Stability of the Mezard-Parisi solution for random manifolds

D. M. Carlucci

Scuola Normale Superiore di Pisa
Piazza dei Cavalieri
Pisa 56126, Italy
Internet: CARLUCCI@UX2SNS.SNS.IT

C. De Dominicis

S.Ph.T., CE Saclay
1991 Gif sur Yvette, France
Internet: CIRANO@AMOCO.SACLAY.CEA.FR

T. Temesvari

Institute for Theoretical Physics
Eötvös University
1088 Budapest, Hungary
Internet: TEMTAM@HAL9000.ELTE.HU

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Abstract

The eigenvalues of the Hessian associated with random manifolds are constructed for the general case of $R$ steps of replica symmetry breaking. For the Parisi limit $R \to \infty$ (continuum replica symmetry breaking) which is relevant for the manifold dimension $D < 2$, they are shown to be non negative.
Resumé

Les valeurs propres de la Hessienne, associée avec une variété aleatoire, sont construites dans le cas général de $R$ étapes de brisure de la symmetrie des repliques. Dans la limite de Parisi, $R \to \infty$ (brisure continue de la symmetrie des repliques) qui est pertinente pour la dimension de la variété $D < 2$, on montre qu’elles sont non negative.

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1 Introduction

The behaviour of fluctuating manifolds pinned by quenched random impurities encompasses a rich variety of physical situations, from the directed polymers in a random potential (with a manifold dimension $D = 1$) to interfacial situations ($D = d - 1$, $d$ being the total dimension of the space). The directed polymer problem, for example, is itself related to surface growth, turbulence of the Burgers equation and spin glass. The literature concerning various aspects is wide and flourishing, a good deal of it being quoted in some recent reviews\cite{1, 2}. The approaches taken have been quite diverse and it is not the purpose of this paper to describe them. This paper is a first step in an approach to random manifolds that uses field theoretic techniques in the replica formalism. Field theoretic techniques have been used with success (in their functional renormalization group variety) by Balents and Fisher\cite{3} to describe the large $N$ ($N = d - D$ is the transverse dimension) for the $4 - \epsilon$ dimensional manifold. In the context of directed polymers Hwa and Fisher\cite{4}, using more conventional field theory, have emphasized the role of large but rare fluctuations. Alternately a unified description of the general random manifold has been developed by Mézard and Parisi\cite{5}(MP), where, via the use of the replica technique, they have laid down the first step (mean field and self-consistent Hartree-Fock level) of a systematic field theory $1/N$ expansion\cite{5}. This is the approach we want to pursue here and this paper may be considered as a first sequel to MP.

In this paper we study the eigenvalues of the Hessian associated with the $1/N$ expansion as laid down by MP. We do it in a general form by keeping discrete the $R$ steps of replica symmetry breaking, and we show that the MP continuous solution ($R = \infty$, as is appropriate for $D < 2$) does not generate negative eigenvalues.

The general discrete expression derived here allows to control carefully the limiting process to the MP continuous solution. It will be also of use in further discussions of the solutions for the case $D > 2$ (and in particular $D \to 4$) an unclear case to date.

In section 2 we summarize the MP approach down to the quadratic fluctuation terms defining the Hessian (see also \cite{7}). Section 3 is devoted to the introduction of a discrete Fourier-like transform on the replica overlaps. In section 4 we briefly discuss the equation of state at mean field level. In section 5 we discuss and exhibit the replicon eigenvalues. Section 6-8 are devoted to the tougher case of the longitudinal-anomalous eigenvalues.

2 Summarizing MP approach

We thus consider the MP random manifold model defined by

\footnote{The intricacies of the first loop correction were briefly considered in further work\cite{6}}
\[ -\mathcal{H}(\vec{\omega}) = -\frac{1}{2} \int dx^D \left[ \sum_{\mu=1}^{D} \left( \frac{\partial \vec{\omega}(x)}{\partial x_\mu} \right)^2 + 2V(x, \vec{\omega}(x)) + \mu \vec{\omega}^2(x) \right] \]  

(1)

where \( d = N + D \) is the dimension of the space, \( N \) the dimension of \( \vec{\omega} \), i.e. of the tranverse space, and \( D \) the dimension of the vector \( x \), the longitudinal directions. The quenched random potential \( V \) has a gaussian distribution of zero mean and correlation

\[ V(x, \vec{\omega})(x', \vec{\omega}') = -\delta^{(D)}(x - x') N f \left( \frac{[\vec{\omega} - \vec{\omega}']^2}{N} \right) \]  

(2)

with

\[ f(y) = \frac{g}{2(1-\gamma)} (\theta + y)^{1-\gamma} \]  

(3)

where the function is regularized at small arguments by the cut-off \( \theta \).

Averaging over the random quenched variable \( V \) via replicas replaces (1) by

\[ -\mathcal{H}_{\text{rep}}(\vec{\omega}_a) = -\frac{1}{2} \int dx^D \left[ \sum_a \left\{ \sum_{\mu=1}^{D} \left( \frac{\partial \vec{\omega}_a(x)}{\partial x_\mu} \right)^2 + \mu \vec{\omega}_a^2(x) \right\} + \beta \sum_{a,b} N f \left\{ \frac{[\vec{\omega}_a(x) - \vec{\omega}_b(x)]^2}{N} \right\} \right] \]  

(4)

In studying (4) MP have used first a variational approach (identical to the Hartree-Fock approximation), which becomes exact in the large \( N \) limit. Then they introduced auxiliary fields with the advantage that it generates correction to Hartree-Fock, namely

(i) The zero loop term embodying the effect of quadratic fluctuation, i.e. the logarithm of the determinant of the Hessian matrix

(ii) \( \frac{1}{N} \) corrections in a systematic way.

In this work we are interested in (i) studying the Hessian and its eigenvalues around the mean field (Hartree-Fock or variational) solutions. We leave the study of (ii) for a separate publication.

Following MP we introduce the \( n(n+1)/2 \) component auxiliary fields

\[ r_{ab}(x) = \frac{1}{N} \vec{\omega}_a(x) \vec{\omega}_b(x) \]  

(5)

and its associated Lagrange multipliers \( s_{ab}(x) \). We need to compute the generating function

\(^2\)Note that in ref.[8] the case 1–RSB, relevant for short ranged correlations, is extensively studied both for its stability and its \( 1/N \) corrections (given in closed form).
\[ Z^n = \int \prod_{a=1}^{n} d[\bar{\omega}_a] \int \prod_{a \leq b}^{n} d[r_{ab}] d[s_{ab}] \times \]

\[ \times \exp \left\{ -\frac{\beta}{2} \sum_{ab} \int dx s_{ab}(x) \left[ N r_{ab}(x) - \bar{\omega}_a(x) \bar{\omega}_b(x) \right] \right\} \times \]

\[ \times \exp \left\{ -\frac{\beta}{2} \int dx \sum_{a=1}^{n} \left[ \sum_{\mu=1}^{D} \left( \frac{\partial \bar{\omega}_a(x)}{\partial x_{\mu}} \right)^2 + \mu \bar{\omega}_a^2(x) \right] - \frac{\beta^2}{2} \int dx \sum_{ab} N f (r_{aa}(x) + r_{bb}(x) - 2r_{ab}(x)) \right\} \]

now quadratic in \( \bar{\omega}'s \), hence after integrating over \( \bar{\omega}_a \)

\[ \tilde{Z^n} = \int \prod_{a \leq b} d[r_{ab}] d[s_{ab}] \exp \left[ N G\{r; s\} \right] \]

\[ G\{r; s\} = -\frac{\beta}{2} \sum_{ab} \int dx \left[ s_{ab}(x) r_{ab}(x) + \beta f (r_{aa}(x) + r_{bb}(x) - 2r_{ab}(x)) \right] + S[s] \]

\[ \exp S[s] = \int \prod_{a} d[\bar{\omega}_a] \exp \left\{ -\frac{\beta}{2} \sum_{ab} \int dx \left[ \bar{\omega}_a(x) \left( (-\nabla^2 + \mu) \delta_{ab} - s_{ab}(x) \right) \bar{\omega}_b(x) \right] \right\} \]

Assuming an uniform saddle point (for large N) with

\[ r_{ab}(x) = \rho_{ab} + \delta r_{ab}(x) \]

\[ s_{ab}(x) = \sigma_{ab} + \delta s_{ab}(x) \]  

one can expand around it. MP thus obtain

(i) The mean field contribution \( G^{(0)}\{r; s\} \)

(ii) The stationarity condition (vanishing of linear terms) that defines \( \rho \) and \( \sigma \) (identical to the variational answer\footnote{Only in the large \( N \) limit. If \( N \) is finite, in the variational answer \( f(y) \) is replaced by \( f(y) = \frac{1}{\Gamma(N/2)} \int_{0}^{\infty} d\alpha \alpha^{N/2-1} e^{-\alpha} f(2\alpha y/N) \) with \( f \to f \) as \( N \to \infty \).})

- Stationarity with respect to \( s_{ab}(x) \):
\[
\rho_{ab} = \frac{1}{\beta} \sum_{p} G_{ab}(p)
\]  

where

\[
(G^{-1})_{ab} = (p^2 + \mu) \delta_{ab} - \sigma_{ab}
\]

- Stationarity with respect to \( r_{ab}(x) \):

\begin{align*}
\sigma_{ab} &= 2 \beta f' \left( \rho_{aa} + \rho_{bb} - 2 \rho_{ab} \right) \\
\sigma_{aa} &= - \sum_{b \neq a} \sigma_{ab}
\end{align*}

(iii) The Hessian matrix of the quadratic form in \( \delta_{s_{ab}} \) (after integration of the quadratic terms in \( \delta_{r_{ab}} \) which in particular says that \( \delta_{s_{aa}} = \sum_{a \neq b} s_{ab} \), i.e.

\[
\mathbb{Z}^{\infty} \sim \exp \left[ NG^0 \{ \rho; \sigma \} \right] \int \prod_{a < b} d[\delta_{s_{ab}}] \exp \left\{ - \frac{1}{2} \sum_{a < b} \sum_{c < d} \delta_{s_{ab}}(p) M_{abcd}(p) \delta_{s_{cd}}(-p) \right\}
\]

\[
M_{abcd}(p) = - \frac{\delta_{ab,cd}}{2 f''(\rho_{aa} + \rho_{bb} - 2 \rho_{ab})} \sum_{q} \left\{ G_{ac}(q) + G_{bd}(q) - G_{ad}(q) - G_{bc}(q) \right\} \times \left\{ G_{ac}(p - q) + G_{bd}(p - q) - G_{ad}(p - q) - G_{bc}(p - q) \right\}
\]

3 Discrete Fourier Transform

We shall work in a formalism where the level \( R \) of replica symmetry breaking is kept discrete (\( R = 0 \) replica symmetric, \( R = 1 \) one step of breaking, \( R = \infty \) Parisi breaking, i.e. continuous overlaps). In the end, if necessary, we shall let \( R \to \infty \).

For an observable \( a_{\alpha \beta} \), with \( a_{\alpha \beta} = a_{r} \) for an overlap \( \alpha \cap \beta = r \), define the discrete Fourier transform as

\[
\hat{a}_k = \sum_{r=k}^{R+1} p_r \left( a_r - a_{r-1} \right)
\]

and its inverse

\[
a_r = \sum_{k=0}^{r} \frac{1}{p_k} (\hat{a}_k - \hat{a}_{k+1})
\]
Here the $p_r$’s are the size of the successive Parisi boxes, $p_0 \equiv n$, $p_{R+1} \equiv 1$. The above transform has the properties

$$a_r - a_{r-1} = \frac{1}{p_r}(\hat{a}_r - \hat{a}_{r+1}) \quad (17)$$

$$\sum_{\gamma} a_{\alpha \gamma} b_{\gamma \beta} = c_{\alpha \beta} \quad (18)$$

$$\hat{a}_k \hat{b}_k = \hat{c}_k \quad (19)$$

In particular

$$\hat{(a^{-1})}_k = \frac{1}{a_k} \quad (20)$$

All this is but a discrete version of the operations introduced in MP for the continuum ($R \to \infty$). Note that in the above all quantities with indices outside the range $(0, 1, \ldots, R+1)$ are to be taken identically null. If the limit $R \to \infty$ is understood as

$$\hat{a}_r \to \tilde{a}(u)$$

$$a_r \to a(u) \quad (21)$$

then the connexion with MP is, in their notations,

$$\tilde{a}(u) = \bar{a} - \langle a \rangle - [a](u)$$

$$[a](u) = - \int_0^u dt a(t) + ua(u) \quad (22)$$

4 The equation of state

With the above definitions we have

$$\rho_r = \frac{1}{\beta} \sum_{p} G_r(p) \quad (23)$$

$$\hat{G}_k(p) = \frac{1}{p^2 + \mu - \sigma_k} \quad (24)$$
and the equation of state

\[ \sigma_r = 2\beta f' \left( \frac{2}{\beta} \sum_q [G_{R+1}(q) - G_r(q)] \right) \quad r = 0, 1, \ldots, R \quad (25) \]

\[ 0 = - \sum_{t=0}^{R+1} p_t (\sigma_t - \sigma_{t-1}) \quad (26) \]

The last equation, identical to (12), also writes

\[ \hat{\sigma}_0 = \sum_{r=0}^{R+1} p_t (\sigma_t - \sigma_{t-1}) = 0 \quad (27) \]

To solve for the equation of state one uses the transform that allows to write

\[ [G_r(q) - G_{r-1}(q)] = \frac{1}{p_r} [\hat{G}_r(q) - \hat{G}_{r+1}(q)] \quad (28) \]

and

\[ y_r = \frac{2}{\beta} \sum_q [G_{R+1}(q) - G_r(q)] = \frac{2}{\beta} \sum_q \left\{ \sum_{k=r+1}^{R} \frac{1}{p_k} (\hat{\sigma}_k - \hat{\sigma}_{k+1}) \hat{G}_k(q) \hat{G}_{k+1}(q) + \hat{G}_{R+1}(q) \right\} \quad (29) \]

The equation of state then writes, with

\[ f'(y_r) = \frac{g}{2} [\theta + y_r]^{-\gamma}, \]

\[ \sigma_r = 2\beta f'(y_r) \quad (30) \]

\[ \left[ \frac{1}{g\beta} \sum_{k=0}^{r} \frac{1}{p_k} [\hat{\sigma}_k - \hat{\sigma}_{k+1}] \right]^{-\frac{1}{2}} = \theta + \frac{2}{\beta} \sum_q \left\{ \sum_{r=k+1}^{R} \frac{1}{p_k} (\hat{\sigma}_k - \hat{\sigma}_{k+1}) \hat{G}_k(q) \hat{G}_{k+1}(q) + \hat{G}_{R+1}(q) \right\}. \quad (31) \]

Here we have

\[ \hat{G}_k(q) = \frac{1}{q^2 + \mu - \hat{\sigma}_k} \quad (32) \]

where \( \mu \) is an infrared cut-off, which can be set to zero in appropriate situations. In the continuum limit
\(- \hat{\sigma}_k \rightarrow [\sigma](x)\)

\(p_k \rightarrow x\)

(in the Parisi gauge) easily restituting the MP solution as given for the case of noise with long range correlations.

For later use, let us take a one step difference in the equation of state

\[\sigma_r - \sigma_{r-1} = \frac{1}{p_r} (\hat{\sigma}_r - \hat{\sigma}_{r+1}) = 2 \beta \left[ f'(y_r) - f'(y_{r-1}) \right] \tag{33}\]

with

\[y_{r-1} - y_r = \frac{2}{\beta} \sum_q [G_r(q) - G_{r-1}(q)] = \frac{2}{\beta} \sum_q \frac{1}{p_r} [\hat{G}_r(q) - \hat{G}_{r+1}(q)] \tag{34}\]

i.e.

\[= \frac{2}{\beta} \left( \frac{\hat{\sigma}_r - \hat{\sigma}_{r+1}}{p_r} \right) \sum_q \hat{G}_r(q)\hat{G}_{r+1}(q) \tag{35}\]

For small \(y_{r-1} - y_r\) (infinitesimal in the continuum) one gets

\[1 = -4 \sum_q \hat{G}_r(q)\hat{G}_{r+1}(q) f''(y_r) \tag{36}\]

a formula used below.

## 5 Hessian Eigenvalues: the replicon sector

The Hessian \([4]\) written in terms of overlaps becomes

\[M^{ab;cd}(p) = M^{ab}_{rs}(p) \tag{37}\]

with

\[r = a \cap b \quad s = c \cap d\]

\[u = \max(a \cap c \quad a \cap d) \tag{38}\]

\[v = \max(b \cap c \quad b \cap d)\]
In ultrametric space, only three of the four overlaps are distinct. In the replicon subspace \( r = s \), and we have two independent overlaps \( u, v \geq r + 1 \). In the longitudinal anomalous sector, if \( r \neq s \), one can keep \( \max(u, v) \) to parametrize \( M \).

For the replicon sector, and on general ground\([9, 10, 11]\), the eigenvalues are given by the double discrete Fourier transform

\[
\lambda_p(r; k, l) = \sum_{u=k}^{R+1} \sum_{v=k}^{R+1} p_u p_v \left( M^r_{u,v} - M^r_{u-1,v} - M^r_{u,v-1} + M^r_{u-1,v-1} \right)
\]

(39)

for \( k, l \geq r + 1 \).

Using the Hessian expression

\[
M^r_{u,v} = -\frac{\delta^r_{u,v}}{2f''_r} - \sum_q \left[ G_u(q) + G_v(q) - 2G_r(q) \right] \left[ G_u(p-q) + G_v(p-q) - 2G_r(p-q) \right]
\]

(40)

Hence, for the double transform

\[
\lambda_p(r; k, l) = -\frac{1}{2f''_r} - \sum_q \left[ \hat{G}_k(q)\hat{G}_l(p-q) + \hat{G}_l(q)\hat{G}_k(p-q) \right]
\]

(41)

and using \([13]\), valid in particular in the continuum,

\[
\lambda_p(r; k, l) = \sum_q \left\{ 2\hat{G}_r(q)\hat{G}_{r+1}(q) - \left[ \hat{G}_k(q)\hat{G}_l(p-q) + \hat{G}_l(q)\hat{G}_k(p-q) \right] \right\}
\]

\( l, k \geq r + 1 \) (42)

The propagators \( \hat{G}_r(q) \) are decreasing functions of \( |q| \) and of \( r \) (since \(-\hat{\sigma}_r \) is an increasing function). Hence the most dangerous eigenvalue is

\[
\lambda_p(r; r + 1, r + 1) = 2 \sum_q \left\{ \hat{G}_r(q)\hat{G}_{r+1}(q) - \hat{G}_{r+1}(q)\hat{G}_{r+1}(p-q) \right\}
\]

(43)

or in the continuum limit

\[
\lambda_p(x; x, x) = 2 \frac{1}{(2\pi)^D} \int dq^D \left\{ \frac{1}{q^2 + \mu - \hat{\sigma}(x)} - \frac{1}{(p-q)^2 + \mu - \hat{\sigma}(x)} \right\} \frac{1}{q^2 + \mu - \hat{\sigma}(x)}
\]

(44)

For \( D < 4 \), one can set the ultraviolet cut-off to infinity and rewrite the dangerous replicon eigenvalues, in the continuum limit, as
\[ \lambda_p(x;x,x) = \sum_q \left[ \hat{G}_\nu(q) - \hat{G}_\nu(p-q) \right]^2 \] (45)

where the marginal stability is obviously displayed\(^4\).

### 6 Hessian eigenvalues: the longitudinal anomalous sector

The analysis of longitudinal anomalous sector is notoriously more difficult. It has been considerably simplified by the introduction\([12,10,11]\) of kernels, i.e. by going to a block-diagonal representation of the ultrametric matrices. Take in (37) a component of the longitudinal anomalous sector \(M_{r,s}^t\), \(r \neq s\), which, given that only three overlaps can be distinct, can be written\(^5\)

\[ M_{r,s}^t = \max(u;v) \]

To \(M_{r,s}^t\) is associated a kernel (in fact a Fourier transform) \(K_{k}^{r,s}\), where \(k = 0,1,2,\ldots,R+1\), \(k = 0\) corresponding to the longitudinal sector. Likewise for the inverse \(M^{-1}\) of the mass operator matrix we have a kernel \(F_{k}^{r,s}\). The relationship between them is the Dyson’s equation\([10,11]\)

\[ \hat{F}_k^{r,s} = K_k^{r,s} - \frac{1}{2} \sum_{t=0}^{R+1} K_k^{r,t} \frac{\Delta_k(t)}{\Lambda_k(t)} \hat{F}_k^{t,s} \] (46)

\[ \hat{F}_k^{r,s} = -\Lambda_k(r) F_k^{r,s} \Lambda_k(s) \] (47)

Here the \(\Lambda_k(r)\) are special replicon eigenvalues defined by

\[ \Lambda_k(r) = \begin{cases} 
\lambda(r;k,r+1) & k \geq r + 1 \\
\lambda(r;r+1,r+1) & k < r + 1 
\end{cases} \] (48)

with \(\lambda\) as in (39) and

\(^4\)It is worth mentioning at this point that the limit \(D \to 4\) is delicate to take. Indeed solving for the equation of state (30-32), one finds that the breakpoint \(x_1\) \(\dot{\sigma}(x) = 0\) for \(x > x_1\) crosses its boundary value \(x_1 = 1\), at some intermediate dimension \(D_0(\gamma)\), \(2 < D_0 < 4\). This delicate point is left out for further consideration. See also [13].

\(^5\)Except for the components \(M_{u,v}^{t,s}\) that also incorporate a replicon contribution.
The kernel $K_{r,s}^{t,k}$ itself is a transformed of the mass operator matrix $M_{r,s}^{t,k}$. It writes as a generalized discrete Fourier transform

$$K_{r,s}^{t,k} = \left( \sum_{r=k}^{r+1} + \sum_{s=0}^{s+1} + 4 \sum_{t=0}^{R+1} \right) p_t \left( M_{r,s}^{t,k} - M_{r,s}^{t,k-1} \right)$$

(50)

where we have displayed the case $k \leq r \leq s$ (for different orderings the summation $\sum_{t=k}^{R+1}$ is only kept in the allowed summands). Knowing $K_{r,s}^{t,k}$, one then computes $F_{r,s}^{t,k}$ via (46, 47) and $(M^{-1})_{r,s}^{t,k}$ via an inverse transform. That is if one is able to solve the integral equation (46) which is only possible with especially simple kernels $K_{r,s}^{t,k}$. This turned out to be the case when studying the bare propagator of the standard spin glass [10] where

$$\frac{1}{4} K_{r,s}^{t,k} = A_k(\min(r; s))$$

(51)

The same functional form also appears for the spin glass model introduced by Th.Nieuwenhuizen [14].

For the system studied here, let us construct the kernel $K_{r,s}^{t,k}$ with the matrix $M_{r,s}^{t,k}$ as given by the Hessian (14) manifestly symmetric in $r, s$. Let us choose $r < s$ and we then find

$$\frac{1}{4} K_{r,s}^{t,k}(p) = - \sum_q \sum_{t=\max(s+1,k)}^{R+1} p_t \left[ G_t(q)G_t(p-q) - G_{t-1}(q)G_{t-1}(p-q) \right] -$$

$$- G_s(q) [G_t(p-q) - G_{t-1}(p-q)] - [G_t(q) - G_{t-1}(q)] G_s(p-q)$$

(52)

which also rewrite as

$$\frac{1}{4} K_{r,s}^{t,k} = \left\{ \begin{array}{ll}
B_k(s) & r < s < k - 1 \\
B_{s+1}(s) & s \geq k - 1
\end{array} \right.$$  

(53)

where

$$B_k(s) = - \sum_q \left\{ \left( G(q)\hat{G}(p-q) \right)_k - G_s(q)\hat{G}_k(p-q) - \hat{G}_k(q)G_s(p-q) \right\}$$

(54)

The important point is that the kernel only depends on $\max(r, s)$.
\[ \frac{1}{4} K_{k}^{r,s} = B_{k}(\max(r; s)) \]  

thus rendering soluble the integral equation (46).

7 Solution of the Dyson equation: the propagator kernel

In the case of \( \frac{1}{4} K_{k}^{r,s} = A_{k}(\min(r; s)) \) the solution of the Dyson equation (46) has been given (in the continuum limit) in [10]. We wish to extend it here to the case \( \frac{1}{4} K_{k}^{r,s} = B_{k}(\max(r; s)) \).

In this section we work for a given value of \( k \), i.e. we remain in a sub-block of the longitudinal anomalous sector (\( k = 0 \) for the longitudinal).

The propagator kernel of eqs (46, 47) is obtained as

\[ \frac{1}{4} F_{k}^{r,s} = \varphi_{k}^{+}(\min(r; s))\varphi_{k}^{-}(\max(r; s)) \]

\[ \Lambda_{k}(r)\Delta_{k}(s) \]  

(56)

where \( \Delta_{k} \) is a Wronskian and \( \varphi_{k}^{\pm} \) the regular and irregular solution of a discrete Sturm-Liouville equation and \( \Lambda_{k} \) the special replicon eigenvalues of (48). Namely we have for the Wronskian

\[ \Delta_{k} = \frac{\varphi_{k}^{+}(s)D_{k}\varphi_{k}^{+}(s) - \varphi_{k}^{-}(s)D_{k}\varphi_{k}^{-}(s)}{D_{k}B_{k}(s)} \]  

(57)

with

\[ D_{k}f(s) \equiv f_{s+1} - f_{s} \]  

(58)

The functions \( \varphi^{\pm} \) are given by

\[ \frac{1}{2} \varphi_{k}^{+}(s) = 1 + \sum_{t=0}^{s} [B_{k}(t) - B_{k}(s)] \frac{\Delta_{k}(t)}{\Lambda_{k}(t)} \varphi_{k}^{+}(t) \]  

(59)

\[ \frac{1}{2} \varphi_{k}^{-}(s) = -\sum_{t=s}^{R} [B_{k}(t) - B_{k}(s)] \frac{\Delta_{k}(t)}{\Lambda_{k}(t)} \varphi_{k}^{-}(t) - B_{k}(s)C_{k} \]  

(60)

with \( B_{k}(s) \) as in (53). Note that, contrary to the solution obtained for (53), where \( C_{k} = 1 \), here, \( \varphi_{k}^{-} \) and \( \varphi_{k}^{+} \) remain coupled via \( C_{k} \)

\[ C_{k} \equiv \sum_{t=0}^{R} \frac{\Delta_{k}(t)}{\Lambda_{k}(t)} \varphi_{k}^{+}(t) \]

In terms of \( \varphi_{k}^{+} \) the Wronskian can also be written as
\[
\Delta_k = 4C_k \left[ 1 + \sum_{s=0}^{R} B_k(s) \frac{\Delta_k(s)}{\Lambda_k(s)} \phi_k^+(s) \right].
\]

8 Stability discussion in the LA sector

Given the explicit expressions (53) of \(B_k(s)\) in terms of \(\hat{G}_t\) (or \(G_t\)) itself a positive definite function, decreasing with increasing \(t\)

\[
D_t G_t = \frac{1}{p_{t+1}} \frac{\hat{\sigma}_{t+1} - \hat{\sigma}_{t+2}}{G_{t+1}^{-1} G_{t+2}^{-1}}
\]

one can easily show that

\[
(i) \quad B(r) < 0 \quad (63)
\]

\[
(ii) \quad D_r B(r) \equiv B(r+1) - B(r) > 0 \quad (64)
\]

Under these conditions we can now give a lower bound for the longitudinal anomalous eigenvalues. We can write out the determinant of the eigenvalues restricted to the sub-block \(k\):

\[
\det \left( M_{rs}^{(k)} - \lambda I \right) = \prod_{r=0}^{R} [\Lambda_k(r) - \lambda] \left[ 1 + \sum_{t=0}^{R} B_k(t) \frac{\Delta_k(t)}{\Lambda_k(t)} - \lambda \phi_k^+(t) \right]
\]

which is directly related to the Wronskian (apart from the \(C_k\) factor).

Note that, from equation (59), one also obtains

\[
D_s \phi_k^+(s) = -2D_s B_k(s) \sum_{t=0}^{R} \frac{\Delta_k(t)}{\Lambda_k(t)} - \lambda \phi_k^+(t).
\]

With the boundary value \(\phi_k^+(0) = 2\) and \(D_s B(s) > 0\), one infers that, in case of \(\lambda < \Lambda_k(t)\) for all \(t\),

\[
\phi_k^+(s) > 0 \quad (67)
\]

\[
D_s \phi_k^+(s) > 0 \quad (68)
\]

Consider now in equation (65) a small value of \(\lambda\) such that \(\lambda < \Lambda_k(t)\) for all \(t\). For that value \(\lambda\) the determinant of (65) cannot vanish, it is always positive on account of (63, 67). Hence the spectrum of the longitudinal anomalous sector is certainly bounded by the bottom values
of the replicon spectrum $\lambda_{p=0}(r; r+1, r+1)$. This establishes stability, a marginal stability as in the continuum limit where $\lambda_{p=0}(x; x, x) = 0$.

9 Conclusions

We thus have set up for random manifolds the formalism to discuss the stability of mean field solutions. For the well established continuous ($R = \infty$) solution, adequate for $D < 2$, we are able to prove its marginal stability. The formalism used here allows a discussion of any type (discrete, continuous or mixed) of mean field solutions.

It is worth noticing that, when discussing the stability, the only place where properties of the noise correlation function may play a direct role (outside of the fully continuous $R = \infty$ solutions) is in (33-36). There the equation of state is used to find the alternative analytical form for $f''$, that cast in evidence the non-negative character of the replicon eigenvalues.

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