Optimizing certain combinations of spectral and linear/distance functions over spectral sets

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Abstract

In the settings of Euclidean Jordan algebras, normal decomposition systems (or Eaton triples), and structures induced by complete isometric hyperbolic polynomials, we consider the problem of optimizing a certain combination (such as the sum) of spectral and linear/distance functions over a spectral set. To present a unified theory, we introduce a new system called Fan-Theobald-von Neumann system which is a triple $(V, W, \lambda)$, where $V$ and $W$ are real inner product spaces and $\lambda : V \to W$ is a norm preserving map satisfying a Fan-Theobald-von Neumann type inequality together with a condition for equality. In this general setting, we show that optimizing a certain combination of spectral and linear/distance functions over a set of the form $E = \lambda^{-1}(Q)$ in $V$, where $Q$ is a subset of $W$, is equivalent to optimizing a corresponding combination over the set $\lambda(E)$ and relate the attainment of the optimal value to a commutativity concept. We also study related results for convex functions in place of linear/distance functions. Particular instances include the classical results of Fan and Theobald, von Neumann, results of Tam, Lewis, and Bauschke et al., and recent results of Ramírez et al. As an application, we present a commutation principle for variational inequality problems over such a system.

Key Words: Fan-Theobald-von Neumann system, Euclidean Jordan algebra, normal decomposition system, Eaton triple, hyperbolic polynomial, spectral set, eigenvalue map, strong operator commutativity, variational inequality problem

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1 Introduction

Let \( \mathcal{V} \) and \( \mathcal{W} \) be two real inner product spaces, \( \lambda : \mathcal{V} \to \mathcal{W} \) be a map, \( Q \) be a subset of \( \mathcal{W} \), and \( E := \lambda^{-1}(Q) \). In analogy with certain concepts in Euclidean Jordan algebras, we say that \( \lambda \) is an eigenvalue map and \( E \) is a spectral set; a function \( \Phi : \mathcal{V} \to \mathbb{R} \) is said to be a spectral function if it is of the form \( \Phi = \phi \circ \lambda \) for some function \( \phi : \mathcal{W} \to \mathbb{R} \). Given a real valued function on \( \mathcal{V} \), we are interested in reformulating the problem of optimizing (that is, finding the infimum/supremum/minimum/maximum of) this function over \( E \) equivalently as a problem of optimizing a related function over \( \lambda(E) \) with the expectation that the latter problem is relatively easy to solve and/or gives some information on the former problem. For example, if \( E \) is a spectral set and \( \Phi \) is a spectral function on \( \mathcal{V} \), then a mere change of variable will show that such a reformulation holds. The main objective of this paper is to show that in some settings, such a reformulation can be carried out for certain combinations of \( \Phi \) and linear/distance/convex functions.

To elaborate, let \( c \in \mathcal{V} \) and consider the linear function \( f(x) := \langle c, x \rangle \) and the distance function \( g(x) := ||c - x|| \) over \( \mathcal{V} \). Define the corresponding linear and distance functions over \( \mathcal{W} \): \( f^*(w) = \langle \lambda(c), w \rangle \) and \( g^*(w) = ||\lambda(c) - w|| \). Then, under certain conditions on the triple \( (\mathcal{V}, \mathcal{W}, \lambda) \) – thus defining a new system called Fan-Theobald-von Neumann system – we will show that

\[
\sup_{E} (f + \Phi) = \sup_{\lambda(E)} (f^* + \phi) \quad \text{and} \quad \inf_{E} (g + \Phi) = \inf_{\lambda(E)} (g^* + \phi),
\]

with similar statements where ‘supremum’ is replaced by ‘maximum’ and ‘infimum’ by ‘minimum’. Additionally, we relate the attainment in each of these problems to a condition of the form

\[
\langle c, x \rangle = \langle \lambda(c), \lambda(x) \rangle,
\]

thus defining the concept of commutativity in this system. We also handle problems of the form \( \inf_{E}(f + \Phi) \) and \( \sup_{E}(g + \Phi) \) in a similar way and consider replacing the sum by other combinations, and replacing linear and distance functions by convex functions. By specializing, we show that all these hold in Euclidean Jordan algebras [10], normal decomposition systems [20] (in particular, Eaton triples [7]), and certain structures induced by complete hyperbolic polynomials [2].

As a simple illustration, consider the following optimization problem stated in the setting of \( \mathcal{S}^n \), the (Euclidean Jordan) algebra of all \( n \times n \) real symmetric matrices: for a given \( C \in \mathcal{S}^n (n \geq 2) \), find

\[
\max \left \{ \langle C, X \rangle : X \succeq 0, 1 \leq \lambda_{\text{max}}(X) \leq 2 \right \},
\]

where \( X \succeq 0 \) means that \( X \) is positive semidefinite and \( \lambda_{\text{max}}(X) \) is the maximum eigenvalue of \( X \). Here, the objective function is linear and the constraint set (defined by eigenvalues) is nonconvex. This problem turns out to be equivalent to finding

\[
\max \left \{ \langle \lambda(C), q \rangle : q = (q_1, q_2, \ldots, q_n) \in \mathcal{R}_+^n : q_1 \geq q_2 \geq \cdots \geq q_n, 1 \leq q_1 \leq 2 \right \},
\]
where $\lambda(C)$ is the vector of eigenvalues of $C$ written in the decreasing order. Clearly, the latter problem, stated in $\mathcal{R}^n$, is easier to solve than the former. In this reformulation, $\mathcal{V} = \mathcal{S}^n$, $\mathcal{W} = \mathcal{R}^n$, $\lambda : \mathcal{V} \to \mathcal{R}^n$ is the eigenvalue map that takes any real symmetric matrix to its vector of eigenvalues written in the decreasing order, $E = \{X \in \mathcal{S}^n : X \succeq 0, 1 \leq \lambda_{\max}(X) \leq 2\}$, and $\Phi = 0$. Another illustrative example in the setting of $\mathcal{S}^n$ is the problem $\inf_E (f + \Phi)$, where $f(X) = \langle C, X \rangle$, $\Phi(X) = -\log \det(X)$, and $E$ is an appropriate spectral set.

A Fan-Theobald-von Neumann system (FTvN system, for short) introduced in this paper is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces and $\lambda : \mathcal{V} \to \mathcal{W}$ is a norm preserving map satisfying the property

$$\max \left\{ \langle c, x \rangle : x \in [u] \right\} = \langle \lambda(c), \lambda(u) \rangle \quad (c, u \in \mathcal{V}),$$

with $[u] := \lambda^{-1} \{\lambda(u)\}$ denoting the (so-called) $\lambda$-orbit of $u$, see Section 2 for an elaborated version and a formulation in terms of the distance function. The above property can be regarded as a combination of Fan-Theobald-von Neumann type inequality, namely, $\langle c, x \rangle \leq \langle \lambda(c), \lambda(x) \rangle$, and a “commutativity” condition for equality. Perhaps, the simplest nontrivial example of such a system is the triple $(\mathcal{V}, \mathcal{R}^n, \lambda)$, where $\mathcal{V}$ is a real inner product space and $\lambda$ denotes the corresponding norm; in this case, the defining property reduces to the Cauchy-Schwarz inequality together with a condition for equality. When $\mathcal{V}$ is a Euclidean Jordan algebra of rank $n$ carrying the trace inner product with $\lambda : \mathcal{V} \to \mathcal{R}^n$ denoting the eigenvalue map, the triple $(\mathcal{V}, \mathcal{R}^n, \lambda)$ becomes a FTvN system. More generally, if $\mathcal{V}$ is a finite dimensional real vector space and $p$ is a real homogeneous polynomial of degree $n$ on $\mathcal{V}$ that is hyperbolic with respect to an element $e \in \mathcal{V}$, complete, and isometric [2], then $(\mathcal{V}, \mathcal{R}^n, \lambda)$ becomes a FTvN system, where for any element $x$ in $\mathcal{V}$, $\lambda(x)$ denotes the vector of roots of the univariate polynomial $t \to p(te - x)$ written in the decreasing order. Also, when $(\mathcal{V}, \mathcal{G}, \gamma)$ is a normal decomposition system (in particular, an Eaton triple) with $\mathcal{G}$ denoting a closed subgroup of the orthogonal group of a real inner product space $\mathcal{V}$ and $\gamma : \mathcal{V} \to \mathcal{V}$ satisfying some specified properties, the triple $(\mathcal{V}, \mathcal{W}, \gamma)$ becomes a FTvN system, where $\mathcal{W} := \text{span}(\gamma(\mathcal{V}))$.

The motivation for our work comes from several results mentioned below.

- For two $n \times n$ complex Hermitian matrices $C$ and $A$ with eigenvalues $c_1 \geq c_2 \geq \cdots \geq c_n$ and $a_1 \geq a_2 \geq \cdots \geq a_n$, a classical result of Fan [9] (see also [37, 35]) states that

$$\max \left\{ \text{tr}(CUAU^*) : U \in C^{n \times n} \text{ is unitary} \right\} = \sum_{i=1}^{n} c_i a_i,$$

where ‘tr’ refers to the trace. By working in the Euclidean Jordan algebra $\mathcal{H}^n$ (of all $n \times n$ complex Hermitian matrices) with $\langle X, Y \rangle = \text{tr}(XY)$ and $\lambda(X)$ denoting the eigenvalues of $X$ written in the decreasing order, this result can be viewed as describing the maximum of
the linear function $f(X) := \langle C, X \rangle$ over the eigenvalue orbit $E = \lambda^{-1}(\{\lambda(A)\}) = \{X \in \mathcal{H}^n : \lambda(X) = \lambda(A)\}$ with the optimal value given by $\langle \lambda(C), \lambda(A) \rangle$. Furthermore, the attainment is described in the form of simultaneous value eigenvalue decomposition.

- A result of von Neumann deals with two $n \times n$ complex matrices $C$ and $A$ with singular values $c_1 \geq c_2 \geq \cdots \geq c_n$ and $a_1 \geq a_2 \geq \cdots \geq a_n$. It asserts that
  \[ \max \left\{ \text{Re} \text{tr}(CUA^*V) : U, V \in \mathbb{C}^{n \times n} \text{ are unitary} \right\} = \sum_{i=1}^{n} c_i a_i. \]

By working in the normal decomposition system (Eaton triple) corresponding to the space $M_n$ (of all $n \times n$ complex matrices) with $\lambda(X)$ denoting the singular values of $X$ written in the decreasing order, and $\langle X, Y \rangle = \text{Re} \text{tr}(XY^*)$, we can view this result as describing the maximum of the linear function $f(X) := \langle C, X \rangle$ over the singular value orbit $E = \lambda^{-1}(\{\lambda(A)\}) = \{X \in M_n : \lambda(X) = \lambda(A)\}$ with the optimal value given by $\langle \lambda(C), \lambda(A) \rangle$. Here, the attainment is described in the form of simultaneous order singular decomposition.

- In [3], Chu and Driessel considered the problems of minimizing the distance function $||C - X||$ over the eigenvalue orbit of a matrix $A$ in $\mathcal{H}^n$ and over the singular value orbit of a matrix $A$ in $M_n$. Their results were refined by Tam ([35], Corollaries 2.2 and 2.3), Tam and Hill ([36], Theorem 27) by working in the setting of Lie algebras/Eaton triples. Related works [24, 17] deal with distance to the convex hull of eigenvalue/singular value orbits.

- In the setting of a normal decomposition system $(\mathcal{V}, \mathcal{G}, \gamma)$, Lewis ([20], Proposition 2.3 and Theorem 2.4) describes the property
  \[ \max_{A \in \mathcal{G}} \langle Ax, y \rangle = \langle \gamma(x), \gamma(y) \rangle \]
  with a condition for equality: $\langle x, y \rangle = \langle \gamma(x), \gamma(y) \rangle$ if and only if there exists an $A \in \mathcal{G}$ such that $x = A\gamma(x)$ and $y = A\gamma(y)$. This property can be viewed as a statement on maximizing a linear function over an orbit of the form $\{Ax : A \in \mathcal{G}\}$.

- In the setting of a triple $(\mathcal{V}, \mathcal{R}^n, \lambda)$ induced by a complete hyperbolic polynomial $p$ on a finite dimensional real vector space $\mathcal{V}$, Bauschke et al., [2] introduce the concept of ‘isometric hyperbolic polynomial’ and state a result ([2], Proposition 5.3) describing the maximum of a linear function over a $\lambda$-orbit. In this result, the optimality condition is given in the form $\lambda(x + y) = \lambda(x) + \lambda(y)$.

- In the setting of a simple Euclidean Jordan algebra $\mathcal{V}$ of rank $n$ carrying the trace inner product with $\lambda : \mathcal{V} \to \mathcal{R}^n$ denoting the eigenvalue map, it is known that for any $c \in \mathcal{V}$,
  \[ \max \left\{ \langle c, x \rangle : x \in \mathcal{J}^k(\mathcal{V}) \right\} = \lambda_1(c) + \lambda_2(c) + \cdots + \lambda_k(c), \]
  \[ (3) \]
  where $\mathcal{J}^k(\mathcal{V})$ is the set of all idempotents of rank $k$ in $\mathcal{V}$, $1 \leq k \leq n$, see [28], Theorem 17.
This result can be viewed as a statement on maximizing a linear function over the \( \lambda \)-orbit of (any) one idempotent of rank \( k \).

- In [31], Ramírez, Seeger, and Sossa formulate a commutation principle in the setting of Euclidean Jordan algebras: If \( a \) is a local optimizer of the problem
\[
\min/\max \{ h(x) + \Phi(x) : x \in E \},
\]
where \( E \) is a spectral set, \( \Phi \) is a spectral function, and \( h \) is Fréchet differentiable, then \( a \) and \( h'(a) \) operator commute. Based on this, they present a commutation principle for a variational inequality problem and consider the problem of describing the distance to a spectral set. See [12] for a slight weakening of the conditions and a similar result proved in the setting of normal decomposition systems. See also [30] for certain elaborations and applications.

Our contributions in this paper are as follows. Motivated by the above results, we formulate the definition of a Fan-Theobald-von Neumann system and study some of its basic properties. We show that Euclidean Jordan algebras, normal decomposition systems (in particular, Eaton triples), and structures induced by complete isometric hyperbolic polynomials are particular instances. In this general framework, we describe results of the form (1) which extend/recover many of the above mentioned results. We also introduce the concept of commutativity in a FTvN system that encompasses (or extends) the concepts of simultaneous order eigenvalue/singular value decompositions and operator commutativity. Additionally, we present a commutation principle for a variational inequality problem in a FTvN system that even strengthens the commutation principle of Ramírez, Seeger, and Sossa ([31], Proposition 8) stated in the setting of Euclidean Jordan algebras.

The outline of the paper is as follows. In Section 2, we describe FTvN systems and present some basic properties. Section 3 deals with optimizations problems coming from certain combinations of spectral and linear/distance/convex functions, and a commutation principle for variational inequality problems. Section 4 deals with Euclidean Jordan algebras and some specialized results. Section 5 deals with an FTvN system induced by hyperbolic polynomials. In Section 6, we cover normal decomposition systems and Eaton triples.

## 2 Fan-Theobald-von Neumann system

Motivated by the results of Fan, Theobald, and von Neumann mentioned in the Introduction, we now formulate the definition of a Fan-Theobald-von Neumann system. Let \( \mathcal{V} \) and \( \mathcal{W} \) be two real inner product spaces where, for convenience, we use the same inner product (and norm) notation. Let \( \lambda : \mathcal{V} \to \mathcal{W} \) be a map. We define the \( \lambda \)-orbit of an element \( u \in \mathcal{V} \) as the set
\[
[u] := \{ x \in \mathcal{V} : \lambda(x) = \lambda(u) \}.
\]
We have the following elementary result.

**Proposition 2.1** Suppose \(|\langle \lambda(x) \rangle| = |x|\) for all \(x \in \mathcal{V}\). Then, for any \(u, c \in \mathcal{V}\), the following are equivalent:

(a) \(\max \left\{ \langle c, x \rangle : x \in [u] \right\} = \langle \lambda(c), \lambda(u) \rangle\).

(b) \(\min \left\{ ||c - x|| : x \in [u] \right\} = ||\lambda(c) - \lambda(u)||\).

**Proof.** Fix \(u, c \in \mathcal{V}\) and let \(x \in [u]\). Then, \(||x|| = ||\lambda(x)|| = ||\lambda(u)||\). Hence,

\[ ||c - x||^2 = ||c||^2 + ||x||^2 - 2\langle c, x \rangle = ||\lambda(c)||^2 + ||\lambda(u)||^2 - 2\langle c, x \rangle. \]

As \(x\) varies over \([u]\), we have

\[ \min ||c - x||^2 = ||\lambda(c)||^2 + ||\lambda(u)||^2 - 2\max \langle c, x \rangle. \]

Comparing this to

\[ ||\lambda(c) - \lambda(u)||^2 = ||\lambda(c)||^2 + ||\lambda(u)||^2 - 2\langle \lambda(c), \lambda(u) \rangle \]

we see that (a) holds if and only if (b) holds.

\[ \Box \]

Note that condition (a) deals with the inner product induced linear function \(x \rightarrow \langle c, x \rangle\) and attainment of its maximum over the \(\lambda\)-orbit \([u]\). We now define a Fan-Theobald-von Neumann system as a triple \((\mathcal{V}, \mathcal{W}, \lambda)\) where \(||\lambda(x)|| = ||x||\) for all \(x \in \mathcal{V}\) and condition (a) in the above proposition holds for all \(c, u \in \mathcal{V}\). An equivalent definition is given in the following expanded form.

**Definition 2.2** A Fan-Theobald-von Neumann system (FTvN system, for short) is a triple \((\mathcal{V}, \mathcal{W}, \lambda)\), where \(\mathcal{V}\) and \(\mathcal{W}\) are real inner product spaces and \(\lambda : \mathcal{V} \rightarrow \mathcal{W}\) is a map satisfying the following conditions:

(A1) \(||\lambda(x)|| = ||x||\) for all \(x \in \mathcal{V}\).

(A2) \(\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle\) for all \(x, y \in \mathcal{V}\).

(A3) For any \(c \in \mathcal{V}\) and \(q \in \lambda(\mathcal{V})\), there exists \(x \in \mathcal{V}\) such that

\[ \lambda(x) = q \quad \text{and} \quad \langle c, x \rangle = \langle \lambda(c), \lambda(x) \rangle. \]  

(A4) \(\lambda\) is a linear isometry on \(\mathcal{V}\). Cartesian product of a finite number of FTvN systems can be made into an FTvN system in an obvious
way (by considering the sum of inner products and creating a $\lambda$ in a componentwise manner). We remark that conditions (A1) and (A2) need not imply (A3); see Section 4 for an example.

In the next several results, we state some basic properties that hold in a FTvN system. Some of these are elementary, and some proofs are modeled after similar ones existing in the literature [20, 2].

In a FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, for any $c \in \mathcal{V}$, we define

$$\tilde{\lambda}(c) := -\lambda(-c).$$

**Proposition 2.3** In a FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, the following statements hold for all $x, y, c, u \in \mathcal{V}$:

(a) $\lambda(\alpha x) = \alpha \lambda(x)$ for all $\alpha \geq 0$.

(b) $(\tilde{\lambda}(c), \lambda(x)) \leq \langle c, x \rangle \leq (\lambda(c), \lambda(x))$.

(c) $||\lambda(c) - \lambda(x)|| \leq ||c - x|| \leq ||\tilde{\lambda}(c) - \lambda(x)||$.

(d) $\min \left\{ \langle c, x \rangle : x \in [u] \right\} = (\tilde{\lambda}(c), \lambda(u))$.

(e) $\max \left\{ ||c - x|| : x \in [u] \right\} = ||\tilde{\lambda}(c) - \lambda(u)||$.

**Proof.** We will use conditions (A1) – (A3) in Definition 2.2.

(a) Let $\alpha \geq 0$. Using (A1) and (A2), we have

$$||\lambda(\alpha x) - \alpha \lambda(x)||^2 = ||\lambda(\alpha x)||^2 + \alpha^2 ||\lambda(x)||^2 - 2\alpha \langle \lambda(\alpha x), \lambda(x) \rangle \leq ||\alpha x||^2 + \alpha^2 ||x||^2 - 2\alpha \langle \alpha x, x \rangle = 0,$$

leading to the given statement.

(b) The first inequality is obtained from (A2) by putting $y = -c$. The second inequality is just (A2) with $y = c$.

(c) Since $||\lambda(c) - \lambda(x)||^2 - ||c - x||^2 = 2[\langle c, x \rangle - \langle \lambda(c), \lambda(x) \rangle]$ and $||\tilde{\lambda}(c) - \lambda(x)||^2 - ||c - x||^2 = 2[\langle c, x \rangle - \langle \tilde{\lambda}(c), \lambda(x) \rangle]$, the inequalities in (c) follow from Item (b).

(d) This is seen by replacing $c$ by $-c$ in Item (a) of Proposition 2.1, which holds because it is equivalent to (A2) and (A3).

(e) Let $x \in [u]$. We have, as in the proof of Proposition 2.1

$$||c - x||^2 = ||c||^2 + ||x||^2 - 2\langle c, x \rangle = ||\tilde{\lambda}(c)||^2 + ||\lambda(u)||^2 - 2\langle \lambda(u), x \rangle.$$

Then, as $x$ varies over $[u]$, we have

$$\max ||c - x||^2 = ||\tilde{\lambda}(c)||^2 + ||\lambda(u)||^2 - 2\min \langle c, x \rangle$$

which, by Item (d), equals $||\tilde{\lambda}(c)||^2 + ||\lambda(u)||^2 - 2\langle \tilde{\lambda}(c), \lambda(u) \rangle = ||\tilde{\lambda}(c) - \lambda(u)||^2$. 

Note: All linear functions from $\mathcal{V}$ to $\mathbb{R}$ considered in this paper are of the form $x \mapsto \langle c, x \rangle$ for
some \( c \in \mathcal{V} \). (These are continuous linear functionals on \( \mathcal{V} \) and when \( \mathcal{V} \) is a Hilbert space, every continuous linear functional arises this way.) As the map \( \lambda \) is Lipschitz continuous (see Item (c) in the above proposition) and norm preserving, every \( \lambda \)-orbit is closed in \( \mathcal{V} \) and lies on a sphere with origin as the center; it is compact when \( \mathcal{V} \) is finite dimensional. Throughout this paper, depending on the context, we use the same notation to denote an optimization problem as well as its optimal value. For example, \( \sup_E f \) denotes the problem of finding/describing the supremum of \( f \) over the set \( E \) as well as the supremum value.

In the setting of a FTvN system, the following statements are simple consequences of convexity/concavity of a linear function and convexity of a distance function; they are based on Propositions 2.1 and 2.3. Here, for a set \( S \), we let \( \text{conv}(S) \) denote the convex hull of \( S \).

- \( \max \left\{ \langle c, x \rangle : x \in \text{conv}([u]) \right\} = \max \left\{ \langle c, x \rangle : x \in [u] \right\} = \langle \lambda(c), \lambda(u) \rangle \) and
- \( \min \left\{ \langle c, x \rangle : x \in \text{conv}([u]) \right\} = \min \left\{ \langle c, x \rangle : x \in [u] \right\} = \langle \lambda(c), \lambda(u) \rangle \).
- \( \max \left\{ \| c - x \| : x \in \text{conv}([u]) \right\} = \max \left\{ \| c - x \| : x \in [u] \right\} = \| \lambda(c) - \lambda(u) \| \).

**Theorem 2.4** (Sublinearity theorem) Let \( (\mathcal{V}, \mathcal{W}, \lambda) \) be a FTvN system. Then, for any \( w \in \lambda(\mathcal{V}) \), the function \( x \mapsto \langle w, \lambda(x) \rangle \) is sublinear, that is, for all \( c, x, y \in \mathcal{V} \), we have:

\[
\langle \lambda(c), \lambda(x + y) \rangle \leq \langle \lambda(c), \lambda(x) \rangle + \langle \lambda(c), \lambda(y) \rangle.
\]  

(5)

Consequently, \( \| \lambda(x + y) \| \leq \| \lambda(x) + \lambda(y) \| \).

**Proof.** Fix \( c, x, y \in \mathcal{V} \). For any \( z \in [c] \), we have \( \lambda(z) = \lambda(c) \) and

\[
\langle z, x + y \rangle = \langle z, x \rangle + \langle z, y \rangle \leq \langle \lambda(z), \lambda(x) \rangle + \langle \lambda(z), \lambda(y) \rangle = \langle \lambda(c), \lambda(x) \rangle + \langle \lambda(c), \lambda(y) \rangle.
\]

Taking the maximum over \( z \) and noting \( \max \left\{ \langle z, x + y \rangle : z \in [c] \right\} = \langle \lambda(c), \lambda(x + y) \rangle \) we get (5). Now, letting \( c = x + y \) in (5), we have

\[
\| \lambda(x + y) \|^2 = \langle \lambda(x + y), \lambda(x + y) \rangle \leq \langle \lambda(x + y), \lambda(x) + \lambda(y) \rangle \leq \| \lambda(x + y) \| \| \lambda(x) + \lambda(y) \|,
\]

leading to \( \| \lambda(x + y) \| \leq \| \lambda(x) + \lambda(y) \| \). \( \square \)

The concept of commutativity, defined below, is central to the study of FTvN systems.

**Definition 2.5** In a FTvN system \( (\mathcal{V}, \mathcal{W}, \lambda) \) we say that elements \( x, y \in \mathcal{V} \) commute if

\[
\langle x, y \rangle = \langle \lambda(x), \lambda(y) \rangle.
\]

As we shall see later, this concept, specialized to Euclidean Jordan algebras, is related to (in fact, stronger than) operator commutativity, which, in the settings of \( \mathcal{S}^n \) and \( \mathcal{H}^n \) reduces to the commu-
tativity of two matrices. In the presence of (A2), we can now interpret condition (A3) in Definition 2.2: Every element \( c \) in \( V \) commutes with some element in any given \( \lambda \)-orbit. Alternatively, defining

\[
C(x) := \{ y \in V : y \text{ commutes with } x \},
\]

(A3) says that \( C(c) \cap [u] \neq \emptyset \) for all \( c, u \in V \). We also note that in the optimization problem

\[
\max \left\{ \langle c, x \rangle : x \in [u] \right\}
\]

whose optimal value is \( \langle \lambda(c), \lambda(u) \rangle \), the objective function attains its maximum at an \( x^* \) if and only if \( x^* \) commutes with \( c \). Because of this, the concept of commutativity can be viewed as (part of) an optimality condition. This may explain why commutativity comes up in various optimization and variational inequality settings.

The following result describes commutativity in alternate ways.

**Proposition 2.6** In a FTvN system \((V, W, \lambda)\), the following are equivalent:

(a) \( x \) and \( y \) commute, that is, \( \langle x, y \rangle = \langle \lambda(x), \lambda(y) \rangle \).

(b) \( ||\lambda(x) - \lambda(y)|| = ||x - y|| \).

(c) \( ||\lambda(x + y)|| = ||\lambda(x) + \lambda(y)|| \).

(d) \( \lambda(x + y) = \lambda(x) + \lambda(y) \).

**Proof.** Using (A1), we get the equalities

\[
||\lambda(x) - \lambda(y)||^2 - ||x - y||^2 = 2[\langle x, y \rangle - \langle \lambda(x), \lambda(y) \rangle] = ||\lambda(x + y)||^2 - ||\lambda(x) + \lambda(y)||^2.
\]

The equivalence \( (a) \iff (b) \iff (c) \) follows.

(c) \( \Rightarrow \) (d): When (c) holds, we have the equality in the Cauchy-Schwarz inequality (6). Hence, one of the vectors in the (in)equality is a nonnegative multiple of the other vector. Since \( ||x + y|| = ||\lambda(x + y)|| = ||\lambda(x) + \lambda(y)|| \), (d) follows.

Finally, (d) \( \Rightarrow \) (c) \( \Rightarrow \) (a). \( \square \)

Arguments similar to the above will show that

\[
x \text{ commutes with } -c \iff \langle c, x \rangle = \langle \lambda(c), \lambda(x) \rangle \iff ||c - x|| = ||\lambda(c) - \lambda(x)||.
\]

(7)

For ease of reference, we collect various statements equivalent to (A3).

**Proposition 2.7** In a FTvN system, the following hold:

(a) For any \( c \in V \) and \( q \in \lambda(V) \), there exists \( x \in V \) such that \( \lambda(x) = q \) and satisfying one/all of the following conditions:

\[
\langle c, x \rangle = \langle \lambda(c), \lambda(x) \rangle, \quad ||c - x|| = ||\lambda(c) - \lambda(x)||, \quad \lambda(c + x) = \lambda(c) + \lambda(x).
\]
(b) For any $c \in V$ and $q \in \lambda(V)$, there exists $x \in V$ such that $\lambda(x) = q$ and satisfying one/both of the following conditions:

$$\langle c, x \rangle = \langle \tilde{\lambda}(c), \lambda(x) \rangle, \quad ||c - x|| = ||\tilde{\lambda}(c) - \lambda(x)||.$$  

Proof. (a) This follows from (A3) and Proposition 2.6.

(b) We replace $c$ in (A3) by $-c$ and use (7). □

A simple consequence of the above result is the following.

**Corollary 2.8** If $(V, W, \lambda)$ is a FTvN system, then $\lambda(V)$ is a convex cone in $W$; It is closed when $V$ is finite dimensional.

Proof. In view of Item (a) in Proposition 2.3, $\lambda(V)$ is a cone. If $\lambda(u)$ and $\lambda(v)$ are two elements in $\lambda(V)$, then, applying Item (a) in the above proposition with $q = \lambda(u)$ and $c = v$, we get an $x \in V$ such that $\lambda(x) = q = \lambda(u)$ and $\lambda(x + c) = \lambda(x) + \lambda(c) = \lambda(u) + \lambda(v)$. Hence, $\lambda(u) + \lambda(v) \in \lambda(V)$. Thus, $\lambda(V)$ is a convex cone. Finally, if $V$ is finite dimensional, we can use (A1) and the continuity of $\lambda$ to show that $\lambda(V)$ is closed. □

Motivated by certain concepts in Euclidean Jordan algebras, we now introduce the following.

**Definition 2.9** Let $(V, W, \lambda)$ be a FTvN system.

- A set $E$ in $V$ is called a spectral set if it is of the form $E = \lambda^{-1}(Q)$ for some $Q \subseteq W$.
- A function $\Phi : V \to \mathbb{R}$ is said to be a spectral function if it is of the form $\Phi = \phi \circ \lambda$ for some $\phi : W \to \mathbb{R}$.

It is clear that a spectral set is a union of $\lambda$-orbits. An intrinsic test for a spectral set is the validity of the following implication:

$$[x \in E, \lambda(x) = \lambda(y)] \Rightarrow y \in E.$$  

(Then, we can let $Q := \lambda(E)$ so that $E = \lambda^{-1}(Q)$.) Also, a (real valued) function on $V$ is a spectral function if and only if it is a constant on any $\lambda$-orbit.

Previously, we listed some elementary examples. In the FTvN system $(V, W, \lambda)$, where $W = \mathbb{R}$ and $\lambda(x) = ||x||$, two elements commute if and only if one of them is a nonnegative (scalar) multiple of the other. Also, $\lambda$-orbits are spheres centered at the origin and spectral functions are radial. In the FTvN system $(V, V, S)$, where $S$ is a linear isometry, any two elements commute. In fact, due to Item (a) in Proposition 2.3 and Item (d) in Proposition 2.6 every FTvN system where any two elements commute arises this way.
In the subsequent sections, we will provide nontrivial examples of FTvN systems. In particular, we will show/see the following:

- If $\mathcal{V}$ is a Euclidean Jordan algebra of rank $n$ carrying the trace inner product and $\lambda : \mathcal{V} \to \mathbb{R}^n$ denotes the eigenvalue map, then the triple $(\mathcal{V}, \mathbb{R}^n, \lambda)$ becomes a FTvN system. In this setting, a set in $\mathcal{V}$ is a spectral set if it is of the form $\lambda^{-1}(Q)$ for some (permutation invariant) set $Q$ in $\mathbb{R}^n$; a function $\Phi : \mathcal{V} \to \mathbb{R}$ is a spectral function if it is of the form $\phi \circ \lambda$ for some (permutation invariant) function $\phi : \mathbb{R}^n \to \mathbb{R}$. When $\mathcal{V}$ is simple, these are precisely sets and functions that are invariant under automorphisms of $\mathcal{V}$. Commutativity of elements $x$ and $y$ in the FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$ means that there is a Jordan frame $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ in $\mathcal{V}$ such that $x$ and $y$ have simultaneous order diagonal decomposition with respect to $\mathcal{E}$, that is, $x = \lambda_1(x)e_1 + \lambda_2(x)e_2 + \cdots + \lambda_n(x)e_n$ and $y = \lambda_1(y)e_1 + \lambda_2(y)e_2 + \cdots + \lambda_n(y)e_n$. This will be referred to as the **strong operator commutativity** in the algebra $\mathcal{V}$. The algebras of $n \times n$ real/complex Hermitian matrices are primary examples of Euclidean Jordan algebras of rank $n$.

- If $\mathcal{V}$ is a finite dimensional real vector space and $p$ is a real homogeneous polynomial of degree $n$, hyperbolic with respect to a vector $e \in \mathcal{V}$, and additionally complete and isometric [2], then $(\mathcal{V}, \mathbb{R}^n, \lambda)$ becomes a FTvN system, where $\lambda(x)$ denotes the vector of roots of the univariate polynomial $t \to p(te - x)$ written in the decreasing order. In this setting, elements $x$ and $y$ commute if and only if $\lambda(x + y) = \lambda(x) + \lambda(y)$ (which is part of the definition of $p$ being ‘isometric’).

- If $(\mathcal{V}, \mathcal{G}, \gamma)$ is a normal decomposition system, then with $\mathcal{W} = \gamma(\mathcal{V}) - \gamma(\mathcal{V})$ and $\lambda = \gamma$, the triple $(\mathcal{V}, \mathcal{W}, \lambda)$ becomes a FTvN system. Here $\mathcal{V}$ is a real inner product space and $\mathcal{G}$ is a closed subgroup of the orthogonal group of $\mathcal{V}$. Spectral sets and (real valued) functions are those that are invariant under elements of $\mathcal{G}$. In this setting, $x$ and $y$ commute in $(\mathcal{V}, \mathcal{W}, \lambda)$ if and only if there exists $A \in \mathcal{G}$ such that $x = A\gamma(x)$ and $y = A\gamma(y)$. The space of all $n \times n$ complex matrices is a primary example of a normal decomposition system.

- If $(\mathcal{V}, \mathcal{G}, F)$ is an Eaton triple, then with $\mathcal{W} := F - F$ and $\lambda(x)$ denoting the unique element in $\text{Orb}(x) \cap F$, the triple $(\mathcal{V}, \mathcal{W}, \lambda)$ becomes a FTvN system. Here, $\mathcal{V}$ is a finite dimensional real inner product space, $\mathcal{G}$ is a closed subgroup of the orthogonal group of $\mathcal{V}$ and $F$ is a closed convex cone in $\mathcal{V}$. (It is known that every Eaton triple is a normal decomposition system.)

We end this section with a remark about the ‘completion’ of a FTvN system. Given a FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, let $\overline{\mathcal{V}}$ and $\overline{\mathcal{W}}$ be the completions of the inner product spaces $\mathcal{V}$ and $\mathcal{W}$ respectively. Since $\lambda : \mathcal{V} \to \mathcal{W}$ is Lipschitz (see Proposition [2,3]), there is a unique extension $\overline{\lambda} : \overline{\mathcal{V}} \to \overline{\mathcal{W}}$. Using elementary arguments and the Eberlein-Smulian Theorem [27] (that in a Banach space, weak compactness is the same as weak sequential compactness), one can show that $(\overline{\mathcal{V}}, \overline{\mathcal{W}}, \overline{\lambda})$ is a
FTvN system.

3 Equivalent formulations of certain optimization problems over spectral sets

Throughout this section, we consider a FTvN system \((V, W, \lambda)\); let \(E\) be a spectral set in \(V\) and \(\Phi\) be a spectral function on \(V\) with \(\Phi = \phi \circ \lambda\) for some \(\phi : W \to \mathbb{R}\). Our goal is to reformulate an optimization problem over \(E\) as a problem over \(\lambda(E)\). In this section, we present several results dealing with combinations of linear/distance/convex functions and spectral functions. We start with an elementary result.

**Proposition 3.1** Suppose \(A\) and \(B\) are two sets in \(\mathbb{R}\) with \(B \subseteq A\). Then the following statements hold:

(i) If every element of \(A\) is less than or equal to some element of \(B\), then \(\sup A \leq \sup B\). In this setting, attainment of one supremum implies that of the other. Moreover, if \(\sup A\) is attained at \(\pi\), then \(\pi \in B\) and \(\sup B\) is also attained at \(\pi\).

(ii) If every element of \(A\) is greater than or equal to some element of \(B\), then \(\inf A \leq \inf B\). In this setting, attainment of one infimum implies that of the other. Moreover, if \(\inf A\) is attained at \(a\), then \(a \in B\) and \(\inf B\) is also attained at \(a\).

**Proof.** (i) The inclusion \(B \subseteq A\) implies that (in the extended real number system) \(\beta := \sup B \leq \sup A =: \alpha\). On the other hand, for any \(a \in A\), there is a \(b \in B\) such that \(a \leq b \leq \beta\). This implies that \(\alpha \leq \beta\). Hence, \(\alpha = \beta\). Now suppose \(\alpha\) is attained at \(\pi \in A\). Then, there is a \(\overline{b} \in B\) such that \(\overline{b} \leq \beta\). But then, \(\alpha = \pi \leq \overline{b} \leq \beta = \alpha\) showing \(\pi = \overline{b} \in B\) and \(\overline{b} = \beta\). Finally, if \(\sup B\) is attained at \(b_0\), then \(b_0 \in A\) (recall \(B \subseteq A\)) and \(b_0 = \beta = \alpha\); thus, \(\sup A\) is attained at \(b_0\).

(ii) The proof is similar to that of (i). \(\square\)

A simple example that illustrates Item (i) above is: \(A\) is the interval \((0, 1)\) in \(\mathbb{R}\) and \(B\) is the set of all rationals in \(A\).

3.1 Optimizing a combination of a linear function and a spectral function over a spectral set

We fix a \(c \in V\) and define, for \(x \in V\) and \(w \in W\),

\[
 f(x) := \langle c, x \rangle, \quad f^*(w) := \langle \lambda(c), w \rangle, \quad \text{and} \quad f_*(w) := \langle \tilde{\lambda}(c), w \rangle.
\]

In view of Item (b) in Proposition 2.3 we have

\[
 f_*(\lambda(x)) \leq f(x) \leq f^*(\lambda(x)). \tag{8}
\]
We show below that for any spectral function $\Phi$ and any spectra set $E$, $\sup_{E} (f + \Phi) = \sup_{\lambda(E)} (f^* + \phi)$ and $\inf_{E} (f + \Phi) = \inf_{\lambda(E)} (f_* + \phi)$, with attainments leading to commutativity relations. We derive this as a special case of a broader result dealing with a certain combination of $f$ and $\Phi$ instead of the sum of $f$ and $\Phi$. The motivation to consider such an extension comes from the work of Niezgoda [30].

Given intervals $I$ and $J$ in $\mathbb{R}$, we say that a function $L : I \times J \to \mathbb{R}$ is strictly increasing in the first variable if for each fixed $s^* \in J$, the function $t \to L(t, s^*)$ is strictly increasing over $I$. Two simple examples are: $L(t, s) = t + s$ on $\mathbb{R} \times \mathbb{R}$ and $L(t, s) = ts$ on $\mathbb{R} \times (0, \infty)$. The definition of $L$ increasing in the second variable is similar.

**Theorem 3.2** Consider $f$, $f^*$, $f_*$, $\Phi$, and $E$ as given above. Suppose the real valued function $L$ (defined on a product of two appropriate intervals in $\mathbb{R}$) is strictly increasing in the first variable. Then, the following statements hold:

(i) \[ \sup_{E} L(f, \Phi) = \sup_{\lambda(E)} L(f^*, \phi). \]

Also, attainment of the supremum in one problem implies that in the other. Moreover, if the supremum of the problem on the left is attained at $\varpi \in E$, then $\varpi$ commutes with $c$ in $(\mathcal{V}, \mathcal{W}, \lambda)$ and the maximum value is given by $L\left( f^*(\lambda(\varpi)), \phi(\lambda(\varpi)) \right)$.

(ii) \[ \inf_{E} L(f, \Phi) = \inf_{\lambda(E)} L(f_*, \phi). \]

Also, attainment of the infimum in one problem implies that in the other. Moreover, if the infimum of the problem on the left is attained at $\varpi \in E$, then $\varpi$ commutes with $-c$ in $(\mathcal{V}, \mathcal{W}, \lambda)$ and the minimum value is given by $L\left( f_*(\lambda(\varpi)), \phi(\lambda(\varpi)) \right)$.

We note that the first variable in $L$ varies over an interval that contains the sets $f(E)$ and $f^*(\lambda(E))$ and the second variable varies over an interval that contains $\Phi(E)$. Also, we write $\sup_{E} L(f, \Phi)$ an abbreviation of $\sup_{x \in E} L(f(x), \Phi(x))$, etc.

**Proof.** (i) Consider the following sets in $\mathcal{R}$:

\[ A := \left\{ L\left( f(x), \Phi(x) \right) : x \in E \right\} \quad \text{and} \quad B := \left\{ L\left( f^*(q), \phi(q) \right) : q \in \lambda(E) \right\}. \]

Because $L$ is increasing in the first variable and (S) holds for any $x \in E$, we see that every element in $A$ is less than or equal to some element of $B$. Also, from (A3), for any $q \in \lambda(E)$, there is an $x \in \mathcal{V}$ such that $\lambda(x) = q$ and $f(x) = \langle c, x \rangle = \langle \lambda(c), \lambda(q) \rangle = f^*(q)$. (As $E$ is a spectral set, $x \in E$.) This shows that $B \subseteq A$. From Item (i) in Proposition 3.1, $\sup_{E} L(f, \Phi) = \sup A = \sup B =$
Moreover, attainment in one problem implies that in the other. Now suppose that \( \sup_E L(f, \Phi) \) is attained at \( \tau \in E \). Then, with \( \lambda(\tau) = \eta \), we have

\[
\sup L(f, \Phi) = L(f(\tau), \Phi(\tau)) \leq L\left(f^*(\eta), \phi(\eta)\right) \leq \sup_{\lambda(E)} L(f^*, \phi) = \sup E L(f, \Phi),
\]

where the first inequality comes from (8) and the assume property of \( L \). It follows that

\[
L(f(\tau), \Phi(\tau)) = L\left(f^*(\eta), \phi(\eta)\right).
\]

Since \( L \) is strictly increasing in the first variable and \( \Phi(x) = \phi(q) \), we must have \( f(\tau) = f^*(\lambda(\tau)) \), proving the commutativity of \( c \) and \( \tau \). Clearly, the maximum value is given by \( L\left(f^*(\lambda(\tau)), \phi((\lambda(\tau)))\right) \).

(ii) The proof is similar to that of (i). Here we consider the sets

\[
A := \left\{ L\left(f(x), \Phi(x)\right) : x \in E \right\} \quad \text{and} \quad B := \left\{ L\left(f_*(q), \phi(q)\right) : q \in \lambda(E) \right\}.
\]

We use (8), Item (b) in Proposition 2.7 and Item (ii) in Proposition 3.1 to get the equality of the two infimums and their attainment. The additional statement regarding the commutativity of \( E \) and \(-c\) comes from the equality \( f(\tau) = f_*(\lambda(\tau)) \).

We specialize the above theorem by putting \( L(t, s) = t + s \) on \( R \times R \).

**Corollary 3.3** In the setting of the above theorem, we have the equalities

\[
\sup_E (f + \Phi) = \sup_{\lambda(E)} (f^* + \phi) \quad \text{and} \quad \inf_E (f + \Phi) = \inf_{\lambda(E)} (f_* + \phi). \tag{9}
\]

Additionally, attainment in these lead to commutativity relations: \( \tau \) commutes with \( c \) in the supremum case and \( \tau \) commutes with \(-c\) in the infimum case.

We end this section with an application to variational inequality problems and state one consequence. Let \( (\mathcal{V}, \mathcal{W}, \lambda) \) be a FTvN system, \( E \) be a set in \( \mathcal{V} \), and \( G : \mathcal{V} \rightarrow \mathcal{V} \) be an arbitrary map. Then, the *variational inequality problem* \( VI(G, E) \) is to find an \( a \in E \) such that

\[
\langle G(a), x - a \rangle \geq 0 \quad \text{for all} \quad x \in E.
\]

When \( E \) is a closed convex cone in \( \mathcal{V} \), \( VI(G, E) \) becomes a *complementarity problem* [4]: Find \( a \in \mathcal{V} \) such that

\[
a \in E, \quad G(a) \in E^*, \quad \text{and} \quad \langle a, G(a) \rangle = 0,
\]

where \( E^* \) is the dual of \( E \) in \( \mathcal{V} \) given by \( E^* := \{ x \in \mathcal{V} : \langle x, y \rangle \geq 0 \ \forall y \in E \} \).

Corollary 3.3 leads to the following commutation principle which can be regarded as a generalization and an improvement of Proposition 8 in [31].
Theorem 3.4 Suppose $E$ is a spectral set and $a$ solves $VI(G, E)$. Then $a$ and $-G(a)$ commute in the given FTvN system.

Proof. If $a$ solves $VI(G, E)$, then $\langle G(a), x \rangle \geq \langle G(a), a \rangle$ for all $x \in E$. Hence, with $c := G(a)$, we see that $a$ is a minimizer of the problem $\min \{\langle c, x \rangle : x \in E \}$. By Corollary 3.3 with $\phi = 0$, $a$ commutes with $-c$ (which is $-G(a)$). \hfill \Box

We now state a result that is similar to (actually generalizes) Theorem 1.3 in [12].

Theorem 3.5 Let $(V, W, \lambda)$ be a FTvN system where $V$ is a Hilbert space, $E$ be a convex spectral set, and $\Phi$ be a convex spectral function. Further, let $L$ (defined on a product of appropriate intervals in $\mathcal{R}$) be strictly increasing in the first variable and increasing in the second variable. Suppose $h : V \to \mathcal{R}$ is Fréchet differentiable and $a$ is a local minimizer of the problem $\min_E L(h, \Phi)$. Then $a$ and $-h'(a)$ commute in $(V, W, \lambda)$.

Note: Because $V$ is a Hilbert space, by the Riesz representation theorem, we can regard the continuous linear functional $h'(a)$ as an element of $V$.

Proof. Take any $x \in [a]$. Since $E$ is spectral and convex, for any $0 \leq t \leq 1$, $y := (1-t)a + tx \in E$. As $a$ is a local minimizer, for all positive $t$ near zero, we have

$$L(h(a), \Phi(a)) \leq L(h(y), \Phi(y)).$$

Fix such a $t$. As $\Phi$ is convex and spectral, $\Phi(y) \leq (1-t)\Phi(a) + t\Phi(x) = (1-t)\Phi(a) + t\Phi(a) = \Phi(a).$ Since $L$ is increasing in the second variable, we have

$$L(h(y), \Phi(y)) \leq L(h(y), \Phi(a)).$$

Thus,

$$L(h(a), \Phi(a)) \leq L(h(y), \Phi(a)).$$

Since $L$ is strictly increasing in the first variable,

$$h(a) \leq h(y) = h((1-t)a + tx).$$

As this holds for all positive $t$ near zero, it follows that $\langle h'(a), x-a \rangle \geq 0$ for all $x \in [a]$. So, $a$ solves $VI(h', [a])$. By the previous result, $a$ and $-h'(a)$ commute in $(V, W, \lambda)$. \hfill \Box

We highlight one special case by taking $L(t, s) = t + s$ and $\Phi = 0$.

Corollary 3.6 Let $(V, W, \lambda)$ be a FTvN system where $V$ is a Hilbert space and $E$ be a convex spectral set. Suppose $h : V \to \mathcal{R}$ is Fréchet differentiable and $a$ is a local minimizer of the problem $\min_E h$. Then $a$ and $-h'(a)$ commute in $(V, W, \lambda)$.
3.2 Optimizing a combination of a distance function and a spectral function over a spectral set

We fix $c \in V$ and define, for $x \in V$ and $w \in W$,

$$g(x) := ||c - x||, \quad g^*(w) := ||\lambda(c) - w||,$$

and $g_*(w) := ||\lambda(c) - w||$.

In view of Item (c) in Proposition 2.3, we have

$$g_*(\lambda(x)) \leq g(x) \leq g^*(\lambda(x)).$$

(10)

Analogous to Theorem 3.2 we have the following.

**Theorem 3.7** Consider $g$, $g^*$, $g_*$, $\Phi$, and $E$ as given above. Suppose the real valued function $L$ (defined on a product of two appropriate intervals in $R$) is strictly increasing in the first variable. Then, the following statements hold:

(i) \[
\sup_{E} L(g, \Phi) = \sup_{\lambda(E)} L(g^*, \phi). \]

Also, attainment of the supremum in one problem implies that in the other. Moreover, if the supremum of the problem on the left is attained at $\overline{\pi} \in E$, then $\overline{\pi}$ commutes with $-c$ in $(V, W, \lambda)$ and the maximum value is given by $L\left(g^*(\lambda(\overline{\pi})), \phi(\lambda(\overline{\pi}))\right)$.

(ii) \[
\inf_{E} L(g, \Phi) = \inf_{\lambda(E)} L(g_*, \phi). \]

Also, attainment of the infimum in one problem implies that in the other. Moreover, if the infimum of the problem on the left is attained at $\overline{\pi} \in E$, then $\overline{\pi}$ commutes with $c$ in $(V, W, \lambda)$ and the minimum value is given by $L\left(g_*(\lambda(\overline{\pi})), \phi(\lambda(\overline{\pi}))\right)$.

**Proof.** (i) The proof is similar to that of Item (i) in Theorem 3.2. We define sets $A$ and $B$ appropriately (by replacing $f$ by $g$), use (10), Item (b) in Proposition 2.7, and Item (i) in Proposition 3.1 to get the equality of the two supremums. The attainment statement comes from the equality $g(\overline{\pi}) = g^*(\lambda(\overline{\pi}))$, which, by (7), gives the commutativity of $\overline{\pi}$ and $-c$.

(ii) The proof is analogous to that of Item (ii) in Theorem 3.2. We replace $f$ by $g$, use (10), Item (a) in Proposition 2.7, and Item (ii) in Proposition 3.1 to get the equality of the two infimums. The attainment statement comes from the equality $g(\overline{\pi}) = g_*(\lambda(\overline{\pi}))$, which gives the commutativity of $\overline{\pi}$ and $c$. \qed

**Remarks.** While comparing Theorems 3.2 and 3.7, the reader will notice that in the supremum case (or the infimum case), commutativity statements are reversed: In the linear case, $\overline{\pi}$ commutes...
with $c$ and in the distance case, $\overline{\tau}$ commutes with $-c$. As we shall see in the next section this has to do with the both $f$ and $g$ being convex and derivatives of $f$ and $g^2$ at $\overline{\tau}$ commuting with $\overline{\tau}$.

We now specialize the above theorem by letting $L(t, s) = t + s$ on $\mathcal{R} \times \mathcal{R}$.

**Corollary 3.8** In the setting of the above theorem, we have the equalities

$$\sup_E (g + \Phi) = \sup_{\lambda(E)} (g^* + \phi) \quad \text{and} \quad \inf_E (g + \Phi) = \inf_{\lambda(E)} (g^* + \phi).$$

Additionally, attainment in these lead to commutativity relations: $\overline{\tau}$ commutes with $-c$ in the supremum case and $\overline{\tau}$ commutes with $c$ in the infimum case.

We end this section by describing the distance between two spectral sets. Consider spectral sets $E$ and $F$ in $\mathcal{V}$. Then,

$$\inf \left\{ \|x - y\| : x \in E, y \in F \right\} = \inf \left\{ \|q - p\| : q \in \lambda(E), p \in \lambda(F) \right\}. $$

This follows from (11) with $\phi = 0$:

$$\inf_{x \in E, y \in F} \|x - y\| = \inf_{x \in E} \inf_{y \in F} \|x - y\| = \inf_{x \in E} \inf_{y \in F} \|\lambda(x) - \lambda(y)\| = \inf_{q \in \lambda(E)} \inf_{p \in \lambda(F)} \|q - p\|.$$

In a similar way, we have

$$\sup_{x \in E} \inf_{y \in F} \|x - y\| = \sup_{q \in \lambda(E)} \inf_{p \in \lambda(F)} \|q - p\|.$$

This leads to the equality of Hausdorff distances

$$d_H(E, F) = d_H(\lambda(E), \lambda(F)), \quad (12)$$

where $d_H(E, F)$ is given by

$$d_H(E, F) := \max \left\{ \sup_{x \in E} \inf_{y \in F} \|x - y\|, \sup_{y \in F} \inf_{x \in E} \|y - x\| \right\},$$

etc.

### 3.3 Optimizing a combination of a convex function and a spectral function over a spectral set

In the previous sections, we considered the problems dealing with linear and distance functions. Noting that these functions are convex, one may raise the question of extending the results of the previous sections to convex functions. In this section, we provide some answers in the setting of finite dimensional spaces.

Consider a FTVN system $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ is finite dimensional. Let $h : \mathcal{V} \to \mathcal{R}$ be a convex function. Then, $h$ is continuous and can be realized as the supremum of affine functions (\S, page
13): For some collection \( \{(c_i, \alpha_i) : i \in I\} \) in \( \mathcal{V} \times \mathcal{R} \),

\[
h(x) = \sup_{i \in I} [(c_i, x) + \alpha_i] \quad (x \in \mathcal{V}).\]

Correspondingly, we define two extended real valued functions on \( \mathcal{W} \):

\[
h^*(w) := \sup_{i \in I} (\langle \lambda(c_i), w \rangle + \alpha_i) \quad (w \in \mathcal{W})
\]

and

\[
h_*(w) := \sup_{i \in I} (\langle \tilde{\lambda}(c_i), w \rangle + \alpha_i) \quad (x \in \mathcal{W}).\]

These functions, as supremums of affine functions, are convex (possibly, extended real valued).

Moreover, from Proposition 2.3, we have the inequalities

\[
\langle \tilde{\lambda}(c_i), \lambda(x) \rangle + \alpha_i \leq \langle c_i, x \rangle + \alpha_i \leq \langle \lambda(c_i), \lambda(x) \rangle + \alpha_i,
\]

and consequently,

\[
h_*(\lambda(x)) \leq h(x) \leq h^*(\lambda(x)) \quad \text{for all } x \in \mathcal{V}. \tag{13}
\]

In the result below, we show that on \( \lambda(\mathcal{V}) \), \( h^* \) is finite valued and is independent of the representation of \( h \). But first, we describe a particular representation of \( h \) based on subdifferentials.

For any \( x^* \in \mathcal{V} \), let \( \partial h(x^*) \) denote the subdifferential of \( h \) at \( x^* \) (which is nonempty, compact and convex). By definition, \( c \in \partial h(x^*) \) means that \( h(x) \geq h(x^*) + \langle c, x - x^* \rangle \) for all \( x \in \mathcal{V} \). Then, letting \( \alpha := h(x^*) - \langle c, x^* \rangle \), we have \( h(x) \geq \langle c, x \rangle + \alpha \) for all \( x \in \mathcal{V} \) with equality at \( x^* \). This gives the representation

\[
h(x) = \sup_{(c, \alpha) \in \Omega} [(c, x) + \alpha], \tag{14}
\]

where

\[
\Omega := \left\{(c, \alpha) \in \mathcal{V} \times \mathcal{R} : \text{for some } x^* \in \mathcal{V}, c \in \partial h(x^*), \alpha = h(x^*) - \langle c, x^* \rangle \right\}.
\]

**Theorem 3.9** Let \( (\mathcal{V}, \mathcal{W}, \lambda) \) be a FTvN system where \( \mathcal{V} \) is finite dimensional. Let \( E \) be a spectral set in \( \mathcal{V} \) and \( \Phi \) be a spectral function on \( \mathcal{V} \). Suppose \( h : \mathcal{V} \to \mathcal{R} \) is convex. By fixing a representation of \( h \), we define the corresponding extended real valued convex function \( h^* \) on \( \mathcal{W} \). Let \( L \) (defined on the product of appropriate intervals in \( \mathcal{R} \)) be strictly increasing in the first variable. Then we have the following:

(a) For any \( q \in \lambda(\mathcal{V}) \), \( h^*(q) = \max \left\{ h(x) : \lambda(x) = q \right\} < \infty \). Hence, on \( \lambda(\mathcal{V}) \), \( h^* \) is independent of the representation of \( h \).

(b) \( \sup_{\lambda(E)} L(h, \Phi) = \sup_{\lambda(E)} L(h^*, \phi) \).
Also, attainment of supremum in one problem implies that in the other. Moreover, if the problem on the left is attained at \( \tau \in E \), then \( \tau \) commutes with every element in the subdifferential of \( h \) at \( \tau \).

(c) \( h^*_*(q) \leq \min \left\{ h(x) : \lambda(x) = q \right\} \quad (q \in \lambda(\mathcal{V})) \).

(d) \( \inf_{\lambda(E)} L(h^*, \phi) \leq \inf_{E} L(h, \Phi) \).

**Proof.** (a) Let \( q \in \lambda(\mathcal{V}) \). Then, the set \( \{x \in V : \lambda(x) = q\} \) is nonempty, closed and bounded (as \( \lambda \) is continuous and norm preserving) in \( \mathcal{V} \). As \( \mathcal{V} \) is finite dimensional, this set is compact in \( \mathcal{V} \). By the continuity of \( h \) we see that \( \max \left\{ h(x) : \lambda(x) = q \right\} \) exists. Now, the inequality \( h(x) \leq h^*(\lambda(x)) \) implies that

\[
\max \left\{ h(x) : \lambda(x) = q \right\} \leq h^*(q).
\]

We recall the representations \( h(x) = \sup_{i \in I} \left[ \langle c_i, x \rangle + \alpha_i \right] \) on \( \mathcal{V} \) and \( h^*(w) := \sup_{i \in I} \left[ \langle \lambda(c_i), w \rangle + \alpha_i \right] \) on \( \mathcal{W} \). By (A3) in Definition 2.2, for every \( i \in I \), there is an \( x_i \in \mathcal{V} \) such that \( \lambda(x_i) = q \) and \( \langle \lambda(c_i), q \rangle = \langle c_i, x_i \rangle \). Consequently,

\[
\langle \lambda(c_i), q \rangle + \alpha_i \leq h(x_i) \leq \max \left\{ h(x) : \lambda(x) = q \right\}.
\]

Taking the supremum over \( i \), we get

\[
h^*(q) \leq \max \left\{ h(x) : \lambda(x) = q \right\}.
\]

We thus have the equality \( h^*(q) = \max \left\{ h(x) : \lambda(x) = q \right\} \). This gives the finiteness of \( h^*(q) \) and shows that \( h^*(q) \) depends on the values of \( h \) alone and not on the representation of \( h \).

(b) Consider the two problems stated in Item (b). Define the sets

\[
A := \left\{ L(h(x), \Phi(x)) : x \in E \right\} \quad \text{and} \quad B := \left\{ L(h^*(q), \phi(q)) : q \in \lambda(E) \right\}.
\]

Because \( L \) is increasing in the first variable and \( \lambda(\mathcal{V}) \) holds, we see that every element of \( A \) is less than or equal to some element of \( B \). Moreover, by Item (a), for each \( q \in \lambda(E) \), \( h^*(q) = h(x) \) for some \( x \in E \) with \( \lambda(x) = q \). Thus \( B \subseteq A \). We now use Proposition 3.1 to see \( \sup A = \sup B \) which gives the equality

\[
\overline{\alpha} := \sup_{E} L(h, \Phi) = \sup_{\lambda(E)} L(h^*, \phi) =: \overline{\beta}.
\]

We also see the attainment of one supremum implying that of the other. To see the commutativity part in (b), suppose that \( \sup_{E} L(h, \Phi) \) is attained at \( \tau \in E \). Let \( \overline{\tau} := \lambda(\tau) \in \lambda(E) \). Then, by (13),

\[
\overline{\alpha} = L(h(\tau), \Phi(\tau)) \leq L(h^*(\overline{\tau}), \phi(\overline{\tau})) \leq \overline{\beta} = \overline{\alpha}.
\]

Because \( L \) is strictly increasing in the first variable, we get

\[
h(\tau) = h^*(\overline{\tau}).
\]
Now, noting that on $\lambda(\mathcal{V})$, $h^*$ is independent of the representation of $h$, we consider the subdifferential representation (14):

$$h(x) = \sup_{(c, \alpha) \in \Omega} [(c, x) + \alpha].$$

Then, for all $q \in \lambda(\mathcal{V})$,

$$h^*(q) = \sup_{(c, \alpha) \in \Omega} [(\lambda(c), q) + \alpha].$$

Now, take any $c \in \partial h(x)$ and let $\alpha := h(x) - \langle c, x \rangle$. Then, $(c, \alpha) \in \Omega$ and

$$h(x) = \langle c, x \rangle + \alpha \leq \langle \lambda(c), \lambda(x) \rangle + \alpha \leq h^*(\pi) = h(x).$$

Hence, $\langle c, x \rangle = \langle \lambda(c), \lambda(x) \rangle$, proving the commutativity of $c$ and $x$.

Finally, as $L$ is increasing in the first variable, Items (c) and (d) are immediate from (13). 

We highlight one special case.

**Corollary 3.10** Suppose $(\mathcal{V}, \mathcal{W}, \lambda)$ is a FTvN system where $\mathcal{V}$ is finite dimensional. Let $E$ be a spectral set and $\Phi$ be a spectral function on $\mathcal{V}$. Suppose $h : \mathcal{V} \to \mathcal{R}$ is convex and $\pi$ is an optimizer of the problem $\max_E (h + \Phi)$. Then, $\pi$ commutes with every element in the subdifferential of $h$ at $\pi$.

**Remarks.** Given the convex function $h$, the construction of $h^*$ is unsatisfactory for two reasons: First, when $h$ is the distance function $g$ considered in the previous section, $h^*$ may be different from $g^*$. Second, unlike for the linear and distance functions, the equality $h^*(q) = \min \{h(x) : \lambda(x) = q\}$ may not hold, see the example below. We can remedy this at this expense of losing convexity by defining:

$$h^{**}(q) := \min \{h(x) : \lambda(x) = q\} \quad (q \in \lambda(\mathcal{V})).$$

Then, one can show that $\inf_E L(h, \Phi) = \inf_{\lambda(\mathcal{E})} L(h^{**}, \phi)$. We omit the details.

**Example 3.11** Let $(\mathcal{V}, \mathcal{W}, \lambda) = (\mathcal{R}^2, \mathcal{R}^2, \lambda)$, where for any $q \in \mathcal{R}^2$, $\lambda(q) = q^+$ (the decreasing rearrangement of $q$, see Section 4.1 for details). Here, $\lambda(q) = -[-(q)^+]) = q^+$ (the increasing rearrangement of $q$). Now, let $c_1 = (1, 0)$ and $c_2 = (-1, 0)$ so that $c_1^+ = (0, 1)$ and $c_2^+ = (-1, 0)$. Consider the convex function

$$h(x) := \max \{\langle c_1, x \rangle, \langle c_2, x \rangle\} = |x_1|,$$

where $x = (x_1, x_2) \in \mathcal{R}^2$. We have, for $w = (w_1, w_2) \in \mathcal{R}^2$,

$$h^*(w) = \max \{\langle c_1^+, w \rangle, \langle c_2^+, w \rangle\} = \max \{w_2, -w_1\}$$
\[ h_{**}(q) = \min\{ h(x) : \lambda(x) = q \} = \min\{|q_1|, |q_2|\} \quad (q \in \lambda(V)). \]

With \( q = (1, -1) \), we have \( \{ x \in \mathbb{R}^2 : \lambda(x) = q \} = \{(1, -1), (-1, 1)\} \); so, \( \min\{ h(x) : \lambda(x) = q \} = 1 \). However, \( h_{**}(q) = -1 \). This shows that the inequality in Item (c) of the above theorem can be strict.

We also note that \( \min_{\lambda(\mathbb{R}^2)} h_{**} = 0 \), while \( \inf_{\lambda(\mathbb{R}^2)} h_{**} = -\infty \). We also observe that \( h_{**} \) is nonconvex.

We end this section by stating a result that is similar to Corollary 3.10, but dealing with the minimum of a convex function.

**Proposition 3.12** Suppose \((V, W, \lambda)\) is a FTvN system, where \( V \) is finite dimensional. Let \( E \) be a convex spectral set in \( V \). Suppose \( h : V \to \mathbb{R} \) is convex and \( x \) is an optimizer of the problem \( \min_E h \). Then, \( x \) commutes with \(-c \) for some element \( c \) in the subdifferential of \( h \) at \( x \).

**Proof.** Let \( \chi \) denote the indicator function of \( E \) (so it takes the value zero on \( E \) and infinity outside of \( E \)). Then, \( x \) is a (global) optimizer of the problem \( \min(h + \chi) \) and so

\[ 0 \in \partial (h + \chi)(x) = \partial h(x) + \partial \chi(x), \]

where the equality comes from the subdifferential sum formula ([33], Theorem 23.8). Hence, there is a \( c \in \partial h(x) \) such that \(-c \in \partial \chi(x)\). This \( c \) will have the property that

\[ \langle c, x - x \rangle \geq 0 \quad \text{for all} \quad x \in E, \]

that is, \( x \) is a minimizer of the problem \( \min\{ \langle c, x \rangle : x \in E \} \). By Corollary 3.3 with \( \phi = 0 \), \( x \) commutes with \(-c \). \( \Box \)

### 4 Euclidean Jordan algebras

In this section, we show that every Euclidean Jordan algebra is a FTvN system and illustrate our previous results. We start with some preliminaries. The Euclidean \( n \)-space \( \mathbb{R}^n \) carries the usual inner product. For any \( q \in \mathbb{R}^n \), we let \( q^\downarrow \) denote the decreasing rearrangement of \( q \) (that is, \( q_1^\downarrow \geq q_2^\downarrow \geq \cdots \geq q_n^\downarrow \)); for any \( Q \subseteq \mathbb{R}^n \), we let \( Q^\downarrow := \{ q^\downarrow : q \in Q \} \). The symbol \( \Sigma_n \) denotes the set of all permutation matrices on \( \mathbb{R}^n \). For a set \( S \) in \( \mathbb{R}^n \), we write \( \Sigma_n(S) := \{ \sigma(s) : \sigma \in \Sigma_n, s \in S \} \). We say that a set \( Q \) in \( \mathbb{R}^n \) is permutation invariant if \( \sigma(Q) = Q \) for all \( \sigma \in \Sigma_n \). (The word symmetric is also used in some literature.)

Let \((V, \circ, \langle \cdot, \cdot \rangle)\) denote a Euclidean Jordan algebra of rank \( n \) [10], where \( x \circ y \) and \( \langle x, y \rangle \) denote, respectively, the Jordan product and inner product of two elements \( x \) and \( y \). It is known [10] that any Euclidean Jordan algebra is a direct product/sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic one of five algebras, three of which are the
algebras of \( n \times n \) real/complex/quaternion Hermitian matrices. The other two are: the algebra of \( 3 \times 3 \) octonion Hermitian matrices and the Jordan spin algebra. By the spectral theorem [10], every element \( x \) in \( V \) has a decomposition \( x = q_1e_2 + q_2e_2 + \cdots + q_ne_n \), where \( q_1, q_2, \ldots, q_n \) are the eigenvalues of \( x \) and \( \{e_1, e_2, \ldots, e_n\} \) is a Jordan frame. (The eigenvalues remain the same in any such representation.) Defining the sum of eigenvalues of \( x \) as the trace of \( x \), we note (the known fact) that the trace inner product \( \langle x, y \rangle = \text{tr}(x \circ y) \) is also compatible with the Jordan product. Henceforth, we assume that the inner product in \( V \) is this trace inner product, that is, \( \langle x, y \rangle = \text{tr}(x \circ y) \). Working with this inner product allows us to say that each Jordan frame is orthonormal.

For any \( x \in V \), we let \( \lambda(x) := (\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)) \) denote the vector of eigenvalues of \( x \) written in the decreasing order. Then \( \lambda : V \to \mathbb{R}^n \) denotes the eigenvalue map. Given a Jordan frame \( E = \{e_1, e_2, \ldots, e_n\} \) in \( V \), we fix its enumeration/listing and define for any \( q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n \),

\[
q \ast E := \sum_{i=1}^{n} q_i e_i.
\]

We note that

\[
\lambda(q \ast E) = q^\dagger.
\]

A set \( E \) in \( V \) is said to be a spectral set if it is of the form \( \lambda^{-1}(Q) \) for some \( Q \subseteq \mathbb{R}^n \). We state the following simple (easily verifiable) result.

**Proposition 4.1** Let \( E = \lambda^{-1}(Q) \) for some \( Q \subseteq \mathbb{R}^n \). Then the following statements hold:

(i) \( x \in E \iff \lambda(x) \in \lambda(E) \).

(ii) \( \lambda(E) = Q \cap Q^\dagger \).

(iii) \( \lambda^{-1}(Q) = \lambda^{-1}(Q \cap Q^\dagger) = \lambda^{-1}(\Sigma_n(Q \cap Q^\dagger)) \).

Because of the third item above, we can always write a spectral set as the \( \lambda \)-inverse image of permutation invariant set. A function \( \Phi : V \to \mathbb{R} \) is a spectral function if it is of the form \( \Phi = \phi \circ \lambda \) for some \( \phi : \mathbb{R}^n \to \mathbb{R} \). Note that we can always rewrite \( \Phi = \phi_1 \circ \lambda \), where \( \phi_1 : \mathbb{R}^n \to \mathbb{R} \) is permutation invariant, that is, \( \phi_1(\sigma(q)) = \phi_1(q) \) for all \( \sigma \in \Sigma_n \) and \( q \in \mathbb{R}^n \). In the case of a simple algebra, spectral sets and functions are precisely those that are invariant under automorphisms of \( V \) [19]. (An automorphism of \( V \) is a linear isomorphism of \( V \) that preserves the Jordan product.)

We say that elements \( x \) and \( y \) operator commute in \( V \) if there is a Jordan frame \( E \) in \( V \) such that

\[
x = q \ast E \quad \text{and} \quad y = p \ast E
\]

for some \( q, p \in \mathbb{R}^n \). It is well-known that this is equivalent to the commutativity of the linear operators \( L_x \) and \( L_y \), where \( L_x(z) = x \circ z \), etc. We say that \( x \) and \( y \) strongly operator commute (or said to be ‘simultaneously order diagonalizable’ [23] or said to have ‘similar joint decomposition’.
in $V$ if there is a Jordan frame $E$ such that
\[ x = \lambda(x) \ast E \quad \text{and} \quad y = \lambda(y) \ast E. \]

The following result extends the Hardy-Littlewood-Pólya rearrangement inequality in $\mathcal{R}^n$ \cite{26} and Fan-Theobald trace inequality for real/complex Hermitian matrices \cite{9, 37} to (general) Euclidean Jordan algebras. One way of proving it to show that the result holds in a simple Euclidean Jordan algebra (see \cite{25, 14}) and then use the above mentioned Hardy-Littlewood-Pólya inequality. For a direct proof, see \cite{1}.

**Theorem 4.2** Let $V$ be a Euclidean Jordan algebra carrying the trace inner product. Then, for $x, y \in V$, we have
\[ \langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle \]
with equality if and only if $x$ and $y$ strongly operator commute.

This leads to the following result.

**Theorem 4.3** Consider a Euclidean Jordan algebra of rank $n$ carrying trace inner product and let $\lambda : V \to \mathcal{R}^n$ denote the eigenvalue map. Then, the triple $(V, \mathcal{R}^n, \lambda)$ is a FTvN system.

**Proof.** We verify conditions $(A1) - (A3)$ in Definition \ref{def:FTvN} For any $c$, consider the spectral decomposition $x = q_1 e_2 + q_2 e_2 + \cdots + q_n e_n$, where $q_1, q_2, \ldots, q_n$ are the eigenvalues of $x$ and $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame. By our assumption that $V$ carries the trace inner product, the Jordan frame is orthonormal. Hence $||x||^2 = \sum_{i=1}^n |q_i|^2 = ||\lambda(x)||^2$. This verifies $(A1)$. Condition $(A2)$ follows from Theorem \ref{thm:hardy-littlewood-polya} To see $(A3)$, let $c \in V$ and $q \in \lambda(V)$. We write the spectral decomposition of $c$ as $c = \lambda(c) \ast E$ for some Jordan frame $E$. Now, as the components of $q$ are decreasing, letting $x := q \ast E$ we see that $\lambda(x) = q$. Since $E$ is orthonormal, $\langle c, x \rangle = \langle \lambda(c), \lambda(x) \rangle$ and so, $(A3)$ is verified. \hfill \Box

The following result shows that strong operator commutativity in the Euclidean Jordan algebra $V$ is equivalent to commutativity in the FTvN system $(V, \mathcal{R}^n, \lambda)$.

**Proposition 4.4** For elements $x$ and $y$ in a Euclidean Jordan algebra $V$ with trace inner product, the following are equivalent:

(i) $x$ and $y$ strongly operator commute in $V$.

(ii) $\lambda(x + y) = \lambda(x) + \lambda(y)$.

(iii) $\langle x, y \rangle = \langle \lambda(x), \lambda(y) \rangle$, that is, $x$ and $y$ commute in the FTvN system $(V, \mathcal{R}^n, \lambda)$.

**Proof.** If (i) holds, then we can write $x = \lambda(x) \ast E$ and $y = \lambda(y) \ast E$ for some Jordan frame $E$. Then, $x + y = [\lambda(x) + \lambda(y)] \ast E$ and (as components of $\lambda(x) + \lambda(y)$ are in decreasing order)
\[ \lambda(x + y) = \lambda(x) + \lambda(y). \] Thus, (i) \Rightarrow (ii).

The equivalence of (ii) and (iii) comes from Proposition \ref{prop:equivalence}.

Finally, the equivalence of (iii) and (i) follows from Theorem \ref{thm:equivalence}.

We now describe two consequences of Theorem \ref{thm:equivalence}. Thanks to the spectral theorem, it is easy to see that an element \( u \) in \( V \) is an idempotent, that is \( u^2 = u \), if and only if zero and one are the only possible eigenvalues of \( u \). For any natural number \( k \), \( 1 \leq k \leq n \), consider an idempotent \( u \) having \( k \) nonzero eigenvalues (in which case, we say that \( u \) has rank \( k \)). Then \( \lambda(u) = (1, 1, \ldots, 1, 0, 0, \ldots, 0) \) in \( \mathbb{R}^n \) and the \( \lambda \)-orbit \([u]\) equals \( J^k(V) \), the set of all idempotents of rank \( k \) in \( V \). For such a \( u \), the statement \( \max \{\langle c, x \rangle : x \in [u] \} = \langle \lambda(c), \lambda(u) \rangle \) extends \( \lambda \)-norms from simple Euclidean Jordan algebras to general Euclidean Jordan algebras.

For the second consequence, consider \( \lambda \)-norms and replace \( c \) by the above \( u \) to get the inequality

\[
\sum_{i=1}^{k} \lambda_i(x + y) \leq \sum_{i=1}^{k} \lambda_i(x) + \sum_{i=1}^{k} \lambda_i(y).
\]

When \( k = n \),

\[
\sum_{i=1}^{n} \lambda_i(x + y) = \langle x + y, e \rangle = \langle x, e \rangle + \langle y, e \rangle = \sum_{i=1}^{n} \lambda_i(x) + \sum_{i=1}^{n} \lambda_i(y),
\]

where \( e \) denotes the unit element in \( V \). These two statements together say (by definition) that \( \lambda(x + y) \) is majorized by \( \lambda(x) + \lambda(y) \). While such a statement is known for simple Euclidean Jordan algebras \( \cite{gurvits} \) and the result for general algebras can be proved by elementary means, for lack of explicit reference, we record this fact below using the standard notation for majorization. As we shall see in the section on hyperbolic polynomials, this is a particular case of a far reaching generalization due to Gurvits.

**Theorem 4.5** In any Euclidean Jordan algebra \( V \), for any two elements \( x \) and \( y \),

\[ \lambda(x + y) \prec \lambda(x) + \lambda(y). \]

### 4.1 Some specialized results in Euclidean Jordan algebras

Since every Euclidean Jordan algebra can now be regarded as a FTvN system, all the results of Section 3 could be stated for Euclidean Jordan algebras. Instead of repeating these, we now state some specialized results.

First consider the Euclidean Jordan algebra \( \mathcal{R}^n \) (with the usual inner product and componentwise product). In this setting, for any \( q \in \mathcal{R}^n \), \( \lambda(q) = q^1 \). Also, as there is only one Jordan frame in \( \mathcal{R}^n \) (up to permutation, namely, the standard coordinate basis), any two elements in \( \mathcal{R}^n \) operator commute, and two vectors \( p \) and \( q \) in \( \mathcal{R}^n \) strongly operator commute if and only if for some permutation matrix \( \sigma \), \( p = \sigma(p^1) \) and \( q = \sigma(q^1) \). This simple observation will allow us to construct an
example of an inner product space satisfying conditions (A1) and (A2) of Definition 2.2, but not (A3): In the Euclidean Jordan algebra (FTvN system) $\mathcal{R}^3$, consider the (sub)space $Z$ generated by vectors $p = (3, 2, 1)$ and $q = (-1, 0, 0)$ and let $\mu$ denote the restriction of $\lambda$ to this subspace. Then, in the triple $(Z, \mathcal{R}^n, \mu)$, conditions (A1) and (A2) hold but not (A3). This example will also show that a subspace of an FTvN system need not be an FTvN system.

We now specialize (9) for $\mathcal{V} = \mathcal{R}^n$, $\Phi = 0$, and $f(x) = \langle p, x \rangle$ where $p \in \mathcal{R}^n$. Let $Q \subseteq \mathcal{R}^n$ so that $E := \lambda^{-1}(Q) = \Sigma_n(Q \cap Q^\perp)$ and $\lambda(E) = Q \cap Q^\perp$. Then,

$$\sup \left\{ \langle p, q \rangle : q \in \Sigma_n(Q \cap Q^\perp) \right\} = \sup \left\{ \langle p^\perp, q \rangle : q \in Q \cap Q^\perp \right\}. \tag{18}$$

In particular, when $Q$ is permutation invariant, we have $Q \cap Q^\perp = Q^\perp$ and $\Sigma_n(Q \cap Q^\perp) = Q$. So

$$\sup \left\{ \langle p, q \rangle : q \in Q \right\} = \sup \left\{ \langle p^\perp, q \rangle : q \in Q^\perp \right\} \quad \text{(Q permutation invariant).} \tag{19}$$

We can now combine (9) with (18) to get a statement in a Euclidean Jordan algebra $\mathcal{V}$: If $Q \subseteq \mathcal{R}^n$ and $E = \lambda^{-1}(Q)$ in $\mathcal{V}$, then for any $c \in \mathcal{V}$,

$$\sup \left\{ \langle c, x \rangle : x \in \lambda^{-1}(Q) \right\} = \sup \left\{ \langle \lambda(c), q \rangle : q \in Q \cap Q^\perp \right\}; \tag{20}$$

Moreover, when $Q$ is permutation invariant, thanks to (19),

$$\sup \left\{ \langle c, x \rangle : x \in \lambda^{-1}(Q) \right\} = \sup \left\{ \langle \lambda(c), q \rangle : q \in Q^\perp \right\} = \sup \left\{ \langle \lambda(c), q \rangle : q \in Q \right\}. \tag{21}$$

Analogous statements can be made for the infimum in place of the supremum.

Similarly, using (11), we have (in the setting of a Euclidean Jordan algebra $\mathcal{V}$),

$$\inf \left\{ \|c - x\| : x \in \lambda^{-1}(Q) \right\} = \inf \left\{ \|\lambda(c) - q\| : q \in Q \cap Q^\perp \right\}, \tag{22}$$

and when $Q$ is permutation invariant,

$$\inf \left\{ \|c - x\| : x \in \lambda^{-1}(Q) \right\} = \inf \left\{ \|\lambda(c) - q\| : q \in Q^\perp \right\} = \inf \left\{ \|\lambda(c) - q\| : q \in Q \right\}. \tag{23}$$

As an illustration, consider the algebra $\mathbb{S}^n (\mathcal{H}^n)$. Here, two matrices $X$ and $Y$ operator commute if and only if $XY = YX$ or, equivalently, there exist an orthogonal (respectively, unitary) matrix $U$ and real diagonal matrices $D_1$ and $D_2$ such that $X = UD_1U^*$ and $Y = UD_2U^*$. They strongly operator commute if and only if the above spectral representations hold when the diagonal vectors of $D_1$ and $D_2$ are, respectively, $\lambda(X)$ and $\lambda(Y)$. To see an explicit example in this setting, consider the problem mentioned in the Introduction: Find

$$\sup \left\{ \langle C, X \rangle : X \succeq 0, 1 \leq \lambda_{\max}(X) \leq 2 \right\},$$

where $C, X \in \mathbb{S}^n$, $X \succeq 0$ means that $X$ is positive semidefinite, and $\lambda_{\max}(X)$ denotes the maximum eigenvalue of $X$. Let $Q := \{q \in \mathcal{R}^n : q \geq 0, 1 \leq \max(q) \leq 2\}$, where $\max(q)$ denotes the maximum of the components of $q$. Clearly, $Q$ is permutation invariant and $\lambda^{-1}(Q) = \{X \succeq 0, 1 \leq \lambda_{\max}(X) \leq 2\}$.
Now, (21) applied to $S^n$ and $Q$ gives
\[ \sup \left\{ \langle C, X \rangle : X \in \lambda^{-1}(Q) \right\} = \sup \left\{ \langle \lambda(C), q \rangle : q \in Q^\perp \right\} = \sup \left\{ \langle \lambda(C), q \rangle : q \in Q \right\}. \]

Here, the set $Q^\perp = \{ q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n : q_1 \geq q_2 \geq \cdots \geq q_n \geq 0 \text{ and } 1 \leq q_1 \leq 2 \}$ is polyhedral and compact. So we have the attainment of supremum in both the problems and, moreover, knowing the components of $\lambda$ allows us to compute this maximum.

To illustrate (23), consider a Euclidean Jordan algebra $V$ and let $E := \lambda^{-1}(Q)$, where $Q$ is a closed permutation invariant set in $\mathbb{R}^n$. Then, $E$ is a closed spectral set in $\mathcal{V}$, $\lambda(E) = Q \cap Q^\perp = Q^\perp$ is a closed set in $\mathbb{R}^n$, and for any $c \in \mathcal{V}$,
\[ \min \left\{ ||c - x|| : x \in \lambda^{-1}(Q) \right\} = \min \left\{ ||\lambda(c) - q|| : q \in Q^\perp \right\} = \min \left\{ ||\lambda(c) - q|| : q \in Q \right\}. \]

Such a result appears in Proposition 1.11 in [34] (or, Proposition 10 in [31]). In a similar vein, we specialize (12) to permutation invariant sets $Q$ and $P$ in $\mathbb{R}^n$:
\[ d_H(\lambda^{-1}(Q), \lambda^{-1}(P)) = d_H(Q^\perp, P^\perp) = d_H(Q, P). \]

(It is interesting to observe that this, together with Proposition 1.1 in [13], shows that the multi-valued map $\lambda^{-1}$ behaves like a linear isometry on the collection of all convex permutation invariant sets.)

Our next result deals with the linear image of a spectral set. Suppose $Q$ is a permutation invariant set in $\mathbb{R}^n$. As observed in [1] [19] [18] [13], many properties of $Q$ carry over to the spectral set $\lambda^{-1}(Q)$. In particular,

(i) If $Q$ is compact/convex, then $\lambda^{-1}(Q)$ is compact/convex, see [11],
(ii) If $Q$ is connected, then $\lambda^{-1}(Q)$ is connected, see [13],
(iii) If $V$ is simple and $Q \cap Q^\perp$ is connected, then $\lambda^{-1}(Q)$ is connected, see [13].

**Theorem 4.6** Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank $n$ with trace inner product. Suppose $Q$ is a compact permutation invariant subset of $\mathbb{R}^n$ and one of the following conditions holds: (i) $Q$ is connected or (ii) $\mathcal{V}$ is simple and $Q \cap Q^\perp$ is connected. Then, for any $c \in \mathcal{V}$,
\[ \left\{ \langle c, x \rangle : x \in \lambda^{-1}(Q) \right\} = [\delta, \Delta], \]

where $\delta = \langle \lambda(a), \lambda(a) \rangle$ for some $a \in \lambda^{-1}(Q)$ that strongly operator commutes with $-c$ and $\Delta = \langle \lambda(c), \lambda(b) \rangle$ for some $b \in \lambda^{-1}(Q)$ that strongly operator commutes with $c$.

**Proof.** By results stated above, $\lambda^{-1}(Q)$ is compact and connected in $\mathcal{V}$. By the continuity of the function $x \to \langle c, x \rangle$, \[ \left\{ \langle c, x \rangle : x \in \lambda^{-1}(Q) \right\} = [\delta, \Delta], \] where $\delta$ and $\Delta$ are, respectively, the minimum and the maximum of $\langle c, x \rangle$ over $\lambda^{-1}(Q)$. By our previous results, they must be of
the form $\delta = \langle \tilde{\lambda}(c), \lambda(a) \rangle$ for some $a \in \lambda^{-1}(Q)$ that strongly operator commutes with $-c$ and $\Delta = \langle \lambda(c), \lambda(b) \rangle$ for some $b \in \lambda^{-1}(Q)$ that strongly operator commutes with $c$.

The following is a simple consequence of the above result. It extends a similar result of Fan, see Corollary 1.6 and Theorem 1.5 in [35].

**Corollary 4.7** Consider a simple algebra $V$ with trace inner product. Then, for any $u, c \in V$,

$$\left\{ \langle c, x \rangle : x \in [u] \right\} = [\delta, \Delta],$$

where $\delta = \langle \tilde{\lambda}(c), \lambda(u) \rangle$ and $\Delta = \langle \lambda(c), \lambda(u) \rangle$.

**Proof.** We let $Q := \{\lambda(u)\}$. Then, condition (ii) in the above theorem applies. \qed

We conclude this section with some remarks about Theorems 3.4 and 3.5. Specialized to a Euclidean Jordan algebra $V$, Theorem 3.4 says that if $E$ is a spectral set in $V$ and $a$ solves the variational inequality problem VI$(G, E)$, then $a$ and $-G(a)$ strongly operator commute. This, in particular, implies that $a$ and $G(a)$ operator commute, thus yielding a result of Ramírez, Seeger, and Sossa ([31], Proposition 8).

Theorem 3.5 stated in the setting of $V$ gives the strong operator commutativity of $a$ and $-h'(a)$. This, in particular, gives the operator commutativity of $a$ and $h'(a)$. We note that if one is concerned with just the operator commutativity of $a$ and $h'(a)$, then, the following result – where no convexity assumptions are made – can be used. This result extends Theorem 1.2 in [12] with a similar/modified proof. Here, weak spectrality refers to invariance under automorphisms of the $V$, see [12].

**Theorem 4.8** Let $V$ be a Euclidean Jordan algebra and suppose that $E$ is a (weakly) spectral set in $V$ and $\Phi$ is a (weakly) spectral function on $V$. Let $L$ (defined on a product of appropriate intervals in $\mathbb{R}$) be strictly increasing in the first variable and $h : V \to \mathbb{R}$ be Fréchet differentiable. If $a$ is a local optimizer of the problem $\min_E L(h, \Phi)$ or $\max_E L(h, \Phi)$, then $a$ and $h'(a)$ operator commute in $V$.

The following example shows that operator commutativity cannot be replaced by strong operator commutativity.

**Example 4.9** In the Euclidean Jordan algebra $\mathbb{R}^2$, let $E = \{(1, 0), (0, 1)\}$. For the function $h(x, y) := \frac{1}{2}x^2 - x + x(y^2 + y)$, $h(1, 0) = -\frac{1}{2}$ and $h(0, 1) = 0$. Also, $h'(x, y) = (x - 1 + y^2 + y, 2xy + x)$. So, $h'(1, 0) = (0, 1)$ and $h'(0, 1) = (1, 0)$. We note that the elements $(1, 0)$ and $(0, 1)$ operator commute in $\mathbb{R}^2$, but not strongly. Thus, if $a$ denotes either a minimizer or a maximizer of $h$ on $E$, then $a$ and $h'(a)$ do not strongly operator commute.
5 FTvN systems induced by hyperbolic polynomials

Hyperbolic polynomials were introduced by Gårding [11] in connection with partial differential equations. They have become important in optimization due to their connection to interior point methods [15, 32] and convex analysis [2].

Let $V$ be a finite dimensional real vector space. With respect to some coordinate system on $V$, let $p : V \rightarrow \mathbb{R}$ be a nonconstant polynomial, homogeneous of degree $n$, where $n$ is a natural number. We say that $p$ is hyperbolic with respect to some $e \in V$ if $p(e) \neq 0$ and for all $x \in V$, the univariate polynomial $t \rightarrow p(te - x)$ has only real roots. We fix such a $p$ and consider, to each $x \in V$, the vector $\lambda(x)$ in $\mathbb{R}^n$ whose entries are the of roots of $p(te - x)$ written in the decreasing order. Then the map $\lambda : V \rightarrow \mathbb{R}^n$ has many interesting properties [2]. Assuming that $p$ is complete, that is, $\lambda(x) = 0 \Rightarrow x = 0$, one can define a norm on $V$ by $||x|| := ||\lambda(x)||$ (the latter norm is the Euclidean norm on $\mathbb{R}^n$) and an inner product on $V$ by $\langle x, y \rangle := \frac{1}{4}[||x + y||^2 - ||x - y||^2]$, see [2]. Then, for all $x, y \in V$,

$$\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle,$$

see [2], Prop. 4.4. In connection with the triple $(V, \mathbb{R}^n, \lambda)$, we have the following result.

**Proposition 5.1** (Proposition 5.3, [2]) Assume that $p$ is complete and consider the associated norm/inner product on $V$. Then, the following are equivalent:

(i) $p$ is ‘isometric’, that is, for all $y, z \in V$, there is an $x \in V$ such that $\lambda(x) = \lambda(z)$ and $\lambda(x + y) = \lambda(x) + \lambda(y)$.

(ii) $\max_{\{x : \lambda(x) = \lambda(u)\}} \langle c, x \rangle = \langle \lambda(c), \lambda(u) \rangle$ for all $c, u \in V$.

Given that $\lambda$ is norm preserving and the inequality $\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle$ holds for all $x, y \in V$, Item (ii) above proves (see Item (a) in Proposition 2.1) the following.

**Theorem 5.2** Let $p$ be a complete and consider the induced triple $(V, \mathbb{R}^n, \lambda)$. Then, $p$ is isometric if and only if $(V, \mathbb{R}^n, \lambda)$ is a FTvN system.

We observe that the condition $\lambda(x + y) = \lambda(x) + \lambda(y)$ that appears in Item (i) of the previous proposition is simply a restatement of the commutativity condition $\langle x, y \rangle = \langle \lambda(x), \lambda(y) \rangle$. There are numerous examples of complete isometric hyperbolic polynomials, see [2]. We provide one more example.

**Example 5.3** Consider a Euclidean Jordan algebra $V$ of rank $n$ with the trace inner product. Then, for any $x \in V$, the determinant of $x$ (written $\det x$) is the product of eigenvalues of $x$ [10]. Let $p(x) := \det x$. With $e$ denoting the unit element in $V$, the roots of the univariate polynomial
t → p(te − x) are precisely the eigenvalues of x. We have already seen (in the previous section) that \((\mathcal{V}, \mathbb{R}^n, \lambda)\) is a FTvN system. Hence the corresponding \(p\) is isometric. As \(\lambda(x) = 0 \Rightarrow x = 0\), this \(p\) is also complete.

**Remarks.** The problem of characterizing the ‘isometric’ property of a complete hyperbolic polynomial seems open. However, the commutativity condition \(\lambda(x + y) = \lambda(x) + \lambda(y)\) can be described as follows. Based on the validity of the Lax conjecture \([23]\), Gurvits \([16]\) has shown that in the canonical setting of \(\mathcal{V} = \mathbb{R}^n\) and \(p(e) = 1\), for any two elements \(x, y\), there exist real symmetric \(n \times n\) matrices \(A\) and \(B\) such that

\[
\lambda(tx + sy) = \lambda(tA + sB),
\]

for all \(t, s \in \mathbb{R}\), where the right-hand side denotes the eigenvalue vector of a symmetric matrix. This result, in addition to showing Lidskii type (majorization) results in the setting of hyperbolic polynomials \([16]\, Corollary 1.3\) hence in all Euclidean Jordan algebras, also yields a description of commutativity:

\[
\lambda(x + y) = \lambda(x) + \lambda(y) \iff \lambda(A + B) = \lambda(A) + \lambda(B) \iff A \text{ and } B \text{ strongly operator commute in } S^n.
\]

6 Normal decomposition systems and Eaton triples

Normal decomposition systems were introduced by Lewis \([20]\) to unify various results of convex analysis. One key assumption in the definition of normal decomposition system (see below) is an inequality that is similar to the one that appears in Theorem 4.2. Another feature is the use of orthogonal transformations in place of Jordan frames.

**Definition 6.1** \([20]\) Let \(\mathcal{V}\) be a real inner product space, \(\mathcal{G}\) be a closed subgroup of the orthogonal group of \(\mathcal{V}\), and \(\gamma : \mathcal{V} \rightarrow \mathcal{V}\) be a map satisfying the following conditions:

(a) \(\gamma\) is \(\mathcal{G}\)-invariant, that is, \(\gamma(Ax) = \gamma(x)\) for all \(x \in \mathcal{V}\) and \(A \in \mathcal{G}\).

(b) For each \(x \in \mathcal{V}\), there exists \(A \in \mathcal{G}\) such that \(x = A\gamma(x)\).

(c) For all \(x, y \in \mathcal{V}\), we have \(\langle x, y \rangle \leq \langle \gamma(x), \gamma(y) \rangle\).

Then, \((\mathcal{V}, \mathcal{G}, \gamma)\) is called a normal decomposition system.

Items (a) and (b) in the above definition show that \(\gamma^2 = \gamma\) and \(||\gamma(x)|| = ||x||\) for all \(x\). We state a few relevant properties.

**Proposition 6.2** \([20]\, Proposition 2.3 and Theorem 2.4\) Let \((\mathcal{V}, \mathcal{G}, \gamma)\) be a normal decomposition system. Then,
(i) For any two elements \( x \) and \( y \) in \( V \), we have
\[
\max_{A \in G} \langle Ax, y \rangle = \langle \gamma(x), \gamma(y) \rangle.
\]
Also, \( \langle x, y \rangle = \langle \gamma(x), \gamma(y) \rangle \) holds for two elements \( x \) and \( y \) if and only if there exists an \( A \in G \) such that \( x = A\gamma(x) \) and \( y = A\gamma(y) \).

(ii) The range of \( \gamma \), denoted by \( F \), is a closed convex cone in \( V \).

\textit{Eaton triples} were introduced and studied in \([7, 5, 6]\) from the perspective of majorization techniques in probability. They were also extensively studied in the papers of Tam and Niezgoda, see the references.

**Definition 6.3** Let \( V \) be a finite dimensional real inner product space, \( G \) be a closed subgroup of the orthogonal group of \( V \), and \( F \) be a closed convex cone in \( V \) satisfying the following conditions:

(a) \( \text{Orb}(x) \cap F \neq \emptyset \) for all \( x \in V \), where \( \text{Orb}(x) := \{Ax : A \in G\} \).

(b) \( \langle x, Ay \rangle \leq \langle x, y \rangle \) for all \( x, y \in F \) and \( A \in G \).

Then, \((V, G, F)\) is called an Eaton triple.

It has been shown (see \([29]\), page 14) that in an Eaton triple \((V, G, F)\), \( \text{Orb}(x) \cap F \) consists of exactly one element for each \( x \in V \). Defining \( \gamma : V \rightarrow V \) such that \( \text{Orb}(x) \cap F = \{\gamma(x)\} \), it has been observed that \((V, G, \gamma)\) is a normal decomposition system. Also, given a finite dimensional normal decomposition system \((V, G, \gamma)\), with \( F := \gamma(V) \), \((V, G, F)\) becomes an Eaton triple. Thus, \textit{finite dimensional normal decomposition systems are equivalent to Eaton triples} \([21, 22, 30]\).

While both appear in various matrix and Lie algebraic settings \([35]\), in this paper, we state our results (only) for normal decomposition systems.

**Theorem 6.4** Let \((V, G, \gamma)\) be a normal decomposition system and \( W := \text{span}(\gamma(V)) \). Then, \((V, W, \gamma)\) is a FTvN system.

**Proof.** We verify conditions \((A1)-(A3)\) in Definition \([2.2]\) the norm preserving property of \( \gamma \) comes from Item (b) in Definition \([6.1]\). Thus, condition \((A1)\) holds. Because of Item (c) in Definition \([6.1]\) we have \((A2)\). Suppose \( c \in V \) and \( q \in \gamma(V) \). Let \( q = \gamma(u) \). From Item (i) in Proposition \([6.2]\) we have, for some \( A \in G \), \( \langle c, Au \rangle = \langle \gamma(c), \gamma(u) \rangle \). Letting \( x := Au \), we observe that \( \gamma(x) = \gamma(Au) = \gamma(u) = q \) and so \( \langle c, x \rangle = \langle \gamma(c), q \rangle \). This verifies condition \((A3)\). \(\square\)

Now, let \((V, G, \gamma)\) be a normal decomposition system. Using the notation of Section 2, for any \( u \in V \),
\[
[u] = \{x : \gamma(x) = \gamma(u)\} = \{Au : A \in G\} = : \text{Orb}(u).
\]
In view of the remarks made after Definition 2.9, we see that in the FTvN system \((\mathcal{V}, \mathcal{W}, \gamma)\), a set \(E\) is spectral if and only if it is \(\mathcal{G}\)-invariant, that is, for all \(A \in \mathcal{G}\), \(A(E) \subseteq E\). Similarly, a function \(\Phi : \mathcal{V} \to \mathcal{R}\) is spectral if and only if it is \(\mathcal{G}\)-invariant, that is, for all \(A \in \mathcal{G}\) and \(x \in \mathcal{V}\), \(\Phi(Ax) = \Phi(x)\). (In some literature, \(\mathcal{G}\)-invariant functions are called orbital functions \([36]\).)

We recall the concept of commutativity in a normal decomposition system.

**Definition 6.5** \([12]\) In a normal decomposition system \((\mathcal{V}, \mathcal{G}, \gamma)\), we say that \(x\) and \(y\) **commute** if there exists an \(A \in \mathcal{G}\) such that \(x = A\gamma(x)\) and \(y = A\gamma(y)\).

Analogous to Proposition 4.4, we have the following.

**Proposition 6.6** For elements \(x\) and \(y\) in a normal decomposition system \((\mathcal{V}, \mathcal{G}, \gamma)\), the following are equivalent:

1. \(x\) and \(y\) commute in the normal decomposition system \((\mathcal{V}, \mathcal{G}, \gamma)\).
2. \(\gamma(x + y) = \gamma(x) + \gamma(y)\).
3. \(\langle x, y \rangle = \langle \gamma(x), \gamma(y) \rangle\), that is, \(x\) and \(y\) commute in the FTvN system \((\mathcal{V}, \mathcal{W}, \gamma)\).

**Proof.** When (i) holds, we can write \(x = A\gamma(x)\) and \(y = A\gamma(y)\) for some \(A \in \mathcal{G}\). Letting \(z = \gamma(x) + \gamma(y)\), we have \(x + y = A(\gamma(x) + \gamma(y)) = Az\). Since the range of \(\gamma\) is a closed convex cone (see item (ii) in Proposition 6.2), we can write \(z = \gamma(u)\) for some \(u \in \mathcal{V}\). Then, \(x + y = A\gamma(u)\).

As \(\gamma^2 = \gamma\), we have \(\gamma(x + y) = \gamma(A\gamma(u)) = \gamma(\gamma(u)) = \gamma(u) = z = \gamma(x) + \gamma(y)\). This proves the implication (i) \(\Rightarrow\) (ii).

The equivalence of (ii) and (iii) comes from Proposition 2.6. Finally, the equivalence of (iii) and (i) follows from Proposition 6.2. \(\square\)

We now specialize some results of Section 3. Consider a normal decomposition system \((\mathcal{V}, \mathcal{G}, \gamma)\), let \(F := \gamma(\mathcal{V})\) (which is a closed convex cone) and \(W := \text{span}(F) (= F - F)\). Consider any \(Q \subseteq \mathcal{V}\). As \(\gamma^2 = \gamma\),

\[
Q \cap F = Q \cap \gamma(Q),
\gamma^{-1}(Q) = \gamma^{-1}(Q \cap F) \text{ and } \gamma(\gamma^{-1}(Q)) = Q \cap \gamma(Q).
\]

From Corollary 3.3 with \(E = \gamma^{-1}(Q)\), \(\Phi = 0\), and \(c \in \mathcal{V}\), we get

\[
\sup \left\{ \langle c, x \rangle : x \in \gamma^{-1}(Q) \right\} = \sup \left\{ \langle \gamma(c), q \rangle : q \in Q \cap \gamma(Q) \right\}.
\]

Moreover, attainment of supremum in one problem implies that in the other and the maximum value is given by \(\langle \gamma(c), \gamma(a) \rangle\) for some \(a \in E\) that commutes with \(c\) in \((\mathcal{V}, \mathcal{G}, \gamma)\).

Similarly, from Corollary 3.8 we get

\[
\inf \left\{ ||c - x|| : x \in \gamma^{-1}(Q) \right\} = \inf \left\{ ||\gamma(c) - q|| : q \in Q \cap \gamma(Q) \right\}.
\]
Moreover, attainment of infimum in one problem implies the attainment in the other, and the minimum value is given by $(\tilde{\gamma}(c), \gamma(a))$ for some $a \in E$ that commutes with $-c$ in $(V, G, \gamma)$.

Specialized, we can now recover the results of von Neumann, Chu and Driessel, Tam mentioned in the Introduction.

The papers by Lewis, Eaton, Eaton and Perlman, Lim et al, Niezgoda, and Tam (see the References) contain numerous examples of normal decomposition systems (Eaton triples) related to matrices, Lie and Euclidean Jordan algebras. We specifically note that the space $M_n$ of $n \times n$ complex matrices [20] and any simple Euclidean Jordan algebra [25] are examples of normal decomposition systems. We now provide an example of a FTvN system that is neither a normal decomposition system nor a Euclidean Jordan algebra. Let $V$ be any real inner product space whose dimension is more than one. On $V$, let $S : V \to V$ be linear and inner product preserving (so it is an isometry, but need not be surjective). We assume that $S$ is different from the identity transformation. Then, with $\gamma(x) := Sx$, the triple $(V, V, \gamma)$ is a FTvN system. We claim that this is not a normal decomposition system. Suppose, if possible, $(V, V, \gamma)$ is a normal decomposition system so that there is a closed subgroup $G$ of the group of orthogonal transformations on $V$ satisfying Definition 6.1. Then $\gamma(Ax) = \gamma(x)$ for all $x \in V$ and $A \in G$; so, $S Ax = Sx$ for all $x$. As $S$ is injective, $Ax = x$ for all $x \in V$ implying that $G$ consists only of the identity transformation. But then, by condition (b) in Definition 6.1, $x = \gamma(x) = Sx$. As $S$ is not the identity transformation, we have a contradiction. Thus, $(V, V, \gamma)$ is not a normal decomposition system. Specializing, let $V = \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation through $90^\circ$ about the origin. Then $(\mathbb{R}^2, \mathbb{R}^2, S)$ is a FTvN system. If this were of the form $(V, \mathbb{R}^n, \lambda)$ for some Euclidean Jordan algebra, then, $V = \mathbb{R}^2$, $n = 2$, and $\lambda = S$. But in this setting, $\lambda^2 = \lambda$. This implies that $S^2 = S$, which is false. Hence, $(\mathbb{R}^2, \mathbb{R}^2, S)$ is a FTvN system which is neither a normal decomposition system nor a Euclidean Jordan algebra.

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