ON SIMPLE REAL LIE BIALGEBRAS

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Abstract. The explicit list of all almost factorizable Lie bialgebra structures on real absolutely simple Lie algebras is given. By “absolutely simple” we mean a real Lie algebra whose complexification is simple.

Introduction

The theory of Poisson-Lie groups occupies a central place in the theory of Poisson manifolds. The category of connected, simply-connected Poisson-Lie groups is equivalent to the category of Lie bialgebras \cite{Dr}. Therefore, a basic problem in the theory of Poisson manifolds is the classification of Lie bialgebras. A fundamental contribution to this question is the Theorem of Belavin and Drinfeld \cite{BD}, which contains the classification of all the simple factorizable complex Lie bialgebras.

A finite-dimensional real Lie algebra is absolutely simple if its complexification is a simple complex Lie algebra; this notation agrees with the tradition in Lie theory to call “absolutely simple” an object that remains simple after any extension of scalars \cite{D, T}. It is well-known that a simple real Lie algebra is either absolutely simple, or it is the realification of a complex simple Lie algebra.

In this paper, we obtain the following result.

Theorem 1. Let \((\mathfrak{g}_0, \delta)\) be an absolutely simple real Lie bialgebra. Then either \((\mathfrak{g}_0, \delta)\) is triangular, or else it is determined by the pair \((\mathfrak{g}_0, r_0)\) as in Table 1, up to isomorphisms of real Lie bialgebras.

Precisely, there exists a unique Cartan subalgebra \(\mathfrak{h}\) of the complexification \(\mathfrak{g}\) of \(\mathfrak{g}_0\); a system of simple roots \(\Delta \subset \Phi(\mathfrak{g}, \mathfrak{h})\); an involution \(\sigma = \varsigma_\mu \) or \(\omega_{\mu, J}\) as in (1.1), (1.2); a BD-triple \((\Gamma_1, \Gamma_2, T)\); \(t \in \mathbb{R}_{>0} \cup i\mathbb{R}_{>0}\); and a continuous parameter \(\lambda \in \mathfrak{h}^{\otimes 2}\); all these data subject to the restrictions in Table 1; such that \((\mathfrak{g}_0, \delta)\) is isomorphic as real Lie bialgebra to \((\mathfrak{g}^\sigma, \partial r_0)\), where

\[
r_0 = t \left( \frac{1}{2} \sum_{\alpha, \beta \in \Delta} \lambda_{\alpha, \beta} h_\alpha \wedge h_\beta + \frac{1}{2} \sum_{\alpha \in \Phi^+} x_{-\alpha} \wedge x_\alpha + \sum_{\alpha, \beta \in \Phi^+, \alpha < \beta} x_{-\alpha} \wedge x_\beta \right).
\]

Here, \(\lambda - \lambda^{21} = \sum_{\alpha, \beta \in \Delta} \lambda_{\alpha, \beta} h_\alpha \wedge h_\beta\), and \(h_\alpha, x_\beta\) have the usual meaning, see subsections 1.A, 1.B.

Two data in the table give rise to isomorphic Lie bialgebras if and only if they belong to the same line and the rest of the data is conjugated by an automorphism of the Dynkin diagram of order 2.

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| $\mathfrak{g}_0$ | $\sigma$ | Type | BD-triple | Continuous parameter | $t \in$ |
|---|---|---|---|---|---|
| $\mathfrak{g}_2$ | $\varsigma$ | All. | All. | $\lambda_{\alpha,\beta} \in \mathbb{R}$ | $\mathbb{R}$ |
| $\mathfrak{su}(n, n+1)$ | $s_\mu$, $\mu \neq \text{id}$ | $A_{2n}$ | $\mu$-stable | $\lambda_{\alpha,\beta} = \lambda_{\mu(\alpha), \mu(\beta)}$ | $\mathbb{R}$ |
| $\mathfrak{su}(n+1, n+1)$ | | $A_{2n+1}$ | | | |
| $\mathfrak{so}(n-1, n+1)$ | | $D_n$ | | | |
| $\mathfrak{EI}$ | | $E_6$ | | | |
| $\mathfrak{g}_c$ | $\omega$ | All. | $\Gamma_1 = \Gamma_2 = 0$ | $\lambda_{\alpha,\beta} \in i\mathbb{R}$ | $i\mathbb{R}$ |
| $\mathfrak{su}(j, n+1-j)$ | $\omega_{j}$ | $A_n$ | $\Gamma_1 = \Gamma_2 = 0$ | $\lambda_{\alpha,\beta} \in i\mathbb{R}$ | $i\mathbb{R}$ |
| $\mathfrak{so}(2j, 2(n-j)+1)$ | | $B_n$ | | | |
| $\mathfrak{sp}(j, n-j)$ | | $C_n$ | | | |
| $\mathfrak{so}(2j, 2n-2j)$ | | $D_n$ | | | |
| $\mathfrak{so}^*(2n)$ | | $E_6$ | | | |
| $\mathfrak{EI}, \mathfrak{EII}$ | | $E_7$ | | | |
| $\mathfrak{EV}, \mathfrak{EVI}, \mathfrak{EVII}$ | | $E_8$ | | | |
| $\mathfrak{EIX}$ | | $F_4$ | | | |
| $\mathfrak{FII}, \mathfrak{FI}$ | | $G_2$ | | | |
| $\mathfrak{G}$ | | | | | |
| $\mathfrak{sl}(n+1, \mathbb{R})$ | $\omega_{J, \mu}$, $\mu \neq \text{id}$ | $A_n$ | $\mu$-antistable | $\lambda_{\alpha,\beta} = -\lambda_{\mu(\alpha), \mu(\beta)}$ | $i\mathbb{R}$ |
| $\mathfrak{sl}(\frac{n+1}{2}, \mathbb{H})$ | | $A_{2n+1}$ | | | |
| $\mathfrak{so}(1, 2n-1)$ | | $D_n$ | | | |
| $\mathfrak{so}(2j+1, 2(n-j)-1)$ | | $D_n$ | | | |
| $\mathfrak{EL, EIV}$ | | $E_6$ | | | |

**Explanation of the table.** We recall the notions of BD-triples and continuous parameter in subsection 1.5; see Definition 2.3 for the notion of $\mu$-stable or antistable. See also Remark 2.5 for a discussion on the involution. The notation for exceptional $\mathfrak{g}_0$’s is standard and goes back to É. Cartan [C].

**Table 1.** Classification of almost-factorizable simple real Lie bialgebras

Some considerations about real simple Lie bialgebras are already present in [LQ, A, CGR, Ch, KRR]. The compact case is well-known, see for example [KS, LW, M]. All real classical $r$-matrices arising from semisimple Lie algebras are listed in [CGR]. A. Panov described in [P1, P2], all possible Manin triples in a suitable class up to gauge and weak equivalence; see also [De]. One might be able to infer our main result from these papers, with extra work. However, we do not follow this path and deduce the classification directly from the Theorem of Belavin and Drinfeld. The key step in our argument is Lemma 2.1, where we attach a Cartan subalgebra and a choice of positive roots to the pair $(\mathfrak{g}_0, r_0)$.

The complexifications of the Lie bialgebras associated to the pairs $(\mathfrak{g}_0, r_0)$ as in Table 1 are factorizable, as defined in [RS]. This is not the case for the real Lie bialgebras themselves; some of them are quasitriangular, some of them are not. We use the term “almost factorizable” to refer to Lie bialgebras that are factorizable after extension of scalars.

The organization of this paper is as follows. In the first section we recall several well-known facts about real Lie algebras (subsections 1.1 and 1.2) and Lie bialgebras (subsection 1.3). The material on real Lie
algebras is standard but we decided to present it for completeness of the arguments leading to the main result. In the second section we prove two lemmas that imply the main result. In the last section we compute the Drinfeld double and the dual Lie bialgebra of a real absolutely simple Lie bialgebra. Our result is a consequence of the determination of the Drinfeld double of a factorizable Lie bialgebra \[\text{RS}\].

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1. Preliminaries

1.1. Conventions. All Lie algebras are finite-dimensional, unless explicitly stated. We shall say that a Lie bialgebra is simple if the underlying Lie algebra is simple.

Let \(g_0\) be a semisimple real Lie algebra and let \(g\) be its complexification. The Killing form on \(g\) or \(g_0\), is denoted by \((\quad ,\quad )\). The Casimir element of \(g\) is denoted by \(\Omega \in g \otimes g\); that is, \(\Omega = \sum x_i \otimes x^i\) where \((x_i)\), \((x^i)\) is any pair of dual basis with respect to the Killing form of \(g\). Clearly \(\Omega \in g_0 \otimes g_0\).

Let \(\mathfrak{h} \subset g\) be a Cartan subalgebra; we denote by \(\Phi = \Phi(g, \mathfrak{h})\) the set of roots. If \(\Delta\) is a system of simple roots, we denote by \(\Phi^+\) the corresponding set of positive roots. We denote by \(\Omega_0\) the component in \(\mathfrak{h} \otimes \mathfrak{h}\) of \(\Omega\); that is, \(\Omega_0 = \sum h_i \otimes h^i\) where \((h_i)\), \((h^i)\) is any pair of dual basis in \(\mathfrak{h}\) with respect to the restriction of the Killing form of \(g\) to \(\mathfrak{h}\). If \(\alpha \in \Phi^+\), we denote by \(h_\alpha \in \mathfrak{h}\) the unique element in \(\mathfrak{h}\) such that \(\alpha(H) = (h_\alpha | H)\) for all \(H \in \mathfrak{h}\).

If \(V\) is a complex vector space and \(\sigma : V \rightarrow V\) is a sesquilinear map with \(\sigma^2 = \text{id}\), then \(\sigma^* : V^* \rightarrow V^*\) denotes the sesquilinear map with \(\sigma^*(\alpha)(h) = \overline{\alpha(\sigma(h))}\), \(\alpha \in V^*, h \in V\).

1.2. Involutions. Let \(g\) be a complex simple Lie algebra, \(\mathfrak{h}\) a Cartan subalgebra of \(g\), \(\Delta \subset \Phi(g, \mathfrak{h})\) a set of simple roots. Let us choose elements \(e_\alpha \in g_\alpha, e_{-\alpha} \in g_{-\alpha}, \alpha \in \Delta\), such that \([e_\alpha, e_{-\alpha}] = h_\alpha\).

Let \(\mu : \Delta \rightarrow \Delta\) be an automorphism of the Dynkin diagram of order 1 or 2. Let \(J\) be any subset of the set \(\Delta^\mu\) of roots fixed by \(\mu\); let \(\chi_J : \Delta \rightarrow \{0, 1\}\) be the characteristic function of \(J\). Then there exist unique sesquilinear Lie algebra involutions \(\varsigma_\mu, \omega_{\mu, J}\) of \(g\) given respectively by

\[
\begin{align*}
\varsigma_\mu(e_\alpha) &= e_{\mu(\alpha)}, \\
\varsigma_\mu(e_{-\alpha}) &= e_{-\mu(\alpha)}, \\
\omega_{\mu, J}(e_\alpha) &= (-1)^{\chi_J(\alpha)}e_{-\mu(\alpha)}, \\
\omega_{\mu, J}(e_{-\alpha}) &= (-1)^{\chi_J(\alpha)}e_{\mu(\alpha)},
\end{align*}
\]

for all \(\alpha \in \Delta\). This follows at once from Serre’s theorem on the presentation of \(g\), see for instance [Kn]. Necessarily, \(\varsigma_\mu(h_\alpha) = h_{\mu(\alpha)}\), \(\omega_{\mu, J}(h_\alpha) = -h_{\mu(\alpha)}\). We shall abbreviate

\[
\varsigma = \varsigma_{\text{id}}, \quad \omega = \omega_{\text{id}, J}, \quad \omega = \omega_{\Delta}.
\]

Thus \(\omega\) is the Chevalley involution of \(g\), with respect to \(\mathfrak{h}\) and \(\Delta\), and the fixed point set of \(\omega\) is a compact form of \(g\), denoted by \(g_c\).

**Lemma 1.1.** Let \(\sigma : g \rightarrow g\) be a sesquilinear Lie algebra involution such that \(\sigma(\mathfrak{h}) = \mathfrak{h}\). Then \(\sigma^*(\Phi) = \Phi\) and \(\sigma(g_\alpha) = g_{\sigma(\alpha)}\).

(a). If \(\sigma^*(\Delta) = \Delta\), there exists a choice of elements \(e_\alpha \in g_\alpha, e_{-\alpha} \in g_{-\alpha}, \alpha \in \Delta\) as above such that \(\sigma = \varsigma_\mu\) for a unique automorphism \(\mu : \Delta \rightarrow \Delta\) of the Dynkin diagram of order 1 or 2.
(b). If \( \sigma(\Delta)^* = -\Delta \), there exists a choice of elements \( e_\alpha \in g_\alpha, e_{-\alpha} \in g_{-\alpha}, \alpha \in \Delta \) as above such that \( \sigma = \omega_{\mu,J} \) for a unique automorphism \( \mu : \Delta \to \Delta \) of the Dynkin diagram of order 1 or 2, and a unique subset \( J \) of \( \Delta^\mu \).

Proof. It is well-known that \( (x|y) = (\sigma(x)|\sigma(y)) \) for all \( x, y \in g \), see for example [11] p. 180. If \( H \in h \) and \( \alpha \in \Phi \), then for all \( X \in g_\alpha \), one has
\[
[H, \sigma(X)] = \sigma([\sigma(H), X]) = \alpha(\sigma(H))\sigma(X) = \sigma^*(\alpha)(H)\sigma(X).
\]
Hence \( \sigma^*(\Phi) = \Phi \) and \( \sigma(g_\alpha) = g_{\sigma^*(\alpha)} \).

Assume that \( \sigma^*(\Delta) = \pm \Delta \). Let \( (a_{\alpha,\beta})_{\alpha,\beta\in \Delta} \) be the Cartan matrix of \( g \). Let \( \mu : \Delta \to \Delta \) be given by \( \mu = \pm \sigma^* \), according to the case. Then
\[
a_{\alpha,\beta} = 2\frac{(\alpha|\beta)}{(\beta|\beta)} = 2\frac{(\sigma^*(\alpha)|\sigma^*(\beta))}{(\sigma^*(\beta)|\sigma^*(\beta))} = a_{\mu(\alpha),\mu(\beta)},
\]
hence \( \mu \) is an automorphism of the Dynkin diagram, and clearly it has order 1 or 2.

Let \( e_\alpha \in g_\alpha, e_{-\alpha} \in g_{-\alpha}, \alpha \in \Delta \) be as above. Let \( c_\alpha \in \mathbb{C} - 0 \) be such that \( \sigma(e_\alpha) = c_\alpha e_{\sigma^*(\alpha)}, \alpha \in \pm \Delta \). Then, for all \( \alpha \in \Delta \), we have
\[
(1.3) \quad c_\alpha e_{-\alpha} = 1,
\]
since \( 1 = (e_\alpha|e_{-\alpha}) = (\sigma(e_\alpha)|\sigma(e_{-\alpha})) = c_\alpha e_{-\alpha}(e_{\sigma^*(\alpha)}|e_{\sigma^*(\alpha)}) = c_\alpha e_{-\alpha}. \) Also, \( \sigma^2 = \text{id} \) implies that
\[
(1.4) \quad c_\alpha^{\sigma^*} = 1,
\]
for all \( \alpha \in \pm \Delta \).

We prove (a). We want to replace \( e_\alpha, e_{-\alpha} \) by \( \tilde{e}_\alpha = d_\alpha e_\alpha, \tilde{e}_{-\alpha} = (d_\alpha)^{-1}e_{-\alpha} \), for well-chosen non-zero scalars \( d_\alpha, \alpha \in \Delta \), such that
\[
\sigma(\tilde{e}_\alpha) = \tilde{e}_{\mu(\alpha)}(\alpha), \quad \sigma(\tilde{e}_{-\alpha}) = \tilde{e}_{-\mu(\alpha)}(\alpha).
\]
But \( \sigma(\tilde{e}_\alpha) = \tilde{d}_\alpha c_\alpha (d_{\mu(\alpha)})^{-1} \tilde{e}_{\mu(\alpha)}(\alpha) \) and \( \sigma(\tilde{e}_{-\alpha}) = (d_{\alpha})^{-1}c_{-\alpha}(d_{\mu(\alpha)})e_{-\mu(\alpha)}(\alpha) \), thus we need to find \( d_\alpha \) such that \( c_\alpha = (d_\alpha)^{-1}d_{\mu(\alpha)}(\alpha) \). The existence of such \( d_\alpha \) is clear, and the uniqueness of \( \mu \) is evident; (a) follows.

We prove (b). We want to replace \( e_\alpha, e_{-\alpha} \) by \( \tilde{e}_\alpha = d_\alpha e_\alpha, \tilde{e}_{-\alpha} = (d_\alpha)^{-1}e_{-\alpha} \), for well-chosen non-zero scalars \( d_\alpha, \alpha \in \Delta \). We have \( \sigma(\tilde{e}_\alpha) = \tilde{d}_\alpha c_\alpha e_{-\mu(\alpha)}(\alpha) = \tilde{d}_\alpha c_\alpha d_{\mu(\alpha)}(\alpha) \) and \( \sigma(\tilde{e}_{-\alpha}) = (d_{\alpha})^{-1}c_{-\alpha} d_{\mu(\alpha)}(\alpha)^{-1}(\alpha) \).

Assume that \( \mu(\alpha) \neq \alpha \); choose \( d_\alpha, d_{\mu(\alpha)}(\alpha) \) such that \( c_\alpha = (d_\alpha d_{\mu(\alpha)}(\alpha))^{-1}. \) Then \( \sigma(\tilde{e}_\alpha) = \tilde{e}_{-\mu(\alpha)}(\alpha) \), but also \( \sigma(\tilde{e}_{-\alpha}) = e_{\mu(\alpha)}(\alpha) \) by (1.3) and \( \sigma(e_{\pm\mu(\alpha)}(\alpha)) = \tilde{e}_{\mp\alpha}(\alpha) \) by (1.3) and (1.4).

Assume that \( \mu(\alpha) = \alpha \). Then \( c_\alpha \in \mathbb{R} \) by (1.3) and (1.4). Let \( J := \{ \alpha \in \Delta : c_\alpha < 0 \} \). Clearly, we can choose \( d_\alpha \) such that \( c_\alpha d_{\alpha}^{-1} = -1 \), resp. 1, if \( \alpha \in J \), resp. if \( \alpha \notin J \).

The uniqueness of \( \mu \) and \( J \) is evident, and (b) follows. \( \square \)

Remark 1.2. The change of generators \( e_\alpha, e_{-\alpha} \) by \( \tilde{e}_\alpha = d_\alpha e_\alpha, \tilde{e}_{-\alpha} = (d_\alpha)^{-1}e_{-\alpha} \) amounts to composing with an automorphism of \( g \) which preserves \( \mathfrak{h} \).
1.3. **Real Lie algebras and Vogan diagrams.** We first briefly recall the main ingredients of the theory of real simple Lie algebras, in particular Vogan diagrams; see [Kn Chapter VI. Next we apply these methods to the study of the real Lie algebras corresponding to the involutions in the preceding subsection.

Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\sigma$ be a sesquilinear Lie algebra involution of $\mathfrak{g}$. Let $\mathfrak{g}_0$ be the real form $\mathfrak{g}_0 = \mathfrak{g}^\sigma$ of $\mathfrak{g}$. Let $\theta_0$ be a Cartan involution of $\mathfrak{g}_0$ and let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ with $\theta(h_0) = h_0$. Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be the complexification of $\theta_0$. Let $\mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0$, respectively $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$, be the Cartan decomposition associated to $\theta_0$, respectively $\theta$. Denote $\mathfrak{t}_0 := \mathfrak{h}_0 \cap \mathfrak{t}_0$, $\mathfrak{a}_0 := \mathfrak{h}_0 \cap \mathfrak{p}_0$. Let $\text{dc}(\mathfrak{h}_0) = \dim(\mathfrak{t}_0)$, $\text{dnc}(\mathfrak{h}_0) = \dim(\mathfrak{a}_0) = \dim \mathfrak{t}_0 - \text{dc}(\mathfrak{h}_0)$. The Cartan subalgebra $\mathfrak{h}_0$ is **maximally compact**, resp. **maximally non-compact** if $\text{dc}(\mathfrak{h}_0)$ is maximal, resp. minimal, among the dc of $\theta_0$-stable Cartan subalgebras of $\mathfrak{g}_0$. The **rank** of the associated symmetric space coincides with $\text{dnc}(\mathfrak{h}_0)$ if $\mathfrak{h}_0$ is a maximally non-compact Cartan subalgebra.

Recall that a root $\alpha \in \Phi$ is called

- **imaginary** if $\alpha$ vanishes on $\mathfrak{a}_0$,
- **real** if $\alpha$ vanishes on $\mathfrak{t}_0$,
- **complex** otherwise.

Also, an imaginary root $\alpha \in \Phi$ is called

- **compact** if $\mathfrak{g}_0 \subset \mathfrak{t}$, i.e. if $\theta$ is the id on $\mathfrak{g}_\alpha$,
- **non-compact** if $\mathfrak{g}_0 \subset \mathfrak{p}$, i.e. if $\theta$ is $-\text{id}$ on $\mathfrak{g}_\alpha$.

A **Vogan diagram** consists of the following data: a Dynkin diagram $\Delta$, an involution $\mu \in \text{Aut}(\Delta)$, and a subset $P$ of roots in $\Delta^\mu$. The roots in $P$ are “painted” in the graphical description of the Vogan diagram. The Vogan diagram of a simple real Lie algebra $\mathfrak{g}_0$ is as follows: the Dynkin diagram corresponds to the complexification $\mathfrak{g}$. Then one fixes a Cartan involution $\theta_0$ and takes a maximally compact $\theta$-stable Cartan subalgebra $\mathfrak{h}_0$, together with a system of simple roots $\Delta \subset \Phi(\mathfrak{g}, \mathfrak{h})$ which is stable under the transpose of $\theta$; $\mu$ is the restriction of $\theta$. Finally, $P$ is the set of non-compact simple roots.

A **normalized Vogan diagram** is a Vogan diagram with at most one painted vertex, i.e. at most one non-compact simple root.

The notion of normalized Vogan diagram helps to take care of redundancies in the classification of real simple Lie algebras. Indeed, the Theorem of Borel and de Siebenthal [Kn Th. 6.96] allows to go from the Vogan diagram of a simple real Lie algebra $\mathfrak{g}_0$ to another one with at most one painted simple root, just by changing appropriately the choice of the system of simple roots.

Let now $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$ and let $\Delta \subset \Phi(\mathfrak{g}, \mathfrak{h})$ be a fixed system of simple roots. Assume that $\sigma$ is a sesquilinear Lie algebra involution of the form $\varsigma_\mu$ or $\omega_{\mu,J}$, for $\mu : \Delta \to \Delta$ an automorphism of the Dynkin diagram of order 1 or 2, and $J$ any subset of the set $\Delta^\mu$. Let $\mathfrak{g}_0$ be the real form $\mathfrak{g}_0 = \mathfrak{g}^\sigma$ of $\mathfrak{g}$. Let $\mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{g}_0$, a Cartan subalgebra of $\mathfrak{g}_0$. Our goal is to determine $\mathfrak{g}_0$, and the isomorphism classes of pairs $(\mathfrak{g}_0, \mathfrak{h}_0)$.

Since $\sigma$ commutes with $\omega$, $\sigma$ preserves $\mathfrak{g}_0$. Let $\theta_0 : \mathfrak{g}_0 \to \mathfrak{g}_0$ be the linear Lie algebra involution given by the restriction of $\omega$. Then $\theta_0$ is a Cartan involution of $\mathfrak{g}_0$ and $\theta(\mathfrak{h}_0) = \mathfrak{h}_0$; its complexification is clearly $\theta = \sigma \omega$. If $\sigma = \omega_{\mu,J}$, the transpose of $\theta$ preserves $\Delta$, and in fact coincides with $\mu$; if $\sigma = \varsigma_\mu$ the transpose of $\theta$ coincides with $-\mu$.

**Lemma 1.3.** Assume that $P = \Delta^\mu - J$ has at most one element. Then the pair $(\mathfrak{g}_0, \mathfrak{h}_0)$ determined by $\sigma$ is as in Table 2.

**Proof.** Let us analyze the different possibilities for $\sigma$. 
(i). $\sigma = \varsigma$. Then $h_0$ is the real span of $h_\alpha, \alpha \in \Delta$; it is a split Cartan subalgebra of $\mathfrak{g}_0$ and this last is the split real form of $\mathfrak{g}$. Also, $dc(h_0) = 0$, $dnc(h_0) = \# \Delta$.

(ii). $\sigma = \varsigma_\mu, \mu \neq id$. Then $h_0 = t_0 \oplus a_0$, where $t_0$ is the real span of $\{i(h_\alpha - h_{\mu(\alpha)}) : \alpha \in \Delta - \Delta^\mu\}$; and $a_0$ is the real span of $\{h_\alpha : \alpha \in \Delta^\mu\} \cup \{h_\alpha + h_{\mu(\alpha)} : \alpha \in \Delta - \Delta^\mu\}$. In this case, $\sigma$ is a Steinberg normal form,
i. e., it leaves a Borel subalgebra invariant. The list of all such is well-known, see for example [W1, W2]. For the sake of completeness we briefly complete the argument.

It is easy to see that there are no imaginary roots; therefore, \( \mathfrak{h}_0 \) is maximally non-compact by [Kn Prop. 6.70]. Also, \( \text{dc}(\mathfrak{h}_0) = \frac{\#(\Delta - \Delta^\mu)}{2} \), \( \text{dnc}(\mathfrak{h}_0) = \#\Delta - \frac{\#(\Delta - \Delta^\mu)}{2} = \#\Delta^\mu + \frac{\#(\Delta - \Delta^\mu)}{2} = \text{rank of the associated symmetric space} \). We can not apply the results in [Kn, Chapter VI]. However, the necessary information can be obtained looking at the classification of symmetric spaces in [Π, Table V, p. 518].

If \( \mathfrak{g} \) is of type \( A_{2n} \), then the rank of the associated symmetric space is \( n \). We conclude that \( \mathfrak{g}_0 = \mathfrak{su}(n, n + 1) \). If \( \mathfrak{g} \) is of type \( A_{2n+1} \), then the rank of the associated symmetric space is \( n + 1 \). We conclude that \( \mathfrak{g}_0 = \mathfrak{su}(n + 1, n + 1) \). If \( \mathfrak{g} \) is of type \( D_n \), \( n \geq 4 \), then the rank of the associated symmetric space is \( n - 1 \). We conclude that \( \mathfrak{g}_0 = \mathfrak{so}(n - 1, n + 1) \). If \( \mathfrak{g} \) is of type \( E_6 \), then the rank of the associated symmetric space is 4. We see that \( \mathfrak{g}_0 \) is of type EII.

(iii). \( \sigma = \omega_J \). Then \( \mathfrak{h}_0 \) is the real span of \( i\hbar_\alpha, \alpha \in \Delta \); it is clearly maximally compact and \( \Delta \) is \( \theta \)-stable. It is easy to see that \( P = \Delta - J \), i. e. that the compact roots are precisely those in \( J \). Also, \( \text{dnc}(\mathfrak{h}_0) = 0 \), \( \text{dc}(\mathfrak{h}_0) = \#\Delta \). We can then apply the results in [Kn, Chapter VI, pages 355 to 362].

(iv). \( \sigma = \omega_{\mu,J}, \mu \neq \text{id} \). Then \( \mathfrak{h}_0 \) is the real span of \( \{i\hbar_\alpha : \alpha \in \Delta^\mu\} \cup \{i(h_\alpha + h_{\mu(\alpha)} : \alpha \in \Delta - \Delta^\mu\} \cup \{h_\alpha - h_{\mu(\alpha)} : \alpha \in \Delta - \Delta^\mu\} \). Clearly, \( \Delta \) is \( \theta \)-stable. It is easy to see that there are no real roots; therefore, \( \mathfrak{h}_0 \) is maximally compact by [Kn Prop. 6.70]. It is easy to see that \( P = \Delta^\mu - J \), i.e. that the compact roots are precisely those in \( J \). Also, \( \text{dnc}(\mathfrak{h}_0) = \frac{\#(\Delta - \Delta^\mu)}{2} \), \( \text{dc}(\mathfrak{h}_0) = \#\Delta - \frac{\#(\Delta - \Delta^\mu)}{2} \). We can then apply the results in [Kn, Chapter VI, pages 355 to 362]. \( \square \)

1.4. Lie bialgebras. Recall that a quasitriangular Lie bialgebra \((\mathfrak{g}, r)\) is called factorizable if \( r + r^{21} \in S^2 \mathfrak{g} \) defines a nondegenerate inner product on \( \mathfrak{g}^* \) [RS].

**Definition 1.4.** We shall say that a real Lie bialgebra \( (\mathfrak{l}_0, \delta) \) is **almost factorizable** if the complexification \( (\mathfrak{l}, \delta) \) is factorizable.

We shall consider the following particular class of almost factorizable Lie bialgebras. A Lie bialgebra \( (\mathfrak{l}_0, \delta) \) is **imaginary factorizable** if the complexification \( (\mathfrak{l}, \delta) \) is factorizable and \( r \in \mathfrak{l} \otimes \mathfrak{l} \) is given by

\[(1.5) \quad r = r_\Lambda + ir_\Omega, \quad \text{where } r_\Lambda \in \Lambda^2(\mathfrak{l}_0), \quad r_\Omega \in S^2(\mathfrak{l}_0).\]

Real factorizable Lie bialgebras are of course almost factorizable, but we shall see that the converse is not true. In fact, real simple Lie bialgebras are triangular, factorizable or imaginary factorizable.

Doubles and duals of imaginary factorizable Lie bialgebras are computed in Proposition 3.21 below.

**Lemma 1.5.** Let \( (\mathfrak{g}_0, \delta) \) be an absolutely simple real Lie bialgebra. Then \( \delta = \partial r_0 \) for a unique \( r_0 \in \Lambda^2(\mathfrak{g}_0) \) and \( (\mathfrak{g}_0, \delta) \) is either triangular or almost factorizable.

**Proof.** Let \( \mathfrak{g} \) be the complexification of \( \mathfrak{g}_0 \). By Whitehead’s Lemma, there exists \( r_0 \in \Lambda^2(\mathfrak{g}_0) \) such that \( \delta(x) = \text{ad}_x r_0 \) for all \( x \in \mathfrak{g}_0 \), and \textit{a fortiori} for all \( x \in \mathfrak{g} \). (Note that \( r_0 \) is uniquely defined by \( \delta \) since \( \Lambda^2(\mathfrak{g}_0)^{\mathfrak{g}_0} = 0 \)). Then \( [r_0^{12}, r_0^{13}] + [r_0^{12}, r_0^{23}] + [r_0^{13}, r_0^{23}] \in \Lambda^3(\mathfrak{g}_0)^{\mathfrak{g}_0} \), and there exists \( c \in \mathbb{C} \) such that \( [r_0^{12}, r_0^{13}] + \)}
continuous parameter $x$ where $r$, $\beta \in \mathfrak{g}$. There exist a Cartan subalgebra (Theorem 2. Let $\Gamma$ be a choice of a set of simple roots. There are quantizations of these Lie bialgebras: these are $*$-Hopf algebras in the factorizable case, and real Hopf algebras in the imaginary factorizable case.

1.5. The theorem of Belavin and Drinfeld. Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Let $\Delta$ be a choice of a set of simple roots.

**Definition 1.6.** A Belavin-Drinfeld triple (BD-triple for short) is a triple $(\Gamma_1, \Gamma_2, T)$ where $\Gamma_1$, $\Gamma_2$ are subsets of $\Delta$ and $T: \Gamma_1 \to \Gamma_2$ is a bijection that preserves the inner product and satisfies the nilpotency condition: for any $\alpha \in \Gamma_1$ there exists a positive integer $n$ for which $T^n(\alpha)$ belongs to $\Gamma_2$ but not to $\Gamma_1$.

Let $(\Gamma_1, \Gamma_2, T)$ be a Belavin-Drinfeld triple. Let $\hat{\Gamma}_1$ be the set of positive roots lying in the subgroup generated by $\Gamma_i$, for $i = 1, 2$. There is an associated partial ordering on $\Phi^+$ given by $\alpha \prec \beta$ if $\alpha \in \hat{\Gamma}_1$, $\beta \in \hat{\Gamma}_2$, and $\beta = T^n(\alpha)$ for a positive integer $n$.

A continuous parameter for the Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, T)$ is an element $\lambda \in \mathfrak{h}^{\otimes 2}$ such that

\[
(T(\alpha) \otimes 1)\lambda + (1 \otimes \alpha)\lambda = 0, \quad \text{for all } \alpha \in \Gamma_1,
\]

and

\[
\lambda + \lambda^{2\mathfrak{g}} = \Omega_0.
\]

Let $\mathfrak{a}_1$, $\mathfrak{a}_2$ be the reductive subalgebras of $\mathfrak{g}$ with Cartan subalgebras generated by $h_\alpha$, $\alpha$ in $\Gamma_1$, resp. in $\Gamma_2$, and with Dynkin diagrams $\Gamma_1$, resp. in $\Gamma_2$. We extend $T$ to a Lie algebra isomorphism $\hat{T}: \mathfrak{a}_1 \to \mathfrak{a}_2$.

**Theorem 2.** (Belavin-Drinfeld, see [BD]). Let $(\mathfrak{g}, \delta)$ be a factorizable complex simple Lie bialgebra. Then there exist a Cartan subalgebra $\mathfrak{h}$, a system of simple roots $\Delta$, a Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, T)$, a continuous parameter $\lambda$ and $t \in \mathbb{C} - 0$ such that the $r$-matrix is given by

\[
r = t \left( \lambda + \sum_{\alpha \in \Phi^+} x_{-\alpha} \otimes x_\alpha + \sum_{\alpha, \beta \in \Phi^+, \alpha \prec \beta} x_{-\alpha} \wedge x_\beta \right),
\]

where $x_\beta \in \mathfrak{g}_\beta$, $\beta \in \pm \Phi^+$, are root vectors normalized by

\[
(x_\beta | x_-\beta) = 1, \quad \text{for all } \beta \in \Phi^+,
\]

and

\[
\hat{T}(x_\beta) = x_{T(\beta)}, \quad \text{for all } \beta \in \Gamma_1.
\]

Clearly, $r + r^{2\mathfrak{g}} = t\Omega$. 

\[
[r_0^{12}, r_0^{23}] + [r_0^{13}, r_0^{23}] = c^2[\Omega_1, \Omega_2].
\]
2. Proof of the main theorem

The proof of the main result is split into two lemmas.

Lemma 2.1. (I). Let \((g_0, \delta)\) be an absolutely simple real Lie bialgebra. Let \(g\) be the complexification of \(g_0\) and let \(\sigma\) be the sesquilinear involution of \(g\) whose fixed-point set is \(g_0\). Assume that \((g_0, \delta)\) is almost factorizable. Then there exist:

- a complex number \(c \neq 0\) with \(c^2 \in \mathbb{R}\); set \(t = 2ic\) and fix the choice \(t \in \mathbb{R}_{>0} \cup i\mathbb{R}_{>0}\);
- a Cartan subalgebra \(h\) of \(g\);
- a system of simple roots \(\Delta\), a Belavin-Drinfeld triple \((\Gamma_1, \Gamma_2, T)\) and a continuous parameter \(\lambda \in h^{\otimes 2}\);

such that

- \(h\) is stable under \(\sigma\) (we denote \(h_0 := h \cap g_0\)).
- \(\sigma^*(\Delta)\) is either \(\Delta\) or \(-\Delta\); furthermore \(\mu := \sigma^* : \Delta \to \pm \Delta\) is an automorphism of the Dynkin diagram. If \(\sigma^*(\Delta) = \Delta\) then there exists an automorphism \(\mu\) of the Dynkin diagram such that, for an appropriate choice of the \(e_\alpha\)'s, \(\sigma\) is \(\varsigma_\mu\).
- If \(\sigma^*(\Delta) = -\Delta\) then there exists an automorphism \(\mu\) of the Dynkin diagram and \(J \subset \Delta^\mu\) such that, for an appropriate choice of the \(e_\alpha\)'s, \(\sigma\) is \(\omega_{\mu,J}\).
- \(\delta = \partial r_0\) where \(r_0 \in \Lambda^2(g_0)\) is given by the formula

\[
(2.1) \quad r_0 = t \left( \frac{1}{2} \left( \lambda - \lambda^{21} \right) + \frac{1}{2} \sum_{\alpha \in \Phi^+} x_{-\alpha} \wedge x_{\alpha} + \sum_{\alpha, \beta \in \Phi^+, \alpha < \beta} x_{-\alpha} \wedge x_{\beta} \right).
\]

(II). Let \((g_0, \delta), (g_0', \delta')\) be two almost factorizable absolutely simple real Lie bialgebras. Let \(\psi : g_0 \to g_0\) be an isomorphism of Lie bialgebras. Let \(\sigma, \sigma'\) be the involutions corresponding to \(g_0, g_0'\). Let \(h', h'_0, \Delta' \subset \Phi(g, h')\); \((\Gamma'_1, \Gamma'_2, T')\), \(\lambda' \in h'^{\otimes 2}, c', \ldots, \) be the corresponding objects for the Lie bialgebra \(g'_0\). Then

- \(\psi(h') = h\).
- If \(\sigma^*(\Delta) = \Delta, \text{ resp. } -\Delta\), then \(\sigma'^*(\Delta') = \Delta', \text{ resp. } -\Delta'\).
- \(\psi\) induces an isomorphism of Dynkin diagrams \(\psi^* : \Delta \to \Delta'\);
- \(\mu' \psi^* = \psi^* \mu, J' = \psi^*(J)\);
- \((\Gamma'_1, \Gamma'_2, T') = (\psi^*(\Gamma_1), \psi^*(\Gamma_2), \psi^*T\psi^{-1})\).

In other words, the Lemma says that, for an appropriate choice of the \(e_\alpha\)'s, \(\sigma\) is either \(\varsigma\) or \(\varsigma_\mu\) with \(\mu \neq \text{id}\), or \(\omega_J\), or \(\omega\), or \(\omega_{\mu,J}\) with \(\mu \neq \text{id}\); and that this does not depend on the isomorphism class as Lie bialgebra.

The uniqueness in Part II uses in an essential way that the parameter \(t\) is in \(\mathbb{R}_{>0} \cup i\mathbb{R}_{>0}\). Changing \(t\) to \(-t\) would affect (2.1) by changing \(\Delta\) to \(-\Delta\), etc.

Proof. (I). By the Theorem of Belavin and Drinfeld, there exist a Cartan subalgebra \(h\) of \(g\), a system of simple roots \(\Delta\), a Belavin-Drinfeld triple \((\Gamma_1, \Gamma_2, T)\), a continuous parameter \(\lambda \in h^{\otimes 2}\) and a non-zero complex number \(c\) (with \(c^2 \in \mathbb{R}\) by Lemma 1.5) such that the complexification \((g, \delta)\) is quasitriangular with \(r\)-matrix \(r\) given by (1.1) and \(t = 2ic\). Then \(\delta = \partial r_0\), where \(r_0\) is given by (2.1); and \(r_0 \in \Lambda^2(g_0)\) by the uniqueness in Lemma 1.5.
Let \( H \) be the image of \( r_0 \) under the Lie bracket \([\cdot, \cdot] : \mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathfrak{g}_0\); then \( H = -2ic \sum_{\alpha \in \Phi^+} h_\alpha \). Indeed, if \( \alpha < \beta \) then \([x_-, x_\beta] = 0\) because \( \alpha \neq \beta \) but both have the same level with respect to \( \Delta \). Note that here the nilpotency condition on the BD-triple is used. Since \( H \) is a regular element of \( \mathfrak{h} \) by [11, 13.3], \( \mathfrak{h} = \text{Cent}_g(H) \) is a Cartan subalgebra of \( \mathfrak{g} \); since \( \mathfrak{h} \) is the complexification of \( \text{Cent}_{\mathfrak{g}_0}(H) \), we see that \( \mathfrak{h} \) is stable under \( \sigma \).

By Lemma 1.1, \( \Phi(g) \) is stable under \( \sigma^* \), and \( \sigma(g_\alpha) = g_{\sigma^*(\alpha)} \), for all \( \alpha \in \Phi(g, \mathfrak{h}) \).

Let
\[
W = \sum_{\alpha, \beta \in \Phi: \alpha + \beta = 0} g_\alpha \otimes g_\beta, \quad U = \sum_{\alpha, \beta \in \Phi: \alpha + \beta \neq 0} g_\alpha \otimes g_\beta.
\]

It is clear that
\[
(\sigma \otimes \sigma) \left( \sum_{\alpha \in \Phi^+} x_- \wedge x_\alpha \right) \in W, \quad (\sigma \otimes \sigma) \left( \sum_{\alpha, \beta \in \Phi^+, \alpha < \beta} x_- \wedge x_\beta \right) \in U.
\]

Assume that \( c \in i\mathbb{R}, i.e. \) that \((g_0, \delta)\) is quasitriangular. It follows that
\[
(\sigma \otimes \sigma) \left( \sum_{\alpha \in \Phi^+} x_- \wedge x_\alpha \right) = \sum_{\alpha \in \Phi^+} x_- \wedge x_\alpha.
\]

Since the elements \( x_- \wedge x_\alpha \) are linearly independent in \( W \), we conclude that \( \sigma^*(\Delta) = \Delta \); \( \mu \) is an automorphism of the Dynkin diagram by Lemma 1.1.

Similarly, if \( c \in \mathbb{R}, i.e. \) if \((g_0, \delta)\) is imaginary factorizable, we conclude that \( \sigma^*(\Delta) = -\Delta \) and \( \mu \) is an automorphism of the Dynkin diagram again by Lemma 1.1.

(II). We assume that \( \mathfrak{g} \) is the complexification of both \( \mathfrak{g}_0, \mathfrak{g}_0' \) (equality, not just isomorphism); \( \psi_0 \) extends to a Lie algebra automorphism \( \psi \) of \( \mathfrak{g} \), and \( \sigma \psi = \psi \sigma' \).

Let \( r_0 \in \Lambda^2(\mathfrak{g}_0), r_0' \in \Lambda^2(\mathfrak{g}_0') \) be such that \( \delta = \partial r_0, \delta' = \partial r_0' \). Thus
\[
r_0' = t' \left( \frac{1}{2}(\lambda' - \lambda)^{21} + \frac{1}{2} \sum_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})^+} x_{\alpha} - x_{\alpha'} + \sum_{\alpha, \beta \in \Phi(\mathfrak{g}, \mathfrak{h}), \alpha < \beta} x_{\alpha} - x_{\beta} \right).
\]

Since \( r_0 \) is unique, \((\psi_0 \otimes \psi_0)(r_0') = r_0 \) and \( a \text{ fortiori} \psi(b_0') = h_0, \psi(h') = h \). Thus
\[
r_0 = t' \left( \frac{1}{2}(\psi \otimes \psi)(\lambda' - \lambda)^{21} + \frac{1}{2} \sum_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})^+} \psi(x_{\alpha}) \wedge \psi(x_{\alpha'}) + \sum_{\alpha, \beta \in \Phi(\mathfrak{g}, \mathfrak{h}), \alpha < \beta} \psi(x_{\alpha}) \wedge \psi(x_{\beta}) \right).
\]

But also \( \psi \) induces a bijection \( \psi^* : \Phi(\mathfrak{g}, \mathfrak{h}) \to \Phi(\mathfrak{g}, \mathfrak{h}') \), and \( \psi(g_{\sigma^*}) = g_\alpha \). Arguing as above, we conclude that
\[
(2.2) \quad t'(\psi \otimes \psi)(\lambda' - \lambda)^{21} = t(\lambda - \lambda)^{21},
\]
\[
(2.3) \quad t' \sum_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})^+} \psi(x_{\alpha}) \wedge \psi(x_{\alpha'}) = t \sum_{\alpha \in \Phi^+} x_{\alpha} \wedge x_{\alpha},
\]
\[
(2.4) \quad t' \sum_{\alpha, \beta \in \Phi(\mathfrak{g}, \mathfrak{h}), \alpha < \beta} \psi(x_{\alpha}) \wedge \psi(x_{\beta}) = t \sum_{\alpha, \beta \in \Phi^+, \alpha < \beta} x_{\alpha} \wedge x_{\beta}.
\]

Now, \((\mathfrak{g}_0, \delta), (\mathfrak{g}_0', \delta')\) are either both quasitriangular, or both imaginary factorizable; that is, are either both \( t, t' \) are in \( \mathbb{R}_{>0} \), or both in \( i\mathbb{R}_{>0} \). By (2.3), \( \Phi(\mathfrak{g}, \mathfrak{h})^+ = \psi^*(\Phi(\mathfrak{g}, \mathfrak{h})^+) \) and hence \( \Delta' = \psi^*(\Delta) \).

Hence, (2.4) implies that \( (\Gamma'_1, \Gamma'_2, T') = (\psi^*(\Gamma_1), \psi^*(\Gamma_2), \psi^* T^* \psi^{-1}) \).

From the equality \( \sigma'^* \psi^* = \psi^* \sigma^* \) we conclude that \( \mu' \psi^* = \psi^* \mu \).
Finally, recall from the proof of the Lemma 1.1 that \( J = \{ \alpha \in \Delta^\mu : \sigma(e_\alpha) = c_\alpha e_{\sigma^* \alpha} \text{ with } c_\alpha < 0 \} \). It follows without difficulties that \( J' = \psi^*(J) \).

\[ \text{Remark 2.2.} \text{ The description of } r_0 \text{ in } (2.1) \text{ depends on the choice of a family } x_\alpha \in \mathfrak{g}_\alpha, \alpha \in \pm \Phi^+, \text{ satisfying (1.10), (1.11). Such a family can be constructed starting from any choice of } x_\alpha \in \mathfrak{g}_\alpha \text{ for } \alpha \in \Gamma_1 - \Gamma_2; \text{ different choices do not affect (2.1).} \]

On the other hand, the choice of elements \( e_\alpha \) in Lemma 1.1 is independent of the \( x_\beta \)'s; the arguments in the preceding and next lemmas do not depend on the explicit form of the \( x_\alpha \) but on which root space they are living in.

\[ \text{Definition 2.3. Let } \mu \text{ be an automorphism of the Dynkin diagram. A BD-triple } (\Gamma_1, \Gamma_2, T) \text{ is } \mu\text{-stable if } \mu(\Gamma_1) = \Gamma_1, \mu(\Gamma_2) = \Gamma_2, \text{ and } T\mu = \mu T. \text{ A BD-triple } (\Gamma_1, \Gamma_2, T) \text{ is } \mu\text{-antistable if } \mu(\Gamma_1) = \Gamma_2, \mu(\Gamma_2) = \Gamma_1, \text{ and } T^{-1}\mu = \mu T. \text{ In particular, if } \mu = \text{id then all BD-triples are } \mu\text{-stable, and the only BD-triple } \mu\text{-antistable has } \Gamma_1 = \Gamma_2 = \emptyset. } \]

\[ \text{Lemma 2.4. Let } \mathfrak{g} \text{ be a simple complex Lie algebra, } \mathfrak{h} \text{ a Cartan subalgebra of } \mathfrak{g}, \Delta \subset \Phi(\mathfrak{g}, \mathfrak{h}) \text{ a system of simple roots, } (\Gamma_1, \Gamma_2, T) \text{ a BD-triple, } \lambda \in \mathfrak{h}^{\otimes 2} \text{ a continuous parameter and } t \text{ a complex number. Write } \]

\( \lambda - \lambda^{21} = \sum_{\alpha, \beta \in \Delta} \lambda_{\alpha, \beta} h_\alpha \wedge h_\beta. \)

By convention, \( \lambda_{\alpha, \beta} = -\lambda_{\beta, \alpha} \) for all \( \alpha, \beta \in \Delta \). Let \( r_0 \) be given by formula (2.1). Let \( \sigma \) be an involution of the form \( \varsigma, \omega, \omega_J, \) or \( \varsigma_{\mu}, \omega_{\mu, J} \) with \( \mu \neq \text{id} \). Let \( \mathfrak{g}_0 \) be the real Lie algebra of vectors fixed by \( \sigma \). Then:

(a). Assume that \( \sigma = \varsigma \). Then \( r_0 \in \Lambda^2(\mathfrak{g}_0) \) if and only if \( t \in \mathbb{R}, \lambda_{\alpha, \beta} \in \mathbb{R} \) for all \( \alpha, \beta \in \Delta \) (no restrictions on the BD-triple).

(b). Assume that \( \sigma = \varsigma_{\mu} \). Then \( r_0 \in \Lambda^2(\mathfrak{g}_0) \) if and only if \( t \in \mathbb{R}, \lambda_{\alpha, \beta} = \lambda_{\mu(\alpha), \mu(\beta)} \) for all \( \alpha, \beta \in \Delta \) and the BD-triple is \( \mu \)-stable.

(c). Assume that \( \sigma = \omega_J \), or \( \sigma = \omega \). Then \( r_0 \in \Lambda^2(\mathfrak{g}_0) \) if and only if \( t \in i\mathbb{R}, \lambda_{\alpha, \beta} \in i\mathbb{R} \), for all \( \alpha, \beta \in \Delta \) and the BD-triple has \( \Gamma_1 = \Gamma_2 = \emptyset \).

(d). Assume that \( \sigma = \omega_{\mu, J} \). Then \( r_0 \in \Lambda^2(\mathfrak{g}_0) \) if and only if \( t \in i\mathbb{R}, \lambda_{\alpha, \beta} = -\lambda_{\mu(\alpha), \mu(\beta)} \), for all \( \alpha, \beta \in \Delta \) and the BD-triple is \( \mu \)-antistable.

\[ \text{Proof.} \text{ Since } \sigma \otimes \sigma \text{ is the sesquilinear involution of } \mathfrak{g} \otimes \mathfrak{g} \text{ corresponding to } \mathfrak{g}_0 \otimes \mathfrak{g}_0, \text{ it is clear that } r_0 \in \Lambda^2(\mathfrak{g}_0) \text{ if and only if } (\sigma \otimes \sigma)(r_0) = r_0. \text{ The lemma follows by a direct computation that we include below for completeness. Note that } \]

\( (\sigma \otimes \sigma)(r_0) = \frac{1}{2} (\sigma \otimes \sigma)(\lambda - \lambda^{21}) + \frac{1}{2} \sum_{\alpha \in \Phi^+} (\sigma \otimes \sigma)(x_{-\alpha} \wedge x_\alpha) + \sum_{\alpha, \beta \in \Phi^+, \alpha < \beta} (\sigma \otimes \sigma)(x_{-\alpha} \wedge x_\beta) = r_0 \)
Thus, \( \text{card} \sum_{\alpha, \beta} \) holds if and only if \( \sum_{\alpha, \beta} \) holds.

In cases (a) and (b), \( (2.6) \) holds if and only if \( \sum_{\alpha, \beta} \) holds if and only if \( \lambda_{\alpha, \beta} \in \mathbb{R} \) -in case (a)– or \( \lambda_{\alpha, \beta} = \frac{\lambda(\alpha, \beta)}{\mu(\alpha, \beta)} \) -in case (b)– for all \( \alpha, \beta \in \Delta \). Finally, in case (b), \( (2.7) \) holds if and only if \( \sum_{\alpha, \beta} \) holds.

It is then easy to see that equality \( (2.7) \) holds if and only if the BD-triple is \( \mu \)-stable. Indeed, let \( \mathcal{P} = \{ (\alpha, \beta) \in \Phi^+ \times \Phi^+ : \alpha \prec \beta \} \). The equality implies that \( \mu \times \mu(\mathcal{P}) = \mathcal{P} \) and that \( \mu(\alpha) \prec \mu(\beta) \) if \( \alpha \prec \beta \). Since \( \alpha \prec T\alpha \) for any \( \alpha \in \Gamma_1 \), we conclude that \( \mu(\Gamma_1) = \Gamma_1 \), and similarly, \( \mu(\Gamma_2) = \Gamma_2 \). For any \( \alpha \in \Gamma_1 \), we set

\[
X(\alpha) = \{ \gamma \in \Gamma_1 : \alpha \prec \gamma \text{ or } \gamma \prec \alpha \text{ or } \gamma = \alpha \} = \{ \gamma \in \Gamma_1 : \gamma = T^m \alpha, \ m \in \mathbb{Z} \}.
\]

Clearly, \( X(\alpha) = X(\mu(\alpha)) \). If card \( X(\alpha) = 1 \), it is easy to see that \( T \mu(\alpha) = T(\alpha) \). Assume that card \( X(\alpha) > 1 \). Now, there exists \( \alpha_0 \) such that \( X(\alpha) = \{ \alpha_0, \alpha_1, \ldots, \alpha_s \} \), where \( \alpha_i < \alpha_{i+1} = T\alpha_i, 0 \leq i < s \).

Thus, \( \mu(X(\alpha)) = \{ \mu(\alpha_0), \mu(\alpha_1), \ldots, \mu(\alpha_s) \} \) and \( T\mu(\alpha_i) = \mu(\alpha_{i+1}) = \mu(T\alpha_i) \) for all \( i \). This shows that \( T\mu = \mu T \).

In cases (c) and (d), \( (2.6) \) holds if and only if \( \lambda_{\alpha, \beta} \in i\mathbb{R} \) -in case (c)– and \( \lambda_{\alpha, \beta} = -\frac{\lambda(\alpha, \beta)}{\mu(\alpha, \beta)} \) -in case (d)– for all \( \alpha, \beta \in \Delta \). Finally, in case (d), \( (2.7) \) holds if and only if \( \sum_{\alpha, \beta} \) holds.

It is easy then to see that equality \( (2.7) \) holds if and only if the BD-triple is \( \mu \)-antistable. Indeed, let \( \mathcal{P}^t = \{ (\beta, \alpha) \in \Phi^+ \times \Phi^+ : \alpha \prec \beta \} \). The equality implies that \( \mu \times \mu(\mathcal{P}) = \mathcal{P}^t \) and that \( \mu(\beta) \prec \mu(\alpha) \) if \( \alpha \prec \beta \). Since \( \alpha \prec T\alpha \) for any \( \alpha \in \Gamma_1 \), we conclude that \( \mu(\Gamma_1) = \Gamma_2 \), and similarly, \( \mu(\Gamma_2) = \Gamma_1 \). Keep the notation for \( X(\alpha) \) as above. For any \( \beta \in \Gamma_2 \), we set

\[
Y(\beta) = \{ \delta \in \Gamma_2 : \beta \prec \delta \text{ or } \delta \prec \beta \text{ or } \delta = \beta \} = \{ \delta \in \Gamma_2 : \delta = T^m \beta, \ m \in \mathbb{Z} \}.
\]

Clearly, \( X(\alpha) = X(\mu(\alpha)) \). If card \( X(\alpha) = 1 \), it is easy to see that \( T^{-1}\mu(\alpha) = T(\alpha) \). Assume that card \( X(\alpha) > 1 \). Now, there exists \( \beta_0 \) such that \( Y(\mu(\alpha)) = \{ \beta_0, \beta_1, \ldots, \beta_s \} \), where \( \beta_i < \beta_{i+1} = T\beta_i, 0 \leq i < s \).

Thus, \( \mu(\alpha_i) = \beta_{s-i} \) and \( T\alpha_i) = \mu(\alpha_{i+1}) = \beta_{s-i-1} = T^{-1}\beta_{s-i} = T^{-1}\mu(\alpha_i) \) for all \( i \). This shows that \( T^{-1}\mu = \mu T \).

We have collected now all the necessary information to prove the main result.

**Proof of Theorem 1.** By Lemmas 2.1 part (I) and 2.4, we know the existence of \( h, \Delta, \sigma \) of the type \( g_{\lambda}^\varphi \) or \( \omega_{\lambda,i} \), \( (\Gamma_1, \Gamma_2, T) \), \( t \in \mathbb{R}_{>0} \cup i\mathbb{R}_{>0} \), and \( \lambda \in \mathfrak{h}^{\otimes 2} \); such that \( (g_0, \delta) \) is isomorphic as real Lie bialgebra to \( (\mathfrak{g}^\varphi, \partial r_0) \), where \( r_0 \) is given by 2.1.
It remains to determine when different data in Table 1 give rise to isomorphic Lie bialgebras. By Lemma 2.1 part (II) and Lemma L.3, isomorphic Lie bialgebras can arise only from data in the same row; then the statement follows from Lemma 2.1 part (II). \hfill \Box

Remark 2.5. Let $h_0 = \text{Cent}_{g_0}(H)$, where $H$ is the image of $r_0$ under the Lie bracket as in the proof of Lemma 2.1. If $\sigma = \omega_{\mu,J}$ or if $\sigma = \omega_J$, then $h_0$ is a maximally compact Cartan subalgebra of $g_0$ and Lemma 2.1 provides a Vogan diagram; however, this Vogan diagram is not normalized. Even if the Theorem of Borel and de Siebenthal [Ku1 Th. 6.96], says that there exists another system of simple roots which contains at most one non-compact root, the corresponding $r$-matrices could give rise to non-isomorphic Lie bialgebras, cf. Lemma 2.1 part (II) again. Such a possibility arises when $\sigma = \omega_J$, or when $\sigma = \omega_{\mu,J}$ and $g_0$ is isomorphic to $so(2j + 1, 2(n - j) - 1)$, $EI$ or $EIV$.

3. Manin triples

In this section, we compute the Manin triples corresponding to the real absolutely simple Lie bialgebras. We keep the notation of the main result: $(g_0, \delta)$ is an absolutely simple, almost factorizable, real Lie bialgebra; $g$ is the complexification of $g_0$; $\sigma$ is the corresponding involution, either of the form $\varsigma_\mu$ or $\omega_{\mu,J}$.

We distinguish two cases:

(a) The bialgebra is factorizable, i.e. the involution $\sigma$ is of the form $\varsigma_\mu$.

(b) The bialgebra is imaginary factorizable, i.e. the involution $\sigma$ is of the form $\omega_{\mu,J}$.

The difference between “factorizable”, case (a), and “imaginary factorizable”, case (b), can be read off also from the double Lie algebra: in the first case it is $g_0 \otimes g_0$, in the second it is the realification $g^R$ of $g$.

3.1. Case (a). In this case, the determination of the Drinfeld double and the dual Lie bialgebra follows from a general result from [RS]. Namely, let $(I, r)$ be a factorizable (real or complex) Lie bialgebra and let $(\ | \ )$ be the nondegenerate inner product on $I$ induced by $r + r^{21} \in S^2(I)$. Let $r_\pm : I^* \to I$ be the maps induced by $r$, given by $r_+(\mu) = (\mu \otimes \text{id})r$, $r_-(\mu) = -(\text{id} \otimes \mu)r$. The factorization map $I : I^* \to I$ is $I = r_+ - r_-$. By hypothesis, $I$ is an isomorphism and $(I(\mu))(I(\tau)) = (\tau, I(\mu)) = \langle \mu, I(\tau) \rangle$.

Let $I'$ be the image of the map $I^* \to I \oplus I$, $\mu \mapsto (r_+(\mu), r_-(\mu))$; it is well-known that $r_\pm$ are Lie algebra maps, and that $I'$ is a Lie subalgebra of $I \oplus I$. Let $\text{diag} I$ be the diagonal Lie subalgebra of $I \oplus I$.

Theorem 3. [RS] The Manin triple corresponding to the factorizable Lie bialgebra $(I, r)$ is $(I \oplus I, \text{diag} I, I')$ where $I \oplus I$ is endowed with the bilinear form $\langle (x, u)|(y, v) \rangle = \langle x|y \rangle - \langle u|v \rangle$, $x, y, u, v \in I$. \hfill \Box

A finer description of $I'$, in terms of the Cayley transform, can be found in [Y Section 2.1]; see also [Y Section 3.1] for the case of complex simple factorizable Lie bialgebras.

3.2. Case (b). We deduce from Theorem 3 the determination of the Manin triples corresponding to imaginary factorizable real Lie bialgebras.

Let $(I_0, \delta)$ be a real Lie bialgebra such that its complexification $(I, \delta)$ is factorizable with $r \in I \otimes I$; let $\sigma : I \to I$ be the involution corresponding to $I_0$. We shall assume that $I_0$ is almost factorizable, see definition L.3. Let $(\ | \ )$ be the nondegenerate inner product on $I$ induced by $r + r^{21} = 2i\Omega \in S^2I$, and let $r_\pm : I^* \to I$ be the maps induced by $r$, as above. We identify $I_0^R$ with a real subspace of $I'$; namely with $\{ \alpha \in I' : \alpha(I_0) \subseteq R \}$. Then $I' = I_0^R \oplus iI_0^R$. Let $\mu \in I'$ and write $\mu = \alpha + i\beta$, with $\alpha, \beta \in I_0^R$. Then $I(\mu) = (-\beta \otimes \text{id})(2\Omega) + i(\alpha \otimes \text{id})(2\Omega)$; in particular $I_0 = I(iI_0^R)$. Hence, if $a = I(i\alpha), b \in I_0$ then $(a|b) = (i\alpha, b) \in iR$; in other words, $(\ | \ )|_{I_0 \times I_0} = iR$. 

Consider the realification \( \mathfrak{l}^\mathbb{R} \) of \( \mathfrak{l} \). To avoid confusions, we denote by \( x \mapsto x' \) the multiplication by \( i \) considered as a real linear endomorphism of \( \mathfrak{l}^\mathbb{R} \). Then \( \mathfrak{l}^\mathbb{R} = \mathfrak{l}_0 \oplus \mathfrak{l}_0' \). The following properties are evident:

\[
x'' = x, \quad [x,y'] = [x',y] = [x,y], \quad [x',y'] = -[x,y], \quad \sigma(x') = -\sigma(x'), \quad x,y \in \mathfrak{l}^\mathbb{R}.
\]

The real bilinear form \( \text{Re}(\quad) : \mathfrak{l}^\mathbb{R} \times \mathfrak{l}^\mathbb{R} \rightarrow \mathbb{R} \) is invariant and non-degenerate; one has

\[
2 \text{Re}(u|v) = (u|v) - (\sigma(u)|\sigma(v)).
\]

Let \( \mathfrak{l}_0^* := r_+ (\mathfrak{l}_0^0) \).

**Proposition 3.1.** The Manin triple corresponding to the Lie bialgebra \((\mathfrak{l}_0, \delta)\) is \((\mathfrak{l}^\mathbb{R}, \mathfrak{l}_0, \mathfrak{l}_0^* \mathfrak{})\) where \( \mathfrak{l}^\mathbb{R} \) is endowed with the bilinear form equal to \( 2 \text{Re}(\quad) \).

**Proof.** Let \( \Psi : \mathfrak{i} \oplus \mathfrak{i} \rightarrow (\mathfrak{l}^\mathbb{C})^\mathbb{C} \), \( \Phi : (\mathfrak{l}^\mathbb{C})^\mathbb{C} \rightarrow \mathfrak{i} \oplus \mathfrak{i} \) be given by

\[
\Psi(x,y) = \frac{1}{2}(x + \sigma(y)) + \frac{i}{2}(-x' + \sigma(y)'), \quad \Phi(u+iv) = (u+iv, \sigma((u-iv)));
\]

\( x,y \in \mathfrak{i}, u,v \in \mathfrak{l}^\mathbb{R} \). Notice the abuse of notation: in the argument of \( \Phi \), \( u+iv \) lives in the complexification of \( \mathfrak{l}^\mathbb{R} \) while in the first component \( u+iv \) lives in \( \mathfrak{i} \). A straightforward computation shows that \( \Phi, \Psi \) are mutually inverse isomorphisms of complex Lie algebras. We claim that

(i) \( \Psi \text{ diag}(\mathfrak{i}) = \mathfrak{l}_0 \oplus i\mathfrak{l}_0 \).

(ii) \( \Psi(\mathfrak{i}^*) = \mathfrak{l}_0^* \oplus i\mathfrak{l}_0^0 \).

(iii) \( \langle \Phi(u+iv) | \Phi(w+iz) \rangle = 2 \text{Re}(u+iv|w+iz) \), for \( u,v,w,z \in \mathfrak{l}^\mathbb{R} \), where the form \( \langle \quad | \quad \rangle \) is the form defined in Theorem 3 and \( 2 \text{Re}(\quad) \) is the complexification of the real form with the same name.

If \( x \in \mathfrak{i} \) then \( \Psi(x,x) = \frac{1}{2}(x + \sigma(x)) - \frac{i}{2} (x' + \sigma(x')) \); this proves (i). Let \( \mu \in \mathfrak{i}^* \); say \( \mu = \alpha + i\beta \), with \( \alpha, \beta \in \mathfrak{l}_0^* \). If we consider \( r_\pm : \mathfrak{i}^* \rightarrow \mathfrak{l}^\mathbb{R} \), we have

\[
\begin{align*}
\sigma (r_+ (\mu)) &= [(\alpha \otimes \text{id}) \Omega + (\beta \otimes \text{id}) \Omega] r_+ (\mu) - [(\alpha \otimes \text{id}) \Omega] (\beta \otimes \text{id}) \Omega + [(\alpha \otimes \text{id}) \Omega + (\beta \otimes \text{id}) \Omega] r_+ (\mu), \\
\sigma (r_- (\mu)) &= [(\alpha \otimes \text{id}) \Omega + (\beta \otimes \text{id}) \Omega] r_- (\mu) - [(\alpha \otimes \text{id}) \Omega] (\beta \otimes \text{id}) \Omega + [(\alpha \otimes \text{id}) \Omega + (\beta \otimes \text{id}) \Omega] r_- (\mu),
\end{align*}
\]

Hence \( \Psi (r_+ (\mu), r_+ (\mu)) = \frac{1}{2} (r_+ (\mu) + \sigma (r_- (\mu))) + \frac{i}{2} (r_- (\mu) + \sigma (r_- (\mu))) = r_+ (\alpha) + ir_+ (\beta) \); this proves (ii). Finally, the verification of (iii) is a straightforward computation using (3.1).

We conclude from the claim, by Theorem 3 that \((\mathfrak{l}^\mathbb{R}, \mathfrak{l}_0, \mathfrak{l}_0^*)\) is a Manin triple. The induced cobracket on \( \mathfrak{l}_0 \) is well the initial one, again by Theorem 3 the proof is finished. \( \square \)

Given a Manin triple \((\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)\), the isotropic Lie subalgebra \( \mathfrak{p}_2 \) is not determined by \( \mathfrak{p}_1 \). Compare Proposition 3.1 with the Manin triple of a compact Lie algebra (with trivial BD-triple) in [LAW] [NI].

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