FROM THE NASH–KUIPER THEOREM OF ISOMETRIC EMBEDDINGS TO THE EULER EQUATIONS FOR STEADY FLUID MOTIONS: ANALOGUES, EXAMPLES, AND EXTENSIONS

SIRAN LI AND MARSHALL SLEMROD

This paper is dedicated to our friend and mentor
Costas Dafermos on the occasion of his 81st birthday

Abstract. Direct linkages between regular or irregular isometric embeddings of surfaces and steady compressible or incompressible fluid dynamics are investigated in this paper. For a surface \((M, g)\) isometrically embedded in \(\mathbb{R}^3\), we construct a mapping which sends the second fundamental form of the embedding to the density, velocity, and pressure of steady fluid flows on \((M, g)\). From the PDE perspectives, this mapping sends solutions to the Gauss–Codazzi equations to the steady Euler equations. Several families of special solutions of physical or geometrical significance are studied in detail, including the Chaplygin gas on standard and flat tori, as well as the irregular isometric embeddings of the flat torus. We also discuss tentative extensions to multi-dimensions.

1. Introduction

In recent years, an important research project attracting enormous attention in PDE, differential geometry, and mathematical hydrodynamics communities is concerned with irregular isometric embeddings of Riemannian manifolds and wild solutions for fluid dynamical PDEs. A sequence of papers has appeared as an outgrowth of the celebrated results of Nash and Kuiper \([41, 38, 37]\) on \(C^1\)-isometric embeddings of Riemannian manifolds into Euclidean spaces. These works are devoted both to sharpening the regularity of embeddings in the Nash-Kuiper theorem \([25, 29]\), and to proving non-uniqueness of solutions for problems arising in continuum mechanics, especially the Euler equations for compressible and incompressible fluid flows \([13, 14, 32]\, amongst many others). See De Lellis–Székelyhidi Jr. \([30]\) for a thorough survey.

In these works, in-depth, far-reaching connections between geometry and fluid — more precisely, between the isometric embedding problem and the fluid dynamical PDE, e.g., the Euler equations — have been established and explored. Such connections are better understood on the methodological level. Indeed, a central technique in common for the above works is convex integration, which enables researchers to establish, in a constructive manner, the existence of irregular isometric embeddings or the non-uniqueness of weak solutions to fluid dynamical PDE. Systematization and further developments of convex integration culminated in Gromov’s \(h\)-principle and partial differential relations, which have become important theories in geometry and topology today. See \([33, 34]\).
The above theories and techniques have also led to important advancements in mathematical hydrodynamics and analysis of PDEs in continuum mechanics. One cornerstone is the final resolution of the Onsager conjecture for incompressible Euler equations. See [13, 14, 30, 32] by Buckmaster, De Lellis, Isett, and Székelyhidi Jr., and the many references cited therein. Let us also mention that recently, complementing the above theoretical developments are the breakthrough computational results of Borrelli–Jabrane–Lazarus–Thibert [4, 8, 9], in which for the first time the convex integration procedure was numerically implemented to produce elegant illustrations of irregular isometric embeddings.

Despite the aforementioned rapid developments in this field of research, the connections between isometric embeddings and fluid dynamics still await better understanding, beyond the level of structural similarities in relevant PDE or shared analytical tools (i.e., convex integration). Indeed, to the best of the authors’ knowledge, in each instance of its applications to continuum mechanics in the literature, the convex integration procedure is carried out on a case by case basis (similarly for the numerical results in [8, 9]); moreover, the geometrical characteristics of convex integration are, to some extent, neglected or suppressed. One exception is the recent work [49] by Theillière, in which a unified, systematic convex integration framework has been proposed, which encompasses several important geometric examples including the $C^1$-Nash embeddings.

In this work, we present some investigations, though of a tentative and preliminary nature, on the “geometry-fluid correspondence” between the isometric embedding problem and fluid dynamical PDE. Our aim is to establish possible direct fluid dynamic analogues of isometric embeddings. That is, we discuss ways of directly translating the isometric embeddings, regular or irregular, into suitable PDE for fluid motions.

Our tentative, partial answers to this problem can be summarised in the sequel:

1. For the convex integration procedure utilised to isometrically embed a surface $(M, g)$ into $\mathbb{R}^3$, each member of the approximating sequence can be identified with a solution of the Euler equations for compressible steady fluid flow on $(M, g)$.
2. Then, upon “renormalisation”, the Nash–Kuiper limit of approximate solutions can be translated into solutions to the steady compressible Euler equation on $(M, g)$.
3. Of particular interest is the sequence of rescaled standard tori $T_c(a, b)$ (see (5.1)), whose metrics converge in $L^\infty$ to the flat metric, in a manner reminiscent of the “Nash wrinkles” in [44]. On each member of the approximating sequence of manifolds, as well as on the limiting manifold, i.e., the flat torus, there are special solutions which have both natural geometrical meaning (geodesics on tori) and physical (Chaplygin gas) significance.

The remaining parts of the paper are organised as follows.

First, §2 contains background materials on the isometric embedding problem. §3 presents a link between smooth isometrically embeddings of a surface $(M, g)$ into $\mathbb{R}^3$ and solutions to the steady compressible Euler equations on $(M, g)$. In §4, two families of special examples for the general theory established in §3 are presented. Next, in §5 we discuss the same fluid dynamical PDE as in earlier sections on the flat torus or on its irregularly isometrically embedded image in $\mathbb{R}^3$. In §6 we discuss one possible correspondence between irregular isometric embeddings and steady Euler equations. In particular, we introduce a renormalisation process of successive rescaling and identify the “renormalised limit” of Nash–Kuiper iterations as weak solutions to
the Euler equations. Finally, some preliminary results on extensions to manifolds of higher dimensions are reported in §7.

Two Appendices A and B discuss, respectively, the geometry of rescaled standard tori and study of Chaplygin gas on higher dimensional manifolds. The appendices supplement §§4 & 5.

2. Geometric Preliminaries

This section collects some basics on Riemannian geometry and the isometric embedding problem. For the moment we restrict our discussions to 2-dimensional Riemannian manifolds, i.e., surfaces.

Let \((M, g)\) be a 2-dimensional Riemannian manifold, and consider an arbitrary local coordinate system \(\{x^i : i = 1, 2\}\). The distance on \(M\) is given by the metric \(g\) (a.k.a. first fundamental form) via \(ds^2 = g_{ij} dx^i \otimes dx^j\). A map \(y : (M, g) \rightarrow \mathbb{R}^3\) is an isometric embedding if \(y\) and \(dy\) are both injective and \(y\) is an isometry:

\[
3 \sum_{k=1}^3 \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j} = g_{ij}
\]

for each \(i, j \in \{1, 2\}\). That is, the intrinsic distance on \(M\) given by \(g\) equals to the ambient Euclidean distance on the image \(y(M) \subset \mathbb{R}^3\).

Isometric embeddings has been a central topic in the development of differential geometry and nonlinear analysis & PDE. See Han–Hong [35] for a thorough introduction. One crucial problem is concerned with the existence of an isometric embedding: given a Riemannian manifold \((M, g)\), find an isometric embedding \(y\) as above. It amounts to solving for the extrinsic geometry — in contrast to the intrinsic geometry, namely the geometric quantities determined by \(g\). In the case that \(M\) is 2-dimensional, the extrinsic geometry of \(y : M \rightarrow \mathbb{R}^3\) is completely characterised by the second fundamental form \(H = \{H_{ij} : 1 \leq i, j \leq 2\}\):

\[
H_{ij} := \frac{\partial^2 y}{\partial x^i \partial x^j} \cdot \nu,
\]

where \(\nu\) is the outward unit normal vector field to \(y(M)\).

There is a well-known necessary condition for the existence of (smooth) isometric embeddings of a surface \((M, g)\) into \(\mathbb{R}^3\): the second fundamental form must satisfy the Gauss and Codazzi equations: for \(i, j, k, l \in \{1, 2\}\), it holds that

\[
R_{ijkl} = H_{ik}H_{jl} - H_{il}H_{jk},
\]

\[
\nabla_i H_{jk} = \nabla_j H_{ik}.
\]

Here \(\nabla\) is the covariant derivative associated to the Levi-Civita connection on \((M, g)\). The connection is fully characterised by the Christoffel symbols

\[
\Gamma^i_{jk} := \frac{1}{2} g^{il} \{\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}\},
\]

where \(g^{-1} = \{g^{ij}\}\). Here and hereafter, we adopt Einstein’s summation convention: repeated lower and upper indices are understood as being summed over. Then, the Riemann curvature
tensor is given by
\[ R_{ijk} := g_{lp} \left\{ \partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^p_{jq} \Gamma^q_{ik} - \Gamma^p_{kq} \Gamma^q_{ij} \right\}. \]

In dimension 2, the only nontrivial component of the Riemann curvature is \( R_{1212} \). One defines the Gauss curvature by
\[ \kappa := \frac{R_{1212}}{\det g}, \]
which is the only intrinsic curvature for a surface. Let us also mention that \( \nabla g = 0 \), commonly known as the Ricci identity.

When \( M \) is simply-connected, the Gauss–Codazzi equations are also sufficient for the existence of isometric embeddings for \( \dim M = 2 \). This is known as the fundamental theorem of surface theory; its proof in the case of lower regularity is given by S. Mardare ([39, 40, 41]).

In the introduction we have discussed the Nash–Kuiper theorem ([44, 38, 37]). The statement is as follows. Note that an improved version of this theorem has been established by Conti–De Lellis–Székelyhidi Jr. ([25, Theorem 6.1]). We shall refer to it in §§5 & 6 below.

**Theorem 2.1.** Let \((M, g)\) be a Riemannian manifold of dimension \( n \). Let \( y_* : M \to \mathbb{R}^{n+k} \) be a smooth embedding such that \( \partial_i y_* \cdot \partial_j y_* < g_{ij} \) (as quadratic forms) and \( k \geq 1 \). Then for any \( \epsilon > 0 \) there exists a \( C^1 \)-isometric embedding \( y : (M, g) \to \mathbb{R}^{n+k} \) such that \( ||y - y_*||_{C^0(M)} \leq \epsilon \).

Typical applications of the Nash–Kuiper Theorem 2.1 include the construction of a \( C^1 \)-isometric embedding (or even a \( C^{1,1-\epsilon} \)-isometric embedding) of the flat 2-torus into \( \mathbb{R}^3 \). Another example is a “corrugated 2-sphere”; that is, a round 2-sphere of radius \( r \) isometrically embedded into \( \mathbb{R}^3 \) while being \( C^0 \)-close to a sphere of a smaller radius \( r_0 \). A visual representation is given by Bartzos–Borrelli–Denis–Lazarus–Rohmer–Thibert [4].

3. From embedded surfaces to Euler: 2D Smooth Solutions

3.1. Gauss-Codazzi and steady Euler equations. This section is mainly concerned with the following question:

Given a smooth surface \((M, g)\) isometrically embedded in \( \mathbb{R}^3 \) with second fundamental form \( H = \{H_{ij}\}_{1 \leq i, j \leq 2} \), can we identify natural variables \( \rho, v, p \) in terms of \( g, H \) which represent, respectively, the density, velocity, and pressure of a steady fluid on \((M, g)\)?

More precisely, consider the following two PDE systems, one geometric and one physical:

1. Let \((M, g)\) be a smooth surface isometrically embedded into \( \mathbb{R}^3 \). Its second fundamental form \( H = \{H_{ij}\}_{1 \leq i, j \leq 2} \) solves the Gauss–Codazzi equations:
   \[ H_{ij} H_{kl} - H_{ik} H_{jl} = R_{ijkl} \quad (3.1) \]
   and
   \[ \nabla_i H^k_j = \nabla_j H^k_i. \quad (3.2) \]

2. The Euler conservation laws of mass and momentum for steady fluid on \((M, g)\):
   \[ \nabla_k (\rho v^k) = 0 \quad (3.3) \]
   and
   \[ \nabla_k p^k = 0, \quad (3.4) \]
where $P^i{}_k$ is the stress-energy tensor:

$$P^i{}_k = \rho v^i v^k + g^i{}_k p. \tag{3.5}$$

These are just the usual fluid equations with covariant differentiation; see for example Anco et al. [2] and Arnold–Khesin [3]. The idea should not be surprising, since it is building block of the “matter” description in general relativity; see Carroll [17].

Our question, formulated in terms of PDE, is as follows: given a smooth solution $H_{ij}$ to the Gauss–Codazzi system (3.1) and (3.2), look for fluid variables $\{\rho, v, p\}$ which satisfy the steady Euler equations (3.3) and (3.4) on $(M,g)$. That is, we hope to provide a direct link from isometric embedding of surfaces to fluid dynamical PDEs. In this section we restrict ourselves to the $C^\infty$-setting; isometric embeddings of weak regularity as in Nash’s setting [44] will be discussed in the subsequent section.

3.2. Step 1: Identifying the stress-energy tensor. To begin with, we rewrite via Ricci’s identity the balance law of linear momentum (3.4) as

$$\nabla_k P^k{}_j = 0. \tag{3.6}$$

Given the solution $\{H^1_{ij}\}_{1 \leq i,j \leq 2}$ to the Codazzi equation (3.2), the following choice of $\{P^1_{ij}\}_{1 \leq i,j \leq 2}$ clearly solves the balance of momentum (3.6):

$$P^1{}_1 = H^2_{2}, \quad P^1{}_2 = -H^2_{1}, \quad P^2{}_1 = -H^1_{2}, \quad P^2{}_2 = H^1_{1}. \tag{3.7}$$

It is important to note that $\{P^1_{ij}\}_{1 \leq i,j \leq 2}$ can be defined globally on $(M,g)$ as a $(1,1)$-tensor. Indeed, the second fundamental form $H = \{H_{ij}\} \in \Gamma(T^*M \otimes T^*M)$ is a contravariant 2-tensor, and let us write $H' := g^{-1}H = \{H'_{ij}\} \in \Gamma(T^*M \otimes TM)$, a $(1,1)$-tensor obtained by raising one index via contraction with metric $g$. Then

$$P' = \{P'_{ij}\} := \text{Adj} H',$$

where the adjugate matrix can be defined globally on $(M,g)$ via multilinear algebra. Note that

$$\det H' = \det P' = \kappa = \text{the Gauss curvature of } (M,g), \tag{3.8}$$

and similarly,

$$\text{tr} H' = \text{tr} P' = m = \text{the mean curvature of } (M,g) \hookrightarrow \mathbb{R}^3. \tag{3.9}$$

Here and hereafter, $\text{tr}$ without subscripts means that the trace is taken with respect to the canonical Euclidean coordinates.

3.3. Step 2: Identifying the pressure. In this step, we shall identify the pressure $p$ with the smaller principal curvature of $(M,g) \hookrightarrow \mathbb{R}^3$. Let us first introduce some notations: we write $m$ for the mean curvature of $M$, namely that

$$m = \text{tr}_g H = H^1_1 + H^2_2.$$

We use the unconventional notation $m$ for the mean curvature since $H$ has already been reserved for the second fundamental form. Also note that $m := \text{tr}_g H$ here instead of $\frac{1}{2}\text{tr}_g H$. In addition, we let $\{\kappa_+, \kappa_-\}$ be the principal curvatures, where $\kappa_+ \geq \kappa_-$ on $M$, and $| \cdot |_g$ denotes the norm of a vector field or 1-form with respect to the metric; i.e.,

$$|v|^2_g := g^{ij}v_i v_j = g_{ij}v^i v^j \equiv v^i v_i.$$
In the previous step we have defined $P$ as a function of $H$. We now require it to satisfy (3.5), which is equivalent to $P_{ij} = \rho v_i v_j + p g_{ij}$. That is, $P = \rho v \otimes v + pg$ as an identity of contravariant 2-forms. We deduce that
\[
\kappa = \det(g^{-1} P) = \det \left[ \rho g^{-1} \cdot (v \otimes v) + p g \right] = p^2 + \text{tr} \left[ \rho g^{-1} \cdot (v \otimes v) \right] + \det \left[ \rho g^{-1} \cdot (v \otimes v) \right] = pp|v|^2 + p^2.
\]
(3.10)
The first line follows from (3.8), the second line holds as $P$ is the stress-energy tensor for steady fluid, the third line is a linear algebra identity, and the last line follows from the definition of norm $\bullet |g$ and that $v \otimes v$ is rank-1, hence having zero determinant. Note that (3.10) generalises the relations previously obtained for the Euclidean case in Chen–Slemrod–Wang [21, Equation (3.2)]; cf. also Acharya–Chen–Li–Slemrod–Wang [1, Eq. (8.4)].

On the other hand, we can compute the mean curvature $m = \text{tr} g H = \text{tr} H'$ from (3.9):
\[
m = H_1^1 + H_2^2 = P_1^1 + P_2^2 = \rho |v|^2 + 2p.
\]
(3.11)
Thus, we find from (3.11) and (3.10) that $m$ and $\kappa$ are related by
\[
p^2 - mp + \kappa = 0.
\]
(3.12)
Let us now consider the principal curvatures $\{\kappa_+, \kappa_-\}$. Then (3.12) can be written as $p^2 - (\kappa_+ + \kappa_-)p + \kappa_+ \kappa_- = 0$, so the roots are $p = \kappa_+$ or $p = \kappa_-$. We shall select
\[
p = \kappa_- = \text{the smaller principal curvature}.
\]
(3.13)
Indeed, such choice is compatible with the meaning of $P$ in fluid dynamics; in particular, with the non-negativity of density function $\rho$. To see this, consider the following three cases:

- **Case 1:** $\kappa_+ > \kappa_-$ and $\kappa_+$ is nonzero. If in this case $p = \kappa_+$, then (3.10) becomes $p[\rho |v|^2 + (p - \kappa_-)] = 0$, which is impossible for $\rho \geq 0$.
- **Case 2:** $\kappa_+ = \kappa_-$. Then the choice for $p$ is non-ambiguous.
- **Case 3:** $\kappa_+ > \kappa_-$ and $\kappa_+$ is vanishing somewhere. By choosing $p = \kappa_-$, (3.10) becomes $p(\rho |v|^2 + p) = 0$, which is permissible as long as $\rho > 0$.

3.4. **Step 3: Reformulation of balance laws.** With the identification of $P$ and $p$ as in (3.7) and (3.13), we shall reformulate in this step the conservation of momentum as a first-order PDE for the $\rho$-variable. In addition, we obtain the PDE for vorticity, which is transported along Lagrangian trajectories on the surface $(M, g)$. This follows from natural considerations in the study of fluid dynamical PDEs and, as will be clear from the later steps, plays a crucial role in our search for special solutions to the steady Euler equation with geometrical significance.

Following conventions in fluid dynamics, we define the vorticity $\omega$ on $(M, g)$:
\[
\omega = \nabla_1 v_2 - \nabla_2 v_1,
\]
which is viewed either as a 2-form or a scalar field, identified via Hodge duality. We also write
\[
q := |v|_g, \quad c := \sqrt{p'(\rho)}, \quad M := q/c
\]
for the flow speed, sonic speed, and Mach number, respectively.
Recall from (3.10), (3.7), and (3.13) that \( \kappa = \kappa_+ - \kappa_- = p\rho q^2 + p^2 \) and \( p = \kappa_- \). Thus
\[
q^2 = \frac{\kappa_+ - \kappa_-}{\rho}.
\] (3.14)

Assume that the conservation of mass (3.3) is satisfied. This will reduce the conservation of momentum (3.4) to
\[
\frac{\rho}{2} v^k \nabla_k \left( \frac{\kappa_+ - \kappa_-}{\rho} \right) + v^k \nabla_k \kappa_- = 0.
\] (3.15)

This equation can be written globally on \((M,g)\):
\[
\frac{\rho}{2} \nabla_v \left( \frac{\kappa_+ - \kappa_-}{\rho} \right) + \nabla_v \kappa_- = 0,
\] (3.16)
where \( \nabla_v \) is the covariant derivative with respect to \( v \in \Gamma(TM) \).

On the other hand, let us consider the vorticity equation. Assuming the conservation of mass (3.3), we rewrite the conservation of momentum (3.4) as
\[
\rho v^k \nabla_k v^i + g^{ik} \nabla_k \kappa_- = 0.
\]

Lowering the index \( i \) (by multiplying with \( g_{i\ell} \) and evaluating separately \( \ell = 1, 2 \)), one obtains
\[
\begin{aligned}
\rho v^1 \nabla_1 v^1 + \rho v^2 \nabla_2 v^1 + \nabla_1 \kappa_- &= 0, \\
\rho v^1 \nabla_1 v^2 + \rho v^2 \nabla_2 v^2 + \nabla_2 \kappa_- &= 0,
\end{aligned}
\]
which is equivalent to
\[
\begin{aligned}
\frac{\rho}{2} \nabla_1 (q^2) - \rho v^2 \omega + \nabla_1 \kappa_- &= 0, \\
\rho v^1 \omega + \frac{\rho}{2} \nabla_2 (q^2) + \nabla_2 \kappa_- &= 0,
\end{aligned}
\]
as \( q^2 := |v|^2_g = v^1 v_1 + v^2 v_2 \). Multiplying respectively the first and the second equations by \(-v^2\) and \(v^1\) and adding them up, we have
\[
\rho q^2 \omega + \frac{\rho}{2} \nabla_{v^+} (q^2) + \nabla_{v^+} \kappa_- = 0,
\] (3.17)
where the differential operator \( \nabla_{v^+} \) is given in local co-ordinates by
\[
\nabla_{v^+} = v_1 \nabla_2 - v_2 \nabla_1.
\]
This operator is globally defined on \((M,g)\). Indeed,
\[
v^1 = Jv
\]
where the \((1,1)\)-tensor field \( J \in \Gamma(T^*M \otimes TM) \) is the almost complex structure on \( M \).

We observe a direct consequence of the derivations above. The terms involving derivatives in (3.16) and (3.17) are of the same form, in view of the Bernoulli law (3.14). We thus have

**Lemma 3.1.** With the choice of variables \( p = \kappa_- \), \( p\rho q^2 + p^2 = \kappa \), and \( q := |v|_g \), conservation of momentum and irrotationality are satisfied whenever the density \( \rho \) solves the following PDE:
\[
\frac{\rho}{2} \mathcal{L} \left( \frac{\kappa_+ - \kappa_-}{\rho} \right) + \mathcal{L} \kappa_- = 0 \quad \text{with} \quad \mathcal{L} \in \{ \nabla_v, \nabla_{v^+} \}.
\]

In §4 below, we shall investigate two families of examples by looking for \( \rho \) which satisfies the identity in Lemma 3.1 for every first-order differential operator \( \mathcal{L} \). However, the general procedure in Step 4 below to solve for \( v \) and \( \rho \) will not make use of this lemma.
3.5. **Step 4: Solving for \( v \) and \( \rho \).** Now let us discuss how to solve for \( v \) and \( \rho \) in terms of \( g \) and \( H \), with the choice of \( p \) and \( P \) as in earlier Steps 1 & 2. Our goal is to find a correspondence

\[
H \rightsquigarrow (\rho, v^1, v^2, p),
\]

such that \( H \) satisfies the Gauss–Codazzi equations (3.1), (3.2) on \((M, g)\), and \((\rho, v, p)\) satisfies the conservation of mass (3.3) and momentum (3.4). For this purpose, we have:

1. identified \( P^{ij} = \rho v^i v^j + g^{ij}p \) as a whole via \( \{P^i_j\} = \text{Adj} \{H^i_j\} \). Then the conservation of momentum is automatically verified, thanks to the Codazzi equation (3.2); and
2. expressed the Gauss equation (3.1) in the form of (3.10): \( \kappa = ppq^2 + p^2 \).

In this way, we have obtained four equations for four unknowns:

\[
\begin{align*}
\rho (v^1)^2 + g^{11}p &= P^{11}, \\
\rho v^1 v^2 + g^{12}p &= P^{12}, \\
\rho (v^2)^2 + g^{22}p &= P^{22}, \\
ppq^2 + p^2 &= \kappa.
\end{align*}
\]

(3.18)

The right-hand sides depend only on the geometric data, which are deemed as given. Moreover, we have also

3. taken the trace of the first three equations in (3.18) to obtain (3.11); that is, \( m = \rho q^2 + 2p \).

This together with the fourth equation in (3.18) leads to the choice of \( p = \kappa_- \) in (3.13). With this choice, the identities \( m = \rho q^2 + 2p \) and \( \kappa = ppq^2 + p^2 \) become equivalent, so the fourth equation in (3.18) is then redundant.

To proceed, we shall solve for \( \rho \) and \( v \) from the first three equations in (3.18); or equivalently,

\[
\rho v_i v_j = f_{ij} := P_{ij} - \kappa_+ g_{ij}.
\]

(3.19)

The crucial observation is that this system has further redundancies. Thus, (3.19) contains at most two independent equations for three unknowns \( \rho, v_1, v_2 \). Therefore, coupling (3.19) with the conservation of mass (3.3) results in a determined (or under-determined) system.

**Lemma 3.2.** The system (3.19) has at most two independent equations for \((\rho, v_1, v_2)\).

**Proof.** We check \( \det f = 0 \). Then \( \rho v_1 v_2 = f_{12} \) can be deduced from \( \rho(v_1)^2 = f_{11} \) and \( \rho(v_2)^2 = f_{22} \).

Indeed, the choice of \( P \) in (3.7) gives us

\[
f = \{f_{ij}\} = \begin{bmatrix}
g_{11}(H^2_2 - \kappa_-) - g_{12}H^1_2 & -g_{11}H^2_2 + g_{12}(H^1_2 - \kappa_-) \\
g_{21}(H^2_2 - \kappa_-) - g_{22}H^1_2 & -g_{21}H^2_2 + g_{22}(H^1_2 - \kappa_-)
\end{bmatrix}.
\]

A similar computation as for (3.10) leads to

\[
\det f = (\det g) \det(P' - \kappa_- I_2) = (\det g) \{\det H' - \kappa_- \text{tr} H' + (\kappa_-)^2\} = (\det g) \{\kappa_- \kappa_+ - \kappa_-(\kappa_+ + \kappa_-) + (\kappa_-)^2\} = 0.
\]

\[\square\]

In particular, \( \{f_{11}, f_{22}\} \) must have the same sign, which may be assumed positive without loss of generality. (If their signs are negative, one just reverses the orientation of \((M, g)\), which
will result in interchanging the inward and outward unit normal vector fields and thus taking \((H, \kappa_\pm)\) to \((-H, -\kappa_\pm)\). Then, for \(\rho > 0\) we set
\[
v_i := \sqrt{\frac{f_{ii}}{\rho}}, \quad i \in \{1, 2\}.
\]
This clearly satisfies (3.19).

Finally, we specify the density \(\rho > 0\) via the continuity equation (3.3). By virtue of (3.20) we have (no summation convention here) that
\[
0 = 2 \sum_{i=1}^{2} \nabla_i \sqrt{\rho f_{ii}} = \frac{1}{\sqrt{\det g}} \sum_{i=1}^{2} \partial_i \sqrt{(\det g) \rho f_{ii}}.
\]
This can be solved locally by the method of characteristics:
\[
\rho(z^1(t), z^2(t)) = \rho(z^1(0), z^2(0)) \exp \left\{-2 \int_0^t \left( \partial_1 \sqrt{(\det g) f_{11}} + \partial_2 \sqrt{(\det g) f_{22}} \right) \text{d}s \right\}, \tag{3.21}
\]
where for each \(i \in \{1, 2\}\)
\[
\frac{\text{d}z^i}{\text{d}t} = \sqrt{(\det g) f_{ii}} \tag{3.22}
\]
as long as \(\rho(z^1(0), z^2(0)) > 0\).

**Remark 3.3.** We emphasise the local nature of the solution formulae (3.21) and (3.22). It is in general not guaranteed that \(\rho\) is defined globally on a non-simply-connected surface \((M, g)\).

4. **Two families of special solutions**

The previous section provides a systematic way of producing fluid variables \((\rho, v, p)\) satisfying the steady Euler equation on \((M, g)\) from the second fundamental form \(H\) of the isometric embedding \((M, g) \hookrightarrow \mathbb{R}^3\). In this subsection, we describe two families of special solutions to the fluid PDE, found with the help of Lemma 3.1.

4.1. **A stronger condition.** Our special solutions in this section shall be found by solving the equation in Lemma 3.1
\[
\frac{\rho}{2} \mathcal{L} \left( \frac{\kappa_+ - \kappa_-}{\rho} \right) + \mathcal{L} \kappa_- = 0 \tag{4.1}
\]
for an arbitrary first-order differential operator \(\mathcal{L}\). In view of Lemma 3.1 if the fluid variables satisfy (4.1) together with the Bernoulli law \(\rho q^2 = \kappa_+ - \kappa_-\) and the conservation of mass, then the fluid is necessarily irrotational, and the conservation of momentum is automatically satisfied. The requirement that (4.1) holds for arbitrary \(\mathcal{L}\) appears over-determined in general. Nevertheless, for certain surfaces \((M, g)\) with very special geometric properties, we may obtain explicit solutions for \(\rho\).

Let us now take one step back: we first consider the PDE system consisting of irrotationality, conservation of mass (3.3), and (3.14), all reproduced below:
\[
\begin{cases}
\omega := \nabla_1 v_2 - \nabla_2 v_1 = 0, \\
\nabla_k (\rho v^k) = 0, \\
\rho = \hat{\rho}(q^2) := \frac{\kappa_+ - \kappa_-}{q^2}.
\end{cases} \tag{4.2}
\]
Here, \(\kappa_{\pm}\) are viewed as given variables and \(\{\rho, v^1, v^2\}\) as unknowns. We refer to the third equation as the *Bernoulli law*, as it expresses density \(\rho\) as a function of the flow speed squared,
thus eliminating the pressure from the balance of momentum. As aforementioned, this system is weaker than (4.1), conservation of mass, and Bernoulli law put together.

**Remark 4.1.** As commented in Chen–Dafermos–Slemrod–Wang [18, p.636], on \( \mathbb{R}^2 \), the system (4.2) is mathematically equivalent to the system of conservation of mass and momentum for smooth flows. The same holds on a surface (\( M, g \)), as irrotationality also persists. Such equivalence breaks down in the presence of shocks, as vorticity may be created and mechanical energy may be converted to heat. The system (4.2) is more popular among the aerodynamists due to its mathematical simplicity and its analogy with incompressible fluids. See also Bers [5].

### 4.2. Example 1: CMC surfaces.

If we set
\[
\rho = \kappa_+ + \kappa_- = \text{mean curvature, } m = \text{constant},
\]
then \( \frac{2}{\rho} \mathcal{L} \left( \frac{\kappa_+ - \kappa_-}{\rho} \right) + \mathcal{L} \kappa_- = \frac{1}{\rho} \mathcal{L}(\kappa_+ + \kappa_-) = 0 \). Conservation of momentum and irrotationality thus hold by Lemma 3.1. Such \( (M, g) \) is known as a CMC (constant mean curvature) surface.

**Remark 4.2.** If an oriented CMC surface \( (M, g) \) has \( m < 0 \), then we should reverse the orientation to make the mean curvature positive, in view of the identification \( \rho = m \) in (4.3).

For the steady Euler equations (3.3) and (3.4) corresponding to the CMC surface here, flow velocity \( v \) is both divergence-free and curl-free; namely, \( d \) and \( d^* \)-free on \( (M, g) \). Thus, the differential 1-form \( v^i := v_i dx^i \) canonically dual to \( v = v^i \partial_i \) is harmonic, hence is smooth by elliptic regularity. By the aforementioned remarks in [18, p.636], the system (4.2) is equivalent to the conservation laws (3.3) and (3.4). We are in the case of incompressible Euler equations.

By now we have not used the Bernoulli law, i.e., the third equation in (4.2). Taking it into consideration, we will show that many CMC surfaces \( (M, g) \) do not admit solutions to the steady Euler equations (3.3) and (3.4), under our specification of the fluid variables from the geometric variables. But, before embarking on ruling out candidates for CMC surfaces, we first present three simple affirmative examples.

**Example 4.3.** Consider the case that both of the principal curvatures \( \kappa_{\pm} \) are constant. Then we have three possible choices for \( (M, g) \): the round sphere, the right cylinder, and the plane.

1. For \( (M, g) = \mathbb{R}^2 \), a harmonic 1-form on \( \mathbb{R}^2 \) is constant. Also \( m = \kappa_+ - \kappa_- = 0 \), so for any constant velocity \( v \) the Bernoulli law is satisfied. Thus any constant vector field \( v \in \Gamma(T\mathbb{R}^2) \) is an Euler solution.

2. For \( (M, g) = r\mathbb{S}^2 \), the sphere \( \{ P \in \mathbb{R}^3 : |P| = r \} \) for any \( r > 0 \) equipped with the round metric, \( v \equiv 0 \) clearly satisfies (4.2). In fact, in Proposition 4.4 below we shall show that the zero solution is the only solution on the sphere.

3. For \( (M, g) = s\mathbb{S}^1 \times \mathbb{R} := \{ (s \cos \theta, s \sin \theta, z) \in \mathbb{R}^3 : \theta \in [0, 2\pi[, z \in \mathbb{R} \} \) for some \( s > 0 \), the cylinder whose inclusion into \( \mathbb{R}^3 \) is an isometric embedding, one has \( \kappa_+ = s^{-1} \) and \( \kappa_- = 0 \). Then the Bernoulli law forces the flow speed \( q \) to be 1 for any \( s > 0 \). There are four solutions: \( v \in \{ \pm \partial_z, \pm \partial\theta \} \). Here \( \partial_z \equiv (0, 0, 1)^T \) and \( \partial\theta \) is the angular vectorfield given by \( \partial\theta|_{(x,y,z)^\top} := (-\dot{y}, \dot{x}, 0)^T \) for any \( (x, y, z)^\top \in s\mathbb{S}^1 \times \mathbb{R} \subset \mathbb{R}^3 \).

Now we discuss the limitations on CMC surfaces imposed by the Bernoulli law together with irrotationality and mass conservation. We have the following
**Proposition 4.4.** Let \((M, g)\) be a smooth surface of constant mean curvature whose corresponding fluid variables satisfy (4.2). Then \((M, g)\) cannot be any surface in the following list, unless we are in the simple cases given by Example 4.3:

1. A minimal surface, i.e., \((M, g)\) on which \(m \equiv 0\);
2. Any simply-connected closed surface;
3. Any surface whose set of umbilical points \(U(M) := \{ x \in M : \kappa_+(x) = \kappa_-(x) \}\) has an accumulation point.

**Proof.** If (1) holds, then by the Bernoulli law \(\kappa_+ \equiv \kappa_-\) on \((M, g)\), as the flow speed \(q\) is finite. This together with \(m \equiv 0\) yields \(\kappa_+ \equiv 0\), hence \((M, g) = \mathbb{R}^2\), which is covered by Example 4.3.

Next, suppose that (3) holds, i.e., the set of umbilical points \(U(M)\) is non-isolated. On this set we have \(\rho = 0/q^2\) by the Bernoulli law. If \(q > 0\) on \(U(M)\), then \(\rho \equiv 0\) on \(M\) due to the CMC condition. This returns to the case (1) above. If instead \(q = 0\) on \(U(M)\) (in which case we view the Bernoulli law on \(U(M)\) as undefined), then \(v = 0\) on \(U(M)\). But since \(v\) is harmonic thanks to irrotationality and conservation of mass, \(v\) is real-analytic on \((M, g)\). Then by analytic continuation \(v \equiv 0\) on \(M\). Then, as the mean curvature \(m = \rho\) is finite, we must have \(\kappa_+ \equiv \kappa_-\) by the Bernoulli law. So \((M, g)\) is totally umbilical, hence is \(\mathbb{R}^2\) or \(r\mathbb{S}^2\) for some \(r > 0\). Both cases have been dealt with in Example 4.3.

Now consider (2), i.e., assume that \(M\) is closed and simply-connected. As \(v^2\) is a harmonic 1-form, it must be constant on \(M\) by de Rham cohomology. But the classification of closed surfaces shows that \(M\) is a topological 2-sphere. Thus, by the hairy ball theorem, we see that \(v \in \Gamma(TM)\) must vanish somewhere. Thus \(v\) is constantly zero. As before, for the Bernoulli law to make sense, we must have \(\kappa_+ \equiv \kappa_-\) on \(M\). This returns to Example 4.3. \(\square\)

Proposition 4.4 rules out many candidates for CMC surfaces in our model from physical considerations. Here we do not intend to mean that these CMC surfaces are unnatural in any sense; we only suggest that the particular mapping we constructed which sends a solution \(\mathbf{H}\) to the Gauss–Codazzi equations (3.1)–(3.2) to the solution \((\rho, v, p)\) to the steady Euler system (4.2) do not work for such CMC surfaces.

On the other hand, there is an abundance of higher-genus CMC surfaces which are not covered by Proposition 4.4, for example, the Wente torus in [50] and the complete CMC surfaces obtained via PDE gluing by Kapouleas and others. See, for example, the recent survey by Breiner–Kapouleas–Kleene [12].

A priori it is unclear if the above CMC surfaces admit solutions to (1.2), as this system is overdetermined in general when \(\rho = \text{constant}\). Nevertheless, assuming existence, we may prove the following weak compactness result.

**Proposition 4.5.** Let \(\{(M_n, g_n)\}\) be a sequence of smooth closed CMC, non-minimal immersed surfaces in \(\mathbb{R}^3\). Assume this sequence converges in the bi-Lipschitz sense to a limiting manifold which is possibly non-smooth. More precisely, there exist a manifold \(M_\infty\) with a Lipschitz metric \(g_\infty\) and bi-Lipschitz homeomorphisms \(\psi_n : (M_\infty, g_\infty) \to (M_n, g_n)\) such that the bi-Lipschitz constants of \(\psi_n\) tend to 1. Suppose that the principal curvatures \(\kappa_{n, \pm}\) of \((M_n, g_n) \hookrightarrow \mathbb{R}^3\) satisfy the integral bound:

\[
\sup_{n \in \mathbb{N}} \int_{M_n} \left| \frac{\kappa_{n,+} - \kappa_{n,-}}{\kappa_{n,+} + \kappa_{n,-}} \right| \, dV_{g_n} \leq K_0 < \infty
\]  

(4.4)
In the above setting, consider a sequence of vector fields \( \{v_n\} \subset \Gamma(TM_n) \) satisfying the system \((4.2)\) of irrotationality, conservation of mass, and Bernoulli law subject to the identification \((4.3)\). Then the corresponding vector fields \( w_n := (\psi_n)^\# v_n \in \text{Lip}(M_\infty; TM_\infty) \) on the limiting manifold converge in the weak \(L^2\)-topology to some \( \overline{w} \in L^2(M_\infty; TM_\infty) \).

Here, the weak limit \( \overline{w} \) is divergence-free and irrotational in the sense of distributions. In addition, \( |\overline{w}|^2_{g_\infty} \) is the distributional weak limit of \( (q_n)^2 := |v_n|^2_{g_n} \) on \((M_\infty, g_\infty)\).

Before giving a proof, we first remark on the statement of Proposition \((4.5)\). Therein, the sequence of CMC surfaces \( \{\{M_n, g_n\}\} \) may be obtained, for example, from scaling the metrics for a given CMC surface. The limiting manifold \((M_\infty, g_\infty)\) is not required to be a Riemannian manifold, but some regularity is assumed. Indeed, it should at least be a Lipschitz manifold with homeomorphisms, the pullback \((\psi_n)^\# v_n \) are well defined Lipschitz vector fields on \(M_\infty\).

One subtlety is that we have assumed \((M_n, g_n)\) to be smooth manifolds (intrinsically), but their CMC immersions in \(\mathbb{R}^3\) may fail to be smooth. This is the case for the Wente torus in \([50]\).

The key point of the theorem is to establish the weak \(L^2\)-convergence \( w_n \rightharpoonup \overline{w} \), such that the limiting vector field \( \overline{w} \) also satisfies the system \((4.2)\) in a “very weak sense”. By this we mean that the irrotationality and conservation of mass hold weakly. Moreover, although the principal curvatures \( \kappa_{n, \pm} \) of each CMC surface \((M_n, g_n) \hookrightarrow \mathbb{R}^3\) may blow up as \( n \to \infty \), the Bernoulli law and \((4.3)\) implies that \( \{(q_n)^2 = \frac{\kappa_{n, +} - \kappa_{n, -}}{\kappa_{n, +} + \kappa_{n, -}}\} \) is uniformly bounded in \(L^1\). Note that for the integral bound condition \((4.4)\) to hold, \((M_n, g_n)\) are necessarily non-minimal.

**Proof.** Let us work with differential 1-forms instead of vector fields. Let \( \alpha_n := (v_n)^f \) be the 1-forms canonically dual to \(v_n\) via metrics \(g_n\). In this case, the irrotationality and the conservation of mass of \(v\) are equivalent to \(d\alpha_n = 0\) and \(d^*\alpha_n = 0\). Equivalently, \(\alpha_n\) are harmonic 1-forms on the smooth Riemannian manifolds \((M_n, g_n)\). Note that they are smooth and are confined in finite-dimensional spaces (dimension = the first betti number of \(M\)).

Consider the Lipschitz 1-forms:

\[
\beta_n := (\psi_n)^\# \alpha_n = (v_n)^f.
\]

We claim that \( V_0 := \sup_{n \in \mathbb{N}} \|\beta_n\|_{L^2(M_n, g_n)} \) is finite. Indeed, by the duality \(TM_n \cong T^*M_n\), Bernoulli law, and the integral bound \((4.4)\), we have

\[
\|\alpha_n\|_{L^2(M_n, g_n)} := \|q_n\|_{L^2(M_n, g_n)} = \sqrt{\frac{\|\kappa_{n, +} - \kappa_{n, -}\|_{L^1(M_n, g_n)}}{\|\kappa_{n, +} + \kappa_{n, -}\|_{L^1(M_n, g_n)}}} \leq \sqrt{K_0}.
\]

So \( V_0 < \infty \) depends only on \(K_0\) and the uniform bound on bi-Lipschitz constants of \(\psi_n\). As \(d\alpha_n\) and the differential commutes with pullbacks, we have that

\[
d\beta_n = 0 \quad \text{on} \quad (M_\infty, g_\infty).
\]

Now let us turn to the divergence of \(v_n\); that is, the co-differential of \(\beta_n\). We have

\[
d^* \beta_n = *d \star \left((\psi_n)^\# \alpha_n\right),
\]

where \(\star\) is the Hodge star. Considering the commutator \(T_n := [\star d \star, (\psi_n)^\#]\), we have

\[
d^* \beta_n = T_n \alpha_n + (\psi_n)^\# (\star d \star \alpha_n) = T_n \alpha_n,
\]

\[12\]
since $d^*\alpha_n = 0$. But $T_n$ is a pseudo-differential operator of order zero with $L^\infty$-coefficients, so
\[ \{d^*\beta_n\} \text{ is bounded in } L^2(M_\infty, g_\infty). \] (4.6)

By a standard application of the div-curl lemma (see Murat [42, 43] and Tartar [47, 48], or its adaptations on manifolds in [19, 20]), (4.5) and (4.6) implies that $\beta_n \to \beta$ in the weak $L^2$-topology to a limiting 1-form $\beta \in L^2(M_\infty; T^*M_\infty)$, such that
\[ \langle \beta_n, \beta_n \rangle_{g_\infty} \to \langle \beta, \beta \rangle_{g_\infty} \text{ in the sense of distributions.} \]

Therefore, by setting $w := (\beta)^b \in L^2(M_\infty; TM_\infty)$ and invoking the assumption that the bi-Lipschitz constants of $\psi_n$ converge to 1, we conclude that
\[ q_n^2 \to |w|_{g_\infty}^2 \text{ in the sense of distributions.} \]

This completes the proof. \hfill \square

Remark 4.6. The Wente torus $T_W$, cf. [20], serves as an illustrative example for Proposition 4.3: its metric is $g = e^{\omega} \delta$ (where $\delta$ is the Euclidean metric) and its principal curvatures are $\kappa_+ = e^{-\omega} \cosh \omega$ and $\kappa_- = e^{-\omega} \sinh \omega$, hence $m = 1$ and $q^3 = e^{-2\omega}$.

Consider the homothety $\iota \sim \eta_\epsilon$ for some $\eta > 0$: here $\iota$ is the isometric immersion of $T_W$ into $\mathbb{R}^3$. Then the metric associated to $\eta_\epsilon$ is $g_{\eta_\epsilon} := \eta \epsilon^{2\omega} \delta$, and its second fundamental form is $\eta H_{ij}$, where $H_{ij}$ is the second fundamental form associated to $\iota$. We are interested in sending $\{\eta_n\} \to \eta_0$, where $\eta_n$ and $\eta_0$ are positive constants. Along this limiting process, the principal curvatures remain unchanged, hence the flow speeds $q_{\eta_n}$ for $(T_W, g_{\eta_n})$ are the same for all $n$, thanks to the Bernoulli law in (4.2). Then, assuming the existence of solutions on $(T_W, g_{\eta_n})$ for each $\eta_n$, we find that the hypotheses in Proposition 4.3 are immediately verified. Indeed, for (4.4), the integral on the left-hand side equals $\|q_{\eta_n}\|_{L^2(T_W, g_{\eta_n})} = (\eta_0)^2 \int_{T_W} e^{-2\omega} dV_g$, where $\omega$ is a smooth function on the compact Wente torus $(T_W, g)$, hence bounded.

4.3. Example 2: Standard torus. In this example, we consider a 2-dimensional connected, closed, orientable manifold $(M, g)$ and look for solutions with $\kappa_+ = \text{constant}$ and $\rho = \overline{\rho}(\kappa_-)$. By purely geometrical arguments, such surfaces can be completely classified: Shiohama–Takagi [46] proved that except for the three surfaces listed in Example 4.3, the only possible such $(M, g)$ is the standard torus; namely, the surface of revolution $T(a, b) = \left\{ \begin{pmatrix} a + b \cos \theta \cos \phi \\ a + b \cos \theta \sin \phi \\ b \sin \theta \end{pmatrix} : 0 \leq \theta, \phi < 2\pi \right\}$. (4.7)

Here $a$ and $b$ with $a > b > 0$ are the major and minor radii of $T(a, b)$. The metric on $T(a, b)$ is the pullback of the Euclidean metric on $\mathbb{R}^3$ by the inclusion $T(a, b) \subset \mathbb{R}^3$, denoted by $g$ in this example.

Note that the standard torus has non-constant mean curvature (and, in addition, sign-changing Gauss curvature), so the corresponding fluid satisfying the steady Euler equations cannot be incompressible. Conversely, if $\rho$ is non-constant, then $\kappa_-$ is non-constant, so we are in neither of the three cases in Example 4.3. Thus the classification result in [46] shows that $(M, g) = T(a, b)$ for some $a > b > 0$. 

13
When \( \rho = \tilde{\rho}(\kappa_-) \), the equation in Lemma 3.1 becomes
\[
\frac{1}{2} \left\{ 1 - \frac{\kappa_+ - \kappa_-}{\rho} \cdot \tilde{\rho}'(\kappa_-) \right\} \mathcal{L}_{\kappa_-} = 0.
\]
Then it suffices to solve the ODE in the \( \kappa_- \) variable:
\[
1 - \frac{\kappa_+ - \kappa_-}{\tilde{\rho}(\kappa_-)} \cdot \tilde{\rho}'(\kappa_-) = 0.
\]
To this end, we make the change of variable
\[
F(\delta) := \log \tilde{\rho}(\kappa_-), \quad \delta := \kappa_+ - \kappa_- > 0.
\]
In the case that \( \kappa_- \) is non-constant, we have
\[
\kappa F'(\kappa) = -1,
\]
thus \( F(z) = C - \log z \) for an arbitrary constant \( C \). Equivalently, the solution is
\[
\tilde{\rho}(\kappa_-) = \frac{C_0}{\kappa_+ - \kappa_-},
\]
where \( C_0 > 0 \) is an arbitrary constant, \( \kappa_+ > \kappa_- \), and \( \kappa_+ = \text{constant} \).

Now let us analyse the explicit solution (4.8) from the perspective of fluid dynamics. Recall from Step 2, (3.13) that \( \kappa_- = p \). So the density \( \rho \) and pressure \( p \) are related by \( \rho = \frac{C_1}{C_2 - p} \) for constants \( C_1 > 0 \) and \( C_2 > p \), or alternatively
\[
p = C_2 - \frac{C_1}{\rho} \quad \text{and} \quad p'(\rho) = \frac{C_1}{\rho^2} =: c^2 \quad (c \text{ is the sonic speed}).
\]
Since we wish \( p \) to be increasing in \( \rho \), we choose \( C_1 > 0 \). This resembles the constitutive relation for the Chaplygin gas; see Chen–Slemrod–Wang [21, p.415]. In this case, the steady fluid motion is sonic: one infers from the Bernoulli law (i.e., the third equation in (4.2)) that
\[
q^2 = \frac{\kappa_+ - \kappa_-}{\rho} = \frac{C_0}{\rho^2} = \frac{d}{dp} \left( \kappa_+ - \frac{C_0}{\rho} \right) = p'(\rho),
\]
hence the Mach number \( M = q/c = 1 \). Note that the relation (4.10) agrees with the classical Bernoulli law for steady flow of polytropic gases (Courant–Friedrichs [26, p.22, (14.05)]):
\[
q^2 + \frac{2}{\gamma - 1} c^2 = \hat{q}^2.
\]
That is,
\[
\hat{q}^2 - p'(\rho) = \hat{q}^2
\]
when applied to the Chaplygin gas, and the Bernoulli constant \( \hat{q} \) is taken to be zero. This of course yields
\[
q^2 \rho^2 = C_1.
\]

We now substitute the Bernoulli law into the conservation of mass (3.3), in provision that the irrotationality condition is already satisfied. The system (4.2) is invariant under the scaling
\[
(\rho, v) \longrightarrow \left( \lambda^2 \rho, \frac{1}{\lambda} v \right) \quad \text{for each constant } \lambda > 0.
\]
Thus, without loss of generality, we may fix \( C_0 = 1 \) in (4.8) or (4.10), which leads to
\[
\rho = \frac{1}{\kappa_+ - \kappa_-} \quad \text{and} \quad |v|_g = q = \kappa_+ - \kappa_-.
\]
As a consequence, $\nabla_k (\rho v^k) = 0$ becomes the equation:

$$\text{div}_g \left( \frac{v}{|v|_g} \right) = 0,$$

(4.12)

where $\text{div}_g : \Gamma(TM) \to C^\infty(M)$, $u \mapsto \nabla_j u^j$ is the Riemannian divergence with respect to $g$.

In summary, we have shown the following

Lemma 4.7. On the standard torus $(M,g) = T(a,b)$ for $a > b > 0$, for the ansatz $\rho = \tilde{\rho}(\kappa_+ - \kappa_-)$, the system (4.2) of irrotationality, conservation of mass, and Bernoulli law is equivalent, modulo a multiplicative constant due to the scaling invariance in (4.11), to the following system:

$$\begin{cases}
\text{div}_g (\frac{v}{q}) = 0,
q = \kappa_+ - \kappa_-,
\rho q = 1.
\end{cases}$$

(4.11)

Remark 4.8. In this lemma, thanks to the Bernoulli law $q^2 = \kappa_+ - \kappa_- \rho$, the flow has no stagnation point (i.e., $q \neq 0$ everywhere), for otherwise there would be an umbilical point on $(M,g)$, which would lead to $(M,g) = \mathbb{R}^2$ or $rS^2$ but not $T(a,b)$ in the presence of one constant principal curvature. See Shiohama–Takagi [46].

Special solutions with geometric significance can be found for the system in Lemma 4.7. Indeed, as computed in Appendix A, by expressing $v \in \Gamma(T T(a,b))$ as

$$v = v^\theta(\theta,\phi)\partial_\theta + v^\phi(\theta,\phi)\partial_\phi$$

in the natural co-ordinate system $\{\partial_\theta, \partial_\phi\}$ for the parametrisation of $T(a,b)$ in (4.7), we may write the first-order system in Lemma 4.7 as follows:

$$\begin{cases}
-2b \sin \theta \cdot v^\theta(\theta,\phi) + (a + b \cos \theta) \left[ \partial_\theta v^\theta(\theta,\phi) + \partial_\phi v^\phi(\theta,\phi) \right] = 0,
(a + b \cos \theta)^2 (v^\theta(\theta,\phi))^2 + b^2 (v^\phi(\theta,\phi))^2 = \frac{a^2}{(a + b \cos \theta)^2}.
\end{cases}$$

(4.13)

(1) By inspection, the ansatz

$$v^\theta \equiv 0, \quad v^\phi(\theta,\phi) = \tilde{v}^\phi(\theta)$$

verifies the first equation in (4.13). Then we can directly determine $\tilde{v}^\phi(\theta)$ from the second equation. In this way, we obtain a solution:

$$v = \pm \frac{a}{b^2 (a + b \cos \theta)^2} \partial_\phi.$$  

(4.14)

Notice that (4.14) yields a well defined vector field $v \in \Gamma(T T(a,b))$, since $\partial_\phi$ is globally defined over $T(a,b)$ and this expression is $2\pi$-periodic in $\theta$. Integral curves of $v$ are precisely the toroidal geodesics. The magnitude of $v$ is constant on each of the toroidal geodesics but varies in the poloidal direction.

(2) On the other hand, the ansatz

$$v^\phi \equiv 0, \quad v^\theta(\theta,\phi) = \tilde{v}^\theta(\theta)$$

also yields a solution. The profile $\tilde{v}^\theta$ is determined by the second equation of (4.13), and one may directly check that it verifies the first equation. The solution is

$$v = \pm \frac{a}{b(a + b \cos \theta)^2} \partial_\theta.$$  

(4.15)
It is rotationally symmetric; i.e., $v$ is completely determined by its profile on any of the generating circles $\{\phi = \text{const.}\}$ of the $T(a, b)$ as a surface of revolution. The integral curves of $v$ are precisely the poloidal geodesics.

In both cases (1) & (2) above, the density is given by

$$\rho = \rho(\theta) = \frac{b(a + b \cos \theta)^2}{a}.$$  \hfill (4.16)

**Remark 4.9.** We emphasise that the topology of $T(a, b)$ plays a crucial role in the discovery of the special solutions (4.14) and (4.15), although in a somewhat fortuitous manner. Indeed, these expressions rely on the properties that the standard torus admits everywhere non-vanishing vector fields, and that the toroidal and poloidal geodesics respectively foliates the whole surface.

In contrast, such properties do not hold on topological 2-sphere $\Sigma$ due to the hairy ball theorem. The Chaplygin-type Bernoulli law

$$\rho q = \text{constant}$$

cannot hold everywhere on $\Sigma$, as stagnation points are unavoidable. Thus, even without assuming the stronger condition (4.11) and only considering the system (4.12) of irrotationality, conservation of mass, and Bernoulli law, it is perhaps undesirable to impose a Chaplygin-type Bernoulli law.

5. **Irrotational Chaplygin gas on the flat torus**

In the above section §4.3, we investigated the irrotational, steady Euler equation for the Chaplygin-type gas (i.e., with the Bernoulli law $pq = \text{constant}$) on the standard torus $T(a, b)$. Now we discuss the analogous issue on the flat torus

$$T^2 = [0, 1] \times [0, 1] / \sim.$$  

Here $\sim$ is the equivalence relation on $[0, 1] \times [0, 1]$ given by $(0, y) \sim (1, y)$ for all $y \in [0, 1]$ and $(x, 0) \sim (x, 1)$ for all $x \in [0, 1]$. The quotient space inherits the Euclidean metric from $[0, 1] \times [0, 1]$ and is a Riemannian manifold.

Our hope is to provide some new insights into the irregular isometric embeddings of $T^2$ into $\mathbb{R}^3$ constructed by Nash [44], Kuiper [38, 37], Borisov [7], Borrelli–Jabrane–Lazarus–Thibert [4, 8], and Conti–De Lellis–Szekelyhidi Jr. [25], among many others. In particular, it is well-known (see the above references) that such irregular isometric embeddings are closely related to the irregular or “wild” solutions to the Euler equations in fluid dynamics, especially in terms of the methods of their constructions. It is thus natural to study the Euler equations on the flat torus $T^2$ and discuss its possible implications on the irregular isometric embeddings.

5.1. **From standard torus to flat torus: convergence of metrics.** Our motivating observation for the developments in this section is as follows. Consider a rescaled version of the standard torus $T(a, b)$ (note that $T_1(a, b) = T(a, b)$):

$$T_c(a, b) = \left\{ \begin{bmatrix} [a + b \cos \left( \frac{\phi}{c} \right)] \cos \phi \\ [a + b \cos \left( \frac{\phi}{c} \right)] \sin \phi \\ b \sin \left( \frac{\phi}{c} \right) \end{bmatrix} : 0 \leq \phi < 2\pi, \ 0 \leq \theta < 2\pi c \right\} \text{ where } c > 0.$$  \hfill (5.1)
We are interested in the limiting process \( b, c \rightarrow 0 \). A straightforward computation yields that the metric \( g = g_{a,b,c} \) for \( T_c(a,b) \) is given by

\[
g = \begin{bmatrix}
[a + b \cos \left( \frac{\theta}{c} \right)]^2 & 0 \\
0 & \frac{b^2}{c^2}
\end{bmatrix}.
\]  

(5.2)

**Remark 5.1.** The rescaled tori \( T_c(a,b) \) are motivated by the “Nash wrinkles”, which are the basic building block for the \( C^1 \)-isometric embeddings \( T^2 \hookrightarrow \mathbb{R}^3 \). Indeed, the Nash wrinkles contain terms of the form \( \epsilon \cos \left( \frac{\theta}{\epsilon} \right) \) or \( \epsilon \sin \left( \frac{\theta}{\epsilon} \right) \) in the approximation sequence to isometric embeddings for \( 0 < \epsilon \ll 1 \). See [Equations (13) & (15)].

If we fix \( b/c = \text{constant} = c_0 \) and send \( b, c \rightarrow 0 \), then:

1. \( g \) tends to the flat metric \( \overline{g} = \begin{bmatrix} a^2 & 0 \\ 0 & (c_0)^2 \end{bmatrix} \) in \( L^\infty \);
2. the first derivative \( \partial g \) converging in the weak-* topology of \( L^\infty \) to zero.

The second property holds since

\[
\partial_\theta g_{11} = -2 \left[ a + b \cos \left( \frac{\theta}{c} \right) \right] \sin \left( \frac{\theta}{c} \right),
\]

where \( b \downarrow 0 \) and \( \sin(\theta/c) \rightharpoonup 0 \) weakly-* in \( L^\infty \) by the Riemann–Lebesgue lemma as \( c \downarrow 0 \). Therefore, we expect the behaviour of the solutions to the Euler equations (in the form of (4.2)) along this limiting process to contain certain persistent information about the steady irrotational flows, which in turn will single out “good” flows on the flat torus.

To this end, we carry out the process in §4.3 for the rescaled standard torus \( T_c(a,b) \). The computations are exactly parallel (see Appendix A). The principal curvatures of \( T_c(a,b) \) are

\[
\kappa_+ = \frac{1}{b} \quad \text{and} \quad \kappa_- = \frac{\cos \left( \frac{\theta}{c} \right)}{a + b \cos \left( \frac{\theta}{c} \right)},
\]

and as before (see (3.13)) we take \( p = \kappa_- \). By seeking solutions of the form \( \rho = \tilde{\rho}(\kappa_-) \) we again arrive at the Bernoulli law of Chaplygin-type gas. Therefore, we once more obtain the PDE system in Lemma 4.7, reproduced below:

\[
\begin{aligned}
\text{div}_g \left( \frac{\nabla}{\theta} \right) &= 0, \\
q &= \kappa_+ - \kappa_-,
\end{aligned}
\]

Thus

\[
q = \frac{a}{b \left[ a + b \cos \left( \frac{\theta}{c} \right) \right]^2} \quad \text{and} \quad \rho = \frac{b \left[ a + b \cos \left( \frac{\theta}{c} \right) \right]^2}{a}.
\]

(5.3)

Moreover, rewriting the above system in local co-ordinates (see (4.13) and ensuing developments for \( T(a,b) = T_1(a,b) \)), we obtain special solutions analogous to those in (4.14) and (4.15):

\[
v = \pm \frac{ac}{b^2 \left[ a + b \cos \left( \frac{\theta}{c} \right) \right]} \partial_\theta \quad \text{or} \quad \pm \frac{a}{b \left[ a + b \cos \left( \frac{\theta}{c} \right) \right]^2} \partial_\theta.
\]

(5.4)

In the limit \( b, c \downarrow 0 \) while keeping \( b/c = \text{constant} \), we see that \( q \rightarrow \infty \) and \( \rho \rightarrow 0 \), both in the pointwise sense. In terms of geometric variables, we have \( \kappa_+ \nearrow +\infty \) and \( \kappa_- \rightarrow 0 \) in the weak-* topology of \( L^\infty \), thanks to the presence of rapid oscillations. Moreover, in view of the
identification between the stress-momentum tensor $P$ and the second fundamental form $H$ (see (3.7), $H \approx \rho v \otimes v + g\kappa$), the extrinsic geometry of the associated isometric embeddings blows up pointwise in the limit. Nevertheless, the relation $\rho q = 1$ remains to hold everywhere.

The above investigations hint at the following:

On the flat torus $T^2 = [0, 1] \times [0, 1] / \sim$, it is natural to consider the steady, irrotational Euler equations for the Chaplygin gas, namely that

$$\begin{cases}
\omega = 0, \\
\text{div}_g(\rho v) = 0, \\
\rho q = 1.
\end{cases} \tag{5.5}$$

See §5.3 below for further remarks.

5.2. On the flat torus. Now we turn to the study of (5.5) on $T^2$. Substituting the Bernoulli law $\rho q = 1$ into the conservation of mass, we obtain

$$\text{div}_g \left( \frac{v}{q} \right) = 0 \quad \text{on } T^2. \tag{5.6}$$

In local co-ordinates the Riemannian divergence is given by

$$\text{div}_g w = \frac{1}{\sqrt{\det g}} \partial_1 \left( \sqrt{\det g} w^1 \right)$$

for any vector field $w$. But $\det g$ is constant for the flat metric, so the equation becomes

$$\partial_1 \left( \frac{v^1}{|v|_g} \right) + \partial_2 \left( \frac{v^2}{|v|_g} \right) = 0.$$

Clearly, we have the following special solutions which are also irrotational:

$$v = f \left( x^1 \right) \partial_1 \quad \text{or} \quad v = h \left( x^2 \right) \partial_2, \tag{5.7}$$

where $f, h \in C^\infty([0, 1])$ are arbitrary 1-periodic smooth functions without zeros. These are well defined global vector fields on $T^2$, and they represent horizontal or vertical flows with arbitrary smooth, stagnation point-free velocity profiles. Throughout we identify $T^2 = [0, 1] \times [0, 1] / \sim$, with $\partial_1, \partial_2$ tangent to each copy of $[0, 1]$.

In view of the irregular isometric embedding $\iota : T^2 \hookrightarrow \mathbb{R}^3$ remarked at the beginning of this section (cf. [44] [38] [37] [7] [4] [8] [25]), we obtain a simple recipe for irregular steady Euler solutions which exhibit fractal patterns of motion. One may take $\iota \in C^{1,\beta}(T^2, \mathbb{R}^3)$; the best $\beta$ known to date is $1/7 - \epsilon$. See [7] [25].

**Theorem 5.2.** Let $T$ be the embedded $C^{1,\beta}$-surface in $\mathbb{R}^3$ which is an isometric copy of the flat torus $T^2$. There exists a $C^{1,\beta}$-vector field $V \in \Gamma(T\mathbb{R}^3)$ whose restriction to $T$ satisfies the Euler equation (5.5) for steady, irrotational Chaplygin gas.

**Proof.** Let $v$ be the smooth shear flows defined in (5.7) on $T^2$, and let $\iota : T^2 \hookrightarrow \mathbb{R}^3$ be an $C^{1,\beta}$-isometric embedding. The pushforward $\iota_* v$ defines a solution on $\iota(T)$, which can be extended to a $C^{1,\beta}$-vector field $V$ by Whitney embedding. \qed

**Remark 5.3.** The velocity field $v \in \Gamma(TT^2)$ is “intrinsically smooth”, in the sense that it is a smooth vector field on $T^2$, which is itself a smooth Riemannian manifold. The irregularity arises only from the extrinsic geometry; namely, the isometric embedding $\iota : T^2 \hookrightarrow \mathbb{R}^3$. 

18
5.3. Geometric remarks. Earlier in this section we presented some investigations on the irregular solutions to the steady motion of Chaplygin gas on the flat torus. Such discussions are of a tentative and preliminary nature; we shall collect here a few related remarks.

Our first observation is that the smooth solutions \( v \in \Gamma(\mathbb{T}^2) \) in \((5.7)\), or the \( C^{1,\beta}_0 \)-solutions in Theorem 5.2, cannot arise from the limiting process introduced in \((5.1)\). Indeed, as computed in \((5.3)\), the flow speed becomes unbounded as \( b, c \searrow 0 \). This is because \( q = \kappa_+ - \kappa_- \), where the larger principal curvature \( \kappa_+ = 1/b \) blows up, while the smaller principal curvature is oscillatory (weakly-\( \ast \) converges to zero in \( L^\infty \)).

For the limiting process in \((5.1)\) we have shown that the metrics \( g = g_{a,b,c} \) for \( \mathbb{T}_c((a,b)) \) converge in the \( L^\infty \)-norm to the flat metric \( \overline{g} \) on \( \mathbb{T}^2 \) (modulo scaling by constants). This, however, does not imply that the sequence of manifolds \( \{ \mathbb{T}_c((a,b)) \} \) converges to \( \mathbb{T}^2 \) in the Hausdorff–Gromov sense. Indeed, \( \mathbb{T}_c((a,b)) \) converge to the central toroidal circle \( C_a = \left\{ \begin{bmatrix} a \cos \phi \\ a \sin \phi \\ 0 \end{bmatrix} : 0 \leq \phi < 2\pi \right\} \) as \( b \searrow 0 \). (The limit \( c \searrow 0 \) has the effect that one traverses the poloidal circles, i.e., in the \( \theta \)-direction, increasingly rapidly.) One possible heuristic for this phenomenon is to view \( C_a \) as a “degenerate flat torus”, namely that \( \mathbb{T}^2 \cong C_a \times S^1 \) with the \( S^1 \) factor collapsing to a point. For \( 0 < b \approx c \ll 1 \) we may view the two special solutions found in \((5.4)\) as being concentrated on the central toroidal circle \( C_a \) and on the vanishing poloidal direction \( \partial_\theta \), respectively. This is vaguely reminiscent of the 5-dimensional Kaluza–Klein theory of cosmology (see e.g., \([51]\)), which has an extra dimension curled up into a small loop in addition to the 4-dimensional spacetime.

In view of the previous paragraph, it is natural to consider other approximating sequences of metrics to the flat torus, as we are not enforcing Gromov–Hausdorff convergence of manifolds. For example, one may take on an arbitrary surface \( M \) the metrics (as long as they exist on \( M \)):

\[
g_n = \begin{bmatrix} 1 + \epsilon_1(n) & 0 \\ 0 & 1 + \epsilon_2(n) \end{bmatrix}, \quad \epsilon_1(n), \epsilon_2(n) \to 0 \text{ in suitable sense as } n \to \infty.
\]

On an arbitrary closed (i.e., compact, without boundary) manifold of arbitrary dimensions, one can always solve (in the distributional sense) the system of irrotationality and conservation of mass for Chaplygin gas, away from the set of stagnation points. If there exist zero-free harmonic 1-forms, then one can further require such solutions to be zero-free. This is summarised as the following theorem, whose proof is presented in Appendix B.

**Theorem 5.4.** Let \((M, g)\) be any closed Riemannian manifold. Let \( h \in \Omega^1(M) \) be any nontrivial harmonic 1-form. There exist \( \rho : M \to [0, \infty[ \) and \( v \in L^1(M; TM) \) which are weak solutions to the steady Euler equations away from the set of stagnation points of \( v \):

\[
\begin{align*}
\omega &= 0, \\
\text{div}_g(\rho v) &= 0, \\
\rho q &= \text{constant},
\end{align*}
\]

such that \( v^\sharp \), the 1-form canonically dual to \( v \), lies in the same cohomology class as \( h \). If, in addition, \((M, g)\) admits an everywhere non-vanishing harmonic 1-form, then \( v \) can be chosen without stagnation points.
We have described a connection between the smooth, steady compressible Euler equations and the isometric embedding of surfaces. Such constructions cannot be directly applied to irregular (non-$C^2$) embeddings obtained from the Nash–Kuiper Theorem 2.4 since neither the second fundamental form nor the Gauss curvature are well defined in the little regularity setting.

In this section, we establish a link between irregular isometric embeddings and compressible Euler equations. We bypass the aforementioned hindrance by a “renormalisation” process, based on explicit estimates in the corrugation/convex integration process of Conti–De Lellis–Székelyhidi Jr. [25]. Indeed, in [25] the authors provided a constructive proof for a version of the Nash–Kuiper theorem, which yields the best Hölder exponent $\alpha$ up to date. In what follows, unless otherwise specified, $\| \bullet \|_k$ for $k = 0, 1, 2, \ldots, \infty$ denotes the $C^k$-norm.

**Theorem 6.1** (Theorem 2 in [25]). Let $M$ be an $n$-dimensional compact manifold with a Riemannian metric $g$ in $C^\beta(M)$ and $m \geq n + 1$. Then there is a constant $\delta_0 > 0$ such that if $u \in C^2(M; \mathbb{R}^m)$ and $\alpha$ satisfy
\[
\| \partial_i u \cdot \partial_j u - g_{ij} \|_0 \leq \delta_0^2 \quad \text{and} \quad 0 < \alpha < \min \left\{ \frac{1}{2(\alpha + 1)n_*}, \frac{\beta}{2} \right\}
\]
where $n_* = n(n + 1)/2$, then there exists a map $v \in C^{1,\alpha}(M; \mathbb{R}^m)$ such that
\[
\partial_i v \cdot \partial_j v = g_{ij} \quad \text{and} \quad \| v - u \|_1 \leq \text{const.} \max_{i,j} \sqrt{\| \partial_i u \cdot \partial_j u - g_{ij} \|}.
\]

The proof of Theorem 6.1 involves the construction of a sequence of smooth maps $\{u_q\}_{q \in \mathbb{N}}$ via the “steps” and “stages” of Nash–Kuiper [14, 38, 37], whose limit as $q \to \infty$ yields the desired irregular isometric embedding. The following estimates are given in [25] §1.6.1:
\[
\begin{align*}
\max_{i,j} \| \partial_i u_q \cdot \partial_j u_q - g_{ij} \| & \leq \delta_q^2, \quad (6.1) \\
\| u_q \|_2 & \leq \mu_q, \quad (6.2) \\
\| u_{q+1} - u_q \|_1 & \leq C\delta_q, \quad (6.3)
\end{align*}
\]
where $\{\delta_q\}$ decreases exponentially while $\{\mu_q\}$ increases exponentially, provided that $\delta_0$ has been appropriately chosen. (6.2) implies that any limit of $u_q$ fails to be $C^2$-regular. To get a $C^{1,\alpha}$-limit with the range of $\alpha$ as stated in Theorem 6.1 the authors of [25] made a delicate choice of $\delta_q, \mu_q$; for our purpose below, we only need the existence of such parameters (but not the explicit choices) as in (6.1)–(6.3).

We propose the following renormalisation process.

Let $\{u_q\}$ be the sequence of smooth embeddings constructed in Theorem 6.1. They are not isometric to the prescribed metric $g$ on $M$ in general (in fact, they are “short”), and they take values in $\mathbb{R}^3$. We shall reserve the symbol $\varepsilon$ for the Euclidean metric. Our proposed renormalisation process consists of the following ingredients:

1. Set $g^{(0)} := u_q^\# \varepsilon$, the pullback metric; assume that $u_q$ are 1-periodic.
2. Let $\kappa^{(0)}$ be the Gauss curvature of $g^{(0)}$, and let $\kappa_1^{(0)}, \kappa_2^{(0)}$ be the principal curvatures.
3. Let $H^{(0)}$ be the second fundamental form of $u_q : (M, g^{(0)}) \to (\mathbb{R}^3, \varepsilon)$.
4. Let $(M, g)$ be the flat 2-torus, which serves as the tentative limiting manifold.
By construction, \( u_q \) are isometric embeddings \( \left( M, g^{(q)} \right) \hookrightarrow (\mathbb{R}^3, \xi) \). Thus \( \left( H^{(q)}, \kappa^{(q)} \right) \) satisfies the Gauss–Codazzi equations with respect to \( g^{(q)} \). Introduce the following new variables:

\[
\begin{align*}
\gamma^{(q)} & := \frac{\kappa^{(q)}}{\eta_q}, \\
\gamma_i^{(q)} & := \frac{\kappa_i^{(q)}}{\eta_q} \quad \text{for } i \in \{1, 2\},
\end{align*}
\]  

where as in \( (6.2) \),

\[
\eta_q := \| u_q \|_2 \leq \mu_q. 
\]  

To proceed, dividing by \( \eta_q^2 \) on both sides of \( (3.1) \), we obtain that

\[
h_{11}^{(q)} h_{22}^{(q)} - h_{12}^{(q)} h_{21}^{(q)} = \det g^{(q)} \gamma^{(q)}. 
\]  

From the Codazzi equation \( (3.2) \) one deduces that

\[
\partial_i H_{jk} - \partial_j H_{ik} = \Gamma^l_{ik} H_{jl} - \Gamma^l_{jk} H_{il},
\]

utilising the identity \( \nabla_k \phi_{ij} = \partial_k \phi_{ij} - \Gamma^l_{ik} \phi_{jl} - \Gamma^l_{jk} \phi_{il} \) for covariant 2-tensor \( \phi = \{ \phi_{ij} \} \). Thus

\[
\partial_i h_{jk}^{(q)} - \partial_j h_{ik}^{(q)} = \Gamma^l_{ik} h_{jl}^{(q)} - \Gamma^l_{jk} h_{il}^{(q)} := Q^{(q)}_{i,j,k}. 
\]  

Here \( \Gamma^i_{jk} \) are the Christoffel symbols of the Levi-Civita connection of \( \left( M, g^{(q)} \right) \), thus

\[
\left| Q^{(q)}_{i,j,k} \right| \lesssim O(\eta_q).
\]

The process of Nash–Kuiper iterations can be viewed as defined 1-periodically in the (global) co-ordinates \( (x_1, x_2) \in \mathbb{R}^2 \). This crucially relies on our choice that \( (M, g) \) is the flat torus.

Let us introduce the following change of the variables. We set

\[
z_j^{(q)} := \eta_q x_j
\]

and

\[
\hat{h}_{ij}^{(q)} \left( z_1^{(q)}, z_2^{(q)} \right) := h_{ij}^{(q)}(x_1, x_2)
\]

for \( i, j \in \{1, 2\} \). As a consequence, we may further express the renormalised Gauss and Codazzi equations \( (6.6) \) and \( (6.7) \) as

\[
\begin{align*}
\hat{h}_{11}^{(q)} \hat{h}_{22}^{(q)} - \hat{h}_{12}^{(q)} \hat{h}_{21}^{(q)} & = \det g^{(q)} \hat{\gamma}^{(q)}, \\
\frac{\partial}{\partial z_i} \hat{h}_{jk}^{(q)} - \frac{\partial}{\partial z_j} \hat{h}_{ik}^{(q)} & = \frac{Q^{(q)}_{i,j,k}}{\eta_q}.
\end{align*}
\]  

Here and hereafter we have assumed a slight abuse of notations: we designate

\[
g^{(q)} \left( z_1^{(q)}, z_2^{(q)} \right) \equiv g^{(q)}(x_1, x_2),
\]

\[
\hat{\gamma}^{(q)} \left( z_1^{(q)}, z_2^{(q)} \right) \equiv \gamma^{(q)}(x_1, x_2),
\]

\[
Q^{(q)}_{i,j,k} \left( z_1^{(q)}, z_2^{(q)} \right) \equiv Q^{(q)}_{i,j,k}(x_1, x_2),
\]

and the like. Notice that \( \hat{h}^{(q)} \) is defined on \( \mathbb{R}^2 \) with period \( \eta_q \); so in \( (6.8) \) we may drop the superscript \( (q) \) in the derivatives. That is, one has \( \partial_i \hat{h}_{jk}^{(q)} - \partial_j \hat{h}_{ik}^{(q)} = Q^{(q)}_{i,j,k}/\eta_q \).
Consider now the renormalised equations (6.8) and (6.9). The variables \( h^{(q)}, \gamma^{(q)}, g^{(q)} \) are uniformly bounded in \( q \), thanks to the estimates and definitions in (6.1), (6.4), and (6.5). Applying the theory of compensated compactness to the Gauss–Codazzi system (cf. Chen–Slemrod–Wang [21] and Chen–Li [19] [20], based on foundational works of Murat [42] [43] and Tartar [47] [48]; cf. also Dafermos [27]), we may deduce the existence of measures \( \tilde{h}_{ij} \) and \( \gamma \) satisfying

\[
\tilde{h}_{ij}^{(q)} \xrightarrow{\ast} \tilde{h}_{ij}, \quad \gamma^{(q)} \xrightarrow{\ast} \gamma, \quad \frac{Q_{i,j,k}^{(q)}}{\eta_q} \xrightarrow{\ast} F_{i,j,k} \quad \text{as} \ q \to \infty; \quad (6.10)
\]

\[
\partial_1 \tilde{h}_{22} - \partial_2 \tilde{h}_{12} = F_1, \quad -\partial_1 \tilde{h}_{12} + \partial_2 \tilde{h}_{11} = F_2; \quad (6.11)
\]

\[
\tilde{h}_{11} \tilde{h}_{22} - \tilde{h}_{12} \tilde{h}_{21} = \gamma. \quad (6.12)
\]

Here (6.10) is understood in the weak-* topology of \( L^\infty(\mathbb{R}^2) \), after passing to subsequences if necessary. The source term \( F_{i,j,k} \) is given by

\[
F_{i,j,k} = \begin{cases} F_1 & \text{if } (i, j, k) = (1, 2, 2), \\ F_2 & \text{if } (i, j, k) = (1, 2, 1), \\ 0 & \text{if else.} \end{cases}
\]

The equalities (6.11) and (6.12) are understood in the sense of distributions and a.e.; they follow from the normalised equations (6.7) and (6.6), as well as the estimate (6.1) obtained in [25].

**Lemma 6.2.** In the above setting, we have \( F_1 = F_2 = \gamma = 0 \).

**Proof.** By the renormalised Codazzi equations (6.7) and (6.9) we have

\[
\partial_i \tilde{h}_{jk}^{(q)} - \partial_j \tilde{h}_{ik}^{(q)} = \frac{(q)\Gamma_{ik}^{l} \tilde{h}_{jl}^{(q)}}{\eta_q} - \frac{(q)\Gamma_{jk}^{l} \tilde{h}_{il}^{(q)}}{\eta_q} = \frac{Q_{i,j,k}^{(q)}}{\eta_q}. \quad (6.13)
\]

As \( |\Gamma_{jk}^{i}| \lesssim \mathcal{O}(\eta_q) \), this quantity is uniformly bounded in \( L^\infty(\mathbb{R}^2) \). Thus, invoking the Sobolev embedding lemma and relabelling the indices, \( \{ \partial_i \tilde{h}_{jk}^{(q)} - \partial_j \tilde{h}_{ik}^{(q)} \} \) lies in a compact subset of \( H^{-1}(\mathbb{R}^2) \). On the other hand, applying the same arguments as for the renormalised Gauss–Codazzi equations (6.8), (6.9) and recalling the definition of Riemann curvature, we may redefine the independent variables (by periodicity) to deduce that

\[
\partial_j \left( \frac{(q)\Gamma_{ik}^{j}}{\eta_q} \right) - \partial_i \left( \frac{(q)\Gamma_{jk}^{i}}{\eta_q} \right) = \frac{(q)R_{kji}^{l}}{\eta_q} - \frac{(q)\Gamma_{jp}^{l} \Gamma_{ik}^{p}}{\eta_q} + \frac{(q)\Gamma_{jp}^{l} \Gamma_{ip}^{q} \Gamma_{jk}^{p}}{\eta_q}. \quad (6.14)
\]

Here \( (q)R_{kji}^{l} \) is the Riemann curvature tensor of the Riemannian manifold \( (M, u^\# q) \). This quantity is also confined in a compact subset of \( H^{-1}(\mathbb{R}^2) \).

Therefore, by a standard application of the div-curl lemma (see [42] [43] [47] [48]) using the differential constraints (6.13) and (6.14), one may infer that

\[
F_{i,j,k} = \Gamma_{ik}^{j} \tilde{h}_{jl}^{(q)} - \Gamma_{jk}^{l} \tilde{h}_{il}^{(q)}, \quad (6.15)
\]

where \( \Gamma_{jk}^{i} \) is the \( L^\infty \)-weak-* limit of \( (q)\Gamma_{jk}^{i}/\eta_q \). Moreover, by definition one has

\[
(q)\Gamma_{jk}^{i} = \frac{1}{2} g^{ijp} \left\{ \partial_j g_{pk}^{(q)} + \partial_k g_{pj}^{(q)} - \partial_p g_{jk}^{(q)} \right\}. \quad (6.16)
\]
Then, taking an arbitrary test function $\Phi \in C_0^\infty(\mathbb{R}^2)$ and integrating by parts with respect to the Lebesgue measure, we get
\[
\frac{1}{\eta_q} \int_{\mathbb{R}^2} g^{(q)ip}(\partial_j g^{(q)}_{pk}) \Phi \, dx = \frac{1}{\eta_q} \int_{\mathbb{R}^2} \Phi \delta^{ip} (\partial_j g^{(q)}_{pk}) \, dx + \frac{1}{\eta_q} \int_{\mathbb{R}^2} \Phi \left\{ g^{(q)ip} - \delta^{ip} \right\} (\partial_j g^{(q)}_{pk}) \, dx
\]
\[
= - \frac{1}{\eta_q} \int_{\mathbb{R}^2} (\partial_j \Phi) g^{(q)}_{pk} \, dx + \int_{\mathbb{R}^2} \left\{ g^{(q)ip} - \delta^{ip} \right\} \Phi \left( \frac{\partial_j g^{(q)}_{pk}}{\eta_q} \right) \, dx
\]
\[
=: I + II.
\]
Here $I \to 0$ since the integral $\int_{\mathbb{R}^2} (\partial_j \Phi) g^{(q)}_{pk} \, dx$ is uniformly bounded while $\eta_q \nearrow \infty$. In addition, $II \to 0$ since $g^{(q)ip} - \delta^{ip} \to 0$ in the $C^1$-topology while $\left\{ \eta_q^{-1} \Phi \left( \frac{\partial_j g^{(q)}_{pk}}{\eta_q} \right) \right\}$ is uniformly bounded.

Applying similar arguments to the two other terms on the right-hand side of (6.16), we deduce that
\[
\int_{\mathbb{R}^2} \frac{(q)\Gamma_{jk}^{i}}{\eta_q} \Phi \, dx \to 0 \quad \text{as } q \nearrow \infty. \quad (6.17)
\]

It thus follows that $\Gamma_{jk}^{i} \equiv 0$, hence $F_{i,j,k} \equiv 0$ for all $i, j, k \in \{1, 2\}$.

To see $\gamma = 0$, notice that $\gamma$ is equal to the distributional limit of
\[
\gamma^{(q)} = \frac{1}{\eta_q g_{ij}^{(q)}} \left\{ \partial_1^{(q)} \Gamma_{22}^{j} - \partial_2^{(q)} \Gamma_{12}^{j} + \partial_1^{(q)} \Gamma_{1k}^{j} \Gamma_{22}^{k} - \partial_2^{(q)} \Gamma_{2k}^{j} \Gamma_{12}^{k} \right\}.
\]
The first term on the right-hand side converges to zero as $q \nearrow \infty$ in the sense of distributions, since for any test function $\Phi \in C_0^\infty(\mathbb{R}^2)$, it holds that
\[
\frac{1}{\eta_q} \int_{\mathbb{R}^2} \Phi g_{ij}^{(q)} \left( \partial_1^{(q)} \Gamma_{22}^{j} \right) \, dx = - \frac{1}{\eta_q} \int_{\mathbb{R}^2} \partial_1 \Phi g_{ij}^{(q)} \Gamma_{22}^{j} \, dx - \frac{1}{\eta_q} \int_{\mathbb{R}^2} \Phi \partial_1 g_{ij}^{(q)} \Gamma_{22}^{j} \, dx
\]
\[
\to 0,
\]

by the definition of $\eta_q$ and the identity $\Gamma_{jk}^{i} \equiv 0$ established above. The same argument applies to the second term. As a consequence, we have that
\[
\gamma = \lim_{q \to \infty} \frac{(q)\Gamma_{1k}^{i} \Gamma_{22}^{k} - (q)\Gamma_{2k}^{i} \Gamma_{12}^{k}}{\eta_q^2}, \quad (6.18)
\]

where the limit is understood in the sense of distributions.

Now we apply once again the classical div-curl lemma (see [42, 43, 47, 48]). Consider the vector fields on $\mathbb{R}^2$:
\[
V_q := \eta_q^{-1} \left( (q)\Gamma_{1k}^{i}, (q)\Gamma_{2k}^{i} \right)^\top
\]
and
\[
W_q := \eta_q^{-1} \left( - (q)\Gamma_{22}^{i}, (q)\Gamma_{12}^{i} \right)^\top.
\]
The curl of $V_q$ and the divergence of $W_q$ are uniformly bounded in $L^\infty(\mathbb{R}^2)$. Thus, as $\Gamma_{jk}^{i} \equiv 0$, we have
\[
\gamma = \Gamma_{1k}^{i} \Gamma_{22}^{k} - \Gamma_{2k}^{i} \Gamma_{12}^{k} = 0.
\]

Hence the lemma is proved. \qed
By virtue of (6.11), (6.12) and Lemma 6.2 we define as in §3 the fluid variables \( \{ \rho, v^1, v^2, p \} \) via the following relations:

\[
\begin{align*}
\rho (v^1)^2 + p &= \bar{h}_{22}, \\
\rho v^1 v^2 &= -\bar{h}_{12}, \\
\rho (v^2)^2 + p &= \bar{h}_{11}.
\end{align*}
\]  

(6.19)

The limiting equations (6.9) and (6.8) thus become

\[
\begin{align*}
\partial_1 (\rho (v^1)^2 + p) + \partial_2 (\rho v^1 v^2) &= 0, \\
\partial_1 (\rho v^1 v^2) + \partial_2 (\rho (v^2)^2 + p) &= 0, \\
\rho p \left[ (v^1)^2 + (v^2)^2 \right] + p^2 &= 0.
\end{align*}
\]  

(6.20)

With \( \bar{h} \) given as above, we may solve for \( p \) in terms of \( \bar{h} \). Indeed, the first and the third equations of (6.19) yield that

\[
\rho p \left[ (v^1)^2 + (v^2)^2 \right] + 2\rho = \bar{h}_{11} + \bar{h}_{22}.
\]

Together with (6.6) and the last equation in (6.20), one deduces that

\[
p = 0 \quad \text{or} \quad p = \bar{h}_{11} + \bar{h}_{22}.
\]  

(6.21)

With either choice of \( p \), as in §3 we may solve for \( v^1, v^2, \) and \( \rho \) in the following way. Denote

\[
\begin{align*}
\bar{f}_{11} &:= \bar{h}_{11} - p, \\
\bar{f}_{22} &:= \bar{h}_{22} - p.
\end{align*}
\]

and solve \( v \) from

\[
\begin{align*}
v_1 &= \left( \frac{\bar{f}_{11}}{\rho} \right)^{\frac{1}{2}}, \\
v_2 &= \left( \frac{\bar{f}_{22}}{\rho} \right)^{\frac{1}{2}}.
\end{align*}
\]  

(6.23)

In terms of \( \bar{h}_{ij} \), (6.22) and (6.23) are equivalent to

\[
\begin{align*}
\partial_1 \sqrt{\rho \bar{h}_{11}} + \partial_2 \sqrt{\rho \bar{h}_{22}} &= 0 \quad \text{or} \quad \partial_1 \sqrt{-\rho \bar{h}_{22}} + \partial_2 \sqrt{-\rho \bar{h}_{11}} = 0
\end{align*}
\]  

(6.24)

and

\[
\begin{align*}
(v_1, v_2) &= \left( \left[ \frac{\bar{h}_{11}}{\rho} \right]^{\frac{1}{2}}, \left[ \frac{\bar{h}_{22}}{\rho} \right]^{\frac{1}{2}} \right) \\
(v_1, v_2) &= \left( \left[ \frac{-\bar{h}_{22}}{\rho} \right]^{\frac{1}{2}}, \left[ \frac{-\bar{h}_{11}}{\rho} \right]^{\frac{1}{2}} \right)
\end{align*}
\]  

(6.25)

with respect to the choices of the two roots of \( p \) in (6.21).

If \( h_{11}, h_{22} > 0 \), we choose the pressureless case \( p = 0 \), and if \( h_{11}, h_{22} < 0 \), we choose \( p = \bar{h}_{11} + \bar{h}_{22} \). Notice that the two solutions in (6.24) and (6.25) can be transformed into each other by identifying

\[
(\bar{h}_{11}, \bar{h}_{22}) \mapsto (-\bar{h}_{22}, -\bar{h}_{11}).
\]

We may thus regard them as dynamically indistinguishable. That is, both choices in (6.21) describe the motion of a pressureless gas. Furthermore, if \( \bar{h}_{11}, \bar{h}_{22} \) are smooth, we may use the method of characteristics to locally define \( \rho \) and thus accomplish our goal of delivering the fluid variables \( \{ \rho, v, p \} \). In general, however, we only have \( \bar{h}_{11}, \bar{h}_{22} \in L^\infty(M) \), so the existence (even definition) of the solution is not obvious.
To summarise this section, we have derived the following

**Theorem 6.3.** The formulae (6.21), (6.22), and (6.23) provide expressions for the fluid variables \(\{p,v,\rho\}\) of the renormalised limit of the Nash–Kuiper iterations. They describe a weak solution to the steady pressureless gas equations, under the assumption that a solution \(\rho\) exists to the continuity equation (6.24).

7. **Extension to Multi-Dimensions**

Now we aim at extending our earlier investigations to multi-dimensions. That is, we hope to establish a link between the isometrically embedded smooth submanifolds in \(\mathbb{R}^{n+k}\) (for arbitrary \(n,k\)) and the solutions to steady compressible Euler equations. Let \((\Sigma,g)\) be an \(n\)-dimensional Riemannian manifold isometrically embedded in \(\mathbb{R}^{n+k}\). The extrinsic geometry is characterised by the second fundamental form \(H_{ij}\) and the affine connection on the normal bundle \(\{A_{ij}\}\). In this section, \(i,j,k,l \in \{1,\ldots,n\}\) are indices for the tangent bundle \(T\Sigma\), and \(\nu,\mu,\eta \in \{n+1,\ldots,n+k\}\) are indices for the normal bundle \(T^\perp \Sigma\). In this case, the fundamental equations for the existence of isometric embeddings are the Gauss, Codazzi and Ricci equations as follows:

\[
\begin{align*}
H_{ij}^l H_{kl}^m - H_{ik}^l H_{jl}^m &= R_{ijkl}, \\
\nabla_i H_{j}^{\mu l} - \nabla_j H_{i}^{\mu l} + A_{\mu j}^{\nu l} H_{i}^{\nu l} - A_{\nu j}^{\nu i} H_{i}^{\nu l} &= 0, \\
\nabla_i A_{\nu j}^{\nu i} - \nabla_j A_{\mu j}^{\mu i} + A_{\nu j}^{\eta i} A_{\mu j}^{\eta j} - A_{\mu j}^{\eta i} A_{\nu j}^{\eta j} &= g^{pq} (H_{ij}^{l} H_{jq}^{p} - H_{ij}^{l} H_{pq}^{i}).
\end{align*}
\]

The mean curvature vector \(\vec{m} = (m^1,\ldots,m^k)\) is given by

\[
m^\nu := g^{ij} H_{ij}^{\nu} = H_{i}^{\nu i}.
\]

On \((\Sigma,g)\) we also have the steady compressible Euler equation. The continuity equation is

\[
\nabla_k (\rho v^k) = 0.
\]

Moreover, for the stress-energy tensor

\[
P^{ij} = \rho v^i v^j + g^{ij} p,
\]

there holds the balance of linear momentum:

\[
\nabla_j P^j_i = \Pi_i,
\]

The unknowns of the Euler equations are the fluid variables \(\rho, v = \{v^i\}_{i=1}^n\), and \(p\). Here \(\Pi_i\) is a given body force on \(\Sigma\).

Assume that \(H_{ij}\) is a smooth solution for the Gauss–Codazzi–Ricci equations (7.1)–(7.3). We shall identify the fluid variables as functions of \(H_{ij}\).

To begin with, consider the **contracted Codazzi equation**:

\[
\nabla_i H_{j}^{n+1,j} - \nabla_j H_{i}^{n+1,j} = \Pi_i,
\]

where

\[
\Pi_i := A_{n+1,j}^{\nu j} H_{i}^{\nu j} - A_{n+1,j}^{\nu i} H_{j}^{\nu j}.
\]
This is obtained by setting $\mu = n + 1$ and multiplying by $\delta j^i$ in (7.2). We choose the following relation between the stress-energy tensor and second fundamental forms:

$$P_{ij}^j := -H_{ij}^{n+1} + m^{n+1}\delta j^i. \quad (7.7)$$

Then

$$\nabla j P_{ij}^j = -\nabla i H_{ij}^{n+1} + \nabla i m^{n+1} + \Pi_i = \Pi_i.$$ i.e., the balance law for linear momentum (7.6) is satisfied.

By contracting with the metric tensor, we see that (7.7) is equivalent to

$$H_{ij}^{n+1} = m^{n+1}g_{ij} - P_{ij}. \quad (7.8)$$

We can deduce the constitutive relation for $P_{ij}$ from the Gauss equation (7.1). Indeed, define

$$L_{iljk} := \sum_{\mu=n+2}^{n+k} H_{ij}^\mu H_{kl}^\mu - H_{ik}^\mu H_{jl}^\mu. \quad (7.9)$$

Then one may compute

$$R_{iljk} = (-P_{ij} + m^{n+1}g_{ij})(-P_{kl} + m^{n+1}g_{kl})$$

$$- (-P_{ik} + m^{n+1}g_{ik})(-P_{jl} + m^{n+1}g_{jl}) + L_{iljk}$$

$$= \left\{P_{ij}P_{kl} - P_{ik}P_{jl}\right\} - m^{n+1}\left\{P_{kl}g_{ij} + P_{ij}g_{kl} - P_{ik}g_{jl} - P_{jl}g_{ik}\right\}$$

$$+ [m^{n+1}]^2\left\{g_{ij}g_{kl} - g_{ik}g_{jl}\right\} + L_{iljk}.$$ Substituting (7.5) into the above expression, one obtains that

$$R_{iljk} = \left\{(\rho v_i v_j + pg_{ij})(\rho v_k v_l + pg_{kl}) - (\rho v_i v_k + pg_{ik})(\rho v_j v_l + pg_{jl})\right\}$$

$$- m^{n+1}\left\{g_{ij}(\rho v_k v_l + pg_{kl}) + g_{kl}(\rho v_i v_j + pg_{ij}) - g_{ij}(\rho v_i v_k + pg_{ik}) - g_{ik}(\rho v_j v_l + pg_{jl})\right\}$$

$$+ [m^{n+1}]^2\left\{g_{ij}g_{kl} - g_{ik}g_{jl}\right\} + L_{iljk}$$

$$= (p - m)A_{iljk} + (p - m)^2B_{iljk} + L_{iljk},$$

where

$$\begin{aligned}
A_{iljk} &= \rho\left[v_i v_j g_{kl} + v_k v_l g_{ij} - v_i v_k g_{jl} - v_j v_l g_{ik}\right], \\
B_{iljk} &= g_{ij}g_{kl} - g_{ik}g_{jl}.
\end{aligned} \quad (7.10)$$

These terms have a natural geometric structure: for 2-tensors $T = T_{ij}$ and $S = S_{ij}$, denote by $T \otimes S$ the 4-tensor

$$(T \otimes S)_{iljk} := T_{ij}S_{kl} - T_{ik}S_{jl}.$$ Then

$$\begin{aligned}
A &= 2\rho(\sigma \otimes \sigma), \\
B &= g \otimes g,
\end{aligned}$$

where

$$\sigma := \frac{1}{2}(g \otimes (v \otimes v) + (v \otimes v) \otimes g)$$

is the symmetrisation of $g$ and $v \otimes v$. 

26
Now we express \( p \) in terms of the geometric quantities. This is done by contracting the Riemann curvature. First, we compute the Ricci curvature tensor \( \text{Ric} \):

\[
\text{Ric}_{i k} = g^{j i} R_{i j k l}
\]

\[
= (p - m) g^{j i} A_{i j k} + (p - m)^2 g^{j i} B_{i j k} + g^{j i} L_{i j k l}
\]

\[= (p - m) g^{j i} v_k v_l + (p - m)^2 (n-1) g_{i j} v_k v_l + (p - m) (n-2) g_{i j} v_k v_l + g^{j i} L_{i j k l}.
\]

Here \( |v|^2 := g^{j i} v_j v_i \). Contracting once more yields

\[
\text{scal} = g^{k l} \text{Ric}_{k l} = (n-1)(p - m) \left[ 2 \rho |v|^2 + n(p-m) \right] + s,
\]

where

\[
s := g^{i j} g^{k l} L_{i j k l}.
\]

(7.13)

Thanks to (7.5) and (7.7), the momentum energy density can be expressed in terms of \( p \) and \( m \):

\[
\rho |v|^2 g_{i j} = P_{n+1}^{i j} - np
\]

\[= -H_{n+1}^{i j} + nm^{n+1} - np
\]

\[= (n-1)m^{n+1} - np.
\]

(7.14)

Now we can conclude that \( p \) satisfies the quadratic equation:

\[-n(n-1)p^2 + 2(n-1)^2 m^{n+1} p + (n-1)(n-2)(m^{n+1})^2 - \text{scal} + s = 0.
\]

(7.15)

It has real roots

\[p = \frac{(n-1)^2 m^{n+1} + \sqrt{(n-1)^4 m^{2n+2} + n(n-1)^2 (n-2)(m^{n+1})^2 + n(n-1)(s - \text{scal})}}{n(n-1)}
\]

where the discriminant

\[\Delta = (n-1)^4 (m^{n+1})^2 + n(n-1)^2 (n-2)(m^{n+1})^2 + n(n-1)(s - \text{scal}) \geq 0.
\]

Similar to §3, we can solve for the velocity from

\[v^k = \sqrt{\frac{f_{i j}}{\rho}}
\]

(7.16)

(no summation convention), where \( \rho \) is determined by the continuity equation (7.4), and

\[f_{i j} := -H_{n+1}^{i j} + (m^{n+1} - p) g_{i j}.
\]

(7.17)

This can be done whenever for all \( q, k \) there holds

\[\rho v^k v_k = -H_{n+1}^{q k} + (m^{n+1} - p) g_{q k}.
\]

(7.18)

By considering \( \rho(v_q)^2 \rho(v_k)^2 = (v_q v_k)^2 \), we can re-express (7.18) as the following consistency conditions (C1), (C2):

For all \( k, q \), the “principal matrices” \( G^{-1} C \) have a common eigenvalue \( \lambda \),

\[G = \begin{bmatrix} g_{k k} & g_{k q} \\ g_{q k} & g_{q q} \end{bmatrix}, \quad C = \begin{bmatrix} H_{n+1}^{k k} & H_{n+1}^{k q} \\ H_{n+1}^{q k} & H_{n+1}^{q q} \end{bmatrix},
\]

(7.19)
as well as
\[ \lambda = m^{n+1} - p \] with \( p \) satisfying Equation (7.13). (C2)

At this point we could repeat the renormalisation procedure in §6 when \( M \) is the flat \( n \)-torus. The arguments would be similar to those in §6 but now applied to the Gauss–Codazzi–Ricci equations in higher dimensions and codimensions (cf. e.g., Chen–Slemrod–Wang [22] and Chen–Li [19]): this is because the quadratic terms \( H \otimes H, A \otimes A, H \otimes A, \Gamma \otimes A, \) and \( \Gamma \otimes H \) are of the order \( O(\eta_q^2) \), where \( \eta_q \) is the \( C^2 \)-norm of the \( q \)-th approximate solution \( u_q \), and \( \Gamma \) is the Levi-Civita connection of \((M, u_q^#)\) as before. However, the consistency conditions (C1) & (C2) would still have to be satisfied for the weak-⋆ limit equations.

As the closing remark, correspondences between solutions to fluid dynamical PDE and solutions to geometric PDE (assuming sufficient regularity) have been studied in recent physics literature. When the geometric PDE are the vacuum Einstein equations for a Ricci-flat hypersurface, by Damour [28] and Bredberg–Strominger [11], rather amazingly, the fluid dynamic PDE formally are the classical non-steady Navier–Stokes equations of incompressible fluids. See also [6, 15, 24, 10, 16, 36] and the references cited therein; this list is by no means exhaustive. It is interesting to further explore such geometry-fluid correspondences using analytic methods.

A. GEOMETRY OF THE STANDARD TORUS \( \mathbf{T}(a,b) \)

In this appendix, we collect some basic facts about the standard torus, which are used in §4.3. The standard torus \( \mathbf{T}(a,b) \) of major radius \( a \) and minor radius \( b \) is the parametric surface in \( \mathbb{R}^3 \) given by (4.7), reproduced below:
\[
\mathbf{T}(a,b) = \left\{ \begin{pmatrix} (a + b \cos \theta) \cos \phi \\ (a + b \cos \theta) \sin \phi \\ b \sin \theta \end{pmatrix} : 0 \leq \theta, \phi < 2\pi \right\}.
\]
More generally, we also consider the scaled version:
\[
\mathbf{T}_c(a,b) = \left\{ \begin{pmatrix} [a + b \cos \left( \frac{\theta}{c} \right)] \cos \phi \\ [a + b \cos \left( \frac{\theta}{c} \right)] \sin \phi \\ b \sin \left( \frac{\theta}{c} \right) \end{pmatrix} : 0 \leq \phi < 2\pi, 0 \leq \theta < 2\pi c \right\} \quad \text{where} \ c > 0. \quad (A.1)
\]
Note that \( \mathbf{T}_1(a,b) = \mathbf{T}(a,b) \).

The Riemannian metric (i.e., the first fundamental form) of \( \mathbf{T}_c(a,b) \) is
\[
g = \begin{bmatrix} [a + b \cos \left( \frac{\theta}{c} \right)]^2 & 0 \\ 0 & \frac{b^2}{c^2} \end{bmatrix}, \quad (A.2)
\]
and the second fundamental form \( H = \{H_{ij}\} \) is
\[
H = \begin{bmatrix} - \left[ a + b \cos \left( \frac{\theta}{c} \right) \right] \cos \left( \frac{\phi}{c} \right) & 0 \\ 0 & -\frac{b}{c^2} \end{bmatrix}. \quad (A.3)
\]
From these we can compute the Gauss curvature
\[
\kappa = \frac{\det H}{\det g} = \frac{\cos \left( \frac{\phi}{c} \right)}{b \left[ a + b \cos \left( \frac{\theta}{c} \right) \right]} \quad (A.4)
\]
and the mean curvature
\[
m = \frac{H_{11}g_{22} - 2H_{12}g_{12} + H_{22}g_{11}}{\det g}
\]
\[
= -\frac{a + 2b \cos \left(\frac{\theta}{c}\right)}{b \left[ a + b \cos \left(\frac{\theta}{c}\right) \right]}. \tag{A.5}
\]
Then, the principal curvatures \(\kappa_\pm\) are the roots for the quadratic polynomial:
\[
q(s) := s^2 - ms + \kappa,
\]
and hence
\[
s = -\frac{\cos \left(\frac{\theta}{c}\right)}{a + b \cos \left(\frac{\theta}{c}\right)} \text{ or } -\frac{1}{b}.
\]

One issue arises here: we have chosen \(p = \kappa_-\), the smaller principal curvature in (3.13). But the smaller solution to \(q(s) = 0\) above is \(s = -1/b\), while it is non-physical to keep the pressure fixed for a compressible fluid. Moreover, throughout §4.3 we have assumed the \(\kappa_+\) is constant. This issue is resolved as in Shiohama–Takagi [46, p.479]: by inverting the orientation we replace the unit normal vector field \(e_3\) by \(-e_3\). In this way, (A.3) and (A.5) are replaced by
\[
H = \begin{bmatrix}
a + b \cos \left(\frac{\theta}{c}\right) & 0 \\
0 & \frac{b}{c^2}
\end{bmatrix}
\]
and
\[
m = \frac{a + 2b \cos \left(\frac{\theta}{c}\right)}{b \left[ a + b \cos \left(\frac{\theta}{c}\right) \right]}.
\tag{A.6}
\]
The principal curvatures are
\[
\kappa_+ = \frac{1}{b} \quad \text{and} \quad \kappa_- = \frac{\cos \left(\frac{\theta}{c}\right)}{a + b \cos \left(\frac{\theta}{c}\right)} = p. \tag{A.7}
\]
Also note that our convention for mean curvature is \(m = \text{tr}_g H\) instead of \(m = \frac{1}{\dim M} \text{tr}_g H\). Thus, the round 2-sphere has \(m = 2\) instead of 1.

As a result, the flow speed, which is equal to the difference between the principal curvatures modulo a constant, is
\[
q = |v|_g = \kappa_+ - \kappa_- = \frac{a}{b \left[ a + b \cos \left(\frac{\theta}{c}\right) \right]}.
\tag{A.8}
\]
The fluid density as chosen in §4.3 (also modulo a constant) is
\[
\rho = \frac{1}{\kappa_+ - \kappa_-} = \frac{b \left[ a + b \cos \left(\frac{\theta}{c}\right) \right]}{a}.
\]
See (4.10) and (4.8). For the purpose of solving the system (4.2) for irrotationality, conservation of mass, and Bernoulli’s law, we may choose as above the normalisation constants for both \(q\) and \(\rho\) to be 1, thanks to the scaling of these PDE.

Now let us write the equation
\[
\text{div}_g \left( \frac{v}{|v|_g} \right) = 0
\]
in local co-ordinates on \(T_c(a,b)\). The natural co-ordinate system \(\{\partial_\theta, \partial_\phi\}\) comes from the parametrisation of \(T_c(a,b)\) in \((A.1)\). Then for \(v \in \Gamma(TT_c(a,b))\) we may write
\[
v = v^\theta(\theta, \phi) \partial_\theta + v^\phi(\theta, \phi) \partial_\phi. \tag{A.9}
\]
Expressing the Riemannian divergence as
\[
\text{div}_g \left( \frac{v}{q} \right) = \frac{1}{\sqrt{\det g}} \partial_i \left( \sqrt{\det g} \frac{v^i}{q} \right)
\]
with Einstein’s summation convention over \( i \in \{\theta, \phi\} \), we obtain
\[
- \frac{2b}{c} \sin \left( \frac{\theta}{c} \right) v^\theta(\theta, \phi) + \left[ a + b \cos \left( \frac{\theta}{c} \right) \right] \left\{ \partial_\theta v^\theta(\theta, \phi) + \partial_\phi v^\phi(\theta, \phi) \right\} = 0. \tag{A.10}
\]
In addition, we have the constraint \(|v|_g = q\), where \( q \) is determined by the geometric parameters of the torus \( \mathbf{T}_c(a, b) \) as in (A.8). This is expressed in local co-ordinates by
\[
\left[ a + b \cos \left( \frac{\theta}{c} \right) \right]^2 \left( v^\theta(\theta, \phi) \right)^2 + \frac{b^2}{c^2} \left( v^\phi(\theta, \phi) \right)^2 = \frac{a^2}{b^2 \left[ a + b \cos \left( \frac{\theta}{c} \right) \right]^2}. \tag{A.11}
\]

**B. Irrotational Chaplygin gas on surfaces**

In this appendix, we prove Theorem 5.4 (reproduced below) concerning the Chaplygin gas on an arbitrary non-simply-connected closed Riemannian manifold \((M, g)\). As for aerodynamists’ convention, the system of irrotationality, conservation of mass, and Bernoulli’s law will be considered.

**Theorem B.1.** Let \((M, g)\) be any closed Riemannian manifold. Let \(h \in \Omega^1(M)\) be any nontrivial harmonic 1-form. There exist \(\rho : M \to [0, \infty]\) and \(v \in L^1(M; TM)\) which are weak solutions to the steady Euler equations away from the set of stagnation points of \(v\):

\[
\begin{align*}
\omega &= 0, \\
\text{div}_g (\rho v) &= 0, \\
\rho q &= \text{constant},
\end{align*}
\tag{B.1}
\]

such that \(v^\theta\), the 1-form canonically dual to \(v\), lies in the same cohomology class as \(h\).

If, in addition, \((M, g)\) admits an everywhere non-vanishing harmonic 1-form, then \(v\) can be chosen without stagnation points.

Here, in arbitrary dimensions, the vorticity \(\omega\) is defined as the 2-form:

\[
\omega = d \left( v^\theta \right).
\]

By writing \(v \in L^1(M; TM)\) we mean that \(v\) is a vector field on \(M\) with \(L^1\)-regularity. In general, for a vector bundle \(E \to M\), we write \(X(M; E)\) for \(X = L^p, W^{k,p}, C^{k,\alpha}, \ldots\) to denote the space of \(E\)-sections with indicated regularity \(X\).

**Remark B.2.** The case \(M = \{q = 0\}\) is ruled out by restricting to the nontrivial cohomology class \([h] \neq [0]\). In 2 dimensions such \(h\) exists if and only if \(M\) is not a topological 2-sphere, by Hodge decomposition and de Rham’s theorem. See, e.g., Petersen [15] §7.2, Theorem 47 on p.205, and Appendix, Theorem 89 on p.386.

Note that if \((M, g)\) is the Euclidean space, then the irrotationality of \(v\) implies that \(v = \nabla \psi\) for a stream function \(\psi\). Then (1.12) becomes the 1-Laplace equation for the scalar field \(\psi\):

\[
\text{div} \left( \frac{\nabla \psi}{|\nabla \psi|} \right) = 0,
\]
which is the Euler–Lagrange equation for the energy functional $I[ψ] := \int |\nabla ψ| \, dx$. When $M$ is not simply-connected, such stream function does not exist in general. We shall make use of a variational argument adapted from Evans [31 §3.1].

**Proof.** We divide our arguments into three steps.

1. Our strategy is as follows. First, by the scaling invariance property of the system (B.1) (see (4.11)), we may assume that the constant therein is 1. Substituting the Bernoulli’s law $ρq = 1$ into the second equation, we see that (B.1) is equivalent to

   \[
   \begin{align*}
   ω &= 0, \\
   \text{div}_g (\frac{u}{q}) &= 0.
   \end{align*}
   \]

   Denote by $α := v^♯ ∈ Ω^1(M, g)$, the differential 1-form canonically isomorphic to $v$. That $v$ is irrotational is equivalent to $dα = 0$, namely that $α$ is a closed 1-form. Thus, for the given harmonic 1-form $h$, it is natural to solve for \( \text{div}_g (\frac{u}{q}) = 0 \) in the cohomology class

   \[ [h] := \{ β ∈ Ω^1(M, g) : β − h = dχ \text{ for some } χ ∈ Ω^0(M, g) \}. \]

   For regularity considerations, we shall relax to a larger class:

   \[ \left[[h]\right] := \{ β ∈ L^∞(M; TM) : β − h = dχ \text{ in the sense of distributions for } χ ∈ D'(M) \}. \]

   When $β ∈ [[h]]$ let us still say that $β$ is cohomologous to $h$ (in the setting of little regularity), as in Ciarlet–Gratie–Mardare [23 §3].

   For this purpose, the Hodge decomposition implies that there exist a scalar field $ψ$ and a harmonic 1-form $h ∈ Ω^1(M, g)$ such that $α = dψ + h$. By raising and lowering indices, (4.12) is equivalent to

   \[
   d^* \left( \frac{α}{|α|_g} \right) = d^* \left( \frac{dψ + h}{|dψ + h|_g} \right) = 0.
   \]

   It then remains to establish the following

   **Claim A:** Fix any harmonic 1-form $h ∈ Ω^1(M, g)$. There exists $α ∈ L^∞(M; T^* M)$ cohomologous to $h$ which is a weak solution to (B.4). The notion of weak solution is understood with its domain away from the zeros of $α$ as in Remark B.2. That is, $α ∈ [[h]]$ is a weak solution if and only if

   \[ \int_{M \setminus \{α = 0\}} \left< \frac{α}{|α|_g}, \ dψ \right>_g \ dV_g = 0 \quad \text{for any } ψ ∈ C^1_0(M \setminus \{α = 0\}). \]

   Indeed, assuming Claim A, we may take $v = α^♭$ to solve for (B.2), and then recover the Bernoulli law by setting

   \[ \rho = \begin{cases} 
   (|v|_g)^{-1} & \text{when } v \neq 0; \\
   \infty & \text{elsewhere.}
   \end{cases} \]

   As a side remark, for many surfaces, e.g., $M = T_c(a, b)$ as in Appendix A, there exist everywhere non-vanishing harmonic 1-forms. Indeed, in the parametrisation $A.1$ for the torus, the space of harmonic vector fields is spanned by $\{∂_θ, ∂_φ\}$. A linear combination $h = a\, dθ + b\, dφ$ is non-vanishing for generic $a, b ∈ \mathbb{R}$. Also, if $(M, g)$ has positive Gauss curvature, then any nontrivial harmonic 1-form has no zero by Bochner’s theorem. See, e.g., Petersen [43 §7.3, Theorem 48 and Corollary 18 on p.208].
2. To prove Claim A above, we resort to a variational formulation of (B.4), motivated by Evans [31 §3.1]. Consider a subclass of [[h]] defined in (B.3):

$$A_h := \{ \beta \in [[h]] : \| \beta \|_{L^\infty(M,g)} \leq 10 \| h \|_{L^\infty(M,g)} \}.$$  

We shall first establish

**Claim B**: (B.4) is the Euler–Lagrange equations for the minimisation problem $\inf_{A_h} I$, where

$$I[\beta] := \| \beta \|_{L^1(M,g)} := \int_M |\beta|_g \, dV_g.$$  

**Proof of Claim B.** Let us compute the first variation of $I$. In fact, to move $\alpha$ within the cohomology class of $h$, we take perturbations of the form $\{ \alpha + t\eta \}_{\delta_0 < t < \delta_0}$ for any arbitrary $\eta \in C^\infty(M)$ whose $C^1$-norm is bounded from above by a uniform constant $c_0$, once $\delta_0$ is fixed. By Hodge decomposition, there is a scalar field $\psi$ such that $\alpha = d\psi + h$. Then

$$\frac{d}{dt} I[\alpha + t\eta] = \int_M \left\{ \frac{d}{dt} [h + d\psi + t\eta]_g \right\} \, dV_g = \int_M \left[ \frac{2}{|h + d\psi + t\eta|_g} \right] \, dV_g.$$  

So the minimality condition $\frac{d}{dt} I[\alpha + t\eta]_{t=0} = 0$ leads to

$$\int_M \left( \frac{h + d\psi}{|h + d\psi|_g} , \frac{\eta}{g} \right) \, dV_g = 0.$$  

Integration by parts gives us

$$\int_M \eta \, d^* \left( \frac{h + d\psi}{|h + d\psi|_g} \right) \, dV_g = 0.$$  

As $\eta$ is arbitrary in the class $\{ \eta \in C^1(M) : \| \eta \|_{C^1} \leq c_0 \}$, we obtain (B.4). □

3. With Claim B at hand, let us conclude Claim A by establishing the existence of minimiser to the functional $I$ over $A_h$.

**Proof of Claim A.** We shall follow the direct method of calculus of variations.

First of all, note that $h \in A_h$ with $I[h] < \infty$, so $i_0 := \inf_{\alpha \in A_h} I[\alpha]$ is finite. Let $\{ \alpha_j \}_{j \in \mathbb{N}} \subset A_h$ be such that $I[\alpha_j] = \| \alpha_j \|_{L^1(M,g)} \rightarrow i_0$ as $j \rightarrow \infty$. If $i_0 = 0$, then $\alpha_j \rightarrow 0$ (the zero 1-form) in the $L^1$-norm. But by assumption $h$ is not cohomologous to 0. Thus we have $0 < i_0 < \infty$.

Now, by convergence of $I[\alpha_j]$ we have that $\{ \alpha_j \}$ is uniformly bounded in $L^1$. In addition, it follows from the definition of $A_h$ that $\| \alpha_j \|_{L^\infty(M,g)} \leq 10 \| h \|_{L^\infty(M,g)}$, so $\{ \alpha_j \}$ is equi-integrable in $L^1$, so by the Dunford–Pettis theorem there exists a subsequence $\{ \alpha_{j_k} \}$ $L^1$-weakly convergent to $\alpha_*$. By lower semi-continuity of the $L^1$-norm with respect to the weak topology, we get

$$i_0 \leq I[\alpha_*] = \| \alpha_* \|_{L^1(M,g)} \leq \liminf_{k \rightarrow \infty} \| \alpha_{j_k} \|_{L^1(M,g)} = i_0.$$  

In addition, the weak $L^1$-convergence implies that there is a further subsequence $\{ \alpha_{j_{k_\ell}} \}$ converging a.e. to $\alpha_*$. So $\| \alpha_* \|_{L^\infty(M,g)} \leq 10 \| h \|_{L^\infty(M,g)}$. Also, the weak convergence $\alpha_{j_k} \rightharpoonup \alpha_*$
in $L^1$ implies convergence as 1-currents (i.e., in the sense of distributions), so $\alpha_*$ is cohomologous to $h$ in the little regularity setting. Therefore, $\alpha_* \in A_h$. We can now conclude that $i_0 := \inf_{\alpha \in A_h} I[\alpha] = I[\alpha_*]$.

Finally, if the harmonic 1-form $h$ is everywhere non-vanishing, we may repeat the above arguments with the admissible class $A_h$ (defined in (B.5)) replaced by

$$A'_h := \left\{ \beta \in [[h]] : \frac{1}{10} \operatorname{ess} \inf_M |h|_g \leq |\beta|_g \leq 10\|h\|_{L^\infty(M,g)} \text{ a.e.} \right\}. \quad (B.7)$$

All the arguments go through; in particular, by the a.e. convergence of a subsequence, the limiting form $\alpha_*$ is in $A'_h$, hence $\alpha_* \neq 0$ a.e.

Thus, one may choose a nowhere vanishing representative of $\alpha_*$ to conclude the proof. □

Acknowledgement. This work has been done during SL’s stay as a CRM–ISM postdoctoral fellow at the Centre de Recherches Mathématiques, Université de Montréal, McGill University, and Concordia University. SL would like to thank these institutions for their hospitality. SL also thanks the Shanghai Frontier Research Institute for Modern Analysis for its support during the finalisation of the paper.

MS is indebted to Vincent Borrelli and Amit Acharya for many helpful discussions. MS was supported in part by Simons Collaborative Research Grant 232531.

References

[1] A. Acharya, G.-Q. Chen, S. Li, M. Slemrod and D. Wang, Fluids, elasticity, geometry, and the existence of wrinkled solutions, Arch. Ration. Mech. Anal., 226 (2017), 1009–1060.

[2] S. C. Anco, A. Dar, N. Tufail, Conserved integrals for inviscid compressible fluid flow in Riemannian manifolds, Proc. A., 471 (2015), no. 2182, 20150223, 24 pp.

[3] V. I. Arnold, B. A. Khesin, Topological methods in hydrodynamics, Applied Mathematical Sciences, 125. Springer-Verlag, New York, 1998. xvi+374 pp.

[4] E. Bartzos, V. Borrelli, R. Denis, F. Lazarus, D. Rohmer and B. Thibert, An Explicit Isometric Reduction of the Unit Sphere into an Arbitrarily Small Ball, Found. Comput. Math., 18 (2018), 1015–1042.

[5] L. Bers, Results and conjectures in the mathematical theory of subsonic and transonic gas flows, Comm. Pure Appl. Math. 7 (1954), 79–104.

[6] S. Bhattacharyya, V.E. Hubeny, S. Minwalla, and M. Rangamani, Nonlinear fluid dynamics from gravity, J. High Energy Phys. 02 (2008), Article Number 045.

[7] J. F. Borisov, $C^{1,\alpha}$-isometric immersions of Riemannian spaces, Doklady 163, 869–871, 1965.

[8] V. Borrelli, S. Jabrane, F. Lazarus and B. Thibert, Isometric embeddings of the square flat torus in ambient space, Ensaios Matemáticos [Mathematical Surveys], 21. Sociedade Brasileira de Matemática, Rio de Janeiro, 2013. ii+91 pp.

[9] V. Borrelli, S. Jabrane, F. Lazarus, and B. Thibert, Flat tori in three-dimensional space and convex integration, Proc. Natl. Acad. Sci. USA, 109 (2012), 7218–7223.

[10] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger, From Navier–Stokes to Einstein, J. High Energy Phys. 7 (2012), Article Number 146.

[11] I. Bredberg, A. Strominger, Black holes as incompressible fluids on the sphere, J. High Energy Phys. 5 (2012), Article Number 043.

[12] C. Breiner, N. Kapouleas, and S. Kleene, Conservation laws and gluing constructions for constant mean curvature (hyper)surfaces, To appear in Notices Amer. Math. Soc. (2022).

[13] T. Buckmaster, C. De Lellis, P. Isett, and L. Székelyhidi, Jr., Anomalous dissipation for 1/5-Hölder Euler flows, Ann. of Math., 182 (2015), 127–172.
[14] T. Buckmaster, C. De Lellis, L. Székelyhidi Jr., Dissipative Euler flows with Onsager-critical spatial regularity, *Comm. Pure Appl. Math.*, 69 (2016), 1613–1670.

[15] R.-G. Cai, L. Li, and Y.-L. Zhang, Non-relativistic fluid dual to asymptotically AdS gravity at finite cutoff surface, *Journal of High Energy Physics* 07 (2011), Article number 027.

[16] R.-G. Cai, T.-J. Li, Y.-H. Qi, and Y.-L. Zhang, Incompressible Navier–Stokes equations from Einstein gravity with Chern–Simons term, *Phys. Rev. D* 86 (2012), Article Number 086008.

[17] S. Carroll, *Spacetime and geometry: An introduction to general relativity*. Addison Wesley, San Francisco, CA, 2004. xiv+513 pp.

[18] G.-Q. Chen, C. M. Dafermos, M. Slemrod, and D. Wang, Two-dimensional sonic-subsonic flow, *Commun. Math. Phys.* 271 (2007), 635–647.

[19] G.-Q. Chen, S. Li, Global weak rigidity of the Gauss–Codazzi–Ricci equations and isometric immersions of Riemannian manifolds with lower regularity, *J. Geom. Anal.*, 28 (2018), 1957–2007.

[20] G.-Q. Chen, M. Slemrod, D. Wang, Isometric embeddings and compensated compactness, *Commun. Math. Phys.* 294 (2010), 411–437.

[21] G.-Q. Chen, M. Slemrod, D. Wang, Weak continuity of the Gauss–Codazzi–Ricci system for isometric embedding, *Proc. Amer. Math. Soc.*, 138 (2010), 1843–1852.

[22] G.-Q. Chen, M. Slemrod, D. Wang, Isometric immersions and compensated compactness, *Arch. Ration. Mech. Anal.*, 241 2 (2021), 579–641.

[23] P. G. Gratie, C. Mardare, and C. Mardare, A Cesàro–Volterra formula with little regularity, *J. Math. Pures Appl.*, 93 (2010), 41–60.

[24] G. Compère, P. McFadden, K. Skenderis, and M. Taylor, The holographic fluid dual to vacuum Einstein gravity, *Journal of High Energy Physics* 07 (2011), Article number 50.

[25] C. De Lellis, D. Inauen, L. Székelyhidi, Jr., H-principle and rigidity for $C^{1,\alpha}$ isometric embeddings, in: Nonlinear partial differential equations, pp.83–116, *Abel Symp.*, 7, Springer, Heidelberg, 2012.

[26] R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*. Reprinting of the 1948 original. Applied Mathematical Sciences, Vol. 21. Springer-Verlag, New York-Heidelberg, 1976. xv+464 pp.

[27] C. De Lellis, L. Székelyhidi, Jr., High dimensionality and h-principle in PDE, *Bull. Amer. Math. Soc.*, 54 (2017), 173–245.

[28] L. C. Evans, The 1-Laplacian, the ∞-Laplacian and differential games, *Perspectives in nonlinear partial differential equations*, pp.245–254, *Contemp. Math.*, 446, Amer. Math. Soc., Providence, RI, 2007.

[29] P. Isett, *Hölder continuous Euler flows in three dimensions with compact support in time*, Annals of Mathematics Studies, 196. Princeton University Press, Princeton, NJ, 2017. x+201 pp.

[30] M. Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Fundamental Principles of Mathematical Sciences], 325. Springer-Verlag, Berlin, 2016. x+363 pp.

[31] M. Gromov, Geometric, algebraic, and analytic descendants of Nash isometric embedding theorems, *Bull. Amer. Math. Soc.*, 54 (2017), 173–245.

[32] N. Kuiper, On $C^{1,\alpha}$-isometric imbeddings, I, II, *Nederl. Akad. Wetensch. Proc. Ser. A*, 58 = *Indag. Math.*, 17 (1955), 545–556; 683–689.

[33] N. Kuiper, Isoemetric and short imbeddings, *Nederl. Akad. Wetensch. Proc. Ser. A*, 62 = *Indag. Math.*, 21 (1959), 11–25.

[34] S. Mardare, The fundamental theorem of surface theory for surfaces with little regularity, *J. Elasticity*, 73 (2003), 251–290.
[40] S. Mardare, On Pfaff systems with $L^p$ coefficients and their applications in differential geometry, *J. Math. Pures Appl.*, **84** (2005), 1659–1692.

[41] S. Mardare, On systems of first order linear partial differential equations with $L^p$ coefficients, *Adv. Diff. Eqns.*, **12** (2007), 301–360.

[42] F. Murat, Compacité par compensation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **5** (1978), 489–507.

[43] F. Murat, Compacité par compensation. II, in: *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis* (Rome, 1978), pp. 245–256, Pitagora, Bologna (1979).

[44] J. Nash, $C^1$ isometric imbeddings, *Ann. of Math.* **60** (1954), 383–396.

[45] P. Petersen, *Riemannian geometry. Second edition*. Graduate Texts in Mathematics, 171. Springer, New York, 2006. xvi+401 pp.

[46] K. Shiohama and R. Takagi, A characterization of a standard torus in $\mathbb{E}^3$, *J. Differential Geometry* **4** (1970), 477–485.

[47] L. Tartar, Compensated compactness and applications to partial differential equations, in: *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium*, vol. 4, pp. 136–212, *Res. Notes in Math.*, vol. 39. Pitman, Boston (1979).

[48] L. Tartar, The compensated compactness method applied to systems of conservation laws, in: *Systems of Nonlinear Partial Differential Equations* (Oxford, 1982), *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, vol. 111, pp. 263–285. Reidel, Dordrecht (1983).

[49] M. Theillière, Convex integration theory without integration, *Math. Z.* **300** (2022), 2737–2770.

[50] H. C. Wente, Counterexample to a conjecture of H. Hopf, *Pacific J. Math.* **121** (1986), 193–243.

[51] P. S. Wesson, *Five-dimensional physics: classical and quantum consequences of Kaluza–Klein cosmology*. World Scientific, 2006.