Existence and uniqueness of local weak solution of d-dimensional tropical climate model without thermal diffusion in inhomogeneous Besov space

Baoquan Yuan* and Ying Zhang
School of Mathematics and Information Science, Henan Polytechnic University, Henan, 454000, China.
(bqyuan@hpu.edu.cn, yz503576344@163.com)

Abstract
This paper studies the existence and uniqueness of local weak solutions to the d-dimensional tropical climate model without thermal diffusion. We establish that, when \( \alpha = \beta \geq 1, \eta = 0, \) any initial data \((u_0, v_0) \in B^{1+\frac{d}{2}-2\alpha}_{2,1}(\mathbb{R}^d) \) and \( \theta_0 \in B^{1+\frac{d}{2}-\alpha}_{2,1}(\mathbb{R}^d) \) yields a unique weak solution.

Key words: Tropical climate model; local weak solution; inhomogeneous Besov space; existence and uniqueness.

MSC(2010): 35Q35, 35D30, 76D03.

1 Introduction
This paper focuses on a d-dimensional tropical climate model, which can be written as:

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \mu (-\Delta)^\alpha u + \nabla \cdot (v \otimes v) + \nabla p &= 0, \\
\partial_t v + u \cdot \nabla v + \nu (-\Delta)^\beta v + v \cdot \nabla u + \nabla \theta &= 0, \\
\partial_t \theta + u \cdot \nabla \theta + \eta (-\Delta)^\gamma \theta + \nabla \cdot v &= 0, \\
\nabla \cdot u &= 0,
\end{aligned}
\]

(1.1)

for \( t \geq 0, x \in \mathbb{R}^d, d \geq 2. \) We denote \( u \) the barotropic mode and \( v \) the first baroclinic mode of the velocity, respectively, and the scalars \( \theta \) and \( p \) represent the temperature and the pressure. \( \mu, \nu, \eta, \alpha, \gamma, \beta \geq 0 \) are real parameters. Here \( v \otimes v \) is the standard tensor notation and the fractional Laplacian operator \((-\Delta)^\alpha\) is defined via the Fourier transform

\[
(-\Delta)^\alpha f(\xi) = |\xi|^{2\alpha} \hat{f}(\xi).
\]

By performing a Galerkin truncation to the hydrostatic Boussinesq equations, Feireisl-Majda-Pauluis in [1] derived a version of (1.1) without any Laplacian terms. As we all know, the tropical climate model is a coupling system between the barotropic mode and the first baroclinic mode.

*Corresponding Author: B. Yuan
of the velocity and the typical mesospheric temperature, which contains much richer structures than the N-S equations (see, [2, 3, 4, 5, 6, 7]) or the MHD equations (see, [8, 9, 10, 11, 12]). They are not merely a combination of three parallel the N-S type equations but an interactive and integrated system. Let us briefly recall some works on the tropical climate model (1.1) firstly. When $\alpha = \beta = 1$, Li and Titi [13] established the global well-posedness of strong solutions on the assumption that the initial data $(u_0, v_0, \theta_0) \in H^1(\mathbb{R}^d)$, for $d = 2$. Later, inspired by [13], Wan [14] proved the global well-posedness of solution with some damping terms under small initial data when $\mu = 0$. Dong et al. [15] obtained the global regularity for the 2D tropical climate model without thermal diffusion when $\alpha + \beta = 2$, $1 < \beta \leq \frac{3}{2}$, and $\alpha = 2, \nu = 0$ respectively. Ye [16] proved that the global regularity result of the two-dimensional zero thermal diffusion tropical climate model with fractional dissipation holds true as long as $\alpha + \beta \geq 2$ with $1 < \alpha < 2$. One can see [17, 18, 19, 20] for some more recent results on the global regularity issue for the 2D tropical climate models.

It is worth particularly mentioning that system (1.1) and the MHD equations are very similar in terms of the structure of the equations. When $\theta$ is a constant, the system (1.1) reduces to the MHD-type equations. For MHD equations, Jiu et al.[21] established the local existence and uniqueness of weak solutions with the minimal initial regularity assumption and for the largest possible range of $\alpha$’s. Naturally, we wonder that whether the tropical climate model can use the minimum initial regularity hypothesis to obtain the existence and uniqueness of the weak solution in the maximum possible range of $\alpha, \beta$.

Inspired by [21, 22], the main goal of this paper is to establish the unique weak solutions to (1.1) in a weakest possible functional setting for the largest possible ranges of $\alpha$ and $\beta$. The difficulty of this paper lies in the treatment of nonlinear terms, which is due to the lack of free divergence conditions of $\nu$ and the absence of thermal diffusion in the equation of $\theta$. It is worth mentioning that the uniqueness can no longer be treated by estimating the difference in $L^2$ norm, the difficulty lies in the lack of thermal diffusion, which makes it hard to estimate the nonlinear terms $\int \tilde{u} \cdot \nabla \theta_1 \cdot \tilde{\theta}$. To bypass this difficulty, we introduce the Chemin-Lerner type Besov space and use Osgood lemma to prove the uniqueness.

Our precise result is stated in the following theorem.

**Theorem 1.1.** Let $d \geq 2$ and consider the system (1.1) with $1 \leq \alpha = \beta < 1 + \frac{d}{2}$, $\eta = 0$. Assume the initial data $(u_0, v_0, \theta_0)$ satisfy

$$(u_0, v_0) \in B^{1+\frac{d}{2}-2\alpha}_{2,1}(\mathbb{R}^d), \quad \theta_0 \in B^{1+\frac{d}{2}-\alpha}_{2,1}(\mathbb{R}^d) \text{ and } \nabla \cdot u_0 = 0.$$  

Then the system (1.1) has a unique weak solution $(u, v, \theta)$ on $[0, T]$ satisfying

$$u \in L^\infty(0, T; B^{1+\frac{d}{2}-2\alpha}_{2,1}(\mathbb{R}^d)) \cap L^1(0, T; B^{1+\frac{d}{2}}_{2,1}(\mathbb{R}^d)),$$

$$v \in L^\infty(0, T; B^{1+\frac{d}{2}-2\alpha}_{2,1}(\mathbb{R}^d)) \cap L^1(0, T; B^{1+\frac{d}{2}}_{2,1}(\mathbb{R}^d)),$$

$$\theta \in L^\infty(0, T; B^{1+\frac{d}{2}-\alpha}_{2,1}(\mathbb{R}^d)).$$

**Remark 1.1.** In the TCM equations (1.1) there is a linear term $\nabla \theta$ in the equation of $\nu$, which has different scaling index with $u$ and $v$, therefore we use the inhomogeneous Besov spaces in Theorem 1.1.

**Remark 1.2.** In this paper, the regularity indices in these Besov spaces appear to be optimal and can not be lowered. When $\alpha = \beta = 1$, the 2D tropical climate model with the standard Laplacian dissipation has a unique weak solution for $(u_0, v_0) \in B^{1}_{2,1}(\mathbb{R}^d), \theta_0 \in B^{1}_{2,1}(\mathbb{R}^d)$, which is also obtained in [23].

**Remark 1.3.** Due to the coupling property on the structure of the system (1.1), this paper considers the case of $\alpha = \beta$. It is known that the larger $\alpha$ or $\beta$ is, the higher the regularity of
the weak solution. However, the condition of \( \alpha = \beta < 1 + \frac{d}{4} \) in Theorem 1.1 is only restricted by the method in this paper (see, (3.12)). In addition, because the temperature equation has no dissipation and \( v \) lacks the divergence free condition, the case of \( \alpha = \beta < 1 \) can not be obtained by the method in this paper.

In the rest of this paper, the letter \( C \) denotes a generic constant whose exact values may change from line to line, but do not depend on particular solutions or functions.

## 2 Several tool lemmas

In this section we present several tool lemmas which serves as a preparation for the proofs of our main results. First we introduce the paraproduct decomposition of two functions \( u \) and \( v \).

**Definition 2.1.** In terms of the inhomogeneous dyadic block operators, we can write the standard product in terms of the paraproducts, namely

\[
wv = \sum_{k \geq 1} S_{k-1} u \Delta_k v + \sum_{k \geq 1} S_{k-1} v \Delta_k u + \sum_{k \geq 1} \Delta_k u \tilde{\Delta}_k v,
\]

where \( \tilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1} \). This is the so-called Bony decomposition (see [24]).

We state the following Chemin-Lerner type Besov spaces introduced in [24].

**Definition 2.2.** Let \( s \in \mathbb{R}, 1 \leq p, q, r \leq \infty \) and \( T \in (0, \infty) \). The functional space \( \dot{L}^r(0,T;B^{s}_{p,q}(\mathbb{R}^d)) \) consists of tempered distributions \( f \), which is defined by

\[
\dot{L}^r(0,T;B^{s}_{p,q}(\mathbb{R}^d)) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \| f \|_{\dot{L}^r(0,T;B^{s}_{p,q}(\mathbb{R}^d))} < \infty \right\},
\]

where

\[
\| f \|_{\dot{L}^r(0,T;B^{s}_{p,q})} = \left\| 2^{js} \| \Delta_j f \|_{L^r(0,T)} \right\|_{L^1}.
\]

By Minkowski’s inequality,

\[
\dot{L}^r(0,T;B^{s}_{p,q}) \subseteq L^r(0,T;B^{s}_{p,q}), \quad \text{if} \quad r > q.
\]

\[
\dot{L}^r(0,T;B^{s}_{p,q}) \supseteq L^r(0,T;B^{s}_{p,q}), \quad \text{if} \quad r < q.
\]

\[
\dot{L}^r(0,T;B^{s}_{p,q}) = L^r(0,T;B^{s}_{p,q}), \quad \text{if} \quad r = q.
\]

Now, we state the bounds for the triple products involving Fourier localized functions. The detailed proof of the following lemma can refer to [21].

**Lemma 2.1.** Let \( j \in \mathbb{Z} \) be an integer. Let \( \Delta_j \) be a dyadic block operator (either inhomogeneous or homogeneous). For any vectors field \( u, v, w \) with \( \nabla \cdot u = 0 \), we have

\[
| \int_{\mathbb{R}^d} \Delta_j (v \cdot \nabla u) \cdot \Delta_j w \, dx | \leq C \| \Delta_j u \|_{L^2}(2^j \sum_{m \leq j-1} 2^{\frac{j}{2} m} \| \Delta_m v \|_{L^2} \sum_{|j-k| \leq 2} \| \Delta_k u \|_{L^2}) + \sum_{|j-k| \leq 2} \| \Delta_k v \|_{L^2} \sum_{m \leq j-1} 2^{1+\frac{j}{2} m} \| \Delta_m u \|_{L^2} \sum_{|j-k| \leq 2} \| \Delta_k u \|_{L^2},
\]

\[
| \int_{\mathbb{R}^d} \Delta_j (u \cdot \nabla v) \cdot \Delta_j w \, dx | \leq C \| \Delta_j u \|_{L^2}(\sum_{m \leq j-1} 2^{1+\frac{j}{2} m} \| \Delta_m u \|_{L^2} \sum_{|j-k| \leq 2} \| \Delta_k v \|_{L^2}) + \sum_{|j-k| \leq 2} \| \Delta_k u \|_{L^2} \sum_{m \leq j-1} 2^{1+\frac{j}{2} m} \| \Delta_m v \|_{L^2} \sum_{|j-k| \leq 2} \| \Delta_k u \|_{L^2}).
\]
Lemma 2.2. For $t > 0$, let $u$ satisfy

$$\|u\|_{L^1(B_{2,1}^{1+\frac{d}{2}})} < \infty,$$

then we have

$$\|u\|_{L^1(B_{2,1}^{1+\frac{d}{2}})} \leq C \|u\|_{L^1(B_{2,\infty}^{1+\frac{d}{2}})} \log \left( e + \frac{\|u\|_{L^1(B_{2,\infty}^{1+\frac{d}{2}})}}{\|u\|_{L^1(B_{2,\infty}^{1+\frac{d}{2}})}} \right).$$

(2.5)

We now state the Osgood lemma which will be used to prove the uniqueness of the weak solution (see [24]).

Lemma 2.3. Let $0 < a < 1$, $f$ be a measurable function, $\phi$ a locally integrable function and $\varphi$ a positive, continuous and nondecreasing function. Assume that, for some nonnegative real number $c$, the function $f$ satisfies

$$f(t) \leq c + \int_{t_0}^t \phi(\tau) \varphi(f(\tau)) d\tau.$$

If $c$ is positive, then we have

$$-\psi(f(t)) + \psi(c) \leq \int_{t_0}^t \phi(\tau) d\tau, \quad \psi(x) = \int_x^a \frac{dr}{\varphi(r)}.$$

If $c = 0$ and $\varphi$ satisfies

$$\int_0^a \frac{dr}{\varphi(r)} = \infty,$$

then we have $f \equiv 0$.

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Our main effort is to construct a successive approximation sequence and show that the limit of a subsequence actually solves (1.1) in the weak sense.

Proof for the existence part of the Theorem 1.1. We consider a successive approximation sequence $(u^{(n)}, v^{(n)}, \theta^{(n)})$ satisfying

$$\begin{aligned}
\partial_t u^{(n+1)} + \mu(-\Delta)^{\alpha} u^{(n+1)} &= \mathcal{P}[-u^{(n)} \cdot \nabla u^{(n+1)} - \nabla \cdot (u^{(n)} \otimes v^{(n)})], \\
\partial_t v^{(n+1)} + \nu(-\Delta)^{\alpha} v^{(n+1)} &= -u^{(n)} \cdot \nabla v^{(n+1)} - \nabla \cdot (v^{(n)} \otimes \theta^{(n)}) - v^{(n)} \cdot \nabla u^{(n)}, \\
\partial_t \theta^{(n+1)} &= -u^{(n)} \cdot \nabla \theta^{(n)} - \nabla \cdot v^{(n)}, \\
\nabla \cdot u^{(n+1)} &= 0, \\
u^{(n+1)}(x, 0) &= S_{n+1} u_0, \quad v^{(n+1)}(x, 0) = S_{n+1} v_0, \quad \theta^{(n+1)}(x, 0) = S_{n+1} \theta_0,
\end{aligned}
$$

(3.1)
where \( P = I - \nabla(-\Delta)^{-1}{\text{div}} \) is the standard Leray projection and \( S_\alpha \) is the standard inhomogeneous low frequency cutoff operator. For \( T > 0 \) sufficiently small and \( 0 < \delta < 1 \) (to be determined later), we set

\[
M = 2\left(\|u_0\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} + \|v_0\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} + \|\theta_0\|_{B_{2,1}^{1+\frac{d}{2}-\alpha}}\right),
\]

\[
Y \equiv \left\{(u, v, \theta) \mid \|u\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq M, \quad \|v\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq M,
\right.
\]
\[
\|\theta\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-\alpha})} \leq M, \quad \|u\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}})} \leq \delta, \quad \|v\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}})} \leq \delta \right\}. \tag{3.2}
\]

We show that \((u^{(n)}, v^{(n)}, \theta^{(n)})\) has a subsequence that converges to the weak solution of (3.1). This process consists of three main steps. The first step is to show that \((u^{(n)}, v^{(n)}, \theta^{(n)})\) is uniformly bounded in \(Y\). The second step is to extract a strongly convergent subsequence by Aubin-Lions Lemma. While the last step is to show that the limit is indeed a weak solution of (3.1). The most important point is to show the uniform bound for \((u^{(n)}, v^{(n)}, \theta^{(n)})\) in \(Y\) by induction.

Recall that \((u_0, v_0) \in B_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d), \theta_0 \in B_{2,1}^{1+\frac{d}{2}-\alpha}(\mathbb{R}^d), \) according to (3.1),

\[
u^{(1)} = S_1u_0, \quad \nu^{(1)} = S_1v_0, \quad \theta^{(1)} = S_1\theta_0.
\]

Clearly,

\[
\|u^{(1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} = \|S_1u_0\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} \leq M,
\]

\[
\|v^{(1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} = \|S_1v_0\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} \leq M,
\]

\[
\|\theta^{(1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-\alpha})} = \|S_1\theta_0\|_{B_{2,1}^{1+\frac{d}{2}-\alpha}} \leq M.
\]

If \( T > 0 \) is sufficiently small, then

\[
\|u^{(1)}\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}})} \leq T\|S_1u_0\|_{B_{2,1}^{1+\frac{d}{2}}} \leq TC\|u_0\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} \leq \delta,
\]

\[
\|v^{(1)}\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}})} \leq T\|S_1v_0\|_{B_{2,1}^{1+\frac{d}{2}}} \leq TC\|v_0\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} \leq \delta.
\]

Assuming that \((u^{(n)}, v^{(n)}, \theta^{(n)})\) obeys the bounds defined in \(Y\), namely

\[
\|u^{(n)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq M, \quad \|v^{(n)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq M, \quad \|\theta^{(n)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-\alpha})} \leq M,
\]

\[
\|u^{(n)}\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}})} \leq \delta, \quad \|v^{(n)}\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}})} \leq \delta,
\]

we prove that \((u^{(n+1)}, v^{(n+1)}, \theta^{(n+1)})\) obeys the same bounds for the aforementioned \( T > 0 \) and \( M > 0 \), namely

\[
\|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq M, \quad \|v^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq M, \quad \|\theta^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-\alpha})} \leq M,
\]

\[
\|u^{(n+1)}\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}})} \leq \delta, \quad \|v^{(n+1)}\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}})} \leq \delta.
\]
3.1 The estimates of $\|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{\theta}{2}-2\alpha})}$ and $\|v^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{\theta}{2}-2\alpha})}$ and $\|\theta^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{\theta}{2}-2\alpha})}$

**3.1.1 The estimate of $\|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{\theta}{2}-2\alpha})}$**

Let $j \geq 0$ be an integer. Applying $\Delta_j$ to (3.1) and then dotting the equation with $\Delta_j u^{(n+1)}$, we obtain

$$\frac{d}{dt} \|\Delta_j u^{(n+1)}\|_{L^2}^2 + \mu \|\Lambda^\alpha \Delta_j u^{(n+1)}\|_{L^2}^2 = A_1 + A_2,$$  \hspace{1cm} (3.3)

where

$$A_1 = - \int \Delta_j (u^{(n)} \cdot \nabla u^{(n+1)}) \cdot \Delta_j u^{(n+1)} dx,$$

$$A_2 = - \int \Delta_j (\nabla \cdot (u^{(n)} \otimes v^{(n)})) \cdot \Delta_j u^{(n+1)} dx.$$

The dissipative part of (3.3) admit lower bounds

$$\mu \|\Lambda^\alpha \Delta_j u^{(n+1)}\|_{L^2}^2 \geq C_0 2^{2j\alpha} \|\Delta_j u^{(n+1)}\|_{L^2}^2,$$

where $C_0 > 0$ is a constant. According to (2.3) of Lemma 2.1, $A_1$ can be bounded by

$$|A_1| \leq C \|\Delta_j u^{(n+1)}\|_{L^2} \left( \sum_{m \leq j-1} 2^{(1+\frac{\theta}{2})m} \|\Delta_m u^{(n)}\|_{L^2} \sum_{|j-k| \leq 2} \|\Delta_k u^{(n+1)}\|_{L^2} + \sum_{|j-k| \leq 2} \|\Delta_k u^{(n)}\|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{\theta}{2})m} \|\Delta_m u^{(n+1)}\|_{L^2} + \sum_{k \geq j-4} 2^j 2^{\frac{\theta}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\tilde{\Delta}_k u^{(n+1)}\|_{L^2} \right).$$  \hspace{1cm} (3.4)

According to (2.4) of Lemma 2.1, $A_2$ can be bounded by

$$|A_2| \leq C \|\Delta_j u^{(n+1)}\|_{L^2} 2^j \|\Delta_j (v^{(n)} \otimes u^{(n)})\|_{L^2} \leq C \|\Delta_j u^{(n+1)}\|_{L^2} 2^j \left( \sum_{m \leq j-1} 2^{\frac{\theta}{2}m} \|\Delta_m v^{(n)}\|_{L^2} \sum_{|j-k| \leq 2} \|\Delta_k v^{(n)}\|_{L^2} + \sum_{|j-k| \leq 2} \|\Delta_k v^{(n)}\|_{L^2} \sum_{m \leq j-1} 2^{\frac{\theta}{2}m} \|\Delta_m v^{(n)}\|_{L^2} + \sum_{k \geq j-4} 2^j 2^{\frac{\theta}{2}k} \|\Delta_k v^{(n)}\|_{L^2} \|\tilde{\Delta}_k v^{(n)}\|_{L^2} \right).$$  \hspace{1cm} (3.5)

Inserting the estimates (3.4) and (3.5) into (3.3) and eliminating $\|\Delta_j u^{(n+1)}\|_{L^2}$ from both sides of the inequality, we get

$$\frac{d}{dt} \|\Delta_j u^{(n+1)}\|_{L^2} + C_0 2^{2j\alpha} \|\Delta_j u^{(n+1)}\|_{L^2} \leq J_1 + \cdots + J_5,$$  \hspace{1cm} (3.6)
where

\[ J_1 = C \| \Delta_j u^{(n+1)} \|_{L^2} \sum_{m \leq j-1} 2^{(1 + \frac{d}{2})m} \| \Delta_m u^{(n)} \|_{L^2}, \]
\[ J_2 = C \| \Delta_j u^{(n)} \|_{L^2} \sum_{m \leq j-1} 2^{(1 + \frac{d}{2})m} \| \Delta_m u^{(n+1)} \|_{L^2}, \]
\[ J_3 = C 2^j \sum_{k \geq j-4} 2^{\frac{d}{2}k} \| \Delta_k u^{(n)} \|_{L^2} \| \Delta_k u^{(n+1)} \|_{L^2}, \]
\[ J_4 = C 2^j \| \Delta_j v^{(n)} \|_{L^2} \sum_{m \leq j-1} 2^{\frac{d}{2}m} \| \Delta_m v^{(n)} \|_{L^2}, \]
\[ J_5 = C 2^j \sum_{k \geq j-4} 2^{\frac{d}{2}k} \| \Delta_k v^{(n)} \|_{L^2} \| \Delta_k v^{(n)} \|_{L^2}. \]

Integrating (3.6) in time yields

\[ \| \Delta_j u^{(n+1)} \|_{L^2} \leq e^{-c_0 2^{2\alpha_j} t} \| \Delta_j u^{(n+1)} \|_{L^2} + \int_0^t e^{-c_0 2^{2\alpha_j} (t-\tau)} (J_1 + \cdots + J_5) d\tau. \] (3.7)

For \( j = -1 \),

\[ \| \Delta_{-1} u^{(n+1)} \|_{L^2} \leq \| \Delta_{-1} u^{(n+1)} \|_{L^2} + \int_0^t (J_1 + \cdots + J_5) d\tau. \] (3.8)

Taking the \( L^\infty(0, T) \) of (3.7) and (3.8), then multiplying by \( 2^{(1 + \frac{d}{2} - 2\alpha)j} \) and summing up the resulting inequalities with respect to \( j \), it holds that

\[ \| u^{(n+1)} \|_{L^\infty(0, T; B_{2,1}^{1 + \frac{d}{2} - 2\alpha})} \leq \| u_0^{(n+1)} \|_{B_{2,1}^{1 + \frac{d}{2} - 2\alpha}} + \sum_{j \geq -1} 2^{(1 + \frac{d}{2} - 2\alpha)j} \int_0^T (J_1 + \cdots + J_5) d\tau, \] (3.9)

where we have used the fact

\[ e^{-c_0 2^{2\alpha_j} (t-\tau)} \leq 1. \]

Now, we estimate the terms involving \( J_1 \) through \( J_5 \). By Hölder’s inequality, \( J_1 \) can be estimated as follows

\[ \sum_{j \geq -1} 2^{(1 + \frac{d}{2} - 2\alpha)j} \int_0^T J_1 d\tau \]
\[ \leq C \int_0^T \sum_{j \geq -1} 2^{(1 + \frac{d}{2} - 2\alpha)j} \| \Delta_j u^{(n+1)} \|_{L^2} \sum_{m \leq j-1} 2^{(1 + \frac{d}{2})m} \| \Delta_m u^{(n)} \|_{L^2} d\tau \]
\[ \leq C \| u^{(n+1)} \|_{L^\infty(0, T; B_{2,1}^{1 + \frac{d}{2} - 2\alpha})} \| u^{(n)} \|_{L^1(0, T; B_{2,1}^{1 + \frac{d}{2}})} \]
\[ \leq C \delta \| u^{(n+1)} \|_{L^\infty(0, T; B_{2,1}^{1 + \frac{d}{2} - 2\alpha})}, \] (3.10)
The term with $J_2$ admits the same bound. We have

\[
\sum_{j \geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^T J_2 \, d\tau \\
\leq C \int_0^T \left( \sum_{j \geq -1} 2^{(1+\frac{d}{2})j} \| \Delta_j u^{(n)} \|_{L^2} \right) \sum_{m \leq j-1} 2^{2\alpha(m-j)} 2^{(1+\frac{d}{2}-2\alpha)m} \| \Delta_m u^{(n+1)} \|_{L^2} \, d\tau \\
\leq C \int_0^T \| u^{(n)} \|_{B_{2+1, \frac{d}{2}}} \| u^{(n+1)} \|_{B_{2+1, \frac{d}{2}-2\alpha}} \, d\tau \\
\leq C \| u^{(n)} \|_{L^1(0,T;B_{2+1, \frac{d}{2}})} \| u^{(n+1)} \|_{L^\infty(0,T;B_{2+1, \frac{d}{2}-2\alpha})} \\
\leq C \delta \| u^{(n+1)} \|_{L^\infty(0,T;B_{2+1, \frac{d}{2}-2\alpha})}, \tag{3.11}
\]

The term involving $J_3$ is bounded similarly

\[
\sum_{j \geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^T J_3 \, d\tau \\
\leq \int_0^T \left( \sum_{j \geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} 2^j \sum_{k \geq j-4} 2^{\frac{d}{2}k} \| \Delta_k u^{(n)} \|_{L^2} \| \Delta_k u^{(n+1)} \|_{L^2} \right) \, d\tau \\
\leq \int_0^T \sum_{j \geq -1} \sum_{k \geq j-4} 2^{2(\frac{d}{2}-2\alpha)(j-k)} 2^{(1+\frac{d}{2})k} \| \Delta_k u^{(n)} \|_{L^2} \| \Delta_k u^{(n+1)} \|_{L^2} \, d\tau \\
\leq C \| u^{(n)} \|_{L^1(0,T;B_{2+1, \frac{d}{2}})} \| u^{(n+1)} \|_{L^\infty(0,T;B_{2+1, \frac{d}{2}-2\alpha})} \\
\leq C \delta \| u^{(n+1)} \|_{L^\infty(0,T;B_{2+1, \frac{d}{2}-2\alpha})}, \tag{3.12}
\]

where we have used Young’s inequality for series convolution and we need $\alpha < 1 + \frac{d}{4}$. The term with $J_4$ is bounded by

\[
\sum_{j \geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^T J_4 \, d\tau \\
\leq C \int_0^T \left( \sum_{j \geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} 2^j \| \Delta_j v^{(n)} \|_{L^2} \right) \sum_{m \leq j-1} 2^{\frac{d}{2}m} \| \Delta_m v^{(n)} \|_{L^2} \, d\tau \\
\leq C \int_0^T \sum_{j \geq -1} 2^{(1+\frac{d}{2})j} \| \Delta_j v^{(n)} \|_{L^2} \sum_{m \leq j-1} 2^{(2\alpha-1)(m-j)} 2^{(1+\frac{d}{2}-2\alpha)m} \| \Delta_m v^{(n)} \|_{L^2} \, d\tau \\
\leq C \int_0^T \| v^{(n)} \|_{B_{2+1, \frac{d}{2}}} \| v^{(n)} \|_{B_{2+1, \frac{d}{2}-2\alpha}} \, d\tau \\
\leq C \| v^{(n)} \|_{L^1(0,T;B_{2+1, \frac{d}{2}})} \| v^{(n)} \|_{L^\infty(0,T;B_{2+1, \frac{d}{2}-2\alpha})} \\
\leq C \delta M, \tag{3.13}
\]
The term with $J_5$ is estimated as follows

$$
\sum_{j \geq -1} 2^{j(1+\frac{d}{2}-2\alpha)} \int_0^T J_5 d\tau
\leq C \int_0^T \sum_{j \geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \sum_{k \geq j-4} 2^j \|\Delta_k v^{(n)}\|_{L^2} \|\Delta_k v^{(n)}\|_{L^2} d\tau
\leq \int_0^T \sum_{j \geq -1} \sum_{k \geq j-4} 2^{(2+\frac{d}{2}-2\alpha)(j-k)} 2^{(1+\frac{d}{2}-2\alpha)k} \|\Delta_k v^{(n)}\|_{L^2} 2^{(1+\frac{d}{2})k} \|\Delta_k v^{(n)}\|_{L^2} d\tau
\leq C \|v^{(n)}\|_{L^1(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} \|v^{(n)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})}
\leq C \delta M. \tag{3.14}
$$

Collecting the estimates (3.10) – (3.14) and inserting them into (3.9), we have

$$
\|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq \|u_0^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} + C \delta \|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})} + C \delta M. \tag{3.15}
$$

### 3.1.2 The estimate of $\|v^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha})}$

Let $j \geq 0$ be an integer. Applying $\Delta_j$ to (3.1)$_2$ and then dotting the equation with $\Delta_j v^{(n+1)}$, we have

$$
\frac{1}{2} \frac{d}{dt} \|\Delta_j v^{(n+1)}\|_{L^2}^2 + \nu \|\Lambda^a \Delta_j v^{(n+1)}\|_{L^2}^2 = B_1 + B_2 + B_3, \tag{3.16}
$$

where

$$
B_1 = - \int \Delta_j (u^{(n)} \cdot \nabla v^{(n+1)}) \cdot \Delta_j v^{(n+1)} dx,
B_2 = - \int \Delta_j (v^{(n)} \cdot \nabla u^{(n)}) \cdot \Delta_j v^{(n+1)} dx,
B_3 = - \int \Delta_j (\nabla \cdot \theta^{(n)}) \cdot \Delta_j v^{(n+1)} dx.
$$

The dissipative part of (3.16) admit lower bounds

$$
\nu \|\Lambda^a \Delta_j v^{(n+1)}\|_{L^2}^2 \geq C_0 2^{2aj} \|\Delta_j v^{(n+1)}\|_{L^2}^2,
$$

where $C_0 > 0$ is a constant. Using (2.3) of Lemma 2.1, $B_1$ can be bounded by

$$
|B_1| \leq C \|\Delta_j v^{(n+1)}\|_{L^2} \left( \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2} \sum_{|j-k| \leq 2} \|\Delta_k v^{(n+1)}\|_{L^2} + \sum_{|j-k| \leq 2} \|\Delta_k u^{(n)}\|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m v^{(n+1)}\|_{L^2} + \sum_{k \geq j-4} 2^j 2^j \|\Delta_k u^{(n)}\|_{L^2} \|\Delta_k v^{(n+1)}\|_{L^2} \right). \tag{3.17}
$$
And by (2.2) of Lemma 2.1, $B_2$ can be bounded by
\[
|B_2| \leq C \| \Delta_j v^{(n+1)} \|_{L^2} \left( \sum_{m \leq j - 1} 2^{\frac{3}{2}m} \| \Delta_m v^{(n)} \|_{L^2} \sum_{|j-k| \leq 2} \| \Delta_k u^{(n)} \|_{L^2} \right) \\
+ \sum_{|j-k| \leq 2} \| \Delta_k v^{(n)} \|_{L^2} \sum_{m \leq j - 1} 2^{1+\frac{3}{2}m} \| \Delta_m u^{(n)} \|_{L^2} \\
+ \sum_{k \geq j-4} 2^j 2^{\frac{3}{2}k} \| \Delta_k v^{(n)} \|_{L^2} \| \Delta_k u^{(n)} \|_{L^2} \right).
\]

(3.18)

By Hölder’s inequality and Bernstein’s inequality, it follows
\[
|B_3| = \left| - \int \Delta_j (\nabla \cdot \theta^{(n)}) \cdot \Delta_j v^{(n+1)} \, dx \right| \\
\leq C \| \Delta_j v^{(n+1)} \|_{L^2} \| \Delta_j (\nabla \cdot \theta^{(n)}) \|_{L^2} \\
\leq C 2^j \| \Delta_j v^{(n+1)} \|_{L^2} \| \Delta_j \theta^{(n)} \|_{L^2}.
\]

(3.19)

Inserting the estimates (3.17), (3.18) and (3.19) into the equality (3.16), then eliminating \( \| \Delta_j v^{(n+1)} \|_{L^2} \) from both sides of the inequality, we get
\[
\frac{d}{dt} \| \Delta_j v^{(n+1)} \|_{L^2} + C_0 2^{2\alpha j} \| \Delta_j v^{(n+1)} \|_{L^2} \leq K_1 + \cdots + K_7,
\]

(3.20)

where
\[
K_1 = C \| \Delta_j u^{(n+1)} \|_{L^2} \sum_{m \leq j - 1} 2^{1+\frac{3}{2}m} \| \Delta_m u^{(n)} \|_{L^2},
\]
\[
K_2 = C \| \Delta_j u^{(n)} \|_{L^2} \sum_{m \leq j - 1} 2^{1+\frac{3}{2}m} \| \Delta_m u^{(n+1)} \|_{L^2},
\]
\[
K_3 = C 2^j \sum_{k \geq j-4} 2^{\frac{3}{2}k} \| \Delta_k u^{(n)} \|_{L^2} \| \Delta_k v^{(n+1)} \|_{L^2},
\]
\[
K_4 = C 2^j \| \Delta_j u^{(n)} \|_{L^2} \sum_{m \leq j - 1} 2^{\frac{3}{2}m} \| \Delta_m u^{(n)} \|_{L^2},
\]
\[
K_5 = C \| \Delta_j v^{(n)} \|_{L^2} \sum_{m \leq j - 1} 2^{1+\frac{3}{2}m} \| \Delta_m u^{(n)} \|_{L^2},
\]
\[
K_6 = C 2^j \sum_{k \geq j-4} 2^{\frac{3}{2}k} \| \Delta_k v^{(n)} \|_{L^2} \| \Delta_k u^{(n)} \|_{L^2},
\]
\[
K_7 = C 2^j \| \Delta_j \theta^{(n)} \|_{L^2}.
\]

Integrating (3.20) in time yields, for any \( t \leq T \)
\[
\| \Delta_j v^{(n+1)} \|_{L^2} \leq e^{-c_1 2^{2\alpha j} t} \| \Delta_j v_0^{(n+1)} \|_{L^2} + \int_0^t e^{-c_1 2^{2\alpha j} (t-\tau)} (K_1 + \cdots + K_7) d\tau.
\]

(3.21)

For \( j = -1 \), arguing similarly as deriving (3.7)-(3.8), we shall omit the details of this case in the following discussion. Taking the \( L^\infty(0,T) \) of (3.21), multiplying by \( 2^l (1+\frac{3}{2}-2\alpha) \) and summing over \( j \), we deduce
\[
\| v^{(n+1)} \|_{L^\infty(0,T;B^{\frac{3}{2}-2\alpha}_{2,1})} \leq \| v_0^{(n+1)} \|_{B^{\frac{3}{2}-2\alpha}_{2,1}} + \sum_{j \geq -1} 2^{1+\frac{3}{2}(1-2\alpha)j} \int_0^T (K_1 + \cdots + K_7) d\tau.
\]

(3.22)
The terms involving $K_1$ through $K_7$ can be bounded as follows. Firstly,

$$
\sum_{j \geq -1} 2^{\left(1 + \frac{d}{2} - 2\alpha\right)j} \int_0^T K_1 d\tau \\
\leq C \int_0^T \sum_{j \geq -1} 2^{\left(1 + \frac{d}{2} - 2\alpha\right)j} \|\Delta_j u^{(n+1)}\|_{L^2} \sum_{m \leq j-1} 2^{\left(1 + \frac{d}{2}\right)m} \|\Delta_m u^{(n)}\|_{L^2} d\tau \\
\leq C \|u^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-2\alpha}_2)} \|u^{(n)}\|_{L^1(0,T;B^{1+\frac{d}{2}}_2)} \\
\leq C\delta \|v^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-2\alpha}_2)}. \quad (3.23)
$$

The terms involving $K_2$ and $K_3$ obey the same bound

$$
\sum_{j \geq -1} 2^{\left(1 + \frac{d}{2} - 2\alpha\right)j} \int_0^T K_2 d\tau \\
\leq C \int_0^T \sum_{j \geq -1} 2^{\left(1 + \frac{d}{2} - 2\alpha\right)j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m \leq j-1} 2^{\left(1 + \frac{d}{2} - 2\alpha\right)m} 2^{2\alpha(m-j)} \|\Delta_m v^{(n+1)}\|_{L^2} d\tau \\
\leq C \|u^{(n)}\|_{L^1(0,T;B^{1+\frac{d}{2}}_2)} \|v^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-2\alpha}_2)} \\
\leq C\delta \|v^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-2\alpha}_2)}, \quad (3.24)
$$

and

$$
\sum_{j \geq -1} 2^{\left(1 + \frac{d}{2} - 2\alpha\right)j} \int_0^T K_3 d\tau \\
\leq C \int_0^T \sum_{j \geq -1} 2^{\left(1 + \frac{d}{2} - 2\alpha\right)j} 2^j \|\Delta_j u^{(n)}\|_{L^2} \|\Delta_j v^{(n+1)}\|_{L^2} d\tau \\
\leq C \|u^{(n)}\|_{L^1(0,T;B^{1+\frac{d}{2}}_2)} \|v^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-2\alpha}_2)} \\
\leq C\delta \|v^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-2\alpha}_2)}. \quad (3.25)
$$

The term with $K_4$ is bounded by

$$
\sum_{j \geq -1} 2^{\left(1 + \frac{d}{2} - 2\alpha\right)j} \int_0^T K_4 d\tau \\
\leq C \int_0^T \sum_{j \geq -1} 2^{\left(1 + \frac{d}{2} - 2\alpha\right)j} 2^j \|\Delta_j u^{(n)}\|_{L^2} \sum_{m \leq j-1} 2^{\frac{d}{2}m} \|\Delta_m v^{(n)}\|_{L^2} d\tau \\
\leq C \int_0^T \sum_{j \geq -1} 2^{\left(1 + \frac{d}{2}\right)j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m \leq j-1} 2^{\left(2\alpha - 1\right)(m-j)} 2^{\left(1 + \frac{d}{2} - 2\alpha\right)m} \|\Delta_m v^{(n)}\|_{L^2} d\tau \\
\leq C \|u^{(n)}\|_{L^1(0,T;B^{1+\frac{d}{2}}_2)} \|v^{(n)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-2\alpha}_2)} \\
\leq C\delta M. \quad (3.26)
$$
The terms related to $K_5$ admit the same bound as $K_4$

$$
\sum_{j \geq -1} 2^{(1+\frac{3}{2}-2\alpha)j} \int_{0}^{T} K_5 d\tau \\
\leq C \int_{0}^{T} \sum_{j \geq -1} 2^{(1+\frac{3}{2}-2\alpha)j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m \leq j-1} 2^{(\frac{3}{2}+1)m} \|\Delta_m u^{(n)}\|_{L^2} d\tau \\
\leq C \|u^{(n)}\|_{L^1(0,T;B^{\frac{1}{2}}_{2,1} + 2\alpha)} \|v^{(n)}\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1} + 2\alpha)} \\
\leq C\delta M. \quad (3.27)
$$

For the term with $K_6$ we write

$$
\sum_{j \geq -1} 2^{(1+\frac{3}{2}-2\alpha)j} \int_{0}^{T} K_6 d\tau \\
\leq C \int_{0}^{T} \sum_{j \geq -1} 2^{(1+\frac{3}{2}-2\alpha)j} 2^{j} \sum_{k \geq j-4} 2^{\frac{3}{2}k} \|\Delta_k v^{(n)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} d\tau \\
\leq C \|u^{(n)}\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1} + 2\alpha)} \|u^{(n)}\|_{L^1(0,T;B^{\frac{1}{2}}_{2,1} + 2\alpha)} \\
\leq C\delta M. \quad (3.28)
$$

The term with $K_7$ is bounded by

$$
\sum_{j \geq -1} 2^{(1+\frac{3}{2}-2\alpha)j} \int_{0}^{T} K_7 d\tau \\
\leq C \int_{0}^{T} \sum_{j \geq -1} 2^{(1+\frac{3}{2}-2\alpha)j} 2^{j} \|\Delta_j \theta^{(n)}\|_{L^2} d\tau \\
\leq C \int_{0}^{T} \sum_{j \geq -1} 2^{-(\alpha-1)j} 2^{(1+\frac{3}{2}-\alpha)j} \|\Delta_j \theta^{(n)}\|_{L^2} d\tau \\
\leq CT \|\theta^{(n)}\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1} + 2\alpha)} \leq CTM, \quad (3.29)
$$

where we need $\alpha \geq 1$. Collecting the estimates (3.23) – (3.29) and inserting them into (3.22), it holds that

$$
\|u^{(n+1)}\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1} + 2\alpha)} \leq \|v_0^{(n+1)}\|_{B^{\frac{1}{2}}_{2,1} + 2\alpha} + C\delta \|v^{(n)}\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1} + 2\alpha)} + C\delta M + CTM. \quad (3.30)
$$

### 3.1.3 The estimate of $\|\theta^{(n+1)}\|_{L^\infty(0,T;B^{\frac{1}{2}}_{2,1} + 2\alpha)}$

Let $j \geq -1$ be an integer. Applying $\Delta_j$ to (3.1)\textsubscript{3} and then dotting the equation with $\Delta_j \theta^{(n+1)}$, we have

$$
\frac{1}{2} \frac{d}{dt} \|\Delta_j \theta^{(n+1)}\|_{L^2}^2 = C_1 + C_2, \quad (3.31)
$$
where

\[
C_1 = -\int \Delta_j (u^{(n)} \cdot \nabla \theta^{(n+1)}) \cdot \Delta_j \theta^{(n+1)} \, dx,
\]

\[
C_2 = -\int \Delta_j (\nabla \cdot v^{(n)}) \cdot \Delta_j \theta^{(n+1)} \, dx.
\]

Making use of (2.3) in Lemma 2.1, it holds that

\[
|C_1| \leq C\|\Delta_j \theta^{(n+1)}\|_{L^2} \left( \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2} \sum_{|j-k| \leq 2} \|\Delta_k \theta^{(n+1)}\|_{L^2} \\
+ \sum_{|j-k| \leq 2} \|\Delta_k u^{(n)}\|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m \theta^{(n+1)}\|_{L^2} \\
+ \sum_{k \geq j-4} 2^j 2^{\frac{d}{2} k} \|\Delta_k u^{(n)}\|_{L^2} \|\Delta_k \theta^{(n+1)}\|_{L^2} \right). \quad (3.32)
\]

By Hölder’s inequality and Bernstein’s inequality, \(C_2\) can be bounded by

\[
|C_2| = \left| -\int \Delta_j (\nabla \cdot v^{(n)}) \cdot \Delta_j \theta^{(n+1)} \, dx \right| \\
\leq C\|\Delta_j \theta^{(n+1)}\|_{L^2} \|\Delta_j (\nabla \cdot v^{(n)})\|_{L^2} \\
\leq C2^j \|\Delta_j \theta^{(n+1)}\|_{L^2} \|\Delta_j v^{(n)}\|_{L^2}. \quad (3.33)
\]

Inserting the estimates (3.32) and (3.33) into the equality (3.31), then eliminating \(\|\Delta_j \theta^{(n+1)}\|_{L^2}\) from both sides of the inequality, we get

\[
\frac{d}{dt} \|\Delta_j \theta^{(n+1)}\|_{L^2} \leq I_1 + \cdots + I_4, \quad (3.34)
\]

where

\[
I_1 = C\|\Delta_j \theta^{(n+1)}\|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2},
\]

\[
I_2 = C\|\Delta_j u^{(n)}\|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m \theta^{(n+1)}\|_{L^2},
\]

\[
I_3 = C2^j \sum_{k \geq j-4} 2^{\frac{d}{2} k} \|\Delta_k u^{(n)}\|_{L^2} \|\Delta_k \theta^{(n+1)}\|_{L^2},
\]

\[
I_4 = C2^j \|\Delta_j v^{(n)}\|_{L^2}.
\]

Integrating (3.34) in time yields, for any \(t \leq T\)

\[
\|\Delta_j \theta^{(n+1)}\|_{L^2} \leq \|\Delta_j \theta_0^{(n+1)}\|_{L^2} + \int_0^T (I_1 + \cdots + I_4) \, d\tau. \quad (3.35)
\]

Taking the \(L^\infty(0, T)\) of (3.35), multiplying by \(2^{(1+\frac{d}{2} - \alpha)j}\) and summing over \(j\), one has

\[
\|\theta^{(n+1)}\|_{L^\infty(0, T; B^{1+\frac{d}{2} - \alpha}_{2, 1})} \leq \|\theta_0^{(n+1)}\|_{B^{1+\frac{d}{2} - \alpha}_{2, 1}} + \sum_{j \geq 1} 2^{(1+\frac{d}{2} - \alpha)j} \int_0^T (I_1 + \cdots + I_4) \, d\tau. \quad (3.36)
\]

We estimate the terms related to \(I_1 - I_4\) respectively. The first term can be estimated as

\[
\sum_{j \geq 1} 2^{(1+\frac{d}{2} - \alpha)j} \int_0^t I_1 \, d\tau \leq C \|\theta^{(n+1)}\|_{L^\infty(0, T; B^{1+\frac{d}{2} - \alpha}_{2, 1})} \|u^{(n)}\|_{L^1(0, T; B^{1+\frac{d}{2}}_{2, 1})} \leq C\delta \|\theta^{(n+1)}\|_{L^\infty(0, T; B^{1+\frac{d}{2} - \alpha}_{2, 1})}, \quad (3.37)
\]
The terms with $I_2$ and $I_3$ can be bounded similarly
\[ \sum_{j \geq 1} 2^{(1+\frac{d}{2}-\alpha)j} \int_0^t I_2 d\tau \leq C\delta \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}-\alpha}_{2,1})}, \tag{3.38} \]
\[ \sum_{j \geq 1} 2^{(1+\frac{d}{2}-\alpha)j} \int_0^t I_3 d\tau \leq C\delta \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-\alpha}_{2,1})}, \tag{3.39} \]

For the term with $I_4$ we arrive at
\[ \sum_{j \geq 1} 2^{(1+\frac{d}{2}-\alpha)j} \int_0^t I_4 d\tau \leq C \int_0^t \sum_{j \geq 1} 2^{(1+\frac{d}{2}-\alpha)j} 2^j \|\Delta_j v^{(n)}\|_{L^4 d\tau} \]
\[ \leq C\|v^{(n)}\|_{L^1(0,T;B^{\frac{d}{2}}_{2,1})} \leq C\delta. \tag{3.40} \]

Collecting the estimates (3.37) – (3.40) and inserting them in (3.36), one gets
\[ \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \leq \|\theta_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} + C\delta\|\theta^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-\alpha}_{2,1})} + C\delta. \tag{3.41} \]

Thus, combing the estimates (3.15), (3.30) and (3.41), we obtain
\[ \|(u^{(n+1)},v^{(n+1)})\|_{L^\infty(0,T;B^{\frac{d}{2}-2\alpha}_{2,1})} + \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-\alpha}_{2,1})} \leq \|(u_0^{(n+1)},v_0^{(n+1)})\|_{B^{\frac{d}{2}-2\alpha}_{2,1}} + \|\theta_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} + C\delta\|\theta^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-\alpha}_{2,1})} + C\delta M + C\delta + CT M. \]

Choosing $C\delta \leq \min\left(\frac{1}{8},\frac{M}{8}\right)$ and $CT \leq \frac{1}{8}$, we have
\[ \|(u^{(n+1)},v^{(n+1)})\|_{L^\infty(0,T;B^{\frac{d}{2}-2\alpha}_{2,1})} + \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-\alpha}_{2,1})} \leq \frac{M}{2} + \frac{1}{8} \|(u_0^{(n+1)},v_0^{(n+1)})\|_{L^\infty(0,T;B^{\frac{d}{2}-2\alpha}_{2,1})} + \frac{1}{8} \|\theta_0^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} + \frac{3M}{8}. \]

by simplification it follows
\[ \|(u^{(n+1)},v^{(n+1)})\|_{L^\infty(0,T;B^{\frac{d}{2}-2\alpha}_{2,1})} \leq M, \quad \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-\alpha}_{2,1})} \leq M. \]

According to the property of Chemin-Lerner type Besov spaces, it implies
\[ \|(u^{(n+1)},v^{(n+1)})\|_{L^\infty(0,T;B^{\frac{d}{2}-2\alpha}_{2,1})} \leq M, \quad \|\theta^{(n+1)}\|_{L^\infty(0,T;B^{1+\frac{d}{2}-\alpha}_{2,1})} \leq M. \]
3.2 The estimates of \( \| (u^{(n+1)}, v^{(n+1)}) \|_{L^1(0,T;B^{1+\frac{3}{2}}_2)} \)

3.2.1 The estimate of \( \| u^{(n+1)} \|_{L^1(0,T;B^{1+\frac{3}{2}}_2)} \)

We multiply (3.7) by \( 2^{(\frac{3}{2}+1)j} \), sum \( j \) over \( j \geq 0 \) and integrate in time on \([0,T]\) to obtain

\[
\sum_{j \geq 0} 2^{(1+\frac{3}{2})j} \| \Delta_j u^{(n+1)} \|_{L^1(0,T;L^2)} \leq \int_0^T \sum_{j \geq 0} 2^{(1+\frac{3}{2})j} \int_0^T e^{-c_0 2^{2\alpha_j} t} \| \Delta_j u_0^{(n+1)} \|_{L^2} dt \\
+ \int_0^T \sum_{j \geq 0} 2^{(1+\frac{3}{2})j} \int_0^T e^{-c_0 2^{2\alpha_j} (s-t)} (J_1 + \cdots + J_5) dr ds. \tag{3.42}
\]

Clearly

\[
\int_0^T \sum_{j \geq 0} 2^{(1+\frac{3}{2})j} e^{-c_0 2^{2\alpha_j} t} \| \Delta_j u_0^{(n+1)} \|_{L^2} = C \sum_{j \geq 0} 2^{(1+\frac{3}{2} - 2\alpha_j)} (1 - e^{-c_0 2^{2\alpha_j} T}) \| \Delta_j u_0^{(n+1)} \|_{L^2}.
\]

Since \( u_0 \in B^{1+\frac{3}{2}-2\alpha}_2 \), it follows from the Dominated Convergence Theorem that

\[
\lim_{T \to 0} \sum_{j \geq 0} 2^{(1+\frac{3}{2} - 2\alpha_j)} (1 - e^{-c_0 2^{2\alpha_j} T}) \| \Delta_j u_0^{(n+1)} \|_{L^2} = 0.
\]

For \( j = -1 \), we multiply (3.8) by \( 2^{-(1+\frac{3}{2})} \) and integrate in time on \([0,T]\) to get

\[
2^{-(1+\frac{3}{2})} \| \Delta_{-1} u^{(n+1)} \|_{L^1(0,T;L^2)} \leq 2^{-2\alpha} \int_0^T \sum_{j \geq 0} 2^{(1+\frac{3}{2} - 2\alpha)} \| \Delta_{-1} u_0^{(n+1)} \|_{L^2} dt \\
+ 2^{-2\alpha} \int_0^T 2^{-(1+\frac{3}{2} - 2\alpha)} \int_0^t (J_1 + \cdots + J_5) dr ds. \tag{3.43}
\]

Clearly

\[
2^{-2\alpha} \int_0^T 2^{-(1+\frac{3}{2} - 2\alpha)} \| \Delta_{-1} u_0^{(n+1)} \|_{L^2} dt \leq 2^{-2\alpha} T \| u_0^{(n+1)} \|_{B^{1+\frac{3}{2}-2\alpha}_2}.
\]

Therefore, we can choose \( T \) sufficiently small such that

\[
\int_0^T \sum_{j \geq 0} 2^{(1+\frac{3}{2})j} e^{-c_0 2^{2\alpha_j} t} \| \Delta_j u_0^{(n+1)} \|_{L^2} dt + 2^{-2\alpha} \int_0^T 2^{-(1+\frac{3}{2} - 2\alpha)} \| \Delta_{-1} u_0^{(n+1)} \|_{L^2} dt \leq \frac{\delta}{2}.
\]

Collecting (3.42) and (3.43), by Young’s inequality for the time convolution and the following fact

\[
\int_0^T e^{-c_0 2^{2\alpha_j} s} ds \leq C(1 - e^{-c_2 T}) 2^{-2\alpha_j},
\]

we arrive at

\[
\| u^{(n+1)} \|_{L^1(0,T;B^{1+\frac{3}{2}}_2)} \leq \frac{\delta}{2} + CT \int_0^T \sum_{j \geq -1} 2^{(\frac{3}{2}+1 - 2\alpha_j)} (J_1 + \cdots + J_5) dt. \tag{3.44}
\]
We estimate the terms involving $J_1$-$J_5$ nextly. Arguing similarly as deriving (3.10)-(3.14)

$$CT \int_0^T \sum_{j \geq -1} 2^{(1+\frac{3}{2}-2\alpha)j}(J_1 + \cdots + J_5) \, d\tau \leq CT\delta \|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{3}{2}-2\alpha})} + CT\delta M. \tag{3.45}$$

Inserting the estimate (3.45) into (3.44), we get

$$\|u^{(n+1)}\|_{L^1(0,T;B_{2,1}^{1+\frac{3}{2}})} \leq \frac{\delta}{2} + CT\delta M.$$ 

Choosing $T$ sufficiently small such that $CT \leq \frac{1}{2M}$, it holds that

$$\|u^{(n+1)}\|_{L^1(0,T;B_{2,1}^{1+\frac{3}{2}})} \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

### 3.2.2 The estimate of $\|\psi^{(n+1)}\|_{L^1(0,T;B_{2,1}^{1+\frac{3}{2}})}$

We multiply (3.21) by $2^{(1+\frac{3}{2})j}$, sum $j$ over $j \geq 0$ and integrate in time $t$ on $[0,T]$ to obtain

$$\sum_{j \geq 0} 2^{(1+\frac{3}{2})j}\|\psi^{(n+1)}\|_{L^1(0,T;L^2)} \leq \int_0^T \sum_{j \geq 0} 2^{(1+\frac{3}{2})j} e^{-c_02^{2\alpha}jt} \|\Delta_j \psi_0^{(n+1)}\|_{L^2} \, dt$$

$$+ \int_0^T \sum_{j \geq 0} 2^{(1+\frac{3}{2})j} \int_0^t e^{-c_02^{2\alpha}(s-t)}(K_1 + \cdots + K_7) \, ds \, dt. \tag{3.46}$$

Clearly

$$\int_0^T \sum_{j \geq 0} 2^{(1+\frac{3}{2})j} e^{-c_02^{2\alpha}jt} \|\Delta_j \psi_0^{(n+1)}\|_{L^2} \, dt = C \sum_{j \geq 0} 2^{(1+\frac{3}{2}-2\alpha)j}(1-e^{-c_02^{2\alpha}jT})\|\Delta_j \psi_0^{(n+1)}\|_{L^2}.$$

For $j = -1$, the method is similar to we did with $\Delta_{-1}u^{(n+1)}$. Then we have

$$\|\psi^{(n+1)}\|_{L^1(0,T;B_{2,1}^{1+\frac{3}{2}})} \leq \frac{\delta}{2} + CT \int_0^T \sum_{j \geq -1} 2^{(1+\frac{3}{2}-2\alpha)j}(K_1 + \cdots + K_7) \, d\tau. \tag{3.47}$$

The terms involving $K_1$-$K_7$ can be estimated as follows. Arguing similarly as deriving (3.23)-(3.29)

$$CT \int_0^T \sum_{j \geq -1} 2^{(1+\frac{3}{2}-2\alpha)j}(K_1 + \cdots + K_7) \, d\tau \leq CT\delta \|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{1+\frac{3}{2}-2\alpha})} + CT\delta M + CT^2 M. \tag{3.48}$$

Inserting (3.48) into (3.47), one gets

$$\|\psi^{(n+1)}\|_{L^1(0,T;B_{2,1}^{1+\frac{3}{2}})} \leq \frac{\delta}{2} + CT\delta M + CT^2 M.$$ 

Choosing $T$ sufficiently small such that $CT \leq \min\left(\frac{1}{4M}, \frac{\delta}{4T M}\right)$ we obtain

$$\|\psi^{(n+1)}\|_{L^1(0,T;B_{2,1}^{1+\frac{3}{2}})} \leq \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4} = \delta.$$
3.3 Proof of the existence part

The uniform bounds above allow us to extract a weakly convergent subsequence depending on \( T \). There exists \((u, v, \theta) \in Y\) such that a subsequence of \((u^n, v^n, \theta^n)\) (still denoted by \((u^n, v^n, \theta^n)\)) satisfies

\[
(u^n, v^n) \rightharpoonup (u, v) \quad \text{in} \quad L^\infty(0, T; B^{1+\frac{\delta}{2}-2\alpha}_{2,1}(\mathbb{R}^d)) \cap L^1(0, T; B^{1+\frac{\delta}{2}-\alpha}_{2,1}(\mathbb{R}^d)),
\]

\[
\theta^n \rightharpoonup \theta \quad \text{in} \quad L^\infty(0, T; B^{1+\frac{\delta}{2}-\alpha}_{2,1}(\mathbb{R}^d)),
\]

where \( \rightharpoonup \) denote the weak* convergence. Moreover, we can show by making use of the equation (3.1) that \((\partial_t u^n, \partial_t v^n, \partial_t \theta^n)\) is uniformly bounded

\[
\partial_t u^n \in L^1(0, T; B^{1+\frac{\delta}{2}-2\alpha}_{2,1}(\mathbb{R}^d)) \cap L^{\frac{8}{3}}(0, T; B^{1+\frac{\delta}{2}-\frac{8}{3}\alpha}_{2,1}(\mathbb{R}^d)), \tag{3.49}
\]

\[
\partial_t v^n \in L^1(0, T; B^{1+\frac{\delta}{2}-2\alpha}_{2,1}(\mathbb{R}^d)) \cap L^{\frac{8}{3}}(0, T; B^{1+\frac{\delta}{2}-\frac{8}{3}\alpha}_{2,1}(\mathbb{R}^d)), \tag{3.50}
\]

\[
\partial_t \theta^n \in L^2(0, T; B^{1+\frac{1}{2}-2\alpha}_{2,1}(\mathbb{R}^d)). \tag{3.51}
\]

For any positive integer \( m \), we denote \( B_m \) the ball in \( \mathbb{R}^d \) of radius \( m \) and centered at the origin. By Aubin-Lions Lemma, there exists a subsequence still denoted by \((u^n, v^n, \theta^n)\), has the following strongly convergent property,

\[
(u^n, v^n) \to (u, v) \quad \text{in} \quad L^2(0, T; B^{1\gamma}_{2,1}(B_m)), \quad \text{for} \quad 1 + \frac{d}{2} - \frac{8}{3}\alpha < \gamma_1 < 1 + \frac{d}{2} - \alpha,
\]

\[
\theta^n \to \theta \quad \text{in} \quad L^2(0, T; B^{1\gamma}_{2,1}(B_m)), \quad \text{for} \quad 1 + \frac{d}{2} - 2\alpha < \gamma_2 < 1 + \frac{d}{2} - \alpha.
\]

According to the Cantor diagonal argument in \( n \) and \( m \), there exists a subsequence still denoted by \((u^n, v^n, \theta^n)\), such that

\[
(u^n, v^n) \to (u, v) \quad \text{in} \quad L^2(0, T; B^{1}_{2,1}(\mathbb{R}^d)),
\]

\[
\theta^n \to \theta \quad \text{in} \quad L^2(0, T; B^{2}_{2,1}(\mathbb{R}^d)).
\]

This strong convergence property would allow us to show that \((u, v, \theta)\) is indeed a weak solution of (1.1), which completes the proof for the existence part of Theorem 1.1. \( \square \)

Remark 3.1. A sketch proof of the estimates (3.49), (3.50) and (3.51) is as follows.

Firstly, we prove \((\partial_t u^n, \partial_t v^n) \in L^1(0, T; B^{1+\frac{\delta}{2}-2\alpha}_{2,1}(\mathbb{R}^d))\). According to (3.1)

\[
\int_0^t \| \partial_t u^{(n+1)} \|_{B^{1+\frac{\delta}{2}-2\alpha}_{2,1}(\mathbb{R}^d)} \, d\tau \leq \int_0^t \| (\partial_t)^{\alpha} u^{(n+1)} \|_{B^{1+\frac{\delta}{2}-2\alpha}_{2,1}(\mathbb{R}^d)} \, d\tau + \int_0^t \| u^{(n)} \cdot \nabla u^{(n+1)} \|_{B^{1+\frac{\delta}{2}-2\alpha}_{2,1}(\mathbb{R}^d)} \, d\tau
\]

\[
+ \int_0^t \| \nabla \cdot (v^{(n)} \otimes u^{(n)}) \|_{B^{1+\frac{\delta}{2}-2\alpha}_{2,1}(\mathbb{R}^d)} \, d\tau,
\]

\[
\int_0^t \| \partial_t v^{(n+1)} \|_{B^{1+\frac{\delta}{2}-2\alpha}_{2,1}(\mathbb{R}^d)} \, d\tau \leq \int_0^t \| (\partial_t)^{\alpha} v^{(n+1)} \|_{B^{1+\frac{\delta}{2}-2\alpha}_{2,1}(\mathbb{R}^d)} \, d\tau + \int_0^t \| u^{(n)} \cdot \nabla v^{(n+1)} \|_{B^{1+\frac{\delta}{2}-2\alpha}_{2,1}(\mathbb{R}^d)} \, d\tau
\]

\[
+ \int_0^t \| v^{(n)} \cdot \nabla u^{(n)} \|_{B^{1+\frac{\delta}{2}-2\alpha}_{2,1}(\mathbb{R}^d)} \, d\tau + \int_0^t \| \nabla \theta^{(n)} \|_{B^{1+\frac{1}{2}-2\alpha}_{2,1}(\mathbb{R}^d)} \, d\tau.
\]
The estimation of the right hand side terms is similar, we take \( \int_0^t \| u^{(n)} \cdot \nabla u^{(n+1)} \|_{L^2} \) as an example to give the proof.

\[
\begin{align*}
\int_0^t \| u^{(n)} \cdot \nabla u^{(n+1)} \|_{L^2} \, d\tau & \leq \int_0^t \sum_{j \geq -1} 2^{(1+\frac{d}{2}-\frac{\alpha}{2})j} \left( \sum_{m \leq j-1} 2^{j2^m} \| \Delta_m u^{(n)} \|_{L^2} \right) \sum_{|j-k| \leq 2} \| \Delta_k u^{(n+1)} \|_{L^2} \\
& + \sum_{|j-k| \leq 2} \| \Delta_k u^{(n)} \|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \| \Delta_m u^{(n+1)} \|_{L^2} \\
& + \sum_{k \geq j-4} 2^{j2^k} \| \Delta_k u^{(n)} \|_{L^2} \| \Delta_k u^{(n+1)} \|_{L^2} \right) \, d\tau.
\end{align*}
\]

The method is similar to the estimates (3.10)-(3.12), we have

\[
\int_0^t \| u^{(n)} \cdot \nabla u^{(n+1)} \|_{L^2} \, d\tau \leq C \| u^{(n)} \|_{L^\infty(0,T;B^1_{2,1}+\frac{d}{2}-\frac{\alpha}{2})} \| u^{(n+1)} \|_{L^1(0,T;B^1_{2,1}+\frac{d}{2})} \leq C\delta M.
\]

Secondly, we prove \((\partial_t u^n, \partial_t v^n) \in L^{\frac{d}{2}}(0,T;B^1_{2,1}+\frac{d}{2}-\frac{\alpha}{2})\). The methods are the same as above, now we only give the estimate of nonlinear term \( \int_0^t \| \nabla \cdot (v^{(n)} \otimes v^{(n)}) \|_{L^2} \, d\tau \),

\[
\begin{align*}
\int_0^t \| \nabla \cdot (v^{(n)} \otimes v^{(n)}) \|_{L^2} \, d\tau & \leq \int_0^t \left( \sum_{j \geq -1} 2^{(1+\frac{d}{2}-\frac{\alpha}{2})j} \sum_{m \leq j-1} 2^{j2^m} \| \Delta_m v^{(n)} \|_{L^2} \right) \sum_{|j-k| \leq 2} \| \Delta_k v^{(n)} \|_{L^2} \\
& + \sum_{k \geq j-4} 2^{j2^k} \| \Delta_k v^{(n)} \|_{L^2} \| \Delta_k v^{(n)} \|_{L^2} \right) \, d\tau.
\end{align*}
\]

Respectively, the terms on the right hand side can be bounded as follows

\[
\begin{align*}
\int_0^t \left( \sum_{j \geq -1} 2^{(1+\frac{d}{2}-\frac{\alpha}{2})j} \sum_{m \leq j-1} 2^{j2^m} \| \Delta_m v^{(n)} \|_{L^2} \right) \sum_{|j-k| \leq 2} \| \Delta_k v^{(n)} \|_{L^2} \right) \, d\tau & \leq \int_0^t \left( \sum_{j \geq -1} \sum_{m \leq j-1} 2^{(2\alpha-1)(m-j)} 2^{(1+\frac{d}{2}-\frac{\alpha}{2})j} \| \Delta_j v^{(n)} \|_{L^2} \sum_{|j-k| \leq 2} \| \Delta_k v^{(n)} \|_{L^2} \right) \, d\tau \\
& \leq C \| v^{(n)} \|_{L^\infty(0,T;B^1_{2,1}+\frac{d}{2}-\frac{\alpha}{2})} \| v^{(n)} \|_{L^\infty(0,T;B^1_{2,1}+\frac{d}{2}-\frac{\alpha}{2})},
\end{align*}
\]

and

\[
\begin{align*}
\int_0^t \left( \sum_{j \geq -1} 2^{(1+\frac{d}{2}-\frac{\alpha}{2})j} \sum_{k \geq j-4} \| \Delta_k v^{(n)} \|_{L^2} \right) \, d\tau & \leq \int_0^t \left( \sum_{j \geq -1} \sum_{k \geq j-4} 2^{(2+\frac{d}{2}-\frac{\alpha}{2})j} \| \Delta_k v^{(n)} \|_{L^1} \right) \, d\tau \\
& \leq \int_0^t \left( \sum_{k \geq j-4} 2^{(2+\frac{d}{2}-\frac{\alpha}{2})(j-k)} 2^{(1+\frac{d}{2}-\frac{\alpha}{2})k} \| \Delta_k v^{(n)} \|_{L^2} \sum_{k \geq j-4} 2^{(1+\frac{d}{2}-\frac{\alpha}{2})k} \| \Delta_k v^{(n)} \|_{L^2} \right) \, d\tau \\
& \leq C \| v^{(n)} \|_{L^\infty(0,T;B^1_{2,1}+\frac{d}{2}-\frac{\alpha}{2})} \| v^{(n)} \|_{L^\infty(0,T;B^1_{2,1}+\frac{d}{2}-\frac{\alpha}{2})},
\end{align*}
\]

18
where we need \( \alpha < \frac{3}{4} + \frac{3d}{4} \). Collecting the estimates (3.53)-(3.54) and inserting them in (3.52), we know that \((u, v) \in L^\infty(0, T; B^{1+\frac{4}{d}-2\alpha}_{2,1}) \cap L^1(0, T; B^{2+\frac{4}{d}-2\alpha}_{2,1})\), it implies

\[
\int_0^t \| \nabla \cdot (v^{(n)} \otimes v^{(n)}) \|_{B^{1+\frac{4}{d}-2\alpha}_{2,1}} \, d\tau \leq C \| v^{(n)} \|_{L^\infty(0, T; B^{1+\frac{4}{d}-2\alpha}_{2,1})} \| v^{(n)} \|_{L^1(0, T; B^{2+\frac{4}{d}-2\alpha}_{2,1})} \leq C \delta M^2,
\]

where we have used the following interpolation relation

\[
\| f \|_{L^2(0, T; B^{\frac{4}{d}-\alpha}_{2,1})} \leq C \| f \|_{L^\infty(0, T; B^{1+\frac{4}{d}-2\alpha}_{2,1})} \| f \|_{L^1(0, T; B^{2+\frac{4}{d}-2\alpha}_{2,1})}.
\]

Therefore, it completes the estimate of \( \partial_t u^n \). For \( \int_0^t \| \partial_t v^{(n+1)} \|_{B^{1+\frac{4}{d}-\alpha}_{2,1}} \, d\tau \), there have a linear term \( \int_0^t \| \nabla \theta^{(n)} \|_{B^{1+\frac{4}{d}-\alpha}_{2,1}} \, d\tau \) on the right hand side, it can be bounded by

\[
\int_0^t \| \nabla \theta^{(n)} \|_{B^{1+\frac{4}{d}-\alpha}_{2,1}} \, d\tau \leq \int_0^t \| \theta^{(n)} \|_{B^{2+\frac{4}{d}-\alpha}_{2,1}} \, d\tau
\]

\[
\leq \int_0^t \left( \sum_{j \geq -1} 2^{(2j+\frac{4}{d}-\alpha)} \| \Delta_j \theta^{(n)} \|_{L^2} \right) \frac{\delta \tau}{2} \leq C \| \theta^{(n)} \|_{L^\infty(0, T; B^{1+\frac{4}{d}-\alpha}_{2,1})} < \infty,
\]

where we need \( \alpha > \frac{3d}{4} \). Finally, we prove \( \partial_t \theta^n \in L^2(0, T; B^{1+\frac{4}{d}-2\alpha}_{2,1})\).

\[
\int_0^t \| \partial_t \theta^{(n+1)} \|_{B^{1+\frac{4}{d}-2\alpha}_{2,1}} \, d\tau \leq \int_0^t \| u^{(n)} \cdot \nabla \theta^{(n+1)} \|_{B^{1+\frac{4}{d}-2\alpha}_{2,1}} \, d\tau + \int_0^t \| \nabla \cdot v^{(n)} \|_{B^{2+\frac{4}{d}-2\alpha}_{2,1}} \, d\tau.
\]

For \( 1 - \alpha \leq 0 \),

\[
\int_0^t \| \nabla \cdot v^{(n)} \|_{B^{2+\frac{4}{d}-2\alpha}_{2,1}} \, d\tau \leq C \int_0^t \| v^{(n)} \|_{B^{1+\frac{4}{d}-\alpha}_{2,1}} \, d\tau \leq C \delta M,
\]

where we have used the interpolation between \( \| v^{(n)} \|_{L^\infty(0, T; B^{1+\frac{4}{d}-\alpha}_{2,1})} \) and \( \| v^{(n)} \|_{L^1(0, T; B^{1+\frac{4}{d}-\alpha}_{2,1})} \).

Now, we give the estimate of \( \int_0^t \| u^{(n)} \cdot \nabla \theta^{(n+1)} \|_{B^{1+\frac{4}{d}-2\alpha}_{2,1}} \, d\tau \).

\[
\int_0^t \| u^{(n)} \cdot \nabla \theta^{(n+1)} \|_{B^{1+\frac{4}{d}-2\alpha}_{2,1}} \, d\tau
\]

\[
\leq \int_0^t \left( \sum_{j \geq -1} 2^{(1+\frac{4}{d}-2\alpha)} \| \Delta_j u^{(n)} \|_{L^2} \sum_{m \leq j-1} \| \Delta_m \theta^{(n+1)} \|_{L^2} \right) \frac{\delta \tau}{2}
\]

\[
+ \sum_{|j-k| \leq 2} \| \Delta_k u^{(n)} \|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{4}{d}-2\alpha)} \| \Delta_m \theta^{(n+1)} \|_{L^2}
\]

\[
+ \sum_{k \geq j-4} 2^{j+\frac{4}{d}} \| \Delta_k u^{(n)} \|_{L^2} \| \Delta_k \theta^{(n+1)} \|_{L^2} \right) \leq \int_0^t \frac{\delta \tau}{2}.
\]

(3.55)
In this section we prove the uniqueness of the solutions constructed in Subsection 3.4. Uniqueness of weak solutions

The terms on the right hand side can be bounded respectively by

\[
\int_0^t \left( \sum_{j \geq -1} 2^{(1+\frac{4}{d}-2\alpha)j} \sum_{m \leq j-1} 2^j \frac{1}{2} \frac{1}{2} \| \Delta_m u^{(n)} \|_{L^2} \sum_{|j-k| \leq 2} \| \Delta_k \theta^{(n+1)} \|_{L^2} \right)^2 \, d\tau
\]
\[
\leq \int_0^t \left( \sum_{j \geq -1} \sum_{m \leq j-1} 2^{(\alpha-1)(m-j)} 2^{(1+\frac{4}{d}-\alpha)j} \| \Delta_j \theta^{(n+1)} \|_{L^2} 2^{(1+\frac{4}{d}-\alpha)m} \| \Delta_m u^{(n)} \|_{L^2} \right)^2 \, d\tau
\]
\[
\leq C \| u^{(n)} \|_{L^2(0,T;B_{2,1}^{1+\frac{4}{d}-\alpha})}^2 \| \theta^{(n+1)} \|_{L^\infty(0,T;B_{2,1}^{1+\frac{4}{d}-\alpha})}^2,
\]

(3.56)

\[
\int_0^t \left( \sum_{j \geq -1} 2^{(1+\frac{4}{d}-2\alpha)j} \sum_{k \geq j-1} 2^j \frac{1}{2} \frac{1}{2} \| \Delta_k u^{(n)} \|_{L^2} \| \Delta_k \theta^{(n+1)} \|_{L^2} \right)^2 \, d\tau
\]
\[
\leq \int_0^t \left( \sum_{j \geq -1} \sum_{k \geq j-1} 2^{(1+\frac{4}{d}-2\alpha)(j-k)} 2^{(1+\frac{4}{d}-\alpha)k} \| \Delta_k u^{(n)} \|_{L^2} 2^{(1+\frac{4}{d}-\alpha)k} \| \Delta_k \theta^{(n+1)} \|_{L^2} \right)^2 \, d\tau
\]
\[
\leq C \| u^{(n)} \|_{L^2(0,T;B_{2,1}^{1+\frac{4}{d}-\alpha})}^2 \| \theta^{(n+1)} \|_{L^\infty(0,T;B_{2,1}^{1+\frac{4}{d}-\alpha})}^2,
\]

(3.57)

\[
\int_0^t \| u^{(n)} \cdot \nabla \theta^{(n+1)} \|_{B_{2,1}^{1+\frac{4}{d}-2\alpha}}^2 \, d\tau \leq C \| u^{(n)} \|_{L^2(0,T;B_{2,1}^{1+\frac{4}{d}-\alpha})}^2 \| \theta^{(n+1)} \|_{L^\infty(0,T;B_{2,1}^{1+\frac{4}{d}-\alpha})}^2
\]
\[
\leq C \delta^2 M^2.
\]

To sum up, we proved that \((\partial_t u^n, \partial_t v^n, \partial_t \theta^n)\) is uniformly bounded

\[
\partial_t u^n \in L^1(0,T;B_{2,1}^{1+\frac{4}{d}-\alpha}) \cap L^\infty(0,T;B_{2,1}^{1+\frac{4}{d}-\alpha}),
\]
\[
\partial_t v^n \in L^1(0,T;B_{2,1}^{1+\frac{4}{d}-\alpha}) \cap L^\infty(0,T;B_{2,1}^{1+\frac{4}{d}-\alpha}),
\]
\[
\partial_t \theta^n \in L^2(0,T;B_{2,1}^{1+\frac{4}{d}-2\alpha}).
\]

3.4 Uniqueness of weak solutions

In this section we prove the uniqueness of the solutions constructed in Subsection 3.3.

Proof. Assume that \((u_1, v_1, \theta_1)\) and \((u_2, v_2, \theta_2)\) are two solutions of the TCM equations (3.1) with the same initial data \((u_0, v_0, \theta_0)\). Denote

\[
(u, v, \theta) = (u_1 - u_2, v_1 - v_2, \theta_1 - \theta_2),
\]
then \((\tilde{u}, \tilde{v}, \tilde{\theta})\) satisfies

\[
\begin{aligned}
\partial_t \tilde{u} + \mu(-\Delta)\tilde{u} &= -P[(u_2 \cdot \nabla \tilde{v} + \tilde{u} \cdot \nabla u_1) + \nabla \cdot (v_2 \otimes \tilde{v}) + \nabla \cdot (\tilde{v} \otimes v_1)], \\
\partial_t \tilde{v} + \nu(-\Delta)\tilde{v} &= -(u_2 \cdot \nabla \tilde{v} + \tilde{u} \cdot \nabla v_1) - \nabla \tilde{\theta} - (v_2 \cdot \nabla \tilde{u} + \tilde{v} \cdot \nabla u_1), \\
\partial_t \tilde{\theta} &= -(u_2 \cdot \nabla \tilde{\theta} + \tilde{u} \cdot \nabla \theta_1) - \nabla \cdot \tilde{v}, \\
\nabla \cdot \tilde{u} &= 0.
\end{aligned}
\] (3.59)

Let \(j \geq -1\) be an integer (for \(j = -1\), the method is same as deriving (3.43)-(3.44)), applying \(\Delta_j\) to (3.59), and dotting the equation with \(\Delta_j \tilde{u}\), we have

\[
\frac{1}{2} \frac{d}{dt}\|\Delta_j u^{(n+1)}\|^2_{L^2} + \mu \|\Lambda^\alpha \Delta_j u^{(n+1)}\|^2_{L^2} = -\int \Delta_j (u_2 \cdot \nabla \tilde{u}) \cdot \Delta_j \tilde{u} dx - \int \Delta_j (\tilde{u} \cdot \nabla u_1) \cdot \Delta_j \tilde{u} dx + \int \Delta_j (\nabla \cdot (v_2 \otimes \tilde{v})) \cdot \Delta_j \tilde{u} dx + \int \Delta_j (\nabla \cdot (\tilde{v} \otimes v_1)) \cdot \Delta_j \tilde{u} dx.
\]

Thanks to (2.2)-(2.4) of Lemma 2.1 and eliminating \(\|\Delta_j \tilde{u}\|_{L^2}\) from both sides of the inequality, then integrating in time, we arrive at

\[
\|\Delta_j \tilde{u}\|_{L^2} \leq \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} \left( \tilde{A}_1 + \|\Delta_j (\tilde{u} \cdot \nabla u_1)\|_{L^2} \right) d\tau, \quad (3.60)
\]

where

\[
\tilde{A}_1 = C \left( \|\Delta_j \tilde{u}\|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{p})m} \|\Delta_j u_2\|_{L^2} + \|\Delta_j u_2\|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{p})m} \|\Delta_j \tilde{u}\|_{L^2} \right. \\
\left. + 2^j \sum_{k \geq j-4} 2^{\frac{d}{p} k} \|\Delta_k u_2\|_{L^2} \|\Delta_k \tilde{u}\|_{L^2} \right).
\]

Taking the \(L^p\)-norm in time, for \(1 \leq p < \infty\)

\[
\|\Delta_j \tilde{u}\|_{L^p_t(L^2)} \leq e^{-c_0 2^{2\alpha j}} \left( \|\tilde{A}_1\|_{L^1} + \|\Delta_j (\tilde{u} \cdot \nabla u_1)\|_{L^1_t(L^2)} \right. \\
\left. + \|\Delta_j (\nabla \cdot (v_2 \otimes \tilde{v}))\|_{L^1_t(L^2)} + \|\Delta_j (\nabla \cdot (\tilde{v} \otimes v_1))\|_{L^1_t(L^2)} \right), \quad (3.61)
\]

where we have used Young’s inequality for the time convolution. Multiplying (3.61) by \(2^{(\frac{d}{p} - 2\alpha p)j}\) and taking the supremum with respect to \(j\), we have

\[
\|\tilde{u}\|_{L^p_t(B^\frac{d}{p} - 2\alpha p + \frac{2\alpha}{p}_s)} \leq C \sup_j 2^{(\frac{d}{p} - 2\alpha)j} \left( \|\tilde{A}_1\|_{L^1} + \|\Delta_j (\tilde{u} \cdot \nabla u_1)\|_{L^1_t(L^2)} \right. \\
\left. + \|\Delta_j (\nabla \cdot (v_2 \otimes \tilde{v}))\|_{L^1_t(L^2)} + \|\Delta_j (\nabla \cdot (\tilde{v} \otimes v_1))\|_{L^1_t(L^2)} \right). \quad (3.62)
\]

Thanks to the product estimates (3.10)-(3.12), it can be derived

\[
\sup_j 2^{(\frac{d}{p} - 2\alpha)j} \|\tilde{A}_1\|_{L^1} \leq C \|u_2\|_{L^1_t(B^\frac{d}{p} + \frac{2\alpha}{p})} \|\tilde{u}\|_{L^\infty_t(B^\frac{d}{p} - 2\alpha)}, \\
\sup_j 2^{(\frac{d}{p} - 2\alpha)j} \|\Delta_j (\tilde{u} \cdot \nabla u_1)\|_{L^1_t(L^2)} \leq C \|u_1\|_{L^1_t(B^\frac{d}{p} + \frac{2\alpha}{p})} \|\tilde{u}\|_{L^\infty_t(B^\frac{d}{p} - 2\alpha)}.
\]
Similar as the product estimates (3.13)-(3.14), we obtain

\[
\sup_j 2^{(\frac{d}{2}-2\alpha)j} \| \Delta_j (\nabla \cdot (v_2 \otimes \tilde{v})) \|_{L^1_t(L^2)} \leq C \| v_2 \|_{L^2_t(B^{1+\frac{d}{2}-\alpha}_{2,1})} \| \tilde{v} \|_{L^\infty_t(B^{\frac{d}{2}-\alpha}_{2,\infty})},
\]

\[
\sup_j 2^{(\frac{d}{2}-2\alpha)j} \| \Delta_j (\nabla \cdot (\tilde{v} \otimes v_1)) \|_{L^1_t(L^2)} \leq C \| v_1 \|_{L^2_t(B^{1+\frac{d}{2}-\alpha}_{2,1})} \| \tilde{v} \|_{L^\infty_t(B^{\frac{d}{2}-\alpha}_{2,\infty})}.
\]

Then taking \( p = \infty, p = 1 \) and \( p = 2 \) in (3.62), we infer from the estimates above

\[
\| \tilde{v} \|_{L^\infty_t(B^{\frac{d}{2}-2\alpha}_{2,\infty})} + \| \tilde{v} \|_{L^1_t(B^{\frac{d}{2}}_{2,\infty})} + \| \tilde{v} \|_{L^\infty_t(B^{\frac{d}{2}-\alpha}_{2,\infty})} 
\leq C \left( \left\| \left( u_1, u_2 \right) \right\|_{L^1_t(B^{1+\frac{d}{2}}_{2,1})} \| \tilde{v} \|_{L^\infty_t(B^{\frac{d}{2}-2\alpha}_{2,\infty})} + \left\| \left( v_1, v_2 \right) \right\|_{L^1_t(B^{1+\frac{d}{2}-\alpha}_{2,1})} \| \tilde{v} \|_{L^\infty_t(B^{\frac{d}{2}-\alpha}_{2,\infty})} \right),
\]

(3.63)

By similar arguments as deriving (3.60) we arrive at

\[
\| \Delta_j \tilde{v} \|_{L^2} \leq \int_0^t e^{-c_0 2^{2\alpha_j}(t-s)} \left( \tilde{B}_1 + \| \Delta_j (\tilde{u} \cdot \nabla v_1) \|_{L^1_t(L^2)} + \| \Delta_j (v_2 \cdot \nabla \tilde{u}) \|_{L^1_t(L^2)} 
+ \| \Delta_j (\tilde{v} \cdot \nabla u_1) \|_{L^1_t(L^2)} + \| \Delta_j (\nabla \tilde{\theta}) \|_{L^1_t(L^2)} \right),
\]

where

\[
\tilde{B}_1 = C \left( \| \Delta_j \tilde{v} \|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \| \Delta_m u_2 \|_{L^2} + \| \Delta_j u_2 \|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \| \Delta_m \tilde{v} \|_{L^2} 
+ 2^j \sum_{k \geq j-4} 2^{\frac{d}{2}k} \| \Delta_k u_2 \|_{L^2} \| \Delta_k \tilde{v} \|_{L^2} \right).
\]

Taking the \( L^p \)-norm in time, for \( 1 \leq p \leq \infty \). And applying Young's inequality for the time convolution to get

\[
\| \Delta_j \tilde{v} \|_{L^p_t(L^2)} \leq e^{-c_0 2^{2\alpha_j} t} \| L^p \left( \| \tilde{B}_1 \|_{L^1} + \| \Delta_j (\tilde{u} \cdot \nabla v_1) \|_{L^1_t(L^2)} + \| \Delta_j (v_2 \cdot \nabla \tilde{u}) \|_{L^1_t(L^2)} 
+ \| \Delta_j (\tilde{v} \cdot \nabla u_1) \|_{L^1_t(L^2)} + \| \Delta_j (\nabla \tilde{\theta}) \|_{L^1_t(L^2)} \right). \]

(3.64)

Multiplying (3.64) by \( 2^{(\frac{d}{2}-2\alpha_j+\frac{2\alpha}{p})j} \) and taking the supremum with respect to \( j \), we deduce

\[
\| \tilde{v} \|_{L^p_t(B^{\frac{d}{2}-2\alpha}+\frac{2\alpha}{p}_{2,\infty})} \leq C \sup_j 2^{(\frac{d}{2}-2\alpha_j)j} \left( \| \tilde{B}_1 \|_{L^1_t} + \| \Delta_j (\tilde{u} \cdot \nabla v_1) \|_{L^1_t(L^2)} + \| \Delta_j (v_2 \cdot \nabla \tilde{u}) \|_{L^1_t(L^2)} 
+ \| \Delta_j (\tilde{v} \cdot \nabla u_1) \|_{L^1_t(L^2)} + \| \Delta_j (\nabla \tilde{\theta}) \|_{L^1_t(L^2)} \right). \]

(3.65)

The estimates of the first two terms on the right hand side of (3.65) are bounded as follows

\[
\sup_j 2^{(\frac{d}{2}-2\alpha_j)j} \| \tilde{B}_1 \|_{L^1_t} \leq C \| u_2 \|_{L^2_t(B^{1+\frac{d}{2}}_{2,1})} \| \tilde{v} \|_{L^\infty_t(B^{\frac{d}{2}-2\alpha}_{2,\infty})},
\]

\[
\sup_j 2^{(\frac{d}{2}-2\alpha_j)j} \| \Delta_j (\tilde{u} \cdot \nabla v_1) \|_{L^1_t(L^2)} \leq C \| v_1 \|_{L^2_t(B^{1+\frac{d}{2}}_{2,1})} \| \tilde{u} \|_{L^\infty_t(B^{\frac{d}{2}-\alpha}_{2,\infty})},
\]

Thanks to the product estimates (3.26)-(3.28), it holds that

\[
\sup_j 2^{(\frac{d}{2}-2\alpha_j)j} \| \Delta_j (v_2 \cdot \nabla \tilde{u}) \|_{L^1_t(L^2)} \leq C \| v_2 \|_{L^2_t(B^{1+\frac{d}{2}-\alpha}_{2,1})} \| \tilde{u} \|_{L^\infty_t(B^{\frac{d}{2}-\alpha}_{2,\infty})},
\]

\[
\sup_j 2^{(\frac{d}{2}-2\alpha_j)j} \| \Delta_j (\tilde{v} \cdot \nabla u_1) \|_{L^1_t(L^2)} \leq C \| u_1 \|_{L^2_t(B^{1+\frac{d}{2}-\alpha}_{2,1})} \| \tilde{v} \|_{L^\infty_t(B^{\frac{d}{2}-\alpha}_{2,\infty})}.
\]
The last term can be bounded by
\[ \sup_j 2^{(\frac{d}{2} - 2\alpha)j} \| \Delta_j (\nabla \tilde{\theta}) \|_{L^1_t(B_{2,\infty}^d)} \leq C \| \tilde{\theta} \|_{L^1_t(B_{2,\infty}^{d+\frac{d}{2} - 2\alpha})}. \]

Then taking \( p = \infty, p = 1 \) and \( p = 2 \) in (3.65), we infer from the estimates above
\[
\begin{align*}
\| \tilde{v} \|_{L^1_t(B_{2,\infty}^{d+\frac{d}{2} - 2\alpha})} + \| \tilde{v} \|_{L^1_t(B_{2,\infty}^{d+\frac{d}{2} - 2\alpha})} + \| \tilde{v} \|_{L^1_t(B_{2,\infty}^{d+\frac{d}{2} - 2\alpha})} \\
\leq C \left( \|(u_1, u_2, v_1)\|_{L^1_t(B_{2,1}^{d+\frac{d}{2} - 2\alpha})} \|(\tilde{u}, \tilde{v})\|_{L^1_t(B_{2,\infty}^{d+\frac{d}{2} - 2\alpha})} \\
+ \|v_2\|_{L^2_t(B_{2,1}^{d+\frac{d}{2} - 2\alpha})} \|\tilde{u}\|_{L^2_t(B_{2,\infty}^{d+\frac{d}{2} - 2\alpha})} + C \|\tilde{\theta}\|_{L^1_t(B_{2,\infty}^{d+\frac{d}{2} - 2\alpha})} \right). \tag{3.66}
\end{align*}
\]

Taking \( T_1 \) small enough such that, for any \( 0 < t \leq T_1 \),
\[
C \left( \|(u_1, u_2, v_1)\|_{L^1_t(B_{2,1}^{d+\frac{d}{2} - 2\alpha})} + \|(v_1, v_2)\|_{L^2_t(B_{2,1}^{d+\frac{d}{2} - 2\alpha})} \right) \leq \frac{1}{4}. \tag{3.67}
\]

Thus, from (3.63), (3.66) and (3.67), it yields
\[
\| \tilde{u} \|_{L^1_t(B_{2,\infty}^{d+\frac{d}{2} - 2\alpha})} + \| \tilde{v} \|_{L^1_t(B_{2,\infty}^{d+\frac{d}{2} - 2\alpha})} \leq C \| \tilde{\theta} \|_{L^1_t(B_{2,\infty}^{d+\frac{d}{2} - 2\alpha})}. \tag{3.68}
\]

Nextly, we have to estimate \( \| \tilde{\theta} \|_{L^1_t(B_{2,\infty}^{d+\frac{d}{2} - 2\alpha})} \). As in (3.35), we have
\[
\| \Delta_j \tilde{\theta} \|_{L^2_t} \leq \int_0^t \left( \tilde{C}_1 + \| \Delta_j (\tilde{u} \cdot \nabla \theta_1) \|_{L^2_t} + \| \Delta_j (\nabla \cdot \tilde{v}) \|_{L^2_t} \right) d\tau, \tag{3.69}
\]
where
\[
\tilde{C}_1 = C \left( \| \Delta_j \tilde{\theta} \|_{L^2_t} \sum_{m \leq j - 1} 2^{(1 + \frac{d}{2})m} \| \Delta_m u_2 \|_{L^2_t} + \| \Delta_j u_2 \|_{L^2_t} \sum_{m \leq j - 1} 2^{(1 + \frac{d}{2})m} \| \Delta_m \tilde{\theta} \|_{L^2_t} \\
+ 2^j \sum_{k \geq j - 4} 2^{\frac{d}{2}k} \| \Delta_k u_2 \|_{L^2_t} \| \Delta_k \tilde{\theta} \|_{L^2_t} \right).
\]

Multiplying (3.69) by \( 2^{(1 + \frac{d}{2} - 2\alpha)j} \) and taking the supremum with respect to \( j \), it can be derived
\[
\| \tilde{\theta} \|_{B_{2,\infty}^{d+\frac{d}{2} - 2\alpha}} \leq C \sup_j 2^{(1 + \frac{d}{2} - 2\alpha)j} \left( \| \tilde{C}_1 \|_{L^1_t} + \| \Delta_j (\tilde{u} \cdot \nabla \theta_1) \|_{L^1_t(L^2)} + \| \Delta_j (\nabla \cdot \tilde{v}) \|_{L^1_t(L^2)} \right). \tag{3.70}
\]

The first two terms on the right hand side can be bounded as
\[
\begin{align*}
\sup_j 2^{(1 + \frac{d}{2} - 2\alpha)j} \| \tilde{C}_1 \|_{L^1_t(B_{2,1}^{d+\frac{d}{2} - 2\alpha})} & \leq C \| u_2 \|_{L^1_t(B_{2,1}^{d+\frac{d}{2} - 2\alpha})} \| \tilde{\theta} \|_{L^\infty_t(B_{2,\infty}^{d+\frac{d}{2} - 2\alpha})}, \\
\sup_j 2^{(1 + \frac{d}{2} - 2\alpha)j} \| \Delta_j (\tilde{u} \cdot \nabla \theta_1) \|_{L^1_t(L^2)} & \leq C \| \theta_1 \|_{L^\infty_t(B_{2,1}^{d+\frac{d}{2} - 2\alpha})} \| \tilde{u} \|_{L^1_t(B_{2,1}^{d+\frac{d}{2} - 2\alpha})}.
\end{align*}
\]

The last term on the right of (3.70) can be bounded by
\[
\begin{align*}
\sup_j 2^{(1 + \frac{d}{2} - 2\alpha)j} \| \Delta_j (\nabla \cdot \tilde{v}) \|_{L^1_t(L^2)} & \leq C \| \tilde{v} \|_{L^1_t(B_{2,1}^{d+\frac{d}{2} - 2\alpha})}.
\end{align*}
\]

Noting the (3.67) we have
\[
\| \tilde{\theta} \|_{B_{2,\infty}^{d+\frac{d}{2} - 2\alpha}} \leq C \| \theta_1 \|_{L^\infty_t(B_{2,1}^{d+\frac{d}{2} - 2\alpha})} \| \tilde{u} \|_{L^1_t(B_{2,1}^{d+\frac{d}{2} - 2\alpha})} + C \| \tilde{v} \|_{L^1_t(B_{2,1}^{d+\frac{d}{2} - 2\alpha})}. \tag{3.71}
\]

23
Making use of Lemma 2.2, one gets
\[
\|\tilde{u}\|_{L^1_t(L^\frac{d}{d+1}(B_{2,1}^d,\infty))} \leq C \|\tilde{u}\|_{L^1_t(B_{2,1}^d,\infty)} \log \left( e + \frac{\|\tilde{u}\|_{L^1_t(B_{2,1}^{d+1}}}{\|\tilde{u}\|_{L^1_t(B_{2,1}^d,\infty)}} \right),
\]
\[
\|\tilde{v}\|_{L^1_t(L^\frac{d}{d+1}(B_{2,1}^d,\infty))} \leq C \|\tilde{v}\|_{L^1_t(B_{2,1}^d,\infty)} \log \left( e + \frac{\|\tilde{v}\|_{L^1_t(B_{2,1}^{d+1}}}{\|\tilde{v}\|_{L^1_t(B_{2,1}^d,\infty)}} \right).
\]

Combining (3.68) and (3.71) we arrive at
\[
\|(\tilde{u}, \tilde{v})\|_{L^1_t(B_{2,1}^{d+1}(B_{2,1}^d,\infty))} \leq CM \int_0^t \|(\tilde{u}, \tilde{v})\|_{L^1_t(B_{2,1}^d,\infty)} \, d\tau \leq CM \int_0^t \|(\tilde{u}, \tilde{v})\|_{L^1_t(B_{2,1}^{d+1}(B_{2,1}^d,\infty))} \log \left( e + \frac{\|(\tilde{u}, \tilde{v})\|_{L^1_t(B_{2,1}^{d+1}(B_{2,1}^d,\infty))}}{\|(\tilde{u}, \tilde{v})\|_{L^1_t(B_{2,1}^d,\infty)}} \right) d\tau.
\]
Noticing that \(\|(\tilde{u}, \tilde{v})\|_{L^1_t(B_{2,1}^{d+1}(B_{2,1}^d,\infty))} < \infty\) and \(\int_0^a \frac{d\tau}{\tau \log(e+C\tau)} = \infty\) (0 < a < 1), applying the Osgood Lemma 2.3, for any \(t \leq T_1\), it can be derived
\[
\|\tilde{u}\|_{L^1_t(B_{2,1}^{d+1}(B_{2,1}^d,\infty))} = \|\tilde{v}\|_{L^1_t(B_{2,1}^{d+1}(B_{2,1}^d,\infty))} = 0,
\]
and by (3.71), it follows \(\|\tilde{\theta}\|_{B_{1,\infty}^{1+\frac{d}{d-2\alpha}}} = 0\), which completes the proof of the uniqueness part of Theorem 1.1.

Acknowledgements The research of B Yuan was partially supported by the National Natural Science Foundation of China (No. 11471103).

References

[1] Frierson D, Majda A, Pauluis O. Large scale dynamics of precipitation fronts in the tropical atmosphere: a novel relaxation limit. Commun Math. Sci., 2004, 2: 591-626, doi: 10.4310/CMS.2004.v2.n4.a3

[2] Fan J S, Alzahrani F, Hayat T, Nakamura G, Zhou Y. Global regularity for the 2D liquid crystal model with mixed partial viscosity. Anal. Appl., 2015, 13: 185-200, doi: 10.1142/S0219530514500481

[3] Fan J S, Ozawa T. Regularity criterion for the incompressible viscoelastic fluid system. Houston J. Math, 2011, 37: 627-636, doi: 10.1515/GMJ.2011.0002

[4] Hopf E. Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. Math Nachr., 1951, 4: 213-231.

[5] Ericksen J L. Conservation laws for liquid crystals. Trans. Soc. Rheol., 961, 5: 23-34, doi: 10.1122/1.548883

[6] Kato T, Fujita H. On the nonstationary Navier-Stokes system. Rend. Semi. Mat. Univ. Padova, 1962, 32: 243-260.

[7] Giga Y, Miyakawa T. Solutions in \(L^r\) of the Navier-Stokes initial value problem. Arch. Rational Mech. Anal., 1985, 89: 267-281, doi: 10.1007/BF00276875
[8] Cao C S, Wu J H. Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. *Adv. Math.*, 2011, 226: 1803-1822, doi: 10.1016/j.aim.2010.08.017

[9] Cao C S, Regmi D, Wu J H. The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion. *J. Differential Equations*, 2013, 254: 2661-2681, doi: 10.1016/j.jde.2013.02.016

[10] Tran C V, Yu X W, Zhai Z C. On global regularity of 2D generalized magnetohydrodynamic equations. *J. Differential Equations*, 2013, 254: 4194-4216, doi: 10.1016/j.jde.2013.01.002

[11] Jia X J, Zhou Y. Regularity criteria for the 3D MHD equations involving partial components. *Nonlinear Anal. Real World Appl.*, 2012, 13: 410-418, doi: 10.1016/j.nonrwa.2011.07.055

[12] He C, Xin Z P. On the regularity of weak solutions to the magnetohydrodynamic equations. *J. Differential Equations*, 2005, 213: 235-254, doi: 10.1016/j.jde.2004.07.002

[13] Li J K, Titi E S. Global well-posedness of strong solutions to a tropical climate model. *Discrete Contin. Dyn. Syst.*, 2016, 36: 4495-4516, doi: 10.3934/dcds.2016.36.4495

[14] Wan R H. Global small solutions to a tropical climate model without thermal diffusion. *J. Math. Phys.*, 2016, 57: 13pp, doi: 10.1063/1.4941039

[15] Dong B Q, Wang W J, Wu J H, Zhang H. Global regularity results for the climate model with fractional dissipation. *Discrete Contin. Dyn. Syst. Ser. B*, 2019, 24: 211-229, doi: 10.3934/dcdsb.2018102

[16] Ye Z. Global regularity of 2D tropical climate model with zero thermal diffusion. *ZAMM Z. Angew. Math. Mech.*, 2020, 100: 20pp, doi: 10.1002/zamm.201900132

[17] Ye Z. Global regularity for a class of 2D tropical climate model. *J. Math. Anal. Appl.*, 2016, 446: 307-321, doi: 10.1016/j.jmaa.2016.08.053

[18] Dong B Q, Wu J H, Ye Z. Global regularity for a 2D tropical climate model with fractional dissipation. *J. Differential Equations*, 2019, 266: 6346-6382.

[19] Ma C C, Wan R H. Spectral analysis and global well-posedness for a viscous tropical climate model with only a damp term. *Nonlinear Anal. Real World Appl.*, 2018, 39: 554-567, doi: 10.1016/j.nonrwa.2017.08.004

[20] Dong B Q, Wang W J, Wu J H, Ye Z, Zhang H. Global regularity for a class of 2D generalized tropical climate models. *J. Differential Equations*, 2019, 266: 6346-6382.

[21] Jiu Q S, Suo X X, Wu J H, Yu H. Unique weak solutions of the non-resistive magnetohydrodynamic equations with fractional dissipation. *Comm. Math. Sci.*, 2020, 18: 987-1022, doi: 10.4310/CMS.v18.n4.a5

[22] Said O, Wu J H. Unique weak solutions of the 4-dimensional micropolar equation with fractional dissipation. *Math. Methods Appl. Sci.*, 2020, doi: 10.1002/mma.6740

[23] Li J L, Zhai X P, Yin Z Y. On the global well-posedness of the tropical climate model. *ZAMM Z. Angew. Math. Mech.*, 2019, 99:17pp, doi: 10.1002/zamm.201700306

[24] Bahouri H, Chemin J. *Fourier Analysis and Nonlinear Partial Differential Equations*. Springer, Heidelberg, 2011, ISBN: 978-3-642-16829-1