A strongly indefinite Choquard equation with critical exponent due to the Hardy-Littlewood-Sobolev inequality

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Abstract
In this paper we are concerned with the following nonlinear Choquard equation

\[-\Delta u + V(x)u = \left( \int_{\mathbb{R}^N} \frac{G(y, u)}{|x-y|^\mu} dy \right) g(x, u) \quad \text{in} \quad \mathbb{R}^N,\]

where \(N \geq 4, 0 < \mu < N\) and \(G(x, u) = \int_0^u g(x, s)ds\). If 0 lies in a gap of the spectrum of 
\(-\Delta + V\) and \(g(x, u)\) is of critical growth due to the Hardy-Littlewood-Sobolev inequality, we
obtain the existence of nontrivial solutions by variational methods. The main result here extends
and complements the earlier theorems obtained in [1,17,25].

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indefinite problem.

1 Introduction and main results

In this article we are going to study a nonlocal equation of the form

\[
\begin{cases}
-\Delta u + V(x)u = \left( \int_{\mathbb{R}^N} \frac{G(y, u)}{|x-y|^\mu} dy \right) g(x, u) & \text{in} \quad \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]

(1.1)

where \(N \geq 4, 0 < \mu < N\) and \(G(x, u) = \int_0^u g(x, s)ds\). This type of nonlocal equation is closely related
to the Choquard-Pekar equation

\[- \Delta u + u = \left( \int_{\mathbb{R}^N} \frac{|u(y)|^p}{|x-y|^\mu} dy \right) |u|^{p-2} u \quad \text{in} \quad \mathbb{R}^N.
\]

(1.2)

For \(N = 3, p = 2\) and \(\mu = 1\), the study of equation (1.2) goes back to the work [28] by S. Pekar
in 1954, there the author used the equation to describe a polaron at rest in the quantum theory. In
1976, to model an electron trapped in its own hole, P. Choquard considered equation (1.2) as a certain
approximation to Hartree-Fock theory of one component plasma, see [20]. In some particular cases,
equation (1.2) is also known as the Schrödinger-Newton equation which was introduced by R. Penrose
in [29] to investigate the selfgravitational collapse of a quantum mechanical wave function.

Mathematically, for \(N = 3, p = 2\) and \(\mu = 1\), the existence of ground states of equation (1.2) was
obtained in [20,22] by variational methods. Involving the qualitative properties of the ground stats,

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the uniqueness was proved in [20] and the nondegeneracy was established in [18, 33]. For equation (1.2) with general \( p \) and \( \mu \), the regularity, positivity, radial symmetry and decay property of the ground states were proved in [11, 23, 24].

To study equation (1.1) variationally, we will use the following Hardy-Littlewood-Sobolev inequality frequently, see [21].

**Proposition 1.1.** Let \( t, r > 1 \) and \( 0 < \mu < N \) satisfying \( 1/t + \mu/N + 1/r = 2 \), \( f \in L^r(\mathbb{R}^N) \) and \( h \in L^t(\mathbb{R}^N) \). There exists a sharp constant \( C(t, N, \mu, r) \), independent of \( f, h \), such that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} \, dx \, dy \leq C(t, N, \mu, r) |f||h|. \tag{1.3}
\]

If \( t = r = 2N/(2N - \mu) \), then there is equality in (1.3) if and only if \( f \equiv (\text{const})h \) and

\[
h(x) = A(\gamma^2 + |x - a|^2)^{-(2N - \mu)/2}
\]

for some \( A \in \mathbb{C}, 0 \neq \gamma \in \mathbb{R} \) and \( a \in \mathbb{R}^N \).

Notice that, by the Hardy-Littlewood-Sobolev inequality and the Sobolev imbedding, the integral

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^t |u(y)|^t}{|x-y|^\mu} \, dx \, dy
\]

is well defined if

\[
\frac{2N - \mu}{N} \leq t \leq \frac{2N - \mu}{N - 2}.
\]

In this way we call \( \frac{2N - \mu}{N} \) the lower critical exponent and \( 2^*_\mu = \frac{2N - \mu}{N - 2} \) the upper critical exponent. We refer the readers to [2, 4, 5, 12, 16, 20] and the references therein for recent progress on the study of the subcritical Choquard equation. The critical problem for the Choquard equation is an interesting topic and has attracted a lot of attention recently. The lower critical exponent case was studied in [27], some existence and nonexistence results were established under suitable assumptions on the potential \( V(x) \). For the upper critical exponent case, a critical Choquard type equation on a bounded domain of \( \mathbb{R}^N, N \geq 3 \) was investigated in [14, 15], where the authors generalized the well-known results obtained in [4, 8]. If the problem was set on the whole plane, a critical Choquard equation in the sense of the Trudinger-Moser inequality was considered in [2]. We need to make a further remark about the critical Choquard equation. Consider the Choquard equation with constant coefficient \( \lambda \) and pure critical term

\[
-\Delta u + \lambda u = \left( \int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} \, dy \right) |u|^{2^*_\mu - 2} u \quad \text{in} \quad \mathbb{R}^N. \tag{1.4}
\]

Similar to the observation made in [3] for the local case, we find by following the steps in [14] that a solution \( u \) of (1.4) satisfies the Pohozaev type identity

\[
\frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{\lambda N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx = \frac{2N - \mu}{2} \cdot 2^*_\mu \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^\mu} \, dx \, dy.
\]

Then we can deduce that

\[
\lambda \int_{\mathbb{R}^N} |u|^2 \, dx = 0
\]

and proves that there are no nontrivial solutions with \( \lambda \neq 0 \). Therefore the existence of solutions for the Choquard equation with upper critical exponent in \( \mathbb{R}^N, N \geq 3 \), is an interesting problem.

If the potential \( V(x) \) is a continuous periodic function, the spectrum of the Schrödinger operator \(-\Delta + V\) is purely continuous and consists of a union of closed intervals. If \( \inf_{\mathbb{R}^N} V(x) > 0 \) and \( \frac{2N - \mu}{N} \leq p < \frac{2N - \mu}{N - 2} \), since the energy functional is invariant under translation, the existence of ground
states by applying the Mountain Pass Theorem, see [11] for example. If \( V(x) \) changes sign, the operator 
\(-\Delta + V\) has essential spectrum below 0 and then equation (1.1) becomes strongly indefinite. In contrast 
to the positive definite case, it becomes more complicated to study the strongly indefinite Choquard equation 
due to the appearance of convolution part. For \( p = 2 \) and \( \mu = 1 \), the existence of one nontrivial solution was obtained in [7] by reduction arguments. For a general class of subcritical 
Choquard type equation

\[ -\Delta u + V(x)u = \left( \int_{\mathbb{R}^N} W(x-y)|u(y)|^p dy \right) |u|^{p-2}u \quad \text{in} \quad \mathbb{R}^3, \tag{1.5} \]

the existence of solutions was obtained in [11] by applying a generalized linking theorem in [32], where \( W(x) > 0 \) belongs to a wide class of functions. The author also proved the existence of infinitely many 
geometrically distinct weak solutions by applying an abstract critical point theorem established in [5].

Since the Choquard equation equipped with nonlocal type nonlinearities can be regarded as a 
generalization of the local Schrödinger equation, we would also like to refer the readers to [1, 3, 10, 13, 14, 17, 18, 19, 30, 31, 32, 34, 35] for the study of the periodic Schrödinger equation with critical or 
subcritical local nonlinearities. Among them, by supposing that 0 lies in a gap of the spectrum of the operator 
\(-\Delta + V\), in [9] the authors considered

\[ -\Delta u + V(x)u = K(x)|u|^{2^*_\mu - 2}u + f(x, u) \quad \text{in} \quad \mathbb{R}^N \tag{1.6} \]

and obtained the existence of nontrivial solutions for equation (1.6) by careful energy estimates and 
linking arguments.

Inspired by [1, 3, 9], the aim of the present paper is to study the existence of nontrivial solutions 
for the critical Choquard equation

\[
\begin{cases}
-\Delta u + V(x)u = \left( \int_{\mathbb{R}^N} \frac{K(y)|u(y)|^{2^*_\mu} + F(y, u)}{|x-y|^{\mu}} dy \right) \left( K(x)|u|^{2^*_\mu - 2}u + \frac{1}{2^*_\mu} f(x, u) \right) \quad \text{in} \quad \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\tag{1.7}
\]

where \( N \geq 4, \ 0 < \mu < 4 \), \( F(x, u) = \int_0^u f(x, s)ds \) and \( 2^*_\mu = (2N - \mu)/(N - 2) \) is the upper critical 
exponent in the sense of the Hardy-Littlewood-Sobolev inequality. We suppose that the functions 
\( V(x), K(x) \) and \( f(x, u) \) satisfy the following assumptions:

(K1) \( V, K \in C(\mathbb{R}^N), f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}), K(x) > 0 \) in \( \mathbb{R}^N \) and \( V, K, f \) are 1-periodic in \( x_j \) for 
\( j = 1, \cdots, N; \)

(K2) There exist \( \frac{2N-\mu}{N} < q \leq p < 2^*_\mu \) and \( c > 0 \) such that for all \( (x, u) \in \mathbb{R}^N \times \mathbb{R}; \ |f(x, u)| \leq c(|u|^{q-1} + |u|^{p-1}); \)

(K3) There exists \( \theta > 1 \) such that: for every \( u \neq 0, 0 < \theta F(x, u) \leq uf(x, u) \) on \( \mathbb{R}^N \times \mathbb{R}; \)

(K4) \( 0 \notin \sigma(-\Delta + V) \) and \( \sigma(-\Delta + V) \cap (-\infty, 0) \neq \emptyset \), where \( \sigma \) denotes the spectrum in \( L^2(\mathbb{R}^N) \).

Remark 1.2. (i) Let \( G(x, u) = K(x)|u|^{2^*_\mu} + F(x, u) \) and \( g(x, u) = \partial_u G(x, u) \), in some parts of the 
paper equation (1.7) are written as

\[
\begin{cases}
-\Delta u + V(x)u = \frac{1}{2^*_\mu} \left( \int_{\mathbb{R}^N} \frac{G(y, u)}{|x-y|^{\mu}} dy \right) g(x, u) \quad \text{in} \quad \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N).
\end{cases}
\tag{1.8}
\]

Assumption (K3) implies the existence of constant \( \theta > 1 \) such that the Ambrosetti-Rabinowitz condition 
for nonlocal problem holds:
0 < \theta G(x, u) \leq ug(x, u) \text{ on } \mathbb{R}^N \times \mathbb{R}, \text{ for every } u \neq 0.

(ii) For the local Schrödinger equation, instead of assumption \((K_3)\), the authors in [9] introduced the assumption:

\[ 0 \leq 2F(x, u) \leq uf(x, u) \text{ on } \mathbb{R}^N \times \mathbb{R}. \]

However, for the nonlocal Choquard equation, a similar assumption

\[ (K'_3) \quad 0 \leq F(x, u) \leq uf(x, u) \text{ on } \mathbb{R}^N \times \mathbb{R} \]

is not enough to ensure the boundedness of the \((PS)\) sequences. And so, we exclude the case of \(f = 0\) in our result.

Now we are ready to state the main result of this paper.

**Theorem 1.3.** Suppose that assumptions \((K_1) - (K_4)\) are satisfied, \(0 < \mu < 4 \leq N\) and \(K(x_0) = \max_{\mathbb{R}^N} K(x)\). If \(K(x) - K(x_0) = o(|x-x_0|^2)\) as \(x \to x_0\) and \(V(x_0) < 0\), then, equation (1.7) has at least one nontrivial solution.

**Remark 1.4.** (i) The existence result obtained in Theorem 1.3 extends the earlier results for the local critical Schrödinger equation in [9] to the case of nonlocal Choquard equation.

(ii) We believe that the same existence result still holds if assumption \((K_3)\) is replaced by assumption \((K'_3)\), in this case \(f = 0\) may not be excluded.

**Remark 1.5.** As was commented in [8] [30], the case \(N = 3\) is much more complicated and remains open. For the case \(N \geq 4\), to overcome the difficulties caused by the negative essential spectrum of \(-\Delta + V\) and the loss of compactness due to the critical growth, the embedding property (Lemma 2.2) for elements of the negative space \(E^-\) plays an important role. However, this embedding property does not work very well for the case \(N = 3\) in the compactness arguments.

The paper is organized as follows: In Section 1 we introduce the background and the progress of the study of the nonlocal Choquard equation. The main existence result of the paper is given at the end of the section. In Section 2 we check that the energy functional of equation (1.7) satisfies the geometry conditions of the generalized linking theorem and analyze the behavior of the \((PS)\) sequences. In Section 3 we apply the methods in [8] to give an estimate of the linking value and prove the existence of nontrivial solution by variational methods.

## 2 Linking geometry and \((PS)_c\) sequences

Throughout this paper, we will denote by \(C, C_1, C_2, \cdots\) the different positive constants and \(| \cdot |_q\) the \(L^q(\mathbb{R}^N)\)-norm. Here \(| \cdot |_{m,q}\) will be used to denote the \(W^{m,q}(\mathbb{R}^N)\)-norm, \(m \in \mathbb{Z}^+\) and \(q \in [1, \infty)\). The working space \(E = H^1(\mathbb{R}^N)\) is equipped with the norm \(\|u\|^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)dx\). To prove the main results by variational arguments, we define the energy functional associated to (1.8) by

\[ J_K(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx - \frac{1}{2} \cdot \frac{2^*}{2^* - \mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(x, u)G(y, u)}{|x-y|^\mu}dxdy. \]

Then the Hardy-Littlewood-Sobolev inequality implies that the functional \(J_K\) belongs to \(C^1(H^1(\mathbb{R}^N), \mathbb{R})\) with

\[ \langle J'_K(u), \varphi \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla \varphi + V(x)u \varphi)dx - \frac{1}{2} \cdot \frac{2^*}{2^* - \mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u)g(x, u)\varphi(x)}{|x-y|^\mu}dxdy \]

\(\forall \varphi \in C_0^\infty(\mathbb{R}^N)\). Consequently, \(u\) is a weak solution of equation (1.8) if and only if \(u\) is a critical point of the functional \(J_K\).
Let $\mathcal{L}: D(\mathcal{L}) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be the operator defined by $\mathcal{L}(u) := -\Delta u + V(x)u$. By Lemma 2.1 in [9] and assumption $(K_1)$, we know that $\mathcal{L}$ is a closed operator with domain $D(\mathcal{L}) = H^2(\mathbb{R}^N)$, the spectrum of $\mathcal{L}$ is purely continuous and consists of a union of closed intervals. Let $(E(\lambda))_{\lambda \in \mathbb{R}}$ be the spectral family of $\mathcal{L}$, then for a fixed $\alpha$, $E(\alpha)L^2$ is the subspace of $L^2$ corresponding to $\lambda \leq \alpha$. The following two lemmas are borrowed from [9], they are very important in proving the main result.

**Lemma 2.1.** ([9]) If $V \in L^\infty(\mathbb{R}^N)$ satisfies $(K_4)$, then $|u|_{1,\infty} \leq c_0 |u|_2$ for some constant $c_0 > 0$ and all $u \in E(0)L^2$.

**Lemma 2.2.** ([9]) If $V \in L^\infty(\mathbb{R}^N)$, then for each $\alpha \in \mathbb{R}$ there exist constants $c_1$ and $c_2 = c_2(\alpha)$ such that $|u|_q \leq c_1 |u|_{2,2} \leq c_2 |u|_2$ whenever $u \in E(\alpha)L^2$. Here $q = \frac{2N}{N-2}$ if $N > 4$, $q$ may be taken arbitrarily large if $N = 4$ and $q = \infty$ if $N < 4$.

Let $E^- := E(0)L^2 \cap H^1(\mathbb{R}^N)$ and $E^+ := (I - E(0))L^2 \cap H^1(\mathbb{R}^N)$. Since $0$ lies in a gap of the spectrum of $\mathcal{L}$, the quadratic form $\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx$ is positive definite on $E^+$ and negative definite on $E^-$. Furthermore we can introduce a new inner product $\langle \cdot, \cdot \rangle$ in $E$ such that the corresponding norm $\| \cdot \|_V$ is equivalent to $\| \cdot \|$ and $\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx = \|u^+\|_V^2 - \|u^-\|_V^2$, where $u^\pm \in E^\pm$. Set

$$\Phi(u) := \frac{1}{2} \cdot 2 \cdot \frac{1}{\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G(x, u)G(y, u) \frac{1}{|x - y|^{\mu}} dxdy,$$

it is obvious that $\Phi \geq 0$. Then the functional $J_K$ can be rewritten as

$$J_K(u) = \frac{1}{2} \|u^+\|_V^2 - \frac{1}{2} \|u^-\|_V^2 = \Phi(u).$$

It follows from the Hardy-Littlewood-Sobolev inequality and Fatou’s lemma that $\Phi$ is weakly sequentially lower semicontinuous. Notice that

$$\langle \Phi'(u), \varphi \rangle = \frac{1}{2} \cdot 2 \cdot \frac{1}{\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G(y, u)g(x, u) \varphi(x) \frac{1}{|x - y|^{\mu}} dxdy = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{K(y)u(y)u^2 + F(y, u)}{|x - y|^{\mu}} dy \right) \left( K(x)|u(x)|^{2\mu}u(x) + \frac{1}{2} f(x, u) \right) \varphi(x) dx,$$

since $f(x, u)$ is of subcritical growth in the sense of the Hardy-Littlewood-Sobolev inequality, to show that $\Phi$ is weakly sequentially continuous, we only need to check that if $u_n \to u$ in $E$ then

$$\int_{\mathbb{R}^N} \frac{K(y)|u_n(y)|^{2\mu}K(x)|u_n(x)|^{2\mu}u_n(x) \varphi(x)}{|x - y|^{\mu}} dxdy \to \int_{\mathbb{R}^N} \frac{K(y)|u(y)|^{2\mu}K(x)|u(x)|^{2\mu}u(x) \varphi(x)}{|x - y|^{\mu}} dxdy$$

for any $\varphi \in E$, as $n \to +\infty$. In fact, by the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$ to $L^{\frac{2N}{\mu}}(\mathbb{R}^N)$, we know

$$\int_{\mathbb{R}^N} \frac{K(y)|u_n(y)|^{2\mu}}{|x - y|^{\mu}} dy \to \int_{\mathbb{R}^N} \frac{K(y)|u(y)|^{2\mu}}{|x - y|^{\mu}} dy \text{ in } L^{\frac{2N}{\mu}}(\mathbb{R}^N),$$

since $|u_n|^{2\mu} \to |u|^{2\mu}$ in $L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$ as $n \to +\infty$. Combining with the fact that

$$K(x)|u_n|^{2\mu}u_n \to K(x)|u|^{2\mu}u \text{ in } L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$$

as $n \to +\infty$, we find

$$\left( \int_{\mathbb{R}^N} \frac{K(y)|u_n(y)|^{2\mu}}{|x - y|^{\mu}} dy \right) K(x)|u_n(x)|^{2\mu}u_n(x) \to \left( \int_{\mathbb{R}^N} \frac{K(y)|u(y)|^{2\mu}}{|x - y|^{\mu}} dy \right) K(x)|u(x)|^{2\mu}u(x) \text{ in } L^{\frac{2N}{\mu}}(\mathbb{R}^N).$$

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as \( n \to +\infty \). Thus, for any \( \varphi \in E \),
\[
\int_{\mathbb{R}^N} \frac{K(y)|u_n(y)|^{2^*_N}K(x)|u_n(x)|^{2^*_N-2}u_n(x)\varphi(x)}{|x-y|^\mu} \, dx \, dy \to \int_{\mathbb{R}^N} \frac{K(y)|u(y)|^{2^*_N}K(x)|u(x)|^{2^*_N-2}u(x)\varphi(x)}{|x-y|^\mu} \, dx \, dy
\]
as \( n \to +\infty \).

In order to look for nontrivial critical points of the functional \( J_K \), we will apply an abstract critical point theorem in [5, 13]. Let \( z_0 \in E^+ \setminus \{0\} \),
\[
M := \{ u = u^- + sz_0 : u^- \in E^-, s \geq 0 \text{ and } \|u\|_V \leq R \}
\]
and denote the boundary of \( M \) in \( E^- \oplus \mathbb{R}z_0 \) by \( \partial M \). In the following lemma we check that the functional \( J_K \) satisfies the geometric structure of the generalized Linking Theorem:

**Lemma 2.3.** The functional \( J_K \) satisfies the following properties:
(i) There exist \( \varpi, \rho > 0 \) such that for any \( u \in E^+ \cap \partial B(0, \rho) \) it results that \( J_K(u) \geq \varpi \).
(ii) There exist \( R > \rho \) (\( R \) depending on \( z_0 \)) such that \( J_K(u) \leq 0 \) for any \( u \in \partial M \).

**Proof.** (i) It follows from assumption \((K_2)\) that there exists \( C \) such that \( |G(x, u)| \leq C(|u|^q + |u|^{2^*_N}) \). By the Sobolev embedding and the Hardy-Littlewood-Sobolev inequality, for all \( u \in E^+ \setminus \{0\} \) we have
\[
J_K(u) \geq \frac{1}{2}\|u^+\|_V^2 - C_0|u^+|_V^{2^*_N} - C_1|u|_V^{2q_N}/2^* \geq \frac{1}{2}\|u^+\|_V^2 - C_2|u|_V^{2^*_N} - C_3\|u\|_V^{2q_N},
\]
Since \( 2 < 2q < 2 \cdot 2^*_N \), we can choose some \( \varpi, \rho > 0 \) such that \( J_K(u) \geq \varpi \) for \( u \in E^+ \cap \partial B(0, \rho) \).

(ii) For any \( u \in E \) with \( \|u\|_V > 1 \) and for any \( t > 0 \), we define \( g(t) := \Phi(\frac{tu}{\|u\|_V}) > 0 \). By Remark 1.2 we have for any \( u \in E \setminus \{0\} \), \( \langle \Phi'(u), u \rangle \geq 2\theta \Phi(u) > 0 \). Thus, we can get for any \( t > 0 \)
\[
\frac{g'(t)}{g(t)} \geq \frac{2\theta}{t}.
\]
Integrating it over \([1, \|u\|_V]\) we know
\[
\Phi(u) \geq \Phi(\frac{u}{\|u\|_V})\|u\|_V^{2q}.
\]
(2.1)

Following [1], for \( \beta \in (0, 1) \), we set \( \gamma = \sin(\text{arctan} \, \beta) \in (0, 1) \) and
\[
P = \{ u \in E : u^+ = sz^0, \|z\|_V^0 \geq \gamma, \|u\|_V = 1 \},
\]
then it is easy to prove that \( \inf_{u \in P} \Phi(u) := \varrho > 0 \). For \( u \in E^- \oplus \mathbb{R}z^0 \) satisfy \( \|u\|_V > 1 \). (1) If \( \|u^+\|_V \geq \beta \) then \( \|u^+\|_V = \sin(\text{arctan} \, \frac{u^+}{\|u^-\|_V}) \geq \gamma \) and therefore \( \frac{u}{\|u\|_V} \in P \). Consequently, \( \Phi(u) \geq \varrho \|u\|_V^{2q} \) and thus
\[
J_K(u) \leq \frac{1}{2}\|u^+\|_V^2 - \varrho\|u\|_V^{2q},
\]
the claim is proved if \( \|u\|_V \) is large enough. (2) If \( \frac{\|u^+\|_V}{\|u^-\|_V} \leq \beta \) we have
\[
J_K(u) \leq \frac{1}{2}(\|u^+\|_V^2 - \|u^-\|_V^2) \leq \frac{1 - \beta^2}{2(1 + \beta^2)}\|u\|_V^2,
\]
(2.2)
the claim is proved since \( 0 < \beta < 1 \).
\[
\square
\]

**Remark 2.4.** Applying the generalized Linking Theorem in [5, 13], we know that there exists a \((PS)\) sequence \( \{u_n\} \subset E \) for the functional \( J_K \) such that \( J_K(u_n) \to c^* \in [\varpi, \sup_{M} J_K] \).
Lemma 2.5. The $(PS)_{c^*}$ sequence $\{u_n\}$ obtained in Remark 2.4 is bounded.

Proof. Let $\{u_n\}$ be the $(PS)_{c^*}$ sequence obtained in Remark 2.4. By Remark 1.2 we have

$$C_0(1 + \frac{||u_n||_V}{n}) \geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u_n)g(x, u_n)u_n(x)}{|x-y|^\mu}dxdy \quad (2.3)$$

for almost all $n$.

Note that $|g(x, u)| \leq C(|u|^{q-1} + |u|^{2q-1})$. Let $\chi \in C^\infty(\mathbb{R})$ such that $\chi(t) = 1$ if $|t| \geq 2$ whereas $\chi(t) = 0$ if $|t| \leq 1$. We define

$$g_1(x, u) := \chi(u)g(x, u), \quad g_2(x, u) := g(x, u) - g_1(x, u),$$

$$\kappa := \frac{2\mu}{2\mu - 1}, \quad q' := \frac{q}{q-1}.$$  

As a consequence of assumption (K$_2$), we have, for some $C_1 > 0$,

$$|g_1(x, u)| \leq C_1|u|^{2q-1} \quad \text{and} \quad |g_2(x, u)| \leq C_1|u|^{q-1}, \quad (2.4)$$

then we deduce from assumption (K$_3$) that

$$|g_1(x, u)|^\kappa \leq C_1 g_1(x, u)u \quad \text{and} \quad |g_2(x, u)|^{q'} \leq C_1 g_2(x, u)u.$$ 

Taking this fact and using the Hölder inequality, we know

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u_n)g_1(x, u_n)\varphi(x)}{|x-y|^\mu}dxdy \right| \leq \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u_n)|g_1(x, u_n)|^{\kappa}dx}{|x-y|^\mu} \right)^\frac{1}{\kappa} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)^{2q'}dx}{|x-y|^\mu} \right)^\frac{1}{q'} \quad (2.5)$$

By the semigroup property of the Riesz potential and the Hölder inequality,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u_n)}{|x-y|^\mu}d\varphi(x)^{2q'}dx$$

$$\quad = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{G(y, u_n)}{|x-y|^\mu}d\varphi(x) \right)^{2}dx \quad \left( \int_{\mathbb{R}^N} \frac{\varphi(x)^{2q'}dx}{|x-y|^\mu} \right)^\frac{1}{q'}$$

$$\quad \leq \left( \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{G(y, u_n)}{|x-y|^\mu}d\varphi(x) \right)^{2} \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{\varphi(x)^{2q'}dx}{|x-y|^\mu} \right)^\frac{1}{q'} \right)^\frac{1}{2}$$

$$\quad = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(x, u_n)G(y, u_n)}{|x-y|^\mu}dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)^{2q'}\varphi(y)^{2q'}dx}{|x-y|^\mu} \right)^\frac{1}{2}$$

$$\quad \leq C_3 \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x, u_n)|u_n(x)|}{|x-y|^\mu}dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)^{2q'}\varphi(y)^{2q'}dx}{|x-y|^\mu} \right)^\frac{1}{2}.$$  

Thus, by (2.5), we have

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u_n)g_1(x, u_n)\varphi(x)}{|x-y|^\mu}dxdy \right| \leq C_4 \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u_n)g(x, u_n)|u_n(x)|}{|x-y|^\mu}dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)^{2q'}\varphi(y)^{2q'}dx}{|x-y|^\mu} \right)^{\frac{1}{2}}.$$
and therefore
\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u_n)g_1(x, u_n)\varphi(x)}{|x-y|^\mu} \, dx \, dy \right| \leq C_5 \left( \langle \Phi'(u_n), u_n \rangle \right)^{\frac{1}{\mu} + \frac{1}{2q}} \| \varphi \|_V.
\]
Similarly, we know
\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u_n)g_2(x, u_n)\varphi(x)}{|x-y|^\mu} \, dx \, dy \right| \leq C_6 \left( \langle \Phi'(u_n), u_n \rangle \right)^{\frac{1}{\mu} + \frac{1}{2q}} \| \varphi \|_V.
\]
Recall that
\[
|\langle \Phi'(u_n), \varphi \rangle| \leq \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u_n)g_1(x, u_n)\varphi(x)}{|x-y|^\mu} \, dx \, dy \right| + \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u_n)g_2(x, u_n)\varphi(x)}{|x-y|^\mu} \, dx \, dy \right|,
\]
since \( \frac{1}{\mu} + \frac{1}{2q} + \frac{1}{2\mu} \in \left( \frac{1}{2}, 1 \right) \), we know
\[
\| \Phi'(u_n) \|_{H^{-1}(\mathbb{R}^N)} \leq C_7 \left( 1 + \langle \Phi'(u_n), u_n \rangle \right). \tag{2.6}
\]
Then, by (2.3) and (2.6), we have
\[
\| \Phi'(u_n) \|_{H^{-1}(\mathbb{R}^N)} \leq C_8 (1 + \frac{\| u_n \|_V}{n}), \tag{2.7}
\]
where \( H^{-1}(\mathbb{R}^N) \) denotes the dual space of \( E \). Using this fact, for large \( n \), we have
\[
\| u_n^+ \|_V^2 = \langle J'_K(u_n), u_n^+ \rangle + \frac{1}{2\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u_n)g_1(x, u_n)u_n^+(x)}{|x-y|^\mu} \, dx \, dy
\leq \| u_n^+ \|_V + C_9 \left( 1 + \frac{\| u_n \|_V}{n} \right) \| u_n^+ \|_V
\]
and
\[
\| u_n^- \|_V^2 = -\langle J'_K(u_n), u_n^- \rangle - \frac{1}{2\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y, u_n)g_2(x, u_n)u_n^-(x)}{|x-y|^\mu} \, dx \, dy
\leq \| u_n^- \|_V + C_{10} \left( 1 + \frac{\| u_n \|_V}{n} \right) \| u_n^- \|_V.
\]
Consequently,
\[
\| u_n \|_V^2 \leq \| u_n \|_V + C_{11} \left( 1 + \frac{\| u_n \|_V}{n} \right) \| u_n \|_V,
\]
this implies that the sequence \( \{ u_n \} \) is bounded in \( E \).
\[
\square
\]
As [14], let \( S_{H,L} \) be the best constant defined as
\[
S_{H,L} := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^\mu} \, dx \, dy \right)^{\frac{\mu-2}{\mu}}}.
\tag{2.8}
\]
Lemma 2.6. ([14]) The constant \( S_{H,L} \) defined in (2.8) is achieved if and only if
\[
u = C \left( \frac{b}{b^2 + |x-a|^2} \right)^{\frac{N-2}{2}},
\]
where \( C > 0 \) is a fixed constant, \( a \in \mathbb{R}^N \) and \( b \in (0, \infty) \) are parameters. What’s more,
\[
S_{H,L} = \frac{S}{C(N, \mu)^{\frac{N-2}{N-\mu}}},
\]
where \( S \) is the best Sobolev constant.
Proof. We sketch the proof here for the completeness. On one hand, by the Hardy-Littlewood-Sobolev inequality, we know

\[ S_{H,L} \geq \frac{1}{C(N, \mu) \frac{2}{2-N}} \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|u|^2} dx = \frac{S}{C(N, \mu) \frac{2}{2-N}}, \]

where \( S \) is the best Sobolev constant.

On the other hand, notice that the equality in the Hardy-Littlewood-Sobolev inequality holds if and only if

\[ h(x) = C \left( \frac{b}{b^2 + |x-a|^2} \right)^{\frac{2-N}{2}}, \]

where \( C > 0 \) is a fixed constant, \( a \in \mathbb{R}^N \) and \( b \in (0, \infty) \) are parameters. Thus

\[ \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2\mu} |u(y)|^{2\mu}}{|x-y|^\mu} \ dx \, dy \right)^{\frac{N-2}{2-N}} = C(N, \mu) \frac{2}{2-N} |u|^2, \]

if and only if

\[ u = C \left( \frac{b}{b^2 + |x-a|^2} \right)^{\frac{N-2}{2}}. \]

Then, by the definition of \( S_{H,L} \), for this \( u = C \left( \frac{b}{b^2 + |x-a|^2} \right)^{\frac{N-2}{2}} \), we know

\[ S_{H,L} \leq \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \ dx}{\int_{\mathbb{R}^N} \frac{|u(x)|^{2\mu} |u(y)|^{2\mu}}{|x-y|^\mu} \ dx \, dy} = \frac{1}{C(N, \mu) \frac{2}{2-N}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \ dx}{|u|^2}. \]

It is well-known that the function \( u = C \left( \frac{b}{b^2 + |x-a|^2} \right)^{\frac{N-2}{2}} \) is a minimizer for \( S \), thus we get

\[ S_{H,L} \leq \frac{S}{C(N, \mu) \frac{2}{2-N}}. \]

From the arguments above, we know that \( S_{H,L} \) is achieved if and only if \( u = C \left( \frac{b}{b^2 + |x-a|^2} \right)^{\frac{N-2}{2}} \) and

\[ S_{H,L} = \frac{S}{C(N, \mu) \frac{2}{2-N}}. \]

Let \( U(x) := \frac{N(N-2)}{(1+|x|^2)^{\frac{N-2}{2}}} \) be a minimizer for \( S \), see [23] for example, direct calculation shows that

\[ \bar{U}(x) = \frac{N(N-2)}{(1+|x|^2)^{\frac{N-2}{2}}} C(N, \mu) \frac{2-N}{2} \frac{N(N-2)}{(1+|x|^2)^{\frac{N-2}{2}}} \]

is the unique minimizer for \( S_{H,L} \) that satisfies

\[ -\Delta u = \left( \int_{\mathbb{R}^N} \frac{|u(y)|^{2\mu}}{|x-y|^\mu} \ dy \right) |u|^{2\mu-2} u \quad \text{in} \quad \mathbb{R}^N. \]

\[ \square \]

Lemma 2.7. If \( 0 < \mu < 4 \) and

\[ 0 < c^* < c_K := \frac{N + 2 - \mu}{4N - 2\mu} |K|_\infty \frac{2}{2-N} \frac{2}{2-N} S_{H,L} \frac{2}{2-N} \]

(2.10)

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then the (PS)$_c$ sequence \( \{u_n\} \) cannot be vanishing: there exist \( r, \eta > 0 \) and a sequence \( \{y_n\} \in \mathbb{R}^N \) such that

\[
\lim_{n \to \infty} \sup \int_{B(y_n, r)} u_n^2 \, dx \geq \eta,
\]

where \( B(x, r) \) denotes the open ball centered at \( x \) with radius \( r \).

**Proof.** We argue by contradiction. If \( \{u_n\} \) is vanishing, then Compactness Lemma [Lemma 1.21, [34]] implies that

\[
u_n \to 0 \quad \text{in} \quad L^r(\mathbb{R}^N),
\]

where \( 2 < r < 2^* \). By choosing \( t, s \) close to \( \frac{2N}{N - p} \) such that

\[
\frac{1}{t} + \frac{\mu}{N} + \frac{1}{s} = 2,
\]

and applying the Hardy-Littlewood-Sobolev inequality again, we know

\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y)|^2 \cdot f(x, u_n(x)u_n(x)}{|x - y|^\mu} \, dx \, dy \right| \leq C |u_n|_{L^t(\mathbb{R}^N)} \left( |u_n|_{L^p} + |u_n|_{L^q} \right).
\]

Thus, we can get

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y)|^2 \cdot f(x, u_n(x)u_n(x)}{|x - y|^\mu} \, dx \, dy \to 0,
\]

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y)|^2 \cdot F(x, u_n)}{|x - y|^\mu} \, dx \, dy \to 0,
\]

as \( n \to +\infty \). Similarly, we have

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, u_n) f(x, u_n(x)u_n(x)}{|x - y|^\mu} \, dx \, dy \to 0,
\]

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, u_n) F(x, u_n)}{|x - y|^\mu} \, dx \, dy \to 0,
\]

as \( n \to +\infty \). Then we can conclude that

\[
J_K(u_n) - \frac{1}{2} J'_K(u_n, u_n) = \left( \frac{1}{2} - \frac{1}{2 \cdot 2^*_\mu} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x) |u_n(x)|^{2^*_\mu} K(y) |u_n(y)|^{2^*_\mu} - 2 u_n(x) u_n(y) \cdot \mu}{|x - y|^\mu} \, dx \, dy + o_n(1) \to c^*,
\]

as \( n \to +\infty \). By Lemma [2.2] there exists \( C_1 > 0 \) such that \( |u_n|_s \leq C_1 |u_n|_2 \), where \( s = \frac{2N}{N - p} \) if \( N > 4 \) and \( s \) may be taken arbitrarily large if \( N = 4 \). Then there exists \( C_2 > 0 \) such that \( |u_n|_s \leq C_2 \|u_n\|_V \). Noting that \( \{u_n\} \) is bounded in \( E \), so \( |u_n|_s \) is also bounded. Let \( r \) be such that \( \frac{2^* N}{N - p} + \frac{2N}{2^* - N} = 1 \). Then, by \( 0 < \mu < 4 \), we have \( 2 < r < 2^* \) (for \( N = 4 \), \( s \) needs to be larger than 4). By the Hardy-Littlewood-Sobolev inequality and the Hölder inequality, similar to the proof of (2.11) and (2.12), we obtain from \( J'_K(u_n, u_n) = o_n(1) \) that

\[
\|u_n\|_V^2 = - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x) |u_n(x)|^{2^*_\mu} K(y) |u_n(y)|^{2^*_\mu} - 2 u_n(x) u_n(y) \cdot \mu}{|x - y|^\mu} \, dx \, dy + o_n(1)
\]

\[
\leq C_3 |u_n|_{L^2}^{2^*_\mu} \left( \int_{\mathbb{R}^N} |u_n|^{2^* \frac{2N - \mu}{2N - p - \mu}} \, dx \right) \left( \int_{\mathbb{R}^N} |u_n|^{2^* - 2 \cdot \mu} \, dx \right) + o_n(1)
\]

\[
\leq C_4 |u_n|_{L^2}^{2^*_\mu} \left( \int_{\mathbb{R}^N} |u_n|^{\frac{N - p + 2}{N - N - 2}} \, dx \right) + o_n(1)
\]

\[
\to 0,
\]

as \( n \to +\infty \). Then we know

\[
\|u_n\|_V = \|u_n\|_V + o_n(1).
\]

Since \( \{u_n\} \) is a Palais-Smale sequence for \( J_K \) such that \( J_K(u_n) \to c^* \), we obtain

\[
c^* = \frac{1}{2} \|u_n\|_V^2 - \frac{1}{2 \cdot 2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x) |u_n(x)|^{2^*_\mu} K(y) |u_n(y)|^{2^*_\mu} - 2 u_n(x) u_n(y) \cdot \mu}{|x - y|^\mu} \, dx \, dy + o_n(1)
\]

(2.15)
and
\[ \|u_n\|_V = \int_{R^N} \int_{R^N} K(x)|u_n(x)|^2 \frac{K(y)|u_n(y)|^2}{|x-y|^\mu} dxdy + o_n(1). \quad (2.16) \]

Thus by (2.14) and (2.16) we get
\[ \|u_n\|_V \leq |K|^2 \int_{R^N} \int_{R^N} |u_n(x)|^2 |u_n(y)|^2 \frac{1}{|x-y|^\mu} dxdy + o_n(1) \]
\[ \leq |K|^2 S_{H,L}^{-2} \|u_n\|_V^2 + o_n(1). \quad (2.17) \]

If \( \|u_n\|_V \to 0 \) as \( n \to +\infty \), it follows from (2.14), (2.15) and (2.16) that \( c^* = 0 \). This is a contradiction, since \( c^* > 0 \). Therefore \( \|u_n\|_V \to 0 \). So by (2.17) we get \( \|u_n\|_V \geq |K|^{-\frac{N-2-\mu}{2N-\mu}} S_{H,L}^{-\frac{2N-\mu}{N-\mu}} \). Then from (2.14), (2.16) and (2.17) we easily conclude that \( c^* \geq \frac{N-\mu+2}{4N-2} |K|^{-\frac{2N-4}{N-\mu}} S_{H,L}^{-\frac{2N-\mu}{N-\mu}} \). This contradicts with our assumption. Hence \( \{u_n\} \) is non-vanishing. \( \square \)

3 Proof of the Main Result

In this section we will prove the existence of solutions for equation (1.8). Consider a cut-off function \( \psi \in C_0^\infty (R^N) \) such that
\[ \psi(x) = 1 \text{ if } |x| \leq \delta, \quad \psi(x) = 0 \text{ if } |x| \geq 2\delta \]
for some \( \delta > 0 \). We define, for \( \varepsilon > 0 \),
\[ U_\varepsilon(x) := \varepsilon \frac{C}{\varepsilon} U\left(\frac{x}{\varepsilon}\right), \]
\[ u_\varepsilon(x) := \psi(x)U_\varepsilon(x). \quad (3.1) \]

From [8], [14] and Lemma 1.46 of [34], we know that as \( \varepsilon \to 0^+ \),
\[ \int_{R^N} |u_\varepsilon|^2 dx = C(N, \mu) \frac{N-2}{N-\mu} \frac{N}{2} S_{H,L}^{N} + O(\varepsilon^N), \quad (3.2) \]
\[ \int_{R^N} |\nabla u_\varepsilon|^2 dx = C(N, \mu) \frac{N-2}{N-\mu} \frac{N}{2} S_{H,L}^{N} + O(\varepsilon^{N-2}), \quad (3.3) \]
\[ \int_{R^N} |\nabla u_\varepsilon| dx = O(\varepsilon^{\frac{N-2}{2}}), \quad \int_{R^N} |u_\varepsilon| dx = O(\varepsilon^{\frac{N-2}{2}}) \quad (3.4) \]
and
\[ \int_{R^N} |u_\varepsilon|^2 dx = \begin{cases} \frac{d^2}{\varepsilon^2} |\ln \varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \\ \frac{d^2}{\varepsilon^2} + O(\varepsilon^{N-2}) & \text{if } N \geq 5, \end{cases} \quad (3.5) \]
where \( d \) is a positive constant.

For the convolution part, we have

Lemma 3.1.
\[ \int_{R^N} \int_{R^N} \frac{|u_\varepsilon(x)|^2 |u_\varepsilon(y)|^2}{|x-y|^\mu} dxdy \geq C(N, \mu) \frac{N}{2} S_{H,L}^{N-\mu} + O(\varepsilon^{N-\frac{\mu}{2}}). \quad (3.6) \]

Proof. In fact, using the Hardy-Littlewood-Sobolev inequality, on one hand, we get
\[
\left( \int_{R^N} \int_{R^N} \frac{|u_\varepsilon(x)|^2 |u_\varepsilon(y)|^2}{|x-y|^\mu} dxdy \right)^{\frac{N-\mu}{N}} \leq C(N, \mu) \frac{N-\mu}{N} |u_\varepsilon|^2,
\]
\[ = C(N, \mu) \frac{N-\mu}{N} \left( C(N, \mu) \frac{N-2}{N-\mu} \frac{N}{2} S_{H,L}^{N} + O(\varepsilon^N) \right)^{\frac{N-2}{N}} \quad (3.7) \]
\[ = C(N, \mu) \frac{N-\mu}{N} \frac{N}{2} S_{H,L}^{N-\mu} + O(\varepsilon^{N-2}). \]
On the other hand, 
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_x(x)|^{2\mu} |u_x(y)|^{2\mu}}{|x-y|^\mu} \, dx \, dy 
\geq \int_{B_\delta} \int_{B_\delta} \frac{|u_x(x)|^{2\mu} |u_x(y)|^{2\mu}}{|x-y|^\mu} \, dx \, dy 
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_x(x)|^{2\mu} |U_x(y)|^{2\mu}}{|x-y|^\mu} \, dx \, dy - 2 \int_{\mathbb{R}^N \setminus B_\delta} \int_{B_\delta} \frac{|U_x(x)|^{2\mu} |U_x(y)|^{2\mu}}{|x-y|^\mu} \, dx \, dy \tag{3.8}
\]

where 
\[
D \equiv \int_{\mathbb{R}^N \setminus B_\delta} \int_{B_\delta} \frac{|U_x(x)|^{2\mu} |U_x(y)|^{2\mu}}{|x-y|^\mu} \, dx \, dy, \quad E \equiv \int_{\mathbb{R}^N \setminus B_\delta} \int_{B_\delta} \frac{|U_x(x)|^{2\mu} |U_x(y)|^{2\mu}}{|x-y|^\mu} \, dx \, dy.
\]

By direct computation, we know 
\[
D \equiv \int_{\mathbb{R}^N \setminus B_\delta} \int_{B_\delta} \frac{|U_x(x)|^{2\mu} |U_x(y)|^{2\mu}}{|x-y|^\mu} \, dx \, dy 
= \varepsilon^{2N-\mu} [N(N-2)]^{2N-\mu} \varepsilon^{N-2N} \int_{\mathbb{R}^N \setminus B_\delta} \int_{B_\delta} \frac{1}{(\varepsilon^2 + |x|^2)^N} |x-y|^\mu (\varepsilon^2 + |y|^2)^{2N-2} \, dx \, dy 
\leq O(\varepsilon^{2N-\mu}) \left( \int_{\mathbb{R}^N \setminus B_\delta} \frac{1}{(\varepsilon^2 + |x|^2)^N} \, dx \right)^{2N-\mu} \left( \int_{B_\delta} \frac{1}{(\varepsilon^2 + |y|^2)^N} \, dy \right)^{2N-\mu} \tag{3.9}
\]

and 
\[
E \equiv \int_{\mathbb{R}^N \setminus B_\delta} \int_{\mathbb{R}^N \setminus B_\delta} \frac{|U_x(x)|^{2\mu} |U_x(y)|^{2\mu}}{|x-y|^\mu} \, dx \, dy 
= \varepsilon^{N-2N} [N(N-2)]^{2N-\mu} \varepsilon^{N-2N} \int_{\mathbb{R}^N \setminus B_\delta} \int_{\mathbb{R}^N \setminus B_\delta} \frac{1}{(\varepsilon^2 + |x|^2)^{2N-\mu} |x-y|^\mu (1 + |x|^2)^{2N-2} \, dx \, dy 
\leq \varepsilon^{N-2N} [N(N-2)]^{2N-\mu} \varepsilon^{N-2N} \int_{\mathbb{R}^N \setminus B_\delta} \int_{\mathbb{R}^N \setminus B_\delta} \frac{1}{|x|^2 |x-y|^\mu |y|^{2N-\mu} \, dx \, dy \tag{3.10}
\]

It follows from (3.8) to (3.10) that 
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_x(x)|^{2\mu} |u_x(y)|^{2\mu}}{|x-y|^\mu} \, dx \, dy \geq C(N, \mu)^N S_{H,L}^{2N-\mu} - O(\varepsilon^{2N-\mu}) - O(\varepsilon^{2N-\mu}) \tag{3.11}
\]
We define the linear space 
\[ G_\varepsilon := E^- \oplus \mathbb{R}u_\varepsilon = E^- \oplus \mathbb{R}u_\varepsilon^+ \]
and set 
\[ m_\varepsilon := \max_{u \in G_\varepsilon, \|u\|_{K^{NL}} = 1} \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx \right), \]
where \( u_\varepsilon \) is defined in (3.11) and \( \|u\|_{K^{NL}} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)|u(x)|^{2^*_\mu}K(y)|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy \right)^{\frac{1}{2^*_\mu}}. \)

We may assume, without loss of generality, that \( K(0) = \|K\|_\infty \) and \( V(0) < 0 \). Moreover, \( \delta \) in the definition of \( u_\varepsilon(x) \) may be chosen so that \( V(x) \leq -\xi \) for some \( \xi > 0 \) and \( |x| \leq \delta \).

**Lemma 3.2.** If \( \varepsilon > 0 \) is small enough, then \( \sup_{G_\varepsilon} J_K < c_K \), where \( c_K \) is defined in Lemma 2.1. In particular, if \( z_0 = u_\varepsilon^+ \) with \( \varepsilon \) small enough then \( c^* \leq \sup_M J_K < c_K \).

**Proof.** We introduce the functional
\[ I_K(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V|u|^2)dx - \frac{1}{2} \cdot 2^*_\mu \int_{\mathbb{R}^N} \frac{K(x)|u(x)|^{2^*_\mu}K(y)|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy. \]

Since \( I_K(u) \geq J_K(u) \) for all \( u \), it suffices to show that \( \sup_{G_\varepsilon} I_K < c_K \). By a direct computation, for all \( u \in E \setminus \{0\} \), we have
\[ \max_{t \geq 0} I_K(tu) = \frac{N + 2 - \mu}{4N - 2\mu} \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + V|u|^2)dx \right)^{\frac{2N-\mu}{N-\mu+2}} \]
whenever the integral in the numerator above is positive, and the maximum is 0 otherwise. It is easy to see from (3.12) that if
\[ m_\varepsilon < \|K\|_{\infty}^{\frac{2N-\mu}{N-\mu+2}} S_{H,L} \]
then \( \sup_{G_\varepsilon} J_K \leq \sup_{G_\varepsilon} I_K < c_K \). So it remains to prove that (3.13) is satisfied for all small \( \varepsilon > 0 \).

Suppose \( u \in G_\varepsilon \) with \( \|u\|_{K^{NL}} = 1 \) and write \( u = u^- + su_\varepsilon \). By direct computation, we have
\[ |u_\varepsilon|^\frac{N-\mu+2}{2N(N-\mu)(N-2)} \leq O(\varepsilon^\frac{N-2}{2}). \]

By convexity, Lemma 2.1, the Hardy-Littlewood-Sobolev inequality and the Hölder inequality, we obtain
\[ 1 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)|u(x)|^{2^*_\mu}K(y)|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy \]
\[ = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)|u^-(x) + su_\varepsilon(x)|^{2^*_\mu}K(y)|u^-(y) + su_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy \]
\[ \geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)|su_\varepsilon(x)|^{2^*_\mu}K(y)|su_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy + 2 \cdot 2^*_\mu \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)|su_\varepsilon(x)|^{2^*_\mu-1}u^-(x)K(y)|su_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy \]
\[ \geq s^{2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)|u_\varepsilon(x)|^{2^*_\mu}K(y)|u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy - 2 \cdot 2^*_\mu s^{2^*_\mu-1} \|K\|_{L^\infty}^{2^*_\mu} \left( \int_{\mathbb{R}^N} |u_\varepsilon|^{2^*_\mu-1}K_{L^\infty}^{2^*_\mu-1} \right) \left( \int_{\mathbb{R}^N} |u_\varepsilon|^{\frac{N-\mu+2}{2N(N-\mu)(N-2)}}dxdy \right)^\frac{2N-\mu}{2N} \]
\[ \geq s^{2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)|u_\varepsilon(x)|^{2^*_\mu}K(y)|u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy - C_1 s^{2^*_\mu-1} \|u_\varepsilon\|_{L^\infty}^{\frac{N-\mu+2}{2N(N-\mu)(N-2)}} \cdot \|u^-\|_{L^{\infty}}^{\frac{N-\mu+2}{2N(N-\mu)(N-2)}} \]
\[ \geq s^{2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)|u_\varepsilon(x)|^{2^*_\mu}K(y)|u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy - C_2 s^{2^*_\mu-1} O(\varepsilon^{\frac{N-2}{2}}) \]
(3.15)
where (3.2) and (3.14) are used. Since (3.15) implies that $s < C_3$ for some constant $C_3 > 0$, we deduce from (3.15) that

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x)|su_\varepsilon(x)|^{2^*_p} K(y)|su_\varepsilon(y)|^{2^*_p} dx dy \leq 1 + O(\varepsilon^{\frac{N-2}{2}})|u^-|_2.
$$

Let

$$
A_\varepsilon := \frac{\int_{\mathbb{R}^N} (|\nabla u_\varepsilon|^2 + V|u_\varepsilon|^2) dx}{\|u_\varepsilon\|^2_{KL}}.
$$

(3.16)

It follows from Lemma 2.1 that

$$
\int_{\mathbb{R}^N} (\nabla u^- \nabla u_\varepsilon + V(x)u^- u_\varepsilon) dx \leq C_4 \left( \int_{\mathbb{R}^N} |\nabla u_\varepsilon| dx + \int_{\mathbb{R}^N} |u_\varepsilon| dx \right) |u^-|_2,
$$

then we know

$$
m_\varepsilon \leq -\|u^-\|^2_V + A_\varepsilon \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)|su_\varepsilon(x)|^{2^*_p} K(y)|su_\varepsilon(y)|^{2^*_p}}{|x - y|^\mu} dx dy \right)^{\frac{N-2}{2N-p}} + C_4 \left( \int_{\mathbb{R}^N} |\nabla u_\varepsilon| dx + \int_{\mathbb{R}^N} |u_\varepsilon| dx \right) |u^-|_2
$$

$$
\leq -C_5 |u^-|^2 + A_\varepsilon \left( 1 + |u^-|_2 O(\varepsilon^{\frac{N-2}{2}}) \right)^{\frac{N-2}{2N-p}} + C_4 \left( \int_{\mathbb{R}^N} |\nabla u_\varepsilon| dx + \int_{\mathbb{R}^N} |u_\varepsilon| dx \right) |u^-|_2
$$

$$
\leq -C_5 |u^-|^2 + A_\varepsilon \left( 1 + |u^-|_2 O(\varepsilon^{\frac{N-2}{2}}) \right) + O(\varepsilon^{\frac{N-2}{2}})|u^-|_2.
$$

Noting that $K(x) - K(0) = o(|x|^2)$ as $x \to 0$, we obtain

$$
\|u_\varepsilon\|^2_{KL} = |K|^2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^{2^*_p} |u_\varepsilon(y)|^{2^*_p}}{|x - y|^\mu} dx dy + 2 |K| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(K(x) - K(0))|u_\varepsilon(x)|^{2^*_p} |u_\varepsilon(y)|^{2^*_p}}{|x - y|^\mu} dx dy
$$

$$
+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(K(x) - K(0))|u_\varepsilon(x)|^{2^*_p} |u_\varepsilon(y)|^{2^*_p}}{|x - y|^\mu} dx dy
$$

$$
\geq |K|^2 C(N, \mu) \frac{2^{2-N}}{S_{H,L}} + o(\varepsilon^2).
$$

By the proof of Proposition 4.2 in [9], we know

$$
\int_{\mathbb{R}^N} V(x)|u_\varepsilon|^2 dx \leq \begin{cases} -\varepsilon |\ln \varepsilon| & \text{if } N = 4, \\ -\varepsilon^2 & \text{if } N \geq 5. \end{cases}
$$

If $N \geq 5$, we have

$$
m_\varepsilon \leq -C_5 |u^-|^2 + \int_{\mathbb{R}^N} (|\nabla u_\varepsilon|^2 + V|u_\varepsilon|^2) dx
$$

$$
\frac{\|u_\varepsilon\|^2_{KL}}{(1 + |u^-|_2 O(\varepsilon^{\frac{N-2}{2}})) + O(\varepsilon^{\frac{N-2}{2}})|u^-|_2}
$$

$$
\leq -C_5 |u^-|^2 + C(N, \mu) \frac{2^{2-N} + N}{S_{H,L}} + O(\varepsilon^{N-2} - \varepsilon^2) \left( 1 + |u^-|_2 O(\varepsilon^{\frac{N-2}{2}}) + O(\varepsilon^{\frac{N-2}{2}})|u^-|_2 \right)
$$

$$
\leq |K|^{\frac{2-N}{2N-p}} S_{H,L} - \varepsilon^2 + O(\varepsilon^{N-2}) - C_5 |u^-|^2 + O(\varepsilon^{\frac{N-2}{2}})|u^-|_2
$$

for $\varepsilon > 0$ sufficiently small. Since

$$
-C_5 |u^-|^2 + O(\varepsilon^{\frac{N-2}{2}})|u^-|_2 \leq O(\varepsilon^{N-2}),
$$

(3.17)

we know

$$
m_\varepsilon \leq |K|^{\frac{2-N-4}{2N-p}} S_{H,L} - \varepsilon^2 + O(\varepsilon^{N-2}) < |K|^{\frac{2-N-4}{2N-p}} S_{H,L}
$$

for $\varepsilon > 0$ sufficiently small.
If $N = 4$, by \[ \text{(1.17)} \], we have
\[
m_\varepsilon \leq -\|u^\varepsilon\|^2_V + \frac{\int_{\mathbb{R}^N} (|\nabla u_\varepsilon|^2 + V|u_\varepsilon|^2) dx}{\|u_\varepsilon\|^4_{KNL}} \left( 1 + |u^\varepsilon|_2O(\varepsilon) \right) + O(\varepsilon)|u^\varepsilon|_2^2
\]
\[
\leq -C_5|u^\varepsilon|_2^2 + \frac{1}{|K|_\varepsilon^{-\mu}} \frac{C(4, \mu) \frac{2N-4}{N-2} S_{H,L}^2 + O(\varepsilon^2) - d\varepsilon^2 |\ln \varepsilon|}{C(4, \mu)^2 S_{H,L}^2 + o(\varepsilon^2)} \left( 1 + |u^\varepsilon|_2O(\varepsilon) \right) + O(\varepsilon)|u^\varepsilon|_2^2
\]
\[
\leq |K|_\varepsilon^{-\mu} S_{H,L} - d\varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) - C_5|u^\varepsilon|_2^2 + O(\varepsilon)|u^\varepsilon|_2^2
\]
\[
\leq |K|_\varepsilon^{-\mu} S_{H,L} - d\varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2)
\]
\[
< |K|_\varepsilon^{-\mu} S_{H,L}
\]
for $\varepsilon > 0$ sufficiently small. The result then follows. \[ \square \]

**Proof of Theorem 1.3.** By Remark 2.4 and Lemma 3.2, there exists a $(PS)_c$ sequence $\{u_n\}$ of the functional $J_K$ with $c^* \leq \frac{2N-4}{N-2} |K|_\varepsilon^{-\mu} S_{H,L}^2$. Applying Lemma 2.7, we know that the sequence $\{u_n\}$ cannot be vanishing. And so, there exist $r, \eta > 0$ and a sequence $\{y_n\} \in \mathbb{R}^N$ such that
\[
\lim_{n \to \infty} \sup_{B(y_n, r)} u_n^2 dx \geq \eta,
\]
where $B(x, r)$ denotes the open ball centered at $x$ with radius $r$. We may assume $y_n \in \mathbb{Z}^N$ by taking a larger $r$ if necessary. Let $\tilde{u}_n(x) := u_n(x + y_n)$. Since $J_K$ is invariant with respect to the translation of $x$ by elements of $\mathbb{Z}^N$, $\| \tilde{u}_n \|_V = \| u_n \|_V$ and $\| J_K(\tilde{u}_n) \|_{H^{-1}(\mathbb{R}^N)} = \| J'_K(u_n) \|_{H^{-1}(\mathbb{R}^N)}$. Hence $\tilde{u}_n \to \tilde{u}$, after passing to a subsequence, such that $J'_K(\tilde{u}) = 0$. Since $\lim_{n \to \infty} \sup_{B(0, r)} \tilde{u}_n^2 dx \geq \eta$, we know $\tilde{u} \neq 0$. Thus $\tilde{u}$ is a nontrivial solution of equation \[ \text{(1.7)}. \]

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