HOMOTOPY INVARIANT COMMUTATIVE ALGEBRA OVER FIELDS

J.P.C. GREENLEES

Abstract. This article grew out of lectures given in the programme ‘Interactions between Representation Theory, Algebraic Topology and Commutative Algebra’ (IRTATCA) at the CRM (Barcelona) in Spring 2015. They give some basic homotopy invariant definitions in commutative algebra and illustrate their interest by giving a number of examples.

Contents

Part 1. Context
1. Introduction
2. Ring and module spectra I (motivation)
3. Ring and module spectra II (construction)
4. The three examples
5. Finiteness conditions

Part 2. Morita equivalences
6. Morita equivalences
7. Proxy-smallness in the examples
8. Exterior algebras

Part 3. Regular rings
9. Regular ring spectra
10. Finite generation

Part 4. Hypersurface rings
11. Complete intersections
12. Hypersurfaces in algebra
13. Bimodules and natural endomorphisms of $R$-modules
14. Hypersurface ring spectra
15. s-hypersurface spaces and z-hypersurface spaces
16. Growth of z-hypersurface resolutions

Part 5. Gorenstein rings
17. The Gorenstein condition
18. Gorenstein duality

I am grateful to the CRM and the Simons Foundation for the opportunity to spend time in the stimulating environment at the CRM during the IRTATCA programme, and to give these lectures.
Part 1. Context

1. Introduction

1.A. The lectures. The purpose of these lectures is to illustrate how powerful it can be to formulate ideas of commutative algebra in a homotopy invariant form. The point is that it shows the concepts are robust, in the sense that they are invariant under deformations. They can then be applied to derived categories of rings, but of course derived categories of rings are just one example of a homotopy category of a stable model category. In short, these ideas are powerful in classical algebra, in representation theory of groups, in classical algebraic topology and elsewhere. We will focus on the fact that this includes many striking new examples from topology.

Indeed, our principal example will be that arising from a topological space $X$, which is to say we will work over the ring $R = C^*(X; k)$ of cochains on a space $X$ with coefficients in a field $k$. Of course this is a rather old idea, applied in rational homotopy theory to give a very rich theory and many interesting examples. The key is that $R$ needs to be commutative: classically, one needs to work over the rationals so that there is a commutative model for cochains (coming from piecewise polynomial differential forms). It is well known that there is no natural model for $C^*(X; k)$ which is a commutative differential graded algebra (commutative DGA) if $k$ is not of characteristic 0, since the Steenrod operations (which do not usually vanish) are built out of the non-commutativity. However, there is a way round this, since we can relax our requirements for what we consider a model. Instead of requiring DGAs, we permit $R$ to be a commutative ring spectrum in the sense of homotopy theory (see Section 3 for a brief introduction). There is then a commutative model for cochains; we will continue to use the notation $C^*(X; k)$ even though it is now a ring spectrum rather than a DGA.

This may be our principal example, but actually it is a very restricted one. It is like restricting graded commutative rings to be negatively graded $k$-algebras. We will hint at a number of examples coming from derived algebraic geometry.

The main point of the lectures is to give some interesting examples, and the main source of examples will be modular representation theory and group cohomology. In fact, the focus
will be on examples picked out by requiring the ring spectrum to be well behaved in terms of commutative algebra: we look for regular, complete intersection or Gorenstein ring spectra. As one expects from commutative algebra, regular ring spectra are very restricted and can sometimes even be classified. One also expects to be able to parametrise complete intersections in some sense, although our present account just mentions a few isolated examples. Finally, just as is the case for conventional commutative rings, the class of Gorenstein ring spectra has proved ubiquitous. We shall see examples from representation theory, from chromatic stable homotopy theory and from rational homotopy theory, and one of the distinctive contributions is to emphasize duality.

1.B. 54321. It may be worth bearing in mind the following correspondences, which suggest useful ways to think about the material here.

**The pentagon:** We may summarize sources examples and the relationship between them in a diagram. The emphasis here will be on commutative algebra and spaces.

```
Commutative Algebra

DAG

Chromatic homotopy theory

Spaces and DGAs over $\mathbb{Q}$

Representation theory of $G$

Topological spaces
```

**The square:** It is worth bearing in mind the dual points of view of geometric objects and rings of functions on them, and the way the global picture is built up out of points. Our present account focuses on the local rings at the bottom right, but it can be illuminating to consider the wider context.

```
Geometry & Algebra

Global

$\mathcal{X}$ $\hookrightarrow$ $\mathcal{O}_{\mathcal{X}}$

Local

$\mathcal{X}_x$ $\hookrightarrow$ $\mathcal{O}_{\mathcal{X},x}$

$X$ $\mapsto$ $C^*(X; k)$
```

**The three classes:** our focus is on the following hierarchy of local rings.

```
Regular $\hookrightarrow$ Complete Intersection $\hookrightarrow$ Gorenstein
```

**The dual pair:** even though our focus is on commutative rings, we make constant use of the freedom to use Morita theory to transpose properties into the non-commutative context.

```
Commutative $\leftrightarrow$ Non-commutative
```
It is the purpose of the present article to give an account of these objects and points of view as a

unified whole.

1.C. Organization. The article is organized into five parts. In Part 1 we introduce the context of ring spectra and modules over them, together with our three main classes of examples (commutative rings, spaces and representation theory) and the basic finiteness conditions.

In Part 2 (Sections 6 to 8) we describe the basic Morita equivalences between commutative and non-commutative contexts, discuss our fundamental finiteness condition and illustrate its power by considering the richness of exterior algebras.

We then turn to our hierarchy of local rings. In Part 3 (Sections 9 and 10) we consider regular ring spectra; the principle of considering definitions in three styles (structural, growth or module theoretic) recurs later, and the good behaviour of modules over regular rings is used later to give a definition of ‘finitely generated’ modules for many ring spectra.

In Part 4 (Sections 11 to 16) we consider hypersurface ring spectra. We spend a little time recalling the classical algebraic theory before introducing the definitions for ring spectra, illustrating it with examples and explaining how the different versions are related.

Finally in Part 5 (Sections 17 to 23) we consider Gorenstein ring spectra, and the duality properties they often enjoy. We describe the basic tools for Gorenstein ring spectra and illustrate them by explaining many known examples (especially those from representation theory and rational homotopy theory). The rich and attractive dualities that appear make this a highlight.

A series of appendices recalls notation and basic properties of local cohomology and derived completion.

Each part starts with a more detailed description of its contents.

1.D. Sources. Much of this account comes from joint work [33, 32, 21, 22, 55, 54] and other conversations with D.J.Benson, W.G.Dwyer, K.Hess, S.B.Iyengar and S. Shamir, and I would like to thank them for their collaboration and inspiration.

We summarize some relevant background, but commutative algebraists may find the articles [52] and [53] from the 2004 Chicago Workshop useful. General background in topology may come from [84] and in conventional commutative algebra from [83, 28].

It is worth emphasizing that this is a vast area, and these lectures will just focus on a tiny part corresponding to local $k$-algebras. In particular we will not touch on arithmetic or geometric aspects of the theory, such as the Galois theory of J. Rognes, Brauer groups, or derived algebraic geometry.

1.E. Conventions. We’ll generally denote rings and ring spectra by letters like, $Q, R$ and $S$. Ring homomorphisms will generally go in reverse alphabetical order, as in $S \rightarrow R \rightarrow Q$. Modules will be denoted by letters like $L, M, N, \ldots$.

The ring of integers (initial amongst conventional rings) is denoted $\mathbb{Z}$. The sphere spectrum (initial amongst ring spectra) is denoted $S$. 

4
After the initial sections we will use the same notation for a conventional ring and the associated Eilenberg-MacLane spectrum. Similarly for modules.

Generally, $M \otimes_R N$ will denote the left derived tensor product, and $\text{Hom}_R(M, N)$ the module of derived homomorphisms. Similarly, fibre sequences, cofibre sequences, pullbacks and pushouts will be derived.

1.F. **Grading conventions.** We will have cause to discuss homological and cohomological gradings. Our experience is that this is a frequent source of confusion, so we adopt the following conventions. First, we refer to lower gradings as *degrees* and upper gradings as *codegrees*. As usual, one may convert gradings to cogradings via the rule $M_n = M^{-n}$. Thus both chain complexes and cochain complexes have differentials of degree $-1$ (which is to say, of codegree $+1$). This much is standard. However, since we need to deal with both homology and cohomology it is essential to have separate notation for homological suspensions ($\Sigma^n$) and cohomological suspensions ($\Sigma_i$): these are defined by

$$(\Sigma^n M)_n = M_{n-i} \quad \text{and} \quad (\Sigma_i M)^n = M^{n-i}.$$ 

Thus, for example, with reduced chains and cochains of a based space $X$, we have

$$C_*(\Sigma^i X) = \Sigma^i C_*(X) \quad \text{and} \quad C^*(\Sigma^i X) = \Sigma_i C^*(X).$$

2. **Ring and module spectra I (motivation)**

Most of the generalities can take place in any category with suitable formal structure: we need a symmetric monoidal product so that we can discuss rings and modules over them, and we need to be able to form tensor products over a commutative ring, and we will want a homotopy theory which is stable so that we can form a triangulated homotopy category. In fact we will work with a category of spectra in the sense of homotopy theory; the construction is sketched in Appendix [3] and the present section gives a brief orientation. A more extensive introduction designed for commutative algebraists is given in [52].

2.A. **Additive structure.** We may start from the observation that the homotopy theory of highly connected spaces is simpler than that of general spaces. By suspending a space we may steadily simplify the homotopy theory, but because cohomology theories have suspension isomorphisms, we do not lose any additive cohomological information: spectra capture the limit of this process. Thus spectra are a sort of abelianization of spaces where behaviour has a more algebraic formal flavour. Associated to any based space $X$ there is a suspension spectrum, and arbitrary spectra can be built from those of this form. The other important source of spectra is as the representing objects for cohomology theories. If $E^*(X)$ is a contravariant functor of a based space $X$ which satisfies the Eilenberg-Steenrod axioms, then there is a spectrum $E$ so that the equation

$$E^*(X) = [X, E]^*$$

holds. This has the usual benefits that one can then apply geometric constructions to cohomology theories and one can argue more easily by universal examples.
2.B. **Multiplicative structure.** Having taken the step of representing cohomology theories by spectra, one may ask if good formal behaviour of the functor $E^*(\cdot)$ is reflected in the representing spectrum $E$. For our purposes the most important piece of structure is that of being a commutative ring, and we would like to say that a cohomology theory whose values on spaces are commutative rings is represented by a spectrum which is a commutative ring in the category of spectra. This is true remarkably often.

In order to do homotopy theory we need a Quillen model structure on the category of spectra, and to have commutative rings in this setting we need a symmetric monoidal smash product, with unit the sphere spectrum $S$ (the suspension spectrum of the two point space $S^0$), so that the two structures are compatible in a way elucidated by Schwede and Shipley [91]. In retrospect it seems strange that such models were not constructed until the 1990s, but several such models are now known, and they give equivalent theories. We sketch the construction of symmetric spectra in Section 3.

In this context, it makes sense to ask for a cohomology theory to be represented by a commutative ring spectrum $R$ (i.e., $R$ comes with a multiplication map $\mu : R \wedge R \to R$ and a unit map $S \to R$ making $R$ into a commutative monoid). Many of the important examples do have this structure. The most obvious example from this point of view is the sphere spectrum $S$ itself. This is the initial ring in the category of spectra, and the smash product plays the role of tensor product over $S$. We will describe a number of examples below, but for the present we continue with the general formalism just assuming that $R$ is a commutative ring spectrum.

2.C. **Modules.** We may consider module spectra over $R$, and there is a model category structure on $R$-modules; furthermore, since $R$ is commutative, there is a tensor product of $R$-modules formed in the usual way from the tensor product over the initial ring $S$ (i.e., as the coequalizer of the two maps $M \otimes_S R \otimes_S N \to M \otimes_S N$). From the good formal properties of the original category, this category of $R$-modules is again a model category with a compatible symmetric monoidal product. This has an associated homotopy category $\text{Ho}(R \text{- mod})$ and will be the context in which we work.

2.D. **Reverse approach.** Commutative algebraists may approach spectra from the algebraic direction. Traditional commutative algebra considers commutative rings $A$ and modules over them, but some constructions make it natural to extend further to considering chain complexes of $A$-modules; the need to consider robust, homotopy invariant properties leads to the derived category $\text{D}(A)$. Once we admit chain complexes, it is natural to consider the corresponding multiplicative objects, differential graded algebras (DGAs). Although it may appear inevitable, the real justification for this process of generalization is the array of naturally occurring examples.

The use of spectra is a natural extension of this process. Shipley has shown [92] that associated to any DGA $A$ there is a ring spectrum $HA$, so that the derived category $\text{D}(A)$ is equivalent to the homotopy category of $HA$-module spectra. Accordingly we can view ring spectra as generalizations of DGAs and categories of module spectra as flexible generalizations of the derived category. Ring spectra extend the notion of rings, module spectra
extend the notion of (chain complexes of) modules over a ring, and the homotopy category
of module spectra extends the derived category. Many ring theoretic constructions extend
to ring spectra, and thus extend the power of commutative algebra to a vast new supply
of naturally occurring examples. Even for traditional rings, the new perspective is often
enlightening, and thinking in terms of spectra makes a number of new tools available. Once
again the only compelling justification for this inexorable process of generalization is the
array of interesting, naturally occurring examples, some of which will be described later in
these lectures.

3. Ring and module spectra II (construction)

The purpose of this section is to outline the construction of symmetric spectra. The details
will not be needed for the lectures, and readers familiar with spectra can comfortably omit
it. The point is to reassure readers that spectra are rather concrete objects.

3.A. Naive spectra. This subsection is designed to explain the idea behind spectra: where
they came from and why they were invented. Those already familiar spectra should skip
directly to Subsection 3.B which describes symmetric spectra.

The underlying idea is that spectra are just stabilised spaces and the bonus is that they
represent cohomology theories. This definition is perfectly good for additive issues, but it
does not have a symmetric monoidal smash product, so is not adequate for commutative
algebra.

Definition 3.1. A spectrum $E$ is a sequence of based spaces $E_k$ for $k \geq 0$ together with
structure maps

$$\sigma : \Sigma E_k \to E_{k+1},$$

where $\Sigma E_k = S^1 \wedge E_k$ is the topological suspension. A map of spectra $f : E \to F$ is a
sequence of maps so that the squares

$$\begin{array}{ccc}
\Sigma E_k & \xrightarrow{\Sigma f_k} & \Sigma F_k \\
\downarrow & & \downarrow \\
E_{k+1} & \xrightarrow{f_{k+1}} & F_{k+1}
\end{array}$$

commute for all $k$.

Example 3.2. If $X$ is a based space we may define the suspension spectrum $\Sigma^\infty X$ to have
$k$th term $\Sigma^k X$ with the structure maps being the identity.

Remark: It is possible to make a definition of homotopy immediately, but this does not work
very well for arbitrary spectra. Nonetheless it will turn out that for finite CW-complexes $K$,
maps out of a suspension spectrum can be easily described. We reserve $[\cdot, \cdot]$ for homotopy
classes of maps between spectra (sometimes called ‘stable maps’), so we write $[A, B]_{un}$ for
homotopy classes of (unstable) maps from a based space $A$ to a based space $B$. With this
notation, we have

$$[\Sigma^\infty K, E] = \lim_{k \to \infty} [\Sigma^k K, E]_{un}.$$
In particular
\[ \pi_n(E) := [\Sigma \infty S^n, E] = \lim_{\to k} [S^{n+k}, E_k]_{un}. \]
For example if \( E = \Sigma \infty L \) for a based space \( L \), we obtain a formula for the stable homotopy groups of \( L \)
\[ \pi_n(\Sigma \infty L) = \lim_{\to k} [\Sigma^k S^n, \Sigma^k L]_{un}, \]
By the Freudenthal suspension theorem, this is the common stable value of the groups \([\Sigma^k S^n, \Sigma^k L]_{un}\) for large \( k \). Thus spectra have captured stable homotopy groups.

**Construction 3.3.** We can suspend spectra by any integer \( r \), defining \( \Sigma^r E \) by
\[
(\Sigma^r E)_k = \begin{cases} 
E_{k-r} & k - r \geq 0 \\
pt & k - r < 0.
\end{cases}
\]
Notice that if we ignore the first few terms, \( \Sigma^r \) is inverse to \( \Sigma^{-r} \). Homotopy groups involve a direct limit and therefore do not see these first few terms. Accordingly, once we invert homotopy isomorphisms, the suspension functor becomes an equivalence of categories. Because suspension is an equivalence, we say that we have a stable category.

**Example 3.4.** (*Sphere spectra*) We write \( S = \Sigma \infty S^0 \) for the 0-sphere because of its special role, and then for an arbitrary integer \( r \) we define
\[
S^r = \Sigma^r S.
\]
Note that \( S^r \) now has meaning for a space and a spectrum for \( r \geq 0 \), but since we have an isomorphism \( S^r \cong \Sigma \infty S^r \) of spectra for \( r \geq 0 \) the ambiguity is not important. We extend this ambiguity, by often suppressing \( \Sigma \infty \).

**Example 3.5.** *Eilenberg-MacLane spectra.* An Eilenberg-MacLane space of type \((M, k)\) for an abelian group \( M \) and \( k \geq 0 \) is a CW-complex \( K(M, k) \) with \( \pi_k(K(M, k)) = M \) and \( \pi_n(K(M, k)) = 0 \) for \( n \neq k \); any two such spaces are homotopy equivalent. It is well known that in each degree ordinary cohomology is represented by an Eilenberg-MacLane space. Indeed, for any CW-complex \( X \), we have \( H^k(X; )M = [X, K(M, k)]_{un} \). In fact, this sequence of spaces, as \( k \) varies, assembles to make a spectrum.

To describe this, first note that the suspension functor \( \Sigma \) is defined by smashing with the circle \( S^1 \), so it is left adjoint to the loop functor \( \Omega \) defined by \( \Omega X := \text{map}(S^1, X) \) (based loops with the compact-open topology). In fact there is a homeomorphism
\[
\text{map}(\Sigma W, X) = \text{map}(W \wedge S^1, X) \cong \text{map}(W, \text{map}(S^1, X)) = \text{map}(W, \Omega X)
\]
This passes to homotopy, so looping shifts homotopy in the sense that \( \pi_n(\Omega X) = \pi_{n+1}(X) \). We conclude that there is a homotopy equivalence
\[
\tilde{\sigma} : K(M, k) \tilde{\cong} \Omega K(M, k + 1),
\]
and hence we may obtain a spectrum
\[
HM = \{K(M, k)\}_{k \geq 0}
\]
where the bonding map

\[ \sigma: \Sigma K(M, k) \to K(M, k + 1) \]

is adjoint to \( \tilde{\sigma} \). Thus we find

\[ [\Sigma^r \Sigma^\infty X, HM] = \lim_{\to k} [\Sigma^r \Sigma^k X, K(M, k)]_{un} = \lim_{\to k} H^k(\Sigma^r \Sigma^k X; M) = H^{-r}(X; M). \]

In particular the Eilenberg-MacLane spectrum has homotopy in a single degree like the spaces from which it was built:

\[ \pi_k(HM) = \begin{cases} M & k = 0 \\ 0 & k \neq 0. \end{cases} \]

3.B. Symmetric spectra [74, 89]. Symmetric spectra give an elementary and combinatorial construction of a symmetric monoidal category of spectra. This is excellent for an immediate access to the formal properties, but to be able to calculate with symmetric spectra and to relate them to the rest of homotopy theory one needs to understand the construction of the homotopy category. This is somewhat indirect, and Subsection 3.A was intended as a motivational substitute.

It is usual to give a fully combinatorial construction of symmetric spectra, by basing them on simplicial sets rather than on topological spaces.

**Definition 3.6.** (a) A symmetric sequence is a sequence

\[ E_0, E_1, E_2, \ldots, \]

of pointed simplicial sets with basepoint preserving action of the symmetric group \( \Sigma_n \) on \( E_n \).

(b) We may define a tensor product \( E \otimes F \) of symmetric sequences \( E \) and \( F \) by

\[ (E \otimes F)_n := \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q), \]

where the subscript + indicates the addition of a disjoint basepoint.

It is elementary to check that this has the required formal behaviour.

**Lemma 3.7.** The product \( \otimes \) is symmetric monoidal with unit

\[ (S^0, *, *, *, \ldots). \]

**Example 3.8.** The sphere is the symmetric sequence \( S := (S^0, S^1, S^2, \ldots) \). Here \( S^1 = \Delta^1/\Delta^1 \) is the simplicial circle and the higher simplicial spheres are defined by taking smash powers, so that \( S^n = (S^1)^\wedge n \); this also explains the actions of the symmetric groups.

It is elementary to check that the sphere is a commutative monoid in the category of symmetric sequences.

**Definition 3.9.** A symmetric spectrum \( E \) is a left \( S \)-module in symmetric sequences.

Unwrapping the definition, we see that this means \( E \) is given by

1. a sequence \( E_0, E_1, E_2, \ldots \) of simplicial sets,
2. maps \( \sigma: S^1 \wedge E \to E_{n+1} \), and
3. basepoint preserving left actions of \( \Sigma_n \) on \( E_n \) which are compatible with the actions in the sense that the composite maps \( S^p \wedge E_n \to E_{n+p} \) are \( \Sigma_p \times \Sigma_n \) equivariant.
Definition 3.10. The smash product of symmetric spectra is

\[ E \wedge_S F := \text{coeq}(E \otimes S \otimes F \longrightarrow E \otimes F). \]

Proposition 3.11. The tensor product over \( S \) is a symmetric monoidal product on the category of symmetric spectra.

It is now easy to give the one example most important to commutative algebraists.

Example 3.12. For any abelian group \( M \), we define the Eilenberg-MacLane symmetric spectra. For a set \( T \) we write \( M \otimes T \) for the \( T \)-indexed sum of copies of \( M \); this is natural for maps of sets and therefore extends to an operation on simplicial sets. We may then define the Eilenberg-MacLane symmetric spectrum \( HM := (M \tilde{\otimes} S^0, M \tilde{\otimes} S^1, M \tilde{\otimes} S^2, \ldots) \), where \( \tilde{\otimes} \) means the tensor product of an abelian group and a simplicial set with the basepoint generator set to zero. It is elementary to check that if \( R \) is a commutative ring, then \( HR \) is a monoid in the category of \( S \)-modules, and if \( M \) is an \( R \)-module, \( HM \) is a module over \( HR \).

Next one may define the stable model structure on symmetric spectra; this is done in several steps (Section 3 of [74]), but we may summarise it by saying that the weak equivalences (“stable equivalences”) are maps \( f : X \longrightarrow Y \) which induce isomorphisms in all cohomology theories. The homotopy category of symmetric spectra is obtained by inverting stable equivalences.

4. The three examples

The point of the present section is to introduce our three basic classes of examples. Classical commutative local Noetherian rings, rational cochains on simply connected spaces and mod \( p \) cochains on classifying spaces of finite groups.

The importance of the category of symmetric spectra [74] is that it admits a symmetric monoidal smash product compatible with the model structures. Given this, we can start to do algebra with spectra: choose a ring spectrum \( R \) (i.e., a monoid in the category of spectra), form the category of \( R \)-modules or \( R \)-algebras and then pass to homotopy. We may then attempt to use algebraic methods and intuitions to study \( R \) and its modules.

4.A. Classical commutative algebra. We explain why the classical derived category is covered by the context of spectra.

In Section 3 we described a functorial construction for symmetric spectra taking an abelian group \( M \) and giving the Eilenberg-MacLane spectrum \( HM \), which is characterised up to homotopy by the property that \( \pi_0(HM) = \pi_0(HM) = M \). For symmetric spectra the construction is lax symmetric monoidal, so that if \( A \) is a commutative ring, \( HA \) is a commutative ring spectrum. Furthermore the construction gives a functor \( A \)-modules \( \longrightarrow HA \)-modules. Passage to homotopy groups gives a functor \( \text{Ho}(HA - mod) \rightarrow \text{Ho}(A\text{-modules}) = D(A) \) and in fact the model categories are equivalent.

Theorem 4.1. (Shipley [92] 1.1, 2.15) There is a Quillen equivalence between the category of differential graded \( \mathbb{Z} \)-algebras and the category of \( H\mathbb{Z} \)-algebras in spectra.
If we choose a DGA $A$ and the corresponding $HZ$-algebra $HA$, there is a Quillen equivalence between the category of $A$-modules and the category of $HA$-modules, and hence in particular a triangulated equivalence
\[ D(A) = \text{Ho}(A\text{-modules}) \simeq \text{Ho}(HA\text{-modules}) = D(HA) \]
of derived categories.

The commutative case is complicated by the fact that there are no model structures on commutative algebras in general. However, if $A$ is commutative, $HA$ may be taken to be commutative and the Quillen equivalence between $A$-modules and $HA$-modules, tensor products over $A$ and over $HA$ correspond.

We sometimes use Shipley’s result to excuse the omission of the letter $H$ indicating spectra. In this translation homology in the classical context of chain complexes corresponds to homotopy in the context of spectra: $H_\ast(M) = \pi_\ast(HM)$.

Now that we can view classical commutative rings as commutative ring spectra, we can attempt to extend classical notions to the context of spectra. From one point of view, we should first attempt to understand the analogues of local rings before attempting to look at more geometrically complicated ones. Accordingly, in most of the lectures we will assume the commutative ring $A$ is local, with residue field $k$.

4.B. Cochains on a space. In the category of spectra, we may solve the commutative cochain problem. More precisely, for any space $X$ and a commutative ring $k$, we may form the function spectrum $C^\ast(X; k) = \text{map}(\Sigma^\infty X_+, Hk)$. It is obviously an $Hk$-module, and we may combine the commutative multiplication on $Hk$ with the cocommutative diagonal on $X_+$ to see that $C^\ast(X; k)$ is a commutative $Hk$-algebra. The notation is chosen because it is a model for the cochains in the sense
\[ \pi_\ast(\text{map}(\Sigma^\infty X, Hk)) = H^\ast(X; k). \]
The commutative algebra of $C^\ast(X; k)$ is one of the main topics for these lectures, and we will omit the coefficient ring $k$ when it is clear from the context.

We then use algebraic behaviour of this commutative ring to pick out interesting classes of spaces. In accordance with the principle that $C^\ast(X; k)$ is a sort of ring of functions on $X$, we simplify terminology and say that $X$ has a property $P$ over $k$ if the commutative ring spectrum $C^\ast(X; k)$ has the property $P$.

We will focus particularly on two special cases.

Example 4.2. (Rational case.) We will refer to the special case $k = \mathbb{Q}$ with $X$ simply connected as the rational case. Indeed this may be treated by classical means with the Thom-Sullivan construction of piecewise polynomial forms giving a commutative DGA model for $C^\ast(X; \mathbb{Q})$. There is an enormous literature studying rational homotopy theory by homotopy invariant commutative algebra methods. The book [42] of Félix, Halperin and Thomas is an excellent starting point. The ‘Looking glass’ paper [12] expounds the philosophy that algebra and topology are imperfect reflections of each other and gives numerous profound examples of it. Because we permit more general ring spectra, we are more restricted in theorems but have a larger range of examples.
**Example 4.3.** *(Modular representation theory.)* We will also focus particularly on the case \(k = \mathbb{F}_p\), when \(X = BG\) for a finite group \(G\). We recall that the classifying space \(BG\) of principal \(G\)-bundles is characterised (when \(G\) is finite) by the fact that it has fundamental group \(G\) and all other homotopy groups trivial. The interest in this special case comes from the fact that its coefficient ring
\[
\pi_*(C^*(BG; \mathbb{F}_p)) = H^*(BG; \mathbb{F}_p) = \text{Ext}_{\mathbb{F}_p G}^*(\mathbb{F}_p, \mathbb{F}_p)
\]
is the group cohomology ring.

There is an operation of \(p\)-completion due to Bousfield-Kan [26] which behaves well for a large class of spaces (the \(p\)-good spaces). In particular, all connected spaces with finite fundamental group are \(p\)-good, which will cover our examples. For \(p\)-good spaces \(C^*(X; \mathbb{F}_p) = C^*(X^p; \mathbb{F}_p)\), and accordingly we will generally assume that \(X\) is \(p\)-complete when \(k = \mathbb{F}_p\). The space \(X = BG\) is already \(p\)-complete if \(G\) is a \(p\)-group, but in general \(BG^p\) has infinitely many homotopy groups and its fundamental group is the largest \(p\)-quotient of \(G\).

5. **Finiteness conditions**

The point of this section is to introduce a variety of finiteness conditions we may impose. The essential limitation of all we do is that it is based on local rings in commutative algebra. We will not discuss the new and interesting features that can arise when there are many maximal ideals.

**Context 5.1.** The main input is a map \(R \rightarrow k\) of ring spectra with notation suggested by the case when \(R\) is commutative local ring with residue field \(k\).

For the most part, we work in the homotopy category \(\text{Ho}(R\text{-mod})\) of left \(R\)-modules.

5.A. **New modules from old.** Two construction principles will be important to us. The terminology comes from Dror-Farjoun [30], but in our stable context the behaviour is rather simpler.

If \(M\) is an \(R\)-module we say that \(X\) is *built* from \(M\) (and write \(M \vdash X\)) if \(X\) can be formed from \(M\) by completing triangles, taking coproducts and retracts (i.e., \(X\) is in the localizing subcategory generated by \(M\)). We refer to objects built from \(M\) as \(M\)-cellular, and write \(\text{Cell}(R, M)\) for the resulting full subcategory of \(\text{Ho}(R\text{-mod})\). An \(M\)-cellular approximation of \(X\) is a map \(\text{Cell}_M(X) \rightarrow X\) where \(\text{Cell}_M(X)\) is \(M\)-cellular and the map is an \(\text{Hom}_R(M, \cdot)\)-equivalence.

We say that \(X\) is *finitely built* from \(M\) (and write \(M \vDash X\)) if only finitely many steps and finite coproducts are necessary (i.e., \(X\) is in the thick subcategory generated by \(M\)).

Finally, we say that \(X\) is *cobuilt* from \(M\) if \(X\) can be formed from \(M\) by completing triangles, taking products and retracts (i.e., \(X\) is in the colocalizing subcategory generated by \(M\)).

5.B. **Finiteness conditions.** We say \(M\) is *small* if the natural map
\[
\bigoplus_{\alpha} [M, N_{\alpha}] \rightarrow [M, \bigoplus_{\alpha} N_{\alpha}]
\]
is an isomorphism for any set of $R$-modules $N_\alpha$. Smallness is equivalent to being finitely built from $R$. It is easy to see that any module finitely built from $R$ is small. For the reverse implication we use the fact that we can build an $R$-cellular approximation $\text{Cell}(R, M) \to M$; this is an equivalence, and by smallness, $M$ is a retract of a finitely built subobject of $\text{Cell}(R, M)$.

We sometimes require that $k$ itself is small, but this is an extremely strong condition on $R$ and it is important to develop the theory under a much weaker condition.

**Definition 5.2.** [33] We say that $k$ is proxy-small if there is an object $K$ with the following properties

- $K$ is small ($R \models K$)
- $K$ is finitely built from $k$ ($k \models K$) and
- $k$ is built from $K$ ($K \vdash k$).

**Remark 5.3.** Note that the second and third condition imply that the $R$-module $K$ generates the same category as $k$ using triangles and coproducts: $\text{Cell}(R, K) = \text{Cell}(R, k)$.

One of the main messages of [33] is that we may use the condition that $k$ is proxy-small as a substitute for the Noetherian condition in the conventional setting. This rather weak condition allows one to develop a very useful theory applicable in a large range of examples.

We can illustrate this by looking at the proxy-small condition in the classical case.

**Example 5.4.** *(Conventional commutative algebra)* Take $R$ to be a commutative Noetherian local ring in degree 0, with maximal ideal $m$ and residue field $k$.

By the Auslander-Buchsbaum-Serre theorem, $k$ is small if and only if $R$ is a regular local ring, confirming that the smallness of $k$ is a very strong condition. On the other hand, $k$ is always proxy-small: we may take $K = K(\alpha)$ to be the Koszul complex for a generating sequence $\alpha$ for $m$ (see Appendix B).

It is shown in [32] that $\text{Cell}(R, k)$ consists of objects whose homology is $m$-power torsion.

**Part 2. Morita equivalences**

Section 6 introduces the basic apparatus of the Morita equivalences that concern us, along with their relation with torsion functors and completion. Section 7 discusses the rather weak condition that $k$ is proxy-small in our examples, and shows it provides an inclusive framework. Section 8 illustrates the power of the Morita framework by discussing the classification of ring spectra and DGAs with coefficient rings exterior on one generator.

6. **Morita equivalences**

Morita theory studies objects $X$ of a category $\mathcal{C}$ by considering maps from a test object $k$. More precisely, $X$ is studied by considering $\text{Hom}(k, X)$ as a module over the endomorphism ring $\text{End}(k)$. In favourable circumstances this may give rather complete information. This is an instance of the philosophy that one gains insights by looking at rings of operations.

In the classical situation, $\mathcal{C}$ is an abelian category with infinite sums and $k$ is a small projective generator, and we find $\mathcal{C}$ is equivalent to the category of $\text{End}(k)$-modules [14] II
We will work with a stable model category rather than in an abelian category, and \( k \) will not necessarily be either small or a generator. The fact that the objects of the categories are spectra is unimportant except for the formal context it provides. See [88] for a more extended account from the present point of view.

In fact, two separate Morita equivalences play a role: two separate categories of modules over a commutative ring are both shown to be equivalent to a category of modules over the same non-commutative ring.

This section is based on [32], with augmentations from [33].

6.A. First variant. To start with we introduce the ring spectrum \( E = \text{Hom}_R(k, k) \) of (derived) endomorphisms of \( k \). This is usually not a commutative ring. Morita theory considers the relationship between the categories of left \( R \)-modules and of right \( E \)-modules. We have the adjoint pair

\[
E : R \text{-mod} \rightleftarrows \text{mod-}E : T
\]

defined by

\[
T(X) := X \otimes_E k \text{ and } E(M) := \text{Hom}_R(k, M).
\]

It is clear from the definition that \( E(M) \) is an \( E \)-module, since \( E = \text{Hom}_R(k, k) \) acts on the right through the factor \( k \) in \( EM = \text{Hom}_R(k, M) \). By the same token, it is clear that the ring

\[
\hat{R} = \text{Hom}_E(k, k)
\]

acts on the left of \( TX = X \otimes_E k \) through its action on \( k \). To see that this gives an \( R \)-module we note that there is a ring map

\[
R \longrightarrow \text{Hom}_E(k, k) = \hat{R}
\]

by ‘right multiplication’ of \( R \) on \( k \); thus \( TX \) is an \( R \)-module by restriction.

**Theorem 6.1.** If \( k \) is small, this adjunction gives equivalence

\[
\text{Cell}(R, k) \simeq \text{mod-}E
\]

between the derived category of \( R \)-modules built from \( k \) and the derived category of \( E \)-modules.

**Proof:** To see the unit \( X \longrightarrow ETX = \text{Hom}_R(k, X \otimes_E k) \) is an equivalence, we note it is obviously an equivalence for \( X = E \) and hence for any \( X \) built from \( E \), by smallness of \( k \). The argument for the counit is similar. \( \square \)

**Remark 6.2.** If \( k \) is not small, the unit of the adjunction may not be an equivalence. For example if \( R = \Lambda(\tau) \) is exterior on a generator \( \tau \) of degree 1 then \( E \simeq k[x] \) is polynomial on a generator of degree \(-2\). As an \( R \)-module, \( k \) is of infinite projective dimension and hence it is not small. In this case all \( R \)-modules are \( k \)-cellular, so that \( \text{Cell}(R, k) = R \text{-mod} \). Furthermore, the only subcategories of \( R \)-modules closed under coproducts and triangles are the trivial category and the whole category. On the other hand the category of torsion \( E \)-modules is a proper non-trivial subcategory closed under coproducts and triangles.
Exchanging roles of the rings, so that \( R = k[x] \) and \( \mathcal{E} \cong \Lambda(\tau) \), we see \( k \) is small as a \( k[x] \)-module and \( \text{Cell}(k[x], k) \) consists of torsion modules. Thus we deduce

\[
\text{tors-}k[x]-\text{mod} \cong \text{mod-}\Lambda(\tau).
\]

In fact the counit

\[
TEM = \text{Hom}_R(k, M) \otimes_{\mathcal{E}} k \longrightarrow M
\]

of the adjunction is of interest much more generally. Notice that any \( \mathcal{E} \)-module (such as \( \text{Hom}_R(k, M) \)) is built from \( \mathcal{E} \), so the domain is \( k \)-cellular. We say \( M \) is \textit{effectively constructible} from \( k \) if the counit is an equivalence, because \( TEM \) gives a concrete and functorial model for the cellular approximation to \( M \). Under the much weaker assumption of proxy smallness we obtain a very useful conclusion linking Morita theory to commutative algebra.

**Lemma 6.3.** Provided \( k \) is proxy-small, the counit

\[
TEM = \text{Hom}_R(k, M) \otimes_{\mathcal{E}} k \longrightarrow M
\]

is \( k \)-cellular approximation, and hence in particular any \( k \)-cellular object is effectively constructible from \( k \).

**Proof:** We observed above that the domain is \( k \)-cellular. To see the counit is a \( \text{Hom}_R(k, \cdot) \)-equivalence, consider the evaluation map

\[
\gamma : \text{Hom}_R(k, X) \otimes_{\mathcal{E}} \text{Hom}_R(Y, k) \longrightarrow \text{Hom}_R(Y, X).
\]

This is an equivalence if \( Y = k \), and hence by proxy-smallness it is an equivalence if \( Y = K \). This shows that the top horizontal in the diagram

\[
\begin{array}{ccc}
\text{Hom}_R(k, X) \otimes_{\mathcal{E}} \text{Hom}_R(K, k) & \longrightarrow & \text{Hom}_R(K, X) \\
\cong & & \downarrow \cong \\
\text{Hom}_R(K, \text{Hom}_R(k, X) \otimes_{\mathcal{E}} k) & \longrightarrow & \text{Hom}_R(K, X)
\end{array}
\]

is an equivalence. The left hand-vertical is an equivalence since \( K \) is small. Thus the lower horizontal is an equivalence, which is to say that the counit

\[
TEX = \text{Hom}_R(k, X) \otimes_{\mathcal{E}} k \longrightarrow X
\]

is a \( K \)-equivalence. By proxy-smallness, this counit map is a \( k \)-equivalence.

**Examples 6.4.** (i) If \( R \) is a commutative Noetherian local ring, we recall in Appendix B that the \( k \)-cellular approximation of a module \( M \) is \( \Gamma_m M = K_\infty(m) \otimes_R M \), where \( K_\infty(m) \) is the stable Koszul complex, so we have

\[
TEM \cong K_\infty(m) \otimes_R M.
\]

(ii) If \( R = C^*(X; k) \) it is not easy to say what the \( k \)-cellular approximation is in general, but any bounded below module \( M \) is cellular.

15
6.B. **Second variant.** There is a second adjunction between the derived categories of left $R$-modules and of right $E$-modules. In the first variant, $k$ played a central role as a left $R$-module and a left $E$-module. In this second variant

$$k^\# := \text{Hom}_R(k, R)$$

plays a corresponding role: it is a right $R$-module and a right $E$-module. We have the adjoint pair

$$E' : R-\text{mod} \quad \dashv \quad \text{mod-}E : C$$

defined by

$$E'(M) := k^\# \otimes_R M \quad \text{and} \quad C(X) := \text{Hom}_E(k^\#, X).$$

**Remark 6.5.** If $k$ is small then

$$E'(M) = \text{Hom}_R(k, R) \otimes_R M \simeq \text{Hom}_R(k, M) = EM,$$

so the two Morita equivalences consider the left and right adjoints of the same functor.

The unit of the adjunction $M \longrightarrow CE'(M)$ is not very well behaved, and the functor $CE'$ is not even idempotent in general.

6.C. **Complete modules and torsion modules.** Even when we are not interested in the intermediate category of $E$-modules, several of the composite functors give interesting endofunctors of the category of $R$-modules.

**Lemma 6.6.** If $k$ is proxy-small then $k$-cellular approximation is smashing:

$$\text{Cell}_k M \simeq (\text{Cell}_k R) \otimes_R M,$$

so that

$$TEM = TE'M.$$

**Proof:** We consider the map

$$\lambda : (\text{Cell}_k R) \otimes_R M \longrightarrow R \otimes_R M = M$$

and show it has the properties of cellularization.

To start with, since $R$ builds $M$, it follows that $(\text{Cell}_k R) \otimes_R R$ builds $(\text{Cell}_k R) \otimes_R M$, and hence $(\text{Cell}_k R) \otimes_R M$ is $k$-cellular.

Since $k$ is proxy small, the map $\lambda$ is $k$-equivalence if and only if it is a $K$-equivalence. We may consider the class of objects $M$ for which $\lambda$ is a $K$-equivalence. Since $K$ is small, it is closed under arbitrary coproducts as well as triangles. Since the class obviously contains $R$ itself, it contains all modules $M$.

We therefore see by 6.3 and 6.6 that if $k$ is proxy-small

$$\text{Cell}_k(M) = T E M = T E'M.$$

This is the composite of two left adjoints, focusing attention on its right adjoint $CEM$, and we note that

$$CE(M) = \text{Hom}_R(k^\#, \text{Hom}_R(k, M)) = \text{Hom}_R(TER, M).$$

By analogy with Subsection II of Appendix B, we may make the following definition.
**Definition 6.7.** The completion of an $R$-module $M$ is the map

$$M \rightarrow \text{Hom}_R(TER, M) = CEM.$$ 

We say that $M$ is complete if the completion map is an equivalence.

**Remark 6.8.** By 6.6 we see that completion is idempotent.

We adopt the notation

$$\Gamma_k M := TE'M$$

and

$$\Lambda^k M := CEM.$$ 

This is by analogy with the case of commutative algebra through the approach of Appendix C, where $\Gamma_k = \Gamma_I$ is the total right derived functor of the $I$-power torsion functor and $\Lambda^k = \Lambda^I$ is the total left derived functor of the completion functor (see [2] [3] for the context of commutative rings).

It follows from the adjunctions described earlier in this section that $\Gamma_k$ is left adjoint to $\Lambda^k$ as endofunctors of the category of $R$-modules:

$$\text{Hom}_R(\Gamma_k M, N) = \text{Hom}_R(M, \Lambda^k N)$$

for $R$-modules $M$ and $N$. Slightly more general is the following observation.

**Lemma 6.9.** If $k$ is proxy-small, $\Gamma_k$ and $\Lambda^k$ give an adjoint equivalence

$$\text{Cell}(R, k) \simeq \text{comp} - R\text{-mod},$$

where $\text{comp} - R\text{-mod}$ is the triangulated subcategory of $R\text{-mod}$ consisting of complete modules.

**Proof:** We have

$$TE'M \simeq TEM \simeq \Gamma_k M \simeq \Gamma K M,$$

and

$$CEM \simeq \text{Hom}_R(\Gamma_k R, M) \simeq \text{Hom}_R(\Gamma_k R, M),$$

so it suffices to prove the result when $k$ is small. When $k$ is small the present adjunction is the composite of two adjoint pairs of equivalences. We have seen this for the first variant, and the second variant is proved similarly by arguing that the unit and counit are equivalences. \hfill \Box

**6.D. dc-completeness.** We have explained the importance of passing from $R$ to $\mathcal{E} = \text{Hom}_R(k, k)$, and we have also given an example where we then took $\mathcal{E}$ as our input ring, meaning that we need to consider the ring $\hat{R} = \text{Hom}_\mathcal{E}(k, k)$. Furthermore, the double centralizer ring homomorphism

$$\kappa : R \rightarrow \text{Hom}_\mathcal{E}(k, k) = \hat{R}$$

played a role at the very start, in defining the $R$-module structure on $TX$.

We say that $R$ is dc-complete if $\kappa$ is an equivalence.
7. Proxy-smallness in the examples

We establish the appropriate finiteness and completeness properties in our principal examples.

7.A. Commutative local rings. When $R$ is a conventional commutative local Noetherian ring, we have seen that $k$ is always proxy-small, and that the Koszul complex for a set of generators of $\mathfrak{m}$ provides the small proxy for $k$.

Next, we note that

$$
\pi_*(E) = \pi_*(\text{Hom}_R(k, k)) = \text{Ext}_R^*(k, k),
$$

which is a ring whose importance is very familiar. We will see in Section 8 that as a DGA, the endomorphisms $E = \text{Hom}_R(k, k)$ contain considerably more information.

We also note that in this case, $R$ is dc-complete if and only if it is $\mathfrak{m}$-complete in the usual sense. Indeed, we calculate

$$
R^\wedge_{\mathfrak{m}} = \varprojlim R/\mathfrak{m}^s = \varprojlim_{s} \text{Hom}_E(\text{Hom}_R(R/\mathfrak{m}^s, k), k) = \text{Hom}_E(\varprojlim_{s} \text{Hom}_R(R/\mathfrak{m}^s, k), k) = \text{Hom}_E(k, k)
$$

Here the first equivalence uses the Mittag-Leffler condition on $\{R/\mathfrak{m}^s\}_s$, and the last condition uses the fact that $\varprojlim_{s} \text{Hom}_R(R/\mathfrak{m}^s, k) \simeq \Gamma_{\mathfrak{m}} k \simeq k$.

More generally, if $R$ is a commutative Noetherian ring and $I$ is an ideal, we may consider $R \to R/I$ (i.e., replacing the field $k$ by a more complicated ring). It turns out that the double centralizer map is $I$-adic completion whenever $R/I$ is proxy-small.

7.B. Cohains on a space. In the setting $R = C^*(X; k)$ of cochains on a space, the Eilenberg-Moore spectral theorem shows that

$$
\mathcal{E} = \text{Hom}_{C^*(X; k)}(k, k) \simeq C_*(\Omega X; k), H_*(\mathcal{E}) = H_*(\Omega X; k)
$$

provided that either (Case 0) $X$ is simply connected or (Case $p$) $X$ is connected, $\pi_1(X)$ is a finite $p$-group $k = \mathbb{F}_p$ and $X$ is $p$-complete [31]. This immediately gives a very rich source of examples that we will revisit frequently.

Provided we have either Case 0 or Case $p$, then $C^*(X)$ is dc-complete. Indeed, the Rothenberg-Steenrod equivalence

$$
\text{Hom}_{C_*(\Omega X)}(k, k) \simeq C_*(X)
$$

holds for any connected space $X$. Accordingly, provided the Eilenberg-Moore equivalence holds, the double centralizer map $\kappa$ is an equivalence.

Finally we may consider proxy-smallness.

**Lemma 7.1.** (i) If $\dim_k H_*(\Omega Y)$ is finite then $k$ is small over $C^*(Y)$.

(ii) If $Y$ falls under Case 0 or Case $p$, there is a map $X \to Y$ with homotopy fibre $F$, and $\dim_k H_*(\Omega Y)$ and $\dim_k H^*(F)$ are finite, then $k$ is proxy-small over $C^*(X)$. 

18
**Proof:** Since \( \dim_k H_*(\Omega Y) \) is finite, 
\[
k \models C_*(\Omega Y).
\]
Applying \( \text{Hom}_{\mathbb{C}}(\Omega Y)(\cdot, k) \) we see 
\[
C^*(Y) \simeq \text{Hom}_{\mathbb{C}}(\Omega Y)(k, k) \models \text{Hom}_{\mathbb{C}}(\Omega Y)(C_*(\Omega Y), k) \simeq k.
\]
This proves Part (i).

For Part (ii), let \( R = C^*(X), S = C^*(Y) \) and note \( C^*(X) \otimes_{C^*(Y)} k \simeq C^*(F) \) by the Eilenberg-Moore theorem. Now take \( K = C^*(F) \). We note that \( k \models K \) by hypothesis. Since \( S \models k \) by Part (i), we may apply \( R \otimes_S (\cdot) \) and deduce 
\[
R = R \otimes_S S \models R \otimes_S k = C^*(F).
\]
Finally, \( C^*(F) \) builds \( k \) by killing homotopy groups, since \( F \) is connected. \( \square \)

**Corollary 7.2.** (Case 0) If \( X \) is simply connected and \( k = \mathbb{Q} \) and if \( H^*(X) \) is Noetherian then \( \mathbb{Q} \) is proxy small over \( C^*(X; \mathbb{Q}) \).

(Case \( p \)) If \( k = \mathbb{F}_p \) and \( X \) is the \( p \)-completion of \( BG \) then \( \mathbb{F}_p \) is proxy small over \( C^*(X; \mathbb{F}_p) \).

**Proof:** For Case 0, since \( H^*(X) \) is Noetherian, we may choose a polynomial subring \( P = \mathbb{Q}[V] \) on finitely many even degree generators. We have \( \mathbb{Q}[V] = H^*(Y) \) where \( Y = K(V) \) is the corresponding product of even Eilenberg-MacLane spaces. The inclusion \( P = K[V] \rightarrow H^*(X) \) is realized by a map \( X \rightarrow Y \). The space \( \Omega Y \) is a product of finitely many odd Eilenberg-MacLane spaces, and rationally this is a product of spheres, so Part (i) of Lemma \[7.1\] applies. Furthermore since \( H^*(X) \) is a finitely generated \( P \) module, it has a finite resolution by finitely generated projectives and the spectral sequence for calculating \( \pi_*(C^*(X) \otimes_{C^*(Y)} \mathbb{Q}) = H^*(F) \) has finite dimensional \( E_2 \) term and so the hypotheses of Part (ii) of Lemma \[7.1\] hold.

For Case \( p \) we note that the \( p \)-completion of \( Y = BU(n) \) satisfies the conditions of Part (i) of Lemma \[7.1\]. Now, given \( G \), choose \( n \) so that \( G \) admits a faithful representation in \( U(n) \) and apply Part (ii) of Lemma \[7.1\] to the \( p \)-completion of the fibration \( U(n)/G \rightarrow BG \rightarrow BU(n) \). \( \square \)

8. Exterior algebras

There are many ways to use Morita theory, but there is a very elementary and striking one which illustrates that it is significant in even in the very simple case when \( \pi_*(\mathcal{E}) \) is an exterior algebra on one generator.

8.A. Exterior algebras over \( \mathbb{F}_p \) on a generator of degree \(-1\). To start with, we note that is easy to see that if \( \mathcal{E} \) is an \( \mathbb{F}_p \)-algebra with \( \pi_*(\mathcal{E}) = \Lambda_{\mathbb{F}_p}(\tau) \) exterior on one generator, then \( \mathcal{E} \) is formal. However if \( \pi_*(\mathcal{E}) = \Lambda_{\mathbb{F}_p}(\tau) \) but \( \mathcal{E} \) itself is only known to be a \( \mathbb{Z} \)-algebra, the situation is considerably more complicated.
We may use Morita theory to give a classification \([36]\) in the special case that the exterior generator is in degree \(-1\). We will return to discuss the more general case in Subsection \(8.B\).

**Theorem 8.1.** Differential graded algebras \(\mathcal{E}\) with \(H_*(\mathcal{E})\) exterior over \(\mathbb{F}_p\) on a single generator of degree \(-1\) (up to quasi-isomorphism) are in bijective correspondence with complete discrete valuation rings with residue field \(\mathbb{F}_p\) (up to isomorphism of rings).

**Proof:** The idea is to associate to any such \(\mathcal{E}\) the endomorphism DGA \(R = \text{Hom}_\mathcal{E}(\mathbb{F}_p, \mathbb{F}_p)\). Evidently the spectral sequence for calculating \(H_*(R)\) collapses at \(E_2\) with value \(\mathbb{F}_p[x]\) for an element \(x\) of total degree 0. This shows \(R\) is a filtered ungraded ring with this as associated graded ring. One argues that it is commutative and complete with residue field \(\mathbb{F}_p\). If one starts with \(R\), its maximal ideal is principal, generated by an element \(x\) and we may form a complex \(\mathbb{F}_p = (R \xrightarrow{x} R)\). The DGA \(\mathcal{E} = \text{Hom}_R(\mathbb{F}_p, \mathbb{F}_p)\) has homology exterior on a generator of degree \(-1\), and the double centralizer completion map 

\[ R \rightarrow \text{Hom}_\mathcal{E}(\mathbb{F}_p, \mathbb{F}_p) \]

is an equivalence. \(\square\)

### 8.B. General algebras with exterior homotopy

Let us consider ring spectra \(\mathcal{E}\) with \(\pi_*(\mathcal{E}) = \Lambda_{\mathcal{E}}(\tau)\) where \(\tau\) is of degree \(d\). We have ring maps \(\mathbb{S} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/p\), and we have already observed that if we restrict attention to \(\mathbb{Z}/p\)-algebras, there is a unique such \(\mathcal{E}\). If \(d = -1\) we classified \(\mathbb{Z}\)-algebras in Subsection \(8.A\) and found there were a lot of them. If \(d = 0\) the answers for \(\mathbb{S}\)-algebras and \(\mathbb{Z}\)-algebras are the same by Shipley’s Theorem, and there is a unique such algebra.

For \(d \geq 1\) there is an obstruction theory, and the answer is given by Dugger and Shipley \([38]\). Indeed, we are considering square zero extensions of \(\mathbb{Z}/p\) by the bimodule \(\mathbb{Z}/p\) in degree \(d\), and hence (with \(A = \mathbb{S}\) or \(\mathbb{Z}\)), the \(\mathbb{A}\)-algebras \(\mathcal{E}\) are classified by the Hochschild cohomology group \(HH^{d+2}(\mathbb{Z}/p|\mathbb{A}; \mathbb{Z}/p)\), with orbits under \(\text{Aut}(\mathbb{Z}/p)\) giving isomorphic algebras \(\mathcal{E}\). In fact the Hochschild cohomology rings are polynomial and divided power algebras on a single degree 2 generator:

\[ HH^*(\mathbb{Z}/p|\mathbb{S}; \mathbb{Z}/p) = \mathbb{Z}/p[x^2_2] \quad \text{and} \quad HH^*(\mathbb{Z}/p|\mathbb{Z}; \mathbb{Z}/p) = \Gamma_{\mathbb{Z}/p}[x^2_2]. \]

We conclude that for any odd \(d \geq 1\) there is single formal algebra and for any even \(d \geq 2\) there are precisely two isomorphism classes of \(\mathbb{Z}\)-algebras and precisely two isomorphism classes of \(\mathbb{S}\)-algebras. In each case one of them is formal and one is not.

The really striking phenomenon is that restriction along \(\mathbb{S} \rightarrow \mathbb{Z}\) does not induce a bijection of isomorphism types. Indeed, one may check that the generator \(x^2_2\) restricts non-trivially (so we may take \(x^2_2\) to be that restriction). It follows that there is a bijection until we reach \((x^2_2)^p\), which restricts to zero in \(\Gamma(x^2_2)\). In other words, the two \(\mathbb{Z}\)-algebras \(\mathcal{E}, \mathcal{E}'\) with \(d = 2p - 2\) become equivalent as \(\mathbb{S}\)-algebras: these two inequivalent DGAs are topologically equivalent. It is hard to be explicit when \(d\) is not small, but if \(p = 2\) and \(d = 2\), the DGA \(\mathbb{Z}[e \mid de = 2]/(e^3, 2e^2)\) is not formal as a \(\mathbb{Z}\)-algebra but it is formal as an \(\mathbb{S}\)-algebra. Finally, the \(\mathbb{S}\)-algebra \(\mathcal{E}\) with \(d = 2p - 2\) which is not realizable as a \(\mathbb{Z}\)-algebra, is the truncation of the \(d\)th Morava \(K\)-theory.
8.C. **Classification of free rational $G$-spectra.** In another direction, working over the rationals we note that for any compact Lie group $G$ the DGA $C_*(G)$ has homology $H_*(G)$ exterior on odd degree generators. It might not be apparent that it is formal. However, we may consider $R = \text{Hom}_{C_*(G)}(k, k) \cong C^*(BG)$; since this is commutative and has cohomology which is polynomial on even degree generators, it is formal and hence equivalent to $H^*(BG)$. It follows that $C_*(G) \cong H_*(G)$. This is the starting point for classifying free rational $G$-spectra [61, 62].

**Part 3. Regular rings**

We have now set up the basic machinery and the rest of the lectures will investigate homotopy invariant counterparts of classical definitions in commutative algebra of local rings. The justification consists of the dual facts that the new definition reduces to the old in the classical setting and that the new definition covers and illuminates examples in new contexts. Regular local rings are the most basic and best behaved objects, so it makes sense to start with these.

Section 9 recalls the three styles of definition from commutative algebra and explains how to give homotopy invariant versions, and then illustrates them in our main examples. Section 10 explains how we can use this class of ring spectra to give a homotopy invariant counterpart of the notion of a finitely generated module, at least for ‘normalizable’ ring spectra.

9. **Regular ring spectra**

In commutative algebra there are three styles for a definition of a regular local ring: ideal theoretic, in terms of the growth of the Ext algebra and a version for modules. We begin in Subsection 9.A by recalling the commutative algebra, where the conditions are equivalent. We then turn to other contexts, where the conditions may differ, and consider which one is most appropriate.

9.A. **Commutative algebra.** The following definitions are very familiar; we have introduced a slightly more elaborate terminology to smooth the transition to other contexts. The prefix s- signifies that the definition is structural, the prefix g- signifies that the definition is in terms of the growth rate, and the prefix m- signifies that the definition is in terms of the module category.

**Definition 9.1.** (i) A local Noetherian ring $R$ is $s$-regular if the maximal ideal is generated by a regular sequence.

(ii) A local Noetherian ring $R$ is $g$-regular if $\text{Ext}_R^*(k, k)$ is finite dimensional.

(iii) A local Noetherian ring $R$ is $m$-regular if every finitely generated module is small in the derived category $D(R)$.

These are equivalent by the Auslander-Buchsbaum-Serre theorem; one might think of this as the first theorem of homotopy invariant commutative algebra.
9.B. **Regularity for ring spectra.** The one definition that is easy to adapt is g-regularity, at least when \( k \) is a field. This is a basic input to the entire theory. It is also convenient to have a name for when the coefficient ring is regular.

**Definition 9.2.** (i) We say that \( R \) is \( c \)-regular when the coefficient ring \( \pi_*(R) \) is regular.

(ii) We say that \( R \) is \( g \)-regular if \( \pi_*(\text{Hom}_R(k, k)) \) is finite dimensional over \( k \).

It is obvious from the spectral sequence

\[
\text{Ext}^*_R(k, k) \Rightarrow \pi_*(\text{Hom}_R(k, k))
\]

that \( c \)-regular implies \( g \)-regular. The converse is very far from being true, as we shall see shortly.

It is more subtle to consider \( s \)-regular ring spectra.

The \( s \)-regularity condition on rings states that \( \mathfrak{m} \) is generated by a regular sequence \( x_1, \ldots, x_r \), so that there are short exact sequences

\[
R \xrightarrow{x_1} R \rightarrow R/(x_1)
\]

\[
R/(x_1) \xrightarrow{x_2} R/(x_1) \rightarrow R/(x_1, x_2)
\]

and

\[
R/(x_1, \ldots, x_{r-1}) \xrightarrow{x_r} R/(x_1, \ldots, x_{r-1}) \rightarrow R/\mathfrak{m} = k.
\]

In other words, we may start with \( R \), and successively factor out a regular element until we get to \( k \).

If we now remember the degrees, a single instance is the (additive) exact sequence of modules

\[
\Sigma^n R \xrightarrow{x} R \rightarrow R/(x).
\]

In other words, the ring \( R/(x) \) is equivalent to the Koszul complex \( K(x) = (R \xrightarrow{x} R) \). If we think multiplicatively this gives a cofibre sequence\(^1\)

\[
R \rightarrow K(x) \rightarrow K(x) \otimes_R k = \Lambda(\tau)
\]

of rings, where \( \tau \) is a generator of degree \( n + 1 \). The definition is now obtained by iterating this construction.

**Definition 9.3.** We say that \( R \) is \( s \)-regular if there are cofibre sequences of rings

\[
R = R_0 \rightarrow R_1 \rightarrow \Lambda_1, R_1 \rightarrow R_2 \rightarrow \Lambda_2, \ldots, R_{r-1} \rightarrow R_r \rightarrow \Lambda_r,
\]

with \( R_r \simeq k \) and \( \pi_*(\Lambda_i) \) exterior over \( k \) on one generator.

**Lemma 9.4.** If \( R \) is \( s \)-regular then it is also \( g \)-regular.

---

\(^1\)If \( S \rightarrow R \) is a map of rings and we have a map \( R \rightarrow k \), the cofibre ring is \( Q = R \otimes_Q k \). This corresponds to the fact that in algebraic geometry and topology this cofibre ring is often the ring of functions on the geometric fibre.
Proof: We need to work our way along the sequence of cofibrations. Since $R_x \simeq k$ is obviously g-regular, it suffices to observe that given a cofibre sequence $C \to B \to \Lambda$ with $B$ g-regular, then $C$ is also g-regular. Suppose then that $\text{Hom}_B(k, k)$ is finitely built from $k$. Since $k \models \Lambda$ it follows that

$$k \models \text{Hom}_B(k, k) \models \text{Hom}_B(\Lambda, k) \simeq \text{Hom}_k(k \otimes_B \Lambda, k).$$

However

$$k \otimes_B \Lambda \simeq k \otimes_B B \otimes_C k \simeq k \otimes_C k,$$

so that

$$\text{Hom}_k(k \otimes_B \Lambda, k) \simeq \text{Hom}_k(k \otimes_C k, k) \simeq \text{Hom}_C(k, k),$$

showing $\text{Hom}_C(k, k)$ is finitely built by $k$ as required. \qed

However, there is no good notion of m-regularity for ring spectra. The problem is that we do not have a homotopy invariant definition of finite generation in general. However, we will turn this around, and in Section 10 we will show that one can use the supposed equivalence of g-regularity and the putative m-regularity conditions to define finite generation.

9.C. Regularity for rational spaces. We have seen in Subsection 7.B, that for simply connected rational spaces $X$, if $R = C^*(X; \mathbb{Q})$ we have $\pi_*(\mathcal{E}) = H_*(\Omega X)$. Accordingly a g-regular rational space is one with $H_*(\Omega X)$ finite dimensional.

Proposition 9.5. A simply connected rational space $X$ is g-regular if and only if it is a finite product of even Eilenberg-MacLane spaces. Its cohomology is therefore polynomial on even degree generators and it is also s-regular.

Proof: To start with, we note that an even Eilenberg-MacLane space $X = K(\mathbb{Q}, 2n)$ is g-regular since its loop space is $K(\mathbb{Q}, 2n - 1) \simeq S^{2n-1}$. On the other hand an odd Eilenberg-MacLane space $X = K(\mathbb{Q}, 2n + 1)$ is not g-regular since its loop space is $K(\mathbb{Q}, 2n)$ which has homology polynomial on one generator of degree $2n$.

Finally, since $\Omega X$ splits as a product of Eilenberg-MacLane space for any rational space, we see $H_*(\Omega X)$ is finite dimensional if and only if the homotopy groups of $X$ are finite dimensional and in even degree. Building $X$ from its Postnikov tower we see inductively that the cohomology of its Postnikov sections are entirely in even degrees (and polynomial). Since the homotopy of $X$ is in even degrees the k-invariants are in odd degrees and hence zero. This shows that $X$ is a finite product of even Eilenberg-MacLane spaces. \qed

9.D. g-regularity for p-complete spaces. Of the commutative algebra definitions, the only one with a straightforward counterpart for $C^*(X)$ is g-regularity. We have seen in Subsection 7.B, that for connected p-complete spaces $X$ with finite fundamental group, if $R = C^*(X; \mathbb{Q})$ we have $\pi_*(\mathcal{E}) = H_*(\Omega X)$. Accordingly a g-regular rational space is one with $H_*(\Omega X)$ finite dimensional.
Example 9.6. For any finite $p$-group $G$, the classifying space $X = BG$ is $p$-complete, and hence satisfies our hypotheses without further completion. Thus $\Omega X$ is equivalent to the finite discrete space $G$, so $\mathcal{E} \simeq kG$ and $X$ is $g$-regular.

On the other hand the coefficient ring $H^*(BG)$ is very rarely regular, so this gives many examples of $g$-regular ring spectra which are not $c$-regular.

Example 9.7. Extending this idea, if $G$ is any compact Lie group with component group a finite $p$-group we have

$$\Omega(BG^\wedge) \simeq (\Omega BG)^\wedge \simeq G^\wedge,$$

so that $X = (BG)^\wedge$ is again $g$-regular over $\mathbb{F}_p$.

In fact $X = (BG)^\wedge$ is $g$-regular if and only if $\pi_0(G)$ is $p$-nilpotent, and we will see explicit examples later where completed classifying spaces are not $g$-regular.

In fact this is the starting point of Dwyer and Wilkerson’s project \cite{37} to capture properties of groups at a single prime homotopically. They define a connected $p$-complete space $X$ to be the classifying spaces of a connected $p$-compact group if $H_*(\Omega X; \mathbb{F}_p)$ is finite dimensional (i.e., if $C^*(X; \mathbb{F}_p)$ is g-regular). In other words, a $p$-compact group is an $\mathbb{F}_p$-finite loop space whose classifying space is $p$-complete and g-regular.

A major programme involving many people has given a classification of connected $p$-compact groups following the lines of that for compact Lie groups. This starts with the theorem of Dwyer and Wilkerson that connected $p$-compact groups have a maximal torus, and has finally culminated in the classification of connected $p$-compact groups in terms of the associated $p$-adic root data (Anderson, Grodal, Møller and Viruel \cite{5} for odd primes, and \cite{4} for $p = 2$). At the prime 2 there is only one simple 2-compact group which is not obtained by completing a compact connected Lie group. At odd primes there are many exotic examples, because of the many $p$-adic reflection groups. It is remarkable that objects defined purely in terms of a finiteness condition can be completely classified.

9.E. s-regularity for $p$-complete spaces. Looking back at the definition of s-regularity for ring spectra, the basic ingredient is to consider the multiplicative sequence

$$R \longrightarrow K(x) \longrightarrow K(x) \otimes_R k = \Lambda(\tau)$$

where $\tau$ is a generator of degree $n + 1$. Since we are working with $k$-algebras, exterior algebras are formal, so if $n \geq 0$, we have $\Lambda(\tau) \simeq C^*(S^{n+1})$. Thus if $R = C^*(X)$, the Eilenberg-Moore theorem shows that the most obvious source of such cofibre sequence is a spherical fibration

$$X \leftarrow X_1 \leftarrow S^{n+1}.$$

It is natural when talking about spaces to restrict the cofibrations to be of this type

Definition 9.8. A space $X$ is $s$-regular if there are $k$-complete fibrations

$$X = X_0 \leftarrow X_1 \leftarrow S^{n_1}, X_1 \leftarrow X_2 \leftarrow S^{n_2}, \ldots, X_{r-1} \leftarrow X_r \leftarrow S^{n_r}$$

with $X_r \simeq \ast$ and $n_i \geq 1$.

Examples of this type seem very rare, but do arise from the classical infinite families of compact Lie groups.
Example 9.9. $X = BU(n)$ is s-regular, in view of the fibrations

$$BU(n) \leftarrow BU(n - 1) \leftarrow U(n)/U(n - 1) \cong S^{2n-1}. $$

Similarly for $BSO(n), BSpin(n), BSp(n)$.

10. Finite generation

In conventional algebra, finite generation is one of the most important finiteness conditions. However it is not all obvious how best to translate this into a homotopy invariant context. In this section we describe an approach we have found useful, which is based on the idea that finite generation and smallness coincide over regular rings.

10.A. Finiteness conditions. If $R$ is a conventional Noetherian commutative ring and $M$ is a module over it, there are a number of natural finiteness conditions. If $M$ has a finite resolution by finitely generated projectives then it is finitely built from $R$ and hence small in the derived category; the converse is also true, since one may factor the identity map through a finite truncation of the projective resolution. These two equivalent homotopy invariant finiteness conditions on $M$ are extremely useful.

On the other hand, it is not so clear how to give a homotopy invariant version of the notion of being finitely generated. In this section we discuss one useful method. It is based on the fact that for a regular conventional ring $Q$ finitely generated modules coincide with small modules.

10.B. Normalizable ring spectra. In commutative algebra, it is natural to assume rings are Noetherian, and one of the most useful consequences for $k$-algebras is Noether normalization, stating that a Noetherian $k$-algebra is a finitely generated module over a polynomial subring.

Definition 10.1. We will say that a ring spectrum $R$ is $g$-normalizable, if there is a $g$-regular ring spectrum $S$ with $S_*$ Noetherian, and a ring map $S \rightarrow R$ making $R$ into small $S$-module. In this case $S \rightarrow R$ is called a $g$-normalization and $R \otimes_S k$ is its Noether fibre.

If $S$ can be chosen so that its coefficient ring $S_*$ is regular, we say $R$ is $c$-normalizable.

As usual, with coefficient-level conditions, c-normalizations are a bit rigid, but they do give a template for comparison.

Lemma 10.2. (i) If $S_*$ is regular then an $S$-module $N$ is small if and only if $N_*$ is a finitely generated $S_*$-module.

(ii) If $R$ is $c$-normalizable then $R_*$ is Noetherian.

Proof: Part (ii) follows from Part (i), so we prove Part (i).

It is easy to see that we have a cofibre sequence $X \rightarrow Y \rightarrow Z$ of $S$-modules, if two terms have finitely generated homotopy, so does the third. Thus any small module has finitely generated homotopy.

For the converse, we start by noting that projectives are realizable. Next, if $P_*$ is projective, $\text{Hom}_S(P, M) = \text{Hom}_{S_*}(P_*, M_*)$. Now for any $M$ we may choose a projective $S_*$-module mapping onto $M_*$ and realize it by a map $P \rightarrow M$ with fibre $F$. If $M_*$ was not projective
to start with, $F_*$ will be of lower projective dimension. Since $S_*$ is regular, this process terminates.

**Example 10.3.** (Venkov) If $G$ is a compact Lie group (for example a finite group), then $C^*(BG)$ is c-normalizable. Indeed, we may choose a faithful representation $G \to U(n)$, giving a fibration $U(n)/G \to BG \to BU(n)$.

It is an unpublished consequence of work of Castellana and Ziemianski that every $p$-compact group has a faithful linear representation, which is to say that if $B\Gamma$ is regular there is a map $BT \to BSU(n)$, for some $n$, whose homotopy fibre is $\mathbb{F}_p$-finite. This means that every $p$-compact group is c-normalizable.

**10.C. Finitely generated modules.** We are now equipped to give our definition.

**Definition 10.4.** Suppose that $R$ is a commutative ring spectrum and $M$ is an $R$-module. If we are given a normalization $\nu : S \to R$, we say that $M$ is $\nu$-finitely generated if $M$ is small over $S$.

If $R$ is g-normalizable, we say that $M$ is finitely generated if $M$ is $\nu$-finitely generated for every g-normalization $\nu$.

Some elementary consequences follow as in Lemma 10.2.

**Lemma 10.5.** (i) If $R$ is g-normalizable and $M$ is finitely generated then $M_*$ is a finitely generated $R_*$-module.

(ii) If $\nu : S \to R$ is a c-normalization, then $M$ is $\nu$-finitely generated if and only if $M_*$ is finitely generated over $R_*$.

**Remark 10.6.** (a) If $R$ is c-normalizable, it is natural to refer to the class of modules $M$ for which $M_*$ if finitely generated over $R_*$ as c-finitely generated.

(b) This raises the question of whether there are c-finitely generated modules which are not finitely generated. In other words, if $R$ is a c-normalizable g-regular ring and $M$ is an $R$-module with $M_*$ finitely generated over $R_*$ does it follow that $M$ is small? It would suffice to show that $R$ has a c-normalization $S$ which is an $S$-module retract of $R$, as happens when $S_* \to R_*$ is a monomorphism and $R_*$ is Gorenstein.

**Proof:** (i) If $M$ is small over $S$ for some normalization, then $M_*$ is finitely generated over $S_*$ and hence over $R_*$. 

(ii) If $M_*$ is finitely generated over $R_*$ it is finitely generated over $S_*$. Since $S_*$ is regular, this means $M$ is small over $S$ and hence finitely generated.

**10.D. Singularity categories.** In conventional commutative algebra the Buchweitz singularity category $D_{\text{sing}}(R)$ is defined to be the quotient of the bounded derived category by the subcategory of perfect complexes.

If we are given a normalization $\nu$ of a commutative ring spectrum $R$, it is evident that the category of $\nu$-finitely generated modules is a triangulated subcategory of the category
of $R$-modules, and similarly for the finitely generated modules. Evidently for any particular $g$-normalization $\nu$ we have full and faithful embeddings

$$D_{\text{small}}(R) \subseteq D_{fg}(R) \subseteq D_{\nu-fg}(R) \subseteq D_{c-fg}(R)$$

of triangulated categories each closed under retracts. We could consider any one of the quotients, but the most natural is the initial quotient

$$D_{\text{sing}}(R) = D_{fg}(R)/D_{\text{small}}(R).$$

This has the merit that if $R$ is itself $g$-regular, the quotient is trivial, and in many cases one can show that $D_{fg}(R) = D_{c-fg}(R)$.

The singularity quotients are understood in many algebraic cases, but for the present we will restrict ourselves to an example where we can see from first principles that the quotient is non-trivial. See [63] for further discussion.

**Example 10.7.** If $R = C^*(BA_4)$ with $k = \mathbb{F}_2$ then $M = C^*(BV_4)$ is $c$-finitely generated since $H^*(BV_4)$ is finitely generated over $H^*(BA_4)$ by Venkov’s Theorem. However if we let $F = \text{fib}(BV_4 \to BA_4)$ there is a fibre sequence $\Omega S^3 \to F \to SO(3)/V_4$ which may be used to check $H^*(F)$ is non-zero in infinitely many degrees. Thus $C^*(BV_4)$ is not small over $C^*(BA_4)$.

**Part 4. Hypersurface rings**

This is the second of a series of parts that take particular classes of commutative local rings, and identifies ways of giving homotopy invariant counterparts of the definitions, which apply to ring spectra. The justification consists of the dual facts that the new definition reduces to the old in the classical setting and that the new definition covers and illuminates examples in new contexts.

Although we restrict attention to hypersurface rings here, we begin in Section 11 with a brief sketch of relevant facts about complete intersections in general to set the context. In Section 12 we describe the algebraic theory of hypersurface rings in a bit more detail. In Section 13 we explain the relevance of bimodules and Hochschild cohomology. In view of the discussion in Part 3, it is then quite routine in Section 14 to provide the definitions of ci ring spectra, and we illustrate some of the results with examples. In Sections 15 and 16 we explain how the three homotopy invariant versions of the ci condition are related to each other.

**11. Complete intersections**

To understand the significance of hypersurfaces, we should first say a word about complete intersections. There is a general framework for discussing these in the homotopy invariant context [55, 22], but there are few examples beyond the hypersurface case. The general theory is in some sense obtained by iterating the case of hypersurfaces, but there are a number of different ways of iterating it, and some work better than others.
11.A. **Classical complete intersections.** These lectures will focus on hypersurfaces, but it is helpful to set the more general context with a brief discussion of complete intersections in general. Just as for regular rings there are three styles of definition: (s) a structural one (g) one involving growth and (m) one involving the homological algebra of modules. These all have counterparts in the homotopy invariant setting which we will introduce in due course.

We start with the structural definition. The best behaved subvarieties of affine space are those which are specified by the right number of equations: if they are of codimension \( c \) then only \( c \) equations are required. On the same basis, a commutative local ring \( R \) is a **structurally complete intersection (sci)** if its completion is the quotient of a regular local ring \( S \) by a regular sequence, \( f_1, f_2, \ldots, f_c \) (this would normally just be called ‘ci’, but the longer name eases comparison with the case of ring spectra). We will suppose that \( R \) is complete, so that

\[
R = S/(f_1, \ldots, f_c).
\]

The smallest possible value of \( c \) (as \( S \) and the regular sequence vary) is called the **codimension** of \( R \). A hypersurface is the special case when \( c = 1 \) so that a single equation is used.

When it comes to growth, if \( R \) is ci of codimension \( c \), one may construct a resolution of any finitely generated module growing like a polynomial of degree \( c - 1 \). In particular the ring \( \text{Ext}_R^*(k, k) \) has polynomial growth (we say that \( R \) is **gci**). Perhaps the most striking result about ci rings is the theorem of Gulliksen \[67\] which states that this characterises ci rings so that the sci and gci conditions are equivalent for local rings.

With a little care, one may construct the resolutions in an eventually multi-periodic fashion: the projective resolution eventually has the pattern of the tensor product of \( c \) periodic exact sequences. In fact the construction is essentially independent of the module and the calculation can be phrased in terms of the Hochschild cohomology of \( R \). This opens the way to the theory of support varieties for modules over a ci ring \[9\].

11.B. **Homotopy invariant versions, and Levi’s groups.** One may give homotopy invariant versions of all three characterizations of the ci condition:

- **(sci):** the ‘regular ring modulo regular sequence’ condition,
- **(mci):** the ‘modules have eventually multiperiodic resolutions’ and
- **(gci):** polynomial growth of the Ext algebra

We note that the Avramov-Quillen characterization of ci rings in terms of André-Quillen homology does not work for cochains on a space in the mod \( p \) context since Mandell has shown \[80\] that the topological André-Quillen cohomology vanishes very generally in this case.

It involves some work to describe the sci and mci definitions, but if we take \( R = C^*(X) \) for a \( p \)-complete space Subsection \[7.3\] shows \( H_*(\text{Hom}_{C^*(X)}(k, k)) = H_*(\Omega X) \), so it is easy to understand the gci condition. Taking \( X = (BG)^\wedge_p \) we may consider what this means. Of course if \( G \) is a \( p \)-group, \( X = BG \) so \( \Omega X \simeq G \) and \( H_*(\Omega X) \) is simply the group ring \( kG \) in degree 0 and therefore finite dimensional. More generally, it is known to be of polynomial growth in certain cases (for instance \( A_4 \) or \( M_{11} \) in characteristic 2) and R. Levi \[75, 76, 77, 78\] has proved there is a dichotomy between small growth and large growth, and given examples
where the growth is exponential. Evidently groups whose \( p \)-completed classifying spaces have loop space homology that has exponential growth cannot be spherically resolvable, so Levi’s groups disproved a conjecture of F.Cohen.

**Example 11.1.** (a) It is amusing to consider the \( p \)-completed classifying space \( BG^p \) for \( G = (C_p \times C_p) \rtimes C_3 \) where \( C_3 \) acts via \((1, 0) \mapsto (0, 1) \mapsto (-1, -1)\). When \( p = 3 \), \( G \) is a \( p \)-group so the space is \( g \)-regular. When \( p = 2 \) the group is the alternating group \( A_4 \), which we will see below is a hypersurface. If \( p \geq 5 \) then Levi shows \( H_*(\Omega(BG^p)) \) has exponential growth by showing that \( \Omega(S^5 \cup_p e^6) \) is a retract.

(b) If the reader prefers an example with a purely algebraic treatment, Benson shows in [15, Example 2.2] that the group \((C_3)^2 \rtimes C_2\) has a 3-completed classifying space whose loop space has exponential growth using the methods sketched in Subsection 14.E below.

12. **Hypersurfaces in algebra**

In this section we recall some standard constructions for hypersurface algebras. We suppose \( R = S/(f) \) is a hypersurface ring, where \( S \) is a regular ring and \( f \) is a nonzero element of degree \( d \). Thus we have a short exact sequence

\[
0 \rightarrow \Sigma^d S \xrightarrow{f} S \rightarrow R \rightarrow 0
\]

of \( S \)-modules for a regular local ring \( S \). There are two basic constructions that we need to generalize.

12.A. **The degree 2 operator.** We describe a construction of a cohomological operator due to Gulliksen [68, 7]. We will do this for a single module, but it is apparent that the construction is essentially independent of the module, and in fact it lifts to Hochschild cohomology.

Given an \( R \)-module \( M \) we may apply \( (\cdot) \otimes_S M \) to the defining sequence to obtain the short exact sequence

\[
0 \rightarrow \text{Tor}_1^S(R, M) \rightarrow \Sigma^d M \xrightarrow{f} M \rightarrow R \otimes_S M \rightarrow 0.
\]

Since \( f = 0 \) in any \( R \)-module, we conclude

\[
\text{Tor}_1^S(R, M) \cong \Sigma^d M, R \otimes_S M \cong M,
\]

and

\[
\text{Tor}_i^S(R, M) = 0 \text{ for } i \geq 2.
\]

Next, choose a free \( S \)-module \( F \) with an epimorphism to \( M \), giving a short exact sequence

\[
0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0
\]

of \( S \)-modules. Applying \( (\cdot) \otimes_S R \), we obtain

\[
0 \rightarrow \text{Tor}_1^S(R, M) \rightarrow \mathcal{K} \rightarrow \mathcal{F} \rightarrow M \rightarrow 0.
\]

Since \( \text{Tor}_1^S(R, M) \cong \Sigma^d M \), this gives an element

\[
\chi_f \in \text{Ext}_R^2(M, \Sigma^d M),
\]
or a map

$$\chi_f : M \longrightarrow \Sigma^{d+2}M$$

in the derived category of $R$-modules. We sketch below how this construction lifts to give an element

$$\chi_f \in HH^{d+2}(R|S),$$

and hence in particular that it gives a natural transformation of the identity functor (i.e., an element of the centre $Z\mathcal{D}(R)$ of $\mathcal{D}(R)$).

12.B. **The eventually periodic resolution.** Continuing with the above discussion, we may show that all modules $M$ have free resolutions over $R$ which are eventually periodic of period 2.

Indeed, $M$ has a finite free $S$-resolution

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

Adding an extra zero term if necessary, we suppose for convenience that $n$ is even. Now apply $(\cdot) \otimes_S R$ to obtain a complex

$$0 \longrightarrow \overline{F}_n \longrightarrow \overline{F}_{n-1} \longrightarrow \cdots \longrightarrow \overline{F}_1 \longrightarrow \overline{F}_0 \longrightarrow M \longrightarrow 0.$$

Since $\operatorname{Tor}^S_i(R, M) = 0$ for $i \geq 2$, this is exact except in homological degree 1, where it is $\operatorname{Tor}^S_1(R, M) \cong \Sigma^d M$. Splicing in a second copy of the resolution, we obtain a complex

$$
\begin{array}{c}
0 \\
\oplus \\
\Sigma^d F_{n-1} \\
\oplus \\
\Sigma^d F_{n-2} \\
\oplus \\
\Sigma^d F_{n-3} \\
\vdots \\
\Sigma^d F_0
\end{array}
\longrightarrow
\begin{array}{c}
\overline{F}_n \\
\overline{F}_{n-1} \\
\overline{F}_2 \\
\overline{F}_1 \\
\overline{F}_0 \\
M
\end{array}
\longrightarrow
0
$$

which is exact except in the second row, in homological degree 4, where the homology is again $M$. We may repeatedly splice in additional rows to obtain a free resolution

$$\cdots \longrightarrow G_3 \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

over $R$. Remembering the convention that $n$ is even, provided the degree is at least $n$, the modules in the resolution are (up to suspension by a multiple of $d$)

$$G_{2i} = \overline{F}_n \oplus \Sigma^d \overline{F}_{n-2} \oplus \cdots \oplus \Sigma^{d(n-2)/2} \overline{F}_2 \oplus \Sigma^{dn/2} \overline{F}_0$$

in even degrees and

$$G_{2i+1} = \overline{F}_{n-1} \oplus \Sigma^d \overline{F}_{n-3} \oplus \cdots \oplus \overline{F}_3 \oplus \Sigma^{d(n-2)/2} \overline{F}_1$$

in odd degrees.

12.C. **Smallness.** We may reformulate the eventual periodicity of the previous subsection in homotopy invariant terms.

**Lemma 12.1.** If $R$ is a hypersurface $R = S/(f)$ and $M$ is a finitely generated $R$-module then the mapping cone of $\chi_f : M \longrightarrow \Sigma^{d+2}M$ is small.
Proof: From the Yoneda interpretation, we notice that
\[ \chi_f : M \longrightarrow \Sigma^{d+2}M \]
is realized by the quotient map factoring out the first row subcomplex
\[ 0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0. \]
Thus the short exact sequence
\[ 0 \longrightarrow F_\bullet \longrightarrow G_\bullet \longrightarrow \Sigma^{d+2}G_\bullet \longrightarrow 0 \]
of \(R\)-free chain complexes realizes the triangle
\[ \Sigma^1 M/\chi \longrightarrow M \longrightarrow \Sigma^{d+2}M. \]
\[ \square \]

12.D. Matrix factorizations. Because of the importance of matrix factorizations and their prominence in the IRTATCA programme, it is worth a brief subsection to make the connection. We have considered resolutions of \(R\)-modules \(M\), and we note that if \(M\) is of projective dimension 1 over \(S\) then the \(S\)-resolution
\[ 0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow 0 \]
gives a periodic resolution
\[ 0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \]
over \(R\).
Since \(f = 0\) on \(M\), multiplication by \(f\) on the \(S\)-resolution is null-homotopic and so there is a diagram
\[ \begin{array}{ccc}
0 & \leftarrow & M \\
\downarrow 0 & & \downarrow f \\
0 & \leftarrow & M \\
\end{array} \]
In short we have two maps \(A : F_1 \longrightarrow F_0\) and \(B : F_0 \longrightarrow F_1\) with \(AB = f \cdot \text{id}, BA = f \cdot \text{id}\).
Since \(F_0\) and \(F_1\) are free modules, we can choose bases and represent \(A\) and \(B\) by matrices, so this structure is known as a matrix factorization.

If we suppose \(S\) is of dimension \(d\), \(R\) is of dimension \(d - 1\). We note that the homological dimension of \(M\) over \(S\) can be expressed as an invariant of \(M\) as an \(R\)-module. In our case \(M\) is of projective dimension 1 over \(S\) and by the Auslander-Buchsbaum formula, it is of depth \(d - 1\) as an \(R\)-module, which is to say it is a maximal Cohen-Macaulay module. For a module \(M\) of depth \(d - 1 - i\) (i.e., of projective dimension \(i + 1\) over \(S\)) the above discussion applies to its \(i\)th syzygy.

13. Bimodules and natural endomorphisms of \(R\)-modules

To describe the m-version of the definition of hypersurface rings, we need to briefly discuss bimodules.
13.A. The centre of the derived category of $R$-modules. If $R$ is a commutative Noetherian ring and $M$ is a finitely generated $R$-module with an eventually $n$-periodic resolution, there is a map $M \to \Sigma^n M$ in the derived category whose mapping cone is a small $R$-module. If $R$ is an ungraded hypersurface ring, all finitely generated modules have such resolutions with $n = 2$. The lesson learnt from commutative algebra is that to use this to characterize hypersurfaces we need to look at all such modules $M$ together, and ask for a natural transformation $1 \to \Sigma^n 1$ of the identity functor. By definition the centre $ZD(R)$ is the graded ring of all such natural transformations. There are various ways of constructing elements of the centre, and various natural ways to restrict the elements we consider. Some of these work better than others, and it is the purpose of this section is to introduce these ideas.

13.B. Bimodules. We consider a map $S \to R$, where $S$ is regular and $R$ is small over $S$. We may then consider $R_e = R \otimes_S R$, and $R_e$-modules are $(R|S)$-bimodules. The Hochschild cohomology ring is defined by

$$HH^*(R|S) = Ext^*_R (R, R).$$

If $f : X \to Y$ is a map of $(R|S)$-bimodules, for any $R$-module $M$ we obtain a map $f \otimes 1 : X \otimes_R M \to Y \otimes_R M$ of (left) $R$-modules.

The simplest way for us to use this is that if we have isomorphisms $X \cong R$ and $Y \cong \Sigma^n R$ as $R$-bimodules, the map $f \otimes 1 : M \to \Sigma^n M$ is natural in $M$ and therefore gives an element of codegree $n$ in $ZD(R)$: we obtain a map of rings

$$HH^n(R) = \text{Hom}_{R_e} (R, \Sigma^n R) \to ZD(R)^n.$$

Continuing, if $X \models_{R_e} Y$ then $X \otimes_R M \models_{R} Y \otimes_R M$. In particular, if $X = R$ builds a small $R_e$-module $Y$ then

$$M = R \otimes_R M \models_{R} Y \otimes_R M = R_e \otimes_R M = R \otimes_S M.$$  

Thus if $M$ is finitely generated (i.e., small over $S$), this shows $M$ finitely builds a small $R$-module.

It is sometimes useful to restrict the maps permitted in showing that $X \models_{R_e} Y$. If we restrict to using maps of positive codegree coming from Hochschild cohomology, we write $X \models_{hh} Y$. If we permit any maps of positive codegree from the centre $ZD(R_e)$ we write $X \models_z Y$. Finally, we could relax further and require only that all the maps involved in building are endomorphisms of non-zero degree for some object and write $X \models_e Y$. It is this last condition that turns out to be most convenient for complete intersections of larger codimension.

14. Hypersurface ring spectra

We are now ready to describe the $s$-, $g$- and $m$- versions of the hypersurface condition for ring spectra. As usual we begin with the template in commutative algebra, go on to describe it in general and then make it concrete for spaces.
14.A. The definition in commutative algebra. In commutative algebra there are three styles for a definition of a hypersurface ring: ideal theoretic, in terms of the growth of the Ext algebra and a derived version. See [21] for a more complete discussion.

Definition 14.1. (i) A local Noetherian ring $R$ is an $s$-hypersurface ring if $R = S/(f)$ some regular ring $S$ and some $f \neq 0$.

(ii) A local Noetherian ring $R$ is a $g$-hypersurface if the dimensions $\text{dim}_k(\text{Ext}^n_R(k, k))$ are bounded independently of $n$.

(iii) A local Noetherian ring $R$ is a $z$-hypersurface if there is an elements $z \in ZD(R)$ of non-zero degree so that $M/z$ is small for all finitely generated modules $M$. Similarly $R$ is an $hh$-hypersurface if the element $z$ can be chosen to come from Hochschild cohomology.

Theorem 14.2. ([7], transcribed into the present language in [21]) For a local Noetherian ring the $s$, $z$- and $g$-hypersurface conditions are all equivalent.

14.B. Definitions for ring spectra. The appropriate definition of an $s$-hypersurface can be seen from the discussion of $s$-regular spectra: the point is that regular elements correspond to maps with exterior cofibres.

Definition 14.3. (i) A ring spectrum $R$ is a $c$-hypersurface if $R_\ast$ is a hypersurface ring.

(ii) The ring spectrum $R$ is an $s$-hypersurface if there is a normalization $S \to R$ with $\pi_\ast(R \otimes_S k) = \Lambda_k(\gamma)$.

(iii) The ring spectrum $R$ is a $g$-hypersurface if $\text{dim}_k(\pi_n(\text{Hom}_R(k, k)))$ is bounded independently of $n$.

(iv) The $c$-normalizable ring spectrum $R$ is a $z$-hypersurface if there is an element $z \in ZD(R)$ of non-zero degree so that $M/z$ is small for all finitely generated modules $M$. Similarly $R$ is an $hh$-hypersurface if the element $z$ can be chosen to come from Hochschild cohomology.

One immediate source of examples comes from $c$-hypersurfaces.

Example 14.4. If $R$ is a $c$-hypersurface then the spectral sequence

$$\text{Ext}^\ast_{R_\ast}(k, k) \Rightarrow \pi_\ast(\text{Hom}_R(k, k))$$

shows that it is a $g$-hypersurface.

14.C. Definitions for spaces. In view of the fact that regular elements correspond to spherical fibrations, adapting the above definitions for spaces is straightforward.

Definition 14.5. (i) A space $X$ is a $c$-hypersurface if $H^\ast(X)$ is a hypersurface ring.

(ii) A space $X$ is an $s$-hypersurface (or spherical hypersurface) if it is the total space of a spherical fibration over a connected $g$-regular space $B\Gamma$: there is a $g$-regular space $B\Gamma$ and a fibration

$$S^n \to X \to B\Gamma$$

for some $n$.

(iii) A space $X$ is a $g$-hypersurface space if $H^\ast(X)$ is Noetherian and $\text{dim}_k H_n(\Omega X)$ is bounded independently of $n$. 

33
(iv) A space $X$ is a $z$-hypersurface space if $X$ is $g$-normalizable and there is an element $z \in ZD(C^*(X))$ of non-zero degree so that $C^*(Y)/z$ is small for all finitely generated $C^*(Y)$. It is an $hh$-hypersurface if $z$ comes from Hochschild cohomology. (The direct transcription for ring spectra would require that this holds for all finitely generated modules and not just those of the particular form $C^*(Y)$).

**Remark 14.6.** Other variants have arisen, such as $\omega$sci where we are permitted to use loop spaces on spheres rather than spheres. These conditions arose in Levi’s work. This is evidently a weakening of sci which still implies gci.

For $g$-normalizable spaces $X$, we have [55, 22] the implications

$$s - \text{hypersurface} \Rightarrow z - \text{hypersurface} \Rightarrow g - \text{hypersurface}.$$  

Over $\mathbb{F}_p$ the final implication may be reversed; the proof is given in [22] and relies on [43].

14.D. **An example.** To start with we may consider the space $BA_4$ at the prime 2. We will observe directly that it is a hypersurface according to any one of the definitions.

To start with, we note that

$$H^*(BA_4) = H^*(BV_4)^{A_4/V_4} = k[x_2, y_3, z_3]/(r_6)$$

where $r_6 = x_2^3 + y_2^3 + y_3^3 + z_2^3$. This shows that $BA_4$ is actually a c-hypersurface and hence also a $g$-hypersurface. Indeed, the Eilenberg-Moore spectral sequence shows that the loop space homology will eventually have period dividing 4.

In fact we see that $BA_4$ is an $s$-hypersurface space at 2. The direct symmetries of a tetrahedron give a homomorphism $A_4 \rightarrow SO(3)$ and hence a map $BA_4 \rightarrow BSO(3)$. The fibre is $SO(3)/A_4$, and at the prime 2 this is $S^3$, so there is a 2-adic fibration

$$S^3 \rightarrow BA_4 \rightarrow BSO(3).$$

In Subsection 14.G we will also describe a representation theoretic approach to showing that $BA_4$ is a $g$-hypersurface, which will show that its ultimate period is exactly 4.

14.E. **Squeezed homology.** Since we are working with groups, it is illuminating to recall Benson’s purely representation theoretic calculation of the loop space homology $H_*(\Omega(BG_\mathbb{F}_p^\wedge))$ [15]. In fact he defines the squeezed homology groups $H\Omega_*(G; k)$ algebraically and proves

$$H_*(\Omega(BG_\mathbb{F}_p^\wedge)) \cong H\Omega_*(G; k).$$

In more detail, $H\Omega_*(G; k)$ is the homology of

$$\cdots \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0,$$

a so-called squeezed resolution of $k$. The sequence of projective $kG$-modules $P_i$ is defined recursively as follows. To start with $P_0 = P(k)$ is the projective cover of $k$. Now if $P_i$ has been constructed, take $N_i = \ker(P_i \rightarrow P_{i-1})$ (where we take $P_{-1} = k$), and $M_i$ to be the smallest submodule of $N_i$ so that $N_i/M_i$ is an iterated extension of copies of $k$. Now take $P_{i+1}$ to be the projective cover of $M_i$.  

34
14.F. **Trivial cases.** Note that if $G$ is a $p$-group, we have $\Omega(BG_p^\wedge) \simeq G$ so that $H_*(\Omega BG) \cong kG$ and since $k$ is the only simple module, $M_0 = 0$ and we again find $H\Omega_*(G) = kG$.

We would expect the next best behaviour to be when $H^*(BG)$ is a hypersurface. Indeed, if $H^*(BG)$ is a polynomial ring modulo a relation of codegree $d$, the Eilenberg-Moore spectral sequence

$$\text{Ext}_{H^*(BG)}^*(k, k) \Rightarrow H^*(\Omega(BG_p^\wedge))$$

shows that there is an ultimate periodicity of period $d - 2$. The actual period therefore divides $d - 2$. We now return to the example of $A_4$.

14.G. **$A_4$ revisited.** We described the homotopy theoretic proof that $BA_4$ is an $s$-hypersurface space above.

Here we sketch a purely algebraic proof from [15]. To start with, we would like to see algebraically that $H_*(\Omega(BA_4) \wedge P^_, k)$ is eventually periodic.

This case is small enough to be able to compute products in $H_*(\Omega BG_p^\wedge, k)$ using squeezed resolutions, and we get

$$H_*(\Omega BG_p^\wedge, k) = \Lambda(\alpha) \otimes k(\beta, \gamma)/(\beta^2, \gamma^2)$$

with $|\alpha| = 1$ and $|\beta| = |\gamma| = 2$. Beware that $\beta$ and $\gamma$ do not commute, so that a $k$-basis for $H_*(\Omega BG_p^\wedge, k)$ is given by alternating words in $\beta$ and $\gamma$ (such as $\beta\gamma\beta$ or the empty word), and $\alpha$ times these alternating words.

There are three simple modules. Indeed, the quotient of $A_4$ by its normal Sylow 2-subgroup is of order 3; supposing for simplicity that $k$ contains three cube roots of unity $1, \omega, \bar{\omega}$, the simples correspond to how a chosen generator acts. The projective covers of the three simple modules are

$$P(k) = \begin{array}{c} k \cr k \cr \omega \end{array}, \quad P(\omega) = \begin{array}{c} k \cr \omega \cr \bar{\omega} \end{array}, \quad P(\bar{\omega}) = \begin{array}{c} \bar{\omega} \cr k \cr \omega \end{array}.$$  

With this information to hand, the reader may verify the assertion about the loop space homology ring by using squeezed resolutions.

15. **$s$-hypersurface spaces and $z$-hypersurface spaces**

In the algebraic setting the remarkable fact is that modules over hypersurfaces have eventually periodic resolutions, and hence that they are hhci of codimension 1. The purpose of this section is to sketch a similar result for spaces. The argument is given in detail in [22].

**Theorem 15.1.** If $X$ is an $s$-hypersurface space with fibre sphere of dimension $\geq 2$ then $X$ is a $z$-hypersurface space.

15.A. **Split spherical fibrations.** The key in algebra was to consider bimodules, for which we consider the (multiplicative) exact sequence

$$R \longrightarrow R^e \longrightarrow R^e \otimes_R k,$$
where the first map is a monomorphism split by the map $\mu$ along which $R$ acquires its structure as an $R^e$-module structure. This corresponds to the pullback fibration

$$X \leftarrow X \times_{B^\Gamma} X \leftarrow S^n,$$

split by the diagonal

$$\Delta : X \rightarrow X \times_{B^\Gamma} X$$

along which the cochains on $X$ becomes a bimodule. To simplify notation, we consider a more general situation: a fibration

$$B \leftarrow E \leftarrow S^n$$

with section $s : B \rightarrow E$. The case of immediate interest is $B = X$, $E = X \times_{B^\Gamma} X$, where a $C^*(E)$-module is a $C^*(X)$-bimodule.

Note that by the third isomorphism theorem for fibrations, there is a fibration

$$\Omega S^n \rightarrow B \stackrel{s}{\rightarrow} E.$$ 

This gives the required input for the following theorem. The strength of the result is that the cofibre sequences are of $C^*(E)$-modules.

**Theorem 15.2.** Suppose given a fibration $\Omega S^n \rightarrow B \stackrel{s}{\rightarrow} E$ with $n \geq 2$.

(i) If $n$ is odd, then there is a cofibre sequence of $C^*(E)$-modules

$$\Sigma_{n-1}C^*(B) \leftarrow C^*(B) \leftarrow C^*(E).$$

(ii) If $n$ is even, then there are cofibre sequences of $C^*(E)$-modules

$$C \leftarrow C^*(B) \leftarrow C^*(E)$$

and

$$\Sigma_{2n-2}C^*(B) \leftarrow C \leftarrow \Sigma_{n-1}C^*(E).$$

In particular the fibre of the composite

$$C^*(B) \rightarrow C \rightarrow \Sigma_{2n-2}C^*(B)$$

is a small $C^*(E)$-module constructed with one cell in codegree 0 and one in codegree $n - 1$.

**Remark 15.3.** Note that in either case we obtain a cofibre sequence

$$K \leftarrow C^*(B) \leftarrow \Sigma_aC^*(B)$$

of $C^*(E)$-modules with $K$ small.

The strategy is to first prove the counterparts in cohomology by looking at the Serre spectral sequence of the fibration from Part (i) and then lift the conclusion to the level of cochains.
16. Growth of z-hypersurface resolutions

In this brief section we prove perhaps the simplest implication between the hypersurface conditions: a z-hypersurface is a g-hypersurface.

**Lemma 16.1.** If \( R \) is a z-hypersurface and \( k \) is a field then the vector spaces \( \pi_n(\mathcal{E}) \) are of bounded dimension and \( R \) is a g-hypersurface.

**Proof:** Since \( R \) is a z-hypersurface, then in particular there is a projective resolution of \( k \) which is eventually periodic. In other words there is a triangle

\[
\Sigma^n k \twoheadrightarrow k \rightarrow L
\]

with \( n \neq 0 \) and \( L \) small over \( R \). Applying \( \text{Hom}_R(\cdot, k) \) we find a triangle

\[
\Sigma^{-n} \mathcal{E} \leftarrow \mathcal{E} \leftarrow \text{Hom}_R(L, k).
\]

Since \( L \) is finitely built from \( R \), \( \text{Hom}_R(L, k) \) is finite dimensional over \( k \), and hence only nonzero in a finite range of degrees (say \([-N, N]\)). Outside that range we have \( \pi_{s+n}(\mathcal{E}) \cong \pi_{s}(\mathcal{E}) \), so every homotopy group is isomorphic as a \( k \)-vector space to one in the range \([-N - n, N + n]\) and the bound is the largest of these dimensions. \( \square \)

**Remark 16.2.** Essentially the same argument shows that a cofibre sequence \( \Sigma^n A \rightarrow A \rightarrow B \) with two terms the same means that the growth rate of \( A \) is at most one more than that of \( B \).

Part 5. Gorenstein rings

This is the final part taking a particular classes of commutative local rings, and giving homotopy invariant counterparts of the definitions. The justification consists of the dual facts that the new definition reduces to the old in the classical setting and that the new definition covers and illuminates new examples. The generalization of the Gorenstein condition is perhaps the most successful of the three, since there are so many Gorenstein ring spectra, and this approach provides consequences that are both unexpected and very concrete.

In Section 17 we introduce the homotopy invariant version of the Gorenstein condition an in Section 18 the associated Gorenstein duality property and its implications for coefficients. We then provide some basic tools for dealing with the Gorenstein condition: ascent and descent in Section 19 and Morita invariance in Section 20. We then turn to examples in earnest, with a discussion of Gorenstein duality for group cohomology in Section 21 for rational homotopy theory in Section 22 and a brief pointer to other examples in Section 23.

17. The Gorenstein condition

In this section we quickly recall the definition of a Gorenstein local ring in a form which also makes sense for ring spectra with a map to \( k \).
17.A. **Gorenstein local rings.** The usual definition of a Gorenstein local ring is that $R$ is of finite injective dimension over itself. But it is then proved that in fact it is then of injective dimension equal to $r = \text{Krull dimension}(R)$ and that $\text{Ext}_R^*(k, R) = \text{Ext}_R^*(k, R) = k$. Conversely, if this holds, the ring is Gorenstein. There are various other characterizations of Gorenstein rings, including a duality statement that we will discuss shortly, but this is enough to suggest the definition for ring spectra.

17.B. **Gorenstein ring spectra.** Ultimately, we want to consider duality phenomena modelled on those in commutative algebra of Gorenstein local rings, so we will develop the theory for spectra in parallel. Corresponding to the Noetherian condition we restrict the class of ring spectra to those which are proxy-small in the sense of Definition 5.2. We begin with the core Gorenstein condition and move onto duality in due course. These definitions come from [33].

17.C. **The Gorenstein condition.** We say that $R \longrightarrow k$ is *Gorenstein* of shift $a$ (and write shift$(R) = a$) if we have an equivalence 

$$\text{Hom}_R(k, R) \simeq \Sigma^a k$$

of $R$-modules.

**Remark 17.1.** We will say that it is c-Gorenstein if $R_\ast \longrightarrow k$ is a Gorenstein local ring. As usual the spectral sequence

$$\text{Ext}_{R_\ast}^{*, *}(k, R_\ast) \Rightarrow \pi_\ast(\text{Hom}_R(k, R))$$

shows that a c-Gorenstein ring spectrum is Gorenstein, but we will give many examples of Gorenstein ring spectra which are not c-Gorenstein.

18. **Gorenstein duality**

Although the Gorenstein condition itself is convenient to work with, the real reason for considering it is the duality property that it implies. To formulate this, we use local cohomology in the sense of Grothendieck, and the reader may wish to refer to Appendix A for the basic definitions.

18.A. **Classical Gorenstein duality.** In classical local commutative algebra, the Gorenstein duality property is that all local cohomology is in a single cohomological degree, where it is the injective hull $I(k)$ of the residue field. To give a formula, we write $\Gamma_m M$ for the $m$-power torsion in an $R$-module $M$, and $H_m^*(M)$ for the local cohomology of $M$, recalling Grothendieck’s theorem that if $R$ is Noetherian, $H_m^*(M) = \mathbb{R}^*_m \Gamma_m(M)$. The Gorenstein duality statement for a local ring of Krull dimension $r$ therefore states

$$H_m^*(R) = H_m^*(R) = I(k).$$

If $R$ is a $k$-algebra, $I(k) = R^\sim = \Gamma_m \text{Hom}_k(R, k)$. 

38
18.B. **Gorenstein duality for \( k \)-algebra spectra.** Turning to ring spectra, we will treat the case that \( R \) is a \( k \)-algebra. This simplifies things considerably, and covers many interesting examples. The more general case requires a discussion of Matlis lifts, for which we refer the reader to [33].

In the case of \( k \)-algebras, we may again define \( R^\vee = \text{Cell}_k(\text{Hom}_k(R, k)) \) and observe this has the Matlis lifting property

\[
\text{Hom}_R(T, R^\vee) \simeq \text{Hom}_k(T, k)
\]

for any \( T \) built from \( k \).

In particular, if \( R \) is Gorenstein of shift \( a \) we have equivalences of \( R \)-modules

\[
\text{Hom}_R(k, \text{Cell}_k R) \simeq \text{Hom}_R(k, R) \overset{(G)}{\simeq} \Sigma^a k \overset{(M)}{\simeq} \text{Hom}_R(k, \Sigma^a R^\vee),
\]

where the equivalence (G) is the Gorenstein property and the equivalence (M) is the Matlis lifting property. We would like to remove the \( \text{Hom}_R(k, \cdot) \) to deduce

\[
\text{Cell}_k R \simeq \Sigma^a R^\vee.
\]

Morita theory (specifically Lemma [6,3]) says that if \( R \) is proxy-regular we may make this deduction provided \( R \) is orientably Gorenstein in the sense that the right actions of \( E = \text{Hom}_R(k, k) \) on \( \Sigma^a k \) implied by the two equivalences (G) and (M) agree.

We note that \( \text{Cell}_k(R) \) is a covariant functor of \( R \) whilst \( R^\vee \) is a contravariant functor of \( R \), so an equivalence between them is a form of duality: when it holds we say \( R \) satisfies **Gorenstein duality**.

18.C. **Automatic orientability.** There are a number of important cases where orientability is automatic because \( E \) has a unique action on \( k \), and in this case the Gorenstein condition automatically implies Gorenstein duality.

The first case of this is when \( R \) is a classical commutative local ring, although of course we knew already that in this case the Gorenstein condition is equivalent to Gorenstein duality.

From our present point of view, we see this as a consequence of connectivity: \( E \) (whose homology is \( \text{Ext}^*_R(k, k) \)) has a unique action on \( k \). The same argument applies when the ring spectrum is both a \( k \)-algebra and connected.

**Proposition 18.1.** Suppose \( R \) is a proxy-regular, connected \( k \)-algebra and \( \pi_s(R) \) is Noetherian with \( \pi_0(R) = k \) and maximal ideal \( m \) of positive degree elements. If \( R \) is Gorenstein of shift \( a \), then it is automatically orientable and so has Gorenstein duality.

**Proof:** First we argue that if \( R \) is Gorenstein, it is automatically orientable. Indeed, we show that \( E \) has a unique action on \( k \). Since \( R \) is a \( k \)-algebra, the action of \( E \) on \( k \) factors through

\[
E = \text{Hom}_R(k, k) \longrightarrow \text{Hom}_k(k, k) = k,
\]

so since \( k \) is an Eilenberg-MacLane spectrum, the action is through \( \pi_0(E) \). Now we observe that since \( R \) is connected, \( \text{Ext}^*_R(k, k) \) is in degrees \( \leq -s \), so that the spectral sequence for calculating \( \pi_s(\text{Hom}_R(k, k)) \) shows \( E \) is coconnective with \( \pi_0(E) = k \) which must act trivially on \( k \).
We may go a little further to the nilpotent case.

**Lemma 18.2.** If $X$ is connected with $\pi_1(X)$ a finite $p$-group and $k$ is of characteristic $p$ then if $C^*(X)$ is Gorenstein it automatically has Gorenstein duality.

**Proof:** Again we find $E$ has a unique action on $k$. Since $E$ is a $k$-algebra, it acts through $\pi_0(E) = H_0(\Omega X) = k[\pi_1(X)]$. By the characteristic assumption, this has a unique action on $k$. □

18.D. **The local cohomology theorem.** In many cases (see Remark D.2 of Appendix Dy) one can give an algebraic description of the $k$-cellularization and infer algebraic consequences of Gorenstein duality. For simplicity we restrict to local $k$-algebras, although the methods apply more generally.

**Lemma 18.3.** If $R_\ast$ is a Noetherian $k$-algebra, with maximal ideal $m$ and residue field $k$, then $k$-cellularization coincides with the derived $m$-power torsion functor. Accordingly, if $R$ has Gorenstein duality, there is a local cohomology spectral sequence

$$H^s_m(R_\ast) \Rightarrow \Sigma^s R_\ast^\vee.$$

If $R$ has Gorenstein duality, Lemma 18.3 shows that the ring $\pi_* (R)$ has very special properties (even if it falls short of being Gorenstein), studied in [70]. Some of these properties were first observed by Benson and Carlson [16, 17] for group cohomology (corresponding to the special case of the ring spectrum $R = C^*(BG)$, which we will see below has Gorenstein duality).

To start with, we note that the spectral sequence collapses if $R_\ast$ is Cohen-Macaulay to show $H^r_m(R_\ast) \cong \Sigma^{a+r} R_\ast^\vee$ (where $r$ is the Krull dimension of $R_\ast$). Thus the coefficient ring $R_\ast$ is also Gorenstein.

The spectral sequence also collapses if $R_\ast$ is of Cohen-Macaulay defect 1, to give an exact sequence

$$0 \longrightarrow H^r_m(R_\ast) \longrightarrow \Sigma^{a+r} R_\ast^\vee \longrightarrow \Sigma H^{r-1}_m(R_\ast) \longrightarrow 0.$$

In general, local duality lets one deduce that the cohomology ring $R_\ast$ is always generically Gorenstein.

The collapse of the local cohomology theorem in the case of Cohen-Macaulay defect $\leq 1$ has very concrete consequences in that the Hilbert series of $R_\ast$ satisfies a suitable pair of functional equations.

**Corollary 18.4.** [56] Suppose $R$ has Gorenstein duality of shift $a$, that $\pi_*(R)$ is Noetherian of Krull dimension $r$ and Hilbert series $p(s) = \sum_i \dim_k(R_i) s^i$.

If $\pi_*(R)$ is Cohen-Macaulay it is also Gorenstein, and the Hilbert series satisfies

$$p(1/s) = (-1)^r s^{a-r} p(s).$$

If $\pi_*(R)$ is almost Cohen-Macaulay it is also almost Gorenstein, and the Hilbert series satisfies

$$p(1/s) - (-1)^r s^{a-r} p(s) = (-1)^{r-1} (1 + s) q(s) \text{ and } q(1/s) = (-1)^{r-1} s^{a-r+1} q(s).$$
In any case $\pi_*(R)$ is Gorenstein in codimension 0 and almost Gorenstein in codimension 1.

19. Ascent, descent and arithmetic of shifts

Very commonly we have a map $\theta : S \to R$ of ring spectra, and we wish to relate properties of the two rings. With language from the geometric counterpart, a theorem stating that if $R$ is Gorenstein then $S$ is Gorenstein is called a descent theorem and a theorem stating that if $S$ is Gorenstein then $R$ is Gorenstein is called an ascent theorem. Typically the hypotheses are either about relative properties of $\theta$ or in terms of the cofibre of $\theta$.

19.A. Relatively Gorenstein maps. We say that $\theta$ is relatively Gorenstein of shift $a$ if

$$\text{Hom}_S(R, S) \simeq \Sigma^a R,$$

and then write $a = \text{shift}(R|S)$. This is quite a strong version of the condition, since we have asked for a single untwisted integer suspension.

We make the elementary observation that for any ring map $\theta : S \to R$

$$\text{Hom}_R(k, \text{Hom}_S(R, S)) \simeq \text{Hom}_R(R \otimes_S k, S) \simeq \text{Hom}_S(k, S).$$

Thus we conclude that if $S \to R$ is relatively Gorenstein then $R$ is Gorenstein if and only if $S$ is Gorenstein, and in that case

$$\text{shift}(S) = \text{shift}(R) + \text{shift}(R|S).$$

19.B. Two chromatic examples. There are a number of examples where known facts amount to proving that a map of ring spectra is relatively Gorenstein, and the resulting descent theorems prove things of great interest. In these examples one shows that $\theta$ is relatively Gorenstein and $R$ is c-Gorenstein; we then reach the interesting conclusion that $S$ is Gorenstein.

Example 19.1. (Complexification in K-theory) (i) Periodic complex K-theory is represented by the ring spectrum $KU$ and periodic real K-theory by $KO$. The coefficient ring of complex K-theory is $KU_* = \mathbb{Z}[v, v^{-1}]$ where $v$ is of degree 2. Complexification gives a ring map

$$S = KO \to KU = R$$

and we observe that it is relatively Gorenstein. Indeed, Wood’s theorem states $KO \wedge \mathbb{C}P^2 \simeq \Sigma^2 KU$. Since

$$\mathbb{C}P^2 = S^2 \cup_{\eta} e^4$$

we see that there is a cofibre sequence

$$\Sigma KO \to KO \to KU$$

of $KO$-modules. It follows by applying $\text{Hom}_{KO}(\cdot, KO)$ that

$$\text{Hom}_{KO}(KU, KO) \simeq \Sigma^{-2} KU$$

so that $KO \to KU$ is relatively Gorenstein of shift $-2$.

(ii) We may take connective covers to obtain a ring map

$$S = ko \to ku = R.$$
Now $ku_* = \mathbb{Z}[v]$ and killing homotopy groups gives a ring map
$$ku \rightarrow \mathbb{Z}$$
which is evidently $c$-Gorenstein (and hence Gorenstein) of shift $-3$. The connective version of Wood’s theorem gives a cofibre sequence
$$\Sigma ko \rightarrow ko \rightarrow ku$$
of $ko$-modules. It follows by applying $\text{Hom}_{ko}(\cdot, ko)$ that
$$\text{Hom}_{ko}(ku, ko) \cong \Sigma^{-2} ku$$
so that $ko \rightarrow ku$ is also relatively Gorenstein of shift $-2$. Hence we deduce that $ko \rightarrow \mathbb{Z}$ is Gorenstein of shift $-5$. We note that the ring $ko_* = \mathbb{Z}[\eta_1, \alpha_4, \beta_8]/(\eta^3, \eta \alpha, \alpha^2 = 4 \beta, 2 \eta)$ is fairly complicated, and it is easy to check that $ko \rightarrow \mathbb{Z}$ is not $c$-Gorenstein.

In this discussion we have taken $k = \mathbb{Z}$, which is not a field. If the reader prefers, we can instead work over the field $\mathbb{F}_p$ and use the fact that for any prime $p$ the map $ku \rightarrow \mathbb{F}_p$ is $c$-Gorenstein of shift $-4$. Since $ko \rightarrow ku$ is relatively Gorenstein of shift $-2$ we conclude $ko \rightarrow \mathbb{F}_p$ is Gorenstein of shift $-6$. □

**Example 19.2. (Topological modular forms)** Precisely similar statements hold for the ring spectrum $tmf$ of topological modular forms at various primes. This uses results of Hopkins-Mahowald as proved by Matthew [82] (see [54] for a slightly expanded discussion).

(3) Localized at the prime 3, there is a map $tmf \rightarrow tmf_0(2)$ to the ring spectrum of topological modular forms with the indicated level structure. The counterpart of Wood’s theorem is the fact that $tmf_0(2) \cong tmf \wedge (S^0 \cup \alpha_1 \varepsilon^4 \cup \alpha_1 \varepsilon^8)$, so that
$$\text{Hom}_{tmf}(tmf_0(2), tmf) \cong \Sigma^{-8} tmf_0(2),$$
and the map is relatively Gorenstein of shift $-8$. Since $tmf_0(2)_* = \mathbb{Z}_{(3)}[c_2, c_4]$ (where $|c_i| = 2i$) we see that $tmf_0(2) \rightarrow \mathbb{Z}_{(3)}$ is $c$-Gorenstein (and hence Gorenstein) of shift $-14$. Hence we deduce by Gorenstein descent that $tmf \rightarrow \mathbb{Z}_{(3)}$ is Gorenstein of shift $-22$.

(2) Localized at the prime 2, there is a map $tmf \rightarrow tmf_1(3)$ to the ring spectrum of topological modular forms with the indicated level structure. The Here $tmf_1(3)$ is a form of $BP(2)$ and $tmf_1(3) \cong tmf \wedge DA(1)$ so that
$$\text{Hom}_{tmf}(tmf_1(3), tmf) \cong \Sigma^{-12} tmf_1(3),$$
and the map is relatively Gorenstein of shift $-12$. Since $tmf_1(3)_* = \mathbb{Z}_{(2)}[\alpha_1, \alpha_3]$ (where $|\alpha_i| = 2i$) we see that $tmf_1(3) \rightarrow \mathbb{Z}_{(2)}$ is $c$-Gorenstein (and hence Gorenstein) of shift $-10$. Hence we deduce by Gorenstein descent that $tmf \rightarrow \mathbb{Z}_{(2)}$ is Gorenstein of shift $-22$. □

Abstracting this slightly, we have
$$S \rightarrow S \rightarrow R \rightarrow k$$
and these examples were all very special in that there was an equivalence
$$R \cong S \otimes_S L$$
for a self-dual finite complex $L$. This means in turn that the cofibre of $\theta : S \to R$
\[ R \otimes_S k \simeq k \otimes_S L, \]
with homotopy $H_*(L; k)$ a Poincaré duality algebra. In the next subsection we show that
this weaker condition is often sufficient to give an ascent theorem.

19.C. Gorenstein Ascent. We suppose that $S \to R \to Q$ is a cofibre sequence of
commutative algebras with a map to $k$, and we now consider the Gorenstein ascent question.
When does the fact that $S$ is Gorenstein imply that $R$ is Gorenstein? It is natural to assume
that $Q$ is Gorenstein, but it is known this is not generally sufficient.

In effect the Gorenstein Ascent theorem will state that under suitable hypotheses (see
Section 19.E) there is an equivalence
\[ \text{Hom}_R(k, R) \simeq \text{Hom}_Q(k, \text{Hom}_S(k, S) \otimes_k Q). \]
When this holds, it follows that if $S$ and $Q$ are Gorenstein, so is $R$ and
\[ \text{shift}(R) = \text{shift}(S) + \text{shift}(Q). \]

19.D. Arithmetic of shifts. We summarize the behaviour of Gorenstein shifts in the ideal
situation when ascent and descent both hold. If all rings and maps are Gorenstein of the
indicated shifts
\[ S \xrightarrow{s} R \xrightarrow{\mu} \lambda \xrightarrow{q} Q \]
then $r = s + q, \mu = -q$ and $\lambda = s$

19.E. When does Gorenstein ascent hold? The core of our results about ascent come from
[33]. Indeed, the proof of [33, 8.6] gives a sufficient condition for Gorenstein ascent in
the commutative context.

Lemma 19.3. If $S$ and $R$ are commutative and the natural map $\nu : \text{Hom}_S(k, S) \otimes_S R \to \text{Hom}_S(k, R)$ is an equivalence then
\[ \text{Hom}_R(k, R) \simeq \text{Hom}_Q(k, \text{Hom}_S(k, S) \otimes_k Q). \]
In this case, if $S$ and $Q$ are Gorenstein, so is $R$, and the shifts add up: $\text{shift}(R) = \text{shift}(S) + \text{shift}(Q). \quad \square$

Now that we have a sufficient condition for Gorenstein ascent, we want to identify cases
in which it is satisfied. The map
\[ \nu : \text{Hom}_S(k, S) \otimes_S M \to \text{Hom}_S(k, M), \]
is clearly an equivalence when $M = R$ and hence for any module finitely built from $R$. This
shows that the hypotheses are satisfied when $R$ is small over $S$ (or equivalently, when $Q$ is
finitely built from $k$) so that ascent holds in this case. This is already a very useful result.

Example 19.4. If we have a fibration $F \to E \to B$ of spaces to which the Eilenberg-
Moore theorem applies in the sense that $C^*(F) = C^*(E) \otimes_{C^*(B)} k$, then if $B$ is Gorenstein
and $F$ is an orientable manifold, it follows from Gorenstein Ascent that $E$ is Gorenstein.
There are other important cases where \( \nu \) is an equivalence. Indeed, we can exploit the fact that the hypothesis on \( \nu \) in Lemma 19.3 only depends on \( R \) as a module over \( S \) to show that \( \nu \) is an equivalence when \( R \) is suitably approximated as an inverse limit. This is a central ingredient in proving Gorenstein duality for many topological Hochschild homology spectra [54].

19.F. **Local duality.** To start with, we clarify terminology. The Gorenstein duality property we have been discussing is a rare and special thing. On the other hand, local duality (as in the title of this subsection) is a tool available very generally. Local duality is based on Noether normalization, which means that every well behaved local ring \( R \) is finite as a module over a Gorenstein ring \( S \). Local duality is the property inherited by \( R \) as a consequence of the existence of \( S \).

Accordingly, we assume for the rest of this section that we are given a map \( S \rightarrow R \) so that \( R \) is a small \( S \)-module. Notice that this means that for an \( R \)-module \( M \), its \( k \)-cellularization as an \( R \)-module is its \( k \)-cellularization as an \( S \)-module and similarly for \( k \)-completions. This is reflected in our notation. We also note that \( R^\vee = \text{Hom}_S(R, S^\vee) \) so that the Matlis lifts are related by coextension of scalars.

Traditionally, local duality is thought of as saying that Matlis dual of local cohomology (embodied in \( \text{Hom}_R(\text{Cell}_k R, R^\vee) \approx (R^\vee)_k^\wedge \)) is isomorphic to a completed Ext group (embodied by \( \text{Hom}_S(R, S)_k^\wedge \)). Directly translating this into our context we reach a more inscrutable definition.

**Definition 19.5.** We say that \( S \rightarrow R \) has local duality of shift \( b \) if there is an equivalence of \( R \)-modules

\[
\text{Hom}_S(R, S)_k^\wedge = \Sigma^b(R^\vee)_k^\wedge.
\]

We now have two properties the map \( S \rightarrow R \) may or may not have: the relative Gorenstein property and local duality. We also have Gorenstein duality for \( S \rightarrow k \) and for \( R \rightarrow k \). It is valuable to disentangle the relationships between them.

In fact we have three complete \( R \)-modules.

- \( R_k^\wedge \)
- \( (R^\vee)_k^\wedge \)
- \( \text{Hom}_S(R, S)_k^\wedge \)

The equivalence of each of the possible pairs has a name. In the following diagram, the label “relGor” means “\( k \)-completion of relatively Gorenstein”, “locD” means “local duality”, and “GorD” means “Gorenstein duality”; the superscripts indicate the suspension necessary to get from the tail of the arrow to the head of the arrow.

\[
\begin{array}{ccc}
R_k^\wedge & \xrightarrow{\text{relGor}} & \text{Hom}_S(R, S)_k^\wedge \\
& \xrightarrow{\text{locD}} & \xrightarrow{\text{GorD}} (R^\vee)_k^\wedge \\
S_k^\wedge & \xrightarrow{\text{GorD}} & (S^\vee)_k^\wedge \\
\end{array}
\]

This makes it clear that any two of relGor, locD and RGorD implies the third, and that \( a + c = b \).
It remains to observe that locD follows from SGorD by coextension of scalars from S-modules to R-modules.

**Lemma 19.6.** If $S$ has Gorenstein duality of shift $b$ then $S \to R$ has local duality of shift $b$.

**Proof:** We apply $\text{Hom}_S(R, \cdot)$ to the equivalence $S^b_k \simeq \Sigma^b(S^\vee)_k$ and then use the following lemma to move the completions to the outside. \hfill $\square$

**Lemma 19.7.** For any $S$-module $N$, we have an equivalence $\text{Hom}_S(R, N^\wedge_k) \simeq \text{Hom}_S(R, N)_k^\wedge$

**Proof:** This is a formality:

$$\text{Hom}_S(R, \text{Hom}_S(\text{Cell}_kS, N)) \simeq \text{Hom}_S(R \otimes _S \text{Cell}_kS, N) \simeq \text{Hom}_R(\text{Cell}_kR, \text{Hom}_S(R, N))$$

\hfill $\square$

This approach can be used to show that Gorenstein duality localizes \cite{20}.

### 20. Morita invariance of the Gorenstein condition

We show that the Gorenstein condition is Morita invariant in many useful cases, provided $R$ is a $k$-algebra. This allows us to deduce striking consequences from well-known examples of Gorenstein rings. For instance we can deduce the local cohomology theorem for finite $p$-groups from the fact that $kG$ is a Frobenius algebra.

**Theorem 20.1.** Suppose $R$ is a $k$-algebra, and that $\mathcal{E}$ and $R$ are Matlis reflexive. Then $\text{Hom}_\mathcal{E}(k, \mathcal{E}) \simeq \text{Hom}_R(k, R)$, and hence $\mathcal{E}$ is Gorenstein $\iff$ $R$ is Gorenstein.

**Proof:** We use the fact that (in the notation of Section\cite{6}) $E(R^\vee) = k^\vee$, so that by Lemma \cite{6.3} we have

$$R^\vee = TE(R^\vee) = Tk^\vee = k^\vee \otimes _\mathcal{E} k.$$ 

We also note that

$$\text{Hom}_R(k \otimes_R k^\vee, k) \simeq \text{Hom}_k(k, \text{Hom}_R(k, k)) \simeq \mathcal{E}$$

so that

$$\mathcal{E}^\vee = k \otimes_R k^\vee.$$ 

Next, note that the expression $k \otimes_R k^\vee \otimes _\mathcal{E} k$ makes sense, where the right $\mathcal{E}$-module structure on the first two factors comes from $k^\vee$. The key equality in the proof is simply the associativity isomorphism

$$\mathcal{E}^\vee \otimes _\mathcal{E} k = k \otimes_R k^\vee \otimes _\mathcal{E} k = k \otimes_R R^\vee.$$ 

Now we make the following calculation,
21. Gorenstein duality for group cohomology

This section describes the key example. The first sign of this duality was in Benson-Carlson duality [16, 17], which in particular shows that the Hilbert series of the group cohomology ring $H^*(BG)$ satisfies a functional equation if it is Cohen-Macaulay or a pair of functional equations if it is almost Cohen-Macaulay. An algebraic construction of the local cohomology spectral sequence was given in [49]; this was inspired by the topological construction using equivariant topology in [47], and a proof using structured equivariant spectra first appears in [18]. The method described here comes from [33].

21.A. $p$-groups. If $G$ is a $p$-group and $k$ is of characteristic $p$ we note that $BG$ is $p$-complete and therefore $\Omega(BG^\wedge_p) \simeq G$ and the Morita pair is

$$R = C^*(BG; k)$$

and

$$E = C_*(\Omega BG) = kG.$$ 

Since $kG$ is a Frobenius algebra, it is Gorenstein of shift 0, and so by Morita invariance of the Gorenstein condition (Theorem 20.1), $C^*(BG)$ is also Gorenstein of shift 0.

21.B. General finite groups. Now if $G$ is an arbitrary finite group, we may choose a faithful representation $\rho : G \rightarrow SU(n)$ for some $n$ and consider the fibration

$$BSU(n) \leftarrow BG \leftarrow SU(n)/G.$$ 

Since $BSU(n)$ is simply connected, the Eilenberg-Moore theorem gives a cofibre sequence

$$C^*(BSU(n)) \rightarrow C^*(BG) \rightarrow C^*(SU(n)/G).$$

Now note that $H^*(BSU(n))$ is polynomial on $c_2, \ldots, c_n$ and therefore $C^*(BSU(n))$ is $c$-Gorenstein and therefore Gorenstein (with shift $2(2 + 3 + \cdots + n) - (n - 1) = \dim(SU(n))$). On the other hand $SU(n)/G$ is an orientable manifold of the same dimension as $SU(n)$ and therefore $C^*(SU(n)/G)$ is Gorenstein. By Gorenstein Ascent (Example 19.4), $C^*(BG)$ is Gorenstein of shift 0 as required.

21.C. The local cohomology theorem. As described in Lemma 18.3 we thus obtain the local cohomology theorem

$$H^*_m(H^* BG) \Rightarrow H_*(BG)$$

for group cohomology.
As described in Corollary 18.4, we then obtain functional equations in many cases. We warn that $t$ is of codegree 1 (unlike $s$ in the corollary, which was of degree 1). If $H^*(BG)$ is Cohen-Macaulay, it is Gorenstein and its Hilbert series satisfies

$$p(1/t) = (-1)^rt^rp(t).$$

If $H^*(BG)$ is almost Cohen-Macaulay it is also almost Gorenstein, and the Hilbert series satisfies

$$p(1/t) - (-1)^rt^rp(t) = (-1)^{r-1}(1 + t)q(t) \text{ and } q(1/t) = (-1)^{r-1}t^{-r+1}q(t).$$

In any case $H^*(BG)$ is Gorenstein in codimension 0 and almost Gorenstein in codimension 1.

We will describe a number of 2-group examples with $k$ of characteristic 2.

21.D. **The elementary abelian group of rank** $r$. The group of order 2 may be viewed as a subgroup of the non-zero real numbers. It therefore acts diagonally on $n\mathbb{R} = \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ and freely on the unit sphere $S(n\mathbb{R})$. Hence

$$BC_2 = S(\infty \mathbb{R})/C_2 = \mathbb{R}P^\infty.$$  

Similarly, if $G \cong C_2 \times \cdots \times C_2$ is of rank $r$ we see that

$$BG = \mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty,$$

and

$$H^*(BG) = k[x_1, \ldots, x_r].$$

This is visibly of dimension $r$ and depth $r$ and

$$p(t) = \frac{1}{(1-t)^r}.$$  

The functional equation is easily checked, and the reader may wish to check directly that the local cohomology of $H^*(BG)$ is $H_*(BG)$ (up to a shift).

21.E. **The quaternion group of order 8**. If $G = Q_8$ is quaternion of order 8 we note that it acts freely on the unit 3-sphere in the quaternion algebra $S(\mathbb{H})$, and in fact $H^*(S(\mathbb{H})/G) = k[x, y]/(x^3, x^2 + xy + y^2, y^3)$. It is then clear that $G$ acts freely on the contractible space $S(\infty \mathbb{H})$ and that

$$H^*(BG) = H^*(S(\mathbb{H})/G)[z]$$

where $z$ is of codegree 4. This is visibly of dimension 1 and depth 1 and

$$p(t) = \frac{1 + 2t + 2t^2 + t^3}{1 - t^4}.$$  

The functional equation is easily checked. It is also easy to check directly that the local cohomology of $H^*(BG)$ is $H_*(BG)$ (up to a shift). Although there are many other groups for which we could calculate the cohomology it is convenient to simply refer to the invaluable Jena database \[45\] for the cohomology of small $p$-groups.

47
21.F. The dihedral group of order 8. If $G = D_8$ is dihedral of order 8

$$H^*(BG) = k[x_1, y_1, z_2]/(xy).$$

It is quick to check that this is of dimension 2 and depth 2 and

$$p(t) = \frac{1}{(1-t)^2}.$$ 

The functional equation is easily checked. It is an easy exercise to check directly that the local cohomology of $H^*(BG)$ is $H^4_*(BG)$ (up to a shift).

21.G. The semi-dihedral group of order 16. If $G = SD_{16}$ is the semi-dihedral group of order 16

$$H^*(BG) = k[x_1, y_1, z_3, t_4]/(xy, x_3, xz, z_2^2 + ty^2).$$

One may check that this is of dimension 2 and depth 1 and is included as the first example which is not Cohen-Macaulay. Its Hilbert series is

$$p(t) = \frac{1}{(1-t)^2(1+t^2)},$$

and one may check that the almost Cohen-Macaulay functional equations are satisfied with shift 0. It is presumably a coincidence that it also satisfies the functional equation of a Cohen-Macaulay graded ring of dimension 2 with (homological) shift 2.

21.H. Group Number 7 of order 32. The 7th group in the Small Groups library list of the 51 groups of order 32 is the only one whose cohomology is neither Cohen-Macaulay nor almost Cohen-Macaulay. The ring has a minimal presentation with 8 generators and 18 relations, so it won’t be recorded here. The important facts for us are that it is of dimension 3 and depth 1 and has Hilbert series

$$p(t) = \frac{1 - t + t^2}{(1-t)^3(1+t^2)}.$$ 

By coincidence this satisfies the functional equation for a Gorenstein graded ring with shift 0.

Taking the description of its cohomology given in [86] one may easily calculate its local cohomology. In fact it has a polynomial subring $P = k[z_1, x_2, s_4]$ over which it is a direct sum of 5 submodules, namely $P \oplus \Sigma_3 P \oplus \Sigma_4 P \oplus M \oplus N$ where

$$M = \text{cok}(\Sigma_4 P \xrightarrow{(x,z)} \Sigma_2 P \oplus \Sigma_3 P)$$

and

$$N = \Sigma_1 P/(x, z).$$

The free submodule is of depth 3, $M$ is of depth 2. Finally

$$H^1_*(H^*(BG)) = H^1_*(N)$$

is of dimension 1 over $k$ in degrees 3, 7, 11, 15, \ldots. Knowing the dimensions of $H_*(BG)$ from $p(t)$ we see that the differential

$$d_2 : H^1_*(H^*(BG)) \rightarrow \Sigma^{-1} H^3_*(H^*(BG))$$

is of dimension 1 over $k$ in degrees 3, 7, 11, 15, \ldots. Knowing the dimensions of $H_*(BG)$ from $p(t)$ we see that the differential

$$d_2 : H^1_*(H^*(BG)) \rightarrow \Sigma^{-1} H^3_*(H^*(BG))$$
is a monomorphism (and so non-zero in infinitely many degrees).

21.I. Other classes of groups. We have described the fact that $C^*(BG)$ is Gorenstein for finite groups $G$. It is shown in [33] that $C^*(BG)$ is Gorenstein of shift equal to $\dim(G)$ whenever $G$ is a compact Lie group, provided that either $k$ is of characteristic 2 or the group of components is of odd order. In general there is a twisting by the representation $H^0(G)$, which is trivial in the aforementioned cases.

If $G$ is a virtual duality group of dimension $n$ there is a form of Gorenstein duality with shift $-n$ for $C^*(BG)$. Now there is a twisting by $H^n(G; kG)$, which is usually of infinite dimension. The algebraic proof is given in [19], along with a topological proof for arithmetic groups using equivariant topology. A proof along the present lines can be given for these groups as follows. We choose a normal subgroup $N$ of finite index in $G$ which is a duality group, and a contractible space $X$ on which $G$ acts with finite isotropy. Now $N$ acts freely on $X$ so $X/N$ is a manifold with boundary. Now let $Q = G/N$, and consider the fibration

$$BQ \leftarrow BG \leftarrow X/N,$$

which we may obtain from the equivalence

$$BG \simeq EQ \times_Q (X/N).$$

We attempt to apply Gorenstein Ascent. In the situation that $X/N$ has empty boundary (i.e., $G$ is a virtual Poincaré duality group) we infer $C^*(BG)$ is Gorenstein as required.

22. Gorenstein duality for rational spaces

This section discusses Gorenstein duality for rational spaces. Félix-Halperin-Thomas [42] have considered the Gorenstein condition in depth, so the distinctive feature here is the concentration on Gorenstein duality as in [33].

For spaces with finite dimensional cohomology $X$ is Gorenstein if and only if $H^*(X)$ is Gorenstein, but in general there are Gorenstein spaces for which $H^*(X)$ is not Gorenstein and we make explicit the local cohomology theorem and its consequences for $H^*(X)$.

22.A. Fundamentals. We specialize some of the above results to $C^*(X; \mathbb{Q})$, beginning with the zero dimensional case.

Lemma 22.1. [42, 3.6] If $H^*(A)$ is finite dimensional then $A$ is Gorenstein if and only if $H^*(A)$ is a Poincaré duality algebra.

Proof: If $H^*(A)$ is a Poincaré duality algebra of formal dimension $n$ then it is a zero dimensional Gorenstein ring with $a$-invariant $-n$, so $A$ is Gorenstein with shift $-n$ by the previous corollary.

Conversely, if $A$ is Gorenstein of shift $a$, we have a Gorenstein duality spectral sequence. Since $H^*(A)$ is finite dimensional, it is all torsion. Accordingly, $H^a_n(H^*(A)) = H^*(A)$, and the spectral sequence reads

$$H^*(A) = \Sigma^a H^*(A)^\vee$$

and $H^*(A)$ is a Poincaré duality algebra of formal dimension $-a$. \hfill \Box
By applying Gorenstein Ascent (Example 19.4) one may use these to construct other examples which are Gorenstein but not c-Gorenstein.

**Proposition 22.2.** [42, 4.3] Suppose we have a fibration \( F \to E \to B \) with \( F \) finite. If \( F \) and \( B \) are Gorenstein with shifts \( f \) and \( b \) then \( E \) is Gorenstein with shift \( e = f + b \).

**Lemma 22.3.** If \( X \) is a simply connected rational space with \( H^*(X) \) Noetherian then if \( C^*(X; \mathbb{Q}) \) is Gorenstein of shift \( a \), then it is automatically orientable and so has Gorenstein duality of shift \( a \).

**Proof:** We see that \( \mathbb{Q} \) is proxy-small by Corollary 7.2. We may now apply the same argument as for Proposition 18.1 to see there is a unique action of \( C_*(\Omega X; \mathbb{Q}) \) on \( \mathbb{Q} \) and deduce automatic Gorenstein duality. □

### 22.B. Examples

The basic results of the previous subsection allow us to construct innumerable examples. For example any finite Postnikov system is Gorenstein [42, 3.4], so that in particular any sci space is Gorenstein. A simple example will illustrate the duality.

**Example 22.4.** We construct a rational space \( X \) in a fibration
\[
S^3 \times S^3 \to X \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty,
\]
so that \( X \) is Gorenstein. We will calculate \( H^*(X) \) and observe that it is not Gorenstein.

Let \( V \) be a graded vector space with two generators \( u, v \) in degree 2, and let \( W \) be a graded vector space with two generators in degree 4. The two 4-dimensional cohomology classes \( u^2, uv \) in \( H^*(KV) = \mathbb{Q}[u, v] \) define a map \( KV \to KW \), and we let \( X \) be the fibre, so we have a fibration
\[
S^3 \times S^3 \to X \to KV
\]
as required. By [33], this is Gorenstein with shift \(-4\) (being the sum of the shift (viz \(-6\)) of \( S^3 \times S^3 \) and the shift (viz \(2\)) of \( KV \)).

It is amusing to calculate the cohomology ring of \( X \). It is \( \mathbb{Q}[u, v, p]/(u^2, uv, up, p^2) \) where \( u, v \) and \( p \) have degrees 2, 2 and 5. The dimensions of its graded components are 1, 0, 2, 0, 1, 1, 1, 1, 1, \ldots (i.e., its Hilbert series is \( p_X(t) = (1 + t^5)/(1 - t^2)^2 + t^3 \), where \( t \) is of codegree 1).

In calculating local cohomology it is useful to note that \( m = \sqrt{v} \). The local cohomology is \( H^0_m(H^*(X)) = \Sigma^2_2 \mathbb{Q} \) in degree 0 (so that \( H^*(X) \) is not Cohen-Macaulay) and as a \( \mathbb{Q}[v] \)-module \( H^1_m(H^*(X)) = \mathbb{Q}[v] \otimes (\Sigma^{-3} \mathbb{Q} \oplus \Sigma^2 \mathbb{Q}) \). Since there is no higher local cohomology the local cohomology spectral sequence necessarily collapses, and the resulting exact sequence
\[
0 \to H^i_m(H^*(X)) \to \Sigma^{-i} H^*(A)^\vee \to \Sigma^{-2} \mathbb{Q} \to 0
\]
is consistent.

Since the Cohen-Macaulay defect here is 1, we have a pair of functional equations
\[
p_X(1/t) - (-t)t^{-4}p_X(t) = (1 + t)\delta(t)
\]
and
\[
\delta(1/t) = t^4\delta(t).
\]
Indeed, the first equation gives $\delta(t) = t^{-2}$, which is indeed the Hilbert series of $H^0_m(H^*(X))^\vee$, and it obviously satisfies the second equation.

23. The ubiquity of Gorenstein ring spectra

We have considered a number of examples, in some cases giving rather complete proofs. This short section points out that the ideas can be extended rather easily. In fact there is a sense in which all the examples come from the first three by using Gorenstein Ascent and Morita invariance of the Gorenstein condition.

- Gorenstein commutative local rings $R$ (shift $-\text{Krulldim}(R)$).
- $C^*(M)$ for closed manifolds $M$ (orientable if $M$ is orientable; shift $-\dim(M)$).
- $C_*(G)$ for compact Lie groups $G$ (orientable if $G$ acts trivially on $H_*(G)$; shift $\dim(G)$).
- $C^*(BG)$ for compact Lie groups $G$ (orientable if $G$ acts trivially on $H_*(G)$; shift $\dim(G)$).
- $C^*(BG)$ for virtual duality groups $G$ (dualizing module $H_*(G;kG)$; shift $-\text{formal-dim}(G)$).
- $C^*(E G \times_G M)$ Borel construction on a $G$-manifold $M^m$ for $G^g$ compact Lie (orientable if $G$, $M$ and the action are; shift $g - m$). An amusing instance of this arises from toric geometry; if $K$ is a simplicial complex one may construct the so-called moment angle complex $Z_K$ which has an action of a torus $G$ whose Borel cohomology is the Stanley-Reisner ring $k[K]$. If $K$ is a simplicial sphere $Z_K$ is a manifold and $k[K]$ is Gorenstein. See [29] for definitions and proofs.

- Chromatic examples. The coefficient ring $R$ (or ring spectrum) is contravariant in the geometric object making it like cochains, but the ring spectrum $R$ is often connective (unlike cochains). With this caveat, the examples are parallel to the $C^*(BG)$ example with the compact Lie group $G$ replaced by an algebraic group $G$, so that $R$ plays the role of the ring spectrum of functions on $G$. The Morita counterpart of $R = O_G$ would be the ring of operations $C_*(G;k) = \text{Hom}_R(k,k)$.

- Hochschild homology. In [54] several classes of examples are identified where the Hochschild homology with field coefficients inherits Gorenstein properties. The context is that we are given maps $S \longrightarrow R \longrightarrow k$ of commutative rings, so that we can define the Hochschild homology

$$HH_\bullet(R|S;k) = R \otimes_{R \otimes S} R k,$$

and it is a ring spectrum with a map to $k$. The results then say that (under substantial additional hypotheses) if $R$ is Gorenstein of shift $a$ and $HH_\bullet(k|S;k)$ is Gorenstein of shift $b$ then $HH_\bullet(R|S;k)$ is Gorenstein of shift $b - a$. The proof comes out of Gorenstein Ascent, with the hypotheses designed to allow the application of Lemma 19.3.
Appendix A. Algebraic definitions: Local and Čech cohomology and homology

Related surveys are given in [59, 60]. The material in this section is based on [66, 58, 48]. Background in commutative algebra can be found in [83, 28].

I. The functors. Suppose to begin with that $R$ is a commutative Noetherian ring and that $I = (\alpha_1, \ldots, \alpha_n)$ is an ideal in $R$. We shall be concerned especially with two naturally occurring functors on $R$-modules: the $I$-power torsion functor and the $I$-adic completion functor.

The $I$-power torsion functor $\Gamma_I$ is defined by
\[ M \mapsto \Gamma_I(M) = \{ x \in M \mid I^k x = 0 \text{ for } k >> 0 \}. \]
We say that $M$ is an $I$-power torsion module if $M = \Gamma_I M$. It is easy to check that the functor $\Gamma_I$ is left exact.

The $I$-adic completion functor is defined by
\[ M \mapsto M_\wedge = \lim \leftarrow_k M/I^k M. \]
The Artin-Rees lemma implies that $I$-adic completion is exact on finitely generated modules, but it is neither right nor left exact in general.

II. The stable Koszul complex. We begin with a sequence $\alpha_1, \ldots, \alpha_n$ of elements of $R$ and define various chain complexes. In Subsection [III] we explain why the chain complexes only depend on the radical of the ideal $I = (\alpha_1, \ldots, \alpha_n)$ generated by the sequence, in Subsection [IV] we define associated homology groups, and in Subsection [V] we give conceptual interpretations of this homology under Noetherian hypotheses.

We begin with a single element $\alpha \in R$, and an integer $s \geq 0$, and define the $s$th unstable Koszul complex by
\[ K_s^\bullet(\alpha) = (\alpha^s : R \longrightarrow R) \]
where the non-zero modules are in cohomological degrees 0 and 1. These complexes form a direct system as $s$ varies,
\[
\begin{align*}
K^\bullet_1(\alpha) &= ( R \xrightarrow{\alpha} R ) \\
&\downarrow \quad = \downarrow \quad \downarrow \alpha \\
K^\bullet_2(\alpha) &= ( R \xrightarrow{\alpha^2} R ) \\
&\downarrow \quad = \downarrow \quad \downarrow \alpha \\
K^\bullet_3(\alpha) &= ( R \xrightarrow{\alpha^3} R ) \\
&\downarrow \quad = \downarrow \quad \downarrow \alpha 
\end{align*}
\]
and the direct limit is the flat stable Koszul complex
\[ K^\bullet_\infty(\alpha) = (R \longrightarrow R[1/\alpha]). \]

When defining local cohomology, it is usual to use the complex $K^\bullet_\infty(\alpha)$ of flat modules. However, we shall need a complex of projective $R$-modules to define the dual local homology modules. Accordingly, we take a particularly convenient projective approximation $PK^\bullet_\infty(\alpha)$ to $K^\bullet_\infty(\alpha)$. Instead of taking the direct limit of the $K^\bullet_s(\alpha)$, we take their homotopy direct
limit. This makes the translation to the topological context straightforward. More concretely, our model for $PK^\bullet(\alpha)$ is displayed as the upper row in the homology isomorphism

$$PK^\bullet(\alpha) = \left( R \oplus R[x] \overset{(1,\alpha x-1)}{\longrightarrow} R[x] \right)$$

$$K^\bullet(\alpha) = \left( R \overset{g}{\longrightarrow} R[1/\alpha] \right),$$

where $g(x^i) = 1/\alpha^i$. Like $K^\bullet(\alpha)$, this choice of $PK^\bullet(\alpha)$ is non-zero only in cohomological degrees 0 and 1.

The stable Koszul cochain complex for a sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$ is obtained by tensoring together the complexes for the elements, so that

$$K^\bullet(\alpha) = K^\bullet(\alpha_1) \otimes_R \cdots \otimes_R K^\bullet(\alpha_n),$$

and similarly for the projective complex $PK^\bullet(\alpha)$.

III. Invariance statements. We list some basic properties of the stable Koszul complex, leaving proofs to the reader.

Lemma A.1. If $\beta$ is in the ideal $I = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then $K^\bullet(\alpha)[1/\beta]$ is exact.

Note that, by construction, we have an augmentation map

$$\varepsilon : K^\bullet(\alpha) \longrightarrow R.$$

Using this to compare different stable Koszul complexes and Lemma A.1 to see when they are equivalences we deduce an important invariance statement.

Corollary A.2. Up to quasi-isomorphism, the complex $K^\bullet(\alpha)$ depends only on the radical of the ideal $I$.

In view of Corollary A.2 it is reasonable to write $K^\bullet(I)$ for $K^\bullet(\alpha)$. Since $PK^\bullet(\alpha)$ is a projective approximation to $K^\bullet(\alpha)$, it too depends only on the radical of $I$. We also write $K^\bullet_s(I) = K^\bullet_s(\alpha_1) \otimes \cdots \otimes K^\bullet_s(\alpha_n)$, but this is an abuse of notation since even its homology groups do depend on the choice of generators.

IV. Local homology and cohomology. The local cohomology and homology of an $R$-module $M$ are then defined by

$$H^s_I(R; M) = H^s(PK^\bullet(\alpha) \otimes M)$$

and

$$H^s_s(R; M) = H_s(\text{Hom}(PK^\bullet(\alpha), M)).$$

Note that we could equally well use the flat stable Koszul complex in the definition of local cohomology, as is more usual. Lemma A.1 shows that $H^s_I(M)[1/\beta] = 0$ if $\beta \in I$, so $H^s_I(M)$ is an $I$-power torsion module and supported over $I$.

It is immediate from the definitions that local cohomology and local homology are related by a third quadrant universal coefficient spectral sequence

$$E_2^{s,t} = \text{Ext}_R^s(H_I^{-t}(R), M) \Rightarrow H^{t-s}_I(R; M),$$

where $g(x^i) = 1/\alpha^i$. Like $K^\bullet(\alpha)$, this choice of $PK^\bullet(\alpha)$ is non-zero only in cohomological degrees 0 and 1.
with differentials $d_r : E^{s,t}_r \to E^{s+r,t-r+1}_r$.

V. Derived functors. We gave our definitions in terms of specific chain complexes. The meaning of the definitions appears in the following two theorems.

**Theorem A.3** (Grothendieck [66]). *If $R$ is Noetherian, then the local cohomology groups calculate the right derived functors of the left exact functor $M \mapsto \Gamma_I(M)$. In symbols,*

$$H^i_I(R; M) = (R^n\Gamma_I)(M).$$

□

This result may be used to give an explicit expression for local cohomology in familiar terms. Indeed, since $\Gamma_I(M) = \lim \Hom(R/I^r, M)$, and the right derived functors of the right-hand side are obvious, we have

$$(R^n\Gamma_I)(M) \cong \lim_{\to} \Ext^i_R(R/I^r, M).$$

The description in terms of the stable Koszul complex is usually more practical.

**Theorem A.4** (Greenlees-May [58]). *If $R$ is Noetherian, then the local homology groups calculate the left derived functors of the (not usually right exact) $I$-adic completion functor $M \mapsto M_I$. Writing $L^I_n$ for the left derived functors of $I$-adic completion, this gives*

$$H^i_I(R; M) = L^I_n(M).$$

□

The conclusions of Theorems A.3 and A.4 are true under much weaker hypotheses [58, 87].

VI. The shape of local cohomology. One is used to the idea that $I$-adic completion is often exact, so that $L^I_0$ is the most significant of the left derived functors. However, it is the top non-vanishing right derived functor of $\Gamma_I$ that is the most significant. Some idea of the shape of these derived functors can be obtained from the following result. Observe that the complex $PK^\bullet_\infty(\alpha)$ is non-zero only in cohomological degrees between 0 and $n$, so that local homology and cohomology are zero above dimension $n$. A result of Grothendieck’s usually gives a much better bound. We write $\dim(R)$ for the Krull dimension of $R$ and $\depth_I(M)$ for the $I$-depth of a module $M$ (the length of the longest $M$-regular sequence from $I$).

**Theorem A.5** (Grothendieck [65]). *If $R$ is Noetherian of Krull dimension $d$, then*

$$H^i_I(M) = 0 \quad \text{and} \quad H^1_I(M) = 0 \quad \text{if } i > d.$$  

If $e = \depth_I(M)$ then

$$H^i_I(M) = 0 \quad \text{if } i < e.$$  

If $R$ is Noetherian, $M$ is finitely generated, and $IM \neq M$, then

$$H^e_I(M) \neq 0.$$  

□

Grothendieck’s proof of vanishing begins by noting that local cohomology is sheaf cohomology with support. It then proceeds by induction on the Krull dimension and reduction to the irreducible case. The statement about depth is elementary, and proved by induction on the length of the $I$-sequence (see [83, 16.8]).

The Universal Coefficient Theorem gives a useful consequence for local homology.
Corollary A.6. If $R$ is Noetherian and $\text{depth}_1(R) = \dim(R) = d$, then
$$L^i_0 M = \text{Ext}_R^{d-i}(H^d_1(R), M). \quad \Box$$

For example if $R = \mathbb{Z}$ and $I = (p)$, then $H^*(\mathbb{Z}) = H^1_1(\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. Therefore the corollary states that
$$L^0_0 M = \text{Ext}_\mathbb{Z}(\mathbb{Z}/p\mathbb{Z}, M) \quad \text{and} \quad L^1_0 M = \text{Hom}_\mathbb{Z}(\mathbb{Z}/p\mathbb{Z}, M),$$
as was observed in Bousfield-Kan [26, VI.2.1].

VII. Čech homology and cohomology. We have motivated local cohomology in terms of $I$-power torsion, and it is natural to consider the difference between the torsion and the original module. In geometry this difference would be more fundamental than the torsion itself, and local cohomology would then arise by considering functions with support.

To construct a good model for this difference, observe that $\varepsilon : K^*_\infty(\alpha) \to R$ is an isomorphism in degree zero and define the flat Čech complex $\check{C}^*(I)$ to be the complex $\Sigma(\ker \varepsilon)$. Thus, if $i \geq 0$, then $\check{C}^i(I) = K^{i+1}(I)$. For example, if $I = (\alpha, \beta)$, then
$$\check{C}^*(I) = ( R[1/\alpha] \oplus R[1/\beta] \to R[1/(\alpha\beta)] ).$$
The differential $K^0(I) \to K^1(I)$ specifies a chain map $R \to \check{C}^*(I)$ whose fibre is exactly $K^*_\infty(I)$. Thus we have a fibre sequence
$$K^*_\infty(I) \to R \to \check{C}^*(I).$$
We define the projective version $P\check{C}^*(I)$ similarly, using the kernel of the composite of $\varepsilon$ and the quasi-isomorphism $PK^*_\infty(I) \to K^*_\infty(I)$; note that $P\check{C}^*(I)$ is non-zero in cohomological degree $-1$.

The Čech cohomology of an $R$-module $M$ is then defined by
$$\check{C}H^*_1(R; M) = H^*(\check{C}^*(I) \otimes M).$$

VIII. Čech cohomology and Čech covers. To explain why $\check{C}^*(I)$ is called the Čech complex, we describe how it arises by using the Čech construction to calculate cohomology from a suitable open cover. More precisely, let $Y$ be the closed subscheme of $X = \text{Spec}(R)$ determined by $I$. The space $V(I) = \{ \varphi | \varphi \supset I \}$ decomposes as $V(I) = V(\alpha_1) \cap \ldots \cap V(\alpha_n)$, and there results an open cover of the open subscheme $X - Y$ as the union of the complements $X - Y_i$ of the closed subschemes $Y_i$ determined by the principal ideals $(\alpha_i)$. However, $X - Y_i$ is isomorphic to the affine scheme $\text{Spec}(R[1/\alpha_i])$. Since affine schemes have no higher cohomology,
$$H^*(\text{Spec}(R[1/\alpha_i]); \tilde{M}) = H^0(\text{Spec}(R[1/\alpha_i]); \tilde{M}) = M[1/\alpha_i],$$
where $\tilde{M}$ is the sheaf associated to the $R$-module $M$. Thus the $E_1$ term of the Mayer-Vietoris spectral sequence for this cover collapses to the chain complex $\check{C}^*(I)$, and
$$H^*(X - Y; \tilde{M}) \cong \check{C}H^*_1(M).$$
Appendix B. Spectral analogues of the algebraic definitions

We now transpose the algebra from Appendix A into the context of spectra. It is convenient to note that it is routine to extend the algebra to graded rings, and we will use this without further comment below. We assume the reader is already comfortable working with ring spectra, but there is an introduction in Sections 2 and 3, and one at greater length in [52].

We replace the standing assumption that $R$ is a commutative $\mathbb{Z}$-algebra by the assumption that it is a commutative $S$-algebra, where $S$ is the sphere spectrum. The category of $R$-modules is now the category of $R$-module spectra. Since the derived category of a ring $R$ is equivalent to the derived category of the associated Eilenberg-MacLane spectrum [92], the work of Appendix A can be reinterpreted in the new context. To emphasize the algebraic analogy, we write $\otimes_R$ and $\text{Hom}_R$ for the smash product over $R$ and function spectrum of $R$-maps and 0 for the trivial module. In particular $X \otimes_S Y = X \wedge Y$ and $\text{Hom}_S(X, Y) = F(X, Y)$.

This section is based on [47, 48].

I. Koszul spectra. For $\alpha \in \pi_* R$, we define the stable Koszul spectrum $K_\infty(\alpha)$ by the fibre sequence

$$K_\infty(\alpha) \to R \to R[1/\alpha],$$

where $R[1/\alpha] = \text{holim}(R \xrightarrow{\alpha} R \xrightarrow{\alpha} \cdots)$. Analogous to the filtration by degree in chain complexes, we obtain a filtration of the $R$-module $K_\infty(\alpha)$ by viewing it as

$$\Sigma^{-1}(R[1/\alpha] \cup CR).$$

Next we define the stable Koszul spectrum for the sequence $\alpha_1, \ldots, \alpha_n$ by

$$K_\infty(\alpha_1, \ldots, \alpha_n) = K_\infty(\alpha_1) \otimes_R \cdots \otimes_R K_\infty(\alpha_n),$$

and give it the tensor product filtration.

The topological analogue of Lemma A.1 states that if $\beta \in I$ then

$$K_\infty(\alpha_1, \ldots, \alpha_n)[1/\beta] \simeq 0;$$

this follows from Lemma A.1 and the spectral sequence (3) below. We may now use precisely the same proof as in the algebraic case to conclude that the homotopy type of $K_\infty(\alpha_1, \ldots, \alpha_n)$ depends only on the radical of the ideal $I = (\alpha_1, \cdots, \alpha_n)$. We therefore write $K_\infty(I)$ for $K_\infty(\alpha_1, \ldots, \alpha_n)$.

II. Localization and completion. With motivation from Theorems A.3 and A.4 we define the homotopical $I$-power torsion (or local cohomology) and homotopical completion (or local homology) modules associated to an $R$-module $M$ by

$$\Gamma_I(M) = K_\infty(I) \otimes_R M \quad \text{and} \quad \Lambda_I(M) = M^\wedge_I = \text{Hom}_R(K_\infty(I), M).$$

In particular, $\Gamma_I(R) = K_\infty(I)$.

Because the construction follows the algebra so precisely, it is easy give methods of calculation for the homotopy groups of these $R$-modules. We use the product of the filtrations of the $K_\infty(\alpha_i)$ given above and obtain spectral sequences

$$E^2_{s,t} = H^{-s-t}_I(R_\ast; M_\ast) \Rightarrow \pi_{s+t}(\Gamma_I M)$$
with differentials \( d^r : E_{s,t}^r \to E_{r-s,t+r-1}^r \) and
\[
E_2^{s,t} = H_{-s,t}^I(R^s; M^r) \Rightarrow \pi_{-(s+t)}(M_t^i)
\]
with differentials \( d_r : E_{r,s}^r \to E_{r+s,t-r+1}^r \).

III. The Čech spectra. Similarly, we define the Čech spectrum by the cofibre sequence
\[
K_\infty(I) \longrightarrow R \longrightarrow \check{C}(I).
\]
We define the homotopical localization (or Čech cohomology) and Čech homology modules associated to an \( R \)-module \( M \) by
\[
M[I^{-1}] = \check{C}(I) \otimes_R M \quad \text{and} \quad \Delta^I(M) = \text{Hom}_R(\check{C}(I), M).
\]
In particular, \( R[I^{-1}] = \check{C}(I) \). Once again, we have spectral sequences for calculating their homotopy groups from the analogous algebraic constructions.

IV. Basic properties. We can now give topological analogues of some basic pieces of algebra that we used in Section A. Recall that the algebraic Koszul complex \( K^\bullet(I) \) is a direct limit of unstable complexes \( K_s^\bullet(I) \) that are finite complexes of free modules with homology annihilated by a power of \( I \). We say that an \( R \)-module \( M \) is an \( I \)-power torsion module if its \( R^\bullet \)-module \( M^\bullet \) of homotopy groups is an \( I \)-power torsion module; equivalently, \( M^\bullet \) must have support over \( I \).

Lemma B.1. The \( R \)-module \( K_\infty(I) \) is a homotopy direct limit of finite \( R \)-modules \( K_s(I) \), each of which has homotopy groups annihilated by some power of \( I \). Therefore \( K_\infty(I) \) is an \( I \)-power torsion module.

The following lemma is an analogue of the fact that \( \check{C}^\bullet(I) \) is a chain complex which is a finite sum of modules \( R[1/\alpha] \) for \( \alpha \in I \).

Lemma B.2. The \( R \)-module \( \check{C}(I) \) has a finite filtration by \( R \)-submodules with subquotients that are suspensions of modules of the form \( R[1/\alpha] \) with \( \alpha \in I \).

These lemmas are useful in combination.

Corollary B.3. If \( M \) is an \( I \)-power torsion module then \( M \otimes_R \check{C}(I) \simeq 0 \); in particular \( K_\infty(I) \otimes_R \check{C}(I) \simeq 0 \).

Proof: Since \( M[1/\alpha] \simeq 0 \) for \( \alpha \in I \), Lemma B.2 gives the conclusion for \( M \).

Appendix C. Completion at ideals and Bousfield localization

Bousfield localizations include both completions at ideals and localizations at multiplicatively closed sets, but one may view these Bousfield localizations as falling into the types typified by completion at \( p \) and localization away from \( p \). Thinking in terms of \( \text{Spec}(R^\bullet) \), this is best viewed as the distinction between localization at a closed set and localization at the complementary open subset. In this section we deal with the closed sets and with the open sets in Section D. This appendix is based on [57, 58, 48].
I. **Homotopical completion.** As observed in the proof of Lemma B.1, we have \( \lim_{\mathcal{S}} \Sigma^{-1}R/\alpha^s \) and therefore

\[
M_\alpha^\wedge = \operatorname{Hom}_R(\lim_{\mathcal{S}} \Sigma^{-1}R/\alpha^s, M) \simeq \lim_{\mathcal{S}} M/\alpha^s.
\]

If \( I = (\alpha, \beta) \), then

\[
M_I^\wedge = \operatorname{Hom}_R(K_\infty(\alpha) \otimes_R K_\infty(\beta), M)
\]

\[
= \operatorname{Hom}_R(K_\infty(\alpha), \operatorname{Hom}_R(K_\infty(\beta), M))
\]

and so on inductively. This should help justify the notation \( M_I^\wedge = \operatorname{Hom}_R(K_\infty(I), M) \).

When \( R = \mathbb{S} \) is the sphere spectrum and \( p \in \mathbb{Z} \cong \pi_0(\mathbb{S}) \), \( K_\infty(p) \) is a Moore spectrum for \( \mathbb{Z}/p^\infty \) in degree \(-1\) and we recover the usual definition

\[
X_p^\wedge = F(\Sigma^{-1}\mathbb{S}/p^\infty, X)
\]

of \( p \)-completions of spectra as a special case, where \( F(A, B) = \operatorname{Hom}_g(A, B) \) is the function spectrum. The standard short exact sequence for the calculation of the homotopy groups of \( X_p^\wedge \) in terms of ‘Ext completion’ and ‘Hom completion’ follows directly from Corollary A.6.

Since \( p \)-completion has long been understood to be an example of a Bousfield localization, our next task is to show that completion at \( I \) is a Bousfield localization in general.

II. **Bousfield’s terminology.** Fix an \( R \)-module \( E \). A spectrum \( A \) is \( E \)-acyclic if \( A \otimes_R E \simeq 0 \); a map \( f : X \longrightarrow Y \) is an \( E \)-equivalence if its cofibre is \( E \)-acyclic. An \( R \)-module \( M \) is \( E \)-local if \( E \otimes_R T \simeq 0 \) implies \( \operatorname{Hom}_R(T, M) \simeq 0 \). A map \( Y \longrightarrow L_EY \) is a Bousfield \( E \)-localization of \( Y \) if it is an \( E \)-equivalence and \( L_EY \) is \( E \)-local. This means that \( Y \longrightarrow L_EY \) is terminal among \( E \)-equivalences with domain \( Y \), and the Bousfield localization is therefore unique if it exists. Similarly, we may replace the single spectrum \( E \) by a class \( \mathcal{E} \) of objects \( E \), and require the conditions hold for all such \( E \).

The following is a specialization of a change of rings result to the ring map \( \mathbb{S} \longrightarrow R \).

**Lemma C.1.** Let \( \mathcal{E} \) be a class of \( R \)-modules. If an \( R \)-module \( N \) is \( \mathcal{E} \)-local as an \( R \)-module, then it is \( \mathcal{E} \)-local as an \( \mathbb{S} \)-module.

**Proof:** If \( E \wedge T = E \otimes_\mathbb{S} T \simeq * \) for all \( E \), then \( E \otimes_R (R \otimes_\mathbb{S} T) \simeq 0 \) for all \( E \) and therefore \( F(T, N) = \operatorname{Hom}_\mathbb{S}(T, N) \simeq \operatorname{Hom}_R(R \otimes_\mathbb{S} T, N) \simeq 0 \). \( \square \)

III. **Homotopical completion is a Bousfield localization.** The class that will concern us most is the class \( I \text{-Tors} \) of finite \( I \)-power torsion \( R \)-modules \( M \). Thus \( M \) must be a finite cell \( R \)-module, and its \( R_\ast \)-module \( M_\ast \) of homotopy groups must be a \( I \)-power torsion module.

**Theorem C.2.** For any finitely generated ideal \( I \) of \( R_\ast \) the map \( M \longrightarrow M^\wedge \) is Bousfield localization in the category of \( R \)-modules in each of the following equivalent senses:

(i) with respect to the \( R \)-module \( \Gamma_I(R) = K_\infty(I) \).

(ii) with respect to the class \( I \text{-Tors} \) of finite \( I \)-power torsion \( R \)-modules.

(iii) with respect to the \( R \)-module \( K_s(I) \) for any \( s \geq 1 \).
Furthermore, the homotopy groups of the completion are related to local homology groups by a spectral sequence
\[ E^2_{s,t} = H^I_s(M_\ell) \Longrightarrow \pi_{s+t}(M_\ell). \]
If \( R_\ast \) is Noetherian, the \( E^2 \) term consists of the left derived functors of \( I \)-adic completion:
\[ H^I_s(M_\ell) = L^I_s(M_\ell). \]

Proof: We begin with (i). Since
\[ \text{Hom}_R(T, M_\ell) \simeq \text{Hom}_R(T \otimes_R K_\infty(I), M), \]
it is immediate that \( M_\ell \) is \( K_\infty(I) \)-local. We must prove that the map \( M \to M_\ell \) is a
\( K_\infty(I) \)-equivalence. The fibre of this map is \( \text{Hom}_R(\check{C}(I), M) \), so we must show that
\[ \text{Hom}_R(\check{C}(I), M) \otimes_R K_\infty(I) \simeq 0. \]
By Lemma B.7, \( K_\infty(I) \) is a homotopy direct limit of terms \( K_s(I) \). Each \( K_s(I) \) is in \( I^{-\text{Tors}} \),
and we see by their definition in terms of cofibre sequences and smash products that their
duals \( DK_s(I) \) are also in \( I^{-\text{Tors}} \), where \( DM = \text{Hom}_R(M, R) \). Since \( K_s(I) \)
is a finite cell \( R \)-module,
\[ \text{Hom}_R(\check{C}(I), M) \otimes_R K_s(I) = \text{Hom}_R(\check{C}(I) \otimes_R DK_s(I), M), \]
and \( \check{C}(I) \otimes_R DK_s(I) \simeq 0 \) by Corollary B.3. Parts (ii) and (iii) are similar but simpler. For
(iii), observe that we have a cofibre sequence \( R/\alpha^s \to R/\alpha^{2s} \to R/\alpha^8 \), so that all of the
\( K_{js}(I) \) may be constructed from \( K_s(I) \) using a finite number of cofibre sequences. \( \square \)

Appendix D. Localization away from ideals and Bousfield localization

In this section we turn to localization away from the closed set defined by an ideal \( I \). First, observe that, when \( I = (\alpha) \), \( M[I^{-1}] \) is just \( R[\alpha^{-1}] \otimes_R M = M[\alpha^{-1}] \). However, the higher
Čech cohomology groups give the construction for general finitely generated ideals a quite
different algebraic flavour, and \( M[I^{-1}] \) is rarely a localization of \( M \) at a multiplicatively
closed subset of \( R_\ast \). This appendix is based on [18].

I. The Čech complex as a Bousfield localization. To characterize this construction as
a Bousfield localization, we consider the class \( I^{-\text{Inv}} \) of \( R \)-modules \( M \) for which there is an
element \( \alpha \in I \) such that \( \alpha : M \to M \) is an equivalence.

Theorem D.1. For any finitely generated ideal \( I = (\alpha_1, \ldots, \alpha_n) \) of \( R_\ast \), the map \( M \to M[I^{-1}] \) is Bousfield localization in the category of \( R \)-modules in each of the following equivalent senses:

(i) with respect to the \( R \)-module \( R[I^{-1}] = \check{C}(I) \).
(ii) with respect to the class \( I^{-\text{Inv}} \).
(iii) with respect to the set \( \{ R[1/\alpha_1], \ldots, R[1/\alpha_n] \} \).

Furthermore, the homotopy groups of the localization are related to Čech cohomology groups
by a spectral sequence
\[ E^2_{s,t} = \check{C}H^{-s,-t}(M_\ell) \Longrightarrow \pi_{s+t}(M[I^{-1}]). \]
If $R_\star$ is Noetherian, the $E^2$ term can be viewed as the cohomology of $\text{Spec}(R_\star) \setminus V(I)$ with coefficients in the sheaf associated to $M_\star$.

**Remark D.2.** One may also characterize the map $\Gamma_I(M) \to M$ by a universal property analogous to that of the cellular approximation in spaces: it is the $K_1(I)$-cellularization of $M$.

Indeed, on the one hand, $\Gamma_I(M)$ is constructed from $K_1(I)$ by [B.1] and on the other hand, the map induces an equivalence of $\text{Hom}_R(K_1(I), \cdot)$ since, by Lemma [B.2] $\text{Hom}_R(K_1(I), \mathcal{C}(I)) \simeq 0$.

**References**

[1] J.F. Adams “Stable homotopy and generalized homology.” Chicago University Press 1974.
[2] L.Alonso Tarrio, A.Jeremias Lopez and J. Lipman “Local homology and cohomology on schemes.” Ann.Scient. Ec.Norm.Sup. 30 (1997) 1-39
[3] L.Alonso Tarrio, A.Jeremias Lopez and J. Lipman “Studies in duality on Noetherian formal schemes and non-Noetherian ordinary schemes” Cont. Math. 244 AMS (1999) x+126
[4] K.K.S.ANDERSEN and J.GRODAL “The classiﬁcation of 2-compact groups” J American Math Soc 22 (2009), 387-436
[5] K.K.S.ANDERSEN, J.GRODAL, J.M.MOLLER and M.VIRUEL “The classiﬁcation of $p$-compact groups for $p$ odd” Annals of Maths, 167 (2008), 95-210
[6] L.L.AVRAMOV “Modules of ﬁnite virtual projective dimension.” Invent. Math. 96 (1989), no. 1, 71–101.
[7] L.L.AVRAMOV “Inﬁnite free resolutions,” Six lectures on commutative algebra (Bellaterra, 1996), 1–118, Progr. Math., 166, Birkhäuser, Basel, 1998.
[8] L.L. Avramov “Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology.” Ann. of Math. (2) 150 (1999), no. 2, 455–487.
[9] L.L.AVRAMOV and R.-O. Buchweitz “Support varieties and cohomology over complete intersections.” Invent. Math. 142 (2000), no. 2, 285–318.
[10] L.L. Avramov and H.-B. Foxby “Locally Gorenstein homomorphisms.” American J. Math. 114 (1992) 1007-1047
[11] L.L. Avramov and H.-B. Foxby “Ring homomorphism and ﬁnite Gorenstein dimension” PLMS 75 (1997) 241-270
[12] L.L.AVRAMOV and S. HALPERIN “Through the looking glass: a dictionary between rational homotopy theory and local algebra.” Algebra, algebraic topology and their interactions (Stockholm, 1983), 1-27, Lecture Notes in Maths., 1183, Springer, Berlin, 1986.
[13] H.Bass “On the ubiquity of Gorenstein rings” Math. Z. 82 1963 8–28.
[14] H.Bass “Algebraic K-theory.” Benjamin (1968) xx+762pp
[15] D.J.Benson “An algebraic model for chains on $\Omega BG_p$” Trans. Amer. Math. Soc. 361 (2009), no. 4, 2225-2242.
[16] D.J.Benson and J.F.Carlson “Projective resolutions and Poincaré duality complexes.” Trans. American Math. Soc. 342 (1994) 447-488
[17] D.J.Benson and J.F.Carlson “Functional equations for Poincaré series in group cohomology.” Bull. London Math. Soc. 26 (1994) 438-448.
[18] D.J.Benson and J.P.C.Greenelee “Commutative algebra for the cohomology of classifying spaces of compact Lie groups.” J.Pure and Applied Algebra 122 (1997) 41-53
[19] D.J.Benson and J.P.C.Greenelee “Commutative algebra in the cohomology of virtual duality groups.” J. Algebra 192 (1997) 678-700.
[20] D.J.Benson and J.P.C.Greenelee “Localization and duality in topology and modular representation theory.” J. Pure Appl. Algebra 212 (2008), no. 7, 17161743.
[21] D.J.Benson and J.P.C.Greenelee “Complete intersections and derived categories.” Preprint (2009) 17pp.
[22] D.J. Benson, J.P.C. Greenlees and S. Shamir “Complete intersections and mod p cochains.” AGT 13 (2013) 61-114, DOI 10.2140/agt.2013.13.61, arXiv:1104.4244
[23] J.M. Boardman “Stable homotopy theory” Mimeographed notes 1966–1970.
[24] A.K. Bousfield. “The localization of spaces with respect to homology.” Topology 14 (1975), 133-150.
[25] A.K. Bousfield. “The localization of spectra with respect to homology.” Topology 18 (1979), 257-281.
[26] A.K. Bousfield and D.M. Kan. “Homotopy limits, completions and localizations.” Springer Lecture notes in mathematics Vol. 304. 1972.
[27] E.H. Brown “Abstract homotopy theory.” Trans. Amer. Math. Soc. 119 (1965) 79–85.
[28] W. Bruns and Herzog “Cohen-Macaulay rings.” Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993. xii+403 pp.
[29] V.Buchstaber and T. Panov “Torus actions and their applications in topology and combinatorics” AMS (2002)
[30] E. Dror-Farjoun “Cellular spaces, null spaces and homotopy localization.” Lecture Notes in Mathematics, 1622. Springer-Verlag, Berlin, 1996. xiv+199 pp.
[31] W.G. Dwyer “Strong convergence of the Eilenberg-Moore spectral sequence” Topology 13 (1974) 255-265
[32] W.G. Dwyer and J.P.C. Greenlees “Complete modules and torsion modules” American J. Math. 124 (2002) 199-220
[33] W.G. Dwyer, J.P.C. Greenlees and S.B. Iyengar “Duality in algebra and topology.” Advances in Maths 200 (2006) 357-402
[34] W.G. Dwyer and J.P.C. Greenlees and S.B.Iyengar “Finiteness conditions in derived categories of local rings.” Comm. Math. Helv. 81 (2006) 383-432
[35] W.G. Dwyer and J.P.C. Greenlees and S.B. Iyengar “Gross-Hopkins duality and the Gorenstein condition”. J. K-Theory 8 (2011), no. 1, 107133.
[36] W.G. Dwyer and J.P.C. Greenlees and S.B. Iyengar “DG algebras with exterior homology.” Bull. Lond. Math. Soc. 45 (2013), no. 6, 12351245.
[37] W.G. Dwyer and C. Wilkerson “Homotopy fixed-point methods for Lie groups and finite loop spaces” Ann. Math. 139 (1994) 395-442
[38] D. Dugger and B.E. Shipley “K-theory and derived equivalences.” Duke Math. J. 124 (2004), no. 3, 587–617
[39] A.D. Elmendorf, J.P.C. Greenlees, I. Kriz and J.P. May. “Commutative algebra in stable homotopy theory and a completion theorem.” Math. Res. Letters 1 (1994) 225-239.
[40] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. “Rings, Modules and Algebras in Stable Homotopy Theory”, Volume 47 of Amer. Math. Soc. Surveys and Monographs. American Mathematical Society, 1996.
[41] A. D. Elmendorf, and J. P. May “Algebras over equivariant sphere spectra” J. Pure and Applied Algebra 116 (1997) 139-149
[42] Y. Félix, S. Halperin and J.-C. Thomas “Gorenstein spaces” Advances in Mathematics, 71, (1988) 92-112
[43] Yves Félix, Stephen Halperin, and Jean-Claude Thomas, “Elliptic Hopf algebras”, J. London Math. Soc. (2) 43 (1991), no. 3, 545–555.
[44] P.G. Goerss and J.F. Jardine “Simplicial homotopy theory.” Progress in Mathematics, 174. Birkhauser Verlag, Basel, 1999. xvi+510 pp. ISBN: 3-7643-6064-X
[45] D.J. Green and S. King “The cohomology of finite p-groups’ http://users.minet.uni-jena.de/cohomology
[46] J.P.C. Greenlees “Commutative algebra in group cohomology.” J. Pure and Applied Algebra 98 (1995) 151-162
[47] J.P.C. Greenlees “K-homology of universal spaces and local cohomology of the representation ring” Topology 32 (1993) 295-308.
[48] J.P.C. Greenlees “Tate cohomology in commutative algebra.” J. Pure Appl. Algebra 94 (1994), no. 1, 59–83.
[49] J.P.C. Greenlees “Commutative algebra for group cohomology.” JPAA 98 (1995) 151-162
[50] J.P.C. Greenlees “Tate cohomology in axiomatic stable homotopy theory.” Cohomological methods in homotopy theory (Bellaterra, 1998), 149–176, Progr. Math., 196, Birkhuser, Basel, 2001.
[51] J.P.C. Greenlees “Local cohomology in equivariant topology.” Local cohomology and its applications (Guanajuato, 1999), 1–38, Lecture Notes in Pure and Appl. Math., 226, Dekker, New York, 2002.

[52] J.P.C. Greenlees “Spectra for commutative algebraists.” Proceedings of the 2004 Chicago Summer School Contemporary Mathematics 436 (2007) 149-173, arXiv:math/0609452

[53] J.P.C. Greenlees “First steps in brave new commutative algebra” Proceedings of the 2004 Chicago Summer School Contemporary Mathematics, 436 (2007) 239-275, arXiv:math/0609453

[54] J.P.C. Greenlees “Ausoni-Bökstedt duality for topological Hochschild homology” Journal of Pure and Applied Algebra 220 (2016), pp. 1382-1402

[55] J.P.C. Greenlees, K. Hess and S. Shamir “Complete intersections in rational homotopy theory.” JPAA 217 (2013) 636-663, arXiv:0906.3247

[56] J.P.C. Greenlees and G. Lyubeznik “Rings with a local cohomology theorem with applications to cohomology rings of groups.” JPAA 149 (2000) 267-285

[57] J.P.C. Greenlees and J.P. May. “Completions of G-spectra at ideals of the Burnside ring.” Proc. Adams Memorial Symposium, Volume II, CUP (1992), 145-178.

[58] J.P.C. Greenlees and J.P. May. “Derived functors of I-adic completion and local homology.” J. Algebra 149 (1992), 438-453.

[59] J.P.C. Greenlees and J.P. May “Equivariant stable homotopy theory”. Handbook of algebraic topology, 277–323, North-Holland, Amsterdam, 1995.

[60] J.P.C. Greenlees and J.P. May “Completions in algebra and topology.” Handbook of algebraic topology, 255–276, North-Holland, Amsterdam, 1995.

[61] J.P.C. Greenlees and B.E. Shipley “An algebraic model for free rational G-spectra for compact connected Lie groups G.” Math Z 269 (2011) 373-400, DOI 10.1007/s00209-010-0741-2

[62] J.P.C. Greenlees and B.E. Shipley “An algebraic model for free rational G-spectra.” Bull. LMS 46 (2014) 133-142, DOI 10.1112/blms/bdt066, arXiv:1101.4818

[63] J.P.C. Greenlees and G. Stevenson “Morita theory and singularity categories” In preparation (2015) 17pp

[64] J.P.C. Greenlees and V. Stojanoska “Local and global duality in chromatic homotopy theory.” In preparation (2015) 37pp

[65] A. Grothendieck. “Sur quelques points d’algèbre homologique”. Tohoku Mathematical Journal 9 (1957), 119-221.

[66] A. Grothendieck (notes by R. Hartshorne). “Local cohomology.” Springer Lecture notes in mathematics Vol. 41. 1967.

[67] T.H. Gulliksen “A homological characterization of local complete intersections.” Compositio Math. 23 (1971), 251–255.

[68] T.H. Gulliksen “A change of ring theorem with applications to Poincaré series and intersection multiplicity.” Math. Scand. 34 (1974), 167–183.

[69] T.H. Gulliksen “On the deviations of a local ring.” Math Scand 47 (1980) 5-20

[70] T.H. Gulliksen and G. Levin “Homology of local rings.” Queen’s Paper in Pure and Applied Mathematics, No. 20 Queen’s University, Kingston, Ont. 1969 x+192 pp.

[71] R. Hartshorne. “Algebraic Geometry”. Springer-Verlag. 1977.

[72] M.J. Hopkins. “Global methods in homotopy theory.” Proc. of the 1985 LMS Symposium on Homotopy Theory. London Math. Soc. 1987, 73-96.

[73] M. Hovey “Model categories.” Mathematical Surveys and Monographs, 63. American Mathematical Society, Providence, RI, 1999. xii+209 pp. ISBN: 0-8218-1359-5

[74] M. Hovey, B.E. Shipley and Jeff Smith “Symmetric spectra.” J. Amer. Math. Soc. 13 (2000), no. 1, 149–208.

[75] R. Levi “On finite groups and homotopy theory.” Mem. Amer. Math. Soc. 118 (1995), no. 567, xiv+100 pp.

[76] R. Levi “A counter-example to a conjecture of Cohen.” Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guxols, 1994), 261–269, Progr. Math., 136, Birkhuser, Basel, 1996.

[77] R. Levi “On homological rate of growth and the homotopy type of $\Omega BG_{p}^\wedge$.” Math. Z. 226 (1997), no. 3, 429–444.

62
[78] R.Levi “On p-completed classifying spaces of discrete groups and finite complexes.” J. London Math. Soc. (2) 59 (1999), no. 3, 1064–1080.
[79] L.G.Lewis, J.P.May and M.Steinberger (with contributions by J.E.McClure) “Equivariant stable homotopy theory” Lecture notes in mathematics 1213, Springer-Verlag (1986).
[80] M.A.Mandell “$E_n$ algebras and p-adic homotopy theory.” Topology 40 (2001), no. 1, 43–94.
[81] M.A.Mandell, J.P.May, S.Schwede, and B.E.Shipley “Model categories of diagram spectra.” Proc. London Math. Soc. (3) 82 (2001), no. 2, 441–512.
[82] A. Mathew “The homology of tmf” Homology, Homotopy and Applications 18 (2016) (to appear) arXiv:1305.6100.
[83] H. Matsumura. “Commutative ring theory.” Cambridge Univ. Press. 1986.
[84] J.P.May “A concise course in algebraic topology.” University of Chicago Press, Chicago, IL, 1999. x+243 pp. ISBN: 0-226-51182-0; 0-226-51183-9
[85] J.Rognes “Galois extensions of structured ring spectra. Stably dualizable groups.” Mem. Amer. Math. Soc. 192 (2008), no. 898, viii+137 pp.
[86] D.J. Rusin “The cohomology of the groups of order 32.” Math. Comp. 53 (1989), no. 187, 359385.
[87] P. Schenzel “Preregular sequences, local cohomology, and completion.” Math. Scand. 92 (2003), no. 2, 161–180.
[88] S. Schwede, “Morita theory in abelian, derived and stable model categories.” Structured ring spectra, 33–86, London Math. Soc. Lecture Note Ser., 315, Cambridge Univ. Press, Cambridge, 2004.
[89] S. Schwede “Symmetric spectra” Book in preparation, available from author’s webpage.
[90] S. Schwede and B.E.Shipley “Stable model categories are categories of modules.” Topology 42 (2003), no. 1, 103–153.
[91] S. Schwede and B.E.Shipley “Algebras and modules in monoidal model categories.” Proc. London Math. Soc. (3) 80 (2000), no. 2, 491–511.
[92] B.E.Shipley “$HZ$-algebra spectra are differential graded algebras.” American J. Math 129 (2007) 351-379
[93] E.H.Spanier and J.H.C.Whitehead “A first approximation to homotopy theory.” Proc. Nat.Acad. Sci. U. S. A. 39, (1953). 655–660

The University of Sheffield, Sheffield, S3 7RH UK
E-mail address: j.greenlees@sheffield.ac.uk

63