Interference of Identical Particles and the Quantum Work Distribution

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Quantum mechanical particles in a confining potential interfere with each other while undergoing thermodynamic processes far from thermal equilibrium. By evaluating the corresponding transition probabilities between many-particle eigenstates we obtain the quantum work distribution function, for identical Bosons and Fermions, which we compare with the case of distinguishable particles. We find that the quantum work distributions for Bosons and Fermions significantly differ at low temperatures, while, as expected, at high temperatures the work distributions converge to the classical expression. These findings are illustrated with two analytically solvable examples, namely the time-dependent infinite square well and the parametric harmonic oscillator.

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I. INTRODUCTION

In the past two decades, nonequilibrium work relations [1], including the Jarzynski Equality [2, 3] and the Crooks Fluctuation Theorem [4, 5], have attracted a lot of attention. These two nonequilibrium work theorems together with other exact relations concerning entropy production in arbitrary far-from-equilibrium processes, collectively known as Fluctuation Theorems [6–11], have shed new light on our understanding of nonequilibrium thermodynamics beyond the close-to-equilibrium regime. The validity of the classical version of these relations has been tested experimentally in various systems [12–16]. In recent years, the quantum version [17–20] of these relations has also been proposed and studied extensively [21–23]. In the quantum regime, the so-called two-time energy measurement approach has proven to be effective. Within this approach quantum work performed by a thermally isolated system is determined by two projective energy measurements. On the other hand, the analysis of the characteristic function, i.e., the Fourier transform of the work density [20], has opened new, alternative avenues to experimentally test quantum work theorems [24, 25].

Previous studies of quantum work relations have been mostly focused on single-particle quantum systems, such as, dragged harmonic oscillators [26, 27], parametric harmonic oscillators [28–30], a single particle in a time-dependent piston [31], two-level systems [32], and the parametric Morse oscillator [33]. However, the interplay of quantum work and quantum statistical properties, e.g., the Fermi-Dirac statistics or the Bose-Einstein statistics have not been fully explored yet. Interference [34] of identical particles will undoubtedly influence the thermodynamic properties of many-particle systems.

The difference between the Bose-Einstein distribution and the Fermi-Dirac distribution for identical particles in single-particle eigenstates can be interpreted as a manifestation of the “static” effect of the interference. Furthermore, in nonequilibrium processes, the transition probabilities between many-particle eigenstates for Bosons and Fermions exhibit interference, as well. This effect can be regarded as the “dynamic” effect of interference, which profoundly influences the work distribution in nonequilibrium processes.

In this article, we extend our previous studies [28, 29, 31, 35, 36] to multi-particle systems. We will show that for noninteracting particles, the transition amplitudes between many-particle eigenstates can be constructed from the transition amplitudes between single-particle eigenstates. From these we obtain the work distribution for arbitrary far-from-equilibrium processes. In practice, however, we will see that for Fermions the work distribution function is relatively easy to compute, whereas for Bosons, the work distribution function is mathematically more involved.

Our findings will be illustrated by two exactly solvable examples—identical particles confined by a quantum piston and by a harmonic potential. For these model systems we will highlight the significant difference between Bosons and Fermions at low temperatures. On the other hand, in the limit of high temperatures and slow driving we will rediscover the work distribution function for classical particles [37].

Only recently, a “correspondence principle” for work distributions [38] has been proposed, which indicates that the quantum distribution converges towards the classical distribution in the semiclassical limit $\hbar \to 0$. Motivated by this result we demonstrate analogously that in the high temperature limit $\beta \to 0$, the work dis-

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tribution functions for both Bosons and Fermions converge towards that of classical distinguishable particles, which has been previously seen in the single particle case [28, 29, 31, 39, 40]. In other words, we demonstrate that in the limit of high temperature, \( \beta \to 0 \), the quantum work distribution obeys a "correspondence principle" in the quantum statistical sense independent of the nature of the particles (in contrast to the "correspondence principle" in the quantum mechanical sense, where \( \hbar \to 0 \) is required).

Finally, we emphasize that we restrict ourselves to non-interacting, spinless identical particles and the nonrelativistic regime. In particular, the particle number is conserved which corresponds to a canonical ensemble. Similar systems have also been studied by Nakamura and his collaborators in Refs. [41, 42], but for grandcanonical ensembles, and their focus is put on the averaged quantities instead of fluctuations.

The paper is organized as follows: In Sec. II, we construct the work distribution for multi-particle systems undergoing nonequilibrium processes. Our findings are illustrated with a 1D box system and a 1D harmonic oscillator, and we numerically compute the work distribution. Section III is dedicated to the convergence of the quantum work distribution for noninteracting Bosons and Fermions in the limit of high temperature. Finally, Sec. IV concludes the discussion with remarks on various properties of the work distribution.

II. QUANTUM WORK DISTRIBUTION FOR IDENTICAL PARTICLES

In the study of quantum processes operating far from thermal equilibrium one of the key quantities is the work distribution. To the best of our knowledge, previous analyses of multi-particle systems have been restricted to quasistatic processes [37], perturbative and sudden [43] compression of classical particles [44] in a piston, or sudden quenches of spin chains [45–47]. In particular, analytical results for the work distribution are only known for classical particles [37, 43, 44], whereas the effect of quantum interference is yet to be explored.

In the following we will explicitly construct the quantum work distribution, \( \mathcal{P}(W) \), for systems of many noninteracting particles (identical or distinguishable), while special focus will be put on the effect of interference on \( \mathcal{P}(W) \). To this end, we have to evaluate the transition probabilities between many-particle eigenstates [34, 48, 49], first. In a second step we will illustrate our findings numerically for two simple model systems, namely a 1D piston system and a 1D harmonic oscillator.

A. General expression

Consider a system of \( N \) noninteracting identical particles (either Bosons or Fermions) in a 1D potential. Let us denote the multi-particle eigenstates at the initial and the final instants of a process by \( \{|i_k^{\lambda_0} : n_{i_k}\}\) and \( \{|f_l^{\lambda_f} : n_{f_l}\}\). Here \( \lambda_0 \) and \( \lambda_f \) denote the initial and the final value of a work parameter with \( \lambda(0) = \lambda_0 \) and \( \lambda(\tau) = \lambda_f \); \( i_k^{\lambda_0} \) \( f_l^{\lambda_f} \) is the quantum number of the single-particle state and \( n_{i_k} \) \( n_{f_l} \) is the occupation number of the particles in the \( i_k \)th \( f_l \)th eigenstate.

Commonly [2–5], the system under study is initially prepared in a thermal state at inverse temperature \( \beta \), which corresponds here to a canonical ensemble. Then the initial probability to find the system in state \( \{|i_k^{\lambda_0} : n_{i_k}\}\) is given by

\[
P\left(\{|i_k^{\lambda_0} : n_{i_k}\}\right) = \frac{1}{Z^{\lambda_0}} \exp \left[ -\beta \left( \sum_k n_{i_k} E_{i_k}^{\lambda_0} \right) \right], \tag{1}
\]

where the partition function \( Z^{\lambda_0} \) reads

\[
Z^{\lambda_0} = \sum_{\{i_k : n_{i_k}\}} \exp \left[ -\beta \left( \sum_k n_{i_k} E_{i_k}^{\lambda_0} \right) \right]. \tag{2}
\]

Here we observe the first effect of the quantum statistics. For Fermions we have \( n_{i_k} \equiv 1, \forall k \), due to the Pauli exclusive principle, whereas for Bosons \( n_{i_k} \) can be an arbitrary positive integer with \( n_{i_k} \leq N \). The total number of particles, however, is conserved in either case, and we have \( \sum_k n_{i_k} = N \). Finally, \( E_{i_k}^{\lambda_0} \) denotes the \( i_k \)th initial eigenenergy.

After the preparation of the system a projective energy measurement is performed. Then, the external control parameter \( \lambda_t \) is varied according to some protocol with \( \lambda_t=0 = \lambda_0 \) and \( \lambda_t=\tau = \lambda_f \), and the total system evolves under unitary dynamics. At \( t = \tau \) a second projective energy measurement is performed, which induces the system to “collapse” into a final multi-particle eigenstate \( \{|f_l^{\lambda_f} : n_{f_l}\}\) [50]. The work performed during one realization of this protocol is given by

\[
W\left(\{|i_k^{\lambda_0} : n_{i_k}\}\to\{|f_l^{\lambda_f} : n_{f_l}\}\right) = \sum_l n_{f_l} E_{f_l}^{\lambda_f} - \sum_k n_{i_k} E_{i_k}^{\lambda_0}, \tag{3}
\]

and we denote by \( P\left(\{|i_k^{\lambda_0} : n_{i_k}\}\to\{|f_l^{\lambda_f} : n_{f_l}\}\right) \) the transition probabilities between many particle eigenstates. Thus, the work distribution,

\[
\mathcal{P}(W) = \left\langle \delta \left[ W - W\left(\{|i_k^{\lambda_0} : n_{i_k}\}\to\{|f_l^{\lambda_f} : n_{f_l}\}\right) \right] \right\rangle \tag{4}
\]

can be written as [17, 18, 20]
\[ P(W) = \sum_{\{i_k : n_{ik}\}} \sum_{\{f_l : n_{fl}\}} P\left(\{i_k^\lambda : n_{ik}\}\right) P\left(\{f_l^\lambda : n_{fl}\}\right) \delta\left[ W - \left(\sum_l n_{fl} E_{fl}^\lambda - \sum_k n_{ik} E_{ik}^\lambda\right)\right]. \]

(5)

The latter expression clearly indicates that to calculate the quantum work distribution expressions for the transition probabilities are necessary. Luckily this quantity has been studied extensively in recent years \[34, 48, 49\], and we will here briefly review how to construct the transition probabilities for distinguishable particles, \[ P_D \left(\{i_k^\lambda\} \rightarrow \{f_k^\lambda\}\right) \] (\(i_k^\lambda\) and \(f_k^\lambda\) denote the quantum number of the initial and the final states of the kth particle, respectively), as well as for Bosons and for Fermions, \[ P_{B/F} \left(\{i_k^\lambda : n_{ik}\} \rightarrow \{f_l^\lambda : n_{fl}\}\right), \] \[34, 48, 49\].

If the transition amplitude between single particle eigenstates can be expressed as \( \langle f_l^\lambda \mid \hat{U} \mid i_k^\lambda \rangle \), where \( \hat{U} \) is the unitary evolution operator corresponding to the Schrödinger equation \( i\hbar \partial_t \hat{U} = H(t) \hat{U} \), the transition probabilities between multi-particle eigenstates can be written as \[34, 48, 49\]

\[ P_D \left(\{i_k^\lambda\} \rightarrow \{f_k^\lambda\}\right) = \prod_{k=1}^{N} \left| \langle f_k^\lambda \mid \hat{U} \mid i_k^\lambda \rangle \right|^2, \quad (k = 1, 2, \ldots, N), \]

\[ P_{B/F} \left(\{i_k^\lambda : n_{ik}\} \rightarrow \{f_l^\lambda : n_{fl}\}\right) = \prod_{l=1}^{L} \frac{1}{n_{fl}} \prod_{k=1}^{K} \frac{1}{n_{ik}} \times \]

\[
\left| \langle f_1^\lambda \mid \hat{U} \mid i_1^\lambda \rangle \cdots \langle f_1^\lambda \mid \hat{U} \mid i_1^\lambda \rangle \langle f_2^\lambda \mid \hat{U} \mid i_2^\lambda \rangle \cdots \langle f_2^\lambda \mid \hat{U} \mid i_2^\lambda \rangle \cdots \langle f_L^\lambda \mid \hat{U} \mid i_L^\lambda \rangle \right|^2, \quad (6)
\]

where the matrix element \( \langle f_l^\lambda \mid \hat{U} \mid i_k^\lambda \rangle \) occupies a block of size \( n_{fl} \times n_{ik} \). Due to the conservation of the particle number we have as before, \( N = \sum_{l=1}^{L} n_{fl} = \sum_{k=1}^{K} n_{ik} \). Furthermore, \( \zeta = -1 \) and \( \zeta = 1 \) in Eq. (6) correspond to Fermions and Bosons, respectively. For Fermions the transition amplitude is equal to the determinant of the matrix (6), whereas for Bosons, the transition amplitude is given by the permanent \[34, 48, 49\].

Generally, to compute the transition probabilities merely the transition amplitudes between single particle eigenstates \( \langle f_l^\lambda \mid \hat{U} \mid i_k^\lambda \rangle \) are necessary. However, we will see shortly that, in practice, the calculation of the permanent, the case of Bosons, is much more involved than the calculation of the determinant, for Fermions.

We now can proceed to compute the quantum work distribution for specific many-particle systems. For the sake of simplicity we will continue our discussion for two analytically solvable examples. For single-particle systems analogous studies include the 1D piston system with a moving wall \[31, 51\] and the 1D harmonic oscillator with a time-dependent angular frequency \[28, 29, 52\].

**B. Case one: particles in one dimensional piston**

The paradigm system in statistical mechanics is undoubtedly the classical ideal gas confined by a piston. As a quantum analog can be considered quantum particles in an infinite square well. The dynamics of a single particle in this “quantum piston” has been studied extensively in various contexts, see for instance Refs. [51, 53]
and references therein. When the piston is pulled or compressed at a constant velocity, analytical solutions to the transition amplitudes between the initial and the final energy eigenstates \( \left\langle f_{i}^{\lambda_{t}} \right| \hat{U} \left| i_{k}^{\lambda_{o}} \right\rangle \) can be obtained analytically [31, 51]. Specifically, for a quantum piston expanding at a constant velocity \( v \) from an initial length \( \lambda(0) = \lambda_{o}, \lambda(t) = \lambda_{o} + vt \), a set of independent solutions to the time-dependent Schrödinger equation can be written as [51]

\[
\Phi_{j}(x, t) = \exp \left[ \frac{i}{\hbar \lambda(t)} \left( \frac{1}{2} M v x^{2} - E_{j}^{\lambda_{o}} \lambda_{o} t \right) \right] \phi_{j}(x, \lambda(t)),
\]

where \( j = 1, 2, 3, \ldots \) and \( M \) is the mass of the particle, \( E_{j}^{\lambda_{o}} = j^{2}\pi^{2} \hbar^{2}/2M\lambda_{0}^{2} \) is the \( j \)th eigenenergy and \( \phi_{j}(x, \lambda) \) is the \( j \)th energy eigenstate of a particle in an infinite square potential

\[
\phi_{j}(x, \lambda) = \sqrt{\frac{2}{\lambda}} \sin \left( \frac{j \pi x}{\lambda} \right).
\]

A general solution of the time-dependent Schrödinger equation takes the form

\[
\Psi(x, t) = \sum_{j}^{\infty} c_{j} \Phi_{j}(x, t),
\]

where the time-independent coefficients \( c_{j} \) are set by the initial wave function

\[
c_{j} = \int_{0}^{\lambda_{0}} dx \Phi_{j}^{*}(x, 0) \Psi(x, 0).
\]

For initial conditions \( \Psi(x, 0) = \phi_{i_{k}}(x, \lambda_{0}) \equiv \left\langle x \right| i_{k}^{\lambda_{o}} \right\rangle \) these coefficients are (setting \( \hbar = 1 \) and \( M = 1 \))

\[
c_{j}(i_{k}) = \frac{2}{\lambda_{0}} \int_{0}^{\lambda_{0}} dx \exp \left( -i \frac{v x^{2}}{2\lambda_{0}} \right) \times \sin \left( \frac{j \pi x}{\lambda_{0}} \right) \sin \left( \frac{i k \pi x}{\lambda_{0}} \right),
\]

and the time evolution matrix elements to the state \( \left| f_{i}^{\lambda_{t}} \right\rangle \) at the final instant \( t = \tau \) become

\[
\left\langle f_{i}^{\lambda_{t}} \right| \hat{U} \left| i_{k}^{\lambda_{o}} \right\rangle = \sum_{j=1}^{\infty} c_{j}(i_{k}) \int_{0}^{\lambda_{t}} dx \Phi_{j}(x, \tau) \phi_{j}^{*}(x, \lambda_{t}).
\]

Substituting the transition amplitude (12) into Eq. (6) we obtain the transition probabilities between the multi-particle eigenstates in the piston.

We plot numerical results of the work distribution for the case of two or three identical particles in Figs. 1–4. For the sake of clarity we plot the cumulative distributions,

\[
\rho(W) = \int_{W}^{W'} dW' \mathcal{P}(W')
\]
rather than the quantum work distributions $\mathcal{P}(W)$. In all plots we compare the results for distinguishable (Boltzmann) particles (red lines) with those for Fermions (black lines) and for Bosons (blue lines).

We start with the case of slow expansion in Fig. 1 with $v = 0.1$. We observe that in the limit of low temperature the work distributions for Bosons and for distinguishable particles are identical. This can be understood by noting that (i) at $T = 0$ both Bosons and distinguishable particles occupy only the single-particle ground state, and (ii) at $T = 0$ the transition probabilities between many-particle states for Bosons and for distinguishable particles are identical (all the transition probabilities for distinguishable particles, which correspond to the same state for Bosons, should be summed up). We also observe that in the limit of low temperature, the work distributions for Bosons and for Fermions differ significantly due to the static interference of identical particles.

At intermediate temperatures, e.g., from $\beta^{-1} = 10$ to $\beta^{-1} = 20$, the work distribution function for distinguishable particles locates between that for Bosons and that for Fermions. In contrast to distinguishable particles, there is an effective "attractive" interaction among Bosons, while there is an effective "dispersive" interaction among Fermions. As a result, Fermions perform more work than distinguishable particles on the piston during an expanding process, while Bosons perform less. By further increasing the initial temperature, the cumulative work distribution functions become smoother and smoother, and the work distribution functions for Bosons and for Fermions show a tendency of convergence.

In the limit of high temperature (e.g., in Fig. 1(d) $\beta^{-1} = 100$ can already be regarded as the limit of high temperature), the work distribution functions for the three kinds of particles collapse onto the same curve. In Fig. 2 all parameters are the same as those in Fig. 1 except that the speed of the expansion of the piston is higher. We observe that the faster the speed of the expansion the earlier, i.e., at lower temperature, the convergence of the work distribution functions for the three kinds of particles.

In Figs. 3-4 we plot the cumulative work distribution (13) for the case of three identical particles at low temperatures. The difference of the work distribution functions between Bosons and Fermions for three particles are more prominent than those for two particles. It can be inferred that, with the increase of the particle number, the distinguishability of the work distributions of Bosons and Fermions at low temperature will becomes even more significant. This can be understood by considering that, at low temperature, the system will stay in a state close to the many-particle ground state. For Bosons and Fermions the ground states are a Bose condensate and a Fermi sea, respectively.

By further increasing the particle number, the complexity of the calculation of the transition probabilities between many-particle eigenstates increases exponentially with the particle number. Also, with the in-

![Fig. 2: (color online) Cumulative work distribution(13) for two Bosons (blue, dotted line), Fermions (black, dashed line) and distinguishable particles (red, solid line) in a rapidly expanding piston with $\lambda_0 = 1$ and $\lambda_T = 2$.](image-url)
crease of the temperature, the number of eigenstates, which will be visited during the work process, increases dramatically. Therefore, we restricted ourselves to two and three particles and to rather low temperatures. However, from the work distribution functions for three particles at $\beta^{-1} = 0$, $\beta^{-1} = 10$ and $\beta^{-1} = 20$ (see Figs. 3-4), one already observes the tendency to converge by raising the temperature.

FIG. 3: (color online) Cumulative work distribution (13) for three Bosons (blue, dotted line), Fermions (black, dashed line) and distinguishable particles (red, solid line) in a medium speed expanding piston with $\lambda_0 = 1$ and $\lambda_T = 2$.

FIG. 4: (color online) Cumulative work distribution (13) for three Bosons (blue, dotted line), Fermions (black, dashed line) and distinguishable particles (red, solid line) in a rapidly expanding piston with $\lambda_0 = 1$ and $\lambda_T = 2$.

C. Case two: particles in one dimensional harmonic potential

As a second case study of pedagogical value we analyze the 1D harmonic oscillator. Specifically, we consider the Hamiltonian

$$H_s(x, t) = \frac{1}{2M} \frac{d^2}{dx^2} + \frac{1}{2} M \omega_t^2 x^2,$$

(14)

where $M = 1$ is again the single particle mass and we identify the work parameter with the angular frequency, $\lambda_t = \omega_t$. This system can be solved analytically, see for instance [52, 54], and has been developed as the prototypical example in quantum thermodynamics, [21, 27–30, 36, 55–64].

It has been shown that the single-particle time evolution operator can be written in position space as,

$$U_t(x, ; x_0) = \sqrt{\frac{M}{2\pi i\hbar X_t}} \times \exp \left( \frac{iM}{2\hbar X_t} \left( \dot{X}_t x^2 - 2xx_0 + Y_t x_0^2 \right) \right)$$

(15)
where $X_t$ and $Y_t$ are solutions of the classical, force free equation of motion,
\[ \ddot{\xi}_t + \omega_t^2 \xi_t = 0 \]  
with $X_{t=0} = 0$, $\dot{X}_{t=0} = 1$ and $Y_{t=0} = 1$, $\dot{Y}_{t=0} = 0$. From the latter the single-particle propagator can be obtained in energy representation by evaluating
\[
\langle f^\lambda \ | \ U \ | i^\lambda_0 \rangle = \int \mathrm{d}x \int \mathrm{d}x_0 \psi_f(x) U_t(x; x_0) \psi_i(x_0),
\]
where $\psi_\nu(x), (\nu = i, f)$ are the instantaneous eigenstates of the time-dependent Hamiltonian $H_s(x, t)$ (14). The result is a rather lengthy expression [29], which we summarize in Appendix A.

In Figs. 5-8 we plot the cumulative work distribution (13) for the linear quench,
\[
\omega_t^2 = \omega_0^2 + (\omega_t^2 - \omega_0^2) t/\tau.
\]
Generally, the same features as those in the case of the 1D piston system can be observed: In the limit of low temperature, the work distribution functions for Bosons and for Fermions differ significantly, while for high temperatures the three distributions converge.

An interesting feature to note is that the interference seems to play much less of a role for the harmonic oscillator than for the piston. In particular, the work distributions for Bosons, Fermion, and distinguishable particles start converging at much lower temperatures. This can be understood by noticing that the energy levels of the harmonic oscillator are much denser than the ones of the square well potential. Thus, interference effects are “smeared out” already at finite but low temperatures.

### III. QUANTUM WORK AT HIGH TEMPERATURE

At low temperatures the thermal distributions for Bosons and for Fermions significantly differ. This is due to “static interference” expressed by the fact that the many-particle eigenstates can be expressed in terms of a determinant (for Fermions) and a permanent (for Bosons) in the second quantization formalism [65]. Note that the Pauli exclusion principle states that two identical Fermions cannot occupy the same single-particle state, while for Bosons there is no such constraint.

At high temperatures, however, the thermal states for Bosons and for Fermions become indistinguishable, which can be interpreted as a consequence of the static “correspondence principle” in quantum statistical sense for $\beta \rightarrow 0$.

As we discussed in Sec. II the transition probabilities between many-particle eigenstates for Bosons and for Fermions (6) can also be expressed in terms of determinants (for Fermions) and permanents (for Bosons) [34, 48, 49]. This effect can be interpreted as a “dynamic

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**FIG. 5:** (color online) Cumulative work distribution (13) for two Bosons (blue, dotted line), Fermions (black, dashed line) and distinguishable particles (red, solid line) in a slowly quenched harmonic potential (14) with $\omega_0 = 1$, and $\omega = \sqrt{2}$. 
FIG. 6: (color online) Cumulative work distribution \((13)\) for two Bosons (blue, dotted line), Fermions (black, dashed line) and distinguishable particles (red, solid line) in a rapidly quenched harmonic potential \((14)\) with \(\omega_0 = 1\), and \(\omega_\tau = \sqrt{2}\).

FIG. 7: (color online) Cumulative work distribution \((13)\) for three Bosons (blue, dotted line), Fermions (black, dashed line) and distinguishable particles (red, solid line) in a slowly quenched harmonic potential \((14)\) with \(\omega_0 = 1\), and \(\omega_\tau = \sqrt{2}\).

The characteristic function of the work distribution for a many-Boson system \(G_B(\mu)\) and a many-Fermion system \(G_F(\mu)\) can be expressed as \([20]\)
\[
G_B/F(\mu) = \text{tr}_{\mathcal{H}_{B/F}^{\omega_0}} \{ \mathcal{D}_{B/F}^{\lambda_0} \exp(i\mu H^\lambda_{\mathcal{H}_{B/F}}) \exp(-i\mu H^{\lambda_0}) \},
\]
where
\[
\mathcal{D}_{B/F}^{\lambda_0} = \frac{\exp(-\beta H^{\lambda_0})}{Z_{B/F}^{\lambda_0}}
\]
is the initial thermal equilibrium distribution with the partition function
\[
Z_{B/F}^{\lambda_0} = \text{tr}_{\mathcal{H}_{B/F}^{\omega_0}} \{ \exp(-\beta H^{\lambda_0}) \}.
\]
Furthermore, \(\mathcal{H}_{B/F}^{\lambda_0}\) describes the Hilbert space of the many-Boson/Fermion system with the work parameter

effect” of interference, which is independent of the initial temperature. It is thus neither obvious nor ad hoc clear whether the work distribution functions for many Bosons will converge towards that of many Fermions at high temperatures.

In the previous section, we discussed the numerical results for the quantum work distribution in two simple model systems. Our numerical results strongly suggest that at high temperatures the work distribution functions for both Bosons and Fermions do converge to that
equals to $\lambda_0$; $H^\lambda_H$ describes the Hamiltonian of the many-particle system in Heisenberg picture with the work parameter equal to $\lambda$; and $H^\lambda_s$ is the Hamiltonian of the many-particle system in Schrödinger picture with the work parameter equal to $\lambda_0$.

After a straightforward calculation the characteristic function of the work distribution for a system consisting of many Bosons or many Fermions can be expressed as the characteristic functions of a single particle system (see Appendix B) [66]

$$G_{B/F}(\mu) = \frac{\sum_{(1^{\nu_1}, 2^{\nu_2}, \ldots, N^{\nu_N})}^{N!} \prod_{k=1}^{N} \left[ \pm \text{tr}_{H^\lambda_s} \left\{ \left[ G^\lambda_0(\mu) \exp \left( -\beta H^\lambda_s \right) \right]^k \right\} \right]^{\nu_k}}{\sum_{(1^{\nu_1}, 2^{\nu_2}, \ldots, N^{\nu_N})}^{N!} \prod_{k=1}^{N} \left[ \pm \text{tr}_{H^\lambda_0} \left\{ \exp \left( -k \beta H^\lambda_0 \right) \right\} \right]^{\nu_k}},$$

(22)

Here

$$G^\lambda_0(\mu) = \exp \left( i \mu H^\lambda_{H,s} \right) \exp \left( -i \mu H^\lambda_s \right),$$

(23)

and $H^\lambda_{H,s}$ represents the Hamiltonian for a single particle in Heisenberg’s picture; analogously $H^\lambda_s$ is the Hamiltonian for a single particle in Schrödinger’s picture; $H^\lambda_0$ denotes the Hilbert space of a single particle system when the work parameter is equal to $\lambda_0$; $(1^{\nu_1}, 2^{\nu_2}, \ldots, N^{\nu_N})$ describes a cycle notation, which corresponds uniquely to a permutation [66] $(k^{\nu_k}$ means that there are $\nu_k$ $k$-cycles, $\nu_k \geq 0$, and $\sum_{k=1}^{N} k \times \nu_k = N$. The definition of $k$-cycles can be found in section 1.1 of Ref. [66]).

Equation (22) constitutes one of our main results. It is further tested and verified for the ideal quantum gas in Appendix C.

In the limit of high temperatures, the dominant contribution in $G_{B/F}(\mu)$ (see Eq. (22)) in both the numerator and the denominator stems from the trivial identity permutation, which is $(1^N, 2^0, \ldots, N^0)$ in cycle notation [67]. If we keep only the leading term in both the denominator and the numerator, the characteristic function (22) can be simplified to read

$$G_{B/F}(\mu) \approx \frac{\text{tr}_{H^\lambda_0} \left\{ G^\lambda_0(\mu) \exp \left( -\beta H^\lambda_s \right) \right\}^N}{\text{tr}_{H^\lambda_0} \left\{ \exp \left( -\beta H^\lambda_s \right) \right\}}.$$  

(24)

The latter expression is the characteristic function of the work distribution for distinguishable particles. Thus, we have demonstrated that, in the high temperature limit, the characteristic function of work distribution functions for both Bosons and Fermions converge towards that of distinguishable particles, and hence to each other. Since there is a one-to-one map between the work distribution and its corresponding characteristic function [20], we conclude that in the high temperature limit, the work distributions for Bosons and Fermions converge.

It is worth emphasizing that this result does not depend on the specifics of the model, and, hence, holds for any system of many noninteracting, identical particles.

FIG. 8: (color online) Cumulative work distribution (13) for three Bosons (blue, dotted line), Fermions (black, dashed line) and distinguishable particles (red, solid line) in a rapidly quenched harmonic potential (14) with $\omega_0 = 1$, and $\omega_\tau = \sqrt{2}$.

IV. DISCUSSION AND CONCLUSION

In this paper we have studied the effect of indistinguishability (quantum interference of identical, noninteracting particles) on the quantum work distribution. We have found that the work distribution can be computed from the time evolution matrix for single particles. Then, the transition amplitudes between multi-particle states are given by the Slater-determinant (for Fermions) or the
A. Determinants and permanents

Generally, the computation of the permanent of a matrix is much more involved than the computation of the determinant – despite the apparent similarity of the definitions [34]: In particular, the determinant obeys several algebraic rules and symmetries, e.g., the product rule $\det(AB) = \det(A) \det(B)$ and the invariance under unitary transformation, which allow the determinant to be evaluated in polynomial time. For a $N \times N$ matrix, e.g., the elementary Gaussian algorithm needs $O(N^3)$ operations [34]. Although the permanent has a similar structure the omission of the alternating sign makes all the difference, and all known strategies for an efficient evaluation of the determinant fail for the permanent. In general, a permanent can only be computed in exponential time, even when applying Ryser’s algorithm [34] – the most efficient algorithm known to date. Therefore, the development of both exact and approximate algorithms for computing the permanent of a matrix is an active area of research.

For our problem this means that the computation of the work density for Fermions is much more feasible than for Bosons. The exponential increase of the complexity of computing the permanent limits the study of the work distribution to at most 25 Bosons [34].

This restriction might be lifted by a novel development in the field of quantum information known as Boson Sampling – a shortcut to quantum computing [68]. In this technique the bosonic distribution is obtained from interfering photons in a random optical network. However, practical applications of Boson Sampling are still under active research [69].

B. Quasistatic limit

Our expression for the characteristic function (22) is valid for both quasistatic and nonquasistatic processes. To the best of our knowledge previous studies have been restricted to perturbative treatments [43], quasistatic changes [37], and the piston system [44].

For the sake of completeness we, thus, briefly show how the expression for quasistatic processes [37] can be obtained from our general formula (22). For very slow driving one can assume that the energy levels remain almost constant. Thus the characteristic function for Bosons/Fermions, $G_{B/F}(\mu)$, can be written as

$$G_{B/F}(\mu) = \frac{\sum_{\{1^{\nu_1},2^{\nu_2},\ldots,N^{\nu_N}\}} \prod_{k=1}^{N} \left[ \pm \sum_{i=1}^{\infty} \exp \left( i\mu E_i^{\lambda_i} \right) \exp \left( -k(i\mu + \beta)E_i^{\lambda_i} \right) \right]^{\nu_k}}{\sum_{\{1^{\nu_1},2^{\nu_2},\ldots,N^{\nu_N}\}} \prod_{k=1}^{N} \left[ \pm \sum_{i=1}^{\infty} \exp \left( -k\beta E_i^{\lambda_i} \right) \right]^{\nu_k}}$$  \hspace{1cm} (25)

and analogously for distinguishable particles

$$G(\mu) = \left[ \frac{\sum_{i=1}^{\infty} \exp \left( i\mu E_i^{\lambda_i} \right) \exp \left( -k(i\mu + \beta)E_i^{\lambda_i} \right) \right]^{N}}{\sum_{i=1}^{\infty} \exp \left( -\beta E_i^{\lambda_i} \right) \right]^{N}}$$ \hspace{1cm} (26)

We further assume the quasistatic work to be proportional to the initial eigenenergies

$$W = E_i^{\lambda_i} - E_i^{\lambda_0} = \alpha E_i^{\lambda_0} \hspace{1cm} \forall i \hspace{1cm} (27)$$

which is justified, for instance, for a particle in a 1D piston or in a 1D harmonic potential. For this kind of systems we also can assume that each eigenenergy can be written as a power of the quantum number, and we have

$$E_i^{\lambda_0} = E_0 \times i^p.$$ \hspace{1cm} (28)

Under these assumptions the numerator of the characteristic function for a single particle can be approximately expressed as

$$\sum_{i=1}^{\infty} \exp \left( i\mu \alpha E_i^{\lambda_i} \right) \exp \left( -\beta E_i^{\lambda_i} \right) \approx \int_{-\infty}^{\infty} dW \exp \left( -\beta W/\alpha \right) \exp \left( i\mu W \right) \left( W/\alpha E_0 \right)^{1/p} \Theta(\alpha W),$$ \hspace{1cm} (29)

where $\Theta(\cdot)$ is the Heaviside step function.

Specifically, if the system is a particle in a 1D piston, the quasistatic work distribution becomes

$$P_s(W) = \frac{\exp \left( -\beta W/\alpha \right) \left( W/\alpha E_0 \right)^{1/2} \Theta(\alpha W)}{\int_{-\infty}^{\infty} dW \exp \left( -\beta W/\alpha \right) \left( W/\alpha E_0 \right)^{1/2} \Theta(\alpha W)} \hspace{1cm} (30)$$

$$= \frac{\beta}{|\alpha|^{1/2}} \left( \frac{\beta W}{\alpha} \right)^{1/2} \exp \left( -\beta W/\alpha \right) \Theta(\alpha W).$$

For $N$ distinguishable particles, the work distribution function can be obtained by replacing the $1/2$ with $N/2$
and due to the additivity of the Gamma distribution we obtain
\[ P(W) = \frac{\beta}{\alpha |\Gamma(N/2)|} \left( \frac{\beta W}{\alpha} \right)^{N/2 - 1} \exp \left( -\frac{\beta W}{\alpha} \right) \Theta(\alpha W). \]  
(31)

The latter result coincides with the expression for quasistatically compressing (or expanding) \( N \)-particle gas inside a piston, which was previously derived in Ref. [37].

C. Fluctuation theorems

We have seen in the above discussion that quantum interference effects the structure of the work distribution. It is worth emphasizing, however, that the quantum Jarzynski equality [2, 23] and the quantum Crooks fluctuation theorem [5, 23, 70] remain valid as the validity of these two nonequilibrium work relations does not depend on the details of the model or the quench protocol. Note that even if the initial \( N \)-particle states occupy different single particle states (as is allowed for both Bosons and Fermions) the average of the exponentiated work will be identical, and only the distribution differs. This is due to the particles being noninteracting but interfering [34].

As a case study we have numerically calculated the work distribution function for two and three identical particles in the 1D piston and 1D harmonic potential, and have demonstrated our theoretical findings.

In the limit of high temperature, the work distribution functions for Bosons and Fermions converge, and we have given a heuristic analysis for this observation by utilizing the representation theory of the symmetric group. Therefore, our study suggests a dynamic “correspondence principle” of work distribution functions in the quantum statistical sense.

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Appendix A: single particle transition matrix for the harmonic oscillator

The parametric harmonic oscillator with Hamiltonian (14) has been extensively studied in the literature. For the sake of completeness we summarize in this appendix several expressions that were used to create the plots in Fig. 5-8, and we also a correct minor typographical error that appeared in a previous publication [29].

The single particle transition matrix has been derived in Ref. [29]
\[ U_{m,n}^\tau = \sqrt{\frac{\omega_i \omega_f}{2 \sigma}} \left( \frac{\zeta^m \zeta^*}{\sigma_{n+1} \Gamma(n+1)} \right) \sum_{l=0}^{\min(m,n)} \frac{[-2i\sqrt{2/(Q^* - 1)}]^l}{l! [(n-l)/2]! [(m-l)/2]!}. \]  
(A1)

According to the selection rule \( m = n \pm 2k \), \( l \) runs over even numbers only, if \( m, n \) are even, and over odd numbers only, if \( m, n \) are odd.

The explicit expression for the matrix elements \( U_{m,n}^\tau \) then reads for even elements

\[ U_{2\mu,2\nu}^\tau = \sqrt{\frac{2\nu! 2\mu!}{22\nu+2\mu-1!}} \sqrt{\frac{\zeta^{2\nu} \zeta^{*2\nu}}{\sigma^{2\nu+2\mu+1}}} \frac{\sqrt{\omega_i \omega_f}}{\Gamma(\mu+1) \Gamma(\nu+1)} {}_2F_1 \left( -\mu, -\nu; \frac{1}{2}; \frac{2}{1-Q^*} \right). \]  
(A2)

and for odd elements

\[ U_{2\mu+1,2\nu+1}^\tau = -\sqrt{\frac{8i (2\nu+1)! (2\mu+1)!}{(Q^* - 1)^2 22\nu+2\mu+1}} \sqrt{\frac{\zeta^{2\nu+1} \zeta^{*2\mu+1}}{\sigma^{2\nu+2\mu+1}}} \frac{\sqrt{\omega_i \omega_f}}{\Gamma(\mu+1) \Gamma(\nu+1)} {}_2F_1 \left( -\mu, -\nu; \frac{3}{2}; \frac{2}{1-Q^*} \right). \]  
(A3)
We have here introduced the hypergeometric function \( _2F_1 \) [71] in order to simplify the sums and write the matrix elements \( U_{m,n} \) in closed form. \( \Gamma(x) \) denotes the Gamma function.

We further introduced the complex parameters,

\[
\begin{align*}
\zeta &= \omega_r \omega_0 X_r - \omega_0 i \dot{X}_r + \omega_r i Y_r + \dot{Y}_r \\
|\zeta|^2 &= 2 \omega_0 \omega_r (Q^* - 1) \\
\sigma &= \omega_r \omega_0 X_r - \omega_0 i \dot{X}_r - \omega_r i Y_r - \dot{Y}_r \\
|\sigma|^2 &= 2 \omega_0 \omega_r (Q^* + 1)
\end{align*}
\]

with

\[
Q^* = \frac{1}{2 \omega_0 \omega_r} \left[ \omega_0^2 \left( \omega_r^2 X^2_r + \dot{X}_r^2 \right) \left( \omega_r^2 Y^2_r + \dot{Y}_r^2 \right) \right]
\]

and where \( X_t \) and \( Y_t \) are solutions of the classical, force free equation of motion,

\[
\ddot{X}_t + \omega_t^2 X_t = 0
\]

with \( X_{t=0} = 0, \dot{X}_{t=0} = 1 \) and \( Y_{t=0} = 1, \dot{Y}_{t=0} = 0 \).

---

**Appendix B: derivation of the characteristic function of work distribution functions (22)**

The characteristic function of the work distribution can be obtained by using the representation theory of the symmetric group (see section 4.4 of Ref. [66]).

From Eq. (19) it follows that the characteristic function for a many-particle system can be expressed as

\[
G_{B/F}(\mu) = \frac{\text{tr}_{\mathcal{H}^{\lambda_0}_{B/F}} \{ \exp \left( i \mu H^\lambda_{B/F} + \exp (-i \mu H^\lambda_{B/F}) \exp (-\beta H^\lambda_{B/F}) \} \}}{\text{tr}_{\mathcal{H}^{\lambda_0}_{B/F}} \{ \exp (-\beta H^\lambda_{B/F}) \}}.
\]

Using the representation theory of the symmetric group, we further have

\[
G_{B/F}(\mu) = \frac{1}{N!} \sum_{P \in S_N} (\pm)^{p(P)} \text{tr}_{\mathcal{H}^{\lambda_0}_{B/F}} \{ \exp \left( i \mu H^\lambda_{B/F} + \exp (-i \mu H^\lambda_{B/F}) \exp (-\beta H^\lambda_{B/F}) \} \}
\]

where the equality sign holds only for non-interacting many-particle systems, and \( S_N \) is a group comprised of all permutation operators on \( \mathcal{H}^{\lambda_0} = \mathcal{H}^\lambda_s \otimes \mathcal{H}^\lambda_s \otimes \cdots \otimes \mathcal{H}^\lambda_s \). The elements of the group are denoted by \( P \), and \( p(P) \) is the transposition number of \( P \); \( \mathcal{M}^{B/F}_{(1^\nu_1,2^\nu_2,\ldots,N^\nu_N)} \) is the number of permutations belonging to \((1^\nu_1,2^\nu_2,\ldots,N^\nu_N)\) type, which has been studied in combinatorial mathematics and satisfies

\[
\mathcal{M}^{F}_{(1^\nu_1,2^\nu_2,\ldots,N^\nu_N)} = (-)^{p(P)} \mathcal{M}^{B}_{(1^\nu_1,2^\nu_2,\ldots,N^\nu_N)}
\]

and

\[
\mathcal{M}^{B}_{(1^\nu_1,2^\nu_2,\ldots,N^\nu_N)} = \frac{N!}{\prod_{i=1}^{\nu_1} \nu_1! \prod_{i=1}^{\nu_2} \nu_2! \cdots \prod_{i=1}^{\nu_N} \nu_N!}
\]

Substituting Eqs. (B3) and (B4) into Eq. (B2) we finally obtain the characteristic function (22).

Before concluding this section, we would like to point out that the relation

\[
\text{tr}_{\mathcal{H}^{\lambda_0}_{B/F}} \left( \hat{A} \right) = \frac{1}{N!} \sum_{P \in S_N} (\pm)^{p(P)} \text{tr}_{\mathcal{H}^{\lambda_0}_{B/F}} \left( \hat{A} P \right)
\]

holds true even when the particles are interacting. Here \( \hat{A} \) is an operator of multi-particle system.

---

**Appendix C: derivation of the equation of state of the ideal quantum gas from the characteristic function (22)**

In this appendix we derive the equation of state of the ideal quantum gas inside a piston from the characteristic function (22). This derivation is used as a self-consistent check to support the validity of Eq. (22). For a 1D piston system we define \( \alpha' = \frac{\lambda_0^2}{\lambda_0' + 1} \) and \( \alpha'' = \frac{\lambda_0'}{\lambda_0'' + 1} \) as the ratio of the work over the initial eigenenergy of the system for the forward \((\lambda_0 \to \lambda')\) and the reverse \((\lambda' \to \lambda_0)\) process. By making use of Eqs. (26) and (31), it can be checked that the work distribution function for Bosons or Fermions undergoing the quasistatic process satisfies the Crooks Fluctuation Theorem [4, 5]
where $\lambda_T$ is the thermal wave length

$$\lambda_T = \sqrt{\frac{2\pi \hbar^2}{M}}. \quad (C2)$$

From Eq. (C1) we obtain the difference of the free energy (the Jarzynski Equality [2]) of the quantum gas by utilizing the Crooks Fluctuation Theorem [4]

$$\exp (-\beta \Delta F_{B/F}) = \frac{\sum_{(1^N, 1^N, \ldots, N^N)} N! \prod_{k=1}^{N} k^{\nu_k} \prod_{l=1}^{N} \nu_l \pm \frac{\beta_0}{\lambda_T} \sum_{k=1}^{N} \nu_k}{\sum_{(1^N, 1^N, \ldots, N^N)} N! \prod_{k=1}^{N} k^{\nu_k} \prod_{l=1}^{N} \nu_l \pm \frac{\beta_0}{\lambda_T} \sum_{k=1}^{N} \nu_k} \mathcal{P}_{B/F}^R(-W) \exp(\beta W), \quad (C1)$$

where $m = N/\lambda$ ($m_0 = N/\lambda_0$) is the particle density. In the following we calculate the pressure of this system by utilizing the thermodynamic relation $p = -(\partial F/\partial \lambda)_T$

$$p_{B/F} = - \left( \frac{\partial \Delta F_{B/F}}{\partial \lambda} \right)_T \approx \frac{N\beta^{-1}}{\lambda + N\lambda_T/2\beta} \approx m\beta^{-1} \left( 1 \mp \frac{m\lambda_T}{2\beta} \right). \quad (C6)$$

For a $d$-dimensional system, the equation of state of the quantum gas inside a piston can be obtained in a similar way

$$p_{B/F} \approx m\beta^{-1} \left( 1 \mp \frac{m\lambda_T^d}{2\beta + d\beta} \right). \quad (C7)$$

This equation of state (C7) agrees with the result derived from the grandcanonical ensemble formulation [72]. The derivation of the equation of state of ideal quantum gases is an evidence supporting the validity of the characteristic function of the work distribution function (22). In fact, we have also checked that the first three virial coefficients are exactly the same as those obtained from the grand canonical ensemble formulation in the thermodynamic limit, which further convinces us the validity of Eq. (22).

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