A SIMPLE BIJECTION BETWEEN BINARY TREES AND COLORED TERNARY TREES

Yidong Sun
Department of Mathematics, Dalian Maritime University, 116026 Dalian, P.R. China
sydmath@yahoo.com.cn

Abstract. In this short note, we first present a simple bijection between binary trees and colored ternary trees and then derive a new identity related to generalized Catalan numbers.

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1. Introduction

Recently, Mansour and the author [1] obtained an identity involving 2-Catalan numbers and 3-Catalan numbers, i.e.,

\[ \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{1}{3p+1} \binom{3p+1}{p} \binom{n+p}{3p} = \frac{1}{n+1} \binom{2n}{n}. \]  

In this short note, we first present a simple bijection between binary trees and ternary trees and then derive a general identity, i.e.,

\[ \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{m}{3p+m} \binom{3p+m}{p} \binom{n+p+m-1}{n-2p} = \frac{m}{2n+m} \binom{2n+m}{n}. \]

2. A bijective algorithm for binary and ternary trees

A colored ternary tree is a complete ternary tree such that all its vertices are signed a nonnegative integer called color number. Let \( T_{n,p} \) denote the set of colored ternary trees \( T \) with \( p \) internal vertices such that the sum of all the color numbers of \( T \) is \( n-2p \). Define \( T_n = \bigcup_{p=0}^{\lfloor n/2 \rfloor} T_{n,p} \). Let \( B_n \) denote the set of complete binary trees with \( n \) internal vertices. For any \( B \in B_n \), let \( P = v_1v_2 \cdots v_k \) be a path of length \( k \) of \( B \) (viewing from the root of \( B \)).

\( P \) is called a \( R \)-path, if (1) \( v_i \) is the right child of \( v_{i-1} \) for \( 2 \leq i \leq k \) and (2) the left child of \( v_i \) is a leaf for \( 1 \leq i \leq k \). In addition, \( P \) is called a maximal \( R \)-path if there exists no vertex \( u \) such that \( uP \) or \( Pu \) forms a \( R \)-path. \( P \) is called an \( L \)-path, if \( v_k \) is the left child of \( v_{k-1} \) for \( 2 \leq i \leq k \). \( P \) is called a maximal \( L \)-path if there exists no vertex \( u \) such that \( uP \) or \( Pu \) forms an \( L \)-path.

Note that the definition of \( L \)-path is different from that of \( R \)-path. Hence, if \( P \) is a maximal \( R \)-path, then (1) the right child \( u \) of \( v_k \) must be a leaf or the left child of \( u \) is not a leaf; (2) \( v_1 \) must be a left child of its father (if exists) or the father of \( v_1 \) has a left child which is not a leaf. If \( P \) is a maximal \( L \)-path, then (1) \( v_k \) must be a leaf which is also a left child of \( v_{k-1} \); (2) \( v_1 \) must be the right child of its father (if exists).
Theorem 2.1. There exists a simple bijection $\phi$ between $B_n$ and $T_n$.

Proof. We first give the procedure to construct a complete binary tree from a colored complete ternary tree.

Step 1. For each vertex $v$ of $T \in T_n$ with color number $c_v = k$, remove the color number and add a $R$-path $P = v_1v_2 \cdots v_k$ of length $k$ to $v$ such that $v$ is a right child of $v_k$ and $v_1$ is a child of the father (if exists) of $v$, and then annex a left leaf to $v_i$ for $1 \leq i \leq k$. See Figure 1(a) for example.

Step 2. Let $T^*$ be the tree obtained from $T$ by Step 1. For any internal vertex $v$ of $T^*$ which has out-degree 3, let $T_1, T_2$ and $T_3$ be the three subtree of $v$. Remove the subtree $T_1$ and $T_2$, annex a left child $v'$ to $v$ and take $T_1$ and $T_2$ as the left and right subtree of $v'$ respectively. See Figure 1(b) for example.

It is clear that any $T \in T_n$, after Step 1 and 2, generates a binary tree $B \in B_n$.

Conversely, we can obtain a colored ternary tree from a complete binary tree as follows.

Step 3. Choose any maximal $L$-path of $B \in B_n$ of length $k \geq 3$, say $P = v_1v_2 \cdots v_k$, then each $v_{2i-1}$ absorbs its left child $v_{2i}$ for $1 \leq i \leq [(k-1)/2]$. See Figure 2(a) for example.

Step 4. Choose any maximal $R$-path of $T'$ derived from $B$ by Step 3, say $Q = u_1u_2 \cdots u_k$, let $u$ be the right child of $u_k$, then $u$ absorbs all the vertex $u_1, u_2, \ldots, u_k$ and assign the color number $c_u = k$ to $u$. See Figure 2(b) for example. Hence we get a colored ternary tree.
Given a complete ternary tree $T$ with $p$ internal vertices, there are totally $3p + 1$ vertices, choose $n - 2p$ vertices repeatedly, define the color number of a vertex to be the times of being chosen. Then there are $\binom{n + p}{n - 2p}$ colored ternary trees in $\mathcal{S}_n$ generated by $T$. Note that $\frac{1}{3p + 1}\binom{3p + 1}{p}$ and $\frac{1}{2n + 1}\binom{2n + 1}{n}$ count the number of complete ternary trees with $p$ internal vertices and complete binary trees with $n$ internal vertices respectively [2]. Then the bijection $\phi$ immediately leads to (1.1).

To prove (1.2), consider the forest of colored ternary trees $F = (T_1, T_2, \ldots, T_m)$ with $T_i \in \mathcal{S}_{n_i}$ and $n_1 + n_2 + \cdots + n_m = n$, define $\phi(F) = (\phi(T_1), \phi(T_2), \ldots, \phi(T_m))$, then it is clear that $\phi$ is a bijection between forests of colored ternary trees and forests of complete binary trees. Note that there are totally $m + 3p$ vertices in a forest $F$ of complete ternary trees with $m$ components and $p$ internal vertices, so there are $\binom{m + n + p - 1}{n - 2p}$ forests of colored ternary trees with $m$ components, $p$ internal vertices and the sum of color numbers equal to $n - 2p$. It is clear [2] that $\frac{m}{3p + m}\binom{3p + m}{p}$ counts the number of forests of complete ternary trees with $p$ internal vertices and $m$ components, and that $\frac{m}{3n + m}\binom{2n + m}{n}$ counts the number forests of complete binary trees with $n$ internal vertices and $m$ components. Then the above bijection $\phi$ immediately leads to (1.2).

### 3. Further Comments

It is well known [2] that the $k$-Catalan number $C_{n,k} = \frac{1}{kn + 1}\binom{kn + 1}{n}$ counts the number of complete $k$-ary trees with $n$ internal vertices, whose generating function $C_k(x)$ satisfies

$$C_k(x) = 1 + xC_k(x).$$

Let $G(x) = \frac{1}{1-x}C_3\left(\frac{x^2}{(1-x)^3}\right)$, then one can deduce that

$$G(x) = \frac{1}{1-x}\frac{1}{C_3}\left(\frac{x^2}{(1-x)^3}\right)$$

$$= \frac{1}{1-x}\left(1 + \frac{x^2}{(1-x)^3}C_3\left(\frac{x^2}{(1-x)^3}\right)\right)$$

$$= \frac{1}{1-x}\left(1 + x^2G^3(x)\right),$$

which generates that $G(x) = C_3(x)$ which is the generating function for Catalan numbers.

By Lagrange inversion formula, we have

$$C_3^n(x) = \sum_{p \geq 0} \frac{m}{3p + m}\binom{3p + m}{p} x^p, \quad C_2^n(x) = \sum_{n \geq 0} \frac{m}{2n + m}\binom{2n + m}{n} x^n.$$

Then

$$G^n(x) = \sum_{p \geq 0} \frac{m}{3p + m}\binom{3p + m}{p} \frac{x^{2p}}{(1-x)^{3p+m}}$$

$$= \sum_{n \geq 0} \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{m}{3p + m}\binom{3p + m}{p}\binom{n + p + m - 1}{n - 2p} x^n.$$

Comparing the coefficient of $x^n$ in $C_2^n(x)$ and $G^n(x)$, one obtains Theorem 2.1.
Similarly, let \( F(x) = \frac{1}{1-x} C_k \left( \frac{x^{k-1}}{(1-x)^k} \right) \), then \( F(x) = \frac{1+x F(x)}{1-x^{k-1} F(x)} \), using Lagrange inversion formula for the case \( k = 5 \), one has

\[
\sum_{p=0}^{\lfloor n/4 \rfloor} \frac{m}{5p + m} \binom{5p + m}{p} \binom{n + p + m - 1}{n - 4p} = \sum_{p=0}^{\lfloor n/2 \rfloor} (-1)^p \frac{m}{m + n} \binom{m + n + p - 1}{p} \binom{m + 2n - 2p - 1}{n - 2p},
\]

which, in the case \( m = 1 \), leads to

\[
\sum_{p=0}^{\lfloor n/4 \rfloor} \frac{1}{4p + 1} \binom{5p}{p} \binom{n + p}{5p} = \sum_{p=0}^{\lfloor n/2 \rfloor} (-1)^p \frac{1}{n + 1} \binom{n + p}{n} \binom{2n - 2p}{n}.
\]

One can be asked to give a combinatorial proof of (3.1) or (3.2).

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**References**

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[2] R. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Univ. Press, Cambridge, 1999.