Mean curvature flow of surfaces in a hyperkähler 4-manifold ✩

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Abstract

In this paper, we firstly prove that every hyper-Lagrangian submanifold \( L_{2n}^n(n > 1) \) in a hyperkähler 4\( n \)-manifold is a complex Lagrangian submanifold. Secondly, we study the geometry of hyper-Lagrangian surfaces and demonstrate an optimal rigidity theorem with the condition on the complex phase map of self-shrinking surfaces in \( \mathbb{R}^4 \). Last but not least, we show that the mean curvature flow from a closed surface with the image of the complex phase map contained in \( S^2 \setminus S^1 \) in a hyperkähler 4-manifold does not develop any Type I singularity.

Keywords: hyper-Lagrangian, self-shrinker, rigidity, mean curvature flow, singularity

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1. Introduction

Let \( \bar{M} \) be a closed \( m \)-dimensional differential manifold and \( (N, h) \) be an \( n \)-dimensional Riemannian manifold which can be embedded into some Euclidean space. The mean curvature flow (MCF) in \( N \) is a smooth one-parameter family of immersions \( F_t = F(\cdot, t) : \bar{M}^m \to N^n \) with the corresponding image \( \bar{M}_t = F_t(\bar{M}) \) such that

\[
\begin{align*}
\frac{\partial}{\partial t} F(x, t) &= H(x, t), \quad (x, t) \in \bar{M} \times (0, T); \\
F(x, 0) &= F_0(x), \quad x \in \bar{M},
\end{align*}
\]

is satisfied, where \( \mathbf{H}(x, t) \) is the mean curvature vector of the isometric immersion \( \bar{M}_t \) in \( N \) at \( F(x, t) \) in \( N^n \). The MCF (1.1) is a (degenerate) quasilinear parabolic evolution equation. By using the DeTurck’s trick (cf. [18]), one can prove that the MCF (1.1) has a smooth solution for short time interval \([0, T)\). Moreover, the maximum existence time \( T \) satisfying (cf. [33, Theorem 8.1])

\[
\lim_{t \to T} \sup_{\bar{M}_t} |\mathbf{B}| = \infty,
\]

where \( \mathbf{B}(x, t) \) is the second fundamental form of the isometric immersion \( \bar{M}_t \) in \( N \) at \( F(x, t) \). There are many significant works on MCF, see the references (not exhaustive): [4, 9, 23–26, 30, 31, 33–40, 43, 49, 50, 56–60] and the references therein.

Brakke [3] firstly studied the motion of a submanifold moving by its mean curvature from the viewpoint of geometric measure theory. In Huisken’s seminal paper [33], he showed that the closed convex hypersurfaces in Euclidean space \( \mathbb{R}^{m+1}(m > 1) \) contracts to a single point under the MCF in finite time and the normalized flow (area is fixed) converges to a sphere of the same area in infinite time. Later, Huisken [34] generalized his results to closed and

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uniformly convex hypersurfaces in a complete Riemannian manifold with bounded geometry. As time evolves, the MCF may develop singularities which can be classified as Type I and Type II according to the blow up rate of the second fundamental form with respect to time $t$. And Huisken [35] proved that after appropriate rescaling near the Type I singularity the hypersurfaces converge to a self-similar solution of the MCF.

In the past twenty years, the MCF of higher codimension has made much progress. And symplectic MCF and Lagrangian MCF are two important class among them. Chen-Li [7] studied the symplectic MCF from a closed symplectic surface in a Kähler-Einstein 4-manifold, by establishing a new monotonicity formula, and using blow up argument, they proved that the MCF has no Type I singularity if the initial symplectic surface is closed in a Kähler-Einstein surface with nonnegative scalar curvature. Almost at the same time, Wang [55] demonstrated the same conclusion by removing the condition on the curvature of the ambient manifold. Smoczyk [47] showed that the Lagrangian condition is preserved by the MCF when the ambient space is a Calabi-Yau $2n$-manifold (which is a closed $2n$-dimensional Riemannian manifold with holonomy contained in $SU(n)$). Afterwards, Wang [55] observed that almost calibrated Lagrangian submanifolds in a Calabi-Yau manifold can not develop Type I singularities. Chen-Li [8] manifested that in this setting the tangent cone of the MCF at a singular point $(X_0, T)$ (here $T$ is the first blow up time of the MCF) is an integer rectifiable stationary Lagrangian varifold. Furthermore, Neves [46] studied finite time singularities for zero-Maslov class Lagrangian submanifolds in $\mathbb{C}^n$, a more general condition than being almost calibrated. As a consequence, he showed that the Lagrangian MCF with zero-Maslov class does not develop any Type I singularity. On the other hand, self-shrinkers are Type I singularity models of the MCF, and there is a multitude of excellent work on the classification and uniqueness problem for self-shrinkers (see e.g. [1, 5, 6, 12–15, 17, 19–22, 27, 28, 32, 44, 48, 51–54]).

In this paper, we shall focus on the case where the ambient space is a hyperkähler manifold. A hyperkähler $4n$-manifold $M$ is a Riemannian manifold with holonomy contained in $Sp(n)$. It admits a 2-sphere-family of complex structures $J$ and the associated holomorphic symplectic form $\Omega_1 \in \Omega^{2,0}(M, J)$. Leung-Wan [45] firstly introduced the concept of hyper-Lagrangian manifolds which is a generalization of complex Lagrangian submanifolds: A submanifold $L^{2n} \subset M^{4n}$ is called hyper-Lagrangian if each tangent space $T_xL \subset T_xM$ is a complex Lagrangian subspace with respect to $\Omega_{J(x)}$ with varying $J(x) \in \mathbb{S}^2$. There are many restrictions for the hyper-Lagrangian submanifold, e.g., every hyper-Lagrangian submanifold is a Kähler manifold with holomorphic normal bundle (cf. [45, Corollary 4.2]). One can check that every oriented surface immersed in a 4-hyperkähler manifold is automatically hyper-Lagrangian. Unfortunately, up to now, we do not know any nontrivial examples of hyper-Lagrangian submanifolds $L^{2n}$ in $M^{4n}$ for $n > 1$. Therefore, it is very important for us to construct nontrivial examples of these submanifolds. Along this direction, we give the following restriction for the hyper-Lagrangian submanifold.

**Theorem 1.1.** Every hyper-Lagrangian submanifold $L^{2n}(n > 1)$ in a hyperkähler $4n$-manifold is a complex Lagrangian submanifold.

The authors [45] showed that the **complex phase map** $J : L \to \mathbb{S}^2, x \mapsto J(x)$ satisfies the evolving harmonic map heat flow along the MCF and the hyper-Lagrangian condition is preserved under the mean curvature flow. Moreover, they demonstrated that the MCF does not develop Type I singularities if the image of $J$ of the initial closed hyper-Lagrangian submanifold is contained in an open hemisphere. When $n = 1$, their results are in accordance with [7, Theorem 4.7] and [55, Theorem A]. In addition, the method of the proof of [45, Theorem 5.1] could also be applied to almost calibrated Lagrangian submanifolds in a Calabi-Yau manifold and arrived at the same conclusion we mention previously. Recently, Kunikawa-Takahashi [42] proved the longtime existence and convergence under the condition that the initial hyper-Lagrangian submanifold has sufficiently small twistor energy. Due to Theorem 1.1, it suffices to study surfaces in a hyperkähler $4$-manifold. Notice that closed Hyperkähler $4$-manifolds are coincide with Calabi-Yau $4$-manifolds since $Sp(1) = SU(2)$.

As we mention above, the problem of singularities is an extremely crucial topic in the MCF, we are mainly interested in the geometry of hyper-Lagrangian surfaces and the corresponding mean curvature flow. Note that in a hyperkähler $4$-manifold, one can check that a surface being symplectic is equivalent to the condition that the image of the complex phase map is contained in an open hemisphere while a surface being Lagrangian is equivalent to the condition that the image of the complex phase map is contained in a great circle. Moreover, a Lagrangian surface being almost calibrated is equivalent to the condition that the image of the complex phase map is contained in an open half great circle. Recall that when Jost-Xin-Yang [41] studied the regularity of harmonic maps into spheres $\mathbb{S}^n$, they
assumed that the image of harmonic maps is contained in $S^n \setminus S_{+}^{n-1}$, which is the maximal open convex supporting subset of $S^n$. Accordingly, it is natural to restrict the image of $J$ in $S^2 \setminus S_{+}^1$, when we consider the MCF from a closed surface in a hyperkähler $4$-manifold, which can be regarded as a generalization of both symplectic and almost calibrated Lagrangian MCF in a hyperkähler $4$-manifold. In order to study the existence of the Type I singularity of this MCF, we firstly study the geometry of the Type I singularity, namely, the self-shrinking surface in $\mathbb{R}^4$, and we find that its complex phase map is a generalized harmonic map (cf. [11]). Based on this observation, by using integral method, we obtain

**Theorem 1.2.** Every complete proper self-shrinking surface in $\mathbb{R}^4$ with the image of the complex phase map contained in $S^2 \setminus S_{+}^1$ is a plane.

This theorem improves and generalizes the result of [2]. We also give an example to illustrate that the restriction on the image of $J$ is optimal. Furthermore, we show that if the image of the complex phase map of the initial closed surface is contained in $S^2 \setminus S_{+}^1$, then the image of the complex phase map of the evolved surface is contained in some fixed compact subset of $S^2 \setminus S_{+}^1$, under the MCF. Consequently, by using Theorem 1.2 and applying the blow up analysis of MCFs, we prove the nonexistence of the Type I singularity of the MCF.

**Theorem 1.3.** Let $\Sigma_0$ a closed surface immersed in hyperkähler $4$-manifold $M$. Let $\Sigma_t \subset M(t \in [0, T)$ for some $T > 0$ be a family of surfaces given by the mean curvature flow. Suppose that the image of the complex phase map $J : \Sigma_0 \to S^2$ is contained in $S^2 \setminus S_{+}^1$, then the mean curvature flow has no Type I singularity.

As a consequence, if the image of $J$ for the initial surface is contained in a great circle avoid a point, then the Lagrangian MCF has no Type I singularity in a hyperkähler $4$-manifold. The restriction on the image of $J$ is sharp in Theorem 1.3, see Example 5.2 in section 5.

The article will be organized as follows. We shall give some preliminaries in Section 2. In Section 3, we firstly give an equivalent condition of the hyper-Lagrangian, from which it is easy to see that any surface in hyperkähler $4$-manifold is hyper-Lagrangian, then we prove that every hyper-Lagrangian submanifolds $L^{2n}(n > 1)$ in a hyperkähler manifold $M^{4n}$ must be complex Lagrangian (Theorem 1.1). Subsequently, we study the geometry of the hyper-Lagrangian surfaces in Section 4. Finally, in Section 5, we demonstrate some rigidity theorems of self-shrinking surfaces and translating soliton surface in $\mathbb{R}^4$ (Theorem 1.2, Theorem 5.4), after that, we show that the MCF from a closed surface with the image of the complex phase map $J$ contained in $S^2 \setminus S_{+}^1$ does not develop any Type I singularity (Theorem 1.3).

2. Preliminaries

In this section, we set some notations that will be used throughout the paper and recall some relevant definitions and results.

Let $M^{4n}$ be a $4n$-dimensional hyperkähler manifold, i.e., there exists two covariant constant anti-commutative almost complex structures $J_1, J_2$, i.e., $J_1, J_2$ are parallel with respect to the Levi-Civita connection and $J_1J_2 = -J_2J_1$. Denote $J_3 := J_1J_2$, then the following quaternionic identities hold

$$J_1^2 = J_2^2 = J_3^2 = J_1J_2J_3 = -1.$$

Every $SO(3)$ matrix preserves the quaternionic identities, i.e., $\bar{J}^\alpha := \sum_{\alpha=1}^3 a_\alpha b^\alpha$ satisfies the quaternionic identities

$$\bar{J}_1^2 = \bar{J}_2^2 = \bar{J}_3^2 = \bar{J}_1\bar{J}_2\bar{J}_3 = -1.$$

In particular, for every unit vector $(a_1, a_2, a_3) \in \mathbb{R}^3$, we get a covariant constant almost complex structure $\sum_{\alpha=1}^3 a_\alpha J_\alpha$, and this implies that $\{M, \sum_{\alpha=1}^3 a_\alpha J_\alpha\}$ is a Kähler manifold.

Let $\bar{J} = \sum_{\alpha=1}^3 a_\alpha J_\alpha$ be an almost complex structure on $M$. Let $\omega_J$ be the Kähler form with respect to $\bar{J}$, then the associated symplectic 2-form $\Omega_J \in \Omega^{2,0}(M, \bar{J})$ is given by

$$\Omega_J = \omega_K + \sqrt{-1} \omega_{KJ},$$
where $K = \sum_{\alpha=1}^{3} \mu_\alpha J_\alpha$ is an almost complex structure which is orthogonal to $\hat{J}$ in the sense that $\sum_{\alpha=1}^{3} \lambda_\alpha \mu_\alpha = 0$. If $\hat{J}$ is parallel, then $\Omega_J$ is holomorphic with respect to the covariant constant almost complex structure $\hat{J}$.

Let $\omega_J$ be the Kähler form associated with the almost complex structure $J_\alpha$, then $(M, J_1)$ is a Kähler manifold and

$$\Omega_J = \omega_1 + \sqrt{-1} \omega_2 \in H^{2,0}(M, J_1)$$

is the associated holomorphic symplectic 2-form. We say that a submanifold $L^{2n}$ of $M^{4n}$ is complex Lagrangian if for some covariant constant complex structure $\hat{J}$ of $M$ such that the associated holomorphic symplectic 2-form $\Omega_J$ vanished everywhere on $L$. Without loss of generality, assume $\hat{J} = J_1$, then $L$ is a Kähler submanifold of the Kähler manifold $(M, J_1)$. In particular, $L$ is a minimal submanifold of $M$. Moreover, both $L \subset (M, J_2)$ and $L \subset (M, J_3)$ are Lagrangian immersions.

We say that $L^{2n}$ is a hyper-Lagrangian submanifold of $M^{4n}$ if there is an almost complex structure $\hat{J} = \sum_{\alpha=1}^{3} \lambda_\alpha J_\alpha$ such that the associated symplectic 2-form $\Omega_J$ vanished everywhere on $L$. The map

$$J : L \longrightarrow \mathbb{S}^2, \quad x \mapsto J(x) := (\lambda_1, \lambda_2, \lambda_3)$$

is called the complex phase map. In other words, $L$ is hyper-Lagrangian iff each $T_x L$ is a complex Lagrangian subspace of $T_x M$. Here we say that $T_x L$ is a complex Lagrangian subspace of $T_x M$ if for some complex structure $\hat{J} = \sum_{\alpha=1}^{3} \lambda_\alpha J_\alpha$ we have

$$\tilde{g}(K_1, \cdot)|_{T_x L} = 0,$$

for all almost complex structures $K = \sum_{\alpha=1}^{3} \mu_\alpha J_\alpha$ which are orthogonal to $\hat{J}$. Therefore, $L$ is complex Lagrangian iff $L$ is hyper-Lagrangian with constant complex phase map.

The complex phase map $J$ defines an almost complex structure $\hat{J} = \sum_{\alpha=1}^{3} \lambda_\alpha J_\alpha|_{T_x L}$ on $L$ and an almost complex structure $\hat{J}^\perp = \sum_{\alpha=1}^{3} \lambda_\alpha J_\alpha|_{T^\perp_x L}$ on $T^\perp_x L$. Denoted $\nabla, \nabla^\perp$ and $\nabla^\parallel$ by the Levi-Civita connections on $T M, T L$ and $T^\perp L$ respectively. Denoted $\tilde{R}, R$ and $R^\perp$ by the Riemannian curvatures on $T M, T L$ and $T^\perp L$ respectively.

### 3. Every hyper-Lagrangian submanifold but surface is complex Lagrangian

In this section, we shall give a proof of Theorem 1.1. In particular, every hyper-Lagrangian submanifold $L^{2n}$ in a hyperkähler manifold $M^{4n}$ is minimal when $n > 1$.

First, we have the following

**Lemma 3.1.** $L$ is hyper-Lagrangian iff

$$J_\alpha|_{T x L} = \lambda_\alpha \hat{J}, \quad \alpha = 1, 2, 3,$$

iff

$$J_\alpha|_{T^\perp x L} = \lambda_\alpha \hat{J}^\perp, \quad \alpha = 1, 2, 3.$$

**Proof.** Under the orthogonal decomposition $T_x M = T_x L \oplus T^\perp_x L$, we write

$$J_\alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ -B^T_\alpha & C_\alpha \end{pmatrix}, \quad \alpha = 1, 2, 3.$$

Let $A = (a_{\beta \gamma})_{\beta \gamma = 1}^{3} \in SO(3)$ where $a_{\alpha \beta} = a_{\beta \alpha}$. Set $\tilde{J}_\beta = \sum_{\alpha=1}^{3} a_{\beta \alpha} J_\alpha$. Since $L$ is hyper-Lagrangian, we get

$$J_2 = \begin{pmatrix} 0 & \sum_{\alpha=1}^{3} a_{2\alpha} B_\alpha \\ -\sum_{\alpha=1}^{3} a_{2\alpha} B^T_\alpha & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & \sum_{\alpha=1}^{3} a_{3\alpha} B_\alpha \\ -\sum_{\alpha=1}^{3} a_{3\alpha} B^T_\alpha & 0 \end{pmatrix},$$

or equivalently,

$$A_\alpha = \lambda_\alpha \tilde{J}, \quad \alpha = 1, 2, 3,$$

which is also equivalent to

$$C_\alpha = \lambda_\alpha \tilde{J}^\perp, \quad \alpha = 1, 2, 3.$$

□

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Remark 3.1. This Lemma claims that every surface immersed in a hyperkähler 4-manifold is automatically hyper-Lagrangian.

The first restriction of hyper-Lagrangian submanifolds is the following ([45, Corollary 4.2]).

**Lemma 3.2.** If \( L \) is hyper-Lagrangian, then \((L, \bar{J})\) is a Kähler manifold with a holomorphic normal bundle.

**Proof.** We will give an alternative proof here. By Lemma 3.1, for all \( X \in \Gamma(TL) \), we have
\[
\lambda_a \bar{J}X = (J_a X)^\top, \quad a = 1, 2, 3.
\]

For all \( Y \in \Gamma(TL) \), we get
\[
Y(\lambda_a)\bar{J}X + \lambda_a \left( \nabla_Y \bar{J} X + \lambda_a \nabla_Y (J a X) + \lambda_a \nabla_Y (\bar{J} X) \right) = \nabla_Y (\lambda_a \bar{J} X)
\]
\[
= \nabla_Y (J_a X)^\top - \mathbf{B} \left( Y, (J_a X)^\top \right) = \bar{\nabla}_Y \left( \sum_{j=1}^{2n} \left( J_a X, e_j \right) e_j \right) - \lambda_a \mathbf{B} \left( Y, \bar{J} X \right)
\]
\[
= \left( \bar{\nabla}_Y (J_a X)^\top \right) + \sum_{j=1}^{2n} \left( J_a X, \mathbf{B} (Y, e_j) \right) e_j + \sum_{j=1}^{2n} \left( J_a X, e_j \right) \mathbf{B} (Y, e_j) - \lambda_a \mathbf{B} \left( Y, \bar{J} X \right)
\]
\[
= (J_a \nabla_Y X + J_a \mathbf{B} (Y, X))^\top + \mathbf{A}^{J(X)^\top} (Y),
\]
where \( \{e_j\}_{1 \leq j \leq 2n} \) is a local orthonormal frame of \( TL \) and \( \mathbf{A} \) is the shape operator. Thus,
\[
Y(\lambda_a)\bar{J}X + \lambda_a \left( \nabla_Y \bar{J} X \right) = (J_a \mathbf{B} (Y, X))^\top + \mathbf{A}^{J(X)^\top} (Y).
\]

Since \( \sum_{a=1}^{3} \lambda_a^2 = 1 \), we get
\[
\left( \nabla_Y \bar{J} \right) X = (J_a \mathbf{B} (Y, X))^\top + \mathbf{A}^{J(X)^\top} (Y) = 0.
\]
Therefore, \( \nabla \bar{J} = 0 \) which implies that \((L, \bar{J})\) is a Kähler manifold. Similarly, one can prove that \( \nabla^\perp \bar{J}^\perp = 0 \). \( \square \)

**Lemma 3.3.** If \( L \) is hyper-Lagrangian, then
\[
\langle \mathbf{B} (X, Y), J_a Z \rangle = \langle \mathbf{B} (X, Z), J_a Y \rangle - X(\lambda_a) \langle \bar{J} Y, Z \rangle, \quad \forall X, Y, Z \in TL.
\] (3.1)

or equivalently
\[
Y(\lambda_a) \bar{J}X = (J_a \mathbf{B} (Y, X))^\top + \mathbf{A}^{J(X)^\top} (Y), \quad \forall X, Y \in TL.
\] (3.2)

Moreover,
\[
\mathbf{B} \left( X, \bar{J} Y \right) = \bar{J}^\perp \mathbf{B} (X, Y) + \sum_{a=1}^{3} X(\lambda_a) J_a Y.
\] (3.3)

**Proof.** Without loss of generality, we may assume \( \nabla_X = \nabla_Y = \nabla Z = 0 \) at a considered point. We shall compute at this considered point,
\[
\langle \mathbf{B} (X, Y), J_a Z \rangle = \left\langle \bar{\nabla}_X Y, J_a Z \right\rangle
\]
\[
= \bar{\nabla}_X \langle Y, J_a Z \rangle - \langle Y, J_a \bar{\nabla}_X Z \rangle
\]
\[
= X \left( \lambda_a \left\langle Y, J Z \right\rangle \right) - \langle Y, J_a \mathbf{B} (X, Z) \rangle
\]
\[
= X(\lambda_a) \left\langle Y, J Z \right\rangle + (J_a Y \mathbf{B} (X, Z))
\].
Here the last two equalities followed from Lemma 3.1 and Lemma 3.2 respectively. For the second claim, we compute

\[
B(X, JY) = \left(\tilde{\nabla}_X (JY)\right)^\perp
= \sum_{a=1}^{3} \left(\tilde{\nabla}_X (\lambda_a J_a Y)\right)^\perp
= \sum_{a=1}^{3} X(\lambda_a) J_a Y + \sum_{a=1}^{3} \lambda_a J_a B(X, Y) + \sum_{a=1}^{3} \lambda_a J_a \nabla_X Y
= \sum_{a=1}^{3} X(\lambda_a) J_a Y + B(X, Y).
\]

Now we can give the following

Proof of Theorem 1.1. For each \(\alpha \in \{1, 2, 3\}\), on one hand, according to (3.2), we have

\[
2nY(\lambda_a) = \sum_{j=1}^{2n} \left(\langle J_a B(Y, e_j), J e_j \rangle\right) + \sum_{j=1}^{2n} \left(\langle A^{(J, e_j)^\perp}(Y), \tilde{J} e_j \rangle\right)
= \sum_{j=1}^{2n} \langle J_a B(Y, e_j), J e_j \rangle + \sum_{j=1}^{2n} \langle B(Y, J e_j), J_a e_j \rangle
= 2 \sum_{j=1}^{2n} \langle B(Y, J e_j), J_a e_j \rangle
= 2 \sum_{j=1}^{2n} \langle A^{(J, e_j)^\perp}(J e_j), Y \rangle.
\]

Thus

\[
n\nabla \lambda_a = \sum_{j=1}^{2n} A^{(J, e_j)^\perp}(J e_j). \tag{3.4}
\]

On the other hand, from (3.1), we derive

\[
\sum_{j=1}^{2n} \langle B(J e_j, X), J_a e_j \rangle = \sum_{j=1}^{2n} \langle B(J e_j, e_j), J_a X \rangle + \sum_{j=1}^{2n} \langle \nabla \lambda_a, J e_j \rangle \langle X, J e_j \rangle
= \langle \nabla \lambda_a, X \rangle.
\]

Thus,

\[
\nabla \lambda_a = \sum_{j=1}^{2n} A^{(J, e_j)^\perp}(J e_j). \tag{3.5}
\]

Combining (3.4) with (3.5), we conclude that

\[
(n - 1)\nabla \lambda_a = 0, \quad \alpha = 1, 2, 3.
\]

Consequently, if \(n > 1\), then \(dJ = 0\) which implies that the complex phase map \(J\) is a constant map. In particular, \(L\) is complex Lagrangian when \(n > 1\). \(\square\)
4. Hyper-Lagrangian surfaces

Let $\Sigma$ be a closed surface immersed in a hyperkähler 4-manifold $M$. As shown in the previous section, we know that $\Sigma$ is a hyper-Lagrangian surface in $M$ with holomorphic norm bundle $T^*\Sigma$. Leung and Wang obtained the following formula ([45, equation (1.1)])

$$\partial J = \frac{\sqrt{-1}}{2} \eta \Omega J.$$  \hfill (4.1)

Introduce the curvature form $H \in \Gamma \left( T^*\Sigma \otimes J^{-1}T^2 \right)$ as follows:

$$H(X) := \langle (H, J_1 X), (H, J_2 X), (H, J_3 X) \rangle \in T J_0 \mathbb{S}^2, \quad \forall X \in T\Sigma.$$

Recall the complex structure $J$ on $T^2$: for every tangent vector field $(a, b, c) \in T^2 \subset T\mathbb{R}^3$,

$$J_S(a, b, c) = (\lambda_1, \lambda_2, \lambda_3) \times (a, b, c)$$

as follows:

$$= \begin{pmatrix} \lambda_2 & \lambda_3 & \lambda_1 \\ b & c & a \\ c & a & b \end{pmatrix}.$$

We can reformulate (4.1) as follows

**Lemma 4.1.**

$$\partial J = -\frac{\sqrt{-1}}{4} H - \frac{1}{4} J_{\mathbb{S}^2} \circ H.$$ \hfill (4.2)

Consequently,

$$|\partial J|^2 = \frac{1}{4} |H|^2.$$

In particular, $\Sigma$ is minimal if $J$ is anti-holomorphic.

**Proof.** For every tangent vector $X \in T\Sigma$ and normal vector $V \in T^*\Sigma$, we have

$$\langle V, J_1 \dot{J} X \rangle = \lambda_2 \langle V, J_3 X \rangle - \lambda_3 \langle V, J_2 X \rangle,$$

$$\langle V, J_2 \dot{J} X \rangle = -\lambda_1 \langle V, J_3 X \rangle + \lambda_3 \langle V, J_1 X \rangle,$$

$$\langle V, J_3 \dot{J} X \rangle = \lambda_1 \langle V, J_2 X \rangle - \lambda_2 \langle V, J_1 X \rangle.$$

In other words,

$$\left\langle \left( \langle V, J_1 \dot{J} X \rangle, \langle V, J_2 \dot{J} X \rangle, \langle V, J_3 \dot{J} X \rangle \right) \right\rangle = J_{\mathbb{S}^2} \left( \langle V, J_1 X \rangle, \langle V, J_2 X \rangle, \langle V, J_3 X \rangle \right).$$ \hfill (4.3)

Thus,

$$H \circ J = J_{\mathbb{S}^2} \circ H.$$

According to (3.3), we get

$$H = \sum_{a=1}^3 J^a J_a \nabla \lambda_a.$$

It follows that

$$\langle H, J_a X \rangle = \sum_{b=1}^3 \langle J_b \nabla \lambda_b, J_a X \rangle = -\sum_{b=1}^3 J_b \nabla \lambda_b, J_a \dot{J} X \rangle = \langle \sum_{b=1}^3 J_b \nabla \lambda_b, J_a \dot{J} X \rangle.$$
Combining with (4.3), we get

\[ H(X) = J_{\Sigma} \left( \sum_{\beta=1}^{3} J_{\beta} \nabla \lambda_{\beta}, J_{1} X \right), \]

\[ \left( \sum_{\beta=1}^{3} J_{\beta} \nabla \lambda_{\beta}, J_{2} X \right), \]

\[ \left( \sum_{\beta=1}^{3} J_{\beta} \nabla \lambda_{\beta}, J_{3} X \right) \].

Direct calculation yields

\[ \left( \sum_{\beta=1}^{3} J_{\beta} \nabla \lambda_{\beta}, J_{1} X \right) = \langle \nabla \lambda_{1} + \alpha_{2} J \nabla \lambda_{3} - \lambda_{1} J \nabla \lambda_{2}, X \rangle, \]

\[ \left( \sum_{\beta=1}^{3} J_{\beta} \nabla \lambda_{\beta}, J_{2} X \right) = \langle \nabla \lambda_{2} + \alpha_{3} J \nabla \lambda_{1} - \lambda_{2} J \nabla \lambda_{3}, X \rangle, \]

\[ \left( \sum_{\beta=1}^{3} J_{\beta} \nabla \lambda_{\beta}, J_{3} X \right) = \langle \nabla \lambda_{3} + \alpha_{1} J \nabla \lambda_{2} - \lambda_{3} J \nabla \lambda_{1}, X \rangle, \]

which implies

\[ \left( \sum_{\beta=1}^{3} J_{\beta} \nabla \lambda_{\beta}, J_{1} X \right), \left( \sum_{\beta=1}^{3} J_{\beta} \nabla \lambda_{\beta}, J_{2} X \right), \left( \sum_{\beta=1}^{3} J_{\beta} \nabla \lambda_{\beta}, J_{3} X \right) = dJ(X) - J_{\Sigma} \circ dJ(JX). \]

Therefore, we have

\[ H = dJ \circ \tilde{J} + J_{\Sigma} \circ dJ. \]

By the definition

\[ \partial J = \frac{1}{4} \left( dJ - J_{\Sigma} \circ dJ \circ \tilde{J} \right) - \frac{\sqrt{-1}}{4} \left( dJ \circ \tilde{J} + J_{\Sigma} \circ dJ \right). \]

Hence we obtain

\[ \partial J = - \frac{\sqrt{-1}}{4} H - \frac{1}{4} J_{\Sigma} \circ H. \]

**Theorem 4.2.** Let \( \Sigma \) be a closed surface immersed in a hyperkähler 4-manifold \( M \), then

\[ \tau(J) = J_{\Sigma} \left( \text{div} \left( J_{1} H \right)^{\top}, \text{div} \left( J_{2} H \right)^{\top}, \text{div} \left( J_{3} H \right)^{\top} \right). \]

Moreover,

\[ \det (dJ) = \kappa + \kappa^{\perp}, \]

where

\[ \kappa = R(e_{1}, e_{2}, e_{1}, e_{2}), \quad \kappa^{\perp} = \left( R^{\perp}(e_{1}, e_{2}) v_{2}, v_{1} \right). \]

Here \( e_{1}, e_{2}, v_{1}, v_{2} \) determines the orientation of M. As a consequence,

\[ 2 \deg(J) = \chi(T \Sigma) + \chi(T^{\perp} \Sigma). \]
Proof. Since
\[ \partial J = \frac{1}{4} \left( dJ - J_{g^2} \circ dJ \circ dJ \right) = \sqrt{-1} \frac{1}{4} \left( dJ \circ J + J_{g^2} \circ dJ \right), \]
we have
\[ 2 \sum_{j=1}^{2} \left( \nabla_{e_j} \partial J \right) (e_j) = \frac{1}{4} \left( 1 - \sqrt{-1} J_{g^2} \right) \tau (J). \]
Moreover,
\[
\text{div} \left( J_{a} \mathbf{H}^{T} \right) = 2 \sum_{j=1}^{2} \left\{ \nabla_{e_j} J_{a} \mathbf{H}^{T} , e_j \right\}
= 2 \sum_{j=1}^{2} e_j \left\langle J_{a} \mathbf{H} , e_j \right\rangle - 2 \sum_{j=1}^{2} \left\langle J_{a} \mathbf{H} , \nabla_{e_j} e_j \right\rangle
= 2 \sum_{j=1}^{2} \left\langle J_{a} \nabla_{e_j} \mathbf{H} , e_j \right\rangle - 2 \sum_{j=1}^{2} \left\langle J_{a} \mathbf{H}^{T} (e_j) , e_j \right\rangle + 2 \sum_{j=1}^{2} \left\langle J_{a} \mathbf{H} , \mathbf{H} \right\rangle
= 2 \sum_{j=1}^{2} \left\langle J_{a} \nabla_{e_j} \mathbf{H} , e_j \right\rangle.
\]
Thus,
\[ \text{div} \left( J_{a} \mathbf{H}^{T} \right) = 2 \sum_{j=1}^{2} \left\langle J_{a} \nabla_{e_j} \mathbf{H} , e_j \right\rangle. \quad (4.5) \]
From (4.2), we get
\[ \tau (J) = J_{g^2} \left( \text{div} \left( J_{1} \mathbf{H}^{T} \right) , \text{div} \left( J_{2} \mathbf{H}^{T} \right) , \text{div} \left( J_{3} \mathbf{H}^{T} \right) \right). \]
According to Lemma 4.1,
\[
\det (dJ) = |\partial J|^2 - |\partial J|^2
= 2 |\partial J|^2 - \frac{1}{2} |dJ|^2
= \frac{1}{2} |\mathbf{H}|^2 - \frac{1}{2} |dJ|^2. \quad (4.6)
\]
By using (3.3), we have
\[ |dJ(X)|^2 |Y|^2 = \left| \mathbf{B} \left( X , \tilde{J} Y \right) - \tilde{J}^{\flat} \mathbf{B} \left( X , Y \right) \right|^2. \]
It follows that
\[
|dJ|^2 = \frac{1}{2} \sum_{j,k=1}^{2} \left| \mathbf{B} \left( e_j , J e_k \right) - J^{\flat} \mathbf{B} \left( e_j , e_k \right) \right|^2
= \frac{1}{2} \left| \mathbf{B} \left( e_j , \tilde{J} e_k \right) \right|^2 + \frac{1}{2} |\tilde{J}^{\flat} \mathbf{B} (e_j , e_k)|^2 - \sum_{j,k=1}^{2} \left\langle \mathbf{B} \left( e_j , \tilde{J} e_k \right) , J^{\flat} \mathbf{B} \left( e_j , e_k \right) \right\rangle.
\]
Set $e_2 = J e_1 = \bar{J}_1 e_1, v_1 = \bar{J}_2 e_1$ and $v_2 = \bar{J}^2 v_1 = J_3 e_1$, then by applying the Gauss equation and Ricci equation, we obtain

\[
|dJ|^2 - |H|^2 = |B|^2 - |H|^2 - \sum_{j,k=1}^2 \left\langle B(e_j, J e_k), \bar{J}^j B(e_j, e_k) \right\rangle
\]

\[
= |B|^2 - |H|^2 + \sum_{j,k=1}^2 \left\langle B(e_j, J e_k), v_1 \right\rangle \left\langle B(e_j, e_k), v_2 \right\rangle - \sum_{j,k=1}^2 \left\langle B(e_j, J e_k), v_2 \right\rangle \left\langle B(e_j, e_k), v_1 \right\rangle
\]

\[
= |B|^2 - |H|^2 + \sum_{j=1}^2 \left( R^j (J e_j, e_j) v_2, v_1 \right) - \sum_{j=1}^2 \bar{R} (J e_j, e_j, v_2, v_1)
\]

\[
= |B|^2 - |H|^2 - 2 \left( R^j (e_j, e_2) v_2, v_1 \right) - 2 \bar{R} (e_1, e_2, J_2 e_1, J_2 e_2)
\]

\[
= |B|^2 - |H|^2 - 2 \left( R^j (e_j, e_2) v_2, v_1 \right) - 2 \bar{R} (e_1, e_2, e_1, e_2)
\]

\[
= -2 R(e_1, e_2, e_1, e_2) - 2 \left( R^j (e_j, e_2) v_2, v_1 \right).
\]

Namely,

\[
|dJ|^2 - |H|^2 = -2\kappa - 2\kappa^+.
\] (4.7)

Substituting (4.7) into (4.6), we derive

\[
\text{det} (dJ) = \kappa + \kappa^+.
\]

By applying the Gauss-Bonnet formula, we get

\[
4\pi \deg(J) = \int_{\Sigma} \text{det} (dJ) = \int_{\Sigma} \kappa + \int_{\Sigma} \kappa^+ = 2\pi \chi (T\Sigma) + 2\pi \chi (T^+\Sigma),
\]

i.e.,

\[
2 \deg(J) = \chi (T\Sigma) + \chi (T^+\Sigma).
\]

Corollary 4.3. Let $\Sigma$ be a closed surface immersed in a hyperkähler 4-manifold $M$. Assume

\[
\text{div} \left( (J_{\alpha} H)^{\alpha} \right) = 0, \quad \alpha = 1, 2, 3,
\]

and $2\chi (T\Sigma) + \chi (T^+\Sigma) > 0$, then $J$ is holomorphic.

Proof. According to (4.4), the assumption means that $J$ is a harmonic map. Then this Corollary is a consequence of the following observation: if $J$ is not holomorphic, then

\[
\Delta \log |\partial J| = \text{det} (dJ) + \kappa
\]

\[
= 2\kappa + \kappa^+,
\]

holds when $\partial J \neq 0$. □

Moreover,
Corollary 4.4. Let $\Sigma$ be a closed surface immersed in a hyperkähler 4-manifold $M$. Assume
\[ \text{div} \left( \left( J_\alpha \mathbf{H} \right)^\top \right) = 0, \quad \alpha = 1, 2, 3, \]
and $\chi (T^\perp \Sigma) < 0$, then $L$ is minimal.

Proof. It is a consequence of Lemma 4.1 and the following observation: if $J$ is not anti-holomorphic, then
\[ \Delta \log |\partial J| = - \det (dJ) + \kappa \]
holds when $\partial J \neq 0$. $\square$

5. Nonexistence of Type I singularity of MCF

In this section, we consider the mean curvature flow from a closed surface in a hyperkähler 4-manifold $M$, i.e., we consider
\begin{align}
\frac{\partial F}{\partial t} &= H, \quad \Sigma \times [0, T); \\
F(\cdot, t) &= F_0(\cdot), \quad \Sigma.
\end{align}
(5.1)

Here $F_0 : \Sigma \rightarrow M$ is an isometric immersion. This flow blows up when
\[ \limsup_{t \rightarrow T} \max_{\Sigma} |B| = \infty. \]

We say that the mean curvature flow $F$ has Type I singularity at $T > 0$ if
\[ \limsup_{t \rightarrow T} \sqrt{T - t} \max_{\Sigma} |B| \leq C, \]
for some positive constant $C$.

We shall need the following theorem which is owed to Leung-Wan (see [45, Theorem 3.4]), here we would like to give an alternative proof.

Theorem 5.1 ([45]). The complex phase maps of the mean curvature flow (5.1) $J : \Sigma_t \rightarrow \mathbb{S}^2$ form an evolving harmonic map heat flow, i.e.,
\[ \frac{\partial J}{\partial t} = \tau(J), \]
where $\tau(J)$ is the tension field of $J$ with respect to the induced metric $g_t$ on $\Sigma_t$.

Proof. Let $\{e_i\}$ be a local orthonormal evolving frame field on $\Sigma_t$, then both $[e_i, \partial_t]$ and $[\tilde{J}e_i, \partial_t]$ are local tangent vector fields and
\[ \lambda_\alpha \delta_{ij} = \left\langle J_\alpha e_i, \tilde{J} e_j \right\rangle. \]
Differentiate with respect to $t$ on both sides of the above equality,
\[ \frac{\partial \lambda_\alpha}{\partial t} \delta_{ij} = \frac{\partial}{\partial t} \left\langle J_\alpha e_i, \tilde{J} e_j \right\rangle = \left\langle J_\alpha \tilde{\nabla}_\alpha e_i, \tilde{J} e_j \right\rangle + \left\langle J_\alpha e_i, \tilde{\nabla}_\alpha \tilde{J} e_j \right\rangle = \lambda_\alpha \left\langle J \left( \tilde{\nabla}_\alpha e_i \right)^\top, \tilde{J} e_j \right\rangle + \lambda_\alpha \left\langle J_\alpha e_i, \left( \tilde{\nabla}_\alpha e_j \right)^\top \right\rangle + \lambda_\alpha \left\langle J_\alpha e_i, \left( \tilde{\nabla}_\alpha \tilde{J} e_j \right)^\top \right\rangle \]
\[ = \lambda_\alpha \left\langle J \left( \tilde{\nabla}_\alpha e_i \right)^\top, \tilde{J} e_j \right\rangle + \lambda_\alpha \left\langle J_\alpha e_i, \left( \tilde{\nabla}_\alpha e_j \right)^\top \right\rangle + \lambda_\alpha \left\langle J_\alpha e_i, \left( \tilde{\nabla}_\alpha \tilde{J} e_j \right)^\top \right\rangle. \]
Since \( \sum_{\alpha=1}^{3} A_{\alpha}^2 = 1 \), we get

\[
0 = \left< \nabla_{\partial_{\alpha}} e_i, J e_j \right> + \left< J e_i, \left( \nabla_{\partial_{\alpha}} J e_j \right)^{\top} \right>.
\]

Thus

\[
\frac{\partial \lambda_{\alpha}}{\partial t} \delta_{ij} = \left< J_{\alpha} \left( \nabla_{e_i} \partial_{\alpha} \right)^{\top}, J e_j \right> + \left< J_{\alpha} e_i, \left( \nabla_{J e_j} \partial_{\alpha} \right)^{\top} \right>.
\]

Consequently,

\[
2 \frac{\partial \lambda_{\alpha}}{\partial t} = \sum_{j=1}^{2} \left< J_{\alpha} \left( \nabla_{e_i} \partial_{\alpha} \right)^{\top}, J e_j \right> + \sum_{j=1}^{2} \left< J_{\alpha} e_i, \left( \nabla_{J e_j} \partial_{\alpha} \right)^{\top} \right> = 2 \sum_{j=1}^{2} \left< J_{\alpha} e_i, \left( \nabla_{J e_j} \partial_{\alpha} \right)^{\top} \right>.
\]

By (4.5) and (4.3), we get

\[
(\tau(J))^\alpha = - \sum_{j=1}^{2} \left< \nabla_{e_i} H, J_{\alpha} e_j \right> = - \sum_{j=1}^{2} \left< \nabla_{J e_j} H, -J_{\alpha} e_j \right> = \sum_{j=1}^{2} \left< \nabla_{J e_j} H, J_{\alpha} e_j \right>
\]

Hence

\[
\frac{\partial \lambda_{\alpha}}{\partial t} - (\tau(J))^\alpha = \sum_{j=1}^{2} \left( \left< \nabla_{J e_j} \left( \frac{\partial F}{\partial t} \right) - H, J_{\alpha} e_j \right> \right) = 0, \quad \lambda = 1, 2, 3.
\]

Namely,

\[
\frac{\partial J}{\partial t} = \tau(J).
\]

Next, we show that the complex phase map is a generalized harmonic map (cf. [11])

**Theorem 5.2.** Let \( X : \Sigma^{2} \longrightarrow \mathbb{R}^{3} \) be a self-shrinker, i.e., \( H = -\frac{1}{2} X^{\top} \), then the complex phase map \( J : \Sigma \longrightarrow \mathbb{S}^{2} \) satisfies

\[
\tau(J) = \frac{1}{2} dJ(X^{\top}).
\]

**Proof.** Since

\[
\nabla_{e_i} X^{\top} = - A X^{\top}(e_i) - B (e_i, X^{\top}),
\]

we get

\[
\sum_{j=1}^{2} \left< \nabla_{J e_j} H, J_{\alpha} e_j \right> = - \frac{1}{2} \sum_{j=1}^{2} \left< \nabla_{J e_j} X^{\top}, J_{\alpha} e_j \right> = - \frac{1}{2} \sum_{j=1}^{2} \left< \nabla_{J e_j} X^{\top}, J_{\alpha} e_j \right> - \frac{1}{2} \sum_{j=1}^{2} \left< A X^{\top}(J e_j), J_{\alpha} e_j \right> = \frac{1}{2} \sum_{j=1}^{2} \left< A X^{\top}(J e_j) + B (J e_j, X^{\top}) - \frac{1}{2} \sum_{j=1}^{2} \left< A X^{\top}(J e_j), J_{\alpha} e_j \right> = \frac{1}{2} \sum_{j=1}^{2} \left< A (J e_j)^{\top}(J e_j), X \right> = \frac{1}{2} \left< \nabla_{J_{\alpha}} X, X \right>.
\]

\[\square\]
The last equality follows from (3.5). Applying (4.5), we conclude
\[
\tau(J) = J_2 \left( \text{div} \left( J_1 \mathbf{H} \right), \text{div} \left( J_2 \mathbf{H} \right), \text{div} \left( J_3 \mathbf{H} \right) \right) \\
= J_2 \left( \sum_{j=1}^2 \langle J_1 \nabla_{e_j}^\perp \mathbf{H}, e_j \rangle, \sum_{j=1}^2 \langle J_2 \nabla_{e_j}^\perp \mathbf{H}, e_j \rangle, \sum_{j=1}^2 \langle J_3 \nabla_{e_j}^\perp \mathbf{H}, e_j \rangle \right) \\
= - J_2 \left( \sum_{j=1}^2 \langle \nabla_{e_j}^\perp \mathbf{H}, J_1 e_j \rangle, \sum_{j=1}^2 \langle \nabla_{e_j}^\perp \mathbf{H}, J_2 e_j \rangle, \sum_{j=1}^2 \langle \nabla_{e_j}^\perp \mathbf{H}, J_3 e_j \rangle \right) \\
= \left( \sum_{j=1}^2 \nabla_{e_j}^\perp \mathbf{H}, J_1 e_j \right), \sum_{j=1}^2 \langle \nabla_{e_j}^\perp \mathbf{H}, J_2 e_j \rangle, \sum_{j=1}^2 \langle \nabla_{e_j}^\perp \mathbf{H}, J_3 e_j \rangle \right) \\
= \frac{1}{2} \text{d} J \left( X^\top \right).
\]

\( \square \)

Using Theorem 5.2 and integral method, we shall prove Theorem 1.2.

Denote \( \overline{S} \triangleq \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_1 = 0, x_2 \geq 0\} \) and put \( \mathcal{V} \triangleq \mathbb{S}^2 \setminus \overline{S} \).

**Proof of Theorem 1.2.** Let \( X : \Sigma \longrightarrow \mathbb{R}^4 \) be a complete proper self-shrinker surface. Firstly, we shall prove that when the image of the complex phase map \( J \) is contained in an open hemisphere, then \( \Sigma \) must be a plane.

Indeed, according to Theorem 5.2, we know that
\[
\tau(J) = \frac{1}{2} \text{d} J \left( X^\top \right).
\]

Let \( \rho \) be the distance function on \( \mathbb{S}^2 \), and define \( \psi := (1 - \cos \rho) \). Let \( u := \psi \circ J \). Then
\[
\Delta \psi u = \sum_{j=1}^2 \text{Hess}(\psi) \left( \text{d} J \left( e_j \right), \text{d} J \left( e_j \right) \right) + \text{d} \varphi \left( \tau(J) + \text{d} J \left( \frac{X^\top}{2} \right) \right) = (\cos \rho) |\text{d} J|^2 \geq 0.
\]

with the equality holds iff \( \text{d} J = 0 \). Since
\[
\Delta \psi u = \Delta u - \frac{1}{2} \left( X^\top, \nabla u \right) = e^{-\frac{\rho}{2}} \text{div} \left( e^{-\frac{\rho}{2}} \nabla u \right).
\]

It follows that
\[
\text{div} \left( e^{-\frac{\rho}{2}} \nabla u \right) \geq 0.
\]

For every compactly supported Lipschitz function \( \eta \) on \( \mathbb{R}^4 \), since \( \Sigma \) is proper, we know that \( \eta|_{\Sigma} \) is also compactly supported in \( \Sigma \). Consequently,
\[
\int_{\Sigma} \text{div} \left( e^{-\frac{\rho}{2}} \nabla u \right) \eta^2 u \geq 0.
\]

which implies that
\[
\int_{\Sigma} \nabla u \eta^2 e^{-\frac{\rho}{2}} + 2 \int_{\Sigma} \langle \nabla u, \nabla \eta \rangle \eta e^{-\frac{\rho}{2}} \leq 0.
\]

Notice that
\[
|\nabla (\eta u)|^2 = |\nabla \eta|^2 u^2 + \eta^2 |\nabla u|^2 + 2 \langle \nabla u, \nabla \eta \rangle \eta u,
\]

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we obtain
\[ \int_{\Sigma} |\nabla(\eta u)|^2 e^{\frac{u^2}{e}} \leq \int_{\Sigma} |\nabla \eta|^2 u^2 e^{\frac{-u^2}{e}}. \]

For every \( R > 0 \), choose
\[
\eta(y) = \begin{cases} 
1, & |y| \leq R; \\
\frac{2R - |y|}{R}, & R < |y| < 2R; \\
0, & |y| \geq 2R.
\end{cases}
\]

Then \( \nabla \eta = 0 \) for \( |x| < R \) or \( |x| > 2R \). For \( R < |x| < 2R \),
\[ |\nabla \eta(x)| \leq \frac{1}{R}. \]

Consequently,
\[
\int_{\Sigma \cap B_R} |\nabla u|^2 e^{\frac{u^2}{e}} \leq \frac{1}{R^2} \int_{\Sigma \cap (B_{2R} \setminus B_R)} u^2 e^{\frac{-u^2}{e}} \leq \frac{1}{R^2} \int_{\Sigma \cap (B_{2R} \setminus B_R)} e^{\frac{-u^2}{e}}.
\]

Since \( \Sigma \) is proper, we have
\[ \int_{\Sigma} e^{\frac{-u^2}{e}} < \infty. \]

Letting \( R \to \infty \), we get
\[ \int_{\Sigma} |\nabla u|^2 e^{\frac{u^2}{e}} = 0. \]

This implies that \( u \equiv \text{constant} \), namely \( J \equiv \text{constant} \). Hence \( \Sigma \) is minimal and \( X^\perp = 0 \). The fact \( X = X^\perp \) gives \( B(X, \cdot) = 0 \). By the minimal condition, we know that \( \Sigma \) is totally geodesic. Since \( \Sigma \) is complete, we conclude that \( \Sigma \) is a plane.

Now we consider the projection \( \pi \) from \( \mathbb{S}^2 \) onto \( \mathbb{D}^2 \) (here \( \mathbb{D}^2 \) is a 2-dimensional closed unit disk)
\[ \pi : \mathbb{S}^2 \to \mathbb{D}^2, \quad (x_1, x_2, x_3) \mapsto (x_1, x_2). \]

Then \( x \in \mathcal{V} \) if and only if \( \pi(x) \) is contained in the domain obtained by removing the radius connecting \((0,0)\) and \((0,1)\) from the closed unit disk. Therefore for any \( x \in \mathcal{V} \), there exists a unique \((0,1]-\)valued function \( r \) and a unique \((0,2\pi)-\)valued function \( \varphi \) on \( \mathcal{V} \), such that
\[ \pi(x) = (r \sin \varphi, r \cos \varphi). \]

Direct computation gives us (see [41, formula (2.12)])
\[ \text{Hess} \varphi = -r^{-1}(d\varphi \otimes dr + dr \otimes d\varphi). \]

It follows that
\[ \Delta_{\mathbb{S}^2}(\varphi \circ J) = \text{Hess}_\varphi(dJ(e_1), dJ(e_1)) = -2(r \circ J)^{-1} \left\langle \nabla (r \circ J), \nabla (\varphi \circ J) \right\rangle. \]

Thus
\[
div \left( (r \circ J)^2 e^{\frac{-u^2}{e}} \nabla (\varphi \circ J) \right) = (r \circ J)^2 \left( e^{\frac{-u^2}{e}} \nabla (\varphi \circ J) + \left\langle \nabla (r \circ J)^2, e^{\frac{-u^2}{e}} \nabla (\varphi \circ J) \right\rangle \right)
= e^{\frac{-u^2}{e}} (r \circ J)^2 \Delta_{\mathbb{S}^2}(\varphi \circ J) + \left\langle \nabla (r \circ J)^2, e^{\frac{-u^2}{e}} \nabla (\varphi \circ J) \right\rangle.
\]
\[-2 \left( (r \circ J) \nabla (r \circ J), e^{\frac{\theta}{4}} \nabla (\varphi \circ J) \right) + \left( \nabla (r \circ J)^2, e^{\frac{\theta}{4}} \nabla (\varphi \circ J) \right) = 0.\]

Let \( \eta \) be as before, multiplying \( \eta^2 \cdot (\varphi \circ J) \) with both sides of the above equality,

\[0 = \int_{\Sigma} \eta^2 \cdot (\varphi \circ J) \text{div} \left( (r \circ J)^2 e^{-\frac{\theta}{4}} \nabla (\varphi \circ J) \right) \]

\[= \int_{\Sigma} \text{div} \left( \eta^2 \cdot (\varphi \circ J) (r \circ J)^2 e^{-\frac{\theta}{4}} \nabla (\varphi \circ J) \right) - \int_{\Sigma} \left( \nabla (\eta^2 \cdot (\varphi \circ J)), \nabla (\varphi \circ J) \right) (r \circ J)^2 e^{-\frac{\theta}{4}} \]

\[= - \int_{\Sigma} \eta^2 |\nabla (\varphi \circ J)|^2 (r \circ J)^2 e^{-\frac{\theta}{4}} - 2 \int_{\Sigma} \left( (\varphi \circ J) \nabla \eta, \eta \nabla (\varphi \circ J) \right) (r \circ J)^2 e^{-\frac{\theta}{4}} \]

\[\leq - \int_{\Sigma} \eta^2 |\nabla (\varphi \circ J)|^2 (r \circ J)^2 e^{-\frac{\theta}{4}} + 2 \int_{\Sigma} (\varphi \circ J)^2 |\nabla \eta|^2 (r \circ J)^2 e^{-\frac{\theta}{4}} \]

\[+ \frac{1}{2} \int_{\Sigma} \eta^2 |\nabla (\varphi \circ J)|^2 (r \circ J)^2 e^{-\frac{\theta}{4}}.\]

Therefore we obtain

\[\int_{\Sigma} \eta^2 |\nabla (\varphi \circ J)|^2 (r \circ J)^2 e^{-\frac{\theta}{4}} \leq 4 \int_{\Sigma} |(\varphi \circ J)|^2 |\nabla \eta|^2 (r \circ J)^2 e^{-\frac{\theta}{4}}.\]

It follows that

\[\int_{\Sigma \cap B_R} |\nabla (\varphi \circ J)|^2 (r \circ J)^2 e^{-\frac{\theta}{4}} \leq \int_{\Sigma} \eta^2 |\nabla (\varphi \circ J)|^2 (r \circ J)^2 e^{-\frac{\theta}{4}} \]

\[\leq 4 \int_{\Sigma} (\varphi \circ J)^2 |\nabla \eta|^2 (r \circ J)^2 e^{-\frac{\theta}{4}} \leq \frac{16 \pi^2}{R^2} \int_{\Sigma \cap (B_{2R})} e^{-\frac{\theta}{4}}.\]

Letting \( R \to \infty \), we then derive \( |\nabla (\varphi \circ J)| \equiv 0 \). Thus \( \varphi \circ J \equiv \varphi_0 \in (0, 2\pi) \).

Denote \( b_0 := (\sin \varphi_0, \cos \varphi_0, 0) \), note that for any \( p \in \Sigma, J(p) = ((r \circ J(p)) \sin (\varphi \circ J(p)), (r \circ J(p)) \cos (\varphi \circ J(p)), x_3) \), then for any \( p \in \Sigma \), the inner product of \( J(p) \) and \( b_0 \) in \( \mathbb{R}^3 \) is \( (J(p), b_0) = r(J(p)) (\sin^2 \varphi_0 + \cos^2 \varphi_0) = r(J(p)) > 0 \). This implies that the image of \( J \) is contained in an open hemisphere centered at \( b_0 \). Hence \( \Sigma \) is a plane. \(\square\)

**Example 5.1** (Cylinder). Consider the cylinder

\[\Sigma^2 := \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \mid (x^1)^2 + (x^2)^2 = 1, x^3 = 0\} \subset \mathbb{R}^4,\]

which is a nontrivial self-shrinker. It is easy to see that \( \nu := x^1 \frac{\partial}{\partial x^0} + x^2 \frac{\partial}{\partial x^0} \) and \( \frac{\partial}{\partial x^0} \) are normal vectors of \( \Sigma \) in \( \mathbb{R}^4 \) and \( e_1 := -x^2 \frac{\partial}{\partial x^4} + x^1 \frac{\partial}{\partial x^4} \), \( e_2 := \frac{\partial}{\partial x^4} \) are tangent vectors of \( \Sigma \). Let \( J \) be the almost complex structure on \( \Sigma \) with \( Je_1 = e_2, Je_2 = -e_1 \).

In \( \mathbb{R}^4 \), under the natural basis \( \left\{ \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\} \), we have

\[
J_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad J_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad J_3 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Direct computation gives us

\[
J_1 e_1 = x^2 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial x^2}, \quad J_1 e_2 = -\frac{\partial}{\partial x^3}, \quad J_2 e_1 = x^2 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial x^2}, \quad J_2 e_2 = \frac{\partial}{\partial x^4}, \quad J_3 e_1 = x^2 \frac{\partial}{\partial x^3} - x^1 \frac{\partial}{\partial x^2}, \quad J_3 e_2 = \frac{\partial}{\partial x^2}
\]
Therefore we have
\[ J_{1|k} = 0, \]
\[ J_{2|k} e_1 = x^2 e_2 = x^2 J e_1, \quad J_{2|k} e_2 = \frac{\partial}{\partial x^1}, v = -x^2 e_1 = x^2 J e_2, \]
\[ J_{3|k} e_1 = -x^1 e_2 = -x^1 J e_1, \quad J_{3|k} e_2 = \frac{\partial}{\partial x^2}, v = x^1 e_1 = -x^1 J e_2. \]

Thus the complex phase map \( J \) can be represented by \((0, x^2, -x^1)\). Note that \((x^1)^2 + (x^2)^2 = 1\), this implies the image of \( J \) is a great circle. Clearly, even we add a point to \( \nu \), it will contain a great circle. Hence this example illustrates that the image restriction of the complex phase map in Theorem 1.2 is optimal.

**Corollary 5.3.** Let \( X : \Sigma^2 \to \mathbb{R}^4 \) be a complete proper symplectic self-shrinking surface, then \( \Sigma \) must be a plane.

**Proof.** The symplectic condition implies that the image of the complex phase map is contained in an open hemisphere. \( \square \)

**Remark 5.1.** Arezzo-Sun [2] proved that complete proper symplectic self-shrinker surfaces in \( \mathbb{R}^4 \) must be a plane under different conditions on the second fundamental form, flat normal bundle or bounded geometry (see [2, Main Theorems 3, 4, 5]).

For the rigidity of translating soliton, we have the following

**Theorem 5.4.** Let \( X : \Sigma^2 \to \mathbb{R}^4 \) be a complete translating soliton surface with flat normal bundle. Assume the image of the complex phase map is contained in a regular ball in \( \mathbb{S}^3 \), i.e., a geodesic ball \( B_k(q) \) disjoint from the cut locus of \( q \) and \( R < \frac{\pi}{2} \), then \( \Sigma \) has to be a plane.

**Proof.** Since we can view \( \Sigma \) as a hyper-Lagrangian submanifold in \( \mathbb{R}^4 \) with respect to some almost complex structure \( J \), let \( \{ e_1, e_2 = \tilde{J} e_1 \} \) be a local orthonormal frame field on \( \Sigma \) such that \( \nabla e_i = 0 \) at the considered point. Denote \( v_1 = J_2 e_1, v_2 = J_2 e_2 = \tilde{J} J_2 e_1, \) then \( \{ v_1, v_2 \} \) is a local orthonormal frame bundle along \( \Sigma \). Recall the translating soliton equation \( H = -V_0^\perp \), here \( V_0 \) is a fixed unit vector. Denote \( V := V_0^\perp \), we obtain

\[
L_v e_i = \nabla_v e_i - \nabla_v (-V) = -\left( \nabla_v e_i \right) \nabla_v e_i + \nabla_v \left( \left( V_0, e_i \right) e_j \right) \\
= - \left( V_0, e_i \right) B \left( e_j, e_i \right) + \left( V_0, B \left( e_j, e_i \right) e_j + \left( V_0, e_j \right) B \left( e_i, e_j \right) \right) e_j \\
= -H^* h_0^\nu_0 e_j.
\]

It follows that
\[
-\frac{1}{2} \left( L_v g \right) (e_i) = -\frac{1}{2} \left( L_v g \right) (e_i, e_j) e_j = \frac{1}{2} g \left( L_v v_0, e_j \right) e_j + \frac{1}{2} g \left( L_v e_j, e_i \right) e_j = -H^* h_0^\nu_0 e_j.
\]

The Gauss equation and the above equality imply that
\[
\text{Ric}_v (e_i) = \text{Ric}(e_i) - \frac{1}{2} \left( L_v g \right) (e_i) = \left( H^* h_j^0 - h_0^0 h_{ik}^0 \right) e_j - H^* h_j^0 e_j = -\sum_{a, jk} h_{ik}^0 h_{jk}^0 e_j.
\]

From (3.3), we obtain
\[
\left\langle \left( dJ (e_i), dJ (e_j) \right) \right\rangle = \left\langle B \left( e_i, \tilde{J} e_1 \right) - \tilde{J}^* B \left( e_i, e_1 \right), B \left( e_j, \tilde{J} e_1 \right) - \tilde{J}^* B \left( e_j, e_1 \right) \right\rangle \\
= \left( B \left( e_i, e_2 \right) - \tilde{J}^* B \left( e_i, e_1 \right), v_1 \right) \left( B \left( e_j, e_2 \right) - \tilde{J}^* B \left( e_j, e_1 \right), v_1 \right) \\
+ \left( B \left( e_i, e_2 \right) - \tilde{J}^* B \left( e_j, e_1 \right), v_2 \right) \left( B \left( e_j, e_2 \right) - \tilde{J}^* B \left( e_i, e_1 \right), v_2 \right) \\
= \left( h_{i2}^0 + h_{j1}^0 \right) (h_{i2}^0 + h_{j1}^0) + \left( h_{i2}^0 - h_{j1}^0 \right) \left( h_{i2}^0 - h_{j1}^0 \right) \\
\]

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\[= \sum_{a=1}^{2} \sum_{k=1}^{2} h^a_{jk} h^a_{jk} + h^2_2 h^2_{j} + h^2_1 h^2_j - h^2_2 h^2_{j1} - h^2_1 h^2_{j2}.\]

We conclude that
\[\langle dJ(e_i), dJ(e_j) \rangle \left(\langle dJ(e_i), dJ(e_j) \rangle - \sum_{a=1}^{2} \sum_{k=1}^{2} h^a_{jk} h^a_{jk} \right)\]
\[= \langle dJ(e_i), dJ(e_j) \rangle \left( h^2_2 h^2_{j} + h^2_1 h^2_j - h^2_2 h^2_{j1} - h^2_1 h^2_{j2} \right)\]
\[= 2 \langle dJ(h^2_2 e_i), dJ(h^2_2 e_j) \rangle - \langle dJ(h^2_2 e_i), dJ(h^2_2 e_j) \rangle\]
\[= 2 \langle dJ(A^1(e_2)), dJ(A^2(e_2)) \rangle - 2 \langle dJ(A^1(e_1)), dJ(A^2(e_2)) \rangle.\]

Since the normal bundle is flat, by the Ricci equation, the coefficients of the second fundamental form \(h^a_{ij}\) satisfy
\[\sum_{a=1}^{2} (h^a_{i2} h^a_{j2} - h^a_{i2} h^a_{j2}) = 0,
\]
which means that two \((2 \times 2)\) matrices
\[(h^1_i), (h^2_i)\]
can be diagonalized simultaneously at a fixed point.

Therefore for any \(p \in \Sigma\), we can choose a local frame field \(e_1, e_2\) around \(p\) such that \(h^a_{ij} = \Lambda^a \delta_{ij}\) at \(p\), i.e.,
\[A^a(e_i) = \Lambda^a e_i, \quad i, \alpha = 1, 2.\]

Hence at the \(p\),
\[\langle dJ(e_1), dJ(e_2) \rangle = \Lambda^1_2 \Lambda^2_1 - \Lambda^1_1 \Lambda^2_2.\]

Thus,
\[\langle dJ(e_i), dJ(e_j) \rangle \left(\langle dJ(e_i), dJ(e_j) \rangle - \sum_{a=1}^{2} \sum_{k=1}^{2} h^a_{jk} h^a_{jk} \right)\]
\[= 2 \left( \Lambda^1_2 \Lambda^2_1 - \Lambda^1_1 \Lambda^2_2 \right) \langle dJ(e_1), dJ(e_2) \rangle \leq 0.\]

It is easy to see that the curvature tensor of \(\mathbb{S}^2\) satisfies
\[\sum_{i,j} R^2_{ij}(dJ(e_i), dJ(e_j), dJ(e_i), dJ(e_j)) = |dJ|^2 - \sum_{i,j} \langle dJ(e_1), dJ(e_2) \rangle^2.\]

From the translator equation, we get
\[\langle \nabla_{J_0}^+ V_0, J_0 e_j \rangle = - \langle \nabla_{J_e}^+ V_0^+ , J_0 e_j \rangle = - \langle \nabla_{J_e} V_0^+, J_0 e_j \rangle - \langle A^{\nu 0}(Je), J_0 e_j \rangle\]
\[= - \langle \nabla_{J_0} V_0, J_0 e_j \rangle + \langle \nabla_{J_e} V_0^+, J_0 e_j \rangle - \Lambda_0 \langle H, V_0 \rangle = \langle \nabla_{J_0} V_0, J_0 e_j \rangle - \Lambda_0 \langle H, V_0 \rangle.\]

Since
\[\langle \nabla_{J_0} V_0^+, J_0 e_j \rangle = \langle V_0, B(J_0, e_k) \rangle \langle e_k, J_0 e_j \rangle + \langle V_0, e_k \rangle \langle B(J_e, e_k), J_0 e_j \rangle\]
\[= \Lambda_0 \langle V_0, H \rangle + \langle A(\nu e)^+ (Je), V_0^+ \rangle.\]

Therefore
\[(\tau(J))^a = \langle \nabla_{J_0}^+ H, J_0 e_j \rangle = \langle A^{a(\nu e)^+}(Je), V_0^+ \rangle = \langle \nabla_{J_0}^+, V_0 \rangle = \langle \nabla_{J_0}^+, V \rangle.\]
Namely,
\[ \tau(J) + dJ(-V) = 0. \]
Thus \( J \) is a \(-V\)-harmonic map, then the Bochner formula (see [10, Lemma 1]) gives us
\[
\frac{1}{2} \Delta_{\mathcal{V}} |dJ|^2 = |\nabla dJ|^2 + \sum_i (dJ(Ric_{\mathcal{V}}(e_i)), dJ(e_i)) - \sum_{i,j} R^g_{ij}(dJ(e_i), dJ(e_j), dJ(e_i), dJ(e_j))
\]
\[
= |\nabla dJ|^2 - \sum_{i,j} h^a_ib^a_{jk}(dJ(e_i), dJ(e_j)) - (|dJ|^4 - \sum_{i,j} (dJ(e_i), dJ(e_j))^2)
\]
\[
\geq |\nabla dJ|^2 - |dJ|^4.
\]

Let \( \rho \) be the distance function on \( S^2 \), and \( h \) the Riemannian metric of \( S^2 \). Define \( \psi = 1 - \cos \rho \), then \( \text{Hess}(\psi) = (\cos \rho)h \).

Since for any \( X = (x_1, ..., x_4) \in \mathbb{R}^4 \), let \( r = |X| \), then we have
\[
\nabla r^2 = 2X^\top, \quad |\nabla r| \leq 1
\]
\[
\Delta r^2 = 4 + 2 \langle H, X \rangle \leq 4 + 2r.
\]

Since \( J(\Sigma) \subset B_R(q) \subset S^2 \), note that \( R < \frac{\pi}{2} \), so we can choose a constant \( b \), such that \( \psi(R) < b < 1 \). Let \( B_R(o) \) be the ball centered at \( o \) with radius \( a \) in \( \mathbb{R}^4 \). Define \( f : \Sigma \cap B_R(o) \rightarrow \mathbb{R} \) by
\[
f = \frac{(a^2 - r^2)^2 |dJ|^2}{(b - \psi \circ J)^2}.
\]

Then by a similar proof of [10, Theorem 2], we conclude that
\[
|dJ|^2 \leq \max \left\{ \frac{64r^2}{C_5(a^2 - r^2)^2(b - \psi \circ J)^2}, \frac{32r^2}{C_4(a^2 - r^2)^2} + \frac{8(r + 2)}{C_4(a^2 - r^2)} \right\},
\]
where \( C_4 \) is a positive constant. From this we can obtain the upper bound of \( f \). Hence at every point of \( \Sigma \cap B_R(o), \) we have
\[
|dJ|^2 \leq \frac{C_5}{a^2}.
\]
(5.2)

Here \( C_5 \) is a positive constant depending only on \( R \). For any fixed \( x \) and letting \( a \to \infty \) in (5.2), we then derive that \( dJ = 0 \), namely, \( J \) must be constant. It follows that \( H \equiv 0 \). Hence \( \Sigma \) is a plane. \( \square \)

**Remark 5.2.** (1) Let \( \alpha \) be the Kähler angle of the translator, **Theorem 5.4** implies that the complete symplectic translating soliton surface with flat normal bundle and \( \cos \alpha \) has a positvie lower bound has to be a plane. Han-Sun [29] showed that if \( \cos \alpha \) has a positive lower bound, then complete symplectic translating soliton surfaces with bounded second fundamental form and nonpositive normal curvature must be a plane, which indicated that when the normal bundle is flat, such translator is a plane (see [29, Main Theorem 1]). In this case, we could remove the condition on the boundedness of the second fundamental form.

(2) The restriction on the image of the complex phase map in **Theorem 5.4** is necessary. For example, the “grim reaper” \( (x, y, -\ln \cos x, 0), |x| < \pi/2, y \in \mathbb{R} \) is a translating soliton to the symplectic MCF which translates in the direction of the constant vector \((0, 0, 1, 0)\), and \( J = (\cos x, 0, -\sin x), |x| < \pi/2 \) can not contained in any regular ball of \( S^2 \). One can check that \( |B|^2 = |H|^2 = |dJ|^2 = \cos^2 x \). In particular, both the tangent bundle and the normal bundle are flat.

Now we are at a position to give a proof of **Theorem 1.3**.

**Proof of Theorem 1.3.** Firstly for the compact subset \( K_1 := J(\Sigma_0) \subset V \), there is a positive and strictly convex smooth function \( \rho \) on \( K_1 \) (cf. [41]). Choose a domain \( U \Subset V \) such that \( K_1 \subset U \) and \( \rho \) is a strictly convex on \( U \). Put
\(c := \max_k \rho\) and consider the function \(u := \rho \circ J\). Then \(u\) is well defined in \(\Sigma \times [0, t_0]\) for small \(t_0 > 0\). According to Theorem 5.1, along the mean curvature flow, the complex map satisfies

\[
\frac{\partial J}{\partial t} = \tau(J).
\]

Thus,

\[
\frac{\partial u}{\partial t} = \Delta u \leq 0.
\]

As a consequence, \(u \leq c\) in \(\Sigma \times [0, t_0]\).

Let \(U_{\varepsilon}\) be a \(\varepsilon\)-neighborhood of \(U\). We claim that

**Claim.** There is a \(\varepsilon_0 > 0\) depending only on \(U\) such that for \(0 < \varepsilon < \varepsilon_0\), we have

\[
\{x \in U : \rho(x) \leq c\} \subseteq \{x \in \overline{U}_{\varepsilon} : \rho(x) \leq c\}.
\]

Indeed, for every \(x \in \partial U\), define \(r(x) < \pi/2\) to be the largest number of \(r\) such that \(B_r^\Sigma(x) \subset V\). Since \(\rho\) is strictly convex on \(\bar{U}\), we can take \(0 < \varepsilon < \varepsilon_0\), such that \(\rho(y) \leq c\). Applying the maximum principle, we can choose \(x \in \partial U\) with \(\rho(x) = c\). Let \(\gamma : [0, 1] \rightarrow U_{\varepsilon} \) be the shortest geodesic from \(x\) to \(\gamma\). Since \(\rho\) is strictly convex, we know that \(f := \rho \circ \gamma\) is also a strictly convex function on \([0, 1]\). Moreover \(f'(0) > 0\) which is impossible by the maximum principle. Thus the Claim holds.

Let \(\tau \in (0, T]\) be the maximum time such that

\[
J(\Sigma_\tau) \subset \{x \in \bar{U} : \rho(x) \leq c\}, \quad \forall t \leq \tau.
\]

If \(\tau < T\), then \(J(\Sigma_\tau) \subset \{x \in \bar{U} : \rho(x) \leq c\}\). Applying the maximum principle and the above claim, we can extend \(\tau\) to some \(\tau' > \tau\) which is a contradiction. Thus we obtain that

\[
J(\Sigma_\tau) \subset \{x \in \bar{U} : \rho(x) \leq c\}, \quad \forall t \leq T.
\]

Denote \(K := \{x \in \bar{U} : \rho(x) \leq c\}\). Clearly, \(K_1 \subset K\) and \(K\) is compact.

Suppose that the mean curvature flow has a Type I singularity at \(T\). Assume

\[
e_k = |B| (x_k, t_k) = \max_{\Sigma_k} |B|,
\]

and \(x_k \rightarrow p \in \Sigma, t_k \rightarrow T, F(x_k, t_k) \rightarrow q \in M\) as \(k \rightarrow \infty\). Set

\[
F_k(x, t) := e_k \left( F(x, e_k^{-1} t + t_k) - q \right).
\]

Denote by \(\Sigma_k^1 := F_k(\cdot, t)(\Sigma)\), then

\[
\epsilon_k^2 g_{ij} = \epsilon_k^2 g_{ij}, \quad (g^1)^{ij} = e_k^{-2} [^0] g^{ij}.
\]

Direct computation gives us

\[
\frac{\partial F_k}{\partial t} = e_k^{-2} \frac{\partial F}{\partial t} = H_k, \quad \Delta_e F_k = e_k^{-2} \Delta F, \quad |B_k|^2 = e_k^{-2} |B|^2.
\]

Thus

\[
|B_k| \leq 1, \quad |B_k(x_k, 0)| = 1.
\]

Therefore there exists a subsequence of \(F_k\), we still denote it by \(F_k\), such that \(F_k \rightarrow F_\infty\) as \(k \rightarrow \infty\) in any ball \(B_k(0) \subset \mathbb{R}^d\), and \(F_\infty\) satisfies

\[
\frac{\partial F_\infty}{\partial t} = H_\infty, \quad |B_\infty| \leq 1, \quad \text{and} \quad |B_\infty(p, 0)| = 1.
\]
Using the blow up analysis of the mean curvature flow (cf. [2, 7]), the blow up limit \( \Sigma \) is a self-shrinker and complete. By the monotonicity formula, it is easy to see that \( \Sigma \) has polynomial volume growth (see [17, Lemma 2.9 and Corollary 2.13]), then by [16, Theorem 4.1], \( \Sigma \) is proper.

As the complex phase map is rescaling invariant, we conclude that \( J(\Sigma^0) \subset K \) since \( J(\Sigma) \subset K \). Then by an elementary topology argument, we get \( J(\Sigma) \subset K \). Hence we obtain a complete proper self-shrinker in \( \mathbb{R}^4 \) with the image of the complex phase map contained in \( K \). Applying Theorem 1.2, \( \Sigma \) must be a plane, which contradicts with \( \|B_\omega(p,0)\| > 0 \) in (5.3). Thus we complete the proof. \( \square \)

Remark 5.3. By a similar method of the proof in Theorem 1.3, we can also demonstrate that the following holds:

Let \( \Sigma_0 \) be a closed hypersurface immersed in Euclidean space \( \mathbb{R}^{n+1} \), and \( \Sigma_t \subset \mathbb{R}^{n+1}(t \in [0, T) \) for some \( T > 0 \) \) a family of hypersurfaces given by the mean curvature flow. Suppose that the Gauss image of \( \Sigma_0 \) is contained in \( \mathbb{S}^n \setminus \mathbb{S}_+^{n-1} \), then the mean curvature flow does not develop any Type I singularity.

The following example shows that the restriction on the image of the complex phase map is sharp in Theorem 1.3.

Example 5.2. Let \( \gamma : \mathbb{S}^1 \rightarrow \mathbb{C} \) be an immersed curve with \( 0 \notin \gamma \) and define

\[ F : \mathbb{T}^2 \rightarrow \mathbb{C}^2, \quad (x, y) \rightarrow (\gamma(x) \cos(y), \gamma(x) \sin(y)). \]

Then \( F \) is a Lagrangian immersion. Since the initial surface is closed, we know that the mean curvature flow always blows up at a finite time (cf. [49, Proposition 3.10]).

Denote \( \Sigma : = F(\mathbb{T}^2) \). Firstly, we have the following fact: the Maslov index of the Lagrangian immersion \( \Sigma \rightarrow \mathbb{C}^2 \) is zero (i.e., \( \Sigma \) is of zero-Maslov class) iff

\[ \text{Ind}_\gamma(0) = \frac{1}{2\pi} \int \kappa, \]

where \( \kappa \) is the curvature of the curve \( \gamma \) in \( \mathbb{C} \). We will give some more details as follows. The induced metric \( g \) on \( \Sigma \) is

\[ g = |\gamma'(x)|^2 \, dx^2 + |\gamma(x)|^2 \, dy^2. \]

Choose an orientation on \( \Sigma \) as following:

\[ d\mu_\Sigma = |\gamma(x)| \, |\gamma'(x)| \, dx \wedge dy. \]

Then the pullback of the holomorphic symplectic 2-form \( \Omega \) is

\[ \Omega_{\Sigma} = \gamma(x)\gamma'(x) \, dx \wedge dy = \frac{\gamma(x)\gamma'(x)}{|\gamma(x)||\gamma'(x)|} \, d\mu_\Sigma = \frac{\gamma'(x)}{|\gamma'(x)|} \, d\mu_\Sigma. \quad (5.4) \]

Thus, \( \Sigma \) is of Maslov class iff the winding number of the curve \( x \mapsto \gamma(x)\gamma'(x) \) around \( 0 \) is zero iff the degree of the map \( x \mapsto \frac{\gamma(x)}{|\gamma'(x)|} \) is zero.

For the immersed curve \( \gamma \) in \( \mathbb{C} \), the unit tangent vector is

\[ \vec{e} = \frac{\gamma'}{|\gamma'|}, \]

and the unit outward normal vector is

\[ \vec{n} = -\sqrt{-1}\vec{e} = -\sqrt{-1}\frac{\gamma'}{|\gamma'|}. \]

Thus, the curvature vector is

\[ \kappa = \nabla_\vec{e}\vec{e} = \kappa \vec{n} \]

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where
\[
\kappa = \frac{\sqrt{-1} (y'' \bar{y}' - \bar{y}' y'')}{2|y'|^2} = -\text{Im}(\ln y').
\]

Notice that the winding number of the curve \( \gamma \) around 0 is
\[
\text{Ind}_\gamma(0) = \frac{1}{2\pi} \int_\gamma \frac{d\gamma}{\gamma}.
\]
An immediately consequence is that the degree of the Gauss map \( \bar{n} : \mathbb{S}^1 \to \mathbb{S}^1 \) is
\[
\text{deg} \bar{n} = -\frac{1}{2\pi} \int \kappa = \frac{1}{2\pi} \int_\gamma \frac{d\gamma'}{\gamma'} = \text{Ind}_\gamma(0).
\]
Therefore, according to (5.4), \( \Sigma \) is of zero-Maslov class if
\[
\text{Ind}_\gamma(0) = \frac{1}{2\pi} \int \kappa = \frac{1}{2\pi} \int (\kappa (y^2)) = 0.
\]
The mean curvature vector \( \mathbf{H} \) of \( \Sigma \) in \( C^2 \) is
\[
\mathbf{H} = \frac{1}{|y(x)||y'(x)|} \left( \left( \frac{|y'(x)|}{|y(x)|} \right)^{-1} y'(x) \right) \cos(y), \left( \frac{|y'(x)|}{|y(x)|} \right)^{-1} y'(x) \sin(y)
\]
\[
- \frac{1}{|y(x)||y'(x)|} \left( \left( \frac{|y'(x)|}{|y(x)|} \right)^{-1} y'(x) \right) \gamma(x) \cos(y), \left( \frac{|y'(x)|}{|y(x)|} \right)^{-1} y'(x) \gamma(x) \sin(y).
\]
We compute
\[
\frac{1}{|y||y'|} (|y| |y'| y') = \frac{\gamma' |y|}{|y'| |y|^2} \left( \frac{\gamma' y' + y'' y'}{2y} + \frac{y'' y'}{2y} - \frac{\gamma'}{y} \right) - \frac{y}{|y|^2}
\]
\[
= \kappa(y) \sqrt{1 + \frac{\gamma'}{|y'| |y|^2} \left( \frac{\gamma'}{2y} + \frac{\gamma'' y'}{2y^2} \right)}
\]
\[
= \kappa(y) - \frac{\gamma^2}{|y|^2}.
\]
Thus,
\[
\mathbf{H} = \left( \left[ \kappa(y(x)) - \frac{\gamma(x)^2}{|y(x)|^2} \right] \cos(y), \left[ \kappa(y(x)) - \frac{\gamma(x)^2}{|y(x)|^2} \right] \sin(y) \right).
\]
Therefore the MCF (5.1) is reduced to (cf. [46])
\[
\begin{align*}
\frac{d\gamma}{dt} &= \kappa(y) - \frac{\gamma}{|y|^2}, \quad \mathbb{S}^1 \times [0, T); \\
\gamma(\cdot, 0) &= \gamma_0(\cdot), \quad \mathbb{S}^1.
\end{align*}
\]
When \( \gamma_0 : \mathbb{S}^1 \to \mathbb{S}^1, x \mapsto x \), then the solution of (5.5) is
\[
\gamma(x, t) = 2 \sqrt{T - t} \gamma_0(x), \quad 0 \leq t < T.
\]
One can check that the Maslov index of \( \Sigma_0 \) is not zero since
\[
\text{Ind}_{\gamma_0}(0) = \frac{1}{2\pi} \int_{\gamma_0} \kappa(y_0) = 2.
\]
In particular, the image of the complex phase map is a great circle. Moreover, along the mean curvature flow
\[
| \mathbf{H} |^2 = \frac{1}{2(T - t)} \cdot \mathbf{H} = \frac{1}{2(T - t)} \mathbf{F}, \quad 0 \leq t < T.
\]
Corollary 5.5. If the image of the complex phase map for the initial closed surface is contained in $S^1 \setminus \{q\}$, where $q \in S^3$, then there does not exist any Type I singularity of the Lagrangian mean curvature flow in a Calabi-Yau 4-manifold.

Proof. The condition on the image of the complex phase map implies that the initial surface is Lagrangian, and Smoczyk [47] proved that the Lagrangian is preserved under the MCF. Note that a Calabi-Yau 4-manifold is hyperkähler. Then the conclusion followed by applying Theorem 1.3. □

Remark 5.4. Under the assumption of Corollary 5.5, one can check that the initial surface is zero-Maslov class. Neves [46] proved that the Lagrangian MCF with zero-Maslov class has no Type I singularity, which is a more general condition than belonging to $S^1 \setminus \{q\}$. In our case, we give an alternative proof comparing to Neve's.

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