SCHATTEN CLASS TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACES OF TUBE DOMAINS OVER SYMMETRIC CONES

BENOÎT FLORENT SEHBA

ABSTRACT. We prove some characterizations of Schatten class of Toeplitz operators on Bergman spaces of tube domains over symmetric cones for small exponents.

1. Introduction

All over the text, Ω will denote an irreducible symmetric cone in \(\mathbb{R}^n\), and \(\mathcal{D} = \mathbb{R}^n + i\Omega\) the tube domain over \(\Omega\). As in [10] we denote by \(r\) the rank of the cone \(\Omega\) and by \(\Delta\) the associated determinant function in \(\mathbb{R}^n\). We recall that for \(n \geq 3\), when \(r = 2\), as example of symmetric cones, we have the Lorentz cone \(\Lambda_n\) which is defined by

\[
\Lambda_n = \{(y_1, \cdots, y_n) \in \mathbb{R}^n : y_1^2 - \cdots - y_n^2 > 0, \ y_1 > 0\};
\]

its associated determinant function is given by the Lorentz form

\[
\Delta(y) = y_1^2 - \cdots - y_n^2.
\]

As usual, we denote by \(\mathcal{H}(\mathcal{D})\) the space of holomorphic functions on \(\mathcal{D}\).

Let \(1 \leq p < \infty\). For \(\nu \in \mathbb{R}\) we denote by \(L^p_\nu(\mathcal{D}) = L^p(D, \Delta^{\nu-\frac{n}{2}}(y)dx\,dy)\) the space of functions \(f\) satisfying the condition

\[
\|f\|_{p,\nu} = \|f\|_{L^p_\nu(\mathcal{D})} := \left(\int_{\mathcal{D}} |f(x + iy)|^p \Delta^{\nu-\frac{n}{2}}(y)dx\,dy\right)^{1/p} < \infty.
\]

The weighted Bergman space \(A^p_\nu(\mathcal{D})\) is the closed subspace of \(L^p_\nu(\mathcal{D})\) consisting of holomorphic functions in \(\mathcal{D}\). Following [10], this space is not trivial (i.e. \(A^p_\nu(\mathcal{D}) \neq \{0\}\)) only if \(\nu > \frac{n}{2} - 1\). The orthogonal projection of the Hilbert space \(L^2_\nu(\mathcal{D})\) onto its closed subspace \(A^2_\nu(\mathcal{D})\) is called the weighted Bergman projection and denoted \(P_\nu\). We recall that \(P_\nu\) is given by

\[
P_\nu f(z) = \int_{\mathcal{D}} K_\nu(z, w)f(w)dV_\nu(w),
\]

with \(K_\nu(z, w) = c_\nu \Delta^{-(\nu+\frac{n}{2})}(z-\overline{w})/i\). We recall that \(K_\nu\) is the reproducing kernel of \(A^2_\nu(\mathcal{D})\) (see [10]). For simplicity, we used the notation \(dV_\nu(w) := \Delta^{\nu-\frac{n}{2}}(v)du\,dv\), where \(w = u + iv \in \mathcal{D}\).

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For $\mu$ a positive Borel measure on $D$, the Toeplitz operator $T_\mu$ is the operator defined for functions $f$ with compact support by
\begin{equation}
T_\mu f(z) := \int_D K_\nu(z, w) f(w) d\mu(w),
\end{equation}
where $K_\nu$ is the weighted Bergman kernel.

Schatten class $S_p$ ($0 < p \leq \infty$) criteria of the Toeplitz operators have been considered by many authors on some bounded domains of $\mathbb{C}^n$ (see [1, 8, 12, 19, 20] and the references therein). For unbounded domains, Schatten classes have been also characterized in Fock spaces by several authors (see for example [11, 14] and the references therein). In [13], we extended these results for $1 \leq p \leq \infty$ to weighted Bergman spaces of tube domains over symmetric cones. To be more precise, let us introduce more notations.

For $\delta > 0$, we denote by
\begin{equation}
B_\delta(z) = \{w \in D : d(z, w) < \delta\}
\end{equation}
the Bergman ball centered at $z$ with radius $\delta$, $d$ is the Bergman distance on $D$. For $\nu > \frac{n}{r} - 1$ and $w \in D$, the normalized reproducing kernel of $A^2_\nu(D)$ at $w$ is given by
\begin{equation}
k_\nu(\cdot, w) = \frac{K_\nu(\cdot, w)}{\|K_\nu(\cdot, w)\|_{2, \nu}} = \Delta^{-\frac{n}{2}} \left( \frac{\cdot - \bar{w}}{i} \right) \Delta^{\frac{n}{2} + \frac{n}{p}}(\Im w).
\end{equation}

Let $\mu$ be a positive measure on $D$. The Berezin transform of the measure $\mu$ is the function $\tilde{\mu}$ defined on $D$ by
\begin{equation}
\tilde{\mu}(w) := \int_D |k_\nu(z, w)|^2 d\mu(z), \quad w \in D.
\end{equation}
The Berezin transform of a function $f$ is defined to be the Berezin transform of the measures $d\mu(z) = f(z)dV_\nu(z)$ (for more on the Berezin transform, see [13]). For $z \in D$ and $\delta \in (0, 1)$, we define the average of the positive measure $\mu$ at $z$ by
\begin{equation}
\hat{\mu}_\delta(z) = \frac{\mu(B_\delta(z))}{V_\nu(B_\delta(z))}.
\end{equation}

The following was obtained in [13].

**Theorem 1.1.** Let $\mu$ be a positive Borel measure on $D$, and $\nu > \frac{n}{r} - 1$. Then for $p \geq 1$, the following assertions are equivalent

\begin{enumerate}[(i)]
\item The Toeplitz operator $T_\mu$ belongs to the Schatten class $S_p(A^2_\nu(D))$.
\item For any $\delta$-lattice ($\delta \in (0, 1)$) $\{\zeta_j\}_{j \in \mathbb{N}}$ in the Bergman metric of $D$, the sequence $\{\hat{\mu}_\delta(\zeta_j)\}$ belongs to $l^p$, that is
\begin{equation}
\sum_j \left( \frac{\mu(B_\delta)}{\Delta^{\nu + n/r}(\Im \zeta_j)} \right)^p < \infty.
\end{equation}
\item For any $\beta \in (0, 1)$, the function $z \mapsto \hat{\mu}_\beta(z)$ belongs to $L^p(D, d\lambda)$, with $d\lambda$ the invariant measure on $D$.
\item $\tilde{\mu} \in L^p(D, d\lambda)$.
\end{enumerate}

Our first concern in this note is for the extension of the above result to the range $0 < p < 1$. We prove that the equivalences (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) still hold for $\frac{2}{\nu + \frac{n}{p}} < p < 1$. This cut-off is due to integrability conditions of the
determinant function. The equivalence with the last assertion in the above result still also holds if we restrict to \(\frac{2\nu}{\nu+\frac{n}{r}} < p < 1\). The last cut-off point is also due to integrability conditions of the determinant function and one can prove that it is sharp. Our result is then as follows.

**Theorem 1.2.** Let \(\mu\) be a positive Borel measure on \(D\), and \(\nu > \frac{n}{r} - \frac{1}{2}\). Then for \(\frac{2\nu}{\nu+\frac{n}{r}} < p < 1\), the following assertions are equivalent

(i) The Toeplitz operator \(T_\mu\) belongs to the Schatten class \(S_p(A^2_\nu(D))\).

(ii) For any \(\delta\)-lattice \((\delta \in (0,1))\) \(\{\zeta_j\}_{j \in \mathbb{N}}\) in the Bergman metric of \(D\), the sequence \(\{\mu_\delta(\zeta_j)\}\) belongs to \(l^p\), that is

\[
\sum_j \left(\frac{\mu(B_j)}{\Delta^{\nu+n/r}(\zeta_j)}\right)^p < \infty.
\]

(iii) For any \(\beta \in (0,1)\), the function \(z \mapsto \hat{\mu}_\beta(z)\) belongs to \(L^p(D, d\lambda)\).

Moreover, if \(p > \frac{2\nu}{\nu+\frac{n}{r}}\), then the above assertions are equivalent to the following

(iv) \(\hat{\mu} \in L^p(D, d\lambda)\).

The main difficulty in the proof of the above theorem is the implication (i)\(\Rightarrow\)(ii). The idea in [20] is to replace the measure \(\mu\) by a measure supported on a disjoint union of Bergman balls, then split the associated Toeplitz operator into its diagonal and off-diagonal part. It is not hard to prove that the Schatten norm of the diagonal part dominates the \(l^p\)-norm of the sequence \(\{\mu_\delta(\zeta_j)\}\). The difficulty is to prove that the latter norm dominates (up to a pretty small constant) the Schatten norm of the off-diagonal operator. A part of the techniques in [20] uses the fact that the unit ball is bounded, and so it cannot be used in our setting. We overcome this difficulty by using a technical lemma originally due to D. Békollé and A. Temgoua [7]. Considered even in the unit ball, our contribution heavily simplifies the proof of K. Zhu in [20].

We are also interested here in some other possible equivalent characterizations of Schatten class Toeplitz operators. For this, we denote by \(\Box_z\) the natural extension to \(\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n\) of the wave operator \(\Box_x\) on the cone:

\[
\Box_z = \Delta \left(1 - \frac{1}{i} \frac{\partial}{\partial z}\right)
\]

which is the differential operator of degree \(r\) defined by the equality:

\[
\Box_z [e^{i(z|\xi)}] = \Delta(z)e^{i(z|\xi)}, \quad z \in \mathbb{C}^n, \xi \in \mathbb{R}^n.
\]

We recall (see [5]) that \(\Box_z\) acts on the Bergman kernel as follows

\[
\Box_z K_\nu(z, w) = C_\nu K_{\nu+1}(z, w).
\]

Let \(m\) be a positive integer. For simplicity, we use the following notation for higher order derivatives of the Bergman kernel,

\[
K^{\nu, m}_z(w) := \Box_z^m K_\nu(z, w)
\]

and

\[
k^{\nu, m}_z(\cdot) := \frac{K^{\nu, m}_z(z, \cdot)}{\|K^{\nu, m}_z(z, \cdot)\|_{2, \nu}}.
\]
Define the quantity
\[ \tilde{\mu}^m(z) := \langle T_\mu k_{\nu,m}^\nu, k_{\nu,m}^\nu \rangle = \int_D |k_{\nu,m}^\nu(w)|^2 d\mu(w). \]

We also have the following equivalent characterization.

**Theorem 1.3.** Let \( \nu > \frac{1}{p} - 1 \), and \( \frac{p-1}{\nu + 2} \leq p < \infty \). Assume \( \mu \) is a positive measure on \( D \). Then the Toeplitz operator \( T_\mu \) belongs to the Schatten class \( S_p(A^{2\nu}_2(D)) \) if and only if for each (or some) integer \( m \geq 0 \) with \( p(\nu + \frac{1}{2} + 2m) > 2\frac{1}{p} - 1 \), \( \tilde{\mu}^m \in L^p(D, d\lambda) \) where \( d\lambda \) is the invariant measure on \( D \).

Again the condition \( p \geq \max\{\frac{2p-1}{\nu + 2}, \frac{p-1}{\nu + 2} + 2m\} \) is due to integrability conditions of the determinant function.

For the proof of Theorem 1.3, we derive the necessary condition for \( 1 \leq p < \infty \) and the sufficient condition for \( 0 < p < 1 \) from a more general result for any positive operator. The proof of the other parts essentially uses the properties of Bergman balls and the \( \delta \)-lattices. We also refer to [15, 16] for this type of results.

We are essentially motivated here by the idea of extending the results in [13] on Schatten class \( S_p(A^{2\nu}_2(D)) \) for \( P \geq 1 \), to the case \( 0 < p < 1 \), and settling the problem of the characterization of Schatten class \( S_p(A^{2\nu}_2(D)) \) for \( 1 \leq p < 2 \) for the Cesàro-type operator introduced in [13].

The paper is organized as follows. In the next section, we present some useful tools and results needed in the proofs of the above results. The proof of Theorem 1.2 is given in Section 3. In Section 4, we provide characterization of Schatten class for general positive operators. We prove Theorem 1.3 in Section 5. In the last section, we apply our results to extend to the range \( 1 \leq p < 2 \), the characterization of Schatten class \( S_p(A^{2\nu}_2(D)) \) for the Cesàro-type operator obtained in [13].

As usual, given two positive quantities \( A \) and \( B \), the notation \( A \lesssim B \) (resp. \( A \gtrsim B \)) means that there is an absolute positive constant \( C \) such that \( A \leq CB \) (resp. \( A \geq CB \)). When \( A \lesssim B \) and \( B \lesssim A \), we write \( A \asymp B \) and say \( A \) and \( B \) are equivalent. Finally, all over the text, \( C, C_k, C_{k,j} \) will denote positive constants depending only on the displayed parameters but not necessarily the same at distinct occurrences. The same remark holds for lower case letters.

## 2. Preliminary results

In this section, we give some fundamental facts about symmetric cones, Berezin transform and related results.

### 2.1. Symmetric cones, Bergman metric and estimations of the determinant function.

It is well known that a symmetric cone \( \Omega \) induces in \( V \equiv \mathbb{R}^n \) a structure of Euclidean Jordan algebra, in which \( \overline{\Omega} = \{x^2 : x \in V\} \). Let \( e \) be the identity element in \( V \). Denote by \( G(\Omega) \) the group of transformations of \( \mathbb{R}^n \) leaving invariant \( \Omega \). We recall that the group \( G(\Omega) \) acts transitively on \( \Omega \). We denote by \( H \) the subgroup of \( G(\Omega) \) that acts simply transitively on \( \Omega \), that is for \( x, y \in \Omega \) there is a unique \( h \in H \) such that...
y = hx. Observe that if we still denote by $\mathbb{R}^n$ the group of translation by vectors in $\mathbb{R}^n$, then the group $G(D) = \mathbb{R}^n \times H$ acts simply transitively on $\mathcal{D}$.

Recall that $\delta > 0$,

$$B_\delta(z) = \{w \in \mathcal{D}: d(z, w) < \delta\}$$

is the Bergman ball centered at $z$ with radius $\delta$, where $d(\cdot, \cdot)$ is the Bergman distance (for a definition, see for example [13]). It well known that the measure $d\lambda(z) = \Delta^{-2n/r}(3z)dV(z)$ is an invariant measure on $\mathcal{D}$ under the actions of $G(\mathcal{D}) = \mathbb{R}^n \times H$.

We recall the following (see [2, Theorem 5.4]).

**Lemma 2.1.** For any $\delta \in (0, 1)$, there exists a sequence $\{\zeta_j\}$ of points of $\mathcal{D}$ called $\delta$-lattice such that then

(i) the balls $B_j' = B_\delta(\zeta_j)$ are pairwise disjoint;

(ii) the balls $B_j = B_\delta(\zeta_j)$ cover $\mathcal{D}$. Moreover, there is an integer $N$ (depending only on $\mathcal{D}$) such that each point of $\mathcal{D}$ belongs to at most $N$ of these balls.

**Remark 2.2.** Let $A > 0$ be fixed. Then any $\delta$-lattice $\{\zeta_j\}$ admits a decomposition into a finite number of sequences $\{\zeta_{jk}\}$ satisfying $d(\zeta_{jk_1}, \zeta_{jk_2}) \geq A$ for $j_1 \neq j_2$.

We observe that

$$\int_{B_j} dV_\nu(z) \approx \int_{B_j'} dV_\nu(z) \approx C_\delta \Delta^{\nu+n/r}(3\zeta_j).$$

We refer to [2, Theorem 5.6] for the following sampling theorem.

**Lemma 2.3.** Let $\{\zeta_j\}_{j \in \mathbb{N}}$ be a $\delta$-lattice in $\mathcal{D}$, $\delta \in (0, 1)$. Then the following assertions hold.

1. There is a positive constant $C_\delta$ such that every $f \in A^2_\nu(\mathcal{D})$ satisfies

$$||\{f(\zeta_j)\Delta^{\frac{\nu+n}{2}}(3\zeta_j)\}||_{l^p} \leq C_\delta ||f||_{p, \nu}.$$

2. Conversely, if $\delta$ is small enough, there is a positive constant $C_\delta$ such that every $f \in A^2_\nu(\mathcal{D})$ satisfies

$$||f||_{p, \nu} \leq C_\delta ||\{f(\zeta_j)\Delta^{\frac{\nu+n}{2}}(3\zeta_j)\}||_{l^p}.$$

We have the following atomic decomposition with change of weight which is derived from [13, Theorem 3.2].

**Theorem 2.4.** Let $\mu, \nu > \frac{n}{r} - 1$. Assume that the operator $P_\mu$ is bounded on $L^2_\nu(\mathcal{D})$ and let $\{\zeta_j\}_{j \in \mathbb{N}}$ be a $\delta$-lattice in $\mathcal{D}$. Then the following assertions hold.

1. For every complex sequence $\{\lambda_j\}_{j \in \mathbb{N}}$ in $l^2$, the series

$$\sum_j \lambda_j K_\mu(z, \zeta_j) \Delta^{\nu+n/2 - \frac{1}{2}(\nu+n)}(3\zeta_j)$$

is convergent in $A^2_\nu(\mathcal{D})$. Moreover, its sum $f$ satisfies the inequality

$$||f||_{2, \nu} \leq C_\delta ||\{\lambda_j\}||_{l^2},$$
Lemma 2.9. \( \text{Siegel domains of type II.} \)

Lemma 2.8. \( \text{Corollary 3.4} \) and the above Korányi’s lemma. 

\( (2.3) \) \( \| \{ \lambda_j \} \|_2 \leq C_\delta \| f \|_2, \nu \)

where \( C_\delta \) is a positive constant.

The following consequence of the mean value theorem (see [2]) is needed.

**Lemma 2.5.** There exists a constant \( C > 0 \) such that for any \( f \in H(D) \) and \( \delta \in (0, 1] \), the following holds

\( (2.2) \)

\[ |f(z)|^p \leq C \delta^{-n} \int_{B_\delta(z)} |f(\zeta)|^p \frac{dV(\zeta)}{\Delta^{2n/r}(3\zeta)}. \]

We recall the following integrability conditions for the determinant function (see [2] Lemma 3.20).

**Lemma 2.6.** Let \( \alpha \) be real. Then the function \( f(z) = \Delta^{-\alpha}((z+it)^d) \), with \( t \in \Omega \), belongs to \( L_\nu^p(D) \) if and only if \( \nu > \frac{n}{r} - 1 \) and \( p\alpha > \nu + \frac{n}{r} - 1 \). In this case,

\[ ||f||_{p,\nu} = C_{\alpha,p} \Delta^{-\alpha + \frac{n}{r} + \nu}(t). \]

We will be using the following Korányi’s lemma.

**Lemma 2.7.** [3] Theorem 1.1 For every \( \delta > 0 \), there is a constant \( C_\delta > 0 \) such that

\[ \left| \frac{K(\zeta, z)}{K(\zeta, w)} - 1 \right| \leq C_\delta d(z, w) \]

for all \( \zeta, z, w \in D \), with \( d(z, w) \leq \delta \).

We close this subsection by recalling the following consequence of [2] Corollary 3.4 and the above Korányi’s lemma.

**Lemma 2.8.** Let \( \nu > \frac{n}{r} - 1 \), \( \delta > 0 \) and \( z, w \in D \). There is a positive constant \( C_\delta \) such that for all \( z \in B_\delta(w) \),

\[ V_\nu(B_\delta(w))|k_\nu(z, w)|^2 \leq C_\delta. \]

If \( \delta \) is sufficiently small, there is \( C > 0 \) such that for all \( z \in B_\delta(w) \),

\[ V_\nu(B_\delta(w))|k_\nu(z, w)|^2 \geq (1 - C\delta). \]

The following was first proved [7] Lemma 5.1 in the case of homogeneous Siegel domains of type II.

**Lemma 2.9.** Let \( 0 < \delta \leq 1 \), \( \alpha, \beta \in \mathbb{R} \) with \( \alpha > 2\frac{n}{r} - 1 \), \( \beta > 2\frac{n}{r} - 1 \) and \( \alpha > \beta + \frac{n}{r} - 1 \). Then for any \( \epsilon > 0 \), there exists \( A_\epsilon > 0 \) such that if \( \{ z_j = x_j + iy_j \} \) is a sequence of points of \( D \) in a \( \delta \)-lattice satisfying \( \inf_{j \neq k} d(z_j, z_k) \geq A_\epsilon \), then for any integer \( j \), the following estimate holds

\( (2.3) \)

\[ \sum_{\{k: k \neq j\}} |\Delta^{-\alpha}(z_k - \bar{z}_j)|\Delta^\beta(y_k) \leq \epsilon \Delta^{-\alpha + \beta}(y_j). \]
Proof. Let us give ourself $A > 0$. Thanks to Remark 2.2 we may assume that the sequence $\{z_j = x_j + iy_j\}$ is such that $d(z_j, z_k) \geq A$ for all $j \neq k$. We first observe with Lemma 2.5 that

$$|\Delta^{-\alpha}(z_k - \bar{z}_j)| \leq (\delta/3)^{-n} \int_{B'_k} |\Delta^{-\alpha}(w - \bar{z}_j)| \frac{dV(w)}{\Delta^{\frac{2n}{\alpha}}(3w)}.$$ 

It follows that

$$S := \sum_{\{k: k \neq j\}} |\Delta^{-\alpha}(z_k - \bar{z}_j)| |\Delta^\beta(y_k)|$$

$$\leq C(\delta/3)^{-n} \sum_{\{k: k \neq j\}} \int_{B'_k} |\Delta^{-\alpha}(w - \bar{z}_j)| |\Delta^\beta(3w)| \frac{dV(w)}{\Delta^{\frac{2n}{\alpha}}(3w)}$$

$$\leq C(\delta/3)^{-n} \int_{B} |\Delta^{-\alpha}(w - \bar{z}_j)| |\Delta^\beta(3w)| \frac{dV(w)}{\Delta^{\frac{2n}{\alpha}}(3w)}$$

where $B := \bigcup_{k \neq j} B'_k$.

Now observe that if $w \in B$, then $w \in B'_k$ for some $k$ and so

$$d(w, z_k) < \frac{\delta}{2} < \frac{A}{2}$$

and for $j \neq k$,

$$d(w, z_j) \geq d(z_j, z_k) - d(w, z_k) > A - \frac{A}{2} = \frac{A}{2}.$$ 

Let $g \in G(\mathcal{D})$ be the transformation such that $g(ie) = z_j$, and put $w = g(\zeta)$. Observe that for any $s \in \mathbb{R}$,

$$\Delta^s(w - \bar{z}_j) = \Delta^s(g(\zeta + ie)) = (Detg)^{\frac{s}{2}} \Delta^s(\zeta + ie) = \Delta^s(3z_j) \Delta^s(\zeta + ie),$$

and

$$\Delta^s(3w) = (Detg)^{\frac{s}{2}} \Delta^s(3\zeta) = \Delta^s(3z_j) \Delta^s(3\zeta)$$

and

$$dV(w) = (Detg)^2 dV(\zeta) = \Delta^{\frac{2n}{\alpha}}(3z_j) dV(\zeta).$$

It follows that

$$S \leq C(\delta/3)^{-n} \int_{d(z_j, w) > A/2} |\Delta^{-\alpha}(w - \bar{z}_j)| \Delta^\beta(3w) \frac{dV(w)}{\Delta^{\frac{2n}{\alpha}}(3w)}$$

$$\leq C(\delta/3)^{-n} \Delta^{-\alpha + \beta}(y_j) \int_{d(ie, \zeta) > A/2} |\Delta^{-\alpha}(\zeta + ie)| \Delta^\beta(3\zeta) \frac{dV(\zeta)}{\Delta^{\frac{2n}{\alpha}}(3\zeta)}.$$ 

From the assumptions on $\alpha$ and $\beta$ together with Lemma 2.6, one has that the integral

$$\int_{\Omega} |\Delta^{-\alpha}(\zeta + ie)| \Delta^\beta(3\zeta) \frac{dV(\zeta)}{\Delta^{\frac{2n}{\alpha}}(3\zeta)}$$

converges. Hence, there exists $A_\varepsilon > 0$ such that for all $A \geq A_\varepsilon$, the following inequality holds

$$\int_{d(ie, \zeta) > A/2} |\Delta^{-\alpha}(\zeta + ie)| \Delta^\beta(3\zeta) \frac{dV(\zeta)}{\Delta^{\frac{2n}{\alpha}}(3\zeta)} \leq \frac{\varepsilon}{C(\delta/3)^{-n}}$$

with $C$ as in the above estimate of $S$. The proof is complete. 
\[ \square \]
2.2. Averaging functions and Berezin transform. The following was proved in [13] for $1 \leq p \leq \infty$. A careful observation of the proof of [13, Lemma 2.9] shows that the result extends to $0 < p < 1$.

**Lemma 2.10.** Let $0 < p \leq \infty$, $\nu \in \mathbb{R}$, and $\delta, \beta \in (0, 1)$. Let $\mu$ be a positive Borel measure on $\mathcal{D}$. Then the following assertions are equivalent.

(i) The function $\mathcal{D} \ni z \mapsto \frac{\mu(B_\delta(z))}{\Delta^{\nu + \frac{\nu}{p}}(3z)}$ belongs to $L^p_\nu(\mathcal{D})$.

(ii) The function $\mathcal{D} \ni z \mapsto \frac{\mu(B_\beta(z))}{\Delta^{\nu + \frac{\nu}{p}}(3z)}$ belongs to $L^p_\nu(\mathcal{D})$.

Note that the above lemma allows flexibility on the choice of the radius of the ball. This fact is quite useful as seen in [13].

We have the following result.

**Lemma 2.11.** Let $0 < p \leq 1$, $\nu > \frac{n}{r} - 1$, $\beta, \delta \in (0, 1)$. Let $\{\zeta_j\}_{j \in \mathbb{N}}$ be a $\delta$-lattice in $\mathcal{D}$, and let $\hat{\mu}_\beta$ and $\tilde{\mu}$ be in this order, the average function and the Berezin transform associated to the weight $\nu$. Then the following assertions are equivalent.

(i) $\hat{\mu}_\beta \in L^p(\mathcal{D}, d\lambda)$.

(ii) $\{\hat{\mu}_\delta(\zeta_j)\}_{j \in \mathbb{N}} \in l^p$.

If moreover, $p > \frac{2\nu - 1}{n + \nu}$, then the above assertions are equivalent to

(iii) $\tilde{\mu} \in L^p(\mathcal{D}, d\lambda)$.

**Proof.** The equivalence (i)$\iff$(ii) follows as in [13, Lemma 2.12]. That (iii)$\Rightarrow$(i) follows from the fact that for any $\delta \in (0, 1)$, there exists a constant $C_\delta > 0$ such that for any $z \in \mathcal{D}$,

$$\hat{\mu}_\delta(z) \leq C_\delta \tilde{\mu}(z)$$

(see [13, Lemma 2.8]). To finish the proof, let us prove that (ii)$\Rightarrow$(iii). First using Lemma 2.7, we obtain

$$\tilde{\mu}(z) := \int_D |K_\nu(z, w)|^2 \Delta^{\nu + \frac{\nu}{p}}(z)d\mu(w)$$

$$\leq \sum_k \int_{B_k} |K_\nu(z, w)|^2 \Delta^{\nu + \frac{\nu}{p}}(z)d\mu(w)$$

$$\leq C \sum_k |K_\nu(z, \zeta_k)|^2 \Delta^{\nu + \frac{\nu}{p}}(z)\Delta^{\nu + \frac{\nu}{p}}(\zeta_k)\tilde{\mu}(\zeta_k).$$

As $0 < p \leq 1$, it follows that

$$(\tilde{\mu}(z))^p \leq C \sum_k |K_\nu(z, \zeta_k)|^{2p} \Delta^{p(\nu + \frac{\nu}{p})}(z)\Delta^{p(\nu + \frac{\nu}{p})}(\zeta_k) (\tilde{\mu}(\zeta_k))^p.$$
Hence using that \( p > \frac{2n - 1}{n + 2} \) together with Lemma 2.6, we obtain

\[
L := \int\limits_D (\tilde{\mu}(z))^p d\lambda(z)
\]

\[
\leq C \sum_k \Delta^{p(n + \frac{\nu}{2})}(\tilde{\mu}(\zeta_k))^p \int\limits_D |K_\nu(z, \zeta_k)|^{2p} \Delta^{p(n + \frac{\nu}{2}) - 2n} dV(z)
\]

\[
\leq C \sum_k (\tilde{\mu}(\zeta_k))^p < \infty.
\]

The proof is complete. \( \square \)

2.3. Schatten class operators. In this subsection, \( \mathcal{H} \) is a Hilbert space with associated norm \( \| \cdot \| \). The spaces of bounded and compact linear operators on \( \mathcal{H} \) are denoted \( B(\mathcal{H}) \) and \( K(\mathcal{H}) \) respectively. We recall that any positive operator \( T \in K(\mathcal{H}) \), then one can find an orthonormal set \( \{e_j\} \) of \( \mathcal{H} \) and a sequence \( \{\lambda_j\} \) that decreases to 0 such that

\[
(2.4) \quad Tf = \sum_{j=0}^\infty \lambda_j \langle f, e_j \rangle e_j, \quad f \in \mathcal{H}.
\]

For \( 0 < p < \infty \), we say a compact operator \( T \) with a decomposition as above belongs to the Schatten-Von Neumann \( p \)-class \( S_p := S_p(\mathcal{H}) \), if

\[
\|T\|_{S_p} := \left( \sum_{j=0}^\infty |\lambda_j|^p \right)^{\frac{1}{p}} < \infty.
\]

For \( p = 1 \), we denote by \( S_1 = S_1(\mathcal{H}) \) the trace class. We recall that for \( T \in S_1 \), the trace of \( T \) is defined by

\[
Tr(T) = \sum_{j=0}^\infty \langle Te_j, e_j \rangle
\]

where \( \{e_j\} \) is any orthonormal basis of the Hilbert space \( \mathcal{H} \).

It is known that a compact operator \( T \) on \( \mathcal{H} \) belongs to the Schatten class \( S_p \) if and only if the positive operator \( (T^*T)^{1/2} \) belongs to \( S_p \), where \( T^* \) denotes the adjoint of \( T \). In this case, we have \( \|T\|_{S_p} = \|(T^*T)^{1/2}\|_{S_p} \). It is also well known that a positive \( T \) belongs to \( S_p \) if and only if the operator \( T^p \) belongs to the trace class \( S_1 \). In this case, \( \|T\|_{S_p} = \|T^p\|_{S_1} \).

We also recall that if \( T \) is a compact operator on \( \mathcal{H} \), and \( p \geq 1 \), then that \( T \in S_p \) is equivalent to

\[
\sum_j |\langle Te_j, e_j \rangle|^p < \infty
\]

for any orthonormal set \( \{e_j\} \) in \( \mathcal{H} \) (see \[18\]).

The following can be found in \[18\].

Lemma 2.12. Suppose that \( T \) is a positive operator on \( \mathcal{H} \), and that \( \{e_j\} \) is an orthonormal basis on \( \mathcal{H} \). Then if \( 0 < p < 1 \) and

\[
\sum_{j=1}^\infty \langle Te_j, e_j \rangle^p < \infty,
\]
then $T$ belongs to $S_p$.

We also observe the following (see [19])

**Lemma 2.13.** Let $T$ be any bounded operator on $\mathcal{H}$ and assume that $A$ is bounded surjective operator on $\mathcal{H}$. Then $T$ belongs to $S_p$ if and only if the operator $A^*TA$ belongs to $S_p$.

Finally, we will need the following result (see [12])

**Lemma 2.14.** Let $T$ be any bounded operator on $\mathcal{H}$ and let $\{e_k\}$ be an orthonormal basis of $\mathcal{H}$. Then for any $0 < p \leq 2$, we have

$$\|T\|_{S_p}^p \leq \sum_k \sum_j |\langle Te_k, e_j \rangle|^p.$$

3. Schatten class membership of Toeplitz operators

The aim of this section is to give criteria for Schatten class membership of Toeplitz operators on the weighted Bergman space $A^2_\nu(D)$.

3.1. **Proof of Theorem 1.2.** We start by proving the following.

**Lemma 3.1.** Let $\mu$ be a positive Borel measure on $D$, and $\nu > \frac{n}{r} - 1$. Assume that $\frac{n-1}{\nu + 1} < p < 1$. Suppose that for any $\delta$-lattice ($\delta \in (0, 1)$) $\{\zeta_j\}_{j \in \mathbb{N}}$ in the Bergman metric of $D$, the sequence $\{\mu_\delta(\zeta_j)\}$ belongs to $l^p$, that is

$$\sum_j \left( \frac{\mu(B_j)}{\Delta^{\nu + n/r}(3\zeta_j)} \right)^p < \infty.$$  

Then the Toeplitz operator $T_\mu$ belongs to the Schatten class $S_p(A^2_\nu(D))$. Moreover,

$$\|T_\mu\|_{S_p}^p \lesssim \sum_j \left( \frac{\mu(B_j)}{\Delta^{\nu + n/r}(3\zeta_j)} \right)^p.$$

**Proof.** Let $\sigma$ be large enough so that $P_\sigma$ is bounded on $L^2_\nu(D)$. Thanks to Lemma 2.10, we can suppose that $\delta$ is small enough so that any $f \in A^2_\nu(D)$ can represented as in Theorem 2.4. That is

$$f(z) = \sum_j \lambda_j K_\sigma(z, \zeta_j) \Delta^{\sigma + \frac{n}{2} - \frac{1}{2}(\nu + \frac{n}{2})}(3\zeta_j)$$

with $||\{\lambda_j\}||_{l^2} \simeq ||f||_{2,\nu}$.

Let $\{e_k\}_{k \geq 1}$ be a fixed orthonormal basis on $A^2_\nu(D)$. Consider the operator $S : A^2_\nu(D) \to A^2_\nu(D)$ defined by

$$S(e_k) = f_k$$

where

$$f_k(z) = K_\sigma(z, \zeta_k) \Delta^{\sigma + \frac{n}{2} - \frac{1}{2}(\nu + \frac{n}{2})}(3\zeta_k).$$

Then it follows from Theorem 2.4 that $S$ is a bounded and surjective operator on $A^2_\nu(D)$. We know from Lemma 2.13 that $T_\mu$ belongs to $S_p(A^2_\nu(D))$ if and
only if $T = S^* T_p S$ belongs to $S_p(A_2^2(\mathcal{D}))$. It follows from Lemma 2.12 that we only have to prove that

$$L := \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle^p < \infty.$$ 

We first observe that

$$\langle Te_k, e_k \rangle = \langle T \mu f_k, f_k \rangle = \int_{\mathcal{D}} |f_k(z)|^2 d\mu(z).$$

Hence using Lemma 2.8, we obtain

$$\langle Te_k, e_k \rangle \leq \sum_{j=1}^{\infty} \int_{B_\delta(\zeta_j)} |f_k(z)|^2 d\mu(z) \leq C \sum_{j=1}^{\infty} |f_k(\zeta_j)|^2 \mu(B_\delta(\zeta_j)).$$

Recalling that $0 < p < 1$, we then obtain

$$\langle Te_k, e_k \rangle^p \leq C \sum_{j=1}^{\infty} |f_k(\zeta_j)|^{2p} \mu(B_\delta(\zeta_j))^p \leq \sum_{j=1}^{\infty} |f_k(\zeta_j)|^{2p} \Delta^{\nu + \frac{\mu}{p}} (\Im \zeta_j)(\hat{\mu}(\zeta_j))^p.$$ 

Thus

$$L := \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle^p \leq C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |f_k(\zeta_j)|^{2p} \Delta^{\nu + \frac{\mu}{p}} (\Im \zeta_j)(\hat{\mu}(\zeta_j))^p \leq C \sum_{j=1}^{\infty} \Delta^{p(\nu + \frac{\mu}{p})} (\Im \zeta_j)(\hat{\mu}(\zeta_j))^p \sum_{k=1}^{\infty} |f_k(\zeta_j)|^{2p}.$$ 

Using the fact that each point in $\mathcal{D}$ belongs to at most $N$ balls $B_k$ and the condition $\frac{p-1}{\nu + \frac{\mu}{p}} < p < 1$, we obtain using Lemma 2.6 the following for the inner sum

$$L_j := \sum_{k=1}^{\infty} |f_k(\zeta_j)|^{2p} \leq C \sum_{k=1}^{\infty} \int_{B_{\delta}(\zeta_k)} |K_\sigma(\zeta_j, z)|^{2p} \Delta^{2p(\sigma + \frac{\nu}{p} - \frac{1}{2} (\nu + \frac{\mu}{p}))} (\Im z) dV(z) \leq CN \Delta^{-p(\nu + \frac{\mu}{p})} (\Im \zeta_j).$$
Using the latter, we conclude that
\[ L := \sum_{k=1}^{\infty} \langle T e_k, e_k \rangle^p \]
\[ \leq C \sum_{j=1}^{\infty} (\hat{\mu}_\delta(\zeta_j))^p < \infty. \]

We next prove the reverse of the above result.

**Lemma 3.2.** Let \( \mu \) be a positive measure on \( \mathcal{D} \). Assume that \( T_\mu \in \mathcal{S}_p(A^2_\nu(\mathcal{D})) \) for some \( 0 < p < 1 \). Let \( \{\zeta_j\}_{j \in \mathbb{N}} \) be a \( \delta \)-lattice in \( \mathcal{D} \). Then the sequence \( \{\hat{\mu}_\delta(\zeta_j)\} \) belongs to \( \ell^p \). Moreover,
\[ \sum_j (\hat{\mu}_\delta(\zeta_j))^p \lesssim \|T_\mu\|_{\mathcal{S}_p}^p. \]

**Proof.** We start by considering \( \sigma \) large enough so that \( \sigma + \frac{2}{r} \) and \( \sigma + \frac{2}{r} - \frac{1}{2}(\nu + \frac{2}{r}) \) satisfy the conditions in Lemma 2.9. Let \( \varepsilon > 0 \), and let \( A_\varepsilon \) be as in Lemma 2.9. Following Remark 2.2, we may assume that our sequence \( \{\zeta_j\} \) is such that \( d(\zeta_j, \zeta_k) > A_\varepsilon \) for \( j \neq k \). We further assume that \( A_\varepsilon \) is large enough so that corresponding balls \( B_k \) are disjoint. Consider the following measure:
\[ d\omega(z) = \sum_k \chi_{B_k}(z) d\mu(z). \]

Then \( 0 \leq \omega \leq \mu \), \( \omega = \mu \) on each ball \( B_k \). We also have the inequality
\[ \|T_\omega\|_{\mathcal{S}_p} \leq \|T_\mu\|_{\mathcal{S}_p}. \]

Now as in the proof of the previous result, we fix an orthonormal basis \( \{e_k\} \) of \( A^2_\nu(\mathcal{D}) \) and consider the same operator \( S \) defined on \( A^2_\nu(\mathcal{D}) \) by
\[ S(e_k) = f_k \]
with
\[ f_k(z) = K_\sigma(z, \zeta_k) \Delta^{\sigma + \frac{n}{r} - \frac{1}{2}(\nu + \frac{2}{r})}(3\zeta_j). \]

We recall with Theorem 2.4 that \( S \) is bounded and surjective on \( A^2_\nu(\mathcal{D}) \). Put again \( T = S^* T_\omega S \). Then as \( T_\omega \in \mathcal{S}_p(A^2_\nu(\mathcal{D})) \), \( T \) also belongs to \( \mathcal{S}_p(A^2_\nu(\mathcal{D})) \) and we have
\[ \|T\|_{\mathcal{S}_p} \leq \|S\|^2 \|T_\omega\|_{\mathcal{S}_p} \leq C\|T_\mu\|_{\mathcal{S}_p}. \]

The main idea of the proof is to show that the \( l^p \)-norm of the sequence \( \{\hat{\mu}_\delta(\zeta_j)\} \) is up to a constant a lower bound for \( \|T_\mu\|_{\mathcal{S}_p} \). For this we decompose \( T \) as \( T = D + R \), where \( D \) is the positive diagonal operator on \( A^2_\nu(\mathcal{D}) \) given by
\[ Df := \sum_k \langle Te_k, e_k \rangle \langle f, e_k \rangle e_k, \quad f \in A^2_\nu(\mathcal{D}) \]
and \( R = T - D \). We observe that
\[ \|T\|_{\mathcal{S}_p}^p \geq \frac{1}{2} \|D\|_{\mathcal{S}_p}^p - \|R\|_{\mathcal{S}_p}^p. \]

Hence, if we can prove that
\[ \|D\|_{\mathcal{S}_p}^p \geq c_1 \sum_j (\hat{\mu}_\delta(\zeta_j))^p \]
and
\[ \|R\|_{S_p}^p \leq c_2 \sum_j (\hat{\mu}_\delta(\zeta_j))^p \]
with \(c_2\) as small as we want, then the proof will be completed.

We start by estimating the diagonal operator \(D\). As \(D\) is positive, we have
\[
\|D\|_{S_p}^p = \sum_k \langle T e_k, e_k \rangle^p = \sum_k \langle T \omega f_k, f_k \rangle^p \\
= \sum_k \left( \int_D |f_k(z)|^2 d\omega(z) \right)^p \\
\geq \sum_k \left( \int_{B_k} |f_k(z)|^2 d\omega(z) \right)^p \\
= \sum_k \left( \int_{B_k} |f_k(z)|^2 d\mu(z) \right)^p \\
\times \sum_k (\hat{\mu}_\delta(\zeta_k))^p.
\]
That is
\[ \|D\|_{S_p}^p \geq c_1 \sum_j (\hat{\mu}_\delta(\zeta_j))^p. \]

We now turn to the estimation of \(\|R\|_{S_p}^p\). First, using Lemma 2.12, we obtain
\[
\|R\|_{S_p}^p \leq \sum_k \sum_j |\langle Re_j, e_k \rangle|^p \\
= \sum_{\{j,k;j\neq k\}} |\langle T \omega f_j, f_k \rangle|^p \\
= \sum_{\{j,k;j\neq k\}} \left( \int_D |f_j(z) f_k(z) d\omega(z)| \right)^p \\
\leq \sum_{\{j,k;j\neq k\}} \left( \int_D |f_j(z)||f_k(z)| d\omega(z) \right)^p.
\]
As the balls \(B_l\) are disjoint, using Lemma 2.8, we obtain
\[
\int_D |f_j(z)||f_k(z)| d\omega(z) = \sum_l \int_{B_l} |f_j(z)||f_k(z)| d\mu(z) \\
\leq C \sum_l |f_j(\zeta_l)||f_k(\zeta_l)| \mu(B_l) \\
\leq \sum_l |f_j(\zeta_l)||f_k(\zeta_l)| \Delta^{\nu+\frac{p}{2}(3\zeta_l)\hat{\mu}_\delta(\zeta_l)}.
\]
As $0 < p < 1$, it follows that
\[ \|R\|_{S_p}^p \leq C \sum_l \Delta^{p(\nu + \frac{n}{r})} (\Im \zeta_l) (\hat{\mu}_\delta(\zeta_l))^p L_l \]
where
\[ L_l := \sum_{\{j,k: j \neq k\}} |f_j(\zeta_l)|^p |f_k(\zeta_l)|^p \]
\[ = \sum_{\{j,k: j \neq k\}} |\Delta^{-p(\mu + \frac{n}{r})} (\frac{\zeta_j - \zeta_l}{i})| \Delta^{p(\mu + \frac{n}{r} - \frac{1}{2}(\nu + \frac{n}{r}))} (\Im \zeta_l) \times |\Delta^{-p(\mu + \frac{n}{r})} (\frac{\zeta_k - \zeta_l}{i})| \Delta^{p(\mu + \frac{n}{r} - \frac{1}{2}(\nu + \frac{n}{r}))} (\Im \zeta_k) \]
\[ = 2L_1^l + L_2^l \]
with
\[ L_1^l := \Delta^{-\frac{n}{2}(\nu + \frac{n}{r})} (\Im \zeta_l) \sum_{\{j: j \neq l\}} |\Delta^{-p(\mu + \frac{n}{r})} (\frac{\zeta_j - \zeta_l}{i})| \Delta^{p(\mu + \frac{n}{r} - \frac{1}{2}(\nu + \frac{n}{r}))} (\Im \zeta_j) \]
and
\[ L_2^l := \sum_{\{j,k: j \neq l \neq k \text{ and } j \neq k\}} |\Delta^{-p(\mu + \frac{n}{r})} (\frac{\zeta_j - \zeta_l}{i})| \Delta^{p(\mu + \frac{n}{r} - \frac{1}{2}(\nu + \frac{n}{r}))} (\Im \zeta_j) \times |\Delta^{-p(\mu + \frac{n}{r})} (\frac{\zeta_k - \zeta_l}{i})| \Delta^{p(\mu + \frac{n}{r} - \frac{1}{2}(\nu + \frac{n}{r}))} (\Im \zeta_k). \]
Using Lemma [2,9] we obtain
\[ L_1^l \leq \varepsilon \Delta^{-p(\nu + \frac{n}{r})} (\Im \zeta_l) \]
and
\[ L_2^l \leq \varepsilon^2 \Delta^{-p(\nu + \frac{n}{r})} (\Im \zeta_l). \]
It follows that
\[ \|R\|_{S_p}^p \leq c_2 (2\varepsilon + \varepsilon^2) \sum_j (\hat{\mu}_\delta(\zeta_j))^p. \]
Hence
\[ \|T\|_{S_p}^p \geq \left[ \frac{c_1}{2} - c_2 (2\varepsilon + \varepsilon^2) \right] \sum_j (\hat{\mu}_\delta(\zeta_j))^p. \]
Taking $\varepsilon$ small enough so that $\frac{c_1}{2} - c_2 (2\varepsilon + \varepsilon^2) > 0$, we conclude that
\[ \sum_j (\hat{\mu}_\delta(\zeta_j))^p < \infty. \]
The proof is complete. \square

We can now prove Theorem 1.2.

Proof of Theorem 1.2 We start by proving the necessity of the condition $p > \frac{2n}{\nu + \frac{n}{r}}$ in assertion (iv). We recall that $e$ is the identity element of
V. We may suppose that $\mu(B_1(ie)) > 0$ (if not change the radius of the Bergman ball). Then using Lemma 2.7 we obtain

$$\tilde{\mu}(z) = \int_{D} |k_\nu^\nu(w)|^2 d\mu(w) \geq \int_{B_1(ie)} |k_\nu^\nu(w)|^2 d\mu(w) \geq C\mu(B_1(ie))|\Delta^{-(\nu+\fract{1}{r})}(\fract{z}{i} + e)|^2 \Delta^{\nu+\fract{1}{r}}(\fract{z}{i}).$$

It follows that if $\tilde{\mu}(z) \in L^p(D, d\lambda)$, then we should have

$$\int_{D} |\Delta^{-(\nu+\fract{1}{r})}(\fract{z}{i} + e)|^2 p(\nu+\fract{1}{r})(\fract{z}{i})\frac{dV(z)}{\Delta^{\frac{1}{2}}(\fract{z}{i})} < \infty$$

which by Lemma 2.6 is possible only if $p(\nu + \fract{1}{r}) > 2\fract{1}{r} - 1$.

Now the equivalences $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ are from Lemma 2.11. The equivalence $(i) \Leftrightarrow (ii)$ is derived from Lemma 3.1 and Lemma 3.2. The proof is complete.

3.2. Schatten class for general operators. We consider here Schatten class criteria for an arbitrary operator defined on $A^2_0(D)$ with values in a Hilbert space $\mathcal{H}$. We denote by $\mathcal{B}(A^2(D), \mathcal{H})$ the set of bounded operators from $A^2_0(D)$ to $\mathcal{H}$. To avoid any confusion, we denote by $\langle \cdot, \cdot \rangle_\mathcal{H}$ and $\langle \cdot, \cdot \rangle_\nu$ the inner products in $\mathcal{H}$ and $A^2_0(D)$ respectively. We start with the Hilbert-Schmidt class $\mathcal{S}_2 := \mathcal{S}_2(A^2_0(D), \mathcal{H})$.

**Proposition 3.3.** Let $T \in \mathcal{B}(A^2(D), \mathcal{H})$ then

$$||T||^2_{\mathcal{S}_2(A^2_0(D), \mathcal{H})} = C_{n,m} \int_{D} ||T(k_\nu^\nu)||^2_{\mathcal{H}} d\lambda(z),$$

for every integer $m \geq 0$.

**Proof.** This result was proved in [17]. As the definition of Bergman spaces here is quite different, let us give a proof here for completeness. Let $\{e_j\}$ is an orthonormal basis of $\mathcal{H}$, then

$$\int_{D} ||T(k_\nu^\nu)||^2_{\mathcal{H}} \Delta^{2m+\nu}(\fract{z}{i}) dV(z) = \sum_{j=0}^{\infty} \int_{D} ||T(k_\nu^\nu, e_j)||^2_{\mathcal{H}} \Delta^{2m+\nu}(\fract{z}{i}) dV(z)$$

$$= \sum_{j=0}^{\infty} \int_{D} ||T(k_\nu^\nu, T^* e_j)||^2_{\mathcal{H}} \Delta^{2m+\nu}(\fract{z}{i}) dV(z)$$

$$= \sum_{j=0}^{\infty} \int_{D} ||k_\nu^\nu T^* e_j||^2_{\mathcal{H}} \Delta^{2m+\nu}(\fract{z}{i}) dV(z)$$

$$= \sum_{j=0}^{\infty} \int_{D} ||T^* e_j||^2_{\mathcal{H}} d\lambda(z)$$

$$= C_{n,m} \sum_{j=0}^{\infty} \int_{D} ||T^* e_j||^2_{A^2_0}$$

$$= C_{n,m} ||T^*||^2_{A^2_0} = C_{n,m} ||T||^2_{\mathcal{S}_2}.$$
In the fourth equality, we used the fact that \( \Box z^m \) is an isometric (up to constant \( C_{n,m} \)) isomorphism from \( A_n^2(D) \) onto \( A_{2m+\nu}(D) \).

We will deduce some results from the above one. The first one is the following which follows as in [17, Lemma 3.2].

**Proposition 3.4.** Suppose that \( T \in B(A_n^2(D), \mathcal{H}) \). Let \( m \geq 0 \) be an integer. Then

i) if \( T \in S_p(A_n^2(D), \mathcal{H}) \) for \( 2 < p < \infty \), then

\[
\int_D \| T(k_z^{\nu,m}) \|_H^p d\nu(z) \leq C_{n,m}\| T \|_{S_p(A_n^2(D), \mathcal{H})}^p.
\]

ii) If for \( 0 < p < 2 \),

\[
\int_D \| T(k_z^{\nu,m}) \|_H^p d\nu(z) < \infty,
\]

then \( T \in S_p(A_n^2(D), \mathcal{H}) \). Moreover,

\[
\| T \|_{S_p(A_n^2(D), \mathcal{H})}^p \leq C_{n,m}\int_D \| T(k_z^{\nu,m}) \|_H^p d\nu(z).
\]

We next have the following which is in fact implicit in the proof of the above result in [17].

**Proposition 3.5.** Suppose that \( T \in B(A_n^2(D)) \) is a positive operator. Let \( m \geq 0 \) be an integer. Then,

i) if \( T \in S_p(A_n^2(D)) \) for \( 1 \leq p < \infty \), then

\[
\int_D |\langle T(k_z^{\nu,m}), k_z^{\nu,m} \rangle \|_\nu^p d\nu(z) \leq C_{n,m}\| T \|_{S_p(A_n^2(D))}^p,
\]

ii) If for \( 0 < p \leq 1 \),

\[
\int_D |\langle T(k_z^{\nu,m}), k_z^{\nu,m} \rangle \|_\nu^p d\nu(z) < \infty,
\]

then \( T \in S_p(A_n^2(D)) \). Moreover,

\[
\| T \|_{S_p(A_n^2(D))}^p \leq C_{n,m}\int_D |\langle T(k_z^{\nu,m}), k_z^{\nu,m} \rangle \|_\nu^p d\nu(z).
\]

**Proof.** From the proof of Proposition 3.3, we have that if \( T \in S_1(A_n^2(D)) \) is a positive operator, then

\[
Tr(T) = \| T^{1/2} \|_{S_2}^2 = C_{n,m}\int_D |\langle T(k_z^{\nu,m}), k_z^{\nu,m} \rangle \|_\nu d\nu(z).
\]

Recalling that \( \| T \|_{S_p}^p = Tr(T^p) \), the proof follows from the fact that for any unit vector (see [13]) in \( g \in L^2(D) \), we have

\[
\langle Tg, g \rangle_\nu^p \leq \langle T^p g, g \rangle_\nu, \quad \text{if} \quad p \geq 1
\]

and

\[
\langle T^p g, g \rangle_\nu \leq \langle Tg, g \rangle_\nu^p \quad \text{if} \quad 0 < p \leq 1.
\]

\( \Box \)
Hence it follows from this and Lemma 2.8 that we recall our notation.

\[ \int_D \langle T_\mu(k_z^{\nu,m}), k_z^{\nu,m} \rangle d\lambda(z) \leq C_{n,m} \|T_\mu\|^p_{\mathcal{S}_p(A^2_\nu(D))}. \]

ii) If for \(0 < p \leq 1\),

\[ \int_D \langle T_\mu(k_z^{\nu,m}), k_z^{\nu,m} \rangle^p d\lambda(z) \leq \infty, \]

then \(T_\mu \in \mathcal{S}_p(A^2_\nu(D))\). Moreover,

\[ \|T_\mu\|^p_{\mathcal{S}_p(A^2_\nu(D))} \leq C_{n,m} \int_D \langle T_\mu(k_z^{\nu,m}), k_z^{\nu,m} \rangle^p d\lambda(z). \]

We next prove the following sufficient condition.

**Lemma 3.7.** Let \(1 \leq p < \infty\), and let \(m \geq 0\) be an integer. Assume that \(\mu\) is a positive measure on \(D\). Then if the Toeplitz operator \(T_\mu\) satisfies

\[ \int_D \langle T_\mu(k_z^{\nu,m}), k_z^{\nu,m} \rangle d\lambda(z) \leq \infty, \]

then \(T_\mu \in \mathcal{S}_p(A^2_\nu(D))\). Moreover,

\[ \|T_\mu\|^p_{\mathcal{S}_p(A^2_\nu(D))} \leq C_{n,m} \int_D \langle T_\mu(k_z^{\nu,m}), k_z^{\nu,m} \rangle^p d\lambda(z). \]

**Proof.** As \(p \geq 1\), we only need to prove that there is a positive constant \(C\) such that for any orthonormal sequence \(\{e_k\}\) on \(A^2_\nu(D)\),

\[ \sum_k \langle T_\mu e_k, e_k \rangle^p \leq C \int_D \langle T_\mu(k_z^{\nu,m}), k_z^{\nu,m} \rangle^p d\lambda(z). \]

We recall our notation

\[ \tilde{\mu}^m(z) := \langle T_\mu k_z^{\nu,m}, k_z^{\nu,m} \rangle = \int_D |k_z^{\nu,m}(w)|^2 d\mu(w). \]

We start by noting that by Lemma 2.5, we have

\[ |e_k(z)|^2 \leq C \delta^{-n} \int_{B_i(z)} |e_k(w)|^2 d\lambda(w). \]

It follows from this and Lemma 2.8 that

\[ |e_k(z)|^2 \leq C \int_D |e_k(w)|^2 |k_z^{\nu,m}(w)|^2 dV_\nu(w) \]

Hence

\[ \langle T_\mu e_k, e_k \rangle = \int_D |e_k(z)|^2 d\mu(z) \leq C \int_D |e_k(w)|^2 \tilde{\mu}^m(w) dV_\nu(w). \]
and so using Hölder’s inequality, that the $e_k$s are orthonormal and
\[ \sum_k |e_k(z)|^2 \leq \|K_z\nu\|_{2,\nu}^2, \]
we obtain
\[ \sum_k \langle T\mu e_k, e_k \rangle_p \leq \sum_k \left( \int_D |e_k(w)|^2 \tilde{\mu}_m(w) d\nu(w) \right)^p \]
\[ \leq \int_D (\tilde{\mu}_m(w))^p \left( \sum_k |e_k(w)|^2 \right) d\nu(w) \]
\[ \leq \int_D (\tilde{\mu}_m(w))^p \|K_w\nu\|_{2,\nu}^2 d\nu(w) \]
\[ \lesssim \int_D (\tilde{\mu}_m(w))^p d\lambda(w). \]

The proof is complete. \qed

We now prove the following necessary condition.

**Lemma 3.8.** Let $m \geq 0$ be an integer such that $\max\{\frac{2n-1}{n+1} \cdot \frac{1}{p}, \frac{2n-1}{n+1} \cdot \frac{3}{p}\} < p \leq 1$. Assume that $\mu$ is a positive measure on $D$. Then if the Toeplitz operator $T\mu$ belongs to the Schatten class $S_p(A^2_\nu(D))$, then
\[ \int_D \langle T\mu (k^{\nu,m}z), k^{\nu,m}_\nu z \rangle_p d\lambda(z) < \infty. \]
Moreover,
\[ \int_D \langle T\mu (k^{\nu,m}z), k^{\nu,m}_\nu z \rangle_p d\lambda(z) \lesssim \|T\mu\|^p_{S_p(A^2_\nu(D))}. \]

**Proof.** Assume that the Toeplitz operator $T\mu$ belongs to the Schatten class $S_p(A^2_\nu(D))$. Then by Lemma 3.1, this implies that for any $\delta$-lattice $\{\zeta_k\}$ of points of $D$, the sequence $\{\hat{\mu}\delta(\zeta_k)\}$ belongs to $l^p$ with
\[ \sum_k (\hat{\mu}\delta(\zeta_k))^p \lesssim \|T\mu\|^p_{S_p}. \]
It follows that to prove the above lemma, it is enough to prove that there is positive constant $C$ such that for any $\delta$-lattice $\{\zeta_k\}$ of points of $D$,
\[ \int_D \langle T\mu (k^{\nu,m}z), k^{\nu,m}_\nu z \rangle_p d\lambda(z) \leq C \sum_k (\hat{\mu}\delta(\zeta_k))^p. \]
Recalling that $0 < p < 1$ and using Lemma 2.8, we first obtain

\[
L := \int_D (\tilde{\mu}^m(z))^p \, d\lambda(z)
\]

\[
= \int_D \left( \int_D |\nu^{\nu,m}(w)|^2 d\mu(w) \right)^p \, d\lambda(z)
\]

\[
\leq \int_D \left( \sum_k \int_{B_k} |\nu^{\nu,m}(w)|^2 d\mu(w) \right)^p \, d\lambda(z)
\]

\[
\leq C \int_D \left( \sum_k |\nu^{\nu,m}(z_k)|^2 \mu(B_k) \right)^p \, d\lambda(z)
\]

\[
\leq C \int_D \left( \sum_k \frac{\Delta^{p(\nu+2m+\frac{2m}{7})} \eta(z)}{\Delta^{2p} \eta(z)} \right) \, dV(z)
\]

\[
\leq C \sum_k \left( \frac{\mu(B_k)}{\Delta^{p(\nu+2m+\frac{2m}{7})} \eta(z) \Delta^{2p} \eta(z)} \right) \left( \nu^{\nu,m}(\zeta_k) \right)^2 \Delta^{2p} \eta(z) \, dV(z).
\]

The condition on $p$ and Lemma 2.6 give us

\[
\int_D |\nu^{\nu,m}(z_k)|^{2p} \Delta^{p(\nu+2m+\frac{2m}{7})-2p} \, dV(z) = C \Delta^{-p(\nu+2m)} \eta(z_k).
\]

We then conclude that

\[
\int_D (\tilde{\mu}^m(z))^p \, d\lambda(z) \leq C \sum_k \left( \frac{\mu(B_k)}{\Delta^{p(\nu+2m+\frac{2m}{7})} \eta(z) \Delta^{2p} \eta(z)} \right) \left( \nu^{\nu,m}(\zeta_k) \right)^2 \Delta^{2p} \eta(z) \, dV(z).
\]

The proof is complete. \hfill \square

Theorem 1.3 clearly follows from Corollary 3.6, Lemma 3.7 and Lemma 3.8.

4. Application to Cesàro-type operators

We consider the following equivalence class

\[
N_n := \{ F \in \mathcal{H}(D) : \square^n F = 0 \}
\]

and set

\[
\mathcal{H}_n(D) = \mathcal{H}(D)/N_n.
\]

For $g \in \mathcal{H}(D)$, we define the operator $T_g$ as follows: for $f \in \mathcal{H}(D)$, $T_g f$ is the equivalence class of the solutions of the equation

\[
\square^n F = f \square^n g.
\]

The operator $T_g$ was called in [13] Cesàro-type operator and it was remarked in the same paper that its definition does not depend on the choice of the representative of the class of the symbol.

We consider in this part, criteria for Schatten class membership of the Cesàro-type operator above on the weighted Bergman space $A^2_p(D)$. In fact, a characterization of Schatten classes for this operator was obtained in [13] for the range $2 \leq p \leq \infty$. Our aim here is to extend this result to the range
1 ≤ p < 2. We refer to [8,12,19,20] for the corresponding results on some classical domains.

Let us recall that the Besov space \( B^p(D) \) is the subset of \( H_n(D) \) consisting of functions \( f \) such that \( \Delta^n \square^n f \in L^p(D,d\lambda) = L^p(D,\frac{dV(z)}{\Delta^2(3z)}) \). For more on Besov spaces of tube domains over symmetric cones, we refer the reader to [3,4].

We now obtain the following.

**Theorem 4.1.** Let \( 1 \leq p < 2 \), \( \nu > \frac{p}{n} - 1 \). If \( g \) is a given holomorphic function in \( D \), then the Cesàro-type operator \( T_g \) belongs to \( S_p(A^2_p(D)) \) if and only if \( g \in B^p(D) \).

**Proof.** Let us first assume that \( T_g \in S_p(A^2_p(D)) \). Then by Proposition 3.5, we have that

\[
\int_D |\langle T_g(k^{\nu,n}_z), k^{\nu,n}_z \rangle| p d\lambda(z) \leq C \|T_g\|^p_{S_p(A^2_p(D))}.
\]

Using reproducing formula, we obtain

\[
\|T_g\|^p_{S_p(A^2_p(D))} \geq C \int_D |\langle T_g(k^{\nu,n}_z), k^{\nu,n}_z \rangle| p d\lambda(z)
\]

\[
= C_n \int_D |\langle \square^n(T_gk^{\nu,n}_z), k^{\nu,n}_z \rangle| p d\lambda(z)
\]

\[
= C_n \int_D |\langle \square^n g(z), k^{\nu,n}_z \rangle| p |\Delta \square^{(\nu+\frac{2}{r}+2n)}(z)| d\lambda(z)
\]

\[
= C_n \int_D |\langle \square^n g(z), k^{\nu,n}_z \rangle| p |\Delta \square^{(\nu+\frac{2}{r}+2n)}(z)| d\lambda(z)
\]

\[
= C_n \int_D |\Delta^n(3z) \square^n g(z)|^p d\lambda(z).
\]

Hence \( g \in B^p(D) \) if \( T_g \in S_p(A^2_p(D)) \).

Now assume that \( g \in B^p(D) \). We consider the following measure

\[
d\mu(z) = |\square^n g(z)|^2 \Delta^{2n+\nu-n/r}(3z) dV(z).
\]

We first observe the following. Let \( \{\zeta_k\} \) be a \( \delta \)-lattice of points of \( D \). Using Lemma 2.3, we obtain

\[
\int_D |\Delta^n(3z) \square^n g(z)|^p d\lambda(z) \leq \sum_j \left( |\square^n g(\zeta_j)|^2 \Delta^{2n}(3\zeta_j) \right)^{p/2}
\]

\[
\times \sum_j \left( \int_{B_j} |\square^n g(z)|^2 \Delta^{2n}(3z) \frac{dV(z)}{\Delta^{2n/r}(3z)} \right)^{p/2}
\]

\[
\times \sum_j \left( \frac{1}{\Delta^{\nu+n/r}(3\zeta_j)} \int_{B_j} d\mu(z) \right)^{p/2}
\]

\[
= \sum_j \left( \frac{\mu(B_j)}{\Delta^{\nu+n/r}(3\zeta_j)} \right)^{p/2}.
\]
That is
\begin{equation}
\int_{\mathcal{D}} |\Delta^n(\Im z)\Box^n g(z)|^p d\lambda(z) \asymp \sum_j \left( \frac{\mu(B_j)}{\Delta^{\nu+n/r}(\Im \zeta_j)} \right)^{p/2}.
\end{equation}

We next observe the following. Let \{e_j\} be any orthonormal basis of \(A^2_{\nu}(\mathcal{D})\). Then using Hölder’s inequality, we obtain
\[
\langle T_g e_j, e_j \rangle_{\nu} = \langle \Box^n T_g e_j, e_j \rangle_{\nu+n} = \langle \Delta^n \Box^n T_g e_j, e_j \rangle_{\nu} \\
\leq \int_{\mathcal{D}} |e_j(z)|^2 d\mu(z) = \langle T_\mu e_j, e_j \rangle_{\nu}.
\]

It follows that
\begin{equation}
\sum_j |\langle T_g e_j, e_j \rangle_{\nu}|^p \leq \sum_j |\langle T_\mu e_j, e_j \rangle_{\nu}|^{p/2}.
\end{equation}

Hence, as \(p \geq 1\), to prove that \(T_g\) belongs to \(S_p(A^2_{\nu}(\mathcal{D}))\), it suffices by (4.2) to prove that
\[
\sum_j |\langle T_\mu e_j, e_j \rangle_{\nu}|^{p/2} < \infty.
\]

This follows as in the proof of Lemma 3.1, using that as \(g \in B^p(\mathcal{D})\), we have by (4.1) that
\[
\sum_j \left( \frac{\mu(B_j)}{\Delta^{\nu+n/r}(\Im \zeta_j)} \right)^{p/2} < \infty.
\]

The proof is complete. \(\square\)

References

[1] H. Arroussi, I. Park, J. Pau, Schatten class Toeplitz operators acting on large weighted Bergman spaces. Studia Math. 229 (2015), no. 3, 203-221.
[2] D. Bekollé, A. Bonami, G. Garrigós, C. Nana, M. Peloso and F. Ricci, Lecture notes on Bergman projectors in tube domains over cones: an analytic and geometric viewpoint, IMHOTEP 5 (2004), Exposé I, Proceedings of the International Workshop in Classical Analysis, Yaoundé 2001.
[3] D. Bekollé, A. Bonami, G. Garrigós and F. Ricci, Littlewood-Paley decompositions related to symmetric cones and Bergman projections in tube domains, Proc. London Math. Soc. 89 (2004), 317-360.
[4] D. Bekollé, A. Bonami, G. Garrigós, F. Ricci and B. Sehba, Hardy-type inequalities and analytic Besov spaces in tube domains over symmetric cones, J. Reine Angew. Math. 647 (2010), 25-56.
[5] D. Bekollé, A. Bonami, M. Peloso and F. Ricci, Boundedness of weighted Bergman projections on tube domains over tight cones, Math. Z. 237 (2001), 31-59.
[6] D. Bekollé, H. Ishi and C. Nana, Kordányi’s Lemma for homogeneous Siegel domains of type II. Applications and extended results, Bull. Aust. Math. Soc. 90 (2014), 77-89.
[7] D. Bekollé and A. Temgoua, Molecular decompositions and interpolation, Integ. Equat. Oper. Theor. 31 (1998), 150-177.
[8] O. Constantin, Carleson embeddings and some classes of operators on weighted Bergman spaces, J. Math. Anal. Appl. 365 (2010) 668-682.
[9] D. Debertol, *Besov spaces and boundedness of weighted Bergman projections over symmetric tube domains*, Dottorato di Ricerca in Matematica, Università di Genova, Politecnico di Torino, (April 2003).

[10] J. Faraut, A. Korányi, *Analysis on symmetric cones*, Clarendon Press, Oxford, (1994).

[11] J. Isralowitz, J. Virtanen, L. Wolf, Schatten class Toeplitz operators on generalized Fock spaces. J. Math. Anal. Appl. 421 (2015), no. 1, 329-337.

[12] D. Luecking, *Trace ideal criteria for Toeplitz operators*, J. Funct. Anal. 73 (2) (1987) 345-368.

[13] C. Nana, B. F. Sehba, *Carleson embeddings and two operators on Bergman spaces of tube domains over symmetric cones*, Integral Equations Operator Theory 83 (2015), no. 2, 151-178.

[14] R. Olivier, D. Pascuas, Toeplitz operators on doubling Fock spaces. (English summary) J. Math. Anal. Appl. 435 (2016), no. 2, 14261457.

[15] J. Pau, *A remark on Schatten class of Toeplitz operators on Bergman spaces*, Proceed. Amer. Math. Soc. 142 (8) (2014), 2763-2768.

[16] B. F. Sehba *Bergman type operators in tubular domains over symmetric cones*, Proc. Edin. Math. Soc. 52 (2) (2009), 529-544.

[17] B. F. Sehba *Hankel operators on Bergman spaces of tube domains over symmetric cones*, Integr. equ. oper. theory 62 (2008), 233-245.

[18] K. Zhu, *Operator theory in function spaces*, Marcel Dekker, New York 1990.

[19] K. Zhu *Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domain*, J. Oper. Theory 20 (1988), 329-357.

[20] K. Zhu, *Schatten class Toeplitz operators on the weighted Bergman spaces of the unit ball*, New York J. Math. 13 (2007), 299-316.

Department of Mathematics, University of Ghana, P. O. Box LG 62 Legon, Accra, Ghana

E-mail address: bfsehba@ug.edu.gh