BOUNDEDNESS AND PERSISTENCE OF POPULATIONS IN ADVECTIVE LOTKA–VOLTERRA COMPETITION SYSTEM

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ABSTRACT. We are concerned with a two–component reaction–advection–diffusion Lotka–Volterra competition system with constant diffusion rates subject to homogeneous Neumann boundary conditions. We first prove the global existence and uniform boundedness of positive classical solutions to this system. This result complements some of the global existence results in [Y. Lou, M. Winkler, and Y. Tao, SIAM J. Math. Anal., 46 (2014), 1228–1262], where one diffusion rate is taken to be a linear function of the population density. Our second result proves that the total population of each species admits a positive lower bound, under some conditions of system parameters (e.g., when the intraspecific competition rates are large). This result of population persistence indicates that the two competing species coexist over the habitat in a long time.

1. Introduction. This paper considers a reaction–advection–diffusion system of $(u, v) = (u(x,t), v(x,t))$ in the following form

\[
\begin{align*}
  u_t & = \nabla \cdot (D_1 \nabla u + \chi \phi(u) \nabla v) + (a_1 - b_1 u - c_1 v)u, & x \in \Omega, t > 0, \\
  v_t & = D_2 \Delta v + (a_2 - b_2 u - c_2 v)v, & x \in \Omega, t > 0, \\
  \partial_\nu u & = \partial_\nu v = 0, & x \in \partial \Omega, t > 0, \\
  u(x,0) & = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in \Omega,
\end{align*}
\]

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with piecewise smooth boundary $\partial \Omega$. System parameters $D_i$, $a_i$, $b_i$, $c_i$, $i = 1, 2$ and $\chi$ are all positive constants, and $\phi(u)$ is a smooth function. (1.1) was proposed in [20] to study the spatial–temporal behaviors of population distributions of the two competing species $u$ and $v$. It is assumed that $v$ moves over the habitat randomly and $u$ moves through a combination of random and directed dispersal. In particular, from the viewpoint of mathematical modeling, $u$ retreats from the region of $v$ to avoid competition if $\chi \phi(u) > 0$ and $u$ invades the region of $v$ to seek competition if $\chi \phi(u) < 0$. The population kinetics are chosen to be the classical Lotka–Volterra type; $\nu$ is unit outer normal to $\partial \Omega$ and the Neumann boundary condition means that the domain is enclosed hence there is no population flux across the boundary; the initial data $u_0$ and $v_0$ are non–negative functions. See [20] for the derivation and biological

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justifications and significance of (1.1). We also want to draw the attention of the reader to a parallel work [13].

For \( \Omega = (0, L) \), global existence and boundedness of classical solutions to (1.1) with \( \phi(u) = u \) are proved by [20]. For \( \Omega \) being a bounded domain in \( \mathbb{R}^N \), \( N \geq 2 \), global existence and boundedness are also obtained for the parabolic–elliptic counterpart of (1.1), provided that \( \frac{\chi}{\nu} \) is sufficiently large. When the sensitivity function \( \phi \) only depends on \( v \) and satisfies \( \phi(v) \leq C_0(v + 1)^\kappa \) for some \( C_0 > 0 \) and \( \kappa < -1 \), global existence and boundedness of (1.1) are proved in [22]. On the other hand, from the viewpoint of mathematical modeling, (1.1) is biologically significant in that it gives rise to the hope of modeling a spatial segregation of these two species by adding even one single advection to the purely diffusive system. For instance, the authors of [20] also show that in 1D (1.1) admits transition layer steady states which model the segregation phenomenon through interspecific competitions. Through bifurcation theorems and singular perturbation methods, a shadow system of (1.1) is further studied by [24] to model the aforementioned segregation in the limit of large advection and small diffusion rates; moreover, the numerics there indicate that this system admits very interesting and complicated spatial–temporal dynamics, even over 1D domains.

One of the goals of this current work is to prove the global existence and boundedness of the fully parabolic system (1.1) over multi–dimensional domain under the following assumption

\[
\phi(u) \leq K u^m, \forall u > 0, \tag{1.2}
\]

where \( K \) and \( m \) are positive constants, the former being arbitrary and the latter satisfying some technical conditions to be made precise. Our first main result reads as follows.

**Theorem 1.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N, N \geq 2 \). Suppose that \( \phi(u) \) satisfies (1.2) with some \( m < \frac{N+2}{2N} \), then for any nonnegative \( u_0 \in C^0(\Omega) \) and \( v_0 \in C^1(\bar{\Omega}) \), there exists a unique pair \((u, v)\) of nonnegative functions belonging to \( C^0(\Omega \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)) \) which solves (1.1) classically in \( \Omega \times (0, \infty) \). Moreover, the solution is bounded by a positive constant \( C \) in the following sense

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C, \forall t \in (0, \infty). \tag{1.3}
\]

We would like to point out that the condition \( m < \frac{N+2}{2N} \) is merely technical and we have to leave the global well–posedness open of (1.1) in this paper if this condition fails–see [21] for the work on (1.1) with nonlinear diffusion for instance.

Before presenting our second main result, we remind the reader that in the absence of advection, i.e., when \( \chi = 0 \), (1.1) reduces to the following classical diffusive Lotka–Volterra competition system

\[
\begin{align*}
    u_t &= D_1 \Delta u + (a_1 - b_1 u - c_1 v)u, & x \in \Omega, t > 0, \\
    v_t &= D_2 \Delta v + (a_2 - b_2 u - c_2 v)v, & x \in \Omega, t > 0, \\
    \partial_n u &= \partial_n v = 0, & x \in \partial \Omega, t > 0, \\
    u(x, 0) &= u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega,
\end{align*}
\tag{1.4}
\]

the spatial–temporal dynamics of which have been extensively studied over the past few decades. For example, the global existence and boundedness of (1.4) are proved in [3] using maximum principles. Existence or non–existence of stable nonconstant positive steady states is investigated in [11, 12, 15, 16] etc. Extinction through competition is studied in [7, 8, 10, 17]. Generally speaking, species \( u \) and \( v \) can coexist over the habitat and both persist if the interspecific competition is weak.
However, if the interspecific competition is strong, the dynamics depend on the initial data and it is possible that initially superior species will dominate over and eventually wipe out their interspecific competitors. This phenomenon is referred to as the extinction through competition. We want to point out that large time behaviors of (1.4) under Dirichlet boundary conditions have been studied by various authors [2, 4, 5, 6].

For a better presentation of our second goal in this paper, and without losing much generality, we choose $D_1 = D_2 = 1$ and $\phi(u) = u$ in (1.1) and have

$$
\begin{cases}
u_t = \nabla \cdot (\nabla u + \chi u \nabla v) + (a_1 - b_1 u - c_1 v)u, & x \in \Omega, t > 0, \\
v_t = \Delta v + (a_2 - b_2 u - c_2 v)v, & x \in \Omega, t > 0, \\
\partial_\nu u = \partial_\nu v = 0, & x \in \partial \Omega, t > 0, \\
u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega.
\end{cases}
$$

(1.5)

We shall show that under the following assumption

$$
\frac{c_1}{c_2} + \left(\frac{a_2}{c_2}\right)^2/27 < \frac{a_1}{a_2} < \frac{b_1}{b_2},
$$

(1.6)

$u$ and $v$ in (1.5) will coexist over the habitat and both species survive in a long time, hence the phenomenon of extinction through competition does not occur. To be precise, let us denote the spatial mean value of a function by

$$
\bar{f} := |\Omega|^{-1} \int_\Omega f(x)dx,
$$

then the second main result of this paper is the following Theorem.

Theorem 1.2. Suppose that $\Omega$ is a bounded convex domain in $\mathbb{R}^N$, $N \geq 1$ with smooth boundary $\partial \Omega$. Assume that condition (1.6) holds. Let $(u, v)$ be a nonnegative classical solution to (1.5) in $\Omega \times (0, \infty)$ with the initial data $(u_0, v_0) \in (C^0(\Omega))^2$ satisfying

$$
\bar{u}_0 < \frac{a_2}{b_2}, \quad \bar{v}_0 < \frac{a_1}{c_1} - \frac{a_2}{27c_1} \left(\frac{a_2}{c_2}\right)^2,
$$

(1.7)

and for any two finite numbers $L_1$ and $L_2$

$$
\int_\Omega \ln u_0(x)dx \geq -L_1, \int_\Omega \ln v_0(x)dx \geq -L_2,
$$

(1.8)

then there exist two positive constants $m_1$ and $m_2$ such that

$$
\int_\Omega u(x, t)dx \geq m_1 > 0, \text{ for all } t > 0
$$

(1.9)

and

$$
\int_\Omega v(x, t)dx \geq m_2 > 0, \text{ for all } t > 0.
$$

(1.10)

Remark 1. In Theorem 1.2, we assume that the solution $(u, v)$ to (1.5) is global in $\Omega \times (0, \infty)$. This is true when $N = 1$, or $N = 2$ with $b_1$ being sufficiently large according to [20].

Remark 2. It is interesting to understand if the condition (1.7) is optimal. Though the latter inequality is technically assumed, it seems that $\frac{a_1}{a_2} < \frac{b_1}{b_2}$ is necessary. For example, if $\frac{a_1}{a_2} > \frac{b_1}{b_2}$, we can easily show that the semi–steady–state $(\bar{u}_0, 0)$ is locally stable. Therefore, if we choose $(u_0, v_0)$ to be around this semi–steady–state, $(u, v) \to (\bar{u}_0, 0)$ as $t \to \infty$ and Theorem 1.2 does not hold any more. Condition (1.8) holds when both $u_0$ and $v_0$ are strictly positive.
Ecologically, (1.9) and (1.10) mean that the competing species $u$ and $v$ exist over the habitat and the total population of both species persist in a long time. We observe that (1.6) holds for large $\chi$ if the intraspecific competition rates $b_1$ and $c_2$ are large, with the rest parameters being fixed. This indicates that weak directed dispersal intensity supports the coexistence of interspecific competing species in model (1.5).

2. Local existence and preliminary results. To establish the global existence and boundedness of classical solutions to (1.1), we start with the following results on existence and extension properties of its local solutions due to the classical theories of Amann [1].

**Proposition 1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial \Omega$. Then for any initial data satisfying $u_0 > 0$, $v_0 \geq 0$ in $\Omega$ and $(u_0, v_0) \in C^2(\Omega) \times W^{1,p}(\Omega)$ for some $p > N$, there exists $T_{max} \in (0, \infty]$ such that system (1.1) has a unique solution $(u(x, t), v(x, t))$ in $\Omega \times [0, T_{max})$ which is classical and satisfies $(u, v) \in C^0(\Omega \times [0, T_{max})) \times C^2,1(\Omega \times [0, T_{max}))$. $T_{max}$ is called the maximal existence time and if $\sup_{s \in (0, t)} \|u(\cdot, s)\|_{L^\infty}$ is bounded for $t \in (0, T_{max})$, then $T_{max} = \infty$ and $(u, v)$ is a global solution to (1.1); if $T_{max} < \infty$ then

$$\lim_{t \to T_{max}} \sup_{\Omega} \|u(\cdot, t)\|_{L^\infty} = \infty.$$ 

**Proof.** It is easy to see that (1.1) is a triangular system. Therefore the existence of maximal classical solutions follows from Theorem 14.4 and Theorem 14.6 in [1]. Moreover, the extensibility property follows from Theorem 15.5 in [1]. □

Next we collect some basic properties of solutions to (1.1).

**Lemma 2.1.** Assume that $(u_0, v_0)$ satisfy the conditions in Proposition 1. Suppose that $(u, v)$ is a classical solution of (1.1) in $\Omega \times (0, T_{max})$ for some $T_{max} > 0$, then for any $T \in (0, T_{max})$ and all $(x, t) \in \Omega \times (0, T)$, $u(x, t) \geq 0$ and

$$0 \leq v(x, t) \leq K_0 := \max \left\{ \|v_0\|_{L^\infty}, \frac{a_2}{c_2} \right\}, \quad (2.1)$$

moreover, the following inequalities hold

$$\int_{\Omega} u(x, t) dx \leq M_0 := \max \left\{ \int_{\Omega} u_0, \frac{a_1 |\Omega|}{b_1} \right\}, \forall t \in (0, T), \quad (2.2)$$

$$\int_{\Omega} v(x, t) dx \leq M_1 := \max \left\{ \int_{\Omega} v_0, \frac{a_2 |\Omega|}{c_2} \right\}, \forall t \in (0, T) \quad (2.3)$$

and

$$\int_{t_0}^{t_0 + T} \int_{\Omega} u^2(x, t) dx dt \leq \frac{M_0(a_1 T + 1)}{b_1}, \int_{t_0}^{t_0 + T} \int_{\Omega} v^2(x, t) dx dt \leq \frac{M_1(a_2 T + 1)}{c_2} \quad (2.4)$$

if $0 < t_0, t_0 + T < T_{max}$.

**Proof.** First of all, the nonnegativity of $u$ and (2.1) follow upon direct applications of parabolic maximum principle. From the $u$–equation, we have

$$\frac{d}{dt} \int_{\Omega} u dx \leq a_1 \int_{\Omega} udx - b_1 \int_{\Omega} u^2 dx \leq a_1 \int_{\Omega} udx - \frac{b_1}{|\Omega|} \left( \int_{\Omega} u dx \right)^2, \quad (2.5)$$

thanks to the inequality $(\int_{\Omega} u dx)^2 \leq |\Omega| \int_{\Omega} u^2 dx$. Then by an ODE comparison argument we have (2.2). Similarly we can show (2.3).
From the first inequality in (2.5) we also get
\[
\int_{\Omega} u^2 \, dx \leq -\frac{1}{b_1} \frac{d}{dt} \int_{\Omega} u \, dx + \frac{a_1}{b_1} \int_{\Omega} u \, dx,
\] (2.6)
and by integrating (2.6) over \((t_0, t_0 + T)\) we obtain
\[
\int_{t_0}^{t_0 + T} \int_{\Omega} u^2(x, t) \, dx \, dt \leq -\frac{1}{b_1} \int_{t_0}^{t_0 + T} \int_{\Omega} u(x, t_0 + T) \, dx + \frac{1}{b_1} \int_{t_0}^{t_0 + T} \int_{\Omega} u(x, t_0) \, dx
\]
\[
+ \frac{a_1}{b_1} \int_{t_0}^{t_0 + T} \int_{\Omega} u(x, t) \, dx \, dt \leq \left(\frac{a_1 T}{b_1} + 1\right) M_0,
\]
Similarly, we can show that the inequality holds for \(v\) and (2.4) is verified. \(\square\)

**Remark 3.** (2.1)–(2.4) also hold for (1.5).

According to Proposition 1 and (2.1), in order to prove Theorem 1.1, we only need to prove the \(L^\infty\)-boundedness of \(u\). In light of the well-known results on global existence of reaction–advection–diffusion system (e.g. Lemma A.1 in [18]), it is sufficient to prove the boundedness of \(\|\nabla v\|_{L^\infty}\). To this end we first convert \(v\)-equation into the following abstract form
\[
v(\cdot, t) = e^{D_2(\Delta-1)t} v_0 + \int_0^t e^{D_2(\Delta-1)(t-s)} \left(D_2 v(\cdot, s) + g(u(\cdot, s), v(\cdot, s))\right) \, ds,
\] (2.7)
where \(g(u, v) = (a_2 - b_2 u - c_2 v)v\), then we can employ the smoothing properties of the operator \(-\Delta + 1\) to prove the following Lemma for future reference. Here we include its proof for the completeness.

**Lemma 2.2.** Let \((u, v)\) be the classical solution of (1.1) in \(\Omega \times (0, T_{\max})\). For \(1 \leq p, q \leq \infty\), there exists a positive constant \(C\) dependent on \(\|v_0\|_{L^q(\Omega)}\) and \([\Omega]\) such that
\[
\|v(\cdot, t)\|_{W^{1,q}} \leq C \left(1 + \max_{s \in (0,t)} \|u(\cdot, s)\|_{L^p}\right), \forall t \geq 0,
\] (2.8)
where \(q \in \left(1, \frac{Np}{N-p}\right)\) if \(p \in [1, N)\), \(q \in [1, \infty)\) if \(p = N\) and \(q = \infty\) if \(p > N\).

**Proof.** After applying the \(L^p-L^q\) estimates between semigroups \(\{e^{t\Delta}\}_{t \geq 0}\) in Lemma 1.3 of [23] on (2.7), thanks to (2.1), we can find positive constants \(C_1, C_2\) and \(C_3\) such that
\[
\|v(\cdot, t)\|_{W^{1,q}}
\]
\[
= \left\| e^{D_2(\Delta-1)t} v_0 + \int_0^t e^{D_2(\Delta-1)(t-s)} \left(D_2 v(\cdot, s) + g(u(\cdot, s), v(\cdot, s))\right) \, ds \right\|_{W^{1,q}}
\]
\[
\leq C_1 \|v_0\|_{L^p} + C_1 \int_0^t e^{-D_2\nu(t-s)} (1 + (t-s)^{-\frac{1}{2}-\frac{N}{2}q}(\frac{N}{2}q-\frac{1}{2}))(\|u(\cdot, t)\|_{L^p} + 1) \, ds
\]
\[
\leq C_2 + C_3 \int_0^t e^{-D_2\nu(t-s)} (1 + (t-s)^{-\frac{1}{2}-\frac{N}{2}q}(\frac{N}{2}q-\frac{1}{2})) \|u(\cdot, s)\|_{L^p} \, ds
\]
\[
\leq C_2 + C_3 \int_0^t e^{-D_2\nu(t-s)} (1 + (t-s)^{-\frac{1}{2}-\frac{N}{2}q}(\frac{N}{2}q-\frac{1}{2})) \, ds \sup_{s \in (0,t)} \|u(\cdot, s)\|_{L^p},
\] (2.9)
where $\nu$ is the first Neumann eigenvalue of $-\Delta$. On the other hand, we see that under the conditions in Lemma 2.2

$$\sup_{t \in (0, \infty)} \int_0^t e^{-\nu(t-s)} (t-s)^{-\frac{1}{2} - \frac{\nu}{2} (\frac{1}{2} - \frac{\nu}{4})} ds < \infty,$$

therefore (2.8) follows from (2.9).

In the sequel, for better organizations of the manuscript and consistency of notations, we shall, in all space integrations, skip the differential $dx$ without confusing the reader. The following results are immediate conclusions from (2.2) and Lemma 2.2.

**Corollary 1.** Let $(u, v)$ be a solution of (1.1), then for each $s \in [1, \frac{N}{N-4})$, there exists a positive constant $C(s)$ such that

$$\|v(\cdot, t)\|_{W^{1,2}(\Omega)} \leq C(s), \forall t \in (0, T_{\max}). \quad (2.10)$$

For any $N \geq 2$ we readily see that $\frac{N}{N-4} \leq 2$. Indeed the following lemma states that $s = 2$ can be achieved in (2.10).

**Lemma 2.3.** There exists a positive constant $C$ such that

$$\|\nabla v(\cdot, t)\|_{L^2(\Omega)} \leq C, \forall t \in (0, T_{\max}). \quad (2.11)$$

**Proof.** Using $v$-equation, we have from the integration by parts that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 = \int_{\Omega} \nabla v \cdot \nabla v_t$$

$$= \int_{\Omega} \nabla v \cdot \nabla [D_2 \Delta v + (a_2 - b_2u - c_2v)v]$$

$$= -D_2 \int_{\Omega} |\Delta v|^2 + a_2 \int_{\Omega} |\nabla v|^2 + \int_{\Omega} b_2uv \Delta v - 2c_2 \int_{\Omega} v|\nabla v|^2$$

$$\leq -D_2 \int_{\Omega} |\Delta v|^2 + a_2 \int_{\Omega} |\nabla v|^2 + \frac{D_2}{2} \int_{\Omega} |\Delta v|^2 + \mu \int_{\Omega} u^2$$

$$= -\frac{D_2}{2} \int_{\Omega} |\Delta v|^2 + a_2 \int_{\Omega} |\nabla v|^2 + \mu \int_{\Omega} u^2, \quad (2.12)$$

where we denote $\mu := \frac{b_2^2}{4\mu} \frac{|\nabla v|_{L^\infty(\Omega)}^2}{2D_2}$. Thanks to the Sobolev interpolation inequality and boundedness of $\|v\|_{L^\infty}$, there exist positive constants $C_4$ and $C_5$ such that

$$\left( a_2 + \frac{1}{2} \right) \int_{\Omega} |\nabla v|^2 \leq \frac{D_2}{2} \int_{\Omega} |\Delta v|^2 + C_4 \int_{\Omega} u^2 \leq \frac{D_2}{2} \int_{\Omega} |\Delta v|^2 + C_5$$

through which (2.12) entails that

$$\frac{d}{dt} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\nabla v|^2 \leq 2\mu \int_{\Omega} u^2 + C_5, \forall t \in (0, T_{\max}). \quad (2.13)$$

Multiplying (2.13) by $\frac{b_1}{4\mu}$ and adding it to the first inequality in (2.5), we can get that for all $t \in (0, T_{\max})$

$$\frac{b_1}{4\mu} \left( \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\nabla v|^2 \right) + \left( \frac{d}{dt} \int_{\Omega} u + \int_{\Omega} u \right)$$

$$\leq (a_1 + 1) \int_{\Omega} u - \frac{b_1}{2} \int_{\Omega} u^2 + \frac{b_1}{4\mu} C_5 \leq C_6,$$

then we can conclude from Grönwall’s lemma that $\|\nabla v(\cdot, t)\|_{L^2(\Omega)}$ is bounded. \qed
3. Existence and boundedness of global solutions. This section is devoted to the proof of Theorem 1.1. The main vehicle of our approach is to prove the boundedness of $\|\nabla v\|_{L^\infty}$ which, thanks to Lemma 2.2, reduces to prove that $\|u\|_{L^p}$ is bounded for some $p > N$.

3.1. A priori estimates and global existence. For any $p > 1$, we test the $u$–equation in (1.1) to obtain that

$$\frac{1}{p} \frac{d}{dt} \int_\Omega u^p = D_1 \int_\Omega u^{p-1} \Delta u + \int_\Omega u^{p-1} \nabla \cdot (\chi \phi(u) \nabla v) + \int_\Omega u^p (a_1 - b_1 u - c_1 v)$$

$$= - D_1 (p - 1) \int_\Omega u^{p-2} |\nabla u|^2 - (p - 1) \int_\Omega \chi \phi(u) u^{p-2} \nabla u \cdot \nabla v$$

$$+ \int_\Omega u^p (a_1 - b_1 u - c_1 v), \quad (3.1)$$

which, in light of (1.2) and the following fact due to Cauchy–Schwartz

$$- (p - 1) \int_\Omega \chi \phi(u) u^{p-2} \nabla u \cdot \nabla v$$

$$\leq \frac{D_1 (p - 1)}{2} \int_\Omega u^{p-2} |\nabla u|^2 + \frac{(p - 1) \chi^2}{2D_1} \int_\Omega \phi^2 u^{p-2} |\nabla v|^2$$

$$\leq \frac{D_1 (p - 1)}{2} \int_\Omega u^{p-2} |\nabla u|^2 + \frac{(p - 1) \chi^2 K^2}{2D_1} \int_\Omega u^{p+2m-2} |\nabla v|^2,$$

gives rise to

$$\frac{1}{p} \frac{d}{dt} \int_\Omega u^p + \frac{D_1 (p - 1)}{2p^2} \int_\Omega |\nabla u|^2 - \frac{b_1}{2} \int_\Omega u^{p+1}$$

$$\leq \frac{(p - 1) \chi^2 K^2}{2D_1} \int_\Omega u^{p+2m-2} |\nabla v|^2 + C_7,$$  \quad (3.2)

where $C_7$ is positive constant.

Lemma 3.1. Let $(u, v)$ be the solution of (1.1), then for any $p > 1$ there exists a positive constant $C(p)$ such that

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C(p), \forall t \in (0, T_{\max}). \quad (3.3)$$

Assuming that Lemma 3.1 holds, we present the proof of our main result.

Proof of Theorem 1.1. By taking $p$ large but fixed in (3.3), we have from Lemma 2.2 that $\|\nabla v(\cdot, t)\|_{L^\infty} < C$ for all $t \in (0, T_{\max})$ and this, together with standard Moser–Alikakos $L^p$ iteration (cf. Lemma A1 in [18]), gives rise to the boundedness of $\|u(\cdot, t)\|_{L^\infty} < C$ for all $t \in (0, T_{\max})$. Therefore we must have that $T_{\max} = \infty$ and the local solution $(u(x, t), v(x, t))$ is global. Finally, one can apply the classical parabolic theory to prove that the classical solution satisfies the regularity properties. This completes the proof of Theorem 1.1.

3.2. Proof of Lemma 3.1. We are left to prove Lemma 3.1 to conclude this section. In order to estimate $\int_\Omega u^p$ through (3.2), we shall work on the combined integrals $\int_\Omega u^p + \int_\Omega |\nabla v|^2$ as in [18].
Proof of Lemma 3.1. For any $q > 1$, we have from the $v$–equation in (1.1) and integration by parts that
\[
\frac{1}{2q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} = \int_{\Omega} |\nabla v|^{2q-2} \nabla v \cdot \nabla v_t \tag{3.4}
\]
\[
= \int_{\Omega} |\nabla v|^{2q-2} \nabla v \cdot \nabla (D_2 \Delta v) + \int_{\Omega} |\nabla v|^{2q-2} \nabla v \cdot \nabla [(a_2 - b_2 u - c_2 v)].
\]
In view of the pointwise identity
\[
\nabla v \cdot \nabla \Delta v = \frac{1}{2} \Delta |\nabla v|^2 - |D^2 v|^2,
\]
upon an integration by parts, the first term in the second line of (3.4) becomes
\[
\int_{\Omega} |\nabla v|^{2q-2} \nabla v \cdot \nabla (D_2 \Delta v) = \frac{D_2}{2} \int_{\Omega} |\nabla v|^{2q-2} \Delta |\nabla v|^2 - D_2 \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2
\]
\[
= \frac{D_2}{2} \int_{\partial \Omega} |\nabla v|^{2q-2} \frac{\partial |\nabla v|^2}{\partial n} - \frac{D_2}{2} \int_{\partial \Omega} \nabla |\nabla v|^{2q-2} \cdot \nabla |\nabla v|^2 - D_2 \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2. \tag{3.5}
\]
To estimate the boundary integral in (3.5), we have from Lemma 2.2–2.4 in [9] and fractional Gagliardo–Nirenberg inequality that for some positive constants $C_\Omega$ that vary from line to line below
\[
\int_{\partial \Omega} |\nabla v|^{2q-2} \frac{\partial |\nabla v|^2}{\partial n} \leq C_\Omega \|
abla v\|^q_{L^2(\partial \Omega)} \|
abla v\|^q_{L^r(\Omega)} \leq C_\Omega \|
abla v\|^q_{W^{1, \frac{r}{r-1}}(\Omega)} \leq \left( C_\Omega \|
abla |\nabla v|^q\|_{L^2(\Omega)} \right)^2 + C_\Omega \|
abla |\nabla v|^q\|_{L^2(\Omega)}^2 + C_8,
\]
where $r \in (0, \frac{5}{2})$ and $h_1 := \frac{q}{2} - (\frac{5}{2} + r) \frac{1}{2} \in (0, 1)$, and we have used the fact (2.11) in Lemma 2.3. By Young’s inequality, for any $\epsilon > 0$ we have that
\[
\int_{\partial \Omega} |\nabla v|^{2q-2} \frac{\partial |\nabla v|^2}{\partial n} \leq \epsilon \int_{\Omega} |\nabla |\nabla v|^q| + C_\epsilon.
\]
On the other hand, since
\[
\frac{1}{2} \nabla |\nabla v|^{2q-2} \cdot \nabla |\nabla v|^2 = \frac{q-1}{2} |\nabla v|^{2q-4} |\nabla v|^2 |\nabla |\nabla v|^2|^2,
\]
we have in (3.5) that
\[
\int_{\Omega} |\nabla v|^{2q-2} \nabla v \cdot \nabla (D_2 \Delta v) \leq \frac{3D_2(q-1)}{8} \int_{\Omega} |\nabla v|^{2q-4} |\nabla |\nabla v|^2|^2
\]
\[
- D_2 \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 + C_9. \tag{3.6}
\]
For the second term in the second line of (3.4), we have from the integration by parts that
\[ \int_\Omega |\nabla v|^{2q-2} \nabla \cdot \nabla [(a_2 - b_2 u - c_2 v) v] \]
\[ = - \int_\Omega (a_2 - b_2 u - c_2 v) v \nabla \cdot (|\nabla v|^{2q-2} \nabla v) \]
\[ = - \int_\Omega (a_2 - b_2 u - c_2 v) v |\nabla v|^{2q-2} \nabla v \]
\[ - (q - 1) \int_\Omega (a_2 - c_2 v) v |\nabla v|^{2q-4} \nabla |\nabla v|^2 \cdot \nabla v \]
\[ = - \int_\Omega (a_2 - c_2 v) v |\nabla v|^{2q-2} \Delta v - (q - 1) \int_\Omega (a_2 - c_2 v) v |\nabla v|^{2q-4} |\nabla v|^2 \cdot \nabla v \]
\[ + b_2 \int_\Omega \nabla v |\nabla v|^{2q-2} \Delta v + b_2 (q - 1) \int_\Omega \nabla v |\nabla v|^{2q-4} |\nabla v|^2 \cdot \nabla v. \]  
(3.7)

To estimate the last four terms in (3.7), we first apply the fact \(|\Delta v|^2 \leq N|D^2 v|^2\) and Cauchy–Schwarz inequality to show that
\[ - \int_\Omega (a_2 - c_2 v) v |\nabla v|^{2q-2} \Delta v \]
\[ \leq \frac{D_2}{2N} \int_\Omega |\nabla v|^{2q-2} \Delta v|^2 + \frac{N}{2D_2} \int_\Omega (a_2 - c_2 v)^2 v^2 |\nabla v|^{2q-2} \]
\[ \leq \frac{D_2}{2} \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 + C_{10} \int_\Omega |\nabla v|^{2q-2} \]  
(3.8)

where \(C_{10} = \frac{N}{D_2} \|(a_2 - c_2 v)^2 v^2\|_{L^\infty}\). Similarly for the rest terms we can estimate
\[ -(q - 1) \int_\Omega (a_2 - c_2 v) v |\nabla v|^{2q-4} |\nabla v|^2 \cdot \nabla v \leq \frac{D_2(q - 1)}{16} \int_\Omega |\nabla v|^{2q-4} |\nabla v|^2 \]
\[ + C_{11} \int_\Omega |\nabla v|^{2q-2} \]  
(3.9)

with \(C_{11} = \frac{4(q - 1)}{D_2} \|(a_2 - c_2 v)^2 v^2\|_{L^\infty}\),
\[ b_2 \int_\Omega \nabla v |\nabla v|^{2q-2} \Delta v \leq \frac{D_2}{2} \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 + C_{12} \int_\Omega u^2 |\nabla v|^{2q-2} \]
(3.10)

with \(C_{12} = \frac{4D_2 N}{D_2} \|u^2\|_{L^\infty}\) and
\[ b_2(q - 1) \int_\Omega \nabla v |\nabla v|^{2q-4} |\nabla v|^2 \cdot \nabla v \leq \frac{D_2(q - 1)}{16} \int_\Omega |\nabla v|^{2q-4} |\nabla v|^2 \]
\[ + C_{13} \int_\Omega u^2 |\nabla v|^{2q-2} \]  
(3.11)

with \(C_{13} = \frac{4D_2(q - 1)}{D_2} \|u^2\|_{L^\infty}\). By substituting (3.8)–(3.11) into (3.7) and using (3.6), we infer from (3.4) that
\[ \frac{1}{2q} \frac{d}{dt} \int_\Omega |\nabla v|^{2q} \leq - \frac{D_2(q - 1)}{4} \int_\Omega |\nabla v|^{2q-4} |\nabla |\nabla v|^2|^2 \]
\[ + (C_{12} + C_{13}) \int_\Omega u^2 |\nabla v|^{2q-2} + C_9. \]  
(3.12)
Here we note that
\[ |\nabla v|^{2q - 4} |\nabla |\nabla v|^q|^2 = \frac{4}{q^2} |\nabla |\nabla v|^q|^2 \]
and for some \( C_{14} > 0 \)
\[ (C_{10} + C_{11}) \int_\Omega |\nabla v|^{2q - 2} \leq \left( 1 - \frac{1}{2q} \right) \int_\Omega |\nabla v|^{2q} + C_{14}. \]
From these facts, (3.12) entails that
\[ \frac{1}{2q} \frac{d}{dt} \int_\Omega |\nabla v|^{2q} + \left( \frac{1}{2q} - 1 \right) \int_\Omega |\nabla v|^{2q} + \frac{D_2(q - 1)}{q^2} \int_\Omega |\nabla |\nabla v|^q|^2 \leq (C_{12} + C_{13}) \int_\Omega u^2 |\nabla v|^{2q - 2} + C_9 + C_{14}. \]

For any fixed \( q > 1 \), we invoke Gagliardo–Nirenberg–Sobolev inequality to estimate that with \( b_2 := \frac{2 - \frac{2}{q} - \frac{q}{2}}{\frac{3}{2} - \frac{q}{2}} \in (0, 1) \)
\[ \int_\Omega |\nabla v|^{2q} = \left\| \nabla v \right\|_{L^2(\Omega)}^{2q} \leq C_{15} \left\| \nabla \nabla v \right\|_{L^2(\Omega)}^{2b_2} \cdot \left\| \nabla v \right\|_{L^q(\Omega)}^{2(1 - b_2)} + C_{16} \left\| \nabla v \right\|_{L^q(\Omega)}^{2} \leq C_{17} \left\| \nabla \nabla v \right\|_{L^2(\Omega)}^{2b_2} + C_{18} \leq \epsilon \int_\Omega |\nabla v|^{2q} + C_{19}, \] (3.14)
where \( \epsilon > 0 \) is any positive constant and the last identity follows from the fact \( \left\| \nabla v \right\|_{L^q(\Omega)}^{2} = \left\| \nabla v \right\|_{L^2(\Omega)}^{2q} \) and Lemma 2.3. Therefore (3.13) implies that
\[ \frac{1}{2q} \frac{d}{dt} \int_\Omega |\nabla v|^{2q} + \frac{1}{2q} \int_\Omega |\nabla v|^{2q} + \frac{D_2(q - 1)}{2q^2} \int_\Omega |\nabla |\nabla v|^q|^2 \leq C_{20} \int_\Omega u^2 |\nabla v|^{2q - 2} + C_{21}, \]
with \( C_{20} = C_{12} + C_{13} \) and \( C_{21} = C_9 + C_{14} + C_{19} \). Combining this inequality with (3.2) yields
\[ \frac{d}{dt} \left( \frac{1}{p} \int_\Omega u^p + \frac{1}{2q} \int_\Omega |\nabla v|^{2q} \right) + \frac{D_1(p - 1)}{2p^2} \int_\Omega |\nabla u^\frac{p}{2}|^2 \]
\[ + \frac{D_2(q - 1)}{2q^2} \int_\Omega |\nabla |\nabla v|^q|^2 \leq C_{22} \int_\Omega u^{p^m - 2} |\nabla v|^2 + C_{23} \int_\Omega u^2 |\nabla v|^{2q - 2} + C_{24}. \] (3.15)
To further estimate (3.15), we use the Young’s inequality to obtain
\[ C_{22} \int_\Omega u^{p^m - 2} |\nabla v|^2 \leq \frac{b_1}{8} \int_\Omega (u^{p^m - 2})^{\frac{p^m - 1}{p^m - 2}} + C_{25} \int_\Omega |\nabla v|^2 \]
\[ = \frac{b_1}{8} \int_\Omega u^{p^m - 2} + C_{25} \int_\Omega |\nabla v|^2 \] (3.16)
and
\[ C_{23} \int_{\Omega} u^2 |\nabla v|^{2q-2} \leq \frac{b_1}{8} \int_{\Omega} (u^2)^{\frac{q+1}{2}} + C_{26} \int_{\Omega} |\nabla v|^{2(q-1)\frac{q+1}{p+1}} = \frac{b_1}{8} \int_{\Omega} u^{p+1} + C_{26} \int_{\Omega} |\nabla v|^{2(q-1)(p+1)\frac{q+1}{p+1}}. \]  
(3.17)

Therefore, (3.15) implies
\[ \frac{d}{dt} \left( \frac{1}{p} \int_{\Omega} u^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} \right) + \frac{D_1 (p-1)}{2p^2} \int_{\Omega} |\nabla u^p|^{2q} + \frac{D_2 (q-1)}{2q^2} \int_{\Omega} |\nabla |\nabla v|^q|^{2q} \leq C_{25} \int_{\Omega} |\nabla v|^{\kappa_1} + C_{26} \int_{\Omega} |\nabla v|^{\kappa_2} + C_{27}, \]
where we put
\[ \kappa_1 := \frac{2(p+1)}{3-2m}, \quad \kappa_2 := \frac{2(q-1)(p+1)}{p-1}. \]  
(3.19)

For each \( \kappa_i, i = 1, 2 \), we invoke Gagliardo–Nirenberg–Sobolev inequality to estimate
\[ \int_{\Omega} |\nabla v|^{\kappa_i} = \| |\nabla v|^q \big\|^\kappa_i_{L^\frac{q}{\kappa_i}(\Omega)} \]
\[ \leq C_{28}(q) \| |\nabla |\nabla v|^q|^{\frac{q\kappa_i}{q\theta_i}} \cdot \| |\nabla v|^q \big\|^\frac{\kappa_i}{q\theta_i}(L^\frac{q}{\kappa_i}(\Omega)) + \| |\nabla v|^{\kappa_i} \big\|^\kappa_i_{L^2(\Omega)} \]
\[ = C_{28}(q) \| |\nabla |\nabla v|^q|^{\frac{q\kappa_i}{q\theta_i}} \cdot \| |\nabla v|^{\kappa_i(1-\theta_i)} \big\|^\frac{\kappa_i}{q(1-\theta_i)}_{L^2(\Omega)} + \| |\nabla v|^{\kappa_i} \big\|^\kappa_i_{L^2(\Omega)} \]
\[ \leq C_{29} \| |\nabla |\nabla v|^q|^{\frac{q\kappa_i}{q\theta_i}} \|_{L^2(\Omega)} + C_{30}, \]  
(3.20)

where
\[ \theta_i := \frac{\frac{q}{2} - \frac{q}{\kappa_i}}{\frac{q}{2} - \left(1 - \frac{1}{N}\right)} = \frac{\frac{1}{2} - \frac{1}{\kappa_i}}{\frac{1}{2} - \frac{1}{q} (1 - \frac{1}{N})}, \quad i = 1, 2. \]

Since \( m < \frac{N+2}{2N} \), by straightforward calculations we can choose \( p \) and \( q \) to be sufficiently large satisfying
\[ \frac{p+1}{3-2m} - \frac{1}{N-1} < q < \frac{N(p+1)}{2(N-1)} - \frac{1}{N-1}. \]  
(3.21)

such that \( \theta_i \in (0, 1) \) and \( \frac{q}{\theta_i} \theta_i \in (0, 2) \) for each \( i = 1, 2 \). Therefore we can obtain from Cauchy–Schwarz that there exists some \( C_{31} > 0 \) such that
\[ C_{25(26)} \int_{\Omega} |\nabla v|^{\kappa_1(\kappa_2)} \leq \frac{D_2 (q-1)}{4q^2} \| |\nabla |\nabla v|^q|^{2} \|_{L^2(\Omega)} + C_{31}. \]  
(3.22)

Since \( \int_{\Omega} u^p \leq \frac{b_4}{4} \int_{\Omega} u^{p+1} + C_{32} \), we put
\[ y(t) := \frac{1}{p} \int_{\Omega} u^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} \]
and conclude from (3.18) that \( y'(t) + y(t) \leq C_{33} \). Solving this ODI using Grönwall’s lemma gives rise to (3.3). \( \square \)
4. **Population persistence of both species.** We proceed to prove Theorem 1.2 which states that both species persist for all the time and extinction through competition does not occur in (1.5) under conditions (1.6)–(1.8). Here one does not require the advection coefficient \( \chi \) to be positive or negative, which models the repulsion and attraction between the competing species respectively. Moreover (1.7) holds for large \( \chi \) if both \( b_2 \) and \( c_1 \) are sufficiently large.

Our proof of Theorem 1.2 is motivated by and follows [19], by combining several preliminary results. We start with the following lemma.

**Lemma 4.1.** Suppose that \( \Omega \) is a bounded domain in \( \mathbb{R}^N, N \geq 1 \) and let \((u, v)\) be a positive classical solution of (1.5) in \( \Omega \times (0, \infty) \). Assume that condition (1.6) holds, then there exists a positive constant \( A_1 \) such that for all \( t \geq 0 \)

\[
\frac{d}{dt} \left\{ \int_{\Omega} \ln u - \frac{\chi^2}{8} \int_{\Omega} v^2 \right\} \geq -b_1 \int_{\Omega} u + A_1. \tag{4.1}
\]

**Proof.** We test the \( u \)-equation in (1.5) by \( \frac{1}{u} \) and have from the integration by parts that

\[
\frac{d}{dt} \int_{\Omega} \ln u = \frac{|\nabla u|^2}{u^2} + \chi \int_{\Omega} \frac{\nabla u \cdot \nabla v}{u} + a_1|\Omega| - b_1 \int_{\Omega} u - c_1 \int_{\Omega} v. \tag{4.2}
\]

Applying Cauchy–Schwarz inequality gives us

\[
\int_{\Omega} \frac{|\nabla u|^2}{u^2} + \frac{\chi^2}{4} \int_{\Omega} |\nabla v|^2 \geq -\chi \int_{\Omega} \frac{\nabla u \cdot \nabla v}{u},
\]

therefore (4.2) entails

\[
\frac{d}{dt} \int_{\Omega} \ln u \geq -\frac{\chi^2}{4} \int_{\Omega} |\nabla v|^2 + a_1|\Omega| - b_1 \int_{\Omega} u - c_1 \int_{\Omega} v. \tag{4.3}
\]

By straightforward calculations involving \( v \)-equation, we have

\[
-\int_{\Omega} |\nabla v|^2 = \frac{1}{2} \frac{d}{dt} \int_{\Omega} v^2 - a_2 \int_{\Omega} v^2 + b_2 \int_{\Omega} uv^2 + c_2 \int_{\Omega} v^3 \\
\geq \frac{1}{2} \frac{d}{dt} \int_{\Omega} v^2 - a_2 \int_{\Omega} v^2 + c_2 \int_{\Omega} v^3. \tag{4.4}
\]

Multiplying (4.4) by \( \frac{\chi^2}{4} \) and then adding it to (4.3), we arrive at the following

\[
\frac{d}{dt} \left\{ \int_{\Omega} \ln u - \frac{\chi^2}{8} \int_{\Omega} v^2 \right\} \geq -b_1 \int_{\Omega} u + A_1,
\]

with

\[
A_1 = \min_{v \geq 0} \left\{ a_1|\Omega| - c_1 \int_{\Omega} v + \frac{\chi^2}{4} \int_{\Omega} (c_2 v^3 - a_2 v^2) \right\}
\]

and \( A_1 > 0 \) in light of (1.6), (1.7) and the fact \( \frac{\chi^2}{4} \int_{\Omega} (c_2 v^3 - a_2 v^2) \geq -\frac{a_2^2 |\Omega|^2}{2c_2^2} \).

**Lemma 4.2.** Let \((u, v)\) be a positive classical solution of (1.5) in \( \Omega \times (0, \infty) \). Let \( L \) be any given positive constant and assume that there exists \( t_0 > 0 \) such that \( \int_{\Omega} \ln u(\cdot, t_0) dx \geq -L \), then for any \( T > 0 \) satisfying

\[
T \geq \frac{2}{A_1} \left( M_0 + L + \frac{\chi^2 K^2_0 |\Omega|}{8} \right), \tag{4.5}
\]
where $A_1$, $K_0$ and $M_0$ are given in (4.1), (2.1) and (2.2) respectively, we have that
\[
\int_{t_0}^{t_0+T} \int_{\Omega} u(x,t) dx dt \geq \frac{A_1}{2b_1} T. \tag{4.6}
\]

Proof. Integrating (4.1) over $(t_0,t_0+T)$, we have that
\[
\int_{\Omega} \ln u(\cdot,t_0+T) - \int_{\Omega} \ln u(\cdot,t_0) - \frac{\chi^2}{8} \int_{\Omega} v^2(\cdot,t_0+T) + \frac{\chi^2}{8} \int_{\Omega} v^2(\cdot,t_0)
\geq -b_1 \int_{t_0}^{t_0+T} \int_{\Omega} u + A_1 T,
\]
which, in light of the fact that $\ln \xi < \xi$ for all $\xi > 0$, entails that
\[
b_1 \int_{t_0}^{t_0+T} \int_{\Omega} u \geq A_1 T - \int_{\Omega} \ln u(\cdot,t_0) + \int_{\Omega} \ln u(\cdot,t_0+T)
\geq A_1 T - M_0 - L - \frac{\chi^2 K_0^2 |\Omega|}{8}. \tag{4.7}
\]

Now for $T$ being sufficiently large as in (4.5), we have
\[
\int_{t_0}^{t_0+T} \int_{\Omega} u \geq \frac{A_1}{b_1} T - \frac{1}{b_1} \left( M_0 + L + \frac{\chi^2 K_0^2 |\Omega|}{8} \right)
\]
from which (4.6) readily follows. \qed

Lemma 4.3. Let $(u,v)$ be a positive classical solution of (1.5) in $\Omega \times (0,\infty)$. Suppose that $\int_{\Omega} u(x,t_0) dx > -L$ hold for some $t_0 > 0$ and $L > 0$. Denote
\[
\eta := \min \left\{ \frac{A_1}{4b_1}, M_0 \right\} \tag{4.8}
\]
and define
\[
S_1 := \left\{ t \in (t_0,t_0+T) \mid \int_{\Omega} u(x,t) dx \geq \eta \right\}. \tag{4.9}
\]
Then for any $T > 0$ satisfying (4.5), we have that $|S_1| \geq \frac{\eta T}{M_0}$.

Proof. Denote the complement of set $S_1$ by
\[
S_1^c := (t_0,t_0+T) \setminus S_1,
\]
then we have that
\[
\int_{t_0}^{t_0+T} \int_{\Omega} u = \int_{S_1} \int_{\Omega} u + \int_{S_1^c} \int_{\Omega} u \leq M_0 |S_1| + \eta T
\]
which in light of (4.6) and (4.8) implies that
\[
M_0 |S_1| \geq \left( \frac{A_1}{2b_1} - \eta \right) T \geq \eta T. \tag{4.10}
\]
\qed
For fixed positive constants $K$ and $M$, we define the following sets

$$S_2 := \{ t \in (t_0, t_0 + T) \mid \int_{\Omega} u^2(x, t) dx \leq K \} \quad (4.10)$$

and

$$S_3 := \{ t \in (t_0, t_0 + T) \mid \int_{\Omega} \frac{\nabla u^2}{u^2} dx \leq M \} \quad (4.11)$$

with their complements being $S_i^c := (t_0, t_0 + T) \backslash S_i$, $i = 2, 3$.

**Lemma 4.4.** Let $(u, v)$ be a classical positive solution of (1.5) in $\Omega \times (0, \infty)$. Assume that all conditions in Lemma 4.3 hold. Let the positive constants in (4.10) and (4.11) be chosen such that

$$K > \frac{8a_1 M_0^2}{b_1 K}$$

and

$$M > \frac{16b_1 M_0^2}{\eta}.$$ 

Then for any $T$ satisfying (4.5) and the following condition

$$T > \frac{4M_0}{\eta} \left( \frac{M_0}{b_1 K} + \frac{2(M_0 + L + \frac{\chi^2 K_0^2 |\Omega|}{4})}{M} \right),$$

we have that

$$|S_1 \cap S_2 \cap S_3| > \frac{\eta}{2M_0} T.$$ 

**Proof.** The definitions of $S_2$ and $S_3$ imply

$$\int_{t_0}^{t_0+T} \int_{\Omega} u^2 \geq \int_{S_2} \int_{\Omega} u^2 \geq K|S_2^c|$$

and

$$\int_{t_0}^{t_0+T} \int_{\Omega} \frac{\nabla u^2}{u^2} \geq \int_{S_3^c} \int_{\Omega} \frac{\nabla u^2}{u^2} \geq M|S_3^c|. (4.15)$$

Thanks to (2.4), (4.14) entails that

$$|S_2^c| \leq \frac{(a_1 T + 1)M_0}{b_1 K},$$

therefore

$$|S_2| = T - |S_2^c| \geq \left( 1 - \frac{a_1 M_0}{b_1 K} \right) T - \frac{M_0}{b_1 K}. (4.16)$$

On the other hand, by applying the Cauchy–Schwartz

$$\frac{1}{2} \int_{\Omega} \frac{\nabla u^2}{u^2} + \frac{\chi^2}{2} \int_{\Omega} |\nabla v|^2 \geq -\chi \int_{\Omega} \frac{\nabla u \cdot \nabla v}{u},$$

to (4.2) and then using (4.7), we obtain from the definition of $A_1$ that

$$\frac{1}{2} \int_{t_0}^{t_0+T} \int_{\Omega} \frac{\nabla u^2}{u^2} \leq b_1 \int_{t_0}^{t_0+T} \int_{\Omega} u + \int_{\Omega} \ln u(\cdot, t_0 + T) - \int_{\Omega} \ln u(\cdot, t_0)$$

$$- \frac{\chi^2}{4} \int_{\Omega} v^2(\cdot, t_0 + T) + \frac{\chi^2}{4} \int_{\Omega} v^2(\cdot, t_0) - A_1 T$$

$$\leq b_1 M_0 T + M_0 + L + \frac{\chi^2 K_0^2 |\Omega|}{4},$$

thanks to the facts (2.2), (2.4), $\ln \xi < \xi$ for all $\xi > 0$ and $\int_{\Omega} \ln u(x, t_0) dx > -L$. Now we have from (4.17) that

$$|S_3^c| \leq \frac{2(b_1 M_0 T + M_0 + L + \frac{\chi^2 K_0^2 |\Omega|}{4})}{M},$$
Choosing inequality holds Lemma 4.5. Let give another important about this set in the following lemma. ST conditions in Lemma 4.4 and This finishes the proof.

conclude from Lemma 4.3, (4.16) and (4.18) that

\[ |S_3| = T - |S_3^*| \geq \left( 1 - \frac{2b_1 M_0}{M} \right) T - \frac{2(M_0 + L + \frac{x^2 K_0^2 |\Omega|}{4})}{M}. \tag{4.18} \]

Finally in light of the conditions on \( K \) and \( M \), as well as (4.5) and (4.12), we conclude from Lemma 4.3, (4.16) and (4.18) that

\[ |S_1 \cap S_2 \cap S_3| = |(S_1 \cap S_2) \cap S_3| \]
\[ = |S_1 \cap S_2| + |S_3| - |(S_1 \cap S_2) \cup S_3| \]
\[ = |S_1| + |S_2| + |S_3| - |S_1 \cup S_2| - |(S_1 \cap S_2) \cup S_3| \]
\[ \geq |S_1| + |S_2| + |S_3| - 2T \]
\[ \geq \left( \frac{\eta M_0}{M_0} - \frac{a_1 M_0}{b_1 K} - \frac{2b_1 M_0}{M} \right) T - \left( \frac{M_0}{b_1 K} + \frac{2(M_0 + L + \frac{x^2 K_0^2 |\Omega|}{4})}{M} \right) \]
\[ \geq \left( \frac{\eta M_0}{8 M_0} - \frac{\eta}{8 M_0} - \frac{\eta}{8 M_0} \right) T - \left( \frac{M_0}{b_1 K} + \frac{2(M_0 + L + \frac{x^2 K_0^2 |\Omega|}{4})}{M} \right) \]
\[ \geq \frac{\eta}{2 M_0} T. \]

This finishes the proof. \( \square \)

From now one, we assume that the positive constants \( K \) and \( M \) satisfy the conditions in Lemma 4.4 and \( T \) satisfies both (4.5) and (4.12). Therefore according to Lemma 4.4, the set \( S_1 \cap S_2 \cap S_3 \cap (t_0 + \frac{\eta}{2 M_0} T, t_0 + T) \) is always nonempty. We give another important about this set in the following lemma.

**Lemma 4.5.** Let \( \Omega \) be a convex domain in \( \mathbb{R}^N, N \geq 1 \). Suppose that all conditions in Lemma 4.4 hold. Then there exists a positive constant \( C_{\eta,K} \) depending on \( \eta \) and \( K \) such that for any fixed \( t_0^* \in S_1 \cap S_2 \cap S_3 \cap (t_0 + \frac{\eta}{2 M_0} T, t_0 + T) \) the following inequality holds

\[ \int_{\Omega} \ln u(x,t_0^*) dx \geq |\Omega| \ln \frac{\eta}{2 |\Omega|} - \sqrt{C_{\eta,K} M |\Omega|} := -L_0^*. \tag{4.19} \]

Here both \( C_{\eta,K} \) and \( L_0^* \) are independent of \( t_0^* \).

**Proof.** Since that \( t_0^* \in (S_1 \cap S_2) \), in the light of Lemma 4.1 in [19] we have that

\[ \left\{ x \in \Omega | u(x,t_0^*) \geq \frac{\eta}{2 |\Omega|} \right\} \geq \frac{\eta^2}{4 K}. \tag{4.20} \]

Choosing \( \varepsilon = \frac{\eta^2}{4 K} \) and \( \delta = \frac{\eta}{2 |\Omega|} \) in Lemma 4.3 in [19], we can find a positive constant \( C(\varepsilon) = C_{\eta,K} \) such that

\[ \int_{\Omega} \ln u(x,t_0^*) dx \geq |\Omega| \ln \delta - \sqrt{C(\varepsilon)|\Omega|} \int_{\Omega} \frac{\nabla u(x,t_0^*)^2}{u^2(x,t_0^*)} dx. \tag{4.21} \]

On the other hand, in light of \( t_0^* \in S_3 \), we infer from (4.11) and (4.21) that

\[ \int_{\Omega} \ln u(x,t_0^*) dx \geq |\Omega| \ln \delta - \sqrt{C(\varepsilon)|\Omega|} \]

and this completes the proof. \( \square \)
Lemma 4.6. Suppose that all conditions in Lemma 4.5 hold. Then there exists a sequence \( \{ t_k \}_{k=1}^{\infty} \subset (0, \infty) \) with \( t_0 = 0 \), \( t_k \to \infty \) as \( k \to \infty \) and \( t_k + \frac{n}{2M_0} T < t_{k+1} < t_k + T \) such that
\[
\int_{\Omega} u(x, t_k) dx \geq \eta^*, \quad \text{for each } k \in \mathbb{N}^+.
\]
(4.22)
where \( \eta^* = \max \{ \eta, e^{-L_0^*} \} \) and \( L_0^* \) is given by (4.19).

Proof. Our proof is based on recursive applications of Lemma 4.5. Under the conditions in Lemma 4.4, we readily see that the set \( S_1 \cap S_2 \cap S_3 \cap (t_0 + \frac{n}{2M_0} T, t_0 + T) \) is always nonempty for any \( t_0 \) such that \( \int_{\Omega} u(x, t_0) dx > -L_0^* \). Choosing \( t_0 = 0 \) in Lemma 4.5, we can find some \( t_1 = t_0^* \in (\frac{n}{2M_0} T, T) \) such that
\[
\int_{\Omega} \ln u(x, t_1) dx \geq -L_0^*,
\]
where \( L_0^* \) is given by (4.19) and it is independent of \( t_0^* \). Now put \( t_0^* \) as \( t_1 \) in Lemma 4.5, then we can find some \( t_2 = t_1^* \in (t_1 + \frac{n}{2M_0} T, t_1 + T) \) such that
\[
\int_{\Omega} \ln u(x, t_2) dx \geq -L_0^*,
\]
where again \( L_0^* \) is independent of \( t_1^* \).

We can repeat this process as follows: for any \( t_k \), there always exists \( t_{k+1} = *t_k^* \in (t_k + \frac{n}{2M_0} T, t_k + T) \) such that
\[
\int_{\Omega} \ln u(x, t_{k+1}) dx \geq -L_0^*.
\]
(4.22)

Obviously the sequence \( \{ t_k \}_{k=1}^{\infty} \) is increasing and \( t_k \to \infty \) as \( k \to \infty \) thanks to \( \eta > \frac{3M_b}{T} \) and (4.8). Each \( t_k \) is contained in \( S_1 \) hence \( \int_{\Omega} u(x, t_k) dx \geq \eta \). Moreover applying Jensen’s inequality on \( \int_{\Omega} \ln u(x, t_k) dx > -L_0^* \) implies that \( \int_{\Omega} u(x, t_k) dx \geq e^{-L_0^*} \). Together with the fact that \( t_k \in S_1 \) for each \( k \in \mathbb{N} \), this proves (4.22).

Finally we give the proof of the second main result of our paper.

Proof of Theorem 1.2. We first prove (1.9). From the \( u \)-equation, we have the following differential inequality
\[
\frac{d}{dt} \int_{\Omega} u(\cdot, t) \leq a_1 \int_{\Omega} u(\cdot, t), \quad \text{for all } t > 0.
\]
(4.23)

For each \( k \in \mathbb{N} \), we solve (4.23) over \( (t, t_{k+1}) \) to have that
\[
\int_{\Omega} u(\cdot, s) \geq \left( \int_{\Omega} u(\cdot, t) \right) e^{-a_1 (t_{k+1} - t)} \geq \eta^* e^{-a_1 T}, \quad \text{for all } t \in (t_k, t_{k+1})
\]
which implies
\[
\int_{\Omega} u(\cdot, t) \geq \eta^* e^{-a_1 T}, \quad \text{for all } t \in \cup_{k=0}^{\infty}(t_k, t_{k+1}).
\]
(4.24)

On the other hand, since each \( t_k \) is contained in \( S_1 \) defined in (4.9), we conclude that (1.9) holds with \( m_1 = \max \{ \eta, \eta^* e^{-a_1 T} \} \).

By the same arguments for (1.9) we can show (1.10). For the sake of completeness we now give its proof but meanwhile we shall only sketch it for the sake of simplify.

First of all, as for (4.2), we can show that there exists a positive constant \( A_2 \) such that
\[
\frac{d}{dt} \int_{\Omega} \ln v \geq \int_{\Omega} \frac{|\nabla v|^2}{v^2} - c_2 \int_{\Omega} v + A_2,
\]
(4.25)
where $A_2 = \min_{u \geq 0} \{a_2|\Omega| - b_2 \int_{\Omega} u \} > 0$ thanks to (1.6), (1.7) and (2.2).

Let $\tilde{L}$ be an arbitrary constant and suppose that there exists $\tilde{t}_0 \geq 0$ such that
\[
\int_{\Omega} \ln v(x, \tilde{t}_0) dx \geq -\tilde{L},
\]  
(4.26) then for any $\tilde{T} > \frac{2(M_1 + \tilde{L})}{A_2}$ we have
\[
\int_{\tilde{t}_0}^{\tilde{t}_0 + \tilde{T}} \int_{\Omega} v \geq \frac{A_2}{2c_2} \tilde{T}.
\]  
(4.27)

Similarly, we introduce the sets
\[
\tilde{S}_1 := \{ t \in (\tilde{t}_0, \tilde{t}_0 + \tilde{T}) | \int_{\Omega} v(x, t) \geq \tilde{\eta} \},
\]
\[
\tilde{S}_2 := \{ t \in (\tilde{t}_0, \tilde{t}_0 + \tilde{T}) | \int_{\Omega} v^2(x, t) \leq \tilde{K} \},
\]
\[
\tilde{S}_3 := \{ t \in (\tilde{t}_0, \tilde{t}_0 + \tilde{T}) | \int_{\Omega} \frac{|
abla u|^2}{v^2} \leq \tilde{M} \},
\]
and their complements
\[
\tilde{S}_i' := (\tilde{t}_0, \tilde{t}_0 + \tilde{T}) \setminus \tilde{S}_i, i = 1, 2, 3,
\]
where $\tilde{\eta}, \tilde{K}$ and $\tilde{M}$ are positive constants.

In particular, fixing
\[
\tilde{\eta} := \min \left\{ \frac{A_2}{4c_2}, M_1 \right\}
\]
and choosing $\tilde{K}, \tilde{M}$ large, dependent on $\tilde{\eta}$, for any sufficiently large $\tilde{T}$ dependent on $\tilde{\eta}, \tilde{K}$ and $\tilde{M}$, we can show that $|S_1 \cap S_2 \cap S_3| > \frac{\tilde{\eta}}{2M_1} \tilde{T}$ hence the set
\[
S_1 \cap S_2 \cap S_3 \cap (\tilde{t}_0 + \frac{\eta}{2M_1} \tilde{T}, \tilde{t}_0 + \tilde{T}) \neq \emptyset.
\]
Furthermore for any $\tilde{t}^* \in (\tilde{S}_1 \cap \tilde{S}_2 \cap \tilde{S}_3)$, one has that
\[
\int_{\Omega} \ln v(x, \tilde{t}^*) dx \geq |\Omega| \ln \frac{\tilde{\eta}}{2|\Omega|} - \sqrt{C_{\tilde{\eta}, \tilde{K}, \tilde{M}} |\Omega|} := -\tilde{L}^*.
\]  
(4.28)

In light of this estimate and the definitions of $\tilde{S}_i, i = 1, 2, 3$, we can show that there exists a positive sequence $\{\tilde{t}_k\}_{k \in \mathbb{N}^+} \subset (0, \infty)$ with $\tilde{t}_k \to \infty$ as $k \to \infty$ and $\tilde{t}_k < \tilde{t}_{k+1} < \tilde{t}_k + \tilde{T}$ such that
\[
\int_{\Omega} v(\cdot, \tilde{t}_k) \geq \tilde{\eta}^* = \max \{\tilde{\eta}, e^{-\tilde{L}^*}\}, \text{ for all } k \in \mathbb{N}^+.
\]  
(4.29)

By the same arguments, we can show that
\[
\int_{\Omega} v(x, t) dx \geq m_2 := \max \{\tilde{\eta}, \tilde{\eta}^* e^{-a_2 \tilde{T}}\}.
\]

The proof of Theorem 1.2 completes.
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