Tests for Comparing Several Two-Parameter Exponential Distributions Based on Uncensored/Censored Samples

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ABSTRACT

The problems of comparing several exponential distributions based on type II censored samples are considered. Likelihood ratio tests (LRTs) for comparing several location parameters, comparing several scale parameters and for homogeneity of distributions are derived. The LRTs for all three problems are exact as their null distributions do not depend on any unknown parameters. Algorithms are provided to estimate the exact p-values or the percentiles of null distributions. Approximations to the null distributions that are accurate even for small sample sizes are provided. For testing the equality of scale parameters, the proposed LRT is compared with the tests based on union-intersection method and an iterative procedure. Comparison studies indicate that the LRT is more powerful than the existing ones for most parameter values. The methods are illustrated using an example involving elapsed times from randomization to diagnosis of a serious infection of chronic granulomatous disease that were collected from three different hospitals.

1. INTRODUCTION

The problem of testing equality of several distributions within a family of distributions arises in many practical situations. If independent samples are available from different sources or collected from different life testing experiments, a homogeneity test may be used to check if the samples can be modeled by a single distribution. For example, in a placebo controlled randomized clinical trial of gamma interferon in chronic granulomatous disease (CGD), the data were collected on elapsed time (in days) from randomization to diagnosis of a serious infection [1]. The samples were taken from three hospitals, and the Kolmogorov–Smirnov test indicates that all three samples satisfy the model assumption of two-parameter exponential distribution. If a homogeneity test indicates that all three samples can be modeled by a single exponential distribution, then better inferences can be made on the basis of the pooled sample. For two-parameter exponential distributions, the problem of testing the equality of parameters of several distributions was considered by [2]. These authors have proposed an exact iterated test procedures for testing the equality of the location parameters under some conditions and for testing the equality of the scale parameters. Abebe [3] have addressed the problem of comparing several location parameters with more than one control. Li et al. [4] have proposed parametric bootstrap simultaneous confidence intervals for pairs of means. Recently, Krishna and Goel [5] have proposed some Bayesian solutions for estimating location parameters when the data are randomly censored.

To describe the problems that will be addressed in this paper and other related problems, we first note that the two-parameter exponential distribution has the probability density function (pdf) given by

\[ f(x|a, b) = \frac{1}{b} \exp\left(-\frac{x-a}{b}\right), \quad x > a > 0, \quad b > 0, \]  

where \( a \) is the location parameter (also called threshold parameter) and \( b \) is the scale parameter. If it is assumed that \( a = 0 \), then the resulting distribution is referred to as the one-parameter exponential distribution with mean \( b \). Let us denote the two-parameter exponential distribution by \( \text{exp}(a, b) \).

In life testing, the experiments are often discontinued as soon as a fixed number of items fail, yielding a type II censored sample. Let \( X_{i1}, \ldots, X_{in_i} \) be a type II censored sample from a life test of \( n_i \) items whose lifetimes follow an \( \text{exp}(a_i, b_i) \) distribution, \( i = 1, \ldots, k \). The problems that we consider here are (i) testing the equality of the scale parameters \( b_i \)’s, (ii) testing the equality of the location parameters \( a_i \)’s and (iii) the homogeneity test, that is, testing \( (a_i, b_i) = \ldots = (a_k, b_k) \). The LRTs that are proposed in the sequel are applicable to the case where \( r_i = n_i \) for some or all \( i = 1, \ldots, k \). In the following section we provide some preliminary results that are needed to derive the

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To test the equality of scale parameters, consider the hypotheses

\[ H_0 : b_1 = ... = b_k \quad \text{vs.} \quad H_a : b_i \neq b_j \text{ for some } i \neq j. \]

In the following, we first describe the LRT followed by an iterative test by [2], and a test based on union-intersection principle proposed in [7].

2. PRELIMINARIES

Let \( X_{(1)} < ... < X_{(r)} \) be a type II censored sample obtained from a life test on \( n \) items whose lifetime have an \( \exp (a, b) \) distribution, where \( a \) is the threshold parameter and \( b \) is the scale parameter. Let \( \overline{X}_r \) denote the mean of the uncensored lifetimes in the sample.

The MLEs (see [6]) based on the censored sample are given by

\[
\hat{a} = X_{(1)} \quad \text{and} \quad \hat{b} = \left[ r \left( \overline{X}_r - X_{(1)} \right) + (n - r) \left( X_{(r)} - X_{(1)} \right) \right] / r.
\]

Furthermore, the MLEs are independent with

\[
\hat{a} \sim \frac{b}{2n} \chi^2_2 + a \quad \text{and} \quad \hat{b} \sim \frac{b}{2r} \chi^2_{2r-2}.
\]

For the uncensored case, that is, when \( r = n \), the MLEs are given by

\[
\hat{a} = X_{(1)} \quad \text{and} \quad \hat{b} = \overline{X}_n - X_{(1)},
\]

and

\[
\hat{a} \sim \frac{b}{2n} \chi^2_2 + a \quad \text{and} \quad \hat{b} \sim \frac{b}{2n} \chi^2_{2n-2}.
\]

Log-likelihood function

Let \( a = (a_1, \ldots, a_k) \) and \( b = (b_1, \ldots, b_k) \). The log-likelihood function based on the \( k \) independent censored samples can be expressed as

\[
l(a, b) = \sum_{i=1}^{k} \left\{ -r_i \ln b_i - \sum_{j=1}^{r_i} \frac{X_{(j)} - a_i}{b_i} - (n_i - r_i) \frac{X_{(r_i)} - a_i}{b_i} \right\} + C,
\]

where \( C \) is a constant term. Following (2), the MLEs of \( a_i \) and \( b_i \) can be expressed as

\[
\hat{a}_i = X_{(i)} \quad \text{and} \quad \hat{b}_i = \left[ r_i \left( \overline{X}_i - X_{(i)} \right) + (n_i - r_i) \left( X_{(r_i)} - X_{(1)} \right) \right] / r_i, \quad i = 1, \ldots, k,
\]

and \( l(a, b) \) at \( (\hat{a}, \hat{b}) \) can be simplified as

\[
l(\hat{a}, \hat{b}) = - \sum_{i=1}^{k} r_i \ln(\hat{b}_i) - R + C,
\]

where \( R = \sum_{i=1}^{k} r_i \).

3. TESTS FOR THE EQUALITY OF SCALE PARAMETERS

To test the equality of scale parameters, consider the hypotheses

\[ H_0 : b_1 = ... = b_k \quad \text{vs.} \quad H_a : b_i \neq b_j \text{ for some } i \neq j. \]

In the following, we first describe the LRT followed by an iterative test by [2], and a test based on union-intersection principle proposed in [7].
3.1. The Likelihood Ratio Test

Let \( b \) denote the common unknown parameter under \( H_0 \) in (7). Then the log-likelihood function under this null hypothesis can be expressed as

\[
l(a, b) = \sum_{i=1}^{k} \left( -r_i \ln b - \frac{r_i}{b} \sum_{j=1}^{r_i} \frac{X_{ij} - a_i}{b} - (n_i - r_i) \frac{X_{i(\hat{a})} - a_i}{b} \right) + C,
\]

and is maximized at

\[
\hat{a}_i = X_{i(1)} \quad \text{and} \quad \hat{b}_c = \frac{1}{R} \sum_{i=1}^{k} \left( r_i \left( \frac{X_{i(1)} - X_{i(\hat{a})}}{r_i} \right) + (n_i - r_i) \left( \frac{X_{i(\hat{a})} - X_{i(\hat{a})}}{r_i} \right) \right) = \frac{\sum_{i=1}^{k} r_i \hat{b}_i}{R},
\]

where \( R = \sum_{i=1}^{k} r_i \) and \( \hat{b}_i \)'s are given in (5). Thus, letting \( \Lambda_b \) to denote the \(-2 \ln(\text{LRT statistic})\), it can be shown that

\[
\Lambda_b = -2 \sum_{i=1}^{k} r_i \ln \left( \frac{\hat{b}_i}{\sum_{i=1}^{k} r_i \hat{b}_i} \right),
\]

where \( \hat{b}_i \) is given in (5). We note that the distribution of \( \Lambda_b \) under \( H_0 : b_1 = \ldots = b_k \) does not depend on any parameter, and

\[
\Lambda_b \sim -2 \sum_{i=1}^{k} r_i \ln \left( \frac{W_i}{\sum_{i=1}^{k} r_i W_i} \right),
\]

where \( W_i \)'s are independent \( \chi^2_{2r_i - 2} \) random variables. Notice also that the null distribution does not depend on the sample sizes and they depend only on \( r_1, \ldots, r_k \). For a given \( r_1, \ldots, r_k \), the exact percentiles of \( \Lambda_b \) can be estimated using Monte Carlo simulation as shown in the following algorithm.

**Algorithm 1**

For a given \( r_1, \ldots, r_k \) and \( \hat{b}_1, \ldots, \hat{b}_k \),

1. Compute \( \Lambda_b = -2 \sum_{i=1}^{k} r_i \ln \left( \frac{\hat{b}_i}{\sum_{i=1}^{k} r_i \hat{b}_i} \right) \).
2. Generate \( W_i = .5 \chi^2_{2r_i - 2} / r_i \), \( i = 1, \ldots, k \).
3. Compute \( \Lambda_b^* = -2 \sum_{i=1}^{k} r_i \ln \left( \frac{W_i}{\sum_{i=1}^{k} r_i W_i} \right) \).
4. Repeat steps 2 and 3, for a large number of times, say, 100,000.
5. The proportion of \( \Lambda_b^* \)'s that are greater than \( \Lambda_b \) is a Monte Carlo estimate of the p-value.
6. The 100 \((1 - \alpha)\) percentile of these 100,000 \( \Lambda_b^* \)'s is a Monte Carlo estimate of the 100 \((1 - \alpha)\) percentile of \( \Lambda_b \).

The test statistic \( \Lambda_b \) is very similar to the one for testing \( b_1 = \ldots = b_k \) of several one-parameter exponential distributions, except that \( \hat{b}_1 \) in our present problem is distributed as \( \chi^2_{2r_1 - 2} / (2r_1) \) whereas in the one-parameter case it is distributed like \( \chi^2_{2} / (2r_1) \). This fact suggest that, for large samples, the null distribution of \( \Lambda_b \) can be approximated by \( \chi^2_{k-1} \) distribution. For large samples, the null hypothesis of equal scale parameters is rejected if \( \Lambda_b > \chi^2_{k-1,1-\alpha} \). A better approximation to the null distribution of \( \Lambda_b \) can be obtained by the moment matching method. Specifically, we determine the value of \( Q_b \) so that \( E(\Lambda_b) = E(Q_b \chi^2_{k-1}) \) so that

\[
\frac{\Lambda_b}{Q_b} \sim \chi^2_{k-1}, \quad \text{approximately}.
\]
Using the result that \( E(\ln \chi^2_m) = \psi(m/2) + \ln(2) \), where \( \psi \) is the digamma function, and the expression (10), we see that

\[
E(A_k) = 2E\left[ \sum_{i=1}^{k} r_i \ln \left( \frac{\sum_{j=1}^{k} r_j W_j}{R} \right) - \sum_{i=1}^{k} r_i \ln(W_i) \right] = 2 \left[ \sum_{i=1}^{k} r_i [\psi(R - k) - \psi(r_i - 1)] + \sum_{i=1}^{k} r_i \ln(r_i/R) \right].
\] (11)

Thus, \( Q_k = E(A_k)/(k - 1) \), and the null hypothesis of equal shape parameters is rejected if \( A_k > Q_k \chi^2_{k-1} \).

To judge the accuracy of the above approximation, we estimated the upper percentiles of \( A_k \) in (10) using simulation with 1,000,000 runs and the percentiles of \( Q_k \chi^2_{k-1} \) for \( k = 3 \) and 5, and some small to large values of \( (r_1, \ldots, r_k) \). Recall that the distribution of \( A_k \) does not depend on sample sizes, and so percentiles were estimated only for some values of \( (r_1, \ldots, r_k) \). These percentiles are reported in Table 1. Comparison of Monte Carlo estimates of the percentiles and those of \( Q_k \chi^2_{k-1} \) shows that the latter approximate percentiles are very close to those based on simulation even when \( (r_1, \ldots, r_k) = (3, 3, 2, 2, 2) \). Thus, the improved approximation can be safely used for any sample sizes with two or more uncensored data. The usual \( \chi^2_1 \) approximation may be used if \( r_i \geq 100 \) for all \( i \); see the last row of Table 1.

### 3.2. An Iterative Test

We shall now describe a test for equality of scale parameters proposed by [2], which is based on an iterative procedure. Consider a sequence of nested hypotheses

\[ H_2 : b_1 = b_2, \quad H_3 : b_1 = b_2 = b_3, \ldots, \quad H_k : b_1 = \ldots = b_k. \]

A test for \( H_i \) is made if and only if \( H_{i-1} \) is accepted for \( i = 3, \ldots, k \). The null hypothesis (7) of equality scale parameters is accepted if and only if \( H_2, \ldots, H_k \) are accepted.

The test statistic for \( H_i : b_1 = \ldots = b_i \) is given by

\[
F_i = \frac{r_i b_i/(2r_i - 2)}{\sum_{j=1}^{i-1} r_j b_j / S_{i-1}}
\] (12)

where \( S_{i-1} = \sum_{j=i}^{k-1} (2r_j - 2), i = 2, \ldots, k \). For a given level \( \alpha \), choose \( 0 < \eta_i < \alpha \) so that \( \alpha = 1 - \prod_{i=2}^{k} (1 - \eta_i) \). The iterative test rejects \( H_0 : b_1 = \ldots = b_k \) if

\[
F_i < F_{2r_i-2, S_{i-1}; \eta_i/2} \quad \text{or} \quad F_i > F_{2r_i-2, S_{i-1}; 1-\eta_i/2} \quad \text{for any } i \in \{2, \ldots, k\},
\]

where \( F_{m,n,q} \) denotes the 100 \( q \) percentile of an \( F \) distribution with the numerator df \( m \) and the denominator df \( n \). If we choose \( \eta_2 = \ldots = \eta_k \), then \( \eta_i = 1 - (1 - \alpha)^{1/(k-1)} \), \( i = 2, \ldots, k \). Notice that the individual test for \( H_i \) has size \( \eta_i \), but the overall test for \( H_0 : b_1 = \ldots = b_k \) has size \( \alpha \).

| Table 1 | Exact percentiles of \( A_k \) based on simulation and the percentiles of \( Q_k \chi^2_{k-1} \). |
|---------|---------------------------------|---------|---------------------------------|---------|
| \( k = 3 \) | \( r_1, r_2, r_3 \) | 99  | 95  | 90  | \( k = 5 \) | \( r_1, \ldots, r_5 \) | 99  | 95  | 90  |
| \[ (2, 2, 2) \] | 21.53 (22.18) | 14.23 (14.43) | 11.03 (11.09) | \[ (2, \ldots, 2) \] | 30.73 (31.46) | 22.21 (22.48) | 18.29 (18.43) |
| \[ (3, 3, 3) \] | 15.12 (15.31) | 9.92 (9.96) | 7.63 (7.66) | \[ (3, 3, 3, 2) \] | 26.69 (26.71) | 19.07 (19.08) | 15.64 (15.65) |
| \[ (4, 3, 4) \] | 14.08 (14.10) | 9.17 (9.17) | 7.05 (7.05) | \[ (4, 3, 3, 4) \] | 20.30 (20.30) | 14.51 (14.51) | 11.91 (11.90) |
| \[ (3, 2, 6) \] | 18.90 (19.13) | 12.40 (12.45) | 9.55 (9.57) | \[ (3, 2, 3, 2) \] | 26.63 (26.64) | 19.06 (19.04) | 15.65 (15.61) |
| \[ (5, 4, 3) \] | 13.97 (13.97) | 9.08 (9.09) | 6.99 (6.99) | \[ (5, 4, 3, 2) \] | 23.45 (23.07) | 16.61 (16.48) | 13.57 (13.52) |
| \[ (8, 7, 10) \] | 10.90 (10.90) | 7.10 (7.09) | 5.45 (5.45) | \[ (8, 7, 6, 20) \] | 16.06 (16.08) | 11.48 (11.49) | 9.41 (9.42) |
| \[ (10, 10, 10) \] | 10.46 (10.49) | 6.82 (6.82) | 5.24 (5.24) | \[ (4, 3, 2, 5) \] | 23.41 (23.07) | 16.62 (16.48) | 13.59 (13.52) |
| \[ (20, 20, 20) \] | 9.82 (9.81) | 6.37 (6.38) | 4.90 (4.90) | \[ (20, 20, 20) \] | 14.14 (14.12) | 10.09 (10.09) | 8.27 (8.28) |
| \[ (30, 30, 30) \] | 9.62 (9.60) | 6.25 (6.25) | 4.80 (4.80) | \[ (30, 30, 30) \] | 13.85 (13.83) | 9.86 (9.88) | 8.10 (8.10) |
| \[ (50, 50, 50) \] | 9.49 (9.44) | 6.14 (6.14) | 4.72 (4.72) | \[ (50, 50, 50) \] | 13.62 (13.60) | 9.74 (9.72) | 7.97 (7.97) |
| \[ (100, 100, 100) \] | 9.33 (9.32) | 6.07 (6.07) | 4.66 (4.66) | \[ (100, 100, 100) \] | 13.43 (13.44) | 9.61 (9.60) | 7.88 (7.87) |

\( \chi^2_{k-1} \) percentiles: 9.21 | 5.99 | 4.61 | \( \chi^2_{k-1} \) percentiles: 13.28 | 9.49 | 7.78

Note: Percentiles of \( Q_k \chi^2_{k-1} \) are given in parentheses.
3.3. Union-Intersection Test

Shah and Rathod [7] have proposed the following test based on the union-intersection principle. The test rejects the null hypothesis of equality of scale parameters if

$$U = \max \{ \hat{b}_1, ..., \hat{b}_3 \} > u_{1-\alpha},$$  \hspace{1cm} (13)

where $\hat{b}_i$ is the MLE defined in (5) and $u_{1-\alpha}$ is the 100 $(1 - \alpha)$ percentile of $U$.

Shah and Rathod have derived analytical expressions to the distribution function of $U$ for $k \leq 3$ and $r_1 = r_2 = r_3$. Even for these special cases, calculation of the critical value $u_{1-\alpha}$ is numerically involved. As the null distribution of the test statistic $U$ does not depend on any parameters, the critical value $u_{1-\alpha}$ can be easily estimated using Monte Carlo simulation for any $k \geq 2$ and $(r_1, ..., r_k)$.

4. THE LRT FOR LOCATION PARAMETERS

Consider testing

$$H_0 : a_1 = ... = a_k \quad \text{vs.} \quad H_a : a_i \neq a_j \text{ for some } i \neq j.$$  \hspace{1cm} (14)

Let $a$ denote the common unknown parameter under the $H_0$. Then the log-likelihood function under $H_0$ in (14) can be expressed as

$$l(a, b) = k \left( -r_i \ln b_i - n_i \ln \left( \frac{X_{(i)} - a}{\hat{b}_i} \right) - (n_i - r_i) \left( \frac{X_{(i)} - a}{\hat{b}_i} \right) \right) + C,$$

where $C$ is a constant term. The log-likelihood function is maximized at

$$\hat{a} = X_{(1)} = \min \{ X_{(1)}, ..., X_{(k)} \}$$  \hspace{1cm} (15)

and

$$\hat{b}_i = \frac{1}{r_i} \left[ \frac{\sum_{j=1}^{r_i} (X_{(j)} - X_{(1)})}{r_i} + (n_i - r_i)(X_{(i)} - X_{(1)}) \right]$$

$$= \frac{1}{r_i} \left[ n_i \hat{b}_i + n_i (X_{(i)} - X_{(1)}) \right], \quad i = 1, ..., k,$$  \hspace{1cm} (16)

where $\hat{b}_i$’s are as defined in (5). Letting $\hat{b} = (\hat{b}_1, ..., \hat{b}_k)$, we have

$$l(\hat{a}, \hat{b}) = -k \sum_{i=1}^{k} r_i \ln(\hat{b}_i) - R + C.$$  \hspace{1cm} (17)

Let $\Lambda_a$ denote $-2 \ln(\text{LRT statistic})$. Then, using $l(\hat{a}, \hat{b})$ defined in (6), it can be easily verified that

$$\Lambda_a = -2 \left[ l(\hat{a}, \hat{b}) - l(\hat{a}, \hat{b}) \right]$$

$$= -2 \sum_{i=1}^{k} r_i \ln \left( \frac{\hat{b}_i}{\hat{b}_i + n_i (U_{(i)} - X_{(1)})/r_i} \right).$$  \hspace{1cm} (18)

As $\Lambda_a$ is invariant under the transformation $X_{ij} \rightarrow c_i X_{ij} + d$, where $c_i$’s and $d$ are positive constants, its null distribution does not depend on any parameters. Specifically, the null distribution of $\Lambda_a$ can be evaluated empirically assuming that $a_1 = ... = a_k = 0$ and $b_1 = ... = b_k = 1$.

Under this assumption, using the distributional results in (3), we see that

$$\Lambda_a \sim -2 \sum_{i=1}^{k} r_i \ln \left( \frac{W_i}{W_i + n_i(U_{(i)} - X_{(1)})} \right),$$  \hspace{1cm} (19)

where $W_i = r_i \hat{b}_i \sim \chi^2_{2r_i-2}/2$, $U_i = \chi^2_{2}/(2n_i)$ and $U_{(1)} = \min \{ U_1, ..., U_k \}$ and $W_i$’s and $U_i$’s are mutually independent. For an observed value of the LRT statistic $\Lambda_a$, the p-value and the percentiles of $\Lambda_a$ can be estimated using the following algorithm.
Algorithm 2

1. For a given set of \( k \) samples, compute the LRT statistic \( \Lambda_n \) in (18).
2. Generate \( U_i = \chi^2_i/n_i \) and \( W_i = \chi^2_i - \alpha, i = 1, \ldots, k \).
3. Set \( U^{\kappa}_{(1)} = \min U_{1}, \ldots, U_{k} \).
4. Compute \( \Lambda_n^* = -2 \sum_{i=1}^{k} r_i \ln \left( \frac{W_i}{W_i + r_i (U^{\kappa}_{(1)} - U_i)} \right) \).
5. Repeat steps 2–4 for a large number of times, say, 100,000.
6. Proportion of \( \Lambda_n^* \)'s that are greater than \( \Lambda_n \) is an estimate of the \( p \)-value.
7. The 100(1−\( \alpha \)) percentile of the 100,000 \( \Lambda_n^* \)'s is a Monte Carlo estimate of the 100(1−\( \alpha \)) percentile of \( \Lambda_n \).

In Table 2, we present the estimated percentiles of \( \Lambda_n \) (based on 1,000,000 runs) for \( k = 3 \) and 5, and for some values for \( (r_1, \ldots, r_k) \) ranging from very small to large. Notice that these are exact percentiles, except for the simulation errors.

To find a closed-form approximate distribution for \( \Lambda_n \), we note that Wilks’ theorem is not applicable to find an asymptotic \( \chi^2 \) null distribution, because the sample space of a two-parameter exponential distribution depends on an unknown parameter. However, our extensive simulation studies indicated that, for large samples, the null distribution of \( \Lambda_n \) is close to \( \chi^2_{2k-2} \) distribution; see the last row in Table 2. This \( \chi^2_{2k-2} \) approximation is satisfactory only when all \( r_i \)'s are 100 or more. This approximation can be improved along the lines of the preceding section. Specifically, we determine \( Q_\alpha \) so that \( E(\Lambda_n) = Q_\alpha E(\chi^2_{2k-2}) \), and then approximate the distribution of \( \Lambda_n \) by \( Q_\alpha \chi^2_{2k-2} \). Toward this, we note that exact calculation of \( E(\Lambda_n) \) seems to be difficult, and so we resort to use an approximation of \( E(\Lambda_n) \). It is shown in Appendix that

\[
E(\Lambda_n) \approx -2 \sum_{i=1}^{k} r_i \left[ \psi(r_i - 1) - \ln(\bar{r}) + \frac{1}{2r_i^2} \left( r_i + \frac{n_i^2}{\sum_{j=1}^{k} n_j} \right)^2 \right],
\]

where \( \bar{r} = r_i - n_i/\sum_{j=1}^{k} n_j \). Finally, letting \( Q_\alpha = E(\Lambda_n)/(2k - 2) \), we propose the distribution of \( Q_\alpha \chi^2_{2k-2} \) as an approximate null distribution of \( \Lambda_n \).

Monte Carlo estimates (based on one million runs) of the percentiles of \( \Lambda_n \) based on Algorithm 2 and those based on the above approximation are given in Table 2 for \( k = 3 \) and 5 and for some sample sizes and \( (r_1, \ldots, r_k) \). Comparison of the percentiles based on Monte Carlo

Table 2 | Monte Carlo estimates of the percentiles of \( \Lambda_n \) based on (19) and those of \( Q_\alpha \chi^2_{2k-2} \).

| \((r_1, r_2, r_3)\) | \((n_1, n_2, n_3) = (9, 10, 8)\) | \((r_1, r_2, r_3)\) | \((n_1, n_2, n_3) = (9, 10, 8)\) |
|---|---|---|---|
| (3, 2, 3) | 22.43 (22.74) | 15.89 (16.25) | 12.96 (13.33) |
| (3, 3, 4) | 19.21 (19.38) | 13.69 (13.85) | 11.21 (11.36) |
| (4, 5, 4) | 17.38 (17.44) | 12.40 (12.46) | 10.18 (10.22) |
| (5, 5, 5) | 16.68 (16.26) | 11.88 (11.88) | 9.75 (9.74) |
| (6, 7, 8) | 15.59 (15.51) | 11.12 (11.08) | 9.11 (9.09) |
| (9, 10, 8) | 14.99 (14.94) | 10.69 (10.68) | 8.77 (8.75) |

| \((r_1, r_2, r_3)\) | \((n_1, n_2, n_3) = (20, \ldots, 20)\) | \((r_1, r_2, r_3)\) | \((n_1, n_2, n_3) = (20, \ldots, 20)\) |
|---|---|---|---|
| (10, 10, 10) | 14.77 (14.72) | 10.53 (10.52) | 8.64 (8.63) |
| (15, 10, 20) | 14.33 (14.26) | 10.23 (10.19) | 8.39 (8.35) |
| (20, 20, 20) | 14.00 (13.95) | 9.98 (9.97) | 8.17 (8.18) |

| \((r_1, r_2, r_3)\) | \((n_1, n_2, n_3) = (300, 200, 100)\) | \((n_1, n_2, n_3) = (300, 200, 100)\) |
|---|---|---|
| (30, 40, 50) | 13.61 (13.58) | 9.73 (9.71) | 7.97 (7.96) |
| (50, 50, 50) | 13.56 (13.53) | 9.68 (9.67) | 7.93 (7.93) |
| (100, 100, 100) | 13.39 (13.40) | 9.59 (9.58) | 7.86 (7.85) |
| (120, 100, 100) | 13.37 (13.38) | 9.54 (9.56) | 7.81 (7.84) |

\(\chi^2_{2k-2}\) percentiles | 13.28 | 9.49 | 7.78 |

\(\chi^2_{2k-2}\) percentiles | 20.09 | 15.51 | 13.36 |

Note: The numbers in parentheses are the approximate percentiles of \( \Lambda_n \) based on \( Q_\alpha \chi^2_{2k-2} \).
Thus, the percentiles of $\Lambda$ follow a Wilks’ theorem is not applicable to find a large sample approximate percentiles are very satisfactory and are safe to use in practical applications.

### 5. HOMOGENEITY TEST

For the homogeneity test, the hypotheses of interest are

$$H_0 : (a_1, b_1) = ... = (a_k, b_k) \quad \text{vs.} \quad H_a : (a_i, b_i) \neq (a_j, b_j) \text{ for some } i \neq j.$$  

(20)

Let $(a, b)$ denote the common unknown parameter under $H_0$ in (20). Then the log-likelihood function under the null hypothesis (20) can be expressed as

$$l(a, b) = \sum_{i=1}^{k} \left( -r_i \ln b - \sum_{j=1}^{r_i} \frac{X_{ij}}{b} - (n_i - r_i) \frac{X_{i(1)} - a}{b} \right) + C,$$

where $C$ is a constant term. The log-likelihood function is maximized at

$$\hat{a} = X_{(1)} = \min \{X_{1(1)}, ..., X_{k(1)}\}$$  

(21)

and

$$\hat{b} = \frac{1}{R} \sum_{i=1}^{k} \left[ \sum_{j=1}^{r_i} (X_{ij} - X_{(1)}) + (n_i - r_i) (X_{i(1)} - X_{(1)}) \right]$$

$$= \frac{1}{R} \sum_{i=1}^{k} [r_i \hat{b}_i + n_i (X_{i(1)} - X_{(1)})]$$  

(22)

where $b_i$ is defined in (5) and $R = \sum_{i=1}^{k} r_i$. Let $\Lambda_h$ denote the $-2 \ln\text{LRT statistic}$. It can be easily verified that

$$\Lambda_h = -2 \sum_{i=1}^{k} r_i \ln \left( \frac{\hat{b}_i}{\hat{b}} \right)$$

$$= -2 \sum_{i=1}^{k} r_i \ln \left[ \frac{1}{R} \sum_{i=1}^{k} r_i \hat{b}_i + \frac{1}{R} \sum_{i=1}^{k} n_i (X_{i(1)} - X_{(1)}) \right].$$

(23)

Since the above statistic is location-scale invariant, its null distribution does not depend on any parameters, and so the null distribution can be evaluated empirically assuming, without loss of generality, that $a_1 = ... = a_k = 0$ and $b_1 = ... = b_k = 1$. Under this assumption and using the distributional results in (3), we see that

$$\Lambda_h \sim -2 \sum_{i=1}^{k} r_i \ln \left[ \frac{1}{R} \left( \sum_{i=1}^{k} r_i W_i + \sum_{i=1}^{k} n_i (U_i - U_{(1)}) \right) \right]$$

(24)

where $W_i = \chi^2_{2(r_i - 1)}/(2r_i)$, $U_j = \chi^2_{2}/(2n_j)$ and $U_{(1)} = \min \{U_1, ..., U_k\}$ and the random variables $W_i’s$ and $U_i’s$ are mutually independent. Thus, the percentiles of $\Lambda_h$ can be estimated using Monte Carlo simulation as in Algorithms 1 and 2.

Alternatively, a closed-form approximation to the null distribution of $\Lambda_h$ can be obtained along the lines of Section 3. As noted earlier, Wilks’ theorem is not applicable to find a large sample approximate $\chi^2$ distribution. On the basis of simulation studies we found that $\Lambda_h \sim \chi^2_{3(k-1)}$ for large samples. This approximation can be improved along the lines of Section 3. That is, we determine the value of $Q_h$ so that $E(\Lambda_h) = E(Q_h \chi^2_{3(k-1)})$ and $\Lambda_h \sim Q_h \chi^2_{3(k-1)}$, approximately. To find $E(\Lambda_h)$, we note the $\hat{b}_i$ follows a $\chi^2_{2(r_i - 1)}/(2r_i)$ distribution and $\hat{b}$ in (22) follows a $\chi^2_{2R}/(2R)$. Using this distributional results in (23), it can be verified that

$$E(\Lambda_h) = 2 \sum_{i=1}^{k} r_i [\psi(R - 1) - \psi(r_i - 1) + \ln(r_i/R)].$$

(25)

Thus, $Q_h = E(\Lambda_h)/(3(k - 1)) and

$$\Lambda_h \sim Q_h \chi^2_{3(k-1)}, \text{ approximately.}$$
To judge the accuracy of the above approximation, we estimated the upper percentiles of \( \Lambda_k \) in (24) using simulation with 1,000,000 runs and the percentiles of \( Q_k \chi^2_{(k-1)} \) for \( k = 3 \) and \( 4 \), and some small to large values of sample sizes. These percentiles are reported in Table 3. Comparison of Monte Carlo estimates of the percentiles and those of \( Q_k \chi^2_{2} \) shows that the latter approximate percentiles are very close to those based on simulation even when \((r_1, r_2, r_3) = (3, 2, 2)\). Thus, the improved approximation can be safely used for all values of \( r_i \)'s greater than or equal to three.

6. POWER STUDIES

6.1. Power Comparison of the Tests for Scale Parameters

To compare the LRT, iterative test (ITER) and the union-intersection test (UIT), we estimated the powers of these tests for \( k = 3 \) and \( 5 \), and some moderate sample sizes using simulation. As all the tests are scale invariant, the powers were estimated assuming, without loss of generality, that \( b_1 = 1 \) and \( b_1 > ... > b_k \). The estimated powers at the level .05 are reported in Table 4. Examination of reported powers indicates that the UIT performs better than others only when the sample sizes are equal, one of the parameters is maximum and only one of the parameters is minimum; see the cases of \((r_1, r_2, r_3) = (10, 10, 10), (b_1, b_2, b_3) = (1, 1, 4)\) and \((1,1,3), (r_1, ..., r_5) = (10, ..., 10)\).

### Table 3 | Monte Carlo estimates of the percentiles of \( \Lambda_k \) in (9) and those of \( Q_k \chi^2_{2} \).

| \((r_1, r_2, r_3)\) | \(\chi^2_{(k-1)}\) percentiles | \(\chi^2_{(k-1)}\) percentiles |
|-----------------|---------------------------------|---------------------------------|
|                 | 99                              | 95                              | 90                              |
|                 | 16.81                           | 12.59                           | 10.64                           |
|                 | 21.67                           | 16.92                           | 14.68                           |

### Table 4 | Powers of the iterative test (ITER), UIT and the LRT for testing the equality of the scale parameters.

| Sample Sizes | \((b_1, b_2, b_3)\) |
|--------------|---------------------|
|              | ITER | UIT | LRT | ITER | UIT | LRT | ITER | UIT | LRT |
| (1, 1, 1)    | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 |
| (1, .8, .6)  | .121 | .134 | .138 | .182 | .108 | .311 | .082 | .089 | .091 |
| (1, .8, .5)  | .203 | .228 | .228 | .342 | .196 | .377 | .153 | .168 | .193 |
| (1, .5, .5)  | .265 | .287 | .310 | .311 | .293 | .375 | .173 | .190 | .210 |
| (1, .4, .6)  | .374 | .367 | .377 | .377 | .411 | .392 | .207 | .253 | .292 |
| (1, .4, .7)  | .374 | .367 | .377 | .375 | .419 | .409 | .367 | .419 | .384 |
| (1, .4, .8)  | .385 | .388 | .387 | .392 | .457 | .703 | .369 | .410 | .431 |
| (1, .4, .9)  | .391 | .395 | .422 | .409 | .424 | .504 | .547 | .620 | .675 |
| (1, 1, 1)    | .443 | .463 | .453 | .703 | .473 | .614 | .863 | .879 | .862 |
| (1, 1, 1)    | .437 | .473 | .508 | .504 | .484 | .709 | .874 | .945 | .917 |
| (1, 1, 1)    | .611 | .615 | .601 | .614 | .677 | .917 | .910 | .949 | .972 |
| (1, 1, 1)    | .556 | .627 | .655 | .709 | .652 | .937 | .934 | .959 | .979 |
| (1, 1, 1)    | .706 | .721 | .710 | .917 | .775 | .922 | .939 | .957 | .980 |
| (1, 1, 1)    | .796 | .851 | .859 | .937 | .887 | .995 | .937 | .966 | .985 |
| (1, 1, 1)    | .819 | .870 | .890 | .922 | .932 | .896 | .957 | .981 | .994 |
| (1, 1, 1)    | .940 | .941 | .936 | .995 | .989 | .969 | .958 | .978 | .993 |
(b_1, ... , b_3) = (1, .4, 1, 1, 1), (1,2,1,1,1). In these cases, the UIT has larger power than those of the other tests. However, for (r_1, ... , r_3) = (5, 15, 10, 5, 15) and (b_1, ... , b_3) = (1, .4, 1, 1, 1), the power of the UIT is much smaller than those of other two tests. Specifically, we see under the column of (r_1, ... , r_3) = (5, 15, 10, 5, 15) the powers of UIT are much smaller than the powers of other two tests. Between the LRT and ITER, the ITER is more powerful than the LRT in some cases; see the powers for the case (r_1, ... , r_3) = (5, 15, 10, 5, 15). However, the LRT has appreciably larger powers than the ITER over larger parameter space, and so the LRT maybe preferred to other two tests.

### 6.2. Power of the LRT for Location Parameters

To judge the power properties of the LRT for the equality of location parameters, we estimated the powers by Monte Carlo simulation and reported them in Table 5. The powers were estimated for (b_1, b_2, b_3) = (3, 4, 5) and (2,2,2) and sample sizes (n_1, n_2, n_3) = (15, 15, 15) and (20,20,20). Given a set of sample sizes, we evaluated the powers at (r_1, r_2, r_3) = (5, 4, 5) and (12,11,12) and for values of (a_1, a_2, a_3) so that a_1 = 4 > a_2 > a_3. We first observe from Table 5 that the powers are increasing with increasing sample sizes. For example, see the powers in the columns of (n_1, n_2, n_3) = (15, 15, 15) and (r_1, r_2, r_3) = (5, 4, 5) and (n_1, n_2, n_3) = (20, 20, 20) and (r_1, r_2, r_3) = (5, 4, 5). We also observe that the powers for larger values of (r_1, r_2, r_3) are larger than those of smaller values of (r_1, r_2, r_3) while the sample sizes and other parameters are fixed. For example, see the powers in the columns of (n_1, n_2, n_3) = (15, 15, 15), (r_1, r_2, r_3) = (5, 4, 5) and (12,11,12) while the other values are fixed. For example, see the powers in the columns of (n_1, n_2, n_3) = (20, 20, 20), (r_1, r_2, r_3) = (5, 4, 5) and (12,11,12). Finally, we also notice that the powers for the case of (b_1, b_2, b_3) = (3, 4, 5) are smaller than corresponding powers for (b_1, b_2, b_3) = (2, 2, 2). This is expected because the variance of an exp(a, b) distribution is b^2, and so the powers for the set of populations with larger variances are expected to be smaller than those for populations with smaller variances while all other parameters and sample sizes are fixed. Thus, the LRT for the equality of location parameters possess all natural properties of an efficient test.

### 6.3. Powers of the Homogeneity Test

Powers of the LRT for testing H_0 : (a_1, b_1) = ... = (a_k, b_k) were estimated for sample sizes (n_1, n_2, n_3) = (15, 15, 15) and (20,20,20), and for each set of sample sizes, (r_1, r_2, r_3) = (7, 8, 11) and (14,13,10). The estimated powers are reported in Table 6. On the basis powers in Table 6, we see that the power properties of the homogeneity test is very similar to the LRT for location parameters discussed in the preceding section. In particular, the power is increasing with increasing sample sizes while other parameters are fixed and also increasing with increasing values of r's while other values are fixed. For example, see the columns under (n_1, n_2, n_3) = (15, 15, 15) (r_1, r_2, r_3) = (7, 8, 11) and (14,13,10) in Table 6. We also notice that the power is increasing with increasing disparities among location parameters and/or scale parameters. Thus, the LRT has all natural properties of an efficient test.

### 7. AN EXAMPLE

The data were collected from a placebo controlled randomized clinical trial of gamma interferon in CGD. The data were subsets of large data given in Appendix D of [1]. They represent elapsed time (in days) from randomization to diagnosis of a serious infection. The samples were taken from three hospitals with ID codes 258, 204 and 332. We applied Kolmogorov-Smirnov test for the exponential distribution by [8], and the test indicates that all three samples satisfy the model assumption of two-parameter exponential distribution.

#### Table 5  Powers of the LRT for equality of location parameters.

| (n_1, n_2, n_3) | (r_1, r_2, r_3) | (r_1, r_2, r_3) |
|-----------------|-----------------|-----------------|
| (a_1, a_2, a_3) | (b_1, b_2, b_3) | (b_1, b_2, b_3) |
| (4, 4, 4)       | (.049, .051)    | (.049, .050)    |
| (4, 3, 4)       | (.072, .080)    | (.104, .105)    |
| (4, 3, .5, 3.5)| (.140, .151)    | (.233, .234)    |
| (4, 3, 6, 4)    | (.166, .195)    | (.340, .341)    |
| (4, 3, 6)       | (.176, .198)    | (.518, .519)    |
| (4, 3, 3)       | (.315, .450)    | (.681, .682)    |
| (4, 3, 3.5)     | (.249, .301)    | (.513, .514)    |
| (4, 3, 3)       | (.459, .618)    | (.839, .840)    |
| (4, 4, 3)       | (.699, .836)    | (.953, .954)    |
Notice that the samples are uncensored with \((n_1, n_2, n_3) = (10, 14, 9)\). The MLEs are \(\hat{\alpha}_1 = X_{a(1)}, \hat{\beta}_1 = \bar{X}_1 - X_{a(1)}, i = 1, 2, 3\), and the MLEs for the three samples are reported in Table 7.

For testing \(H_0: \alpha_1 = \alpha_2 = \alpha_3\) vs. \(H_1: \alpha_1 \neq \alpha_2\), for some \(i \neq j\), the LRT statistic \(\Lambda_3 = 9.63\) and the 95th percentile of the null distribution is 10.51. To find the approximate 95th percentile, we evaluated \(\sum_{i=1}^{3} Q_{\alpha_i} = E(\Lambda_3)/(2k - 2) = 1.105\) and \(1.105 \times X_{0.95}^2 = 1.105 \times 9.488 = 10.48\). Notice that the exact and the approximate percentiles are practically the same. Furthermore, we estimated the p-value using Algorithm 2 with 1000000 runs as \(.069\) and the p-value based on the \(Q_{\alpha}X_3^2\) distribution is also \(.069\). Thus, at 5% level, the location parameters are not significantly different.

For testing \(H_0: \beta_1 = \beta_2 = \beta_3\) vs. \(H_1: \beta_1 \neq \beta_2\), the statistic \(\Lambda_3 = 2.17\) and the p-value based on Algorithm 1 is \(.385\). If we use the level of significance \(.05\), then the exact critical value is 6.83 and \(Q_{\beta}X_{0.95}^2 = 1.138 \times 5.99\) is 6.82, which is very close to the exact one. To apply the UIT, we estimated the 5% critical value as 3.056, and the test statistic \(\max \{\beta_1, \beta_2, \beta_3\} \min \{\beta_1, \beta_2, \beta_3\} = 194.5/104.6 = 1.859\). To apply the iterative test procedure, we calculated the individual test statistics \(F_1 = .944\) and \(F_2 = .563\). We chose the levels for the individual test as \(\eta_1 = \eta_2 = \eta_3 = .01266\) so that \(\alpha = 1 - (1 - 2 \times .01266)^2 = .05\). For these levels, the (lower, upper) critical values are \((.383, 2.84)\) for \(F_2\) and \((.349, 2.36)\) for \(F_3\). Note that both test statistics \(F_2\) and \(F_3\) fall in their corresponding acceptance intervals, and so the null hypothesis of equality of scale parameters is accepted. Thus, all three tests indicate that \(H_0: \beta_1 = \beta_2 = \beta_3\) is tenable.

For testing \(H_0: (a_1, b_1) = (a_2, b_2)\) vs. \(H_1: a_1 \neq a_2\), the statistic \(\Lambda_3\) in (23) is 9.68 and the p-value based on (24) is \(.187\). If we use the level of significance \(.05\), then the exact critical value is 13.93; the approximate critical value \(Q_{ab}X_{0.95}^2 = 1.101 \times 12.59 = 13.86\). Thus, homogeneity of exponential distributions is tenable. This means that all three samples may be pooled and the pooled sample can be modeled by a single two-parameter exponential distribution.

### 8. CONCLUDING REMARKS

Although several tests for comparing parameters of several two-parameter exponential distributions were proposed in the literature, none of them is based on the likelihood approach. As the sample space of a two-parameter exponential distribution depends on an unknown parameter, it does not satisfy all the regularity conditions. So an LRT statistic for comparing parameters does not have the asymptotic chi-square distribution with the degrees of freedom determined by the difference between the dimensions of the parameter spaces under the alternative and null hypotheses. We have shown in this article that the null distributions of the LRT statistics for all three problems do not depend on any unknown parameters, and so the LRTs are exact. However, calculation of the percentiles of the LRT statistics involves simulation. Even though calculation of percentiles based on Monte Carlo simulation is not difficult, we provided closed-form approximate
chi-square distributions for all three problems. These approximate chi-square null distributions are not only simple, but also very accurate even for small samples. These approximate null distributions maybe warranted in situations where the number of distributions to be compared is large.

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REFERENCES

1. T.R. Fleming, D.P. Harrington, Counting Processes and Survival Analysis, John Wiley & Sons, 2011.
2. R.V. Hogg, E.A. Tanis, J. Am. Stat. Assoc. 58 (1963), 435–443.
3. A. Abebe, Stat. Methods Appl. 3 (2009), 359–374.
4. J. Li, W. Song, J. Shi, Stat. Probab. Lett. 106 (2015), 39–45.
5. H. Krishna, N. Goel, Comput. Stat. 33 (2018), 249–275.
6. B. Epstein, M. Sobel, Ann. Math. Stat. 25 (1954), 373–381.
7. S.M. Shah, V.R. Rathod, Can. J. Stat. 4 (1976), 119–126.
8. H.W. Lilliefors, J. Am. Stat. Assoc. 64 (1969), 387–389.
9. J.D. Gibbons, S. Chakraborti, Nonparametric Statistical Inference, Springer, Berlin, Heidelberg, 2011.
APPENDIX

Let $Z_i = W_i + n_i(U_i - U_{(i)})$, $\Theta_i = E(Z_i)$, $i = 1, \ldots, k$ and $g(x) = \ln(x)$. Recall that $W_i$‘s and $U_i$‘s are independent with

$$W_i \sim \frac{1}{2} \chi^2_{2i-2} \quad \text{and} \quad U_i \sim \frac{1}{2n_i} \chi^2_i, \quad i = 1, \ldots, k.$$

To find an approximation to $E \ln(Z_i)$, we shall use the result (see Section 2.9, [9]) that

$$E \ln(Z_i) \approx \ln(\Theta_i) + \frac{\operatorname{Var}(Z_i)}{2\Theta_i^2}.$$

Recall that $n_i U_i \sim \chi^2_i/2$, or $n_i U_i$ follows a standard exponential distribution. So

$$P(U_{(i)} \leq u) = 1 - \prod_i P(n_i U_i \geq n_i u) = 1 - \exp\left(-\sum_{i=1}^k n_i u\right),$$

which is the distribution function of an exponential distribution with mean $1/\sum_{i=1}^k n_i$. Thus,

$$E(U_{(i)}) = \frac{1}{\sum_{i=1}^k n_i} \quad \text{and} \quad \operatorname{Var}(U_{(i)}) = \frac{1}{\left(\sum_{i=1}^k n_i\right)^2} \tag{A.1}$$

Using the above expectation, we have

$$E(Z_i) = E(W_i) + n_i E(U_i - U_{(i)}) = r_i - \frac{n_i}{\sum_{j=1}^k n_j}, \quad i = 1, \ldots, k.$$

To find the variance,

$$\operatorname{Var}(Z_i) = \operatorname{Var}(W_i) + n_i^2 \left[\operatorname{Var}(U_i) + \operatorname{Var}(U_{(i)}) - 2 \operatorname{Cov}(U_i, U_{(i)})\right]. \tag{A.2}$$

To find the $\operatorname{Cov}(U_i, U_{(i)})$, we find the joint distribution of $U_i$ and $U_{(i)}$ as

$$F_{U_i, U_{(i)}}(u, v) = P(U_i \leq u, U_{(i)} \leq v)$$

$$= P(U_i \leq u) - P(U_i \leq u, U_{(i)} > v)$$

$$= P(U_i \leq u) - P(U_i \leq u, U_i \geq v, \ldots, U_k \geq v)$$

$$= P(U_i \leq u) - \prod_{j \neq i} P(n_j U_j \geq n_j v)$$

$$= (1 - e^{-n_i u}) - (e^{-n_i v} - e^{-n_i u}) e^{-\sum_{j \neq i} n_j v}.$$

By taking the derivative with respect to $(u, v)$, we find the joint density as

$$f_{U_i, U_{(i)}}(u, v) = n_i e^{-n_i u} \sum_{j \neq i} n_j e^{-\sum_{j \neq i} n_j v},$$

which shows that $U_i$ and $U_{(i)}$ are independent. Using this fact in (A.2), we find

$$\operatorname{Var}(Z_i) = r_i + \frac{n_i^2}{\left(\sum_{j=1}^k n_j\right)^2}.$$

Thus,

$$E \ln(Z_i) \approx \ln(\Theta_i) - \frac{\operatorname{Var}(Z_i)}{2\Theta_i^2}$$

$$= \ln(\Theta_i) - \frac{1}{2\Theta_i^2} \left(r_i + \frac{n_i^2}{\left(\sum_{j=1}^k n_j\right)^2}\right).$$
where $\bar{\theta}_i = E(Z_i) = r_i - \frac{n_i}{\sum_{j=1}^k n_j}$, $i = 1, \ldots, k$. Using this approximation, we see that

$$E(\Lambda_\psi) = -E \left[ \sum_{i=1}^k r_i \ln \left( \frac{X_{2n_i - 2/2}}{Z_i} \right) \right]$$

$$= -2 \sum_{i=1}^k r_i [\psi(r_i - 1) - E\ln(Z_i)]$$

$$\approx -2 \sum_{i=1}^k r_i \left[ \psi(r_i - 1) - \ln(\bar{\theta}_i) + \frac{1}{2\bar{\theta}_i^2} \left( r_i + \frac{n_i^2}{\left( \sum_{j=1}^k n_j \right)^2} \right) \right]. \quad (A.3)$$