SYMPLECTIC INVARIANCE OF RATIONAL SURFACES ON KÄHLER MANIFOLDS

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Abstract. Kollár and Ruan proved symplectic deformation invariance for uniruledness of Kähler manifolds. Zhiyu Tian proved the same for rational connectedness in dimension \( \leq 3 \). Kollár conjectured this in all dimensions. We prove Kollár’s conjecture, as well as existence of a covering family of rational surfaces, for all Kähler manifolds that are symplectically deformation equivalent to \( G/P \) or to a low degree complete intersection in such.

1. Statement of Results

Theorem 1.5 produces rational curves and surfaces via Gromov-Witten theory for \( G/P \). Theorem 1.18 extends this to complete intersections.

Question 1.1. For a connected, compact manifold \( X \) with (integrable) complex structure \( J \) and with the symplectic form \( \omega \) of a Kähler metric, which holomorphic properties depend only on \( \omega \)? Which depend only on the deformation class of \( \omega \)?

The (topological) complex vector bundle \( T^{1,0}_{X,\omega} \) underlying the holomorphic tangent bundle \( T^{1,0}_{X,J} \) depends only on the deformation class, as do its Chern classes. Gromov-Witten invariants also depend only on the deformation class.

Theorem 1.2. \([\text{Kol98}, \text{Theorem 4.2.10}]\), \([\text{Rua99}, \text{Proposition 4.9}]\) For connected, compact, Kähler manifolds, uniruledness is invariant under symplectic deformation.

Conjecture 1.3 (Kollár). For connected, compact, Kähler manifolds, rational connectedness is invariant under symplectic deformation.

The best result on this conjecture is a theorem of Zhiyu Tian.

Theorem 1.4 (Zhiyu Tian). \([\text{Tia12}]\) In complex dimension \( \leq 3 \), rational connectedness is preserved by symplectic deformation equivalence.

The formulation below has no explicit mention of Gromov-Witten invariants.

Theorem 1.5 (Symplectic rational curves and surfaces on \( G/P \)). 1. Every generalized complex flag variety \( G/P \) is a fiber type Fano manifold; the Mori cone is a free \( \mathbb{Z}_{\geq 0} \)-semigroup generated by extremal rays \( \mathbb{R}_{>0} \cdot \beta_i \) of classes \( \beta_i \) of free rational curves.

2. Kollár’s conjecture holds for \( G/P \): every Kähler manifold \( Z \) that is symplectically deformation equivalent to \( G/P \) is rationally connected.

3. Finally, \( Z \) has a covering family of rational surfaces, except if \( Z \cong \mathbb{C}P^1 \).

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The basic notions of Mori theory depend sensitively on $J$ rather than $\omega$.

**Definition 1.6** (Mori cone). The Mori cone, $\text{NE}^J(X) \mathbb{R}$, is the closure in $H_2(X; \mathbb{R})$ of the $\mathbb{R}_{\geq 0}$-semigroup $\text{NE}^J(X) \mathbb{R}$ spanned by classes of $J$-holomorphic curves. These classes are $J$-**effective**, and the $\mathbb{Z}_{\geq 0}$-semigroup spanned by these classes is $\text{NE}^J(X)$. A class $\beta \in \text{NE}^J(X)$ is $J$-**extremal** if $0 < \langle c_1(T^{1,0}_{X,\omega}), \beta \rangle \leq 1 + \dim C(X)$ and if, for some Kähler $[D] \in H^2(X; \mathbb{R})$, the function

$$\text{NE}^J(X) \mathbb{R} \setminus \{0\} \to \mathbb{R}, \; \alpha \mapsto \langle c_1(T^{1,0}_{X,\omega}), [D], \alpha \rangle / \langle [D], \alpha \rangle,$$

has a nonnegative maximum precisely on the ray $\mathbb{R}_{>0} \cdot \beta$. An element of a $\mathbb{Z}_{>0}$-semigroup $N$ in $H_2(X; \mathbb{Z})$ is $J$-**reducible** if it equals a nonzero sum of elements of $N$ plus a nonzero sum of classes of $J$-holomorphic spheres. Otherwise, it is $J$-**irreducible**. A nonzero element of $N$ is **decomposable** if it equals a sum of two nonzero classes in $N$. Otherwise, it is **indecomposable**.

Our focus is exclusively on extremal rays whose associated contraction is fiber type. This occurs if and only if the ray is generated by a $J$-extremal class that is free.

**Definition 1.7** (Free classes). A non-constant $J$-holomorphic map $u : \mathbb{C}P^1 \to X$ is $J$-**free**, resp. $J$-**very free**, if the holomorphic vector bundle $u^*T^{1,0}_{X,J}$ on $\mathbb{C}P^1$ is semi-ample, resp. ample. Then, the class $u_*[\mathbb{C}P^1]$ is $J$-**free**, resp. $J$-**very free**.

Denote the $\mathbb{Z}_{>0}$-span of such classes by $\text{NE}^J_*(X)$. A $J$-extremal class is a $J$-**ruling class** if the associated contraction is a ruling by conics. These classes generate the $\mathbb{Z}_{>0}$-semigroup $\text{NE}^J_*(X)$.

The next theorem uses symplectic “freeness” via Gromov-Witten invariants (for $\omega$).

**Definition 1.8** (Symplectically free classes). A class $\beta \in H_2(X; \mathbb{Z})$ is symplectically effective in genus $g$ if there exists $n \in \mathbb{Z}_{\geq 0}$ such that the full Gromov-Witten invariant is not identically zero,

$$\text{GW}_{X,\omega}^{g,n,\beta} : H^*(\mathfrak{M}_{g,n}; \mathbb{Q}) \otimes H^*(X^n; \mathbb{Q}) \to \mathbb{Q}.$$  

Denote by $\eta_X$ the Poincaré dual of the homology class of a point. The class $\beta$ is symplectically pseudo-free if there exists $n \in \mathbb{Z}_{\geq 0}$, if there exists $u \in H^*(\mathfrak{M}_{0,n+1}; \mathbb{Q})$, and if there exists $v \in H^*(X^n; \mathbb{Q})$ with $\text{GW}_{0,n+1,\beta}(u \otimes w) \neq 0$ for $w := pr^n_1\eta_X \sim pr^n_2v \in H^*(X \times X^n; \mathbb{Q})$.

Denote the $\mathbb{Z}_{>0}$-span of such classes by $\text{NE}_{\omega,p,f}^J(X)$. The normal degree is $m_\beta = m_\beta(X) := \langle c_1(T^{1,0}_{X,\omega}), \beta \rangle - 2$. If $\beta$ is free, then $m_\beta \geq 0$.

For $m_\beta \geq 0$, the first gravitational descendant is

$$f_\beta = f_\beta(X) := \langle r_\beta(\eta_X) \rangle^{X,\omega}_{0,\beta} = \text{GW}_{0,1,\beta}(c_1(\psi)^{m_\beta} \otimes \eta_X).$$

If $0 \leq m_\beta \leq \dim C(X) - 1$ and if $f_\beta > 0$, then $\beta$ is symplectically free. Denote the $\mathbb{Z}_{>0}$-span of such classes by $\text{NE}_{\omega,p,f}^J(X)$.

If $m_\beta = 0$, if $f_\beta = 1$, and if there exists a symplectically pseudo-free class that is $\mathbb{Q}$-linearly independent from $\beta$, then $\beta$ is symplectically ruling. Denote the $\mathbb{Z}_{>0}$-span of such classes by $\text{NE}_{\omega,r}^J(X)$.

If $m_\beta(X) > 0$ and $f_\beta(X) > 0$, then the second gravitational descendant and quotient invariant are

$$s_\beta = s_\beta(X) := \left(r_{m_\beta - 1}(\eta_m), c_2(T^{1,0}_{X,\omega}) + \frac{m_\beta}{2(m_\beta + 2)}c_1(T^{1,0}_{X,\omega})^2\right)^{X,\omega}_{0,\beta}, \; q_\beta(X) := \frac{s_\beta}{f_\beta}.$$
A symplectically free class $\beta$ is **symplectically 2-free** if $m_\beta > 0$ and $s_\beta > 0$.

The relevance of $f_\beta$ and $s_\beta$ is a sharp version of the Theorem 1.2. Kollár and Ruan prove that for every $J$-free $\beta \in \text{NE}_J^\omega(X)$ that is $J$-irreducible, the evaluation map,

$$\text{ev} : \overline{M}_{0,1}((X, J), \beta) \to X,$$

has fiber $F = \text{ev}^{-1}(\{q\})$ that is smooth, projective, with pure dimension $m_\beta$ and parameterizing only maps with smooth domain for every $q$ contained in a dense “Zariski” open $X_\beta \subset X$. Thus, the $J$-irreducible $J$-free classes equal the $J$-irreducible symplectically pseudo-free classes, $\beta \in \text{NE}_{s.p.f}^\omega(X)$.

**Theorem 1.9** (Relative ampleness of $\psi$). 1. The $J$-irreducible $J$-free classes also equal the $J$-irreducible symplectically free classes, $\beta \in \text{NE}_{s.f}^\omega(X)$, and these exist if and only if $\text{NE}_{s.f}^\omega(X)$ is nonzero.

2. For every such class $\beta$, the $\psi$-line bundle is ample on the fiber $F$ of $\text{ev}$ over every point of $X_\beta$. If $m_\beta > 0$, the two leading coefficients of the Hilbert polynomial are,

$$\chi(F, \psi^d) = \frac{f_\beta}{m_\beta!}d^m + \frac{s_\beta}{2(m_\beta - 1)!}q^m - 1 + \cdots + \frac{f_\beta d^m}{m_\beta!} \left(1 + \frac{m_\beta q^d}{2d} + \cdots \right).$$

3. For every $[D_0], \ldots, [D_r] \in H^2(X; \mathbb{R})$, there is an equality of Gysin classes,

$$\pi_*u^*(D_0 \cdots \sim D_r) = (D_0, \beta) \cdots (D_r, \beta) c_1(\psi)^{r} \in H^{2r}(F_{\beta, x}; \mathbb{R}),$$

thus

$$\langle \tau_0(\eta_X), D_0 \cdots \sim D_r, -\rangle_{0, \beta}^X = (D_0, \beta) \cdots (D_r, \beta) \langle \tau_0(\eta_X), -\rangle_{0, \beta}^X.$$

In particular, $X$ is uniruled if and only if $\text{NE}_{s.f}^\omega(X)$ is nonzero.

The general form of Theorem 1.9 is as follows.

**Theorem 1.10** (Rational curves and surfaces via $\text{NE}_{s.f}^\omega(X)$). 1. For Kähler $X$ and $Z$ that are symplectically deformation equivalent, both are rationally connected whenever $\text{NE}_{s.f}^\omega(X)$ is nonzero and the $\mathbb{Q}$-span of $\text{NE}_{s.p.f}^\omega(X)$ equals $H_2(X; \mathbb{Q})$.

2. A Kähler $Z$ that is symplectically equivalent, resp. symplectically deformation equivalent, to $X$ is covered by rational surfaces if $\text{NE}_{s.f}^\omega(X)$ is nonzero and if at least one nonzero $\omega$-minimal class, resp. every nonzero indecomposable class, in $\text{NE}_{s.f}^\omega(X)$ is symplectically ruling or symplectically 2-free.

3. Conversely, for a $J$-irreducible $\beta \in \text{NE}_{s.f}^\omega(X)$ with $m_\beta > 0$, if a general $\text{ev}$-fiber $F$ has $c_1(T_F) \in H^2(F; \mathbb{R})$ nonzero and pseudo-effective, then $\beta$ is symplectically 2-free.

**Question 1.11.** Does every rationally connected $X$ have a nonzero 2-point invariant $\text{GW}_{0,n+2,\beta}^X(u \otimes w)$ for some $\beta \in H_2(X; \mathbb{Z})$, for some $u \in H^\omega(\mathfrak{h}_{0,n+2}; \mathbb{Q})$, and for some $v \in H^\omega(X^n; \mathbb{Q})$, where $w := pr_1^*(\eta_{X \times X}) \sim pr_2^*(v) \in H^\omega(X^2 \times X^n; \mathbb{Q})$?

**Remark 1.12.** Every Hirzebruch surface $X = \Sigma_n$, satisfies (1) above. However, for $n \geq 4$, for every $\beta$ such that the 2-point evaluation map to $X \times X$ is dominant, every fiber has excess irreducible components. Thus the **proof method** of Theorem 1.2 does not imply nonzero 2-point invariants.

Larger than the class of generalized complex flag varieties $G/P$ are the smooth complete intersections, $X = Y_1 \cap \cdots \cap Y_c$, of analytic hypersurfaces $Y_i$ in $G/P$. 


Theorem 1.18 (Complete intersections). 1. Assume that the compact, Kähler manifold $Y$ has a submersive contraction of ruling classes, $\pi: Y \to Y'$. Let $X = \pi^{-1}(X')$ be the inverse image of a complete intersection $X' = Y'_1 \cap \cdots \cap Y'_c$ of positive, complex analytic hypersurfaces $Y'_j \subset Y'$ with $\dim \mathbb{C}(X') \geq 3$.

2. Assume further that $Y$ is integrally fiber type Fano, resp. simplicially Fano. Then also $X$ is integrally fiber type Fano, resp. simplicially Fano, and the pushforward map induces an isomorphism $\operatorname{NE}_f(X) \to \operatorname{NE}_f(Y)$ if and only if $m_{\beta_i}(Y, X) \leq m_{\beta_i}(Y)$ for every primitive generator $\beta_i \in \operatorname{NE}_f(Y)$ of an extremal ray. In this case, there are identities

$$m_{\beta_i}(X) = m_{\beta_i}(Y) - m_{\beta_i}(Y, X), \quad f_{\beta_i}(X) = f_{\beta_i}(Y) \cdot f_{\beta_i}(Y),$$

3. For $X$ and $Y$ satisfying the conditions in 1 and 2, a Kähler $Z$ that is symplectically equivalent to $X$, resp. symplectically deformation equivalent to $X$, is covered by rational surfaces if at least one $\omega$-minimal $\beta_i \in \operatorname{NE}^s_f(Y)$, resp. every indecomposable $\beta_i \in \operatorname{NE}^s_f(Y)$, satisfies either

- (i) $\beta_i$ is symplectically ruling, or
- (ii) $\beta_i$ is symplectically 2-free, $m_{\beta_i}(Y, X) < m_{\beta_i}(Y)$, and $s_{\beta_i}(Y, X) < s_{\beta_i}(Y)$.

In Case (i), $\beta_i$ is symplectically ruling for $X$. In Case (ii), $\beta_i$ is symplectically 2-free for $X$ and

$$q_{\beta_i}(X) = q_{\beta_i}(Y) - q_{\beta_i}(Y, X), \quad s_{\beta_i}(X) = f_{\beta_i}(Y, X) \cdot (s_{\beta_i}(Y) - s_{\beta_i}(Y, X)).$$

Proposition 1.19 (\operatorname{NE}^s_f(Y) and fiber type Fano manifolds). 1. A Kähler $X$ is fiber type Fano if and only if $\operatorname{NE}^s_f(X)_R$ is dual to the closure of the Kähler cone.

2. Then it is integrally fiber type Fano if and only if a general fiber of the contraction of each primitive, $J$-extremal $\beta_i$ has Fano index equal to the pseudo-index, $m_{\beta_i} + 2$. 

Question 1.13. Which complete intersections $X$ in $G/P$ are fiber type Fano? Which have the above property of existence of a covering family of rational surfaces?
Question 1.20. Is a Fano manifold simplicial whenever it is integrally fiber type?

All evidence points to a positive answer. I am grateful to Cinzia Casagrande who taught me about positive results in this direction. Wiśniewski bounds the number of extremal rays of fiber type Fano manifolds, [Wis91]. Druel proves that when this bound is attained, then the Mori cone is simplicial, [Dru16]. On studies this problem when the number of extremal rays is one smaller that the maximal bound, [Ou18]. Finally, Casagrande proves a positive answer whenever the complex dimension is \( \leq 5 \), [Cas08].

2. Approaches to uniruledness, rational connectedness, and rational surfaces

Gromov-Witten invariants of an almost complex, symplectic manifold rely on a 
\emph{virtual structure}. This is a trace of transversality on the solution space of the \( \partial \) -equation defining \( J \)-holomorphic curves. This virtual structure is independent of deformations of \((J, \omega)\).

The Kollár-Ruan Theorem establishes transversality of the space of \( J \)-holomorphic spheres containing a general point of a Kähler manifold when the free curve class is \( J \)-irreducible, i.e., when “bubbling” does not occur. Thus, a Kähler manifold is uniruled if and only if there exists a symplectically pseudo-free curve class. Therefore, uniruledness is independent of deformations of \((J, \omega)\).

Unfortunately, transversality fails for \( J \)-holomorphic spheres containing two or more general points. Zhiyu Tian’s approach to Kollár’s conjecture compensates for non-transversality via an explicit study of the Mori program applied to a Kähler \( X \) that is symplectically deformation equivalent to a rationally connected, projective manifold. Since divisorial and flipping contractions are birational transformations, the work of Hu-Li-Ruan on symplectic birational cobordism invariance of Gromov-Witten invariants plays a key role, [HLR08]. Zhiyu Tian’s tour-de-force proof of Kollár’s conjecture in dimension 3 uses both Gromov-Witten theory and the explicit classification of threefold extremal contractions by Mori and Mukai, [MM82].

There are several theorems giving alternative characterizations and extensions of uniruledness with applications to other questions in symplectic topology. For instance, for symplectic manifolds with a Hamiltonian \( S^1 \)-action (an analogue of Kähler manifolds with a nontrivial holomorphic \( C^* \)-action), McDuff uses Seidel elements to prove uniruledness via a ring-theoretic property of the quantum cohomology ring, [McD09]. Also, McLean proves that for smooth affine varieties, uniruledness of projective compactifications is a symplectic invariant of the underlying symplectic manifold of the smooth affine variety, [McL14].

Following Voisin, [Voi08], and Zhiyu Tian, [Tia12], our approach to Kollár’s conjecture instead relies on the maximally rationally connected fibration. For a compact, Kähler manifold \((X, J, \omega)\), denote the Barlet space by \( \text{Barlet}(X, J) \), cf. [Bar75].

**Definition 2.1.** A \textbf{Zariski open subset} of \( X \) is the open complement \( X^o \) of a closed complex analytic subspace of \( X \).

A \textbf{meromorphic function} on \( X \) is an equivalence class of pairs \((X^o, \phi)\) of a dense Zariski open subset \( X^o \) of \( X \) and a holomorphic function,

\[
X \ni X^o \xrightarrow{\phi} Q^o \subseteq Q,
\]
to a dense Zariski open subset $Q^o$ of a compact complex analytic space $Q$ in Fujiki class $C$ such that the closure of the graph of $\phi$ is a complex analytic subspace of $X \times Q$.

An **almost holomorphic fibration** of $X$ is a finite (and proper) holomorphic map with normal domain,

$$\Phi : Q \to \text{Barlet}(X, J),$$

such that the pullback universal cycle in $X \times Q$ is the closure of the graph of a meromorphic function,

$$\phi : X \supseteq X^o \to Q^o \subseteq Q,$$

that is a proper holomorphic submersion between dense Zariski opens $X^o$ of $X$ and $Q^o$ of $Q$.

A **rational quotient** of $(X, J)$ is an almost holomorphic fibration with the following property. For every pair

$$(\pi : C \to M, u : C \to X),$$

of (1) a proper, flat, holomorphic map $\pi$ to a normal, connected analytic space $M$ whose fibers are trees of rational curves and (2) a holomorphic map $u$ such that the composition $\phi \circ u$ is defined and dominant, there exists a dense, Zariski open $M^o \subset M$ and a commutative diagram of complex analytic spaces,

$$
\begin{array}{ccc}
C^o & \xrightarrow{u^o} & X^o \\
\pi^o \downarrow & & \downarrow \phi^o \\
M^o & \xrightarrow{v^o} & Q^o
\end{array}
$$

where $C^o$ equals $\pi^{-1}(M^o)$.

**Remark 2.2.** The existence of the rational quotient of each connected, compact Kähler manifold was proved in [Cam92]. When $(X, J)$ is a complex projective manifold, this is the same notion as the **maximally rationally connected fibration**, which was constructed for normal projective schemes over arbitrary fields (including positive characteristic fields) in [KMM92].

Since the fiber dimension of $\phi$ is at least $m_\beta + 1$ for every $J$-irreducible $\beta \in \text{NE}_{m, f}(X)$, the fiber dimension is at least the maximum of all $m_\beta$.

**Notation 2.3.** Denote by $m$, resp. by $m'$, the maximum, resp. minimum, of $m_\beta$ for every $J$-irreducible $\beta \in \text{NE}_{m, f}(X)$.

The pullback by $\phi$ (rather, the corresponding graph closure considered as a correspondence) is a $\mathbb{Z}$-linear map of cohomology that is compatible with the pullback map on global sections of reflexive coherent analytic sheaves,

$$H^\ell(Q; \mathbb{Z}) \to H^\ell(X; \mathbb{Z}),$$

resp. $H^0(Q, (\Omega^{\otimes \ell}_{Q/\mathbb{C}})^{\vee \vee}) \to H^0(X, \Omega^{\otimes \ell}_{X/\mathbb{C}})$.

**Proposition 2.4.** A compact, connected, Kähler manifold $X$ is rationally connected if for every non-uniruled $Q$ as above with $1 \leq \dim_C(Q) \leq \dim_C(X) - m - 1$, for at least one $\ell$ positive, $h^0(Q, (\Omega^{\otimes \ell}_{Q/\mathbb{C}})^{\vee \vee}) > h^0(X, \Omega^{\otimes \ell}_{X/\mathbb{C}})$.

**Proof.** By [GHS03], the quotient $Q$ is either a point or it is non-uniruled of positive dimension. Since $\phi$ is a holomorphic submersion on a dense open $X^o$, the pullback map on holomorphic contravariant tensors is injective. \qed
This is particularly useful if the dimension inequalities imply that \( Q \) has small dimension, e.g., if \( Q \) is a curve. In this case, \( h^{\ell,0}(Q) \) is positive for some \( \ell > 0 \). If \( b_\ell(X) \leq 1 \), this contradicts the Hodge inequality \( b_\ell \geq 2h^{\ell,0} \). Conjecturally, every non-uniruled \( Q \) of positive dimension has positive \( h^0(Q,\mathcal{O}_Q^{\otimes \ell}) \) for all \( \ell \gg 0 \). The variant of this technique proved at the beginning of Section 5 is even simpler, and it is not contingent upon conjectures.

2.1. Rational surfaces. There is no transversality for the spaces of holomorphic rational surfaces in a Kähler manifold: the \( \partial \)-equation is over-determined. However, the virtual structure on the space of \( J \)-holomorphic spheres determines a virtual first Chern class on each fiber \( F \) of the evaluation map,

\[
ev : \overline{\mathcal{M}}_{0,1}((X,J),\beta) \rightarrow X.
\]

This is computed in [dJS17]. If \( F \) is pure of complex dimension \( m_\beta \), the virtual first Chern class equals the first Chern class of the dual of the cotangent complex of \( F \) (which is perfect of amplitude \([−1,0])\). When the fiber \( F \) is also smooth, this virtual first Chern class equals the actual first Chern class of \( F \).

The formula from [dJS17] for the virtual first Chern class is a pullback of an element of \( H^*(\mathcal{M}_{0,1};\mathbb{Q}) \otimes H^*(X;\mathbb{Q}) \). Thus, for each element \( \gamma \in H^*(\mathcal{M}_{0,1};\mathbb{Q}) \otimes H^*(X;\mathbb{Q}) \) in complementary degree \( 2(m_\beta − 1) \), the Gromov-Witten invariant of the cup product of the virtual first Chern class and \( \gamma \) is a gravitational descendant. Thus, it is a symplectic deformation invariant.

When the pullback of \( \gamma \) is the Poincaré dual cohomology class of the curve class of a moving family of curves in \( F \), this gravitational descendant equals the anticanonical degree of a moving family of curves in \( F \). Whenever this degree is positive, \( F \) is uniruled by Mori’s theorem. Finally, for a big divisor class \([D] \in H^2(F;\mathbb{Q}) \), the class \( \gamma = [D]^{m_\beta − 1} \) is a \( \mathbb{Q}_{>0} \)-multiple of the Poincaré dual of a moving curve class on \( F \). Thus, \( F \) is uniruled if the gravitational descendant is positive for some big divisor class \([D] \). This is the essence of our approach: find enough pairs \( (\beta,[D]) \) of an indecomposable \( \beta \in \text{NE}_{X,\omega}^+(X) \) and an element \([D] \in H^2(\mathcal{M}_{0,1} \times X;\mathbb{Q}) \) with positive gravitational descendant so that for every Kähler structure \((J,\omega)\) in the deformation class, there exists at least one such pair with \( \beta \) a \( J \)-irreducible class and with \([D] \) a big class on \( F \).

The earlier work, [dJS07], followed a similar strategy to prove uniruledness of all spaces \( \mathcal{M}_{0,0}((X,J),\beta) \), not merely those with \( \beta \) a \( J \)-irreducible element of \( \text{NE}_{X,\omega}^+(X) \). In that case, the class \([D] \) is a “quasi-map divisor class”. The pairing of the virtual first Chern class with the appropriate power \([D]^n \) is positive if \( c_1(T_{X,\omega}^{1,0})^2 \), resp. \( ch_2(T_{X,\omega}^{1,0}) \), has positive pairing, resp. nonnegative pairing, with the homology class of every \( J \)-holomorphic, irreducible surface in \((X,J)\). The Kähler \( X \) is \textbf{Fano} if \( c_1(T_{X,\omega}^{1,0}) \) is positive, and it is \textbf{weakly 2-Fano} if also \( ch_2(T_{X,\omega}^{1,0}) \) has nonnegative pairing with \( J \)-holomorphic, irreducible surfaces.

**Theorem 2.5.** [dJS07] For every weakly 2-Fano manifold \( X \) with pseudo-index \( \geq 3 \), free rational curves parameterized by an irreducible component \( M_j \) of \( \mathcal{M}_{0,0}((X,J),\beta) \) move in pencils on a covering family of rational surfaces provided that every component \( M_j \) intersecting \( M_i \) parameterizes some maps with irreducible domain.
There are many examples of such $X$. The best classification results are due to Araujo and Castravet, [AC12], [AC13]. They also prove a sharper version of this theorem. Although the transversality hypothesis on $M_j$ is valid in many cases, there are also cases where it fails. Moreover, the cone of $J$-effective homology classes of surfaces is certainly not a symplectic deformation invariant. Thus, neither the hypotheses nor conclusion of this theorem are invariant under deformation of $J$.

Our method here produces a covering family of rational surfaces from positivity of gravitational descendants, precisely by restricting to indecomposable $\beta \in \text{NE}_\omega(X)$ and by replacing the quasi-map divisor $[D]$ by the psi class $\psi$.

2.2. Next steps. This approach is the first step in a larger program to construct “very twisting” ruled surfaces on Kähler manifolds with sufficient positivity of $c_1(T_{X,\omega}^{1,0})$ and $c_2(T_{X,\omega}^{1,0})$. These surfaces come from covering families of rational curves in parameter spaces $\overline{\mathcal{M}}_{0,1}(X, J, \beta)$ for $J$-irreducible classes $\beta$, just as the rational surfaces produced in this paper. However, the very twisting ruled surfaces also satisfy additional constraints. By [dJHS11] and [Zhu17], very twisting surfaces are the key ingredient in proving existence of rational sections for fibrations over a surface whose general fiber is deformation equivalent to $(X, J)$. By [HJS06], cf. also [DeL15] and [Min16], existence of very twisting surfaces also implies the Weak Approximation Conjecture of Hassett and Tschinkel for $(X, J)$, cf. [HT06] and [Has10]. In each earlier paper, [dJHS11], [Zhu17], [DeL15], and [Min16], the construction of each very twisting surface uses special properties of the particular Kähler manifold $(X, J, \omega)$. The goal here is to construct covering ruled rational surfaces in the most robust manner possible: using Gromov-Witten invariants rather than special properties.

Finally, a leading open case for existence of very twisting ruled surfaces is when $(X, J, \omega)$ is the wonderful compactification orbifold $\hat{G}$ of a simply connected semisimple complex Lie group. Existence of very twisting ruled surfaces in this case is the key missing ingredient in a type-free proof of Serre’s “Conjecture II” over function fields for all simply connected, semisimple algebraic groups (not merely the split and quasi-split groups). The generalized complex flag manifolds appear as closed orbits in the orbit decomposition of $\hat{G}$. It would be useful to either compute directly the relevant gravitational descendants $f_\beta$ and $s_\beta$ for $\hat{G}$, or to use virtual localization to reduce this to the computation of gravitational descendants on those generalized complex flag manifolds appearing as closed orbits in $\hat{G}$.

2.3. Changes from previous versions. A previous version of this paper included a description of the “quasi-map divisor class” $[D]$, as well as extensions of the main theorems to include the case when the big divisor class on the ev-fiber $F$ is the quasi-map class rather than the psi class. As mentioned, those extensions are not invariant under deformation of the almost complex structure $J$, both because transversality of the moduli space of $J$-holomorphic curves is not invariant, and because the cone of $J$-effective surfaces is not invariant. This version focuses on consequences of positivity of the psi class and results that are invariant under deformation of $J$.
3. Gromov-Witten invariants

The numbers $m_{g,n}$, $f_{g,n}$, and $s_{g,n}$ are defined via Gromov-Witten invariants. Here is a quick review of the relevant aspects of Gromov-Witten invariants for compact, Kähler manifolds.

For integers $g,n \geq 0$, denote by $\mathcal{M}_{g,n}$ both the Artin $\mathbb{C}$-stack (for $\mathbb{C}$-schemes with the fppf topology) and the associated complex analytic stack parameterizing flat, proper families of $n$-pointed curves

$$(C,(p_1,\ldots,p_n)),$$

that are prestable, i.e., $C$ is a proper, connected, reduced, at-worst-nodal curve of arithmetic genus $g$ together with an ordered $n$-tuple $(p_1,\ldots,p_n)$ of distinct, smooth points of $C$. For every homology class $\beta \in H_2(X;\mathbb{Z})$, the virtual dimension of the space of genus-$g$, $n$-pointed stable maps of class $\beta$ is

$$d = q_{g,n,\beta}^X \omega = \langle c_1(T_{X,\omega}^{1,0}),\beta \rangle + (\dim \mathbb{C}(X) - 3)(1 - g) + n.$$

For every connected, closed, symplectic manifold $(X,\omega)$, the Gromov-Witten invariant is a $\mathbb{Q}$-linear functional on cohomology,

$$\text{GW}^X_{g,n,\beta} : H^{2d}(\mathcal{M}_{g,n} \times X^n;\mathbb{Q}) \rightarrow \mathbb{Q}.$$

For $n \geq 1$, for every $i = 1,\ldots,n$, denote by $\psi_i$ the $i$th psi class, i.e., the relative dualizing sheaf of the 1-morphism forgetting the $i$th marked point,

$$\pi_i : \mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,n-1}.$$

For every $n \geq 0$, for every $n$-tuple of nonnegative integers, $\underline{m} = (m_1,\ldots,m_n)$, for every $n$-tuple $(\gamma_1,\ldots,\gamma_n) \in H^*(X,\mathbb{Q})^n$ of homogeneous classes of degrees $\text{deg}(\gamma_i) = e_i$ with $2|m| + |\underline{m}|$ equal to $2d$, the associated gravitational descendant equals

$$\langle \tau_{m_1}(\gamma_1),\ldots,\tau_{m_n}(\gamma_n) \rangle_{g,n,\beta}^X \omega := \text{GW}^X_{g,n,\beta} \left( \prod_{i=1}^{n} q^i c_1(\psi_i)^{m_i} \sim \text{pr}_i^* \gamma_i \right).$$

In particular,

$$f_{g,n} = \langle \tau_{m_n}(\eta_M) \rangle_{g,n,\beta}^X \omega, \quad s_{g,n} = \langle \tau_{m_1-1}(\eta_M),\text{ch}_2(T_{X,\omega}^{1,0}) + \frac{m_{g,n}}{2(m_{g,n} + 2)^2} c_1(T_{X,\omega}^{1,0}) \rangle_{g,n,\beta}^X \omega.$$

For a connected, closed, symplectic manifold $(X,\omega,J)$ with an $\omega$-tame almost complex structure $J$, one construction of the Gromov-Witten invariant uses the $\mathbb{Q}$-linear functional of pairing against a virtual fundamental class (in homology) for elements of the cohomology of a moduli space $\overline{\mathcal{M}}_{g,n}((X,J),\beta)$. Here $\overline{\mathcal{M}}_{g,n}((X,J),\beta)$ is the moduli stack parameterizing genus-$g$, $n$-pointed $J$-holomorphic stable maps of class $\beta$ to $X$,

$$(C,(p_1,\ldots,p_r),u : C \rightarrow X).$$

The datum $(C,(p_1,\ldots,p_r))$ is an object of $\mathcal{M}_{g,n}$. The map $u$ is $J$-holomorphic with pushforward class $u_*[C] = \beta$. Finally, the datum is stable, i.e., the log dualizing sheaf $\omega_C(p_1 + \cdots + p_r)$ on $C$ is ample on every irreducible component $C_i$ such that $u_*[C_i]$ vanishes.

The forgetful map associates to each stable map the underlying object of $\mathcal{M}_{g,n}$,

$$\Phi : \overline{\mathcal{M}}_{g,n}((X,J),\beta) \rightarrow \mathcal{M}_{g,n}, \quad (C,(p_1,\ldots,p_r),u : C \rightarrow X) \mapsto (C,(p_1,\ldots,p_r)).$$
Similarly, the evaluation map is the map
\[ \text{ev} : \overline{M}_{g,n}((X, J), \beta) \to X^\circ, \quad (C, (p_1, \ldots, p_r), u : C \to X) \mapsto (u(p_1), \ldots, u(p_n)) \].

There are natural embeddings of \( \overline{M}_{g,n}((X, J), \beta) \) into (infinite-dimensional) function spaces of stable \( L^\circ_1 \) maps that are not necessarily \( J \)-holomorphic, cf. [LT98] and [Zin11], and this gives \( \overline{M}_{g,n}((X, J), \beta) \) a topological and metric structure for which \( \Phi \) and \( \text{ev} \) are continuous. When \( (X, \omega, J) \) is \( \text{Kähler} \), then \( \overline{M}_{g,n}((X, J), \beta) \) is even a complex analytic Deligne-Mumford stack whose coarse moduli space is in Fujiki class \( C \) and compact. In particular, it has finitely many irreducible components, each of which is a compact complex analytic space in Fujiki class \( C \).

Li and Tian give an analytic construction of the virtual fundamental class in the homology of \( \overline{M}_{g,n}((X, J), \beta) \). There is a construction within algebraic geometry if \( X \) is projective using perfect obstruction theories. That construction also computes the tangent bundle on the smooth locus of the moduli space by [dJS17]. The algebraic construction applies for classes \( \beta \) that are \( \text{ev}-\text{dominant} \), since the rational quotient is relatively projective,
\[ \phi : X \supseteq X^\circ \to Q^\circ \subseteq Q. \]

**Definition 3.1.** A class \( \beta \in H_2(X; \mathbb{Z}) \) is **ev-dominant** if the associated evaluation map is surjective,
\[ \text{ev}_{0,1,\beta} : \overline{M}_{0,1}((X, J), \beta) \to X. \]

An irreducible component of \( \overline{M}_{0,0}((X, J), \beta) \) is **ev-dominant** if the inverse image in \( \overline{M}_{0,1}((X, J), \beta) \) surjects to \( X \) under the evaluation map.

**Proposition 3.2.** 1. A connected, compact, \( \text{Kähler} \) manifold is projective if it is rationally connected, e.g., this holds for the fibers of \( \phi \).
2. For every \( \beta \in H_2(X; \mathbb{Z}) \), shrinking \( Q^\circ \) and \( X^\circ \) if necessary, there exists a Zariski open \( \overline{M}_{0,0}((X, J), \beta)^o \) of \( \overline{M}_{0,0}((X, J), \beta) \) whose inverse image \( \overline{M}_{0,1}((X, J), \beta) \) equals \( \text{ev}^{-1}(X^\circ) \) and such that there is a commutative diagram,
\[ \begin{array}{ccc}
\overline{M}_{0,1}((X, J), \beta)^o & \longrightarrow & X^\circ \\
\downarrow & & \downarrow \\
\overline{M}_{0,0}((X, J), \beta)^o & \longrightarrow & Q^\circ 
\end{array} \]
that identifies \( \overline{M}_{0,0}((X, J), \beta)^o \) with the relative moduli space \( \overline{M}_{0,0}(X^o/Q^o, \beta) \).
3. The \( Q^\circ \)-fibers of \( \overline{M}_{0,0}((X, J), \beta)^o \) are algebraic Deligne-Mumford stacks whose coarse moduli spaces are complex projective varieties. Up to shrinking \( Q^\circ \) further, both \( \overline{M}_{0,0}((X, J), \beta)^o \) and the coarse moduli space are flat and projective over \( Q^\circ \).

**Proof.** The fibers of \( \phi \) are connected, compact \( \text{Kähler} \) manifolds that are rationally connected, where the restriction of \( \omega \) is a \( \text{Kähler} \) class in \( H^2_\omega \). For every such manifold, all small perturbations of the \( \text{Kähler} \) class in \( H^2_\omega \) are symplectic classes. Every rationally connected manifold has vanishing \( h^{\ell,0} \) for every \( \ell > 0 \), so that \( H^2_\omega \) equals \( H^{1,1} \). Thus, sufficiently small perturbations of the \( \text{Kähler} \) class are \( \text{Kähler} \) \( (1,1) \)-classes in \( H^2_\omega \). Since \( H^2_\omega \) is dense in \( H^2_\omega \), some of these \( \text{Kähler} \) \( (1,1) \)-classes are rational. Therefore, every rationally connected, \( \text{Kähler} \) manifold is a complex projective manifold by the Kodaira embedding theorem.
By the defining property of the rational quotient, up to shrinking $Q^o$, every $\beta$-curve that intersects $X^o$ is contained in a fiber of $\phi$. Thus, the relative moduli space equals the open subset $\overline{M}_{0,0}(\mathcal{X}, J, \beta)^o$ parameterizing maps that intersect $X^o$, i.e., the Zariski open complement of the closed analytic subspace of $\overline{M}_{0,0}(\mathcal{X} \setminus X^o, \beta)$ of $\overline{M}_{0,0}(\mathcal{X}, \beta)$.

Since $X^o$ is relatively projective over $Q^o$, also $\overline{M}_{0,0}(\mathcal{X}^o/Q^o, \beta)$ is $Q^o$-relatively an algebraic Deligne-Mumford stack whose coarse moduli space is relatively projective. By generic flatness, up to shrinking $Q^o$, these are even $Q^o$-flat.

Because $\overline{M}_{0,0}(\mathcal{X}^o/Q^o, \beta)$ is an algebraic Deligne-Mumford stack over $Q^o$ whose coarse moduli space is projective, and both are flat over $Q^o$, the algebraic results of [dJS17] apply. This gives a formula for the canonical divisor class on the maximal open that has pure dimension $d$.

4. Ampleness of the Psi Class

One of the fundamental facts about the evaluation map for genus 0 stable maps is that obstructedness of the evaluation map is independent of the choice of the marked point, at least when the domain is irreducible. This follows from facts about vanishing of higher cohomology of coherent sheaves on a genus 0 curve. The fastest way to formulate these facts is via the “universal extension” of the structure by the relative dualizing sheaf; roughly just the direct sum of two copies of the dual Serre twisting sheaf $O(-1)$.

**Definition 4.1.** [dJS17, Section 3] For every proper, flat map of complex analytic spaces, $\pi : C \to S$, whose fibers are reduced, at-worst-nodal, connected, genus-0 curves, the **universal extension** of the relative dualizing sheaf $\omega_\pi$ is the short exact sequence of coherent analytic sheaves on $C$,

$$\Xi_\pi : 0 \to \omega_\pi \to E_\pi \to \mathcal{O}_C \to 0,$$

unique up to $O^*_S$ and compatible with arbitrary base change, such that the coherent analytic sheaves $\pi_* E_\pi$ and $R^1 \pi_* E_\pi$ are both zero. For every finite rank, locally free $\mathcal{O}_C$-module $\mathcal{F}$, for the associated short exact sequence

$$Hom_{\mathcal{O}_C}(\mathcal{F}, \Xi_\pi) : 0 \to Hom_{\mathcal{O}_C}(\mathcal{F}, \omega_\pi) \to Hom_{\mathcal{O}_C}(\mathcal{F}, E_\pi) \to Hom_{\mathcal{O}_C}(\mathcal{F}, \mathcal{O}_C) \to 0,$$

the **adjoint map** is the connecting map of higher direct image sheaves,

$$\delta_{\Xi, \mathcal{F}} : \pi_* \mathcal{F}^\vee \to R^1 \pi_* Hom_{\mathcal{O}_C}(\mathcal{F}, \omega_\pi),$$

whose fiber map at each $s \in S$, via Serre duality, equals

$$\delta_{\Xi, \mathcal{F}, s} : (\pi_* \mathcal{F}^\vee)_s \to H^0(C_s, \mathcal{F}_s)^\vee.$$

Up to taking duals, vanishing of a higher direct image sheaf involving the universal extension allows us to encode both vanishing of higher direct image of our original sheaf, and also surjectivity of the adjoint map.

**Lemma 4.2.** For every finite rank, locally free $\mathcal{O}_C$-module $\mathcal{F}$, the higher direct image sheaf $R^1 \pi_* Hom_{\mathcal{O}_C}(\mathcal{F}, E_\pi)$ is zero if and only if both $R^1 \pi_* (\mathcal{F}^\vee)$ is zero and for every $s \in S$, the adjoint map $\delta_{\Xi, \mathcal{F}, s}$ is surjective.
Proof. Since $R^q\pi_*$ of a coherent sheaf vanishes for every $q \geq 2$, the long exact sequence of higher direct images associated to the short exact sequence $\text{Hom}_{\mathcal{O}_C}(\mathcal{F}, \Xi_{\pi})$ yields

$$\pi_*(\mathcal{F}^\vee) \xrightarrow{\delta} R^1\pi_*(\mathcal{F}^\vee \otimes \omega_\pi) \to R^1\pi_*\text{Hom}_{\mathcal{O}_C}(\mathcal{F}, E_{\pi}) \to R^1\pi_*(\mathcal{F}^\vee) \to 0.$$ 

Thus, $R^1\pi_*\text{Hom}_{\mathcal{O}_C}(\mathcal{F}, E_{\pi})$ vanishes if and only if $R^1\pi_*(\mathcal{F}^\vee)$ vanishes and $\delta_{\Xi,F}$ is surjective. Finally, $\delta_{\Xi,F}$ is surjective if and only if for every $s \in S$, the fiber map $\delta_{\Xi,F,s}$ is surjective. □

The locus of maps from a family of smooth curves has an ev-obstructed locus that can be detected after base change, e.g., to a relative Hom space.

**Definition 4.3.** [Dou66, Section 10] For $\pi$ as above, for every smooth map of complex analytic spaces, $\rho : \mathcal{Y} \to S$, the relative Hom space,

$$H_{\pi,\rho} = \text{Hom}_S(C, \mathcal{Y}) \to S,$$

is the maximal open subset of the $S$-relative Douady space of the fiber product $C \times_S \mathcal{Y}$ parameterizing holomorphic, flat families of closed analytic subspaces of $C \times_S \mathcal{Y}$ that are proper over the base and such that the first projection map to the pullback of $C$ is an isomorphism.

Inverting this isomorphism and composing with the second projection map gives a universal $S$-morphism,

$$u : H_{\pi,\rho} \times_S C \to \mathcal{Y}.$$ 

With respect to this relative Hom space, the ev-obstructed locus can be formulated in terms of the associated extension sheaf.

**Definition 4.4.** The associated extension sheaf is the coherent analytic sheaf on $H_{\pi,\rho} \times_S C$,

$$E_{\pi,\rho} := \text{Hom}_{\mathcal{O}}(u^*\Omega_\rho, \text{pr}_2^*E_{\pi}).$$

Relative to the first projection morphism,

$$\tilde{\pi} : H_{\pi,\rho} \times_S C \to H_{\pi,\rho},$$

the obstruction sheaf is the coherent analytic sheaf on $H_{\pi,\rho}$,

$$O_{\pi,\rho} := R^1\tilde{\pi}_*E_{\pi,\rho}.$$ 

The non-free locus $H_{\pi,\rho}^{nf}$ is the support of $O_{\pi,\rho}$, and the free locus $H_{\pi,\rho}^{f}$ is the open complement in $H_{\pi,\rho}$ of this closed analytic subspace. For every morphism of complex analytic spaces,

$$\zeta : T \to H_{\pi,\rho},$$

the pullback non-free locus $T^{nf}$, resp, the pullback free locus $T^{f}$, is the inverse image under $\zeta$ of $H_{\pi,\rho}^{nf}$, resp. of $H_{\pi,\rho}^{f}$.

**Lemma 4.5.** If $\pi$ is smooth, then the open subset $H_{\pi,\rho}^{f} \times_S C$ of $H_{\pi,\rho} \times_S C$ is the smooth locus of $u$. The restriction of $u$ to every analytic subspace of $H_{\pi,\rho}^{nf} \times_S C$ is non-submersive, compatibly with arbitrary base-change of $S$. 

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Proof. Denote the dual \( \text{Hom}_{\mathcal{O}_Y}(\Omega_{\rho}, \mathcal{O}_Y) \) by \( T_{\rho} \). By Lemma 4.2, the open \( H_{\pi,\rho} \) is contained in the open complement \( H_{\pi,\rho}^2 \) of the support of \( R^1\pi_*u^*T_{\rho} \).

The open \( H_{\pi,\rho}^2 \) is the maximal open that is smooth over \( S \) of relative dimension equal to the expected dimension, and the \( S \)-relative tangent sheaf on \( H_{\pi,\rho}^2 \) equals the restriction of

\[ \pi_*\text{Hom}_{\mathcal{O}}(u^*\Omega_{\rho}, \mathcal{O}) \]

Moreover, \( H_{\pi,\rho}^2 \) is the maximal open in \( H_{\pi,\rho} \) such that for every point \( s \in S \) and for every point \([v : C_s \rightarrow Y_s]\) in the fiber \( H_{\pi,\rho,s} \), the induced map of Zariski tangent spaces,

\[ H^0(C_s, v^*T_{Y_s}) \rightarrow H^0(C_s, v^*\Omega_{Y_s})^\vee \]

is surjective.

The fiber \( C_s \) is isomorphic to \( \mathbb{CP}^1 \). Thus the locally free sheaf \( v^*\Omega_{Y_s} \) is isomorphic to a direct sum of invertible sheaves \( \mathcal{O}(-a_j) \) for integers \( a_j \in \mathbb{Z} \). Of course the induced map,

\[ H^0(\mathbb{CP}^1, \mathcal{O}(a_j)) \rightarrow H^0(\mathbb{CP}^1, \mathcal{O}(-a_j))^\vee, \]

is surjective if and only if \( a_j \geq 0 \). Thus, \( \pi^{-1}H_{\pi,\rho}^2 \) is the unique open subscheme of \( H_{\pi,\rho}^2 \) whose points \((s, [v])\) are precisely those maps such that \( v^*T_{Y/S} \) is globally generated.

Let \((s, [v])\) be a point in \( H_{\pi,\rho}^2 \). Since constant morphisms have \( v^*T_{Y/S} \) isomorphic to a direct sum of copies of the structure sheaf \( \mathcal{O}_{\mathbb{CP}^1} \), which is globally generated, the morphism \( v \) is not a constant morphism. Thus, the derivative map,

\[ dv : T_{\mathbb{CP}^1} \rightarrow v^*T_{Y/S}, \]

is an injective homomorphism of coherent analytic sheaves on \( \mathbb{CP}^1 \). Since \( T_{\mathbb{CP}^1} \) is globally generated, the derivative map of \( v \) on the factor \( T_{\mathbb{CP}^1} \) factors through the adjointness homomorphism \( \alpha \),

\[ \alpha : H^0(\mathbb{CP}^1, v^*T_{\mathbb{CP}^1}) \otimes C \mathcal{O}_{\mathbb{CP}^1} \rightarrow v^*T_{\mathbb{CP}^1}. \]

Since \((s, [v])\) is not in \( H_{\pi,\rho}^2 \), the cokernel of \( \alpha \) is a locally free sheaf of positive rank. Thus, for every \( q \in \mathbb{CP}^1 \), the image of the derivative map of \( u \) at \((s, [v], q)\) is not surjective. Thus, the restriction of \( u \) to each complex analytic subvariety of \( H_{\pi,\rho} \times S \) \( C \) containing \((s, [v], q)\) is not submersive. \( \square \)

For every \( \text{ev}-\text{dominant} \) class \( \beta \), denote by \( X'_{\beta} \) the maximal open subset (possibly empty) over which the following evaluation map is smooth, i.e., submersive,

\[ \text{ev}_{0,1,\beta} : \overline{\mathcal{M}}_{0,1}(\{(X,J), \beta\}) \rightarrow X. \]

Denote by \( X_{\beta} \) the open subset of \( X'_{\beta} \) over which the evaluation map is smooth, and every fiber of the evaluation map parameterizes only maps with smooth domain.

**Proposition 4.6.** For every connected, compact Kähler manifold \((X, J, \omega)\), for every \( J \)-irreducible, \( \text{ev}-\text{dominant} \) class \( \beta \), the complements of both \( X'_{\beta} \) and \( X_{\beta} \) are proper, closed, analytic subvarieties of \( X \).
Proof. The critical locus of $\text{ev}_{0,1,\beta}$ in $\overline{M}_{0,1}((X,J),\beta)$ is a closed analytic subspace. Since the evaluation morphism is proper and holomorphic, the image of this closed analytic subspace is a closed analytic subspace of $X$. The goal is to prove that this closed analytic subspace is proper, i.e., it does not equal all of $X$. By definition, $X'_{\beta}$ is the open complement of this closed analytic subspace.

By Proposition 3.2 there exists a nowhere dense, closed analytic subvariety $Z$ of $X$ whose open complement $X^o$ is submersive and relatively projective over $Q^o$, and such that the open complement $\overline{M}_{0,0}(Z,\beta)$ of $\overline{M}_{0,0}(X,J),\beta)$ equals the relative moduli space $\overline{M}_{0,0}(X^o/Q^o,\beta)$. Since $Z$ is nowhere dense, it suffices to prove that the critical set of $\text{ev}_{0,1,\beta}$ in $X^o$ is nowhere dense.

Denote by $\Delta^o \subset \overline{M}_{0,1}((X,J),\beta)^o$ the boundary divisor. As a closed analytic subspace of $\overline{M}_{0,1}((X,J),\beta)^o$, this is proper over $Q^o$. Hence it is a union of finitely many compact complex analytic spaces each bimeromorphic to a connected Kähler manifold that is proper over $Q^o$. Each of these finitely many “irreducible components” is in the image $\Delta_{\beta',\beta''}$ of the natural 1-morphism,

$$\overline{M}_{0,2}((X,J),\beta') \times_{\text{ev}_2,X,\text{ev}} \overline{M}_{0,1}((X,J),\beta'' \to \overline{M}_{0,1}((X,J),\beta).$$

for an ordered pair $(\beta',\beta'')$ of nonzero homology classes with $\beta' + \beta'' = \beta$. Thus, there are finitely many such pairs $(\beta',\beta'')$ such that the images $\Delta_{\beta',\beta''}$ cover all of $\Delta^o$. The relative complement of $X_{\beta}$ in $X'_{\beta}$ is the intersection of $X_{\beta}$ with the union over these finitely many pairs $(\beta',\beta'')$ of the image $Z_{\beta'}$ of the proper holomorphic map,

$$\overline{M}_{0,2}((X,J),\beta') \times_{\text{ev}_2,X,\text{ev}} \overline{M}_{0,1}((X,J),\beta'') \to \overline{M}_{0,2}((X,J),\beta') \to X.$$

Thus, it suffices to prove that $X'_{\beta}$ is nonempty.

Since $\beta$ is a $J$-irreducible ev-dominant class, neither of $\beta'$ nor $\beta''$ is ev-dominant. Thus $Z_{\beta'}$ is a proper closed analytic subspace of $X$. Similarly, the image $Z_{\beta',\beta''}$ of the following evaluation map is a proper, closed analytic subspace of $X$,

$$\text{pr}_1 \circ \text{ev}_{0,2,\beta'} \circ \text{pr}_1 : \overline{M}_{0,2}((X,J),\beta') \times_{\text{ev}_2,X,\text{ev}} \overline{M}_{0,1}((X,J),\beta'') \to X.$$

The union in $X$ of $Z$ and the proper, closed analytic subspaces $Z_{\beta'}$ and $Z_{\beta',\beta''}$ for the finitely many pairs $(\beta',\beta'')$ is a proper, closed analytic subspace $Z'$ of $X$, and $X'_{\beta}$ equals $X' \setminus Z'$. Over the dense, Zariski open complement $U$ of $Z'$, the evaluation morphism

$$ev = ev_{0,1,\beta} : \overline{M}_{0,1}((X,J),\beta) \to X,$$

parameterizes only stable maps with irreducible domain.

For the universal family of genus-0 stable maps of class $\beta$,

$$\pi : \overline{M}_{0,1}((X,J),\beta) \to \overline{M}_{0,0}(X,J),\beta), \quad ev : \overline{M}_{0,1}((X,J),\beta) \to X,$$

for the holomorphic submersion that is the projection morphism,

$$\rho : X \times \overline{M}_{0,0}(X,J),\beta) \to \overline{M}_{0,0}(X,J),\beta),$$

the universal map defines a section of the relative Douady space,

$$\zeta : \overline{M}_{0,0}(X,J),\beta) \to H_{\pi,\rho}.$$
Denote by $\overline{M}_{0,0}((X, J), \beta)^{nf}$ the pullback under $\zeta$ of the non-free locus $H^{nf}_{\pi, \rho}$. This is a closed, analytic subvariety of $\overline{M}_{0,0}((X, J), \beta)$. Thus, the inverse image under $\pi$,

$$\overline{M}_{0,1}((X, J), \beta)^{nf} \subset \overline{M}_{0,1}((X, J), \beta),$$

is a closed analytic subvariety. Since $\text{ev}$ is a proper map of complex analytic spaces, the image of $\overline{M}_{0,1}((X, J), \beta)^{nf}$ in $X$ is a closed analytic subvariety $W$ of $X$. By Lemma 4.5, the intersection of $W$ with the open $U$ is precisely the image in $U$ of the non-smooth locus of the restricted morphism,

$$\text{ev} : \text{ev}^{-1}(U) \to U.$$ 

This morphism is everywhere non-submersive on the non-smooth locus. Thus the closed analytic subspace $U \cap W$ of $U$ is proper. The dense open complement in $U$ of this closed analytic subspace is contained in $X_\beta$. Therefore $X_\beta$ is nonempty. □

Ampleness of the psi class in Theorem 1.9 follows from the identity of the Gysin classes. This identity, in turn, follows from simple facts about the low degree cohomology groups and Picard group of the total space of a $\mathbb{CP}^1$-fibration.

**Definition 4.7.** For every proper, holomorphic submersion, $\pi : C \to S$, of connected complex analytic spaces whose fibers are connected, genus-0 curves, of $\pi$, the normalized section bundle is $\pi^* \psi(\sigma)$, where $\psi$ is $\sigma^* \omega_\pi$.

The pullback by $\sigma$ of the normalized section bundle is isomorphic to $\mathcal{O}_S$. Thus, there is a complex

$$\mathbb{Z}[\pi^* \psi(\sigma)] \to \text{Pic}(C) \xrightarrow{\sigma^*} \text{Pic}(S).$$

Similarly, via the first Chern class, there is a complex of singular cohomology groups,

$$\mathbb{Z} \cdot c_1(\pi^* \psi(\sigma)) \to H^2(C; \mathbb{Z}) \xrightarrow{\sigma^*} H^2(S; \mathbb{Z}).$$

**Lemma 4.8.** Both of the complexes above are short exact sequences that are split by the pullback map $\pi^*$.

**Proof.** Pullback by the inclusion of the fiber establishes left exactness. Since $\pi \circ \sigma$ is the identity, the composition $\sigma^* \circ \pi^*$ is the identity. This establishes right exactness as well as the splitting property. It remains to prove exactness in the middle.

Up to twisting by an appropriate integer power of $\pi^* \psi(\sigma)$, every holomorphic invertible sheaf $\mathcal{L}$ on $C$ has relative degree 0 over $S$. By cohomology and base change, and by vanishing of $h^1(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1})$, the higher direct images $R^q \pi_* \mathcal{L}$ are zero for $q > 0$, and $\pi_* \mathcal{L}$ is an invertible sheaf $\mathcal{A}$. Moreover, by adjointness of $\pi_*$ and $\pi^*$, there is a natural morphism of coherent sheaves,

$$\pi^* \mathcal{A} = \pi^* \pi_* \mathcal{L} \to \mathcal{L},$$

compatible with arbitrary base change of $S$. In particular, restricting to fibers of $\pi$, this morphism is an isomorphism. Therefore, the invertible sheaf $\mathcal{L}$ is trivial if and only if $\mathcal{A} = s^* \mathcal{L}$ is also trivial. That establishes exactness in the middle for the sequence of Picard groups. Exactness in the middle for cohomology follows from the Leray-Serre spectral sequence: existence of the adjusted section class proves vanishing of the transgression map to $H^3(S; \mathbb{Z})$. □
Corollary 4.9. For every integer \( r \geq 1 \), with the same hypotheses as above,
\[
c_1(\pi^*\psi(\alpha))^r = \sigma^*(c_1(\psi)^r - 1) + \pi^*c_1(\psi)^r.
\]
For cohomology classes \( D_i \in H^2(C; \mathbb{Z}), i = 0, \ldots, r \), in the kernel of \( \sigma^* \) having \( \pi \)-relative degrees \( \langle D_i, \beta \rangle \in H^2(\mathbb{C}P^1; \mathbb{Z}) = \mathbb{Z} \), the Gysin pushforward equals
\[
\pi_*(D_0 \sim \cdots \sim D_r) = \langle D_0, \beta \rangle \cdots \langle D_r, \beta \rangle c_1(\psi)^r.
\]
By multilinearity, the same holds for \( D_i \in H^2(C; \mathbb{Q}) \), resp. \( D_i \in H^2(C; \mathbb{R}) \).

**Proof.** The first formula is proved by induction on \( r \). For \( r = 1 \), this is the Whitney sum formula for \( c_1 \). By way of induction, let \( r \geq 1 \) be an integer such that the formula holds. By the Projection Formula, \( \sigma^*\sigma_*(\alpha) \) equals \(-c_1(\psi) - \alpha \). Thus,
\[
\sigma^*\sigma_*(c_1(\psi)^{r-1}) + \sigma^*\pi^*c_1(\psi)^r = -c_1(\psi) - c_1(\psi)^{r-1} + c_1(\psi)^r = 0.
\]
Therefore,
\[
c_1(\pi^*\psi(\alpha))^{r+1} = (\sigma_*(1) + \pi^*c_1(\psi)) - c_1(\pi^*\psi(\alpha))^{r+1} = \\
\sigma_*(\sigma^*c_1(\pi^*\psi(\alpha))^{r+1}) + \pi^*c_1(\psi) - (\sigma_*(c_1(\psi)^{r-1}) + \pi^*c_1(\psi)^r) = \\
\sigma_*(c_1(\psi)^{r-1}) + \pi^*c_1(\psi)^r + \pi^*c_1(\psi)^{r+1}.
\]
So the formula is proved by induction on \( r \).

Since every \( D_i \) is in the kernel of \( \sigma^* \), by Lemma 4.8, \( D_i \) equals \( \langle D_i, \beta \rangle c_1(\pi^*\psi(\alpha)) \). Thus \( D_0 \sim \cdots \sim D_r \) equals \( \langle D_0, \beta \rangle \cdots \langle D_r, \beta \rangle c_1(\pi^*\psi(\alpha))^{r+1} \). Combined with the formula from the last paragraph and the Projection Formula, this gives the formula for the Gysin pushforward class. \( \square \)

**Proposition 4.10.** For every integer \( r \geq -1 \), with the same hypotheses as above, the \( q \)-th higher direct image under \( \pi \) of the \( r \)-times twisted invertible sheaf \( \mathcal{F}_r := [\pi^*\psi(\alpha)]^{\otimes q} \) vanishes for \( q > 0 \). Also, the pushforward \( \mathcal{E}_r := \pi_*\mathcal{F}_r \) is locally free of rank \( r+1 \) and compatible with arbitrary base change of \( S \). For every integer \( r \geq 0 \), there is a short exact sequence,
\[
0 \longrightarrow \psi \otimes_{\mathcal{O}_S} \mathcal{E}_{r-1} \rightarrow \mathcal{E}_r \rightarrow \mathcal{O}_S \rightarrow 0,
\]
where the second homomorphism is evaluation along \( \sigma \).

**Proof.** All parts are proved by induction on \( r \). When \( r \) equals \(-1 \), since all cohomology on fibers is zero, the result follows by cohomology and base change. For every \( r \geq 0 \), restriction to \( \sigma \) defines a short exact sequence,
\[
0 \longrightarrow \pi^*(\psi) \otimes_{\mathcal{O}_C} \mathcal{F}_{r-1} \longrightarrow \mathcal{F}_r \longrightarrow \sigma_*\mathcal{O}_S \longrightarrow 0.
\]
By the long exact sequence of higher direct images, \( R^q\pi_*\mathcal{F}_r \) vanishes if both \( R^q\pi_*\sigma_*\mathcal{O}_S \) vanishes and \( R^q\pi_*\mathcal{F}_{r-1} \) vanishes. Since \( \sigma \circ \pi \) is the identity, of course \( R^q\pi_*\mathcal{O}_S \) vanishes for all \( q > 0 \). By the induction hypothesis, also \( R^q\pi_*\mathcal{F}_{r-1} \) vanishes for all \( q > 0 \). Thus, also \( R^q\pi_*\mathcal{F}_r \) vanishes for all \( q > 0 \). Thus, the long exact sequence reduces to a short exact sequence,
\[
0 \longrightarrow \psi \otimes_{\mathcal{O}_S} \mathcal{E}_{r-1} \rightarrow \mathcal{E}_r \rightarrow \mathcal{O}_S \rightarrow 0.
\]
By the induction hypothesis, the first term is locally free of rank \( r \), so that also the third term is locally free of rank \( r+1 \). \( \square \)
Notation 4.11. For every connected, compact Kähler manifold \((X, J, \omega)\), for every \(J\)-irreducible, ev-dominant class \(\beta\), for the evaluation morphism
\[ ev : \mathcal{M}_{0,1}(X, J, \beta) \to X, \]
denote the inverse image open subset \(ev^{-1}(X_\beta)\) by \(\mathcal{M}_{0,1}^\beta((X, J), \beta)\). Denote by \(\pi^\beta : C^\beta_0 \to \mathcal{M}_{0,1}^\beta((X, J), \beta)\) the following fiber product,
\[
\begin{array}{ccc}
C^0_\beta & \xrightarrow{\text{pr}_1} & \mathcal{M}_{0,1}((X, J), \beta) \\
\downarrow \pi^\beta & & \downarrow \pi \\
\mathcal{M}_{0,1}((X, J), \beta) & \xrightarrow{\pi|_\beta} & \mathcal{M}_{0,0}((X, J), \beta)
\end{array}
\]
The graph of the inclusion morphism \(\mathcal{M}_{0,1}((X, J), \beta) \hookrightarrow \mathcal{M}_{0,1}((X, J), \beta)\) is a section of \(\pi^\beta_0\), denoted
\[ \sigma^\beta_0 : \mathcal{M}_{0,1}((X, J), \beta) \to C^\beta_0. \]
The composition \(ev \circ \text{pr}_1\) is a morphism of complex analytic spaces, denoted
\[ u^\beta_0 : C^\beta_0 \to X. \]
For every complex analytic space \(S\) and for every morphism of complex analytic spaces \(\zeta : S \to \mathcal{M}_{0,1}((X, J), \beta)\), denote the pullback diagram by
\[ \zeta : (C_\zeta \xleftarrow{\pi} S, \sigma_\zeta : S \to C_\zeta, u_\zeta : C_\zeta \to X). \]

Proposition 4.12. For every connected, compact Kähler manifold \((X, J, \omega)\), for every \(J\)-irreducible, ev-dominant class \(\beta\), for every morphism \(\zeta : S \to \mathcal{M}_{0,1}((X, J), \beta)\) of complex analytic spaces such that \(ev \circ \zeta\) is constant with image \(x \in X\), the pullback class \(u^*_\zeta[\omega]\) equals \(\langle \omega, \beta \rangle c_1(\pi^*\psi(\omega))\). If there exists a section \(\tilde{\sigma} : S \to C_\zeta\) that is disjoint from \(\sigma\) and with \(u_\zeta \circ \tilde{\sigma}\) constant, then \(\zeta\) is locally constant. Conversely, if \(S\) is compact of (pure) complex dimension \(r\) and if \(\zeta\) is generically finite to its (closed analytic) image, then \(\int_S \langle c_1(\psi)^r \rangle\) is positive.

Proof. Since \(u_\zeta \circ \sigma\) equals \(ev \circ \zeta\), also \(\sigma^* u^*_\zeta\) is the zero homomorphism. Thus, by Lemma 4.8, \(u^*_\zeta[\omega]\) equals a multiple of the first Chern class of the adjusted section bundle. Restricting to a fiber of \(\pi_\zeta\), that multiple equals \(\langle \omega, \beta \rangle\).

If there exists a section \(\tilde{\sigma}\) that is disjoint from \(\sigma\), then \(\tilde{\sigma}^* u^*_\zeta[\omega]\) equals \(\langle \omega, \beta \rangle c_1(\psi)\) in \(H^2(S; \mathbb{R})\). If \(u_\zeta \circ \tilde{\sigma}\) is constant, then \(c_1(\psi)\) is contained in the torsion kernel of \(H^2(S; \mathbb{Z}) \to H^2(S; \mathbb{R})\). This torsion class determines a finite, unbranched cover of \(S\). Up to replacing \(S\) by this cover, \(c_1(\psi)\) is zero. By the Lefschetz \((1, 1)\) Theorem, the holomorphic invertible sheaf \(\psi\) is trivial. By Lemma 4.8, the invertible sheaf \(\mathcal{O}_{C_\zeta}(\tilde{\Omega})\) equals \(\mathcal{O}_{C_\zeta}(\sigma)\). Thus, the two members \(\sigma\) and \(\tilde{\sigma}\) span a pencil of divisors associated to this common invertible sheaf, and that pencil defines an isomorphism of \(C_\zeta\) with \(\mathbb{CP}^1 \times S\) over \(S\) such that the two sections \(\sigma\) and \(\tilde{\sigma}\) are the zero and infinity sections. Since \(u^*_\zeta[\omega]\) is the pullback of a class via the projection
\[ pr_{\mathbb{CP}^1} : C_\zeta \to \mathbb{CP}^1, \]
there exists a unique morphism \(u : \mathbb{CP}^1 \to X\) such that \(u_\zeta\) equals \(u \circ pr_{\mathbb{CP}^1}\). Thus, the morphism \(\zeta\) is constant.

By Corollary 4.9,
\[ u^*_\zeta[\omega]^r = (\langle \omega, \beta \rangle^r (\sigma^* c_1(\psi)^{r-1}) + \pi^* \zeta^* c_1(\psi)^r), \]
where $\sigma_*$ is the Gysin pushforward homomorphism. Assume now that $S$ is compact and connected of dimension $r$ and that $\zeta$ is generically finite to its image. To prove that the integral is positive, it suffices to replace $S$ by the image of $\zeta$, since the integral on $S$ equals the generic degree of $\zeta$ times the integral on the image of $S$. Thus, assume that $S$ is a closed analytic subspace of $F_{\beta,x}$. The claim is that the associated morphism

$$u_\zeta : C_\zeta \rightarrow X$$

has finite fibers except over $x$. For any $\tilde{x} \in X \setminus \{x\}$, for the inverse image closed analytic subspace $\tilde{S} = u_\zeta^{-1}(\{\tilde{x}\})$ of $C_\zeta$, for the composition

$$\zeta \circ \pi_\zeta : \tilde{S} \rightarrow S \rightarrow F_{\beta,x},$$

the base change family has a section $\tilde{\sigma}$ with $u_\zeta \circ \tilde{\sigma}$ constant to $\tilde{x}$. By the previous paragraph, $\tilde{\zeta}$ is locally constant, i.e., $\tilde{S}$ is a finite set. This proves the claim.

Since $C_\zeta$ is a compact complex analytic space of pure dimension $r + 1$, and since $u_\zeta$ is generically finite, the integral

$$\int_{C_\zeta} u_\zeta^* [\omega]^{r+1}$$

is positive. Since $\zeta^* c_1(\psi)^{r+1}$ is zero on the $r$-dimensional complex analytic space $S$, the inductive formula above gives

$$\int_{C_\zeta} u_\zeta^* [\omega]^{r+1} = \langle \omega, \beta \rangle^{r+1} \int_{C_\zeta} s_\star \zeta^* c_1(\psi)^r = \langle \omega, \beta \rangle^{r+1} \int_S \zeta^* c_1(\psi)^r.$$

Thus, the integral of $\zeta^* c_1(\psi)^r$ is positive. □

**Corollary 4.13.** For every connected, compact Kähler manifold $(X, J, \omega)$, for every $J$-irreducible, ev-dominant class $\beta$, for the Barlet space $\text{Barlet}(X, J)$, and for the family $\xi$ of 1-cycles in $(X, J)$ that is the image of

$$\left(\pi_\beta^0, u_\beta^0\right) : C_\beta^0 \rightarrow \overline{\mathcal{M}}_{0,1}(\{X, J\}, \beta) \times X,$$

the associated morphism of complex analytic spaces,

$$(\text{ev}, \xi) : \overline{\mathcal{M}}_{0,1}(\{X, J\}, \beta) \rightarrow X_\beta \times \text{Barlet}(X, J),$$

is finite. In particular, for every $x \in X_\beta$, the fiber $F_{\beta,x}$ of $\text{ev}$ is in Fujiki class $C$.

**Proof.** By Proposition 4.12, for every $x \in X_\beta$, for every genus-0, 1-pointed stable map,

$$(C, p, v : C \rightarrow X_\beta)$$

parameterized by a point of $F_{\beta,x}$, for every $\tilde{x} \in v(C) \setminus \{x\}$ there are at most finitely many other points of $F_{\beta,x}$ parameterizing maps with image containing $\tilde{x}$. In particular, there are at most finitely many points whose image cycle coincides with $v_\star [C]$. Thus the morphism $(\text{ev}, \xi)$ has finite fibers. Since $\text{ev}$ is a proper morphism of complex analytic spaces, the morphism $(\text{ev}, \xi)$ is a finite morphism of complex analytic spaces. By $[\text{Fuj82}]$, every irreducible component of the Barlet space of the Kähler manifold $(X, J)$ is in Fujiki class $C$. Since $F_{\beta,x}$ admits a finite morphism of complex analytic spaces to a compact complex analytic space in Fujiki class $C$, also $F_{\beta,x}$ is in Fujiki class $C$. □
Definition 4.14. For each normalization $Q \to \text{Barlet}(X, J)$ of an irreducible, reduced closed analytic subspace of the Barlet space, the associated cycle $Z \subset Q \times X$ is the pullback to $Q$ of the universal cycle. The isomorphic open $X_Q \subset X$ is the maximal open subscheme of $X$ (possibly empty) over which the projection $\text{pr}_X : Z \to X$ is an isomorphism. The fundamental locus $\text{Fund}(Q)$ is the image in $Q$ of the closed analytic subspace $Z \setminus \text{pr}_X^{-1}(X_Q)$ of $Z$, and $Q^o$ is the open complement.

Remark 4.15. The family of cycles is an almost holomorphic fibration in the sense of Definition 2.1 if and only if $X_Q$ and $Q^o$ are nonempty.

For an almost holomorphic fibration, the complement $Q^o$ of the fundamental locus is a dense open of $Q$, there is a unique open $X_Q^o \subset X_Q$ such that $\text{pr}_Q^{-1}(Q^o)$ equals $\text{pr}_X^{-1}(X_Q^o)$, and the composition $\phi = \text{pr}_Q \circ \text{pr}_X^{-1}$,

$$\phi : X_Q^o \to \text{pr}_X^{-1}(X_Q^o) \to Q^o$$

is a surjective, proper morphism of irreducible, normal complex analytic spaces of Fujiki class C whose fibers are all pure-dimensional. By Sard’s theorem, the image in $Q^o$ of the non-smooth morphism of $\phi$ is a proper closed analytic subspace whose open complement $Q^o_{ss}$ is smooth, and such that the restriction of $\phi$ to the inverse image $X_Q^{ss} = \phi^{-1}(Q^o_{ss})$,

$$\phi^{ss} : X_Q^{ss} \to Q^o_{ss},$$

is a surjective, proper, smooth morphism, i.e., a “holomorphic fibration” (in the sense of Ehresmann’s theorem). Thus, $\phi$ is an “almost holomorphic fibration”.

For every connected, compact Kähler manifold $(X, J, \omega)$, for every $J$-irreducible, ev-dominant class $\beta$, recall that $X_\beta \subset X$ is the maximal open over which $\text{ev}_{0,1,\beta}$ is smooth and disjoint from the boundary $\Delta$. By Proposition 4.6, this is a dense open whose closed complement is an analytic subvariety of $X$. Also recall that $\overline{\mathcal{M}}_{0,1}((X, J), \beta) = \text{ev}^{-1}(X_\beta)$ is defined to be the inverse image of $X_\beta$ under $\text{ev}_{0,1,\beta}$, so that the restriction of the evaluation morphism,

$$\text{ev} : \overline{\mathcal{M}}_{0,1}((X, J), \beta) \to X_\beta,$$

is smooth and parameterizes only free 1-pointed maps with irreducible domain.

Proposition 4.16. For every connected, compact Kähler manifold $(X, J, \omega)$, for every $J$-irreducible, ev-dominant class $\beta$, the restriction of $\psi$ to $\overline{\mathcal{M}}_{0,1}((X, J), \beta)$ is ev-ample. Equivalently, for every $x \in X_\beta$, the fiber $F_{\beta,x}$ of $\text{ev}$ over $x$ is a complex projective manifold on which the restriction of $\psi$ is ample.

Proof. By openness of ampleness, ev-relative ampleness of the restriction of $\psi$ is equivalent to ampleness of $\psi$ on every fiber $F_{\beta,x}$ of $\text{ev}$.

Let $Q \to \text{Barlet}(X, J)$ be the rational quotient of $(X, J)$. Then the open $X_Q^{ss}$ is dense in $X$. By Proposition 4.6, also $X_\beta$ is a dense open in $X$. Thus, the intersection $X_\beta \cap X_Q^{ss}$ is a dense open in $X$.

For every $x \in X_Q^{ss} \cap X_\beta$, every map parameterized by $F_{\beta,x}$ has irreducible domain and $\text{ev}$ is smooth at this genus-0, 1-pointed map, i.e., the map is a free map from $\mathbb{CP}^1$. Moreover, the image contains $x$, a point of $X_Q^{ss}$. Thus, these free maps all have image in the fiber $X_\beta$ of $\phi$ over $q = \phi(x)$. So $F_{\beta,x}$ is an open and closed subspace of the space of stable maps in $X_\beta$. Also $X_q$ is projective by Proposition 3.2. Thus, $F_{\beta,x}$ is a projective manifold as well: the morphism from $F_{\beta,x}$ to the Chow scheme.
of $X_\beta$ is finite, and the components of the Chow scheme of a projective scheme are themselves projective. For all $x \in X_\beta$, by Corollary 4.13, the fiber $F_{\beta,x}$ is a compact, complex manifold in Fujiki class C. Thus, the standard results of Hodge theory still apply for the proper, holomorphic submersion,

$$\text{ev} : \mathcal{M}_{0,1}((X, J), \beta) \to X_\beta.$$

In particular, the Hodge numbers $h^{p,q}$ of the fiber $F_{\beta,x}$ are themselves projective. For all $x \in X_\beta$, the fiber $F_{\beta,x}$ is projective for the general $x \in X_\beta$, the fiber is Moishezon for every $x \in X_\beta$. Finally, by Proposition 4.12, the restriction of $\psi$ satisfies the hypotheses of the Nakai-Moishezon criterion for ampleness of the holomorphic invertible sheaf $\psi$ on the compact, Moishezon manifold $F_{\beta,x}$. Therefore $\psi$ is ample on $F_{\beta,x}$. □

**Corollary 4.17.** With the same hypotheses as in Proposition 4.16, for every integer $r \geq 0$, for every $x \in X_\beta$, the restriction to $F_{\beta,x}$ of the pushforward $E_r := \pi_*([\pi^*\psi(\sigma)]^\otimes r)$ is a locally free sheaf of rank $r + 1$ with a natural surjection to $O_{F_{\beta,x}}$ (evaluation at the marked point). The kernel of this surjection is a locally free sheaf of rank $r$ that is ample when $r \geq 1$. The restriction to $F_{\beta,x}$ of the twist $\psi \otimes E_r$ is an ample locally free sheaf of rank $r + 1$.

**Proof.** By Proposition 4.10 the pushforward is a locally free sheaf of rank $r + 1$ with a natural surjection to the structure sheaf. Ampleness of the kernel and ampleness of the twist by $\psi$ are proved by induction on $r$. When $r = 0$, then $E_0$ equals the structure sheaf, so that the twist by $\psi$ equals $\psi$. This is ample by the previous result. Thus, assume that $r \geq 1$, and assume that the result is true for $r - 1$.

The kernel of the natural surjection equals $\psi \otimes E_{r-1}$, and this is ample by the induction hypothesis. Since $E_r$ is an extension of the structure sheaf by an ample locally free sheaf, also $\psi \otimes E_r$ is an extension of $\psi$ by the twist of an ample locally free sheaf by $\psi$. By the previous result, $\psi$ is ample. The tensor product of an ample locally free sheaf by an ample invertible sheaf is ample. Finally, an extension of an ample sheaf by an ample sheaf is ample. Altogether, $\psi \otimes E_r$ is ample. Thus the result holds by induction on $r$. □

**Proof of Theorem 1.9.** By definition,

$$f_\beta = \langle \tau_{m,\omega}(\eta_X) \rangle_{0,1,\beta}^{X_\omega} = GW_{0,1,\beta}^{X_\omega}(q^*c_1(\psi_1)^{m,\omega} \sim \text{pr}_1^*(\eta_X)) = GW_{0,1,0,\beta}^{X_\omega}(q^*c_1(\psi_1)^{m,\omega}).$$

Thus, if $\beta$ is symplectically free, then also $\beta$ is symplectically pseudo-free. Similarly, if $\beta$ is symplectically free, then the pairing of $\text{ev}_{0,1,\beta}^*(\eta_X)$ with the virtual fundamental class is nonzero, so that $\text{ev}_{0,1,\beta}$ is surjective (else choose a representative for the Poincaré dual homology class of $\eta_X$ that is contained in the complement of the image). By Sard’s Theorem, for a proper, holomorphic map between compact complex analytic varieties, the image in the target of the submersive locus in the domain is an open (possibly empty) that is the complement of a closed analytic subvariety of the target. If $\beta$ is free, then the submersive locus of $\text{ev}_{0,1,\beta}$ is nonempty, so that the image of the submersive locus is a dense open subset of $X$. Since $\text{ev}_{0,1,\beta}$ is proper, the image of $\text{ev}_{0,1,\beta}$ is closed, and it contains this dense open. Therefore $\text{ev}_{0,1,\beta}$ is surjective, i.e., $\beta$ is ev-dominant.

Next, assume that $\beta$ is ev-dominant and $J$-irreducible. By Proposition 4.16 $\beta$ is free. The goal is to prove that $\beta$ is symplectically free (closing the circle for all notions of “free” for $J$-irreducible classes). Also, the goal is to prove the formula for the Gysin class $\pi_*u^*(D_0 \sim \cdots \sim D_r)$ in $H^{2r}(F_{\beta,x}; \mathbb{R})$. By Proposition 4.10, the open...
subspace $X_\beta$ is a dense open whose complement is a complex analytic subspace. In particular, this open is nonempty. By definition, for every $x \in X_\beta$, the fiber $F_{\beta,x}$ is a compact complex manifold of dimension $m_\beta$. By Proposition 4.10, the restriction of $\psi$ on this fiber is ample. Also, for $D_i \in H^2(X;\mathbb{R})$, $i = 0, \ldots, r$, the pullback of each $D_i$ by $u_\beta^i$ is a class whose further pullback by $\sigma_\beta^i$ vanishes when restricted to $F_{\beta,x}$, since this is also the pullback by the (constant) evaluation morphism. Thus, Corollary 4.9 gives the formula for the Gysin pushforward of the cup product of $\omega$ connected. Thus the restriction of $\psi$ is ample. By definition, for every $x \in X_\beta$, the fiber $F_{\beta,x}$ is a regular fiber of the evaluation morphism, $ev : \overline{M}_{0,1}((X,J),\beta) \to X$, the pushforward of the fundamental class equals the cap product of $ev^*\eta_X$ with the virtual fundamental class of $\overline{M}_{0,1}((X,J),\beta)$. Thus, the positive integral equals the gravitational descendant, 

\[
\langle r_{m_\beta}(\eta_X) \rangle_{0,\beta}^{X,\omega} = \int_{[\overline{M}_{0,1}((X,J),\omega)]^{vir}} c_1(\psi)^{m_\beta} \sim ev^*\eta_X.
\]

Therefore $\beta$ is a symplectically free class.

The next goal is to prove that the set of $\omega$-degrees of ev-dominant classes is discrete. Thus, there exists an $\omega$-minimal, hence J-irreducible, element of $NE^\omega_\gamma(X)$ if and only if $NE^\omega_\gamma(X)$ is nonzero. Thus, assume that there exists an ev-dominant class. Then the fibers $X_q$ of the rational quotient are nontrivial. For a very general fiber $X_q$, every genus-0 stable map whose image intersects $X_q$ factors through $X_q$. In particular, every ev-dominant class in $X$ is the pushforward from $X_q$ of an ev-dominant class in $X_q$. Thus, to prove discreteness of the $\omega$-degrees of ev-dominant class in $X$, it is equivalent to prove discreteness of the $\omega$-degrees of ev-dominant classes in $X_q$.

Every general fiber $X_q$ is a connected projective manifold that is rationally connected. Thus the restriction of $\omega$ is $\mathbb{R}$-ample, i.e., it equals a linear combination with positive real coefficients of ample integer divisors, say

\[
r_1[D_1] + \cdots + r_\ell[D_\ell], \quad 0 < r_1 \leq \cdots \leq r_\ell.
\]

For every real number $R > 0$, for the integer part $N = \lfloor R/r_1 \rfloor$, there are at most $N^\ell/\ell!$ ordered $\ell$-tuples of non-negative integers $(n_1, \ldots, n_\ell)$ with $r_1 n_1 + \cdots + r_\ell n_\ell$ bounded by $R$. Thus, the set of effective curve classes in $X_q$ with $\omega$-degree bounded by $R$ is finite. Thus, the set of $\omega$-degrees of ev-dominant curve classes is discrete.

By discreteness of the $\omega$-degrees of ev-dominant classes, for every ev-dominant class $\beta$, there are at most finitely many ev-dominant classes $\beta'$ with $\beta$ equal to $\beta' + \beta''$ for a class $\beta''$ that is a (nonzero) sum of classes of genus-0 maps (not necessarily ev-dominant). In particular, one of these finitely many classes $\beta'$ has minimal $\omega$-degree, and thus that class is a J-irreducible, ev-dominant class, which is then symplectically free. Thus, every ev-dominant class $\beta$ is the sum of a J-irreducible,
symplectically free class \( \beta' \) and a sum (possibly zero) of classes of genus-0 maps
(not necessarily ev-dominant). In particular, if \( \text{NE}_{s.f.}^\omega(X) \) is nonzero, then there
exists a \( J \)-irreducible, symplectically free class \( \beta \).

Finally, for every \( J \)-irreducible, symplectically free \( \beta \), for every \( q \in X_\beta \), by
Proposition 4.16, the restriction of \( \psi \) to the fiber \( F = \text{ev}^{-1}(\{q\}) \) is ample. Also
\( F \) has pure dimension \( m_\beta \). Assume that \( m_\beta > 0 \). Then by [dJS17], the
\( \psi \)-degree of the first Chern class of \( F \) equals the second gravitational descendant,
\[
\sigma_\beta = \sigma_\beta(X) := \left( \tau_{m_\beta - 1}(\eta_m), \text{ch}_2(T_{X,\omega}) + \frac{m_\beta}{2(m_\beta + 2)} c_1(T_{X,\omega})^2 \right)_{0,\beta}. 
\]

By the asymptotic Riemann-Roch formulas of [KM83], [Luo89], and [Mat 91], even if \( \psi \) is merely big and nef, there is an asymptotic formula for \( d \gg 0 \),
\[
\begin{aligned}
\eta_0(F; \psi^d) &= \frac{f_{\beta}}{m_\beta!} d^{m_\beta} + \frac{s_\beta}{2(m_\beta - 1)!} d^{m_\beta - 1} + \ldots = \frac{f_{\beta} d^{m_\beta}}{m_\beta!} \left( 1 + \frac{m_\beta q_\beta}{2d} + \ldots \right).
\end{aligned}
\]

5. **Proof of Theorem 1.10**

Let \( (X, J, \omega) \) be a connected, compact Kähler manifold. Denote the rational quotient
by
\[
Q \to \text{Barlet}(X), \quad X \supseteq X^o \xrightarrow{\phi} Q^o \subseteq Q.
\]

**Proof of (1) of Theorem 1.10**. If \( \text{NE}_{s.f.}^\omega(X) \) is nonzero, then by Theorem 1.9 the
rational quotient \( \phi \) has positive fiber dimension.

Since \( Q \) is finite over the Barlet space, it is in Fujiki class \( C \). Thus, if \( Q \) is not a point, then there exists a nonzero element in \( H^2(Q; \mathbb{Q}) \). The pullback in \( H^2(X; \mathbb{Q}) \)
is orthogonal to every homology class \( \beta \) of a curve contained in a fiber of the
rational quotient. In particular, it is orthogonal to every connected tree of rational
curves such that at least one component is a general free curve. Thus, the class
is orthogonal to every symplectically pseudo-free class. When these classes span
\( H_2(X; \mathbb{Q}) \), then the pullback is zero. Therefore \( Q \) is a point. \( \square \)

The remainder of Theorem 1.10 follows from the next three propositions.

**Proposition 5.1.** For every \( \omega \)-minimal symplectically free class \( \beta \) that is symplec-
tically \( 2 \)-free, for every point \( x \in X_\beta \), at least one connected component of \( F_{\beta,x} \) is
uniruled. Moreover, each surface in \( X \) swept out by a rational curve \( R \) in \( F_{\beta,x} \) is a
rational surface on which the \( \beta \)-curves parameterized by \( R \) move in the pencil \( R \).
Conversely, if \( F_{\beta,x} \) is weakly Fano, then \( \beta \) is symplectically \( 2 \)-free.

**Proof.** By Proposition 4.6, the \( \beta \)-generic locus \( X_\beta \) is a dense open whose com-
plement is a proper closed analytic subspace of \( X \). For the inverse image open
\( \overline{\mathcal{M}}_{0,1}((X, J), \beta) := \text{ev}^{-1}(X_\beta) \), the restricted evaluation map,
\[
\text{ev} : \overline{\mathcal{M}}_{0,1}((X, J), \beta) \to X_\beta,
\]
is a proper, smooth morphism of complex analytic spaces. Every fiber has pure
dimension \( m_\beta \), and the restriction of \( \psi \) is \( \text{ev} \)-relatively ample. Thus, the \( \text{ev} \)-relative
spaces of genus-0, 1-pointed stable maps to fibers of \( \psi \) with specified curve class
are themselves proper over \( \overline{\mathcal{M}}_{0,1}((X, J), \beta) \) (for the evaluation map). To prove
uniruledness of at least one connected component of the fiber $F_{\beta,x}$ of $\text{ev}$ over every point $x \in X_{\beta}$, it suffices to prove uniruledness for general $x \in X_{\beta}$.

Denote by $Q \to \text{Barlet}(X,J)$ the rational quotient of $(X,J)$. Denote by $\phi^{sm}$ the associated maximally-extended proper, smooth morphism of complex analytic spaces defined on dense open subspaces of $X$, resp, of $Q$,

$$(X \supseteq) X^{sm}_Q \xrightarrow{\phi} Q^{sm} (\subseteq Q),$$

such that all fibers of $\phi^{sm}$ are rationally connected and every free rational curve that intersects $X^{sm}_Q$ is contained in a fiber of $\phi^{sm}$. The restriction of $\omega$ to the dense open $X^{sm}_Q$ is Kähler. Since the fibers of $\phi^{sm}$ are also rationally connected, compact, complex manifolds, they are projective, i.e., $\phi^{sm}$ is (weakly) projective.

For each point $q \in Q^{sm}$, for each point $x$ in the fiber $X_q = \phi^{-1}(q)$, the fiber $F_{\beta,x}$ of $\text{ev}$ equals the fiber over $x$ of

$$\text{ev}_q : \overline{M}_{0,1}((X_q,J_q),\beta) \to X_q,$$

since every free curve in $X$ that contains $x$ is contained in $X_q$. Since $X_q$ is a fiber of the proper proper morphism $\phi^{sm}$, there is a normal bundle short exact sequence of holomorphic vector bundles on $X_q$,

$$0 \to T^1_{X_q,J_q} \to T^1_{X,J_q}|_{X_q} \to C^{\text{dim}(Q)}_{X_q} \to 0,$$

so that $\text{ch}_r$ of the tangent bundle of $X_q$ equals the restriction of $\text{ch}_r$ of the tangent bundle of $X$ for every $r \geq 1$.

By [JS17], the relative first Chern class of $\text{ev}_q$ equals

$$-\text{ev}_q^* c_1(T^1_{X_q,J_q}) - \psi + u^* \left( \text{ch}_2(T^1_{X_q,J_q}) + \frac{1}{2(m_\beta + 2)} c_1(T^1_{X_q,J_q})^2 \right).$$

The first summand vanishes when restricted to a fiber of $\text{ev}_q$. By Corollary 4.9, we have a divisor class relation,

$$-\psi + u^* \left( \frac{1}{2(m_\beta + 2)} c_1(T^1_{X_q,J_q})^2 \right) = u^* \left( \frac{m_\beta}{2(m_\beta + 2)^2} c_1(T^1_{X_q,J_q})^2 \right),$$

i.e.,

$$u^* \left( c_1(T^1_{X_q,J_q})^2 \right) = \langle c_1(T^1_{X_q,J_q}), \beta \rangle^2 \psi.$$

Thus, for every $x \in X_q$, the restriction to the fiber $F_{\beta,x}$ of $\text{ev}_q$ of the relative first Chern class of $\text{ev}_q$ gives the first Chern class of the fiber,

$$c_1(T^1_{F_{\beta,x}}) = \pi_* u^* \left( \text{ch}_2(T^1_{X_q,J_q}) + \frac{m_\beta}{2(m_\beta + 2)^2} c_1(T^1_{X_q,J_q})^2 \right).$$

Since $\psi$ is ample on $F_{\beta,x}$, there exists an integer $d > 0$ such that $\psi^\otimes d$ is very ample. Thus, $F_{\beta,x}$ is covered by smooth, projective curves $A$ that are complete intersections of $m_\beta - 1$ divisors in the linear system of the very ample invertible sheaf $\psi^\otimes d$. The total degree on $A$ of the restriction of the first Chern class equals

$$\langle c_1(T^1_{F_{\beta,x}}), [A] \rangle = d^{m_\beta - 1} \int F_{\beta,x} c_1(\psi)^{m_\beta - 1} \sim c_1(T^1_{F_{\beta,x}}) =$$

$$d^{m_\beta - 1} \left\langle \tau_{m_\beta - 1}(\eta_X), \text{ch}_2(T^1_{X_q,J_q}) + \frac{m_\beta}{2(m_\beta + 2)^2} c_1(T^1_{X_q,J_q})^2 \right\rangle_{0,\beta}.$$
In particular, the total degree on $A$ of the first Chern class is positive if and only if $s_{\beta}$ is positive.

If the fiber $F_{\beta,x}$ is “Fano in the weak sense” that the first Chern class equals a nonzero, pseudo-effective divisor class, then the intersection number of that divisor class with the moving curve class $[A]$ is strictly positive. Then $s_{\beta}$ is also positive.

Conversely, if $s_{\beta}$ is positive, then the first Chern class of $F_{\beta,x}$ has positive degree on at least one connected component of $A$. By Bertini’s Connectedness Theorem, each connected component of $A$ is the intersection of $A$ with a connected component of $F_{\beta,x}$. By Mori’s theorem, [Mor79], this connected component of $F_{\beta,x}$ is uniruled. By Corollary 4.13, for every rational curve $R$ in $F_{\beta,x}$, the union in $X$ of the curves parameterized by $R$ is a surface that is the image of a generically finite morphism from a $\mathbb{CP}^1$-bundle over $R$. Such a unirational surface is itself a rational surface. Moreover, the fiber $\mathbb{CP}^1$ over each point of $R$ moves in a pencil of such curves on the surface by varying the point of $R$.

**Proposition 5.2.** For every minimal symplectically free class $\beta$, if $\beta$ is symplectically ruling, then $F_{\beta,x}$ is a singleton for every point $x \in X_{\beta}$. The corresponding genus-0, $\beta$-curve indexed by $x \in X_{\beta}$ move in a pencil of rational curves on a rational surface. Thus $X$ is swept out by rational surfaces. Conversely, if $F_{\beta,x}$ is a singleton for a general $x \in X_{\beta}$, and if the corresponding genus-0, $\beta$-curve moves on a rational surface, then $\beta$ is symplectically ruling.

**Proof.** By Proposition 4.16, the $\beta$-generic locus $X_{\beta}$ is a dense open whose complement is a proper closed analytic subspace of $X$. Then $m_{\beta}$ equals 0 and $f_{\beta}$ equals 1 if and only if for every $x \in X_{\beta}$, the fiber $F_{\beta,x}$ is a singleton set. In this case, the restriction of the universal curve over $\overline{\mathcal{M}}_{0,1}(X,J,\beta) = X_{\beta}$ is a family of 1-cycles in $X$ parameterized by $X_{\beta}$. This family of 1-cycles defines a meromorphic map from $X$ to the Barlet space of $X$.

$$R \to \text{Barlet}(X), \quad X \supset X_{\beta} \overset{\epsilon}{\to} R^o \subseteq R.$$  

As the normalization of the image of a meromorphic map between compact, complex analytic spaces in Fujiki’s class, the image $R$ is an irreducible, normal, compact, complex analytic space in Fujiki’s class. Up to blowing up $R$, assume that the MRC fibration of $R$ is everywhere holomorphic. By construction, the general fibers of $\epsilon$ are the genus-0, $\beta$-curves in $X$ parameterized by $X_{\beta}$. In particular, the only ev1-dominant curve classes that are contracted by $\epsilon$ are in the $\mathbb{Q}$-span of $\beta$.

If there exists a symplectically pseudo-free class $\gamma$ that is not in the $\mathbb{Q}$-span of $\beta$, then the image class in $R$ is ev1-dominant. By Part 1 of Theorem 1.9, the Kähler manifold $R$ has a minimal free class (which is also symplectically free). For a general point of $R$, for a minimal free curve $C$ containing that point, the inverse image of $C$ under $\epsilon$ is a rational surface in $X$ on which the unique genus-0, $\beta$-curve moves in a pencil (indexed by $C$).

Conversely, assume that a general fiber of $\epsilon$ moves in a pencil on a rational surface. The goal is to construct a symplectically pseudo-free class $\gamma$ on $X$ that is $\mathbb{Q}$-linearly independent from $\beta$. In fact, we will construct a symplectically pseudo-free class $\gamma$ such that the pushforward class $\gamma_{R} = \epsilon_* \gamma$ is nonzero.

Since $X$ is covered by rational surfaces, and every pair of points of a rational surface is contained in a chain of rational curves in that rational surface, not all rational
curves of this surface can equal fibers of $\phi$. Thus, the image $R$ is uniruled by free bases. By the proof of Theorem 1.2, there exists a minimal free curve class $\gamma_R$ on $R$. Denote by $\omega_R$ a Kähler class on $R$. By the original proof of Theorem 1.2 for $m_{\gamma,R} = \langle c_1(T^1_R), \gamma_R \rangle - 2$, we have positivity of the Gromov-Witten invariant

$$\langle \eta_R, [\omega_R] \sim [\omega_R], \ldots, [\omega_R] \sim [\omega_R]/R, \omega_R, \rangle,$$

with $m_{\gamma,R}$ insertions of $[\omega_R] \sim [\omega_R]$. More precisely, when restricted to the fiber of the MRC fibration of $R$ containing a general point $q$, the class $[\omega_R]$ is an $\mathbb{R}_{>0}$-linear combination of ample $\mathbb{Z}$-divisor classes, so that we can represent the restriction of $[\omega_R] \sim [\omega_R]$ to the fiber as an $\mathbb{R}_{>0}$-linear combination of Poincaré duals of homology classes of moving, codimension-2 complex analytic subvarieties of the fiber. Thus, choosing these subvarieties generically, the Gromov-Witten invariant is an $\mathbb{R}_{>0}$-linear combination of the homology classes of finitely many finite subsets of $\mathcal{M}_{0,m,\beta,R+1}((Q, J_R), \gamma_R)$, each of which equals the finite set of all genus-0, $\gamma_R$-curves containing $q$ and intersecting each of $m_{\gamma,R}$ specified codimension-2 closed, analytic subvarieties of the MRC fiber of $q$.

For each genus-0, $\gamma_R$-curve $C$ in one of these finite sets, the inverse image $\Sigma$ of $C$ under $\epsilon$ is a conic bundle over $C$. By Tsen’s Theorem, this conic bundle admits holomorphic sections. For every class $\gamma$ of a section, the pushforward under $\gamma$ equals $\gamma_R$. Since the image of the MRC fibration of $R$ is not uniruled, the composition of $\phi$ and the MRC fibration of $R$ is an almost holomorphic fibration of $X$. In particular, every chain of genus-0 curves in $X$ that intersects $\epsilon^{-1}(q)$ maps under $\phi$ to a chain of genus-0 curves in the MRC fiber of $q$. Thus, every chain $C'$ of genus-0 curves whose total class equals $\gamma$ maps under $\epsilon$ to a chain of genus-0 curves whose total class equals $\gamma_R$. If the chain $C'$ intersects $\epsilon^{-1}(q)$ as well as the $\epsilon$-inverse images of the specified collection of codimension-2 complex analytic subvarieties of the MRC fiber of $q$, then the image of $C'$ equals $C$. Thus, $C'$ is divisor in $\Sigma$.

For a divisor class $\gamma$ in $\Sigma$ of a complete linear system $\mathbb{P}^{n_\gamma}$ of moving section curves, for a general point $x \in \epsilon^{-1}(q)$ and for $n_\gamma - 1$ additional auxiliary points $y_2, \ldots, y_{n_\gamma}$, there is a unique curve $D$ in the complete linear system that contains $x$ and the points $y_i$, and $\epsilon$ maps $D$ isomorphically to $C$. Of course the restriction of $\omega$ to $\Sigma$ is an $\mathbb{R}_{>0}$-linear combination of ample divisor classes. Thus, the restriction of $[\omega] \sim [\omega]$ to $\Sigma$ is represented by an $\mathbb{R}_{>0}$-linear combination of Poincaré duals of homology classes of general points of $\Sigma$. Altogether, we have positivity of the degree of the Gromov-Witten invariant,

$$\langle \eta_X, \epsilon^*(\omega_R) \sim [\omega_R], \ldots, \epsilon^*(\omega_R) \sim [\omega_R], \omega \sim [\omega], \ldots, \omega \sim [\omega]/X, X \omega \rangle,$$

with $m_{\gamma,R}$ of the insertions $\phi^*(\omega_R) \sim [\omega_R]$ and with $n_\gamma - 1$ of the insertions $[\omega] \sim [\omega]$. Indeed, the homology class of this Gromov-Witten invariant equals an $\mathbb{R}_{>0}$-linear combination of finitely many homology classes of finite subsets of $\mathcal{M}_{0,m,\gamma,n_\gamma}(X, J, \gamma)$ parameterizing genus-0, $\gamma$-curves $D$ in $X$ that (1) map to one of the finitely many genus-0, $\gamma_R$-curves in $R$ that contains $x$, itself mapping to $q$, that (2) intersects the inverse images under $\epsilon$ of each of the $m_{\gamma,R}$ specified codimension-2 complex analytic subvarieties, and that (3) contains each of the $n_\gamma - 1$ general points $y_i$ in $\Sigma$. Thus, $\gamma$ is a symplectically pseudo-free class, and the pushforward $\gamma_R = \epsilon_* \gamma$ is nonzero, so that $\gamma$ is $\mathbb{Q}$-linearly independent from $\beta$. Therefore $\beta$ is symplectically ruling. □
Part 3 of Theorem 1.10 follows by the converse direction of Proposition 5.1. Similarly, if \( Z \) is symplectically equivalent to \( X \), then Part 2 of Theorem 1.10 follows from Proposition 5.1 and 5.2. Finally, if \( Z \) is symplectically deformation equivalent to \( X \), then the cone of symplectically free classes of \( Z \) equals the cone of symplectically free classes of \( X \). Assume that this cone is nonzero, and assume that every indecomposable element of this common cone is symplectically 2-free or symplectically ruling (both of these properties depend only on the symplectic deformation class). Then for the symplectic form \( \omega' \) on \( Z \), there exists an \( \omega' \)-minimal class in the symplectically free cone, by Theorem 1.9. Since this is a \( \omega' \)-minimal class, in particular it is an indecomposable class of the cone. Thus, this \( \omega' \)-minimal class is either symplectically 2-free or symplectically ruling. By Propositions 5.1 and 5.2, the Kähler manifold \( Z \) is swept out by rational surfaces.

6. Proof of Theorem 1.10

The following propositions adapt to projective homogeneous varieties of arbitrary Picard rank some results valid in Picard rank 1 from [dIHS11, Sections 14 and 15]. For every connected, complex Lie group \( G \), the solvable radical \( R_{\text{solv}}(G) \), resp. the unipotent radical \( R_{\text{uni}}(G) \), is the maximal connected, normal, solvable, complex Lie subgroup of \( G \), resp. it is the kernel \( R_{\text{uni}}(G) \) in \( R_{\text{solv}}(G) \) of the initial homomorphism of complex Lie groups from \( R_{\text{solv}}(G) \) to a group of multiplicative type,

\[
\chi : R_{\text{solv}}(G) \to M(G) \cong \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times.
\]

There exists a connected, complex Lie subgroup \( L(G) \) such that the induced homomorphism \( L(G) \to G/R_{\text{uni}}(G) \) is an isomorphism. This is a Levi factor of \( G \), and \( L(G) \) is unique up to conjugation by \( R_{\text{uni}}(G) \). The group \( G \) is semisimple if \( R_{\text{solv}}(G) \) is trivial. This is true for the commutator subgroup \([L(G), L(G)]\), which is called a semisimple factor of \( G \).

Let \( G \) be a connected, simply connected, semisimple, complex Lie group. Let \( B \subset G \) be a Borel subgroup \( B \), i.e., a connected, solvable, complex Lie subgroup of \( G \) that is maximal with these properties. Let \( T \subset B \) be a maximal torus, i.e., a connected, Abelian, complex Lie subgroup of \( B \) of multiplicative type \( \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times \) that is maximal with these properties. Denote by \( N_G(T) \) the normalizer of \( T \) in \( G \), and denote by \( W = W_{G,T} \) the associated Weyl group, i.e., the finite quotient subgroup \( N_G(T)/T \). Via conjugation, \( W \) acts on the set of all subgroups \( H \) of \( G \) that contain \( T \). Denote by \( W_H \) the stabilizer subgroup of \( H \) in \( W \). In particular, \( W_B \) is the trivial subgroup.

The set of positive simple roots is (naturally bijective to) the subset \( \Delta = \Delta_{G,T,B} \) of \( W_{G,T} \) of all elements \( s \in W \) such that the complex Lie subgroup \( H_s \) generated by \( B \) and \( sBs^{-1} \) has either \( \text{SL}_2 \) or \( \text{PGL}_2 \) for its semisimple factor. The positive simple roots give generators of \( W \) each having order 2, and the minimal word length of an element with respect to the generating set \( \Delta \) is the Coxeter length of the element.

Let \( P \) be a connected, proper, complex Lie subgroup of \( G \) containing \( B \). The quotient complex manifold \( Y = G/P \) is projective. For every \( W_P \)-double coset \( [w] \in W_P \), the corresponding Schubert cell is \( (PwP)/P \) in \( G/P \). The closure of this Schubert cell is a Schubert variety, and the homology class is
a Schubert class in $H_*(G/P; \mathbb{Z})$. The natural (left) $G$-action on $Y$ induces a diagonal $G$-action on $Y \times Y$.

**Theorem 6.1** (Bruhat Decomposition). There are finitely many $G$-orbits on $Y \times Y$, each of the form $G \cdot (E_w \times \{P/P\})$ for a unique $P$-orbit $E_{P,w} = PuP/P$ in $Y = G/P$, indexed by $W_P$-double cosets $[w] \in W_P \backslash W/W_P$. Each Schubert cell $E_{P,w}$ is a locally closed subvariety that is algebraically isomorphic to affine space $\mathbb{C}^\ell$, where $\ell$ is the least Coxeter length of a representative of the double $W_P$-coset. The homology classes of the Schubert varieties $\overline{E}_{P,w}$ form an additive basis for $H_*(G/P; \mathbb{Z})$ that is self-dual under Poincaré duality.

In particular, the 1-dimensional Schubert varieties $\beta_i$ are in bijection with the positive simple roots in $\Delta_P := \Delta \setminus (\Delta \cap W_P)$, and the corresponding Schubert classes form an additive basis for $H_2(G/P; \mathbb{Z})$. The Schubert classes of codimension-1 Schubert varieties form a dual basis $D_i$ for $H^2(G/P; \mathbb{Z})$.

**Proposition 6.2.** For $G/P$, the pseudo-effective cone equals the effective cone equals the base-point free cone equals the nef cone, and all of these equal the free $\mathbb{Z}_{\geq 0}$-semigroup with simplicial generators $D_i$. Dually, the Mori cone equals the movable cone equals the free $\mathbb{Z}_{\geq 0}$-semigroup with simplicial generators $\beta_i$. The symplectically free cone equals the Mori cone, so that $G/P$ is integrally simply Fano. Also the $\beta_i$-generic locus equals all of $G/P$.

**Proof.** First of all, for every effective divisor (possibly empty), the $G$-translates of the divisor form a base-point free subset of the complete linear system. Thus, the complete linear system of every effective divisor is base-point free, i.e., the associated invertible sheaf is globally generated. Thus, the divisor class is nef. Dually, the Mori cone equals the movable cone. In particular, every pseudo-effective divisor class has nonnegative intersections with every 1-dimensional Schubert variety $\beta_i$. Thus, the divisor class equals a nonnegative linear combination of the dual basis $D_i$. Therefore the pseudo-effective cone equals the free $\mathbb{Z}_{\geq 0}$-semigroup generated by the classes $[D_i]$ which equals the base-point free cone. Dually, the Mori cone is the free $\mathbb{Z}_{\geq 0}$-semigroup generated by the classes $\beta_i$.

Every symplectically free class is symplectically pseudo-free, which is $\text{ev}_1$-dominant, and hence is an effective curve class. Thus the symplectically free cone is contained in the Mori cone. To prove these cones are equal for $G/P$, it suffices to prove that every $\beta_i$ is symplectically free. By the $G$-translation action, every effective curve is $\text{ev}_1$-dominant. Since the classes $\beta_i$ are the simplicial generators of the Mori cone, these classes are $J$-irreducible. Thus, by Theorem 1.20 every $\beta_i$ is symplectically free. Since the $\beta_i$-free locus is a dense Zariski open that is $G$-invariant, it equals the entire homogeneous variety $G/P$. □

**Proposition 6.3.** Every contraction of $G/P$ is $G$-equivariant, of the form $G/P \rightarrow G/Q$ for a parabolic subgroup $Q$ of $G$ that is uniquely determined by the classes $\beta_i$ contracted to points, or equivalently, by the positive simple roots in $\Delta_P \cap W_Q$. A class $\beta_i$ with $m_{\beta_i}$ equal to 0 is symplectically ruling unless $G/P$ equals $\mathbb{CP}^1$. The contraction that contracts precisely the symplectically ruling classes $\beta_i$ is a Zariski locally trivial fiber bundle. In particular, it is submersive.

**Proof.** The translation action of $G$ on $G/P$ induces a $G$-action on the projective linear system of any divisor. (In fact, since $G$ is assumed to be simply connected,
For every class $\beta_i$ with $m_{\beta_i} = 0$, since the $\beta_i$-free locus equals all of $G/P$, the evaluation morphism,

$$\text{ev}_1 : \overline{M}_{0,1}(G/P, \beta_i) \to G/P,$$

is surjective and everywhere smooth of relative dimension $m_{\beta_i} = 0$, i.e., it is a finite unbranched covering. Yet $G/P$ is simply connected. Thus the $\mathbb{CP}^1$-bundle structure,

$$\overline{M}_{0,1}(G/P, \beta_i) \to \overline{M}_{0,0}(G/P, \beta_i),$$

induces a $\mathbb{CP}^1$-bundle contraction of $G/P$, necessarily of the form $G/P \to G/P_i$ where $P_i$ is the minimal parabolic containing $P$ such that $\beta_i$ is not contained in $\Delta_{P_i}$. In particular, every curve whose class is in the extremal ray spanned by $\beta_i$ is contracted in $G/P_i$, so that the $\beta_i$-curves are precisely the fibers of this contraction. Therefore, the degree $f_{\beta_i}$ equals 1; there is a unique $\beta_i$-curve containing each point $x$ of $G/P$, namely the fiber of the contraction containing $x$. If $\dim(G/P) \geq 2$, then also $\dim(G/P_i) \geq 1$. Since $G/P_i$ is uniruled, by the proof of Theorem 1.10 the class $\beta_i$ is symplectically ruling.

Let $D$ be the semi-ample divisor class that equals the sum of $D_j$ over all $j$ with $\beta_j$ not equal to a symplectically ruling class. The $G$-equivariant contraction of the corresponding complete linear system, say $\pi : Y \to \overline{Y}$, is of the form $G/P \to G/Q_i$. Thus it is a Zariski locally trivial fiber bundle. In particular, $H_2$ of the fiber equals the kernel of the pushforward map,

$$H_2(G/P, \mathbb{Z}) \to H_2(G/Q_i, \mathbb{Z}).$$

By the construction of $G/Q_i$, this kernel is precisely the span of those $\beta_i$ that are symplectically ruling. Thus, $G/P$ is strongly simplicially Fano. \hfill $\square$

**Proposition 6.4.** Assume that $\dim(G/P) \geq 2$. Every $\beta_i$ that is not symplectically ruling is symplectically 2-free. In fact, both $\overline{M}_{0,1}(G/P, \beta_i)$ and $\overline{M}_{0,2}(G/P, \beta_i)$ have only finitely many $G$-orbits, both of these are connected, Kähler manifolds, and the first Chern class of each is represented by a nonzero, effective divisor.

*Proof.** Every $\beta_i$-curve is contained in a fiber $Z = gP_i/P$ of the associated extremal contraction $G/P \to G/P_i$ of $\beta_i$. This fiber, in turn, is a projective homogeneous space of the semisimple Levi factor $G_i$ of $P_i$, which has Picard rank 1; say $G_i/R_i$ for a maximal parabolic subgroup $R_i$ of $G_i$. The ample generator of the Picard group $D_1$ of $G_i/R_i$ has intersection pairing 1 with $\beta_i$. Since $D_1$ is effective, it is base-point free. Since $D_1$ is ample, the associated contraction of $G_i/R_i$ is finite. Since it is $G_i$-equivariant, the contraction is everywhere smooth, i.e., it is a finite covering map. However, every projective homogeneous space, including the image
of the contraction, is simply connected. Thus, the contraction is an isomorphism, i.e., $D_i$ is very ample. Since $\beta_i$-curves have intersection number 1 with $D_i$, these curves are mapped to lines under the projective embedding of the complete linear system of $D_i$.

Every line in projective space is uniquely determined by two distinct points on the line. Denote by $\sigma$ the diagonal section of the projection $\Phi_k$ forgetting the $k^{th}$ marked point, for $k = 1, 2$.

$$\Phi_k : \overline{M}_{0,2}(G/P, \beta_i) \to \overline{M}_{0,1}(G/P, \beta_i), \quad \sigma : \overline{M}_{0,1}(G/P, \beta_i) \to \overline{M}_{0,2}(G/P, \beta_i).$$

Denote by $\mathcal{M}_{0,2}(G/P, \beta_i)$ the open complement in $\overline{M}_{0,2}(G/P, \beta_i)$ of the image of $\sigma$. Since every line is uniquely determined by two distinct points on that line, the following evaluation morphism is injective,

$$ev_2 : \mathcal{M}_{0,2}(G/P, \beta_i) \to G/P \times G/P.$$ This morphism is also $G$-equivariant.

By the Bruhat decomposition, $G/P \times G/P$ has only finitely many $G$-orbits. Thus, also the $G$-invariant subvariety $\mathcal{M}_{0,2}(G/P, \beta_i)$ has only finitely many $G$-orbits. Since $\Phi_k$ is $G$-equivariant and surjective, also $\overline{M}_{0,1}(G/P, \beta_i)$ has only finitely many $G$-orbits. Thus, the image of $\sigma$ has only finitely many $G$-orbits. Altogether, both $\overline{M}_{0,2}(G/P, \beta_i)$ and $\overline{M}_{0,1}(G/P, \beta_i)$ have only finitely many $G$-orbits.

Since the $\beta_i$-free locus equals all of $G/P$, every $\beta_i$-curve is free, so that both of $\overline{M}_{0,2}(G/P, \beta_i)$ and $\overline{M}_{0,1}(G/P, \beta_i)$ are smooth, i.e., they are compact, Kähler manifolds (possibly disconnected). Also, the first evaluation morphism

$$ev_1 \circ \Phi_1 : \overline{M}_{0,2}(G/P, \beta_i) \to \overline{M}_{0,1}(G/P, \beta_i) \to G/P, \quad (C, q_1, q_2, u : C \to G/P) \mapsto u(q_1),$$

is generically smooth (by Sard’s Theorem), and $G$-equivariant, hence everywhere smooth. Since $G/P$ is simply connected, the restriction of the evaluation morphism to every connected component of $\overline{M}_{0,2}(G/P, \beta_i)$ has smooth, connected fibers. For that connected component, denote by $F_{\beta_i,x}$ the fiber in that connected component of the evaluation morphism over the special point $x = P/P$. This is a smooth, connected Kähler manifold with an induced action of $P$. The second evaluation morphism,

$$ev_1 \circ \Phi_2 : \overline{M}_{0,2}(G/P, \beta_i) \to \overline{M}_{0,1}(G/P, \beta_i) \to G/P, \quad (C, q_1, q_2, u : C \to G/P) \mapsto u(q_1),$$

maps $F_{\beta_i,x}$ $P$-equivariantly to an irreducible, closed, $P$-invariant subset $C_{\beta_i,x}$ of $G/P$. As with the Bruhat decomposition on all of $G/P$, the homology of $C_{\beta_i,x}$ is a free $\mathbb{Z}$-module on the Schubert classes of those Schubert varieties $F_{\beta_i,w}$ that happen to intersect $C_{\beta_i,x}$ (and thus are contained in $C_{\beta_i,x}$). In particular, $H_2(C_{\beta_i,x}; \mathbb{Z})$ equals the free $\mathbb{Z}$-module on those 1-dimensional Schubert varieties $\beta_j$ that are contained in $C_{\beta_i,x}$. Since $C_{\beta_i,x}$ is a union of curves in the homology class $\beta_i$, the image of the morphism,

$$H_2(C_{\beta_i,x}; \mathbb{Z}) \to H_2(G/P; \mathbb{Z}),$$

contains the class of $\beta_i$. Thus, among the free generators $\beta_j$ of $H_2(C_{\beta_i,x}; \mathbb{Z})$, which also are linearly independent in $H_2(G/P; \mathbb{Z})$, there must be the class $\beta_i$. Therefore, some $\beta_i$-curve parameterized by $F_{\beta_i,x}$, i.e., containing $x = P/P$, also contains a point of $\beta_i$ different from $x$. Since every $\beta_i$-curve is uniquely determined by two distinct points on that curve, one of the $\beta_i$-curves parameterized by $F_{\beta_i,x}$ is the Schubert variety $\beta_i$. Since every connected component of the fiber over $x$ of $ev_1 \circ \Phi_1$
contains the special point $[\beta_i]$, there is only one connected component. Therefore the compact, Kähler manifolds $\overline{M}_{0,0}(G/P, \beta_i)$, $\overline{M}_{0,1}(G/P, \beta_i)$, and $\overline{M}_{0,2}(G/P, \beta_i)$ are each connected.

Associated to the natural $G$-action on the connected, Kähler manifold $\overline{M}_{0,r}(G/P, \beta_i)$, there is an induced map from the Lie algebra $\mathfrak{g}$ of $G$ to the tangent bundle,

$$\mu : \mathfrak{g} \otimes \mathcal{C}_\overline{M}_{0,r}(Y, \beta_i) \to T_{\overline{M}_{0,r}(Y, \beta_i)}.$$  

For each of $r = 1, 2$, since there are only finitely many $G$-orbits each of which is a locally closed subvariety, there is a unique open $G$-orbit on $\overline{M}_{0,r}(Y, \beta_i)$. Thus, $\mu$ is generically surjective.

For a general choice of a $\mathbb{C}$-linear subspace $V$ of $\mathfrak{g}$ of dimension equal to the dimension of $\overline{M}_{0,r}(Y, \beta_i)$, the restriction $t_V$ of $t$ to $V \otimes \mathcal{C}_\overline{M}_{0,r}(Y, \beta_i)$ is generically an isomorphism. Thus, the first Chern class of $\overline{M}_{0,r}(Y, \beta_i)$ is represented by the effective divisor that is the degeneracy locus $D$ of $t_V$. Since $\overline{M}_{0,r}(Y, \beta_i)$ has an almost homogeneous action of $G$, this is a projective variety that is unirational. Hence the tangent bundle is not trivial, so that $D$ is nonempty.

Consider the evaluation morphism,

$$\text{ev}_1 : \overline{M}_{0,1}(Y, \beta_i) \to Y.$$  

The restriction of $\text{ev}_1$ to $D$ is dominant. By Grothendieck’s Generic Freeness Theorem, i.e., by “generic flatness”, this morphism is flat when restricted over a particular dense Zariski open subset of $Y$. Thus, for a general $x \in Y$, the intersection of the fiber $F_{\beta_i,x}$ of $\text{ev}$ with $D$ is a nonempty divisor in $F_{\beta_i,x}$ that represents the first Chern class of $F_{\beta_i,x}$, i.e., $F_{\beta_i,x}$ is weakly Fano. By the converse direction of Theorem 1.10 the class $\beta_i$ is symplectically 2-free. Since this holds for every simplicial generator $\beta_i$ that is not symplectically ruling, the projective homogeneous variety $G/P$ is symplectically 2-free. \hfill \Box

Returning to the proof of Theorem 6.4 Part 1 follows from Proposition 6.2. Theorem 1.10 and Proposition 6.2 give Part 2. Finally, Theorem 1.10 and Proposition 6.2 give Part 3.

7. PROOF OF PROPOSITION 1.19 AND THEOREM 1.18

Proof of Proposition 1.19 1. By Part 1 of Theorem 1.9 for every fiber type Fano manifold, the Mori cone $\text{NE}^\omega_{\text{eff}}(X)$ equals $\text{NE}^\omega_f(X)$, and this Mori cone is dual to the Kähler cone. Conversely, since $\text{NE}^\omega_{\text{eff}}(X)$ is contained in the Mori cone, if this cone is dual to the Kähler cone, then it equals the Mori cone. In particular, since $\text{NE}^\omega_{\text{eff}}(X)$ is generated by classes on which $c_1(T_{X,\omega})$ has positive degree, this implies that $X$ is Fano. The Mori cone of a Fano manifold is rational polyhedral. By Part 1 of Theorem 1.9 the extremal rays are symplectically free.

2. The only way that a fiber type Fano manifold can fail to be integrally fiber type Fano is if the minimal free class $\beta_i$ generating an extremal, fiber type contraction is a multiple $r > 1$ of some integral homology class $\gamma$. But then this forces $m_{\beta_i} + 2 = \langle c_1(T_{X,\omega}), \beta_i \rangle$ to be $r$ times the integer $\langle c_1(T_{X,\omega}), \gamma \rangle$, which in turn is divisible by the Fano index (by Poincaré duality). Thus, the Fano pseudo-index is strictly greater than the Fano index. Conversely, if the fiber type Fano manifold is integrally fiber type, then for the Poincaré dual homology class $\gamma$ to the ample generator of the
Picard $[H]$ group of the fiber, the class $\gamma$ is represented by a free curve class (which is even symplectically free). The Fano index $i$ of the fiber is the unique integer such that the restriction of $c_1(T^{1,0}_{X,\omega})$ equals $i[H]$. Since $\langle [H], \gamma \rangle$ equals 1, the Fano pseudo-index $\langle c_1(T^{1,0}_{X,\omega}), \gamma \rangle$ equals the Fano index $i$. \hfill \Box

I am very grateful to the anonymous reader who recommended replacing moving lemmas in an earlier draft by a more natural argument in the proof of Theorem 1.18. The argument used here to avoid moving lemmas is inspired by \[Min16, Chapter 3\].

**Definition 7.1.** Let $(Y, \mathcal{O}_Y)$ be a compact, complex analytic space of pure dimension $n$. Let $b$ be an integer with $1 \leq b \leq n$. A **flag of pseudodivisors** of length $b$ is an ordered $b$-tuple of nested, closed analytic subspaces,

$$((Z_1, \mathcal{O}_{Z_1}), \ldots, (Z_b, \mathcal{O}_{Z_b})), $$

i.e., closed immersions of complex analytic spaces,

$$(e_j, e_j^\#) : (Z_j, \mathcal{O}_{Z_j}) \hookrightarrow (Z_{j-1}, \mathcal{O}_{Z_{j-1}}),$$

for $j = 1, \ldots, b$, with $(Z_0, \mathcal{O}_{Z_0})$ defined to equal $(Y, \mathcal{O}_Y)$, together with an assignment to every $j = 0, \ldots, b - 1$ of an ordered pair

$$(\mathcal{L}_j, s_j : \mathcal{O}_{Z_j} \to \mathcal{L}_j),$$

of an invertible $\mathcal{O}_{Z_j}$-module $\mathcal{L}_j$ together with a global section whose associated zero scheme in $(Z_j, \mathcal{O}_{Z_j})$ equals $(Z_{j+1}, \mathcal{O}_{Z_{j+1}})$. The flag is **reduced** if every $Z_j$ is a reduced complex analytic space. The flag is **co-Stein** if for every $j = 0, \ldots, b - 1$, the open subspace $Z_j^\circ := Z_j \setminus Z_{j+1}$ is a Stein analytic space. The flag is **ample** if $\mathcal{L}_j$ is ample on $Z_j$ for every $j = 0, \ldots, b - 1$. The flag is **locally complete intersection** if every $(Z_j, \mathcal{O}_{Z_j})$ is local complete intersection of pure dimension $n - j$.

**Lemma 7.2.** Every ample flag of pseudodivisors is co-Stein. For every ample flag of pseudodivisors, if $(Y, \mathcal{O}_Y)$ is a local complete intersection, then the flag is locally complete intersection if and only if $(Z_b, \mathcal{O}_{Z_b})$ has pure dimension $n - b$. In this case, for every $c = 0, \ldots, n - b + 1$, if the singular locus of $(Z_b, \mathcal{O}_{Z_b})$ has dimension $\leq n - b - c$, then for every $i = 0, \ldots, b$, also the singular locus of $(Z_j, \mathcal{O}_{Z_j})$ has dimension $\leq n - j - c$. In particular, if $(Z_b, \mathcal{O}_{Z_b})$ is reduced, then the flag is reduced.

**Proof.** For each $j = 0, \ldots, b - 1$, denote by $Z_j'$ the union of all irreducible components of $Z_j$ that are not completely contained in $Z_{j+1}$. By hypothesis, $Z_j'$ is a complex projective algebraic variety and $Z_j' \cap Z_{j+1}$ is an ample hypersurface in $Z_j'$. Thus, the open complement $Z_j'' = Z_j' \setminus (Z_j' \cap Z_{j+1})$ is a Stein analytic space. Thus, the flag is co-Stein.

The local complete intersection result is proved by induction on $b$. When $b = 0$, there is nothing to prove. Thus, by way of induction, assume that $b \geq 1$ and assume the result is true for $b - 1$.

For $j = 1, \ldots, b$, by the Principal Ideal Theorem, every irreducible component of $Z_j$ has dimension $\geq n - j$. For $j = 1, \ldots, b - 1$, if an irreducible component of $Z_j$ has dimension $d_j \geq 1$, then the zero scheme $Z_{j+1}$ is nonempty and has dimension $\geq d_j - 1$. Thus, since $n - (b - 1)$ is $\geq 1$, if there is any irreducible component of $Z_{b-1}$ that has dimension $d_{b-1} \geq n - (b - 1) + 1$, then also $Z_b$ has an irreducible
Similarly, if the singular locus of $Z_{b-1}$ has dimension $\geq n - (b - 1) - c + 1$, which is $\geq 2$ since $c \leq n - b + 1$, then the intersection of $Z_b$ with that singular locus is nonempty and has dimension $\geq n - b - c + 1 > n - b - c$. That contradicts the hypothesis that the singular locus of $Z_b$ has pure dimension $n - j$.

Therefore, again by induction, for every $j = 1, \ldots, b$ the zero scheme of $Z_j$ has dimension $\leq n - j - c$. In particular, setting $c$ equal to $n - b + 1$, if $(Z_b, \mathcal{O}_{Z_b})$ is reduced, then also every $(Z_i, \mathcal{O}_{Z_i})$ is reduced.

**Theorem 7.3.** [HT90, Theorems 3.2.1, 3.4.1] For every compact, complex analytic space $(Y, \mathcal{O}_Y)$ of pure dimension $n$, for every reduced, co-Stein flag of $b \leq n$ pseudo-divisors in $(Y, \mathcal{O}_Y)$, for every $j = 1, \ldots, b$, the relative homotopy group $\pi_i(Z_{j-1}, Z_j)$ is zero for $i \leq n - j$. By the Hurewicz Theorem, also $H_i(Z_{j-1}, Z_j; \mathbb{Z})$ is zero for $i \leq n - j$. Thus, each induced homomorphism,

$$\pi_i(Z_j) \to \pi_i(Z_{j-1}), \quad H_i(Z_j; \mathbb{Z}) \to H_i(Z_{j-1}; \mathbb{Z})$$

is an isomorphism for every $i < n - j$, and is an epimorphism for $i = n - j$.

**Proof.** By hypothesis, every inclusion $Z_j \hookrightarrow Z_{j-1}$ is a hypersurface whose open complement is Stein and local complete intersection of pure dimension $n - j + 1$. Thus, the result follows from [HT90, Theorems 3.2.1 and 3.4.1].

Let $(Y, J_Y, \omega_Y)$ be a compact Kähler manifold that has a submersive contraction of ruling classes,

$$\pi : Y \to Y'.$$

**Proposition 7.4.** The fibers $Y_q$ of $\pi$ are connected and simply connected. Thus, for every submanifold $X'$ of $Y'$ and its inverse image $X = \pi^{-1}(X')$, the Serre spectral sequence for $H_2$ degenerates to a short exact sequence,

$$0 \to H_2(Y_q; \mathbb{Z}) \to H_2(X; \mathbb{Z}) \to H_2(X'; \mathbb{Z}) \to 0.$$

**Proof.** Since $\pi$ is a contraction, the fibers are connected. By hypothesis, the morphism $\pi$ is everywhere submersive. Thus, $Y_q$ is a connected, compact, Kähler manifold.

Now we repeat the proof of Part 1 of Theorem [1.10]. The Kähler manifold $Y_q$ has a rational quotient. Either $Y_q$ is rationally connected or the the pullback of a Kähler form from the target of the rational quotient is a nonzero element of $H^2$. Since $H_2(Y_q; \mathbb{Z})$ is generated by classes of free rational curves, each of which is contracted by the rational quotient, the pairing of this element of $H^2$ is zero with all of $H_2(Y_q; \mathbb{Z})$, contradicting Poincaré duality. Thus, the rational quotient is a point, i.e., $Y_q$ is rationally connected. Every connected, compact, Kähler manifold that is rationally connected is also simply connected. [Cam91]. Therefore, every fiber $Y_q$ is simply connected.

Since the fibers of $\pi$ are connected and simply connected, the portion of the Serre spectral sequence for the homology $X$ computing $H_2(X; \mathbb{Z})$ degenerates to the stated short exact sequence.
As in Definition 7.1, let \(((Z_1, \mathcal{O}_{Z_1}), \ldots, (Z_c, \mathcal{O}_{Z_c}))\) and \(((L_0, s_0), \ldots, (L_{c-1}, s_{c-1}))\) be a flag of pseudodivisors in a Kähler manifold \(Y'\). Denote \(Z_c\) by \(X'\).

**Proposition 7.5.** If the flag of pseudodivisors is ample and locally complete in intersection, and if \(X'\) a reduced complex analytic space with \(\dim(X') \geq 3\), then the pushforward map on homology is an isomorphism,

\[
H_2(X'; \mathbb{Z}) \xrightarrow{\cong} H_2(Y'; \mathbb{Z}).
\]

For a submersive contraction \(\pi\) as in Proposition 7.4, for \(X := \pi^{-1}(X')\), also the following pushforward map is an isomorphism,

\[
H_2(X; \mathbb{Z}) \xrightarrow{\cong} H_2(Y; \mathbb{Z}).
\]

**Proof.** The first isomorphism follows from the Lefschetz hyperplane theorem, Theorem 7.3. Thus, in the commutative diagram of short exact sequences,

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H_2(F; \mathbb{Z}) & \longrightarrow & H_2(X; \mathbb{Z}) & \longrightarrow & H_2(X'; \mathbb{Z}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_2(F; \mathbb{Z}) & \longrightarrow & H_2(Y; \mathbb{Z}) & \longrightarrow & H_2(Y'; \mathbb{Z}) & \longrightarrow & 0
\end{array}
\]

the first and the third vertical homomorphisms are isomorphisms. By the Snake Lemma, also the middle vertical homomorphism is an isomorphism. □

**Proposition 7.6.** For every compact Kähler manifold \(Y\), for every codimension \(c\) complex submanifold \(X\) that is a complete intersection \(Y_1 \cap \cdots \cap Y_c\) of nef divisors \(Y_j\) in \(Y\), every free class in \(X\) pushes forward to a free class in \(Y\). In particular, every decomposition of a \(J\)-irreducible ev-dominant class on \(Y\) as a (possibly zero) sum of genus-0 classes and the pushforward of an ev-dominant class on \(X\) is a trivial decomposition, i.e., the ev-dominant class on \(Y\) equals the pushforward of the ev-dominant class on \(X\).

**Proof.** This is well-known. The normal bundle of \(X\) in \(Y\) equals the restriction to \(X\) of the direct sum of invertible sheaves \(\bigoplus_j \mathcal{O}_Y(Y_j)\). Since each of the divisors \(Y_j\) is nef, the restriction of each of these invertible sheaves to every irreducible curve has nonnegative degree. Thus, the restriction to a free curve in \(X\) of the normal bundle is a direct sum of invertible sheaves of nonnegative degrees, i.e., it is semi-positive. Since the restrictions to the free curve of both the tangent bundle of \(X\) and the normal bundle are semi-positive, also the restriction of the tangent bundle of \(Y\) is semi-positive. Thus, the curve is free in \(Y\).

Every ev-dominant class on \(X\) is a sum of a free class on \(X\) and a (possibly zero) sum of genus 0 curves. By the previous paragraph, the pushforward to \(Y\) of the free class is free on \(Y\), hence ev-dominant. Thus, if the class on \(Y\) is a \(J\)-irreducible ev-dominant class, then it equals the pushforward of the free class on \(X\). □

Let \((Y, J_Y, \omega_Y)\) be a Kähler manifold. As in Definition 7.1, let \(((Z_1, \mathcal{O}_{Z_1}), \ldots, (Z_c, \mathcal{O}_{Z_c}))\) and \(((L_0, s_0), \ldots, (L_{c-1}, s_{c-1}))\) be a flag of pseudodivisors in \(Y\). Denote \(Z_c\) by \(X\). Let \(\beta\) be a curve class on \(Y\) that is in the pushforward of \(H_2(X; \mathbb{Z})\) so that every integer,

\[
m_j := \langle c_1(L_j), \beta \rangle,
\]

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is defined. Define $m_{\beta}(Y, X) := \sum_{j=1}^{c} m_{j}$, and define $m_{j}^{+}(Y, X) := \sum_{j=1}^{c} \max(m_{j}, 0)$. Recall the notation for the evaluation morphism,

$$\text{ev}_{0,1,\beta} : \overline{\mathcal{M}}_{0,1}(Y, \beta) \rightarrow Y.$$  

**Lemma 7.7.** Assume that the fiber of $\text{ev}_{0,1,\beta}$ over a point $x$ of $X$ is disjoint from the boundary. For every $j = 1, \ldots, c$, if $m_{j} \leq 0$, then the fiber $F_{\beta,x}^{Z_{j}}$ for $Z_{j}$ equals the fiber $F_{\beta,x}^{Z_{j}-1}$ for $Z_{j}-1$. If $m_{j} \geq 0$, then $F_{\beta,x}^{Z_{j}}$ arising from an ample flag of $m_{j}$ pseudodivisors in $F_{\beta,x}^{Z_{j}-1}$ whose sequence of invertible sheaves equals

$$(\psi^{\otimes 1}, \psi^{\otimes 2}, \ldots, \psi^{\otimes m_{j}}).$$

If $m_{j}^{+}(Y, X) \leq m_{\beta}(Y)$, then $F_{\beta,x}^{X} is nonempty, and every irreducible component has dimension $\geq m_{\beta}(Y) - m_{j}^{+}(Y, X)$.

**Proof.** Denote $F_{\beta,x}^{Z_{j}-1}$ by $S$, denote the universal family of curves over $S$ by $\pi : C \rightarrow S$. Denote the constant section of $\pi$ by $\sigma : S \rightarrow C$. Denote the evaluation morphism by

$$u : C \rightarrow Z_{j-1}.$$  

Then $u^{*}s_{j}$ is a global section of $u^{*}L_{j}$ that evaluates to zero under restriction to $\sigma(S)$. By Corollary 4.9, $u^{*}L_{j}$ is isomorphic to $F_{mj} = [\pi^{*}(\psi)\otimes m_{j}]$. Thus, $u^{*}s_{j}$ is a global section of the pushforward $E_{m_{j}} = \pi_{*}F_{m_{j}}$ that evaluates to zero in the structure sheaf. By the proof of Corollary 4.11, the kernel of evaluation is an ample locally free sheaf of rank $m_{j}$, and it admits a filtration by locally free sheaves whose associated graded sheaves are $(\psi^{\otimes 1}, \ldots, \psi^{\otimes m_{j}})$. The images of $u^{*}s_{j}$ in this associated graded sheaves define a sequence of pseudodivisors whose common zero locus is $F_{\beta,x}^{Z_{j}}$.

Concatenating these sequences of pseudodivisors, and using the fact that the zero scheme of an ample invertible sheaf on a proper analytic space of dimension $\geq 1$ is nonempty, if $m_{\beta}(Y, X)$ is less than the dimension $m_{\beta}(Y)$ of $F_{\beta,x}^{X}$, then also $F_{\beta,x}^{X}$ is nonempty of dimension $\geq m_{\beta}(Y) - m_{\beta}(Y, X)$. \hfill $\square$

**Proposition 7.8.** With hypotheses as in Lemma 7.7 also assume that $X$ is a codimension-$c$ intersection of nef hypersurfaces $Y_{j}$ in $Y$, so that also the hypotheses of Proposition 7.6 hold and $m_{\beta}^{+}(Y, X)$ equals $m_{\beta}(Y, X)$.

1. The subspace $F_{\beta,x}^{X}$ is nonempty for general $x \in X$ if and only if $m_{\beta}(Y, X) \leq m_{\beta}(Y)$.

2. In this case, for $x \in X$ general, $F_{\beta,x}^{X}$ is smooth of pure dimension $m_{\beta}(X) = m_{\beta}(Y) - m_{\beta}(Y, X)$ and parameterizes only free, irreducible maps to $X$.

3. Considered as a closed subspace of $F_{\beta,x}^{Y}$, the manifold $F_{\beta,x}^{X}$ arises from an ample, locally complete intersection flag of pseudodivisors as in Lemma 7.7. 

4. The pushforward to $F_{\beta,x}^{Y}$ of the pullback to $F_{\beta,x}^{X}$ of each cycle class $w$ equals

$$w \cdot \prod_{j=1}^{c} (m_{j}) c_{1}(\psi)^{m_{j}}(Y, X).$$  

Also, the pushforward to $F_{\beta,x}^{Y}$ of the first Chern class of the normal sheaf of $F_{\beta,x}^{X}$ in $F_{\beta,x}^{Y}$ equals

$$\left(\sum_{j=1}^{c} (m_{j}(1 + m_{j})/2)c_{1}(\psi)\right) \cdot \prod_{j=1}^{c} (m_{j}) c_{1}(\psi)^{m_{j}}(Y, X).$$
Proof. If \(m_\beta(Y,X) \leq m_\beta(Y)\), then by Lemma 7.7, \(F^{X}_{\beta,x}\) is nonempty for every \(x \in X\). Conversely, assume that \(F^{X}_{\beta,x}\) is nonempty for general \(x \in X\). Since \(F^{Y}_{\beta,x}\) parameterizes only irreducible curves, the same holds for the subspace \(F^{X}_{\beta,x}\). Thus, the curves parameterized by \(F^{X}_{\beta,x}\) are all free, irreducible curves. Therefore \(F^{X}_{\beta,x}\) is a manifold of pure dimension equal to the expected dimension \(m_\beta(X) = \langle c_1(T_{X,J}^{1,0}), \beta \rangle - 2\). By adjunction applied to the flag of pseudodivisors defining \(X\) in \(Y\), \(m_\beta(X)\) equals \(m_\beta(Y) - m_\beta(Y,X)\). Since this integer is nonnegative, \(m_\beta(Y,X) \leq m_\beta(Y)\).

By Proposition 7.6, the subspace \(F^{X}_{\beta,x}\) is contained in the maximal open subspace of \(F^{Y}_{\beta,x}\) that parameterizes free curves, and this open has pure dimension equal to \(m_\beta(Y)\). By Lemma 7.7, the ample flag of pseudodivisors in \(F^{Y}_{\beta,x}\) from Lemma 7.7 is locally complete intersection, i.e., each pseudo-divisor is an honest divisor. Pullback of cycles to a divisor follows by pushforward is simply cup product with the first Chern class of the divisor. Iterating through all divisors in the flag gives the formula for the pushforward to \(F^{Y}_{\beta,x}\) of the pullback of cycles to \(F^{X}_{\beta,x}\). Finally, since \(F^{X}_{\beta,x}\) arises from a locally complete intersection flag of divisors, the conormal sheaf has an associated filtration whose associated graded sheaves are the invertible sheaves that equal the restriction of the duals of the invertible sheaves from the flag of pseudodivisors. By the Whitney sum formula, the first Chern class of the conormal sheaf equals the sum of all of the first Chern classes of these restrictions. This gives the second formula.

\(\square\)

**Proof of Theorem 1.18.** 1. By Proposition 7.5, the pushforward map on \(H_2\) is an isomorphism.

2. First, by Proposition 7.6, the pushforward to \(Y\) of the free cone of \(X\) is contained in the free cone of \(Y\). The goal is to characterize when the pushforward cone is the entire free cone of \(Y\).

Let \(\beta_i \in H_2(Y,\mathbb{Z})\) be a primitive generator of an extremal ray of \(\text{NE}_d^f(Y)_{\mathbb{R}}\). By hypothesis, \(\beta_i\) is free: \(\beta_i\) is one of the extremal rays of the free cone. Since \(\beta_i\) is a primitive generator of an extremal ray, it is also \(J\)-irreducible in the Mori cone. Thus, every \(\beta_i\)-curve in \(Y\) is irreducible, i.e., the hypothesis of Lemma 7.7 holds. By Proposition 7.8, this class is the pushforward of a free curve class from \(X\) if and only if \(m_{\beta_i}(Y,X) \leq m_{\beta_i}(Y)\), in which case \(m_{\beta_i}(Y)\) equals \(m_{\beta_i}(Y) - m_{\beta_i}(Y,X)\). Thus, the pushforward map identifies free cones if and only if every \(m_{\beta_i}(Y,X) \leq m_{\beta_i}(Y)\). In this case, by Part 4 of Proposition 7.8 also \(f_{\beta_i}(X)\) equals \(f_{\beta_i}(Y) \cdot f_{\beta_i}(Y,X)\). Finally, if \(m_{\beta_i}(X)\) is positive, then Part 4 of Proposition 7.8 also gives the identities,

\[
q_{\beta_i}(X) = q_{\beta_i}(Y) - q_{\beta_i}(Y,X), \quad s_{\beta_i}(X) = f_{\beta_i}(Y,X) \cdot (s_{\beta_i}(Y) - s_{\beta_i}(Y,X)).
\]

3. This follows from the identities above and Part 2 of Theorem 1.10. \(\square\)

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