A note on the third way consistent deformation of Yang-Mills theory

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Abstract

Three-dimensional Yang-Mills theory allows for a deformation quadratic in the field strengths which can not be integrated to a local action without auxiliary fields. Yet, its covariant divergence consistently vanishes after iterating the equation, realizing a spin-1 analogue of ‘minimal massive gravity’, which has been dubbed ‘third way consistent’. In this note, we show that after dualization of the three-dimensional gauge fields, the model possesses a natural action as a Chern-Simons coupled gauged sigma model. In this dual formulation, coupling to matter and to gravity becomes straightforward. As a direct application, we derive the coupling of the model to $\mathcal{N} = 1$ supergravity.
1 Introduction

In [1] a three-dimensional gauge theory defined by the following field equations
\[ \varepsilon^{\mu\nu\rho} \partial_\mu \tilde{F}_\nu^A + \mu \tilde{F}_\rho^A = -\frac{1}{2m} \varepsilon^{\mu\nu\rho} f_{BC}^A \tilde{F}_\mu^B \tilde{F}_\nu^C, \] (1.1)
with non-abelian Yang-Mills field strength
\[ \tilde{F}_\mu^A = \frac{1}{2} \varepsilon_{\mu\nu\rho} F^{\nu\rho A} = \frac{1}{2} \varepsilon_{\mu\nu\rho} \left( 2 \partial^\nu A^\rho A + f_{BC}^A A^\nu B A^\rho C \right), \] (1.2)
was considered. Here, the \( f_{BC}^A \) are structure constants of a non-abelian gauge group \( G \), and \( m \) and \( \mu \) are constants. This is a deformation of (topologically massive) Yang-Mills theory (TMYM) as the l.h.s. of (1.1) can be derived from its Lagrangian
\[ \mathcal{L}_{\text{TMYM}} = -\frac{1}{4} \text{tr} \left[ F^{\mu\nu} F_{\mu\nu} \right] + \frac{\mu}{4} \varepsilon^{\mu\nu\rho} \text{tr} \left[ F_{\mu\nu} A_\rho \right] . \] (1.3)

However, in presence of the r.h.s., the equations (1.1) can no longer be integrated to a gauge-invariant local action without auxiliary fields. Therefore, consistency of (1.1) is not automatic. Yet, rather surprisingly, the gauge covariant divergence of (1.1) vanishes on-shell upon iterating the equation. This novel way of satisfying the consistency requirement is referred to as ‘third-way consistent’ and reviewed in [4,5]. The mechanism was first discovered in the context of three-dimensional, higher curvature gravity theories [6] and other three-dimensional gravity examples were found later [7,8]. Recently, third-way consistent \( p \)-form theories in arbitrary dimensions have been constructed [9] where a generalization of (1.1) with higher derivative terms is also given. Moreover, in [10] the \( \mathcal{N} = 1 \) off-shell supersymmetric extension of (1.1) has been obtained so that bosonic and fermionic field equations are mapped to each other under supersymmetry. The existence of a supersymmetric version of (1.1) calls for a better understanding of this model. It was observed in [9] that the equations (1.1) are actually dual to a principal chiral sigma model. In this paper we will make this correspondence more concrete and use this fact to construct the coupling of (1.1) to \( \mathcal{N} = 1 \) supergravity. After decoupling gravity, the model reduces to the supersymmetric system of [10].

The key to a dual formulation of (1.1) is the on-shell duality between vector and scalar fields in three dimensions. This is a standard feature for the dynamics of abelian vector fields satisfying the free Maxwell equations, but the duality can be extended to non-abelian gauge groups and is central to the construction of gauged supergravities in three dimensions [11]. In particular, the topologically massive Yang-Mills Lagrangian (1.3) has a dual description as a gauged sigma model on the flat target space \( g = \text{Lie } G \), upon introducing an additional set of vector fields \( B_\mu \) gauging the shift symmetry on the scalar fields [12]. Specifically, the dual Lagrangian is given by
\[ \mathcal{L}_{\text{TMYM}} = -\frac{1}{2} \text{tr} \left[ \partial_\mu \phi \partial^\mu \phi \right] + \varepsilon^{\mu\nu\rho} \text{tr} \left[ B_\mu F_{\nu\rho} \right] + 2 \mu \varepsilon^{\mu\nu\rho} \text{tr} \left[ A_\mu \left( \partial_\nu A_\rho + \frac{1}{3} [A_\nu, A_\rho] \right) \right], \] (1.4)
with scalar fields \( \phi \) in the adjoint representation of \( g \), and their covariant derivatives defined as
\[ \partial_\mu \phi = \partial_\mu \phi + [A_\mu, \phi] + B_\mu . \] (1.5)

\(^1\)Equation (1.1) with some particular choices of these constants appeared earlier in [2,3].
In the dual formulation (1.4), all degrees of freedom are carried by the scalar fields \( \phi \), while the vector fields appear with a Chern-Simons rather than a Yang-Mills coupling. The full gauge algebra of (1.4) is given by \( g \oplus n \) where \( n \) denotes a set of nilpotent generators transforming in the adjoint representation of \( g \).

In this note, we point out that a very similar construction can be given for the dual formulation of (1.1). In this case, the scalar sigma model is built on the curved target space \( G \), with gauging of the full \( g \oplus g \) algebra of isometries. The Lagrangian is given by the gauged sigma model coupled to \( G \hat{G} \) Chern-Simons vector fields, see (2.5) below. The rest of this note is organized as follows. In section 2, we spell out the details of the dual formulation of (1.1). In section 3 we derive the coupling of (1.1) to matter and to gravity directly in the dual formulation. In section 4 finally, we use the dual formulation together with the results of \cite{13} to work out the coupling of (1.1) to \( N = 1 \) supergravity.

## 2 Dual formulation

The dual formulation of (1.1) is a gauged sigma model with Chern-Simons gauge fields. Explicitly, we let the scalar fields of the model parametrize a \( G \)-valued matrix \( U \) with covariant derivatives given by

\[
\mathcal{D}_\mu U = \partial_\mu U + \beta B_\mu U - \alpha U A_\mu,
\]

i.e. the vector fields \( A_\mu \) and \( B_\mu \) gauge the commuting right and left group action on \( U \) with coupling constants \( \alpha \) and \( \beta \), respectively. The left invariant currents

\[
\mathcal{J}_\mu = U^{-1} \mathcal{D}_\mu U \in g,
\]

satisfy the integrability relations

\[
\mathcal{D}[\mathcal{J}_\mu, \mathcal{J}_\nu] + \frac{1}{2} [\mathcal{J}_\mu, \mathcal{J}_\nu] = -\frac{\alpha}{2} F_{\mu\nu} + \frac{\beta}{2} U^{-1} H_{\mu\nu} U,
\]

with the field strengths given by

\[
F_{\mu\nu} = 2 \partial_{[\mu} A_{\nu]} + \alpha [A_\mu, A_\nu],
\]

\[
H_{\mu\nu} = 2 \partial_{[\mu} B_{\nu]} + \beta [B_\mu, B_\nu].
\]

The model is given by coupling the gauged sigma model with target space \( G \) built from (2.2) to a \( G \times G \) Chern-Simons term for the vector fields

\[
\mathcal{L}_\sigma = -\frac{1}{2} \text{tr} [\mathcal{J}_\mu \mathcal{J}^\mu] - g_1 \varepsilon^{\mu\nu\rho} \text{tr} \left[ A_\mu \left( \partial_\nu A_\rho + \frac{\alpha}{3} [A_\nu, A_\rho] \right) \right] - g_2 \varepsilon^{\mu\nu\rho} \text{tr} \left[ B_\mu \left( \partial_\nu B_\rho + \frac{\beta}{3} [B_\nu, B_\rho] \right) \right],
\]

with coupling constants \( g_1 \) and \( g_2 \), respectively. Variation of (2.5) in particular yields the first-order field equations

\[
\mathcal{J}_\mu = \frac{g_1}{\alpha} \varepsilon_{\mu\nu\rho} F^{\nu\rho} = -\frac{g_2}{\beta} \varepsilon_{\mu\nu\rho} U^{-1} H^{\nu\rho} U.
\]
Plugging these relations back into the integrability relations \((2.3)\) yields
\[
\varepsilon^{\mu\nu\rho} \mathcal{G}_{[\mu} \tilde{F}_{\nu]} + \frac{g_1}{\alpha} \varepsilon^{\mu\nu\rho} \left[ \tilde{F}_{\mu} \tilde{F}_{\nu} \right] = -\frac{1}{2} \left( \frac{\alpha^2}{g_1} + \frac{\beta^2}{g_2} \right) \tilde{F}^\rho.
\] (2.7)
Setting \(\alpha = \beta = 1\), these equations precisely reproduce the deformed Yang-Mills equations \((1.1)\) with the translation of parameters
\[
m = \frac{1}{2g_1}, \quad \mu - m = \frac{1}{2g_2}.
\] (2.8)

Since both left and right group multiplication of \(U\) are local symmetries, we may fix part of the gauge symmetry by setting
\[
U = I, \tag{2.9}
\]
which breaks the gauge group down to the diagonal \(G_{\text{diag}} \subset G \times G\). With the translation of constants \((2.8)\), the action \((2.5)\) then reduces to
\[
\mathcal{L}_0 = -\frac{1}{2} \text{tr} \left[ (B_\mu - A_\mu)(B^\mu - A^\mu) \right] - \frac{1}{2m} \varepsilon^{\mu
u\rho} \text{tr} \left[ A_\mu \left( \partial_\nu A_\rho + \frac{1}{3} [A_\nu, A_\rho] \right) \right]
\]
\[
- \frac{1}{2(\mu - m)} \varepsilon^{\mu
u\rho} \text{tr} \left[ B_\mu \left( \partial_\nu B_\rho + \frac{1}{3} [B_\nu, B_\rho] \right) \right], \tag{2.10}
\]
which (up to a global factor) precisely reproduces the massive Chern-Simons action found in \([1]\). We note in passing that while the Lagrangian \((2.10)\) appears degenerate at the special value \(m = \mu\) \([1]\), in this case an action for the system is straightforwardly obtained from the Lagrangian \((2.5)\) upon setting \(\beta = 0 = g_2\), such that the vector fields \(B_\mu\) disappear from the action. Its gauge fixed version then reduces to the massive Chern-Simons Lagrangian
\[
\mathcal{L}_{0, \mu = m} = -\frac{1}{2} \text{tr} \left[ A_\mu A_\mu \right] - \frac{1}{2m} \varepsilon^{\mu
u\rho} \text{tr} \left[ A_\mu \left( \partial_\nu A_\rho + \frac{1}{3} [A_\nu, A_\rho] \right) \right]. \tag{2.11}
\]

Note that the dual formulation \((2.5)\) as well as its gauge fixed version \((2.10)\) make manifest the duality symmetry of the system under simultaneous exchange of the vector fields \(A_\mu\) and \(B_\mu\) together with the parameters
\[
g_1 \leftrightarrow g_2 \iff m \leftrightarrow \mu - m. \tag{2.12}
\]
Since only the vector fields \(A_\mu\) appear in the original deformed Yang-Mills equations \((1.1)\), this duality can be employed in order to map solutions to \((1.1)\) with given values of \(m\) and \(\mu\) into new solutions to the same equations with different values of the parameters related by \((2.12)\).

Let us also note, that the equations \((1.1)\) admit a smooth limit to a deformation of the Maxwell equations (i.e. to abelian field strengths \(F_\mu A_\nu = 2 \varepsilon_{\mu\nu\rho} A_\rho\)) while keeping the structure constants \(f_{BC A}\) on the r.h.s. of \((1.1)\), parametrizing the non-linear deformation of the topologically massive Maxwell equations as was considered in \([9]\). This limit is described by rescaling vector fields as
\[
A_\mu A_\nu \rightarrow m A_\mu A_\nu, \tag{2.13}
\]
dividing a resulting overall factor \(m\) from \((1.1)\), and finally sending \(m \rightarrow 0\). It is however not clear how this limit can be implemented directly on the level of the action \((2.5)\).

Finally, the limit of \((2.5)\) to the dual formulation of the undeformed Yang-Mills theory \((1.4)\) requires flattening of the sigma model target space, such that in particular its isometry group contracts from \(G \times G\) to \(G \times N\).
3 Coupling to matter and gravity

The coupling of the deformed Yang-Mills system (1.1) to gravity and lower-spin matter has been determined in [1] by a careful analysis of the consistency conditions of the coupled system. Within the dual formulation (2.5), this coupling is straightforward. Simply adding a matter Lagrangian \( \mathcal{L}_{\text{mat}} \) to (2.5)

\[
\mathcal{L} = \mathcal{L}_\sigma + \mathcal{L}_{\text{mat}},
\]

the first-order field equations (2.6) change into

\[
\mathcal{J}_\mu = \frac{g_1}{\alpha} \varepsilon_{\mu\nu\rho} F^{\nu\rho} + j_\mu, \quad j_\mu = \frac{1}{\alpha} \frac{\partial \mathcal{L}_{\text{mat}}}{\partial A^\mu}.
\]

For simplicity, we assume that the matter is only charged under the right factor of the gauge group \( G \times G \), i.e. that \( \mathcal{L}_{\text{mat}} \) does not depend on \( B_\mu \). After plugging these equations into the integrability relations (2.3), the equations (2.7) are modified according to

\[
\varepsilon^{\mu\nu\rho} \mathcal{D}_\rho \tilde{F}_\mu + \frac{g_1}{\alpha} \varepsilon^{\mu\nu\rho} \left[ \tilde{F}_\mu, \tilde{F}_\nu \right] + \frac{1}{2} \left( \frac{\alpha^2}{g_1} + \frac{\beta^2}{g_2} \right) \tilde{F}^\rho = -\frac{\alpha}{2g_1} \varepsilon^{\mu\nu\rho} \mathcal{D}_\nu j_\rho - \frac{\alpha}{4g_1} \varepsilon^{\mu\nu\rho} [j_\mu, j_\nu]
\]

\[
- \frac{\alpha \beta^2}{4g_1g_2} j^\rho - \varepsilon^{\mu\nu\rho} \left[ \tilde{F}_\mu, j_\nu \right],
\]

where the r.h.s. directly reproduces (after setting \( \alpha = \beta = 1 \), rescaling \( j_\mu \rightarrow -4g_1g_2 j_\mu \), and translating parameters by (2.8)) the matter coupling constructed in [1] but with the opposite sign of the second term on the r.h.s. which agrees with [9].

Similarly, coupling of the system (2.5), (3.1) to three-dimensional gravity is straightforward. The full energy-momentum tensor \( T_{\mu\nu} \) is given by

\[
T_{\mu\nu} = -\frac{1}{2m^2} \text{tr} \left[ \tilde{F}_\mu \tilde{F}_\nu \right] + \frac{1}{4m^2} g_{\mu\nu} \text{tr} \left[ \tilde{F}_\rho \tilde{F}^\rho \right] - \frac{1}{m} \text{tr} \left[ \tilde{F}_\mu j_\nu \right] + \frac{1}{2m} g_{\mu\nu} \text{tr} \left[ j_\rho \tilde{F}^\rho \right]
\]

\[
- \frac{1}{2} \text{tr} \left[ j_\mu j_\nu \right] + \frac{1}{4} g_{\mu\nu} \text{tr} \left[ j_\rho j^\rho \right] + \sqrt{|g|} \left( 1 - \frac{1}{2} \frac{\partial \mathcal{L}_{\text{mat}}}{\partial g^{\mu\nu}} \right),
\]

with the first six terms derived from \( \sqrt{|g|} \left( 1 - \frac{1}{2} \frac{\partial \mathcal{L}_{\text{mat}}}{\partial g^{\mu\nu}} \right) \) after using (3.2). In absence of \( \mathcal{L}_{\text{mat}} \), i.e. for \( j_\mu = 0 \), this precisely reproduces the energy-momentum tensor given in [1]. The full expression (3.4) directly extends this result to the presence of additional matter.

4 Coupling to \( \mathcal{N} = 1 \) supergravity

The \( \mathcal{N} = 1 \) supersymmetrization of the system (1.1) was constructed in [10] on the level of the field equations. In terms of the dual formulation (2.5), we can trace this back to the known structures of a supersymmetric sigma model with target space \( G \) [14]. As a new application, we will employ the dual formulation (2.5) in order to work out the coupling of the system (1.1) to \( \mathcal{N} = 1 \) supergravity. The coupling of a general gauged sigma-model to three-dimensional supergravity has been constructed in [13]. In particular, any Riemannian target space allows for a coupling to \( \mathcal{N} = 1 \) supergravity. The fermionic field content in this case is given by \( n(= \text{dim} G) \) spin-1/2 fermions \( \chi^i \) together with a gravitino \( \psi_\mu \).
To translate to the notation from [13], we parametrize the group manifold G by coordinates \( \phi^i \) and define the left and right invariant vector fields \( L_A, R_A \) satisfying the differential relations
\[
\nabla_i L_{A_j} = -\frac{1}{2} f^{ABC}_A L_{B_i} L_{C_j}, \quad \nabla_i R_{A_j} = \frac{1}{2} f^{ABC}_A R_{B_i} R_{C_j}.
\]
The metric on the group manifold can be expressed as
\[
g_{ij} = L_{A_i} L^A_i = R_{A_i} R^A_i.
\]
The currents of the gauged sigma model (2.2) are then given by
\[
J^A_i = L^A_i \partial_i \phi^i = L^A_i \partial_i \phi^i - \alpha A^A_\mu + \beta B^A_\mu (L^A_i R^B_i) .
\]
On the fermionic side, we define the fermions \( \chi^A = L^A_i \chi^i \) transforming under the right action of G, with covariant derivatives
\[
\partial_\mu \chi^A = (\partial_\mu + \frac{1}{4} \omega_\mu ^a \gamma^a ) \chi^A + \alpha f^{BC}_A L A^B \chi^C.
\]
In terms of these objects, we can obtain the coupling of the gauged sigma model (2.5) to \( \mathcal{N} = 1 \) supergravity from the general result of [13] (after a suitable rescaling of the vector fields) as
\[
\sqrt{-g} \mathcal{L} = -\frac{1}{4} \epsilon^{\mu \nu \rho \sigma} \left( \epsilon^a_\mu R_{\nu a} + \bar{\psi}_\nu \partial_\nu \psi_\mu \right) - 4 C^2 - g_1 \epsilon^{\mu \nu \rho \sigma} \left( \partial_\mu A^A_\nu + \frac{\beta}{2} f^{BC}_A A^B_\nu A^C_\rho \right)
- g_2 \epsilon^{\mu \nu \rho \sigma} B^A_\mu \left( \partial_\nu B^A_\rho + \frac{\alpha}{2} f^{BC}_A B^B_\nu B^C_\rho \right) - \frac{1}{2} J^A_\mu J^A_\mu - \frac{1}{2} \chi^A \partial_\mu \chi^A
+ \frac{1}{4} \epsilon^{\mu \nu \rho \sigma} B^A_\mu \left( \partial_\nu B^A_\rho + \frac{\alpha}{2} f^{BC}_A B^B_\nu B^C_\rho \right) \partial_\mu \chi^A
- \frac{1}{4} \left( \frac{a_1}{4} + \frac{a_2}{4} - 2C \right) \bar{\chi}^A \chi^A + \frac{1}{16} \left( \bar{\chi}^A \chi^A \right)^2 - \frac{1}{8} \left( \bar{\chi}^A \gamma^\mu \gamma^\nu \chi^A \right) \left( \bar{\gamma}^\mu \gamma^\nu \chi^A \right)
+ \frac{1}{32} f^{AB}_C f^{CDE} \bar{\chi}^A \gamma^B \bar{\chi}^C \gamma^D \chi^A ,
\]
with a cosmological constant \( C \). The supersymmetry variations are given by
\[
\delta e^a_\mu = \frac{1}{2} \epsilon^a_\mu \psi_\mu , \quad \delta \phi^i = \frac{1}{2} \epsilon^i \chi^i ,
\]
\[
\delta A^A_\mu = \frac{\alpha}{4 g_1} \bar{\chi}^A \gamma^\mu \epsilon , \quad \delta B^A_\mu = - \frac{\beta}{4 g_2} \left( R^A_i L^B_i \right) \bar{\chi}^B \gamma^\mu \epsilon ,
\]
\[
\delta \psi_\mu = (\partial_\mu + \frac{1}{4} \omega_\mu ^a \gamma^a ) \epsilon - \frac{1}{8} \bar{\chi}^A \gamma^\mu \gamma^\nu \chi^A \gamma^\mu \epsilon + C \gamma^\mu \epsilon ,
\]
\[
\delta \chi^A = \frac{1}{2} J^A_\mu \gamma^\mu \epsilon - \frac{1}{4} \left( \bar{\psi}_\mu \chi^A \right) \gamma^\mu \epsilon - \frac{1}{4} f^{AB}_C \left( \epsilon \bar{\chi}^B \right) \chi^C .
\]
and leave the Lagrangian (4.5) invariant. The supersymmetric extension of the Yang-Mills system (1.1) is then obtained in full analogy to (2.7), and (3.3). Variation of (4.5) w.r.t. the vector fields, yields the first-order duality equations
\[
J^A_\mu = g_1 \epsilon^{\mu \nu \rho \sigma} F^A_{\nu \rho} \gamma^\mu \epsilon + \frac{1}{2} \bar{\chi}^A \gamma^\nu \gamma^\mu \psi_\nu - \frac{1}{4} f^{A BC} \bar{\chi}^B \gamma^\mu \chi^C
= - \frac{g_2}{\beta} \epsilon^{\mu \nu \rho \sigma} H^A_{\nu \rho} \gamma^\mu \epsilon + \frac{1}{2} \bar{\chi}^A \gamma^\nu \gamma^\mu \psi_\nu + \frac{1}{4} f^{A BC} \bar{\chi}^B \gamma^\mu \chi^C .
\]
Upon plugging these equations into the integrability relations (2.3) of the scalar currents, and translating the coupling constants via (2.8) together with \( \alpha = \beta = 1 \), we finally obtain the desired supersymmetric extension of (1.1)
\[
\epsilon^{\mu \nu \rho} \partial_\mu \tilde{F}^A_{\nu \rho} + \mu \tilde{F}^A_{\nu \rho} = - \frac{1}{2 \mu} \epsilon^{\mu \nu \rho} f^{BC}_A \tilde{F}^B_{\nu \rho} \tilde{F}^C_{\nu \rho} - \epsilon^{\mu \nu \rho} f^{BC}_A \tilde{F}^B_{\nu \rho} \tilde{F}^{(+) A}_\nu - m \epsilon^{\mu \nu \rho} \partial_\mu \tilde{F}^{(+) A}_\nu
+ m (m - \mu) \left( \tilde{F}^{(+) A}_\nu - \tilde{F}^{(-) A}_\nu \right) - \frac{m}{2} \epsilon^{\mu \nu \rho} f^{BC}_A \tilde{F}^{(+) B}_\mu \tilde{F}^{(+) C}_\rho ,
\]
with matter currents bilinear in the fermions

\[ j^{(\pm)}_{\mu}^{A} = \frac{1}{2} \bar{\chi}^{A} \gamma^{\mu} \gamma_{\mu} \psi + \frac{1}{4} f^{BC} A \bar{\chi}^{C} \gamma_{\mu} \chi^{C}. \quad (4.9) \]

The appearance of a second current \( j^{(-)}_{\mu} \) as opposed to the general form of (3.3) is due to the appearance of the vectors \( B_{\mu}^{A} \) via the fermion couplings to \( J_{\mu}^{A} \) in the matter couplings of (4.5).

In order to complete the supersymmetric system, it remains to compute the remaining equations of motion from (4.5) and replace all the appearing scalar currents \( J_{\mu}^{A} \) and field strengths \( H_{\mu \nu}^{A} \) by means of the duality equations (4.7). For the gravitino and the spin 1/2 field equations, this yields after some straightforward computation (and notably some Fierzing) the deformed Rarita-Schwinger equation

\[ \epsilon^{\mu \rho} \partial_{\nu} \psi_{\rho} = -\frac{1}{2m} \gamma^{\nu} \gamma^{A} \bar{F}_{\nu}^{A} + C \gamma^{\mu} \psi_{\nu} - \frac{1}{4} f_{BC} A \chi^{A} (\bar{\chi} B \gamma^{\mu} \chi^{C}) - \frac{1}{4} (\bar{\chi} A \chi^{C}) \psi_{\mu}, \quad (4.10) \]

as well as the deformed Dirac equation

\[ \gamma^{\mu} \partial_{\nu} \chi^{A} = \frac{1}{2m} \gamma^{\mu} \gamma^{B} \bar{F}_{\nu}^{B} C + \frac{1}{2m} \gamma^{\mu} \gamma^{\nu} \psi_{\mu} \bar{F}_{\nu}^{A} + (C - \mu) \chi^{A} + \frac{1}{4} f_{BC} A (\bar{\chi} B \gamma^{\nu} \chi^{C}) \]

\[- \frac{1}{4} \chi^{A} (\bar{\psi} \psi_{\mu}) - \frac{1}{2} \gamma^{\mu} \chi^{B} (\bar{\chi} D \gamma^{\nu} \chi^{C}) f_{BC} A f_{DE} C + \frac{1}{4} \chi^{A} (\bar{\chi} B \chi^{B}). \quad (4.11) \]

Finally, the Einstein field equations are of the standard form with the energy-momentum tensor given by (3.4) and again all scalar currents eliminated by (4.7). By construction, the system of (4.8), (4.10), (4.11), and the Einstein equations yields a supersymmetric system, with the equations transforming into each other under the transformations (4.6) of the remaining fields

\[ \delta e_{\mu}^{\alpha} = \frac{1}{2} \bar{\epsilon} \gamma^{\alpha} \psi_{\mu}, \]

\[ \delta A_{\mu}^{A} = \frac{m}{2} \bar{\chi}^{A} \gamma_{\mu} \epsilon, \]

\[ \delta \psi_{\mu} = (\partial_{\mu} + \frac{1}{2} \omega_{\mu}^{a} \gamma_{a}) \epsilon - \frac{1}{8} \bar{\chi}^{A} \gamma^{\nu} \chi^{A} \gamma_{\mu \nu} \epsilon + C \gamma_{\mu} \epsilon, \]

\[ \delta \chi^{A} = \frac{1}{2m} \bar{F}_{\mu}^{A} B \gamma^{\mu} \epsilon + \frac{1}{4} (\bar{\chi}^{A} \gamma^{\nu} \psi_{\mu}) \gamma_{\nu} \epsilon. \quad (4.12) \]

A direct construction of this system starting from the supersymmetrization of (4.8) would have represented a formidable technical challenge with an uncertain outcome. In the dual picture, the entire system is straightforwardly obtained by variation of the supersymmetric action (4.5).

As a consistency check, we may study the rigid limit of the model. In order to decouple gravity, we first set \( \psi_{\mu} = C = 0 \), and then scale fields and structure constants with a constant \( k \) as \( \chi^{A} \rightarrow k \chi^{A} \), \( A_{\mu}^{A} \rightarrow k A_{\mu}^{A} \), \( f_{BC} A \rightarrow f_{BC} A/k \). After cancelling the overall \( k \) and taking the limit \( k \rightarrow 0 \) together with a few redefinitions\(^2\) one arrives precisely at the field equations of \( \mathcal{N} = 1 \) off-shell supersymmetric massive Yang-Mills theory constructed in [10]. This limit can also be taken at the action (4.5) by setting \( g_{\mu \nu} = \eta_{\mu \nu} \) and making additional scalings \( \phi^{i} \rightarrow k \phi^{i} \), \( B_{\mu}^{A} \rightarrow kB_{\mu}^{A} \).

Let us finally note that the field equations (4.8), (4.10), (4.11), as well as the supersymmetry transformations (4.12) are compatible with the limit \( m \rightarrow 0 \) after rescaling the vector fields as (2.13). This is the limit to abelian field strengths \( F_{\mu \nu}^{A} = 2 \partial_{[\mu} A_{\nu]}^{A} \) while keeping all the remaining structure constants \( f_{BC} A \) in the various couplings [9].

\(^2\)Explicitly, \( m \rightarrow m/4 \), \( \mu \rightarrow 2 \mu \), \( j^{(\pm)}_{\mu}^{A} \rightarrow -2 j^{(\pm)}_{\mu}^{A}/m \).
5 Conclusions

In this note, we have shown that the dual formulation (2.5) of the deformed Yang-Mills equations (1.1) provides a natural explanation for many features of the model and in particular allows for a straightforward coupling of the model to $\mathcal{N} = 1$ supergravity. The dual formulation also shows that the construction cannot be extended to higher supersymmetry $\mathcal{N} > 1$, since for a sigma model higher supersymmetry requires further constraints on the scalar target space [15] which are not satisfied by the real compact Lie groups $G$ that appear here. In particular, an extension to $\mathcal{N} = 2$ would require $G$ to be a Kähler manifold, thus the gauge group to be complex [16].

Among further applications, it would be interesting to work out the analogous dual models to the higher dimensional third way consistent $p$-form theories of [9]. Higher derivative extensions of (1.1) were constructed in [9] and finding supersymmetric versions of these would also be desirable.

Another interesting extension of the present construction is its analogue in the context of third way consistent gravitational theories, such as minimal massive gravity [6], and extensions thereof [7, 8]. We will come back to this in [17].

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