Occupation time of Lévy processes with jumps rational Laplace transforms

Djilali Ait-Aoudia*

Abstract

We are interested in occupation times of Lévy processes with jumps rational Laplace transforms. The corresponding boundary value problems via the Feynman-Kac representation are solved to obtain an explicit formula for the joint distribution of the occupation time and the terminal value of the Lévy processes with jumps rational Laplace transforms.

Keywords: rational Laplace transforms jump-diffusion process; occupation times.

AMS MSC 2010: 60J60; 60G51.

Submitted to ECP on September 14, 2017, final version accepted on September 17, 2018.

1 Introduction

The occupation time is the amount of time a stochastic process stays within a certain range. It is an interesting topic for stochastic processes. Many explicit results on Laplace transforms for occupation times have been obtained for some well-known examples of Lévy process. For a standard Brownian motion $W = \{W_t : t \geq 0\}$, P. Lévy’s arcsine law is a well-known result. It states the following, let $\Gamma^+(t)$ be the time $W$ spends above 0 up to time $t$:

$$\Gamma^+(t) = \int_0^t 1_{\{W_s > 0\}} ds.$$ 

Lévy [10] (for more details see Chapter IV of [16]) showed that for each $t > 0$ the variable $\Gamma^+(t)/t$ follows the arcsine law:

$$P\left(\frac{\Gamma^+(t)}{t} \in du\right) = \frac{du}{\pi \sqrt{u(1-u)}}, \quad 0 < u < 1.$$ 

This result was then extended to a Brownian motion with drift by Akahori [2] and Takács [14]. After that, the investigation on occupation times of Lévy processes has made much great progress. For recent works in this topic, see [1], [3], [12], [9], [15] and the references therein for more details.

In this paper, we are interested in the joint Laplace transforms of $X = (X_t)_{t \geq 0}$ and its occupation times, i.e,

$$E_x \left[ e^{-\beta \int_0^\infty 1_{\{\alpha < X_t < M\}} dt + \gamma X_\alpha} \right],$$ 

where $\alpha > 0$, $\beta > 0$, $\gamma$ is some suitable constant and $e_\alpha$ is an independent (of $X$) exponential random variable with rate $\alpha > 0$ and $X = (X_t)_{t \geq 0}$ is a Lévy process with jumps

*HEC Montréal, CANADA. E-mail: djilali.ait-aoudia@hec.ca
Occupation time of Lévy processes with jumps rational Laplace transforms

rational Laplace transforms proposed by Lewis and Mordecki [11], see also Kuznetsov [8]. And the purpose is deriving formulas for

$$\psi(x) = \int_0^\infty ae^{-\alpha T} E_x \left[ e^{-\beta f_0 T} 1_{(b < x_t < a)} d t + \gamma X_T \right] d T.$$  \hspace{1cm} (1.2)

This extends recent results obtained in Ait-Aoudia and Renaud [1], (Theorem 2) on the processes with hyper-exponential jumps. More precisely, to find an explicit formula for the function $\psi(x)$ in Equation (1.2), the corresponding boundary value problem via the Feynman-Kac representation is considered. By direct calculation, the associated ordinary integro-differential equation (OIDE) is transformed into a homogeneous ordinary differential equation (ODE) of higher order, which is then solved in closed form and its solution equals to $\psi(x)$.

Results obtained here can be applied to price occupation time derivatives as in Cai et al. [3], in which the authors have noted that there are several products in the real market with payoffs depending on the occupation times of an interest rate or a spread of swap rates. For other investigations, see, e.g., [15], [17] and [18].

The rest of the paper is organized as follows. In section 2, we introduce the jump-diffusion process having jumps with rational Laplace transform. Section 3 contains our main results.

2 The model

A Lévy jump-diffusion process $X = \{X_t, \ t \geq 0\}$ is defined as

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$  \hspace{1cm} (2.1)

where $\mu \in \mathbb{R}$ and $\sigma > 0$ represent the drift and volatility of the diffusion part respectively, $W = \{W_t, \ t \geq 0\}$ is a (standard) Brownian motion, $N = \{N_t, \ t \geq 0\}$ is a homogeneous Poisson process with rate $\lambda$ and $\{Y_i, i = 1, 2, \ldots\}$ are independent and identically distributed random variables supported in $\mathbb{R} \setminus \{0\}$; moreover, $\{W_t, t \geq 0\}$, $\{N_t, t \geq 0\}$ and $\{Y_i, i = 1, 2, \ldots\}$ are mutually independent; finally, the probability density function (pdf) of $Y_1$ is given by

$$f(y) = \sum_{j=1}^{m} \sum_{i=1}^{n_j} p_{ij} \frac{\eta_j^i y^{i-1}}{(i-1)!} e^{-\eta_j y} 1_{y > 0} + \sum_{j=1}^{n} q_j \frac{\theta_j (y)^{j-1} (j-1)!}{(i-1)!} \rho y^{j} 1_{y < 0},$$  \hspace{1cm} (2.2)

where, $p_{ij}, q_j \geq 0$ and they are such $\sum_{j=1}^{m} \sum_{i=1}^{n_j} p_{ij} + \sum_{j=1}^{n} q_j = 1$. The parameters $\eta_j$ and $\theta_j$ can in principle take complex values (see [11]) with

$$0 < \eta_1 < \text{Re}(\eta_2) < \cdots < \text{Re}(\eta_m),$$

$$0 < \theta_1 < \text{Re}(\theta_2) < \cdots < \text{Re}(\theta_n).$$

Another important tool to establish the key result of the article is the infinitesimal generator of $X$. Note that $X$ is a Markovian process and its infinitesimal generator is given by

$$Lh(x) := \lim_{t \searrow 0} \frac{\mathbb{E}[h(X_t) | X_0 = x] - h(x)}{t} = \mu h'(x) + \sigma^2 \frac{h''(x)}{2} + \lambda \left( \int_{-\infty}^{+\infty} h(x + y) f(y) dy - h(x) \right).$$  \hspace{1cm} (2.3)

for any bounded and twice continuously differentiable function $h$.

Throughout the rest of the paper, the law of $X$ such that $X_0 = x$ is denoted by $P_x$ and the corresponding expectation by $E_x$; we write $P$ and $E$ when $x = 0$. The Lévy exponent
We first show that
\[ G(\xi) = \frac{\ln E[\exp(\xi X_t)]}{t} \]
\[ = \mu\xi + \frac{\sigma^2}{2}\xi^2 + \lambda \left( E[e^{\xi Y_1}] - 1 \right) \]
\[ = \mu\xi + \frac{\sigma^2}{2}\xi^2 + \lambda \left( \sum_{j=1}^{m_j} \sum_{i=1}^{m_j} \frac{p_{ij}(\eta_j)}{(\eta_j - \xi)^i} + \sum_{j=1}^{n_j} \sum_{i=1}^{n_j} \frac{q_{ij}(\theta_j)}{(\xi + \theta_j)^i} - 1 \right). \]
Accordingly, \( G \) is a rational function on \( \mathbb{C} \). The equation \( G(\xi) - \alpha = 0 \) with \( \alpha > 0, \sigma > 0 \) and \( \mu \in \mathbb{R} \) yields \( S = M + N + 2 \) zeros with \( M = \sum_{i=1}^{m_j} m_i \) and \( N = \sum_{j=1}^{n_j} m_j \) (see [8] for details). Let us denote the zeros of \( G(\xi) - \alpha \) in the half-plane \( \text{Re}(\xi) > 0 \) \{Re(\xi) < 0\} as \( \rho_{1,\alpha}, \rho_{2,\alpha}, \ldots, \rho_{M+1,\alpha}, \hat{\rho}_{1,\alpha}, \hat{\rho}_{2,\alpha}, \ldots, \hat{\rho}_{N+1,\alpha} \).

3 Main results
Throughout this paper \( X = \{X_t, t \geq 0\} \) will be a Lévy process of the type described before, that is with jumps rational Laplace transforms. The time spent by \( X \) between the lower barrier \( h \) and the upper barrier \( H \), from time 0 to time \( T \), is given by
\[ \int_0^T 1_{(h < X_t < H)} dt. \]
Our main objective is to obtain the joint distribution of \( \int_0^T 1_{(h < X_t < H)} dt \) and \( X_{e_x} \), where \( e_x \) is an independent (of \( X \)) exponential random variable with rate \( \alpha > 0 \). In order to do so, we will compute the following joint Laplace-Carson transform with respect to \( T \): for each \( x \in \mathbb{R} \), set
\[ \psi(x) = E_x \left[ e^{-\beta \int_0^T 1_{(h < X_t < H)} dt + \gamma X_{e_x}} \right], \] (3.1)
where \( \beta > 0, \alpha > 0 \) and we assume that \( 0 < \gamma < \min(\eta_1, \theta_1) \) and \( G(\gamma) < \alpha \). Clearly, we have
\[ \psi(x) = \int_0^\infty e^{-\alpha T} E_x \left[ e^{-\beta \int_0^T 1_{(h < X_t < H)} dt + \gamma X_T} \right] dT. \] (3.2)
By the Feynman-Kac formula (see, for instance, [13] Theorem 1.4.3) we have that \( \psi(x) \) must satisfy
\[ (\mathcal{L} - \alpha - \beta 1_{(h < x < H)}) \psi(x) = -\alpha e^{\gamma x}, \quad x \in \mathbb{R}. \] (3.3)
Now, our goal is to solve the boundary problem (3.3) and find explicit formulae for \( \psi(x) \).

We first show that \( \psi \) satisfies an integro-differential equation and then derive an ordinary differential equation for \( \psi \). Based on the ODE, we show \( \psi \) can be written as a linear combination of known exponential functions.

Let \( \mathcal{P}_0(\zeta) = \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} (-\zeta + \eta_j) \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} (\zeta + \theta_j) \), then \( \mathcal{P}_0(\zeta) = \mathcal{P}_0(\zeta)(G(\zeta) - \alpha) \) is a polynomial whose zero coincide with those of \( G(\zeta) - \alpha \). Also, denote by \( D_\alpha \) the differential operator such that its characteristic polynomial is \( \mathcal{P}_0(\zeta) \).

The following Lemma will be needed for our proof of Proposition 3.2.

Lemma 3.1. Let \( \delta^{(k)} \) indicate the \( k \)-th derivative with respect to \( x \) of any differentiable function. Let \( \phi \) be a bounded and continuous function on \( \mathbb{R} \) and for \( \delta > 0 \), we define two functions \( F^+ \) and \( F^- \) such that
\[ F^+(i, \delta, x) = \left( \frac{d}{dx} + \delta \right)^{(i)} e^{-\delta x} \int_{-\infty}^{x} \phi(y)(x-y)^{i-1} e^{\delta y} dy, \] (3.4)
\[ F^-(i, \delta, x) = \left( -\frac{d}{dx} + \delta \right)^{(i)} e^{\delta x} \int_{x}^{+\infty} \phi(y)(y-x)^{i-1} e^{-\delta y} dy, \] (3.5)
with \( (\pm \frac{d}{dx} + \delta)^{(i)} \) be the Generalized Leibniz operator such that
which is the desired result.

Then for all $i \geq 1$,

$$F^+(i, \delta, x) = (i - 1)! \phi(x).$$  \hfill (3.6)

\textbf{Proof.} We need only to prove first part of the Lemma, the proof of the second part is similar. We proceed by induction on $i$. For $i = 1$, we have

$$F^+(1, \delta, x) = \left( \frac{d}{dx} + \delta \right) e^{-\delta x} \int_{-\infty}^{x} \phi(y) e^{\delta y} dy = \phi(x).$$

Moreover, for all $i \geq 2$,

$$\left( \frac{d}{dx} + \delta \right) e^{-\delta x} \int_{-\infty}^{x} \phi(y)(x - y)^{i-1} e^{\delta y} dy = i e^{-\delta x} \int_{-\infty}^{x} \phi(y)(x - y)^{i-1} e^{\delta y} dy + (i - 1) e^{-\delta x} \int_{-\infty}^{x} \phi(y)(x - y)^{i-2} e^{\delta y} dy$$

$$= (i - 1) e^{-\delta x} \int_{-\infty}^{x} \phi(y)(x - y)^{i-2} e^{\delta y} dy.$$  \hfill (3.6)

It follows inductively that for all $i \geq 2$,

$$F^+(i, \delta, x) = \left( \frac{d}{dx} + \delta \right)^{(i)} e^{-\delta x} \int_{-\infty}^{x} \phi(y)(x - y)^{i-1} e^{\delta y} dy$$

$$= (i - 1) \left( \frac{d}{dx} + \delta \right)^{(i-1)} e^{-\delta x} \int_{-\infty}^{x} \phi(y)(x - y)^{i-2} e^{\delta y} dy$$

$$= (i - 1)! F^+(i - 1, \delta, x)$$

$$= (i - 1)! \phi(x),$$

which is the desired result. \hfill \Box

We may now state.

\textbf{Proposition 3.2.} Suppose a bounded solution $\psi$ defined on $\mathbb{R}$ to the boundary value problem (3.3) exists. Then on $\mathbb{R} \setminus \{h, H\}$, $\phi(x) = \psi(x) + \alpha e^{\gamma x} / (\Gamma(\gamma) - \alpha - \beta 1_{(h < x < H)})$ is infinitely differentiable and satisfies the ODE

$$D_\alpha \phi = 0, \text{ on } \mathbb{R} \setminus \{h, H\},$$  \hfill (3.7)

with $\alpha = \alpha + \beta$ on $(h, H)$ and $\alpha = \alpha$ on $(-\infty, h) \cup (H, +\infty)$. Hence,

$$\psi(x) = \begin{cases} \sum_{k=1}^{M+1} Q_k^L e^{\beta_k \cdot x} - \frac{\alpha e^{\gamma x}}{\zeta(\gamma) - \alpha}, & x \leq h, \\ \sum_{k=1}^{M+1} Q_k^H e^{\beta_k \cdot x} + \sum_{k=1}^{N+1} Q_k^L e^{\beta_k \cdot x} - \frac{\alpha e^{\gamma x}}{\zeta(\gamma) - \alpha - \beta}, & h < x < H, \\ \sum_{k=1}^{N+1} Q_k^L e^{\beta_k \cdot x} - \frac{\alpha e^{\gamma x}}{\zeta(\gamma) - \alpha}, & x \geq H, \end{cases}$$  \hfill (3.8)

for some constants $Q_k^L, Q_k^H, Q_k^L$ and $Q_k^U$.
Occupation time of Lévy processes with jumps rational Laplace transforms

Proof. Using the same idea as in Chen et al.[6] (see also, Cai et al.[3]). Applying the infinitesimal generator $\mathcal{L}$ to the function $\phi$, we obtain

$$\mathcal{L}\phi(x) = \frac{\sigma^2}{2} \phi''(x) + \mu \phi'(x) + \lambda \sum_{i=1}^{n} q_{ij} \frac{1}{(i-1)!} e^{-\theta_j x} \int_{-\infty}^{x} \phi(y)(x-y)^{i-1} e^{\theta_j y} dy + \lambda \sum_{j=1}^{m} \sum_{i=1}^{m_j} p_{ij} \frac{1}{(i-1)!} e^{\eta_j x} \int_{x}^{\infty} \phi(y)(y-x)^{i-1} e^{-\eta_j y} dy - \lambda \phi(x).$$

Next, $\phi$ will be shown to satisfy an ODE. Thanks to Lemma (3.1), we get for $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, m_j$,

$$\left( \frac{d}{dx} + \theta_j \right)^{(i)} e^{-\theta_j x} \int_{-\infty}^{x} (x-y)^{i-1} \phi(y) e^{\theta_j y} dy = (i-1)! \phi(x).$$

Similarly, we obtain for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, n_j$,

$$\left( -\frac{d}{dx} + \eta_j \right)^{(i)} e^{\eta_j x} \int_{x}^{\infty} (y-x)^{i-1} \phi(y) e^{-\eta_j y} dy = (i-1)! \phi(x).$$

Now, since $\mathcal{L} e^{\gamma x} = G(\gamma) e^{\gamma x}$ then from (3.3), it easily follows that for $x \in (h, H)$,

$$(\mathcal{L} - \alpha - \beta 1_{(h,x<H\cup H,+)}) \phi(x) = (\mathcal{L} - \alpha - \beta) \phi(x)$$

$$= (\mathcal{L} - \alpha - \beta) (\psi(x) + \frac{\alpha e^{\gamma x}}{G(\gamma) - \alpha - \beta})$$

$$= (\mathcal{L} - \alpha - \beta) \psi(x) + \frac{\alpha \mathcal{L} e^{\gamma x}}{G(\gamma) - \alpha - \beta} - \frac{\alpha(\alpha + \beta)e^{\gamma x}}{G(\gamma) - \alpha - \beta}$$

$$= -\alpha e^{\gamma x} + \frac{\alpha G(\gamma)e^{\gamma x}}{G(\gamma) - \alpha - \beta} - \frac{\alpha(\alpha + \beta)e^{\gamma x}}{G(\gamma) - \alpha - \beta}$$

$$= -\alpha e^{\gamma x} + \alpha e^{\gamma x} = 0.$$

(3.9)

The same computation will yield, for $x \in (\infty, h) \cup (H, +\infty)$,

$$(\mathcal{L} - \alpha - \beta 1_{(h,x<H\cup H,+)}) \phi(x) = (\mathcal{L} - \alpha) \phi(x)$$

$$= (\mathcal{L} - \alpha) (\psi(x) + \frac{\alpha e^{\gamma x}}{G(\gamma) - \alpha})$$

$$= -\alpha e^{\gamma x} + \alpha e^{\gamma x} = 0.$$

(3.10)

Thanks to Proposition 3.3 in the work of Chen et al.[5], $\phi$ is infinitely differentiable on $\mathbb{R} \setminus \{h, H\}$ and for $x \in \mathbb{R} \setminus \{h, H\}$,

$$0 = \prod_{j=1}^{m} \prod_{i=1}^{m_j} \left( -\frac{d}{dx} + \eta_j \right)^{(i)} \prod_{j=1}^{m} \prod_{i=1}^{m_j} \left( \frac{d}{dx} + \theta_j \right)^{(i)} (\mathcal{L} - \alpha) \phi(x)$$

$$= \prod_{j=1}^{m} \prod_{i=1}^{m_j} \left( -\frac{d}{dx} + \eta_j \right)^{(i)} \prod_{j=1}^{m} \prod_{i=1}^{m_j} \left( \frac{d}{dx} + \theta_j \right)^{(i)} \left( \frac{\sigma^2}{2} \frac{d^2}{dx^2} + \mu \frac{d}{dx} - \lambda - \alpha \right) \phi(x)$$

$$+ \lambda \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \prod_{i=1}^{m_k} \prod_{j=1}^{m_j} \frac{1}{(i-1)!} \left( -\frac{d}{dx} + \eta_k \right)^{(i)} p_{ij} \left( \eta_j \right)^{(i)} (i-1)! \phi(x).$$

ECP 23 (2018), paper 68. http://www.imstat.org/ecp/
Occupation time of Lévy processes with jumps rational Laplace transforms

\[
\psi(x) = \left\{ \begin{array}{ll}
\sum_{k=1}^{M^+} Q_k^L e^{\phi_k^{-x}} + \sum_{k=1}^{N^+} Q_k^L e^{\phi_k^{-x}} - \frac{\alpha e^{-x}}{G(\gamma)-\alpha}, & x \leq h, \\
\sum_{k=1}^{N^+} Q_k^U e^{\phi_k^{-x}} + \sum_{k=1}^{M^+} Q_k^U e^{\phi_k^{-x}} - \frac{\alpha e^{-x}}{G(\gamma)-\alpha}, & h < x < H, \\
\sum_{k=1}^{N^+} Q_k^U e^{\phi_k^{-x}} + \sum_{k=1}^{M^+} Q_k^U e^{\phi_k^{-x}} - \frac{\alpha e^{-x}}{G(\gamma)-\alpha}, & x \geq H,
\end{array} \right.
\]  

(3.12)

Furthermore, we can argue that the coefficients \(Q_{0,k}^L\) and \(Q_{0,k}^U\) should be zero. In fact, we know that

\[
\lim_{x \to \pm \infty} \frac{\psi(x)}{e^{\gamma x}} = +\infty,
\]

which implies \(Q_{0,k}^L\) and \(Q_{0,k}^U\) must be zero and the proof is complete.

**Proposition 3.3.** Suppose that \(\psi\) is a bounded solution to the boundary value problem (3.3) and,

\[
\psi(x) = \left\{ \begin{array}{ll}
\sum_{k=1}^{M^+} Q_k^L e^{\phi_k^{-x}} - \frac{\alpha e^{-x}}{G(\gamma)-\alpha}, & x \leq h, \\
\sum_{k=1}^{M^+} Q_k^L e^{\phi_k^{-x}} + \sum_{k=1}^{N^+} Q_k^U e^{\phi_k^{-x}} - \frac{\alpha e^{-x}}{G(\gamma)-\alpha}, & h < x < H, \\
\sum_{k=1}^{N^+} Q_k^U e^{\phi_k^{-x}} - \frac{\alpha e^{-x}}{G(\gamma)-\alpha}, & x \geq H,
\end{array} \right.
\]  

(3.13)

for somme constants \(Q_k^L, Q_k^0, Q_k^U\) and \(Q_k^U\). Then the constant vector

\[
Q = (Q_1^L, Q_1^U, Q_2^L, Q_2^U, \ldots, Q_{M+1}^L, Q_{M+1}^U), i = 1, \ldots, M + 1, j = 1, \ldots, N + 1
\]

satisfies a linear system

\[
AQ = V.
\]  

(3.14)
Here $V$ is an $2S = 2(M + N + 2)$-dimensional vector:

$$V = (c_1 - c_0) \begin{pmatrix} V_1(h) & V_2(h) & V_3(h) & V_1(H) & V_2(H) & V_3(H) \end{pmatrix}^T$$

(3.15)

where

$$c_1 = \frac{\alpha}{G(\gamma) - \alpha}, \quad c_0 = \frac{\alpha}{G(\gamma) - \alpha - \beta}$$

$$V_1(s) = e^{\gamma s} \begin{pmatrix} 1 & \gamma \end{pmatrix}$$

$$V_2(s) = e^{\gamma s} \begin{pmatrix} \frac{1}{(\eta_1 - \gamma)} & \frac{1}{(\eta_1 - \gamma)^2} & \cdots & \frac{1}{(\eta_m - \gamma)^{m-1}} \\ \frac{1}{(\eta_1 - \gamma)^2} & \frac{1}{(\eta_1 - \gamma)^3} & \cdots & \frac{1}{(\eta_m - \gamma)^m} \end{pmatrix}$$

$$V_3(s) = e^{\gamma s} \begin{pmatrix} \frac{1}{(\eta_1 - \gamma)} & \frac{1}{(\eta_2 - \gamma)^2} & \cdots & \frac{1}{(\eta_n - \gamma)^n} \end{pmatrix},$$

and $A$ is an $2S \times 2S$ matrix

$$A = \begin{pmatrix} BO_1 & \hat{B}D_1 \\ BD_2 & BO_2 \end{pmatrix}$$

(3.16)

where $O_1, D_1, O_2$ and $D_2$ are four $S \times S$ diagonal matrices given by the formulas

$$O_1 = \text{diag} \left( e^{\rho_1 h}, \ldots, e^{\rho_{M+1} h}, e^{\rho_{M+1} h}, \ldots, e^{\rho_{N+1} h} \right)$$

$$D_1 = \text{diag} \left( 0, \ldots, 0, e^{\rho_1 h}, \ldots, e^{\rho_{N+1} h} \right)$$

$$O_2 = \text{diag} \left( e^{\rho_{M+1} h}, \ldots, e^{\rho_{M+1} h}, e^{\rho_{M+1} h}, \ldots, e^{\rho_{N+1} h} \right)$$

$$D_2 = \text{diag} \left( 0, \ldots, 0, e^{\rho_{M+1} h}, \ldots, e^{\rho_{N+1} h} \right),$$

and $B$ and $\hat{B}$ are given by

$$B = \Theta_{M,N} \left[ (\rho_1, \ldots, \rho_{M+1}); (\hat{\rho}_1, \ldots, \hat{\rho}_{N+1}) \right]$$

$$\hat{B} = \Theta_{N,M} \left[ (\hat{\rho}_1, \ldots, \hat{\rho}_{N+1}); (\rho_1, \ldots, \rho_{M+1}) \right],$$

$\Theta_{i,j}$ is defined such that for all $\kappa = [(u_1, \ldots, u_i); (v_1, \ldots, v_j)]$

$$\Theta_{i,j} = \begin{pmatrix} 1 & \cdots & 1 & -1 & \cdots & -1 \\ u_1 & \cdots & u_1 & -v_1 & \cdots & -v_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (\eta_{m} - u_1) & \cdots & (\eta_{m} - u_1) & (\eta_{m} - v_1) & \cdots & (\eta_{m} - v_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \eta_{m} - u_1 & \cdots & \eta_{m} - u_1 & \eta_{m} - v_1 & \cdots & \eta_{m} - v_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (\eta_{m} - u_1) & \cdots & (\eta_{m} - u_1) & (\eta_{m} - v_1) & \cdots & (\eta_{m} - v_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \eta_{m} - u_1 & \cdots & \eta_{m} - u_1 & \eta_{m} - v_1 & \cdots & \eta_{m} - v_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (\eta_{m} - u_1) & \cdots & (\eta_{m} - u_1) & (\eta_{m} - v_1) & \cdots & (\eta_{m} - v_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \eta_{m} - u_1 & \cdots & \eta_{m} - u_1 & \eta_{m} - v_1 & \cdots & \eta_{m} - v_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \eta_{m} - u_1 & \cdots & \eta_{m} - u_1 & \eta_{m} - v_1 & \cdots & \eta_{m} - v_1 \end{pmatrix}$$

(3.17)
Occupation time of Lévy processes with jumps rational Laplace transforms

Proof. We suppose that \( \psi \) is a bounded solution to the boundary value problem (3.3) and

\[
\psi(x) = \begin{cases}
  w_1(x) = \sum_{k=1}^{M+1} Q_k^L e^{\rho_k \alpha + \beta x} - \frac{\alpha e^{\gamma x}}{\alpha - \beta}, & x \leq h, \\
w_2(x) = \sum_{k=1}^{M+1} Q_k^L e^{\rho_k \alpha + \beta x} + \sum_{k=1}^{N+1} Q_k^s e^{\rho_k \alpha + \beta x} - \frac{\alpha e^{\gamma x}}{\alpha - \beta}, & h < x < H, \\
w_3(x) = \sum_{k=1}^{N+1} Q_k^s e^{\rho_k \alpha + \beta x} - \frac{\alpha e^{\gamma x}}{\alpha - \beta}, & x \geq H.
\end{cases}
\]

(3.18)

Then equation (3.3) can be rewritten as three separate equations in the regions \((-\infty, h), (h, H)\) and \((H, +\infty)\). For \( x < h \),

\[
-\alpha e^{\gamma x} = \frac{\sigma^2}{2} w_1''(x) + \mu w_1'(x) - (\lambda + \alpha) w_1(x),
\]

and for \( x > H \),

\[
-\alpha e^{\gamma x} = \frac{\sigma^2}{2} w_3''(x) + \mu w_3'(x) - (\lambda + \alpha) w_3(x),
\]

(3.19)

(3.20)

Now, observe that \( G(\rho_k, \alpha) - \alpha = 0 \) for all \( k \) and

\[
\int \limits_z^\infty y^{i-1} e^{-by} dy = b^{-i} \Gamma(i, zb) = b^{-i}(i-1)! e^{-zb} \sum_{l=0}^{i-1} \frac{(zb)^l}{l!} = b^{-i}(i-1)! e^{-zb} \left[ 1 + O(z^i) \right],
\]

with \( \Gamma(i, u) \) is the incomplete gamma function (see [7], p. 342).

Consequently, substituting \( w_1(x), w_2(x) \) and \( w_3(x) \) into (3.19) and (3.20) yields that for any \( x < h \)

\[
0 = \sum_{j=1}^{m} \sum_{i=1}^{m_j} p_{ij} e^{\eta_j (x-h)} \left\{ \sum_{k=1}^{M+1} Q_k^L \left( \eta_j - \rho_k, \alpha \right) e^{\rho_k \alpha + \beta x} + Q_k^s \left( \eta_j - \rho_k, \alpha + \beta \right) e^{\rho_k \alpha + \beta x} \right\}.
\]

ECP 23 (2018), paper 68.

http://www.imstat.org/ecp/
Occupation time of Lévy processes with jumps rational Laplace transforms

\[
+ \sum_{k=1}^{M+1} Q_k \left( \frac{(\eta_j)_i^+ e^{\hat{\rho}_k,\alpha,\beta}}{(\eta_j - \hat{\rho}_k,\alpha,\beta)^{\gamma h}} \right) \left[ 1 + o((x-h)^{\gamma h}) \right] - (c_1 - c_0) \frac{e^{\gamma h}}{(\eta_j - \gamma)^{\gamma h}} + \sum_{k=1}^{m} \sum_{j=1}^{m_j} p_{ij} e^{u(x-h)} \left\{ \sum_{k=1}^{M+1} \left[ Q_k \left( \frac{(\eta_j)_i^+ e^{H\hat{\rho}_k,\alpha,\beta}}{(\eta_j - \hat{\rho}_k,\alpha,\beta)^{\gamma h}} \right) + Q_k \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha}}{(\eta_j - \rho_k,\alpha)^{\gamma h}} \right) \right] + \sum_{k=1}^{N+1} Q_k^0 \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha,\beta}}{(\eta_j - \rho_k,\alpha,\beta)^{\gamma h}} \right) \left[ 1 + o((x-H)^{\gamma h}) \right] - (c_1 - c_0) \frac{e^{\gamma h}}{(\eta_j - \gamma)^{\gamma h}} \right\},
\]

and, for \( x > H \)

\[
0 = \sum_{j=1}^{n} \sum_{i=1}^{n_j} q_{ij} e^{\theta_j(x-h)} \left\{ \left( \sum_{k=1}^{M+1} Q_k^L \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha,\beta}}{(\eta_j - \rho_k,\alpha,\beta)^{\gamma h}} \right) + Q_k^L \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha}}{(\eta_j - \rho_k,\alpha)^{\gamma h}} \right) \right) + \sum_{k=1}^{N+1} Q_k^0 \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha,\beta}}{(\eta_j - \rho_k,\alpha,\beta)^{\gamma h}} \right) \left[ 1 + o((x-h)^{\gamma h}) \right] - (c_1 - c_0) \frac{e^{\gamma h}}{(\gamma + \theta_j)^{\gamma h}} \right\} + \sum_{j=1}^{n} \sum_{i=1}^{n_j} q_{ij} e^{\theta_j(x-h)} \left\{ \left( \sum_{k=1}^{M+1} Q_k^U \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha,\beta}}{(\eta_j - \rho_k,\alpha,\beta)^{\gamma h}} \right) + Q_k^U \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha}}{(\eta_j - \rho_k,\alpha)^{\gamma h}} \right) \right) + \sum_{k=1}^{N+1} Q_k^0 \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha,\beta}}{(\eta_j - \rho_k,\alpha,\beta)^{\gamma h}} \right) \left[ 1 + o((x-H)^{\gamma h}) \right] - (c_1 - c_0) \frac{e^{\gamma h}}{(\gamma + \theta_j)^{\gamma h}} \right\}.
\]

Therefore, the constant vector \( Q \) or, in other words, the coefficients \( \{ Q_k^L, k = 1, \ldots, M + 1 \}, \{ Q_k^U, k = 1, \ldots, M + 1 \}, \{ Q_k^L, k = 1, \ldots, N + 1 \} \) and \( \{ Q_k^U, k = 1, \ldots, N + 1 \} \) satisfy the following: For \( j = 1, \ldots, m, i = 1, \ldots, m_j \),

\[
0 = \sum_{k=1}^{M+1} Q_k^L \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha,\beta}}{(\eta_j - \rho_k,\alpha,\beta)^{\gamma h}} \right) + Q_k^L \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha}}{(\eta_j - \rho_k,\alpha)^{\gamma h}} \right) + \sum_{k=1}^{N+1} Q_k^0 \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha,\beta}}{(\eta_j - \rho_k,\alpha,\beta)^{\gamma h}} \right) \left[ 1 + o((x-h)^{\gamma h}) \right] - (c_1 - c_0) \frac{e^{\gamma h}}{(\eta_j - \gamma)^{\gamma h}}.
\]

and for \( j = 1, \ldots, n, i = 1, \ldots, n_j \),

\[
0 = \sum_{k=1}^{M+1} Q_k^L \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha,\beta}}{(\eta_j + \rho_k,\alpha,\beta)^{\gamma h}} \right) + Q_k^L \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha}}{(\eta_j + \rho_k,\alpha)^{\gamma h}} \right) + \sum_{k=1}^{N+1} Q_k^0 \left( \frac{(\eta_j)_i^+ e^{H\rho_k,\alpha,\beta}}{(\eta_j + \rho_k,\alpha,\beta)^{\gamma h}} \right) \left[ 1 + o((x-h)^{\gamma h}) \right] - (c_1 - c_0) \frac{e^{\gamma h}}{(\gamma + \theta_j)^{\gamma h}}.
\]

In addition, we can also obtain another four equations from the fact that \( \psi(x) \) is continuously differentiable at \( x = h \) and \( x = H \):

\[
\begin{align*}
\sum_{k=1}^{M+1} Q_k e^{\rho_k,\alpha,\beta} - c_1 e^{\gamma h} &= \sum_{k=1}^{M+1} Q_k e^{\rho_k,\alpha,\beta} + \sum_{k=1}^{N+1} Q_k e^{\hat{\rho}_k,\alpha,\beta} - c_0 e^{\gamma h}, \\
\sum_{k=1}^{N+1} Q_k e^{H\rho_k,\alpha,\beta} - c_1 e^{\gamma H} &= \sum_{k=1}^{M+1} Q_k e^{H\rho_k,\alpha,\beta} + \sum_{k=1}^{N+1} Q_k e^{H\hat{\rho}_k,\alpha,\beta} - c_0 e^{\gamma H}, \\
\sum_{k=1}^{M+1} Q_k e^{\rho_k,\alpha,\beta} - c_1 e^{\gamma h} &= \sum_{k=1}^{M+1} Q_k e^{\rho_k,\alpha,\beta} + \sum_{k=1}^{N+1} Q_k e^{\hat{\rho}_k,\alpha,\beta} - \gamma c_0 e^{\gamma h}, \\
\sum_{k=1}^{N+1} Q_k e^{H\rho_k,\alpha,\beta} - c_1 e^{\gamma H} &= \sum_{k=1}^{M+1} Q_k e^{H\rho_k,\alpha,\beta} + \sum_{k=1}^{N+1} Q_k e^{H\hat{\rho}_k,\alpha,\beta} - \gamma c_0 e^{\gamma H}.
\end{align*}
\]

ECP 23 (2018), paper 68.
Occupation time of Lévy processes with jumps rational Laplace transforms

\[
\sum_{k=1}^{N+1} Q_k^U \hat{p}_{k,\alpha} e^{H \hat{p}_{k,\alpha}} - c_1 \gamma e^{\gamma H} = \sum_{k=1}^{M+1} Q_k^0 \hat{p}_{k,\alpha+\beta} e^{H \hat{p}_{k,\alpha+\beta}} + \sum_{k=1}^{N+1} Q_k^I \hat{p}_{k,\alpha+\beta} e^{H \hat{p}_{k,\alpha+\beta}} - c_0 \gamma e^{\gamma H}.
\]

Consequently, since all of these equations are linear with respect to the undetermined parameters, it follows that the constant vector \( Q = (Q_i^U, Q_i^0, Q_i^I, i = 1, \ldots, M + 1, j = 1, \ldots, N + 1) \) satisfies a linear system (3.14) which completes the proof. \( \square \)

**Proposition 3.4** (Uniqueness of the solution of the OIDE (3.3)). A bounded solution to the problem OIDE (3.3), if it exists, must be unique. More precisely, suppose \( \psi(x) \) solves the OIDE (3.3) and \( \sup_{x \in \mathbb{R}} |\psi(x)| \leq C < \infty \) for some constant \( C > 0 \). Then we must have

\[
\psi(x) = \int_0^\infty e^{-\alpha s} E_x \left[ e^{-\beta \int_0^s 1_{(h < X_t < H)} dt + \gamma X_s} \right] ds.
\]

(3.21)

**Proof.** Using the same idea as in Cai and Kou [4] (Theorem 4.1). Applying Ito’s formula to the process \( \{\psi(X_t) e^{-\alpha t} f_0^1 1_{(h < X_t < H)}\} \) we obtain that the process

\[
M_t := \psi(X_t) e^{-\alpha t - \beta \int_0^t 1_{(h < X_s < H)} ds} - \int_0^t \left[ -\alpha - \beta 1_{(h < X_s < H)} \right] \psi(X_s) + \mathcal{L} \psi(X_s) e^{-\alpha u - \beta \int_0^u 1_{(h < X_u < H)} du} ds,
\]

is a local martingale starting from \( M_0 = \psi(x) \). Because \( \psi(x) \) solves the OIDE (3.3), we have that

\[
M_t = \psi(X_t) e^{-\alpha t - \beta \int_0^t 1_{(h < X_s < H)} ds} + \int_0^t e^{-\alpha s - \beta \int_0^s 1_{(h < X_u < H)} du} + \gamma X_s ds.
\]

Since \( G(\gamma) < \alpha \), it follows from Fubini’s theorem that

\[
E[M_t] \leq C + \alpha \int_0^t E[e^{-\alpha s + \gamma X_s}] ds = C + \alpha \int_0^t e^{s(\alpha - G(\gamma))} ds = \frac{C + \alpha e^{(-\alpha + G(\gamma)) t} - 1}{-\alpha + G(\gamma)} < \infty.
\]

So, using Lebesgue’s dominated convergence theorem, we have that \( \{M_t, t \geq 0\} \) is actually a positive martingale. In particular

\[
\psi(x) = M_0 = E_x \left[ \lim_{t \to +\infty} M_t \right] = \int_0^\infty e^{-\alpha s} E_x \left[ e^{-\beta \int_0^s 1_{(h < X_t < H)} dt + \gamma X_s} \right] ds,
\]

(3.22)

which ends the proof. \( \square \)

**Lemma 3.5.** For a given value of \( \alpha > 0 \) the matrix \( A \) given by Equation (3.16) is invertible.

**Proof.** Assume that \( AC = 0 \) for some vector \( C = (C_1, C_2, \ldots, C_{2S})^T \). Consider the function \( V(x) = \sum_{k=1}^{2S} C_k e^{p_k x} \) for \( x \in \mathbb{R} \setminus \{h, H\} \), with \( p_1, \ldots, p_{2S} \) be the distinct zeros of the equation \( G(x) - \hat{\alpha} = 0 \) with \( \hat{\alpha} = \alpha + \beta \) on \( (h, H) \) and \( \hat{\alpha} = \alpha \) on \( (-\infty, h) \cup (H, +\infty) \). Since \( AC = 0 \) and \( V(x) \) is a solution to the boundary value problem

\[
\begin{cases}
(L - \alpha - \beta) \phi(x) = 0, & x \in (h, H), \\
(L - \alpha) \phi(x) = 0, & x \in (-\infty, h] \cup [H, +\infty).
\end{cases}
\]

(3.23)
Theorem 3.6. \( x < h \) shows that for any somme constants

Proof. The fact that

From the uniqueness of solutions to the boundary value problem \((3.23)\), \( V(x) \equiv 0 \) on \( x \in \mathbb{R} \setminus \{h, H\} \).

Now, evaluating \( \sum_{k=1}^{2S} C_k e^{\rho_k x} \) at \( x = 0, 1, \ldots, 2S - 1 \), we obtain

Because the \( e^{\rho_k} \), are distinct, the Vandermonde matrix in equation \((3.24)\) is invertible. Consequently \( C = 0 \) and \( A \) is invertible. \( \square \)

In the following, \( y \) is written for the usual inner product of the vector \( y \) and \( z \) in \( \mathbb{C}^S \) and for every real value \( x \), \( e^L(y) \) is \( \left[ e^{\rho_1 x}, \ldots, e^{\rho_{M+1} x} \right] \), \( e^0(y) \) is \( \left[ e^{\rho_1 x}, \ldots, e^{\rho_{M+1} x} \right] \), \( e^{-x} \) and \( e^H(y) \) is \( \left[ e^{\rho_1 x}, \ldots, e^{\rho_{N+1} x} \right] \), where \( \rho_1, \ldots, \rho_{M+1}, \rho_N \) are the \( S = N + M + 2 \) roots of the equation \( G(\zeta) = \alpha \). Our main result is the following:

**Theorem 3.6.** For any \( \beta \geq 0, \alpha > 0 \) and \( 0 < \gamma < \min(\eta_1, \theta_1) \) such that \( \alpha > G(\gamma) \), the following assertions are equivalent:

(a) \( \psi(x) = \int_0^\infty e^{-\alpha t} E_x \left[ e^{-\gamma t} \mathbb{1}_{\{h < x < H\}} \right] dt \).

(b) The function \( \psi(x) \) solve the boundary problem \((3.3)\).

(c) The function \( \psi(x) \) is given by the formula

\[
\psi(x) = \begin{cases} 
Q^L \cdot e^L(x) - \frac{\alpha e^{\gamma x}}{G(\gamma) - \alpha}, & \text{if } x \leq h, \\
Q^0 \cdot e^0(x) - \frac{\alpha e^{\gamma x}}{G(\gamma) - \alpha - \beta}, & \text{if } h < x < H, \\
Q^U \cdot e^U(x) - \frac{\alpha e^{\gamma x}}{G(\gamma) - \alpha}, & \text{if } x \geq H,
\end{cases}
\]

with \( (Q^L, Q^0, Q^U) = A^{-1}V \) and \( A \) and \( V \) are given by the formulas \((3.16)\) and \((3.15)\), respectively.

Proof. The fact that (b) implies (c) is straightforward consequence of Proposition 3.3. Conversely, consider the function

\[
W(x) = \begin{cases} 
\sum_{k=1}^{M+1} Q_k^L e^{\rho_{k, x}} - \frac{\alpha e^{\gamma x}}{G(\gamma) - \alpha}, & x \leq h, \\
\sum_{k=1}^{M+1} Q_k^0 e^{\rho_{k, x}} + \sum_{k=1}^{N+1} Q_k^U e^{\rho_{k, x}} - \frac{\alpha e^{\gamma x}}{G(\gamma) - \alpha - \beta}, & h < x < H, \\
\sum_{k=1}^{N+1} Q_k^U e^{\rho_{k, x}} - \frac{\alpha e^{\gamma x}}{G(\gamma) - \alpha}, & x \geq H,
\end{cases}
\]

for some constants \( Q_k^L, Q_k^0, Q_k^U \) and \( Q_k^U \). Then the same reasoning as in Proposition 3.3 shows that for any \( x < h \),

\[
(\mathcal{L} - \alpha) W(x) + \alpha e^{\alpha x} = \sum_{j=1}^{m_j} \sum_{i=1}^{m_i} p_{ij} e^{\eta_j (x-h)} \left\{ \left( \sum_{k=1}^{M+1} Q_k^L (\eta_j - \rho_{k, x}) e^{\rho_{k, x}} \right) + Q_k^0 (\eta_j - \rho_{k, x} + \beta) e^{\rho_{k, x}+\beta} \right\}
\]

\[
+ \sum_{j=1}^{m_j} \sum_{i=1}^{m_i} p_{ij} e^{\eta_j (x-H)} \left\{ \left( \sum_{k=1}^{M+1} Q_k^1 (\eta_j - \rho_{k, x} + \beta) e^{\rho_{k, x}+\beta} \right) + Q_k^U (\eta_j - \rho_{k, x}) e^{\rho_{k, x}} \right\}
\]

ECP 23 (2018), paper 68. Page 11/13 http://www.imstat.org/ecp/
Occupation time of Lévy processes with jumps rational Laplace transforms

\[ + \sum_{k=1}^{N+1} Q_0^k \left( \eta_j \right)^{i} e^{H \rho_{k,\alpha + \beta}} \left( \eta_j - \rho_{k,\alpha + \beta} \right)^{i} \left[ 1 + o((x-H)^i) \right] - (c_1 - c_0) e^{\eta_j \gamma} \left( \eta_j - \gamma \right)^{i} \].

Using the fact that \( Q = (Q_L^k, Q_0^k, k = 1, \ldots, M + 1, Q_1^l, Q_1^U, l = 1, \ldots, N + 1) = A^{-1}V \) where \( A \) and \( V \) are given by the formulas (3.16) and (3.15), respectively, we get that for any \( x < h \) the function \( W(x) \) solves the boundary value problem (3.3). Applying the same reasoning for \( x \in (h, H) \) and \( x \in (H, +\infty) \), we consequently have (c) implies (b).

Let us finally assume that (a) holds. Then by Feynman-Kac formula, the function \( \psi(x) \) solve the boundary problem (3.3); hence (b) holds. Conversely, thanks to Proposition 3.4, the boundary problem (3.3) has a unique solution, consequently (b) implies (a). The proof is complete. \( \square \)

4 Conclusion

The main result of this paper is an explicit representation for the joint distribution of the occupation time and the terminal value of the Lévy processes with jumps rational Laplace transforms. The corresponding boundary value problem via the Feynman-Kac representation is considered. By direct calculation, the associated ordinary integro-differential equation (OIDE) is transformed into a homogeneous ordinary differential equation (ODE) of higher order, which is then solved in closed form to obtain an explicit formula for the joint distribution of the occupation time and the terminal value of the Lévy processes with jumps rational Laplace transforms.

References

[1] Ait-Aoudia, D. and Renaud, J.F.: Pricing occupation-time options in a mixed-exponential jump-diffusion model. Applied Mathematical Finance. 23, (2016), 1–22. MR-3500525
[2] Akahori, J.: Some formulae for a new type of path-dependent option. Ann. Appl. Probab. 5, (1995), 383–388. MR-1336874
[3] Cai, N., Chen, N. and Wan, X.: Occupation times of jump-diffusion processes with double exponential jumps and the pricing of options. Math. Oper. Res. 35, (2010), 412–437. MR-2674727
[4] Cai, N. and Kou, S.G.: Pricing Asian options under a hyper-exponential jump diffusion model. Oper. Res. (2012), 60, 64–77. MR-2911657
[5] Chen, Y.T., Lee, C.F. and Sheu, Y.C.: An ODE approach for the expected discounted penalty at ruin in jump diffusion model. Finance and Stochastics. (2007), 11, 323–355. MR-2322916
[6] Chen, Y.T., Sheu, Y.C. and M.C. Chang.: A note on first passage functionals for Hyper-Exponential jump-diffusion process. Electron. Comm. Probab. (2013), 18, 1–8. MR-3011529
[7] Gradshteyn, I. S. and Ryzhik, I.: Table of Integrals, Series, and Products, Sixth Edition Academic Press. (2000). MR-1773820
[8] Kuznetsov, A.: On the distribution of exponential functionals for Lévy processes with jumps of rational transform. Stochastic Process. Appl. (2012), 122, 654–663. MR-2868934
[9] Landriault, D., Renaud, J.F. and Zhou, X.: Occupation times of spectrally negative Lévy processes with applications. Stochastic Process. Appl. (2011), 121, 2629–2641. MR-2832417
[10] Lévy, P.: Sur certains processus homogène. Compos. Math. (1939), 7, 283–339. MR-0000919
[11] Lewis, A. L. and Mordecki, E.: Wiener–Hopf factorization for Lévy processes having positive jumps with rational transforms. J. Appl. Probab. (2008), 45, 118–134. MR-2409315
[12] Li, B. and Zhou, X.: The joint Laplace transforms for diffusion occupation times. Adv. in Appl. Probab. (2013), 45, 1049–1067. MR-3161296
[13] Sepp, A.: Affine Models in Mathematical Finance: an Analytical Approach. PhD thesis. (2007), University of Tartu.
Occupation time of Lévy processes with jumps rational Laplace transforms

[14] Takács. L.: On a generalization of the arc-sine law. Ann. Appl. Probab. (1996), 6, 1035–1040. MR-1410128

[15] Wu. L. and Zhou. J.: Occupation times of hyper-exponential jump diffusion processes with application to price step options. Comput. Appl. Math. (2016), 294, 251–274. MR-3406981

[16] Yen. J.Y. and Yor. M.: Local Times and Excursion Theory for Brownian Motion. A Tale of Wiener and Itô Measures. Lecture Notes in Mathematics. (2013), 2088. Springer, Berlin. MR-3134857

[17] Zhou. J, Wu. L and Bai. Y.: Occupation times of Lévy-driven Ornstein-Uhlenbeck processes with two-sided exponential jumps and applications. Statistics and Probability Letters. (2017), 125, 80–90. MR-3626071

[18] Zhou. J. and Wu. L.: The distribution of refracted Lévy processes with jumps having rational Laplace transforms. J. Appl. Probab. (2017), 54, 1167–1192. MR-3731290

Acknowledgments. The author is grateful to the anonymous referees for various helpful comments and suggestions on an earlier version, which help to improve the structure and text of the paper, and he would like to thank the Editor for handing this paper.
Advantages of publishing in EJP-ECP

• Very high standards
• Free for authors, free for readers
• Quick publication (no backlog)
• Secure publication (LOCKSS\(^1\))
• Easy interface (EJMS\(^2\))

Economical model of EJP-ECP

• Non profit, sponsored by IMS\(^3\), BS\(^4\), ProjectEuclid\(^5\)
• Purely electronic

Help keep the journal free and vigorous

• Donate to the IMS open access fund\(^6\) (click here to donate!)
• Submit your best articles to EJP-ECP
• Choose EJP-ECP over for-profit journals

\(^1\)LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
\(^2\)EJMS: Electronic Journal Management System http://www.vtex.lt/en/ejms.html
\(^3\)IMS: Institute of Mathematical Statistics http://www.imstat.org/
\(^4\)BS: Bernoulli Society http://www.bernoulli-society.org/
\(^5\)Project Euclid: https://projecteuclid.org/
\(^6\)IMS Open Access Fund: http://www.imstat.org/publications/open.htm