POINTWISE CONVERGENCE OF THE SCHRÖDINGER FLOW

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ABSTRACT. In this paper we address the question of the pointwise almost everywhere limit of nonlinear Schrödinger flow to the initial data, in both the continuous and the periodic settings. Then we show how, in some cases, certain smoothing effects for the non-homogeneous part of the solution can be used to upgrade to an uniform convergence to zero of that part and we discuss the sharpness of the results obtained. We also use randomization techniques to prove almost everywhere convergence with much less regularity of the initial data, hence showing how more generic results can be obtained.

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1. Introduction

In this work, we are concerned with the question of almost everywhere convergence of solutions to the nonlinear Schrödinger equation (NLS) to initial data. More precisely, let $u(x,t)$ be a solution to

$$\begin{cases}
    i\partial_t u + \Delta u = \mathcal{N}(u), \\
    u(x,0) = f(x),
\end{cases}$$

where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and $\mathcal{N}$ is a power type nonlinearity. If $f \in H^s$, for what $s$ do we have that $u(x,t) \to f(x)$ as $t \to 0$ for (Lebesgue) almost every $x$?

In the linear Euclidean setting, namely $\mathcal{N} = 0$ and $x \in \mathbb{R}^d$, this question was first posed by Carleson [11], who showed that almost everywhere (a.e.) convergence holds for $f \in H^{1/4}$. Dahlberg–Kenig [15] showed that this one dimensional result is sharp; in fact they proved that $s \geq 1/4$ is a necessary condition for a.e. convergence on $\mathbb{R}^d$, $d \geq 1$. Since then, the higher dimensional problem has been studied by many authors [14, 10, 38, 46, 3, 31, 44, 45, 42, 26, 6, 27, 16, 28, 20]. Recently, Bourgain [7] proved that $s \geq \frac{d}{d+2}$ is a necessary condition for a.e. pointwise convergence to the data (see also [29] for an alternative counterexample). This has been proved to be sharp, up to the endpoint, by Du–Guth–Li [19] in the $\mathbb{R}^d$ case, and by Du–Zhang [18] in higher dimensions.

In the linear periodic setting, namely $\mathcal{N} = 0$ and $x \in \mathbb{T}^d$, much less is known. The only result appears to be that of Mouya–Vega [32] when $d = 1$, (sufficiency of $s > \frac{1}{3}$ and necessity of $s \geq \frac{1}{4}$), which method of proof, based on Strichartz estimates, has been extended to higher dimensions by Wang–C. Zhang [47]. Together with recent improvements in periodic Strichartz estimates [8], one can show that $s > \frac{d}{2(d+1)}$ is a sufficient condition for almost everywhere convergence to initial data. We refer to Section 3.1 for more details. In Section 3.1 we also show that almost everywhere convergence fails when $s < \frac{d}{2(d+1)}$, see Proposition 3.2. At the moment, in the periodic case almost sure convergence when $s \in \left[\frac{d}{2(d+1)}, \frac{d}{d+2}\right]$ remains an open question.

In the first part of this paper we extend these results to the nonlinear setting. Hereafter $\Omega$ denotes either $\mathbb{T}$ or $\mathbb{R}$. We define

$$s_{\Omega^d} := \begin{cases} 
    \frac{d}{d+2} & \text{if } \Omega = \mathbb{T}, \\
    \frac{d}{2(d+1)} & \text{if } \Omega = \mathbb{R}.
\end{cases}$$

Summarizing the results mentioned above, one has

$$\lim_{t \to 0} e^{it\Delta} f(x) = f(x) \quad \text{for a.e. } x \in \Omega^d$$

for all $f \in H^s(\Omega^d)$ with $s > s_{\Omega^d}$. If $\Omega^d = \mathbb{R}$ we only need $s \geq s_{\mathbb{R}} = \frac{1}{4}$.

In the following theorem we prove that a similar result is true for solutions to NLS with power nonlinearities.

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1 Although in this paper we only consider rational tori, this particular result holds for any torus since it is based on Strichartz estimates, now available for any torus thanks to [8].

2 We prove this failure only for rational tori, our proof does not extend to the irrational case.
Pointwise Convergence of the Schrödinger Flow

**Theorem 1.1.** Let \( N(z) = \pm |z|^{p-1} z \) with \( p \geq 3 \). If \( f \in H^s(\Omega^d) \) with

\[
s > \max \left( s_{\Omega^d}, \frac{d}{2} - \frac{2}{p-1} \right),
\]

and \( u \) is the corresponding solution to (1), then

\[
\lim_{t \to 0} u(x, t) = f(x) \quad \text{for a.e. } x \in \Omega^d.
\]

If \( \Omega^d = \mathbb{R} \) and \( p < 9 \) we can relax the condition \( s > s_\mathbb{R} = \frac{1}{4} \) to \( s \geq \frac{1}{4} \). Moreover if we consider the cubic equation \( (p = 3) \) and \( d = 1 \) or \( \Omega^d = \mathbb{R}^2 \) we have for \( s > \frac{d}{6} \)

\[
\lim_{t \to 0} u(x, t) - e^{it\Delta} f(x) = 0 \quad \text{for every } x \in \Omega^d.
\]

and the convergence is uniform\(^3\).

**Remark 1.2.** The result is sharp in the following sense. The conditions \( p \geq 3 \) and

\[
s > \max \left( 0, \frac{d}{2} - \frac{2}{p-1} \right),
\]

ensure that the flow is locally well defined, in fact \( s_c := \frac{d}{2} - \frac{2}{p-1} \) is the critical exponent. The extra assumption \( s > s_{\Omega^d} \) ensures that the linear part \( e^{it\Delta} f \) of the flow converges pointwise a.e. to the initial datum \( f \). This condition is sharp if \( \Omega = \mathbb{R} \) (modulo endpoints when \( d \geq 2 \)) and we do not expect improved convergence to the data when we introduce a nonlinearity. Moreover by the proof of Theorem 1.1 it will be clear that any improvement of the exponent \( s_{\Omega^d} \) into the linear setting would provide an analogous improvement of Theorem 1.1 as well. More precisely, if we define

\[
s_{\Omega^d}^* := \inf \left\{ s : \lim_{t \to 0} e^{it\Delta} f(x) = f(x) \quad \text{for a.e. } x \in \Omega^d, \quad \forall f \in H^s(\Omega^d) \right\},
\]

we can replace the assumption (3) by

\[
s > s_{\Omega^d}^* \quad \text{and} \quad s > \frac{d}{2} - \frac{2}{p-1},
\]

and we can relax \( s > s_{\Omega^d}^* \) to \( s \geq s_{\Omega^d}^* \) if the inf in (6) is a min.

To prove (4) we consider (smooth) approximations of the solutions of NLS obtained by truncating (1) on the first \( N \) Fourier modes. Since for this class of solutions we have pointwise convergence to the initial data, we are able to rewrite the convergence problem as an \( L^2_{x,\text{loc}} \) bound for a suitable maximal function, adapted to the nonlinear setting; see Proposition 3.3. It is worth mentioning that (already in the linear setting) the maximal function approach is the most powerful tool to study a.e. pointwise convergence to the initial data. In order to obtain a good enough bound, in Proposition 3.3 we embed the restriction space \( X_{\delta}^{s, \frac{1}{2}^+} \) into the space

\[
\left\{ F(x, t) : \| \sup_{0 \leq t \leq \delta} |F(x, t)|\|_{L^2_{x,\text{loc}}} < \infty \right\},
\]

for \( s > s_{\Omega^d} \); see Proposition 2.2 (\( \delta > 0 \) will be the local existence time). This embedding allows us to use Strichartz estimates to conclude the proof; see Section 3.2.

---

\(^3\)A proof of (5) in the case \( d = 1 \) is in [21, 13] Here we extend the result to \( \Omega^d = \mathbb{R}^2 \).
For the cubic NLS we can prove stronger results, taking advantage of the algebraic structure of the nonlinearity $N(z) = \pm |z|^2 z$. A first example of this phenomenon is already in the statement (5). Let us consider $x \in \mathbb{R}^2$ to fix the notations (similar observations can be made if $x \in \Omega$). Since for $s > s_{\mathbb{R}^2} = \frac{1}{8}$ one has $e^{it\Delta}f(x) \to f(x)$ as $t \to 0$ for a.e. $x \in \mathbb{R}^2$, we see that (5) is clearly stronger than (4). In fact, we can show that for any $t \in \mathbb{R}$, the function

$$x \in \mathbb{R}^2 \to u(x, t) - e^{it\Delta}f(x) \in \mathbb{C}$$

is continuous. Moreover the map

$$(8) \quad t \in \mathbb{R} \to u(x, t) - e^{it\Delta}f \in C_x(\mathbb{R}^2) \quad \text{(with the sup}_{x \in \mathbb{R}^2} \text{ norm)}$$

is also continuous. This stronger convergence result is a consequence of a smoothing effect associated to the cubic nonlinearity on $\mathbb{R}^2$, that we prove in Corollary 2.6. A similar smoothing effect has been noted in $1d$ in both the periodic $^4$ [21] and non-periodic [13] settings: the nonlinearity turns out to be $\sigma$–smoother than the initial datum in $H^s$, with $\sigma < \max(2s, 1/2)$. We strengthen and extend this smoothing effect on $\mathbb{R}^d$, $d = 1, 2$ to $\sigma < \max(2s, 1)$; see Corollary 2.6. Using these facts and the Sobolev embedding $H^{\frac{d}{2}+}(\Omega) \to L^\infty(\Omega)$, we see that the ($1d$ analog of) property (8) is satisfied by initial data in $H^s(\Omega)$ with $s > 1/6$. On the other hand, in [36] (see also [29]) it has been observed that for any $s < 1/4$ there are initial data such that $\limsup_{t \to 0} |e^{it\Delta}f(x)| = \infty$ for $x$ in a set of strictly positive measure. This construction, done for $\Omega = \mathbb{R}$, is based on the Dahlberg–Kenig counterexample and can be repeated also in the periodic setting. Combining this with the above mentioned smoothing effect of the $1d$ cubic NLS, it is immediate to see that the convergence statement (4) fails for $s \in (1/6, 1/4)$ for the one-dimensional cubic NLS ($p = 3$). However, as observed in Remark 1.2, we expect (4) to fail for any $s < s_{\Omega}^*$ and for any $p \geq 3$.

In the second part of this paper we continue the analysis of the cubic NLS, showing that a.e. pointwise convergence to the data is generically true for initial data which are less smooth than the data postulated in Theorem 1.1. In the periodic setting, we consider

$$(9) \quad f^\omega(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n^\omega}{\langle n \rangle^{\frac{s}{2} + \alpha}} e^{in \cdot x}, \quad \alpha > 0,$$

where $g_n^\omega$ are independent (complex) standard Gaussian variables and we define $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{d}{2}}$. Note that $f^\omega(x)$ is a function in $\bigcap_{\alpha < \alpha} H^\alpha(\mathbb{T}^d)$ almost surely with respect to $\omega$. Thus we are working at the $H^\alpha$–level. Moreover $f^\omega$ is $\omega$–almost surely a continuous function; see Section 4.1 for details.

In the following statement we consider the Wick ordered cubic NLS (73) in $\mathbb{T}^d$, $d = 1, 2$. Since once we fix the initial datum $f \in L^2$, solutions to this equation are related to that of the cubic NLS by multiplication with a factor $e^{it\mu}$ with $\mu = \int_{\mathbb{T}^d} |f|^2$, the study of a.e. pointwise convergence of (73) turns out to be completely equivalent to that of the cubic NLS.

---

$^4$In fact in the periodic setting one has to consider the Wick ordered equation; see the paragraph before Theorem 1.3 and Section 4.3 for details.
Theorem 1.3. Let $f^\omega$ be defined in (9) for $\alpha > 0$, and hence $f^\omega \in \bigcap_{s<\alpha} H^s(\mathbb{T}^d)$ and $f^\omega$ continuous $\omega$–almost surely. Then $\omega$–almost surely
\begin{equation}
\lim_{t \to 0} e^{it\Delta} f^\omega(x) = f^\omega(x) \quad \text{for every } x \in \mathbb{T}^d.
\end{equation}
and the convergence is uniform. Let $d = 1, 2$ and let $u$ be the solution to the Wick ordered cubic NLS (73) with random initial data $f^\omega$ as above. Then $\omega$–almost surely
\begin{equation}
\lim_{t \to 0} u(x, t) = f^\omega(x) \quad \text{for a.e. } x \in \mathbb{T}^d.
\end{equation}
Furthermore, if $\alpha > \frac{d-1}{2}$, then $\omega$–almost surely
\begin{equation}
\lim_{t \to 0} u(x, t) - e^{it\Delta} f^\omega(x) = 0 \quad \text{for every } x \in \mathbb{T}^d.
\end{equation}
and the convergence is uniform.

Remark 1.4. Notice that if $d = 1$ combining (10) and (12) we get in fact a stronger convergence statement than (11), namely the convergence occurs at any $x$ and uniformly ($\omega$–almost surely). If $d = 2$ the combination of (10) and (12) gives this stronger convergence only for data that are in $H^{\frac{4d}{d+1}}(\mathbb{T}^2)$, while by (11) we see that a.e. convergence occurs for initial data that are merely in $H^{\frac{4d}{d+1}}(\mathbb{T}^2)$ ($\omega$–almost surely).

We also obtain results for randomized initial data on Euclidean spaces. We fix an initial datum $f \in H^s(\mathbb{R}^d)$, and then we obtain from it a collection of randomized data $f^\omega$ via the integer–tiling randomization method described in Section 4.2. This randomization satisfies analogous properties to the one described above in the periodic setting, namely $f^\omega \in \bigcap_{s<\alpha} H^s(\mathbb{R}^d)$ and the $f^\omega$ are continuous functions, $\omega$–almost surely; moreover, it should be noted that randomization does not improve smoothness; see Section 4.2 for details. We then have the following theorem.

Theorem 1.5. Fix $f \in H^s(\mathbb{R}^d)$ with $s > 0$. Let $f^\omega$ be a randomization of $f$ as defined in (71). Then $\omega$–almost surely
\begin{equation}
\lim_{t \to 0} e^{it\Delta} f^\omega(x) = f^\omega(x) \quad \text{for every } x \in \mathbb{R}^d
\end{equation}
and the convergence is uniform. Let then $u$ be the solution to the cubic NLS with initial data $f^\omega$. If $d = 1, 2$ and $s > \frac{d}{2(d+1)}$, then $\omega$–almost surely
\begin{equation}
\lim_{t \to 0} u(x, t) = f^\omega(x) \quad \text{for almost every } x \in \mathbb{R}^d.
\end{equation}

Remark 1.6. Notice that the randomization gives convergence $e^{it\Delta} f^\omega \to f^\omega$ for any $x \in \Omega^d$ ($\omega$–almost surely); see (10), (13). In the deterministic case one has convergence at any $x \in \mathbb{R}^d$ only for $s > \frac{d}{2}$. In the range $s \in \left( \frac{d}{2(d+1)}, \frac{d}{4} \right)$ is still open the problem of determining the (worst possible) Hausdorff dimension of the set where the convergence to $H^s(\mathbb{R}^d)$ initial data fails. This problem, introduced in [37], has been solved in [1] when $s \in \left[ \frac{d}{4}, \frac{d}{2} \right]$. In the range $s \in \left( \frac{d}{2(d+1)}, \frac{d}{4} \right)$ the best positive and negative results to date are in [18] and [28, 29], respectively.

The proofs of Theorem 1.3 and Theorem 1.5 rely upon a combination of the following two facts. First, the randomization improves the integrability of the randomized function. This allows us to deduce uniform convergence of the linear propagator $e^{it\Delta} f^\omega$ to the initial data $f^\omega$, $\omega$–almost surely; see Propositions 4.1 and
4.5. Second, we deduce a smoothing effect associated to the cubic nonlinearity. This allows us to control the nonlinear (Duhamel) contribution $u(x, t) - e^{i t \Delta} f^2$. While in the Euclidean case (Theorem 1.5), we use the deterministic smoothing effect given in Corollary 2.6, in the periodic case (Theorem 1.3) the proof is much more involved. In fact, we follow the argument of Bourgain in [5] and we start with the Wick–reordering of the nonlinearity. Then using more probabilistic arguments and Jarnick’s theorem (namely counting lattice points on convex arcs), as in [5], we obtain in Proposition 4.6 a precise quantification of the amount of smoothing. In our argument a quantification of the smoothing is necessary because we need to be sure that the nonlinear (Duhamel) contribution sits in $X^{s, \frac{1}{2}+}_d$ with $s > s_{\Omega d}$ (we consider $d = 1, 2$), so that we can conclude the proof by implementing techniques from Theorem 1.1.

1.1. Notations. For a fixed $p \in \mathbb{R}$ we often use the notation $p_+ := p + \varepsilon$ and $p_- := p - \varepsilon$, where $\varepsilon$ is any sufficiently small strictly positive real number. Let $A, B > 0$. As usual we write $A \lesssim B$ if $A \leq C B$ where $C > 0$ is a constant which only depends on fixed parameters. We write $A \gtrsim B$ if $B \lesssim A$ and $A \sim B$ when $A \lesssim B$ and $A \gtrsim B$. We write $A \ll B$ if $A \leq c B$ for $c > 0$ sufficiently small and $A \gg B$ if $B \ll A$. We denote $A \wedge B := \min(A, B)$ and $A \vee B := \max(A, B)$.

2. Preliminaries

Let us recall that $\Omega$ denotes either $T$ or $\mathbb{R}$. We denote by $B_p$ a ball of radius $\rho > 0$ centered at a generic point of $\Omega^d$ or $\mathbb{Z}^d$. The following Strichartz estimates are the main tool to study the nonlinear Schrödinger flow:

$$\left\| e^{it \Delta} f(x) \right\|_{L^p_x L^q_u(\Omega^d \times [0, 1])} \lesssim N^{\frac{d + 2}{p} - \frac{d + 2}{q}} \| f \|_{L^2_x(\Omega^d)}, \quad p \geq 2 \left( \frac{d + 2}{d} \right), \quad \text{supp} \, \hat{f} \subseteq B_N .$$

These estimates were proved in [40] for $\Omega = \mathbb{R}$ and in [8] for $\Omega = T$. The additional factor $N^{0+}$ is removable except when $\Omega = T$ and $p = 2 \left( \frac{d + 2}{d} \right)$; see [4, 25] and the references therein. However, we never use this finer information. If $\Omega = \mathbb{R}$ one can extend the time integration to $\mathbb{R}$. However, we only need local–in–time estimates.

Hereafter $\delta \in (0, 1]$. The main tool used in the study of the a.e. pointwise convergence of solutions to linear Schrödinger equation to the initial data is the following maximal estimate

$$\left\| \sup_{0 \leq t \leq \delta} \left| e^{it \Delta} f(x) \right| \right\|_{L^2_x(L^1_t(B_1))} \lesssim \| f \|_{H^s_x(\Omega^d)} .$$

The validity of this estimate is equivalent to the fact that $e^{it \Delta} f(x) \to f(x)$ as $t \to 0$ for almost every (with respect to the Lebesgue measure) $x \in B_1$. One implication of this statement is elementary: the other is a consequence of the Stein–Nikishin maximal principle [39, 34]. Inequality (16) holds for all $s > s_{\Omega d}$ where $s_{\Omega d}$ is defined in (2); see the introduction and the forthcoming Proposition 3.1.

The main result of this section (Lemma 2.2) is that given a function

$$F : (x, t) \in \Omega^d \times [0, \delta] \to F(x, t) \in \mathbb{C}$$
we can bound the $L^2_x(B_1)$ norm of the associated maximal function $\sup_{0 \leq t \leq \delta} |F(x, t)|$ with an appropriate $X^{s,b}_\delta$ norm of $F$; see also [4]. This embedding is used to obtain convergence results for solutions to the nonlinear Schrödinger equation.

We recall that

$$\|F\|_{X^{s,b}_\delta} := \inf_{G = F \text{ on } t \in [0, \delta]} \|G\|_{X^{s,b}_\delta},$$

where

$$\|F\|^2_{X^{s,b}_\delta} := \int_\mathbb{R} \sum_{n \in \mathbb{Z}^d} \langle \tau - |n|^2 \rangle^{2b} \langle n \rangle^{2s} |\hat{F}(n, \tau)|^2 d\tau \quad \text{if } \Omega = \mathbb{T},$$

$$\|F\|^2_{X^{s,b}_\delta} := \int_\mathbb{R} \int_\mathbb{R}d \langle \xi \rangle^{2b} \langle |\xi|^2 \rangle^{2s} |\hat{F}(\xi, \tau)|^2 d\xi d\tau \quad \text{if } \Omega = \mathbb{R},$$

$$\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}},$$

and $\hat{F}$ is the space-time Fourier transform of $F$.

The next lemma shows how to embed $X^{s,b}_\delta$, $b > \frac{1}{2}$, into several functional spaces. The proof can be found in [43, Lemma 2.9], in the case $\Omega = \mathbb{R}$. The argument adapts to $\Omega = \mathbb{T}$.

**Lemma 2.1.** Let $b > \frac{1}{2}$ and let $Y_\delta$ be a Banach space of functions

$$F : (x, t) \in \Omega^d \times [0, \delta] \to F(x, t) \in \mathbb{C}.$$ 

If

$$(17) \quad \| e^{i\alpha t} e^{it\Delta} f(x) \|_{Y_\delta} \leq C_\delta \| f \|_{H^s(\Omega^d)}, \quad \alpha \in \mathbb{R},$$

then

$$\|F\|_{Y_\delta} \leq C_\delta \|F\|_{X^{s,b}_\delta}.$$ 

Using Lemma 2.1 with

$$\|F\|_{Y_\delta} = \left\| \sup_{0 \leq t \leq \delta} |F(x, t)| \right\|_{L^2_x(B_1)}$$

and the fact that the maximal estimate (16) hold for $s > s_{\Omega^d}$ ($s \geq \frac{1}{4}$ if $\Omega^d = \mathbb{R}$), we have the following

**Lemma 2.2.** Let $b > \frac{1}{2}$ and $s > s_{\Omega^d}$ defined in (2). We have

$$(18) \quad \left\| \sup_{0 \leq t \leq \delta} |F(x, t)| \right\|_{L^2_x(\Omega^d)} \lesssim \|F\|_{X^{s,b}_\delta}.$$ 

If $\Omega^d = \mathbb{R}$ we can relax $s > s_{\mathbb{R}} = \frac{1}{4}$ to $s \geq \frac{1}{4}$.

We also recall several useful estimates, repeatedly used in the paper, that can be obtained using the Strichartz estimates (15) and Lemma 2.1. Let $P_N$ be the frequency projection into the annulus of size $N$, namely $\{ N/2 < |\xi| \leq N \}$. Let $P_A$ be the frequency projection into the set $A$. We denote by $Q_N$ a (frequency) cube of side length $N$, centered at any point. Then in the statement of the lemma below we use $\Gamma_N$ to denote either $Q_N$ or the (frequency) annulus of size $N$. Thus $P_{\Gamma_N}$ is either $P_{Q_N}$ or $P_N$.  


Lemma 2.3. Let \( p \geq 2 \left( \frac{d+2}{d} \right) \). Then

\[
\| \mathcal{P}_N F \|_{L^p_{x,t}(\Omega^d \times [0,\delta])} \lesssim N^{d - \frac{d+2}{p} + s +} \| \mathcal{P}_N F \|_{X_s^{\frac{d}{2} - \frac{d+2}{p} +}},
\]

(19)

\[
\| \mathcal{P}_N F \|_{X^{0,-\frac{1}{2} +}} \lesssim N^{0 +} \| \mathcal{P}_N F \|_{L^2_{x,t}(\Omega^d \times [0,\delta])}.
\]

(20)

Let also \( s > \frac{d}{2} - \frac{d+2}{p} \), then

\[
\| F \|_{L^p_{x,t}(\Omega^d \times [0,\delta])} \lesssim N^{d - \frac{d+2}{p} + s +} \| F \|_{X^{0,-\frac{1}{2} +}}.
\]

(21)

Proof. By dyadic decomposition in frequency, summing a geometric series, and then using Plancherel, we see that (15) implies

\[
\| e^{it\Delta} f \|_{L^p_{x,t}(\Omega^d \times [0,\delta])} \lesssim \| f \|_{H^{d - \frac{d+2}{p} +}}.
\]

This and Lemma 2.1 imply

\[
\| F \|_{L^p_{x,t}(\Omega^d \times [0,\delta])} \lesssim \| F \|_{X^{0,-\frac{1}{2} +}}.
\]

(22)

Now the estimate (19) is an immediate consequence of (22). Letting \( p = 2 \left( \frac{d+2}{d} \right) \) in (19) and interpolating it with \( \| \cdot \|_{X_s^{0,0}} = \| \cdot \|_{L^2_{x,t}(\Omega^d \times [0,\delta])} \) we get

\[
\| \mathcal{P}_N F \|_{L^p_{x,t}(\Omega^d \times [0,\delta])} \lesssim N^{0 +} \| \mathcal{P}_N F \|_{X_s^{0,-\frac{1}{2} +}}.
\]

Dualizing this yields (20). The estimate (21) follows from (19) by taking \( p = 2 \left( \frac{d+2}{d} \right) \) and \( \mathcal{P}_N = \mathcal{P}_N \), after again performing a dyadic frequency decomposition, summing a geometric series, and using Plancherel.

\[ \square \]

We also recall some well known properties of the \( X_s^{a,b} \) spaces that are repeatedly used in the paper; see for example [24]. Hereafter \( \eta \) is a smooth cut-off of the unit interval.

Lemma 2.4. Let \( s \in \mathbb{R} \). Then

\[
\| \eta(t)e^{it\Delta} f(x) \|_{X^{\frac{d}{2} - \frac{d+2}{2p} +}} \lesssim \| f \|_{H^{s}(\Omega^d)},
\]

(23)

\[
\left\| \eta(t) \int_0^t e^{i(t-t')\Delta} F(\cdot,t') dt' \right\|_{X^{\frac{d}{2} - \frac{d+2}{2p} +}} \lesssim \| F \|_{X^{s,-\frac{1}{2} +}}.
\]

(24)

\[
\| F \|_{X^{s,-\frac{1}{2} +}} \lesssim \delta^{0 +} \| F \|_{X^{s,-\frac{1}{2} +}}.
\]

(25)

We end this section with a bilinear estimate in the space \( \mathbb{R}^2 \) and in \( \mathbb{R} \). We first prove the result in \( \mathbb{R}^2 \). This is the harder case, and a proof has already appeared in [12]; we report it below for completeness. The analogous result in \( \mathbb{R} \) is easier and can be proved with similar techniques. These bilinear estimates are used to obtain smoothing results for the nonlinear (Duhamel) part of the solution.
We start with some notation. For dyadic numbers $M_0, M_1, M_2$ we set $M_* = \min(M_0, M_1, M_2)$ and $M^* = \max(M_0, M_1, M_2)$. We use the notation $f_{\mathcal{X}(|\mu| \sim M)} := f_M$ for the restriction to a dyadic annulus. We then define

$$\int_{\tau_0 + \tau_1 + \tau_2 = 0, \mu_0 + \mu_1 + \mu_2 = 0} = \int \cdots$$

and

$$C_+(f; g, h) = \int f(\mu_0, \tau_0) g(\mu_1, \tau_1) \frac{h(\mu_2, \tau_2)}{2} \left(1 + |\tau_1 - |\mu_1|^2\right)^b \left(1 + |\tau_2 \pm |\mu_2|^2\right)^b d\tau_1 d\tau_2 d\mu_1 d\mu_2,$$

with $b > \frac{1}{2}$.

Lemma 2.5 ([12, Lemma 1]). Assume we are in $\mathbb{R}^2$. The following estimates hold

$$|C_+(f_{\mathcal{X}_0}; g_{M_1}, h_{M_2})| \lesssim \left(\frac{M_*}{M}\right)^{\frac{1}{2}} \|f_{\mathcal{X}_0}\|_{L^2} \|g_{M_1}\|_{L^2} \|h_{M_2}\|_{L^2},$$

$$|C_-(f_{\mathcal{X}_0}; g_{M_1}, h_{M_2})| \lesssim \left(\frac{M_1 \wedge M^*}{M_2 \vee M^*}\right)^{\frac{1}{2}} \|f_{\mathcal{X}_0}\|_{L^2} \|g_{M_1}\|_{L^2} \|h_{M_2}\|_{L^2}.$$

We then have the following corollary.

Corollary 2.6. Assume we are in $\mathbb{R}^d$, $d = 1, 2$, $b > b' > \frac{1}{2}$ and $s \geq 0$. Then if $\sigma < \min(2s, 1)$ we have

$$\|\|u\|^2 u\|_{X^{s, b'}} \lesssim \|u\|^2_{X^{s, b}}.$$

Proof. We describe the proof in dimension $d = 2$, which is the hardest case. At the end of the proof we comment on the case $d = 1$. Using a dyadic decomposition and duality we need to estimate for $v \in X^{0, 1-b'}$

$$M_0^{s+b} \left| \int u_{M_1} \bar{u}_{M_2} u_{M_3} \bar{v}_{M_0} dx dt \right|.$$

Without loss of generality we can assume that $M_1 \geq M_3$. Also, below we present the calculation as if $v \in X^{0, b'}$, but we can adjust this by possibly losing an $\varepsilon$ on the highest frequency, and this can be done by assuming that $b' < b$. We then consider two cases, when $M_1 \geq M_2$ or when $M_1 \leq M_2$. In the first case the most dangerous situation is when $M_0 \sim M_1$. In this case we have

$$M_0^{s+b} \left| \int u_{M_1} \bar{u}_{M_2} u_{M_3} \bar{v}_{M_0} dx dt \right| \lesssim M_0^{s+b} \|u_{M_1}\|_{L^2} \|\bar{u}_{M_2}\|_{L^2} \|u_{M_3}\|_{L^2} \|\bar{v}_{M_0}\|_{L^2}.$$

After renormalizing and using the lemma above with $C_+$ we can continue with

$$\lesssim M_0^{s+b} M_1^{b} M_3^{\frac{1}{2}} M_2^{\frac{1}{2}} M_0^{\frac{3}{2}} \|u_{M_1}\|_{X^{0, b}} \|\bar{u}_{M_2}\|_{X^{0, b}} \|u_{M_3}\|_{X^{0, b}} \|\bar{v}_{M_0}\|_{X^{0, b}}$$

$$\lesssim M_0^{s-1} M_1^{b-s} M_2^{\frac{1}{2}-s} \|u_{M_1}\|_{X^{\sigma, b}} \|\bar{u}_{M_2}\|_{X^{\sigma, b}} \|u_{M_3}\|_{X^{\sigma, b}} \|\bar{v}_{M_0}\|_{X^{\sigma, b}}.$$

If $s \leq \frac{1}{2}$ then we need $\sigma - 2s < 0$ and hence $\sigma < 2s$. If $s > \frac{1}{2}$ then we need $\sigma < 1$. Assume now that $M_1 \leq M_2$ and that again $M_0 \sim M_2$. With a similar argument we estimate

$$M_0^{s+b} \left| \int u_{M_1} \bar{u}_{M_2} u_{M_3} \bar{v}_{M_0} dx dt \right| \lesssim M_0^{s+b} \|u_{M_1}\|_{L^2} \|\bar{u}_{M_2}\|_{L^2} \|u_{M_3}\|_{L^2} \|\bar{v}_{M_0}\|_{L^2}.$$

After renormalizing and using the lemma above with $C_-$ we obtain a similar result.
The previous argument (and Lemma 2.5) adapts to the case $d = 1$; see also [24]. Alternatively one can deduce them from the case $d = 2$ using Lemmata 3.1 and 3.6 in [41].

\[ \text{□} \]

3. Deterministic Results

3.1. The Linear Schrödinger Equation on $\mathbb{T}^d$.

In this section we focus on maximal estimates of the linear Schrödinger flow

\begin{equation}
(27) \quad \left\| \sup_{0 \leq t \leq 1} |e^{it\Delta} f(x)| \right\|_{L^2_t(L^2_x)} \lesssim \| f \|_{H^s_x(\mathbb{T}^d)}.
\end{equation}

As mentioned in Section 2, it is standard that this estimate implies pointwise convergence $e^{it\Delta} f(x) \to f(x)$ as $t \to 0$ for almost every $x \in \mathbb{T}^d$. The problem of identifying the minimal regularity $s$ for which (27) holds is still open. The following result has been proved in [32] when $d = 1$ and in [47] when $d \geq 2$. The proof is based on Strichartz estimates. However, comparing with [47], the exponent in the next Proposition is better for $d \geq 3$ due to the use of the optimal periodic Strichartz estimates from [8]. Thus, we recall the proof for the sake of completeness.

**Proposition 3.1.** The inequality (27) holds for all $s > \frac{d}{d+2}$.

**Proof.** By dyadic frequency decomposition (here $N \in 2^\mathbb{N}$) it is sufficient to prove that

\begin{equation}
(28) \quad \left\| \sup_{0 \leq t \leq 1} |e^{it\Delta} P_N f| \right\|_{L^2_t(L^2_x)} \lesssim N^s \| P_N f \|_{L^2_x}
\end{equation}

holds for all $s > \frac{d}{d+2}$; we recall that $P_N$ is the frequency projection into the annulus of size $N$.

We actually prove the stronger estimate

\begin{equation}
(29) \quad \left\| \sup_{0 \leq t \leq 1} |e^{it\Delta} P_N f| \right\|_{L^{\frac{d+2}{d+1}}_t(L^2_x)} \lesssim N^{\frac{d-2s}{d+2}} \| P_N f \|_{L^2_x}
\end{equation}

We use the following inequality (see [26]), that holds by the Fundamental Theorem of Calculus and Hölder’s inequality,

\begin{equation}
(30) \quad \sup_{0 \leq t \leq 1} |\phi(t)| \lesssim |\phi(0)| + \alpha^{\frac{d}{2} - 1} \| \partial_t \phi(t) \|_{L^p_t([0,1])} + \alpha^{\frac{d}{2}} \| \phi(t) \|_{L^p_t([0,1])},
\end{equation}

with $\phi(t) = e^{it\Delta} P_N f(x)$. The parameter $\alpha > 0$ will be chosen later in such a way as to equalize the second and third term on the right hand side of (30). Since $\partial_t e^{it\Delta} P_N f(x) = i\Delta e^{it\Delta} P_N f(x)$ we obtain

\begin{equation}
(31) \quad \sup_{0 \leq t \leq 1} |e^{it\Delta} P_N f(x)|
\end{equation}

\[ \lesssim \| P_N f(x) \| + \alpha^{\frac{d}{2} - 1} \| \Delta e^{it\Delta} P_N f(x) \|_{L^p_t([0,1])} + \alpha^{\frac{d}{2}} \| e^{it\Delta} P_N f(x) \|_{L^p_t([0,1])} \]

\[ \lesssim \| P_N f(x) \| + \alpha^{\frac{d}{2} - 1} N^2 \| e^{it\Delta} P_N f(x) \|_{L^p_t([0,1])} + \alpha^{\frac{d}{2}} \| e^{it\Delta} P_N f(x) \|_{L^p_t([0,1])}. \]
Specifying $\alpha = N^2$ and taking the $L^p_\gamma(\mathbb{T}^d)$ norm of (31) we arrive at

\begin{equation}
\left\| \sup_{0 \leq t \leq 1} |e^{it\Delta} P_N f| \right\|_{L^p(\mathbb{T}^d)} \lesssim \| P_N f \|_{L^p(\mathbb{T}^d)} + N^{\frac{2}{d}} \| e^{it\Delta} P_N f \|_{L^p(\mathbb{T}^d \times [0,1])}
\end{equation}

\begin{equation}
\lesssim N^{\frac{2}{d} - \frac{1}{p}} \| P_N f \|_{L^2(\mathbb{T}^d)} + N^{\frac{2}{d}} \| e^{it\Delta} P_N f \|_{L^p(\mathbb{T}^d \times [0,1])},
\end{equation}

where in the second estimate we used Bernstein’s inequality. Letting $p = 2 \left( \frac{d+2}{d} \right)$ and using the Strichartz estimates (15) we obtain (29).

\[ \square \]

In the proof of Proposition 3.1 one can replace $\mathbb{T}^d$ by $\mathbb{R}^d$. However, in the latter case more powerful techniques are available.

We now analyze the sharpness of the maximal estimate (27). We exhibit a counterexample which shows that in fact the maximal estimate (27) fails for sufficiently rough data. The following Proposition has been proved in [32] in the case $d = 1$ using different counterexamples. The proof we give relies upon the Galilean invariance rather than Gauss sums as in [32]. This last approach is somehow related to the pseudoconformal invariance of the Schrödinger equation. The equivalence between these symmetries in the convergence problem has been already observed when the (linear) problem was settled on $\mathbb{R}^d$, comparing the counterexamples in [16, 28] and [7, 29]. In the case $d = 1$, the following statement can be also proved by adapting the Dahlberg–Kenig counterexample. Similarly one could adapt all the counterexamples on $\mathbb{R}^d$.

**Proposition 3.2.** The inequality (27) fails for all $s < \frac{d}{4(d+1)}$.

**Proof.** Let $\kappa \in (0, \frac{1}{d+1})$, $N \gg 1$ and $D = \lfloor N^{1-\kappa} \rfloor$. We first focus on the family of initial data

\begin{equation}
f(x) = \sum_{k \in \mathbb{Z}^d \atop |k| \leq N/D} e^{iDk \cdot x}
\end{equation}

and the corresponding solutions

\begin{equation}
e^{it\Delta} f(x) = \sum_{|k| \leq N/D} e^{i(Dk \cdot x - D^2|k|^2 t)}.
\end{equation}

Notice that

\begin{equation}
e^{it\Delta} f(x) \sim \sum_{|k| \leq N/D} 1 \sim (N/D)^d \sim N^{dk} \quad \text{if} \quad (x,t) \in X \times T,
\end{equation}

where

\[ X := \lfloor N^{1-\kappa} \rfloor^{-1} \mathbb{Z}^d + B \left( 0, \frac{1}{10N} \right), \quad T := \lfloor N^{1-\kappa} \rfloor^{-2} \mathbb{Z}. \]

This is because we can write elements $t \in T$ as $\lfloor N^{1-\kappa} \rfloor^{-2} \tau$ with $\tau \in \mathbb{Z}$, so that $\lfloor N^{1-\kappa} \rfloor^2 |k|^2 t = \lfloor N^{1-\kappa} \rfloor^2 |k|^2 \lfloor N^{1-\kappa} \rfloor^{-2} \tau = |k|^2 \tau \in \mathbb{Z}$ and we can write elements $x \in X$ as $x = \lfloor N^{1-\kappa} \rfloor^{-1} \ell + \varepsilon$, with $\ell \in \mathbb{Z}^d$, $\varepsilon \in \mathbb{R}^d$ with $|\varepsilon| \leq \frac{1}{10N}$, so that

\[ \lfloor N^{1-\kappa} \rfloor k \cdot x = \lfloor N^{1-\kappa} \rfloor k \cdot (\lfloor N^{1-\kappa} \rfloor^{-1} \ell + \varepsilon) = k \cdot \ell + \lfloor N^{1-\kappa} \rfloor k \cdot \varepsilon \in \mathbb{Z}^d + B \left( 0, \lfloor N^{1-\kappa} \rfloor |k| |\varepsilon| \right). \]
and
\[ N^{1-\kappa} |k| |\varepsilon| \leq N^{1-\kappa} N^{\frac{1}{10N}} \leq \frac{1}{10}. \]

Let now \( \Theta \in \mathbb{Z}^d \). We consider the modulated initial data
\[ f_{\Theta}(x) = e^{ix \cdot \Theta} f(x), \]
where \( f \) is chosen as in (33). The corresponding solutions are
\[ e^{it\Delta} f_{\Theta}(x) = e^{i(x \cdot \Theta - \frac{t}{|\Theta|^2}) (e^{it\Delta} f)(x - 2t \Theta)}. \]

Thus, recalling (35), we have
\[ \sup_{0 \leq t \leq 1/D} |e^{it\Delta} f_{\Theta}(x)| \gtrsim N^{d\kappa} \text{ if } x \in \bigcup_{t \in T \cap [0,1/D]} X + \Theta t, \]

We take \( \Theta = (1, D, \ldots, D) \). Since the set \( \bigcup_{t \in T \cap [0,1/D]} X + \Theta t \) is equidistributed in \( \mathbb{T}^d \), its measure is of order \( 1 \wedge (DD^d N^{-d}) \sim 1 \wedge N^{1-\kappa(d+1)} \), the first factor being the cardinality of \( T \cap [0,1/D] \), the second the cardinality of the small balls of \( X \cap \mathbb{T}^d \), and the last the volumes of these balls. Thus the set \( \bigcup_{t \in T \cap [0,1/D]} X + \Theta t \) has full measure for all large \( N \) since we assume \( \kappa < 1/(d+1) \). By (36) and noting that
\[ \|f_{\Theta}\|_{L^2(T^d)} = \|f\|_{L^2(T^d)} \sim (N/D)^{d/2} \sim N^{d\frac{p}{2}}, \]
the maximal inequality (27) implies that
\[ N^{d\kappa} \ll N^s N^{d\frac{p}{2}}. \]

Letting \( N \to \infty \) this leads to a contradiction if \( s < \frac{d}{2p} \). Since we have restricted to \( \kappa < \frac{1}{d+1} \) we have disproved the inequality (27) for all \( s < \frac{d}{2(d+1)}. \)

\[ \square \]

### 3.2. The NLS Equation on \( \mathbb{T}^d \) and \( \mathbb{R}^d \) (Theorem 1.1).

In this section we prove Theorem 1.1. Thus we focus on the NLS equation (1) on \( \Omega^d \), where \( \Omega = \mathbb{R} \) or \( \Omega = \mathbb{T} \). The nonlinearity is \( N(z) = \pm |z|^{p-1}z \) with \( p \geq 3 \). We focus on initial data in \( H^s(\Omega^d) \) with
\[ s > \max \left( 0, \frac{d}{2} - \frac{2}{p-1} \right). \]

For such data the flow is locally well defined; see [23, 43] for the non-periodic case and [4] for the periodic one. Let \( \Phi_t^N \) be the flow associated to the truncated NLS equation
\[ i\partial_t \Phi_t^N f + \Delta \Phi_t^N f = P_{\leq N} N(\Phi_t^N f), \]
with initial datum \( \Phi_0^N f := P_{\leq N} f \). As usual \( P_{\leq N} \) denotes the frequency projection on the ball of radius \( N \) centered in the origin. We write \( \Phi_t f := \Phi_{t|N}^N f \) for the flow of the NLS equation with initial datum \( f = P_{\infty} f \). We also denote \( P_{>N} := P_{\infty} - P_{\leq N} \) and as already mentioned \( P_N := P_{\leq N} - P_{\leq N/2} \).

The following maximal estimate ensures a.e. pointwise convergence to the data. This is the nonlinear analog of the maximal estimate (16).
Proposition 3.3. Let \( f \in L^2(\Omega^d) \) be such that for some \( \delta > 0 \) we have
\[
\lim_{N \to \infty} \sup_{0 \leq t \leq \delta} \| \Phi_t f(x) - \Phi_t^N f(x) \|_{L^2(B_1)} = 0
\]
for any \( B_1 \subset \Omega^d \). Then \( \Phi_t f(x) \to f(x) \) as \( t \to 0 \) for almost every \( x \in \Omega^d \).

From the proof it will be clear that in (39) we can replace the \( L^2 \) norm with the (smaller) \( L^1 \) norm. However is usually convenient to work in \( L^2 \) setting.

Proof. To prove Proposition 3.3 we decompose the difference as follows:
\[
|\Phi_t f(x) - f(x)| \leq |\Phi_t f(x) - \Phi_t^N f(x)| + |\Phi_t^N f(x) - P_{\leq N} f(x)| + |P_{> N} f(x)|
\]
and pass to the limit \( t \to 0 \). The second term on the right hand side is zero. In fact, since \( P_{\leq N} f \in H^s \) for all \( s > 0 \), we are allowed to take \( s > \frac{d}{2} \) so that the map \( t \in \mathbb{R} \to \Phi_t^N f \in H^s_x \) is continuous. Then Sobolev embedding (in the \( x \) variable) gives
\[
\lim_{t \to 0} \sup_{x \in \Omega^d} |\Phi_t^N f(x) - P_{\leq N} f(x)| = 0.
\]
So we arrive at
\[
\lim_{t \to 0} \sup_{x \in \Omega^d} |\Phi_t f - f| \leq \lim_{t \to 0} \sup_{x \in \Omega^d} |\Phi_t f - \Phi_t^N f| + |P_{> N} f|.
\]
Let \( \lambda > 0 \). Using the Chebyshev inequality
\[
|\{x \in B_1 : \limsup_{t \to 0} |\Phi_t f - f| > \lambda \}| \leq |\{x \in B_1 : \sup_{0 \leq t \leq \delta} |\Phi_t f - \Phi_t^N f| > \lambda/2\}|
\]
\[
+ |\{x \in B_1 : |P_{> N} f| > \lambda/2\}|
\]
\[
\lesssim \lambda^{-2} \left( \sup_{0 \leq t \leq \delta} |\Phi_t f - \Phi_t^N f| \right)
\]
\[
+ \|P_{> N} f\|_{L^2(B_1)}^2 \right),
\]
where \( \cdot \) is the Lebesgue measure. On the other hand we have \( \|P_{> N} f\|_{L^2(\Omega^d)} \to 0 \) as \( N \to \infty \) (since \( f \in L^2(\Omega^d) \)) and
\[
\lim_{N \to \infty} \sup_{0 \leq t \leq \delta} \|\Phi_t f - \Phi_t^N f\|_{L^2(B_1)} = 0
\]
by assumption (39). Thus we arrive to
\[
|\{x \in B_1 : \limsup_{t \to 0} |\Phi_t f - f| > \lambda\}| = 0
\]
and the statement follows taking the union over \( \lambda > 0 \) and covering \( \Omega^d \) with a countable collection of balls \( B_1 \).

We combine the following lemma with the embedding contained in Lemma 2.2 to verify the maximal estimate hypothesis of Proposition 3.3 in concrete situations.

---

\(^5\)Hereafter we remove the \( x \) variable in the argument of decompositions like (40) to simplify the notation.
Lemma 3.4. Let $p \geq 3$ and $s > \max(0, \frac{d}{2} - \frac{2}{p-1})$. Then

\begin{equation}
\|N(u) - N(v)\|_{X^s_d} \lesssim \left( \|u\|_{X^{s+\frac{d}{2}}_d}^{p-1} + \|v\|_{X^{s+\frac{d}{2}}_d}^{p-1} \right) \|u - v\|_{X^{s+\frac{d}{2}}_d}.
\end{equation}

We postpone the proof of Lemma 3.4 to the end of the section.

We denote $R_0 = \|f\|_{H^s(\Omega^d)}$. Recall that $\eta$ is a smooth cut-off of $[0, 1]$. Taking $\delta = \delta(R_0) < 1$ sufficiently small and combining (23), (24), (25) and Lemma 3.4 one can show that the map

\begin{equation}
\Gamma(u(x, t)) = \eta(t)e^{it\Delta} P_{\leq N} f(x) - i\eta(t) \int_0^t e^{i(t-t')\Delta} P_{\leq N} N(u(x, t')) dt'
\end{equation}

is a contraction on the ball $\{ u : \|u\|_{X^{s+\frac{d}{2}}_d} \leq 2R_0 \}$, for all $N \in 2^\mathbb{N} \cup \{\infty\}$. This is a standard argument, so we omit the proof. Moreover, a similar computation is part of the proof of Theorem 1.1. However, we stress that the value of $\delta$ is uniform in $N \in 2^\mathbb{N} \cup \{\infty\}$. In particular we have

\begin{equation}
\|\Phi^N_t f(x)\|_{X^{s+\frac{d}{2}}_d} \leq 2R_0, \quad \text{for all } N \in 2^\mathbb{N} \cup \{\infty\}.
\end{equation}

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We first prove the a.e. convergence statement (4). By Lemma 2.2 we have

\begin{equation*}
\left\| \sup_{0 \leq t \leq \delta} |\Phi_t f(x) - \Phi^N_t f(x)| \right\|_{L^2(B_1)} \lesssim \|\Phi_t f - \Phi^N_t f\|_{X^{s+\frac{d}{2}}_d}.
\end{equation*}

Thus using Proposition 3.3 it suffices to show that the right hand side goes to zero as $N \to \infty$. For $t \in [0, \delta]$ we have (see (42))

\begin{equation*}
\Phi_t f(x) - \Phi^N_t f(x) = \eta(t)e^{it\Delta} P_{> N} f(x) - i\eta(t) \int_0^t e^{i(t-t')\Delta} \left( N(\Phi_t f(x)) - P_{\leq N} N(\Phi^N_t f(x)) \right) dt'.
\end{equation*}

Then using (23) and (24) we have

\begin{equation}
\|\Phi_t f - \Phi^N_t f\|_{X^{s+\frac{d}{2}}_d} \lesssim \|P_{> N} f\|_{H^s(\Omega^d)} + \|N(\Phi_t f) - P_{\leq N} N(\Phi^N_t f)\|_{X^{s-\frac{d}{2}}_d}.
\end{equation}

To handle the nonlinear contribution we further decompose

\begin{equation*}
N(\Phi_t f) - P_{\leq N} N(\Phi^N_t f) = P_{\leq N} (N(\Phi_t f) - N(\Phi^N_t f)) + P_{> N} N(\Phi_t f)
\end{equation*}

so that

\begin{equation}
\|\Phi_t f - \Phi^N_t f\|_{X^{s+\frac{d}{2}}_d} \lesssim \|P_{> N} f\|_{H^s(\Omega^d)} + \|P_{> N} N(\Phi_t f)\|_{X^{s-\frac{d}{2}}_d} + \|P_{\leq N} (N(\Phi_t f) - N(\Phi^N_t f))\|_{X^{s-\frac{d}{2}}_d}.
\end{equation}

Then by (25), Lemma 3.4, and (43), we get

\begin{equation}
\|P_{\leq N} (N(\Phi_t f) - N(\Phi^N_t f))\|_{X^{s-\frac{d}{2}}_d} \lesssim \delta^{0+} R_0^{p-1} \|\Phi_t f - \Phi^N_t f\|_{X^{s+\frac{d}{2}}_d},
\end{equation}

\end{equation*}
where we recall $R_0 = \|f\|_{H^s(\Omega^s)}$. Plugging (46) into (45), taking $\delta = \delta(R_0)$ small enough and absorbing
\[ \delta^{0+}R_0^{-1} \|\Phi_t f - \Phi_t^N f\|_{X^s_{\delta} \frac{1}{2} +} \leq \frac{1}{2}\|\Phi_t f - \Phi_t^N f\|_{X^s_{\delta} \frac{1}{2} +} \]
into the left hand side, we arrive to
\[ ||\Phi_t f - \Phi_t^N f\|_{X^s_{\delta} \frac{1}{2} +} \lesssim \|P > N f\|_{H^s(\Omega^s)} + \|P > N N(\Phi_t f)\|_{X^s_{\delta} \frac{1}{2} +} \]
The right hand side of (47) goes to zero as $N \to \infty$ since $f \in H^s(\Omega^s)$ and $N(\Phi_t f) \in X^s_{\delta} \frac{1}{2} +$; in fact applying Lemma 3.4 with $v = 0$ and recalling (43) we have
\[ \|N(\Phi_t f)\|_{X^s_{\delta} \frac{1}{2} +} \lesssim \|\Phi_t f\|^p_{X^s_{\delta} \frac{1}{2} +} \lesssim R_0^p. \]
This concludes the proof of (4).

To prove (5) it is enough to show that if $d = 1, 2$ and $s > d/6$ then
\[ \left\| \int_0^t e^{i(t-t')\Delta}[\Phi_t f(x)]^2 \Phi_t f(x) dt' \right\|_{X^s_{\delta} \frac{1}{2} +} \lesssim \|\Phi_t f(x)\|^3_{X^s_{\delta} \frac{1}{2} +} \lesssim R_0^3. \]
Indeed then we would have $\Phi_t f(x) - e^{it\Delta} f(x) \in X^s_{\delta} \frac{1}{2} +$ and we can use $X^s_{\delta} \frac{1}{2} + \hookrightarrow C_c([0, \delta]; H^{s+\frac{1}{2}}(\Omega^d))$ and $H^{s+\frac{1}{2}}(\Omega^d) \hookrightarrow C_c(\Omega^d)$ (Sobolev embedding) to get (5). On the other hand (48) follows by (24), Corollary 2.6 and (43), so we are done.

We conclude this section with the proof of Lemma 3.4 and the statement of a similar one – an analog for functions with frequencies restricted to dyadic annuli. These kind of results are now very well understood; however we report the proof for the sake of completeness.

**Proof of Lemma 3.4.** We consider the case $\Omega = \mathbb{T}$. The proof in the case $\Omega = \mathbb{R}$ requires some modification. It is in fact easier, since there is no loss in the endpoint case $p = 2(\frac{4+2s}{d+2})$ of the Strichartz estimates (15). We abbreviate everywhere in the proof
\[ L^q_{x,t}(\mathbb{T}^d \times [0, \delta]) \rightarrow \mathcal{N}(z) = |z|^{p-1}z \] and using the Fundamental Theorem of Calculus we can represent
\[ \mathcal{N}(u) - \mathcal{N}(v) = \int_0^1 \frac{d}{d\rho}(\mathcal{N}(v + \rho(u - v)))d\rho \]
\[ = (u - v) \int_0^1 (\partial_x \mathcal{N})(v + \rho(u - v))d\rho + (u - v) \int_0^1 (\partial_x \mathcal{N})(v + \rho(u - v))d\rho \]
\[ =: (u - v)\zeta_1(u, v) + (u - v)\zeta_2(u, v). \]
Notice that $\partial_x \mathcal{N}$ and $\partial_x \mathcal{N}$ are continuous functions since $p \geq 3$. For simplicity we only show that the $X^s_{\delta} \frac{1}{2} +$ norm of $(u - v)\zeta_1(u, v)$ is bounded by the right hand side of (41). The proof that the same holds for $(u - v)\zeta_2(u, v)$ is identical. Let decompose dyadically
\[ \|(u - v)\zeta_1(u, v)\|_{X^s_{\delta} \frac{1}{2} +}^2 = \sum N^{2s} \|P_N((u - v)\zeta_1(u, v))\|_{X^s_{\delta} \frac{1}{2} +}^2 \]
and estimate

\begin{equation}
N^* \| P_N((u - v)\zeta_1(u, v)) \|_{X_{s,1}^0, s} \lesssim N^* \| P_N((u - v) P_{<N} \zeta_1(u, v)) \|_{X_{s,1}^0, s} + \sum_{N_1 \gtrsim N} N^* \| P_N((u - v) P_{N_1} \zeta_1(u, v)) \|_{X_{s,1}^0, s}.
\end{equation}

We first focus on the second term on the right hand side of (51). This one is the easiest to bound, since the restriction on frequencies \(N_1 \gtrsim N\) gives a gain once we estimate the norm of \(P_{N_1} \zeta_1(u, v)\). Using (20), Hölder’s inequality, and (19) we get

\begin{equation}
\sum_{N_1 \gtrsim N} N^* \| P_N((u - v) P_{N_1} \zeta_1(u, v)) \|_{X_{s,1}^0, s} \lesssim \sum_{N_1 \gtrsim N} N^{s+} \| (u - v) P_{N_1} \zeta_1(u, v) \|_{L_x^2 L_t^{2(d+2)}} + \sum_{N_1 \gtrsim N} N^{s+} \| u - v \|_{L_x^{2(d+2)} L_t^{2(d+2)+}} \| P_{N_1} \zeta_1(u, v) \|_{L_x^{2(d+2)} L_t^{2(d+2)+}}.
\end{equation}

Recalling the definition of \(\zeta_1(u, v)\) and using Minkowski’s inequality and \(L^p\) estimates for nonlinear operators of power type (see for instance [17, Proposition 2.3]) we have

\begin{equation}
\| P_{N_1} \zeta_1(u, v) \|_{L_x^{2(d+2)} L_t^{2(d+2)+}} \lesssim N_1^{s+} \left( \| u \|_{L_x^{p-2} L_t^{2(d+2)+}} \| v \|_{L_x^{p-2} L_t^{2(d+2)+}} \right)
\times \sum_{N_2} \min \left( 1, \frac{N_2}{N_1} \right) \left( \| P_{N_2} u \|_{L_x^{(p-2)(d-2s)} L_t^{2(d+2)}} + \| P_{N_2} v \|_{L_x^{(p-2)(d-2s)} L_t^{2(d+2)}} \right).
\end{equation}

Notice that (19) gives

\begin{equation}
\| P_{N_2} F(x, t) \|_{L_x^{(p-2)(d-2s)} L_t^{2(d+2)}} \lesssim N_2^{0-} \| P_{N_2} F(x, t) \|_{X_{s,1}^\frac{p}{2}+}.
\end{equation}

Indeed, if \(\frac{2(d+2)}{4-(p-2)(d-2s)} \geq \frac{2(d+2)}{d}\) we can use (19) and we get a factor \(N_2^{(p-1)(d-2s)-\frac{4}{d}}\). Since \(\frac{(p-1)(d-2s)-\frac{4}{d}}{2} < 0\) for \(s > \frac{d}{2} - \frac{2}{p-1}\) we get (54). If \(\frac{2(d+2)}{4-(p-2)(d-2s)} \leq \frac{2(d+2)}{d}\) we can bound the \(L_x^{(p-2)(d-2s)}\) norm with the \(L_x^{2(d+2)}\) norm and use (19), to get a factor \(N_2^{-s+}\). Again, since \(s > 0\) we get (54). Using (54) and (21) the estimate (53) becomes

\begin{equation}
\| P_{N_1} \zeta_1(u, v) \|_{X_{s,1}^0} \lesssim N_1^{s+} \left( \| u \|_{X_{s,1}^{p-2}} \| v \|_{X_{s,1}^{p-2}} \right)
\times \sum_{N_2} \min \left( 1, \frac{N_2}{N_1} \right) \left( \| P_{N_2} u \|_{X_{s,1}^{\frac{p}{2}+}} + \| P_{N_2} v \|_{X_{s,1}^{\frac{p}{2}+}} \right)
\lesssim N_1^{0-} \left( \| u \|_{X_{s,1}^{p-1}} + \| v \|_{X_{s,1}^{p-1}} \right).
\end{equation}
Plugging (55) into (52) we obtain (recall $N_1 \gtrsim N$)

$$
\sum_{N_1 \gtrsim N} N^s \| P_N ((u - v) P_{N_1} \zeta_1 (u, v)) \|_{X^{0, \frac{1}{2}+}}
\lesssim \sum_{N_1 \gtrsim N} N_1^{0-} \| u - v \|_{X^{0, \frac{1}{2}+} \cdot X^{0, \frac{1}{2}+}} \left( \| u \|_{X^{0, \frac{1}{2}+}}^{p-1} + \| v \|_{X^{0, \frac{1}{2}+}}^{p-1} \right).
$$

Summing the square of this inequality over $N$, we have handled the contribution of the second term on the right hand side of (51). To handle the first term we note that

$$
N^s \| P_N ((u - v) P_{\ll N} \zeta_1 (u, v)) \|_{X^{0, \frac{1}{2}+}} \lesssim N^s \| (P_{\sim N} (u - v)) P_{\ll N} \zeta_1 (u, v) \|_{X^{0, \frac{1}{2}+}}
$$
and decompose

$$
(P_{\sim N} (u - v)) P_{\ll N} \zeta_1 (u, v) = \sum_{N_1 \ll N} \sum_{Q_{N,N_1}} (P_{Q_{N,N_1}} (u - v)) P_{N_1} \zeta_1 (u, v)
$$

where $Q_{N,N_1}$ is a partition of the annulus of size $N$ into cubes of side $N_1$ (this is possible since $N_1 < N$). In the second identity we used that the support of $(P_{Q_{N,N_1}} F) P_{N_1} G$ is contained in $100Q_{N,N_1}$. Since for different $N, N_1$ the projections $P_{Q_{N,N_1}}$ are (almost) orthogonal, squaring (57) we get

$$
N^{2s} \| P_N ((u - v) P_{\ll N} \zeta_1 (u, v)) \|^2_{X^{0, \frac{1}{2}+}}
= N^{2s} \sum_{N_1 \ll N} \sum_{Q_{N,N_1}} \| P_{100Q_{N,N_1}} ((P_{Q_{N,N_1}} (u - v)) P_{N_1} \zeta_1 (u, v)) \|^2_{X^{0, \frac{1}{2}+}}.
$$

Proceeding exactly as before we get

$$
\| P_{100Q_{N,N_1}} ((P_{Q_{N,N_1}} (u - v)) P_{N_1} \zeta_1 (u, v)) \|_{X^{0, \frac{1}{2}+}} \lesssim N_1^{0-} \| P_{Q_{N,N_1}} (u - v) \|_{X^{0, \frac{1}{2}+}} \left( \| u \|_{X^{0, \frac{1}{2}+}}^{p-1} + \| v \|_{X^{0, \frac{1}{2}+}}^{p-1} \right);
$$
notice that since the side of $Q_{N,N_1}$ is $N_1$ we had only powers of $N_1$ in this computation. Thus

$$
N^{2s} \| P_N ((u - v) P_{\ll N} \zeta_1 (u, v)) \|^2_{X^{0, \frac{1}{2}+}} \lesssim
\left( \| u \|_{X^{0, \frac{1}{2}+}}^{p-1} + \| v \|_{X^{0, \frac{1}{2}+}}^{p-1} \right)^2 \sum_{N_1 \ll N} \sum_{Q_{N,N_1}} N_1^{0-} \| P_{Q_{N,N_1}} (u - v) \|_{X^{0, \frac{1}{2}+}}^2.
$$
Summing the square of \((61)\) over \(Q_{N,N_1}\) (recall that these cubes are a partition of the annulus of size \(N\)) and later over \(N_1\), we obtain (after taking the square root)

\[ N^s \| P_N((u-v)P_{\leq N} \xi_1(u,v)) \|_{X^{\frac{3}{2}+ \frac{1}{2}+ \frac{s}{2}}} \lesssim \left( \|u\|_{X^s_X}^{p-1} + \|v\|_{X^s_X}^{p-1} \right) \| P_N(u-v) \|_{X^{s, \frac{1}{2}+}} \]

which gives the correct control also on the first term on the right hand side of \((51)\). This concludes the proof.

Later we will also need the following Lemma, whose proof is a straightforward adaptation of the previous argument.

Lemma 3.5. Let \(s > 0\) and \(\delta > 0\). Let \(M_1 \geq M_2 \geq M_3\) be dyadic scales. Then

\[ \| (P_{M_1} F)(P_{M_2} G)(P_{M_3} H) \|_{X^{s, \frac{3}{2}+ \frac{1}{2}+ \frac{s}{2}}} \lesssim \| P_{M_1} F \|_{X^{s, \frac{1}{2}+}} \| P_{M_2} G \|_{X^{0+, \frac{1}{2}+ \frac{s}{2}}} \| P_{M_3} H \|_{X^{0, \frac{1}{2}+}}. \]

4. Probabilistic Results

4.1. The Linear Schrödinger Equation on \(\mathbb{T}^d\) with Random Data. Here we prove almost surely uniform convergence of the randomized Schrödinger flow to the initial datum, at the \(H^{0+}\) level. More precisely, we show that \(e^{it\Delta} f^\omega \to f^\omega\) as \(t \to 0\) uniformly over \(x \in \mathbb{T}^d\) and \(\omega\)–almost surely for data \(f^\omega\) defined as

\[ f^\omega(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n^\omega}{\langle n \rangle^{\frac{d}{2}+\alpha}} e^{in \cdot x}, \quad x \in \mathbb{T}^d, \]

where \(\alpha > 0\) and each \(g_n^\omega\) is complex and independently drawn from a standard normal distribution\(^6\). We have that almost surely in \(\omega\)

\[ f^\omega \in \bigcap_{s<\alpha} H^s(\mathbb{T}^d). \]

This is an immediate consequence of \((69)\) below, taking the union over \(\varepsilon > 0\). Moreover, the \(f^\omega\) are \(\omega\)–almost surely continuous functions\(^7\). This is a consequence of the higher integrability property \((66)\) below. Indeed, taking \(p > \frac{d}{2}\) the \((66)\) immediately gives uniform convergence as \(N \to \infty\) of the sequence \(P_{\leq N} f^\omega\), with probability larger than \(1-\varepsilon\). So the limit \(f^\omega\) is continuous with the same probability, and the almost sure continuity follows taking the union over \(\varepsilon > 0\).

Now we prove the first part of Theorem 1.3, namely

Proposition 4.1. Let \(\alpha > 0\). For \(\omega\)–almost every \(f^\omega\) of the form \((9)\) we have that

\[ e^{it\Delta} f^\omega(x) \to f^\omega(x) \quad \text{as } t \to 0 \]

for every \(x \in \mathbb{T}^d\) and uniformly.

\(^6\)The argument we present works for independent \(g_n^\omega\) drawn from any distribution with sufficiently strong decay properties. We present the standard normal case for definiteness.

\(^7\)In fact they belong to \(\bigcap_{s<\alpha} C^s(\mathbb{T}^d)\) \(\omega\)-almost surely, but we will never need this stronger information.
This proposition proves the first part of Theorem 1.3. Its proof appears at the end of this section after we establish few lemmata.

We start recalling the following well–known large-deviation bound:

**Lemma 4.2 ([9, Lemma 3.1]).** There exists a constant $C$ such that

$$\frac{1}{2}\sum_{n \in \mathbb{Z}^d} g_n^\omega a_n \leq Cr_{2}^{\frac{1}{2}}\|a_n\|_{\ell^2(\mathbb{Z}^d)}$$

for all $r \geq 2$ and $\{a_n\} \in \ell^2(\mathbb{Z}^d)$.

Using (64) with $a_n = e^{i\langle n \cdot x \rangle}a_n^{\frac{d}{2}-\alpha}$ we obtain for $r \geq 2$ that for $f^\omega$ an in (63)

$$\|P_N f^\omega\|_{L^r_\omega} \leq Cr_{2}^{\frac{1}{2}}N^{-\alpha}.$$

From this, we also have improved $L^p$ estimates for randomized data.

**Lemma 4.3.** Let $p \in [2, \infty)$. Assume $f^\omega$ is as in (63). There exists constants $C$ and $c$, such that

$$\mathbb{P}(\|P_N f^\omega\|_{L^p(T^d)} > \lambda) \leq Ce^{-\epsilon^2 N^2 \lambda^2}.$$  

In particular, for any $\epsilon > 0$ sufficiently small, we have

$$\|P_N f^\omega\|_{L^p(T^d)} \lesssim N^{-\alpha} (-\ln \epsilon)^{1/2}$$

with probability at least $1 - \epsilon$. Thus

$$\|P_N f^\omega\|_{L^p(T^d)} \lesssim N^{-\alpha + \frac{d}{2}} (-\ln \epsilon)^{1/2}$$

with probability at least $1 - \epsilon$.

**Proof:** We prove (65), then (66) follows by Bernstein inequality. By Minkowski’s inequality and Lemma 4.2 above, we have for any $r \geq p \geq 2$

$$\left(\mathbb{E}(\|P_N f^\omega\|_{L^p(T^d)}^r)\right)^{\frac{1}{r}} \leq \|P_N f^\omega\|_{L^r(T^d)} \leq C_0 N^{-\alpha} r^{\frac{d}{2}}.$$

Then by Chebyshev’s inequality we have

$$\mathbb{P}(\|P_N f^\omega\|_{L^p(T^d)} > \lambda) \leq \left(\frac{C_0 r^{\frac{d}{2}}}{N^\alpha \lambda}\right)^r.$$  

If $\left(\frac{N^\alpha \lambda}{\epsilon C_0}\right)^2 \geq p$, let $r = \left(\frac{N^\alpha \lambda}{\epsilon C_0}\right)^2$. Then, setting $c = (\epsilon C_0)^{-2}$, we have

$$\mathbb{P}(\|P_N f^\omega\|_{L^p(T^d)} > \lambda) \leq e^{-r} = e^{-c N^2 \lambda^2}.$$

Otherwise, if $\left(\frac{N^\alpha \lambda}{\epsilon C_0}\right)^2 \leq p$, we can choose $C = e^p$. Then

$$\mathbb{P}(\|P_N f^\omega\|_{L^p(T^d)} > \lambda) \leq 1 = Ce^{-p} \leq Ce^{-c N^2 \lambda^2},$$

as desired. This completes the proof. \qed
Using (64) with \( a_n = e^{in \cdot x - it|n|^2} \langle n \rangle^{-\frac{d}{2} - \alpha} \) we get for \( r \geq 2 \)

\[
\|e^{it\Delta} P_N f^\omega\|_{L^r_x(T^d \times [0,1])} \leq Cr^{\frac{d}{r}} N^{-\alpha},
\]

with a constant uniform in \( t \in \mathbb{R} \). Proceeding as in the proof of Lemma 4.3 we also obtain improved Strichartz estimates for randomized data.

**Lemma 4.4.** Let \( p \in [2, \infty) \). Assume \( f^\omega \) is as in (63). Then we have, for some constants \( C \) and \( c \), the bound

\[
P \left( \|e^{it\Delta} P_N f^\omega\|_{L^p_x(T^d \times [0,1])} > \lambda \right) \leq Ce^{-cN^2\lambda^2}.
\]

In particular, for any \( \varepsilon > 0 \) sufficiently small, we have

\[
\|e^{it\Delta} P_N f^\omega\|_{L^p_x(T^d \times [0,1])} \lesssim N^{-\alpha} (-\ln \varepsilon)^{1/2}
\]

with probability at least \( 1 - \varepsilon \). Thus

\[
\|e^{it\Delta} P_N f^\omega\|_{L^p_x(T^d \times [0,1])} \lesssim N^{-\alpha} + \frac{d}{p} (-\ln \varepsilon)^{1/2}
\]

with probability at least \( 1 - \varepsilon \).

Later we will also need the following uniform bound (with high probability) for the \( H^s \) norm of \( f^\omega \) with \( s < \alpha \). This is a well known fact that we recall applying again (64) with \( a_n = e^{in \cdot x} \langle n \rangle^{-\frac{d}{2} - \alpha + s} \), so that we get for \( r \geq 2 \)

\[
\|P_N(D)^s f^\omega\|_{L^r_x} \leq Cr^{\frac{d}{r}} N^{s-\alpha}, \quad s < \alpha.
\]

Here \( (D) \) denotes the Fourier multiplier operator \( \langle n \rangle \). Proceeding as in the proof of Lemma 4.3 we also obtain

\[
P \left( \|\langle D \rangle^s P_N f^\omega\|_{L^2_x(T^d)} > \lambda \right) \leq Ce^{-cN^2(\alpha - s)^2}, \quad s < \alpha
\]

and in particular, for any \( \varepsilon > 0 \) sufficiently small

\[
\|f^\omega\|_{H^s_x(T^d)} \lesssim (-\ln \varepsilon)^{1/2} \quad s < \alpha,
\]

with probability at least \( 1 - \varepsilon \).

We thank Chenjie Fan for sharing with us the argument we used in the next proof, and that strengthens our original a.e. convergence to a uniform one.

**Proof of Proposition 4.1.** Let us decompose

\[
|e^{it\Delta} f^\omega - f^\omega| \leq |e^{it\Delta} P_{\geq N} f^\omega| + |e^{it\Delta} P_{\leq N} f^\omega - P_{\leq N} f^\omega| + |P_{> N} f^\omega|.
\]

We fix \( \lambda > 0 \) and \( \varepsilon > 0 \) sufficiently small. Using (66) and (68) we see that

\[
\|e^{it\Delta} P_{> N} f^\omega\|_{L^\infty_x(T^d \times [0,1])} + \|P_{> N} f^\omega\|_{L^p_x(T^d)} < \lambda/2
\]

holds for all \( N \) sufficiently large, depending on \( \lambda, \varepsilon \), with probability larger than \( 1 - \varepsilon \). Let us fix such \( N^* = N^*(\lambda, \varepsilon) \). Since

\[
e^{it\Delta} P_{\leq N^*} f^\omega - P_{\leq N^*} f^\omega = \sum_{|n| \leq N^*} (e^{-int^2} - 1)e^{in \cdot x} \hat{f^\omega}(n),
\]

using (66) again we can find \( t^* \) sufficiently small, depending only on \( N^* \) and \( \varepsilon \), such that for all \( t \in (0, t^*) \) we have

\[
\|e^{it\Delta} P_{\leq N^*} f^\omega - P_{\leq N^*} f^\omega\|_{L^\infty_x(T^d)} < \lambda/2
\]

with probability larger than \( 1 - \varepsilon \). Thus given \( \lambda > 0 \), we have found \( t^* = t^*(\lambda, \varepsilon) \) such that \( \|e^{it\Delta} f^\omega - f^\omega\|_{L^\infty_x(T^d)} < \lambda \) for all \( t \in (0, t^*) \). This implies that given any
\( \varepsilon > 0 \) sufficiently small we have that \( e^{it\Delta} f^\omega \) converges uniformly to \( f^\omega \) as \( t \to 0 \), with probability larger than \( 1 - 2\varepsilon \). Thus the statement follows taking the union over \( \varepsilon > 0 \).

\[ \square \]

### 4.2. The Linear Schrödinger Equation on \( \mathbb{R}^d \) with Random Data.

For the linear Schrödinger equation on \( \mathbb{R}^d \), randomization arguments similar to those in Section 4.1 can be applied. We use an integer tiling–type randomization, of the type introduced in [48, 30, 2]. To begin, we construct a partition of unity on \( \mathbb{R}^d \).

As before, \( \eta \) is a smooth cut-off of the unit interval. Specifically, let \( \eta : \mathbb{R}^d \to [0, 1] \) be a smooth function such that \( \text{supp} \eta \subset \{ \xi : |\xi| \leq 2 \} \) and \( \eta(\xi) = 1 \) for all \( |\xi| \leq 1 \).

Then for \( n \in \mathbb{Z}^d \), define

\[ \psi_n(\xi) = \frac{\eta(\xi - n)}{\sum_{\ell \in \mathbb{Z}^d} \eta(\xi - \ell)}. \]

Observe that \( \psi_n \) is smooth function supported on \( \{ \xi : |\xi - n| \leq 2 \} \) and we have \( \sum_n \psi_n(\xi) = 1 \) for all \( \xi \).

Fix \( f \in H^s(\mathbb{R}^d) \) with \( s \geq 0 \). We construct a randomization \( f^\omega \) of \( f \) as follows. Let \( g_n^\omega \) be a collection of independent (complex) standard Gaussian variables\(^8\), and define \( f^\omega \) by

\[ \hat{f^\omega}(\xi) = \sum_{n \in \mathbb{Z}^d} g_n^\omega \psi_n(\xi) \hat{f}(\xi). \]

One should note that randomization does not improve smoothness; see for example Remark 1.2 in [9].

By arguments almost identical to those for the periodic case, we have for any \( p \in [2, \infty) \)

\[ \| P_N e^{it\Delta} f^\omega \|_{L^p(\mathbb{R}^d \times [0, 1])} \lesssim N^{-\alpha} (-\ln \varepsilon)^{1/2} \| f \|_{L^2(\mathbb{R}^d)}, \]

\[ \| P_N f^\omega \|_{L^p(\mathbb{R}^d)} \lesssim N^{-\alpha} (-\ln \varepsilon)^{1/2} \| f \|_{L^2(\mathbb{R}^d)}, \]

for \( \omega \) in a set of probability at least \( 1 - \varepsilon \). Moreover \( f^\omega \) are \( \omega \)-almost surely continuous (in fact \( C^s \)) and in \( H^s \). More precisely one has uniform bounds for the \( H^s \) norm

\[ \| f^\omega \|_{H^s(\mathbb{R}^d)} \lesssim (-\ln \varepsilon)^{1/2} \| f \|_{H^s(\mathbb{R}^d)} \]

for \( \omega \) in a set of probability at least \( 1 - \varepsilon \). For proofs of these estimates, see for instance [35, Lemmata 2.1 & 2.3].

Thus the Bernstein inequality gives

\[ \| P_N e^{it\Delta} f^\omega \|_{L^\infty(\mathbb{R}^d \times [0, 1])} \lesssim N^{-\alpha + \frac{1}{2}} (-\ln \varepsilon)^{1/2} \| f \|_{L^2(\mathbb{R}^d)}, \]

\[ \| P_N f^\omega \|_{L^p(\mathbb{R}^d)} \lesssim N^{-\alpha + \frac{1}{2}} (-\ln \varepsilon)^{1/2} \| f \|_{L^2(\mathbb{R}^d)}. \]

Using these estimates and proceeding exactly as in the periodic case, we can establish the first part of Theorem 1.5, namely

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\(^8\)As in the periodic case, this argument works for any independent random variables whose distribution functions decay sufficiently rapidly. We work with the standard normal distribution for the sake of definiteness.
Proposition 4.5. Let $s > 0$ and $f \in H^s(\mathbb{R}^d)$. For $\omega$–almost every $f^{\omega}$ of the form (71) we have
\[ e^{it\Delta} f^{\omega}(x) \to f^{\omega}(x) \quad \text{as } t \to 0 \]
for every $x \in \mathbb{R}^d$ and uniformly.

4.3. The Cubic NLS Equation on $T^d$ ($d = 1, 2$) with Random Data (Theorem 1.3).

In this section, we consider the cubic Wick-ordered NLS on $T^d$ ($d = 1, 2$) as in the work of Bourgain in [5]. More precisely, letting
\[ N(u) := \pm u (|u|^2 - 2\mu), \quad \mu := \int_{T^d} |u(x,t)|^2 \, dx, \]
we consider the initial value problem
\[ \begin{cases} 
  i\partial_t u + \Delta u = N(u), & x \in T^d \\
  u(x,0) = f^{\omega}(x). 
\end{cases} \tag{73} \]
We are interested again in randomized initial data, i.e. $f^{\omega}$ is taken to be of the form (63). Recall (see (69)) that such data is $\omega$–almost surely in $H^s$ for all $s < \alpha$ and
\[ \|f^{\omega}\|_{H^s} \lesssim (-\ln \varepsilon)^{1/2}, \quad s < \alpha, \tag{74} \]
with probability at least $1 - \varepsilon$, for all $\varepsilon \in (0, 1)$ sufficiently small. Since we work with any $\alpha > 0$, we are considering initial data in $H^0$. We approximate equation (73) as in (38), for all $N \in 2\mathbb{N} \cup \{\infty\}$. Recall that $\Phi_t^N f^{\omega}$ denotes the associated flow, with initial datum
\[ \Phi_0^N f^{\omega} := P_{\leq N} f^{\omega} = \sum_{|n| \leq N} \frac{g_n^\omega}{2^{n+\alpha}} e^{in \cdot x}. \]
We write $\Phi_t f^{\omega} = \Phi_t^\infty f^{\omega}$ for the flow of (73) with datum $f^{\omega} = P_\infty f^{\omega}$.

Proposition 4.6. Let $d = 1, 2$ and $\alpha > 0$. Let $N \in 2\mathbb{N} \cup \{\infty\}$. For all $\sigma \in [0, \frac{1}{2})$, the following holds. Assume
\[ u = u(I) + u(II), \quad u(I) = e^{it\Delta} P_{\leq N} f^{\omega}, \quad \|u(II)\|_{X^s_{\beta, \frac{1}{4}+}} < 1 \tag{75} \]
and the same for $v$. Then
\[ \|N(u)\|_{X^s_{\beta, -\frac{1}{4}+}} \lesssim (-\ln \varepsilon)^{3/2} \tag{76} \]
\[ \|N(u) - N(v)\|_{X^s_{\beta, -\frac{1}{4}++}} \lesssim (-\ln \varepsilon) \|u - v\|_{X^s_{\beta, \frac{1}{2}+}} \tag{77} \]
for initial data of the form (63), with probability at least $1 - \varepsilon$, for all $\varepsilon \in (0, 1)$ sufficiently small. If we take $u$ as in (75) and we instead assume
\[ v = v(I) + u(II), \quad v(I) = e^{it\Delta} f^{\omega}, \quad \|u(II)\|_{X^s_{\beta, \frac{1}{4}+}} < 1, \]
we have
\[ \|N(u) - N(v)\|_{X^s_{\beta, \frac{1}{4}+}} \lesssim N^{-\alpha}. \tag{78} \]
Remark 4.7. Recall that $\alpha$ indicates the regularity of the initial datum. We are denoting by $\sigma$ the amount of smoothing we can prove for the Wick–ordered cubic nonlinearity $N$. More precisely, since the initial data (63) belongs to $H^{\alpha+}$, we can interpret this statement as saying that, with arbitrarily large probability, $N$ is $\sigma+$ smoother than $f^\omega$. Since $\sigma < \frac{1}{2}$ is permissible, we reach $\frac{1}{2}$–smoothing for $N$ and, combining with (24), also for the Duhamel contribution $\Phi^N_t f^\omega - e^{it\Delta} P_{\leq N} f^\omega$.

We postpone the proof of Proposition 4.6 to the end of the section. Let us fix $\alpha > 0$. Using (24), (25) and Proposition 4.6 one can show that for all $\delta > 0$ sufficiently small the following holds. For all $N \in 2^\mathbb{N} \cup \{\infty\}$, the map

$$\Gamma^N(u) := \eta(t)e^{it\Delta} P_{\leq N} f^\omega - i\eta(t) \int_0^t e^{i(t-s)\Delta} P_{\leq N} N(u(s)) \, ds$$

is a contraction on the set

$$\left\{ e^{it\Delta} P_{\leq N} f^\omega + g, \|g\|_{X^{\alpha+\sigma}_{\delta}+} < 1 \right\}$$

equipped with the $X^{\alpha+\sigma}_{\delta}+$ norm, outside an exceptional set (we call it a $\delta$–exceptional set) of initial data of probability smaller than $e^{-\delta^{-\gamma}}$, with $\gamma > 0$ a given small constant. Notice that this holds uniformly over $N \in 2^\mathbb{N} \cup \{\infty\}$. Again, this is a standard routine calculation that we omit. Thus we have, outside the $\delta$–exceptional set

$$\|\Phi^N f^\omega - \eta(t)e^{it\Delta} P_{\leq N} f^\omega\|_{X^{\alpha+\sigma}_{\delta}+} < 1, \quad N \in 2^\mathbb{N} \cup \{\infty\}.$$  \hfill (80)

We only explain how to find the relation between the local existence time $\delta$ and the size of the exceptional set. Given any $\varepsilon \in (0, 1)$ sufficiently small, using (24), (25) and Proposition 4.6, we have

$$\|\Gamma^N(u) - \eta(t)e^{it\Delta} P_{\leq N} f^\omega\|_{X^{\alpha+\sigma}_{\delta}+} \lesssim \delta^{0+} (-\ln \varepsilon)^{3/2},$$

for all $f^\omega$ outside an exceptional set of probability smaller than $\varepsilon$. Letting $\delta$ such that $\varepsilon = e^{-\delta^{-\gamma}}$ with $\gamma > 0$ a fixed small constant, we have $C\delta^{0+} (-\ln \varepsilon)^{3/2} < 1$ for all $\delta > 0$ sufficiently small. Note that the measure $e^{-\delta^{-\gamma}}$ of the $\delta$–exceptional set converges to zero as $\delta \to 0$. We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Notice that (10) is the content of Proposition 4.1. To prove (11), let us assume that we have proved

$$\lim_{N \to \infty} \left\| \sup_{0 \leq t \leq \delta} \|\Phi_t f^\omega(x) - \Phi^N_t f^\omega(x)\|_{L^2_\omega(T^2)} \right\|_{L^2(T^2)} = 0$$  \hfill (81)

for all $f^\omega$ outside a $\delta$–exceptional set $A_\delta$. This means that given $f^\omega$ we can find, $\omega$–almost surely, a $\delta_\omega$ such that (81) is satisfied. Indeed, if we could not do so, this would mean that $f^\omega \in \bigcap_{\delta > 0} A_\delta$, and the probability of this event is zero, since $P(A_\delta) \to 0$ as $\delta \to 0$. So, using Proposition 3.3, we have $\omega$–almost surely

$$\lim_{t \to 0} \Phi_t f^\omega(x) - f^\omega(x) = 0, \quad \text{for a.e. } x \in T^2,$$

as claimed. It remains to prove (81). We decompose

$$|\Phi_t f^\omega - \Phi^N_t f^\omega| \leq |e^{it\Delta} P_{> N} f^\omega| + |\Phi_t f^\omega - e^{it\Delta} f^\omega - (\Phi^N_t f^\omega - e^{it\Delta} P_{\leq N} f^\omega)|,$$
Thus, recalling the decay of the high frequency linear term given by (77), it remains to show that

\[
\lim_{N \to \infty} \left\| \sup_{0 \leq t \leq \delta} \left| \Phi_t f^\omega - e^{it \Delta} f^\omega - (\Phi_t^N f^\omega - e^{it \Delta} P_{\leq N} f^\omega) \right| \right\|_{L^2(T^2)} = 0,
\]

for all \( f^\omega \) outside a \( \delta \)-exceptional set.

For any \( \alpha > 0 \), we can choose \( \sigma \) sufficiently close to \( \frac{1}{2} \) that

\[
\frac{s_T}{s_T^2} < \frac{1}{2} < \alpha + \sigma.
\]

Thus, using the \( X^{s,b} \) space embedding from Lemma 2.2, it suffices to prove

\[
\lim_{N \to \infty} \left\| w - w^N \right\|_{X^{\alpha+\frac{1}{2}+}} = 0,
\]

where

\[
w^N := \Phi_t^N f - e^{it \Delta} P_{\leq N} f^\omega,
\]

\( w := w^\infty \).

Notice that by (80) we have

\[
\left\| w^N \right\|_{X^{\alpha+\frac{1}{2}+}} < 1, \quad N \in 2^N \cup \{\infty\}.
\]

Since for \( t \in [0, \delta] \) we have

\[
w - w^N = -i\eta(t) \int_0^t e^{i(t-t') \Delta} \left( \mathcal{N}(\Phi_t f^\omega) - \mathcal{N}(\Phi_t^N f^\omega) \right) dt',
\]

using (24), (25), we get

\[
\left\| w - w^N \right\|_{X^{\alpha+\frac{1}{2}+}} \lesssim \delta^{0+} \left\| \mathcal{N}(\Phi_t f) - \mathcal{N}(\Phi_t^N f) \right\|_{X^{\alpha+\frac{1}{2}+}}.
\]

We decompose

\[
\mathcal{N}(\Phi_t f) - P_{\leq N} \mathcal{N}(\Phi_t^N f) = P_{\leq N} \left( \mathcal{N}(e^{it \Delta} P_{\leq N} f^\omega + w) - \mathcal{N}(e^{it \Delta} P_{\leq N} f^\omega + w^N) \right) + \text{Remainders},
\]

where

\[
\text{Remainders} := P_{\leq N} \left( \mathcal{N}(e^{it \Delta} P_{\leq N} f^\omega + w) - \mathcal{N}(e^{it \Delta} P_{\leq N} f^\omega + w) \right) + P_{> N} \mathcal{N}(\Phi_t f).
\]

Notice that by (76), (78) we have

\[
\left\| \text{Remainders} \right\|_{X^{\alpha+\frac{1}{2}+}} \to 0 \quad \text{as } N \to \infty,
\]

with probability at least \( 1 - \varepsilon \). Using (77) we can estimate

\[
\left\| P_{\leq N} \left( \mathcal{N}(e^{it \Delta} P_{\leq N} f^\omega + w) - \mathcal{N}(e^{it \Delta} P_{\leq N} f^\omega + w) \right) \right\|_{X^{\alpha+\frac{1}{2}+}} \lesssim (-\ln \varepsilon) \left\| w - w^N \right\|_{X^{\alpha+\frac{1}{2}+}}
\]

and (86), (87), (89) give

\[
\left\| w - w^N \right\|_{X^{\alpha+\frac{1}{2}+}} \lesssim \delta^{0+} (-\ln \varepsilon) \left\| w - w^N \right\|_{X^{\alpha+\frac{1}{2}+}} + \left\| \text{Remainders} \right\|_{X^{\alpha+\frac{1}{2}+}}
\]
with probability at least $1 - \varepsilon$. Since with our choice of $\varepsilon = e^{-\delta \gamma}$ we have $C\delta^{3/2}(-\ln \varepsilon)^{3/2} < 1$, we can absorb the first term on the right hand side into the left hand side and we still have that (88) holds outside a $\delta$–exceptional set. Thus letting $N \to \infty$ the proof of (11) is complete.

To prove (12) outside the exceptional set, we proceed as we did after (48), and we see that it is enough to show that $\Phi_t f^\omega - f^\omega \in X^{\frac{3}{4}+\frac{1}{2}}_\delta$. This is done exactly as before, except that here we have to require

$$\frac{d}{2} < \alpha + \sigma.$$ 

Since we can take $\sigma < \frac{1}{2}$ for $d = 1, 2$, we see that the previous condition is satisfied as long as $\alpha > \frac{d-1}{2}$. Once we have shown (12) outside the $\delta$–exceptional set, we also have it with probability 1 taking the intersection on $\delta > 0$, as noted at the beginning of the proof.

We are now ready to prove the smoothing estimates given in Proposition 4.6.

**Proof of Proposition 4.6.** Notice that the Wick–ordered nonlinearity can be written as

$$\mathcal{N}(u(x, \cdot)) = \sum_{n_2 \not\equiv n_1, n_3} \tilde{u}(n_1)\tilde{u}(n_2)\tilde{u}(n_3)e^{i(n_1-n_2+n_3)\cdot x} - \sum_n \tilde{u}(n)|\tilde{u}(n)|^2e^{in\cdot x}$$

where we are looking at the nonlinear term for fixed time and $\tilde{u}(\cdot)$ denotes the space Fourier coefficients. From (91), exploiting the symmetry $n_1 \leftrightarrow n_3$, we also have the identity

$$\mathcal{N}(u(x, \cdot)) - \mathcal{N}(v(x, \cdot))$$

$$= \sum_{n_2 \not\equiv n_1, n_3} (\tilde{u}(n_1) - \tilde{v}(n_1))\tilde{u}(n_2)\tilde{u}(n_3)e^{i(n_1-n_2+n_3)\cdot x} - \sum_n (\tilde{u}(n) - \tilde{v}(n))|\tilde{u}(n)|^2e^{in\cdot x}$$

$$+ \sum_{n_2 \not\equiv n_1, n_3} (\tilde{u}(n_3) - \tilde{v}(n_3))\tilde{v}(n_2)\tilde{v}(n_1)e^{i(n_1-n_2+n_3)\cdot x} - \sum_n (\tilde{u}(n) - \tilde{v}(n))|\tilde{v}(n)|^2e^{in\cdot x}$$

$$+ \sum_{n_2 \not\equiv n_1, n_3} (\tilde{u}(n_2) - \tilde{v}(n_2))\tilde{v}(n_1)\tilde{v}(n_3)e^{i(n_1-n_2+n_3)\cdot x} - \sum_n (\tilde{u}(n) - \tilde{v}(n))\tilde{u}(n)\tilde{v}(n)e^{in\cdot x}.$$ 

Using (92) (and recalling again the symmetry $n_1 \leftrightarrow n_3$), it is clear that we can reduce to proving the (more general) Lemma 4.8 given below. It implies the desired statement since each summation in the above decomposition can be controlled by letting

$$u_j(n_j) = u(n_j), \quad v(n_j), \quad \text{or} \quad u(n_j) - v(n_j).$$

The proof of Lemma 4.8 below follows closely the arguments introduced by Bourgain in [5]. We still display the details since we need to show explicitly how the gain of regularity depends on the exponent $\alpha$ in the definition of $f^\omega$ in (63). One will note though that the proof of Lemma 4.8 reported here is much easier than the one presented in [5] since in our case $f^\omega$ is more regular, namely we consider $\alpha > 0$ instead of $\alpha = 0$. 

\[\square\]
Lemma 4.8. Let $d = 1, 2$ and $\alpha > 0$. Let $N \in 2^\mathbb{N} \cup \{\infty\}$. For all $\sigma \in [0, \frac{1}{2})$ the following holds. Assume for $j = 1, 2, 3$

$$u_j(t) = e^{it\Delta}P_{\leq N}f^\omega, \quad \|u_j(t)\|_{X^\sigma_t} < 1.$$  

Let $J_j \in \{I, II\}$, $j = 1, 2, 3$. Then, for all $\varepsilon \in (0, 1)$ sufficiently small we have the following

$$\|\mathcal{N}(u_1(J_1), \overline{u_2}(J_2), u_3(J_3))\|_{X^\sigma_t} \lesssim (-\ln \varepsilon)^{3/2},$$

and more precisely

$$\|\mathcal{N}(u_1(I), \overline{u_2}(J_2), u_3(J_3))\|_{X^\sigma_t} \lesssim (-\ln \varepsilon) \|u_1(I)\|_{X^\sigma_t},$$

$$\|\mathcal{N}(u_1(I), \overline{u_2}(II), u_3(J_3))\|_{X^\sigma_t} \lesssim (-\ln \varepsilon) \|u_2(II)\|_{X^\sigma_t},$$

with probability at least $1 - \varepsilon$. Moreover, if in (93) we replace for some $j = j^*$ the projection operator $P_{\leq N}$ by $P_{> N}$, then the estimate (94) with $J_j = I$ holds with an extra factor $N^{-\sigma}$ on the right hand side.

Notice that by the symmetry $n_1 \leftrightarrow n_3$ the estimate (95) implies an analogous estimate for $u_3(III)$.

Before we pass to the proof we should remark that Lemma 4.8 proves an almost sure gain of smoothness of $\sigma = \frac{1}{2} -$ for the nonhomogeneous part of the solution of (73) with initial data $f^\omega \in H^{\alpha -}$, $\alpha > 0$. This smoothing effect should be compared to the one recorded in Corollary 2.6 proved in a deterministic manner. There we proved that if the initial data is in $H^{0+}$ then basically there is only a $0 + +$ smoothing.

Proof. We prove (94), (95), (96) in the case $N = \infty$. It is then immediate to adapt the proof to $N \in \mathbb{N}$ and to prove the second part of the statement. Moreover, we first give the proof in dimension $d = 2$, which is the hardest case. At the end of the proof we explain how to handle the case $d = 1$. We split the nonlinearity into two parts:

$$\mathcal{N}_1(u_1(J_1), \overline{u_2}(J_2), u_3(J_3)) = \sum_{n_2 \neq n_1, n_3} u_1(J_1)(n_1)\overline{u_2}(J_2)(n_2)u_3(J_3)(n_3)e^{i(n_1 - n_2 + n_3)x},$$

$$\mathcal{N}_2(u_1(J_1), \overline{u_2}(J_2), u_3(J_3)) = \sum_n u_1(J_1)(n)\overline{u_2}(J_2)(n)u_3(J_3)(n)e^{inx}.$$

We prove (94), (95), (96) for $\mathcal{N}_1$, which is the most challenging contribution. The proof for $\mathcal{N}_2$ is elementary, so we leave the details to the reader. We decompose over dyadic scales $N_1, N_2, N_3$ in the following way:

$$\|\mathcal{N}_1(u_1(J_1), \overline{u_2}(J_2), u_3(J_3))\|_{X^\sigma_t} \lesssim \sum_{N_1, N_2, N_3} \|\mathcal{N}_1(P_{N_1} u_1(J_1), P_{N_2} \overline{u_2}(J_2), P_{N_3} u_3(J_3))\|_{X^\sigma_t}$$

$$=: \sum_{M_1, M_2, M_3} \|\mathcal{N}_1(P_{M_1} u_1(J_1), P_{M_2} \overline{u_2}(J_2), P_{M_3} u_3(J_3))\|_{X^\sigma_t}$$

Before we pass to the proof we should remark that Lemma 4.8 proves an almost sure gain of smoothness of $\sigma = \frac{1}{2} -$ for the nonhomogeneous part of the solution of (73) with initial data $f^\omega \in H^{\alpha -}, \alpha > 0$. This smoothing effect should be compared to the one recorded in Corollary 2.6 proved in a deterministic manner. There we proved that if the initial data is in $H^{0+}$ then basically there is only a $0 + +$ smoothing.
where, following [5], we denoted with $M_1, M_2, M_3$ the decreasing order of $N_1, N_2, N_3$. Notice that in this way $w_1$ denotes the $u_j$ supported on the largest frequency. We estimate this sum by first doing some reductions and then considering several cases. First we show that we can reduce to considering the case where the highest–frequency function is a random linear flow; i.e.

\[(97)\quad w_1(J_2) = w_1(I).\]

Indeed if $w_1(J_1) = w_1(II)$ we get, using (62)

\[(98)\quad \|N_1(P_{M_1} w_1(II), P_{M_2} w_2(J_2), P_{M_3} w_3(J_3))\|_{X^{0+\sigma, \frac{1}{2}+}} \lesssim \|P_{M_1} w_1(II)\|_{X^{0+\sigma, \frac{1}{2}+}} \|P_{M_2} w_2(J_2)\|_{X^{0+\sigma, \frac{1}{2}+}} \|P_{M_3} w_3(J_3)\|_{X^{0+\sigma, \frac{1}{2}+}} \lesssim \|w_1(II)\|_{X^{0+\sigma, \frac{1}{2}+}} \|w_2(J_2)\|_{X^{0+\sigma, \frac{1}{2}+}} \|w_3(J_3)\|_{X^{0+\sigma, \frac{1}{2}+}}.
\]

On the other hand, recalling (93) and (74) we have

\[(99)\quad\|w_3(J_3)\|_{X^{0+\sigma, \frac{1}{2}+}} < 1, \quad \|w_3(I)\|_{X^{0+\sigma, \frac{1}{2}+}} \lesssim \|f\|_{H^{\alpha\sigma}} \lesssim (-\ln \varepsilon)^{1/2},\]

where the second inequality holds with probability at least $1 - \varepsilon$. In this inequality we also used (23) and (74). Thus, when $w_1(J_1) = w_1(II)$ the estimates (94), (95), (96) follow summing the square of (98) over $M_1, M_2, M_3$, factorizing the sum, and then using Plancherel and (99).

Then we perform a second reduction to remove frequencies which are far from the paraboloid. More precisely, we denote with $P_A$ the space-time Fourier projection into the set $A$ and our goal is to reduce

\[(100)\quad \sum_{N,M_1,M_2,M_3} N^{2\alpha+2\sigma} \|P_N N_1(P_{M_1} w_1(I), P_{M_2} w_2(J_2), P_{M_3} w_3(J_3))\|_{X^{0+\sigma, \frac{1}{2}+}}^2 = \sum_{N,M_1,M_2,M_3} N^{2\alpha+2\sigma} \|P_N N_1(P_{M_1} w_1(I), P_{M_2} w_2(J_2), P_{M_3} w_3(J_3))\|_{X^{0+\sigma, \frac{1}{2}+}}^2
\]

to

\[(101)\quad \sum_{N,M_1,M_2,M_3} N^{2\alpha+2\sigma} \|P_N P_{\{|\tau - |n|^2| \leq N^{1+\frac{1}{4}\gamma}} N_1(P_{M_1} w_1(I), P_{M_2} w_2(J_2), P_{M_3} w_3(J_3))\|_{X^{0+\sigma, \frac{1}{2}+}}^2
\]

(\(\frac{1}{4}\gamma\) is removable, however it does not create any problems and facilitates the computations). To obtain this reduction, it is sufficient to show that projection of the nonlinearity onto the complementary set is appropriately bounded; i.e. that

\[(102)\quad \sum_{N,M_1,M_2,M_3} N^{2\alpha+2\sigma} \|P_N P_{\{|\tau - |n|^2| > N^{1+\frac{1}{4}\gamma}} N_1(P_{M_1} w_1(I), P_{M_2} w_2(J_2), P_{M_3} w_3(J_3))\|_{X^{0+\sigma, \frac{1}{2}+}}^2 \lesssim (-\ln \varepsilon) \|w_2(J_2)\|_{X^{0+\sigma, \frac{1}{2}+}}^2 \|w_3(J_3)\|_{X^{0+\sigma, \frac{1}{2}+}}^2
\]
on a set of probability larger than $1 - \varepsilon$. Indeed, recalling (99) and summing over $N$, this would imply the validity of (94), (95), (96) for this term. We could have required a weaker bound than (102), replacing the $X^{0+\sigma, \frac{1}{2}+}$ norm with an $X^{0+\sigma, \frac{1}{2}+}$
norm if $J_2 = I I$ and with a $(- \ln \varepsilon)$ factor if $J_2 = I$. However, we are able to prove the stronger estimate (102). To do so we bound

\begin{equation}
\sum_{M_1, M_2, M_3} N^{2\alpha+2\sigma} \| P_N P_{\{ \tau - |n| \tau > N \}} M_1 (P_{M_1} w_1(I), P_{M_2} w_2(J_2), P_{M_3} w_3(J_3)) \|_{X^{\alpha, \frac{3}{4} + \frac{\varepsilon}{2}}}^2 \\
\sim N^{2\alpha+2\sigma} \sum_{M_1, M_2, M_3} \int \frac{N_I(\tau + |n|^2)^{1/2}}{\tau + |n|^2} |N_I(n, \tau)|^2 \, d\tau \\
\lesssim N^{2\alpha -} \sum_{M_1, M_2, M_3} \| P_N M_1 (P_{M_1} w_1(I), P_{M_2} w_2(J_2), P_{M_3} w_3(J_3)) \|_{L^{2, \alpha}_{x,t}}^2.
\end{equation}

Then using Hölder’s inequality, the improved Strichartz inequality (67) for randomized functions (for the $L^p$ norm of $w_1(I)$), and the Strichartz inequality (19) (for the $L^p$ norms of $w_2(J_2)$ and $w_3(J_3)$), we obtain

\begin{equation}
\| P_N M_1 (P_{M_1} w_1(I), P_{M_2} w_2(J_2), P_{M_3} w_3(J_3)) \|_{L^{2, \alpha}_{x,t}}^2 \\
\leq \| P_{M_1} w_1(I) \|_{L^{2, \alpha}_{x,t}}^2 \| P_{M_2} w_2(J_2) \|_{L^{2, \alpha}_{x,t}}^2 \| P_{M_3} w_3(J_3) \|_{L^{2, \alpha}_{x,t}}^2, \\
\lesssim (- \ln \varepsilon) M_1^{-2\alpha} \| P_{M_2} w_2(J_2) \|_{L^{2, \alpha}_{x,t}}^2 \| P_{M_3} w_3(J_3) \|_{L^{2, \alpha}_{x,t}}^2, \\
\lesssim (- \ln \varepsilon) M_1^{-2\alpha} \| P_{M_2} v_2(J_2) \|_{X^{\alpha, \frac{3}{4} + \frac{\varepsilon}{2}}}^2 \| P_{M_3} v_3(J_3) \|_{X^{\alpha, \frac{3}{4} + \frac{\varepsilon}{2}}}^2,
\end{equation}

where we are taking $q \gg 1$ sufficiently large. This holds on a set of probability larger than $1 - \varepsilon$. Since for $M_1 \ll N$ the Fourier support of the nonlinear term is disjoint from the annulus of size $N$ (recall $M_1 \geq M_2 \geq M_3$), once we plug (104) into (103) the factor $N^{2\alpha -}$ is absorbed by $M_1^{-2\alpha}$ and we can rewrite the remaining factor as $M_1^{-1} N^{0 -}$. Thus, summing over $N, M_1, M_2, M_3$ we obtain (102). So we have reduced to (101).

To handle this term we need a more explicit expression for the functions $w_j$. If we consider functions of the form $w(I)$ (here we omit the subscript $j$ to simplify the notation) we already know

\begin{equation}
w(I)(x, t) = \sum_m \frac{g^m}{\langle m \rangle^{1 + \sigma}} e^{i m \cdot x - i |m|^2 t}.
\end{equation}

We can obtain a similar expression for $w(II)$. Namely, we can write

\begin{equation}
w(II)(x, t) = \int \phi(\lambda) \sum_m b_m(m) e^{i m \cdot x - i |m|^2 t} \, d\lambda,
\end{equation}

where $\phi$ satisfies

\begin{equation}
\int |\phi(\lambda)| \, d\lambda \lesssim \| w(II) \|_{X^{\alpha, \frac{3}{4} + \frac{\varepsilon}{2}}},
\end{equation}

and the coefficients $b_m(m)$ satisfy

\begin{equation}
\sum_m \langle m \rangle^{2\alpha + 1 - |b_m(m)|^2} = 1.
\end{equation}
To prove (106)–(108) we change variables by setting \(\tau' = \tau + |m|^2\):

\[
\begin{align*}
  w(\Pi)(x, t) &= \sum_{m} \int e^{ix \cdot m + it \cdot \tau} \overline{w(\Pi)}(m, \tau) \, d\tau \\
  &= \sum_{m} \int e^{it \cdot \tau} e^{im \cdot x - |m|^2 t} \overline{w(\Pi)}(m, \tau - |m|^2) \, d\tau \\
  &= \int \left( \sum_{\ell} e^{2\pi \ell} |\overline{w(\Pi)}(\ell, \tau' - |\ell|^2)|^2 \right)^{\frac{1}{2}} e^{it \cdot \tau} \sum_{m} e^{im \cdot x - |m|^2 t} b_{\lambda'}(m) \, d\tau,
\end{align*}
\]

where we have defined

\[
 b_{\lambda}(m) := \frac{\overline{w(\Pi)}(m, \lambda - |m|^2)}{\left( \sum_{\ell} e^{2\pi \ell} |\overline{w(\Pi)}(\ell, \lambda - |\ell|^2)|^2 \right)^{\frac{1}{2}}}.
\]

Thus (106) holds with

\[
\phi(\lambda) := \left( \sum_{\ell} e^{2\pi \ell} |\overline{w(\Pi)}(\ell, \lambda - |\ell|^2)|^2 \right)^{\frac{1}{2}} e^{it \cdot \lambda}.
\]

Notice that (108) is immediate by the definition (109). The property (107) follows by the Cauchy–Schwartz inequality and changing variables \(\lambda' = \lambda - |\ell|^2\):

\[
\int |\phi(\lambda)| \, d\lambda \leq \left( \int \frac{d\lambda}{(|\lambda| + \tau)^{1+}} \right)^{\frac{1}{2}} \left( \langle \lambda \rangle^{1+} \langle \ell \rangle^{2+} |\overline{w(\Pi)}(\ell, \lambda - |\ell|^2)\rangle^2 \, d\lambda \right)^{\frac{1}{2}} \\
\lesssim \left( \langle \lambda' + |\ell|^2 \rangle^{1+} \langle \ell \rangle^{2+} |\overline{w(\Pi)}(\ell, \lambda')\rangle^2 \right)^{\frac{1}{2}} = \|w(\Pi)\|_{s \to \sigma + \frac{1}{2}+}.
\]

We now come back to the \(u\) functions and introduce the notation

\[
a_{u(J), \lambda}(m) := \begin{cases} 
  \frac{q_{m}^{\lambda}}{(m)^{1+}} & \text{if } J = I, \\
  b_{\lambda}(m) & \text{if } J = II.
\end{cases}
\]

Recalling (105) and (106), we have

\[
\begin{align*}
P_{N} P_{\{\tau - |n|^2 \leq N^{2+}\}} \mathcal{N}_{1}(P_{N_{1}}, u_{1}(I), P_{N_{2}}, u_{2}(J_{2}), P_{N_{3}}, u_{3}(J_{3})) \\
= \int P_{N} P_{\{\tau - |n|^2 \leq N^{2+}\}} \left( \sum_{|n|_{j} = N_{j}} e^{ix \cdot (n_{1} - n_{2} + n_{3})} e^{-it(|n_{1}|^2 - |n_{2}|^2 + |n_{3}|^2)} \right) \\
\times \prod_{j=1,2,3} a_{u(J_{j}), \lambda}(n_{j}) \delta_{J_{j}} \left( \phi(\lambda_{j}) \, d\lambda_{j} \right),
\end{align*}
\]

where

\[
\delta_{J_{j}} \left( \phi(\lambda_{j}) \, d\lambda_{j} \right) = \begin{cases} 
  1 & \text{if } J_{j} = I, \\
  \phi(\lambda_{j}) \, d\lambda_{j} & \text{if } J_{j} = II.
\end{cases}
\]
Thus using Minkowski’s inequality and recalling (107), we see that (101) satisfies the desired inequalities (94), (95), (96) as long as we can bound

\[ N^{2\alpha + 2\sigma} \left\| \sum_{N_1, N_2, N_3} P_{N} \mathbb{P} \left\{ \tau - |n|^2 \leq N^{\frac{3}{2}} \right\} \left( \sum_{|n_j| \sim N_j} e^{i x (n_1 - n_2 + n_3)} e^{-i t (|n_1|^2 - |n_2|^2 + |n_3|^2)} \right) \times \prod_{j=1,2,3} a_{u_j(J_j), \lambda_j} \right\|_{X^{0, \frac{1}{2} + \varepsilon}}^2 \lesssim (- \ln \varepsilon)^3 N^{0^+}, \]

uniformly in \( \lambda_j \), for all \( \varepsilon \in (0, 1) \) sufficiently small, on a set of probability larger than \( 1 - \varepsilon \). All the following estimates are indeed uniform in \( \lambda_j \) and the exceptional set on which (112) could be not satisfied is independent of \( \lambda_j \). We omit the subscript \( \lambda_j \) to simplify the notation.

Since

\[ \mathcal{F} \left( \eta(t) e^{i x (n_1 - n_2 + n_3)} e^{-i t (|n_1|^2 - |n_2|^2 + |n_3|^2)} \right) (n, \tau) = \sum_{n_1 - n_2 + n_3 = n} \hat{\eta}(\tau + |n_1|^2 - |n_2|^2 + |n_3|^2), \]

where \( \mathcal{F} \) is the space-time Fourier transform, and \( \hat{\eta} \) is rapid decreasing, we reduce (112) to showing that

\[ N^{2\alpha + 2\sigma} \sum_{N_1, N_2, N_3} \int_{\frac{3}{2}}^{\infty} \frac{\chi_{\left\{ \tau + |n|^2 \leq N^{\frac{3}{2}} \right\}}}{(\tau - |n|^2)^{1 - \varepsilon}} \left( \sum_{|n_j| \sim N_j, n_2 \neq n_1, n_3} \prod_{j=1,2,3} a_{u_j(J_j)} (n_j) \right)^2 d\tau \lesssim (- \ln \varepsilon)^3 N^{0^+}. \]

Let

\[ \mu' := \tau - |n|^2 \]

and

\[ \mu = |n|^2 + |n_1|^2 - |n_2|^2 + |n_3|^2. \]

Notice that

\[ |\mu' + \mu| = |\tau + |n_1|^2 - |n_2|^2 + |n_3|^2| \leq N^{0^+}. \]

Since the \( d\tau \) integration region can be written as \( |\mu'| \lesssim N^{\frac{3}{2}} \) and we have

\[ |\mu| \leq |\mu'| + |\mu' + \mu| \leq |\mu'| + N^{0^+} \]

and

\[ |\mu'| \leq |\mu| + |\mu + \mu'| \leq |\mu| + N^{0^+}, \]

we can also rewrite this integration region as

\[ |\mu| \lesssim N^{\frac{3}{2}}. \]

Thus integrating over \( d\tau \) we reduce (114) to the estimate
Recalling that
\[ n, \mu \]

denote the size of the indices of the Fourier coefficients of \( u \)

\[ (116) \quad R_n(n_1, n_2, n_3) := \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : |n_j| \sim N_j, j = 1, 2, 3, \\
\quad n_2 \neq n_1, n_3, n_1 - n_2 + n_3 = n, \mu = |n|^2 + |n_1|^2 - |n_2|^2 + |n_3|^2 \}. \]

The set \( R_n(\cdot) \) depends on \( \mu \) also (like all the sets we define below). However we omit this dependence to simplify the notation. Notice that in the definition of \( R_n(\cdot) \) the condition

\[ |n|^2 + |n_1|^2 - |n_2|^2 + |n_3|^2 = \mu \]

can be equivalently replaced by

\[ 2(n_1 - n_2) \cdot (n_3 - n_2) = \mu. \]

Recalling that \( M_1, M_2, M_3 \) is the decreasing order of \( N_1, N_2, N_3 \), we now notice that we must have \( N_1 \sim M_1 \) or \( N_3 \sim M_1 \). Indeed, if we assume that both \( N_1 \ll M_1 \) and \( N_2 \ll M_1 \) we must have \( N_2 \sim M_1 \sim N \) and \( \mu \sim N^2 \), which contradicts the fact that \( \mu \sim N^\frac{3}{2} \). Since the roles of \( N_1 \) and \( N_3 \) are symmetric (they are always the size of the indices of the Fourier coefficients of \( u_1, u_3 \)), hereafter we assume that \( N_1 = M_1 \sim N \) and so \( u_1 = w_1 \);

recall that \( u_1 \) is the \( u_j \) supported on the largest frequency, and we have previously reduced to considering \( u_1(J_1) = w_1(I) \); see (97). Thus, the argument above allows us to further reduce (115) to showing that

\[ (117) \quad N^{2\alpha + 2\sigma} \sum_{N_1, N_2, N_3} \sup_{|n| \leq N_1 \atop |\mu| \leq N_2 \atop |n| \sim N_1} \left| \sum_{R_n(n_1, n_2, n_3)} \frac{g_{n_1}^\omega}{(n_1)^{1+\alpha}} a_{u_2(n_2)} a_{u_3(n_3)}(n_3) \right|^2 \lesssim (-\ln \varepsilon)^3 N^{0-}. \]

To estimate (117) we can now distinguish few last possibilities. It is useful to denote

\[ S(n_1, n_2, n_3) := \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : |n_j| \sim N_j, j = 1, 2, 3, \\
\quad n_2 \neq n_1, n_3, \mu = 2(n_1 - n_2) \cdot (n_3 - n_2) \}. \]

Case \( J_2 = J_3 = I \). We must show that

\[ (119) \quad N^{2\alpha + 2\sigma} \sum_{N_1, N_2, N_3} \sup_{|n| \leq N_1 \atop |\mu| \leq N_2 \atop |n| \sim N_1} \left| \sum_{R_n(n_1, n_2, n_3)} \frac{g_{n_1}^\omega}{(n_1)^{1+\alpha}} \frac{g_{n_2}^\omega}{(n_2)^{1+\alpha}} \frac{g_{n_3}^\omega}{(n_3)^{1+\alpha}} \right|^2 \lesssim (-\ln \varepsilon)^3 N^{0-}. \]

Recalling that
\[ \mathbb{E}(g_{n_j}^\omega g_{n_j'}^\omega) = 0, \quad \mathbb{E}(g_{n_j}^\omega g_{n_{j'}}^\omega) = \begin{cases} 0 & \text{if } j \neq j' \\ 1 & \text{if } j = j' \end{cases}, \]

we have denoted
\[ (115) \quad N^{2\alpha + 2\sigma} \sum_{N_1, N_2, N_3} \sup_{|n| \leq N_1 \atop |\mu| \leq N_2 \atop |n| \sim N_1} \left| \sum_{R_n(n_1, n_2, n_3)} \prod_{j=1, 2, 3} a_{u_j(n_j)}(n_j) \right|^2 \lesssim (-\ln \varepsilon)^3 N^{0-}, \]
along with the fact that the sum is restricted over \( n_1, n_3 \neq n_2 \) and symmetric under \( n_1 \leftrightarrow n_3 \), we get
\[
\mathbb{E} \left( \sum_{R_\sigma(n_1,n_2,n_3)} \frac{g_{n_1}^\omega}{(n_1)^{1+\alpha}} \frac{g_{n_2}^\omega}{(n_2)^{1+\alpha}} \frac{g_{n_3}^\omega}{(n_3)^{1+\alpha}} \right)^2 = 2 \sum_{R_\sigma(n_1,n_2,n_3)} \frac{1}{(n_1)^{2\alpha+2}} \frac{1}{(n_2)^{2\alpha+2}} \frac{1}{(n_3)^{2\alpha+2}}.
\]

In the following bound we first restrict the summation over \((n_1, n_2, n_3) \in R_\sigma(n_1, n_2, n_3)\) such that \(n_1 \neq n_3\) (with a small abuse of notation we do not introduce additional notation for this restriction). In this case
\[
\mathbb{E} \left( \sum_{|n| \sim N_1} \sum_{R_\sigma(n_1,n_2,n_3)} \frac{g_{n_1}^\omega}{(n_1)^{1+\alpha}} \frac{g_{n_2}^\omega}{(n_2)^{1+\alpha}} \frac{g_{n_3}^\omega}{(n_3)^{1+\alpha}} \right)^2 \lesssim \sum_{|n| \sim N_1} \sum_{R_\sigma(n_1,n_2,n_3)} \frac{1}{(n_1)^{2\alpha+2}} \frac{1}{(n_2)^{2\alpha+2}} \frac{1}{(n_3)^{2\alpha+2}} \lesssim N_1^{-2a-2} N_2^{-2a-2} N_3^{-2a-2} \#S(n_1, n_2, n_3) \lesssim N_1^{-2a-1} N_2^{-2a} N_3^{-2a},
\]
where we used that if \(n_1 \neq n_3\), then
\[
\#S(n_1, n_2, n_3) \lesssim N_1^2 N_2^2 N_3^2;
\]
this is because once we have fixed \(n_2, n_3\) in \(N_2^2 N_3^2\) possible ways, we remain with at most \(N_1\) choices for \(n_1\) by the relation \(\mu = 2(n_1 - n_2) \cdot (n_3 - n_2)\). If we sum over \((n_1, n_2, n_3) \in R_\sigma(n_1, n_2, n_3)\) such that \(n_1 = n_3\), the restriction \(\mu = 2|n_1 - n_2|^2\) implies that once we have chosen \(n_2\) in \(N_2^2\) possible ways, we remain with \(\lesssim \mu^{0+} \lesssim N_1^{0+}\) choices for \(n_1 = n_3\) (since a circle of radius \(\mu\) contains \( \mu^{0+}\) integer points). This gives an even better bound than the one above. Summing the (120) over \(N_2, N_3\) and recalling that \(N_1 \sim N\), we have bounded the \(L^p_\mu\) norm of the left hand side of (119) by
\[
N^{2\alpha+2\sigma} \sum_{N_1} N_1^{-2\alpha-1} \lesssim N^{2\sigma-1} \lesssim N^{0-},
\]
where we used \(\sigma < \frac{1}{2}\). Using the hypercontractivity of the Gaussian variables, we can upgrade this to an \(L^p_\mu\) bound, for any \(p < \infty\), with a constant \(Cp^{3/2}\) (see [22, Proposition 4.5] for details). Proceeding as in the proof of Lemma 4.3, this implies (119) for all \(\omega\) outside a set of probability smaller than \(\varepsilon\), as required.

**Case** \(J_2 = J_3 = II\). We show that
\[
N^{2\alpha+2\sigma} \sum_{N_1, N_2, N_3} \sup_{|\mu| \leq N_1^{1+\alpha}} \sum_{|n| \sim N_1} \left| \sum_{R_\sigma(n_1,n_2,n_3)} \frac{g_{n_1}^\omega}{(n_1)^{1+\alpha}} b_2(n_2) b_3(n_3) \right|^2 \lesssim (- \ln \varepsilon) N^{0-},
\]
which clearly implies (117). We denote
\[ R_{n,n_2}(n_1, n_3) := \{(n_1, n_3) \in \mathbb{Z}^2 : (n_1, n_2, n_3) \in R_n(n_1, n_2, n_3)\}, \]
and for \( j = 2, 3 \)
\[ \|b_j\|_{\overline{R}_{n_2}^j}^2 := \sum_{|n_j| \sim N_j} |b_j(n_j)|^2. \]

Notice that by (108) (and recalling the change in notations) we have for \( \sigma < 1/2 \)
\[ \sum_{N_j} N_j^{2\alpha + 2\sigma} \|b_j\|_{\overline{R}_{n_2}^j}^2 \lesssim 1, \quad \sum_{N_j \leq N} N_j^{2\alpha + 1} \|b_j\|_{\overline{R}_{n_2}^j}^2 \lesssim N^{0+}. \]

Hereafter all the sums over indexes \( n_j \) are restricted to \( n_j \sim N_j \). We omit this fact in the subscripts to simplify the notation. We estimate
\[ \left( \sum_n \left( \sum_{R_n(n_1,n_2,n_3)} \frac{g_{n_2}^n}{(n_1)^{1+\alpha}} b_2(n_2) b_3(n_3) \right)^2 \right)^{1/2} \]
\[ \lesssim \|b_2\|_{\overline{R}_{n_2}^2}^2 \sum_{n_2} \left( \sum_{R_{n_2}(n_1,n_3)} \frac{g_{n_2}^n}{(n_1)^{1+\alpha}} b_3(n_3) \right)^2, \]
where we have used the Cauchy–Schwartz inequality with respect to \( n_2 \) and (108).

We further denote
\[ S_{n_2}(n_1, n_3) := \{(n_1, n_3) \in \mathbb{Z}^2 : (n_1, n_2, n_3) \in S(n_1, n_2, n_3)\}. \]

We recall the estimate
\[ \# S_{n_2}(n_1, n_3) \lesssim N_1^{0+} \quad \text{(Lemma 1 part (i) in [5]).} \]

Thus, summing the (123) over \( |n| \sim N_1 \) yields
\[ \left( \sum_n \left( \sum_{R_n(n_1,n_2,n_3)} \frac{g_{n_2}^n}{(n_1)^{1+\alpha}} b_2(n_2) b_3(n_3) \right)^2 \right)^{1/2} \]
\[ \lesssim \|b_2\|_{\overline{R}_{n_2}^2}^2 \sum_{n_2} \left( \sum_{S_{n_2}(n_1,n_3)} \frac{|b_3(n_3)|}{(n_1)^{1+\alpha}} \right)^2 \]
\[ \lesssim (-\ln \epsilon) \|b_2\|_{\overline{R}_{n_2}^2}^2 \sum_{n_2} \left( \sum_{S_{n_2}(n_1,n_3)} \frac{|b_3(n_3)|}{(n_1)^{1+\alpha}} \right)^2 \]
\[ \lesssim (-\ln \epsilon) \|b_2\|_{\overline{R}_{n_2}^2}^2 N_1^{0+} \sum_{n_2} \sum_{S_{n_2}(n_1,n_3)} \frac{|b_3(n_3)|^2}{(n_1)^{2\alpha + 2}} \]
\[ \lesssim (-\ln \epsilon) \|b_2\|_{\overline{R}_{n_2}^2}^2 N_1^{0+} N_1^{-2\alpha - 2} \sum_{S(n_1,n_2,n_3)} |b_3(n_3)|^2, \]
where we used (124) and the fact that
\[ \sum_{S(n_1,n_2,n_3)} = \sum_{n_2} \sum_{S_{n_2}(n_1,n_3)}. \]
To justify the previous computation, in particular the factor of $(-\ln \epsilon)$, we should first average over $d\omega$ and then use the hypercontractivity of the Gaussian variables. Since this works exactly as in the previous case ($J_2 = J_3 = I$), we omit the details. We do the same in (129). Denoting

$$S_{n_3}(n_1, n_2) := \{(n_1, n_2) \in \mathbb{Z}^3 : (n_1, n_2, n_3) \in S(n_1, n_2, n_3)\},$$

we recall that

$$\#S_{n_3}(n_1, n_2) \lesssim N_1^{1+}N_2$$  \hspace{1cm} (Lemma 2 part (i) in [5] switching $n_1$ and $n_3$).

Since

$$\sum_{S(n_1, n_2, n_3)} = \sum_{n_3} \sum_{S_{n_3}(n_1, n_2)},$$

we have by (126)

$$\sum_{S(n_1, n_2, n_3)} |b_3(n_3)|^2 \lesssim N_1^{1+}N_2 \sum_{n_3} |b_3(n_3)|^2 \lesssim N_1^{1+}N_2 \|b_3\|_{L^2_{n_3}}^2.$$

Plugging (127) into (125) we see that (121) is satisfied as long as

$$N^{2\alpha + 2\sigma} \sum_{N_1, N_2, N_3} \frac{N_1^{-2\alpha - 1 + 0+} N_2 \|b_2\|_{L^2_{n_2}}^2 \|b_3\|_{L^2_{n_3}}^2}{N_3} \lesssim N^{0-}.$$

Recalling (122) and the fact that $N \sim N_j$, $j = 2, 3$, this is immediately verified for $\sigma < \frac{1}{2}$.

Case $J_2 = I, J_3 = II$. We show that

$$N^{2\alpha + 2\sigma} \sum_{N_1, N_2, N_3} \sup_{|n| \leq N_3^{1+}} \sum_{|n| \sim N_1} \sum_{R_n(n_1, n_2, n_3)} \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{g_{n_2}^\omega}{\langle n_2 \rangle^{1+\alpha}} b_3(n_3) \lesssim (-\ln \epsilon)^2 N^{0-},$$

which clearly implies (117). Since

$$\#R_n(n_1, n_2, n_3) \lesssim N_2 N_3^{0+}$$  \hspace{1cm} (Lemma 1 in [5]),

we can estimate using the Cauchy–Schwarz inequality:

$$\sum_{R_n(n_1, n_2, n_3)} \left| \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{g_{n_2}^\omega}{\langle n_2 \rangle^{1+\alpha}} b_3(n_3) \right|^2 \lesssim (-\ln \epsilon)^2 N_2 N_3^{0+} \sum_{R_n(n_1, n_2, n_3)} \frac{|b_3(n_3)|^2}{\langle n_1 \rangle^{2+2\alpha} \langle n_2 \rangle^{2+2\alpha}} \lesssim (-\ln \epsilon)^2 N_2 N_3^{0+} \sum_{R_n(n_1, n_2, n_3)} \frac{|b_3(n_3)|^2}{N_1^{-2\alpha} N_2^{-1+2\alpha} N_3^{0+}} \sum_{R_n(n_1, n_2, n_3)} |b_3(n_3)|^2.$$

Summing this over $|n_1| \sim N_1$ yields

$$\sum_{|n_1| \sim N_1} \left( \sum_{R_n(n_1, n_2, n_3)} \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{g_{n_2}^\omega}{\langle n_2 \rangle^{1+\alpha}} b_3(n_3) \right)^2.$$
\[ (-\ln \varepsilon)^2 N_1^{-2-2\alpha} N_2^{-1-2\alpha} N_3^{0+} \sum_{|n_1| \sim N_1} \sum_{R_n(n_1, n_2, n_3)} |b_3(n_3)|^2 \]

\[ \lesssim (-\ln \varepsilon)^2 N_1^{-2-2\alpha} N_2^{-1-2\alpha} N_3^{0+} \sum_{S(n_1, n_2, n_3)} |b_3(n_3)|^2 . \]

Then since

\[ \sum_{S(n_1, n_2, n_3)} = \sum_{n_3} \sum_{S(n_1, n_2, n_3)} \]

and

\[ (131) \quad \#S_{n_3}(n_1, n_2) \lesssim N_1^{1+} N_2, \quad \text{(Lemma 2 part (i) in [5])}, \]

we have

\[ (132) \quad \sum_{S(n_1, n_2, n_3)} |b_3(n_3)|^2 N_1^{1+} N_2 \sum_{n_3} |b_3(n_3)|^2 \lesssim N_1^{1+} N_2 \|b_3\|_{L_N^2}^2 . \]

Plugging (132) into (130) we see that the left hand side of (128) is bounded by

\[ N^{2\alpha+2\sigma} \sum_{N_1, N_2, N_3} N_1^{-2\alpha-1+0+} N_2^{-2\alpha} N_3^{0+} \|b_3\|_{L_N^2}^2 \lesssim N^{\alpha-} \]

as required, where we used \( \sigma < \frac{1}{2}, \) (122), and the fact that \( N \sim N_1 \gtrsim N_3. \)

**Case J_2 = II, J_3 = I.** We proceed exactly as in the case \( J_2 = I, J_3 = II, \) but we exchange the roles of \( n_2 \) and \( n_3. \) Notice that everything works symmetrically under \( n_2 \leftrightarrow n_3 \) except the fact that the sets \( S_{n_3}(n_1, n_2) \) and \( S_{n_2}(n_1, n_3) \) do not coincide. However, in the previous argument, we only needed the estimate (131). Here we instead use

\[ \#S_{n_2}(n_1, n_3) \lesssim N_1^{1+} N_3, \quad \text{(Lemma 2 part (ii) in [5])}, \]

whose right hand side is indeed the same as that of (131) after interchanging \( N_2 \leftrightarrow N_3. \) This concludes the proof of (94), (95), (96) in dimension \( d = 2. \)

The case \( d = 1 \) is much easier. One can easily check that the previous argument indeed adapts and simplifies. We omit the details. However we explicitly compute the contribution from the 1d version of case \( J_2 = J_3 = I. \) This computation shows that with this argument we cannot get more than \( \frac{1}{4} \)- smoothing. That is, we need \( \sigma < \frac{1}{2} \) even in the one dimensional case. Indeed, we must show the analog of (119) for the \( d = 1 \) case, which is

\[ (133) \quad N^{2\alpha+2\sigma} \sum_{N_1, N_2, N_3} \sup_{|n| \sim N_1} \left| \sum_{R_n(n_1, n_2, n_3)} \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{\alpha+\alpha}} \frac{g_{n_2}^\omega}{\langle n_2 \rangle^{\alpha+\alpha}} \frac{g_{n_3}^\omega}{\langle n_3 \rangle^{\alpha+\alpha}} \right|^2 \lesssim (-\ln \varepsilon)^3 N^{\alpha-} , \]

where now \( n_j \in \mathbb{Z} \) and the set \( R_n \) is defined as in (116), with obvious modifications (the same is true for the set \( S(\cdot) \) below; see (118)). Recall also that \( N_1 \sim N. \) Proceeding as above we get

\[ (134) \quad \mathbb{E} \left( \sum_{|n| \sim N_1} \left| \sum_{R_n(n_1, n_2, n_3)} \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{\alpha+\alpha}} \frac{g_{n_2}^\omega}{\langle n_2 \rangle^{\alpha+\alpha}} \frac{g_{n_3}^\omega}{\langle n_3 \rangle^{\alpha+\alpha}} \right|^2 \right) \]
\[ \lesssim \sum_{|n| \sim N_1} \sum_{R(n_1,n_2,n_3)} \frac{1}{\langle n_1 \rangle^{2\alpha+1}} \frac{1}{\langle n_2 \rangle^{2\alpha+1}} \frac{1}{\langle n_3 \rangle^{2\alpha+1}} \]

\[ \lesssim \sum_{S(n_1,n_2,n_3)} \sum_{R(n_1,n_2,n_3)} \frac{1}{\langle n_1 \rangle^{2\alpha+1}} \frac{1}{\langle n_2 \rangle^{2\alpha+1}} \frac{1}{\langle n_3 \rangle^{2\alpha+1}} \]

\[ \sim \sum_{S(n_1,n_2,n_3)} N_1^{-2\alpha-1} N_2^{-2\alpha-1} \]

\[ \lesssim N_1^{-2\alpha-1} N_2^{-2\alpha-1} \# S(n_1,n_2,n_3) \]

\[ \lesssim N_1^{-2\alpha-1} N_2^{-2\alpha-1} N_3^{-2\alpha} \]

where we used the bound

\[ \# S(n_1,n_2,n_3) \lesssim N_2 N_3. \]

This holds because once we have fixed \( n_2, n_3 \) in \( N_2 N_3 \) possible ways, then \( n_1 \) is given by the relation \( \mu = 2(n_1 - n_2)(n_3 - n_2) \); recall \( n_2 \neq n_3 \). Summing the (134) over \( N_2, N_3 \) and recalling \( N \sim N_1 \), we can bound the \( \omega \)-expectation of the left hand side of (133) by

\[ N^{2\alpha+2\sigma} \sum_{N_1} N_1^{-2\alpha-1} \lesssim N^0, \]

but we have to use \( \sigma < \frac{1}{2} \) in this case also.

\[
\Box
\]

The quintic NLS on \( \mathbb{T} \). Here we explain how one can prove an analogous \( \frac{1}{2} \)-smoothing result for the quintic NLS \((p = 5)\) on \( \mathbb{T} \), after removing certain bad resonances from the nonlinearity, as we have done using the Wick order in the cubic case. We plan to study this problem in detail in a future work. We consider

\[ \mathcal{N}(u) := \pm u \left( |u|^5 - 3\mu \right), \quad \mu = \int_{\mathbb{T}} |u(x,t)|^4 \, dx, \]

and

\[
\begin{cases}
  i\partial_t u + \Delta u = \mathcal{N}(u), & x \in \mathbb{T}, \\
  u(x,0) = f^\omega(x),
\end{cases}
\]

with randomized initial data

\[ f^\omega(x) = \sum_{n \in \mathbb{Z}} \frac{g_n^\omega}{\langle n \rangle^{2\alpha}} e^{i n \cdot x}. \]

Recall that such data is \( \omega \)-almost surely in \( H^s \) for all \( s < \alpha \) (namely we work at \( H^{0+} \) level) and satisfies a uniform bound for these \( H^s \) norms with arbitrarily high probability; see (74). Proceeding as for the cubic equation before, and focusing only on the fully random evolution, namely the case \( J_j = I \) for \( j = 1, \ldots, 5 \), we reduce to proving the following fact. We fix

\[ 0 < \sigma < \frac{1}{2}. \]

Then with probability at least \( 1 - \varepsilon \), we have

\[ \text{For more information about why this is the relevant nonlinear term to consider in the quintic case, consult [33].} \]
\[(138) \quad N^{2\alpha+2\sigma} \sum_{N_1, \ldots, N_5} \sup_{|n| \sim N_1} \sum_{|n| \sim N_1, \ldots, n_5} \left| \sum_{R_n(n_1, \ldots, n_5)} \frac{g_n^\omega}{(n_1)^{\frac{1}{2}+\alpha}} \frac{g_n^{\alpha}}{(n_2)^{\frac{1}{2}+\alpha}} \frac{g_n^{\alpha}}{(n_3)^{\frac{1}{2}+\alpha}} \times \frac{g_n^{\alpha}}{(n_4)^{\frac{1}{2}+\alpha}} \frac{g_n^{\alpha}}{(n_5)^{\frac{1}{2}+\alpha}} \right|^2 \lesssim (-\ln \varepsilon)^5 N^{0-},\]

where
\[(139) \quad R_n(n_1, \ldots, n_5) := \{ (n_1, \ldots, n_5) \in \mathbb{Z}^5 : |n_j| \sim N_j, j = 1, \ldots, 5, \]
\[n_2, n_4 \neq n_1, n_3, n_5, n_1 - n_2 + n_3 - n_4 + n_5 = n, \]
\[\mu = |n|^2 + |n_j|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 \}

and we have assumed \( N_1 = \max\{N_1, \ldots, N_5\} \), so that we can restrict to the case \( N \sim N_1 \). However, the argument below adapts immediately to the case in which the largest frequency is \( N_2 \) (all the other cases are clearly symmetric). Again, averaging in \( d\omega \) and upgrading the corresponding estimate to any \( L^p, p \in [2, \infty) \) by hypercontractivity, we reduce to proving, uniformly over \( \mu \), the following:
\[(140) \quad N^{2\alpha+2\sigma} \sum_{N_1, \ldots, N_5} \sum_{|n| \sim N_1} \sum_{R_n(n_1, \ldots, n_5)} \prod_{j=1, \ldots, 5} \frac{1}{(n_j)^{1+2\alpha}} \lesssim N^{0-}.
\]

In fact, we have
\[N^{2\alpha+2\sigma} \sum_{N_1, \ldots, N_5} \sum_{|n| \sim N_1} \sum_{R_n(n_1, \ldots, n_5)} \prod_{j=1, \ldots, 5} \frac{1}{(n_j)^{1+2\alpha}} \lesssim N^{2\alpha+2\sigma} \sum_{N_1, \ldots, N_5} \#R(n_1, \ldots, n_5) N_1^{-(1+2\alpha)} N_2^{-(1+2\alpha)} N_3^{-(1+2\alpha)} N_4^{-(1+2\alpha)} N_5^{-(1+2\alpha)}\]

and since \( \#R(n_1, \ldots, n_5) \lesssim N_5 N_4 N_3 N_2 \) and \( \sigma < \frac{1}{2} \), the estimate (140) is proved.

4.4. The Cubic NLS Equation on \( \mathbb{R}^d \) (\( d = 1, 2 \)) with Random Data (Theorem 1.5).

We prove Theorem 1.5. Given \( f \in H^s(\mathbb{R}^d) \) with \( s > 0 \), we are considering the randomized initial data \( f^{\omega} \) defined in (71). Remember that these functions are typically more integrable than \( f \). They are \( \omega \)-almost surely in \( L^p \) for any \( p \in [2, \infty) \). On the other hand, they are not more regular than \( f \), but they rather have comparable \( H^s \) norms; see (72). We approximate the equation (here \( \mathcal{N}(z) = \pm |z|^2 z \) as in (38), for all \( N \in 2\mathbb{N} \cup \{\infty\} \), and \( \Phi_t^N f^{\omega} \) denotes the associated flow, with initial datum \( f^{\omega} \).

**Proof of Theorem 1.5.** Notice that (13) is the content of Proposition 4.5. To prove (14), proceeding exactly as in the proof of Theorem 1.3, it suffices to show that
\[(141) \quad \lim_{N \to \infty} \left\| \sup_{|\xi| \leq 1} |\Phi_t f^{\omega}(x) - \Phi_t^N f^{\omega}(x)| \right\|_{L^2(\mathbb{T}^d)} = 0 \]

for all \( f^{\omega} \) outside a \( \delta \)-exceptional set \( A_\delta \). This can be done using Corollary 2.6 exactly as in the proof of (81) using Proposition 4.6. In fact, since Corollary 2.6 is
a deterministic statement, we can actually prove (141) for all \( \omega \). To do so we need to require (compare with (83))

\[
(142) \quad s_{R^d} = \frac{d}{2(d+1)} < s + \sigma, \quad d = 1, 2.
\]

Since we used Corollary 2.6, we are allowed to take \( \sigma < \min(2s, 1) \) and we see that (142) is satisfied for all \( s > \frac{d}{2(d+1)} \). This completes the proof.

\[\square\]

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