CORRESPONDENCE PRINCIPLE FOR THE SPIN LIGHT THEORY

V A Bordovitsyn, O A Konstantinova
Tomsk State University, Lenin Ave. 36, 634050 Tomsk, Russia
E-mail: vabord@sibmail.com, olgakonst@sibmail.com

Abstract. In this work a precise quantum analogue of the spin precession of the classical theory is obtained. As an example a neutral Dirac’s particle (neutron) is considered. It is shown that classical and quantum theories lead to an adequate description of the spin precession.

1. Introduction
It is known that the classical Bargmann-Michel-Telegdi (BMT) equation precisely describes a relativistic electrons motion in the external electromagnetic fields [1] (see also [2]). This very equation is frequently used in the precision experiments of measuring the anomaly magnetic momentum of electron and other relativistic particles [3, 4]. Together with this the most precise spin precession description is possible only using the Dirac’s equation of the Relativistic Quantum Theory. Furthermore spin processes are described in terms of the quantum transitions with a spin direction change. This method is quite different from the classical method of description of the spin precession [2, 5, 9]. Thus a question of equivalence of the results of both theories arises.

The goal of this work is to find a connection between quantum spin-flip transitions and classical spin precession theory. Particularly considering a precession of the neutral Dirac’s particle (neutron) the equivalence of both methods is shown. To begin, we consider the BMT equation for the neutron moving in the homogeneous magnetic field. To compare the results we solve the Dirac-Pauli equation for the neutron with the same initial conditions configuration. Then we find the average values of the spin. The non-stationary wave function is based on the spin-flip states of the particle (originally this idea belongs to [6], see also [7]).

2. Spin precession in the classical theory
Let us learn the spin precession itself by using a neutron which has an anomalous magnetic moment (the electric charge is equal to zero) \( \mu = -1.91 \mu_n \), where \( \mu_n = e_0 \hbar / 2m_p c \) - nuclear Bohr’s magneton, \( m_p \) is the mass of the proton, the rest of the values are equal to universally recognized.

a) Spin precession description on the basis of the tensor BMT equation
The neutron’s motion is considered in the homogeneous magnetic field. Thus the charge motion and the spin precession are independent. The tensor BMT equation [1] for neutron is [2]

\[
\frac{d\Pi^{\mu\nu}}{d\tau} = \frac{[\mu]}{s\hbar} \left( H^{[\mu\nu} \Pi_{\rho]} + \frac{1}{c^2} \nu^{[\mu} \Pi^{\nu]} H_{\rho\sigma} \nu^{\sigma} \right)
\]

Here \( \Pi^{\mu\nu} = (\Phi, \Pi) \) is a space-like spin tensor, which satisfies the condition \( \Pi^{\mu\nu} \nu_\nu = 0 \), \( H^{\mu\nu} = (-E, H) \) is an external electromagnetic field tensor, \( \nu^\mu = (\nu_0, \nu) = c\gamma(1, \beta) \) - four-dimensional vector of the velocity, \( \gamma = \sqrt{1 - \beta^2}, s = 1/2 \). Here a square brackets denote an antisymmetrization: \( a^{[a} b^{c]} = a^a b^c - a^c b^a \).
Let us assume $E = 0, \ H = (0, 0, H)$. Due to the space-likeness condition $\Phi = [\beta \Pi]$, (1.1) turns to

$$\frac{d\Pi}{d\tau} = -\frac{|\mu|}{\hbar} \left( \Pi + \frac{1}{c^2} v(\Pi v) \right) \mathbf{H}. \quad (1.2)$$

It should be noted due to (1.1) and (1.2) we can use a dimensionless form. So, tensor $\Pi^{\mu \nu}$ has an invariant

$$\frac{1}{2} \Pi^{\mu \nu} \Pi_{\mu \nu} = \zeta^2, \quad (1.3)$$

where $\zeta$ is a unit spin vector in the rest frame.

In the case of the uniform motion of the neutron $\beta = \beta(\sin \alpha, 0, \cos \alpha)$, equation system (1.2) takes on the following form

$$\frac{d\Pi^x}{d\tau} = -h\Pi_y, \quad \frac{d\Pi^y}{d\tau} = h\left\{ \gamma^2 (1 - \beta^2 \cos^2 \alpha) \Pi_x + \gamma^2 \beta^2 \cos \alpha \sin \alpha \Pi_z \right\}, \quad \frac{d\Pi^z}{d\tau} = 0; \quad (1.4)$$

where $h = |\mu| H / \hbar$. We will search the solution with an initial condition $\Pi_i = (0, \Pi_1, 0)$. Then we have

$$\Pi_x = -\frac{\Pi}{\gamma \sqrt{1 - \beta^2 \cos^2 \alpha}} \sin (\omega t + \eta), \quad \Pi_y = \Pi_1 \cos (\omega t + \eta), \quad \Pi_z = 0; \quad (1.5)$$

spin precession frequency $\omega$ has a form

$$\omega = \frac{|\mu| H}{\sigma \hbar} \sqrt{1 - \beta^2 \cos^2 \alpha}. \quad (1.6)$$

Amplitude $\Pi_1$ can be obtained using (1.3). According to the initial condition $\Pi_1 = \gamma \zeta$ and

$$\Pi_x = -\frac{\zeta}{\sqrt{1 - \beta^2 \cos^2 \alpha}} \sin \omega t, \quad \Pi_y = \gamma \zeta \cos \omega t, \quad \Pi_z = 0; \quad (1.7)$$

b) Spin precession description on the basis of the Lorentz transformations

Let us consider spin in the rest frame and magnetic field - in the laboratory system. BMT equation in the rest frame is the follows expression

$$\frac{d\zeta^x}{d\tau} = -\frac{|\mu|}{\sigma \hbar} [\zeta, \mathbf{H}_0], \quad (1.8)$$

where $\zeta$ is the unit spin vector. The magnetic field in the laboratory system turns to

$$\mathbf{H}_0 = \gamma \left\{ \mathbf{E} - \frac{\gamma}{\gamma + 1} \mathbf{B}(\mathbf{B}\mathbf{H}) \right\}. \quad (1.9)$$

Taking the initial condition into consideration (1.8) can be written as

$$\frac{d\zeta_x}{d\tau} = -h \left( \gamma \sin^2 \alpha + \cos^2 \alpha \right) \zeta_y, \quad \frac{d\zeta_y}{d\tau} = h \left( \zeta_x (\gamma \sin^2 \alpha) + \zeta_z (\gamma - 1) \sin \alpha \cos \alpha \right), \quad \frac{d\zeta_z}{d\tau} = -h \zeta_y (\gamma - 1) \sin \alpha \cos \alpha, \quad (1.10)$$

2
and the solution is equal to
\[
\begin{align*}
\zeta_x &= -\zeta \gamma \sin^2 \alpha + \cos^2 \alpha \sin(\omega t + \eta), \\
\zeta_y &= \zeta \cos(\omega t + \eta), \\
\zeta_z &= -\zeta \frac{(y-1) \cos \alpha \sin \alpha}{\gamma \sqrt{1 - \beta^2 \cos^2 \alpha}} \sin(\omega t + \eta)
\end{align*}
\] (1.11)

with the same precession frequency as in (1.6).

It is easy to check these results by placing (1.11) into Lorentz transformations for the spin
\[
\Pi = \gamma \zeta - \gamma^2 (\gamma + 1) \beta (\beta, \xi).
\] (1.12)

Thus we obtain the solutions (1.7). And the inverse Lorentz transformation
\[
\zeta = \frac{1}{\gamma} \Pi + \gamma^2 (\gamma + 1) \beta (\beta, \Pi),
\] (1.13)
naturally leads to (1.11).

3. Spin precession in the quantum theory

To make sure that the classical method is absolutely correct let us discover the neutron motion in the same initial conditions, but using the quantum theory of the relativistic particles.

Dirac’s equation in the orthogonal fields takes a form [1]
\[
\left( m_0 c - i \gamma \mu p_\mu - \frac{[\mu]}{2c} \sigma_{\alpha \beta} H^{\alpha \beta} \right) \Psi = 0.
\] (2.1)

Here \( p_\mu = -ih \partial_\mu \) is an operator of the four-dimensional impulse, \( \gamma \mu = i \rho_3 (1, \alpha) \), \( \sigma^{\mu \nu} = (-i \alpha, \sigma) \) - Dirac’s matrixes, \( \Psi(r, t) \) - four-dimensional spinor, which has a form of the non-stationary wave function
\[
\Psi(r, t) = L^{3/2} \left( \begin{array}{c} c_1 \\ c_2 e^{i \phi} \\ c_3 \\ c_4 e^{-i \phi} \end{array} \right) \exp \left( \frac{im_0 c}{\hbar} \left[ (b \cdot r) - \gamma \zeta c t \right] \right) = \psi_\zeta(r) \exp(-\frac{im_0 c}{\hbar} \gamma \zeta t),
\] (2.2)

where \( b^\mu = \gamma (1, \beta) \) is a dimensionless analogue of the four-dimensional impulse of the particle.

Let us assume in the following \( \nu = \mu H / m_0 c^2 \), \( b_\phi = b_\perp + i b_\parallel = b_\perp e^{i \phi} \). Placing (2.2) into (2.1) we obtain a homogeneous system of equations with constant coefficients
\[
\begin{align*}
(\gamma \zeta - 1 - \nu) c_1 - b_\perp c_4 - b_\parallel c_3 &= 0, \\
(\gamma \zeta - 1 + \nu) c_2 - b_\perp c_1 + b_\parallel c_4 &= 0, \\
(\gamma \zeta + 1 + \nu) c_3 - b_\perp c_2 - b_\parallel c_1 &= 0, \\
(\gamma \zeta + 1 - \nu) c_4 + b_\perp c_3 + b_\parallel c_2 &= 0.
\end{align*}
\] (2.3)

Using the condition of nontriviality of this system of the homogeneous equations with constant coefficients (the determinant of the system is equal to zero) we find the following
\[
\gamma \zeta = \sqrt{\beta^2 + \left( \sqrt{1 + \beta^2} + \zeta \nu \right)^2}.
\] (2.4)

Due to the normalization \( c^+ c = 1 \) coefficients \( c_i \) of (2.3) take a form of
where we denote: $q = \sqrt{1 + \gamma^2 b_{-1}^2} = \gamma \sqrt{1 - \beta^2 \cos^2 \alpha}$, and as usual $\gamma = 1/\sqrt{1 - \beta^2}$.

In the quantum theory the spin precession is described as a non-stationary process using a wave function. This wave function is a superposition of the spin stations $\zeta = \pm 1$:

$$
\tilde{\Psi}(r, t) = A\psi_{\uparrow}(r)\exp\left(-i\frac{m_0 c^2}{\hbar} \gamma r t\right) + B\psi_{\downarrow}(r)\exp\left(-i\frac{m_0 c^2}{\hbar} \gamma r t\right), \quad A^+ A + B^+ B = 1. \quad (2.6)
$$

The quantum number $\zeta = \pm 1$ characterizes the spin projection on the magnetic field direction. Numeric coefficients $A$ and $B$ are determined by the initial conditions, which are given by the projection of the spin operator (see [1]):

$$
\Pi = \rho_{2}\left[\sigma, b\right] + \sigma, \quad (2.7)
$$
on the arbitrary direction $n = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$:

$$
\left(\Pi n\right)\tilde{\Psi}(r, 0) = \lambda \tilde{\Psi}(r, 0), \quad (2.8)
$$

where

$$
\tilde{\Psi}(r, 0) = A\psi_{\uparrow}(r) + B\psi_{\downarrow}(r). \quad (2.9)
$$

It should be noted that the spin projection itself isn’t a motion integral. The initial conditions correspond to the non-stationary wave function with $t = 0$.

For $b = b(\sin \alpha, 0, \cos \alpha)$ the matrix $(\Pi_1 n)$ in the case $\Pi_1 = (0, \Pi_1, 0), \theta = \pi/2, \varphi = \pi/2$ takes a form of

$$
\Pi_1 = \begin{pmatrix}
0 & -i & -ib \cos \alpha & ib \cos \alpha \\
1 & 0 & ib \cos \alpha & ib \cos \alpha \\
ib \cos \alpha & -ib \cos \alpha & 0 & -i \\
-ib \cos \alpha & ib \cos \alpha & i & 0
\end{pmatrix} \quad (2.10)
$$

If the initial orientation coincides with the $Y$ axis we have

$$
\lambda = \zeta \gamma \quad A = \frac{i}{\sqrt{2}} \quad B = \frac{\zeta}{\sqrt{2}} \quad (2.11)
$$

Note here that we considered this very case in the classical method.

The calculation of the average values of the spin operator (2.7) needs to determine matrix elements.
The average values $\langle \hat{\Pi} \rangle_t$ depend on the initial orientation of the spin. The initial conditions we consider give the following relations

$$
\langle \hat{\Pi}_x \rangle_t = -\frac{\zeta \sin \omega t}{\sqrt{1 - \beta^2 \cos^2 \alpha}}, \quad \langle \hat{\Pi}_y \rangle_t = \zeta \cos \omega t, \quad \langle \hat{\Pi}_z \rangle_t = 0.
$$

(2.13)

It is obvious that these solutions are in agreement with the solutions of the tensor BMT equation (1.7). The frequency of the spin precession corresponds to the quantum spin-flip transition $\zeta \rightarrow -\zeta$. It is equals to

$$
\omega = \frac{mc^2}{\hbar} (\gamma_\zeta - \gamma_{-\zeta}) = \zeta \frac{2|\mu|H}{\hbar} \sqrt{1 - \beta^2 \cos^2 \alpha},
$$

(2.14)

and (1.6) is also the same.

4. Conclusion

Thus, in this work, we show the equivalence on the classical and the quantum theories relative to the spin precession description. The consequence of our equations gives the main factor of the spin-flip transitions. It is the crosscut part of the classical spin vector. It should be noted that this conclusion can be proved using the solutions of the Heisenberg equations of the spin operator’s motion [8].

This paper is one more confirmation of the fundamental conformity principle and the unity of the laws of nature in the description of different physical phenomena. The method at hand can be used to prove the calculations of spin processes.

The work was partially supported by the Federal Targeted Program «Scientific and scientific – pedagogical personnel of innovative Russia», contract № 02.740.11.0238; № II789.

References
[1] Bargmann V, Michel L, Telegdi V L 1959 Phys. Rev. Lett. 2 435.
[2] Synchrotron Radiation Theory and it’s Development 1999 In Memory of I M Ternov. Ed. V A Bordovitsyn. (World Scientific, Singapore).
[3] Methods of the determination of the main characteristics atomic nuclei and elementary particles. Mass, spin, parity, polarization and lifetime of the particles 1963 Yuan and Chien-Shiung Wu (Publ. Acad. Press, New York).
[4] Tests of Fundamental Physical Theories from Measurements of Free Charged Leptons 1978 Field J H, Picasso E, Combley F (Geneva: CERN).
[5] Introduction to the spin particles physics 1997 Ternov I M (M. MGU press).
[6] Ternov I M, Bagrov V G, Hapaev A M 1965 Zh. Exp. Teor. Fiz. 48 921.
[7] Bordovitsyn V A, Gushchina V S, Myagkii A N, 1998 Nucl. Instrum. Methods A 405 256.
[8] Bordovitsyn V A, Torres R 1986 Sov. Fiz. J. 29 117.