EQUIVALENCE OF TWO DEFINITIONS OF SET-THEORETIC YANG-BAXTER HOMOLOGY

JÓZEF H. PRZYTYCKI AND XIAO WANG

Abstract. In 2004, Carter, Elhamdadi and Saito defined a homology theory for set-theoretic Yang-Baxter operators (we will call it the “algebraic” version in this article). In 2012, Przytycki defined another homology theory for pre-Yang-Baxter operators which has a nice graphic visualization (we will call it the “graphic” version in this article). We show that they are equivalent. The “graphic” homology is also defined for pre-Yang-Baxter operators, and we give some examples of its one-term and two-term homologies. In the two-term case, we have found torsion in homology of Yang-Baxter operator that yields the Jones polynomial.

1. Introduction

The Yang-Baxter equation was introduced independently by C.N. Yang (1967) [Yan] and R.J. Baxter (1972) [Bax]. It is well known that a certain solution of Yang-Baxter equation give rise to the Jones polynomial [Jon-2]. In 2004, Carter, Elhamdadi and Saito defined a (co)homology theory for set-theoretic Yang-Baxter operators, from which they gave a way to generate link invariants, cocycle invariants [CES]. In 2012, Przytycki gave a graphical definition of homology for a pre-Yang-Baxter operator [Prz-2]. We provide the definitions of two homology theories for set-theoretic Yang-Baxter operators in Section 2 and show their equivalence in Section 3. In Section 4, we give definitions of one-term and two-term homology of pre-Yang-Baxter operators, and show examples, in particular, we find torsion in two-term homology of Yang-Baxter operator that yields the Jones polynomial. We start from basic definitions.

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**Definition 1.1.** Let $k$ be a commutative ring and $X$ be a set. Consider $V = kX$, the free $k$–module generated by $X$. If a $k$–linear map, $R : V \otimes V \to V \otimes V$, satisfies the following equation

\[(R \otimes Id_V) \circ (Id_V \otimes R) \circ (R \otimes Id_V) = (Id_V \otimes R) \circ (R \otimes Id_V) \circ (Id_V \otimes R),\]

then we say $R$ is a pre-Yang-Baxter operator. The equation above is called a Yang-Baxter equation. If, in addition, $R$ is invertible, then we say $R$ is a Yang-Baxter operator. From now on, we assume $k$ is a commutative ring with identity whenever we deal with Yang-Baxter operators.

**Definition 1.2.** Let $X$ be a set. If $R : X \times X \to X \times X$ is a function that satisfies

\[(R \times Id_X) \circ (Id_X \times R) \circ (R \times Id_X) = (Id_X \times R) \circ (R \times Id_X) \circ (Id_X \times R),\]

then we say $R$ is a pre-set-theoretic Yang-Baxter operator and the equation above is a set-theoretic Yang-Baxter equation. If, in addition, $R$ is invertible, then we say $R$ is a set-theoretic Yang-Baxter operator.

Pre-set-theoretic Yang-Baxter operator leads to Yang-Baxter operator by putting $V = kX$, and extending $R : X \times X \to X \times X$ to $R : V \otimes V \to V \otimes V$. This Yang-Baxter operator is still called pre-set-theoretic Yang-Baxter operator.

2. **Set-theoretic Yang-Baxter homology theories**

Given a pre-set-theoretic Yang-Baxter operator $R$, we have two approaches to homology built on $R$. The “algebraic” version defined in [CES] and the “Graphic” version in [Prz-2]. We discuss them in the next two subsections. We prove the equivalence of them in Section 3.

We first review the “algebraic” version of set-theoretic Yang-Baxter homology theory based on [CES] and then introduce the “graphic” version of set-theoretic Yang-Baxter homology theory.

### 2.1. “Algebraic” homology of Carter, Elhamdadi, and Saito.

**Definition 2.1.** The set $X$ together with a set-theoretic Yang-Baxter operator $R : (X, R)$, is called in [CES] a Yang-Baxter set. We represent a function $R$ by $R(x_1, x_2) = (R_1(x_1, x_2), R_2(x_1, x_2))$.

We use the following notation. Let $I^n$ be the $n$–dimensional cube $I^n (I = [0, 1])$ regarded as a CW (cubical) complex, where $n$ is a positive integer\(^1\). Denote the $k$–skeleton by $I^n_k$ with orientation given by the order of coordinate axes. In particular, every 2–face can be written as

\[d_i^k : X_{k-1} \to X_k \text{ defined by } d_i^k(x_1, x_2, ..., x_{k-1}) = (x_1, x_2, ..., x_{i-1}, e, x_i, ..., x_{k-1});\]

they satisfy $d_i^k d_j^k = d_j^k d_i^k$ where $i < j$.\(^1\)
\[ \epsilon_1 \times \cdots \times \epsilon_{i-1} \times I_i \times \epsilon_{i+1} \times \cdots \times \epsilon_{j-1} \times I_j \times \epsilon_{j+1} \times \cdots \times \epsilon_n, \]

for some \(1 \leq i < j \leq n\), where \(\epsilon_k = 0\) or \(1\), and \(I_i, I_j\) denote two copies of \(I\) at the \(i\)th, \(j\)th positions, respectively.

**Definition 2.2.** The Yang-Baxter coloring of \(I_n\) by a Yang-Baxter set \((X, R)\) is a map \(L : E(I_n) \rightarrow X\), where \(E(I_n)\) denotes the set of edges of \(I_n\), with each edge oriented as above, such that

\[
\begin{align*}
L(\epsilon_1 \times \cdots \times I_i \times \cdots \times 0_j \times \cdots \times \epsilon_n) &= x, \\
L(\epsilon_1 \times \cdots \times I_i \times \cdots \times I_j \times \cdots \times \epsilon_n) &= y,
\end{align*}
\]

then

\[
\begin{align*}
L(\epsilon_1 \times \cdots \times 0_i \times \cdots \times I_j \times \cdots \times \epsilon_n) &= R_1(x, y), \\
L(\epsilon_1 \times \cdots \times I_i \times \cdots \times 1_j \times \cdots \times \epsilon_n) &= R_2(x, y),
\end{align*}
\]

**Definition 2.3.** The initial path in \(I_n\) is the sequence of edges of \(I_n\), \((e_1, \ldots, e_n)\), where

\[e_1 = I_1 \times 0_2 \times \cdots \times 0_n,\]
\[e_2 = 1_1 \times I_2 \times 0_3 \times \cdots \times 0_n,\]
\[\vdots\]
\[e_n = 1_1 \times 1_2 \cdots \times 1_{n-1} \times I_n.\]

**Lemma 2.4.** (S.Carter, M. Elhamdadi, and M.Saito, 2004):

Let \((X, R)\) be a Yang-Baxter set, and \((e_1, \ldots, e_n)\) be the initial path of \(I_n\). For any \(n\)-tuple of elements of \(X\), \((x_1, \ldots, x_n)\), there exists a unique Yang-Baxter coloring \(L\) of \(I_n\) by \((X, R)\) such that \(L(e_i) = x_i\) for all \(i = 1, \ldots, n\).

This lemma gives the following two properties.

1. Each edge has the color uniquely induced by the \(n\)-tuple associated to the initial path of \(I_n\).
(2) Each $k$-face $J$ of $I_n$ has its induced initial path determined by the order of coordinates. Therefore, we can associate to it the $k$-tuple $(y_1, \ldots, y_k)$ determined by colors on its induced initial path. Denote this situation by $L(J) = (y_1, \ldots, y_k)$.

From these two facts, we have a way to map an $n$-tuple to $(n - 1)$-tuple through the face maps in cubic homology theory. Recall that $\partial^c_n$ denotes the $n$-dimensional boundary map in the cubical homology theory. Thus $\partial^c_n(I_n) = \sum_{i=1}^{2n} \epsilon_i J_i$, where $J_i$ is an $(n - 1)$-face and $\epsilon_i = \pm 1$ depending on whether the orientation of $J_i$ matches the induced orientation. For the induced orientation, we take the convention that the inward pointing normal to an $(n - 1)$-face appears last in a sequence of vectors that specifies an orientation, and the orientation of the $(n-1)$-face is chosen so that this sequence agrees with the orientation of the $n$-cube.

Let $(X, R)$ be a Yang-Baxter set. Let $C^Y_B(X)$ be the free abelian group generated by $n$-tuples $(x_1, \ldots, x_n)$ of elements of $X$. Define a homomorphism $\partial^A_n : C^Y_B(X) \to C^Y_B(X)$ by $\partial^A_n((x_1, \ldots, x_n)) = L(\partial^c_n(I_n)) = \sum_{i=1}^{2n} \epsilon_i L(J_i)$. We have $\partial^A_{n-1} \circ \partial^A_n = 0$, and $(C^Y_B(X), \partial^A_n)$ is a chain complex. As usual, we can define $H^A_n = \ker \partial^A_n / \text{im} \partial^A_{n+1}$ to be the “algebraic” version of Yang-Baxter homology group [CES].

2.2. “Graphic” approach to Yang-Baxter homology. In this homology theory, the chain groups are the same as before, that is $C^Y_B(X) = ZX^n$. We define the boundary homomorphism $\partial^G_n : C^Y_B(X) \to C^Y_B(X)$ as follow, $\partial^G_n = \sum_{i=1}^n (-1)^i d_{i,n}$, where $d_{i,n} = d^l_{i,n} - d^r_{i,n}$. We can interpret the face maps through Figure 2.1.

![Figure 2.1](image-url)  

Figure. 2.1; a face map $d_{i,n}$

The meaning of $d^l_{i,n}$, is illustrated in Figure 2.2; $d^r_{i,n}$ can be described similarly.
We have an $n$--tuple as an input and each strand carries the corresponding element of the $n$--tuple. We track down the graph from top to bottom, and at each crossing we apply the fixed Yang-Baxter operator with input the ordered pair consists of two elements carried by the two strands right above the crossing (as in Figure 2.3).

Then the left strand after the crossing carries the $R_1$ function value and the right strand after the crossing carries the $R_2$ function value. In the end, we ignore the element carried by the left most strand, and this procedure generates an $(n - 1)$--tuple consisting of the $n - 1$ elements carried by the other $n - 1$ strands at the bottom.

One can easily check $(X^n, d^G_{i,n})$ form a pre-cubic set, which implies that $(C^YB(X), \partial^G_n)$ is a chain complex. We define $H^G_n = \ker \partial^G_n / \text{im} \partial^G_{n+1}$ as the “graphic” version of Yang-Baxter homology group.
3. Equivalence of two homology theories

**Theorem 3.1.** The “algebraic” and “graphic” definitions of Yang-Baxter homology coincide. More generally, chain complexes leading to above homologies are isomorphic.

**Proof.** Consider an \( n \)-dimensional cube \( I^n \) and denote it as \( I_1 \times \cdots \times I_n \). For any coloring, say \( L(I_1 \times \cdots \times I_n) = (x_1, \ldots, x_n) \), and an \((n-1)\)-face \( J = I_1 \times \cdots \times I_{i-1} \times I_i \times I_{i+1} \times \cdots \times I_n \), we are going to demonstrate that \( L(J) = d_{i,n}^H(x_1, \ldots, x_n) \). To see this, we need to calculate the coloring of the initial path \((a_1, \ldots, a_{n-1})\) of this \((n-1)\)-face. By definition,

\[
\begin{align*}
a_1 &= I_1 \times 0_2 \times 0_3 \times \cdots \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\
a_2 &= I_1 \times I_2 \times 0_3 \times \cdots \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\
& \quad \vdots \\
a_{i-1} &= I_1 \times I_2 \times I_3 \times \cdots \times I_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\
a_i &= I_1 \times I_2 \times I_3 \times \cdots \times I_{i-1} \times I_i \times I_{i+1} \times 0_{i+2} \times \cdots \times 0_n \\
a_{i+1} &= I_1 \times I_2 \times I_3 \times \cdots \times I_{i-1} \times I_i \times 0_{i+1} \times I_{i+2} \times \cdots \times 0_n \\
& \quad \vdots \\
a_{n-2} &= I_1 \times I_2 \times I_3 \times \cdots \times I_{i-1} \times I_i \times 0_{i+1} \times \cdots \times I_{n-1} \times 0_n \\
a_{n-1} &= I_1 \times I_2 \times I_3 \times \cdots \times I_{i-1} \times I_i \times 0_{i+1} \times \cdots \times I_{n-1} \times I_n
\end{align*}
\]

We need another sequence \((b_2, \ldots, b_i)\), where

\[
\begin{align*}
b_2 &= I_1 \times 0_2 \times 0_3 \times \cdots \times 0_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\
b_3 &= I_1 \times I_2 \times 0_3 \times \cdots \times 0_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\
& \quad \vdots \\
b_{i-1} &= I_1 \times I_2 \times I_3 \times \cdots \times I_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\
b_i &= I_1 \times I_2 \times I_3 \times \cdots \times I_{i-2} \times I_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n
\end{align*}
\]

For example, for \( j = i \), the edges of the square are

\[
\begin{align*}
e_{i-1} &= I_1 \times I_2 \times I_3 \times \cdots \times I_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\
b_i &= e_i = I_1 \times I_2 \times I_3 \times \cdots \times I_{i-2} \times I_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\
b_{i-1} &= 0_{i-1} \times I_1 \times I_2 \times I_3 \times \cdots \times I_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n \\
a_{i-1} &= I_1 \times 0_2 \times I_3 \times \cdots \times 0_{i-2} \times 0_{i-1} \times I_i \times 0_{i+1} \times \cdots \times 0_n
\end{align*}
\]

Since

\[
L(e_{i-1}) = x_{i-1}, \quad L(b_i) = x_i
\]
we have

\[
L(a_{i-1}) = R_2(L(e_{i-1}), L(b_i)) = R_2(x_{i-1}, x_i) \\
L(b_{i-1}) = R_1(L(e_{i-1}), L(b_i)) = R_1(x_{i-1}, x_i)
\]

Once we know the color of \( b_j \), we know the colors of \( b_{j-1} \), and \( a_{j-1} \) (at each iteration we deal with a square in Figure 3.1). Thus, recursively, we know all the colors of \( a_j \)’s i.e. we get an \((n - 1)\)-tuple.

In general, we have

\[
L(a_j) = \begin{cases} 
R_2(L(e_j), L(b_{j+1})) & 1 \leq j \leq i - 1 \\
x_{j+1} & i \leq j \leq n - 1 
\end{cases}
\]

\[
L(b_j) = \begin{cases} 
R_1(L(e_j), L(b_{j+1})) & 2 \leq j \leq i - 1 \\
x_j & j = i 
\end{cases}
\]

We can see that this \((n - 1)\)-tuple is the same as the one given by \( d_{t,n}^i \) (compare with Figure 2.2).

Similarly, if we consider \( J = \prod_{i=1}^{n} I_{i-1} \times 0 \times 0 \times i \times 0 \times \cdots \times 0 \times I_n \), we can show \( L(J) = d_{t,n}^i(x_1, \ldots, x_n) \).

As for the sign, we can directly calculate that, for \( L(\prod_{i=1}^{n} I_{i-1} \times \epsilon_i \times I_{i+1} \times \cdot \cdot \cdot \times 0 \times 1 \times \cdot \cdot \cdot \times I_{n}) \), the sign is \((-1)^{n-i}(1 - 2\epsilon_i)\).

Thus, the boundary map of “algebraic” version and the boundary map of “graphic” version only differ by a global sign, therefore, the considered chain complexes are isomorphic and they give isomorphic homology groups. \( \square \)

**Example 3.2.** Comparison of the face maps corresponding to the 2-face \( I_1 \times I_2 \times 1_3 \) of a cube and \( d_{3,3}^i \).
From the figure above, we can see a “curtain-like” object similar to the figure on the right hand side “climbing” on the faces of the cube. For higher dimensions, the similar consideration holds. Therefore this also gives a way to visualize the equivalence.

4. Homology for unital Yang-Baxter operators

This “graphic” definition of Yang-Baxter homology was motivated by the homology theory of self-distributive systems [Prz-1, Prz-2], for example shelves, racks, and quandles. Similarly, we can define one-term and two-term homology not only for set-theoretic Yang-Baxter operators but also for pre-Yang-Baxter operators. Furthermore, we can extend the definition of pre-Yang-Baxter homology to pre-Yang-Baxter operators with Yang-Baxter wall (see Definition 4.2). We will give the general definitions below and discuss some properties of these homology theories.

4.1. One-term Yang-Baxter homology. Let $k$ be a commutative ring with identity, $V = kX$ be a free $k$ module with basis $X$ and $M$ be a right $k$–module. We define in Definition 4.2 the one-term pre-Yang-Baxter chain complex $C^{YB} = (C_n, M, \partial_n)$ from the pre-simplicial module $(C_n, M, d_i)$.

First we recall the notion of a pre-simplicial module.

**Definition 4.1.** The pre-simplicial module $\mathcal{M}$ is a collection of modules $M_n, n \geq 0$, together with maps, called face maps or face operators, $d_i : M_n \to M_{n-1}, \ 0 \leq i \leq n$,

such that:

$$d_id_j = d_{j-1}d_i, \ 0 \leq i < j \leq n,$$
we define a chain complex with chain modules $M_n$ and a boundary map $\partial_n : M_n \to M_{n-1}$ given by:

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i$$

We are ready to define the pre-Yang-Baxter pre-simplicial module.

**Definition 4.2.** ([Leb, Prz-2]) Consider a linear map $R^W : M \otimes V \to M$, such that $R^W \circ (R^W \otimes id_V) = R^W \circ (R^W \otimes id_V) \circ (id_M \otimes R)$ as shown graphically in Figure 4.1, we call this the left wall condition.

Let $C_n = M \otimes V^\otimes n$ and the face map $d_i = d_{i,n} : C_n \to C_{n-1}$ is defined by

$$d_i = (R^W \otimes id^{\otimes n-1}) \circ (id_M \otimes R \otimes id^{\otimes n-2}) \circ (id_M \otimes \ldots \otimes R \otimes id^{\otimes n-i})$$

We can interpret the face maps through Figure 4.2

$(C_n, M, d_i)$ is a pre-simplicial module, and $(C_n, M, \partial_n)$ is the one-term pre-Yang-Baxter chain complex. It’s homology is called the one-term pre-Yang-Baxter homology $H_n(R, R^W)$.

**Proposition 4.3.** Let $(C_n, M, \partial_n)$ be the chain complex of the one-term pre-Yang-Baxter homology. For a fixed element $v \in V$, consider
the map \( f_n : C_n \to C_n \), defined by \( f_n(a) = d_{n+1,n+1}(a \otimes v) \), where \( a \in C_n \), then this is a chain map. We have

1. \( (f_n)_*(H_n) = 0 \).
2. If there is an element \( v \in V \) such that \( f_n \) is invertible, then the one-term pre-Yang-Baxter homology is trivial.

**Proof.** We construct a chain homotopy \( P_n \) between \((-1)^{n+1}f_n \) and zero map, in particular showing that \( f_n \) is a chain map. The chain homotopy \( P_n = C_n \to C_{n+1} \) is defined by \( P_n(v_i \otimes \ldots \otimes v_i) = (-1)^n v_i \otimes \ldots \otimes v_i \otimes v \). We check: \( \partial_{n+1}P_n + P_{n-1}\partial_n = \sum_{i=1}^{n+1}(-1)^i d_i,n+1 P_n + \sum_{j=1}^{n}(-1)^{j} P_n d_{i,n} = (-1)^{n+1} d_{n+1,n+1} P_n = (-1)^{n+1}f_n \). (1) follows because \( \{f_n\} \) is chain homotopic to the 0 map, therefore, \( (f_n)_* \) is the 0 map on homology. (2) follows since if \( f_n \) is invertible, so is \((-1)^{n+1}f_n \), thus \( H_n(C_*) \) is isomorphic to \( H_n(C_*) \) through zero map. This shows that \( H_n(C_*) = 0 \).

In the case that \( M = k \), and of \( V \) acting on \( k \) trivially, the condition making \((V^\otimes n, d_{i,n})\) a pre-simplicial module is equivalent to that the sum of each column of the \( R \) matrix is 1, which we call the column unital condition (e.g. stochastic matrices satisfy the condition).

**Corollary 4.4.** Let \( M = k \), \( V \) act on \( k \) trivially, and \( R \) be a set-theoretic Yang-Baxter operator. If for any pair \((B, D)\), there is a unique \( A \) such that \( R(A, B) = (R_1(A, B), R_2(A, B)) = (C, D) \), then conditions in Proposition 4.3 hold. In particular, biracks satisfy this condition (see Definition 3.1 condition 3, that is right invertibility, in \([\text{CES}]\)).

**Proof.** We need to show that for any \( n \)-tuple \((y_1, \ldots, y_n)\) in \( C_n \), there exists unique \( n \)-tuple \((x_1, \ldots, x_n)\) in \( C_n \) such that \( f_n((x_1, \ldots, x_n)) = d_{n+1,n+1}((x_1, \ldots, x_n, v)) = (y_1, \ldots, y_n) \). Since we know \( y_n \) and \( v \), we get the values of \( x_n \) and \( R_1(x_n, v) \) uniquely. Once we have \( R_1(x_n, v) \), together the value of \( x_{n-1} \), we get the value of \( x_{n-1} \) and \( R_1(x_{n-1}, R_1(x_n, v)) \) uniquely. Thus by this iteration, we get the \( n \)-tuple \((x_1, \ldots, x_n)\) uniquely and this shows \( f_n \) is invertible.

**Example 4.5.** (Compare \([\text{CES}]\)) Let \( F \) be a commutative ring with identity. Let \( k = F[[s^\pm 1, t^\pm 1]]/(1 - s)(1 - t) \), then

\[
R(x, y) = (R_1(x, y), R_2(x, y)) = ((1 - s)x + ty, sx + (1 - t)y)
\]

is a set-theoretic Yang-Baxter operator satisfying the conditions in Corollary 4.4. This holds because for any given \( y \) and \( a = R_2(x, y) = sx + (1 - t)y \), we can solve \( x = s^{-1}(a - (1 - t)y) \). Thus the one-term homology of this operator is trivial.
Remark 4.6. Pre-Yang-Baxter coming from racks \((X, \ast)\) where \(R(a, b) = (b, a \ast b)\) satisfies the conditions in Proposition 4.3. Thus it has zero one-term homology (see [Prz-1]).

4.2. Two-term Yang-Baxter homology. Let \(k\) be a commutative ring with identity, \(V = kX\) be a free \(k\) module with basis \(X\), \(M\) be a right \(k\)–module and \(N\) be a left \(k\)–module. We define in Definition 4.8 the two-term pre-Yang-Baxter chain complex \(C^{\text{YB}} = (C_n, M, N, \partial_n)\) from the pre-cubical module \((C_n, M, N, d^i)\).

Definition 4.7. The pre-cubical module \(\mathcal{M}\) is a collection of modules \(M_n, n \geq 0\), together with maps, called face maps or face operators,

\[
d^i : M_n \to M_{n-1}, \quad 1 \leq i \leq n, \epsilon = 0, 1
\]

such that:

\[
d^i d^j = d^j d^i_{j-1}, \quad 1 \leq i < j \leq n, \epsilon, \delta = 0, 1
\]

we define a chain complex with chain groups \(M_n\) and a boundary map \(\partial_n : M_n \to M_{n-1}\) given by:

\[
\partial_n = \sum_{i=1}^{n} (-1)^i(d^0_i - d^1_i).
\]

We are ready to define the pre-Yang-Baxter pre-cubical module.

Definition 4.8. ([Leb][Prz-2]) Consider a linear map \(R^W_l : M \otimes V \to M\), such that \(R^W_l \circ (R^W_l \otimes id_V) = R^W_l \circ (R^W_l \otimes id_V) \circ (id_M \otimes R)\), and a linear map \(R^W_r : V \otimes N \to N\), such that \(R^W_r \circ (id_V \otimes R^W_r) = R^W_r \circ (id_V \otimes R^W_r) \circ (R \otimes id_N)\) we call them the left and right wall conditions (graphically see Figure 4.3 and Figure 4.4).

\[
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\]

Figure 4.3: left wall condition
Let $C_n = M \otimes V^\otimes n \otimes N$ and face maps $d'_i : C_n \to C_{n-1}$ are given by

$$d'_i = (R_i^W \otimes \text{id}^\otimes n-2 \otimes \text{id}_N) \circ (\text{id}_M \otimes \text{id} \otimes R \otimes \text{id}^\otimes n-1 \otimes \text{id}_N) \circ (\text{id}_M \otimes \text{id} \otimes R \otimes \text{id}^\otimes n-3 \otimes \text{id}_N) \circ \cdots \circ (\text{id}_M \otimes \text{id}^\otimes n-i \otimes \text{id}_N)$$

and

$$d''_i = (\text{id}_M \otimes \text{id}^\otimes n-1 \otimes R^W_i) \circ (\text{id}_M \otimes \text{id}^\otimes n-2 \otimes R \otimes \text{id}_N) \circ (\text{id}_M \otimes \text{id}^\otimes n-3 \otimes R \otimes \text{id} \otimes \text{id}_N) \circ \cdots \circ (\text{id}_M \otimes \text{id}^\otimes n-i-1 \otimes \text{id}_N)$$

We can interpret the face maps $d'_i$, $d''_i$ and their difference through Figure 4.5.

$$(C_n, M, N, d'_i)$$

is a pre-cubical module, and $$(C_n, M, N, \partial_n)$$ is the two-term pre-Yang-Baxter chain complex.

In the case that $M = k = N$, and the action of $V$ on $k$ is the trivial action, the conditions making $(V^\otimes n, d''_i)$ a pre-cubical module is equivalent to saying that $R$ has the column unital condition.

**Example 4.9.** We give a family of unital Yang-Baxter operator $R : V \otimes V \to V \otimes V$, where $V = k \{v_1, \ldots, v_m\}$, $k = \mathbb{Q}[y, y^{-1}]$ and $m$
is a positive integer. For any given \( m \), \( R \) can be represented by its coefficients,

\[
R_{ijkl} = \begin{cases} 
1, & \text{if } i=j=k=l; \\
1, & \text{if } l=i>j=k; \\
y^2, & \text{if } l=i<j=k; \\
1 - y^2, & \text{if } k=i<j=l; \\
0, & \text{otherwise.}
\end{cases}
\]

These family of Yang-Baxter operators are unital, for example when \( m = 2 \), it is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - y^2 & 1 & 0 \\
0 & y^2 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Computation shows interesting pattern in the two-term Yang-Baxter homology of this family of Yang-Baxter operators.

**Conjecture 4.10.** ([Wan]) When \( m = 2 \), \( H_n = k^2 \bigoplus (k/(1-y^2))^{a_n} \bigoplus (k/(1-y^4))^{s_{n-2}} \), where \( s_n = \sum_{i=1}^{n+1} f_i \) is the partial sum of Fibonacci sequence, where \( f_1 = 1 = f_2 \) and \( a_n \) is given by \( 2^n = 2 + a_{n-1} + s_{n-2} + a_n + s_{n-1} \) with \( a_1 = 0 \). We verified the conjecture for \( n \leq 10 \).

What fascinates us is that this family of Yang-Baxter operators come from the Yang-Baxter operators giving \( sl_m \) polynomial invariants of links (substitutions to the Homflypt polynomial). For example, when \( m = 2 \), the matrix is

\[
\begin{bmatrix}
-q & 0 & 0 & 0 \\
0 & q^{-1} - q & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -q
\end{bmatrix}
\]

If we divide elements in each column by the sum of those elements and make the substitution \( y = (1 + q^{-1} - q)^{-1/2} \), we will get the matrix in Example 4.9. In general, we can get our family in Example 4.9 in a similar way (normalizing columns) and again they are Yang-Baxter operators. More interestingly, this new family of Yang-Baxter operators also provide \( sl_m \) polynomial invariants of links [Wan]. This fact is implicit in [Jon-2].

### 4.3. Cocycle invariant from Yang-Baxter homology.

In the paper [CES] where Carter, Elhamdadi and Saito define their set-theoretic Yang-Baxter homology, they also define a 2-cocycle link invariant from the set-theoretic Yang-Baxter homology. This can also be done in the
Two definitions of set-theoretic Yang-Baxter homology

In [Prz-3], it demonstrates the third Reidemeister move preserve the (co)homology of the (co)cycle constructed form a knot diagram. See Figure 4.6 and [Prz-3] for details.

Figure. 4.6; relation between third Reidemeister move and 2-(co)cycle condition

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