A MONOTONE DISCRETIZATION FOR INTEGRAL FRACTIONAL LAPLACIAN ON BOUNDED LIPSCHITZ DOMAINS: POINTWISE ERROR ESTIMATES UNDER HÖLDER REGULARITY ∗

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Abstract. We propose a monotone discretization for the integral fractional Laplace equation on bounded Lipschitz domains with the homogeneous Dirichlet boundary condition. The method is inspired by a quadrature-based finite difference method of Huang and Oberman, but is defined on unstructured grids in arbitrary dimensions with a more flexible domain for approximating singular integral. The scale of the singular integral domain not only depends on the local grid size, but also on the distance to the boundary, since the Hölder coefficient of the solution deteriorates as it approaches the boundary. By using a discrete barrier function that also reflects the distance to the boundary, we show optimal pointwise convergence rates in terms of the Hölder regularity of the data on both quasi-uniform and graded grids. Several numerical examples are provided to illustrate the sharpness of the theoretical results.

Key words. Monotone discretization, bounded Lipschitz domains, unstructured grids, pointwise error estimate, Hölder regularity

AMS subject classifications. 35R11, 65N06, 65N12, 65N15

1. Introduction. It is known that the fractional Laplacian can be defined in many ways [22, 23], which are not equivalent on the bounded Lipschitz domain Ω ⊂ R^n. In this work, we focus on the integral fractional Laplace equation of form [15]

\begin{equation}
\begin{aligned}
\mathcal{L}u := (-\Delta)^s u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega^c := \mathbb{R}^n \setminus \Omega,
\end{aligned}
\end{equation}

(1.1)

where \((-\Delta)^s\) is the integral fractional Laplacian of order \(s \in (0, 1)\), defined by

\begin{equation}
(-\Delta)^s u(x) := C_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy.
\end{equation}

(1.2)

The normalization constant is given by \(C_{n,s} := \frac{2^{2s}\Gamma(\frac{n}{2}+s)}{\pi^{n/2}\Gamma(1-s)}\). In probability, the fractional Laplacian is the infinitesimal generator of a symmetric 2s-stable Lévy process [3]. Meanwhile, the fractional Laplacian has been used in place of the integer-order Laplacian in many applications, including financial asset prices [31], fractional conservation law [16], geophysical fluid dynamics [13].

Besides the non-locality and singularity in the kernel, another main feature of integral fractional Laplacian is that the solutions to (1.1) exhibit an algebraic boundary singularity regardless of the domain regularity. As a well-known fact, even if domain is smooth, the unique solution to (1.1) has only the optimal \(C^s\)-Hölder regularity in \(\Omega\) provided that \(f \in L^\infty(\Omega)\), and develops a singularity of the form \(\text{dist}(x, \partial\Omega)^s\) near \(\partial\Omega\). These properties are not limited to (1.1), but widely appear in a class of nonlocal elliptic equations, see a survey in [28]. Moreover, when the data has better regularity, the higher-order interior Hölder regularity of \(u\) requires the use of weighted Hölder norms, in which the distance to the boundary is also involved [29].

∗The work of Shuonan Wu is supported in part by the National Natural Science Foundation of China grant No. 11901016 and the startup grant from Peking University.

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Numerical studies of (1.1) and related fractional-order problems have experienced some rapid developments in recent years. Based on the variational formula, a finite element (FE) discretization using piecewise linear continuous function space was developed and analyzed in [1]. Significantly, the error estimates in energy norm on quasi-uniform and graded grids hinge on the standard and weighted Sobolev regularity, respectively. Other works related to the conforming FE discretizations include error estimate in $H^1$ norm in the case $s > \frac{1}{2}$ [8], local energy estimate [10], multilevel solver [11], and extensions to non-homogeneous Dirichlet problem [2] and eigenvalue problem [9]. A nonconforming FE discretization, based on the Dunford-Taylor representation, was proposed and analyzed in [7]. Discretizations of the spectral fractional Laplacian can be found in [26, 14]. We refer to [6, 23] for the survey of existing numerical methods for fractional Laplacian.

The maximum principle, even in the case of weak solutions to (1.1), is valid [28]. In view of numerical stability, it is desirable that the resulting discrete system also satisfies a similar maximum principle at discrete level — known as the monotonicity. As in the case of the Laplacian $-\Delta$, the monotonicity of FE discretizations essentially relies on some grid conditions [32], for instance, the Delaunay triangulation in 2D. Such grid conditions, however, are extremely difficult to obtain and verify for the FE discretizations of integral fractional Laplacian mainly due to its nonlocal nature.

On the other hand, the finite difference (FD) methods have natural advantages in constructing monotone schemes [24, 21]. For instance, a criterion for easy verification of monotonicity was proposed in [27] in the FD setting. In [20], Huang and Oberman first proposed a quadrature-based FD method for solving 1D integral fractional Laplacian. The monotonicity was proved when using piecewise linear function space, which yields convergence. In this case, the accuracy of the scheme in $L^\infty$ norm was shown to be $O(h^{2-2s})$ under the assumption that the solution of (1.1) belongs to $C^4$, which cannot be guaranteed in general. In fact, a simple numerical example in Table 1 shows that the convergence rate in $L^\infty$ behaves like $O(h^{\min\{s, 2-2s\}})$, where $h$ represents the uniform grid size. The bottleneck of $O(h^s)$ convergence rate at most also exists in other FD methods for (1.1) [17, 18].

| Value of $s$ | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 | 0.67 | 0.70 | 0.80 | 0.90 |
|--------------|------|------|------|------|------|------|------|------|------|------|
| Order (in $h$) | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 | 0.67 | 0.64 | 0.42 | 0.21 |

Table 1: $\Omega = (-1, 1)$, $f = 1$. The pointwise convergence order of Huang-Oberman scheme [20] is $\approx \min\{s, 2-2s\}$.

The primal goal of this paper is to develop a monotone scheme with optimal convergence rates for (1.1) on bounded Lipschitz domains. To this end, we first triangulate the domain $\Omega$ by a shape regular grid, and construct the associated piecewise linear continuous function space where the approximate solution lies. Inspired by [20], the integral fractional Laplacian (1.2) is divided into the singular part and the tail part for every interior grid node $x_i$. Under some symmetry conditions for the singular integral domain $\Omega_i$, the singular integral can be approximated by a local differential operator to which the standard monotone FD discretization can be applied. This, together with a monotone discretization of the tail part and a discrete barrier function, leads to the discrete comparison principle.

To achieve the optimal convergence rates, several key ideas are introduced and utilized in this paper. First, we show the relationship between high-order Hölder constant of solution and distance to boundary, which indicates that the scale of $\Omega_i$,...
denoted by \( H_i \), should also depend on \( \text{dist}(x_i, \partial \Omega) \). Second, upon the ratio between \( \text{dist}(x_i, \partial \Omega) \) and local grid size, we introduce the \( \delta \)-interior nodes in (4.1) where the consistency can be proved. This, together with a discrete barrier function and the discrete comparison principle, leads to the optimal error estimates. The analysis only requires the local quasi-uniformity hence can be applied to both quasi-uniform and graded grids. Moreover, the analysis only requires the minimal Hölder regularity of the data, and interestingly, the Hölder regularity indeed influences the optimal scale of \( \Omega_i \), see Theorem 6.1 and Theorem 6.3.

It is interesting to note that, there is a strong connection between our work and the monotone two-scale methods for solving Monge-Ampère equations [25]. We note that the two-scale methods are also important for problems in which preserving monotonicity and comparison is of relevance [24, 21]. In our work, the scale of \( \Omega_i \) can be viewed as the second scale in addition to grid size, which depends on the distance to boundary due to the PDE theory of (1.1). Moreover, it is known that monotonicity is one of the essential ingredients in devising convergent schemes for fully nonlinear PDEs [5]. Our work, in this sense, is expected to have broad prospects in solving nonlinear problems involving integral fractional Laplacian.

The rest of the paper is organized as follows. In Section 2, we review the regularity results for the integral fractional Laplace equation (1.1). In Section 3, we introduce a monotone discretization and prove the discrete comparison principle. The consistency error is divided into singular and tail parts, which are estimated in Section 4 and Section 5 separately. In Section 6, combining the discrete comparison principle and the consistency error, we establish the pointwise error estimates for the numerical scheme. Finally, in Section 7, several numerical examples are exhibited to illustrate the theoretical results.

2. Preliminary results. In this section, we present some preliminary results in the analysis setting. For \( \beta > 0 \), we denote by \( |\cdot|_{C^{\beta}(U)} \) the \( C^{\beta}(U) \) seminorm. More precisely, we will write \( \beta = k + \beta' \) with \( k \) integer and \( \beta' \in (0, 1] \), then

\[
|w|_{C^{\beta}(U)} = |w|_{C^{k, \beta'}(U)} := \sup_{x,y \in U, x \neq y} \frac{|D^k w(x) - D^k w(y)|}{|x-y|^\beta},
\]

\[
|w|_{C^{\beta}(U)} := \sum_{\ell=0}^{k} \left( \sup_{x \in U} |D^\ell w(x)| \right) + |w|_{C^{\beta}(U)}.
\]

Next, we summarize the Hölder regularity results for (1.1) given in [29]. To begin with, we state the definition of Lipschitz domain (cf. [19, Definition 1.2.1.1]).

DEFINITION 2.1 (Lipschitz domain). Let \( \Omega \) be an open subset of \( \mathbb{R}^n \). We say that \( \Omega \) is a Lipschitz domain if for every \( x \in \partial \Omega \) there exists a neighborhood \( V \) of \( x \) in \( \mathbb{R}^n \) and new orthogonal coordinates \( \{z_1, \cdots, z_n\} \) such that

(a) \( V \) is a hypercube in the new coordinates:

\[
V = \{(z_1, \cdots, z_n) : -a_j < z_j < a_j, \; 1 \leq j \leq n\};
\]

(b) there exists a Lipschitz function \( \varphi \), defined in

\[
V' := \{(z_1, \cdots, z_{n-1}) : -a_j < z_j < a_j, \; 1 \leq j \leq n-1\}
\]

and such that

\[
|\varphi(z')| \leq a_n/2 \text{ for every } z' = (z_1, \cdots, z_{n-1}) \in V',
\]

\[
\Omega \cap V = \{z = (z', z_n) : z_n < \varphi(z')\},
\]

\[
\partial \Omega \cap V = \{z = (z', z_n) : z_n = \varphi(z')\}.
\]
For $\sigma > C$ where $m$ follows. For the sake of expository simplicity, we adopt $\beta > 0$ and $\sigma \geq -\beta$, we define the seminorm

\[w_{\beta;U}^{(\sigma)} := \sup_{x,y \in U} \left( (\delta(x,y))^{\beta + \sigma} \frac{|D^k w(x) - D^k w(y)|}{|x - y|^\beta} \right) \quad \forall w \in C^\beta(U) := C^{k,\beta'}(U).\]

For $\sigma > -1$, the associated norm $\| \cdot \|_{\beta;U}^{(\sigma)}$ is defined as follows:

1. For $\sigma \geq 0$,

\[
(2.2a) \quad \|w\|_{\beta;U}^{(\sigma)} := \frac{1}{k} \sum_{\ell=0}^k \sup_{x \in U} \left( (\delta(x))^{\ell+\sigma} \frac{|D^\ell w(x)|}{|x|^\beta} \right) + \|w|_{\beta;U}^{(\sigma)};
\]

2. For $0 > \sigma > -1$,

\[
(2.2b) \quad \|w\|_{\beta;U}^{(\sigma)} := \|w\|_{C^{-\sigma}(U)} + \sum_{\ell=1}^k \sup_{x \in U} \left( (\delta(x))^{\ell+\sigma} \frac{|D^\ell w(x)|}{|x|^\beta} \right) + \|w|_{\beta;U}^{(\sigma)}.
\]

The following result is the starting point of this work.

**Proposition 2.3** (Proposition 1.4 in [29]). **Let $\Omega$ be a bounded domain, and $\beta > 0$ be such that neither $\beta$ nor $\beta + 2s$ is an integer. Let $f \in C^\beta(\Omega)$ be such that $\|f\|_{\beta;\Omega}^{(s)} < \infty$, and $u \in C^s(\mathbb{R}^n)$ be a solution of (1.1). Then, $u \in C^{\beta + 2s}(\Omega)$ and

\[
\|u\|_{\beta + 2s;\Omega}^{(s)} \leq C(\|u\|_{C^s(\mathbb{R}^n)} + \|f\|_{\beta;\Omega}^{(s)}),
\]

where $C$ is a constant depending only on $\Omega$, $s$ and $\beta$.

**Remark 2.4** (blow-up behavior). We further note that the blow-up behavior of the constant $C$ in (2.3) is $|\beta - m|^{-1}$ or $|\beta + 2s - m|^{-1}$ as $\beta$ or $\beta + 2s$ approaching to some integer $m$, which can be found in the proof of [30, Proposition 2.5 & Proposition 2.7].

As a corollary, the dependence of the $\delta(x)$ on the Hölder norm of $u$ is given as follows. For the sake of expository simplicity, we adopt $\beta$ as the Hölder index of $u$. We also assume $\Omega$ to be a bounded Lipschitz domain with exterior ball condition in what follows.

**Corollary 2.5** ($\delta$-dependence in Hölder norm). **Let $\Omega$ be a bounded Lipschitz domain with exterior ball condition, and $\beta > 2s$ be such that neither $\beta - 2s$ nor $\beta$ is an integer. Let $f \in L^\infty(\Omega)$ be such that $\|f\|_{\beta - 2s;\Omega}^{(s)} < \infty$. Then, $u \in C^\beta(\Omega)$ and

\[
\|u\|_{C^\beta(\{x \in \Omega: \delta(x) \geq \rho\})} \leq C \rho^{\beta - \beta} \quad \forall \rho \in (0, \text{diam}(\Omega)),
\]

where $C$ is a constant depending only on $\Omega$, $s$, $\beta$, $\|f\|_{L^\infty(\Omega)}$ and $\|f\|_{\beta - 2s;\Omega}^{(s)}$.\]
Proof. We denote $U_{\rho} := \{ x \in \Omega : \delta(x) \geq \rho \}$. Using Proposition 2.3 and the definition of weighted seminorm (2.1), we have

\[
|u|_{C^s(U_{\rho})} = \rho^{s-\beta} \sup_{x,y \in U_{\rho}} \rho^{\beta-s} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^s} \\
\leq \rho^{s-\beta} \sup_{x,y \in U_{\rho}} \delta(x,y)^{\beta-s} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^s} \\
\leq \rho^{s-\beta}|u|_{C^s(U_{\rho})} \leq C(\Omega, s, \beta) \rho^{s-\beta} \left( \|u\|_{C^s(\mathbb{R}^n)} + \|f\|_{B_{-2s,\Omega}} \right) \\
\leq C(\Omega, s, \beta, \|f\|_{L^\infty(\Omega)}, \|f\|_{B_{-2s,\Omega}}) \rho^{s-\beta}.
\]

Here, we use Proposition 2.2 in the last step. Similarly, for any $1 \leq \ell \leq k < \beta$,

\[
\sup_{x \in U_{\rho}} |D^\ell u(x)| \leq \rho^{s-\ell} \|u\|_{B_{-s,\Omega}} \leq C(\Omega, s, \beta, \|f\|_{L^\infty(\Omega)}, \|f\|_{B_{-2s,\Omega}}) \rho^{s-\beta},
\]

where we use $\rho^{s-\ell} \leq \text{diam}(\Omega)^{\beta-\ell}$ in the last step. This completes the proof. \qed

At this point, we recall a useful estimate corresponding to the integrability of kernel outside the domain (cf. [19, Eq. (1,3,2,12)]): There exist two constants $0 < C_1 \leq C_2$ such that

\[
(2.5) \quad C_1(\Omega, s) \delta(x)^{-2s} \leq \int_{\Omega^c} \frac{1}{|x-y|^{n+2s}} \, dy \leq C_2(\Omega, s) \delta(x)^{-2s}.
\]

In the last of this section, we present an elementary result of the center difference in Hölder seminorm along the direction $\theta \in S^{n-1}$, where $S^{n-1}$ denotes the unit $(n-1)$-sphere.

**Lemma 2.6** (FD in Hölder seminorm). Let $\rho > 0$ and $u \in C^\beta(B_\rho(x))$.

1. If $\beta \leq 2$, then there exists $C > 0$ for any $\theta \in S^{n-1}$,

\[
|2u(x) - u(x + \rho\theta) - u(x - \rho\theta)| \leq C\rho^\beta |u|_{C^\beta(B_\rho(x))}.
\]

2. If $2 < \beta \leq 4$, then there exists $C > 0$ for any $\theta \in S^{n-1}$,

\[
\left| 2u(x) - u(x + \rho\theta) - u(x - \rho\theta) + \rho^2 \frac{\partial^2 u}{\partial \theta^2}(x) \right| \leq C\rho^\beta |u|_{C^\beta(B_\rho(x))}.
\]

Proof. Recall that $\beta = k + \beta'$ for integer $k \leq 3$ and $\beta' \in (0, 1]$, we have by Taylor’s expansion with Lagrange remainder term that, for some $\xi \in B_\rho(x)$,

\[
u(x + \rho\theta) = u(x) + \sum_{j=1}^{k-1} \frac{1}{j!} \frac{\partial^j u}{\partial \theta^j}(x) \rho^j + \frac{1}{k!} \frac{\partial^k u}{\partial \theta^k}(\xi) \rho^k
\]

\[
= u(x) + \sum_{j=1}^{k} \frac{1}{j!} \frac{\partial^j u}{\partial \theta^j}(x) \rho^j + \frac{1}{k!} \left( \frac{\partial^k u}{\partial \theta^k}(\xi) - \frac{\partial^k u}{\partial \theta^k}(x) \right) \rho^k.
\]

The last term can be controlled by using the definition of Hölder seminorm, which completes the proof. \qed
3. Monotone discretization. For ease of discretization and the sake of simplicity of the exposition, we shall henceforth consider \( \Omega \) to be a bounded open polytope. We assume \( \Omega \) admits an admissible triangulation \( \mathcal{T}_h \), i.e., \( \cup_{T \in \mathcal{T}_h} T = \overline{\Omega} \). Let \( \mathcal{N}_h \) denote the node set of \( \mathcal{T}_h \), \( \mathcal{N}_h^0 := \{ x_i \in \mathcal{N}_h : x_i \in \partial \Omega \} \) be the collection of boundary nodes, and \( \mathcal{N}_h^0 := \mathcal{N}_h \setminus \mathcal{N}_h^0 \). We require that the family of triangulation under consideration satisfies shape regularity and local quasi-uniformity. Note that the latter can be deduced by the former when \( n \geq 2 \) [12]. More precisely, let \( h_T \) and \( \rho_T \) respectively be the diameter of \( T \) and diameter of the largest ball contained in \( T \), the conditions of triangulation read

\[
\begin{align*}
\exists \lambda_1 &> 0 \ s.t. \ h_T \leq \lambda_1 \rho_T \ \forall T \in \mathcal{T}_h, \\
\exists \lambda_2 &> 0 \ s.t. \ h_T \leq \lambda_2 h_T' \ \forall T, T' \in \mathcal{T}_h \text{ with } T \cap T' \neq \emptyset.
\end{align*}
\]

For each \( x_i \in \mathcal{N}_h^0 \), we define \( \omega_i = \cup_{T \ni x_i} T \), and denote by \( h_i \) the radius of inscribed sphere centered at \( x_i \) for \( \omega_i \). Clearly, for any \( T \ni x_i \),

\[
2 \lambda_1 \lambda_2 h_i \geq \lambda_1 \lambda_2 \rho_T \geq \lambda_2 h_T \geq \max_{T \ni x_i} h_T'.
\]

Let \( V_h \) be the space of continuous piecewise linear functions over \( \mathcal{T}_h \) that vanish in \( \Omega^c \), that is,

\[
V_h := \{ v : v|_T \in P_1(T) \ \forall T \in \mathcal{T}_h, \ v|_{\Omega^c} = 0 \}.
\]

Following [20], we divide the integral of fractional Laplacian into two parts: The singular part and the tail part.

3.1. Discretization of singular integral. For each \( x_i \in \mathcal{N}_h^0 \), we take a proper scale \( H_i \) such that \( B_{H_i}(x_i) \subset \Omega \). The standard centered difference of the \( \Delta u(x_i) \) with spacing \( H_i \) is denoted by

\[
\Delta_{FD} u(x_i; H_i) := \frac{\sum_{j=1}^{n} u(x_i + H_i e_j) - 2u(x_i) + u(x_i - H_i e_j)}{H_i^2},
\]

where \( e_j \) is the unit vector of the \( j \)-th coordinate.

For the discretization of singular integral, we consider a star-shaped domain \( \Omega_i \) centered at \( x_i \), i.e., by using the multi-dimensional polar coordinate

\[
\Omega_i := \{ x_i + \rho \theta : \theta \in S^{n-1}, \rho \in [0, \rho_i(\theta)), \rho_i(\theta) > 0 \}.
\]

The domain \( \Omega_i \) is assumed to satisfy the following conditions:

1. Interior of \( \Omega_i : \Omega_i \subset \Omega, \forall x_i \in \mathcal{N}_h^0 \).
2. Symmetry:

\[
\begin{align*}
\forall x_i \in \mathcal{N}_h^0, \forall \sigma \in S_n, \\
& x_i + \pm \rho_i(\theta) \rho_i(\theta) > 0 \text{ for any permutation } \sigma.
\end{align*}
\]

3. Quasi-uniformity: there exist positive constants \( c_\rho, \tilde{c}_\rho \), such that

\[
\rho_i(\theta) \geq c_\rho \rho_i(\theta) \geq \tilde{c}_\rho H_i \ \forall x_i \in \mathcal{N}_h^0, \theta \in S^{n-1}.
\]

\[
\rho_i(\theta) \geq c_\rho \rho_i(\theta) \geq \tilde{c}_\rho H_i \ \forall x_i \in \mathcal{N}_h^0, \theta \in S^{n-1}.
\]
Using the symmetry of \(\Omega_i\) and polar coordinate transformation \(y \mapsto x_i + z = x_i + \rho \theta\) which satisfies \(dz = \rho^{n-1} d\rho d\theta\), we have

\[
\int_{\Omega_i} \frac{v(x_i) - v(y)}{|x_i - y|^{n+2s}} \, dy = \frac{1}{2} \int_{\Omega_i-x_i} \frac{2v(x_i) - v(x_i + z) - v(x_i - z)}{|z|^{n+2s}} \, dz \\
= \frac{1}{2} \int_{\Omega-x_i} \int_0^{\rho_i(\theta)} \frac{2v(x_i) - v(x_i + \rho \theta) - v(x_i - \rho \theta)}{\rho^{1+2s}} \, d\rho \, d\theta.
\]

(3.7)

Suppose the function \(v \in C^2(\Omega_i)\), the above integral can be approximated by the following second-order differential operator

\[
\frac{1}{2} \int_{\Omega-x_i} \int_0^{\rho_i(\theta)} \frac{2v(x_i) - v(x_i + \rho \theta) - v(x_i - \rho \theta)}{\rho^{1+2s}} \, d\rho \, d\theta \\
\approx -\frac{1}{2} D^2 u(x_i) : \int_{\Omega-x_i} \int_0^{\rho_i(\theta)} \rho^{1-2s} \, d\rho \, \theta \otimes \theta \, d\theta \\
= -\frac{H_i^{2-2s}}{4(1-s)} D^2 u(x_i) : \int_{\Omega-x_i} (\rho_i(\theta) \theta)_{2-2s} \theta \otimes \theta \, d\theta.
\]

Thanks to the symmetry of \(\Omega_i\), the last integral can be simplified by the lemma below.

**Lemma 3.1** (symmetric integral). *Under the conditions* (3.5) – (3.6),

\[
\frac{1}{4(1-s)} \int_{\Omega-x_i} \left( \frac{\rho_i(\theta)}{H_i} \right)^{2-2s} \theta \otimes \theta \, d\theta = \frac{1}{4n(1-s)} \int_{\Omega-x_i} \left( \frac{\rho_i(\theta)}{H_i} \right)^{2-2s} \, d\theta \quad I_n,
\]

where \(I_n\) is the identity matrix. Moreover, \(\kappa_{n,s,i}\) is a uniformly bounded constant.

**Proof.** It is readily seen that \(\rho_i(\theta(\sigma(1)), \ldots, \theta(\sigma(n)))\) are the same. For any fixed permutation \(\sigma\), let \(\theta^\sigma_{j_1, \ldots, j_n}\) := \((-1)^{j_1} \theta(\sigma(1)), \ldots, (-1)^{j_n} \theta(\sigma(n))\), then

\[
\sum_{j_1, \ldots, j_n=0}^{1} \theta^\sigma_{j_1, \ldots, j_n} \otimes \theta^\sigma_{j_1, \ldots, j_n} = 2^n \text{diag}(\theta^{2\sigma(1)}, \ldots, \theta^{2\sigma(n)}),
\]

which together with \(\theta^\sigma_{j_1, \ldots, j_n} \in S^{n-1}\) yields

\[
\sum_{\sigma} \sum_{j_1, \ldots, j_n=0}^{1} \theta^\sigma_{j_1, \ldots, j_n} \otimes \theta^\sigma_{j_1, \ldots, j_n} = \frac{2^n n!}{n} I_n.
\]

Hence, the contribution of \(\theta \otimes \theta\) in the integral is identically to \(\frac{1}{n} I_n\), which leads to (3.8). The quasi-uniformity (3.6) implies that \(\kappa_{n,s,i} = O(1)\).

Combining (3.7) – (3.8), the discretization of the singular integral is defined by

\[
L^S_{n}[u](x_i) := -\kappa_{n,s,i} \frac{\Delta_{FD} u(x_i; H_i)}{H_i^{2s}} \\
= -\kappa_{n,s,i} \sum_{j=1}^{n} \frac{u(x_i + H_i e_j) - 2u(x_i) + u(x_i - H_i e_j)}{H_i^{2s}}.
\]

(3.9)
Then, we have a similar definition was first proposed by Huang and Oberman \cite{HuangOberman2009} for consistency, we adopt $x_i$ on local grid size and the distance of $x_i$ to the boundary. Therefore, for the sake of consistency, we adopt $L_h^S$ to indicate the essential dependence on $h$.

Remark 3.3 (examples of $\Omega_i$). The simplest example that satisfies the condition \eqref{eq:domain} is $\Omega_i = B_{H}(x_i)$, which gives $\kappa_{n,s,i} = \frac{[n^{s-1}]}{4n(1-s)} = \frac{\omega_n}{4(1-s)}$, where $\omega_n = \frac{2^{n/2}}{\pi(n/2)}$ is the volume of unit ball in $\mathbb{R}^n$. Another example is the $n$-dimensional cube centered at $x_i$ with scale $H_i/\sqrt{n}$, namely

$$\Omega_i = \{x_i + (z_1, \ldots, z_n) : |z_i| < \frac{H_i}{\sqrt{n}}\}.$$ Clearly, $\Omega_i \subset B_{H}(x_i) \subset \Omega$. A direct calculation shows that in 2D,

$$\kappa_{n,s,i} = \frac{1}{1-s} \int_0^{\pi/2} \left( \frac{1}{\sqrt{2} \cos \theta} \right)^{2-2s} d\theta = \begin{cases} \frac{\pi}{2} \left( \frac{1}{s+1} B_1(s+1, \frac{1}{2}) - B_1(s, \frac{1}{2}) \right) & s \neq 0.5, \\ \frac{\pi}{4} \tan^{-1}(\tan(\frac{x}{8})) & s = 0.5. \end{cases}$$

Here, $B_x(a,b)$ is the incomplete beta function.

### 3.2. Monotonicity

We seek $u_h \in V_h$ such that for $x_i \in \mathcal{N}_h$, \eqref{eq:discrete_laplacian}

$$L_h[u_h](x_i) := -\kappa_{n,s,i} \frac{\Delta_F D u_h(x_i; H_i)}{H_i^{2s}} + \int_{\Omega_i} \frac{u_h(x_i) - u_h(x)}{|x_i - y|^{n+2s}} dy = f(x_i).$$

A similar definition was first proposed by Huang and Oberman \cite{HuangOberman2009} on 1D uniform grids with $H_i = h$. Any function in $V_h$ has a pointwise definition and hence $L_h$ is well defined. The tail of integral $L_h^T[u_h]$ renders integral of piecewise linear functions outside $\Omega_i$.

We now show that \eqref{eq:discrete_laplacian} is monotone and prove the discrete comparison principle.

**Lemma 3.4 (monotonicity).** Let $v_h, w_h \in V_h$. If $v_h - w_h$ attains a non-negative maximum at an interior node $x_i \in \mathcal{N}_h$, then $L_h[v_h](x_i) \geq L_h[w_h](x_i)$.

**Proof.** If $v_h - w_h$ attains a non-negative maximum at $x_i \in \mathcal{N}_h$, then $v_h(x_i) \geq w_h(x_i)$, and

$$v_h(x_i) - v_h(y) \geq w_h(x_i) - w_h(y) \quad \forall y \in \Omega.$$ Then, we have

$$L_h^S[v_h](x_i) = -\kappa_{n,s,i} \frac{\Delta_F D v_h(x_i; H_i)}{H_i^{2s}} \geq -\kappa_{n,s,i} \frac{\Delta_F D w_h(x_i; H_i)}{H_i^{2s}} = L_h^S[w_h](x_i),$$

$$L_h^T[v_h](x_i) = \int_{\mathbb{R}^n \setminus \Omega_i} \frac{v_h(x_i) - v_h(y)}{|x_i - y|^{n+2s}} dy \geq \int_{\mathbb{R}^n \setminus \Omega_i} \frac{w_h(x_i) - w_h(y)}{|x_i - y|^{n+2s}} dy = L_h^T[w_h](x_i),$$

which implies $L_h[v_h](x_i) \geq L_h[w_h](x_i)$, as asserted. $\square$

**Lemma 3.5 (discrete barrier function).** Let $b_h \in V_h$ satisfy \eqref{eq:discrete_barrier}

$$b_h(x_i) := 1 \quad \forall x_i \in \mathcal{N}_h.$$
Denoting $\delta_i$ as the shorthand of $\delta(x_i)$, then we have

\begin{equation}
L_h[b_i](x_i) \geq C\delta_i^{-2s} \quad \forall x_i \in \mathcal{N}_h^0,
\end{equation}

where the constant $C$ depends only on $s$ and $\Omega$.

**Proof.** Since $L_h^s[b_i] \geq 0$, it suffices to prove $L_h^s[b_i](x_i) \geq C\delta_i^{-2s}$. By the definition of $b_i$,

\begin{equation*}
L_h^s[b_i](x_i) = \int_{\Omega^c \setminus \Omega} \frac{b_i(x_i) - b_i(y)}{|x_i - y|^{n+2s}} dy \geq \int_{\Omega^c} \frac{b_i(x_i) - b_i(y)}{|x_i - y|^{n+2s}} dy
\end{equation*}

\begin{equation*}
= \int_{\Omega^c} \frac{1}{|x_i - y|^{n+2s}} dy \geq C\delta_i^{-2s},
\end{equation*}

where (2.5) is used in the last step.  

**Lemma 3.6 (discrete comparison principle).** Let $v_h, w_h \in \mathbb{V}_h$ be such that

\begin{equation}
L_h[v_h](x_i) \geq L_h[w_h](x_i) \quad \forall x_i \in \mathcal{N}_h^0.
\end{equation}

Then, $v_h \geq w_h$ in $\Omega$.

**Proof.** Since $v_h, w_h \in \mathbb{V}_h$, it suffices to prove $v_h(x_i) \geq w_h(x_i)$ for all $x_i \in \mathcal{N}_h^0$.

The proof splits into two steps according to whether the inequality (3.14) is strict or not.

**Case 1: Strict inequality.** That is, $L_h[v_h](x_i) > L_h[w_h](x_i), \forall x_i \in \mathcal{N}_h^0$. We assume by contradiction that there exists an interior node $x_k \in \mathcal{N}_h^0$ such that $v_h(x_k) < w_h(x_k)$ and

\begin{equation*}
v_h(x_k) - w_h(x_k) \leq v_h(x_i) - w_h(x_i) \quad \forall x_i \in \mathcal{N}_h.
\end{equation*}

Reasoning as in Lemma 3.4 we obtain $L_h[v_h](x_k) \leq L_h[w_h](x_k)$, which contradicts to the strict inequality at $x_k$.

**Case 2: Non-strict inequality.** Let $b_h \in \mathbb{V}_h$ be the discrete barrier function defined in Lemma 3.5, which satisfies

\begin{equation*}
L_h[b_h](x_i) \geq C\delta_i^{-2s} \geq C_0,
\end{equation*}

where $C_0$ is a fixed positive constant. For arbitrary $\varepsilon > 0$, the function $v_h + \varepsilon b_h$ satisfies $v_h + \varepsilon b_h = w_h$ on $\partial \Omega$, and

\begin{equation*}
L_h[v_h + \varepsilon b_h](x_i) \geq L_h[v_h](x_i) + \varepsilon C_0 > L_h[w_h](x_i) \quad \forall x_i \in \mathcal{N}_h^0.
\end{equation*}

Applying Case 1 we deduce

\begin{equation*}
v_h + \varepsilon b_h \geq w_h \quad \forall \varepsilon > 0.
\end{equation*}

Taking the limit as $\varepsilon \to 0$ leads to the asserted inequality.  

In the following sections, we consider the error estimate of (3.11) with the choice

\begin{equation}
H_i = h^{\alpha}_i \min\{\delta_i^{1-\alpha}, \delta_0^{1-\alpha}\},
\end{equation}

where $\alpha \in [\frac{1}{2}, 1]$ is a parameter that will be determined later, $\delta_i$ is the shorthand of $\delta(x_i)$, and $\delta_0 > 0$ is a fixed constant. The definition of $h_i$ implies that $h_i \leq \delta_i$, or $H_i \leq h^{\alpha}_i \delta_i^{1-\alpha} \leq \delta_i$ and hence $B_{H_i}(x_i) \subset \Omega$.  

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We first divide the consistency error into three components

\[
\mathcal{L}_h[\mathcal{I}_h u](x_i) - \mathcal{L}[u](x_i) = \mathcal{L}_h^S[u](x_i) - \int_{\Omega_i} \frac{u(x_i) - u(y)}{|x_i - y|^{n+2s}} dy \\
\text{(3.16)}
\]

\[
+ \mathcal{L}_h^S[\mathcal{I}_h u](x_i) - \mathcal{L}_h^S[u](x_i) \\
+ \int_{\Omega_h} \frac{u(y) - \mathcal{I}_h u(y)}{|x_i - y|^{n+2s}} dy,
\]

where \( \mathcal{I}_h \) stands for the nodal interpolation. A simple observation is that the first component does not entail \( \mathcal{I}_h \), which is estimated without the underlying grids in Section 4. We then estimate the other two components respectively using the information of underlying grids in Section 5.

### 4. Consistency error I: Singular part

In this section, we quantify the operator consistency error of \( \mathcal{L}_h^S \) in conjunction with the Hölder regularity results.

#### 4.1. \( \delta \)-interior region

As shown in Corollary 2.5, the Hölder constant of \( u \) at \( x \) depends on \( \delta(x) \). It is therefore useful to define the node \( x_i \) such that all the points around \( x_i \) share a similar \( \delta(\cdot) \). Recall that \( \lambda \) in (3.2) and \( \bar{c}_S \) in (3.6), let us define a set of \( \delta \)-interior nodes

\[
\mathcal{N}_h^{0,\delta} := \left\{ x_i \in \mathcal{N}_h^0 : \frac{\delta_i}{h_i} \geq C_\delta \right\}, \quad C_\delta := \max \left\{ (2\max\{\bar{c}_S,1\})^{1/\alpha}, 2\lambda \right\}.
\]

We also denote

\[
\bar{\Omega}_i := \Omega_i \cup B_{H_i}(x_i) \subset \Omega.
\]

We have the following lemma.

**Lemma 4.1** (uniform \( \delta \) in \( \bar{\Omega}_i \)). For any \( x_i \in \mathcal{N}_h^{0,\delta} \), it holds

\[
\frac{1}{2} \delta_i \leq \delta(x) \leq \frac{3}{2} \delta_i \quad \forall x \in \bar{\Omega}_i \cup \omega_i.
\]

**Proof.** From the definition of \( \mathcal{N}_h^{0,\delta} \) and shape regularity (3.2), we have

\[
\delta_i = \delta_0 \delta_i^1 - \alpha \geq 2\max\{\bar{c}_S,1\} h_0 \delta_i^{1-\alpha} \geq 2\max\{\bar{c}_S,1\} H_i, \\
\delta_i \geq 2\lambda h_i \geq 2\max_{T \ni x_i} h_T.
\]

Using (3.6) (quasi-uniformity condition of \( \Omega_i \)), we have

\[
|\delta(x) - \delta_i| \leq \max\{\bar{c}_S,1\} H_i \leq \frac{1}{2} \delta_i \quad \forall x \in \Omega_i \cup B_{H_i}(x_i), \\
|\delta(x) - \delta_i| \leq \max_{T \ni x_i} h_T \leq \frac{1}{2} \delta_i \quad \forall x \in \omega_i,
\]

which completes the proof.

We define the \( \delta \)-interior region and \( \delta \)-interior triangulation as

\[
\Omega_h^{0,\delta} := \bigcup_{x_i \in \mathcal{N}_h^{0,\delta}} \omega_i, \quad \mathcal{T}_h^{0,\delta} := \{ T \in \mathcal{T}_h : \exists x_i \in \mathcal{N}_h^{0,\delta} \text{ such that } x_i \in T \}.
\]

(4.2)

As a direct consequence of Lemma 4.1, for any \( T \in \mathcal{T}_h^{0,\delta} \), it holds that \( \frac{1}{2} \delta(x') \leq \delta(x) \leq 3\delta(x') \) for any \( x, x' \in T \).
4.2. Consistency of $L_h^S$. Thanks to Corollary 2.5 ($\delta$-dependence in Hölder norm), we have the following lemmas.

**Lemma 4.2 ($\delta$-interior consistency of $L_h^S$).** Let $\hat{\delta} := \min\{\delta, 4\}$, and $\beta > 2s$ be such that neither $\hat{\delta} - 2s$ nor $\hat{\delta}$ is an integer. Let $f \in L^\infty(\Omega)$ be such that $\|f\|_{L^\infty(\Omega)} < \infty$. Then, the solution of (1.1) satisfies

$$L_h^S[u](x) \leq \kappa_{n,s,i} H_i^{\beta-2s} \sum_{j=1}^n [2u(x_i) - u(x_i + H_i e_j) - u(x_i - H_i e_j)]$$

$$\leq C|u|_{C^\delta(B_{H_i}(x_i))} H_i^{\beta-2s}.$$ 

Using (3.6), (3.7) and Lemma 2.6, we obtain

$$\left| \int_{\Omega_i} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy \right| = \frac{1}{2} \left| \int_{\Omega_i - x_i} \frac{2u(x) - u(x + z) - u(x - z)}{|z|^{n+2s}} dz \right|$$

$$\leq C|u|_{C^\delta(\Omega_i)} \int_{\Omega_i - x_i} \left| \frac{z}{|z|^{n+2s}} \right| dz$$

$$= C|u|_{C^\delta(\Omega_i)} \int_{S^{n-1}} \int_0^{\rho_i(\theta)} \rho^{\beta-2s} \rho \, dS_\theta$$

$$= C|u|_{C^\delta(\Omega_i)} \int_{S^{n-1}} \left( \frac{\rho_i(\theta)}{H_i} \right)^{\beta-2s} \, dS_\theta$$

$$\leq C|u|_{C^\delta(\Omega_i)} H_i^{\beta-2s}.$$ 

By Lemma 4.1 (uniform $\delta$ in $\hat{\Omega}_i \cup \omega_i$) and Corollary 2.5 ($\delta$-dependence in Hölder norm), we have that for any $\delta$-interior node $x_i \in \mathcal{N}_h^{0,\delta}$

$$\max \left( |u|_{C^\delta(\Omega_i)}, |u|_{C^\delta(B_{H_i}(x_i))} \right) \leq |u|_{C^\delta(\Omega_i)} \leq C(\hat{\delta}) \delta_i^{\beta-\hat{\delta}},$$

which leads to the desired result by combining the above two estimates.

**Case 2:** $\beta > 2$. By triangle inequality, we have

$$\left| L_h^S[u](x) - \int_{\Omega_i} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy \right| \leq \left| L_h^S[u](x) + \kappa_{n,s,i} \Delta u(x_i) H_i^{2-2s} \right|$$

$$+ \kappa_{n,s,i} \Delta u(x_i) H_i^{2-2s} + \int_{\Omega_i} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy := E_1 + E_2.$$ 

Thanks to Lemma 2.6 and Corollary 2.5, the estimate of $E_1$ is standard:

$$E_1 = \kappa_{n,s,i} \left| \frac{\Delta_F D_u(x_i; H_i)}{H_i^{2-2s}} - \Delta u(x_i) \right| H_i^{2-2s}$$

$$\leq C|u|_{C^\delta(B_{H_i}(x_i))} H_i^{\beta-2s} \leq C(\hat{\delta}) \delta_i^{\beta-\hat{\delta}} H_i^{\beta-2s} \quad \forall x_i \in \mathcal{N}_h^{0,\delta}. $$
Using the definition of \( \kappa_{n,s,i} \) in (3.8),

\[
\kappa_{n,s,i} \Delta u(x_i) H_i^{2-2s} = \frac{1}{4(1-s)} \frac{D^2 u(x_i) \cdot \int_{S^{n-1}} \rho_i^{2-2s}(\theta) \otimes \theta \, dS_\theta}{\int_{S^{n-1}} \rho_i^{1-2s} D^2 u(x_i) : \theta \otimes \theta \, d\rho \, dS_\theta},
\]

together with (3.7), yields

\[
E_2 = \left| \frac{1}{2} \int_{S^{n-1}} \int_0^{\rho_i(\theta)} \frac{2u(x_i) - u(x_i + \rho \theta) - u(x_i - \rho \theta) - \frac{\partial^2 u(x_i)}{\partial \rho^2} u(x_i) \rho^2}{\rho^{1+2s}} \, d\rho \, dS_\theta \right|
\leq C \int_{S^{n-1}} \int_0^{\rho_i(\theta)} |u|_{C^2(B_{\rho}(x_i))} \rho^{2-1-2s} \, d\rho \, dS_\theta \leq C(\beta) \delta_i^{s-\beta} H_i^{\beta-2s} \quad \forall x_i \in \mathcal{N}_{h}^{0, \delta}.
\]

This completes the proof. \( \square \)

**Lemma 4.3** (global consistency of \( \mathcal{L}_h^s \)). Let \( \beta > 2s \) and \( f \in L^\infty(\Omega) \) be such that \( \|f\|_{\beta-2s; \Omega} < \infty \). Then, the solution of (1.1) satisfies

\[
\|u - f\|_{\beta-2s} \leq C \delta_i^{-s} \quad \forall x_i \in \mathcal{N}_h^0.
\]

Here, the constant \( C \) depends on \( \Omega, s, \beta, \|f\|_{L^\infty(\Omega)} \) and \( \|f\|_{\beta-2s; \Omega} \), but will not blow up as \( \beta - 2s \) or \( \beta \) being an integer.

**Proof.** We notice that for any \( \rho \geq 0 \), it holds that \( \text{dist}(B_{\rho}(x_i), \partial \Omega) + \rho \geq \delta_i \). Since \( \beta > 2s \), there exists \( \beta_0 \in (2s, \min\{1 + s, \beta\}) \), such that neither \( \beta_0 \) nor \( \beta_0 - 2s \) is an integer. Then, we have \( u \in C^{\beta_0}(\Omega_i) \), whence

\[
\left| \int_{\Omega_i} \frac{u(x_i) - u(y)}{|x_i - y|^{n+2s}} \, dy \right| = \frac{1}{2} \left| \int_{S^{n-1}} \rho_i(\theta) \, \frac{2u(x_i) - u(x_i + \rho \theta) - u(x_i - \rho \theta) - \frac{\partial^2 u(x_i)}{\partial \rho^2} u(x_i) \rho^2}{\rho^{1+2s}} \, d\rho \, dS_\theta \right|
\leq C \int_0^{\delta_i} |u|_{C^{\beta_0}(B_{\rho}(x_i))} \rho^{2-1-\beta_0} \rho^{1+2s} \, d\rho \quad \text{(by Lemma 2.6)}
\leq C(\beta_0) \int_0^{\delta_i} (\delta_i - \rho)^{s-\beta_0} \rho^{\beta_0 - 1 - 2s} \, d\rho \quad \text{(by Corollary 2.5)}
= C(\beta_0) \delta_i^{-s} B(s - \beta_0 + 1, \beta_0 - 2s).
\]

Here, \( B(a, b) \) is the beta function. Moreover, using Proposition 2.2 and Corollary 2.5, we take \( \beta_1 \in (2s, \min\{1 + s, \beta\}) \), such that neither \( \beta_1 \) nor \( \beta_1 - 2s \) is an integer, to obtain

\[
|\mathcal{L}_h^s[u](x_i)| = \kappa_{n,s,i} \left| \frac{\Delta_{FD} u(x_i; H_i)}{H_i^2} \right| H_i^{2-2s}
\leq C \min \left\{ |u|_{C^{\beta_1}(B_{H_i}(x_i))} H_i^{\beta_1 - s}, |u|_{C^{\beta_1}(B_{H_i}(x_i))} H_i^{\beta_1 - 2s} \right\}
\leq \delta_i^{-s} \min \left\{ c_1 \left( \frac{H_i}{\delta_i} \right)^{-s}, c_2(\beta_1) \left( 1 - \frac{H_i}{\delta_i} \right)^{s-\beta_1} \left( \frac{H_i}{\delta_i} \right)^{\beta_1 - 2s} \right\}
\leq \max\{c_1, c_2(\beta_1)\} 2^{s} \delta_i^{-s}.
\]

Here, the last step is deduced by considering whether \( \frac{H_i}{\delta_i} \in [1/2, 1] \) or \( \frac{H_i}{\delta_i} \in (0, 1/2) \).

The proof is thus complete. \( \square \)
5. Consistency error II: interpolation and tail part. In this section, we quantify the consistency errors of interpolation and $L^2_h$ in (3.16) on the following graded grids with parameter $h$: There is a number $\mu \geq 1$ such that for any $T \in \mathcal{T}_h$,

\begin{equation}

h_T \approx \begin{cases} 
    h^\mu & \text{if } T \cap \partial \Omega \neq \emptyset, \\
    h \text{dist}(T, \partial \Omega)^{\lambda_1} & \text{if } T \cap \partial \Omega = \emptyset,
\end{cases}

\end{equation}

where $X \lesssim Y$ means that there exists a constants $C > 0$ such that $X \leq CY$, and $X \approx Y$ means that $X \lesssim Y$ and $Y \lesssim X$.

As a direct consequence,

\begin{equation}

\lambda_0^{-1} h^\mu \leq h_T \quad \forall T \in \mathcal{T}_h.
\end{equation}

We note that the quasi-uniform grid corresponds to the case in which $\mu = 1$, thus the results in this section are applicable to the quasi-uniform grids. Further, the construction (5.1) yields a total number of degrees of freedom (cf. [4])

\begin{equation}

N := \dim \mathcal{V}_h \approx \begin{cases} 
    h^{-n} & \text{if } \mu < \frac{n}{n-1}, \\
    \log h \log h^{-n} & \text{if } \mu = \frac{n}{n-1}, \\
    h^{((1-n)\mu)} & \text{if } \mu > \frac{n}{n-1}.
\end{cases}
\end{equation}

5.1. Characterization of $\delta$-interior region. We begin with the characterization of the $\delta$-interior region $\Omega^{0,\delta}_h$ defined in (4.2). The first observation is that any point outside $\Omega^{0,\delta}_h$ should be very close to $\partial \Omega$, which is rigorously stated in the following lemma.

**Lemma 5.1 ($\delta$ outside $\Omega^{0,\delta}_h$).** On the graded grids, there exists $C_6 > 0$ such that

\begin{align}
    \text{dist}(T, \partial \Omega) & \leq C_6 h^\mu \quad \forall T \in \mathcal{T}_h \setminus \mathcal{T}^{0,\delta}_h, \\
    \delta(y) & \leq C_6 h^\mu \quad \forall y \in \Omega \setminus \Omega^{0,\delta}_h.
\end{align}

**Proof.** For any $T \in \mathcal{T}_h \setminus \mathcal{T}^{0,\delta}_h$, the case in which $T \cap \partial \Omega \neq \emptyset$ is trivial since the left hand side vanishes. Below we consider the case in which $T \cap \partial \Omega = \emptyset$. Let $x_j$ be any vertex of $T$. From the definition of $\delta$-interior triangulation (4.2), we see that $x_j \notin \mathcal{N}_h^{0,\delta}$, whence from the local quasi-uniformity (3.1b),

\[ \text{dist}(T, \partial \Omega) \leq \delta(x_j) \leq C_6 h_j \leq \lambda_2 C_6 h_T, \]

where $C_6$ is defined in (4.1). Together with the graded grid condition (5.1), we obtain

\[ \text{dist}(T, \partial \Omega) \leq \lambda_0 \lambda_2 h h \text{dist}(T, \partial \Omega)^{(1-\mu)/\mu}, \]

which leads to (5.4a). Due to (5.1), we then have $h_T \leq C h^\mu$ for $T \in \mathcal{T}_h \setminus \mathcal{T}^{0,\delta}_h$ and hence (5.4b). The proof is thus complete.

**Lemma 5.2 (inside-outside distance).** On the graded grids, there exists $C_d > 0$ such that

\begin{equation}

|x_i - y| \geq (1 + C_d)^{-1} \delta_i \quad \forall x_i \in \mathcal{N}^{0,\delta}_h, y \in \Omega \setminus \Omega^{0,\delta}_h.
\end{equation}

**Proof.** In view of (3.2) and (5.2), we define $C_d := \lambda_0 \lambda C_6$. Next, we prove the assertion by a classified discussion on $\delta_i$. Specifically, if $\delta_i \geq (1 + C_d)(\lambda_0 \lambda)^{-1} h^\mu$, then by (5.4b),

\[ |x_i - y| \geq \delta_i - \delta(y) \geq \delta_i - C_6 h^\mu \geq [1 - (1 + C_d)^{-1} C_d] \delta_i = (1 + C_d)^{-1} \delta_i. \]
Otherwise, by the definition of $\Omega_h^{\beta, \delta}$ in (4.2), the sphere centered at $x_i$ with radius $h_i$ contains in $\Omega_h^{\beta, \delta}$. Therefore, using (3.2) and (5.2),

$$|x_i - y| \geq h_i \geq \lambda^{-1} \lambda_0 h_k > (1 + C_d)^{-1} \delta_i.$$ 

This finishes the proof. □

5.2. Consistency of interpolation. Following the standard polynomial approximation theory [12, Section 4], for any $T \in \mathcal{T}_h$ and $v \in C^0(T)$ ($\beta \geq 0$), it holds that

$$\|v - I_h v\|_{L^\infty(T)} \leq C\|v\|_{C^\beta(T)} h_T^{\tilde{\beta}}, \quad \tilde{\beta} := \min\{\beta, 2\}. \quad (5.6)$$

We recall the symbols $\hat{\beta} := \min\{\beta, 4\}$ and $\tilde{\beta} := \min\{\beta, 2\}$, which will be used many times in this Section.

We first exploit the approximation error in the $\delta$-interior region. Thanks to Lemma 4.1 (uniform $\delta$ in $\tilde{\Omega} \cup \omega_i$) and Corollary 2.5 ($\delta$-dependence in Hölder norm), we conclude from (5.6) that, if neither $\beta - 2s$ nor $\tilde{\beta}$ is an integer, the solution of (1.1) satisfies

$$|u(y) - I_h u(y)| \leq C(\tilde{\beta}) \delta \|v\|_{C^\beta(\tilde{\omega})} h^{\tilde{\beta}} \delta^s \delta y^{s - \tilde{\beta}} \quad \forall y \in \tilde{\Omega}_h^{\beta, \delta}, \quad (5.7)$$

where the constant depends on $\Omega$, $s$, $\tilde{\beta}$, $\|v\|_{L^\infty(\Omega)}$ and $\|f\|_{L^\infty(\Omega)}$. In case that $s - \tilde{\beta}/\mu < 0$, we can take $\tilde{\beta}_1 \in (\mu s, \tilde{\beta})$, such that neither $\tilde{\beta}_1$ nor $\tilde{\beta}_1 - 2s$ is an integer, then the fact $h^\mu \leq \delta(y)$ for all $y \in \tilde{\Omega}_h^{\beta, \delta}$ leads to $|u(y) - I_h u(y)| \leq C(\tilde{\beta}_1) h^{\tilde{\beta}_1 - \beta} \|v\|_{C^\beta(\tilde{\omega})} h^{\tilde{\beta}_1 - \beta} \delta y^{s - \tilde{\beta}_1} \leq C h^{\mu s}$. On the other hand, the global $C^s$ Hölder regularity (Proposition 2.2) together with Lemma 5.1 ($\delta$ outside $\Omega_h^{\beta, \delta}$) yield

$$\|u - I_h u\|_{L^\infty(T)} \leq C h^{\mu s} \quad \forall T \in \mathcal{T}_h \setminus T_h^{\beta, \delta}, \quad (5.8)$$

where $C$ depends on $\Omega$, $s$ and $\|f\|_{L^\infty(\Omega)}$. As a consequence of (5.7), (5.8), we have that for any $\mu \geq 1$,

$$\|u - I_h u\|_{L^\infty(\Omega)} \leq \max\{C h^{\mu s}, C(\tilde{\beta}) h^{\tilde{\beta}}\}, \quad (5.9)$$

where it is assumed again that neither $\tilde{\beta} - 2s$ nor $\tilde{\beta}$ is an integer.

We are now in the position to discuss the consistency of interpolation in (3.16).

**Lemma 5.3** (consistency of interpolation). Let $\beta > 2s$ be such that neither $\tilde{\beta} - 2s$ nor $\tilde{\beta}$ is an integer. Let $f \in L^\infty(\Omega)$ be such that $\|f\|_{L^\infty(\Omega)} \leq \infty$. On the graded grids (5.1) with any $\mu \geq 1$, the solution of (1.1) satisfies

$$\|L_h^S [I_h u](x_i) - L_h^S [u](x_i)\| \leq C(\tilde{\beta}) h^{\tilde{\beta}} \delta^s \delta_i H_i^{-2s} \quad \forall x_i \in \mathcal{N}_h^{\beta, \delta}, \quad (5.10a)$$

$$\|L_h^S [I_h u](x_i) - L_h^S [u](x_i)\| \leq \max\{C h^{\mu s}, C(\tilde{\beta}) h^{\tilde{\beta}}\} H_i^{-2s} \quad \forall x_i \in \mathcal{N}_h^{\beta, \delta}, \quad (5.10b)$$

where the constant $C(\tilde{\beta})$ depends also on $\Omega$, $s$, $\|f\|_{L^\infty(\Omega)}$ and $\|f\|_{L^\infty(\Omega)}$.

**Proof.** Thanks to Lemma 4.1 and 5.7, we have for any $x_i \in \mathcal{N}_h^{\beta, \delta}$,

$$\|L_h^S [I_h u - u](x_i)\| \leq \kappa_{n.s.i} H_i^{-2s} \Delta_F D(I_h u - u)(x_i; H_i)$$

$$\leq C\|u - I_h u\|_{L^\infty(\Omega)} H_i^{-2s} \leq C(\tilde{\beta}) h^{\tilde{\beta}} \delta^s \delta_i H_i^{-2s},$$

which gives (5.10a). Similarly, the global approximation result (5.9) leads to (5.10b). □
5.3. Consistency of $L_h^T$. We discuss the consistency of the tail integral $L_h^T$, which is also considered in two cases: $\mathcal{N}_h^0$ and $\mathcal{N}_h^{0,\delta}$. We first consider the former case.

**Lemma 5.4** (global consistency of $L_h^T$). Let $\beta > 2s$ be such that neither $\tilde{\beta} - 2s$ nor $\tilde{\beta}$ is an integer. Let $f \in L^\infty(\Omega)$ be such that $\|f\|_{\tilde{\beta}-2s,\Omega} < \infty$. On the graded grids, the solution of (1.1) satisfies

\[(5.11) \quad \left| \int_{\Omega_i} \frac{u(y) - I_h u(y)}{|x_i - y|^{n+2s}} \, dy \right| \leq \max\{Ch^s, C(\tilde{\beta})h^\tilde{\beta}\} H_i^{-2s} \quad \forall x_i \in \mathcal{N}_h^0,
\]

where the constant $C(\tilde{\beta})$ depends also on $\Omega$, $\tilde{s}$, $\|f\|_{L^\infty(\Omega)}$ and $\|f\|_{\tilde{\beta}-2s,\Omega}$.

**Proof.** Using the global approximation result (5.9), we obtain for all $x_i \in \mathcal{N}_h^0$,

\[
\left| \int_{\Omega_i} \frac{u(y) - I_h u(y)}{|x_i - y|^{n+2s}} \, dy \right| \leq \max\{Ch^s, C(\tilde{\beta})h^\tilde{\beta}\} \int_{\Omega_i} \frac{1}{|x_i - y|^{n+2s}} \, dy \leq \max\{Ch^s, C(\tilde{\beta})h^\tilde{\beta}\} \int_{S = 1}^{\infty} \int_{\rho \in (\tilde{s}, \delta)} \frac{1}{\rho^{n+2s}} \, d\rho \, d\theta \leq \max\{Ch^s, C(\tilde{\beta})h^\tilde{\beta}\} H_i^{-2s},
\]

where the quasi-uniformity on $\rho_1(\theta)$ (3.6) is used in the last step. \(\square\)

The consistency on the $\delta$-interior nodes would be more delicate. Let $x_i \in \mathcal{N}_h^{0,\delta}$. Since $u - I_h u$ vanishes on $\Omega_i^c$, we confine the domain of integration of $L_h^T$ in (3.16) as $\Omega_i^c \cap \Omega$, which can be partitioned into three subdomains (see Figure 1a)

\[
(5.12) \quad R_{1,i} := \Omega_i^c \cap (\Omega \setminus \Omega_h^{0,\delta}),
R_{2,i} := \Omega_i^c \cap (\Omega_h^{0,\delta} \cap B(1+c_d)^{-1}\delta_i(x_i)),
R_{3,i} := \Omega_i^c \cap (\Omega_h^{0,\delta} \setminus B(1+c_d)^{-1}\delta_i(x_i)).
\]

**Lemma 5.5** (consistency of $L_h^T$ on $R_{1,i}$). On the graded grids (5.1) with any $\mu \geq 1$, there exists $h_0 > 0$ such that when $h < h_0$, the solution of (1.1) satisfies

\[(5.13) \quad \left| \int_{R_{1,i}} \frac{u(y) - I_h u(y)}{|x_i - y|^{n+2s}} \, dy \right| \leq Ch^{\mu(s+1)} \delta_i^{1-2s} \quad \forall x_i \in \mathcal{N}_h^{0,\delta},
\]

where the constant depends on $\Omega$, $\tilde{s}$, $\|f\|_{L^\infty(\Omega)}$.

**Proof.** We apply the approximation result outside $\Omega_h^{0,\delta}$ (5.8) to obtain

\[
\left| \int_{R_{1,i}} \frac{u(y) - I_h u(y)}{|x_i - y|^{n+2s}} \, dy \right| \leq Ch^s \int_{R_{1,i}} \frac{1}{|x_i - y|^{n+2s}} \, dy.
\]

Then, the desired result (5.13) can be obtained by showing

\[(5.14) \quad \int_{R_{1,i}} \frac{1}{|x_i - y|^{n+2s}} \, dy \leq Ch^{\mu} \delta_i^{1-2s}.
\]
Let us consider first the case when $\Omega = \mathbb{R}^n_+ := \{(z', z_n) : z' \in \mathbb{R}^{n-1}, z_n > 0\}$. Invoking Lemma 5.1 (δ outside $\Omega^{0, \delta}_h$), we consider $z_i^* := (0, \cdots, 0, C_i \delta_i)$ and $R_{1,i} \subset \{(z', z_n) : 0 \leq z_n \leq C_2 h^n\}$ (see Figure 1b). We also have $C_1 \delta_i - C_2 h^n \geq C_3 \delta_i$ in light of Lemma 5.2 (inside-outsise distance). Then, the integral in (5.14) has the estimate
\[
\int_0^{C_2 h^n} dz_n \int_{\mathbb{R}^{n-1}} \frac{1}{|z^*_i - z|^{n+2s}} dz' \\
\leq \int_0^{C_2 h^n} dz_n \int_{|z'| > \delta_i} \frac{1}{|z^*_i - z|^{n+2s}} dz' + \int_0^{C_2 h^n} \frac{1}{|C_1 \delta_i - z_n|^{n+2s}} dz_n \int_{|z'| \leq \delta_i} dz' \\
\leq Ch^\mu \delta_i^{-1-2s} + C|C_1 \delta_i - C_2 h^n|^{1-n-2s} - (C_1 \delta_i)^{1-n-2s} \delta_i^{-1} \leq Ch^\mu \delta_i^{-1-2s},
\]
where we use $(1 - C_2 h^n)^{1-n-2s} - 1 \lesssim \frac{h^\mu}{\delta_i}$ in the last step.

We conclude by extending this result to a general bounded Lipschitz domain $\Omega$. We use the notation as in Definition 2.1 and confine the integral domain to $V \cap R_{1,i}$, the general case is then applied by a standard partition of unity. Define a point $z_i^* := (0, \cdots, 0, C_i \delta_i)$ in the z-coordinate. Thanks again to Lemma 5.2, for any $y \in V \cap R_{1,i}$ (denoted by z for change of coordinates), $|x_i - y| \geq (1+C_d)^{-1} \delta_i \gtrsim |z_i^* - z|$. Therefore, by bi-Lipschitz changes of coordinates on $z$, we have
\[
\int_{V \cap R_{1,i}} \frac{1}{|x_i - y|^{n+2s}} dy \leq C \int_{V \cap R_{1,i}} \frac{1}{|z_i^* - z|^{n+2s}} dz \leq Ch^\mu \delta_i^{-1-2s}.
\]
Here, the constant $C$ depends on the Lipschitz constant of $\partial \Omega$, which is uniformly bounded due to the partition of unity. This proves (5.14) and thus (5.13).

**Lemma 5.6** (consistency of $\mathcal{L}^T_h$ on $R_{2,i}$). Let $\beta > 2s$ be such that neither $\beta - 2s$ nor $\beta$ is an integer. Let $f \in L^\infty(\Omega)$ be such that $\|f\|_{\beta-2s;\Omega} < \infty$. On the graded grids (5.1) with any $\mu \geq 1$, the solution of (1.1) satisfies
\[
\int_{R_{2,i}} \frac{u(y) - I_h u(y)}{|x_i - y|^{n+2s}} dy \leq C(\beta) h^{\beta - \delta - \frac{\delta}{\beta - 2s} + 2s - 2s} \forall x_i \in N_{h}^{0, \delta},
\]
Fig. 1: Three subdomains of $\Omega^c \cap \Omega$. The proofs of Lemma 5.5 and Lemma 5.7 use (b) and (c), respectively.
where the constant \( C(\tilde{\beta}) \) depends also on \( \Omega, s, \|f\|_{L^\infty(\Omega)} \) and \( \|f\|_{\tilde{\beta} - 2s; \Omega}^{(s)} \).

**Proof.** Note that all \( y \in R_{2,i} \), the distance satisfies

\[
\delta_i - |x_i - y| \leq \delta(y) \leq \delta_i + |x_i - y| \leq C\delta_i.
\]

Case 1: \( \mu \leq \tilde{\beta}/s \). Using (5.7), the multi-dimensional polar coordinate, and the quasi-uniformity of \( \Omega_i \) (3.6), we have

\[
\left| \int_{R_{2,i}} \frac{u(y) - \mathcal{I}_h u(y)}{|x_i - y|^{n+2s}} dy \right| \leq C(\tilde{\beta})h^{\tilde{\beta}} \int_{R_{2,i}} \frac{1}{\delta(y)^{\tilde{\beta}/n - s}} |x_i - y|^{n+2s} dy
\]

\[
\leq C(\tilde{\beta})h^{\tilde{\beta}} \int_{R_{2,i}} \frac{1}{(\delta_i - |x_i - y|)^{\tilde{\beta}/n - s}} |x_i - y|^{n+2s} dy
\]

\[
= C(\tilde{\beta})h^{\tilde{\beta}} \int_{S^{n-1}} \frac{1}{(1+\rho_i)^{-1/2}} \frac{1}{(\delta_i - \rho)^{\tilde{\beta}/n - s}} \rho^{1+2s} d\rho dS_\theta
\]

\[
\leq C(\tilde{\beta})h^{\tilde{\beta}} \delta_i^{-s/2} \int_{S^{n-1}} \frac{1}{(1+\rho_i)^{-1/2}} \frac{1}{(1+t)^{\tilde{\beta}/n - s}} t^{1+2s} dt dS_\theta
\]

\[
\leq C(\tilde{\beta})h^{\tilde{\beta}} \delta_i^{-s/2} \frac{1}{\tilde{\beta}} H_i^{-2s},
\]

where the last step uses the fact that \( H_i \leq \delta_i \).

Case 2: \( \mu > \tilde{\beta}/s \). In this case, we use the \( \delta(y) \leq C\delta_i \) to obtain

\[
\left| \int_{R_{2,i}} \frac{u(y) - \mathcal{I}_h u(y)}{|x_i - y|^{n+2s}} dy \right| \leq C(\tilde{\beta})h^{\tilde{\beta}} \int_{R_{2,i}} \frac{1}{\delta(y)^{\tilde{\beta}/n - s}} |x_i - y|^{n+2s} dy
\]

\[
\leq C(\tilde{\beta})h^{\tilde{\beta}} \delta_i^{-s/2} \int_{R_{2,i}} \frac{1}{|x_i - y|^{n+2s}} dy \leq C(\tilde{\beta})h^{\tilde{\beta}} \delta_i^{-s/2} H_i^{-2s},
\]

This completes the proof. \( \square \)

**Lemma 5.7** (consistency of \( \mathcal{L}_h^T \) on \( R_{3,i} \)). Let \( \tilde{\beta} > 2s \) be such that neither \( \tilde{\beta} - 2s \) nor \( \tilde{\beta} \) is an integer. Let \( f \in L^\infty(\Omega) \) be such that \( \|f\|_{\tilde{\beta} - 2s; \Omega}^{(s)} < \infty \). On the graded grids (5.1) with any \( \mu \geq 1 \), there exists \( h_0 > 0 \) such that when \( h < h_0 \), the solution of (1.1) satisfies, \( \forall x_i \in \mathcal{N}_h^{n, \delta} \),

\[
(5.16)
\]

\[
\left| \int_{R_{3,i}} \frac{u(y) - \mathcal{I}_h u(y)}{|x_i - y|^{n+2s}} dy \right| \leq \begin{cases} 
C(\tilde{\beta})h^{\mu(s+1)} \delta_i^{-1-2s} & \text{if } \tilde{\beta} > \mu(1+s), \\
C(\tilde{\beta})h^{\mu(s+1)} |\log h| \delta_i^{-1-2s} & \text{if } \tilde{\beta} = \mu(1+s), \\
C(\tilde{\beta})h^{\beta} \delta_i^{-s} & \text{if } \mu(1+s) > \tilde{\beta} > \mu s, \\
C(\tilde{\beta})h^{\beta} \delta_i^{-2s} & \text{if } \mu s \geq \tilde{\beta},
\end{cases}
\]

where the constant \( C(\tilde{\beta}) \) depends also on \( \Omega, s, \|f\|_{L^\infty(\Omega)} \) and \( \|f\|_{\tilde{\beta} - 2s; \Omega}^{(s)} \). Moreover, the constant \( C(\tilde{\beta}) \) behaves as \( |\mu(1+s) - \tilde{\beta}|^{-1} \) when \( \tilde{\beta} \to \mu(1+s)^{-} \).

**Proof.** Since \( R_{3,i} \subset \Omega_h^{n, \delta} \), we apply (5.7) to obtain

\[
\left| \int_{R_{3,i}} \frac{u(y) - \mathcal{I}_h u(y)}{|x_i - y|^{n+2s}} dy \right| \leq C(\tilde{\beta})h^{\tilde{\beta}} \int_{R_{3,i}} \frac{1}{\delta(y)^{\tilde{\beta}/n - s}} |x_i - y|^{n+2s} dy,
\]
which boils down to the estimate of the right integral. Similar to the estimate of integral on $R_{1,i}$, we consider first the case when $\Omega = \mathbb{R}^n_+$ but the integral domain is restricted by $z_n \leq M$ for $M > 0$ sufficiently large. Specifically, it holds that $R_{3,1} \subset \{(z', z_n) : z_n \geq C_2 h^\mu\}$ (This is because $C_\delta > 1$ from (4.1), and hence $\Omega \bigcap \Omega^{h, \delta}_n$ has at least one layer of elements with size $h^\mu$). Moreover, we consider $z'_i = (0, \cdots, 0, a)$ and a cylinder $C_i$ with size $\delta_i$; (see Figure 1c)

$$C_i := \{(z', z_n) : |z_n - a| < C_1 \delta_i, |z'| < C_1 \delta_i\}.$$ 

Upon the relationship between $\tilde{\beta}$, $\mu(1 + s)$ and $\mu s$, we consider the estimate of the following integral in three cases:

$$I := \int_{\mathbb{R}^{n-1} \times [C_2 h^\mu, M]\backslash \{C_i\}} \frac{1}{z_n^{\frac{s}{n}} |z'_i|^n - z |^{n+2s}} \, dz.$$  

(5.17)

Case 1: $\tilde{\beta} \geq \mu(1 + s)$. Then, we divide $I$ into two components:

$$I = \left( \int_{C_2 h^\mu}^{M} \int_{|z'| > C_1 \delta_i} \frac{1}{z_n^{\frac{s}{n}} |z'_i|^n - z |^{n+2s}} \, dz' \right) \, \, dz_n$$

$$+ \int_{[C_2 h^\mu, M]\backslash[a-C_1 \delta_i, a+C_1 \delta_i]} \int_{|z'| < C_1 \delta_i} \frac{1}{z_n^{\frac{s}{n}} |z'_i|^n - z |^{n+2s}} \, dz' \, dz_n$$

$$:= I_1 + I_2.$$ 

The estimates of $I_1$ and $I_2$ are given as

$$I_1 \leq \int_{C_2 h^\mu}^{M} \frac{1}{z_n^{\frac{s}{n}} |z'_i|^n - z |^{n+2s}} \, dz_n \int_{|z'| > C_1 \delta_i} \frac{1}{|z'|^{n+2s}} \, dz'$$

$$\leq C \delta_i^{-1-2s} \int_{C_2 h^\mu}^{M} \frac{1}{z_n^{\frac{s}{n}} |z'_i|^n - z |^{n+2s}} \, dz_n \leq \left\{ \begin{array}{ll}
C h^{\mu(s+1)-\tilde{\beta} \delta_i^{-1-2s}} & \text{if } \tilde{\beta} > \mu(1 + s), \\
C |\log h| \delta_i^{-1-2s} & \text{if } \tilde{\beta} = \mu(1 + s),
\end{array} \right.$$

$$I_2 \leq \int_{[C_2 h^\mu, M]\backslash[a-C_1 \delta_i, a+C_1 \delta_i]} \int_{|z'| < C_1 \delta_i} \frac{1}{z_n^{\frac{s}{n}} |z'_i|^n - z |^{n+2s}} \, dz' \, dz_n$$

$$\leq C \delta_i^{-1-2s} \int_{C_2 h^\mu}^{M} \frac{1}{z_n^{\frac{s}{n}} |z'_i|^n - z |^{n+2s}} \, dz_n \leq \left\{ \begin{array}{ll}
C h^{\mu(s+1)-\tilde{\beta} \delta_i^{-1-2s}} & \text{if } \tilde{\beta} > \mu(1 + s), \\
C |\log h| \delta_i^{-1-2s} & \text{if } \tilde{\beta} = \mu(1 + s).
\end{array} \right.$$

Here, the constant blows up as $\tilde{\beta} \to \mu(1 + s)^+$. 

Case 2: $\mu(1 + s) > \tilde{\beta} > \mu s$. In this case, $z_n^{\frac{s}{n}} |z'_i|^n - z |^{n+2s}$ is unbounded but integrable at $z_n = 0$. We take $C_2 > 0$ so that $\tilde{C}_2 \delta_i \geq C_2 h^\mu$ (guaranteed by Lemma 5.2). Therefore, we divide $I$ into three components,

$$I = \left( \int_{\mathbb{R}^{n-1} \times [C_2 \delta_i, M]\backslash \{C_i\}} \frac{\tilde{C}_2 \delta_i}{z_n^{\frac{s}{n}} |z'_i|^n - z |^{n+2s}} \, dz' \right) \, \, dz_n$$

$$+ \int_{[C_2 h^\mu, \tilde{C}_2 \delta_i]\backslash[a-C_1 \delta_i, a+C_1 \delta_i]} \int_{|z'| < C_1 \delta_i} \frac{1}{z_n^{\frac{s}{n}} |z'_i|^n - z |^{n+2s}} \, dz' \, dz_n$$

$$:= J_0 + J_1 + J_2.$$
Since \( z_n \geq C_2 \delta_i \) in the integral domain of \( J_0 \), we have
\[
J_0 \leq C \delta_i^{-\frac{2}{\beta} + s} \int_{R^{n-1} \setminus [C_2 \delta_i, M] \setminus \mathcal{C}_i} \frac{1}{|z_n^* - z|^{n+2s}} \, dz' \, dz_n \leq C \delta_i^{-\frac{2}{\beta} - s}.
\]
The estimates of \( J_\ell (\ell = 1, 2) \) are similar to the Case 1, i.e.,
\[
J_\ell \leq C \delta_i^{-1-2s} \int_{C_2 h^\mu}^\infty \frac{1}{z_n^{\beta-s}} \, dz_n \leq C \delta_i^{-\frac{\beta}{\mu} - s} \quad \ell = 1, 2,
\]
where the constant blows up as \( \beta \to \mu(1+s)^{-} \).

Case 3: \( \mu s \geq \beta \). In this case, \( z_n^{-\frac{2}{\beta} - s} \) is bounded, and hence \( I \leq C(M) \delta_i^{-2s} \).
Combining Case 1-3, we arrive at
\[
(5.18) \quad h^{\bar{\beta}} I \leq \begin{cases} 
C h^{\mu(s+1)} \delta_i^{-1-2s} & \text{if } \bar{\beta} > \mu(1+s), \\
C h^{\mu(s+1)} \log h^{-1-2s} & \text{if } \bar{\beta} = \mu(1+s), \\
C h^{\beta} \delta_i^{-\frac{\beta}{\mu} - s} & \text{if } \mu(1+s) > \bar{\beta} > \mu s, \\
C(M) h^{\beta} \delta_i^{-2s} & \text{if } \mu s \geq \bar{\beta}.
\end{cases}
\]

We generalize this result to a bounded Lipschitz domain \( \Omega \). Using the notation as in Definition 2.1, there exists \( \{V_k\}_{k=1}^K \) such that \( \partial \Omega \subset \cup_{k=1}^K V_k \). Notice that \( \delta(y) \gtrsim 1 \) for \( y \in R_{3,1} \setminus \cup_{k=1}^K V_k \), then
\[
(5.19) \quad h^{\bar{\beta}} \int_{R_{3,1} \setminus \cup_{k=1}^K V_k} \frac{1}{\delta(y)^{\frac{\beta}{\mu} - s} |x_i - y|^{n+2s}} \, dy \leq C h^{\bar{\beta}} \delta_i^{-2s} \leq C h^{\bar{\beta}} I.
\]
For any \( V_k \cap R_{3,1} \), using the bi-Lipschitz changes of coordinates, and noticing that the cylinder \( \mathcal{C}_i \) is equivalent (mutually bounded up to constant) to \( B_{(1+C_d)}^{-1} \delta_i(z_i^*) \), then
\[
(5.20) \quad h^{\bar{\beta}} \sum_{k=1}^K \int_{R_{3,1} \cap V_k} \frac{1}{\delta(y)^{\frac{\beta}{\mu} - s} |x_i - y|^{n+2s}} \, dy \leq C h^{\bar{\beta}} I,
\]
where the constant \( M \) in (5.18) can be taken as \( \mathcal{O}(\text{diam}(\Omega)) \). Combining (5.19) and (5.20) yields (5.16), as asserted.

Combining Lemmas 5.5-7, and noticing that \( h^{\mu} \delta_i^{-1} \lesssim 1 \) for the \( \delta \)-interior nodes, we then have the \( \delta \)-interior consistency of \( \mathcal{L}_h^\Gamma \) as follows.

**Lemma 5.8** (\( \delta \)-interior consistency of \( \mathcal{L}_h^\Gamma \)). Let \( \beta > 2s \) be such that neither \( \bar{\beta} - 2s \) nor \( \bar{\beta} \) is an integer. Let \( f \in L^\infty(\Omega) \) be such that \( \|f\|_{\bar{\beta} - 2s; \Omega} < \infty \). On the graded grids (5.1) with any \( \mu \geq 1 \), there exists \( h_0 > 0 \) such that when \( h < h_0 \), the solution of (1.1) satisfies
\[
\left| \int_{\Omega_h^\Gamma} \frac{u(y) - \mathcal{I}_h \mathcal{I}_h u(y)}{|x_i - y|^{n+2s}} \, dy \right| \leq C(\bar{\beta}) h^{\beta} \delta_i^{-\frac{\beta}{\mu} - s} \, H_{-2s}^\Gamma
\]
(5.21) + \begin{cases} 
C(\bar{\beta}) h^{\mu(s+1)} \delta_i^{-1-2s} & \text{if } \bar{\beta} > \mu(1+s), \\
C(\bar{\beta}) h^{\mu(s+1)} \log h^{-1-2s} & \text{if } \bar{\beta} = \mu(1+s), \\
C(\bar{\beta}) h^{\beta} \delta_i^{-\frac{\beta}{\mu} - s} & \text{if } \mu(1+s) > \bar{\beta} > \mu s, \\
C(\bar{\beta}) h^{\beta} \delta_i^{-2s} & \text{if } \mu s \geq \bar{\beta},
\end{cases} \quad \forall x_i \in \mathcal{N}_h^{0, \delta},
where the constant $C(\bar{\beta})$ depends also on $\Omega$, $s$, $\|f\|_{L^\infty(\Omega)}$ and $\|f\|_{\beta-2s;\Omega}^{(s)}$. Moreover, the constant $C(\bar{\beta})$ behaves as $|\mu(1+s) - \bar{\beta}|^{-1}$ when $\bar{\beta} \to \mu(1+s)^-$. 

6. Pointwise error estimate. In this section, we establish the convergence rate of the proposed scheme under the choice of $H_i$ given in (3.15), i.e.,

$$H_i = h^\alpha \min\{\delta_i^{1-\alpha}, \delta_0^{1-\alpha}\}.$$ 

Thanks to Lemma 5.1 ($\delta$ outside $\Omega^0_{h\delta}$) and (5.1), we have

$$H_i \approx \begin{cases} h^\alpha \approx \delta_i & \text{if } x_i \in \mathcal{N}_h^0 \setminus \mathcal{N}_{h\delta}^0, \\ h^\alpha \delta_i^{1-\alpha} & \text{if } x_i \in \mathcal{N}_{h\delta}^0. \end{cases}$$

(6.1)

Note that the (6.1) also holds for $\delta_i \geq \delta_0$, where $H_i = h^\alpha \delta_i^{1-\alpha} \approx h^\alpha \approx h^\alpha \delta_i^{1-\alpha}$. 

In light of Corollary 2.5 ($\delta$-dependence in Hölder norm) and Remark 2.4 (blow-up behavior), we define the following two indices:

$$\bar{\beta} := \begin{cases} 1 & \text{if } \hat{\beta} - 2s \text{ or } \hat{\beta} \text{ is an integer}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\bar{\beta} := \begin{cases} 1 & \text{if } \hat{\beta} - 2s \text{ or } \hat{\beta} \text{ is an integer}, \\ 0 & \text{otherwise}, \end{cases}$$

(6.2)

where we recall that $\hat{\beta} = \min\{\beta, 4\}$ and $\bar{\beta} = \min\{\beta, 2\}$.

We now derive pointwise error estimates for the solution of (1.1). We proceed as follows. We apply the global and $\delta$-interior consistency results respectively to control the consistency errors outside and inside $\Omega^0_{h\delta}$. The combination of consistency error and Lemma 3.5 (discrete barrier function) will conclude the argument, thanks to Lemma 3.6 (discrete comparison principle).

THEOREM 6.1 (Convergence rates in terms of $h$). Let $\Omega$ be a bounded Lipschitz domain with exterior ball condition. Let $f \in L^\infty(\Omega)$ be such that $\|f\|_{\beta-2s;\Omega}^{(s)} < \infty$. Then, on the graded grids with any $\mu \geq 1$ and $H_i$ chosen as in (3.15), we have

$$\|u - u_h\|_{L^\infty(\Omega)} \leq C \max\{h^{\mu s}, |\log h|^{\bar{\beta}} h^{(\hat{\beta} - 2s)\alpha}, |\log h|^{\bar{\beta}} h^{\hat{\beta} - 2s\alpha}\},$$

(6.3)

where the constant depends on $\Omega$, $s$, $\beta$, $\|f\|_{L^\infty(\Omega)}$ and $\|f\|_{\beta-2s;\Omega}^{(s)}$. Moreover, the optimal $\alpha$ and corresponding convergence rate are

$$\alpha_{\text{opt}} := \frac{\bar{\beta}}{\beta} \in \left[\frac{1}{2}, 1\right], \quad \|u - u_h\|_{L^\infty(\Omega)} \leq C \max\{h^{\mu s}, |\log h|^{\max\{\bar{\beta}, \bar{\beta}\}} h^{\hat{\beta} - 2s\bar{\beta}/\beta}\}.$$ 

(6.4)

Proof. We consider the consistency error $L_h[\mathcal{L}_h u](x_i) - \mathcal{L}[u](x_i)$ in two cases.

Case 1: $x_i \in \mathcal{N}_h^0 \setminus \mathcal{N}_{h\delta}^0$. Applying Lemma 4.3 (global consistency of $L_h^\delta$), Lemma 5.3 (consistency of interpolation) and Lemma 5.4 (global consistency of $L_h^\delta$), we have

$$|L_h[\mathcal{L}_h u](x_i) - \mathcal{L}[u](x_i)| \leq C \delta_i^{-s} + \max\{Ch^{\mu s}, C(\bar{\beta})h^{\bar{\beta}}\} H_i^{-2s} \forall x_i \in N_h^0 \setminus N_h.$$ 

If $\bar{\beta}$ or $\beta - 2s$ is an integer, we use Remark 2.4 (blow-up behavior) to obtain that, for arbitrary small $\varepsilon > 0$,

$$C(\bar{\beta} - \varepsilon) h^{\bar{\beta} - \varepsilon} \leq \frac{C}{\varepsilon} h^{\bar{\beta} - \varepsilon}.$$
Taking $\varepsilon = |\log h|^{-1}$ leads to the bound $C|\log h| h^{3\beta}$. Then, using $\delta_i \approx h^{\mu} \approx H_i$ for $x_i \in \mathcal{N}_h^{0} \setminus \mathcal{N}_h^{0,\delta}$ (see (6.1)), we have

\begin{equation}
|\mathcal{L}_h[I_h u](x_i) - \mathcal{L}[u](x_i)| \leq C \max\{h^{\mu s}, |\log h|^{3\beta}h^{3\beta}\} \delta_i^{-2s} \quad \forall x_i \in \mathcal{N}_h^{0} \setminus \mathcal{N}_h^{0,\delta}.
\end{equation}

Case 2: $x_i \in \mathcal{N}_h^{0,\delta}$. We consider the three components of the $\delta$-interior consistency errors. (2-a) In view of Lemma 4.2 (2-interior consistency of $\mathcal{L}_h^{S}$) and (6.1), we have

\begin{equation}
|\mathcal{L}_h^{S}[u](x_i) - \int_{\Omega_i} \frac{u(x_i) - u(y)}{|x_i - y|^{n+2s}} \, dy| \leq C(\beta) h^{\beta - 2s} \delta_i^{-2s} \\
\approx C(\beta) h^{\beta - 2s} \delta_i^{(\beta - 2s)\alpha} \delta_i^{-2s}.
\end{equation}

If $(\beta - 2s)\alpha \leq \mu s$, then $\delta_i^{(\beta - 2s)\alpha} \delta_i^{-2s} \leq 1$ due to $\delta_i \leq 1$. Hence, we have the upper bound $C(\beta) h^{\beta - 2s} \delta_i^{-2s}$, which turns out to be $C|\log h|^{3\beta}h^{3\beta} \delta_i^{-2s}$. Otherwise, if $(\beta - 2s)\alpha > \mu s$, there exists $\beta_0 \in [2s + \frac{\mu s}{\alpha}, \beta]$, such that neither $\beta_0 - 2s$ nor $\beta_0$ is an integer. Then,

\begin{equation}
C(\beta) h^{\beta - 2s} \delta_i^{(\beta - 2s)\alpha} \delta_i^{-2s} \leq C(\beta) h^{\mu s} (h^{\mu \beta_0} - 1) \delta_i^{-2s} \\
\leq C(\beta) h^{\mu s} \delta_i^{-2s}.
\end{equation}

Here, $h^{\mu \delta_i^{-1}} \leq 1$ is used, since $h^{\mu} \approx H_i \leq \delta_i$. Summing up two cases, we have

\begin{equation}
|\mathcal{L}_h^{S}[I_h u](x_i) - \int_{\Omega_i} \frac{u(x_i) - u(y)}{|x_i - y|^{n+2s}} \, dy| \leq C \max\{h^{\mu s}, |\log h|^{3\beta}h^{3\beta}\} \delta_i^{-2s}.
\end{equation}

(2-b) Similarly, in view of Lemma 5.3 (consistency of interpolation) and (6.1), we have for any $x_i \in \mathcal{N}_h^{0,\delta}$,

\begin{equation}
|\mathcal{L}_h^{S}[I_h u](x_i) - \mathcal{L}_h^{S}[u](x_i)| \leq C(\beta) h^{\beta - 2s} \delta_i^{-2s} \\
\approx C(\beta) h^{\beta - 2s} \delta_i^{-2s} \\
\leq C \max\{h^{\mu s}, |\log h|^{3\beta}h^{3\beta}\} \delta_i^{-2s}.
\end{equation}

(2-c) For the $\delta$-interior consistency of $\mathcal{L}_h^{S}$ (Lemma 5.8), the first term on the right hand side of (5.21) is the same as the case (2-b). For the other term, since $\beta \geq \mu s$, there exists $\beta_0 \in (\mu s, \mu (s + 1))$ such that neither $\beta_0 - 2s$ nor $\beta_0$ is an integer. Then,

\begin{equation}
C(\beta_0) h^{\beta_0} \delta_i^{-2s} \delta_i^{-2s} = C(\beta_0) h^{\mu s} (h^{\mu \beta_0} - 1) \delta_i^{-2s} \\
\leq C h^{\mu s} \delta_i^{-2s}.
\end{equation}

Otherwise $\beta \leq \mu s$, then $C(\beta) h^{\beta} \delta_i^{-2s} \leq |\log h|^{3\beta}h^{3\beta} \delta_i^{-2s}$. As a result,

\begin{equation}
\left| \int_{\Omega_i} \frac{u(y) - I_h u(y)}{|x_i - y|^{n+2s}} \, dy \right| \leq C \max\{h^{\mu s}, |\log h|^{3\beta}h^{3\beta}\} \delta_i^{-2s}.
\end{equation}

Combining (6.6)-(6.8), the consistency error on $\delta$-interior nodes $x_i \in \mathcal{N}_h^{0,\delta}$ is given as

\begin{equation}
|\mathcal{L}_h[I_h u](x_i) - \mathcal{L}[u](x_i)| \leq C \max\{h^{\mu s}, |\log h|^{3\beta}h^{3\beta}\} \delta_i^{-2s}.
\end{equation}
The numerical solution satisfies \( L_h[u_h](x_i) = f(x_i) = L[u](x_i), \forall x_i \in N_h^0 \). Invoking the discrete barrier function \( b_h \) defined in Lemma 3.5, the estimates (6.5) and (6.9) yield

\[
|L_h[I_h u - u_h](x_i)| \\
\leq C \max \{ h^{\mu_1}, |\log h|\hat{h}^{(\hat{\beta} - 2s)\alpha}, |\log h|\hat{h}^{\hat{\beta} - 2s} \} L_h[b_h](x_i) \quad \forall x_i \in N_h^0,
\]

whence, by Lemma 3.6 (discrete comparison principle)

\[
\max_{x_i \in N_h^0} |I_h u(x_i) - u_h(x_i)| \leq C \max \{ h^{\mu_1}, |\log h|\hat{h}^{(\hat{\beta} - 2s)\alpha}, |\log h|\hat{h}^{\hat{\beta} - 2s} \}.
\]

The desired estimate (6.3) then follows from the approximation result (5.9).

The optimal \( \alpha \) satisfies \((\hat{\beta} - 2s)\alpha_{opt} = \hat{\beta} - 2s\alpha_{opt} \), which gives \( \alpha_{opt} = \hat{\beta}/\hat{\beta} \). It is straightforward to see that \( \alpha_{opt} \in \left[ \frac{1}{2}, 1 \right] \) from the definitions of \( \hat{\beta} \) and \( \hat{\beta} \). Taking the optimal \( \alpha \) into (6.3) leads to (6.4). The proof is therefore complete.

Remark 6.2 (Huang-Oberman’s work [20] revisited). In [20], the 1D uniform grid with \( H = h \) was applied, namely \( \mu = 1 \) and \( \alpha = 1 \). Then, (6.3) implies the pointwise error \( \mathcal{O}(\max \{ h^{\mu_1}, |\log h|\hat{h}^{(\hat{\beta} - 2s)} \}) \), which is observed in Table 1 since \( \hat{\beta} \leq 2 \).

Using the relationship between \( h \) and the total number of degrees of freedom (5.3), we obtain the following theorem.

**Theorem 6.3** (Convergence rates in terms of \( N \)). Let \( \Omega \) be a bounded Lipschitz domain with exterior ball condition. Let \( f \in L^\infty(\Omega) \) be such that \( \| f \|_{\beta+2s;\Omega} < \infty \). Then, on the graded grids (5.1) and \( H_t \) chosen as in (3.15), \( \alpha \) as \( \alpha_{opt} = \hat{\beta}/\hat{\beta} \), and

\[
\mu \left\{ \begin{array}{ll}
\in \left[ \frac{\beta - 2s}{\hat{\beta}}, \frac{n}{n-1} \right) & \text{if } \left( \frac{n}{n-1} + \frac{2\beta}{\hat{\beta}} \right) s > \hat{\beta}, \\
\in \left[ \frac{\beta - 2s}{\hat{\beta}}, \frac{n}{n-1} \right) & \text{if } \left( \frac{n}{n-1} + \frac{2\beta}{\hat{\beta}} \right) s \leq \hat{\beta}.
\end{array} \right.
\]

Then, the convergence rate

\[
\| u - u_h \|_{L^\infty(\Omega)} \leq C \left\{ \begin{array}{ll}
(\log N)^{\max(\hat{\kappa}, \hat{\kappa})} N^{-\frac{1}{2}(\beta - \frac{2s}{\hat{\beta}})} & \text{if } \left( \frac{n}{n-1} + \frac{2\beta}{\hat{\beta}} \right) s > \hat{\beta}, \\
(\log N)^{\max(\hat{\kappa}, \hat{\kappa}) + \frac{n}{n-1}} N^{-\frac{s}{\hat{\beta}}} & \text{if } \left( \frac{n}{n-1} + \frac{2\beta}{\hat{\beta}} \right) s = \hat{\beta}, \\
(\log N)^{\frac{\hat{\kappa}}{\hat{\beta}}} N^{-\frac{s}{\hat{\beta}}} & \text{if } \left( \frac{n}{n-1} + \frac{2\beta}{\hat{\beta}} \right) s < \hat{\beta},
\end{array} \right.
\]

where the constant depends on \( \Omega \), \( s \), \( \beta \), \( \| f \|_{\beta+2s;\Omega} \) and \( \| f \|_{\beta+2s;\Omega} \).

7. **Numerical Experiments.** In this section, we present some numerical experiments in both one and two-dimensional domains, which illustrate the sharpness of our theoretical estimates. In all of the experiments below, we set \( \Omega = B_1(0) \subset \mathbb{R}^n \), \( n = 1, 2 \) and \( f \equiv 1 \), so that we have an explicit solution

\[
u(x) = \frac{2^{-2s}(n/2)}{(n/2 + s)(1 + s)} (1 - |x|^2)^{s} \quad \forall x \in \Omega.
\]

This corresponds to smooth right hand side and the discussion of Section 6 applies with \( \hat{\beta} = 2 \), \( \hat{\beta} = 4 \), \( \hat{\kappa} = \hat{\kappa} = 1 \), and \( \alpha_{opt} = \frac{2}{\hat{\beta}} = \frac{1}{2} \).
7.1. 1D test. When \( n = 1 \), we always have the relation \( N \approx h^{-1} \) for any \( \mu \geq 1 \) due to (5.3). In the approximation of the singular integral, the domain \( \Omega_i \) in (3.4) is taken as the open interval centered at \( x_i \) with radius \( H_i \), namely, \( \Omega_i = (x_i - H_i, x_i + H_i) \). On any element \( T \) for which \( T \cap \Omega_i^c \neq \emptyset \), the intersection is still an interval so that the weight can be calculated explicitly, see [20, Section 3]. In this case, the convergence rate estimate (6.4) given by Theorem 6.1 turns out to be

\[
\|u - u_h\|_{L^\infty(\Omega)} \leq C \max\{h^{\mu s}, \log h|2^{-s}|\}.
\]

We start with the numerical tests on quasi-uniform grids (\( \mu = 1 \)). The convergence rates for several values of \( s \) are listed in Table 2a, and the computational errors for \( s = 0.3, s = 0.6 \), and \( s = 0.9 \) are shown in Figure 2b. In all cases, we see good agreement with the convergence rate \( O(h^s) \) predicted by (7.1).

| \( s \) | Rate | \( s \) | Rate |
|---|---|---|---|
| 0.1 | 0.10 | 0.6 | 0.60 |
| 0.2 | 0.20 | 0.7 | 0.71 |
| 0.3 | 0.30 | 0.8 | 0.81 |
| 0.4 | 0.40 | 0.9 | 0.94 |

(a) Convergence rates

(b) \( \log(L^\infty\text{-error}) - \log(h) \)

Fig. 2: Convergence rates for problem (1.1) using uniform grids in 1D case.

We next consider numerical approximations using graded grids that satisfy (5.1) with \( \mu = (2 - s)/s \). We would expect a convergence rate of order \( O(h^{2-s}) \) (up to a logarithmic factor) according to (7.1). In Figure 3 we display the computational rates of convergence for \( s = 0.4, 0.6, 0.8 \), which are in good agreement with the theory.

Fig. 3: Convergence rates for graded grids with \( \mu = 2-s/s \) in 1D case. Convergence rate of \( 2-s \) is observed.

Next, we plot the \( L^\infty \)-errors for both uniform and graded grids in Figure 4. We observe that the \( L^\infty \)-error increases rapidly near the boundary on quasi-uniform grids, due to the poor Hölder continuity near the boundary. The graded grids will alleviate this effect. Further, the error behaviors make it possible to establish some improved interior (or local) pointwise error estimates, which belong to our future work.

7.2. 2D test. In the 2D test, \( \Omega_i \) is taken as the square centered at \( x_i \) with side length \( \sqrt{2}H_i \), see Remark 3.3 (examples of \( \Omega_i \)). An immediate benefit is the convenient
numerical integration on $\Omega^c_i$ on unstructured grids in (3.11). More precisely, the intersection of $T \cap \Omega^c_i$, if not empty, will become a polygon denoted by $P$. Therefore, the calculation of weight turns out to be the approximation of $C_{2,s} \int_P \phi(y)|y|^{-2-2s} dy$, where $\phi$ is a linear function. Let $F(y) := \frac{C_{2,s}}{|y|^{2s}}|y|^{-2s}$ so that $\Delta F(y) = C_{2,s}|y|^{-2-2s}$. Then, after integration by parts twice, we obtain

$$C_{2,s} \int_P \phi(y)|y|^{-2-2s} dy = \int_{\partial P} \phi \frac{\partial F}{\partial n} ds - \int_{\partial P} F \frac{\partial \phi}{\partial n} ds.$$ 

The integral has been transformed into one-dimensional intervals, where the high-order numerical quadrature can be applied.

![Fig. 5: Convergence rates for problem (1.1) using quasi-uniform grids in 2D case.](image)

![Fig. 6: Convergence rates for problem (1.1) using graded grids in 2D case.](image)

We next explore the sharpness of our estimates derived in Section 6. On a sequence of quasi-uniform grids (Figure 5a), the plots of $L^\infty$-errors for several values of $s$ are given in Figure 5b. In light of (5.3), we have the relationship $h \approx N^{-\frac{1}{2}}$ for quasi-uniform grids ($\mu = 1$). From (6.4) in Theorem 6.1, the theoretical convergence rate $O(h^s)$ or $O(N^{-\frac{1}{2}})$ coincides with the numerical tests.
In the last test, we consider the computation on graded grids (Figure 6a), where the errors are computed with respect to the total number of degrees of freedom $N$. According to Theorem 6.3, the expected convergence rates are $O(N^{-\frac{3}{2}})$ for $s > \frac{2}{3}$, and $O(N^{-s})$ for $s \leq \frac{2}{3}$, up to a logarithmic factor. In Figure 6b we display the computational rates of convergence for $s = 0.3, 0.6, 0.9$, which are in good agreement with theory.

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