Measurement uncertainty relation for three observables

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In this work we establish rigorously a measurement uncertainty relation (MUR) for three unbiased qubit observables, which was previously shown to hold true under some presumptions. The triplet MUR states that the uncertainty, which is quantified by the total statistic distance between the target observables and the jointly implemented observables, is lower bounded by an incompatibility measure that reflects the joint measurement conditions. We derive a necessary and sufficient condition for the triplet MUR to be saturated and the corresponding optimal measurement. To facilitate experimental tests of MURs we propose a straightforward implementation of the optimal joint measurements. The exact values of incompatibility measure are analytically calculated for some symmetric triplets when the corresponding triplet MURs are not saturated. We anticipate that our work may enrich the understanding of quantum incompatibility in terms of MURs and inspire further applications in quantum information science. This work presents a complete theory relevant to a parallel work [Y.-L. Mao, et al., Testing Heisenberg’s measurement uncertainty relation of three observables, arXiv:2211.09389] on experimental tests.

I. INTRODUCTION

One of the most distinguishing features of quantum theory, from a modern point of view, is the incompatibility [1, 2] of quantum measurements whose outcomes cannot be read out simultaneously. From a practical point of view the quantum incompatibility, just like entanglement [3, 4], can be established as a non-classical resource and has found numerous applications in quantum information science including Bell’s nonlocality, quantum steering, quantum contextuality, and quantum state discrimination. From a fundamental point of view, the quantum incompatibility quantifies Bohr’s complementarity and lies at the heart of all kinds of Heisenberg’s uncertainty relations [5].

By employing different measures for uncertainty (or errors or disturbance) such as mean square errors, trace distance, or entropies, various kinds of uncertainty relations have been established [6–15]. These uncertainty relations can be classified into two different kinds, namely, the preparation uncertainty relations (PUR, also known as the Heisenberg-Robertson uncertainty relation) [16, 17] and the measurement uncertainty relations (MUR) [18–45]. While the PURs prohibit us from preparing quantum states with definite values for incompatible observables, the MURs capture the essence of quantum incompatibility, namely, quantum measurements may disturb each other, which was the main concern in the original Heisenberg’s gedanken experiment of microscope [29, 30]. Recent years have witnessed a number of MURs of two observables being conceived and verified in experiments [19, 21–44, 46].

The quantum incompatibility can be quantified most naturally in the framework of joint measurement. For example, we can approximately measure two incompatible observables by performing a pair of compatible, i.e., jointly measurable, quantum observables by introducing some errors. The resulting MUR indicates that the total error, which may be quantified in, e.g., a worst case scenario, characterizes the performance of the measurement device in the spirit of Bush, Lahti, and Werner (BLW) [30], is lower bounded by some measure of incompatibility. The optimal errors, however, are in general achieved on different states.

An operational and meaningful lower bound for the total error was proposed independently in Refs.[40, 47], via the distance, e.g., relative entropy or statistic distance, from the given pair of incompatible measurements to the set of all the jointly measurable pairs, calculated on the same state. For relative entropy approach the lower bounds as well as optimal measurements are in general not analytically tractable. Moreover a relation of the lower bound to the joint measurability condition is missing. For the statistic distance measure of total error, an elegant form of MUR is proposed with a presumption that approximate joint measurements are restricted to the unbiased triplets [48, 49]. Thus a MUR for a triplet of observables is still missing and how to find the optimal measurements to attain the lower bound and their experimental implementations remain to be outstanding problems.

In this paper we at first re-establish the MUR for triplet of unbiased qubit observables by considering the most general jointly measurable triplet as approximations. And then we provide a necessary and sufficient condition of attainability as well as the optimal measurement. Third, we propose a straightforward implementation of the optimal joint measurement by measuring randomly four ideal qubit observables. Lastly, we derive analytically the incompatibility measure for two symmetric triplets after showing that the optimal jointly measurable triplet shares the same graded symmetries with the original triplet of incompatible observables.
II. TRIPLET MUR AND AN IMPLEMENTATION

For a qubit the most general measurement or observable with two outcomes is represented by positive operator valued measure (POVM) \( \{N_{\pm}\} \) with

\[
N_{\pm} = \frac{I \pm (x + \vec{n} \cdot \vec{\sigma})}{2} := N_{\pm}(x, \vec{n}),
\]

where we have denoted explicitly its dependence on the biasedness \( x \) and Bloch vector \( \vec{n} \) satisfying \( |x| + |\vec{n}| \leq 1 \). The observables with vanishing biasedness, i.e. \( x = 0 \), are referred to as unbiased. In general, some POVMs are called jointly measurable if there exists a parent POVM with multiple outcomes such that each observable in the given set arises as a marginal measurement or equivalently from a post-measurement processing [50]. The exact joint measurement conditions are known in a few special cases [51–54] and in general the problem can be cast into semidefinite programms [55, 56].

In particular, a set \( \mathcal{N} = \{N_{j}\}_{j=1}^{3} \) of three general two-outcome qubit observables \( N_{j} = \{N_{\pm j} = N_{\pm}(x_{j}, \vec{n}_{j})\} \) is jointly measurable if there exists a parent measurement \( \{M(\mu_{1}, \mu_{2}, \mu_{3})\} \) with 8 outcomes, labelled with binary vector \( \mu = (\mu_{1}, \mu_{2}, \mu_{3}) \) for \( \mu_{j} = \pm 1 \), such that the given three general observables arise as marginals, i.e., for each \( j = 1, 2, 3 \) it holds

\[
N_{\mu_{j}|\omega} = \int \sum_{h_{j}} M(\mu_{1}, \mu_{2}, \mu_{3}).
\]

Equivalently, \( \mathcal{N} \) is jointly measurable if there exists a parent measurement \( \{M_{\omega}\} \) together with a set of post-measurement processing \( \{p_{j}(\pm|\omega)\} \), i.e., a set of probability distributions for each \( j \) and outcome \( \omega \), such that

\[
N_{\pm j} = \sum_{\omega} p_{j}(\pm|\omega) M_{\omega}.
\]

The necessary and sufficient conditions for a triplet of unbiased qubit observables to be compatible is explicitly given by [53]

\[
\sum_{k=0}^{3} |q_{k} - \tilde{q}_{j}| \leq 4,
\]

where

\[
\tilde{q}_{j} = \sum_{j=1}^{3} \gamma_{jk} \vec{n}_{j}, \quad \gamma_{jk} = (-1)^{j+k} (\frac{1}{2})^{j+k}
\]

with \( \tilde{q}_{j} \) being the Fermat-Torricelli (FT) point of \( \{q_{k}\}_{k=0}^{3} \), the vector that minimize the distance sum as given in the left-hand-side of inequality Eq. (4). We note that \( \{q_{k}\}_{k=0}^{3} = \{\mu \vec{n}_{1} + \nu \vec{n}_{2} + \mu \nu \vec{n}_{3} | \mu, \nu = \pm\} \).

**Lemma** If a triplet \( \{N_{\pm}(x_{j}, \vec{n}_{j})\}_{j=1}^{3} \) is jointly measurable, then the corresponding unbiased triplet \( \{N_{\pm}(0, \vec{n}_{j})\}_{j=1}^{3} \) is also jointly measurable.

**Proof** As the triplet \( \{N_{\pm}(x_{j}, \vec{n}_{j})\} \) is jointly measurable, there exists a joint measurement

\[
8M(\mu_{1}, \mu_{2}, \mu_{3}) = 1 + \sum_{j=1}^{3} \mu_{j}(x_{j} + \vec{n}_{j} \cdot \vec{\sigma})
\]

\[
+ \sum_{j>k} \mu_{j} \mu_{k}(\vec{z}_{jk} + \vec{\xi}_{jk} \cdot \vec{\sigma}) - \mu_{1} \mu_{2} \mu_{3}(z + \vec{z} \cdot \vec{\sigma})
\]

for some real \( \{z, \vec{z}_{jk}\} \) and vectors \( \{\vec{\xi}_{jk}, \vec{\xi}\} \) \( (j > k = 1, 2, 3) \) with the given triplet as marginals. The positivity requirements \( M(\mu_{1}, \mu_{2}, \mu_{3}) \geq 0 \), i.e., demand that

\[
1 + \sum_{j=1}^{3} \mu_{j} x_{j} + \sum_{j>k} \mu_{j} \mu_{k} \vec{z}_{jk} - \mu_{1} \mu_{2} \mu_{3} \vec{z}
\]

\[
\geq \left| \sum_{j=1}^{3} \mu_{j} \vec{n}_{j} + \sum_{j>k} \mu_{j} \mu_{k} \vec{\xi}_{jk} - \mu_{1} \mu_{2} \mu_{3} \vec{\xi} \right|.
\]

By summing up over all \( \mu_{j} = \pm \) we obtain

\[
8 \geq \sum_{\mu_{j}, \mu_{k=1}^{3}} \left| \sum_{j=1}^{3} \mu_{j} \vec{n}_{j} + \sum_{j>k} \mu_{j} \mu_{k} \vec{z}_{jk} - \mu_{1} \mu_{2} \mu_{3} \vec{z} \right|
\]

\[
= \frac{1}{2} \sum_{\mu_{j}, \mu_{k=1}^{3}} \left| \sum_{j=1}^{3} \mu_{j} \vec{n}_{j} \pm \sum_{j>k} \mu_{j} \mu_{k} \vec{z}_{jk} - \mu_{1} \mu_{2} \mu_{3} \vec{z} \right|
\]

\[
\geq \sum_{\mu_{j}, \mu_{k=1}^{3}} \left| \sum_{j=1}^{3} \mu_{j} \vec{n}_{j} - \mu_{1} \mu_{2} \mu_{3} \vec{z} \right|
\]

\[
= 2 \sum_{\mu_{j}, \mu_{k=1}^{3}} \left| \sum_{j=1}^{3} \mu_{j} \vec{n}_{j} - \vec{q}_{j} \right| \geq 2 \sum_{\mu_{j}, \mu_{k=1}^{3}} \left| \sum_{j=1}^{3} \mu_{j} \vec{n}_{j} - \vec{q}_{j} \right|
\]

where the first equality is due to an average over \( \vec{n} \) and \( -\vec{n} \) while the first inequality is because of triangle inequality and the last inequality comes from the definition of FT point \( \vec{q}_{j} \) of four vectors \( \{\mu_{1} \vec{n}_{1} + \mu_{2} \vec{n}_{2} + \mu_{3} \vec{n}_{3} | \mu_{1,2,3}=\pm\} \). This means that the unbiased triplet \( \{N_{\pm}(0, \vec{n}_{j})\} \) is also jointly measurable. \( \square \)

Let \( \mathcal{M} = \{M_{j}\}_{j=1}^{3} \) with \( M_{j} = \{M_{\pm j} := N_{\pm}(0, \vec{n}_{j})\} \) be a triplet of unbiased qubit observables. A jointly measurable triplet of general observables \( \mathcal{N} = \{N_{\pm j} := N_{\pm}(x_{j}, \vec{n}_{j})\} \) is performed and the total uncertainty measured by the statistics distance defines the incompatibility of the triplet

\[
\Delta_{\mathcal{M}} := \min_{\mathcal{N}} \max_{\rho} \frac{1}{3} \sum_{j=1}^{3} d_{\rho}(M_{j}; N_{j})
\]

where

\[
d_{\rho}(M_{j}; N_{j}) = 2 \sum_{\pm} \left| \text{Tr} \rho M_{\pm j} - \text{Tr} \rho N_{\pm j} \right|
\]

An elegant lower bound of \( \Delta_{\mathcal{M}} \) was proposed in [48] which is intimately related to the joint measurability condition. However, in deriving their result a strong presumption that the optimal measurements are unbiased was introduced. We strengthen this measurement uncertainty relation by considering the most general form of jointly measurable triplet by proving that the optimal measurement is actually unbiased.
Theorem 1 (Triplet MUR) For an unbiased triplet $\mathcal{M} = \{M_j\}_{j=1}^3$, by performing the most general measurements $\{N_j\}_{j=1}^3$ that are jointly measurable, it holds MUR

$$\Delta_{\mathcal{M}} \geq \frac{1}{2} \sum_{k=0}^3 |\vec{p}_k - \vec{p}_f| - 2 := 2\delta$$

(7)

where $\{\vec{p}_k = \sum_j \gamma_{jk} \vec{m}_j\}$ with $\vec{p}_f$ being its FT point. The lower bound is saturated if and only if

$$\delta \leq \min_k |\vec{p}_k - \vec{p}_f|.$$  

(8)

If the condition is met, the optimal set of jointly measurable triplet reads

$$\vec{m}_j = \vec{m}_j + \frac{\delta}{4} \sum_{k=1}^3 \gamma_{jk} \frac{|\vec{p}_f - \vec{p}_k|}{|\vec{p}_f - \vec{p}_k|} (k = 1, 2, 3).$$  

(9)

Proof To lower bound the incompatibility we calculate

$$\begin{aligned}
\frac{1}{2} \Delta_{\mathcal{M}} &= \min_{N_j} \max_{\vec{p}_k} \sum_{j=1}^3 \sum_{\pm} |\text{Tr} (M_{\pm j}) - N_{\pm j}| \\
&= \min_{\{x_j\}, \vec{m}_j} \sum_{j=1}^3 |\vec{f} \cdot (\vec{m}_j - \vec{n}_j) - x_j| \\
&= \min_{\{x_j\}, \vec{m}_j} \max_{\mu_{j_1}, \mu_{j_2}, \mu_{j_3} = \pm 1} \sum_{j=1}^3 \mu_j (\vec{f} \cdot (\vec{m}_j - \vec{n}_j) - x_j) \\
&= \min_{\{x_j\}, \vec{m}_j} \max_{\mu_{j_1}, \mu_{j_2}, \mu_{j_3} = \pm 1} \sum_{j=1}^3 (\sum_j \mu_j \vec{m}_j - \sum_j \mu_j \vec{n}_j) - \sum_j \mu_j x_j \\
&= \min_{\{x_j\}, \vec{m}_j} \max_{\mu_{j_1}, \mu_{j_2}, \mu_{j_3} = \pm 1} \sum_{j=1}^3 (\mu \vec{m}_1 + \nu \vec{m}_2 + \mu \nu \vec{m}_3 - (\mu \vec{n}_1 + \nu \vec{n}_2 + \mu \nu \vec{n}_3) + \mu x_1 + \nu x_2 + \mu \nu x_3) \\
&\geq \min_{\{\vec{q}_k\}} \max_{k=0,1,2,3} |\vec{p}_k - \vec{q}_k| \\
&\geq \min_{\{\vec{q}_k\}} \frac{1}{4} \sum_{k=1}^3 |\vec{p}_k - \vec{q}_k| \\
&\geq \min_{\{\vec{q}_k\}} \frac{1}{4} \sum_{k=1}^3 |\vec{p}_k - \vec{q}_f| - |\vec{q}_k - \vec{q}_f| \\
&\geq \min_{\{\vec{q}_k\}} \frac{1}{4} \sum_{k=1}^3 (|\vec{p}_k - \vec{q}_f| - |\vec{q}_k - \vec{q}_f|) \\
&\geq \frac{1}{4} \sum_{k=1}^3 |\vec{p}_k - \vec{q}_f| - \frac{1}{4} \sum_{k=1}^3 |\vec{q}_k - \vec{q}_f| \\
&\geq \frac{1}{4} \sum_{k=0}^3 |\vec{p}_k - \vec{q}_f| - \frac{1}{4} \sum_{k=0}^3 |\vec{q}_k - \vec{q}_f| \\
&\geq \frac{1}{4} \sum_{k=0}^3 |\vec{p}_k - \vec{q}_f| - 1 = \delta
\end{aligned}$$

(10a-10k)

Here we have used the fact that the absolute value can be rewritten as $|A| = \max_x \{ \pm A \}$ in deriving Eq. (10c) and the fact that $\sum_j \mu_j \vec{m}_j = \mu_1 \mu_2 \mu_3 \sum_{i,j,k} \text{cyclic} \mu_i \mu_j \vec{m}_k$ and redefine $\mu = \mu_2 \mu_3, \nu = \mu_1 \mu_3$ in deriving Eq. (10e). In the first inequality yielding condition 0 we have used Lemma to take $x_j = 0$ while keeping the resulting unbiased set still jointly measurable with a no larger total uncertainty. In derive condition 2 we have taken $\vec{q}_f$ to be the FT point of $\{\vec{q}_k\}$. In condition 4 we have used the property of the FT point $\vec{p}_f$ for $\{\vec{p}_k\}$ while the joint measurement condition is used in deriving condition 5.

In order to saturate the MUR in Eq. (7) we have only to let all the inequalities become equalities in the above derivation. It is straightforward to see that in order to saturate Eq. (10f), i.e., the condition 0, the optimal measurement has to be unbiased. For the other inequalities we have, respectively,

1. $|\vec{p}_k - \vec{q}_k|$ is independent of $k$
2. $\vec{p}_k, \vec{q}_k, \vec{q}_f$ are linearly dependent for each $k$
3. $|\vec{p}_k - \vec{q}_f| \geq |\vec{q}_k - \vec{q}_f|$ for each $k$
4. $\vec{q}_f$ coincides with $\vec{p}_f$
5. $\sum_k |\vec{q}_k - \vec{q}_f| = 4$

From conditions 2 and 4 it follows that we can have linear expansions $\vec{q}_k = (1 - \alpha_k) \vec{p}_k + \alpha_k \vec{q}_f$ with some real coefficient $\alpha_k$ (arbitrary for now) for each $k = 0, 1, 2, 3$. As a result of condition 1 we have

$$\delta = |\vec{p}_k - \vec{q}_k| = |\alpha_k| \cdot |\vec{p}_k - \vec{p}_f|$$

is independent of $k$. Therefore we can assume $|\vec{p}_k - \vec{p}_f| > 0$ for all $k$ because otherwise $|\vec{p}_k - \vec{q}_k|$ would become zero for all $k$ (condition 1) so that the triplet $\mathcal{M}$ is also joint.
measurable, a contradiction. Therefore from conditions 3 and 4, i.e.,
\[ |\tilde{q}_k - \tilde{q}_j| = |\tilde{q}_k - \tilde{p}_j| = |1 - \alpha_k| \cdot |\tilde{p}_k - \tilde{p}_j| \leq |\tilde{p}_k - \tilde{p}_j| \]
we obtain $|1 - \alpha_k| \leq 1$ so that we have $0 \leq \alpha_k \leq 2$. Thus
\[ \alpha_k = \frac{\delta}{|\tilde{p}_k - \tilde{p}_j|}. \]

From condition 5 it follows
\[ 4 = \sum_k |\tilde{q}_k - \tilde{q}_j| = \sum_k |\tilde{q}_k - \tilde{p}_j| = \sum_k |\tilde{p}_k - \tilde{p}_j| = \sum_k (|\tilde{p}_k - \tilde{p}_j| - \delta) \geq \sum_k (|\tilde{p}_k - \tilde{p}_j| - \delta) = 4(\delta + 1) - 4\delta = 4 \]
That is, the inequality is in fact an equality, meaning that for all $k = 0, 1, 2, 3$ it holds
\[ \frac{1}{4} \sum_k |\tilde{p}_k - \tilde{p}_j| - 1 = \delta = |\tilde{p}_k - \tilde{p}_j|. \]

Sufficiency. Suppose the condition is satisfied we have $|\tilde{p}_k - \tilde{p}_j| > 0$ for all $k$ and need to show that the unbiased triplet defined in Eq.(9) is jointly measurable and saturate the MUR. By construction we have
\[ \tilde{q}_k = \sum_{j=1}^3 \gamma_{jk} \tilde{R}_j = \tilde{p}_k + \alpha_k (\tilde{p}_j - \tilde{p}_k), \quad \alpha_k = \frac{\delta}{|\tilde{p}_j - \tilde{p}_k|}. \]
Condition ensures that $\alpha_k \leq 1$ so that
\[ \sum_k \frac{\tilde{p}_j - \tilde{q}_k}{|\tilde{p}_j - \tilde{q}_k|} = \sum_k (1 - \alpha_k) \cdot \frac{\tilde{p}_j - \tilde{p}_k}{|\tilde{p}_j - \tilde{p}_k|} = \sum_k \left( \frac{\tilde{p}_j - \tilde{p}_k}{|\tilde{p}_j - \tilde{p}_k|} \right) = 0 \]
showing that $\tilde{q}_j = \tilde{p}_j$. As a result we have the joint measurement condition
\[ \sum_{k=0}^3 |\tilde{q}_k - \tilde{p}_j| = \sum_{k=0}^3 (1 - \alpha_k)|\tilde{p}_k - \tilde{p}_j| = 4. \]
Furthermore, we actual have qubit observables, i.e., $|\tilde{R}_k| \leq 1$ which follows from joint measurement condition
\[ |\tilde{R}_j| = \frac{1}{4} \left( \sum_k \gamma_{jk} (\tilde{q}_k - \tilde{p}_j) \right) \leq \frac{1}{4} \sum_k |\tilde{q}_k - \tilde{p}_j| = 1 \]
And the optimal lower bound can be attained in the pure state with Bloch vector $\vec{f} = \vec{e}_0$ where
\[ \vec{e}_k := \frac{\tilde{p}_j - \tilde{p}_k}{|\tilde{p}_j - \tilde{p}_k|}, \quad (k = 0, 1, 2, 3) \]
satisfying $\sum_k \vec{e}_k = 0$ by the definition of the FT point. In fact we have the total uncertainty
\[ \sum_{j=1}^3 |\vec{f} \cdot (\vec{m}_j - \vec{n}_j)| = \frac{\delta}{4} \sum_{j=1}^3 |\vec{e}_0 \cdot \sum_{k=0}^3 \gamma_{jk} \vec{e}_k| \]
\[ = \frac{\delta}{2} \sum_{j=1}^3 |\vec{e}_0 \cdot (\vec{e}_0 + \vec{e}_j)| = \frac{\delta}{2} \sum_{j=1}^3 (1 + \vec{e}_0 \cdot \vec{e}_j) = \delta \]
which means that the MUR is saturated. □

As an optimal measurement always lies on the boundary, i.e., saturating the joint measurement condition, we may accomplish the optimal joint measurement in a single-qubit experiment [31].

**Theorem 2** (Implementation) A jointly measurable triplet of unbiased qubit observables $\{N_i\}$ that saturates the joint measurement condition Eq.(4) can be implemented by the following parent measurement $\{P_i, O_k\}$ with $O_k = \{O_{\mu |k}\}$ where
\[ P_k = \frac{\tilde{q}_k - \tilde{q}_f}{4}, \quad O_{\mu |k} = \frac{1}{2} \left( 1 + \mu \frac{\tilde{q}_k - \tilde{q}_f}{|\tilde{q}_k - \tilde{q}_f|} \vec{\sigma} \right), \]
with outcome labeled with $\mu_k = \pm 1$ for each $k = 0, 1, 2, 3$ and post-measurement processing $p_j(\mu |k, \mu_k) = \frac{1 + \gamma_{jk} \mu_k}{2}$.

**Proof** The measurement can be implemented by first randomly choose $k = 0, 1, 2, 3$ according to distribution $P_k = \frac{\tilde{q}_k - \tilde{q}_f}{4}$ and then perform a measurement $\{O_k = \{O_{\pm |k}\}\}$ with
\[ O_{\pm |k} = \frac{1 \pm \tilde{g}_k \cdot \vec{\sigma}}{2}, \quad \tilde{g}_k = \frac{\tilde{q}_k - \tilde{q}_f}{|\tilde{q}_k - \tilde{q}_f|}. \]
In fact by using post-measurement processing
\[ \sum_{k, \mu_k} P_k \frac{1 + \gamma_{jk} \mu_k}{2} = N_{\mu |j} \]
we obtain the desired marginals
\[ \sum_{k, \mu_k} p_j(\mu |k, \mu_k) M(k, \mu_k) = \sum_k P_k \frac{1 + \gamma_{jk} \mu_k}{2} = N_{\mu |j} \]
The standard parent measurement with 8 outcomes can be constructed as
\[ M(\omega_1, \omega_2, \omega_3) = \sum_{k, \mu_k} P_k \frac{1 + \gamma_{jk} \mu_k}{2} O_{\mu |k} \]
\[ = \frac{1 + \sum_j \omega_j \tilde{R}_j \cdot \vec{\sigma} - \omega_1 \omega_2 \omega_3 \tilde{q}_f \cdot \vec{\sigma}}{8}. \]
with $\omega_j = \pm$, which coincides with the joint measurement given in Ref. [53]. □

As an immediate result, when the MUR is attained the optimal joint measurement Eq.(9), can be readily implemented by performing 4 ideal qubit measurements along directions $\vec{e}_k$ with probability $\frac{1}{4} (|\tilde{p}_k - \tilde{p}_j| - \delta)$ for $k = 0, 1, 2, 3$. 
In order to show that the optimal joint measurement also has the same symmetry it suffices to show for each joint measurement there exists another joint measurement having the symmetry and in the mean time with a no larger uncertainty. In fact, if \( \{ \vec{n}_i \} \) is an arbitrary jointly measurable triplet of unbiased observables, then we have

\[
2 \max_{\mu} |\vec{n}_\mu - \vec{n}_\mu'|
= \max_{\mu} |\vec{n}_\mu - \vec{n}_\mu'| + \max_{\mu} |g \cdot (\vec{n}_\mu - \vec{n}_\mu')|
= \max_{\mu} |\vec{n}_\mu - \vec{n}_\mu'| + \max_{\mu} |\vec{n}_{\mu,\omega} - g \cdot \vec{n}_\mu'|
= \max_{\mu} |\vec{n}_\mu - \vec{n}_\mu'| + \max_{\mu} |\vec{n}_\mu' - g \cdot \vec{n}_{\mu,\omega}|
\geq 2 \max_{\mu} \left| \vec{n}_\mu - \frac{\vec{n}_\mu + g \cdot \vec{n}_{\mu,\omega}}{2} \right|.
\]

(12)

Therefore if we consider another triplet of unbiased observables along directions

\[
\vec{n}'_j = \frac{\vec{n}_j + \omega_j g \cdot \vec{n}_{\sigma(j)}}{2}
\]

we have \( g \cdot \vec{n}'_j = \omega_j \vec{n}'_{\sigma(j)} \) (by noting \( \sigma^2(j) = j \) and \( \omega_j = \omega_{\sigma(j)} \)), i.e., the averaged triplet \( \{ \vec{n}'_i \} \) shares the same symmetry with triplet \( \mathcal{M} \). In order to show that \( \{ \vec{n}'_i \} \) is also jointly measurable we note that the joint measurement condition for \( \{ \vec{n}'_i \} \) can be equivalently written as

\[
\sum_{\mu} |\vec{n}_\mu - \mu_0 \vec{q}_f| \leq 8 \iff \sum_{\mu} |g \cdot \vec{n}_{\mu,\omega} - \omega_0 \mu_0 g \cdot \vec{q}_f| \leq 8, \quad (13)
\]

where we have denoted \( \mu = (\mu_1, \mu_2, \mu_3) \) and \( \mu_0 = \mu_1 \mu_2 \mu_3 \) and \( \omega_0 = \omega_1 \omega_2 \omega_3 \). By summing up these two inequalities and using triangle inequality we have

\[
8 \geq \sum_{\mu} \left| \vec{n}_\mu - \mu_0 \vec{q}_f - \omega_0 g \cdot \vec{q}_f \right| \geq \sum_{\mu} \left| \vec{n}'_\mu - \mu_0 \vec{q}_f \right|,
\]

where

\[
\vec{n}'_\mu = \sum_j \mu_j \vec{n}'_j = \frac{\vec{n}_\mu + g \cdot \vec{n}_{\mu,\omega}}{2}
\]

with \( \vec{q}_f \) being the FT point of \( \{ \vec{q}'_f \} \) = \( \{ \vec{n}'_i \} |\mu_0 = 1 \} \), which gives the desired joint measurability of \( \{ \vec{n}'_i \} \). In sum, starting form an arbitrary jointly measurable triplet \( \{ \vec{n}_i \} \) the newly introduced triplet \( \{ \vec{n}'_i \} \) is also jointly measurable, shares the same symmetry as \( \mathcal{M} \), and, due to Eq.(12), has a no larger total uncertainty. Therefore the optimal measurement can be taken to share the same symmetry as \( \mathcal{M} \) without loss of generality.

We note that Eq.(13) actually proves that if the jointly measurable triplet \( \{ \vec{n}_i \} \) possesses some graded symmetry \( g \), i.e., \( g \cdot \vec{n}_i = \omega_j \vec{n}_{\sigma(j)} \), then the FT point also shares the same graded symmetry in the sense that \( g \cdot \vec{p}_j = \omega_0 \vec{p}_j \).

### IV. TWO ANALYTICAL EXAMPLES

Some triplets of unbiased observables might be determined completely by their symmetry. For example the

[Diagram of measurement setup and symmetry considerations]
triplet of three orthogonal observables, e.g., \(\{\sigma_x\}\), is completely determined by 3 reflections over planes \(XY, YZ, ZX\) (upto some scalings). In this case, the symmetry theorem above therefore enables us to determine the optimal measurements completely (upto some scalings). In other cases however the symmetry might determine partially the triplet. And in this case the symmetry also help to simplify the optimization. Our first example is the triplet \(M_1\) along directions

\[
\vec{m}_1 = (\sin \theta, 0, \cos \theta), \quad \vec{m}_2 = (1, 0, 0), \quad \vec{m}_3 = (0, 1, 0)
\]

The reflection \(\tau_{YZ}\) over \(YZ\) plane and the reflections \(\tau_{\pm}\) over planes passing \(\hat{x}\) and angle bisectors \(\vec{m}_{\pm} = \vec{m}_1 \pm \vec{m}_2\) of \(\vec{m}_{1,2}\) generate the graded symmetry group of this triplet. In fact, we have \(\tau_{XZ} \cdot \vec{m}_3 = -\vec{m}_3\) while preserving directions \(\vec{m}_{1,2}\) and \(\tau_{\pm} \cdot \vec{m}_k = \mp \vec{m}_{3-k}\) with \(k = 1, 2\) while preserving \(\vec{m}_3\). Using theorem above we see that the optimal joint measurement \(\{\vec{n}_i\}\) should also have such a symmetry. From \(\tau_{XY} \cdot \vec{n}_3 = -\vec{n}_3\) it follows that \(\vec{n}_3\) should be also along the direction given by the eigenvector of \(\tau_{YZ}\) corresponding eigenvalue \(-1\), which infers \(\vec{n}_3 \propto \vec{m}_3\). Similarly from \(\tau_{YZ} \cdot \vec{n}_k = \vec{n}_k\) it follows that \(\vec{n}_{1,2}\) also lie on the \(YZ\) plane. Finally, from symmetry \(\tau_{\pm} \cdot \vec{n}_k = \pm \mp \vec{n}_{3-k}\) we see that \(\vec{n}_{1,2}\) have the same length and are also symmetric about the angle bisector \(\vec{m}_1\). As a result the optimal joint measurement must be of form

\[
\vec{n}_3 = n_3 \vec{m}_3, \quad \vec{n}_1 = \frac{\beta_+ \vec{m}_+ + \beta_- \vec{m}_-}{2}, \quad \vec{n}_2 = \frac{\beta_+ \vec{m}_+ - \beta_- \vec{m}_-}{2},
\]

where

\[
\vec{m}_{\pm} = \frac{\vec{m}_1 \pm \vec{m}_2}{|\vec{m}_1 \pm \vec{m}_2|}, \quad \vec{m}_3 = \frac{\vec{m}_3}{|\vec{m}_3|}
\]

with 3 suitable constants \(n_3, \beta_\pm\) satisfying joint measurement condition \(|\beta_+| + |\beta_-| \leq 2\sqrt{1 - n_3^2}\). Now the incompatibility \(\Delta_{\perp}\) can be calculated as

\[
\min_{n_3} \max_{\beta_\pm} \left| \sqrt{(1 - n_3)^2} + (\beta_\pm - m_{\pm})^2 \right| = \min_{n_3} \max_{\beta_\pm} \left| (1 - n_3)^2 + (|\beta_\pm| - m_{\pm})^2 \right|
\]

\[
= \min_{n_3} \left( \sqrt{(1 - n_3)^2} + (d_+ - \sqrt{1 - n_3^2} + |\kappa|)^2 \right)
\]

\[
= \min_{n_3} \left( \sqrt{(1 - n_3)^2} + (d_+ - \sqrt{1 - n_3^2})^2 \right)
\]

with equality holding for \(\theta = \theta_1 \approx 71.53^\circ\). The second bound therefore can be implicitly given by a parametric curve \(\Delta_{opt}(\theta)\)

\[
\Delta_{opt} = \frac{1 - \cos t}{\cos t} \sqrt{1 + 3 \cos^2 t}
\]

\[
\theta = \arcsin \left( \frac{1}{8} \tan^2 t (1 + 3 \cos t)^2 - 1 \right)
\]

To summarize the compatibility reads

\[
\Delta_{\perp} = \begin{cases} 
2\sqrt{2 + \cos \bar{\sigma}} - 2 & \theta_0 \leq \theta \leq \theta_1 \\
2\sqrt{3 + \cos \bar{\sigma}} - 2\sqrt{\cos \bar{\sigma}} - 2\sin \bar{\sigma} & \theta_0 \leq \theta \leq \theta_1 \\
\Delta_{opt}(\theta) & \theta \geq \theta_1
\end{cases}
\]

(14)

Respectively, there are three optimal measurements for three ranges of \(\theta\):

\(M_1 \theta \leq \theta_0\). In this case the condition of Theorem 2 is met and the MUR can be saturated so that the optimal measurement, as given by Theorem 2, reads

\[
\vec{n}_3 = \frac{\vec{m}_3}{d_+ + 1}, \quad \vec{n}_{1,2} = \frac{\beta_+ \vec{m}_+ + \beta_- \vec{m}_-}{2}
\]

where \(\beta_\pm = \frac{d_+}{d_+ + 1} \pm d_-\).

\(M_2 \theta_0 \leq \theta \geq \theta_0\). In this case MUR cannot be saturated and the optimal measurement reads

\[
\vec{n}_3 = \sqrt{1 - d_+^2} \vec{m}_3, \quad \vec{n}_1 = \vec{n}_2 = d_- \vec{m}_+
\]
where$j$for that the MUR illustrated in Fig. 2. Firstly, let us see under what conditions with $0$ giving rise to the lower optimal triplet $\mathcal{M}_\gamma$ together with its symmetries and the optimal triplet $\mathcal{N}_\gamma$ with the same symmetries.

**M3** $\theta \geq \theta_1$. The optimal joint measurement reads

$$\tilde{n}_3 = \sqrt{1 - (d_\perp - \kappa)^2} \tilde{m}_3, \quad \tilde{n}_1 = \tilde{n}_2 = (d_\perp - \kappa)\hat{m}_+$$

with $\kappa \geq 0$ minimizing

$$\sqrt{(1 - \sqrt{1 - (d_\perp - \kappa)^2})^2 + (m_- + 2\kappa)^2}.$$

If we introduce $\cos 2\gamma = \tilde{m}_1 \cdot \tilde{m}_2 = \sin \theta$ and three corresponding regions for different optimal measurements are determined by which one of the following three intervals

$$0 < \Gamma_0 < \Gamma_1 < 90^\circ$$

that $|\gamma - 45^\circ|$ falls into, where

$$\Gamma_j = 45^\circ - \frac{180^\circ}{2\pi} \arcsin \cos \theta_j, \quad (j = 0, 1).$$

Our second example is the following highly symmetric triplet $\mathcal{M}_\gamma$ of unbiased observables along directions

$$\tilde{m}_j = \hat{e}_j \cos \gamma + \hat{e}_j \sin \gamma, \quad \hat{e}_j \cdot \hat{e}_k = -\frac{1}{2}, \quad \hat{e}_j \cdot \hat{e}_j = 0,$$

with $0 \leq \gamma \leq \frac{\pi}{2}$ and $\hat{e}_j$ is the unit vector along $Z$ direction, as illustrated in Fig. 2. Firstly, let us see under what conditions the MUR $\Delta_\gamma \geq 2\delta_\perp$ for this triplet can be saturated. Four FT anchors are $\tilde{p}_0 = 3 \cos \gamma \hat{z}$ and $\tilde{p}_j = -\cos \gamma \hat{z} + 2 \sin \gamma \hat{e}_j$ for $j = 1, 2, 3$ with FT point reads $\tilde{p}_j = \alpha \hat{e}$ (by symmetry) where

$$\alpha = \begin{cases} \sin \gamma / \sqrt{2} - \cos \gamma & 0 \leq \tan \gamma \leq 4 \sqrt{2} \\ 3 \cos \gamma & |\tan \gamma| \geq 4 \sqrt{2} \\ -\sin \gamma / \sqrt{2} - \cos \gamma & 0 \geq \tan \gamma \geq -4 \sqrt{2} \end{cases}$$

(a detailed derivation given later) giving rise to the lower bound $\Delta_\perp \geq 2\delta_\perp$ where

$$\delta_\perp = \begin{cases} |\sin \gamma| + \sqrt{2} \sin \gamma - 1 & |\tan \gamma| \leq 4 \sqrt{2} \\ 3 \sqrt{4 \cos^2 \gamma + \sin^2 \gamma - 1} & |\tan \gamma| \geq 4 \sqrt{2} \end{cases}$$

In the case of $\gamma \leq \gamma_0 = \arctan 2\sqrt{2} \approx 70.53^\circ$, which is determined by the condition $\delta_{\gamma} = \min_k |\tilde{p}_k - \tilde{p}_j|$, namely, $\sin \gamma / \sqrt{2} - \cos \gamma = 1/3$, the attainability condition of Theorem 1 is satisfied so that the MUR can be attained.

In the case of $\gamma > \gamma_0$ MUR cannot be attained. In order to find the optimal measurement for the incompatibility we consider the symmetry of triplet $\mathcal{M}_\gamma$. Let $T_j$ be the reflection over the plane orthogonal to $\hat{z} \times \hat{e}_j$ passing the origin. Then we have $T_j \cdot \tilde{m}_j = \tilde{m}_j$ with $(j, k, l)$ cyclic. This symmetry is obeyed by the optimal joint measurement triplet according to Theorem 3. As a result the optimal triplet $\mathcal{N}_\gamma$ should be along directions

$$\tilde{n}_j = x \hat{z} + y \hat{e}_j, \quad j = 1, 2, 3. \quad (15)$$

Next we shall prove that the triplet above is jointly measurable if and only if

$$y \leq \begin{cases} \frac{1-x}{\sqrt{2}} & |x| \geq \frac{1}{\sqrt{2}} \\ \frac{x}{\sqrt{2}} - x & 4 \sqrt{2} y \geq y \\ 3x + 4 \sqrt{2} x |y| \leq y \\ -\frac{x}{\sqrt{2}} - x & 4 \sqrt{2} y \leq -y \end{cases}$$

which is minimized by (via derivative as a function of $\alpha$)

$$\alpha = \begin{cases} -\frac{x}{\sqrt{2}} - x & 4 \sqrt{2} y \geq y \\ 3x + 4 \sqrt{2} x |y| \leq y \\ 6 \sqrt{4 \cos^2 \gamma + \sin^2 \gamma} - 4 \sqrt{2} x |y| \leq y \end{cases}$$

giving rise to the joint measurement condition

$$4 \geq \sum_{k=0}^3 |\tilde{q}_k - \tilde{q}_j| = \begin{cases} 4|x| + 4 \sqrt{2} y & 4 \sqrt{2} x |y| \geq y \\ 6 \sqrt{4 \cos^2 \gamma + \sin^2 \gamma} - 4 \sqrt{2} x |y| \leq y \end{cases},$$

which leads to the desired condition Eq. (16) for $x, y$.

According to Theorem 3 the optimal incompatibility can be obtained by considering only triplet of from Eq. (15), which now becomes

$$\Delta_\gamma = \min_{x, y} \{3 |\cos \gamma - x|, \sqrt{(\cos \gamma - x)^2 + 4(\sin \gamma - y)^2} \}$$

with minimum taken over region determined by Eq. (16). By min-max inequality we have

$$\Delta_\gamma \geq \min_{x, y} \sqrt{(\cos \gamma - x)^2 + 4(\sin \gamma - y)^2}$$

whose minimum is attained at (by derivative over the boundary $y = \frac{x}{3} \sqrt{1 - 9x^2}$)

$$(\cos \gamma - x) y = 16 (\sin \gamma - y) x. \quad (17)$$

Furthermore in order for $\Delta_\gamma$ to attain this lower bound we must also have

$$3 |\cos \gamma - x| \leq \sqrt{(\cos \gamma - x)^2 + 4(\sin \gamma - y)^2} \quad (18)$$
FIG. 3. Exact values of incompatibility for two triplets of ideal qubit measurements with symmetries, (a) triplet $M_\perp$ with $|\cos 2\gamma| = \vec{m}_1 \cdot \vec{m}_2$ and (b) highly symmetric triplet $M_Y$ of unbiased observables (see main text for details) according to analytical results computed with Eqs. (14) and (19), respectively.

which is possible as long as $\gamma \geq \gamma_1$ where

$$\gamma_1 = \arctan \frac{1}{79} \sqrt{2} \left( 5\sqrt{337} + 129 \right) \approx 75.80^\circ$$

is determined by Eq.(17) joint measurability condition $y = \frac{2}{3} \sqrt{1 - 9x^2}$ together with the equality in Eq.(18).

Though not all conditions in Eq.(10) can be satisfied simultaneously, some of them, e.g., condition 1, can be satisfied, which leads to

$$\Delta_\gamma \geq \min_{x,y} |\cos \gamma - x| + \sqrt{(\cos \gamma - x)^2 + 4(\sin \gamma - y)^2}$$

which is attained by $(x, y)$ equalizing Eq.(18) and $y = \frac{2}{3} \sqrt{1 - 9x^2}$, giving rise to

$$\Delta_\gamma = \frac{\sin \gamma}{\sqrt{2}} + 2|\cos \gamma| - \sqrt{\frac{2}{3} - (\sin \gamma - \sqrt{2}|\cos \gamma|)^2}$$

This happens if $\gamma_0 \leq \gamma \leq \gamma_1$. To summarize, the incompatibility reads

$$\Delta_\gamma = \begin{cases} 
2\cos \gamma + 2\sqrt{2}\sin \gamma - 2 \\
\sqrt{2}\sin \gamma + 4\cos \gamma - 2\sqrt{\frac{2}{3} - (\sin \gamma - \sqrt{2}\cos \gamma)^2} \\
\min_{x^2 + y^2 = \frac{1}{3}} 2\sqrt{(\cos \gamma - x)^2 + 4(\sin \gamma - 2y)^2}
\end{cases}$$

for three intervals $0 < \gamma_0 < \gamma_1 < 90^\circ$ divided by $\gamma_0 \approx 70.53^\circ$ and $\gamma_1 \approx 75.80^\circ$. In Fig. 3 we plot the exact incompatibility measures of these two symmetric triplets, which fit perfect with the numerical results.

V. CONCLUSIONS AN DISCUSSIONS

Quantum incompatibility reflects the basic fact that some quantum measurements may disturb each other, prevent-
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