QUANTUM MATRIX BALL: DIFFERENTIAL AND INTEGRAL CALCULI

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1 Introduction

The first step in studying q-analogues of irreducible bounded symmetric domains was made in [19]. This work considers the simplest among those q-analogues, the quantum matrix balls. In this special case, we present here the proofs of the main results formulated in [18, 19], and produce, in particular, an explicit formula for the positive invariant integral, and thus prove its existence.

The initial six sections of this work use $\mathbb{C}(q^{1/s})$ as the ground field, the field of rational function of a single indeterminate $q^{1/s}$, with $s$ being some natural number. The subsequent sections already assume $q$ to be a number $q \in (0, 1)$, and use $\mathbb{C}$ as a ground field.

We assume a knowledge of the basic notions of quantum group theory [8], and, in particular, the notion of a universal R-matrix introduced by V. Drinfeld. Some of the general properties of a universal R-matrix to be alluded below, could be easily deduced from the explicit formula for R. This very well known multiplicative formula for a universal R-matrix is presented in Appendix 1.

2 The covariant algebra $\mathbb{C}[\text{Mat}_{mn}]_q$

Recall the definition of the quantum universal enveloping algebra $U_q\mathfrak{sl}_N$, introduced by V. Drinfeld and M. Jimbo. Let $(a_{ij})_{i,j=1,...,N-1}$ be the Cartan matrix given by

$$a_{ij} = \begin{cases} 2 & , \ i - j = 0 \\ -1 & , \ |i - j| = 1 \\ 0 & , \ \text{otherwise} \end{cases} .$$

(2.1)

The algebra $U_q\mathfrak{sl}_N$ is determined by the generators $E_i, F_i, K_i, K_i^{-1}, i = 1, \ldots, N-1$, and the relations
\[ K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i E_j = q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i \]

\[ E_i F_j - F_j E_i = \delta_{ij}(K_i - K_i^{-1})/(q - q^{-1}) \]

\[ E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_i E_i^2 = 0, \quad |i - j| = 1 \quad (2.2) \]

\[ F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_i F_i^2 = 0, \quad |i - j| = 1 \]

\[ [E_i, E_j] = [F_i, F_j] = 0, \quad |i - j| \neq 1. \]

The comultiplication \( \Delta \), the antipode \( S \), and the counit \( \varepsilon \) are determined by

\[ \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i, \quad (2.3) \]

\[ S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1}, \quad (2.4) \]

\[ \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1. \]

We consider in the sequel only \( U_q \mathfrak{sl}_N \)-modules of the form

\[ V = \bigoplus_{\mu \in \mathbb{Z}^{N-1}} V_\mu, \quad V_\mu = \{ v \in V | K_i v = q^{\mu_i} v, i = 1, \ldots, N \}. \]

This agreement allows one to introduce the linear operators \( H_j, X_j^\pm, j = 1, \ldots, N - 1 \), by setting up

\[ H_j v = \mu_j v, \quad v \in V_\mu, \quad E_j = X_j^+ q^{\frac{1}{2} H_j}, \quad F_j = q^{-\frac{1}{2} H_j} X_j^-. \quad (2.5) \]

Note that the classical universal enveloping algebra can be derived from \( U_q \mathfrak{sl}_N \) via

\[ q = e^{-h/2}, \quad K_i = e^{-h H_i/2} \quad (2.6) \]

and the subsequent passage to a limit as \( h \to 0 \).

Turn to a construction of q-analogue of the matrix space \( \text{Mat}_{mn} \), \( m, n \in \mathbb{N} \). Everywhere in the sequel \( N = m + n \). We follow [19] in equipping all the \( U_q \mathfrak{sl}_N \)-modules with the grading

\[ \deg v = j \Leftrightarrow H_0 v = 2j v, \quad \text{where} \]

\[ H_0 = \frac{2}{m + n} \left( m \sum_{j=1}^{n-1} j H_j + n \sum_{j=1}^{m-1} j H_{N-j} + mn H_n \right). \]

(The coefficients in the latter identity are chosen so that \( [H_0, X_n^\pm] = \pm 2 X_n^\pm, [H_0, X_j^\pm] = 0 \) for \( j \neq n \).)

In what follows \( V_k \) will stand for the homogeneous components of a graded vector space \( V \), and \( V^* \) for the dual graded vector space: \( (V^*)_k \overset{\text{def}}{=} (V_k)^* \).

Remind some notions of the theory of Hopf algebras [1].

Let \( A \) be a Hopf algebra and \( F \) an algebra equipped also by a structure of \( A \)-module. \( F \) is said to be an \( A \)-module (covariant) algebra if the multiplication \( F \otimes F \to F \), \( f_1 \otimes f_2 \mapsto f_1 f_2 \), is a morphism of \( A \)-modules. In the case of a unital algebra \( F \), an additional assumption is introduced that the embedding \( \mathbb{C} \hookrightarrow F \), \( 1 \mapsto 1 \), is a morphism of \( A \)-modules. A duality argument allows one also to introduce a notion of \( A^{op} \)-module (covariant) coalgebra.

The notion of a covariant (bi-)module over a covariant algebra and a covariant (bi- )comodule over a covariant coalgebra are introduced in a similar way.
Let $\lambda = (\lambda_1, \ldots, \lambda_{N-1}) \in \mathbb{Z}^{N-1}$, $\lambda_j \geq 0$ for $j \neq n$. Consider a generalized Verma module $V_\lambda(\lambda)$. It is a $U_q\mathfrak{sl}_N$-module with a single generator $v_\lambda(\lambda)$ and the relations $E_i v_\lambda(\lambda) = 0$, $K_i^{\pm 1} v_\lambda(\lambda) = q^\pm \lambda_i v_\lambda(\lambda)$, $i = 1, \ldots, n$, $F^{\lambda_j+1}_j v_\lambda(\lambda) = 0$, $j \neq n$.

The $U_q\mathfrak{sl}_N$-module $V_\lambda(0)$ will be equipped with a structure of covariant coalgebra: $\Delta_+: v_\lambda(0) \mapsto v_\lambda(0) \otimes v_\lambda(0)$, and the $U_q\mathfrak{sl}_N$-module $V_\lambda(\lambda)$ with a structure of a covariant bicomodule: $\Delta_L : v_\lambda(\lambda) \mapsto v_\lambda(0) \otimes v_\lambda(\lambda)$; $\Delta_R : v_\lambda(\lambda) \mapsto v_\lambda(\lambda) \otimes v_\lambda(0)$.

In our work [19], a dual algebra $C$ was implicit in [19] instead of $\mathbb{C}[\text{Mat}_{mn}]_q$, since the exposition of that work was not restricted to the special case of matrix balls.

The principal purpose of this section is to describe $\mathbb{C}[\text{Mat}_{mn}]_q$ in terms of generators and relations.

Consider the Hopf subalgebra $U_q\mathfrak{sl}_n \subset U_q\mathfrak{sl}_N$ generated by $E_i$, $F_i$, $K_i^{\pm 1}$, $i = 1, 2, \ldots, n-1$, and the Hopf subalgebra $U_q\mathfrak{sl}_m \subset U_q\mathfrak{sl}_N$ generated by $E_{n+i}$, $F_{n+i}$, $K_{n+i}^{\pm 1}$, $i = 1, 2, \ldots, m-1$. It follows from the definitions that the homogeneous component $\mathbb{C}[\text{Mat}_{mn}]_{q,1} = \{ f \in \mathbb{C}[\text{Mat}_{mn}]_q \mid \deg f = 1 \}$ is a $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-module. Prove that this module splits into the tensor product of a $U_q\mathfrak{sl}_n$-module related to the vector representation and a $U_q\mathfrak{sl}_m$-module related to the covector representation.

Consider the $U_q\mathfrak{sl}_n$-module $U$ and the $U_q\mathfrak{sl}_m$-module $V$, determined in the bases $\{ u_a \}_{a=1,\ldots,n}$, $\{ v^\alpha \}_{\alpha=1,\ldots,m}$ by

\[
X^+_i u_a = \begin{cases} u_{a+1} & \text{if } a = i, \\ 0 & \text{otherwise} \end{cases}, \quad X^+_i v^\alpha = \begin{cases} v^\alpha & \text{if } \alpha = m-i+1, \\ 0 & \text{otherwise} \end{cases} \]

\[
X^-_i u_a = \begin{cases} u_{a-1} & \text{if } a = i+1, \\ 0 & \text{otherwise} \end{cases}, \quad X^-_i v^\alpha = \begin{cases} v^\alpha & \text{if } \alpha = m-i, \\ 0 & \text{otherwise} \end{cases} \]

\[
H_i u_a = \begin{cases} 0 & \text{if } a = i, \\ u_a & \text{if } a = i+1, \\ -u_a & \text{otherwise} \end{cases}, \quad H_i v^\alpha = \begin{cases} 0 & \text{if } \alpha = m-i+1, \\ v^\alpha & \text{if } \alpha = m-i, \\ -v^\alpha & \text{otherwise} \end{cases} \]

**Proposition 2.1** There exists a unique collection $\{ z^\alpha_a \}_{a=1,\ldots,n; \alpha=1,\ldots,m}$ of elements of $\mathbb{C}[\text{Mat}_{mn}]_{q,1}$ such that the map $i : u_a \otimes v^\alpha \mapsto z^\alpha_a$, $a = 1, \ldots, n; \alpha = 1, \ldots, m$ admits an extension up to an isomorphism of $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-modules $i : U \otimes V \mapsto \mathbb{C}[\text{Mat}_{mn}]_{q,1}$, and $F_n z^m = q^{1/2}$.

**Proof.** Consider the maximum length elements for the permutation group $S_N$ and for its subgroup $S_n \times S_m$

\[
w_0 = (N, N-1, \ldots, 2, 1), \quad w'_0 = (n, n-1, \ldots, 1, N, N-1, \ldots, n+1).\]

Impose the notation $M = N(N-1)/2$, $M' = M - mn$, $s_j = (j, j+1)$. Let $w_0 = s_{i_1} s_{i_2} \ldots s_{i_{M'}}$ be such a reduced decomposition for $w_0$ that $s_{i_1} s_{i_2} \ldots s_{i_{M'}} = w'_0$. Consider the Hopf subalgebra $U_q\mathfrak{R}_- \subset U_q\mathfrak{sl}_N$ generated by $\{ F_j \}_{j=1,\ldots,N-1}$, and the base $\{ F^k_{\beta_1} F^k_{\beta_2} \ldots F^k_{\beta_{M'}} \}_{k_{1,\ldots,k_{M'}} \in \mathbb{Z}^+}$ in the vector space $U_q\mathfrak{R}_-$ associated to the above reduced decomposition.
The reader is referred to the Appendix 1 for a description of this base, together with the associated base of the graded vector space $V_-(0): \{ \tilde{F}_{k_1}^m, \tilde{F}_{k_2}^{m-1} \ldots \tilde{F}_{k_{M'+1}}^{m+1}, v_-(0) \}$, with $(k_{M'+1}, k_{M'+2}, \ldots, k_M) \in \mathbb{Z}_+^{mn}$. Hence, the dimensionalities of the weight subspaces are just the same as in the classical $(q = 1)$ case. In particular,

$$\dim V_-(0)_{-k} \overset{\text{def}}{=} \dim \{ v \mid H_0 v = -2kv \} = \binom{mn + k - 1}{k}. \quad (2.7)$$

Note that $\dim V_-(0)_{-1} = mn$, and $v' = F_n v_-(0)$ is a non-zero primitive vector:

$$E_j v' = 0, \quad H_j v' = \begin{cases} -2v', & j = n \\ v', & |j - n| = 1 \\ 0, & |j - n| > 1 \end{cases}, \quad j = 1, \ldots, N - 1.$$

Hence, the $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m$-module $U \otimes V$ is isomorphic to the $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m$-module $(V_-(0))_1^* \simeq \mathbb{C}[\text{Mat}_{mn}]_{q,1}$. Of course, the isomorphism $i: U \otimes V \to \mathbb{C}[\text{Mat}_{mn}]_{q,1} -\text{module}$ is unique up to a multiple from the ground field, and the elements $z^\alpha_a = i(u_a \otimes v^\alpha)$, $a = 1, \ldots, n$, $\alpha = 1, \ldots, m$, satisfy all the requirements of our proposition, except, possibly, the last property $F_n z_n^m = q^{l/2}$. One can readily choose the above multiple in the definition of $i$, which provides this property unless $F_n z_n^m = 0$. In the latter case one has $F_n (E_{i_1}^{k_1} E_{i_2}^{k_2} \ldots E_{i_l}^{k_l} z_n^m) = 0$ for all $i_1, \ldots, i_l$ different from $n$ and all $k_1, k_2, \ldots, k_l \in \mathbb{Z}_+$. Hence $F_n \mathbb{C}[\text{Mat}_{mn}]_{q,1} = 0$, and thus $F_n v_-(0) = 0$. That is, $\dim V_-(0) = 1$. On the other hand, it follows from (2.7) that $\dim V_-(0) = \infty$. This contradiction shows that $F_n z_n^m \neq 0$.

Proposition 2.2 $z^\alpha_a, \ a = 1, \ldots, n, \ \alpha = 1, \ldots, m,$ generate the algebra $\mathbb{C}[\text{Mat}_{mn}]_q$.

Proof. By a virtue of (2.7), it suffices to prove that for all $k \in \mathbb{Z}_+$, one can choose \( \binom{mn + k - 1}{k} \) linear independent vectors among the monomials $z^{\alpha_1}_a z^{\alpha_2}_a \ldots z^{\alpha_k}_a \in \mathbb{C}[\text{Mat}_{mn}]_{q,k}$. An application of the standard argument (see [4, chapter 5]) reduces this statement to its classical analogue.

Consider the ring $A = \mathbb{C}[q^{1/4}, q^{-1/4}]$ and the $A$-algebra $U_A$ generated by the elements $E_i, F_i, K_i^{\pm 1}, L_i = K_i - K_i^{-1} / q - q^{-1}$. This is a Hopf algebra: $\Delta(L_i) = L_i \otimes K_i + K_i^{-1} \otimes L_i$, $S(L_i) = -L_i$, $\epsilon(L_i) = 0$, $i = 1, \ldots, N - 1$. Let $V_A = U_A v_-(0)$, and $F_A \subset \mathbb{C}[\text{Mat}_{mn}]_q$ be the minimal $A$-module which contains all the monomials $z^{\alpha_1}_a z^{\alpha_2}_a \ldots z^{\alpha_k}_a$. It follows from the definitions that the value of a linear functional $z^\alpha_a$ on a vector $v \in V_A$ is in $A$. Hence, a similar statement is also valid for all the monomials $z^{\alpha_1}_a z^{\alpha_2}_a \ldots z^{\alpha_k}_a$, $k \in \mathbb{Z}_+$, and thus for all $f \in F_A$. By means of a specialization $q = 1$ we get (see [4, chapter 5], [3]):

$$F_A \to \mathbb{C}[z^1_1, z^2_1, \ldots, z^m_1], \quad V_A \to \mathbb{C} \left[ \frac{\partial}{\partial z^1_1}, \frac{\partial}{\partial z^2_1}, \ldots, \frac{\partial}{\partial z^m_1} \right].$$

What remains is to apply the non-degeneracy of the natural pairing for the graded vector spaces $\mathbb{C}[z^1_1, \ldots, z^m_1], \mathbb{C} \left[ \frac{\partial}{\partial z^1_1}, \ldots, \frac{\partial}{\partial z^m_1} \right]$ and the fact that the dimensionalities of the corresponding homogeneous components are in both cases $\binom{mn + k - 1}{k}$. \qed
Proposition 2.3

\[ z_{a_1}^{\alpha_1} z_{a_2}^{\alpha_2} - q z_{a_2}^{\alpha_2} z_{a_1}^{\alpha_1} = 0, \quad a_1 = a_2 \quad \& \quad \alpha_1 < \alpha_2 \quad \text{or} \quad a_1 < a_2 \quad \& \quad \alpha_1 = \alpha_2, \quad \text{(2.8)} \]

\[ z_{a_1}^{\alpha_1} z_{a_2}^{\alpha_2} - z_{a_2}^{\alpha_2} z_{a_1}^{\alpha_1} = 0, \quad \alpha_1 < \alpha_2 \quad \& \quad a_1 > a_2, \quad \text{(2.9)} \]

\[ z_{a_1}^{\alpha_1} z_{a_2}^{\alpha_2} - z_{a_2}^{\alpha_2} z_{a_1}^{\alpha_1} = (q - q^{-1}) z_{a_1}^{\alpha_1} z_{a_2}^{\alpha_2}, \quad \alpha_1 < \alpha_2 \quad \& \quad a_1 < a_2. \quad \text{(2.10)} \]

Proof. The validity of (2.8) – (2.10) follows from their validity at the 'classical limit' \( q = 1 \) and the \( U_q sl_n \otimes U_q sl_m \)-invariance of the associated subspace in \( \mathbb{C}[\text{Mat}_{mn}]_{q,1}^\otimes \). Let \( M = \mathbb{C}[\text{Mat}_{mn}]_{q,1}^\otimes \) and \( M_A \subset M \) be the \( A \)-module generated by \( \{ z_a^{\alpha} \}, \quad a = 1, \ldots, n, \quad \alpha = 1, \ldots, m. \) Remind that \( M \) is a module over the Hopf algebra \( U_q sl_n \otimes U_q sl_m \). By a virtue of proposition 2.3, the map \( \Phi : M \rightarrow \mathbb{C} \) is unambiguously determined by its specialization at \( q = 1 \). Consider two such submodules. The first \( U_q sl_n \otimes U_q sl_m \)-submodule is the kernel of the multiplication operator \( \mathbb{C}[\text{Mat}_{mn}]_{q,1}^\otimes \rightarrow \mathbb{C}[\text{Mat}_{mn}]_{q,2}, \quad f_1 \otimes f_2 \mapsto f_1 f_2 \). Another \( U_q sl_n \otimes U_q sl_m \)-submodule is the linear span of the elements given by the left hand sides of (2.8) – (2.10). Their specializations at \( q = 1 \) coincide, and hence the \( U_q sl_n \otimes U_q sl_m \)-submodules themselves are the same. \( \Box \)

Proposition 2.4 The relation list (2.8) – (2.10) is complete.

Proof. Consider a graded unital algebra \( F \) determined by degree 1 generators \( \{ u_a^{\alpha} \}, \quad a = 1, \ldots, n, \quad \alpha = 1, \ldots, m, \) and the relations (2.8) – (2.10) with the letter 'z' being replaced by the latter 'u'. It is an easy exercise to compute the dimensionalities of the homogeneous components \( F^{(k)} = \{ f \in F \mid \deg f = k \} \). Specifically,

\[ \dim F^{(k)} = \binom{mn + k - 1}{k}. \quad \text{(2.11)} \]

By a virtue of proposition 2.3, the map \( u_a^{\alpha} \mapsto z_a^{\alpha}, \quad a = 1, \ldots, n, \quad \alpha = 1, \ldots, m, \) admits an extension up to a homomorphism \( F \rightarrow \mathbb{C}[\text{Mat}_{mn}]_q \). It follows from proposition 2.2 that this homomorphism is onto. What remains is to apply the relations (2.8), (2.11) to establish the coincidence of the dimensionalities of the graded components:

\[ \dim \mathbb{C}[\text{Mat}_{mn}]_{q,k} = \dim F^{(k)}, \quad k \in \mathbb{Z}_+. \quad \Box \]

To conclude, note that the commutation relations (2.8) – (2.10) were used in a different context by a large number of authors [4].
3 Differential calculus

With [13] as a background, we describe a differential calculus on the quantum space of matrices. An advantage of our approach is that it discovers an additional surprising symmetry of the standard bicovariant differential calculus on this quantum space [18].

Consider the vector $v' = F_n v_-(0)$. It follows from the definitions that

$$E_j v' = 0, \quad H_j v' = -a_{jn} v', \quad j = 1, \ldots, N - 1, \quad (3.1)$$

$$F_i^{-a_{in} + 1} v' = 0, \quad i \neq n. \quad (3.2)$$

Here $(a_{ij})$ is a Cartan matrix [2.1]. Now (3.1), (3.2) imply

**Proposition 3.1** Consider the $U_q\mathfrak{sl}_N$-module $V_-(\lambda')$ with the highest weight $\lambda' = (-a_{1n}, -a_{2n}, \ldots, -a_{N-1,n})$. The map $v_-(\lambda') \mapsto F_n v_-(0)$ admits a unique extension up to a morphism $\delta_- : V_-(\lambda') \to V_-(0)$ of $U_q\mathfrak{sl}_N$-modules.

The graded vector space $\bigwedge^1(\operatorname{Mat}_{mn})_q$ dual to $V_-(\lambda')$ is a covariant bimodule over $\mathbb{C}[\operatorname{Mat}_{mn}]_q$. The adjoint to $\delta_-$ operator $d : \mathbb{C}[\operatorname{Mat}_{mn}]_q \to \bigwedge^1(\operatorname{Mat}_{mn})_q$ is called a differential. It follows from the definitions (see [13]) that

$$d(f_1 f_2) = df_1 \cdot f_2 + f_1 \cdot df_2, \quad f_1, f_2 \in \mathbb{C}[\operatorname{Mat}_{mn}]_q.$$ 

Describe the $\mathbb{C}[\operatorname{Mat}_{mn}]_q$-bimodule $\bigwedge^1(\operatorname{Mat}_{mn})_q$ in terms of generators and relations. Remind that $\deg \omega = j \iff H_0 \omega = 2j \omega$. Let $\bigwedge^1(\operatorname{Mat}_{mn})_{q,1} = \{ \omega \in \bigwedge^1(\operatorname{Mat}_{mn})_q | \deg \omega = 1 \}$.

**Lemma 3.2** $dz^\alpha_\alpha$, $\alpha = 1, \ldots, m$, $a = 1, \ldots, n$, constitute a base of the vector space $\bigwedge^1(\operatorname{Mat}_{mn})_{q,1}$.

**Proof.** Since $z^\alpha_\alpha$, $\alpha = 1, \ldots, m$, $a = 1, \ldots, n$, form a base of the vector space $\mathbb{C}[\operatorname{Mat}_{mn}]_{q,1}$, it suffices to prove that the linear map

$$d : \mathbb{C}[\operatorname{Mat}_{mn}]_{q,1} \to \bigwedge^1(\operatorname{Mat}_{mn})_{q,1}$$

is one-to-one. Consider the adjoint linear operator

$$\delta_- : V_-(\lambda')_{-1} \to V_-(0)_{-1}.$$ 

It follows from the definition of the $U_q\mathfrak{sl}_N$-module $V_-(\lambda')$ that the $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-module $V_-(\lambda')_{-1}$ is simple. One can easily deduce from proposition A1.2 that it is determined by the same relations as the $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-module $U^* \otimes V^*$. It was also shown in proposition 2.2 that the $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-module $V_-(0)_{-1}$ is simple as well. On the other hand, $\delta_-|_{V_-(\lambda')_{-1}}$ is a non-zero morphism of $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-modules: $\delta_- v_-(\lambda') = F_n v_-(0) \neq 0$. Hence the restrictions of $\delta_-$ and $d$ onto the corresponding homogeneous components are one-to-one. 

□

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Lemma 3.3 The $\mathbb{C}[\text{Mat}_{mn}]_q$-bimodule $\wedge^1(\text{Mat}_{mn})_q$ is a free left $\mathbb{C}[\text{Mat}_{mn}]_q$-module:

$$\wedge^1(\text{Mat}_{mn})_q = \bigoplus_{a=1}^m \bigoplus_{\alpha=1}^n \mathbb{C}[\text{Mat}_{mn}]_q dz^\alpha_a,$$

and a free right $\mathbb{C}[\text{Mat}_{mn}]_q$-module:

$$\wedge^1(\text{Mat}_{mn})_q = \bigoplus_{a=1}^m \bigoplus_{\alpha=1}^n dz^\alpha_a \mathbb{C}[\text{Mat}_{mn}]_q.$$

Proof. We are about to prove that the maps

$$\mathbb{C}[\text{Mat}_{mn}]_q \otimes \wedge^1(\text{Mat}_{mn})_q, 1 \rightarrow \wedge^1(\text{Mat}_{mn})_q, f \otimes \omega \mapsto f \omega,$$

$$\wedge^1(\text{Mat}_{mn})_q, 1 \otimes \mathbb{C}[\text{Mat}_{mn}]_q \rightarrow \wedge^1(\text{Mat}_{mn})_q, \omega \otimes f \mapsto \omega f$$

are one-to-one. Their injectivity can be easily derived from a similar result in the case $q = 1$ (cf. the proof of proposition 2.2). What remains is to use the coincidence of the dimensionalities of homogeneous components of the graded vector spaces $\wedge^1(\text{Mat}_{mn})_q$, $\mathbb{C}[\text{Mat}_{mn}]_q \otimes \wedge^1(\text{Mat}_{mn})_q$, $\wedge^1(\text{Mat}_{mn})_q \otimes \mathbb{C}[\text{Mat}_{mn}]_q$. (The dimensionalities of homogeneous components $\mathbb{C}[\text{Mat}_{mn}]_q, k \in \mathbb{Z}_+$, were computed before using a basis in $V_-(0)$ formed by homogeneous elements. The dimensionalities of homogeneous components of $\wedge^1(\text{Mat}_{mn})_q$ could be found in a similar way: a basis in $V_-(\lambda')$ could be constructed via an application of an appropriate reduced decomposition of the complete permutation $w_0 \in S_N$, together with the associated basis in $U_q\mathfrak{sl}_N$ (see Appendix 1).)

Of course, the covariant $\mathbb{C}[\text{Mat}_{mn}]_q$-bimodule $\wedge^1(\text{Mat}_{mn})_q$ is not free. The elements $dz^\alpha_a$ are its generators. Find a complete list of relations.

Let $U_q\mathfrak{sl}_N^{op}$ be the Hopf algebra which differs from $U_q\mathfrak{sl}_N$ by a replacement of its comultiplication $\Delta$ with an opposite $\Delta^{op}$. The structure of a $\mathbb{C}[\text{Mat}_{mn}]_q$-bimodule $\wedge^1(\text{Mat}_{mn})_q$ has been defined in \cite{19} via an application of a duality argument and the following morphisms in the category of modules over the Hopf algebra $U_q\mathfrak{sl}_N^{op}$:

$$\Delta^L_\gamma : V_-(\lambda') \rightarrow V_-(0) \otimes V_-(\lambda'); \quad \Delta^L_\gamma : v_-(\lambda') \mapsto v_-(0) \otimes v_-(\lambda');$$

$$\Delta^R_\gamma : V_-(\lambda') \rightarrow V_-(\lambda') \otimes V_-(0); \quad \Delta^R_\gamma : v_-(\lambda') \mapsto v_-(\lambda') \otimes v_-(0).$$

Let $P : V_-(\lambda') \otimes V_-(0) \rightarrow V_-(0) \otimes V_-(\lambda')$ be the ordinary flip of tensor multiples: $P(v' \otimes v'') = v'' \otimes v'$. Define an operator $\tilde{R}_{V_-(\lambda')V_-(0)} : V_-(\lambda') \otimes V_-(0) \rightarrow V_-(0) \otimes V_-(\lambda')$ via the universal R-matrix (see Appendix 1) by $\tilde{R}_{V_-(\lambda')V_-(0)} = R_{V_-(0)V_-(\lambda')}P$.

Lemma 3.4 $\tilde{R}_{V_-(\lambda')V_-(0)} \Delta^R_\gamma = \Delta^L_\gamma$.

Proof. It is well known that the operator $P \cdot R_{V_-(\lambda')V_-(0)}$ is a morphism in the category of $U_q\mathfrak{sl}_N$-modules. Hence, the operator $R_{V_-(0)V_-(\lambda')}P$ is a morphism in the category of $U_q\mathfrak{sl}_N^{op}$-modules. What remains is to apply the identity $\tilde{R}_{V_-(\lambda')V_-(0)} v_-(\lambda') \otimes v_-(0) = v_-(0) \otimes v_-(\lambda')$, which follows from the property (A1.6) of the universal R-matrix. \qed
The universal R-matrix satisfies the identity $S \otimes S(R) = R$, with $S$ being the antipode of the Hopf algebra in question (see [4]). Hence, the adjoint to $\hat{R}_{\lambda^{-1}} V_{\lambda}(0)$ operator is of the form

$$
\hat{R}_{C[\text{Mat}_{mn}]} \otimes \Lambda^1(\text{Mat}_{mn}) = PR_{C[\text{Mat}_{mn}]} \otimes \Lambda^1(\text{Mat}_{mn})
$$

with $P : C[\text{Mat}_{mn}] \otimes \Lambda^1(\text{Mat}_{mn}) \to \Lambda^1(\text{Mat}_{mn}) \otimes C[\text{Mat}_{mn}]$ being the ordinary flip of tensor multiples.

**Corollary 3.5** For all $\alpha, \beta = 1, \ldots, m, a, b = 1, \ldots, n$,

$$
z_b^\beta dz_a^\alpha = m_R PR_{C[\text{Mat}_{mn}]} \Lambda^1(\text{Mat}_{mn}) (z_b^\beta \otimes dz_a^\alpha),
$$

with $m_R : \Lambda^1(\text{Mat}_{mn}) \otimes C[\text{Mat}_{mn}] \to \Lambda^1(\text{Mat}_{mn})$, $m_R : \omega \otimes f \mapsto \omega f$.

**Proof.** It suffices to pass in the statement of lemma 3.4 to dual graded vector spaces and to adjoint operators. \[\square\]

Simplify (3.4) by computing $R_{C[\text{Mat}_{mn}]} \Lambda^1(\text{Mat}_{mn}) (z_b^\beta \otimes dz_a^\alpha)$ via an application of the multiplicative formula for the universal R-matrix.

**Lemma 3.6** $H_0$ is orthogonal to all the vectors $H_j$, $j \neq n$, with respect to the bilinear invariant scalar product $(H_i, H_j) = a_{ij}$, $i, j = 1, \ldots, N - 1$.

**Proof.** The invariant scalar product $(H_0, H_j)$ is given by $\text{tr} \pi_1 (H_0) \pi_1 (H_j)$, with $\pi_1$ being the vector representation of the Lie algebra $\mathfrak{sl}_N$. What remains is to compute this trace using the standard basis $\{e_j\}_{j=1}^N$ and the relation

$$
\pi_1 (H_0) e_j = \begin{cases} 
2 & \text{if } j \leq n \\
-m e_j & \text{if } j > n
\end{cases}
$$

Consider the $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m$-modules $L' = C[\text{Mat}_{mn}]_{q,1}$, $L'' = \Lambda^1(\text{Mat}_{mn})_{q,1}$ (the homogeneous components of the graded vector spaces $C[\text{Mat}_{mn}]_q$, $\Lambda^1(\text{Mat}_{mn})_q$). Let $R_{L' L''}$ stand for the linear operator in $L' \otimes L''$ determined by the action of the universal R-matrix of the Hopf algebra $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m$.

**Lemma 3.7** For all $\alpha, \beta = 1, \ldots, m, a, b = 1, \ldots, n$,

$$
R_{C[\text{Mat}_{mn}]} \Lambda^1(\text{Mat}_{mn}) (z_b^\beta \otimes dz_a^\alpha) = \text{const} \cdot R_{L' L''} (z_b^\beta \otimes dz_a^\alpha),
$$

with $\text{const}$ being independent of $a, b, \alpha, \beta$.

---

1This universal R-matrix is a tensor product of the universal R-matrices (A1.6) for $U_q \mathfrak{sl}_n$ and $U_q \mathfrak{sl}_m$. 

---

8
Proof. Apply to both sides of (3.3) the multiplicative formula for the universal R-matrix (A1.6). The ‘redundant’ exponential multiples in the left hand side of the resulting identity can be omitted since
\[ \exp_q((q^{-1} - q)E_{\beta j} \otimes F_{\beta j})z_b^\beta \otimes dz_a^\alpha = z_b^\beta \otimes dz_a^\alpha \]
for all \( \alpha, \beta = 1, \ldots, m, a, b = 1, \ldots, n \), \( j > \frac{m(m - 1)}{2} + \frac{n(n - 1)}{2} \). What remains is to compare the multiple \( q^{-t_0} \) related to the Hopf algebra \( U_q\mathfrak{sl}_N \) to a similar multiple related to the Hopf subalgebra \( U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m \). It follows from lemma 3.6 and the description of \( t_0 \) in terms of the orthogonal basis of the Cartan subalgebra (see Appendix 1) that their actions on the subspace \( \mathbb{C}[\text{Mat}_{mn}]_{q,1} \otimes \wedge^1 (\text{Mat}_{mn})_{q,1} \) differ only by a constant multiple. \( \square \)

Let
\[
\hat{R}_{UU}^{b'a'} = \begin{cases}
q^{-1}, & a = b = a' = b' \\
1, & a \neq b \; \& \; a = a' \; \& \; b = b' \\
q^{-1} - q, & a < b \; \& \; a = b' \; \& \; b = a' \\
0, & \text{otherwise}
\end{cases}
\]

\[
\hat{R}_{VV}^{\beta'\alpha'} = \begin{cases}
q^{-1}, & \alpha = \beta = \alpha' = \beta' \\
1, & \alpha \neq \beta \; \& \; \alpha = \alpha' \; \& \; \beta = \beta' \\
q^{-1} - q, & \alpha < \beta \; \& \; \alpha = \beta' \; \& \; \beta = \alpha' \\
0, & \text{otherwise}
\end{cases}
\]

**Proposition 3.8** For all \( \alpha, \beta = 1, \ldots, m, a, b = 1, \ldots, n \),
\[
z_b^\beta dz_a^\alpha = \sum_{\alpha', \beta' = 1}^m \sum_{a', b' = 1}^n \hat{R}_{VV}^{\beta'\alpha'} \hat{R}_{UU}^{b'a'} \cdot d\beta' \cdot \alpha'.
\]

**Proof.** Consider the operators in \( U \otimes U \) and \( V \otimes V \) determined by the actions of the universal R-matrices for Hopf algebras \( U_q\mathfrak{sl}_n \) and \( U_q\mathfrak{sl}_m \) respectively. It is well known (see [3, 4]) that these operators coincide up to constant multiples with the operators \( \hat{R}_{UU}, \hat{R}_{VV} \) given by the matrices \( \hat{R}_{UU}^{b'a'}, \hat{R}_{VV}^{\beta'\alpha'} \). Hence, by virtue of (3.4), (3.5), and proposition 2.1, one has
\[
z_b^\beta dz_a^\alpha = \text{const}_1 \sum_{\alpha', \beta' = 1}^m \sum_{a', b' = 1}^n \hat{R}_{VV}^{\beta'\alpha'} \hat{R}_{UU}^{b'a'} \cdot d\beta' \cdot \alpha'.
\]

What remains is to prove that \( \text{const}_1 = 1 \). This is due to
\[
\langle z_m^m d\lambda_m, F_n v_-(\lambda') \rangle = q^{-2} \langle d\lambda_m, z_m^m, F_n v_-(\lambda') \rangle \neq 0.
\]

The latter relation could be easily deduced from the definitions (just as it was done in the special case \( m = n = 1 \) described in details in [19]). \( \square \)

We have described an order one differential calculus on the quantum matrix space in terms of generators and relations. Consider the associated universal full differential calculus (see, for instance [19]). Proposition 3.8 implies
Corollary 3.9

\[ dz_b^\beta dz_a^\alpha = - \sum_a^{m} \sum_b^{n} \hat{R}_{VV}^{\beta \alpha} \hat{R}_{UU \psi \alpha} d\hat{z}_a^\alpha \cdot d\hat{z}_b^\beta. \]

The differential algebra \( \land (\text{Mat}_{mn})_q \) described here in terms of generators and relations is well known [4]. Our approach to its construction made it possible to discover a hidden symmetry of this differential algebra (see [18]). While producing the covariant algebras \( \mathbb{C}[\text{Mat}_{mn}]_q \land (\text{Mat}_{mn})_q \), the generalized Verma modules \( V_-(\lambda) \ni v_-(\lambda) \) with highest weights were implemented. It was demonstrated in [19] that, after replacing them by the generalized Verma modules \( V_+(\lambda) \ni v_+(\lambda) \) with lowest weights, it is possible to produce a covariant algebra of 'antiholomorphic polynomials' and the associated differential algebra \( \land (\text{Mat}_{mn})_q \).

4 Covariant \(*\)-algebra \( \text{Pol} (\text{Mat}_{mn})_q \)

Remind [1] that in the case of involutive algebras the definition of an \( A \)-module algebra includes the following compatibility axiom for involutions:

\[(af)^* = (S(a))^* f^*, \quad a \in A, \ f \in F \quad (4.1)\]

Let \( U_q \mathfrak{su}_{nm} \) stand for the \(*\)-Hopf algebra \((U_q \mathfrak{sl}_N, \ast)\) given by

\[(K_j^{\pm 1})^* = K_j^{\pm 1}, \quad E_j^* = \begin{cases} K_j F_j, \ j \neq n \\ -K_j F_j, \ j = n \end{cases}, \quad F_j^* = \begin{cases} E_j K_j^{-1}, \ j \neq n \\ -E_j K_j^{-1}, \ j = n \end{cases} \]

with \( j = 1, \ldots, N - 1 \). In terms of the 'generators' \( H_j, X_j^{\pm 1} \) (i.e. for operators from the class of *-representations of \( U_q \mathfrak{su}_{nm} \) described in section 2) one has

\[ H_j^* = H_j, \quad (X_j^\pm)^* = \begin{cases} X_j^{\mp}, \ j \neq n \\ -X_j^{\mp}, \ j = n \end{cases}, \quad j = 1, \ldots, N - 1. \]

A standard method of quantum group theory was used in [19] to equip each of the spaces

\[ \text{Pol} (\text{Mat}_{mn})_q \overset{\text{def}}{=} \mathbb{C}[\text{Mat}_{mn}]_q \otimes \mathbb{C}[\overline{\text{Mat}}_{mn}]_q, \quad \Omega (\text{Mat}_{mn})_q \overset{\text{def}}{=} \land (\text{Mat}_{mn})_q \otimes \land (\overline{\text{Mat}}_{mn})_q \]

with a structure of \( U_q \mathfrak{sl}_{nm} \)-module algebra (covariant algebra). The subalgebras

\[ \mathbb{C}[\overline{\text{Mat}}_{mn}]_q \otimes 1 \subset \text{Pol} (\text{Mat}_{mn})_q, \quad 1 \otimes \mathbb{C}[\text{Mat}_{mn}]_q \subset \text{Pol} (\text{Mat}_{mn})_q \]

are conjugate \((* : \mathbb{C}[\text{Mat}_{mn}]_q \to \mathbb{C}[\overline{\text{Mat}}_{mn}]_q)\); they are q-analogues of subalgebras of holomorphic and antiholomorphic polynomials respectively.

It follows from the definitions of [19] and proposition [2.2] that \( \{ z^\alpha_a \}, \alpha = 1, \ldots, m, \ a = 1, \ldots, n, \) generate the \(*\)-algebra \( \text{Pol} (\text{Mat}_{mn})_q \), and the complete relation list consists of (2.8) – (2.10), together with the following R-matrix commutation relation (cf. (3.4)):

\[(z^\beta_b)^* z^\alpha_a = mPR_{\mathbb{C}[\overline{\text{Mat}}_{mn}]_q \mathbb{C}[\overline{\text{Mat}}_{mn}]_q} z^\alpha_a \otimes z^\beta_b, \quad (4.2)\]
with $m : \text{Pol}(\text{Mat}_{mn})^q \to \text{Pol}(\text{Mat}_{mn})_q$, $P$ the flip of tensor multiples, and $R_{C[\text{Mat}_{mn}]_q} C[\text{Mat}_{mn}]_q$ the linear operator in $C[\text{Mat}_{mn}]_q \otimes C[\text{Mat}_{mn}]_q$ determined by the universal R-matrix.

Simplify the expression $R_{C[\text{Mat}_{mn}]_q} C[\text{Mat}_{mn}]_q (z_b^\beta)^* \otimes z_a^\alpha$ and thus the right hand side of (4.2).

Denote by $U_q \mathfrak{su}_n \otimes U_q \mathfrak{su}_m$ the subalgebra of the *-Hopf algebra $U_q \mathfrak{su}_{nm}$ generated by $E_j, F_j, K_j, K_j^{-1}$ with $j \neq n$.

Now an application of proposition 2.1 makes it easy to prove the following

Lemma 4.1 The sesquilinear form in $C[\text{Mat}_{mn}]_{q,1}$ given by $(z_b^\beta, z_a^\alpha) = \delta_{ab} \delta^{\alpha\beta}$, $a, b = 1, \ldots, n$, $\alpha, \beta = 1, \ldots, m$, is $U_q \mathfrak{su}_n \otimes U_q \mathfrak{su}_m$-invariant: $(\xi z_b^\beta, z_a^\alpha) = (z_b^\beta, \xi z_a^\alpha)$ for all $\xi \in U_q \mathfrak{su}_n \otimes U_q \mathfrak{su}_m$, $a = 1, \ldots, n$, $\alpha, \beta = 1, \ldots, m$.

Note that $(z_b^\beta)^*, b = 1, \ldots, n$, $\beta = 1, \ldots, m$, form a base for the homogeneous component $C[\text{Mat}_{mn}]_{q,-1}$ of the graded vector space $C[\text{Mat}_{mn}]_q$.

Corollary 4.2 The linear functional $\mu$ on $C[\text{Mat}_{mn}]_{q,-1} \otimes C[\text{Mat}_{mn}]_{q,1}$ given by $\mu((z_b^\beta)^* \otimes z_a^\alpha) = \delta_{ab} \delta^{\alpha\beta}$, is invariant (i.e. $\mu((\xi (z_b^\beta)^* \otimes z_a^\alpha)) = \varepsilon(\xi) \mu((z_b^\beta)^* \otimes z_a^\alpha)$ for all $\xi \in U_q \mathfrak{su}_n \otimes U_q \mathfrak{su}_m$, $a = 1, \ldots, n$, $\alpha, \beta = 1, \ldots, m$).

Proof. Let $L = C[\text{Mat}_{mn}]_{q,1}$. Consider the antimodule $\overline{L}$ which is still $L$ as an Abelian group, but the actions of the ground field and $U_q \mathfrak{su}_n \otimes U_q \mathfrak{su}_m$ are given by $(\lambda, v) \mapsto \overline{\lambda} v$, $(\xi, v) \mapsto S(\xi) * v$, $\xi \in U_q \mathfrak{su}_n \otimes U_q \mathfrak{su}_m, v \in L$. It follows from lemma 4.1 that the linear functional $\overline{L} \otimes L \to C(q^{1/2})$ corresponding to the sesquilinear form in $L$, is invariant. The relationship of an invariant integral and an invariant form is discussed, for example, in [13].

Let $L' = C[\text{Mat}_{mn}]_{q,-1}, L'' = C[\text{Mat}_{mn}]_{q,1}$, and $R_{L' L''}$ is the linear operator in $L' \otimes L''$ given by the action of the universal R-matrix of the Hopf algebra $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m \subset U_q \mathfrak{sl}_N$.

Lemma 4.3 For all $a, b = 1, \ldots, n$, $\alpha, \beta = 1, \ldots, m$,

$$R_{C[\text{Mat}_{mn}]_q} C[\text{Mat}_{mn}]_q ((z_b^\beta)^* \otimes z_a^\alpha) = \text{const}_1 \cdot R_{L' L''}((z_b^\beta)^* \otimes z_a^\alpha) + \text{const}_2 \cdot \delta_{ab} \delta^{\alpha\beta},$$

with $\text{const}_1$ and $\text{const}_2$ being independent of $a, b, \alpha, \beta$.

Proof. Reproduce essentially the proof of lemma 3.7 to establish the existence of such element $\text{const}_1$ of the ground field that for all $a, b, \alpha, \beta$ one has

$$R_{C[\text{Mat}_{mn}]_q} C[\text{Mat}_{mn}]_q ((z_b^\beta)^* \otimes z_a^\alpha) - \text{const}_1 \cdot R_{L' L''}((z_b^\beta)^* \otimes z_a^\alpha) \in C[\text{Mat}_{mn}]_{q,0} \otimes C[\text{Mat}_{mn}]_{q,0}.$$
In virtue of general properties of a universal R-matrix [4] this linear functional belongs to the subspace of $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-invariant linear functionals. This subspace is one-dimensional due to the simplicity of the $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_m$-module $\mathbb{C}[\text{Mat}_{mn}]_{q,1}$. What remains is to apply corollary 4.2.

We need an explicit form of the operator $R_{U'/U''}$.

Let $*: U \to \overline{U}$, $*: V \to \overline{V}$, be the identical maps between the above $U_q\mathfrak{sl}_n$-module $U$ and $U_q\mathfrak{sl}_m$-module $V$ onto the associated antimodules. Let $R_{U'}$, $R_{V'}$ stand for the operators in $U \otimes U$, $V \otimes V$ respectively, given by the actions of the universal R-matrices of the Hopf algebras $U_q\mathfrak{sl}_n$ and $U_q\mathfrak{sl}_m$.

**Lemma 4.4** For all $a, b = 1, \ldots, n$, $\alpha, \beta = 1, \ldots, m$, $$R_{U'} u^*_b \otimes u_a = \text{const}' \cdot \begin{cases} u^*_a \otimes u_a - (q^{-2} - 1) \sum_{k > a} u^*_k \otimes u_k, & a \neq b \\ u^*_a \otimes u_a - (q^{-2} - 1) \sum_{k > a} u^*_k \otimes u_k, & a = b \end{cases}$$ $$R_{V'} (v^*_\beta)^* \otimes v^*_\alpha = \text{const}'' \cdot \begin{cases} (v^*_\alpha)^* \otimes v^*_\alpha - (q^{-2} - 1) \sum_{k > \alpha} (v^*_k)^* \otimes v^*_k, & \alpha \neq \beta \end{cases}$$ with $\text{const}', \text{const}''$ being independent of $a, b, \alpha, \beta$.

**Proof.** It suffices to prove the first identity. Consider the linear operator $PR_{U'}: \overline{U} \otimes U \to U \otimes \overline{U}$, with $P$ being the flip of tensor multiples. It follows from the general properties of the universal R-matrix that this operator is a morphism of $U_q\mathfrak{sl}_n$-modules. Besides, it follows from (A1.6) that $PR_{U'} u^*_n \otimes u_n = \text{const}' \cdot u_n \otimes u^*_n$ since $u_n$ is the lowest weight vector of the $U_q\mathfrak{sl}_n$-module $U$.

On the other hand, it is well known (see, for example, [21]) that the operators defined by the right hand sides of the identities in the statement of our lemma possess the same properties. What remains is to use the fact that each morphism of $U_q\mathfrak{sl}_n$-modules $\overline{U} \otimes U \to U \otimes \overline{U}$ which annihilates $u^*_n \otimes u_n$, is identically zero (this vector does not belong to any of the two simple components of the $U_q\mathfrak{sl}_n$-module $U \otimes U$, and hence it generates this module).

Lemmas 4.3, 4.4 allow one to deduce all the relations between $(z^\beta)_b$, $z^\alpha_a$ up to two constants. These will be computed by means of the following

**Lemma 4.5** $R_{C[\text{Mat}_{mn}]_q \mathbb{C}[\text{Mat}_{mn}]_q} (z^m_n)^* \otimes z^m_n = q^2 (z^m_n)^* \otimes z^m_n + 1 - q^2$.

**Proof.** We are about to apply the explicit formula (A1.6) for the universal R-matrix.

Prove that $H_j z^m_n = \begin{cases} 2 z^m_n, & j = n \\ -z^m_n, & |j - n| = 1 \\ 0, & \text{otherwise} \end{cases}$. The two latter relations follow from the definitions of $z^\alpha_a$, see section 2. The first relation follows from $H_0 z^m_n = 2 z^m_n$:

$$2 z^m_n = \frac{2}{m + n}(-m(n - 1) - n(m - 1)) z^m_n + \frac{2mn}{m + n} H_n z^m_n.$$
Corollary 4.6 \((z_n^m)^* z_n^m = q^2 z_n^m (z_n^m)^* + 1 - q^2\).

Let
\[
\hat{R}_{UU}^\mathbb{V}^\mathbb{V} = \begin{cases} 
q^{-1}, & a \neq b & b = b' & a = a' \\
1, & a = b = a' = b' \\
-(q^2 - 1), & a = b & a' = b' & a' > a \\
0, & \text{otherwise}
\end{cases}
\]
\[
\hat{R}_{VV}^\mathbb{V}^\mathbb{V} = \begin{cases} 
q^{-1}, & a \neq \beta & \beta = \beta' & \alpha = \alpha' \\
1, & \alpha = \beta = \alpha' = \beta' \\
-(q^2 - 1), & \alpha = \beta & \alpha' = \beta' & \alpha' > \alpha \\
0, & \text{otherwise}
\end{cases}
\]

Proposition 4.7 For all \(a, b = 1, \ldots, n, \alpha, \beta = 1, \ldots, m\),
\[
(z_b^\beta)^* \cdot z_a^\alpha = q^2 \cdot \sum_{a', b'=1 \atop \alpha', \beta'=1}^{n \atop m} \hat{R}_{UU}^\mathbb{V}^\mathbb{V}_{ba}^\mathbb{V}^\mathbb{V} \cdot \hat{R}_{VV}^\mathbb{V}^\mathbb{V} \cdot z_a^\alpha (z_b^\beta)^* + (1 - q^2)\delta_{ab}\delta_{\alpha\beta}.
\]

Proof. The desired commutation relation with indefinite coefficients instead of \(q^2\) and \(1 - q^2\) follows from lemmas 4.3, 4.4. The values of those coefficients can be found via an application of corollary 4.6.

An application of the operators \(\partial, \bar{\partial}\) (see [19]) yields

Corollary 4.8 For all \(a, b = 1, \ldots, n, \alpha, \beta = 1, \ldots, m\),
\[
d(z_b^\beta)^* \cdot z_a^\alpha = q^2 \cdot \sum_{a', b'=1 \atop \alpha', \beta'=1}^{n \atop m} \hat{R}_{UU}^\mathbb{V}^\mathbb{V}_{ba}^\mathbb{V}^\mathbb{V} \cdot \hat{R}_{VV}^\mathbb{V}^\mathbb{V} \cdot z_a^\alpha (z_b^\beta)^*,
\]
\[
(z_b^\beta)^* \cdot dz_a^\alpha = q^2 \cdot \sum_{a', b'=1 \atop \alpha', \beta'=1}^{n \atop m} \hat{R}_{UU}^\mathbb{V}^\mathbb{V}_{ba}^\mathbb{V}^\mathbb{V} \cdot \hat{R}_{VV}^\mathbb{V}^\mathbb{V} \cdot z_a^\alpha (z_b^\beta)^*,
\]
\[
d(z_b^\beta)^* \cdot dz_a^\alpha = -q^2 \cdot \sum_{a', b'=1 \atop \alpha', \beta'=1}^{n \atop m} \hat{R}_{UU}^\mathbb{V}^\mathbb{V}_{ba}^\mathbb{V}^\mathbb{V} \cdot \hat{R}_{VV}^\mathbb{V}^\mathbb{V} \cdot z_a^\alpha (z_b^\beta)^*.
\]
5 The quantum group $SL_N$

Remind that currently we use $\mathbb{C}(q^{1/s})$, $s \in \mathbb{N}$, as a ground field. Later on, we shall, keeping the notation, pass to $\mathbb{C}$ as a ground field.

Consider the Hopf algebra $\mathbb{C}[SL_N]_q$ of regular functions on the quantum group $SL_N$ (see [6, 13]). This algebra is determined by its generators $\{t_{ij}\}_{i,j=1,\ldots,N}$, the commutation relations analogous to (2.8) – (2.10), and the relation $\det_q T = 1$. (Here $\det_q T$ is a $q$-determinant of the matrix $T = (t_{ij})_{i,j=1,\ldots,N}$:

$$\det_q T = \sum_{s \in S_N} (-q)^{l(s)} t_{1s(1)} t_{2s(2)} \cdots t_{Ns(N)},$$

with $l(s) = \text{card}\{(i,j) \mid i < j \quad \& \quad s(i) > s(j)\}$.) Comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$ are defined as follows:

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}, \quad \varepsilon(t_{ij}) = \delta_{ij}, \quad S(t_{ij}) = (-q)^{i-j} \det_q T_{ji}.$$

Here $i,j = 1,\ldots,N$, and the matrix $T_{ji}$ is derived from $T$ by obliterating its $j$-th line and $i$-th column.

Just as in section 2, we consider the vector representation $\pi_1$ of the Hopf algebra $U_q \mathfrak{sl}_N$ and the basis $\{u_k\}_{k=1,\ldots,N}$ in the space of this representation. We also need a well known [4] non-degenerate pairing of Hopf algebras $\mathbb{C}[SL_N]_q \times U_q \mathfrak{sl}_N \to \mathbb{C}(q^{1/s})$ in which a pair $(t_{ij},\xi)$, $i,j = 1,\ldots,N$, is sent to the corresponding matrix element of the operator $\pi_1(\xi)$.

This pairing is used to equip $\mathbb{C}[SL_N]_q$ with a structure of $U_q \mathfrak{sl}_N^{\text{op}} \otimes U_q \mathfrak{sl}_N$-module algebra as follows:

$$\langle (\eta \otimes \xi) f, \zeta \rangle = \langle f, S(\eta) \zeta \xi \rangle \quad (5.1)$$

for all $f \in \mathbb{C}[SL_N]_q$, $\xi,\eta,\zeta \in U_q \mathfrak{sl}_N$ (see the definition of $U_q \mathfrak{sl}_N^{\text{op}}$ in section 3).

We have described a $q$-analogue for the action of $SL_N \times SL_N$ on its homogeneous space $SL_N$

$$(g_1,g_2) : \quad g \mapsto g_1 g g_2^{-1}, \quad g,g_1,g_2 \in SL_N.$$

The action by ‘right shifts’ is crucial in what follows, so we write $\xi f$ instead of $(1 \otimes \xi) f$, $\xi \in U_q \mathfrak{sl}_N$, $f \in \mathbb{C}[SL_N]_q$. One has:

$$X^+_i t_{jk} = \begin{cases} t_{j\cdot k-1} & , \quad k = i + 1 \\ 0 & , \quad \text{otherwise} \end{cases}, \quad X^-_i t_{jk} = \begin{cases} t_{j \cdot k+1} & , \quad k = i \\ 0 & , \quad \text{otherwise} \end{cases},$$

$$H_i t_{jk} = \begin{cases} t_{jk} & , \quad k = i \\ -t_{jk} & , \quad k = i + 1 \\ 0 & , \quad \text{otherwise} \end{cases}.$$

The generators $t_{ij}$, $i,j = 1,\ldots,N$, of $\mathbb{C}[SL_N]_q$ are just the matrix elements of $\pi_1$. We are about to introduce the notation for matrix elements of other fundamental representations of the quantum group $SL_N$. Let $k$ be a natural number which does not
exceed \( N \), and consider the representation \( \pi_1^{\otimes k} \) of \( U_q\mathfrak{sl}_N \). Associate to each collection \( J \) of natural numbers \( j_1 < j_2 < \ldots < j_k \leq N \) the vector

\[
u_J = \nu_{j_1} \wedge \nu_{j_2} \wedge \ldots \wedge \nu_{j_k} = \sum_{s \in S_k} (-q)^{l(s)} \nu_{j_{s(1)}} \otimes \nu_{j_{s(2)}} \otimes \ldots \otimes \nu_{j_{s(k)}}.
\]

These vectors form a basis in the space of the representation \( \pi_1^{\wedge k} \) of \( U_q\mathfrak{sl}_N \). The matrix elements of \( \pi_1^{\wedge k} \) with respect to the basis \( \left\{ \nu_J \right\} \) are of the form

\[
t^k_{IJ} \overset{\text{def}}{=} \sum_{s \in S_k} (-q)^{l(s)} t_{i_{s(1)}j_{s(1)}} \cdot t_{i_{s(2)}j_{s(2)}} \cdot \ldots \cdot t_{i_{s(k)}j_{s(k)}},
\]

with \( I = (i_1, i_2, \ldots, i_k) \), \( J = (j_1, j_2, \ldots, j_k) \).

Consider the element

\[
t \overset{\text{def}}{=} t^m_{\{1, 2, \ldots, m\}\{n, n+2, \ldots, N\}},
\]

\[\text{Lemma 5.1}\]

\[
t_{ij} \cdot t = \begin{cases} qt \cdot t_{ij}, & i \leq m \; \& \; j \leq n \\ q^{-1}t \cdot t_{ij}, & i > m \; \& \; j > n \\ t \cdot t_{ij}, & \text{otherwise} \end{cases}, \quad (5.4)
\]

\[
X^+_n t = t^m_{\{1, 2, \ldots, m\}\{n, n+2, \ldots, N\}}, \quad X^-_n t = 0, \quad H_n t = -t.
\]

**Proof.** The latter three equalities follow directly from the definitions. The commutation relations (5.4) is well known [4]; we present the proof for the reader’s convenience in section 6. \(\Box\)

**Corollary 5.2** For any polynomial \( f \in \mathbb{C}[t] \) one has:

\[
X^+_j f(t) = \begin{cases} t^m_{\{1, 2, \ldots, m\}\{n, n+2, \ldots, N\}} \cdot \frac{f(q^{-1}t) - f(t)}{q^{-1}t - t}, & j = n \\ 0, & j \neq n \end{cases}, \quad (5.5)
\]

\[
H_j f(t) = \begin{cases} -t \frac{df(t)}{dt}, & j = n \\ 0, & j \neq n \end{cases}, \quad X^-_j f(t) = 0, \; j = 1, \ldots, N - 1. \quad (5.6)
\]

Let \( \mathbb{C}[SL_N]_{q,t} \) stand for the localization of \( \mathbb{C}[SL_N]_{q} \) with respect to the multiplicative system \( t, t^2, t^3, \ldots \).

\( \mathbb{C}[SL_N]_{q,t} \) has no zero divisors, its generators are \( t^{-1}, t_{ij}, i, j = 1, \ldots, N \), and the relation list includes all the relations which determine \( \mathbb{C}[SL_N]_{q} \) and

\[
t^{-1} \cdot t^m_{\{1, 2, \ldots, m\}\{n, n+2, \ldots, N\}} - 1 = 0
\]

\[
t^m_{\{1, 2, \ldots, m\}\{n, n+2, \ldots, N\}} \cdot t^{-1} - 1 = 0
\]

(5.7)
Apply the relations (5.7), (5.9) to equip \( \mathbb{C}[SL_N]_{q,t} \) with a structure of covariant algebra in such a way that the canonical embedding \( \mathbb{C}[SL_N]_q \hookrightarrow \mathbb{C}[SL_N]_{q,t} \) becomes a morphism of \( U_q\mathfrak{sl}_N \)-modules:

\[
X_j^+(t^{-1}) = \begin{cases} -q \cdot t^{\lambda_m} \cdot t^{-2} & j = n, \\ 0 & j \neq n \end{cases}
\]

\[
H_j(t^{-1}) = \begin{cases} t^{-1} & j = n, \\ 0 & j \neq n \end{cases}
\]

\[X_j^-(t^{-1}) = 0, \ j = 1, \ldots, N - 1.\]

These rules are well defined, as one can easily see by applying \( X_j^+ \), \( H_j \), \( j = 1, \ldots, N - 1 \), to the left hand sides of (5.7).

**Remark 5.3.** One can also extend the structure of \( U_q\mathfrak{sl}_N^{\text{op}} \)-module algebra in the same way. Thus, \( \mathbb{C}[SL_N]_{q,t} \) becomes a \( U_q\mathfrak{sl}_N^{\text{op}} \otimes U_q\mathfrak{sl}_N \)-module algebra.

The following results of the present section are essentially due to M. Noumi [12]. They will be also refined in a subsequent section.

Introduce the notation \( J_{aa} = \{ n + 1, n + 2, \ldots, N \} \setminus \{ N + 1 - \alpha \} \cup \{ a \} \).

**Proposition 5.4 (cf. [12])** The map \( i : z_0^\alpha \mapsto t^{-1} \cdot t_{\{1, 2, \ldots, m\}, j_{\alpha}}, \ \alpha = 1, \ldots, m, \ a = 1, \ldots, n, \ \text{admits a unique extension up to an embedding of} \ U_q\mathfrak{sl}_N \)-module algebras \( i : \mathbb{C}[\text{Mat}_{mn}]_q \hookrightarrow \mathbb{C}[SL_N]_{q,t} \).

**Proof.** We embed the algebras in question into the vector space \( (\tilde{w}_0 \cdot U_q\mathfrak{sl}_N)^* \), with \( \tilde{w}_0 \in \mathbb{C}[SL_N]^*_q \) being the maximum length element of the quantum Weyl group (see Appendix 1). More exactly, we restrict ourselves to the subspace \( \mathbb{F} \in (\tilde{w}_0 \cdot U_q\mathfrak{sl}_N)^* \) generated by \( U_q\mathfrak{b}_- \)-finite weight vectors

\[
\{ f \in (\tilde{w}_0 \cdot U_q\mathfrak{sl}_N)^* \mid \dim(U_q\mathfrak{b}_- f) < \infty, \ H_i f = \mu_i f, \ \mu_i \in \mathbb{Z}, \ i = 1, 2, \ldots, N - 1 \},
\]

with \( U_q\mathfrak{b}_- \) being the standard Borel subalgebra of \( U_q\mathfrak{sl}_N \). \( \mathbb{F} \) is equipped with a structure of \( U_q\mathfrak{sl}_N \)-module algebra by the following identities derived from (A1.9):

\[
\langle f_1 f_2, \tilde{w}_0 \xi \rangle = \langle R_{\mathbb{FF}} \Delta(\xi) f_1 \otimes f_2, \tilde{w}_0 \otimes \tilde{w}_0 \rangle,
\]

\[
\langle f, \tilde{w}_0 \eta \rangle = \langle f, \tilde{w}_0 \xi \rangle, \ \xi, \eta \in U_q\mathfrak{sl}_N, \ \ f, f_1, f_2 \in \mathbb{F}.
\]

(Here \( R_{\mathbb{FF}} \) is the linear operator in \( \mathbb{F} \otimes \mathbb{F} \) determined by the universal R-matrix (see Appendix 1)).

By a virtue of (A1.9), there exists an embedding of covariant algebras \( \mathbb{C}[SL_N]_q \hookrightarrow \mathbb{F} \). It is worthwhile to note that the element \( t \in \mathbb{F} \) is invertible. (The proof of invertibility requires some additional constructions. Given weight vector \( f \) in a \( U_q\mathfrak{sl}_N \)-module \( \mathbb{C}[SL_N]_q \), the sequence \( c_k(f) = \langle ft^k, \tilde{w}_0 \rangle \) satisfies a difference equation of order one derived from (A1.9):

\[
\langle ft^k, \tilde{w}_0 \rangle = \langle R_{\mathbb{C}[SL_N]_q, \mathbb{C}[SL_N]_q} (ft^{k-1} \otimes t), \tilde{w}_0 \otimes \tilde{w}_0 \rangle, \quad k \in \mathbb{N}.
\]
That is, by a virtue of (A1.6),
\[
\langle f t^k, \bar{w}_0 \rangle = \langle q^{-\frac{H_0 \otimes H_0}{(H_0, H_0)}} (f t^{k-1} \otimes t), \bar{w}_0 \otimes \bar{w}_0 \rangle, \quad k \in \mathbb{N}.
\]
This allows one to extend the linear functional $\bar{w}_0$ from $\mathbb{C}[SL_N]_q$ onto $\mathbb{C}[SL_N]_{q,t}$. We shall use in the sequel exactly this extension, together with the following pairing of $\mathbb{C}[SL_N]_{q,t}$ and $\bar{w}_0 U_q \mathfrak{sl}_N$:
\[
\langle f, \bar{w}_0 \xi \rangle \overset{\text{def}}{=} \langle \xi f, \bar{w}_0 \rangle, \quad \xi \in U_q \mathfrak{sl}_N, \quad f \in \mathbb{C}[SL_N]_{q,t}.
\]
Its non-degeneracy follows from the fact that its restriction onto $\mathbb{C}[SL_N]_q \times \bar{w}_0 U_q \mathfrak{sl}_N$ is non-degenerate. In fact, if $\langle f, \bar{w}_0 \xi \rangle = 0$ for all $\xi \in U_q \mathfrak{sl}_N$, then $\langle f \cdot t^j, \bar{w}_0 \xi \rangle = 0$ since $\langle f \cdot t^j, \bar{w}_0 \xi \rangle = \langle R_{\mathbb{C}[SL_N]_{q,t}}(f \otimes t^j), \bar{w}_0 \otimes \bar{w}_0 \rangle$. Thus we get an embedding $\mathbb{C}[SL_N]_{q,t} \hookrightarrow (\bar{w}_0 U_q \mathfrak{sl}_N)^*$. What remains is to note that the image of $t^{-1}$ under this embedding is in $\mathbb{F}$.

Consider the onto linear map $j : \bar{w}_0 U_q \mathfrak{sl}_N \rightarrow V_-(0), j : \bar{w}_0 \xi \mapsto S(\xi) v_-(0), \xi \in U_q \mathfrak{sl}_N$, with $v_-(0)$ being the generator of the $U_q \mathfrak{sl}_N$-module $V_-(0)$ dual to $\mathbb{C}[\text{Mat}_{mn}]_q$ (see [19]). It is easy to prove that the adjoint linear map $j^* : \mathbb{C}[\text{Mat}_{mn}]_q \rightarrow \mathbb{F}$ is an embedding of $U_q \mathfrak{sl}_N$-module algebras. Let us agree not to distinguish between the $U_q \mathfrak{sl}_N$-module algebras $\mathbb{C}[\text{Mat}_{mn}]_q$, $\mathbb{C}[SL_N]_{q,t}$ and their images under the above embedding into $\mathbb{F}$. In view of propositions 2.1, 2.2, it suffices to prove that $t^{-1} t_{\{1,2,...,m\},J_{\alpha \alpha}} \in \mathbb{C}[\text{Mat}_{mn}]_q$, $\alpha = 1, 2, \ldots, m$, $a = 1, 2, \ldots, n$. For that, we need only to establish that $t^{-1} t_{\{1,2,...,m\},J_{\alpha \alpha}}$ are orthogonal to the kernel of $j$ with respect to the above pairing. This kind of orthogonality follows from
\[
((K_j^{\pm 1} - 1) \otimes 1) t^{-1} t_{\{1,2,...,m\},J_{\alpha \alpha}} = (F_j \otimes 1) t^{-1} t_{\{1,2,...,m\},J_{\alpha \alpha}} = 0,
\]
\[
(E_i \otimes 1) t^{-1} t_{\{1,2,...,m\},J_{\alpha \alpha}} = 0, \quad j = 1, 2, \ldots, N - 1, \quad i = 1, 2, \ldots, n - 1, n + 1, \ldots, N,
\]
in view of the definitions of the $U_q \mathfrak{sl}_N$-module $V_-(0)$ and $\bar{w}_0$.

Let us agree not to distinguish the elements of $\mathbb{C}[\text{Mat}_{mn}]_q$ and their images under the embedding $i : \mathbb{C}[\text{Mat}_{mn}]_q \hookrightarrow \mathbb{C}[SL_N]_{q,t}$.

**Lemma 5.5** For all $1 \leq a < b \leq n$, $1 \leq \alpha < \beta \leq m$,
\[
t^{-1} t_{\{1,2,...,m\}\{a,b,...,N+1-\beta,...,N+1-\alpha,...,N\}} = z_\alpha^\beta z_\beta^\alpha - q z_\alpha^\beta z_\beta^\alpha.
\]

**Proof.** In the same way as in the proof of proposition 5.4, one can establish that $t^{-1} t_{\{1,2,...,m\}\{a,b,...,N+1-\beta,...,N+1-\alpha,...,N\}} \in \mathbb{C}[\text{Mat}_{mn}]_q \subset \mathbb{F}$. What remains is to express this element in terms of the generators of $\mathbb{C}[\text{Mat}_{mn}]_q$. It is a weight vector of the $U_q \mathfrak{sl}_N$-module $\mathbb{C}[\text{Mat}_{mn}]_q$. A computation of the weight yields
\[
t^{-1} t_{\{1,2,...,m\}\{a,b,...,N+1-\beta,...,N+1-\alpha,...,N\}} = c_1 z_\alpha^\beta z_\beta^\alpha + c_2 z_\alpha^\beta z_\beta^\alpha,
\]
with $c_1$, $c_2$ being the elements of the ground field $\mathbb{C}(q^{1/s})$. When computing the constants $c_1$, $c_2$, one can restrict oneself to the special case $m = n = 2$ by passing from the algebra $\mathbb{C}[SL_N]_{q,t}$ to the corresponding factor algebra. In the special case $m = n = 2$ the result in question is accessible via a direct calculation [2].

\[\square\]
Corollary 5.6

\[ X^+_n z^\alpha_a = \begin{cases} -q^{-1/2} z^m_a z^\alpha_m, & a \neq n \quad \& \quad \alpha \neq m \\ -q^{-1}/2(z^m_a)^2, & a = n \quad \& \quad \alpha = m \\ -z^m_a z^\alpha_m, & \text{otherwise} \end{cases} \]

The latter relation, together with proposition 2.1 and the relations

\[ X^-_n z^\alpha_a = \begin{cases} \frac{q}{2}, & a = n \quad \& \quad \alpha = m \\ 0, & \text{otherwise} \end{cases} \]

\[ H_n z^\alpha_a = \begin{cases} 2z^\alpha_a, & a = n \quad \& \quad \alpha = m \\ z^\alpha_a, & a = n \quad \& \quad \alpha \neq m \quad \text{or} \quad a \neq n \quad \& \quad \alpha = m \\ 0, & \text{otherwise} \end{cases} \]

describe the action of \( U_q \mathfrak{sl}_N \) in \( \mathbb{C}[\text{Mat}_{mn}]_q \).

6 The quantum principal homogeneous space

In the case \( q = 1 \) the matrix ball \( \mathbb{U} \) as a homogeneous space of the group \( SU_{mn} \) is isomorphic to \( S(U_n \times U_m) \setminus SU_{mn} \). A straightforward generalization of this statement is derived via a replacement of \( S(U_n \times U_m) \setminus SU_{mn} \) by \( S(U_n \times U_m) \setminus \tilde{X} \), with \( \tilde{X} \) being some principal homogeneous space of the group \( SU_{mn} \). We construct a quantum principal homogeneous space in such a way that the isomorphism \( \mathbb{U} \simeq S(U_n \times U_m) \setminus \tilde{X} \) is valid in the quantum case.

Consider the element \( \tilde{w}_0 \in \mathbb{C}[SL_N]^* \) of the quantum Weyl group (see Appendix 1). It follows from the invertibility of this element that the pairing of \( \mathbb{C}[SL_N]_q \) and \( \tilde{w}_0 U_q \mathfrak{su}_{mn} \) is non-degenerate, and hence there exists a unique antilinear operator \( * \) in \( \mathbb{C}[SL_N]_q \) such that

\[ \langle f^*, \tilde{w}_0 \xi \rangle = \overline{\langle f, \tilde{w}_0(S(\xi))^* \rangle} \quad (6.1) \]

for all \( f \in \mathbb{C}[SL_N]_q, \xi \in U_q \mathfrak{su}_{mn} \).

Proposition 6.1 The map \( * \) is an antilinear involution:

\[ f^{**} = f, \quad (f_1 f_2)^* = f_2^* f_1^*, \quad f, f_1, f_2 \in \mathbb{C}[SL_N]_q. \quad (6.2) \]

Proof. It follows from the well known properties of a universal \( R \)-matrix (see [4]) and the definition of involution \( * \) that

\[ R^{\otimes *} = R_{21}, \quad S \otimes S(R) = R, \quad (6.3) \]

with \( R_{21} \) is derivable from \( R \) via a displacement of tensor multiples. The second one of the identities (6.2) follows from the relations (A1.9) and (6.3), and the first one follows from \( (S((\xi)^*))^* = \xi \), which is valid for all \( \xi \in U_q \mathfrak{su}_{mn} \). \( \square \)
Evidently, (1.1) is valid for the *-algebra $\text{Pol}(\widetilde{X})_q = (\mathbb{C}[SL_N]_q, \star)$. That is, $\text{Pol}(\widetilde{X})_q$ is a covariant *-algebra. We call it the polynomial algebra on the quantum principal homogeneous space. (In the classical case $q \to 1$ one has $\widetilde{X} = \tilde{w}_qSU_{mn} \subset SL_N$.) We are to produce an explicit formula for $t'_{ij}$, $i, j = 1, \ldots, N$. Let us consider the covector representation $\pi_{N-1}$ of $U_q\mathfrak{sl}_N$ defined in the base $v^1, v^2, \ldots, v^N$ by the formulae from section 2. In the next lemma we express all the matrix elements $t'_{ij} \in \mathbb{C}[SL_N]_q$ of $\pi_{N-1}$

$$
\pi_{N-1}(\xi) v^j = \sum_i \langle t'_{ij}, \xi \rangle v^i
$$

in terms of generators $t_{ij}$.

**Lemma 6.2** For all $i, j = 1, \ldots, N$,

$$
t'_{ij} = \text{det}_q T_{i'j'}, \quad \text{with} \quad i' = N + 1 - i, \quad j' = N + 1 - j.
$$

**Proof.** Let $L$ be the linear span of $t'_{ij} \in \mathbb{C}[SL_N]_q$. Evidently, $L$ is a simple $U_q\mathfrak{sl}_N^\text{op} \otimes U_q\mathfrak{sl}_N$-module, and the map $t'_{ij} \mapsto \text{det}_q T_{i'j'}, i, j = 1, \ldots, N$, admits an extension up to an endomorphism $\varphi$ of this simple module. Hence, $\varphi = \text{const} \cdot 1$. On the other hand, $
\langle t_{ij},1 \rangle = \langle t'_{ij},1 \rangle = \delta_{ij}.\n$ Thus we have $\langle t'_{N1},1 \rangle = \langle \text{det}_q T_{N1},1 \rangle = 1$, and so $\varphi = 1$ and $t'_{ij} = \text{det}_q T_{i'j'}$ for all $i, j = 1, \ldots, N$.

Compare the matrices of the operators $\pi_1(w_0)$ and $\pi_{N-1}(w_0)$.

**Lemma 6.3** For all $i, j = 1, \ldots, N$,

$$
\langle t_{ij}, \tilde{w}_0 \rangle = \langle t'_{ij}, \tilde{w}_0 \rangle = \text{const} \cdot (-q)^{-i} \cdot \delta_{i+j,N+1}, \quad (6.4)
$$

with $\text{const}$ being an element of the ground field independent of $i, j$.

**Proof.** In the case $N = 2$ the desired statement is well known [22, 19]. The general case is reducible to this one via (A.10), (A.11) and the following reduced decompositions of the full permutation $w_0 = (N, N - 1, \ldots, 2, 1) \in S_N$:

$$
w_0 = s_{N-1}(s_{N-2}s_{N-1})\ldots(s_1s_2\ldots s_{N-1}) = s_1(s_2s_1)\ldots(s_{N-1}s_{N-2}\ldots s_1).
$$

(Note that for any orthogonal basis $\{I_k\}_{k=1}^{N-1}$ in the Cartan subalgebra one has

$$
\pi_1 \left( \sum_k \frac{I_k^2}{(I_k, I_k)} \right) = \text{const}' \cdot I, \quad \pi_{N-1} \left( \sum_k \frac{I_k^2}{(I_k, I_k)} \right) = \text{const}'' \cdot I,
$$

with $I$ being the identity operator. This is because the element $\sum_k \frac{I_k^2}{(I_k, I_k)}$ is an invariant of the Weyl group action. That is, by a slight misuse of the notation,

$$
\langle t_{ij}, \sum_k \frac{I_k^2}{(I_k, I_k)} \rangle = \text{const}' \cdot \delta_{ij}, \quad \langle t'_{ij}, \sum_k \frac{I_k^2}{(I_k, I_k)} \rangle = \text{const}'' \cdot \delta_{ij}.
$$

In this setting $\text{const}' = \text{const}''$ since $\langle t'_{ij}, \xi \rangle = \langle t_{ij}, \omega(\xi) \rangle$, $\xi \in U_q\mathfrak{sl}_N$, $i, j = 1, \ldots, N$, with $\omega : U_q\mathfrak{sl}_N \to U_q\mathfrak{sl}_N$ being the automorphism given by $\omega(K_j^{\pm1}) = K_j^{\mp1}$, $\omega(E_j) = E_{N-j}$, $\omega(F_j) = F_{N-j}$. \hfill \Box
Lemma 6.4
1) There exists a unique antilinear involution $\#$ in $\mathbb{C}[SL_N]_q$ such that for all $f \in \mathbb{C}[SL_N]_q$, $\xi \in U_q\mathfrak{su}_{nm}$,
$$
\langle f^#, \xi \rangle = \langle f, (S(\xi))^\ast \rangle.
$$
2) For all $i,j = 1, \ldots, N$,
$$
i_{ij}^# = \text{sign} \left( (n - i + \frac{1}{2})(n - j + \frac{1}{2}) \right) (-q)^{j-i} \text{det}_q T_{ij}.
$$

Proof. The desired statement is a well known fact in quantum group theory. It follows from the results of [13] where the involution (6.5) was initially considered. (It was also noted in this paper that the $\ast$-algebra $\mathbb{C}[SU_{nm}]_q \overset{\text{def}}{=} (\mathbb{C}[SL_N]_q, \#)$ is a Hopf $\ast$-algebra.)

Proposition 6.5 For all $i,j = 1, \ldots, N$,
$$
t_{ij}^* = \text{sign} \left( (i - m - \frac{1}{2})(n - j + \frac{1}{2}) \right) (-q)^{j-i} \text{det}_q T_{ij}.
$$

Proof. The pairing considered above is non-degenerate and allows one to embed $U_q\mathfrak{sl}_N$ into $\mathbb{C}[SL_N]_q^*$. Let $L$ be the antirepresentation of $\mathbb{C}[SL_N]_q^*$ in the space $\mathbb{C}[SL_N]_q$ given by
$$
\langle L(\xi)f, \eta \rangle = \langle f, \xi \eta \rangle, \quad f \in \mathbb{C}[SL_N]_q, \ \xi, \eta \in \mathbb{C}[SL_N]_q^*.
$$
Now compare the definitions for involutions $\ast$ and $\#$ to get
$$
t_{ij}^* = L(\bar{w}_0^{-1}) \cdot (L(\bar{w}_0)t_{ij}^#), \quad i,j = 1, \ldots, N.
$$
On the other hand, it follows from lemma 6.3 that
$$
L(\bar{w}_0)t_{ij} = \text{const} \cdot (-q^{-1})^i \cdot t_{N+1-i,j},
$$
$$
L(\bar{w}_0^{-1})t_{ij}^* = \frac{1}{\text{const}} \cdot (-q)^{N-i+1} \cdot t_{N+1-i,j}^*.
$$
and, by a virtue of lemmas 6.2, 6.4,
$$
t_{ij}^# = \text{sign} \left( (n - i + \frac{1}{2})(n - j + \frac{1}{2}) \right) (-q)^{j-i} t_{N+1-i,N+1-j}^*.
$$
Hence,
$$
t_{ij}^* = \text{const} \cdot (-q^{-1})^i L(\bar{w}_0^{-1})t_{N+1-i,j}^# =
$$
$$
= \text{const} \cdot (-q^{-1})^i L(\bar{w}_0^{-1}) \cdot \text{sign} \left( (i - m - \frac{1}{2})(n - j + \frac{1}{2}) \right) (-q)^{i+j-N-1} \cdot t_{i,N+1-j}^* =
$$
$$
= (-q^{-1})^i \cdot \text{sign} \left( (i - m - \frac{1}{2})(n - j + \frac{1}{2}) \right) (-q)^{i+j-N-1} \cdot (-q)^{N-i+1} \cdot t_{N+1-i,N+1-j}^*.
$$
What remains is to apply lemma 6.2. □
We use in the sequel the results of Ya. Soibelman concerning the quantum group $SU_N$ \cite{20, 23}. Remind the definitions.

Equip the Hopf algebras $U_q\mathfrak{sl}_N$ and $C[SL_N]_q$ with antilinear involutions given by

$$(K^\pm_j)^\ast = K_j^\pm, \quad E_j^\ast = K_j F_j, \quad F_j^\ast = E_j K_j^{-1}, \quad j = 1, \ldots, N - 1,$$

$$\langle f^\ast, \xi \rangle = \overline{\langle f, (S(\xi))^\ast \rangle}, \quad f \in C[SL_N]_q, \quad \xi \in U_q\mathfrak{sl}_N.$$ 

The associated Hopf $\ast$-algebras are denoted by

$$U_q\mathfrak{su}_N \overset{\text{def}}{=} (U_q\mathfrak{sl}_N, \ast), \quad C[SU_N]_q = (C[SL_N]_q, \ast).$$

Similarly to (6.5), (6.6), one has

$$t_{ij}^\ast = (-q)^{j^{-i}} \det_{ij} T_{ij}, \quad (6.7)$$

together with the following

**Lemma 6.6** For all $k = 1, \ldots, N - 1$,

$$\left( t^\ast_{\{1, \ldots, k\}\{N-k+1, \ldots, N\}} \right)^\ast = (-q)^{k(N-k)} \cdot t^{\ast(N-k)}_{\{1, \ldots, N\}\{1, \ldots, N-k\}}.$$ 

Introduce the notation $t = t^\ast_{\{1,2,\ldots,m\}\{n+1, n+2, \ldots, N\}}$, $x = (-q)^{mn} \cdot t^\ast_{\{1,2,\ldots,m\}\{n+1, n+2, \ldots, N\}}$, $t^{\ast n}_{\{m+1, m+2, \ldots, N\}\{1,2,\ldots,n\}}$ for the elements of a crucial importance in the function theory in quantum matrix ball. Note that in view of (6.6) one has $x = tt^\ast$.

**Lemma 6.7**

$$t_{ij} \cdot t = \begin{cases} q t \cdot t_{ij} & , \quad i \leq m \quad \& \quad j \leq n \\ q^{-1} t \cdot t_{ij} & , \quad i > m \quad \& \quad j > n \\ t \cdot t_{ij} & , \quad \text{otherwise} \end{cases}, \quad (6.8)$$

$$t_{ij} \cdot t^\ast = \begin{cases} q t^\ast \cdot t_{ij} & , \quad i \leq m \quad \& \quad j \leq n \\ q^{-1} t^\ast \cdot t_{ij} & , \quad i > m \quad \& \quad j > n \\ t^\ast \cdot t_{ij} & , \quad \text{otherwise} \end{cases}. \quad (6.9)$$

**Proof.** (6.8) can be easily verified in the special case $i \in \{m, m+1\}$, $j \in \{n, n+1\}$. The biinvariance of $t$

$$(1 \otimes E_i) t = (1 \otimes F_i) t = (E_j \otimes 1) t = (F_j \otimes 1) t = 0, \quad i \neq n, \quad j \neq m$$

allows one to reduce the general case to the above special case via an application of the operators

$$1 \otimes E_i, \quad 1 \otimes F_i, \quad i \neq n; \quad E_j \otimes 1, \quad F_j \otimes 1, \quad j \neq m$$

to each side of (6.8). A similar argument proves also (6.9). 

Now lemmas (6.6, 6.7) imply
Corollary 6.8 \( x = t^* t = t^* t \), and for every polynomial \( f \in \mathbb{C}[x] \),
\[
 t_{ij} \cdot f(x) = \begin{cases} 
 f(q^2 x) t_{ij} & , \ i \leq m \ \text{&} \ j \leq n \\
 f(q^{-2} x) t_{ij} & , \ i > m \ \text{&} \ j > n \\
 f(x) t_{ij} & , \ \text{otherwise}
\end{cases} (6.10)
\]

Proposition 6.9 For every polynomial \( f \in \mathbb{C}[x] \),
\[
 X_n^+ f(x) = (X_n^+ x) \cdot \frac{f(q^{-2} x) - f(x)}{q^{-2} x - x}, (6.11)
\]
\[
 X_n^- f(x) = \frac{f(q^{-2} x) - f(x)}{q^{-2} x - x} \cdot (X_n^- x). (6.12)
\]

Proof. It follows from the explicit formulae which define the action of the operators \( X_n^+, H_n \) on the generators \( t_{ij} \) of \( \mathbb{C}[SL_N]^*_q \) and the covariance of the latter algebra that
\[
 X_n^+ x = q^{-1/2} (-q)^{mn} \cdot t^{\wedge n}_{\{1,2,..,m\}{n,n+2,..,N}} \cdot t^{\wedge n}_{\{m+1,m+2,..,N\}{1,2,..,n}}; (6.13)
\]
\[
 X_n^- x = q^{-1/2} (-q)^{mn} \cdot t^{\wedge n}_{\{1,2,..,m\}{n+1,n+2,..,N}} \cdot t^{\wedge n}_{\{m+1,m+2,..,N\}{1,2,..,n-1,n+1}}. (6.14)
\]

It follows from (6.10), (6.13), (6.14) that
\[
 (X_n^+ x) \cdot x = q^2 x \cdot (X_n^+ x), \quad (X_n^- x) \cdot x = q^{-2} x \cdot (X_n^- x).
\]

Hence, (6.11), (6.12) are valid for all monomials \( f = x^k, k \in \mathbb{Z}_+ \).

Let \( \text{Pol}(\widetilde{X})_{q,x} \) stand for a localization of the integral domain \( \text{Pol}(\widetilde{X})_q \) with respect to the multiplicative system \( x, x^2, x^3, \ldots \). An involution in \( \text{Pol}(\widetilde{X})_{q,x} \) is imposed in a natural way: \( (x^{-1})^* = x^{-1} \).

Apply (6.11), (6.12) to equip \( \text{Pol}(\widetilde{X})_{q,x} \) with a structure of \( U_q \mathfrak{su}_{nm} \)-module algebra in such a way that the canonical embedding \( \text{Pol}(\widetilde{X})_q \hookrightarrow \text{Pol}(\widetilde{X})_{q,x} \) becomes a morphism of \( U_q \mathfrak{su}_{nm} \)-modules:

\[
 H_j(x^{-1}) = 0, \quad X_j^+(x^{-1}) = \begin{cases} 
 -q^2 (X_j^+ x) x^{-2} & , \ j = n \\
 0 & , \ j \neq n
\end{cases},
\]
\[
 X_j^-(x^{-1}) = \begin{cases} 
 -q^2 x^{-2} (X_j^- x) & , \ j = n \\
 0 & , \ j \neq n
\end{cases}.
\]

This structure of \( U_q \mathfrak{sl}_N \)-module algebra is well defined, as one can easily verify just as for a similar statement in the previous section. The relation \( (\xi f)^* = (S(\xi))^* f^* \), \( \xi \in U_q \mathfrak{sl}_N \), is valid both for \( f = x^{-1} \) and \( f \in \text{Pol}(\widetilde{X})_q \). Hence it is valid for all \( f \in \text{Pol}(\widetilde{X})_{q,x} \).

Just as in remark 5.3, note that \( \text{Pol}(\widetilde{X})_{q,x} \) is a \( U_q \mathfrak{sl}_N^\text{op} \otimes U_q \mathfrak{sl}_N \)-module algebra.
Proposition 6.10 The map

\[ \mathcal{I} : z^\alpha_a \mapsto t^{-1}t_{(1,2,\ldots,m)J_{aa}} \tag{6.15} \]

with \( J_{aa} = \{n + 1, n + 2, \ldots, N\} \setminus \{N + 1 - \alpha\} \cup \{a\} \), is uniquely extendable up to an embedding of \( U_q\mathfrak{su}_{nm}\)-module *-algebras \( \mathcal{I} : \text{Pol}(\text{Mat}_{mn})_q \hookrightarrow \text{Pol}(\widetilde{X})_{q,x} \).

**Proof.** The uniqueness of the embedding \( \mathcal{I} \) follows from proposition 2.2. Prove its existence. Consider the embedding of \( U_q\mathfrak{sl}_N \)-module algebras \( i : \mathbb{C}[\text{Mat}_{mn}]_q \hookrightarrow \mathbb{C}[SL_N]_{q,t} \) (see section 5) and a similar embedding \( \widetilde{i} : \mathbb{C}[\text{Mat}_{mn}]_q \hookrightarrow \mathbb{C}[SL_N]_{q,t} \).

We use the embeddings \( \mathbb{C}[\text{Mat}_{mn}]_q \subset \text{Pol}(\text{Mat}_{mn})_q, \mathbb{C}[SL_N]_{q,t} \subset \text{Pol}(\widetilde{X})_{q,x} \).

Let \( \mathcal{I} \) be such a linear operator \( \mathcal{I} : \text{Pol}(\text{Mat}_{mn})_q \to \text{Pol}(\widetilde{X})_{q,x} \) that \( \mathcal{I} : f_- \cdot f_+ \mapsto \widetilde{i}(f_-) \cdot i(f_+), f_- \in \mathbb{C}[\text{Mat}_{mn}]_q, f_+ \in \mathbb{C}[\text{Mat}_{mn}]_q \). By our construction, \( \mathcal{I} \) is a morphism of \( U_q\mathfrak{su}_{nm}\)-modules and satisfies (6.15). Prove that it is a homomorphism of *-algebras. It suffices to show that (in the notation of section 4)

\[ (\mathcal{I}z^\beta_b)^*(\mathcal{I}z^\alpha_a) = mPR_{\mathbb{C}[\text{Mat}_{mn}]_q/\mathbb{C}[\text{Mat}_{mn}]_q}( (\mathcal{I}z^\beta_b)^* \otimes (\mathcal{I}z^\alpha_a) ) \]

for all \( a, b = 1, 2, \ldots, n; \alpha, \beta = 1, 2, \ldots, m \). For that, it suffices to establish

\[ \mathcal{I}((z^\beta_b)^t)^* t^{jk} \mathcal{I}(z^\alpha_a) = q^{\text{const} - k}mPR_{\mathbb{C}[SL_N]_{q,t}/\mathbb{C}[SL_N]_{q,t}}( (\mathcal{I} ((z^\beta_b)^t)^* t^{jk} \mathcal{I}(z^\alpha_a) ) ) \]

with \( j, k \in \mathbb{Z} \), and \( \text{const} \) being determined by the equation \( H_0 \cdot H_0(t \otimes t^*) = \text{const} \cdot (H_0, H_0) \cdot t \otimes t^* \). We may restrict ourselves to the special case \( j, k \in \mathbb{N} \) since

\[ t^{-k}(q^{\text{const} - k}mPR_{\mathbb{C}[SL_N]_{q,t}/\mathbb{C}[SL_N]_{q,t}}( (\mathcal{I} ((z^\beta_b)^t)^* t^{jk} \mathcal{I}(z^\alpha_a) ) ) t^{-j} \]

is a Laurent polynomial of \( q^{k/s}, q^{j/s} \). In the above special case \( \mathcal{I}((z^\beta_b)^t)^* t^{jk} \mathcal{I}(z^\alpha_a) \), \( a, b = 1, 2, \ldots, n; \alpha, \beta = 1, 2, \ldots, m \), are the matrix elements of finite dimensional representations of \( U_q\mathfrak{sl}_N \). What remains is to apply the well known R-matrix commutation relations between those matrix elements, (A1.6), and the relations \( (F_i \otimes 1)(t^k \mathcal{I}(z^\alpha_a)) = (E_i \otimes 1)(\mathcal{I}((z^\beta_b)^t)^* t^{jk} \mathcal{I}(z^\alpha_a)) \) = 0, \( a, b = 1, 2, \ldots, n; \alpha, \beta = 1, 2, \ldots, m; i \neq m \).

So we need only to prove the injectivity of the homomorphism \( \mathcal{I} \) via passage to the improper specialization \( q = 1 \) and observing that the corresponding homomorphism in the case \( q = 1 \) is injective (see the proof of proposition 2.2 where a similar well known argument was used).

To conclude, extend the embedding \( \mathbb{C}[SL_N]_q \hookrightarrow (\bar{w}_0 U_q\mathfrak{sl}_N)^* \) initially constructed in the proof of proposition 5.4 up to an embedding \( \text{Pol}(\widetilde{X})_{q,x} \hookrightarrow (\bar{w}_0 U_q\mathfrak{sl}_N)^* \). One can verify in the same way as in section 5 that for every weight vector \( f \in \text{Pol}(\widetilde{X})_{q,x} \), the numbers \( c_{jk} = \langle t^{jk} f^k, \bar{w}_0 \rangle \), \( j, k \in \mathbb{Z}_+ \), satisfy the system of order one difference equations

\[
\begin{cases}
\langle t^{jk} f^k, \bar{w}_0 \rangle = q^{-1/m_{n} \otimes \bar{w}_0} t^{*} \otimes t^{*j-1} f^k, \bar{w}_0 \otimes \bar{w}_0 \\
\langle t^{jk} f^k, \bar{w}_0 \rangle = q^{-1/m_{n} \otimes \bar{w}_0} t^{*j} f^k \otimes t, \bar{w}_0 \otimes \bar{w}_0
\end{cases}
\]

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Extend the linear functional \( \tilde{w}_0 \) onto \( \text{Pol}(\tilde{X})_{q,x} \) via the above system of difference equations and set up \( \langle f, \tilde{w}_0 \xi \rangle = \langle \xi f, \tilde{w}_0 \rangle, \xi \in U_q \mathfrak{sl}_N, f \in \text{Pol}(\tilde{X})_{q,x} \) (see the proof of proposition 5.4).

A non-degenerate pairing supplies an embedding \( \text{Pol}(\tilde{X})_{q,x} \hookrightarrow (\tilde{w}_0 U_q \mathfrak{sl}_N)^\ast \). According to an approach of V. Drinfeld, we use the term 'function on a quantum group' to denote the linear functions on \( U_q \mathfrak{sl}_N \) (more exactly, on \( U_h \mathfrak{sl}_N \), see [6]). The basic distinction of our approach to construction of algebras of functions on quantum homogeneous spaces is in replacing the space \( U_q \mathfrak{sl}_N \) with \( \tilde{w}_0 U_q \mathfrak{sl}_N \).

7 Algebras of finite functions

From now on we assume that \( 0 < q < 1 \) and use \( \mathbb{C} \) as a ground field. We also keep the above notation for the Hopf algebras and covariant algebras involved, together with their descriptions in terms of generators and relations.

Among the principal tools in harmonic analysis, one should mention the algebra of finite functions and the \( L^2 \) space being its completion. Now turn to a construction of covariant \( \ast \)-algebra \( D(\tilde{X})_q \) of finite functions on the quantum principal homogeneous space \( \tilde{X} \).

We refer to Appendix 2 for a construction of a \( \ast \)-representation \( \tilde{\Pi} \) of \( \text{Pol}(\tilde{X})_q \) with \( \tilde{\Pi}(x) \neq 0 \) and \( \text{spec} \tilde{\Pi}(x) = q^{-2Z_+} \). For a function \( f \) on \( q^{-2Z_+} \) with a finite support, a bounded linear operator \( f(\tilde{\Pi}(x)) \) is well defined. Our immediate intention is to add the elements of the form \( f(x), \text{supp} f(x) \subset q^{-2Z_+} \), to \( \text{Pol}(\tilde{X})_q \).

Consider the algebra of polynomial functions and the algebra of functions with finite support inside \( q^{2Z} \); let \( F \) stand for their sum. Note that \( F \) admits the involution \( f(x) \mapsto \overline{f(x)} \).

Consider the \( \mathbb{C}[x] \)-bimodules \( \text{Pol}(\tilde{X})_q, F \), and form their tensor product \( F = \text{Pol}(\tilde{X})_q \otimes_{\mathbb{C}[x]} F \). One can deduce from (6.10) that there exists a canonical isomorphism \( F \simeq F \otimes_{\mathbb{C}[x]} \text{Pol}(\tilde{X})_q \). This isomorphism, together with the multiplication laws

\[
\text{Pol}(\tilde{X})_q \otimes \text{Pol}(\tilde{X})_q \to \text{Pol}(\tilde{X})_q, \quad F \otimes F \to F,
\]

is used to equip \( F \) with such structure of \( \ast \)-algebra that the embeddings

\[
\text{Pol}(\tilde{X})_q \hookrightarrow F, \quad F \hookrightarrow F.
\]

are homomorphisms of algebras.

**Proposition 7.1** There exists a unique extension of the structure of covariant \( \ast \)-algebra from \( \text{Pol}(\tilde{X})_q \) onto \( F \) such that for any function \( f \in F \), (6.11) and (6.12) are valid.

**Proof.** The uniqueness of the extension is obvious. The existence is due to the fact that for any function \( f \in F \) and any finite subset \( M \subset q^{2Z} \), there exists a polynomial \( \psi \in \mathbb{C}[x] \) with \( f(x) = \psi(x) \) for all \( x \in M \). \( \square \)
Proposition 7.2 Let $J$ be the bilateral ideal in $\mathcal{F}$ generated by such elements $f \in \mathcal{F}$ that $\text{supp} f(x) \subset q^{2N}$. Then $J$ is a submodule of the $U_q \mathfrak{su}_{nm}$-module $\mathcal{F}$.

Proof. It suffices to apply (6.11), (6.12), and the relations $X_j f(x) = 0$ for $j \neq n$, $H_i f(x) = 0$ for $i = 1, \ldots, N - 1$. □

Corollary 7.3 The quotient algebra $\mathcal{F}/J$ is a covariant $\ast$-algebra.

Remark 7.4. The algebra of functions $f(x)$ with finite support $\text{supp} f \subset q^{-2N}$ is obviously embedded into $\mathcal{F}/J$. Among the above functions, a principal position is occupied by the function

$$f(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases}.$$ 

This element of $\mathcal{F}/J$ is denoted in the sequel by $f_0$. The next proposition is a straightforward consequence of our definitions.

Proposition 7.5 In $\mathcal{F}/J$, the following relations are valid:

\begin{align*}
    t_{ij} f_0 &= 0 \quad \text{for} \quad i > m \& j > n, \quad (7.1) \\
    f_0 t_{ij} &= 0 \quad \text{for} \quad i \leq m \& j \leq n, \quad (7.2) \\
    t_{ij} f_0 &= f_0 t_{ij} \quad \text{for} \quad i > m \& j \leq n \quad \text{or} \quad i \leq m \& j > n, \quad (7.3) \\
    x f_0 &= f_0 x = f_0, \quad f_0 = f_0^2 = f_0^\ast.
\end{align*}

($f_0$ can be treated as a q-analogue of the delta-function $\delta(x - 1)$ on $\tilde{X}$).

One can use (6.11) – (6.14) to deduce

Proposition 7.6 In the covariant $\ast$-algebra $\mathcal{F}/J$ one has

\begin{align*}
    K_n f_0 &= f_0, \\
    E_n f_0 &= -q^{3/2} \frac{1 - q^{2t(1,2,\ldots,m)\{n,n+2,\ldots,N\}}} {1 - q^{2t(1,2,\ldots,m)\{n,n+2,\ldots,N\}}} t^\ast f_0, \\
    F_n f_0 &= -q^{3/2} \frac{1 - q^{2t(1,2,\ldots,m)\{n,n+2,\ldots,N\}}} {q^{-2} - 1} f_0 t \left( t(1,2,\ldots,m)\{n,n+2,\ldots,N\} \right)^\ast,
\end{align*}

while for $j \neq n$

\begin{align*}
    K_j f_0 &= f_0, \quad E_j f_0 = F_j f_0 = 0.
\end{align*}
Consider a covariant $*$-algebra $\text{Fun}(\tilde{X})_q \subset \mathcal{F}/J$, generated by $\{t_{ij}\}_{i,j=1,\ldots,N}$ and $f_0$.\footnote{We do not discuss in the present work whether or not this inclusion is proper.} Replace in the above observations the ring of polynomials $\mathbb{C}[x]$ with the ring $\mathbb{C}[x, x^{-1}]$ to get a covariant $*$-algebra $\text{Fun}(\tilde{X})_{q,x} \supset \text{Fun}(\tilde{X})_q$.

The term 'finite functions' is reserved for the elements of the bilateral ideal $D(\tilde{X})_q$ of $\text{Fun}(\tilde{X})_q$ generated by $f_0$. It is evident that

$$\text{Fun}(\tilde{X})_q = \text{Pol}(\tilde{X})_q + D(\tilde{X})_q, \quad \text{Fun}(\tilde{X})_{q,x} = \text{Pol}(\tilde{X})_{q,x} + D(\tilde{X})_q,$$

and $\text{Fun}(\tilde{X})_q, \text{Fun}(\tilde{X})_{q,x}, D(\tilde{X})_q$ are covariant $*$-algebras.

Define a covariant $*$-algebra $\text{Fun}(\mathbb{U})_q$ by its generators $f_0, z^\alpha_a, a = 1, \ldots, n, \alpha = 1, \ldots, m$, and the relations (motivated by the relations in $\text{Fun}(\tilde{X})_q$). The list of relations that determine $\text{Fun}(\mathbb{U})_q$ includes all the relations of $\text{Pol}(\text{Mat}_{mn})_q$, and additionally

$$f_0 = f_0^2 = f_0^*, \quad (z^\alpha_a)^* f_0 = f_0 (z^\alpha_a) = 0, \quad a = 1, \ldots, n, \quad \alpha = 1, \ldots, m.$$

The structure of a covariant $*$-algebra is imposed by

$$(K_j^{\pm 1} - 1) f_0 = E_j f_0 = F_j f_0 = 0 \quad \text{for} \quad j \neq m.$$

(To verify the correctness of the latter definition, one has to apply corollary 5.6 and the fact that $z^m_n$ quasi-commutes with $(z^\alpha_a)^*$ for $(a, \alpha) \neq (n, m)$.) Of course, the bilateral ideal $D(\mathbb{U})_q \subset \text{Fun}(\mathbb{U})_q$ generated by $f_0$, is a covariant $*$-algebra.

Evidently, the map

$$i : f_0 \mapsto f_0, \quad i : z^\alpha_a \mapsto t^{-1} t_{(1,2,\ldots,m)j_{mn}}, \quad a = 1, \ldots, n; \quad \alpha = 1, \ldots, m$$

admits a unique extension up to a homomorphism of covariant $*$-algebras $i : \text{Fun}(\mathbb{U})_q \to \text{Fun}(\tilde{X})_{q,x}$.

**Remark 7.7.** We have extended the structure of $U_q\mathfrak{sl}_N$-module algebra from $\mathbb{C}[\mathfrak{sl}_N]_q$ onto $\text{Fun}(\tilde{X})_{q,x}$. In a similar way, the structure of $U_q\mathfrak{so}^p_N$-module algebra admits an extension as well (see section 5), as one can observe from the following analogues of (6.11), (6.12):

$$(H_m \otimes 1) f(x) = 0, \quad (X^+_m \otimes 1) f(x) = \frac{f(q^{-2}x) - f(x)}{q^{-2}x - x} (X^+_m \otimes 1) x,$$

$$(X^-_m \otimes 1) f(x) = (X^-_m \otimes 1) x \cdot \frac{f(q^{-2}x) - f(x)}{q^{-2}x - x},$$

$$(H_j \otimes 1) f(x) = (X^-_j \otimes 1) f(x) = 0 \quad \text{for} \quad j \neq m.$$
\((X_m \otimes 1)f_0 = -\frac{1}{q^2 - 1} \cdot ((X_m^- \otimes 1)x)f_0.\)

Turn to a construction of quantum analogues for homogeneous spaces \(S(U_n \times U_m) \backslash SU_{nm}\) and \((SU_n \times SU_m) \backslash SU_{nm}\).

Remind that a vector \(v \in V\) is called an \(A\)-invariant if \(V\) is a module over the Hopf algebra \(A\) with a counit \(\varepsilon\) and \(\alpha \cdot v = \varepsilon(\alpha) \cdot v\) for all \(\alpha \in A\).

Let \(U_q\mathfrak{sl}_m \times \mathfrak{gl}_n\) be the Hopf subalgebra generated by \(K_{m,1}^\pm\), \(\{K_j^\pm, E_j, F_j\}_{j \neq m}\) and \(U_q\mathfrak{sl}_m \otimes U_q\mathfrak{sl}_n \subset U_q\mathfrak{sl}_m \times \mathfrak{gl}_n\) – a Hopf subalgebra generated by \(\{K_j^\pm, E_j, F_j\}_{j \neq m}\).

Consider the covariant \(*\)-algebra \(\mathcal{P}(\hat{X})_q\). Denote the subalgebra of its \(U_q\mathfrak{sl}_m \times \mathfrak{gl}_n\)-invariants by \(\mathcal{P}(\hat{X})_q\), and the subalgebra of \(U_q\mathfrak{sl}_m \otimes U_q\mathfrak{sl}_n\)-invariants by \(\mathcal{P}(\hat{X})_q\). Evidently, they are covariant \(*\)-algebras and

\[\mathcal{P}(\hat{X})_q = \{ f \in \mathcal{P}(\hat{X})_q \mid (H_n \otimes 1)f = 0 \}.\]

\(\mathcal{P}(\hat{X})_q \), \(\mathcal{P}(\hat{X})_q\) substitute the algebras of polynomial functions on the homogeneous spaces

\[X = S(U_m \times U_n) \backslash \hat{X} \sim S(U_n \times U_m) \backslash SU_{nm},\]
\[\hat{X} = (SU_m \times SU_n) \backslash \hat{X} \sim (SU_n \times SU_m) \backslash SU_{nm}.\]

A replacement of \(\mathcal{P}(\hat{X})_q\) in the above observations by any of the following covariant \(*\)-algebras \(\mathcal{P}(\hat{X})_q, \mathcal{P}(\hat{X})_q, \mathcal{P}(\hat{X})_q, D(\hat{X})_q\), allows one to produce the covariant \(*\)-algebras \(\mathcal{P}(\hat{X})_q \subset \mathcal{P}(\hat{X})_q, \mathcal{P}(\hat{X})_q \subset \mathcal{P}(\hat{X})_q, \mathcal{P}(\hat{X})_q \subset \mathcal{P}(\hat{X})_q, D(\hat{X})_q \subset D(\hat{X})_q\). (In every pair a smaller subalgebra is distinguished from a larger one by the equation \((H_n \otimes 1)f = 0\).)

Note that the element \(x\) is a \(U_q\mathfrak{sl}_m \times \mathfrak{gl}_n\)-invariant. Therefore the algebras \(\mathcal{P}(\hat{X})_q \subset \mathcal{P}(\hat{X})_q\) are derivable from \(\mathcal{P}(\hat{X})_q \subset \mathcal{P}(\hat{X})_q\) via a localization with respect to the multiplicative system \(x^\mathbb{N}\).

8 Canonical isomorphism \(D(\mathbb{U})_q \simeq D(\mathcal{X})_q\)

Section 7 contains a construction of a morphism of covariant \(*\)-algebras \(i : \mathcal{P}(\mathbb{U})_q \rightarrow \mathcal{P}(\hat{X})_q\). Our immediate purpose is to prove its injectivity and to describe the image of \(D(\mathbb{U})_q\) with respect to the embedding into \(\mathcal{P}(\hat{X})_q\).

Proposition 8.1 The least subalgebra in \(\mathcal{P}(\hat{X})_q\) which contains \(x\) and \(if, f \in \mathcal{P}(\text{Mat}_{nm})_q\), is \(\mathcal{P}(X)_{q,x}\).

**Proof.** The least subalgebra of \(\mathcal{P}(\hat{X})_q\) which contains \(t, t^{-1}, t^*, t^{* - 1}\) and \(if, f \in \mathcal{P}(\text{Mat}_{nm})_q\), is a \(U_q\mathfrak{sl}_N\)-module subalgebra. Hence it coincides with \(\mathcal{P}(\hat{X})_q\), by a virtue of lemma 8.2 to be proved below. Thus every \(\psi \in \mathcal{P}(\hat{X})_q\) can be written in the form \(\psi = \sum_{j=1}^\infty \psi_j t^j + \psi_0 + \sum_{j=1}^\infty \psi_{-j} t^{*j}\) with coefficients \(\{\psi_j\}_{j=-\infty}^\infty\)
from a subalgebra $F \subset \text{Pol}(X)_{q,x}$ spanned by $x$ and $if$, $f \in \text{Pol}(\text{Mat}_{mn})_q$. The above representation for $\psi$ may be treated as an expansion of $\psi$ in eigenvectors of $K_{m}^{\pm 1} \otimes 1$. On the other hand, $\text{Pol}(X)_{q,x} = \{f \in \text{Pol}(\hat{X})_{q,x} \mid (K_{m}^{\pm 1} \otimes 1) f - f = 0\}$. Thus, $\psi \in \text{Pol}(X)_{q,x}$ implies $\psi_{j} = 0$ for all $j \neq 0$. That is, $\psi \in \text{Pol}(X)_{q,x}$ implies $\psi \in F$. □

**Lemma 8.2** The least $U_{q}\mathfrak{sl}_{N}$-module subalgebra of $\text{Pol}(\hat{X})_{q}$ which contains both $t_{\{1,2,\ldots,m\}\{n+1,n+2,\ldots,N\}}$ and $t_{\{m+1,m+2,\ldots,N\}\{1,2,\ldots,n\}}$ is $\text{Pol}(\hat{X})_{q}$.

**Proof.** Consider the subalgebra $\hat{F}_{+} \subset \mathbb{C}[\text{SL}_{N}]_{q}$ generated by $t_{ij}$ with $i \leq m$. Just as in the case $q = 1$, it has simple $U_{q}\mathfrak{sl}_{m}^{\text{op}} \otimes U_{q}\mathfrak{sl}_{N}$-isotypical components whose generators are $t_{1N}^{a_{1}} \cdot \cdots \cdot \left(t_{\{1,2\}\{N-1\}}^{a_{2}} \cdots \cdot \left(t_{\{1,2,\ldots,m\}\{n+1,n+2,\ldots,N\}}^{a_{m}} \right)^{a_{m}} \cdot a_{1}, a_{2}, \ldots , a_{m} \in \mathbb{Z}^{+}$.

(A classical result of theory of invariants is applied here (see, for example, [4, 24]), together with the fact that the dimensionality of a simple module with highest weight $\lambda$ is independent of $q \in (0, 1]$ (see [3])). The above monomials are $U_{q}\mathfrak{sl}_{m}^{\text{op}}$-invariant if and only if $a_{1}, a_{2}, \ldots , a_{m-1} = 0$. Hence generators of the $U_{q}\mathfrak{sl}_{N}$-module

$$
\hat{F}_{+} = \{ f \in \hat{F}_{+} \mid (E_{j} \otimes 1) f = (F_{j} \otimes 1) f = (K_{j}^{\pm 1} \otimes 1) f - f = 0, j = 1, \ldots , m - 1 \}
$$

are the vectors \( \left(t_{\{1,2,\ldots,m\}\{n+1,n+2,\ldots,N\}}^{a_{m}} \right)^{a_{m}} , a_{m} \in \mathbb{Z}^{+} \). That is, $U_{q}\mathfrak{sl}_{N}$-module algebra $\hat{F}_{+}$ is generated by $t_{\{1,2,\ldots,m\}\{n+1,n+2,\ldots,N\}}^{a_{m}}$.

Consider the subalgebra $\hat{F}_{-} \subset \mathbb{C}[\text{SL}_{N}]_{q}$ generated by $t_{ij}$, $i > m$, and the subalgebra $\hat{F}_{-} = \hat{F}_{-} \cap \text{Pol}(\hat{X})_{q}$. Just as in the case of $\hat{F}_{+}$, one can prove that $t_{\{m+1,m+2,\ldots,N\}\{1,2,\ldots,n\}}^{a_{m}}$ generates $\hat{F}_{-}$ as a $U_{q}\mathfrak{sl}_{N}$-module algebra. What remains is to apply $\mathbb{C}[\text{SL}_{N}]_{q} = \hat{F}_{+} \cdot \hat{F}_{-}$, $\text{Pol}(\hat{X})_{q} = \hat{F}_{+} \cdot \hat{F}_{-}$. □

**Theorem 8.3** $i : D(\mathbb{U}) \to D(X)_{q}$ is an isomorphism.

**Proof.** Evidently, $D(\hat{X})_{q} = \hat{F}_{+} \cdot f_{0} \cdot \hat{F}_{-}$. Thus, by proposition 7.3, $D(\hat{X})_{q} = \hat{F}_{+} \cdot f_{0} \cdot \hat{F}_{-}$. On the other hand, by a virtue of lemma 8.2, $\hat{F}_{+}$ is a subalgebra generated by the elements $t_{\{1,2,\ldots,m\}\{I\}}^{a_{m}} ; \text{card}(I) = m$, while $\hat{F}_{-}$ is a subalgebra generated by the elements $t_{\{m+1,m+2,\ldots,N\}\{J\}}^{a_{m}} ; \text{card}(J) = n$. Hence, in view of $xf_{0} = f_{0}x = f_{0}$, one has

$$
D(\hat{X})_{q} = \sum_{j>0} i^{j} D(X)_{q} + D(X)_{q} + \sum_{j>0} D(X)_{q} t^{j} = D(X)_{q} = (i\mathbb{C}[\text{Mat}_{mn}]_{q}) f_{0} \left(i\mathbb{C}[\text{Mat}_{mn}]_{q}\right) = iD(\mathbb{U}).
$$

That is, we have proved that $i$ is onto. To prove its injectivity, introduce a vector space $\mathcal{H} = \mathbb{C}[\text{Mat}_{mn}]_{q} \cdot f_{0} \subset D(\mathbb{U})_{q}$, together with a representation $\Theta$ of $\text{Fun}(\mathbb{U})_{q}$ in $\mathcal{H}$, given by $\Theta(\psi) : f \mapsto \psi f$, $\psi \in \text{Fun}(\mathbb{U})_{q}$, $f \in \mathcal{H}$.

Equip $\mathcal{H}$ with such a sesquilinear form (scalar product) that

$$
(\psi_{1}f_{0}, \psi_{2}f_{0}) = f_{0} \psi_{2}^{\ast} \psi_{1}f_{0}, \quad \psi_{1}, \psi_{2} \in \mathbb{C}[\text{Mat}_{mn}]_{q}.
$$

This is well defined, as one can see from $f_{0} \cdot \text{Pol}(\text{Mat}_{mn})_{q}f_{0} = \mathbb{C}f_{0}$.
Lemma 8.4 The scalar product (8.1) in $H$ is positive definite and $\Theta$ is a $*$-representation of $\text{Pol} (\text{Mat}_{mn})_q$ in the pre-Hilbert space $H$.

Proof. Remind that the space $\tilde{H}$ of the $*$-representation $\Pi$ (see Appendix 2) is a graded vector space: $\tilde{H} = \bigoplus_{j=0}^{\infty} \tilde{H}_j$, and $\tilde{H}_0 = \mathbb{C} \cdot e_0$. For all $v \in \tilde{H}$ and all $a, \alpha$, one has
\[
\deg(\Pi(z^\alpha) v) = \deg v + 1, \quad \deg(\Pi(z^\alpha)^* v) = \deg v - 1
\]
by a virtue of (6.10). Hence, $(\Pi(\psi)e_0, e_0) = f_0 \psi f_0$ for all $\psi \in \text{Pol}(\text{Mat}_{mn})_q$. Thus, the map\[
j : H \to \tilde{H}, \quad j : \Theta(\psi)f_0 \mapsto \Pi(\psi)e_0, \quad \psi \in \mathbb{C}[\text{Mat}_{mn}]_q
\]
is well defined and intertwines the scalar products.

Now apply propositions A2.2.3 and 8.1 to conclude that the map $j$ is onto. Hence $j(\mathbb{C}[\text{Mat}_{mn}]_q \cdot f_0) = H_k$ for all $k \in \mathbb{Z}_+$. On the other hand, $\dim \tilde{H}_k = \dim \mathbb{C}[\text{Mat}_{mn}]_q k$. Therefore, $j$ is one-to-one, and the representations $\Theta$ and $\Pi$ are unitarily equivalent. \[\Box\]

Remark 8.5. There exists a unique extension of $\tilde{\Pi}$ onto $\text{Fun}(\tilde{X})_q$ such that $\tilde{\Pi}(f_0)$ is the projection onto $\tilde{H}_0$ with kernel $\bigoplus_{k \neq 0} \tilde{H}_k$. One can observe from the proof of lemma 8.4 that the representations $\Theta$ and $\Pi = \tilde{\Pi} \circ i$ of $\text{Fun}(U)_q$ are unitarily equivalent.

Turn back to proving the injectivity of $i$. By a virtue of remark 8.5, it suffices to prove that the homomorphism $\Theta : D(U)_q \to \text{End}(H)$ is an embedding. The linear map\[
m : \mathbb{C}[\text{Mat}_{mn}]_q \cdot f_0 \otimes f_0 \cdot \mathbb{C}[\text{Mat}_{mn}]_q \to D(U)_q, \quad m : f_1 \otimes f_2 \mapsto f_1 f_2
\]
is one-to-one, as one can easily deduce via an application of the well known diamond lemma [3] to producing monomial bases in the vector spaces $\mathbb{C}[\text{Mat}_{mn}]_q \cdot f_0$, $f_0 \cdot \mathbb{C}[\text{Mat}_{mn}]_q$, $D(U)_q$. Thus,
\[
D(U)_q \simeq (\mathbb{C}[\text{Mat}_{mn}]_q \cdot f_0) \otimes (f_0 \cdot \mathbb{C}[\text{Mat}_{mn}]_q) \simeq H \otimes H^*,
\]
and the representation $\Theta : D(U)_q \to \text{End}(H)$ reduces to the canonical linear map $H \otimes H^* \to \text{End}(H)$. What remains is to observe that this map is an embedding. The theorem 8.3 is proved. \[\Box\]

Remark 8.6. One can easily deduce a description of the image $i(D(U)_q)$ in $\text{End}(H)$. Equip $H$ with a gradation\[
H = \bigoplus_{k=0}^{\infty} H_k, \quad H_k = \mathbb{C}[\text{Mat}_{mn}]_{q,k} \cdot f_0,
\]
and let $\text{End}(H)_f$ stand for the algebra of finite dimensional finite degree operators in $H$. Since $\dim H_k < \infty$, $k < \infty$, one has $\text{End}(H)_f \simeq H \otimes H^*$. Hence, $\Theta$ provides a 'canonical' isomorphism of algebras $D(U)_q \to \text{End}(H)_f$. 

\[\text{That is, } (v, v) > 0 \text{ for all } v \neq 0.\]
Let $P_k$ be the projection in $\mathcal{H}$ onto the homogeneous component $\mathcal{H}_k = \mathbb{C}[\text{Mat}_{mn}]_{q,k} \cdot f_0$ with kernel $\bigoplus_{j \neq k} \mathbb{C}[\text{Mat}_{mn}]_{q,j} \cdot f_0$, and let $\mathbb{C}[\text{Mat}_{mn}]_{q,k} \cdot \mathbb{C}[\text{Mat}_{mn}]_{q,-l}$ be the linear span of
\[
\{ f_+ \cdot f_- \in \text{Pol}(\text{Mat}_{mn})_q | f_+ \in \mathbb{C}[\text{Mat}_{mn}]_q, f_- \in \mathbb{C}[\text{Mat}_{mn}]_q \}.
\]

**Lemma 8.7** For all $k, l \in \mathbb{Z}_+$, the map
\[
\mathbb{C}[\text{Mat}_{mn}]_{q,k} \cdot \mathbb{C}[\text{Mat}_{mn}]_{q,-l} \to \text{Hom}(\mathcal{H}_l, \mathcal{H}_k); \quad f \mapsto P_k \Theta(f)|_{\mathcal{H}_l}
\]
is one-to-one.

**Proof.** Both $\mathcal{H}_k, \mathcal{H}_l$ are finite dimensional Hilbert spaces. Arguing just as in the proof of theorem 8.3, we get
\[
\mathbb{C}[\text{Mat}_{mn}]_{q,k} \cdot \mathbb{C}[\text{Mat}_{mn}]_{q,-l} \cong \mathbb{C}[\text{Mat}_{mn}]_{q,k} \otimes \mathbb{C}[\text{Mat}_{mn}]_{q,-l} \cong \\
\mathbb{C}[\text{Mat}_{mn}]_{q,k} \cdot f_0 \otimes f_0 \cdot \mathbb{C}[\text{Mat}_{mn}]_{q,-l} \cong \mathcal{H}_k \otimes \mathcal{H}_l \cong \text{Hom}(\mathcal{H}_l, \mathcal{H}_k).
\]
What remains is to use the fact that the resulting linear map
\[
\mathbb{C}[\text{Mat}_{mn}]_{q,k} \cdot \mathbb{C}[\text{Mat}_{mn}]_{q,-l} \ni f \mapsto f_0 \cdot (\psi_+ f_0, f_0 - f_0)
\]
coincides with the operator described in the statement of this lemma. (In fact, let $f = f_+ f_+, f_+ \in \mathbb{C}[\text{Mat}_{mn}]_{q,k}, f_- \in \mathbb{C}[\text{Mat}_{mn}]_{q,l}$. Then for all $\psi_+ \in \mathbb{C}[\text{Mat}_{mn}]_{q,k}, \psi_- \in \mathbb{C}[\text{Mat}_{mn}]_{q,l}$ one has
\[
(\psi_+ f_0, P_k \Theta(f_+ f_+) \psi_- f_0) = (\psi_+ f_0, f_+ f_- \psi_- f_0) = (f_+ \psi_+ f_0, f_- \psi_- f_0) = \\
= ((\psi_+ f_0, f_0) f_0, (\psi_- f_0, f_0) f_0) = (\psi_+ f_0, f_0) \cdot (\psi_- f_0, f_0).
\]

**Proposition 8.8** The homomorphism $\Theta : \text{Pol}(\text{Mat}_{mn})_q \to \text{End}(\mathcal{H})$ is an embedding.

**Proof.** Equip the vector space $\text{Pol}(\text{Mat}_{mn})_q$ with a bigradation
\[
\text{Pol}(\text{Mat}_{mn})_q = \bigoplus_{k,l=0}^{\infty} \mathbb{C}[\text{Mat}_{mn}]_{q,k} \cdot \mathbb{C}[\text{Mat}_{mn}]_{q,-l}
\]
as one can easily verify, this is well defined). We need also a standard partial order relation on $\mathbb{Z}_+^2$:
\[
(k_1, l_1) \leq (k_2, l_2) \iff k_1 \leq k_2 \text{ and } l_1 \leq l_2.
\]
Assume that our statement is wrong and $\Theta(f) = 0$ for some $f \in \text{Pol}(\text{Mat}_{mn})_q, f \neq 0$. Consider a homogeneous component $f_{k,l}$ of $f$ with minimal bidegree $(k,l)$.
(Such homogeneous component certainly exists, but it is not unique for a given $f \in \text{Pol}(\text{Mat}_{mn})_q$). Let $\mathcal{H}_j = \mathbb{C}[\text{Mat}_{mn}]_{q,j} \cdot f_0$, and $P_k : \mathcal{H} \to \mathcal{H}_k$ be the projection onto $\mathcal{H}_k$ with kernel $\bigoplus_{j \neq k} \mathcal{H}_j$. Since $f_{k,l}$ is of a minimal bidegree, one has $P_k \Theta(f_{k,l})|_{\mathcal{H}_l} = P_k \Theta(f)|_{\mathcal{H}_l} = 0, f_{k,l} \neq 0$, which contradicts the statement of lemma 8.7. \qed
Corollary 8.9  

i) The morphism $i : \text{Pol}(\text{Mat}_{mn})_q \to \text{Fun}(\tilde{X})_{q,x}$ is an embedding.

ii) The restriction of $\Pi$ onto the subalgebra $\text{Pol}(X)_{q,x}$ is a faithful $*$-representation of this subalgebra.

Proof. The first statement follows from the equivalence of $\Theta$ and $\Pi = \bar{\Pi} \circ i$, and the faithfulness of $\Theta$ established in proposition 8.8.

Now turn to proving the second statement. Suppose that $\psi \in \text{Pol}(X)_{q,x}$ and $\bar{\Pi}(\psi) = 0$. By proposition 8.1 and the relations (6.10), there exist such elements $\psi_1, \psi_2, \ldots, \psi_M \in \text{Pol}(\text{Mat}_{mn})_q$ that $\psi = \sum_{k=0}^{M} i(\psi_k) x^k$. In [17] an element $y \in \text{Pol}(\text{Mat}_{mn})_q$ is found with the property $iy = x^{-1}$. Hence $\psi = i(\Psi) x^M$ with $\Psi = \sum_{k=0}^{M} \psi_k y^{M-k}$. It follows from $\bar{\Pi}(\psi) = 0$ that $\bar{\Pi}(\Pi) \cdot \left(\bar{\Pi}(x)^{M}\right) = 0$. Observe that $\bar{\Pi}(x)$ is invertible, and so $\bar{\Pi}(\Psi) = 0, \Psi = 0, \psi = 0$. $\blacksquare$

Proposition 8.10  
The morphism of covariant algebras $i : \text{Fun}(U)_q \to \text{Fun}(\tilde{X})_{q,x}$ is an embedding.

Proof. It was proved before that $i$ is an embedding while restricted onto the subalgebras $D(U)_q$ and $\text{Pol}(\text{Mat}_{mn})_q$.

Let $i(f_1 + f_2) = 0$, $f_1 \in D(U)_q$, $f_2 \in \text{Pol}(\text{Mat}_{mn})_q$. By a virtue of Remark 8.6, $\Theta(f_1) H \subset \bigoplus_{j=0}^{M-1} H_j$ for some $M \in \mathbb{N}$. It follows that $\Theta(f_2) H \subset \bigoplus_{j=0}^{M-1} H_j$. Hence all the elements of $\mathbb{C}[\text{Mat}_{mn}]_{q,-M} f_2$ are in the kernel of $\Theta$. By proposition 8.8, $\mathbb{C}[\text{Mat}_{mn}]_{q,-M} f_2 = 0$. We claim this implies $f_2 = 0$. In fact, the invertibility of the linear maps $R_{UU}, R_{VV}$ for all $q \in (0,1)$ allows one to apply diamond lemma to prove that

$$\text{Pol}(\text{Mat}_{mn})_q = \bigoplus_{k,l=0}^{\infty} \mathbb{C}[\text{Mat}_{mn}]_{q,-l} \cdot \mathbb{C}[\text{Mat}_{mn}]_{q,k},$$

via producing bases of lexicographically ordered monomials in each of the subspaces $\mathbb{C}[\text{Mat}_{mn}]_{q,-l}, \mathbb{C}[\text{Mat}_{mn}]_{q,k}$. If $\psi$ is the first (lowest) element of such basis in $\mathbb{C}[\text{Mat}_{mn}]_{q,-M}$, then obviously $\psi f_2 = 0$ implies $f_2 = 0$. $\blacksquare$

9  An invariant integral

Consider the Hopf subalgebra $U_q \mathfrak{p}_+ \subset U_q \mathfrak{sl}_N$ generated by $K_n^{\pm 1}$, $E_n$, and $K_j^{\pm 1}$, $E_j$, $F_j$, $j \neq n$. By a virtue of the relation $E_n f_0 = -\frac{q^{1/2}}{1-q^2} z^m f_0$ from section 7, one has

Proposition 9.1  
$\mathcal{H}$ is a $U_q \mathfrak{p}_+$-submodule of the $U_q \mathfrak{p}_+$-module $D(U)_q$. 

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Let \( \Gamma \) stand for the associated representation of \( U_q\mathfrak{p}_+ \) in \( \mathcal{H} \).

We use in what follows the standard scalar product in the Cartan subalgebra \( \mathfrak{h} \):

\[
(H_i, H_j) = \begin{cases}
2 & , i = j \\
-1 & , |i - j| = 1 \\
0 & , \text{otherwise}
\end{cases}
\]

and the element \( \tilde{\rho} \in \mathfrak{h} \) given by \( (\tilde{\rho}, H_i) = 1, i = 1, \ldots, N - 1 \).

**Theorem 9.2** The linear functional

\[
\nu : D(\mathbb{U})_q \to \mathbb{C}, \quad \int f d\nu \overset{\text{def}}{=} \text{tr}(\Theta(f)\Gamma(e^{h\tilde{\rho}}))
\]

is well defined, \( U_q\mathfrak{sl}_N \)-invariant and positive \( ^4 \).

**Proof.** One can deduce that \( \nu \) is well defined and positive from the results of section 8 since \( \mathcal{H} \) is a pre-Hilbert space, the \( * \)-representation \( \Theta \) is faithful, and \( \Theta(f), f \in D(\mathbb{U})_q \), are finite dimensional finite degree operators. The proof of \( U_q\mathfrak{sl}_N \)-invariance of \( \nu \) is just the same as that in the special case \( m = n = 1 \) \( ^{13} \).

Specifically, by a virtue of \( \square \) for \( F = D(\mathbb{U}), U_q\mathfrak{sl}_N \)-invariance of this integral follows from its \( U_q\mathfrak{p}_+ \)-invariance and its realness: \( \int f* d\nu = \overline{\int f d\nu}, f \in D(\mathbb{U})_q \). So what remains is to prove the \( U_q\mathfrak{p}_+ \)-invariance of the integral we deal with. It follows from the covariance of \( D(\mathbb{U})_q \) that the linear map \( D(\mathbb{U})_q \otimes \mathcal{H} \to \mathcal{H}, f \otimes v \mapsto f v, f \in D(\mathbb{U})_q, v \in \mathcal{H} \), is a morphism of \( U_q\mathfrak{p}_+ \)-modules. Hence the associated linear map \( D(\mathbb{U})_q \to \mathcal{H} \otimes \mathcal{H}^* \) is also a morphism of \( U_q\mathfrak{p}_+ \)-modules (see \( ^{13} \) proposition 1.2). So one needs to use the fact that the square of the antipode \( S \) is an inner automorphism \( S^2(\xi) = e^{h\xi} \cdot e^{-h\tilde{\xi}}, \xi \in U_q\mathfrak{sl}_N \), and to apply the general argument given below (it is well known from the theory of Hopf algebras \( ^{14} \)) to the \( U_q\mathfrak{p}_+ \)-module \( \mathcal{H} \).

Let \( A \) be a Hopf algebra and \( \Gamma \) its representation in a vector space \( V \). Then \( V, V^*, V^{**}, \ldots \) are \( A \)-modules, while the standard embedding \( i_0 : V \hookrightarrow V^{**} \), is not in general a morphism of \( A \)-modules (unless \( S^2 = \text{id} \)). Let \( u \in A \) be such that \( S^2(\xi) = u \cdot \xi \cdot u^{-1} \) for all \( \xi \in A \). Then the embedding \( i_1 = i_0 \Gamma(u) : V \hookrightarrow V^{**} \) is a morphism of \( A \)-modules since \( i_0 S^2(\xi)v = \xi i_0 v \) for all \( v \in V, \xi \in A \).

We observe that the composition of the linear map \( i_1 \otimes \text{id} : V \otimes V^* \to V^{**} \otimes V^* \) and the canonical pairing \( V^{**} \otimes V^* \to \mathbb{C} \) is a morphism of \( A \)-modules, i.e. an invariant integral. This invariant integral can be written in the form \( \text{tr}_q(A) = \text{tr}(A \Gamma(u)), A \in V \otimes V^* \subset \text{End}_\mathbb{C}(V) \) via an application of the canonical embedding \( V \otimes V^* \hookrightarrow \text{End}_\mathbb{C}(V) \). \( \square \)

To conclude, we apply theorem 9.2 for producing a positive invariant integral on the quantum principal homogeneous space.

A passage from functions on \( X \) to functions on \( X \) could be done via averaging with respect to an action of the compact group \( S(U_m \times U_n) \). We do this in the quantum case.

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\(^4\)Positive in the sense that \( \int f d\nu > 0 \) for all non-zero non-negative elements of the \( * \)-algebra \( D(\mathbb{U})_q \).
Consider the bilateral ideal \( J \subset \mathbb{C}[SL_m] \) generated by \( t_{kl} \) with \( k \leq m \) \& \( l > m \) or \( k > m \) \& \( l \leq m \), and the canonical onto morphism

\[
j : \mathbb{C}[SL_m] \to \mathbb{C}[S(GL_m \times GL_n)],
\]

with \( \mathbb{C}[S(GL_m \times GL_n)] = \mathbb{C}[SL_m]/J \). Introduce the notation

\[
\mathbb{C}[S(U_m \times U_n)] = (\mathbb{C}[S(GL_m \times GL_n)],*)
\]

for the 'algebra of regular functions on the compact quantum group \( S(U_m \times U_n) \).\footnote{It is easy to prove that \( J^* \subset J \). For example, obviously, \( t_{1N}^* \in J \), \( t_{N1}^* \in J \), and for other generators of \( J \) the covariance argument is applicable.}

**Lemma 9.3** The composition \( \tilde{\Delta} \) of homomorphisms \( \Delta : \mathbb{C}[SL_m] \to \mathbb{C}[SL_m] \otimes \mathbb{C}[SL_m], j \otimes \text{id} : \mathbb{C}[SL_m] \otimes \mathbb{C}[SL_m] \to \mathbb{C}[S(GL_m \times GL_n) \otimes \mathbb{C}[SL_m] \) is a homomorphism of *-algebras \( \tilde{\Delta} : \text{Pol}(X) \to \mathbb{C}[S(U_m \times U_n)] \otimes \text{Pol}(X) \).

**Proof.** By the definition of involution in the *-algebra \( \mathbb{C}[SU_n] \), \( j \otimes \text{id}(\Delta) \) is a homomorphism of *-algebras \( \mathbb{C}[SU_n] \to \mathbb{C}[S(U_m \times U_n)] \otimes \mathbb{C}[SU_n] \). What remains is to apply the relations (6.6), (6.7).

Extend \( \tilde{\Delta} \) up to a homomorphism of *-algebras \( \tilde{\Delta} : \text{Fun}(X) \to \mathbb{C}[S(U_m \times U_n)] \otimes \text{Fun}(X) \) via \( \tilde{\Delta}f_0 = 1 \otimes f_0 \). (The existence and uniqueness of such extension follows from the definitions of \( f_0 \) and the *-algebra \( \text{Fun}(X) \).)

Let \( \tilde{\mu} : \mathbb{C}[S(U_m \times U_n)] \to \mathbb{C} \) be the invariant integral on the compact quantum group \( S(U_m \times U_n) \) normalized by \( \int_{S(U_m \times U_n)} 1d\tilde{\mu} = 1 \) \footnote{It is easy to prove that \( J^* \subset J \). For example, obviously, \( t_{1N}^* \in J \), \( t_{N1}^* \in J \), and for other generators of \( J \) the covariance argument is applicable.}, and \( \tilde{\nu} : D(X) \to \mathbb{C} \) an invariant integral transferred from \( \nu : D(U) \to \mathbb{C} \), \( \int_U f d\nu = \text{tr}(\Theta(f)\Gamma(e^{h\rho})) \) via the canonical isomorphism \( D(U) \simeq D(X) \).

**Proposition 9.4** The linear functional \( (\tilde{\mu} \otimes \tilde{\nu})\tilde{\Delta} : D(X) \to \mathbb{C} \) is positive and \( U_q\mathfrak{sl}_N \)-invariant.

**Proof.** The scalar product \( (f_1, f_2) = \tilde{\mu} \otimes \tilde{\nu}(f_2^* f_1) \) in \( \mathbb{C}[S(U_m \times U_n)] \otimes D(X) \) is positive definite, as one can see from theorem \( \text{[1,2]} \) and the orthogonality relations for a compact quantum group \( \text{[4]} \). Hence \( \tilde{\mu} \otimes \tilde{\nu}(f^* f) > 0 \) for \( f \neq 0 \), and the positivity of the linear functional \( (\tilde{\mu} \otimes \tilde{\nu})\tilde{\Delta} \) now follows from the injectivity of \( \tilde{\Delta} : D(X) \to \mathbb{C}[S(U_m \times U_n)] \otimes D(X) \). The \( U_q\mathfrak{sl}_N \)-invariance follows from the fact that the 'averaging operator' \( (\tilde{\mu} \otimes \text{id})\tilde{\Delta}D(X) \to D(X) \) is a morphism of \( U_q\mathfrak{sl}_N \)-modules. \( \square \)
10 Concluding notes

A number of commutation relations obtained in sections 1 - 6 are known within a different approach to function theory on quantum complex manifolds [5]. The conceptual equivalence of both approaches is due to the isomorphism of quantum homogeneous spaces $U$ and $X$ (see section 8).

There is a wide class of quantum homogeneous spaces for which our explicit formula for invariant integral is plausible. All the related covariant algebras are derivable via a factorization from the quantum universal enveloping algebra $U_q\mathfrak{sl}_N$ equipped with the adjoint action $[4]$. As one can observe already in the case of quantum disc (see [16]), the algebras of functions considered in the present work are not in the above wide class of covariant algebras, although being derivable from those by passage to a limit. That kind of passage to a limit was investigated before by Berezin within his approach to quantization of bounded symmetric domains $[1, 2]$.

In the first six sections of the present work, $\mathbb{C}(q^{1/s})$ worked as a ground field, with $s$ being a natural number whose value was not specified precisely. It follows from the subsequent descriptions of the covariant algebras $\text{Pol} (\text{Mat}_{mn})_q$, $\text{Fun}(\tilde{X})_q$ (in terms of their generators and relations) that the ground field could be chosen to be $\mathbb{C}(q^{1/2})$.

Appendix 1. Universal R-matrix and quantum Weyl group

The subject of this appendix is to remind some well known results of quantum group theory. We follow S. Levendorskii and Ya. Soibelman [10, 11]. A large part of these results were independently obtained by A. Kirillov and N. Reshetikhin [9, 4]. A more general and rather complete exposition of the background on quantum group theory can be found in a remarkable survey of M. Rosso [14].

Consider a reduced decomposition $w_0 = s_{i_1} \cdot s_{i_2} \ldots s_{i_M} = N(N - 1)/2$, of the longest permutation $w_0 = (N, N - 1, \ldots, 2, 1) \in S_N$. Our purpose is to associate to each such reduced decomposition a linear order relation on the set of positive roots of the Lie algebra $\mathfrak{sl}_N$, and then a basis in the vector space $U_q\mathfrak{sl}_N$. Remind also that the simple roots $\alpha_i, i = 1, \ldots, N - 1$, are given by $\alpha_i(H_i) = a_{ij}, i, j = 1, \ldots, N - 1$, with $(a_{ij})$ being the Cartan matrix (2.1). We use the following linear order relation on the set of positive roots:

$$\beta_1 = \alpha_1, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \beta_3 = s_{i_1}s_{i_2}(\alpha_{i_3}), \quad \ldots, \quad \beta_M = s_{i_1} \ldots s_{i_{M-1}}(\alpha_{i_M}).$$

The work [10] associates to the generators $S_i, i = 1, \ldots, N - 1$, of the Weyl group the automorphisms $T_i$ of $U_q\mathfrak{sl}_N$, which differ inessentially from Lusztig automorphisms (see [4, 14]). Note that, in particular,

$$T_i(K_j) = \begin{cases} K_j^{-1}, & i = j \\ K_iK_j, & |i - j| = 1 \\ K_j, & \text{otherwise} \end{cases}$$  (A1.1)
Associate to each simple root $\alpha_i$ the generators $E_i, F_i$ of $U_q\mathfrak{sl}_N$. The above map defined on the family of simple roots is extendable onto the set of all positive roots: $E_{\beta_s} = T_i T_{i+1} \ldots T_{i-s-1}(E_i)$, $F_{\beta_s} = T_i T_{i+1} \ldots T_{i-s-1}(F_i)$.

**Proposition A1.1.** $E_{\beta_1}^{k_1} \cdot E_{\beta_2}^{k_2} \cdot \ldots \cdot E_{\beta_M}^{k_M}, (k_1, k_2, \ldots, k_M) \in \mathbb{Z}_+^M$, constitute a basis in the vector space $U_q \mathfrak{M}_+$. $F_{\beta_1}^{j_1} \cdot F_{\beta_2}^{j_2} \cdot \ldots \cdot F_{\beta_M}^{j_M}, (j_1, j_2, \ldots, j_M) \in \mathbb{Z}_+^M$, constitute a basis in the vector space $U_q \mathfrak{M}_-$. $T_i$ since $\mathcal{K} \mathcal{M} \mathcal{N} \mathcal{E} \mathcal{V} \mathcal{S} \mathcal{K}$ decompositions for the element $\mathcal{V} \mathcal{S} \mathcal{K} \mathcal{A}$, we are about to apply corollary A1.2 to constructing the bases of the vector spaces $\mathcal{V} \mathcal{S} \mathcal{K} \mathcal{A}$, hence $\mathcal{K} \mathcal{V} \mathcal{S} \mathcal{K}$ is determined by the element $\mathcal{H} \mathcal{D} \mathcal{N} \mathcal{D} \mathcal{E} \mathcal{D} \mathcal{M} \mathcal{A} \mathcal{L} \mathcal{A}$ defined in section 2. Hence

$$T_i(U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m) = U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m, \quad i \neq n,$$

since $T_i U_q \mathfrak{M} \subset U_q \mathfrak{M}$, and

$$U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m \cap U_q \mathfrak{M} = \{ \xi \in U_q \mathfrak{M} | \deg(\xi) = 0 \}. \quad (A1.3)$$

Impose the notation $M' = M - mn - \frac{m(m-1)}{2} + \frac{n(n-1)}{2}$. It follows from (A1.3) that $\deg(\bar{F}_{\beta_j}) = 0$ for $j \leq M'$. Just in the same way as in the case $q = 1$ one deduces that $\deg(\bar{F}_{\beta_j}) \in \{-1, 0\}$. Thus $\deg(\bar{F}_{\beta_j}) = -1$ for $j > M'$, and

$$\deg(\bar{F}_{\beta_M} \cdot \bar{F}_{\beta_{M-1}} \cdot \ldots \cdot \bar{F}_{\beta_1}) = - \sum_{j=M+1}^{M'} k_j. \quad (A1.4)$$

Now it follows from (A1.4) that the elements

$$\bar{F}_{\beta_M}^{k_M} \bar{F}_{\beta_{M-1}}^{k_{M-1}} \ldots \bar{F}_{\beta_1}^{k_1}, \quad (k_1, \ldots, k_M) \in \mathbb{Z}_+^M,'$$

constitute a basis in the vector space $U_- = U_q \mathfrak{M} \cap (U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_m)$, while

$$\bar{F}_{\beta_M}^{k_M} \bar{F}_{\beta_{M-1}}^{k_{M-1}} \ldots \bar{F}_{\beta_1}^{k_1}, \quad \text{with} \quad \sum_{j=1}^{M'} k_j > 0, \quad (A1.5)$$

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form a basis in $U_q\mathfrak{H}_- \cdot U_-$. Therefore, the vectors (A1.5) form a basis in the kernel of the linear map $U_q\mathfrak{H}_- \to V_-(0)$, $\xi \mapsto \xi v_-(0)$. Thus the vectors

$$
\tilde{F}_{\beta_M}^{k_M} \tilde{F}_{\beta_{M-1}}^{k_{M-1}} \cdots \tilde{F}_{\beta_{M'}}^{k_{M'+1}} \cdot v_-(0), \quad (k_{M'+1}, \ldots, k_M) \in \mathbb{Z}_{+}^{mn},
$$

constitute a basis in the vector space $V_-(0)$. By a virtue of (A1.4),

$$
\deg(\tilde{F}_{\beta_M}^{k_M} \tilde{F}_{\beta_{M-1}}^{k_{M-1}} \cdots \tilde{F}_{\beta_{M'}}^{k_{M'+1}} \cdot v_-(0)) = - \sum_{j=M'+1}^{M} k_j.
$$

**Remark A1.3.** In section 2 the Hopf algebra $U_A \subset U_q\mathfrak{sl}_N$ was considered over the ring $A = \mathbb{C}[q^{1/s}, q^{-1/s}]$. It follows from the definition of the automorphisms $T_i$ (see [10]) that $T_i U_A = U_A$ for all $i \neq n$. Hence all the basis vectors (A1.2) of the vector space $U_q\mathfrak{sl}_N$ are in the lattice $U_A$.

Let $V_1$, $V_2$ be $U_q\mathfrak{sl}_N$-modules satisfying the integrity condition for weights as in section 2. Our additional assumption is that either $V_1$ possesses a highest weight or $V_2$ possesses a lowest weight. Under a suitable choice of the ground field $\mathbb{C}(q^{1/s})$ the formula below determines a linear operator $R_{V_1,V_2}$ in $V_1 \otimes V_2$:

$$
R = \exp_{q^2} \left( (q^{-1} - q) E_{\beta_1} \otimes F_{\beta_1} \right) \cdot \ldots \cdot \exp_{q^2} \left( (q^{-1} - q) E_{\beta_M} \otimes F_{\beta_M} \right) q^{-t_0}, \quad (A1.6)
$$

with $\exp_{q^2}(u) = \sum_{k=0}^{\infty} \frac{u^k}{(k)_{q^2}}!$,

$$
(k)_{q^2}! = \prod_{j=1}^{k} \frac{1 - q^{2j}}{1 - q^2}, \quad (A1.7)
$$

$$
t_0 = \sum_{i,j=1}^{N-1} c_{ij} H_i \otimes H_j, \text{ and } (c_{ij})_{i,j=1,\ldots,N-1} \text{ being the inverse matrix with respect to the Cartan matrix } (a_{ij})_{i,j=1,\ldots,N-1}.
$$

Now we use the relation $\alpha_i(H_j) = a_{ij}$, $i, j = 1, \ldots, N - 1$, to get a different description of $t_0$:

$$
\alpha_i \otimes \alpha_j(t_0) = a_{ij}, \quad i, j = 1, \ldots, N - 1.
$$

Also, an application of the bilinear scalar product $(H_i, H_j) = a_{ij}$, $i, j = 1, \ldots, N - 1$, in the Cartan subalgebra, we get the third description of $t_0$:

$$
(t_0, H_i \otimes H_j) = (H_i, H_j); \quad i, j = 1, \ldots, N - 1.
$$

That is, $t_0 = \sum_{k=1}^{N-1} \frac{I_k \otimes I_k}{(I_k, I_k)}$ for any orthogonal basis of the Cartan subalgebra.

Consider the covariant algebra $\mathbb{C}[SL_N]_q$ as in section 5. It is well known that

$$
\mathbb{C}[SL_N]_q \simeq \bigoplus_{\lambda} \text{End}(V_{\lambda})^*, \quad (A1.8)
$$

with $V_{\lambda}$ being the simple $U_q\mathfrak{sl}_N$-modules from the class described in section 2. Hence the operator $R_{\mathbb{C}[SL_N]_q\mathbb{C}[SL_N]_q}$ is well defined.
We follow V. Drinfeld’s approach in defining such \( \tilde{w}_0 \in \mathbb{C}[SL_N]_q^* \) that
\[
\langle f_1 \cdot f_2, \tilde{w}_0 \rangle = \langle R_{\mathbb{C}[SL_N]_q} \mathbb{C}[SL_N]_q(f_1 \otimes f_2), \tilde{w}_0 \otimes \tilde{w}_0 \rangle
\]  
(A1.9)
for all \( f_1, f_2 \in \mathbb{C}[SL_N]_q \).

Consider the Hopf algebra \( \mathbb{C}[SL_2]_q \). Remind the relation \( t_{12}t_{21} = t_{21}t_{12} \). It is well known that every element of this algebra admits a unique decomposition as follows
\[
f = \sum_{j=1}^{\infty} t_{22}^j \cdot f_j(t_{12}, t_{21}) + f_0(t_{12}, t_{21}) + \sum_{j=1}^{\infty} f_{-j}(t_{12}, t_{21}) \cdot t_{11}^j,
\]
with \( f_j \) being polynomials in two commuting indeterminates.

Consider the element \( s \in \mathbb{C}[SL_2]_q \) given by
\[
s(f) = f_0(q, -1), \quad f \in \mathbb{C}[SL_2]_q
\]
(it is a \( q \)-analogue of the Weyl element \((0 1 \quad \quad -1 0)\)).

Associate to each \( j = 1, \ldots, N-1 \) the onto homomorphism \( \varphi_j : \mathbb{C}[SL_N]_q \to \mathbb{C}[SL_2]_q \),
\[
\varphi_j(t_{ik}) = \begin{cases} 
\delta_{ik}, & i \notin \{j, j+1\} \text{ or } k \notin \{j, j+1\} \\
t_{i-j+1, k-j+1}, & \text{otherwise}
\end{cases}
\]

Consider a reduced decomposition \( w_0 = s_{i_1} s_{i_2} \ldots s_{i_M}, \ M = N(N-1)/2, \) of the longest permutation \( w_0 \in S_N \), together with the element
\[
\overline{w}_0 = \overline{s}_{i_1} \overline{s}_{i_2} \ldots \overline{s}_{i_M} \in \mathbb{C}[SL_N]_q^*,
\]
(A1.10)
with \( \overline{s}_j = \overline{s} \circ \varphi_j, \ j = 1, \ldots, N-1 \). It is well known that \( \overline{w}_0 \) is independent of the choice of reduced decomposition. Now we are in a position to define \( \tilde{w}_0 \) by
\[
\tilde{w}_0 = \overline{w}_0^{-1} \cdot q^{-\frac{1}{2} \sum_k I_k^2/(I_k, I_k)},
\]
(A1.11)
with \( \{I_k\}_{k=1}^{N-1} \) being an orthogonal basis of the Cartan subalgebra. (It follows from [1] that this is well defined and (A1.9) holds.)

Note that in [13, 14] the ‘quantum simple maps’ have been used to produce automorphisms \( T_i \) of \( U_q \mathfrak{sl}_N \) involved into the definition of \( E_{\beta_j}, F_{\beta_j} \). Specifically, one has
\[
T_i(\xi) = \overline{s}_i \cdot \xi \cdot \overline{s}_i^{-1}, \quad \xi \in U_q \mathfrak{sl}_N, \quad i = 1, \ldots, N - 1.
\]
Hence \( T_i \), \( i = 1, \ldots, N - 1 \), are extendable by a continuity from the weakly dense subalgebra \( U_q \mathfrak{sl}_N \subset \mathbb{C}[SL_N]_q^* \) onto the entire \( \mathbb{C}[SL_N]_q^* \).
Appendix 2. On some ∗-representation of $\text{Pol}(\tilde{X})_q$

§1. The construction of a ∗-representation $\prod$

In section 6 $\mathbb{C}[SL_N)_q$ was equipped with involutions $\ast$ and $\ast \ast$. In view of (3.6), (3.7) one has

$$t_{ij}^* = \lambda_1(i)\lambda_2(j)t_{ij}^*,$$  \hspace{1cm} i, j = 1, \ldots, N,  \hspace{1cm} (A2.1.1)

with

$$\lambda_1(k) = \text{sign}(k - m - 1/2), \quad \lambda_2(k) = \text{sign}(n - k + 1/2).$$  \hspace{1cm} (A2.1.2)

A representation $\pi$ of $\mathbb{C}[SL_N]$ in a pre-Hilbert space determines a ∗-representation of $\text{Pol}(\tilde{X})_q$ if and only if $\pi(t_{ij})^* = \lambda_1(i)\lambda_2(j)\pi(t_{ij})$ for all $i, j = 1, \ldots, N$.

Our purpose is to produce such a ∗-representation $\prod$ of $\text{Pol}(\tilde{X})_q$ that $\prod(x) \neq 0$. The method we apply is well known in quantum group theory [1].

Let $\Lambda' = (\lambda'(1), \lambda'(2), \ldots, \lambda'(N))$, $\Lambda'' = (\lambda''(1), \lambda''(2), \ldots, \lambda''(N))$ be two sequences whose entries are ±1. Suppose we are given a representation $\pi$ of $\mathbb{C}[SL_N]$ in a pre-Hilbert space. $\pi$ is said to be of type $(\Lambda', \Lambda'')$ if

$$\pi(t_{ij})^* = \lambda'(i)\lambda''(j)\pi(t_{ij}).$$

(In the special case of the sequences $(\Lambda_1, \Lambda_2)$ determined by (A2.1.2) one has the class of all ∗-representations of $\text{Pol}(\tilde{X})_q$).

**Proposition A2.1.1.** Suppose that the representations $\pi'$ and $\pi''$ are of types $(\Lambda', \Lambda'')$ and $(\Lambda'', \Lambda''')$ respectively. Then their tensor product $\pi' \otimes \pi''$ is of type $(\Lambda', \Lambda''')$.

**Proof.** An application of the relation $(\lambda''(k))^2 = 1$ and the fact that the comultiplication $\Delta : \mathbb{C}[SU_N] \to \mathbb{C}[SU_N]^\otimes 2$ is a homomorphism of ∗-algebras yields

$$(\pi' \otimes \pi''(t_{ij}))^* = \sum_{k=1}^N \pi'(t_{ik})^* \otimes \pi''(t_{kj})^* = \lambda'(i)\lambda''(j) \sum_{k=1}^N (\lambda''(k))^2 \pi'(t_{ik}) \otimes \pi''(t_{kj}) = \lambda'(i)\lambda''(j)^2 \pi'(\delta_{ik}) \otimes \pi''(\delta_{kj}).$$  \hspace{1cm} \Box

**Example A2.1.2.** In the special case $m = n = 1$ one has $\Lambda_1 = (-1, 1)$, $\Lambda_2 = (1, -1)$, $t_{11}' = -t_{11}^*$, $t_{12}' = t_{12}^*$, $t_{21}' = t_{21}^*$, $t_{22}' = -t_{22}^*$. Let $\{e_j\}_{j \in \mathbb{Z}^+}$ be such an orthogonal basis of a pre-Hilbert space that $(e_0, e_0) = 1$, $(e_j, e_j) = (q^{-2j} - 1)(q^4 - 1) \ldots (q^{-2j} - 1)$ for $j \in \mathbb{N}$. The following formulae determine a representation $\pi_+$ of type $(\Lambda_1, \Lambda_2)$:

$$\pi_+(t_{12})e_j = q^{-j}e_j, \quad \pi_+(t_{21})e_j = -q^{-(j+1)}e_j, \quad \pi_+(t_{22})e_j = (1 - q^{-2j})e_{j-1}.  \hspace{1cm} (A2.1.3)$$

**Example A2.1.3.** Let $N \geq 2$, $k \in \{1, \ldots, N - 1\}$, and the pair $(\Lambda', \Lambda'')$ possesses the properties: $\lambda'(j) = \lambda''(j)$ for $j \notin \{k, k + 1\}$, $\lambda'(k) = -1$, $\lambda''(k) = 1$, $\lambda'(k + 1) = 1$, $\lambda''(k + 1) = -1$. Consider the homomorphism of algebras

$$\psi_k = \psi_{(k,k+1)} : \mathbb{C}[SL_N] \to \mathbb{C}[SL_2]_q, \quad \psi_k(t_{ij}) = \left\{ \begin{array}{ll} t_{i-k+1,j-k+1}, & i, j \in \{k, k+1\} \\ \delta_{ij}, & \text{otherwise} \end{array} \right..$$
It is well known that $\psi_k(f^*) = (\psi_k(f))^*$ for all $f \in \mathbb{C}[SL_N]$. On can readily deduce from the definitions that the representation $\pi_+ \circ \psi_k$ of $\mathbb{C}[SL_N]$ is of type $(\Lambda', \Lambda^\prime)$. 

Now turn to a construction of a $\ast$-representation $\tilde{\Pi}$ of $\text{Pol}(\tilde{X})$, that is, a representation of $\mathbb{C}[SL_N]$ of type $(\Lambda_1, \Lambda_2)$. Consider the element

$$
\begin{pmatrix}
1 & 2 & \ldots & m & m + 1 & m + 2 & \ldots & N \\
(n + 1) & n + 2 & \ldots & N & 1 & 2 & \ldots & n
\end{pmatrix}
$$

of the symmetric group $S_N$, together with its reduced decomposition $\sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_{mn}$. Let $s_0 = e$, $s_1 = \sigma_1$, $s_2 = \sigma_1 \cdot \sigma_2$, \ldots, $s_{mn} = \sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_{mn}$. Our construction involves the sequence $\Lambda^{(0)}$, $\Lambda^{(1)}$, \ldots, $\Lambda^{(mn)}$, given by

$$
\Lambda^{(j)} = (\lambda_2(s_{mn-j}(1)), \lambda_2(s_{mn-j}(2)), \ldots, \lambda_2(s_{mn-j}(N))).
$$

Evidently, $\Lambda^{(0)} = \Lambda_1$, $\Lambda^{(mn)} = \Lambda_2$, and the sequences in each pair $(\Lambda^{(j)}, \Lambda^{(j+1)})$, $j = 1, \ldots, mn - 1$, differ only by a permutation of some two neighbour terms $+1$, $-1$. Just as in Example A2.1.3, construct representations of types $(\Lambda^{(j)}, \Lambda^{(j+1)})$. Their tensor product is of type $(\Lambda_1, \Lambda_2)$ due to proposition A2.1.1. Hence this tensor product $\tilde{\Pi}$ is a $\ast$-representation of $\text{Pol}(\tilde{X})$.

We are interested in considering the restriction of $\tilde{\Pi}$ onto the subalgebra $\text{Pol}(X)$ (see section 7).

§2. A faithful irreducible $\ast$-representation of $\text{Pol}(\tilde{X})$

Lemma A2.2.1. Let $v$ be such a vector in the space of a $\ast$-representation $\rho$ of $\text{Pol}(\tilde{X})$ that

$$
\begin{align*}
\rho(t^m_{\{m+1,m+2,\ldots,N\}\{1,2,\ldots,n\}})v &= c \cdot v, \quad c \in \mathbb{C}; \\
\rho(t^m_{\{m+1,m+2,\ldots,N\}\{1,2,\ldots,n\}}^\prime) &\neq \{1,2,\ldots,n\}.
\end{align*}
$$

Then $|c| = q^{mn}$.

Proof. There is a relation in $\text{Pol}(\tilde{X})$ between $t^m_{\{1,2,\ldots,m\}}$ and $t^m_{\{m+1,m+2,\ldots,N\}}$ derived from $\det_q T = 1$, $T = (t_{ij})_{i,j=1,\ldots,N}$, via a $q$-analogue of the Laplace formula. By a virtue of (A2.2.1),

$$
\rho\left((-q)^{mn}t^m_{\{1,2,\ldots,m\}\{n+1,n+2,\ldots,N\}} \cdot t^m_{\{m+1,m+2,\ldots,N\}\{1,2,\ldots,n\}}\right)v = v.
$$

On the other hand,

$$
t^m_{\{1,2,\ldots,m\}\{n+1,n+2,\ldots,N\}} = (-q)^{mn}(t^m_{\{m+1,m+2,\ldots,N\}\{1,2,\ldots,n\}})^\ast.
$$

So

$$
q^{2mn} \left\| \rho\left(t^m_{\{m+1,m+2,\ldots,N\}\{1,2,\ldots,n\}}\right)v \right\|^2 = \left\| v \right\|^2,
$$

that is,

$$
\left\| \rho\left(t^m_{\{m+1,m+2,\ldots,N\}\{1,2,\ldots,n\}}\right)v \right\| = q^{-mn} \left\| v \right\|. \quad \square
$$
Impose the notation
\[ e_k = e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_m}, \quad k = (k_1, k_2, \ldots, k_m) \in \mathbb{Z}_+^m \]
for basis vectors of the space of \( \widetilde{\Pi} \).

**Example A2.2.2.** Let \( m = n = 2 \). It follows from the definitions that
\[ \psi_2 \otimes \psi_3 \otimes \psi_1 \otimes \psi_2(t_{13}t_{24} - qt_{14}t_{23}) = t_{12} \otimes t_{12} \otimes t_{12} \otimes t_{12}. \]
(In fact,
\[ \psi_2 \otimes \psi_3 \otimes \psi_1 \otimes \psi_2(t_{13}) = 1 \otimes 1 \otimes t_{12} \otimes t_{12}, \]
\[ \psi_2 \otimes \psi_3 \otimes \psi_1 \otimes \psi_2(t_{24}) = t_{12} \otimes t_{12} \otimes 1 \otimes 1, \]
\[ \psi_2 \otimes \psi_3 \otimes \psi_1 \otimes \psi_2(t_{14}) = 0. \]
Hence for all \( k = (k_1, k_2, k_3, k_4) \)
\[ \widetilde{\Pi}(t_{13}t_{24} - qt_{14}t_{23})e_k = q^{-(k_1+k_2+k_3+k_4)}e_k. \tag{A2.2.2} \]

It is easy to extend (A2.2.2) onto the case of arbitrary \( m, n \in \mathbb{N} \).

Consider the element \( u = (m+1, m+2, \ldots, N, 1, 2, \ldots, m) \in S_N \), together with its reduced decomposition of the form \( u = \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_m \).

\[ \sigma_k = \left( m - \left[ \frac{k-1}{n} \right] \right) + \left\{ \frac{k-1}{n} \right\} n, m - \left[ \frac{k-1}{n} \right] + \left\{ \frac{k-1}{n} \right\} n + 1 \]
(here \( [\cdot], \{\cdot\} \) stand for integral and fractional parts of a real number, respectively). For example, in the case \( m = 2, n = 3 \), one has \( u = (3, 4, 5, 1, 2) \), and the above reduced decomposition acquires the form \( u = (2, 3)(3, 4)(4, 5)(1, 2)(2, 3)(3, 4) \). It is easy to show that \( \widetilde{\Pi} \pi_{12}^{\otimes mn} \circ \Psi \), with \( \Psi : \mathbb{C}[SL_N]_q \to \mathbb{C}[SL_2]_q^{\otimes mn}, \) \( \Psi = \psi_{\sigma_1} \otimes \psi_{\sigma_2} \otimes \ldots \otimes \psi_{\sigma_m} \) (we use here the notation from Example(A2.1.3)).

**Lemma A2.2.2.** For all \( k \in \mathbb{Z}_+^m \)
\[ \widetilde{\Pi} \left( t_{\{1,\ldots,m\}\{n+1,\ldots,N\}} \right) e_k = q^{-\sum_{j} k_j} e_k. \tag{A2.2.3} \]

**Proof.** Let \( n \) be fixed. We use an induction in \( m \) to show that for \( i \leq m \)
\[ \Psi(= \Psi_m) : t_{ij} \mapsto \begin{cases} 1 \otimes \ldots \otimes 1 \otimes t_{12} \otimes \ldots \otimes t_{12} \otimes 1 \otimes \ldots \otimes 1, & j > i + n \\ (m-i)n & \end{cases}, \quad j = i + n. \]

In the case \( m = 1 \) the statement is evident (since \( \Psi_1(t_{1,n+1}) = \psi_1 \otimes \psi_2 \otimes \ldots \otimes \psi_n \left( \sum t_{i1} \otimes t_{i2} \otimes \ldots \otimes t_{n-1,n+1} \right) = \psi_1 \otimes \psi_2 \otimes \ldots \otimes \psi_n (t_{12} \otimes t_{23} \otimes \ldots \otimes t_{n,n+1}) = t_{12} \otimes t_{12} \otimes \ldots \otimes t_{12} \).
Now we are to make the induction passage from \((m - 1)\) to \(m\). Let \(\Phi_m = \psi_{\sigma_1} \otimes \psi_{\sigma_2} \otimes \ldots \otimes \psi_{\sigma_n} : \mathbb{C}[SL_N]_q \rightarrow \mathbb{C}[SL_2]_q\), \(\Psi_m = \psi_{\sigma_{n+1}} \otimes \psi_{\sigma_{n+2}} \otimes \ldots \otimes \psi_{\sigma_m} : \mathbb{C}[SL_N]_q \rightarrow \mathbb{C}[SL_2]_q^{\otimes (m-1)}\), i.e. \(\Psi_m = \Phi_m \otimes \Psi'_{m-1}\).

Obviously, the subalgebra of \(\mathbb{C}[SL_N]_q\) generated by \(t_{ij}\) with \(i,j < N\), is isomorphic to \(\mathbb{C}[SL_{N-1}]_q\). If we agree to identify in what follows \(\mathbb{C}[SL_{N-1}]_q\) with this subalgebra, one can claim that

\[
\Psi_{m-1}(t_{ij}) = \Psi'_{m-1}(t_{ij}), \quad (i, j < N).
\]

Besides that, \(\Psi'_{m-1}(t_{iN}) = \Psi_{m-1}(t_{N_i}) = \delta_{iN} 1 \otimes 1 \otimes \ldots \otimes 1\). These facts are to be used in the passage from \((m - 1)\) to \(m\). We start this passage with considering the special case of elements of the last column:

\[
\Psi_m(t_{iN}) = \Phi_m \otimes \Psi'_{m-1} \left( \sum_{k=1}^{N} t_{ik} \otimes t_{kN} \right) = \sum_{k=1}^{N} \Phi_m(t_{ik}) \otimes \Psi'_{m-1}(t_{kN}) = \Phi_m(t_{iN}) \otimes \Psi'_{m-1}(t_{NN}) = \Phi_m(t_{iN}) \otimes 1 \otimes \ldots \otimes 1.(m-1)n
\]

In the case \(i < m\) it is easily deducible from the definition of \(\Phi_m\) that \(\Phi_m(t_{iN}) = 0\) (the cycle \(\sigma_1 \sigma_2 \ldots \sigma_n = (m, m + 1)(m + 1, m + 2) \ldots (N - 1, N)\) can not send \(N\) to \(i\)). If \(i = m\), then

\[
\Phi_m(t_{mN}) = \psi_{\sigma_1} \otimes \psi_{\sigma_2} \otimes \ldots \otimes \psi_{\sigma_n} (t_{mN}) = \psi_m \otimes \psi_{m+1} \otimes \ldots \otimes \psi_{N-1} \left( \sum t_{mi} \otimes t_{i2} \otimes \ldots \otimes t_{ia-1N} \right) = \psi_m \otimes \psi_{m+1} \otimes \ldots \otimes \psi_{N-1} (t_{m,m+1} \otimes t_{m+1,m+2} \otimes \ldots \otimes t_{N-1,N}) = t_{12} \otimes t_{12} \otimes \ldots \otimes t_{12}.
\]

Thus we have done an induction passage for elements of the last column. What remains is to consider the case \(j \leq N - 1\), \(i + n \leq j\) (note that \(i + n \leq N - 1\) implies \(i < m\)). One has

\[
\Psi_m(t_{ij}) = \Phi_m \otimes \Psi'_{m-1} \left( \sum_{k=1}^{N} t_{ik} \otimes t_{kj} \right),
\]

and, since \(\Psi'_{m-1}(t_{Nj}) = 0\) for \(j < N\),

\[
\Psi_m(t_{ij}) = \sum_{k=1}^{N-1} \Phi_m(t_{ik}) \otimes \Psi'_{m-1}(t_{kj}). \quad (*)
\]

Consider the element \(\Phi_m(t_{ik})\):

\[
\Phi_m(t_{ik}) = \psi_m \otimes \psi_{m+1} \otimes \ldots \otimes \psi_{N-1} \left( \sum t_{ij} \otimes t_{i2j} \otimes \ldots \otimes t_{jn-1k} \right).
\]

Since \(i < m\), one has \(\psi_m(t_{ij}) \neq 0\) only for \(j_1 = i\). Similarly, \(j_2 = j_3 = \ldots = k = i\). Hence, in (*) only one term "survives":

\[
\Psi_m(t_{ij}) = \Phi_m(t_{ii}) \otimes \Psi_{m-1}(t_{ij}).
\]
The induction hypothesis implies \( \Psi_{m-1}(t_{ij}) = 0 \) for \( j > i + n \), and hence for such \( i \) and \( j \) that \( \Psi_m(t_{ij}) = 0 \). What remains is to consider the case \( i + n = j \). If so, again the induction hypothesis yields

\[
\Psi_{m-1}(t_{ij}) = 1 \otimes 1 \otimes \ldots \otimes 1 \otimes t_{12} \otimes \ldots \otimes t_{ij} \otimes 1 \otimes 1 \otimes \ldots \otimes 1, \]

i. e.

\[
\Phi_m(t_{ii}) \otimes \Psi_{m-1}(t_{ij}) = 1 \otimes 1 \otimes \ldots \otimes 1 \otimes t_{12} \otimes 1 \otimes 1 \otimes \ldots \otimes 1 =
\]

\[
= 1 \otimes 1 \otimes \ldots \otimes 1 \otimes t_{12} \otimes 1 \otimes 1 \otimes \ldots \otimes 1. \]

This completes the induction passage. □

By a virtue of (A2.2.3) the operator \( \tilde{\Pi}(x) \) is invertible, and hence the representation \( \tilde{\Pi} \) admits a unique extension onto the \( \ast \)-algebra \( \text{Pol}(\bar{X})_{q,x} \). Let \( \tilde{\Pi} = \tilde{\Pi} \circ i \) be the \( \ast \)-representation of \( \text{Pol} (\text{Mat}_{mn})_q \) deduced from the \( \ast \)-homomorphism \( \tilde{\Pi} : \text{Pol} (\text{Mat}_{mn})_q \to \text{Pol}(\bar{X})_{q,x} \) described in section 6. Equip the pre-Hilbert representation space \( \tilde{H} \) of \( \tilde{\Pi} \) and the algebras \( \text{Pol}(\bar{X})_q, \text{Pol}(\text{Mat}_{mn})_q \) with the gradations:

\[
\tilde{H} = \bigoplus_{j=0}^{\infty} \tilde{H}_j, \quad \tilde{H}_j = \{ v \in \tilde{H} | \tilde{\Pi}(x)v = q^{-2j}v \},
\]

\[
deg(t_{ij}) = \begin{cases} 1, & i \leq m \text{ & } j \leq n \\ -1, & i > m \text{ & } j > n \\ 0, & \text{otherwise} \end{cases}
\]

\[
deg(z_a^m) = 1, \quad \deg(z_a^n)^* = -1.
\]

It follows from the commutation relations (6.8), (6.9) that \( \tilde{\Pi} \) allows one to equip \( \tilde{H} \) with the structure of a graded \( \text{Pol}(\bar{X})_q \)-module, and the representation \( \Pi \) with a structure of a graded \( \text{Pol}(\text{Mat}_{mn})_q \)-module.

**Proposition A2.2.3.** The graded \( \text{Pol}(X)_q \)-module \( \tilde{H} \) is simple.□

**Proof.** Let \( L \subset \tilde{H} \) be a nontrivial graded submodule. It follows from lemma A2.2.2 that the operators \( \tilde{\Pi}(t) \) and \( \tilde{\Pi}(t^*) \) are the same. Also, \( \tilde{\Pi}(x)L \subset L \) implies \( \tilde{\Pi}(t)L \subset L, \tilde{\Pi}(t^*)L \subset L \). Hence \( \tilde{\Pi}(f)L \subset L \) for all \( f \in \text{Pol}(\bar{X})_q \) by lemma 8.2. In particular, \( t_{1,2}^{m,n}L \subset L \) for all \( m \)-element subsets \( J \subset \{1,2,\ldots,N\} \). On the other hand, in this case there exists a non-zero vector \( v \in L \) such that \( \tilde{\Pi}(t_{1,2}^{m+1,N+1})v = 0 \) for all \( n \)-element subsets \( I \subset \{1,2,\ldots,N\} \) different from \( \{1,2,\ldots,n\} \). By a virtue of lemmas A2.2.1 and A2.2.2, the subspace of all such vectors is one-dimensional. Hence, \( v = \text{const} \cdot e_0, e_0 \in L, L = \tilde{H} \). A contradiction. □

We prove in section 8 that the restriction of \( \tilde{\Pi} \) onto \( \text{Pol}(\bar{X})_q \) is a faithful representation of this subalgebra.

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6That is, it has no nontrivial graded submodules

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