INVARIA NT S OF COMMUTING MATRICES

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1. An open question

1.1. Introduction. Consider the space of pairs \((X, Y)\) of \(n \times n\) matrices over a field \(F\) of characteristic 0, and denote by \(R_n(X, Y)\) the polynomial algebra on this space.

An intriguing, difficult open question in commutative algebra asks whether the ideal \(I_n\), generated by the entries of the equation \([X, Y] = XY - YX = 0\) is a prime ideal in \(R_n(X, Y)\). Since it is known that the variety of pairs of commuting matrices is irreducible (cf. [4]) this means that the ideal of this variety is generated by these natural quadratic equations.

Several attempts to this have been made but, as far as I understand this issue, no conclusive proof has been reached except for small \(n \leq 4\) (Wallach private communication). In [3] the authors study a related problem, that is they study pairs of matrices with commutator of rank 1.

From this they deduce that, for the case of 2 commuting matrices, the ring of invariants of the natural action by conjugation on \(R_n(X, Y)/I_n\) is an integral domain. The purpose of this paper is to comment two papers, one by Domokos [2] and one of Francesco Vaccarino [11] where they prove the same statement but for the algebra of \(m\)-commuting matrices, for all \(m\). This is in fact a consequence of a stronger theorem[1] which gives an isomorphism of this ring of invariants with the invariants of diagonal matrices under the symmetric group. The method of proof in the two papers is quite different. The two papers are clearly independent and although published with some time gap I have been informed that the two preprints appeared on the archive more or less at the same time.

The approach of Vaccarino is through polynomial maps and the one of Domokos via classical invariant theory.

I want to point out some simply related approach which may apply to classical Lie algebras, since the same question can be asked for all simple Lie algebras (cf. [6], [10] and [12]) but I have made no attempt in this direction. It also gives some supplementary information on these algebras.

1.2. Commuting matrices. We work over a field \(F\) of characteristic 0, for instance \(\mathbb{Q}\). Let us fix a positive integer \(n\), for all \(m\) in fact for \(m = \infty\) let us consider the ring of polynomials in the entries \(x_{h,k}^i, h, k = 1, \ldots, n; i = 1, \ldots, m\) of \(m \times n\) matrix variables \(X_i := (x_{h,k}^i)\), modulo the ideal generated by the entries of the commuting equations \([X_i, X_j] = 0\). Denote by \(A_{n,m}\) this ring and \(A_n = A_{n,\infty}\). On this ring acts the linear group \(GL(m, F)\) by linear transformations on the variables \(X_i\) and the group \(GL(n, F)\) by conjugation; the two actions commute.

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One may consider in the ring $M_n(A_{n,m})$ the $m$ generic commuting matrices $\xi_i$ images of the generic matrices $X_i$.

Consider now the ring $B_{n,m}$ ($B_m = B_{n,\infty}$) of polynomials in $m$ vector variables $x_i = (x_{i1}, \ldots, x_{in})$ which we also consider as $m$ diagonal matrices. On this ring acts the linear group $GL(m,F)$ by linear transformations on the variables $x_i$ and the symmetric group $S_n$ by conjugation; the two actions commute. We can also identify $B_{n,m} = B_n^{\otimes m}$ where $B_n = F[x_1, \ldots, x_n]$ is the polynomial ring in $n$ variables.

We have a natural restriction map $\pi : A_{n,m} \to B_{n,m}$ compatible under the action of $GL(m,F)$ and $S_n$, which induces thus a natural map of invariants

$$\tilde{\pi} : A_{n,m}^{GL(n,F)} \to B_{n,m}^{S_n} = (F[x_1, \ldots, x_m]^{\otimes n})^{S_n}.$$  

**Theorem 1.** [Domokos, Vaccarino] The map $\tilde{\pi}$ is an isomorphism.

**Proof.** The proof from [2] is similar to the proof I want to present, the one in [11] is based on the Theory of polynomial maps.

One first considers the isomorphism $F[x_1, \ldots, x_m] \to F[\xi_1, \ldots, \xi_m]$ to the ring of generic commuting matrices, when we compose this map with the determinant we have a polynomial map, homogeneous of degree $n$:

$$(2) \quad D : F[x_1, \ldots, x_m] \to F[\xi_1, \ldots, \xi_m] \xrightarrow{\det} A_{n,m}^{GL(n,F)}$$

This map is multiplicative in the sense that $D(ab) = D(a)D(b)$, hence by a general theorem of Roby, [3] it factors through a homomorphism of the $n^{th}$ symmetric tensors $\hat{D} : (F[x_1, \ldots, x_m]^{\otimes n})^{S_n} \to A_{n,m}^{GL(n,F)}$. So the point is just to prove that this map is inverse to the restriction. The composition $\pi \circ \hat{D}$ is clearly identity, so $\hat{D}$ is injective and $\tilde{\pi}$ surjective the issue is in the other order $\hat{D} \circ \pi$. In [11] this is proved by showing a correspondence between generators of the two algebras. For matrix invariants it is known (even before imposing the commutative law), that invariants are generated by the coefficients of the characteristic polynomials of primitive monomials in the variables $X_i$, see [1], a similar statement holds for the symmetric group and one verifies the correspondence.

Here I want to point out a slightly different approach, which consists in applying the classical Arhonold method, of reducing to multilinear elements. As I will point out at the end this also gives some normal form for the invariants. Remark that $\tilde{\pi}$ is $GL(m,F)$ equivariant for all $m$, thus it is enough to prove the statement for $m = \infty$ and, by the classical method of Arhonold, it is enough to prove that $\tilde{\pi}$ is an isomorphism when restricted to the multilinear elements.

We thus have to gather some information on these elements.

Let us start with the multilinear elements of $B_{n,m}^{S_n}$. The multilinear elements of $B_{n,m}$ have as basis the monomials

$$x_{1,i_1} \cdots x_{m,i_m}; \quad 1 \leq i_j \leq n, \forall j.$$  

Thus these monomials can be thought of as maps from $[m] := \{1, 2, \ldots, m\}$ to $[n]$.

These monomials are permuted by the symmetric group so that the space of multilinear elements of $B_{n,m}^{S_n}$ has as basis the orbits of $S_n$ on such a space of functions.

To any function $f : [m] \to [n]$, is associated the partition of $[m]$ into its fibers which is a partition of $[m]$ into at most $n$ subsets. This partition is independent of the orbit. Conversely given a partition $\Lambda$ of $[m]$ into at most $n$ subsets, we have a natural ordering of these sets by associating to each subset its minimal element and then ordering them according to the minimal element as $S_1, S_2, \ldots, S_k$. We then associate to this the function $f_{\Lambda} : [m] \to [n]$ which takes the value $i$ on $S_i$. In its orbit under $S_n$ this function is leading in
a suitable lexicographic order. Hence it is easily seen that in this way we have established a 1–1 correspondence between maps $f : [m] \to [n]$ up to symmetry and decompositions $\Lambda$ of $[m]$ into at most $n$ subsets.

Let us now turn our attention to $A_{n,m}^{GL(n,F)}$. By classical invariant theory multilinear invariants of $m$ matrix variables are spanned by the functions $\phi_\sigma$ where $\sigma \in S_m$ is a permutation and

$$\phi_\sigma(x_1, x_2, \ldots, x_m) = tr(x_{i_1}x_{i_2}\ldots x_{i_h}) tr(x_{j_1}x_{j_2}\ldots x_{j_k}) \ldots tr(x_{s_1}x_{s_2}\ldots x_{s_m})$$

there the various factors correspond to the decomposition into cycles of $\sigma$ (see for instance [5]).

When we are working with $n \times n$ matrices we have a basic formal identity (equivalent to the Cayley–Hamilton identity)

$$\sum_{\sigma \in S_{n+1}} \epsilon_\sigma \phi_\sigma(x_1, x_2, \ldots, x_{n+1}) = 0.$$ We deduce from this that

$$\phi_1(x_1, x_2, \ldots, x_{n+1}) = \prod_{i=1}^{n+1} tr(x_i) = - \sum_{\sigma \in S_{n+1}, \sigma \neq 1} \epsilon_\sigma \phi_\sigma(x_1, x_2, \ldots, x_{n+1})$$

and then each summand of the R.H.S. of this equality is the product of at most $n$ factors of the form $tr(M)$ for $M$ some monomial.

Therefore by a repeated use of this identity we deduce that every invariant for $n \times n$ matrices is a linear combination of products with at most $n$ factors of the form $tr(M)$ for $M$ some monomial.

We apply this in particular to the multilinear invariants for commuting matrices, in this case the invariant $t_\Lambda := \prod_{i \in S} t_i$ we have that these multilinear elements span linearly all multilinear invariants. Thus we can prove that they form a basis and that $\tilde{\pi}$ is an isomorphism as soon as we show that $\tilde{\pi}$ is surjective (proved already in [12]).

For this order lexicographically the monomials $x_{1,i_1} \ldots x_{m,i_n}$ which can be just thought of as words of length $m$ in the letters $1,2,\ldots,n$. Observe that the leading term of $t_\Lambda$ is the monomial associated to $f_\Lambda$ and the claim follows. □

Let us draw some consequences, let $J$ be the nil ideal of $A_n$ clearly it is stable under both groups and we have

**Corollary 1.3.** $J$ does not contain any element which is invariant under $GL(n,F)$. We should notice that if one is interested in describing these algebra, in detail, one method is to describe them as representation of $GL(m)$ ($m$ is the number of copies) and this is done by describing each homogeneous component of degree $k$ as a direct sum of Schur functors $S_\lambda(F^k)$ where $\lambda \vdash k$ is a partition of $k$. 

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The description of homogeneous component of degree $k$ passes through the description of the multilinear part of degree $k$ for the case of $k$ copies as representation of the symmetric group implies the description of the homogeneous part of the same degree as representation of the linear group $GL(k)$.

In fact by the Theory of Schur functors, a partition $\lambda \vdash k$ defines a Schur functor $S_\lambda(W)$ on vector spaces homogeneous of degree $k$. When we apply this to an $k$-dimensional space $W = F^k$ with a prescribed basis we may consider inside the space $M_\lambda(F^k)$ of multilinear elements, which by definition are the invariants for the torus of diagonal matrices in that basis with determinant 1. It then is true that $M_\lambda(F^k)$ is the irreducible representation of the symmetric group $S_k$, permuting that basis, associated to the same partition $\lambda$.

Of course $S_\lambda(W) = 0$ if the dimension of $W$ is strictly smaller than the height of $\lambda$ so when $m < k$ not all the representations appearing contribute. In our case it would remain to describe the decomposition into irreducible representations of the permutation representations, which we have seen describe the multilinear invariants, of decompositions $\lambda$ of $[m]$ into at most $n$ subsets. The orbits of this permutation action are parametrized by partitions of $m$ of height $\leq n$, to each such partition corresponds the permutation representations given on cosets of the corresponding Young subgroup. These are well understood, there is an extensive literature, see for instance [9].

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