Construction of local Lyapunov functions around non-hyperbolic equilibria by verified numerics for two dimensional cases

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Abstract

Numerical verification methods are proposed in order to construct local Lyapunov functions around non-hyperbolic equilibria of dynamical systems described by ODEs in two dimensional space. The normal form theory in dynamical systems gives basic ideas of these methods. To prove negative definiteness of polynomials of higher degree than two, a new theorem on interval arithmetic is also proposed.

Keywords numerical verification, dynamical systems, Lyapunov functions, non-hyperbolic equilibria

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1. Introduction

There are a plenty of researches in combination of verified numerics and dynamical systems. One can find a long list in [1]. Among them, the paper [2] by K.Matsue et al. treats construction of local Lyapunov functions around hyperbolic equilibria, and there are some applications of their methods [3]. The present paper is one of their successors to give new methods to construct local Lyapunov functions around non-hyperbolic equilibria using verified computation. We believe this is an important work since we derive somewhat general methods even though the methods can be applied to only two dimensional dynamical systems at this moment. Note that local Lyapunov functions do not always exist around non-hyperbolic equilibria, and it has been said that there is no general way to obtain a local Lyapunov function even if it exists.

Our methods are based on [2] and some ideas come from normal form theory in the field of dynamical systems. Normal form theory is a technique for nonlinear dynamical systems described by ordinary differential equations in order to transform the systems into certain standard forms. One can remove the inessential part of higher-order nonlinearities by using a particular class of coordinate transformations.

We also give a new numerical verification method in order to prove the negative definiteness of polynomials of higher degree than four, which will be necessary to specify a domain of a local Lyapunov function.

The structure of the present paper is as follows.

- In the third section, we classify our problems of two dimensions into a few cases and give sufficient conditions to construct local Lyapunov functions for the first case.
- The numerical verification method to prove the negative definiteness of polynomials of higher degree than four is derived in the section four.
- We show the effectivity of our methods by a numerical example in the section five.

2. Our approach to construct local Lyapunov functions

Consider the differential equation

\[ \dot{x} = f(x), \]  

where \( f \in C^r(\mathbb{R}^n, \mathbb{R}^n) \) with \( r \geq 4 \). The solution of (1) at time \( t \) with an initial value \( x \in \mathbb{R}^n \) at \( t = 0 \) is represented by \( \varphi(t, x) \in \mathbb{R}^n \).

We assume that (1) has a non-hyperbolic equilibrium \( x^* \). This means \( Df^* \), the Jacobian matrix of \( f(x) \) at \( x = x^* \), has at least one eigenvalue whose real part is 0.

**Definition 1** Let \( U \subset \mathbb{R}^n \) be an open subset. A Lyapunov function \( L : U \rightarrow \mathbb{R} \) is a \( C^4 \)-function satisfying the following conditions.

(1) \( (dL/dt)(\varphi(t, x))|_{t=0} \leq 0 \) holds for each solution orbit \( \{\varphi(t, x)\} \) through \( x \in U \).

(2) \( (dL/dt)(\varphi(t, x))|_{t=0} = 0 \) implies \( \varphi(t, x) \equiv x^* \in U \).

When the domain \( U \) is a bounded neighborhood of \( x^* \), we say \( L(x) \) is a local Lyapunov function for \( x^* \). Note that our Lyapunov function can take negative values in order to treat saddle equilibria.
Our main purpose is to construct a Lyapunov function for a non-hyperbolic equilibrium \( x^* \) and to specify a certain closed set \( D_L \subset U \) including \( x^* \) as an inner point. Verified numerics is applied to the specification.

First of all, we transform (1) to the following system.

\[
\dot{v} = Jv + p^{(2)}(v) + p^{(3)}(v) + \mathcal{O}(\|v\|^4). \tag{2}
\]

Here \( J \) is a real block diagonal matrix with each block being a real Jordan block, \( p^{(k)} \in \mathbb{R}^n \) is a vector whose elements are polynomials of \( v \in \mathbb{R}^n \) with terms of degree \( k \), and \( \mathcal{O}(\|v\|^4) \) means terms of degree four and higher degrees. The polynomials are written by

\[
p^{(2)}(v) = \begin{pmatrix} v^T P_1 v \\ \vdots \\ v^T P_n v \end{pmatrix}, \quad p^{(3)}(v) = \begin{pmatrix} v^T \dot{P}_1(v)v \\ \vdots \\ v^T \dot{P}_n(v)v \end{pmatrix},
\]

where \( P_i \in \mathbb{R}^{n \times n} \) is a real symmetric matrix, and \( \dot{P}_i(v) \in \mathbb{R}^{n \times n} \) is also a real symmetric matrix with the \((j,k)\) element \( v^T \dot{P}^{(j)}_{k}(v) \) for \( P^{(j)}_{k} \in \mathbb{R}^n \).

The matrix \( J \) is obtained by

\[
J = X^{-1}DX^*X,
\]

using an \( n \times n \) real invertible matrix \( X \) and has a form of

\[
J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix},
\]

where \( J_k \) is a Jordan block with real eigenvalues of \( Df^* \) or a \( 2 \times 2 \) matrix of the form

\[
J_k = \begin{pmatrix} \alpha_k & -\beta_k \\ \beta_k & \alpha_k \end{pmatrix},
\]

with real numbers \( \alpha_k \) and \( \beta_k \).

The system (2) is obtained from (1) by the transformation \( x = x^* + Xv \). At this moment, we have to assume that the transformation can be carried out without rounding errors, though we are struggling to relieve this restriction. This means that any floating point approximation to the matrix \( X \) should not be used in our verification process. Note that on the other hand we can use approximations to \( X \) in the cases of hyperbolic equilibria [2].

Let us describe our approach.

1. Apply the following transformation to the system (2).

\[
v = u + q^{(2)}(u) + q^{(3)}(u). \tag{3}
\]

The definitions of \( q^{(2)}(u) \) and \( q^{(3)}(u) \) are

\[
q^{(2)}(u) = \begin{pmatrix} u^T Q_1 u \\ \vdots \\ u^T Q_n u \end{pmatrix}, \quad q^{(3)}(u) = \begin{pmatrix} u^T \dot{Q}_1(u)u \\ \vdots \\ u^T \dot{Q}_n(u)u \end{pmatrix},
\]

where \( Q_i \in \mathbb{R}^{n \times n} \) is a real symmetric matrix, and \( \dot{Q}_i(u) \in \mathbb{R}^{n \times n} \) is also a real symmetric matrix with the \((j,k)\) element \( u^T \dot{Q}^{(j)}_{k}(u) \) for \( Q^{(j)}_{k} \in \mathbb{R}^n \).

Then we have another system

\[
\dot{u} = J(u) + D^{(2)}(u) + D^{(3)}(u) + \mathcal{O}(\|u\|^4), \tag{4}
\]

where

\[
D^{(2)}(u) = (Jq^{(2)}(u) - Dq^{(2)}(u)J)u + p^{(2)}(u), \\
D^{(3)}(u) = (Jq^{(3)}(u) - Dq^{(3)}(u)J)u + Dq^{(2)}D^{(2)}(u) + p^{(3)}(u), \\
p^{(3)} = p^{(2)} + 2 \left( (q^{(2)})^T P_1 \right) u.
\]

Here \( Dq^{(2)}(u) \) and \( Dq^{(3)}(u) \) denote the derivatives of \( q^{(2)}(u) \) and \( q^{(3)}(u) \) with respect to \( u \), respectively.

2. Try to construct a local Lyapunov function of the form

\[
L(u) = a^T u + u^T Y u, \tag{5}
\]

where \( a \in \mathbb{R}^n \) and \( Y \in \mathbb{R}^{n \times n} \) which is a real symmetric matrix.

3. Derive \( L(x) \) from \( L(u) \) as a candidate for a local Lyapunov function around \( x^* \) as a polynomial up to degree four using inverse transformations:

\[
u = v - q^{(2)}(v - q^{(2)}(v)) - q^{(3)}(v) + \mathcal{O}(\|v\|^4), \tag{6}
\]

and

\[
v = X^{-1}(x - x^*). \tag{7}
\]

Substituting (6) into (5), it is seen that the terms of higher degree than four contains the terms from \( \mathcal{O}(\|v\|^4) \) in (6). This implies these terms belong to the inessential part. Thus we omit the terms of higher degree than four in \( L(x) \).

4. Specify the closed domain \( D_L \) by using verified numerics where

\[
\frac{d}{dt} L(f(t,x))|_{t=0} < 0, \quad \forall x \in D_L \setminus \{x^*\}
\]

holds. Note that this process verifies that \( L(x) \) is indeed a local Lyapunov function within \( D_L \).

The normal form theory simplifies \( D^{(2)}(u) \) and \( D^{(3)}(u) \) in (4) as possible by choosing appropriate \( q^{(2)}(u) \) and \( q^{(3)}(u) \). In our approach, we try to select \( q^{(2)}(u) \) and \( q^{(3)}(u) \) to control \( D^{(2)}(u) \) and \( D^{(3)}(u) \) in order that \((dL/dt)(f(t,u))|_{t=0} < 0 \) holds for any \( u \in B_r \setminus \{0\} \) for a ball \( B_r \) centered at 0 with some radius \( r > 0 \). Here \( f(t,u) \) is the solution of the system (4) with the initial point \( u \in \mathbb{R}^n \). We have

\[
\frac{d}{dt} L(f(t,u))|_{t=0} = a^T J u + E^{(2)}(u) + E^{(3)}(u) + E^{(4)}(u) + \mathcal{O}(\|u\|^5), \tag{8}
\]

where

\[
E^{(2)}(u) = a^T D^{(2)}(u) + u^T (J^T Y + Y J) u, \tag{9}
\]

\[
E^{(3)}(u) = a^T D^{(3)}(u) + 2u^T Y D^{(2)}(u), \tag{10}
\]

\[
E^{(4)}(u) = a^T \mathcal{O}(\|u\|^4) + 2u^T Y D^{(3)}(u). \tag{11}
\]

In order that \((dL/dt)(f(t,u))|_{t=0} < 0 \) holds in \( B_r \setminus \{0\} \), there are necessary conditions.
The matrix $J$
In the following we only explain the case (A) due to the generality. Here $\rho_E$ that matrix for some $u \in B_r \setminus \{0\}$.

Note that
- If $a \neq 0$ we cannot control the fourth degree term $E^{(4)}(u)$.

We divide our situation into three cases.

(A) The matrix $J$ is not singular and the vector $a$ should be taken as $0$.
In this case the expressions (9), (10) and (11) are changed into

$$E^{(2)}(u) = u^T(J^TY + YJ)u, \quad E^{(3)}(u) = 2u^TYD^{(2)}(u), \quad E^{(4)}(u) = 2u^TYD^{(3)}(u),$$

respectively. Since the matrix $J$ has pure imaginary eigenvalues in this case, the real symmetric matrix $Y$ cannot be chosen such that $E^{(2)}(u) < 0$ in $B_r \setminus \{0\}$. This means that $E^{(3)}(u)$ should be 0 for any $u \in \mathbb{R}^n$ and $E^{(4)}(u)$ should be negative for any $u \in B_r \setminus \{0\}$. We derive sufficient conditions for $E^{(3)}(u) = 0$ and $E^{(4)}(u) < 0$ in the next section.

(B) The matrix $J$ is singular and the vector $a$ is taken as $a = 0$.
Since $a \neq 0$, we cannot control $E^{(3)}(u)$. This means $E^{(2)}(u)$ should be negative for any $u \in B_r \setminus \{0\}$. The singularity of the matrix $J$ leads to certain conditions such that $E^{(2)}(u) < 0$ in $B_r \setminus \{0\}$.

(C) The matrix $J$ is singular and the vector $a$ is taken as $0$.
We do not yet investigate this case sufficiently, and this is one of our future works.

In the following we only explain the case (A) due to the page restriction.

3. Classification of 2 dimensional cases
The matrix $J$ takes one of the following form.

$$J_1 = \begin{pmatrix} 0 & -\rho \\ \rho & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 \\ 0 & \rho \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 0 & \rho \\ 0 & 0 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ Here $\rho \in \mathbb{R}$ is not 0, and we take $\rho = 1$ without loss of generality.

Case (A) $J = J_1$ and $a = 0$. The real symmetric matrix $Y$ should be taken as

$$Y = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

for $\alpha \in \mathbb{R}$, which gives $E^{(2)}(u) = 0$ in $B_r \setminus \{0\}$. Otherwise we cannot achieve $E^{(2)}(u) \leq 0$ in $B_r \setminus \{0\}$. In order to control $E^{(3)}(u)$ we take an arbitrary $\alpha \neq 0$.

Now we define $Q_1$ and $Q_2$ appeared in $q^{(2)}(u)$ such that $E^{(3)}(u) = 0$ holds, which implies $D^{(2)}(u) = 0$. From straightforward calculation, we have

$$Q_1 = -\frac{1}{3} \left[ P_2 + 2J_1^TP_2J_1 + (J_1^TP_1 + P_1J_1) \right],$$

$$Q_2 = \frac{1}{3} \left[ P_1 + 2J_1^TP_1J_1 + (J_1^TP_2 + P_2J_1) \right].$$

The next step is to define $q^{(3)}(u)$ such that $E^{(4)}(u) < 0$ holds in $B_r \setminus \{0\}$. Using $D^{(2)}(u) = 0$, we have

$$E^{(4)}(u) = 2u^T (J_1q^{(3)}(u) - Dq^{(3)}(u)J_1u + p^{(3)}(u)).$$

(15)

Based on the above expression, we take the following procedure to obtain sufficient conditions to construct local Lyapunov functions.

1. Choose a basis $\{q_i(u)\}_{i=1,...,8}$ of the linear space:

$$V^{(3)} = \{ q \in C(\mathbb{R}^2, \mathbb{R}) \mid q = (q_1,q_2)^T, q_i \text{ consists of terms of degree three w.r.t. } u \in \mathbb{R}^2 \}.$$ 

An example of the basis is $\{ \hat{h}_1(u), \ldots, \hat{h}_4(u) \}$ with

$$\hat{h}_1(u) = u_1^3 - 3u_1u_2^2, \quad \hat{h}_2(u) = u_2^3 - 3u_1^2u_2,$$

$$\hat{h}_3(u) = u_1^2u_2, \quad \hat{h}_4(u) = u_1u_2^2.$$ 

The corresponding $q_i(u)$ are

$$q_1(u) = -\begin{pmatrix} 0 \\ u_1^3 \end{pmatrix}, \quad q_2(u) = \begin{pmatrix} u_2^3 \\ 0 \end{pmatrix},$$

$$q_3(u) = \frac{1}{4} \begin{pmatrix} u_1^3 \\ u_2^3 \end{pmatrix}, \quad q_4(u) = -\frac{1}{4} \begin{pmatrix} u_1^3 \\ u_2^3 \end{pmatrix}.$$ 

3. Check whether

$$u^T p^{(3)}(u) \not\in U^{(4)} = \text{span} \{ \hat{h}_1(u), \ldots, \hat{h}_4(u) \} \quad (16)$$

holds or not.

Note that
- The dimension of $U^{(4)}$ is four, even though a basis of polynomials of degree four with respect to $u \in \mathbb{R}^2$ has five linear independent functions.
- For any polynomial $h(u) \in U^{(4)}$, we have $h(u_1) < 0$ and $h(u_2) > 0$ for some $u_1, u_2 \in B_r$. Therefore if and only if

$$\left\{ u^T p^{(3)}(u), \hat{h}_1(u), \ldots, \hat{h}_4(u) \right\}$$

gives a basis of polynomials of degree four, then we can take $E^{(4)}(u)$ of (15) as a polynomial of degree four such that $E^{(4)}(u) < 0$ holds in $B_r \setminus \{0\}$. This means that (16) is a sufficient condition to have a local Lyapunov function $L(u) = au^T u$. If (16) holds, then we derive an appropriate $q^{(3)}(u)$ using $q_1(u), \ldots, q_4(u)$.
Case (B) $J = J_i$, $i = 2, 3, 4$ and $a \neq 0$. We omit this case since we have the page restriction. A forthcoming full version paper will include detail explanation of the case (B).

4. Verification of negative definiteness

Let us consider how to verify the negative definiteness of $(dL/dt)(\varphi(t, x))|_{t=0}$ for $x \in D_L$.

Note that essential part of $(dL/dt)(\varphi(t, x))|_{t=0}$ may be represented by terms of degree four in our case (A). In this case, the method in [2] to check negative definiteness using eigenvalues cannot be applied. Instead we adopt the following lemma.

**Lemma 2** Consider a homogeneous form of terms of degree $2m$ with respect to $x \in \mathbb{R}^n$ together with interval coefficients $[c_a] \subset \mathbb{R}$ as

$$[H](x) = \sum_{|\alpha| = 2m} [c_a] x^\alpha. \quad (17)$$

The right-hand side is calculated by interval arithmetic, then $[H](x) \subset \mathbb{R}$ results a closed interval. Here $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}^+ \cup \{0\})^n$, $|\alpha| = \sum_{k=1}^{n} \alpha_k$, and $x^\alpha = x_1^\alpha_1 \cdots x_n^\alpha_n$ for $x = (x_1, \ldots, x_n)^T$.

Define $\overline{H}(x) := \max \{ h | h \in [H](x) \}$. If

$$\overline{H}(y) < 0 \quad (18)$$

holds for any $y \in \{ x | \| y \| = 1 \}$ with some vector norm $\| \cdot \|$, then $\overline{H}(x) < 0$ holds for any $x \in \mathbb{R}^n \setminus \{0\}$.

**Proof** Take arbitrary $c_a \in [c_a]$ and fix them. Define a homogeneous form of $2m$ th degree by

$$H(x) = \sum_{|\alpha| = 2m} c_a x^\alpha. \quad \text{Then the following holds for any } x \in \mathbb{R}^n \setminus \{0\},$$

$$\frac{1}{\|x\|^{2m}} H(x) = H \left( \frac{x}{\|x\|} \right).$$

Since the right-hand side is negative from the assumption of the lemma, we have $H(x) < 0$ for any $c_a \in [c_a]$, which indicates the conclusion of the lemma.

(QED)

We will show how to apply Lemma 4.1 to our problems by a numerical example in the section 5.

5. Numerical Example

Our example problem in case (A) is as follows.

$$x = J_i x + [x_1^2, x_2^3]^T,$$

where $dx/dt$. We take $v = x$ in this case.

According to the process described in the section 3, we have

$$Q_1 = \begin{pmatrix} 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix},$$

and

$$u^T \phi^{(3)}(u) = \frac{4}{3} u_1^3 u_2 + u_2^4 \not\in U^{(4)}.$$