Rethinking the Mathematical Framework and Optimality of Set-Membership Filtering

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Abstract—Set-Membership Filter (SMF) has been extensively studied for state estimation in the presence of bounded noises with unknown statistics. Since it was first introduced in the late 1960s, the studies on SMF have used the set-based description as its mathematical framework. One important issue that has been overlooked is the optimality of SMF. In fact, the optimality has never been rigorously established. In this work, we put forward a new mathematical framework for SMF using concepts of uncertain variables. We first establish two basic properties of uncertain variables, namely, the law of total range (a non-stochastic version of the law of total probability) and the equivalent Bayes’ rule. This enables us to put forward, for the first time, an optimal SMFing framework. Furthermore, we obtain the optimal SMF under a non-stochastic Markovness condition, which is shown to be fundamentally equivalent to the Bayes filter. Note that the classical SMF in the literature is only equivalent to the optimal SMF we obtained under the non-stochastic Markovness condition. When this condition is violated, we show that the classical SMF is not optimal and it only gives an outer bound on the optimal estimation.

Index Terms—Set-membership filtering, optimality, uncertain variables, nonlinear systems, law of total range, Bayes’ rule for uncertain variables.

I. INTRODUCTION

A. Motivation and Related Work

The filtering problems in the state-space description are concerned with estimating the state information in the presence of noises, and thus are widely considered in control systems, telecommunications, navigation, and many other important fields [1], [2]. When the statistics of the noises are known, the corresponding solution method is called stochastic filter.

A famous optimal filtering framework for Hidden Markov Models (HMMs) is the Bayes filter [1]–[3] which provides the complete solution (i.e., the posterior distribution) to the filtering problem. As a special case, if the noises are white Gaussian in linear systems, the corresponding Bayes filter is known as the Kalman filter [4]. Note that in the Bayes filter, the white noise assumption plays an important role in supporting the optimality, since otherwise the HMM condition can hardly be guaranteed.

When the noises have unknown statistics but known ranges, the corresponding solution method is called non-stochastic filter. In the late 1960s, Witsenhausen proposed a famous filtering framework for linear systems [5], which is also suitable for nonlinear systems, known as the Set-Membership Filter (SMF). Similarly to the Bayes filter, the SMF also has the prediction step (using the set image under system function, which becomes the Minkowski sum for linear systems) and the update step (using the set intersection). Under this filtering framework, the followed-up studies focused on how to derive the exact or approximate the solution for different scenarios. More specifically, there are mainly two types of SMFs in the literature:

• **Ellipsoidal SMF.** This type of SMFs approximates the Minkowski sum and set intersection using ellipsoidal outer bounds. In [9], a continuous-discrete ellipsoidal SMF was proposed to deal with linear systems with ellipsoidal noises, where the SMF has a similar structure to the Kalman filter. With a similar system setting, both SMFing and smoothing problems were investigated in [10] by solving corresponding Riccati equations. In [11] and [12], algorithms were provided for minimizing the volume of the outer bounds on the Minkowski sum and intersection of ellipsoids. Nevertheless, minimizing the volume can result in a very narrow ellipsoid with an unacceptably large diameter. Thus, the semi-axes of ellipsoids were constrained, e.g., via the trace of the matrix in the quadratic form. In [13], a volume-minimizing and a trace-minimizing ellipsoidal outer bounds (each described by two ellipsoids) were derived for the linear discrete-time SMF, and the description of outer bounds were generalized to multiple ellipsoids in [14]. Note that the ellipsoidal SMF is computationally cheaper but usually less accurate than the polytopic SMF discussed below.

• **Polytopic SMF.** This type of SMFs describes or outer bounds the prediction and the update using convex polytopes. Different from the ellipsoidal SMF, the polytopic SMF can derive the exact solution for linear filtering problems. This is because polytopes are closed under Minkowski sum and set intersection. Nevertheless, the complexity is unacceptable for deriving exact solutions. Noticing this fact, researchers used different subclasses of convex polytopes to give the outer bounds. In [15], the recursive optimal bounding parallelopotope algorithm was proposed. In [16], a zonotopic SMF was designed for linear discrete-time systems by using singular-value-decomposition-based approximation. In [17], a zonotopic SMF was proposed for nonlinear discrete-time systems, where the volume of the zonotopic outer bound is mini-

1Interval observers [6]–[8] are not included in the SMF, since the basic idea in the update step is based on designing an observer, which is different from that of the SMFing framework discussed in this paper.
mized by using convex optimization, which was improved in [18] by using the DC (Difference of Two Convex functions) programming. In [19] and [20], the zonotopic SMFs were given for linear systems under P-radius-based and weighted-Frobenius-norm criteria, respectively, which efficiently balanced the complexity and the accuracy of the zonotopic outer bounds. In [21], the constrained zonotope was proposed and applied to the linear polytope-SMF; the proposed method can balance the complexity and the accuracy, and is closed under linear transformations, Minkowski sums, and set intersections. In [22], an SMF was proposed for nonlinear systems by combining the interval arithmetic and constrained zonotopes. In [23], a zonotopic SMF was studied for nonlinear systems which has advantages in handling high dimensionality.

All the above-mentioned studies on SMFs used the set-based description as its mathematical framework. We argue that this framework has the optimality issue: for stochastic filters, we know that even with the same marginal distributions, the white noises and correlated noises in linear systems lead to different optimal estimations (which should result in different optimal estimations) were not distinguished; thus, the prior studies overlooked the condition (as shown later in Section II-A) under which the SMFs are optimal, and the optimal SMFing framework has not been rigorously established.

Departing from the conventional/suboptimal set-based SMFing description, in this article, we aim to establish the optimal SMFing framework in a completely different way.

B. Our Contributions

In this work, we put forward a new mathematical framework for SMFing for nonlinear systems, based on the concepts of uncertain variables proposed by Nair in the early 2010s [24]. Similarly to the Bayesian filtering, our filtering framework recursively derives the non-stochastic prior and posterior. The prior, posterior and noises are modeled as uncertain variables. These properties enable us to establish two new and fundamental properties of uncertain variables, the first one is called the law of total prior, posterior and noises are modeled as uncertain variables. The conditional range of uncertain variables allows us to establish the optimal SMFing framework in the literature and the optimal SMF obtained based on the new framework.

C. Paper Organization and Notation

In Section II, we provide a self-contained introduction to the uncertain variables (see Section II-A), and derive the law of total range and Bayes’ rule for uncertain variables (see Section II-B). By describing the uncertainties as uncertain variables, an optimal SMFing framework was established for nonlinear systems in Section III. In Section IV, numerical examples are provided to corroborate our theoretical results. Finally, the concluding remarks are given in Section V.

Throughout this paper, \( \mathbb{R} \), \( \mathbb{N}_0 \), and \( \mathbb{Z}_+ \) denote the sets of real numbers, non-negative integers, and positive integers, respectively. \( \mathbb{R}^n \) stands for the \( n \)-dimensional Euclidean space.

II. UNCERTAIN VARIABLES: PRELIMINARIES AND NEW RESULTS

In this work, the uncertainties are with known ranges but unknown probability distributions. To model the uncertainties rigorously, we introduce the “uncertain variable” framework established in [24] and derive two important properties which will constitute the foundation of the optimal SMFing framework.

A. Preliminaries of Uncertain Variables

Consider a sample space \( \Omega \). A measurable function \( x: \Omega \to \mathcal{X} \) from the sample space \( \Omega \) to a measurable set \( \mathcal{X} \) is called an uncertain variable [24]. We define a realization of \( x \) as \( x(\omega) =: x \), and sometimes we write it as \( x = x(\omega) \) for conciseness.

Different from random variables which can be described by probability distributions, an uncertain variable (say \( x \)) does not have any information on the probability, but it can be described by its range \( [x] \):

\[
[x] := \{x(\omega) : \omega \in \Omega\}. \tag{1}
\]

Similar to the probability distribution for multiple random variables, the range can also be defined w.r.t. multiple uncertain variables.

**Definition 1** (Joint Range, Conditional Range, Marginal Range [24]). Let \( x \) and \( y \) be two uncertain variables. The joint range of \( x \) and \( y \) is

\[
[x, y] := \{(x(\omega), y(\omega)) : \omega \in \Omega\}. \tag{2}
\]

The conditional range of \( x \) given \( y = y \) is

\[
[x|y] := \{x(\omega) : y(\omega) = y, \omega \in \Omega\} = \{x(\omega) : \omega \in \Omega_{y=y}\}, \tag{3}
\]

where \( \Omega_{y=y} := y^{-1}(\{y\}) = \{\omega : y(\omega) = y, \omega \in \Omega\} \) is the preimage of \( y(\omega) = y : \omega \in \Omega \), and \( [y|x] \) is defined in a similar way. The marginal range of \( x \) is \([x]\) expressed by (1).

In analogy with the joint probability distribution, the joint range can be fully determined by the conditional and marginal ranges [24], i.e.,

\[
[x, y] = \bigcup_{y \in [y]} ([x]|y) \times \{y\} = \bigcup_{x \in [x]} \{x\} \times [y|x], \tag{4}
\]

2Unfortunately, such non-stochastic correlations can be neither captured by the set-based description nor characterized by the statistical dependence.
where $\times$ is the Cartesian product.

Next, we introduce the definition of unrelatedness \cite{24}, which is a non-stochastic analogue of statistical independence.

**Definition 2** (Unrelatedness and Conditional Unrelatedness \cite{24}). **Uncertain variables** $u_1, \ldots, u_r$ are unrelated if

$$[u_1, \ldots, u_r] = [u_1] \times \cdots \times [u_r].$$

They are conditionally unrelated given $v$ if

$$[u_1, \ldots, u_r|v] = [u_1|v] \times \cdots \times [u_r|v], \quad \forall v \in [v].$$

If the uncertain variables are not unrelated, we say they are related.

Based Definition 2, we have the following properties for unrelatedness and conditional unrelatedness \cite{24}:

i) $u_1$ and $u_2$ are unrelated if and only if (iff)

$$[u_1|u_2] = [u_1], \quad \forall u_2 \in [u_2].$$

ii) $u_1$ and $u_2$ are conditionally unrelated given $v$ iff

$$[u_1|u_2, v] = [u_1|v], \quad \forall (u_2, v) \in [u_2, v].$$

**B. Law of Total Range and Bayes’ Rule for Uncertain Variables**

In this subsection, we establish two properties, namely, the law of total range and Bayes’ rule for uncertain variables, as the non-stochastic counterparts of the law of total probability and Bayes’ rule. They establish a mathematical foundation of the optimal SMF which will be introduced in Section III.

**Lemma 1** (Law of Total Range).

$$[x] = \bigcup_{y \in [y]} [x|y], \quad [y] = \bigcup_{x \in [x]} [y|x].$$

**Proof:** See Appendix A.

The law of total range links the marginal range and the conditional range. An illustrative example is given in Fig. 1.

With (9), we know that $[x|y] \subseteq [x]$ which implies observations can reduce uncertainty.

**Lemma 2** (Bayes’ Rule for Uncertain Variables).

$$[x|y] = \left\{x : [y|x] \cap [y] \neq \emptyset, x \in [x]\right\}.$$

**Proof:** See Appendix B.

Bayes’ rule for uncertain variables reflects the fundamental relationship among the prior range $[x]$, the likelihood range $[y|x]$, and the posterior range $[x|y]$. An illustrative example is given in Fig. 1.

**III. THE OPTIMAL FILTERING FRAMEWORK**

Now, we model the SMFing problem in the framework of uncertain variables. Consider the following nonlinear system:

$$x_{k+1} = f_k(x_k, w_k),$$

$$y_k = g_k(x_k, v_k),$$

for time $k \in \mathbb{N}_0$, where (11) and (12) are called the state equation and the measurement equation, respectively. The state equation describes how the system state $x_k$ (with its realization $x_k \in [x_k] \subseteq \mathbb{R}^n$) changes over time, where $w_k$ is the process/dynamical noise (with its realization $w_k \in [w_k] \subseteq \mathbb{R}^p$), and $f_k : [x_k] \times [w_k] \rightarrow [x_{k+1}]$ stands for the system transition function. The measurement equation gives how the system state is measured, where $v_k$ represents the measurement (with its realization, called observed measurement, $y_k \in [y_k] \subseteq \mathbb{R}^m$) and $v_k$ (with its realization $v_k \in [v_k] \subseteq \mathbb{R}^q$) stands for the measurement noise, and $g_k : [x_k] \times [v_k] \rightarrow [y_k]$ is the measurement function.

By (10), the posterior range $[x_k|y_0:k] := [x_k|y_0, \ldots, y_k]$ is the set of all possible $x_k$ given the measurements up to $k$, i.e., $y_0:k$. Thus, it gives the complete solution of the filtering problem. Theorem 1 provides the optimal SMFing framework for deriving $[x_k|y_0:k]$.

**Theorem 1** (Optimal Set-Membership Filter). For the system described by (11) and (12), the optimal SMF is obtained by the following steps:

- **Initialization.** Set the initial prior range $[x_0]$.
- **Prediction.** For $k \in \mathbb{Z}_+$, given the posterior range $[x_k|y_0:k-1]$ in the previous time step, the prior range $[x_k|y_0:k-1]$ is predicted by the law of total range that

$$[x_{k+1}|y_0:k-1] = \bigcup_{x_k \in [x_k|y_0:k-1]} f_k(x_k, w_k).$$

- **Update.** For $k \in \mathbb{N}_0$, given the observed measurement $y_k$ and the prior range $[x_k|y_0:k-1]$, the posterior range $[x_k|y_0:k]$ is updated by Bayes’ rule for uncertain vari-
ables that
\[ \{ x_{k} \in [x_{k}|y_{0:k-1}] : g_{k}(x_{k}, [v_{k}|x_{k}, y_{0:k-1}]) \cap \{ y_{k} \} \neq \emptyset \} \]  
where we define \([x_{0}] := [x_{0}|y_{0}|0] \) and \([v_{0}|x_{0}] = [v_{0}|x_{0}, y_{0} - 1] \) for consistency.

Proof: See Appendix [2].

In general, it is not easy to obtain \([w_{k-1}|x_{k-1}, y_{0:k-1}] \) and \([v_{k}|x_{k}, y_{0:k-1}] \) in Theorem [1]. They depend on how the process noises, the measurement noises, and the initial prior \(w_{0:k}, v_{0:k}, x_{0} \) are related. However, if the noises and the initial state are unrelated (see Assumption 1), the optimal filter is easy to derive (see Theorem 2).

**Assumption 1 (Unrelated Noises and Initial State).** \( \forall k \in \mathbb{N}_{0}, w_{0:k}, v_{0:k} \) are unrelated.

**Theorem 2 (Optimal SMF Under Assumption 1).** For the system described by (11) and (12), the optimal SMF under Assumption 1 is given by the following steps:

- **Initialization.** Set the initial prior range \([x_{0}]\).
- **Prediction.** For \( k \in \mathbb{Z}_{+} \), given \([x_{k-1}|y_{0:k-1}] \) derived in the previous time step \( k-1 \), the prior range is
  \[ [x_{k}|y_{0:k-1}] = f_{k-1}([x_{k-1}|y_{0:k-1}], [w_{k-1}]). \]
- **Update.** For \( k \in \mathbb{N}_{0} \), given the observed measurement \( y_{k} \) and the prior range \([x_{k}|y_{0:k-1}] \), the posterior range is
  \[ [x_{k}|y_{0:k}] = \bigcup_{y_{k} \in [y_{k}]} g_{k}^{-1}(\{ y_{k} \}) \cap [x_{k}|y_{0:k-1}], \]
  where \( g_{k}^{-1}(\cdot) \) is the inverse map of \( g_{k}(\cdot, v_{k}) \).

Proof: See Appendix 1.

**Remark 1 (Fundamental Equivalence Between SMF Under Assumption 1 and Bayes Filter).** The Bayes filter [1] is based on the stochastic Hidden Markov Model (HMM) with
\[ p(x_{k}|x_{0:k-1}, y_{0:k-1}) = p(x_{k}|x_{0:k-1}), \]
\[ p(y_{k}|x_{0:k}, y_{0:k-1}) = p(y_{k}|x_{k}), \]
where \( p(\cdot|\cdot) \) is the conditional distribution of the random variable \( a \) given the realization \( b = b(\omega) \). For the optimal SMF, the system described by (11) and (12) under Assumption 1 satisfies the following non-stochastic HMM [1]:
\[ [x_{k}|x_{0:k-1}, y_{0:k-1}] = [x_{k}|x_{0:k-1}], \]
\[ [y_{k}|x_{0:k}, y_{0:k-1}] = [y_{k}|x_{k}]. \]
These two HMMs are equivalent, since \( p(\cdot) \) and \([\cdot] \) describe the uncertainties for random variables and uncertain variables, respectively. Furthermore, (11) reflects the conditional independence between \( x_{k} \) and \( x_{0:k-2}, y_{0:k-1} \) given \( x_{k-1} \); while (19) indicates the conditional unrelatedness [3] between them. Similar observation can be obtained between (18) and (20).

3Equations (19) and (20) can be proved by using Lemma 4 and the same technique in (24) of Appendix B.

In the Bayes filter, the prediction step is based on the Chapman-Kolmogorov equation, i.e., the law of total probability combined with the Markov property (17) that
\[ p(x_{k}|y_{0:k-1}) = \int p(x_{k}|x_{k-1})p(x_{k-1}|y_{0:k-1})dx_{k-1}. \]

In the optimal SMF under Assumption 1, the prediction step is given by the law of total range [9] and the non-stochastic Markov property (19) that
\[ [x_{k}|y_{0:k-1}] = \bigcup_{x_{k-1} \in [x_{k-1}|y_{0:k-1}]} [x_{k}|x_{k-1}]. \]

For the update steps, the Bayes filter derives the posterior distribution \( p(x_{k}|y_{0:k}) \) by Bayes’ rule, while the optimal SMF gets the posterior range \([x_{k}|y_{0:k}]\) by Bayes’ rule for uncertain variables (10).

Further, if the system is linear, the optimal SMF under Assumption 1 is obtained in Corollary 1.

**Corollary 1.** For the linear system described by
\[ x_{k+1} = Ax_{k} + Bw_{k}, \]
\[ y_{k} = Cy_{k} + Dw_{k}, \]
where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{m \times n}, \) and \( D \in \mathbb{R}^{m \times q}, \) the optimal SMF under Assumption 1 has the following steps:
- **Initialization.** Set the initial prior range \([x_{0}]\).
- **Prediction.** For \( k \in \mathbb{Z}_{+} \), the prior range is
  \[ [x_{k}|y_{0:k-1}] = A[x_{k-1}|y_{0:k-1}] \oplus D[w_{k-1}], \]
where \( \oplus \) stands for the Minkowski sum [8].
- **Update.** For \( k \in \mathbb{N}_{0} \), given \( y_{k} \), the posterior range is
  \[ [x_{k}|y_{0:k}] = X_{k}(C, y_{k}, D[v_{k}]) \bigcup [x_{k}|y_{0:k-1}], \]
where we define \([x_{0}] := [x_{0}|y_{0}] \) for consistency, and \( X_{k}(C, y_{k}, D(v_{k})) = \{ x_{k}: y_{k} = Cx_{k} + Dw_{k}, v_{k} \in [v_{k}] \} \).

**Remark 2 (The Existing SMFing Framework).** The classical SMF in the literature is under the set-based description: e.g., [23], [25] for nonlinear filters and [3], [9] for linear filters, which are equivalent to the filters obtained in Theorem 2 and Corollary 1 respectively. However, when Assumption 1 is violated, the property of non-stochastic HMM described by (19) and (20) can hardly be guaranteed. Without this property, the classical SMF is not optimal any more, i.e., it cannot give the exact set of all possible states determined by the optimal SMF in Theorem 1.

Although the classical SMF does not give the optimal solution for state estimation when Assumption 1 is violated, the following theorem tells that it is still useful in giving a more conservative estimate.

**Theorem 3 (Outer Bound).** Let \([x_{0}^{*}|y_{0:k}]\) and \([x_{k}|y_{0:k}]\) be the posterior ranges derived by Theorem 1 and Theorem 2 respectively. Then, \([x_{0}^{*}|y_{0:k}] \subseteq [x_{0}|y_{0:k}] \) holds.

4The RHS of (22) is \( f_{k-1}(\{ x_{k-1} \mid y_{0:k-1}, \} , [w_{k-1}]) \) as stated in Theorem 2. But the Bayes filter does not have such an elegant expression for general nonlinear systems.

5Given two sets \( S_{1} \) and \( S_{2} \) in Euclidean space, the Minkowski sum of \( S_{1} \) and \( S_{2} \) is \( S_{1} \oplus S_{2} = \{ s_{1} + s_{2} : s_{1} \in S_{1}, s_{2} \in S_{2} \} \).
we can derive the posterior range by the following steps:

\[
\text{Algorithm 1 Classical SMF}
\]

1: Initialization: \( [x_0] = [a_0^*, b_0^*] = [0, 1] \); \{Comments: \([x_k][y_0,k] = [a_k, b_k] (k \in \mathbb{Z}_+).\}
2: loop
3: if \( k > 0 \) then
4: \( [a_k, b_k] = \sin([a_{k-1}, b_{k-1}] + [w_{k-1}, w_{k-1}] + [w_{k-1}, w_{k-1}]) \)
5: end if
6: if \( y_k \neq 0 \) then
7: \( [a_k, b_k] = \min\{0.5y_k, y_k\}, \max\{0.5y_k, y_k\} \}
8: else if \( y_k = 0 \) then
9: \( [a_k, b_k] = [a_k, b_k] \)
10: end if
11: \( k = k + 1 \);
12: end loop

IV. NUMERICAL EXAMPLES

In this section, we illustrate the performance gap between the optimal SMF (in Theorem 1) and the classical SMF (equivalent to the filter design in Theorem 2) through two numerical examples, where the gap is acceptable for the first example but is unacceptable for the second example.

A. Nonlinear System with Related Process and Measurement Noises

Consider the nonlinear system described by

\[
x_{k+1} = \sin(x_k) + x_k + w_k,
\]

\[
y_k = v_k x_k,
\]

where \([x_0] = [0, 1], [w_k] = [0, 1], \) and \([v_k] = [1, 2], \forall k \in \mathbb{N}_0, w_{0,k}, v_{0,k}, x_0\) are unrelated, except that the process noise \(w_{k-1}\) and the multiplicative measurement noise \(v_k\) have the relationship \([v_k][w_{k-1}] = [0, 2 - w_{k-1}] (k \in \mathbb{Z}_+).\)

If we ignore this relatedness, Theorem 2 will give the classical SMF in Algorithm 1, where Line 4 gives the prediction relationship \(J_k\) of the update step. However, we would not use (13) directly, because in the update step, \(|\cdot\|\) = \([0, 1] \) \(U(0, 1)\) \{Comments: \(x \sim U(a, b)\) means \(x\) is a realization of a random variable uniformly distributed in \([a, b]\).\}

Now, design the optimal SMF from Theorem 1 and its state is denoted by \([x^*_k]\) to be distinguished from the ranges in Algorithm 1. For \( k = 0 \), the posterior range \([x^*_0][y_0,k]\) is identical to that derived in Algorithm 1 since \([x^*_0] = [x_0]\) in the initialization step and their update steps are the same. For \( k > 1\), assume the posterior range \([x^*_{k-1}][y_0,k-1] := [a^*_{k-1}, b^*_{k-1}] \) has already been derived at \( k-1\). Since \( w_{k-1} \) is only related to \( v_k \), we have \([w_{k-1}][x_{k-1}, y_0,k] = [w_{k-1}]\) in (12) of the prediction step. Similarly, we have \([v_k][x_k, y_0,k-1] \in [v_k][x_k]\) in (14) of the update step. However, we would not use (14) to obtain \([x^*_k][y_0,k-1]\) directly, because in the update step, \([v_k][x_k]\) is not explicit which cannot help to derive \([x^*_k][y_0,k]\). Instead, we can rewrite (14) as:

\[
\begin{align*}
x_k &= f_{k-1}(x_{k-1}, w_{k-1}) \in [v_k][w_{k-1}], \\
&= g^{-1}_{k-1,x_k}(\{y_k\}) \in [v_k][w_{k-1}], \\
&= g^{-1}_{k-1,x_k}(\{y_k\}) \in [v_k][w_{k-1}], \\
&= g^{-1}_{k-1,x_k}(\{y_k\}) \in [v_k][w_{k-1}], \\
&= g^{-1}_{k-1,x_k}(\{y_k\}) \in [v_k][w_{k-1}],
\end{align*}
\]

where \( g^{-1}_{k-1,x_k}(\cdot) \) is the inverse map of \( g_k(x_k, \cdot) \). From (29), we can derive the posterior range by the following steps:

\[\text{Algorithm 2 Approximation of the Optimal SMF}\]

1: Initialization: \([x_0]\) = \([0, 1]\), \(N = 10000\); \{Comments: \(N\) is the number of random samples for \([x^*_k][y_0,k] = [a^*_k, b^*_k] (k \in \mathbb{Z}_+).\}
2: loop
3: if \( k = 0 \) then
4: \( [a^*_0, b^*_0] = [y_0] \sim \mathcal{U}(0, 1) \)
5: else if \( k > 0 \) then
6: \( i = 1; \)
7: \( \text{while } i \leq N \text{ do} \)
8: \( x_k \sim \mathcal{U}(0, 1); \text{\{Comments: } x \sim \mathcal{U}(a, b) \text{ means } x \text{ is a realization of a random variable uniformly distributed in } [a, b].\}
9: \( x_k \sim g_{k-1,x_k}(\{y_k\}) \in [v_k][w_{k-1}]; \text{\{Comments: } g_k(x_k, \cdot) \text{ is the inverse map of } g_k(x_k, \cdot) \text{ in (14).\}
10: \( i = i + 1; \)
11: \( \text{end while} \)
12: \( a^*_k = \min\{x^*_k\}, b^*_k = \max\{x^*_k\}; \)
13: \( \text{end if} \)
14: \( k = k + 1; \)
15: \( \text{end loop} \)

Fig. 2 shows the averaged interval-length ratio defined in the caption. This ratio reflects the conservativeness of the classical SMF in estimating the posterior range that: a small ratio means the posterior range of the classical SMF is much larger than that of the optimal SMF, and thus the classical SMF is conservative as it includes unnecessary states. We can see that the ratio is 1 at \( k = 0 \). This is because the classical
SMF in Algorithm 1 is optimal at \( k = 0 \). Nevertheless, the ratio decreases with \( k \) and stays around 84\% for \( k \geq 5 \).

### B. Linear System with Identical Process Noise

Consider the linear system described by

\[
\begin{align*}
\mathbf{x}_{k+1} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \mathbf{w}_k, \\
\mathbf{y}_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_k + \mathbf{v}_k,
\end{align*}
\]

(30)

where \( \mathbb{E}[\mathbf{x}_0] = [-10, 10] \times [-10, 10] \), \( \mathbb{E}[\mathbf{w}] = [-1, 1] \), and \( \mathbb{E}[\mathbf{v}_k] = [-1, 1], \forall k \in \mathbb{N}_0 \). \( \mathbf{w}, \mathbf{v}_{0:k} \) are unrelated.

If we replace \( \mathbf{w} \) with \( \mathbf{w}_k \) and assume Assumption 1 holds, Theorem 2 will give the classical SMF in Corollary 1 with

\[
\begin{align*}
A &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \\
B &= \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \\
C &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
D &= 0.
\end{align*}
\]

(32)

Note that there are various ways to implement Corollary 1. We employ the projection-based method in [26] to give the accurate \( [\mathbf{x}_k; \mathbf{y}_0; k] \).

Now we design the optimal SMF using Theorem 1. Since \( \mathbf{v}_k \) is unrelated to \( \mathbf{x}_0, \mathbf{w}, \mathbf{v}_{0:k-1} \). Theorem 1 implies the update step in the optimal SMF is the same as that in Corollary 1. With \( \mathbb{E}[[\mathbf{w}|x_{k-1}; y_0; k-1]] \subseteq \mathbb{E}[[\mathbf{w}]] \) in (13), we know that the process noise \( \mathbf{w} \) can be gradually estimated, just as a system state. Regrading \( \mathbf{w} \) as a state, we rewrite (30) and (31) as

\[
\begin{align*}
\mathbf{x}_{k+1} &= \begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_k, \\
\mathbf{y}_k &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}_k + \mathbf{v}_k,
\end{align*}
\]

(33)

(34)

where \( \tilde{\mathbf{x}}_k = [\mathbf{x}_k^T, \mathbf{w}_k]^T = [\mathbf{x}^{(1)}_k, \mathbf{x}^{(2)}_k, \mathbf{w}_k]^T \). Thus, an optimal filter can be derived from Corollary 1 with

\[
\begin{align*}
A &= \begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \\
B &= 0, \\
C &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \\
D &= 0.
\end{align*}
\]

(35)

Similarly to Section IV-A, we denote \( [\tilde{\mathbf{x}}_k; \mathbf{y}_0; k] \) as the optimal posterior range, which can be derived by the projection of \( [\tilde{\mathbf{x}}_k; \mathbf{y}_0; k] \) to the \( x^{(1)}, x^{(2)} \)-plane.

Fig. 5 shows the differences between the posterior ranges of the optimal SMF and the classical SMF at \( k = 0, 10, 20 \), respectively. We can see that \( [\tilde{\mathbf{x}}_k; \mathbf{y}_0; k] = [\tilde{\mathbf{x}}_0; \mathbf{y}_0; 0] \) at \( k = 0 \) [see Fig. 3(a)]. \( [\tilde{\mathbf{x}}_k; \mathbf{y}_0; 10] \) is smaller than \( [\tilde{\mathbf{x}}_k; \mathbf{y}_0; 10] \) at \( k = 10 \) [see Fig. 3(b)]. The area ratio is 27.1\%; finally, \( [\tilde{\mathbf{x}}_k; \mathbf{y}_0; 20] \) becomes much smaller than \( [\tilde{\mathbf{x}}_k; \mathbf{y}_0; 20] \) at \( k = 20 \) [see Fig. 3(c)]. The area ratio is 1.04\%, which means approximately 99\% of the estimated range by the classical SMF is excluded by the optimal SMF.

### V. Conclusion

In this work, we have studied the optimal SMFing problem for nonlinear discrete-time systems. By using a new mathematical framework based on the concepts and properties of uncertain variables, we have put forward an optimal SMFing framework. Then, we have obtained the optimal SMF under the unrelativeness assumption between the noises and initial prior (to guarantee a non-stochastic HMM), and revealed the fundamental equivalence between the SMF and the Bayes filter. We have also shown that the classical SMF in the literature must rely on the non-stochastic Markovness condition to guarantee optimality. When the Markovness is violated, the SMF in the literature is not optimal and can only provide an outer bound on the optimal estimation. To corroborate our theoretical results, we have designed the optimal SMFs for two specific examples with related noises, and the (potentially very significant) performance gap between the classical SMF and the optimal SMF has been clearly illustrated.

### Appendix A

#### Proof of Lemma 1

We only prove \( \|\mathbf{x}\| = \bigcup_{y \in [\mathbf{y}]\setminus \{\mathbf{y}\}} \{y(\omega) : \omega \in \Omega_{\mathbf{y}=y}\} \) and the proof for \( \|\mathbf{y}\| = \bigcup_{x \in [\mathbf{x}]\setminus \{\mathbf{x}\}} \{y(\omega) : \omega \in \Omega_{\mathbf{y}=y}\} \) is similar. From (3), we have

\[
\bigcup_{y \in [\mathbf{y}]} \{y(\omega) : \omega \in \Omega_{\mathbf{y}=y}\} = \{y(\omega) : \omega \in \Omega\} = \mathbb{E}[[\mathbf{y}]],
\]

(36)

where \( (a) \) is from \( \mathbb{E}[[\mathbf{y}]] \mathbf{x}(\Omega_{\mathbf{y}=y}) = \mathbf{x}(\bigcup_{y \in [\mathbf{y}]} \Omega_{\mathbf{y}=y}) = \mathbf{x}(\Omega) \).

### Appendix B

#### Proof of Lemma 2

Firstly, we define \( [\mathbf{x}, \mathbf{y}] \) as

\[
[\mathbf{x}, \mathbf{y}] : = \{y(\omega), y(\omega)) : y(\omega) = y, \omega \in \Omega\} = \{y(\omega), y(\omega)) : \omega \in \Omega_{\mathbf{y}=y}\}.
\]

(37)

With (3), we have

\[
[\mathbf{x}, \mathbf{y}] = [\mathbf{x}|\mathbf{y}] \times \{y\},
\]

(38)

and conversely we have

\[
[\mathbf{x}|\mathbf{y}] = \text{Proj}_{(x, y) \rightarrow x} [\mathbf{x}, \mathbf{y}],
\]

(39)

where \( \text{Proj}_{(x, y) \rightarrow x} (\cdot) \) is a projection from the space w.r.t. \( (x, y) \) to the subspace w.r.t. \( x \) that \( \text{Proj}_{(x, y) \rightarrow x}(\mathcal{S}_x, \mathcal{S}_y) = \mathcal{S}_x \) for sets \( \mathcal{S}_x \) and \( \mathcal{S}_y \).

Secondly, we prove the following equation holds

\[
[\mathbf{x}, \mathbf{y}] = [\mathbf{x}, \mathbf{y}] \cap ([\mathbf{x}] \times \{y\}).
\]

(40)

With the RHS of the first equality in (4), the RHS of (40) can be rewritten as

\[
\begin{align*}
= & \bigcup_{y \in [\mathbf{y}]} ([\mathbf{x}|\mathbf{y}] \times \{y\}) \cap ([\mathbf{x}] \times \{y\}) \\
= & \bigcup_{y \in [\mathbf{y}]} ([\mathbf{x}|\mathbf{y}] \cap ([\mathbf{x}] \times \{y\}) \\
= & \bigcup_{y \in [\mathbf{y}]} ([\mathbf{x}|\mathbf{y}] \cap ([\mathbf{x}] \cap \{y\}) \\
= & [\mathbf{x}|\mathbf{y}] \cap \{y\}.
\end{align*}
\]

(41)

\[\text{We note that (38)}\] combining with \( [\mathbf{x}, \mathbf{y}] = \bigcup_{y \in [\mathbf{y}]} [\mathbf{x}, \mathbf{y}] \) gives the first equality in (4)
where (a) follows \((S_1 \times S_2) \cap (S_3 \times S_4) = (S_1 \cap S_3) \times (S_2 \cap S_4)\) for sets \(S_1, \ldots, S_4\). Equality (b) is established by (9) (which implies \([x,y] \subseteq [x]\)) and \(S \times \emptyset = \emptyset\) for set \(S\). Then, (c) follows from (38).

Thirdly, we prove a projection-based version of Bayes’ rule
\[
[x|y] = \text{Proj}_{(x,y) \rightarrow x} \left( \bigcup_{x \in [x]} \left( \{x\} \times ([|x|] \cap [y]) \right) \right). \tag{42}
\]
With (40) and the RHS of the second equality in (4), we get
\[
[x, y] = \bigcup_{x \in [x]} \left( \{x\} \times ([|x|] \cap [y]) \right) = \bigcup_{x \in [x]} \left( \{x\} \times ([|x|] \cap [y]) \right) = \bigcup_{x \in [x]} \left( \{x\} \times ([|x|] \cap [y]) \right). \tag{43}
\]
By (39) and (43), (42) is obtained.

Finally, we prove that (10) and (12) are equivalent. Let \(T_1\) and \(T_2\) denote the RHS of (42) and the RHS of (10), respectively. \(\forall x' \in T_1\), we have \([x'] \cap [y] \neq \emptyset\), since otherwise \([x'] \cap [y] \neq \emptyset\), which means \(x' \notin S_1\). Observing that \(x' \notin [x]\), we get \(x' \notin T_2\), and thus \(T_1 \subseteq T_2\). Conversely, \(\forall x'' \in T_2\), we have \(x'' \in [x]\) and \([|x''|] \cap [y] \neq \emptyset\). Hence, \(x'' \in T_1\) and therefore \(T_1 \subseteq T_2\). Combining it with \(T_1 \subseteq T_2\), we get \(T_1 = T_2\).

**APPENDIX C**

**PROOF OF THEOREM 1**

We divide the proof of Theorem 1 into two parts, the prediction step and the update step.

For the prediction step, the law of total range in (9) gives
\[
[x_k|y_0:k-1] = \bigcup_{x_{k-1} \in [x_{k-1}|y_0:k-1]} [x_k|x_{k-1}, y_0:k-1]. \tag{44}
\]
From (11), the following holds
\[
[x_k|x_{k-1}, y_0:k-1] = [f(x_{k-1}, w_{k-1})|x_{k-1}, y_0:k-1] = [f(x_{k-1}, w_{k-1})|x_{k-1}, y_0:k-1] \tag{45}
\]
where (a) follows from (9) that
\[
[x_k|x_{k-1}, y_0:k-1] = [f(x_{k-1}, w_{k-1})|x_{k-1}, y_0:k-1] = [f(x_{k-1}, w_{k-1})|x_{k-1}, y_0:k-1]. \tag{46}
\]
Combining (44) with (45), we get (13).

For the update step, we prove it with Bayes’ rule for uncertain variables. From (10), we have
\[
[x_k|y_0:k] = \left\{ x_k \in [x_k|y_0:k-1] : [y_k|x_k, y_0:k-1] \cap [y_k] \neq \emptyset \right\}. \tag{47}
\]
Similarly to dealing with \([x_k|x_{k-1}, y_0:k-1]\) in the prediction step [see (45)], we have
\[
[y_k|x_k, y_0:k-1] = g_k(x_k, [v_k|x_k, y_0:k-1]). \tag{48}
\]
Thus, the RHS of (47) can be rewritten as (14).

**APPENDIX D**

**PROOF OF THEOREM 2**

Before start, we need the following two lemmas.

**Lemma 3** (Function of Conditional Range). Given uncertain variables \(u_1\) and \(u_2\) and map \(h\), \([h(u_1)|u_2] = h([u_1]|u_2)\) holds.

**Proof:** \([h(u_1)|u_2] = \{h(u_1(\omega)) : \omega \in \Omega_{u_1=u_2}\} = h(\{u_1(\omega) : \omega \in \Omega_{u_2=u_2}\}) = h([u_1]|u_2)\).

**Lemma 4** (Invariance of Unrelatedness under Maps). If \(u_1\) and \(u_2\) are unrelated, then \(h_1(u_1)\) and \(h_2(u_2)\) are also unrelated, i.e.,
\[
[h_1(u_1)|h_2(u_2)] = [h_1(u_1)], \quad \forall u_2 \in [u_2]. \tag{49}
\]
Proof: By Lemma 3, the LHS and RHS of equation (49) can be written as $h_1([\mathbf{u}_1][h_2(\mathbf{u}_2)])$ and $h_1([\mathbf{u}_1])$, respectively. Since a sufficient condition to $h_1([\mathbf{u}_1][h_2(\mathbf{u}_2)]) = h_1([\mathbf{u}_1])$ is $\{\mathbf{u}_1| h_2(\mathbf{u}_2)\} = \{\mathbf{u}_1\}$, we need to prove that (50) holds for $\mathbf{u}_2 \in \{\mathbf{u}_2\}$.

∀$\mathbf{u}_2 \in \{\mathbf{u}_2\}$, we have

$$\{\mathbf{u}_1| h_2(\mathbf{u}_2)\} = \{\mathbf{u}_1(\omega) : \omega \in \Omega_{h_2(\mathbf{u}_2)}\},$$  

(51)

$$\{\mathbf{u}_1| h_2(\mathbf{u}_2)\} = \{\mathbf{u}_1(\omega) : \omega \in \Omega_{h_2(\mathbf{u}_2) = h_2(\mathbf{u}_2)}\}. $$  

(52)

As $\Omega_{h_2(\mathbf{u}_2)} = h_2^{-1}(\{h_2(\mathbf{u}_2)\})$ and $h_2^{-1}(\{h_2(\mathbf{u}_2)\}) \supseteq \{\mathbf{u}_2\}$, we get $\Omega_{h_2(\mathbf{u}_2) = h_2(\mathbf{u}_2)} \supseteq \Omega_{h_2(\mathbf{u}_2) = h_2(\mathbf{u}_2)}$, which implies $\{\mathbf{u}_1| h_2(\mathbf{u}_2)\} \supseteq \{\mathbf{u}_1| h_2(\mathbf{u}_2)\}$. Thus (50) is established by

$$\{\mathbf{u}_1\} \supseteq \{\mathbf{u}_1| h_2(\mathbf{u}_2)\} \supseteq \{\mathbf{u}_1| h_2(\mathbf{u}_2)\},$$  

(53)

where (a) follows from the fact that $\forall \mathbf{u}_2 \in \{\mathbf{u}_2\}$. \{\mathbf{u}_1| h_2(\mathbf{u}_2)\} = \{\mathbf{u}_1\} for unrelated $\mathbf{u}_1$ and $\mathbf{u}_2$ (see (7)). Therefore, (49) holds, and combining it with (7), we know that $h_1(\mathbf{u}_1)$ and $h_2(\mathbf{u}_2)$ are also unrelated.

Now we prove the prediction and update steps in Theorem 2 respectively. In the prediction step, for $[w_k-1|x_k-1, y_0:k-1]$ in (15), we know that the collection of $\mathbf{x}_k-1, y_0:k-1$ is a function of $w_0:k-2, v_0:k-1, x_0 = \mathbf{x}_k-1$, i.e., $\mathbf{x}_k-1, y_0:k-1 = \xi(\mathbf{w}_k-1)$. By Assumption 1, $w_0:k-1$ and $\mathbf{x}_k-1$ are unrelated. Thus, applying Lemma 4 we get

$$[w_k-1|x_k-1, y_0:k-1] = \{w_k-1|\xi(x_k-1)\} = [w_k-1],$$  

(54)

where $\mathbf{x}_k-1$ is the realization of $w_k-1$. With (54), (16) becomes (15).

In the update step, we can use a similar technique in (54) to obtain $[v_k|x_k, y_0:k-1] = [v_k]$. Then, (14) becomes

$$\begin{cases} 
\{x_k \in [\mathbf{x}_k]|y_0:k-1\} : g_k(x_k, [v_k]) \cap \{y_k\} \neq \emptyset \\
\bigcup_{v_k \in [v_k]} \{x_k \in [\mathbf{x}_k]|y_0:k-1\} : \{y_k\} = g_k^{-1}([v_k])\} 
\end{cases},$$  

(55)

$$\begin{cases} 
\bigcup_{v_k \in [v_k]} \{g_k^{-1}(\{y_k\})\} \cap [\mathbf{x}_k]|y_0:k-1\} = \text{RHS of (16).} 
\end{cases}$$

where (a) is from $g_k(x_k, [v_k]) = \bigcup_{v_k \in [v_k]} \{g_k(x_k, v_k)\}$ and the fact that if $\{g_k(x_k, v_k)\} \cap \{y_k\} \neq \emptyset$ if $\{g_k(x_k, v_k)\}$, in which $g_k^{-1}(\{y_k\}) = \{x_k : g_k(x_k, v_k) = y_k\}$.  

Appendix E

Proof of Theorem 3

In the initialization step, $[x_0^\ast] = [x_0]$ holds. In the update step at $k = 0$, since (9) implies $[\mathbf{x}/y] \subseteq [\mathbf{x}]$, we have $[v_0,x_0] \subseteq [v_0]$ in (14). Thus,

$$\begin{cases} 
[x_0^\ast]|y_0\} = \{x_0 \in [x_0^\ast] : g_0(x_0, [v_0]|x_0) \cap \{y_0\} \neq \emptyset \\
\bigcup_{x_0 \in [x_0^\ast]} \{g_0(x_0, [v_0]) \cap \{y_0\} \neq \emptyset \} = \{x_0|y_0\} 
\end{cases},$$  

(56)

where (a) follows from (16) and (55). Similarly, in the prediction step at $k = 1$, we have $[w_1|x_0, y_0] \subseteq [w_1]$, which implies $[x_1|y_0] \subseteq [x_1|y_0]$. Proceeding forward, we get $[x_k^\ast]|y_0:k \subseteq [x_k]|y_0:k$ for $k \in \mathbb{N}$.