Dressing Method and the Coupled KP Hierarchy

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Abstract

The coupled KP hierarchy, introduced by Hirota and Ohta, are investigated by using the dressing method. It is shown that the coupled KP hierarchy can be reformulated as a reduced case of the 2-component KP hierarchy.

1 Introduction

During the last decade, mathematical structure of soliton equations have been extensively studied especially in case of the Kadomtsev-Petviashvili (KP) hierarchy. Sato found that solution space can be parameterized by universal Grassmann manifold [1]. It has been revealed that the KP hierarchy has a class of solutions that are represented as quotients of determinants [2], i.e., soliton equations can be reduced to algebraic identities of determinants. Hirota and Ohta investigated another class of soliton equations whose solutions are expressed as Pfaffians rather than determinants [3]. Their equations can be considered as a coupled version of the KP hierarchy and reduced to algebraic identities of Pfaffians.

In ref. [3], the derivation of the coupled KP (cKP) hierarchy is based on the bilinear formulation. They have shown how the Pfaffian \( \tau \)-functions solve the equations of lower order, by direct substitution using quadratic identities of Pfaffians. However, no proof has been given explicitly for general cases. Lax-type or Zakharov-Shabat-type representations of the hierarchy have not been presented. Since the cKP hierarchy is related to the matrix integrals of orthogonal or symplectic type [4], it may be worth studying the mathematical structures of the hierarchy.

As an example of the equation in the coupled hierarchy, we consider the cKP equations [3],

\[
\begin{align*}
(4u_t - uu_x - 12u_{xxx})_x - 3u_{yy} + 12(v\tilde{v})_{xx} &= 0, \\
2v_t + 6uv_x + v_{xxx} + 3v_{xy} + 6v \int_x^y u_y dx &= 0, \\
2\tilde{v}_t + 6u\tilde{v}_x + \tilde{v}_{xxx} - 3\tilde{v}_{xy} - 6\tilde{v} \int_x^y u_y dx &= 0,
\end{align*}
\]

(1)

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where a subscript indicates partial differentiation with regard to some variable. Note that our notation is slightly different from that of [3] up to suitable scaling.

Making a change of the dependent variables,

\[ u = (\log \tau)_{xx}, \quad v = \sigma / \tau, \quad \dot{v} = \ddot{\sigma} / \tau, \]  

we have the following bilinear equations,

\[ (D_1^4 - 4D_1D_3 + 3D_2^2)\tau \cdot \tau = 24\ddot{\sigma} \sigma, \]
\[ (D_1^2 + 2D_3 + 3D_1D_2)\sigma \cdot \tau = 0, \]
\[ (D_1^2 + 2D_3 - 3D_1D_2)\ddot{\sigma} \cdot \tau = 0. \]  

Here the Hirota bilinear operators are defined by

\[ D_m^k D_n^l f \cdot g = \left( \frac{\partial}{\partial t_m} - \frac{\partial}{\partial t'_m} \right)^k \left( \frac{\partial}{\partial t_n} - \frac{\partial}{\partial t'_n} \right)^l f(t_1, t_2, \ldots) g(t_1', t_2', \ldots) \bigg|_{t'=t}, \]

and \( x = t_1, \ y = t_2, \ t = t_3. \)

The bilinear equations (3) have a class of solutions that are expressed as Pfaffians. Throughout this paper, we will use a frequently used notation for Pfaffians [3, 5, 6, 7]:

\[ (1, 2, \ldots, 2N) = \sum_{j_1 < j_2 < \cdots < j_{2N-1} \atop j_1 < j_2 < \cdots < j_{2N-1} < j_{2N}} \text{sgn} \left( \begin{array}{cccc} 1 & 2 & \cdots & 2N \end{array} \right)_{j_1 \ j_2 \ \cdots \ j_{2N}} \times (j_1, j_2, j_3) \cdots (j_{2N-1}, j_{2N}), \]

where \((i, j)\) denotes the \((i, j)\)-element of the Pfaffian which satisfies \((j, i) = -(i, j)\). A class of solution, which Hirota and Ohta called Gram-type, is then given by

\[ \tau^G = (1, 2, \ldots, 2N), \]
\[ \sigma^G = (c_1, c_0, 1, 2, \ldots, 2N), \]
\[ \ddot{\sigma}^G = (d_0, d_1, 1, 2, \ldots, 2N). \]  

The elements \((i, j)\) are of the following form:

\[ (i, j) = c_{ij} + \int^x (f_i g_j - f_j g_i) dx, \quad c_{ji} = -c_{ij}, \]
\[ (d_n, i) = \partial_x^n f_i(x), \quad (c_n, i) = \partial_x^n g_i(x), \]
\[ (d_n, d_m) = (c_n, c_m) = (c_n, d_m) = 0, \]  

where \(\partial_x\) denotes differentiation with respect to \(x\). The functions \(f_k, g_k\ (k = 1, \ldots, 2N)\) satisfy the dispersion relations,

\[ \partial_n f_k(x, t) = \partial_x^n f_k(x, t), \quad \partial_n g_k(x, t) = (-1)^{n-1} \partial_x^n g_k(x, t), \]  

where \(\partial_n\) denotes differentiation with respect to the \(n\)-th “time” variable \(t_n\ (n = 1, 2, \ldots)\). If we substitute \(\tau^G, \sigma^G\) and \(\ddot{\sigma}^G\) into the bilinear equations (3), they are reduced to quadratic identities of Pfaffians [3].
On the other hand, the dressing method of Zakharov and Shabat provides another powerful tool to treat wide class of soliton equations \cite{8}. Using this technique, one can construct solutions and Lax pair at the same time. The relationship between Sato’s viewpoint and the dressing method has been investigated by Pöppe and Sattinger \cite{9}. The dressing method can be applied to the equations other than the KP hierarchy. Hirota applied the dressing method to the BKP hierarchy \cite{7}. He introduced a modified version of the Gel’fand-Levitan-Marchenko equation. The aim of the present article is to apply the dressing method to the cKP hierarchy. We will show that the cKP hierarchy can be obtained as a reduced case of the 2-component KP hierarchy.

2 Dressing method for the multi-component KP hierarchy

Dressing method for the single-component KP hierarchy has been introduced by Pöppe and Sattinger \cite{9}. In this section, we generalize their results to the multi-component case. Consider the differential operators with $r \times r$-matrix coefficients,

$$\frac{\partial}{\partial t_n^{(a)}} - E_a \partial_x^n, \quad (E_a)_{i,j} = \delta_{a,i} \delta_{i,j} \quad (a = 1, \ldots, r; n = 1, 2, \ldots).\quad(7)$$

These “bare” operators generate commuting flows of the multi-component KP hierarchy via the “dressing equations”,

$$\left(\frac{\partial}{\partial t_n^{(a)}} - B_n^{(a)}\right) (1 + \hat{K}_\pm) = (1 + \hat{K}_\pm) \left(\frac{\partial}{\partial t_n^{(a)}} - E_a \partial_x^n\right).\quad(8)$$

The operators $\hat{K}_\pm$ are Volterra integral operators with the matrix kernels $K_\pm(x, z)$, s.t.

$$\hat{K}_+ \psi(x) = \int_x^{\infty} K_+(x, z) \psi(z) dz, \quad \hat{K}_- \psi(x) = \int_{-\infty}^x K_-(x, z) \psi(z) dz.$$ 

The operators $B_n^{(a)}$ are differential operators whose coefficients are $r \times r$ matrices and depend on infinitely many variables $t_n^{(a)} = (t_n^{(a)}) \quad (a = 1, \ldots, r; n = 1, 2, \ldots)$. They can be obtained as

$$B_n^{(a)} = \left[(1 + \hat{K}_+) E_a \partial_x^n (1 + \hat{K}_+)^{-1}\right]_+,\quad(9)$$

where $[\cdot]_+$ denotes differential operator part. The multi-component KP hierarchy is obtained from the commutation relations,

$$\left[\frac{\partial}{\partial t_n^{(a)}}, \frac{\partial}{\partial t_m^{(b)}} - B_n^{(b)}\right] = 0 \quad (a, b = 1, \ldots, r; n = 1, 2, \ldots).$$

We note that the coefficients of the operators $B_n^{(a)}$ are written in terms of the kernel $K_+(x, z)$.

If the operator $(1 + \hat{K}_+)$ is invertible, one can define the integral operator $\hat{F}$ as

$$(1 + \hat{F}) = (1 + \hat{K}_+)^{-1}(1 + \hat{K}_-).\quad(10)$$
An easy calculation shows that the operator \( \hat{F} \) commutes with the bare operators \( \mathcal{F} \). The commutativity reduce to the following linear equations for the kernel \( \mathcal{F}(x, z; t) \):

\[
\frac{\partial}{\partial t_n} \mathcal{F}(x, z) - E_a \partial_x^n \mathcal{F}(x, z) + (-1)^n \partial_x^n \mathcal{F}(x, z) E_a = 0. \tag{11}
\]

On the contrary, if a operator \( \hat{F} \) is given, one can construct \( \hat{K}_\pm \) by the Volterra decomposition \( (10) \). In what follows, we shall assume the existence and the uniqueness of the decomposition \( (10) \). The integral kernel \( K_+(x, z) \) can be constructed by solving the Gel’fand-Levitan-Marchenko (GLM) equation,

\[
K_+(x, z) + F(x, z) + \int_x^\infty K_+(x, y) F(y, z) dy = 0. \tag{12}
\]

It is then easy to recover \( B^{(a)}_n \) by \( (8) \). From the commutativity of \( \left[ \partial/\partial t_n \right] (\mathcal{F} - B^{(a)}_n) \) and \( \hat{F} \), it follows that each of \( B^{(a)}_n \) is a pure differential operator \( \mathcal{F} \).

The bare wave function

\[
w^{(0)} = t \left( \exp [\xi(x, t^{(1)}; k_1)], \ldots, \exp [\xi(x, t^{(M)}; k_M)] \right),
\]

with \( \xi(x, t^{(a)}; k) = xk + \sum_{j=1}^{\infty} t_j^{(a)} k^j \), satisfies the infinite set of the equations

\[
\left( \frac{\partial}{\partial t_n^{(a)}} - E_a \partial_x^n \right) w^{(0)} = 0.
\]

A wave function for the multi-component KP hierarchy is given by

\[
w(x, t; k) = (1 + \hat{K}_+) w^{(0)},
\]

which satisfies the linear problems

\[
\left( \frac{\partial}{\partial t_n^{(a)}} - B^{(a)}_n \right) w(t; k) = 0.
\]

Next we discuss the adjoint of the multi-component KP hierarchy, which is obtained by dressing the operators

\[
\frac{\partial}{\partial t_n^{(a)}} + (-1)^n E_a \partial_x^n, \quad (a = 1, \ldots, r; n = 1, 2, \ldots) \tag{13}
\]

In this case, we introduce the bare wave function of the adjoint hierarchy by

\[
w^{*(0)} = t \left( \exp [-\xi(x, t^{(1)}; k_1)], \ldots, \exp [-\xi(x, t^{(M)}; k_M)] \right).
\]

We denote the adjoint of \( \hat{F} \) by \( \hat{F}^* \):}

\[
\hat{F}^* \psi(x) = \int_{-\infty}^\infty t F(z, x) \psi(z) dz.
\]

The commutativity of these two operators,

\[
\left[ \frac{\partial}{\partial t_n^{(a)}} + (-1)^n E_a \partial_x^n, \hat{F}^* \right] = 0,
\]
results in the differential equation (11).

Since the adjoint of eq. (10) is

\[(1 + \tilde{F}^*) = (1 + \tilde{K}^{-}_\pm)(1 + \tilde{K}^{*}_\pm)^{-1},\]

the operators corresponding to \((1 + \tilde{K}^\pm)\) for the adjoint hierarchy are \((1 + \tilde{K}^{*\pm})^{-1}\), respectively. Dressing the adjoint bare operators (13) by these operators, we obtain commuting flows of the adjoint hierarchy:

\[
\left(\frac{\partial}{\partial t^{(a)}_n} + B^{(a)*}_n\right)(1 + \tilde{K}^{*}_\pm)^{-1} = (1 + \tilde{K}^{*\pm})^{-1}\left(\frac{\partial}{\partial t^{(a)}_n} + (-1)^n E_a \partial_x^n\right).
\]

The operators \(B^{(a)*}_n\) are obtained as

\[
B^{(a)*}_n = (-1)^n \left[(1 + \tilde{K}^{*}_\pm)^{-1} E_a \partial_x^n (1 + \tilde{K}^{*}_\pm)\right]_+.
\]

The adjoint wave function \(w^*\) is then constructed as

\[
w^*(x, t; k) = (1 + \tilde{K}^{*}_\pm)^{-1} w^{*(0)}(0),
\]

which satisfies the adjoint linear problems,

\[
\left(\frac{\partial}{\partial t^{(a)}_n} + B^{(a)*}_n\right) w^*(t; k) = 0.
\]

### 3 Reduction to the coupled KP hierarchy

Hereafter we only consider the 2-component case \((r = 2)\). We shall impose the following condition on the operator \(\tilde{F}\):

\[
\tilde{F}^* = -J \tilde{F} J, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

In terms of the matrix kernel \(F(x, z)\), this condition is written as

\[
^t F(z, x) = -J F(x, z) J.
\]

The condition (13) induces some symmetries of the Volterra operators. For example, the operator \(\tilde{K}^+_\pm\) obeys the following relations,

\[
^t K^+_\pm(z, x) = -J K^+_\pm(x, z) J.
\]

This can be proved by using the fact that the formal solution of the GLM equation (12) is expressed with an infinite series,

\[
K(x, z) = \sum_{j=1}^{\infty} (-1)^j \int_x^\infty \cdots \int_x^\infty F(x, y_1) F(y_1, y_2) \cdots F(y_{j-1}, z) dy_1 \cdots dy_{j-1}.
\]

From the symmetry (17), we know that the matrix \(K^+_\pm(x, z)\) is reduced to scalar if we set \(z = x\), i.e.,

\[
K^{(11)}_+(x, x) = K^{(22)}_+(x, x), \quad K^{(12)}_+(x, x) = K^{(21)}_+(x, x) = 0.
\]
Furthermore, applying the symmetry (13) to eq. (14), we find that

\[(1 + \hat{F}^\ast) = (1 - J \hat{K}_+ J)^{-1} (1 - J \hat{K}^\ast J)\]

Assuming the uniqueness of the decomposition, we obtain

\[(1 + \hat{K}^\ast) = (1 - J \hat{K} J)^{-1}, \quad (1 + \hat{K}^\ast) = (1 - J \hat{K}_- J)^{-1}.\]  

(19)

From the eqs. (9), (14), (19), it follows that

\[B_{(1)}^n \star n = \begin{pmatrix} -1 \end{pmatrix}^n + 1 J B_{(2)}^n \star n J, \quad B_{(2)}^n \star n = \begin{pmatrix} -1 \end{pmatrix}^n + 1 J B_{(1)}^n \star n J\]

Let us now consider the time evolutions. Under the condition (16), the adjoint of eq. (11) leads to

\[\partial_t F(x,z) + \begin{pmatrix} -1 \end{pmatrix}^n + 1 J E_a J \partial^n_x F(x,z) + \partial^n_z F(x,z) J E_a J = 0.\]

Using the formulas

\[J E_1 J = -E_2, \quad J E_2 J = -E_1,\]

we can show that the kernel should satisfy the following equations:

\[\left\{ \frac{\partial}{\partial t_n^{(1)}} + \begin{pmatrix} -1 \end{pmatrix}^n \frac{\partial}{\partial t_n^{(2)}} \right\} F(x,z; t) = 0 \quad (n = 1, 2, \ldots).\]  

(20)

If we make a change of the variables

\[t_n = \{t_n^{(1)} - (-1)^n t_n^{(2)}\}/2, \quad \tilde{t}_n = \{t_n^{(1)} + (-1)^n t_n^{(2)}\}/2,\]

eq. (20) shows that \(F\) should depends only on \(t_n\), and not on \(\tilde{t}_n\). The corresponding bare operators are

\[A_n^{(0)} = \begin{pmatrix} \frac{\partial}{\partial t_n^{(1)}} - E_1 \partial_x^n \end{pmatrix} - \begin{pmatrix} -1 \end{pmatrix}^n \begin{pmatrix} \frac{\partial}{\partial t_n^{(2)}} - E_2 \partial_x^n \end{pmatrix}\]

\[= \frac{\partial}{\partial t_n} + \begin{pmatrix} -1 & 0 \\ 0 & (-1)^n \end{pmatrix} \partial_x^n.\]

We then consider the dressing equation for these operators:

\[\partial_n - B_n \ast (1 + \hat{K}_+) = (1 + \hat{K}_+) A_n^{(0)},\]

i.e., \(B_n = B_n^{(1)} - (-1)^n B_n^{(2)}\). Under the condition (16), it can be seen that the operators \(B_n^\ast\) satisfy \(B_n^\ast = J B_n J\).

For \(n = 2, 3\), explicit form of the operators \(B_2, B_3\) are

\[B_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x^2 + 2 \begin{pmatrix} u & \tilde{v} \\ -\tilde{v} & -u \end{pmatrix},\]  

(21)

\[B_3 = \partial_x^3 + 3u \partial_x + 3u \tilde{u} + 3 \begin{pmatrix} w_x & \tilde{v}_x \\ v_x & \tilde{w}_x \end{pmatrix},\]  

(22)
with
\[\tilde{u} = K_+^{(11)}(x, x), \quad u = \partial_x \tilde{u}, \quad \left(\begin{array}{c} w \\ \tilde{v} \end{array}\right) = \partial_x K_+(x, z)|_{z=x}.\] (23)

The commutativity \(\partial/\partial t_2 - B_2, \partial/\partial t_3 - B_3 = 0\) is equivalent to the Zakharov-Shabat (ZS) equation,
\[\frac{\partial B_3}{\partial t_2} - \frac{\partial B_2}{\partial t_3} = [B_2, B_3].\] (24)
Collecting the coefficients of \(\partial_x\) in (24), we have
\[u_{tt} = w_{xx} - w_{xx}, \quad u_x = 2u\tilde{u} + w_x + \tilde{w}_x.\] (25)
The cKP equations (1) are obtained from the constant term of (24). In addition, one can show by a direct calculation that the relation \(B_3^{*} = JB_3J\) is consistent with (25).
To construct special solutions for the cKP hierarchy, we first consider solutions for the 2-component KP hierarchy. We assume that the \((i, j)\)-element of the matrix kernel \(F(x, z)\) is of the form,
\[\left[F(x, z)\right]_{ij} = F_{ij}(x, z) = \sum_{1 \leq k, l \leq 2N} f_{ij}^{(k)}(x)C^{(j)}_{kl}g_j^{(l)}(z),\]
where \(C^{(j)} = \left[C_{kl}^{(j)}\right]_{k,l=1,...,2N} (j = 1, 2)\) are invertible matrices and the functions \(f_{ij}^{(k)}(x), g_j^{(k)}(z)\) obey the following equations:
\[
\frac{\partial}{\partial t^{(a)}_n} f_{ij}^{(k)}(x, t^{(1)}, t^{(2)}) = \delta_{ai}\delta_{t^{(2)}_n} f_{ij}^{(k)}(x, t^{(1)}, t^{(2)}),
\]
\[
\frac{\partial}{\partial t^{(a)}_n} g_i^{(k)}(x, t^{(1)}, t^{(2)}) = (-1)^{n+1}\delta_{ai}\delta_{t^{(1)}_n} g_i^{(k)}(x, t^{(1)}, t^{(2)}).
\]
Under these assumptions, the kernel \(F(x, z)\) satisfies the linear differential equation (11).
If we impose the following conditions,
\[
f_k(x) = f_{11}^{(k)}(x, t, \tilde{t} = 0) = f_{12}^{(k)}(x, t, \tilde{t} = 0) = g_2^{(k)}(x, t, \tilde{t} = 0),
\]
\[-g_k(x) = f_{22}^{(k)}(x, t, \tilde{t} = 0) = f_{21}^{(k)}(x, t, \tilde{t} = 0) = g_1^{(k)}(x, t, \tilde{t} = 0),
\]
\[C^{(1)} = C^{-1}, \quad C^{(2)} = -C^{-1}, \quad tC = -C;\]
the functions \(f_k(x)\) and \(g_k(x)\) satisfy the dispersion relations (3). In this case, the kernel \(F(x, z)\) can be written as
\[F(x, z) = -J t\Xi(x)C^{-1}\Xi(z),\] (26)
with
\[t\Xi(x) = \left(\begin{array}{cccc} g_1(x) & g_2(x) & \cdots & g_{2N}(x) \\ f_1(x) & f_2(x) & \cdots & f_{2N}(x) \end{array}\right),\]
and satisfies the condition (16).
Next we will construct \(K_+(x, z)\) explicitly by the use of the GLM equation. If we set
\[K_+(x, z) = -J tK(x)\Xi(z), \quad tK(x) = \left(\begin{array}{cccc} K_1(x) & K_2(x) & \cdots & K_{2N}(x) \\ \bar{K}_1(x) & \bar{K}_2(x) & \cdots & \bar{K}_{2N}(x) \end{array}\right),\]
then the GLM equation is reduced to

\[ A(x)K(x) = \Xi(x), \tag{27} \]

where the anti-symmetric matrix \( A(x) \) is defined as

\[ A(x) = C + \int_x^\infty \Xi(y)J^t\Xi(y)dy = \left[ C_{ij} - \int_x^\infty (f_ig_j - f_jg_i)dy \right]_{i,j=1,\ldots,2N}. \tag{28} \]

Solving eq. (27), we get

\[ K_+(x,z) = J^t\Xi(x)A^{-1}\Xi(z). \tag{29} \]

This relation, together with (23), reproduces the Gram-type solution of Hirota and Ohta, given by eqs. (2), (4), (5). A proof can be found in the appendix.

4 Further reduction to the coupled KdV equations

As is mentioned in [3], the cKP equations (1) are reduced to the coupled Korteweg-de Vries (cKdV) equations [10, 11, 12, 13, 14]

\[ 4u_t - uu_x - 12u_{xxx} + 24vv_x = 0, \]
\[ 2v_t + 6uv_x + v_{xxx} = 0, \tag{30} \]

by neglecting the \( y \)-dependency and putting \( v = \tilde{v} \). However, Hirota and Ohta have not discussed under what condition one can neglect the \( y \)-dependency.

In terms of the kernel \( F(x,z) \), the condition becomes quite simple; If \( F(x,z) \) is independent of \( y = t_2 \), the integral operator \( \hat{F} \) commutes with \((E_1 - E_2)\partial_z\), i.e.,

\[ (E_1 - E_2)\partial_z^2F(x,z) = \partial_z^2F(x,z)(E_1 - E_2). \tag{31} \]

In this case, the ZS equation (24) are reduced to

\[ \frac{\partial B_2}{\partial t_3} = [B_3, B_2]. \tag{32} \]

Furthermore, if \( F(x,z) \) has the symmetry

\[ F(x,z) = PF(x,z)P, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{33} \]

the corresponding \( K_+(x,z) \) also satisfy the same equation since \( K_+(x,z) \) can be expressed as [15], and hence \( v = \tilde{v} \). Then eq. (32) gives the desired cKdV equation. We note that the Lax-type formulation (32) coincides with that of [14], and is different from that of [11, 12, 13] where they used differential operators with scalar coefficients.

As an example of solutions, we choose

\[ C_{ij} = \begin{cases} 1 & (i = j - 1), \\ -1 & (i = j + 1), \\ 0 & (\text{otherwise}), \end{cases} \]


which satisfy eq. (6). If we assume \( b_j = a_j \) \((j = 1, \ldots, 2N)\) and set \( t_{4n} = 0 \), the kernel (26) obeys the conditions (51), (53). Then \( K_\pm(x, z) \) gives a Pfaffian expression for the \( N \)-soliton solution of the cKdV equations.

If we assume
\[
\begin{align*}
\bar{p}_j &= ip_j, & b_j &= i\bar{a}_j \quad (j = 1, \ldots, 2N), \\
t_{2k-1} &\in \mathbb{R}, & t_{2k} &\in i\mathbb{R} \quad (k = 1, \ldots, N),
\end{align*}
\]
where \( \bar{a} \) denotes the complex conjugation, the kernel (26) obeys the condition
\[
\overline{F(x, z)} = PF(x, z)P.
\]

In this case, \( u \) is real-valued and the relation \( \bar{v} = \bar{\bar{v}} \) holds. Then the cKP equations (1) are reduced to a complex cKdV equations
\[
\begin{align*}
4u_t - uu_x - 12u_{xxx} + 12(|v|^2)_x &= 0, \\
2v_t + 6uv_x + v_{xxx} &= 0,
\end{align*}
\]
which have been discussed by Wu et al. [14].

5 Concluding remarks

We have applied the dressing method to the cKP hierarchy and obtained the Zakharov-Shabat representations of the equations. The hierarchy is a reduced case of the 2-component KP hierarchy. We also show how the cKP equation (1) is reduced to the cKdV equations (30) and the complex cKdV equation (34).

Recently, Adler et al. discussed the relationship between matrix integrals and the “Pfaff lattice” [15, 16, 17]. Their Pfaff lattice seems to be related to the coupled KP hierarchy. It is expected that the Pfaff lattice could be derived as a reduction of the 2-component Toda lattice hierarchy along the lines of this paper. We hope to report them in near future.

Appendix

In this appendix, we prove that the formula (29) is equivalent to the Gram-type Pfaffian solution given by eqs. (2), (4), (5).

First we prepare several formulas for Pfaffians [3, 4]:
\[
(1, 2, \ldots, 2N) = \sum_{j=2}^{2N} (-1)^j(1, j)(1, \ldots, \hat{j}, \ldots, 2N), \quad (35)
\]
\[
(a, b, 1, 2, \ldots, 2N) = (a, b)(1, 2, \ldots, 2N) + \sum_{1 \leq i < j \leq 2N} \{(a, i)(b, j) - (a, j)(b, i)\} \times (-1)^{i+j}(1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, 2N), \quad (36)
\]
where  \( \hat{i} \) means the elimination of \( i \). The identity (35) implies that the inverse of an antisymmetric matrix \( A = [(i, j)]_{i,j=1,...,2N} \) \( ((i, j) = -(j, i)) \) is written by using Pfaffians [4], i.e.,

\[
[A^{-1}]_{ij} = (-1)^{i+j} \frac{(1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, 2N)}{(1, 2, \ldots, 2N)} \quad (1 \leq i < j \leq 2N).
\]  

(37)

Using eqs. (36), (37), we can rewrite the matrix kernel (29) as

\[
\begin{align*}
K_+^{(11)}(x, z) &= -\sum_{i,j=1}^{2N} f_i(x) [A(x)^{-1}]_{ij} g_j(z) = -\frac{(d_0, \tilde{c}_0, 1, \ldots, 2N)}{(1, \ldots, 2N)}, \\
k_+^{(12)}(x, z) &= -\sum_{i,j=1}^{2N} f_i(x) [A(x)^{-1}]_{ij} f_j(z) = -\frac{(d_0, \tilde{d}_0, 1, \ldots, 2N)}{(1, \ldots, 2N)}, \\
k_+^{(21)}(x, z) &= \sum_{i,j=1}^{2N} g_i(x) [A(x)^{-1}]_{ij} g_j(z) = \frac{(c_0, \tilde{c}_0, 1, \ldots, 2N)}{(1, \ldots, 2N)}, \\
k_+^{(22)}(x, z) &= \sum_{i,j=1}^{2N} g_i(x) [A(x)^{-1}]_{ij} f_j(z) = \frac{(c_0, \tilde{d}_0, 1, \ldots, 2N)}{(1, \ldots, 2N)},
\end{align*}
\]  

(38)

where we have introduced new type of Pfaffians:

\[
(d_m, \tilde{d}_n) = (d_m, \tilde{c}_n) = (c_m, \tilde{c}_n) = (\tilde{c}_m, d_n) = 0,
\]

\[
(\tilde{d}_0, j) = \partial^j g_j(z), \quad (\tilde{c}_0, j) = \partial^j f_j(z).
\]

Next we consider differential rules of the Pfaffians. The \( (i, j) \)-element of the matrix (38) satisfies the differential equation

\[
\partial_x(i, j) = (c_0, d_0, i, j).
\]

Using the identity (35), we can prove the following formulas by induction:

\[
\begin{align*}
\partial_x(1, 2, \ldots, 2N) &= (c_0, d_0, 1, 2, \ldots, 2N) \\
&= -(d_0, \tilde{c}_0, 1, 2, \ldots, 2N)|_{z=x}, \\
\partial_x(c_0, \tilde{c}_0, 1, 2, \ldots, 2N) &= (c_1, \tilde{c}_0, 1, 2, \ldots, 2N), \\
\partial_x(d_0, \tilde{d}_0, 1, 2, \ldots, 2N) &= (d_1, \tilde{d}_0, 1, 2, \ldots, 2N).
\end{align*}
\]

Applying these formulas to (38), we find that

\[
\begin{align*}
K_+^{(11)}(x, x) &= \frac{\partial}{\partial x} \log(1, 2, \ldots, 2N), \\
\partial_x K_+^{(12)}(x, z) |_{z=x} &= \frac{(d_0, d_1, 1, 2, \ldots, 2N)}{(1, 2, \ldots, 2N)}, \\
\partial_x K_+^{(21)}(x, z) |_{z=x} &= \frac{(c_1, \tilde{c}_0, 1, 2, \ldots, 2N)}{(1, 2, \ldots, 2N)},
\end{align*}
\]

which are the desired results.
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