On norm continuity, differentiability and compactness of perturbed semigroups

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Abstract
The main purpose of this paper is to treat semigroup properties like norm continuity, compactness and differentiability for perturbed semigroups in Banach spaces. In particular, we investigate three large classes of perturbations: Miyadera–Voigt, Desch–Schappacher and Staffans–Weiss perturbations. Our approach is mainly based on feedback theory of Salamon–Weiss systems. Our results are applied to abstract boundary integro-differential equations in Banach spaces.

Keywords Operator semigroup · Unbounded perturbation · Norm continuity · Compactness · Differentiability · Bergman space · Feedback theory · Integro-differential equations

1 Introduction
In this paper we investigate classical properties like norm continuity, compactness and differentiability for some classes of perturbed semigroups. To be more precise, let $X, Z$ be Banach spaces and $(A, D(A))$ a generator of a strongly continuous semigroup $\mathbb{T} := (T(t))_{t \geq 0}$ on $X$ such that $D(A) \subset Z \subset X$. We introduce a linear operator $L \in \mathcal{L}(\tilde{Z}, \tilde{X})$ where $\tilde{Z}$ and $\tilde{X}$ are Banach spaces carefully chosen in such a way that

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A + L with appropriate domain is well defined and generates a strongly continuous semigroup \( T^{cl} := (T^{cl}(t))_{t \geq 0} \) on \( X \) (this notation will be justified in Sect. 2). Now the problem to be treated is: do the two semigroups generated by \( A \) and \( A + L \) share the aforementioned properties? As a matter of fact this problem has already been considered by many authors who have some partial answers (depending on the type of perturbations).

The class of bounded perturbations, i.e., the case when \( X = \bar{Z} = \bar{X} \) (so that \( L \in \mathcal{L}(X) \)), is mainly treated by Phillips [23]. He proved that if \( (T(t))_{t \geq 0} \) is norm continuous (resp. compact) for \( t > 0 \), then the operator \((A + L, D(A))\) generates a strongly continuous semigroup \( (T^{cl}(t))_{t \geq 0} \) on \( X \) which is norm continuous (resp. compact) for \( t > 0 \), as well. On the other hand, Phillips constructed a semigroup \( (T(t))_{t \geq 0} \) which is norm continuous for \( t > t_0 \) with \( t_0 > 0 \) (i.e., eventually norm continuous) but the semigroup \( (T^{cl}(t))_{t \geq 0} \) is not norm continuous for \( t > t_0 \). Thus, eventual norm continuity and eventual compactness are, in general, not preserved even under bounded perturbations. It is shown in [7, Proposition III.1.14] that in the case of compact perturbation operator \( L \in \mathcal{L}(X) \), the eventual norm continuity is preserved for the perturbed semigroup whenever the initial semigroup is eventually norm continuous. In 1983, Pazy [21] (see also [24]) showed that the eventual differentiability of the semigroup \( (T(t))_{t \geq 0} \) is not translated to the perturbed semigroup generated by \( A + L \) even if \( L \) is a bounded perturbation. As shown in [4–6, 14, 33], extra conditions on the semigroup \( T \) are needed to assure the preservation of the eventual differentiability for the semigroup generated by \( A + L \).

Let us now analyze in profile the case of unbounded perturbations. Three large classes of unbounded perturbations will be investigated. The first class: choose \( \bar{Z} = Z \) and \( \bar{X} = X \); then we say that \( L \in \mathcal{L}(Z, X) \) is a Miyadera–Voigt perturbation for \( A \) if there exist \( a > 0 \) and \( \gamma \in (0, 1) \) such that for any \( x \in D(A) \) we have

\[
\int_0^a \| L(T(t)x) \| dt \leq \gamma \| x \|.
\]

In this case the operator \((A + L, D(A))\) generates a strongly continuous semigroup \( (T^{cl}(t))_{t \geq 0} \) on \( X \). As an example of application we cite the case of delay equations [3, 12]. In general, we do not have preservation of the aforementioned regularities under Miyadera–Voigt perturbations. However, for delay evolution equations with \( L' \)-history spaces, it is shown in [3] (resp. [19]) that if the free delay equation is governed by an immediately norm continuous (resp. immediately compact) semigroup, then the delay semigroup in product spaces associated with the delay equation is eventually norm continuous (resp. eventually compact). Moreover, some results on eventual differentiability for delay equations are obtained in [5]. The second class of unbounded perturbations is the case \( \bar{Z} = X \) and \( \bar{X} = X_{-1} \), so that \( L \in \mathcal{L}(X, X_{-1}) \), where \( X_{-1} \) is the completion of \( X \) with respect to the norm \( \| x \|_{-1} := \|(\lambda I - A)^{-1}x\| \), for \( x \in X \) and some (hence all) \( \lambda \in \rho(A) \). The semigroup \( \bar{T} \) can be extended to a strongly continuous semigroup \( \bar{T}_{-1} := (T_{-1}(t))_{t \geq 0} \) on \( X_{-1} \), whose generator \( A_{-1} : X \to X_{-1} \) is the extension of \( A \) to \( X \). In this case we say that \( L \) is a Desch–Schappacher perturbation for \( A \) if there exists \( t_0 > 0 \) such that
\[ \int_0^{t_0} T_{-1}(t_0 - s)\mathbb{L}f(s)ds \in X, \quad \forall f \in L^1([0, t_0], X). \]

It is well known (see [7, Chap. III-3-a]) that for a such \( \mathbb{L} \), the part of the operator \( A_{-1} + \mathbb{L} \) on \( X \) (denoted by \( A^{cl} := (A_{-1} + \mathbb{L}|_X) \)) generates a strongly continuous semigroup \( \mathbb{T}^{cl} \) on \( X \). For this kind of perturbations, Mátrai [20] has shown that \( \mathbb{T}^{cl} \) is immediately norm-continuous whenever so is the semigroup \( \mathbb{T} \) (see also Jung [17]).

To the best of our knowledge, there are no results concerning the differentiability under Desch–Schappacher perturbations. This is one of the objectives of this paper. In fact, we will show that if the generator \( A \) satisfies the so-called Pazy condition [22, p. 57] [see also (36) below], then the semigroup generated by \( A^{cl} \) is differentiable. Finally, let us discuss another more general class of perturbations. To that purpose, let \( A_m : Z \subset X \to X \) be a differential linear closed operator and \( G, M : Z \to U \) boundary linear operators, where \( U \) is a (boundary) Banach space. We consider the linear operator on \( X \),

\[ \mathcal{A} := A_m, \quad D(\mathcal{A}) = \{ f \in Z, \quad Gf = Mf \}. \]

We assume that \( A := A_m \) with domain \( D(A) = \ker(G) \) is a generator of a strongly continuous semigroup \( \mathbb{T} \) on \( X \). Observe that the operator \( \mathcal{A} \) is obtained by perturbing the domain \( D(A) \) of \( A \) by an unbounded perturbation \( M \). Based on feedback theory of regular linear systems ([31], see also the next section for definitions), the authors of [13] introduced sufficient conditions on \( G \) and \( M \) for which \( (\mathcal{A}, D(\mathcal{A})) \) is a generator of a strongly continuous semigroup \( \mathbb{T}^{cl} \) on \( X \). In fact, they proved that there exists a space \( \tilde{Z} \) such that \( D(A) \subsetneq Z \subset \tilde{Z} \subsetneq X \) and an operator \( \mathbb{L} \in \mathcal{L}(\tilde{Z}, X_{-1}) \) such that the operator \( (\mathcal{A}, D(\mathcal{A})) \) coincides with the following one

\[ A^{cl} := A_{-1} + \mathbb{L}, \quad D(A^{cl}) = \{ x \in \tilde{Z} : (A_{-1} + \mathbb{L})x \in X \}. \]

In this case, the operator \( \mathbb{L} \in \mathcal{L}(\tilde{Z}, X_{-1}) \) is called a Staffans–Weiss perturbation (see Theorem 1). The main objective of this work is to prove that immediate norm continuity and compactness are preserved under Staffans–Weiss perturbation operators, see Theorems 4 and 5. A special case of these results is when the operator \( M \) is bounded, i.e., \( M \in \mathcal{L}(X) \), so that we are in the Desch–Schappacher perturbations setting.

As source of applications of our abstract results, we will consider regularity of solutions of the following intergo-differential equation

\[
\begin{aligned}
\dot{x}(t) &= A_m x(t) + \int_0^t k(t - s)Px(s)ds, \quad t \geq 0 \\
Gx(t) &= Mx(t), \quad t \geq 0, \\
x(0) &= x,
\end{aligned}
\]

where \( A_m, G \) and \( M \) are as above, \( P : Z \to X \) is an admissible observation operator for \( A \) and the kernel \( k(\cdot) \) belongs to an appropriate Bergman space (see Sect. 5).

For the reader’s convenience, we briefly recall the relevant background from [31] (and also [13]) and related works and introduce (much of) our notation in Sect. 2.
Section 3 is on the study of immediate norm continuity and compactness of semigroups under Staffans–Weiss perturbation operators. In Sect. 4 we investigate the eventual differentiability of semigroups under Desch–Schappacher perturbations. The last section is concerned with the study of a class of integro-differential equations in Banach spaces.

2 Staffans–Weiss perturbation theorem

In this section, we shall recall the recent concept of Staffans–Weiss perturbations. The origin of these perturbations is the feedback theory of well-posed and regular linear systems introduced mainly by Salamon, Staffans and Weiss, see, e.g., [9, 25, 27, Chap. 7] and [31].

Throughout this section $X$ and $U$ are Banach spaces (with norms denoted, for simplicity, by the same symbol $\| \cdot \|$) and $p > 1$ is a real number. Let $Z$ be another Banach space such that $Z \subset X$ (with continuous and dense embedding). We now consider a differential operator $A_m : Z \rightarrow X$ and a trace operator $G : Z \rightarrow U$ assumed to be surjective. We also assume that the following operator generates a strongly continuous semigroup $T(t)_{t \geq 0}$ on $X$ of type $\omega_0(A)$.

We denote by $\rho(A)$ the resolvent set of $A$, $\sigma(A) = \mathbb{C} \setminus \rho(A)$ the spectrum of $A$, and $R(\lambda, A) := (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$, the resolvent operator of $A$. The graph norm with respect to $A$ is $\|x\|_A := \|x\| + \|Ax\|$ for $x \in D(A)$. It is well-known that $X_A := (D(A), \| \cdot \|_A)$ is a Banach space and $X_A \hookrightarrow X$ (densely and continuously).

We consider the control problem

$$
\begin{align*}
\dot{z}(t) &= A_m z(t), & t \geq 0, \\
Gz(t) &= u(t), & t \geq 0, \\
y(t) &= Mz(t), & t \geq 0, \\
z(0) &= x,
\end{align*}
$$

where $u : [0, +\infty) \rightarrow U$ is a control function (boundary control). It is shown by Greiner [11] that the restriction of $G$ to $\ker(\lambda - A_m)$ for $\lambda \in \rho(A)$ is invertible. As known, the inverse

$$
D_\lambda := \left( G_{\ker(\lambda - A_m)} \right)^{-1} \in \mathcal{L}(U, Z)
$$

is called the Dirichlet operator. We introduce the following operator

$$
B := (\lambda - A_-1)D_\lambda \in \mathcal{L}(U, X_-1), \quad \lambda \in \rho(A),
$$

(this operator does not depend of $\lambda$, due to the resolvent equation) and select

$$
C = M_1 \in \mathcal{L}(X_A, U),
$$
where \( t : D(A) \to Z \) is the continuous injection. From [28, Sect. 10.1], [8, 13], one can see that the control problem (2) is reformulated as to

\[
\begin{cases}
\dot{z}(t) = Az(t) + Bu(t), & t \geq 0, \\
z(0) = x, & \\
y(t) = Cz(t), & t \geq 0.
\end{cases}
\]  

(6)

The state of (6) (and hence of (2)) is given by

\[
z(t) = T(t)x + \int_0^t T_{-1}(t-s)Bu(s)ds
\]

\[\ := T(t)x + \Phi_t u, \]  

(7)

for any \( t \geq 0, \ x \in X \) and \( u \in L^p([0, +\infty), U) \). We have

\[\Phi_t \in \mathcal{L}(L^p([0, t], U), X_{-1}), \quad \forall t > 0.\]

Observe that the solution \( z(t) \) takes value in \( X_{-1} \). We then have the following definition.

**Definition 1** The operator \( B \in \mathcal{L}(U, X_{-1}) \) is called an admissible control operator for \( A \) if there exists \( t_0 > 0 \) such that \( \text{Range}(\Phi_{t_0}) \subset X \). In this case we also say that the pair \((A, B)\) is admissible or sometimes well-posed.

If \((A, B)\) is well-posed then we have \( \Phi_t \in \mathcal{L}(L^p([0, t], U), X) \) for any \( t > 0 \) and the solution of (2) satisfies \( z \in C([0, +\infty), X) \), see [30, 28, Chap. 4]. The family \( (\Phi_t)_{t \geq 0} \), satisfies for all \( t, \tau \geq 0 \),

\[
\Phi_{t+\tau}u = T(t)\Phi_{\tau}(u_{|[0, t]}) + \Phi_t u(\cdot + \tau)
\]

(8)

for any \( u \in L^p([0, t + \tau], U) \). In addition

\[
\|\Phi_{t_1}\| \leq \|\Phi_{t_2}\|
\]

(9)

for any \( 0 \leq \tau_1 \leq \tau_2 \). Moreover, for all \( \omega > \omega_0(A) \), there exists a constant \( c > 0 \) such that

\[
\|R(\lambda, A_{-1})B\| \leq \frac{c}{(\text{Re}\lambda - \omega)^{\frac{1}{p}}}
\]

(10)

for any \( \lambda \in \mathbb{C} \) such that \( \text{Re}\lambda > \omega \) (see [27, Proposition 4.2.9] and [28, Proposition 4.4.6]). We need the following definition

**Definition 2** \( C \in \mathcal{L}(X_A, U) \) is called an admissible observation operator for \( A \) (we also say that \((C, A)\) is admissible or well-posed) if
\[
\int_0^\tau \|CT(s)x\|^p \, ds \leq \gamma^p(\tau)\|x\|^p
\] (11)

for all \(x \in D(A)\) and for some constants \(\tau > 0\) and \(\gamma := \gamma(\tau) > 0\).

To state our main results in next sections, we need to define the concept of zero class admissible observation operators which was first introduced in [34], in order to provide conditions for exact observability of semigroup systems. This concept was further developed in [16].

**Definition 3** The operator \(C \in \mathcal{L}(X_A, U)\) is said to belong to the zero class of admissible observation operators for \(A\) (\(C\) is zero-class admissible) if the best constant \(\gamma(\tau)\), given by (11), satisfies \(\gamma(\tau) \to 0\) as \(\tau \to 0\).

Obviously, bounded observation operators \(C \in \mathcal{L}(X, U)\) are zero-class admissible. Let \((C, A)\) be admissible. Due to (11) and the density of the domain \(D(A)\) in \(X\), the observation function \(y\) can be extended to a \(p\)-locally integrable function for any initial condition \(x \in X\). Next we recall the representation of \(y(\cdot)\) using an extension operator of \(C\). We then need the following concept.

**Definition 4** The Yosida extension of an operator \(C \in \mathcal{L}(D(A), U)\) with respect to \(A\) is the operator defined by

\[
C_Ax := \lim_{\lambda \to +\infty} C\lambda R(\lambda, A)x,
\]

\[
D(C_A) := \{x \in X, \lim_{\lambda \to +\infty} C\lambda R(\lambda, A)x \text{ exists in } U\}.
\]

Now if \((C, A)\) is admissible, the representation theorem of Weiss [29] shows that \(\text{Range}(T(t)) \subset D(C_A)\) for a.e. \(t > 0\). On the other hand, for any \(x \in X\) and a.e. \(t \geq 0\), we have

\[
y(t) = C_AT(t)x.
\]

A necessary condition for the well-posedness of \((C, A)\) is: for all \(\omega \in \mathbb{C}\) with \(\omega > \omega_0(A)\), there exists a constant \(b > 0\) such that

\[
\|CR(\lambda, A)\| \leq \frac{b}{(\text{Re}\lambda - \omega)^{1-\frac{1}{p}}}, \quad \text{Re}\lambda > \omega,
\] (12)

see [27, Proposition 4.4.9] and [28, Theorem 4.3.7] for more details.

Define the space

\[
W^{2,p}_{0,\text{loc}}([0, +\infty), U) := \left\{ v \in W^{2,p}_{\text{loc}}([0, +\infty), U) : v(0) = v'(0) = 0 \right\},
\]

which is dense in \(L^p_{\text{loc}}([0, +\infty), U)\). Without loss of generality, we may assume that \(0 \in \rho(A)\), so \(B = (-A_{-1})D_0\) (see (4)). Now an integration by parts yields, for any \(u \in W^{2,p}_{0,\text{loc}}([0, +\infty), U)\),
The system \((A, B, C)\) is called well-posed on \(X, U, U\) if

1. \((A, B)\) is well-posed on \(X, U, U\),
2. \((C, A)\) is well-posed on \(X, U, U\), and
3. there exist constants \(\tau > 0\) and \(\kappa_\tau > 0\) such that

\[
\|F_\infty u\|_{L^p([0, \tau], U)} \leq \kappa_\tau \|u\|_{L^p([0, \tau], U)}, \quad u \in W^{2,p}_{0,\text{loc}}([0, +\infty), U). \quad (13)
\]

The first consequence of the well-posedness of the system \((A, B, C)\) is that the operator \(F_\infty\) can be extended to a linear bounded operator on \(L^p_{\text{loc}}([0, +\infty), U)\), denoted again by \(F_\infty\). Hence, the observation function of the system (2) is a function in \(L^p_{\text{loc}}([0, +\infty), U)\).

Now in order to give a representation of the observation function \(y(t)\) in terms of the operator \(C\) and the state \(z(t)\) of the system (2), we need the following important subclass of well-posed systems.

Definition 6 A well-posed system \((A, B, C)\) is regular on \(X, U, U\) (with feedthrough \(D = 0\)) if

\[
\lim_{t \to 0^+} \frac{1}{t} \int_0^t (F_\infty u_0)(\sigma)d\sigma = 0
\]

exists in \(U\), for the constant control function \(u_0(t) = v, v \in U, t \geq 0\).

We mention that (see [32]) if \((A, B, C)\) is regular, then \(\text{Range}(\Phi_t) \subset D(C_A)\) and \((F_\infty u)(t) = C_A \Phi_t u\) for all \(u \in L^p_{\text{loc}}([0, +\infty), U)\) and a.e. \(t \geq 0\). In addition, the observation function of the system (2) is given by

\[
y(t) = C_A z(t), \quad \text{a.e. } t > 0
\]

for any initial condition \(x \in X\) and any control function \(u \in L^p_{\text{loc}}([0, +\infty), U)\), see [32] for more details. Moreover, if a system \((A, B, C)\) is regular, then its transfer function is given by

\[
H(\lambda) := C_A R(\lambda, A_-^1)B, \quad \text{Re} \lambda > \omega_0(A).
\]

According to Weiss [32], there exists \(\gamma > 0\) such that

\[
\sup_{\text{Re} \lambda > \gamma} \|H(\lambda)\| < +\infty. \quad (14)
\]
In the rest of this section we shall present a perturbation theorem associated with regular linear systems, due to Weiss \[32\] in the Hilbert setting and Staffans \[27, Chap. 7\] for the Banach cases. To that purpose we need the following definition.

**Definition 7** Let \((A, B, C)\) be a regular linear system on \(X, U, U\). The identity operator \(I_U : U \to U\) is called an admissible feedback if \(I - F_\infty\) has uniformly bounded inverse.

**Theorem 1** Assume that the system \((A, B, C)\) is regular on \(X, U, U\) and \(I_U\) is an admissible feedback. Let \(C \mathcal{A}\) be the Yosida extension of \(C\) with respect to \(A\). Then the operator \(A^{\mathcal{A}} := A_{-1} + BC\), \(D(A^{\mathcal{A}}) := \{x \in D(C_A) : (A_{-1} + BC)x \in X\}\), generates a \(C_0\)-semigroup \(\mathcal{T}^{\mathcal{A}} := (T^{\mathcal{A}}(t))_{t \geq 0}\) on \(X\) such that

\[
\begin{align*}
\text{(i)} & \quad \text{Range}(T^{\mathcal{A}}(t)) \subset D(C_A) \text{ for a.e. } t > 0, \\
\text{(ii)} & \quad \text{for any } x \in X \text{ and } t \geq 0, \\
T^{\mathcal{A}}(t)x &= T(t)x + \int_0^t T_{-1}(t - s)BC_A T^{\mathcal{A}}(s)xds,
\end{align*}
\]

(15)

Let us denote by \(X_A^{\mathcal{A}}\) (resp. \(X_{-1}^{\mathcal{A}}\)) the extrapolation space associated with \(X\) and \(A\) (resp. \(X\) and \(A^{\mathcal{A}}\)). Obviously these spaces are different. In [31], Weiss constructed subspaces \(W_A\) and \(W_{A^{\mathcal{A}}}\) of \(X_A^{\mathcal{A}}\) and \(X_{-1}^{\mathcal{A}}\), respectively, such that \(Jx := \lim_{\lambda \to \infty} \lambda R(\lambda, A_{-1})x\) in \(X_{-1}^{\mathcal{A}}\) defines an isomorphism \(J : W_A \to W_{A^{\mathcal{A}}}\). Obviously, \(Jx = x\) in \(X\). We obtain

\[
\int_0^t T_{-1}(t - s)BC_A T^{\mathcal{A}}(s)xds = \int_0^t T_{-1}^{\mathcal{A}}(t - s)BC_A T(s)xds,
\]

(17)

see [31, pp. 54–55] and [26, Remark 4.6(b)] for more detail. Hence, from (17), the perturbed semigroup \((T^{\mathcal{A}}(t))_{t \geq 0}\) satisfies also the following variation of constants formula

\[
T^{\mathcal{A}}(t)x = T(t)x + \int_0^t T_{-1}^{\mathcal{A}}(t - s)BC_A T(s)xds
\]

(18)

for any \(t \geq 0\) and \(x \in X\).

**Remark 1** Consider the boundary value problem
On norm continuity, differentiability and compactness of...

\[ \begin{aligned}
\dot{z}(t) &= A_m z(t), \quad t \geq 0, \\
G z(t) &= M z(t), \quad t \geq 0, \\
z(0) &= x,
\end{aligned} \tag{19} \]

where \( A_m : Z \to X \) and \( G, M : Z \to U \) are defined at the beginning of this section. The problem (19) can be viewed as a partial differential equation where the boundary operator \( G \) is perturbed by another unbounded trace operator \( M \). This system can also be reformulated as the following Cauchy problem in \( X \),

\[ \begin{aligned}
\dot{z}(t) &= \mathcal{A} z(t), \quad t \geq 0, \\
z(0) &= x,
\end{aligned} \tag{20} \]

where

\[ \mathcal{A} := A_m, \quad D(\mathcal{A}) = \{ x \in Z, \quad G f = M f \}. \tag{21} \]

Then the problem (19) is well-posed if the operator \( (\mathcal{A}, D(\mathcal{A})) \) is a generator of a \( C_0 \)-semigroup on \( X \). Let the operator \( A, B \) and \( C \) be defined by (1), (4) and (5), respectively. It is shown in [13] that if \((A, B, C)\) is regular and \( I_U \) is an admissible feedback, then \( \mathcal{A} \) coincides with the operator \( A^{cl} \) defined by (15). Now according to Theorem 1, the operator \( \mathcal{A} \) generates a \( C_0 \)-semigroup \( \mathcal{T} := (\mathcal{T}(t))_{t \geq 0} \) on \( X \) given by

\[ \mathcal{T}(t)x = T(t)x + \int_0^t T_{-1}(t-s)BC_A \mathcal{A} x ds, \]

for all \( t \geq 0 \) and \( x \in X \). On the other hand it is shown in [13] that for any \( \lambda \in \rho(A) \) we have

\[ \lambda \in \rho(\mathcal{A}) \iff 1 \in \rho(D_{\lambda} M) \iff 1 \in \rho(M D_{\lambda}). \]

Moreover, for \( \lambda \in \rho(A) \cap \rho(\mathcal{A}) \),

\[ R(\lambda, \mathcal{A}) = (I - D_{\lambda} M)^{-1} R(\lambda, A). \tag{22} \]

3 Immediate norm continuity and compactness of perturbed semigroups

In this section, we will work under the assumptions of Theorem 1 (and also of Remark 1). We suppose that the semigroup \( \mathbb{T} \) is norm continuous or compact and then study if the perturbed semigroup \( \mathbb{T}^{cl} \) inherits such properties.

Let us first introduce a short proof of a result proved in [20] for Desch–Schappacher perturbations (i.e., in the case when the observation operator \( C \in \mathcal{L}(X, U) \) in Theorem 1 or the boundary operator \( M \in \mathcal{L}(X, U) \) in Remark 1). Before doing so, we recall from [10] the following characterization of immediately norm continuity for strongly continuous semigroups in Banach spaces.
Theorem 2  A strongly continuous semigroup \((T(t))_{t \geq 0}\) on a Banach \(X\) is continuous in the operator norm for \(t > 0\) if and only if for all \(\tau > 0\) the operator

\[
K : L^r([0, \tau], X) \longrightarrow L^r([0, \tau], X), \quad (Kf)(t) = \int_0^t T(t-s)f(s)ds
\]

satisfies the following Riesz condition \((R_r)\) for some (all) \(r \in (1, \infty)\) i.e.:

\[
(R_r) \quad \int_0^\tau \| (Kf)(t+h) - (Kf)(t) \|^r dt \to 0 \text{ as } h \to 0
\]

uniformly for \(f \in L^r([0, \tau], X)\) with \(\|f\|_r \leq 1\).

We are now in the position to give a new proof of [20, Theorem 6].

Theorem 3  Let the control system \((A, B)\) be well-posed on \(X, U\) and \(C \in \mathcal{L}(X, U)\). Assume that the semigroup \(\mathcal{T} = (T(t))_{t \geq 0}\) is immediately norm continuous on \(X\). Then the semigroup \(\mathcal{T}^{cl} = (T^{cl}(t))_{t \geq 0}\) is immediately norm continuous on \(X\).

Proof  Let \(0 < h < \tau\) and \(f \in L^p([0, \tau], X)\) with \(\|f\|_p \leq 1\). We put

\[
(K^{cl}f)(t) := \int_0^t T^{cl}(t-s)f(s)ds.
\]

We will prove that if \(K\) satisfies \((R_p)\), then \(K^{cl}\) also verifies \((R_p)\). In fact, by using (18), a change of variables and Fubini’s theorem, we obtain

\[
(K^{cl}f)(t) = (Kf)(t) + \Phi_t^J CKf, \tag{23}
\]

where we set

\[
\Phi_t^J v(\cdot) = \int_0^t T^{cl}_{-1}(t-s)JBv(s)ds, \quad t \geq 0, v \in L^p([0, t], U).
\]

In view of (8) and (23),

\[
(K^{cl}f)(t+h) - (K^{cl}f)(t) = (Kf)(t+h) - (Kf)(t) + \Phi_t^J CKf - \Phi_t^{J+h} CKf
\]

\[
= (Kf)(t+h) - (Kf)(t) + T^{cl}(t)\Phi_t^J CKf + \Phi_t^J C((Kf)(\cdot + h) - Kf).
\]

By admissibility of \(JB\) for \(\mathcal{T}^{cl}\), we obtain
\[
\int_0^\tau \|(K^\text{cl} f)(t + h) - (K^\text{cl} f)(t)\|^p dt \leq c_p \int_0^\tau \|(Kf)(t + h) - (Kf)(t)\|^p dt \\
+ c_p \int_0^\tau \| T^\text{cl}(t) \Phi^I_h CKf\|^p dt \\
+ c_p \int_0^\tau \| \Phi^I_t C[(Kf)(\cdot) + h) - Kf\|^p dt.
\]

Let \( M \geq 1 \) and \( \tilde{\omega} \in \mathbb{R} \) such that \(\|T^\text{cl}(t)\| \leq Me^{\tilde{\omega}t} \) for any \( t \geq 0 \) and \( p > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). By using (9),

\[
\int_0^\tau \| T^\text{cl}(t) \Phi^I_h C(Kf)(\cdot)\|^p dt \leq M \tau e^{\tilde{\omega}\tau} \| \Phi^I_h \|^p \| CKf\|^p_{L^p([0, h], \mathbb{U})} \\
\leq M \tau e^{\tilde{\omega}\tau} \| \Phi^I_h \|^p \beta \| f\|_p \\
\leq M \tau e^{\tilde{\omega}\tau} \| \Phi^I_h \|^p \beta \| f\|_p \xrightarrow{h \to 0} 0
\]

uniformly in \( f \) such that \( \| f\|_p \leq 1 \), where \( \beta \) is a constant independent of \( f \). On the other hand, by using (9),

\[
\int_0^\tau \| \Phi^I_t C[(Kf)(\cdot) + h) - Kf\|^p dt \leq \| \Phi^I_t \|^p \| C\|^p \int_0^\tau \|(Kf)(t + h) - (Kf)(t)\|^p dt,
\]

which goes to 0 as \( h \to 0 \) uniformly in \( \| f\|_p \leq 1 \), due to the norm continuity of \( \mathbb{T} \) and Theorem 2, respectively. This shows that

\[
\int_0^\tau \|(K^\text{cl} f)(t + h) - (K^\text{cl} f)(t)\|^p dt \xrightarrow{h \to 0} 0
\]

uniformly in \( \| f\|_p \leq 1 \). Now according to Theorem 2, \( T^\text{cl} \) is immediately norm continuous. \(\square\)

In the following result we generalize the results shown in [20] from Miyadera–Voigt and Desch–Schappacher perturbations to those of Staffans–Weiss.

**Theorem 4** Let assumptions of Theorem 1 be satisfied with \( C \) is a zero-class admissible. In addition we suppose that the semigroup \( \mathbb{T} = (T(t))_{t \geq 0} \) is immediately norm continuous on \( X \). Then the perturbed semigroup \( \mathbb{T}^\text{cl} = (T^\text{cl}(t))_{t \geq 0} \) is immediately norm continuous on \( X \) as well.

**Proof** By assumption, for any \( t > 0 \) we have

\[
\lim_{h \to 0} \| T(t + h) - T(t)\| = 0. \tag{24}
\]

Now let us prove that the perturbed semigroup \( T^\text{cl} \) introduced in Theorem 1 also has the above property. Due to (18), we have
\[ T^{cl}(t)x = T(t)x + \Phi^I_t[C_A T(\cdot)x] \] \tag{25}

for all \( t \geq 0 \) and \( x \in X \), where

\[ \Phi^I_t = \int_0^t T^{-1}_{t-s} JBv(s)ds, \quad t \geq 0, \quad v \in L^p([0, +\infty), U). \]

According to (24), it suffices to prove that the map \( t \mapsto \Phi^I_t[C_A T(\cdot)] \) is norm continuous. In fact, fix \( t_0 > 0 \) and choose arbitrary \( h, \delta \in \mathbb{R} \) such that \( 0 < |h| < \delta < t \). We then have

\[
\begin{align*}
\Phi^I_{t+h}[C_A T(\cdot)x] - \Phi^I_t[C_A T(\cdot)x] &= \int_0^{\delta+h} T^{-1}_{t+s}(t+s)JB\tilde{C}(s)xds + \int_{\delta+h}^{t+h} T^{-1}_{t+s}(t+s)JB\tilde{C}(s)xds \\
& \quad - \int_0^{\delta} T^{-1}_{t+s}(t+s)JB\tilde{C}(s)xds - \int_{\delta}^{t} T^{-1}_{t+s}(t+s)JB\tilde{C}(s)xds \\
& = T^{cl}(t-h)[\Phi^I_{\delta+h}C_A T(\cdot)x - \Phi^I_{\delta}C_A T(\cdot)x] \\
& \quad + \int_{\delta}^{t} T^{-1}_{t+s}(t+s)JB\tilde{C}(s)xds \\
& := I_1(x, h, t) + I_2(x, h, t), \tag{26}
\end{align*}
\]

where

\[
I_1(x, h, t) := T^{cl}(t-h)[\Phi^I_{\delta+h}C_A T(\cdot)x - \Phi^I_{\delta}C_A T(\cdot)x],
\]

\[
I_2(x, h, t) := \int_{\delta}^{t} T^{-1}_{t+s}(t+s)JB\tilde{C}(s)xds.
\]

By admissibility of \( JB \) for \( T^{cl} \), we have

\[
\| \Phi^I_t \| \leq \| \Phi^I_t \| \| v \|_{L^p([0, \tau], U)}
\]

for any \( \tau \geq 0 \) and \( v \in L^p([0, +\infty), U) \). Let \( \tilde{M} \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( \| T^{cl}(t) \| \leq \tilde{M}e^{\omega t} \) for any \( t \geq 0 \). We estimate

\[
\begin{align*}
\| I_1(x, h, t) \| &= \| T^{cl}(t-h)[\Phi^I_{\delta+h}C_A T(\cdot)x - \Phi^I_{\delta}C_A T(\cdot)x] \| \\
& \leq \tilde{M}e^{\omega(t-h)}(\| \Phi^I_{\delta+h} \| \| C_A T(\cdot)x \|_{L^p([0, \delta+h], U)} \\
& \quad + \| \Phi^I_{\delta} \| \| C_A T(\cdot)x \|_{L^p([0, \delta], U)}) \\
& \leq 2\tilde{M}e^{\omega t}\| \Phi^I_{2\delta} \| \| C_A T(\cdot)x \|_{L^p([0, 2\delta], U)} \\
& \leq 2\tilde{M}e^{\omega t}\| \Phi^I_{2\delta} \| \| x \|,
\end{align*}
\tag{27}
\]

where \( \gamma(2\delta) \to 0 \) as \( \delta \to 0 \), by (9) and zero-class admissibility of \( C \) for \( T \). We then have
We now show that \( f \) to prove that the operators \( \Phi_t[f] \) → 0 as \( \delta \to 0 \). By admissibility of \( JB \) for \( \mathbb{T}^{cl} \), a change of variables and admissibility of \( C \) for \( \mathbb{T} \), we obtain

\[
\|I_2(x, h, t)\| = \left\| \int_0^t T_{-\delta}^{cl}(t-s)JBC_A[T(s+h)x-T(s)x]\| \delta, \|s\| ds \right\|
\]

\[
\leq \|\Phi_t^{I}\| \left( \int_{-\delta}^{t} \|C_A [T(s+\delta)h]x-T(\delta)x\|^{p}\right)^{1/p} ds
\]

\[
\leq \gamma(t) \|\Phi_t^{I}\| \|T(\delta + h)x - T(\delta)x\|
\]

Combining (26), (28) and (29), we have

\[
\|\Phi_{t+h}^{J} [C_A T(\cdot)] - \Phi_{t}^{J} [C_A T(\cdot)]\| \leq \omega_t(\delta) + \gamma(t) \|\Phi_t^{I}\| \|T(\delta + h) - T(\delta)\|
\]

The fact that the semigroup \( \mathbb{T} \) is immediately norm continuous implies that

\[
\lim_{h \to 0} \|\Phi_{t+h}^{J} [C_A T(\cdot)] - \Phi_{t}^{J} [C_A T(\cdot)]\| \leq \omega_t(\delta).
\]

By letting \( \delta \to 0 \), we obtain

\[
\lim_{h \to 0} \|\Phi_{t+h}^{J} [C_A T(\cdot)] - \Phi_{t}^{J} [C_A T(\cdot)]\| = 0.
\]

This ends the proof. \( \square \)

The next result is about immediate compactness of perturbed semigroups.

**Theorem 5** Let assumptions of Theorem 1 be satisfied with \( C \) is a zero-class admissible. In addition we suppose that the semigroup \( \mathbb{T} = (T(t))_{t \geq 0} \) is immediately compact on \( X \). Then perturbed semigroup \( \mathbb{T}^{cl} = (T^{cl}(t))_{t \geq 0} \) is immediately compact on \( X \) as well.

**Proof** According to (25) and the immediate compactness of the semigroup \( \mathbb{T} \), it suffices to prove that the operators \( \Phi_t^{I} [C_A T(\cdot)] \) are compact for any \( t > 0 \). For this, we shall use an approximation argument. Take \( \epsilon > 0 \) and define

\[
\Phi_{\epsilon, cl}^{I} [C_A T(\cdot)x] = \int_{\epsilon}^{t} T_{-\epsilon}^{cl}(t-s)JBC_A T(s)x ds, \quad t > \epsilon.
\]

We now show that \( \Phi_{\epsilon}^{I} [C_A T(\cdot)] \) approaches \( \Phi_{t}^{I} [C_A T(\cdot)] \) uniformly as \( \epsilon \to 0 \). By admissibility of \( JB \) for \( \mathbb{T}^{cl} \) and \( C \) for \( \mathbb{T} \), we obtain
\[ \| \Phi^I_{\epsilon,t}(C_A T(x)) - \Phi^I_{\epsilon,0}(C_A T(x)) \| = \left\| T^{cl}(t - \epsilon) \int_0^\epsilon T^{cl}_{-1}(\epsilon - s) JBC_A T(s) x ds \right\| \]
\[ \leq \| T^{cl}(t - \epsilon) \| \| \Phi^I_{\epsilon} \| \left[ \int_0^\epsilon \| C_A T(s) \|^p ds \right]^{\frac{1}{p}} \]
\[ \leq \gamma(\epsilon) \| T^{cl}(t - \epsilon) \| \| \Phi^I_{\epsilon} \| \| x \|. \]

This shows that \[ \| \Phi^I_{\epsilon,t}(C_A T(x)) - \Phi^I_{\epsilon,0}(C_A T(x)) \| \longrightarrow 0 \] as \( \epsilon \to 0 \). Thus it suffices to show that \( \Phi^I_{\epsilon,t}(C_A T(x)) \) is compact for \( t > \epsilon \). Let us consider a sequence \( (x_n) \subset X \) with \( \| x_n \| \leq 1 \). Since \( T(\epsilon) \) is compact, then there exists a subsequence \( x_{\phi(n)} \) such that
\[ T(\epsilon)x_{\phi(n)} \longrightarrow y \in X \quad \text{as} \quad n \to \infty. \]

Hence
\[ \Phi^I_{\epsilon,t}(C_A T(x_{\phi(n)})) = \int_0^t T^{cl}_{-1}(t - s) JBC_A T(s - \epsilon) T(\epsilon)x_{\phi(n)} ds 
= \int_0^t T^{cl}_{-1}(t - s) JBC_A T(s - \epsilon) \left[ T(\epsilon)x_{\phi(n)} - y \right] ds 
+ \int_0^{t-\epsilon} T^{cl}_{-1}(t - s) JBC_A T(s) y ds 
= \Phi^I_{\epsilon,0}(C_A T(\cdot - \epsilon)(T(\epsilon)x_{\phi(n)} - y)) + \Phi^I_{t-\epsilon}(C_A T(\cdot)y). \]

On the other hand,
\[ \| \Phi^I_{\epsilon,t}(C_A T(\cdot - \epsilon)(T(\epsilon)x_{\phi(n)} - y)) \| 
= \left\| \int_0^t T^{cl}_{-1}(t - s) JBC_A T(s - \epsilon) \left[ T(\epsilon)x_{\phi(n)} - y \right] \|_{\epsilon,t} ds \right\| 
\leq \| \Phi^I_{\epsilon,0} \| \left[ \int_0^t \| C_A T(s - \epsilon) \left[ T(\epsilon)x_{\phi(n)} - y \right] \|^p ds \right]^{\frac{1}{p}} 
\leq \| \Phi^I_{\epsilon,0} \| \left[ \int_0^t \| C_A T(s) \left[ T(\epsilon)x_{\phi(n)} - y \right] \|^p ds \right]^{\frac{1}{p}} 
\leq \gamma(t) \| \Phi^I_{\epsilon,0} \| \| T(\epsilon)x_{\phi(n)} - y \| \longrightarrow 0 \quad \text{as} \quad n \to \infty. \]

We see that the sequence \( \Phi^I_{\epsilon,t}(C_A T(\cdot)x_{\phi(n)}) \) converges to \( \Phi^I_{t-\epsilon}(C_A T(\cdot)y) \in X \) which means that \( \Phi^I_{\epsilon,t}(C_A T(\cdot)) \) are compact for \( t > \epsilon \). This ends the proof.

**Remark 2** As consequences of Theorems 4 and 5, we have

1. If \( (A, B) \) is well-posed (with \( B \in \mathcal{L}(U, X_{-1}) \)) and \( C \) is bounded (i.e., \( C \in \mathcal{L}(X, U) \)), then assumptions of Theorem 1 are satisfied and \( C \) is a zero class observation.
operator. Then $\mathbb{T}^{cl}$ is immediately norm continuous (resp. compact) whenever the semigroup $\mathbb{T}$ is.

2. If $(A, C)$ is well-posed (with $C \in \mathcal{L}(D(A), U)$) and $B$ is bounded (i.e., $B \in \mathcal{L}(U, X)$), then assumptions of Theorem 1 are satisfied. Let $\Phi'_r$ as in the proof of Theorem 4. Using the boundedness of $B$ and Hölder inequality, we obtain

$$\|\Phi'_r\| \leq \|B\| \frac{1}{r}$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Then by (27) we have $\|I_t(x, h, t)\| \to 0$ as $h \to 0$ uniformly in $x$.

Then without assuming zero class property for $C$, we obtain that $\mathbb{T}^{cl}$ is immediately norm continuous (resp. compact) whenever so is the semigroup $\mathbb{T}$.

In the following theorem we extend the result proven in [18] from Miyadera–Voigt perturbation to that of Staffans–Weiss.

**Theorem 6** Let assumptions of Theorem 1 be satisfied, with $C$ being zero-class admissible. Then $T^{cl}(t) - T(t)$ is compact for $t > 0$ if and only if $T^{cl}(t) - T(t)$ is norm continuous for $t \geq 0$ and $R(\lambda, A^{cl}) - R(\lambda, A)$ is compact for $\lambda \in \rho(A^{cl}) \cap \rho(A)$.

**Proof** Sufficiency can be obtained by the same arguments as in [18]. Let us now prove necessity. The compactness of $R(\lambda, A^{cl}) - R(\lambda, A)$ is obtained by taking the Laplace transform of compact operators $T^{cl}(t) - T(t)$ and using the result [22, Chap 3, Theorem 3.3]. On the other hand, for $t > 0$ and $h$ near of zero, we have

$$T^{cl}(t) - T(t) = T^{cl}(t + h) - T(t + h) - T^{cl}(t) + T(t)$$

$$= T^{cl}(h)(T^{cl}(t) - T(t)) + (T^{cl}(h) - T(h))T(t) - T^{cl}(t) + T(t)$$

$$= (T^{cl}(h) - I)[T^{cl}(t) - T(t)] + [T^{cl}(h) - T(h)]T(t).$$

Now the compactness of $T^{cl}(t) - T(t)$ implies that

$$\|(T^{cl}(h) - I)[T^{cl}(t) - T(t)]\| \to 0 \quad \text{as} \quad h \to 0.$$

Moreover, using admissibility of $JB$ for $T^{cl}$ and admissibility of $C$ for semigroup $\mathbb{T}$, we obtain

$$\|T^{cl}(t)x - T(t)x\| = \left\| \int_0^h T_{-1}^{cl}(h - s)JBC_A T(s)xd\right\|$$

$$\leq \|\Phi'_r\| \left( \int_0^h \|C_A T(s)x\|^p ds \right)^{1/p}$$

$$\leq \gamma(h)\|\Phi'_r\||x|$$

converges to 0 as $h \to 0$ for every $x \in X$. Hence,

$$\|T^{cl}(h) - T(h)\| \to 0 \quad \text{as} \quad h \to 0.$$
This ends the proof.

\[ \square \]

**Remark 3** In Theorems 4–6, we can replace the condition \( C \) is zero class observation operator by a similar dual concept on admissible control operator \( B \). In fact, let \( B \in \mathcal{L}(U, X_{-1}) \) be an admissible control operator for \( A \) with control maps \( \Phi_i, i \geq 0 \). For any \( \tau > 0 \), there exists \( c(\tau) > 0 \) such that

\[ \left\| \Phi_\tau u \right\| \leq c(\tau)\| u \|_p \]  

(33)

for all \( u \in L^p([0, +\infty), U) \). Now we say that \( B \) is a zero class control operator if the constant \( c(\tau) \rightarrow 0 \) as \( \tau \rightarrow 0 \). This notion is used in [15] to study input-to-state stability for the infinite-dimensional systems. Let the assumptions of Theorem 1 be satisfied. From (17), we have

\[ \Phi_\tau^t C_A T(\cdot)x = \Phi_\tau^t C_A T^{cl}(\cdot)x \]

for any \( \tau \geq 0 \) and \( x \in X \). According to (33) and the admissibility of \( C_A \) for \( T^{cl} \), there exists a constant \( \tilde{\gamma} > 0 \) such that

\[ \left\| \Phi_\tau^t C_A T(\cdot)x \right\| \leq c(\tau)\tilde{\gamma}\| x \|. \]  

(34)

Thus if in Theorems 4–6 instead of \( C \) being a zero class observation operator we assume that \( B \) is a zero class control operator, then we obtain the same results. In fact, in the proof of these theorems we replace the fact that \( \gamma(\tau) \rightarrow 0 \) as \( \tau \rightarrow 0 \) by \( c(\tau) \rightarrow 0 \) as \( \tau \rightarrow 0 \), due to (27), (30), (33) and (34).

**Example 1** Consider a one-dimensional heat equation with mixed boundary conditions

\[ \begin{align*}
\frac{\partial z}{\partial t}(t, x) &= \frac{\partial^2 z}{\partial x^2}(t, x), \quad 0 < x < \pi, t \geq 0; \\
\frac{\partial z}{\partial x}(t, 0) + z(t, 0) &= 0, \quad z(t, \pi) = 0, \quad t \geq 0; \\
z(0, x) &= \varphi(x), \quad 0 < x < \pi.
\end{align*} \]  

(35)

In order to use our abstract results, we select

\[ X := L^2([0, \pi]), \quad Z := \{ f \in H^2([0, \pi]) : f(\pi) = 0 \}, \quad \partial X := \mathbb{C} \]

and operators

\[ A_m f = f'' \quad \text{and} \quad M f = -f(0), \quad \text{for} \quad f \in Z. \]

We know that the operator

\[ A \varphi := A_m \varphi, \quad D(A) = \{ f \in Z : f'(0) = 0 \} \]
On norm continuity, differentiability and compactness of…

generates an immediately norm continuous (and even compact) $C_0$-semigroup $\mathbb{T} := (T(t))_{t \geq 0}$ on $X$ (note that $A$ is self-adjoint). On the other hand, $G$ is surjective. So that the Dirichlet operator $D_\lambda$ exists for any $\lambda \in \rho(A)$, see the beginning of this section. As we are in the Hilbert setting, the extrapolation space $X_{-1}$ of $X$ associated with $A$ is isomorphic to the topological dual space $(D(A^*))'$, where $A^*$ is the adjoint operator of $A$. We now put for any $\lambda \in \rho(A)$,

$$B := (\lambda - A_{-1})D_\lambda \in \mathcal{L}(\mathbb{C}, D(A^*))'.$$

A straightforward computation shows that the adjoint operator of $B$ is given by

$$B^* \varphi = -\varphi(0), \quad \varphi \in D(A^*) = D(A).$$

In addition, $B^*$ is an admissible observation operator for the adjoint semigroup $\mathbb{T}^* := (T^*(t))_{t \geq 0}$. Hence, by duality, $B$ is an admissible control operator for the semigroup $\mathbb{T}$, (see [28, p. 126]). In addition, $B^*$ is a zero-class observation operator (see [16, Example 3.8]).

Moreover, by computation the Dirichlet operator is given by

$$(D_0 u)(x) = (x - \pi) \cdot u, \quad \text{for} \quad 0 \leq x \leq \pi;$$

$$(D_\lambda u)(x) = \frac{\sinh(\sqrt{\lambda}(x - \pi))}{\sqrt{\lambda} \cosh(\sqrt{\lambda} \pi)} \cdot u, \quad \text{for} \quad 0 \leq x \leq \pi \quad \text{and} \quad \lambda > 0.$$

It is clear that $\text{Range}(D_\lambda) \subset Z$ and the transfer function $H(\lambda) := MD_\lambda$ is uniformly bounded on half plane. Thus, $(A, B, B^*)$ generates a regular system on $L^2([0, \pi]), \mathbb{C}, \mathbb{C}$. It is well known that $A$ generates a compact semigroup in $X$. Hence by Theorem 5 the semigroup solution of heat equation with mixed boundary (35) is compact in $X$.

4 The eventual differentiability under Desch–Schappacher perturbations

In this section, we still assume that the assumptions of Theorem 1 are satisfied (hence the perturbed semigroup $\mathbb{T}^{cl}$ exists), and then discuss conditions for which there exists $\tau > 0$ such that for any $x \in X$, the map $t \in (\tau, \infty) \to T^{cl}(t)x \in X$ is differentiable.

In the following theorem we generalize the result of Pazy on the stability of differentiability under bounded perturbations (see [21, Corollary 4.1]) to Desch–Schappacher perturbations.

**Theorem 7** Let the control system $(A, B)$ be well-posed on $X$, $U$ and $C \in \mathcal{L}(X)$. Assume that the generator $(A, D(A))$ satisfies the condition

$$\tau_0 := \limsup_{|\tau| \to +\infty} \log |\tau| \|R(\mu + i\tau, A)\| < \infty,$$

(36)
for some $\mu > \omega_0(A)$. Then the assumption of Theorem 1 are satisfied and the perturbed semigroup $T^{cl}$ is eventually differential from a time $2\tau_0$. On the other hand, if $\tau_0 = 0$, then the semigroup $T^{cl}$ is immediately differentiable.

**Proof** The part of $A_{-1} + BC$ on $X$ denoted by $A^{cl}$ generates a strongly continuous semigroup $T^{cl}$ given by the variation of constants formula (16). Taking Laplace transform on both sides of this formula we obtain

$$R(\lambda, A^{cl}) = R(\lambda, A) + R(\lambda, A_{-1})BCR(\lambda, A^{cl}), \quad \lambda \in \rho(A).$$

Let $\omega > \omega_0(A)$. According to (10) there exists a constant $c > 0$ such that for any $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > \omega$,

$$||R(\lambda, A^{cl})|| \leq ||R(\lambda, A)|| + \frac{c||C||}{(\text{Re}\lambda - \omega)\frac{1}{p}} ||R(\lambda, A^{cl})||.$$

Now, for $\mu > \omega + (2c||C||)^p := \omega'$, and $\tau \in \mathbb{R}$, we obtain

$$||R(\mu + i\tau, A^{cl})|| \leq 2||R(\mu + i\tau, A)||. \quad (37)$$

Thus the result follows from [21, Theorem 2.2 and Corollary 2.2].

**Remark 4** Assume that $A : D(A) \subseteq X \rightarrow X$ is a generator of a $C_0$-semigroup $T$ on $X$, $B \in \mathcal{L}(X)$ and $C \in \mathcal{L}(D(A), X)$ such that $(C, A)$ is well-posed. Then the conditions of Theorem 1 are verified and the operator $A^{cl} = A + BC$ with domain $D(A^{cl}) = D(A)$ generates a $C_0$-semigroup $T^{cl}$ on $X$ (see also [12]). Moreover,

$$R(\lambda, A^{cl}) = R(\lambda, A) + R(\lambda, A^{cl})BCR(\lambda, A), \quad \lambda \in \rho(A).$$

Let $\mu \in \mathbb{R}$ such that $\mu > \omega > \omega_0$ and $\tau \in \mathbb{R}$. According to (12), there exists a constant $c > 0$ such that

$$||R(\mu + i\tau, A^{cl})|| \leq ||R(\mu + i\tau, A)|| + \frac{c||B||}{(\mu - \omega)^{1 - \frac{1}{p}}} ||R(\mu + i\tau, A^{cl})||.$$

Now for $\mu > \omega + (2c||B||)^{\frac{1}{p}}$, we have

$$||R(\mu + i\tau, A^{cl})|| \leq 2||R(\mu + i\tau, A)||.$$

As $A$ satisfies the condition (36), the generator of the perturbed semigroup $T^{cl}$ also satisfies this condition and hence $T^{cl}$ is a differential semigroup by [21, Theorem 2.2 and Corollary 2.2].

We also have the following observation about immediate norm continuity for perturbed semigroups on Hilbert spaces.

**Remark 5** Assume that we work in the Hilbert setting and let us be in the situations of Theorem 7 and/or Remark 4. In both cases we have proved that the estimate (37)
for the generators $A$ and $A^\text{cl}$. It is well-known (see e.g. [7, p. 115]) that $\mathbb{T}$ is im-
mediately norm continuous on a Hilbert space $X$ if and only if $\|\mathcal{R}(\mu + i\tau, A)\| \to 0$ as $\tau \to \pm\infty$. Now the inequality (37) shows that the immediate norm continuity is sta-
ble under Miyadera–Voigt and/or Desch–Schappacher perturbations.

5 Application to boundary integro-differential equations

Let $X$ and $Z$ be Banach spaces with $Z \hookrightarrow X$ continuous and dense embedding. Let $A_m : Z \to X$ be a closed linear (differential) operator and an application $k : \mathbb{R}^+ \to \mathbb{C}$ a measurable function. Consider the following boundary integro-differential equation

\[
\begin{aligned}
\dot{x}(t) &= A_m x(t) + \int_0^t k(t-s)Px(s)ds, \quad t \geq 0 \\
Gx(t) &= Mx(t), \quad t \geq 0, \\
x(0) &= x,
\end{aligned}
\]

where the initial condition $x \in X$ and the boundary operators $G : Z \to \partial X$ and $M : Z \to \partial X$ are linear.

The objective of this section is to study the well-posedness of the Eq. (38) and establish regularity of the solution. In the spirit of Greiner [11] and Salamon [25], we introduce the hypothesis

(H1) $A := A_m$ with domain $D(A) = \ker G$ generates a $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$.
(H2) The operator $G : Z \to \partial X$ is surjective.

As discussed in the introductory section, let $D_\lambda$ be the Dirichlet operator associated with $A_m$ and $G$ and set $B := (\lambda - A_{-1})D_\lambda$ for $\lambda \in \rho(A)$. On the other hand, we select

\[ C := M_{|D(A)|}, \quad \text{and} \quad \mathbb{P} := P_{|D(A)|}. \]

We further assume that

(H3) $(A, B, C)$ is a regular system on $X, \partial X, \partial X$ and $I : \partial X \to \partial X$ is a feedback admissible with $C$ is a zero-class admissible.
(H4) $(A, B, \mathbb{P})$ is a regular system on $X, \partial X, X$.

In order to study the existence and regularity of the solution of the integro-differential Eq. (38), we need to introduce a Bergman space. Let then $h : \mathbb{R}^+ \to \mathbb{R}^+$ be an admis-
sible function (i.e., $h$ is increasing, convex and $h(0) = 0$). Hereafter, we assume that for $s > 1$,

\[
\int_0^1 h(\sigma)^{1-s} d\sigma < \infty.
\]

Let $p, q \in (1, +\infty)$ be such that
We define the sector
\[ \Sigma_h := \{ \sigma + i\tau \in \mathbb{C}, \sigma > 0 \text{ and } |\tau| < h(\sigma) \} \]

The Bergman space is defined by
\[ \mathcal{B}^q_h(\Sigma_h, X) := \left\{ f : \Sigma_h \to X \text{ holomorphic such that } \|f\|_{\mathcal{B}^q_h(\Sigma_h, X)} < \infty \right\} \]

with the norm
\[ \|f\|_{\mathcal{B}^q_h(\Sigma_h, X)} := \left( \int_{\Sigma_h} \int \|f(\sigma + i\tau)\|^q d\sigma d\tau \right)^{\frac{1}{q}} < \infty. \]

We shall assume
\[ (H5) \quad k(\cdot) \in \mathcal{B}^q_h(\Sigma_h, \mathbb{C}). \]

According to [1], the following translation semigroup on \( \mathcal{B}^q_h(\Sigma_h, X) \),
\[ (S(t)f)(z) := f(t + z) \]

with generator
\[ \frac{d}{dz}f = f', \quad D\left( \frac{d}{dz} \right) := \left\{ f \in \mathcal{B}^q_h(\Sigma_h, X), f' \in \mathcal{B}^q_h(\Sigma_h, X) \right\} \]

is analytic. We first use product spaces to reformulate the Eq. (38) as an abstract boundary value problem. In fact, consider the Banach space
\[ \mathcal{X} := X \times \mathcal{B}^q_h(\Sigma_h, X) \quad \text{with norm} \quad \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\| := \|x\| + \|f\|_{\mathcal{B}^q_h(\Sigma_h, X)}. \]

Moreover we consider the space
\[ \mathcal{Z} := Z \times D\left( \frac{d}{dz} \right). \]

The Eq. (38) can be rewritten as
\[
\begin{cases}
\dot{z}(t) = \mathcal{A}_m z(t) + \mathcal{P} z(t), \quad t \geq 0 \\
\mathcal{G} z(t) = \mathcal{M} z(t), \quad t \geq 0, \\
z(0) = z \in \mathcal{X}.
\end{cases}
\]  

(40)

where \( \mathcal{A}_m, \mathcal{P} : \mathcal{X} \to \mathcal{X} \) are given by
and the boundary operators $\mathcal{G}, \mathcal{M} : \mathcal{L} \rightarrow \partial X$ are defined by

$$\mathcal{G} := \begin{pmatrix} G & 0 \end{pmatrix}, \quad \mathcal{M} := \begin{pmatrix} M & 0 \end{pmatrix}.\]$$

Now consider the operator

$$\mathcal{A} := \mathcal{A}_m, \quad D(\mathcal{A}) = \{ x \in Z : Gx = Mx \} \times D\left( \frac{d}{dz} \right).$$

The boundary problem (40) is reformulated again as a Cauchy problem

$$\begin{cases}
\dot{z}(t) = \mathcal{A}z(t) + \mathcal{P}z(t), & t \geq 0, \\
z(0) = z. &
\end{cases}$$

(41)

**Lemma 1** Let the assumptions (H1), (H2) and (H3) be satisfied. Then the operator $(\mathcal{A}, D(\mathcal{A}))$ is the generator of a strongly continuous semigroup $(\mathcal{R}(t))_{t \geq 0}$ on $\mathcal{X}$, given by

$$\mathcal{R}(t) = \begin{pmatrix} T^{cl}(t) & 0 \\
0 & S(t) \end{pmatrix}, \quad t \geq 0,$$

where $(T^{cl}(t))_{t \geq 0}$ is the strongly continuous semigroup on $X$ generated by the operator

$$A^{cl} := A_{-1} + BC_A \quad \text{with} \quad D(A^{cl}) = \{ x \in D(C_A) : (A_{-1} + BC_A)x \in X \}. \quad (42)$$

**Proof** According to Remark 1, the following operator

$$A^{cl} := A_m, \quad D(A^{cl}) = \{ x \in Z : Gx = Mx \},$$

coincides with the operator defined by (42), which is a generator of a strongly continuous semigroup $T^{cl} := (T^{cl}(t))_{t \geq 0}$, due to (H3) and Theorem 1. With these we have

$$\mathcal{A} = \begin{pmatrix} A^{cl} & 0 \\
0 & \frac{d}{dz} \end{pmatrix}, \quad D(\mathcal{A}) = D(A^{cl}) \times D\left( \frac{d}{dz} \right).$$

This ends the proof.

**Theorem 8** Let the assumptions (H1)–(H5) be satisfied. Then the operator $(\mathcal{A} + \mathcal{P}, D(\mathcal{A}))$ is a generator of a strongly continuous semigroup on $\mathcal{X}$. 

**Proof** According to [12], it suffices to prove that $\mathcal{P}$ is an admissible operator for $\mathcal{A}$. Let $\left( f \right) \in D(\mathcal{A})$. As $\mathcal{R}(t) f \in D(\mathcal{A})$, then
\( T^{cl}(t)x \in D(A^{cl}) \) and \( S(t)f \in D\left( \frac{d}{dz} \right) \),

for any \( t \geq 0 \). According to [13, Lemma 3.6] and (39), we have

\[
P A \mathcal{R}(t)(x) = \begin{pmatrix} f(t) \\ k(\cdot) P^A A T^{cl}(t)x \end{pmatrix}, \quad t \geq 0.
\]

Here \( P_A \) is the Yosida extension of \( P \) relatively to \( A \). We recall that from feedback theory and the condition (H4), the operator \( P_A \) is an admissible observation operator for \( T^{cl} \). Then for constants \( \lambda > 0 \) and \( p > 1 \), there exist constants \( \gamma > 0 \) and \( c_p > 0 \) such that

\[
\int_0^a \left\| \mathcal{R}(t)(x) \right\|_p^p \, dt \leq c_p \left( \int_0^a \| f(t) \|_X^p \, dt + \int_0^a \| k(\cdot) P_A T^{cl}(t)x \|_B^p(\Sigma_h, X) \, dt \right)
\]

\[
\leq c_p \int_0^a \| f(t) \|_X^p \, dt + \int_0^a \left( \int_0^a \| k(\sigma + i\tau) P_A T^{cl}(t)x \|_q^q \, d\sigma d\tau \right)^{\frac{p}{q}} \, dt
\]

\[
\leq c_p \int_0^a \| f(t) \|_X^p \, dt + c_p \gamma^p \| k \|_B^p(\Sigma_h, C) \int_0^a \| P_A T^{cl}(t)x \|^p \, dt
\]

\[
\leq c_p \int_0^a \| f(t) \|_X^p \, dt + c_p \gamma^p \| k \|_B^p(\Sigma_h, C) \| x \|^p.
\]

On the other hand, using Cauchy formula, Jensen’s inequality and similar arguments as in [2, Lemma 4.3], one can see that there exists a constant \( \kappa > 0 \) such that

\[
\int_0^a \| f(t) \|_X^p \, dt \leq \kappa \| f \|_B^p(\Sigma_h, X).
\]

Now by taking \( \theta := c_p \max\{ \kappa, \gamma^p \| k \|_B^p(\Sigma_h, C) \} \), we obtain

\[
\int_0^a \left\| \mathcal{R}(t)(x) \right\|_p^p \, dt \leq \theta (\| x \| + \| f \|_B^p(\Sigma_h, X))^p.
\]

This ends the proof. \( \square \)

**Theorem 9** Let the assumptions (H1)–(H5) be satisfied. Moreover, assume that the operator \( A \) generates an immediately norm continuous semigroup on \( X \). Then the operator \( (\mathcal{A} + \mathcal{P}, D(\mathcal{A})) \) generates an immediately norm continuous semigroup on \( \mathcal{X}^c \) as well.

**Proof** Theorem 4 shows that the operator \( A^{cl} \) generates an immediately norm continuous semigroup \( T^{cl} \) on \( X \). On the other hand, according to [1], we know that the shift semigroup \( S \) is analytic in the Bergman space \( B^p_0(\Sigma_h, X) \), hence it is an immediately norm continuous semigroup. This show that the semigroup \( (\mathcal{R}(t))_{t \geq 0} \) generated by \( \mathcal{A} \)
is immediately norm continuous. As $\mathcal{P}$ is a Miyadera–Voigt perturbation for $\mathcal{A}$, then by Remark 2 the operator $(\mathcal{A} + \mathcal{P}, D(\mathcal{A}))$ generates an immediately norm continuous semigroup on $\mathcal{X}$.

**Theorem 10** Assume that $M \in \mathcal{L}(X, \partial X)$ and the assumptions (H1)–(H5) are satisfied. If for some $\mu > \omega_0(A)$ we have

$$\tau_0 := \lim \sup_{|\tau| \to \infty} \log(|\tau|) ||R(\mu + i\tau, A)|| < \infty,$$

(43)

then the semigroup generated by $(\mathcal{A} + \mathcal{P}, D(\mathcal{A}))$ is eventually differentiable.

**Proof** As $A$ satisfies the Pazy condition (43), then by the proof of Theorem 7 the operator $A^{cl}$ also satisfies the Pazy condition. We know from [1] that the shift semigroup $(S(t))_{t \geq 0}$ is analytic on the Bergman space $B^q_h(\Sigma_h, X)$. This implies that the operator $\mathcal{A}$ satisfies the Pazy condition. Now as $(\mathcal{P}, \mathcal{A})$ is admissible (see the proof of Theorem 8), then by Remark 4, the operator $(\mathcal{A} + \mathcal{P}, D(\mathcal{A}))$ also satisfies the Pazy condition. It follows from [21, Theorem 2.2] that the operator $(\mathcal{A} + \mathcal{P}, D(\mathcal{A}))$ generates an eventually differentiable semigroup on $\mathcal{X}$.

**Remark 6** In the case of $M \equiv 0$, the boundary integro-differential Eq. (38) becomes

$$\begin{cases}
\dot{x}(t) = Ax(t) + \int_0^t k(t-s)Px(s)ds, & t \geq 0, \\
x(0) = x \in X
\end{cases}$$

(44)

In this case Bartà [2] showed the differentiability of the solutions by assuming a smooth regularity on the kernel $k$ that is $k' \in B^q_h(\Sigma_h, \mathbb{C})$. However, in our results this extra condition on $k$ is not needed any more. On the other hand, the approach of Bartà is based on small perturbations, and cannot be extended to Desch–Schappacher perturbations. We think that our approach based on feedback theory of regular linear systems is the right way to solve such problems.

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