Investigation on vibration of the functionally graded material–stepped cylindrical shell coupled with annular plate in thermal environment

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Abstract
This article is concerned with thermal vibration behaviors of the functionally graded material–stepped cylindrical shell coupled with annular plate, including free vibration, transient response, and steady state response. The stepped cylindrical shell is divided into $N_s$ segments at locations of thickness and radius variations, which is coupled with $N_p$ annular plates. The boundary and coupling conditions are achieved by introducing the artificial virtual spring technology. Under the framework of FSDT, the displacement function of arbitrary shell segment and annular plate is expanded with Chebyshev polynomials and Fourier series for circumferential direction. Compared with results obtained by the finite element method and the references, a series of numerical examples and validations are presented to verify the convergence and accuracy of the current method. The effects of the relevant parameters containing the geometric parameters, boundary conditions, various loadings, and the thermal environment are investigated in detail.

Keywords
functionally graded material–stepped cylindrical shell coupled with annular plate, free vibration, transient response, steady state response, thermal environment, Rayleigh–Ritz method

Introduction
The cabin structures of high-speed trains, rockets, and submarines are subjected to the extreme environment such as high speed, extreme high and low temperature, high pressure, erosion, dynamic loads, and even attacks. Therefore, ensuring the stability and safety of the structure in the aforementioned extreme conditions is of great importance in practical engineering fields. Functionally graded material (FGM)–stepped cylindrical shell coupled with annular plate combines the advantages of different materials and the advantages of the coupled structure, which has excellent mechanical properties and stable chemical properties. For this reason, the FGM-stepped cylindrical shell coupled with annular plate is widely applied in many engineering fields, especially in some occasions with extreme high speed, high pressure, and temperature. Subsequently, the relevant investigation on vibration of the shell–plate combinations has attracted extensive attention.
Substantial investigations on cylindrical shell and annular plate have been conducted. Then, some researches have been published on the coupling structure derived from shell and annular plate, and most of them are about shell–shell or cylindrical shell–rectangular combinations. Based on the above study, there have been researches on vibration of the cylindrical shell coupled with annular plate. Ma presented a unified solution for vibration analysis of cylindrical shell coupled with annular plate. By means of a Fourier–Ritz method, displacements of the cylindrical shell and annular plate can be obtained, regardless of the boundary constraints and continuity conditions. Cheng analyzed the free vibration of a structure coupled by a finite cylindrical shell and a circular plate at one end using the Rayleigh–Ritz method together with the artificial spring technique. Cao took an improved Fourier series method into analysis for vibration of a cylindrical shell–circular plate coupled annular plate–cylindrical shell. The wave-based method was introduced by Xie to observe the free and forced vibration of elastically coupled thin annular plate and cylindrical shell structures. Liu took the wave-based method to investigate the free vibration characteristics of functionally graded cylindrical shell under arbitrary boundary conditions, with consideration of the influences of power law exponents and geometric parameters. Qin investigated free vibrations of rotating thin cylindrical shell with an annular moderately thick plate with Chebyshev–Ritz method. The effects of geometric parameters and boundary conditions and stiffness coupling springs were considered. Zhang presented the free and forced vibration analysis of circular cylindrical double-shell structures under arbitrary boundary conditions.

Efforts have also been made to study the influence of temperature on vibration of FGM structures. Aris and Yang study the nonlinear vibration of FGM-truncated conical shell in thermal environment. Then, Fu investigated FGM conical shell subjected to parametric excitation and nonlinear thermal loading. Li presented a semi-analytical method for free vibration of spiral-stiffened multilayer functionally graded cylindrical shells under the thermal environment. Considering the elastic foundation, investigated the free vibration of FGM cylindrical shell in the thermal environment. Li presented a general approach for the thermal vibration of FGM porous–stepped cylindrical shell with characteristic orthogonal polynomials. Ninh analyzed the FGM convex–concave shells with electro-thermal-mechanical loads surrounded by the Pasternak foundation. Minh investigated the free vibration of cracked FGM plate with nonlinear varying thickness under thermal environment. Wang analyzed the effects of volume fraction, distribution pattern, geometrical characteristics, and temperature on the buckling behaviors of the functionally graded carbon nanotube–reinforced composite quadrilateral plate with the first order shear deformation theory (FSDT) and Moving Least Squares (MLS) theory. Based on the three-dimensional theory, Yang investigated the thermal response of FG annular plates. With the generalized differential quadrature method, Javani analyzed the large amplitude forced vibration reduced by rapid heating of the surface of the FG annular plate. Lal also employed the generalized differential quadrature method for analysis of free axisymmetric vibrations of functionally graded circular plates, which is under nonlinear temperature variation. Yang and Fang observed the vibration behaviors of beam and plate under thermal environment. Yang, Zhong, and Tran also carried out parametric analysis for FGM plate.

From the above analysis, it is seen that the FGM-stepped cylindrical shell coupled with annular plate accounting for temperature has not been attempted. Nevertheless, the FGM-stepped cylindrical shell coupled with annular plate is one of extensively used engineering structures because of its superior mechanical and chemical properties. The main challenge to analyze the vibration of FGM-stepped cylindrical shell coupled with annular plate is the complexity of matching the coupling conditions between the substructures which increases the difficulty of the corresponding theoretical modeling. Furthermore, taking the effects of temperature into account also complicates the theoretical formulation. In addition, most of the aforementioned theses about shell–plate combination are carried out with the finite element method (FEM). FEM is efficient and accurate when subjected to vibration of structures. However, ensuring the accuracy of the high-order vibration with FEM leads much computational cost, and remeshing the FE mesh is essential and cumbersome to deal with the elastic boundary and coupling conditions, and unacceptable growth of calculating costs may be caused. Therefore, a general and accurate approach for analysis of the vibration of the FGM-stepped cylindrical shell coupled with annular plate with temperature is of great significance on both theoretical and practical sides.

The main objective of the current article is to present a general approach for analysis of vibration of the FGM-stepped cylindrical shell and annular plate in the thermal environment. The Chebyshev–Ritz method is employed to analyze the characteristics of free and forced vibration of the coupled annular plate–stepped cylindrical shell in the framework of FSDT. The boundary and coupling conditions are achieved by introducing the artificial virtual spring technology. By setting the cylindrical shell segments to the same thickness and radius and setting the inner diameter of the ring plate to zero, the cylindrical shell of uniform thickness coupled with circular plate can be treated as a special shape of the current model. Compared with those obtained by FEM and the references, a series of numerical examples and validations are presented to verify the convergence and accuracy of the current method. Furthermore, the effects of the relevant parameters containing the geometric parameters, boundary conditions, various loadings, and the thermal environment are investigated in detail.
Theoretical analysis

Establishment of the model

As illustrated in Figure 1, the model for analysis is established in the framework of the first-order shear deformation theory. The stepped functionally graded shell is coupled with a several annular plates through a set of elastic springs. The stepped functionally graded cylindrical shell consists of $M$ sub-shell segments, of which the length and thickness are respectively denoted with $L_i$ and $h_i$ ($i = 1, 2, \ldots, M$). $N$ denotes the number of plates which are connected to shell varying for requirements of investigation. In this analysis, the shell segments and annular plates are separately located at a cylindrical coordinate system $(x, \theta, z)$ and polar coordinate system $(r, \theta, z)$. With taking artificial spring technique into application, arbitrary boundary conditions are achieved easily by regulating the stiffness of the corresponding springs. Since the specific implementation of the artificial spring technology was introduced by Refs. 19 and 53, it is not repeated here for sake of brevity.

Material properties

The FGM-stepped cylindrical shell coupled with annular plate is made of a mixture of metal and ceramic materials in this analysis. Since the properties of metal and ceramic materials are sensitive to the environmental temperature, the material properties continuously vary through the thickness from the top surface to the bottom surface of the substructures. For simplicity, the material properties of ceramic and metal are separately denoted by $P_c$ and $P_m$, which can be obtained by the following function of temperature

$$P_c(P_m) = C_0\left(c_0 + c_1 T + c_2 T^2 + c_3 T^3\right)$$

Figure 1. The universal model of the functionally graded material–stepped cylindrical shell coupled with annular plate in thermal environment: (a) Geometry model; (b) Coupling springs; and (c) Boundary constraints.
where $T$ indicates temperature shown by adopting Kelvin (K). The material properties of the model for analysis contain not only Young’s modulus $E$, thermal expansion coefficient $\alpha$, mass density $\rho$, thermal conductivity $\kappa$, and Poisson’s ratio $\nu$ of the constituent materials, but also the volume fraction $V_c$, and $V_m$, $c_0$, $c_{-1}$, $c_1$, $c_2$, $c_3$ are coefficients of temperature $T$ and related to the constituent materials. The relationships are given by

$$P = P_c V_c + P_m V_m, \quad V_c + V_m = 1 \quad (2)$$

with the assumption of a simple power law distribution of ceramic, the volume fraction $V_c$ is written as

$$V_c = (0.5 + z/h)\gamma \quad (3)$$

where $p$ represents the power law exponent in the interval $[0, +\infty]$.

**Equations of motion**

As the above mentioned, the coupled structure herein comprises the stepped cylindrical shell segments and annular plates. In order to obtain the total displacement and strain of the coupled structure, displacements of constituent parts should be presented at the first. For the cylindrical shell segments (annular plate), the displacement components of arbitrary point in $x$, $\theta$, and $z$ $(r, \theta,$ and $z)$ axial directions are separately indicated by $U_i(p)$, $V_i(p)$, and $W_i(p)$, the subscript $c$ and $p$ represent the cylindrical shell segment and annular plate. In the framework of the FSDT, they can be given as

$$U_i(p) = u_i(p) + z\phi_i(\theta, p)$$

$$V_i(p) = v_i(p) + z\psi_i(\theta, p)$$

$$W_i(p) = w_i$$

where the superscript $i$ represents the $i$th shell segment or annular plate; $u_i(p)$, $v_i(p)$, $w_i$ are the displacements in the midplane of the segment; $\phi_i(\theta, p)$ and $\psi_i(\theta, p)$ separately represent the transverse normal rotations about the $x$ and $\theta$ $(r$ and $\theta)$ axes. According to the relationships between strains and displacement components of shell segment and annular plate, the expression of strains of the $i$th cylindrical shell segment can be given by

$$\varepsilon_{cxi}(pr) = \varepsilon_{cxi}(pr) + zk_i \varepsilon_{cxi}(pr) = \varepsilon_{cxi}(pr) + zk_i \varepsilon_{cxi}(pr)$$

$$\varepsilon_{cz}(pr) = \varepsilon_{cz}(pr) + zk_i \varepsilon_{cz}(pr)$$

$$\varepsilon_{czi}(pr) = \varepsilon_{czi}(pr) + zk_i \varepsilon_{czi}(pr)$$

where $\varepsilon_{cxi}(pr)$, $\varepsilon_{czi}(pr)$, and $\varepsilon_{czi}(pr)$ denote the membrane strains corresponding to $u_i(p)$, $v_i(p)$, and $w_i$, and $\phi_i(\theta, p)$, $\psi_i(\theta, p)$, and $\psi_i(\theta, p)$ indicate the normal and shear strains of the middle surface in the $x-\theta$ $(r-\theta)$ plane; $k_i \varepsilon_{cxi}(pr)$, $k_i \varepsilon_{czi}(pr)$, and $k_i \varepsilon_{czi}(pr)$ represent the changes of curvatures, of which expressions are given by Ref. 41. Based on the generalized Hooke’s law, the relationship between stress and strain of the $i$th cylindrical shell segment and annular plate can be written as

$$\begin{bmatrix}
\sigma_{cxi}(pr) \\
\sigma_{czi}(pr) \\
\sigma_{czi}(pr) \\
\sigma_{czi}(pr) \\
\sigma_{czi}(pr)
\end{bmatrix} = \begin{bmatrix}
Q_{11} & Q_{12} & 0 & 0 & 0 \\
Q_{12} & Q_{22} & 0 & 0 & 0 \\
0 & 0 & Q_{66} & 0 & 0 \\
0 & 0 & 0 & Q_{44} & 0 \\
0 & 0 & 0 & 0 & Q_{55}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{cxi}(pr) - \Delta T_0 \alpha \\
\varepsilon_{czi}(pr) - \Delta T_0 \alpha \\
\varepsilon_{czi}(pr) \\
\varepsilon_{czi}(pr) \\
\varepsilon_{czi}(pr)
\end{bmatrix}$$

where $\sigma_{cxi}(pr)$, $\sigma_{czi}(pr)$, $\sigma_{czi}(pr)$, and $\sigma_{czi}(pr)$ represent the stresses corresponding to strains exhibited by equation (6); $\Delta T_0$ denotes the temperature difference between the inner and outer surface of the $i$th cylindrical shell segment and
annular plate; $Q_{mn}$ indicates the material stiffness which is derived from Young’s modulus $E$ and Poisson’s ratio $\nu$ for isotropic materials, the specific expression is as follows

$$Q_{11} = Q_{22} = \frac{E}{1 - \nu^2}Q_{12} = \frac{\nu E}{1 - \nu^2}$$

$$Q_{66} = Q_{44} = Q_{55} = \frac{E}{2(1 + \nu)}$$

Then, the resultant stresses and moments of the $i$th cylindrical shell segment and annular plate can be obtained according to equation (8)

$$\begin{bmatrix}
N_{cxi(\text{pr})}^i \\
N_{cxi(\text{ph})}^i \\
N_{cxi(\text{ph})}^i \\
M_{cxi(\text{pr})}^i \\
M_{cxi(\text{ph})}^i \\
Q_{cxi(\text{pr})}^i \\
Q_{cxi(\text{ph})}^i \\
Q_{cxi(\text{pr})}^i \\
Q_{cxi(\text{ph})}^i \\
\end{bmatrix} = \begin{bmatrix}
A_{11}^i & A_{12}^i & 0 & B_{11}^i & B_{12}^i & 0 & 0 & 0 \\
A_{12}^i & A_{11}^i & 0 & B_{12}^i & B_{11}^i & 0 & 0 & 0 \\
0 & 0 & A_{66}^i & 0 & 0 & B_{66}^i & 0 & 0 \\
B_{11}^i & B_{12}^i & 0 & D_{11}^i & D_{12}^i & 0 & 0 & 0 \\
B_{12}^i & B_{11}^i & 0 & D_{12}^i & D_{11}^i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \kappa A_{66}^i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa A_{66}^i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
\frac{\partial}{\partial z} N_{cxi(\text{pr})}^i \\
\frac{\partial}{\partial z} N_{cxi(\text{ph})}^i \\
\frac{\partial}{\partial z} N_{cxi(\text{ph})}^i \\
\frac{\partial}{\partial z} M_{cxi(\text{pr})}^i \\
\frac{\partial}{\partial z} M_{cxi(\text{ph})}^i \\
\frac{\partial}{\partial z} Q_{cxi(\text{pr})}^i \\
\frac{\partial}{\partial z} Q_{cxi(\text{ph})}^i \\
\frac{\partial}{\partial z} Q_{cxi(\text{pr})}^i \\
\frac{\partial}{\partial z} Q_{cxi(\text{ph})}^i \\
\end{bmatrix}$$

where $N_{cxi(\text{pr})}^i$, $N_{cxi(\text{ph})}^i$, and $N_{cxi(\text{ph})}^i$ represent the stress resultants of the $i$th cylindrical shell segment and annular plate; $M_{cxi(\text{pr})}^i$, $M_{cxi(\text{ph})}^i$, and $M_{cxi(\text{ph})}^i$ indicate the bending and twisting resultants; $Q_{cxi(\text{pr})}^i$ and $Q_{cxi(\text{ph})}^i$ denote the transverse shear force resultants; $\kappa$ is the shear correction factor; $A_{mn}^i$, $B_{mn}^i$, $D_{mn}^i$ separately denote the extensional, coupling, and bending stiffness, which can be obtained by

$$\begin{align*}
(A_{mn}^i, B_{mn}^i, D_{mn}^i) &= \int_{-h/2}^{h/2} Q_{ma}^i(z) (1, z, z^2) \, dz \\
(11)
\end{align*}$$

**Energy equation**

As shown in Figure 1, artificial virtual spring technology is introduced into the coupled model to meet the displacement and physical admissible coordination conditions at coupling locations. Arbitrary coupling conditions can be assumed as

$$u_{c}^{i}|_{z = L_{i}} = -w_{p}^{i}, v_{c}^{i}|_{z = L_{i}} = v_{p}^{i}, w_{c}^{i}|_{z = L_{i}} = u_{p}^{i}, \phi_{cx}^{i}|_{z = L_{i}} = \phi_{p}\frac{\partial}{\partial z}$$

$N_{cxi|z = L_{i}}^{i} = -Q_{cxi|z = L_{i}}^{i} N_{cxi|z = L_{i}}^{i} = N_{p\theta\theta} \frac{\partial}{\partial z} Q_{cxi|z = L_{i}}^{i} = \frac{\partial}{\partial z} N_{cxi|z = L_{i}}^{i} = M_{p\theta\theta} \frac{\partial}{\partial z} M_{cxi|z = L_{i}}^{i} = M_{p\theta\theta} \frac{\partial}{\partial z} M_{cxi|z = L_{i}}^{i} = M_{p\theta\theta}$

The arbitrary elastic boundary conditions at the coupling locations can be also indicated by

$$k_{\theta}^{i} \left( \phi_{c\theta}^{i}|_{z = L_{i}} - \phi_{p\theta}^{i} \right) - M_{\theta\theta}^{i}|_{z = L_{i}} = 0, k_{\theta}^{i} \left( \phi_{c\theta}^{i}|_{z = L_{i}} - \phi_{p\theta}^{i} \right) - M_{\theta\theta}^{i}|_{z = L_{i}} = 0$$

On basis of the aforementioned analyses, the strain energy stored by coupled spring associated with cylindrical shell segments and annular plate segments is obtained as
\[ V_{is} = \frac{1}{2} \int_{-h/2}^{h/2} \int_{0}^{h} \left\{ k_{is-w} \left( w_c^{i+1} - w_c^i \right)^2 + k_{is-\theta} \left( \phi_c^{i+1} - \phi_c^i \right)^2 + k_{is-\theta} \left( \phi_{c,\theta}^{i+1} - \phi_{c,\theta}^i \right) \right\} r_c d\theta_c dz_c \]  

(12)

where subscripts \( i \) and \( s \) separately indicate the \( i \)th annular plate and \( s \)th cylindrical shell segment which is coupled by the connecting springs.

As illustrated in Figure 1 and the mathematically expressions about continuity conditions and boundary conditions, based on the artificial virtual spring technique, the energy stored by springs among the cylindrical shell segments can be indicated by

\[ V_{i-i+1} = \frac{1}{2} \int_{-h/2}^{h/2} \int_{0}^{h} \left\{ k_{cw} \left( u_c^{i+1} - u_c^i \right)^2 + k_{cv} \left( \psi_c^{i+1} - \psi_c^i \right)^2 + k_{cw} \left( w_c^{i+1} - w_c^i \right)^2 + k_{cx} \left( \phi_{cx}^{i+1} - \phi_{cx}^i \right) + k_{cx} \left( \phi_{c,\theta}^{i+1} - \phi_{c,\theta}^i \right) \right\} r_c d\theta_c dz_c \]  

(13)

where \( k_{cw}, k_{cv}, k_{cw}, k_{cx}, \) and \( K_{ax} \) represent the stiffness of connecting springs between the adjacent cylindrical shell segments.

The kinetic energy and strains energy of components and boundary springs can be written as

\[ T_i = T_c + T_p \]  

(14)

\[ U_i = U_c + U_p + U_{bc} + U_{hp} + V_{cp} + V_{pc} + U_{cT} - U_{pT} \]

where \( T_i \) denotes the total kinetic energy of the coupled annular plate–stepped cylindrical shell; \( T_c \) and \( T_p \) represent the kinetic energy of cylindrical shell segments and annular plates, respectively; \( U_i \) is the total strain energy of the coupled stepped cylindrical shell; \( U_c \) and \( U_p \) are the strain energy of cylindrical shell segments and annular plates; \( U_{bc} \) and \( U_{hp} \) are the strain energy of boundary springs of cylindrical shell segments and annular plates; \( V_{cp} \) denotes the strain energy of connecting springs between cylindrical shell segments and annular plates; \( V_{pc} \) represents strain energy of connecting springs between the adjacent cylindrical shell segments; \( U_{cT} \) and \( U_{pT} \) represent the energy of cylindrical shell and annular plate about the thermal environment. The specific expressions of those symbols above about cylindrical shell and annular plate are given in the Appendix 1.

The thermal stress resultant is written as

\[ N_{ct} = \sum_{i=1}^{M} N_{ct}^i, N_{pt} = \sum_{j=1}^{N} N_{pt}^j \]  

(15)

\[ N_{ct}^i = \int_{-h/2}^{h/2} (Q_{1z} \alpha_1 + Q_{1z} \alpha_2) \Delta T dz \]

\[ N_{pt}^j = \int_{-h/2}^{h/2} (Q_{1z} \alpha_1 + Q_{1z} \alpha_2) \Delta T dz \]

Rayleigh–Ritz energy method is utilized to analyze the characteristics of vibration and transient response of the cylindrical shell coupled with annular plate. The Lagrangian energy function is expressed as

\[ L = T_i - U_i + W_{ext} \]  

(16)

where \( W_{ext} \) represents the work done by the external force. This term is
\[
W_{ext} = \oint_{\Gamma} \left( f_x u'_x + f_y v'_y + f_w w'_w + m_x \phi'_x + m_y \phi'_y \right) r' dx d\theta
\]

(17)

\[
F = [f_x, f_y, f_w, m_x, m_y]^T
\]

where \( f_x, f_y, \) and \( f_w \) represent the distributed forces in the corresponding directions; \( m_x \) and \( m_y \) are the moment components about \( x \) and \( \theta \) axes. As a method used for investigation on the vibration and dynamic analyses of shell and plate structures,\(^{55,56}\) Chebyshev–Lagrangian method is employed here to study characteristics of vibration and transient response of the model established here.

The Chebyshev polynomials of second kind are defined as

\[
T_0(x) = 1, \quad T_1(x) = 2x, \quad T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x), \quad (m \geq 2)
\]

(18)

The admissible displacement exponents of arbitrary shell segment and annular plate can be expanded by means of Chebyshev polynomials as

\[
H = \sum_{m=0}^{m_{max}} \sum_{n=0}^{N} G T_m(\xi) \cos(n\theta)
\]

(19)

\[
H = [u'_x, v'_x, w'_x, \phi'_x, \phi'_y, u'_y, v'_y, w'_y, \phi'_y, \phi'_y]^T
\]

where \( \mathbf{G} = [A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}, A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}]^T \) the coefficients of the Chebyshev expansions; \( T_m \) denotes the \( m \) order Chebyshev polynomials; \( n \) is the circumferential wave number. Based on the above discussion, the following equation of the eigenvalue is obtained by minimizing the Lagrangian function with respect to the coefficients.

\[
(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{G} = \mathbf{F}
\]

(20)

where \( \mathbf{K} \) and \( \mathbf{M} \) separately denote the stiffness and mass matrix of the cylindrical shell coupled with annular plate; \( \mathbf{G} \) is the coefficients vector of the displacement; \( \mathbf{F} \) is the column vector containing the external force contributions. The specific expressions of \( \mathbf{K} \) and \( \mathbf{M} \) are shown in Appendix 2.

**Numerical results and discussion**

In this section, the free vibration, transient response, and steady state response of the FGM-stepped cylindrical shell coupled with annular plate are analyzed based on the theoretical method presented in theoretical analysis. In order to realize various boundary constraints, the technique of artificial virtual springs is introduced. The boundary constraints considered in this paper are shown in Table 1. Unless otherwise stated, the geometric parameters and material properties of the coupled annular plate–stepped cylindrical shell are: \( L = 5 \text{ m}; R = 1 \text{ m}; r_p = 0.3 \text{ m}; M = 2; N = 1; L = L_1 + L_2; L_1 = L_2 = 2.5 \text{ m}; \mu = 0.28; \)

\( h_i = 0.1 \text{ m}; h_p = 0.15 \text{ m}; \) reference temperature \( T_0 \) is set as 300 K; the temperature \( T_m = T_c = T = 300 \text{ K}; \) \( p_p = 0.5. \) Where \( L \) and \( R \) are the length and radius of cylindrical shell, respectively; \( L_1 \) and \( L_2 \) are the length of the first subsection and the second subsection of the cylindrical shell; \( r_p \) represents the inner diameter of the annular plate; \( M \) and \( N \) are the number of cylindrical shell segments and annular plate; \( \rho \) is the density of the materials; \( \mu \) is the Poisson’s ratio; \( h_i \) and \( h_p \) represent the thicknesses of the \( i \)th cylindrical shell segment and annular plate; \( T_m \) and \( T_c \) represent the temperature of ceramics and metals materials; \( p_p \) is used to reveal the relative position of the annular plate in the \( x \)-direction of cylindrical shell. What is

| BC  | \( k_r \) | \( k_\theta \) | \( k_w \) | \( K_r \) | \( K_\theta \) |
|-----|----------|----------|----------|----------|----------|
| C   | \( 10^{14} \) | \( 10^{14} \) | \( 10^{14} \) | \( 10^{14} \) | \( 10^{14} \) |
| SD  | 0        | \( 10^{14} \) | \( 10^{14} \) | 0        | \( 10^{14} \) |
| F   | 0        | 0        | 0        | 0        | 0        |
| EI  | \( 10^6 \) | \( 10^6 \) | \( 10^6 \) | \( 10^6 \) | \( 10^6 \) |
more, the annular plate is clamped (C) in its inner boundary in current study. The temperature-dependent coefficients of material properties are exhibited in Table 2.

**Convergence and validation studies**

As a foundation of the current research, convergence and validation cases should be conducted at the first. The geometrical parameters and material properties of the stepped cylindrical shell coupled with annular plate are: \( h_i = h_p = 0.1 \) m. Figure 2 presents the convergence behavior of frequency parameter with respect to the truncated number, the axial halfwave number \( m \) is chosen from 1 to 3, and the circumferential wave number \( n \) is chosen from 1 to 4. The cylindrical shell is clamped at its two ends (C–C). It is observed from Figure 2 that the current results converge rapidly with the increments of truncated number \( m_i \). And furthermore, compared results in case of \( m_i = 8 \) and 10, the frequency parameter is almost stable. Thus, the truncated number \( m_i \) is chosen to be eight in the following calculation. Table 3 shows the comparison of the results with the

| Materials | Properties | \( p_0 \) | \( p_{-1} \) | \( p_1 \) | \( p_2 \) | \( p_3 \) |
|-----------|------------|---------|---------|---------|---------|---------|
| Si3N4     | \( E \)    | \( 3.4843 \times 10^{11} \) | 0       | \( -3.0700 \times 10^{-4} \) | \( 2.1600 \times 10^{-7} \) | \( -8.9460 \times 10^{-11} \) |
|           | \( \alpha \) | \( 5.8723 \times 10^{-6} \) | 0       | \( 9.0950 \times 10^{-4} \) | 0       | 0       |
|           | \( \nu \)   | 0.28    | 0       | 0       | 0       | 0       |
|           | \( \kappa \) | 9.19    | 0       | 0       | 0       | 0       |
|           | \( \rho \)  | 2370    | 0       | 0       | 0       | 0       |
| SUS304    | \( E \)    | \( 2.0104 \times 10^{11} \) | 0       | \( 3.0790 \times 10^{-4} \) | \( -6.5340 \times 10^{-7} \) | 0       |
|           | \( \nu \)   | \( 1.2330 \times 10^{-5} \) | 0       | \( 8.0860 \times 10^{-4} \) | 0       | 0       |
|           | \( \nu \)   | 0.28    | 0       | 0       | 0       | 0       |
|           | \( K \)     | 12.04   | 0       | 0       | 0       | 0       |
|           | \( \rho \)  | 8166    | 0       | 0       | 0       | 0       |

Figure 2. Convergence behavior of frequency parameter with respect to the truncated number: (a) \( n = 1 \); (b) \( n = 2 \); (c) \( n = 3 \); and (d) \( n = 4 \) (C–C; \( h_i = h_p = 0.1 \)).
Table 3. Compared of the 10th frequency parameter of the functionally graded material–stepped cylindrical shell coupled with annular plate with different boundary cases ($T_0 = 300K; m_i = 8; n = 0:1:8$).

| BC | Temperature | Method | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|----|-------------|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| C–C | dT = 0 K   | Present | 0.512 | 0.512 | 0.532 | 0.532 | 0.537 | 0.537 | 0.565 | 0.565 | 0.641 | 0.641 |
|     |             | FEM    | 0.510 | 0.510 | 0.529 | 0.529 | 0.536 | 0.536 | 0.564 | 0.564 | 0.640 | 0.640 |
|     |             | Error (%) | 0.477 | 0.477 | 0.470 | 0.470 | 0.250 | 0.250 | 0.278 | 0.278 | 0.139 | 0.139 |
|     | dT = 50 K  | Present | 0.510 | 0.510 | 0.530 | 0.530 | 0.535 | 0.535 | 0.563 | 0.563 | 0.640 | 0.640 |
|     |             | FEM    | 0.510 | 0.510 | 0.529 | 0.529 | 0.536 | 0.536 | 0.564 | 0.564 | 0.640 | 0.640 |
|     |             | Error (%) | 0.042 | 0.042 | 0.061 | 0.061 | 0.089 | 0.089 | 0.068 | 0.068 | 0.010 | 0.010 |
| F–C | dT = 0 K   | Present | 0.178 | 0.178 | 0.229 | 0.229 | 0.376 | 0.376 | 0.376 | 0.376 | 0.507 | 0.507 |
|     |             | FEM    | 0.181 | 0.181 | 0.230 | 0.230 | 0.377 | 0.377 | 0.382 | 0.382 | 0.507 | 0.507 |
|     |             | Error (%) | 2.000 | 2.000 | 0.418 | 0.418 | 0.246 | 0.246 | 1.561 | 1.561 | 0.004 | 0.004 |
|     | dT = 50 K  | Present | 0.179 | 0.179 | 0.230 | 0.230 | 0.375 | 0.375 | 0.382 | 0.382 | 0.507 | 0.507 |
|     |             | FEM    | 0.178 | 0.178 | 0.229 | 0.229 | 0.376 | 0.376 | 0.376 | 0.376 | 0.507 | 0.507 |
|     |             | Error (%) | 0.691 | 0.691 | 0.431 | 0.431 | 0.066 | 0.066 | 1.539 | 1.539 | 0.074 | 0.074 |
| SD–SD | dT = 0 K | Present | 0.139 | 0.139 | 0.451 | 0.451 | 0.472 | 0.472 | 0.487 | 0.487 | 0.517 | 0.517 |
|      |             | FEM    | 0.139 | 0.139 | 0.448 | 0.448 | 0.470 | 0.470 | 0.484 | 0.484 | 0.515 | 0.515 |
|      |             | Error (%) | 0.035 | 0.035 | 0.679 | 0.679 | 0.382 | 0.382 | 0.615 | 0.615 | 0.399 | 0.399 |
|      | dT = 50 K  | Present | 0.137 | 0.137 | 0.448 | 0.448 | 0.470 | 0.470 | 0.484 | 0.484 | 0.515 | 0.515 |
|      |             | FEM    | 0.139 | 0.139 | 0.448 | 0.448 | 0.470 | 0.470 | 0.484 | 0.484 | 0.515 | 0.515 |
|      |             | Error (%) | 0.985 | 0.985 | 0.143 | 0.143 | 0.039 | 0.039 | 0.153 | 0.153 | 0.000 | 0.000 |

Figure 3. Frequency parameter with respect to different types of subsections: (a) C–C; (b) C–E1; (c) F–C; and (d) SD–SD.
present method and the finite element method, considering four boundary conditions. The temperature differences are chosen to be 0 K and 50 K. In Table 3, the maximum relative error between results with the present method and FEM reaches rarely 2%, proving the validity of the present method.

Free vibration analysis

This subsection is concerned with the analysis for free vibration of the FGM-stepped cylindrical shell coupled with annular plate, which is subjected to various boundary conditions, including $C–C$, $C–E1$, $F–C$, and $SD–SD$ boundary conditions. The influence of various types of segments of the cylindrical shell on the frequency parameter of the coupled annular plate–stepped cylindrical shell is exhibited in Figure 3; details about four types of subsections of the cylindrical shell are shown in Table 4. It is noted that in order to facilitate comparison, the total length of the shell segments of the same thickness is chosen 2.5 m. Frequency parameter shown in Figure 3 increases as the number of the circumferential wave number $n$ is increased and as well as the axial halfwave number $m$. From Figure 3, we can see that the various types of shell segments make effects on frequency parameter of the coupled system. For one thing, frequency of Type 3 is lower than Type 1 when the circumferential number $n$ is below 4, based on the same distribution of the shell segments with thickness of 0.15 and 0.1. This suggests that more shell segments benefit the free vibration of the coupled annular plate–stepped cylindrical shell in term of low frequency. For another, shell segments of Type 4 provide the minimum frequency parameter, indicating that different shell segments distributions have an effect on the frequency of structural free vibration. Figure 4 presents the tendency of frequency parameter of the coupled annular plate–stepped cylindrical shell as the length of cylindrical shell varies from 1 m to 5 m, which is clamped at the ends. Apparently, the frequency parameter of the coupled annular plate–

| Type | 1   | 2   | 3   | 4   |
|------|-----|-----|-----|-----|
| $\text{Number of subsections (M)}$ | 2   | 3   | 3   | 3   |
| $h_i (m)$ | $h_1 = 0.1, h_2 = 0.15$ | $h_1 = 0.1 = h_3, h_2 = 0.15$ | $h_1 = 0.1 = h_2, h_3 = 0.15$ | $h_1 = 0.15, h_2 = 0.1 = h_3$ |
| $L_i (m)$ | $L_1 = L_2 = 2.5$ | $L_1 = L_3 = 1.25, L_2 = 2.5$ | $L_1 = L_2 = 1.25, L_3 = 2.5$ | $L_1 = 2.5, L_2 = L_3 = 1.25$ |

Figure 4. Frequency parameter with respect to length of shell: (a) $n = 1$; (b) $n = 2$; (c) $n = 3$; and (d) $n = 4$ ($C–C$).
Figure 5. Frequency parameter with respect to relative position of circular plate (a) $n = 1$; (b) $n = 2$; (c) $n = 3$; and (d) $n = 4$ ($C-EI$).

Figure 6. Frequency parameter with respect to temperature (a) $n = 1$; (b) $n = 2$; (c) $n = 3$; and (d) $n = 4$ ($C-EI$).
stepped cylindrical shell decreases with increasing the length $L$ of the cylindrical shell. The relation between the frequency parameter of the FGM-stepped cylindrical shell coupled with annular plate and the relative position $p_p$ of the annular plate is revealed in Figure 5. For the sake of diversity of numerical data, the boundary condition is chosen as $C–E1$. From Figure 5, it can be concluded that the frequency parameter is sensitive to the relative position of the annular plate. On the one hand, except for $n = 1$ shown in Figure 5(a), when $m$ is the same, the frequency parameter changes with the relative position in the same law, and as $n$ increases, the influence of the relative position becomes smaller. On the other hand, apart from $m = 3$ and $n = 1$ at the same time, setting the relative position $p_p$ as 0.1 can provide lower frequency parameter of the FGM-stepped cylindrical shell coupled with annular plate regardless of high frequency or low frequency. Figure 6 exhibits the frequency parameter of the FGM-stepped cylindrical shell coupled with annular plate with respect to temperature. The boundary condition is chosen as $C–C$, thicknesses of cylindrical shell and annular plate are both chosen to be 0.1. It is observed from Figure 6 that the frequency parameter of the FGM-stepped cylindrical shell coupled with annular plate decreases linearly with increasing temperature.

**Transient response analysis**

In this subsection, numerical examples are selected to illustrate the contribution of geometric parameters and temperature of the FGM-stepped cylindrical shell coupled with annular plate. To begin with, the validation case is presented to confirm the correctness of the current method. Figure 7 presents four types of loading employed in this article: (a) Rectangular pulse; (b) Triangular pulse; (c) Half-sine pulse; (d) Exponential pulse, which are expressed in detail by Ref. 57. As is shown in Figure 8, comparison of transient response of the FGM-stepped cylindrical shell coupled with annular plate with the present method and the finite element method (FEM) is exhibited. Consider the four types of point excitation loading, the boundary constraints are $C–C$, time for loading ($t_l$) is 0.01 s; time for analysis ($t_0$) is 0.02 s; the positions of loading $p_l$ and measure $p_m$ point are severally chosen to be (3, 0) and (2, 0) in the $x$-$\theta$ coordinate system, temperature is chosen to be 300 K. It is visibly seen that results obtained with FEM and the present method share excellent agreement, validating the correctness of the present method employed in this article.

Figure 9 shows the contributions of various types of loading to acceleration of transient response of the FGM-stepped cylindrical shell coupled with annular plate. The boundary condition, time for loading and analysis are chosen from Figure 8; position of loading is (2, 0) in the $x$-$\theta$ coordinate system, while the position of measure point is selected as (1, 0); temperature is 300 K. From Figure 9, the transient response of the FGM-stepped cylindrical shell coupled with annular plate decreases linearly with increasing temperature.

![Figure 7](image-url)
varies evidently owe to different excitations. In Figure 10, four types of cylindrical shell segments are taken into account to study the effect of the number and distributions of segments on the transient response. The boundary conditions are chosen as $C-C$ and $SD-SD$, the positions of loading $p_l$ and measure $p_m$ point, temperature, time for loading and analysis are same as Figure 9. It is visibly seen from Figure 10 that the displacement with types 1 and 3 varies more intensely as time varies from 0 to 0.02 s, and curves of types 1 and 3 are very close. Besides, the curves of displacement gradually become gentle when time is over 0.01 s. In other words, the transient response is hardly related to the number of cylindrical shell segments, while the distribution of shell segments makes more influence. With purpose that analyzes the effect caused by the inner diameter of the annular plate on the transient response of the FGM-stepped cylindrical shell coupled with annular plate, Figure 11 illustrates the changes of transient response considering the inner diameter of annular plate. The boundary conditions are chosen as $F-C$ and $SD-SD$; the type of loading is triangular; $p_l = (2, 0)$; $p_m = (1, 0)$; temperature is 300 k; the inner diameter of annular plate is $r_p$ selected to be 0, 0.1, 0.2, 0.3. It is noted that the annular plate would be circular plate as the inner diameter is selected to be 0. As seen from Figure 11, the trend of four curves with respect to time is similar, and the most change of displacement occurs as the $r_p$ is 0, curves with $r_p$ of 0.1, 0.2, and 0.3 are very close, though the displacement provided by $r_p$ of 0.3 is minimum. In a word, inner diameter makes little difference on transient response of the FGM-stepped cylindrical shell coupled with annular plate as the $r_p$ is not 0 and range in a certain extent.

Figure 12 shows the transient response of the FGM-stepped cylindrical shell coupled with annular plate with respect to temperature. The boundary conditions are $C-C$ and $SD-SD$; $p_l$, $p_m$, $t_l$, and $t_0$ are chosen from Figure 11; loading is rectangular. It is seen from Figure 12 that the differences of transient responses with $T = 300$ K, 340 K, and 400 k are small and the displacement under $T = 400$ K is the lowest. This suggests that temperature does not work much for transient response of the FGM-stepped cylindrical shell coupled with annular plate. For the completeness, the influence caused by the length $L$ of cylindrical shell is revealed in Figure 13. Various boundary conditions are considered, for instance, $C-C$, $F-C$, $C-F$,
Figure 9. Transient response of the functionally graded material–stepped cylindrical shell coupled with annular plate with respect to various loadings (C–C, $p_i = (2, 0)$; $p_m = (1, 0)$; $t_i = 0.01$; $t_0 = 0.02$).

Figure 10. Transient response of the functionally graded material–stepped cylindrical shell coupled with annular plate with respect to four types of cylindrical shell segments: (a) C–C; and (b) SD–SD. ($p_i = (2, 0)$; $p_m = (1, 0)$; $t_i = 0.01$; $t_0 = 0.02$).

Figure 11. Transient response of the functionally graded material–stepped cylindrical shell coupled with annular plate with respect to $r_p$: (a) F–C; and (b) SD–SD (triangular, $p_i = (2, 0)$; $p_m = (1, 0)$; $t_i = 0.01$; $t_0 = 0.02$).
The loading is exponential for sake of diversity of numerical example; $p_l$, $p_m$, and temperature are chosen from Figure 11; $t_l$ and $t_0$ are 0.01 s and 0.012 s. The results show that the longer cylindrical shell leads to more visible displacement of the FGM-stepped cylindrical shell coupled with annular plate and does harm to the FGM-stepped cylindrical shell coupled with annular plate.
Steady state response analysis

Steady state response of the FGM-stepped cylindrical shell coupled with annular plate is analyzed in this subsection. It is worth noting that the validation case should be conducted as the basis of the following work. Figure 14 presents the

![Figure 14](image1)

**Figure 14.** Comparison of steady state response of the functionally graded material–stepped cylindrical shell coupled with annular plate: (a) measure point A (2, 0); and (b) measure point B (1, 0) (C–C; \( h_i = 0.1, h_p = 0.1, p_p = 0.5; p_i = (3, 0) \)).

![Figure 15](image2)

**Figure 15.** Steady state response of the functionally graded material–stepped cylindrical shell coupled with annular plate with respect to length of shell: (a) C–C; (b) F–C; (c) SD–SD; and (d) F–E1 (\( h_i = 0.1, h_p = 0.1; p_i = (3, 0); p_m = (2, 0) \)).
comparison of steady state response of the FGM-stepped cylindrical shell coupled with annular plate with the present method. In order to avoid occasionality of calculation, measure point position is chosen as (2, 0) and (1, 0) in the $x-\theta$ coordinate system; loading position is (3, 0); thicknesses of cylindrical shell and annular plate are both 0.1 m; the boundary condition is $C-C$. As is seen from Figure 14, the results with the present method and FEM share good agreement. In other words, the present method is suitable for the steady state response calculation of the FGM-stepped cylindrical shell coupled with annular plate owe to its correctness.

Next, effects of parameters of the geometry, relative position of the annular plate, and temperature are analyzed. Figure 15 shows the steady state response of the FGM-stepped cylindrical shell coupled with annular plate with respect to various length $L$ of cylindrical shell. $C-C$, $F-C$, $SD-SD$, and $F-E1$ boundary conditions are employed to ensure the comprehensiveness of the numerical solutions; thicknesses of annular plate and cylindrical shell are both 0.1 m. Positions of loading and measure point are (3, 0) and (2, 0) in the $x-\theta$ coordinate system. From Figure 15, it can be found that displacement is sensitive to the length $L$ of the cylindrical shell. As length $L$ of cylindrical shell increases, peaks of displacement are more concentrated in the frequency region from 300 Hz to 650 Hz. As depicted in Figure 16, the steady state response of the FGM-stepped cylindrical shell coupled with annular plate of uniform thickness with respect to the inner diameter $r_p$ of annular plate is revealed. Boundary conditions and geometric parameters are same as Figure 15. On the one hand, it is visible that the peaks of displacement shift to low frequency region with increment of $r_p$. On the other hand, the number of peaks of displacement in the frequency region from 300–650 Hz does not change as the inner diameter $r_p$ increases. Figure 17 shows the influence of different types of stepped cylindrical shell segments on steady state response of the FGM-stepped cylindrical shell coupled with annular plate. The positions of loading and measure point are same as Figure 15; $F-C$ boundary condition is selected. It is visible curves of Type 1 and Type 3 almost coincide because the difference between Types 1 and 3 is merely number of shell segments. In addition, it is

![Figure 16](image-url). Steady state response of the functionally graded material–stepped cylindrical shell coupled with annular plate with respect to inner diameter of the circular plate: (a) $F-E1$; (b) $C-C$; (c) $SD-SD$; and (d) $F-C$ ($h_p = h_1 = 0.1$).
found that displacements of Types 1 and 3 are lower than those of Types 2 and 4, but the number of peaks reverses. In a word, shell segments number plays little role in steady state response of the FGM-stepped cylindrical shell coupled with annular plate, while distribution of shell segments affects. This is in harmony with discussion about transient response.

Figure 17. Steady state response of the functionally graded material–stepped cylindrical shell coupled with annular plate with respect to various types of cylindrical shell segments: F–C (\(h_p = 0.1\); \(p_l = (3, 0)\); \(p_m = (2, 0)\)).

Figure 18. Steady state response of the functionally graded material–stepped cylindrical shell coupled with annular plate with respect to the relative position of the circular plate: (a) C–C; (b) F–C (c) SD–SD; and (d) F–E1 (\(p_l = (3, 0)\); \(p_m = (2, 0)\); \(h = h_p = 0.1\)).
Results considering the relative position of annular plate \((p_p)\) are exhibited in Figure 18. Different relative positions are taken into calculation to evaluate the contribution to steady state response of the FGM-stepped cylindrical shell coupled with annular plate of uniform thickness. Boundary conditions, thicknesses of cylindrical shell, and circular plate are chosen from Figure 15. Comparing the curves with each other in Figure 18(a) and (c), it can be concluded that curves of symmetrical position of annular plate overlap. The reason is that the constraints at the ends of cylindrical shell are uniform as \(C–C\) and \(SD–SD\) boundary conditions are utilized. In addition, peaks of displacement of \(p_p = 0.5\) are least in the frequency interval and the displacement is almost minimum. This indicates that better steady state response properties can be achieved by placing the annular plate in the middle of the cylindrical shell.

For sake of completeness and comprehensiveness of numerical results, steady state response behavior in the thermal environment is depicted in Figure 19. Boundary conditions, loading and measure and geometric parameters are same as Figure 15. As seen from Figure 19, larger difference between \(T\) and reference \(T_0\) makes the peaks of displacement move to higher frequency region.

**Conclusions**

In this article, an analysis model is established to investigate the FGM-stepped cylindrical shell coupled with annular plate in the thermal environment. According to the Chebyshev polynomial of first kind, the displacement fields are expanded. Compared with these results from the FEM model and the reference, sets of numerical examples are conducted to confirm the correctness and accuracy of the current method. Consequently, comprehensive and detailed investigations for influence of major factors are presented, including the geometric parameters, boundary conditions, loadings, coupling position, and temperature. And then, corresponding conclusions are obtained as the following:
(1) For free vibration: more shell segments benefit free vibration of the FGM-stepped cylindrical shell coupled with annular plate at low frequency; frequency parameter of the shell–plate combination decreases with increasing temperature and the length $L$.

(2) For transient response: transient response of the FGM-stepped cylindrical shell coupled with annular plate varies evidently owing to different excitations; meanwhile, the number of cylindrical shell segments and inner diameter $r_p$ which is not equal to 0 do hardly affect the transient response of the shell–plate combination; higher temperature leads larger displacement, but it does not work much; longer cylindrical does harm to the transient response of the shell–plate combination.

(3) For steady state response: as length $L$ of cylindrical shell of uniform thickness increases, peaks of displacement are more concentrated in the frequency region from $300$ Hz to $650$ Hz; with increments of $r_p$, peaks of displacement shift to low frequency region; shell segments number plays little role in steady state response of the coupled structure, while distribution of shell segments works; better steady state response properties can be obtained by placing the annular plate in the middle of the cylindrical shell; lager difference between $T$ and reference $T_0$ makes the peaks of displacement move to higher frequency region.

Declaration of conflicting interests

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Appendix I

The specific expressions of the total kinetic energy of the coupled stepped cylindrical shell–annular plate are given by

\[
U_c = \sum_{i=1}^{M} U_{ci}, U_p = \sum_{j=1}^{N} U_{pj}, V_{cp} = \sum_{i=1}^{N} V_{ci}, V_{cc} = \sum_{j=1}^{N} V_{pj}
\]

\[
V_{pc} = \sum_{i=1}^{N} V_{pc}, U_{cT} = \sum_{i=1}^{M} U_{ciT}, U_{pT} = \sum_{j=1}^{N} U_{pjT}
\]

\[
T_c = T_c + T_p
\]

\[
T_c = \sum_{i=1}^{N} T_{ci}, T_p = \sum_{j=1}^{N} T_{pj}
\]
The total stiffness and mass matrices are

\[ K = \text{diag}[K(c), K(p)] \]
\[ M = \text{diag}[M(c), M(p)] \]

where \( K(c), K(p) \) indicate the stiffness matrices of cylindrical shell and annular plate; \( M(c), M(p) \) represent the mass matrices of cylindrical shell and annular plate, respectively.
The specific expressions of $K'$ can be written as

$$K' = \begin{bmatrix}
K'_{1,1} & K'_{1,2} & K'_{1,3} & K'_{1,4} & K'_{1,5} \\
K'_{2,1} & K'_{2,2} & K'_{2,3} & K'_{2,4} & K'_{2,5} \\
K'_{3,1} & K'_{3,2} & K'_{3,3} & K'_{3,4} & K'_{3,5} \\
K'_{4,1} & K'_{4,2} & K'_{4,3} & K'_{4,4} & K'_{4,5} \\
K'_{5,1} & K'_{5,2} & K'_{5,3} & K'_{5,4} & K'_{5,5}
\end{bmatrix}$$

$$K'_{i,j} = K'_{i,j}^{ui} + K'_{i,j}^{vi}$$

$$K'_{i,j}^{ui} = \frac{L_i}{2} \int_{-1}^{1} \int_{0}^{2\pi} \left( \frac{4}{L_i^2} A_{11} \dot{\hat{u}}_i \dot{\hat{v}}_i + A_{66} \dot{\hat{u}}_i \dot{\hat{v}}_i + \frac{4}{L_i^2} A_{16} \dot{\hat{u}}_i \dot{\hat{v}}_i \right) d\theta_d d\zeta_i$$

$$K'_{i,j}^{vi} = \frac{L_i}{2} \int_{-1}^{1} \int_{0}^{2\pi} \left( \frac{2}{L_i} A_{11} \dot{\hat{u}}_i \dot{\hat{v}}_i + \frac{4}{L_i} A_{66} \dot{\hat{u}}_i \dot{\hat{v}}_i + \frac{4}{L_i} A_{16} \dot{\hat{u}}_i \dot{\hat{v}}_i \right) d\theta_d d\zeta_i$$
\[
K^{vi}_{1,1} = \int_{-h/2}^{h/2} \int_{0}^{2\pi} \left( k_n u_i^T u_i \big|_{x=0} + k_n u_i^T u_i \big|_{x=L_i} \right) dz d\theta \\
K^{vi}_{2,2} = \int_{-h/2}^{h/2} \int_{0}^{2\pi} \left( k_\theta v_i^T v_i \big|_{x=0} + k_\theta v_i^T v_i \big|_{x=L_i} \right) dz d\theta \\
K^{vi}_{1,3} = \int_{-h/2}^{h/2} \int_{0}^{2\pi} \left( k_n w_i^T w_i \big|_{x=0} + k_n w_i^T w_i \big|_{x=L_i} \right) dz d\theta \\
K^{vi}_{2,4} = \int_{-h/2}^{h/2} \int_{0}^{2\pi} \left( k_\phi \phi_i^T \phi_i + k_\phi \phi_i^T \phi_i \big|_{x=L_i} \right) dz d\theta \\
K^{vi}_{2,5} = \int_{-h/2}^{h/2} \int_{0}^{2\pi} \left( k_\phi \phi_i^T \phi_i + k_\phi \phi_i^T \phi_i \big|_{x=L_i} \right) dz d\theta
\]

The mass matrix \( M \) (c) is given by

\[
M(c) = \text{diag} [M^1, M^2, M^3 \ldots M^N]
\]

\[
M' = \begin{bmatrix}
M'_{1,1} & 0 & 0 & M'_{1,4} & 0 \\
0 & M'_{2,2} & 0 & 0 & M'_{2,5} \\
0 & 0 & M'_{3,3} & 0 & 0 \\
M'_{4,4} & 0 & 0 & M'_{4,4} & 0 \\
0 & M'_{2,5} & 0 & 0 & M'_{5,5}
\end{bmatrix}
\]

\[
M'_{1,1} = \frac{L_i}{2} \int_{-1}^{1} \int_{0}^{\pi} l_n u_i^T u_i d\zeta d\theta, \quad M'_{2,2} = \frac{L_i}{2} \int_{0}^{2\pi} \int_{1}^{1} l_\phi \phi_i^T \phi_i d\zeta d\theta, \quad M'_{3,3} = \frac{L_i}{2} \int_{0}^{2\pi} \int_{1}^{1} l_\phi \phi_i^T \phi_i d\zeta d\theta, \quad M'_{4,4} = \frac{L_i}{2} \int_{0}^{2\pi} \int_{1}^{1} l_\phi \phi_i^T \phi_i d\zeta d\theta, \quad M'_{5,5} = \frac{L_i}{2} \int_{0}^{2\pi} \int_{1}^{1} l_\phi \phi_i^T \phi_i d\zeta d\theta
\]

\[
(l_n, l_\phi, l_\theta) = \int_{-h/2}^{h/2} \rho(z) \left( 1, z, z^2 \right) dz
\]

Also, the stiffness and mass matrix of the \( j \)th annular plate can be derived from

\[
K(p) = \text{diag} [K^1_i, K^2_i, K^3_i \ldots K^N_i], M(p) = \text{diag} [M^1_i, M^2_i, M^3_i \ldots M^N_i]
\]

The expressions of \( K^j_i \) and \( M^j_i \) are similar to those of cylindrical shell segment exhibited above, there is no need to repeat the equations.
\[ K'_j = \begin{bmatrix} K'_{1,1} & K'_{1,2} & K'_{1,3} & K'_{1,4} & K'_{1,5} \\ K'_{2,1} & K'_{2,2} & K'_{2,3} & K'_{2,4} & K'_{2,5} \\ K'_{3,1} & K'_{3,2} & K'_{3,3} & K'_{3,4} & K'_{3,5} \\ K'_{4,1} & K'_{4,2} & K'_{4,3} & K'_{4,4} & K'_{4,5} \\ K'_{5,1} & K'_{5,2} & K'_{5,3} & K'_{5,4} & K'_{5,5} \end{bmatrix} \]

\[ K'_{a,b} = K'^{uj}_{a,b} + K'^{ij}_{a,b} + K'^{ij}_{a,b} \]

\[ K'^{ij}_{1,1} = \int_0^{2\pi} \int_{\gamma_p} \left( A_{11} \frac{\partial u_j^T}{\partial r} + A_{22} \frac{u_j^T}{r} + A_{60} \frac{\partial v_j}{\partial \theta} + A_{12} \frac{\partial u_j^T}{\partial \theta} \right) dr d\theta \]

\[ K'^{ij}_{1,2} = \int_0^{2\pi} \int_{\gamma_p} \left( A_{22} \frac{\partial u_j^T}{\partial r} + A_{12} \frac{\partial u_j^T}{\partial \theta} \right) dr d\theta \]

\[ K'^{ij}_{1,3} = 0 \]

\[ K'^{ij}_{1,4} = \int_0^{2\pi} \int_{\gamma_p} \left( B_{22} \left( \frac{u_j^T}{r} \frac{\partial \varphi_j}{\partial r} + B_{12} \frac{\partial u_j^T}{\partial \theta} \frac{\partial \varphi_j}{\partial r} + B_{66} \frac{\partial u_j^T}{\partial \theta} \right) \right) dr d\theta \]

\[ K'^{ij}_{1,5} = \int_0^{2\pi} \int_{\gamma_p} \left( B_{22} \frac{\partial u_j^T}{\partial r} \frac{\partial \varphi_j}{\partial \theta} + B_{12} \frac{\partial u_j^T}{\partial \theta} \frac{\partial \varphi_j}{\partial \theta} + B_{66} \frac{\partial u_j^T}{\partial \theta} \right) dr d\theta \]

\[ K'^{ij}_{2,3} = 0 \]

\[ K'^{ij}_{2,4} = \int_0^{2\pi} \int_{\gamma_p} \left( B_{22} \frac{\partial v_j}{\partial r} \frac{\partial \varphi_j}{\partial \theta} + B_{12} \frac{\partial v_j}{\partial \theta} \frac{\partial \varphi_j}{\partial \theta} + B_{66} \frac{\partial v_j}{\partial \theta} \right) dr d\theta \]

\[ K'^{ij}_{2,5} = \int_0^{2\pi} \int_{\gamma_p} \left( B_{22} \frac{\partial v_j}{\partial r} \frac{\partial \varphi_j}{\partial \theta} + B_{12} \frac{\partial v_j}{\partial \theta} \frac{\partial \varphi_j}{\partial \theta} + B_{66} \frac{\partial v_j}{\partial \theta} \right) dr d\theta \]

\[ K'^{ij}_{3,4} = \int_0^{2\pi} \int_{\gamma_p} A_{55} \frac{\partial \varphi_j}{\partial r} dr d\theta \]

\[ K'^{ij}_{3,5} = \int_0^{2\pi} \int_{\gamma_p} A_{44} \frac{\partial \varphi_j}{\partial \theta} dr d\theta \]

\[ K'^{ij}_{4,4} = \int_0^{2\pi} \int_{-1}^{1} \left( A_{55} \frac{\partial \varphi_j}{\partial r} + D_{11} \frac{\partial \varphi_j}{\partial r} + D_{11} \frac{\partial \varphi_j}{\partial \theta} + D_{12} \frac{\partial \varphi_j}{\partial \theta} \right) d\zeta d\theta \]

\[ K'^{ij}_{4,5} = \int_0^{2\pi} \int_{\gamma_p} \left( D_{22} \frac{\partial \varphi_j}{\partial r} + D_{12} \frac{\partial \varphi_j}{\partial \theta} + D_{66} \frac{\partial \varphi_j}{\partial \theta} \right) dr d\theta \]

\[ K'^{ij}_{5,5} = \int_0^{2\pi} \int_{\gamma_p} \left( D_{22} \frac{\partial \varphi_j}{\partial r} + D_{12} \frac{\partial \varphi_j}{\partial \theta} + D_{66} \frac{\partial \varphi_j}{\partial \theta} \right) dr d\theta \]
\[
K_{1,1}^{V_j} = \int_0^{2\pi} \left( r_p k_{r_p} \mathbf{u}_j^T \mathbf{u}_j + R k_{r_p} \mathbf{u}_j^T \mathbf{u}_j \right) d\theta + \int_{r_p}^{R} \left( k_{r_{\theta=0}}^w \mathbf{u}_j^T \mathbf{u}_j + k_{r_{\theta=2\pi}}^w \mathbf{u}_j^T \mathbf{u}_j \right) dr
\]

\[
K_{1,2}^{V_j} = K_{1,3}^{V_j} = K_{1,4}^{V_j} = K_{2,3}^{V_j} = K_{2,4}^{V_j} = K_{3,4}^{V_j} = K_{4,5}^{V_j} = K_{5,5}^{V_j} = 0
\]

\[
K_{2,2}^{V_j} = \int_0^{2\pi} \left( r_p k_{r_p} \mathbf{v}_j^T \mathbf{v}_j + R k_{r_p} \mathbf{v}_j^T \mathbf{v}_j \right) d\theta + \int_{r_p}^{R} \left( k_{r_{\theta=0}}^w \mathbf{v}_j^T \mathbf{v}_j + k_{r_{\theta=2\pi}}^w \mathbf{v}_j^T \mathbf{v}_j \right) dr
\]

\[
K_{3,3}^{V_j} = \int_0^{2\pi} \left( r_p k_{r_p} \mathbf{w}_j^T \mathbf{w}_j + R k_{r_p} \mathbf{w}_j^T \mathbf{w}_j \right) d\theta + \int_{r_p}^{R} \left( k_{r_{\theta=0}}^w \mathbf{w}_j^T \mathbf{w}_j + k_{r_{\theta=2\pi}}^w \mathbf{w}_j^T \mathbf{w}_j \right) dr
\]

\[
K_{4,4}^{V_j} = \int_0^{2\pi} \left( r_p k_{r_p} \mathbf{\Phi}_j^T \mathbf{\Phi}_j + R k_{r_p} \mathbf{\Phi}_j^T \mathbf{\Phi}_j \right) d\theta + \int_{r_p}^{R} \left( k_{r_{\theta=0}}^w \mathbf{\Phi}_j^T \mathbf{\Phi}_j + k_{r_{\theta=2\pi}}^w \mathbf{\Phi}_j^T \mathbf{\Phi}_j \right) dr
\]

\[
K_{5,5}^{V_j} = \int_0^{2\pi} \left( r_p k_{r_p} \mathbf{\Psi}_j^T \mathbf{\Psi}_j + R k_{r_p} \mathbf{\Psi}_j^T \mathbf{\Psi}_j \right) d\theta + \int_{r_p}^{R} \left( k_{r_{\theta=0}}^w \mathbf{\Psi}_j^T \mathbf{\Psi}_j + k_{r_{\theta=2\pi}}^w \mathbf{\Psi}_j^T \mathbf{\Psi}_j \right) dr
\]

\[
K_{1,1}^{c_{ij}} = \int_{r_p}^{R} \left( k_{c_{ij}} \mathbf{u}_j \bigg|_{\theta=0} - k_{c_{ij}} \mathbf{u}_j \bigg|_{\theta=2\pi} \right) dr
\]

\[
K_{2,2}^{c_{ij}} = \int_{r_p}^{R} \left( k_{c_{ij}} \mathbf{v}_j \bigg|_{\theta=0} - k_{c_{ij}} \mathbf{v}_j \bigg|_{\theta=2\pi} \right) dr
\]

\[
K_{3,3}^{c_{ij}} = \int_{r_p}^{R} \left( k_{c_{ij}} \mathbf{w}_j \bigg|_{\theta=0} - k_{c_{ij}} \mathbf{w}_j \bigg|_{\theta=2\pi} \right) dr
\]

\[
K_{4,4}^{c_{ij}} = \int_{r_p}^{R} \left( k_{c_{ij}} \mathbf{\Phi}_j \bigg|_{\theta=0} - k_{c_{ij}} \mathbf{\Phi}_j \bigg|_{\theta=2\pi} \right) dr
\]

\[
K_{5,5}^{c_{ij}} = \int_{r_p}^{R} \left( k_{c_{ij}} \mathbf{\Psi}_j \bigg|_{\theta=0} - k_{c_{ij}} \mathbf{\Psi}_j \bigg|_{\theta=2\pi} \right) dr
\]

\[
K_{1,2}^{c_{ij}} = K_{1,3}^{c_{ij}} = K_{1,4}^{c_{ij}} = K_{2,3}^{c_{ij}} = K_{2,4}^{c_{ij}} = K_{3,4}^{c_{ij}} = K_{4,5}^{c_{ij}} = K_{5,5}^{c_{ij}} = 0
\]

\[
\mathbf{M}_1^I = \int_0^{2\pi} \int_{r_p}^{R} I_0 \mathbf{u}_j^T \mathbf{u}_j d\theta dr, \quad \mathbf{M}_2^I = \int_0^{2\pi} \int_{r_p}^{R} I_0 \mathbf{v}_j^T \mathbf{v}_j d\theta dr, \quad \mathbf{M}_3^I = \int_0^{2\pi} \int_{r_p}^{R} I_0 \mathbf{w}_j^T \mathbf{w}_j d\theta dr
\]

\[
\mathbf{M}_4^I = \int_0^{2\pi} \int_{r_p}^{R} I_1 \mathbf{\Phi}_j^T \mathbf{\Phi}_j d\theta dr, \quad \mathbf{M}_5^I = \int_0^{2\pi} \int_{r_p}^{R} I_1 \mathbf{\Psi}_j^T \mathbf{\Psi}_j d\theta dr
\]