High Temperature Superconductivity and Effective Gravity

Djordje Minic* and Jean J. Heremans†

Department of Physics, Virginia Tech
Blacksburg, VA 24061, U.S.A.

Abstract

We argue that an approach involving effective gravity could play a crucial role in elucidating the properties of the high temperature superconducting materials. In particular we propose that the high critical temperature might be naturally explained in a framework constructed as a direct condensed matter analog of the Randall-Sundrum approach to a geometrization of the hierarchy problem in high-energy physics.
1 Introduction

The problem of high temperature superconductivity is one of the most outstanding puzzles in contemporary physics [1]. Recently much activity has been devoted to the possible relevance of the gauge theory/gravity duality (i.e. the anti de Sitter space/conformal field theory correspondence) to this remarkable problem [2]. The general concept of duality, on which a lot of recent progress in quantum field theory and string theory hinges, has also been used recently in various condensed matter settings [3, 4]. Furthermore, there has been a lot of work regarding the induced gravitational processes in various condensed matter systems (see for example [5], [6] for some recent samplings of the literature).

In this note we observe that if indeed effective gravity could arise in condensed matter systems then there are possible dramatic effects associated with its geometric physics. For example, widely separated energy scales can be related through a warping (red-shift) factor. This possibility has been widely discussed in the recent high-energy literature. In particular, in the context of the Randall-Sundrum scenario [7], a geometric warping factor is used to naturally bump up the 1TeV scale to the Planck scale, thus evading the hierarchy problem.

The main aim of this note is to point out a close analogy between the naturalness problem associated with the Higgs mass from the realm of particle physics and the naturalness problem associated with the high critical temperature ($T_C$) of high temperature superconductors. In view of the recent geometric proposals for dealing with the particle physics naturalness problem, i.e. the Randall-Sundrum scenarios (which indeed have not been tested experimentally, and are, at the moment, only theoretical suggestions) we point out, that a similar geometric mechanism can be naturally used in a condensed matter context. We propose a condensed matter analog of the Randall-Sundrum scenario for a geometrically induced hierarchy of scales. We argue that in the context of superconductivity, such a scenario can lead to the high $T_C$ of cuprate (Cu-O) and oxypnictide (Fe-As) superconductors, both representing families of layered anisotropic compounds with complex normal-state Fermi surfaces. Our discussion is based entirely on effective field theory (i.e. a gravitationally dressed Landau-Ginsburg approach) and does not address deeper issues concerning the microscopic mechanism of high $T_C$ superconductivity. Nevertheless, the new effective description may point toward previously unexplored avenues for understanding the underlying microscopic physics. Because our approach concentrates on the subsection of the phase diagram containing the strange metal/superconducting phase, we ask how the usual canonical Wilsonian thinking associated with naturalness of Fermi liquids should be modified for a geometric scenario akin to Randall-Sundrum to be realized in an effective Landau-Ginsburg description of the superconducting phase. Thus our proposal can be summarized as: a) a description of high $T_C$ superconductors based on a gravitationally dressed version of the canonical Landau-Ginsburg Lagrangian (and the gravitationally dressed Fermi liquid as its normal phase) presented in Section 2 and b) a specific proposal for the geometry of this induced gravitational background, which is appropriate for the anisotropic geometric properties of the known high $T_C$ materials presented in Sections 3 and 4. The main result of the paper is presented in Section 4. The specific geometry, as in the Randall-Sundrum scenario, uses a slice of anti de Sitter space (AdS) and that is the gravitational metric that appears in the gravitationally dressed...
Landau-Ginsburg Lagrangian for the high-temperature superconductors.

Note that, even though a comparison can be drawn with the approach used for the currently pursued AdS superconductors [2] (which indeed build on an AdS black hole, more specifically the Abelian Higgs model in that background) we do not rely on the same gravitational description. Our background, like the original Randall-Sundrum description, represents a slice of the AdS space. A dual description of our proposal can be in principle envisioned, and it is natural to conclude that this would involve a non-relativistic AdS/conformal field theory correspondence [4].

2 Fermi liquids vs. non-Fermi liquids

As noticed by many authors, various condensed matter systems, such as superfluids or Bose-Einstein condensates, exhibit physical phenomena that can be interpreted by invoking effective gravity [5]. In these contexts, which can be viewed as analog models of gravity, one speaks for example of an effective acoustic metric, acoustic black holes (i.e. “dumb holes”) or emergent relativity.

Given the intricacies of the physics of high temperature superconducting materials and the complexity of their phase diagram, one might ask whether the strong electron correlations responsible for these phenomena can induce effective gravitational effects, thus opening a possibility for a geometric explanation of some of the outstanding puzzles.

We concentrate on one such puzzle: the actual high value of the critical temperature. In particular, given the fact that in recent years high energy physicists have invoked gravity to explain the unnatural mass scale hierarchy we propose a condensed matter analog of the same mechanism. In the context of high temperature superconductivity we argue for a geometric reinterpretation of the hierarchy that exists between the ordinary critical temperatures for low-temperature superconductors and their high temperature relatives. This geometric reinterpretation may point to a new effective gravitational physics which may be responsible, among other things, for the high critical temperature of the high temperature superconducting materials.

A geometrically induced hierarchy of scales forces a revisiting of the Wilsonian approach to the Fermi surface, to ascertain consequences for Fermi Liquid theory in the renormalization group approach, and for the emergence of non-Fermi Liquid behavior. From the effective field theory standpoint, the question can be formulated: how does the low energy Wilsonian action with effective gravity modify the usual scalings of the effective low energy field theory of the Fermi surface?

2.1 The Fermi liquid

To set the stage for our discussion let us summarize the classic effective field theory of Landau Fermi liquids as most succinctly presented by Polchinski [8].

One starts with a natural (i.e. not-finely tuned) Fermi surface and decomposes the
momenta into the Fermi momentum and a component orthogonal to the Fermi surface

\[ \vec{p} = \vec{k} + \vec{l} \]  

and then one considers scaling of energy and momentum towards the Fermi surface, in other words

\[ E \rightarrow sE, \quad \vec{k} \rightarrow \vec{k}, \quad \vec{l} \rightarrow s\vec{l}. \]  

The lowest order action, to quadratic order, is then given as

\[ S_{FL} = \int L_{FL}(\psi, \partial \psi) \equiv \int dtd^3\vec{p}[i\psi^*(\vec{p})\partial_t \psi(\vec{p}) - (E(\vec{p}) - E_F(\vec{p}))\psi^*(\vec{p})\psi(\vec{p})]. \]  

Close to the Fermi surface

\[ E(\vec{p}) - E_F(\vec{p}) \sim lv_f, \quad v_F = \partial_{\vec{p}}E \]  

so that after the agreed renormalization orthogonal to the Fermi surface (note that also \( t \rightarrow s^{-1}t \)) one obtains

\[ \psi \rightarrow s^{-1/2}\psi. \]  

Note that this scaling leads immediately to the usual two point function for a free quasiparticle with a single particle pole. Now, by considering 4-Fermi interactions one can see that for generic momenta the 4-Fermi interaction scales as a positive power of \( s \) and is thus irrelevant at low energy. The measure over time contributes one negative power, the measure over the momenta orthogonal to the Fermi surface contributes 4 powers and the 4-Fermi interaction contributes 4/2 negative powers. The delta function over the 4 momenta generically does not scale. So the overall generic scaling for the 4-Fermi vertex is indeed

\[ s^{-1+4-4/2} = s^1. \]  

This is valid, except if the momenta are paired. In that case the scaling goes as \( s^0 \), because now the delta function depends only on the sum of momenta orthogonal to the Fermi surface and due to

\[ \delta(sl) \rightarrow s^{-1}\delta(l) \]  

the 4-fermi interaction indeed scales as \( s^0 \) and is marginal. (This encapsulates the usual Cooper pairing phenomenon.) A one loop calculation of the beta function does ensure the asymptotic freedom provided the interaction is perturbatively attractive, as is the case for phonon-electron interaction, which leads to the strong coupling regime (with bound states forming) in the infrared. The resulting wave-function describing the superconducting state is of a BCS kind

\[ \Psi_{BCS} \sim \prod_k (\alpha + \beta a_k^* a_{-k}) \Psi_{FL} \]  

where \( \alpha \) and \( \beta \) are the usual variational parameters and \( a^* \) denote the creation operators of the electron quasiparticle.
2.2 The Fermi liquid coupled to effective gravity

In what follows we outline an intuitive argument for the crucial relevance of an effective gravitational description of the normal state of the high $T_C$ materials. How could effective gravity arise from the dynamics of the Fermi surface? First, we note that experiments indicate a highly irregular Fermi surface in the normal state, resulting from the microscopic physics [1]. This irregularity in turn could lead to an effective gravitational description. As a concrete model, let us view the Fermi surface as an incompressible fluid in momentum space. Then in analogy with the discussion of induced gravity in fluid dynamics ([9], [5]) we can envision generating an effective metric, which in the case of real irrotational fluids is precisely the acoustic metric [9]. This effective metric is generated from the fluctuations of the fluid density $\rho$ and the velocity potential $\phi$ (where the velocity $\vec{v} = \nabla \phi$). The underlying space-time action of the moving fluid is

$$ S = \int dx^4 \left[ \rho \dot{\phi} + \frac{1}{2} \rho (\nabla \phi)^2 + U(\rho) \right] $$

(9)

with $U(\rho)$ denoting the effective potential. Upon variation of this action one obtains the equations of motion for $\rho$ and $\phi$ (the Euler continuity equation and the Bernoulli energy balance equation). When these equations of motion are perturbed around the equilibrium values $\rho_0$ and $\phi_0$ one is led to the equations for the fluctuations of the velocity potential $\varphi$. In particular, the equation for the fluctuations of the velocity potential can be written in a geometric form [9]

$$ \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b \varphi) = 0. $$

(10)

The effective space-time metric has the canonical ADM form of [9] and [3] (apart from a conformal factor) and has the Lorentzian signature

$$ ds^2 = \frac{\rho_0}{c} [c^2 dt^2 - \delta_{ij} (dx^i - v^i dt)(dx^j - v^j dt)] $$

(11)

where $c$ is the sound velocity$^1$ and $v^i$ are the components of the fluid’s velocity vector.

Note that the collective dynamics of the Fermi surface can be considered as a “bosonization” of the Fermi liquid [10] and in that approach the quasiparticle excitations can be represented by a collective mode of the “bosonized” Fermi liquid. That approach runs into trouble with the essential difference between fermions and bosons in spatial dimensions one (where the bosonization can be used because of the very special kinematics) and spatial dimensions above one, where such efforts are largely prohibitive$^2$. In contrast, in our proposal the non-Fermi liquid behavior originates from the gravitational dressing, itself caused by the non-trivial geometry and topology of the Fermi surface, an experimental fact. Thus we argue that the new quasiparticles are the usual fermions, but propagating in a non-trivial gravitational background and hence dressed by the gravitational fluctuations. In other words, the collective motion of the Fermi surface is of an effective gravitational kind (i.e. not spin 0 but spin 2) and the usual fermionic quasiparticles are now coupled to this collective spin

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1For a plasmon velocity for a system of charged fermions
2For a sample of the literature on higher dimensional bosonizations please consult [10].
2 bosonic mode. Thus we propose that the effective theory of the strange normal state of high $T_C$ superconductors is a gravitationally dressed Fermi liquid, i.e. the usual Fermi liquid albeit coupled to gravity

$$S_{NFL} = \int \sqrt{-g} L_{FL}(\psi, \nabla \psi).$$  

(12)

Applying this approach to the Fermi surface in momentum space, by relying on the induced effective diffeomorphism invariance, we are led to conclude that the effective action for the fermionic quasiparticle around the Fermi surface should be a gravitationally dressed action, in which the naive scaling dimensions discussed above can be changed by adding gravitational dressing. In general this would mean that instead of $\psi \to s^{-1/2} \psi$ as discussed above, we should have

$$\psi \to s^{-1/2 + \alpha} \psi$$  

(13)

where $\alpha$ denotes gravitational dressing. Thus the fermionic two point function can be changed to scale as $s^{-1+2\alpha}$, indicative of a non-Fermi liquid behavior.

Naturally, the anomalous dimensions of Fermi propagators have been considered previously [1, 12, 14], and our proposal is not unique in placing emphasis on the anomalous fermionic propagators. Also, the phenomenology implied by anomalous fermionic propagators, including non-Fermi liquid behavior such as a resistivity linear in temperature $T$ [12] is inherent to our proposal as well. But the proposal that the anomalous nature of the fermionic two-point function follows from the concept of a normal state based on a gravitationally dressed Fermi liquid Lagrangian describing the anomalous normal state is, to our knowledge, new.

In this scenario, the normal state is described by a gravitationally dressed Fermi-liquid theory. Similarly, the BCS wave-function would be gravitationally dressed. \textit{Note that our discussion is couched in terms of an effective field theory and therefore does not carry information about the underlying microscopic mechanism.} In particular, the specific nature of the pairing mechanism and its relationship with the effective gravitational description is outside the scope of the present approach. The approach generally does conclude that the usual effective Landau-Ginsburg theory describing the physics of the condensate is gravitationally dressed as follows

$$S_{LGo} = \int d^3 x \sqrt{-g} [g^{\mu \nu} \nabla_\mu H^* \nabla_\nu H - V(|H|^2)]$$  

(14)

where the complex order parameter is denoted by $H$ and $V(|H|^2)$ is an effective potential (for example, of a quartic type). In what follows, most important is that such a gravitationally dressed Landau-Ginsburg theory leads to a simple geometric mechanism for the high $T_C$ of high temperature superconductors.

\textsuperscript{3}Such dressing can be explicitly computed in simple cases such as the coupling of 2d gravity to 2d matter as discussed in [11].

\textsuperscript{4}In the marginal Fermi liquid theory, through couplings with an effective, induced gauge field, in the context of quantum critical fixed points, and many other proposals [1]
3 Effective gravity and energy scale hierarchy

To implement a condensed matter analog of the gravitationally induced energy scale hierarchy, we first present a brief summary of the Randall-Sundrum proposal \cite{7} which precisely accomplishes this goal. The exponential hierarchy is induced by a warped extra dimensional geometry. The actual example envisions two three-branes (i.e. two 3+1 dimensional worlds), separated by a distance of \( r_c \) in one extra (fifth) spatial dimension. This bulk 4+1 dimensional space represents a portion of the 4+1 dimensional AdS space, whose cosmological constant is determined by another parameter of the model, \( k \). One of the branes carries the degrees of freedom of the Standard Model. The other carries the Planck scale degrees of freedom.

This set-up can be carried over from the 3+1 to the 2+1 dimensional context. The 2+1 dimensional context carries relevance, because the high \( T_C \) materials are layered anisotropic compounds where the Cu or Fe planes can be identified with 2+1 dimensional branes. The extra anisotropic dimension is to be identified with the actual anisotropy of these materials. The crucial assumption in this set-up is that effective gravity is induced, perhaps by the internal dynamics of geometrically very complicated Fermi surfaces. The warped geometry of the extra dimension could originate from the inter-layer coupling. We address the question of how the effective gravitational description could be induced from the dynamics of the Fermi surface in Section 4.

Returning to the Randall-Sundrum scenario in the condensed matter situation, the 3+1 bulk metric reads

\[
d s^2 = r_c^2 d u^2 + e^{-2 k r_c} u \eta_{\mu \nu} d x^\mu d x^\nu. \tag{15}\]

Here \( u \) denotes the extra anisotropic spatial direction of size \( l_c \), and \( \mu, \nu = 0, 1, 2 \) are the planar space-time indices. Also, \( \eta_{\mu \nu} \) is the flat planar space-time (Minkowski) metric. The gravitational fluctuations \( h_{\mu \nu} \) around the flat Minkowski metric define the background metric \( g^0_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \) \cite{7}. The low-energy effective action for a Higgs field (order parameter), i.e. the Landau-Ginsburg action, coupled to effective gravity is

\[
S_{LG_g} = \int d^3 x \sqrt{-g} [g^{\mu \nu} \nabla_\mu H^* \nabla_\nu H - \lambda (|H|^2 - v_0^2)^2] \tag{16}\]

where \( g_{\mu \nu} = e^{-2 k r_c} g^0_{\mu \nu} \) \cite{7} and \( g \) is the determinant of \( g_{\mu \nu} \). After rescaling the Higgs field (the Landau-Ginsburg order parameter) \( H \rightarrow e^{k r_c \pi} H \) the effective action becomes

\[
S_{LG_g} = \int d^3 x \sqrt{-g^0} [g^0_{\mu \nu} \nabla_\mu H^* \nabla_\nu H - \lambda (|H|^2 - e^{-2 k r_c \pi} v_0^2)^2], \tag{17}\]

Note that we can take \( l_c = \pi \) as in \cite{7}. Thus the physical mass scale is given by a blue-shifted symmetry-breaking scale (where \( v_0 \) is the minimum of the potential)

\[
v = v_0 e^{-k r_c \pi} \tag{18}\]

or alternatively, the relevant mass scales are related as

\[
m = m_0 e^{-k r_c \pi}. \tag{19}\]
This is the key result. Thus in order to obtain \( m_0 \sim 10^{19}\text{GeV} \) and \( m \sim 1\text{TeV} \) one only needs to require \( kr_c \sim 50 \). Note that even though this result has been derived in an effective field theory, there exists an ultraviolet (UV) completion of this scheme from the point of view of string theory \[13\].

The question arises why the Randall-Sundrum warp factor should occur in the interlayer coupling. We notice that the physics of interlayer tunneling considered in the past \[13\] naturally involves exponential wave functions and thus can be directly compared to the geometric exponential factors that appear in the Randall-Sundrum case. Thus the proposed geometrization captures the relevant interlayer physics, at least on the level of an effective, and not microscopic description. Here we emphasize the weak nature of the inter-layer coupling as opposed to the strong intra-layer coupling, the latter leading to non-Fermi liquid behavior and induced gravitational dressing as argued in the previous section. Finally we remark that in principle the geometric argument applies to both cuprates and oxypnictides, and indeed stresses the very features they bear in common.

4 Why high \( T_c \)?

Here we present a simple computation of \( T_c \) based on the gravitationally dressed Landau-Ginsburg effective description and the geometric argument presented above. This constitutes the main result of this note.

Remembering the usual BCS dispersion relation given a gap \( \Delta \)

\[
E = \sqrt{\Delta^2 + k^2}
\]  \hspace{1cm} (20)

in which the gap plays the role of the Higgs mass (given the well-known analogy with the relativistic dispersion relation for a particle of mass \( m \), i.e. \( E = \sqrt{m^2 + k^2} \)), we see that there exists a distinct possibility of similarly warping the gap, as in Eq. (5), and thus warping the \( T_c \).

Normally \( T_c \) is related to the gap evaluated at low momenta \[15\]

\[
T_c = \text{const} \Delta(0).
\]  \hspace{1cm} (21)

In the superconductors considered, one obtains a geometrically shifted temperature so that the following relation results

\[
T_{HT_c} = T_c e^{kr_c\pi}.
\]  \hspace{1cm} (22)

Here \( T_c \) denotes the low critical temperature of conventional superconductors - typically a few K. \( T_{HT_c} \) denotes the critical temperature of high \( T_c \) superconductors - equal to around 50 – 150 K.

Therefore, a high \( T_c \) naturally arises from the above exponential factor that is responsible for a hierarchy of energy scales. An exponential factor multiplying the usual 1-10 K critical temperature (determined by the value of the BCS-like gap at zero momenta), produces a new

\[5\]In particular, the complex issue of stability of the Randall-Sundrum scenario requires a UV complete description.
critical temperature in the 50-150 K range, or higher. A representative sample of, from top to bottom, cuprate, oxypnictide (Fe-As) \[18\], BCS type II, and BCS type I superconducting critical temperatures is collected in the table below. For example, transitioning from the lowest (Al) to the highest (Tl compound) requires only an argument of 5 in the exponent.

| Compound              | \( T_C \) (K) |
|-----------------------|---------------|
| \( \text{Tl}_2\text{Ba}_2\text{Ca}_2\text{Cu}_3\text{O}_{10} \) | 125           |
| \( \text{Bi}_2\text{Sr}_2\text{Ca}_2\text{Cu}_3\text{O}_{10} \) | 110           |
| \( \text{YBa}_2\text{Cu}_3\text{O}_7 \)          | 92            |
| \( \text{La}_{2-x}\text{Sr}_x\text{CuO}_4 \)        | 38            |
| \( \text{SmFeAsO}_{0.85}\text{F}_{0.15} \)         | 42            |
| \( \text{CeFeAsO}_{0.84}\text{F}_{0.16} \)         | 41            |
| \( \text{LaFeAsO}_{0.89}\text{F}_{0.11} \)         | 26            |
| \( \text{Nb}_3\text{Sn} \)                        | 18            |
| \( \text{NbTi} \)                                 | 10            |
| \( \text{Pb} \)                                   | 7.2           |
| \( \text{Al} \)                                   | 1.1           |

5 Conclusions

In this note we have proposed a condensed matter analog of the Randall-Sundrum scenario for a geometrically induced hierarchy of scales. We have argued that in the context of the physics of high temperature superconductivity, such a scenario can be responsible for the high \( T_C \) of cuprate and oxypnictide superconductors. The conjecture spontaneously leads to emphasis on the unusual features shared by cuprates and oxypnictides. Needless to say we have only scratched the surface of the manifold of issues relevant for high \( T_C \) superconductivity. Of primary importance remain a specific microscopic mechanism as well as a more detailed phenomenology derived from the gravitationally dressed Landau-Ginsburg approach. Apart from advancing understanding of the unusual normal as well as the superconducting phase, this new approach can illuminate other parts of the phase diagrams of high temperature superconductors. For instance, of future interest are the relevance of spin-spin interaction \[16\] for the effective gravitational description proposed in this note, a study whether the \( d \)-wave nature of the order parameter \[1\] naturally follows from a gravitationally dressed Landau-Ginsburg description, an understanding of the pseudo-gap region of the phase diagram, and finally connecting the recent discussion of Mottness \[17\] to our context. These and other questions are left for more detailed future work.

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