VOLUME-CONSTRAINT LOCAL ENERGY-MINIMIZING SETS IN A BALL

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ABSTRACT. In this paper, we prove a Poincaré inequality for any volume-constraint local energy-minimizing sets, provided its singular set is of Hausdorff dimension at most \( n - 3 \). With this inequality, we prove that the only volume-constraint local energy-minimizing sets in the Euclidean unit ball, whose singular set is closed and of Hausdorff dimension at most \( n - 3 \), are totally geodesic balls or spherical caps intersecting the unit sphere with constant contact angle; for stable sets in a wedge-shaped domain or in a half space, provided the same condition of the singular set, must be spherical. In particular, they are smooth.

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1. Introduction

The study of equilibrium shapes of a liquid confined in a given container has a long history. Since the work of Gauss, this subject has been studied through the introduction of a free energy functional. Precisely, for a liquid occupies a region \( E \) inside a given container \( \Omega \), its free energy is given by

\[
\sigma (P(E;\Omega) - \beta P(E;\partial \Omega)) + \int_E g(x)dx,
\]

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where $P$ is the perimeter functional as in (2.1).

Mathematically, we assume $\Omega \subset \mathbb{R}^n$ is a fixed bounded open set with smooth boundary and $E$ is a set of finite perimeter in $\Omega$. Here $\sigma \in \mathbb{R}_+$ denotes the surface tension at the interface between this liquid and other medium filling $\Omega$. $\beta \in \mathbb{R}$ is called relative adhesion coefficient between the fluid and the container, which satisfies $|\beta| < 1$ due to Young’s law, $g$ is typically assumed to be the gravitational energy, whose integral is called the potential energy. The free energy functional is usually minimized under a prescribed volume constraint. The existence of global minimizers of the free energy functional under volume constraint is easy to be shown by the direct method in calculus of variations, see for example [Mag12, Theorem 19.5].

For simplicity, we assume throughout this paper that $\sigma = 1, g = 0$, that is, we consider the energy functional

$$F_\beta(E; \Omega) = P(E; \Omega) - \beta P(E; \partial \Omega), \quad |\beta| < 1. \quad (1.1)$$

Provided $\partial E \cap \Omega$ is sufficiently smooth, the boundary of stationary points of $E$ for the corresponding variational problems are capillary hypersurfaces $\partial E \cap \Omega$, namely, constant mean curvature hypersurfaces intersecting $\partial \Omega$ at constant contact angle $\theta = \pi - \arccos \beta$. For the reader who is interested in the physical consideration of capillary surfaces, we refer to Finn’s celebrated monograph [Fin86] for a detailed account.

When $\beta = 0$, $F_\beta(E; \Omega)$ reduces to the perimeter functional $P(E; \Omega)$ of $E$ in $\Omega$. The structure and regularity of local minimizers of $P(E; \Omega)$ under volume constraint has been studied by Gonzalez-Massari-Tamanini [GMT83] and Gruter [GJ86; Gr87]. It was shown that for any local minimizer $E$, $\partial E \cap \Omega$ is smooth in $\Omega$ except for a singular set of Hausdorff dimension at most $n - 8$. Moreover, Sternberg-Zumbrun [SZ98] has derived a Poincaré-type inequality for any local minimizer $E$, provided the singular set in $\partial E \cap \Omega$ is of Hausdorff dimension at most $n - 3$. This Poincaré-type inequality can be used to prove the connectivity of stable solutions in convex domains, as well as smoothness of local minimizers in $\mathbb{B}^n (n-$dimensional Euclidean unit ball) under the condition $|E| < \frac{1}{n-1} H^{n-1}(E \cap S^{n-1})$, where $S^{n-1}$ denotes the $(n - 1)$-dimensional unit sphere. Such condition has been recently verified by Barbosa [Bar18]. Sternberg-Zumbrun [SZ98] have conjectured all the local minimizers in a convex domain are smooth. On the other hand, they constructed a local minimizer with singularity in a non-convex domain [SZ18]. Recently, Wang-Xia [WX19] classified all local minimizers in $\mathbb{B}^n$ to be either totally geodesic balls or spherical caps intersecting $S^{n-1}$ orthogonally. In particular, they proved the smoothness of local minimizers in $\mathbb{B}^n$. The classification for $n = 3$ has been proved by Nunes [Nun17]. Note that for $n = 3$, the local minimizers are a priori known to be smooth by virtue of [GMT83; Gr87].

We remark that, in the smooth setting, that is, provided $\partial E \cap \Omega$ is $C^2$, the Poincaré-type inequality is just the second variational formula for $P(E; \Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega)$ under volume constraint and the stability problem has been first investigated by Ros-Vergasta [RV95].

In this paper, we study the general case $|\beta| < 1$. In smooth setting,

$$F_\beta(E; \Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega) - \beta \mathcal{H}^{n-1}(\partial E \cap \partial \Omega).$$

As we have already mentioned, the boundary of stationary points of $E$ are capillary hypersurfaces $\partial E \cap \Omega$. The second variational formula for $F_\beta(E; \Omega)$ under volume constraint has been derived by Ros-Souam [RS97]. Wang-Xia [WX19] classified all smooth stable solutions in $\mathbb{B}^n$ to be either totally geodesic balls or spherical caps intersecting $S^{n-1}$ at constant contact angle $\theta = \arccos(-\beta)$. In general, as in the case of $\beta = 0$, the local minimizers of $F_\beta(E; \Omega)$ under volume constraint are not known to be smooth. Therefore, one needs to study the regularity of local minimizers as well as Sternberg-Zumbrun’s Poincaré-type inequality in the case $|\beta| < 1$. 
The main purpose is to study the Poincaré-type inequality for local minimizers of $F_\beta(E;\Omega)$ under volume constraint in the case $|\beta| < 1$.

1.1. Notation. In all follows, we denote by $\mathcal{H}^k$ the $k$-dimensional Hausdorff measure in $\mathbb{R}^n$, by $\omega_n$ the volume of $n$-dimensional Euclidean unit ball, by $\overline{E}$ the topological closure of a set $E$, by $\text{Int}(E)$ the topological interior of $E$, by $E^c$ the topological complement of $E$ and by $\partial E$ the topological boundary of $E$.

For the constrained isoperimetric problems, the container $\Omega \subset \mathbb{R}^n$ is assumed to be an open domain possibly unbounded with smooth boundary $\partial \Omega$. Let $E \subset \Omega$ be a set with finite volume and perimeter, let $M$ denote the closed set $\partial E \cap \overline{\Omega}$, let $\text{reg}M$ denotes the $C^2$ part of $M$ in $\overline{\Omega}$, let $\text{sing}M = M \setminus \text{reg}M$ denote the singular part of $M$; let $B^+$ denote the set $\text{Int}(\partial E \cap \partial \Omega)$, which is open and smooth; let $\Gamma$ denote the closed set $M \cap \partial B^+$, namely, $\Gamma = \partial M \cap \partial \Omega = \partial B^+$. Throughout this paper, $B_M$ denotes the second fundamental form of $\text{reg}M$ in $\mathbb{R}^n$ with respect to $-\nu_M$ and $B_{\partial \Omega}$ denotes the second fundamental form of $\partial \Omega$ with respect to the inwards pointing unit normal $-\nu_{B^+}$, $|B_M|^2 = \sum_{i=1}^{n-1} \kappa_i^2$, where $\{\kappa_i\}$ are the principal curvatures of $M$. When taking an orthonormal basis $\{\tau_i\}_{i=1}^{n-1}$ on $TM$, the mean curvature $H$ of $M$ with respect to $B_M$ is given by $H = \sum_{i=1}^{n-1} B_M(\tau_i, \tau_i)$.

We introduce the following admissible family of sets of finite perimeter for the study of fixed-volume variation.

**Definition 1.1.** For some $T > 0$, a family of sets of finite perimeter in $\Omega$, denoted by $\{E_t\}_{t \in (-T,T)}$, with each $E_t$ of finite perimeter and $E_0 = E$, is called admissible, if:

1. $\chi_{E_t} \to \chi_E$ in $L^1(\Omega)$ as $t \to 0$,
2. $t \to F_\beta(E_t;\Omega)$ is twice differentiable at $t = 0$,
3. $|E_t| = |E|$ for all $t \in (-T,T)$.

The stationary and stable sets in our settings are defined in the following sense.

**Definition 1.2.** For a set of finite perimeter $E \subset \Omega \subset \mathbb{R}^n$, let $F_\beta(t) := F_\beta(E_t)$. $E$ is said to be stationary for energy functional $F_\beta$ under volume constraint if $F'_\beta(0) = 0$ for all admissible families $\{E_t\}$. A stationary set $E$ is called stable if $F''_\beta(0) \geq 0$ for all admissible families $\{E_t\}$.

Our main results are as follows.

1.2. Main results.

**Theorem 1.1.** Let $E \subset \Omega$ be set of finite perimeter, which is stable for $F_\beta, |\beta| < 1$, under volume constraint, as in Definition 1.2. Using the notations in Section 1.1, if the singular set $\text{sing}M$ is closed and satisfying $\mathcal{H}^{n-3}(\text{sing}M) = 0$. Then, we have: the mean curvature of $\partial \Omega$ is a constant $H$ and $\nu_M \cdot \nu_{B^+} = -\beta$ along $\partial \Omega$. Moreover, for any smooth function $\zeta : \text{reg}M \to \mathbb{R}^1$ with

\[
\int_M \zeta(x) d\mathcal{H}^{n-1}(x) = 0, \tag{1.2}
\]

the following Poincaré-type inequality holds,

\[
J(\zeta) := \int_{M \cap \Omega} (|\nabla^M \zeta|^2 - |B_M|^2 \zeta^2) d\mathcal{H}^{n-1}(x) - \int_{M \cap \partial \Omega} q\zeta^2 d\mathcal{H}^{n-2} \geq 0 \tag{1.3}
\]

\[\text{Namely, } B_M(X,Y) = \langle \nabla_X \nu_M, Y \rangle \text{ for any } X,Y \in TM.\]
where

\[ q = \frac{1}{\sin \theta} B_{\partial \Omega}(\nu^B_1, \nu^B_1) - \cot \theta B_M(\nu^M_1, \nu^M_1). \tag{1.4} \]

Equivalently,

\[ -\int_{M \cap \Omega} (\Delta_M \zeta + \|B_M\|^2 \zeta^2) \, d\mathcal{H}^{n-1}(x) + \int_{M \cap \partial \Omega} (\nabla^M \zeta \cdot \nu^M_1 - q \zeta^2) \, d\mathcal{H}^{n-2} \geq 0, \tag{1.5} \]

where \( \nabla^M \) denotes the tangential gradient with respect to \( M \) and \( \Delta_M \) denotes the tangential Laplacian with respect to \( M \).

**Remark 1.1.** When \( \beta = 0 \), the inequality (1.3) or (1.5) reduces to the Poincaré-type inequality by Sternberg-Zumbrun [SZ98, Theorem 2.2]. When \( M \) is \( C^2 \), the inequality (1.3) or (1.5) has been derived by Ros-Souam [RS97].

**Remark 1.2.** We remark that the interior regularity result of Gonzalez-Massari-Tamanini [GMT83] tells that \( \mathcal{H}^{n-8+\gamma}(\text{sing}(\partial E \cap \Omega)) = 0 \) for any \( \gamma > 0 \). When \( \beta = 0 \), the boundary regularity result of Grüter and Jost [GJ86; Grü87] tells that \( \text{sing}(\partial E \cap \Omega) \) is closed and \( \mathcal{H}^{n-8+\gamma}(\text{sing}(\partial E \cap \Omega)) = 0 \) for any \( \gamma > 0 \). However, it seems unclear if one has the boundary regularity as Grüter and Jost for general \( |\beta| < 1 \), see for example [KT17].

We will follow closely the proof of Sternberg-Zumbrun [SZ98], where the case \( \beta = 0 \) has been handled. The key step is the construction of smooth cut-off functions vanishing near \( \text{sing} M \). These cut-off functions were first introduced in [SZ99]. With the help of such smooth cut-off functions, we can define smooth test functions which coincide with \( \zeta \nu_M \) outside some neighborhood of \( \text{sing} M \). To construct such smooth cut-off functions, we use the stationary condition to associate \( M \) to a general varifold inside \( \Omega \) intersecting \( \partial \Omega \) with fixed contact angle, which was introduced by Kagaya-Tonegawa [KT17]. By virtue of this, we can use monotonicity formulas, both inside \( \Omega \) and near the boundary \( \partial \Omega \), to obtain a uniform control on the ratio \( \mathcal{H}^{n-1}(B_r(x) \cap M) / r^{n-1} \) for every small ball \( B_r(x) \) which together form a covering of \( \text{sing} M \). This uniform bound enables us to construct the smooth cut-off functions that we will be needed.

By using the Poincaré-type inequality (1.5), we can get the regularity and classification of stable sets in the unit ball \( \mathbb{B}^n \) whose singular set is of Hausdorff dimension at most \( n - 3 \).

**Theorem 1.2.** Let \( E \subset \mathbb{B}^n \) be stable as in Definition 1.2. If the singular set \( \text{sing} M \) is closed and satisfying \( \mathcal{H}^{n-3}(\text{sing} M) = 0 \). Then \( M \) is either a totally geodesic ball or spherical cap, in particular, \( M \) is smooth.

As we mentioned before, when \( M \) is \( C^2 \), this result was proved by Wang-Xia [WX19, Theorem 3.1]. The key ingredient of their proof is a special choice of test function \( \varphi \in C^2(M) \) with \( \int_M \varphi = 0 \), which depends on a conformal Killing vector field in \( \mathbb{R}^n \) tangential to \( S^{n-1} \). With the help of the Poincaré-type inequality (1.5), we can use their test function in \( \text{reg} M \) and their proof still works for the case when \( \mathcal{H}^{n-3}(\text{sing} M) = 0 \). Moreover, we construct a volume preserving perturbation which strictly decreases the perimeter. This contradicts to the fact that \( E \) is capillary stable, and hence rule out the singularities.

On the other hand, we can also prove that the stable measure-theoretic capillary hypersurface in a wedge-shaped domain must be spherical and smooth, which generalizes the results of the smooth case [LX17; Sou21] to the measure-theoretic sense.
Theorem 1.3. Let $\Omega \subset \mathbb{R}^n$ be a wedge-shaped domain\(^2\) with planar boundaries $P_1, \ldots, P_L$. Let $E \subset \Omega$ be a set with finite volume and perimeter, which is stable as in Definition 6.1. If the singular set $\text{sing} M$ is closed with $\mathcal{H}^{n-3}(\text{sing} M) = 0$ and $|k| \leq 1$\(^3\), then $M$ must be a spherical cap, in particular, $M$ is smooth.

1.3. Organization of the paper. The paper is organized as follows. In Section 2 we recall some background materials from Geometric Measure Theory. In Section 3 we prove that the regular part of the boundary of any stationary set is capillary(Proposition 3.1). In Section 4, we first construct smooth cut-off functions(Lemma 4.1), which are crucial for deducing the Poincaré-type inequality, then we finish the proof of (Theorem 1.1). In Section 5, as an application of the Poincaré-type inequality, we give the classification of stable sets in $\mathbb{B}^n$(Theorem 1.2). In Section 6, as another application, we prove that the stable measure-theoretic capillary hypersurface in a wedge-shaped domain must be smooth and spherical(Theorem 1.3), in particular, the case when the wedge-shaped domain is a half space is included(Corollary 6.1).

2. Preliminaries

Throughout this paper, we denote by $|E|$ the $n$-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$, $\chi_E$ denotes the indicator function of $E$. Also, $B_r(x)$ denotes a $n$-dimensional open ball in $\mathbb{R}^n$ with radius $r$ and centered at $x$. $\mathbb{B}^n, S^{n-1}$ denote the $n$-dimensional unit ball, $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$, respectively. $\text{div}, \nabla$ will always denote the divergence operator, gradient operator in $\mathbb{R}^n$, respectively.

2.1. Sets of finite perimeter. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, namely,

$$P(E) := \int_{\mathbb{R}^n} |D\chi_E| = \sup \left\{ \int_{\mathbb{R}^n} \text{div} X d\mathcal{L}^n : X \in C^1_c(\mathbb{R}^n; \mathbb{R}^n), |X| \leq 1 \right\} < \infty.$$ 

Suppose $\Omega \subset \mathbb{R}^n$ is open and $E \subset \mathbb{R}^n$ is a Borel set. $E$ is called a set of finite perimeter in $\Omega$ if:

$$P(E; \Omega) := \int_{\Omega} |D\chi_E| = \sup \left\{ \int_{E} \text{div} X d\mathcal{L}^n : X \in C^1_c(\Omega; \mathbb{R}^n), |X| \leq 1 \right\} < \infty. \quad (2.1)$$

Let $\mu_E$ denote the Gauss-Green measure of $E$, $\partial^* E$ denotes the reduced boundary of $E$, and $\nu_E$ denote the (measure-theoretic )outer unit normal to $E$. For a detailed account for Gauss-Green measure and reduced boundary, we refer to [Mag12, Chapter 15]. The well-known De Giorgi’s structure theorem is stated as follows.

**Theorem 2.1** ([Mag12, Theorem 15.9]). If $E$ is a set of finite perimeter in $\mathbb{R}^n$, then the Gauss-Green measure $\mu_E$ of $E$ satisfies

$$\mu_E = \nu_E \mathcal{H}^{n-1} \cdot \partial^* E$$

and the generalized Gauss-Green formula holds:

$$\int_E \text{div} X d\mathcal{H}^n = \int_{\partial^* E} X \cdot \nu_E d\mathcal{H}^{n-1}, \quad \forall X \in C^1_c(\mathbb{R}^n; \mathbb{R}^n). \quad (2.2)$$

--\(^2\)See Section 6 for a precise definition of wedge-shaped domain
--\(^3\)k is defined in (6.20), see also [LX17, Remark 6].
Without loss of generality, we can assume that \( \text{spt} \mu_E = \partial E \). Indeed, for any set of finite perimeter \( E \), it is equivalent (modulo a volume zero set) to a Borel set \( F \) such that \( \text{spt} \mu_F = \partial F \) (c.f., [Mag12, Proposition 12.19]). With this modification, combined with the fact that \( \text{spt} \mu_E = \partial^* E \), we have

\[
\partial E = \partial^* E. \tag{2.3}
\]

We need the following first variation formula of perimeter.

**Theorem 2.2** ([Mag12, Theorem 17.5]). If \( A \) is an open set in \( \mathbb{R}^n \), \( E \) is a set of finite perimeter, \( \{f_t\}_{|t| < \epsilon} \) is a local variation in \( A \), then

\[
P(f_t(E); A) = P(E; A) + t \int_{\partial^* E} \operatorname{div} E X d\mathcal{H}^{n-1} + O(t^2) \tag{2.4}
\]

where \( X \) is the initial velocity of \( \{f_t\}_{|t| < \epsilon} \) and \( \operatorname{div} E X : \partial^* E \to \mathbb{R} \), defined by

\[
\operatorname{div} E X(x) = \operatorname{div} X(x) - \nu_E(x) \cdot \nabla X(x) \nu_E(x), \quad x \in \partial^* E
\]

is a Borel function called the boundary divergence of \( X \) on \( E \). Here, \( \{f_t\}_{|t| < \epsilon} \) is called a local variation in \( A \) if it is a one parameter family of diffeomorphisms in \( \mathbb{R}^n \) and

\[
f_0(x) = x, \quad \forall x \in \mathbb{R}^n, \quad \{x \in \mathbb{R}^n : f_t(x) \neq x\} \subset A \quad \forall |t| < \epsilon.
\]

### 2.2. Rectifiable varifold

First we recall the basic concepts of rectifiable set and rectifiable varifold, we refer to [Sim83; De 08; Mag12] for a detailed account.

#### 2.2.1. Rectifiable set

A Borel set \( M \subset \mathbb{R}^n \) is a locally \( \mathcal{H}^{n-1} \)-rectifiable set if \( M \) can be covered, up to a \( \mathcal{H}^{n-1} \)-negligible set, by countably many Lipschitz images of \( \mathbb{R}^{n-1} \) to \( \mathbb{R}^n \), and if \( \mathcal{H}^{n-1} \cup M \) is locally finite on \( \mathbb{R}^n \). \( M \) is called \( \mathcal{H}^{n-1} \)-rectifiable if in addition, \( \mathcal{H}^{n-1}(M) < \infty \); \( M \) is said to be normalized, if \( M = \text{spt}(\mathcal{H}^{n-1} \cup M) \). In this paper, we always assume that a rectifiable set is normalized.

#### 2.2.2. Rectifiable varifold

Let \( M \) be a countably \((n-1)\)-rectifiable, \( \mathcal{H}^n \)-measurable set in \( \mathbb{R}^n \), let \( \theta \) be a positive locally \( \mathcal{H}^{n-1} \)-integrable function on \( M \). A rectifiable \((n-1)\)-varifold is denoted by \( \nu(M, \theta) \), and is defined to be the equivalent class of all pairs \((\tilde{M}, \tilde{\theta})\), where \( \tilde{M} \) is countably \((n-1)\)-rectifiable with \( \mathcal{H}^{n-1}(\tilde{M} \setminus M) \cup (M \setminus \tilde{M}) = 0 \), and \( \tilde{\theta} = \theta \) for \( \mathcal{H}^{n-1} \)-a.e. on \( M \cap \tilde{M} \). Associated to \( V = \text{var}(M, \theta) \), the weight measure of \( V \), denoted by \( \mu_V \), and is defined by

\[
\mu_V := \mathcal{H}^{n-1} \setminus \theta
\]

where we adopt the convention that \( \theta \equiv 0 \) on \( \mathbb{R}^n \setminus M \).

**Definition 2.1.** Let \( V = \nu(M, \theta) \) be a rectifiable \((n-1)\)-varifold in the open set \( \Omega \subset \mathbb{R}^n \), we say that \( V \) has generalized mean curvature vector \( \mathbf{H} \) if

\[
\int_M \operatorname{div}_M X d\mu_V = -\int_M \langle X, \mathbf{H} \rangle d\mu_V, \quad \forall X \in C^1_c(\Omega; \mathbb{R}^n). \tag{2.5}
\]

For a rectifiable varifold whose generalized mean curvature is bounded, we have the following monotonicity formula.

**Theorem 2.3** ([Sim83, Theorem 17.6]). Suppose \( \Omega \subset \mathbb{R}^n \) is open, for any \( \xi \in \Omega \) and some \( R > 0 \) s.t. \( B_R(\xi) \subset \Omega \). If \( V \) has generalized mean curvature \( \mathbf{H} \) as in Definition 2.1, with \( |\mathbf{H}| \leq \Lambda \), then

\[
F(r)^{\mu_V(B_r(\xi))} - F(s)^{\mu_V(B_s(\xi))} = G(s, r) \int_{B_r(\xi) \setminus B_s(\xi)} \frac{|D^2 r|^2}{r^{n-1}} d\mu_V(x), \tag{2.6}
\]

where \( F(r) = \mu_V(B_r(\xi)) \), and \( G(s, r) = F(r)^{\mu_V(B_r(\xi))} - F(s)^{\mu_V(B_s(\xi))} \).
for all $0 < s < r < R$, where $F(r) \in [e^{-\Delta r}, e^{\Delta r}]$ for all $0 < r < R$, and $G(s, r) \in [e^{-\Delta R}, e^{\Delta R}]$ for all $0 < s < r < R$, $\tau(x) := |x - \xi|$.

We will need the following Schätzle’s strong maximum principle for multiplicity one rectifiable varifold (Schätzle’s strong maximum principle for rectifiable varifold), which is a very deep result and turns out to be powerful for singularity analysis when M.G. Delgadino and F. Maggi proved that the only stationary points for the Euclidean Isoperimetric problem must be finite union of Euclidean balls, c.f., [DM19, Section 3, conclusion of the proof].

2.2.3. Schätzle’s strong maximum principle.

**Theorem 2.4** (Schätzle’s strong maximum principle for rectifiable varifold). Let $M$ be a normalized locally $H^{n-1}$-rectifiable set with distributional mean curvature vector $H \in L^p(\mathcal{H}^{n-1} \cdot M; \mathbb{R}^n)$ for some $p > \max\{2, n\}$.

Pick a direction $\nu \in \mathbb{S}^{n-1}$ and a number $h_0 \in \mathbb{R}$, consider a connected open set $U \in \nu^\perp$ such that

$$\varphi(z) = \inf\{h > h_0 : z + h\nu \in M\}, \quad z \in U,$$

(2.7)
satisfies $\varphi(z) \in (h_0, \infty)$ for every $z \in U$.

For another function $\eta \in W^{2,p}(U; (h_0, \infty))$, whose graph lies below the graph of $\varphi$ on $U$ and touches at a point $z_0$, namely, $\eta \leq \varphi$ on $U$ and $\eta(z_0) = \varphi(z_0)$ for some $z_0 \in U$, then it cannot be that

$$-\text{div}\left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}}\right)(z) \leq H_M(z + \varphi(z)\nu) \cdot \frac{-\nabla \varphi(z) + \nu}{\sqrt{1 + |\nabla \varphi(z)|^2}}$$

(2.8)

for $\mathcal{H}^{n-1}$-a.e. $z \in U$, unless $\eta = \varphi$ on $U$.

2.3. General varifolds. Now we recall the basic concepts of general varifold, for more detailed account, we refer to [All72; All75; Sim83; GJ86].

In what follows, we consider only the $(n - 1)$-dimensional varifold in $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be open, set $G_{n-1}(U) := U \times G(n, n-1)$. A general $(n-1)$-varifold in $U$ is a Radon measure on $G_{n-1}(U)$. Let $V_{n-1}(U)$ denote the set of all general $(n-1)$-varifolds in $U$. For $V \in V_{n-1}(U)$, let $\mu_V$ denote the weight of $V$, which is a Radon measure on $U$ defined by

$$\mu_V(\phi) := \int_{G_{n-1}(U)} \phi(x)dV(x,S) \quad \forall \phi \in C_c(U),$$

then for $\phi = \chi_K$ for some compact $K \subset U$,

$$\mu_V(K) = \int_{G_{n-1}(U) \cap G_{n-1}(K)} 1dV(x,S) = V(\pi^{-1}(K)),$$

here $\pi : U \times G(n, n-1) \to U; (x, S) \mapsto x$, is the canonical projection.

Let $\delta V$ be the first variation of $V$, which is a linear functional on $C^1_c(U; \mathbb{R}^n)$, and is given by

$$\delta V(g) = \int_{G_{n-1}(U)} Dg(x) \cdot SdV(x,S) = \int_{G_{n-1}(U)} \text{div}_Sg(x)dV(x,S) \quad \forall g \in C^1_c(U; \mathbb{R}^n).$$

\footnote{Although Schätzle's strong maximum principle for multiplicity one rectifiable is sufficient in our approach, it is still worth mentioning that Schätzle’s strong maximum principle indeed works for integer multiplicity rectifiable varifolds with sufficiently summable (distributional) mean curvature vector. In that case, the distributional mean curvature vector does not play a role as the mean curvature of some graph function, but rather the mean curvature of multiple sheets which overlaps in a possibly complicated way.}
Let $||\delta V||$ be the total variation when it exists. Indeed, if $V$ is of locally bounded variation in $U$, by Riesz representation theorem, there exists a Radon measure $||\delta V||$ on $U$ and a vector valued $||\delta V||$-measurable function $\nu$ with $|\nu| = 1$ for $||\delta V||$-a.e in $U$, such that

$$
\delta V(g) = \int_{G_{n-1}(U)} \text{div}_S g(x) dV(x, S) \equiv -\int_U \nu \cdot gd||\delta V||.
$$

(2.9)

Here $||\delta V||$ is characterized by

$$
||\delta V||(W) := \sup_{g \in C^1_c(\Omega;\mathbb{R}^n), |g| \leq 1, \text{spt}|g| \subset W} |\delta V(g)|
$$

for any open $W \subset\subset U$.

Furthermore, if $||\delta V|| << \mu_V$, we know that

$$
D_{\mu_V}||\delta V|| = \lim_{r \to 0^+} \frac{||\delta V||(B_r(x))}{\mu_V(B_r(x))} := H(x)
$$

exists for $\mu_V$-a.e. and (2.9) can be written as

$$
\delta V(g) = -\int_U \nu \cdot gd||\delta V|| = -\int_U H(x) \nu \cdot gd\mu_V = -\int_U H(x) \cdot gd\mu_V.
$$

Here the vector field $H(x) := H(x)\nu(x)$ is called the generalized mean curvature vector of $V$.

For a $\mathcal{H}^{n-1}$ measurable countably $(n-1)$-rectifiable set $M \subset U$ with locally finite $\mathcal{H}^{n-1}$ measure, a natural $(n-1)$-varifold $|M| \in V_{n-1}(U)$ is defined by

$$
|M|(\varphi) := \int_M \varphi(x, \text{Tan}(M, x)) d\mathcal{H}^{n-1}(x), \quad \forall \varphi \in C^c(G_{n-1}(U)),
$$

(2.10)

where $\text{Tan}(M, x)$ is the approximate tangent space of $M$ at $x$, which exists $\mathcal{H}^{n-1}$-a.e. on $M$.

In particular, for some $K \subset\subset G_{n-1}(U)$,

$$
|M|(K) = |M|(\chi_K) = \int_M \chi_K(x, \text{Tan}(M, x)) d\mathcal{H}^{n-1}(x) = \int_{\{x \in M : (x, \text{Tan}(M, x)) \cap K \neq \emptyset\}} d\mathcal{H}^{n-1}(x).
$$

In this case, the weight measure $\mu_{|M|} = \mathcal{H}^{n-1} \cap M$, since for any Borel set $E \subset U \subset \mathbb{R}^n$,

$$
\mu_{|M|}(E) = |M|(\pi^{-1}(E)) = \int_{\{x \in M : (x, \text{Tan}(M, x)) \cap E \times G(n, n-1) \neq \emptyset\}} 1 d\mathcal{H}^{n-1}(x)
$$

$$
= \int_{E \cap M} d\mathcal{H}^{n-1}(x) = \mathcal{H}^{n-1} \cap M(E).
$$

Also, by setting $\varphi_g(x, S) := Dg(x) \cdot S = \text{div}_S g(x)$, we have

$$
\delta|M|(g) := |M|(\varphi_g) = \int_M \varphi_g(x, \text{Tan}(M, x)) d\mathcal{H}^{n-1}(x) = \int_M \text{div}_M g(x) d\mathcal{H}^{n-1}(x).
$$

2.4. General varifold with constant contact angle. We briefly introduce the following definitions and propositions for general varifold with constant contact angle, which were first introduced and studied in a recent work [KT17].

**Definition 2.2** ([KT17, Definition 3.1]). Given $V \in V_{n-1}(\Omega)$ with $\mu_V(\Omega) < \infty$, given a $\mathcal{H}^{n-1}$-measurable set $B^+ \subset \partial \Omega$ and $\theta \in [0, \pi]$, $V$ is said to have a fixed contact angle $\theta$ at $\partial \Omega$ at the boundary of $B^+$ if the following two conditions hold.

1. The generalized mean curvature vector $H$ exists, i.e., for any $g \in C^1_c(\Omega; \mathbb{R}^n)$, we have

$$
\int_{G_{n-1}(\Omega)} \text{div}_S g(x) dV(x, S) = -\int_{\Omega} H(x) \cdot g(x) d\mu_V(x);
$$

2. Given $g \in C^1_c(\Omega; \mathbb{R}^n)$, we have

$$
\int_{\partial \Omega} \delta V(g) = \int_{\partial \Omega} \varphi \cdot H \cdot g d\mu_V = \theta \int_{\partial \Omega} \varphi(x) g(x) d\mu_V(x).
$$

(2.11)
(2) For any \( g \in C^1(\Omega; \mathbb{R}^n) \) with \( g \cdot \nu_{\partial \Omega} = 0 \) on \( \partial \Omega \), we have
\[
\int_{G_{n-1}(\Omega)} \text{div}_{G_n} g(x) dV(x, S) + \cos \theta \int_{B^+} \text{div}_{\partial \Omega} g(x) d\mathcal{H}^{n-1}(x) = -\int_{\Omega} H(x) \cdot g(x) d\mu_V(x),
\]
where \( \text{div}_{\partial \Omega} \) is the tangential divergence on \( \partial \Omega \) and \( \nu_{\partial \Omega} \) is the outward unit normal vector on the boundary \( \partial \Omega \).

Before we state the monotonicity formula, we need some further notations. Let us set
\[
\kappa := ||\text{principal curvature of } \partial \Omega||_{L^\infty(\partial \Omega)},
\]
for \( s > 0 \), define a subset \( N_s \) of \( \mathbb{R}^n \) as
\[
N_s := \{ x \in \mathbb{R}^n : \text{dist}(x, \partial \Omega) < s \},
\]
Since \( \Omega \) is bounded and \( \partial \Omega \) is \( C^2 \), there exists \( s_0 \in (0, \kappa^{-1}] \), depending only on \( \partial \Omega \), s.t., for any \( x \in N_{s_0} \), there exists a unique closest point \( \xi(x) \in \partial \Omega \), namely, \( \text{dist}(x, \partial \Omega) = |x - \xi(x)| \). By the projection \( \xi(x) \), we define the reflection point \( \tilde{x} \) of \( x \) with respect to \( \partial \Omega \) as \( \tilde{x} := 2\xi(x) - x \).

For any \( a \in \mathbb{R}^n \), the reflection ball \( \tilde{B}_r(a) \) of \( B_r(a) \) with respect to \( \partial \Omega \) is defined as
\[
\tilde{B}_r(a) = \{ x \in \mathbb{R}^n : |\tilde{x} - a| < r \}.
\]

**Proposition 2.1 ([KT17, Theorem 3.2]).** Given \( V \in V_{n-1}(\Omega) \) with \( \mu_V(\Omega) < \infty \), given \( \mathcal{H}^{n-1} \)-measurable set \( B^+ \subset \partial \Omega \), and \( \theta \in (0, \frac{\pi}{2}] \), if \( V \) has a fixed contact angle \( \theta \) with \( \partial \Omega \) at the boundary of \( B^+ \) in the sense of Definition 2.2. Assume that for some \( p > n \) and \( \Gamma > 0 \), we have
\[
\left( \frac{1}{\omega_{n-1}} \int_{N_{s_0} \cap \Omega} 2 |H(x)|^p d\mu_V(x) \right)^{\frac{1}{p}} \leq \Gamma.
\]
Then there exists a constant \( C_1 \geq 0 \), depending only on \( n \), s.t., for any \( x \in N_{s_0/6} \cap \overline{\Omega} \),
\[
\begin{aligned}
\left\{ \frac{\mu_V(B_r(x)) + \mu_V(\tilde{B}_r(x)) - 2\beta \mathcal{H}^{n-1}_{\partial B^+}(B_r(x))}{\omega_{n-1} \Gamma^{n-1}} \right\} \left( 1 + C_1 \kappa r (1 + \frac{1}{p - n + 1}) + \frac{\Gamma r^{1-n}}{p - n + 1} \right)
\end{aligned}
\]
is a non-decreasing function of \( r \) in \((0, s_0/6)\), here \( \beta = -\cos \theta \).

For those \( \theta \in \left( \frac{\pi}{2}, \pi \right] \), we consider \( \partial \Omega - B^+ \) and \( -\beta = -\cos \theta \) instead of \( B^+ \) and \( \beta = -\cos \theta \), respectively, then the same claim holds.

**Remark 2.1.** For convenience to prove our main results, here we use \( \beta \) to replace \( \sigma \) in [KT17, Theorem 3.2], notice that \( \beta = -\sigma \).

3. Characterization of stationary points

In this section, we give a characterization of the stationary points of the free-energy functional (1.1).

First we prove a topological lemma.

**Lemma 3.1.** \( \Omega \subset \mathbb{R}^n \) is a bounded open set with \( C^2 \) boundary \( \partial \Omega \), \( E \subset \Omega \) is a set of finitc perimeter. Using the notations in Section 1.1, if the singular set satisfies \( \mathcal{H}^{n-3}(\text{sing} M) = 0 \). Then, \( B^+ = \partial E \cap \partial \Omega \) is \( C^2 \) in \( \partial \Omega \) off a singular set \( \text{sing} B^+ \) with \( \mathcal{H}^{n-3}(\text{sing} B^+) = 0 \). Moreover, \( \text{sing} B^+ \subset \text{sing} M \).
Proof. By definition, \( \text{int} (B^+) = \text{int} (\partial E \cap \partial \Omega) \) is \( C^2 \), and hence \( \text{sing} B^+ \subset \partial B^+ \). Notice that \( \partial B^+ \subset M \), this is a simple topological fact, but we present the proof for the sake of clarity.

First we need the following fact: for any two sets \( C, D \),
\[
\partial (C \cap D) = \partial (\text{int} (C \cap \text{int} D)) = (\overline{C \cap D}) \cap ((\text{int} C \cap \text{int} D)^c) = (C \cap \overline{D} \cap \text{int} C) \cup (\overline{C} \cap \text{int} D).\]

Thus
\[
\partial B^+ = \partial (\partial E \setminus (\partial E \cap \Omega)) \subset (\partial (\partial E \cap \overline{\partial E \cap \Omega})) \cup (\partial (\partial E \cap \Omega)^c \cap \overline{\partial E}).
\]

Notice that \( \partial (\partial E) = \emptyset, \overline{\partial E} = \partial E \) and \( \partial (\partial E \cap \Omega)^c = \partial (\partial E \cap \Omega) \), and hence
\[
\partial B^+ \subset (\partial (\partial E \cap \Omega) \cap \partial E).
\]

Notice also that
\[
\partial (\partial E \cap \Omega) \subset \overline{(\partial E \cap \Omega)} = M.
\]

Finally, we have:
\[
\partial B^+ \subset (\partial (\partial E \cap \Omega) \cap \partial E) \subset (M \cap \partial E) \subset M.
\]

Moreover, by above argument we see that \( \text{sing} B^+ \subset \partial B^+ \subset M \), by definition of \( \text{sing} M \), we have:
\[
\text{sing} B^+ \subset \text{sing} M,
\]
which also implies \( \mathcal{H}^{n-3}(\text{sing} B^+) = 0. \) \( \square \)

Now we show the first part of Theorem 1.1 about the geometry of the measure-theoretic capillary hypersurface \( M \).

**Proposition 3.1.** Let \( E \subset \Omega \) be a set of finite perimeter, which is stationary for \( F_\beta \) under volume constraint, as in Definition 1.2. Using notations in Section 1.1, if the singular set satisfies \( \mathcal{H}^{n-2}(\text{sing} M) = 0^5 \). Then \( E \) satisfies

i. **(Constant mean curvature)** On \( \text{reg} M \), the mean curvature of \( M \) is constant, denoted by \( H_M \),

ii. **(Constant contact angle)** On \( \text{reg} M \cap \partial \Omega \), the measure-theoretic hypersurface \( m \) intersects \( \partial \Omega \) with a constant contact angle \( \theta (\cos \theta = -\beta) \), i.e.,
\[
\langle \nu_M, \nu_{B^+} \rangle = -\beta = -\langle \nu^M_T, \nu^{B^+}_T \rangle. \tag{3.1}
\]

Moreover, \( \text{reg} M \cap \partial \Omega \) is locally an analytic hypersurface with constant mean curvature, relatively open in \( \partial E \cap \Omega \).

---

5Notice that the here \( \text{sing} M \) is not required to be closed, also, the Hausdorff dimension at most \( (n-2) \) is sufficient here.
Remark 3.1. Notice that by virtue of $H^{n-3}(\text{sing } M) = 0$, the integrals $\int_{M \cap \Omega} dH^{n-1}(x)$ and $\int_{\text{reg } M \cap \Omega} dH^{n-1}(x)$ or $\int_{M \cap \partial \Omega} dH^{n-2}(x)$ and $\int_{\text{reg } M \cap \partial \Omega} dH^{n-2}(x)$ are exactly the same things. Also, by Lemma 3.1 and notice that $\Gamma := \partial B^+ \cap M$, we have:

\[
\int_{B^+} X \cdot dH^{n-1}(x) = \int_{\text{reg } B^+} X \cdot dH^{n-1}(x) = \int_{\text{reg } B^+ \cap \Gamma} X \cdot dH^{n-2}(x).
\]

Here we use $\text{reg } B^+$ to denote the $C^2$ part of $B^+$.

Proof. We argue as in [SZ98], first we construct a family of admissible sets as in Definition 1.1, we start from any variation which preserves volume at first order and maps boundary to the boundary, and then do some modifications inside $\Omega$ such that the volume of $E_t$ defined below preserves for a short time $(-T, T)$. The method is fairly well-known and, for example, is applied in the proof of Young’s law [Mag12, Theorem 19.8].

First, let $X \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$ be any vector field satisfying

\[
\int_M X \cdot \nu_M dH^{n-1}(x) = 0, \quad (3.2)
\]

\[
X(x) \in T_x(\partial \Omega), \quad \forall x \in \partial \Omega. \quad (3.3)
\]

By solving the Cauchy’s problems

\[
\frac{\partial}{\partial t} \Psi(t, x) = X(\Psi(t, x)), \quad x \in \mathbb{R}^n, \quad (3.4)
\]

\[
\Psi(0, x) = x, \quad x \in \mathbb{R}^n, \quad (3.5)
\]

we obtain a local variation $\{\Psi_t\}_{|t| < T}$ for some small $T > 0$, having $X$ as its initial velocity.

Let $E_t := \Psi_t(E)$, we see that $\Psi_t(\Omega) \subset \Omega$ by (3.3), hence $E_t \subset \Omega$.

Let $V(t) := |E_t|$, following the same computations in the proof of [SZ98, Theorem 2.2] we have:

1. $V'(0) = 0$,
2. $V''(0) = \int_M \text{div } X(\nu_M) dH^{n-1}(x)$.

Now we do some modifications inside $\Omega$ to obtain a new family of admissible sets $\{\tilde{E}_t\}_{|t| < T}$.

Fix any $x \in \text{reg } M \cap \Omega$, by regularity, $\partial E$ can be locally written as the graph of some smooth function $u_0 : D' \to \mathbb{R}^1$, where $D' \subset \mathbb{R}^{n-1}$ is a fixed open set near $x$. Since $X \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$ satisfies (3.4), we can find a much smaller number, still denoted by $T$, such that not only $\partial E$, but also $\partial \tilde{E}_t$ for all $t \in (-T, T)$, can be written as a graph of a smooth function $u : D' \times (-T, T) \to \mathbb{R}^1$. 

\[\text{Figure 1. Capillary surface}\]
Then we do some modifications on \( \partial E_t \) over \( D' \) as follows, for any smooth function \( g : D' \times (-T, T) \to \mathbb{R}^1 \) satisfying
\[
\int_{D'} g(x', t)dx' = \begin{cases} 
\frac{V(0)-V(t)}{t^2} & t \neq 0, \\
-\frac{1}{2}V''(0) & t = 0 
\end{cases} 
\tag{3.6}
\]
Moreover, \( g \mid_{\partial D'} = 0 \) for any \( t \in (-T, T) \).

We point out that such smooth function does exist since \( V''(0) = 0 \), the Taylor expansion of \( V(t) \) at \( t = 0 \) is: \( V(t) = V(0) + \frac{1}{2}t^2V''(0) + o(t^2) \), hence such smooth function indeed exists.

Now, we define a new family of sets \( \{E_t\} \) by replacing the boundary portion of \( \partial E_t \) given by \( \{(x', u(x', t)) : x' \in D'\} \) with new boundary for \( x' \in D' \), denoted by \( \{\tilde{E}_t\} \), and given by
\[
\{(x, u(x', t) + t^2g(x', t)) : x' \in D'\}. \tag{3.7}
\]
Let \( \tilde{V}(t) := |E_t| \), since \( E_t \) and \( \tilde{E}_t \) coincide outside \( D' \), we have
\[
\tilde{V}(t) - V(t) = \int_{D'} [(u(x', t) + t^2g(x', t)) - (u(x', t))] \, dx' = t^2 \int_{D'} g(x', t)dx'.
\]
By (3.6)
\[
\tilde{V}(t) - V(t) = V(0) - V(t), \quad \forall t \in (-T, T),
\]
which reveals the fact that
\[
\tilde{V}(t) = V(0), \quad \forall t \in (-T, T).
\]
Thus, \( \{\tilde{E}_t\} \) is admissible as in Definition 1.1.

In the following, by the fact that \( E \) is stationary in the sense as Definition 1.2, we will deduce that \( \text{reg} M \cap \Omega \) is of constant mean curvature and on \( \text{reg} M \cap \partial \Omega, \nu_M \cdot \nu_{B^+} = -\beta \). Set
\[
\tilde{F}_{\beta}(t) := P(\tilde{E}_t; \Omega) - \beta P(\tilde{E}_t; \partial \Omega).
\]
Since \( E_t \) and \( \tilde{E}_t \) coincide outside \( D' \), we have:
\[
\tilde{F}_{\beta}(t) - F_{\beta}(t) = \tilde{F}_{\beta}(t) - F_{\beta}(t) \mid_{D'} = \int_{D'} \left( \sqrt{1 + |
abla x' (u + t^2 g)|^2} - \sqrt{1 + |
abla x' (u)|^2} \right) \, dx'. \tag{3.8}
\]
Hence
\[
\tilde{F}_{\beta}'(0) - F_{\beta}'(0) = 0.
\]
Since \( E \) is stationary as in Definition 1.2, we find that
\[
F_{\beta}'(0) = \tilde{F}_{\beta}'(0) = 0.
\]
Using the first variation formula of perimeter of \( E \) (Theorem 2.2), we obtain
\[
F_{\beta}'(0) = \int_{M \cap \Omega} \text{div}M X(x) \, dH^{n-1}(x) - \beta \int_{B^+} \text{div}_{B^+} X(x) \, dH^{n-1}(x). \tag{3.9}
\]
More precisely, we set
\[
A(t) := P(E_t; \Omega), B(t) := P(E_t; \partial \Omega).
\]
We shall use Theorem 2.2 for \( E \) in some open set \( U \subset \mathbb{R}^n \) s.t., \( \Omega \subset U, \text{spt} X \subset U \). Notice that \( E_t \subset \Omega \), hence
\[
P(E_t; U) = P(E_t; \Omega) + P(E_t; \partial \Omega).
\]
With the help of the first variation formula inside $\Omega$ (c.f., [SZ98, (2.23)]) and Remark 3.1, we deduce that

$$A'(0) = \int_{M \cap \Omega} \text{div}_M X(x) d\mathcal{H}^{n-1}(x),$$

$$B'(0) = \int_{B^+} \text{div}_{B^+} X(x) d\mathcal{H}^{n-1}(x).$$

From stationary, back to (3.9) we have

$$\int_{M \cap \Omega} \text{div}_M X(x) d\mathcal{H}^{n-1}(x) - \beta \int_{B^+} \text{div}_{B^+} X(x) d\mathcal{H}^{n-1}(x) = 0 \quad (3.10)$$

for any $X$ satisfying (3.2),(3.3).

For such $X$ on $\text{reg} M$, it is decomposed to

$$X = X^T + X^\perp$$

where $X^\perp = (X \cdot \nu_M) \nu_M$, $X^T = X - X^\perp$. Then

$$\text{div}_M X^\perp = D_{\tau_i} ((X \cdot \nu_M) \nu_M) \cdot \tau_i = X \cdot [(\tau_i \cdot D_{\tau_i} \nu_M) \nu_M] = X \cdot H\nu_M. \quad (3.11)$$

By divergence theorem and (3.3), we have

$$\int_{M \cap \Omega} \text{div}_M X^T d\mathcal{H}^{n-1}(x) = \int_{\Gamma} X^T \cdot \nu^M_{\Gamma} d\mathcal{H}^{n-2}(x) = \int_{\Gamma} X \cdot \nu^M_{\Gamma} d\mathcal{H}^{n-2}(x),$$

$$\int_{B^+} \text{div}_{B^+} X d\mathcal{H}^{n-1}(x) = \int_{\Gamma} X \cdot \nu^B_{\Gamma} d\mathcal{H}^{n-2}(x).$$

Thus, back to (3.10), we have

$$\int_{M \cap \Omega} H(x) X \cdot \nu_M(x) d\mathcal{H}^{n-1}(x) + \int_{\Gamma} X \cdot (\nu^M_{\Gamma} - \beta \nu^B_{\Gamma}) (x) d\mathcal{H}^{n-2}(x) = 0 \quad (3.12)$$

for any $X$ satisfying (3.2),(3.3).

For those $X \in C^2_c(\Omega; \mathbb{R}^n)$ with (3.2) holds, it is apparent that (3.3) holds for such $X$, and (3.12) reduce to

$$\int_{M \cap \Omega} H(x) X \cdot \nu_M(x) d\mathcal{H}^{n-1}(x) = 0.$$

From this we deduce the condition of constant mean curvature $H$ throughout $\text{reg} M \cap \Omega$ by [SZ98], namely, for some constant $H$,

$$H(x) = H, \quad \forall x \in \text{reg} M \cap \Omega.$$

Back to (3.12), we have

$$H \int_{M \cap \Omega} X \cdot \nu_M(x) d\mathcal{H}^{n-1}(x) + \int_{\Gamma} X \cdot (\nu^M_{\Gamma} - \beta \nu^B_{\Gamma}) (x) d\mathcal{H}^{n-2}(x) = 0.$$

By (3.2), we obtain

$$\int_{\Gamma} X \cdot (\nu^M_{\Gamma} - \beta \nu^B_{\Gamma}) (x) d\mathcal{H}^{n-2}(x) = 0 \quad (3.13)$$

for any $X$ satisfying (3.2),(3.3).

From (3.13), we conclude that $\text{reg} M$ intersect at $\partial \Omega$ with constant angle $\theta$, s.t: $\cos \theta = -\beta$, i.e., $\nu_M \cdot \nu_B = -\beta$ on $\text{reg} M \cap \partial \Omega$. 
We claim that (3.13) holds for any $X_0 \in C^2_c(\mathbb{R}^n; \mathbb{R}^n)$ satisfying (3.3). (i.e., we rule out the restriction (3.2)). Indeed, for any $X_0 \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$ satisfying (3.3), there exists $s > 0$ and $S_0 \in C^\infty(\Omega; \mathbb{R}^n)$ such that $X := S_0 + sX_0 \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ satisfies (3.2),(3.3).

By (3.13), we conclude that
\[
\int_{\Gamma} X_0 \cdot \left( \nu^M_{\Gamma} - \beta \nu^B_{\Gamma} \right)(x) dH^{n-2}(x) = 0
\]
holds for any $X_0 \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$ and $X_0(x) \in T_x(\partial \Omega)$ for any $x \in \partial \Omega$.

Notice that for any such $X_0$, we have
\[
X_0 \cdot \nu^M_{\Gamma} = X_0 \cdot \left( \left( \nu^M_{\Gamma} \cdot \nu^B_{\Gamma} \right) \nu^B_{\Gamma} \right).
\]
Hence we have
\[
\int_{\Gamma} X_0 \cdot \left( \nu^M_{\Gamma} \cdot \nu^B_{\Gamma} - \beta \right) \nu^B_{\Gamma}(x) dH^{n-2}(x) = 0
\]  \hspace{1cm} (3.14)
for any $X_0 \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$ satisfying (3.3).

For any $X \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$, since $X$ can be written as tangential part and normal part with respect to $\partial \Omega$, denoted by $X = X^T + X^\perp$, the following is valid:
\[
\int_{\Gamma} X^\perp \cdot \left( \nu^M_{\Gamma} \cdot \nu^B_{\Gamma} - \beta \right) \nu^B_{\Gamma}(x) dH^{n-2}(x) = 0,
\]
and $X^T \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$, $X^T(x) \in T_x(\partial \Omega)$, $\forall x \in \partial \Omega$.

This shows that (3.14) holds for any $X \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)$. By virtue of fundamental lemma of the calculus of variations, we have $-\nu_M \cdot \nu_B^+(x) = \nu_M^+ \cdot \nu_B^+(x) = \beta$ for any $x \in \Gamma$.

4. Poincaré type inequality

When $M$ is pointwise $C^2$, this result is well-known and is obtained by testing the second variation formula via $\zeta \nu_M$, where $\zeta$ is smooth everywhere on $M$ and $\int_M \zeta dH^{n-1} = 0$, and $\nu_M$ is the outward pointing unit normal on $M$. For our case, the main difficulty is that near the singularities, we are not able to define the normal, however, we still want a smooth vector field defined on the whole $M$ and the corresponding variation preserves the volume at the first order, with the help of (3.8), we can invoke the stability condition.

In order to prove this Poincaré type inequality, we shall need the following lemma, which was used by Sternberg and Zumbrun in [SZ99] to study the local minimizers. This lemma enables us to construct smooth cut-off functions, which play essential roles when we are dealing with singularities.

**Lemma 4.1.** Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^2$ boundary $\partial \Omega$, let $E$ be stationary in the sense of Definition 1.2, set $M := \overline{\Omega} \cap \partial E$ and assume that $M$ is $C^2$ in $\overline{\Omega}$ off a singular set $\text{sing}M$ which is closed and satisfying $H^{n-3}(\text{sing}M) = 0$. Then for any small $\epsilon > 0$, there exist open sets $S'_\epsilon \subset S_\epsilon \subset \mathbb{R}^n$, with $\text{sing}M \subset S'_\epsilon$ and $S_\epsilon \subset \{x : \text{dist}(x, \text{sing}M) < \epsilon\}$, and there exists a smooth cut-off function $\varphi_\epsilon \in C^\infty(\mathbb{R}^n)$ such that $0 \leq \varphi_\epsilon(x) \leq 1$ with
\[
\varphi_\epsilon(x) = \begin{cases} 
0 & x \in S'_\epsilon, \\
1 & x \in \mathbb{R}^n \setminus S_\epsilon,
\end{cases}
\]
and
\[
\int_M |\nabla^M \varphi_\epsilon(x)|^2 dH^{n-1}(x) \leq C_2 \epsilon
\]
for some positive constant $C_2$ independent of $\epsilon$.

**Remark 4.1.** In contrast to [SZ98, Theorem 2.2], here we additionally assume that $\text{sing}M$ is closed. This is due to the lack of Allard type theorem ([GJ86, Theorem 4.13]) for boundary regularity as pointed out in Remark 1.2.

**Proof of Lemma 4.1.** First, since $\text{sing}M$ is closed and is a subset of a bounded set $\bar{\Omega}$, we have $\text{sing}M$ is compact.

Fix some $\epsilon > 0$, since $\mathcal{H}^{n-3}(\text{sing}M) = 0$, we can cover $\text{sing}M$ with finitely many balls $\mathcal{G} := \{B_{r_i}(z_i)\}_{i=1}^{N_1}$ where $z_i \in \bar{\Omega}$, $\sum_{i=1}^{N_1} r_i^{n-3} < \epsilon$ satisfying $r_i < s_0/6$, here $s_0/6$ is given in Chapter 2$(r_i < s_0/6)$ is valid for $\epsilon$ small enough). Set $\mathcal{I} := \{B_{r_i}(z_i) \in \mathcal{G} : z_i \in \Omega \setminus N_{s_0/6}\}$, $\mathcal{B} := \mathcal{G} \setminus \mathcal{I}$.

In terms of $\mathcal{I}$, we consider a natural $(n-1)$-rectifiable varifold related to $M$. Set $V_1 := \nu(M, \theta)$, where $\theta \equiv 1$ on $M$, and $\theta \equiv 0$ on $\mathbb{R}^n \setminus M$. For such $E$, by Proposition 3.1, $\text{reg}M \cap \Omega$ is of constant mean curvature $H$, for any $X \in C^1_c(\Omega; \mathbb{R}^n)$, arguing as in (3.11), we obtain

$$\int_M \text{div}_M X d\mu_V = \int_M \text{div}_M X d\mathcal{H}^{n-1} = \int_M X \cdot H d\mathcal{H}^{n-1} = \int_M X \cdot H d\mu_V$$

where $H(x) = H\nu_M(x)$. This implies that $V_1$ has generalized mean curvature vector $H$ and $|H| \leq |H|$. Set $\tau_1 := \max\{r_i : B_{r_i}(z_i) \in \mathcal{I}\}$, $R_1 := \min\{|\text{dist}(z_i, \partial \Omega) : B_{r_i}(z_i) \in \mathcal{I}\}$. Using the monotonicity formula (2.6) with $\Lambda, R$ replaced by $|H|, R_1$, respectively, then for any $B_{r_i}(z_i) \in \mathcal{I}$, and for any $r \leq \tau_1$, we have

$$F(r) \frac{\mathcal{H}^{n-1}(M \cap B_{r}(z_i))}{r^{n-1}} \leq F(R_1) \frac{\mathcal{H}^{n-1}(M \cap B_{R_1}(z_i))}{R_1^{n-1}}.$$  

Notice that by definition of $\mathcal{I}$, we have $R_1 > s_0/6$. Set $C_3 := e^{2|H|diam(\Omega)} \frac{\mathcal{H}^{n-1}(\partial M)}{(s_0/6)^n}$, we have

$$\mathcal{H}^{n-1}(M \cap B_{r_i}(z_i)) \leq C_3 r_i^{n-1}$$  \hspace{1cm} (4.1)

for any $B_{r_i}(z_i) \in \mathcal{I}$.

In terms of $\mathcal{B}$, we consider a natural $(n-1)$-varifold $|M| \in V_{n-1}(\Omega)$ as (2.10), which is induced by $V_1 = \nu(M, \theta)$. Suppose that $\beta \in (-1, 0)$ in the following argument, for $\beta \in [0, 1)$, consider $\Omega \setminus E$ instead. For such $E$, by Proposition 3.1, for any $X \in C^1_c(\Omega; \mathbb{R}^n)$, arguing as (3.11) to obtain

$$\delta|M|(X) = \int_M \text{div}_M X d\mathcal{H}^{n-1}$$

$$= \int_M \text{div}_M (X^T + X^\perp) d\mathcal{H}^{n-1}$$

$$= \int_{\Gamma} X \cdot \nu_M^T d\mathcal{H}^{n-2} - \int_M X \cdot (-H) d\mathcal{H}^{n-1}$$

$$= \int_{\Gamma} X \cdot (\nu_M^T \cdot \nu_B^{B^+}) \nu_B^{B^+} d\mathcal{H}^{n-2} - \int_{\Omega} X \cdot (-H) d\mu_{|M|}$$

$$= \beta \int_{\Gamma} X \cdot \nu_B^{B^+} d\mathcal{H}^{n-2} - \int_{\Omega} X \cdot (-H) d\mu_{|M|}$$

$$= \beta \int_{B^+} \text{div}_{\partial \Omega} X d\mathcal{H}^{n-1} - \int_{\Omega} X \cdot (-H) d\mu_{|M|}.$$
For any \( X \in C_c^1(\Omega; \mathbb{R}^n) \), we deduce that

\[
\delta |M| (X) = \int_M \text{div}_M X d\mathcal{H}^{n-1} = -\int_\Omega X \cdot (-H) d\mu_M.
\]

Hence \( |M| \) is a \((n-1)\)-varifold with generalized mean curvature vector \(-H(x) = -H \nu_M\). Furthermore, \( |M| \) has a fixed contact angle \( \theta \) with \( \partial \Omega \) at the boundary of \( B^+ \) as in Definition 2.2.

Set \( p := 2n - 1, \Gamma := \left( \frac{2}{\omega_{n-1}} \mathcal{H}^{n-1}(M) \right)^{\frac{1}{n-1}} H, r_2 := \max \{ r_i : B_{r_i}(z_i) \in \mathcal{B} \} \). By Proposition 3.1 and the fact that \( E \) is a set of finite perimeter, we see that (2.11) holds for such \( |M| \), by the monotonicity formula (2.12),

\[
\frac{\mathcal{H}^{n-1}(M \cap B_r(x))}{\omega_{n-1} r^{n-1}} \left( 1 + C_1 \kappa \frac{n+1}{n} \right) \\
\leq \frac{2}{\omega_{n-1}(s_0/6)^{n-1}} \mathcal{H}^{n-1}(M) + \mathcal{H}^{n-1}(B^+) \left( 1 + C_1 \kappa (s_0/6 \frac{n+1}{n}) \right) + \frac{\Gamma}{n} \frac{(s_0/6)^{n-1}}{n} := C_4
\]

holds for any \( r < \frac{s_0}{6} \) and any \( x \in N_{s_0/6} \cap \Omega \).

In particular,

\[
\mathcal{H}^{n-1}(M \cap B_{r_i}(z_i)) \leq C_5 r_i^{n-1} \tag{4.2}
\]

holds for any \( B_{r_i}(z_i) \in \mathcal{B} \).

Now, set \( \mu := \min \{ R_i/2, \epsilon/2, s_0/6 \} \), and we further require that \( r_i < \mu \) for all \( i \). Set \( C_0 := \max \{ C_3, C_5 \} \) (notice that \( C_3, C_5 \) are independent of \( \epsilon \)). By (4.1), (4.2), we have

\[
\mathcal{H}^{n-1}(M \cap B_{r_i}(z_i)) \leq C_0 r_i^{n-1} \tag{4.3}
\]

for all \( B_{r_i}(z_i) \in \mathcal{G} \).

For each \( i \), let \( \varphi_i \in C^\infty(\mathbb{R}^n) \) satisfies \( 0 \leq \varphi_i \leq 1 \) with

\[
\varphi_i(x) = \begin{cases} 
0 & \forall x \in B_{r_i}(z_i), \\
1 & \forall x \in \mathbb{R}^n \setminus B_{2r_i}(z_i),
\end{cases}
\]

and \( |\nabla \varphi_i(x)| \leq \frac{2}{r_i} \) for all \( x \in \mathbb{R}^n \).

Then we define \( \tilde{\varphi}_\epsilon \) by

\[
\tilde{\varphi}_\epsilon(x) := \min_i \varphi_i(x).
\]

It follows that \( \tilde{\varphi}_\epsilon \) is piecewise smooth with \( 0 \leq \tilde{\varphi}_\epsilon \leq 1 \), and

\[
\tilde{\varphi}_\epsilon(x) = \begin{cases} 
0 & \text{on } \bigcup_i B_{r_i}(z_i) \supset \text{sing} M \\
1 & \text{on } \mathbb{R}^n \setminus \bigcup_i B_{2r_i}(z_i)
\end{cases} \tag{4.4}
\]
By (4.3) and \(\sum_{i=1}^{N_1} r_i^{n-3} < \epsilon\), we have
\[
\int_M |\nabla^M \tilde{\varphi}_\epsilon(x)|^2 d\mathcal{H}^{n-1}(x) \leq \int_M |\nabla \tilde{\varphi}_\epsilon(x)|^2 d\mathcal{H}^{n-1}(x) \leq \sum_i \int_M |\nabla \varphi_i(x)|^2 d\mathcal{H}^{n-1}(x)
= \sum_i \int_{B_{2r_i}(z_i) - B_{r_i}(z_i)} |\nabla \varphi_i(x)|^2 d\mathcal{H}^{n-1}(x)
\leq \sum_i \frac{4}{r_i^2} \mathcal{H}^{n-1}(M \cap B_{2r_i}(z_i))
\leq 2^{n+1} C_0 \sum_i r_i^{n-3} < C \epsilon.
\] (4.5)

Similarly, we readily see that
\[
\int_M |\nabla^M \tilde{\varphi}_\epsilon| d\mathcal{H}^{n-1} < C \epsilon,
\] (4.6)
and
\[
\int_{\partial M} |\nabla^M \tilde{\varphi}_\epsilon| d\mathcal{H}^{n-2} \leq \sum_i \int_{B_{2r_i}(z_i) \setminus B_{r_i}(z_i)} |\nabla \varphi_i| d\mathcal{H}^{n-2} \leq \sum_i \frac{2}{r_i} \mathcal{H}^{n-2}(\partial M \cap B_{2r_i}(z_i))
\leq \sum_i \frac{2}{r_i} \mathcal{H}^{n-2}(M \cap B_{2r_i}(z_i)) \leq 2^{n-1} C_0 \sum_i r_i^{n-3} < C \epsilon.
\] (4.7)

Finally, we mollify \(\tilde{\varphi}_\epsilon\) to obtain a smooth function \(\varphi_\epsilon\), which still satisfies estimates of the form (4.5), (4.6) and (4.7) with the constant \(C\) replaced by some constant \(C_2\) (which is independent of \(\epsilon\)). Since \(\tilde{\varphi}_\epsilon\) satisfies (4.4), let \(S_\epsilon', S_\epsilon\) denote the sets such that
\[
\varphi_\epsilon(x) = \begin{cases} 
0 & x \in S_\epsilon', \\
1 & x \in \mathbb{R}^n \setminus S_\epsilon.
\end{cases}
\]

We see that \(\varphi_\epsilon\) is the desired smooth cut-off function. \(\square\)

In order to derive the second variation formula, we need the following Lemma. Ros and Souam ([RS97]) derived this Lemma when \(M\) is a \(C^2\) hypersurface. Since we consider \(\text{reg}M\) (which is \(C^2\)), the proof of this Lemma is essentially the same with Ros and Souam’s result, and hence omitted.

Lemma 4.2 ([RS97], Lemma 4.1). For a set \(E \subset \Omega\) which is stable as in Definition 1.2 and for any \(C^2\) variation \(\Psi_t\) whose initial velocity \(X := \frac{d}{dt} |_{t=0} \Psi_t\) satisfies (3.2) and (3.3). Let \(X_t(x) := \frac{d}{ds} |_{s=t} \Psi_s(x)\) denote the velocity of the variation at \(t\), for ease of notation, we use \(X\) to represent the initial velocity \(X_t|_{t=0}\). Let \(\nabla^M, \nabla\) denote the gradient on \(M, \partial M\), respectively, and \(X_0 (\text{resp. } X_1)\) the tangent part of \(X\) to \(M\) (resp. to \(\partial M\)). Let also \(S_0, S_1, S_2\) denote respectively the shape operator of \(M\) in \(\mathbb{R}^n\) with respect to \(-\nu_M\), of \(\partial M\) in \(M\) with respect to \(\nu^M_T\) and of \(\partial M\) in \(\partial B^+\) with respect to \(\nu^{B^+}_T\). Let \(\cdot\) denote the Euclidean inner product, let \(f = X \cdot (-\nu_M)\) be a \(C^1\) function defined on \(\text{reg}M\) and \(\frac{\partial f}{\partial \nu_T} = \nu^M_T \cdot \nabla^M f\). Then, for \(q\) as in (1.4), we have
\begin{enumerate}
\item \((-\nu_M)' = -\nabla^M f\) - \(S_0(X_0),\)
\item \((\nu^M_T)' = (\frac{\partial f}{\partial \nu_T} + B_M(X_0, \nu^M_T)) (-\nu_M) + fS_0(\nu^M_T) - fB_M(\nu^M_T, \nu^M_T) \nu^M_T - S_1(X_1) + \frac{\beta}{\sqrt{1-\beta^2}} \nabla B^+ f,\)
\item \((\nu^{B^+}_T)' = -B_{\partial M}(X, \nu^{B^+}_T) \nu_{B^+} - S_2(X_1) + \frac{1}{\sqrt{1-\beta^2}} \nabla B^+ f,\)
\end{enumerate}
(4) \[ X' \cdot \left( \nu^M_t - \beta \nu^{B^+}_t \right) + X \cdot \left( (\nu^M_t)' - (\beta \nu^{B^+}_t)' \right) = f \frac{\partial f}{\partial \nu^M_t} - qf^2. \]

Here we denote by a "prime" the first derivative \( \frac{d}{dt} \mid_{t=0} \) in the Euclidean space \( \mathbb{R}^n \). We use notations in \textbf{Section 1.1} (see also \textbf{Figure 1}).

\textbf{Lemma 4.3.} For a set \( E \subset \Omega \) which is stable as in \textbf{Definition 1.2} and for any \( C^2 \) variation \( \Psi_t \) whose initial velocity \( X := \frac{d}{dt} \mid_{t=0} \Psi_t \) satisfies (3.2) and (3.3), we have the following second variation formula

\[ F''_\beta(0) = -\int_M \left( f \Delta_M f + \|B_M\|^2 f^2 \right) d\mathcal{H}^{n-1} + \int_{\partial M} f \left( \frac{\partial f}{\partial \nu^M_t} - qf \right) d\mathcal{H}^{n-2} + HV''(0) \quad (4.8) \]

where

\[ f = X \cdot \nu_M \text{ is a function defined on } \text{reg}M, \]

\( q \) is given by (1.4),

\( H \) is the constant mean curvature as in \textbf{Proposition 3.1}.

\textbf{Proof.} Set \( A(t) = P(E_t; \Omega), B(t) = P(E_t; \partial \Omega), M_t := \Psi_t(\text{reg}M), B^{+}_t := \Psi_t(B^+), \Gamma_t := \partial M_t, V(t) := |E_t| \). Also, let \( \nu_M, \nu^{B^+}_M \) denote the outer unit normal of \( M_t, B^+_t \), respectively; Let \( \nu^M_{\Gamma_t}, \nu^{B^+}_M \) denote the outer unit conormal of \( M_t, B^+_t \) at \( \Gamma_t \), respectively. For simplicity, we still use \( \Gamma \) as in \textbf{Section 1.1} to represent \( \Gamma_{\mid t=0} \). In the following integration argument, we use \textbf{Remark 3.1}.

Then at time \( t \), by \textbf{Theorem 2.2}, we have

\[ A'(t) = \int_{M_t} \text{div}_M X \mid_y d\mathcal{H}^{n-1}(y) \]

\[ = \int_{M_t} H(t) X \cdot \nu_M \mid_y d\mathcal{H}^{n-1}(y) + \int_{\Gamma_t} X \cdot \nu^M_{\Gamma_t} \mid_y d\mathcal{H}^{n-2}(y). \]

\[ B'(t) = \int_{B^{+}_t} \text{div}_{B^{+}_t} X \mid_y d\mathcal{H}^{n-1}(y) = \int_{\Gamma_t} X \cdot \nu^{B^+}_M \mid_y d\mathcal{H}^{n-2}(y). \]

By the area formula

\[ \int_{M_t=\Psi_t(\text{reg}M)} H(t) X \cdot \nu_M \mid_y d\mathcal{H}^{n-1}(y) = \int_{\text{reg}M} H(t) X \cdot \nu_M \mid_{\Psi_t(x)} J\Psi_t(x)d\mathcal{H}^{n-1}(x). \]

Similarly

\[ \int_{\Gamma_t=\Psi_t(\Gamma)} X \cdot \left( \nu^M_{\Gamma_t} - \beta \nu^{B^+}_{\Gamma_t} \right) \mid_y d\mathcal{H}^{n-2}(y) = \int_{\Gamma} X \cdot \left( \nu^M_{\Gamma_t} - \beta \nu^{B^+}_{\Gamma_t} \right) \mid_{\Psi_t(x)} J\Psi_t(x)d\mathcal{H}^{n-2}(x). \]

Thus

\[ F''_\beta(t) = A'(t) - \beta B'(t) = \int_M H(t) X \cdot \nu_M \mid_{\Psi_t(x)} J\Psi_t(x)d\mathcal{H}^{n-1}(x) \]

\[ - \int_{\Gamma} X \cdot \left( \nu^M_{\Gamma_t} - \beta \nu^{B^+}_{\Gamma_t} \right) \mid_{\Psi_t(x)} J\Psi_t(x)d\mathcal{H}^{n-2}(x). \]
Hence
\[
\mathcal{F}''(0) = \int_M H'(0) X \cdot \nu_M \mid_x \, d\mathcal{H}^{n-1}(x) \\
+ H \frac{d}{dt} \mid_{t=0} \left( \int_M X_t \cdot \nu_{M_t} \mid_{\Phi_t(x)} J\Phi_t(x) \, d\mathcal{H}^{n-1}(x) \right) \\
+ \int_{\Gamma} \left( \frac{d}{dt} \mid_{t=0} X(\Phi_t(x)) \cdot (\nu^M_{\Gamma_t} - \beta \nu^{B^+}_{\Gamma_t} \mid_x) \right) \, d\mathcal{H}^{n-2}(x) \\
+ \int_{\Gamma} X(x) \cdot (\nu^M_{\Gamma_t} - \beta \nu^{B^+}_{\Gamma_t} \mid_x) \frac{d}{dt} \mid_{t=0} J\Phi_t(x) \, d\mathcal{H}^{n-2}(x). 
\]

(4.10)

For the second term in (4.10), notice that
\[
V'(t) = \int_{E_t} \text{div} X \mid_y \, d\mathcal{H}^n(y) = \int_{M_t=\Phi_t(M)} X \cdot \nu_{M_t} \mid_y \, d\mathcal{H}^{n-1}(y) \\
= \int_M (X \cdot \nu_{M_t} \mid_{\Phi_t(x)}) J\Phi_t(x) \, d\mathcal{H}^{n-1}(x) 
\]
where we use the divergence theorem in the first equality and the area formula in the last equality.

Thus we have
\[
H \frac{d}{dt} \mid_{t=0} \left( \int_M X \cdot \nu_{M_t} \mid_{\Phi_t(x)} J\Phi_t(x) \, d\mathcal{H}^{n-1}(x) \right) = HV''(0). 
\]

For the fifth term, notice that on \( \Gamma \), by Proposition 3.1, \((\nu^M_{\Gamma_t} - \beta \nu^{B^+}_{\Gamma_t}) \mid_x \perp T_x \partial \Omega \) and \( X \in T_x \partial \Omega \), hence
\[
X \cdot (\nu^M_{\Gamma_t} - \beta \nu^{B^+}_{\Gamma_t}) \mid_x = 0, \quad \forall x \in \Gamma. 
\]

We deduce that
\[
\mathcal{F}''(0) = -\int_M (f \Delta_M f + \|B_M\|^2 f^2) d\mathcal{H}^{n-1}(x) + HV''(0) \\
+ \int_{\Gamma} \left( \frac{d}{dt} \mid_{t=0} X(\Phi_t(x)) \cdot (\nu^M_{\Gamma_t} - \beta \nu^{B^+}_{\Gamma_t} \mid_x) \right) \, d\mathcal{H}^{n-2}(x) \\
+ \int_{\Gamma} X(x) \cdot \left( \frac{d}{dt} \mid_{t=0} (\nu^M_{\Gamma_t} - \beta \nu^{B^+}_{\Gamma_t} \mid_{\Phi_t(x)}) \right) \, d\mathcal{H}^{n-2}(x). 
\]

(4.11)

Notice that the last two terms in (4.11) are computed in Lemma 4.2 (4).
Combining the computations above, we have
\[
\mathcal{F}''(0) = -\int_M (f \Delta_M f + \|B_M\|^2 f^2) d\mathcal{H}^{n-1}(x) + HV''(0) + \int_{\partial M} \left( f \frac{\partial f}{\nu^M_{\Gamma_t}} - qf^2 \right) \, d\mathcal{H}^{n-2} 
\]

(4.12)

where \( q \) is given by (1.4). □
Proof of Theorem 1.1. With the help of the smooth cut-off functions, we are able to finish the proof. Fix a small \( \epsilon > 0 \), consider any smooth function \( \zeta : \operatorname{reg} M \to \mathbb{R}^1 \) satisfying \( \int_M \zeta d\mathcal{H}^{n-1}(x) = 0 \).

First, we define a smooth function \( \tilde{\zeta}_\epsilon : M \to \mathbb{R}^1 \) by \( \tilde{\zeta}_\epsilon := \varphi_\epsilon \cdot \zeta \). By virtue of Lemma 4.1, we have:

1. \( \tilde{\zeta}_\epsilon \equiv 0 \) on \( S'_\epsilon \),
2. \( \tilde{\zeta}_\epsilon \equiv \zeta \) on \( M \setminus S_\epsilon \),
3. \( \tilde{\zeta}_\epsilon \to \zeta \) in \( H^1(\operatorname{reg} M) \) as \( \epsilon \to 0 \).

Notice that

\[ \int_M \zeta - \tilde{\zeta}_\epsilon d\mathcal{H}^{n-1} = \int_{S^\epsilon} (1 - \varphi_\epsilon) \zeta d\mathcal{H}^{n-1} \leq \sup_{S_\epsilon} |\zeta| \mathcal{H}^{n-1} \left( \bigcup_{i=1}^{N_1} B_{r_i}(z_i) \right) \leq C_\eta \epsilon^{n-1} \]

where \{ \{ B_{r_i}(z_i) \}_{i=1}^{N_1} \}, \( N_1 \) are given in the proof of Lemma 4.1, and we use the fact that \( S_\epsilon \subset \bigcup_{i=1}^{N_1} (B_{r_i}(z_i)) \). Also, the last inequality holds due to the fact that \( r_i < \frac{\epsilon}{2} \).

With this estimate, we can now smoothly modify \( \tilde{\zeta}_\epsilon \) on \( M \setminus S_\epsilon \), and obtain a new smooth function, denoted by \( \zeta_\epsilon \), with:

1. \( \zeta_\epsilon \equiv 0 \) on \( S'_\epsilon \),
2. \( |\zeta_\epsilon - \zeta| < \epsilon \) on \( M \setminus S_\epsilon \),
3. \( \zeta_\epsilon \to \zeta \) in \( H^1(\operatorname{reg} M) \) as \( \epsilon \to 0 \),
4. \( \int_M \zeta_\epsilon d\mathcal{H}^{n-1} = 0 \).

To be precise, set \( f(\epsilon) := \int_M (\zeta - \tilde{\zeta}_\epsilon) d\mathcal{H}^{n-1} \), then we can find a smooth function \( \eta : \mathbb{R}^n \to \mathbb{R}^1 \) s.t.,

1. \( \int_M \eta d\mathcal{H}^{n-1} = 1 \),
2. \( \eta \equiv 0 \) on \( S'_\epsilon \).

Then, set \( \eta_\epsilon := f(\epsilon) \eta \), we have

1. \( \int_M \eta_\epsilon d\mathcal{H}^{n-1} = \int_M f(\epsilon) \eta d\mathcal{H}^{n-1} = f(\epsilon) \),
2. \( \eta_\epsilon \equiv 0 \) on \( S'_\epsilon \),
3. \( |\eta_\epsilon| \mathcal{H}(M) \leq |f(\epsilon)| \sup_M |\eta| \leq C \epsilon^{n-1} \).

Thus, set \( \zeta_\epsilon := \tilde{\zeta}_\epsilon + \eta_\epsilon \), and we have the desired smooth function.

Moreover, we extend \( \zeta_\epsilon \) smoothly to the whole \( \mathbb{R}^n \) with only the requirement that \( \nabla \zeta_\epsilon \cdot \nu_M = 0 \) on \( M \setminus S'_\epsilon \).

Then, take any smooth vector field \( \nu_M^\epsilon \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n) \) satisfying

1. \( |\nu_M^\epsilon| = 1 \) in a neighborhood of \( M \setminus S'_\epsilon \),
2. \( \nu_M^\epsilon = \nu_M \) on \( M \setminus S'_\epsilon \).

Also, take any smooth vector field \( N^\epsilon \in C^\infty_c(\mathbb{R}^n; \mathbb{R}^n) \) satisfying

1. \( |N^\epsilon| = 1 \) in a neighborhood of \( (M \setminus S'_\epsilon) \cap \partial \Omega \),
2. \( N^\epsilon = \nu_M^\epsilon \) on \( (M \setminus S'_\epsilon) \cap \partial \Omega \),
3. \( N^\epsilon(x) \in T_x M \) for any \( x \in M \).

Then, let \( X_\epsilon \in C^2(\mathbb{R}^n; \mathbb{R}^n) \) be a vector field satisfying

1. \( X_\epsilon = \zeta_\epsilon \left( \frac{\beta}{\sqrt{1 - \beta^2}} N^\epsilon + \nu_M^\epsilon \right) \) in some neighborhood of \( M \),
2. \( X_\epsilon(x) \in T_x(\partial \Omega) \) for all \( x \in \partial \Omega \).
Notice that such $X_\epsilon$ exists since these conditions can both be satisfied by virtue of the fact that $M$ intersects $\partial \Omega$ with the constant contact angle $\theta$, where $\cos \theta = -\beta$ and $\frac{\beta}{\sqrt{1-\beta^2}} = \cot (\pi - \theta)$, and hence on $\text{reg} M \cap \partial \Omega$, $\frac{\beta}{\sqrt{1-\beta^2}} N^\epsilon(x) + \nu_M^\epsilon(x) \in T_x \partial \Omega$.

Also,

$$\int_M X_\epsilon \cdot \nu_M d\mathcal{H}^{n-1} = \frac{\beta}{\sqrt{1-\beta^2}} \int_M \zeta_\epsilon N^\epsilon \cdot \nu_M d\mathcal{H}^{n-1} + \int_M \zeta_\epsilon \nu_M^\epsilon \cdot \nu_M d\mathcal{H}^{n-1} = 0.$$ 

Hence $X_\epsilon$ satisfies (3.2),(3.3). Following the same argument as in the proof of Proposition 3.1, we obtain an admissible family of sets $\{\tilde{E}_t\}$ by a smooth modification through the graph function(denoted by $g_\epsilon$) inside $\Omega$.

By stability of $E$, a direct computation of (3.8) yields

$$0 \leq \mathcal{F}_\beta''(0) = \mathcal{F}_\beta''(0) + 2 \int_{D'} \frac{\nabla x'u_\epsilon(x',0) \cdot \nabla x'g_\epsilon(x',0)}{\sqrt{1 + |\nabla x'u_\epsilon(x',0)|^2}} dx'$$

$$= \mathcal{F}_\beta''(0) + 2H \int_{D'} g_\epsilon(x',0) dx'$$

$$= \mathcal{F}_\beta''(0) - HV_\beta''(0)$$

where we use the definition of $g_\epsilon$ in the last inequality. Here $\nabla x', u_\epsilon, g_\epsilon$ are defined in the proof of Proposition 3.1, precisely, (3.6) and the paragraph before (3.6).

Combining with (4.8), we have

$$-\int_M (f \Delta_M f + ||B_M||^2 f^2) d\mathcal{H}^{n-1} + \int_{\partial M} f \left( \frac{\partial f}{\partial \nu_M} - qf \right) d\mathcal{H}^{n-2} \geq 0 \quad (4.13)$$

where $f = X_\epsilon \cdot \nu_M$, $q$ is given by (1.4).

In terms of $f$, we have the following observations

$$\zeta_\epsilon N^\epsilon \cdot \nu_M = 0 \quad \text{on reg} M,$$

$$\zeta_\epsilon \nu_M^\epsilon \cdot \nu_M = \zeta_\epsilon \quad \text{on reg} M.$$ 

Thus

$$f = \zeta_\epsilon \quad \text{on reg} M.$$ 

By plugging $f$ into (4.13), we obtain

$$-\int_M (\zeta_\epsilon \Delta_M \zeta_\epsilon + ||B_M||^2 \zeta_\epsilon^2) d\mathcal{H}^{n-1} + \int_{\partial M} \zeta_\epsilon \left( \frac{\partial \zeta_\epsilon}{\partial \nu_M} - q\zeta_\epsilon \right) d\mathcal{H}^{n-2} \geq 0.$$ 

By divergence theorem and the fact that $\nabla \zeta_\epsilon \cdot \nu_M = 0$ on $M - S'_{\epsilon'}$, we have

$$\int_M (||\nabla M \zeta_\epsilon||^2 - ||B_M||^2 \zeta_\epsilon^2) d\mathcal{H}^{n-1} - \int_{\partial M} q\zeta_\epsilon d\mathcal{H}^{n-2} \geq 0.$$

Sending $\epsilon \searrow 0$, by virtue of $\zeta_\epsilon \to \zeta$ in $H^1(\text{reg} M)$ as $\epsilon \to 0$, we deduce that (1.3) holds for any smooth $\zeta$ satisfying $\int_M \zeta(x) d\mathcal{H}^{n-1}(x) = 0$.

Finally, by divergence theorem again, we obtain (1.5).
5. Stable measure-theoretic capillary hypersurface in a Euclidean ball

Following the same approach in [WX19], we prove the rigidity result on the regular part of stable $E \subset \mathbb{B}^n$ with the help of the Poincare-type inequality (1.3). Moreover, we will rull out the singularities by stability of $E$. Throughout Section 5 and Section 6, we use $\langle \cdot, \cdot \rangle$ to denote the Euclidean inner product in $\mathbb{R}^n$.

First, fix $a \in \mathbb{R}^n$, we consider a vector field $X_a$ in $\mathbb{R}^n$ defined by

$$X_a(x) := \langle x, a \rangle x - \frac{1}{2}(|x|^2 + 1)a, \quad x \in \mathbb{R}^n.$$ 

$X_a$ is a conformal Killing vector field in $\mathbb{R}^n$ which is tangent to $\mathbb{S}^{n-1}$.

**Proposition 5.1** ([WX19, Proposition 3.1]).

(1) $\mathcal{L}_{X_a} \delta = \langle x, a \rangle \delta$, or equivalently

$$\frac{1}{2}[\nabla_i(X_a)_j + \nabla_j(X_a)_i] = \langle x, a \rangle \delta_{ij}.$$ 

(2) $\langle X_a, x \rangle |_{\partial \mathbb{B}^n} = 0$.

Here $\mathcal{L}$ denotes the Lie derivative, $\delta$ denotes the canonical Riemannian metric in $\mathbb{R}^n$, $\nabla_i(X)_j := \delta(\nabla e_i X, e_j)$, where $\{e_i\}_{i=1,\ldots,n}$ denote the coordinate vectors of $\mathbb{R}^n$.

The following are some properties which we will use in the proof of the rigidity of $M$, and due to the fact that we study these properties only on $\text{reg} M$, which admits the unit normal $\nu_M$ point wisely, so the proof of the following properties is exactly the same as [WX19], hence omitted.

**Proposition 5.2.** For a stable $E$ as in Definition 1.2 in the unit ball $\mathbb{B}^n$ with $\text{sing} M$ closed and $\mathcal{H}^{n-3}(\text{sing} M) = 0$, using the notations in Section 1.1 (see also Figure 1) we have:

(1) on $\text{reg} M \cap \Omega$, the mean curvature of $M$ is constant, denoted by $H$; On $\text{reg} M \cap \partial \Omega$, we have: $\nu_M \cdot \nu_{B^+} = -\beta = \cos \theta$. \(^6\)

(2) on $\text{reg} M$, the following Minkowski type formula holds:

$$\int_{\text{reg} M} (n - 1) \langle x - \cos \theta \nu_M, a \rangle d\mathcal{H}^{n-1} = \int_{\text{reg} M} H \langle X_a, \nu_M \rangle d\mathcal{H}^{n-1}. \quad (5.1)$$

(3) along $\text{reg} M \cap \partial \Omega$, we have

$$\nabla_{\nu_M^B} \langle x - \cos \theta \nu_M, a \rangle = q \langle x - \cos \theta \nu_M, a \rangle, \quad (5.2)$$

$$\nabla_{\nu_M^B} \langle X_a, \nu_M \rangle = q \langle X_a, \nu_M \rangle, \quad (5.3)$$

where $q$ is given by (1.4).

(4) on $\text{reg} M$, the following identities hold:

$$\Delta_M x = -H \nu_M,$$

$$\Delta_M \frac{1}{2}|x|^2 = (n - 1) - H \langle x, \nu_M \rangle,$$

$$\Delta_M \nu_M = \nabla^M H - ||B_M||^2 \nu_M,$$

$$\Delta_M \langle x, \nu_M \rangle = \langle x, \nabla^M H \rangle + H - ||B_M||^2 \langle x, \nu_M \rangle,$$

$$\Delta_M \langle X_a, \nu_M \rangle = \langle X_a, \nabla^M H \rangle + \langle x, a \rangle H - ||B_M||^2 \langle X_a, \nu_M \rangle (n - 1) \langle \nu_M, a \rangle.$$ 

\(^6\) Notice that in [WX19], $\theta$ therein is the angle between the outer unit conormal, namely, $\langle \nu^M, \nu^M_{B^+} \rangle = \cos \theta$. In contrast to this, in our case, $\langle \nu^M, \nu^M_{B^+} \rangle = -\cos \theta$.

\(^7\) To prove the Minkowski type formula, an approximation argument is needed, we will present a detailed proof in Proposition 6.3.
Proof of Theorem 1.2. By Proposition 3.1, we know that $\text{reg} M$ intersects $\partial \mathbb{B}^n$ with a constant angle $\theta \in (0, \pi)$, and is of constant mean curvature $H$. In the following, we suppose that $H \geq 0$ on $\text{reg} M \cap \mathbb{B}^n$. Otherwise, if $H < 0$, we consider $\mathbb{B}^n \setminus E \subset \mathbb{B}^n$ with $|\mathbb{B}^n \setminus E| = |(1 - \sigma)|\mathbb{B}^n|$, which is stable as in Definition 1.2. The reason that $\mathbb{B}^n \setminus E$ is stable is as follows:

We consider the functional $\mathcal{F}_{-\beta}$ instead of $\mathcal{F}_{\beta}$, then we have

$$
\mathcal{F}_{-\beta}(\mathbb{B}^n \setminus E_t) = \mathcal{P}(\mathbb{B}^n \setminus E_t; \mathbb{B}^n) - (\beta) \mathcal{P}(\mathbb{B}^n \setminus E_t; \mathbb{B}^n).
$$

By Proposition 5.2, a direction computation yields

$$
\mathcal{F}_{\beta}(E_t) = \mathcal{P}(E_t; \mathbb{B}^n) - (\beta) [\mathcal{P}(\mathbb{B}^n) - \mathcal{P}(E_t; \mathbb{B}^n)]
$$

$$
= \mathcal{F}_{\beta}(E_t) + \beta \mathcal{P}(\mathbb{B}^n).
$$

Since $E$ is stable with respect to the functional $\mathcal{F}_{\beta}$, we see that $\mathbb{B}^n \setminus E$ is stable with respect to the functional $\mathcal{F}_{-\beta}$. Then by Proposition 3.1, $M$ as the boundary of $\mathbb{B}^n \setminus E$ is of constant contact angle $\pi - \theta$ and constant mean curvature $-H \geq 0$.

For the $C^2$ part $\text{reg} M$, we argue as [WX19] with some modifications, first we construct a special test function to use the stability condition. Precisely, we define a test function pointwisely along $\text{reg} M$ in the following manner: For each $a \in \mathbb{R}^n$, on $\text{reg} M$, we define:

$$
\phi_a := (n - 1) \langle x - \cos \theta \nu_M, a \rangle - H \langle X_a, \nu_M \rangle.
$$

By (5.1) and the fact that $\int_{\text{reg} M} d\mathcal{H}^{n-1} = \int_M d\mathcal{H}^{n-1}$, we have

$$
\int_M \phi_a d\mathcal{H}^{n-1} = 0.
$$

(5.4)

By virtue of Theorem 1.1, Proposition 5.1 and Proposition 5.2 and following the same argument as in [WX19, Proposition 3.5, Theorem 3.1], we obtain:

$$
\int_M \left( (n - 1)|x|^2 - (n - 1) \cos \theta \langle x, \nu_M \rangle - \frac{1}{2}(|x|^2 - 1)H \langle x, \nu_M \rangle \right)
\cdot [(n - 1)||B_M||^2 - H^2] d\mathcal{H}^{n-1} \leq 0.
$$

(5.5)

Consider the following function $\Phi$ defined on $\text{reg} M$ by

$$
\Phi := \frac{1}{2}(|x|^2 - 1)H - (n - 1) (\langle x, \nu_M \rangle - \cos \theta).
$$

By Proposition 5.2, a direction computation yields

$$
\Delta_M \Phi = (n - 1)||B_M||^2 - H^2 \langle x, \nu_M \rangle, \quad \forall x \in \text{reg} M.
$$

(5.6)

Using the divergence theorem and following the same argument as in the proof of [WX19, Theorem 3.1], we have:

$$
0 \geq \int_M \left( (n - 1)|x|^2 - (n - 1) \cos \theta \langle x, \nu_M \rangle - \frac{1}{2}(|x|^2 - 1)H \langle x, \nu_M \rangle \right)
\cdot [(n - 1)||B_M||^2 - H^2] + \Delta_M (\frac{1}{2}\Phi^2) d\mathcal{H}^{n-1}
$$

$$
= \int_M \left( (n - 1)|x|^2 + (n - 1) \cos \theta \langle x, \nu_M \rangle - \frac{1}{2}(|x|^2 - 1)H \langle x, \nu_M \rangle \right)
\cdot [(n - 1)||B_M||^2 - H^2] + \Phi \Delta_M \Phi + |\nabla M \Phi|^2 d\mathcal{H}^{n-1}
$$

$$
= \int_M (n - 1)|x|^2 (n - 1)||B_M||^2 - H^2 + |\nabla M \Phi|^2 d\mathcal{H}^{n-1} \geq 0,
$$

(5.7)

where $x^T$ is the tangential part of $x$ with respect to $\text{reg} M$ and in the last inequality we use the fact that $(n - 1)||B_M||^2 - H^2$ is non-negative by virtue of Cauchy’s inequality.
From (5.7) we see that equalities hold throughout the computations, thus $|\nabla M \Phi|^2 \equiv 0$ on $\text{reg}M$, and $|x^T|^2 ((n-1)||B_M||^2 - H^2) \equiv 0$ on $\text{reg}M$. Since $|\nabla M \Phi|^2 \equiv 0$, we see that $\Delta_M \Phi \equiv 0$ on $\text{reg}M$, namely, (5.6) reduces to

$$\langle x, \nu_M \rangle ( (n-1)||B_M||^2 - H^2) = 0, \quad \forall x \in \text{reg}M.$$ 

Together with $|x^T|^2 ((n-1)||B_M||^2 - H^2) \equiv 0$, we obtain

$$|x|^2 ((n-1)||B_M||^2 - H^2) \equiv 0, \quad \forall x \in \text{reg}M,$$

which reveals the fact that $(n-1)||B_M||^2 = H^2, \quad \forall x \in \text{reg}M$.

By virtue of Cauchy’s inequality, the equality holds if and only if $\text{reg}M$ is umbilical in $\mathbb{B}^n$. It follows that away from $\text{sing}M$, locally $M$ is flat or spherical. Next we prove that $\text{sing}M$ is empty and hence $M$ is either a totally geodesic ball or a spherical cap.

Indeed, since $\text{reg}M$ is of constant mean curvature $H$ (assume $H > 0$, for the case $H = 0$, the proof is essentially the same), if $\text{sing}M \neq \emptyset$, then $M$ is the union of finitely many spherical caps with the same radius $r = \frac{1}{H}$. Here the reason that spherical caps are finite is that they are of the same radius and $P(M; \mathbb{B}^n) < \infty$. In this case, fix a singular point $x \in M$, locally near $x$, we can find a smooth volume-preserving perturbation which decreases the perimeter of $E$ strictly. Precisely, we can remove a portion with small volume $v$ near the regular part of a spherical cap, which is the intersection of two balls with the same radius, and add a portion with a flat boundary and volume $v$ near the singular point. This perturbation preserves the volume and strictly decreases the perimeter. Indeed, the removing near the regular part does not change the perimeter, while the perimeter near the singular point strictly decreases after the perturbation, see Figure 2.

6. **Stable measure-theoretic capillary hypersurface in a Wedge-shaped domain**

In light of the arguments for the smooth stable capillary hypersurface [LX17; Sou21], we proceed the rigidity results to the stable measure-theoretic capillary hypersurfaces.

Let us first set things up, let $\Omega$ be a smooth, unbounded domain in $\mathbb{R}^n (n \geq 3)$, $\partial \Omega$ consists of a finite family of hyperplanes $P_1, \ldots, P_L$, for some integer $L \geq 1$. Let $n_1, \ldots, n_L$ be the exterior unit normal to $P_i$ in $\Omega$. We call such $\Omega$ a wedge-shaped domain when $\Omega$ satisfies that:
{n_1, \ldots, n_L}$ are linearly independent. Up to translation, we can assume that the origin $O \in \mathbb{R}^n$ is in the intersection $\bigcap_{i=1}^L P_i$.

Let $E \subseteq \Omega$ be a set with finite volume and perimeter, we consider the free energy functional $F_L(E; \Omega)$ given by

$$F_L(E; \Omega) = P(E; \Omega) - \sum_{i=1}^L \beta_i P(E; P_i),$$

(6.1)

where for each $i$, $\beta_i \in (-1, 1)$ is a prescribed constant.

In this situation, the definition of stationary sets of $F_L$ under volume constraint in $\Omega$ is brought up easily.

**Definition 6.1.** Let $\Omega$ be a wedge-shaped domain in $\mathbb{R}^n$. For a set of finite perimeter $E \subseteq \Omega$, let $F_L(E; \Omega)$ be a set of finite perimeter, which is stationary for the energy functional $F_L$ under volume constraint if $F_L(0) = 0$ for all admissible families $\{E_i\}$. Moreover, a stationary set $E$ is called stable if $F_L(0) \geq 0$ for all admissible families $\{E_i\}$.

Naturally, we can generalize Proposition 3.1 to this case,

**Proposition 6.1.** Let $E$ be a set of finite perimeter, which is stationary for $F_L$ under volume constraint, as in Definition 6.1, with $\mathcal{H}^{n-2}(\text{sing } M) = 0$. Let $\theta \in (0, \pi)$ such that $\cos \theta = -\beta$. Let $M$ denote the set $\partial E \cap \overline{\Omega}$, let $B^+_i$ denote the set $\text{Int}(\partial E \cap P_i)$, which is open and smooth; let $\Gamma_i$ denote the closed set $M \cap \partial B^+_i$, namely, $\Gamma_i = \partial M \cap P_i = \partial B^+_i$. Throughout this section, let $\nu_{M, \Gamma_i}$ denote the outer unit normal of $M, B^+_i$, respectively, when they exist; $\nu_{\Gamma_i}^M, \nu_{\Gamma_i}^{B^+_i}$ denote the exterior unit conormal of $\partial E \cap \Omega$ in $M, B^+_i$, respectively.

Then $E$ satisfies

i. *(Constant mean curvature)* On $\partial^* E \cap \Omega$, the mean curvature of $\partial^* E$ is constant, denoted by $H_M$.

ii. *(Constant contact angle)* On $\partial^* E \cap P_i$, the measure-theoretic hypersurface $\partial^* E$ intersects $P_i$ with a constant contact angle $\theta_i(\cos \theta_i = -\beta_i)$, i.e.,

$$\langle \nu_{M, \Gamma_i}, \nu_{\Gamma_i}^{B^+_i} \rangle = -\beta_i = -\langle \nu_{\Gamma_i}^M, \nu_{\Gamma_i}^{B^+_i} \rangle$$

(6.2)

Moreover, $\partial^* E \cap \Omega$ is locally an analytic hypersurface with constant mean curvature, relatively open in $\partial E \cap \Omega$.

**Proof.** When $M$ is $C^2$, this result has been derived in [LX17, Section 2]. For the non-smooth case, the proof is essentially the same with Proposition 3.1 and hence omitted. $\square$

**Remark 6.1.** It is possible that $E$ intersects $\partial \Omega$ in a trivial way, namely, $\mathcal{H}^{n-1}(\partial^* E \cap P_i) = 0$ for each $i \in \{1, \ldots, L\}$. In this situation, the free energy functional reduces to the perimeter functional $P(E; \Omega)$ and $E$ is a stationary point of the isoperimetric problem, which is well-studied by D.M. Delgadino and F. Maggi in [DM19]. Thus in the following we assume that $E$ intersects $\partial \Omega$ in a non-trivial way, i.e., there exists some integer $0 < K \leq L$ such that $P(E; P_i) \neq 0$ for each $i \leq K$.

---

8Notice that this implies: for any combination $\{n_1, \ldots, n_r\}$ ($1 \leq r \leq L$), the normals are linearly independent.

9It follows that for each $i$ and for any $x \in P_i$, $\langle x, n_i \rangle = 0$, which turns out to be useful in the proof of Proposition 6.3.

10It follows that $E$ is bounded, namely, in this section, we mainly concerned with bounded measure-theoretic capillary hypersurfaces in $\Omega$. 
In this situation, by a minor modification of the proof of Theorem 1.1, we conclude that,

**Proposition 6.2.** Let \( \Omega \subset \mathbb{R}^n \) be a wedge-shaped domain, let \( E \subset \Omega \) be set with finite volume and perimeter, which is stable for \( \mathcal{F}_L \) under volume constraint, as in Definition 6.1. Using the notations in Proposition 6.1, if the singular set \( \text{sing} M \) is closed and satisfying \( \mathcal{H}^{n-3}(\text{sing} M) = 0 \). Then, for any smooth function \( \zeta : \text{reg} M \to \mathbb{R}^1 \) with

\[
\int_M \zeta(x) d\mathcal{H}^{n-1}(x) = 0,
\]

the following Poincaré-type inequality holds,

\[
J(\zeta) := \int_{M \cap \Omega} (|\nabla^M \zeta|^2 - |B_M|^2 \zeta^2) d\mathcal{H}^{n-1}(x) - \sum_{i=1}^K \int_{\Gamma_i} q_i \zeta^2 d\mathcal{H}^{n-2} \geq 0
\]

where

\[
q_i = -\cot \theta_i B_M(\nu^M_{i1}, \nu^M_{i1}).
\]

Equivalently,

\[
-\int_{M \cap \Omega} (\zeta \Delta_M \zeta + |B_M|^2 \zeta^2) d\mathcal{H}^{n-1}(x) + \int_{\Gamma_i} (\zeta \nabla^M \zeta \cdot \nu^M_{i1} - q_i \zeta^2) d\mathcal{H}^{n-2} \geq 0,
\]

Here \( \nabla^M \) denotes the tangential gradient with respect to \( M \) and \( \Delta_M \) denotes the tangential Laplacian with respect to \( M \).

In order to use the Poincaré-type inequality (6.6), we need a test function which is smooth on \( \text{reg} M \), having vanishing integration. To this end, we first set up a lemma which gives a good characterization of the capillary geometry. Thanks to this lemma, we can derive a Minkowski-type formula, which gives us a desired test function to test the stability. Although these formulas are well-known and widely used in the smooth setting (see for example [AS16; LX17; Sou21]), yet we still need the useful smooth cut-off functions to derive the proof.

**Lemma 6.1.** Let \( E \subset \Omega \) be a set with finite volume and perimeter with \( \text{sing} M \) closed and \( \mathcal{H}^{n-2}(\text{sing} M) = 0 \). Let \( \nu^M_i \) denote the exterior unit conormal to \( \partial M \) in \( M \), which is well-defined for \( \mathcal{H}^{n-2} \)-a.e. \( x \in \partial M \). Then

\[
(n - 1) \int_M \nu_M d\mathcal{H}^{n-1} = \int_{\partial M} \{ \langle x, \nu^M \rangle \nu_M - \langle x, \nu_M \rangle \nu^M \} d\mathcal{H}^{n-2}.
\]

**Proof.** Let \( \bar{a} \) be any constant vector field in \( \mathbb{R}^n \), consider the following vector field on \( M \),

\[
Y = \langle \bar{a}, \nu_M \rangle x^T - \langle x, \nu_M \rangle \bar{a}^T,
\]

which is a well-defined \( C^2 \) vector field on \( \text{reg} M \), here \( x^T = x - \langle x, \nu_M \rangle \nu_M \) is the orthogonal projection of \( x \) onto the approximate tangent space \( T_x M \), \( \bar{a}^T \) is understood similarly. Notice aslo that \( |Y| \) is bounded on \( \text{reg}(M) \) by some constant \( C_7 \) since \( \bar{a} \) is a constant vector field and \( M \) is bounded.

For each \( \epsilon > 0 \), we have \( \varphi_\epsilon, S'_\epsilon, S_\epsilon \) from Lemma 4.1. Let \( \tilde{Y}_\epsilon : \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^2 \) vector field satisfying

\[
|\tilde{Y}_\epsilon| \leq C_7 \text{ in a neighborhood of } M \setminus S'_\epsilon, \quad \tilde{Y}_\epsilon = Y \text{ on } M \setminus S'_\epsilon.
\]

---

11 Notice that the boundaries of \( \Omega \) are planar, thus \( B_{P_i} \equiv 0 \).

12 The structural lemma holds for any container \( \Omega \) (could be unbounded) with \( C^2 \) boundary.
Then let \( Y_\epsilon \in C^2(\mathbb{R}^n; \mathbb{R}^n) \) be a vector field satisfying
\[
Y_\epsilon = \varphi_\epsilon \tilde{Y}_\epsilon.
\] (6.9)

Thus we find
\[
Y_\epsilon = \begin{cases} 
0 & \text{on } S'_\epsilon, \\
\varphi_\epsilon \tilde{Y}_\epsilon & \text{on } S_\epsilon \setminus S'_\epsilon, \\
\tilde{Y} & \text{on } M \setminus S_\epsilon.
\end{cases}
\]

Notice that on \( M \setminus S'_\epsilon \),
\[
\begin{align*}
\text{div}_M (x^T) &= \text{div}_M (x - \langle x, \nu_M \rangle \nu_M) = (n - 1) - H \langle x, \nu_M \rangle, \quad (6.10) \\
\text{div}_M (\tilde{a}^T) &= \text{div}_M (\tilde{a} - \langle \tilde{a}, \nu_M \rangle \nu_M) = -H \langle x, \nu_M \rangle. \quad (6.11)
\end{align*}
\]

Thus we have, on \( M \setminus S'_\epsilon \)
\[
\begin{align*}
\text{div}_M (Y_\epsilon) &= \varphi_\epsilon (\langle \tilde{a}, \nu_M \rangle \text{div}_M (x^T) + \langle \tilde{a}, \nabla_x x^T \nu_M \rangle - \langle x, \nu_M \rangle \text{div}_M (\tilde{a}^T) - \langle \tilde{a}^T, \nu_M \rangle - \langle x, \nabla_{\tilde{a}^T} \nu_M \rangle) + \langle \nabla^M \varphi_\epsilon, \tilde{Y}_\epsilon \rangle \\
&= \varphi_\epsilon ((n - 1) \langle \tilde{a}, \nu_M \rangle - H \langle x, \nu_M \rangle \langle \tilde{a}, \nu_M \rangle + \langle \tilde{a}^T, \nabla_x x^T \nu_M \rangle + H \langle \tilde{a}, \nu_M \rangle \langle x, \nu_M \rangle - \langle x^T, \nabla_{\tilde{a}^T} \nu_M \rangle) + \langle \nabla^M \varphi_\epsilon, \tilde{Y}_\epsilon \rangle \\
&= (n - 1) \varphi_\epsilon \langle \tilde{a}, \nu_M \rangle + \langle \nabla^M \varphi_\epsilon, \tilde{Y}_\epsilon \rangle,
\end{align*}
\]
where in the second equality, we use (6.10), (6.11); in the last equality, we use the fact that
\[
\langle \tilde{a}^T, \nabla_x x^T \nu_M \rangle = \langle x^T, \nabla_{\tilde{a}^T} \nu_M \rangle = B_M (x^T, \tilde{a}^T),
\]
here \( h_M \) denotes the second fundamental form of \( \text{reg}(M) \), which is well-defined for \( \mathcal{H}^{n-1} \)-a.e.

Finally, integrating \( \text{div}_M (Y_\epsilon) \) on \( M \setminus S'_\epsilon \) and using divergence theorem, we have
\[
\begin{align*}
\int_{M \cap (S_\epsilon \setminus S'_\epsilon)} (n - 1) \varphi_\epsilon \langle \tilde{a}, \nu_M \rangle + \langle \nabla^M \varphi_\epsilon, \tilde{Y}_\epsilon \rangle \ d\mathcal{H}^{n-1} &+ \int_{M \setminus S_\epsilon} \text{div}_M (Y_\epsilon) \ d\mathcal{H}^{n-1} \\
&= \int_{\partial M \setminus S_\epsilon} \langle Y_\epsilon, \nu_M^M \rangle \ d\mathcal{H}^{n-2} + \int_{\partial M \cap (S_\epsilon \setminus S'_\epsilon)} \varphi_\epsilon \langle \tilde{Y}_\epsilon, \nu_M^M \rangle \ d\mathcal{H}^{n-2} + \int_{\partial S'_\epsilon} 0 \ d\mathcal{H}^{n-2}. \quad (6.12)
\end{align*}
\]

By Lemma 4.1, (4.6), (4.7) and Remark 3.1, sending \( \epsilon \searrow 0 \), we find
\[
(n - 1) \int_M \langle \tilde{a}, \nu_M \rangle \ d\mathcal{H}^{n-1} = \int_{\partial M} \{ \langle x, \nu_M^M \rangle \langle \nu_M, \tilde{a} \rangle - \langle x, \nu_M \rangle \langle \nu_M^M, \tilde{a} \rangle \} \ d\mathcal{H}^{n-2}
\]
Since \( \tilde{a} \) is taken to be any constant vector field in \( \mathbb{R}^n \), we have (6.7).

Exploiting Lemma 6.1, we can obtain the following Minkowski type formula.

**Proposition 6.3** (Minkowski type formula). Let \( E \) be as in Proposition 6.1. Then
\[
\int_M \{ (n - 1) - H \langle x, \nu_M \rangle + (n - 1) \langle \nu_M, k \rangle \} \ d\mathcal{H}^{n-1} = 0, \quad (6.13)
\]
\[
\int_{\Gamma_i} \{ (n - 2) - H_{\Gamma_i} \langle x, \nu_{\Gamma_i}^{B_i} \rangle \} \ d\mathcal{H}^{n-2} = 0. \quad (6.14)
\]
Proof. For each $\epsilon > 0$, we have $\varphi_\epsilon, S'_\epsilon, S_\epsilon$ from Lemma 4.1. Let $\tilde{Y}_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field satisfying

$$|\tilde{Y}_\epsilon| \leq C_8 \text{ in a neighborhood of } M \setminus S'_\epsilon, \quad \tilde{Y}_\epsilon = (x - \langle x, \nu_M \rangle \nu_M) \text{ on } M \setminus S'_\epsilon.$$ 

Then let $Y_\epsilon \in C^2(\mathbb{R}^n; \mathbb{R}^n)$ be a vector field satisfying

$$Y_\epsilon = \varphi_\epsilon \tilde{Y}_\epsilon. \quad (6.15)$$

Integrating $\text{div}_M (Y_\epsilon)$ over $M \setminus S'_\epsilon$ and using divergence theorem, we have

$$\sum_{i=1}^K \left( \int_{(\partial M \cap P_i) \setminus S_\epsilon} \langle x, \nu_{T_i}^M \rangle \, d\mathcal{H}^{n-2} + \int_{(M \cap P_i \cap (S_\epsilon \setminus S'_\epsilon))} \varphi_\epsilon \langle x, \nu_{T_i}^M \rangle \, d\mathcal{H}^{n-2} + \int_{S'_\epsilon} 0 \, d\mathcal{H}^{n-2} \right)$$

$$= \int_{S_\epsilon \setminus S'_\epsilon} \langle \nabla M \varphi_\epsilon, x^T \rangle \, d\mathcal{H}^{n-1} + \int_{M \setminus S'_\epsilon} \varphi_\epsilon ((n - 1) - H \langle x, \nu_M \rangle) \, d\mathcal{H}^{n-1}, \quad (6.16)$$

By Lemma 4.1, (4.6) and (4.7), sending $\epsilon \searrow 0$, we find

$$\sum_{i=1}^K \int_{\partial M \cap P_i} \langle x, \nu_{T_i}^M \rangle \, d\mathcal{H}^{n-2} = \int_M \{ (n - 1) - H \langle x, \nu_M \rangle \} \, d\mathcal{H}^{n-1}. \quad (6.17)$$

On the other hand, by Proposition 6.1 we have on each $\text{reg} M \cap \partial P_i$,

$$\cos \theta_i \nu_M + \sin \theta_i \nu_{T_i}^M = n_i, \quad (6.18)$$

it follows that

$$\cos \theta_i \langle x, \nu_M \rangle + \sin \theta_i \langle x, \nu_{T_i}^m \rangle = \langle x, n_i \rangle. \quad (6.19)$$

Enlightened by [LX17, (9)], we define

$$k = \sum_{i=1}^K c_i n_i, \quad (6.20)$$

where the constants $c_i$ are such that $\langle k, n_i \rangle = -\cos \theta_i$, this implies

$$\cos \theta_i \langle \nu_M, k \rangle + \sin \theta_i \langle \nu_{T_i}^M, k \rangle = \langle n_i, k \rangle = -\cos \theta_i. \quad (6.21)$$

---

13Here $C_8$ is a constant independent of $\epsilon$, by virtue of the fact that $M$ is bounded, and hence $|x - \langle x, \nu_M \rangle \nu_M|$ is bounded.
By (6.7), inner product with \( k \), exploiting (6.19) and (6.21), we have

\[
(n - 1) \int_M \langle \nu_M, k \rangle \, d\mathcal{H}^{n-1} = \sum_{i=1}^{K} \int_{\partial M \cap P_i} \left\{ \langle x, \nu^M_{\Gamma_i} \rangle \langle \nu_M, k \rangle - \langle x, \nu_M \rangle \langle \nu^M_{\Gamma_i}, k \rangle \right\} \, d\mathcal{H}^{n-2}
\]

\[
= \sum_{i=1}^{K} \int_{\partial M \cap P_i} \left\{ \langle x, \nu^M_{\Gamma_i} \rangle \langle \nu_M, k \rangle + \langle x, \nu_M \rangle \frac{\cos \theta_i}{\sin \theta_i} \left( 1 + \langle \nu_M, k \rangle \right) \right\} \, d\mathcal{H}^{n-2}
\]

\[
= \sum_{i=1}^{K} \int_{\partial M \cap P_i} \left\{ \langle x, \nu^M_{\Gamma_i} \rangle \langle \nu_M, k \rangle + \frac{1}{\sin \theta_i} \left( \langle x, \nu \rangle - \sin \theta_i \langle x, \nu^M_{\Gamma_i} \rangle \right) \left( 1 + \langle \nu_M, k \rangle \right) \right\} \, d\mathcal{H}^{n-2}
\]

\[
= - \sum_{i=1}^{K} \int_{\partial M \cap P_i} \langle x, \nu^M_{\Gamma_i} \rangle \, d\mathcal{H}^{n-2},
\]

where in the last equality, we use Footnote 9.

Combining with (6.17) and (6.22), we see that

\[
\int_M \{ (n - 1) - H \langle x, \nu_M \rangle + (n - 1) \langle \nu_M, k \rangle \} \, d\mathcal{H}^{n-1} = 0.
\]

On the other hand, we consider a smooth vector field \( X(x) = x \), integrating \( \text{div}_{\Gamma_i} X \) on each \( \Gamma_i \), the divergence theorem gives

\[
(n - 2)\mathcal{H}^{n-2} (\Gamma_i) = \int_{\Gamma_i} \text{div}_{\Gamma_i} X \, d\mathcal{H}^{n-2} = \int_{\Gamma_i} H_{\Gamma_i} \left\langle x, \nu^B_{\Gamma_i} \right\rangle \, d\mathcal{H}^{n-2},
\]

where \( H_{\Gamma_i} \) denotes the mean curvature of \( \Gamma_i \) in \( M \cap P_i \) with respect to \(-\nu^B_{\Gamma_i} \); in the last equality, we decompose \( X \) to the tangent part and normal part with respect to \( \Gamma_i \), and we use the fact that \( \Gamma_i = \partial M \cap P_i = \partial B^z_{\Gamma_i} \) is closed.

Rearranging (6.24), we obtain (6.14). This completes the proof.

\[
\square
\]

**Remark 6.2.** In the proof of Proposition 6.3, we use the fact that \( H_M \neq 0 \), which can be proved by Schätzle’s strong maximum principle (Theorem 2.4). Indeed, we know from Proposition 6.1 that \( H_M \) is a constant on \( M \), if \( H_M = 0 \), then \( \frac{g(M, 1)}{2} \) is a stationary multiplicity 1 \( n \)-rectifiable varifold on \( \mathbb{R}^n \). Without loss of generality, we may assume that \( P(E; P_1) \neq 0 \). Up to a rotation, we may assume that \( P_1 = x \in \mathbb{R}^n : x_n = 0 \). Pick \( \nu = e_n \), \( h_0 = 0 \) and \( U = \{ x \in \mathbb{R}^n : x_n = 0 \} \). Consider the constant function \( \eta(x_1, \ldots, x_{n-1}) = 0 \), it is easily seen that the graph of \( \eta \) is exactly the supporting hyperplane \( P_1 \cap U \). From definition of \( E \), we see that \( M \) lies above \( U \) and touches \( U \) at the points of \( \partial^* E \cap \partial \Omega \). Since \( H_M = 0 \) and the mean curvature of a hyperplane is also 0, we deduce from Theorem 2.4 that \( M \) must coincide with \( \partial E \cap P_1 \), which leads to a contradiction.

Thanks to the Minkowski-type formula, we find a test function \( \zeta = (n - 1) - H \langle x, \nu_M \rangle + (n - 1) \langle k, \nu_M \rangle \), which is smooth on \( \text{reg} M \) by virtue of Proposition 6.1, and has vanishing integration, i.e., \( \int_{\text{reg} M} \zeta = 0 \), by virtue of Proposition 6.3. Here \( k \) is given in (6.20).
Using \( \zeta \) to test the stability (1.5) and noticing that each \( P_i \) is planar \(^{14}\), thus we obtain

\[
- \int_M \left( \Delta_M \zeta + ||B_M||^2 \zeta \right) d\mathcal{H}^{n-1} + \sum_{i=1}^K \int_{M \cap P_i} \left( \langle \nabla^M \zeta, \nu^{M}_{\Gamma_i} \rangle + \cot \theta_i B_M (\nu^{M}_{\Gamma_i}, \nu^{M}_{\Gamma_i}) \right) \zeta d\mathcal{H}^{n-2} \geq 0.
\]

(6.25)

We need the following formulas and computations in the proof of Theorem 1.3, which are well-known to experts (see for example [LX17; WX19; Sou21]). It is worth mentioning that the proof do not depend on the divergence theorem, thus we can readily see that these formulas hold on \( \text{reg} \, M \).

**Lemma 6.2.** Let \( E \) be as in Theorem 1.3, then on \( \text{reg} \, M \), we have

(1) \([LX17, \text{Lemma 6}]\) For each \( i \), on \( \text{reg} \, M \cap P_i \), \(^{15}\)

\[
\langle \nabla^M \zeta, \nu^{M}_{\Gamma_i} \rangle + \cot \theta_i B_M (\nu^{M}_{\Gamma_i}, \nu^{M}_{\Gamma_i}) = 0.
\]

(6.26)

(2) \([WX19, \text{Proposition 3.4}]\) on \( \text{reg} \, M \),

\[
\Delta_M \langle x, \nu_M \rangle + ||B_M||^2 \langle x, \nu_M \rangle = H,
\]

\[
\Delta_M \langle \nu_M, k \rangle + ||B_M||^2 \langle \nu_M, k \rangle = 0.
\]

We are ready to give the proof of Theorem 1.3. The proof follows the ones in [LX17; Sou21], since we will use the divergence theorem in the non-smooth setting, for the sake of clarification, we present the proof.

**Proof of Theorem 1.3.** Our starting point is Proposition 6.1. Also, by virtue of the Minkowski-type formula Proposition 6.3, we can use \( \zeta = (n-1) - H \langle x, \nu_M \rangle + (n-1) \langle k, \nu_M \rangle \) to test the Poincaré-type inequality (6.6).

By virtue of Lemma 6.2 (1), (6.25) reduces to

\[
- \int_M \left( \Delta_M \zeta + ||B_M||^2 \zeta \right) d\mathcal{H}^{n-1} \geq 0.
\]

(6.27)

By a direct computation and Lemma 6.2 (2), we get

\[
\Delta_M \zeta = - H^2 - ||B_M||^2 (\zeta - (n-1)).
\]

(6.28)

Thus (6.27) becomes

\[
- \int_M \left\{ (n-1)||B_M||^2 - H^2 \right\} \zeta d\mathcal{H}^{n-1} \geq 0,
\]

(6.29)
equivalently,

\[
- \int_M \left\{ (n-1)||B_M||^2 - H^2 \right\} ((n-1) - H \langle x, \nu_M \rangle + (n-1) \langle k, \nu_M \rangle) d\mathcal{H}^{n-1} \geq 0.
\]

(6.30)

We will prove that

\[
\int_M \left\{ (n-1)||B_M||^2 - H^2 \right\} H \langle x, \nu_M \rangle d\mathcal{H}^{n-1} = 0.
\]

(6.31)
To this end, we explore the function \( \Phi = \left( \frac{H}{n-1} x - \nu_M, \frac{H}{n-1} x - \nu_M \right) \), which is smooth on \( \text{reg} \, M \) by virtue of Proposition 6.1.

\(^{14}\)Hence the second fundamental form \( B_{P_i} = 0 \).

\(^{15}\)Notice that in [LX17], \( \theta_i \) therein is the angle between the outer unit conormal, namely, \( \langle \nu^{M}_{\Gamma_i}, \nu^{M}_{\Gamma_i} \rangle = \cos \theta_i \).

In contrast to this, in our case, \( \langle \nu^{M}_{\Gamma_i}, \nu^{M}_{\Gamma_i} \rangle = - \cos \theta_i \).
On reg $M$, by virtue of Proposition 5.2 (4), a direct computation gives

$$\Delta_M \Phi = \frac{2H}{n-1} \langle x, \nu_M \rangle \left( \|B_M\|^2 - \frac{H^2}{n-1} \right), \quad (6.32)$$

$$\langle \nabla^M \Phi, \nu^M_{\Gamma_i} \rangle = \frac{2H}{n-1} \left( \frac{H}{n-1} - B_M(\nu^M_{\Gamma_i}, \nu^M_{\Gamma_i}) \right) \langle x, \nu^M_{\Gamma_i} \rangle. \quad (6.33)$$

For each $\epsilon > 0$, we have $\varphi_\epsilon, S'_\epsilon, S_\epsilon$ from Lemma 4.1. Let $\tilde{Y}_\epsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth vector field satisfying

$$\tilde{Y}_\epsilon = \nabla^M \Phi \text{ on } M \setminus S'_\epsilon.$$ 

Then let $Y_\epsilon \in C^2(\mathbb{R}^n; \mathbb{R}^n)$ be a vector field satisfying

$$Y_\epsilon = \varphi_\epsilon \tilde{Y}_\epsilon. \quad (6.34)$$

Integrating $\text{div}_M(\nabla^M \Phi)$ over $M \setminus S'_\epsilon$ and using divergence theorem, we get

$$\int_{M \cap (S \setminus S'_\epsilon)} \varphi_\epsilon \Delta_M \Phi + \left\langle \nabla^M \Phi, \tilde{Y}_\epsilon \right\rangle d\mathcal{H}^{n-1} + \int_{M \setminus S_\epsilon} \Delta_M \Phi d\mathcal{H}^{n-1} = \sum_{i=1}^K \int_{\partial M \cap P_i \setminus S_i} \langle Y_i, \nu^M_{\Gamma_i} \rangle d\mathcal{H}^{n-2} + \int_{\partial M \cap P_i \cap (S \setminus S'_\epsilon)} \varphi_\epsilon \left\langle \tilde{Y}_\epsilon, \nu^M_{\Gamma_i} \right\rangle d\mathcal{H}^{n-2} + \int_{\partial S'_\epsilon} 0 d\mathcal{H}^{n-2}. \quad (6.35)$$

By virtue of the estimates in Lemma 4.1, sending $\epsilon \searrow 0$ we find

$$\int_M \Delta_M \Phi d\mathcal{H}^{n-1} = \sum_{i=1}^K \int_{\Gamma_i} \left\langle \nabla^M \Phi, \nu^M_{\Gamma_i} \right\rangle d\mathcal{H}^{n-2}, \quad (6.36)$$

by virtue of (6.32) and (6.33), this reads

$$\int_M \left( \|B_M\|^2 - \frac{H^2}{n-1} \right) \langle x, \nu_M \rangle d\mathcal{H}^{n-1} = \sum_{i=1}^K \int_{\Gamma_i} \left( \frac{H}{n-1} - B_M(\nu^M_{\Gamma_i}, \nu^M_{\Gamma_i}) \right) \langle x, \nu^M_{\Gamma_i} \rangle d\mathcal{H}^{n-2} \quad (6.37)$$

Arguing as [LX17, (24)], we obtain, on each reg $M \cap P_i$,

$$B_M(\nu^M_{\Gamma_i}, \nu^M_{\Gamma_i}) = H - \sin \theta_i H_{\Gamma_i}, \quad (6.38)$$

where $H_{\Gamma_i}$ is as in Proposition 6.3.

Combining (6.37) with (6.38), and using the fact that $\langle x, \nu^M_{\Gamma_i} \rangle = -\cos \theta_i \langle x, \nu^{B^+}_{\Gamma_i} \rangle$, we obtain

$$\int_M \left( \|B_M\|^2 - \frac{H^2}{n-1} \right) \langle x, \nu_M \rangle d\mathcal{H}^{n-1} = \sum_{i=1}^K \left\{ \cos \theta_i \int_{\Gamma_i} \left( \frac{n-2}{n-1} H - \sin \theta_i H_{\Gamma_i} \right) \langle x, \nu^{B^+}_{\Gamma_i} \rangle d\mathcal{H}^{n-2} \right\} \quad (6.39)$$

We claim that the RHS of (6.39) vanishes, which shows (6.31).

Indeed, by virtue of (6.24), we have

$$\int_{\Gamma_i} \left( \frac{H}{n-1} - \sin \theta_i H_{\Gamma_i} \right) \langle x, \nu^{B^+}_{\Gamma_i} \rangle d\mathcal{H}^{n-2} = \int_{\Gamma_i} \frac{H}{n-1} \langle x, \nu^{B^+}_{\Gamma_i} \rangle d\mathcal{H}^{n-2} - \sin \theta_i \mathcal{H}^{n-2}(\Gamma_i). \quad (6.40)$$
On the other hand, integrating the smooth vector field $\Delta_M x$ on $M$, using divergence theorem and Proposition 5.2 (4), we obtain

$$-H \int_M \nu_M dH^{n-1} = \sum_{i=1}^K \int_{\Gamma_i} \nu_i^M dH^{n-2} - \sum_{i=1}^K \int_{\Gamma_i} \sin \theta_i n_i - \cos \theta_i \nu_{\Gamma_i}^{B^+} dH^{n-2}. \quad (6.41)$$

Using Lemma 6.1 with $M$ replaced by $\Gamma_i$, and noticing that $\Gamma_i$ is closed without boundary, we obtain

$$\int_{\Gamma_i} \nu_{\Gamma_i}^{B^+} dH^{n-2} = 0, \quad (6.42)$$

which implies

$$-H \int_M \nu_M dH^{n-1} = \sum_{i=1}^K \int_{\Gamma_i} \sin \theta_i n_i dH^{n-2}. \quad (6.43)$$

Using Lemma 6.1 again for $M$ and noticing that on $\Gamma_i$, $\nu_M = \sin \theta_i n_i - \cos \theta_i \nu_{\Gamma_i}^{B^+}$, $\nu_M = \sin \theta_i \nu_{\Gamma_i}^{B^+} + \cos \theta_i n_i$ and $\langle x, \nu_{\Gamma_i}^M \rangle = -\cos \theta_i \langle x, \nu_{\Gamma_i}^{B^+} \rangle$, we obtain

$$(n-1) \int_M \nu_M dH^{n-1} = \sum_{i=1}^K \int_{\Gamma_i} \{\langle x, \nu_{\Gamma_i}^M \rangle \nu_M - \langle x, \nu_M \rangle \nu_{\Gamma_i}^M \} dH^{n-2} = -\sum_{i=1}^K \int_{\Gamma_i} \langle x, \nu_{\Gamma_i}^{B^+} \rangle n_i dH^{n-2}. \quad (6.44)$$

By (6.43), (6.44) and the linearly independence of $n_i$, we see that

$$H \int_{\Gamma_i} \langle x, \nu_{\Gamma_i}^{B^+} \rangle dH^{n-2} = (n-1) \sin \theta_i H^{n-2}(\Gamma_i). \quad (6.45)$$

In particular, this shows $\langle n-1 \rangle = 0$, and hence (6.31).

Thus (6.30) becomes

$$-\int_M \{\langle n-1 \rangle ||B_M||^2 - H^2 \} (1 + \langle k, \nu_M \rangle) dH^{n-1} \geq 0. \quad (6.46)$$

We are now in the position to conclude the proof. Notice that since $|k| \leq 1$, we have $1 + \langle k, \nu_M \rangle \geq 0$, it follows that

$$\int_M \{\langle n-1 \rangle ||B_M||^2 - H^2 \} dH^{n-1} \leq 0. \quad (6.47)$$

On the other hand, the Schwarz inequality implies that on $\text{reg} M$,

$$(n-1) ||B_M||^2 \geq H^2. \quad (6.48)$$

Thus we conclude that on $\text{reg} M$,

$$(n-1) ||B_M||^2 = H^2. \quad (6.49)$$

By virtue of Proposition 6.1, $\text{reg} M$ is locally an analytic umbilical hypersurface in $\mathbb{R}^n$, which means it must be locally spherical. Moreover, if $\text{sing} M \neq \emptyset$, we can find a volume-preserving perturbation as in the proof of Theorem 1.2 (see also Figure 2), which strictly decreases the

\footnote{It is worth mentioning that Lemma 6.1 is applicable for $\Gamma_i$, since we have $H^{n-3}(\text{sing}) M = 0$, and hence the approximation argument is still workable.}$^16$

\footnote{Namely, $\partial \Gamma_i = \emptyset$.}$^17
perimeter (and hence free energy functional). This contradicts to the fact that $E$ is stable as in Definition 6.1.

\begin{remark}
As considered by R. Souam in [Sou21], when each $\theta_i$ is close enough to $\frac{\pi}{2}$, then it must be that $|k| \leq 1$ due to the continuity of $k$. In particular, this generalizes the results for the free boundary capillary hypersurface in a wedge of R. López [Lóp14, Theorem 1] to the measure-theoretic settings. As argued in Remark 4.1, for the free boundary measure-theoretic capillary hypersurface, the condition $\text{sing} M$ is closed can be removed, since it is an immediate result of [GJ86, Theorem 4.13].
\end{remark}

\begin{remark}
We remark that Schätzle’s strong maximum principle Theorem 2.4 provides an alternative approach for the singularity analysis, see [DM19, Theorem 1, conclusion of the proof] for a detailed argument. Using this method, one can prove that $M$ must be union of finitely many spherical caps and complete spheres (could be mutually tangent), then using stability and the perimeter decreasing perturbation, one can conclude that these spherical caps and complete spheres could not be mutually tangent. It is worth mentioning that the maximum principle’s approach does not depend on the stability of $E$, it works successfully for the stationary points.

When $L = K = 1$, the wedge-shaped domain $\Omega$ is indeed a half space of $\mathbb{R}^n$, in this regard, Theorem 1.3 becomes,

\begin{corollary}
Let $\Omega \subset \mathbb{R}^n$ be a half space. Let $E \subset \Omega$ be a set with finite volume and perimeter, which is stable as in Definition 1.2. If the singular set $\text{sing} M$ is closed with $H^{n-3}(\text{sing} M) = 0$, then $M$ must be a spherical cap, in particular, $M$ is smooth.
\end{corollary}

\begin{proof}
The proof is essentially the same with Theorem 1.3, it suffice to show that $|k| \leq 1$. Indeed, up to translation and rotation, we may assume that $\Omega$ is the upper half space $\{x \in \mathbb{R}^n : x_n \geq 0\}$. In this case, $n = -e_n$, and it follows that $k$ in (6.20) is just $k = \cos \theta e_n$ and we definitely have $|k| \leq 1$. This completes the proof.
\end{proof}

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\footnote{See Remark 6.1 for the definition of $K, L$.}
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