Sylvester double sums, subresultants and symmetric multivariate Hermite interpolation

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Abstract
Sylvester doubles sums, introduced first by Sylvester (see \cite{8, 9}), are symmetric expressions of the roots of two polynomials. Sylvester’s definition of double sums makes no sense in the presence of multiple roots, since the definition involves denominators that vanish when there are multiple roots. The aim of this paper is to give a new definition of Sylvester double sums making sense in the presence of multiple roots, which coincides with the definition by Sylvester in the case of simple roots, to prove that double sums indexed by \((a, b)\) are equal up to a constant if they share the same value for \(a + b\), as well a proof of the relationship between double sums and subresultants, i.e. that they are equal up to a constant. In the simple root case, proofs of these properties are already known (see \cite{6, 1, 7}). The more general proofs given here are using generalized Vandermonde determinants and symmetric multivariate Hermite interpolation as well as an induction on the length of the remainder sequence of \(P\) and \(Q\).

Introduction
We consider \(P\) of degree \(p\) and \(Q\) of degree \(q < p\) and their multisets of roots. In Section 1 we give a general definition of Sylvester double sums, valid also when there are multiple roots, and prove that it coincides with Sylvester’s definition in the special case where all roots are simple.
In Section 2 we consider generalized Vandermonde determinants (also called confluent Vandermonde determinants, see \cite{4}) and connect them with the Sylvester double sums.
In Section 3 we introduce an Hermite interpolation for multivariate symmetric polynomials and use it to obtain an expression of the Sylvester double sum in a special case.
Following a technique inspired by \cite{5}, we introduce in Section 4 multi Sylvester double sums. We study their properties, using the Hermite interpolation for
symmetric multivariate polynomials, and obtain results on Sylvester double sums as corollaries. In subsection 4.1 we compute the multi Sylvester double sums and Sylvester double sums for indices \((a, b)\) with \(a + b \geq q\). In subsection 4.2 we prove that Sylvester double sums indexed by \(a, b\), depend only (up to a constant) on \(j = a + b < p\). This was already known in the simple case but even in this case our proof is new.

In Section 5 we give a relationship between the Sylvester double sums of \(P, Q\) and those of \(Q, R\) where \(R\) is the opposite of the remainder of \(P\) by \(Q\) in the Euclidean division. Finally we prove in Section 6 that Sylvester double sums coincide (up to a constant) with subresultants, by an induction on the length of the remainder sequence of \(P\) and \(Q\).

1 Sylvester double sums

Let \(K\) be a field of characteristic 0. Let \(P = \{x_1, \ldots, x_p\}\) and \(Q = \{y_1, \ldots, y_q\}\) \((q < p)\) be two finite sets of elements of \(K\). We use the following notation: \(A \subset_a P\) (resp. \(B \subset_b Q\)) means that \(A\) is a subset of cardinality \(a\) of \(P\) (resp. \(B\) is a subset of cardinality \(b\) of \(Q\)), and we denote

\[
\Pi(A, B) = \prod_{a \in A} (a - b).
\]

Abusing notation we denote \(\Pi(a, B) = \Pi(\{a\}, B)\) and \(\Pi(A, b) = \Pi(A, \{b\})\).

Note that if \(P(X) = \Pi(X, P)\), \(Q(X) = \Pi(X, Q)\), \(\Pi(P, Q)\) is the classical resultant of the monic polynomials \(P\) and \(Q\).

The Sylvester double sums of \((P, Q)\) are usually defined, for \((a, b) \in \mathbb{N}^2\), as the following polynomial in \(K[U]\):

\[
\sum_{A \subset_a P, B \subset_b Q} \Pi(U, A)\Pi(U, B) \frac{\Pi(A, B)\Pi(P \setminus A, Q \setminus B)}{\Pi(A, P \setminus A)\Pi(B, Q \setminus B)} \quad (1)
\]

(see [8, 9]).

This definition of Sylvester double sums makes no sense in the presence of multiple roots for \(P\) and \(Q\), since \(P\) and \(Q\) are multisets and some of the \(\Pi(A, P \setminus A)\) (resp. \(\Pi(B, Q \setminus B)\)) are equal to 0.

In this section, we give a general definition of Sylvester double sums, valid even in the presence of multiple roots and prove that it coincides with the classical one when all the roots are simple.

**Definition 1.** The Vandermonde vector for the indeterminate \(U\), denoted \(v_i(U)\), is

\[
v_i(U) = \begin{bmatrix} 1 \\ U \\ \vdots \\ \vdots \\ U^{i-1} \end{bmatrix}.
\]
Let $\mathbf{T} = (T_1, \ldots, T_i)$ be an ordered list of elements of a field, and $V(\mathbf{T})$ be the Vandermonde determinant $V(\mathbf{T})$ of the Vandermonde matrix $V(\mathbf{T})$, i.e. the $i \times i$ matrix having as column vectors $v_1(T_1), \ldots, v_i(T_i)$. It is well known that

$$V(\mathbf{T}) = \prod_{i \geq k > j \geq 1} (T_k - T_j).$$

Let $X = (X_1, \ldots, X_p)$ and $Y = (Y_1, \ldots, Y_q)$ be two ordered sets of indeterminates.

If $B$ and $A$ are ordered sets, we denote $B \| A$ the ordered set obtained concatenating $B$ and $A$.

If $X'$ is a subset of $X$ of cardinality $a$ (ordered by the order induced by $X$), we write: $X' \subset_a X$. Similarly, if $Y'$ is a subset of $Y$ of cardinality $b$ (ordered by the order induced by $Y$), we write: $Y' \subset_b Y$. We denote $\sigma_X$, the signature of the permutation $s_X$, obtained by putting the elements of $X$ in the order $X \setminus X' \| X'$, and by $\sigma_Y$, the signature of the permutation $s_Y$, obtained by putting the elements of $Y$ in the order $Y \setminus Y' \| Y'$.

For any $(a, b) \in \mathbb{N}^2$, we define the polynomial $F^{a,b}(X, Y)(U)$ in $K[X, Y, U]$, where $U$ is one new indeterminate,

$$F^{a,b}(X, Y)(U) = \sum_{\substack{X' \subset_a X \\subset_b Y \| Y'}} \sigma_X \sigma_Y \ V(Y \setminus Y' \| X \setminus X') V(Y' \| X' \| U) \quad (3)$$

Note that if $a > p$ or $b > q$ then $F^{a,b}(X, Y)(U) = 0$.

**Proposition 2.** The polynomial $F^{a,b}(X, Y)(U)$ is antisymmetric in the variables $X$ and in the variables $Y$.

**Proof.** Let $\tau$ be the transposition exchanging $X_i$ and $X_{i+1}$. Given $X' \subset_a X$, we denote $X'' = \tau(X') \subset_a X$ (with the order induced by $X$). We denote by $\bar{X}$ the ordered list obtained from $X$ by exchanging $X_i$ and $X_{i+1}$ and by $\bar{X}'$ and $\bar{X}''$ the ordered lists having as elements $X', \bar{X}'$, in the order induced by $X$. We denote also by $\tau$ the action of the permutation $\tau$ on any polynomial in $K[X, Y, U]$.

We want to prove

$$\tau(F^{a,b}(X, Y)(U)) = -F^{a,b}(X, Y)(U). \quad (4)$$

Denote

$$F^{X', Y'} = \sigma_{X'} \sigma_{Y'} \ V(Y' \| X' \| U) V(Y \setminus Y' \| X \setminus X').$$

Let

- $C_1 = \{X' \subset_a X \mid X_i \in X', X_{i+1} \in X'\}$,
- $C_2 = \{X' \subset_a X \mid X_i \notin X', X_{i+1} \notin X'\}$,
- $C_3 = \{X' \subset_a X \mid X_i \in X', X_{i+1} \notin X'\}$,
- $C_4 = \{X' \subset_a X \mid X_i \notin X', X_{i+1} \in X'\}$.
We have 4 cases to consider.

- If $X' \in C_1$ then,
  \[ \tau(F^{X',Y}') = \sigma_{X'} \sigma_{Y'} V(Y'\|X') V(Y \setminus Y'\|X \setminus X') \]
  \[ = \sigma_{X'} \sigma_{Y'} V(Y'\|X') V(Y \setminus Y'\|X \setminus X') \]
  
  \[ = -F^{X',Y'} \]
  since $\bar{X} \setminus \bar{X}' = X \setminus X'$.

- If $X' \in C_2$, a similar proof implies that $\tau(F^{X',Y'}) = -F^{X',Y'}$.

- If $X' \in C_3$, then $X'' \in C_4$. Then, $\tau \circ s_{X'} = s_{X''}$, $\bar{X}' = X''$, $\bar{X} = X \setminus X''$, $\bar{X}'' = X'$ and $\bar{X}'' = X \setminus X'$, so that
  \[ \tau(F^{X',Y'}) = \sigma_{X'} \sigma_{Y'} V(Y'\|X'') V(Y \setminus Y'\|X \setminus X'') \]
  \[ = \sigma_{X'} \sigma_{Y'} V(Y'\|X'') V(Y \setminus Y'\|X \setminus X'') \]
  
  \[ = -F^{X',Y'} \]
  \[ \tau(F^{X'',Y'}) = \sigma_{X'} \sigma_{Y'} V(Y''\|X') V(Y'\|X' \setminus X'') \]
  \[ = \sigma_{X'} \sigma_{Y'} V(Y''\|X') V(Y'\|X' \setminus X'') \]
  
  \[ = -F^{X',Y'} \]
  From which we deduce
  \[ \tau(F^{X',Y'} + F^{X'',Y'}) = -\left(F^{X',Y'} + F^{X'',Y'}\right) \]
  So, since we have
  \[ F^{a,b}(X,Y)(U) = \sum_{X' \in C_1 \cup C_2 \atop Y' \in C_b \setminus Y} F^{X',Y'} + \sum_{X' \in C_2 \atop Y' \in C_b \setminus Y} \left(F^{X',Y'} + F^{X'',Y'}\right) \]
  we get (4).

- The proof is the same when $X' \in C_4$ and $X'' \in C_3$.

The exchange between two elements of $Y$ can be treated similarly. \qed

**Lemma 3.** If $A(X,Y)$ in $K[X,Y]$ is antisymmetric with respect to the variables $X$, then $A(X,Y) = S(X,Y)V(X)$ where $S \in K[X,Y]$ is symmetric with respect to the variables $X$.

**Proof.** If $A(X,Y)$ is antisymmetric with respect to $X$ then, for $j < k$, denote $\tau_{j,k}(X)$ the ordered set of variables obtained by transposing $X_j$ and $X_k$.

\[ \frac{A(X,Y) - A(\tau_{j,k}(X,Y))}{X_j - X_k} = \frac{2A(X,Y)}{X_j - X_k} \]

is a polynomial. So $A(X,Y) = S(X,Y)V(X)$, where $S(X,Y)$ is a symmetric polynomial with respect to $X$. \qed
From this lemma and proposition we denote $S^a,b(X, Y)(U)$ the symmetric polynomial with respect to the indeterminates $X$ and with respect to the indeterminates $Y$ satisfying

$$S^a,b(X, Y)(U) = \frac{F^a,b(X, Y)(U)}{V(X)V(Y)}.$$  \hspace{1cm} (5)

**Notation 4.** Let $P$ be a monic polynomial of degree $p$ with coefficients in a field $K$. Let $x_1, \ldots , x_k$ be an ordering of the distinct roots of $P$ in an algebraic closure $C$ of $K$, with $x_i$ of multiplicity $\mu_i$, and denote $P$ the ordered multiset

$$P = \{x_{1,0}, \ldots , x_{1,\mu_1-1}, \ldots , x_{k,0}, \ldots , x_{k,\mu_k-1}\},$$

with $x_{i,j} = (x_i, j)$, $x_i \neq x_i'$ for $i \neq i'$, $\sum_{i=1}^k \mu_i = p$.

Let $Q$ be a monic polynomial of degree $q$ with coefficients $K$. Let $y_1, \ldots , y_\ell$ be an ordering of the distinct roots of $Q$ in $C$ with $y_i$ of multiplicity $\nu_i$, for $i = 1, \ldots , \ell$. We denote $Q$ the ordered multiset of its root

$$Q = \{y_{1,0}, \ldots , y_{1,\nu_1-1}, \ldots , y_{\ell,0}, \ldots , y_{\ell,\nu_\ell-1}\},$$

with $y_{i,j} = (y_i, j)$, $y_i \neq y_i'$ for $i \neq i'$, $\sum_{i=1}^\ell \nu_i = q$.

We introduce a set of variables $X_P = \{X_{1,0}, \ldots , X_{1,\mu_1-1}, \ldots , X_{k,0}, \ldots , X_{k,\mu_k-1}\}$ and a set of variables $Y_Q = \{Y_{1,0}, \ldots , Y_{1,\nu_1-1}, \ldots , Y_{\ell,0}, \ldots , Y_{\ell,\nu_\ell-1}\}$. For a polynomial $f(X_P, X_Q)$ we denote $f(P, Q)$ the result of the substitution of $x_i$ by $x_{i,j}$ and $y_i$ by $y_{i,j}$.

**Definition 5.** The generalized Sylvester double sum of $(P, Q)$ for the exponents $a, b \in \mathbb{N} \times \mathbb{N}$ is defined by

$$\text{Sylv}^a,b(P, Q)(U) = S^a,b(P, Q)(U),$$

identifying variables $X$ to $X_P$ and variables $Y$ to $Y_Q$.

Note that this definition does not depend on the order given for the roots of the polynomial $P$ and of the polynomial $Q$.

This definition of generalized Sylvester double sums for monic polynomials coincides with the usual definition of Sylvester double sums when the polynomials $P$ and $Q$ are without multiple roots, as we see now in Proposition.

**Proposition 6.** If $P, Q$ have only simple roots,

$$\text{Sylv}^a,b(P, Q)(U) = \sum_{A \subseteq P \atop B \subseteq Q} \Pi(U, A)\Pi(U, B) \frac{\Pi(A, B)\Pi(P \setminus A, Q \setminus B)}{\Pi(A, P \setminus A)\Pi(B, Q \setminus B)}$$

We first recall the immediate following result.
Lemma 7.

\[ V(D\|C) = V(C)\Pi(C, D)V(D). \]  \hspace{1cm} (6)

and, as a special case,

\[ V(C\|U) = \Pi(U, C)V(C). \]

Proof of Proposition 6.

\[
\sum_{A \subset a, B \subset b} \Pi(U, A)\Pi(U, B) \frac{\Pi(A, B)\Pi(P \setminus A, Q \setminus B)}{\Pi(A, P \setminus A)\Pi(B, Q \setminus B)} =
\]

\[
= \sum_{A \subset a, B \subset b} \Pi(U, A)\Pi(U, B) \frac{V(B)\Pi(A, B)V(A)\Pi(Q \setminus B)\Pi(P \setminus A, Q \setminus B)V(P \| A)}{V(A)\Pi(A, P \setminus A)V(P \| A)V(P)\Pi(B, Q \setminus B)V(Q \setminus B)}
\]

\[
= \sum_{A \subset a, B \subset b} \sigma_A \sigma_B \frac{V(B)\|U)\Pi(Q \setminus B)\Pi(P \setminus A)}{V(P)\Pi(Q)}
\]

\[
= \frac{F_{a,b}(P, Q)(U)}{V(P)V(Q)} = S_{a,b}(P, Q)(U)
\]

\[
= \text{Sylv}_{a,b}(P, Q)(U)
\]

applying Lemma 7.

2  Sylvester double sums and generalized Vandermonde determinants

If \( f \) is a univariate polynomial in the variable \( U \), we denote

\[ f^{[i]}(U) = \frac{f^{(i)}(U)}{i!}. \]  \hspace{1cm} (7)

If \( f \) is a multivariate polynomial depending on the variable \( U \), we denote

\[ \frac{\partial^{[i]} f}{\partial U^j} = \frac{1}{i!} \frac{\partial^j f}{\partial U^i}. \]  \hspace{1cm} (8)

Let \( P \) be a monic polynomial of degree \( p \), and \( P \) be the ordered multiset of its roots. The notation \( A \subset a, P \) means that \( A \) is an ordered subset of \( P \) of cardinality \( a \) (ordered by the order induced by \( P \)).

Similarly, let \( Q \) be a polynomial of degree \( q \), and \( Q \) be the ordered multi set of its roots. The notation \( B \subset b, Q \) means that \( B \) is a subset of \( Q \) of cardinality \( b \) (ordered by the order induced by \( Q \)).

Note that a subset of a multiset is not in general a multiset since, in a multiset \( P \), if \( x_{i,j} \in P \), then, for every \( 0 \leq k \leq j \), \( x_{i,k} \in P \).

**Notation 8.** Let \( P \) be the ordered multiset of the roots of \( P \) and \( A \subset a, P \).
Lemma 10. The generalized Vandermonde determinant and satisfies the property

\[ F(U) = V[\mathbf{P} || U] \]

Proof. The proof is done by induction on \( p \).

If \( p = 1 \), \( V[\mathbf{P}] = 1 \).

Suppose that

\[ V[\mathbf{P}] = \prod_{1 \leq i < j \leq k} (x_j - x_i)^{\mu_i, \mu_j}. \]

The polynomial \( F(U) = V[\mathbf{P} || U] \) is of degree \( p \), with leading coefficient \( V[\mathbf{P}] \) and satisfies the property

\[ F^{(j)}(x_i) = 0, \quad \text{for all } 1 \leq i \leq k, \text{ for all } 0 \leq j < \mu_i, \]

and

\[ V[\mathbf{P}] = \prod_{1 \leq i < j \leq k} (x_j - x_i)^{\mu_i, \mu_j}. \]
So,

\[ F(U) = V(P) \prod_{i=1}^{k} (U - x_i)^{\mu_i} = \prod_{1 \leq i < j \leq k} (x_j - x_i)^{\mu_i \mu_j} \prod_{i=1}^{k} (U - x_i)^{\mu_i}. \]

Consider \( S(U) = (U - x)P(U) \).

- First case: \( x \) is not a root of \( P \). Let \( S \) the ordered set of roots of the polynomial \( S \) obtained by adding \( x \) at the end of \( P \), so that \( x = x_{k+1} \) with multiplicity 1. Then

\[ V[S] = F(x) = \prod_{1 \leq i < j \leq k} (x_j - x_i)^{\mu_i \mu_j} \prod_{i=1}^{k} (x - x_i)^{\mu_i} = \prod_{1 \leq i < j \leq k+1} (x_j - x_i)^{\mu_i \mu_j}. \]

- Second case: \( x \) is a root of \( P \). So there exists \( 1 \leq j \leq k \) such that \( x = x_j \). Let \( S \) the ordered set of roots of the polynomial \( S \) obtained by inserting \( x_j, \mu_j \) after \( x_j, \mu_j - 1 \) in \( P \), so that \( x_j \) is of multiplicity \( \mu_j + 1 \). Then

\[ V[S] = (-1)^{\mu_{j+1} + \cdots + \mu_k} F[\mu_j](x_j) \]
\[ = (-1)^{\mu_{j+1} + \cdots + \mu_k} V[P] \prod_{i=1 \atop i \neq j}^{k} (x_j - x_i)^{\mu_i} \]
\[ = \prod_{1 \leq i < j \leq k} (x_j - x_i)^{\mu_i (\mu_j + 1)} \quad \Box \]

If \( A \subset P \) is not a multiset, it can happen that \( V[A] = 0 \).

**Example 11.** Taking \( P = \{x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}\} \), \( V[\{x_{1,1}, x_{2,1}\}] = 0 \).

We denote \( \sigma_A \) the signature of the permutation \( s_A \) obtained by putting the elements of \( P \) in the order \( P \setminus A \| A \), and by \( \sigma_B \) the signature of the permutation \( s_B \) obtained by putting the elements of \( Q \) in the order \( Q \setminus B \| B \).

**Proposition 12.**

\[ \text{Sylv}^{a,b}(P,Q)(U) = \sum_{\substack{A \subset P \\text{B} \subset Q}} \sigma_A \sigma_B \frac{V[Q \setminus B \| P \setminus A] V[B \| A \| U]}{V[P] V[Q]} \]

Before proving Proposition 12, we prove the following lemma 13.

**Lemma 13.** If \( h(X_P) = V(X_P)f(X_P) \),

\[ \partial^P h(P) = V[P]f(P) \]
Proof. We first note that
\[ \partial^p V(X_P) = \partial^p V(X_P) f(X_P) + \sum_\ell V_{\ell}(X_P) f_{\ell}(X_P) \]
where \( V_{\ell}(X_P) \) (resp. \( f_{\ell}(X_P) \)) is obtained from \( V(X_P) \) (resp. from \( f(X_P) \)) by partial derivations, one variable \( X_{i,j} \) at least being derived less than \( j \) times (resp. more than \( j \) times). Denoting \( X_{i,j} \) the first variable which is being derived less than \( j \) times in \( V_{\ell}(X_P) \), we define \( j' \) as the order of derivation of \( X_{i,j} \) in \( V_{\ell}(X_P) \). We notice that \( V_{\ell}(X_P) \) is the determinant of a matrix with two equal columns, the one indexed by \( i, j' \) and the one indexed by \( i, j \). Hence \( V_{\ell}(X_P) = 0 \). This proves the claim. \( \square \)

Proof of Proposition 72. Since
\[ F^{a,b}(X_P, Y_Q)(U) = V(X_P)V(Y_Q)S^{a,b}(X_P, Y_Q)(U), \]
using Lemma 13 we obtain
\[ \partial^{|Q|} \partial^{|P|} F^{a,b}(P, Q)(U) = V[P]V[Q]S^{a,b}(P, Q)(U) = V[P]V[Q]Sylv^{a,b}(P, Q)(U). \]
On the other hand, denoting
\[ h_{A,B}(X_P, Y_Q)(U) = V(Y_Q\|X_P\|A)V(Y_B\|X_A\|U), \]
we have
\[ \partial^{|Q|} \partial^{|P|} h_{A,B}(P, Q)(U) = V[Q \| B \| P \| A]V[Q \| B \| A\|U]. \]
Since
\[ F^{a,b}(X_P, Y_Q)(U) = \sum_{A \subseteq P \atop B \subseteq Q} \sigma_A \sigma_B h_{A,B}, \]
we get
\[ \partial^{|Q|} \partial^{|P|} F^{a,b}(P, Q)(U) = \sum_{A \subseteq P \atop B \subseteq Q} \sigma_A \sigma_B V[Q \| B \| P \| A]V[Q \| B \| A\|U]. \square \]

3 Hermite Interpolation for multisymmetric polynomials and application to Sylvester sums

We consider an ordered set of variables \( U = \{ U_1, \ldots, U_{p-a} \} \). If \( A \subset_a P \) we denote as before \( V[A\|U] \) the determinant of the \( p \times p \) matrix \( V(A\|U) \) having as column vectors the \( a \) columns \( v_p^{[j]}(x_i) \) for \( x_{i,j} \in A \) followed by the \( p-a \) columns \( v_p(U_i) \) for \( U_i \in U \) (using notation (2) of definition 11). We remark that \( \partial^{|P\|A\|U]V[A'\|U] \) (identifying \( X_P\|A \) to \( U \)) is the determinant of the \( p \times p \) matrix having as column vectors the \( a \) columns \( v_p^{[j]}(x_i) \) for \( x_{i,j} \in A' \) followed by the \( p-a \) columns \( v_p^{[j]}(U_i) \) for \( U_i \in U \), for \( x_{i,j} \in P \setminus A \).
Proposition 14. The set
\[ B_{\mathcal{P}, a}(U) = \left\{ \frac{V[A \parallel U]}{V[\mathcal{P}]V(U)} \mid A \subset_a \mathcal{P} \right\} \]
is a basis of the vector-space of symmetric polynomials in $U$ of multidegree at most $a, \ldots, a$.

Remark 15. Proposition 14 is a generalization of a similar result in [3, 5] given for Lagrange interpolation of symmetric multivariate polynomials.

The proof of Proposition 14 uses the following Lemma.

Lemma 16.

1. \( V[A \parallel \mathcal{P} \setminus A] = (-1)^{a(p-a)} \sigma_A V[\mathcal{P}] \neq 0 \).

2. If \( A' \neq A \), \( V[A' \parallel \mathcal{P} \setminus A] = 0 \).

Proof.

1. It is clear that \( V[A \parallel \mathcal{P} \setminus A] = (-1)^{a(p-a)} \sigma_A V[\mathcal{P}] \neq 0 \), since \( \sigma_A \) is the signature of the permutation putting \( \mathcal{P} \) in the order \( \mathcal{P} \setminus A \parallel A \).

2. The fact that \( V[A' \parallel \mathcal{P} \setminus A] = 0 \) when \( A' \neq A \) follows from the fact that the matrix \( V[A' \parallel \mathcal{P} \setminus A] \) has two equal columns. \( \square \)

Proof of Proposition 14. Since the number of subsets of cardinality \( a \) of \( \mathcal{P} \) is \( \binom{p}{a} \) and that it is also the dimension of the vector space of symmetric polynomials in \( U \) of multidegree at most \( a, \ldots, a \), it is enough to prove that
\[ \sum_{A' \subset_a \mathcal{P}} c_{A'} \frac{V[A' \parallel U]}{V[\mathcal{P}]V(U)} = 0 \]
implies \( c_A = 0 \) for all \( A \subset_a \mathcal{P} \).

Let us fix \( A \subset_a \mathcal{P} \). Since
\[ \sum_{A' \subset_a \mathcal{P}} c_{A'} V[A' \parallel U] = 0, \]
it follows by derivation that
\[ \sum_{A' \subset_a \mathcal{P}} c_{A'} \sigma_A V[\mathcal{P} \setminus A \parallel A' \parallel U] = 0. \]

When replacing \( U \) by \( \mathcal{P} \setminus A \) we obtain
\[ \sum_{A' \subset_a \mathcal{P}} c_{A'} V[A' \parallel \mathcal{P} \setminus A] = 0. \]

Using Lemma 16 \( c_A = 0. \) \( \square \)
The following Proposition gives the connection between a polynomial and its coordinates in the basis $B_{P,a}(U)$.

**Proposition 17. (Multivariate symmetric Hermite Interpolation)** If

$$g(U) = \sum_{A \subseteq a} g_A \frac{V[A \parallel U]}{V[P \parallel V(U)}}$$

then

$$g_A = (-1)^{a(p-a)} \sigma_A h(P \setminus A)$$

with

$$h(X_P \setminus A) = \sigma^{[P \setminus A]}(V(X_{P \setminus A}) g(X_{P \setminus A})).$$

**Proof.** We have

$$\sum_{A \subseteq a} g_A V[A \parallel U] = V[P \parallel g(U) V(U)].$$

Derivating both sides by $\partial^{[P \setminus A]}$ and substituting $P \setminus A'$ for $U$ we get, using Lemma 16

$$g_A' \sigma_A' (-1)^{a(p-a)} V[P] = V[P \parallel h(P \setminus A'),$$

and finally

$$g_A' = (-1)^{a(p-a)} \sigma_A' h(P \setminus A'). \quad \square$$

**Remark 18.** Proposition [14] is a generalization of a similar result in [3, 5] given for Lagrange interpolation of symmetric multivariate polynomials.

As a corollary of Proposition [17] we recover the classical Hermite Interpolation

**Proposition 19. (Hermite Interpolation)** Given a sequence

$$q = q_{1,0}, \ldots, q_{1,\mu_1-1}, \ldots, q_{k,0}, \ldots, q_{k,\mu_k-1}$$

of $p$ numbers, there is one and only one polynomial of degree at most $p - 1$ satisfying the property

for all $1 \leq i \leq k$, for all $0 \leq j < \mu_i$, $Q^{[j]}(x_i) = q_{i,j}$.

**Proof.** If $a = p - 1$ in Proposition [14] then

$$B_{P,p-1}(U) = \left\{ \frac{V[P \setminus \{x_{i,j}\} \parallel U]}{V[P]} \mid x_{i,j} \in P \right\}$$

is a basis of the vector space of univariate polynomials in $U$ of degree at most $p - 1$.

Note that $(-1)^{p-1} \sigma_{P \setminus \{x_{i,j}\}} = (-1)^{\mu_i+\ldots+\mu_k-j-1}$. So, the family

$$\{( -1)^{\mu_i+\ldots+\mu_k-j-1} q_{i,j} \mid i = 1, \ldots, k, j = 0, \ldots, \mu_i - 1\}$$

is the coordinates in the basis $B_{P,p-1}(U)$ of a polynomial $Q(U)$ (necessarily unique) of degree at most $p - 1$ such that $Q^{[j]}(x_i) = q_{i,j}$, applying Proposition [14]. \square
We have the following proposition

**Proposition 20.**

1. Defining
   \[ f(Y_Q \setminus B) = (-1)^{p(b-Q)} \partial^{Q \setminus B}(V(Y_Q \setminus B) \prod_{y_{i,j} \in Q \setminus B} P(Y_{i,j})) , \]
   we have
   \[ V[Q]Sylv^{0,j}(P, Q)(U) = \sum_{B \subset j} \sigma_B f(Q \setminus B) V[B \parallel U] . \]

2. Defining
   \[ g(X_P \setminus A) = \partial^{P \setminus A}(V(X_P \setminus A) \prod_{x_{i,j} \in P \setminus A} Q(X_{i,j})) , \]
   we have
   \[ V[P]Sylv^{j,0}(P, Q)(U) = \sum_{A \subset j} \sigma_A g(P \setminus A) V[A \parallel U] . \]

The proposition is a direct consequence of the following lemma.

**Lemma 21.**

1. For \( B \subset Q \), we have
   \[ V[Q \setminus B \parallel P] = V[P] f(Q \setminus B) . \]

2. For \( A \subset P \), we have
   \[ V[P \parallel Q \setminus A] = V[Q] g(P \setminus A) . \]

**Proof.** Defining
\[ h(X_P, Y_Q \setminus B) = \partial^{Q \setminus B}(V(Y_Q \setminus B \parallel X_P)) = \partial^{Q \setminus B}(V(X_P) \prod(X_P, Y_Q \setminus B)V(Y_Q \setminus B)) = V(X_P) \partial^{Q \setminus B}(V(Y_Q \setminus B)) \prod(X_P, Y_Q \setminus B)) \]
and applying Lemma 13, we get
\[ \partial^P h(P, Y_Q \setminus B) = V[P] \partial^{Q \setminus B}(V(Y_Q \setminus B) \prod (P, Y_Q \setminus B)) = V[P] \partial^{Q \setminus B}(V(Y_Q \setminus B) \prod_{y_{i,j} \in Q \setminus B} (-1)^p Y_{i,j}) = V[P] f(Y_Q \setminus B) . \]

and finally
\[ V[Q \setminus B \parallel P] = \partial^P h(P, Q \setminus B) = V[P] f(Q \setminus B) . \]

Which is Lemma 21.1. The proof for Lemma 21.2 is similar. \( \square \)
4 Multi Sylvester double sums

The idea of replacing the variable $U$ by a block of indeterminates is directly inspired from [5].

**Definition 22.** The multi Sylvester double sum, for $(a, b)$ a pair of natural numbers with $a + b = j$, is the polynomial $\text{MSylv}^{a,b}(P, Q)(U)$, where $U$ is a block of indeterminates of cardinality $p - j$,

$$
\text{MSylv}^{a,b}(P, Q)(U) = \sum_{A \subseteq P, B \subseteq Q} \sigma_A \sigma_B V[Q \parallel P \setminus A] V[B \parallel A \parallel U] / V[P] V[Q] V(U) \quad (9)
$$

In particular

$$
\text{MSylv}^{j,0}(P, Q)(U) = \sum_{A \subseteq P} \sigma_A V[P \parallel A] V[A \parallel U] / V[Q] V(P) V(U) \quad (10)
$$

The following proposition gives the relation between multi Sylvester double sums and Sylvester double sums.

**Proposition 23.** Denoting $U = U \parallel U'$ with $U'$ a block of $p - j - 1$ indeterminates;

$\text{Sylv}^{a,b}(P, Q)(U)$ is the coefficient of $\prod_{U' \in U'} U'^j$ in $\text{MSylv}^{a,b}(P, Q)(U)$.

It is based on the following Lemma.

**Lemma 24.** $V[A \parallel U] / V(U)$ is the coefficient of $\prod_{U' \in U'} U'^{a}$ in $V[A \parallel U'] / V(U')$.

**Proof.**

$$
\frac{V[A \parallel U] V(U')}{V(U') V(U)} = \frac{\partial^{|A|} V(X_A \parallel U \parallel U')}{V(U') V(U)}
$$

$$
= \frac{\partial^{|A|} (V(X_A \parallel U) \Pi(U', X_A) \Pi(U', U) V(U'))}{V(U) \Pi(U', U) V(U')}
$$

$$
= \frac{\partial^{|A|} (V(X_A \parallel U) \Pi(U', X_A))}{V(U)}
$$

Noting that

$$
\partial^{|A|} (V(X_A \parallel U) \Pi(U', X_A)) = 
\frac{(\partial^{|A|} V(X_A \parallel U)) \times \Pi(U', X_A) + \sum_{i} V_i(X_A, U) \Pi_i(U', X_A)}{V(U)}
$$

where each $\Pi_i(U', X_A)$ is obtained by partial derivation of $\Pi(U', X_A)$ with respect to at least one variable in $X_A$, it is clear that the degree of some $U' \in U'$ in $\Pi_i(U', X_A)$ is less than $a$. The claim follows, substituting $A$ to $X_A$. 

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Proof of Proposition 24. The coefficient of $\prod_{U^j \in U^j} V[B\|A]U$ in $V[B\|A]U$ is $V[B\|A]U$ by Lemma 24. The coefficient of $\prod_{U^j \in U^j} U$ in $\text{MSylv}^{a,b}(P,Q)(U)$ is $\sum_{A \subset P} \sigma \sigma_B V[Q \| B \| P \| A] \prod_{x_i,j \in X_{P\backslash A}} Q(x_{i,j})$ by Proposition 12.

4.1 Computation of (multi) Sylvester double sums for $j \geq q$

Proposition 25. If $q \leq j < p$

$$\text{MSylv}^{j,0}(P,Q)(U) = (-1)^{j(p-j)} \prod_{i=1}^{p-j} Q(U_i)$$

Proof. The polynomial $\prod_{i=1}^{p-j} Q(U_i)$ is a symmetric polynomial in $U$ of multidegree $q, \ldots, q$, so at most $j, \ldots, j$. Its coordinates in the basis $B_{P,j}(U)$ are, for $A \subset j P$, $(-1)^{j(p-j)} \sigma_A g(P \backslash A)$ where

$$g(X_{P\backslash A}) = \delta^{[P\|A]} \left( V(X_{P\backslash A}) \prod_{x_i,j \in X_{P\backslash A}} Q(x_{i,j}) \right)$$

by Proposition 17, and moreover

$$g(P \backslash A) = \frac{V[Q\|P\|A]}{V[Q]}$$

by Lemma 21.2.

So, the polynomials $\text{MSylv}^{j,0}(P,Q)(U)$ and $(-1)^{j(p-j)} \prod_{i=1}^{p-j} Q(U_i)$ have the same coordinates in the basis $B_{P,a}(U)$ and are equal.

As a corollary

Proposition 26.

1. $\text{Sylv}^{p-1,0}(P,Q)(U) = (-1)^{p-1} Q(U)$
2. For any $q < j < p-1$, $\text{Sylv}^{j,0}(P,Q)(U) = 0$
3. $\text{Sylv}^{q,0}(P,Q)(U) = (-1)^{q(p-q)} Q$

Proof.
1. For \( j = p - 1 \) Proposition 25 is exactly
\[
\text{Sylv}^{p-1,0}(P, Q)(U) = (-1)^{p-1}Q(U).
\]

2. If \( q < j < p - 1 \), the coefficient of \( U_j^2 \ldots U_{p-j} \) in \( \prod_{i=1}^{p-j} Q(U_i) \) is equal to 0 so, applying Proposition 25, \( \text{Sylv}^j,0(P, Q)(U) = 0 \).

3. From Proposition 25 and Proposition 23, we know that \( \text{Sylv}^q,0(P, Q)(U) \) is equal to the coefficient of \( U_q^2 \ldots U_{p-q} \) in \( (-1)^{q(p-a)}Q(U) \prod_{i=q+1}^{p-a} Q(U_i) \).

This coefficient is exactly \( (-1)^{a(p-a)}Q(U) \).

**Proposition 27.** Let \( P \) and \( Q \) be two monic polynomials; if \( b \leq q \leq a + b = j < p \) then
\[
\text{MSylv}^{a,b}(P, Q, U) = (-1)^{b(p-j)} \binom{q}{b} \text{MSylv}^j,0(P, Q, U)
\]

**Proof.** Let \( B \subset Q \) and \( U' = (U'_1, \ldots, U'_{p-a}) \); the polynomial \( \frac{V(Q \setminus B || U')}{V(U')} \) is a symmetric polynomial in the indeterminates \( U' \) of degree at most \( q - b \leq a \) in each indeterminate \( U'_i, 1 \leq i \leq p - a \). So, we can write this polynomial in the basis \( B_{p,a}(U') \)
\[
\frac{V(Q \setminus B || U')}{V(U')} = \sum_{A \subset a P} g_A \frac{V[A || U']}{V[P] V(U')}
\]
where, by Proposition 17
\[
g_A = (-1)^{a(p-a)} \sigma_A V(Q \setminus B \setminus P \setminus A).
\]

We deduce from this
\[
V(Q \setminus B || U') = \sum_{A \subset a P} (-1)^{a(p-a)} \sigma_A V(Q \setminus B \setminus P \setminus A) \frac{V[A || U']}{V[P] V(U')}
\]
We replace \( U' \) by \( U' = Y_B || U \), where \( U \) is a set of \( p - j \) indeterminates, derivate with respect to \( \partial^{[B]} \) and replace \( Y_B \) by \( B \); we obtain
\[
V(Q \setminus B || B || U) = \sum_{A \subset a P} (-1)^{a(p-a)} \sigma_A V(Q \setminus B \setminus P \setminus A) \frac{V[B || A || U]}{V[P] V(U')}
\]
\[
= (-1)^{ab} (-1)^{a(p-a)} \sum_{A \subset a P} \sigma_A V(Q \setminus B \setminus P \setminus A) \frac{V[B || A || U]}{V[P] V(U')}
\]
As
\[
V(Q \setminus B || B || U) = \sigma_B V(Q || U),
\]
we have

\[ V(Q\|U) = \sum_{A \subseteq a} (-1)^{a(p-j)} \sigma_A \sigma_B(V|Q\setminus B|P\setminus A)V[B\|A\|U] \]

and

\[ \frac{V(Q\|U)}{V(U)} = \sum_{A \subseteq a} (-1)^{a(p-j)} \sigma_A \sigma_B(V|Q\setminus B|P\setminus A)V[B\|A\|U] \frac{V[P]}{V(U)}. \]

The polynomial \( \frac{V(Q\|U)}{V(U)} \) is a symmetric polynomial in the indeterminates \( U \) of degree at most \( q \leq j \) in each indeterminate \( U_i, 1 \leq i \leq p - j \). So, we can write it in the basis \( B_{P,j}(U') \)

\[ \frac{V(Q\|U)}{V(U)} = \sum_{D \subseteq j} g_D \frac{V[D\|U]}{V[P]V(U)} \]

where by Proposition 17

\[ g_D = (-1)^{j(p-j)} \sigma_D V[P\setminus D] \]

So

\[ \sum_{D \subseteq j} \sigma_D V(Q\|P\setminus D)V[D\|U] = \sum_{A \subseteq a} (-1)^{b(p-j)} \sigma_A \sigma_B V(Q\setminus B\setminus P\setminus A)V[B\|A\|U]. \]

It follows

\[ \sum_{B \subseteq i} \sum_{D \subseteq j} \sigma_D V(Q\|P\setminus D)V[D\|U] \]

\[ = (-1)^{b(p-j)} \sum_{B \subseteq i} \sum_{A \subseteq a} \sigma_A \sigma_B V(Q\setminus B\setminus P\setminus A)V[B\|A\|U] \]

\[ \left( \begin{array}{c} q \\ b \end{array} \right) \sum_{D \subseteq j} \sigma_D V(Q\|P\setminus D)V[D\|U] \]

\[ = (-1)^{b(p-j)} \sum_{B \subseteq i} \sum_{A \subseteq a} \sigma_A \sigma_B V(Q\setminus B\setminus P\setminus A)V[B\|A\|U] \]

\[ \left( \begin{array}{c} q \\ b \end{array} \right) \sum_{D \subseteq j} \sigma_D V(Q\|P\setminus D)V[D\|U] \]

\[ = (-1)^{b(p-j)} \sum_{B \subseteq i} \sum_{A \subseteq a} \sigma_A \sigma_B V(Q\setminus B\setminus P\setminus A)V[B\|A\|U] \]

\[ \left( \begin{array}{c} q \\ b \end{array} \right) \sum_{D \subseteq j} \sigma_D V(Q\|P\setminus D)V[D\|U] \]

\[ = (-1)^{b(p-j)} \sum_{B \subseteq i} \sum_{A \subseteq a} \sigma_A \sigma_B V(Q\setminus B\setminus P\setminus A)V[B\|A\|U] \]

and

\[ \text{MSylv}^{a,b}(P, Q, U) = (-1)^{b(p-j)} \left( \begin{array}{c} q \\ b \end{array} \right) \text{MSylv}^{j,0}(P, Q, U). \]
Corollary 28. For \( q \leq j < p \),

\[
\text{Sylv}^{a,b}(P, Q) = (-1)^{b(p-j)} \binom{q}{b} \text{Sylv}^{j,0}(P, Q)
\]

Proof. Immediate using Proposition 27 and Proposition 23.

Proposition 29.

1. For any \((a, b)\) with \( q = a + b \),

\[
\text{Sylv}^{a,b}(P, Q)(U) = (-1)^{a(p-q)} \binom{q}{a} Q
\]

2. For any \((a, b)\) with \( b \leq q, j = a + b \) with \( q < j < p - 1 \),

\[
\text{Sylv}^{a,b}(P, Q)(U) = 0
\]

3. For any \((a, b)\) with \( b \leq q, a + b = p - 1 \),

\[
\text{Sylv}^{a,b}(P, Q)(U) = (-1)^{a} \binom{q}{b} Q(U)
\]

Proof. Follows from Corollary 28 and Proposition 26.

4.2 Fundamental property of Sylvester double sums

This section is essentially devoted to the proof of Theorem 30, which is a fundamental property of Sylvester double sums: up to a constant Sylvester double sums \( \text{Sylv}^{a,j-a}(P, Q) \) depend only on \( j \). Such a result has been already given for \( q \leq j < p \) by Corollary 28.

Theorem 30. If \( a \in \mathbb{N}, b \in \mathbb{N}, a + b = j < q, b \leq q < p \),

\[
\text{Sylv}^{a,b}(P, Q)(U) = (-1)^{b(p-j)} \binom{j}{b} \text{Sylv}^{j,0}(P, Q)(U).
\]

We, in fact, prove Theorem 30 as a corollary of a multivariate version (Theorem 31). The proof of Theorem 31 uses in an essential way the Exchange Lemma coming from [5].

Theorem 31. If \( a \in \mathbb{N}, b \in \mathbb{N}, a + b = j < q < p \), and \( U \) a set of \( p - j \) indeterminates,

\[
\text{MSylv}^{a,b}(P, Q)(U) = (-1)^{b(p-j)} \binom{j}{b} \text{MSylv}^{j,0}(P, Q)(U)
\]

To prove Theorem 31 for \( j < q \), we need a lemma
Lemma 32. Let $A \subset_a P, B \subset_b Q$ and $U = U_1, \ldots, U_l$ an ordered set of variables. Then $V[B\|A]$ is the coefficient of the leading monomial
\[
\prod_{i=1}^{l} \ell_{i}^{a+b+\ell-i}
\]
of $V[B\|A\|U]$ with respect to the lexicographical ordering.

Proof.
\[
V[B\|A\|U] = \partial^{[A]} \partial^{[B]} (V(Y_B\|X_A\|U))(A, B, U) = V(U) \partial^{[A]} \partial^{[B]} (V(Y_B\|X_A\|U))(A, B)\]
The coefficient of $\prod_{i=1}^{l} \ell_{i}^{a+b+\ell-i}$ in $V[B\|A\|U]$ is the coefficient of $\prod_{i=1}^{l} \ell_{i}^{a+b}$ in $\partial^{[A]} \partial^{[B]} (V(Y_B\|X_A\|U))(A, B, U)$. This coefficient is
\[
\partial^{[A]} \partial^{[B]} V(Y_B\|X_A\|U)(A, B) = V[B\|A];
\]
indeed if any derivation is done on $\Pi(U, X_A\|Y_B)$, with respect to $A$ or $B$, the degree in at least one indeterminate $U_i \in U$ decreases strictly.

Proof of Theorem 31. We have $j < q$. Let $U'$ be a block of $p-b$ indeterminates. On one hand,
\[
\sum_{C \subset_a P} \frac{\Pi(X_P \setminus X_C, Y_Q) \Pi(U', X_C)}{\Pi(X_P \setminus X_C, X_C)} = (-1)^{b(p-b)} \sum_{C \subset_a P} \frac{V(Y_Q\|X_P \setminus X_C)V(X_C\|U')}{V(Y_Q)V(U')V(X_C\|X_P \setminus X_C)}
\]
On the other hand,
\[
\sum_{B \subset_a Q} \frac{\Pi(X_P, Y_Q \setminus Y_B) \Pi(U', Y_B)}{\Pi(Y_B, Y_Q \setminus Y_B)} = \sum_{B \subset_a Q} \frac{V(Y_Q\|Y_B\|X_P)V(Y_B\|U')}{V(X_P)V(U')V(Y_Q\|Y_B)}
\]
From the Exchange Lemma in [5], we can write
\[
\sum_{C \subset_a P} \frac{\Pi(X_P \setminus X_C, Y_Q) \Pi(U', X_C)}{\Pi(X_P \setminus X_C, X_C)} = \sum_{B \subset_a Q} \frac{\Pi(X_P, Y_Q \setminus Y_B) \Pi(U', Y_B)}{\Pi(Y_B, Y_Q \setminus Y_B)}
\]
(11)
So, we deduce from (11)

\[
\sum_{C \subset P} \sigma_C V(Y_Q \| X_P \setminus X_C)V(X_C \| U') = \\
(-1)^{b(p-b)} \sum_{B \subset Q} \sigma_B V(Y_Q \setminus Y_B \| X_P)V(Y_B \| U') V(X_P \| U') V(Y_Q) 
\]  

(12)

\[
\sum_{C \subset P} \sigma_C V(Y_Q \| X_P \setminus X_C)V(X_C \| U') = \\
(-1)^{b(p-b)} \sum_{Y_B \subset Y_Q} \sigma_B V(Y_Q \setminus Y_B \| X_P)V(Y_B \| U') V(X_P \| U') V(Y_Q) 
\]  

(13)

Hence, derivating with respect to \(Q\) and substituting \(Q\) to \(X_Q\),

\[
\sum_{C \subset P} \sigma_C V(Q \| X_P \setminus X_C)V(X_C \| U') = \\
(-1)^{b(p-b)} \sum_{B \subset Q} \sigma_B V(Q \setminus B \| X_P)V(B \| U') \]  

(14)

We fixe \(A \subset P\). The total degree with respect to \(X_A\) of \(V(Q \| B \| X_P \| V(B \| U')\)

is

\[
d_1 = \left( \begin{array}{c}
a \\
2 \\
\end{array} \right) + a(p + q - j). \]

Denoting, for any \(C \subset P\), \(c\) the cardinality of \(A \cap C\), we note that the cardinality of \((P \setminus C) \cap A\) is \(a - c\).

So, the total degree with respect to \(X_A\) of \(V(Q \| X_P \setminus X_C)V(X_C \| U')\) is

\[
d_{2,c} = \left( \begin{array}{c}
a - c \\
2 \\
\end{array} \right) + (a - c)(p + q - j + c) + \left( \begin{array}{c}
c \\
2 \\
\end{array} \right) + c(p - c), \]

i.e.

\[
d_{2,c} = \left( \begin{array}{c}
a \\
2 \\
\end{array} \right) + a(p + q - j) - c(q - j + 2c) \]

and

\[
d_1 - d_{2,c} = c(a - c + q - b) \]

So \(d_{2,c} < d_1\) if \(c > 0\) and \(d_{2,c} = d_1\) if \(c = 0\) i.e. if \(C \subset P \setminus A\). This implies that subsets \(C\) which intersect \(A\) don’t contribute to the homogeneous part of total degree \(d_1\) in \(X_A\) on the left side of (14).

Note that, if \(C \subset P \setminus A\),

\[
V(Q \| X_P \setminus X_C) = \rho_{A,C} V(Q \| X_P \setminus (X_A \cup X_C) \| X_A),
\]

where \(\rho_{A,C}\) is the signature of the permutation \(r_{A,C}\) taking the ordered set \(P \setminus C\) to the ordered set \(P \setminus (A \cup C)\).
We can also write
\[ V[Q \setminus B||X_P] = \sigma_A V[Q \setminus B||X_P \setminus X_A||X_A] \]

If \( X_A = X_{i_1}, \ldots, X_{i_n} \), taking the coefficient of \( \prod_{k=1}^n X_{i_k}^{a-k+p+q-j} \) in both sides of (13) gives, by Lemma 32
\[
\sum_{C \subseteq i P \setminus A} \rho_{A,C} \sigma_C V[Q||B||X_P \setminus (X_A \cup X_C)] V[X_C ||U'] = \\
(-1)^{b(p-b)} \sum_{B \subseteq C Q} \sigma_A \sigma_B V[Q \setminus B||B \setminus P \setminus A||B \setminus U'] V[|C||U'] . \tag{15}
\]

Derivating both sides of (15) with respect to \( \partial[P] \) and replacing \( X_P \) by \( P \), followed by replacing \( U' \) by \( X_A||U \), where \( U \) is a set of \( p-j \) indeterminates, derivating with respect to \( \partial[A] \) and replacing \( X_A \) by \( A \) gives
\[
\sum_{C \subseteq i P \setminus A} \rho_{A,C} \sigma_C V[Q||P \setminus (A \cup C)] V[C||A||U] = \\
(-1)^{b(p-b)} \sum_{B \subseteq C Q} \sigma_A \sigma_B V[Q \setminus B||P \setminus A||B||A||U] . \tag{16}
\]

Summing with respect to \( A \), we get
\[
\sum_{A \subseteq P} \sum_{C \subseteq k P \setminus A} \rho_{A,C} \sigma_C V[Q||P \setminus (A \cup C)] V[C||A||U] = (-1)^{b(p-b)} \text{Sylv}^{a,b}(P, Q)(U).
\]

Denote \( D \) the set \( A \cup C \) ordered by the induced order on \( P \). Let \( t_{A,C} \) be the permutation sending the ordered set \( P \setminus D||D \) to the ordered set \( P \setminus D||C||A \) and \( \tau_{A,C} \) its signature. We deduce
\[
\sum_{A \subseteq P} \sum_{C \subseteq k P \setminus A} \tau_{A,C} \rho_{A,C} \sigma_C V[Q||P \setminus D] V[D||U] = (-1)^{b(p-b)} \text{Sylv}^{a,b}(P, Q)(U).
\]

We remark that \( \tau_{A,C} \rho_{A,C} \sigma_C = (-1)^{ab} \sigma_D \). Indeed, denoting \( i_{A,C} \) the permutation sending the ordered set \( P \setminus D||C||A \) to the ordered set \( P \setminus D||A||C \), with signature \((-1)^{ab}\), and by \( i'_{A,C} \) the permutation sending the ordered set \( (P \setminus C)||C \) to the ordered set \( (P \setminus A)||A||C \), with signature \( \rho_{A,C} \), we have the following sequence of permutations

\[
\begin{align*}
& s_D^{-1} : & P & \leftrightarrow & P \setminus D||D \\
& t_{A,C} : & P \setminus D||D & \leftrightarrow & P \setminus D||C||A \\
& i_{A,C} : & P \setminus D||C||A & \leftrightarrow & P \setminus D||A||C \\
& r_{A,C}^{-1} : & P \setminus D||A||C & \leftrightarrow & P \setminus C||C \\
& s_C : & P \setminus C||C & \leftrightarrow & P
\end{align*}
\]

with \( s_C \circ r_{A,C}^{-1} \circ i_{A,C} \circ t_{A,C} \circ s_D^{-1} = \text{Id} \).
Noting that there are \( \binom{j}{b} \) ways of decomposing \( D \subset_j P \) as \( D = A \cup C \), we get
\[
\text{MSylv}^{a,b}(P, Q)(U) = (-1)^{b(p-j)} \binom{j}{b} \text{MSylv}^{j,0}(P, Q)(U).
\]

**Proof of Theorem 30.** Theorem 30 is an immediate consequence of Theorem 31, by applying Corollary 28.

### 5 Sylvester double sums and remainders

We are now dealing with not necessarily monic polynomials.

**Definition 33.** Let \( P \) be a polynomial of degree \( p \) which leading coefficient is denoted \( \text{lc}(P) \). Let \( Q \) be a polynomial of degree \( q \) which leading coefficient is denoted \( \text{lc}(Q) \).

Let \((a, b)\) be a pair of natural numbers. We define
\[
\text{Sylv}^{a,b}(P, Q)(U) = \text{lc}(P) \frac{q-j}{\text{lc}(Q)} \text{Sylv}^{a,b}(P, Q)(U).
\]

**Remark 34.** Note that if \( a \in \mathbb{N}, b \in \mathbb{N}, b \le q, a + b = j < q \)
\[
\text{Sylv}^{a,b}(P, Q)(U) = (-1)^{b(p-j)} \binom{j}{b} \text{Sylv}^{j,0}(P, Q)(U).
\]

follows immediately from Theorem 30 and Definition 33.

**Proposition 35.** If \( j \in \mathbb{N}, j < q \)
- If \( R = 0 \), \( \text{Sylv}^{j,0}(P, Q)(U) = 0 \).
- If \( R \neq 0 \), \( \text{Sylv}^{j,0}(P, Q)(U) = (-1)^{q(p-q)} \text{lc}(Q) \text{Sylv}^{j,0}(Q, R)(U) \)

The following elementary lemma plays a key role in the proof of Proposition 35.

**Lemma 36.** Write: \( P = CQ - R \) For every \( y_{i,j} \in Q \), \( 0 \le j' < j \),
\[
P^{[j']}_{[y]}(y_i) = -R^{[j']}_{[y]}(y_i).
\]

**Proof of Proposition 35** If \( j \le q \), define for \( B \subset_j A \)
\[
f(Y_{Q \setminus B}) = (-1)^{p(q-j)} \prod_{Y \in Y_{Q \setminus B}} (V(Y_{Q \setminus B}) P(Y)) \]
\[
g(Y_{Q \setminus B}) = (-1)^{(p+1)(q-j)} \prod_{Y \in Y_{Q \setminus B}} (V(Y_{Q \setminus B}) R(Y))
\]

We have, by Lemma 36 \( f(Q \setminus B) = g(Q \setminus B) = (-1)^{(p+1)(q-j)} h(Q \setminus B) \)
\[
V[Q]\text{Sylv}^{0,j}(P,Q)(U) = \text{lc}(Q)^{p-j} \sum_{B \subseteq j} \sigma_B V[B\parallel U]f(Q \setminus B), \text{ using Lemma 21.1}
\]
\[
= (-1)^{(p+1)(q-j)}\text{lc}(Q)^{p-j} \sum_{B \subseteq j} \sigma_B V[B\parallel U]g(Q \setminus B)
\]

If \( R = 0 \), \( \text{Sylv}^{0,j}(P,Q)(U) = 0 \).

If \( R \neq 0 \),

\[
\text{Sylv}^{0,j}(P,Q)(U) = (-1)^{(p+1)(q-j)}\frac{\text{lc}(Q)^{p-j}}{\text{lc}(Q)^{p-j}} \text{Sylv}^{j,0}(Q,R)(U), \text{ by Lemma 21.2}
\]
\[
= (-1)^{(p+1)(q-j)}\text{lc}(Q)^{p-j} \text{Sylv}^{j,0}(Q,R)(U)
\]

The claim follows since

\[
\text{Sylv}^{j,0}(P,Q)(U) = (-1)^{(p-j)} \text{Sylv}^{0,j}(P,Q)(U)
\]

by Theorem 30 and

\[
(-1)^{(p-j)}(-1)^{(p+1)(q-j)} = (-1)^{q(p-q)}.
\]

\[\square\]

6 Sylvester double sums and subresultants

This section is devoted to the proof of the link between double sums and subresultants which is known in the simple case (see [6, 1, 7]).

Notation. \( \varepsilon_k = (-1)^{k(k-1)/2} \). The sign \( \varepsilon_k \) is the signature of the permutation reversing the order i.e. sending 1, 2, \ldots, \( k-1 \), \( k \) to \( k \), \( k-1 \), \ldots, 2, 1. We have also \( \varepsilon_k = 1 \) if \( k \equiv 0, 1 \) mod 4, \( \varepsilon_k = -1 \) if \( k \equiv 2, 3 \) mod 4.

The main theorem of this section is the following.

**Theorem 37.** Let \( a \in \mathbb{N} \), \( b \in \mathbb{N} \), \( b \leq q \), \( a + b = j < p - 1 \)

\[
\text{Sylv}^{a,b}(P,Q)(U) = (-1)^a(p-j)\varepsilon_{p-j} \begin{pmatrix} j \\ a \end{pmatrix} \text{Sres}_j(P,Q)(U).
\]

**Remark 38.** When \( j = p - 1 \), \( \text{Sres}_{p-1}(P,Q)(U) = Q(U) \) by convention; so, as \( \text{Sylv}^{a,b}(P,Q)(U) = (-1)^a \begin{pmatrix} q \\ b \end{pmatrix} Q(U) \) for \( a + b = p - 1 \), we get

\[
\text{Sylv}^{a,b}(P,Q)(U) = (-1)^a \begin{pmatrix} q \\ b \end{pmatrix} \text{Sres}_{p-1}(P,Q)(U)
\]

In order to prove Theorem 37, we use an induction based on Proposition 35.

Before proving theorem 37, we recall the following properties of subresultants.
Lemma 39. Let \( R = -\text{Rem}(P, Q) \).

1. \( q < j < p - 1 \) \( \text{Sres}_j(P, Q)(U) = 0 \)
2. \( j = q \) \( \text{Sres}_q(P, Q)(U) = \varepsilon_{p-q}\text{lc}(Q)^{p-q-1}Q(U) \)
3. \( j = q - 1 \) \( \text{Sres}_{q-1}(P, Q)(U) = \varepsilon_{p-q}\text{lc}(Q)^{p-q+1}R(U) \)
4. \( j < q - 1, R \neq 0 \) \( \text{Sres}_j(P, Q)(U) = \varepsilon_{p-q}\text{lc}(Q)^{p-q}\text{Sres}_j(Q, R)(U) \)
5. \( j < q - 1, R = 0 \) \( \text{Sres}_j(P, Q)(U) = 0 \)

Proof. All items follow from Proposition 24 except for the computation of \( \text{Sres}_{q-1}(P, Q)(U) \) which is clearly equal to \( \varepsilon_{p-q+2}\text{lc}(Q)^{p-q+1}(-R(U)) \) by replacing the row of \( P \) by a row of \( -R \) in the Sylvester-Habicht matrix, and reversing the order of its \( p - q + 2 \) rows. Notice now that \( \varepsilon_{p-q+2} = -\varepsilon_{p-q} \).

Lemma 40. Let \( R = -\text{Rem}(P, Q) \). Let \( j \in \mathbb{N}, j < p - 1 \)

1. \( q < j < p - 1 \) \( \text{Sylv}^{j,0}(P, Q)(U) = 0 \),
2. \( j = q \) \( \text{Sylv}^{q,0}(P, Q)(U) = (-1)^{q(p-q)}\text{lc}(Q)^{p-q-1}Q(U) \)
3. \( j = q - 1 \) \( \text{Sylv}^{q-1,0}(P, Q)(U) = (-1)^{(q-1)(p-q+1)+p-q}\text{lc}(Q)^{p-q+1}R(U) \)
4. \( j < q - 1, R \neq 0 \) \( \text{Sylv}^{j,0}(P, Q)(U) = (-1)^{(p-q)}\text{lc}(Q)^{p-q}\text{Sylv}^{j,0}(Q, R)(U) \),
5. \( j < q - 1, R = 0 \) \( \text{Sylv}^{j,0}(P, Q)(U) = 0 \),

Proof. All items follow from Proposition 24 except for the computation of \( \text{Sylv}^{q-1,0}(P, Q)(U) \). Using Proposition 24 for \( Q, R \) and Proposition 35 it remains to remark that \((q - 1)(p - q + 1) + p - q = q(p - q) + q - 1 \).

Proof of Theorem 17. The statement for \( q \leq j < p - 1 \) follows from Lemma 39,1 and Lemma 40,2 and Theorem 34.

The statement for \( j = q - 1 \) follows from Proposition 39,3, Lemma 40,3 and Lemma 34 since

\[ \varepsilon_{p-q+1} = (-1)^{p-q}\varepsilon_{p-q}. \]

For \( j < q - 1 \) we first prove the special case

\[ \text{Sylv}^{j,0}(P, Q) = (-1)^{j(p-q)}\varepsilon_{p-j}\text{Sres}_j(P, Q) \] (17)

The proof is by induction on the length of the remainder sequence of \( P, Q \).

The basic case is when \( Q \) divides \( P \), i.e. \( R = 0 \), and the claim is true by Lemma 39,5, Lemma 40,5.

Otherwise suppose, by induction hypothesis that

\[ \text{Sylv}^{j,0}(Q, R) = (-1)^{j(q-j)}\varepsilon_{q-j}\text{Sres}_j(Q, R) \] (18)

Using Proposition 39,5 and Proposition 40,5 it remains to note that that

\[ (-1)^{j(p-q)}(-1)^{j(q-j)}(-1)^{q(p-q)} = (-1)^{(q-j)(p-q)} \]

and

\[ \varepsilon_{p-j} = (-1)^{(q-j)(p-q)}\varepsilon_{p-q}\varepsilon_{q-j}. \]
which follows from the fact that reversing \( p - j \) numbers can be done in three steps: reversing the first \( p - q \) ones, then the last \( q - j \) one and placing the last \( q - j \) numbers in front of the \( p - q \) first.

Theorem 37 then follows from Theorem 30.

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