A NEW ALGORITHM FOR FINDING THE NILPOTENCY CLASS OF A FINITE $p$-GROUP DESCRIBING THE UPPER CENTRAL SERIES

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Abstract. In this paper we describe an algorithm for finding the nilpotency class, and the upper central series of the maximal normal $p$-subgroup $\Delta(G)$ of the automorphism group, $\text{Aut}(G)$ of a bounded (or finite) abelian $p$-group $G$. This is the first part of two papers devoted to compute the nilpotency class of $\Delta(G)$ using formulas, and algorithms that work in almost all groups. Here, we prove that for $p \geq 3$ the algorithm always runs. The algorithm describes a sequence of ideals of the Jacobson radical, $J$, and because $\Delta(G) = J + 1$, this sequence induces the upper central series in $\Delta(G)$.

1. Introduction

The automorphism group of an abelian $p$-group, was studied for K. Shoda in 1928, under the advice of Emmy Noether, and he gave the description of the endomorphism ring and a characterization of the automorphism group using a matrix representation over the integer modulo a primary number $\mathbb{Z}_{p^n}$, [9]. The maximal normal $p$-subgroup of the automorphism group of a bounded abelian $p$-group $G$ plays a very important role in the description of the automorphism group, because $\text{Aut}(G)$ decomposes on semidirect product of $\Delta(G)$ in several cases, see [5]. This paper is the first part of two papers devoted to the description of the nilpotency class of $\Delta(G)$ using formulas, and algorithms that work in almost all groups. Here, we prove that for $p \geq 3$ the algorithm always runs. The algorithm describes a sequence of ideals of the Jacobson radical, $J$, and because $\Delta(G) = J + 1$, this sequence induces the upper central series in $\Delta(G)$.

In fact, we obtain the nilpotency class of $\Delta(G)$, and characterize its upper central series (ucs) $\{Z_t\}$. To do this, we use the upper annihilating sequence (uas) $\{J_t\}$, where $J_t = \text{Annihilator of } (J/J_{t-1})$, for all $t$, a sequence of two sided ideals of $J$ in the endomorphism ring $\mathcal{E}(G)$. This sequence was defined and studied for finite groups in [1, 2, 3, 4], and for bounded groups in [6]. Here we obtain new results for the (uas), in Section 3. For most the cases this sequence determine the upper central series. The nilpotency class of a finite $p$-group is usually computing the lower central series, but here we introduce an algorithm for computing the upper central series and of course the nilpotency class. We have a recursive function that
permits the construction of the (ucs). One of the most interesting results is that the description, and the length of the upper central series only depends on the exponents of the group \( G \), and the rank \( r_s \) of the homocyclic subgroup \( G_s \cong (\mathbb{Z}_{p^{n_s}})^{r_s} \) of maximal exponent in the group \( G \). This is true even if the group is infinite but bounded. Here, we introduce a method to construct central series associated to annihilating series, that is, if we have a ring \( R \) with identity 1, and there exists a bilateral ideal \( I \) that has a finite annihilating sequence then \( H = I + 1 \) is a nilpotent group with degree less or equal the length of the upper annihilating sequence of \( I \).

A group \( G \) has the (uas)-(ucs)-property for the Jacobson radical \( \mathcal{J} \) of \( E \), if the upper central series \( (Z_t)_{t\geq 1} \) of \( \Delta(G) \) satisfies the following \( Z_t = (\mathcal{J}^t + 1)\Delta \) for all \( t \) from 1 to \( n(\Delta) \), the nilpotency class of \( \Delta(G) \), and \( Z_\Delta \) is the intersection of the subgroup \( \Delta(G) \) with the center of \( E \).

Associated to the Upper Annihilating Sequence of \( \mathcal{J} \), we have a special function that is defined in a recursive form, because the function calls itself to construct the ideals, we have the initial value, and the condition to stop given by the ideal function associated to the radical \( \mathcal{J} \), and the Theorem 6.5. We call this function Upper Function and we use it to describe an algorithm for finding the Upper Central Series of \( \Delta(G) \), it is done in Section 7. In this paper, the following theorem is proved:

**Theorem 1.1.** Let \( G \cong \bigoplus_{i=1}^{s} G_i \) be a bounded abelian \( p \)-group, where the \( p \)-rank of \( G_i \cong (\mathbb{Z}_{p_n})^{r_i} \) is the ordinal number \( r_i \), where \( 0 < n_1 < \cdots < n_s \). Let \( \{\mathcal{J}_t\} \) be the upper annihilating series of the radical \( \mathcal{J} \) of \( E \), and let \( \{Z_t\} \) be the upper central series of the maximal normal \( p \)-subgroup of \( \text{Aut}(G) \), \( \Delta(G) \). If \( Z = Z \cap \Delta(G) \) and one of the following cases holds:

1. \( p \geq 3 \)
2. \( p = 2, \sigma(G) \geq 2 \),
3. \( p = 2, r_s > 1, \) and \( s \geq 2 \),

then for \( t \geq 1 \), \( Z_t = (\mathcal{J}^t + 1)Z \). The description of the elements in the hypercenters \( Z_t \) is the following:

\[
Z_t = \left\{ C + 1_{rr} \in \Delta(G) \mid \begin{array}{l}
\frac{c_{i,j}^{(l,k)}}{c_{i,i}^{(l,k)}} \equiv 0 \pmod{p^{n_j-a(i,j,t)}} \quad i \neq j; \\
\frac{c_{i,i}^{(l,k)}}{c_{i,i}^{(r,r)}} \equiv 0 \pmod{p^{n_j-a(i,i,t)}} \quad \text{for } i \leq s \\
\frac{c_{i,i}^{(l,k)}}{c_{i,i}^{(k,k)}} \equiv 0 \pmod{p^{n_j-a(i,i,t)}} \quad l \neq k
\end{array} \right\}
\]

where \( C = (c_{i,j}^{(l,k)})_{r \times r} \) is a matrix with entries by columns in \( \mathbb{Z}_{p^{n_j}} \), where \( r = \sum_{i=1}^{s} r_i \).

In the second paper, general formulas for the length of the upper annihilating sequence, and the nilpotency class of \( \Delta \) are given.

2. Preliminaries and Notation

Let \( G = \bigoplus_{i=1}^{s} G_i \) be a decomposition of a bounded abelian \( p \)-group \( G \) into homocyclic subgroups \( G_i \cong (\mathbb{Z}_{p^{n_i}})^{r_i} \), where \( 0 < n_1 < \cdots < n_s \) are integers and \( r_i \) cardinals.

The endomorphism ring \( E \) of \( G \) will be represented as an \( s \times s \) matrix ring \( (E_{ij}) \), where for all \( (i,j) \), \( E_{ij} = \text{Hom}(G_i,G_j) \), considered as an \( E_i \)-\( E_j \)-bimodule. For each endomorphism \( f = (f_{ij})_{s \times s} \in E \) we consider the functions \( f_{ij} \) defined as follows. If \( x \in G \) decomposes as sum of elements \( x_i \in G_i \), then \( f(x) = \sum_i f(x_i) = \sum_i \sum_j x_{ij} \).
where \( f(x_i) = \sum_j x_{ij} \in G_j \), so \( f_{ij}(x_i) = x_{ij} \), and \( f_{ij} \in \mathcal{E}_{ij} \). So, \( \mathcal{E} \cong (\mathcal{E}_{ij}) \). By the above decomposition we have that the matrix \( \mathcal{E}_{ij} \) has entries in \( \mathbb{Z}_{p^n} \), and satisfies the following condition \( \mathcal{E}_{ij} \equiv 0 \pmod{p^{n_i - n_j}} \) if \( i < j \). The center \( \mathcal{Z} \) of \( \mathcal{E} \) is the ring of scalar matrices \( e \mathcal{E} \) where \( c_{ii} \equiv c \pmod{p^n} \), and for \( i < j \), \( c_{ii} \equiv c_{ij} \pmod{p^n} \). The maximal normal \( p \)-subgroup of \( \text{Aut}(G) \) is denoted by \( \Delta(G) = 1 + J \).

Here, we always consider an ideal like a bilateral ideal. \( I \) is an ideal of \( \mathcal{E} \) if and only if \( I = (I_{ij}) \) where each \( I_{ij} \) is a sub-bimodule of \( \mathcal{E}_{ij} \). \( I \) is an ideal of \( \mathcal{E} \) if and only if there exists an ideal function \( \beta \) such that the matrix representation of the ideal is the following \( \mathcal{I} = (p^{\beta(i,j)} \mathcal{E}_{ij}) \), where \( p^{\beta(i,j)} \mathcal{E}_{ij} = I_{ij} \), and \( \mathcal{E}_{ij} \) is a \( r_i \times r_j \)-matrix with entries in the integers modulo \( p^{n_i} \).

\( \mathcal{E} \) has Jacobson radical \( \mathcal{J} = (\mathcal{J}_{ij}) \) where for all \( i \neq j \), \( \mathcal{J}_{ij} = \mathcal{E}_{ij} \) and \( \mathcal{J}_{ii} = p\mathcal{E}_{ii} \). We denote the ideal function associated to the Jacobson Radical by \( n_j - f_{\mathcal{J}}(i,j) \), where \( f_{\mathcal{J}}(i,j) = \begin{cases} n_{ij} \quad \text{if } i \neq j \\ n_j - 1 \quad \text{if } i = j \end{cases} \) \( \min \{i,j\} = i \land j \).

We will say that \( G \) has underlying type \( \underline{\ell}(G) = (n_1, \ldots, n_s) \), with \( n_i < n_{i+1} \), for all \( i \). We denote by \( \sigma(G) \) the minimum gap in the sequence \( n_1 < n_2 < \cdots < n_s \), that is \( \sigma(G) = \min\{n_j - n_{j-1}\} \) for all \( j \). Let \( \ell = \max\{j : n_j - n_{j-1} = \sigma(G)\} \).

If the rank of \( G \) is denoted by \( r = \sum_i r_i \), then let \( G = \bigoplus_{i=1}^{r_1} \bigoplus_{k=1}^{r_2} \langle x_{ik} \rangle \) be a fixed decomposition of \( G \) into cyclic summands, and let \( X = \cup X_i \) where \( x_i = \{x_{1i}, \ldots, x_{ri}\} \) is a basis of \( G_i \). We define a set of endomorphisms in \( \mathcal{J} \), called elementary endomorphisms, that are the common elementary matrices, but consider in the radical \( \mathcal{J} \).

1. For all \( x_{ki}, x_{lj} \in X_i \), \( e_{ii}^{(k,l)} \) maps \( x_{ki} \) to \( px_{li} \) and annihilates the complement of \( \langle x_{ki} \rangle \) with respect to the basis \( X \). The elementary matrix of \( e_{ii}^{(k,l)} \) has \( p \) in the place \( kl \) of the diagonal cell \( ii \). For example, the endomorphism \( e_{11}^{(k,l)} \) has matrix representation

\[
E_{11}^{(k,l)} = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

where the matrix \( E_{11}^{(k,l)} \) is the elementary \( r_1 \times r_1 \)-matrix which has the number 1 in the place \( kl \).

2. For \( i > j \), and for all \( x_{ki} \in X_i \), \( x_{lj} \in X_j \), \( e_{ij}^{(k,l)} \) maps \( x_{ki} \) onto \( x_{lj} \) and annihilates the complement of \( \langle x_{ki} \rangle \). The elementary matrix of \( e_{ij}^{(k,l)} \) has 1 in the place \( kl \) of the cell \( ij \). For example, the endomorphism \( e_{21}^{(k,l)} \) has matrix representation

\[
E_{21}^{(k,l)} = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

where the matrix \( E_{21}^{(k,l)} \) is the elementary \( r_2 \times r_1 \)-matrix which has the number 1 in the place \( kl \).

3. For \( i < j \), and for all \( x_{ki} \in X_i \), \( x_{lj} \in X_j \) with \( i < j \), \( e_{ij}^{(k,l)} \) maps \( x_{ki} \) onto \( p^{n_j - n_i}x_{lj} \) and annihilates the complement of \( \langle x_{ki} \rangle \). The elementary matrix of \( e_{ij}^{(k,l)} \) has \( p^{n_j - n_i} \) in the place \( kl \) of the cell \( ij \). For example, the
endomorphism \( e_{12}^{(k,l)} \) has matrix representation
\[
E_{12}^{(k,l)} = \begin{pmatrix}
0 & p^{n_2-n_1}E_{12}^{kl} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
where the matrix \( E_{12}^{kl} \) is the elementary \( r_1 \times r_2 \)-matrix which has the number 1 in the place \( kl \).

3. **Central Series induced by Annihilating Sequences**

Let \( I \) be an ideal of a ring \( R \). Let \( A_1 = \text{Ann}(I) = \{ a \in I : aI = 0 = Ia \} \) be the annihilator of \( I \). The following definitions appear in [1, 2, 3, 6].

**Definition 3.1.** The upper annihilating sequence, (uas), of the ideal \( I \) is defined by
\[
I_0 = 0, \text{ and for } t \geq 1, \ I_t/I_{t-1} = \text{Ann}(I/I_{t-1}).
\]
In other words, \( I_t = \{ a \in I : aI \subseteq I_{t-1}, Ia \subseteq I_{t-1} \} \). This is an ascending sequence of ideals \( I_t \). If there is a positive integer \( d \) such that
\[
0 = I_0 \subseteq \cdots \subseteq I_{t-1} \subseteq U_t \subseteq \cdots \subseteq I_d = I,
\]
we say that the annihilating length of \( I \) is \( d \).

We make a generalization of the above definition in order to find the ideal function associated to the upper annihilating sequence of the Jacobson radical of \( \mathcal{E} \), This definition will be used in the next section in a ring \( R \) with unit 1.

**Definition 3.2.** We will say that a sequence of ideals \( A_t \) is an annihilating sequence of \( I \) if
\[
A_0 = 0, \ A_1 = \text{Ann}(I) \text{ and for } t \geq 2, \ A_t/A_{t-1} \subseteq \text{Ann}(I/A_{t-1}),
\]
that is \( A_t = \{ a \in I | ab \in A_{t-1}, ba \in A_{t-1}, \forall b \in I \} \). So,
\[
0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{t-1} \subseteq A_t \subseteq \cdots \subseteq A_{l(A_t)} = I.
\]
If for some \( t \) the ideal \( I = A_t \), then the first \( t \) with this property will be called the length of \( A_t \) and denoted by \( l(A_t) \).

It is clear that \( l(I_t) \leq l(A_t) \) for all annihilating sequence \( (A_t)_t \) of \( I \).

**Central series induced by annihilating sequences.** In this section we introduce the mean relation between the upper annihilating series of the radical \( J \), and the central series of the maximal normal \( p \)-subgroup \( \Delta(G) = J + 1 \). This relation is given in the Theorem 3.3.

The length of the (uas) of \( I \) is called the annihilating length of \( I \).

If \( R \) is a ring with 1, and \( I \) is an ideal of \( R \), with finite annihilating length, then \( 1 + I = \Gamma \) is a normal subgroup of the group of units of \( R \), that satisfies the following theorem.

**Theorem 3.3.** Let \( \{ A_t \} \) be an annihilating sequence of the ideal \( I \) of length \( l(A_t) \), and let \( \Gamma = I + 1 \). If \( Z = Z \cap \Gamma \), where \( Z \) is the center of \( R \). Then
\[
\{ 1 \} \leq (A_T + 1)Z \leq \cdots \leq (A_1 + 1)Z \leq \cdots \leq (A_{l(A_t)} + 1)Z = \Gamma
\]
is a central series for \( \Gamma \), and \( \Gamma \) is a nilpotent group. Consequently, the nilpotency class of \( \Gamma \) is less than or equal to the annihilating length of \( I \).
Proof. Let $\Gamma_0 = I$, and $\Gamma_t = (A_t + 1)Z$ for $1 \leq t \leq l(A_t)$. If $\alpha = a + 1 \in \Gamma_t$ and $\beta = b + 1 \in \Gamma = I + 1$, then we will prove that the commutator $[\alpha, \beta] \in \Gamma_{t-1}$. Computing $[\alpha, \beta]$ we have that

$$[\alpha, \beta] = \alpha^{-1} \beta - \alpha^{-1} \beta^{-1} = \alpha^{-1} \beta^{-1} (\alpha \beta - \beta \alpha) + 1 = \alpha^{-1} \beta^{-1} (\alpha + 1)(b + 1) - (b + 1)(\alpha + 1) + 1 = \alpha^{-1} \beta^{-1} ab - ba + 1$$

Denoting $\alpha = (1 + a_t)z$ with $a_t \in A_t$ and $z \in Z$, we have that $\alpha = z + \bar{a}$, with $\bar{a} \in A_t$. On the other hand

$$ab - ba = (\bar{a} + z)(b + 1) - (b + 1)(\bar{a} + z) = \bar{a}b - b\bar{a} = (a, b) \in A_{t-1},$$

because $\bar{a} \in A_t$. So $\alpha^{-1} \beta^{-1} (ab - ba) \in A_{t-1}$ and $[\alpha, \beta] \in \Gamma_{t-1} + 1 \subseteq \Gamma_{t-1}$. \hfill $\square$

Corollary 3.4. Let $\{A_t\}$ be an annihilating sequence of the ideal $I$ of $R$ and let $\Gamma = I + 1$. If $Z = Z \cap \Gamma$, where $Z$ is the center of $R$. Then the central series $(\Gamma_t)_{t \geq 1}$, and $\Gamma_0 = \{1\}$, satisfies that the following property, for all $t > 0$:

If $a + 1 \in \Gamma_t = (A_t + 1)Z$, then $(a, b) \in A_{t-1}$ for all $b + 1 \in \Gamma$.

Of course, these results are true for the endomorphism ring $E$, for the radical $J$, and the maximal normal $p$-subgroup of $AutG$, $\Delta(G) = J + 1$.

4. THE ANNIHILATING FUNCTIONS

In this section we introduce the function, that we call Upper annihilating function which describes the upper annihilating sequence. In particular the function $A^{4.2}$ gives the algorithm to construct the upper central series of $\Delta(G)$. First we compute the annihilator of $J$, $Ann(J) = J_1$.

Lemma 4.1. The annihilator of $J$ is the ideal $J_1$ described by the matrices $(A_{ij})_{s \times s}$, such that $A_{ij} = 0$ for all $(i, j) \neq (s, s)$, and $(A_{i(s, s)}) = p^{n_i-1}E_{ss}$. 

Proof. By definition $J_1 = Ann(J) = \{A \in J | AB = BA = 0, \forall B \in J\}$ using the matrix representation of $E$. Suppose that $A \in J_1$, and $B \in J$, then the condition for the annihilating ideal implies that

\begin{equation}
(i, j) \quad \sum_{k=1}^{s} A_{ik}B_{kj} \equiv \sum_{k=1}^{s} B_{ik}A_{kj} \equiv 0 \pmod{p^n}.
\end{equation}

If $B_{ks} = 0$, for $k \neq s$, and $B_{ss} = pI_{ss}$, then for all $i \neq s$ in $\{1 - (i, s)\}$, we have that $A_{is} \equiv 0 \pmod{p^n}$, and $pA_{ss} \equiv 0 \pmod{p^n}$, so $A_{ss} \equiv 0 \pmod{p^{n-1}}$. Similarly we prove that $A_{ij} \equiv 0 \pmod{p^{n_i}}$, for all $(i, j) \neq (s, s)$. On the other hand, an element $A \in J$ with these conditions satisfies that $A \in J_1$. Then our claim holds. We can observe that the ideal function associated to $J_1 = Ann(J)$ is $\beta(s, s) = n_s - 1$, and $\beta(i, j) = 0$, for $(i, j) \neq (s, s)$. \hfill $\square$

We use the matrix representation of $J = (p^{n_i-n_j} + \delta_{ij}E_{ij})$, where $\delta_{ij}$ is the Kronecker’s delta, and $E_{ij}$ is the set of $r_i \times r_j$-matrices with entries in $Z_{p^{n_j}}$.

Theorem 4.2. The sequence of ideals

$$0 = A_0 \subset \cdots \subset A_{t-1} \subset A_t \subset \cdots \subset A_n = J,$$

is an annihilating sequence of $J$ if, and only if, there exists an ideal function $f$ such that $A_t = (p^{n_i-f(i,j,t)}E_{ij})$, for all $(i, j, t) \in D$, which satisfies:

\begin{enumerate}
  \item $f(s, s, 1) = 1$, and $f(i, j, 1) = 0$,
  \item $f(i, j, t) \leq f(i, j, t-1) + 1$,
\end{enumerate}

\begin{enumerate}
  \item $f(s, s, 1) = 1$, and $f(i, j, 1) = 0$,
  \item $f(i, j, t) \leq f(i, j, t-1) + 1$,
\end{enumerate}
(3) \( f(i-1, j, t) \leq f(i, j, t-1) \leq f(i, j, t) \),
(4) \( f(i, j-1, t) \leq f(i, j, t-1) \leq f(i, j, t) \),
(5) \( f(i+1, j, t) \leq f(i, j, t-1) \),
(6) \( f(i, j+1, t) \leq f(i, j, t-1) \).

We will call this class of functions, the annihilating functions.

Proof. \((\Rightarrow)\) Suppose \( \{A_t\} \) is an annihilating sequence of \( \mathcal{J} \). So, there exists an ideal function associated to this sequence that we denote, \( \beta_t(i, j) = n_i - f(i, j, t) \). We claim that \( f \) is an annihilating function, so we prove the six properties of the function \( f \) in the theorem. By Lemma 4.1 the condition (1) holds. Because \( A_{t-1} \subseteq A_t \), for \( t \geq 2 \), we have that

\[
p^{n_i-f(i,j,t-1)}E_{ij} \subseteq p^{n_i-f(i,j,t)}E_{ij} \Rightarrow f(i,j,t-1) \leq f(i,j,t).
\]

By definition of annihilating sequence, if we take \( A \in A_t \), and \( B \in \mathcal{J} \), then \( AB \in A_{t-1} \). On the other hand, \( AB = (\sum_{k=1}^{s} A_{ik}B_{kj})_{s \times s} \), with \( A_{ik} \in p^{n_i-f(i,k,t)}E_{ik} \), and \( B_{kj} \in p^{n_j-n_k+\delta_{kj}}E_{kj} \). For all \( 1 \leq k \leq s \), we have the following

\[
A_{ik}B_{kj} \in p^{n_i-f(i,k,t)+n_j-n_k+\delta_{kj}}E_{ik}E_{kj} \Rightarrow \]

\[
p^{n_i-f(i,k,t)+n_j-n_k+\delta_{kj}}E_{ij} \subseteq p^{n_i-f(i,j,t-1)}E_{ij}
\]

because \( AB \in A_{t-1} \).

Therefore

\[
n_k - f(i, k, t) + n_j - n_k + \delta_{kj} \geq n_j - f(i, j, t-1) \Rightarrow \]

\[
f(i, k, t) - \delta_{kj} \leq n_k - n_k + f(i, j, t-1).
\]

On the other hand, \( BA \in A_{t-1} \), and \( BA = (\sum_{k=1}^{s} B_{ik}A_{kj})_{s \times s} \), where the matrix \( B_{ik} \in p^{n_j-n_i+n_k+\delta_{ik}}E_{ik} \), and \( A_{kj} \in p^{n_i-f(k,j,t)}E_{kj} \). For all \( 1 \leq k \leq s \), we have the following

\[
n_j - f(k, j, t) + n_k - n_i + \delta_{ik} \geq n_j - f(i, j, t-1) \Rightarrow \]

\[
f(k, j, t) - \delta_{ik} \leq n_k - n_i + f(i, j, t-1).
\]

Using the equation (2) we obtain the following relations

\[
k = j + 1 \Rightarrow f(i, j + 1, t) \leq n_{j+1} - n_j + f(i, j, t-1)
\]

\[
k = j \Rightarrow f(i, j, t) \leq f(i, j, t-1) + 1
\]

\[
k = j - 1 \Rightarrow f(i, j - 1, t) \leq f(i, j, t-1)
\]

Using the equation (3) we obtain the following relations.

\[
k = i + 1 \Rightarrow f(i + 1, j, t) \leq n_{i+1} - n_i + f(i, j, t-1)
\]

\[
k = i \Rightarrow f(i, j, t) \leq f(i, j, t-1) + 1
\]

\[
k = i - 1 \Rightarrow f(i - 1, j, t) \leq f(i, j, t-1)
\]

The condition (1) is proved by Lemma 4.1 So, our claim holds.

\((\Leftarrow)\) It is easy to prove that the chain of ideals \( A_t = (p^{n_i-f(i,j,t)}E_{ij}) \), where \( f \) is an annihilating function, is an annihilating sequence of \( \mathcal{J} \). So, the theorem holds.

It is clear that the upper annihilating sequence has the minimal length between the annihilating sequences. Therefore, the annihilating function associated to the \((\text{us})\) is the minimum annihilating function, that we denote by \( \alpha \). The symbol \( \wedge \) means the minimum number.
Corollary 4.3 (The Upper Function). For \( t \geq 1 \), the upper annihilating function \( \alpha \) is a recursive function on \( t \), defined by the following formulas:

1. \( \alpha(s, s, 1) = 1, \alpha(i, j, 1) = 0, \) for \( (i, j) \neq (s, s) \)
2. \( \alpha(i, j, t) = \bigwedge \{n_j - n_{j-1} + \alpha(i, j-1, t-1); n_i - n_{i-1} + \alpha(i-1, j, t-1);\}
   \( \alpha(i, j, t-1) + 1; \alpha(i + 1, j, t-1); \alpha(i, j + 1, t-1)\}. \)

we consider the function \( \alpha : S \times S \times T \to \mathbb{N} \), where \( S = \{1, \ldots, s\} \), and \( T = \{1, \ldots, d = l(J_t)\} \).

Proposition 4.4. The following properties holds

1. \( \alpha(i + 1, j, t) \leq \alpha(i, j, t) \), for all \( 1 \leq i \leq s - 1 \),
2. \( \alpha(i, j, t) \leq \alpha(i, j, t + 1) \), for all \( t \leq l(J_t) \),
3. \( \alpha(i, j + 1, t) \leq \alpha(i, j, t) \), for all \( 1 \leq j \leq s - 1 \),
4. \( \text{For all } j < s, n_j - \alpha(j, j, t) \leq n_s - \alpha(s, s, t) \).

Proof. Because \( \alpha \) is an annihilating function, it satisfies the properties in Theorem 4.2 so again the properties (1), (2), (3).

The proof of property (4) is the following

\[ \alpha(j + 1, j + 1, t) \leq \alpha(j, j + 1, t) \leq \alpha(j, j, t) \]

by properties (1), and (3). On the other hand, we have that

\[ n_{j+1} - n_j \geq 0, \text{ and } \alpha(j, j, t) - \alpha(j + 1, j + 1, t) \geq 0. \]

Therefore, \( n_j - \alpha(j, j, t) \leq n_{j+1} - \alpha(j + 1, j + 1, t) \), for all \( j < s \), and the property (4) holds.

\( \square \)

Remark 4.5. Using the matrix representation of \( E \), we proved that \( J_t = (p^{n_i - \alpha(i, j, t)} E_{ij}) \).

We want to remark that the integers \( \alpha(i, j, t) \) depend only on the exponents \( (n_1, \ldots, n_s) \) of the group \( G \), but not the ranks of the homocyclic components of \( G \).

In this paper we use the following notation, for \( x \in \mathbb{R}_{\geq 0} \), \([x]\) = the integer part of \( x \), if \( x < 0 \), then \([x]\) means 0.

Lemma 4.6 (Case 1 for \( \alpha \)). If \( \sigma(G) \geq 2 \) then \( \alpha(i, j, t) = [t + i + j - 2s] \).

Proof. If \( \sigma(G) \geq 2 \), we use induction in order to prove that \( \alpha(i, j, t) = [t + i + j - 2s] \).

For \( t = 1 \) is trivial, suppose that the theorem holds for \( t - 1 \), then

\[ \alpha(i, j, t) = \bigwedge \{n_j - n_{j-1} + [t - 2 + j + i - 2s]; n_i - n_{i-1} + [t - 2 + j + i - 2s]; \}
   \[ t - 1 + i + j - 2s] + 1; [t + i + j - 2s]; [t + i + j - 2s]\} = [t + i + j - 2s] \]

\( \square \)

Lemma 4.7 (Case 2 for \( \alpha \)). If \( \sigma(G) = 1 \), and \( n_s - n_{s-1} = 1 \) then

\[ \alpha(i, j, t) = \left[ \frac{t + i + j - 2s + 1}{2} \right]. \]

Proof. We use induction to prove the Lemma, so for \( t = 1 \) is trivial. Suppose the lemma holds for \( t - 1 \), and we have

\[ \alpha(i, j, t) = \bigwedge \left\{ n_j - n_{j-1} + \left[ \frac{t + i + j - 2s - 1}{2} \right]; n_i - n_{i-1} + \left[ \frac{t + i + j - 2s - 1}{2} \right]; \right\} \]

\[ \left[ \frac{t + i + j - 2s}{2} \right] + 1; \left[ \frac{t + i + j - 2s + 1}{2} \right]; \left[ \frac{t + i + j - 2s + 1}{2} \right] \} \]
It is obvious that $\alpha(i, j, t) = \left[ \frac{t + i + j - 2s + 1}{2} \right]$. \square

The third case for $\alpha$ is the following: $\sigma(G) = 1$, and $n_j - n_{j-1} = 1$, but $j < s$. There are formulas for all the cases. In the second paper, general formulas for the length of the upper annihilating sequence, and the nilpotency class of $\Delta$ are given.

**Example 4.8.** Consider a group of type $\mathfrak{A}(G) = (3, 5, 6, 8, 10)$ with $p$-rank $r_i = 1$, for all $i$. The matrices in the annihilating sequence have the entries by columns in $\mathbb{Z}_{p^3}, \mathbb{Z}_{p^5}, \mathbb{Z}_{p^6}, \mathbb{Z}_{p^8},$ and $\mathbb{Z}_{p^{10}}$ respectively. Observe that this is the third case, because $\sigma(G) = 1$, but $n_s - n_{s-1} = 2$. Using the Corollary 1.3 the upper annihilating sequence is the following, we compute the recursive formula for the ideals $\mathcal{J}_t$. We only include the correspondent power of $p$ in the place $(i, j)$, the meaning is: that place is congruent to 0 modulo $p^r$. The matrices represent the sequence of ideals $\mathcal{J}_1 \subset \cdots \subset \mathcal{J}_{14} = \mathcal{J}$.

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p^9 \\
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p^7 & p^9 \\
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p^7 & p^9 \\
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & p^7 & p^9 & p^9 \\
\end{pmatrix},
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p^9 & 0 \\
0 & 0 & p^7 & p^9 & 0 \\
0 & 0 & p^7 & p^9 & 0 \\
0 & p^4 & p^4 & p^6 & p^6 \\
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 & 0 & p^9 \\
0 & 0 & 0 & p^9 & 0 \\
0 & 0 & p^7 & p^9 & 0 \\
0 & 0 & p^7 & p^9 & 0 \\
0 & p^2 & p^4 & p^4 & p^4 \\
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 & 0 & p^9 \\
0 & 0 & 0 & p^9 & 0 \\
0 & 0 & p^7 & p^9 & 0 \\
0 & 0 & p^7 & p^9 & 0 \\
0 & p^2 & p^4 & p^4 & p^4 \\
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 & p^9 & p^9 \\
0 & 0 & p^7 & p^9 & p^9 \\
0 & 0 & p^7 & p^9 & p^9 \\
0 & 0 & p^7 & p^9 & p^9 \\
0 & p^2 & p^4 & p^4 & p^4 \\
\end{pmatrix},
$$

$$
\begin{pmatrix}
0 & 0 & p^5 & 0 & 0 \\
0 & 0 & p^7 & p^8 & 0 \\
0 & p^4 & p^6 & p^8 & 0 \\
0 & p^4 & p^6 & p^8 & 0 \\
0 & p^3 & p^3 & p^3 & p^3 \\
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & p^5 & 0 & 0 \\
0 & 0 & p^7 & p^8 & 0 \\
0 & p^4 & p^6 & p^8 & 0 \\
0 & p^4 & p^6 & p^8 & 0 \\
0 & p^3 & p^3 & p^3 & p^3 \\
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & p^5 & 0 & 0 \\
0 & 0 & p^7 & p^8 & 0 \\
0 & p^4 & p^6 & p^8 & 0 \\
0 & p^4 & p^6 & p^8 & 0 \\
0 & p^3 & p^3 & p^3 & p^3 \\
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & p^5 & 0 & 0 \\
0 & 0 & p^7 & p^8 & 0 \\
0 & p^4 & p^6 & p^8 & 0 \\
0 & p^4 & p^6 & p^8 & 0 \\
0 & p^3 & p^3 & p^3 & p^3 \\
\end{pmatrix},
$$

$$
\begin{pmatrix}
p & p^2 & p^3 & p^4 & p^5 \\
p & p^2 & p^3 & p^4 & p^5 \\
p & p^3 & p^4 & p^5 & p^6 \\
p & p^3 & p^4 & p^5 & p^6 \\
* & p^2 & p^4 & p^4 & p^4 \\
\end{pmatrix}, \quad
\begin{pmatrix}
p & p^2 & p^3 & p^4 & p^5 \\
p & p^2 & p^3 & p^4 & p^5 \\
p & p^3 & p^4 & p^5 & p^6 \\
p & p^3 & p^4 & p^5 & p^6 \\
* & p^2 & p^4 & p^4 & p^4 \\
\end{pmatrix}, \quad
\begin{pmatrix}
p & p^2 & p^3 & p^4 & p^5 \\
p & p^2 & p^3 & p^4 & p^5 \\
p & p^3 & p^4 & p^5 & p^6 \\
p & p^3 & p^4 & p^5 & p^6 \\
* & p^2 & p^4 & p^4 & p^4 \\
\end{pmatrix}, \quad
\begin{pmatrix}
p & p^2 & p^3 & p^4 & p^5 \\
p & p^2 & p^3 & p^4 & p^5 \\
p & p^3 & p^4 & p^5 & p^6 \\
p & p^3 & p^4 & p^5 & p^6 \\
* & p^2 & p^4 & p^4 & p^4 \\
\end{pmatrix},
$$

$$
\begin{pmatrix}
p & p^2 & * & p^3 & p^3 \\
* & p & p & p^2 & p^3 \\
* & p & p & p & p^2 \\
* & * & * & p & p \\
* & * & * & * & p \\
\end{pmatrix}, \quad
\begin{pmatrix}
p & p^2 & * & p^3 & p^3 \\
* & p & p & p & p \\
* & p & p & p & p \\
* & * & * & p & p \\
* & * & * & * & p \\
\end{pmatrix},
$$

with annihilating length $d = 14$. 

$$
\mathcal{J}_{14} = \begin{pmatrix}
p & p^2 & p^3 & p^4 \\
* & p & p & p^2 \\
* & p & p & p \\
* & * & * & * \\
* & * & * & * \\
\end{pmatrix} = \mathcal{J}
$$
5. The (uas)-(ucs)bounded abelian $p$-groups

Let $\{J_t\}$ be the upper annihilating sequence of $\mathcal{J}$. We denote the nilpotency class of $\Delta(G)$ by $n(\Delta)$, and $Z = \Delta(G) \cap Z$, where $Z$ is the center of the ring $\mathcal{E}$, (2b). Denoting by $\Gamma_t = (\mathcal{J}_t + 1)Z$, for $t > 1$, and $\Gamma_0 = \{1\}$, we will prove that $\Gamma_t = Z_t$. By Theorem 6.1, the series $(\Gamma_t)$ is a central series, so we only need to prove that $Z_t \leq \Gamma_t$, for all $t$. The Lemma 5.2 gives a characterization of the elements in the groups $\Gamma_t$. This characterization of $\Gamma_t$ is very important because we will prove that $\Gamma_t = Z_t$ for almost all of groups $G$.

Definition 5.1. We will say that the abelian $p$-group $G$ has the property (uas)-(ucs)–group for the Jacobson radical $\mathcal{J}$ of $\mathcal{E}$, if the upper central series $(Z_t)_{t=0,n(\Delta)}$ of the $p$-subgroup $\Delta(G)$ satisfies the following $Z_t = \Gamma_1$ for all $t$ from 1 to $n(\Delta)$.

If $1_{rr}$ is the identity matrix of size $r \times r$, we use the following notation for the elements of $\Gamma_t$: $C + 1_{rr} \in \Gamma_t$, then $C + 1_{rr} = (c_{ij}^{(l,k)})_{r \times r} + 1_{rr}$. On the other hand $\Gamma_1 = (\mathcal{J}_1 + 1)Z$.

Lemma 5.2 (Characterization of $\Gamma_t$).

(1) $C + 1_{rr} \in \Gamma_t$ if, and only if $C - c_{ss}^{(r,r)}1_{rr} \in J_t$,

(2) $\Gamma_t = \left\{ C + 1_{rr} \in \Delta(G) \mid \begin{array}{l}
    c_{ij}^{(l,k)} \equiv 0 \pmod{p^{n_j \cdot \alpha(i,j,t)}} \quad i \neq j; \\
    c_{ii}^{(l,k)} \equiv c_{ss}^{(r,r)} \pmod{p^{n_i \cdot \alpha(i,i,t)}} \\
    c_{ij}^{(l,k)} \equiv 0 \pmod{p^{n_i \cdot \alpha(i,j,t)}} \quad l \neq k
\end{array} \right\}$

Proof. (1) We know that $C + 1_{rr} \in \Gamma_t = \mathcal{J}_t + Z_\Delta$ if, and only if there exists $\overline{C} \in \mathcal{J}_t$, and a scalar matrix $(pc+1)1_{rr} \in Z$ such that $\overline{C} + (pc+1)1_{rr} = C + 1_{rr}$. If $C + 1_{rr} \in \Gamma_t$, then either for $i \neq j$, or $i = j$, and $l \neq k$, we have that $c_{ij}^{(l,k)} = \overline{c}_{ij}^{(l,k)} \equiv 0 \pmod{p^{n_j \cdot \alpha(i,j,t)}}$. Because $c_{jj}^{(l,0)} = \overline{c}_{jj}^{(l,0)} + pc$, for all $l$, and $j$. We have that $c_{jj}^{(l,k)} - c_{ss}^{(r,r)} = \overline{c}_{jj}^{(l,k)} - c_{ss}^{(r,r)} \equiv 0 \pmod{p^{n_j \cdot \alpha(j,j,t)}}$, by Proposition 4.4, property (4).

Therefore $C - c_{ss}^{(r,r)}1_{rr} \in \mathcal{J}_t$, and $C + 1_{rr} = (C - c_{ss}^{(r,r)}1_{rr}) + (c_{ss}^{(r,r)} + 1)1_{rr} \in \Gamma_t$, if and only if $C - c_{ss}^{(r,r)}1_{rr} \in \mathcal{J}_t$.

(2) It is a consequence of part (1). □

6. Construction of the upper central series using the upper annihilating sequence

This section is devoted to prove Theorem 6.1. In fact, we want to prove $\Gamma_t = Z_t$ for all $t \geq 1$, we know that $\Gamma_t \leq Z_t$, so we need to prove that $Z_t \leq \Gamma_t$. We use induction on $t$, for $t = 1$ is proved in Lemma 6.2, under assumption the property holds for $t-1$, then we prove the property for $t$ holds too. It is done in Lemmas 6.2 and 6.4. We want to remark that in the hypothesis of induction on $t$ is $\Gamma_{t-1} = Z_{t-1}$.

Theorem 6.1. The group $G = \oplus_{i=1}^{s} G_i$, where the $p$-rank of $G_i$ is $r_i$, is a (uas)-(ucs)–group for $\Delta(G)$, in the following cases

(1) $r_s > 1$, and $s \geq 2$, for all prime number $p$,
(2) $\sigma(G) \geq 2$, for all prime number $p$,
(3) $p \geq 3$.

Lemma 6.2. The center $Z_1$ of the subgroup $\Delta(G)$ is equal $\Gamma_1 = (\mathcal{J}_1 + 1)Z$. 
Proof. We know that $\Gamma_1 \leq Z_1$ by Theorem [3,3]. So, we will prove $Z_1 \leq \Gamma_1$. Taking two elements in $\Delta(G)$ and considering $A + 1 \in Z_1$, we have that

$$(A + 1)(B + 1) = (B + 1)(A + 1)$$

$$BA - AB = (A, B) = 0,$$

for all $B + 1 \in \Delta(G) = J + 1$. But

$$(BA - AB)_{ij} = \sum_{k=1}^{s} (B_{ik}A_{kj} - A_{ik}B_{kj}) \equiv 0 \pmod{p^n}.$$  

Taking for the matrix $B = (B_{ij})_{s \times s}$ the elementary matrices $A - a_{ss}^{(r,r)}1_{rr} \in J$ so $A + 1 \in \Gamma_1$. Therefore $\Gamma_1 = Z_1$, and our claim holds.

A trivial consequence is that $\Delta(G)$ is abelian if, and only if $G$ has type $t(G) = (2)$.

In the following Lemma we prove that $Z_t \leq \Gamma_t$, for the cases described in the Lemma [6,4] assuming the condition (6).

Lemma 6.3. The group $G$ is a (uas)-(ucs)-group for $\Delta(G)$ if and only if the following condition holds,

$$(6) \quad \text{If } A + 1 \in Z_t \text{ then } BA - AB = (B, A) \in J_{t-1} \text{ for all } B + 1 \in \Delta(G).$$

Proof. ($\Rightarrow$) Suppose the group is (uas)-(ucs)-group, that is $\Gamma_t = Z_t$ for all $t$, then the condition holds by Corollary [3,4].

($\Leftarrow$) Suppose the group satisfies the condition (6). By definition of $Z_t$, we have that for all $B + 1 \in \Delta(G)$, and $A + 1 \in Z_t$ then

$$[B + 1, A + 1] = C + 1 \in Z_{t-1} = \Gamma_{t-1} = (J_{t-1} + 1)Z = J_{t-1} + Z$$

by induction. This implies that $BA - AB = C(A + 1)(B + 1)$, where $C + 1 \in Z_{t-1} = \Gamma_{t-1}$. Therefore

$$(C(A + 1)(B + 1))_{ij} = \sum_{k=1}^{s} C_{ik} \sum_{l=1}^{s} (A_{kl} + \delta_{kl})(B_{lj} + \delta_{lj}),$$

But $(BA - AB)_{kj} = \sum_{i=1}^{s} (B_{ki}A_{ij} - A_{ki}B_{ij}) \equiv 0 \pmod{J_{t-1}}$, by condition (6).

So,

$$(7) \quad (kj) \quad \sum_{i=1}^{s} (B_{ki}A_{ij} - A_{ki}B_{ij}) \equiv 0 \pmod{p^{n_j - \alpha(k,j,t-1)}}$$

Now, we will prove that $A - a_{ss}^{(r,r)}1_{rr} \in J_t$, and of course $A + 1 \in \Gamma_t$ then the group is (uas)-(ucs)-group. The idea is to select for the element $B$ different matrices, in such a way that all the conditions for the upper annihilating functions are satisfied. In fact, in (7)-(kj), we have the following

| Taking $B = E_{k_i}^{(u,v)}$ | for $i \neq j$, $e_{k_i}^{(u,v)}a_{ij}^{(v,t)} \equiv 0 \pmod{p^{n_j - \alpha(k,j,t)}}$ |
|-----------------------------|---------------------------------------------------------------------------------------------------|
| if $k = i + 1$ then        | $e_{i+1,i}^{(u,v)} = 1$, $a_{ij}^{(v,t)} \equiv 0 \pmod{p^{n_j - \alpha(i+1,j,t)}}$ |
| if $k = i - 1$ then        | $e_{i-1,i}^{(u,v)} = p^{n_i - n_{i-1}}$, $a_{ij}^{(v,t)} \equiv 0 \pmod{p^{n_j - n_{i+1} + n_{i-1} - \alpha(i-1,j,t)}}$ |

| Taking $B = E_{k_j}^{(u,v)}$ | for $i \neq j$, $a_{ki}^{(l,u)}e_{i,j}^{(u,v)} \equiv 0 \pmod{p^{n_j - \alpha(k,j,t)}}$ |
|-----------------------------|---------------------------------------------------------------------------------------------------|
| if $j = i + 1$ then        | $e_{i+1,i}^{(u,v)} = p^{n_i + n_{i+1}}$, $a_{ki}^{(l,u)} \equiv 0 \pmod{p^{n_j - n_{i+1} + n_{i-1} - \alpha(k,i+1,t-1)}}$ |
| if $j = i - 1$ then        | $e_{i-1,i}^{(u,v)} = 1$, $a_{ki}^{(l,u)} \equiv 0 \pmod{p^{n_j - n_{i-1} + n_{i+1} - \alpha(k,i-1,t-1)}}$ |
Similarly, for \( i \neq j \) we obtain that \( a_{ij}^{(l,k)} \equiv 0 \pmod{p^{n_j-1-\alpha(i,j,t-1)}} \), and \( a_{ij}^{(l,k)} \equiv 0 \pmod{p^{n_j-\alpha(i,j,t)}} \) because

\[
\alpha(i,j,t) = \sum_{i,j,t-1} (n_j - n_i - 1 + \alpha(i,j-1,t-1) + n_i - n_{i-1} + \alpha(i-1,j,t-1)),
\]

In the same way for the matrix \( A - a_{ss}^{(r,s)}1_{rr} \), we prove that

\[
a_{jj}^{(l)} - a_{ss}^{(r,s)} \equiv 0 \pmod{p^{n_j-\alpha(j,j,t)}}, \text{ for } j < s
\]

and if the \( p \)-rank \( r_s \) of the last homocyclic group \( G_s \) is greater than 1, we have that

\[
a_{ss}^{(l)} - a_{ss}^{(r,s)} \equiv 0 \pmod{p^{n_s-\alpha(s,s,t)}}, \text{ for } l \leq r.
\]

We want to remark that in the place \( rr \) of the matrix \( A - a_{ss}^{(r,s)}1_{rr} \) we have 0. Therefore \( A - a_{ss}^{(r,s)}1_{rr} \in \mathcal{J}_t, A + 1 \in \mathcal{J}_t + 1 \), and the group is a \((uas)-(ucs)\)-group. So, our claim holds. \( \square \)

**Lemma 6.4.** The condition (6) holds in the following cases

1. \( r_s > 1 \), and \( s \geq 2 \), for all prime number \( p \),
2. \( s(G) \geq 2 \), for all prime number \( p \),
3. otherwise, only for \( p \geq 3 \).

**Proof.** I need to prove that \( B, A \in \mathcal{J}_{t-1} \), for \( A + 1 \in \mathcal{J}_t \), and \( B + 1 \in \Delta(G) \). We have the following expression

\[
(B, A) = C(A + 1)(B + 1) = \sum_{k=1}^{s} \sum_{l=1}^{s} C_{ik} \sum_{i,j}^{s} (A_{kl} + \delta_{kl})(B_{lj} + \delta_{lj}),
\]

we have that \( (B, A) \in \mathcal{J}_{t-1} \) if, and only if \( C \in \mathcal{J}_{t-1} \). We know that for all \( B \) and \( A + 1 \in \mathcal{Z}_t \), we have \( C + 1 \in \mathcal{J}_{t-1} = \Gamma_{t-1} \), by induction. So by Lemma [5,2] we have that:

\[
C + 1 \in \Gamma_{t-1} \iff C - c_{ss}^{(r,r)}1_{rr} \in \mathcal{J}_{t-1}, \text{ and } C \in \mathcal{J}_{t-1} \text{ if and only if } c_{ss}^{(r,r)}1_{rr} \in \mathcal{J}_{t-1}.
\]

But

\[
BA - AB = (C - c_{ss}^{(r,r)}1_{rr} + c_{ss}^{(r,r)}1_{rr})(A + 1)(B + 1).
\]

By \( C + 1 \in \mathcal{Z}_{t-1} = \Gamma_{t-1} \), then \( C - c_{ss}^{(r,r)}1_{rr} \in \mathcal{J}_{t-1} \), so

\[
BA - AB \equiv c_{ss}^{(r,r)}1_{rr}(A + 1)(B + 1) \pmod{\mathcal{J}_{t-1}}.
\]

Because the elementary endomorphism defined in Section 2 form a set of generator for the Jacobson radical \( \mathcal{J} \), we will prove that our claim is true for all the elementary matrix in \( \mathcal{J} \).

\[
(E_{ij}^{(k,u)}, A) \equiv c_{ss}^{(r,r)} \sum_{l=1}^{s} (A_{kl} + \delta_{kl})(E_{lj}^{(k,u)} + \delta_{lj}), \pmod{p^{n_j-\alpha(i,j,t-1)}}.
\]

We want to remark here that in general, all elementary endomorphism have a different matrix \( C + 1 \). The notation of the entries in the matrix \( C \) are the following \( c_{ij}^{(l,k)} \), so if we need to include another notation in these elements, it will be more complicated. Then we prefer to consider only the notation \( c_{ss}^{(r,r)} \) for all the matrices, knowing that in each case we have the possibility to have a different \( C \). We will prove that in all the cases the element \( c_{ss}^{(r,r)} \equiv 0 \pmod{p^{n_s-\alpha(s,s,t-1)}} \) First we prove that for all the elementary matrices such that \( (i, j) \neq (s, s) \) then the entry \( c_{ss}^{(r,r)} \) in
the associated matrix $C$ satisfies the condition for the elements in $J_{l-1}$, for $k \neq r$, and $u \neq r$

| $(E_{ij}^{(k,u)}, A)^{(r,r)}_{ss}$ | $0 \equiv c_{s,s}^{(r,r)} (a_{ss}^{(r,r)} + 1) \pmod{p^{n_s-\alpha(s,s,t-1)}}$ |
|-----------------------------|----------------------------------------------------------------------------------|
| but $(a_{ss}^{(r,r)} + 1)^{-1}$ exists | then $0 \equiv c_{s,s}^{(r,r)} \pmod{p^{n_s-\alpha(s,s,t-1)}}$ |
| then $c_{s,s}^{(r,r)} 1_{rr} \in J_{l-1}$ | and $(E_{ij}^{(k,u)}, A) \in J_{l-1}$ |

Then we have proved the condition for all the entries in the elementary endomorphism, where $k < r$, and $u < r$.

(Case number 1) $r_s > 1$, and $s \geq 2$, for all prime number $p$.
Suppose that $k = r$ and $l \neq r$. Then in the place $(r-1, r-1)$ we have $0 \equiv c_{s,s}^{(r,r)} (a_{ss}^{(r-1,r-1)} + 1) \pmod{p^{n_s-\alpha(s,s,t-1)}}$, and our claim holds because $(E_{ij}^{(r,u)}, A) \in J_{l-1}$. Similarly we prove that $(E_{ls}^{(r,r)}, A) \in J_{l-1}$, even $(E_{ss}^{(r,r)}, A) \in J_{l-1}$. The case is solved.

(Case number 2) $\sigma(G) \geq 2$, for all prime number $p$.
The solution is similar the above case, considering $r_s = 1$, and $\sigma(G) \geq 2$, that are the correspondent unsolve case.

(Case number 3) $p \geq 3$.
We have the cases for giving solution
(a) $r_s = 1$, $\sigma(G) = 1$, and $n_s > n_{s-1} + 1$. The solution is similar the first case,
(b) $r_s = 1$, $\sigma(G) = 1$, and $n_s = n_{s-1} + 1$, we need the condition $p \geq 3$, and only in this case.

We give here the solution:

$$
E_{ij}^{(r,u)}, A)^{(r,r)}_{ss} = e_{s_j}^{(r,u)} a_{is}^{(u,r)} \Rightarrow 
E_{ij}^{(r,u)}, A)^{(r,r)}_{ss} = c_{s,s}^{(r,r)} (a_{ss}^{(r,r)} + 1) \pmod{p^{n_s-\alpha(s,s,t-1)}} \quad (I)
$$

$$
E_{ij}^{(r,u)}, A)^{(r,r)}_{ss} = -a_{is}^{(u,r)} e_{s_j}^{(r,u)} \Rightarrow 
E_{ij}^{(r,u)}, A)^{(r,r)}_{ss} = c_{s,s}^{(r,r)} (a_{ss}^{(r,r)} + 1 + a_{is}^{(u,r)}) \pmod{p^{n_s-\alpha(j,j,t-1)}} \quad (II)
$$

taking $j = s - 1$ we have by Lemma 1.7

$$
n_s - \alpha(s,s,t-1) = n_{s-1} - \alpha(s-1, s-1, t-1)
$$

summand $(I)$ and $(II)$

$$
0 \equiv c_{s,s}^{(r,r)} (a_{js}^{(l,l)} + 1 + a_{is}^{(u,r)} + a_{ss}^{(r,r)} + 1) \pmod{p^{n_s-\alpha(s,s,t-1)}}
$$

because $(a_{js}^{(l,l)} + a_{is}^{(u,r)} + a_{ss}^{(r,r)} + 2)^{-1}$ exists for $p \neq 3$

$$
c_{s,s}^{(r,r)} 1_{ss} \in J_{l-1}, \text{ and } (E_{ij}^{(r,u)}, A) \in J_{l-1}
$$

Similarly we prove that $(E_{ls}^{(l,r)}, A) \in J_{l-1}$, even $(E_{ss}^{(r,r)}, A) \in J_{l-1}$. The case is solved.

\[ \square \]

If $S = \{1, \ldots, s\}$, we denote by $y(G)$ the least $t$ such that $\alpha(i,j,t) \geq f_j(i,j)$, for $(i,j) \in S \times S \setminus \{(s,s)\}$. Clearly $y(G) \leq l(J) = d$

**Theorem 6.5.** If $G$ is a $(uas)$-$(ucs)$--group for $\Delta(G)$ then

\[ n(G) = \left\{ \begin{array}{ll}
\ell(J) & \text{if } r_s \geq 2 \\
y(G) & \text{if } r_s = 1
\end{array} \right. \]

**Proof.** By Theorem 6.3, we know that $n(G) \leq l(J)$.

Suppose $r_s > 2$, and $n(G) = d = l(J)$ then we have that $\alpha(s,s,d-1) < n_s - 1$. 

If $S = \{1, \ldots, s\}$, we denote by $y(G)$ the least $t$ such that $\alpha(i,j,t) \geq f_j(i,j)$, for $(i,j) \in S \times S \setminus \{(s,s)\}$. Clearly $y(G) \leq l(J) = d$

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By definition of elementary endomorphism the matrix $E_{ss}^{(l,k)} \in \mathcal{J}_d = \mathcal{J}$ for $(l,k) \neq (r,r)$. But $E_{ss}^{(l,k)} \notin \mathcal{J}_{d-1}$ because $E_{ss}^{(l,k)} \not\equiv 0 \pmod{p^{n_d-\alpha(s,s,d-1)}}$, because $n_s - \alpha(s,s,d-1) \geq 2$. So, $E_{ss}^{(l,k)} + 1_{ss} \in \Delta(G) - \Gamma_{d-1}$, and as a consequence $\Gamma_{d-1} < \Delta$, and $n(G) = d = l(\mathcal{J})$.

If $r_s = 1$, we prove that $\Delta(G) = \mathcal{J} + 1 \leq \Gamma_y(G)$, because trivially $\Gamma_y(G) \leq \Delta(G)$.

If we consider $C + 1_{ss} \in \Delta(G) = \Gamma_d$, then $C - c_{ss}^{(r,r)}1_{rr} \notin \mathcal{J}_d$, but we assume that $C - c_{ss}^{(r,r)}1_{rr} \notin \mathcal{J}_y(G)$. Therefore $C + 1_{ss} \in \Gamma_y(G)$ by Lemma \ref{lem:example}, then $\Delta(G) \leq \Gamma_y(G)$, and our claim holds. \hfill \Box

7. Algorithm for computing the upper central series, and the nilpotency class of $\Delta(G)$

Let $G$ be a group with the property (uas)-(ucs)-group, that is the hypercenters $Z_t = \Gamma_t = (A_1 + 1)Z$, where $\{\mathcal{J}_t\}$ is the upper annihilating sequence of $\mathcal{J}$, and $Z = \Delta(G) \cap Z$. Let $C + 1 \in Z_t = \Gamma_t$ be the matrix described as follows: $C = (c_{ij}^{(l,k)})_{r \times r}$, with the entries in $\mathbb{Z}_{p^n}$, for $1 \leq j \leq s$.

**Algorithm 7.1.** Input: $f(G) = (n_1, \ldots, n_s)$, and the $p$-ranks $(r_1, \ldots, r_s)$.

(A1) Compute $r = r_1 + \cdots + r_s$.

(A2) Write the matrix $C^t = (c_{ij}^{(l,k)})_{r \times r}$, with the entries described above.

(A3) If $r_s > 1$ for $t = 0$ until least $t$ such that $\alpha(i,j,t) \geq f_T(i,j)$ for all $(i,j)$ do else for $t = 1$ until the least $t$ such that $\alpha(i,j,t) \geq f_T(i,j)$ for all $(i,j) \neq (s,s)$ do

1. $\alpha(i,j,0) = 0$ for all $(i,j)$.

2. Compute the function $\alpha(i,j,t)$ using formulas in Corollary \ref{cor:algorithm}.

(A4) Write on the matrix $C^t$ of $Z_t$, the following:

for $i \neq j$ do $c_{ij}^{(l,k)} = p^{n_j-\alpha(i,j,t-1)}c_{ij}^{(l,k)} \rightarrow p^{n_j-\alpha(i,j,t)}c_{ij}^{(l,k)}$

for $l \neq k$ do $c_{ij}^{(l,k)} = p^{n_j-\alpha(j,j,t-1)}c_{ij}^{(l,k)} \rightarrow p^{n_j-\alpha(j,j,t)}c_{ij}^{(l,k)}$

(A5) Write the relations for the elements in the diagonal for $j \leq s$, and $l \neq r$ do

$c_{ij}^{(l,l)} = p^{n_j-\alpha(j,j,t-1)}c_{ij}^{(l,l)} + p^{r,r}c_{ss}^{(r,r)} \rightarrow c_{ij}^{(l,l)} = p^{n_j-\alpha(j,j,t)}c_{ij}^{(l,l)} + p^{r,r}c_{ss}^{(r,r)}$

and $pc_{ss}^{rr} \rightarrow pc_{ss}^{rr}$.

(A6) Save $C^t$, and write $Z_t = \{C^t + 1_{rr}\}$.

(A7) Save and write the number $t = n(G)$ when the algorithm stop.

Output: $n(G) =$ the nilpotency class of $\Delta(G)$, and the centers $Z_1, \ldots, Z_{n(G)}$.

**Example 7.2.** For a group of type $(3,5,7)$, with ranks $(1,1,2)$, when $p \geq 3$, we have $\Gamma_1 = Z_1$. Applying the Algorithm we found the upper central series of $\Delta(G)$

Then the elements of the upper central series are

$Z_1 = \left\{ \begin{array}{cc} p_{c_{44} + 1} + 1 & 0 \\ 0 & p_{c_{44} + 1} \\ 0 & 0 & p^6c_{33} + pc_{44} + 1 & p^6c_{34} \\ 0 & 0 & 0 & pc_{44} + 1 \\ 0 & 0 & 0 & 0 \\ p_{c_{44} + 1} & 0 & 0 & 0 \\ 0 & p^6c_{33} + pc_{44} + 1 & 0 & p^6c_{34} \\ 0 & 0 & 0 & pc_{44} + 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & p^6c_{33} + pc_{44} + 1 & p^6c_{34} \\ 0 & 0 & 0 & pc_{44} + 1 \end{array} \right\}$,
Example 7.3. For a group of type \((2, 4, 7)\), with ranks \((1, 1, 1)\), when \(p \geq 3\), we have \(\Gamma_i = Z_i\) for \(t = 1\) to \(t = 2(G)\). Applying the Algorithm we found the upper central series of \(\Delta(G)\). Then the elements of the upper central series are

\[
\Delta(G) = Z_6 = \begin{bmatrix}
 p_{c_{11}} + 1 & p_{c_{11}} c_{12} & p_{c_{11}} c_{13} & p_{c_{11}} c_{14} \\
 c_{21} & p_{c_{22}} + 1 & p_{c_{23}} & p_{c_{24}} \\
 c_{31} & c_{32} & p_{c_{33}} + 1 & p_{c_{34}} \\
 c_{41} & c_{42} & p_{c_{43}} & p_{c_{44}} + 1 \\
\end{bmatrix}
\]

The class of nilpotency of the group \(\Delta(G)\) is \(n(G) = 6\).

The class of nilpotency of the group \(\Delta(G)\) is \(n(G) = 5\), we can check that in this case the length of the upper annihilating sequence is \(l(\mathcal{J}_t) = 6\).
A NEW ALGORITHM FOR FINDING THE NILPOTENCY CLASS OF A FINITE $p$-GROUP

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