Duality and Higher Temperature Phases of Large $N$
Chern-Simons Matter Theories on $S^2 \times S^1$

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ABSTRACT: It has been recently demonstrated that the thermal partition function of any large $N$ Chern-Simons gauge theories on $S^2 \times S^1$, coupled to fundamental matter, reduces to a capped unitary matrix model. The matrix models corresponding to several specific matter Chern-Simons theories at temperature $T$ were determined in \cite{1}. The large $N$ saddle point equations for these theories were determined in the same paper, and were solved in the low temperature phase. In this paper we find exact solutions for these saddle point equations in three other phases of these theories and thereby explicitly determine the free energy of the corresponding theories at all values of $T^2/N$. As anticipated on general grounds in \cite{1}, our results are in perfect agreement with conjectured level rank type bosonization dualities between pairs of such theories.
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1. Introduction

Recently, three dimensional field theories have become fascinating in the context of AdS/CFT (or dS/CFT) correspondences which do not necessarily rely on supersymmetry \[2\]. It has been recently conjectured that level \( k \) Chern-Simons theories with matter in the fundamental/bifundamental representations admit a dual description governed by parity violating Vasiliev’s higher spin equations \[3, 4\] (see \[5\] for more detail and references) at finite values of the ’t Hooft coupling \( \lambda = \frac{N}{k} \) \[6, 7\].

Sometimes, analysis of boundary field theories can be applied to study bulk gravity theories through the AdS/CFT correspondence when the direct analysis of the gravity theory is difficult. For example, it is known that the AdS/CFT correspondence maps deconfinement transitions of large \( N \) gauge theories on spheres to gravitational phase transitions involving black hole nucleation \[8\]. This observation has motivated the intensive study of the phase structures as well as the deconfinement phase transitions of large \( N \) \( p+1 \) dimensional Yang Mills theories (coupled to adjoint and fundamental matter) on \( S^p \). Now, we can expect similar application of the Chern-Simons matter theories to know the phase structures of parity violating Vasiliev’s higher spin gravity theories.

From this motivation, in \[1\], the finite temperature phase structure of renormalized level \( k \) \( U(N) \) Chern-Simons (CS) theories coupled to a finite number of fundamental fields on \( S^2 \times S^1 \) in the ’t Hooft limit \( N \to \infty, k \to \infty \) with \( \lambda = \frac{N}{k} \) fixed were studied. The phase structure was studied by using a simple toy large \( N \) Gross-Witten-Wadia (GWW) matrix integral, and it turned out that there is a rich phase structure caused by the summation over the \( U(1) \) flux sectors in the Chern-Simons theory. In particular, the eigenvalues must be discretized and the eigenvalue density function must be saturated from upper bound \( \frac{1}{2\pi \lambda} \). This saturation was first suggested by \[9\]. This discretization generates the new phases so-called the “upper gap phase” and the “two gap phase”, which are absent in Yang-Mills theories. ¹ The level-rank duality has also been confirmed in the GWW toy model in \[1\] (see \[14, 15\] for a relatively recent discussion of level-rank duality and references to earlier work), but the phase structures of the actual Chern-Simons matter theories have not been worked out. Moreover, we have not had the perfect analytic proof of the duality, even in the GWW model.

In this paper we will work out the study of the phase structure of several Chern-Simons matter theories: (1) the CS theory minimally coupled to the fundamental fermion \[1\], (2) the CS theory coupled to massless critical bosons \[16, 17\], and (3) the \( \mathcal{N} = 2 \) supersymmetric CS theory with a single fundamental chiral multiplet \[18, 19\] (SUSY CS matter theory). We will also provide the perfect analytic proof of the level-rank duality (Giveon-Kutasov type duality) on the \( S^2 \times S^1 \) between the regular fermion theory and the critical boson theory. Moreover we will prove the level-rank self-duality in both the SUSY CS matter theory and the GWW matrix integration as well. We will also supply several

¹The same upper bound for the density function appears in two dimensional Yang-Mills theory \( (p = 1) \), in which case the situation becomes similar due to the fact that there are no propagating degrees of freedom for the gauge field. A phase transition relevant to this upper bound was studied in 2d (q-deformed) Yang-Mills theory on \( S^2 \) \[10, 11, 12, 13\], which we will see from Chern-Simons theory.
useful formulae which can be used for any analysis related to the duality in general CS matter theories on $S^2 \times S^1$.

1.1 Outline of this paper

The rest of this paper is organized as follows: In section 2 we will review the Landau-Ginzburg description of the partition function of Chern-Simons theories coupled to fundamental matters as a preliminary, and supply techniques to evaluate eigenvalue densities as well as partition functions by using complex line integration in the complex plane. In section 3, we will elaborate the phase structure of the regular fermion theory and the critical boson theory, confirming that the level-rank duality between them is in agreement. In section 4, we will study the phase structure of the $\mathcal{N} = 2$ supersymmetric CS matter theory and examine the Giveon-Kutasov type duality. We will confirm the self-duality of the theory. In section 5, we will give the analytic proof of the level-rank self-duality in the two gap phase of the GWW toy model. Section 6 is devoted to the summary and discussion.

The content of the appendix is as follows: We put several definitions of the cut regions and cut functions in appendix A. We give the proof of important formulae used in the proof of the duality and others in appendix B. In appendix C, we provide the detailed analysis of the large $\zeta$ behavior of the eigenvalue density functions. In appendix D, we also prove that the eigenvalue distributions at $\zeta \to \infty$ converge to the universal distribution (2.16).

2. Preliminaries

2.1 High temperature effective action of the Chern-Simons-fundamental matter theory on $S^2 \times S^1$

We will consider the level $k$ $U(N)$ fundamental matter Chern-Simons theories in the ’t Hooft large $N$ limit, with $^2$

$$V_2T^2 = N\zeta, \quad \lambda = \frac{N}{k}$$

(2.1)

where $\zeta$ and $\lambda$ are held fixed in the large $N$ limit. These theories have recently been studied intensively at finite $\lambda$ [4, 18, 22, 23, 24, 6, 17, 23, 24, 25, 9], and thermal partition functions of these theories are studied in [29, 1, 5].

We will consider the partition function of these theories

$$Z_{\text{CS}} = \int DAD\mu \ e^{i\frac{k}{4\pi} \text{Tr} (\text{Ad}A + \frac{2}{3} A^3) - S_{\text{matter}}}$$

(2.2)

$^2$We use the dimensional reduction regulation scheme throughout this paper. In this case $|k| = |k_{YM}| + N$, where $k_{YM}$ is the level of the Chern-Simons theory regulated by including an infinitesimal Yang Mills term in the action, and $k$ is the level of the theory regulated in the dimensional reduction scheme [24, 21]. For this reason this class of Chern-Simons theories may be well-defined only in the range $|\lambda| \leq 1$. Hereafter we assume that the ’t Hooft coupling is always positive for simplicity. Our results are easily generalized to the case where $\lambda$ is negative by taking the absolute value of it.
where \( D\mu \) denotes the integration measure of matter fields. To calculate it, we will integrate the matter fields first, and obtain the effective action which depends on the gauge fields. According to the study in [1], the partition function is given as

\[
Z_{CS} = \int DA e^{\frac{i k}{4\pi} \text{Tr} f(\text{Ad}A + \frac{2}{3} A^3) - S_{eff}(U)}
\]

(2.3)

where \( S_{eff}(U) \) is a function of the two dimensional holonomy fields \( U(x) \) around the thermal circle \( S^1 \). Here \( x \) of \( U(x) \) indicates a point on the \( S^2 \), and its eigenvalue is \( e^{i\alpha_m(x)} \) where \( m \) is the index of \( N \times N \) matrix. The effective action \( S_{eff}(U) \) would be obtained by summing up all vacuum graphs which involve at least one matter field. The graphs which do not involve any matter fields are not summed yet, but they are summed at the stage of the path integration in (2.3).

To calculate (2.3), it is useful to observe that \( S_{eff}(U) \) is ultralocal in large \( N \) as well as in high temperature. The \( S_{eff}(U) \) would be expanded as a series of local operators,

\[
S_{eff}(U) = \int d^2x \left( T^2 \sqrt{g} v(U) + v_1(U) \text{Tr} D_i UD^i U + \ldots \right).
\]

(2.4)

Note that the scaling (2.1) converts higher temperature expansion (2.4) into expansion in inverse power of \( N \). Here the first term is leading \( \mathcal{O}(N^2) \) and the second term is \( \mathcal{O}(N^1) \) and terms \( \ldots \) are further suppressed at large \( N \). \(^3\) Then at large \( N \), the effective action is simplified to be the leading term

\[
S_{eff}(U) = \int d^2x T^2 \sqrt{g} v(U).
\]

(2.5)

This (2.3) is originally suggested in [3], which is a consequence of the entire effect of matter loops on gauge dynamics at temperature \( \sqrt{N} \) and at leading order in \( N \). By (2.3), the partition function would be given by

\[
Z_{CS} = \int DA e^{\frac{i k}{4\pi} \text{Tr} f(\text{Ad}A + \frac{2}{3} A^3) - T^2 \int d^2x \sqrt{g} v(U)}
\]

\[= \langle e^{-T^2 \int d^2x \sqrt{g} v(U)} \rangle_{N,k} \]

(2.6)

where

\[
\langle \Psi \rangle_{N,k}
\]

(2.7)

is the expectation value of \( \Psi \) in the pure \( U(N) \) Chern-Simons theory at level \( k \). Since the Chern-Simons theory is topological, expectation values become independent of \( x \), then the \( x \) dependence is removed in the step from the second line to the third line of (2.6). Here \( V_2 \) is the volume of \( S^2 \).

\(^3\)Every term in the effective action (2.4) is of order \( N \) at fixed \( T \), as the action is generated by integrating out fundamental fields.
2.2 Path integration to obtain the partition function, deconfinement phase transition and further phase transition caused by flux

We can perform the calculation of (2.6) in the same manner as [30]. You can see the details of how to compute the path integration in [1]. Keeping the effective potential term $N\zeta v(U)$ intact, we achieve the following value

$$Z_{CS} = \int \prod_{j=1}^{N} d\alpha_j \left( \prod_{m \neq l} 2 \sin \left( \frac{\alpha_m(n_m) - \alpha_l(n_l)}{2} \right) \right) e^{-N\zeta v(U)} \left( \sum_{M_j = -\infty}^{\infty} e^{ikM_j\alpha_j} \right)$$

(2.8)

where $M_j$ are constant units of flux in the $U(1)_j$ ($j = 1, \ldots, N$) factors. As we can see in the last line, the summation over flux $M_j$ in $e^{ikM_j\alpha_j}$ constrains $\alpha_j$ to take discrete values

$$\alpha_j = \frac{2\pi n_j}{k}.$$  (2.9)

Then the partition function would be

$$Z_{CS} = \prod_{m = 1}^{N} \sum_{n_m = -\infty}^{\infty} \left[ \left( \prod_{l \neq m} 2 \sin \left( \frac{\alpha_l(n_l) - \alpha_m(n_m)}{2} \right) \right) e^{-N\zeta v(U)} \right]$$

(2.10)

where the summation over $n_m$ is restricted so that no two $n_m$ are allowed to be equal.

2.3 Eigenvalue density

We will estimate (2.10) in the 't Hooft limit, $k \to \infty, N \to \infty, \lambda = \frac{N}{k}$: fixed. The summation over the eigenvalue (2.11) in the limit is dominated by the saddle point configuration of the eigenvalues at large $N$. The saddle points minimize the following potential

$$V(U) = \sum_{m \neq l} \ln 2 \sin \frac{\alpha_m - \alpha_l}{2}$$

(2.11)

where $V(U) = N\zeta v(U)$. The saddle points are obtained by solutions of the following equation

$$V'(\alpha_m) = \sum_{m \neq l} \cot \frac{\alpha_m - \alpha_l}{2}.$$  (2.12)

In terms of eigenvalue density function $\rho(\alpha)$, the equation would be represented by

$$V'(\alpha_0) = NP \int d\alpha \cot \frac{\alpha_0 - \alpha}{2} \rho(\alpha)$$

(2.13)

where $\rho(\alpha) = \frac{1}{N} \sum_{m=1}^{N} \delta(\alpha - \alpha_m)$.

We can see that this saddle point equation problem is very similar to the Gross-Witten-Wadia (GWW) problem [31, 32, 33] in the Yang-Mills theory on $S^p \times S^1$. In a usual GWW
problem in Yang-Mills theories, a thermal partition function obtained after integrating out all massive modes is given by the integration over the single unitary matrix as

\[ Z_{YM} = \prod_{m=1}^{N} \int_{-\infty}^{\infty} d\alpha_m \left[ \prod_{l \neq m} 2 \sin \left( \frac{\alpha_l - \alpha_m}{2} \right) e^{-V_{YM}(U)} \right], \quad (2.14) \]

and there is a competition between potential \( V_{YM}(U) \) that tends to clump eigenvalues, and the measure factor (Vandermonde determinant) which tends to repel them.\(^4\) At low temperature with small \( \zeta \), the repulsive force by the measure factor is stronger. Then the eigenvalue density would have support everywhere on \( -\pi \leq |\alpha| \leq \pi \). But on the other hand, in the high enough temperature, the attracting force caused by the potential \( V_{YM}(U) \) becomes stronger than the repulsive force, and the eigenvalue will be clumped. Then in the high temperature the eigenvalue distribution has support only on a finite arc, there would be a domain of eigenvalues so-called "lower gap" such that \( \{ \alpha | \alpha \sim \alpha + 2\pi, \rho(\alpha) = 0 \} \), and phase transition would occur. At extremely high temperature \( T \to \infty \), the eigenvalue density function would be clumped to be a delta function \( \rho(\alpha) \sim \delta(\alpha) \).

Also in the current Chern-Simons case (2.10), such a competition between \( V(U) \) and measure term exists. But in this case, unlike the usual Yang-Mills case (2.14), there is a constraint on the eigenvalue (2.9). (2.9) saturates the eigenvalue density from above as

\[ \rho(\alpha) \leq \frac{k}{2\pi} \times \frac{1}{N} = \frac{1}{2\pi \lambda}, \quad (2.15) \]

while the eigenvalue density in the Yang-Mills theory is not bounded from above (is bounded only from below). Saddle points of (2.14) do not always obey the inequality (2.13), then the Yang-Mills solutions violating (2.15) will not be the solution of the Chern-Simons saddle point equations (2.13) and (2.10). Instead, the current Chern-Simons theory admits new classes of solutions saturating the upper bound of the inequality over several arcs along the unit circle. Hence in this Chern-Simons case, there can exist not only the "lower gaps" but also the "upper gaps", which are arcs over which the upper bound of (2.15) are saturated.

Hence the phase structures of the current Chern-Simons theories (2.10) would be different from the one of usual Yang-Mills theory (2.14). In a usual Yang-Mills theory case, there are only two phases, "no gap phase" and "lower gap phase". In the no gap phase, the eigenvalue has support everywhere on the unit circle in the complex plane, on the other hand in the lower gap phase, the eigenvalue has support only on a finite arcs. On the other hand, in the current Chern-Simons case, due to the existence of upper bound (2.13), not only the no gap and lower gap phases, but also "upper gap" and "two gap" phases exist. In the upper gap phase, there is one upper gap where the upper bound of (2.15) is saturated, and the eigenvalue density has support everywhere on the unit circle. In the two gap phase, there is one upper gap as well as one lower gap where the eigenvalue density vanishes.

In the Chern-Simons case, the eigenvalue distribution is in the no gap phase in the low temperature, and if we increase the temperature, eigenvalue density starts to clump

\(^4\)The effective potential \( V_{YM}(U) \) was computed in free gauge theories \( [34, 35] \); it has also been evaluated at higher orders in perturbation theory in special examples \( [36, 37, 38, 39] \). At least in perturbation theory \( [33] \) and perhaps beyond \( [40, 41] \), the potential \( V_{YM}(U) \) is an analytic function of \( U \).
to \( \alpha = 0 \), and then we can expect lower gaps or upper gaps will show up. In the small \('t Hooft coupling region \( \lambda < \lambda_c \), since the upper bound in (2.15) is large enough, lower gaps will show up before the maximum of the eigenvalue density function reaches the upper bound. Then it transits to lower gap phase first. As we increase the temperature further, the maximum of the eigenvalue density \( \rho \) eventually reaches the upper bound \( \frac{1}{2\pi \lambda} \) and it transits to the two gap phase. On the other hand, at large \( \lambda \) with \( \lambda > \lambda_c \), the upper bound in (2.15) is small, then the maximum of the eigenvalue reaches the upper bound before the lower gap shows up. Then in the large \( \lambda \) region, it transits to upper gap phase first. If we increase the temperature further, since the eigenvalue density keeps clumping, lower gaps will show up and it transits to two gap phase. In the large \( \zeta \) limit, the eigenvalue density will have a universal configuration

\[
\rho(\alpha) = \begin{cases} 
\frac{1}{2\pi \lambda} & (|\alpha| < \pi \lambda) \\
0 & (|\alpha| > \pi \lambda)
\end{cases}
\]  

(2.16)

which is the nearest thing to a \( \delta \) function permitted by the effective Fermi statistics of the eigenvalues \( 0 \leq \rho(\alpha) \leq \frac{1}{2\pi \lambda} \). This is in perfect agreement with the results of [9] in which they used the Hamiltonian methods [12].

For your reference, you can see the graph of the eigenvalue density function in each phase listed in Figs. 1(a) \~ 1(d). You can also see the phase diagrams of Chern-Simons matter theories in Figs. 2(a) \~ 4(a) and 6(a).

2.4 How to calculate the eigenvalue density in lower gap, upper gap and two gap phases

In this subsection we will review how to obtain the eigenvalue density in the lower gap, the upper gap and the two gap phases. To calculate the eigenvalue density, we will decompose the eigenvalue density to \( \rho(\alpha) = \rho_0(\alpha) + \psi(\alpha) \). By substituting it into the saddle point equation we can rewrite the saddle point equation as

\[
N\mathcal{P} \int d\alpha \psi(\alpha) \cot \left( \frac{\alpha_0 - \alpha}{2} \right) = U(\alpha_0)
\]

\[
\int d\alpha \psi(\alpha) = A[\rho_0]
\]

\[
U(\alpha) = V'(\alpha) - N\mathcal{P} \int d\theta \rho_0(\theta) \cot \left( \frac{\alpha - \theta}{2} \right)
\]

\[
A[\rho_0] = 1 - \int d\alpha \rho_0(\alpha).
\]

(2.17)

Here \( \psi(\alpha) \) is nonzero in the compliment of both upper gaps and lower gaps. Starting from this, we turn to the complex variables

\[
z = e^{i\alpha}, \quad z_0 = e^{i\alpha_0}, \quad A_i = e^{ia_i}, \quad B_i = e^{ib_i}.
\]

Note that

\[
d\alpha = \frac{dz}{iz}, \quad \cot \left( \frac{\alpha_0 - \alpha}{2} \right) = \frac{i(z_0 + z)}{z_0 - z}.
\]
Figure 1: The eigenvalue distribution $\rho(\alpha)$ in the no gap phase Fig.1(a), in the lower gap phase Fig.1(b), in the upper gap phase Fig.1(c) at $\lambda = 0.51$, and in the two gap phase Fig.1(d). These are the graphs calculated in the GWW toy model. We can see that the eigenvalue density in the upper gap phase Fig.1(c) is saturated from above as $\rho(\alpha) = \frac{1}{2\pi \cdot 0.51}$. We can also see that the two gap density function Fig.1(d) has both a lower gap and an upper gap.

Here (2.17) will be rewritten in terms of these variables as

$$N\mathcal{P} \int \frac{dz}{z} \left( \frac{z_0 + z}{z_0 - z} \psi(z) \right) = U(z_0),$$

$$\int \frac{dz}{iz} \psi(z) = A$$

where the integrals in (2.18) run counterclockwise over the unit circle in the complex plane. Suppose that the solution to this equation, $\psi(z)$, has support on $n$ connected arcs on the unit circle in the complex plane. We denote the beginning and endpoints of these arcs by $A_i = e^{ia_i}$ and $B_i = e^{ib_i}$ ($i = 1 \ldots n$). These are the endpoints of lower gaps or upper gaps as well. Our convention is that the points $A_1, B_1, A_2, B_2 \ldots A_n, B_n$ sequentially follow each other counterclockwise on the unit circle. We refer to the $n$ arcs $(A_i, B_i)$ as ‘cuts’. The arcs $(B_i, A_{i+1})$ (as also $(B_n, A_1)$) are referred to as gaps. By definition, $\psi$ has support only along the cuts on the unit circle.

We use the (as yet unknown) function $\psi(z)$ to define an analytic function $\Phi(u)$ on the complex plane

$$\Phi(u) = \sum_i \int_{A_i}^{B_i} \frac{dz}{iz} \frac{u + z}{u - z} \psi(z)$$

where the integral is taken counterclockwise along the $n$ cuts, $(A_i, B_i)$ of the unit circle in the complex plane. It is easy to see that the analytic function $\Phi(u)$ is discontinuous along
these n cuts. Let $\Phi(z)^+$ denote the limit of $\Phi(u)$ as it approaches a cut from $|u| > 1$, and let $\Phi^-(z)$ denote the limit of $\Phi(u)$ as it approaches a cut from $|u| < 1$. Then

$$
\Phi^+(z) - \Phi^-(z) = 4\pi \psi(z) \quad (2.20)
$$

and

$$
\Phi^+(z) + \Phi^-(z) = 2 \sum_i \mathcal{P} \int_{A_i}^{B_i} \frac{d\omega}{i\omega} \frac{z + \omega}{z - \omega} \phi(\omega) = \frac{2U(z)}{iN} \quad (2.21)
$$

(the first equality is the definition of the principal value, while the second equality follows using (2.18)). Moreover it follows immediately from (2.18) and (2.19) that

$$
\lim_{u \to \infty} \Phi(u) = A. \quad (2.22)
$$

We are now posed with the problem of determining $\Phi(u)$ given its principal value along a cut. To determine the $\Phi(u)$, we introduce following "cut function" $h(u)$ as

$$
h(u) = \sqrt{(A_1 - u)(B_1 - u)(A_2 - u)(B_2 - u) \cdots (A_n - u)(B_n - u)}. \quad (2.23)
$$

We define $h(u)$ to have cuts precisely on the $n$ arcs on the unit circle that extend from $A_i$ to $B_i$. This definition fixes the function $h(u)$ up to an overall sign. This sign will cancel out in our solution for $\Phi$ below, and so is uninteresting. For future use we note that when $u = e^{i\alpha}$,

$$
h^2(u) = \prod_{m=1}^{n} 4e^{i\alpha m + \beta_m} e^{i\alpha} \left( \sin^2 \left( \frac{\alpha_m - \beta_m}{4} \right) - \sin^2 \left( \frac{\alpha}{2} - \frac{\alpha_m - \beta_m}{4} \right) \right). \quad (2.24)
$$

We use the function $h(z)$ to define a new function, $H(z)$, via the equation

$$
\Phi(z) = h(z)H(z).
$$

Using the fact that $h^+(z) = -h^-(z)$ along the cut, (2.21) turns into

$$
H^+(z) - H^-(z) = \frac{2U(z)}{iNh^+(z)}. \quad (2.25)
$$

Here we assume that $H(u)$ is holomorphic except in the cut region. From (2.22) and (2.23), $H(v) = \mathcal{O}(\frac{1}{v^N})$ at large $v$ so that

$$
\int_{C_\infty} dv \frac{H(v)}{2\pi i(v-u)} = 0 \quad (2.26)
$$

where the contour $C_\infty$ runs counterclockwise over a very large circle at infinity. Since $H(v)$ is holomorphic except on the cut, by the Cauchy’s theorem,

$$
0 = \oint_{C_\infty} dv \frac{H(v)}{2\pi i(v-u)} - \oint_{C_{\text{cuts}}} dv \frac{H(v)}{2\pi i(v-u)} - H(u) = -\oint_{C_{\text{cuts}}} dv \frac{H(v)}{2\pi i(v-u)} - H(u). \quad (2.27)
$$

We use analogous notation for other analytic functions below. Let $z$ denote a complex number of unit norm. The symbol $F^+(z)$ will denote the limit of $F(u)$ as $u \to z$ from above (i.e. from $|u| > 1$), while $F^-(z)$ is the limit of the same function as $u \to z$ from below (i.e. from $|u| < 1$). Note that along a cut $F^-(z) = -F^+(z)$.
The contour \( C_{\text{cuts}} \) are the loops enclosing each of the \( n \) cuts \((A_i, B_i)\), but not the point \( u \).
Then from (2.25),
\[
H(u) = -\oint_{C_{\text{cuts}}} \frac{H(v)}{2\pi i(v-u)} = \frac{1}{\pi} \int_{L_{\text{arcs}}} dz \frac{U(z)}{N h^+(z)(z-u)} = \frac{1}{2\pi} \oint_{C_{\text{cuts}}} dv \frac{U(v)}{N h(v)(v-u)}.
\]
(2.28)

In the equation above, \( v \) is a variable on the complex plane while \( z \) is a variable on the unit circle of the complex plane. The integration region \( L_{\text{arcs}} \) runs counterclockwise along \( n \) cuts on the unit circle (This is \textit{not} loop !), integration region for each cut is from \( A_i \) to \( B_i \). The third equality uses \( h^+(z) = -h^-(z) \) together with the assumption that \( U(z) \) has no singularities on the \( n \) cuts.

Now by applying the Cauchy’s theorem for the integrand \( \frac{U(v)}{N h(v)(v-u)} \) again, we will obtain the following
\[
H(u) = \frac{1}{2\pi} \oint_{C_{\text{cuts}}} dv \frac{U(v)}{N h(v)(v-u)}
= \frac{1}{2\pi} \int_{C_{\infty}} \frac{U(z)}{N h(z)(z-u)} - \sum_{m=1}^{r} i \text{Res}_{z=z_k} \frac{U(z)}{N h(z)(z-u)}.
\]
(2.29)
The last term is the sum of the residues of \( U(z) \). Here we have assumed that \( U(z) \) is a meromorphic function of \( z \).

Based on \( H(u) \) in (2.29), and from \( \Phi^+(u) - \Phi^-(u) = 4\pi \psi(u) \), combining with \( \rho(\alpha) = \psi(\alpha) + \rho_0(\alpha) \), we can obtain the eigenvalue density \( \rho(\alpha) \). From next section, we will obtain the phase structure of the regular fermion theory, the critical boson theory and the \( N=2 \) supersymmetric CS matter theory by using the techniques supplied by this subsection.

3. Higher temperature phases of the regular fermion theory and critical boson theory and the duality

3.1 Regular fermion theory

In this subsection we study the level \( k U(N) \) Chern-Simons theory coupled to massless fundamental fermions. The Lagrangian of the theory is presented in equation (2.1) of [6].

From the equation (3.5) of [1], effective potential \( V(U) \) is obtained as
\[
V(U) = -\frac{N^2 \zeta}{6\pi} \left( \frac{\tilde{c}^3}{\lambda} - \tilde{c}^3 + 3 \int_{-\pi}^\pi d\alpha \rho(\alpha) \int_\tilde{c}^\infty dy \left( \ln(1 + e^{-y-i\alpha}) + \ln(1 + e^{-y+i\alpha}) \right) \right)
\equiv V_r[f][\rho, N; \tilde{c}, \zeta],
\]
(3.1)

where \( \tilde{c} \) determines the thermal mass of the fermions.\footnote{\( \Sigma_T = \tilde{c}^2 T^2 \) is the thermal mass of the fundamental fermions. More precisely, the fermionic self energy}

The value of \( \tilde{c} \) is obtained by extremizing \( V(U) \) w.r.t. \( \tilde{c} \) at fixed \( \rho, \zeta \), i.e. \( \tilde{c} \) obeys the equation
\[
\tilde{c} = \lambda \int_{-\pi}^\pi d\alpha \rho(\alpha) \left( \ln 2 \cosh\left( \frac{\tilde{c} + i\alpha}{2} \right) + \ln 2 \cosh\left( \frac{\tilde{c} - i\alpha}{2} \right) \right).
\]
(3.3)
The general form of the free energy of the regular fermion theory on $S^2 \times S^1$ is given as
\[
F_{r.f}^N = V_{r.f}[\rho, N] - N^2 \mathcal{P} \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^\pi d\beta \rho(\alpha)\rho(\beta) \log \left| 2\sin \frac{\alpha - \beta}{2} \right|
\]
\[
= V_{r.f}[\rho, N] + F_2[\rho, N].
\] (3.4)

To obtain the free energy in each phase, we only have to evaluate the eigenvalue density $\rho$ and $\tilde{c}$ in each phase and just substitute into (3.4). Note that $\tilde{c}$ can be determined if the eigenvalue density is determined. So obtaining the eigenvalue density in each phase is equivalent to obtaining the free energy in each phase.

For later use, we will give the form of $V'(z)$ here. We obtain $V'(z)$ from (3.1) as
\[
V'(z) = -\frac{N\zeta}{2\pi} \int_\xi^\infty dy \frac{-ie^{-y}}{z+e^{-y}} + \frac{ie^{-y}}{z^{-1}+e^{-y}}.
\] (3.5)

The phase structure of this theory is depicted in Fig. 2(a), as obtained in following subsections.

### 3.1.1 Lower gap phase

To obtain the eigenvalue density in the lower gap phase, we will employ the cut region and the cut function described in appendix A.1. The function $H(u)$ as well as $\Phi(u) = h(u)H(u)$ in the lower gap phase are obtained by using (2.29) with substituting $\rho_0(\alpha) = 0$. $U(z)$ in (2.29) is identical to $V'(z)$ as
\[
U(z) = V'(z) = -\frac{N\zeta}{2\pi} \int_\xi^\infty dy \ y \left( \frac{-ie^{-y}}{z+e^{-y}} + \frac{ie^{-y}}{z^{-1}+e^{-y}} \right),
\] (3.6)

is given by $\Sigma_T(p) = f(\beta p_s)\gamma^+ + ig(\beta p_s)\gamma^-$, where
\[
f(y) = \frac{\lambda}{y} \int_{-\pi}^{\pi} d\alpha \ \rho(\alpha) \left( \ln 2\cosh \left( \frac{\sqrt{y^2 + c^2 + i\alpha}}{2} \right) + \ln 2\cosh \left( \frac{\sqrt{y^2 + c^2 - i\alpha}}{2} \right) \right)
\]
\[
g(y) = \frac{c^2}{y^2} - f(y)^2.
\] (3.2)
because \( \rho_0(\alpha) = 0 \). By substituting the above into (2.29), we immediately obtain \( H(u) \) as well as \( \Phi(u) = h(u)H(u) \), as

\[
\Phi(u) = \Phi^{r,f}_{lg}(\bar{c}, b, \zeta; u) \equiv \frac{\zeta}{2\pi} \int_{\tilde{c}}^{\infty} dy \frac{y\rho(u)(u+1)(1+e^y)}{\sqrt{(e^y+e^u)(e^y+e^{-u})(u+e^y)(u+e^{-y})}} + \frac{\zeta}{2\pi} \left( \int_{\tilde{c}}^{\infty} dy \frac{e^{-y}}{u+e^{-y}} - \int_{\tilde{c}}^{\infty} dy \frac{e^{-y}}{u-1+e^{-y}} \right).
\]

(3.7)

From \( \Phi^+(u) - \Phi^-(u) = 4\pi \rho(u) \) at the unit circle \( u = e^{i\alpha} \) with \(-b \leq \alpha \leq b\), we obtain the eigenvalue density in the lower gap phase as

\[
\rho(\alpha) = \frac{\zeta}{\sqrt{2\pi}} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2}} \int_{\tilde{c}}^{\infty} dy \frac{y\cos \frac{\alpha}{2} \cosh \frac{\alpha}{2}}{(\cosh y + \cos \alpha) \sqrt{\cosh y + \cos b}}
\]

(3.8)

\[
\equiv \rho^{r,f}_{lg}(\zeta, \lambda; \bar{c}, b; \alpha).
\]

At \( \pi > |\alpha| > b \), \( \rho(\alpha) = 0 \). By substituting the eigenvalue density (3.8) into (3.3) and (3.4), we can obtain \( \bar{c} \) as well as the free energy in the lower gap phase of the regular fermion theory.

The condition \( \lim_{u \to \infty} \Phi(u) = 1 \) requires the following condition

\[
\tilde{M}^{r,f}_{lg}(\zeta, \bar{c}, b) \equiv \frac{\zeta}{2\pi} \int_{0}^{\infty} dx \left( \log x \log \left( \frac{1+x}{x} \right) \right) = 1.
\]

(3.9)

By (3.3) and (3.4), we can obtain \( (b, \bar{c}) \) as functions of \( (\lambda, \zeta) \) as \( (b, \bar{c}) = (b(\lambda, \zeta), \bar{c}(\lambda, \zeta)) \). (3.8), (3.3) and (3.4) provide a complete set of the solutions in the lower gap phase of the regular fermion theory.

3.1.2 Upper gap phase

Next we will search for a solution with no lower gap and one upper gap. The domain of cut and the cut function in this upper gap phase are defined in appendix A.2. To obtain the eigenvalue density based on (2.29), we will take \( \rho_0(\alpha) = \frac{1}{2\pi} \lambda \neq 0 \) at the upper gap region \(-a \leq \alpha \leq a\). Then by using (2.29) with (3.3), we obtain \( H(u) \) as well as \( \Phi(u) \) as

\[
\Phi(u) = \Phi^{r,f}_{ug}(\bar{c}, a, \zeta; u) + \Phi^0_{ug}(\lambda, a; u), \quad \text{where}
\]

\[
\Phi^{r,f}_{ug}(\bar{c}, a, \zeta; u) \equiv \frac{\zeta}{2\pi} \int_{\tilde{c}}^{\infty} dy \frac{y\rho(u)(1-e^y)(1-u)}{\sqrt{(e^y+e^u)(e^y+e^{-u})(e^y+u)(e^y+u)}} + \frac{\zeta}{2\pi} \int_{\tilde{c}}^{\infty} dy \frac{e^{-y}}{u+e^{-y}} - \frac{\zeta}{2\pi} \int_{\tilde{c}}^{\infty} dy \frac{e^{-y}}{u-1+e^{-y}},
\]

(3.10)

\[
\Phi^0_{ug}(\lambda, a; u) \equiv \frac{i}{\pi \lambda} \int_{L_{ugs}} d\omega \frac{1}{h(\omega)h(u) - h(\omega - u)} + iP \int_{L_{cir}} d\omega \rho_0(\omega) \frac{u + \omega}{\omega(\omega - u)},
\]

where \( L_{ugs} \) runs counterclockwise over upper gap region, and \( L_{cir} \) runs counterclockwise over unit circle.

From (3.10), and by taking \( \Phi^+(u) - \Phi^-(u) = 4\pi (\rho(u) - \rho_0(u)) \) at the cut region \( u = e^{i\alpha} \) with \( \pi \geq |\alpha| \geq a \), we can obtain the eigenvalue density function as

\[
\rho(\alpha) = \frac{1}{2\pi} \lambda \sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}} \int_{\tilde{c}}^{\infty} dy \frac{y|\sin \frac{\alpha}{2}| \sinh \frac{y}{2}}{\sqrt{\cosh y + \cos \alpha}} \cosh y}
\]

(3.11)

\[
\equiv \rho^{r,f}_{ug}(\zeta, \lambda; \bar{c}, a; \alpha).
\]
In the region $|\alpha| < |a|$, $\rho(\alpha) = \frac{1}{2\pi \lambda}$. To derive (3.11), we have used a formula

$$i \frac{1}{2\pi^2 \lambda} \int_{L_{ug}} d\omega \frac{1}{h(\omega)(\omega - u)} h(u) = \frac{1}{2\pi \lambda}$$

(3.12)

where the $u = e^{i\alpha}$ is located at the cut region. This is proved in appendix [B.1.1]. By substituting the eigenvalue density (3.11) into (3.3) and (3.4), we can obtain $\tilde{c}$ as well as the free energy in the upper gap phase of the regular fermion theory.

From the condition, $\lim_{u \to \infty} \Phi(u) = 1 - \int_{-\pi}^{\pi} d\alpha \rho_0(\alpha)$, we obtain the equation

$$\tilde{M}_{ug}^{r.f}(\zeta, \tilde{c}, a) \equiv \frac{\zeta}{2\pi} \int_{0}^{e^{-\tilde{c}}} dx \left( \frac{\log x}{x} - \frac{1}{x} \frac{\log x}{\sqrt{x^2 + 2x \cos a + 1}} \right) = 1 - \frac{1}{\lambda}.$$

(3.13)

Here we have used the formula

$$i \frac{1}{\pi \lambda} \int_{L_{ug}} d\omega \frac{1}{h(\omega)} = \frac{1}{\lambda}$$

(3.14)

which is proved in appendix [B.1.1]. By (3.13) and (3.3), we can obtain $(a, \tilde{c})$ as functions of $(\lambda, \zeta)$ as $(a, \tilde{c}) = (a(\lambda, \zeta), \tilde{c}(\lambda, \zeta))$. (3.11), (3.13) and (3.3) provide a complete set of solution in the upper gap phase of the regular fermion theory.

### 3.1.3 Two gap phase

Now we will search for a solution with one lower gap and one upper gap. The details of our two cuts and the cut function $h(u)$ are described in appendix [A.3].

The function $H(u)$ as well as $\Phi(u) = h(u)H(u)$ are obtained by using (2.29),

$$\Phi(u) = \Phi_{tg}^{r.f}(\zeta, a, b; \tilde{c}, a) + \Phi_{tg}^{l.o}(\lambda, a, b; u)$$

where

$$\Phi_{tg}^{r.f}(\zeta, a, b; \tilde{c}, a) \equiv \frac{\zeta}{2\pi} \int_{\tilde{c}}^{\infty} dy \frac{ye^{-y}(u^{-1} - u)}{(u + e^{-y})(u^{-1} + e^{-y})} - \frac{yh(u)e^{y} \left( 2u + e^{y} + e^{-y} \right)}{h(-e^{y}) (u + e^{y})(u + e^{-y})}$$

$$\Phi_{tg}^{l.o}(\lambda, a, b; u) \equiv \frac{i}{\pi \lambda} \int_{L_{ug}} d\omega \frac{h(u)}{h(\omega)(\omega - u)}$$

(3.15)

From $\lim_{u \to \infty} \Phi(u) = 1 - \int_{-\pi}^{\pi} d\alpha \rho_0(\alpha)$, we obtain the following two conditions,

$$\frac{1}{4\pi \lambda} \Upsilon(a, b) = \frac{\zeta}{2\pi} \Upsilon^{r.f}(a, b, \tilde{c}),$$

where

$$\Upsilon(a, b) \equiv \int_{-a}^{a} d\theta \frac{1}{\sqrt{\sin^2 \frac{a}{2} - \sin^2 \frac{\theta}{2}} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\theta}{2}}}$$

(3.16)

$$\Upsilon^{r.f}(a, b, \tilde{c}) \equiv \int_{\tilde{c}}^{\infty} dy \frac{y}{\sqrt{(\cosh y + \cos a)(\cosh y + \cos b)}}$$

$$\frac{1}{4\pi \lambda} \Lambda(a, b) = 1 - \frac{\zeta}{4\pi} \Lambda^{r.f}(a, b, \tilde{c}),$$

where

$$\Lambda(a, b) \equiv \int_{-a}^{a} d\theta \frac{\cos \theta}{\sqrt{\sin^2 \frac{a}{2} - \sin^2 \frac{\theta}{2}} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\theta}{2}}}$$

(3.17)

$$\Lambda^{r.f}(a, b, \tilde{c}) \equiv \int_{\tilde{c}}^{\infty} dy \left( \frac{e^{y} + e^{-y}}{\sqrt{(\cosh y + \cos a)(\cosh y + \cos b)} - 2} \right).$$
By using (3.16), (3.17) and (3.3), \((a, b, \tilde{c})\) are determined as functions of \((\lambda, \zeta)\) as \((a, b, \tilde{c}) = (a(\lambda, \zeta), b(\lambda, \zeta), \tilde{c}(\lambda, \zeta))\).

From (3.15) and by taking \(\Phi^+(u) - \Phi^-(u) = 4\pi(\rho(u) - \rho_0(u))\) at the cuts \(|a| < |\alpha| < |b|\), we obtain the eigenvalue density as

\[
\rho_0 = \rho_{1,tg}(\zeta, a, b, \tilde{c}; \alpha) + \rho_{2,tg}(\lambda, a, b; \alpha),
\]

where

\[
\rho_{1,tg}(\zeta, a, b, \tilde{c}; \alpha) = \frac{4\sin \alpha}{\pi^2} F(a, b; \alpha) \int_{c}^{\infty} dy \sqrt{y e^{-y}} \left( \frac{|\sin \alpha|}{\cos \alpha + \cosh y} \right),
\]

\[
\rho_{2,tg}(\lambda, a, b; \alpha) = \frac{\sin \alpha}{4\pi^2 \lambda} F(a, b; \alpha) I_1(a, b, \alpha),
\]

\[
F(a, b; \alpha) \equiv \sqrt{(\sin^2 \frac{\alpha}{2} - \sin^2 \frac{a}{2})(\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2})},
\]

\[
\nu_r(a, b; y) \equiv \sqrt{(1 + 2e^{-y}\cos a + e^{-2y})(1 + 2e^{-y}\cos b + e^{-2y})},
\]

\[
I_1(a, b; \alpha) \equiv \int_{-a}^{a} d\theta \frac{d\theta}{(\cos \theta - \cos \alpha)\sqrt{(\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2})(\sin^2 \frac{b}{2} - \sin^2 \frac{\theta}{2})}}.
\]

(3.18)

\[
\rho(\alpha) = 0 \text{ in } b < |\alpha| < \pi, \text{ and } \rho(\alpha) = \frac{1}{\pi^2 \lambda} \text{ in } 0 < |\alpha| < a. \text{ During the calculation of (3.18), we have used (3.16) also. Here } \rho_{2,tg} \text{ has the same functional form as the eigenvalue density in the GWW type matrix integration (7.6) in [1]. } \rho_{1,tg} \text{ can be regarded as an additional term depending on the detail of the theory. By substituting the eigenvalue density (3.18) into (3.3) and (3.4), we can obtain } \tilde{c} \text{ as well as the free energy in the two gap phase. } \Upsilon(a, b), \Lambda(a, b) \text{ and } |\sin \alpha|F(a, b, \alpha)I_1(a, b, \alpha) \text{ can be represented by the complex line integral by taking } \omega = e^{i\theta} \text{ as }
\]

\[
\Upsilon(a, b) = -4i \int_{L_{ugs}} d\omega \frac{1}{h(\omega)},
\]

\[
\Lambda(a, b) = -4i \int_{L_{ugs}} d\omega \frac{\omega}{h(\omega)},
\]

\[
|\sin \alpha|F(a, b, \alpha)I_1(a, b, \alpha) = i \int_{L_{ugs}} d\omega \frac{h^+(u)}{h(\omega)} \left( \frac{2}{\omega - u} + \frac{1}{u} \right).
\]

(3.19)

These are useful to discuss the level-rank duality later.

The combination of (3.16), (3.17), (3.3) and (3.18) provides a complete set of solutions in the two gap phase.

At the large \(\zeta\) limit, as we proved in appendix C.1.1, the eigenvalue density approaches the universal distribution (2.16) because \(\tilde{c}\) remains as a finite positive quantity at the limit. In the limit, \(a, b\) and the eigenvalue density behave as

\[
a = \pi \lambda - \frac{\epsilon}{2}, \quad b = \pi \lambda + \frac{\epsilon}{2},
\]

\[
\epsilon = 8 \sin(\pi \lambda) \exp \left( -\frac{\sin(\pi \lambda)}{2} \lambda \zeta \int_{c}^{\infty} dy \frac{y}{\cosh y + \cos \pi \lambda} \right) + \ldots,
\]

\[
\rho(\alpha) = \frac{1}{\pi^2 \lambda} \cos^{-1} \sqrt{\frac{\alpha - a}{b - a}}.
\]

(3.20)
At appendix C.2.1, we have demonstrated the above behavior (3.20). As we can see in the appendix, the eigenvalue density function is dominated by \( \rho_{2.1g} \) at the large \( \zeta \) limit. The form of the eigenvalue density approaches \( \cos^{-1} \sqrt{n} \) function which is the same form as (7.11) in [1] which is the large \( \zeta \) limit of the one of the GWW model. We can also see that the range of the domain of the cut \( \epsilon \) is smaller than the one in the GWW type matrix integration at the same value of \( (\lambda, \zeta) \).

### 3.1.4 Phase transition points

**Phase transition points between no gap and lower gap**  Let us consider the behavior of the lower gap solutions (3.8) and (3.9) at the point \( b = \pi \) which would correspond to the phase transition points from the lower gap to the no gap phase. If we substitute \( b = \pi \) into (3.9), it becomes

\[
1 = -\frac{\zeta}{2\pi} \int_0^{e^{-\epsilon}} dx \frac{2\log x}{1-x} = \frac{\zeta}{\pi} \sum_{n=1}^{\infty} \left(\frac{1 + \tilde{c}n}{n^2} e^{-n\tilde{c}}\right). \tag{3.21}
\]

This is exactly same as the condition for the phase transition from the no gap to the lower gap discussed in [1], which is obtained by substituting \( \alpha = \pi \) into the third line of (6.33) and by requiring \( \rho(\pi) = 0 \). Based on this, if we substitute \( b = \pi \) into (3.8), we can see

\[
\rho(\alpha) = \frac{\zeta}{\sqrt{2\pi}^2} \sqrt{\frac{\sin^2 b}{2} - \sin^2 \frac{\alpha}{2}} \int_\tilde{\epsilon}^{\infty} dy \frac{y \cos \frac{\alpha}{2} \cosh \frac{y}{2}}{(\cosh y + \cos \alpha) \sqrt{(\cosh y + \cos b)}} |_{b=\pi} = \frac{\zeta}{\pi^2} \int_\tilde{\epsilon}^{\infty} dy \left(\frac{1}{2(1-e^{-y})} - \frac{1}{4(1 + e^{-y+i\alpha})} - \frac{1}{4(1 + e^{-y-i\alpha})}\right) \tag{3.22}
\]

\[
= \frac{1}{2\pi} - \frac{\zeta}{2\pi^2} \sum_{n=1}^{\infty} (-1)^n \cos n\alpha \left(\frac{1 + \tilde{c}n}{n^2}\right) e^{-n\tilde{c}}.
\]

In the last step we have used (3.22), the Taylor expansion and

\[
\int_\tilde{\epsilon}^{\infty} dy \ y e^{-ny} = \left(\frac{1 + \tilde{c}n}{n^2}\right) e^{-n\tilde{c}}. \tag{3.23}
\]

The last line of (3.22) is exactly same as the eigenvalue density function in the no gap phase described in the third line of Eq. (6.33) in [1]. So, at the phase transition points, the lower gap solutions are smoothly connected to the no gap phase solutions.

**Phase transition points between no gap and upper gap**  Let us consider the behavior of (3.11) and (3.13) at \( a = 0 \) which would correspond to the phase transition points from the upper gap to the no gap phase. If we substitute \( a = 0 \) into (3.13), it becomes

\[
1 - \frac{1}{\lambda} = \frac{\zeta}{2\pi} \int_0^{e^{-\epsilon}} dx \frac{2\log x}{1+x} = \frac{\zeta}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1 + \tilde{c}n}{n^2} e^{-n\tilde{c}}. \tag{3.24}
\]

This is exactly same as the condition for the phase transition from the no gap to the upper gap phase discussed in [1], which is obtained by substituting \( \alpha = 0 \) into the third line of (6.33) and by requiring \( \rho(0) = \frac{1}{2\pi\lambda} \).
Figure 3: These are the plots of phase transition points in the regular fermion theory. Fig. 3(a) shows the plots of the phase transition points from the lower gap to the two gap phase, and Fig. 3(b) shows the ones between the upper gap and the two gap phase.

Based on this, if we substitute $a = 0$ in (3.11), we can see

$$\rho(\alpha) = \frac{1}{2\pi\lambda} - \frac{\zeta}{\sqrt{2\pi^2}} \sqrt{\left(\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}\right)} \int_{\tilde{\zeta}}^\infty dy \frac{y \sin \frac{\alpha}{2} \sinh \frac{y}{2}}{\sqrt{\cosh y + \cos a(\cos \alpha + \cosh y)}} \bigg|_{a=0}$$

$$= \frac{1}{2\pi\lambda} \frac{\zeta}{\pi^2} \int_{\tilde{\zeta}}^\infty dy \frac{1}{2(1 + e^{-y})} - \frac{1}{4(1 + e^{-(y+i\alpha)})} - \frac{1}{4(1 + e^{-(y-i\alpha)})}$$

$$= \frac{1}{2\pi} - \frac{\zeta}{2\pi^2} \sum_{n=1}^{\infty} (-1)^n \cos n\alpha \left(1 + \tilde{\zeta} c_n n^2\right) e^{-\tilde{\zeta} n}.$$  

(3.25)

In the last step we have used (3.24), the Taylor expansion and the formula (3.23). The last line of (3.25) is exactly same as the eigenvalue density function in the no gap phase, described in the third line of Eq. (6.33) in [1]. So the upper gap solutions are smoothly connected to the ones in the no gap phase at the phase transition points.

Phase transition from the lower gap to the two gap In the lower gap phase at fixed $\lambda$, the phase transition from the lower gap phase to the two gap phase will occur when the maximum of the eigenvalue density $\rho(0)$ reaches $\rho(0) = \frac{1}{2\pi\lambda}$. The condition would be represented as

$$\rho(0) = \frac{\zeta}{\sqrt{2\pi^2}} \sin \frac{b}{2} \int_{\tilde{\zeta}}^\infty dy \frac{y \cosh \frac{y}{2}}{(\cosh y + 1) \sqrt{(\cosh y + \cos b)}} = \frac{1}{2\pi\lambda}.$$  

(3.26)

The combination of (3.26), (3.3) and (3.9) provides the phase transition points from the lower gap to the two gap. By numerical calculations based on (3.24), (3.3) and (3.9), we obtain the phase transition points plotted in Fig. 3(a).

Let us see the phase transition points from the standpoint of the two gap phase. If we substitute $a = 0$ into (3.16), it becomes the same as (3.26), which is the condition for the phase transition points. If we set $a = 0$ at (3.17), it becomes the same as (3.3). Let us
check whether eigenvalue density in the two gap phase (3.18) becomes the one in the lower gap phase (3.8) in $a = 0$ limit. By using (3.26), we can replace $\rho_{2,tg}(\lambda, 0, b; \alpha)$ of (3.18) by

$$
\rho_{2,tg}(\lambda, 0, b; \alpha) = \frac{\zeta}{2\pi^2} \int_{\tilde{c}}^{\infty} dy \frac{y \cos \frac{\alpha}{2} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2}}}{\sqrt{(1 + \cosh y)(\cosh y + \cos b)}}. \tag{3.27}
$$

Then by summing up with $\rho_{1,tg}^{r.f}(\zeta, 0, b; \alpha)$, (3.18) at $a = 0$ becomes (3.8). So we can see that solutions of two gap phase are smoothly connected to solutions of the lower gap phase at the phase transition point.

**Phase transition from the upper gap to the two gap**  In the upper gap phase, if the minimum of the eigenvalue density $\rho(\pi)$ reaches to $\rho(\pi) = 0$ the phase transition from the upper gap phase to the two gap phase will occur. The condition would be represented as

$$
\zeta \cos \frac{a}{2} \int_{\tilde{c}}^{\infty} dy \frac{ye^{-y}}{\sqrt{(e^{-2y} + 2e^{-y}\cos a + 1)(1 - e^{-2y})}} = \frac{1}{2\pi \lambda}. \tag{3.28}
$$

The combination of (3.28), (3.13) and (3.3) provides the phase transition points from the upper gap to the two gap. By numerical calculations based on (3.28), (3.13) and (3.3), we have obtained the phase transition points plotted in Fig. 3(b).

Let us see the phase transition points from the standpoint of the two gap phase. If we substitute $b = \pi$ into (3.16), it becomes (3.28). By using the relationship (3.28), (3.17) at $b = \pi$ is reduced to (3.13). Let us check that the eigenvalue density in the two gap phase (3.18) at $b = \pi$ is smoothly connected to the one in the upper gap phase (3.11). To confirm it, first we apply the following formula

$$
I_1(a, \pi, \alpha) = \frac{\pi}{\cos^2 \frac{\alpha}{2}} \left( \frac{1}{\sin \frac{\alpha}{2} \sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}}} - \frac{1}{\cos \frac{\alpha}{2}} \right), \tag{3.29}
$$

and then by using (3.28) we can rewrite $\rho_{2,tg}$ as

$$
\rho_{2,tg}(\lambda, a, \pi; \alpha) = \frac{1}{2\pi \lambda} - \frac{\zeta}{2\pi^2} \int_{\tilde{c}}^{\infty} dy \frac{ye^{-y} \sin \frac{\alpha}{2} \sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}}}{(1 - e^{-y})\sqrt{1 + 2\cos ae^{-y} + e^{-2y}}} \tag{3.30}
$$

Then by summing up with $\rho_{1,tg}^{r.f}(\zeta, a, \pi, \tilde{c}; \alpha)$, we can see that (3.18) at $b = \pi$ becomes (3.11). So we can see that solutions of two gap phase are smoothly connected to solutions of the upper gap phase.

**Quadruple phase transition point in the regular fermion theory**  There is the quadruple phase transition point

$$
\lambda_{c}^{r.f} = 0.596967, \quad \zeta_{c}^{r.f} = 2.864539029, \quad (\tilde{c}_{c} = 0.644715) \tag{3.31}
$$

at which the no gap, the lower gap, the upper gap and the two gap phase coexist. Let us check whether (3.31) is the quadruple phase transition point or not. Note that (3.21) is
the condition for the phase transition from the no gap to the lower gap phase. Eq. (3.24) is the condition for the phase transition between the no gap and the upper gap, (3.26) is the one between the lower gap and the two gap, and (3.28) is the one between the upper gap and the two gap. So if (3.31) satisfies these four equations simultaneously, it would be the quadruple phase transition point. If we substitute $b = \pi$ into (3.26) it becomes

$$
\zeta \pi^2 \int_0^\infty dy \frac{e^{-y}}{1 - e^{-2y}} = \frac{1}{2\pi \lambda}.
$$

(3.32)

If we substitute $a = 0$ into (3.28), it also becomes the same equation as (3.32). We can also see that if both (3.21) and (3.24) are simultaneously satisfied, (3.32) is also automatically satisfied. So if there is a point satisfying (3.32), the point becomes the quadruple phase transition point. By the study in [1], we already know that only the point (3.31) simultaneously satisfies both conditions (3.21) and (3.24). Hence (3.31) satisfies (3.32). So (3.31) becomes the quadruple phase transition point in the regular fermion theory. Then due to the existence of the quadruple point, the phase structure of the regular fermion theory becomes as in Fig. 2(a). We can also check that it becomes $a = 0$ and $b = \pi$ at (3.31), and then the eigenvalue density functions in each phase (3.8), (3.11) (3.18) and (6.33) of [1] coincide.

3.2 Critical boson theory

In this subsection we will study the higher temperature phases, the lower gap, the upper gap and the two gap phase of the level $kU(N)$ Chern-Simons matter theory coupled to massless critical bosons in the fundamental representation. The low temperature no gap phase is already studied in section 6.2 of [1]. This is a Chern-Simons gauged version of the $U(N)$ Wilson Fisher theory. The Lagrangian of the UV theory is that of massless minimally coupled fundamental bosons deformed by the interaction,

$$
\delta S = \int d^3x A \phi \phi,
$$

(3.33)

where $A$ is a Lagrange multiplier field. The effective potential in this theory is given by (3.12) of [1] and it is

$$
V(U) = - \frac{N^2 \zeta}{6\pi} \sigma^3 + \frac{N^2 \zeta}{2\pi} \int_\sigma^\infty dy \int_\pi^\sigma d\alpha \rho(\alpha) \left( \ln(1 - e^{-y+i\alpha}) + \ln(1 - e^{-y-i\alpha}) \right)
\equiv V^{c.b}[\rho,N],
$$

(3.35)

\footnote{See subsection 4.3 of [1] for details, and in particular eq.(4.35) for the Lagrangian; note [1] employs the symbol $\sigma$ for our field $A$.}

\footnote{Originally in [1], the effective potential is obtained as (3.9) of [1],

$$
v[\rho] = - \frac{N}{6\pi} \left( \sigma^2 + \frac{2(\sigma^2-A\beta^2)^2}{\lambda} \right) + \frac{N}{2\pi} \int_\sigma^\infty ydy \int_\pi^\sigma \rho(\alpha) d\alpha \left( \ln(1 - e^{-y+i\alpha}) + \ln(1 - e^{-y-i\alpha}) \right).
$$

(3.34)

From this, constants $A$ and $\sigma$ are obtained by requirement that they extremize the above (3.34) at constant $\rho$. By this, we yield $A = \sigma^2 T^2$, and the equation (3.36). By substituting these, the effective potential is simplified to be (3.35).}
where \( \sigma \) provides the squared thermal mass as \( \sigma^2 T^2 \). The \( \sigma \) was determined by

\[
\int_{-\pi}^{\pi} \rho(\alpha) \left( \ln \left( 2 \sinh \left( \frac{\sigma - i\alpha}{2} \right) \right) + \ln \left( 2 \sinh \left( \frac{\sigma + i\alpha}{2} \right) \right) \right) = 0. \tag{3.36}
\]

The general form of the free energy of the critical boson theory on \( S^2 \times S^1 \) is given as

\[
F_{\text{c.b}}^N = V_{\text{c.b}}[\rho, N] - N^2 \mathcal{P} \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \, \rho(\alpha) \rho(\beta) \log \left| 2 \sin \frac{\alpha - \beta}{2} \right| \tag{3.37}
\]

\[= V_{\text{c.b}}[\rho, N] + F_2[\rho, N]. \]

To obtain the free energy in each phase, we only have to evaluate the eigenvalue density \( \rho \) and \( \sigma \) in each phase, and substitute to (3.37). Note that \( \sigma \) can be determined if the eigenvalue density is determined. So obtaining the eigenvalue density in each phase is equivalent to obtaining the free energy in each phase.

For later use we will give the form of \( V'(z) \) here. We obtain \( V'(z) \) from (3.35),

\[
V'(z) = \frac{N \zeta}{2\pi} \int_{-\pi}^{\pi} dy \, y \left( \frac{-ie^{-y}}{z^{-1} - e^{-y}} + \frac{ie^{-y}}{z - e^{-y}} \right) \tag{3.38}
\]

where \( z = e^{i\alpha} \).

The phase structure of this theory is depicted in Fig. 4(a). We will elaborate the phase structure in following subsubsections.

### 3.2.1 Lower gap phase

In the lower gap phase of the critical boson theory, we use the same procedure as the one in section 3.1.1 to obtain the eigenvalue density \( \rho \). We use the cut region and the cut function described in appendix A.1. \( U(z) \) is same as \( V'(z) \)

\[
U(z) = V'(z) = \frac{N \zeta}{2\pi} \int_{-\pi}^{\pi} dy \, y \left( \frac{-ie^{-y}}{z^{-1} - e^{-y}} + \frac{ie^{-y}}{z - e^{-y}} \right), \tag{3.39}
\]
because \( \rho_0 = 0 \). By substituting the above into (2.29), we immediately obtain \( H(u) \) as well as \( \Phi(u) \), as

\[
\Phi(u) = \Phi_{c,b}^{v}(\sigma, b, \zeta; u) \equiv \frac{\zeta}{\pi} \int_{0}^{\infty} dy \, y \frac{e^{-y} b(u)(1 - e^{-y})(1 + u^{-1})}{\sqrt{(e^{-y} - e^{ib})(e^{-y} - e^{-ib})(u - e^{-y})(u^{-1} - e^{-y})}}
\]

\[
+ \frac{\zeta}{\pi} \left( \int_{0}^{\infty} dy \frac{e^{-y}}{u - e^{-y}} - \int_{\sigma}^{\infty} dy \frac{e^{-y}}{u^{-1} - e^{-y}} \right).
\]

(3.40)

From \( \Phi^+(u) - \Phi^-(u) = 4\pi \rho(u) \) at the cut \( u = e^{i\alpha} \) with \(-b \leq \alpha \leq b\), we obtain the eigenvalue density in the lower gap phase as

\[
\rho(\alpha) = \frac{\zeta}{\sqrt{2\pi^2}} \sqrt{\frac{\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2}}{2}} \int_{\sigma}^{\infty} dy \, \frac{y \sinh \frac{\alpha}{2}}{\sqrt{\cosh y - \cos b(cosh y - \cos \alpha)}}
\]

\[
\equiv \rho_{c,b}^{v}(\zeta, \lambda; \sigma, b; \alpha).
\]

(3.41)

At the lower gap phase we get \( \rho(\alpha) = 0 \) in \( \pi > |\alpha| > b \). By substituting the eigenvalue density (3.41) into (3.36) and (3.37), we can obtain the free energy in the lower gap phase in the critical boson theory.

The condition \( \lim_{u \to \infty} \Phi(u) = 1 \) requires the following condition

\[
M_{c,b}^{v}(\zeta, \sigma, b) \equiv -\frac{\zeta}{\pi} \int_{0}^{e^{-\sigma}} dx \left( \frac{\log x}{x} - \frac{1 - x}{\sqrt{x^2 - 2x \cos b + 1}} \right) = 1.
\]

(3.42)

By (3.42) and (3.36), we can obtain \( (b, \sigma) \) as functions of \( (\lambda, \zeta) \) as \( (b, \sigma) = (b(\lambda, \zeta), \sigma(\lambda, \zeta)) \). (3.41), (3.42) and (3.36) provide a set of complete solutions in the lower gap phase of the critical boson theory.

### 3.2.2 Upper gap phase

Next we will search for an upper gap solution in the critical boson theory. The cut region and the cut function \( h(u) \) in this case are defined in appendix A.2. Here the procedure to obtain the eigenvalue density function \( \rho \) is the same as the one in subsection 3.1.2. We will take \( \rho_0(\alpha) = \frac{1}{2\pi \lambda} \neq 0 \) at the upper gap region \(-a \leq \alpha \leq a\).

Functions \( H(u) \) and \( \Phi(u) \) are obtained based on (2.24),

\[
\Phi(u) = \Phi_{c,b}^{v}(\sigma, a, \zeta; u) + \Phi_{a,b}^{v}(\lambda, a; u), \quad \text{where}
\]

\[
\Phi_{c,b}^{v}(\sigma, a, \zeta; u) \equiv \frac{\zeta}{\pi} \int_{0}^{\infty} dy \, y e^{-y} h(u)(1 + e^{-y})(1 - u^{-1})
\]

\[
\frac{\sqrt{(e^{-y} - e^{ia})(e^{-y} - e^{-ia})(u - e^{-y})(u - e^{-y})}}{\sqrt{(e^{-y} - e^{ia})(e^{-y} - e^{-ia})(u - e^{-y})(u - e^{-y})}}
\]

\[
+ \frac{\zeta}{\pi} \left( \int_{0}^{\infty} dy \frac{y e^{-y}}{u - e^{-y}} - \int_{\sigma}^{\infty} dy \frac{y e^{-y}}{u^{-1} - e^{-y}} \right).
\]

(3.43)

Here we use the same definition of \( \Phi_{a,b}^{v}(\lambda, a; u) \) as (3.10). From (3.43), by taking \( \Phi^+(u) - \Phi^-(u) = 4\pi(\rho(u) - \rho_0(\alpha)) \) at the cut region \( u = e^{i\alpha} \) with \( \pi \geq |\alpha| \geq a \), we can obtain the eigenvalue density function as

\[
\rho(\alpha) = \frac{1}{2\pi \lambda} - \frac{\zeta}{\sqrt{2\pi^2}} \sqrt{\frac{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}}{2}} \int_{\sigma}^{\infty} dy \frac{y |\sin \frac{\alpha}{2}|}{\sqrt{\cosh y - \cos a(cosh y - \cos \alpha)}}
\]

\[
\equiv \rho_{a,b}^{v}(\zeta, \zeta; \sigma, a; \alpha).
\]

(3.44)
In the region $|\alpha| < |a|$, $\rho(\alpha) = \frac{1}{2\pi x}$. By substituting the eigenvalue density (3.44) into (3.36) and (3.37), we can obtain the free energy in the upper gap phase. To derive (3.44), we have used the formula (3.12).

From the condition, $\lim_{u \to \infty} \Phi(u) = 1 - \int_{-\pi}^\pi d\alpha \rho_0(\alpha)$, we obtain the condition

$$M^{c,b}_{ug}(\zeta, \sigma; a) \equiv -\frac{\zeta}{2\pi} \int_0^\infty dx \left( \frac{\log x}{x} - \frac{1 + x}{x} \frac{\log x}{\sqrt{x^2 - 2x \cos a + 1}} \right) = 1 - \frac{1}{\lambda}. \quad (3.45)$$

Here we have used the formula (3.14). By (3.45) and (3.36), we can obtain $(\lambda, \zeta)$ as functions of $(\lambda, \zeta)$ as $(a, \sigma) = (a(\lambda, \zeta), \sigma(\lambda, \zeta))$. (3.44), (3.45) and (3.36) provide a set of complete solutions in the upper gap phase in the critical boson theory.

### 3.2.3 Two gap phase

We will search for a two gap solution in the critical boson theory. Our two cuts and the cut function are defined in appendix A.3. By the same procedure as the one in section 3.1.3, we can obtain $H(u)$ as well as $\Phi(u)$ as

$$\Phi(u) = \Phi^{c,b}_g(\zeta, a, b, \sigma; u) + \Phi^{\rho_0}_g(\lambda, a; u)$$

$$\Phi^{c,b}_g(\zeta, a, b, \sigma; u) \equiv \frac{\zeta}{2\pi} \int_{\sigma}^\infty dy \left( \frac{ye^{-y}(u^{-1} - u)}{(u - e^{-y})(u^{-1} - e^{-y})} + \frac{ye^{-y}(u + e^{-y} - 2u)}{e^{-y}(u - e^{-y})(e^{-y} - u)} \right). \quad (3.46)$$

Here $\Phi^{\rho_0}_g(\lambda, a; b; u)$ is already defined in (3.15). From $\lim_{u \to \infty} \Phi(u) = 1 - \int_{-\pi}^\pi d\alpha \rho_0(\alpha)$, we obtain following two conditions,

$$\frac{1}{4\pi \lambda} \Upsilon(a, b) = \frac{\zeta}{2\pi} \Upsilon^{c,b}(a, b, \sigma)$$

$$\Upsilon^{c,b}(a, b, \sigma) \equiv \int_{\sigma}^\infty dy \frac{y}{(\cosh y - \cos a)(\cosh y - \cos b)}, \quad (3.47)$$

and

$$\frac{1}{4\pi \lambda} \Lambda(a, b) = 1 + \frac{\zeta}{4\pi} G^{c,b}(a, b, \sigma)$$

$$G^{c,b}(a, b, \sigma) \equiv \int_{\sigma}^\infty dy \frac{e^y + e^{-y}}{(\cosh y - \cos a)(\cosh y - \cos b)} - 2, \quad (3.48)$$

$\Upsilon(a, b)$ and $\Lambda(a, b)$ are already defined in (3.13) and (3.17). By using (3.48), (3.47) and (3.36), $(a, b, \sigma)$ are determined as functions of $(\lambda, \zeta)$ as $(a, \sigma) = (a(\lambda, \zeta), b(\lambda, \zeta), \sigma(\lambda, \zeta))$.

From (3.46) and by taking $\Phi^+(u) - \Phi^-(u) = 4\pi(\rho(u) - \rho_0(u))$ at the cuts $|a| < |\alpha| < |b|$, we obtain the eigenvalue density function as

$$\rho(\alpha) = \rho^{c,b}_g(\alpha) = \rho^{c,b}_{1,1g}(\zeta, a, b, c; \alpha) + \rho^{2,1g}(\lambda, a; b; \alpha), \quad \text{where}$$

$$\rho^{c,b}_{1,1g}(\zeta, a, b, c; \alpha) \equiv -\frac{\zeta}{\pi^2} F(a, b; \alpha), \int_{\xi}^\infty dy \frac{ye^{-y}}{\nu_{c,b}(a, b; y)} \left( \frac{|\sin \alpha|}{\cosh y - \cos \alpha} \right)$$

$$\nu_{c,b}(a, b; y) \equiv \sqrt{(e^{-2y} - 2e^{-y} \cos a + 1)(e^{-2y} - 2e^{-y} \cos b + 1)}. \quad (3.49)$$
Here (3.49) shares the same definition of $\rho_{2, tg}$, $\mathcal{F}$ with (3.18). $\rho_{2, tg}$ is the same functional form as the eigenvalue density in the GWW model (7.6) in [1]. $\rho_{1, c b}^{a, b}$ can be regarded as an additional term depending on the detail of the theory. By substituting the eigenvalue density (3.49) into (3.36) and (3.37), we can obtain the free energy in the two gap phase of the critical boson theory. During the calculation of (3.49), we have used (3.47) also.

The combination of (3.47), (3.48), (3.36) and (3.49) provides a complete set of solutions in the two gap phase.

At large $\zeta$ limit, as we proved in appendix C.1.2, the eigenvalue density approaches the universal distribution (2.16) because $\sigma$ remains finite positive quantity at the limit. In the limit, $a, b$ and the eigenvalue density behave as

$$a = \pi \lambda - \frac{\epsilon}{2}, b = \pi \lambda + \frac{\epsilon}{2},$$

$$\epsilon = 8 \sin(\pi \lambda) \exp \left( -\frac{\sin(\pi \lambda)}{2} \frac{\zeta}{\lambda} \int_{\tilde{c}}^{\infty} dy \frac{y}{\cosh y - \cos \pi \lambda} \right),$$

$$\rho(\alpha) = \frac{1}{\pi^2 \lambda} \cos^{-1} \sqrt{\frac{\alpha - a}{b - a}}. \tag{3.50}$$

At appendix C.2.2, we have demonstrated the above behavior (3.50). As we can see in the appendix, the eigenvalue density is dominated by $\rho_{2, tg}$ at the large $\zeta$ limit. The form of the eigenvalue density approaches $\cos^{-1} \sqrt{\alpha_1}$ function which has the same form as Eq. (7.11) in [1], which is the large $\zeta$ limit of the eigenvalue density of the GWW model. We can also see that the range of the domain of the cut $\epsilon$ is smaller than the one in the GWW type matrix integration at the same value of $(\lambda, \zeta)$.

3.2.4 Phase transition points

Phase transition from the lower gap to the no gap Let us consider the behavior of lower gap solutions (3.42) and (3.41) at $b = \pi$ which would correspond to the phase transition points from the lower gap to the no gap. If we substitute $b = \pi$ into (3.42), it becomes

$$1 = -\frac{\zeta}{\pi} \int_{0}^{e^{-\sigma}} dx \left( \frac{\log x}{1 + x} \right) = \frac{\zeta}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1 + \sigma n}{n^2} e^{-n\sigma}. \tag{3.51}$$

This is exactly same as the condition for the phase transition from the no gap to the lower gap phase which is obtained by substituting $\alpha = \pi$ into eq. (6.24) of [1] and by requiring $\rho(\pi) = 0$. Based on this, if we substitute $b = \pi$ into (3.41), we obtain

$$\rho(\alpha) = \frac{\zeta}{\sqrt{2\pi}^2} \sqrt{\sin^2 \frac{\beta}{2} - \sin^2 \frac{\alpha}{2}} \int_{\sigma}^{\infty} dy \frac{y \sinh \frac{\beta}{2} \cos \frac{\alpha}{2}}{\cosh y - \cos \beta (\cosh y - \cos \alpha)} \Bigg|_{b=\pi}$$

$$= \frac{\zeta}{\pi^2} \int_{\sigma}^{\infty} dy \left( -\frac{1}{2(1 + e^{-y})} + \frac{1}{4(1 - e^{-y + i\alpha})} + \frac{1}{4(1 - e^{-y - i\alpha})} \right)$$

$$= \frac{1}{2\pi} + \frac{\zeta}{2\pi^2} \sum_{n=1}^{\infty} \cos n\alpha \left( \frac{1 + \sigma n}{n^2} \right) e^{-n\sigma}. \tag{3.52}$$

In the last step we have used (3.51), the Taylor expansion and an integration formula (3.23). The last line of (3.52) is exactly same as the eigenvalue distribution in the no gap
Phase, described at Eq. (6.24) in [1]. So, at the phase transition points, the lower gap solutions are smoothly connected to the no gap phase solutions.

**Phase transition from the upper gap to the no gap** Let us consider the behavior of upper gap solutions (3.44) and (3.45) at \(a = 0\), which would correspond to the phase transition points from the upper gap to the no gap phase. If we substitute \(a = 0\) into (3.45), it becomes

\[
1 - \frac{1}{\lambda} = \frac{\zeta}{\pi} \int_0^\infty e^{-\sigma} \log \frac{1}{1-x} = -\frac{\zeta}{\pi} \sum_{n=1}^\infty \left( \frac{1 + \sigma n}{n^2} e^{-n\sigma} \right).
\tag{3.53}
\]

This is exactly the same as the condition for the phase transition from the no gap to the upper gap phase discussed in [1], which is obtained by substituting \(\alpha = 0\) into (6.24) of [1] and by requiring \(\rho(0) = \frac{1}{2\pi \lambda}\).

Based on this, if we substitute \(a = 0\) into (3.44), we can see

\[
\rho(0) = \frac{1}{2\pi \lambda} - \frac{\zeta}{\sqrt{2\pi^2}} \int_{\alpha}^\infty dy \frac{y \sin \frac{\alpha}{2}}{\sqrt{\cosh y - \cos \alpha}} \Bigg|_{a=0} \int_{\sigma}^\infty dy \left( \frac{1}{2(1 - e^{-y})} - \frac{1}{4(1 - e^{-y + i\alpha})} - \frac{1}{4(1 - e^{-y - i\alpha})} \right)
\tag{3.54}
\]

In the last step we have used (3.53), the Taylor expansion and the formula (3.23). The last line of (3.54) is exactly the same as the eigenvalue distribution in the no gap phase, described at eq. (6.24) in [1]. So, at the phase transition points, the upper gap solutions are smoothly connected to the no gap phase solutions.

**Phase transition from the lower gap to the two gap** In the lower gap phase at fixed \(\lambda\), if the maximum of the eigenvalue density \(\rho(0)\) reaches to \(\rho(0) = \frac{1}{2\pi \lambda}\), the phase transition from the lower gap phase to the two gap phase will occur. The conditions would be represented in terms of (3.41) as

\[
\rho(0) = \frac{\zeta}{\pi^2} \sin \frac{b}{2} \int_{\sigma}^\infty dy \frac{y}{\sqrt{2(\cosh y - \cos \beta) \sinh \frac{y}{2}}} = \frac{1}{2\pi \lambda}.
\tag{3.55}
\]

The combination of (3.55), (3.36) and (3.42) provides the phase transition points from the lower gap to the two gap. By numerical calculations based on (3.55), (3.36) and (3.42), we obtain the phase transition points plotted in Fig. 5(a).

Let us see the phase transition points from the standpoint of the two gap phase. If we substitute \(a = 0\) into (3.47), it becomes the same as (3.55), which is the condition of the phase transition points. If we set \(a = 0\) at (3.48), it becomes the same as (3.42). Let us check if the eigenvalue density in the two gap phase (3.49) becomes the one in the lower gap phase (3.41) in \(a = 0\) limit. By using (3.55), we can replace \(\rho_{2tg}(\lambda, 0, b; \alpha)\) of (3.48) by

\[
\rho_{2tg}(\lambda, 0, b; \alpha) = \frac{\zeta}{\pi^2} \int_{\sigma}^\infty dy \frac{y \cos \frac{\alpha}{2} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2}}}{\sqrt{2(\cosh y - \cos \beta) \sinh \frac{y}{2}}}.
\tag{3.56}
\]
Then by summing up with \( \rho^{c,b}_{1,tg}(\zeta, 0, b, \tilde{c}; a) \), (3.49) at \( a = 0 \) becomes (3.41). So, at the phase transition points, we can see that the solutions of two gap phase are smoothly connected to the lower gap phase solutions.

**Phase transition from the upper gap to the two gap** In the upper gap phase, if the minimum of the eigenvalue density \( \rho(\pi) \) reaches to \( \rho(\pi) = 0 \), the phase transition from the upper gap phase to two gap phase will occur. The condition would be represented as

\[
\frac{1}{2\pi \lambda} = \frac{\zeta}{\sqrt{2\pi}^2} \cos \frac{a}{2} \int_{\sigma} \frac{d\sigma}{y} \left( \frac{y \cosh \frac{y}{2}}{\cosh y - \cos a (\cosh y + 1)} \right).
\]  

(3.57)

The combination of (3.57), (3.45) and (3.36) provides the phase transition points from the upper gap to the two gap. By numerical calculations based on (3.57), (3.45) and (3.36), we obtain the phase transition points plotted in Fig. 5(b).

Let us see the phase transition points from the standpoint of the two gap phase. If we substitute \( b = \pi \) into (3.47) it becomes (3.57) which is the phase transition condition. By using (3.57), we can see that (3.48) at \( b = \pi \) is reduced to (3.45). Let us check that the eigenvalue density in the two gap phase (3.49) at \( b = \pi \) is smoothly connected to the one in the upper gap phase (3.44). To confirm it, first we apply the formula (3.29). Then by using (3.57) we can rewrite \( \rho_{2,tg} \) as

\[
\rho_{2,tg}(\lambda, a, \pi; \alpha) = \frac{1}{2\pi \lambda} - \frac{\zeta}{\pi^2} \int_{\sigma} dy \frac{y e^{-y} \sin \frac{\alpha}{2} \sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}}}{(1 + e^{-y}) \sqrt{1 - 2 \cos a e^{-y} + e^{-2y}}},
\]  

(3.58)

Then by summing up with \( \rho^{c,b}_{1,tg}(\zeta, a, \pi, \tilde{c}; a) \), we can see that (3.49) at \( b = \pi \) becomes (3.44). So we can see that the solutions of two gap phase are smoothly connected to solutions of the upper gap phase.

**Quadruple phase transition point in the critical boson theory** There is a quadruple phase transition point

\[
\lambda^{c,b}_c = 0.403033, \quad \zeta^{c,b}_c = 4.24292, \quad (\sigma_c = 0.644715),
\]  

(3.59)
at which the four phases coexist. Let us check whether (3.59) is the quadruple phase transition point or not. Note that (3.51) is the condition for the phase transition from the no gap to the lower gap phase. Eq. (3.53) is the condition for the phase transition between the no gap and the upper gap, (3.55) is the one between the lower gap and the two gap, and (3.57) is the one between the upper gap and the two gap. So if (3.59) satisfies these four equations simultaneously, it becomes the quadruple phase transition point. If we substitute $b = \pi$ into (3.55) it becomes

$$\frac{\zeta}{\pi^2} \int_{\sigma}^{\infty} dy \frac{ye^{-y}}{1 - e^{-2y}} = \frac{1}{2\pi \lambda},$$

(3.60)

and if we substitute $a = 0$ into (3.57), it also becomes the same equation as (3.60). We can also see that if both (3.51) and (3.53) are simultaneously satisfied, (3.60) is also automatically satisfied. So if there is a point satisfying (3.60), the point becomes the quadruple phase transition point. By the study in [1], we already know that only the point (3.59) simultaneously satisfies both condition (3.51) and (3.53). Hence (3.59) satisfies (3.60). So (3.59) becomes the quadruple phase transition point in the critical boson theory. Then due to the existence of the quadruple point, the phase structure of the critical boson theory becomes as in Fig. 4(a). We can also check that it becomes $a = 0$ and $b = \pi$ at (3.59), and then the eigenvalue density functions in each phase (3.41), (3.44), (3.49) and (6.24) of [1] coincide.

### 3.3 Duality between the regular fermion theory and the critical boson theory

In this subsection we will verify the level-rank duality between the level $k \ U(N)$ critical boson theory and the level $k \ U(k-N)$ regular fermion theory. We will show that the quantities at $(\lambda_{c,b}, \zeta_{c,b})$ in the critical boson theory have duality relationships to the quantities at $(\lambda_{r,f}, \zeta_{r,f}) = (1 - \lambda_{c,b}, \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \zeta_{c,b})$ in the regular fermion theory as

$$\rho^{r,f}(\alpha) = \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \left( \frac{1}{2\pi \lambda_{c,b}} - \rho^{c,b}(\alpha + \pi) \right),$$

(3.61)

$$b_{r,f} = \pi - a_{c,b}, \quad a_{r,f} = \pi - b_{c,b}, \quad \tilde{c} = \sigma.$$ 

(3.62)

Here the $\rho^{r,f}$ is the eigenvalue density in the regular fermion theory and $\rho^{c,b}$ is the one in the critical boson theory. The relationship between $(\lambda_{c,b}, \zeta_{c,b})$ and $(\lambda_{r,f}, \zeta_{r,f})$ is

$$\lambda_{r,f} = 1 - \lambda_{c,b}, \quad \zeta_{r,f} = \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \zeta_{c,b}, \quad \left( \frac{N}{k} = \lambda_{c,b}, \quad \frac{k - N}{k} = \lambda_{r,f} \right).$$

(3.63)

If the above duality relationships (3.61) $\sim$ (3.63) are valid, the free energy at $(\lambda_{c,b}, \zeta_{c,b})$ in the critical boson theory agrees with the free energy at $(\lambda_{r,f}, \zeta_{r,f}) = (1 - \lambda_{c,b}, \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \zeta_{c,b})$ in the regular fermion theory. So if we verify the above relationships (3.61) $\sim$ (3.63), it would be the proof of the level-rank duality between the regular fermion theory and the critical boson theory.

In this subsection, we will first demonstrate the relationships (3.61) $\sim$ (3.63) between the lower gap phase of the regular fermion theory and the upper gap phase of the critical boson theory.
boson theory. Next we will verify the relationships between the upper gap phase of the regular fermion theory and the lower gap phase of the critical boson theory. As the third step, we will demonstrate the duality relationships between the two gap phase of both theories. We complete the proof of the relationships (3.61) \sim (3.63) between both theories by these three steps. As the final part of this subsection, we will completely verify the duality by evaluating the free energy in both theories.

### 3.3.1 Between the lower gap phase of regular fermion and upper gap phase of critical boson

Let us define sets of quantities $S_{ub}^{c,b}, S_{lg}^{r,f}$ as

\[
S_{ub}^{c,b} = (\lambda_{c,b}, \zeta_{c,b}; a_{c,b}, \sigma, \rho_{ub}(\alpha)),
\]
\[
S_{lg}^{r,f} = (\lambda_{r,f}, \zeta_{r,f}; b_{r,f}, \tilde{c}, \rho_{lg}(\alpha))
= \left(1 - \lambda_{c,b}, \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \zeta_{c,b} \pi - a_{c,b}, \sigma, \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \left(\frac{1}{2\pi \lambda_{c,b}} - \rho_{ub}(\pi + \alpha)\right)\right).
\]

Note that the eigenvalue density (3.44) and the two equations (3.45), (3.36) provide a complete set of solutions in the upper gap phase of the critical boson model, while the eigenvalue density (3.8) and the two equations (3.9), (3.3) provide a complete set of solutions in the lower gap phase of the regular fermion theory. So we can verify the duality relationships by checking that $S_{lg}^{r,f}$ becomes a solution provided by the three equations in the regular fermion theory if $S_{ub}^{c,b}$ is a solution provided by the three equations in the critical boson side.

First we will confirm the relationship (3.61) between the eigenvalue densities (3.44) and (3.8) under the correspondence (3.63), (3.62). We can show the relationship by directly
substituting the $^{(3.63)}(3.62)$ into (3.8) as

$$
p_{tg}^{r,f} \left( \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \zeta_{c,b}, 1 - \lambda_{c,b}; \sigma, \pi - a_{c,b}; \alpha \right)
= \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \left( \frac{1}{2\pi \lambda_{c,b}} - \rho_{uc,b}^{\zeta_{c,b}} (\zeta_{c,b}, \lambda_{c,b}; \sigma, a_{c,b}; \pi + \alpha) \right).
$$

(3.66)

We can also see that $S_{tg}^{r,f}$ becomes a solution of (3.9) if $S_{ug}^{c,b}$ is a solution of (3.45) by

$$
\tilde{M}_{tg}^{r,f} \left( \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \zeta_{c,b}, \sigma, \pi - a_{c,b} \right) = \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \tilde{M}_{ug}^{c,b} (\zeta_{c,b}, \sigma, a_{c,b})
.$$  

(3.67)

You can check this by a straightforward calculation similar to (3.65).

We will also see that $S_{tg}^{r,f}$ becomes a solution of (3.3) if $S_{ug}^{c,b}$ is a solution of (3.36). This is already verified in section 3.6.2 of [1] in the general case. We can check it as follows:

$$
0 = \int_{-\pi}^{\pi} d\alpha \rho_{uc,b}^{\zeta_{c,b}} (\alpha) \left( \log \left( 2 \sinh \left( \frac{\sigma - i\alpha}{2} \right) \right) + c.c \right)
\Leftrightarrow 0 = \int_{-\pi}^{\pi} d\alpha \left( \frac{1}{2\pi \lambda_{r,f}} - \rho_{tg}^{r,f} (\tilde{\alpha}) \right) \left( \log \left( 2 \cosh \left( \frac{\tilde{\alpha} + i\tilde{\sigma}}{2} \right) \right) + c.c \right)
\Leftrightarrow \tilde{c} = \lambda_{r,f} \int_{-\pi}^{\pi} d\alpha \rho_{tg}^{r,f} (\alpha) \left( \log \left( 2 \cosh \left( \frac{\tilde{\alpha} + i\tilde{\sigma}}{2} \right) \right) + c.c \right).
$$

(3.68)

Here we have used

$$
\int_{0}^{2\pi} d\alpha \left( \log (1 + 2 \cos \alpha e^{-\tilde{\sigma}} + e^{-2\tilde{\sigma}}) = 0 \quad (\tilde{\sigma} > 0) \right)
$$

(3.69)

confirmed by the Taylor expansion and $\int_{0}^{2\pi} d\alpha e^{i\alpha} = 0$

By the above discussion, we can conclude that $S_{tg}^{r,f}$ becomes a solution provided by the three equations (3.8), (3.9) and (3.3) if $S_{ug}^{c,b}$ is a solution provided by the equations (3.44), (3.45), and (3.36). Thus we have verified the duality relationships between the lower gap phase of the regular fermion theory and the upper gap phase of the critical boson theory.

\textsuperscript{9}Steps of the calculation to verify (3.66) are basically combinations of $\cos(x + \pi) = -\cos x$, $\sin \frac{x + \pi}{2} = \cos \frac{x}{2}$, as follows:

$$
\frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \left( \frac{1}{2\pi \lambda_{c,b}} - \rho_{uc,b}^{\zeta_{c,b}} (\zeta_{c,b}, \lambda_{c,b}; \sigma, a_{c,b}; \alpha + \pi) \right)
= \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \left( \sqrt{\frac{\sin^{2} \alpha + \pi}{2} - \sin^{2} \frac{a_{c,b}}{2}} \int_{\alpha}^{\pi} dy \sqrt{\cosh y - \cos a_{c,b}(\cosh y - \cos(\alpha + \pi))} \right)
= \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \left( \sqrt{\frac{\cos^{2} \pi - a_{c,b}}{2} + \cos^{2} \frac{\alpha}{2}} \int_{\alpha}^{\pi} dy \sqrt{\cosh y + \cos(\pi - a_{c,b})(\cosh y + \cos \alpha)} \right)
= \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \left( \sqrt{\frac{\sin^{2} \pi - a_{c,b}}{2} - \sin^{2} \frac{\alpha}{2}} \int_{\alpha}^{\pi} dy \sqrt{\cosh y + \cos(\pi - a_{c,b})(\cosh y + \cos \alpha)} \right)
= \rho_{tg}^{r,f} \left( \frac{\lambda_{c,b}}{1 - \lambda_{c,b}} \zeta_{c,b}, 1 - \lambda_{c,b}; \sigma, \pi - a_{c,b}; \alpha \right).
$$

(3.65)

Basically, almost all steps of calculations to verify the duality relationships are just repeating the similar steps to the above. So from here we will not give a blow-by-blow description of similar steps to avoid being lengthy.
3.3.2 Between upper gap phase of regular fermion and lower gap phase of critical boson

Let us define sets of quantities $S_{c.b}^{tg}, S_{r.f}^{tg}$ as

$$
S_{tg}^{c.b} = (\lambda_{c.b}, \zeta_{c.b}; b_{c.b}, \sigma, \rho_{tg}^{c.b}(\alpha)),
S_{tg}^{r.f} = (\lambda_{r.f}, \zeta_{r.f}; a_{r.f}, \tilde{c}, \rho_{tg}^{r.f}(\alpha))
= \left(1 - \lambda_{c.b}, \frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \zeta_{c.b}; \pi - b_{c.b}, \sigma, \frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \left(\frac{1}{2\pi \lambda_{c.b}} - \rho_{tg}^{c.b}(\pi + \alpha)\right)\right).
$$

(3.70)

By similar reasoning to that in section 3.3.1, we can verify the duality relationships by checking that $S_{tg}^{r.f}$ becomes a solution provided by the three equations (3.11) (3.13) and (3.3) in the regular fermion theory if $S_{tg}^{c.b}$ is a solution provided by the three equations (3.41), (3.42), and (3.36) in the critical boson side.

We will confirm the relationship (3.61) between the eigenvalue densities (3.41) and (3.11) under the correspondence (3.63), (3.62). By a similar calculation to (3.65), we can show the relationship (3.61) as

$$
\rho_{tg}^{r.f} \left(\frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \zeta_{c.b}, 1 - \lambda_{c.b}; \sigma, \pi - b_{c.b}; \alpha\right)
= \frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \left(\frac{1}{2\pi \lambda_{c.b}} - \rho_{tg}^{c.b}(\zeta_{c.b}, \lambda_{c.b}; \sigma, b_{c.b}; \pi + \alpha)\right).
$$

(3.71)

By checking

$$
\tilde{M}_{tg}^{r.f} \left(\frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \zeta_{c.b}, \sigma, \pi - b_{c.b}\right) = -\frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \tilde{M}_{tg}^{c.b} (\zeta_{c.b}, \sigma, b_{c.b})
$$

(3.72)

in a similar way to (3.67), we can also see that $S_{tg}^{r.f}$ becomes a solution of (3.13) if $S_{tg}^{c.b}$ is a solution of (3.42).

By repeating the analogous calculation to (3.68), we can see that $S_{tg}^{r.f}$ becomes a solution of (3.3) if $S_{tg}^{c.b}$ is a solution of (3.36).

So by the above discussion we can conclude that $S_{tg}^{r.f}$ becomes a solution of the three equations (3.11), (3.13) and (3.3) if $S_{tg}^{c.b}$ is a solution of the equations (3.41), (3.42), and (3.36). Thus we have verified the duality relationships between the upper gap phase of the regular fermion theory and the lower gap phase of the critical boson theory.

3.3.3 Between two gap phases of regular fermion and critical boson theory

Let us define sets of quantities $S_{tg}^{c.b}, S_{tg}^{r.f}$ as

$$
S_{tg}^{c.b} = (\lambda_{c.b}, \zeta_{c.b}; a_{c.b}, b_{c.b}, \sigma, \rho_{tg}^{c.b}(\alpha)),
S_{tg}^{r.f} = (\lambda_{r.f}, \zeta_{r.f}; a_{r.f}, b_{r.f}, \tilde{c}, \rho_{tg}^{r.f}(\alpha))
= \left(1 - \lambda_{c.b}, \frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \zeta_{c.b}; \pi - a_{c.b}, \sigma, \frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \left(\frac{1}{2\pi \lambda_{c.b}} - \rho_{tg}^{c.b}(\pi + \alpha)\right)\right).
$$

(3.73)
By similar reasoning to that in section 3.3.1, we can verify the duality relationships by checking that $S_{tg}^{c,f}$ becomes a solution provided by the four equations (3.18), (3.16), (3.17) and (3.3) in the regular fermion theory if $S_{c,b}^{c,b}$ is a solution provided by the four equations (3.49), (3.47), (3.48) and (3.36) in the critical boson side.

**Correspondence (3.61) between the eigenvalue density functions** First we will confirm the relationship (3.61) between the eigenvalue densities (3.49) and (3.18) under the correspondence (3.63), (3.62). We can directly check

$$
\rho_{1,tg}(\lambda_{c.b}, \pi - b_{c.b}, \pi - a_{c.b}; \sigma; \alpha) = -\frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \rho_{1,tg}^{c,b}(\zeta_{c.b}, a_{c.b}, b_{c.b}, \sigma; \pi + \alpha) \quad (3.74)
$$

in a similar calculation to (3.65). From (3.19), we can see

$$
\rho_{1,tg}(1 - \lambda_{c.b}, a_{r.f}, b_{r.f}; \alpha) = \frac{i h^+(u)}{4\pi^2(1 - \lambda_{c.b})} \int_{L_{tg}^{c.f}} \frac{d\omega}{h(\omega)} \left( \frac{\omega + u}{u(\omega - u)} \right),
$$

$$
\rho_{2,tg}(\lambda_{c.b}, \pi - b_{r.f}, \pi - a_{r.f}; \pi + \alpha) = \frac{i h^+(u)}{4\pi^2\lambda_{c.b}} \int_{L_{tg}^{c.f}} \frac{d\omega}{h(\omega)} \left( \frac{\omega + u}{u(\omega - u)} \right). \quad (3.77)
$$

Here $L_{tg}^{r,f}$ denotes the complex line integration along the upper gap running counterclockwise from $e^{-ia_{r.f}}$ to $e^{ia_{r.f}}$. On the other hand $L_{tg}^{r,f}$ denotes the complex line integration along the lower gap running counterclockwise from $e^{ib_{r.f}}$ to $e^{-ib_{r.f}}$. By using the formula proved in appendix B.2.1,

$$
\int_{L_{tg}^{r,f}} \frac{d\omega}{h(\omega)} \frac{\omega + u}{u(\omega - u)} + \int_{L_{tg}^{c.f}} \frac{d\omega}{h(\omega)} \frac{\omega + u}{u(\omega - u)} = \frac{-2\pi i}{h^+(u)}, \quad (3.78)
$$

we can see that

$$
\rho_{2,tg}(1 - \lambda_{c.b}, \pi - b_{c.b}, \pi - a_{c.b}; \alpha) = \frac{1}{2\pi(1 - \lambda_{c.b})} \frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \rho_{2,tg}(\lambda_{c.b}, a_{c.b}, b_{c.b}; \pi + \alpha) \quad (3.79)
$$

where we use $a_{r.f} = \pi - b_{c.b}, b_{r.f} = \pi - a_{c.b}$. From (3.74) and (3.79), we can confirm the relationship (3.61) between the eigenvalue densities (3.49) and (3.18),

$$
\rho_{tg}^{r.f}(\alpha) = \frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \left( \frac{1}{2\pi\lambda_{c.b}} - \rho_{tg}^{c,b}(\alpha + \pi) \right). \quad (3.80)
$$

---

10 We can see it from

$$
I_1(\pi - b, \pi - a; \pi + \alpha) = -2 \int_0^\pi \frac{d\theta}{(\cos \theta - \cos \alpha)\sqrt{\sin^2 \frac{\theta}{2} - \sin^2 \frac{\pi}{2}}(\sin^2 \frac{\theta}{2} - \sin^2 \frac{\pi}{2})}.
$$

$$
\mathcal{F}(\pi - b, \pi - a; \pi + \alpha) = \mathcal{F}(a, b, \alpha).
$$

These provide the complex line integration along the lower gap.
Relationship between (3.47) and (3.16) We will check that $\mathcal{S}_{tg}^{r.f}$ becomes a solution of (3.16) if $\mathcal{S}_{tg}^{c.b}$ is a solution of (3.47). We can easily check

$$\mathcal{Y}^{r.f}(\pi - b_{c,b}, \pi - a_{c,b}, \sigma) = \mathcal{Y}^{c.b}(a_{c,b}, b_{c,b}, \sigma).$$  \hfill (3.81)

From (3.19), by a similar calculation to (3.75), we can see

$$\mathcal{Y}(a_{c,b}, b_{c,b}) = -4i \int_{L_{ug}^{b}} \frac{d\omega}{h(\omega)}, \quad \mathcal{Y}(\pi - b_{c,b}, \pi - a_{c,b}) = 4i \int_{L_{lgs}^{b}} \frac{d\omega}{h(\omega)}. \hfill (3.82)$$

Here $L_{ug}^{b}$ denotes the complex line integration along the upper gap running counterclockwise from $e^{-ia_{c,b}}$ to $e^{ia_{c,b}}$, and $L_{lgs}^{b}$ denotes the one along the lower gap running counterclockwise from $e^{ib_{c,b}}$ to $e^{-ib_{c,b}}$. From the formula proved in appendix B.2.2, we can see

$$\int_{L_{ug}^{b}} d\omega \frac{1}{h(\omega)} = -\int_{L_{lgs}^{b}} d\omega \frac{1}{h(\omega)} \Rightarrow \mathcal{Y}(a_{c,b}, b_{c,b}) = \mathcal{Y}(\pi - b_{c,b}, \pi - a_{c,b}).$$  \hfill (3.83)

Hence by taking into account $\zeta_{c.b} \lambda_{c.b} = (1 - \lambda_{c.b}) \zeta_{r.f}$, we can conclude

$$\frac{1}{4\pi \lambda_{c.b}} \mathcal{Y}(a_{c,b}, b_{c,b}) = \frac{\zeta_{c.b}}{2\pi} \mathcal{Y}^{c.b}(a_{c,b}, b_{c,b}, \sigma) \Rightarrow \frac{1}{4\pi(1 - \lambda_{c.b})} \mathcal{Y}(\pi - b_{c,b}, \pi - a_{c,b}) = \frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \frac{\zeta_{c.b}}{2\pi} \mathcal{Y}^{r.f}(\pi - b_{c,b}, \pi - a_{c,b}, \sigma).$$  \hfill (3.84)

This (3.84) shows that $\mathcal{S}_{tg}^{r.f}$ becomes a solution of (3.16) if $\mathcal{S}_{tg}^{c.b}$ is a solution of (3.47).

Relationship between (3.48) and (3.17) We will verify that $\mathcal{S}_{tg}^{r.f}$ becomes a solution of (3.17) if $\mathcal{S}_{tg}^{c.b}$ is a solution of (3.48). By a straightforward calculation, we can see

$$\mathcal{G}^{r.f}(\pi - b_{c,b}, \pi - a_{c,b}, \sigma) = \mathcal{G}^{c.b}(a_{c,b}, b_{c,b}, \sigma).$$  \hfill (3.85)

From (3.19), by a similar calculation to (3.75), we can see

$$\Lambda(a_{c,b}, b_{c,b}) = -4i \int_{L_{ug}^{b}} \frac{d\omega}{h(\omega)}, \quad \Lambda(\pi - b_{c,b}, \pi - a_{c,b}) = -4i \int_{L_{lgs}^{b}} \frac{d\omega}{h(\omega)}. \hfill (3.86)$$

By using the formula shown in appendix B.2.3,

$$1 = \frac{1}{\pi i} \int_{L_{ug}^{b}} d\omega \frac{\omega}{h(\omega)} + \frac{1}{\pi i} \int_{L_{lgs}^{b}} d\omega \frac{\omega}{h(\omega)},$$  \hfill (3.87)

we can see

$$\frac{1}{4\pi \lambda_{c.b}} \Lambda(a_{c,b}, b_{c,b}) = 1 + \frac{\zeta_{c.b}}{4\pi} \mathcal{G}^{c.b}(a_{c,b}, b_{c,b}, \sigma) \Rightarrow \frac{1}{4\pi(1 - \lambda_{c.b})} \Lambda(\pi - b_{c,b}, \pi - a_{c,b}) = 1 - \frac{\lambda_{c.b}}{1 - \lambda_{c.b}} \frac{\zeta_{c.b}}{4\pi} \mathcal{G}^{r.f}(\pi - b_{c,b}, \pi - a_{c,b}, \sigma).$$  \hfill (3.88)

This (3.88) shows that $\mathcal{S}_{tg}^{r.f}$ becomes a solution of the (3.17) if $\mathcal{S}_{tg}^{c.b}$ is a solution of (3.48).
Correspondence between (3.3) and (3.36): By repeating the analogous calculation to (3.68), we can see that $S^{r,f}_{tg}$ becomes a solution of (3.3) if $S^{c,b}_{tg}$ is a solution of (3.36).

Summary of the proof: So by the above discussion, we can conclude that $S^{r,f}_{tg}$ becomes a solution provided by the four equations (3.18), (3.16), (3.17) and (3.3) in the regular fermion theory if $S^{c,b}_{tg}$ is a solution provided by the four equations (3.48), (3.47), (3.48) and (3.36) at the critical boson side. Thus we have verified the duality relationships at the two gap phase between the regular fermion theory and the critical boson theory.

3.3.4 Relationships between the phase transition points

We have seen that the conditions for the phase transitions (3.21), (3.24), (3.57) etc., can be obtained as a certain limit of corresponding equations between which we have already seen the duality relationships. So we have already shown the duality relationship between the phase transition points, in fact. For example, (3.55) is obtained by $a_{c,b} \rightarrow 0$ limit of the equation (3.47) and (3.28) is obtained by $b_{r,f} \rightarrow \pi$ limit of (3.16). We have already shown that (3.16) maps to (3.47) under the duality, moreover we can see $a_{c,b} = \pi - b_{r,f} = \pi - \pi = 0$. Hence the phase transition points between the upper gap to two gap in the regular fermion theory as (A,B,RF), and the transition points in the critical boson theory as (A,B,CB). We can see following dual relationships represented by these symbols

$$(\text{No,Low,RF}) \leftrightarrow (\text{No,Up,CF}), \quad (\text{No,Up,RF}) \leftrightarrow (\text{No,Low,CF}), \quad (\text{Low,Two,RF}) \leftrightarrow (\text{Up,Two,CF}), \quad (\text{Up,Two,RF}) \leftrightarrow (\text{Low,Two,CF}).$$

(3.89)

3.3.5 Free energy and completing the proof of the duality

Now we have proved that for any value of $(\lambda_{c,b}, \zeta_{c,b})$, there are duality relationships (3.61)~(3.63) between the critical boson theory and the regular fermion theory.

Under these correspondences (3.61)~(3.63), we will show the free energy of the level $k$ $U(N)$ critical boson theory at $(\lambda_{c,b}, \zeta_{c,b})$ (see (3.37)) agrees with the free energy of the level $k$ $U(k-N)$ regular fermion theory at $(\lambda_{r,f}, \zeta_{r,f}) = (1 - \lambda_{c,b}, \frac{\lambda_{c,b} \zeta_{c,b}}{1 - \lambda_{c,b}})$ (see (3.4)),

$$F^N_{c,b} = V^{c,b}[\rho^{c,b}, N] + F_2[\rho^{c,b}, N] = V^{r,f}[\rho^{r,f}, k-N] + F_2[\rho^{r,f}, k-N] = F^{k-N}_{r,f}$$

(3.90)

where $\rho^{r,f}$ can be represented by $\rho^{c,b}$ through the relationship (3.61).

As already shown in (3.37) of [1], by a straightforward calculation similar to (3.68), we can show $V^{c,b}[\rho^{c,b}, N] = V^{r,f}[\rho^{r,f}, k-N]$.

Under (3.61), $F_2[\rho^{r,f}, k-N]$ is expanded in terms of $\rho^{c,b}$ as

$$F_2[\rho^{r,f}, k-N] = -N^2P \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \rho^{c,b}(\pi + \alpha)\rho^{c,b}(\pi + \beta) \log \left| 2 \sin \frac{\alpha - \beta}{2} \right|$$

$$+ 2N^2 \left( \frac{1}{2\pi \lambda_{c,b}} \right)^2 P \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \rho^{c,b}(\pi + \beta) \log \left| 2 \sin \frac{\alpha - \beta}{2} \right|$$

$$- N^2 \left( \frac{1}{2\pi \lambda_{c,b}} \right)^2 P \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \log \left| 2 \sin \frac{\alpha - \beta}{2} \right|. \quad (3.91)$$
By using the periodicity of $\rho^{c.b}(\alpha) = \rho^{c.b}(2\pi + \alpha)$ and by using
\[
\mathcal{P} \int_{-\pi}^{\pi} d\alpha \; \log \left| 2 \sin \frac{\alpha - \beta}{2} \right| = 0,
\]
we can check
\[
F_2[\rho^{r.f}, k - N] = F_2[\rho^{c.b}, N].
\]
So we can see
\[
F^N_{c.b} = F^{k-N}_{r.f},
\]
under the relationships (3.61)–(3.63). Now we have completed the proof of the level-rank duality between the level $k U(N)$ critical boson theory and the level $k U(k - N)$ regular fermion theory.

4. Higher temperature phases of the Supersymmetric Chern-Simons matter theory

Next we will study the higher temperature phases of the $\mathcal{N} = 2$ supersymmetric level $k U(N)$ Chern-Simons matter theory with a single fundamental chiral multiplet. From here we will call the theory “the SUSY CS matter theory” or “the SUSY theory”. The Lagrangian of the theory we study is presented in equations (2.1) and (3.58) of [25] (see also (6.1) and (6.20) of [9]). From (3.21) of [1], the effective potential $V(U)$ in this theory is obtained as
\[
V(U) = -\frac{N^2 \zeta}{6\pi |\lambda|} \left( \tilde{c}^3 - 6|\lambda| \int_{-\pi}^{\pi} d\alpha \rho(\alpha) \text{Re} \int_{\tilde{c}}^{\infty} dy y \log \tanh \frac{y + i\alpha}{2} \right)
\equiv V^{su}[\rho, N],
\]
where the thermal mass (for both the boson as well as the fermion in the supermultiplet) is denoted by $\tilde{c}T$. The constant $\tilde{c}$ above is determined by the requirement that it extremizes $V(U)$; i.e.
\[
\tilde{c} = 2 \left| \text{Re} \int_{-\pi}^{\pi} d\alpha \lambda \rho(\alpha) \log \cosh \frac{\tilde{c} + i\alpha}{2} \right|.
\]
The effective potential can be interpreted as the sum of the regular fermion potential (3.1) and the critical boson potential (3.35). So the potential can be written as
\[
V(U) = V_{r.f}(U) + V_{c.b}(U)
\]
where
\[
V_{r.f}(U) = -\frac{N^2 \zeta}{6\pi} \left( \frac{\tilde{c}^3}{\lambda} - \tilde{c}^3 + 3 \int_{-\pi}^{\pi} d\alpha \rho(\alpha) \int_{\tilde{c}}^{\infty} dy \left( \ln(1 + e^{-y-i\alpha}) + \ln(1 + e^{y+i\alpha}) \right) \right),
\]
\[
V_{c.b}(U) = -\frac{N^2 \zeta}{6\pi} \tilde{c}^3 + \frac{N^2 \zeta}{2\pi} \int_{\tilde{c}}^{\infty} dy \int_{-\pi}^{\pi} d\alpha \ \rho(\alpha) \left( \ln(1 - e^{-y+i\alpha}) + \ln(1 - e^{-y-i\alpha}) \right).
\]
Figure 6: Phase diagram of the SUSY CS matter theory. Here \((\lambda_c, \zeta_c) = (0.5, 1.62509)\) is the quadruple phase transition point where the four phases (no gap, lower gap, upper gap, two gap phases) coexist.

Hence it is convenient to utilize and combine the results in the previous section to obtain the eigenvalue densities and the free energy in the \(N = 2\) supersymmetric case. The general form of the free energy of the supersymmetric CS matter theory on \(S^2 \times S^1\) is given as

\[
F_{su}^N = V^{su}[\rho, N] + F_2[\rho, N]. \tag{4.6}
\]

As in the other CS matter theories, to obtain the free energy in each phase, we only have to evaluate the eigenvalue density \(\rho\) in each phase and substitute them into (4.6) and (4.2). So obtaining the eigenvalue densities in each phase is equivalent to obtaining the free energies in each phase.

For later use we will give the form of \(V'(z)\) here. We obtain \(V'(z) = V_{r,f}'(z) + V_{c,b}'(z)\) from (4.7) and (1.3) as

\[
\begin{align*}
V_{r,f}'(z) &= -\frac{N\zeta}{2\pi} \int_{\tilde{c}}^{\infty} dy \left( \frac{-ie^{-y}}{z+e^{-y}} + \frac{ie^{-y}}{z^{-1}+e^{-y}} \right), \\
V_{c,b}'(z) &= \frac{N\zeta}{2\pi} \int_{\tilde{c}}^{\infty} dy \left( \frac{-ie^{-y}}{z^{-1}-e^{-y}} + \frac{ie^{-y}}{z-e^{-y}} \right). \tag{4.7}
\end{align*}
\]

The phase structure of this theory is depicted in Fig. 6(a), as obtained in following subsubsections.

4.1 Lower gap phase

In the lower gap phase of the SUSY CS matter theory, we employ the same procedure as the one in section 3.1.1 to obtain the eigenvalue density \(\rho\). We use the cut region and the cut function defined in appendix A.1. Because \(\rho_0 = 0\), \(U(z)\) becomes the same as \(V'(z)\), and we can immediately obtain \(H(u)\) as well as \(\Phi(u) = h(u)H(u)\) by substituting the above (4.7) into (2.29). Since (2.29) is linear in \(V'(u) = U(u)\), the \(\Phi(u)\) is obtained as the sum of (3.7) and (3.41) with \(\sigma = \tilde{c}\). So \(\Phi(u)\) is obtained as

\[
\Phi(u) = \Phi_{ig}^{r,f}(\tilde{c}, b, \zeta; u) + \Phi_{ig}^{c,b}(\tilde{c}, b, \zeta; u) \equiv \Phi_{ig}^{su}(\tilde{c}, b, \zeta; u). \tag{4.8}
\]
From $\Phi^+(u) - \Phi^-(u) = 4\pi \rho(u)$ at the cut $u = e^{i\alpha}$ with $-b \leq \alpha \leq b$, we obtain the eigenvalue density in the lower gap phase as

$$
\rho(\alpha) = \rho_{lg}^{r,f}(\zeta, \lambda; \tilde{c}, b; \alpha) + \rho_{lg}^{r,b}(\zeta, \lambda; \tilde{c}, b; \alpha) = \rho_{lg}^{\pi}(\zeta, \lambda; \tilde{c}, b; \alpha),
$$

(4.9)

and $\rho(\alpha) = 0$ for $\pi > |\alpha| > b$. This is the sum of (3.8) and (3.41). By substituting the eigenvalue density (4.9) into (4.2) and (4.6), we can obtain the free energy in the lower gap phase.

The condition $\lim_{u \to \infty} \Phi(u) = 1$ requires following equation

$$
\tilde{M}_{ug}^{su}(\zeta, \tilde{c}, b) \equiv \tilde{M}_{lg}^{r,f}(\zeta, \tilde{c}, b) + \tilde{M}_{lg}^{r,b}(\zeta, \tilde{c}, b) = 1,
$$

(4.10)

where $\tilde{M}_{lg}^{r,f}$ and $\tilde{M}_{lg}^{r,b}$ are defined in (3.9) and (3.42).

By (4.10) and (4.2), we can obtain $(b, \tilde{c})$ as functions of $(\lambda, \zeta)$ as $(b, \tilde{c}) = (b(\lambda, \zeta), \tilde{c}(\lambda, \zeta))$. (4.9), (4.10) and (4.2) provide a set of complete solutions in the lower gap phase of the SUSY theory.

### 4.2 Upper gap phase

To look for the upper gap solution in the current supersymmetric CS matter theory, we only have to follow the same procedure employed in sections 3.1.2 and 3.2.2. The upper cut region as well as the cut function are described in appendix A.2. The function $H(u)$ as well as $\Phi(u)$ are obtained by using (2.29), and the function $\Phi(u)$ here is evaluated as

$$
\Phi(u) = \Phi_{ug}^{\pi}(\lambda, a; u) + \Phi_{ug}^{r,f}(\tilde{c}, a, \zeta; u) + \Phi_{ug}^{r,b}(\tilde{c}, a, \zeta; u).
$$

(4.11)

Here $\Phi_{ug}^{r,f}(\tilde{c}, a, \zeta; u)$ and $\Phi_{ug}^{\pi}(\lambda, a; u)$ are defined in (3.10), $\Phi_{ug}^{r,b}(\tilde{c}, a, \zeta; u)$ is given in (3.43). From (4.11), by taking $\Phi^+(u) - \Phi^-(u) = 4\pi (\rho(u) - \rho_0(u))$ at the cut region $u = e^{i\alpha}$ with $\pi \geq |\alpha| \geq a$, we obtain the eigenvalue density function as

$$
\rho(\alpha) = \rho_{ug}^{r,f}(\zeta, \lambda; \tilde{c}, a; \alpha) + \rho_{ug}^{r,b}(\zeta, \lambda; \tilde{c}, a; \alpha) - \frac{1}{2\pi \lambda} \equiv \rho_{ug}^{\pi}(\zeta, \lambda, \tilde{c}, a; \alpha),
$$

(4.12)

where $\rho_{ug}^{r,f}$ and $\rho_{ug}^{r,b}$ are defined in (3.11) and (3.44) respectively. In the region $|\alpha| < |a|$, $\rho(\alpha) = \frac{1}{2\pi \lambda}$. By substituting the eigenvalue density (4.12) into (4.2) and (4.6), we can obtain the free energy in the upper gap phase.

From the condition, $\lim_{u \to \infty} \Phi(u) = 1 - \int_{-\pi}^{\pi} \frac{1}{\lambda} d\alpha\rho_0(\alpha)$, we obtain the equation

$$
1 - \frac{1}{\lambda} = \tilde{M}_{ug}^{r,f}(\zeta, \tilde{c}, a) + \tilde{M}_{ug}^{r,b}(\zeta, \tilde{c}, a) \equiv \tilde{M}_{ug}^{su}(\zeta, \tilde{c}, a)
$$

(4.13)

where $\tilde{M}_{ug}^{r,f}(\zeta, \tilde{c}, a)$ and $\tilde{M}_{ug}^{r,b}(\zeta, \tilde{c}, a)$ are defined in (3.13) and (3.45) respectively.

By (4.13) and (4.2), we obtain $(a, \tilde{c})$ as functions of $(\lambda, \zeta)$ as $(a, \tilde{c}) = (a(\lambda, \zeta), \tilde{c}(\lambda, \zeta))$. (4.12), (4.13) and (4.2) provide a complete set of solutions in the upper gap phase in the $\mathcal{N} = 2$ supersymmetric CS matter theory.
4.3 Two gap phase

We will search for a two gap phase solution. We only have to follow the same procedure as the one in section 3.1.3 and 3.2.3. We will take the two cuts region and the cut function defined in appendix A.3. By the same procedure as section 3.1.3, we obtain the functions $H(u)$ and $\Phi(u)$ as

$$\Phi(u) = \Phi_{\text{r.f}}(\zeta, a, b, \tilde{c}; u) + \Phi_{\text{c.b}}(\zeta, a, b, \tilde{c}; u) + \Phi_{\rho_0}(\lambda, a, b; u).$$

(4.14)

Here $\Phi_{\text{r.f}}(\zeta, a, b, \tilde{c}; u)$ and $\Phi_{\rho_0}(\lambda, a, b; u)$ are defined in (3.15), and $\Phi_{\text{c.b}}(\zeta, a, b, \tilde{c}; u)$ is given in (3.46).

From this (4.14), by taking $\Phi^+(u) - \Phi^-(u) = 4\pi (\rho(u) - \rho_0(u))$ at the cuts $|a| < |\alpha| < |b|$, we obtain the eigenvalue density function as

$$\rho(\alpha) = \rho_{\text{ru}}(\alpha) = \rho_{2, ty}(\lambda, a, b; \alpha) + \rho_{1, ty}^r(\zeta, a, b, \tilde{c}; \alpha) + \rho_{1, ty}^c(\zeta, a, b, \tilde{c}; \alpha) \equiv \rho_{\text{ru}}(\zeta, \lambda; a, b, \tilde{c}; \alpha).$$

(4.15)

At $b \leq |\alpha| \leq \pi$, $\rho(\alpha) = 0$ and at $|\alpha| \leq a$, $\rho(\alpha) = \frac{1}{2\pi} \rho_{2, ty}(\lambda, a, b; \alpha)$ and $\rho_{1, ty}^r(\zeta, a, b, \tilde{c}; \alpha)$ are defined in (3.18) and $\rho_{1, ty}^c(\zeta, a, b, \tilde{c}; \alpha)$ is defined in (3.49). Similar to (3.18) and (3.49), (4.15) can be interpreted as the sum of GWW type eigenvalue density $\rho_{\text{ru}}$ and other additional terms depending on the details of the theory. By substituting the eigenvalue density (4.15) into (4.2) and (4.6), we can obtain the free energy in the two gap phase of the $\mathcal{N} = 2$ supersymmetric CS matter theory.

From $\lim_{u \to \infty} \Phi(u) = 1 - \int_{-\pi}^\pi d\alpha \rho_0(\alpha)$, we obtain following two conditions,

$$\frac{1}{4\pi} \Upsilon(a, b) = \frac{\zeta}{2\pi} \Upsilon^{\text{ru}}(a, b, \tilde{c})$$

(4.16)

$$\Upsilon^{\text{ru}}(a, b, \tilde{c}) \equiv \Upsilon^r(a, b, \tilde{c}) + \Upsilon^c(a, b, \tilde{c}),$$

and

$$\frac{1}{4\pi} \Lambda(a, b) = 1 + \frac{\zeta}{4\pi} G^{\text{ru}}(a, b, \tilde{c})$$

$$G^{\text{ru}}(a, b, \tilde{c}) \equiv G^c(a, b, \tilde{c}) - G^r(a, b, \tilde{c}).$$

(4.17)

Here $\Upsilon(a, b)$ and $\Upsilon^r$ are defined in (3.16), and $\Upsilon^c$ is in (3.47). $\Lambda(a, b)$ and $G^r$ are defined in (3.17) and $G^c$ is in (3.48).

By using (4.17), (4.16) and (4.2), $(a, b, \tilde{c})$ are determined as functions of $(\lambda, \zeta)$ as $(a, b, \tilde{c}) = (a(\lambda, \zeta), b(\lambda, \zeta), \tilde{c}(\lambda, \zeta))$. The combination of (4.17), (4.16), (4.2) and (4.14) provide a complete set of solutions in the two gap phase of the $\mathcal{N} = 2$ SUSY CS matter theory.

At the large $\zeta$ limit, as we prove in appendix C.1.3, the eigenvalue density approaches the universal distribution (2.16) because $\tilde{c}$ remains finite positive quantity at the limit. In
the limit, \(a, b\) and the eigenvalue density behave as

\[
\begin{align*}
a &= \pi \lambda - \frac{\epsilon}{2}, \\
b &= \pi \lambda + \frac{\epsilon}{2}, \\
\epsilon &= 8 \sin(\pi \lambda) \exp\left(-\sin(\pi \lambda) - \frac{\zeta \lambda}{2} \int_\infty^{\infty} \frac{\cosh y}{\cosh^2 y - \cos^2 \pi \lambda} \right) + \ldots, \\
\rho(\alpha) &= \frac{1}{\pi^2 \lambda} \cos^{-1} \sqrt{\frac{\alpha - a}{b - a}}. \tag{4.18}
\end{align*}
\]

In appendix C.2.3, we have demonstrated the above behavior (4.18). The eigenvalue density function (4.15) is dominated by \(\rho_{2ty}\), and approaches \(\cos^{-1} \sqrt{\alpha} \), which is the same functional form as (7.11) in [1]. The Eq. (7.11) is the large \(\zeta\) limit of the eigenvalue density of the GWW model. We can also see that the range of the domain of the cut \(\epsilon\) is smaller than the one in the regular fermion theory as well as the one in the critical boson theory at the same values of \((\lambda, \zeta)\).

### 4.4 Phase transition points

#### Phase transition from the lower gap to the no gap

Let us consider the behavior of lower gap equations (4.10) and (4.9) at \(b = \pi\) which would correspond to the phase transition points from the lower gap to the no gap. If we substitute \(b = \pi\) into (4.10), it becomes

\[
1 = -2 \frac{\zeta}{\pi} \int_0^{\infty} \frac{\log x}{1 - x^2} = 2 \frac{\zeta}{\pi} \sum_{n=0}^{\infty} \frac{1 + \tilde{c}(2n + 1)}{(2n + 1)^2} e^{-(2n+1)\tilde{c}}. \tag{4.19}
\]

This is exactly same as the condition for the phase transition from the no gap to the lower gap phase which is obtained by substituting \(\alpha = \pi\) into eq. (6.15) of [1] and by requiring \(\rho(\pi) = 0\). By using the condition (4.19), if we substitute \(b = \pi\) into (4.9), we obtain

\[
\rho_{sg}^\alpha(\zeta, \lambda, \tilde{c}, \pi; \alpha) = \frac{1}{2\pi} + \frac{\zeta}{\pi^2} \sum_{n=0}^{\infty} \cos(2n + 1) \alpha \frac{1 + \tilde{c}(2n + 1)}{(2n + 1)^2} e^{-(2n+1)\tilde{c}}. \tag{4.20}
\]

This is the same as the eigenvalue density function at the no gap phase of the supersymmetric CS matter theory, which is described by (6.15) of [1]. Also this can be written as the sum of (3.22) and (3.52). So the lower gap solutions are smoothly connected to the ones in the no gap phase at the phase transition points.

#### Phase transition from the upper gap to the no gap

Let us consider the behavior of upper gap solutions (4.13) and (4.12) at \(a = 0\) which would correspond to the phase transition points from the upper gap to the no gap phase. If we substitute \(a = 0\) into (4.13), it becomes

\[
1 - \frac{1}{\lambda} = \frac{2 \zeta}{\pi} \int_0^{\infty} \frac{\log x}{1 - x^2} = \frac{2 \zeta}{\pi} \sum_{n=0}^{\infty} \frac{1 + \tilde{c}(2n + 1)}{(2n + 1)^2} e^{-(2n+1)\tilde{c}}. \tag{4.21}
\]

This is equivalent to the condition for the phase transition from the no gap to the upper gap phase discussed in [1], which is obtained by substituting \(\alpha = 0\) into (6.15) of [1] and
Figure 7: These are the plots of phase transition points in the SUSY CS matter theory. Fig. 7(a) shows the plots of the ones from the lower gap to the two gap, and Fig. 7(b) is the ones from the upper gap to the two gap phase.

by requiring $\rho(0) = \frac{1}{2\pi\lambda}$. By using (4.21), the eigenvalue density (4.12) at $a = 0$ becomes

$$\rho_{\text{ug}}(\zeta, \lambda, \tilde{c}, 0; \alpha) = \frac{1}{2\pi} + \frac{\zeta}{\pi^2} \sum_{n=0}^{\infty} \cos(2n+1) \alpha \left( \frac{1 + \tilde{c}(2n+1)}{(2n+1)^2} \right) e^{-\frac{2n+1}{2}\tilde{c}}. \quad (4.22)$$

This is the same as the eigenvalue density function at the no gap phase of the SUSY CS matter theory, which is described by (6.15) of [1]. So, at the phase transition points, the upper gap solutions are smoothly connected to the no gap phase solutions.

**Phase transition from the lower gap to the two gap** In the lower gap phase at fixed $\lambda$, if the maximum of the eigenvalue density $\rho(0)$ reaches to $\rho(0) = \frac{1}{2\pi\lambda}$, the phase transition from the lower gap phase to the two gap phase will occur. The conditions are represented in terms of (4.9) as

$$\frac{1}{\lambda} = \sqrt{2\zeta \frac{\pi}{a}} \int_{\tilde{c}}^{\infty} dy \left( \frac{y \sin \frac{b}{2} \cosh \frac{y}{2}}{(\cosh y + 1) \sqrt{(\cosh y + \cos b)}} + \frac{y \sin \frac{b}{2} \sinh \frac{y}{2}}{(\cosh y - 1) \sqrt{(\cosh y - \cos b)}} \right). \quad (4.23)$$

The combination of (4.23), (4.2) and (4.11) provide the phase transition points from the lower gap to the two gap. By numerical calculations based on (4.23), (4.2) and (4.11), we obtain the phase transition points plotted in Fig. 7(a).

Let us examine the phase transition points from the standpoint of the two gap phase. If we substitute $a = 0$ into (4.10), it becomes (4.23), and if we set $a = 0$ into (4.17), it becomes the same as (4.10). Let us check whether the eigenvalue density in the two gap phase (4.11) becomes the one in the lower gap phase (4.9) in $a = 0$ limit. By using (4.23) and by similar calculations to (3.27) and (3.56), we can confirm that (4.11) at $a = 0$ becomes (4.9). So we can see that the solutions in the two gap phase are smoothly connected to the ones in the lower gap phase at the phase transition points.

**Phase transition from the upper gap to the two gap** In the upper gap phase at fixed $\lambda$, if the minimum of the eigenvalue density $\rho(\pi) = \rho(-\pi)$ reaches to $\rho(\pi) = 0$, the
The phase transition from the upper gap to the two gap phase will occur. The condition is represented in terms of (4.12) as
\[
\frac{1}{\lambda} = \frac{\zeta}{\pi} \int_{\tilde{c}}^{\infty} dy \left( \frac{\sqrt{2}y \cos \frac{a}{2} \sinh \frac{b}{2}}{\sqrt{\cosh y + \cos a(\cosh y - 1)}} + \frac{\sqrt{2}y \cos \frac{a}{2} \cosh \frac{b}{2}}{\sqrt{\cosh y - \cos a(\cosh y + 1)}} \right).
\] (4.24)

The combination of (4.24), (4.2) and (4.13) provide the phase transition points from the upper gap to the two gap. By numerical calculations based on (4.24), (4.2) and (4.13), we obtain the phase transition points plotted in Fig. 7(b).

Let us examine the phase transition points from the standpoint of the two gap phase. If we substitute $b = \pi$ into (4.16) and (4.17), they become (4.24) and (4.13) respectively. By using these correspondence and by using the same procedure as the one in section 3.1.3 and 3.2.3, we can also check that (4.15) at $b = \pi$ coincides with (4.12). So, at the phase transition points, the solutions in the two gap phase are smoothly connected to the upper gap solutions.

### Quadruple phase transition points in the SUSY CS matter theory
There is a quadruple phase transition point
\[
\lambda_{c}^{\text{susy}} = 0.5, \quad \zeta_{c}^{\text{susy}} = 1.62509, \quad (\tilde{c}_{c}^{\text{susy}} = 0.542933),
\] (4.25)
at which the four phases coexist. Let us check whether (4.25) is the quadruple point. Note that (4.23) is the equation for the phase transition from the lower gap to the two gap. Eq. (4.24) is the equation for the phase transition from the upper gap to two gap, (4.19) is the one from the no gap to the lower gap, and (4.21) is the one from the no gap to the upper gap. So if the point (4.25) simultaneously satisfies the four equations, it is indeed the quadruple point. We can see that at $a = 0, b = \pi$, (4.23) as well as (4.24) become
\[
-\frac{2\zeta}{\pi^2} \int_{0}^{e^{-x}} dx \frac{\log x}{1 - x^2} = \frac{1}{2\pi \lambda}.
\] (4.26)
We can also see that at $\lambda = \frac{1}{2}$, (4.19) and (4.21) become equivalent to (4.26). By the study at (4.21), we already know that both (4.19) and (4.21) are simultaneously satisfied at the point (4.25). So also (4.26) is satisfied at (4.25). Then we see that (4.25) is the quadruple phase transition point in the SUSY CS matter theory. Due to the existence of the quadruple phase transition point, the phase structure becomes as in Fig. 6(b).

### 4.5 Self-duality of the SUSY CS matter theory
We will study the self-duality of the supersymmetric CS matter theory under the Giveon-Kutasov type level-rank duality [13, 14]. The level $k \, U(N)$ SUSY CS matter theory is dual to the same theory with level $k \, U(k - N)$ gauge group. We will show that there are the following relationships between the quantities in the level $k \, U(N)$ theory at $(\lambda, \zeta)$ and the quantities in the level $k \, U(k - N)$ theory at $(\lambda', \zeta')$:
\[
\bar{\lambda} = 1 - \lambda, \quad \bar{\zeta} = \frac{\lambda}{1 - \lambda}, \quad \left( \frac{N}{k} = \lambda, \quad \frac{k - N}{k} = \bar{\lambda} \right),
\] (4.27)
\[
\bar{\rho}^{su}(\alpha) = \frac{\lambda}{1 - \lambda} \left( \frac{1}{2\pi \lambda} - \rho^{su}(\pi + \alpha) \right),
\]
\[
\bar{b} = \pi - a, \quad \bar{a} = \pi - b, \quad \bar{\tilde{c}} = \tilde{c}.
\]

Quantities in the \( U(k-N) \) theory are denoted with over-lines while the quantities in the \( U(N) \) theory are described without over-lines. First we will show the duality between the quantities in the lower gap phase of the \( U(N) \) theory and the ones in the upper gap phase of the \( U(k-N) \) theory. Next we will demonstrate the duality in the two gap phase between the \( U(N) \) and the \( U(k-N) \) theories. At the end of this subsection, we will show the agreement of the free energy under the duality.

4.5.1 Between the lower gap phase of the \( U(N) \) theory and the upper gap phase of the \( U(k-N) \) theory

Let us define sets of quantities
\[
S_{lg}^{su} = (\lambda, \bar{\lambda}; b, \tilde{c}, \rho_{lg}^{su}),
S_{ug}^{su} = \left( 1 - \lambda, \frac{\lambda}{1 - \lambda} \bar{\lambda}; \bar{b}, \tilde{c}, \frac{\lambda}{1 - \lambda} \rho_{lg}^{su}(\pi + \alpha) \right).
\]

Please note that the the eigenvalue density (4.9) and the two equations (4.10), (4.2) provide a complete set of solutions in the lower gap phase, while the eigenvalue density (4.12) and (4.13), (4.2) provide the one in the upper gap phase. So we can verify the duality relationships by checking that \( S_{ug}^{su} \) becomes a solution provided by the corresponding three equations in the upper gap phase if \( S_{lg}^{c,b} \) is a solution provided by the other three equations in the lower gap.

First we will confirm the relationship (4.28) between the eigenvalue densities (4.9) and (4.12) under the correspondence (4.27), (4.29). Here by utilizing relationship (3.71) and (3.66), we can check
\[
\tilde{M}_{lg}^{su}(\zeta, \tilde{c}, b; \alpha) = -\frac{1 - \lambda}{\lambda} \tilde{M}_{lg}^{su} \left( \frac{\lambda}{1 - \lambda} \zeta, \tilde{c}, b; \pi = \alpha \right).
\]

To show this we have used the combination of (3.67) and (3.68).

We will also see that \( S_{ug}^{c,b} \) becomes a solution of (4.2) if \( S_{lg}^{su} \) is a solution of it. By using (4.31), we can directly check as
\[
\tilde{c} = 2 \left| \text{Re} \int_{-\pi}^{\pi} d\alpha \lambda \rho_{lg}^{su}(\alpha) \log \coth \frac{\tilde{c} + i\alpha}{2} \right|
\]
\[
\Leftrightarrow \tilde{c} = -2 \left| \text{Re} \int_{-\pi}^{\pi} d\alpha (1 - \lambda) \rho_{ug}^{su}(\pi + \alpha) \log \coth \frac{\tilde{c} + i\alpha}{2} \right|
\]
\[
\Leftrightarrow \tilde{c} = -2 \left| \text{Re} \int_{-\pi}^{\pi} d\alpha (1 - \lambda) \rho_{ug}^{su}(\alpha) \log \coth \frac{\tilde{c} + i\alpha}{2} \right|.
\]
Here we have used
\[ \int_{-\pi}^{\pi} d\alpha \log \coth \frac{x + i\alpha}{2} = 0, \quad \coth \left( \frac{x + i\pi}{2} \right) = \tanh \left( \frac{x}{2} \right) = -\coth \left( \frac{x}{2} \right). \] (4.34)

Eq. (4.33) shows that \( S_{tg}^{su} \) becomes a solution of (4.12) if \( S_{tg}^{su} \) is a solution of it.

By the above discussion, we can conclude that \( S_{tg}^{su} \) becomes an upper gap solution provided by the three equations (4.12), (4.13) and (4.2) in the \( U(k-N) \) theory if \( S_{tg}^{su} \) is a lower gap solution provided by the equations (4.9), (4.10), and (4.2) in the \( U(N) \) theory.

Thus we have verified the self-duality relationships between the lower gap phase of the level \( k \) \( U(N) \) SUSY CS matter theory and the upper gap phase of the level \( k \) \( U(k-N) \) SUSY CS matter theory.

### 4.5.2 Self-duality in the two gap phase

Let us define sets of quantities
\[ S_{tg}^{su} = (\lambda, \zeta; b, \tilde{c}, \rho_{tg}^{su}(\alpha)), \]
\[ \bar{S}_{tg}^{su} = (\tilde{\lambda}, \tilde{\zeta}; \tilde{a}, \tilde{\tilde{c}}, \bar{\rho}_{tg}^{su}(\alpha)) \]
\[ = \left( 1 - \lambda, \frac{\lambda}{1 - \lambda} \tilde{\zeta}; \pi - b, \pi - a, \tilde{\tilde{c}}, \frac{1}{2\pi \lambda} - \bar{\rho}_{tg}^{su}(\pi + \alpha) \right). \] (4.35)

Here \( \bar{\rho}_{tg}^{su}(\alpha) \) is originally defined as the eigenvalue density function of the dual variable
\[ \bar{\rho}_{tg}^{su}(\zeta, \lambda; a, b, \tilde{c}; \alpha) \equiv \rho_{tg}^{su}(\lambda - \lambda \zeta, \lambda - 1; \pi - b, \pi - a, \tilde{c}; \alpha). \] (4.36)

Please note that the eigenvalue density (4.15) and the three equations (4.16), (4.17) and (4.2) provide a complete set of solutions in the two gap phase. So we can verify the duality relationships by checking the following: If \( S_{tg}^{su} \) is a two gap phase solution of the \( U(N) \) theory provided by the four equations, also \( S_{tg}^{su} \) becomes a two gap solution of the \( U(k-N) \) theory provided by the same equations.

**Correspondence (4.28) between the eigenvalue density functions**

First we will confirm the relationship (4.28) between the eigenvalue densities \( \rho_{tg}^{su} \) and \( \bar{\rho}_{tg}^{su} \).

By using (3.74) and (3.79), we can see
\[ \bar{\rho}_{tg}^{su}(\zeta, \lambda; a, b, \tilde{c}; \alpha) \equiv \rho_{tg}^{su}(\lambda - \lambda \zeta, \lambda - 1; \pi - b, \pi - a, \tilde{c}; \alpha) \]
\[ = \frac{1}{2\pi(1 - \lambda)} - \frac{\lambda}{1 - \lambda} \rho_{tg}^{su}(\zeta, \lambda; a, b, \tilde{c}; \pi + \alpha). \] (4.37)

So there is a relationship (4.28) between \( \rho_{tg}^{su} \) and \( \bar{\rho}_{tg}^{su} \).

**About (4.16).** Let us check \( S_{tg}^{su} \) becomes a solution of (4.16) if \( S_{tg}^{su} \) is a solution of it. By using \( \lambda \zeta = (1 - \lambda)\zeta \) and (3.83), (3.81) we can see
\[ \frac{1}{4\pi \lambda} \Upsilon(a, b) = \frac{\zeta}{2\pi} \Upsilon^{su}(a, b, \tilde{c}) \]
\[ \Rightarrow \frac{1}{4\pi(1 - \lambda)} \Upsilon(\pi - b, \pi - a) = \frac{\lambda}{1 - \lambda} \frac{\zeta}{2\pi} \Upsilon^{su}(\pi - b, \pi - a, \tilde{c}). \] (4.38)

This shows that \( S_{tg}^{su} \) becomes a solution of (4.16) if \( S_{tg}^{su} \) is a solution of it.
About (4.17). Let us check $\bar{S}_{\text{su}}$ becomes a solution of (4.17) if $S_{\text{su}}$ is a solution of it. By using (3.85), (3.86) and (3.87), we can see

$$\frac{1}{4\pi\lambda}\Lambda(a, b) = 1 + \frac{\zeta}{4\pi}G_{\text{su}}(a, b, \bar{c})$$

$$\Rightarrow \frac{1}{4\pi(1 - \lambda)}\Lambda(\pi - b, \pi - a) = 1 + \frac{\lambda}{1 - \lambda}\frac{\zeta}{4\pi}G_{\text{su}}(\pi - b, \pi - a, \bar{c}).$$

(4.39)

Please note that

$$G_{\text{su}}(\pi - b, \pi - a, \bar{c}) = -G_{\text{su}}(a, b, \bar{c}).$$

(4.40)

This (4.39) shows that $\bar{S}_{\text{su}}$ becomes a solution of the (4.17) if $S_{\text{su}}$ is a solution of it.

About (4.2). By repeating the analogous calculation to (4.33), we can see that $\bar{S}_{\text{su}}$ becomes a solution of the (4.2) if $S_{\text{su}}$ is a solution of it.

Summary By the above discussion we can conclude that $\bar{S}_{\text{su}}$ becomes a solution provided by the four equations (4.15), (4.16), (4.17) and (4.2) in the level $kU(N)$ SUSY CS matter theory if $S_{\text{su}}$ is a solution of the same equations in the level $kU(N)$ SUSY CS matter theory. Thus we have verified the self-duality relationships at the two gap phase between the level $kU(N)$ SUSY CS matter theory and the same theory with level $kU(k-N)$ gauge group.

4.5.3 Relationships between the phase transition points

We have seen that the conditions for the phase transitions (4.19), (4.21), (4.23) and (4.24) can be obtained as a certain limit of corresponding equations between which we have already verified the duality relationships. So, by the same logic as in section 3.3.4, we have already shown the duality relationships between the phase transition points. By using the analogous symbols to the ones used in (3.89), we can see the following duality relationships

$$(\text{No,Low,SUSY}) \Leftrightarrow (\text{No,Up,SUSY}), \quad (\text{Low,Two,SUSY}) \Leftrightarrow (\text{Up,Two,SUSY}). \quad (4.41)$$

4.5.4 Free energy and completing the proof of the duality

Now we have proved that there are self-duality relationships (4.27)~(4.29) in the SUSY CS matter theory. Under these correspondences (4.27)~(4.29), we will show that the free energy of the level $kU(N)$ SUSY CS matter theory at $(\lambda, \zeta)$ agrees with the free energy of the same theory with level $kU(k-N)$ gauge group at $(1 - \lambda, \frac{1}{1-\lambda}\zeta)$, namely,

$$F_{\text{su}}^{k-N} = V_{\text{su}}[\bar{\rho}_{\text{su}}, k - N] + F_{2}[\bar{\rho}_{\text{su}}, k - N] = V_{\text{su}}[\rho_{\text{su}}, N] + F_{2}[\rho_{\text{su}}, N] = F_{\text{su}}^{N}. \quad (4.42)$$

Here we have used (4.6). As already shown in (3.30) of [1], by a straightforward calculation similar to (4.33), we can show $V_{\text{su}}[\bar{\rho}_{\text{su}}, k - N] = V_{\text{su}}[\rho_{\text{su}}, N]$. We can also check that $F_{2}[\bar{\rho}, k - N] = F_{2}[\rho, N]$ by almost the same calculation as (3.91) and (3.92).

So we can see

$$F_{\text{su}}^{N} = F_{\text{su}}^{k-N}, \quad (4.43)$$

under the relationships (4.27)~(4.29). Now we have completed the proof of the level-rank self-duality between the level $kU(N)$ SUSY CS matter theory and the level $kU(k-N)$ SUSY CS matter theory.
5. Analytic proof of the duality in the two gap phase of the GWW type matrix integral

In [1], the phase structure of the toy large $N$ GWW type matrix integral has been already investigated. Although some numerical evidence have been provided, the analytic proof of the level-rank duality in the two gap phase of the GWW model has not been completed. Then we will complete the analytic proof of the duality by using the three formulae (3.78), (3.83) and (3.87).

This proof would be also useful for analysis of general CS matter theories on $S^2 \times S^1$. As we can see at (3.20), (3.50), (4.18), and appendix C.2, the eigenvalue densities converge to the one in the GWW model at large $\zeta$ limit. Moreover, every eigenvalue density of CS matter theories at the two gap phase includes the term $\rho_{2,\xi}$ which is the same functional form as the eigenvalue density in the GWW model. So the duality relationships shown in this section play one of the fundamental roles in establishing the level-rank duality in CS matter theories on $S^2 \times S^1$.

Let us focus on the proof of the level-rank duality in the two gap phase of the GWW model. As described in section C.4 of [1], we can regard $(\lambda, \zeta)$ as functions of the edge of the cuts $(a,b)$. To complete the proof of the self-duality of the GWW type model in terms of this, we should show the following:

$$\lambda(a,b)\zeta(a,b) = \lambda(\pi - a, \pi - b)\zeta(\pi - a, \pi - b), \quad (5.1)$$
$$\lambda(\pi - b, \pi - a) = 1 - \lambda(a,b), \quad (5.2)$$
$$\rho_{t\xi} \left( (1 - \lambda), \frac{\zeta \lambda}{1 - \lambda}, \alpha - \frac{1}{2\pi \lambda} - \rho_{t\xi} (\lambda, \zeta, \alpha + \pi) \right). \quad (5.3)$$

If we can verify the above three equations, it becomes the proof of the level-rank duality in the two gap phase of the GWW type matrix integral.

5.1 Proof of (5.1)

From (C.17) of [1] and by identifying $\omega = e^{i\theta}$, we can see

$$\zeta(a,b)\lambda(a,b) = \frac{1}{i\pi} \int_{L_{ugs}} \frac{d\omega}{h(\omega)}. \quad (5.4)$$

From this, by a direct substitution, we can also see

$$\zeta(\pi - b, \pi - a)\lambda(\pi - b, \pi - a) = -\frac{1}{i\pi} \int_{L_{ugs}} \frac{d\omega}{h(\omega)}. \quad (5.5)$$

So by using (3.83), we can immediately verify (5.1).

5.2 Proof of (5.2)

From (C.17) and by the identification $\omega = e^{i\theta}$, we can see that $\lambda(a,b)$ can be written as

$$\lambda(a,b) = \frac{1}{i\pi} \int_{L_{ugs}} \frac{d\omega}{h(\omega)} \left( \omega - \frac{1}{2}(\cos a + \cos b) \right). \quad (5.6)$$
By the same way, we can also see
\[ \lambda(\pi - b, \pi - a) = \frac{1}{i\pi} \int_{L_{\text{gs}}} \frac{d\omega}{h(\omega)} \left( \omega - \frac{1}{2}(\cos a + \cos b) \right). \] (5.7)

So by using both (3.83) and (3.87), we can show that
\[ \lambda(\pi - b, \pi - a) + \lambda(a, b) = 1. \] (5.8)

This is the proof of (5.2).

5.3 Proof of (5.3)
The eigenvalue density (7.6) of [1] can be described by the complex line integral as
\[ \rho(\lambda(a, b), \zeta(a, b), \alpha) = \frac{i h^+(u)}{4\pi^2\lambda} \int_{L_{\text{gs}}} d\omega \frac{1}{h(\omega)} \left( \frac{2}{\omega - u} + \frac{1}{u} \right), \] (5.9)

here we identify \( \omega = e^{i\theta} \). In the same way, by using (5.2) and (5.1), we can also check that
\[ \rho(\lambda(\pi - b, \pi - a), \zeta(\pi - b, \pi - a), \pi + \alpha) = \frac{i h^+(u)}{4\pi^2(1 - \lambda)} \int_{L_{\text{gs}}} d\omega \frac{1}{h(\omega)} \left( \frac{2}{\omega - u} + \frac{1}{u} \right). \] (5.10)

So by using (5.78), (5.2) and (5.1), we can check
\[ \rho(\lambda, \zeta, \alpha) + \frac{1 - \lambda}{\lambda} \rho \left( 1 - \lambda, \frac{\lambda}{1 - \lambda} \zeta, \pi + \alpha \right) = \frac{1}{2\pi\lambda}. \] (5.11)

This is the proof of (5.3).

Then we have verified (5.1), (5.2) and (5.3). Hence we have completed the analytic proof of the level-rank duality in the two gap phase of the GWW type model.

6. Summary and discussions
In this paper, we have elaborated the higher temperature phases of the Chern-Simons matter theories with fundamental representation on \( S^2 \times S^1 \). Here we have studied the lower gap, the upper gap and the two gap phases of the regular fermion theory, the critical boson theory, and the \( \mathcal{N} = 2 \) supersymmetric Chern-Simons matter theory. We have obtained the eigenvalue density function and the free energy in each phase, and we have also obtained the phase transition points and the phase diagrams. Based on these obtained quantities, we have completely analytically verified the level-rank duality (Giveon-Kutasov type duality) between the regular fermion theory and the critical boson theory, and the self-duality of the SUSY CS matter theory. We have also supplied the analytic proof of the duality in the two gap phase of the GWW type matrix integration which is suggested in [1].

We can see that the proof of the duality in the two gap phase of the GWW model as well as the three formulae (3.78), (3.83) and (3.87) proved in appendix B.2 are based on the cut structure of the two gap phase, which would be shared by all CS matter theories on
Figure 8: Fig. 8(a) is the phase transition points of the SUSY theory v.s the ones in the regular fermion theory. Fig. 8(b) is the phase transition points of the SUSY theory v.s the ones in the critical boson theory. Here bold blue line in the both figure is the plots in the SUSY theory. The red dashed line in Fig. 8(a) is the points in the regular fermion theory and the black dashed line in Fig. 8(b) is the points in the critical boson theory. From these we can see the phase transition points in the SUSY theory locate in lower temperature than the other two theories.

$S^2 \times S^1$. Please note that $\rho_{2, tg}$ in the eigenvalue density functions at the two gap phase has the same functional form as the eigenvalue density of the GWW model, and it is shared by all the eigenvalue densities in the two gap phase of the all CS matter theories dealt with in this paper. The three formulae play the role of establishing the level-rank duality with respect to the $\rho_{2, tg}$. Hence the duality structure of the GWW model and the three formulae would play the most fundamental role in the level-rank duality of general CS matter theories on $S^2 \times S^1$.

It is also interesting to compare the three CS matter theories in this paper. As we have seen at (3.20), (3.50), (4.18), and appendix C.2, in the supersymmetric theory, the range of the domain of the cut $\epsilon$ is smaller than the one in the regular fermion theory as well as the one in the critical boson theory at the same value of $(\lambda, \zeta)$. We can also see that the phase transitions at the SUSY theory occur at lower temperatures than the ones in the other two CS matter theories. (See Fig. 8(a), 8(b).) From this, we can guess that there should be stronger attractive forces between the eigenvalues in the SUSY theory than the one in the other two theories. This may be because the SUSY theory has more degree of freedom of matter fields than the other two theories. Then the effective potential in the SUSY theory obtained after integrating out the matter fields, which causes the attractive force between the eigenvalues, are written by the sum of the ones of the regular fermion and the critical boson theory. Hence the attractive force would be enlarged by the summation.

At the very high temperature limit, the eigenvalue densities of these three theories and the one of the GWW model converge to $\cos^{-1} \sqrt{\alpha}$, and they finally become the same universal distribution (2.10) if we further increase the temperature. We have given the analysis in (3.20), (3.50), (4.18) and appendix C.2. So at the very high temperature limit, the results coincide with the ones in [9]. We can regard their results as the very high
temperature limit of our results.

It is easy and straightforward to study the critical fermion and the regular boson theory on $S^2 \times S^1$ in the same way as this paper.

One of the interesting application of this paper would be to analyze the phase structure of the dual gravity description governed by parity violating Vasiliev’s higher spin equations \[3, 4, 5\] at finite values of the ’t Hooft coupling $\lambda = \frac{N}{\pi}$. The thermal system studied in this paper is dual to the Euclidean Vasiliev system in global $AdS$ space compactified on a circle of circumference $2\pi R = \frac{1}{T}$. The field theory analysis performed in this paper implies that this Vasiliev system admits a classical limit in large $N$ limit with $V^2 T^2 = \zeta N$ with $\zeta$ held fixed. The saddle points obtained in this paper should map to classical solutions of this bulk description; however the equations of motion governing this classical description are not necessarily Vasiliev’s equations. This is because quantum corrections to Vasiliev’s equations proportional $\frac{V^2 T^2}{N}$ are not suppressed compared to classical effects in the combined high temperature and large $N$ limit studied in this paper; such terms could modify Vasiliev’s classical equations. It would be fascinating to determine the effective 3 dimensional bulk equations dual to the large $N$ limit of this paper, and to study the classical solutions dual to the saddle points described in this paper.

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A. Cut function $h(u)$

We will list the cut region and the cut function in this appendix. The cut region in each phase is depicted in Fig.9(a), 9(b) and 9(c).

A.1 Lower gap case

In the lower gap phase, there is a single cut from $A_1 = e^{-ib}$, to $B_1 = e^{ib}$ counterclockwise. As depicted in figure 9(a), the center of the cut is located at the point 1.

Now let us define the cut function $h(u)$ carefully. Let $u = |u|e^{i\theta}$ with $\theta \in (-\pi, \pi]$, then we will define the $h(u)$ as

$$h(u) = \left((u - e^{ib})(u - e^{-ib})\right)^{\frac{1}{2}}. \quad (A.1)$$
\( \rho = \rho \text{GWW} (a) \)
\( \rho = \rho \text{GWW} (b) \)
\( \rho = \rho \text{two cut} \)

Figure 9: Fig.9(a), 9(b), 9(c) show the cut region in the lower gap phase, in upper gap phase, in the two gap phase, respectively.

The single cut of this function is taken to lie in an arc on the unit circle extending from \( A_1 \) to \( B_1 \). The function \( h \) is discontinuous on this arc with

\[
h^\pm(z) = \pm 2e^{\frac{ib\theta}{2}} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\theta}{2}} \tag{A.2}
\]

where \( z = e^{i\theta} \). Away from this cut (i.e. on the gap) on the unit circle

\[
h(z) = 2ie^{\frac{ib\theta}{2}} \sqrt{\sin^2 \frac{\theta}{2} - \sin^2 \frac{b}{2}} \quad \theta \geq 0,
\]
\[
h(z) = -2ie^{\frac{ib\theta}{2}} \sqrt{\sin^2 \frac{\theta}{2} - \sin^2 \frac{b}{2}} \quad \theta \leq 0. \tag{A.3}
\]

Recall that \( \sin^2 \frac{\theta}{2} \geq \sin^2 \frac{b}{2} \) everywhere on the gap. On the real axis \( h(x) \) is everywhere real and is positive for \( x > 1 \), but is negative for \( x < 1 \). In particular \( h(0) = -1 \). In more detail along the real axis

\[
h(x) = \sqrt{(x - \cos b)^2 + \sin^2 b} > 0 \quad x > 1,
\]
\[
h(x) = -\sqrt{(x - \cos b)^2 + \sin^2 b} < 0 \quad x < 1. \tag{A.4}
\]

A.2 Upper gap case

Let us consider the case with one upper gap without lower gap. In this case there is a single cut extending from \( A_1 = e^{ia} \) to \( B_1 = e^{-ia} \) counterclockwise along the unit circle. The cut centers on the point -1 in the complex plane, see Fig. 9(b). We can also regard this region as the complement of the cut in the lower gap phase. This interpretation would have important role to show the duality later.

In this case, the function \( h(z) = \sqrt{(e^{ia} - z)(e^{-ia} - z)} \) has a branch cut running coun-
terclockwise from $e^{ia}$ to $e^{-ia}$. The function $h(u)$ obeys followings:

$$-h^-(e^{ia}) = h^+(e^{ia}) = +2ie^{i\frac{\pi}{2}}\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{a}{2}} \quad \text{if} \quad \pi > \alpha > a,$$

$$-h^-(e^{-ia}) = h^+(e^{-ia}) = -2ie^{i\frac{\pi}{2}}\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{a}{2}} \quad \text{if} \quad -\pi < \alpha < -a,$$

$h(e^{ia}) = 2e^{i\frac{\alpha}{2}}\sqrt{\sin^2\frac{a}{2} - \sin^2\frac{\alpha}{2}} \quad \text{if} \quad a > \alpha > -a,$

$h(e^{-ia}) = 2e^{i\frac{-\alpha}{2}}\sqrt{\sin^2\frac{a}{2} - \sin^2\frac{\alpha}{2}} \quad \text{if} \quad -\pi < \alpha < \pi.$

(A.5)

$$h(x) = +\sqrt{(x - \cos a)^2 + \sin^2 a} \quad \text{for} \quad x > 1,$$

$$-\sqrt{(x - \cos a)^2 + \sin^2 a} \quad \text{for} \quad x < -1,$$

$h(z \to \infty) \to z,$

$h(0) = 1.$

A.3 Two gap case

In the two gap phase, our two cuts extend along unit circle from $A_1 = e^{ia}$ to $B_1 = e^{ib}$, and from $A_2 = e^{-ib}$ to $B_2 = e^{-ia}$ counterclockwise respectively, see the Fig.9(c). The cut function $h(u)$ in this two gap case will be

$$h(u) = \left((u - e^{ia})(u - e^{-ia})(u - e^{ib})(u - e^{-ib})\right)^{\frac{1}{2}}, \quad \text{(A.6)}$$

where this function has branch cuts along the arcs enumerated above. We will summarize some of properties of the analytic function $h(u)$. Along the unit circle outside the cuts, $h$ is analytic as

$$h(e^{i\theta}) = 4e^{i\theta}\sqrt{(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2})(\sin^2\frac{b}{2} - \sin^2\frac{\theta}{2})}, \quad -a < \theta < a, \quad \text{(A.7)}$$

$$h(e^{-i\theta}) = -4e^{-i\theta}\sqrt{(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2})(\sin^2\frac{b}{2} - \sin^2\frac{\theta}{2})}, \quad -\pi < \theta < -b, \quad b < \theta < \pi. \quad \text{(A.8)}$$

Along the cuts, $h$ becomes discontinuous with

$$h^\pm(e^{i\theta}) = \pm 4ie^{i\theta}\sqrt{(\sin^2\frac{\theta}{2} - \sin^2\frac{a}{2})(\sin^2\frac{b}{2} - \sin^2\frac{\theta}{2})}, \quad a < \theta < b, \quad \text{(A.9)}$$

$$h^\pm(e^{-i\theta}) = \mp 4ie^{-i\theta}\sqrt{(\sin^2\frac{\theta}{2} - \sin^2\frac{a}{2})(\sin^2\frac{b}{2} - \sin^2\frac{\theta}{2})}, \quad -b < \theta < -a, \quad \text{(A.10)}$$

where $h = h^+$ on the outside of the circle and $h = h^-$ on the inside. Moreover $h$ is real and positive along the real axis; explicitly for real $x$.

$$h(x) = \sqrt{(1 - 2x \cos a + x^2)(1 - 2x \cos b + x^2)}, \quad -\infty < x < \infty. \quad \text{(A.11)}$$

Note in particular that $h(0) = 1$. At large $u$ we have

$$h(u) \sim +u^2 - u(\cos a + \cos b), \quad u \sim \infty. \quad \text{(A.12)}$$
B. Important formula

B.1 Important formula in the upper gap phase

In this subsection, we will derive several formulae in the upper gap phase with cut function $h(z) = \sqrt{(e^{ia} - z)(e^{-ia} - z)}$. This has a branch cut running counterclockwise from $e^{ia}$ to $e^{-ia}$, and has the property (A.3).

B.1.1 Proof of (3.14)

Here we will prove the formula (3.14). Since $\frac{1}{h(\omega)}$ is everywhere holomorphic except the cuts, from the Cauchy’s theorem

$$0 = \oint_{|\omega| = \infty} \frac{d\omega}{h(\omega)} - \oint_{R_c} \frac{d\omega}{h(\omega)}$$  \hspace{1cm} (B.1)

Note that from $h(\omega) \to \omega$ as $|\omega| \to \infty$,

$$\oint_{|\omega| = \infty} \frac{d\omega}{h(\omega)} = 2\pi i$$  \hspace{1cm} (B.2)

then

$$2\pi i = \oint_{R_c} \frac{d\omega}{h(\omega)} = \oint_{|\omega| = 1 + |\epsilon|} \frac{d\omega}{h(\omega)} - \oint_{|\omega| = 1 - |\epsilon|} \frac{d\omega}{h(\omega)}$$

$$= \oint_{|\omega| = 1 + |\epsilon|} \frac{d\omega}{h(\omega)} + \oint_{|\omega| = 1 - |\epsilon|} \frac{d\omega}{h(\omega)}$$  \hspace{1cm} (B.3)

$$= 2 \int_{L_{\text{Lugs}}} \frac{d\omega}{h(\omega)}.

From first to second line we have used the fact that $\frac{1}{h(\omega)}$ is holomorphic inside the circle $|\omega| = 1 - |\epsilon|$. From the second to third, we have used $\frac{1}{h^+(\omega)} = -\frac{1}{h^-(\omega)}$ in the cut region. We finally obtain

$$\pi i = \int_{L_{\text{Lugs}}} \frac{d\omega}{h(\omega)}$$  \hspace{1cm} (B.4)

B.1.2 Proof of (3.12)

Here we will prove the formula (3.12).

Since $\frac{1}{h(\omega)(\omega - u)}$ is everywhere holomorphic except the cut, from the Cauchy’s theorem

$$0 = \oint_{|\omega| = \infty} \frac{d\omega}{h(\omega)(\omega - u)} - \oint_{R_c} \frac{d\omega}{h(\omega)(\omega - u)}.$$  \hspace{1cm} (B.5)

From $\frac{1}{h(\omega)(\omega - u)} \to \omega^{-2}$, at $|\omega| \to \infty$,

$$\oint_{|\omega| = \infty} \frac{d\omega}{h(\omega)(\omega - u)} = 0.$$  \hspace{1cm} (B.6)
so

$$\begin{align*}
0 &= \oint_{C_{\text{cuts}}} \frac{d\omega}{h(\omega)(\omega - u)} = \oint_{|\omega|=1+|\epsilon|} \frac{d\omega}{h(\omega)(\omega - u)} - \oint_{|\omega|=1-|\epsilon|} \frac{d\omega}{h(\omega)(\omega - u)} \\
&= \oint_{|\omega|=1+|\epsilon|} \frac{d\omega}{h(\omega)(\omega - u)} + \oint_{|\omega|=1-|\epsilon|} \frac{d\omega}{h(\omega)(\omega - u)} \\
&= 2\int_{L_{\text{ugs}}} \frac{d\omega}{h(\omega)(\omega - u)} + \int_{L_{\text{arcs}}} \frac{d\omega}{h^+(\omega)} \left( \frac{1}{(\omega - u)^+} - \frac{1}{(\omega - u)^-} \right).
\end{align*}$$

(B.7)

Here $L_{\text{arcs}}$ denotes the complex line integration counterclockwise along the cuts. Note that on the cut region, \(^{11}\)

$$\frac{1}{(\omega - u)^+} - \frac{1}{(\omega - u)^-} = 2\pi i \delta(\omega - u),$$

(B.8)

so it becomes,

$$\int_{L_{\text{arcs}}} \frac{d\omega}{h^+(\omega)} \left( \frac{1}{(\omega - u)^+} - \frac{1}{(\omega - u)^-} \right) = 2\pi i \frac{h^+(u)}{h^+(u)}.$$

(B.9)

We finally obtain

$$\int_{L_{\text{ugs}}} \frac{d\omega}{h(\omega)(\omega - u)} = -\frac{\pi i}{h^+(u)}.$$

(B.10)

### B.2 Important formulae for the proof of level-rank duality in the two gap phase

In this subsection, we will derive several formulae in the two gap phase with cut function $h(u)$ s.t.

$$h(u) = \left( (u - e^{ia})(u - e^{-ia})(u - e^{ib})(u - e^{-ib}) \right)^{\frac{1}{2}},$$

(B.11)

where this function has branch cuts extending along unit circle from $A_1 = e^{ia}$ to $B_1 = e^{ib}$ and from $A_2 = e^{-ib}$ to $B_2 = e^{-ia}$ running counterclockwise. This function satisfies $[A.7] \sim [A.12]$. The formulae derived here are important for the proof of the duality in the two gap phase.

#### B.2.1 Proof of (3.78)

We will show (3.78),

$$i \frac{h^+(u)}{4\pi^2 \lambda} \left[ \int_{L_{\text{ugs}}} d\omega \ F(\omega) + \int_{L_{\text{arcs}}} d\omega \ F(\omega) \right] = \frac{1}{2\pi \lambda}$$

(B.12)

where we define $F(\omega)$ as

$$F(\omega) = \frac{1}{h(\omega)} \left( \frac{2}{\omega - u} + \frac{1}{u} \right).$$

(B.13)

Throughout this subsection $L_{\text{ugs}}$ is the arc running counterclockwise along the lower gap on the unit circle, i.e, arc $e^{i\theta}$ s.t. $b < |\theta| < \pi$. $L_{\text{arcs}}$ is the arc running counterclockwise along the upper gap on the unit circle, i.e, arc $e^{i\theta}$ s.t. $-a < \theta < a$.

\(^{11}\)We should note from the standpoint of the counterclockwise complex integration, the sign of the delta function is flipped due to the opposite direction of the integration measure direction.
Since \( F(\omega) \) is holomorphic everywhere except the cut region. Applying the Cauchy’s theorem we get
\[
0 = \oint_{|\omega|=\infty} d\omega \, F(\omega) - \oint_{\text{cuts}} d\omega \, F(\omega). 
\] (B.14)
Since \( F(\omega) \to \frac{1}{\omega^2} \) as \(|\omega| \to \infty \), we have
\[
0 = \oint_{|\omega|=\infty} d\omega \, F(\omega) = \oint_{\text{cuts}} d\omega \, F(\omega). 
\] (B.15)
From this, we can rewrite
\[
0 = \oint_{|\omega|=1} d\omega \, F(\omega) = \int_{|\omega|=1-|\epsilon|} d\omega \, F(\omega) = -\oint_{|\omega|=1+|\epsilon|} d\omega \, F(\omega) 
= \oint_{|\omega|=1} \left( F^+(\omega) + F^-(\omega) \right). 
\] (B.16)
From the first line to second line we have used the fact that \( F(\omega) \) is holomorphic inside the circle \(|\omega| = 1 - |\epsilon| \),
\[
\oint_{|\omega|=1-|\epsilon|} d\omega \, F(\omega) = -\oint_{|\omega|=1+|\epsilon|} d\omega \, F(\omega). 
\] (B.17)
Note that at the cuts,
\[
F^+(\omega) + F^-(\omega) = \frac{1}{h^+(\omega)} \left( \frac{2}{(\omega - u)^+} - \frac{2}{(\omega - u)^-} \right) = \frac{1}{h^+(\omega)} (4\pi i \delta(\omega - u)), 
\] (B.18)
and \( F^+(\omega) = F^-(\omega) = F(\omega) \) in the upper gap as well as in the lower gap region. Then
\[
0 = \oint_{|\omega|=1} d\omega \, (F^+(\omega) + F^-(\omega)) 
= \int_{\text{arcs}} d\omega \, \frac{1}{h^+(\omega)} (4\pi i \delta(\omega - u)) + 2 \int_{\text{ugs}} d\omega \, F(\omega) + 2 \int_{\text{lgs}} d\omega \, F^+(\omega) 
= \frac{4\pi i}{h^+(u)} + 2 \left[ \int_{\text{ugs}} d\omega \, F(\omega) + \int_{\text{lgs}} d\omega \, F(\omega) \right]. 
\] (B.19)
So by multiplying \( \frac{h^+(u)}{8\pi^2\lambda} \) to the last line of (B.19), we obtain (B.12).

**B.2.2 Proof of (3.83)**

Let us prove the formula (3.83),
\[
\int_{\text{ugs}} d\omega \, \frac{1}{h(\omega)} = -\int_{\text{lgs}} d\omega \, \frac{1}{h(\omega)}. 
\] (B.20)
Because \( \frac{1}{h(\omega)} \) is holomorphic everywhere except the cuts region, from the Cauchy’s theorem, we get
\[
0 = \oint_{|\omega|=\infty} d\omega \, \frac{1}{h(\omega)} - \oint_{\text{cuts}} d\omega \, \frac{1}{h(\omega)}. 
\] (B.21)
Using \( \frac{1}{h(\omega)} \to \frac{1}{\omega^2} \) as \( |\omega| \to \infty \), we also get
\[
0 = \oint_{|\omega| = \infty} d\omega \frac{1}{h(\omega)} = \oint_{\text{cuts}} d\omega \frac{1}{h(\omega)}.
\]  
\( (B.22) \)

Hence, we can rewrite
\[
0 = \oint_{\text{cuts}} d\omega \frac{1}{h(\omega)} = \oint_{|\omega| = 1+|\epsilon|} d\omega \frac{1}{h(\omega)} - \oint_{|\omega| = 1-|\epsilon|} d\omega \frac{1}{h(\omega)}
\]
\[= \oint_{|\omega| = 1+|\epsilon|} d\omega \frac{1}{h(\omega)} + \oint_{|\omega| = 1-|\epsilon|} d\omega \frac{1}{h(\omega)}
\]
\[= \oint_{|\omega| = 1} d\omega \left( \frac{1}{h^+(\omega)} + \frac{1}{h^-(\omega)} \right)
\]
\[= 2 \left[ \int_{L_{\text{ugs}}} d\omega \frac{1}{h(\omega)} + \int_{L_{\text{lgs}}} d\omega \frac{1}{h(\omega)} \right].
\]  
\( (B.23) \)

From first line to second line, we have used the fact that \( \frac{1}{h(\omega)} \) is holomorphic inside the circle \( |\omega| = 1 - |\epsilon| \),
\[
\oint_{|\omega| = 1-|\epsilon|} d\omega \frac{1}{h(\omega)} = 0 = - \oint_{|\omega| = 1-|\epsilon|} d\omega \frac{1}{h(\omega)}.
\]  
\( (B.24) \)

In the last step from the third to the forth line of \( (B.23) \), we have used that \( \frac{1}{h^+(\omega)} = -\frac{1}{h^-(\omega)} \) in the cuts region while \( \frac{1}{h^+(\omega)} = -\frac{1}{h^-(\omega)} = \frac{1}{h(\omega)} \) in the gaps region. Hence we can see from the last line of \( (B.23) \),
\[
\int_{L_{\text{ugs}}} d\omega \frac{1}{h(\omega)} = - \int_{L_{\text{lgs}}} d\omega \frac{1}{h(\omega)}.
\]  
\( (B.25) \)

This completes the proof of \( (3.83) \).

**B.2.3 Proof of \( (3.87) \)**

Since \( \frac{\omega}{h(\omega)} \) is holomorphic everywhere except the cuts, from the Cauchy’s theorem, we get
\[
0 = \frac{1}{2\pi i} \oint_{|\omega| = \infty} d\omega \frac{\omega}{h(\omega)} = \frac{1}{2\pi i} \oint_{\text{cuts}} d\omega \frac{\omega}{h(\omega)}.
\]  
\( (B.26) \)

Using \( \frac{1}{h(\omega)} \to \frac{1}{\omega^2} \) as \( |\omega| \to \infty \), we also get
\[
\frac{1}{2\pi i} \oint_{\text{cuts}} d\omega \frac{\omega}{h(\omega)} = \frac{1}{2\pi i} \oint_{|\omega| = \infty} d\omega \frac{\omega}{h(\omega)} = \frac{1}{2\pi i} \int_0^{2\pi} d\theta \frac{i\omega^2}{\omega^2} = 1.
\]  
\( (B.27) \)

Hence
\[
1 = \frac{1}{2\pi i} \oint_{\text{cuts}} d\omega \frac{\omega}{h(\omega)} = \frac{1}{2\pi i} \oint_{|\omega| = 1+|\epsilon|} d\omega \frac{\omega}{h(\omega)} - \frac{1}{2\pi i} \oint_{|\omega| = 1-|\epsilon|} d\omega \frac{\omega}{h(\omega)}
\]
\[= \frac{1}{2\pi i} \oint_{|\omega| = 1+|\epsilon|} d\omega \frac{\omega}{h(\omega)} + \frac{1}{2\pi i} \oint_{|\omega| = 1-|\epsilon|} d\omega \frac{\omega}{h(\omega)}
\]
\[= \frac{1}{\pi i} \int_{L_{\text{ugs}}} d\omega \frac{\omega}{h(\omega)} + \frac{1}{\pi i} \int_{L_{\text{lgs}}} d\omega \frac{\omega}{h(\omega)}.
\]  
\( (B.28) \)
From first line to second line we have used the fact that $\omega h(\omega)$ is holomorphic inside the circle $|\omega| = 1 - |\epsilon|$. From the second line to third line we have used the fact that $(\omega h(\omega))^+ = - (\omega h(\omega))^-$ on the cuts region while $(\omega h(\omega))^+ = (\omega h(\omega))^-$ in the gaps region. From the last line of (B.28), we can see

$$1 = \frac{1}{\pi i} \int_{L_{wgs}} d\omega \frac{\omega}{h(\omega)} + \frac{1}{\pi i} \int_{L_{lgs}} d\omega \frac{\omega}{h(\omega)}.$$  \hfill (B.29)

This completes the proof of the formula (3.87)

**C. Behavior of eigenvalue distribution at large $\zeta$**

We expect that as $\zeta \to \infty$, with fixed $\lambda$, the eigenvalue densities would behave as

$$
\rho(\alpha) = \begin{cases} 
\frac{1}{2\pi \lambda} & (|\alpha| < \pi \lambda) \\
0 & (|\alpha| > \pi \lambda).
\end{cases}
$$  \hfill (C.1)

First we will verify that at $\zeta = \infty$, with fixed finite $\lambda$, the eigenvalue density in each CS matter theory must be this function. After that we will elaborate more on the behavior of the eigenvalue densities in this limit more.

**C.1 Proof of the universal distribution of eigenvalue density at $\zeta = \infty$**

To verify it, it is useful to note that $\Upsilon(a,b) = \infty \Rightarrow a = b$, at $\zeta = \infty$ with finite $\lambda$. Here $\Upsilon(a,b)$ is defined in (3.16). Note that since

$$
\frac{1}{\sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2}}} \leq \frac{1}{\cos \frac{\alpha}{2} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2}}},
$$  \hfill (C.2)

for every $|\alpha| \leq a \leq b$, it would be

$$
\Upsilon(a,b) \leq \frac{1}{\sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{a}{2}}} \int_{-\alpha}^{\alpha} da \frac{1}{\cos \frac{\alpha}{2} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2}}} = \frac{1}{2 \cos \frac{\alpha}{2} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2}}}.
$$  \hfill (C.3)

From (C.3) we can see that $\Upsilon = \infty \Rightarrow a = b$. Also in the case of $a = \pi$, because of the constraint $a \leq b \leq \pi$, $a = b$ is automatically satisfied. So we conclude that

$$
\Upsilon(a,b) = \infty \Rightarrow a = b.
$$  \hfill (C.4)

In following subsubsections, we will verify the statement by using (C.4).

**C.1.1 Regular fermion case**

If the $\bar{c}$ remains finite in the limit $\zeta \to \infty$ with fixed $\lambda$, from the third line of (3.16), it follows that $\Upsilon^{\text{F}}(a,b,\bar{c})$ remains non-zero finite. Then from the first line of (3.16), $\Upsilon(a,b)$ becomes infinite. Hence it follows $a = b$ in the limit $\zeta \to \infty$ with fixed $\lambda$ if the $\bar{c}$ remains finite.
To complete the proof we have to show that $\tilde{c}$ remains finite in this limit. $\tilde{c}$ is given by

$$\tilde{c} = f_{\rho,\lambda}(\tilde{c}) \equiv \frac{\lambda}{(1-\lambda)} \int_{-\pi}^{\pi} d\alpha \rho(\alpha) \log(1 + 2 \cos \alpha e^{-\tilde{c}} + e^{-2\tilde{c}}). \quad (C.5)$$

The function $f_{\rho,\lambda}(\tilde{c})$ is positive by the property of $\rho$ as a probability function and by the property that $\rho(\alpha') \leq \rho(\tilde{\alpha})$ if $\pi \geq |\alpha'| \geq |\tilde{\alpha}|$. At positive $\tilde{c}$ we can see the following inequality

$$0 \leq f_{\rho,\lambda}(\tilde{c}) \leq \frac{2\lambda}{(1-\lambda)} \log 2. \quad (C.6)$$

Solution of (C.5) is represented by the intersection point of the line $y = x$ and $y = f_{\rho,\lambda}(x)$ in the $x - y$ plane, which is denoted as $(x, y) = (\tilde{c}, \tilde{c})$. From the inequality (C.6), we immediately see that

$$\tilde{c} \leq \frac{2\lambda}{(1-\lambda)} \log 2. \quad (C.7)$$

$\frac{2\lambda}{(1-\lambda)} \log 2$ is finite quantity for every $0 < \lambda < 1$. From this inequality, we can see that the $\tilde{c}$ remains positive finite even in the limit $\zeta \to \infty$ with fixed $\lambda$. Hence we have shown that the eigenvalue density becomes universal distribution (C.1) in the limit $\zeta \to \infty$.

### C.1.2 Critical boson case

We can show in a way similar to the regular fermion case. If the positive number $\sigma$ remains finite in the limit $\zeta \to \infty$ with fixed $\lambda$, from the second line of (3.47), $Y^{c,b}(a, b, \sigma)$ remains non-zero finite. Then from the first line of (1.47), $Y(a, b)$ becomes infinite. Therefore it follows $a = b$ in the limit $\zeta \to \infty$ with fixed $\lambda$, if $\sigma$ remains finite.

In order to complete the proof, we have to show that the $\sigma$ remains positive finite in the corresponding limit. $\sigma$ is given by the following equation

$$x = g_{\rho}(x) \equiv -\int_{-\pi}^{\pi} d\alpha \rho(\alpha) \log \left(1 - 2 \cos \alpha e^{-x} + e^{-2x}\right). \quad (C.8)$$

Solution of (C.8) is represented by the intersection point of the line $y = x$ and $y = g_{\rho}(x)$ in the $x - y$ plane, which is denoted as $(x, y) = (\sigma, \sigma)$. Note that at $x > 0$,

$$0 \leq g_{\rho}(x) \leq -2 \log \left(1 - e^{-x}\right). \quad (C.9)$$

Suppose that $x = \hat{x} > 0$ is the solution of the equation

$$x = -2 \log \left(1 - e^{-x}\right), \quad (C.10)$$

from the inequality (C.6), we can immediately see that

$$\sigma \leq \hat{x} = \log \left(\frac{3 + \sqrt{5}}{2}\right). \quad (C.11)$$

So $\sigma$ remains finite even in the limit. Then we have shown that the eigenvalue density becomes universal distribution (C.1) in the limit $\zeta \to \infty$ with fixed $\lambda$.

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12 We can not apply this analysis for the case $\lambda = 1$. But in case of $\lambda = 1$, the eigenvalue density is always $\rho(\alpha) = \frac{1}{2\pi}$, which already obeys the universal density. So we do not have to mind the $\lambda = 1$ case.
C.1.3 SUSY CS matter theory case

We can show in a way similar to the previous cases. As in the previous cases, if the positive number \( \tilde{c} \) remains finite in the limit \( \zeta \to \infty \) with fixed \( \lambda \), from the second line of (4.16), \( \Upsilon^{\text{SU}}(a, b, \sigma) \) remains non-zero finite. Then from the first line of (4.16), \( \Upsilon(a, b) \) becomes infinite. Hence it follows \( a = b \) in the limit \( \zeta \to \infty \) with fixed \( \lambda \), if the \( \tilde{c} \) remains finite.

In order to complete the proof, we have to show that the \( \tilde{c} \) remains finite at the corresponding limit. The \( \tilde{c} \) is given by the equation

\[
\tilde{c} = \lambda \int_{-\pi}^{\pi} d\alpha \rho(\alpha) \log \frac{1 + 2 \cos \alpha e^{-\tilde{c}} + e^{-2\tilde{c}}}{1 - 2 \cos \alpha e^{-\tilde{c}} + e^{-2\tilde{c}}}.
\] 

(C.12)

For any \( \tilde{c} > 0 \), \( 0 \leq \lambda \leq 1 \), we see the following inequality,

\[
\lambda \int_{-\pi}^{\pi} d\alpha \rho(\alpha) \log \frac{1 + 2 \cos \alpha e^{-\tilde{c}} + e^{-2\tilde{c}}}{1 - 2 \cos \alpha e^{-\tilde{c}} + e^{-2\tilde{c}}} \leq 2 \log \left| \frac{2}{1 - e^{-x}} \right|. \tag{C.13}
\]

Note that the last term is independent of \( \zeta \). Suppose that \( x = \tilde{c}_o > 0 \) is the solution of the equation

\[
x = 2 \log \left| \frac{2}{1 - e^{-x}} \right|, \tag{C.14}
\]

due to the inequality (C.13), we see

\[
0 < \tilde{c} \leq \tilde{c}_o = \log(3 + 2\sqrt{2}). \tag{C.15}
\]

Hence we conclude the positive \( \tilde{c} \) remains finite even in the limit \( \zeta \to \infty \), then we have shown that the eigenvalue density becomes the universal distribution (C.1) in the limit \( \zeta \to \infty \) with fixed \( \lambda \).

C.2 Behavior of eigenvalue density in large \( \zeta \) limit, with \( 1 \gg \frac{1}{\zeta} > 0 \)

We will elaborate how the eigenvalue density deviates from the universal distribution (C.1) if we gradually decrease the temperature from \( \zeta = \infty \).

C.2.1 Regular fermion theory

To consider the behavior in the regular fermion theory, we should evaluate behavior of the combination of (3.16), (3.17) and (3.3) at large \( \zeta \) where \( b \sim a + \epsilon \) with \( \epsilon \ll 1 \). (As in section C.2 in [1].) In this limit, these three equations can be expanded as in \( \epsilon \) to give

\[
\frac{1}{2\pi \lambda} \left( \frac{1}{\sin(a)} \log \left( \frac{8 \sin(a)}{\epsilon} \right) - \epsilon \frac{\cos(a)}{\sin^2(a)} \left( -1 + \log \left( \frac{8 \sin(a)}{\epsilon} \right) \right) \right) + O(\epsilon^2) 
\]

(C.16)

\[
= \frac{\zeta}{2\pi} \int_{\tilde{c}}^{\infty} dy \left( \frac{y}{\cosh y + \cos a} + \epsilon \frac{y \sin a}{2(\cosh y + \cos a)^2} \right),
\]

and

\[
\frac{1}{2\pi \lambda} \left( 2 \epsilon a + 2 \cot(a) \log \left( \frac{8 \sin(a)}{\epsilon} \right) - \epsilon \frac{1}{\sin^2(a)} \left( -1 + \log \left( \frac{8 \sin(a)}{\epsilon} \right) \right) \right) + O(\epsilon^2) 
\]

(C.17)

\[
= 1 + \frac{\zeta}{2\pi} \int_{\tilde{c}}^{\infty} dy \frac{y \cos a}{\cosh y + \cos a} - \epsilon \frac{\zeta}{4\pi} \int_{\tilde{c}}^{\infty} dy \frac{y \cosh y \sin a}{(\cosh y + \cos a)^2},
\]
and
\[ \tilde{c} = \frac{1}{2\pi (1 - \lambda)} \int_{-\pi \lambda}^{\pi \lambda} d\alpha \log(1 + 2 \cos \alpha e^{-\tilde{c}} + e^{-2\tilde{c}}), \tag{C.18} \]
respectively. From these we can obtain \( \epsilon \) and \( a, b \) as
\[ \epsilon = 8 \sin(\pi \lambda) \exp \left( -\sin(\frac{\pi \lambda}{2}) \lambda \int_{\tilde{c}}^{\infty} dy \frac{y \cosh y + \cos \pi \lambda}{\cosh y + \cos \pi \lambda} \right) + \ldots, \tag{C.19} \]
\[ a = \pi \lambda - \frac{1}{2} \epsilon, \quad b = \pi \lambda + \frac{1}{2} \epsilon. \tag{C.20} \]

Since
\[ Y_{rf} = \int_{\tilde{c}}^{\infty} dy \frac{y \cosh y + \cos \pi \lambda}{\cosh y + \cos \pi \lambda} \sim O(\epsilon^0), \tag{C.21} \]
\( \zeta \) is estimated as \( \zeta \sim -\log \epsilon \) from (C.19).

Let us consider the asymptotic behavior of the eigenvalue density (3.18) in \( \epsilon \to 0 \) limit. Here \( \rho_{2,tg} \) of (3.18) is the same functional form as the one in the GWW type model (7.6) of [1], while \( \rho_{1,r,f} \) of (3.18) is regarded as an additional term appearing in the regular fermion theory. We can see that \( \rho_{1,r,f} \) falls off as \( \rho_{1,r,f} \sim -\epsilon \log \epsilon \to 0 \). On the other hand \( \rho_{2,tg} \) remains finite. So in the limit, \( \rho \) becomes
\[ \rho(\alpha) = \rho_{2,tg}(\lambda, \pi \lambda - \frac{1}{2} \epsilon, \pi \lambda + \frac{1}{2} \epsilon; \alpha) = \frac{1}{\pi^2 \lambda} \cos^{-1} \sqrt{\alpha_1} \tag{C.22} \]
where \( \alpha = a + \alpha_1 \epsilon \) where \( 0 < \alpha_1 < 1 \). This behavior is very similar to the one in the GWW type model, (7.11) of [1].

Although the functional form of \( \rho \) in the large \( \zeta \) limit is almost same as the one in the GWW type model, the range of \( \epsilon \) in the regular fermion theory would be different from \( \epsilon \) in the GWW type model. To evaluate \( \epsilon \), we should calculate the factor \( Y_{rf} \) in (C.19), and compare with \( \epsilon = 8 \sin(\pi \lambda) \exp \left( -\sin(\frac{\pi \lambda}{2}) \lambda \zeta \right) \) in the GWW case. By numerical calculations, we can confirm that
\[ Y_{rf} > 1 \tag{C.23} \]
for every value of \( \lambda \). We plot the function \( Y_{rf} \) in Fig. 11(a). Hence we see that \( \epsilon \) at \( (\lambda, \zeta) \) of the regular fermion theory (C.13) is smaller than \( \epsilon \) at the same \( (\lambda, \zeta) \) of GWW model. This means that the shape of the eigenvalue density function of the regular fermion theory has a sharper cliff than the one in GWW model. (Schematic graph of the eigenvalue density is depicted at Fig. 10(a).)

**C.2.2 Critical boson theory**

Also in the critical boson theory, we consider how the eigenvalue density deviates from universal configuration (C.1) as we decrease the temperature.

To consider the behavior, we should investigate the combination of the conditions (3.47), (3.48), and (3.36) in the large \( \zeta \) where \( b = a + \epsilon \) with \( \epsilon \ll 1 \). In this limit, the asymptotic behavior of these three equations are
\[ \frac{1}{2\pi \lambda} \left( \frac{2}{\sin(a)} \log \left( \frac{8 \sin(a)}{\epsilon} \right) - \epsilon \cos(a) \sin^2(a) \left( -1 + \log \left( \frac{8 \sin(a)}{\epsilon} \right) \right) \right) + O(\epsilon^2). \tag{C.24} \]
Figure 10: Plots of \( \cos^{-1}(\sqrt{\alpha}) \) curves, which are the eigenvalue density functions on the cut domain \( \pi \lambda - \frac{\epsilon}{2} \leq \alpha \leq \pi \lambda + \frac{\epsilon}{2} \) at \( \zeta \to \infty \). (Horizontal axis is \( \alpha \) and vertical axis is the function \( \cos^{-1}\sqrt{\frac{\alpha-\pi \lambda}{\epsilon} + \frac{1}{2}} \). We did not plot precise \( \epsilon \) here.) By using these schematic graphs, we are trying to compare the behaviors of the eigenvalue densities; the one of the GWW, the one of the regular fermion (the critical boson) and the one in the SUSY CS matter theory. Dotted blue line is the one in the GWW model and the bold line is the regular fermion’s one, and the dashed red line is the one in the SUSY CS matter theory. The regular fermion theory has a sharper slope than the one in the GWW model, moreover the SUSY theory has a sharper slope than the one in the regular fermion theory. At fixed \( \lambda \), the center of the cut domain is always fixed at \( \lambda \).

and

\[
\frac{1}{2\pi \lambda} \left( 2a + 2 \cot(a) \log \left( \frac{8 \sin(a)}{\epsilon} \right) - \epsilon \frac{1}{\sin^2(a)} \left( -1 + \log \left( \frac{8 \sin(a)}{\epsilon} \right) \right) \right) + O[\epsilon^2]
\]

\[
= 1 + \frac{\zeta}{2\pi} \int_{\sigma}^{\infty} dy \frac{y \cos a}{\cosh y - \cos a} - \epsilon \frac{\zeta}{4\pi} \int_{\sigma}^{\infty} dy \frac{y \cosh y \sin a}{(\cosh y - \cos a)^2},
\]

(C.25)

and

\[
\sigma = -\frac{1}{2\pi \lambda} \int_{-\pi \lambda}^{\pi \lambda} da \log \left( 1 - 2 \cos \alpha e^{-\sigma} + e^{-2\sigma} \right),
\]

(C.26)

respectively. From these, \( \epsilon, a \) and \( b \) are evaluated as

\[
\epsilon = 8 \sin(\pi \lambda) \exp \left( -\frac{1}{2} \sin(\pi \lambda) \zeta \lambda \int_{\sigma}^{\infty} \frac{dy}{\cosh y - \cos \pi \lambda} \right),
\]

(C.27)

\[
a = \pi \lambda - \frac{\epsilon}{2}, \quad b = \pi \lambda + \frac{\epsilon}{2}.
\]

(C.28)

Since

\[
Y_{cb} = \int_{\sigma}^{\infty} dy \frac{y}{\cosh y - \cos \pi \lambda} \sim O(\epsilon^0),
\]

(C.29)

\( \zeta \) is evaluated as \( \zeta \sim -\log \epsilon \) from (C.27).

Let us consider the asymptotic behavior of the eigenvalue density (3.49) in \( \epsilon \to 0 \) limit. Here \( \rho_{2,tg} \) of (3.49) is the same functional form as the one in the GWW type model (7.6) of [1], while \( \rho_{1,tg} \) of (3.49) is regarded as an additional term appearing in the critical boson theory. We can see that \( \rho_{1,tg}^{lb} \) falls off as \( \rho_{1,tg}^{lb} \sim -\epsilon \log \epsilon \to 0 \). On the other hand \( \rho_{2,tg} \)
remains finite. So in this limit, $\rho$ becomes
\[
\rho(\alpha) = \rho_{2,19}(\lambda, \pi \lambda - \frac{1}{2} \epsilon, \pi \lambda + \frac{1}{2} \epsilon; \alpha) = \frac{1}{\pi^2 \lambda} \cos^{-1}\sqrt{\alpha_1}.
\] (C.30)
This behavior is also very similar to the one in the regular fermion case as well as the one in the GWW type \(7.11\) of [1].

The range of $\epsilon$ in the critical boson theory would also be different from $\epsilon$ in the GWW type model. To evaluate the $\epsilon$, we should calculate the factor $Y_{cb}$ in (C.27), and compare with $\epsilon$ in the GWW case. By numerical calculations, we confirm that
\[
Y_{cb} > 1 \quad \text{(C.31)}
\]
for every value of $\lambda$. (See the plots of $Y_{cb}$ in Fig. 11(a)). This means that the shape of the eigenvalue density function of the critical boson theory has a sharper cliff than the one in GWW model. (Schematic graph of the eigenvalue density is depicted at Fig. 10(a)). We can see that $\epsilon$ in (C.27) as well as $Y_{cb}$ are dual to $\epsilon$ in (C.19) and $Y_{rf}$ under the level-rank duality. We can also see the duality between $Y_{cb}$ and $Y_{rf}$ from the graph Fig. 11(a).

C.2.3 Supersymmetric CS matter theory

Similar to the other cases, we consider the behavior of the combination of (4.16), (4.17) and (4.2) at large $\zeta$. In this limit these behave as
\[
\frac{1}{2\pi \lambda} 2 \frac{1}{\sin(a)} \log \left( \frac{8 \sin(a)}{\epsilon} \right) - \frac{\cos(a)}{\sin^2(a)} \left( -1 + \log \left( \frac{8 \sin(a)}{\epsilon} \right) \right) + O(\epsilon^2)
\]
\[
= \frac{\zeta}{2\pi} \int_{\tilde{c}}^{\infty} dy \left( \frac{2y \cosh y}{\cosh^2 y - \cos^2 a} - \epsilon \frac{2y \sin a \cos a \cosh y}{(\cosh^2 y - \cos^2 a)^2} \right),
\] (C.32)
and
\[
\frac{1}{2\pi \lambda} 2a + 2 \cot(a) \log \left( \frac{8 \sin(a)}{\epsilon} \right) - \epsilon \frac{1}{\sin^2(a)} \left( -1 + \log \left( \frac{8 \sin(a)}{\epsilon} \right) \right) + O(\epsilon^2)
\]
\[
= 1 + \frac{\zeta}{2\pi} \int_{\tilde{c}}^{\infty} dy \frac{2y \cos a \cosh y}{\cosh^2 y - \cos^2 a} - \epsilon \frac{2y \sin a \cosh y}{(\cosh^2 y - \cos^2 a)^2},
\] (C.33)
and
\[
\tilde{c} = \frac{1}{2\pi} \left| \int_{-\pi/2}^{\pi/2} d\alpha \log \frac{1 + 2 \cos \alpha e^{-\tilde{c}} + e^{-2\tilde{c}}}{1 - 2 \cos \alpha e^{-\tilde{c}} + e^{-2\tilde{c}}} \right|. \quad \text{(C.34)}
\]
From these, $\epsilon, a$ and $b$ are evaluated as
\[
\epsilon = 8 \sin(\pi \lambda) \exp \left( -\frac{\sin(\pi \lambda)}{2} \zeta \lambda \int_{\tilde{c}}^{\infty} dy \frac{2y \cosh y}{\cosh^2 y - \cos^2 \pi \lambda} \right) + \ldots,
\] (C.35)
\[
a = \pi \lambda - \frac{\epsilon}{2}, \quad b = \pi \lambda + \frac{\epsilon}{2}.
\] (C.36)
Since
\[
Y_{su} = \int_{\tilde{c}}^{\infty} dy \frac{2y \cosh y}{\cosh^2 y - \cos^2 \pi \lambda} \sim O(\epsilon^0), \quad \text{(C.37)}
\]
Figure 11: The points with black filled circle are plots of $Y = Y_{su}$ in the SUSY CS matter theory. The circle and square plots without being filled are $Y = Y_{rf} = \int_0^\infty dy \frac{y}{\cosh y + \cos \pi \lambda}$ in the regular fermion theory and $Y = Y_{cb} = \int_\pi^\infty dy \frac{y}{\cosh y - \cos \pi \lambda}$ at the critical boson theory respectively. Dotted horizontal line indicates $Y = 1$. We can see that for any $\lambda$, $Y_{su} > Y_{rf} > 1$ and $Y_{su} > Y_{cb} > 1$.

we can see $\zeta \sim - \log \epsilon$.

Let us consider the asymptotic behavior of the eigenvalue density (4.15) in the limit. Here $\rho_{2, tg}$ of (4.13) is the same functional form as the one in the GWW type model (7.6) of [1], while $\rho_{1, tg}^{susy}$ of (4.15) is regarded as an additional term appearing in the SUSY CS matter theory. We can see that $\rho_{1, tg}^{susy}$ falls off as $\rho_{1, tg}^{susy} \sim - \epsilon \log \epsilon \to 0$. On the other hand $\rho_{2, tg}$ remains finite. So in the limit, $\rho$ becomes

$$
\rho(\alpha) = \rho_{2, tg}(\lambda, \pi \lambda - \frac{1}{2} \epsilon, \pi \lambda + \frac{1}{2} \epsilon; \alpha) = \frac{1}{\pi^2 \lambda} \cos^{-1} \sqrt{\alpha_1}. 
$$

(C.38)

This behavior is very similar to other CS matter theories.

The range of $\epsilon$ in the SUSY CS matter theory would be the smallest among the CS matter theories which we have already studied in this paper. We can estimate $\epsilon$ by calculating the factor $Y_{su}$ in (C.33). (The plots of $Y_{su}$ is in Fig. 11(a)). From this, we can see $Y_{su} > Y_{cb}$, $(Y_{rf}) > 1$. This means that $\epsilon$ in the SUSY CS matter theory is the smallest and shape of the eigenvalue density function of the SUSY CS matter theory has the sharpest cliff among the CS matter theories. (Schematic graph of the eigenvalue density is depicted at Fig. 10(a)). We can see that $\epsilon$ at (C.35) as well as $Y_{su}$ are self-dual under the level-rank duality. We can also see the self-duality of $Y_{su}$ from the graph Fig. 11(a).

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