The squashed fuzzy sphere, fuzzy strings and the Landau problem

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Abstract

We discuss the squashed fuzzy sphere, which is a projection of the fuzzy sphere onto the equatorial plane, and use it to illustrate the stringy aspects of noncommutative field theory. We elaborate explicitly how strings linking its two coincident sheets arise in terms of fuzzy spherical harmonics. In the large $N$ limit, the matrix-model Laplacian is shown to correctly reproduce the semi-classical dynamics of these charged strings, as given by the Landau problem.

1 Introduction

Field theory on noncommutative (NC) spaces has been studied intensively from various points of view in the past decades. One of the original motivations was the (naive) hope that the UV-divergences of quantum field theory would be regularized on a noncommutative space, due to the presence of an intrinsic noncommutative scale $\Lambda_{NC}$. This hope turned out not to be vindicated. Rather, NC field theory behaves very differently from ordinary field theory at scales far above $\Lambda_{NC}$, where the basic degrees of freedom display a string-like or dipole-like nature. This is already implicit in the trivial observation that NC fields are matrices or operators, which thus have two indices, and are naturally represented in t’Hooft’s double line notation \cite{1}. Indeed, scalar fields on a noncommutative space arise in string theory as open strings starting and ending on a D-brane with $B$ field \cite{2,3}. This suggests a dipole-like nature of noncommutative fields \cite{1,5}, which is also implicit in the matrix-model realization of noncommutative gauge theory and its relation with string theory \cite{6}, culminating in the remarkable proposals \cite{7,8} that string theory might be defined in terms of matrix models. In particular, the IKKT matrix model is tantamount to noncommutative $\mathcal{N} = 4$ SYM on $\mathbb{R}_\theta^4$.

In the same vein, the interactions determined by the algebra of noncommutative scalar fields with momentum far above $\Lambda_{NC}$ is also very different from the commutative case; this

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can be seen easily on the quantum plane $R^2_\theta$, but also e.g. on the fuzzy sphere $S^2_N$. Since all these high-scale modes are probed in QFT via loop contributions, it should not be surprising that NC quantum field theory (NCQFT) is typically quite different from ordinary QFT, and seems consistent only for very special models$^3$. The stringy nature of NCQFT manifests itself also in the gravitational aspects of noncommutative gauge theory $^{11\,14}$ and the notorious UV/IR mixing $^{15}$.

These insights are very useful also to study NC field theory per se, without any direct relation with string theory. It allows to understand better its intrinsic properties, and suggests a different organization of its fundamental degrees of freedom. In the present paper, we provide a particularly simple and explicit illustration of the stringy nature of noncommutative scalar fields, in the example of noncommutative scalar field theory on the squashed fuzzy sphere $PS^2_N$. This is a noncommutative space obtained by projecting the fuzzy sphere $S^2_N$ on the equatorial plane. It should be viewed as a stack of two coinciding fuzzy disks with opposite (non-constant) Poisson structures, glued together at their boundary. The dipole or string picture discussed above suggests that there should be string-like modes connecting these two sheets, with opposite charges at the ends moving in the fields $B_+$ and $B_-$ on the two sheets. Here $B_+ = -B_-$ corresponds to the symplectic forms i.e. the inverse Poisson structures on the two sheets. At low energies, these should behave like point-like charged objects moving in an effective magnetic field $B_+ - B_-$, which – focusing on the center of the disks in a suitable scaling limit – should reduce to the Landau problem$^4$.

With this in mind, we study free scalar field theory on $PS^2_N$, and identify the low-energy modes and their effective action. We can indeed identify the lowest eigenmodes of the (matrix) Laplacian with string-like modes connecting the opposite sheets, which reproduce precisely the energy levels and degeneracies of the Landau problem. They are identified as fuzzy spherical harmonics $\hat{Y}_l^m$ with large quantum numbers $m \approx \pm l$. For the lowest Landau level, the modes at (or near) the origin can be expressed succinctly in terms of coherent states localized at the origin of the two sheets, thus exhibiting their stringy nature. This is also related to recent results on the low-energy modes of coinciding or intersecting branes on squashed $SU(3)$ branes $^{18}$.

The present paper hence demonstrates how an appropriate organization of the degrees of freedom$^5$ can illuminate the stringy physics hidden in NCFT, which transcends the picture of conventional field theory.

## 2 The Landau levels

We recall the quantum mechanical description of a (spinless) charged particle moving perpendicular to an uniform magnetic field along the $z$-axis,

$$\vec{B} = B\hat{e}_z.$$ 

$^3$This includes the maximally supersymmetric $\mathcal{N} = 4$ Super-Yang-Mills, which is nothing but the IKKT model, and a particular matrix model interpreted as scalar field theory $^{10}$.

$^4$For a treatment of the Landau problem on the fuzzy sphere with monopole charge see $^{16\,17}$. This is not directly related to the problem under investigation here.

$^5$A somewhat related organization of fields in terms of the so-called the matrix base was used in $^{19}$ to analyze perturbation theory for scalar field theory on the quantum plane.
The Hamiltonian for such a set-up is
\[ H = \frac{1}{2\mu} \left( \vec{P} - \frac{q}{c} \vec{A} \right)^2 \] (2.1)
where $\vec{A}$ is the vector potential related to the magnetic field, which has the form
\[ \vec{A} = \frac{B}{2} \begin{pmatrix} -Y \\ X \\ 0 \end{pmatrix} \] (2.2)
in the Landau gauge. Inserting (2.2) into (2.1) and introducing the cyclotron (or Larmor) frequency
\[ \omega_c = -\frac{qB}{\mu c} \] (2.3)
the Hamiltonian can be written as
\[ H = \frac{P_x^2 + P_y^2}{2\mu} + \frac{\mu\omega_c^2}{8} (X^2 + Y^2) + \frac{\omega_c^2}{2} L_z = H_{xy} + \frac{\omega_c}{2} L_z, \]
where $L_z$ is the angular momentum operator in $z$-direction, and $H_{xy}$ is the Hamiltonian of a two dimensional harmonic oscillator with frequency $\frac{\omega_c}{2}$. We can reformulate the problem in terms of the ladder operators
\[ a_r = \frac{1}{2} \left( \beta (X - iY) + \frac{i}{\beta \hbar} (P_x - iP_y) \right), \]
\[ a_l = \frac{1}{2} \left( \beta (X + iY) + \frac{i}{\beta \hbar} (P_x + iP_y) \right), \]
with $\beta = \sqrt{\frac{\mu c^2}{\hbar}}$. These are the annihilation operators of right and left circular quanta respectively. We introduce the number operators
\[ N_r = a_r^\dagger a_r, \]
\[ N_l = a_l^\dagger a_l \]
so that
\[ H_{xy} = (N_r + N_l + 1) \frac{\hbar \omega_c}{2}, \]
\[ L_z = (N_r - N_l) \hbar. \]
Now it is evident that $a_r^\dagger$ ($a_l^\dagger$) create right (left) circular quanta. Both raise the energy by $\frac{\hbar \omega_c}{2}$, but acting with $a_r^\dagger$ increases the additional angular momentum by $\hbar$, while acting with $a_l^\dagger$ decreases the angular momentum by $\hbar$. Thus the Hamiltonian (2.1) has the form
\[ H = \left( N_r + \frac{1}{2} \right) \hbar \omega_c \]
with eigenvalues
\[ E = \left( n_r + \frac{1}{2} \right) \hbar \omega_c \] (2.4)
and eigenfunctions
\[ |\chi_{n_r,n_l}\rangle = \frac{1}{\sqrt{n_r!n_l!}}(a_{r}^\dagger)^{n_r}(a_{l}^\dagger)^{n_l}|\chi_{0,0}\rangle, \quad n_l,n_r \in \mathbb{N}. \]

Note that the energy depends only on \( n_r \) but is independent of \( n_l \), thus the energy states corresponding to a particular Landau level \( n_r \) are infinitely degenerate. It is not hard to see that the wave-functions for the lowest Landau level \( n_r = 0 \) are concentric circles around the origin with radius measured by \( n_l \),

\[ \chi_{0,n_l}(\rho,\varphi) = \frac{\beta}{\sqrt{\pi n_l!}}e^{-in\varphi}e^{-\beta^2 \rho^2} \]  

in polar coordinates.

If we include spin, the Hamiltonian is modified as follows
\[ H = \left( N_r + \frac{1}{2} - \frac{\sigma_z}{2} g \right) \hbar \omega_c \quad (2.6) \]
where \( \sigma_z \) is the spin operator in the \( z \)-direction and \( g \) the g-factor dependent on the type of particle.

3 \ The fuzzy sphere \( S_N^2 \)

The fuzzy sphere \( S_N^2 \) [20, 21] is a quantization of the usual sphere \( S^2 \) with a cutoff in angular momentum, which contains \( N \) quanta of area. The quantization of \( S^2 \) is given by a quantization map \( Q \),
\[ Q : \quad \mathcal{C}_n(S^2) \to \mathcal{M}_N = \text{Mat}(N, \mathbb{C}) \]  
\[ x^a \mapsto X^a = \kappa J^a \]
which maps in particular the embedding functions \( x^a \) on \( S^2 \) to quantized embedding functions \( X^a = \kappa J^a \) on \( S_N^2 \). Here \( J^a \) are the generators of \( \mathfrak{su}(2) \) in the \( N = 2n+1 \)-dimensional irreducible representation, \( \mathcal{C}_n(S^2) \) is the space of polynomials on \( S^2 \) of degree \( \leq n \) and \( \mathcal{M}_N \) is the algebra of complex \( N \times N \) matrices. Since the quadratic Casimir operator has the form
\[ J^2 = C_N \mathbb{1} \quad \text{with} \quad C_N = \frac{1}{4} (N^2 - 1), \]
the radial constraint of a sphere with radius \( r \)
\[ (X^1)^2 + (X^2)^2 + (X^3)^2 = r^2 \]
is recovered if we set
\[ \kappa^2 = \frac{r^2}{C_N}. \]
We introduce a constant which is the analogue of \( \hbar \)
\[ \k = \kappa r = \frac{r^2}{\sqrt{C_N}} \]  
(3.2)
and the commutative limit is given by $k \to 0$ as $N \to \infty$ for fixed radius. The generators $X^a$ of the algebra $\mathcal{M}_N$ satisfy the commutation relations

\begin{align*}
[X^a, X^b] &= i\hbar C^{abc} X^c =: i\Theta^{ab}, \\
C^{abc} &= r^{-1} e^{abc}, \\
(\Theta^{ab})_{S^2_N} &= \frac{k}{r} \begin{pmatrix}
0 & X^3 & -X^2 \\
-X^3 & 0 & X^1 \\
X^2 & -X^1 & 0
\end{pmatrix}.
\end{align*}

(3.3)\hspace{1cm} (3.4)\hspace{1cm} (3.5)

To complete the definition of the quantization map $Q$, we decompose $\mathcal{M}_N$ into irreducible representations under the adjoint action of $\mathfrak{su}(2)$

$$\mathcal{M}_N \cong (N) \otimes (\bar{N}) = (1) \oplus (3) \oplus \ldots \oplus (2N-1) = \left\{ \hat{Y}_0^m \right\} \oplus \ldots \oplus \left\{ \hat{Y}^{N-1}_m \right\}. \quad (3.6)$$

This defines the fuzzy spherical harmonics $\hat{Y}_l^m$, and allows to write down a natural definition for the quantization map $Q$ for polynomial functions of degree less than or equal to $n = 2N+1$:

$$Q : \quad C_n(S^2) \rightarrow \mathcal{M}_N = \text{Mat}(N, \mathbb{C})$$

$$Y_l^m \mapsto \hat{Y}_l^m,$$

compatible with the $\text{SO}(3)$ symmetry. Here $Y_l^m$ are the usual spherical harmonics. In the limit $N \to \infty$, we recover the full algebra of polynomial functions on $S^2$.

The commutation relations (3.5) define a quantization of the Poisson structure

\begin{align*}
\{x^a, x^b\} &= \hbar C^{abc} x^c =: \theta^{ab}, \\
C^{abc} &= r^{-1} e^{abc} \\
\theta^{ab} &= \frac{k}{r} \begin{pmatrix}
0 & x^3 & -x^2 \\
-x^3 & 0 & x^1 \\
x^2 & -x^1 & 0
\end{pmatrix}
\end{align*}

which corresponds to the $\text{SO}(3)$–invariant symplectic 2-form

$$\omega_N = \frac{1}{k} C_{abc} x^a dx^b \wedge dx^c \quad (3.7)$$

and satisfies the flux quantization condition $2\pi N = \int_{S^2} \omega_N$. Thus the fuzzy sphere $S^2_N$ is the quantization of the symplectic manifold $(S^2, \omega_N)$. Furthermore, the Laplace operator on the fuzzy sphere is defined by

$$\Box = \frac{1}{k^2} \sum_{a=1}^{3} [X^a, [X^a, \cdot]] . \quad (3.8)$$

This type of matrix Laplacian arises naturally in the context of Yang-Mills models [14].
3.1 Fuzzy spherical harmonics

The fuzzy spherical harmonics \( \hat{Y}_m^l \) were identified in equation (3.6) as the irreducible representations of \( SU(2) \) acting on the non-commutative algebra \( \mathcal{M}_N \), analogous to the commutative case up to a cutoff. It is easy to see that they are also eigenfunctions of the Laplace operator

\[
\Box \hat{Y}_m^l = \frac{k^2}{k^2} [(l + 1) \hat{Y}_m^l] = \frac{1}{r^2} [(l + 1) \hat{Y}_m^l],
\]

in analogy to the classical case, with the same \( 2l + 1 \)-fold degeneracy. We can get more information on the explicit (matrix) form of the \( \hat{Y}_m^l \) for fixed \( N \) using the representation theory of \( SU(2) \). Consider a basis where the Cartan generator \( H \) of \( SU(2) \) is diagonal. Since \( m \) gives the eigenvalue of \( H \), all the matrices \( \hat{Y}_0^l \) are diagonal, \( \hat{Y}_1^l \) have entries only along the first diagonal above the main diagonal, \( \hat{Y}_2^l \) have entries only along the second diagonal above the main diagonal and so forth. An analogous statement can be made for \( \hat{Y}_{-1}^l \), \( \hat{Y}_{-2}^l \), etc. below the main diagonal. The entries of the matrices are symmetric w.r.t. the anti-diagonal, and their values are decreasing with increasing distance from the anti-diagonal. Clearly the maximal value for \( l \) is \( l_{\max} = N - 1 \), and all matrices with \( |m| > l \) vanish.

4 The squashed fuzzy sphere \( PS^2_N \)

In this section we discuss the squashed fuzzy sphere, which is interpreted as projection of the fuzzy sphere onto the equatorial plane \([13]\). This arises e.g. as building block of cosmological solutions in the IR-regulated IKKT matrix model \([22]\). In particular, we explain how strings linking its two coincident sheets arise in terms of noncommutative functions. The relation of matrix models with noncommutative gauge theory is illustrated by showing how the description of these strings in noncommutative field theory reproduces the semi-classical dynamics of these charged strings as given by the Landau problem.

A projection \( \Pi \) of a classical sphere onto its equatorial plane is achieved simply by replacing the three embedding functions \( x^a : S^2 \hookrightarrow \mathbb{R}^3 \) by only two embedding functions \( x^1 \) and \( x^2 \), dropping \( x^3 \):

\[
S^2 \rightarrow \mathbb{R}^3 \quad \Pi \rightarrow \mathbb{R}^2 \quad p \mapsto x^a(p) \rightarrow x^a(p), \quad a = 1, 2
\]

Here we keep the same space of functions on \( S^2 \), but change the embedding information given by the \( x^a \). After projecting, the two hemispheres are stacked one onto another as two coinciding disks glued at the boundary.

Accordingly, we define the projected or squashed fuzzy sphere \( PS^2_N \) in terms of the two generators \( X^a, \quad a = 1, 2 \). They generate the same algebra of fuzzy functions \( \text{Mat}(N, \mathbb{C}) \) as for \( S^2_N \), but will lead to a different fuzzy Laplacian. It can be viewed as two projected fuzzy disks glued at the boundary. The relation between the fuzzy disk and the fuzzy sphere can be seen explicitly by expressing \( X^3 \) in terms of the two independent generators \( X^1, X^2 \):

\[
(X^1)^2 + (X^2)^2 + (X^3)^2 = r^2 \quad \Rightarrow \quad (X^3)^2 = r^2 - (X^1)^2 - (X^2)^2.
\]

We define

\[
X^3_\pm = \pm \sqrt{r^2 - (X^1)^2 - (X^2)^2}
\]
as positive respectively negative part of $X^3$. Then $X^3_{\pm}$ reduces in the semi-classical (i.e. Poisson) limit to the embedding functions $x^3_{\pm}$ of upper respectively lower hemisphere in $\mathbb{R}^3$. Then the matrix Laplacian on the squashed fuzzy sphere is

$$\Box_S = \frac{1}{k^2} \sum_{i=1}^{2} \left[ X^i, \left[ X^i, \right] \right].$$

(4.4)

4.1 Poisson structure

The commutators of the generators $X^1, X^2$ of the squashed fuzzy sphere define in the semi-classical limit a Poisson structure on the projected disks. This is nothing but the push-forward of the Poisson structure on $S^2$ by $\Pi$. On the upper sheet, we have

$$\{x^1, x^2\} = \frac{k}{r} x^3 (x^1, x^2) = \frac{k}{r} \sqrt{r^2 - (x^1)^2 - (x^2)^2} = \theta_{12}^{12}$$

(4.5)

while on the lower sheet we have

$$\{x^1, x^2\} = -\frac{k}{r} x^3 (x^1, x^2) = -\frac{k}{r} \sqrt{r^2 - (x^1)^2 - (x^2)^2} = \theta_{12}^{12}$$

(4.6)

Thus the Poisson tensor on the two sheets indicated by $\pm$ is given by

$$\theta_{\pm}^{ij} = \pm \frac{\sqrt{r^2 - (x^1)^2 - (x^2)^2}}{r} k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.7)$$

We observe that the two coinciding fuzzy disks have opposite Poisson structure

$$-\theta_{\pm}^{ij} = \theta_{\mp}^{ij}, \quad (4.8)$$

and $\theta_{\pm}$ vanishes as we approach the edge, so that we have a smooth transition

$$r^2 - (x^1)^2 - (x^2)^2 = 0 \quad \Rightarrow \quad \theta_{\pm}^{ij} = \theta_{\mp}^{ij} = 0. \quad (4.9)$$
4.2 Effective gauge fields

In this section, we will obtain an interpretation of the matrix Laplacian \( \square \) in terms of non-commutative gauge theory. This will allow to identify particular functions on the squashed fuzzy sphere as charged strings linking its two sheets, and provide an explicit relation with the energy levels of Landau problem.

To understand this relation, we recall that gauge fields on the Moyal-Weyl quantum plane \( \mathbb{R}^2_\theta \) can be introduced as deformations or fluctuations
\[
X^i = \bar{X}^i - \bar{\theta}^{ij} A_j(\bar{X})
\]  (4.10)
of the generators \( \bar{X}^i \) of \( \mathbb{R}^2_\theta \), which satisfy
\[
[\bar{X}^i, \bar{X}^j] = i \bar{\theta}^{ij} = i k \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right).
\]  (4.11)

The \( X^i \) are known as covariant coordinates \[23\]. Their commutators are given by
\[
[X^i, X^j] = [\bar{X}^i, \bar{X}^j] - \bar{\theta}^{ij} [\bar{X}^i, A_j] + \bar{\theta}^{ij} [\bar{X}^j, A_i] + \bar{\theta}^{ij} \bar{\theta}^{ij'} [A_i, A_j] = -i \bar{\theta}^{ij} \bar{\theta}^{ij'} (\bar{\theta}^{-1}_{ij'} - \partial_i A_{j'} + \partial_{j'} A_i + [A_i, A_{j'}]) = i \bar{\theta}^{ij} - i \bar{\theta}^{ij} \bar{\theta}^{ij'} F_{ij'} \]  (4.12)
where \( F_{ij} \) can be interpreted as field strength of the \( U(1) \) gauge field \( A_i \) on \( \mathbb{R}^2_\theta \). The commutators
\[
[X^i, \phi] = [\bar{X}^i, \phi] - \bar{\theta}^{ij} [\bar{X}^i, A_j] + \bar{\theta}^{ij} [\bar{X}^j, A_i] + \bar{\theta}^{ij} \bar{\theta}^{ij'} [A_i, A_j] = i \bar{\theta}^{ij} (\partial_i + i [A_i, \phi]) \]  (4.13)
defines the covariant derivatives of a scalar field \( \phi \). In the semi-classical limit, the Poisson-brackets of \( x^i \sim X^i \) can be expressed accordingly
\[
\{x^i, x^j\} = \{\bar{x}^i, \bar{x}^j\} - \bar{\theta}^{ij} \{\bar{x}^i, A_j\} + \bar{\theta}^{ij} \{\bar{x}^j, A_i\} + \bar{\theta}^{ij} \bar{\theta}^{ij'} \{A_i, A_j\} = \bar{\theta}^{ij} - \bar{\theta}^{ij} \bar{\theta}^{ij'} F_{ij'} \]  (4.14)
as deformation of the constant Poisson bracket \( \{\bar{x}^i, \bar{x}^j\} = \bar{\theta}^{ij} \) by the field strength \( F_{ij} \)
\[
F_{ij} = \partial_i A_j - \partial_j A_i - \{A_i, A_j\}.
\]
Thus \( \bar{x}^i \) can be viewed as Darboux coordinates on \( (\mathbb{R}^2, \{., .\}) \). The semi-classical version of
\[
x^i = \bar{x}^i - \bar{\theta}^{ij} A_j(\bar{x})
\]  (4.15)
therefore allows to interpret the difference between the \( x^i \) and the Darboux coordinates \( \bar{x}^i \) in terms of a \( U(1) \) gauge field.

We now apply these insights to the example of the squashed fuzzy sphere. To avoid complications due to the finite-dimensional representation, we restrict ourselves to the semi-classical (i.e. Poisson) limit. As we have seen in \[4.8\], the squashed fuzzy sphere decomposes
into an upper and a lower fuzzy disk, which arise by restricting the matrices to the upper and lower blocks defined by the positive and negative spectrum of $X^3$:

$$X^i = \begin{pmatrix} X_+^i & 0 \\ 0 & X_-^i \end{pmatrix} \sim \begin{pmatrix} x_+^i & 0 \\ 0 & x_-^i \end{pmatrix} = \begin{pmatrix} x^i - \bar{\theta}^{ij}A^+_j \\ 0 \\ \bar{x}^a - \bar{\theta}^{ij}A^-_j \end{pmatrix}$$

(4.16)

with $+$ ($-$) indicating the upper (lower) sheet. Note that although the full Poisson structures $\theta_+^{ij}, \theta_-^{ij}$ have opposite sign, the Darboux coordinates $\bar{x}^i$ define the same constant $\bar{\theta}^{ij}$ on the upper and the lower sheet,

$$\{\bar{x}^i, \bar{x}^j\} = \bar{\theta}^{ij} = k\epsilon^{ij}.$$  

(4.17)

This is essential for an interpretation in terms of noncommutative gauge theory on a stack of coinciding branes.

Now we want to find the corresponding gauge fields $A_+^i$ on the two sheets explicitly. From (4.7) we get

$$\{x_+^i, x_+^j\} = \bar{\theta}_+^{ij} = \frac{1}{k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which indeed reduces to (4.17) for $x_1, x_2 \approx \vec{0}$, and vanishes at the edge of the disk. We can rewrite equation (4.14) as

$$\bar{\theta}_+^{-1}\bar{\theta}_-^{-1}\theta^{ij} = -\bar{\theta}_-^{-1} - F_{i'j'}$$

with

$$\bar{\theta}_-^{-1} = \frac{1}{k} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and obtain

$$\bar{\theta}_+^{-1}\bar{\theta}_-^{-1}\theta^{ij} = \frac{1}{kr} \begin{pmatrix} 0 & \sqrt{r^2 - (x_1^1)^2 - (x_2^2)^2} \\ -\sqrt{r^2 - (x_1^1)^2 - (x_2^2)^2} & 0 \end{pmatrix}.$$  

We thus obtain the field strength $F$ on the different sheets as

$$F = \begin{pmatrix} F_+^- & 0 \\ 0 & F_-^+ \end{pmatrix}$$

with

$$F_{ij}^\pm = \frac{1}{k} \begin{pmatrix} 0 & 1 \mp \frac{1}{r}\sqrt{r^2 - (x_1^1)^2 - (x_2^2)^2} \\ 1 \mp \frac{1}{r}\sqrt{r^2 - (x_1^1)^2 - (x_2^2)^2} & 0 \end{pmatrix}.$$  

(4.18)

To obtain explicit expressions for the gauge fields $\tilde{A}_i^\pm$, we have to solve the following differential equations

$$F_{12}^\pm = k^{-1} \left( 1 \pm \frac{1}{r}\sqrt{r^2 - (x_1^1)^2 - (x_2^2)^2} \right) = \partial_1 A_2^\pm - \partial_2 A_1^\pm - \{ A_1^\pm, A_2^\pm \}.$$  

Since $\{ A_1^\pm, A_2^\pm \}$ is of higher order in $k$ than $\partial_1 A_2^\pm - \partial_2 A_1^\pm$, it is negligible in the semi-classical limit, and (4.18) simplifies to

$$F_{12}^\pm \simeq \partial_1 A_2^\pm - \partial_2 A_1^\pm.$$  

(4.19)
The solutions of this differential equations are given by

\[ \vec{A}^\pm (\vec{x}) = \frac{1}{2K} \begin{pmatrix} -x^2 \pm K^1 \\ x^1 \pm K^2 \end{pmatrix} \]  

where

\[ K^1 = \frac{1}{2r} \left( x^2 \sqrt{r^2 - (x^1)^2 - (x^2)^2} + (r^2 - (x^1)^2) \arctan \left( \frac{x^2}{\sqrt{r^2 - (x^1)^2 - (x^2)^2}} \right) \right), \]

\[ K^2 = \frac{-1}{2r} \left( x^1 \sqrt{r^2 - (x^1)^2 - (x^2)^2} + (r^2 - (x^2)^2) \arctan \left( \frac{x^1}{\sqrt{r^2 - (x^1)^2 - (x^2)^2}} \right) \right). \]

It is easy to verify

\[ F^\pm_{12} = \vec{\nabla} \times \vec{A}^\pm. \]

Now consider in more detail the covariant derivative (4.13) acting on general noncommutative scalar fields on the squashed fuzzy sphere including off-diagonal components,

\[ \phi \equiv \Upsilon = \begin{pmatrix} \Upsilon_+ \\ \Upsilon_{12} \\ \Upsilon_{21} \\ \Upsilon_- \end{pmatrix}. \]

We denote the scalar fields on \( PS^2_N \) with \( \Upsilon \) henceforth, to emphasize their stringy nature. Here \( \Upsilon_{\pm} \) correspond to functions on the upper and lower sheet, respectively, while \( \Upsilon_{12}, \Upsilon_{21} \) are naturally interpreted as strings connecting these sheets\(^6\). Then the covariant derivatives acting on the string-like modes is

\[ D_i \Upsilon_{12} = -i \partial_i \Upsilon_{12} - i(A_i^+ - A_i^-) \Upsilon_{12} \]

\[ D_i \Upsilon_{21} = -i \partial_i \Upsilon_{21} + i(A_i^+ - A_i^-) \Upsilon_{21} \]

with

\[ \vec{A}^+ - \vec{A}^- = k^{-1} \begin{pmatrix} K^1 \\ K^2 \end{pmatrix}. \]

We note that the off-diagonal string-like modes couple to the difference \( \vec{A}^+ - \vec{A}^- \) of the gauge fields on the two sheets, and behave like charged objects moving in a background with field strength \( F^+ - F^- \). In particular, the Laplacian (4.4) acting on these fields becomes

\[ \Box_S \Upsilon_{12} = \delta^{ij} D_i D_j \Upsilon_{12} \]

in the semi-classical limit. This is precisely the Hamiltonian for a charged particle moving in a magnetic field, as studied in section\(^2\). We therefore expect that in the pole limit i.e. near the origin \( \vec{x} = 0 \) for \( r \to \infty \), its spectrum should reproduce that of the Landau problem. This will be elaborated below.

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\(^6\)Equivalently, one may consider \( \Upsilon \) as \( u(2) \)-valued noncommutative gauge field on a single sheet.
Pole limit \( \vec{x} = 0 \). Near the pole we can expand the field strength in a Taylor series in \( x^i \).
Neglecting terms suppressed by \( \mathcal{O}(\frac{x^i}{r^2}) \), we obtain
\[
F_{12}^+ = 0, \quad F_{12}^- = \frac{2}{\bar{k}}
\]
and the gauge fields obtained from (4.22) are
\[
\vec{A}^+ - \vec{A}^- = \frac{1}{\bar{k}} \left( \frac{x^2}{-x^1} \right).
\]
In particular, \( (\vec{A}^+ - \vec{A}^-) \) acting on \( Y_{21} \) corresponds to the field strength
\[
F^+ - F^- = -\frac{2}{\bar{k}}
\]
while \( (\vec{A}^- - \vec{A}^+) \) acting on \( Y_{12} \) corresponds to the opposite field strength. Not surprisingly, the two sheets reduce for \( r \to \infty \) to Moyal-Weyl quantum planes \( R^2_{\bar{\theta}} \), with a constant field \( F = \frac{2}{\bar{k}} \) on the lower sheet. We can of course absorb the field strength in any given quantum plane by redefining \( \bar{\theta}^{ij} \), but the difference between the two sheets is unambiguous.

Edge limit \( (x^1)^2 + (x^2)^2 = r^2 \). At the edge, the field strength \( F_{ij} \) becomes
\[
F_{12}^+ = \frac{1}{\bar{k}}
\]
on both sheets, consistent with the fact that the Poisson structures on the upper and lower sheet (4.9) have a smooth transition.

4.3 Fuzzy Laplacian and its eigenfunction

In the previous chapter we arrived at an interpretation of the matrix Laplacian \( \Box_S \) on the squashed fuzzy sphere in the semi-classical limit (4.22). Now we return to the fuzzy case, and study this matrix Laplacian exactly. Comparing it with the Laplacian on the fuzzy sphere (3.8), we can write
\[
\Box_S = \Box - \frac{1}{\bar{k}^2} [X^3, [X^3, .]]
\]
In fact we can immediately write down all the eigenvectors and eigenvalues: they are given by the same fuzzy spherical harmonics \( \hat{Y}_l^m \) which diagonalize \( \Box \), since \( X^3 \) is proportional to \( J^3 \) and \( J^3 \hat{Y}_m^l = m \hat{Y}_m^l \). Thus
\[
\Box_S \hat{Y}_m^l = \frac{1}{r^2} \left( l(l+1) - m^2 \right) \hat{Y}_m^l.
\]
Note that the spectrum of \( \Box_S \) is independent of the matrix dimension \( N \), up to the cutoff.

Since the eigenvalues of \( \Box_S \) are not independent of \( m \) anymore (unlike in \( \Box \)), the degeneracy of each eigenvalue has a more complicated structure than for \( \Box \). Figure 2 shows the lowest eigenvalues and the corresponding states \( \hat{Y}_m^l \).

Thus the \( \hat{Y}_m^l \) span the Hilbert space on the squashed fuzzy sphere. However to identify the string modes \( \hat{Y}_{12} \), we need to identify those \( \hat{Y}_m^l \), which have entries exclusively in the upper right block, because of equation (4.21). In chapter 3.1 we saw that with larger \( m \), the entries of the \( \hat{Y}_m^l \) are farther away from the main diagonal, thus for \( l \approx l_{\text{max}} \) and \( |m| \approx l \) the \( \hat{Y}_m^l \) have entries solely in these string domains. Therefore for \( m > 0 \) the \( \hat{Y}_m^l \) serve as a basis for the \( \hat{Y}_{12} \), when \( m < 0 \) for \( \hat{Y}_{21} \). These are the string modes we are looking for.
Figure 2: The first few eigenvalues of □S corresponding to \( \hat{Y}_m^l \). Each eigenvalue where \( m \neq 0 \) is at least twice degenerated, since \( \hat{Y}_m^l \) and \( \hat{Y}_{-m}^l \) have the same eigenvalues. However, this does not mean that the \( m = 0 \) the eigenvalue is not degenerate, since e.g. \( \hat{Y}_0^2 \) and \( \hat{Y}_{\pm6}^6 \) have the same eigenvalue given by 6.

### 4.4 Semi-classical limit and string states

Having identified the basis for the strings \( \Upsilon_{12} \) (and \( \Upsilon_{21} \)) as \( \hat{Y}_m^l \) (and \( \hat{Y}_{m=-l}^l \)), we want to understand their precise relation with the states of the Landau problem, and relate the spectrum of \( \Box_S \) for these string states in the semi-classical limit.

Recall from chapter 3.1 that the quantum numbers for the fuzzy spherical harmonics for fixed matrix dimension \( N \) are given by

\[
\begin{align*}
  l & = 0, 1, \ldots, l_{\text{max}}, & l_{\text{max}} = N - 1, \\
  m & = l, l-1, \ldots, -l.
\end{align*}
\]

The distribution of the eigenfunctions in figure 2 already suggests which grouping of the \( \hat{Y}_m^l \) might be appropriate. The \( \hat{Y}_m^l \) with fixed difference \( l - m \) lie on certain lines, as illustrated in figure 3. In order to appropriately describe the \( \hat{Y}_m^l \) states with \( l \simeq l_{\text{max}} \) and \( |m| \simeq l \) for large \( N \), we define two new (small) quantum numbers \( L \) and \( M \) as the complement of \( l \) and \( m \) (which are large). Let us distinguish the two cases where \( m \) is positive and negative, since they are correlated to different strings, \( \Upsilon_{12} \) and \( \Upsilon_{21} \) respectively. Thus we define

\[
\begin{align*}
  L & := l_{\text{max}} - l \in \{0, 1, 2, \ldots\} \\
  M & := l - m \in \{0, 1, 2, \ldots\} \quad \text{(for } m > 0\text{)}, \\
  M' & := l + m \in \{0, 1, 2, \ldots\} \quad \text{(for } m < 0\text{)}.
\end{align*}
\]

Since \( \Box_S \) in (4.22) has the form of a squared momentum operator \( (\partial + A)^2 \), we multiply it with a factor of \( \frac{1}{2\mu} \) in order to relate \( \Box_S \) with the Hamiltonian (2.1), where \( \mu \) is the mass of
the particle. Then the eigenvalue equation for $\Box S$ becomes
\[
\frac{1}{2\mu} \Box_S \hat{Y}^l_m = \frac{(l(l+1) - m^2)}{2\mu r^2} \hat{Y}^l_m.
\]

We can now rewrite this in terms of our new quantum numbers $L$ and $M$, and get
\[
l(l+1) - m^2 \quad m \geq 0 \quad \Rightarrow \quad ((N - L - 1)(N - L) - (N - 1 - L - M)^2)
\]
\[
-1 - L - 2M - 2LM - M^2 \ll N \quad 2N \left( M + \frac{1}{2} \right) + O(1)
\]
for $m > 0$, and
\[
l(l+1) - m^2 \quad m < 0 \quad 2N \left( M' + \frac{1}{2} \right) + O(1),
\]
for $m < 0$. Here we assume that $N$ is very large while $M, L$ are small, as appropriate for the flat (pole) limit. Accordingly, we define a new basis of string modes as follows:
\[
\Upsilon^{L,M}_{(12)} = \hat{Y}^l_{l_{\text{max}} - L - M}, \quad l_{\text{max}} - L - M > N/2
\]
\[
\Upsilon^{L,M'}_{(21)} = \hat{Y}^l_{l_{\text{max}} - L - (l_{\text{max}} - L) + M'}, \quad -l_{\text{max}} + L + M' < -N/2
\]
for $L = 0, 1, 2, \ldots$ (4.24)

Thus
\[
\frac{1}{2\mu} \Box_S \Upsilon^{L,M}_{(12)} = \frac{N}{\mu r^2} \left( M + \frac{1}{2} \right) \Upsilon^{L,M}_{(12)} \quad \text{for } L = 0, 1, 2, \ldots
\]
\[
\frac{1}{2\mu} \Box_S \Upsilon^{L,M'}_{(21)} = \frac{N}{\mu r^2} \left( M' + \frac{1}{2} \right) \Upsilon^{L,M'}_{(21)} \quad \text{for } L = 0, 1, 2, \ldots.
\]

We can compare this to the eigenvalue equation (2.4) of the Landau problem in chapter 2
\[
H \left| \chi_{n_r,n_l} \right> = \hbar \omega_c \left( n_r + \frac{1}{2} \right) \left| \chi_{n_r,n_l} \right> \quad \text{for } q < 0 \quad \text{with } n_l = 0, 1, 2, \ldots
\]
\[
H \left| \chi_{n_r,n_l} \right> = \hbar \omega_c \left( n_l + \frac{1}{2} \right) \left| \chi_{n_r,n_l} \right> \quad \text{for } q > 0 \quad \text{with } n_r = 0, 1, 2, \ldots
\]

Note that we have both charged sectors $q = \pm 1$ realized at the same time, by the $\Upsilon_{(12)}$ and $\Upsilon_{(21)}$ respectively. Therefore we can identify
\[
M \equiv n_r
\]
\[
M' \equiv n_l
\]
and
\[
\frac{N}{\mu r^2} = \hbar \omega_c.
\]
Recall that we are using Planck units $\hbar, c = 1$, and the coupling constant is set to $q = \pm 1$. Thus using the definition of $\omega_c$ from (2.3) and $k \simeq \frac{2\pi r}{N}$ from (3.2) for large $N$, we obtain

$$\frac{2}{k} = B$$

in the semi-classical limit. This is indeed precisely the field strength acting on strings $\Upsilon_{12}$ connecting the upper to the lower sheet near the poles as we have seen in (4.23). Therefore we have found complete agreement between the Landau problem and the string states on the squashed fuzzy sphere in the planar limit.

To illustrate this, we display in figure 3 and 4 the eigenfunctions of $\frac{1}{2\mu} \Box_S$ for small and large $N$, and match these with the eigenstates of the Landau problem. The lowest Landau level is given by the wave functions $\chi_{0,n}$, where $n$ is either $n_l$ or $n_r$ depending on the sign of the $\vec{B}$-field. The lowest level of $\frac{1}{2\mu} \Box_S$ is given by the $\hat{Y}_{l_{max}}$ or $\hat{Y}_{l_{max}}$ for very large $l$. Thus the $\chi_{n_r,0}$ should be identified with $\hat{Y}_{l_{max}}$, and the $\chi_{0,n_l}$ with $\hat{Y}_{l_{max}}$. Note that these are the highest and lowest weight states in the algebra of functions $Mat(N, \mathbb{C})$. In particular, $\chi_{n_r=0, n_l=0}$ can be identified with both $\hat{Y}_{l_{max}}$ or $\hat{Y}_{l_{max}}$. The reason for this doubling is that we have both charged sectors $q = \pm 1$ realized at the same time, by the $\Upsilon_{12}$ and $\Upsilon_{21}$ respectively. We can thus identify the states in the various Landau levels as

$$\chi_{0,n_l} \leftrightarrow \hat{Y}_{l_{max}-n_l} = \Upsilon_{n_l,0}^{(12)},$$

$$\chi_{1,n_l} \leftrightarrow \hat{Y}_{l_{max}-n_l-1} = \Upsilon_{n_l,1}^{(12)},$$

$$\chi_{2,n_l} \leftrightarrow \hat{Y}_{l_{max}-n_l-2} = \Upsilon_{n_l,2}^{(12)}$$

(4.26)

and so forth. Similarly for the opposite charges,

$$\chi_{n_r,0} \leftrightarrow \hat{Y}_{l_{max}-n_r-(l_{max}-n_r)} = \Upsilon_{n_r,0}^{(21)},$$

$$\chi_{n_r,1} \leftrightarrow \hat{Y}_{l_{max}-n_r-(l_{max}-n_r)+1} = \Upsilon_{n_r,1}^{(21)}$$

(4.27)
Figure 4: Levels $M = 0, \ldots, 10$ for large $l$, if $N$ is large. For large $N$ these levels are approximately constant over this interval of $l$. Notice that the lowest level $M = 0$ is not 0, but has an offset, compatible with the result in equation (4.25).

and so forth. The quantum number labeling the degenerate states in a Landau level can be identified using the operator $J_z^{(ad)} = \kappa^{-1}[X_3, ]$, which corresponds to angular momentum around the $z$ axis on the fuzzy sphere. We can compute its eigenvalue either directly from the $SU(2)$ quantum numbers

$$J_z^{(ad)}\Upsilon_{n l, 0}^{(12)} = (N - 1 - n_l), \quad J_z^{(ad)}\Upsilon_{n r, 0}^{(21)} = -(N - 1 - n_r)$$

or in the semi-classical limit $N \to \infty$ as follows

$$J_z^{(ad)}\Upsilon_{n l, 0}^{(12)} = \kappa^{-1}(X_3 \Upsilon_{(12)} - \Upsilon_{(12)} X_3)$$

$$= \sqrt{C_N} \left(1 - \frac{1}{2 \rho^2}((x^1)^2 + (x^2)^2) + O\left(\frac{x^4}{\rho^4}\right)\right) \Upsilon_{(12)}$$

$$- \sqrt{C_N} \Upsilon_{(12)} \left(1 + \frac{1}{2 \rho^2}((X^1)^2 + (X^2)^2) + O\left(\frac{x^4}{\rho^4}\right)\right)$$

$$\sim \frac{N}{2}\left(2 - \frac{1}{\rho^2}((x^1)^2 + (x^2)^2) + O\left(\frac{x^4}{\rho^4}\right)\right) \Upsilon_{(12)}.$$ (4.29)

Neglecting the $O(\frac{x^4}{\rho^4})$ terms and taking into account the above identifications, we find

$$((x^1)^2 + (x^2)^2) \Upsilon_{n l, 0}^{(12)} = \frac{r^2}{\sqrt{C_N}} (n_l + 1) \Upsilon_{n l, 0}^{(12)} = \frac{2}{B} (n_l + 1) \Upsilon_{n r, 0}^{(12)}.$$ (4.30)

Hence these states are localized on circles around the origin with radius measured by $n_l$, just like the states $\chi_{0,n_l}$ in the Landau problem (2.5). The analogous statement for $\chi_{n_r, 0}$ completes the identification of the harmonics on the squashed fuzzy sphere with those of the Landau problem.
Figure 5: The strings $\Upsilon_{12}$ and $\Upsilon_{21}$ should be thought of as connecting the upper and lower sheet, creating transitions from one to the other. The solid arrows create transitions from the upper to the lower sheet and should be identified with $\Upsilon_{12}$, while the dashed with $\Upsilon_{21}$. Strings creating transitions from pole to pole will have very large quantum number $|m| \approx l_{\text{max}}$ in terms of $\hat{Y}_{l_{\text{max}} \pm l_{\text{max}}}$. Finally we can exhibit the stringy interpretation of these matrix states as links between the sheets. The states $\hat{Y}^{l_{\text{max}}}_{l_{\text{max}}}$ or $\hat{Y}^{l_{\text{max}}}_{-l_{\text{max}}}$ can be written explicitly as follows

$$\Upsilon_{0,0}^{(12)} = \left| \frac{N - 1}{2} \right\rangle \langle - \frac{N - 1}{2} \right|$$

$$\Upsilon_{0,0}^{(21)} = \left| - \frac{N - 1}{2} \right\rangle \langle N - 1 \right| .$$

Here the extremal weight states $|\pm \frac{N - 1}{2}\rangle$ are the coherent states localized at the north and south pole of the fuzzy sphere, hence at the origin of the two fuzzy disks. This makes the interpretation of the $\Upsilon_{l_{\text{max}}}^{0,0}$ as strings connecting the two sheets at the origin manifest, and vindicates the identification with $\chi_{0,0}$ (4.26), (4.27). Although the expressions of the other states in terms of coherent states is more complicated, it is clear that they can be thought of as slightly extended strings localized at circles around the origin. This is illustrated in figure 5.

### 4.5 Dirac operator

For completeness, we briefly discuss also the Dirac operator on the squashed fuzzy sphere. $\hat{D}$ is naturally defined by

$$\hat{D} = \frac{1}{k} \left( \sigma_1 \otimes [X^1, \cdot] + \sigma_2 \otimes [X^2, \cdot] \right)$$

with $X^1, X^2$ from (4.16). We can compute its square

$$\hat{D}^2 = \frac{1}{k^2} \sigma_1 \sigma_j \otimes [X^i, [X^j, \cdot]]$$

$$= \Box S \otimes \mathbb{1}_2 - \frac{1}{k^2} [X^3, \cdot] \otimes \sigma_3$$

$$= \Box \otimes \mathbb{1}_2 - \frac{1}{k^2} \left( [X^3, \cdot] + \frac{1}{2} \sigma_3 \right)^2 + \frac{1}{4k^2}$$

(4.33)
using the commutation relations of the fuzzy sphere. The last form allows to find immediately the eigenvalues and eigenfunctions following [18]: Decomposing the space of functions Mat\((N, \mathbb{C}) = \bigoplus_{l=0}^{N-1} \mathbb{C}^{2l+1}\) with basis \(|l, m_l\rangle\) and passing to the total angular momentum basis of \(\mathbb{C}^2 \otimes \mathbb{C}^{2l+1}\) labeled by \(j, l, m_j\), the eigenvalues are

\[
E_{jlm_j}^2 = 4l(l+1) - 4m_j^2 + 1 = 4(l + \frac{1}{2} - m_j)(l + \frac{1}{2} + m_j)
\]

(4.34)

where \(m_j = m + s\), and \(s\) is the eigenvalue of \(\frac{1}{2}\sigma_3\). Hence for each \(l \in \{0, 1, 2, ..., N-1\}\), there is pair of zero modes with extremal weights \(m_j = \pm(l + \frac{1}{2})\), which can be written as

\[
\Psi_+ = |\uparrow\rangle|l, l\rangle, \quad \Psi_- = |\downarrow\rangle|l, -l\rangle.
\]

(4.35)

Thus the fermionic zero modes correspond to the extremal weight states in the angular momentum decomposition of Mat\((N, \mathbb{C})\). In particular for \(L = 0\) or \(l = l_{\text{max}}\), these can again be written in terms of coherent states as in (4.31). More generally, these zero modes can be interpreted as fermionic strings, linking the two opposite sheets at or near the origin.

On the other hand, the second form in (4.33) allows to easily take the semi-classical (pole) limit as in the previous section. Using the analogous procedure as for the Laplacian before, we obtain

\[
\frac{1}{2\mu} D^2 \Upsilon^{LM}_{(12)} = \hbar \omega_c \left( M + \frac{1}{2} - \frac{\sigma_3}{2} \right) \Upsilon^{LM}_{(12)}
\]

in the large \(N\) limit using \(\frac{2N}{r^2} = \hbar \omega_c\) for (4.26), and similarly for the \(\Upsilon^{LM}_{(21)}\). Thus for these string states, \(\frac{1}{2\mu} D^2\) reproduces the Hamiltonian of the Landau levels including spin (2.6), for \(g = 1\). In particular, we can understand the above fermionic zero modes as fermions in the lowest Landau level with appropriate orientation of the spin. Remarkably, they are exact zero modes even for finite \(N\). For generalizations we refer the reader to [18].

5 Conclusion

We have identified string-like modes among the fuzzy spherical harmonics, which connect the upper with the lower hemisphere. On the squashed fuzzy sphere, these behave like charged objects moving under the influence of a magnetic field. In the large \(N\) limit, this field becomes approximately constant in the vicinity of the origin resp. the north and south poles, and the lowest string-like modes behave like charged point-like objects. In particular, we have identified the lowest Landau levels among these fuzzy spherical harmonics, providing an organization of the space of functions in terms of string-like modes.

Our results illustrate the well-known fact that noncommutative field theory is much richer than ordinary gauge theory, and behaves more like a string theory rather than a field theory. The present example provides a particularly clear identification of such string modes in a non-trivial background, in a simple finite-dimensional setting. It illustrates how non-trivial backgrounds in matrix models can be understood quantitatively in the semi-classical limit. The present example is related to the new solutions of (deformed) \(N = 4\) SYM and the IKKT matrix model [18, 24], which could be analyzed in a similar way. More generally, a systematic use of analogous string-like modes in the study of noncommutative field theory might help to illuminate various issues and problems in this context.
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