TRANSFORMATION FORMULAS AND THREE-TERM RELATIONS FOR BASIC HYPERGEOMETRIC SERIES

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Abstract. We derive two generalizations of Gasper’s transformation formula for basic hypergeometric series. Using these generalized formulas, we give explicit expressions for the coefficients of three-term relations for the basic hypergeometric series $\phi_1$, which are generalizations of the author’s previous results on three-term relations for $\phi_1$.

1. Introduction

In this paper, we derive two generalizations of Gasper’s [7, p. 200, (20)] transformation formula for basic hypergeometric series, and using these generalized formulas, we give explicit expressions for the coefficients of three-term relations for the basic hypergeometric series $\phi_1$. These expressions are generalizations of the author’s [11] previous results.

The basic hypergeometric series $r+1\phi_r$ is defined by

$$
r+1\phi_r(a, b_1, \ldots, b_r; c_1, \ldots, c_r; q, x) = r+1\phi_r(a; b_1, \ldots, b_r; c_1, \ldots, c_r; q, x),
$$

where $(a)_i$ denotes the $q$-shifted factorial defined by

$$(a)_i = (a; q)_i = (a; q)_0/(aq^i; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j).$$

It is assumed that $|q| < 1$ and none of the denominator parameters $c_1, \ldots, c_r$ are 1 or any negative integer power of $q$.

It is known that for any quadruples of integers $(k, l, m, n)$ and $(k', l', m', n')$, the three basic hypergeometric series

$$
2\phi_1(aq^k, b; q^{k'}, q^m; q, x), \quad 2\phi_1(aq^l, b; q^{l'}, q^m; q, x), \quad 2\phi_1(a, b; q^{m'}, q, x)
$$
satisfy a linear relation with coefficients that are rational functions of $a, b, c, q$, and $x$. We call such a relation the “three-term relation (for $\phi_1$).”

In [11], for any integers $k, l, m, n$, the three-term relation of the following form is considered:

$$
2\phi_1(aq^k, b; q^{k'}, q^m; q, x) = Q_0 \cdot 2\phi_1(aq^l, b; q^{l'}, q^m; q, x) + R_0 \cdot 2\phi_1(a, b; q^{m'}, q, x).
$$

The author obtained explicit expressions for $Q_0$ and $R_0$ using the following transformation formula for basic hypergeometric series obtained by Gasper [7, p. 200, (20)]: If

2010 Mathematics Subject Classification. 33D15.

Key words and phrases. Basic hypergeometric series; $q$-differential equation; Three-term relation; Contiguous relation; Transformation formula; Summation formula.
Theorem 1. For any integers $k, l, m, n$, with $|a^{-1}q^{m+1-\infty}| < 1$, then

\[
\begin{align*}
\phi_{r+1}\left(\frac{a, b, c_{1}q^{n_{1}}, \ldots, c_{r}q^{n_{r}}}{b_{q^{m+1}}, c_{1}, \ldots, c_{r}}; q, a^{-1}q^{m+1-\infty}\right)
\end{align*}
\]

(1.2) gives explicit expressions for \(Q\). Here we present our main results. Note that without loss of generality, it is sufficient to consider \((1.2)\) for only cases satisfying \(k \leq l\), because \(\phi_{1}(a, b; c; q, x)\) is symmetric with respect to the exchange of \(a\) and \(b\).

The following theorem asserts the uniqueness of the pair \((Q, R)\) satisfying \((1.3)\), and gives explicit expressions for \(Q\) and \(R\).

**Theorem 1.** For any integers \(k, l, m, n\), there is a unique pair \((Q, R)\) of rational functions of \(a, b, c, q,\) and \(x\) satisfying the three-term relation \((1.3)\). When \(k \leq l\), these functions can be expressed as

\[
\begin{align*}
Q &= Q(k, l, m, n) = \frac{(1-a)(1-b)c}{(q-c)(1-c)} x^{1-\max(m,0)} P\left(k, l, \min(m,0) : a, b, c, q, x\right), \\
R &= R(k, l, m, n) = -\frac{x^{1-\max(m,0)}(q)_{\min(m,0)}}{(abq/c)_{\max(k+l-m+n,0)-1}} P\left(k-1, l-1, \min(m,0) : aq, bq, cq, x\right).
\end{align*}
\]

Here, \(P\) is the polynomial in \(x\) defined by

\[
\begin{align*}
P_{n}\left(k, l, \min(m,0) : a, b, c, q, x\right) \\
:= \sum_{j=0}^{d} \sum_{i=0}^{\max(n,0)} \frac{(q^{n})_{j}}{(q)_{i}} \left(\begin{array}{c}
\mu \left(A_{j+i+m-\max(m,0)} - B_{j-i-\max(m,0)}\right) x^{i}, & k + l - m + n \geq 0, \\
\end{array}\right)
\]

\[
\begin{align*}
+ \sum_{j=0}^{d} \sum_{i=0}^{\min(n,0)} \frac{(q^{n})_{j}}{(q)_{i}} \left(\begin{array}{c}
\mu \left(\tilde{A}_{j+i+m-\max(m,0)} - \tilde{B}_{j-i-\max(m,0)}\right) x^{i}, & k + l - m + n < 0,
\end{array}\right)
\end{align*}
\]

1.1. Main results of this paper. Here we present our main results. Note that without loss of generality, it is sufficient to consider \((1.3)\) for only cases satisfying \(k \leq l\), because \(\phi_{1}(a, b; c; q, x)\) is symmetric with respect to the exchange of \(a\) and \(b\).

Throughout this paper, unless explicitly stated otherwise, we assume that

\[
a, b, c, \frac{a}{b}, \frac{c}{b} \notin q^\mathbb{Z} \cup \{0\}.
\]
where $d := \max \{k + l - m + n, 0\} + \max \{m, 0\} - \min \{n, 0\} - k - 1$ and, for any integer $j$, 

\[
A_j := \frac{(aq/c)_{k-n}(bq/c)_m}{(q^j/c)_{m-j-1}} \phi \left( q^{-j}, c(q^{-j-1}; a, b; c, \frac{aq^{k-j}}{bq^{j-1}}; q, q^{1-n} \right)
\]

\[
B_j := \frac{(aq/c)(bq/c)_{j}}{(q^j/c)_{j}} \phi \left( q^{-j}, c(q^{-j-1}; a, b; c, \frac{aq^{k-j}}{bq^{j-1}}; q, q^{1-n} \right)
\]

\[
\bar{A}_j := \frac{(aq/c)(bq/c)_{j}}{(q^j/c)_{j}} \phi \left( q^{-j}, c(q^{-j-1}; a, b; c, \frac{aq^{k-j}}{bq^{j-1}}; q, q^{1-n} \right)
\]

\[
\bar{B}_j := \frac{(aq/c)(bq/c)_{j}}{(q^j/c)_{j}} \phi \left( q^{-j}, c(q^{-j-1}; a, b; c, \frac{aq^{k-j}}{bq^{j-1}}; q, q^{1-n} \right)
\]

This theorem is a generalization of [11, Lemma 1 and Theorem 2]. (For an expression for $P$ as a sum of products of two $q\phi_1$, see Lemma 5 a generalization of [11, Lemma 3].) From Theorem 1 we obtain the following corollary, a generalization of [11, Corollary 4].

**Corollary 2.** The coefficients of the three-term relation (1.3) satisfy

\[
Q(k-1, l-1, m-1, n) \left|_{(a,b,c)= (aq,bq,cq)} \right. = \frac{(1-aq)(1-bq)x(c-abq)}{(1-c)(1-cq)} R(k, l, m, n).
\]

The following proposition provides an alternative expression for $P$. This proposition is a generalization of [11, Proposition 5].

**Proposition 3.** For any integers $k, l, m, n$, with $k \leq l$, the polynomial $P$ defined in Theorem 1 can be rewritten as follows:

\[
P(k, l; m; a, b, c; q, x) = \mu \sum_{j=0}^{d} \sum_{i=0}^{\max\{n,0\}} \frac{(q^{-j})_i}{(q)_i} \phi \left( q^{-j}, aq^{-j}/b, bq^{-j}/b, c(q^{-j-1}/b); q, q^{1-n} \right),
\]

where $d := \max \{k + l - m + n, 0\} + \max \{m, 0\} - \min \{n, 0\} - k - 1$ and, for any integer $j$,

\[
C_j := \mu_1 \frac{(aq/c)(bq/c)_{j}}{(q^j/c)_{j}} \phi \left( q^{-j}, aq^{-j}/b, bq^{-j}/b, c(q^{-j-1}/b); q, q^{1-n} \right),
\]

\[
D_j := \mu_2 \frac{(aq/c)(bq/c)_{j}}{(q^j/c)_{j}} \phi \left( q^{-j}, aq^{-j}/b, bq^{-j}/b, c(q^{-j-1}/b); q, q^{1-n} \right),
\]

\[
\bar{C}_j := \mu_1 \frac{(aq/c)(bq/c)_{j}}{(q^j/c)_{j}} \phi \left( q^{-j}, aq^{-j}/b, bq^{-j}/b, c(q^{-j-1}/b); q, q^{1-n} \right),
\]

\[
\bar{D}_j := \mu_2 \frac{(aq/c)(bq/c)_{j}}{(q^j/c)_{j}} \phi \left( q^{-j}, aq^{-j}/b, bq^{-j}/b, c(q^{-j-1}/b); q, q^{1-n} \right),
\]

with

\[
\mu_1 := \frac{k^{(m-k-l-n)}a^{m-k-l-n}(c/a)^{k-l}}{(aq/b)^{k-l}},
\]

\[
\mu_2 := \frac{(aq/c)(bq/c)^{m-k-l-n}(c/a)^{k-l}}{(aq/b)^{k-l}}.
\]
Here, \( \mu \) is defined by

\[
\mu := (-1)^{k+l-m-n-1} q^{k(l-1)+(l-1)-(m-1)+m(M-1)-n(N-1)/2} \frac{(q-c)ab^M}{(b-a)c^M (a)_{k(b)_l}}
\]

with \( M := \max \{ k + l - m + n, 0 \} \) and \( N := \min \{ n, 0 \} \).

Note that the number of terms with degree \( j \) in \( P \) appearing in Theorem 1 increases as \( j \) grows larger, whereas the number of terms with degree \( j \) in \( P \) appearing in Proposition 3 decreases as \( j \) grows larger, because, in general, for any non-negative integer \( j \),

\[
\phi_j(q^{-1}, b_1, b_2, c_1, c_2; q, x)
\]

is a sum of \( j + 1 \) terms. This implies, as stated in Section 1.2, that both the expressions for \( P \) given in Theorem 1 and Proposition 3 are useful in the construction of an algorithm for calculating \( P \).

The following lemma provides two generalizations of (1.2), which are used to prove Theorem 1 and Proposition 3. When \( m \geq 0 \) and \( s = 0 \), these formulas are reduced to (1.2).

Lemma 4. Assume that \( m \in \mathbb{Z} \) and \( r, n_1, \ldots, n_s \in \mathbb{Z}_{\geq 0} \). If \( |a^{-1} q^{m+1-n_1-s}| < 1 \), then

\[
\sum_{i=0}^{r} \frac{(q^{-r})_i}{(q)_i} \frac{(q^{1+i}(1/q)_i c_1 q^{n_1+i}/c_1)_{n_1-i} \cdots (q^{1-n_1+i}/c_1)_{n_1-i}}{(q)_i (q^{m-i}/b)_{m+i}} \cdot q^{r+1} \phi_{r+1}
\]

(1.5)

If \( |a^{-1} q^{m+1-n_1-s}| \leq 1 \) and \( \min \{ n_1, \ldots, n_s \} = 0 \), then

\[
\sum_{i=0}^{r} \frac{(q^{-r})_i}{(q)_i} \frac{(q^{1+i}(1/q)_i c_1 q^{n_1+i}/c_1)_{n_1-i} \cdots (q^{1-n_1+i}/c_1)_{n_1-i}}{(q)_i (q^{m-i}/b)_{m+i}} \cdot q^{r+1} \phi_{r+1}
\]

(1.6)

Note that by using Cauchy’s residue theorem, Chu extended (1.2) to the transformation formula [3] (15), which transforms a bilateral basic hypergeometric series \( r+2 \phi_{r+2} \) into an \( r+2 \phi_{r+1} \) series. From his formula, (1.5) can be derived in the following way: If we set \( c = q \), the \( r+2 \phi_{r+2} \) series in his formula reduces to an \( r+2 \phi_{r+1} \) series. Then, setting \( e = q^{-M} \), with \( M \in \mathbb{Z} \), in the resulting formula and relabeling the parameters, we obtain (1.5). In Section 2.1, we present an independent derivation of (1.5).

Also, as corollaries of Lemma 4 we give generalizations of Gasper’s [7] summation formulas for basic hypergeometric series.

2. Transformation and summation formulas for basic hypergeometric series

In this section, we prove Lemma 4. Also, as corollaries of Lemma 4 we give some summation formulas for basic hypergeometric series. In order to prove Lemma 4, we use
the following formula obtained by Andrews [11 p. 621, (4.1)]:
\[
\phi_D\left(\frac{a; b_1, \ldots, b_r}{c}; x_1, \ldots, x_r\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a)_{i+j} (b_1)_{i+j} \cdots (b_r)_{i+j}}{(c)_{i+j}} x_1^i \cdots x_r^j,
\]
provided that \(|x_1, \ldots, x_r, a| < 1\) when each of the series on both sides does not terminate, where \(\phi_D\) is the multiple series defined by
\[
\phi_D\left(\frac{a; b_1, \ldots, b_r}{c}; x_1, \ldots, x_r\right) := \frac{(a)_{\infty} (b_1 x_1)_{\infty} \cdots (b_r x_r)_{\infty}}{(c)_{\infty} (x_1)_{\infty} \cdots (x_r)_{\infty}} r+1 \phi_D\left(\frac{c/a, x_1, \ldots, x_r}{b_1 x_1, \ldots, b_r x_r}; q, a\right).
\]

(2.1) In order to prove Lemma 4, we use (2.1), and prove the equality. Next, we also rewrite both sides of (1.5) by using (2.1), and then, using (2.2), we prove the equality. In Section 2.2, considering the \(m = 0\) and \(m = -1\) cases of each (1.5) and (1.6), we obtain generalizations of Gasper’s [7, p. 197–198, (7), (8), and (15)] summation formulas for basic hypergeometric series.

Note that for any integers \(m\) and \(n\) satisfying \(m > n\), we define a sum \(\sum_{i=m}^{n} f_i\) as zero, where \(\{f_i\}\) is an arbitrary sequence. Also, below, we frequently use the following identities without explicitly stating so:
\[
(a)_0 = (a^{-1} q^{-1} - 1)(a^{-1} q^{-1} - 1)^{-1/2}, \quad (a)_{i+j} = (a)(aq^j), \quad (a)_{-j} = \frac{(a)}{(a^{-1} q^{-i})} (a^{-1} q^{j(1+i/2) - i}).
\]

2.1. Transformation formulas. We prove Lemma 4.

Let us put
\[
\overline{\phi_D}\left(\frac{a; b_1, \ldots, b_r}{c}; x_1, \ldots, x_r\right) := \frac{(c)_{\infty}}{(a)_{\infty}} \phi_D\left(\frac{a; b_1, \ldots, b_r}{c}; x_1, \ldots, x_r\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(c q^{i+j})_{\infty} (b_1)_{i+j} \cdots (b_r)_{i+j}}{(a q^{i+j})_{\infty} (q)_{i+j}} x_1^i \cdots x_r^j.
\]

Then, (2.1) can be rewritten as
\[
(2.3) \quad r+1 \overline{\phi_D}\left(\frac{a; b_1, \ldots, b_r}{c_1, \ldots, c_r}; x; q, x\right) = \frac{(b_1)_{\infty} \cdots (b_r)_{\infty}}{(c_1)_{\infty} \cdots (c_r)_{\infty}} \overline{\phi_D}\left(\frac{x; b_1, \ldots, b_r}{c_1, \ldots, c_r}; q; ax\right),
\]
provided that \(|x, |b_1|, \ldots, |b_r| < 1\) when each of the series on both sides does not terminate.

In order to prove Lemma 4, we use (2.3) and the following two lemmas and a corollary.

Reversing the order of summation in a \(\overline{\phi_D}\) multiple series, we obtain the following lemma:

Lemma 5. If \(m \in \mathbb{Z}, n, c_1, \ldots, c_r \in \mathbb{Z}_{\geq 0}, a \neq q, q^m, |x| < 1\), then
\[
\overline{\phi_D}\left(\frac{a; b, q^{-n_1}, \ldots, q^{-n_r}}{c_1, \ldots, c_r}; x, x_1, \ldots, x_r\right) = (-1)^{n_1+\cdots+n_r} q^{-\left(m(n_1+1)+\cdots+n_r(n_r+1)\right)/2} q^{-m+\cdots+n_r} x_1^{n_1} \cdots x_r^{n_r}
\]
\[
\times \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{k=\max(0,j)}^{\infty} \frac{(b)_{j}(q)_{j+i} \cdots (q^{-n_1})_{i} \cdots (q^{-n_r})_{i}}{(a q^{m+j})_{\infty} (q)_{i+j} \cdots (q)_{i}} x_1^{i} \cdots x_r^{i+j},
\]
where \(I := m + (n_1 + \cdots + n_r) - (i_1 + \cdots + i_r)\).
Corollary 6. Because 1/\(r\)

Proof. From the definition of \(\bar{\Phi}_D\), the left-hand side of (2.4) is equal to

\[
\sum_{i=0}^{n_1} \cdots \sum_{i=0}^{n_r} \sum_{j=0}^{\infty} \frac{(q^{m+1-j_i+\cdots+j_r})_0(b)(q^{-n_1})_{i_1} \cdots (q^{-n_r})_{i_r}}{(aq^{r_i-i+j})(q)_i \cdots (q)_r} x^j x_1^{i_1} \cdots x_r^{i_r}. \tag{2.5}
\]

Replacing \(i_v (v=1, \ldots, r)\) by \(n_v-i_v\), respectively, we find that (2.5) can be rewritten as

\[
(-1)^{n_1+\cdots+n_r} q^{-(n_1(n_1+1)+\cdots+n_r(n_r+1))/2} \sum_{i=0}^{n_1} \cdots \sum_{i=0}^{n_r} (q^{1+j_i+j})_0(b)(q^{-n_1})_{i_1} \cdots (q^{-n_r})_{i_r} x^j x_1^{i_1} \cdots x_r^{i_r}. \tag{2.6}
\]

Moreover, replacing \(j\) by \(j-1\), we find that (2.6) is equal to the right-hand side of (2.4), because 1/(q)_j = 0 holds for any negative integer \(j\). Thus the lemma is proved. \(\square\)

From Lemma 8, we obtain the following corollary:

Corollary 6. If \(m \in \mathbb{Z}, r, n_1, \ldots, n_r \in \mathbb{Z}_{\geq 0}, a \notin q^\mathbb{Z}, (bq^{-m-(n_1+\cdots+n_r)})_{m+(n_1+\cdots+n_r)} \neq 0, \) and \(|x| < 1\), then

\[
\bar{\Phi}_D(a; b, q^{-n_1}, \ldots, q^{-n_r}; q^{m+1} \colon x, x_1, \ldots, x_r) = \frac{(b)_\infty}{(aq^{-m})_\infty} \left(\frac{-1}{q^m n_1+\cdots+n_r(n_r+1)/2} \sum_{i=0}^{n_1} \cdots \sum_{i=0}^{n_r} (aq^{-m})_{i_1+\cdots+i_r} x^j x_1^{i_1} \cdots x_r^{i_r} \right). \tag{2.7}
\]

Proof. If \((bq^{-m-(n_1+\cdots+n_r)})_{m+(n_1+\cdots+n_r)} \neq 0\), then, eliminating terms that are zero on the right-hand side of (2.7), we are able to reduce (2.7) to (2.4). This proves the corollary. \(\square\)

From (2.2), we obtain the following lemma:

Lemma 7. If \(r, n_1, \ldots, n_r, s \in \mathbb{Z}_{\geq 0}\), then

\[
\sum_{i=0}^{s} \frac{(q^{-i})_q}{(q)_i} \sum_{i=0}^{n_1} \cdots \sum_{i=0}^{n_r} \sum_{j=0}^{s-1-i} \frac{(q^{1-s})_s j_i(aq^{-i})(b_1q^j)_i \cdots (b_rq^j)_i}{(cq^1j_1+a_1)(q)_i \cdots (q)_r} (xq^j)^{x_1^{i_1} \cdots x_r^{i_r}} = 0,
\]

where \(I_r := i_1 + \cdots + i_r\).

Proof. Changing the order of summation, we rewrite the left-hand side as

\[
\times_{r+3} \Phi_{r+3}(q^{-s} \colon c, q/a, b_1q^n, \ldots, b_rq^n; c(q^1j_1+a_1), q^{1-i}/a, b_1, \ldots, b_r; q, q).
\]

Then, from (2.2), we find that the sum of the above \(r+3\Phi_{r+3}\) series equals zero. This proves the lemma. \(\square\)

Remark 8. It follows from Lemma 7 that if \(r, s \in \mathbb{Z}_{\geq 0}\) and \(c \notin \{q^{-m} \mid m \in \mathbb{Z}_{\geq 0}\},\) then

\[
\sum_{i=0}^{s} \frac{(q^{-i})_q}{(q)(cq^1)^{i}} \Phi_{s}(q^{-i}; aq^{-i}, b_1q^n, \ldots, b_rq^n; \begin{array}{c} cq^{1-s} \colon x_1^{i}, x_1, \ldots, x_r \end{array}) = 0.
\]
Now, let us prove Lemma 4. It is sufficient to prove this lemma under the additional assumption that \(|b| < 1\), because, from the uniqueness of analytic continuation, we may drop this assumption. Thus, below, we impose the additional assumption.

First, we prove (1.5) by using (2.3) and Corollary 6. From (2.3), the left- and the right-hand sides of (1.5) can be rewritten as

\[
\frac{(b)_{m+1}}{(c_1)_n \cdots (c_r)_n} \Phi_D \left( \frac{a^{-1} q^{m+1-(n_1+\cdots+n_r)}; q^{-m}, \ldots, q^{-n_r} \quad q^{m+1-(n_1+\cdots+n_r)}; b, c_1 q^{n_1}, \ldots, c_r q^{n_r}}{q^{m+1-(n_1+\cdots+n_r)}; q^{-m}, \ldots, q^{-n_r} \quad q^{m+1-(n_1+\cdots+n_r)}; b, c_1 q^{n_1}, \ldots, c_r q^{n_r}} \right).
\]

respectively. Then, from Corollary 6 we find that these two expressions are equal to each other. This completes the proof of (1.5).

Next, we prove (1.6) by using (2.3) and Lemmas 5 and 7. Note that \(1/(q^{-m})_{m-i} = 0\) holds for \(i > m\). From this and (2.3), the right-hand side of (1.6) can be written as

\[
(2.8) \quad - \sum_{i=0}^{\min(m,s)} \frac{(q^{-r}_i)}{(q_i)} q^{(i-i)}/(r^{-m})_{m-i} \Phi_D \left( q^{1-r}/a, q^{r-m_1}, \ldots, q^{r-m} ; bq, \frac{bq}{c_1}, \ldots, \frac{bq}{c_r} \right).
\]

On the other hand, from (2.3), the left-hand side of (1.6) can be written as

\[
(2.9) \quad \sum_{i=0}^{\min(m,s)} \frac{(q^{-r}_i)}{(q_i)} q^{(i-i)}/(r^{-m})_{m-i} \Phi_D \left( q^{1-r}/a, q^{r-m_1}, \ldots, q^{r-m} ; bq, \frac{bq}{c_1}, \ldots, \frac{bq}{c_r} \right).
\]

Below, we show that (2.9) is equal to (2.8). It follows from Lemma 5 that (2.9) equals

\[
(2.10) \quad \sum_{i=0}^{\min(m,s)} \frac{(q^{-r}_i)}{(q_i)} (-1)^{m+1-i} q^{(m-i)(m-i-1)/2+m} b^{1-s} \times \prod_{i=0}^{n_1-i} \cdots \prod_{i=0}^{n_r-i} q^{(r+1)/i} \frac{q^{r-m_1}/(r^{-m})_{m-i}}{q^{r-m_1}/(r^{-m})_{m-i}} \left( bq \right)^{i} \left( \frac{bq}{c_1} \right)^{i_1} \cdots \left( \frac{bq}{c_r} \right)^{i_r},
\]

where \(l' \equiv m+i-s-l\), with \(l := i_1 + \cdots + i_r\). Since

\[
\frac{(q^{-r+1})}{(q^{-r})} = (-1)^{m+1-i} q^{(m-i)(m-i-1)/2+m} \frac{(q^{r+1})}{(q^{-r})} \frac{(q^{r+1})}{(q^{-r})},
\]

where \(\delta(i \leq m) := 1 \leq m \) and \(0 \leq m \), putting

\[
H(i_1, \ldots, i_r, j) := \frac{(q^{1-s})_{i_1} q^{r-i}/(q^{-m})_{i_1} \cdots q^{r-m_1}/(q^{-m})_{i_r} \left( bq \right)^{i_1} \left( \frac{bq}{c_1} \right)^{i_1} \cdots \left( \frac{bq}{c_r} \right)^{i_r}},
\]

we rewrite (2.10) as

\[
(2.11) \quad \sum_{i=0}^{r} \frac{(q^{-r}_i)}{(q_i)} q^{(i-i)/r(m-s-1)} b^{1-s} \prod_{i=0}^{n_1-i} \cdots \prod_{i=0}^{n_r-i} \sum_{j=0}^{\max(0, l')} H(i_1, \ldots, i_r, j).
\]
Moreover, considering cases satisfying $H(i, i_1, \ldots, i_r, j) = 0$, we find that (2.11) equals

\begin{equation}
(2.12) \quad - \frac{b^{1-s}}{(q^{-s})_{s-1}} \sum_{i=0}^{\min[m,s]} \frac{(q^{-s})_i}{(q)_i} q^{n_i} \cdot \frac{(q^{-s})_j}{(q)_j} q^{n_j} \cdot \sum_{i=0}^{n_i} \sum_{j=0}^{n_j} H(i, i_1, \ldots, i_r, j)
\end{equation}

It follows from Lemma 7 that the second multiple series in (2.12) equals

\begin{equation}
- \frac{b^{1-s}}{(q^{-s})_{s-1}} \sum_{i=0}^{\min[m,s]} \frac{(q^{-s})_i}{(q)_i} q^{n_i} \cdot \sum_{i=0}^{n_i} \sum_{j=0}^{n_j} H(i, i_1, \ldots, i_r, j)
\end{equation}

This implies that (2.12) equals

\begin{equation}
(2.13) \quad - \frac{b^{1-s}}{(q^{-s})_{s-1}} \sum_{i=0}^{\min[m,s]} \frac{(q^{-s})_i}{(q)_i} q^{n_i} \cdot \sum_{i=0}^{n_i} \sum_{j=0}^{n_j} H(i, i_1, \ldots, i_r, j)
\end{equation}

It is easily verified that (2.13) is equal to (2.8). This completes the proof of (1.6).

2.2. Summation formulas. By using Lemma 3 we derive some summation formulas. Below, it is assumed that $r, n_1, \ldots, n_r, s \in \mathbb{Z}_{\geq 0}$.

By setting $m = 0$ in (1.5), we find that if $|a^{-1}q^{-(n_1+\cdots+n_r)-s}| < 1$, then

\begin{equation}
(2.14) \quad r+2\phi_{r+1} \left( \frac{a, b, c_1q^{n_1}, \ldots, c_rq^{n_r}}{bq, c_1, \ldots, c_r} : q, a^{-1}q^{-(n_1+\cdots+n_r)-s} \right)
= \frac{(q)_{n_1} (bq/a)_{n_1} (c_1/b)_{n_1} \cdots (c_r/b)_{n_r} b^{n_1+\cdots+n_r}}{(q/a)_{s} (bq)_{s} (c_1)_{s} \cdots (c_r)_{s} b^{s}}.
\end{equation}

(This follows also from the $c = q$ case of Chu’s [3] (14) bilateral summation formula.) By setting $m = 0$ in (1.6), we find that if $|a^{-1}q^{-(n_1+\cdots+n_r)+s+\min[r-2,0]i}| < 1$ and $\min\{n_1, \ldots, n_r, s\} = s$, then

\begin{equation}
(2.15) \quad \sum_{i=0}^{\infty} \frac{(q^{-s})_i}{(q)_i} q^{(q^{-1}/a)_{s} (q^{1-n_1}/c_1)_{n_1-1} \cdots (q^{1-n_r}/c_r)_{n_r-1}}
\times r+2\phi_{r+1} \left( \frac{aq^{1-n_1}/c_1, \ldots, c_rq^{n_r}}{bq, c_1q^{n_1}, \ldots, c_rq^{n_r}} : q, a^{-1}q^{-(n_1+\cdots+n_r)+s+(r-2)k} \right)
= \frac{(bq/a)_{s} (bq^{1-n_1}/c_1)_{n_1-1} \cdots (bq^{1-n_r}/c_r)_{n_r-1} b^{s}}{(b)_{s}}.
\end{equation}

When $s = 0$, both (2.14) and (2.15) reduce to the formula [7, p. 197, (7)].

It follows from the $m = -1$ case of (1.5) that if $|a^{-1}q^{-(n_1+\cdots+n_r)-s}| < 1$, then

\begin{equation}
(2.16) \quad r+1\phi_{r+1} \left( \frac{a, c_1q^{n_1}, \ldots, c_rq^{n_r}}{c_1, \ldots, c_r} : q, a^{-1}q^{-(n_1+\cdots+n_r)-s} \right) = 0.
\end{equation}

Substituting $a = q^{-(n_1+\cdots+n_r)-s}$ into (2.16) gives the formula [7, p. 198, (15)], which is introduced in (2.2) and used to prove (1.6). It follows from the $m = -1$ case of (1.6) that if
\[ q^{-1}q^{-\{n_1+\cdots+n_r\}+s+\min\{r-2,0\}} < 1 \text{ and } \min\{n_1,\ldots,n_r,s\} = s, \text{ then} \]

\[
(2.17) \quad \sum_{r=0}^{s} \frac{(q^{-s})_r}{(q)_r} q^{n_r} (q^{-1}/a)_{n_1} (q^{-n_1}/c_1)_{n_1} \cdots (q^{-n_r}/c_r)_{n_r} (q/b)_i \times r+2 \phi_r+1 (aq', bq', c_1 q^{\phi_1}, \ldots, c_r q^{\phi_r}; q, a^{-1}, q^{-\{n_1+\cdots+n_r\}+s+(r-2)s}) = 0.
\]

When \( s = 0 \), both (2.16) and (2.17) reduce to the formula [7, p. 197, (8)].

Of course, considering limit cases and inversions of (2.14)–(2.17) as in [7], we are able to derive generalizations of Gasper’s [7] other summation formulas.

3. Three-term relations for the basic hypergeometric series \(_2\phi_1\)

In this section, following the notation, the method, and the process described in [11], we prove Theorem [11], Corollary [2], and Proposition [3]. In Sections 3.1–3.6, we prove Theorem [11] in the following way: In Section 3.1, following the method described in [11, Section 2] (that is, employing the method due to Vidūnas [12, Section 3]), we prove the uniqueness of the pair \((Q, R)\) satisfying (1.3). In Section 3.2, we introduce a series \(_2\phi_1\) and rewrite the three-term relation (1.3) into a three-term relation for \(_2\phi_1\) with coefficients \(Q\) and \(R\). Then, comparing these two three-term relations, we obtain expressions for \(Q\) and \(R\) in terms of \(\hat{Q}\) and \(\hat{R}\), respectively. In Section 3.3, we introduce four local solutions \(y_i\) \((i = 1, 2, 3, 4)\) of a \(q\)-differential equation \(E_q(a, b, c)\) defined by \(L_y(x) = 0\), where

\[
L := (1 - T_q)(1 - cq^{-1}T_q) - x(1 - aT_q)(1 - bT_q)
\]

with \(T_q : x \mapsto qx\). Also, in Section 3.4, we introduce eight \(q\)-differential operators called “contiguity operators,” and combining these operators, we obtain a \(q\)-differential operator \(\theta(k, l, m; n)\). Then, in Section 3.5, operating \(\theta(k, l, m; n)\) on \(y_i\)’s and using the uniqueness proved in Section 3.1, we obtain linear equations for \(\hat{Q}\) and \(\hat{R}\). Solving these equations, we are able to express each \(\hat{Q}\) and \(\hat{R}\) as a ratio of infinite series defined as a sum of products of \(y_i\)’s. In Section 3.6, using these expressions and Lemma [4], we obtain expressions for \(\hat{Q}\) and \(\hat{R}\) in terms of \(P\) and thereby complete the proof of Theorem [11]. In Section 3.7, to prove Proposition [5] we first define a polynomial \(\hat{P}\) and obtain an expression for \(Q\) in terms of \(\hat{P}\) that differs from the expression in terms of \(P\) given in Theorem [11]. Then, employing the uniqueness of \(Q\), by comparing these expressions for \(Q\), we are able to complete the proof. In Section 3.8, by using Theorem [11] we prove Corollary [2].

3.1. Uniqueness of the pair \((Q, R)\) We prove by contradiction that the pair \((Q, R)\) satisfying (1.3) is uniquely determined by \((k, l, m, n)\).

Let us assume that there are two distinct pairs \((Q_1, R_1)\) and \((Q_2, R_2)\) of rational functions satisfying (1.3). Then, we have

\[
(Q_1 - Q_2) \cdot _2\phi_1\left( \frac{aq', bq'}{cq}; q, x \right) = (R_2 - R_1) \cdot _2\phi_1\left( \frac{a, b}{c}; q, x \right).
\]

Therefore, it turns out that \(_2\phi_1\left(aq', bq'; cq; q, x\right)/_2\phi_1(a, b; c; q, x)\) is a rational function of \(a, b, c, q\), and \(x\). However, this leads to a contradiction (see the last eight lines of [11, Section 2.1] for detail). Thus the uniqueness of the pair \((Q, R)\) satisfying (1.3) is proved.
3.2. Rewriting of the three-term relation. We introduce a series \( \tilde{\phi}_1 \) and rewrite the three-term relation (1.3) into a three-term relation for \( \tilde{\phi}_1 \).

Let \( \tilde{\phi}_1 \) be the series defined by
\[
\tilde{\phi}_1(a, b, c; x) := \frac{(q)_{\infty}(c)_{\infty}}{(a)_\infty(b)_\infty} \phi_1(a, b, c; q, x).
\]
Then, the three-term relation (1.3) can be rewritten as
\[
\tilde{\phi}_1(aq^{k}, bq^{n}; xq^{n}) = \tilde{Q} \cdot \tilde{\phi}_1(aq, bq; cx) + \tilde{R} \cdot \tilde{\phi}_1(a, b; x).
\]  
(3.1)

Comparing (1.3) with (3.1), we find that \( \tilde{Q} \) and \( \tilde{R} \) can be expressed in terms of \( \tilde{Q} \) and \( \tilde{R} \) as
\[
\tilde{Q} = \frac{(cq)_{n-1}}{(aq)_{k-1}(bq)_{l-1}} \tilde{Q},
\]
(3.2)
\[
\tilde{R} = \frac{(c)_{m}}{(a)_{l}(b)_{l}} \tilde{R}.
\]
(3.3)

Hence, we investigate \( \tilde{Q} \) and \( \tilde{R} \). Note that as a consequence of Section 3.1, it follows that the pair \( (\tilde{Q}, \tilde{R}) \) satisfying (3.1) is uniquely determined by \( (k, l, m, n) \).

3.3. Local solutions of \( E_{q}(a, b, c; x) \). We introduce four local solutions of \( E_{q}(a, b, c; x) \).

Let \( y_{i} (i = 1, 2, 3, 4) \) be the functions defined by
\[
y_{1}(a, b, c; x) = y_{1}(a, b, c; x) := \tilde{\phi}_1(a, b, c; x),
\]
\[
y_{2}(a, b, c; x) = y_{2}(a, b, c; x) := x^{1-\gamma} \tilde{\phi}_1(aq/c, bq/c; q^{2}/c; x),
\]
\[
y_{3}(a, b, c; x) = y_{3}(a, b, c; x) := a^{1-\alpha} \tilde{\phi}_1(aaq, bq, cq/c; abx),
\]
\[
y_{4}(a, b, c; x) = y_{4}(a, b, c; x) := b^{1-\alpha} \tilde{\phi}_1(abx, bq, cq/c; abx),
\]
where \( a = q^{n}, b = q^{l}, \) and \( c = q^{m} \).

Then, using the general theory of linear difference equations mentioned in [4] p. 62, Theorem 2.15, we are able to verify the following lemma:

**Lemma 9.** Under the assumption (1.4), \( y_{i}(a, b, c; x) (i = 1, 2) \) are linearly independent solutions around \( x = 0 \) of \( E_{q}(a, b, c; x) \), and \( y_{i}(a, b, c; x) (i = 3, 4) \) are linearly independent solutions around \( x = \infty \).

**Remark 10.** The Casoratians are given by
\[
\det \begin{pmatrix}
y_{1} & y_{2} \\
T_{q^{y_{1}}} & T_{q^{y_{2}}}
\end{pmatrix} = \frac{(q)_{n}(c)_{n}(q/c)_{n}(abx/c)_{n}}{(a)_{n}(b)_{n}(aq/c)_{n}(bq/c)_{n}(x)_{\infty}} x^{1-\gamma},
\]
\[
\det \begin{pmatrix}
y_{3} & y_{4} \\
T_{q^{y_{3}}} & T_{q^{y_{4}}}
\end{pmatrix} = \frac{(a)_{n}(b)_{n}(aq/c)_{n}(bq/c)_{n}(c)_{n}(abx)_{\infty}}{(a)_{n}(b)_{n}(aq/c)_{n}(bq/c)_{n}(c)_{n}(abx)_{\infty}} x^{-\alpha}. 
\]
For these expressions, see Remark [13 3.20], and [3.21].
3.4. Contiguity operators. We introduce eight $q$-differential operators that each increase or decrease $a, b, c,$ or $x$ by $q$ times, and then, combining these operators, we obtain a linear isomorphism $\theta(k, l, m; n)$, which sends $a, b, c,$ and $x$ to $aq^k, bq^l, cq^m,$ and $xq^n$, respectively.

Let $H_j$ and $B_j$ $(j = 1, 2, 3, 4)$ be the $q$-differential operators defined by

$$
H_1(a, b, c; x) := 1 - aT_q, \\
H_2(a, b, c; x) := 1 - bT_q, \\
H_3(a, b, c; x) := (c - a)(c - b)x^{-1}\left[c^2 + abx - (a + b)cx - c(c - ab)T_q\right], \\
H_4(a, b, c; x) := T_q, \\
B_1(a, b, c; x) := [(q - a)(c - a)]^{-1}\left[q - a(q + c) + a^2x + a(c - ab)T_q\right], \\
B_2(a, b, c; x) := [(q - b)(c - b)]^{-1}\left[q - b(q + c) + b^2x + b(c - ab)T_q\right], \\
B_3(a, b, c; x) := 1 - cq^{-1}T_q, \\
B_4(a, b, c; x) := (q - x)^{-1}\left[c + q - (a + b)x - (c - ab)T_q\right].
$$

Then, the operators $H_j$ $(j = 1, 2, 3)$ increase the parameters $a, b,$ and $c$ by $q$ times, respectively, while the operators $B_j$ $(j = 1, 2, 3)$ decrease these parameters by $q$ times. Also, the operators $H_4$ and $B_4$ shift the variable $x$ by $q$ and $q^{-1}$ times, respectively. Hence, we call these eight operators “contiguity operators.” The definitions of $H_1, H_2,$ and $B_3$ are easily derived from Heine’s \cite{8} p. 287, Formula 2 and p. 292, Formulas 42–44 three-term relations and the definition of $H_4$ immediately comes from the definition of $T_q$. Using the method described in \cite{9} p. 46, Remark 2.1.4, we are able to derive $B_1, B_2, H_3,$ and $B_4$ from $H_1, H_2, B_3,$ and $H_4,$ respectively.

In fact, performing direct calculations, we obtain the following lemma:

Lemma 11. Let us write $y_i(a, b; c; x), y_i(aq^k, b; c; x), y_i(a, bq^l; c; x), y_i(a, b; cq^m; x),$ and $y_i(a, b; c; xq^n)$ as $y_i$, $y_i(aq^k)$, $y_i(bq^l)$, $y_i(cq^m)$, and $y_i(xq^n)$, respectively. Let $H_j$ and $B_j$ denote $H_j(a, b, c; x)$ and $B_j(a, b, c; x)$, respectively. Then, for $i = 1$ and 2, we have

$$
H_1y_i = y_i(aq), \\
H_2y_i = y_i(bq), \\
H_3y_i = y_i(cq), \\
H_4y_i = y_i(xq).
$$

Also, for $i = 3$ and 4, we have

$$
B_1y_i = -aq^{-1}y_i, \\
B_2y_i = -bq^{-1}y_i, \\
B_3y_i = -cq^{-1}y_i, \\
B_4y_i = -xq^{-1}y_i.
$$

Moreover, for $i = 1, 2, 3,$ and 4, we have

$$
H_1y_i = y_i(a), \\
H_2y_i = y_i(b), \\
H_3y_i = y_i(c), \\
H_4y_i = y_i(x).
$$

Let $S_q(a, b, c; x)$ denote the solution space of $E_q(a, b, c; x)$ on a simply connected domain in $\mathbb{C}\setminus\{0\}$. Then, from Lemmas 9 and 11 under the assumption 14, combining $H_j$ and $B_j$ $(j = 1, 2, 3, 4)$, we obtain a linear isomorphism

$$
\theta(k, l, m; n) : S_q(a, b, c; x) \rightarrow S_q(aq^k, bq^l, cq^m; xq^n)
$$
for any integers \(k, l, m, \) and \(n\). Note that the definition of \(\theta(k, l, m; n)\) does not depend on the order of composition of contiguity operators. Here, we set the order as follows:

\[
S_q(a, b, c; x) \rightarrow \cdots \rightarrow S_q(aq^k, b, c; x) \rightarrow \cdots \rightarrow S_q(aq^k, bq^l, c; x) \rightarrow \cdots \\
\rightarrow S_q(aq^k, bq^l, cq^m; x) \rightarrow \cdots \rightarrow S_q(aq^k, bq^l, cq^m; xq^n).
\]

### 3.5. Expressions for each \(\tilde{Q}\) and \(\tilde{R}\) as a ratio of infinite series.

We express each \(\tilde{Q}\) and \(\tilde{R}\) as a ratio of infinite series defined as a sum of products of \(y_i\) \((i = 1, 2, 3, 4)\).

Put \(\Delta := x^3(1 - T_q)\). By direct calculations, we obtain the following lemma:

**Lemma 12.** Let \(y_i(a, b, c)\) denote \(y_i(a, b; c; x)\). Then, we have

\[
\Delta y_i(a, b, c) = \begin{cases} y_i(aq, bq, cq), & i = 1, 2, \\ -abc^{-1}y_i(aq, bq, c), & i = 3, 4. \end{cases}
\]

Let \(Q(a, b, c, q, x)\) denote the field generated by \(a, b, c, q,\) and \(x\) over \(Q\), and let us write the ring of polynomials in \(T_q\) over \(Q(a, b, c, q, x)\) as \(Q(a, b, c, q, x)[T_q]\), where \(T_q\) and \(x\) are not commutative because \(T_qx = qxT_q\). Then, regarding \(\theta(k, l, m; n)\) as an element of \(Q(a, b, c, q, x)[T_q]\), we express \(\theta(k, l, m; n)\) as

\[
\theta(k, l, m; n) = p(T_q) \cdot L + \tilde{Q} \cdot T_q + \tilde{R},
\]

where \(\tilde{Q}, \tilde{R} \in Q(a, b, c, q, x)\) and \(p(T_q) \in Q(a, b, c, q, x)[T_q]\). From \(T_q = 1 - x\Delta\), this expression can be rewritten as

\[
\theta(k, l, m; n) = p(T_q) \cdot L - x\tilde{Q} \cdot \Delta + (\tilde{Q} \cdot R).
\]

From Lemmas 9 and 12 operating (3.4) on \(y_1(a, b; c; x)\) gives

\[
2\tilde{y}_1(aq^k, bq^l, cq^m; x) = -x\tilde{Q} \cdot 2\tilde{y}_1(aq, bq, cq; x) + (\tilde{Q} \cdot R) \cdot 2\tilde{y}_1(a, b, c; x).
\]

Since the pair \((\tilde{Q}, \tilde{R})\) satisfying (3.1) is unique as a consequence of Section 3.1, comparing (3.5) with (3.1), we find that \(-x\tilde{Q} = \tilde{Q}\) and \(\tilde{Q} + \tilde{R} = \tilde{R}\). Namely, we obtain

\[
\theta(k, l, m; n) = p(T_q) \cdot L + \tilde{Q} \cdot \Delta + \tilde{R}.
\]

From Lemmas 9 and 12 operating (3.6) on \(y_i(a, b; c; x)\) \((i = 1, 2, 3, 4)\) gives

\[
y_i(aq^k, bq^l, cq^m; x) = \tilde{Q} \cdot y_i(aq, bq, cq; x) + \tilde{R} \cdot \tilde{y}_i(a, b, c; x), \quad i = 1, 2,
\]

\[
\lambda \cdot \tilde{y}_i(aq^k, bq^l, cq^m; x) = -abc^{-1}\tilde{Q} \cdot \tilde{y}_i(aq, bq, cq; x) + \tilde{R} \cdot \tilde{y}_i(a, b, c; x), \quad i = 3, 4,
\]

where \(\lambda := (-1)^{k+l+m}d^kb^lc^m \cdot q^{(l-1) + (k-l-1) + (m-1)}/2\). Solving two equations (3.7) for \(\tilde{Q}\) and \(\tilde{R}\), we have

\[
\tilde{Q} = \tilde{Q}(k, l, m, n) = Y^\left(k, l \; m \; n \; a, b \; c \; x\right) Y^\left(1, 1 \; 0 \; a, b \; c \; x\right),
\]

\[
\tilde{R} = \tilde{R}(k, l, m, n) = -Y^\left(k - 1, l - 1 \; m - 1 \; aq, bq \; cq \; x\right) Y^\left(1, 1 \; 0 \; a, b \; c \; x\right),
\]

where \(Y\) is the infinite series defined by

\[
Y^\left(k, l \; m \; n \; a, b \; c \; x\right) := y_1(aq^k, bq^l, cq^m; xq^n) y_2(a, b, c; x) - y_2(aq^k, bq^l, cq^m; xq^n) y_1(a, b, c; x).
\]
On the other hand, solving two equations (3.8) for $\tilde{Q}$, we have

\[(3.12) \quad \tilde{Q} = \tilde{Q}(k, l, m, n) = -Ac(ab)^{-1} \cdot \tilde{Y}^{k, l}_{m} \left( a, b ; x \right) \sqrt[3]{n} \tilde{Y}^{1, 1}_{1} ; 0 ; a, b ; c ; x \]

where $\tilde{Y}$ is the infinite series defined by

\[(3.13) \quad \tilde{Y}^{k, l}_{m} \left( a, b ; c ; x \right) := y_{3} \left( aq^{k}, b_{q}^{l} ; xq^{m} \right) y_{4} \left( a_{q}^{k}, b_{q}^{l} ; x_{q}^{m} \right) y_{3} \left( a, b ; c ; x \right).

In the next two sections, we use (3.9)–(3.11) for proving Theorem 1 and use (3.12) and (3.13) for proving Proposition 8.

Remark 13. The Casoratians noted in Remark 10 are appearing in the denominators of (3.9), (3.10), and (3.12) as follows:

\[\begin{align*}
\tilde{Y}^{1, 1}_{1} ; 0 ; a, b ; c ; x &= -\text{det} \left( \begin{array}{cc} y_{1} & y_{2} \\ \Delta y_{1} & \Delta y_{2} \end{array} \right) = x^{-1} \cdot \text{det} \left( \begin{array}{cc} y_{1} & y_{2} \\ T_{y_{1}} & T_{y_{2}} \end{array} \right), \\
\tilde{Y}^{1, 1}_{1} ; 0 ; a, b ; c ; x &= c(ab)^{-1} \cdot \text{det} \left( \begin{array}{cc} y_{3} & y_{4} \\ \Delta y_{3} & \Delta y_{4} \end{array} \right) = -c(abx)^{-1} \cdot \text{det} \left( \begin{array}{cc} y_{3} & y_{4} \\ T_{y_{3}} & T_{y_{4}} \end{array} \right).
\end{align*}\]

3.6. Expressions for $Q$ and $R$. We complete the proof of Theorem 1.

To prove Theorem 1, we give four lemmas. From Lemma 4, we obtain the following lemma:

Lemma 14. For any integers $k, l, m$, and $n$, with $k \leq l$, the following statements are true:

(i) \hspace{1cm} \[k + l - m + n \geq 0, n \geq 0 \Rightarrow \sum_{i=0}^{n} \frac{(q^{-n})_{i}}{(q)_{i}} q^{mi}(A_{j-i} - B_{j-i-m}) = 0, \; j \geq l + n;\]

(ii) \hspace{1cm} \[k + l - m + n \geq 0, n \leq 0 \Rightarrow A_{j} - B_{j-m} = 0, \; j \geq l;\]

(iii) \hspace{1cm} \[k + l - m + n \leq 0, n \geq 0 \Rightarrow \tilde{A}_{j} - \tilde{B}_{j-m} = 0, \; j \geq m - k;\]

(iv) \hspace{1cm} \[k + l - m + n \leq 0, n \leq 0 \Rightarrow \sum_{i=0}^{n} \frac{(q^{-n})_{i}}{(q)_{i}} (\tilde{A}_{j-i} - \tilde{B}_{j-i-m}) = 0, \; j \geq m - k - n.\]

Proof. Using (1.6), we prove (i). Replacing $m, n_{1}, n_{2}, s, a, b, c_{1}, c_{2}$ in the $r = 2$ case of (1.6) by $j, k, j-l, n, q^{m-1}, c_{q}^{m-1}, c_{q}^{m-1}/a, c_{q}^{m-1}/b$, respectively, we find that if $k + l - m + n \geq 0$ and $\min \{ j - k, j - l \} \geq n \geq 0$, then

\[\begin{align*}
\sum_{i=0}^{n} \frac{(q^{-n})_{i}}{(q)_{i}} q^{m}(aq/c)_{j-i-m}(bq/c)_{j-i-m} & \frac{q^{2}/c}_{m-1} (aq/c)_{k-m}(bk/c)_{k-m} \\
\times 4\phi_{3} \left( q^{-(j-i-m)}, c_{q}^{-(j-i-m)-1}, c_{q}^{m-k}/a, c_{q}^{m-j}/b, cq^{m}, cq^{-j-i-m}/a, cq^{m-j-i-m}/b ; q, q^{k+l-m+n+1} \right) \\
= -\sum_{i=0}^{n} \frac{(q^{-n})_{i}}{(q)_{i}} q^{m}(aq/c)_{m-1}(aq^{m-j-i-1}/b)^{j-i} q^{j-i} a^{j-i} b^{j-i} c_{q}^{m-j-i} a q^{m-j-i} q^{l}.
\end{align*}\]

Moreover, multiplying both sides of this by $(aq/c)_{k-m}(bk/c)_{k-m}/(q^{2}/c)_{m-1}, we have

\[\sum_{i=0}^{n} \frac{(q^{-n})_{i}}{(q)_{i}} q^{m}B_{j-i-m} = \sum_{i=0}^{n} \frac{(q^{-n})_{i}}{(q)_{i}} q^{m}A_{j-i}.\]

Thus (i) is proved.

In almost the same way, using (1.5), we are able to prove (ii) and (iii), and using (1.6), we are able to prove (iv). \[\square\]
To be used below, we introduce a formula:

\[(3.14) \quad \sum_{i=0}^{n} \frac{(q^{-n})_i}{(q)_i} x^i = (xq^{-n})_n, \quad n = 0, 1, \ldots .\]

This is a special case of the $q$-binomial theorem

\[\phi_0 \left( a \frac{1}{1-q}; x \right) = \sum_{i=0}^{\infty} \frac{(a)_{i}}{(q)_{i}} x^i = \frac{(ax)_{\infty}}{(x)_{\infty}}, \quad |x| < 1.\]

(See [2] pp. 488–490, Theorem 10.2.1 and [6] Section 1.3 for simple proofs of the $q$-binomial theorem.) We also use the following product formula:

\[(3.15) \quad z\phi_1 \left( \frac{a, b}{c} ; q, gx \right) z\phi_1 \left( \frac{d, e}{f} ; q, hx \right) = \sum_{j=0}^{\infty} \frac{(a)_{j}(b)_{j}}{(q)_{j}(c)_{j}} g^j \frac{1}{4} \phi_3 \left( q^{-j}, q^{-j}/c, d, e, qch \frac{abq}{abg} ; q, j; q, j; q, j \right), \quad |gx| < 1, |hx| < 1.\]

Collecting the coefficients of $x^i$ on the left-hand side gives this formula.

From Lemma [14] (3.14), and (3.15), we obtain the following lemma:

**Lemma 15.** When $k \leq l$, the polynomial $P$ defined in Theorem [1] can be rewritten in the following form:

\[(3.16) \quad P \left( k, l \frac{m}{n} ; a, b \frac{c}{c} ; q, x \right) = \begin{cases} \left( x \right)_{\max(n,0)} \left( x \right)_{\max(m,0)} \left( x^{-m} \sum_{j=0}^{\infty} A_j x^j - \sum_{j=0}^{\infty} B_j x^j \right), & k + l - m + n \geq 0, \\ \left( x \right)_{\min(n,0)} \left( x \right)_{\min(m,0)} \left( x^{-m} \sum_{j=0}^{\infty} \tilde{A}_j x^j - \sum_{j=0}^{\infty} \tilde{B}_j x^j \right), & k + l - m + n < 0. \end{cases}\]

Moreover, each of the infinite series $\sum_{j=0}^{\infty} A_j x^j$, $\sum_{j=0}^{\infty} B_j x^j$, $\sum_{j=0}^{\infty} \tilde{A}_j x^j$, and $\sum_{j=0}^{\infty} \tilde{B}_j x^j$ can be rewritten as a product of two $2\phi_1$ series as follows, respectively:

\[
2\phi_1 \left( \frac{aq/c}{q^2/c} ; q, \frac{aq/c}{q^2/c} \right) 2\phi_1 \left( \frac{bq/c}{c} : q, \frac{bq/c}{c} : q, x \right), \\
2\phi_1 \left( \frac{aq/c}{q^2/c} ; q, \frac{aq/c}{q^2/c} \right) 2\phi_1 \left( \frac{bq/c}{c} : q, \frac{bq/c}{c} : q, x \right),
\]

\[
2\phi_1 \left( \frac{aq/c}{q^2/c} ; q, \frac{aq/c}{q^2/c} \right) 2\phi_1 \left( \frac{bq/c}{c} : q, \frac{bq/c}{c} : q, x \right),
\]

\[
2\phi_1 \left( \frac{aq/c}{q^2/c} ; q, \frac{aq/c}{q^2/c} \right) 2\phi_1 \left( \frac{bq/c}{c} : q, \frac{bq/c}{c} : q, x \right).
\]

**Proof.** From Lemma [14] we obtain

\[(3.17) \quad P \left( k, l \frac{m}{n} ; a, b \frac{c}{c} ; q, x \right) = \begin{cases} \sum_{j=0}^{\infty} \frac{(q^{-j})_{j}}{(q)_{j}} \left( A_{j+i-m-\max(m,0)} - B_{j-i-\min(m,0)} \right) x^j, & k + l - m + n \geq 0, \\ \sum_{j=0}^{\infty} \frac{(q^{-j})_{j}}{(q)_{j}} \left( \tilde{A}_{j+i-m-\max(m,0)} - \tilde{B}_{j-i-\min(m,0)} \right) x^j, & k + l - m + n < 0. \end{cases}\]
Also, from (3.14), we have
\[ (3.18) \sum_{i=0}^{\max\{n,0\}} \frac{(q^a)^i}{(q)^i} (xq^n)^i = (x)_{\max\{n,0\}}, \quad \sum_{i=0}^{\min\{n,0\}} \frac{(q^a)^i}{(q)^i} x^i = (xq_{\min\{n,0\}})_{-\min\{n,0\}}. \]

Using (3.17), (3.18), and the fact that \( A_j, B_j, \tilde{A}_j, \) and \( \tilde{B}_j \) are equal to zero for any negative integer \( j \), we obtain (3.16). Also, it follows from (3.15) that the statement following (3.16) is true. Thus the lemma is proved. \( \square \)

Heine \cite{8} p. 325, Formula XVIII] obtained the following \( q \)-Euler transformation formula:
\[ (3.19) \sum_{j=0}^{\infty} \frac{(abx/c)_{\infty}}{(x)_{\infty}} 2\phi_1\left( \frac{c}{a}, \frac{c}{b} : q, \frac{ab}{c} x \right). \]

From Lemma 15 and (3.19), we obtain the following lemma:

**Lemma 16.** When \( k \leq l \), the series \( Y \) defined by (3.11) can be expressed as
\[ Y\left( \begin{array}{ccc} k & l & n; & a & b \cr m & c & x \end{array} \right) = \frac{(abq^{m(n-1)+l-m+n,0})}{(cx^{m(n-1)+l-m+n,0})_\infty} \Phi\left( \begin{array}{ccc} k & l & n; & a & b \cr m & c & x \end{array} \right), \]
where \( \lambda_1 := (q^a)_\infty (q^{c/2})_\infty \mid (a)_\infty (q)_{\infty} (bq/c)_\infty \).

**Proof.** In the cases of \( k + l - m + n \geq 0 \), we apply (3.19) to \( y_1(abq^k, bq^l, cq^{m-n}) \) \((i = 1, 2) \) in (3.11). On the other hand, in the cases of \( k + l - m + n < 0 \), we apply (3.19) to \( y_1(a, b; c; x) \) \((i = 1, 2) \) in (3.11). Then, from Lemma 15 we obtain Lemma 16. \( \square \)

From Lemma 16 and the definition of \( P \), we have
\[ (3.20) Y\left( \begin{array}{ccc} 1 & 1 & 0; & a & b \cr 1 & c & x \end{array} \right) = \lambda_1 \frac{q-c\theta}{c} \left( \frac{abq/c}{x} \right)_{\infty}. \]

Therefore, from (3.9), (3.10), (3.20), and Lemma 16, we obtain the following lemma:

**Lemma 17.** Assume that \( k \leq l \). Then, the coefficients of (3.1) can be expressed as
\[ \tilde{Q}(k, l, m, n) = \frac{c(a)_k(b)_l}{(q-c)c_{m} (ab/c)_\max(k+l-m+n,0)-1} \left( x \right)_{\min\{n,0\}} \Phi\left( \begin{array}{ccc} k & l & n; & a & b \cr m & c & x \end{array} \right), \]
\[ \tilde{R}(k, l, m, n) = \frac{c(a)_k(b)_l}{(q-c)c_{m} (ab/c)_\max(k+l-m+n,0)-1} \left( x \right)_{\min\{n,0\}} \Phi\left( \begin{array}{ccc} k & l & n; & a & b \cr m & c & x \end{array} \right). \]

From (3.2), (3.3), and Lemma 17, we are able to complete the proof of Theorem 1

### 3.7. Alternative expressions for \( Q \) and \( R \)

We complete the proof of Proposition 3

Let \( P \) be the polynomial in \( x \) defined by
\[ P\left( \begin{array}{ccc} k & l & n; & a & b \cr m & c & x \end{array} \right) = \sum_{j=0}^{d} \sum_{i=0}^{\max\{n,0\}} \frac{(q^a)^i}{(q)^i} (C_{j-i} - D_{j-i+k}) x^{d-j}, \quad k + l - m + n \geq 0, \]
\[ + \sum_{j=0}^{d} \sum_{i=0}^{\min\{n,0\}} \frac{(q^a)^i}{(q)^i} (C_{j-i} - D_{j-i+k}) x^{d-j}, \quad k + l - m + n < 0, \]
where \( d, C_j, D_j, \tilde{C}_j, \) and \( \tilde{D}_j \) are defined as in Proposition 3.

To prove Proposition 3, we give four lemmas. By using Lemma 4 as in the proof of Lemma 14, we obtain the following lemma:
Lemma 18. For any integers \(k, l, m,\) and \(n,\) with \(k \leq l,\) the following statements are true:

\[
k + l - m + n \geq 0, \quad n \geq 0 \implies \sum_{j=0}^{n} \frac{(q^{-x})^i}{(q)_i} (C_{j-i} - D_{j+i-k-l}) = 0, \quad j \geq \max\{l, l-m\} + n;
\]

\[
k + l - m + n \geq 0, \quad n \leq 0 \implies C_j - D_{j+i-k-l} = 0, \quad j \geq \max\{-k, m-k\} - l;
\]

\[
k + l - m + n \leq 0, \quad n \geq 0 \implies \bar{C}_j - \bar{D}_{j+i-k-l} = 0, \quad j \geq \max\{-k, m-k\} - n.
\]

Also, in the same way as in the proof of Lemma 15 by using (13.14), (13.15), and Lemma 18 we obtain the following lemma:

Lemma 19. When \(k \leq l,\) the polynomial \(\bar{P}\) can be rewritten in the following form:

\[
\bar{P}(k, l, m ; n ; a, b, c ; q, x)
\]

\[
= \begin{cases} 
\phi_1(b, bq/c ; q, cq/abx) + \phi_1(q/a, c/a ; q, q/ab), & \text{if } k + l - m + n \geq 0, \\
\phi_1(q/a, c/a ; q, q/ab) + \phi_1(q/b, c/b ; q, q/x), & \text{if } k + l - m + n < 0,
\end{cases}
\]

where \(d, \mu_1, \) and \(\mu_2\) are defined as in Proposition 3.

Moreover, in the same way as in the proof of Lemma 16 by applying (13.19) to (13.13) and using Lemma 19 we obtain the following lemma:

Lemma 20. The series \(\bar{Y}\) defined by (3.13) can be expressed as

\[
\bar{Y}(k, l, m ; a, b, c ; x) = \lambda_2 \frac{x^{-a-b-k-d}(q^{1-min(n,0)}/x)_{\infty}}{(cq^{l-max(k+l-m+n,0)}/(abx)_{\infty})} \bar{P}(k, l, m ; n ; a, b, c ; q, x),
\]

where \(\lambda_2 := (ab)^{-a-b+l}(q^{2}(aq/b)_{\infty}(bq/a)/((a)_{\infty}(b)_{\infty}(aq/c)_{\infty}(bq/c)_{\infty}).\)

From Lemma 20 and the definition of \(\bar{P},\) we obtain

\[
(3.21) \quad \bar{Y}(1, 1 ; a, b ; c ; x) = \lambda_2 \frac{c(b-a) x^{-a-b-l}(q/x)_{\infty}}{a^2 b^2 (c/(abx)_{\infty})}.\]

Therefore, from (3.2), (3.12), (3.21), and Lemma 20 we obtain the following lemma:

Lemma 21. For any integers \(k, l, m,\) and \(n,\) with \(k \leq l,\) the coefficient \(Q\) of (1.3) can be expressed as

\[
Q = Q(k, l, m, n) = \mu \frac{(1-a)(1-b)c}{(q-c)(1-c)} x^{1-max(m,0)/max(k+l-m+n,0)} \bar{P}(k, l, m ; n ; a, b, c ; q, x),
\]

where \(\mu\) is defined as in Proposition 5.
From the uniqueness of the pair \((Q, R)\) proved in Section 3.1, we find that \(Q\) appearing in Theorem 1 and \(Q\) appearing in Lemma 21 are equal to each other. Therefore, by comparing the expressions for \(Q\) given in Theorem 1 and Lemma 21, we have \(P = \mu \tilde{P}\). This completes the proof of Proposition 3.

**Remark 22.** We consider the degree of \(P\). From Proposition 3, we find that for any integers \(k, l, m,\) and \(n\), with \(k \leq l\), the degree of the polynomial \(P\) is no more than \(d\), where \(d := \max\{k + l - m + n, 0\} + \max\{m, 0\} - \min\{n, 0\} - k - 1\), and the coefficient of \(x^d\) in \(P\) equals \(\mu(C_0 - D_0)\) if \(k = l\); \(\mu C_0\) if \(k < l\). Namely, the coefficient of \(x^d\) in \(P\) equals

\[
\mu \begin{cases} 
\frac{c_k}{a_b} q^k (m-k-l-n+1) & a_m(aq/c)_{k-m} - b_m(bq/c)_{k-m}, \quad k = l, \\
\frac{c_k}{a_b} q^k (m-k-l-n+1) & a_m(aq/c)_{k-m} (aq/b)_{k-l}, \quad k < l,
\end{cases}
\]

where \(\mu\) is defined as in Proposition 3.

### 3.8. Relation between \(Q\) and \(R\)

We prove Corollary 2.

From Theorem 1, we have

\[
Q(k - 1, l - 1, m - 1, n)_{(a,b,c)=(aq,bq,cq)} = \frac{(1 - aq)(1 - bq)c}{(1 - c)(1 - cq)} x^{l\max(m-1,0)}(x)_{\min(m,0)} P(k - 1, l - 1, m - 1; n; aq, bq; cq, x)
\]

\[
= \frac{(1 - aq)(1 - bq)x(c - abq)x}{(1 - c)(1 - cq)} R(k, l, m, n).
\]

Thus Corollary 2 is proved.

**Acknowledgments** We are deeply grateful to Prof. Hiroyuki Ochiai for helpful comments. Also, we would like to thank Naoya Yamaguchi for his constructive suggestions.

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