Twisted Alexander modules of hyperplane arrangement complements

Eva Elduque

Received: 13 May 2020 / Accepted: 16 February 2021 / Published online: 26 February 2021
© The Royal Academy of Sciences, Madrid 2021

Abstract
We study torsion properties of the twisted Alexander modules of the affine complement $M$ of a complex essential hyperplane arrangement, as well as those of punctured stratified tubular neighborhoods of complex essential hyperplane arrangements. We investigate divisibility properties between the twisted Alexander polynomials of the two spaces, compute the (first) twisted Alexander polynomial of a punctured stratified tubular neighborhood of an essential line arrangement, and study the possible roots of the twisted Alexander polynomials of both the complement and the punctured stratified tubular neighborhood of an essential hyperplane arrangement in higher dimensions. We apply our results to distinguish non-homeomorphic homotopy equivalent arrangement complements. We also relate the twisted Alexander polynomials of $M$ with the corresponding twisted homology jump loci.

Keywords Hyperplane arrangements · Twisted Alexander polynomials · Homology jump loci

Mathematics Subject Classification 32S22 · 32S20 · 14J17

1 Introduction
The twisted Alexander polynomial was first used to study plane algebraic curves by Cogolludo and Florens in [2]. In their paper, they refine Libgober’s divisibility results regarding the classical Alexander polynomial ([15]), and use the twisted Alexander polynomials to distinguish Zariski pairs (pairs of plane curves with homeomorphic tubular neighborhoods but non-homeomorphic complements) that the classical Alexander polynomial cannot distinguish.

Cohen and Suciu study the multivariable twisted Alexander polynomials of the boundary manifold of a line arrangement in [3], and use the non-twisted version to obtain a complete description of the first characteristic variety of the fundamental group of the boundary manifold. Hironaka [10] and Florens-Guerville-Marco [8] have studied relationships between the
topology of a line arrangement complement and that of the boundary manifold of such an arrangement.

In [17], Maxim and Wong investigated torsion properties for the twisted Alexander modules of the affine complements of complex hypersurfaces in general position at infinity. They did so by using the complement of the link at infinity, which fibers over a circle and “dominates” the hypersurface complement in the sense that the homotopy type of the hypersurface complement can be obtained from it by adding cells of dimension greater or equal than the middle dimension. They were also able to describe a polynomial such that the roots of the (one-variable) twisted Alexander polynomials of the hypersurface complements were roots of it. This polynomial came from studying the twisted Alexander modules of the complement of the link at infinity.

Kohno and Pajitnov showed in [13] that complex essential hyperplane arrangements also had a similar structure. Hyperplane arrangements are not necessarily in general position at infinity, but there is a different space $V$ that plays a similar role as the one that the complement of the link at infinity plays in the case of hypersurfaces in general position at infinity studied by Maxim and Wong. This space is the boundary of a certain neighborhood of the arrangement, it fibers over a circle, and “dominates” the arrangement complement in the sense that the homotopy type of the arrangement complement can be obtained from it by adding cells of the middle dimension.

In this paper, we follow Maxim and Wong’s approach of using a “dominating” space to study the torsion properties for the twisted Alexander modules in the case of complements of complex essential hyperplane arrangements, which are not necessarily in general position at infinity. We do so by using the structure proved by Kohno and Pajitnov. We investigate divisibility properties between the twisted Alexander polynomials of arrangement complements and those of punctured tubular neighborhoods of arrangements, compute the (first) twisted Alexander polynomial of a punctured stratified tubular neighborhood of an essential line arrangement, and study the possible roots of the twisted Alexander polynomials of both the complement and the punctured stratified tubular neighborhood of an essential hyperplane arrangement in higher dimensions. To be able to do so, we define and study the topology of a punctured stratified tubular neighborhood $W^*$ of the arrangement. In particular, we prove that $W^*$ shares useful “dominating” properties with Kohno and Pajitnov’s $V$, but, unlike $V$, it is well suited for our computations due to the stratified nature of its definition.

In the last section (Sect. 5) we give two applications of our results. The first application (Example 1) uses twisted Alexander polynomials to distinguish two non-homeomorphic homotopy equivalent line arrangement complements. The result in Example 1 was first proved by Jiang and Yau in [11] as a corollary of their powerful result that states that homeomorphic complex projective line arrangement complements have isomorphic intersection posets, and later reproved by Cohen and Suciu in [3] using multivariable Alexander polynomials. The interesting part about our proof given in Example 1 is that it does not rely on the heavy machinery of Jiang and Yau and it uses an a priori easier invariant than Cohen and Suciu, since multivariable Alexander polynomials are in principle harder to compute than univariable (twisted) ones due to the fact that they do not live in a PID. The second application relates the zeros of twisted Alexander polynomials to the twisted homology jump loci of rank one $C$-local systems.

The rest of this section is structured as follows. In Sect. 1.1 we introduce important notation that will be used throughout the paper regarding the topology of hyperplane arrangement complements. In Sect. 1.2 we recall the definition of twisted Alexander modules and polynomials. Once the main objects of study have been introduced, we give an overview of the
main results in the paper in Sect. 1.3, describing the structure of Sects. 2, 3 and 4. Finally, in Sect. 1.4, we recall some facts about the Reidemeister torsion that will be used in Sect. 4.1.

1.1 Setup

Let $H_j$ be a complex hyperplane in $\mathbb{C}^n$ given by the zero locus of an affine linear map $\xi_j : \mathbb{C}^n \to \mathbb{C}$, where $j = 1, \ldots, m$.

**Definition 1** The hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_m\}$ is called **essential** if the maximal codimension of a non-empty intersection of a subfamily of $\mathcal{A}$ is $n$.

Let $\{H_1, \ldots, H_m\}$ be an essential hyperplane arrangement, let $H = \bigcup_{j=1}^{m} H_j$ be the union of the hyperplanes, and let $M = \mathbb{C}^n \setminus H$ be its complement in $\mathbb{C}^n$.

**Remark 1** Every hyperplane arrangement complement is homotopy equivalent to the complement of an essential one in an affine space of less or equal dimension ([13, Proposition 6.1]), so we do not lose information by restricting ourselves to the study of essential hyperplane arrangements.

Now, we need to identify and name certain loops in $\pi_1(M)$ that will be used throughout the paper. For a complete algorithm describing a presentation of $\pi_1(M)$, we refer the reader to [1]. It is a well known fact that $\pi_1(M)$ is generated by a choice of meridians $a_j$ around each hyperplane $H_j$, for $j = 1, \ldots, m$. These meridians $a_1, \ldots, a_m$ have a canonical (positive) orientation induced by the complex structure.

For the rest of Sect. 1.1, we will deal with the case where $\mathcal{A}$ is a line arrangement (that is, $n = 2$). Let $P_1, \ldots, P_s$ be the singular points of $H$, and let $d_k$ be the number of lines in $\mathcal{A}$ passing through $P_k$.

**Definition 2** We denote by $M_k$ the local complement

$$M_k := M \cap \mathbb{B}_{k}^4$$

where $\mathbb{B}_{k}^4$ is a small enough 4-ball in $\mathbb{C}^2$ centered at the point $P_k$, for $k = 1, \ldots, s$.

Note that $M_k$ is homotopy equivalent to $M \cap S_k^3$, where $S_k^3$ is the boundary of $\mathbb{B}_k^4$. In fact, $M \cap S_k^3 = S_k^3 \setminus L_k$, where $L_k$ is a Hopf link with $d_k$ components. Also $M_k$ is naturally homeomorphic to a central line arrangement complement $U_k \subset \mathbb{C}^2$ consisting on $d_k$ distinct lines passing through the origin.

**Definition 3** We denote by $\beta_k$ the loop in $\pi_1(M_k)$ corresponding via the homeomorphism described above to a meridian about the line at infinity with **negative** orientation in $U_k$.

**Remark 2** Two meridians about the same line with the same orientation are not necessarily the same elements in $\pi_1(M)$, but they are conjugate to one another. The above definition of $\beta_k$ for all $k = 1, \ldots, s$ is well defined only up to conjugation in $\pi_1(M_k)$, but this will suffice for our purposes. Abusing notation and disregarding base points, we will look at the $\beta_k$’s inside of $\pi_1(M)$ via the maps $\pi_1(M_k) \longrightarrow \pi_1(M)$ induced by inclusion.
Remark 3 $\beta_k$ can be taken to be the composition of $d_k$ loops $\gamma_1 \cdot \ldots \cdot \gamma_{d_k}$, where each one of these loops is a certain positively oriented meridian about each of the $d_k$ lines in $A$ going through $P_k$. If the reader wishes to know what line of $A$ corresponds to each of the $\gamma$'s in a given example, they can do so using Arvola's presentation for $\pi_1(\mathcal{M})$ ([1]). For the purposes of this paper, we just need to know that a presentation for the fundamental group of $\mathcal{M}_k$ (and $S^3_k \setminus \mathcal{A}_k$) is given by

$$\langle \beta_k, \gamma_1, \ldots, \gamma_{d_k-1} | [\beta_k, \gamma_l] \quad \text{for} \quad l = 1, \ldots, d_k - 1 \rangle \tag{1}$$

(see [17, Lemma 2.7]).

1.2 General construction of Alexander modules and polynomials

Let $\mathbb{F}$ be a field, and let $V$ be a finite dimensional $\mathbb{F}$-vector space. Let $X$ be a path-connected finite CW complex, let $\rho : \pi_1(X) \to \text{GL}(V)$ be a linear representation, and let $\varepsilon : \pi_1(X) \to \mathbb{Z}$ be a group homomorphism. Together, $\rho$ and $\varepsilon$ define the homological twisted Alexander modules $H_i^{\varepsilon, \rho}(X, \mathbb{F}[t^{\pm 1}])$, as in [17, Section 2.1].

Definition 4 The $i$-th (homological) twisted Alexander module $H_i^{\varepsilon, \rho}(X, \mathbb{F}[t^{\pm 1}])$ of $(X, \varepsilon, \rho)$ is the $i$-th homology of the complex of $\mathbb{F}[t^{\pm 1}]$-modules

$$C_i^{\varepsilon, \rho}(X, \mathbb{F}[t^{\pm 1}]) := (\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} V) \otimes_{\mathbb{F}[\pi_1(X)]} \mathcal{C}_+(\tilde{X}, \mathbb{F}).$$

Here, $\mathcal{C}_+(\tilde{X}, \mathbb{F})$ is the cellular homology complex of the universal cover $\tilde{X}$ of $X$, seen as a free left $\mathbb{F}[\pi_1(X)]$-module via the action given by deck transformations. We regard $\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} V$ as a right $\mathbb{F}[\pi_1(X)]$-module, with the right action given by

$$(p(t) \otimes v) \cdot \alpha = (p(t) \cdot t^{\varepsilon(\alpha)}) \otimes (v \cdot \rho(\alpha))$$

for every $p(t) \in \mathbb{F}[t^{\pm 1}]$, $v \in V$ and $\alpha \in \pi_1(X)$, where $v$ is regarded as a row vector and $\rho(\alpha)$ as a square matrix.

Together, $\varepsilon$ and $\rho$ define a tensor representation

$$\varepsilon \otimes \rho : \pi_1(X) \to \text{Aut}_{\mathbb{F}[t^{\pm 1}]}(\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} V)$$

$$\alpha \mapsto p(t) \otimes v \mapsto (p(t) \cdot t^{\varepsilon(\alpha)}) \otimes (v \cdot \rho(\alpha))$$

which gives rise to a local system of $\mathbb{F}[t^{\pm 1}]$ modules $\mathcal{L}_{\varepsilon, \rho}$.

Remark 4 There is an $\mathbb{F}[t^{\pm 1}]$-module isomorphism $H_i^{\varepsilon, \rho}(X, \mathbb{F}[t^{\pm 1}]) \cong H_i(X, \mathcal{L}_{\varepsilon, \rho})$ (see [17, Section 4.4]). We will use both the chain complex definition and properties of homology with local systems when it is most convenient.

Since $X$ is a finite CW-complex and $V$ is finite dimensional over $\mathbb{F}$, $\mathcal{C}_+(\tilde{X}, \mathbb{F})$ is a complex of finitely generated free left $\mathbb{F}[\pi_1(X)]$-modules. Thus, the twisted (homological) Alexander modules are finitely generated $\mathbb{F}[t^{\pm 1}]$-modules over the principal ideal domain $\mathbb{F}[t^{\pm 1}]$, and therefore have a direct sum decomposition into cyclic modules.

Definition 5 The $i$-th (homological) twisted Alexander polynomial of $(X, \varepsilon, \rho)$ is defined as the order of the torsion part of the $i$-th twisted Alexander module $H_i^{\varepsilon, \rho}(X, \mathbb{F}[t^{\pm 1}])$. We denote this polynomial by $\Delta_i^{\varepsilon, \rho}(X)$, and it is an element in $\mathbb{F}[t^{\pm 1}]$ that is well defined up to multiplication by a unit of $\mathbb{F}[t^{\pm 1}]$. If $H_i^{\varepsilon, \rho}(X, \mathbb{F}[t^{\pm 1}])$ is free, then $\Delta_i^{\varepsilon, \rho}(X) = 1$ by convention.

Equivalently, $\Delta_i^{\varepsilon, \rho}(X)$ can be defined as a generator of the first non-zero Fitting ideal of the $\mathbb{F}[t^{\pm 1}]$-module $H_i^{\varepsilon, \rho}(X, \mathbb{F}[t^{\pm 1}])$. 
Let $\varepsilon : \pi_1(M) \to \mathbb{Z}$ be a fixed group homomorphism. Throughout this paper, we will assume that $\varepsilon$ is an epimorphism. As we already pointed out, $\pi_1(M)$ is generated by a choice of positively oriented meridians $a_j$ around each hyperplane $H_j$. In fact, $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^m$ is the free abelian group generated by the classes of those meridians. Hence, $\varepsilon$ is completely determined by the value it takes in those oriented meridians. We will denote by $\varepsilon_j := \varepsilon(a_j)$ for all $j = 1, \ldots, m$. In this paper, we will study the twisted Alexander modules $H^{\varepsilon, \rho}_i(M, \mathbb{F}[t^{\pm 1}])$, and unless stated otherwise, we require $\varepsilon$ to be a positive epimorphism, that is, $\varepsilon_j > 0$ for all $1 \leq j \leq m$. The “positive” condition (also required in [17]) provides a natural general framework in which the results of Sect. 3 (regarding torsion properties of twisted Alexander modules of hyperplane arrangements) and Sect. 4 (regarding divisibility results) hold true, and we should remark that those results do not hold true for all epimorphisms $\varepsilon$. In Remarks 7, 9 and 11 we specify where this “positive” condition is used and how it can be relaxed. Note that, as we can see in Remark 11, the set of non-positive $\varepsilon$ for which our methods in Sects. 3 and 4 are still valid depend on the arrangement complement.

Throughout the paper, we will use the following notation.

**Notation 1** Let $\gamma \in \pi_1(M)$. We denote by $\det_{\varepsilon, \rho}(\gamma)$ the determinant $\det(t^{\varepsilon(\gamma)} \rho(\gamma) - \text{Id}) \in \mathbb{F}[t^{\pm 1}]$.

### 1.3 Overview of the main results

In this paper, we study the twisted Alexander modules and the twisted Alexander polynomials of both $M$ and a punctured stratified tubular neighborhood $W^*$ of $H$, as defined explicitly in Definition 7. The motivation behind studying $W^*$ and its relationship with $M$ comes from the fact that, roughly speaking, $W^*$ is constructed by gluing tubular neighborhoods of the different strata of the arrangement and then removing the arrangement, so it is well suited for Mayer-Vietoris type computations.

In Sect. 2, we start by recalling a result from [13] (Theorem 1 in this paper) involving a space $V$ which is the boundary of a certain neighborhood of the arrangement $H$ and fibers over a circle. We show that $V$, $W^*$ and $M$ are related through inclusions $V \hookrightarrow W^* \hookrightarrow M$, and using the relationship between the topologies of $V$ and $M$ given by Theorem 1, we are able to relate the topologies of $M$ and $W^*$, as stated in the main result in that section, namely Theorem 2.

**Theorem 2** $M$ has the homotopy type of $W^*$ with cells of dimension $\geq n$ attached.

In Sect. 3 we study the torsion properties of the twisted Alexander modules of both $M$ and $W^*$, using the space $V$ and the fibration structure we know by Theorem 1 to do so. The fact that $V$ fibers over a circle allows us to show that the twisted Alexander modules of $V$ are torsion (see Theorem 3). The main result in this section is Theorem 4, which is the hyperplane arrangement extension of [17, Theorem 4.1, Corollary 4.4]. Note that [17, Theorem 4.1, Corollary 4.4] applies to Alexander modules of complements of hypersurfaces in general position at infinity, and hyperplane arrangements are not necessarily in general position at infinity.

**Theorem 4** The twisted Alexander modules $H^{\varepsilon, \rho}_i(M, \mathbb{F}[t^{\pm 1}])$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules for every $0 \leq i \leq n - 1$, they are trivial modules for $i > n$, and $H^{\varepsilon, \rho}_n(M, \mathbb{F}[t^{\pm 1}])$ is a free $\mathbb{F}[t^{\pm 1}]$-module of rank $(-1)^n \cdot \dim_{\mathbb{F}}(\mathbb{N}) \cdot \chi(M)$. 

Springer
The proof of this last result uses the relationship between the topologies of \(V, W^*\) and \(M\) proved in Sect. 2 to translate the torsion properties of the twisted Alexander modules of \(V\) to those of \(M\) and \(W^*\). In the proof of this last result, and making use of Theorem 2, we will also arrive at Corollary 2, which gives us a divisibility result.

Corollary 2 \(H^{ε, ρ}_i (M, \mathbb{F}[t^{±1}])\) and \(H^{ε, ρ}_i (W^*, \mathbb{F}[t^{±1}])\) are torsion \(\mathbb{F}[t^{±1}]\)-modules for any \(0 ≤ i ≤ n − 1\). Moreover, their twisted Alexander polynomials \(Δ^{ε, ρ}_i (M)\) and \(Δ^{ε, ρ}_i (W^*)\) coincide for \(0 ≤ i < n − 1\), and \(Δ^{ε, ρ}_n (M)\) divides \(Δ^{ε, ρ}_n (W^*)\).

Finally, in Sect. 4 we study the twisted Alexander polynomials of \(W^*\), which also give us information about the twisted Alexander polynomials of \(M\) by the divisibility result of Sect. 3, namely Corollary 2. This is easier to do in the case of line arrangements (Sect. 4.1), where we are able to find an explicit formula for \(Δ^{ε, ρ}_1 (W^*)\) as in Theorem 5 below. We recall first the notation used in this statement, which we defined in Sect. 1.1. \(a_i\) is a positively oriented meridian around the line \(H_i\) for all \(i = 1, \ldots, m\). If \(d_k\) is the number of lines in the arrangement \(A\) containing a singular point \(P_k\), then \(β_k\) is the composition of \(d_k\) positively oriented meridians, one for each of the lines of \(A\) containing \(P_k\), in a specific order (see Definition 3 and Remark 3). Also, \(\det_{ε, ρ}(γ) := \det(ε(γ) ρ(γ) − \text{Id})\) (Notation 1).

**Theorem 5** Let \(A = \{H_1, \ldots, H_m\} \subset \mathbb{C}^2\) be an essential line arrangement, \(H = \bigcup_{i=1}^{m} H_i, M = \mathbb{C}^2\setminus H\) and let \(W^*\) be a punctured stratified tubular neighborhood of \(H\). Let \(P_1, \ldots, P_s\) be the singular points of \(H\), let \(s_i\) be the number of singular points of \(H\) on \(H_i\), and let \(d_k\) be the number of lines of \(A\) going through the singular point \(P_k\). Then, we have

1. \(Δ^{ε, ρ}_1 (W^*) = \left(\prod_{k=1}^{s} \det_{ε, ρ}(β_k)^{d_k−2}\right) \cdot \left(\prod_{i=1}^{m} \det_{ε, ρ}(a_i)^{s_i−1}\right) \cdot Δ^{ε, ρ}_0 (M)\).
2. \(Δ^{ε, ρ}_1 (M)\) divides \(\left(\prod_{k=1}^{s} \det_{ε, ρ}(β_k)^{d_k−2}\right) \cdot \left(\prod_{i=1}^{m} \det_{ε, ρ}(a_i)^{s_i−1}\right) \cdot \gcd_{i=1,\ldots,m} \{\det_{ε, ρ}(a_i)\}\).

Note that Corollary 2 and Theorem 5 refine the divisibility result obtained in [2, Theorem 1.1] (where \(ρ\) is further assumed to be unitary), since the polynomial which \(Δ^{ε, ρ}_1 (M)\) divides in [2, Theorem 1.1] also contains contributions from the link at infinity. In some cases, we will be able to further refine the bound for \(Δ^{ε, ρ}_1 (M)\) given by Theorem 5, part 2, as shown in the other main result of Sect. 4.1, namely Theorem 6. The notation \(β_{k,i}\) in the statement of the following theorem has the same meaning as \(β_k\) in Theorem 5, switching \(P_k\) for \(P_k^i\) and \(d_k\) for \(d_k^i\).

**Theorem 6** Let \(B\) be the subset of lines of an essential line arrangement \(A = \{H_1, \ldots, H_m\}\) such that for each line in \(B\) no other line in \(A\) is parallel to it. Suppose that \(B = \{H_1, \ldots, H_l\} \neq \emptyset\). Let \(s_i\) be the number of singular points of the union of lines in \(A\) that are contained in \(H_i\), which we denote by \(P_1^i, \ldots, P_{s_i}^i\), and let \(d_k^i\) be the number of lines of \(A\) going through the singular point \(P_k^i\). Then, \(Δ^{ε, ρ}_1 (M)\) divides \(\left(\gcd_{r=1,\ldots,m} \{\det_{ε, ρ}(a_r)\}\right) \cdot \gcd_{i=1,\ldots,l} \left\{\prod_{k=1}^{s_i} \det_{ε, ρ}(β_{k,i})^{d_k^i−2}\right\} \cdot \det_{ε, ρ}(a_i)^{s_i−1}\).
In the higher dimensional case discussed in Sect. 4.2, we consider the natural stratification of \( H \) to obtain an open cover of \( W^* \). More precisely, let \( s_k \) be the number of \( k \)-dimensional strata in this stratification, for \( k = 0, \ldots, n - 1 \), and let
\[
\{ S^k_l \mid k = 0, \ldots, n - 1; l = 1, \ldots, s_k \}
\]
be an open cover of \( W^* \) such that each one of the open sets \( S^k_l \) fibers over the corresponding stratum \( \Sigma^k_l \) of dimension \( k \) of \( H \). Then, we use the Mayer-Vietoris cohomology spectral sequence for the twisted Alexander modules associated to this open cover to get a bound for the twisted Alexander polynomials \( \Delta^\epsilon,\rho_i(M) \), and arrive at the following result, which generalizes Theorem 5.

**Theorem 7** Let \( \mathcal{A} = \{H_1, \ldots, H_m\} \) be an essential hyperplane arrangement in \( \mathbb{C}^n \), with the natural induced stratification \( \{ \Sigma^k_l \mid k = 0, \ldots, n - 1; l = 1, \ldots, s_k \} \), and let \( M \) be the complement of that arrangement in \( \mathbb{C}^n \). For every \( k \) and \( l \), let \( F_l^k \), \( k \) be the fiber of the fibration \( S^k_l \to \Sigma^k_l \) and let \( \gamma_{\infty}(F_l^k) \) be a meridian around the hyperplane at infinity in \( \mathbb{C}P^{n-k} \) with positive orientation, where \( F_l^k \) is naturally seen in \( \mathbb{C}P^{n-k} \). Then, for any \( i = 0, \ldots, n - 1 \), the zeros of the \( i \)-th Alexander polynomial of \( M \) (i.e. \( \Delta^\epsilon,\rho_i(M) \)) are among those of
\[
\prod_{k=0}^{n-1} \prod_{l=1}^{s_k} \det_{\epsilon,\rho}(\gamma_{\infty}(F_l^k)).
\]

### 1.4 Reidemeister torsion

In Sect. 4.1, we will be using the **torsion** \( \tau(C_*) \) of a finite chain complex \( C_* \) of finite dimensional vector spaces over a field \( \mathbb{K} \), as defined in [19, Section 3] (but we use multiplicative notation instead of additive notation, unlike in [19]). The torsion \( \tau(C_*) \) is an element of \( \mathbb{K}^*/\{\pm 1\} \), and depends on a choice of bases for both the chain complex and its homology. In particular, if \( C_* \) is acyclic, then \( \tau(C_*) \) only depends on a choice of bases for \( C_* \). The actual definition of the torsion is not going to be relevant in this paper. The torsion behaves well with respect to short exact sequences, as illustrated in the following result.

**Lemma 1** ([19]) Let
\[
0 \to C' \to C \to C'' \to 0
\]
be a short exact sequence of based finite chain complexes of finite dimensional vector spaces, with compatible bases. Let \( \mathcal{H} \) be the associated long exact sequence in homology, viewed as a based acyclic complex, the bases being the fixed bases of the homology of \( C' \), \( C \), and \( C'' \). Then,
\[
\tau(C) = \tau(C') \tau(C'') \tau(\mathcal{H})
\]
where the torsion is taken with respect to the fixed bases.

Let \((X, \rho, \epsilon)\) be as in Definition 4. By tensoring \( C^{\epsilon,\rho}_*(X, \mathbb{F}[t^\pm 1]) \) with the field of rational functions \( \mathbb{F}(t) \), we construct a finite chain complex of based finite dimensional vector spaces over \( \mathbb{F}(t) \), which we call \( C^{\epsilon,\rho}_*(X, \mathbb{F}(t)) \).
Definition 6 ([12, Section 3]) We denote by \( \tau_{\varepsilon, \rho}(X) \) the twisted Reidemeister torsion of \((X, \varepsilon, \rho)\), which is defined as

\[
\tau_{\varepsilon, \rho}(X) = \tau(C^\varepsilon_\rho(X, \mathbb{F}(t))).
\]

In this definition we have not specified a choice of bases of \( C^\varepsilon_\rho(X, \mathbb{F}(t)) \), but we will only consider bases of the form \( b \otimes c_i \), where \( b \) is a basis of \( V \) as a vector space over \( \mathbb{F} \) and \( c_i \) is a “geometric” basis of \( C_i(\tilde{X}, \mathbb{F}) \) as a free left \( \mathbb{F}[\pi_1(X)] \)-module, that is, a basis obtained by lifting \( i \)-cells of \( X \) for all \( i \). We also have not specified a choice of bases of the homology of \( C^\varepsilon_\rho(X, \mathbb{F}(t)) \), but in this paper we will only deal with the torsion of acyclic complexes, so we will not need to. The following result explains the indeterminacy of the torsion of such complexes.

Lemma 2 ([12, Section 3]) Suppose that \( C^\varepsilon_\rho(X, \mathbb{F}(t)) \) is acyclic. Then, \( \tau_{\varepsilon, \rho}(X) \) is independent of the choice of bases up to multiplication by a unit of \( \mathbb{F}[t^{\pm 1}] \).

In light of this last result, we will always consider \( \tau_{\varepsilon, \rho}(X) \) to be an element of \( \mathbb{F}(t) \) up to multiplication by a unit of \( \mathbb{F}[t^{\pm 1}] \).

We end this section by stating the relation between the twisted Reidemeister torsion and the twisted Alexander polynomials.

Lemma 3 ([12, Theorem 3.4]) Suppose that \( C^\varepsilon_\rho(X, \mathbb{F}(t)) \) is acyclic, and let \( \tau_{\varepsilon, \rho}(X) \) be the twisted Reidemeister torsion of \((X, \varepsilon, \rho)\). Then,

\[
\tau_{\varepsilon, \rho}(X) = \frac{\prod_i \Delta_i^{\varepsilon_\rho}(X)}{\prod_i \Delta_i^{\varepsilon_\rho}(X)},
\]

up to multiplication by a unit of \( \mathbb{F}[t^{\pm 1}] \).

2 The homotopy type of \( M \)

We will study the topology of \( M \) with the help of two functions \((f_\varepsilon \text{ and } g_\varepsilon)\) defined from the fixed positive epimorphism \( \varepsilon : \pi_1(M) \to \mathbb{Z} \) as follows:

\[
f_\varepsilon : M \to \mathbb{R}, \quad z \mapsto \prod_{j=1}^m |\xi_j(z)|^{\varepsilon_j},
\]

\[
g_\varepsilon : M \to \mathbb{R}/2\pi\mathbb{Z} \cong S^1, \quad z \mapsto \arg \left( \prod_{j=1}^m \xi_j(z)^{\varepsilon_j} \right) = \sum_{j=1}^m \varepsilon_j \cdot \arg(\xi_j(z)).
\]

Let \( \delta > 0 \) small enough, let \( V := f_\varepsilon^{-1}(\delta) \). The following result can be found in [13, Theorem 2.3].

Theorem 1 For every \( \delta > 0 \) small enough, we have that

1. \( V \) is a smooth manifold of dimension \( 2n - 1 \).
2. The inclusion \( V \hookrightarrow f_\varepsilon^{-1}((0, \delta]) \) is a homotopy equivalence.
3. The map $g_{\varepsilon|V} : V \to S^1$ is a fiber bundle, and the fiber $F$ has the homotopy type of a finite CW-complex of dimension $n - 1$.

4. $M$ has the homotopy type of $V$ with $|\chi(M)|$ cells of dimension $n$ attached.

**Remark 5** Note that the space $V$ depends on both $\delta$ and the homomorphism $\varepsilon$.

Theorem 1 gives us some good properties of $f_{\varepsilon}^{-1}((0, \delta])$, which we will be using in Sect. 3. However, those properties alone will not be enough for us to compute possible roots of the twisted Alexander polynomials of $M$. The rest of this section is devoted to describe a different neighborhood of the arrangement with a nice stratification and prove some properties about it that will come in handy in Sect. 4.

We stratify our hyperplane arrangement in the natural way: two points $P$ and $P'$ in $H$ lie in the same stratum if the collections of hyperplanes in the arrangement containing $P$ and $P'$ coincide. Each stratum is a smooth submanifold of $\mathbb{C}^n$. We define a neighborhood $W$ of $H$ inductively as follows. Let $\Sigma_k$ the union of strata of dimension $k$ in $H$. For each stratum of dimension 0, we pick a ball of radius $\delta_0$ around it, and call $W(\delta_0)$ the union of those balls. Now, we take a tubular neighborhood of $\Sigma_1 \setminus W(\frac{\delta_0}{2})$ of radius $\delta_1 < \delta_0$, and define $W(\delta_0, \delta_1)$ as the union of $W(\delta_0)$ with this tubular neighborhood that we have just described. Now, we take a tubular neighborhood of $\Sigma_2 \setminus W(\frac{\delta_0}{2}, \frac{\delta_1}{2})$ of radius $\delta_2 < \delta_1$ and create $W(\delta_0, \delta_1, \delta_2)$. We proceed inductively until we reach $W := W(\delta_0, \ldots, \delta_{n-1})$.

Note that, when all of the $\delta$’s are small enough, all of these neighborhoods that we have defined are homeomorphic. From now on, we will assume that all of the $\delta$’s are small enough, and will not specify them.

**Definition 7** We call $W$ a stratified tubular neighborhood of $H$. Let $W^* = W \setminus H$. We call $W^*$ a punctured stratified tubular neighborhood of $H$.

**Remark 6** $W^*$ is homotopy equivalent to $\partial W$.

The following theorem relates the topologies of $W^*$ and $M$.

**Theorem 2** $M$ has the homotopy type of $W^*$ with cells of dimension $\geq n$ attached.

The proof of this theorem is an immediate consequence of the following proposition.

**Proposition 1** Let $j : W^* \hookrightarrow M$ be the inclusion. Then

1. $j_* : \pi_i(W^*) \to \pi_i(M)$ is an isomorphism for $i < n - 1$.

2. $j_* : \pi_{n-1}(W^*) \to \pi_{n-1}(M)$ is an epimorphism.

**Proof** The outline of the proof is going to be the following. First, we will find two stratified tubular neighborhoods $W$ and $W'$ of $H$ and a $\delta > 0$ such that $W' \subset f_{\varepsilon}^{-1}((0, \delta]) \subset W$. Then, we will get the result about $W^*$ from the information about $f_{\varepsilon}^{-1}((0, \delta])$ that we know from Theorem 1 and the fact that the inclusion $W' \setminus H \hookrightarrow W \setminus H$ is a homotopy equivalence.

Let us start with a stratified tubular neighborhood $W = W(\delta_0, \ldots, \delta_{n-1})$, and let $\delta'$ be the minimum of the $\delta_i$’s. We have that every point that is at distance less than $\delta'$ of $H$ is contained in $W$. Also note that the factor $|\xi_j(z)|^{f_j(z)}$ of $f_j(z)$ is proportional to a positive power of the distance of a point $z$ to the hyperplane $H_j = \{x \in \mathbb{C}^n | \xi_j(x) = 0\}$ for all $j = 1, \ldots, m$. Hence, for sufficiently small $\delta$, $f_{\varepsilon}^{-1}((0, \delta])$ will be contained in the set of points on $\mathbb{C}^n$ that
are at distance less than \( \delta' \) of \( H \), which is in turn contained in \( W \). Thus, we have found \( \delta \) such that \( f^{-1}_e([0, \delta]) \subset W \).

Let us find a stratified tubular neighborhood \( W' \) of \( H \) such that \( W' \subset f^{-1}_e([0, \delta]) \) to complete the first part of our outline of the proof. This \( W' \) is constructed by taking the union of tubular neighborhoods of open sets of the strata like in Definition 7, but not requiring those tubular neighborhoods to have a fixed radius. These “generalized” stratified tubular neighborhoods are still homotopy equivalent to the ones in Definition 7. It is straightforward to see that we can find one such \( W' \) inside of \( f^{-1}_e([0, \delta]) \). In particular, we have that \( W' \setminus H \subset f^{-1}_e((0, \delta]) \subset W \setminus H = W^* \).

Let us look at the following diagram, where all of the arrows are induced by inclusions.

\[
\pi_i(W' \setminus H) \xrightarrow{a_i} \pi_i(f^{-1}_e((0, \delta])) \xrightarrow{b_i} \pi_i(W^*) \xrightarrow{c_i} \pi_i(M)
\]  

Since the inclusion from \( W' \setminus H \) to \( W^* \) is a homotopy equivalence, we have that \( b_i \circ a_i \) is an isomorphism for all \( i \). In particular, \( b_i \) is an epimorphism for all \( i \). Also, by Theorem 1, parts 2 and 4, we have that \( c_i \circ b_i \) is an isomorphism if \( i < n - 1 \) and an epimorphism if \( i = n - 1 \). In particular, \( b_i \) is a monomorphism if \( i < n - 1 \), and \( c_i \) is an epimorphism for \( i \leq n - 1 \). This concludes the proof of the second assertion of the proposition.

Since we already know that \( b_i \) is an epimorphism for all \( i \) and a monomorphism if \( i < n - 1 \), we find that \( b_i \) is an isomorphism if \( i < n - 1 \). Since \( c_i \circ b_i \) is an isomorphism for \( i < n - 1 \), we get that \( c_i \) is an isomorphism for \( i < n - 1 \), and this concludes the proof of the first assertion of the proposition. \( \square \)

**Remark 7** The definition of the space \( V \) used in the proof of Proposition 1 depends on a positive epimorphism \( \varepsilon \). In fact, \( V \) is not the boundary of a neighborhood of the hyperplane arrangement if \( \varepsilon \) is not positive. However, Theorem 2 and Proposition 1 do not depend on a positive epimorphism \( \varepsilon \), as one can pick any positive epimorphism to construct an auxiliary \( V \) in the proof of Proposition 1.

## 3 Torsion properties of the twisted Alexander modules

From now on, we fix \( \delta > 0 \) small enough so that Theorem 1 holds. Let \( j : V \hookrightarrow M \) be the inclusion, and \( j_* : \pi_1(V) \to \pi_1(M) \) be the map it induces on fundamental groups. Abusing notation, we will also denote by \( \varepsilon \) and \( \rho \) the induced maps on \( \pi_1(V) \) that we get by composing \( j_* \) with \( \varepsilon \) and \( \rho \) respectively.

**Proposition 2** Let \( n \geq 2 \). The inclusion map \( j : V \hookrightarrow M \) induces isomorphisms of \( \mathbb{F}[t^{\pm 1}] \)-modules

\[
H^\varepsilon_\rho_i(V, \mathbb{F}[t^{\pm 1}]) \xrightarrow{\cong} H^\varepsilon_\rho_i(M, \mathbb{F}[t^{\pm 1}])
\]

for any \( i < n - 1 \), and an epimorphism of \( \mathbb{F}[t^{\pm 1}] \)-modules

\[
H^\varepsilon_\rho_{n-1}(V, \mathbb{F}[t^{\pm 1}]) \twoheadrightarrow H^\varepsilon_\rho_{n-1}(M, \mathbb{F}[t^{\pm 1}]).
\]

**Proof** We consider two cases: \( n > 2 \) and \( n = 2 \).

Suppose that \( n > 2 \). By Theorem 1, part 4, the homotopy type of the space \( M \) is obtained from \( V \) by attaching cells of dimension \( n \), so \( j_* \) is an isomorphism of fundamental groups. Since \( j_* : \pi_1(V) \to \pi_1(M) \) is an isomorphism, the universal cover of \( V \) is included in the
universal cover of $M$. Since $V$ and $M$ (up to homotopy equivalence) have the same skeleton up to dimension $n-1$ and the $n$-th skeleton of $V$ is contained in the $n$-th skeleton of $M$, the same statements hold for the skeletons of their universal covers. Hence, the chain complexes $C^\epsilon_\rho(V, \mathbb{F}[t^{\pm 1}])$ and $C^\epsilon_\rho(M, \mathbb{F}[t^{\pm 1}])$ coincide for $\star = 0, \ldots, n-1$, and $j$ induces an inclusion $C^\epsilon_\rho(V, \mathbb{F}[t^{\pm 1}]) \hookrightarrow C^\epsilon_\rho(M, \mathbb{F}[t^{\pm 1}])$. The result follows from this observation.

Now, let us consider the case $n = 2$. In this case, applying Theorem 1, part 4, only tells us that $j_\ast$ is an epimorphism between the fundamental groups. We have that ker $\epsilon \circ j_\ast$ is a normal subgroup of $\pi_1(V)$. Let $V_{\text{ker} j_\ast}$ be the covering space associated to ker $j_\ast$, and note that $\pi_1(V)/\ker j_\ast \cong \pi_1(M)$.

We construct the chain complex

$$D_\ast := (\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} V) \otimes_{\mathbb{F}[\pi_1(V)/\ker j_\ast]} C_\ast(V_{\text{ker} j_\ast}, \mathbb{F}).$$

The inclusion $V \hookrightarrow M$ induces a map $V_{\text{ker} j_\ast} \hookrightarrow \tilde{M}$, where $\tilde{M}$ is the universal cover of $M$. Since the homotopy type of the space $M$ is obtained from $V$ by attaching cells of dimension $\geq 2$, this map induces isomorphisms

$$D_i \cong C^\epsilon_\rho_i(M, \mathbb{F}[t^{\pm 1}]) = (\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} V) \otimes_{\mathbb{F}[\pi_1(M)]} C_i(\tilde{M}, \mathbb{F})$$

for $i = 0, 1$ and a monomorphism

$$D_2 \hookrightarrow C^\epsilon_\rho_2(M, \mathbb{F}[t^{\pm 1}]).$$

Thus, we have an isomorphism

$$H_0(D_\ast) \cong H^\epsilon_0(M, \mathbb{F}[t^{\pm 1}])$$

and an epimorphism

$$H_1(D_\ast) \rightarrow H^\epsilon_1(M, \mathbb{F}[t^{\pm 1}]).$$

By [5, Section 2.5, p. 50], the homology of $D_\ast$ is the same as the homology of $C^\epsilon_\rho(V, \mathbb{F}[t^{\pm 1}])$. The result follows from this observation.

**Remark 8** Using the discussion following diagram (2) in the proof of Proposition 1, and repeating the same steps in the proof of Proposition 2, we can conclude that the same results hold for the maps $H^\epsilon_\rho_i(V, \mathbb{F}[t^{\pm 1}]) \rightarrow H^\epsilon_\rho_i(W^*, \mathbb{F}[t^{\pm 1}])$ and $H^\epsilon_\rho_i(W^*, \mathbb{F}[t^{\pm 1}]) \rightarrow H^\epsilon_\rho_i(M, \mathbb{F}[t^{\pm 1}])$ induced by inclusion.

The following corollary is a direct consequence of Proposition 2 and Remark 8.

**Corollary 1** Let $n \geq 2$. For any $0 \leq i \leq n-1$, if $H^\epsilon_\rho_i(V, \mathbb{F}[t^{\pm 1}])$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module, then so are $H^\epsilon_\rho_i(M, \mathbb{F}[t^{\pm 1}])$ and $H^\epsilon_\rho_i(W^*, \mathbb{F}[t^{\pm 1}])$.

Now, we will show that the hypothesis of Corollary 1 is actually satisfied.

**Theorem 3** Let $n \geq 2$. Then, $H^\epsilon_\rho_i(V, \mathbb{F}[t^{\pm 1}])$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module for all $i \geq 0$.

**Proof** Note that $(g_\epsilon)_\ast = \epsilon$. Let $V^\epsilon \xrightarrow{P_1} V$ be the covering space induced by ker $\epsilon$. Recall that by Theorem 1, part 3, the map $(g_\epsilon)_{|V} : V \rightarrow S^1$ is a fiber bundle. We call the fiber $F$.

The covering space $V^\epsilon \xrightarrow{P_2} V$ is the pullback by $(g_\epsilon)_{|V}$ of the universal cover $\mathbb{R} \xrightarrow{P_2} S^1$, and we have the following commutative diagram of the pullback.
Note that $V^e \to \mathbb{R}$ is a fiber bundle over a contractible space with fiber $F$, so $V^e$ is homeomorphic to $F \times \mathbb{R}$, and therefore homotopically equivalent to $F$.

Let $L_\rho$ be the local system of $\mathbb{F}$-vector spaces given by the representation of $\pi_1(V^e)$ induced by $\rho$. By [12, Theorem 2.1], we have that

$$H_i^{e,\rho}(V, \mathbb{F}[t^{\pm1}]) \cong H_i(V^e, L_\rho)$$

as $\mathbb{F}[t^{\pm1}]$-modules for all $i \geq 0$. Since $V^e$ is homotopy equivalent to $F$, which by Theorem 1, part 3, has the homotopy type of a finite CW-complex, we have that the $H_i(V^e, L_\rho)$ are finite dimensional $\mathbb{F}$-vector spaces for all $i$, and thus $H_i^{e,\rho}(V, \mathbb{F}[t^{\pm1}])$ are torsion $\mathbb{F}[t^{\pm1}]$-modules for all $i$.

Let us recall the following fact, which can be found in [12].

**Proposition 3** Let $X$ be a finite CW-complex. If $\epsilon$ is non-trivial, then

$$H_0^{e,\rho}(X, \mathbb{F}[t^{\pm1}])$$

is a torsion $\mathbb{F}[t^{\pm1}]$-module.

Now, we are ready to prove the main result in this section.

**Theorem 4** The twisted Alexander modules $H_i^{e,\rho}(M, \mathbb{F}[t^{\pm1}])$ are torsion $\mathbb{F}[t^{\pm1}]$-modules for every $0 \leq i \leq n - 1$, they are trivial modules for $i > n$, and $H_n^{e,\rho}(M, \mathbb{F}[t^{\pm1}])$ is a free $\mathbb{F}[t^{\pm1}]$-module of rank $(-1)^n \cdot \dim_{\mathbb{F}}(V) \cdot \chi(M)$

**Proof** The space $M$ is an affine variety of complex dimension $n$, so it is homotopy equivalent to a finite CW complex of real dimension $n$ ([4,18]). Thus $H_i^{e,\rho}(M, \mathbb{F}[t^{\pm1}]) = 0$ for $i > n$. This also implies that $H_n^{e,\rho}(M, \mathbb{F}[t^{\pm1}])$ is a free module, since it is the kernel of a morphism of free $\mathbb{F}[t^{\pm1}]$-modules.

Now, let us prove that the twisted Alexander modules $H_i^{e,\rho}(M, \mathbb{F}[t^{\pm1}])$ are torsion $\mathbb{F}[t^{\pm1}]$-modules for every $0 \leq i \leq n - 1$. If $n = 1$, this is true by Proposition 3. Suppose that $n \geq 2$. In that case, by Corollary 1, we just need to show that $H_1^{e,\rho}(V, \mathbb{F}[t^{\pm1}])$ is a torsion $\mathbb{F}[t^{\pm1}]$-module for every $0 \leq i \leq n - 1$, which is true by Theorem 3.

Finally, let us compute the rank of $H_n^{e,\rho}(M, \mathbb{F}[t^{\pm1}])$. We abuse notation and call $L_{\epsilon,\rho}$ the local system of vector spaces over the field of rational functions $\mathbb{F}(t)$ defined by the tensor representation induced by $\epsilon$ and $\rho$ (instead of the local system of $\mathbb{F}[t^{\pm1}]$-modules induced by $\epsilon$ and $\rho$). By [5, Proposition 2.5.4], we have that

$$\text{rank}_{\mathbb{F}(t^{\pm1})} H_n^{e,\rho}(M, \mathbb{F}[t^{\pm1}]) = (-1)^n \chi(M, L_{\epsilon,\rho})$$

$$= (-1)^n \text{rank}_{\mathbb{F}(t^{\pm1})}(\mathbb{F}(t^{\pm1}) \otimes_{\mathbb{F}} V) \cdot \chi(M) = (-1)^n \dim_{\mathbb{F}}(V) \cdot \chi(M).$$

**Remark 9** (Relaxing the assumptions on $\epsilon$, I) The space $V$ depends on the positive epimorphism $\epsilon$, but $W^\rho$ does not. This dependence on $\epsilon$ came in handy in the proof of Theorem 4, although it can be proved that the Alexander modules $H_i^{e,\rho}(W^\rho, \mathbb{F}[t^{\pm1}])$ are torsion for all $i \geq 0$ directly when $\epsilon$ is positive, as we will see in Sect. 4.2. In fact, the result about $H_i^{e,\rho}(W^\rho, \mathbb{F}[t^{\pm1}]) \to H_i^{e,\rho}(M, \mathbb{F}[t^{\pm1}])$ from Remark 8 holds for any epimorphism $\epsilon$,

 Springer
not just positive ones, since it only relied on Theorem 2 (see also Remark 7). Hence, if $H_i^{ε,ρ}(W^*, \mathbb{F}[t^{±1}])$ is torsion for some $0 \leq i \leq n - 1$ for a specific non-positive epimorphism $ε$, so is $H_i^{ε,ρ}(M, \mathbb{F}[t^{±1}])$, and we also have the divisibility result of Corollary 2 below between the corresponding $i$-th twisted Alexander polynomials of both spaces.

We end this section with the result that we will use in Sect. 4, which is a consequence of everything we have discussed in this section.

**Corollary 2** $H_i^{ε,ρ}(M, \mathbb{F}[t^{±1}])$ and $H_i^{ε,ρ}(W^*, \mathbb{F}[t^{±1}])$ are torsion $\mathbb{F}[t^{±1}]$-modules for any $0 \leq i \leq n - 1$. Moreover, their twisted Alexander polynomials $Δ_i^{ε,ρ}(M)$ and $Δ_i^{ε,ρ}(W^*)$ coincide for $0 \leq i < n - 1$, and $Δ_{n-1}^{ε,ρ}(M)$ divides $Δ_{n-1}^{ε,ρ}(W^*)$.

### 4 Roots of twisted Alexander polynomials

#### 4.1 Line arrangement case ($n = 2$)

Let $A = \{H_1, \ldots, H_m\} \subset \mathbb{C}^2$ be an essential line arrangement. Note that, in the line arrangement case, the only two twisted Alexander polynomials that we will be considering are those in degree 0 and 1.

The 0-th case is always easy to compute, not just in dimension 2. The 0-th and first twisted Alexander polynomials of any finite CW complex can be computed from a presentation of the fundamental group using Fox Calculus ([12, Section 4]). In particular, if $M$ is the complement of a complex hyperplane arrangement $\{H_1, \ldots, H_m\}$, after a Lefschetz type argument we can in principle use a presentation of $π_1(M)$ to compute the 0-th and first twisted Alexander polynomials of $M$. Let us consider the map of $\mathbb{F}[t^{±1}]$-modules

$$\partial : (\mathbb{F}[t^{±1}] \otimes \mathbb{V})^m \to \mathbb{F}[t^{±1}] \otimes \mathbb{V}$$

given by the column matrix with entries

$$t^{ε(\rho)}(a_i) - \text{Id} \in \mathcal{M}_{\dim_{\mathbb{F}} \mathbb{V} \times \dim_{\mathbb{F}} \mathbb{V}}(\mathbb{F}[t^{±1}]), \quad i = 1, \ldots, m$$

where $a_1, \ldots, a_m$ are the generators of $π_1(M)$ as described in Sect. 1.1, and $a_i$ is a positively oriented meridian around $H_i$ for all $i = 1, \ldots, m$. The 0-th twisted Alexander polynomial $Δ_0^{ε,ρ}(M)$ is just a generator of the Fitting ideal of the cokernel of $\partial$, so it is the greatest common divisor of the minors of size $\dim_{\mathbb{F}} \mathbb{V}$ of the column matrix we just described (see [12, Section 4]). Hence, we have the following result. Recall that $\det_{ε,ρ}(γ) := \det(t^{ε(γ)}\rho(γ) - \text{Id})$ (Notation 1).

**Proposition 4** $Δ_0^{ε,ρ}(M)$ is the greatest common divisor of the minors of size $\dim_{\mathbb{F}} \mathbb{V}$ of the column matrix with entries

$$t^{ε(\rho)}(a_i) - \text{Id} \in \mathcal{M}_{\dim_{\mathbb{F}} \mathbb{V} \times \dim_{\mathbb{F}} \mathbb{V}}(\mathbb{F}[t^{±1}])$$

for $i = 1, \ldots, m$. In particular,

$$Δ_0^{ε,ρ}(M) \text{ divides } \gcd_{i=1,\ldots,m} \{\det_{ε,ρ}(a_i)\}$$

Now, let us study the first twisted Alexander polynomials of $M$. We have the following result. Recall the notation used: if $d_k$ is the number of lines in the arrangement $A$ containing
a singular point \( P_k \), then \( \beta_k \) in the statement of Theorem 5 is the composition of \( d_k \) positively oriented meridians, one for each of the lines of \( A \) containing \( P_k \), in a specific order (see Definition 3 and Remark 3).

**Theorem 5** Let \( A = \{ H_1, \ldots, H_m \} \subset \mathbb{C}^2 \) be an essential line arrangement, \( H = \bigcup_{i=1}^m H_i \), \( M = \mathbb{C}^2 \setminus H \) and let \( W^* \) be a punctured stratified tubular neighborhood of \( H \). Let \( P_1, \ldots, P_s \) be the singular points of \( H \), let \( s_i \) be the number of singular points of \( H \) on \( H_i \), and let \( d_k \) be the number of lines of \( A \) going through the singular point \( P_k \). Then, we have

1. \( \Delta_{1}^{\epsilon, \rho}(W^*) = \left( \sum_{k=1}^{s} \det_{\epsilon, \rho}(\beta_k)^{d_k-2} \right) \cdot \left( \prod_{i=1}^{m} \det_{\epsilon, \rho}(a_i)^{s_i-1} \right) \cdot \Delta_{0}^{\epsilon, \rho}(M) \).
2. \( \Delta_{1}^{\epsilon, \rho}(M) \) divides

\[
\left( \sum_{k=1}^{s} \det_{\epsilon, \rho}(\beta_k)^{d_k-2} \right) \cdot \left( \prod_{i=1}^{m} \det_{\epsilon, \rho}(a_i)^{s_i-1} \right) \cdot \gcd_{i=1,\ldots,m} \{ \det_{\epsilon, \rho}(a_i) \}.
\]

**Proof** We will use techniques coming from [2, Theorem 5.6], although, in our case, the work done in Sects. 2 and 3 allows us to not have to deal with contributions coming from the line at infinity, unlike in [2]. Let \( F = H \setminus \bigcup_{k=1}^{s} (H \cap \mathbb{B}_k^4) \) be the surface obtained by removing small balls \( \mathbb{B}_k^4 \) around the singular points \( P_k \). Note that what we are really removing from our surface is a 2-dimensional open disk \( D_i^k \) from every line \( H_i \) in \( A \) containing \( P_k \).

Let \( N = F \times S^1 \). \( N \) should be thought of as the boundary of a tubular neighborhood around the non-singular part of \( H \). We have that \( \partial N = \partial F \times S^1 \), and since \( \partial F \) is a union of disjoint \( S^1 \)'s (one from every disk \( D_i^k \) removed), then \( \partial N \) is a union of disjoint tori \( \bigcup_{k,i} T_i^k \) (again, one from every disk \( D_i^k \) removed). Let us fix a point \( f^i_k \) in the \( S^1 \) corresponding to the boundary of the disk \( D_i^k \) for every such disk removed.

Let \( L_k \) be the link of the singularity at the point \( P_k \) (which is a Hopf link with \( d_k \) components), and let \( S_k^3 \) be the boundary of \( \mathbb{B}_k^4 \). We consider the space

\[
X = N \cup \left( \bigcup_{k,i} T_i^k \right) \left( \bigcup_{k=1}^{s} S_k^3 \setminus L_k \right) \subset M
\]

where the gluing is done as follows. A meridian around the \( i \)-th component of \( L_k \) (the one corresponding to the line \( H_i \), which we will denote by \( L_i^k \)) is glued to \( \{ f^i_k \} \times S^1 \subset N \), and \( L_i^k \) is glued to the \( S^1 \) corresponding to the boundary of \( D_i^k \).

By the definition of the stratified tubular neighborhood \( W \), we have that \( X \) is homotopy equivalent to \( \partial W \). By Corollary 2, the first twisted Alexander polynomial of the line arrangement complement \( M \) divides the first twisted Alexander polynomial of \( \partial W \) (which is homotopy equivalent to \( W^* \), see Remark 6), so our goal now is to compute \( \Delta_{1}^{\epsilon, \rho}(X) \).

Notice that \( N \) has \( m \) connected components, one for every line \( H_i \) in our arrangement. That is, if we define \( F_i = F \cap H_i \), then \( N = \bigcup_{i=1}^{m} F_i \times S^1 \). Notice that \( F_i \) is just a complex line \( H_i \) (or a real plane) with \( s_i \) disks removed, one for every singular point of \( H \) in \( H_i \). Thus, \( F_i \) is homotopy equivalent to a wedge sum of \( s_i \) circles, and hence \( F_i \times S^1 \) (and \( N \)) is homotopy equivalent to a 2-dimensional CW-complex.
It is also well known ([14, Lemma 2]) that $S_3^3 \setminus L_k$ has the homotopy type of a 2-dimensional CW-complex as well. The space $X$ also has the homotopy type of a 2-dimensional CW-complex by how it is constructed.

We have the following Mayer-Vietoris short exact sequence of complexes with coefficients in $\mathbb{F}(t)$.

$$0 \to \bigoplus_{k,i} C_\ast^{\varepsilon, \rho}(T_i^k, \mathbb{F}(t)) \to \left( \bigoplus_k C_\ast^{\varepsilon, \rho}(S_3^3 \setminus L_k, \mathbb{F}(t)) \right) \oplus C_\ast^{\varepsilon, \rho}(N, \mathbb{F}(t)) \to C_\ast^{\varepsilon, \rho}(X, \mathbb{F}(t)) \to 0$$

Let $\mathcal{H}$ be the Mayer-Vietoris long exact sequence of the twisted homology groups (seen as a complex). We will consider the twisted Reidemeister torsion $\tau_{\varepsilon, \rho}$ (as defined in Definition 6) of all the pieces involved in this short exact sequence, namely $N$, $\bigcup_{k=1}^s S_3^3 \setminus L_k$, and their intersection $\bigcup_{k,i} T_i^k$.

As pointed out in Lemma 2, the twisted Reidemeister torsion for acyclic complexes is independent of the choice of bases up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$, and, as we will see in the proof of Proposition 5, we only consider the twisted Reidemeister torsion of acyclic complexes in this proof. Since all of those pieces (including $X$) have the homotopy type of a 2-dimensional CW-complex, then the only non-trivial Alexander polynomials are those in degree 0 and 1 for all of those spaces, and by Lemma 3, we have that

$$\tau_{\varepsilon, \rho}(\cdot) = \frac{\Delta_{1}^{\varepsilon, \rho}(\cdot)}{\Delta_{0}^{\varepsilon, \rho}(\cdot)}$$

for all of the relevant spaces in this problem ($X$, $N$, $\bigcup_{k,i} T_i^k$ and $\bigcup_{k=1}^s S_3^3 \setminus L_k$).

By Lemma 1, we have that

$$\left( \prod_{k=1}^s \tau_{\varepsilon, \rho}(S_3^3 \setminus L_k) \right) (\tau_{\varepsilon, \rho}(N)) = \left( \prod_{k,i} \tau_{\varepsilon, \rho}(T_i^k) \right) \tau_{\varepsilon, \rho}(X) \tau(\mathcal{H})$$

(3)

were $\tau(\mathcal{H})$ is the torsion of a complex. \qed

Now, we use the following result.

**Proposition 5** $\mathcal{H}$ is the trivial complex. In particular, $\tau(\mathcal{H}) = 1$

**Proof** We need to show that the complexes $C_\ast^{\varepsilon, \rho}(T_i^k, \mathbb{F}(t))$ (for every $k$ and $i$), $C_\ast^{\varepsilon, \rho}(S_3^3 \setminus L_k, \mathbb{F}(t))$ (for every $k$), $C_\ast^{\varepsilon, \rho}(N, \mathbb{F}(t))$ and $C_\ast^{\varepsilon, \rho}(X, \mathbb{F}(t))$ are acyclic. By the long exact sequence in homology, it suffices to show that three out of those four are acyclic.

By [17, Proposition 2.9], since $\mathbb{F}(t)$ is flat over $\mathbb{F}[t^{\pm 1}]$ and $\varepsilon(\beta_k) \neq 0$ (in fact, $\varepsilon(\beta_k) > 0$ by Remark 3), we have that

$$C_\ast^{\varepsilon, \rho}(S_3^3 \setminus L_k, \mathbb{F}(t))$$

is acyclic.

Let us now see that $C_\ast^{\varepsilon, \rho}(N, \mathbb{F}(t))$ is acyclic, or equivalently, that $H_i^{\varepsilon, \rho}(F_i \times S^1, \mathbb{F}(t)) = 0$ for all $i = 1, \ldots, m$ and $j \geq 0$. We can compute $H_0^{\varepsilon, \rho}(F_i \times S^1, \mathbb{F}(t))$ and $H_1^{\varepsilon, \rho}(F_i \times S^1, \mathbb{F}(t))$ directly using Fox Calculus ([12, Section 4]), a technique that only requires a presentation of the presentation of $F_i$.\qed
the fundamental group. Recall that $F_i$ is homotopy equivalent to a wedge sum of $s_i$ circles, and let $b_{i1}, \ldots, b_{is_i}$ be loops around the respective circles. With this notation, we see that
\[ \pi_1(F_i \times S^1) = \langle b_{i1}, \ldots, b_{is_i}, a_i \mid [b_{ij}, a_i], \; j = 1, \ldots, s_i \rangle. \] (4)

In this presentation we are abusing notation, since the base point of $a_i$ is not in $F_i \times S^1$. By $a_i$ in this presentation, we mean a loop contained in $F_i \times S^1$ that is isotopic to $a_i$ in $M$ after a change of base points.

Using that $\varepsilon(a_i)$ is not 0 for any $i = 1, \ldots, m$ in a routine Fox Calculus computation using this presentation, we get that $H_j^{\varepsilon,\rho}(F_i \times S^1, \mathbb{F}(t)) = 0$ for $j = 0, 1$.

To finish proving that $H^\varepsilon_{j,\rho}(F_i \times S^1, \mathbb{F}(t)) = 0$ for all $j$, we just have to show it for $j = 2$, since $F_i \times S^1$ is homotopy equivalent to a 2-dimensional CW-complex. This 2-dimensional CW-complex is the cartesian product of a wedge sum of $s_i$'s and an $S^1$. Thus, it has one 0-cell, $(s_i + 1)$ 1-cells, and $s_i$ 2-cells. Hence, an Euler characteristic argument tells us that $H_2^{\varepsilon,\rho}(F_i \times S^1, \mathbb{F}(t)) = 0$, concluding our proof of the acyclicity of $C_{\varepsilon,\rho}^*(N, \mathbb{F}(t))$.

The only thing left to prove here is that $C^{\varepsilon,\rho}_*(T^k_i, \mathbb{F}(t))$ is acyclic (for every $k$ and $i$). This is just a computation that follows the same steps as what we did for $C^{\varepsilon,\rho}_*(F_i \times S^1, \mathbb{F}(t))$, so we will omit it. It also relies on the fact that $\varepsilon(\gamma_i) \neq 0$, for every meridian $\gamma_i$ around $H_i$ and for all $i = 1, \ldots, m$.

Now, using this result, equation (3) becomes
\[ \left( \prod_{k=1}^s \tau_{\varepsilon,\rho}(S_k^3 \setminus L_k) \right) (\tau_{\varepsilon,\rho}(N)) = \left( \prod_{k,i} \tau_{\varepsilon,\rho}(T^k_i) \right) \tau_{\varepsilon,\rho}(X). \] (5)

We want to compute $\Delta^{\varepsilon,\rho}_1(X)$. By Proposition 2, we have that $\Delta^{\varepsilon,\rho}_0(X) = \Delta^{\varepsilon,\rho}_0(M)$, and we know $\Delta^{\varepsilon,\rho}_0(M)$ by Proposition 4. Hence, to compute $\Delta^{\varepsilon,\rho}_1(X)$, it suffices to compute $\tau_{\varepsilon,\rho}(X) = \frac{\Delta^{\varepsilon,\rho}_1(X)}{\Delta^{\varepsilon,\rho}_0(X)}$. By the equation relating the torsions that we just found, it suffices to compute the twisted Reidemeister torsion for the other pieces.

**Proposition 6**

1. $\tau_{\varepsilon,\rho}(N) = \prod_{i=1}^m \det_{\varepsilon,\rho}(a_i)^{s_i-1}$

2. $\tau_{\varepsilon,\rho}\left( \bigsqcup_{k,i} T^k_i \right) = 1$

3. $\tau_{\varepsilon,\rho}\left( \bigsqcup_{k=1}^s S_k^3 \setminus L_k \right) = \prod_{k=1}^s \det_{\varepsilon,\rho}(\beta_k)^{d_k-2}$

**Proof** First of all, by the multiplicativity of the torsion (which can be inferred from Lemma 3), we have that
\[ \tau_{\varepsilon,\rho}(N) = \prod_{i=1}^m \tau_{\varepsilon,\rho}(F_i \times S^1), \quad \tau_{\varepsilon,\rho}\left( \bigsqcup_{k,i} T^k_i \right) = \prod_{k,i} \tau_{\varepsilon,\rho}(T^k_i). \]

Using the presentations given in Eqs. (1) and (4) and Fox Calculus ([12, Section 4]), we can compute the twisted Reidemeister torsion of all the spaces involved, namely
\[
\begin{align*}
\tau_{\varepsilon,\rho}(F_i \times S^1) &= \det_{\varepsilon,\rho}(a_i)^{s_i-1} & \text{for all } i = 1, \ldots, m \\
\tau_{\varepsilon,\rho}(T^k_i) &= 1 & \text{for all } k, i \\
\tau_{\varepsilon,\rho}(S_k^3 \setminus L_k) &= \det_{\varepsilon,\rho}(\beta_k)^{d_k-2} & \text{for all } k = 1, \ldots, s & [17, \text{Proposition 2.9}].
\end{align*}
\]
Now, we can use Proposition 6 and Eq. (5) to get

\[ \tau_{\varepsilon, \rho}(X) = \prod_{k=1}^{s} \det_{\varepsilon, \rho}(\beta_k^{d_k - 2}) \prod_{i=1}^{m} \det_{\varepsilon, \rho}(a_i)^{s_i - 1} \]

where this equality is defined up to multiplication by a unit of \( \mathbb{F}[t^{\pm 1}] \).

Hence \( \Delta_{1}^{\varepsilon, \rho}(X) = \left( \prod_{k=1}^{s} \det_{\varepsilon, \rho}(\beta_k^{d_k - 2}) \right) \cdot \left( \prod_{i=1}^{m} \det_{\varepsilon, \rho}(a_i)^{s_i - 1} \right) \cdot \Delta_{0}^{\varepsilon, \rho}(M) \)

so, \( \Delta_{1}^{\varepsilon, \rho}(X) \) divides

\[ \left( \prod_{k=1}^{s} \det_{\varepsilon, \rho}(\beta_k^{d_k - 2}) \right) \cdot \left( \prod_{i=1}^{m} \det_{\varepsilon, \rho}(a_i)^{s_i - 1} \right) \cdot \gcd \left\{ \det_{\varepsilon, \rho}(a_i) \right\} . \]

Now, by Corollary 2 and the fact that \( W^* \) is homotopy equivalent to \( X \), the proof of Theorem 5 is complete. \( \square \)

**Remark 10** (Twisted Alexander polynomials of the boundary manifold) Let \( \mathcal{L} = \{ l_1, \ldots, l_m \} \subset \mathbb{C}^2 \) be an essential line arrangement, and let \( l_0 = \mathbb{CP}^2 \setminus \mathbb{C}^2 \) be the line at infinity. We consider the projective line arrangement \( \mathcal{L}' = \mathcal{L} \cup \{ l_0 \} \subset \mathbb{CP}^2 \). The boundary manifold \( B \) of the affine arrangement \( \mathcal{L} \) is the boundary of the manifold obtained by gluing balls around the singular points of the arrangement \( \mathcal{L}' \) and tubes around the smooth part of the lines, similar to what we did in the construction of \( W^* \).

We have that the map induced by inclusion \( \pi_0(B) \longrightarrow \pi_0(M) \) is an isomorphism, since both spaces are connected, and \( \pi_1(B) \longrightarrow \pi_1(M) \) is an epimorphism, by a Lefschetz type argument. Thus, the homotopy type of \( M \) is obtained from \( B \) by adjoining cells of dimension \( \geq 2 \), which as we have seen in the proof of Proposition 2 implies that

\[ \Delta_{0}^{\varepsilon, \rho}(M) = \Delta_{0}^{\varepsilon, \rho}(B) \]

and that

\[ \Delta_{1}^{\varepsilon, \rho}(M) \text{ divides } \Delta_{1}^{\varepsilon, \rho}(B), \]

provided that \( H_{1}^{\varepsilon, \rho}(B, \mathbb{F}[t^{\pm 1}]) \) is a torsion \( \mathbb{F}[t^{\pm 1}] \)-module. Moreover, following the proof of Theorem 5, we conclude that \( H_{1}^{\varepsilon, \rho}(B, \mathbb{F}[t^{\pm 1}]) \) is indeed torsion (because \( C_{x}^{\varepsilon, \rho}(B, \mathbb{F}(t)) \) is acyclic) and

\[ \frac{\Delta_{1}^{\varepsilon, \rho}(B)}{\Delta_{0}^{\varepsilon, \rho}(B)} = \left( \prod_{k=1}^{s} \det_{\varepsilon, \rho}(\beta_k^{d_k - 2}) \right) \cdot \left( \prod_{i=0}^{m} \det_{\varepsilon, \rho}(a_i)^{\tilde{s}_i - 2} \right) \]

where \( s \) is the number of singular points of the projective arrangement, the \( \beta_k \)'s are certain distinguished loops near each of the singular points as in Definition 3, \( \tilde{s}_i \) is the number of singular points of the projective arrangement on the line \( l_i \), and \( a_i \) is a positively oriented meridian around the line \( l_i \). This resembles the result obtained in a different way by Cohen and Suciu in [3, Theorem 5.2] for multivariable twisted Alexander polynomials, and it agrees with the divisibility result obtained from [2, Theorem 5.6] in the case where \( \rho \) is a unitary representation.
Note that, if \( s_i \) is the number of singular points of the affine arrangement on the line \( l_i \), for \( i = 1, \ldots, m \), then \( \tilde{s}_i = s_i + 1 \), so we can see that
\[
\Delta_i^{\varepsilon, \rho}(W^*) \text{ divides } \Delta_i^{\varepsilon, \rho}(B),
\]
and we conclude that the punctured stratified tubular neighborhood \( W^* \) constitutes a better bound than the boundary manifold \( B \) for the roots of the first twisted Alexander polynomial of \( M \).

**Remark 11** (Relaxing the assumptions on \( \varepsilon, \Pi \)) For Proposition 4 we do not need that \( \varepsilon \) be a positive epimorphism, just that it is a non-trivial map. Proposition 5 holds if \( \varepsilon \) takes non-zero values on the \( a_i \)'s and the \( \beta_k \)'s. Since \( W^* \) is homotopy equivalent to \( X \), \( H_i^{\varepsilon, \rho}(W^*, \mathbb{F}[i^{\pm 1}]) \) is torsion for all \( i \geq 0 \) if and only if \( C_*^{\varepsilon, \rho}(X, \mathbb{F}(t)) \) is acyclic, which follows from Proposition 5. Hence, for Theorem 5 to hold, it suffices to assume that \( \varepsilon \) takes non-zero values on the distinguished loops that appear in the formula of the twisted Alexander polynomial of \( W^* \), as one can see in the proof (recall Remark 9).

In some cases, we can refine the result given by Theorem 5 as follows. The notation \( \beta_{k, i} \) in the statement of the following theorem has the same meaning as \( \beta_k \) in Theorem 5, switching \( P_k \) for \( P_k^i \) and \( d_k \) for \( d_k^i \).

**Theorem 6** Let \( B \) be the subset of lines of an essential line arrangement \( A = \{ H_1, \ldots, H_m \} \) such that for each line in \( B \) no other line in \( A \) is parallel to it. Suppose that \( B = \{ H_1, \ldots, H_l \} \neq \emptyset \). Let \( s_i \) be the number of singular points of the union of lines in \( A \) that are contained in \( H_l \), which we denote by \( P_1^i, \ldots, P_{s_i}^i \), and let \( d_i^k \) be the number of lines of \( A \) going through the singular point \( P_k^i \). Then, \( \Delta_i^{\varepsilon, \rho}(M) \) divides
\[
\left( \gcd_{r=1}^{s_m} \det_{\varepsilon, \rho}(a_r) \right) \cdot \gcd_{i=1}^{l} \left\{ \prod_{k=1}^{s_i} \det_{\varepsilon, \rho}(\beta_{k,i})^{d_k^i-2} \cdot \det_{\varepsilon, \rho}(a_i)^{\delta_i-1} \right\}.
\]

**Proof** Let \( 1 \leq i \leq l \). Let \( B_{k,i}^i \) be a small ball around the singular point \( P_k^i \), with boundary \( S_{k,i}^3 \), and let \( L_{k,i} \subset S_{k,i}^3 \) be the link of the singularity of \( H \) at the point \( P_k^i \). We will follow the proof of Theorem 5 and use the notation introduced there, but this time we define \( X_i \) (instead of \( X \)) as the result of gluing \( F_i \times S^1 \) and \( \bigcup_{k=1}^{s_i} S_{k,i}^3 \setminus L_{k,i} \) along the correspondig tori.

\( X_i \) is connected, so the map induced by inclusion \( \pi_0(X_i) \rightarrow \pi_0(M) \) is an isomorphism. Moreover, since no other line in \( A \) is parallel to \( H_l \), we can see by a Lefschetz type argument that the map \( \pi_1(X_i) \rightarrow \pi_1(M) \) induced by inclusion is an epimorphism. Thus, the homotopy type of \( M \) is obtained from \( X_i \) by adjoining cells of dimension \( \geq 2 \), which as we have seen in the proof of Proposition 2, implies that
\[
H_i^{\varepsilon, \rho}(X_i, \mathbb{F}[i^{\pm 1}]) \rightarrow H_i^{\varepsilon, \rho}(M, \mathbb{F}[i^{\pm 1}])
\]
is an epimorphism. Moreover, following the proof of Theorem 5, we can show that all of the complexes involved in the Mayer-Vietoris short exact sequence of complexes with coefficients in \( \mathbb{F}(t) \) except for \( C_*^{\varepsilon, \rho}(X_i, \mathbb{F}(t)) \) are acyclic, so the long exact sequence in homology will tell us that \( C_*^{\varepsilon, \rho}(X_i, \mathbb{F}(t)) \) is acyclic as well. In particular, \( H_i^{\varepsilon, \rho}(X_i, \mathbb{F}[i^{\pm 1}]) \) is torsion, and \( \Delta_i^{\varepsilon, \rho}(M) \) divides \( \Delta_i^{\varepsilon, \rho}(X_i) \) for all \( i = 1, \ldots, l \).

Following the proof of Theorem 5, we get that
\[
\Delta_i^{\varepsilon, \rho}(X_i) = \left( \prod_{k=1}^{s_i} \det_{\varepsilon, \rho}(\beta_{k,i})^{d_k^i-2} \right) \cdot \det_{\varepsilon, \rho}(a_i)^{\delta_i-1} \cdot \Delta_0^{\varepsilon, \rho}(M)
\]
for all \( i = 1, \ldots, l \), and the result follows immediately by Proposition 4. \( \square \)

### 4.2 Higher-dimensional case

Let \( \mathcal{A} = \{ H_1, \ldots, H_m \} \) be an essential hyperplane arrangement in \( \mathbb{C}^n \). We consider the natural stratification of \( H = \bigcup_{i=1}^{m} H_i \), the one in which two points \( P \) and \( P' \) in \( H \) lie in the same stratum if the collections of hyperplanes in the arrangement containing \( P \) and \( P' \) coincide. Let \( \Sigma^k_1, \ldots, \Sigma^k_{s_k} \) be the collection of connected strata of (complex) dimension \( k \). For each stratum, we define the multiplicity \( m(\Sigma^k_i) \) as the number of hyperplanes in \( \mathcal{A} \) containing a point from this stratum.

**Remark 12** By Corollary 2, the zeros of the \( i \)-th twisted Alexander polynomials of our arrangement complement \( M \) are among the zeros of the \( i \)-th twisted Alexander polynomial of a punctured stratified tubular neighborhood \( W^* \) of the arrangement, for \( i = 0, \ldots, n - 1 \). This observation prompts us to study what the zeros of the twisted Alexander polynomials of \( W^* \) could be.

Let

\[
\{ S^k_l \mid k = 0, \ldots, n-1; l = 1, \ldots, s_k \}
\]

be a collection of open subsets of \( W^* \), each of which fibers over the corresponding stratum \( \Sigma^k_l \), and chosen so that their union is \( W^* \). These open subsets can be taken to be the tubular neighborhoods of open subsets of the strata that appeared in the construction of \( W \) (before Definition 7) minus \( H \). The fiber \( F_{l,k} \) of \( S^k_l \rightarrow \Sigma^k_l \) is a central hyperplane arrangement complement consisting on \( m(\Sigma^k_l) \) hyperplanes in \( \mathbb{C}^{n-k} \). As it is pointed out in [16, p. 5], if \( k_1 \geq k_2 \), then \( S^k_{l_1} \cap S^k_{l_2} \) is not empty if and only if the stratum \( \Sigma^k_{l_1} \) is in the closure of the stratum \( \Sigma^k_{l_2} \) (which in particular implies that \( k_1 > k_2 \)), and, in this intersection, the fibration that we consider is the one from \( S^k_{l_1} \rightarrow \Sigma^k_{l_1} \) restricted to it.

Let \( W := V^* = \text{Hom}_F(V, F) \) be the dual vector space of \( V \), and let

\[
\rho^* : \pi_1(M) \rightarrow \text{GL}(W)
\]

be the dual representation of \( \rho : \pi_1(M) \rightarrow \text{GL}(V) \), given by

\[
(w \cdot \alpha)(v) = w(v \cdot \alpha^{-1})
\]

for every \( w \in W, \alpha \in \pi_1(M) \) and \( v \in V \).

We consider the involution given by

\[
\tau : F[t^{\pm 1}] \rightarrow F[t^{\pm 1}]
\]

\[
t \mapsto \bar{t} := t^{-1}
\]

and we define the conjugate \( F[t^{\pm 1}] \)-module structure of an \( F[t^{\pm 1}] \)-module as the one obtained by composing the \( F[t^{\pm 1}] \)-module structure with the involution \( \tau \). Then, as justified in [17, p. 6 and p. 17], we have that

\[
\overline{H}^i(W^*, \mathcal{L}_{\varepsilon, \rho^*}) \cong H^i \left( \text{Hom}_{F[t^{\pm 1}]}(C^*_{\varepsilon, \rho^*}(W^*, F[t^{\pm 1}]), F[t^{\pm 1}]) \right)
\]

for all \( i \), where \( \overline{H}^i(W^*, \mathcal{L}_{\varepsilon, \rho^*}) \) means the \( F[t^{\pm 1}] \)-module \( H^i(W^*, \mathcal{L}_{\varepsilon, \rho^*}) \) with the conjugate module structure, and \( \mathcal{L}_{\varepsilon, \rho^*} \) is the local system of \( F[t^{\pm 1}] \)-modules induced by the tensor

\[\text{ Springer}\]
representation \( \varepsilon \otimes \rho^* \). Therefore, the Universal Coefficient Theorem applied to the principal ideal domain \( \mathbb{F}[t^{\pm 1}] \) yields

\[
\overline{H}^i(W^*, \mathcal{L}_{\varepsilon, \rho^*}) \cong \text{Hom}_{\mathbb{F}[t^{\pm 1}]}(H^i_{\varepsilon, \rho}(W^*, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]) \\
\oplus \text{Ext}_{\mathbb{F}[t^{\pm 1}]}(H^i_{\varepsilon, \rho}(W^*, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]).
\]

Now, applying Corollary 2 we get that \( H^i_{\varepsilon, \rho}(W^*, \mathbb{F}[t^{\pm 1}]) \) is a torsion \( \mathbb{F}[t^{\pm 1}] \)-module for all \( i \leq n - 1 \), so by the UCT, we get that

\[
\overline{H}^i(W^*, \mathcal{L}_{\varepsilon, \rho^*}) \cong \text{Ext}_{\mathbb{F}[t^{\pm 1}]}(H^i_{\varepsilon, \rho}(W^*, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]) \cong H^i_{\varepsilon, \rho}(W^*, \mathbb{F}[t^{\pm 1}])
\]

for all \( i \leq n - 1 \).

**Remark 13** In fact, we will see later on that \( H^i_{\varepsilon, \rho}(W^*, \mathbb{F}[t^{\pm 1}]) \) is a torsion \( \mathbb{F}[t^{\pm 1}] \)-module for all \( i \), so

\[
\overline{H}^i(W^*, \mathcal{L}_{\varepsilon, \rho^*}) \cong H^i_{\varepsilon, \rho}(W^*, \mathbb{F}[t^{\pm 1}])
\]

for all \( i \).

Hence, the order of \( H^i(W^*, \mathcal{L}_{\varepsilon, \rho^*}) \) is \( \Delta_{\varepsilon, \rho}(W^*)^{(i)} = \Delta_{\varepsilon, \rho}(W^*)(t^{-1}) \) for all \( i \). As indicated in Remark 12, we are interested in studying the zeros of \( \Delta_{\varepsilon, \rho}(W^*) \) for all \( i \), or equivalently, the inverses of the zeros of the order of \( H^{i+1}(W^*, \mathcal{L}_{\varepsilon, \rho^*}) \) for all \( i \).

Let us consider the Mayer-Vietoris spectral sequence ([9, II.5.4]) of the sheaf \( \mathcal{L}_{\varepsilon, \rho^*} \) associated to the open covering

\[\{S^k_l \mid k = 0, \ldots, n - 1; l = 1, \ldots, s_k\}\].

The first page of this spectral sequence is

\[E^p,q_1 = \bigoplus H^q(S^k_{l_1} \cap \ldots \cap S^k_{l_{p+1}}, \mathcal{L}_{\varepsilon, \rho^*}),\] (6)

where the direct sum is taken over all the possible non-empty intersections of \( p + 1 \) open sets of our covering. This spectral sequence converges to \( H^{p+q}(W^*, \mathcal{L}_{\varepsilon, \rho^*}) \).

Let us study the elements in the first page of our Mayer-Vietoris spectral sequence, namely, the elements of the form

\[H^q(S^k_{l_1} \cap \ldots \cap S^k_{l_{p+1}}, \mathcal{L}_{\varepsilon, \rho^*}),\]

with \( S^k_{l_1} \cap \ldots \cap S^k_{l_{p+1}} \neq \emptyset \). Reordering, we can assume that \( k_1 > \ldots > k_{p+1} \), and that \( \Sigma^k_{l_j} \subseteq \Sigma^k_{l_i} \) for every \( j = 2, \ldots, p + 1 \). Let \( f : S^k_{l_1} \to \Sigma^k_{l_1} \) be the fibration. We consider a good cover \( \mathcal{U} \) of \( f(S^k_{l_1} \cap \ldots \cap S^k_{l_{p+1}}) \), which is an open set of the manifold \( \Sigma^k_{l_1} \), and thus it is a manifold. By good cover we mean an open cover where all open sets and all finite intersections of those open sets are contractible. Let

\[\mathcal{V} := \{f^{-1}(A) \mid A \in \mathcal{U}\}\]

We consider the Mayer-Vietoris spectral sequence of the sheaf \( \mathcal{L}_{\varepsilon, \rho^*} \) associated to the open covering \( \mathcal{V} \). The first page of this spectral sequence is

\[E^p,q_1 = \bigoplus H^q(B_{l_1} \cap \ldots \cap B_{l_{p+1}}, \mathcal{L}_{\varepsilon, \rho^*})\] (7)
where the direct sum is taken over all the possible non-empty intersections $B_{r_1} \cap \ldots \cap B_{r_{p+1}}$ of $p + 1$ open sets of our covering $\mathcal{V}$. This spectral sequence converges to $H^{p+q}(S_{l_i}^{k_1} \cap \ldots \cap S_{l_{p+1}}^{k_{p+1}}, \mathcal{L}_{\varepsilon, \rho^*})$.

Note that, since non-empty finite intersections of open sets in $U$ are contractible, and the map $f$ restricted to $S_{l_i}^{k_1} \cap \ldots \cap S_{l_{p+1}}^{k_{p+1}}$ is a locally trivial fibration with fiber $F_{l_1, k_1}$, which is the complement of a central hyperplane arrangement consisting on $m(\Sigma_{l_i}^{k_1})$ hyperplanes in $\mathbb{C}^{n-k_1}$, then $B_{r_1} \cap \ldots \cap B_{r_{p+1}}$ is homeomorphic to the product of $F_{l_1, k_1}$ and a contractible open set, and thus it is homotopy equivalent to $F_{l_1, k_1}$. Thus,

$$H^q(B_{r_1} \cap \ldots \cap B_{r_{p+1}}, \mathcal{L}_{\varepsilon, \rho^*}) \cong H^q(F_{l_1, k_1}, \mathcal{L}_{\varepsilon, \rho^*}).$$

Note that central hyperplane arrangements are homotopy equivalent to the complement in $S^{2n-1}$ of their links at infinity, so by [17, proof of Theorem 4.1], $H^{k, \rho^*}(F_{l_1, k_1}, \mathbb{F}[t^{\pm 1}])$ (and consequently $H^q(F_{l_1, k_1}, \mathcal{L}_{\varepsilon, \rho^*})$) are torsion for all $q$. This implies that all of the elements in spectral sequence (7) are torsion modules. In fact, by [17, Theorem 4.11], the zeros of the order of $H^q(F_{l_1, k_1}, \mathcal{L}_{\varepsilon, \rho^*})$ are among those of the order of the cokernel of the endomorphism

$$t^{-\varepsilon(\gamma_{\infty}(F_{l_1, k_1}))} \rho^*(\gamma_{\infty}(F_{l_1, k_1}))^{-1} - \text{Id} \in \text{End}(\mathbb{F}[t^{\pm 1}] \otimes \mathbb{F} \forall)$$

where $\gamma_{\infty}(F_{l_1, k_1})$ is a loop around the hyperplane at infinity in $\mathbb{CP}^{n-k_1}$. Note that $\varepsilon$ restricted to $\pi_1(F_{l_1, k_1})$ is not necessarily an epimorphism, but the proof of [17, Theorem 4.11] does not require it.

Let us try to describe these “loops at infinity” that we are using in more detail. Let $H_{l_1}, \ldots, H_{l_{m(\Sigma_{l_i}^{k_1})}}$ be the hyperplanes going through the stratum $\Sigma_{l_i}^{k_1}$ associated to the fiber $F_{l_1, k_1}$. We have that $\gamma_{\infty}(F_{l_1, k_1})$ has an expression of the form

$$(\gamma_{l_1} \cdot \ldots \cdot \gamma_{l_{m(\Sigma_{l_i}^{k_1})}})^{-1}$$

where $\gamma_{l_1}, \ldots, \gamma_{l_{m(\Sigma_{l_i}^{k_1})}}$ are an appropriate choice of meridians around each component of the central hyperplane arrangement given by $F_{l_1, k_1}$ in the appropriate order.

Note that $\varepsilon(\gamma_{\infty}(F_{l_1, k_1})) < 0$, so the order of the cokernel of

$$t^{-\varepsilon(\gamma_{\infty}(F_{l_1, k_1}))} \rho^*(\gamma_{\infty}(F_{l_1, k_1}))^{-1} - \text{Id}$$

is exactly

$$\text{det}(t^{-\varepsilon(\gamma_{\infty}(F_{l_1, k_1}))} \rho^*(\gamma_{\infty}(F_{l_1, k_1}))^{-1} - \text{Id})$$

By the discussion above, the zeros of the order of $H^q(F_{l_1, k_1}, \mathcal{L}_{\varepsilon, \rho^*})$ are among those of

$$\text{det}(t^{-\varepsilon(\gamma_{\infty}(F_{l_1, k_1}))} \rho^*(\gamma_{\infty}(F_{l_1, k_1}))^{-1} - \text{Id}).$$

Using the spectral sequence (7), we get that the zeros of the order of

$$H^q(S_{l_i}^{k_1} \cap \ldots \cap S_{l_{p+1}}^{k_{p+1}}, \mathcal{L}_{\varepsilon, \rho^*})$$

are among the zeros of $\text{det}(t^{-\varepsilon(\gamma_{\infty}(F_{l_1, k_1}))} \rho^*(\gamma_{\infty}(F_{l_1, k_1}))^{-1} - \text{Id})$.

Now, by using spectral sequence (6), we see that $H^i(W^*, \mathcal{L}_{\varepsilon, \rho^*})$ is torsion for all $i$. By the Universal Coefficient Theorem, this means that $H^i_{\varepsilon, \rho^*}(W^*, \mathbb{F}[t^{\pm 1}])$ is also torsion for all $i$, as
anticipated in Remark 13. Moreover, the zeros of the order of $H^q(W^*, \mathcal{L}_{e, \rho}^*)$ are among the zeros of

$$
\prod_{k=0}^{n-1} \prod_{l=1}^{s_k} \det(t^{-e}(\gamma_{\infty}(F_{l,k})) \rho^*(\gamma_{\infty}(F_{l,k})))^{-1} - \text{Id})
$$

for all $q$.

Hence, by Remark 12 and Remark 13, and the fact that $\rho^*(\alpha)^{-1} = \rho(\alpha)^T$ for every $\alpha \in \pi_1(M)$ (seen as matrices in $\text{GL}_n(\mathbb{F})$), we get that the zeros of $\Delta_{e, \rho}^i(M)$ are among the zeros of

$$
\prod_{k=0}^{n-1} \prod_{l=1}^{s_k} \det(t^{e}(\gamma_{\infty}(F_{l,k})) \rho(\gamma_{\infty}(F_{l,k}))) - \text{Id})
$$

Thus, we have arrived to the following result. For the notation used in it, recall that $\{S^k_l \mid k = 0, \ldots, n-1; l = 1, \ldots, s_k\}$ is an open cover of $W^*$, constructed so that there exist fibrations $S^k_l \to \Sigma^k_l$ from the open sets to their corresponding $k$-dimensional strata of $H = \bigcup_{i=1}^{m} H_i$, and such that the fibers $F_{l,k}$ are central hyperplane arrangement complements in $\mathbb{C}^{n-k}$.

**Theorem 7** Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ be an essential hyperplane arrangement in $\mathbb{C}^n$, with the natural induced stratification $\{\Sigma^k_l \mid k = 0, \ldots, n-1; l = 1, \ldots, s_k\}$, and let $M$ be the complement of that arrangement in $\mathbb{C}^n$. For every $k$ and $l$, let $F_{l,k}$ be the fiber of the fibration $S^k_l \to \Sigma^k_l$ and let $\gamma_{\infty}(F_{l,k})$ be a meridian around the hyperplane at infinity in $\mathbb{C}^{n-k}$ with positive orientation, where $F_{l,k}$ is naturally seen in $\mathbb{C}\mathbb{P}^{n-k}$. Then, for any $i = 0, \ldots, n-1$, the zeros of the $i$-th Alexander polynomial of $M$ (i.e. $\Delta_{e, \rho}^i(M)$) are among those of

$$
\prod_{k=0}^{n-1} \prod_{l=1}^{s_k} \det_{e, \rho}(\gamma_{\infty}(F_{l,k})).
$$

We can see that this result generalizes the one obtained in the line arrangement case. The meridians around the hyperplane at infinity in the line arrangement case are $\beta^1_k (k = 1, \ldots, s)$, which correspond to the 0-dimensional strata (the singular points); and $a^1_i$ $(i = 1, \ldots, m)$, which correspond to the 1-dimensional strata.

**5 Applications**

**5.1 Topology of a hyperplane arrangement complement via twisted Alexander polynomials.**

In the following example, we discuss how twisted Alexander polynomials can give us information about the topology of the complement of a line arrangement. In particular, they can be used to distinguish the homeomorphism type of certain line arrangement complements that are homotopy equivalent.

**Example 1** Let us consider a pair of line arrangements (the Falk arrangements) $\mathcal{A}_1$ and $\mathcal{A}_2$ shown in Fig. 1, which are given by the zeros of $p_1(x, y)$ and $p_2(x, y)$ respectively, where
Fig. 1 The real part of the Falk Arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$

$$p_1(x, y) = (x + 1)(x - 1)(x + y)(x - y)$$

$$p_2(x, y) = (x + 1)(x - 1)(y + 1)(y - 1)(x - y - 1).$$

In [7], Falk showed that the complements of these two arrangements are homotopy equivalent even though they are combinatorially quite different as line arrangements of 6 lines in $\mathbb{CP}^2$ (including the line at infinity). These two complements are not homeomorphic, as shown by Jiang and Yau in [11]. In [3], Cohen and Suciu reproved that the complements are not homeomorphic by showing that the boundary manifolds of $\mathcal{A}_1$ and $\mathcal{A}_2$ are not homotopy equivalent, which they did by showing that their corresponding multivariable Alexander polynomials had a different number of distinct factors.

We will show that the boundary manifolds of $\mathcal{A}_1$ and $\mathcal{A}_2$ are not homotopically equivalent by showing that certain (one-variable) twisted Alexander polynomials of their boundary manifolds have a different number of distinct roots with multiplicity, thus reproving the result by Jiang and Yau by using simpler invariants.

**Proof** (The boundary manifolds of $\mathcal{A}_1$ and $\mathcal{A}_2$ are not homotopically equivalent) Let $B_j$ be the boundary manifold of $\mathcal{A}_j$, for $j = 1, 2$. Let $M_j$ be the complement in $\mathbb{C}^2$ of the arrangement $\mathcal{A}_j$, for $j = 1, 2$. We will argue by contradiction. Let us assume that there exists a homotopy equivalence

$$h : B_1 \rightarrow B_2$$

We denote by $h_*$ the map that $h$ induces on fundamental groups. Let $i_2 : B_2 \hookrightarrow M_2$ be the inclusion and $(i_2)_*$ the map it induces on fundamental groups. Let

$$\varepsilon : \pi_1(M_2) \rightarrow \mathbb{Z}$$

be an epimorphism, and let

$$\rho : \pi_1(M_2) \rightarrow \mathbb{C}^*$$

be a one dimensional representation. Restricting ourselves to one dimensional representations makes computing twisted Alexander polynomials so much easier, since they factor through the abelianization of $\pi_1(M_2)$ and we do not have to care about the conjugation of meridians due to the braiding in the fundamental group.

We also use the letters $\varepsilon$ and $\rho$ to denote the maps from $\pi_1(B_2)$ (and from $\pi_1(B_1)$ respectively) induced by precomposing $\varepsilon$ and $\rho$ by $(i_2)_* : \pi_1(B_2) \rightarrow \pi_1(M_2)$ (and by
If we pick a given root of the term \((\rho(a_1\cdots a_j))\), then (recall Remark 11). This choice of \(C\) up to multiplication by a unit of \(\mathbb{C}[t^\pm 1]\). Thus, the set of non-zero roots with multiplicity corresponding to both sides should be the same. We will show that for some choice of \(\varepsilon\) and \(\rho\), they are not, which will conclude our proof.

Let \(a_i^j\) be a meridian around the line given by the \(i\)-th factor of \(p_j(x, y)\), and let \(a_0^j = \left(\prod_{i=1}^{5} a_i^j\right)^{-1}\), for \(j = 1, 2\) and \(i = 1, \ldots, 5\). The loop \(a_0^j\) is not necessarily a meridian around the line at infinity (due to the order chosen in the multiplication), but will have the same image by \(\varepsilon\) and \(\rho\), than any root of \(\rho(\prod_{i=1}^{5} a_i^j)\). Note that, if the lines \(l_{i_1}, l_{i_2}\) and \(l_{i_3}\) intersect in a triple point, then \(a_0^j\) will have the same image by \(\varepsilon\) and \(\rho\) than the corresponding \(\beta_k\).

We choose \(\varepsilon : \pi_1(M_2) \to \mathbb{Z}\) to be an epimorphism such that all of the loops involved in the formulas for \(\frac{\Delta_{1,\rho}(B_2)}{\Delta_{0,\rho}(B_2)}\) and \(\frac{\Delta_{1,\rho}(B_1)}{\Delta_{0,\rho}(B_1)}\) given by Remark 10 have a non-zero image by \(\varepsilon\) (recall Remark 11). This choice of \(\varepsilon\) depends on \(h_*,\) and generically, this condition on \(\varepsilon\) is satisfied. In that case, we have that

\[
\frac{\Delta_{1,\rho}(B_2)}{\Delta_{0,\rho}(B_2)} = \left(\prod_{i=1}^{4} \rho(a_i^2)t^{e(a_i^2)} - 1\right)^2 \cdot \left(\rho(a_5^2)t^{e(a_5^2)} - 1\right)^3.
\]

so, up to multiplication by a unit of \(\mathbb{C}[t^\pm 1]\),

\[
\frac{\Delta_{1,\rho}(B_2)}{\Delta_{0,\rho}(B_2)} = \left(\prod_{i=1}^{4} \rho(a_i^2)t^{e(a_i^2)} - 1\right)^2 \cdot \left(\rho(a_5^2)t^{e(a_5^2)} - 1\right)^3. \tag{8}
\]

Also, we have that

\[
\frac{\Delta_{1,\rho}(B_1)}{\Delta_{0,\rho}(B_1)} = \left(\prod_{i=0}^{5} \rho(a_i^1)t^{e(a_i^1)} - 1\right)^2 \cdot (\rho(a_{012}^1)t^{e(a_{012}^1)} - 1)(\rho(a_{345}^1)t^{e(a_{345}^1)} - 1).
\]

Note that, up to multiplication by a unit of \(\mathbb{C}[t^\pm 1]\), the last two factors are the same, so

\[
\frac{\Delta_{1,\rho}(B_1)}{\Delta_{0,\rho}(B_1)} = \left(\prod_{i=0}^{5} \rho(a_i^1)t^{e(a_i^1)} - 1\right)^2 \cdot (\rho(a_{345}^1)t^{e(a_{345}^1)} - 1)^3. \tag{10}
\]

Now that we have fixed \(\varepsilon\), we can choose \(\rho\) so that any root of \((\rho(a)t^{e(a)} - 1)\) is different than any root of \((\rho(b)t^{e(b)} - 1)\) for different loops \(a\) and \(b\) involved in the formula (8). That way, if we pick a given root of the term \((\rho(a_5^2)t^{e(a_5^2)} - 1)^3\), we know it only appears 3 times as
a root of $\Delta_i^{e,\rho}(B_2)$. On the other hand, no non-zero root of $\Delta_i^{e,\rho}(B_1)$ can have odd multiplicity, so we have reached a contradiction. \hfill \Box

5.2 Twisted jump loci vs. twisted Alexander polynomials

Let $A = \{H_1, \ldots, H_m\}$ be an essential hyperplane arrangement in $\mathbb{C}^n$, let $H = \bigcup_i H_i$, and let $M = \mathbb{C}^n \setminus H$ be the arrangement complement in $\mathbb{C}^n$. Let $\mathbb{V}$ be an $n$-dimensional vector space over $\mathbb{C}$, and let

$$\rho : \pi_1(M) \longrightarrow \text{GL}(\mathbb{V})$$

be a representation. We denote by $V_\rho$ the corresponding $\mathbb{V}$-local system on $M$.

**Definition 8** The rank 1 homology jump loci of $M$ twisted by $\rho$ are defined to be

$$V_i^k(M, \rho) = \{ \eta \in \text{Hom}(\pi_1(M), \mathbb{C}^*) \mid \dim_{\mathbb{C}} H_i(M, L_\eta \otimes V_\rho) \geq k \}$$

for all $i, k \geq 0$, where $L_\eta$ is the rank 1 local system on $M$ defined by $\eta$.

There exists a natural isomorphism

$$(\mathbb{C}^*)^m \cong \text{Hom}(\pi_1(M), \mathbb{C}^*)$$

that takes any tuple $(z_1, \ldots, z_m) \in (\mathbb{C}^*)^m$ to the unique morphism that sends the positively oriented meridians around the line $H_i$ to $z_i$ for all $i = 1, \ldots, m$. In this way, we can see the homology jump loci $V_i^k(M, \rho)$ inside of $(\mathbb{C}^*)^m$.

**Remark 14** Suppose that $\rho$ is a one dimensional representation, which corresponds to $(\rho_1, \ldots, \rho_m)$ in $(\mathbb{C}^*)^m$ via the natural isomorphism described above. Let

$$V_i^k(M) := \{ \eta \in \text{Hom}(\pi_1(M), \mathbb{C}^*) \mid \dim_{\mathbb{C}} H_i(M, L_\eta) \geq k \}$$

be the untwisted homology jump loci. Then, we have the following correspondence between subsets of $(\mathbb{C}^*)^m$:

$$(x_1, \ldots, x_m) \in V_i^k(M, \rho) \iff (x_1\rho_1, \ldots, x_m\rho_m) \in V_i^k(M).$$

Let $\varepsilon : \pi_1(M) \longrightarrow \mathbb{Z}$ be an epimorphism. It induces the following map

$$\varepsilon^* : \mathbb{C}^* \longrightarrow \text{Hom}(\pi_1(M), \mathbb{C}^*)$$

$$a \longmapsto h_a \circ \varepsilon$$

where $h_a : \mathbb{Z} \longrightarrow \mathbb{C}^*$ is the only group homomorphism taking 1 to $a$. Since $\varepsilon$ is an epimorphism, we have that the image of $\varepsilon^*$ is naturally isomorphic to $\mathbb{C}^*$. With this notation, we have the following result that relates the zeros of twisted Alexander polynomials of $M$ and the twisted rank 1 homology jump loci.

**Proposition 7** Let $F = \mathbb{C}$. Then,

$$\{a \in \mathbb{C}^* \mid \Delta_i^{e,\rho}(M)(a) \cdot \Delta_i^{e,\rho_1}(M)(a) = 0 \} = V_i^1(M, \rho) \cap \text{Im}(\varepsilon^*)$$

for $0 \leq i \leq n - 1$, and

$$\{a \in \mathbb{C}^* \mid \Delta_n^{e,\rho_1}(M)(a) = 0 \} = V_n^{\dim_{\mathbb{C}} \mathbb{V} \cdot |\chi(M)| + 1}(M, \rho) \cap \text{Im}(\varepsilon^*)$$

where $V_i^k(M, \rho) \cap \text{Im}(\varepsilon^*)$ is seen as a subset of $\mathbb{C}^*$. 
Proof} We will follow the notation in [6, Theorem 4.5], where the non-twisted case is
discussed.

Let \( a \in \mathbb{C}^* \). The homomorphism \( \varepsilon^*(a) \) defines a 1-dimensional local system, which we
will call \( \mathcal{L}_a \). We consider the following short exact sequence of vector spaces over \( \mathbb{C} \):

\[
0 \rightarrow \mathbb{C}[t^{\pm 1}] \xrightarrow{t^{-a}} \mathbb{C}[t^{\pm 1}] \xrightarrow{t^{=a}} \mathbb{C} \rightarrow 0
\]

Tensoring by \( V \), we obtain the following short exact sequence of vector spaces over \( \mathbb{C} \):

\[
0 \rightarrow \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} V \xrightarrow{f} \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} V \xrightarrow{g} \mathbb{C} \otimes_{\mathbb{C}} V \rightarrow 0 \tag{11}
\]

The vector space \( \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} V \) can be given the structure of a right \( \mathbb{C}[\pi_1(M)] \)-module, as we
described in Definition 4. Moreover, \( \mathbb{C} \otimes_{\mathbb{C}} V \cong V \) can also be given the structure of a right
\( \mathbb{C}[\pi_1(M)] \)-module, with the right action given by

\[
v \cdot \alpha = a^{\varepsilon(\alpha)} v \cdot \rho(\alpha)
\]

for every \( v \in V \) and \( \alpha \in \pi_1(X) \), where \( v \) is regarded as a row vector and \( \rho(\alpha) \) as a square
matrix.

We can check that both \( f \) and \( g \) respect the right \( \mathbb{C}[\pi_1(M)] \)-module structure, so the short
exact sequence (11) is also a short exact sequence of right \( \mathbb{C}[\pi_1(M)] \)-modules.

Let \( \tilde{M} \) be the universal cover of \( M \). We have that \( C_i(\tilde{M}, \mathbb{C}) \) is a free left \( \mathbb{C}[\pi_1(M)] \)-module
for all \( i \in \mathbb{Z} \), as explained in Definition 4. In particular, it is flat, so we can tensor (11) by
\( C_i(\tilde{M}, \mathbb{C}) \) to get

\[
0 \rightarrow C_i^{\varepsilon,\rho}(M, \mathbb{C}[t^{\pm 1}]) \rightarrow C_i^{\varepsilon,\rho}(M, \mathbb{C}[t^{\pm 1}]) \rightarrow V \otimes_{\mathbb{C}[\pi_1(M)]} C_i(\tilde{M}, \mathbb{C}) \rightarrow 0
\]

These short exact sequences for \( i \in \mathbb{Z} \) extend to a short exact sequence of complexes (i.e. they are compatible with the differentials), so we get the corresponding long exact sequence in homology, namely

\[
\ldots \rightarrow H^i_\varepsilon(M, \mathbb{C}[t^{\pm 1}]) \xrightarrow{t^{-a}} H^i_\varepsilon(M, \mathbb{C}[t^{\pm 1}]) \rightarrow H_i(M, \mathcal{L}_a \otimes V_\rho)
\]

\[
\rightarrow H_{i-1}^i(M, \mathbb{C}[t^{\pm 1}]) \rightarrow \ldots \tag{12}
\]

By Theorem 4 and the fact that \( \mathbb{C}[t^{\pm 1}] \) is a principal ideal domain, we get that, for
\( 0 \leq i \leq n - 1 \), the twisted Alexander modules have a primary decomposition of the form

\[
H^i_\varepsilon(M, \mathbb{C}[t^{\pm 1}]) \cong \mathbb{C}[t^{\pm 1}]/(t^{(1)} - b_1)^{r_1} \oplus \ldots \oplus \mathbb{C}[t^{\pm 1}]/(t^{(1)} - b_{r_i})^{r_i}.
\]

Let \( N(a, i) \) be the number of direct summands in the \( (t - a) \)-torsion part of
\( H^i_\varepsilon(M, \mathbb{C}[t^{\pm 1}]) \). We have that

\[
N(a, i) = \dim_{\mathbb{C}} \ker \left( H^i_\varepsilon(M, \mathbb{C}[t^{\pm 1}]) \xrightarrow{t^{-a}} H^i_\varepsilon(M, \mathbb{C}[t^{\pm 1}]) \right).
\]

Let us consider (12) as a long exact sequence of vector spaces. By a dimension counting
argument, we deduce that

\[
\dim_{\mathbb{C}} H_i(M, \mathcal{L}_a \otimes V_\rho) = N(a, i) + N(a, i - 1)
\]

for \( 0 \leq i \leq n - 1 \). Note that \( a \) is a zero of \( \Delta^i_\varepsilon(M) \) if and only if \( N(a, i) \geq 1 \). Thus

\[
\{a \in \mathbb{C}^* \mid \Delta^i_\varepsilon(M)(a) \cdot \Delta^{i-1}_\varepsilon(M)(a) = 0 \} = V^{i-1}_\varepsilon(M, \rho) \cap \text{Im}(e^*)
\]

for \( 0 \leq i \leq n - 1 \).
Taking into account that \( H_n^{ε,ρ}(M, \mathbb{C}[t^{±1}]) \) is a free \( \mathbb{C}[t^{±1}] \)-module of dimension \( \dim_\mathbb{C} \mathcal{V} \cdot |\chi(M)| \) (Theorem 4), by a dimension counting argument in the long exact sequence (12), we have that

\[
\dim_\mathbb{C} H_n(M, \mathcal{L}_a \otimes V_ρ) = \dim_\mathbb{C} \mathcal{V} \cdot |\chi(M)| + N(a, n - 1).
\]

Thus,

\[
\{ a \in \mathbb{C}^n \mid \Delta_n^{ε,ρ}(M)(a) = 0 \} = \mathcal{V}_n^{\dim_\mathbb{C} \mathcal{V} \cdot |\chi(M)| + 1}(M, ρ) \cap \text{Im}(ε^*).
\]

\( \square \)

The result that we just proved, along with the main results of Sect. 4, can give us some information about the rank 1 twisted homology jump loci of \( M \). More specifically, the following corollaries follow from Proposition 4, Theorem 5, Theorem 6 and Theorem 7 respectively.

**Corollary 3** Let \( \mathbb{F} = \mathbb{C} \). Using the same notation and under the same assumptions of Proposition 4, we have that the points of

\[
\mathcal{V}_0^1(M, ρ) \cap \text{Im}(ε^*)
\]

are in one-to-one correspondence with the common roots of all of the dimension \( \dim_\mathbb{C} \mathcal{V} \) minors of the column matrix with entries

\[
t^{ε(a_i)}(ρ) | t = a \in \mathcal{M}(\mathcal{V} \otimes \mathcal{V})(\mathbb{C}[t^{±1}]), \quad i = 1, \ldots, m.
\]

**Corollary 4** Let \( \mathbb{F} = \mathbb{C} \). Using the same notation and under the same assumptions of Theorem 5, we have that both

\[
\mathcal{V}_1^1(M, ρ) \cap \text{Im}(ε^*) \quad \text{and} \quad \mathcal{V}_2^{\dim_\mathbb{C} \mathcal{V} \cdot |\chi(M)| + 1}(M, ρ) \cap \text{Im}(ε^*)
\]

are contained in

\[
\left\{ a \in \mathbb{C}^n \mid \left( \prod_{k=1}^s \det_ε,ρ(β_k)^{d_k - 2} \right) \mid t = a \cdot \left( \prod_{i=1}^m \det_ε,ρ(α_i)^{s_i - 1} \right) \mid t = a \cdot \Delta_0^{ε,ρ}(M)(a) = 0 \right\}.
\]

**Corollary 5** Let \( \mathbb{F} = \mathbb{C} \). Using the same notation and under the same assumptions of Theorem 6, we have that both

\[
\mathcal{V}_1^1(M, ρ) \cap \text{Im}(ε^*) \quad \text{and} \quad \mathcal{V}_2^{\dim_\mathbb{C} \mathcal{V} \cdot |\chi(M)| + 1}(M, ρ) \cap \text{Im}(ε^*)
\]

are contained in

\[
\bigcap_{i=1}^l \left\{ a \in \mathbb{C}^n \mid \Delta_0^{ε,ρ}(M)(a) \cdot \left( \prod_{k=1}^{s_i} \det_ε,ρ(β_{k,i})^{d_k - 2} \right) \mid t = a \cdot \left( \det_ε,ρ(α_i)^{s_i - 1} \right) \mid t = a = 0 \right\}.
\]

**Corollary 6** Let \( \mathbb{F} = \mathbb{C} \). Using the same notation and under the same assumptions of Theorem 7, we have that

\[
\mathcal{V}_i^1(M, ρ) \cap \text{Im}(ε^*)
\]

for \( 0 \leq i \leq n - 1 \), and

\[
\mathcal{V}_n^{\dim_\mathbb{C} \mathcal{V} \cdot |\chi(M)| + 1}(M, ρ) \cap \text{Im}(ε^*)
\]

are all contained in

\[
\left\{ a \in \mathbb{C}^n \mid \prod_{k=0}^{n-1} \prod_{l=1}^{s_k} \det_ε,ρ(γ_∞(F_{l,k})) \mid t = a = 0 \right\}.
\]
Acknowledgements The author would like to thank Laurențiu Maxim for all of his guidance and support during this project, as well as the referee for their helpful comments. She would also like to acknowledge Enrique Artal-Bartolo, José Ignacio Cogolludo-Agustín, and Miguel Ángel Marco-Buzunáriz for the interesting discussions we had about this topic.

References

1. Arvola, W.A.: The fundamental group of the complement of an arrangement of complex hyperplanes. Topology 31(4), 757–765 (1992)
2. Cogolludo Agustín, J.I., Florens, V.: Twisted Alexander polynomials of plane algebraic curves. J. Lond. Math. Soc. (2) 76(1), 105–121 (2007)
3. Cohen, D.C., Suciu, A.I.: The boundary manifold of a complex line arrangement. In: Groups, homotopy and configuration spaces, Geom. Topol. Monogr., vol. 13, pp. 105–146. Geom. Topol. Publ., Coventry (2008)
4. Dimca, A.: Singularities and topology of hypersurfaces. Universitext. Springer-Verlag, New York (1992)
5. Dimca, A.: Sheaves in topology. Universitext. Springer-Verlag, Berlin (2004)
6. Dimca, A., Némethi, A.: Hypersurface complements, Alexander modules and monodromy. In: Real and complex singularities, Contemp. Math., vol. 354, pp. 19–43. Am. Math. Soc., Providence, RI (2004)
7. Falk, M.: Homotopy types of line arrangements. Invent. Math. 111(1), 139–150 (1993)
8. Florens, V., Guerville-Ballé, B., Marco-Buzunariz, M.A.: On complex line arrangements and their boundary manifolds. Math. Proc. Cambridge Philos. Soc. 159(2), 189–205 (2015)
9. Godement, R.: Topologie algébrique et théorie des faisceaux. Actualités Sci. Ind. No. 1252. Publ. Math. Univ. Strasbourg. No. 13. Hermann, Paris (1958)
10. Hironaka, E.: Boundary manifolds of line arrangements. Math. Ann. 319(1), 17–32 (2001)
11. Jiang, T., Yau, S.S.T.: Intersection lattices and topological structures of complements of arrangements in CP^2. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 26(2), 357–381 (1998)
12. Kirk, P., Livingston, C.: Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants. Topology 38(3), 635–661 (1999)
13. Kohno, T., Pajitnov, A.: Circle-valued Morse theory for complex hyperplane arrangements. Forum Math. 27(4), 2113–2128 (2015)
14. Libgober, A.: On the homotopy type of the complement to plane algebraic curves. J. Reine Angew. Math. 367, 103–114 (1986)
15. Libgober, A.: The topology of complements to hypersurfaces and nonvanishing of a twisted de Rham cohomology. In: Singularities and complex geometry (Beijing, 1994), AMS/IP Stud. Adv. Math., vol. 5, pp. 116–130. Am. Math. Soc., Providence, RI (1997)
16. Libgober, A.: Eigenvalues for the monodromy of the Milnor fibers of arrangements. In: Trends in singularities, Trends Math., pp. 141–150. Birkhäuser, Basel (2002)
17. Maxim, L., Wong, K.T.: Twisted Alexander invariants of complex hypersurface complements. Proc. R. Soc. Edinburgh Sect. A 148(5), 1049–1073 (2018)
18. Milnor, J.: Morse theory. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J. (1963)
19. Milnor, J.: Whitehead torsion. Bull. Am. Math. Soc. 72, 358–426 (1966)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.