Quantum theory without Hilbert spaces

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Abstract

Quantum theory does not only predict probabilities, but also relative phases for any experiment, that involves measurements of an ensemble of systems at different moments of time. We argue, that any operational formulation of quantum theory needs an algebra of observables and an object that incorporates the information about relative phases and probabilities. The latter is the (de)coherence functional, introduced by the consistent histories approach to quantum theory. The acceptance of relative phases as a primitive ingredient of any quantum theory, liberates us from the need to use a Hilbert space and non-commutative observables. It is shown, that quantum phenomena are adequately described by a theory of relative phases and non-additive probabilities on the classical phase space. The only difference lies on the type of observables that correspond to sharp measurements. This class of theories does not suffer from the consequences of Bell’s theorem (it is not a theory of Kolmogorov probabilities) and Kochen-Specker’s theorem (it has distributive ”logic”). We discuss its predictability properties, the meaning of the classical limit and attempt to see if it can be experimentally distinguished from standard quantum theory. Our construction is operational and statistical, in the spirit of Kopenhagen, but makes plausible the existence of a realist, geometric theory for individual quantum systems.

I Introduction

The first completed and consistent formulation of quantum theory was the Kopenhagen interpretation. Its attitude was primarily operational: it considered quantum mechanics as a theory that provides the probabilities for measurement outcomes. Measurements are thought as the point of encounter between the classical world of apparatuses and observers and the microscopic world of atoms. All predictions were phrased in terms of ensembles and ensemble averages. Kopenhagen would not deal with individual systems.

Such a description, however successful, did not answer the important question: how is an individual system described and what are its properties? This was Einstein’s demand for ”elements of reality” and Bohm’s later emphasis on
the need for “ontology in quantum theory”. This question is of utmost importance in physics: in spite of the remarkable success of quantum mechanics, it was thought imperative, that new frameworks have to be devised in order to address it.

There were two main directions: either supplement quantum theory with extra variables that would describe individual systems (“hidden variable theories”), or keep the existing formalism and try to interpret it in a realist sense, as though it refers to properties of individual systems (state vector reduction [1], relative state formulation [2, 3], consistent histories [4, 5, 6, 7,...]). Both approaches come to insurmountable obstacles due to two theorems: Bell’s [8] and Kochen and Specker’s [9].

Bell’s theorem and the subsequent experiments [10] prevent hidden variable theories from being local, while the Kochen-Specker’s theorem forbids realist theories from asserting the existence of definite (i.e. non-contextual) properties for individual physical systems. Hence from the first category only non-local theories, such as the precious but inelegant Bohmian mechanics [11, 12] have survived. From the second one, all approaches have to accept that properties of quantum theory are contextual - something very disturbing for any theory that pertains to ”objectively” describe physical phenomena. It would, then, be fair to say, that seventy two years later, the Copenhagen interpretation has survived all assaults. This is due to its balance and moderation: it claims little and does not refrain from admitting ignorance (but will not admit incompleteness, either).

Kopenhagen quantum theory is essentially a model for describing experiments and is based on the notion of probability. We count the number of events in an ensemble of physical systems and define probabilities for these events from their relative frequency. But counting occurrences of events does not exhaust the physical or observable content of quantum theory. Relative phases are observable, as was first shown by Bohm and Aharonov [13] and these phases cannot be solely described by probabilistic concepts. What is more, the consistent history approach has emphasised, that when we examine properties of systems at more than one moments of time, we cannot use standard probability theory. The physical probabilities correspond to a non-additive measure. This non-additivity comes from the presence of interference phases. In fact, these are of the same origin as the Bohm-Aharonov phases; they are all generated by the Berry connection [14].

The relative phases are important and ever present; they are also measurable. This is what we show in detail in section 2. All possible measurement outcomes, whether of phases or of probabilities can be encoded in an object, that was first introduced in the consistent histories approach: the decoherence functional. We further argue that one should not think of quantum theory as a probability theory. Probability theory, its axioms and its concepts are mathematical constructs; there is no a priori reasons for them to describe physical reality. One has to provide physical arguments, before any mathematical model is chosen to be applied in a physical situation. In particular, an operationalist attitude to probability has to admit that certain mathematical axioms (additivity of probabilities) are not warranted by any operational procedure.

Quantum theory is not adequately described solely in terms of probabilistic
concepts. A theory for the quantum phenomena ought to provide values for both "probabilities" and relative phases. Once we admit this, there is no reason to be constrained by the Hilbert space formalism. All that is needed for an operational formulation of a quantum theory is an algebra of observables and the decoherence functional, which contains the information about all possible measurements. (We think, that a more appropriate name for the functionality of the latter, would be the coherence functional.) The algebra of observables does not have to be non-commutative. In fact, as we show in section 3, one can consider that our observables are functions on the classical phase space and use the Wigner transform to construct a theory that fully reproduces the predictions of quantum theory. Its only disagreement is in what each theory considers as sharp measurements: in our theory sharp measurements correspond to subsets of a space $\Omega$ (the phase space), while in standard quantum theory they are projection operators. The quantum behaviour is all contained in the relative phases of the coherence functional.

We are careful to define our theory as an operational one, in the spirit of Copenhagen. But it easy to see that any theory starting therefrom would be able to sidestep both Bell’s and Kochen-Specker’s theorems. We avoid the former, because we do not have a probability theory - our primitive concepts are intensities and relative phases. And we avoid the latter, because the "logic" of the theory is distributive, as a corollary of the commutativity of the algebra of observables. Classical probabilities and, as a limiting case, determinism, are obtained as approximate theories. In some regime, we can describe our experiments using classical probability theory (if we coarse-grain enough to suppress the relative phases) and if one wishes one might use rules and interpretation of classical probability, in order to to make predictions. These are not unambiguous, but at least clearer than the quantum ones.

This theory is operationally equivalent to quantum theory. But it may lead to more than that. Having abandoned the Hilbert space one has much more freedom in trying to construct a quantum theory for individual systems. In fact, one has the freedom to look for a geometric origin of quantum phenomena, and exploit all the technical machinery of the quantisation approaches in doing so. This is something suggested by some recent results in the consistent histories programme \[14, 15\]. The ideal result of such an attempt would be to understand the physical reason, why relative phases appear and how they are related to statistics.

II Filter measurements

Our first aim is to recover the Copenhagen quantum theory. We, therefore, take a minimalist stance towards what a physical theory should do. We just demand that it can provide an adequate model for describing our possible experiments. For this reason we find convenient to idealise experiments in the fashion of von Neumann and Jauch \[16\] or the operational approach to quantum theory \[17, 18\]. We start by stating our basic operational concepts.

First, we have the sources $S$: they prepare an ensemble of physical systems (which we will call particles). This ensemble we will call a beam. By its intensity we mean the number of particles it contains, which is assumed to be as large as
we want. An experiment that can be modeled corresponds to a beam passing through a sequence of filters: these are experimental setups, that allow a particle to pass only if it is found to satisfy a certain property. The filter then typically reduces the intensity of a beam. We choose to idealise measurements by filters rather than by pointer devices, because the latter are more difficult to visualise when we have successive measurements. A pointer device can be viewed as a collection of mutually incompatible filters, placed at (roughly) the same point in the path of the beam. The result of a beam passing through a series of filter we shall call an experimental history or short a history.

We will also assume that we have some way of determining the intensity of each beam, by measuring the number of particles that were incident on a detector. The relative intensity of the beam after passing through a number of filters \( C, D, \ldots, E \), we will call the probability determined through this experiment. No mathematical meaning is to be given to this word: in particular it does not refer to Kolmogorov probability. It denotes the ratio of intensities, and the choice for this word implies the frequency interpretation of probabilities, which is the natural in an operational setting.

We will also assume that we have devices, such as beam splitters and beam recombinators. We also have screens, upon which we can see interference patterns.

The experiments idealised here refer to ensembles rather than individual particles. A non-deterministic theory can only model ensembles, at least when no further information about the physical content of the system is assumed. One model might be able to make predictions about the individual system, if we decide to push the frequency interpretation of probability to its limits. Nonetheless, this is not necessary and we should attempt this only if we have a good grasp of the meaning of the laws that govern the measurement of beams.

II.1 Classical beams

Let us consider experiments, where the beam is assumed to satisfy classical probability theory. The physical system is characterised by a number of parameters that determine points on a space \( \Omega \). Then the beam will be described by a normalised, positive function on \( \Omega \), \( \rho(x), x \in \Omega \). Each filter will be described by a function \( \chi(x) \), that truncates all values of \( x \) outside the range that characterises the filter. A perfect filter has for \( \chi \) the characteristic function of a subset of \( \Omega \). An imperfect filter does not perform according to its specification all the time, it should therefore be described by a smeared characteristic function. We shall write as \( C(x) \) the characteristic function of a subset \( C \) of \( \Omega \).

Consider a particular two-filter experiment as in figure 1. A beam \( \rho \) leaves the source and passes through two filters \( C \) and \( D \) at times \( t_1 \) and \( t_2 \) respectively. After passing \( C \) the beam has become \( C\rho \) and after \( D \) it is \( D\rho \). The detector \( I \) then measures the intensity of the beam which is equal to \( \sum_x (D\rho)(x) \). This way we measure the probability for the history “\( C \) and then \( D \)” This we shall denote as \( p(C; t_1; D; t_2) \).

Note, that we have assumed that the probabilities for the beam have no self-dynamics.

Suppose we carry these experiments with many different filters. In particu-
Figure 1: An experiment with classical beams. The beam described by $\rho$, leaves the source $S$, passes through the filters $C$ and $D$ and its intensity is measured by the detector $I$.

lar, we can consider a class of filters $C_i$, each of them corresponding to a value $\lambda_i$ of an observable $A$. Then, to this observable we can assign a function

$$A(x) = \sum_i \lambda_i C_i(x)$$  \hspace{1cm} (II. 1)

Suppose we perform a number of experiments for all possible filter configurations “$C_i$ and then $C_j$” and that we measure the resulting quantities $p_{ij} = p(C_i, t_1; C_j, t_2)$. Then the correlation function for the observable $A$ can be reconstructed as

$$\langle A_{t_1} A_{t_2} \rangle = \sum_{ij} \lambda_i \lambda_j p_{ij}$$  \hspace{1cm} (II. 2)

In general, the information for all possible measurements at all possible times is encoded in the stochastic probability measure $d\mu(x(\cdot))$, on the space of all paths $x(\cdot)$ from $\mathbb{R}$ to $\Omega$. From this, we can predict or reconstruct any probability or correlation function of the theory. Conversely, a sufficiently large number of beam experiments suffices to reconstruct (with any desired accuracy) the stochastic probability measure.

## II.2 Quantum beams: intensities

Let us now try to repeat the same type of experiments in the quantum case. To a beam we associate a vector $|\psi\rangle$ in a Hilbert space $H$. According to the rules of quantum theory a filter will correspond to a projection operator $P$. Let us then put two filters $P$ and $Q$ from which the beam will pass, at time $t_1$ from $P$ and at time $t_2$ from $Q$. Then after passing from $Q$ the beam will be described by the vector $|f\rangle = QP|\psi\rangle$.

\footnote{The determination of an observable, that corresponds to a given set of filters is an important fact that a theory describing the physical system has to provide. In practice, it is determined by reference to other physical systems and can be argued to be eventually tied to measurement of time or space, or to the counting of numbers.}
Figure 2: Measuring intensity of quantum beams. The beam $|\psi\rangle$ leaves the source $S$, passes through filters $P$ and $Q$ and its intensity is measured at $I$. The setup is identical to the one for the classical case.

From the rules of quantum theory we know that the relative intensity of this beam will be

$$\frac{\langle f|f\rangle}{\langle \psi|\psi\rangle} = \frac{\langle \psi|PQPP|\psi\rangle}{\langle \psi|\psi\rangle}$$

(II. 3)

This gives the probability $p(P, t_1; Q, t_2)$ that the history "first $P$ and then $Q$" will be realised. Now consider the case of three distinct experiments. In the first we take the first filter to be $P_1$ and the second $Q$. In the second, we replace $P_1$ with $P_2$, where $P_2$ is another filter such that $P_1P_2 = 0$ (no beam can pass those two filters in succession) and $P_1 + P_2 = 1$ (there does not exist any other filter $R$ such that both $P_1R = P_2R = 0$). In the third experiment we put no filter before $Q$. We can then measure the beam intensities in three cases and when we compare them we find that in general

$$p(P_1, t_1; Q, t_2) + p(P_2, t_2; Q, t_2) \neq p(Q, t_2)$$

(II. 4)

This means that we cannot use standard probability theory, in order to construct a model that predicts the beam intensities. The Kolmogorov additivity condition that a probability theory has to satisfy fails.

In general, the correct formula for the intensities would have to take into account the self-dynamics for the system that constitutes the beam. This would be generated by a Hamiltonian operator $\hat{H}$ and would give for the vector $|f\rangle$, that describes the beam incident on the detector at time $t_f$

$$|f\rangle = e^{-i\hat{H}(t_f-t_2)}Qe^{-i\hat{H}(t_2-t_1)}Pe^{-i\hat{H}t_1}|\psi\rangle$$

(II. 5)

This gives the following expressions for the intensity of the beam after passing through two filters

$$p(P_1, t_1; Q, t_2) = \langle \psi|e^{i\hat{H}_1}Pe^{i\hat{H}(t_2-t_1)}Qe^{-i\hat{H}(t_2-t_1)}Pe^{-i\hat{H}_1}|\psi\rangle$$

(II. 6)
If we restrict to filters \( P_i \), each corresponding to a value \( \lambda_i \) of an observable \( A \), then we can write the statistical correlation function for \( A \)

\[
<A_{t_1}, A_{t_2}> = \sum_{ij} \lambda_i \lambda_j p(P_i, t_1; P_j, t_2)
\]

This is a real number: it is different from the quantum mechanical correlation functions, which are complex valued.

\[
G_A(t_1, t_2) = \langle \psi| e^{iH_{t_1}} A e^{-iH_{t_1}} e^{iH_{t_2}} A e^{-iH_{t_2}} |\psi\rangle
\]

Clearly this correlation function, cannot be determined by measurement of intensities as described so far.

### II.3 Quantum beams: relative phases

Quantum beams contain more information than their intensities. Their relative phase is also of physical interest. In other words, when a beam passes through a succession of filters, it undergoes more changes, than the ones encoded in its intensity. And these changes are measurable only through procedures of combination of beams and comparison of beams. In effect if the beam passes through the filters \( P \) and \( Q \) the \( (\text{complex}) \) quantity \( \langle \psi|QP|\psi\rangle \) can be experimentally determined.

Let us give a hypothetical example of how to measure this phase structure. First, let us assume that each source \( S \) that prepares a beam \( |\psi\rangle \) comes from its manufacturer together with a set of fine filters labeled \( |\psi\rangle\langle\psi| \). When the beam exits the second filter \( Q \) it enters this fine filter becoming

\[
\langle \psi|QP|\psi\rangle|\psi\rangle
\]

The measurement of the relative intensity of this beam gives as an outcome a positive number less than one

\[
r(P, t_1; Q, t_2) = |\langle \psi|QP|\psi\rangle|\]

But we can also perform another measurement. Let us consider a source emitting a beam \( |\psi\rangle + |\phi\rangle \). And let us assume we can monitor its wave pattern at a screen SC and store it in memory. Then we carry another experiment where \( |\psi\rangle + |\phi\rangle \) passes through a beam splitter and splits into its \( |\psi\rangle \) and \( |\phi\rangle \) component. The component \( |\phi\rangle \) is kept as a reference beam, but the \( |\psi\rangle \) component has to pass through filters \( P, Q \) and \( |\psi\rangle\langle\psi| \) as before. The resulting beam \( \langle \psi|QP|\psi\rangle|\psi\rangle \) is recombined with \( |\phi\rangle \) and we can see its interference pattern on the screen. When we compare this interference pattern with the previous one, we notice a phase shift equal to the phase of \( \langle \psi|QP|\psi\rangle \). This we will call \( e^{i\theta(P; T_1; Q; t_2)} \). It is, in fact, a generalisation of Berry's phase \( \theta \).

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2The comparison of interference patterns is, for instance, the way the Bohm-Aharonov phase was originally measured \[\theta\]. It should be remembered, that the measurement of phases is not something that is adequately described by the practical rule that "measured quantities correspond to self-adjoint operators". This point was, in fact, the original motivation for Bohm and Aharonov’s work. The relative phases are obtained by comparing patterns, rather than reading pointers in devices. In a sense, a phase measurement is a non-local way of extracting information from a quantum system.
Figure 3: Determining $|\langle \psi |QP|\psi \rangle |$ through measurement of intensities. A beam $|\psi \rangle$ leaves the source $S$, passes through filters $P$ and $Q$ and then through $|\psi \rangle\langle \psi |$, before it is measured at $I$.

Figure 4: Measuring the relative phase of the beam. A source $S$ emits a beam $|\psi \rangle + |\phi \rangle$, which a beam splitter splits into $\psi$ and $|\phi \rangle$. The component $|\psi \rangle$ passes through filters $P$ and $Q$ and then through $|\psi \rangle\langle \psi |$, before it is recombined with $|\phi \rangle$ and their interference pattern measured at the screen $SC$. Comparing this pattern, with the pattern of $|\psi \rangle + |\phi \rangle$ (which we have from a previous experiment) enables the determination of the relative phase between the history with filters $P$, $Q$ and the history with no filter, for the beam $|\psi \rangle$. 
There are more efficient ways to measure the complex number $\langle \psi | QP | \psi \rangle$, without needing the fine filter $| \psi \rangle \langle \psi |$. We can for instance consider it as correlated with another system, which is sufficiently controlled, and find the relative phase by the change in certain transition rates. This is how the Berry phase was measured in NMR interferometry [21]. Or perhaps the phase could be obtained by studying interference of the beam $QP | \psi \rangle$ with a large number of reference beams $| \phi_i \rangle$. The exact procedure of measurement makes little difference in the conceptual description, though. The important statement is that, any experiment that measures phases must necessarily compare or interfere two beams.

This means that the phase is not absolute; therefore, it cannot be read from intensity measurements of a single beam. In fact we can find the relative phase between two beams that have passed through different filters: it can be read from the comparison of their interference patterns with a reference beam $| \phi \rangle$. This phase is equal to the one we would measure for a single beam passing successively through $P_1, Q_1, Q_2, P_2$, i.e. adding the filters of the second beam in reverse order. (If $H \neq 0$, the placement of the filters is also important). As far as the correlation functions are concerned, we can easily see that the quantum mechanical correlation functions 3 can be measured as

$$\langle A(t_1)A(t_2) \rangle = \sum_{ij} \lambda_i \lambda_j [re^{i\theta}](P_i, t_1; P_j, t_2) \quad (\text{II. 11})$$

III Models for quantum theory

How do we encode the results of filter measurements? Let us denote by $\alpha$ the series of filters $P_1, \ldots, P_n$. We can define the operator

$$C_\alpha = e^{iHt_1}P_1 e^{-iHt_1} \cdots e^{iHt_n}P_ne^{-iHt_n} \quad (\text{III. 1})$$

For a pair of histories $\alpha$ and $\beta$, we can write the object

$$d(\alpha, \beta) = \langle \psi | C_\alpha^\dagger C_\beta | \psi \rangle = Tr(\rho C_\alpha^\dagger C_\beta) \quad (\text{III. 2})$$

If $\alpha \neq \beta$ this gives the relative phase between the two histories. If $\alpha = \beta$, this gives the relative intensity of the final beam.

III.1 The consistent histories approach

This object was introduced by Gell-Mann and Hartle in the context of the consistent histories approach to quantum theory. They called it the decoherence functional. Its function is to determine, when classical probability can be used to describe a quantum system.

3 To the best of my knowledge, the only experimental determination of correlation functions in quantum theory is in the case of the electromagnetic field. There, correlations of photon number can be measured. But, since for the EM field, the photon number commutes with the Hamiltonian, the quantum mechanical and the statistical correlation functions coincide and are real-valued.
The consistent histories is a realist formulation of quantum theory for individual systems. Its main tenet is that probabilities can be defined in a consistent set, i.e., a set of exhaustive and exclusive histories that satisfy

\[ d(\alpha, \beta) = 0, \]  

(III. 3)

for \( \alpha \neq \beta \). Propositions in this set can then be described by classical logic and one can make predictions and retrodictions within this set. But all such predictions are contextual: they have to make reference to a given consistent set. Otherwise, one might obtain contradictory inferences \[22, 23\]. This pathology is typical in all realist interpretational schemes for quantum theory: it is of the same nature as the Kochen-Specker paradox \[26\].

All treatments of consistent histories in the literature focus on its status as a realist interpretation. However, the formalism makes sense even if it is interpreted solely as an operationalist one, i.e., a generalisation of the Copenhagen interpretation dealing with time-ordered series of measurements. For this reason we make a distinction in this paper, between the histories formalism and the consistent histories approach, which is the realist interpretation of the formalism in terms of consistent sets.

The consistent histories interpretation ignores the values of the off-diagonal elements of the decoherence functional. It focuses on the probability aspect of quantum theory (only intensities) and ignores the phase aspect, even if there exist meaningful operational schemes of measuring them. This we think is a severe omission: not because it affects the logical consistency of the theory. The consistent histories framework claims that it describes adequately all measurement situations. The point is, rather, that it contradicts the motivation of the scheme as providing a realist description of physical phenomena: The reasonable expectation one would have for a realist formulation of quantum theory is that it should explain the predictions of ensemble measurements, in terms of properties of the individual systems.

The consistent histories scheme, as any scheme that is based solely on probabilities, cannot do that for the relative phase measurements. An insistence that only probabilities are physically relevant, inevitably gives physical meaning to only a part of the information contained in the decoherence functional.

### III.2 The filters’ algebra

Our stance, so far, is operational: filter measurements allow us to determine (in principle) all possible values of the decoherence functional. As such it is the object that incorporates all information about measurements, as the stochastic

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4 We should remark that this pathology does not lie at the level of logical consistency of the theory \[24, 25\], but at the epistemic level, since it forces a redefinition of fundamental notions (truth or property of a system), which is much weaker than what is employed in scientific practice.

5 To be precise, if we are to stay purely at the level of probabilities, the real part of the decoherence functional is sufficient and necessary, since it contains the information about the non-additivity of the probability measure: \[ 2\text{Red}(\alpha, \beta) = d(\alpha + \beta, \alpha + \beta) - d(\alpha, \alpha) - d(\beta, \beta) \]. In this sense, probabilities are exactly half the content of the decoherence functional. Nonetheless, Gell-Mann and Hartle have identified the full decoherence functional as the physically relevant object, using arguments based on persistency of records.

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measure does in the case of classical beams. One could still question, whether 
there is any more information we can get through a particular implementation 
of our filter-measurement scheme. Is it possible that the comparison of three 
histories gives results that are not contained in the measurement of intensities 
and comparison of two histories? Sorkin has demonstrated that in standard 
quantum theory the answer is negative [27, 28]. All information obtained from 
the comparison of three histories is entirely attributed to the interference be-
tween all pairs of them. And this stems from the fact that quantum theory is 
based on the complex numbers rather than any other algebraic field.

We are then interested in finding a mathematical model that describes this 
set of experiments. What would be the possible ingredients? First we need 
to identify the mathematical objects that correspond to filters $P$. The simpler 
condition is to consider that the filters correspond to idempotent members of 
an associative algebra ($P^2 = P$). We shall denote the algebra by $A$ and by $I(A)$ 
the set of its idempotent elements. The algebra has to have a unity 1, which 
corresponds to no filter and a zero 0, which is the completely opaque filter. 
There is then the following correspondence between mathematical objects and 
physical operations.

- The idempotency condition $P^2 = P$ means that the (almost instantane-
ous) succession of two identical filters changes the beam in an identical 
fashion as one single filter does. 
The condition $P_1 P_2 = P_1$ implies that the filter $P_1$ is finer (or more re-
strictive) than the filter $P_2$. (Conversely $P_2$ is coarser than $P_1$). This 
is represented as $P_1 \leq P_2$, and defines a partial ordering in the space of 
filters.

- The condition $P_1 P_2 = 0$ corresponds to two incompatible filters: if we put 
them in succession they act like the opaque filter.

- For incompatible filters $P_1 + P_2$ denotes a filter that is coarser than $P_1$ 
and $P_2$ and we cannot construct any finer filter that has this property.

- If the algebra is not Abelian, neither addition nor multiplication of generic 
filters in the algebra have a natural interpretation in terms of simple op-
erations.

- If among a class of filters $P_i$, each of them corresponds to a value $\lambda_i$ of an 
observable $A$, this observable is represented as an element $A = \sum \lambda_i P_i$ of 
the algebra $A$.

An experiment consisting of a sequence of filters $P_1, \ldots, P_n$, placed at mo-
ments $t_1, \ldots, t_n$, can be described by the element $P_1 \otimes \cdots \otimes P_n$ of the tensor 
product algebra $\otimes_i A_{t_i}$, where $A_{t_i}$ is a copy of the filter algebra labeled by time 
$t_i$. Elements of this tensor product algebra can be viewed as corresponding to (possibly) time-averaged observables for this system. The algebra $A_h = \otimes_t A_t$ 
includes all possible filter sequences $\alpha$ we can construct for the system $\alpha$.

\[\text{footnote}^{6}\] The tensor product over $t$ of the filter algebras can have many interpretations. We can 
consider it as a space containing all finite tensor products of algebras $A$ [30], or a genuine 
tensor product over all values of $t \in \mathbb{R}$ (which makes the algebra too large). Or a construction 
of an object that resembles a tensor product over a continuous variable [31, 32]. For simplicity
We shall call any finite filter sequence a history: it will be represented by small greek letters. As a mathematical object it is represented by an element of $A_h$ that can be written as a tensor product of idempotent elements of $\mathcal{A}$, i.e. $\alpha = P_1 \otimes P_2 \otimes \ldots \otimes P_n$.

The moments of time \{t_1, \ldots, t_n\} upon which a filter has been set defined the temporal support of this history. Two histories are incompatible, if at some time $t \in T_1 \cap T_2$ the corresponding filters are incompatible. Consider two histories $\alpha_1$ and $\alpha_2$ have temporal supports $T_1$ and $T_2$. We will, then, denote by $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ the same histories, but viewed as having temporal support $T_1 \cup T_2$ (we just append the trivial filter at their non-common points). A history $\alpha$ is finer than $\beta$ (denoted $\alpha \leq \beta$), if at all $t \in T_1 \cup T_2$ the filters of $\tilde{\alpha}$ are finer than the filters of $\tilde{\beta}$.

Two incompatible histories with the same temporal support can be added to give an idempotent elements of $A_h$, but it is not necessary that they form a history (i.e an idempotent element of the form $P_1 \otimes \ldots \otimes P_n$). In the consistent histories scheme, they are said to correspond to propositions about the possible outcomes of the experiment. A case where $\alpha + \beta$ corresponds to a history is the following: when $\alpha$ and $\beta$ are constructed from filters that are at all time-points, but one, in which their filters are incompatible, e.g. if $\alpha = P_1 \otimes Q$, $\beta = P_2 \otimes Q$, with $P_1 P_2 = 0$, then $\alpha + \beta = (P_1 + P_2) \otimes Q$ corresponds to a history. This would not be true for $\alpha = P_1 \otimes Q$ and $\beta = P_2 \otimes Q'$, with $[Q,Q'] \neq 0$. Another case is when the two histories are disjoint because they have disjoint temporal supports. This is, for instance, the case of the history $\alpha = P_t$ and the history $\beta = Q_{t_2} \otimes Q_{t_3}$, with (say) $t_1 < t_2 < t_3$. Then we define the “addition” as $\alpha + \beta := P_t \otimes Q_{t_2} \otimes Q_{t_3}$ is another filter history on the Hilbert space $H_t \otimes H_{t_2} \otimes H_{t_3}$. We shall call two filter histories $\alpha$ and $\beta$ operationally additive, if $\alpha + \beta$ is a filter history.

If we want to consider experiments carried out simultaneously on two different physical systems, each characterised by a filter algebra $A_1$ and $A_2$, the filter algebra for the total system is $A_1 \otimes A_2$. This tensor product is distinct from the one used earlier to construct a space, where possible histories might be embedded.

### III.3 The coherence functional

To each source we then assign an object $d$, which is a complex valued map of pairs of measurements. It incorporates all information about all possible filter measurements for this beam. Unlike the consistent histories scheme, we want to emphasise the overall importance of the relative phases. We think it would be more precise in our scheme to call this object the coherence functional. Its diagonal elements $d(\alpha, \alpha)$ measure the relative intensity of the final beam, while its off-diagonal the relative phase between the beam having passed through different series of filters. As such it can be extended to a functional over $A_h \times A_h$.

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[1] Isham, C. (1979). The quantum logic reformulation of the consistent histories approach. The constructions presented in sections 3.2 and 3.3 are a rephrasing of this quantum logic scheme in an operational language.

[2] We shall consider that $t$ takes values in a finite set with a large number of elements. As such there is no problem in the construction of this object.
Quantum theory suggests that this functional ought to satisfy the following properties \[32\]

1. **Positivity**: \(d(\alpha, \alpha) \geq 0\); all beams have positive intensity.

2. **Normalisation**: \(d(1, 1) = 1\); in absence of filters the intensity does not change.

3. **Hermiticity**: \(d(\alpha, \beta) = d^{\ast}(\beta, \alpha)\); the reverse order of comparing two histories, gives the opposite number for the relative phase.

4. **Additivity**: \(d(\alpha + \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma)\), if \(\alpha\) and \(\beta\) are disjoint histories; this is suggested by the corresponding structure in quantum theory.

5. **Triviality**: \(d(0, \alpha) = 0\); from an opaque filter no beam can pass.

These conditions on the coherence functional are natural consequences of the operations we aim to describe. Only number 4. is a mathematical assumption, that is not intuitively evident, but clearly suggested as fundamental from quantum theory. Note that the histories \(\alpha, \beta, \alpha + \beta, \ldots\) in properties 1-5 refer to idempotent elements of \(A_{h}\) of the form \(P_{1} \otimes \ldots \otimes P_{n}\), that correspond to filter measurements. But they hold for general indempotent elements of \(A_{h}\). There exist, though, properties of the coherence functional in standard quantum theory, that cannot be extended this way \[32\]. The most important are

6. **Boundedness**: \(|d(\alpha, \beta)| \leq 1\); this is suggested by the operational meaning of \(|d(\alpha, \beta)|\) as an intensity. Clearly the intensity of the final beam cannot be larger than the original one’s.

7. **Subadditivity**: if \(\alpha\) and \(\beta\) are operationally additive, then \(|d(\alpha, \alpha)| \leq |d(\alpha + \beta, \alpha + \beta)|\); from a coarser filter the beam will exit with higher intensity.

**Correlations**  As we mentioned, one can view the coherence functional as a functional on \(A_{h} \times A_{h}\). As such it can be defined on pairs of observables. In fact one can show that

\[
d(A_{t_1} \otimes \ldots \otimes A_{t_r}, A'_{t_1'} \otimes \ldots \otimes A'_{t_r'}) = G^{nr}_{A}(t_1, \ldots, t_r; t'_1, \ldots, t'_r) \tag{III. 4}
\]

where by \(G^{nr}_{A}\) we denote the \(rs\) correlation function, where \(r\) denotes the number of indices that are time-ordered and \(s\) the number of indices that are anti-time ordered. The generator of these functions is known as the closed-time-path generating functional associated to \(A\) \[34, 35\]. As a functional on \(A_{h}\) it is identical to the coherence functional. For our purposes, it is sufficient to remark, that the knowledge of the coherence functional allows us to fully reproduce any correlation function of the theory \[36\].
III.4 Conditioning

When we want to go beyond descriptions of ensembles and actually predict properties of an individual system, we need to incorporate the information we obtained through experiments into the mathematical object that allows us to make predictions. This process is known as conditioning. In classical probability theory, it is implemented through conditional probability. In standard quantum theory, it is commonly known as "wave packet reduction", even though in a strict sense it does not correspond to a physical process. If a measurement corresponding to $P$ is realised then the state of the systems transforms as $\rho \rightarrow P\rho P/\text{Tr}(\rho P)$.

The consistent histories approach defines conditioning through probability theory. That is, in a consistent set that classical probability holds, one can use its rules to define conditional probabilities. This choice comes from the insistence of the consistent histories approach on probabilities as the basic objects of the theory. From our perspective it is not necessary to use concepts of probability theory to define conditioning. In fact, conditioning is primarily an algebraic rather than a probabilistic concept. If we have a system described by the coherence functional $d$ and we have found that a filter history $\alpha$ has been realised, then we condition as follows.

First, we restrict to all filter histories $\beta$ operationally compatible with $\alpha$. This means that if $\alpha$ is a series of filters, then any allowable setup has to include the filters of $\alpha$ or filters that are compatible and coarser from the filters of $\alpha$. In this sense operational compatibility means that there exists a filter history $\gamma$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$, such that there is no other filter $\gamma'$ satisfying this property with $\gamma' \leq \gamma$. We shall denote such a filter history $\gamma = \alpha \circ \beta$. Then we define the conditioned coherence functional for pairs of filter histories

$$d_\alpha(\beta, \beta') = d(\alpha \circ \beta, \alpha \circ \beta')/d(\alpha, \alpha) \quad \text{(III. 5)}$$

We divide over $d(\alpha, \alpha)$ in order to satisfy the normalisation condition. For the simple case that $\alpha$ consists for a simple filter $P$ before all measurements we have that $d_\alpha$ is given by the standard expression (3.2), with the substitution $\rho \rightarrow P\rho P/\text{Tr}(\rho P)$. If $\alpha$ consists of a single filter $P$ at time $t_f$ after all measurements, the coherence functional is given by

$$d(\alpha, \beta) = \text{Tr}(C_\alpha^\dagger \rho C_\beta \rho_{f}) \text{Tr}(\rho_{f}) \quad \text{(III. 6)}$$

where $\rho_{f} = e^{iH_f t_f} P e^{-iH_f t_f}/\text{Tr}P$.

In our operational perspective, conditioning means that we restrict our experiments, to having certain filter configurations that correspond to $\alpha$ fixed. If one wished to make predictions about individual systems, according to the consistent histories interpretation, one can use the conditioned coherence functional to define consistent sets and seek for histories with probability one in these sets.

III.5 The relative phase theory

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8In the language of lattices, we demand that the meet of $\alpha$ and $\beta$ exists is a filter history itself.
III.5.1 Standard quantum theory as a model

The specification of an algebra and of a class of coherence functionals constitutes a model theory that aims to describe the filter experiments we described. Quantum theory takes for $\mathcal{A}$ the set of bounded operators $B(H)$ on a complex Hilbert space $H$. A filter then corresponds to a projection operator on $H$. When we combine the algebras to form the history algebra, we have $\mathcal{A}_h = \otimes_t B(H_t) = B(\otimes_t H_t)$. This means that histories can be thought as (homogeneous) projectors on $\otimes_t H_t$, while general (time-averaged) observables can be identified with self-adjoint operators on $\otimes_t H_t$.

The algebra $\mathcal{A}$ is non-Abelian. Two projectors do not generically commute, hence two filters need not be compatible. Clearly, when we study two systems we have $B(H_1) \otimes B(H_2) = B(H_1 \otimes H_2)$ and the combined system is characterised by the Hilbert space $H_1 \otimes H_2$. The coherence functional is given by equation (3.2). It is constructed out of two ingredients: one that characterises the nature of the physical systems (the Hamiltonian) and one that characterises the way the source $S$ was constructed. This second piece is represented by a Hilbert space vector or more generally by a density matrix.

III.5.2 Abandoning the Hilbert space

Quantum mechanics assumes that the algebra of filters is non-commutative, corresponding to the bounded operators in a complex Hilbert space. If one takes quantum theory to be primarily a theory of probabilities, then non-commutativity as a requirement has a deep physical meaning. The product of two non-commuting self-adjoint operators is not self-adjoint, hence the correlation between them is inevitably complex. The non-commutativity of two operators is connected with the need of complex numbers in quantum theory, since both are necessary for the uncertainty principle to hold.

Non-commutativity is therefore an integral part of quantum theory, when this is viewed from the perspective of probabilities. But we have emphasised that quantum theory is not only a theory of probabilities, rather it also includes information from relative phases. We have tried to develop a chain of arguments leading to a scheme, where these phases enter in the formulation of quantum theory as primitive ingredients: in the coherence functional. We are then led to the natural question, whether non-commutative observables and more generally the Hilbert space structure are at all necessary in this scheme and whether one can substitute them with a simpler structure without compromising the predictive power of quantum theory.

The answer to this last question is affirmative: one can do without the Hilbert space structure. In fact, we can get a theory that is based on commuting observables (simple functions), that fully reproduces the predictions of quantum theory. We define the space of filters as in classical probability theory. They ought to correspond to idempotent elements of $B(\Omega)$, the set of bounded functions on some (measurable) space $\Omega$. A filter is, then, represented by a characteristic function of a subset of $\Omega$. Clearly a history corresponds to an idempotent element of $B(\Omega)$.
element of $\otimes \omega(t) = B(\times \omega(t))$. Hence a history corresponds to an element of $\times \omega(t)$, or rather a subset $\omega(t)$ of it consisting of some suitable maps from $\mathbb{R}$ to $\omega$. Two subsystems are combined by $B(\omega_1) \otimes B(\omega_2) = B(\omega_1 \times \omega_2)$, hence they are represented in a space $\omega_1 \times \omega_2$. And then we define a coherence functional as a bilinear functional

$$d : B(\omega_h) \times B(\omega_h) \rightarrow \mathbb{C}$$

(III. 7)

The essence of this construction is that the filters and the observables are identical to the ones of classical theory and the distinct quantum mechanical behaviour is encoded in the introduction of the coherence functional (and hence the relative phases) as the primitive elements of the theory. This class of theories we shall call relative phase theories.

Commutative variables are considered as fundamental in hidden variable theories, as for instance in Bohm’s mechanics. Another instance, is Nelson’s stochastic mechanics [33], which tries to reproduce quantum mechanics by a stochastic process on configuration space. Besides problems of locality, this construction cannot reproduce the unequal-time correlation functions of quantum theory. In fact, no additive probability measure can.

Our proposal is closer in spirit and substantially influenced by to Sorkin’s quantum measure theory [27, 28]. We have a different attitude, though: we do not agree that measure-theoretical ideas are themselves sufficient to describe individual quantum systems. At the level of formalism, the main difference is that Sorkin insists on the probability structure (his quantum measure is the real part of the coherence functional), thereby downplaying the importance of relative phases. The reader is also referred to [29], which states different motivations for this line of reasoning.

III.5.3 Construction of a theory

In order to construct a relative phase theory for a physical system, we would have to specify the space $\Omega$ of elementary alternatives and find a procedure that will enable us to write a large class of physically relevant coherence functionals on this space. The question then arises, whether there exist such constructions that reproduce the predictions of quantum theory.

In standard quantum theory it is known that one can obtain full information about a physical system, solely through (unsharp) phase space measurements. The reason for this is that every physical system we know, has a phase space structure incorporated in its quantum description. This comes from the canonical commutation relations, or stated differently, from the representation of the canonical group on the system’s Hilbert space. This suggests that the phase

10The canonical group is classically identified as a group acting transitively by canonical transformations on a classical phase space. In the corresponding quantisation scheme it is required, that the Hilbert space of the quantum theory ought to carry a unitary, irreducible representation of the canonical group. Conversely, if a Hilbert space of a quantum theory admits an irreducible, unitary representation $U(g)$ of some group $G$, we can construct the coherent states $U(g)|0\rangle$, where $|0\rangle$ is a fiducial vector. If we define an equivalence relation on $G$ such that $g \sim g'$, if $U(g)|0\rangle$ and $U(g')|0\rangle$ just differ by a phase, the resulting space $G/\sim$ can be viewed as a phase space for the classical system. It carries a symplectic structure, which it inherits from the imaginary part of the Hilbert space’s inner product. It also carries a metric structure coming from the real part of the inner product.
space ought to be identified with the space $\Omega$ of a relative phase theory. And the representation of the canonical group gives a way to construct coherence functional on phase space, that correspond to the standard quantum mechanical ones. The procedure by which this is effected is known as the Wigner transform.

Systems that have a linear phase space have as canonical group the Weyl group. For the system case of an one-dimensional system (particle at a line), its Lie algebra is determined by

$$[\hat{q}, \hat{p}] = i$$  \hspace{1cm} (III. 8)

Let us denote by $\hat{U}(\chi, \xi) = e^{i\hat{q}\xi + i\hat{p}\chi}$ the unitary operator representing one element of this group. Then we can define the operator

$$\hat{\Delta}(q, p) = \int d\chi d\xi e^{-iq\xi - ip\chi} \hat{U}(\chi, \xi)$$  \hspace{1cm} (III. 9)

The operators $\hat{\Delta}$ provides a linear map from the space of operators on the Hilbert space $H$ to the phase space $\Gamma$ of our system

$$\hat{A} \mapsto F_A(q, p) = Tr(\hat{A}\hat{\Delta}(q, p))$$  \hspace{1cm} (III. 10)

This map is trace preserving, in the sense that

$$Tr\hat{A} = \int \frac{dq dp}{2\pi} F_A(q, p)$$  \hspace{1cm} (III. 11)

$$Tr(\hat{A}\hat{B}) = \int \frac{dq dp}{2\pi} F_A(q, p)F_B(q, p)$$  \hspace{1cm} (III. 12)

But it does not preserve the multiplication; the condition $P^2 = P$ is not preserved in phase space: a projection operator is not mapped into a characteristic function. A sharp quantum mechanical filter is not sharp on phase space: this is a manifestation of the uncertainty principle. Since the Hilbert space, in which histories live is constructed out of tensor products of the single-time Hilbert space, the Wigner transform can be employed to pass from the Hilbert space $\otimes_i H_t$, $i = 1, \ldots, n$, we define a function $F_A$ on $\times_i \Gamma_t$ as

$$F_A(q_1, p_1, t_1; \ldots; q_n, p_n, t_n) = Tr_{\otimes_i H_t} A(\hat{\Delta}(q_1, p_1) \otimes \ldots \otimes \hat{\Delta}(q_n, p_n))$$  \hspace{1cm} (III. 13)

Given any discrete-time history with support $\{t_1, \ldots, t_r\}$ we can define the operator $C_m$ as

$$\hat{C}_m = e^{i\hat{H}_t t_1} \hat{\Delta}(q_1, p_1)e^{-i\hat{H}_t t_1} \ldots e^{i\hat{H}_t m} \hat{\Delta}(q_m, p_m)e^{-i\hat{H}_t m}$$  \hspace{1cm} (III. 14)

and define the following object that has support on a pair of an $n$-point and an $m$-time temporal support.

$$W_{n,m}[q_1, p_1, t_1; \ldots; q_n, p_n, t_n; q'_1, p'_1, t'_1; \ldots; q'_m, p'_m, t'_m] = Tr\left(\hat{C}_m^\dagger \hat{\rho}_0 \hat{C}_n\right)$$  \hspace{1cm} (III. 15)
Then it corresponds to a coherence functional on $\otimes_i B(\Omega_t) \times \otimes_i B(\Omega'_t)$. To a pair of functions $A \in \otimes_i B(\Omega_t)$ and $B \in \otimes_i B(\Omega'_t)$ it assigns the complex number

$$d(A, B) = \int \frac{dq_1 dp_1}{2\pi} \cdots \frac{dq_n dp_n}{2\pi} \frac{dq'_1 dp'_1}{2\pi} \cdots \frac{dq'_m dp'_m}{2\pi} A(q_1, p_1, t_1; \ldots; q_n, p_n, t_n) B(q'_1, p'_1, t'_1; \ldots; q'_m, p'_m, t'_m) \times W_{n,m}[q_1, p_1, t_1; \ldots; q_n, p_n, t_n | q'_1, p'_1, t'_1; \ldots; q'_m, p'_m, t'_m]$$

(III. 16)

In fact, one can prove that these discrete time definitions allow for a definition of the coherence functional for continuous-time histories [36]. There are two structures on phase space, that are relevant in this case: a $U(1)$ connection form $pdq$ that is responsible for the introduction of the complex numbers in the coherence functional and the Moyal bracket. The later is represents the algebra of operator commutators on phase space. The Moyal bracket $\{\cdot, \cdot\}$ between a pair of functions on phase space $f$ and $g$ reads

$$\{f, g\} = 2i f \sin \left( \frac{1}{2} \{\cdot, \cdot\} \right) g,$$

(III. 17)

where by $\{\cdot, \cdot\}$ we denote the Poisson bracket as a bilinear operator: $f \{\cdot, \cdot\} g = \{f, g\}$. The theory, thus constructed gives the same values for the correlation function of any observable. Any quantum mechanical operator corresponds to a function on $\Omega$. Its correlation functions are identical with the quantum mechanical ones, by virtue of equation (3.16). But in the relative phase theory the sharp filters are different from the corresponding quantum mechanical ones. Hence the predictions about outcomes of idealised, "precise" measurements would differ and so would the corresponding relative phases. Note, that even at a single moment of time, the theory is not described by a probability distribution. This is equivalent to the well known fact, that the Wigner function is in general non-positive. In terms of filter measurements, this means that the relative phase between two single-time histories with incompatible filters is generally non-zero. In standard quantum theory (and for sharp filters) this vanishes identically.

This construction preserves naturally all properties of the coherence functional, since it amounts to substituting in equation (3.2) general positive operators in place of projectors. In particular, property 6. also holds since a characteristic function corresponds to a positive operator with norm less than unity, which is a sufficient for a proof.

The Wigner transform can be generalised for other quantum mechanical systems. For spin systems, it arises from the study of the representations of $SU(2)$ [11] while for bosonic fields one uses the infinite-dimensional Weyl group as the canonical group. Even for fermion fields there exists a phase space description [37]. Hence, for all systems of physical interest one can construct a relative phase theory, that completely reproduces the predictions of quantum mechanics.

11The phase space for a single fermionic oscillator out of which the field is constructed is the two sphere $S^2$.  

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We have to add here, that the Wigner transform is not the only possible group-theoretic construction, that can give predictions that reproduce the ones of quantum theory. There exist also the $P$ and $Q$ transformations that are based on coherent states. The $Q$-transform of an operator $A$ is its expectation value on the coherent state basis $Q_A(x, \xi) = \langle x\xi | A | x\xi \rangle$, while its $P$-transform is a function $P_A(x, \xi)$ defined by $A = \int dx\xi P_A(x, \xi) | x\xi \rangle \langle x\xi |$. The following property holds

$$Tr(AB) = \int dx\xi Q_A(x, \xi) P_B(x, \xi) = \int dx\xi Q_B(x, \xi) P_A(x, x)$$

This implies, that we could construct then a relative phase theory by considering either the $P$-transform of the coherence functional together with the $Q$-transform of operators for the filter algebra, or conversely the $Q$-transform for the coherence functional and the $P$-transform of the operators of the filter algebra. And perhaps this does not exhaust the possible ways of constructing a relative phase theory on phase space, that reproduces the correlation functions of quantum theory. Which one of them is the correct description is something that quantum theory itself cannot answer. We would need a theory for the individual system.

IV General properties

To summarise: in the previous sections we argued that, at least at an operational level, quantum phenomena can be described in terms of a filter algebra $\mathcal{A}$ and a coherence functional that gives the relative phase and probability content of this theory. And then we proposed, that

i. Quantum theory can be described by a commutative filter algebra, corresponding to functions on some space $\Omega$.

ii. The space $\Omega$ can be identified with the phase space of the corresponding classical theory.

In general we can use group theoretic constructions (like the Wigner transform) to construct a coherence functional on $\Omega$, that reproduces the predictions of quantum theory. We are now going to analyse the distinct structures and general properties that our proposal implies.

IV.1 Quantum logic

We said earlier that the space of filters has a partial ordering relation. We also said that in special cases it is operationally meaningful to consider the addition as corresponding to a conjunction of propositions. In standard quantum theory, filters correspond to projection operators on a Hilbert space, and it turns out that the partial ordering structure is complete and forms a lattice. This lattice has a number of operations that are algebraically identical to the conjunction ($\land$), disjunction ($\lor$) and negation of logical propositions. It is therefore often
said that the lattice of projectors on the Hilbert space is the logic of the quantum theory, which is distinct from classical logic. Indeed, it is stated, that a projection operator corresponds to a proposition about a property of an individual physical system. This is motivated by the analogue in classical probability, that a statement about a physical system can always be phrased as a statement that the system lies in a subset of the sample space $\Omega$. And the set of subsets of $\Omega$ has the structure of a Boolean lattice.

The interpretation of the algebraic structure of a lattice as the logic of individual systems is often questioned, due to the fact that the lattice of projectors is not distributive. This means that if $\lor$ and $\land$ represent the algebraic operations of disjunction and conjunction between projectors it is not always true that

$$P \land (Q \lor R) = (P \lor Q) \land (P \lor R) \quad \text{(IV. 1)}$$

The failure of distributivity is the cause of the Kochen-Specker theorem: one cannot consistently assign true or false values to all propositions. In other words the lattice of projectors cannot be taken to describe properties of an individual system at a single moment of time. Properties of such systems are at most contextual. We referred earlier a manifestation of this pathology in the consistent histories approach. If contextuality is viewed as referring to our perspective of the system, the notion of objectivity is lost. If the context, it refers to, is a concrete experimental setup, there is little gain from the Copenhagen interpretation. For instance, in consistent histories there are very few propositions, beside measurement outcomes that can be considered as “true” for a physical system \[22\].

We have stressed throughout, that we deliberately take an operational stance. In fact, we are rather wary to consider, that the models for beam measurements have validity as description for individual systems, and even more wary to talk about logic of such systems. However, unlike the standard quantum theory, should we wish to do so, we can. We are not constrained by the Kochen-Specker theorem. Our filters correspond to subsets of $\Omega$, hence to a distributive lattice. The "logical" structure of our theory is identical to the one of classical probability and therefore completely unambiguous, as far as definability of properties is concerned \[12\].

### IV.2 Predictability

Usually predictability refers to our ability to make predictions about an individual system, when we have some probabilistic description of it. In classical probability, one says that if the probability of an event is 1, then almost surely this event will be realised for any individual system. In quantum theory predictability is a more complicated issue; the Copenhagen interpretation does not
care to address it: an operational treatment needs not be concerned with individuals. And with good reason: prediction and retrodiction are obscure in all realist schemes. Amongst such schemes, the consistent histories provides the most solid treatment. It states, that if in a consistent set the probability of an history is equal to one (or if the conditional probability of an event is one and the condition is satisfied), then this event is predicted (or retrodicted) by the theory. Unfortunately, (at least) retrodiction is pathological. In different consistent sets one can retrodict mutually exclusive propositions \[23\]. This is again the problem of contextuality.

Now, in the relative phase theory we can have a consistent histories formulation of predictability (after all the formalism satisfies the Gell-Mann - Hartle - Isham axioms \[7, 30\]). Let us take two histories \(\alpha\) and \(\beta\) that are disjoint and exhaustive \((\alpha + \beta = 1)\). Consider that \(d(\alpha, \alpha) = 1\). Would it mean that we can definitely predict that \(\alpha\) will be realised\(^\dagger\)? Since we have \(d(\alpha + \beta, \alpha + \beta) = 1 = d(\alpha, \alpha)\), we would have that \(d(\beta, \beta) = -2\text{Red}(\alpha, \beta)\). Hence it is not as though the intensity of the beam passing through \(\beta\) vanishes. The possibility \(\beta\) cannot be ruled out unless \(\text{Red}(\alpha, \beta) = 0\). This is the consistency condition. In this case, our chosen set of histories is described by a probability measure. Then one can use a rule of inference that states: whenever a probability for an event is unity, then this event is predicted (or retrodicted) within our consistent set. The dependence of predictions upon choice of sets would carry out here as well.

We would then need to check whether there are incompatible predictions in different consistent sets. We will examine the most elementary case: we take a coherence functional, perhaps conditioned with respect to some experimental data. Then consider three disjoint alternatives \(\alpha, \beta, \gamma\), that are exhaustive \((\alpha + \beta + \gamma = 1)\). Let us assume that \(\{\alpha, \beta + \gamma\}\) forms a consistent set in which \(\alpha\) is predicted. This means that \(d(\alpha, \alpha) = 1, d(\beta, \beta) = 0, \text{Red}(\alpha, \beta) = 0\). And let us also assume that \(\{\alpha + \beta, \gamma\}\) forms a consistent set for which \(\gamma\) is predicted. This is clearly a situation, where we get incompatible inferences in different consistent sets. We can verify, that for this to occur we need have \(\text{Red}(\alpha, \gamma) \leq -\frac{1}{2}\). This means that the contextuality of propositions is not necessarily a consequence of non-distributivity. In our case, it would come from the non-additivity of the probability measure, together with the prediction rule.

Of course, since we have a distributive logic, the incompatible inferences are much more controllable. They cannot arise if \(\text{Red}(\alpha, \beta) \geq -\frac{1}{2}\). One could be tempted to impose this restriction on allowable physical theories, or use it to define strongly predictable quantum theories. But we believe, that the notion of prediction based on consistent sets is rather artificial for the nature of quantum theory. Fundamentally, the relative phase theory is not a probability theory and an attempt to force it into the strict axiomatic framework of Kolmogorov probability, does violence to its nature. We are therefore very skeptical, whether this formalism can be extended to describe properties of an individual system and make predictions about it. We are content with the description of beam experiments and anything that can be modeled upon them. Our attitude is very much into the tradition initiated by Bohr; but unlike him we have no reason to

\(^\dagger\)The coherence functional could have been obtained by the incorporation of a number of experiment outcomes, using conditioning as described in section 3.4.
believe that our formalism is complete. We have defined it for measurements of ensembles and it is for measurements of ensembles only that it is good.

The most we could say about the decoherence condition, is that there is a degree of coarse-graining in the filters, that would allow us to ignore all phase information and describe the filter experiments using another model: classical probability theory. And within this approximation, one could sometimes interpret the theory in such a way as to make some predictions about an individual system. But probability theory arises as an approximation, not as a fundamental set of concepts that interpret our theory.

IV.2.1 The classical limit

The observables of our theory are functions on a space $\Omega$, which we can take to be the classical phase space of the system. When sufficient coarse-graining is allowed as to make the approximation by probability theory sufficiently good, the system will be described by a stochastic process on phase space. Typically, for sufficient coarse-graining, this process is almost deterministic giving rise to the Hamilton equations of motion.

There is no ambiguity what the classical limit of the theory will be and that it is independent of the degree of coarse-graining. This is unlike the consistent histories approach, where the non-distributivity of the lattice of propositions makes in principle possible the existence of very different and incompatible classical limits (what Gell-Mann and Hartle call quasiclassical domains). In fact, such quasiclassical domains are generally unstable, something that diminishes predictability even at the semiclassical level.

To summarise: we do not believe that there is yet a meaningful way to make predictions about individual systems. Only in the case, that we consider sufficient coarse-graining, so that the decoherence condition approximately holds, we can approximate our theory by classical probability theory. This, together with a set of assumptions that have to do with the meaning of probabilities in a classical setting, might allow us to make predictions about an individual system. But it is by means of an approximation that we can make predictions, not by means of a fundamental law of nature.

IV.3 The Bell-Wigner theorem

Any new interpretation of quantum theory that tries to do without the Hilbert space has to phase the restraining demands of Bell’s theorem and all its generalisations. Bell’s theorem is usually taken to imply that realism and locality is not compatible with quantum theory. By realism, one usually means the specification of hidden variables, i.e. variables that characterise more precisely the state of the system.

Our recipe, that the underlying structure of quantum theory is the classical phase space implies the use of the Cartesian product to combine the phase space of such systems. This might lead to a hasty conclusion that such a construction is forbidden by Bell’s theorem. This is not true. The most general proof of Bell’s theorem is based on the assumption that the ensemble description of hidden variables that characterise individual systems, is given by a probability
distribution that satisfies the Kolmogorov axioms. This is exactly what we explicitly renounce in this paper. We do not believe that probability theory is a priori of relevance to a physical theory. Probability theory is a branch of mathematics that has provided useful models for certain physical phenomena and there is no reason to expect that it would be relevant for the totality of them.

Quantum theory also shows that it is not only frequencies of occurrences that we measure, but also relative phases, which do not fit any concepts of probability theory. Once we remove this condition, Bell's theorem is not constraining. In this regard let us note the following points. Bell's original proof did not use a probability distribution. He assumed definite values for each particle of the correlated pair, which essentially implied determinism. As such, it is not constraining for our formulation. There exist more general versions of Bell's theorem that employ probability distributions and constrain stochastic hidden variable theories. They are not relevant to our case. There is also the Greenberger-Holt-Zeilinger (GHZ) argument \[41\], which pertains to refer to properties of individual systems and distinguishes the prediction of hidden variable theories from the outcome of a single measurement. In their proof they use an assumption from probability theory: if the “probability” of an event is equal to 1 then this will be true. As we showed in the previous section, this is valid only if the probability satisfies the Kolmogorov additivity condition. As such it cannot be used against a hidden variable theory that is not modeled this way.

Overall, we believe that we should make the qualification that: Bell's theorem forbids not local realism in general, but local realist theories that are modeled by classical probability of the Kolmogorov type. In the light of our discussion, it is a much weaker restriction and has fewer metaphysical implications, than what is usually claimed.

We should note that there exists a group theoretic justification for the use of Cartesian product to construct the phase space of the combined system. If \(\Gamma_1\) and \(\Gamma_2\) are phase spaces and \(G_1\) and \(G_2\) the corresponding canonical groups, then \(G_1 \times G_2\) is the canonical group of \(\Gamma_1 \times \Gamma_2\). If \(G_1\) is irreducibly represented on a Hilbert space \(H_1\) and \(G_2\) on a Hilbert space \(H_2\), then \(G_1 \times G_2\) is irreducibly represented on \(H_1 \otimes H_2\) and a Wigner transform can be naturally constructed to pass from the quantum theory on \(H_1 \otimes H_2\) to the relative phase theory on \(\Gamma_1 \times \Gamma_2\). This construction allows us to fully reproduce the correlation functions of the quantum theory: in fact, it was done by Agarwal in reference \[44\].

\[14\] Probability theory is not as primitive a structure as arithmetic or geometry is. These can be argued to underlie most, if not all, of our physical concepts and form an irreducible mathematical background from which to view physical phenomena. Unlike them, probability theory arose much later and its relation to physics was not immediately evident. People became confident with its use in physics after the success of statistical mechanics, and this description was put in a solid mathematical footing by Kolmogorov. And still, the use of probability theory had to be justified by physical arguments: the ergodic postulate. We should also add, that Kolmogorov probability, which is a measure-theoretical, description of probabilities is but one formalisation of the intuitive notion of probability. Its abstractness and the rich mathematical structure of measure theory, make it extremely useful, but its applicability to a particular situation should be a matter of the physics of the concrete physical system. It is easy to think of counterexamples in classical physics, of ensembles with random behaviour, that cannot be described by a probability measure \[38\].
IV.4 The uncertainty principle

The Wigner transform in standard quantum theory yields a non-positive function on phase space (the Wigner function) corresponding to a density matrix. It does not define a probability distribution. This is usually taken to be a consequence of the noncommutativity of position and momentum and of the fact that one cannot measure position and momentum with infinite accuracy. These remarks are correct, when they refer to beams, i.e. ensembles. But it is often stated that there is no meaning to a sharp phase space point for an individual system. This is completely unwarranted by the precise formulation of the uncertainty principle: $\Delta q$ and $\Delta p$ are defined as deviations for a series of measurements of position and momentum respectively in an ensemble of physical systems.

In our scheme the uncertainty principle still holds, since it refers to the comparison of results of measurements with filters, that correspond to phase space observables. Both $\Delta q$ and $\Delta p$ can be measured as correlation functions by a sequence of series experiments. However, in principle, we can have filters that correspond to a phase space area less than $\hbar$. The beam passing through them, will develop a much stronger phase shift, compared to the case, where it passes from a coarser filter. In effect, if the Wigner function of the beam oscillates around zero, taking negative values, at some phase space scale, it is filters with accuracy in this scale, that will give rise to a strong interference behaviour for the beam. Nonetheless, both intensities and relative phases are well predicted by the theory.

IV.5 Elements of reality

We have refrained from considering our formulation as anything more, but an operational one, in the spirit of Copenhagen. We can, then, have no claim about existence or not of elements of reality for an individual quantum system.

However, if this were our true aim, we would not have bothered to move beyond Hilbert space quantum theory. The main reason for abandoning Hilbert space for the phase space is that the latter is more amenable to our intuition for individual systems and has a rich geometric structure. We would like to formulate a geometrical theory on phase space for individual systems, that will explain the role of the coherence functional and its relation to statistics of ensembles. In a sense, we want to ask the following question: what geometric behaviour on the phase space leads to a statistical behaviour like the one given by a coherence functional? In a sense, this question is analogous to the one about the validity of the description of macroscopic systems by statistical mechanics and probabilistic concepts. There, an answer, was the property of ergodicity or quasiergodicity. Would it be possible to find a similar answer in the quantum case?

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15The operator form of the uncertainty principle has a different meaning from the original derivation of Heisenberg. Heisenberg discussed the physical mechanisms by which measurements of a single physical system are prevented from giving a simultaneous accurate reading of position and momentum. But neither Heisenberg’s derivation implies the non-definability of sharp phase space properties. In standard quantum theory this is only implied by the Kochen-Specker theorem.
The only thing we can say right now is that the coherence functional is built from geometric objects on the phase space: bundles and connections \[14, 36\]. These structures arise naturally in the geometric quantisation scheme \[42\] and the group theoretic approach to quantisation \[43\]. The temporal logic histories programme \[30, 31, 45\] also suggests that these geometric structures are related to the temporal structure of this theory \[15, 46\]. These are interesting links that will set the tone of the work to follow.

What the present construction has accomplished is the removal of the constraints of Bell and Kochen-Specker’s theorems. This makes plausible the existence of a geometric description for individual systems that reproduces the ensemble predictions of Copenhagen quantum theory. In this respect, our stance is very much a continuation of Bohm’s programme. But all our motivation and structural insight has come from the consistent histories approach to quantum theory.

IV.6 Experiments

The only difference between our scheme and standard quantum theory, lies in the specification of the sharp filters. In an ideal world, where all filters could be assumed perfect, we would be able to explicitly distinguish between the predictions of the relative phase theory and standard quantum theory.

Unfortunately, this is not true. Even in standard quantum theory, a realistic filter is best described by a positive operator (sometimes known as an effect). On the other hand, it is the construction of the filter that determines, what exactly it will let pass and one would need to have a detailed specification of its physics, before estimating what the function that characterises it would be. For these reasons, it is very difficult to imagine realistic experiments, that would be able to distinguish between these theories. Such an experiment would be of high importance, as it could be said to separate between classical and quantum logic.

In general, the difference between the Weyl transform of a projection operator and a characteristic functions on phase space is to be found in a phase space region of the size of \(\hbar\). This means that the distinguishability between classical and quantum sharp filters is particularly acute in particle systems. There, the phase space is non-compact, hence the difference of filters at the order of \(\hbar\) is practically impossible to detect.

However, for spin systems the phase space is a two-sphere and its volume with respect to the natural metric induced by the representation of \(SU(2)\) (see footnotes 10 and 11) is of the order of \(\hbar\). In this case, one might expect a significant deviation in the measurement of intensities or relative phases. Of course, this deviation would have to correspond to the same correlation functions for the observables. There is an important difference between classical filters and projection operators, which comes from the fact that the Wigner transform does not preserve positivity. The Wigner transform of a projection operator is a non-positive function, while a classical filter, even if not sharp, cannot help not be positive. The negative values for the symbol of the projector are particularly distinctive in the case of spin systems. In this case, we have the symbol corresponding to a projection on a vector characterised by spin in the \(J\)
direction

\[ F_P(\theta, \phi) = \frac{1}{4\pi} (1 + 2\sqrt{3} \hat{n} \cdot J), \quad (IV. 2) \]

where \( \hat{n} \) is the unit vector in the direction specified by \( \theta \) and \( \phi \). We see that it takes negative values in a significant portion of the sphere. Our initial thought for a distinguishing feature was to exploit this lack of positivity in order to establish a bound between the predictions of quantum theory and relative phase theory for the statistical correlation functions of the observables. Two theories that have the same quantum mechanical correlation functions do not have the same statistical ones, because in the latter there is a different combination of probabilities for elementary outcomes.

Unfortunately, there are two flaws in this line of reasoning. First, it is not absolutely necessary that the transforms of the quantum mechanical projectors take negative values. We could have chosen to define the relative phase theory with respect to the Q-transform for the observables and the P for the coherence functional. The Q-transform of a projector is always positive, so the distinguishing feature between classical and quantum filters would not appear. So, these distinctions would be model-dependent, rather than relevant to the basic structure of a relative phase theory.

But there is a second reason, which we believe is potentially more important, as it might provide the beginning of an explanation, why the Hilbert space description is so natural in quantum theory. Any physical filter is made out of matter and has to interact with the measured physical system. One can, to first approximation, ignore all backreaction effects. But even if the self-dynamics of the measured system is zero, there will be always be a non-zero Hamiltonian evolution, due to the coupling with the degrees of freedom of the filter (in the case of spin consider the use of a Stern-Gerlach device as part of a filter). The point is that (at least in the Wigner picture) the classical filters are not robust under quantum mechanical evolution. This is to say, that evolution as given by the Moyal bracket does not preserve positivity of the filters. In this sense, a classical filter is an approximation to a realistic filter. It is an idealisation for a non-material way of blocking the beams. This idealisation is in essence a fundamental part in a mathematical formalisation of the notion of experiment, as carried out in Kolmogorov probability theory.

An analogy with classical probability might be indicative of what we have in mind. In analogy with the Heisenberg picture one can consider that the probability distribution of the beam stays the same, but the observables - filters evolve in time. Hence, one has a differential equation \( \frac{\partial}{\partial t} P = L P \), where \( L \) is a differential operator, that incorporated the dynamics; in a measurement situation it would contain a term for the interaction between the beam and the device. Now, in the case that all eigenvalues of \( L \) are negative and if we assume that the process lasts long, the smallest (in absolute magnitude) eigenvalue dominates and \( P \) goes to its corresponding eigenfunction. Hence, the effective filter associated with a measurement is a function that solely depends upon the dynamics. It will not be negative-valued though, because physical evolution operators preserve positivity. Different measurement devices correspond to different forms of \( L \) and different eigenfunctions.

This description is meaningful also in the quantum case. The (Heisenberg
type) equations of motion on the phase space $\Gamma$ read for an observable $A$

$$\dot{A} = \{A, H\} := \mathcal{L}_H A$$  \hspace{1cm} (IV. 3)

where $H$ is the Wigner transform of the Hamiltonian and $\mathcal{L}_H$ is an operator on the space of phase space functions. We repeat that in a measurement situation $H$ ought to include the action of the classical device on the quantum system.

The space of phase space functions can be made into a Hilbert space $L^2(\Gamma, d\mu)$, through the introduction of the natural measure on $\Gamma$. Hence $\mathcal{L}_H$ is an operators on $L^2(\Gamma, d\mu)$. The classical analysis would be also valid in this case. If $\mathcal{L}_H$ has only non-positive eigenvalues, in the long time limit, any observable would converge to an eigenfunction of $\mathcal{L}_H$, often a zero eigenstate. In the canonical Hilbert space picture this would imply that the effective filter, would be described by an operator commuting with the Hamiltonian $H$. In the minimal coarse-graining case, this would be a projector onto an eigenstate of the Hamiltonian. Hence, if we assume that the time-scale of a measurement is much larger than the natural time scale associated to the Hamiltonian, it is natural mathematically to consider that the effective filter, would correspond to an eigenstate of the (interaction) Hamiltonian.

In light of these considerations, it is possible, that the role of quantum mechanical filters is more important in realistic measurement situations, because of the nature of the averaged dynamics in the ensemble. This would imply, that the underlying “logic” of the theory can be distributive, but in ensemble measurements, projectors on a Hilbert space might provide a more realistic description, when the interaction dynamics and the finite-time interval of a measurement process is taken into account.

But the argumentation, we have presented, is far from complete and at best only suggestive. The fact is, we do not yet have a complete picture of how the standard Hilbert space would naturally arise from the theory on phase space. However, this discussion suggests that the use of Hilbert space vectors might be a consequence not only of the details of an underlying theory, but also from the basic operations one need to perform before setting an experiment. This would be something very desirable, since it would be a justification of the fact that Copenhagen quantum theory is the most efficient way to describe statistical outcomes of ensembles.

In any case, these two arguments make unlikely, that it is possible to differentiate between the statistical predictions of quantum theory and a relative phase theory. It is not impossible in principle, but it is difficult to establish a compelling argument that would sharply distinguish between those two. Perhaps, we shall be able to devise one, when we construct a more definite theory, about the quantum behaviour of individual systems on phase space. The few specifications we have given for a relative phase theory here, do not constrain much in a degree sufficient to phrase definite statements, about predictions distinct from quantum theoretical ones.
V Appraisal

Let us first summarise the arguments, that are central to the thesis of this paper:

1. There is no *a priori* reason to assume that an ensemble of physical systems, is describable by a probabilistic model that satisfies Kolmogorov’s axioms. One has to give a physical reason, in order to justify this assumption.

2. Quantum theory is not a theory of probabilities only: it also predicts relative phases between different histories. These phases are measurable.

3. Any theory purporting to describe quantum phenomena needs to specify an algebraic structure for the observables and a bilinear coherence functional that contains probability and phase information.

4. Once we accept phases as primitive ingredients of the theory, there is no compelling physical reason to employ the Hilbert space formalism. Observables can be defined in a purely classical fashion, as functions on some space of elementary alternatives.

5. The most conservative approach is to take for observables functions on the classical phase space. The Wigner transform is one way that can be used to construct a coherence functional that fully reproduces the predictions of standard quantum theory.

6. The resulting theory can be called a hidden variables theory, but does not suffer from Bell’s theorem, because it is not a probability theory. It neither suffers from the Kochen-Specker theorem, because its logic is distributive.

7. This construction suggests, that one should look on a geometric superstructure on the classical phase space, in order to construct a viable quantum theory for individual systems.

A significant part in our argumentation has been the identification of an axiomatic framework distinct from classical probability, that would also be operationally meaningful. This framework, we claimed, could substitute Hilbert space quantum theory; we, therefore, focused on the mathematical possibility of reproducing quantum mechanical predictions by a different model theory. For this reason our constructions were mainly technical and no physical principles were evoked. This is what we need in order to establish a framework, that would allow us to derive these results without any reference to quantum theory.

The second point is that there is no easy way to recover the Hilbert space of the standard theory. We would expect a description in terms of Hilbert space vectors and the Schrödinger equation to arise in a simple manner and to be related with the operational procedure of measurements. We have not been able to find an intuitively simple way to do so. It seems that the properties 1-7 for the coherence functional provide little restriction or insufficient guidelines.
for this purpose. An improved theory should have additional assumptions, but these would only come from a detailed construction of a theory for the individual system. Axioms of statistical nature are, perhaps, not sufficient by themselves to explain the structure of standard quantum theory.

In any case, we do not purport to have a definite theory yet. This work is more an indication of possibilities, rather than a completed framework. We showed, that the Hilbert space formulation and concepts are not necessary in a formulation of quantum theory, as far as statistical properties of measurements are concerned. This removes any impossibility objections and makes plausible, that the relative phase theories we described, are the statistical limit of a geometric theory of individual systems. Our main motive is the hope for a realist formulation of quantum theory, which would tell us more about elements of reality than Kopenhagen does.

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