Central Charge and the Andrews-Bailey Construction

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Abstract

From the equivalence of the bosonic and fermionic representations of finitized characters in conformal field theory, one can extract mathematical objects known as Bailey pairs. Recently Berkovich, McCoy and Schilling have constructed a ‘generalized’ character formula depending on two parameters $\rho_1$ and $\rho_2$, using the Bailey pairs of the unitary model $M(p-1, p)$. By taking appropriate limits of these parameters, they were able to obtain the characters of model $M(p, p+1)$, $N = 1$ model $SM(p, p+2)$, and the unitary $N = 2$ model with central charge $c = 3(1 - \frac{2}{p})$. In this letter we computed the effective central charge associated with this ‘generalized’ character formula using a saddle point method. The result is a simple expression in dilogarithms which interpolates between the central charges of these unitary models.

1 Introduction

More than a decade since its creation, two dimensional conformal field theory (CFT) [1] and its integrable perturbations [2] still remain as one of the most active research topics in modern physics. A current focus is in the study of various bases of the Hilbert space in CFT. Different choices of the basis would lead to a different representation for the partition function of the CFT defined on a compact manifold such as a torus or a cylinder. This partition function is usually written in terms of characters of the Virasoro or some extended algebras. The ‘bosonic’ form of the character formula is well known for quite some time [3]. Recently the Stony Brook group have constructed numerous new character formulae based on fermionic quasi-particles [4–6]. For several CFTs, more than one ‘fermionic’ expression exist for the same conformal character. In these cases, the different expressions are related to the different integrable perturbations of the same CFT. These developments all lend supports to the idea of a massless scattering S-matrix description of CFT [7–10]. The construction of the quasi-particle basis of the Hilbert space is also apparently related to the problem of diagonalizing the infinite set of local Integrals of Motion in CFT [11–13]. For a description in terms of other bases see [14–15].

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The equivalence of the bosonic and fermionic character formulae give rise to beautiful $q$-series identities of the Rogers-Ramanujan type [13]. To prove some of these identities, the method of finitization of character formula was employed in [17]. The basic idea is that the equivalence between bosonic and fermionic finitized characters would also imply the equivalence of the $q$-series [18]. In [19], several classes of $q$-series identities were proven using Andrews’ generalization [20] of Bailey’s lemma [21]. The key observations in [19] was that Bailey pairs can be extracted from finitized characters such as those of [17], and several series of CFT are ‘linked’ on a so-called Bailey’s chain [20]. The equivalence proof for all members of a series is a straightforward application of Bailey’s lemma, once a proof is established for a single member [20].

In a remarkable paper [23], the procedure of [19] was repeated using a more general form of the Andrews-Bailey construction which contains two parameters $\rho_1$ and $\rho_2$ [20]. From the (dual) Bailey pairs for the unitary minimal model $M(p-1, p)$, the more general construction gives a ‘generalized’ character formula (equation (4.19) of [23]) depending on the two parameters. Three specializations of these parameters lead to known results:

(I): $\rho_1 \to \infty$, $\rho_2 \to \infty$;
(II): $\rho_1 \to \infty$, $\rho_2 = \text{finite}$;
(III): $\rho_1 = \text{finite}$, $\rho_2 = \text{finite}$.

In the first case, the ‘generalized’ character formula becomes the character for the next model in the unitary series, i.e. $M(p, p+1)$ with central charge $1 - \frac{6}{p(p+1)}$. Case (II) leads to fermionic character formula for $N = 1$ supersymmetric model $SM(p, p+2)$ with central charge $\frac{3}{2} - \frac{12}{p(p+1)}$, while case (III) gives the fermionic character of unitary $N = 2$ model with central charge $c = 3(1 - \frac{3}{p})$. It is amazing that this ‘generalized’ character formula connects the unitary models with their supersymmetric counterparts. In fact, this construction can also be applied to any minimal models $M(p, p')$ [6, 13, 24]. A natural question (raised in [23]) which needs addressing is whether this construction has any connection to massless renormalization group flows between these CFT [7, 25].

In this letter, we shall attempt to understand this ‘generalized’ character formula for the unitary series by computing the associated ‘generalized’ effective central charge. In §2 we give a brief review of the construction detailed in [23] and establish our notations. The ‘generalized’ effective central charge is calculated in §3 via a saddle point approximation following [4, 26, 27]. A discussion of our result is given in §4 where we speculate on a possible interpretation of the ‘generalized’ character formula.

2 The Andrews-Bailey construction of the unitary models

The unitary CFT $M(p-1, p)$ is the continuum limit of the $(p-1)$-states RSOS lattice model [28] at its critical point between Regime III and IV [29]. The equivalence of the associated bosonic and fermionic finitized characters can be written as [17]

$$B_{r,s}^{(L,p)} = F_{r,s}^{(L,p)}.$$  (2.1)

\(^{1}\)To extract Bailey pairs, the finitized characters must have the general bosonic form of [22]. We would to thank B. McCoy for this comment.
The bosonic side has the form

\[ B^{(L,p)}_{r,s}(q) = \sum_{j=-\infty}^{\infty} \left( q^{j(p+1)+pr-(p-1)s} \left[ \frac{L}{2}(L+s+r) - pj \right]_q - q^{(p-s)(j(p-1)-r)} \left[ \frac{L}{2}(L-s-r) + pj \right]_q \right), \tag{2.2} \]

where \([n]\) denotes the integer part of \(n\), and

\[
\left[ \begin{array}{c} n \\ m \end{array} \right] = \left\{ \begin{array}{cl} \frac{(q)_n}{(q)_m(q)_{n-m}} & \text{for } 0 \leq m \leq n \\ 0 & \text{otherwise}, \end{array} \right. \tag{2.3} \]

is the usual \(q\)-binomial coefficient with

\[
(a)_n = \frac{(a)_\infty}{(aq^n)_\infty}; \quad (a)_\infty = \prod_{l=0}^{\infty} (1 - aq^l). \tag{2.4} \]

One should note that this finitized character is equal to the off-critical corner transfer matrix of the RSOS model in its low-temperature Regime III, defined on a square lattice of size \(L\) \([23]\). In the limit \(L \to \infty\), (2.2) becomes the character formula for the irreducible representation generated by the primary field \(\Phi^{(L,p)}\), with normalization \(\chi_{r,s}(L) = 1 + \sum_{N=1}^{\infty} a_N q^N\).

There are two forms for the fermionic character \(F^{(L,p)}_{r,s}\), depending on the finite parameter \(L\). Let \(C_{p-3}\) and \(I_{p-3}\) stand for respectively the Cartan and incidence matrices of the Lie algebra \(A_{p-3}\). Furthermore denote the \(i\) unit vector in \(\mathbb{R}^{p-3}\) as \(\vec{e}_i\), and set \(\vec{e}_i = 0\) for \(i < 1\) or \(i > (p-3)\). Then

\[
F^{(L,p)}_{r,s}(q) = q^{-\frac{1}{4}(s-r)(s+r-1)} \sum_{\vec{m} \in 2Z^{p-3+Q_{r,s}}} q^{\frac{1}{2}\vec{m}^T C_{p-3}\vec{m} - \frac{1}{2}\vec{A}_{r,s}\vec{m}} \prod_{i=1}^{p-3} \left[ \frac{1}{2} (I_{p-3}\vec{m} + \vec{u}_{r,s} + L\vec{e}_1)_i \right]_q, \tag{2.5} \]

where \(\vec{m}^T = (m_1, \ldots, m_{p-3})\). When \(L + r - s\) is even

\[
\vec{A}_{r,s} = \vec{e}_{s-1}, \quad \vec{u}_{r,s} = \vec{e}_{s-1} + \vec{e}_{p-r-1} \tag{2.6} \]

\[
\vec{Q}_{r,s} = (r - 1) \sum_{i=1}^{p-3} \vec{e}_i + (\vec{e}_{s-2} + \vec{e}_{s-4} + \ldots) + (\vec{e}_{p-r} + \vec{e}_{p+2-r} + \ldots); \]

and when \(L + r - s\) is odd,

\[
\vec{A}_{r,s} = \vec{e}_{p-s-1}, \quad \vec{u}_{r,s} = \vec{e}_{p-s-1} + \vec{e}_r \tag{2.7} \]

\[
\vec{Q}_{r,s} = (s - 1) \sum_{i=1}^{p-3} \vec{e}_i + (\vec{e}_{r-1} + \vec{e}_{r-3} + \ldots) + (\vec{e}_{p-s} + \vec{e}_{p+2-s} + \ldots). \]

These two forms yield the same \(q\)-series in the limit \(L \to \infty\). Proofs of the fermionic sums are given in \([22][22]\).

Two sequences \(\{\alpha_n\}\) and \(\{\beta_n\}\) form a (bilateral) Bailey pair relative to \(a\) if they satisfy the relation

\[
\beta_n = \sum_{j=-\infty}^{n} \frac{\alpha_j}{(q)_{n-j}(aq)_{n+j}}. \tag{2.8} \]
If we set \( L = 2l + r - s + 2x \), then from (2.1) we can read off a (bilateral) Bailey pair relative to \( a = q^{r-s+2x} \) as

\[
\alpha_n = \begin{cases} 
q^{(jp(p-1)+pr-(p-1)s)} & \text{for } n = pj - x \\
-q^{(jp-s)(j(p-1)-r)} & \text{for } n = pj - r - x \\
0 & \text{otherwise} 
\end{cases} \quad (2.9a) \\
\beta_n = \begin{cases} 
\frac{1}{(aq)_{2n}} F_{r,s}^{(2n+r-s+2x,p)}(q) & \text{for } n \geq 0 \\
0 & \text{otherwise.} 
\end{cases} \quad (2.9b)
\]

An important step in the Andrews-Bailey construction of the unitary models is to define another Bailey pair relative to \( \alpha \)

\[
A_n = \begin{cases} 
q^{j(p-1) + pr - (p-1)s} & \text{for } n = pj - x \\
-q^{j(p-1) + pr - (p-1)s} & \text{for } n = pj - r - x \\
0 & \text{otherwise} 
\end{cases} \quad (2.10a) \\
B_n = \begin{cases} 
\frac{1}{(aq)_{2n}} q^{n^2} a^n F_{r,s}^{(2n+r-s+2x,p)}(q^{-1}) & \text{for } n \geq 0 \\
0 & \text{otherwise}, 
\end{cases} \quad (2.10b)
\]

which are dual to (2.9) \( \{2,3\} \). Here

\[
F_{r,s}^{(L,p)}(q^{-1}) = q^{\frac{1}{2}(s-r)(s-r-1)} \sum_{\vec{m} \in \mathbb{Z}^{p-3} + \vec{Q}_{r,s}} q^{\frac{1}{2} \vec{m}^T C_{r,s} \vec{m} + \frac{1}{4} (\vec{A}_{r,s} - \vec{u}_{r,s} - L \vec{e}_1) \vec{m}} \\
\times \prod_{i=1}^{p-3} \left[ \frac{1}{2} (I_{p-3} \vec{m} + \vec{u}_{r,s} + L \vec{e}_1)_i \right] _q . \quad (2.11)
\]

This dual transformation \((q \rightarrow q^{-1})\) takes us from the \( M(p-1,p) \) finitized characters to the \( M(1,p) \) finitized characters \([3,32] \). The non-unitary minimal model \( M(1,p) \) has actually zero operator content in the usual range of \( r \) and \( s \), but admits nontrivial finitizations.

The Andrews-Bailey construction tells us that if (2.10) is a (bilateral) Bailey pair, then

\[
\alpha'_n = \left( \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1 \rho_2)_n}{(aq/\rho_1)_n(aq/\rho_2)_n} \right) A_n \\
\beta'_n = \sum_{m=-\infty}^{n} \left( \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1 \rho_2)_n}{(aq/\rho_1)_n(aq/\rho_2)_n} \right) B_m \quad (2.12a, b)
\]

also forms a (bilateral) Bailey pair with respect to \( a \). Now using the defining relation (2.8) with this new Bailey pair and taking the limit \( n \rightarrow \infty \), one easily obtains the formula

\[
\frac{(aq/\rho_1)_\infty (aq/\rho_2)_\infty}{(aq)_\infty (aq/\rho_1 \rho_2)_\infty} \sum_{j=\infty}^{j=-\infty} q^{(jp-s)+x(s-r-x)} \\
\times \left( \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1 \rho_2)_n}{(aq/\rho_1)_n(aq/\rho_2)_n} \right) \frac{(aq/\rho_1 \rho_2)_n}{(aq/\rho_1 \rho_2)_n} \\
= \sum_{m=0}^{\infty} (\rho_1)_n(\rho_2)_n(aq/\rho_1 \rho_2)^n q^{n^2 a^n F_{r,s}^{(2n+r-s+2x,p)}(q^{-1})} . \quad (2.13)
\]

\[\text{\footnote{Although one can extend these ranges to generate interesting CFTs [33,34].}}\]

\[\text{\footnote{In fact, up to some prefactors, (2.11) is the finitization of the \( \varphi_n \) parafermions which describe the critical point between Regime I and II of the RSOS model [28,35]. We would like to thank O. Foda and O. Warnaar for pointing this out.}}\]
We shall refer to the expression in (2.13) as the 'generalized' character formula, and we will compute the effective central charge associated with it in the next section. In the limiting case (I) (and setting \( x = 0 \)), (2.13) becomes the character formula for \( \chi_{s,r}^{(p,p+1)} \), where \( 1 \leq r \leq (p-2) \) and \( 1 \leq s \leq (p-1) \). Similarly, the 'generalized' character (2.13) yields characters for the \( N = 1 \) supersymmetric model \( SM(p,p+2) \) in the case (II), while case (III) leads to characters of the \( N = 2 \) model with central charge \( c = 3(1 - \frac{2}{p}) \) [23]. Repeating the Andrews-Bailey construction starting from (2.1) and taking \( L = 2l + r - s + 2x + 1 \), another 'generalized' character can be obtained. The latter becomes the character formula for \( \chi_{s,r+2}^{(p,p+1)} \) in case (I), and gives more characters for the supersymmetric models in the cases of (II) and (III) (please see [23] for more details). Since this second 'generalized' character leads to the same central charge as (2.13), it will not be considered further in this work.

3 Effective central charge

In this section, we shall calculate the asymptotic behavior of (2.13) as \( q \to 1^- \). This method of computing the effective central charge for CFT fermionic characters are by now standard [1, 20, 21, 31]. Therefore we will be brief with the procedure, but detailing in places where our calculation differs from the norm. Firstly we shall predict the asymptotic growth of the 'generalized' character using a physical argument. Subsequently we will derive this asymptotic behavior directly from the character formula.

3.1 Asymptotic behavior of the 'generalized' character

Characters in CFT admit an interpretation as the partition function of the model defined on a cylinder with conformal boundary conditions on its rims [32]. The modular invariance property of these character formulae give us precise information about their asymptotic behavior. The working assumption in this section will be that (2.13) also gives the cylindrical partition function for some quantum field theory with appropriate boundary conditions labeled \( a \) and \( b \). Noted that \( a \) and \( b \) depend on the values of \( r \) and \( s \), as well as \( \rho_1 \) and \( \rho_2 \). Define the modular parameter

\[
q = e^{2\pi i \tau}, \quad \tilde{q} = e^{-2\pi i / \tau}, \quad \text{with} \quad \tau = \frac{iR}{2\pi L} \tag{3.1}
\]

for a cylinder of length \( L \) and circumference \( R \). If we take the (imaginary) time coordinate to be in the \( R \) direction and space in the \( L \) direction, the generalized character (2.13) can be written as

\[
\chi_{s,r}^{(p)}(\rho_1, \rho_2; \tau) = \text{Tr}_P e^{-R \mathcal{H}_{ab}(\rho_1, \rho_2)/L} = \text{Tr}_P q^\mathcal{H}_{ab}(\rho_1, \rho_2), \tag{3.2}
\]

where \( \mathcal{H}_{ab} \) is the (normalized) dimensionless Hamiltonian of the field theory with open boundary conditions \( a \) and \( b \). The trace is taken over the sector of the Hilbert space with boundary condition \( P \) along the circumference of the cylinder. For the cases (1.1), \( \mathcal{H}_{ab} \) becomes \( L_0 - \Delta(s,r) \) where \( \Delta(s,r) \) is the appropriate conformal dimension for each unitary model. If instead we take space to be compactified in the \( R \) direction and time to evolve in the \( L \) direction, then the partition function will be

\[
\chi_{s,r}^{(p)}(\rho_1, \rho_2; \tau) = \langle a | e^{-L \mathcal{H}_{P}(\rho_1, \rho_2)/R} | b \rangle, \tag{3.3}
\]

\(^4\)To be more precise, there should be a pre-factor on the right hand side of (3.3) involving powers of \( q \) due to normalization. However it becomes irrelevant to our analysis in the limit \( q \to 1^- \).
where $\mathcal{H}_P$ is the dimensionless Hamiltonian with closed boundary condition $P$. Here $\langle a \rangle$ and $| b \rangle$ represent the boundary states at the ends of the cylinder. In the limit $L \to \infty$, the inner product (3.3) is dominated by the ground state of $\mathcal{H}_P$ with energy $E_0$

$$\lim_{L \to \infty} \chi_{s,r}^{(p)}(\rho_1, \rho_2; \tau) \sim g_a g_b q^{E_0/4\pi^2}, \quad (3.4)$$

where we denote the contributions from each boundary as $g_a$ and $g_b$. (3.4) is our prediction of the asymptotic behavior of (2.13).

The fermionic form of the generalized character (2.13) is most suitable for taking the asymptotic limit $q \to 1^-$. The important thing to notice here is that for fixed $p$, $E_0$ depends only on the parameters $\rho_1$ and $\rho_2$, and is independent of $r$ and $s$. Standard arguments\footnote{With our choice of parameterization, the limits (1.1) actually become (I): $\rho_1 \to -\infty$, $\rho_2 \to -\infty$ and (II): $\rho_1 \to -\infty$, $\rho_2 =$ finite. However we can still obtain the same conformal characters from (2.13).} (related to the $r$ and $s$ independence of $E_0$) give us the freedom to remove the restriction $\bar{Q}_{r,s}$ and linear terms in the exponent of $q$ from the fermionic sum in this limit. Thus to compute the asymptotic behavior of the generalized character, i.e. to obtain the leading exponent of $\tilde{q}$, we could just concentrate on the simplest case of the identity representation $\chi_{1,1}^{(p)}(\rho_1, \rho_2; \tau)$. To implement the special limits (1.1) in this case, let us parameterize $\rho_1$ and $\rho_2$ as

$$\rho_1 = -\frac{q^{1/2}}{A}, \quad \rho_2 = -\frac{q^{1/2}}{B}, \quad (3.5)$$

thus we have:

\begin{align*}
(I): & \quad A = 0, B = 0; \\
(II): & \quad A = 0, B = 1; \\
(III): & \quad A = 1, B = 1,
\end{align*} \quad (3.6a)

with $x = 0$ and $a = 1$ for all three cases. The limits (II) and (III) lead to the Neveu-Schwarz characters for the supersymmetric models. We will not consider the Ramond sector, although it can be treated by a straightforward generalization of the computation presented here. A convenient parameterization of the ground state energy $E_0$ is

$$E_0(A, B|p) = -\frac{\pi^2}{6} \tilde{c}(A, B|p). \quad (3.7)$$

From (3.4), we shall interpret $\tilde{c}$ as the ‘generalized’ effective central charge, and expect that in the limits (3.6), it will take on the values of $1 - \frac{6}{p(p+1)}$, $\frac{3}{2} - \frac{12}{p(p+1)}$ and $c = 3(1 - \frac{2}{p})$ respectively.

### 3.2 Effective central charge

After all the simplifications mentioned above, the $q$-series we shall consider is

$$\tilde{\chi}_{1,1}^{(p)}(A, B; q) = \sum_{n=0}^{\infty} \sum_{m_1, \ldots, m_{p-3}=0} \left( -\frac{q^{1/2}}{A} \right)_n \left( -\frac{q^{1/2}}{B} \right)_n (AB)^n q^{\vec{m}^T C_{p-3} \vec{m} - 2nm_1 + n^2} \left( q \right)_{2n} \times \prod_{i=1}^{p-3} \left[ (I_{p-3} \vec{m} + n\vec{e}_1)_i \right] q. \quad (3.8)$$
If the coefficients in this series \( \chi_{1,1}^{(p)} = \sum a_M q^M \) behave like \( a_M \sim e^{2\pi \sqrt{M\epsilon/6}} \) for large \( M \), then as \( q \to 1^- \), \( \chi_{1,1}^{(p)} \) diverges like \( \tilde{q}^{-\epsilon/4} \). In other words one can obtain the ‘generalized’ central charge \( \tilde{c} \) from the asymptotic growth of the coefficient \( a_M \). The latter is computed by applying the saddle point method to

\[
a_{M-1} = \oint \frac{dq}{2\pi i} \chi_{1,1}^{(p)}(A, B; q) q^{-M} = \oint \frac{dq}{2\pi i} \sum_{n} \sum_{\vec{m}} f(n, \vec{m}; q). \tag{3.9}
\]

The saddle point occurs at the point where the derivatives of

\[
\log f(n, \vec{m}; q) \approx \int_{0}^{n} \log \left( 1 + \frac{q^k}{A} \right) dk + \int_{0}^{n} \log \left( 1 + \frac{q^k}{B} \right) dk + n \log(AB)
\]

\[
- \int_{0}^{2n} \log(1 - q^k) dk + (a^2 - 2mn_1 + \vec{m}^T C_{p-3} \vec{m} - M) \log q
\]

\[
+ \sum_{i=1}^{p-3} \left( (I_{p-3} \vec{m} + n\vec{e}_i) - (I_{p-3} \vec{m} + n\vec{e}_i - 2\vec{m}_i) - \int_{0}^{2\vec{m}_i} \log(1 - q^k) dk \right) \tag{3.10}
\]

with respect to \( n, m_1, \ldots, m_{p-3} \) and \( q \) are all zero. In deriving the expression in (3.10), sums such as \( \log \{ (q)_n \} \) and \( \log \{ (-q^{2}/A)_n \} \) were approximated by integrals. There are several ways to make this approximation. Ultimately, the difference between the various approximation schemes is equivalent to a difference in the linear terms in the exponent of \( q \), and do not influence the quadratic terms. Since the asymptotic growth is not expected to depend on the linear terms as explained above, we have the freedom to use the following two (different) approximations:

\[
\log \{ (q)_n \} \sim \int_{0}^{n} \log(1 - q^k) dk \quad \text{and} \quad \tag{3.11a}
\]

\[
\log \left\{ \left( -\frac{q^2}{A} \right)_n \right\} \sim \int_{0}^{n} \log \left( 1 + \frac{q^k}{A} \right) dk. \tag{3.11b}
\]

This combination of approximations was chosen to simplify the algebra after differentiation.

Let us define

\[
v_i = q^{-2m_i} \quad \text{and} \quad w_i = q^{(I_{p-3} \vec{m} + n\vec{e}_i)}. \tag{3.12}
\]

The differentiation with respect to \( m_i \) and \( n \) produced the following set of relations for their saddle point values \( \vec{m}_i \) and \( \vec{n} \):

\[
(1 - y_i)^2 = \prod_{j=1}^{p-3} y_j^{L_{ij}}, \tag{3.13a}
\]

\[
q^{2n\delta_i} (1 - x_i)^2 = \prod_{j=1}^{p-3} x_j^{L_{ij}}, \tag{3.13b}
\]

\[
(1 - q^{2\vec{n}})^2 = (A + q^{\vec{n}})(B + q^{\vec{n}}) q^{2\vec{n}} x_1, \tag{3.13c}
\]

where

\[
x_i = \frac{(1 - \vec{w}_i)\vec{e}_i}{1 - \vec{e}_i\vec{w}_i} \quad \text{and} \quad y_i = \frac{(1 - \vec{w}_i)}{1 - \vec{e}_i\vec{w}_i}. \tag{3.14}
\]

It is easy to show that in the special cases of (3.6), (3.13) reduces to a system of algebraic equations governed by the algebras \( A_{p-2}, A_{p-1} \) and \( D_{p-1} \) respectively. For
these algebras, the corresponding systems of equations are solved in the literature, and are known to be related to the Thermodynamic Bethe Ansatz (TBA) approach (see for example \[37–41\]). Here we can easily write down the solution for $y_i$ as

$$y_i = \frac{\sin^2(1 + i)\frac{\pi}{p}}{\sin^2\frac{\pi}{p}}.$$  \hspace{1cm} (3.15)

One can also show that

$$x_i = \frac{\sin^2(p - 1 - i)\theta}{\sin^2\theta}$$  \hspace{1cm} (3.16)

satisfies (3.13) with the closure conditions

$$x_{p-2} = 1,$$  \hspace{1cm} (3.17a)

$$x_0 = \frac{\sin^2(p - 1)\theta}{\sin^2\theta} = q^{-2n},$$  \hspace{1cm} (3.17b)

$$x_{-1} = \frac{\sin^2\frac{p\theta}{2}}{\sin^2\theta} = (1 + Aq^{-n})(1 + Bq^{-n}).$$  \hspace{1cm} (3.17c)

The parameter $\theta$ is related to $A$ and $B$ by the relation

$$(A + B) \sin\theta + AB(p - 1) \theta = \sin(p + 1)\theta.$$  \hspace{1cm} (3.18)

To compute $\log f(n, \vec{m}; q)$ at the stationary point with respect to $m_i$ and $n_i$, we first rewrite it using the relations

$$\int_0^\infty \log (1 - q^k) \, dk = \frac{1}{\log q} \left[ L(1 - q^2) + \frac{1}{2} \log (1 - q^2) \log q^2 \right];$$  \hspace{1cm} (3.19a)

$$\int_0^\infty \log \left(1 + \frac{q^k}{A}\right) \, dk = \frac{1}{\log q} \left[ L\left(\frac{q^2}{q^2 + A}\right) - L\left(\frac{1}{1 + A}\right)\right] + \frac{1}{2} \log A \log \left(\frac{1 + A}{q^2 + A}\right) + \frac{i}{2} \log \left(1 + \frac{q^2}{A}\right).$$  \hspace{1cm} (3.19b)

The Rogers dilogarithm in (3.19) is defined by \[42\]

$L(z) = Li_2(z) + \frac{1}{2} \log z \log (1 - z)$; $Li_2(z) = -\int_0^z \log(1 - w) \, dw$  \hspace{1cm} (3.20)

and $L(1) = \frac{\pi^2}{6}$. The five terms relation for the dilogarithm in our case can be written as

$$L(1 - w_i) - L(1 - v_i w_i) - L(1 - v_i^{-1}) = L(1 - y_i^{-1}) - L(1 - x_i^{-1}).$$  \hspace{1cm} (3.21)

Hence we have

$$\log f(n, \vec{m}; q) \bigg|_{\vec{m} = \vec{m}} \approx -M \log q - \frac{\pi^2 \tilde{c}(A, B|p)}{6 \log q},$$  \hspace{1cm} (3.22)

where

$$\tilde{c}(A, B|p) = \frac{1}{L(1)} \left[ L\left(\frac{1}{1 + A}\right) + L\left(\frac{1}{1 + B}\right) + L\left(1 - \frac{1}{x_0}\right) - L\left(1 + \frac{1}{\sqrt{x_0} A}\right) - L\left(1 + \frac{1}{\sqrt{x_0} B}\right) + \sum_{i=1}^{p-3} \left[ L\left(1 - \frac{1}{x_i}\right) - L\left(1 - \frac{1}{y_i}\right)\right] + \frac{1}{2} \log A \log \left(\frac{1 + \sqrt{x_0} A}{1 + A}\right) + \frac{1}{2} \log B \log \left(\frac{1 + \sqrt{x_0} B}{1 + B}\right)\right].$$  \hspace{1cm} (3.23)
By differentiating (3.22) with respect to $q$, we found the saddle point value of $q$ to be

$$\bar{q} = e^{-\sqrt{\frac{\pi^2}{6M}}}.$$  

(3.24)

This leads to the expected asymptotic behavior of $a_M$ for large $M$, and hence we can interpret $\tilde{c}(A,B|p)$ as a ‘generalized’ effective central charge for (2.13). The sums in (3.23) can be further simplified using dilogarithm sum rules \[42,43\] to yield

$$\sum_{i=1}^{p-3} L\left(1 - \frac{1}{y_i}\right) = (p - 5 + \frac{6}{p})L(1); \quad \text{and}$$

$$L\left(1 - \frac{1}{y_0}\right) + \sum_{i=1}^{p-3} \left[L\left(1 - \frac{1}{x_i}\right)\right] = (p - 1)L(1) - p(p - 1)\theta^2$$

$$+ 2Li_2\left(-\frac{\sin(p - 1)\theta}{\sin \theta}, p\theta\right) + \log\left(\frac{\sin(p - 1)\theta}{\sin \theta}\right) \log\left(\frac{\sin p\theta}{\sin \theta}\right),$$

(3.25)

where

$$Li_2(r, \theta) = \text{Re}\{Li_2(re^{i\theta})\} = -\frac{1}{2} \int_0^r \frac{\log(1 - 2x \cos \theta + x^2)}{x} dx.$$  

(3.26)

The resultant expression for the ‘generalized’ central charge is

$$\tilde{c}(A,B|p) = \frac{1}{L(1)} \left[Li_2(-A) + Li_2(-B) - Li_2\left(-A\frac{\sin(p - 1)\theta}{\sin \theta}\right)\right]$$

$$- Li_2\left(-B\frac{\sin(p - 1)\theta}{\sin \theta}\right) + (4 - \frac{6}{p})L(1) - p(p - 1)\theta^2$$

$$+ 2Li_2\left(-\frac{\sin(p - 1)\theta}{\sin \theta}, p\theta\right).$$

(3.27)

The simple expression in (3.28) is the main result of this letter. It gives the effective central charge associated with the ‘generalized’ character formula (2.13) in terms of dilogarithm functions. The expressions in (3.23) and (3.28) are valid for $A \geq 0$ and $B \geq 0$.

### 3.3 Special cases

Consider the domain $A = 0$, and $B = \frac{\sin(p + 1)\theta}{\sin \theta}$ follows from (3.13). To implement the special case (I), we take the limit $B \to 0$, thus yielding $\theta = \frac{\pi}{p+1}$. Consequently by using the identity \[42\]

$$Li_2(2 \cos \theta, \theta) = \left(\frac{\pi}{2} - \theta\right)^2,$$

we obtained $\tilde{c}(0,0|p) = 1 - \frac{6}{p(p+1)}$, which is the central charge of the unitary model $M(p, p + 1)$. In the limit (II), taking $B = 1$ we found $\theta = \frac{\pi}{p+2}$. Using the limit

$$Li_2\left(-\frac{\sin(p - 1)\theta}{\sin \theta}\right) + 2Li_2\left(-\frac{\sin(p - 1)\theta}{\sin \theta}, p\theta\right) \bigg|_{\theta = \frac{\pi}{p+2}} = \frac{2}{3}\pi^2 - 2\left(\frac{2p + 1}{p + 2}\right)^2 \pi^2,$$

we recover the central charge of the $N = 1$ unitary model $\tilde{c}(0,1|p) = \frac{3}{2} - \frac{12}{p(p+2)}$. 

9
It is interesting that \( \tilde{c}(0, B|p) \) is a smooth monotonic function of \( B \) between the above two limits. In particular for the case of \( p = 3 \), we have a function which connects the central charges of the Ising and Tricritical Ising model.\(^6\) Hence it is desirable to compare \( \tilde{c}(0, B|3) \) with the known ground state scaling function \( C(r) \) obtained from TBA \(^7\). Recall that the latter is a function of a scaling parameter \( r \), with UV limit \( (r \to 0) \tfrac{7}{10} \) and IR limit \( (r \to \infty) \tfrac{1}{10} \) respectively. Therefore to compare the two expressions, we need to find a parameterization of the variable \( B \) in terms of \( r \). This can always be done since one can in principle invert the function \( \tilde{c}(0, B|3) \) to obtain the parameterization \( B(r) = \tilde{c}^{-1}(C(r)) \). However we were unable to express \( B(r) \) in a simple and closed form. This is perhaps not surprising since \( C(r) \) is written as an integral involving two pseudo-energies \( \varepsilon_1 \) and \( \varepsilon_2 \), who in turn are given by two coupled integral equations involving \( r \). Only in the UV or IR limits do we get a simplification of the integral equations, which then allow us to write \( C \) in terms of dilogarithms \(^7\). Hence the parameterization \( B(r) \), which yields an expression for \( C(r) \) in terms of dilogarithms for general \( r \), is likely to be complicated. It is also unclear at this stage whether this parameterization admits any physical interpretations.

Now let us focus on the other domain \( A = B \). The relation \((3.18)\) tells us \( A = \frac{\cos((p+1)q/2)}{\cos(pq/2)} \). Of course in the limiting case \((I)\), \( A \to 0 \), we found \( \theta = \frac{\pi}{p+1} \) as before. Once again \( \tilde{c}(A, A|p) \) is a smooth monotonic function of \( A \). For the \( N = 2 \) supersymmetric limit \((III)\), taking \( A \to 1 \), we get \( \theta = 0 \) and \( \tilde{c}(1, 1|p) = 3(1 - \frac{2}{p}) \) as expected. Hence \( \tilde{c}(A, B|p) \) indeed give us a function which interpolates between the central charges of an unitary model and its supersymmetric counterparts.

### 4 Discussion

In this work, we have studied the asymptotic behavior of the ‘generalized’ character formula \( \chi_{s,r}^{(p)}(\rho_1, \rho_2; q) \) \((2.13)\) in the limit \( q \to 1^- \). In this limit, we show that the \( q \)-series diverges like \( \tilde{q}^{-\pi} \) and we found a simple expression \((3.28)\) for the ‘generalized’ effective central charge \( \tilde{c} \) in terms of dilogarithms. In the limiting cases \((I)\), \((II)\) and \((III)\), \( \tilde{c} \) yields the central charges of the unitary models and their supersymmetric counterparts.

Having stated our conclusion, we shall take the liberty to indulge in a some (pure) speculations. Of course it is not surprising that we can find a function which reproduces the correct central charges in the various limits. Indeed \( \chi_{s,r}^{(p)}(\rho_1, \rho_2; q) \) also becomes the corresponding CFT characters in these limits. But what is \textit{a priori} not expected from the Andrews-Bailey construction is that the ‘generalized’ character \((2.13)\) would exhibit the asymptotic behavior found in \( \S 3 \). The prediction for this behavior was based on the assumption that \((2.13)\) gives the partition function for some quantum field theory. This field theory must be invariant under interchanging the roles of space and time. This suggests that \( \chi_{s,r}^{(p)}(\rho_1, \rho_2; q) \), when multiplied by a suitable factor \( q^{D_{s,r}^{(p)}(\rho_1, \rho_2|q)} \), may be modular covariant. It would be very interesting to show directly from \((2.13)\) that

\[
q^{D_{s,r}^{(p)}(\rho_1, \rho_2|p)} \chi_{s,r}^{(p)}(\rho_1, \rho_2; q) = \sum_{s',r'} q^{D_{s',r'}^{(p)}(\rho_1, \rho_2|p)} q^{D_{s',r'}^{(p)}(\rho_1, \rho_2|p)} \chi_{s',r'}^{(p)}(\rho_1, \rho_2; q)
\]

for some ‘generalized’ S-matrix. Presumably the elements of this S-matrix (if it exits) can be calculated from the non-perturbative corrections to the saddle point \(14, 15\). This computation would be much more involved than that in \( \S 3 \) since the elements of \( S \) depend on \( r \) and \( s \).

\(^6\)In this case, the ‘generalized’ character \((2.13)\) take us from the Tricritical Ising character \( \chi_{1,1} + \chi_{1,4} \) to the Ising character \( \chi_{1,1} \).
Another interesting puzzle is the nature of the quantum field theory with the Hamiltonian $H_P$ discussed in §3. From (3.7), the ground state energy of $H_P(\rho_1, \rho_2|p)$ is proportional to $\tilde{c}(A, B|p)$ which is written in terms of dilogarithms. This seems to indicate that $H_P(\rho_1, \rho_2|p)$ is the Hamiltonian for a (maybe irrational) CFT which interpolates between the $N = 0, N = 1$ and $N = 2$ unitary models. It is known in some cases that by varying the closed boundary condition $P$ around the cylinder, one can interpolate continuously between several CFTs [10]. Examples include the $Q$-Potts and $O(n)$ models [17], and the Ising model with defect lines (see for example [15]). Whether one can obtain the ‘generalized’ character (2.13) by varying the cylindrical boundary conditions of the $N = 2$ unitary model is worth investigating.

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