Fast Non-Parametric Learning to Accelerate Mixed-Integer Programming for Online Hybrid Model Predictive Control

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Abstract: Today’s fast linear algebra and numerical optimization tools have pushed the frontier of model predictive control (MPC) forward, to the efficient control of highly nonlinear and hybrid systems. The field of hybrid MPC has demonstrated that exact optimal control law can be computed, e.g., by mixed-integer programming (MIP) under piecewise-affine (PWA) system models. Despite the elegant theory, online solving hybrid MPC is still out of reach for many applications. We aim to speed up MIP by combining geometric insights from hybrid MPC, a simple-yet-effective learning algorithm, and MIP warm start techniques. Following a line of work in approximate explicit MPC, the proposed learning-control algorithm, LNMS, gains computational advantage over MIP at little cost and is straightforward for practitioners to implement.

1. INTRODUCTION

The presence of hybrid dynamical systems, defined as systems whose state evolution is governed not only by continuous dynamics (flow) but also by discrete events (jump), is ubiquitous in physical systems. Consider, for instance, a legged robot that dynamically navigates the environment. It must not only coordinate the motion of its joints but decide when to make and break contact at its end-effectors. This inherent coupling of discrete events and continuous decision-making gives the robot tremendous flexibility and capability of efficiently handling many tasks. However, it challenges optimization-based control designs such as model predictive control (MPC). One of the greatest challenges is perhaps the combinatorial growth of computational complexity caused by the hybrid mode switches.

This paper aims to answer an intuitive question: what information can be learned when solving hybrid MPC problems over and over again? Our main insight is: The geometric structure of hybrid MPC solutions can directly be induced by a nonparametric learning algorithm. It learns to predict mode sequences from previously visited states.

Hence, instead of storing exact solution structures, e.g., polyhedral partitions, we store samples of visited states to greatly speed up hybrid MPC. This is done at little extra computation effort.

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Consider the running example of the simple cart and wall. How-
ever, solving MIP online involves computationally heavy
processes such as Branch-and-Bound with exponential
time-complexity. Despite the speed-up by several orders
of magnitude in previous years Bertsimas et al. (2016),
solving MIP for online control is still impractical.

We hope to contribute in the following aspects:

- We propose a simple-yet-effective fast learning algo-
rithm to warm-start mixed-integer programs for MPC
in PWA hybrid systems. As learning progresses, it
greatly reduces computational cost.
- The properties of the proposed algorithm are ex-
  ploited to allow i) post-processing of solutions im-
  proving towards optimality and early-termination of
  MIP; ii) straightforward implementation for practi-
  tioners.

**Notation**
In this paper, \( x \in \mathbb{R}^d \) denotes a column vector
of dimension \( d \) and \( x^\top \) its transpose. \( N \) denotes the horizon
of an optimal control problem (OCP) and \( n \) the number of samples
of the learning algorithm. \( x_p \) is a parameter (e.g. 
current state estimation) of an optimization problem. \( e_i \)
denotes the one-hot vector of suitable dimension where the
\( i \)-th element is 1 and rest are zeros. A sequence of modes
of a discrete dynamical system is denoted as \( \mathcal{M} = \{m_t\}_{t=1}^N \)
where each \( m_t = (\mu_1^t, \mu_2^t, \ldots, \mu_M^t) \) is a system mode at
time \( t \) and \( \mu_i \) a binary variable. A feasible solution of an
optimization problem is one at which all constraints are
satisfied.

### 2. PRELIMINARIES

#### 2.1 Model predictive control for piece-wise affine hybrid systems

In this paper, we consider a discrete-time optimal control
problem (OCP). The goal is to compute a sequence of control actions \( \{u_t\}_{t=0}^N \)
to steer the system state \( x \) to the origin.

\[
\begin{align*}
\text{minimize} & \quad \sum_{t=0}^{N-1} (x_t^\top Q x_t + u_t^\top R u_t) + x_N^\top P x_N, \\
\text{subject to} & \quad x_{t+1} = f(x_t, u_t), \quad t = 0, \ldots, N-1, \\
& \quad h_t(x_t, u_t) \leq 0, \quad t = 0, \ldots, N-1, \\
& \quad x_0 = x_p,
\end{align*}
\]

where \( x_p \) is the current state, \( x_t^\top Q x_t + u_t^\top R u_t \) is the stage
cost and \( x_N^\top P x_N \) the terminal cost. \( f \) is the dynamics
that describes the evolution of the system over time and \( h_t \)'s
are the constraints (e.g., bounds on control input \( u_t \)).

Model predictive control (MPC) Richalet et al. (1978);
Rawlings and Mayne (2009) solves OCP (1) at every
sampling time and implement the first control input \( u_0 \).

Consider the running example of the simple cart and wall
in Fig. 1 again. In this case, there is no single dynamics
function \( f \) that governs on the whole state-space — this is
a hybrid system. One approach to solve hybrid MPC is to
formulate the dynamics constraints using piece-wise affine
(PWA) formulations (Sontag (1981)). In fact, PWA models
can describe general nonlinear dynamics while enjoying the
advantage of having a wide range of linear control tools,
\[
x_{t+1} = A^i x_t + B^i u_t, \quad \text{for } x \in C_i, i = 1 \ldots, n_M. \tag{2}
\]

Mathematically, solving OCP (1) under PWA dynamics
(2) is typically formulated as an mixed-integer program
(MIP), due to the presence of both continuous and discrete
variables. We consider the big-M formulation of MIP:

\[
\begin{align*}
|x_{t+1} - A^i x_t - B^i u_t| & \leq (1 - \mu_i) M \\
b(x_t, u_t) & \leq (1 - \mu_i) M \\
\sum_i \mu_i^t & = 1, \quad \mu_i \in \{0, 1\}, \quad \forall i, t,
\end{align*}
\]

where \( \mu_i^t, i = 1, \ldots, n_M \) are the auxiliary integer variables.
At time \( t \), it is easy to see that the system is in
mode \( i: m_t = e_i \) if and only if \( \mu_i^t = 1 \). A mode sequence
\( M = \{m_t\}_{t=1}^N \) thus corresponds to a set of integer decision
variables of the hybrid OCP (1),(3).

We use \( \mathcal{X}^M \) to denote the feasible region (not to be
confused with critical region) of an OCP given \( M \): the
set of parameters \( x_p \) where a mode sequence \( M \) is feasible
(i.e., \( \text{OCP (1)}(3) \) has solution with this fixed \( M \)). It is easy
to see that one mode sequence may not be feasible for the
whole state space in hybrid systems. For example, consider
a PWA control law of the cart example in Fig. 2 (a), the
mode sequence associated with the region (colored red)
in the upper right corner is not feasible once we cross the
boundary of this region. This corresponds to a different set of
integer solution to \( \text{OCP (1)}(3) \). The geometric property of
feasible regions can be summarized as follows: i) Feasible
regions are convex polyhedra. ii) Each feasible region
 corresponds to a integer solution to the hybrid OCP\( (1) (3) \)
(and hence a mode sequence). iii) They may or may not
overlap.

One insight commonly exploited is that, the hybrid
OCP (1) becomes a continuous (convex) optimization
problem if the integer variables in (3) are fixed to a feasible
mode sequence \( M \). Thus, the problem can be solved by
extremely efficient algorithms. Hence, if we store the
feasible region \( \mathcal{X}^M \) and associated \( M \) offline, during online
runtime of MPC, we only need to look up which \( \mathcal{X}^M \),
the current state \( x_p \) belongs to and fix the mode sequence
to associated \( M \).

#### 2.2 Voronoi tesselation and nearest neighbor classification

One of the simplest yet effective machine learning al-
gerithms is nearest neighbor (NN) classification [Cover
et al. (1967)]. It stores all data in terms of input-output
pairs. For a new data-point the nearest neighbor is
retrieved and its associated output is returned. Let \( D \) be
the set of data points with entries \( x_i \in \mathbb{R}^d \). The runtime
complexity depends on the underlying storage structure:
with tree structures [Omohundro (1989)] the retrieval is
\( O(d \log(|D|)) \) (while building the indexing structure
requires \( O(d|D| \log(|D|)) \) operations). Interestingly, the
explicit MPC approach in Jones et al. (2006) uses Voronoi
diagram as a way to reduce complexity of online execution.
Their online look-up complexity is logarithmic in number
of regions whereas our setting scales logarithmically with
the number of samples.

Fig. 2 (b,c,d) is an intuitive example of NN classification
resulting in the Voronoi diagram approximating the true
region partition. Such approximation is the key motivation
to our method: we directly store sampled points rather

3.1 Approximate hybrid MPC by learning from nearest neighbor mode sequences

Drawing from geometric insights discussed in Sec. 2, we propose a concise approximation approach based on the nearest neighbor (NN) rule. Our main idea is simple: use Voronoi tessellation induced by the NN classifier to approximate feasible regions for the hybrid MPC control law. The data for the NN classifier is given by the mode sequence $\mathcal{M}$ based on the sampled states.

The learning-control algorithm works as follows. We query the NN classifier for the mode sequence $\mathcal{M}$ for a given state $x_n$ based on which feasible region it falls in (a simple NN look-up). We then use $\mathcal{M}$'s corresponding integer solution to warm-start the MIQP (1),(3). If the modes are feasible, the computation is reduced to a quadratic program that can be solved with state-of-the-art solvers in the order of microsecond, e.g. Houska et al. (2011). The implication of this proposition is that, with increasing number of sampling points, we always able to return feasible mode sequences for feasible states. Let $P_n(\text{error})$ denote the probability that the NN classifier in Algorithm 1 predicts an infeasible mode sequence (classification error). Use the NN-convergence bound $P^* \leq \lim_{n \to \infty} P_n(\text{error}) \leq P^* (2 - \frac{c}{\epsilon} P^*)$, where $c$ is the number of classes (c.f. Cover et al. (1967)), and $P_n(\text{error}) \to 0$ as the solver is an oracle. The conclusion follows.

Remark One key feature of NN is that it belongs to the class of “lazy learning” algorithms; it does not require an additional training process such as neural networks do. This important aspect facilitates the fast-learning capability of the learning-controller we shall propose next.

Algorithm 1 LNMS: Online Approximate MPC with Nearest Neighbor Learning

1: Given: sample initial state $x(0) \sim P_0$ where $P_0$ is the initial state distribution. Optional: an initial dataset $\mathcal{D} = D_0$ of sampled points.
2: loop
3: Get current state estimation $x_n$.
4: Query the nearest neighbor classifier (with dataset $\mathcal{D}$) for the mode sequence $\mathcal{M} = \{m_i\}_{i=1}^{N}$ as warm-start solution. Terminate once a feasible solution is found.
5: Solve hybrid OCP (1),(3) with integer variables $\{m_i\}_{i=1}^{N}$ as warm-start solution. Terminate once a feasible solution is found. Add the $(x_n, \mathcal{M}^*)$-pair to the dataset $\mathcal{D}$, where $\mathcal{M}^*$ is the integer solution obtained last step.
6: Apply the first control solution $u^*_0$ to the system.
7: end loop

3.2 Theoretical properties

Let us present the theoretical analysis for Alg. 1. Typical MPC analysis concerns feasibility (whether the OCP has a solution) and stability (whether the closed-loop system can be bounded around a set-point or within a set). For (exact) hybrid MPC, many properties are inherited from nominal MPC, i.e. recursive feasibility (and therefore asymptotic stability) is guaranteed by choosing appropriate terminal cost (Lyapunov function) and terminal constraint (control invariant set). We refer interested readers to Rawlings and Mayne (2009); Borrelli et al. (2017) for detailed theoretical analysis. As Alg. 1 executes the regular hybrid MPC as a warm-start improvement in the case of predicted modes being infeasible, it inherits the feasibility (hence stability) properties from regular hybrid MPC. Obviously, it is of less value if it merely behaves like regular hybrid MPC. We therefore present the following analysis of computational speed-up and an intuitive proof sketch.

In Step. 4 of Alg. 1, a convex program need to be solved if the predicted mode sequence $\mathcal{M}$ is feasible.

Proposition 3.1. Given dataset $\mathcal{D}$ of size $n$ obtained by Alg. 1, let $P_n(\text{error})$ denote the probability of hybrid MPC, whose initial state $x(0) \sim P_0$, having to execute MIP solver in Step 4 of Alg. 1. Then, $P_n^{\text{MIP}} \to 0$ as $n \to \infty$.

Proof sketch: For every feasible state, we consider the MIQP solver to be an oracle. It achieves the minimum possible classification error rate $P^* = 0$ as the solver is always able to return feasible mode sequences for feasible states. Let $P_n(\text{error})$ denote the probability that the NN classifier in Algorithm 1 predicts an infeasible mode sequence (classification error). Use the NN-convergence bound $P^* \leq \lim_{n \to \infty} P_n(\text{error}) \leq P^* (2 - \frac{c}{\epsilon} P^*)$, where $c$ is the number of classes (c.f. Cover et al. (1967)), and $P_n^{\text{MIP}} = P_n(\text{error})$, the conclusion follows. □

3.3 Improving optimality of sampled mode sequences by warm-starting MIP

As an approximate MPC algorithm, LNMS only aims to produce feasible instead of optimal mode sequence prediction. In practice, one may also choose to early
terminate the MIP at a feasible solution due to the fact that most costly computation is to produce a tight dual bound to certify optimality — a good solution may be available much sooner [cf. Bertsimas et al. (2016)]. Both those two sources contribute to suboptimal mode sequences for sampled points.

However, using the instance-based nature of the NN classifier, we can improve on the resulting controller by simply “relabeling” the stored samples via warm-starting techniques of MIP.

Intuitively, this process picks a point from the stored samples and feed its mode sequence as warm-start to the MIP solver, see Alg. 2. As it is already feasible, the solver will always return a mode sequence that has the same or lower cost. Hence we have the intuitive results in Proposition 3.2 (omitting the straightforward proof). We provide examples of this process in Sec. 4 (Fig. 3(d), Fig. 4(c)).

Algorithm 2 LNMS: Solution optimality improvement

1: loop
2: Pick a state-label (x0, M)-pair from dataset D
3: Solve OCP (1) with integer variables M = \{mi\}i=1N as warm-start incumbent solution. Optionally, early terminate after computational budget reached.
4: Relabel this sample and add the new (x0, M*)-pair to D, where M* is the integer solution from step 3.,
5: end loop

Proposition 3.2. If the re-labeling using Algorithm 2 is done to full MIP optimality for all stored data points, then Algorithm 1, with dataset D and initial state x(0) ∼ P0, recovers the exact optimal solution of hybrid MPC as number of samples |D| → ∞.

4. NUMERICAL EXPERIMENTS

Experiment setup We implement our MPC controller in Python with Gurobi as the optimization solver (both MIP and QP). In NN learning, we use a simple weighted Euclidean distance. It is also possible to consider more general similarity measures such as in kernelized-NN. We consider two examples using a cart with one and two walls, and a pendulum with an elastic wall, see Fig. 6(a).

We first detail our setup in the next two sections. Then, we give the discussion of computational reduction, demonstrating the effectiveness of LNMS in Alg. 1.

4.1 Example 1. Cart-wall contact dynamics

The dynamics equation for the cart-wall example in Fig. 1 are described by the following piece-wise affine (PWA) formulation:

\[
\begin{align*}
    x_{1(t+1)} &= x_{1(t)} + x_{2(t)} \Delta t, \\
    x_{2(t+1)} &= x_{2(t)} + \frac{u}{m} \Delta t, & \text{if} \ x \in C_1 \\
    x_{1(t+1)} &= x_{1(t)}, \\
    x_{2(t+1)} &= -\epsilon x_{2(t)}, & \text{if} \ x \in C_2
\end{align*}
\]

where m is the cart mass (set to 1.0) and ε is the coefficient of restitution (set to 0.9). It could be thought of an actuated version of a bouncing ball — a classic hybrid system. \(C_1 = \{x_1 + x_2 \Delta t < x_{\text{wall}}\}\) denotes the state space where the dynamics is the double integrator and \(C_2 = \{x_1 + x_2 \Delta t \geq x_{\text{wall}}\}\) denotes the state space where the contact with the wall happens. In our case \(x_{\text{wall}} = 0.75\) and the discretization is set to \(\Delta t = 0.01\) for all our experiments. The PWA dynamics for the cart with two walls is a straightforward extension.

We synthesize MPC with horizon \(N = 10\) (no early termination of the MIP). The cost weights in Eqn. (1) are \(Q = I_2, R = 0.001\). \(S = \beta \cdot K\) where \(K\) is the solution to the algebraic Riccati equation for the system in mode \(C_1\), we choose \(\beta = 1000\) for faster convergence behavior to the attractor but this is not a crucial choice. In the first experiment, we use terminal cost instead of terminal set following common practice in MPC applications (cf. Rawlings and Mayne (2009)).

In this experiment, we start from initial states \(x(0)\) randomly sampled within the region \([0.1, 0.75] \times [-10, 10]\). For each initial state, the hybrid OCP (1)(3) is solved by LNMS in Alg. 1. As a result of the non-parametric learning Alg. 1, we obtained a set of samples \(D\). Those samples store the solution structure (feasible mode sequences) of LNMS and are the key to speeding up MIP. They are displayed in Fig. 3 (a)-(c) (gray dots) with different sample size.

We visualize the resulting PWA control law in Fig. 3 (a)-(c). Each color patch indicates a region with different affine control law. Notably, Fig. 3 (a)-(c) shows the evolution of the PWA control law as the learning-controller gathers more samples. We refer to our earlier discussion on Fig. 2 for how to interpret the visualization. We plot the resulting closed-loop state trajectories in Fig. 5 (with different initial states). The trajectories obtained by LNMS are identical to the exact hybrid MPC control law.

Scaling up to complex control law structure It is known that constrained control law with numerous mode switches has complex structure. To demonstrate that LNMS scales to such cases, we consider the cart example with two walls,
higher initial velocities, and an MPC prediction horizon of $N = 25$. Furthermore, we impose control constraint ($|u| \leq 10$). The resulting control law is significantly more complex than the previous example due to a large amount of mode sequences and constraints.

A direct consequence is that the MIP is challenging to solve to full optimality. Therefore, early termination is necessary — we use a 5 sec early-termination threshold. The HMPC control law is shown in Fig. 4(a,b). We observe a more complex structure of the control law — 493 regions of different mode sequences correspond to different PWA control laws.

**Improving solution optimality with Alg. 2**  As discussed, Alg. 1 seeks a feasible solution to speed up and warm-start MIP. The resulting solution might be a sub-optimal control law. However, due to the non-parametric nature of our learning algorithm, we can simply post-process the stored sample points to improve optimality of the learning-controller. We apply the optimality-improvement scheme in Alg. 2 to both the one-wall and two-wall cases. The resulting optimal control law are shown in Fig. 3(d) for one-wall and Fig. 4(c) for two-walls. Interestingly, we observe that the control law in Fig. 4(c) is simplified due to merging via comparing optimal cost-to-go of different PWA control law. This is similar to the producedues in merging piecewise partitions in explicit MPC (cf. Borrelli et al. (2017)).

**4.2 Example 2. Elastic wall with variable duration contact**

The elastic pendulum environment, see Fig. 6(a), is adapted from Marcucci et al. (2017). The dynamics is given by

\[
\begin{align*}
\dot{x} &= \begin{cases} A_1 x + B_1 u, & \text{if } (x, u) \in D_1, \\
A_2 x + B_2 u + c_2, & \text{if } (x, u) \in D_2,
\end{cases}
\end{align*}
\]

with

\[
A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1/l^2 \end{bmatrix},
\]

\[
A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1/l^2 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 \\ kd/2m \end{bmatrix},
\]

\[
D_1 = \{ (x, u) | x_1 \leq d/4, x_1 \leq x \leq x_{\text{max}}, u_{\text{min}} \leq u \leq u_{\text{max}} \},
\]

\[
D_2 = \{ (x, u) | x_1 > d/4, x_1 \leq x \leq x_{\text{max}}, u_{\text{min}} \leq u \leq u_{\text{max}} \}
\]

The key difference between the two examples is that, the cart-wall system makes and breaks contact instantaneously while the pendulum with elastic wall allows variable contact duration, thus resulting in different mode sequences. Due to this difference, we use a control invariant set as terminal set in this experiment 2, avoiding the terminal mode “gets stuck in the wall”.

Similar to the previous experiment, we execute Alg.1 to obtain the LNMS learning-controller and collected data samples. They are plotted in Fig. 6. An example closed-loop trajectory in Fig. 7. We can see that, compared with the previous example, the LNMS trajectories differs from the optimal solution obtained by running MIP to full optimality (no time limit). This is caused by the fact that variable-during contact results in suboptimal mode sequences being reused by Alg. 1. Again, this may be improved by Alg. 2 or possibility a shift-started mode sequence in HMPC.
4.3 Computational complexity reduction

We now discuss how LNMS in Alg. 1 can greatly speed up the HMPC solution method. The results here empirically validate the theory in Sec. 3.2. We compare the solution time of LNMS in Alg. 1 with that of the exact MIQP solution of hybrid MPC (1)(3) and report the relative run-time (time of Alg. 1)/(time of exact MIQP)%). This comparison is shown in Fig. 8 for all experiments.

We observe that, as Alg. 1 collects more data samples, only a small fraction of mixed-integer optimization needs to be solved by branch&bound. As there is a big gap in computation time between branch&bound and continuous programs, the proposed scheme drastically improve the computational efficiency. The actual wall clock time may vary greatly depending on the specific learning and MPC code implementation. Nonetheless, we include the a wall clock-time comparison in Appendix A. This experiment demonstrates the answer to our question posed in Section 1, of what can be learned from HMPC runs in the same environment.

5. RELATED WORK

Explicit MPC [Bemporad et al. (2000, 2002); ToNdel et al. (2003)] seeks to offload online computation to offline. Its insight is that 1) the optimal control law of an OCP with quadratic objective linear constraints is piecewise affine state-feedback, i.e. \( u(x) = F^T x + G_i, x \in \mathcal{R}_i \), where \( \mathcal{R}_i \) are convex polyhedra (referred to as critical regions) and 2) there may be exponentially many such regions need to be computed and stored. In the context of hybrid systems, different regions correspond to different (switching) mode sequences, e.g. in-contact or not-in-contact. It is then possible to carry out the mode sequence enumeration offline and store the state-feedback-affine policy. Despite its elegant theory, explicit MPC may incur considerable complexity in 1) both online look-up and offline computation and 2) storage of the optimal partitions. Many approximate MPC algorithms, e.g. Zeilinger et al. (2011); Domahidi et al. (2011), trade off optimality for fast run-time and feasibility. The goal is to avoid exhaustively computing, storing, and comparing regions.

Motivated by this challenge, many approaches have been proposed in the literature to solve such hybrid control problems. For example, Mordatch et al. (2012) suggest the use of soft contact and dynamics models to smooth the dynamics, thus allowing the use of gradient-based numerical optimization. The work of Posa and Tedrake (2013) exploits homotopy methods to compute solutions that gradually converge from relaxed to accurate ones.

Closely related to this work, the authors of Marcucci et al. (2017) use insights from MPC stability proof to simplify explicit MPC from critical regions to inner approximations of feasible regions, resulting significant computational saving. Our approach further simplifies the region computation and storage by the non-parametric learning algorithm, sidestepping polyhedron operations completely.

Very recently, a few works show promises in applying machine learning to hybrid control. Deits et al. (2018) experimented with value function and policy learning in hybrid systems. In this work, we show our method can in fact serve as an efficient oracle for policy training. This work shares similar features with Hogan et al. (2018) in reducing computational cost by learning mode sequences. However, we exploit the non-forgetting property of non-parametric methods to perform online learning, as well as the geometric structure of hybrid MPC solutions to avoid exhaustive offline MIP runs.

There is a thread of works studying the topic of “learning to branch and bound.” [He et al. (2014); Khalil et al. (2016); Rachelson et al. (2010)]. We share the common thread of using function approximation. However, those learning methods are mostly black-box approaches. In contrast, our insight is rooted in the understanding of HMPC solution structure.

6. DISCUSSION

This paper presents LNMS which learns from prior experiences to accelerate MIP for hybrid MPC. Philosophically speaking, LNMS is between the two extremes: the exact explicit MPC on one end, and black-box (e.g., deep neural network) on the other end. Our choice of non-parametric learning algorithms is motivated by the theoretical understanding of (H)MPC solutions. For example, neural network training is parametric and cannot easily deal with wrong points near the hybrid mode boundaries. It also loses any feasibility and stability of (H)MPC. In contrast, LNMS does not need training and preserves stability of the (H)MPC. It achieves speed-up virtually for free by using adaptive samples while is extremely easy to implement.

It is known that if absolute optimality is the goal instead of feasibility which was the aim of this paper, the
boundaries of optimal mode sequence partition may be quadratic rather than affine (cf. Borrelli et al. (2017)). While LNMS still works, an interesting extension is to generalize the notion of (weighted) Euclidean distance, possibly via kernelized nearest neighbor.

One potential application of LNMS is to serve as an efficient supervised learning oracle for training a imitation learning policy \( \pi : x \rightarrow u \), such as in Deits et al. (2018); Hogan et al. (2018). Their computational cost of training may be greatly reduced by adopting LNMS.

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Appendix A. WALL-CLOCK COMPARISON

This section corresponds to the computation time benchmark in Fig. 8. We show the results using a straightforward implementation of LNMS in Python Sklearn library.

| Number of OCPs | LNMS runtime (s) | Regular MIP |
|----------------|-----------------|-------------|
| 10             | 4.07            | 69.02       |
| 100            | 69.28           | 156.25      |
| 500            | 249.53          | 1186.90     |

We generate OCPs (with different initial states). There is a big gap in solving speed between MIP and continuous programs. Our method is significantly faster (∼ 2 – 17X speed-up). Speed-ups are expected values across all runs. Note, that this is done without code optimization on our side while MIP is solved using Gurobi – further putting LNMS in a disadvantage.