HYPERBOLIC BETA INTEGRALS

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Abstract. Hyperbolic beta integrals are analogues of Euler’s beta integral in which the role of Euler’s gamma function is taken over by Ruijsenaars’ hyperbolic gamma function. They may be viewed as \((q, \tilde{q})\)-bibasic analogues of the beta integral in which the two bases \(q\) and \(\tilde{q}\) are interrelated by modular inversion, and they entail \(q\)-analogues of the beta integral for \(|q| = 1\). The integrals under consideration are the hyperbolic analogues of the Ramanujan integral, the Askey-Wilson integral and the Nassrallah-Rahman integral. We show that the hyperbolic Nassrallah-Rahman integral is a formal limit case of Spiridonov’s elliptic Nassrallah-Rahman integral.

1. Introduction

Euler’s gamma function is defined by

\[
\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx, \quad \text{Re}(z) > 0.
\]

In the fundamental paper [19], Ruijsenaars defined gamma functions of rational, trigonometric, hyperbolic and elliptic type, Euler’s gamma function being of rational type. Accordingly, one expects to have extensive theories on special functions of rational, trigonometric, hyperbolic and elliptic type. The rational and trigonometric special functions are precisely the special functions of hypergeometric and basic hypergeometric type, which have been thoroughly studied (see e.g. [2] and [13]). Systematic studies of hyperbolic and elliptic special functions have commenced only recently, see e.g. [17], [20], [26] for the hyperbolic case and [12], [23], [24] for the elliptic case.

A basic step in the development of special functions of a given type is the derivation of the associated beta integrals. The beta integral of rational type is Euler’s beta integral

\[
\int_0^1 x^{a-1}(1-x)^{b-1} \, dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \text{Re}(a), \text{Re}(b) > 0.
\]

Trigonometric and elliptic analogues of beta integrals (involving the trigonometric and the elliptic gamma function respectively) have been studied in detail. The goal of this paper is to derive hyperbolic analogues of beta integrals.

The importance of beta integrals lies in its variety of applications. The rational beta integral is the normalization constant for the orthogonality measure of the Jacobi polynomials and, more generally, multivariate analogues of the beta integral arise as normalization constants in the theory of zonal spherical functions on compact symmetric spaces. Beta type integrals also appear in number theory, combinatorics, conformal field theories and in certain completely integrable systems.
Trigonometric beta integrals have been studied extensively, see e.g. [4], [5], [6], [7], [16]. In this case the role of Euler’s gamma function, as well as of functions like $x^{a-1}$ and $(1-x)^{b-1}$, are taken over by (quotients of) Ruijsenaars’ [19] trigonometric gamma functions, or equivalently, (quotients of) $q$-Pochhammer symbols. The $q$-Pochhammer symbol is defined for $|q|<1$ by

\[(a; q)_{\infty} = \prod_{j=0}^{\infty} (1 - aq^j),\]

and products of $q$-Pochhammer symbols will be denoted by

\[(a_1, \ldots, a_m; q)_{\infty} = \prod_{j=1}^{m} (a_j; q)_{\infty},\]

(1.2)

\[(a_1z_1^{\pm 1}, \ldots, a_mz_m^{\pm 1}; q)_{\infty} = \prod_{j=1}^{m} (a_jz_j; q)_{\infty} (a_jz_j^{-1}; q)_{\infty}.\]

A trigonometric beta integral which closely resembles (1.1) is the contour integral

\[\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{(-q^{\frac{1}{2}} z^{-1}, -q^{\frac{1}{2}} z; q)_{\infty}}{z} \, dz = \frac{(qa, qb; q)_{\infty}}{(q, qab; q)_{\infty}}\]

(1.3)

for $|a|, |b| < |q^{-\frac{1}{2}}|$, where $\mathcal{C} = \{z \in \mathbb{C} \mid |z| = 1\}$ is the positively oriented unit circle in the complex plane. We call (1.3) the Ramanujan integral, since it is closely related to one of Ramanujan’s trigonometric beta integrals from his lost notebook (see [4] and references therein). A second trigonometric analogue of (1.1) is the Askey-Wilson integral [7].

\[\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{(z^{\pm 2}; q)_{\infty}}{\prod_{j=1}^{4} (t_jz^{\pm 1}; q)_{\infty}} \, dz = \frac{2(t_1t_2t_3t_4; q)_{\infty}}{(q; q)_{\infty} \prod_{1 \leq k < m \leq 4} (tkt_m; q)_{\infty}}\]

(1.4)

for parameters $t_j \in \mathbb{C}$ with $|t_j| < 1$ ($j = 1, \ldots, 4$). The Askey-Wilson integral is the normalization constant for the orthogonality measure of the celebrated Askey-Wilson polynomials [7]. The Nassrallah-Rahman [16] integral is the trigonometric beta integral

\[\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{(z^{\pm 2}, Az^{\pm 1}; q)_{\infty}}{\prod_{j=0}^{4} (t_jz^{\pm 1}; q)_{\infty}} \, dz = \frac{2\prod_{j=0}^{4} (At_j^{-1}; q)_{\infty}}{(q; q)_{\infty} \prod_{0 \leq k < m \leq 4} (tkt_m; q)_{\infty}}\]

(1.5)

with $A = t_0t_1t_2t_3t_4$ and with parameters $t_j \in \mathbb{C}$ satisfying $|t_j| < 1$ ($j = 0, \ldots, 4$). The Nassrallah-Rahman integral is the normalization constant for the biorthogonality measure of a five parameter family of $10\phi_9$ rational functions, see [18]. Note that the Askey-Wilson integral (1.4) is the special case $t_0 = 0$ of the Nassrallah-Rahman integral (1.5). The Nassrallah-Rahman integral is the most general trigonometric analogue of Euler’s beta integral (1.1) known up to date.
One way to introduce Ruijsenaars’ \cite{19} hyperbolic gamma function, which will take over the role of the $q$-Pochhammer symbol in the hyperbolic analogues of \eqref{1.3}, \eqref{1.4} and \eqref{1.5}, is by its explicit infinite product realization, see \cite{22}, \cite{20}. Explicitly, we define for $\tau \in \mathbb{C} \setminus \{0\}$ two deformation parameters $q = q_\tau$ and $\tilde{q} = \tilde{q}_\tau$ by
\begin{equation}
q = \exp(2\pi i \tau), \quad \tilde{q} = \exp(-2\pi i / \tau).
\end{equation}
The two deformation parameters are thus related by the transformation $\tau \mapsto -\frac{1}{\tau}$, which is one of the modular transformations preserving the upper half plane
\[ \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}. \]
We now define for $\tau \in \mathbb{H}$ (so $|q|, |\tilde{q}| < 1$) a $(q, \tilde{q})$-bibasic analogue of the Pochhammer symbol called the $\tau$-shifted factorial by
\begin{equation}
[z;\tau]_\infty = \frac{(\exp(-2\pi i z);\tilde{q})_\infty}{(q^z;q)_\infty}.
\end{equation}
Shintani’s \cite{22} crucial result implies that $[z;\tau]_\infty$ extends continuously to $\tau \in \mathbb{R}_{<0}$ for generic $z \in \mathbb{C}$ (in which case $|q| = 1 = |\tilde{q}|$). The resulting function, which we still denote by $[z;\tau]_\infty$, depends meromorphically on $z \in \mathbb{C}$ and can be expressed explicitly in terms of Barnes’ double gamma function, or equivalently in terms of Ruijsenaars’ \cite{19} hyperbolic gamma function or Kurokawa’s double sine function (see \cite{20} Appendix A). In the appendix, §6, the interrelation with Ruijsenaars’ hyperbolic gamma function is made explicit and relevant properties of the hyperbolic gamma function are translated to the $\tau$-shifted factorial $[z;\tau]_\infty$.

The hyperbolic analogues of \eqref{1.3}, \eqref{1.4} and \eqref{1.5} can now be formulated explicitly as follows. We define the shorthand notations
\begin{align*}
[a_1, \ldots, a_m;\tau]_\infty &= \prod_{j=1}^m [a_j;\tau]_\infty, \\
[a_1 \pm z_1, \ldots, a_m \pm z_m;\tau]_\infty &= \prod_{j=1}^m [a_j \pm z_j;\tau]_\infty [a_j - z_j;\tau]_\infty
\end{align*}
for products of $\tau$-shifted factorials. We use the convention that $q^u = \exp(2\pi i \tau u)$ and $\tilde{q}^u = \exp(-2\pi i u / \tau)$ for $u \in \mathbb{C}$. The hyperbolic analogue of the Ramanujan integral \eqref{1.3} becomes
\begin{equation}
\int_{-i\infty}^{i\infty} \frac{[\frac{1}{2} + \frac{1}{2\tau} + z, \frac{1}{2} + \frac{1}{2\tau} - z;\tau]_\infty}{[\frac{1}{2} + \frac{1}{2\tau} + \alpha + z, \frac{1}{2} + \frac{1}{2\tau} + \beta - z;\tau]_\infty} dz = -q^\frac{1}{2\tau} \tilde{q}^\frac{1}{2\tau} [\frac{1}{\tau} + \alpha, \frac{1}{\tau} + \beta;\tau]_\infty
\end{equation}
for $\text{Re}(\tau \alpha), \text{Re}(\tau \beta) < 0$ and $\text{Re}(\alpha - \frac{1}{2} + \frac{1}{2\tau}), \text{Re}(\beta - \frac{1}{2} + \frac{1}{2\tau}) < 0$, where $\sqrt{\cdot}$ is the branch of the square root on $\mathbb{C} \setminus \mathbb{R}_{<0}$ which takes positive values on $\mathbb{R}_{\geq 0}$ (this branch of the square root will be fixed throughout the paper). Here the deformation parameter $\tau$ may be either from the interior of the second quadrant of the complex plane, so $\text{Re}(\tau) < 0$ and $\text{Im}(\tau) > 0$ (in which case $|q|, |\tilde{q}| < 1$), or from $\mathbb{R}_{<0}$ (in which case $|q| = 1 = |\tilde{q}|$). The rather
mysterious looking $q$ and $\tilde{q}$ powers in the right hand side of (1.8) arise from an application of the modularity of the Dedekind eta function,

$$\frac{(q; q)_\infty}{(\tilde{q}; \tilde{q})_\infty} = \frac{q^{-\frac{i\pi}{12}}}{\sqrt{-i\tau}}, \quad \tau \in \mathbb{H}.$$  

The proof of (1.8) will given in §3. With the same conditions on the deformation parameter $\tau$, the hyperbolic analogue of the Askey-Wilson integral (1.4) becomes

$$\int_{-i\infty}^{i\infty} \frac{[1 \pm 2z; \tau]_\infty}{\prod_{j=1}^{4} [\tau_j \pm z; \tau]_\infty} dz = -\frac{2q^{-\frac{i\pi}{12}}}{\sqrt{-i\tau}} \frac{[\tau_1 + \tau_2 + \tau_3 + \tau_3 - 3; \tau]_\infty}{\prod_{1 \leq k < m \leq 4} [\tau_k + \tau_m - 1; \tau]_\infty}$$  

for parameters $\tau_j \in \mathbb{C}$ satisfying $\text{Re}(\tau_j) < 1$ ($j = 1, \ldots, 4$) and $\text{Re}((\tau_1 + \tau_2 + \tau_3 + \tau_4 - 3)\tau) < 1$. The proof will be given in §4. The hyperbolic analogue of the Nassrallah-Rahman integral (1.5) with deformation parameter $\tau$ satisfying the same conditions as for the hyperbolic Ramanujan integral (1.8), reads

$$\int_{-i\infty}^{i\infty} \frac{[1 \pm 2z; a - 4 \pm z; \tau]_\infty}{\prod_{j=0}^{4} [\tau_j \pm z; \tau]_\infty} dz = -\frac{2q^{-\frac{i\pi}{12}}}{\sqrt{-i\tau}} \frac{\prod_{j=0}^{4} [a - \tau_j - 3; \tau]_\infty}{\prod_{0 \leq k < m \leq 4} [\tau_k + \tau_m - 1; \tau]_\infty}$$  

where $a = \tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4$, with the parameters $\tau_j \in \mathbb{C}$ satisfy $\text{Re}(\tau_j) < 1$ ($j = 0, \ldots, 4$) and $\text{Re}(a - \tau^{-1}) > 4$. The proof will be given in §5. The hyperbolic Askey-Wilson integral (1.9) is formally the limit case $\text{Im}(\tau_0) \to -\infty$ of the hyperbolic Nassrallah-Rahman integral (1.10), see Remark 5.5.

The proofs are based on the observation that a hyperbolic beta integral is essentially the fusion of a trigonometric sum and a trigonometric integral. The technique of fusing trigonometric sums and integrals will be discussed in §2. The hyperbolic Ramanujan integral (1.8) is then derived from fusing Ramanujan’s $_1\Psi_1$ summation formula with the trigonometric Ramanujan integral (1.3), while the hyperbolic Askey-Wilson integral (1.9) is derived from fusing Bailey’s $_6\Psi_6$ summation formula with the trigonometric Askey-Wilson integral (1.4). The hyperbolic analogue of the Nassrallah-Rahman integral is more delicate: it is derived from fusing the trigonometric Nassrallah-Rahman integral with a formula expressing the bilateral very-well-poised $_8\Psi_8$ as a sum of two very-well-poised $_6\phi_7$ series. During the fusion procedure the sum of the $_6\phi_7$ series is corrected in such a way that it becomes summable by Bailey’s summation formula, leading eventually to (1.10).

The fusion technique also reveals a close connection between trigonometric beta integrals and Macdonald-Mehta type integrals. Roughly speaking, Macdonald-Mehta integrals can be obtained by fusing trigonometric beta integrals with the inversion formula for the Jacobi theta function (regarded as a bilateral sum identity). We will encounter several one-variable Macdonald-Mehta type integrals in this way. In particular, we derive in §4 Cherednik’s [8] one-variable Macdonald-Mehta integral as a consequence of the Askey-Wilson integral (1.4).
For elliptic beta integrals, the role of Euler’s gamma function is taken over by Ruijsenaars’ \[19\] elliptic gamma function

\[
\Gamma(z; p_1, p_2) = \prod_{k,m=0}^{\infty} \frac{1 - z^{-1}p_1^{k+1}p_2^{m+1}}{1 - zp_1^kp_2^m},
\]

where \(p_1, p_2 \in \mathbb{C}\) are two arbitrary complex numbers with \(|p_1|, |p_2| < 1\). Spiridonov \[23\] proved the following elliptic analogue of the Nassrallah-Rahman integral,

\[
\frac{1}{2\pi i} \int_T \frac{\prod_{j=0}^{4} \Gamma(t_j z^\pm; p_1, p_2)}{\Gamma(z^\pm, A z^\pm; p_1, p_2)} \frac{dz}{z} = \frac{2}{(p_1; p_1)_\infty (p_2, p_2)_\infty} \prod_{0 \leq k < m \leq 4} \Gamma(t_k t_m; p_1, p_2)
\]

where \(A = t_0 t_1 t_2 t_3 t_4\), with the five parameters \(t_j \in \mathbb{C}\) satisfying \(|t_j| < 1\) and \(|p_1 p_2| < |A|\). Here we used the notations

\[
\Gamma(a_1, \ldots, a_m; p_1, p_2) = \prod_{j=1}^{m} \Gamma(a_j; p_1, p_2),
\]

\[
\Gamma(a_1 z_1^\pm, \ldots, a_m z_m^\pm; p_1, p_2) = \prod_{j=1}^{m} \Gamma(a_j z_j, a_j z_j^{-1}; p_1, p_2)
\]

for products of elliptic gamma functions. The trigonometric Nassrallah-Rahman integral \([1.9]\) is the special case of the elliptic Nassrallah-Rahman integral \([1.12]\) when one of the deformation parameters \(p_j\) is zero. We show in §5.4 that the hyperbolic Nassrallah-Rahman integral \([1.10]\) can also be formally obtained as limit of the elliptic Nassrallah-Rahman integral \([1.12]\). It is based on a remarkable limit from the elliptic gamma function to the hyperbolic gamma function due to Ruijsenaars \[19\]. The rigidity of the elliptic theory seems to prevent the existence of elliptic degenerations, and consequently there do not seem to exist elliptic analogues of the Ramamujan integral and of the Askey-Wilson integral.

**Remark 1.1.** After the appearance of this paper as a preprint, Erik Koelink kindly pointed out to me that Ruijsenaars recently obtained another proof of the hyperbolic Askey-Wilson integral \([1.9]\) as a spin-off of his studies of Hilbert space transforms associated to relativistic hypergeometric functions in his recent paper \[21\]. Subsequently van Diejen and Spiridonov derived multidimensional analogues of the hyperbolic Askey-Wilson and Nassrallah-Rahman integrals \([1.9]\) and \([1.10]\) by completely different methods in \[11\]. Their multidimensional hyperbolic Askey-Wilson integral \([11\] Thm. 5] can be rederived with the methods of this paper by fusing Gustafson’s \[14\] multidimensional Askey-Wilson integral with van Diejen’s \[10\] multidimensional generalization of Bailey’s very-well-poised 6\(\psi_6\) summation formula.

**Notations:** For \(|q| < 1\) we denote

\[
(z; q)_\alpha = \frac{(z; q)_\infty}{(z q^{\alpha}; q)_\infty}.
\]
This reduces to a finite product when \( \alpha \in \mathbb{Z} \). Products of \((z; q)_{\alpha}\) will be denoted in the same way as for \((z; q)_{\infty}\), see (1.2). The \( r+1 \phi_r \) basic hypergeometric series with base \( q \) is defined by

\[
_r \phi_r \left( \frac{a_1, a_2, \ldots, a_{r+1}}{b_1, \ldots, b_r}; q, z \right) = \sum_{m=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_m}{(q, b_1, \ldots, b_r; q)_m} z^m, \quad |z| < 1.
\]

The special case of a very-well-poised \( r+1 \phi_r \) series is defined as

\[
_r W_r \left( a_1; a_4, \ldots, a_{r+1}; q, z \right) = r+1 \phi_r \left( \frac{a_1, qa_1^2 - qa_1^2, a_4, \ldots, a_{r+1}}{a_1, -a_1, qa_1/a_4, \ldots, qa_1/a_{r+1}}; q, z \right)
= \sum_{m=0}^{\infty} \frac{1 - a_1 q^{2m}}{1 - a_1} \left( \frac{a_1, a_4, \ldots, a_{r+1}; q)_m}{(q, qa_1/a_4, \ldots, qa_1/a_{r+1}; q)_m} z^m.
\]

The bilateral basic hypergeometric series \( r \psi_r \) with base \( q \) is defined by

\[
_r \psi_r \left( \frac{a_1, \ldots, a_r}{b_1, \ldots, b_r}; q, z \right) = \sum_{m=-\infty}^{\infty} \left( \frac{a_1, \ldots, a_r; q)_m}{(b_1, \ldots, b_r; q)_m} \right) z^m, \quad \left| \frac{b_1 \cdots b_r}{a_1 \cdots a_r} \right| < |z| < 1.
\]

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**2. Folding and fusion of integrals.**

In this section we explain the principle of folding and fusion of integrals, which is fundamental for the study of hyperbolic beta integrals.

We take \( \tau \in \mathbb{H} \), so that \(|q|, |\tilde{q}| < 1\) for the associated deformation parameters \( q \) and \( \tilde{q} \) (see (1.6)). The (rescaled) Gaussian

\[
G_r(z) = q^\frac{z^2}{2}
\]

is analytic, zero-free and satisfies the quasi-periodicity conditions

\[
G_r(z+1) = q^{\frac{1}{2}} q^{-z} G_r(z), \quad G_r(z - \tau^{-1}) = \tilde{q}^{-\frac{1}{2}} \exp(-2\pi i z) G_r(z).
\]

Suppose that \( \phi(z) \) is a \( \frac{1}{\tau} \)-periodic, meromorphic function and consider the associated function

\[
\tilde{\phi}(z) = \phi(z) G_r(z) = \phi(z) q^{\frac{z^2}{2}}.
\]

By the \( \frac{1}{\tau} \)-periodicity of \( \phi \), we still have the quasi-periodicity

\[
\tilde{\phi}(z - \tau^{-1}) = \tilde{q}^{-\frac{1}{2}} \exp(-2\pi i z) \tilde{\phi}(z).
\]

We now force \( \tilde{\phi}(z) \) to become one-periodic by considering the series

\[
\phi^+(z) = \sum_{m=-\infty}^{\infty} \tilde{\phi}(z + m)
\]
(this resembles the construction of automorphic forms as Poincaré series). We assume that
the function \( \phi \) is such that the series (2.3) converges to a meromorphic function. In this
situation, the resulting bilateral sum \( \phi^+(z) \) is thus an one-periodic meromorphic function
satisfying
\[
\phi^+(z - \tau^{-1}) = \tilde{q}^{\frac{-1}{2}} \exp(-2\pi iz)\phi^+(z).
\]
On the other hand, the Jacobi theta function
\[
\vartheta_{-\frac{1}{2}}(z) = \sum_{m=-\infty}^{\infty} G_{-\frac{1}{2}}(m) \exp(2\pi imz) = \sum_{m=-\infty}^{\infty} \tilde{q}^{\frac{m^2}{2}} \exp(2\pi imz)
\]
is a one-periodic entire function which satisfies the same quasi-periodicity (2.4), hence
\[
\phi^+(z) = \sum_{m=-\infty}^{\infty} \tilde{\phi}(z + m) = \Phi(z)\vartheta_{-\frac{1}{2}}(z)
\]
with \( \Phi \) an elliptic function with respect to the periods 1 and \(-\frac{1}{\tau}\).

If the bilateral series \( \phi^+(z) \) is entire, then the elliptic function \( \Phi(z) \) is automatically a
constant. This well known fact follows from the observation that the Fourier coefficients in
the Fourier expansion \( \phi^+(z) = \sum_n a_n \exp(2\pi inz) \) satisfy the first order recurrence relation
\( a_{n+1} = \tilde{q}^{\frac{1}{2}+n}a_n \) \((n \in \mathbb{Z})\) due to the quasi-periodicity (2.4) of \( \phi^+ \).

Choose now in addition a one-periodic meromorphic function \( \psi(z) \). In the examples
treated in this paper, the integral
\[
\int_0^1 \psi(x)\Phi(x)\vartheta_{-\frac{1}{2}}(x)dx
\]
over the period cycle \([0, 1]\) will be closely related to the trigonometric beta integral (1.3),
(1.4) or (1.5). The integral
\[
\int_{-\infty}^{\infty} \psi(x)\phi(x)q^{\frac{x^2}{2}}dx = \int_{-\infty}^{\infty} \psi(x)\tilde{\phi}(x)dx
\]
may then be viewed as the fusion of the integral (2.7) and the sum (2.6), since folding the
integral and using the one-periodicity of \( \psi \) implies
\[
\int_{-\infty}^{\infty} \psi(x)\phi(x)q^{\frac{x^2}{2}}dx = \sum_{m=-\infty}^{\infty} \int_0^1 \psi(x)\tilde{\phi}(x + m)dx
\]
(2.9)
\[
= \int_0^1 \psi(x)\phi^+(x)dx = \int_0^1 \psi(x)\Phi(x)\vartheta_{-\frac{1}{2}}(x)dx
\]
by the bilateral sum (2.6). In all applications below, (2.9) can be justified by a straightforward
application of Fubini’s theorem. Note that the fused integral (2.8) admits an explicit
evaluation as soon as the folded integral (2.7) admits some explicit evaluation.
Remark 2.1. A summation formula $F = \sum_{m \in \mathbb{Z}^n} f(x + m)$ for some function $f : \mathbb{R}^n \to \mathbb{C}$, with $F$ independent of $x$, implies the identity

$$\iint_{\mathbb{R}^n} f(x) \, dx = F$$

by the folding technique. This observation was extensively used by Gustafson [15] to obtain evaluations of multivariate trigonometric integrals from known multivariate bilateral summation formulas.

The simplest example of fusion and folding of integrals arises when $\phi(z) \equiv 1 \equiv \psi(z)$. Then

$$\phi^+(z) = \sum_{m=-\infty}^{\infty} q^{\frac{(x+m)^2}{2}} = \vartheta(\tau z) q^\frac{x^2}{\tau}$$

is entire, hence

$$\phi^+(z) = C \vartheta_{-\frac{1}{\tau}}(z)$$

(2.10) is the corresponding sum (2.6) for some constant $C$. The integral (2.7) is

$$C \int_{0}^{1} \vartheta_{-\frac{1}{\tau}}(x) \, dx$$

which equals the constant $C$ in view of the explicit Fourier expansion (2.6) of the Jacobi theta function. The fused integral (2.7) thus becomes

$$\int_{-\infty}^{\infty} q^\frac{x^2}{\tau} \, dx,$$

and the above analysis shows that it equals the constant $C$. The fused integral is up to a change of integration variable the Gauss integral, whose well known evaluation is given by

$$\int_{-\infty}^{\infty} q^\frac{x^2}{2} \, dx = \frac{1}{\sqrt{-i\tau}}.$$

We thus conclude that the $C = \frac{1}{\sqrt{-i\tau}}$ in (2.10). Returning to the underlying bilateral sum, we obtain

$$\sum_{m=-\infty}^{\infty} q^{\frac{(x+m)^2}{2}} = \frac{1}{\sqrt{-i\tau}} \vartheta_{-\frac{1}{\tau}}(z),$$

(2.11) which is the well known inversion formula for the Jacobi theta function. The Jacobi inversion formula can be rewritten as

$$\vartheta_{-\frac{1}{\tau}}(z) = \sqrt{-i\tau} q^\frac{x^2}{2} \vartheta_{\tau}(\tau z),$$

or equivalently, with $z$ replaced by $z/\tau$,

$$\vartheta_{-\frac{1}{\tau}}(z/\tau) = \sqrt{-i\tau} q^{-\frac{x^2}{\tau}} \vartheta_{\tau}(z).$$
Note that in this example, we used the explicit evaluation of the fused integral to derive the underlying bilateral summation formula (2.11) explicitly. Conversely, the evaluation of the Gauss integral follows from the Jacobi inversion formula and the elementary integral
\[(2.12)\]
\[\int_0^1 \vartheta_{-\frac{1}{4}}(x) \, dx = 1.\]

In subsequent sections we show how hyperbolic versions of beta integrals can be derived by the fusion procedure. In these cases, \(\psi(x) = \psi_{-\tau^{-1}}(x)\) is a slight modification of the (one-periodic variant) of the integrand of the trigonometric version of the beta integral with respect to base \(\tilde{q}\), while \(\phi(x)\) is roughly \(\psi_\tau(\tau x)^{-1}\) (thus with respect to the base \(q\)); this is precisely what one would expect from the explicit formula (1.7) expressing the hyperbolic gamma function as a quotient of trigonometric gamma functions.

### 3. The hyperbolic Ramanujan integral

In this section we derive a hyperbolic analogue of the Ramanujan integral (1.3). The derivations in this section are based on the \(q\)-binomial theorem
\[(3.1)\]
\[\Phi_0(a; -q, z) = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |q|, |z| < 1,\]
the Jacobi inversion formula (2.11) and the Jacobi triple product identity
\[(3.2)\]
\[\vartheta_\tau(z) = \left(q, -q^{\frac{1}{2}} \exp(2\pi i z), -q^{\frac{1}{2}} \exp(-2\pi i z); q\right)_\infty, \quad |q| < 1.\]

An elementary proof of the \(q\)-binomial theorem can e.g. be found in [13]. Of the many known proofs of the Jacobi triple product identity, we note that there are several which essentially only uses the \(q\)-binomial theorem, see e.g. [1]. We have chosen to give full details of all other identities we encounter since it clarifies the techniques leading to hyperbolic beta integrals.

#### 3.1. Ramanujan’s \(\Psi_1\) summation formula

In this subsection we consider a nontrivial example of the bilateral sum construction (2.6). It leads to an elementary proof of Ramanujan’s \(\Psi_1\) summation formula. This method of proving bilateral sum identities has been applied in several closely related setups, see e.g. [3].

Let \(\tau \in \mathbb{H}\) and \(0 < |a|, |b| < 1\). We consider the \(\frac{1}{\tau}\)-periodic entire function
\[(3.3)\]
\[\phi(z; a, b) = \phi(z) = (-aq^{\frac{1}{2}+z}, -bq^{\frac{1}{2}-z}; q)_\infty\]
and the associated entire function \(\tilde{\phi}(z; a, b) = \tilde{\phi}(z)\) defined by
\[\tilde{\phi}(z; a, b) = \phi(z; a, b) q^{\frac{z^2}{2}}.\]

The function \(\tilde{\phi}\) satisfies (2.2), as well as \(\tilde{\phi}(z+1) = t(z)\tilde{\phi}(z)\) with
\[t(z) = \frac{1 + b^{-1}q^{\frac{1}{2}+z}}{1 + aq^{\frac{1}{2}+z}} b.\]
As explained in §2, we now force $\tilde{\phi}$ to become one periodic by considering the bilateral series

$$\phi^+(z) = \sum_{m=-\infty}^{\infty} \tilde{\phi}(z + m),$$

which is easily seen to converge absolutely and uniformly on compacta of $\mathbb{C}$ to an entire function $\phi^+(z)$ due to the conditions on $a$ and $b$. Thus we conclude

$$\phi^+(z) = C \vartheta_{-\frac{1}{2}}(z)$$

for some constant $C \in \mathbb{C}$. To find an explicit expression for the constant $C$, we first express $\phi^+(z)$ in terms of the $1\Psi_1$ bilateral series. Using $\tilde{\phi}(z + 1) = t(z)\tilde{\phi}(z)$, we can rewrite $\phi^+(z)$ as

$$\phi^+(z) = \sum_{m=-\infty}^{\infty} \tilde{\phi}(z + m) = t^+(z)\tilde{\phi}(z)$$

with

$$t^+(z) = \sum_{m=0}^{m-1} \prod_{k=0}^{m-1} t(z + k) + \sum_{m=\infty}^{m=\infty} \prod_{k=m}^{m-1} \frac{1}{t(z + k)}$$

$$= 1\Psi_1(-b^{-1}q^{\frac{1}{2}+z}; -aq^{\frac{1}{2}+z}; q, b)$$

(3.5)

(3.4)

(3.6)

(3.7)

On the other hand,

$$t^+(z) \mid_{q^z = -q^{\frac{1}{2}a}a^{-1}} = \frac{(q/a; q)_{\infty}}{(b; q)_{\infty}}.$$

We thus conclude that

$$C = \frac{1}{\sqrt{-i\tau}} \frac{(ab; q)_{\infty}}{(a, b; q)_{\infty}}.$$

The resulting bilateral summation formula

$$\phi^+(z; a, b) = \sum_{m=-\infty}^{\infty} \tilde{\phi}(z + m; a, b) = \frac{1}{\sqrt{-i\tau}} \frac{(ab; q)_{\infty}}{(a, b; q)_{\infty}} \vartheta_{-\frac{1}{2}}(z)$$

(3.7)
is Ramanujan’s $\psi_1$ summation formula, written in the convenient form (2.6).

3.2. The Ramanujan integral. We consider in this subsection the natural trigonometric beta integral associated to Ramanujan’s $\psi_1$ sum (3.7).

Again we take $\tau \in \mathbb{H}$, so that $|q| < 1$. Let $T = \{z \in \mathbb{C} \mid |z| = 1\}$ be the positively oriented unit circle in the complex plane. Choose parameters $c, d \in \mathbb{C}$ satisfying $1 < |c| < |q^{-\frac{1}{2}}|$ and $0 < |d| < |q^{-\frac{1}{2}}|$. We compute the trigonometric integral

$$\int_0^1 \frac{\vartheta_\tau(x)}{-q^x c \exp(-2\pi ix), -q^x d \exp(2\pi ix); q}_\infty \, dx = \frac{1}{2\pi i} \int_T \frac{(q, -q^\frac{1}{2} z^{-1}, -q^\frac{1}{2} z; q)_\infty}{(-q^\frac{1}{2} cz^{-1}, -q^\frac{1}{2} dz; q)_\infty} \, dz$$

by shrinking the radius of the integration contour $T$ to zero while picking up residues (the second expression is obtained from the first by the Jacobi triple product identity and a change of integration variable). This computation is a simplified version of a general residual approach to basic contour integrals developed by Slater, see [13, §4.9] for details and references.

By the conditions on the parameters, the poles $-cq^{\frac{1}{2} + m}$ ($m \in \mathbb{Z}_{\geq 0}$) of the integrand lie inside $T$, while the poles $-d^{-1}q^{-\frac{1}{2} - m}$ ($m \in \mathbb{Z}_{\geq 0}$) lie outside $T$. The condition $|c| > 1$ then ensures that

$$\frac{1}{2\pi i} \int_T \frac{(q, -q^\frac{1}{2} z^{-1}, -q^\frac{1}{2} z; q)_\infty}{(-q^\frac{1}{2} cz^{-1}, -q^\frac{1}{2} dz; q)_\infty} \, dz = \sum_{m=0}^{\infty} \text{Res}_{z= -cq^{\frac{1}{2} + m}} \left( \frac{(q, -q^\frac{1}{2} z^{-1}, -q^\frac{1}{2} z; q)_\infty}{(-q^\frac{1}{2} cz^{-1}, -q^\frac{1}{2} dz; q)_\infty} \right),$$

cf. [13, §4.9]. The residues are easily computed, leading to

$$\frac{1}{2\pi i} \int_T \frac{(q, -q^\frac{1}{2} z^{-1}, -q^\frac{1}{2} z; q)_\infty}{(-q^\frac{1}{2} cz^{-1}, -q^\frac{1}{2} dz; q)_\infty} \, dz = \frac{(qc, c^{-1}; q)_\infty}{(qcd; q)_\infty} \sum_{m=0}^{\infty} \frac{(qcd; q)_m}{(q; q)_m} c^{-m}$$

by the $q$-binomial theorem. The conditions $|c| > 1$ and $|d| > 0$ may now be removed by analytic continuation. This proves the Ramanujan integral (1.3),

$$\int_0^1 \frac{\vartheta_\tau(x)}{-q^x c \exp(-2\pi ix), -q^x d \exp(2\pi ix); q}_\infty \, dx = \frac{(qc, qd; q)_\infty}{(qcd; q)_\infty}$$

for $\tau \in \mathbb{H}$ and $|c|, |d| < |q^{-\frac{1}{2}}|$. The connection with the integrals from Ramanujan’s lost notebook will become apparent in the following subsection.

3.3. Fusion. The Ramanujan $\psi_1$ summation formula (3.7) and the trigonometric Ramanujan integral (3.8) can be fused as follows.

**Proposition 3.1.** Let $\tau \in \mathbb{H}$, $0 < |a|, |b| < 1$ and $|c|, |d| < |q^{-\frac{1}{2}}|$. Then

$$\int_{-\infty}^{\infty} \frac{(-aq^{\frac{1}{2} + x}, -bq^{\frac{1}{2} - x}; q}_\infty q^2 z x}{(-q^x c \exp(-2\pi ix), -q^x d \exp(2\pi ix); q)_\infty} \, dx = \frac{1}{\sqrt{-i\tau}} \frac{(ab; q)_\infty (\tilde{q}c, \tilde{q}d; \tilde{q})_\infty}{(\tilde{q}cd; \tilde{q})_\infty (a, b; q)_\infty}.$$
Proof. The integrand $J(x)$ of (3.9) can be written as

$$J(x) = \frac{\tilde{\phi}(x; a, b)}{(-\tilde{q}^{x}c \exp(-2\pi ix), -\tilde{q}^{x}d \exp(2\pi ix); \tilde{q})_{\infty}}.$$  

Note that the denominator is one-periodic. Hence folding the integral gives

$$\int_{-\infty}^{\infty} J(x) \, dx = \int_{0}^{1} \frac{\phi^{+}(x; a, b)}{(-\tilde{q}^{x}c \exp(-2\pi ix), -\tilde{q}^{x}d \exp(2\pi ix); \tilde{q})_{\infty}} \, dx$$

$$= \frac{1}{\sqrt{-i\tau}} \frac{(ab; q)_{\infty}}{(a, b; q)_{\infty}} \int_{0}^{1} \frac{\vartheta_{\frac{1}{2}}(x)}{(-\tilde{q}^{x}c \exp(-2\pi ix), -\tilde{q}^{x}d \exp(2\pi ix); \tilde{q})_{\infty}} \, dx$$

where we used Ramanujan’s $\psi$ summation formula (3.7) for the second equality and the trigonometric beta integral (3.8) for the last identity. □

The proposition is also valid for $a = b = 0$, in which case the resulting integral

$$(3.10) \quad \int_{-\infty}^{\infty} \frac{q^{x^2}}{(-\tilde{q}^{x}c \exp(-2\pi ix), -\tilde{q}^{x}d \exp(2\pi ix); \tilde{q})_{\infty}} \, dx = \frac{1}{\sqrt{-i\tau}} \frac{(\tilde{q}c, \tilde{q}d; \tilde{q})_{\infty}}{(\tilde{q}cd; \tilde{q})_{\infty}} \frac{(a, b; q)_{\infty}}{(a, b; q)_{\infty}}$$

is the fusion of the Jacobi inversion formula (2.11) and the trigonometric Ramanujan integral (3.8). The integral (3.10) is one of Ramanujan’s trigonometric analogues of the beta integral from his lost notebook, see [4], [13, Exerc. 6.15] and references therein. Ramanujan’s second trigonometric analogue of the beta integral is the special case $c = d = 0$ of the proposition,

$$(3.11) \quad \int_{-\infty}^{\infty} \frac{q^{x^2}}{(-aq^{\frac{1}{2}+x}, -bq^{\frac{1}{2}-x}; q)_{\infty}} \, dx = \frac{1}{\sqrt{-i\tau}} \frac{(ab; q)_{\infty}}{(a, b; q)_{\infty}}.$$

This integral is the fusion of Ramanujan’s $\psi$ summation formula (3.7) and the elementary integral (2.12).

3.4. The hyperbolic Ramanujan integral. The integral (3.9) turns out to be more than just a particular $(q, \tilde{q})$-bibiastic extension of the Ramanujan integral. As we will see in this subsection, it is the crucial intermediate step towards the hyperbolic Ramanujan integral.

We specialize the parameters in the fused Ramanujan integral (3.9) to

$$a = q^{\alpha}, \quad b = q^{\beta}, \quad c = \exp(-2\pi i\alpha), \quad d = \exp(-2\pi i\beta).$$

We assume that $\tau \in \mathbb{H}$ and that $\alpha, \beta \in \mathbb{C}$ are such that the corresponding parameters $a, b, c, d$ satisfy the parameter constraints of Proposition 3.1. The fused Ramanujan integral
(3.9) can then be written entirely in terms of the $\tau$-shifted factorial (1.4),

$$
(3.13) \quad \int_{-\infty}^{\infty} \frac{q^{x^2}}{[\frac{1}{2} + \frac{1}{2\tau} + \alpha + x, \frac{1}{2} + \frac{1}{2\tau} + \beta - x; \tau]_{\infty}} \, dx = \frac{1}{\sqrt{-i\tau}} \frac{[\frac{1}{2} + \alpha, \frac{1}{2} + \beta; \tau]_{\infty}}{[\frac{1}{2} + \alpha + \beta; \tau]_{\infty}}.
$$

By the reflection equation (6.13) we can rewrite (3.13) in a form which closely resembles the trigonometric Ramanujan integral (1.3),

$$
(3.14) \quad \int_{-\infty}^{\infty} \frac{[\frac{1}{2} + \frac{1}{2\tau} + x, \frac{1}{2} + \frac{1}{2\tau} - z; \tau]_{\infty}}{[\frac{1}{2} + \alpha + x, \frac{1}{2} + \frac{1}{2\tau} + \beta - x; \tau]_{\infty}} \, dx = \frac{q^{-\frac{1}{2}} q^{\frac{z}{\tau}} [\frac{1}{2} + \alpha, \frac{1}{2} + \beta; \tau]_{\infty}}{\sqrt{-i\tau} [\frac{1}{2} + \alpha + \beta; \tau]_{\infty}}.
$$

One may view (3.14) already as an hyperbolic analogue of the Ramanujan integral. It is though not completely satisfactory, because it does not extend to $\tau \in \mathbb{R}_{<0}$ (in which case the deformation parameters $q$ and $\tilde{q}$ satisfy $|q| = |\tilde{q}| = 1$). To include this important case, we first need to rotate the integration cycle $\mathbb{R}$ to $-i\mathbb{R}$ about the origin.

For the rotation of the integration cycle it is convenient to restrict attention to $\tau \in \mathbb{H} \cap \mathbb{C}_-$, where $\mathbb{C}_-$ (respectively $\mathbb{C}_+$) is the open left (respectively right) half plane in $\mathbb{C}$, and to parameters $\alpha, \beta \in \mathbb{C}$ satisfying

$$
(3.15) \quad 0 < \alpha, \beta < \frac{1}{2}.
$$

With these parameter constraints, the corresponding parameters $a, b, c, d$ (see (6.12)) automatically satisfy the conditions of Proposition 3.1. We write

$$
I(z) = \frac{[\frac{1}{2} + \frac{1}{2\tau} + z, \frac{1}{2} + \frac{1}{2\tau} - z; \tau]_{\infty}}{[\frac{1}{2} + \frac{1}{2\tau} + \alpha + z, \frac{1}{2} + \frac{1}{2\tau} + \beta - z; \tau]_{\infty}}
$$

for the integrand of (3.14). By the zero and pole locations of the $\tau$-shifted factorial (see (6.11) and (6.12) respectively) we conclude that the poles of $I(z)$ are contained in the two sets

$$
(3.16) \quad \begin{align*}
& -\alpha + \frac{1}{2} - \frac{1}{2\tau} + \frac{1}{\tau} Z_{\leq 0} + Z_{\geq 0}, \\
& \beta - \frac{1}{2} + \frac{1}{2\tau} + \frac{1}{\tau} Z_{\geq 0} + Z_{\leq 0}.
\end{align*}
$$

By the conditions $\tau \in \mathbb{H} \cap \mathbb{C}_-$ and (3.15), the first sequence $-\alpha + \frac{1}{2} - \frac{1}{2\tau} + \frac{1}{\tau} Z_{\leq 0} + Z_{\geq 0}$ lies in the interior of the first quadrant of the complex plane (so in $\mathbb{H} \cap \mathbb{C}_+$), while the second sequence $\beta - \frac{1}{2} + \frac{1}{2\tau} + \frac{1}{\tau} Z_{\geq 0} + Z_{\leq 0}$ lies in the interior of the third quadrant of the complex plane (so in $(-\mathbb{H}) \cap \mathbb{C}_-$). Hence the integrand $I(z)$ of (3.14) is analytic in the second and fourth quadrant of the complex plane. We now clockwise rotate the integration contour $\mathbb{R}$ to $-i\mathbb{R}$ in (3.14) about the origin. We thus stay within the union of the second and fourth quadrant and do not pass poles of $I(z)$. To rigorously verify that the integral evaluation does not alter, we need to take the asymptotic behaviour of the integrand $I(z)$ into account.
Lemma 3.2. Suppose that \( \tau \in \mathbb{H} \cap \mathbb{C}_- \) and that the parameters satisfy (3.15). Then there exists a constant \( C \in \mathbb{R}_{>0} \) such that

\[
|I(z)| \leq C|q^{\beta z}| \quad \text{if } \text{Re}(z) \geq 0 \text{ and } \text{Im}(z) \leq 0,
\]

\[
|I(z)| \leq C|q^{-\alpha z}| \quad \text{if } \text{Re}(z) \leq 0 \text{ and } \text{Im}(z) \geq 0.
\]

Proof. Using the reflection equation (6.14) we can write \( I(z) \) as

\[
I(z) = C_1 q^{\beta z} \frac{\left[ \frac{1}{2} + \frac{1}{2\tau} - \beta + z; \tau \right]_\infty}{\left[ \frac{1}{2} + \frac{1}{2\tau} + \alpha + z; \tau \right]_\infty}
\]

(3.17)

\[
= C_2 q^{-\alpha z} \frac{\left[ \frac{1}{2} + \frac{1}{2\tau} - \alpha - z; \tau \right]_\infty}{\left[ \frac{1}{2} + \frac{1}{2\tau} + \beta - z; \tau \right]_\infty},
\]

for certain irrelevant \( z \)-independent nonzero constants \( C_1 \) and \( C_2 \). For the asymptotics in the fourth quadrant we use now the first expression for \( I(z) \) in (3.17). Using the infinite product expression (1.17) for the \( \tau \)-shifted factorial, we may thus write

\[
I(z) = C_1 \frac{(-q^{\frac{1}{2}d^{-1}} \exp(-2\pi iz); q)_\infty}{(-q^{\frac{1}{4}c} \exp(-2\pi iz); q)_\infty} (-q^{\frac{1}{2}+\alpha} q^z; q)_\infty q^{\beta z}
\]

(3.18)

with \( c \) and \( d \) given by (3.12). Now note that \( z \mapsto \exp(-2\pi iz) \) maps the fourth quadrant \( \{z \in \mathbb{C} \mid \text{Re}(z) \geq 0, \text{Im}(z) \leq 0\} \) of the complex plane into the closed unit disc

\[
\mathbb{D} = \{w \in \mathbb{C} \mid |w| \leq 1\}.
\]

Since furthermore \( \left| q^{\frac{1}{2}} \right| < 1 \) and \( |c| = 1 \) by the parameter constraints (3.15), we conclude that

\[
\left| \frac{(-q^{\frac{1}{2}d^{-1}} \exp(-2\pi iz); q)_\infty}{(-q^{\frac{1}{4}c} \exp(-2\pi iz); q)_\infty} \right|
\]

is uniformly bounded on the fourth quadrant of the complex plane. Observe that \( z \mapsto q^z \) maps the fourth quadrant of the complex plane into the closed unit disc \( \mathbb{D} \) since \( \tau \in \mathbb{H} \cap \mathbb{C}_- \). Furthermore, \( \left| q^{\frac{1}{2}+\beta} \right| < 1 \) by the parameter constraints (3.15), hence

\[
\left| \frac{(-q^{\frac{1}{2}+\alpha} q^z; q)_\infty}{(-q^{\frac{1}{2}+\beta} q^z; q)_\infty} \right|
\]

is uniformly bounded on the fourth quadrant of the complex plane. In view of (3.18) this proves the uniform asymptotics for \( I(z) \) with \( z \) in the fourth quadrant of the complex plane. The asymptotics of \( I(z) \) in the second quadrant of the complex plane is determined similarly using the second expression for \( I(z) \) in (3.17).

We have the following direct consequence of Lemma 3.2.

Corollary 3.3. Let \( \tau \in \mathbb{H} \cap \mathbb{C}_- \) and suppose that the parameters \( \alpha, \beta \in \mathbb{C} \) satisfy (3.15). Let \( 0 < \epsilon < 1 \) be the growth coefficient

\[
\epsilon = \exp(-2\pi \min(\alpha, \beta) m)
\]
with $m$ the strictly positive constant

$$m = \min_{\theta \in [-\frac{\pi}{2}, 0]} \left( \text{Im}(\tau e^{i\theta}) \right).$$

Then there exists a constant $C \in \mathbb{R}_{>0}$ such that

$$|I(z)| \leq Ce^{|z|}$$

for $z \in \mathbb{C}$ satisfying $\text{Re}(z) \geq 0$ and $\text{Im}(z) \leq 0$, as well as for $z \in \mathbb{C}$ satisfying $\text{Re}(z) \leq 0$ and $\text{Im}(z) \geq 0$.

We keep the assumptions $\tau \in \mathbb{H} \cap \mathbb{C}_-$ and (3.15). In view of Corollary 3.3 we may apply Cauchy’s theorem to rotate clockwise the integration contour $\mathbb{R}$ in (3.14) to $-i\mathbb{R}$ about the origin without altering its evaluation. Relaxing the parameter constraints leads now to the following main result.

**Theorem 3.4** (Hyperbolic Ramanujan integral). Let $\tau \in \mathbb{C}$ with $\text{Re}(\tau) < 0$ and $\text{Im}(\tau) \geq 0$, and suppose that the parameters $\alpha, \beta \in \mathbb{C}$ satisfy

$$(3.19) \quad \text{Re}(\tau \alpha), \text{Re}(\tau \beta) < 0, \quad \text{Re}(\alpha - \frac{1}{2} + \frac{1}{2\tau}), \text{Re}(\beta - \frac{1}{2} + \frac{1}{2\tau}) < 0.$$

Then

$$(3.20) \quad \int_{-i\infty}^{i\infty} \left[ \frac{1}{2} + \frac{1}{2\tau} + z, \frac{1}{2} + \frac{1}{2\tau} - z; \tau \right]_{\infty} dz = -q^{-\frac{1}{2\tau}} q^{\frac{1}{2\tau}} \left[ \frac{1}{2} + \alpha, \frac{1}{2} + \beta; \tau \right]_{\infty}.$$

**Proof.** Let $\tau \in \mathbb{H} \cap \mathbb{C}_-$. We already argued that (3.20) is valid for parameters $\alpha$ and $\beta$ satisfying the constraints (3.19). By analytic continuation one easily verifies that the integral evaluation (3.20) is valid under the milder constraints (3.19). In fact, in view of (3.15) and (6.16) the requirements that the integrand $I(z)$ decays exponentially for $z \rightarrow \pm i\infty$ and that the sequence of poles $-\alpha + \frac{1}{2} - \frac{1}{2\tau} + \frac{1}{2\tau} \mathbb{Z}_{\geq 0} + \mathbb{Z}_{< 0}$ (respectively $\beta - \frac{1}{2} + \frac{1}{2\tau} + \frac{1}{2\tau} \mathbb{Z}_{\geq 0} + \mathbb{Z}_{< 0}$) of $I(z)$ is contained in $\mathbb{C}_+$ (respectively $\mathbb{C}_-$), lead to the conditions (3.19) on the parameters $\alpha$ and $\beta$.

Let $\tau' \in \mathbb{R}_{<0}$ and choose $\alpha, \beta \in \mathbb{C}$ satisfying (3.19) with $\tau = \tau'$. Let $0 < \delta < \frac{\pi}{2}$ and define the compact subset

$$K = \{ \tau(\phi) \mid \phi \in [0, \delta] \} \subset \mathbb{C}_-$$

with $\tau(\phi) = \exp(-i\phi)$. By choosing $\delta$ small enough we may assume that the parameter constraints (3.19) are valid for all $\tau \in K$. Observe that $K \cap \mathbb{H} = \{ \tau(\phi) \mid 0 < \phi \leq \delta \}$ and that $K \cap \mathbb{R}_{<0} = \tau(0) = \tau'$. By the reflection equation (6.14) and by the asymptotic estimates (6.15) and (6.16) for the $\tau$-shifted factorial, the weight function $I(z)$ in (3.20) has the asymptotic behaviour

$$I(z) = \begin{cases} O(\exp(-2\pi i \tau \alpha z)), & \text{if } \text{Im}(z) \rightarrow \infty, \\ O(\exp(2\pi i \tau \beta z)), & \text{if } \text{Im}(z) \rightarrow -\infty, \end{cases}$$

uniformly for $\tau \in K$ and for $\text{Re}(z)$ in compacta of $\mathbb{R}$. Consequently,

$$|I(ix)| = O(e^{\pi|x|}), \quad x \mapsto \pm \infty.$$
uniformly for $\tau \in K$, with growth exponent
$$
\epsilon = \exp \left( 2\pi \max_{\tau \in K} (\text{Re}(\tau \alpha), \text{Re}(\tau \beta)) \right) < 1
$$
depending only on $\alpha, \beta$ and $K$. Thus if we take $\tau = \tau(\phi) = K \cap \mathbb{H}$ in (3.20), then Lebesgue’s dominated convergence theorem allows us to interchange the limit $\phi \downarrow 0$ with the integral over $i\mathbb{R}$. This yields (3.20) for $\tau = \tau' \in \mathbb{R}_{<0}$.

4. The hyperbolic Askey-Wilson integral

In this section we apply the folding and fusion techniques to derive an hyperbolic analogue of the Askey-Wilson integral. For proofs of the underlying summation formula and the underlying trigonometric Askey-Wilson integral (1.4) we refer now directly to the literature (in practice we often refer to the book [13] of Gasper and Rahman, in which one can find further detailed references to the literature).

4.1. Bailey’s $6\Psi_6$ summation formula. We take $\tau \in \mathbb{H}$, so that $|q| < 1$. For the Askey-Wilson integral and its hyperbolic analogue, the role of the function $\phi$ in the underlying summation formula (see §2) is

$$
\phi(z) = \prod_{j=1}^{4} (s_j q^{\pm z}; q)_{\infty} \left( q^{\frac{1}{2}+z}, -q^{\frac{1}{2}+z}, q^{\frac{3}{2}+z}; q \right)_{\infty}.
$$

As in §2, we define
$$
\tilde{\phi}(z) = \phi(z) q^{\frac{z}{2}}.
$$
We assume for the remainder of this subsection that the parameters $s_j$ are nonzero complex parameters and satisfy $|q^{-3}s_1s_2s_3s_4| < 1$. A direct computation shows that
$$
\tilde{\phi}(z+1) = t(z) \tilde{\phi}(z)
$$
with
$$
t(z) = \frac{s_1s_2s_3s_4 q^3}{q^3} \left( \frac{1 - q^{1+z}}{1 - q^z} \right) \left( \frac{1 + q^{1+z}}{1 + q^z} \right) \prod_{j=1}^{4} \left( \frac{1 - s_j^{-1} q^{1+z}}{1 - s_j q^z} \right).
$$
Due to the conditions on the parameters the bilateral sum
$$
t^+(z) = \sum_{m=0}^{\infty} \prod_{k=0}^{m-1} t(z+k) + \sum_{m=-\infty}^{-1} \prod_{k=m}^{-1} \frac{1}{t(z+k)}
$$
converges absolutely and uniformly on compacta away from the $\mathbb{Z}_{<0}$-translates of the poles and the $\mathbb{Z}_{>0}$-translates of the zeros of $t(z)$, and it extends to a meromorphic function in $z \in \mathbb{C}$. Hence the bilateral sum
$$
\phi^+(z) = \sum_{m=-\infty}^{\infty} \tilde{\phi}(z + m)
$$
converges to a meromorphic function in \( z \in \mathbb{C} \), given explicitly by
\[
\phi^+(z) = t^+(z)\tilde{\phi}(z).
\]
We now take a shortcut compared to our analysis of the Ramanujan integrals by directly applying Bailey’s \( _6\Psi_6 \) summation formula \([13](5.3.1)\) for the explicit evaluation of \( t^+(z) \),
\[
t^+(z) = \frac{(q; q)_{\infty}\prod_{1 \leq k < m \leq 4}(q^{-1}s_k s_m; q)_{\infty}}{(q^{-3}s_1s_2s_3s_4; q)_{\infty}} \frac{(q^{1\pm 2z}; q)_{\infty}}{\prod_{j=1}^{4}(s_j q^{\pm z}; q)_{\infty}}.
\]
Since
\[
(u^2; q)_{\infty} = (u, -u, q^{\frac{1}{2}}u, -q^{\frac{1}{2}}u; q)_{\infty},
\]
the meromorphic function \( \phi^+(z) = t^+(z)\tilde{\phi}(z) \) can now be written as
\[
\phi^+(z) = \frac{1}{\sqrt{-i\tau}} \frac{1}{(q^{-3}s_1s_2s_3s_4; q)_{\infty}} \prod_{1 \leq k < m \leq 4}(q^{-1}s_k s_m; q)_{\infty} \vartheta_{-\frac{1}{2}}(z)
\]
after a straightforward computation using the Jacobi triple product identity and the Jacobi inversion formula. Written in the form \([2.6]\), we thus have \( \phi^+(z) = \Phi(z)\vartheta_{-\frac{1}{2}}(z) \) with the elliptic function \( \Phi(z) \) being the constant
\[
\Phi = \frac{1}{\sqrt{-i\tau}} \prod_{1 \leq k < m \leq 4}(q^{-1}s_k s_m; q)_{\infty}.
\]

4.2. Fusion. In this subsection we fuse the trigonometric Askey-Wilson integral \([14]\) with Bailey’s \( _6\Psi_6 \) summation formula \([13]\). For proofs of the trigonometric Askey-Wilson integral \([14]\), see \([7], [13], \S 6\) and references therein.

**Proposition 4.1.** Let \( \tau \in \mathbb{H} \) and \( s_j, t_k \in \mathbb{C} \) \((j, k = 1, \ldots, 4)\) with \( |q^{-3}s_1s_2s_3s_4| < 1 \) and \( |t_k| < 1 \) \((k = 1, \ldots, 4)\). Then
\[
\int_{-\infty}^{\infty} \frac{\exp(\pm 2\pi i x)}{(q^{1\pm x}, -q^{1\pm x}, q^{\pm 1\pm x}, q)_{\infty}} \prod_{j=1}^{4} \frac{(s_j q^{\pm x}; q)_{\infty}}{(t_j \exp(\pm 2\pi i x); \hat{q})_{\infty}} q^x dx = \frac{2}{\sqrt{-i\tau}} \frac{(t_1 t_2 t_3 t_4; \hat{q})_{\infty}}{(q^{-3}s_1s_2s_3s_4; q)_{\infty}} \prod_{1 \leq k < m \leq 4} \frac{(q^{-1}s_k s_m; q)_{\infty}}{(t_k t_m; \hat{q})_{\infty}}.
\]

**Proof.** The integrand \( J(x) \) can be written as
\[
J(x) = \tilde{\phi}(x) w(x)
\]
with \( \tilde{\phi}(x) \) as in \S 4.1 and with \( w(x) \) the one-periodic function
\[
w(x) = \frac{\exp(\pm 2\pi i x)}{\prod_{j=1}^{4}(t_j \exp(\pm 2\pi i x); \hat{q})_{\infty}}.
\]
Note that \( J(x) \) is regular on \( \mathbb{R} \) since the real zeros of the factor \((q^{1\pm x}, q^{\pm 1\pm x}, q)_{\infty}\) in the denominator of \( J(x) \) are compensated by zeros of the factor \((\exp(\pm 2\pi i x), -\exp(\pm 2\pi i x); \hat{q})_{\infty}\).
in the numerator of $J(x)$. Thus folding the integral and using Bailey’s $6\Psi_6$ sum (1.3), we obtain with $\Phi$ the constant (4.4),
\[
\int_{-\infty}^{\infty} J(x) \, dx = \int_{-\infty}^{\infty} \phi(x) w(x) \, dx \\
= \int_{0}^{1} \phi^+(x) w(x) \, dx = \Phi \int_{0}^{1} \varphi_{-\frac{1}{2}}(x) w(x) \, dx
\]
by Fubini’s theorem. Using now the Jacobi triple product identity and (4.2), the resulting integral over the period cycle $[0, 1]$ is essentially the trigonometric Askey-Wilson integral (1.4) in base $\tilde{q}$. Its evaluation gives the desired result. \hfill \Box

**Remark 4.2.** Under the parameter constraints of Proposition 4.1 we have
\[
\int_{-\infty}^{\infty} f(x) J(x) \, dx = \Phi \int_{0}^{1} f(x) \varphi_{-\frac{1}{2}}(x) w(x) \, dx
\]
for a regular one-period function $f : \mathbb{R} \to \mathbb{C}$, where we used the notations introduced in the proof of Proposition 4.1. The latter integral is essentially the integral of $f$ against the Askey-Wilson measure depending on the four Askey-Wilson parameters $t_j$ ($j = 1, \ldots, 4$) and with base $\tilde{q}$. In particular, the Askey-Wilson polynomials \[7, \text{[13, §7.5]}\]
\[
p_n(x) = 4\phi_3\left(\tilde{q}^{-n}, \tilde{q}^{-n+1}; t_1t_2t_3t_4, t_1 \exp(2\pi ix), t_1 \exp(-2\pi ix) ; \tilde{q}, \tilde{q}\right), \quad n \in \mathbb{Z}_{\geq 0}
\]
form a basis of the space $\mathbb{C}[\cos(2\pi x)]$ of polynomials in $\cos(2\pi x)$, and they are orthogonal with respect to the complex pairing
\[
\langle p_1, p_2 \rangle = \int_{-\infty}^{\infty} p_1(x)p_2(x) J(x) dx.
\]
The “quadratic norms” $\langle p_n, p_n \rangle$ ($n \in \mathbb{Z}_{\geq 0}$) can be explicitly evaluated using the quadratic norm evaluations of the Askey-Wilson polynomials, see \[7\] and \[13, §7.5\] (the special case $n = 0$ is the statement of Proposition 4.1).

The proposition is also valid for $(s_1, s_2, s_3, s_4) = (q, -q, q^{\frac{1}{2}}, 0)$, in which case the resulting integral
\[
\int_{-\infty}^{\infty} \frac{(\exp(\pm 2\pi ix), -\exp(\pm 2\pi ix), \tilde{q}^{\frac{1}{2}} \exp(\pm 2\pi ix); \tilde{q})_\infty}{\prod_{j=1}^{4} (t_j \exp(\pm 2\pi ix); \tilde{q})_\infty} q^{\frac{s_2}{2}} dx
\]
\[
= \frac{2}{\sqrt{-i\tau}} \frac{\left(t_1t_2t_3t_4; \tilde{q}\right)_\infty}{\prod_{1 \leq k < m \leq 4} (t_k t_m; \tilde{q})_\infty}
\]
is the fusion of the Jacobi inversion formula (2.11) and the trigonometric Askey-Wilson integral (1.3). This integral may be viewed as a one-variable Macdonald-Mehta integral, see \[25\] and \[8\]. In particular, the special case $(t_1, t_2, t_3, t_4) = (0, \tilde{q}^{\frac{1}{2}}, \tilde{q}^k, -\tilde{q}^k)$ with $k \in \mathbb{R}_{> 0}$
leads to Cherednik’s integral yields

\[ \int_{-\infty}^{\infty} \frac{\exp(\pm 4\pi i x; \tilde{q}^2)_{\infty}}{(\tilde{q}^{2k} \exp(\pm 4\pi i x; \tilde{q}^2)_{\infty})^{\frac{1}{2}}} \frac{x^2}{\sqrt{-i\tau}} \frac{q^2}{(q^{2k} \tilde{q}^2)_{\infty}} = \frac{2}{\sqrt{-i\tau}} \frac{q^{2k}}{q^{4k} \tilde{q}^2}_{\infty}. \]

In a similar way we would like to put \((t_1, t_2, t_3, t_4) = (1, -1, \tilde{q}^\frac{1}{2}, 0)\) in the fused Askey-Wilson integral (Proposition 4.1), but this is not allowed since the integrand is irregular on the real line for these parameter values. A suitable regularization of this integral yields an integral evaluation which is closely related to a trigonometric beta integral of Askey [3, §3], and which has a similar appearance as Etingof’s [7, §3.5] Macdonald-Mehta type integral with real integration cycle. The result is as follows.

**Corollary 4.3.** Let \(\tau \in \mathbb{H}\) and \(s_j \in \mathbb{C}\) \((j = 1, \ldots, 4)\) with \(|q^{-3}s_1s_2s_3s_4| < 1\). Then

\[ \lim_{\epsilon \to 0} \int_{i\epsilon - \infty}^{i\epsilon + \infty} \frac{\prod_{j=1}^{4}(s_j q^{\pm z}; q)_{\infty}}{(q^{1 \pm z}, -q^{1 \pm z}, q^{\frac{1}{2} \pm z}; q)_{\infty}} q^2 dz = \frac{1}{\sqrt{-i\tau}} \frac{1}{\prod_{k=0}^{4}(q^{-1/s_k s_m}; q)_{\infty}}. \]

**Proof.** The integrand \(\bar{\phi}(z)\) is regular on \(i\epsilon + \mathbb{R}\) for \(\epsilon \in \mathbb{R}_{>0}\) sufficiently small. Folding the integral yields

\[ \int_{i\epsilon - \infty}^{i\epsilon + \infty} \bar{\phi}(z) dz = \int_{0}^{1} \phi^+(i\epsilon + x) dx = \Phi \int_{0}^{1} \vartheta_{-\frac{1}{\tau}}(i\epsilon + x) dx \]

by (4.3) and (4.4). By (2.12) we have

\[ \lim_{\epsilon \to 0} \int_{0}^{1} \vartheta_{-\frac{1}{\tau}}(i\epsilon + x) dx = \int_{0}^{1} \vartheta_{-\frac{1}{\tau}}(x) dx = 1, \]

which completes the proof. \(\square\)

### 4.3. The hyperbolic Askey-Wilson integral

In this section we derive the hyperbolic analogue of the Askey-Wilson integral and, as a special case, the analogue of the Askey-Wilson integral for \(|q| = 1\). The techniques are the same as for the derivation of the hyperbolic Ramanujan integral in §3.4.

For the moment we assume that \(\tau \in \mathbb{H} \cap \mathbb{C}_-\). As in the previous section we write \(q = q_\tau = \exp(2\pi i\tau)\) and \(\tilde{q} = \tilde{q}_\tau = \exp(-2\pi i/\tau)\), and \(q^u = \exp(2\pi i\tau u), \tilde{q}^u = \exp(-2\pi iu/\tau)\) for \(u \in \mathbb{C}\). We fix four parameters \(\tau_j\) \((j = 1, \ldots, 4)\) satisfying the parameter constraints

\[
\tau_1 - \tau_j \in \mathbb{H} \cap \mathbb{C}_+, \quad (1 - \tau_j) \tau \in \mathbb{H} \quad (j = 1, \ldots, 4),
\]

\[
(\tau_1 + \tau_2 + \tau_3 + \tau_4 - 3 - \tau^{-1}) \tau \in \mathbb{H} \cap \mathbb{C}_-.
\]

If we take parameters \(\tau_j \in \mathbb{C}\) satisfying

\[ \text{Re}(\tau_j) < 1, \quad \text{Im}(\tau_j) < 0, \quad \text{Re}(\tau_1 + \tau_2 + \tau_3 + \tau_4) > 3, \]

then sufficient additional conditions on their imaginary parts to satisfy the parameter constraints (4.5) are

\[ \text{Im}(\tau_j) > (1 - \text{Re}(\tau_j)) \frac{\text{Im}(\tau)}{\text{Re}(\tau)}, \quad \text{Im}(\tau_1 + \tau_2 + \tau_3 + \tau_4) > \text{Re}(\tau_1 + \tau_2 + \tau_3 + \tau_4 - 3) \frac{\text{Re}(\tau)}{\text{Im}(\tau)}. \]
The existence of parameters satisfying these inequalities follows from the fact that the right hand sides of the inequalities in (4.6) are strictly negative.

For fixed parameters $\tau_j$ satisfying the parameter constraints (4.5) we associate eight parameters $s_j, t_j$ ($j = 1, \ldots, 4$) by

$$s_j = q^{\tau_j}, \quad t_j = \exp(-2\pi i \tau_j), \quad (j = 1, \ldots, 4).$$

These eight parameters satisfy the parameter requirements of Proposition 4.1. The resulting fused Askey-Wilson integral can then be written in terms of the $\tau$-shifted factorial (1.7) as

$$\int_{-\infty}^{\infty} \left[1 \pm x, \frac{1}{2} \pm x, 1 + \frac{1}{2\tau} \pm x; \tau\right]_{\infty} q^{\frac{x^2}{2}} dx = \frac{2}{\sqrt{-i\tau}} \prod_{1 \leq k < m \leq 4} [\tau_k + \tau_m - 1; \tau)]_{\infty}.$$

Using the reflection equation (6.14) and

$$\left[\frac{1}{2} \pm x, 1 \pm x, \frac{1}{2} + \frac{1}{2\tau} \pm x, 1 + \frac{1}{2\tau} \pm x; \tau\right]_{\infty} = [1 \pm 2x; \tau]_{\infty}$$

(which follows easily from (1.7) and (4.2)), we can rewrite (4.8) as

$$\int_{-\infty}^{\infty} \left[1 \pm 2x; \tau\right]_{\infty} \frac{dx}{\prod_{j=1}^{4} [\tau_j \pm x; \tau]_{\infty}} = \frac{2q^{-\frac{1}{4}q^{2} - \frac{1}{4}q^{2}\left(q^{-z} - q^{z}\right)} \left(\exp(2\pi iz) - \exp(-2\pi iz)\right)}{\prod_{1 \leq k < m \leq 4} [\tau_k + \tau_m - 1; \tau)]_{\infty}},$$

which now has a very similar appearance as the trigonometric Askey-Wilson integral (1.4). The numerator of the integrand

$$I(z) = \frac{[1 \pm 2z; \tau]_{\infty}}{\prod_{j=1}^{4} [\tau_j \pm z; \tau]_{\infty}}$$

can be simplified as follows.

**Lemma 4.4.**

$$[1 \pm 2z; \tau]_{\infty} = -i q^{\frac{1}{4}q^{2} - \frac{1}{2}q^{2}\left(q^{2} - q^{-2}\right)} \left(\exp(2\pi iz) - \exp(-2\pi iz)\right).$$

**Proof.** By the reflection equation (6.14) we can write

$$[1 \pm 2z; \tau]_{\infty} = q^{-\frac{1}{4}q^{2} - \frac{1}{2}q^{2}\left(q^{2} - q^{-2}\right)} \left(\frac{2z + 1; \tau]_{\infty}}{2z + \tau^{-2} \tau^{-1}; \tau)]_{\infty}.$$  

Now apply the functional equations (6.10) to the numerator and denominator to get rid of the $\tau$-shifted factorials. Simplifying the resulting elementary expression gives the desired result. \qed

By the zero and pole locations of the $\tau$-shifted factorial (see (6.11) and (6.12) respectively) we conclude that the poles of $I(z)$ are contained in the eight discrete sets

$$\pm \left(\tau_j + \frac{1}{\tau} \mathbb{Z}_{\geq 0} + \mathbb{Z}_{< 0}\right) \quad (j = 1, \ldots, 4).$$

By the parameter constraints $\tau \in \mathbb{H} \cap \mathbb{C}_-$ and (4.5), the four sets in (4.12) with plus sign are contained in the interior of the third quadrant of the complex plane, and consequently
the four sets with minus sign are contained in the interior of the first quadrant of the complex plane.

**Lemma 4.5.** Suppose that \( \tau \in \mathbb{H} \cap \mathbb{C}_- \) and that the parameters \( \tau_j \) \( (j = 1, \ldots, 4) \) satisfy (4.5). Then there exists a constant \( C \in \mathbb{R}_{>0} \) such that

\[
|I(z)| \leq C \left| q^{(\tau_1 + \tau_2 + \tau_3 + \tau_4 - 3 - \tau^{-1})z} \right|
\]

for \( z \in \mathbb{C} \) satisfying \( \text{Re}(z) \geq 0 \) and \( \text{Im}(z) \leq 0 \).

**Proof.** By Lemma 4.4 the reflection equation (6.11) and by (1.7) we can rewrite the integrand \( I(z) \) as

\[
I(z) = C \prod_{j=1}^{4} \left( \frac{\bar{q} \exp(2\pi i \tau_j) \exp(-2\pi i z); \bar{q}}{\exp(-2\pi i \tau_j) \exp(-2\pi i z); \bar{q}} \right)_{\infty} \left( q^{\tau_j} q^z; q \right)_{\infty} \times (1 - \exp(-4\pi i z)) \left( 1 - q^{2z} \right) q^{(\tau_1 + \tau_2 + \tau_3 + \tau_4 - 3 - \tau^{-1})z}
\]

for some irrelevant nonzero constant \( C \). The proof is now analogous to the proof of Lemma 3.2 since \( |\exp(-2\pi i \tau_j)| < 1 \) and \( |q^{1-\tau_j}| < 1 \) by the conditions (4.5) on the parameters. \( \square \)

Note that a bound for the weight function \( I(z) \) for \( z \in \mathbb{C} \) satisfying \( \text{Re}(z) \leq 0 \) and \( \text{Im}(z) \geq 0 \) can be immediately deduced from Lemma 4.5 using (4.10) to \( \exp(-\pi i \tau) \) about the origin without altering its evaluation. Relaxing the parameter constraints leads now to the following result.

**Corollary 4.6.** Let \( \tau \in \mathbb{H} \cap \mathbb{C}_- \) and suppose that the parameters \( \tau_j \) satisfy the parameter constraints (4.5). Let \( 0 < \epsilon < 1 \) be the growth coefficient \( \epsilon = \exp(-2\pi m) \) with \( m \) the strictly positive constant

\[
m = \min_{\theta \in [-\pi, 0]} \left( \text{Im}((\tau_1 + \tau_2 + \tau_3 + \tau_4 - 3 - \tau^{-1})\tau e^{i\theta}) \right).
\]

Then there exists a constant \( C \in \mathbb{R}_{>0} \) such that

\[
|I(z)| \leq C e^{\epsilon |z|}
\]

for \( z \in \mathbb{C} \) satisfying \( \text{Re}(z) \geq 0 \) and \( \text{Im}(z) \leq 0 \), as well as for \( z \in \mathbb{C} \) satisfying \( \text{Re}(z) \leq 0 \) and \( \text{Im}(z) \geq 0 \).

We keep the assumptions \( \tau \in \mathbb{H} \cap \mathbb{C}_- \) and (4.5). In view of Corollary 4.6 we may apply Cauchy’s theorem to rotate clockwise the integration contour \( \mathbb{R} \) in (4.10) to \( -i\mathbb{R} \) about the origin without altering its evaluation. Relaxing the parameter constraints leads now to the following result.

**Theorem 4.7** (Hyperbolic Askey-Wilson integral). Let \( \tau \in \mathbb{C} \) with \( \text{Re}(\tau) < 0 \) and \( \text{Im}(\tau) \geq 0 \), and suppose that the parameters \( \tau_j \in \mathbb{C} \) \( (j = 1, \ldots, 4) \) satisfy

\[
\text{Re}(\tau_j) < 1 \quad (j = 1, \ldots, 4), \quad \text{Re}((\tau_1 + \tau_2 + \tau_3 + \tau_4 - 3)\tau) < 1.
\]

Then

\[
\int_{i\infty}^{i\infty} \frac{[1 \pm 2z; \tau]_{\infty}}{i \prod_{j=1}^{4} [\tau_j \pm z; \tau]_{\infty}} \, dz = \frac{2q^{-\frac{\pi}{2} \tau}}{\sqrt{-i\tau}} \prod_{1 \leq k < m \leq 4} [\tau_k + \tau_m - 1; \tau]_{\infty}.
\]
Proof. For $\tau \in \mathbb{H} \cap \mathbb{C}_-$ we already argued that the integral evaluation (4.14) is valid for parameters $\tau_j$ satisfying the parameter constraints (4.5). By analytic continuation one proves that the integral evaluation (4.14) is valid for parameters $\tau_j$ satisfying the milder constraints (4.13). In fact, by (6.15) and (6.16) the open convex parameter domain of parameters $(\tau_1, \tau_2, \tau_3, \tau_4)$ satisfying the constraints (4.13) is the maximal extension of the parameter domain (4.5) such that the associated integrand $I(z)$ decays exponentially as $z \to \pm i\infty$ and such that the sequence of poles $\tau_j + \frac{1}{z} \mathbb{Z}_{\geq 0} + \mathbb{Z}_{< 0}$ $(j = 1, \ldots, 4)$ lie in $\mathbb{C}_-$.

The extension of the integral evaluation (4.14) to $\tau \in \mathbb{R}_{< 0}$ follows now in a similar manner as for the hyperbolic Ramanujan integral (see the proof of Theorem 3.4) using the reflection equation (6.14) and the asymptotics (6.15) and (6.16) of the $\tau$-shifted factorial. □

Remark 4.8. The Askey-Wilson polynomials (see Remark 4.2) do not remain an orthogonal system in the hyperbolic case. Indeed, the Askey-Wilson polynomials are one-periodic hence bounded on $\mathbb{R}$, but they grow exponentially in the imaginary direction (with growth rate $\exp(2\pi n |\text{Im}(z)|)$ where $n$ is the degree of the Askey-Wilson polynomial). The rotation of the integration cycle $\mathbb{R}$ to $-i\mathbb{R}$ in the complex orthogonality relations of Remark 4.2 is only allowed if the integrand has uniform exponential decay in the second and fourth quadrant of the complex plane. For given parameters satisfying (4.5), this will thus only be the case for certain low degree Askey-Wilson polynomials.

On the other hand, in [26] an hyperbolic analogue of the Askey-Wilson function is defined, which is expected to play the role of integral kernel for some generalized $q$-Fourier transform with $|q| = 1$.

5. THE HYPERBOLIC NASSRALLAH-RAHMAN INTEGRAL

5.1. A weak $\Psi_8$ summation formula. We take $\tau \in \mathbb{H}$, so that $|q| < 1$, and we fix five generic nonzero complex parameters $t_j$ $(j = 0, \ldots, 4)$. It is convenient to write $A = t_0 t_1 t_2 t_3 t_4$ for the product of the five parameters. For the hyperbolic analogue of the Nassrallah-Rahman integral, the role of the function $\phi$ in the underlying summation formula (see §2) is

$$
\phi(z) = \frac{\prod_{j=0}^4 (t_j q^{1+z}; q)^{\infty}}{(q^{1+z}, -q^{1+z}, q^{2+z}, Aq^{-4+z}; q)^{\infty}}.
$$

As in §2, we define

$$
\tilde{\phi}(z) = \phi(z) q^{z^2}.
$$

A direct computation shows that

$$
\tilde{\phi}(z + 1) = t(z)\tilde{\phi}(z)
$$

with $t(z)$ given by

$$
t(z) = q \frac{(1 - q^{1+z})(1 + q^{1+z})(1 - Aq^{-4+z})}{(1 - q^2)(1 + q^2)(1 - A^{-1}q^{5+z})} \prod_{j=0}^4 \frac{(1 - t_j^{-1}q^{1+z})}{(1 - t_j q^2)}.
$$
The bilateral sum
\[ t^+(z) = \sum_{m=0}^{\infty} \prod_{k=0}^{m-1} t(z + k) + \sum_{m=-\infty}^{-1} \prod_{k=m}^{-1} \frac{1}{t(z + k)} \]
\[ = s \Psi_8 \left( q^{1+z}, -q^{1+z}, t_0^{-1}q^{1+z}, t_1^{-1}q^{1+z}, t_2^{-1}q^{1+z}, t_3^{-1}q^{1+z}, t_4^{-1}q^{1+z}, Aq^{-4+z} ; q, q \right) \]
converges absolutely and uniformly on compacta away from the \( \mathbb{Z}_{\leq 0} \)-translates of the poles and the \( \mathbb{Z}_{>0} \)-translates of the zeros of \( t(z) \), and it extends to a meromorphic function in \( z \in \mathbb{C} \). Hence the bilateral sum
\[ \phi^+(z) = \sum_{m=-\infty}^{\infty} \tilde{\phi}(z + m) \]
converges to a meromorphic function in \( z \in \mathbb{C} \), given explicitly by
\[ \phi^+(z) = t^+(z)\tilde{\phi}(z). \]
Unfortunately, there is no explicit summation formula on the \( s \Psi_8 \) level. Instead we apply a formula that expresses \( t^+(z) \) as a combination of two one-sided sums which only involve \( z \)-independent summands. This can be done by applying \( [13, (5.6.2)] \) with its seven parameters \( a, b, c, \ldots, g \) specialized to
\[ (a, b, c, d, e, f, g) = (q^2, qt_1^{-1}, qt_2^{-1}, \ldots, qt_0^{-1}, q^{-1}A). \]
After a straightforward but tedious computation using \( [14, 15] \), the Jacobi triple product identity and the Jacobi inversion formula, we then arrive at the explicit expression
\[ t^+(z) = \left\{ C_1 + \frac{(t_0q^{\pm z}, t_0^{-1}q^{\pm z}; q)_\infty}{(A_q^{-4+z}, A^{-1}q^{5\pm z}; q)_\infty} C_2 \right\} \tilde{\phi}(z)^{-1} q^{-\frac{z}{2}}(z) \]
with the \( z \)-independent constants \( C_1 \) and \( C_2 \) given by
\[ C_1 = \frac{1}{\sqrt{-i\tau}} \frac{\prod_{j=1}^{4} (q^{-1} t_0 t_j, q t_0^{-1} t_j; q)_\infty}{(q^{-5} A t_0, q^{-3} A t_0^{-1}, q^{-2} t_0^{-2}; q)_\infty} \]
\times \[ s W_7 \left( q^2 t_0^{-2}, q^2 t_0^{-1} t_1^{-1}, q^2 t_0^{-1} t_2^{-1}, q^2 t_0^{-1} t_3^{-1}, q^2 t_0^{-1} t_4^{-1}, q^{-3} A t_0^{-1}; q, q \right) \]
and
\[ C_2 = \frac{1}{\sqrt{-i\tau}} \frac{\prod_{j=1}^{4} (q^4 A^{-1} t_j, q^{-4} A t_j; q)_\infty}{(q^5 A^{-1} t_0, q^{-3} A t_0^{-1}, q^{-2} A^2; q)_\infty} \]
\times \[ s W_7 \left( q^{-8} A^2, q^{-3} A t_0^{-1}, q^{-3} A t_1^{-1}, q^{-3} A t_2^{-1}, q^{-3} A t_3^{-1}, q^{-3} A t_4^{-1}; q, q \right) \]
The meromorphic function \( \phi^+(z) \) can thus be written as
\[ \phi^+(z) = \Phi(z) q^{-\frac{z}{2}}(z), \quad \Phi(z) = C_1 + \frac{(t_0 q^{\pm z}, t_0^{-1} q^{\pm z}; q)_\infty}{(A q^{-4+z}, A^{-1} q^{5\pm z}; q)_\infty} C_2. \]
The function $\Phi(z)$ is an elliptic function in $z$ with respect to the periods 1 and $-\frac{1}{2}$. For later purposes it is convenient to rewrite $\Phi(z)$ as follows. Fix $\tau_j \in \mathbb{C}$ $(j = 0, \ldots, 4)$ such that

$$t_j = q^{\gamma_j} \quad (j = 0, \ldots, 4)$$

and denote $a = \tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4$, so that $A = q^a$. We define new nonzero complex parameters $\tilde{t}_j$ $(j = 0, \ldots, 4)$ by

$$\tilde{t}_j = \exp(-2\pi i \tau_j) \quad (j = 0, \ldots, 4),$$

and we write $\tilde{A} = \tilde{t}_0 \tilde{t}_1 \tilde{t}_2 \tilde{t}_3 \tilde{t}_4 = \exp(-2\pi ia)$. Then

$$\Phi(z) = C_1 + \left(\tilde{t}_0 \exp(\pm 2\pi iz), \tilde{q} \tilde{t}_0^{-1} \exp(\pm 2\pi iz); \tilde{q}\right)_\infty C_2 q^\gamma,$$

(5.5)

$$\gamma = \left(a - \frac{9}{2} - \frac{1}{2\tau}\right)^2 - \left(\tau_0 - \frac{1}{2} - \frac{1}{2\tau}\right)^2$$

by a direct computation using the Jacobi triple product identity and the Jacobi inversion formula.

5.2. Fusion. In this subsection we fuse the weak $\Psi_8$ summation formula [5,4] with the Nassrallah-Rahman integral [13]. For proofs of the Nassrallah-Rahman integral [13] itself, see [16, 13] (6.4.1) and references therein.

We fix $\tau \in \mathbb{H}$ and generic complex parameters $\tau_j$. As in the previous subsection, we write

$$t_j = q^{\gamma_j}, \quad \tilde{t}_j = \exp(-2\pi i \tau_j) \quad (j = 0, \ldots, 4)$$

and $A = q^a$, $\tilde{A} = \exp(-2\pi ia)$ with $a = \tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4$. We assume throughout this subsection that $\text{Im}(\tau_j) < 0$ for $j = 0, \ldots, 4$ and that $\text{Im}(a) > \text{Im}(\tau^{-1})$. These conditions imply that

$$|\tilde{t}_j| < 1 \quad (j = 0, \ldots, 4), \quad |\tilde{A}| > |\tilde{q}|.$$  

By [14,9], the reflection equation (6.14) and the definition (1.7) of the $\tau$-shifted factorial, we can write

$$\left[1 \pm 2z, a - 4 \pm z; \tau\right]_\infty = q^{-\frac{1}{2}\pi} q^{\frac{1}{2}} \tilde{\phi}(z) v(z)$$

(5.7)

with $\tilde{\phi}(z)$ as in §5.1, and with $v(z)$ the one-periodic function

$$v(z) = \frac{(\exp(\pm 2\pi iz), -\exp(\pm 2\pi iz), \tilde{q}^\frac{1}{2} \exp(\pm 2\pi iz), \tilde{A} \exp(\pm 2\pi iz); \tilde{q})_\infty}{\prod_{j=0}^4 (\tilde{t}_j \exp(\pm 2\pi iz); \tilde{q})_\infty}.$$  

(5.8)

Note that (5.7) is regular on $\mathbb{R}$ in view of the conditions on the parameters $\tau_j$. By Fubini’s theorem, the integral

$$\int_{-\infty}^{\infty} \left[1 \pm 2x, a - 4 \pm x; \tau\right]_\infty dx = q^{-\frac{1}{2}\pi} q^{\frac{1}{2}} \int_{-\infty}^{\infty} \tilde{\phi}(x) v(x) dx$$

(5.9)
can now be folded,
\[
\int_{-\infty}^{\infty} \tilde{\phi}(x)v(x)\,dx = \int_{0}^{1} \tilde{\phi}^+(x)v(x)\,dx
\]
\[
= C_1 \int_{0}^{1} v(x)\vartheta_{\frac{\pi}{4}}(x)\,dx
\]
\[
+ C_2 q^\gamma \int_{0}^{1} v(x) \left( \frac{t_0 \exp(\pm 2\pi ix), \tilde{q} t_0^{-1}\exp(\pm 2\pi ix); \tilde{q}}{\tilde{A}\exp(\pm 2\pi ix), \tilde{q} \tilde{A}^{-1}\exp(\pm 2\pi ix); \tilde{q}} \right)_{\infty} \vartheta_{\frac{\pi}{4}}(x)\,dx
\]
by (5.4) and (5.5). Applying the Jacobi triple product identity and (4.2), the remaining two integrals over the period interval [0, 1] can be evaluated by the trigonometric Nassrallah-Rahman integral (1.3). This gives
\[
\int_{-\infty}^{\infty} \tilde{\phi}(x)v(x)\,dx = \frac{2 \prod_{j=1}^{4} \left( \tilde{A} t_j^{-1}, \tilde{q} t_j^{-1}; \tilde{q} \right)_{\infty}}{\prod_{0 \leq k < m \leq 4} (t_k t_m; \tilde{q})_{\infty}} \left\{ C_1 + C_2 q^\gamma \prod_{j=1}^{4} \left( \frac{t_0 t_j, \tilde{q} t_0^{-1} t_j^{-1}; \tilde{q}}{\tilde{A} t_j^{-1}, \tilde{q} \tilde{A}^{-1} t_j; \tilde{q}} \right)_{\infty} \right\},
\]
with the constants $C_1$ and $C_2$ given by (5.2) and (5.3), respectively. The following key lemma gives the evaluation of the remaining sum.

**Lemma 5.1.**

\[
C_1 + C_2 q^\gamma \prod_{j=1}^{4} \left( \frac{t_0 t_j, \tilde{q} t_0^{-1} t_j^{-1}; \tilde{q}}{\tilde{A} t_j^{-1}, \tilde{q} \tilde{A}^{-1} t_j; \tilde{q}} \right)_{\infty} = \frac{1}{\sqrt{-i\tau}} \frac{\prod_{0 \leq k < m \leq 4} (q^{-1} t_k t_m; q)_{\infty}}{\prod_{j=1}^{4} (q^{-3} A t_j^{-1}; q)_{\infty}}.
\]

**Proof.** Using the Jacobi triple product identity and the Jacobi inversion formula we can write
\[
\prod_{j=1}^{4} \left( \frac{t_0 t_j, \tilde{q} t_0^{-1} t_j^{-1}; \tilde{q}}{\tilde{A} t_j^{-1}, \tilde{q} \tilde{A}^{-1} t_j; \tilde{q}} \right)_{\infty} = q^\delta \prod_{j=1}^{4} \left( \frac{q^{-1} t_0 t_j, q^2 t_0^{-1} t_j^{-1}; q}{{q^{-3} A t_j^{-1}, q^4 A^{-1} t_j; q}} \right)_{\infty}
\]
with $\delta$ given explicitly by
\[
\delta = \frac{1}{2} \sum_{j=1}^{4} \left\{ \left( \tau_0 + \tau_j - \frac{3}{2} - \frac{1}{2\tau} \right)^2 - \left( \alpha - \tau_j - \frac{7}{2} - \frac{1}{2\tau} \right)^2 \right\}.
\]
Writing out the squares for $\gamma$ (5.5) and $\delta$ gives
\[
\gamma = 4\tau^{-1} + 20 - 8\tau_0 + \sum_{j=1}^{4} (\tau_j^2 - (9 + \tau^{-1})\tau_j) + 2 \sum_{0 \leq k < m \leq 4} \tau_k \tau_m = -\delta,
\]
hence
\[
C_1 + C_2 q^\gamma \prod_{j=1}^{4} \left( \frac{t_0 t_j, \tilde{q} t_0^{-1} t_j^{-1}; \tilde{q}}{\tilde{A} t_j^{-1}, \tilde{q} \tilde{A}^{-1} t_j; \tilde{q}} \right)_{\infty} = C_1 + C_2 \prod_{j=1}^{4} \left( \frac{q^{-1} t_0 t_j, q^2 t_0^{-1} t_j^{-1}; q}{{q^{-3} A t_j^{-1}, q^4 A^{-1} t_j; q}} \right)_{\infty}
\]
with $C_1$ and $C_2$ given by (5.2) and (5.3), respectively. The right hand side of (5.11) can be evaluated by Bailey’s summation formula \cite{13} (2.11.7) with the six parameters $a, b, \ldots, f$ specialized to

$$(a, b, c, d, e, f) = (q^2 t_0^{-1}, q^{-3} A t_0^{-1}, q^2 t_0^{-1} t_1^{-1}, q^2 t_0^{-1} t_2^{-1}, q^2 t_0^{-1} t_3^{-1}, q^2 t_0^{-1} t_4^{-1}).$$

This yields the desired result. \hfill \Box

Combining the lemma with (5.7) and (5.10) immediately implies the following result.

**Proposition 5.2.** Let $\tau \in \mathbb{H}$ and $\tau_j \in \mathbb{C}$ with $\text{Im}(\tau_j) < 0$ ($j = 0, \ldots, 4$) and $\text{Im}(a) > \text{Im}(\tau^{-1})$, where $a = \tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4$. Then

$$\int_{-\infty}^{\infty} \frac{[1 + 2x, a - 4 \pm x; \tau]_\infty}{\prod_{j=0}^{4} [\tau_j \pm x; \tau]_\infty} \text{d}x = \frac{2q^{\frac{1}{2}} \tau^{\frac{1}{2}}}{\sqrt{-i\tau}} \frac{\prod_{j=0}^{4} [a - \tau_j - 3; \tau]_\infty}{\prod_{0 \leq k < m \leq 4} [\tau_k + \tau_m - 1; \tau]_\infty}.$$  

(5.12)

**Remark 5.3.** The weak $8\Psi_8$ summation formula (5.4) involves an elliptic function $\Phi(z)$ depending nontrivially on the parameters $t_j$. Consequently, the fusion of the weak $8\Psi_8$ summation formula with the trigonometric Nassrallah-Rahman integral (1.5) only has an explicit evaluation when the parameters in the trigonometric Nassrallah-Rahman integral are matched to the parameters in the weak $8\Psi_8$ summation formula via (5.6). This is an essential difference with the fused Ramanujan integral in §3.3 and the fused Askey-Wilson integral in §4.2.

It is possible to fuse the trigonometric Nassrallah-Rahman integral (1.5) in base $\tilde{q}$ with the Jacobi inversion formula (2.11), yielding the one-variable Macdonald-Mehta type integral

$$\int_{-\infty}^{\infty} \left( \exp(\pm 2\pi i x), - \exp(\pm 2\pi i x), \tilde{q}^2 \exp(\pm 2\pi i x), A \exp(\pm 2\pi i x); \tilde{q} \right)_\infty \text{d}x$$

$$= \frac{2}{\sqrt{-i\tau}} \frac{\prod_{j=0}^{4} (At_j^{-1}; \tilde{q})_\infty}{\prod_{0 \leq k < m \leq 4} (tk tm; \tilde{q})_\infty}$$

(5.13)

with $A = t_0 t_1 t_2 t_3 t_4$ and with parameters $t_j \in \mathbb{C}$ satisfying $|t_j| < 1$ ($j = 0, \ldots, 4$).

### 5.3. The hyperbolic Nassrallah-Rahman integral

Let $\tau \in \mathbb{H} \cap \mathbb{C}_-$ with $\text{Re}(\tau^{-1}) < -1$ and fix five parameters $\tau_j$ ($j = 0, \ldots, 4$) satisfying

$$1 - \tau_j, a - 4 - \tau_j^{-1} \in \mathbb{H} \cap \mathbb{C}_+,$$

(5.14)

$$1 - (1 - \tau_j) \tau_j, (a - 4) \tau \in \mathbb{H}$$

for $j = 0, \ldots, 4$, where $a = \tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4$. If the real parts of the parameters $\tau_j$ satisfy

$$\text{Re}(\tau_j) < 1 \quad (j = 0, \ldots, 4), \quad \text{Re}(a) > 4,$$

then they satisfy (5.14) when the imaginary parts of $\tau_j$ satisfy

$$\text{Im}(\tau_j) < 0, \quad \text{Im}(a) \frac{\text{Re}(\tau)}{\text{Im}(\tau)} < -1 - \text{Re}(\tau^{-1})$$

(5.16)
for \( j = 0, \ldots, 4 \). Note that the condition \( \Re(\tau^{-1}) < -1 \) implies \(-1 - \Re(\tau^{-1}) > 0\), hence there exist parameters \( \tau_j \) satisfying (5.15) and (5.16).

The constraints (5.11) imply the parameter conditions of Proposition 5.2. We now rotate the integration cycle \( \mathbb{R} \) clockwise to \(-i\mathbb{R}\) about the origin in the fused Nassrallah-Rahman integral (5.12). By the zero and pole locations of the \( \tau \)-shifted factorial (see (6.11) and (6.12) respectively) and by Lemma 4.4 the poles of the integrand for (5.12) are contained in the interior of the twelve discrete sets

\[
(5.18) \quad \pm \left( \tau_j + \frac{1}{\tau} \mathbb{Z}_{\geq 0} + \mathbb{Z}_{< 0} \right) \quad (j = 0, \ldots, 4), \quad \pm \left( -a + \frac{1}{\tau} \mathbb{Z}_{> 0} + \mathbb{Z}_{\leq 4} \right).
\]

By the parameter constraints \( \tau \in \mathbb{H} \cap \mathbb{C}_- \), \( \Re(\tau^{-1}) < -1 \) and (5.14), the six sets in (5.18) with plus sign are contained in the interior of the third quadrant of the complex plane, and consequently the six sets in (5.18) with minus sign are contained in the interior of the first quadrant of the complex plane. Furthermore, the integrand \( I(z) \) is even, \( I(-z) = I(z) \), and by (5.14) and (1.7) it can be rewritten as

\[
I(z) = C (1 - q^{\tau z}) \left( 1 - \exp(-4\pi iz) \right) \frac{\left( \exp(-2\pi ia) \exp(-2\pi iz); \tilde{q} \right)_{\infty} \left( q^{a-\tau} q^2; q \right)_{\infty}}{\left( \tilde{q} \exp(2\pi ia) \exp(-2\pi iz); \tilde{q} \right)_{\infty} \left( q^{-4} q^2; q \right)_{\infty}} \times \left\{ \prod_{j=0}^4 \frac{\left( \tilde{q} \exp(2\pi i\tau_j) \exp(-2\pi iz); \tilde{q} \right)_{\infty} \left( q^{\tau_j} q^2; q \right)_{\infty}}{\left( \exp(-2\pi i\tau_j) \exp(-2\pi iz); \tilde{q} \right)_{\infty} \left( q^{1-\tau_j} q^2; q \right)_{\infty}} \right\} z^2 \exp(-2\pi iz)
\]

for some nonzero constant \( C \). With similar arguments as in the proof of Lemma 3.2 and Corollary 3.3 (or Lemma 4.5 and Corollary 4.6) we conclude that for some \( C_1 \in \mathbb{R}_{> 0} \),

\[
|I(z)| \leq C_1 |z|^2
\]

for \( z \in \mathbb{C} \) satisfying \( \Re(z) \geq 0 \) and \( \Im(z) \leq 0 \) as well as for \( z \in \mathbb{C} \) satisfying \( \Re(z) \leq 0 \) and \( \Im(z) \geq 0 \), with growth exponent \( 0 < \epsilon = \exp(-2\pi m) < 1 \),

\[
m = \min_{\theta \in [-\frac{\pi}{2}, 0]} \left( (\tau - 1) e^{i\theta} \right) > 0.
\]

Thus we may apply Cauchy’s theorem to rotate clockwise the integration cycle \( \mathbb{R} \) in (5.12) to \(-i\mathbb{R}\) about the origin without altering its evaluation. Relaxing the parameter constraints leads to the following result.

**Theorem 5.4 (Hyperbolic Nassrallah-Rahman integral).** Let \( \tau \in \mathbb{C} \) with \( \Re(\tau) < 0 \) and \( \Im(\tau) \geq 0 \), and suppose that the parameters \( \tau_j \in \mathbb{C} \) (\( j = 0, \ldots, 4 \)) satisfy

\[
(5.19) \quad \Re(\tau_j) < 1 \quad (j = 0, \ldots, 4), \quad \Re(a - \tau^{-1}) > 4,
\]
where \( a = \tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4 \). Then

\[
\int_{-\infty}^{\infty} \frac{[1 \pm 2z, a - 4 \pm z; \tau]_\infty}{\prod_{j=0}^{4} [\tau_j \pm z; \tau]_\infty} \, dz = -\frac{2 q^{-\frac{1}{2}} \bar{q}^{\frac{3}{4}}}{\sqrt{-i\tau}} \prod_{0 \leq k < m \leq 4} [\tau_k + \tau_m - 1; \tau]_\infty
\]

Proof. For \( \tau \in \mathbb{H} \cap \mathbb{C}_- \) satisfying \( \text{Re}(\tau^{-1}) < -1 \) we have already argued that (5.20) is valid for parameters \( \tau_j \) satisfying the constraints (5.19). By analytic continuation we conclude that (5.20) is valid under the milder parameter constraints (5.19), using similar arguments as in the proof of Theorem 3.4.

The condition \( \text{Re}(\tau^{-1}) < -1 \) can be removed by analytic continuation using the asymptotics (uniform for \( \tau \) in compacta) of the \( \tau \)-shifted factorial, see (6.15) and (6.16). The extension of (5.20) to \( \tau \in \mathbb{R}_{<0} \) then follows as in the proof of Theorem 3.4.

Remark 5.5. The hyperbolic Askey-Wilson integral (4.14) is formally a limit case of the hyperbolic Nassrallah-Rahman integral. For example, let \( \tau \in \mathbb{R}_{<0} \) and choose parameters \( \tau_j \in \mathbb{C} \) \((j = 1, \ldots, 4)\) satisfying (4.13). Concretely, the \( \tau_j \)'s thus satisfy

\[ \text{Re}(\tau_j) < 1, \quad \text{Re}(\tau_1 + \tau_2 + \tau_3 + \tau_4) > 3 + \tau^{-1}. \]

Choose \( \tau'_0 \in \mathbb{R}_{<1} \) such that

\[ \tau'_0 + \text{Re}(\tau_1 + \tau_2 + \tau_3 + \tau_4) > 4 + \tau^{-1}, \]

then any five tuple \((\tau_0, \tau_1, \tau_2, \tau_3, \tau_4)\) with \( \text{Re}(\tau_0) = \tau'_0 \) satisfies the parameter constraints (5.19). We can now formally take the limit \( \text{Im}(\tau_0) \to -\infty \) in the hyperbolic Nassrallah-Rahman integral (5.20) while keeping \( \text{Re}(\tau_0) = \tau'_0 \) fixed. By (6.17), this formal limit of the hyperbolic Nassrallah-Rahman integral is the hyperbolic Askey-Wilson integral (4.14).

5.4. The hyperbolic degeneration of the elliptic Nassrallah-Rahman integral.

We introduce the short hand notation \( \mathcal{I} = (t_0, t_1, t_2, t_3, t_4) \). With the notations of §1, we write

\[
\Delta(z; \mathcal{I}; p_1, p_2) = \frac{\prod_{j=0}^{4} \Gamma(t_j z^{\pm 1}; p_1, p_2)}{\Gamma(z^{\pm 2}, A z^{\pm 1}; p_1, p_2)}
\]

with \( A = t_0 t_1 t_2 t_3 t_4 \) and

\[
N(\mathcal{I}; p_1, p_2) = \frac{2}{(p_1, p_1)_{\infty} (p_2, p_2)_{\infty}} \prod_{0 \leq k < m \leq 4} \Gamma(t_k t_m; p_1, p_2) \prod_{j=0}^{4} \Gamma(A t_j^{-1}; p_1, p_2),
\]

so that

\[
\frac{1}{2\pi i} \int_{\gamma} \Delta(z; \mathcal{I}; p_1, p_2) \frac{dz}{z} = N(\mathcal{I}; p_1, p_2)
\]

is Spiridonov’s [23] elliptic Nassrallah-Rahman integral (1.12), valid for \(|t_j|, |p_k| < 1\) and \(|p_1 p_2| < |A|\).

Fix \( \tau \in \mathbb{R}_{<0} \). We introduce the limiting parameter \( r \in \mathbb{R}_{>0} \) by writing \( \Delta_r(x; \mathcal{I}; \tau) \) and \( N_r(\mathcal{I}; \tau) \) for the weight function \( \Delta(\exp(2\pi i r x); \mathcal{I}; p_1, p_2) \) and the norm \( N(\mathcal{I}; p_1, p_2) \) with parameters

\[
t_j = \exp(2\pi r (\tau_j - 1)), \quad p_1 = \exp(2\pi r / \tau), \quad p_2 = \exp(-2\pi r).
\]
The conditions \(|t_j| < 1\) and \(|p_1 p_2| < |A|\) then translate to the parameter constraints (5.19) for the corresponding hyperbolic Nassrallah-Rahman integral (5.20).

The elliptic beta integral (5.23) can then be rewritten as

\[
\int_{-\frac{\pi}{2r}}^{\frac{\pi}{2r}} \Delta_r(x; \tau) \, dx = r^{-1} N_r(\tau; \tau).
\]

We show in this subsection that the formal limit \(r \downarrow 0\) of (5.24) gives the hyperbolic Nassrallah-Rahman integral (1.10). The limit is based on Ruijsenaars’ [19, Prop. III.12] observation that the hyperbolic gamma function is a limit case of the elliptic gamma function (see (6.18) and (6.19) for the explicit limit transition in the present notations).

We first rewrite the weight function \(\Delta_r(x; \tau)\) and the norm \(N_r(\tau; \tau)\) in terms of the renormalized elliptic gamma function \(\tilde{\Gamma}_r(z; \tau)\) (see (6.18)). This yields the expressions

\[
\Delta_r(x; \tau) = \prod_{j=0}^{4} \frac{\tilde{\Gamma}_r(i - i \tau_j \pm x; \tau)}{\Gamma_r(\pm 2x, 5i - ia \pm x; \tau)} \exp \left( \pi \frac{1}{2r} (1 - \tau) \right),
\]

\[
N_r(\tau; \tau) = \frac{2}{(\exp(-2\pi r); \exp(-2\pi r))_{\infty} (\exp(2\pi r / \tau); \exp(2\pi r / \tau))_{\infty}} \times \frac{\prod_{0 \leq k < m \leq 4} \tilde{\Gamma}_r(2i - i \tau_k - i \tau_m; \tau)}{\prod_{j=0}^{4} \tilde{\Gamma}_r(4i - ia + i \tau_j; \tau)} \exp \left( \frac{5\pi}{12r} (1 - \tau) \right)
\]

with \(a = \tau_0 + \tau_1 + \tau_2 + \tau_3 + \tau_4\), where we use the notations

\[
\tilde{\Gamma}_r(a_1, \ldots, a_m; \tau) = \prod_{j=1}^{m} \tilde{\Gamma}_r(a_j; \tau),
\]

\[
\tilde{\Gamma}_r(a_1 \pm z_1, \ldots, a_m \pm z_m; \tau) = \prod_{j=1}^{m} \tilde{\Gamma}_r(a_j + z_j, a_j - z_j; \tau)
\]

for products of renormalized elliptic gamma functions. Consequently, (5.24) can be rewritten as

\[
\int_{-\frac{\pi}{2r}}^{\frac{\pi}{2r}} \prod_{j=0}^{4} \frac{\tilde{\Gamma}_r(i - i \tau_j \pm x; \tau)}{\Gamma_r(\pm 2x, 5i - ia \pm x; \tau)} \, dx = c_r(\tau) \prod_{0 \leq k < m \leq 4} \tilde{\Gamma}_r(2i - i \tau_k - i \tau_m; \tau) \prod_{j=0}^{4} \tilde{\Gamma}_r(4i - ia + i \tau_j; \tau)
\]

with

\[
c_r(\tau) = \frac{2 \exp \left( \frac{\pi}{12r} (\tau - 1) \right)}{r (\exp(-2\pi r); \exp(-2\pi r))_{\infty} (\exp(2\pi r / \tau); \exp(2\pi r / \tau))_{\infty}}.
\]

The limit \(r \downarrow 0\) of the constant \(c_r(\tau)\) can be computed using the modularity of the Dedekind eta function.

\textbf{Lemma 5.6.}

\[
\lim_{r \downarrow 0} c_r(\tau) = \frac{2}{\sqrt{\tau}}.
\]
Proof. The Dedekind eta function $\eta(\sigma)$ for $\sigma \in \mathbb{H}$ is defined by
\begin{equation}
\eta(\sigma) = \left(\exp(2\pi i\sigma); \exp(2\pi i\sigma)\right)_\infty \exp(\pi i\sigma/12),
\end{equation}
and satisfies
\begin{equation}
\eta\left(-\frac{1}{\sigma}\right) = \eta(\sigma)\sqrt{-i\sigma}.
\end{equation}
Applying (5.27) with $\sigma = ir$ and with $\sigma = -ir/\tau$, we can rewrite $c_r(\tau)$ as
\begin{equation}
c_r(\tau) = \frac{2\exp\left(\frac{\pi i}{12}(1-\tau)\right)}{\sqrt{-\tau} \left(\exp(-2\pi/r); \exp(-2\pi/r)\right)_\infty \left(\exp(2\pi\tau/r); \exp(2\pi\tau/r)\right)_\infty}.
\end{equation}
Since $\tau \in \mathbb{R}_{<0}$ we can take the limit $r \searrow 0$ in the latter expression, which yields the desired result. \hfill \Box

We now further investigate the limit $r \searrow 0$ of (5.25) using (6.19). We associate with $\tau$ the two deformation parameters $q = \exp(2\pi i\tau)$ and $\tilde{q} = \exp(-2\pi i/\tau)$, and we use the standard convention that $q^u = \exp(2\pi iuv)$ and $\tilde{q}^u = \exp(-2\pi iuv)$ for $u \in \mathbb{C}$. For the weight function and norm in (5.25) we obtain by (6.19) we obtain by (6.19)
\begin{equation}
\lim_{\tau \searrow 0} \frac{\prod_{j=0}^4 \Gamma_r(i - it_j \pm x; \tau)}{\prod_{j=0}^4 \Gamma_r(\pm 2x, 5i - ia \pm x; \tau)} = \alpha \left[\prod_{j=0}^4 \Gamma_r(i - it_j \pm x; \tau)\right]_\infty,
\end{equation}
\begin{equation}
\lim_{\tau \searrow 0} \frac{\prod_{0 \leq k < m \leq 4} \Gamma_r(2i - it_k - it_m; \tau)}{\prod_{j=0}^4 \Gamma_r(4i - ia + it_j; \tau)} = \beta \left[\prod_{0 \leq k < m \leq 4} \Gamma_r(2i - it_k - it_m; \tau)\right]_\infty
\end{equation}
with the constants $\alpha$ and $\beta$ given by
\begin{align*}
\alpha &= q^{-\frac{1}{24}}\tilde{q}^{-\frac{1}{4}}\left\{\left(\frac{1}{2\pi} - \frac{1}{2}\right)^2 + \left(\frac{1}{2\pi} + \frac{1}{2} - a\right)^2 - \sum_{j=0}^4 \left(\frac{1}{2\pi} + \frac{1}{2} - t_j\right)^2\right\},
\beta &= q^{-\frac{1}{24}}\tilde{q}^{-\frac{1}{4}}\left\{\sum_{0 \leq k < m \leq 4} \left(\frac{1}{2\pi} + \frac{1}{2} - t_k - t_m\right)^2 - \sum_{j=0}^4 \left(\frac{1}{2\pi} + \frac{1}{2} - a + t_j\right)^2\right\}.
\end{align*}
Writing out the squares leads to
\begin{equation}
\beta = q^{-\frac{1}{24}}\tilde{q}^{-\frac{1}{4}} \exp\left(\frac{\pi i}{4}\right) \alpha.
\end{equation}
Combining Lemma 5.6 (5.28) and (5.29), we can formally take the limit $r \searrow 0$ in (5.25), leading to
\begin{equation}
\int_{-\infty}^{\infty} \frac{2q^{-\frac{1}{24}}\tilde{q}^{-\frac{1}{4}}}{\prod_{j=0}^4 \Gamma_r(i - it_j \pm x; \tau)} dx = \frac{2q^{-\frac{1}{24}}\tilde{q}^{-\frac{1}{4}}}{\prod_{j=0}^4 \Gamma_r(i - it_j \pm x; \tau)} 
\end{equation}
\begin{equation}
\prod_{j=0}^4 \Gamma_r(i - it_j \pm x; \tau) = \prod_{0 \leq k < m \leq 4} \Gamma_r(2i - it_k - it_m; \tau) \prod_{0 \leq k < m \leq 4} \Gamma_r(4i - ia + it_j; \tau) \prod_{0 \leq k < m \leq 4} \Gamma_r(2i - it_k - it_m; \tau).
\end{equation}
Changing the integration variable and using
\begin{equation}
\exp(-3\pi i/4) \sqrt{-\tau} = -\sqrt{-i\tau}
\end{equation}
we arrive at
\begin{align*}
\int_{-\infty}^{\infty} \frac{1}{\prod_{j=0}^4 \Gamma_r(i - it_j \pm x; \tau)} dx &= -\frac{2q^{-\frac{1}{24}}\tilde{q}^{-\frac{1}{4}}}{\sqrt{-i\tau}} \prod_{0 \leq k < m \leq 4} \Gamma_r(2i - it_k - it_m; \tau) \prod_{0 \leq k < m \leq 4} \Gamma_r(4i - ia + it_j; \tau) \prod_{0 \leq k < m \leq 4} \Gamma_r(2i - it_k - it_m; \tau),
\end{align*}
which is the hyperbolic Nassrallah-Rahman integral \(1.10\).

6. Appendix: The hyperbolic gamma function

In this section we discuss the \(\tau\)-shifted factorial \([z; \tau]\) (see \(1.7\)) and its connection with Ruijsenaars' \([19]\) hyperbolic gamma function. For detailed proofs we refer to Ruijsenaars' papers \([19]\) and \([20]\).

We start by recalling the definition of Ruijsenaars' hyperbolic gamma function. We write

\[ C_\pm = \{ z \in \mathbb{C} \mid \text{Re}(z) \gtrless 0 \} \]

for the open right/left half plane. The integral

\[ g(z) = g(a_+, a_-; z) = \int_0^\infty \frac{dy}{y} \left( \frac{\sin(2yz)}{2 \sinh(a_+ y) \sinh(a_- y)} - \frac{z}{a_+ a_- y} \right) \]

defines an analytic function for \((a_+, a_-, z) \in D\), where

\[ D = \{ (a_+, a_-, z) \in \mathbb{C}^3 \mid \text{Re}(a_\pm) > 0, \ |\text{Im}(z)| < \frac{1}{2}(\text{Re}(a_+) + \text{Re}(a_-)) \}. \]

Ruijsenaars' \([19]\) hyperbolic gamma function is now defined as the analytic, zero-free function on \(D\) defined by

\[ \Gamma_h(z) = \Gamma_h(a_+, a_-; z) = \exp(ig(a_+, a_-; z)). \]

The hyperbolic gamma function satisfies the functional equations

\[ \frac{\Gamma_h(z + ia_+/2)}{\Gamma_h(z - ia_+/2)} = 2 \cosh(\pi z/a_+) \]

whenever the left hand side is defined. Hence \(\Gamma_h\) extends to a meromorphic function in the domain \((a_+, a_-, z) \in \mathbb{C}^2 \times \mathbb{C}\), which we again denote by \(\Gamma_h(a_+, a_-; z)\).

The functional equations \(6.3\) imply that the zeros of \(z \mapsto \Gamma_h(a_+, a_-; z)\) are located at

\[ \left( \mathbb{Z}_{\geq 0} + \frac{1}{2} \right) ia_+ + \left( \mathbb{Z}_{\geq 0} + \frac{1}{2} \right) ia_-, \]

and the poles are located at

\[ -\left( \mathbb{Z}_{\geq 0} + \frac{1}{2} \right) ia_+ - \left( \mathbb{Z}_{\geq 0} + \frac{1}{2} \right) ia_. \]

The zeros and poles are simple when \(a_+/a_- \notin \mathbb{Q}_{>0}\).

The explicit integral expression for \(g\) implies

\[ \Gamma_h(a_+, a_-; z) = \Gamma_h(a_-, a_+; z), \]

\[ \Gamma_h(ra_+, ra_-; rz) = \Gamma_h(a_+, a_-; z), \quad r \in \mathbb{R}_{>0}, \]

as well as the reflection equation

\[ \Gamma_h(a_+, a_-; z)\Gamma_h(a_+, a_-; -z) = 1. \]
Note that by the translation invariance \((6.5)\), the hyperbolic gamma function \(\Gamma_h\) essentially only depends on the quotient \(a_+/a_-\) of the two deformation parameters \(a_\pm\).

Some special values of \(\Gamma_h\) are easily computed. Obviously, we have

\[
\Gamma_h(a_+, a_-; 0) = 1. \tag{6.7}
\]

Applying the two functional equations and the reflection equation, we furthermore have

\[
\Gamma_h\left(a_+, a_-; z - \frac{ia_+}{2} + \frac{ia_-}{2}\right) \Gamma_h\left(a_+, a_-; -z - \frac{ia_+}{2} + \frac{ia_-}{2}\right) = \frac{\sinh(\pi z/a_+)}{\sinh(\pi z/a_-)}.
\]

Taking the limit \(z \to 0\) we conclude that

\[
\Gamma_h\left(a_+, a_-; \frac{ia_-}{2} - \frac{ia_+}{2}\right) = \sqrt{\frac{a_-}{a_+}} \tag{6.8}
\]

(to see that the branch of the square root is the right one, note that \(\Gamma_h(a_+, a_-; x) > 0\) for \(a_\pm \in \mathbb{R}_{>0}\) and \(x \in i\mathbb{R}\) in view of \((6.1)\) and \((6.2)\)).

The hyperbolic gamma function can be explicitly expressed as quotient of trigonometric gamma functions when \(\text{Im}(a_+/a_-) \neq 0\). This expression was first obtained by Shintani \cite{22}, see also \cite{20}, Appendix A].

**Proposition 6.1.** Let \(a_\pm \in \mathbb{C}_+\) with \(\text{Im}(a_+/a_-) > 0\). Then

\[
\Gamma_h(a_+, a_-; z) = \frac{(-\exp(\pi i a_+/a_-) \exp(-2\pi z/a_-) \exp(2\pi i a_+/a_-))_\infty}{(-\exp(-\pi i a_-/a_+) \exp(-2\pi z/a_+) \exp(-2\pi i a_-/a_+))_\infty} \times \exp\left(\frac{-\pi i}{24} \left(\frac{a_+}{a_-} + \frac{a_-}{a_+}\right)\right) \exp\left(-\frac{\pi iz^2}{2a_+a_-}\right).
\]

**Proof.** We sketch a proof because the infinite product expression for \(\Gamma_h\) plays such a crucial role in the present paper. Write \(\widehat{\Gamma}_h(z)\) for the right hand side of the desired identity. It is easily verified that \(\widehat{\Gamma}_h(z)\) is meromorphic in \(z\), having the same poles and zeros as \(\Gamma_h(z)\). A direct check shows that \(\widehat{\Gamma}_h(z)\) satisfies the same functional equations \((6.3)\) as \(\Gamma_h(z)\). Thus \(\widehat{\Gamma}_h/\Gamma_h\) is an entire, bounded function, hence a constant. The constant is one since

\[
\widehat{\Gamma}_h\left(\frac{ia_-}{2} - \frac{ia_+}{2}\right) = \sqrt{\frac{a_-}{a_+}},
\]

which follows from the modularity \((5.27)\) of the Dedekind eta function \((5.26)\). \(\square\)

Note that an infinite product expression for \(\Gamma_h(a_+, a_-; z)\) with \(\text{Im}(a_+/a_-) < 0\) can be directly derived from Proposition 6.1 by applying \((6.4)\).

Using Proposition 6.1 the rather harmless looking reflection equation \((6.6)\) turns into a nontrivial infinite product identity. This identity can be proven without referring to the integral representation of \(\Gamma_h\) using Jacobi’s triple product identity, Jacobi’s inversion formula and the modularity of the Dedekind eta function.
Ruijsenaars [20] established rather delicate asymptotic bounds for the hyperbolic gamma function. We formulate here a weaker version of these bounds which suffices for our purposes. It can be stated as

\[
\Gamma_h(a_+, a_-; z) = O \left( \exp \left( \mp \frac{i\pi z^2}{2a_+a_-} \right) \right), \quad \text{Re}(z) \to \pm \infty,
\]

uniformly for \( \text{Im}(z) \) in compacta of \( \mathbb{R} \) and for \( a_\pm \) in compacta of \( \mathbb{C}_+ \). The precise meaning is as follows, cf. [20, Thm. A.1].

**Proposition 6.2.** Let \( K_+ \subset \mathbb{C}_+ \) and \( K \subset \mathbb{R} \) be compact subsets. There exist positive constants \( R = R(K_+, K_-; K) \) and \( C = C(K_+, K_-; K) \), both depending on \( K_\pm \) only, such that

\[
\left| \Gamma_h(a_+, a_-; z) \exp \left( \pm \frac{i\pi z^2}{2a_+a_-} \right) \right| \leq C, \quad \text{Re}(z) \not\to \pm \infty
\]

when \( \text{Im}(z) \in K \) and \( a_\pm \in K_\pm \).

The precise connection between \( \Gamma_h \) and the \( \tau \)-shifted factorial \( [z; \tau]_\infty \) (see (1.7)) is as follows. Fix \( \tau \in \mathbb{C}_- \cap \mathbb{H} \) and write \( q = \exp(2\pi i\tau) \) and \( \tilde{q} = \exp(-2\pi i/\tau) \). Recall the notational convention \( q^u = \exp(2\pi i\tau u) \) and \( \tilde{q}^u = \exp(-2\pi i\tau u) \) for \( u \in \mathbb{C} \). The \( \tau \)-shifted factorial \( [z; \tau]_\infty \) (see (1.7)) can then be expressed in terms of the hyperbolic gamma function by

\[
[z; \tau]_\infty = q^{\left( z - \frac{1}{2} - \frac{1}{2\tau} \right)^2 / 4 q^{-1/8} \tilde{q}^{1/8}} \Gamma_h \left( -\frac{1}{\tau}, 1; i \left( z - \frac{1}{2} - \frac{1}{2\tau} \right) \right).
\]

Indeed, substituting the infinite product expression for \( \Gamma_h \) from Proposition 6.1 in the right hand side of (6.9) gives precisely the expression (1.7) defining the \( \tau \)-shifted factorial. Working with the \( \tau \)-factorial \( [z; \tau]_\infty \) has the advantage that formulas have a similar appearance as in the usual basic hypergeometric (trigonometric) setup. The disadvantage is the loss of symmetry in the parameters \( a_\pm \) (see (6.4)). We end this section by reformulating the above properties of the hyperbolic gamma function in terms of the \( \tau \)-shifted factorial.

Formula (6.9) shows that \( [z; \tau]_\infty \) has a meromorphic continuation to \((z, \tau) \in \mathbb{C} \times \mathbb{C}_-\), which we again denote by \( [z; \tau]_\infty \) (in fact, in view of (1.7) and (6.4), \( [z; \tau]_\infty \) extends to a meromorphic function in \((z, \tau) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{R}_{\geq 0})\)). The functional equations (6.3) become

\[
\begin{align*}
[z + 1; \tau]_\infty &= (1 - q^z)[z; \tau]_\infty, \\
[z - \tau^{-1}; \tau]_\infty &= (1 - \tilde{q}^{-1} \exp(-2\pi iz))[z; \tau]_\infty.
\end{align*}
\]

The zeros of \( z \mapsto [z; \tau]_\infty \) are located at

\[
\frac{1}{\tau} \mathbb{Z}_{\leq 0} + \mathbb{Z}_{> 0},
\]

and the poles are located at

\[
\frac{1}{\tau} \mathbb{Z}_{> 0} + \mathbb{Z}_{\leq 0}.
\]
The zeros and poles are simple when $\tau \notin \mathbb{Q}_{<0}$. Property (6.4) translates into
\begin{equation}
[z; \tau^{-1}]_{\infty} = \left[-\frac{z}{\tau} + \frac{1}{\tau} + 1; \tau\right]_{\infty}
\end{equation}
for $\tau \in \mathbb{C}_{-}$. The reflection equation (6.6) becomes
\begin{equation}
\left[\frac{1}{2} + \frac{1}{2\tau} + z; \frac{1}{2} + \frac{1}{2\tau} - z; \tau\right]_{\infty} = q^{-\frac{1}{2\tau}}q^{\frac{1}{2\tau}q^{-\frac{1}{2}\tau}}
\end{equation}
for $\tau \in \mathbb{C}_{-}$. Finally, the asymptotic bounds for the $\tau$-shifted factorial, deduced from Proposition 6.2, become
\begin{equation}
[z; \tau]_{\infty} = \mathcal{O}\left(q^{\left(z-\frac{1}{2} - \frac{1}{2\tau}\right)^{2}/2}\right), \quad \text{Im}(z) \to \infty,
\end{equation}
uniformly for Re($z$) in compacta of $\mathbb{R}$ and for $\tau$ in compacta of $\mathbb{C}_{-}$, and
\begin{equation}
[z; \tau]_{\infty} = \mathcal{O}(1), \quad \text{Im}(z) \to -\infty,
\end{equation}
uniformly for Re($z$) in compacta of $\mathbb{R}$ and for $\tau$ in compacta of $\mathbb{C}_{-}$. Using the more precise asymptotic estimates for the hyperbolic gamma function in [19, Thm. A.1], we actually have the limit
\begin{equation}
\lim_{\text{Im}(z) \to -\infty} [z; \tau]_{\infty} = 1
\end{equation}
uniformly for Re($z$) in compacta of $\mathbb{R}$ and for $\tau$ in compacta of $\mathbb{C}_{-}$.

We end this section by rewriting the hyperbolic degeneration (see [19, Prop. III.12]) of Ruijsenaars’ elliptic gamma function in our present notations. Ruijsenaars’ elliptic gamma function $G(r, a_{+}, a_{-}; z)$ (see e.g. [19, Prop. III.11]) relates to the elliptic gamma function (1.11) by
\begin{equation}
G(r, a_{+}, a_{-}; z) = \Gamma\left(\exp(2\pi irz - a_{+}r - a_{-}r); \exp(-2a_{+}r), \exp(-2a_{-}r)\right)
\end{equation}
for Re($a_{\pm}r$) > 0. Thus [19, Prop. III.12] becomes
\begin{equation}
\lim_{r \to 0^{+}} \Gamma\left(\exp(2\pi irz); \exp(-2\pi a_{+}r), \exp(-2\pi a_{-}r)\right) \exp\left(\frac{2\pi z - \pi ia_{+} - \pi ia_{-}}{12ira_{+}a_{-}}\right)
= \Gamma_{h}\left(a_{+}, a_{-}; z - \frac{ia_{+}}{2} - \frac{ia_{-}}{2}\right)
\end{equation}
for $a_{\pm} \in \mathbb{R}_{>0}$. We now take $a_{+} = -\frac{1}{\tau}$ and $a_{-} = 1$ with $\tau \in \mathbb{R}_{<0}$, and we denote for simplicity
\begin{equation}
\tilde{\Gamma}_{r}(z; \tau) = \Gamma\left(\exp(2\pi irz); \exp(2\pi r \tau), \exp(-2\pi r)\right) \exp\left(\frac{\pi i(2\tau z + i - i\tau)}{12r}\right).
\end{equation}
By (6.9), we arrive at the limit
\begin{equation}
\lim_{r \to 0^{+}} \tilde{\Gamma}_{r}(z; \tau) = q^{-\left(\frac{1}{12r} - \frac{1}{4} - iz\right)^{2}/4}q^{\frac{1}{24}q^{-\frac{1}{24}q^{-\frac{1}{24}}}\left[\tau^{-1} - iz; \tau\right]_{\infty}}
\end{equation}
with \( q = \exp(2\pi i \tau) \) and \( \tilde{q} = \exp(-2\pi i / \tau) \). Applying the reflection equation (6.14) leads to the limit

\[
\lim_{r \to 0} \tilde{\Gamma}_r(z; \tau) = q^{\left(\frac{1}{4} - \frac{z}{2} - iz\right)^2 / 4 - \frac{1}{48} \tilde{q}^{1 / 48} [1 + iz; \tau]_{\infty}^{-1}
\]

for \( \tau \in \mathbb{R}_{<0} \).

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