ON THE KÄHLER STRUCTURES OVER QUOT SCHEMES

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Abstract. Let $S^n(X)$ be the $n$-fold symmetric product of a compact connected Riemann surface $X$ of genus $g$ and gonality $d$. We prove that $S^n(X)$ admits a Kähler structure such that all the holomorphic bisectional curvatures are nonpositive if and only if $n < d$. Let $Q_X(r,n)$ be the Quot scheme parametrizing the torsion quotients of $O_X^{\otimes r}$ of degree $n$. If $g \geq 2$ and $n \leq 2g - 2$, we prove that $Q_X(r,n)$ does not admit a Kähler structure such that all the holomorphic bisectional curvatures are nonnegative.

1. Introduction

Let $X$ be a compact connected Riemann surface of genus $g$ and gonality $d$. For a positive integer $n$, let $S^n(X)$ denote the $n$-fold symmetric product of $X$. More generally, $Q_X(r,n)$ will denote that Quot scheme that parametrizes all the torsion quotients of $O_X^{\otimes r}$ of degree $n$. So $S^n(X) = Q_X(1,n)$. This $Q_X(r,n)$ is a complex projective manifold.

We prove the following (see Theorem 3.1):

The symmetric product $S^n(X)$ admits a Kähler structure satisfying the condition that all the holomorphic bisectional curvatures are nonpositive if and only if $n < d$.

The “only if” part was proved in [Bi1].

The main theorem of [BR] says the following (see [BR, Theorem 1.1]): If $g \geq 2$ and $n \leq 2(g - 1)$, then $S^n(X)$ does not admit any Kähler metric for which all the holomorphic bisectional curvatures are nonnegative. A simpler proof of this result was given in [Bi2]. Here we prove the following generalization of it (see Proposition 4.1):

Assume that $g \geq 2$ and $n \leq 2(g - 1)$. Then $Q_X(r,n)$ does not admit any Kähler structure such that all the holomorphic bisectional curvatures are nonnegative.

If $r > 1$, the method in [Bi2] give a much weaker version of Proposition 4.1.

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2. Preliminaries

Let $X$ be a compact connected Riemann surface of genus $g$. For any positive integer $n$, consider the Cartesian product $X^n$. Denote by $P_n$ the group of permutations of $\{1, \cdots, n\}$. The group $P_n$ has a natural action on $X^n$. The quotient $X^n/P_n$ will be denoted by $S^n(X)$; it is called the $n$-fold symmetric product of $X$.

Let $\mathcal{O}_X$ denote the sheaf of germs of holomorphic functions on $X$. For a positive integer $r$, consider the sheaf $\mathcal{O}_X^{\oplus r}$ of germs of holomorphic sections of the trivial holomorphic vector bundle on $X$ of rank $r$. For any positive integer $n$, let $Q := Q_X(r,n)$ be the Quot scheme parametrizing all the torsion quotients of degree $n$ of the $\mathcal{O}_X$–module $\mathcal{O}_X^{\oplus r}$. Equivalently, points of $Q$ parametrize coherent analytic subsheaves of $\mathcal{O}_X^{\oplus r}$ of rank $r$ and degree $-n$. This $Q$ is an irreducible smooth complex projective variety of dimension $rn$ [Be, p. 1, Theorem 2].

Note that $Q_X(1,n)$ is identified with the symmetric product $S^n(X)$ by sending a quotient of $\mathcal{O}_X$ to the scheme-theoretic support of it. If we consider $Q_X(1,n)$ as the parameter space for the coherent analytic subsheaves of $\mathcal{O}_X$ of rank 1 and degree $-n$, then the above identification of $Q_X(1,n)$ with $S^n(X)$ sends a subsheaf $\psi : L \hookrightarrow \mathcal{O}_X$ to the divisor of $\psi$.

The gonality of $X$ is the smallest integer $d$ such that there is a nonconstant holomorphic map $X \rightarrow \mathbb{CP}^1$ of degree $d$ (see [Ei, p. 171]). Therefore, the gonality of $X$ is one if and only if $g = 0$. If $g \in \{1, 2\}$, then the gonality of $X$ is two. More generally, the gonality of $X$ is two if and only if $X$ is hyperelliptic of positive genus.

3. Nonpositive holomorphic bisectional curvatures

Theorem 3.1. Let $d$ denote the gonality of $X$. The symmetric product $S^n(X)$ admits a Kähler structure satisfying the condition that all the holomorphic bisectional curvatures are nonpositive if and only if $n < d$.

Proof. If $n \geq d$, then we know that $S^n(X)$ does not admit any Kähler structure such that all the holomorphic bisectional curvatures are nonpositive [Bi1, p. 1491, Proposition 3.2]. We recall that this follows from the fact that there is a nonconstant holomorphic embedding of $\mathbb{CP}^1$ in $S^n(X)$ if $n \geq d$.

So assume that $n < d$.

Let

$$\varphi : S^n(X) \rightarrow \text{Pic}^n(X)$$

be the natural holomorphic map that sends any $\{x_1, \cdots, x_n\} \in S^n(X)$ to the holomorphic line bundle $\mathcal{O}_X(\sum_{i=1}^n x_i)$. We will show that $\varphi$ is an immersion.
Take any point \( x = \{ x_1, \ldots, x_n \} \in S^n(X) \). The divisor \( \sum_{i=1}^n x_i \) will be denoted by \( D \). Let

\[
0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow Q'(\mathcal{I}) := \mathcal{O}_X/\mathcal{O}_X(-D) \longrightarrow 0
\]

be the short exact sequence corresponding to the point \( x \). Tensoring it with \( \mathcal{O}_X(-D)^* = \mathcal{O}_X(D) \) we get the short exact sequence

\[
0 \longrightarrow \text{End}(\mathcal{O}_X(-D)) = \mathcal{O}_X \longrightarrow \text{Hom}(\mathcal{O}_X(-D), \mathcal{O}_X) = \mathcal{O}_X(D) \longrightarrow Q(\mathcal{I}) := \text{Hom}(\mathcal{O}_X(-D), Q'(\mathcal{I})) \longrightarrow 0.
\]

Let

\[
0 \longrightarrow H^0(X, \mathcal{O}_X) \overset{\alpha}{\longrightarrow} H^0(X, \mathcal{O}_X(D)) \overset{\beta}{\longrightarrow} H^0(X, Q(\mathcal{I})) \overset{\gamma}{\longrightarrow} H^1(X, \mathcal{O}_X)
\]

be the long exact sequence of cohomologies associated to this short exact sequence of sheaves.

The holomorphic tangent space to \( S^n(X) \) at \( x \) is

\[ T_x S^n(X) = H^0(X, Q(\mathcal{I})), \]

and the tangent bundle of \( \text{Pic}^n(X) \) is the trivial vector bundle with fiber \( H^1(X, \mathcal{O}_X) \). The differential at \( x \) of the map \( \varphi \) in (3.1)

\[ (d\varphi)(\mathcal{I}) : T_x S^n(X) = H^0(X, Q(\mathcal{I})) \longrightarrow T_{\varphi(\mathcal{I})} \text{Pic}^n(X) = H^1(X, \mathcal{O}_X) \]

satisfies the identity

\[ (d\varphi)(\mathcal{I}) = \gamma, \]

where \( \gamma \) is the homomorphism in (3.3).

Now, \( H^0(X, \mathcal{O}_X) = \mathbb{C} \). Since \( n < d \), it can be shown that

\[ H^0(X, \mathcal{O}_X(D)) = \mathbb{C}. \]

Indeed, \( \dim H^0(X, \mathcal{O}_X(D)) \geq 1 \) because \( D \) is effective. If

\[ \dim H^0(X, \mathcal{O}_X(D)) \geq 2, \]

then considering the partial linear system given by two linearly independent sections of \( \mathcal{O}_X(D) \) we get a holomorphic map from \( X \) to \( \mathbb{C}P^1 \) whose degree coincides with the degree of \( D \). This contradicts the fact that the gonality of \( X \) is strictly bigger than \( n \). Therefore, \( H^0(X, \mathcal{O}_X(D)) = \mathbb{C} \).

Since \( H^0(X, \mathcal{O}_X(D)) = \mathbb{C} \), the homomorphism \( \alpha \) in (3.3) is an isomorphism. Hence \( \beta \) in the exact sequence (3.3) is the zero homomorphism and \( \gamma \) in (3.3) is injective.

Since \( \gamma \) in (3.3) is injective, from (3.4) we conclude that \( \varphi \) is an immersion.

The compact complex torus \( \text{Pic}^n(X) \) admits a flat Kähler metric \( \omega \). The pullback \( \varphi^* \omega \) is a Kähler metric on \( S^n(X) \) because \( \varphi \) is an immersion. Since \( \omega \) is flat, all the holomorphic bisectional curvatures of \( \varphi^* \omega \) are nonpositive. \( \square \)
Lemma 3.2. Take \( r \geq 2 \) and take any positive integer \( n \). The Quot scheme \( Q_X(r, n) \) does not admit any Kähler structure such that all the holomorphic bisectional curvatures are nonpositive.

Proof. Let

\[
(3.5) \quad f : Q_X(r, n) \longrightarrow S^n(X)
\]

be the holomorphic map that sends any quotient \( Q \) of \( O_X^{\oplus r} \) to the scheme theoretic support of \( Q \). Take any \( \underline{x} = \{x_1, \ldots, x_n\} \in S^n(X) \) such that all \( x_i \) are distinct points. Then \( f^{-1}(\underline{x}) \) is isomorphic to \( (\mathbb{C}P^{r-1})^n \). In particular, there are embeddings of \( \mathbb{C}P^1 \) in \( Q_X(r, n) \). This immediately implies that \( Q_X(r, n) \) does not admit any Kähler structure such that all the holomorphic bisectional curvatures are nonpositive. \( \square \)

4. Nonnegative holomorphic bisectional curvatures

In this section we assume that \( g \geq 2 \).

Corollary 4.1. If \( n \leq 2(g - 1) \), then \( Q_X(r, n) \) does not admit any Kähler structure such that all the holomorphic bisectional curvatures are nonnegative.

Proof. Assume that \( Q_X(r, n) \) has a Kähler structure \( \omega \) such that all the holomorphic bisectional curvatures for \( \omega \) are nonnegative. Consequently, tangent bundle \( TQ_X(r, n) \) is nef. See [DPS, p. 305, Definition 1.9] for the definition of a nef vector bundle; nef line bundles are introduced in [DPS, p. 299, Definition 1.2]. Since \( Q_X(r, n) \) is a complex projective manifold, nef bundles on \( Q_X(r, n) \) in the sense of [DPS] coincide with the nef bundles on \( Q_X(r, n) \) in the algebraic geometric sense (see lines 13–14 (from top) in [DPS, p. 296]).

Let

\[
(4.1) \quad \delta := \phi \circ f : Q_X(r, n) \longrightarrow \text{Pic}^n(X)
\]

be the composition, where \( \phi \) and \( f \) are constructed in (3.1) and (3.5) respectively. The homomorphism

\[
H^2(\text{Pic}^n(X), \mathbb{Q}) \longrightarrow H^2(S^n(X), \mathbb{Q}), \quad c \longmapsto \phi^*c
\]

is an isomorphism [Ma, p. 325, (6.3)]. Also, the homomorphism

\[
H^2(S^n(X), \mathbb{Q}) \longrightarrow H^2(Q_X(r, n), \mathbb{Q}), \quad c \longmapsto f^*c
\]

is an isomorphism [BGL, p. 647, Proposition 4.2] (see also the last line of [BGL, p. 647]). Combining these we conclude that the homomorphism

\[
H^2(\text{Pic}^n(X), \mathbb{Q}) \longrightarrow H^2(Q_X(r, n), \mathbb{Q}), \quad c \longmapsto \delta^*c
\]

is an isomorphism, where \( \delta \) is constructed in (4.1).

Therefore, \( \delta \) is the Albanese morphism for \( Q_X(r, n) \). Since the tangent bundle of \( Q_X(r, n) \) is nef, the Albanese map \( \delta \) is a holomorphic surjective submersion onto \( \text{Pic}^n(X) \) [CP], [DPS, p. 321, Proposition 3.9].
Since $\delta$ is surjective, the map $\varphi$ in (3.1) is surjective. Therefore,

$$g = \dim \text{Pic}^n(X) \leq \dim S^n(X) = n.$$  

The map $\varphi$ is a submersion because $\delta$ is a submersion and $f$ is surjective.

We will show that $\varphi$ is not a submersion if $n \leq 2(g - 1)$.

Take any $n \leq 2(g - 1)$. Let $D'$ be the divisor of a holomorphic 1–form on $X$. We note that the degree of $D'$ is $2(g - 1)$. Take an effective divisor $D$ on $X$ of degree $n$ such that $D' - D$ is effective. Writing $D' = x_1 + \ldots + x_{2g-2}$, we may take $D = x_1 + \ldots + x_n$. Substitute this $D$ in (3.2) and consider the corresponding long exact sequence of cohomologies

$$(4.2)\quad H^0(X, Q(\mathcal{E})) \xrightarrow{\gamma} H^1(X, \mathcal{O}_X) \xrightarrow{\gamma'} H^1(X, \mathcal{O}_X(D)) \rightarrow H^1(X, Q(\mathcal{E}))$$

as in (3.3). We note that $H^1(X, Q(\mathcal{E})) = 0$ because $Q(\mathcal{E})$ is a torsion sheaf on $X$. From Serre duality,

$$H^1(X, \mathcal{O}_X(D)) = H^0(X, \mathcal{O}_X(D' - D))^*.$$  

Now $H^0(X, \mathcal{O}_X(D' - D)) \neq 0$ because $D' - D$ is effective. Combining these, we conclude that $\gamma'$ in (4.2) is nonzero. Hence $\gamma$ in (4.2) is not surjective. Therefore, from (3.4) we conclude that the differential $d\varphi$ of $\varphi$ is not surjective at the point $D \in S^n(X)$. In particular, $\varphi$ is not a submersion if $n \leq 2(g - 1)$.

This completes the proof. □

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