Katok’s Entropy Formula of Unstable Metric Entropy for Partially Hyperbolic Diffeomorphisms

Ping Huang$^{1,2}$, Ercai Chen$^{*2,3}$, Chenwei Wang$^1$

1 College of Mathematical and Physical Sciences, Taizhou University, Taizhou 225300, Jiangsu, P.R. China.
2 School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, Jiangsu, P.R.China.
3 Center of Nonlinear Science, Nanjing University, Nanjing 210093, Jiangsu, P.R.China.

e-mail: pinghuang1984@163.com ecchen@njnu.edu.cn chenweiwang01@163.com

Abstract: Katok’s entropy formula is an important formula in entropy theory. This paper is devoted to establishing Katok’s entropy formula of unstable metric entropy which is the entropy caused by the unstable part of partially hyperbolic systems.

Keywords: Katok’s entropy formula, unstable metric entropy, partially hyperbolic systems, measure decomposition

1 Introduction

Let a triple $(X, d, f)$ be a topological dynamical system in the sense that $f : X \to X$ is a continuous map on the compact metric space $X$ with metric $d$. For $x, y \in X$ and $n \in \mathbb{N}$, the Bowen metric $d_n$ is given by

$$d_n(x, y) = \max\{d(f^i(x), f^i(y)) : i = 0, 1, \cdots, n - 1\}.$$ 

Given $\varepsilon > 0$, $x \in X$ and $n \in \mathbb{N}$, let $B_n(x, \varepsilon) = \{y \in X : d_n(x, y) < \varepsilon\}$ denote the $d_n$-ball centered at $x$ with radius $\varepsilon$.

In classical ergodic theory, measure-theoretic entropy and topological entropy are very important determinants of complexity in dynamical systems. And the variational principle reveals the relationship between the measure-theoretic entropy and the topological entropy. Katok’s entropy formula which plays an important role in the study of entropy theory is an equivalent definition of the measure-theoretic entropy in a manner analogous to the definition of the topological entropy.

* Corresponding author
In 1980, Katok [13] introduced the Katok’s entropy formula: for any \( f \)-invariant ergodic Borel probability measure \( \mu \), and \( 0 < \delta < 1 \),
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log N_\mu(n, \varepsilon, \delta)}{n} = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{\log N_\mu(n, \varepsilon, \delta)}{n} = h_\mu(f),
\]
where \( N_\mu(n, \varepsilon, \delta) \) denotes the minimal number of \( d_\mu \)-balls with radius \( \varepsilon \) whose union is a set of \( \mu \)-measure more than or equal to \( 1 - \delta \).

In 2004, using spanning sets, He, Lv and Zhou [6] introduced a definition of measure-theoretic pressure of additive potentials for ergodic measures, and obtained a pressure version of Katok’s entropy formula. In 2009, Zhao and Cao [19] gave a definition of measure-theoretic pressure of sub-additive potentials for ergodic measures, and generalized the above results in [13] and [6]. Moreover, we refer to [3, 2] for more pressure versions of Katok’s entropy formula. In 2009, Zhu [21] established a random version of Katok’s entropy formula. In 2017, Zheng, Chen and Yang [20] introduced an amenable version of Katok’s entropy formula for countable discrete amenable group actions. Huang, Wang and Ye [12] established Katok’s entropy formula for ergodic measures in the case of mean metrics. In 2018, Huang, Chen and Wang [10] constructed Katok’s entropy formula of conditional entropy in mean metrics, where the conditional entropy is with respect to a \( T \)-invariant sub-\( \sigma \)-algebra. In 2019, Huang and Wang [11] established a pressure version of Katok’s entropy formula in the case of mean metrics.

Let \( M \) be an \( n \)-dimensional smooth, connected and compact Riemannian manifold without boundary and \( f : M \to M \) be a \( C^1 \)-diffeomorphism. \( f \) is said to be partially hyperbolic if there exists a nontrivial \( Df \)-invariant splitting \( TM = E^s \oplus E^c \oplus E^u \) of the tangent bundle into stable, center, and unstable distributions, such that all unit vectors \( v^\sigma \in E^\sigma_x \) (\( \sigma = c, s, u \)) with \( x \in M \) satisfy
\[
\| D_x f v^s \| < \| D_x f v^c \| < \| D_x f v^u \|,
\]
and
\[
\| D_x f |_{E^s_x} \| < 1 \quad \text{and} \quad \| D_x f^{-1} |_{E^u_x} \| < 1,
\]
for some suitable Riemannian metric on \( M \). The stable distribution \( E^s \) and unstable distribution \( E^u \) are integrable to the stable and unstable foliations \( W^s \) and \( W^u \) respectively such that \( DW^s = E^s \) and \( DW^u = E^u \).

In 1985, from the measure theoretic point of view, Ledrappier and Young [14] gave a definition of unstable metric entropy for partially hyperbolic diffeomorphisms. And the entropy defined in [14] can be regarded as that given by \( H_\mu(\alpha|f\alpha) \), where \( \alpha \) is an increasing partition (i.e. \( \alpha \geq f\alpha \)) subordinate to the unstable leaves. In 2008, from the topological point of view, Hua, Saghin and Xia [9] provided the unstable volume growth. In 2017, Hu, Hua and Wu [7] introduced the definition of unstable metric entropy \( h^u_\mu(f) \) for any invariant measure \( \mu \), and gave the definition of the unstable topological entropy \( h^u_{\text{top}}(f) \). Similar to that in the classical entropy theory, the corresponding versions of Shannon-McMillan-Breiman theorem, and the variational principle relating \( h^u_\mu(f) \) and \( h^u_{\text{top}}(f) \) are given. The unstable metric entropy \( h^u_\mu(f) \) for an invariant measure \( \mu \) is defined by using
$H_\mu(\bigvee_{i=0}^{n-1} f^{-i}\xi|\eta)$, where $\xi$ is a finite measurable partition of the underlying manifold $M$, and $\eta$ is a measurable partition consisting of local unstable leaves that can be obtained by refining a finite partition into pieces of unstable leaves. In [7], they showed that the unstable metric entropy $h^u_\mu(f)$ is identical to the metric entropy defined in [13]. Hu, Wu, Zhu [8] introduced the notion of unstable topological pressure for a $C^1$-partially hyperbolic diffeomorphism $f : M \to M$ with respect to a continuous function on $M$, obtained a variational principle for this pressure. And they investigated the corresponding so-called $u$-equilibrium. Motivated by the work of Bowen [11] and Pesin [15], Tian and Wu [16] established a concept called Bowen unstable topological entropy which is the unstable topological entropy for any subsets (not necessarily compact or invariant) in partially hyperbolic systems as a Carathéodory dimension characteristic. In particular, they proved the Bowen unstable topological entropy of the whole space coincides with the unstable topological entropy of the system in [7]. And they also constructed some basic results in dimension theory for Bowen unstable topological entropy including a variational principle for any compact (not necessarily invariant) subset between its Bowen unstable topological entropy and unstable metric entropy of probability measures supported on this set. Wang, Wu and Zhu [17] introduced the unstable entropy and unstable pressure for a random dynamical system with $u$-domination. For random diffeomorphisms with domination, they also gave a version of Shannon-McMillan-Brieman theorem for unstable metric entropy, and obtained a variational principle for unstable pressure. In 2019, Wu [18] introduced two notions of local unstable metric entropies and the notion of local unstable topological entropy relative to a Borel cover $\mathcal{U}$ of $M$, and showed that when $\mathcal{U}$ is an open cover with small diameter, the entropies coincide with the unstable metric entropy and unstable topological entropy, respectively. And the unstable tail entropy and the unstable topological conditional entropy were also defined in [18].

In this paper, inspired by the ideas of Katok [13], for ergodic measures, we establish the Katok’s entropy formula of unstable metric entropy $h^u_\mu(f)$ for partially hyperbolic diffeomorphisms.

The following theorems present the main results of this paper.

**Theorem 1.1. (Katok’s entropy formula of unstable metric entropy)**

Let $M$ be an $n$-dimensional smooth, connected and compact Riemannian manifold without boundary and $f : M \to M$ be a $C^1$-partially hyperbolic diffeomorphism. Suppose $\mu$ is an ergodic measure of $f$. Let $\eta \in \mathcal{P}^u_{\varepsilon_0}$, and the measure disintegration of $\mu$ over $\eta$ is

$$\mu = \int \mu^u_x d\mu(x).$$

Then for any $0 < \delta < 1$, we have

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \inf \frac{\log N^u_{\mu^x}(n, \varepsilon, \delta)}{n} = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log N^u_{\mu^x}(n, \varepsilon, \delta)}{n} = h^u_\mu(f)$$

for $\mu$-a.e. $x \in M$, where $N^u_{\mu^x}(n, \varepsilon, \delta)$ denotes the minimal number of $d^u_n$-balls with radius $\varepsilon$ whose union has $\mu^u_x$-measure more than or equal to $1 - \delta$. 

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See Section 2 for the definitions of $d_n^\mu$-ball, $P_{\varepsilon_0}$ and $h_\mu^u(f)$.

Similarly, we have the following result:

**Theorem 1.2.** Let $M$ be an $n$-dimensional smooth, connected and compact Riemannian manifold without boundary and $f : M \to M$ be a $C^1$-partially hyperbolic diffeomorphism. Suppose $\mu$ is an ergodic measure of $f$. Let $\eta \in P_{\varepsilon_0}$, and the measure disintegration of $\mu$ over $\eta$ is

$$
\mu = \int \mu_\eta^u d\mu(x).
$$

Let $\xi \in P_{\varepsilon_0}$ satisfying $H_\mu(\xi|\eta) < \infty$. Then for any $0 < \delta < 1$, we have

$$
\lim_{n \to \infty} \log \frac{N_{\mu_\eta^u}(n, \xi, \delta)}{n} = h_\mu^u(f)
$$

for $\mu$-a.e. $x \in M$, where $N_{\mu_\eta^u}(n, \xi, \delta)$ denotes the minimal number of elements of the partition $\xi_{0}^{n-1}$ whose union has $\mu_\eta^u$-measure more than or equal to $1 - \delta$.

See Section 2 for the definitions of $\xi_{0}^{n-1}$ and $P_{\varepsilon_0}$.

The remainder of this paper is organized as follows. Section 2 gives some preliminaries. Section 3 provides the proof of Theorem 1.1. Finally, we prove Theorem 1.2 in Section 4.

## 2 Preliminaries

Let $M$ be an $n$-dimensional smooth, connected and compact Riemannian manifold without boundary and $f : M \to M$ be a $C^1$-diffeomorphism. From now on we always assume that $f$ is a $C^1$-partially hyperbolic diffeomorphism of $M$, and $\mu$ is an $f$-invariant probability measure.

We say that $\alpha$ is a measurable partition of $M$ if there exists some measurable set $M_0 \subset M$ with full measure such that, restricted to $M_0$,

$$
\alpha = \bigvee_{n=1}^\infty \alpha_n
$$

for some increasing sequence $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots$ of countable partitions. By $\alpha_i \leq \alpha_{i+1}$ we mean that every element of $\alpha_{i+1}$ is contained in some element of $\alpha_i$, or, equivalently, every element of $\alpha_i$ is a union of elements of $\alpha_{i+1}$. Then we say that $\alpha_i$ is coarser than $\alpha_{i+1}$ or, equivalently, $\alpha_{i+1}$ is finer than $\alpha_i$. Represent by $\bigvee_{n=1}^\infty \alpha_n$ the partition whose elements are the non-empty intersections of the form $\cap_{n=1}^\infty A_n$ with $A_n \in \alpha_n$ for each $n$. Equivalently, this is the coarser partition such that

$$
\alpha_i \leq \bigvee_{n=1}^\infty \alpha_n \text{ for every } i.
$$

It is easy to see that every countable partition is measurable. For a measurable partition $\xi$ of $M$ and $x \in M$, denote by $\xi(x)$ the element of $\xi$ containing $x$, and set

$$
\xi_{0}^{n-1} = \xi \bigvee f^{-1}\xi \bigvee \cdots \bigvee f^{-(n-1)}\xi.
$$
Let \( \varepsilon_0 \) be small enough and \( \mathcal{P}_{\varepsilon_0} \) denote the set of finite measurable partitions of \( M \) whose elements have diameters smaller than or equal to \( \varepsilon_0 \), that is, \( \text{diam}\xi := \sup\{\text{diam}A : A \in \xi\} \leq \varepsilon_0 \). For each \( \beta \in \mathcal{P}_{\varepsilon_0} \), we can define a finer partition \( \eta \) such that \( \eta(x) = \beta(x) \cap W^u_{\text{loc}}(x) \) for each \( x \in M \), where \( W^u_{\text{loc}}(x) \) denotes the local unstable manifold at \( x \) whose size is greater than the diameter \( \varepsilon_0 \) of \( \beta \). Note that for each \( x \in M \), \( W^u_{\text{loc}}(x) \subset W^u(x) \), where \( W^u(x) \) is the unstable manifold at \( x \), i.e.

\[
W^u(x) = \{ t \in M : \lim_{n \to \infty} d(f^{-n}x, f^{-n}t) = 0 \},
\]

where \( d \) is the distance on \( M \) generated by the Riemannian metric on \( M \).

Note that if \( \eta(x) = \beta(x) \cap W^u_{\text{loc}}(x) \), \( \eta(y) = \beta(y) \cap W^u_{\text{loc}}(y) \) and \( \eta(x) \cap \eta(y) \neq \emptyset \), then \( \eta(x) = \eta(y) \). In fact, choose any \( z \in \eta(x) \cap \eta(y) \). Since \( z \in \beta(x) \cap \beta(y) \), we obtain \( \beta(x) = \beta(y) \). For any \( t \in \eta(x) \), obviously, \( t \in \beta(x) \), so \( t \in \beta(y) \). And we also have \( \lim_{n \to \infty} d(f^{-n}t, f^{-n}x) = 0 \). Moreover, observe that \( \lim_{n \to \infty} d(f^{-n}y, f^{-n}z) = \lim_{n \to \infty} d(f^{-n}t, f^{-n}x) = \lim_{n \to \infty} d(f^{-n}t, f^{-n}y) = 0 \) and

\[
d(f^{-n}t, f^{-n}y) \leq d(f^{-n}t, f^{-n}x) + d(f^{-n}x, f^{-n}z) + d(f^{-n}z, f^{-n}y),
\]

we have \( \lim_{n \to \infty} d(f^{-n}t, f^{-n}y) = 0 \), so \( t \in W^u(y) \). Noting that \( t \in \beta(y) \) and the size of \( W^u_{\text{loc}}(y) \) is greater than \( \text{diam}(\beta(y)) \), we obtain \( t \in W^u_{\text{loc}}(y) \). Therefore, \( t \in \eta(y) = \beta(y) \cap W^u_{\text{loc}}(y) \). Then, we have \( \eta(x) \subset \eta(y) \). Similarly, \( \eta(y) \subset \eta(x) \). Hence, \( \eta(x) = \eta(y) \).

Clearly, \( \eta \) is a measurable partition satisfying \( \beta \leq \eta \). Let \( \mathcal{P}_{\varepsilon_0}^u \) denote the set of partitions \( \eta \) obtained by this way.

Denote by \( d^u \) the metric induced by the Riemannian structure on the unstable manifold. Given \( \varepsilon > 0 \) and \( x \in M \), let \( B^u(x, \varepsilon) \) denote the open ball centered at \( x \) with radius \( \varepsilon \) in the unstable manifold \( W^u(x) \) with respect to \( d^u \). For \( x \in M \), \( y \in W^u(x) \) and \( n \in \mathbb{N} \), let \( d^u_n(x, y) = \max\{d^u(f^i x, f^i y) : i = 0, 1, \ldots, n - 1\} \). Given \( \varepsilon > 0 \), \( x \in M \) and \( n \in \mathbb{N} \), let \( B^u_n(x, \varepsilon) = \{ y \in W^u(x) : d^u_n(x, y) < \varepsilon \} \) denote the \( d^u_n \)-ball centered at \( x \) with radius \( \varepsilon \), i.e. an \((n, \varepsilon)\) Bowen ball in \( W^u(x) \) centered at \( x \).

The following conclusion is used in the proof of the Lemma 4.4 in [7]. Let \( \gamma > 0 \) be small enough. There exists \( C > 1 \) such that for any \( x \in M \),

\[
d(y, z) \leq d^u(y, z) \leq Cd(y, z)
\]

for any \( y, z \in \overline{W^u(x, \gamma)} \), where \( W^u(x, \gamma) \) is the open ball inside \( W^u(x) \) centered at \( x \) with radius \( \gamma \) with respect to the metric \( d^u \) and \( \overline{W^u(x, \gamma)} \) is the closure of \( W^u(x, \gamma) \).

Then, we can obtain the following two lemmas.

**Lemma 2.1.** For \( \gamma > 0 \) small enough, \( B^u_n(x, \gamma) \subset B_n(x, \gamma) \) for any \( x \in M \).

**Proof.** Observe that

\[
B^u_n(x, \gamma) = \cap_{i=0}^{n-1} f^{-i}B^u(f^{i}x, \gamma).
\]

Noting that for \( \gamma > 0 \) small enough, for each \( x \in M \), \( y, z \in \overline{W^u(x, \gamma)} \), we have \( d(y, z) \leq d^u(y, z) \). For any \( y \in B^u_n(x, \gamma) \), \( f^i y \in B^u(f^i x, \gamma) \subset \overline{W^u(f^i x, \gamma)(i = 0, 1, \ldots, n - 1)} \), so

\[
d(f^i x, f^i y) \leq d^u(f^i x, f^i y) < \gamma,
\]

for \( i = 0, 1, \ldots, n - 1 \). Thus \( y \in B_n(x, \gamma) \). Hence \( B^u_n(x, \gamma) \subset B_n(x, \gamma) \). \( \square \)
Lemma 2.2. Given \( \varepsilon_0 > 0 \) small enough and \( \eta \in \mathcal{P}_{\varepsilon_0} \), there exists \( C > 1 \) such that for each \( x \in M, n \geq 1, 0 < \varepsilon < \varepsilon_0 \) and \( y \in \eta(x) \),

\[
B_n(y, \frac{\varepsilon}{C}) \cap \eta(x) \subset B_n^u(y, \varepsilon) \cap \eta(x).
\]

Proof. Since \( f : M \rightarrow M \) is a \( C^1 \)-diffeomorphism, for any \( x \in M \), there exists \( \lambda > 1 \) such that

\[
d^u(fy, fz) < \lambda d^u(y, z),
\]

for any \( y, z \in W^u(x) \). Note that for \( \gamma > 0 \) small enough, there exists \( C > 1 \) such that for any \( x \in M \),

\[
d(y, z) \leq d^u(y, z) \leq Cd(y, z)
\]

for any \( y, z \in \overline{W^u(x, \gamma)} \). Therefore, for any \( z \in B_n(y, \frac{\varepsilon}{C}) \cap \eta(x) \), we have

\[
d^u(y, z) \leq Cd(y, z) \leq \varepsilon_0.
\]

Since \( y, z \in W^u(x) \), \( d^u(y, z) \leq \varepsilon_0 \) and \( \varepsilon_0 \) is small enough,

\[
d^u(fy, fz) < \lambda d^u(y, z) < \lambda \varepsilon_0.
\]

Since \( \varepsilon_0 \) is small enough, \( \lambda \varepsilon_0 \) can be smaller than \( \gamma \). And note that \( f(y), f(z) \in W^u(fx) \), we have

\[
d^u(fy, fz) \leq Cd(fy, fz) \leq C \cdot \frac{\varepsilon}{C} = \varepsilon.
\]

By induction, we obtain

\[
d^u(f^iy, f^iz) \leq Cd(f^iy, f^iz) < \varepsilon,
\]

for \( i = 0, 1, \cdots, n - 1 \). Therefore \( z \in B_n^u(y, \varepsilon) \cap \eta(x) \). Thus

\[
B_n(y, \frac{\varepsilon}{C}) \cap \eta(x) \subset B_n^u(y, \varepsilon) \cap \eta(x).
\]

Recall that for a measurable partition \( \eta \) of \( M \) and a probability measure \( \nu \) on \( M \), the canonical system of conditional measures for \( \nu \) and \( \eta \) is a family of probability measures \( \{\nu_\eta^x : x \in M\} \) with \( \nu_\eta^x(\eta(x)) = 1 \), such that for every measurable set \( B \subset M \), \( x \mapsto \nu_\eta^x(B) \) is measurable and

\[
\nu(B) = \int_M \nu_\eta^x(B) d\nu(x).
\]

The following notions are standard. The information function of \( \xi \in \mathcal{P}_{\varepsilon_0} \) with respect to \( f \)-invariant probability measure \( \mu \) is defined as

\[
I_\mu(\xi)(x) := -\log \mu(\xi(x)),
\]
and the entropy of partition $\xi$ as

$$H_\mu(\xi) := \int_M I_\mu(\xi)(x) d\mu(x) = -\int_M \log \mu(\xi(x)) d\mu(x).$$

The conditional information function of $\xi \in P_{\epsilon_0}$ given $\eta \in P_{\epsilon_0}^u$ with respect to $\mu$ is defined as

$$I_\mu(\xi|\eta)(x) = -\log \mu_x^n(\xi(x)).$$

Then the conditional entropy of $\xi \in P_{\epsilon_0}$ given $\eta \in P_{\epsilon_0}^u$ with respect to $\mu$ is defined as

$$H_\mu(\xi|\eta) := \int_M I_\mu(\xi|\eta)(x) d\mu(x) = -\int_M \log \mu_x^n(\xi(x)) d\mu(x).$$

**Definition 2.1.** [7] For an $f$-invariant probability measure $\mu$, the conditional entropy of $f$ with respect to $\xi \in P_{\epsilon_0}$ given $\eta \in P_{\epsilon_0}^u$ is defined as

$$h_\mu(f, \xi|\eta) = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\xi_0^{n-1}|\eta).$$

The conditional entropy of $f$ given $\eta \in P_{\epsilon_0}^u$ with respect to $\mu$ is defined as

$$h_\mu(f|\eta) = \sup_{\xi \in P_{\epsilon_0}} h_\mu(f, \xi|\eta).$$

And the unstable metric entropy of $f$ with respect to $\mu$ is defined as

$$h^u_\mu(f) = \sup_{\eta \in P_{\epsilon_0}^u} h_\mu(f|\eta).$$

The following theorems will be used in proving the main results.

**Theorem 2.1.** [7] Suppose $\mu$ is an $f$-invariant probability measure. Then for any $\xi \in P_{\epsilon_0}$ and $\eta \in P_{\epsilon_0}^u$,

$$h^u_\mu(f) = h_\mu(f|\eta) = h_\mu(f, \xi|\eta).$$

**Theorem 2.2.** [7] Suppose $\mu$ is an ergodic measure of $f$. Let $\eta \in P_{\epsilon_0}^u$ be given. Then for any partition $\xi \in P_{\epsilon_0}$ with $H_\mu(\xi|\eta) < \infty$, we have

$$\lim_{n \to \infty} \frac{1}{n} I_\mu(\xi_0^{n-1}|\eta)(x) = h_\mu(f, \xi|\eta),$$

for $\mu$-a.e. $x \in M$.

The result in Theorem 2.2 is a version of Shannon-McMillan-Breiman theorem for the unstable metric entropy.
3 Proof of Theorem 1.1

This section gives the proof of Theorem 1.1.

Lemma 3.1. Suppose $\mu$ is an ergodic measure of $f$. Let $\eta \in \mathcal{P}_{\varepsilon_0}$ be given, and $\xi \in \mathcal{P}_{\varepsilon_0}$ with $H_\mu(\xi|\eta) < \infty$. Then there exists a set $M_1 \subset M$ with $\mu(M_1) = 1$, and for each $x \in M_1$, there exists $G_x \subset \eta(x)$ with $\mu^0_x(G_x) = 1$ such that

$$\lim_{n \to \infty} \frac{\log \mu^0_x(\xi^{n-1}(y))}{n} = h_\mu(f, \xi|\eta)$$

for each $y \in G_x$, where $\mu = \int \mu^0_x \, d\mu(x)$ is the measure disintegration of $\mu$ over $\eta$.

Proof. By Theorem 2.2, there exists a set $M_1 \subset M$ with $\mu(M_1) = 1$, such that

$$\lim_{n \to \infty} \frac{-\log \mu^0_x(\xi^{n-1}(x))}{n} = h_\mu(f, \xi|\eta)$$

for each $x \in M_1$. Then for each $x \in M_1$, we can find a set $G_x \subset \eta(x)$ with $\mu^0_x(G_x) = 1$ such that

$$\lim_{n \to \infty} \frac{-\log \mu^0_x(\xi^{n-1}(y))}{n} = h_\mu(f, \xi|\eta)$$

for each $y \in G_x$. Note that for each $y \in G_x$, $\mu^0_x = \mu^0_y$. Therefore, for each $x \in M_1$ and $y \in G_x$, we have

$$\lim_{n \to \infty} \frac{-\log \mu^0_x(\xi^{n-1}(y))}{n} = h_\mu(f, \xi|\eta).$$

Now, we are going to prove Theorem 1.1.

Proof. (1) Firstly, we are going to show that for every $0 < \delta < 1$, we have

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log N^u_{\mu^2}(n, \varepsilon, \delta)}{n} \leq h^u_\mu(f)$$

for $\mu$-a.e. $x \in M$.

For $0 < \varepsilon < \varepsilon_0$, let us choose a finite partition $\xi \in \mathcal{P}_{\varepsilon_0}$ with $\text{diam}(\xi) < \varepsilon/C$ and $H_\mu(\xi|\eta) < \infty$. Since $\varepsilon_0$ is small enough, by Lemma 2.2, for each $x \in M$, $n \geq 1$ and $y \in \eta(x)$, we have

$$\xi^{n-1}_0(y) \cap \eta(x) \subset B_n(y, \frac{\varepsilon}{C}) \cap \eta(x) \subset B^u_n(y, \varepsilon) \cap \eta(x).$$

Observe that $\mu$ is ergodic. According to Lemma 3.1, there exists a subset $M_1 \subset M$ with $\mu(M_1) = 1$ such that for any $x \in M_1$, there exists a set $G_x \subset \eta(x)$ with $\mu^0_x(G_x) = 1$ such that for any $y \in G_x$,

$$\lim_{n \to \infty} \frac{-\log \mu^0_x(\xi^{n-1}(y))}{n} = h_\mu(f, \xi|\eta).$$
Fix $x \in M_1$. For $n \in \mathbb{N}$ and $\gamma > 0$, set

$$Y_n = \{y \in G_x : \mu_x^n(\xi_0^{n-1}(y)) > \exp(-(h_\mu(f, \xi|\eta) + \gamma)n)\} \subset \bigcup_{V \in \mathcal{J}_n} V \cap \eta(x),$$

where $\mathcal{J}_n = \{V \in \xi_0^{n-1} : \mu_x^n(V) > \exp(-(h_\mu(f, \xi|\eta) + \gamma)n)\}$. Then for each $\gamma > 0$, $\lim_{n \to \infty} \mu_x^n(Y_n) = 1$. Thus, for sufficiently large $n \in \mathbb{N}$, we have $\mu_x^n(Y_n) > 1 - \delta$. Since

$$\# \mathcal{J}_n = \# \{V \in \xi_0^{n-1} : \mu_x^n(V) > \exp(-(h_\mu(f, \xi|\eta) + \gamma)n)\} \leq \exp((h_\mu(f, \xi|\eta) + \gamma)n),$$

the set $Y_n$ contains at most $\exp((h_\mu(f, \xi|\eta) + \gamma)n)$ elements of $\xi_0^{n-1} \cap \eta(x)$, where $\xi_0^{n-1} \cap \eta(x) = \{V \cap \eta(x) : V \in \xi_0^{n-1}\}$. Noting that for any $y \in \eta(x)$, $\xi_0^{n-1}(y) \cap \eta(x) \subset B_n(y, \varepsilon) \cap \eta(x)$, so $Y_n$ can be covered by $d_n^\varepsilon$-balls with radius $\varepsilon$ of the same number. So

$$N_{\mu_2}^u(n, \varepsilon, \delta) \leq \exp((h_\mu(f, \xi|\eta) + \gamma)n).$$

Then for any $\gamma > 0$,

$$\lim_{n \to \infty} \lim_{\varepsilon \to 0} \frac{\log N_{\mu_2}^u(n, \varepsilon, \delta)}{n} \leq h_\mu(f, \xi|\eta) + \gamma.$$

Since $\gamma$ can be taken arbitrarily small and by Theorem 2.1 $h_\mu(f, \xi|\eta) = h_\mu^u(f)$, we obtain

$$\lim_{n \to \infty} \lim_{\varepsilon \to 0} \frac{\log N_{\mu_2}^u(n, \varepsilon, \delta)}{n} \leq h_\mu^u(f)$$

for every $x \in M_1$. Noting that $\mu(M_1) = 1$, we have

$$\lim_{n \to \infty} \lim_{\varepsilon \to 0} \frac{\log N_{\mu_2}^u(n, \varepsilon, \delta)}{n} \leq h_\mu^u(f)$$

for $\mu$-a.e. $x \in M$.

(2) Secondly, we will turn to prove the second part of the theorem: for every $0 < \delta < 1$, we have

$$\lim_{n \to \infty} \lim_{\varepsilon \to 0} \frac{\log N_{\mu_2}^u(n, \varepsilon, \delta)}{n} \geq h_\mu^u(f)$$

for $\mu$-a.e. $x \in M$.

(i) Let $0 < \delta < 1$ be given. Let $\varepsilon > 0$, without loss of generality, we require additionally $\varepsilon^\frac{1}{4} < \frac{1 - \delta}{4}$. Let us choose a partition $\xi \in \mathcal{P}_{\varepsilon_0}$ with $\mu(\partial \xi) = 0$ and $H_\mu(\xi|\eta) < \infty$, where $\partial \xi$ denotes union of the boundaries $\partial B$ of all elements $B \in \xi$. For $\theta > 0$, let

$$U_\theta(\xi) = \{x \in M : \text{ the ball } B(x, \theta) \text{ is not contained in } \xi(x)\},$$

where $\xi(x)$ denotes the element of the partition $\xi$ containing $x$. Since $\bigcap_{\theta > 0} U_\theta(\xi) = \partial \xi$,

we have

$$\mu(U_\theta(\xi)) \to 0, \text{ as } \theta \to 0.$$
Therefore, there exists $0 < \gamma < \varepsilon$ such that $\mu(U_\theta(\xi)) \leq \varepsilon$ for any $0 < \theta \leq \gamma$. Using Birkhoff Ergodic Theorem, for $\mu$-a.e. $y \in M$ there exists $N_1(y) > 0$ such that for any $k \geq N_1(y)$,

$$
\frac{1}{k} \sum_{i=0}^{k-1} \chi_{U_\varepsilon(\xi)}(f^i(y)) \leq \varepsilon,
$$

where $\chi_{U_\varepsilon(\xi)}$ is the characteristic function of the set $U_\varepsilon(\xi)$. For $l \in \mathbb{N}^+$, we define

$$
D_l = \left\{ y \in M : \frac{1}{k} \sum_{i=0}^{k-1} \chi_{U_\varepsilon(\xi)}(f^i(y)) \leq \varepsilon \text{ for any } k \geq l \right\}.
$$

Clearly, the sets $D_l$ are nested and exhaust $M$ up to a set of $\mu$-measure zero. Therefore, there exists $l_0 > 1$ such that $\mu(D_l) \geq 1 - 2\sqrt{\varepsilon}$ for any $l \geq l_0$.

Define $M_l = \{ x \in M : \mu^0_x(D_l) \geq 1 - 2\varepsilon^\frac{1}{4} \}$, then $M_l^C = \{ x \in M : \mu^0_x(D_l^C) \geq 2\varepsilon^\frac{1}{4} \}$, where for any set $A \subset M$, $A^C$ is the complement of $A$. Using Chebyshev’s inequality, we obtain

$$
\mu(M_l^C) = \int \mu^0_x(M_l^C) d\mu(x) \leq \frac{\int \mu^0_x(D_l^C) d\mu(x)}{2\varepsilon^\frac{1}{4}} \leq \frac{\mu(D_l^C)}{2\varepsilon^\frac{1}{4}} \leq \frac{2\sqrt{\varepsilon}}{2\varepsilon^\frac{1}{4}} = \varepsilon^\frac{1}{2},
$$

for any $l \geq l_0$. Thus for any $l \geq l_0$, $\mu(M_l) \geq 1 - \varepsilon^\frac{1}{2}$. The sets $D_l$ are nested, i.e. $D_1 \subset D_2 \subset \cdots$. Then fix some $l_1 > l_0$, for any $x \in M_{l_1}$, $l \geq l_1$ we have

$$
\mu^0_x(D_l) \geq \mu^0_x(D_{l_1}) \geq 1 - 2\varepsilon^\frac{1}{4}. \quad (1)
$$

According to Lemma [3.1], we can find a subset $M_1 \subset M$ with $\mu(M_1) = 1$ such that for any $x \in M_1$, there exists set $G_x$ with $\mu^0_x(G_x) = 1$ such that for any $y \in G_x$,

$$
\lim_{n \to \infty} \frac{-\log \mu^0_x(\xi_0^{n-1}(y))}{n} = h_\mu(f, \xi|\eta).
$$

Let $I = M_1 \cap M_{l_1}$. Clearly, $\mu(I) \geq 1 - \varepsilon^\frac{1}{2}$.

(ii) For $n \in \mathbb{N}$ and given a point $y \in M$, we call the collection

$$
C(n, y) := (\xi(y), \xi(f(y)), \cdots, \xi(f^{n-1}(y))
$$

the $(\xi, n)$-name of $y$. Since each point in one element $V$ of $\xi_0^{n-1}$ has the same $(\xi, n)$-name, we can define

$$
C(n, V) := C(n, y)
$$

for any $y \in V$, which is called the $(\xi, n)$-name of $V$.

For $n \in \mathbb{N}$ and $\xi$, we give a metric $d_n^\xi$ between $(\xi, n)$-names of $y$ and $z$ as follows:

$$
d_n^\xi(C(n, y), C(n, z)) = \frac{1}{n} \#\{0 \leq i \leq n-1 : \xi(f^i(y)) \neq \xi(f^i(z))\}.
$$

It can also be viewed as a semi-metric on $M$. 

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Fix $\hat{x} \in I$ and $l_2 \geq l_1$. According to Lemma 2.1, for $\gamma > 0$ small enough, $B^u_n(y, \gamma) \subset B_n(y, \gamma)$, for any $y \in M$. If $z \in B(y, \gamma)$, then either $y$ and $z$ belong to the same element of $\xi$ or $y \in U_\gamma(\xi)$, $z \notin \xi(y)$. Hence if $y \in D_{l_2}$, $n \geq l_2$ and $z \in B^u_n(y, \gamma)$, the distance $d^\xi_n$ between $(\xi, n)$-names of $y$ and $z$ does not exceed $\varepsilon$, i.e.

$$d^\xi_n(C(n, y), C(n, z)) \leq \varepsilon.$$  

Furthermore, for $y \in D_{l_2}$, $n \geq l_2$, $B^u_n(y, \gamma)$ is contained in the set of points $z$ whose $(\xi, n)$-names are $\varepsilon$-close to the $(\xi, n)$-name of $y$, i.e.

$$B^u_n(y, \gamma) \subset B^\xi_n(y, \varepsilon).$$  

(2)

By Stirling’s formula, there exists a large number $l_3 \in \mathbb{N}$ and for any $n \geq l_3$, it can be shown that the total number $K_n$ of such $(\xi, n)$-names consisting of $B^\xi_n(y, \varepsilon)$ admits the following estimate:

$$K_n \leq \sum_{j=0}^{[ne]} C_j^j (\#\xi - 1)^j \leq \sum_{j=0}^{[ne]} C_{l_3}^j (\#\xi)^j \leq \exp((\varepsilon + \diamond)n),$$

(3)

where

$$\diamond = \varepsilon \log(\#\xi) - \varepsilon \log \varepsilon - (1 - \varepsilon) \log(1 - \varepsilon).$$

For $n \geq \max\{l_2, l_3\}$, set

$$U := \left\{ B^u_n(y_i, \frac{\gamma}{2}) : i = 1, 2, \cdots, N^u_{\mu_\gamma}(n, \frac{\gamma}{2}, \delta) \right\}$$

with $\mu_\gamma(F_n) > 1 - \delta$, where

$$F_n := \bigcup_{i=1}^{N^u_{\mu_\gamma}(n, \frac{\gamma}{2}, \delta)} B^u_n(y_i, \frac{\gamma}{2}).$$

According to (1) and $\varepsilon^{\frac{1}{4}} < \frac{1 - \delta}{4}$, then $\mu_\gamma(F_n \cap D_n) > 1 - \delta - 2\varepsilon^{\frac{1}{4}} > \frac{1 - \delta}{2}$. For $i = 1, 2, \cdots, N^u_{\mu_\gamma}(n, \frac{\gamma}{2}, \delta)$, if $B^u_n(y_i, \frac{\gamma}{2}) \cap D_n \neq \emptyset$, we choose any $z_i \in B^u_n(y_i, \frac{\gamma}{2}) \cap D_n$. Then we apply the relation (2), so we have

$$B^u_n(y_i, \frac{\gamma}{2}) \cap D_n \subset B^u_n(z_i, \gamma) \subset B^\xi_n(z_i, \varepsilon).$$

Thus,

$$F_n \cap D_n \subset S_n,$$

where

$$S_n = \bigcup_{i:B^u_n(y_i, \frac{\gamma}{2}) \cap D_n \neq \emptyset} B^\xi_n(z_i, \varepsilon).$$
Let 
\[ P_n = \left\{ V \in \xi_{0}^{n-1} : d_n^\xi(C(n, V), C(n, z_i)) < \varepsilon, \text{ for some } i = 1, 2, \cdots, N_{\mu^2}^u(n, \frac{\gamma}{2}, \delta) \right\}. \]

It is clear that 
\[ S_n = \bigcup_{V \in P_n} V \]
and 
\[ \#P_n \leq N_{\mu^2}^u(n, \frac{\gamma}{2}, \delta) \cdot K_n. \]

By Lemma 3.1 and Egorov Theorem, there exists a large number \( l_4 > \max\{l_2, l_3\} \) such that, \( \mu^u_2(T_n) \geq \frac{1}{4} \) for each \( n \geq l_4 \), where 
\[ T_n = \{ y \in S_n : \mu^u_2(\xi_{0}^{n-1}(y)) \leq \exp(-(h_{\mu}(f, \xi|\eta) - \varepsilon)n) \}. \]

Write \( t_n := \#\{\xi_{0}^{n-1}(y) : y \in T_n\} \). Then 
\[ \frac{(1 - \delta) \exp((h_{\mu}(f, \xi|\eta) - \varepsilon)n)}{4} \leq t_n \leq \#P_n \leq N_{\mu^2}^u(n, \frac{\gamma}{2}, \delta) \cdot K_n. \]

Hence, we have 
\[ N_{\mu^2}^u(n, \frac{\gamma}{2}, \delta) \geq \frac{(1 - \delta) \exp((h_{\mu}(f, \xi|\eta) - \varepsilon)n)}{4K_n}. \]

Noting that by Theorem 2.1 \( h_{\mu}(f, \xi|\eta) = h_{\mu}^u(f) \), and using (3), we obtain 
\[ \liminf_{n \to \infty} \frac{\log N_{\mu^2}^u(n, \frac{\gamma}{2}, \delta)}{n} \geq h_{\mu}(f, \xi|\eta) - \varepsilon - \limsup_{n \to \infty} \frac{\log K_n}{n} + \lim_{n \to \infty} \frac{1 - \delta}{n} \log \frac{1}{4} \]
\[ \geq h_{\mu}(f, \xi|\eta) - \varepsilon - (\varepsilon + \hat{\diamond}) \]
\[ = h_{\mu}^u(f) - \varepsilon - (\varepsilon + \hat{\diamond}) \]
\[ = h_{\mu}^u(f) - 2\varepsilon - \hat{\diamond}. \]

Let \( \varepsilon \to 0 \). Since \( \gamma < \varepsilon, \lim_{\varepsilon \to 0} \hat{\diamond} = 0, \hat{x} \in I \) and \( \mu(I) \geq 1 - \varepsilon^\frac{1}{2} \), we have 
\[ \lim \liminf_{\gamma \to 0} \frac{\log N_{\mu^2}^u(n, \frac{\gamma}{2}, \delta)}{n} \geq h_{\mu}^u(f), \]
for \( \mu \)-a.e. \( \hat{x} \in M \).

Therefore, for every \( 0 < \delta < 1 \), we obtain 
\[ \lim \liminf_{\varepsilon \to 0} \frac{\log N_{\mu^2}^u(n, \varepsilon, \delta)}{n} \geq h_{\mu}^u(f) \]
for \( \mu \)-a.e. \( x \in M \). \( \square \)
4 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2.

Proof. (1) Firstly, we will prove that for each $0 < \delta < 1$,

$$\limsup_{n \to \infty} \frac{\log N_{\mu_2}^{u}(n, \xi, \delta)}{n} \leq h_\mu^{u}(f)$$

for $\mu$-a.e. $x \in M$.

Noting that $\mu$ is ergodic and according to Lemma 3.1, there exists a subset $M_1 \subset M$ with $\mu(M_1) = 1$ such that for any $x \in M_1$, there exists a set $G_x \subset \eta(x)$ with $\mu_{\eta}(G_x) = 1$ such that for any $y \in G_x$,

$$\lim_{n \to \infty} -\frac{\log \mu^{u}(\xi_{n}^{\eta} - 1(y))}{n} = h_\mu(f, \xi|\eta).$$

Fix $x \in M_1$. For $n \in \mathbb{N}$ and $\gamma > 0$, set

$$Y_n = \{y \in G_x : \mu_{x}^{u}(\xi_{n}^{\eta} - 1(y)) > \exp(-(h_\mu(f, \xi|\eta) + \gamma)n)\} \subset \bigcup_{V \in \mathcal{J}_n} V \cap \eta(x),$$

where $\mathcal{J}_n = \{V \in \xi_{n}^{\eta} - 1 : \mu_{x}^{u}(V) > \exp(-(h_\mu(f, \xi|\eta) + \gamma)n)\}$. Then for each $\gamma > 0$, $\lim_{n \to \infty} \mu_{x}^{u}(Y_n) = 1$. Thus, for sufficiently large $n \in \mathbb{N}$, we have $\mu_{x}^{u}(Y_n) > 1 - \delta$. Since

$$\# \mathcal{J}_n = \# \{V \in \xi_{n}^{\eta} - 1 : \mu_{x}^{u}(V) > \exp(-(h_\mu(T, \xi|\eta) + \gamma)n)\} \leq \exp((h_\mu(f, \xi|\eta) + \gamma)n),$$

the set $Y_n$ contains at most $\exp((h_\mu(f, \xi|\eta) + \gamma)n)$ elements of $\xi_{n}^{\eta} - 1 \cap \eta(x)$, where $\xi_{n}^{\eta} - 1 \cap \eta(x) = \{V \cap \eta(x) : V \in \xi_{n}^{\eta} - 1\}$. Thus

$$N_{\mu_2}^{u}(n, \xi, \delta) \leq \exp((h_\mu(f, \xi|\eta) + \gamma)n).$$

Then for any $\gamma > 0$,

$$\limsup_{n \to \infty} \frac{\log N_{\mu_2}^{u}(n, \xi, \delta)}{n} \leq h_\mu(f, \xi|\eta) + \gamma.$$
for \( \mu \)-a.e. \( x \in M \).

(2) Secondly, we will turn to prove that for every \( 0 < \delta < 1 \),

\[
\liminf_{n \to \infty} \frac{\log N_{\mu_x}^u(n, \xi, \delta)}{n} \geq h_{\mu}^u(f)
\]

for \( \mu \)-a.e. \( x \in M \).

Given \( \eta \in \mathcal{P}_{\varepsilon_0}^u \) and \( \xi \in \mathcal{P}_{\varepsilon_0}^u \) with \( H_{\mu}(\xi|\eta) < \infty \). By Lemma 3.1, we can find a subset \( M_1 \subseteq M \) with \( \mu(M_1) = 1 \) such that for any \( x \in M_1 \), there exists set \( G_x \) with \( \mu_\eta^x \)(G_x) = 1 such that for any \( y \in G_x \),

\[
\lim_{n \to \infty} \frac{\log \mu_\eta^x(\xi_{n-1}(y))}{n} = h_{\mu}(f, \xi|\eta).
\]

Now, fix \( \hat{x} \in M_1 \). For \( n \in \mathbb{N} \), set

\[
U := \left\{ \xi_{n-1}(y_i) : i = 1, 2, \cdots, N_{\mu_{\hat{x}}}^u(n, \xi, \delta) \right\}
\]

with \( \mu_\hat{x}^u(F_n) > 1 - \delta \), where

\[
F_n := \bigcup_{i=1}^{N_{\mu_{\hat{x}}}^u(n, \xi, \delta)} \xi_{n-1}(y_i).
\]

Let \( \varepsilon > 0 \) small enough. By Lemma 3.1 and Egorov Theorem, there exists a large number \( N \in \mathbb{N} \) such that, \( \mu_\eta^{\hat{x}}(T_n) \geq (1 - \delta)/2 \) for each \( n \geq N \), where

\[
T_n = \{ y \in F_n : \mu_\hat{x}^u(\xi_{n-1}(y)) \leq \exp(-(h_{\mu}(f, \xi|\eta) - \varepsilon)n) \}.
\]

Write \( t_n := \#\{\xi_{n-1}(y) : y \in T_n\} \). Then

\[
\frac{(1 - \delta) \exp((h_{\mu}(f, \xi|\eta) - \varepsilon)n)}{2} \leq t_n \leq N_{\mu_\eta^x}^u(n, \xi, \delta).
\]

Noting that by Theorem 2.1, \( h_{\mu}(f, \xi|\eta) = h_{\mu}^u(f) \), we obtain

\[
\liminf_{n \to \infty} \frac{\log N_{\mu_x}^u(n, \xi, \delta)}{n} \geq h_{\mu}(f, \xi|\eta) - \varepsilon + \lim_{n \to \infty} \frac{1}{n} \log \frac{1 - \delta}{2}
\]

\[
= h_{\mu}(f, \xi|\eta) - \varepsilon
\]

\[
= h_{\mu}^u(f) - \varepsilon.
\]

Let \( \varepsilon \to 0 \), we have

\[
\liminf_{n \to \infty} \frac{\log N_{\mu_x}^u(n, \xi, \delta)}{n} \geq h_{\mu}^u(f).
\]

Note that \( \hat{x} \in M_1 \) and \( \mu(M_1) = 1 \). Therefore, for every \( 0 < \delta < 1 \), we obtain

\[
\liminf_{n \to \infty} \frac{\log N_{\mu_x}^u(n, \xi, \delta)}{n} \geq h_{\mu}^u(f)
\]

for \( \mu \)-a.e. \( x \in M \).
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References

[1] R. Bowen, Topological entropy for noncompact sets, Trans. Amer. Math. Soc., 184 (OCT), (1973), 125-136.
[2] Y. Cao, H. Hu, Y. Zhao, Nonadditive measure-theoretic pressure and applications to dimensions of an ergodic Measure, Ergod. Theory Dyn. Syst., 33 (3), (2013), 831-850.
[3] W. Cheng, Y. Zhao, and Y. Cao, Pressures for asymptotically sub-additive potentials under a mistake function, Discrete Contin. Dyn. Syst. A, 32 (2), (2013), 487-497.
[4] H. Furstenberg, Reccurence in ergodic theory and combinatorial number theory, Princeton University Press, Princeton, New Jersey, 1981.
[5] B. Hasselblatt and A. Katok, Principal structures, Handbook of dynamical systems, Vol. 1A, North-Holland, Amsterdam, 2002, 1-203.
[6] L. He, J. Lv and L. Zhou, Definition of measure-theoretic pressure using spanning sets, Acta Math. Sinica, Engl. Ser., 20 (4), (2004), 709-718.
[7] H. Hu, Y. Hua and W. Wu, Unstable entropies and variational principle for partially hyperbolic diffeomorphisms, Adv. Math., 321 (2017), 31-68.
[8] H. Hu, W. Wu and Y. Zhu, Unstable pressure and u-equilibrium states for partially hyperbolic diffeomorphisms, arXiv:1710.02816
[9] Y. Hua, R. Saghin and Z. Xia, Topological entropy and partially hyperbolic diffeomorphisms, Ergodic Theory and Dynamical Systems, 28 (3), (2008), 843-862.
[10] P. Huang, E. Chen and C. Wang, Entropy formulae of conditional entropy in mean metrics, Discrete Contin. Dyn. Syst. Ser. A, 38 (10), 2018, 5129-5144.
[11] P. Huang and C. Wang, Measure-theoretic pressure and topological pressure in mean metrics, Dyn. Syst., 34 (2), (2019), 259-273.
[12] W. Huang, Z. Wang and X. Ye, Measure complexity and Möbius disjointness, Adv. Math., 347 (2019), 827-858.
[13] A. Katok, Lyapunov exponents, entropy and periodic orbits for dffoemorphisms, Publ. IHES 51 (1980), 137-173.
[14] F. Ledrappier and L. Young, The metric entropy of diffeomorphisms: part II: relations between entropy, exponents and dimension, Annals of Mathematics (1985), 540-574.
[15] Y. Pesin and B. Pitskel, Topological pressure and the variational principle for noncompact sets, Funct. Anal. Appl., 18 (1984), 307-318.
[16] X. Tian and W. Wu, Unstable entropies and dimension theory of partially hyperbolic systems, arXiv:1811.03797
[17] X. Wang, W. Wu and Y. Zhu, Unstable entropy and unstable pressure for random diffeomorphisms with domination, arXiv:1811.12674.

[18] W. Wu, Local unstable entropies of partially hyperbolic diffeomorphisms, doi: 10.1017/etds.2019.3.

[19] Y. Zhao and Y. Cao, Measure-theoretic pressure for subadditive potentials, Nonlinear Analysis, 70 (2009), 2237-2247.

[20] D. Zheng, E. Chen and J. Yang, On large deviations for amenable group actions, Discrete Contin. Dyn. Syst. Ser. A, 36 (12), (2017), 7191-7206.

[21] Y. Zhu, Two notes on measure-theoretic entropy of random dynamic systems, Acta Math. Sin., 25 (2009), 961-970.