Bidiagonal triads and the tetrahedron algebra

Darren Funk-Neubauer

Department of Mathematics and Physics, Colorado State University-Pueblo, Pueblo, Colorado, USA

ABSTRACT
We introduce a linear algebraic object called a bidiagonal triad. A bidiagonal triad consists of three diagonalizable linear transformations on a finite-dimensional vector space which satisfy the following condition. The eigenspaces of the three transformations can be ordered so that each transformation raises the eigenspaces of the other two in a block-bidiagonal fashion. We show that each of the three eigenvalue sequences associated to a bidiagonal triad satisfy the same linear recurrence relation. Based on the solutions to this recurrence we define the notion of a reduced bidiagonal triad, and show that every bidiagonal triad is equivalent to a reduced one. We show that every finite-dimensional irreducible representation of the tetrahedron Lie algebra provides numerous examples of reduced bidiagonal triads. Conversely, we show how irreducible representations of the tetrahedron algebra can be constructed starting from a certain type of reduced bidiagonal triad.

1. Introduction
In this paper we introduce a linear algebraic object called a bidiagonal triad (BD triad). We begin with some comments on the origins of this concept. BD triads arose as a way to extend and generalize the concept of a bidiagonal pair (BD pair). A BD pair consists of two diagonalizable linear transformations on a finite-dimensional vector space which satisfy the following condition. The eigenspaces of the two transformations can be ordered so that each transformation raises the eigenspaces of the other in a block-bidiagonal fashion (see Definition 2.4 for the precise definition). BD pairs have been used to study representations of various well-known algebras including \(\mathfrak{sl}_2\), \(U_q(\mathfrak{sl}_2)\), and \(U_q(\mathfrak{b}\mathfrak{sl}_2)\). BD pairs appear implicitly in [5, 12, 13, 22, 27, 36]. However, they were not explicitly defined until [14] which contains a systematic study of BD pairs including a classification theorem. The study of BD pairs revealed that each BD pair has a third transformation associated to it which also acts in a bidiagonal fashion on the eigenspaces of the original two. This led to the concept of a bidiagonal triple (BD triple). A BD triple consists of three diagonalizable linear transformations on a finite-dimensional vector space which satisfy the following condition. The eigenspaces of the three transformations can be ordered so that each transformation bidiagonally raises the eigenspaces of one of the other transformations and bidiagonally lowers the eigenspaces of the remaining transformation (see Definition 2.6 for the precise definition). A systematic study of BD triples including a classification theorem appears in [15]. See [3, 4, 29, 33, 37–41] for more references to BD pairs and triples.

A BD triad is a modification of a BD triple. As with a BD triple, a BD triad consists of three diagonalizable linear transformations on a finite-dimensional vector space. However, in a BD...
triad, the eigenspaces of the three transformations can be ordered so that each transformation bidia
diagonally raises the eigenspaces of the other two transformations (see Definition 2.9 for the pre
cise definition). Whereas a BD triple consists of three bidiagonal raising actions and three bidiagonal lowering actions, a BD triad consists of six bidiagonal raising actions. A BD triad is a more symmetric version of a BD triple in the following sense. If \( A, B, C \) denotes a BD triad then any permutation of \( A, B, C \) is also a BD triad. However, if \( A, B, C \) denotes a BD triple then only the permutations \( (A, B, C), (C, A, B), (B, C, A) \) are BD triples. The permutations \( (A, C, B), (C, B, A), (B, A, C) \) will not produce BD triples. Another key difference between a BD triad and a BD triple is as follows. For a triad the three transformations have a common eigenvector (see Note 2.11 for details). However, for a triple there does not exist a common eigenvector for all three transformations, but any two of the three have a common eigenvector (see Note 2.8 for details).

The four main results of this paper are as follows. First, we show that the three eigenvalue se
quences associated to a BD triad satisfy a linear recurrence relation (see Theorem 4.1). Based on the solutions to this recurrence we define the notion of a reduced BD triad (see Definition 4.2). In our second main result we show that reduced BD triads are canonical in the sense that every BD triad is equivalent to a reduced one (see Definition 2.15 and Theorem 4.3). Versions of Theorems 4.1 and 4.3 for BD triples appear in [15, Theorem 4.3, Theorem 4.6]. Our next two main results concern the relationship between BD triads and representations of the tetrahedron Lie algebra. In our third main result we show that every finite-dimensional irreducible module for the tetrahedron algebra provides numerous examples of reduced BD triads (see Theorem 4.5). Our fourth main result is a partial converse of Theorem 4.5. In Theorem 4.6 we show how an irreducible module for the tetrahedron algebra can be constructed starting from a certain type of reduced BD triad called thin (see Definition 2.14). The stronger non-thin version of Theorem 4.6 is false as we demonstrate with a counterexample in Section 11.

We now offer some background on the tetrahedron algebra (see Definition 3.1). The tetrahe
dron algebra, denoted \( \mathfrak{d} \), was introduced in [17]. \( \mathfrak{d} \) has essentially six generators, each of which can be identified with the six edges of a tetrahedron. In other words, each generator is indexed by a pair of vertices in the tetrahedron. The symmetric group \( S_4 \) acts on \( \mathfrak{d} \) as a group of auto
morphisms [17, Section 2]. \( \mathfrak{d} \) is isomorphic to the three-point \( \mathfrak{sl}_2 \) loop algebra [17, Theorem 11.5]. Any five of the six edges of the tetrahedron generate a subalgebra of \( \mathfrak{d} \) which is isomorphic to the \( \mathfrak{sl}_1 \) algebra [17, Corollary 12.6]. The three surrounding edges in each of the four faces of the tetrahedron form a basis for a subalgebra of \( \mathfrak{d} \) isomorphic to \( \mathfrak{sl}_2 \) [17, Corollary 12.4]. Each of the three pairs of opposite edges of the tetrahedron generate a subalgebra of \( \mathfrak{d} \) isomorphic to the Onsager algebra [17, Corollary 12.5]. \( \mathfrak{d} \) decomposes into the direct sum of these three Onsager subalgebras [17, Theorem 11.6]. For more information on the algebras mentioned in the previous five sentences see the references in [17, Section 1]. The finite-dimensional irreducible \( \mathfrak{d} \)-modules were classified in [16, Theorems 1.7, 1.8]. This classification invokes the classification of modules for the Onsager algebra (see [7–9] as well as the summary in [16, Theorems 1.3, 1.4, 1.6]). For more information on representations of the tetrahedron algebra see [2, 6, 11, 25, 32]. See [18, 30, 31] for how the tetrahedron algebra has been used in algebraic combinatorics and graph theory.

Finite-dimensional irreducible \( \mathfrak{d} \)-modules provide numerous examples of BD pairs, triples, and triads as follows. The actions of the two generators corresponding to any two edges of the tetrahedron that meet in a common vertex act as a BD pair [14, Theorem 5.10], [16, Theorem 3.8], [17, Corollary 12.4]. The actions of the three generators corresponding to the surrounding edges of each face of the tetrahedron act as a BD triple [15, Theorem 4.8], [16, Theorem 3.8], [17, Corollary 12.4]. The actions of the three generators corresponding to any three edges of the tetrahedron that meet in a common vertex act as a BD triad (see Theorem 4.5 of this paper). See also Note 10.4 of this paper. The \( q \)-tetrahedron algebra \( \mathfrak{d}_q \) is a quantum analog of the
tetrahedron algebra [21]. In a similar fashion to $\mathfrak{e}$, finite-dimensional irreducible $\mathfrak{e}_q$-modules provide numerous examples of BD pairs and triples [15, 21].

We now remark on the initial motivation for investigating BD pairs, triples, triads and the tetrahedron algebra. All of these objects originated from the study of tridiagonal pairs. A tridiagonal pair is an ordered pair of diagonalizable linear transformations on a finite-dimensional vector space, each of which acts in a tridiagonal fashion on the eigenspaces of the other (see [20, Definition 1.1] for the precise definition). Tridiagonal pairs originally arose in algebraic combinatorics through the study of a combinatorial object called a P- and Q-polynomial association scheme [20]. Tridiagonal pairs appear in a wide variety of mathematical subjects including representation theory, orthogonal polynomials, special functions, partially ordered sets, statistical mechanics, and classical mechanics. See [14, Introduction] for the appropriate references. A major classification result involving tridiagonal pairs appears in [26]. Finite-dimensional irreducible $\mathfrak{e}$-modules provide numerous examples of tridiagonal pairs. The actions of each pair of generators corresponding to opposite edges of the tetrahedron act as a tridiagonal pair [16, Theorem 1.7, Corollary 2.7]. These tridiagonal pairs are said to have Krawtchouk type [23]. See Note 10.4 for a survey of Leonard pairs. Leonard pair, has also been the subject of much research. The Leonard pairs are classified up to isomorphism [34, Theorem 1.9]. See [35] for a survey of Leonard pairs.

This paper is organized as follows. In Section 2 we recall the definitions of BD pair and triple and then define BD triad along with a number of related definitions. In Section 3 we recall the relationship between $\mathfrak{e}$ and the Lie algebra $\mathfrak{sl}_2$. Section 4 contains statements of the main results of the paper. In Section 5 we develop the tools needed to prove our main results including some additional properties of BD triads. Sections 6 through 10 contain the proofs of the main results stated in Section 4. In Section 11 we provide a counterexample to show that a stronger version of Theorem 4.6 is false.

2. Bidiagonal triads

Notation 2.1. Throughout this paper we adopt the following notation. Let $d$ denote a nonnegative integer. Let $\mathbb{K}$ denote a field. Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. For linear transformations $X : V \to V$ and $Y : V \to V$ we define $[X,Y] := XY - YX$.

Note 2.2. With reference to Notation 2.1 observe that $[X,X] = 0$ and $[X,Y] = -[Y,X]$. Also, $[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]]$. That is, for $c_1, c_2 \in \mathbb{K}$ and a linear transformation $Z : V \to V$ we have $[X, c_1 Y + c_2 Z] = c_1 [X,Y] + c_2 [X,Z]$ and $[c_1 X + c_2 Y, Z] = c_1 [X,Z] + c_2 [Y,Z]$.

We present the following lemma in order to motivate the definitions of bidiagonal pair, triple, and triad.

Lemma 2.3. [15, Lemma 2.2] With reference to Notation 2.1 the following $(i),(ii)$ hold.

(i) Suppose that there exists an ordering $\{Y_i\}_{i=0}^d$ of the eigenspaces of $Y$ with $XY_i \subseteq Y_i + Y_{i+1}$ $(0 \leq i \leq d)$, where $Y_{d+1} = 0$. Then for $0 \leq i \leq d$, the restriction $[X,Y]_{|Y_i}$ maps $Y_i$ into $Y_{i+1}$.

(ii) Suppose that there exists an ordering $\{Y_i\}_{i=0}^d$ of the eigenspaces of $Y$ with $XY_i \subseteq Y_{i-1} + Y_i$ $(0 \leq i \leq d)$, where $Y_{-1} = 0$. Then for $0 \leq i \leq d$, the restriction $[X,Y]_{|Y_i}$ maps $Y_i$ into $Y_{i-1}$. 
**Definition 2.4.** [14, Definition 2.2] A bidiagonal pair (BD pair) on $V$ is an ordered pair of linear transformations $A : V \rightarrow V$ and $A' : V \rightarrow V$ that satisfy the following three conditions.

(i) Each of $A$, $A'$ is diagonalizable.
(ii) There exists an ordering $\{ V_i \}_{i=0}^{d}$ (resp. $\{ V'_i \}_{i=0}^{D}$) of the eigenspaces of $A$ (resp. $A'$) with

\begin{align}
A' V_i &\subseteq V_i + V_{i+1} \quad (0 \leq i \leq d), \\
A V'_i &\subseteq V'_i + V'_{i+1} \quad (0 \leq i \leq D),
\end{align}

where each of $V_{d+1}$, $V'_{D+1}$ is equal to 0.

(iii) The restrictions

\begin{align}
[A', A]^{d-2i}|_{V_i} : V_i &\rightarrow V_{d-i} \quad (0 \leq i \leq d/2), \\
[A, A']^{D-2i}|_{V'_i} : V'_i &\rightarrow V'_{D-i} \quad (0 \leq i \leq D/2),
\end{align}

are bijections.

**Lemma 2.5.** [14, Lemma 2.4] With reference to Definition 2.4, we have $d = D$.

**Definition 2.6.** [15, Definition 2.5] A bidiagonal triple (BD triple) on $V$ is an ordered triple of linear transformations $A : V \rightarrow V$, $A' : V \rightarrow V$, and $A'' : V \rightarrow V$ that satisfy the following three conditions.

(i) Each of $A$, $A'$, $A''$ is diagonalizable.
(ii) There exists an ordering $\{ V_i \}_{i=0}^{d}$ (resp. $\{ V'_i \}_{i=0}^{D}$) (resp. $\{ V''_i \}_{i=0}^{\delta}$) of the eigenspaces of $A$ (resp. $A'$) (resp. $A''$) with

\begin{align}
A' V_i &\subseteq V_i + V_{i+1}, \quad A'' V_i \subseteq V_{i-1} + V_i \quad (0 \leq i \leq d), \\
A'' V'_i &\subseteq V'_i + V'_{i+1}, \quad A V'_i \subseteq V'_{i-1} + V'_i \quad (0 \leq i \leq D), \\
A V''_i &\subseteq V''_i + V''_{i+1}, \quad A' V''_i \subseteq V''_{i-1} + V''_i \quad (0 \leq i \leq \delta),
\end{align}

where each of $V_{d+1}$, $V_{-1}$, $V'_{D+1}$, $V'_{-1}$, $V''_{\delta+1}$, $V''_{-1}$ is equal to 0.

(iii) The restrictions

\begin{align}
[A', A]^{d-2i}|_{V_i} : V_i &\rightarrow V_{d-i} \quad (0 \leq i \leq d/2), \\
[A'', A']^{D-2i}|_{V'_i} : V'_i &\rightarrow V'_{D-i} \quad (0 \leq i \leq D/2), \\
[A, A'']^{\delta-2i}|_{V''_i} : V''_i &\rightarrow V''_{\delta-i} \quad (0 \leq i \leq \delta/2),
\end{align}

are bijections.

**Lemma 2.7.** [15, Lemma 2.6] With reference to Definition 2.6, we have $d = D = \delta$.

**Note 2.8.** With reference to Definition 2.6, we have $V_d = V'_0$, $V'_d = V''_0$, and $V''_d = V_0$ [15, Lemma 5.9]. Thus, in a BD triple there does not exist a common eigenvector for all three transformations, but any two of the three have a common eigenvector.

**Definition 2.9.** A bidiagonal triad (BD triad) on $V$ is an ordered triple of linear transformations $A : V \rightarrow V$, $A' : V \rightarrow V$, and $A'' : V \rightarrow V$ that satisfy the following three conditions.

(i) Each of $A$, $A'$, $A''$ is diagonalizable.
(ii) There exists an ordering $\{ V_i \}_{i=0}^{d}$ (resp. $\{ V'_i \}_{i=0}^{D}$) (resp. $\{ V''_i \}_{i=0}^{\delta}$) of the eigenspaces of $A$ (resp. $A'$) (resp. $A''$) with
A' V_i \subseteq V_i + V_{i+1}, \quad A'' V_i \subseteq V_i + V_{i+1} \quad (0 \leq i \leq d), \quad (11)

A'' V'_i \subseteq V'_i + V'_{i+1}, \quad AV'_i \subseteq V'_i + V'_{i+1} \quad (0 \leq i \leq D), \quad (12)

AV''_i \subseteq V''_i + V''_{i+1}, \quad A'V''_i \subseteq V''_i + V''_{i+1} \quad (0 \leq i \leq \delta), \quad (13)

where each of \( V_{d+1}, V'_{d+1}, V''_{\delta+1} \) is equal to 0.

(iii) The restrictions

\[ [A', A]^{d-2i}|_{V_i} : V_i \rightarrow V_{d-i}, \quad [A'', A]^{d-2i}|_{V_i} : V_i \rightarrow V_{d-i} \quad (0 \leq i \leq d/2), \quad (14)\]

\[ [A'', A]^{D-2i}|_{V'_i} : V'_i \rightarrow V'_{D-i}, \quad [A, A]^{D-2i}|_{V'_i} : V'_i \rightarrow V'_{D-i} \quad (0 \leq i \leq D/2), \quad (15)\]

\[ [A, A'']^{\delta-2i}|_{V''_i} : V''_i \rightarrow V''_{\delta-i}, \quad [A', A'']^{\delta-2i}|_{V''_i} : V''_i \rightarrow V''_{\delta-i} \quad (0 \leq i \leq \delta/2), \quad (16)\]

are bijections.

**Lemma 2.10.** With reference to Definition 2.9, we have \( d = D = \delta \).

**Proof.** Let \( A, A', A'' \) denote a BD triad. Then \( A, A' \) is a BD pair. So by Lemma 2.5 we have \( d = D \). Also, \( A', A'' \) is a BD pair. So by Lemma 2.5 we have \( D = \delta \).

**Note 2.11.** With reference to Definition 2.9, we have \( V_d = V'_d = V''_d \) [14, Lemma 6.6]. Thus, in a BD triad the three transformations have a common eigenvector.

With reference to Definition 2.9 we call \( V \) the vector space underlying \( A, A', A'' \). We say \( A, A', A'' \) is over \( \mathbb{K} \). In view of Lemma 2.10, for the remainder of this paper we use \( d \) to index the eigenspaces of \( A, A', \) and \( A'' \). We call \( d \) the diameter of \( A, A', A'' \). The diameter of a BD pair and BD triple are defined similarly.

For the remainder of this paper we assume that the field \( \mathbb{K} \) is algebraically closed and characteristic zero.

Let \( A, A', A'' \) denote a BD triad of diameter \( d \). An ordering of the eigenspaces of \( A \) (resp. \( A' \) (resp. \( A'' \)) is called standard whenever this ordering satisfies (11) (resp. (12)) (resp. (13)). Let \( \{V_i\}_{i=0}^d \) (resp. \( \{V'_i\}_{i=0}^d \) (resp. \( \{V''_i\}_{i=0}^d \)) denote a standard ordering of the eigenspaces of \( A \) (resp. \( A' \) (resp. \( A'' \)). Then no other ordering of these eigenspaces is standard, and so the BD triad \( A, A', A'' \) uniquely determines these three standard orderings. For \( 0 \leq i \leq d \), let \( \theta_i \) (resp. \( \theta'_i \)) (resp. \( \theta''_i \)) denote the eigenvalue of \( A \) (resp. \( A' \) (resp. \( A'' \)) corresponding to \( V_i \) (resp. \( V'_i \)) (resp. \( V''_i \)). We call \( \{\theta_i\}_{i=0}^d \) (resp. \( \{\theta'_i\}_{i=0}^d \) (resp. \( \{\theta''_i\}_{i=0}^d \)) the first (resp. second) (resp. third) eigenvalue sequence of \( A, A', A'' \). For the remainder of this section, we adopt the notation from this paragraph. All the terms defined in this paragraph have analogs for BD pairs and BD triples, see [14, Section 2] and [15, Section 2].

**Lemma 2.12.** For \( 0 \leq i \leq d \), the spaces \( V_{d-i}, V'_d, V''_d \) all have the same dimension.

**Proof.** Since \( A, A' \) is a BD pair then, by [14, Lemma 2.7], the spaces \( V_{d-i}, V'_d, V''_d \) all have the same dimension. Since \( A', A'' \) is a BD pair then, by [14, Lemma 2.7], the spaces \( V'_d, V''_{d-i}, V''_{d-i} \) all have the same dimension.

**Definition 2.13.** With reference to Lemma 2.12, for \( 0 \leq i \leq d \), let \( \rho_i \) denote the common dimension of \( V_{d-i}, V'_d, V''_{d-i} \). We refer to the sequence \( \{\rho_i\}_{i=0}^d \) as the shape of \( A, A', A'' \).

The shape of a BD pair and BD triple are defined similarly. See [14, Lemma 2.8] and [15, Lemma 2.9].
The following definitions will be used in stating two of our main results (see Theorem 4.6 and Theorem 4.3).

**Definition 2.14.** Let \( \{ \rho_i \}_{i=0}^d \) denote the shape of \( A, A', A'' \). We say that \( A, A', A'' \) is **thin** whenever \( \rho_i = 1 \) for \( 0 \leq i \leq d \).

Similarly, a BD pair or BD triple is called **thin** whenever its shape is \( (1, 1, 1, \ldots, 1) \).

**Definition 2.15.** Let \( \{ f_i \}_{i=0}^d \) and \( \{ g_i \}_{i=0}^d \) each denote a sequence of scalars taken from \( K \). Let \( X : V \to V \) and \( Y : V \to V \) denote linear transformations. Let \( B, B', B'' \) denote a BD triad on \( V \).

(i) We say \( f_i \) is **affine equivalent** to \( g_i \), denoted \( f_i \sim g_i \), whenever there exist \( r, s \) in \( K \) with \( r \neq 0 \) such that \( f_i = r g_i + s \) for \( 0 \leq i \leq d \).

(ii) We say \( X \) is **affine equivalent** to \( Y \), denoted \( X \sim Y \), whenever there exist \( r, s \) in \( K \) with \( r \neq 0 \) such that \( X = r Y + s I \).

(iii) We say \( A, A', A'' \) is **affine equivalent** to \( B, B', B'' \) whenever \( A \sim B, A' \sim B' \), and \( A'' \sim B'' \).

### 3. The tetrahedron algebra

**Definition 3.1.** [17, Definition 1.1] Let \( \mathfrak{g} \) denote the Lie algebra over \( K \) that has generators

\[
\{ X_{ij} \mid i, j \in \mathbb{I}, i \neq j \} \quad \mathbb{I} = \{0, 1, 2, 3\}
\]

and the following relations:

(i) For distinct \( i, j \in \mathbb{I} \),

\[
X_{ij} + X_{ji} = 0.
\]

(ii) For mutually distinct \( h, i, j \in \mathbb{I} \),

\[
[X_{hi}, X_{ij}] = 2X_{hi} + 2X_{ij}.
\]

(iii) For mutually distinct \( h, i, j, k \in \mathbb{I} \),

\[
[X_{hi}, [X_{hi}, X_{ij}]] = 4[X_{hi}, X_{ijk}].
\]

We call \( \mathfrak{g} \) the **tetrahedron algebra** and pronounce the symbol \( \mathfrak{g} \) as “tet.”

The relations in Definition 3.1(iii) are the well-known **Dolan-Grady** relations. See [1, 7–10, 28].

We now comment of how the generators of \( \mathfrak{g} \) correspond to the edges of a tetrahedron. Label the four vertices of a tetrahedron with the elements of \( \mathbb{I} \). The generator \( X_{ij} \) is identified with the directed edge of the tetrahedron pointing from vertex \( i \) to vertex \( j \).

The following definition will be used in stating two of our main results (see Theorem 4.5 and Theorem 4.6).

**Definition 3.2.** We call \( X_{ru}, X_{su}, X_{tu} \) a **corner triad** in \( \mathfrak{g} \) whenever \( X_{ru}, X_{su}, X_{tu} \) are mutually distinct generators of \( \mathfrak{g} \).

The name “corner triad” comes from the fact that the generators \( X_{ru}, X_{su}, X_{tu} \) correspond to the three edges of the tetrahedron that meet at the vertex labeled by \( u \).

The following information concerning \( \mathfrak{g} \)-modules will be useful in proving one of our main results (Theorem 4.5).
Lemma 3.3. [16, Theorem 3.8] Let $V$ denote a finite-dimensional irreducible $\mathfrak{g}$-module. Then the following (i), (ii) hold.

(i) For distinct $r, s \in I$ the generator $X_{rs}$ is diagonalizable on $V$.
(ii) There exists an integer $d \geq 0$ such that for all distinct $r, s \in I$ the eigenvalues for $X_{rs}$ on $V$ are $\{d - 2i\}_{i=0}^d$.

Definition 3.4. [16, Definition 3.9] Let $V$ denote a finite-dimensional irreducible $\mathfrak{g}$-module. By the diameter of $V$ we mean the nonnegative integer $d$ from Lemma 3.3(ii).

Notation 3.5. Let $V$ denote a finite-dimensional irreducible $\mathfrak{g}$-module of diameter $d$. For distinct $r, s \in I$ and $0 \leq i \leq d$ we let $V_{rs}(d - 2i)$ denote the eigenspace of $X_{rs}$ on $V$ corresponding to eigenvalue $d - 2i$.

Note that by Definition 3.1(i) we have

$$X_{sr} = -X_{rs},$$

and so

$$V_{rs}(d - 2i) = V_{sr}(2i - d).$$

We now discuss the Lie algebra $\mathfrak{sl}_2$ and its relationship to $\mathfrak{g}$.

Definition 3.6. Let $\mathfrak{sl}_2$ denote the Lie algebra over $\mathbb{K}$ that has a basis $h, e, f$ and Lie bracket

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$  

The following two lemmas give a description of all finite-dimensional $\mathfrak{sl}_2$-modules.

Lemma 3.7. [19, Theorem 6.3] Each finite-dimensional $\mathfrak{sl}_2$-module $V$ is completely reducible; this means that $V$ is a direct sum of irreducible $\mathfrak{sl}_2$-modules.

The finite-dimensional irreducible $\mathfrak{sl}_2$-modules are described as follows.

Lemma 3.8. [19, Theorem 7.2] There exists a family of finite-dimensional irreducible $\mathfrak{sl}_2$-modules $V(d), \quad d = 0, 1, 2, ...$

with the following properties: $V(d)$ has a basis $\{v_i\}_{i=0}^d$ such that $h.v_i = (d - 2i)v_i$ for $0 \leq i \leq d$, $e.v_i = (i + 1)v_{i+1}$ for $0 \leq i \leq d$, $f.v_i = (i - 1)v_{i-1}$ for $0 \leq i \leq d$, where $v_{d+1} = 0$, and $v_{-1} = 0$. Moreover, every finite-dimensional irreducible $\mathfrak{sl}_2$-module is isomorphic to exactly one of the modules $V(d)$.

Lemma 3.9. [14, Lemma 3.7] Let $V$ denote an $\mathfrak{sl}_2$-module with finite positive dimension. Define

$$V_{\text{even}} := \text{span}\{v \in V \mid h.v = i \cdot v, \quad i \in \mathbb{Z}, \quad i \text{ even}\},$$

$$V_{\text{odd}} := \text{span}\{v \in V \mid h.v = i \cdot v, \quad i \in \mathbb{Z}, \quad i \text{ odd}\}.$$ 

Then $V_{\text{even}}$ and $V_{\text{odd}}$ are $\mathfrak{sl}_2$-modules, and $V = V_{\text{even}} + V_{\text{odd}}$ (direct sum).

Definition 3.10. [14, Definition 3.8] Let $V$ denote an $\mathfrak{sl}_2$-module with finite positive dimension. With reference to Lemma 3.9, we say $V$ is segregated whenever $V = V_{\text{even}}$ or $V = V_{\text{odd}}$.

The following two lemmas explain the relationship between $\mathfrak{sl}_2$ and $\mathfrak{g}$.
Lemma 3.11. [17, Lemma 3.2] The Lie algebra \( \mathfrak{sl}_2 \) is isomorphic to the Lie algebra over \( \mathbb{K} \) that has basis \( X, Y, Z \) and Lie bracket
\[
[X, Y] = 2X + 2Y, \quad [Y, Z] = 2Y + 2Z, \quad [Z, X] = 2Z + 2X.
\]
An isomorphism with the presentation in Definition 3.6 is given by:
\[
X \rightarrow 2e - h, \quad Y \rightarrow -2f - h, \quad Z \rightarrow h.
\]
The inverse of this isomorphism is given by:
\[
e \rightarrow (X + Z)/2, \quad f \rightarrow -(Y + Z)/2, \quad h \rightarrow Z.
\]

Definition 3.12. [15, Definition 3.3] Let \( X, Y, Z \) denote a basis for \( \mathfrak{sl}_2 \) satisfying the relations from Lemma 3.11. We refer to the ordered triple \( X, Y, Z \) as an equitable triple in \( \mathfrak{sl}_2 \).

Lemma 3.13. [17, Proposition 3.6] Let \( h, i, j \) denote mutually distinct elements of \( \mathbb{I} \). Then there exists a unique Lie algebra homomorphism from \( \mathfrak{sl}_2 \) to \( \mathbb{K} \) that sends
\[
X \rightarrow X_{hi}, \quad Y \rightarrow X_{ij}, \quad Z \rightarrow X_{jh}.
\]

4. The main theorems

The four theorems in this section make up the main conclusions of the paper.

Theorem 4.1. Let \( A, A', A'' \) denote a BD triad on \( V \) of diameter \( d \). Let \( \{\theta_i\}_{i=0}^d \), \( \{\theta_i'\}_{i=0}^d \) and \( \{\theta_i''\}_{i=0}^d \) denote the first, second, and third eigenvalue sequences of \( A, A', A'' \) respectively. Suppose that \( d \geq 2 \). Then for \( 1 \leq i \leq d - 1 \) we have
\[
\frac{\theta_{i+1} - \theta_i}{\theta_i - \theta_{i-1}} = \frac{\theta_i'_{i+1} - \theta_i'}{\theta_i' - \theta_{i-1}'} = \frac{\theta_i''_{i+1} - \theta_i''}{\theta_i'' - \theta_{i-1}''} = 1. \tag{19}
\]

We refer to Theorem 4.1 as the recurrence theorem. The following definition will be used to state the next three theorems.

Definition 4.2. Let \( A, A', A'' \) denote a BD triad of diameter \( d \). We say \( A, A', A'' \) is reduced if the first, second, and third eigenvalue sequences of \( A, A', A'' \) are each \( \{2i - d\}_{i=0}^d \).

See [14, Definition 5.7] (resp. [15, Definition 4.5]) for the corresponding definition of a reduced BD pair (resp. BD triple).

Theorem 4.3. Every BD triad is affine equivalent to a reduced BD triad.

We refer to Theorem 4.3 as the reducibility theorem.

Note 4.4. We make the following observation in order to motivate the next two theorems. Let \( X_{ru}, X_{su}, X_{tu} \) denote a corner triad in \( \mathbb{K} \). By Definition 3.1(i),(ii) we have
\[
[X_{ru}, X_{su}] = -2X_{ru} + 2X_{su}, \quad [X_{su}, X_{tu}] = -2X_{su} + 2X_{tu}, \quad [X_{tu}, X_{ru}] = -2X_{tu} + 2X_{ru}.
\]

Let \( A, A', A'' \) denote a reduced BD triad. It is shown in Lemma 5.7 that
\[
[A, A'] = -2A + 2A', \quad [A', A''] = -2A' + 2A'', \quad [A'', A] = -2A'' + 2A.
\]

The following two theorems provide a correspondence between reduced BD triads and irreducible \( \mathbb{K} \)-modules.
Theorem 4.5. Let \( V \) denote a finite-dimensional irreducible \( \mathfrak{R} \)-module. Then each corner triad in \( \mathfrak{R} \) acts on \( V \) in a reduced BD triad.

We refer to the result in Theorem 4.5 as corner triads act as BD triads. The next theorem can be thought of as a partial converse of Theorem 4.5.

Theorem 4.6. Let \( A, A', A'' \) denote a reduced thin BD triad on \( V \). Let \( X_r, X_s, X_t \) denote a corner triad in \( \mathfrak{R} \). Then there exists a \( \mathfrak{R} \)-module structure on \( V \) such that \((X_r - A)V = 0, (X_s - A')V = 0, \) and \((X_t - A'')V = 0\). This \( \mathfrak{R} \)-module structure on \( V \) is irreducible.

We refer to the result in Theorem 4.6 as BD triads act as corner triads. Theorem 4.3 and Theorem 4.6 together show that, in the thin case, every BD triad is affine equivalent to a BD triad of the type constructed in Theorem 4.5.

The \( \mathfrak{R} \)-module constructed in Theorem 4.6 is an evaluation module for \( \mathfrak{R} \) as defined in [25, Definition 5.7]. See also [25, Proposition 6.4]. Evaluation modules are the building blocks for all finite-dimensional irreducible \( \mathfrak{R} \)-modules as described in [25, Theorem 15.2].

In Theorem 4.6 the assumption that \( A, A', A'' \) is thin is needed to guarantee the conclusion of the theorem. In Section 11 we provide a counterexample to demonstrate this.

5. Preliminaries

In this section we develop some tools needed to prove our main results.

Definition 5.1. Suppose that \( d \geq 2 \) and let \( \{\sigma_i\}_{i=0}^d \) denote a sequence of distinct scalars taken from \( K \). This sequence is called \( 1 \)-recurrent whenever

\[
\frac{\sigma_{i+1} - \sigma_i}{\sigma_i - \sigma_{i-1}} = 1 \quad (1 \leq i \leq d-1).
\]

Lemma 5.2. [15, Lemma 5.2] Suppose that \( d \geq 2 \) and let \( \{\sigma_i\}_{i=0}^d \) and \( \{\tau_i\}_{i=0}^d \) each denote a sequence of distinct scalars taken from \( K \). Assume \( \{\sigma_i\}_{i=0}^d \) and \( \{\tau_i\}_{i=0}^d \) are each \( 1 \)-recurrent. Then \( \{\sigma_i\}_{i=0}^d \sim \{\tau_i\}_{i=0}^d \).

Lemma 5.3. [15, Lemma 5.3] Suppose that \( d < 2 \) and let \( \{\sigma_i\}_{i=0}^d \) and \( \{\tau_i\}_{i=0}^d \) each denote a sequence of distinct scalars taken from \( K \). Then \( \{\sigma_i\}_{i=0}^d \sim \{\tau_i\}_{i=0}^d \).

Definition 5.4. [15, Definition 5.4] A decomposition of \( V \) is a sequence \( \{U_i\}_{i=0}^d \) consisting of nonzero subspaces of \( V \) such that \( V = \sum_{i=0}^d U_i \) (direct sum). For any decomposition of \( V \) we adopt the convention that \( U_{-1} := 0 \) and \( U_{d+1} := 0 \).

Lemma 5.5. [15, Lemma 5.5] Let \( \{U_i\}_{i=0}^d \) denote a decomposition of \( V \). Let \( \{\sigma_i\}_{i=0}^d \) and \( \{\tau_i\}_{i=0}^d \) each denote a sequence of distinct scalars taken from \( K \). Let \( X : V \rightarrow V \) (resp. \( Y : V \rightarrow V \)) denote the linear transformation such that for \( 0 \leq i \leq d \), \( U_i \) is an eigenspace for \( X \) (resp. \( Y \)) with eigenvalue \( \sigma_i \) (resp. \( \tau_i \)). Then \( \{\sigma_i\}_{i=0}^d \sim \{\tau_i\}_{i=0}^d \) if and only if \( X \sim Y \).

We now develop some more properties of BD triads.

Lemma 5.6. Let \( A, A', A'' \) denote a BD triad on \( V \) of diameter \( d \). Let \( \{\theta_i\}_{i=0}^d \), \( \{\theta'_i\}_{i=0}^d \), \( \{\theta''_i\}_{i=0}^d \) denote the first, second, third eigenvalue sequences of \( A, A', A'' \) respectively. Let \( r, s, t, u, v, w \) denote scalars in \( K \) with \( r, t, v \) each nonzero. Then \( ra + sI, ta' + uI, va'' + wI \) is a BD triad on \( V \). Moreover, \( \{r\theta_i + s\}_{i=0}^d \), \( \{t\theta'_i + u\}_{i=0}^d \), \( \{v\theta''_i + w\}_{i=0}^d \) are the first, second, third eigenvalue sequences of \( ra + sI, ta' + uI, va'' + wI \) respectively.

Proof. Imitate the proof of [14, Lemma 2.10].
**Lemma 5.7.** Let $A$, $A'$, $A''$ denote a reduced BD triad. Then the following hold.

\[
[A,A'] = -2A + 2A', \quad (20)
\]
\[
[A',A''] = -2A' + 2A'', \quad (21)
\]
\[
[A'',A] = -2A'' + 2A. \quad (22)
\]

**Proof.** We show (20). The proofs of (21) and (22) are similar. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta'_i\}_{i=0}^d$) denote the first (resp. second) eigenvalue sequence of $A$, $A'$, $A''$. By Definition 4.2 we have

\[
\theta_i = 2i - d = \theta'_i \quad (0 \leq i \leq d). \quad (23)
\]

First assume that $d = 0$. Observe that $A = \theta_0 I$ and $A' = \theta'_0 I$. By (23), $\theta_0 = 0$ and $\theta'_0 = 0$. Combining the previous two sentences we obtain (20). Now assume that $d \geq 1$. Observe that $A$, $A'$ is a BD pair. So by [14, Theorem 5.3] there exist scalars $b, x, x', y$ such that

\[
AA' - bA'A - xA - x'A' - yI = 0. \quad (24)
\]

When $d = 1$ then $b = 1$ by [14, Definition 5.5]. When $d \geq 2$ then $b = 1$ by (23) and [14, Lemma 9.1]. Combining this with [14, Theorem 8.1] and (23) we have that $x = -2, x' = 2, y = 0$. Substituting these values into (24) we obtain (20).

### 6. The proofs of the recurrence and reducibility theorems

In this section we prove Theorem 4.1 and Theorem 4.3. First we prove the recurrence theorem.

**Proof of Theorem 4.1.** Let $A$, $A'$, $A''$ denote a BD triad on $V$ of diameter $d$. Let $\{\theta_i\}_{i=0}^d$, $\{\theta'_i\}_{i=0}^d$, and $\{\theta''_i\}_{i=0}^d$ denote the first, second, and third eigenvalue sequences of $A$, $A'$, $A''$ respectively. Suppose that $d \geq 2$. For $1 \leq i \leq d - 1$ abbreviate $\frac{\theta_{i+1} - \theta_{i}}{\theta_{i} - \theta_{i-1}}$, $\frac{\theta'_{i+1} - \theta'_{i}}{\theta'_{i} - \theta'_{i-1}}$, and $\frac{\theta''_{i+1} - \theta''_{i}}{\theta''_{i} - \theta''_{i-1}}$ as $T_i$, $T'_i$, and $T''_i$ respectively. By construction $\{\theta_i\}_{i=0}^d$ is a list of distinct scalars. So $T_i \neq 0$. Similarly, $T'_i \neq 0$ and $T''_i \neq 0$. Observe that $A$, $A'$ is a BD pair. So by [14, Theorem 5.1] we have

\[
T_i = T'_i - 1 \quad (1 \leq i \leq d - 1). \quad (25)
\]

Observe that $A'$, $A''$ is a BD pair. So by [14, Theorem 5.1] we have

\[
T'_i = T''_i - 1 \quad (1 \leq i \leq d - 1). \quad (26)
\]

Observe that $A''$, $A$ is a BD pair. So by [14, Theorem 5.1] we have

\[
T''_i = T_i - 1 \quad (1 \leq i \leq d - 1). \quad (27)
\]

Combining (25), (26), (27) we have that $T_i = T''_i - 1$. So $T_i$ must equal 1 or $-1$. But $\{\theta_i\}_{i=0}^d$ is a list of distinct scalars so $T_i \neq -1$. Thus $T_i = 1$ for $1 \leq i \leq d - 1$. Similarly $T'_i = 1$ and $T''_i = 1$ for $1 \leq i \leq d - 1$.

We now prove the reducibility theorem. The following proof is essentially the same as the proof of [15, Theorem 4.6]. For the sake of completeness and accessibility we reproduce the argument here in full.

**Proof of Theorem 4.3.** Let $A$, $A'$, $A''$ denote a BD triad on $V$ of diameter $d$. Let $\{\theta_i\}_{i=0}^d$, $\{\theta'_i\}_{i=0}^d$, and $\{\theta''_i\}_{i=0}^d$ denote the first, second, and third eigenvalue sequences of $A$, $A'$, $A''$ respectively. Let $\{V_i\}_{i=0}^d$, $\{V'_i\}_{i=0}^d$, and $\{V''_i\}_{i=0}^d$ denote the standard orderings of the eigenspaces of $A$, $A'$, $A''$ respectively. First we show that
each of \( \{\theta_i\}_{i=0}^d, \{\theta'_i\}_{i=0}^d, \{\theta''_i\}_{i=0}^d \) is affine equivalent to \( \{2i - d\}_{i=0}^d. \) \( (28) \)

If \( d < 2 \) then (28) is immediate from Lemma 5.3. Now assume that \( d \geq 2 \). By Theorem 4.1, each of the sequences \( \{\theta_i\}_{i=0}^d, \{\theta'_i\}_{i=0}^d, \{\theta''_i\}_{i=0}^d \) is 1-recurrent. Observe that the sequence \( \{2i - d\}_{i=0}^d \) is also 1-recurrent. So by Lemma 5.2 we obtain (28). Let \( B : V \to V \) (resp. \( B' : V \to V \)) (resp. \( B'' : V \to V \)) be the linear transformation such that for \( 0 \leq i \leq d \), \( V_i \) (resp. \( V''_i \)) is an eigenspace for \( B \) (resp. \( B' \) (resp. \( B'' \)) with eigenvalue \( 2i - d \). From (28) and Lemma 5.5 we have \( A \sim B, A' \sim B', \) and \( A'' \sim B''. \) Combining this with Lemma 5.6 we find that \( B, B', B'' \) is a BD triad on \( V \) of diameter \( d \). Also, \( A, A', A'' \) is affine equivalent to \( B, B', B'' \). Lemma 5.5 and Lemma 5.6 also show that each of the first, second, and third eigenvalue sequences of \( B, B', B'' \) is \( \{2i - d\}_{i=0}^d \). Hence, \( B, B', B'' \) is reduced. So \( A, A', A'' \) is affine equivalent to a reduced BD triad.

\[ \square \]

7. The proof that corner triads act as BD triads

In this section we prove Theorem 4.5.

Note 7.1. Throughout the remainder of the paper we will refer to the base of a BD pair and BD triple. See [14, Definition 5.5] and [15, Definition 2.13] for the definitions of base.

Proof of Theorem 4.5. Let \( V \) denote a finite-dimensional irreducible \( \mathbb{F} \)-module of diameter \( d \). Let \( X_{tu}, X_{su}, X_{ts} \) denote a corner triad in \( \mathbb{F} \). We show \( X_{tu}, X_{su}, X_{ts} \) acts on \( V \) as a reduced BD triad. First, observe that by Lemma 3.3

\[ \text{each of } X_{tu}, X_{su}, X_{ts} \text{ is diagonalizable on } V. \] \( (29) \)

By Lemma 3.13, there exists an \( \mathfrak{sl}_2 \)-module structure on \( V \) such that \((X - X_{tu})V = 0, (Y - X_{tu})V = 0, (Z - X_{tu})V = 0 \) where \( X, Y, Z \) is an equitable triple in \( \mathfrak{sl}_2 \). We now check that this \( \mathfrak{sl}_2 \)-module structure on \( V \) is segregated. By Lemma 3.3 the action of \( X_{tu} \) on \( V \) is diagonalizable with eigenvalues \( \{d - 2i\}_{i=0}^d \). Using the \( \mathfrak{sl}_2 \) isomorphism from Lemma 3.11 we have \( (h - Z)V = 0 \). Thus, \((h - X_{tu})V = 0 \) and so the action of \( h \) on \( V \) has eigenvalues \( \{d - 2i\}_{i=0}^d \). Hence, if \( d \) is even (resp. odd) then \( V = V_{\text{even}} \) (resp. \( V = V_{\text{odd}} \). This shows the \( \mathfrak{sl}_2 \)-module structure on \( V \) is segregated. Applying [14, Theorem 5.10] we have that \( X_{tu}, X_{su} \) acts on \( V \) as a reduced BD pair of diameter \( d \) and base 1. By [14, Definition 5.7] the eigenvalue (resp. dual eigenvalue) sequence of \( X_{tu}, X_{su} \) is \( \{2i - d\}_{i=0}^d \) (resp. \( \{d - 2i\}_{i=0}^d \)). See [14, Section 2] for the definitions of eigenvalue sequence and dual eigenvalue sequence of a BD pair. From this, Definition 2.4(ii), and Notation 3.5 we have

\[
X_{tu}V_{\text{su}}(d - 2i) \subseteq V_{\text{su}}(d - 2i) + V_{\text{su}}(d - 2(i + 1)) \quad (0 \leq i \leq d),
\]

\[
X_{su}V_{tu}(2i - d) \subseteq V_{tu}(2i - d) + V_{tu}(2(i + 1) - d) \quad (0 \leq i \leq d).
\]

Combining the previous sentence with (17) and (18) we have

\[
X_{tu}V_{\text{su}}(2i - d) \subseteq V_{\text{su}}(2i - d) + V_{\text{su}}(2(i + 1) - d) \quad (0 \leq i \leq d), \quad (30)
\]

\[
X_{su}V_{tu}(2i - d) \subseteq V_{tu}(2i - d) + V_{tu}(2(i + 1) - d) \quad (0 \leq i \leq d). \quad (31)
\]

From Definition 2.4(iii) and Notation 3.5 we have

the restriction \( [X_{tu}, X_{su}]_{d-2i} : V_{\text{su}}(d - 2i) \to V_{\text{su}}(2i - d) \) is a bijection for \( 0 \leq i \leq d/2, \)
the restriction \( [X_{tu}, X_{su}]_{d-2i} : V_{\text{su}}(2i - d) \to V_{\text{tu}}(d - 2i) \) is a bijection for \( 0 \leq i \leq d/2, \) (32)

Combining the previous sentence with (17) and (18) we have

the restriction \( [X_{tu}, X_{su}]_{d-2i} : V_{\text{su}}(2i - d) \to V_{\text{su}}(d - 2i) \) is a bijection for \( 0 \leq i \leq d/2, \) (32)
the restriction \([X_{su}, X_{tu}]^{d-2i} : V_{tu}(2i-d) \to V_{tu}(d-2i)\) is a bijection for \(0 \leq i \leq d/2\). (33)

By Lemma 3.13, there exists an \(\mathfrak{sl}_2\)-module structure on \(V\) such that \((X - X_{tu})V = 0, (Y - X_{tu})V = 0\) where \(X, Y, Z\) is an equitable triple in \(\mathfrak{sl}_2\). Repeating the argument from the previous paragraph in this case gives

\[X_{su}V_{ru}(2i-d) \subseteq V_{ru}(2i-d) + V_{ru}(2(i+1)-d), \quad (0 \leq i \leq d),\] (34)

\[X_{ru}V_{su}(2i-d) \subseteq V_{su}(2i-d) + V_{su}(2(i+1)-d), \quad (0 \leq i \leq d),\] (35)

the restriction \([X_{su}, X_{tu}]^{d-2i} : V_{tu}(2i-d) \to V_{tu}(d-2i)\) is a bijection for \(0 \leq i \leq d/2\), (36)

the restriction \([X_{ru}, X_{su}]^{d-2i} : V_{su}(2i-d) \to V_{su}(d-2i)\) is a bijection for \(0 \leq i \leq d/2\). (37)

By Lemma 3.13, there exists an \(\mathfrak{sl}_2\)-module structure on \(V\) such that \((X - X_{tu})V = 0, (Y - X_{tu})V = 0\) where \(X, Y, Z\) is an equitable triple in \(\mathfrak{sl}_2\). Repeating the argument from the first paragraph again in this case gives

\[X_{tu}V_{ru}(2i-d) \subseteq V_{ru}(2i-d) + V_{ru}(2(i+1)-d), \quad (0 \leq i \leq d),\] (38)

\[X_{ru}V_{tu}(2i-d) \subseteq V_{tu}(2i-d) + V_{tu}(2(i+1)-d), \quad (0 \leq i \leq d),\] (39)

the restriction \([X_{tu}, X_{ru}]^{d-2i} : V_{ru}(2i-d) \to V_{ru}(d-2i)\) is a bijection for \(0 \leq i \leq d/2,\) (40)

the restriction \([X_{ru}, X_{tu}]^{d-2i} : V_{tu}(2i-d) \to V_{tu}(d-2i)\) is a bijection for \(0 \leq i \leq d/2\). (41)

Combining (29)–(41) we have that \(X_{tu}, X_{su}, X_{tu}\) acts on \(V\) as a BD triad. It is immediate from (30), (31), (34), (35), (38), (39) that the first, second, and third eigenvalue sequences of \(X_{tu}, X_{su}, X_{tu}\) are each \(\{2i-d\}_{i=0}^{d}\). Therefore, the BD triad \(X_{tu}, X_{su}, X_{tu}\) is reduced.

\[\square\]

8 The raising maps \(R\) and \(r\)

This section and the next two are devoted to the proof of Theorem 4.6. Throughout the remainder of the paper we will refer to the following assumption.

Assumption 8.1. Let \(A, A', A''\) denote a reduced BD triad on \(V\) of diameter \(d\). Let \(\{V_i\}_{i=0}^d, \{V'_i\}_{i=0}^d, \{V''_i\}_{i=0}^d\) denote the standard orderings of the eigenspaces of \(A, A', A''\) respectively.

Definition 8.2. With reference to Assumption 8.1, let \(R : V \to V\) and \(r : V \to V\) denote the following linear transformations:

\[R := A - A'', \quad r := A' - A''.\]

Note 8.3. With reference to Assumption 8.1 note that by Definition 2.9(iii) \(A \neq A''\), and so \(R \neq 0\). Similarly, \(A' \neq A''\) and so \(r \neq 0\). Lastly, \(A \neq A'\) and so \(R \neq r\).

We refer to the linear transformations from Definition 8.2 as raising maps. The following lemma explains this terminology.

Lemma 8.4. With reference to Definition 8.2 the following (i), (ii) hold.

(i) \(rV''_i \subseteq V''_{i+1}\) \((0 \leq i \leq d)\).

(ii) \(rV''_i \subseteq V''_{i+1}\) \((0 \leq i \leq d)\).
Proof. (i) Adopt Assumption 8.1. Observe that $A, -A''$ is a reduced BD pair of base 1. Combining this with [14, Lemma 6.7(i)] we have, for $0 \leq i \leq d$ and $v \in V''_i$,

$$[A, -A'']v = 2(A - (2i - d)I)v = 2(A - A'')v.$$  

Combining this with Lemma 8.4 and (43) we have

$$([R, r] + [R, A''] + [A'', r] + 2R - 2r) V''_i = 0.$$  

Recall for $0 \leq i \leq d$ that $V''_i$ is the eigenspace for $A''$ corresponding to the eigenvalue $2i - d$. Combining this with Lemma 8.4 and (43) we have

$$(R, r) + (2i - d)R - (2i + 2 - d)R + (2i + 2 - d)r - (2i - d)r + 2R - 2r) V''_i = 0.$$  

Simplifying this gives $(Rr - rR)V''_i = 0$ for $0 \leq i \leq d$. Since $A''$ is diagonalizable then $\{V''_i\}_{i=0}^d$ is a decomposition of $V$. Therefore, $Rr - rR = 0$ and the result follows.

Lemma 8.5. With reference to Definition 8.2 we have $Rr = rR$.

Proof. Adopt Assumption 8.1. Observe that $A, -A'$ is a reduced BD pair of base 1. Combining this with [14, Lemma 12.2] we have

$$[A, -A'] = 2A - 2A'.$$  

From Definition 8.2 we have $A = R + A''$ and $A' = r + A''$. Substituting these into (42) and simplifying gives

$$[R, r] + [R, A''] + [A'', r] + 2R - 2r = 0.$$  

Recall for $0 \leq i \leq d$ that $V''_i$ is the eigenspace for $A''$ corresponding to the eigenvalue $2i - d$. Combining this with Lemma 8.4 and (43) we have

$$(R, r) + (2i - d)R - (2i + 2 - d)R + (2i + 2 - d)r - (2i - d)r + 2R - 2r) V''_i = 0.$$  

Simplifying this gives $(Rr - rR)V''_i = 0$ for $0 \leq i \leq d$. Since $A''$ is diagonalizable then $\{V''_i\}_{i=0}^d$ is a decomposition of $V$. Therefore, $Rr - rR = 0$ and the result follows.

Lemma 8.6. Adopt Assumption 8.1 and assume that the BD triad $A, A', A''$ is thin. With reference to Definition 8.2 the following (i),(ii) hold.

(i) The restriction $R|_{V''_i} : V''_i \to V''_{i+1}$ is a bijection for $0 \leq i \leq d - 1$.

(ii) The restriction $r|_{V''_i} : V''_i \to V''_{i+1}$ is a bijection for $0 \leq i \leq d - 1$.

Proof. (i) Observe that $A, -A''$ is a reduced BD pair of base 1. Combining this with [14, Lemma 6.7(i)] and Definition 8.2 we have

$$[A, -A'']|_{V''_i} = 2R|_{V''_i} \quad (0 \leq i \leq d).$$  

Case 1: $0 \leq i < d/2$. Combining [14, Lemma 2.5] with (44) shows that the restriction $R|_{V''_i} : V''_i \to V''_{i+1}$ is an injection. Since $A, A', A''$ is thin then $\dim(V''_i) = 1 = \dim(V''_{i+1})$. Thus, the restriction $R|_{V''_i} : V''_i \to V''_{i+1}$ is also a surjection and hence a bijection.

Case 2: $d/2 \leq i \leq d - 1$. Combining [14, Lemma 2.5] with (44) shows that the restriction $R|_{V''_i} : V''_i \to V''_{i+1}$ is a surjection. Since $A, A', A''$ is thin then $\dim(V''_i) = 1 = \dim(V''_{i+1})$. Thus, the restriction $R|_{V''_i} : V''_i \to V''_{i+1}$ is also an injection and hence a bijection.

(ii) Similar to (i).

Lemma 8.7. Adopt Assumption 8.1 and assume that the BD triad $A, A', A''$ is thin. With reference to Definition 8.2 there exists a nonzero $c \in \mathbb{K}$ such that $r = cR$.

Proof. Since $A''$ is diagonalizable then $\{V''_i\}_{i=0}^d$ is a decomposition of $V$. So to prove the desired result it suffices to show
There exists a nonzero \( c \in \mathbb{K} \) such that \((r - cR)V''_i = 0\) for \( 0 \leq i \leq d \). \hspace{1cm} (45)

Recall that since \( A, A', A'' \) is thin we have

\[
\dim(V''_i) = 1 \quad 0 \leq i \leq d.
\] \hspace{1cm} (46)

We show (45) by induction on \( i \). First we show (45) holds for \( i = 0 \). Let \( v \) denote a nonzero vector in \( V''_0 \). Note that \( \{v\} \) is a basis for \( V''_0 \) by (46). Observe that \( Rw \in V''_0 \) and \( Rw \neq 0 \) by Lemma 8.6(i). Also \( rv \in V''_1 \) and \( rv \neq 0 \) by Lemma 8.6(ii). Combining the previous two sentences with (46) shows there exists a nonzero \( c \in \mathbb{K} \) such that \( rv = cRv \). This shows (45) holds for \( i = 0 \) since \( \{v\} \) is a basis for \( V''_0 \). Now let \( i \geq 1 \) and assume (45) holds for \( i \). Let \( w \) denote a nonzero vector in \( V''_i \). Note that \( Rw \in V''_{i+1} \) and \( Rw \neq 0 \) by Lemma 8.6(i). Combining this with (46) shows \( \{Rw\} \) is a basis for \( V''_{i+1} \). By Lemma 8.5 and the induction hypothesis we have \((r - cR) Rw = R(r - cR)w = 0\). This shows (45) holds for \( i + 1 \) since \( \{Rw\} \) is a basis for \( V''_{i+1} \). \( \square \)

9. The linear transformations \( B, B', B'' \)

**Lemma 9.1.** Adopt Assumption 8.1. Then there exists a linear transformation \( B : V \to V \) such that \( A', -A'', B \) is a reduced BD triple on \( V \) of base 1. Moreover, if the BD triad \( A, A', A'' \) is thin then the BD triple \( A', -A'', B \) is also thin.

**Proof.** Observe that \( A', -A'' \) is a reduced BD pair on \( V \) of base 1. So by [15, Theorem 4.1] there exists a linear transformation \( B : V \to V \) such that \( A', -A'', B \) is a BD triple on \( V \) of base 1. Moreover, \((2i - d)^d_{i=0} \) is both the first and second eigenvalue sequence of \( A', -A'', B \). Let \((\{\overline{p}_i\}^d_{i=0})\) denote the third eigenvalue sequence of \( A', -A'', B \). See [15, Section 2] for the definition of first, second, third eigenvalue sequence of a BD triple. By [15, Lemma 2.12], Lemma 5.2, and Lemma 5.3 we have \((2i - d)^d_{i=0} \sim (\{\overline{p}_i\}^d_{i=0})\). So by Definition 2.15(i) there exist \( r, s \) in \( \mathbb{K} \) with \( r \neq 0 \) such that \( 2i - d = r\overline{p}_i + s \) for \( 0 \leq i \leq d \). Define the linear transformation \( B : V \to V \) as \( B := rB + sI \). Thus, by [15, Lemma 5.7] we have that \( A', -A'', B \) is a reduced BD triple on \( V \) of base 1. Now suppose that the BD triad \( A', A'', B \) is thin. So both the BD triad \( A', A'', B \) and the BD pair \( A', -A'' \) have shape \((1,1,1,...,1)\). From this and [15, Theorem 4.1] the BD triple \( A', -A'', B \) also has shape \((1,1,1,...,1)\). Thus, the BD triple \( A', -A'', B \) is thin.

**Definition 9.2.** Let \( c \) denote the nonzero scalar in \( \mathbb{K} \) from Lemma 8.7. Define the scalar \( a \) in \( \mathbb{K} \) as \( a := 1 - c^{-1} \).

**Note 9.3.** Let \( c \) denote the scalar from Lemma 8.7, and let \( a \) denote the scalar from Definition 9.2. By Note 8.3 and Lemma 8.7 we have that \( c \neq 1 \). Thus, \( a \neq 0 \). Also, it is clear from Definition 9.2 that \( a \neq 1 \).

**Definition 9.4.** With reference to Lemma 9.1 and Definition 9.2, let \( B' : V \to V \) and \( B'' : V \to V \) denote the following linear transformations:

\[
B' := (a^{-1} - 1)^{-1}A'' + (a - 1)^{-1}B,
\] \hspace{1cm} (47)

\[
B'' := (1 - a^{-1})A' - a^{-1}B.
\] \hspace{1cm} (48)

**Lemma 9.5.** Adopt Assumption 8.1 and assume that the BD triad \( A, A', A'' \) is thin. With reference to Lemma 9.1 and Definition 9.4 the following hold.

\[
A = (1 - a)A' + aA'',
\] \hspace{1cm} (49)

\[
A' = (1 - a)^{-1}A'' + (1 - a)^{-1}A,
\] \hspace{1cm} (50)
\[ A'' = a^{-1}A + (1 - a^{-1})A', \]  
\[ B = (a - 1)B' + aA'', \]  
\[ A'' = a^{-1}B + (a^{-1} - 1)B', \]  
\[ B = -aB'' + (a - 1)A', \]  
\[ A' = (a - 1)^{-1}B + (1 - a^{-1})^{-1}B'', \]  
\[ A = (1 - a)B' - aB'', \]  
\[ B' = (a^{-1} - 1)^{-1}B'' + (1 - a)^{-1}A, \]  
\[ B'' = -a^{-1}A + (a^{-1} - 1)B'. \]  

**Proof.** By Definition 9.2 we have \( c = (1 - a)^{-1} \). Combining this with Definition 8.2 and Lemma 8.7 gives \( A' - A'' = (1 - a)^{-1}(A - A'') \). Solving this for \( A \) gives (49). Solving (49) for \( A' \) gives (50). Solving (49) for \( A'' \) gives (51). Solving (47) for \( B \) gives (52). Solving (47) for \( A'' \) gives (53). Solving (48) for \( B \) gives (54). Solving (48) for \( A' \) gives (55). Substituting (53),(55) into (49) and simplifying gives (56). Solving (56) for \( B' \) gives (57). Solving (56) for \( B'' \) gives (58).

**Lemma 9.6.** Adopt Assumption 8.1. With reference to Lemma 9.1 the following hold.

\[ [A'', B] = 2A'' - 2B, \quad (59) \]
\[ [B, A'] = 2B + 2A'. \quad (60) \]

**Proof.** Immediate from Lemma 9.1 and [15, Corollary 8.2(i)].

**Lemma 9.7.** Adopt Assumption 8.1 and assume that the BD triad \( A', A'' \) is thin. With reference to Lemma 9.1 and Definition 9.4 the following hold.

\[ [B', A''] = 2B' + 2A'', \quad (61) \]
\[ [B', B] = 2B' + 2B, \quad (62) \]
\[ [B, B''] = 2B + 2B'', \quad (63) \]
\[ [A', B''] = 2A' - 2B'', \quad (64) \]
\[ [B'', B'] = 2B'' + 2B', \quad (65) \]
\[ [B'', A] = 2B'' + 2A, \quad (66) \]
\[ [A, B'] = 2A - 2B'. \quad (67) \]

**Proof.** Substituting (52) into (59) and simplifying gives (61). Substituting (53) into (59) and simplifying gives (62). Substituting (55) into (60) and simplifying gives (63). Substituting (54) into (60) and simplifying gives (64). Substituting (53),(55) into (21) and simplifying using (62),(63) gives (65). Substituting (51),(54) into (59) and simplifying using (20),(64) gives (66). Substituting (50),(52) into (60) and simplifying using (22),(61) gives (67).

**Lemma 9.8.** Adopt Assumption 8.1 and assume that the BD triad \( A, A', A'' \) is thin. With reference to Lemma 9.1 and Definition 9.4 the following (i)--(vi) hold.
Lemma 10.2. With reference to Notation 2.1 the following (i),(ii) are equivalent.

(i) \([A, [A, [A, B]]] = 4[A, B]\).
(ii) \([B, [B, [B, A]]] = 4[B, A]\).
(iii) \([A', [A', [A', B']]] = 4[A', B']\).
(iv) \([B', [B', [B', A']]] = 4[B', A']\).
(v) \([A'', [A'', [A'', B'']]] = 4[A'', B'']\).
(vi) \([B'', [B'', [B'', A'']]] = 4[B'', A'']\).

Proof. (i) We have
\[
[A, [A, [A, B]]] = [A, [A, [A, (a - 1)B' + aA'']]] \quad \text{by (52)}
\]
\[
= [A, [A, (a - 1)(2A - 2B') + a(-2A + 2A'')]] \quad \text{by (22), (67)}
\]
\[
= [A, -2(a - 1)[A, B'] + 2a[A, A'']] \quad \text{by (22), (67)}
\]
\[
= 4(a - 1)[A, B'] + 4a[A, A''] \quad \text{by (22), (67)}
\]
\[
= 4[A, B] \quad \text{by (52)}.
\]

(ii) We have
\[
[B, [B, [B, A]]] = [B, [B, [B, (1 - a)A' + aA'']]] \quad \text{by (49)}
\]
\[
= [B, [B, (1 - a)(2B' + 2A') + a(2B - 2A'')]] \quad \text{by (59), (60)}
\]
\[
= [B, 2(1 - a)[B, A'] - 2a[B, A'']] \quad \text{by (59), (60)}
\]
\[
= 4(1 - a)[B, A'] + 4a[B, A''] \quad \text{by (59), (60)}
\]
\[
= 4[B, A] \quad \text{by (49)}.
\]

(iii),(v) Similar to (i).
(iv),(vi) Similar to (ii).

\[\Box\]

10. The proof that BD triads act as corner triads

In this section we prove Theorem 4.6.

Lemma 10.1. With reference to Notation 2.1 the following (i),(ii) are equivalent.

(i) \([X, Y] = 2X + 2Y\).
(ii) \([-Y, -X] = -2Y - 2X\).

Proof. Immediate from Note 2.2.

Lemma 10.2. With reference to Notation 2.1 the following (i)–(iv) are equivalent.

(i) \([X, [X, [X, Y]]] = 4[X, Y]\).
(ii) \([X, [X, [X, -Y]]] = 4[X, -Y]\).
(iii) \([-X, [-X, [-X, Y]]] = 4[-X, Y]\).
(iv) \([-X, [-X, [-X, -Y]]] = 4[-X, -Y]\).

Proof. Immediate from Note 2.2.

Lemma 10.3. Let \(X, Y, Z\) denote a thin BD triple on \(V\). Let \(\{Y_i\}_{i=0}^d\) denote the standard ordering of the eigenspaces of \(Y\). Then the following (i),(ii) hold.

(i) The restriction \(\left[X, Y_i\right]_{Y_i} : Y_i \rightarrow Y_{i-j}\) is a bijection for \(0 \leq i \leq d\) and \(0 \leq j \leq i\).
(ii) The restriction \(\left[Z, Y_i\right]_{Y_i} : Y_i \rightarrow Y_{i+j}\) is a bijection for \(0 \leq i \leq d\) and \(0 \leq j \leq d - i\).
\textbf{Proof.} (i) Recall by Lemma 2.3(ii) and (6) that the restriction \([X,Y]|_{Y_i}\) maps \(Y_i \to Y_{i-j}\) for \(0 \leq i \leq d\) and \(0 \leq j \leq i\). We now show that this mapping is a bijection. Since \(X, Y, Z\) is thin we have \(\dim(Y_i) = 1\) for \(0 \leq i \leq d\).

The result is trivially true for \(i = 0\). Now let \(i \geq 1\). Since the composition of two bijections is also a bijection it suffices to show that the restriction \([X,Y]|_{Y_i} : Y_i \to Y_{i-1}\) is a bijection for \(1 \leq i \leq d\). (68)

\textbf{Case 1:} \(1 \leq i \leq d/2\). First we show that the restriction \([X,Y]|_{Y_i} : Y_i \to Y_{i-1}\) is a surjection. Let \(v \in Y_{i-1}\). By (9) there exists \(w \in Y_{d-i+1}\) such that \([X,Y]\)^{d-2i+2} \(w = v\). Observe that \([X,Y]^{d-2i+1}w \in Y_i\) and \([X,Y][X,Y]^\dagger Y_i\) \(w = v\). This shows that the restriction \([X,Y]|_{Y_i} : Y_i \to Y_{i-1}\) is a surjection. From this and (68) we have (69).

\textbf{Case 2:} \(d/2 < i \leq d\). First we show that the restriction \([X,Y]|_{Y_i} : Y_i \to Y_{i-1}\) is an injection. Let \(v \in \ker([X,Y]|_{Y_i})\). So, \([X,Y]v = 0\) and thus \([X,Y]^{2i-d}v = 0\). From this and (9) we have \(v = 0\). This shows that the restriction \([X,Y]|_{Y_i} : Y_i \to Y_{i-1}\) is an injection. From this and (68) we have (69).

(ii) Similar to (i). \(\Box\)

\textbf{Proof of Theorem 4.6.} Adopt Assumption 8.1 and assume that the BD triad \(A, A', A''\) is thin. Let \(X_{rst}, X_{su}, X_{st}\) denote a corner triad in \(\mathfrak{S}\). Let \(B\) denote the linear transformation from Lemma 9.1. Let \(B', B''\) denote the linear transformations from Definition 9.4. Comparing Definition 3.1 with Lemmas 5.7, 9.6, 9.7, 9.8, 10.1, and 10.2 we find there exists a \(\mathfrak{S}\)-module structure on \(V\) such that

\[
(X_{rst} - A)V = 0, \quad (X_{su} + A)V = 0, \quad (X_{st} - A')V = 0, \quad (X_{st} + A')V = 0, \\
(X_{stu} - A'')V = 0, \quad (X_{str} + A'')V = 0, \quad (X_{st} - B)V = 0, \quad (X_{st} + B)V = 0, \\
(X_{rt} - B')V = 0, \quad (X_{tr} + B')V = 0, \quad (X_{sr} - B'')V = 0, \quad (X_{sr} + B'')V = 0.
\]

We now show that this \(\mathfrak{S}\)-module structure on \(V\) is irreducible. Let \(W\) denote a nonzero \(\mathfrak{S}\)-submodule of \(V\). We show that \(W = V\). Let \(0 \neq w \in W\). Since \(A''\) is diagonalizable then \(\{V''_{i}\}_{i=0}^{d}\) is a decomposition of \(V\). So for \(0 \leq i \leq d\) there exists \(v''_{i} \in V''_{i}\) such that \(w = \sum_{i=0}^{d} v''_{i}\). Define \(t := \min\{i \mid 0 \leq i \leq d, \ v''_{i} \neq 0\}\), and observe that

\[
w = \sum_{i=t}^{d} v''_{i}.
\]

Recall by Lemma 9.1 that \(A', -A'', B\) is a thin BD triple and note that \(\{V''_{d-i}\}_{i=0}^{d}\) is the standard ordering of the eigenspaces of \(-A''\). By Lemma 10.3(i) we have that the restriction \([A', -A'']^{d-i}|_{V''_{i}} : V''_{i} \to V''_{d-i}\) is a bijection. Abbreviate the vector \([A', -A'']^{d-i}v''_{i}\) as \(v\). Recall \(0 \neq v''_{i} \in V''_{i}\) by construction, and so

\[
0 \neq v \in V''_{d}.
\]

By Lemma 10.3(ii) we have that for \(0 \leq r \leq d\) the restriction \([B, -A'']|_{V''_{d-r}} : V''_{d-r} \to V''_{d-r}\) is a bijection. Combining this with (71) we have \(0 \neq [B, -A'']^{d-r}v \in V''_{d-r}\) for \(0 \leq r \leq d\). Since the BD triple \(A', -A'', B\) is thin then \(\dim(V''_{d-r}) = 1\) for \(0 \leq r \leq d\). Combining the previous two sentences with the fact that \(A''\) is diagonalizable shows \(\{[B, -A'']^{d-r}v\}_{r=0}^{d}\) is a basis for \(V\).

Since \(W\) is a \(\mathfrak{S}\)-submodule of \(V\) then

\[
A'W \subseteq W, \quad -A''W \subseteq W, \quad BW \subseteq W.
\]
acts on $V$ toward a contradiction, that the conclusion of Theorem 4.6 holds. That is, suppose there exists a $f$ and third eigenvalue sequences are each $\lambda_i$ such that $\sum_{i=1}^{n} \lambda_i = \lambda$. Suppose $W$ is irreducible. We make the following observations. By Theorem 4.5 each of the following

\[
A, A', A'' - A''', B', -B', -B', -A
\]

acts on $V$ as a reduced BD triad. By [15, Theorem 4.8], [16, Theorem 3.8], and [17, Corollary 12.4] each of the following

\[
A', -A'', B, A, -A', B'' A, -A'', -B' B', B, B''
\]

acts on $V$ as a reduced BD triple of base 1. From this and [15, Lemma 5.8] each of the following

\[
A', -A'' - A'', B, A', A', A, -A', -A', B'', B'' A, -A'' - B', -B', A, B, B', B'' B', B''
\]

acts on $V$ as a reduced BD pair of base 1. By [16, Theorems 1.7, 2.4] each of the following

\[
A, B A', B' A'', B''
\]

acts on $V$ as a tridiagonal pair of Krawtchouk type [23].

### 11. A counterexample

In this section we provide a counterexample to show that the thin assumption in Theorem 4.6 is needed. That is, we demonstrate that the conclusion of Theorem 4.6 can fail to hold for BD triads which are not thin. Consider the vector space $K^6$ and let $A : K^6 \to K^6, A' : K^6 \to K^6, A'' : K^6 \to K^6$ denote the linear transformations given by the following matrices:

\[
A = \begin{bmatrix}
-3 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 3 & 0 & 3
\end{bmatrix}, \quad A' = \begin{bmatrix}
-3 & 0 & 0 & 0 & 0 & 0 \\
-2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & -4 & 0 & 1 & 0 & 0 \\
0 & 0 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & -6 & 0 & 3
\end{bmatrix}, \quad A'' = \begin{bmatrix}
-3 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 3
\end{bmatrix}.
\]

It is straightforward to verify that $A, A', A''$ is a reduced BD triad on $K^6$ whose first, second, and third eigenvalue sequences are each $\{2i-3\}_i$. Notice that the shape of $A, A', A''$ is $(1, 2, 2, 1)$, and so this BD triad is not thin. Consider the corner triad $X_{03}, X_{13}, X_{23}$ in $K$. Suppose, toward a contradiction, that the conclusion of Theorem 4.6 holds. That is, suppose there exists a $K$-module structure on $K^6$ such that $(X_{03} - A)K^6 = 0, (X_{13} - A')K^6 = 0, and (X_{23} - A'')K^6 = 0$. Then the matrix representing $X_{03}$ is given by

\[
X_{03} = \begin{bmatrix}
3 & 12 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 8 & 0 & 0 \\
0 & 0 & 1 & 5 & 2 & 0 \\
0 & 0 & 0 & -1 & 0 & 4 \\
0 & 0 & 0 & 0 & -1 & 6 \\
0 & 0 & 0 & 0 & 0 & -3
\end{bmatrix}.
\]
However, direct calculation shows that
\[
[X_{13}, [X_{13}, [X_{13}, X_{02}]]] \neq 4[X_{13}, X_{02}],
\]
which contradicts Definition 3.1(iii). Thus, there does not exist a \( \mathcal{A} \)-module structure on \( \mathbb{K}^6 \) such that
\[
(X_{03} - A)\mathbb{K}^6 = 0, \ (X_{13} - A')\mathbb{K}^6 = 0, \ (X_{23} - A'')\mathbb{K}^6 = 0,
\]
and so the conclusion of Theorem 4.6 does not hold.

**Acknowledgments**

The author would like to thank Professor Paul Terwilliger for his advice and many helpful suggestions. His comments greatly improved this paper. In particular, his comment indicating the relevance of [25] to this paper was especially helpful.

**References**

[1] Ahn, C., Shigemoto, K. (1991). Onsager algebra and integrable lattice models. *Mod. Phys. Lett. A*. 06(38):3509–3515. DOI: 10.1142/S021773239100405X.

[2] Al-Najjar, H., Curtin, B. (2013). A bidiagonal and tridiagonal linear map with respect to eigenbases of equitable basis of \( \mathfrak{sl}_2 \). *Linear Multilinear Algebra*. 61(12):1668–1674.

[3] Al-Najjar, H., Curtin, B. (2020). Linear maps that act tridiagonally with respect to eigenbases of the equit-

able generators of \( U_q(\mathfrak{sl}_2) \). *Mathematics*. 8(9):1546.

[4] Baselhac, P. Freidel-Maillet type presentations of \( U_q(\mathfrak{sl}_2) \). arXiv:0807.3990, in press.

[5] Benkart, G., Terwilliger, P. (2004). Irreducible modules for the quantum affine algebra \( U_q(\mathfrak{sl}_2) \) and its Borel subalgebra. *J. Algebra*. 282:172–194.

[6] Benkart, G., Terwilliger, P. (2007). The universal central extension of the three-point \( \mathfrak{sl}_2 \) loop algebra. *Proc. Amer. Math. Soc*. 135(6):1659–1668.

[7] Date, E., Roan, S.S. (2000). The structure of quotients of the Onsager algebra by closed ideals. *J. Phys. A: Math. Gen*. 33(16):3275–3296. DOI: 10.1088/0305-4470/33/16/316.

[8] Davies, B. (1990). Onsager’s algebra and superintegrability. *J. Phys. A: Math. Gen*. 23(12):2245–2261. DOI: 10.1088/0305-4470/23/12/010.

[9] Davies, B. (1991). Onsager’s algebra and the Dolan-Grady condition in the non-self-dual case. *J. Math. Phys*. 32(11):2945–2950. DOI: 10.1063/1.529036.

[10] Dolan, L., Grady, M. (1982). Conserved charges from self-duality. *Phys. Rev. D*. 25(6):1587–1604. DOI: 10.1103/PhysRevD.25.1587.

[11] Elduque, A. (2007). The \( S_4 \)-action on the tetrahedron algebra. *Proc. Roy. Soc. Edinburgh Sect. A*. 137(6):1227–1248.

[12] Funk-Neubauer, D. (2007). Raising/lowering maps and modules for the quantum affine algebra \( U_q(\mathfrak{sl}_3) \) and its Borel subalgebra. *J. Algebra Appl*. 12(5):1250207.

[13] Funk-Neubauer, D. (2017). Bidiagonal triples. *Linear Algebra Appl*. 521:104–134. DOI: 10.1016/j.laa.2017.01.026.

[14] Hartwig, B. (2007). The tetrahedron algebra and its finite dimensional irreducible modules. *Linear Algebra Appl*. 422(1):219–235. DOI: 10.1016/j.laa.2006.09.024.

[15] Hartwig, B., Terwilliger, P. (2007). The tetrahedron algebra, the Onsager algebra, and the \( \mathfrak{sl}_3 \) loop algebra. *J. Algebra*. 308(2):840–863.

[16] Hou, B., Gao, S. (2015). Hypercube and tetrahedron algebra. *Chin. Ann. Math. Ser. B*. 36(2):293–306. DOI: 10.1007/s11401-015-0906-8.

[17] Humphreys, J. E. (1972). *Introduction to Lie Algebras and Representation Theory*. New York, NY: Springer-Verlag.

[18] Ito, T., Tanabe, K., Terwilliger, P. (2001). Some algebra related to P- and Q-polynomial association schemes. In *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, vol. 56. Providence, RI: American Mathematical Society, p. 167–192.

[19] Ito, T., Terwilliger, P. (2007). The q-tetrahedron algebra and its finite dimensional irreducible modules. *Commun. Algebra*. 35(11):3415–3439. DOI: 10.1080/00927870701509180.
[22] Ito, T., Terwilliger, P. (2007). Tridiagonal pairs and the quantum affine algebra $U_q(\mathfrak{sl}_2)$. Ramanujan J. 13(1–3):39–62.
[23] Ito, T., Terwilliger, P. (2007). Tridiagonal pairs of Krawtchouk type. Linear Algebra Appl. 427(2-3):218–233. DOI: 10.1016/j.laa.2007.07.014.
[24] Ito, T., Terwilliger, P. (2007). Two non-nilpotent linear transformations that satisfy the cubic $q$-Serre relations. J. Algebra Appl. 6(3):477–503. DOI: 10.1142/S021949880700234X.
[25] Ito, T., Terwilliger, P. (2008). Finite irreducible modules for the three-point $\mathfrak{sl}_2$ loop algebra. Commun. Algebra. 36(12):4557–4598.
[26] Ito, T., Terwilliger, P. (2009). Tridiagonal pairs of $q$-Racah type. J. Algebra. 322(1):68–93. DOI: 10.1016/j.jalgebra.2009.04.008.
[27] Klishevich, S., Plyushchay, M. (2002). Nonlinear holomorphic supersymmetry, Dolan-Grady relations and Onsager algebra. Nuclear Phys. B. 628(1–2):217–233. DOI: 10.1016/S0550-3213(02)0071-8.
[28] Lynch, A.G. The positive even subalgebra of $U_q(\mathfrak{sl}_2)$ and its finite-dimensional irreducible modules. arXiv:1506.02086, preprint.
[29] Morales, J.V. (2012). The tetrahedron algebra and Shrikhande graph. Manila J. Sci. 7(2):10–18.
[30] Morales, J.V., Pascasio, A. (2014). An action of the tetrahedron algebra on the standard module for the Hamming graphs and Doob graphs. Graphs and Combinatorics. 30(6):1513–1527. DOI: 10.1007/s00373-013-1366-0.
[31] Neher, E., Savage, A., Senesi, P. (2012). Irreducible finite-dimensional representations of equivariant map algebras. Trans. Amer. Math. Soc. 364(5):2619–2646. DOI: 10.1090/S0002-9947-2011-05420-6.
[32] Nomura, K., Terwilliger, P. (2021). Totally bipartite tridiagonal pairs. ELA. 37:434–491. DOI: 10.13001/ela.2021.5029.
[33] Terwilliger, P. (2001). Two linear transformations each tridiagonal with respect to an eigenbasis of the other. Linear Algebra Appl. 330(1–3):149–203. DOI: 10.1016/S0024-3795(01)00242-7.
[34] Terwilliger, P. (2006). An algebraic approach to the Askey scheme of orthogonal polynomials. In Orthogonal Polynomials and Special Functions, Lecture Notes in Math, vol. 1883. Berlin, Heidelberg: Springer, p. 255–330.
[35] Terwilliger, P. (2013). Finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules from the equitable point of view. Linear Algebra Appl. 439(2):358–400.
[36] Terwilliger, P. (2014). Billiard arrays and finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules. Linear Algebra Appl. 461:211–270.
[37] Terwilliger, P. (2015). Lowering-raising triples and $U_q(\mathfrak{sl}_2)$. Linear Algebra Appl. 486:1–172.
[38] Terwilliger, P. (2017). The Lusztig automorphism of $U_q(\mathfrak{sl}_2)$ from the equitable point of view. J. Algebra Appl. 16(12):1750235.
[39] Yang, Y. (2016). Upper triangular matrices and billiard arrays. Linear Algebra Appl. 493:508–536. DOI: 10.1016/j.laa.2015.12.023.
[40] Yang, Y. (2020). Some $q$-exponential formulas for finite-dimensional $\mathfrak{g}_q$-modules. Algebr. Represent. Theor. 23:467–482.