Integrable models and star structures

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Abstract. We consider the representations of Hopf algebras involved in some physical models, namely, factorizable S-matrix models (FSM’s), one-dimensional quantum spin chains (QSC’s) and statistical vertex models (SVM’s). These physical representations have definite hermiticity assignments and lead to star structures on the corresponding Hopf algebras. It turns out that for FSM’s and the quantum mechanical time-evolution of QSC’s the corresponding stars are compatible with the Hopf structures. However, in the case of statistical models the resulting star structure is not a Hopf one but what we call a twisted star. Real representations of a twisted star Hopf algebra do not close under the usual tensor product of representations. We briefly comment on the relation of these results with the Wick rotation.

1. Preliminary remarks

This paper is devoted to the study of the ∗-structures on Hopf algebras provided by certain families of physical models. The methods and presentation used in describing these systems are the ones usually employed in theoretical physics. However, we believe that the nature of the investigation and the conclusions obtained are of interest to mathematicians, and could trigger some interesting work from the mathematical side.

2. Factorizable S-matrix models and quantum groups

2.1. FSM’s. FSM’s are one of the physical models where the quantum Yang-Baxter equation first appeared in the physics literature [1]. By now there are a number of books and reviews that deal with these systems [2, 3, ...]. In this subsection we will briefly review some basic facts about them, however the reader is referred to the above quoted literature for a complete treatment of this subject.

The idea is to describe some particular scattering processes of particles moving in one spatial dimension (R). Such processes assume the existence of asymptotically free particles in the initial (time \( t \to -\infty \)) and final (\( t \to +\infty \)) states of the system. Such quantum mechanical free particle states are described by elements on a one-particle Hilbert space \( \mathcal{H}^{(1)} \). We denote the state of a free particle with rapidity \( \theta \).

\[ p = m \sinh(\theta) \]
\[ E = m \cosh(\theta) \]
\[ \theta \in \mathbb{R} \] and internal indices \( i = 1, \ldots, I \) by \( |i, \theta \rangle \). Furthermore, one assumes that the states \( |i, \theta \rangle \) form an orthonormal and complete basis of \( H^{(1)} \). The inner product in this basis given by

\[ \langle i, \theta | j, \theta' \rangle = \delta_{ij} \delta(\theta - \theta') . \]

The Hilbert space \( H^{(n)} \) for \( n \) free particles is obtained as the \( n^{th} \) tensor product of the space \( H^{(1)} \), the inner product in \( H^{(n)} \) being the extension of \( (2.2) \) to the tensor product.

It will be useful for our purely algebraic purposes to think \( H^{(1)} \) as giving an \( I \)-dimensional complex vector space \( V(\theta) \) for each value of the rapidity \( \theta \).

Let \( B_{ij}(\theta) \) be the probability amplitude\(^2\) for the scattering of 2 particles, of types \( i \) and \( j \), into two particles of types \( k \) and \( l \). If the initial states have rapidities \( \theta_1, \theta_2 \), then the scattered particles will have the same (but interchanged) rapidities, and the scattering amplitude \( B_{ij}^{kl} \) will only depend on the difference \( \theta_1 - \theta_2 \equiv \theta \), due to Poincaré invariance. Such probability amplitude corresponds to the following matrix element of the unitary evolution operator \( \hat{P} \) of the model:

\[ B_{ij}^{kl}(\theta) \delta(\theta + \theta') = \langle i, \theta | j, \theta' \rangle |k, l, \theta \rangle . \]

Factorizability of the model means that any many-particle scattering amplitude can be written as the product of two-to-two particles scattering amplitudes. Unicity of this factorization requires the following identity for the scattering of three particles:

\[ B_{12}(\theta_{23})B_{23}(\theta_{13})B_{12}(\theta_{12}) = B_{23}(\theta_{12})B_{12}(\theta_{13})B_{23}(\theta_{23}) \]

where \( \theta_{ab} = \theta_a - \theta_b \) are the rapidity differences by pairs of the three particles involved in the equality \((2.4)\). Hence they are not independent, and satisfy

\[ \theta_{12} = \theta_{13} - \theta_{23} . \]

The subindices of \( B \) denote its action on the vector space \( V(\theta_1) \otimes V(\theta_2) \otimes V(\theta_3) \). We define the \( R \)-matrix by

\[ R(\mu) = B(\mu)P , \]

where \( P \) is the permutation operator. Equation \((2.4)\) can be rewritten in terms of \( R \) as

\[ R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu) , \]

where \( \lambda = \theta_{13} , \mu = \theta_{12} \). This last equation, or also Eq. \((2.4)\), is called the Quantum Yang-Baxter equation with spectral parameter. All of the above comes directly from the theory of FSM’s and nothing about QG’s is employed.

### 2.2. QG’s with spectral parameter

Consider a set of elements \( T^i_j(\lambda) \), for every value of the parameter \( \lambda \in \mathbb{C} \) and \( i, j = 1, \cdots, I \). The free algebra they generate can be turned into a bialgebra \( \mathcal{F}_n \) by defining the coproduct as \( \Delta T^i_j(\lambda) = T^k_j(\lambda) \otimes T^i_k(\lambda) \) and the counit as \( \epsilon(T^i_j(\lambda)) = \delta^i_j \). Let us now consider a collection of

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\( \text{Note that Eqs. } (2.1) \) are simply a parametrization of the relativistic energy-momentum relation \( E = \sqrt{p^2 + m^2} \).

\( \text{This means that the probability of such a process is given by the modulus squared of the probability amplitude } B_{ij}^{kl}(\theta). \)
vector spaces \( V(\lambda) \) for every \( \lambda \in \mathbb{C} \), with o.n. basis \( \{ e_i(\lambda), i = 1, \ldots, I \} \), and take the linear map

\[
B(\lambda - \mu) : V(\lambda) \otimes V(\mu) \rightarrow V(\mu) \otimes V(\lambda)
\]

\( (2.8) \)

\[
e_i(\lambda) \otimes e_j(\mu) \rightarrow B_{ij}^{kl}(\lambda - \mu) e_k(\mu) \otimes e_l(\lambda),
\]

to be a spectral parameter dependent solution of the Yang-Baxter equation \( (2.4) \).

Replacing \( (2.10) \) into \( (2.9) \) we reobtain \( (2.7) \), showing that it is indeed a representation.

Moreover, all the \( T^k_j(\lambda) \) commute with the scattering matrix \( \hat{S} \), as operators on the \( n \)-particle Hilbert space given by the (tensor product) representation \( \rho \):

\[
\rho([T^k_j(\lambda)]) \hat{S} = \hat{S} \rho([T^k_j(\lambda)]), \quad \forall \lambda, i, j
\]

This corresponds, physically, to an additional scattering with a particle of rapidity \( \lambda \) and (in, out) indices \( i, j \).

2.3. Relation between FSM’s and QQ G’s with spectral parameter. Using the 2-2 \( S \) matrix of \( (2.3) \) we can build the following representation of the bialgebra of the previous subsection:

\[
\rho : \mathcal{F} \rightarrow T(V(\lambda))
\]

\( (2.10) \)

\[
\rho(T^k_j(\lambda)) = R_{ij}^{kl}(\lambda).
\]

Replacing \( (2.10) \) into \( (2.9) \) we reobtain \( (2.7) \), showing that it is indeed a representation.

Note that from the point of view of FSM’s, this representation \( (2.10) \) is such that

\[
R_{ij}^{kl}(\theta) \delta(\theta + \theta') = B_{ij}^{kl}(\theta) \delta(\theta + \theta') = \langle i, j, \theta | \hat{S} | l, k, \theta' \rangle.
\]

Moreover, all the \( T^k_j(\lambda) \) commute with the scattering matrix \( \hat{S} \), as operators on the \( n \)-particle Hilbert space given by the (tensor product) representation \( \rho \):

\[
\rho([T^k_j(\lambda)]) \hat{S} = \hat{S} \rho([T^k_j(\lambda)]), \quad \forall \lambda, i, j
\]

This corresponds, physically, to an additional scattering with a particle of rapidity \( \lambda \) and (in, out) indices \( i, j \).

Comodule algebra and Zamolodchikov’s operators Let us consider the quantum linear space defined by the quadratic algebra given by the quotient

\[
A = \frac{\langle V(\lambda) \rangle}{R}
\]

of the free algebra \( \langle V(\lambda) \rangle \) by the relation

\[
R : \cup_{\lambda, \mu} [1 \otimes 1 - B(\lambda - \mu)] V(\lambda) \otimes V(\mu).
\]

An \( \mathcal{F} \)-comodule structure on \( V(\lambda) \) is given by the coaction \( \delta_{V(\lambda)} e_i(\lambda) = T^i_j(\lambda) \otimes e_j(\lambda) \).

The definition of the coaction on a tensor product \( V(\lambda) \otimes V(\mu) \) is

\[
\delta_{V(\lambda) \otimes V(\mu)} = (m \otimes I_{V(\lambda)} \otimes I_{V(\mu)}) (I_{\mathcal{F}^*} \otimes \tau \otimes I_{V(\mu)}) (\delta_{V(\lambda)} \otimes \delta_{V(\mu)}),
\]

where \( \tau \) is the flip operator. \( A \) is the algebra obeyed by Zamolodchikov’s operators \( \mathcal{F} \), or the also called spectral parameter dependent quantum plane algebra. Note that this comodule action preserves the quadratic relations of the algebra \( A \), i.e.,

\[
(I_{\mathcal{F}^*} \otimes R) \delta_{V \otimes V} = \delta_{V \otimes V} \circ R.
\]
\[ (B^1)^{ik}_{lj} = R^{il}_{jk} = [\rho(T_j^i)^\dagger]^k_i , \]

we may rewrite (2.12) as

\[ B^1_{ij}(\theta) (B^1)^{mn}_{lk}(\theta) = \delta^m_i \delta^n_j = (B^1)^{ik}_{lj}(\theta) B^1_{mn}^{ij}(\theta) . \]

Using Eq. (2.10), and considering that

\[ \rho(\hat{T}_j^i(\theta)) = \delta^i_j \]

we have the notion of the adjoint of an operator. The physical requirement of unitarity of the \( \hat{S} \) matrix operator together with above mentioned definition of the adjoint leads to a unitary \( B \) matrix. Furthermore, if one assumes \( \rho \) to be a star representation of the bialgebra \( \mathcal{F} \) with a star structure, then one can determine this star on \( \mathcal{F} \). From the discussion above, we have

\[ (2.12) \quad B^1_{ij}(\theta) (B^1)^{mn}_{lk}(\theta) = \delta^m_i \delta^n_j = (B^1)^{ik}_{lj}(\theta) B^1_{mn}^{ij}(\theta) . \]

These last equalities can be fulfilled if

\[ T^k_i(\theta) [T^k_j(\theta)]^* = \delta^i_j \quad \delta^{ij} \mathbf{1} \]

Therefore the algebraic structure provided by this formulation of the FSM's is the one involved in the following proposition.

**Proposition 1.** Let \( A \) denote the \(*\)-bialgebra generated by \( 1 \) and the \( T^i_j(\theta) \)

\[ (i, j = 1, \cdots, N; \ \theta \in \mathbb{R}) \]

satisfying the relations

\[ (2.15) \quad R^p_{ij}(\lambda - \mu) T^k_i(\lambda) T^p_j(\mu) = T^k_j(\mu) T^p_i(\lambda) R^p_{ki}(\lambda - \mu) \]

\[ (2.16) \quad T^k_i(\theta) [T^k_j(\theta)]^* = \delta^i_j \mathbf{1} \]

\[ (2.17) \quad [T^k_i(\theta)]^* T^k_j(\theta) = \delta^{ij} \mathbf{1} , \]

where \( R \) is a solution of the Yang-Baxter equation (2.7). The coproduct and counit are taken to be \(*\)-homomorphisms,

\[ \Delta(a^*) = [\Delta a]^* \]

\[ \epsilon(a^*) = \overline{\epsilon(a)} , \quad a \in A \]

given on the generators by

\[ (2.18) \quad \Delta(T^i_j(\theta)) = T^i_k(\theta) \otimes T^k_j(\theta) , \quad \Delta(1) = 1 \otimes 1 \]

\[ \epsilon(T^i_j(\theta)) = \delta^i_j \]

\[ \epsilon(1) = 1 . \]

For the star structure on the tensor product we consider two possibilities,

i) \((a \otimes b)^* = a^* \otimes b^* ; \quad a, b \in A \) (Hopf star),

ii) \((a \otimes b)^* = b^* \otimes a^* ; \quad a, b \in A \) (twisted star).

Furthermore, we consider a comodule \( V(\theta) \) \((\dim(V(\theta)) = I)\) for each value of \( \theta \). Let \( \{ e_i(\theta) : \quad i = 1, \cdots, I \} \) be a basis of \( V(\theta) \), and define a right coaction on \( V(\theta) \) by

\[ (2.19) \quad \delta e^i(\theta) = e^j(\theta) \otimes T^i_j(\theta) . \]

Then:

\[ \delta e^i(\theta) = e^j(\theta) \otimes T^i_j(\theta) . \]
1. The linear antihomomorphism $S$ defined on the generators by
\begin{equation}
S(T^i_j(\theta)) = [T^i_j(\theta)]^* , \quad S(1) = 1 , \tag{2.20}
\end{equation}
is an antipode for the algebra $A$, which therefore has a Hopf algebra structure.

2. As $A$ is non-cocommutative ($N \geq 2$), only possibility (i) for the star closes without requiring additional relations.

3. The scalar product on $V = \oplus \theta V(\theta)$ determined by
\begin{equation}
(e^i(\theta), e^j(\theta')) = \delta^i_j \delta(\theta - \theta'), \tag{2.21}
\end{equation}
is such that the coaction (2.19) induces a $*$-representation of the dual $\tilde{A}$ of $A$ on $V$. This scalar product is invariant under the action of $\tilde{A}$ in the sense of reference [6].

We refer the reader to [6] for a detailed discussion of twisted and Hopf stars and some of their properties. The proof of this Proposition is an easy calculation. For point (1) it suffices to show that $S$ satisfies the axioms of the antipode on the generators, which involves essentially relations (2.16) and (2.17). Regarding statement (2), applying $\Delta$ to (2.16) and assuming $*$ to be a Hopf star, one sees that the obtained relation is automatically satisfied. On the other hand, in the twisted star case new (additional) relations between the $T^i_j$’s need to be true in order to have a consistent $*$-bialgebra. The proof of (3) is a simple calculation that again involves in an essential way relations (2.16) and (2.17).

In fact, the proof of point (1) can be reversed and used to show that if one starts with an $RTT$ Hopf algebra (an $RTT$ bialgebra as above plus an antipode $S$), a Hopf star $*$ satisfying (2.16) and (2.17) can be immediately defined as an antilinear antimorphism by
\begin{equation}
[T^i_j(\theta)]^* = S(T^i_j(\theta)) , \quad 1^* = 1 . \tag{2.23}
\end{equation}
Hence, we conclude that at least for a certain class of FSM’s (for instance, all those described by an $R$-matrix of the $GL_q(N)$ type) the real structures on the underlying Hopf algebras are Hopf stars. In other words, the combined requirement of having a quantum group symmetry and the physical unitarity of the system (plus some naturalness assumptions, as the star representation condition) restrict the $RTT$ bialgebra to be a true Hopf algebra. An example of a field theoretic model that leads to a factorizable $S$-matrix is given by the sine-Gordon model. The scattering of solitonic and antisolitonic asymptotic states in this model is described by a factorizable and unitary $S$-matrix [2, 3], associated to a quantum group of $SL_q$ type.

3. Integrable quantum spin chains. The $XXZ$-model

In this section we consider the case of integrable one dimensional spin chains. All the reasoning will be exemplified by considering the $XXZ$-model, however most of the steps are quite general and apply to any one dimensional spin chain. We are interested in the representation of the underlying quantum group that this kind of
models provide. We choose to make this representation explicit by obtaining a two dimensional classical vertex model out of the quantum spin chain. Such relation between \( d \)-dimensional quantum systems and \((d+1)\)-dimensional classical systems is quite general \([7]\) and is essentially what is often referred in physics as the path integral formulation of quantum mechanics \([8]\).

Let us consider a linear lattice of \( N \) sites labelled by an index \( k = 1, \cdots, N \). To each site \( k \) of this lattice we associate a complex \( n \)-dimensional vector space \( H(k) \) (\( n = 2 \) for the spin \( 1/2 \) XXZ case). In each of these spaces \( H(k) \) we consider an irreducible \( n \)-dimensional representation \( \sigma_a \) of the generators of the Lie algebra of \( SU(2) \). We choose them so as to satisfy the following algebraic relations:

\[
\begin{align*}
[\sigma_a, \sigma_b] &= 2i \epsilon_{abc} \sigma_c, \\
\{\sigma_a, \sigma_b\} &= 2 \delta_{ab}, \quad \text{for the spin 1/2 case}
\end{align*}
\]

where \( a, b, c = 1, 2, 3 \), \( \epsilon_{abc} \) is the totally antisymmetric tensor with \( \epsilon_{123} = 1 \), and \( \{\cdot, \cdot\} \) stands for the anticommutator. The total vector space of the chain is taken to be \( H = \bigotimes_k H(k) \). We define spin operators acting on this space by \( \sigma_a(k) = 1 \otimes \cdots \otimes 1 \otimes \sigma_a \otimes 1 \otimes \cdots \otimes 1 \). The Hamiltonian is also an operator acting on \( H \). For the case of the spin 1/2 XXZ-model, it is given by

\[
H = \sum_{k=1}^{N} H_{k,k+1},
\]

\[
H_{k,k+1} = \sigma_1(k) \sigma_1(k+1) + \sigma_2(k) \sigma_2(k+1) + J \sigma_3(k) \sigma_3(k+1).
\]

In the above formula for \( H \) we impose periodic boundary conditions, i.e.,

\[
\sigma_a(N+1) = \sigma_a(1), \quad \forall a.
\]

Note that \([H_{k,k+1}, H_{j,j+1}] \neq 0 \) only if \( j = k+1 \) or \( k-1 \).

The quantity of physical interest is the operator

\[
U = \exp \left[ -z H \right].
\]

If we are doing quantum mechanics, we take \( z = it \) (\( t \in \mathbb{R} \) is the time) and \( U \) is called the time-evolution operator. If we are doing statistical mechanics of the QSC, instead, we take \( z = \beta = 1/(k_B T) \) the inverse temperature (\( z \in \mathbb{R} \)), and \( U \) would be the quantum Boltzmann operator of the chain. Using the parameter \( z \) we can analyse simultaneously both cases. Now we may rewrite \( U \) using Trotter’s formula, which is valid for bounded operators \([9]\),

\[
\exp \sum_i A_i = \lim_{L \to \infty} \left[ \prod_i \exp \frac{1}{L} A_i \right]^L.
\]

Hence we have

\[
U = \lim_{L \to \infty} T(\epsilon)^L,
\]
where $T(\epsilon)$ is the transfer matrix

\begin{equation}
T(\epsilon) = \prod_{k=1}^{N} B_{k,k+1} \tag{3.8}
\end{equation}

\begin{equation}
B_{k,k+1} = \exp[-\epsilon H_{k,k+1}] \quad \text{with} \quad \epsilon = \frac{z}{L} . \tag{3.9}
\end{equation}

Therefore we are led to the evaluation of matrix elements of the transfer matrix. In order to do so we choose an orthonormal basis of the $H$ that we denote $\{|j\rangle, j = 1, \cdots, n\}$. Having a basis for each $H(k)$, we construct a basis $\{|i_1, \cdots, i_N\rangle\}$ of $H$ by taking the tensor product, $|i_1, \cdots, i_N\rangle = |i_1\rangle \otimes \cdots \otimes |i_N\rangle$. Note that the operators $B_{k,k+1}$ only act non-trivially on the states of sites $k$ and $k+1$. Thus the only factor of $T(\epsilon)$ that acts non-trivially on the vector space corresponding to site $k$ is the product of operators $B_{k-1,k} B_{k,k+1}$.

We consider next the matrix element

\begin{equation}
\langle i_1, \cdots, i_N|T(\epsilon)|j_1, \cdots, j_N \rangle = \langle i_1, \cdots, i_N\rangle \prod_{k=1}^{N} B_{k,k+1} \langle j_1, \cdots, j_N \rangle . \tag{3.10}
\end{equation}

One could now introduce an identity of the form $1 = \sum_{p_k} |p_k\rangle \langle p_k|$ between the operators $B_{k-1,k} B_{k,k+1}$. However, doing so would break explicit translational invariance along the chain, because the sites 1 and $N$ are treated differently. This problem can be solved if one remarks that Trotter’s formula (3.6) would still be valid if we keep only first order terms in $1/L$ inside the square brackets. This means that only $O(\epsilon)$ terms matter in (3.8). Considering this, it is easy to see that the matrix element (3.10) of the infinitesimal “evolution” (in time or temperature) operator may be expanded in a translation-invariant way as

\begin{equation}
\langle i_1, \cdots, i_N|T(\epsilon)|j_1, \cdots, j_N \rangle = B_{i_j}^{0,k} \langle i, j|B(\epsilon)|k, l \rangle , \tag{3.11}
\end{equation}

Here we made use of the notation

\begin{equation}
B_{i_j}^{kl}(\epsilon) \equiv \langle i, j|B(\epsilon)|k, l \rangle , \tag{3.12}
\end{equation}

for each (any) pair of particles. In fact, the $O(\epsilon^2)$ terms in (3.11) become irrelevant in the $N \to \infty$ limit, as they are associated to the chosen “endpoints” of the chain.

Being $P$ the permutation map on $H(k) \otimes H(k+1)$, we introduce the operator $R(\epsilon) = B(\epsilon) P$, so

\begin{equation}
R_{ij} = B_{ij}^{kl} \tag{3.13}
\end{equation}

Now (3.11) reads

\begin{equation}
\langle i_1, \cdots, i_N|T(\epsilon)|j_1, \cdots, j_N \rangle = R_{i_j}^{0,k} \langle i, j|P(\epsilon)|k, l \rangle \tag{3.14}
\end{equation}

\begin{equation}
\quad \langle j_1, \cdots, j_N|P(\epsilon)|i_1, \cdots, i_N \rangle + O(\epsilon^2) .
\end{equation}

If we now assume that this $R$-matrix satisfies Yang-Baxter equation (this will be proved later for the spin 1/2 XXZ model), we may use it to define an RTT bialgebra through equation (2.9) as in section 2.2. As before, the $R$-matrix itself is a representation of this bialgebra,

\begin{equation}
\langle i|T_{a}^{b}(\lambda)|j \rangle = \rho(T_{a}^{b}(\lambda))_{i}^{j} = R_{a1}^{bj}(\lambda) . \tag{3.15}
\end{equation}
Therefore, the transfer matrix of the spin model can be written in terms of the trace $T^a_a(\lambda) \equiv \sum_a T^a_a(\lambda)$ (trace over the auxiliary space) of the $T^a_a$ operators:

\begin{equation}
\langle i_1, \cdots, i_N | T | j_1, \cdots, j_N \rangle = \langle i_2, \cdots, i_N, i_1 | \rho(T^{p_2}_{p_1}) \rho(T^{p_3}_{p_2}) \cdots \rho(T^{p_N}_{p_{N-1}}) \rho(T^{p_1}_{p_N}) | j_1, \cdots, j_N \rangle = \langle i_1, \cdots, i_N | C^\dagger \rho \otimes (T^p_p) \rho \otimes (T^p_{p_1}) \cdots \rho \otimes (T^p_{p_{N-1}}) \rho \otimes (T^p_{p_N}) | j_1, \cdots, j_N \rangle .
\end{equation}

Here $C$ is the (unitary) cyclic permutation operator,

\[ C | j_1, j_2, \cdots, j_N \rangle = | j_2, \cdots, j_N, j_1 \rangle . \]

An easy calculation tells us that\[^6\]

\[ [C, T^a_a(\mu)] = 0 , \ \forall \mu , \]

which is not true for arbitrary $T^b_a$’s. As the $T^a_a(\mu)$ commute for different values of the parameter, this implies that we have an infinite set (parametrized by $\mu$) of conserved quantities:

\[ [U(z), T^a_a(\mu)] = 0 , \ \forall \mu . \]

Coming back to the $R$ matrix, we will now show that in the spin 1/2 case of the XXZ-model, it can be chosen as a solution of the YB equation. We are interested only in the first order terms in $R$. Using (3.9) we easily get

\[ ||R||^{|k_l|}_{ij}(\epsilon) = \begin{pmatrix} 1 - J\epsilon & 0 & 0 & 0 \\ 0 & 1 + J\epsilon & 0 & 0 \\ 0 & 0 & -2\epsilon & 0 \\ 0 & 0 & 0 & 1 - J\epsilon \end{pmatrix} + O(\epsilon^2) . \]

This matrix is of the “six-vertex model” type[3]. Solutions of this form to the Yang-Baxter equation (2.7) exist and can be parametrized as

\[ ||R_\alpha||^{|k_l|}_{ij}(u) = \begin{pmatrix} \sinh(u + \alpha) & 0 & 0 & 0 \\ 0 & \sinh(u) & 0 & 0 \\ 0 & 0 & \sinh(\alpha) & 0 \\ 0 & 0 & 0 & \sinh(u + \alpha) \end{pmatrix} . \]

Remark now that the YB equation is not reparametrization invariant, and $\epsilon$ could not be the “right” parameter to get a given $R$ matrix to satisfy (2.7). Note also that if $R(\lambda)$ is a solution of the Yang-Baxter equation then $f(\lambda)R(\lambda)$ is also a solution for any scalar function $f$ of the parameter $\lambda$.

Taking into account the above remarks, we see that the matrices $R$ and $R_\alpha$ of equations (3.17) and (3.18) coincide, up to a global factor and $O(\epsilon^2)$ terms, if we take

\[ \epsilon(u) = \frac{1}{2} \frac{\sinh u}{\sinh(\alpha)} \]

\[ J = \cosh \alpha . \]

If we are considering the quantum mechanical time-evolution of the QSC, remembering that $\epsilon = it/L \in i \mathbb{R}$ and $J \in \mathbb{R}$ we see that $u \in \mathbb{R}$. In the statistical case we have $u \in \mathbb{R}$ instead.

\[^6\]We have dropped the $\rho$’s from the formulas.
3.1. Stars. For the quantum mechanical time-evolution of any quantum integrable spin chain the $B$-matrix we obtain is unitary, since it is given by matrix elements in a Hilbert space of the unitary operator $\exp(-iH/L)$, cf. equation (3.9). Therefore, we could repeat exactly the same argument of section 2.4 and conclude that it is natural to endow the $RTT$ bialgebra introduced in (3.15) with a $\Delta$-compatible (untwisted) star operation $\ast$. This is so in the cases where the $R$-matrix allows the existence of an antipode for the bialgebra, as shown by Proposition 1.

If we do a statistical study of the same QSC, now $z = \beta \in \mathbb{R}$, and the operator $B$ is hermitian. The analysis of star operations for this case will be done in the next section, where we will consider SVM’s.

4. Statistical vertex models

We now consider a (classical) statistical vertex model defined for an $N \times M$ square lattice ($N$ “horizontal” sites and $M$ “vertical” ones, and periodic boundary conditions). Each vertex in the lattice has a Boltzmann weight $R_{ij}^{kl}(\beta) = \exp[-\beta E(ij; kl)]$ associated to it, corresponding to the energy of the configuration $(i, j, k, l)$ of the four links converging to the vertex and to an inverse temperature $\beta$. This matrix $R$ of thermal weights is evidently real, and would also be symmetric ($R_{ij}^{kl} = R_{ij}^{kl}$) if we assume the energy of each configuration to be invariant under a diagonal reflection of the vertex. This happens in the so called six and eight vertex models. As we are talking about integrable systems, we also require $R$ to satisfy YB equation (2.7).

We refer the reader to a standard reference such as [3] for a detailed analysis of these vertex models. However, here we need at least to sketch the way of relating them to Quantum Groups to be able to introduce the physically relevant stars.

The basic quantity in a statistical system is always the partition function, $Z = \sum_{\text{configurations } c} \exp[-\beta E(c)]$, as it encodes all the physical information of the system. In this particular case $Z$ may be rewritten as $Z = \sum_{\text{configurations } c} \prod_{\text{vertices } v} R(c[v])$.

Here we can group the Boltzmann weights by rows to build transfer matrices (compare with Eq.(3.14) . . .)

\[ t_{j_1, j_2, \ldots, j_N}^{i_1, \ldots, i_N} = R_{b_{1i_1}}^{b_{1j_1}} R_{b_{2i_2}}^{b_{2j_2}} \cdots R_{b_{Nj_N}}^{b_{Ni_N}}, \]

which may then be thought as matrix elements on an $N$-site Hilbert space of a self adjoint operator (assuming that $R$ is a real, symmetric matrix). Moreover, using $R$ we can introduce an $RTT$ bialgebra exactly as in the previous sections, and so the transfer matrices (14) happen to be matrix elements of $T^a_\alpha$ in an $N$th tensor product representation. In fact, what we have done in the section about QSC’s, was to rewrite the evolution of the quantum system in terms of transfer matrices (4.1) of a (classical) statistical system . . . Here all the same formulas apply, except that now the partition function does not include the cyclic permutation operators $C$ that we previously found in $U$, and that $\epsilon = \beta \in \mathbb{R}$.
As a consequence, if we assume the representation given by Eq. (3.15) to be a star representation of a \( \ast \)-algebra defined by Eq. (2.9), then we can determine this star by writing

\[
\langle i | \rho [ T_b^a (\mu) ] | j \rangle = R_{bi}^{aj}(\mu) = \langle i | \rho [(T_b^a (\mu))^\ast] | j \rangle , \quad \forall i, j .
\]

From this we obtain

\[
[T_b^a (\mu)]^\ast = T_a^b (\mu) ,
\]

as a sufficiency condition. Therefore, the SVM's characterized by a symmetric \( R \) matrix fit in the hypothesis of the following proposition.

**Proposition 2.** Under the hypothesis of Proposition 1 but replacing (2.16) and (2.17) by

\[
[T_b^a (\theta)]^\ast = T_a^b (\theta) ,
\]

we have:

1. As the bialgebra \( A \) is non-cocommutative for \( N \geq 2 \), only possibility (ii) (twisted star) for the \( \ast \)-structure on \( A \otimes A \) is consistent.
2. The following scalar product on \( V = \bigotimes_\theta V(\theta) \) fixed by

\[
(e^i(\theta), e^j(\theta')) = \delta^{ij}(\theta - \theta')
\]

is such that the coaction (2.19) induces a \( \ast \)-representation of the dual \( \hat{A} \) of \( A \) on \( V \).

Hence we conclude that for SVM’s (with a symmetric \( R \)) or for any model with an hermitian \( R \)-matrix (as is the case of statistical QSC’s with a Hamiltonian symmetric in neighboring sites\(^7\)), the star operation on the underlying Hopf algebra is a twisted star.

It is interesting to note that if one performs a Wick rotation \((\epsilon \rightarrow i\epsilon)\) on this statistical system then one obtains an unitary \( R \) matrix that corresponds, as we have seen for the FSM’s, to a Hopf star.

### 4.1. Stars in \( U_q(sl(2)) \) and the star in SVM’s

The star defined by (4.3) is not a Hopf star, but a twisted star instead, for the bialgebra defined by Eq. (2.9) and the coproduct (2.18). We have a classification of the Hopf stars in \( U_q(sl(2)) \) but we do not have a classification of the stars of the bialgebras defined by relations (2.9). However, both are related. For the case of the XXZ-model this relation is given in [10]. The \( R \)-matrix of this model is given by (3.18), or, changing variables \((u = i\gamma \delta, \alpha = i\gamma)\) and basis, by

\[
R(\lambda) = i \begin{pmatrix}
\sin \gamma (\delta + 1) & 0 & 0 & 0 \\
0 & \sin \gamma \delta & \exp(-i\gamma \delta) \sin \gamma & 0 \\
0 & \exp(i\gamma \delta) \sin \gamma & \sin \gamma \delta & 0 \\
0 & 0 & 0 & \sin \gamma (\delta + 1)
\end{pmatrix},
\]

where \( i\delta = \lambda, q = \exp i\gamma \).

Defining the operator valued matrices

\[
L_+ = q^{\frac{1}{2}} \begin{pmatrix}
k^\frac{1}{2} & (q - q^{-1}) x^- \\
0 & k^{-\frac{1}{2}}
\end{pmatrix}, \quad L_- = q^{-\frac{1}{2}} \begin{pmatrix}
k^{-\frac{1}{2}} & 0 \\
-(q - q^{-1}) x^+ & k^\frac{1}{2}
\end{pmatrix},
\]

\(^7\) Even if the \( R_{ij}^{kl} \) matrix is always hermitian, this is not true in general for \( R_{ij}^{kl} \) unless the property \( R_{ij}^{kl} = R_{ij}^{lk} \) holds.
we build using them the $T(\lambda)$ operators:

\begin{equation}
T(\lambda) = e^{\lambda \gamma} L_+ - e^{-\lambda \gamma} L_-
\end{equation}

Replacing (4.8) into (2.7) you get an identity if $x_+, x_-$ and $k$ satisfy the algebraic relations of $U_q(sl(2))$. The $R$ matrix (4.6) gives a representation of this algebra: comparing (4.7) with (4.6) it is given by

\begin{equation}
\begin{align*}
x_- &= \begin{pmatrix} 0 & 0 \\ q^{-\frac{1}{2}} & 0 \end{pmatrix}, & x_+ &= \begin{pmatrix} 0 & q^\frac{1}{2} \\ 0 & 0 \end{pmatrix}, & k &= \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.
\end{align*}
\end{equation}

Taking this to be a star representation of $U_q(sl(2))$ leads to the following twisted star structure:

\begin{equation}
\begin{align*}
x_+^* &= x_-, & k^* &= k^{-1}.
\end{align*}
\end{equation}

Indeed it is very simple, to show that for $|q| = 1$ the Hopf star $x_+^* = x_+, x_-^* = x_-, k^* = k$ can not be implemented in this example. This is so since there is no representation of the algebra $U_q(sl(2))$ by $2 \times 2$ hermitian matrices. Recall that we have a positive definite inner product (the one associated to the quantum version of this model), in the vector space where the linear operators $x_+, x_-$ and $k$ act.

5. Concluding remarks

The star structures of Hopf algebras appearing in physics depend on the model, and are not necessarily Hopf stars. We have seen that for factorizable scattering models and quantum mechanics of QSC there is a compatibility with the Hopf structure. However, for statistical vertex models and statistical physics of QSC this is not the case, and a twisted star is obtained. As we already mentioned, this difference can be traced back to the Wick rotation \[11\] that connects quantum mechanics with statistical physics\[8\]. Twisted stars have different properties than Hopf stars \[6\]; in particular they do not form a tensorial category with respect to the usual tensor product. However, the present analysis shows their physical relevance in the field of integrable models.

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8 In brief, consider the following operator for a quantum mechanical system: $U(z) = \exp(-zH)$ where $H$ is the Hamiltonian of the system (assumed to be time independent) and $z$ a complex number. If you restrict $z$ to the imaginary axis, then $U(z)$ is the quantum mechanical evolution operator. If you “rotate” (Wick rotation) the $z$ variable to the positive real axis you get the central object of quantum statistical mechanics, the quantum Boltzmann weight $Z = \exp(-\beta H)$.
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