SMOOTH LOOPS AND FIBER BUNDLES: THEORY OF PRINCIPAL Q-BUNDLES

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A nonassociative generalization of the principal fiber bundles with a smooth loop mapping on the fiber is presented. Our approach allows us to construct a new kind of gauge theories that involve higher "nonassociative" symmetries.

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1. Introduction

During the last few decades, nonassociative structures have been employed in various fields of modern physics. Among others, one may mention the rise of nonassociative objects such as 3-cocycles, which are linked with violations of the Jacobi identity in anomalous quantum field theory, and quantum mechanics with the Dirac monopole, the appearance of Lie groupoids and algebroids in the context of Yang-Mills theories, and the application of nonassociative algebras to gauge theories on commutative but nonassociative fuzzy spaces.

Nonassociative algebraic structures such as quasigroups and loops have considerable potential interests for mathematical physics, especially in view of the appearance of nonassociative algebras, such as the Mal'cev algebra, related to the problem of the chiral gauge anomalies, the emergence of a nonassociative electric field algebra in a two-dimensional gauge theory, and so on.

Quasigroups and loops have recently been employed in general relativity and for the description of the Thomas precession, coherent states, geometric phases, and nonassociative gauge theories.

In this paper, we give a detailed account of nonassociative fiber bundles leading to "nonassociative" gauge theories. In general terms, a consequence of nonassociativity is that the structure constants of the gauge algebra have to be changed by structure functions.
The paper is organized as follows. In Section 2, we outline the basic constructions of the smooth loops theory. In Section 3, the theory of nonassociative principal fiber bundles (Q-bundles) is introduced.

2. Smooth Loops

2.1. Basic notations

Here, we outline the main results on the algebraic theory of quasigroups and loops. The details may be found in [29,30,31,32,33].

Let \( \langle Q, \cdot \rangle \) be a groupoid, i.e. a set with a binary operation \((a, b) \mapsto a \cdot b\). A groupoid \( \langle Q, \cdot \rangle \) is called a quasigroup if each of the equations \( a \cdot x = b, y \cdot a = b \) has unique solutions: \( x = a \backslash b, y = b / a \). A loop is a quasigroup with a two-sided identity \( a \cdot e = e \cdot a = a, \forall a \in Q \). A loop \( \langle Q, \cdot, e \rangle \) with a smooth operation \( \phi(a, b) := a \cdot b \) is called the smooth loop. We define

\[
L_a b = R_b a = a \cdot b, \quad l_{(a, b)} = L_a^{-1} \circ L_a \circ L_b, \\
\hat{l}_{(a,b)} = L_a \circ L_b \\circ L_a^{-1} b, \quad r_{(b,c)} = R_b^{-1} \circ R_c \circ R_b, \tag{2.1}
\]

where \( L_a \) is a left translation, \( R_b \) is a right translation, \( l_{(a,b)} \) is a left associator, \( \hat{l}_{(a,b)} \) is an adjoint associator and \( r_{(b,c)} \) is a right associator.

Let \( T_e(Q) \) be the tangent space of \( Q \) at the neutral element \( e \). Then for each \( X_e \in T_e(Q) \), we construct a smooth vector field on \( Q \)

\[
X_b = L_b X_e, \quad b \in Q, \quad X_e \in T_e(Q), \quad X_b \in T_b(Q)
\]

where \( L_b : T_e(Q) \rightarrow T_b(Q) \) denotes the differential of the left translation. Notice that \( X_b \) depends smoothly on both variables \( b \in Q \) and \( X_e \in T_e(Q) \), and satisfies

\[
L_{a \cdot b} X_b = \hat{l}_{(a,b)} X_{a \cdot b}. \tag{2.2}
\]

**Definition 1.** A vector field \( X \) on \( Q \) that satisfies the relation \( L_a X_b = \hat{l}_{(a,b)} X_{a \cdot b} \) for any \( a, b \in Q \) is called the left fundamental or left quasi-invariant vector field.

Let \( V \) denote the set of left fundamental vector fields on \( Q \). It becomes a vector space under the operations

\[
(X + Y)_a = X_a + Y_a, \quad (XY)_a = X_a Y_a, \quad a \in Q,
\]

\[
(\alpha X)_a = \alpha X_a, \quad \alpha \in \mathbb{R}.
\]

The following lemma is evident.

**Lemma 2.** The vector space of left fundamental vector fields is isomorphic to the tangent space \( T_e(Q) \) at the neutral element. The isomorphism is defined by the map \( L_{a^{-1}} X_a \rightarrow X_e, X_e \in T_e(Q), X_a \in T_a(Q) \).

The vector space \( V \) can be equipped with a Lie commutator. This leads us to the notion of quasialgebra.

**Definition 3.** We define a quasialgebra \( Q \) on \( Q \) as the vector space of left fundamental vector fields, under the Lie commutator operation.
Let $\Gamma_i = R_i^j \partial/\partial a^j$, $i = 1, 2, \ldots, r$ be a basis of the space of left fundamental vector fields. Then, we have

$$[\Gamma_i, \Gamma_j] = C_{ij}^p(a) \Gamma_p$$

(2.3)

where $C_{ij}^p(a)$ are the structure functions satisfying the modified Jacobi identity

$$C_{ijk,l}^p R_k^n + C_{jki,l}^p R_j^n + C_{lik,j}^p C_{kl}^p + C_{jil,k}^p C_{kl}^p + C_{kil,j}^p C_{jl}^p = 0.$$  

(2.4)

In view of the noncommutativity of the right and left translations, there exists a problem in the definition of the 'adjoint' map of $Q$ on itself. Indeed, in the case of the group $G$, the adjoint map is given by

$$\text{Ad}_g g' = g g' g^{-1} \equiv L g \circ R - 1 g g' \equiv R - 1 g \circ L g.$$  

For quasigroups, the left and right translations do not commute, $L_a \circ R_b \neq R_b \circ L_a$, and it follows that the definitions $\text{Ad}_g g' = L_g \circ R - 1 g'$ and $\tilde{\text{Ad}}_g g' = R - 1 g \circ L g'$ are not equivalent. We introduce a generalized adjoint map of $Q$ on itself in the following way.

**Definition 4.** The map

$$\text{Ad}_b(a) = L_a^{-1} \circ R_b^{-1} \circ L_{a,b} : Q \to Q$$

(2.5)

is called the Ad-map.

**Remark 5.** The Ad-map $\text{Ad}_b(a)$ leaves invariant the neutral element $e$ and generates the map $T_e(Q) \to T_e(Q)$ as follows:

$$\text{Ad}_b(a) := (\text{Ad}_b(a))_e = L_a^{-1} R_b^{-1} L_{a,b}.$$  

(2.6)

It is easy to check that $\text{Ad}_b(e) = \text{Ad}_b := R_b^{-1} \circ L_b$.

**Definition 6.** The vector-valued 1-form $\omega$ defined through the relation

$$\omega(V_a) = V_e \quad \text{where} \quad V_e = L_a^{-1} V_a, \quad V_e \in T_e(Q), \quad V_a \in T_e(Q)$$

(2.7)

where $V$ is a left fundamental vector field, is called the canonical Ad-form.

**Theorem 7.** The canonical Ad-form $\omega$ is a left fundamental form and is transformed under left (right) translations as follows:

$$L_b^* \omega V_a = l_{(b,a)}^* \omega(V_a)$$

(2.8)

$$R_b^* \omega V_a = \text{Ad}_b^{-1}(a) \omega(V_a).$$

(2.9)

**Proof.** For the left translations, we obtain

$$(L_b^* \omega) V_a = \omega(L_b V_a) = L_b^{-1} L_b L_a V_e = l_{(b,a)}^* \omega(V_a),$$
thus (2.8) is true.

Similarly, for the right translations, we find
\[(R_\omega^* b)V_a = \omega(R_{ba} V_a) = L_{a b a}^{-1} R_{b a} L_{a^*}^{-1} V_a \]
\[= L_{a b a}^{-1} R_{b a} L_{a^*} \omega(V_a) = \text{Ad}^{-1}_b(a) \omega(V_a) \] (2.10)
and the proof is completed.

2.1.1. Examples

The examples given below play an important role in the description of generalized coherent states, Thomas precession and nonassociative geometry [7,22,23,24,25,26,27].

Example 8. Loop \((\mathbb{R}/\mathbb{Z}, \ast)\). Let \(\ast\) be the operation defined on \(\mathbb{R}/\mathbb{Z}\) as follows:
\[x \ast y = x + y + f(x) + f(y) - f(x + y) \] (2.11)
where \(f(x) = (1 - \cos 2\pi x)/4\). Then \((\mathbb{R}/\mathbb{Z}, \ast)\) is the analytical loop [34,35].

Example 9. Loop QC. Let \(\mathbb{C}\) be a complex plane and \(\zeta, \eta \in \mathbb{C}\). Then, the loop QC is obtained by introducing on \(\mathbb{C}\) the nonassociative multiplication defined as follows
\[R_\zeta^\eta = L_\zeta \eta = \zeta \eta = \frac{\zeta + \eta}{1 - \zeta \eta}, \quad \zeta, \eta \in \mathbb{C}, \] (2.12)
where a bar denotes complex conjugation. The inverse operation is is found to be
\[L_\zeta^{-1} \eta = \frac{\eta - \zeta}{1 + \zeta \eta}, \quad \zeta, \eta \in \mathbb{QC}. \] (2.13)
The associator \(l_{(\zeta, \eta)}\) is given by
\[l_{(\zeta, \eta)} \xi = \frac{1 - \zeta \eta}{1 - \eta \bar{\zeta}} \] (2.14)
and can also be written as \(l_{(\zeta, \eta)} = \exp(i\alpha)\), where \(\alpha = 2 \arg(1 - \zeta \bar{\eta})\).

The basis of left quasi-invariant vectors and the dual basis of 1-forms are found to be
\[\Gamma_1 = (1 + |\eta|^2) \partial_\eta, \quad \Gamma_2 = (1 + |\eta|^2) \partial_{\bar{\eta}}, \] (2.15)
\[\theta^1 = \frac{d\eta}{1 + |\eta|^2}, \quad \theta^2 = \frac{d\bar{\eta}}{1 + |\eta|^2}. \] (2.16)
The computation of the commutator yields
\[[\Gamma_1, \Gamma_2] = \bar{\eta} \Gamma_2 - \eta \Gamma_1, \] (2.17)
And, for the structure functions, we find \(C_{12}^1 = -\eta, C_{12}^2 = \bar{\eta}\).

The loop thus obtained is related to the two-sphere \(S^2\), which admits a natural quasigroup structure. Namely, \(S^2\) is the local two-parametric Bol loop \(QS^2\) [7,9].
The isomorphism between points of the sphere and the complex plane \( \mathbb{C} \) is established by the stereographic projection from the south pole of the sphere: 
\[ \zeta = e^{i\varphi} \tan(\theta/2). \]

**Example 10.** Loop \( QSU(2) \). Let us consider a set of the unitary matrices \( QSU(2) \subset SU(2) \), where an arbitrary element of \( QSU(2) \) has the form 
\[ U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1, \quad \alpha = \bar{\alpha}. \]

It is convenient to set 
\[ \alpha = \frac{1}{\sqrt{1 + |\eta|^2}}, \quad \beta = \eta \frac{1}{\sqrt{1 + |\eta|^2}}, \]
and write \( U \) as 
\[ U_\eta = \frac{1}{\sqrt{1 + |\eta|^2}} \begin{pmatrix} 1 & \eta \\ \bar{\eta} & 1 \end{pmatrix}. \quad (2.18) \]

For arbitrary elements of \( QSU(2) \), we define the nonassociative binary operation as follows:
\[ U_\eta * U_\zeta = U_\eta U_\zeta \Lambda(\eta, \zeta), \quad (2.19) \]
where, on the right-hand side, the matrices are multiplied in the usual way, and 
\[ \Lambda(\eta, \zeta) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \quad \varphi = 2 \text{arg}(1 - \eta \bar{\zeta}). \]

The set of matrices \( (2.18) \) with the operation \( * \) is called the loop \( QSU(2) \).

The loop \( QSU(2) \) forms the so-called nonassociative representation of \( QS^2 \). Indeed, writing Eq. \( (2.19) \) as 
\[ U_\eta * U_\zeta = \frac{1}{\sqrt{1 + |\eta \zeta|^2}} \begin{pmatrix} 1 & \eta \zeta \\ \bar{\eta} \bar{\zeta} & 1 \end{pmatrix}, \quad (2.20) \]
we find (see Ex. 9)
\[ L_\eta \zeta \equiv \eta \zeta = \frac{\eta + \zeta}{1 - \eta \bar{\zeta}}, \quad l_{(\eta, \zeta)} \xi = \frac{1 - \eta \bar{\zeta}}{1 - \bar{\eta} \xi}, \quad \zeta, \eta, \xi \in \mathbb{C}. \quad (2.21) \]

**Example 11.** Loop \( QH^2 \). This loop is associated with the group \( SU(1,1) \) and its action on the two-dimensional unit hyperboloid \( H^2 \). Let \( D \subset \mathbb{C} \) be the open unit disk, \( D = \{ \zeta \in \mathbb{C} : |\zeta| < 1 \} \). We define the nonassociative binary operation \( * \) as 
\[ \zeta * \eta = \frac{\zeta + \eta}{1 + \zeta \bar{\eta}}, \quad \zeta, \eta \in D. \quad (2.22) \]

The associator \( l_{(\zeta, \eta)} \) on \( QH^2 \) is determined by 
\[ l_{(\zeta, \eta)} \xi = \frac{1 + \zeta \eta}{1 + \eta \bar{\zeta}} \xi, \quad (2.23) \]
and can also be written as $l_{(\zeta, \eta)} = \exp(i\alpha), \quad \alpha = 2 \arg(1 + \zeta \overline{\eta})$. Inside the disk $D$, the set of complex numbers with the operation $*$ forms a two-sided loop $QH^2$, which is isomorphic to the geodesic loop of two-dimensional Lobachevsky space realized as the upper part of the two-sheeted unit hyperboloid. The isomorphism is established by $\zeta = e^{i\phi} \tanh(\theta/2)$, where $(\theta, \phi)$ are inner coordinates on $H^2$.

The basis of left quasi-invariant vectors and the dual basis of 1-forms are found to be

$$\Gamma_1 = (1 - |\eta|^2)\partial_\eta, \quad \Gamma_2 = (1 - |\eta|^2)\overline{\partial}_\eta,$$

and the computation of the commutator gives

$$[\Gamma_1, \Gamma_2] = \eta \Gamma_1 - \overline{\eta} \Gamma_2.$$ \hfill (2.26)

**Example 12.** Loop $QH_R$. Let us consider the quaternionic algebra over the complex field $\mathbb{C}$

$$H_C = \{\alpha + \beta i + \gamma j + \delta k \mid \alpha, \beta, \gamma, \delta \in \mathbb{C}\}$$

with the multiplication operation defined by the property of bilinearity and following rules for $i, j, k$:

$$i^2 = j^2 = k^2 = -1, \quad jk = -kj = i,$$

$$ki = -ik = j, \quad ij = -ji = k.$$

For $q = \alpha + \beta i + \gamma j + \delta k$, the quaternionic conjugation is defined as follows:

$$q^+ = \alpha - \beta i - \gamma j - \delta k.$$ \hfill (2.27)

Furthermore, we restrict ourselves to the set of quaternions $H_R \subset H_C$:

$$H_R = \{\zeta = \zeta^0 + i(\zeta^1 i + \zeta^2 j + \zeta^3 k) : i^2 = -1, \ i \in \mathbb{C}, \ \zeta^0, \zeta^1, \zeta^2, \zeta^3 \in \mathbb{R}\},$$

with the norm $\|\zeta\|^2$ given by

$$\|\zeta\|^2 = \zeta \zeta^+ = (\zeta^0)^2 - (\zeta^1)^2 - (\zeta^3)^2 - (\zeta^4)^2.$$ \hfill (2.28)

Introducing a binary operation

$$\zeta * \eta = (\zeta + \eta)/(1 + K/4 \zeta^+ \eta), \quad \zeta, \eta \in H_R,$$

where $K$ is constant and $/$ denotes the right division, we find that the set of quaternions $H_R$ with the binary operation $*$ forms a loop, which we denote by $QH_R$. The associator is found to be

$$l_{(\zeta, \eta)} = (1 + K/4 \zeta^+ \eta)/(1 + K/4 \zeta^+ \eta).$$ \hfill (2.30)

The loop $QH_R$ is related to the spacetime of constant positive curvature $K > 0$ (de Sitter spacetime), which is locally characterized by the condition \hfill (2.31)

$$R_{\mu\nu\lambda\sigma} = K(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}).$$
2.2. Infinitesimal theory of smooth loops in brief

It is well known that the infinitesimal theory of Lie groups arises from the associativity of the operation

\[ a \cdot (b \cdot c) = (a \cdot b) \cdot c. \]

In what follows, we will show that the quasiassociativity identities

\[ R_c \circ R_b a = R_{R_b}^a c, \quad L_a \circ L_b c = L_{L_b}^a c \quad (2.32) \]

lead to the infinitesimal theory of smooth loops. We start with the definition of a quasigroup of transformations introduced by Batalin [39].

**Definition 13.** Let \( M \) be an \( n \)-dimensional manifold and let the continuous law of transformation be given by \( x' = T_a x, \quad x \in M \), where \( \{a_i\} \) is the set of real parameters, \( i = 1, 2, \ldots, r \). The set of transformations \( \{T_a\} \) forms a \( r \)-parametric quasigroup of transformations (with right map on \( M \)), if:

1) there exists a unit element that is common for all \( x_i \) and corresponds to \( a_i = 0 : T_a x|_{a=0} = x \);

2) the modified composition law holds:

\[ T_a T_b x = T_{\varphi(b,a;x)} x \]

3) the left and right units coincide:

\[ \varphi(a,0;x) = a, \quad \varphi(0,b;x) = b; \]

4) the modified law of associativity is satisfied:

\[ \varphi(\varphi(a,b;x),c;x) = \varphi(a,\varphi(b,c;T_a x);x); \]

4) the inverse transformation of \( T_a \) exists: \( x = T_a^{-1} x' \).

The generators of infinitesimal transformations

\[ \Gamma_i = (\partial (T_a x)/\partial a^i)|_{a=0} \equiv R_i^\alpha \partial/\partial x^\alpha \]

form a quasialgebra and obey the commutation relations

\[ [\Gamma_i, \Gamma_j] = C_{ij}^p (x) \Gamma_p, \quad (2.33) \]

where \( C_{ij}^p (x) \) are the structure functions satisfying the modified Jacobi identity

\[ C_{ij,\alpha}^p R_k^\alpha + C_{jk,\alpha}^p R_i^\alpha + C_{ki,\alpha}^p R_j^\alpha + C_{ij}^l C_{kl}^p + C_{jk}^l C_{il}^p + C_{ki}^l C_{jl}^p = 0. \quad (2.34) \]

**Theorem 14.** Let the given functions \( R_i^\alpha, C_{ij}^p \) obey the equations \( (2.33), (2.4) \). Then, locally the quasigroup of transformations can be reconstructed as the solution of the set of differential equations

\[ \frac{\partial \tilde{x}^\alpha}{\partial a^i} = R_i^\alpha (\tilde{x}) \lambda_i^j (a; x), \quad \tilde{x}^\alpha (0) = x^\alpha, \quad (2.35) \]

\[ \frac{\partial \lambda_i^j}{\partial a^p} - \frac{\partial \lambda_i^k}{\partial a^j} + C_{mn}^i (\tilde{x}) \lambda_m^p \lambda_n^j = 0, \quad \lambda_i^j (0; x) = \delta_i^j. \quad (2.36) \]
Eq. (2.35) is an analog of the Lie equation, and Eq. (2.36) is the generalized Maurer-Cartan equation.

The Batalin approach can be easily extended to the case of smooth loops if we consider the action of the loop on itself. Indeed, for smooth loops, the modified associativity law can be written as follows:

\[ \varphi^i(\varphi(a, b), c) = \varphi^i(a, \tilde{\varphi}(b, c; a)), \quad \varphi^i(a, 0) = a^i, \quad \varphi^i(0, b) = b^i. \] 

(2.37)

The derivation of Eq. (2.37) in \( e \) with respect to \( c^j \) yields

\[ \left. \frac{\partial \varphi^i(a, b, c)}{\partial c^j} \right|_e = \left. \frac{\partial \varphi^i(a, b)}{\partial b^k} \frac{\partial \tilde{\varphi}(b, c; a)}{\partial c^j} \right|_e \] 

(2.38)

Let us introduce

\[ \alpha_j^k(b; a) = \left. \frac{\partial \varphi^k(b, c; a)}{\partial c^j} \right|_e, \quad \omega_p^j = \partial^j_p = \left. \frac{\partial \varphi^i(a, b)}{\partial b^p} \right|_e. \] 

(2.39)

Substituting (2.39) in (2.38), we obtain an analog of Lie equation (see Eq. (2.35))

\[ \frac{\partial \varphi^i(a, b)}{\partial b^j} = \omega_p^j(\varphi(a, b)) \omega_q^p(\varphi(b; a)). \] 

(2.40)

Actually, the parametric operation \( \varphi(b, c; a) \) is not independent, and it can be written in the terms of left translations as follows:

\[ \varphi(b, c; a) = L_b \circ l_{(a, b)}^{-1} c. \]

This yields

\[ \omega_p^j(\varphi(b; a)) = \omega_p^j(b; e). \]

Taking into account that \( \omega_p^j(b; e) = \omega_q^p(b) \), we obtain the generalized Lie equation deduced by Sabinin [33]

\[ \frac{\partial \varphi^i(a, b)}{\partial b^j} = q_p^i(\varphi(a, b)) \omega_q^p(\varphi(b; a)). \] 

(2.41)

3. Principal Q-bundles

3.1. Basic definitions and examples

Let \( \mathcal{M} \) be a manifold and \( \langle Q, \cdot, e \rangle \) be a smooth two-sided loop. A principal loop Q-bundle (a principal bundle with structure loop Q) is a triple \( (P, \pi, M) \), where \( P \) is a manifolds and the following conditions hold:

1. \( Q \) acts freely on \( P \) by the right map: \( (p, a) \in P \times Q \mapsto \tilde{R}_a p \equiv pa \in P \).
2. \( \mathcal{M} \) is the quotient space of \( P \) by the equivalence relation induced by \( Q \), \( \mathcal{M} = P/Q \), and the canonical projection \( \pi : P \to M \) is a smooth map.
3. \( P \) is locally trivial, i.e. any \( x \in M \) has a neighborhood \( U \) and a diffeomorphism \( \Phi \) of \( \pi^{-1}(U) \to U \times Q \) such that, for any point \( p \in \pi^{-1}(U) \), it has the form \( \Phi(p) = (\pi(p), \varphi(p)) \), where \( \varphi \) is the map from \( \pi^{-1}(U) \) to \( Q \) satisfying \( \varphi(\tilde{R}_a p) = R_a \varphi(p) \).
Then, let

Definition 16. Let \( \Phi \) be a local trivialization. This means that there exists a diffeomorphism \( \Phi : \pi^{-1}(U_{\alpha}) \rightarrow U_{\alpha} \times Q \) such that the restriction of the fibration to each open set \( U_{\alpha} \) is trivializable. This means that there exists a diffeomorphism \( \Phi_{\alpha} : \pi^{-1}(U_{\alpha}) \rightarrow U_{\alpha} \times Q \). \( \Phi_{\alpha} \) is a trivialization.

Proof. Let \( \sigma_{\alpha} = \Phi_{\alpha}^{-1}(x,e) \), \( x \in U_{\alpha} \) is called the canonical local section associated with this trivialization.

Let \( u \in P \); then, one can describe the local section as follows: \( \sigma_{\alpha} = \tilde{R}_{u_{\alpha}}^{-1}(u) \), where we set \( u_{\alpha} = \varphi_{\alpha}(u) \).

Proposition 17. The section \( \sigma_{\alpha} \) does not depend on the choice of the point in the fiber.

Proof. Let \( u, p \in P \) and \( u = \tilde{R}_{u_{\alpha}}^{-1}(u, e) \), \( p = \tilde{R}_{q_{\alpha}}^{-1}(p, e) \). There exists an element \( a \in Q \) such that \( p = \tilde{R}_{u_{\alpha}}u \). Then we have

\[
\begin{align*}
p &= \tilde{R}_{q_{\alpha}}(p, e) = \tilde{R}_{q_{\alpha}}u = \tilde{R}_{u_{\alpha}} \circ \tilde{R}_{q_{\alpha}}(u) \\
&= \varphi_{\alpha}^{-1}(R_{a} \circ R_{q_{\alpha}}(u) \varphi_{\alpha}(u)) \\
&= \varphi_{\alpha}^{-1}(R_{a} \circ R_{q_{\alpha}}(u) \varphi_{\alpha}(a)) \\
&= \varphi_{\alpha}^{-1}(R_{a} \circ R_{q_{\alpha}}(u) \varphi_{\alpha}(a)) \\
&= \varphi_{\alpha}^{-1}(R_{a} \circ R_{q_{\alpha}}(u) \varphi_{\alpha}(a)). \quad (3.1)
\end{align*}
\]

Taking into account that \( q_{\alpha}(p) = q_{\alpha}(u) \cdot a \), we obtain \( R_{q_{\alpha}}(p) \sigma'_{\alpha} = R_{q_{\alpha}}(p) \sigma_{\alpha} \). This yields \( \sigma_{\alpha} = \sigma'_{\alpha} \), and, hence, the section \( \sigma_{\alpha} \) does not depend on the point of the fiber, while the dependence on the base, \( \sigma_{\alpha} = \sigma_{\alpha}(x) \), may still hold.

Proposition 18. Let the point \( x \) lie in the intersection of the neighborhoods \( U_{\alpha} \) and \( U_{\beta} \), \( x \in U_{\alpha} \cap U_{\beta} \). Then, the following formula holds:

\[
\sigma_{\alpha}(x) = \tilde{R}_{q_{\alpha}} \sigma_{\beta}(x). \quad (3.2)
\]

Proof. Let \( u = \tilde{R}_{q_{\alpha}} \sigma_{\alpha} \), rewriting this formula in the chart \( U_{\beta} \), we obtain

\[
\begin{align*}
\Phi_{\beta}(\tilde{R}_{q_{\alpha}} \sigma_{\alpha}) &= \Phi_{\beta}(\tilde{R}_{q_{\alpha}} \sigma_{\beta}) = (\pi(u), R_{q_{\beta}} \varphi_{\beta}(\sigma_{\beta})) \\
&= (\pi(u), R_{q_{\beta}} \circ R_{q_{\alpha}}(u) \varphi_{\beta}(\sigma_{\beta})) = (\pi(u), R_{q_{\beta}} \circ R_{q_{\alpha}}(u) \varphi_{\beta}(\sigma_{\beta})) \\
&= (\pi(u), R_{q_{\beta}} \circ R_{q_{\alpha}}(u) \varphi_{\beta}(\sigma_{\beta})) = \Phi_{\beta}(\tilde{R}_{q_{\beta}} \circ \tilde{R}_{q_{\alpha}} \sigma_{\beta}).
\end{align*}
\]
where the relation $\varphi_\beta(\sigma_\beta) = e$ has been used. Finally, we obtain $\sigma_\alpha(x) = \hat{R}_{q_\alpha,\sigma_\beta}(x)$ in the intersection $U_\alpha \cap U_\beta$.

**Definition 19.** The family of maps $\{q_{\beta\alpha}(p) = R_{q_\alpha,\sigma_\beta}^{-1} : \pi^{-1}(U_\alpha \cap U_\beta) \to Q\}$, where $q_\alpha := \varphi_\alpha(p)$, $q_\beta := \varphi_\beta(p)$, $p \in \pi^{-1}(U_\alpha \cap U_\beta)$, is called the family of transition functions of the bundle $P(M,Q)$ corresponding to the trivializing covering $\{U_\alpha, \Phi_\alpha\}_{\alpha \in J}$ of $\mathfrak{M}$.

**Proposition 20.** The transition functions $q_{\beta\alpha}(p)$ change under right translations as

$$R_\alpha q_{\beta\alpha}(p) = r_{(q_\alpha,\sigma_\beta)} q_{\beta\alpha}(p)$$

where $r_{(q_\alpha,\sigma_\beta)} = R_{q_\alpha,\sigma_\beta}^{-1} \circ R_\alpha \circ R_{q_\alpha}$ is the right associator.

**Proof.** According to the definition, we have

$$R_\alpha q_{\beta\alpha}(p) = q_{\beta\alpha}(pa) = R_{q_\alpha,\sigma_\beta}^{-1} \circ R_\alpha q_\beta = R_{q_\alpha,\sigma_\beta}^{-1} \circ R_\alpha \circ R_{q_\alpha} = R_{q_\alpha,\sigma_\beta}^{-1} = r_{(q_\alpha,\sigma_\beta)} q_{\beta\alpha}(p).$$

We find that $q_{\beta\alpha}$ is changed under the right translations, and, therefore, it depends on the points of the fiber: $q_{\beta\alpha} = q_{\beta\alpha}(p)$, $p \in F_x$, $x \in U_\alpha \cup U_\beta$. The transition functions obey the so-called *cocycle condition*:

$$q_{\beta\alpha}(p) \cdot q_\alpha(p) = q_{\beta\gamma}(p) \cdot (q_{\gamma\alpha}(p) \cdot q_\alpha(p)) \quad (3.3)$$

where $x \in U_\alpha \cap U_\beta \cap U_\gamma$ and $p \in F_x$,

which may also be written as

$$q_{\beta\alpha} \cdot q_\alpha = q_{\beta\gamma} \cdot q_{\gamma\alpha} \cdot (r_{(q_\alpha,\sigma_\beta)} q_\alpha).$$

**Definition 21.** Let $P(M,Q,\pi)$ be a principal loop $Q$-bundle and $\{U_\alpha\}$ some covering of the base $\mathfrak{M}$. Then we say that the system $\{\mathfrak{M}, Q, \{U_\alpha\}, q_{\alpha\beta}\}$ is a principal coordinate $Q$-bundle.

Assume that there are two families of local sections, $\sigma_\alpha$ and $\sigma'_\alpha$. These sections are linked by the right translations. This implies that $\exists q_\alpha \in Q : \sigma'_\alpha = R_{q_\alpha, \sigma_\alpha}$. Using the set $\{\sigma_\alpha\}$, let us introduce the new transition functions $q'_{\alpha\beta}$. Then, the following transformation law holds:

$$q'_{\alpha\beta} = L_{q_\alpha}^{-1} \circ R_{q_\beta} q_{\alpha\beta}.$$

**Example 22.** Principal $QS^2$ bundle over $S^1$. Here, we will show that $S^3$ is the $QS^2$ bundle over $S^1$. We consider the unit-three sphere $S^3$ as a subset of $\mathbb{C}^2$:

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}. \quad (3.4)$$

We define the map $\pi : S^3 \to S^1$ by

$$\pi(z_1, z_2) = \left( \frac{z_1 + \bar{z}_1}{2 \sqrt{1 - |z_2|^2}}, \frac{z_1 - \bar{z}_1}{2i \sqrt{1 - |z_2|^2}}, 0 \right). \quad (3.5)$$
Parameterizing $S^3$ by
\[ z_1 = \cos \frac{\theta}{2} e^{i\psi_1}, \quad z_2 = \sin \frac{\theta}{2} e^{i\psi_2} \]
where $0 \leq \theta \leq \pi$ and $\psi_1, \psi_2 \in \mathbb{R}$, we get
\[ \pi(\cos \frac{\theta}{2} e^{i\psi_1}, \sin \frac{\theta}{2} e^{i\psi_2}) = (\cos \psi_1, \sin \psi_1, 0), \] (3.6)
and one can see that $\pi$ indeed maps $S^3$ to $S^1$. By this means, one can see that locally $S^3 \approx S^1 \times S^2$; however, this is not true globally since the spaces $S^1 \times S^3$ and $S^3$ have different second homotopy groups [40]:
\[ \pi_2(S^1 \times S^2) = \mathbb{Z}, \quad \pi_2(S^3) = 0. \] (3.7)

Let $U_- = (-\pi, \pi)$ and $U_+ = (0, 2\pi)$ be an open covering of $S^1$. We define the local trivializations, $\Phi_\pm : \pi^{-1}(U_\pm) \to U_\pm \times QS^2$, as follows:
\[ \Phi_-\left(\cos \frac{\theta}{2} e^{i\psi_1}, \sin \frac{\theta}{2} e^{i\psi_2}\right) = (e^{i\psi_1}, \tan \frac{\theta}{2} e^{i(\psi_2-\psi_1)}) \] (3.8)
\[ \Phi_+\left(\cos \frac{\theta}{2} e^{i\psi_1}, \sin \frac{\theta}{2} e^{i\psi_2}\right) = (e^{i\psi_1}, \tan \frac{\theta + \pi/2}{2} e^{i(\psi_2-\psi_1)}). \] (3.9)

Let us set
\[ \zeta_- = \tan \frac{\theta}{2} e^{i(\psi_2-\psi_1)}, \quad \zeta_+ = \tan \frac{\theta + \pi/2}{2} e^{i(\psi_2-\psi_1)}, \] (3.10)
then the transition function relative to our trivialization determined by
\[ q_{+-} : U_- \cap U_+ \to QS^2 \] (3.11)
and
\[ \zeta_+ = L_{q_{+-}} \zeta_- = \frac{q_{+-} + \zeta_-}{1 - q_{+-} \zeta_-}. \] (3.12)
is given by $q_{+-} = e^{i(\psi_2-\psi_1)}$, and we have $(\cos \psi_1, \sin \psi_1, 0) \to e^{i(\psi_2-\psi_1)} \in QS^2$.

Since the structure loop $QS^2$ acts on the standard fiber $S^2$ by left translations (3.12), we indeed have a $QS^2$ principal bundle over $S^1$. For the right translations of $QS^2$ on $S^3$, we find
\[ \hat{R}_\eta(z_1, z_2) = \left( \frac{z_1(1 - \eta z_2/z_1)}{\sqrt{1 + |\eta|^2}}, \frac{z_2(1 + \eta z_1/z_2)(1 - \eta z_2/z_1)}{\sqrt{1 + |\eta|^2}} \right), \quad \eta \in QS^2. \] (3.13)

**Example 23.** Principal $QS^2$ bundle over $S^2$. We construct the principal $QS^2$ bundle over $S^2$ by taking
Base $\mathcal{M} = S^2$ with coordinates $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$,
Fiber $QS^2$ with coordinates $\zeta \in \mathbb{C}$.

We split $S^2$ into two hemispheres $H_\pm$ with $H_+ \cap H_-$ being a thin strip parameterized by the equatorial angle $\varphi$. Thus, locally the bundle can be described as
\[ H_- \times QS^2 \quad \text{with coordinates } (\theta, \varphi, \zeta_-), \]
\[ H_+ \times QS^2 \quad \text{with coordinates } (\theta, \varphi, \zeta_+). \]

The transition functions must be elements of the loop \( QS^2 \) in order to give a principal \( Q \)-bundle. We choose to relate the \( H_+ \) and \( H_- \) fiber coordinates in \( H_+ \cap H_- \) by the left map (3.12)

\[ L_{q_-} \zeta_- = \frac{q_+ - \zeta_-}{1 - q_+ \zeta_-}. \tag{3.14} \]

Since \( q_+ \) maps \( S^2 \) to \( S^2 \), it is classified by \( \pi_2(S^2) = \mathbb{Z} \). This implies that

\[ \zeta_+ = (L_{q_+})^n \zeta_- \tag{3.15} \]

and the power \( n \) of \( L_{q_+} \) must be an integer to give a well-defined manifold. Let us define \( q^n_+ \) by the following relation: \( (L_{q_+})^n = L_{q^n_+} \); then, we get

\[ q^n_+ = \frac{\zeta_+(1 - \zeta_- \zeta_+) - \zeta_-(1 - \zeta_- \zeta_+)}{1 - \vert \zeta_- \vert^2 \vert \zeta_+ \vert^2}. \tag{3.16} \]

Setting \( q_+ = e^{i\gamma} \tan(\theta/2) \) and using Eq. (3.15), we obtain \( q^n_+ = e^{i\gamma} \tan(n\theta/2) \).

### 3.1.1. Associated bundles

In this section, we will discuss a method of constructing quasigroup fiber bundles that are associated with some principal \( Q \)-bundle. Let \( F \) be a space on which \( Q \) acts (on the left) as the loop of transformations. The basic idea is to construct for a particular principal \( Q \)-bundle \( P(M, Q) \) a fiber bundle with fiber \( F \). First, we need the concept of a \( Q \)-product of a pair of spaces on which \( Q \) acts by the right mappings.

**Definition 24.** Let \( X \) and \( Y \) be any pair of right \( Q \)-spaces. Then, the \( Q \)-product of \( X \) and \( Y \) is the space of orbits of the \( Q \)-mapping on the Cartesian product \( X \times Y \).

This definition implies that there is defined an equivalence relation on \( X \times Y \) in which \( (x, y) \equiv (x', y') \) iff \( \exists q \in Q \) such that \( x' = R_q x \) and \( y' = R_q y \). The \( Q \)-product is denoted \( X \times_Q Y \).

**Definition 25.** Let \( P(\mathfrak{M}, \pi, Q) \) be a principal \( Q \)-bundle and let \( F \) be a manifold on which \( Q \) acts by left maps. Define \( P_F := P \times_Q F \) and the map \( Q \) on \( P_F \) as follows

\[ (p, \xi) \rightarrow (R_a p, \tilde{L}_a^{-1} \xi), \quad p \in P, \xi \in F, \]

where \( \tilde{L}_a \) denotes the left map on \( F \). Then, \( E(\mathfrak{M}, Q, F, P) = P \times_Q F/Q \) is the associated fiber bundle over \( \mathfrak{M} \) with fiber \( F \).

The differential structure over \( E(\mathfrak{M}, Q, F, P) \) is defined as follows. Let \( \pi_E \) be the projection map \( E \rightarrow \mathfrak{M} : (u, \xi) \rightarrow \pi(u) \), that is induced by the projection
\( \pi : P \times \mathcal{M} \to \mathcal{M} \). For any \( x \in \mathcal{M} \), the set \( \pi_E^{-1}(x) \) is called the fiber in \( E \) over \( x \) and the \( Q \)-map on \( \pi_E^{-1}(U) \times F \) is defined by \((x,a,\xi) \to (x,ab,L_b^{-1}\xi)\), where \((x,a,\xi) \in U \times Q \times F, b \in Q, U \subset \mathcal{M} \) and \( \pi_E^{-1}(U) \) is the open manifold in \( E: \pi_E^{-1}(U) \approx U \times F \). Then, the projection \( \pi_E^{-1} \) is a differentiable mapping \( E \to \mathcal{M} \). A map \( \sigma : \mathcal{M} \to E \), such that \( \pi_E \circ \sigma = Id \) is the identity map \( \mathcal{M} \to \mathcal{M} \), is called a section of the fiber bundle \( E(\mathcal{M}, Q, F, P) \).

### 3.2. Connection, curvature and Bianchi identities

#### 3.2.1. Connection on principal \( Q \)-bundles

Let \( P(\mathcal{M}, Q) \) be a principal \( Q \)-bundle over the manifold \( \mathcal{M} \). For any \( u \in P \), we denote a tangent space at \( u \) by \( T_u(P) \) (or \( T_u \)), and the tangent to the fiber passing through \( u \) by \( \mathcal{V}_u \). We call \( \mathcal{V}_u \) a vertical subspace. It is generated by the right translations on the fiber: \( u \to \hat{R}_u u, a \in Q, u \in P \):

\[
X_u = \frac{d}{dt} \hat{R}_u(t)u \bigg|_{t=0}, \quad X_u \in \mathcal{V}_u, a \in Q, u \in P. \tag{3.17}
\]

Let \( \gamma(t) \in \mathcal{M} \) be a smooth curve. A horizontal lift of \( \gamma \) is a curve \( \tilde{\gamma}(t) \in P \) such that \( \pi(\tilde{\gamma}(t)) = \gamma(t) \). Evidently, to determine \( \tilde{\gamma}(t) \), it is sufficient to define at any point a tangent vector \( \tilde{X} \): \( \pi_* \tilde{X} = X \), where \( X \) is the tangent vector to \( \gamma(t) \). A set \( \{X\} \) is called a horizontal subspace \( \mathcal{H}_u \).

**Definition 26.** A connection on a principal \( Q \)-bundle is a smooth assignment to each point \( p \in P \) of a horizontal subspace \( \mathcal{H}_p \) of \( T_p(P) \) such that:

(i) \( T_p = \mathcal{V}_p \oplus \mathcal{H}_p \) (a direct sum).

(ii) The family of horizontal subspaces is invariant under the right map of the loop \( Q \); i.e., for any \( a \in Q, p \in P \)

\[
\mathcal{H}_{\hat{R}_a p} = (\hat{R}_a)_* \mathcal{H}_p,
\]

where \( \hat{R}_a \) denotes the right map by \( a \) on a tangent vector.

The connection allows us to decompose any vector \( Z \in T_p(P) \) in the form \( Z = X + Y \), where \( X = hor Z \in \mathcal{H}_p \) is the horizontal component of the vector \( Z \) and \( Y = ver Z \in \mathcal{V}_p \) is the vertical one. The map \( \varphi \) induces the map of the vertical subspace \( \mathcal{V}_p \) onto the tangent space to \( Q \), \( \mathcal{V}_p \xrightarrow{\varphi_*} T_q(Q), q = \varphi(p) \):

\[
\hat{V}_p = \frac{d}{dt} \hat{R}_u(t)\bigg|_{t=0} \xrightarrow{\varphi_*} \hat{V}_q = \frac{d}{dt} R_{a(t)}q \bigg|_{t=0}. \tag{3.18}
\]

Notice that the vector field \( \hat{V} \) is the left quasi-invariant vector field. Indeed, it can be written as \( \hat{V}_q = (L_q)_* V_e \), where

\[
V_e = \frac{da(t)}{dt} \bigg|_{t=0} \in T_e(Q). \tag{3.19}
\]

**Definition 27.** Let \( V_e \in T_e(Q) \). The vector field \( \hat{V} \) connecting with \( V_e \) by means of \((3.18), \ (3.19)\) is called a fundamental vector field.
The notion of connection may be reformulated in the following way:

**Definition 28.** A connection form on a principal $Q$-bundle is a vector-valued 1-form taking values at $T_e(Q)$, which satisfies:

(i) $\omega(X_p) = X_e$, where $X_p, X_e \in T_e(Q)$ are determined according to (3.18), (3.19).

(ii) $(\hat{R}_a \omega) X_p = \text{Ad}^{-1}(q) \omega(X_p)$, where $q = \varphi(p)$.

(iii) The horizontal subspace $\mathcal{H}_p$ is defined as a kernel of $\omega$:

$$\mathcal{H}_p = \{X_p \in T_p(P) : \omega(X_p) = 0\}.$$

A local 1-form taking values in $T_e(Q)$ can be associated with the given connection form as follows. Let $\sigma : U \subset \mathcal{M} \to \sigma(U) \subset P$, $\pi \circ \sigma = \text{id}$ be a local section of a $Q$-bundle $Q \to P \to \mathcal{M}$ that is equipped with a connection 1-form $\omega$. Define the local $\sigma$-representative of $\omega$ to be the vector-valued 1-form $\omega^U$ (taking values at $T_e(Q)$) on the open set $U \subset \mathcal{M}$ given by $\omega^U := \sigma^* \omega$.

**Theorem 29.** (On reconstruction of the connection form) For a given canonical $\text{Ad}$-form $\hat{\omega}$ defined on $U \subset \mathcal{M}$ with values in $T_e(Q)$ and a given section $\sigma : U \to \pi^{-1}(U)$, there exists one and only one connection 1-form $\omega$ on $\pi^{-1}(U)$ such that $\sigma^* \omega = \hat{\omega}$.

**Proof.** Let $p_0 = \sigma(x)$ and $Z \in T_{p_0}(P)$. We have $Z = X_1 + X_2$, where $X_1 := (\sigma \circ \pi_1) Z$ and $X_2 \in V_{p_0}$, $\pi_2 X_2 = 0$. Define at $p_0$ the 1-form $\omega$ to be the vector-valued 1-form given by $\omega_{p_0} = \hat{\omega}_x(\pi_2 X) + \hat{X}_2$. The continuation of the 1-form $\omega$ onto all points of the fiber is realized by means of right translations, namely, $\forall p \in P, \exists a \in Q : p = \hat{R}_a p_0$. This yields

$$\omega_p((\hat{R}_a)_* X) = \text{Ad}^{-1}(q_0) \omega_{p_0} (X), \quad q_0 = \varphi(p_0).$$

It is easy to see that the 1-form so obtained satisfies all the conditions of Definition 28.\hfill $\square$

This construction can be generalized to the whole base $\mathcal{M}$ and leads to the following theorem.

**Theorem 30.** Any smooth principal $Q$-bundle $(P, \pi, \mathcal{M})$ has a connection.

**Proof.** Let $\hat{\omega}$ be a differentiable 1-form on $\mathcal{M}$ with values in $T_e(Q)$ and $\{U_\alpha, \Phi_\alpha\}$ a family of local trivializations associated with the covering $\{U_\alpha\}$ of $\mathcal{M}$. For each local trivialization, there exists the local connection form $\omega_\alpha$ on $\pi^{-1}(U_\alpha)$ obtained from $\hat{\omega}$ by a choice of a section $\sigma_\alpha$ over $U_\alpha$ (see Theorem 29). Let $\lambda_\alpha$ be a partition of unity subordinate to the covering $\{U_\alpha\}$, $\lambda = 0$ outside of $U_\alpha$ and $\sum_\alpha \lambda_\alpha (x) = 1$ at an arbitrary point $x \in \mathcal{M}$. We form

$$\omega = \sum_\alpha (\lambda_\alpha \circ \pi) \omega_\alpha.$$
This is a 1-form on $P$ with values in $T_e(Q)$ and for each $\alpha$ we have $\omega\big|_{U_\alpha} = (\lambda_\alpha \circ \pi)\omega_\alpha$. It is easily seen that $\omega$ satisfies the conditions of the Definition 28 and thus this is a connection form on $P$.

3.2.2. Connection in the local trivialization

We consider a principal $Q$-bundle $(P, \pi, M)$ over $\mathfrak{M}$ and fix a trivializing covering $\{U_\alpha, \Phi_\alpha\}_{\alpha \in J}$. Let $\tilde{\Pi} : U_\alpha \to U_\alpha \times Q$ by $x \to (x, e)$. A trivialization $\Phi_\alpha$ defines a canonical section $\sigma_\alpha$ by the equation

$$\sigma_\alpha = \Phi_\alpha^{-1} \circ \tilde{\Pi},$$

and vice versa. Further, we denote by $\omega_C$ a canonical Ad-form.

**Definition 31.** Let $\omega_\alpha = \sigma_\alpha^* \omega$, where $\omega$ is the connection form. The form $\omega_\alpha$ on $U_\alpha$ is called the connection form in the local trivialization $\{U_\alpha, \Phi_\alpha\}$.

**Theorem 32.** Let $\{U_\alpha, \Phi_\alpha\}_{\alpha \in J}$ be a family of local trivializations for $P$ with $\bigcup_{\alpha \in J} U_\alpha = M$; then, on $U_\alpha \cap U_\beta$, the local connection forms $\omega_\alpha$ and $\omega_\beta$, corresponding to the same connection $\omega$ on $P$, are related by

$$\omega_\beta = \text{Ad}^{-1}_{q_{\alpha\beta}}(q_{\beta\alpha})\omega_\alpha + l_{(q_{\alpha\beta}, q_{\beta\alpha})} \theta_{\alpha\beta}, \tag{3.20}$$

where the $q_{\alpha\beta}$ are the transition functions, and $\theta_{\alpha\beta} = q_{\alpha\beta}^* \omega_C$ denotes the pullback on $U_\alpha \cap U_\beta$ of the canonical $1$-form $\omega_C$ on $Q$. Vice versa, for any set of the local forms $\{\omega_\alpha\}_{\alpha \in J}$ satisfying \([3.20]\), there exists the unique connection form $\omega$ on $P$ generating this family of local forms, namely $\omega_\alpha = \sigma_\alpha^* \omega$, $\forall \alpha \in J$.

**Proof.** 1. The direct theorem. Let $x \in U_\alpha \cap U_\beta$. Applying (3.2), we obtain $\sigma_\beta(x) = R_{q_{\beta\alpha}} \sigma_\alpha(x), \forall x \in U_\alpha \cap U_\beta$. The map $(\sigma_\beta(x))_*$ transforms any vector $X \in T_x(U_\alpha \cap U_\beta)$ into $(\sigma_\beta)_* X \in T_{\sigma_\beta(x)}(P)$. Using the Leibniz formula (see [11], Ch. I), we get

$$(\sigma_\beta)_* = (R_{q_{\beta\alpha}})_*(\sigma_\alpha)_* + (L_{q_{\beta\alpha}})_*(q_{\alpha\beta})_*,$$

where $q_{\beta\alpha} := \varphi_\beta(\sigma_\alpha)$. Applying $\omega$ to both sides of this relation, we find

$$\omega_\beta(X) := \omega((\sigma_\beta)_* X) = \omega((R_{q_{\beta\alpha}})_*(\sigma_\alpha)_* X) + \omega((L_{q_{\beta\alpha}})_*(q_{\alpha\beta})_* X)$$

$$= \text{Ad}^{-1}_{q_{\beta\alpha}}(q_{\beta\alpha})\omega_\alpha(X) + l_{(q_{\beta\alpha}, q_{\alpha\beta})} \theta_{\alpha\beta} \omega_C(X)$$

$$= \text{Ad}^{-1}_{q_{\beta\alpha}}(q_{\beta\alpha})\omega_\alpha(X) + l_{(q_{\beta\alpha}, q_{\alpha\beta})} \theta_{\alpha\beta} \omega_C(X). \tag{3.21}$$

2. The inverse theorem. Let us define the 1-form $\tilde{\omega}$ as follows:

$$\tilde{\omega} = \text{Ad}^{-1}_{q_{\alpha}}(e)(\pi^* \omega_\alpha) + q_{\alpha}^* \omega_C, \quad q_\alpha := \varphi_\alpha(p). \tag{3.22}$$

Let $X \in T_\alpha(P)$ be an arbitrary vector and $u = \sigma_\alpha(\pi(x))$. Decompose $X$ into horizontal $Y$ and vertical $Z$ components:

$$X = Y + Z, \quad Y = (\sigma_\alpha)_*(\pi_*(X)), \quad \pi_* Z = 0.$$
This yields
\[
\tilde{\omega}(X) = \text{Ad}_{q^{-1}_\alpha}(e)\omega_\alpha(\pi_*X) + (q^*_\alpha \omega_C)(X) = \text{Ad}_{q^{-1}_\alpha}(e)\omega((\sigma_\alpha)_*\pi_*X) + \omega_C((q_\alpha)_*X)
\]
\[
= (\tilde{R}^*_\alpha \sigma^*_\alpha)\omega(\pi_*Y) + \omega_C((q_\alpha)_*Z) = \omega((\sigma_\alpha)_*\pi_*Y) + \omega_C((q_\alpha)_*Z)
\]
\[
= \omega(Y) + \tilde{Z} = \omega(Y) + \omega(Z) = \omega(X),
\]
(3.23)
and one sees that \(\tilde{\omega} = \omega\) at any point of the section \(\sigma_\alpha\). So, these forms are transformed in the same way under the right translations and therefore coincide on \(\pi^{-1}(U_\alpha)\).

**Corollary 33.** For arbitrary sections \(\sigma_1\) and \(\sigma_2\) such that \(\sigma_2 = R_\varphi \sigma_1\) and \(\omega_1 = \sigma^*_1 \omega, \omega_2 = \sigma^*_2 \omega\), the following relation holds:
\[
\omega_2 = \text{Ad}^{-1}_q(q_1)\omega_1 + l_{(q_1,q)_*}(q^*_1 \omega_C),
\]
where \(q_1 := \varphi(\sigma_1)\).

### 3.3. Covariant derivative. Curvature form

Let \(\{x^\mu, y^j\}\) be a local coordinate system in the neighborhood \(\pi^{-1}(U_\alpha)\), where the \(x^\mu\) are coordinates in \(U_\alpha \subset M\) and the \(y^j\) are coordinates in the fiber. Locally \(\pi^{-1}(U_\alpha)\) can be presented as a direct product \(U_\alpha \times Q\). The connection form can be written in the form: \(\omega = \omega^i L_i\), with \(\{L_i\}\) being the basis of left fundamental fields and \(\{\omega^i\}\) the basis of 1-forms. Taking into account Eq. (3.22), we find that in the coordinates \(\{x^\mu, y^j\}\),
\[
\omega^i = (\text{Ad}^{-1}_y(e))_j^i A^i_j(x)dx^\mu + \omega^i_j(y)dy^j
\]
(3.24)
where \(A^i_j(x)dx^\mu = \pi^*(\omega^i)\) and \(\omega^i_j(y)dy^j = (L^{-1}_i)_j^i dy^j\).

**Definition 34.** A **covariant derivative** \(D_\mu\) in the principal \(Q\)-bundle is defined as follows:
\[
D_\mu = \partial_\mu - A^i_\mu(x)L_i,
\]
where the \(L_i = (R_*^\mu)_j^i \partial/\partial y^j\) are generators of the left translations (right quasi-invariant vector fields).

Now, we show that \(\omega(D_\mu) = 0\). Indeed, Eq. (3.24) implies
\[
\omega(D_\mu) = [(\text{Ad}^{-1}_y(e))_j^i A^i_j(x) - \omega^i_j(R_*^\mu)_j^i]L_i.
\]
Noting that \(\omega^i_\mu = (L^{-1}_i)_j^i\), we obtain \(\omega^i_\mu(R_*^\mu)_j^i = (\text{Ad}^{-1}_y(e))_j^i\), and hence \(\omega(D_\mu) = 0\).

The computation of the commutator \([D_\mu, D_\nu]\) yields
\[
[D_\mu, D_\nu] = (\partial_\nu A^i_\mu - \partial_\mu A^i_\nu)L_i + A^i_\mu A^j_\nu [L_i, L_j],
\]
and introducing \([\bar{L}_i, \bar{L}_j] = C^p_{ij}(y)\bar{L}_p\), we get
\[
[D_\mu, D_\nu] = -F^i_{\mu\nu}\bar{L}_i, \quad (3.25)
\]
\[
F^i_{\mu\nu} := \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - A^j_\mu A^i_\nu C^p_{jp}. \quad (3.26)
\]

**Definition 35.** Let \(\Psi\) be a vector-valued \(r\)-form in the principal \(Q\)-bundle. A \((r+1)\)-form \(D\Psi\) defined by
\[
D\Psi(X_1, X_2, \ldots, X_{r+1}) = d\Psi(horX_1, \ldots, horX_{r+1}) \quad (3.27)
\]
is called a covariant differential of the form \(\Psi\).

**Definition 36.** A vector-valued 2-form \(\Omega(X, Y)\) defined as
\[
\Omega(X, Y) = D\omega(X, Y) = d\omega(horX, horY) \quad (3.28)
\]
where \(\omega\) is a connection form, is called a curvature form.

**Lemma 37.** Let \(X, Y\) be horizontal fields, then the following relation holds:
\[
\omega([X, Y]) = -2\Omega(X, Y).
\]

**Proof.** Applying the exterior differentiation to the 1-form \(\omega\), we obtain
\[
2d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).
\]
As \(X, Y \in H_p\), then \(\omega(X) = \omega(Y) = 0\). This yields \(\omega([X, Y]) = -2\Omega(X, Y)\).

**Corollary 38.** The curvature form can be defined as follows:
\[
\Omega(X, Y) = -\frac{1}{2}\omega([hor\bar{Y}, hor\bar{Y}])
\]
where \(\bar{X}, \bar{Y}\) are any continuations of the vectors \(X, Y \in T_p(P)\), respectively.

**Corollary 39.** The 2-form \(\Omega\) is \(Ad\)-form and is transformed under the right translations as
\[
(\tilde{R}_q^*\Omega)(X, Y) = Ad_a^{-1}(q)\Omega|_p(X, Y), \quad \text{where} \ q = \varphi(p). \quad (3.29)
\]

**Theorem 40.** The curvature form \(\Omega\) satisfies the structure equation
\[
\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \quad (3.30)
\]

**Proof.** We are going to prove (3.30), considering all possible pairs \(X, Y\).
1. Let \(X, Y\) be any two horizontal vectors then \(\omega(X) = \omega(Y) = 0\) and we find
\[
d\omega(X, Y) + \frac{1}{2}[\omega(X), \omega(Y)] = d\omega(horX, horY) = \Omega(X, Y)
\]
2. Let $X, Y$ be any two vertical vectors. Without loss of generality, one can assume that $X, Y$ are fundamental vector fields. This implies $\omega(X) = \hat{X}$, $\omega(Y) = \hat{Y}$, $\hat{X}, \hat{Y} \in T_e(Q)$. The computation yields

$$2d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) = X(\hat{Y}) - Y(\hat{X}) - \omega([X,Y])$$

$$= -[\hat{X}, \hat{Y}] = -[\omega(X), \omega(Y)] = d\omega(X,Y) + \frac{1}{2}[\omega(X), \omega(Y)] = 0.$$

As $X, Y$ are the vertical fields, then $\Omega(X,Y) = d\omega(\text{hor}X, \text{hor}Y) = 0$.

3. Let $X$ be a horizontal vector field and $Y$ be a vertical one (or fundamental vector field). From the one side, we have

$$\Omega(X,Y) = d\omega(\text{hor}X, \text{hor}Y) = 0,$$

and from the other side,

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) = 0,$$

since the commutator $[X,Y]$ is the horizontal vector field.

The last statement arises from the following lemma.

**Lemma 41.** Let $Z$ be a fundamental vector field and $X$ be a horizontal vector field; then, the commutator $[X,Z]$ is a horizontal vector field.

**Proof.** The fundamental vector field is induced by the right translations $\tilde{R}_a(t)$. The commutator can be defined (see, e.g., [11]) by

$$[X,Z] = \lim_{t \to 0} \frac{1}{t}((\tilde{R}_a(t))_*X - X).$$

If $X$ is a horizontal vector field, then $(\tilde{R}_a(t))_*X$ is a horizontal vector field, and, hence, $[X,Z]$ is also horizontal.

Let us compute $\Omega_{\mu\nu} = 2\Omega(D_\mu, D_\nu)$. The result is given by

$$\Omega_{\mu\nu} = -\omega([D_\mu, D_\nu]) = F_{\mu\nu}^i \omega(\hat{L}_i) = F_{\mu\nu}^i (Ad_q^{-1}(e))_{ij} \hat{L}_j$$

where $\{\hat{L}_j = \omega(L_j)\}$ is the basis of the left quasi-invariant vector fields at $T_e(Q)$, $q = \varphi(p)$, $p \in P$. Choosing the family of local sections $\sigma_\alpha$ associated with the trivialization $U_\alpha, \Phi_\alpha$ and taking into account that $\varphi_\alpha(\sigma_\alpha) = e$ where $\varphi_\alpha$ is the restriction of $\Phi_\alpha$ on $\pi^{-1}(U_\alpha)$, we obtain

$$\pi^*\Omega_\alpha = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu. \quad (3.33)$$

Introducing $F_{\mu\nu} = F_{\mu\nu}^i L_i$.

Since $\Omega$ is a $Ad$-form, the following transformation law is true:

$$\Omega_{\beta\gamma} = (Ad_{q_{3\alpha}}^{-1}(q_{3\beta})) \Omega_{\alpha} \quad (3.34)$$

**Theorem 42.** Bianchi identity: $D\Omega = 0$. 

Proof. It is sufficient to show that $d\Omega(X,Y,Z) = 0$, if $X,Y,Z$ are horizontal vector fields. Applying the exterior derivative to (3.30), we obtain $d\Omega(X,Y,Z) = 0$, if $X,Y,Z$ are horizontal vector fields.

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