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Duality for convex infinite optimization on linear spaces

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Abstract

This note establishes a limiting formula for the conic Lagrangian dual of a convex infinite optimization problem, correcting the classical version of Karney [Math. Programming 27 (1983) 75-82] for convex semi-infinite programs. A reformulation of the convex infinite optimization problem with a single constraint leads to a limiting formula for the corresponding Lagrangian dual, called sup-dual, and also for the primal problem in the case when strong Slater condition holds, which also entails strong sup-duality.

Key words Convex infinite programming · Lagrangian duality · Haar duality · Limiting formulas

Mathematics Subject Classification Primary 90C25; Secondary 49N15 · 46N10

1 Introduction

Given a real linear space $X$, consider the (algebraic) convex infinite programming (CIP) problem

$$(P) \quad \inf_{x \in X} f(x), \text{ s.t. } f_t(x) \leq 0, \ t \in T,$$

where $T$ is an infinite index set and $f, f_t : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}$, $t \in T$, are convex proper functions. We denote by

$$E := \bigcap_{t \in T} [f_t \leq 0] = \{x \in X : f_t(x) \leq 0, \ t \in T\}$$

the feasible set of $(P)$ and define

$$M := \bigcap_{t \in T} \text{dom } f_t \supset E \text{ and } \Delta := M \cap \text{dom } f.$$
Let $\mathbb{R}^{(T)}_+$ be the positive cone of the space $\mathbb{R}^{(T)}$ of functions $\lambda = (\lambda)_{t \in T} : T \to \mathbb{R}$ whose support $\text{supp} \lambda := \{ t \in T : \lambda_t \neq 0 \}$ is finite and let $0_{\mathbb{R}^{(T)}}$ be its null element. The ordinary Lagrangian function associated to $(P)$ is (see [7], [8], etc.) is $L_0 : X \times \mathbb{R}^{(T)}_+ \to \mathbb{R}$ such that $L_0 (x, \lambda) := f(x) + \sum_{t \in T} \lambda_t f_t(x)$, where

$$
\sum_{t \in T} \lambda_t f_t(x) := \left\{ \begin{array}{ll}
\sum_{t \in \text{supp} \lambda} \lambda_t f_t(x), & \text{if } \lambda \neq 0_{\mathbb{R}^{(T)}}, \\
0, & \text{if } \lambda = 0_{\mathbb{R}^{(T)}}.
\end{array} \right.
$$

A slightly different Lagrangian is the one associated with the cone constrained reformulation of $(P)$, that is [14, page 138], the function $L : X \times \mathbb{R}^{(T)}_+ \to \mathbb{R}$ such that

$$
L(x, \lambda) := \left\{ \begin{array}{ll}
f(x) + \sum_{t \in T} \lambda_t f_t(x), & \text{if } x \in M, \lambda \in \mathbb{R}^{(T)}_+, \\
+\infty, & \text{else}.
\end{array} \right.
$$

We call $L$ the conic Lagrangian of $(P)$.

For each $x \in X$ we have

$$
\sup_{\lambda \in \mathbb{R}^{(T)}_+} L_0 (x, \lambda) = \sup_{\lambda \in \mathbb{R}^{(T)}_+} L(x, \lambda) = f(x) + \delta_E(x),
$$

where $\delta_E$ is the indicator of $E$, that is, $\delta_E (x) = 0$ if $x \in E$ and $\delta_E (x) = +\infty$ otherwise. Consequently,

$$
\inf_{x \in X} \sup_{\lambda \in \mathbb{R}^{(T)}_+} L_0 (x, \lambda) = \inf_{x \in X} \sup_{\lambda \in \mathbb{R}^{(T)}_+} L(x, \lambda) = \inf (P).
$$

The ordinary and conic-Lagrangian dual problems of $(P)$ read, respectively,

$$(D_0) \quad \sup_{\lambda \in \mathbb{R}^{(T)}_+} \inf_{x \in X} \left( f(x) + \sum_{t \in T} \lambda_t f_t(x) \right),$$

and

$$(D) \quad \sup_{\lambda \in \mathbb{R}^{(T)}_+} \inf_{x \in M} \left( f(x) + \sum_{t \in T} \lambda_t f_t(x) \right),$$

and one has

$$
\sup(D_0) \leq \sup(D) \leq \inf (P). \quad (1.1)
$$

Note that, if $\text{dom } f \subset M$, then $\sup(D_0) = \sup(D)$. This is, in particular, the case when the functions $f_t$, $t \in T$, are real-valued. But it may happen that $\sup(D_0) < \sup(D)$ even if $T$ is finite and Slater condition holds. This is the case in the next example.

**Example 1.1** Consider $X = \mathbb{R}^2$, $T = \{1\}$, $f(x_1, x_2) = e^{x_2}$, and

$$
f_1(x_1, x_2) = \left\{ \begin{array}{ll}
x_1, & \text{if } x_2 \geq 0, \\
+\infty, & \text{if } x_2 < 0.
\end{array} \right.
$$

We then have

$$
\max(D_0) = 0 < 1 = \max(D) = \min (P).
$$
Duften [5] observed that a positive duality gap might occur when one considers the ordinary Lagrangian dual \((D_0)\) of \((P)\). The same happens when \((D_0)\) is replaced by \((D)\) even though, according to (1.1), the gap may be smaller. Different ways have been proposed to close the duality gap, e.g., by adding a linear perturbation to the saddle function \(f + \sum_{t \in T} \lambda_t f_t\), and sending it to zero in the limit [5]. Blair, Duften and Jeroslow [1] used the conjugate duality theory to extend the limiting phenomena to the general minimax setting. Pomerol [12] showed that it was possible to obtain similinf sup theorems, including that of [1], by using a slightly more general form of the duality theory. In turn, Karney and Morley [9] proved that, when \(X = \mathbb{R}^n\), either the convex semi-infinite programming (CSIP in brief) problem \((P)\) satisfies some recession condition guaranteeing a zero duality gap or there exists \(d \in \mathbb{R}^n \setminus \{0_n\}\) such that the problem
\[
(P_\varepsilon) \quad \inf_{x \in X} f(x) + \varepsilon \langle d, x \rangle, \quad \text{s.t.} \quad f_t(x) \leq 0, \ t \in T,
\]
satisfies the mentioned recession condition for \(\varepsilon > 0\) sufficiently small, with \((P_\varepsilon)\) enjoying strong duality, and \(\inf (P) = \lim_{\varepsilon \to 0} (P_\varepsilon)\). The theory developed in [9] subsumed the CSIP versions of some results on limiting Lagrangians in [2] and [6]. Three years before, Karney gave, in the CSIP setting, a limiting formula for the dual problem \((D_0^*)\):
\[
\sup (D_0) = \lim_{\varepsilon \to 0} \inf \{ f(x) : f_t(x) \leq \varepsilon, \ t \in T \}. \tag{1.2}
\]
According to [8, Proposition 3.1], this formula comes from [13, Theorem 7] and [2, Corollary 2], and does not require any constraint qualification (other than \(E \neq \emptyset\), or something stronger as \(E \cap \text{dom } f \neq \emptyset\), \(E \subset \text{cl dom } f\), ...). The next example shows that [8, Proposition 3.1] fails even in linear semi-infinite programming, where \(\text{dom } f = X = \mathbb{R}^n\), while [13, Theorem 7] and [2, Corollary 2] hold.

**Example 1.2** Consider the following optimization problem, with \(T = \mathbb{N}\):
\[
(P) \quad \inf_{x \in \mathbb{R}^2} x_2 \quad \text{s.t.} \quad x_1 \leq 0, \quad (t = 1) \\
\quad \quad \quad \quad \quad \quad \quad -x_2 \leq 1, \quad (t = 2) \\
\quad \quad \quad \quad \quad \quad \quad t^{-1}x_1 - x_2 \leq 0, \quad t = 3, 4, ...
\]

Its dual problem \((D_0)\), that is also \((D)\), is equivalent to the Haar dual (see, e.g., [7])
\[
\sup_{\lambda \in \mathbb{R}^{(n)}_+} -\lambda_2 \quad \text{s.t.} \quad \lambda_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{t \geq 3} \lambda_t \begin{pmatrix} -t^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
whose unique feasible solution is \(\lambda \in \mathbb{R}^{(n)}_+\) such that \(\lambda_2 = 1\) and \(\lambda_t = 0\) for \(t \neq 2\). So, \(\max (D_0) = -1\) while \(E = \{(x_1, x_2) : x_1 \leq 0, x_2 \geq 0\}\), so that \(\min (P) = 0\). On the other hand, given \(\varepsilon > 0\),
\[
\{ x \in \mathbb{R}^2 : f_t(x) \leq \varepsilon, \ t \in \mathbb{N} \} = \left\{ x \in \mathbb{R}^2 : x_1 \leq \varepsilon, x_2 \geq -\varepsilon, \frac{x_1}{3} - x_2 \leq \varepsilon \right\},
\]
so that

\[
\min \{ x_2 : f_t(x) \leq \varepsilon, \ t \in \mathbb{N} \} = -\varepsilon
\]

is attained at \( \{(x_1, -\varepsilon) : x_1 \leq 0\} \). Hence,

\[
\max (D_0) = -1 < 0 = \lim_{\varepsilon \downarrow 0} \min \{ x_2 : f_t(x) \leq \varepsilon, \ t \in \mathbb{N} \}.
\]

From [8, Proposition 3.1] Karney obtained, following the suggestion of an unknown referee, the reverse strong duality theorem [8, Theorem 3.2] guaranteeing zero duality gap with primal attainment, i.e.,

\[
\min (P) = \sup (D_0),
\]

under some recession condition. However, he asserted in [8, Section 5] that he had two (longer) unpublished proofs. In either case, his result has been recently proved from a new strong duality theorem for CIP (see [4, Corollary 3.2 and Remark 3.2]).

In this note we show in a simpler way, for general CIP problems, that, under the strong Slater condition

\[
\exists \alpha > 0, \exists a \in \text{dom } f : \ f_t(a) \leq -\alpha, \ \forall t \in T,
\]

(1.2) entails that zero duality gap holds:

\[
\sup (D_0) = \inf (P).
\]

This duality theorem is obtained by studying the Lagrangian dual \( (D_1) \) associated with the representation of \( E \) by a single constraint (the so-called sup-function). Section 2 (resp. Section 3) provides a limiting formula for \( \sup (D) \) (resp. \( \sup (D_1) \)). Under the strong Slater condition, the limiting formula for \( \sup (D_1) \) also holds for \( \inf (P) \) together with the strong duality theorem \( \inf (P) = \max (D_1) \).

2 Conic-Lagrangian duality

Problem \( (D) \) receives a perturbational interpretation (see [3], [14], etc.) in terms of the ordinary value function \( v : \mathbb{R}^T \rightarrow \mathbb{R} \) associated with \( (P) \) defined by

\[
v(y) := \inf \{ f(x) : f_t(x) \leq y_t, t \in T \}, \ \forall y = (y_t)_{t \in T} \in \mathbb{R}^T.
\]

Let us make this approach explicit. The linear space \( Y := \mathbb{R}^T \), equipped with the product topology, is a locally convex Hausdorff topological vector space whose topological dual is \( \mathbb{R}^{(T)} \) via the bilinear pairing

\[
\langle \cdot, \cdot \rangle : Y \times \mathbb{R}^{(T)} \rightarrow \mathbb{R} \text{ such that } \langle y, \lambda \rangle = \sum_{t \in T} \lambda_t y_t.
\]
The Fenchel conjugate of $v$ is (see [3], [14], etc.)

$$-v^*(-\lambda) = \begin{cases} 
\inf_{x \in \Delta} \left( f(x) + \sum_{t \in T} \lambda_t f_t(x) \right), & \text{if } \Delta \neq \emptyset \text{ and } \lambda \in \mathbb{R}^{(T)}_+, \\
-\infty, & \text{if } \Delta = \emptyset \text{ or } \lambda \in \mathbb{R}^{(T)} \setminus \mathbb{R}^{(T)}_+. 
\end{cases} \quad (2.1)$$

If $\Delta \neq \emptyset$ we the have

$$v^{**}(0_Y) = \sup_{\lambda \in \mathbb{R}^{(T)}} -v^*(-\lambda) = \sup_{\lambda \in \mathbb{R}^{(T)}} -v^*(-\lambda) = \sup_{\lambda \in \mathbb{R}^{(T)}_+} \inf_{x \in \Delta} \left( f(x) + \sum_{t \in T} \lambda_t f_t(x) \right) = \sup(D).$$

Note that, if $\Delta = \emptyset$ we have $\text{dom } v = \emptyset$ and $v^{**}(0_Y) = +\infty = \sup(D)$. Therefore, in all cases we have

$$\sup(D) = v^{**}(0_Y) \leq \overline{v}(0_Y) \leq v(0_Y) = \inf(P), \quad (2.2)$$

where $\overline{v}$ is the lower semicontinuous (lsc in brief) hull of $v$ for the product topology on $Y = \mathbb{R}^T$. A neighborhood basis of the origin $0_Y$ is furnished by the family

$$\{V^H_\varepsilon : \varepsilon > 0, H \in \mathcal{F}(T)\},$$

where $\mathcal{F}(T)$ is the class of non-empty finite subsets of $T$, and

$$V^H_\varepsilon := \{y \in Y : |y_t| \leq \varepsilon, t \in H\}.$$

We now give a general explicit formula for $\overline{v}(0_Y)$:

**Lemma 2.1** $\overline{v}(0_Y) = \sup_{\varepsilon > 0, H \in \mathcal{F}(T)} \inf_{x \in M} \{f(x) : f_t(x) \leq \varepsilon, t \in H\}$.

**Proof** For each $\varepsilon > 0$ and $H \in \mathcal{F}(T)$ one has

$$\inf_{y \in V^H_\varepsilon} v(y) = \inf \{f(x) : f_t(x) \leq y_t, t \in H; |y_t| \leq \varepsilon, t \in H\}$$

$$= \inf \{f(x) : f_t(x) \leq \varepsilon, t \in H; f_t(x) < +\infty, t \notin H\}$$

$$= \inf_{x \in M} \{f(x) : f_t(x) \leq \varepsilon, t \in H\}.$$ 

Since $\overline{v}(0_Y) = \sup_{\varepsilon > 0, H \in \mathcal{F}(T)} \inf_{y \in V^H_\varepsilon} v(y)$, we are done. $\square$

**Remark 2.1** From Lemma 2.1 one gets

$$\overline{v}(0_Y) \leq \liminf_{\varepsilon \downarrow 0} \inf \{f(x) : f_t(x) \leq \varepsilon, t \in T\}.$$
Remark 2.2 In the case when the index set $T$ is finite, the formula provided by Lemma 2.1 can be simplified as follows:

$$v(0_Y) = \lim_{\varepsilon \downarrow 0} \inf \{ f(x) : f_t(x) \leq \varepsilon, t \in T \}.$$ 

In such a case we also have $M = \bigcap_{t \in T} \text{dom } f_t$ and

$$v^{**}(0_Y) = \sup_{\lambda \in \mathbb{R}_+} \inf_{x \in M} \left( f(x) + \sum_{t \in T} \lambda_t f_t(x) \right).$$

Proposition 2.1 (Limiting formula for $\sup(D)$) Assume either $\overline{v}(0_Y) \neq +\infty$ or $\sup(D) \neq -\infty$. Then we have

$$\sup(D) = \sup_{\varepsilon > 0, H \in \mathcal{F}(T)} \inf_{x \in M} \{ f(x) : f_t(x) \leq \varepsilon, t \in H \}.$$ 

Proof We know that $\sup(D) = v^{**}(0_Y)$ (see (2.2)). Since the functions $f$ and $f_t$, $t \in T$, are convex, the value function $v$ is convex, too. By [2, Proposition 1], we then have $\sup(D) = \overline{v}(0_Y)$ and Lemma 2.1 concludes the proof. \qed

Remark 2.3 Condition $\overline{v}(0_Y) \neq +\infty$ is in particular satisfied if $\inf(P) \neq +\infty$, that is $E \cap \text{dom } f \neq \emptyset$.

Condition $\sup(D) \neq -\infty$ is satisfied if and only if there exists $\lambda \in \mathbb{R}_{+}^{(T)}$ and $r \in \mathbb{R}$ such that

$$x \in M \implies f(x) + \sum_{t \in T} \lambda_t f_t(x) \geq r.$$

Remark 2.4 By (1.1), (2.1) and (2.2), we have

$$\sup(D_0) \leq \sup(D) \leq \lim_{\varepsilon \downarrow 0} \inf \{ f(x) : f_t(x) \leq \varepsilon, t \in T \}.$$ 

In [8, Proposition 3.1] it is claimed that for $X = \mathbb{R}^n$, $f$ and $f_t$, $t \in T$, are proper, lsc and convex, and $E \neq \emptyset$, it holds that

$$\sup(D_0) = \lim_{\varepsilon \downarrow 0} \inf \{ f(x) : f_t(x) \leq \varepsilon, t \in T \}.$$ 

To the best of our knowledge, this fact has not been proved anywhere. We prove in Proposition 3.2 below an exact formula for its right-hand side.
3 Sup-Lagrangian duality

Let \( h := \sup_{t \in T} f_t \) be the sup-function of \((P)\) which allows to represent its feasible set \( E \) with a single constraint. We associate with \((P)\) another Lagrangian \( L_1 : X \times \mathbb{R}_+ \rightarrow \mathbb{R} \), called sup-Lagrangian, such that

\[
L_1 (x, s) := \begin{cases} 
  f(x) + sh(x), & \text{if } x \in \Delta_1 := \text{dom } f \cap \text{dom } h \text{ and } s \geq 0, \\
  +\infty, & \text{else}.
\end{cases}
\]

Note that \( \Delta_1 \subset \Delta \). For each \( x \in X \) we have

\[
\sup_{s \geq 0} L_1 (x, s) = f(x) + \delta_E (x),
\]

and

\[
\inf_{x \in X} \sup_{s \geq 0} L_1 (x, s) = \inf (P).
\]

The corresponding Lagrangian dual problem, say sup-dual problem, reads

\[(D_1) \sup_{s \geq 0} \inf_{x \in \Delta_1} (f(x) + sh(x)).\]

Let us introduce the sup-value function \( v_1 : \mathbb{R} \rightarrow \mathbb{R} \) associated with \((P)\) via \( L_1 \), namely,

\[
v_1 (r) := \inf \{ f(x) : h(x) \leq r \}, \ r \in \mathbb{R},
\]

which is non-increasing and satisfies

\[
\bar{v}_1 (0) = \lim_{\varepsilon \downarrow 0} v_1 (\varepsilon) = \lim_{\varepsilon \downarrow 0} \inf \{ f(x) : f_t (x) \leq \varepsilon, t \in T \}. \quad (3.1)
\]

**Lemma 3.1** \( \sup (D) \leq \sup (D_1) \leq \inf (P) \).

**Proof** Let us prove the first inequality (the second being obvious). Given \( \lambda \in \mathbb{R}_+^{(T)} \), one has to check that

\[
\inf_{x \in \Delta} \left( f(x) + \sum_{t \in T} \lambda_t f_t (x) \right) \leq \sup(D_1).
\]

If \( \text{supp } \lambda = \emptyset \), then

\[
\inf_{x \in \Delta} \left( f(x) + \sum_{t \in T} \lambda_t f_t (x) \right) = \inf_{x \in \Delta} f \leq \inf_{x \in \Delta_1} f \leq \sup(D_1)
\]

and we are done.
If $\text{supp} \lambda \neq \emptyset$, one has, for $s = \sum_{t \in T} \lambda_t$,

$$
\sup(D_1) \geq \inf_{x \in \Delta_1} (f(x) + sh(x)) \\
\geq \inf_{x \in \Delta_1} (f(x) + s \sum_{t \in T} \frac{\lambda_t}{s} f_t(x)) \\
\geq \inf_{x \in \Delta_1} (f(x) + \sum_{t \in T} \lambda_t f_t(x)) \\
\geq \inf_{x \in \Delta} (f(x) + \sum_{t \in T} \lambda_t f_t(x)) .
$$

\[\square\]

**Proposition 3.1 (Limiting formula for $\sup(D_1)$)** Assume that either $\overline{v}_1(0) \neq +\infty$ or $\sup(D_1) \neq -\infty$. Then we have

$$
\sup(D_1) = \lim_{\varepsilon \downarrow 0} \inf \{ f(x) : f_t(x) \leq \varepsilon, t \in T \} .
$$

**Proof** By (3.1), the right-hand side of (3.1) coincides with $\overline{v}_1(0)$. By definition of $v_1$ we have (as for $v$), $v_1^*(0) = \sup(D_1)$. Since $v_1$ is convex and either $\overline{v}_1(0) \neq +\infty$ or $v_1^*(0) \neq -\infty$, we then have, by [2, Proposition 1], $\sup(D_1) = \overline{v}_1(0)$ and we are done. \[\square\]

**Proposition 3.2 (Limiting formula for $\inf(P)$)** Assume that the strong Slater condition

$$
\exists \alpha > 0, \exists a \in \text{dom } f : \ f_t(a) \leq -\alpha, \ \forall t \in T ,
$$

(3.2) holds. Then we have

$$
\inf(P) = \max \inf_{s \geq 0, x \in \Delta_1} (f(x) + sh(x)) = \lim_{\varepsilon \downarrow 0} \inf \{ f(x) : f_t(x) \leq \varepsilon, t \in T \} .
$$

(3.3)

**Proof** By definition of $h$ we have

$$
\inf(P) = \inf \{ f(x) : h(x) \leq 0 \} .
$$

Note that (3.2) amounts to the usual Slater condition relative to $h$:

$$
\exists a \in \text{dom } f : \ h(a) < 0 .
$$

Since the functions $f$ and $h$ are convex, we then have (see, e.g., [10, Lemma 1])

$$
\inf(P) = \max \inf_{s \geq 0, x \in \Delta_1} (f(x) + sh(x)) = \max(D_1) .
$$

By (3.2) we have $\overline{v}_1(0) \leq v_1(0) < +\infty$. By Proposition 3.1 it follows that

$$
\sup(D_1) = \lim_{\varepsilon \downarrow 0} \inf \{ f(x) : f_t(x) \leq \varepsilon, t \in T \}
$$

and we are done. \[\square\]
Let us revisit Example 1.2, where (3.3) fails. Any candidate \( a \) to be strong Slater point is feasible. Let \( a \) be a feasible solution of \((P)\). Then \( a = (a_1, 0) \), with \( a_1 \leq 0 \), and 
\[ h(a) \geq \sup \{ t^{-1}a_1 : t = 3, 4, \ldots \} = 0. \]
Thus, \( h(a) = 0 \) and the strong Slater constraint qualification (3.2) fails. However, by Proposition 3.1, we have
\[
\sup(D_1) = \lim_{\varepsilon \downarrow 0} \inf \{ f(x) : h(x) \leq \varepsilon \} = \lim_{\varepsilon \downarrow 0} -\varepsilon = 0
\]
and, finally,
\[
-1 = \sup(D_0) = \sup(D) < \sup(D_1) = 0 = \min(P)
= \inf \{ f(x) : h(x) = 0 \} = \lim_{\varepsilon \downarrow 0} \inf \{ f(x) : h(x) \leq \varepsilon \}.
\]

**Remark 3.1** In the case when \( T \) is finite, condition (3.2) reads
\[
\exists a \in \text{dom } f : f_t(a) < 0, \forall t \in T,
\]
that is the familiar Slater constraint qualification. One also has \( \Delta_1 = \left( \bigcap_{t \in T} \text{dom } f_t \right) \cap \text{dom } f \) and, by Proposition 3.2, there exists \( \bar{s} \geq 0 \) such that
\[
\inf(P) = \inf_{x \in \Delta_1} \left( f(x) + \bar{s} h(x) \right) = \inf_{x \in \Delta_1} \sup_{a \in S_T \nu \in \Delta_1} \left( f(x) + \bar{s} \sum_{t \in T} \nu_t f_t(x) \right),
\]
where \( S_T = \{ \nu \in \mathbb{R}_+^T : \sum_{t \in T} \nu_t = 1 \} \) is the unit simplex in \( \mathbb{R}^T \). By the minimax theorem [14, Theorem 2.10.1], with \( A = S_T \) and \( B = \Delta_1 \), there exists \( \nu \in S_T \) such that
\[
\inf(P) = \inf_{x \in \Delta_1} \left( f(x) + \bar{s} \sum_{t \in T} \nu_t f_t(x) \right) \leq \sup(D) \leq \inf(P)
\]
and, consequently, \( \inf(P) = \max(D) \), which is the strong duality theorem [14, Theorem 2.9.3] without assuming a topological structure on the basic linear space \( X \) (see also [11, Remark 8]).

Concerning Example 1.1, let us note that
\[
\max(D_0) = 0 < 1 = \max(D) = \lim_{\varepsilon \downarrow 0} \inf \{ f(x) : f_1(x) \leq \varepsilon \} = \min(P),
\]
which also contradicts [8, Proposition 3.1].

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