$ at the point $o$, we use the hypertangent divisors

$$D_4|P, D_5|P, \ldots, D_{M-3}|P$$

(one less than before). This yields the bound

$$1 \geq \frac{\text{mult}_o Y \cdot 3 \ 5 \ 6}{2 \ 4 \ 5} \cdots \frac{M-2}{M-3},$$ 

whence

$$\frac{\text{mult}_o Y}{\deg Y} \leq \frac{8}{3(M-2)}.$$ 

The right-hand side does not exceed $4/M$ for $M \geq 6$. This proves part (iii).
We now return to the proof of Theorem 1.4.

4.2. The blow-up of a singular point. Assume that the pair \((F, \frac{1}{n} D)\) is non-log-canonical for some divisor \(D \in |nH|\). In other words, the following log Noether–Fano inequality holds for some prime divisor \(E^* \subset \tilde{F}\), where \(\psi: \tilde{F} \to F\) is a birational morphism, and \(\tilde{F}\) is non-singular and projective:

\[
\text{ord}_{E^*} \psi^* D > n(a(E^*) + 1).
\]

Since this inequality is linear in \(D\) and \(n\), we can assume that \(D\) is a prime divisor. Let \(B^* = \psi(E^*) \subset F\) be the centre of the log maximal singularity \(E^*\). We know that \(B^* \subset \text{Sing} F\). In particular, \(\text{codim}(B^* \subset F) \geq 7\). Let \(o \in B^*\) be a point of general position, \(\varphi: F^+ \to F\) its blow-up, and \(E \subset F^+\) the exceptional quadric.

Consider the first hypertangent divisor \(D^2 \in |2H|\) at the point \(o\). By Lemma 3.2, \(D^2\) is irreducible and reduced. By Proposition 3.7, the pair \((F, \frac{1}{2} D^2)\) is log canonical at \(o\). Hence, \(D \neq D_2\).

**Proposition 4.2.** We have

\[
\text{mult}_o D \leq \frac{8}{3} n.
\]

**Proof.** Consider the effective cycle \((D \circ D_2)\) of codimension 2. Clearly,

\[
\frac{\text{mult}_o (D \circ D_2)}{\deg (D \circ D_2)} \geq \frac{3}{2} \frac{\text{mult}_o D}{\deg D}.
\]

But the left-hand side does not exceed \(4/M\) by Proposition 4.1, (i). Since \(\deg D = nM\), this proves the proposition. \(\square\)

We write \(D^+ \sim nH - \nu E\), where \(\nu \leq \frac{4}{3} n\).

Consider the section \(P\) of the hypersurface \(F\) by a general 5-dimensional linear subspace through \(o\). Let \(P^+\) be the strict transform of \(P\) on \(F^+\). Then \(E_P = P^+ \cap E\) is a non-singular three-dimensional quadric. We also put \(D_P = D|_P\). Clearly, the pair \((P, \frac{1}{n} D_P)\) has \(o\) as the isolated centre of a non-log-canonical singularity. Since \(a(E_P) = 2\) and \(D_P^+ \sim nH_P - \nu E_P\) (where \(H_P\) is the class of a hyperplane section of \(P\)) with \(\nu \leq \frac{4}{3} n < 2n\), the pair \((P^+, \frac{1}{n} D_P^+)\) is not log canonical and the union \(\text{LCS}_E(P^+, \frac{1}{n} D_P^+)\) of the centres of all non-log-canonical singularities (of this pair) that intersect \(E_P\) is a connected closed subset of the exceptional quadric \(E_P\). Let \(S_P\) be an irreducible component of this subset. Clearly,

\[
\text{mult}_{S_P} D_P^+ > n.
\]

Furthermore, \(\text{codim}(S_P \subset E_P) \in \{1, 2, 3\}\). Returning to the original pair \((F, \frac{1}{n} D)\), we see that the pair \((F^+, \frac{1}{n} D^+)\) has a non-log-canonical singularity whose centre is a subvariety \(S \subset E\) such that \(S \cap E_P = S_P\) and, in particular, \(\text{codim}(S \subset E) \in \{1, 2, 3\}\).

Note that the case \(\text{codim}(S \subset E) = 3\) is impossible. Indeed, \(S_P\) would then be a point by the connectedness principle and, therefore, \(S\) would be a linear subspace of codimension 3 on the quadric \(E\) of rank at least 8, which is impossible.
It is also easy to exclude the case when \( \text{codim}(S \subset E) = 1 \). Indeed, in this case the divisor \( S \) is cut out on \( E \) by a hypersurface of degree \( d_S \geq 1 \). Let \( H_E \) be the class of a hyperplane section of \( E \). We have \( D^+|_E \sim \nu H_E \), whence
\[
\frac{4}{3} n \geq \nu > n d_S
\]
and, therefore, \( S \) is a hyperplane section of \( E \). Let \( \Delta \in |H| \) be the unique hyperplane section of \( F \) such that \( \Delta \owns o \) and \( \Delta^+ \cap E = S \). The pair \((F^+, \Delta^+)\) is log canonical, whence \( D \neq \Delta \). For the effective cycle \((D \circ \Delta)\) of codimension 2 on \( F \) we have
\[
\text{mult}_o(D \circ \Delta) \geq 2\nu + 2 \text{mult}_S D^+ > 4n
\]
and, therefore,
\[
\frac{\text{mult}_o(D \circ \Delta)}{\deg(D \circ \Delta)} > \frac{4}{M}
\]
contrary to Proposition 4.1. This excludes the case of a divisorial centre.

4.3. The case of codimension 2. From now until the end of the paper we assume that \( \text{codim}(S \subset E) = 2 \).

Lemma 4.1. The subvariety \( S \) is contained in some hyperplane section of \( E \).

Proof. Since \( \text{mult}_S D^+ > n \) and \( D^+|_E \sim \nu H_E \) with \( \nu \leq \frac{4}{3} n \), we have \( L \subset D^+ \) for every secant line \( L \subset E \) of the subvariety \( S \). Let \( \Pi \subset E \) be a linear subspace of maximal dimension in general position. Put \( S_\Pi = S \cap \Pi \). The secant lines of the closed set \( S_\Pi \subset \Pi \) of codimension 2 cannot sweep out \( \Pi \) since \( E \not\subset D^+ \). Hence two possibilities may occur (see [29], Lemma 2.3):

1) the secant lines of \( S_\Pi \) sweep out a hyperplane in \( \Pi \);
2) \( S_\Pi \subset \Pi \) is a linear subspace of codimension 2.

In case 1), the secant lines \( L \subset E \) of the set \( S \) sweep out a divisor on \( E \). This divisor may be only a hyperplane section of \( E \). In case 2), \( S \) contains all its secant lines and is a section of \( E \) by a linear subspace of codimension 2. \( \square \)

We have just shown that one of the following two possibilities occurs: either there is a unique hyperplane section \( \Lambda \) of \( E \) containing \( S \) (case 1)), or \( S = E \cap \Theta \) for a certain linear subspace \( \Theta \) of codimension 2 (case 2)). We study them separately.

Assume that case 1) holds. Then \( S \) is cut out on \( \Lambda \) by a hypersurface of degree \( d_S \geq 2 \). Put
\[
\mu = \text{mult}_S D^+, \quad \gamma = \text{mult}_\Lambda D^+,
\]
where \( \mu > n \) and \( \mu \leq 2\nu \leq \frac{8}{3} n \).

Lemma 4.2. We have
\[
\gamma \geq \frac{2\mu - \nu}{3}.
\]

Proof. This is easily obtained in the same way as in the short proof of Lemma 3.5 in [14], §3.7. Let \( L \subset \Lambda \) be a general secant line of \( S \). Consider a section \( P \) of the hypersurface \( F \) by a general 4-plane in \( \mathbb{P} \) such that \( P \owns o \) and \( P^+ \cap E \) contains the line \( L \). Clearly, \( o \in P \) is a non-degenerate quadratic point and \( E_P = P^+ \cap E \cong \mathbb{P}^1 \times \mathbb{P}^1 \) is a non-singular quadric in \( \mathbb{P}^3 \). Put \( D_P = D|_P \). Clearly, \( \gamma = \text{mult}_L D_P^+ \).
Let $\sigma_L: P_L \to P^+$ be the blow-up of $L$, and let $E_L = \sigma_L^{-1}(L)$ be the exceptional divisor. Since $N_{L/P^+} \cong O \oplus O_L(-1)$, the exceptional surface $E_L$ is a ruled surface of type $\mathbb{F}_1$. Hence

$$\text{Pic } E_L = \mathbb{Z}s \oplus \mathbb{Z}f,$$

where $s$ (resp. $f$) is the class of the exceptional section (resp. of the fibre). Furthermore, $E_L|_{E_L} = -s - f$. Let $D_L$ be the strict transform of $D^+_P$ on $P_L$. Clearly,

$$D_L \sim nH_P - \nu E_P - \gamma E_L,$$

where $H_P$ is the class of a hyperplane section of $P$. Hence

$$D_L|_{E_L} \sim \gamma s + (\gamma + \nu)f.$$

On the other hand, $L$ is a general secant line of $S$ and, therefore, $L$ contains at least two distinct points $p, q \in S$. Hence the divisor $D^+_L$ has multiplicity $\mu$ at the points $p, q \in L$ and, therefore, the effective 1-cycle $D_L|_{E_L}$ contains the corresponding fibres $\sigma_L^{-1}(p)$ and $\sigma_L^{-1}(q)$ over these points with multiplicity $\mu - \gamma$. Thus,

$$\gamma + \nu \geq 2(\mu - \gamma),$$

and the lemma follows. \qed

We now consider the unique hyperplane section $\Delta$ of the hypersurface $F \subset \mathbb{P}$ such that $\Delta \ni o$ and $\Delta^+ \cap E = \Lambda$. Put $D_\Delta = D|_\Delta$. Write

$$D^+|_{\Delta^+} = D^+_\Delta + a\Lambda.$$

Clearly,

$$\text{mult}_o D_\Delta = 2(\nu + a) \geq 2\nu + 2\frac{2\mu - \nu}{3} = \frac{4}{3}(\mu + \nu) > \frac{8}{3}n.$$

We have already mentioned that the subvariety $S$ is cut out on the quadric $\Lambda$ by a hyperplane of degree $d_S \geq 2$. Therefore the divisor $D^+_\Delta \sim nH_\Delta - (\nu + a)\Lambda$ cannot contain $S$ with multiplicity greater than

$$\frac{1}{d_S}(\nu + a) \leq \frac{\nu + a}{2}.$$

Since the pair $(F^+, \frac{1}{n}D^+)$ has a non-log-canonical singularity with centre at $S$, it follows from the inversion of adjunction that so does the pair $(\Delta^+, \frac{1}{n}(D^+_\Delta + a\Lambda))$. We recall that $\text{codim}(S \subset \Delta^+) = 2$. Consider the blow-up $\sigma_S: \tilde{\Delta} \to \Delta^+$ of the subvariety $S$ and denote the exceptional divisor $\sigma_S^{-1}(S)$ by $E_S$. The following fact is well known.

**Proposition 4.3.** For some irreducible divisor $S_1 \subset E_S$ such that the projection $\sigma_S|_{S_1}$ is birational, we have

$$\text{mult}_S(D^+_\Delta + a\Lambda) + \text{mult}_{S_1}(\tilde{D}_\Delta + a\tilde{\Lambda}) > 2n,$$

where $\tilde{D}_\Delta$ (resp. $\tilde{\Lambda}$) is the strict transform of $D^+_\Delta$ (resp. $\Lambda$) on $\tilde{\Delta}$. 


Proof. See [26], Proposition 9. □

Put \( \mu_S = \text{mult}_S D^+_\Delta \) and \( \beta = \text{mult}_S \tilde{D}_\Delta \). We first consider the case of general position: \( S_1 \neq E_S \cap \tilde{\Lambda} \). Then \( S_1 \not\subset \tilde{\Lambda} \) and the inequality (4.2) takes the form

\[
\mu_S + \beta + a > 2n.
\]

Since \( \mu_S \geq \beta \), we certainly have \( 2\mu_S + a > 2n \). On the other hand, we mentioned above that \( 2\mu_S \leq \nu + a \). As a result, we obtain the bound

\[
\nu + 2a > 2n.
\]

Therefore \( \text{mult}_o D_\Delta > \nu + 2n > 3n \). However, \( D_\Delta \sim nH_\Delta \) is an effective divisor on the hyperplane section \( \Delta \) and, by Proposition 4.1, (ii), we have

\[
\frac{\text{mult}_o D_\Delta}{\deg D_\Delta} \leq \frac{3}{M}.
\]

The resulting contradiction excludes the case of general position. Therefore we are left with only one option: \( S_1 = E_S \cap \tilde{\Lambda} \).

In this case, the inequality (4.2) takes the form

\[
\mu_S + \beta + 2a > 2n.
\]

This inequality is weaker than its analogue in the case of general position, but as a compensation we obtain the additional inequality

\[
2\mu_S + 2\beta \leq \nu + a
\]

(the restriction \( D^+_\Delta|_\Lambda \) is cut out by a hyperplane of degree \( \nu + a \) and contains the divisor \( S \sim d_S H_\Lambda \) with multiplicity at least \( \mu_S + \beta \)). Combining the two last estimates, we obtain the inequality

\[
\nu + 5a > 4n,
\]

whence \( 5(\nu + a) > 8n \) and, therefore, \( \text{mult}_o D_\Lambda > \frac{16}{5} n \). We have already mentioned that this contradicts Proposition 4.1, (ii). This completes the exclusion of case 1).

Therefore case 2) occurs: \( S = E \cap \Theta \), where \( \Theta \subset \mathbb{P}^{M-1} \) is a linear subspace of codimension 2. Let \( P \subset F \) be a section of \( F \) by the linear subspace of codimension 2 in \( \mathbb{P} \) which is uniquely determined by the conditions \( P \ni o \) and \( P^+ \cap E = S \). Furthermore, let \( |H - P| \) be the pencil of hyperplane sections of \( F \) containing \( P \). The following assertions hold for a general hyperplane section \( \Delta \in |H - P| \).

a) The divisor \( D \) does not contain \( \Delta \) as a component. Hence the effective cycle \( (D \circ \Delta) = D_\Delta \) of codimension 2 on \( F \) is well defined and may be regarded as an effective divisor \( D_\Delta \in |nH_\Delta| \) on the hypersurface \( \Delta \subset \mathbb{P}^{M-1} \).

b) For the strict transform \( D^+_\Delta \) on \( F^+ \) we have \( \text{mult}_S D^+_\Delta = \text{mult}_S D^+ \).

Of course, the divisor \( D_\Delta \) may contain \( P \) as a component. Write \( D_\Delta = G + aP \), where \( a \in \mathbb{Z}_+ \) and \( G \) is an effective divisor that does not contain \( P \) as a component,
We put \( m = n - a \). The effective cycle \( G_P = (F \circ P) \) of codimension 2 on \( \Delta \) is well defined and may be regarded as an effective divisor \( G_P \in |mH_P| \) on the hypersurface \( P \subseteq \mathbb{P}^{M-2} \). The divisor \( G_P \) satisfies the inequality
\[
\text{mult}_o G_P \geq 2(\nu - a) + 2 \text{mult}_S G^+ > 4m.
\]
This is impossible by Proposition 4.1, (iii).

Thus the assumption that the pair \( (F, \frac{1}{n}D) \) is not log canonical for some divisor \( D \sim nH \) leads to a contradiction.

This completes the proof of Theorem 1.4.

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Received 2/JUL/14

Translated by THE AUTHOR