A higher chromatic analogue of the image of J

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Abstract

We prove a higher chromatic analogue of Snaith’s theorem which identifies the K-theory spectrum as the localisation of the suspension spectrum of $CP^\infty$ away from the Bott class; in this result, higher Eilenberg-MacLane spaces play the role of $CP^\infty = K(Z,2)$. Using this, we obtain a partial computation of the part of the Picard-graded homotopy of the $K(n)$-local sphere indexed by powers of a spectrum which for large primes is a shift of the Gross-Hopkins dual of the sphere. Our main technical tool is a $K(n)$-local notion generalising complex orientation to higher Eilenberg-MacLane spaces. As for complex-oriented theories, such an orientation produces a one-dimensional formal group law as an invariant of the cohomology theory. As an application, we prove a theorem that gives evidence for the chromatic redshift conjecture.

1 Introduction

The stable homotopy groups of a space $X$ are defined as the colimit

$$\pi^S_j(X) = \lim_{m \to \infty} \pi_{j+m}(\Sigma^m X) = \lim_{m \to \infty} \pi_j(\Omega^m \Sigma^m X) = \pi_j(QX).$$

where $QX = \lim_{\to} \Omega^m \Sigma^m X$.

The J-homomorphism $J : \pi_j(O) \to \pi^S_j(S^0)$ may be regarded as a first approximation to the stable homotopy groups of $S^0$; here $O$ denotes the infinite orthogonal group $\lim_{\to} O(m)$. It is induced in homotopy by the limit over $m$ of maps

$$J_m : O(m) \to \Omega^m S^m,$$

where for a matrix $M \in O(m)$ regarded as a linear transformation $M : \mathbb{R}^m \to \mathbb{R}^m$, $J_m(M) = M \cup \{\infty\} : S^m \to S^m$. There is an analogous function from the infinite unitary group $U$: $J_U : U \to QS^0$ given by composition with the forgetful map $U \to O$. The homotopy groups of the domains are computable via Bott periodicity, and are

| $j$ mod 8 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------|---|---|---|---|---|---|---|---|
| $\pi_j(O)$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}$ |
| $\pi_j(U)$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |

The work of Adams [Ada66] shows that for an odd prime $p$, in dimensions 3 mod 4, the $p$-torsion of the image in $\pi^S_j(S^0)$ of the cyclic group $\pi_j(O)$ is isomorphic to $\mathbb{Z}/p^{k+1}$ when we can write $j + 1 = 2(p - 1)p^k m$ with $m$ coprime to $p$. When $j$ cannot be written in this form, the $p$-torsion in the image of $J$ is zero. If we are working away from $p = 2$, this computation may be done using $U$.
in place of $O$. A computation of the 2-torsion in the image of $J$ follows from Quillen’s proof of the Adams conjecture [Qui71].

As this result indicates, the burden of computation in stable homotopy theory often encourages one to work locally at a prime $p$. That is, one computes the $p$-power torsion in $\pi_n^S(X)$, and then assembles the results together into an integral statement. One of the deeper insights of homotopy theorists in the last half century is that one may, following Bousfield [Bou79], do such computations and assemble the results together into an integral statement. One of the deeper insights of homotopy theorists is that one may, following Bousfield [Bou79], do such computations and assemble the results together into an integral statement.

Related to the theories local to any cohomology theory $E^*$ to get more refined computations. Morava’s extraordinary $K$-theories $K(n)$ and $E$-theories $E_n$ [Mor85] have proven particularly suited to this purpose; see, e.g., [MRW77] [HS98].

When $n = 1$, $K(1)$ is identified with (a split summand of) mod $p$ $K$-theory. The fact that $\pi_*(U) = \pi_{*+1}(K)$ for $* > 0$ suggests that Adams’ computation of the $p$-torsion in the image of $J$ is related to $K(1)$-local homotopy theory. This is in fact the case; the localisation map

$$\pi_*^S(S^0) \to \pi_*(L_{K(1)}S^0)$$

carries $J$ isomorphically onto the codomain in positive degrees.

One substantial difference between the stable homotopy category and its $K(n)$-local variant is the existence of exotic invertible elements. In the stable homotopy category, the only spectra which admit inverses with respect to the smash product are spheres; thus the Picard group of equivalence classes of such spectra is isomorphic to $\mathbb{Z}$. In contrast, the Picard group of the $K(n)$-local category, $\text{Pic}_n$, includes $p$-complete factors as well as torsion (see, e.g., [HMS94, GHMR12]).

Our main result is a computation of part of the Picard graded homotopy of the $K(n)$-local sphere $S := L_{K(n)}S^0$ analogous to the image of $J$ computation. Throughout this paper $p$ will denote an odd prime; when localising with respect to $K(n)$, the prime $p$ is implicitly used.

**Theorem 1.1.** Let $\ell \in \mathbb{Z}$, and write $\ell = p^km$, where $m$ is coprime to $p$. Then the group $[S(\det_{\pm})^{\ell(p-1)}, L_{K(n)}S^1]$ contains a subgroup isomorphic to $\mathbb{Z}/p^{k+1}$. Furthermore, if $n^2 < 2p - 3$, there is an exact sequence

$$0 \to \mathbb{Z}/p^{k+1} \to [S(\det_{\pm})^{\ell(p-1)}, L_{K(n)}S^1] \to N_{k+1} \to 0$$

where $N_{k+1} \leq \pi_{-1}(S)$ is the subgroup of $p^{k+1}$-torsion elements.

This is proved as Corollary 5.10. Here, $S(\det_{\pm}) \in \text{Pic}_n$ is closely related to a spectrum $S(\det)$ introduced by Goerss et al. in [GHMR12]; it is defined below. When $n = 1$ and $p > 2$, $S(\det_{\pm})$ may be identified as $L_{K(1)}S^2$, and so this result recovers the constructive part of the classical image of $J$ computation. If $n$ is odd, $S(\det_{\pm})$ may generally be identified as a shift of the Brown-Comenetz dual of the $n^{th}$ monochromatic layer of the sphere spectrum if $\max\{2n + 2, n^2\} < 2p - 1$ (see [HG94]) (if $n$ is even, these spectra differ by a two-torsion element of $\text{Pic}_n$). This identification fails for small primes; see, e.g., [GH12] for the case $n = 2$ and $p = 3$.

### 1.1 The invertible spectrum $S(\det_{\pm})$

Morava’s $E$-theories are Landweber exact cohomology theories $E_n$ associated to the universal deformation of the Honda formal group $\Gamma_n$ from $\mathbb{F}_{p^n}$ to $\mathbb{W}(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}][[u_{n+1}]].$ When $n = 1$, $E_1$ is precisely $p$-adic $K$-theory. The Goerss-Hopkins-Miller theorem [GH04] [GH05] equips the spectrum $E_n$ with an action of the Morava stabiliser group

$$G_n := \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \rtimes \text{Aut}(\Gamma_n)$$

2
which lifts the defining action in homotopy. The work of Devinatz-Hopkins [DH04], Davis [Dav06, Dav09], and Behrens-Davis [BD10] then allows one to define continuous homotopy fixed point spectra with respect to closed subgroups of $G_n$ in a consistent way. The homotopy fixed point spectrum of the full group is the $K(n)$-local sphere: $E_n^{hG_n} \simeq L_{K(n)}S^0$.

The automorphism group $\text{Aut}(\Gamma_n)$ is known to be the group of units of an order of a rank $n^2$ division algebra over $\mathbb{Q}_p$; the determinant of the action by left multiplication defines a homomorphism $\det : \text{Aut}(\Gamma_n) \to \mathbb{Z}_p^\times$. Extend this to a homomorphism $\det_{\pm} : G_n \to \mathbb{Z}_p^\times$ by sending the Frobenius generator of $\text{Gal}(F_{p^n}/F_p)$ to $(-1)^{n-1}$. We will write $S^\pm_{G_n}$ for the kernel of this map. We may define the homotopy fixed point spectrum $E_n^{hS^\pm_{G_n}}$ for the restricted action of this subgroup. We must point out: in many references, such as [HG94, GHMR12], a different extension $\det : G_n \to \mathbb{Z}_p^\times$ is used; there $\text{Gal}(F_{p^n}/F_p)$ is in the kernel of $\det$. In an earlier version of this article, we incorrectly used this older form of $\det$; we thank Charles Rezk for clarifying this point.

The spectrum $E_n^{hS^\pm_{G_n}}$ retains a residual action of $\mathbb{Z}^\times_p = G_n/S^\pm_{G_n}$; for an element $k \in \mathbb{Z}_p^\times$, we will write the associated map as $\psi_k : E_n^{hS^\pm_{G_n}} \to E_n^{hS^\pm_{G_n}}$.

The reader is encouraged to think of these automorphisms as analogues of Adams operations. Noting that $\mathbb{Z}^\times_p = \mu_{p-1} \times (1 + p\mathbb{Z}_p)^\times$ is topologically cyclic with generator $g = \zeta_{p-1}(1+p)$, we define $F_\gamma$ as the homotopy fibre of $\psi^g-\gamma : E_n^{hS^\pm_{G_n}} \to E_n^{hS^\pm_{G_n}}$ for any $\gamma \in \mathbb{Z}_p^\times$. These spectra are always invertible, and in fact the construction $\gamma \mapsto F_\gamma$ defines a homomorphism $\mathbb{Z}_p^\times \to \text{Pic}_n$. When $\gamma = 1$, the associated homotopy fibre defines the homotopy fixed point spectrum for the action of $\mathbb{Z}_p^\times$, and so

$$F_1 = (E_n^{hS^\pm_{G_n}})^{h\mathbb{Z}_p^\times} \simeq E_n^{hG_n} \simeq L_{K(n)}S^0$$

In contrast, one defines $S(\det_{\pm}) := F_g$.

1.2 A Snaith theorem

The K-theory spectrum admits a remarkable description due to Snaith [Sna79]. He shows that the natural inclusion of $\mathbb{C}P^\infty$ into $BU \times \mathbb{Z}$ localises to an equivalence

$$\Sigma^\infty \mathbb{C}P^\infty[\beta^{-1}] \simeq K.$$ 

Here the Bott map $\beta : S^2 \to \Sigma^\infty \mathbb{C}P^\infty_+$ is a reduced, stable form of the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^\infty$.

**Theorem 1.2.** There is a map $\rho_n : S(\det_{\pm}) \to L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n + 1)_+$ and an equivalence of $E_\infty$-ring spectra

$$L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n + 1)_+ [\rho_n^{-1}] \to E_n^{hS^\pm_{G_n}}.$$

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1 All homotopy fixed point spectra considered in this article will be of the continuous sort.
We will refer to the equivalent ring spectra of the theorem as $R_n$. One inverts the map $\rho_n$ in the same fashion as for normal homotopy elements of a ring spectrum, via a telescope construction. A consequence of this theorem is that the homotopy of $R_n$ is $S(\det_+)$-periodic. When $n = 1$, $R_1$ is simply $p$-adic K-theory, and this result is familiar as Bott periodicity. Indeed, the map $\rho_1$ is the $K(1)$-localisation of $\beta$, and so we refer to $\rho_n$ as a higher Bott map.

The proof of this theorem uses the computations of Ravenel-Wilson [RW80] of the Morava K-theories of Eilenberg-MacLane spaces, as well as the $E_\infty$ obstruction theory developed by Goerss-Hopkins in [GH04]. It is proven as Corollary 3.24 below.

### 1.3 Higher orientation for $K(n)$-local cohomology theories

The map $\beta$ is traditionally used to define the notion of complex orientation for a cohomology theory; the associated formal group law is an important invariant of the cohomology theory, and central to the structure of an infinite loop space. We will write $K$ a delooping $BGL_1(A)$.

For an $A$-ring spectrum $A$, the space $GL_1(A)$ of units (the invertible components in $\Omega^\infty A$) admits a delooping $BGL_1(A)$. If $A$ is in fact an $E_\infty$ ring spectrum, its multiplication equips $BGL_1(A)$ with the structure of an infinite loop space. We will write $K(A)$ for the algebraic $K$-theory spectrum of $A$. There is a map of $E_\infty$ ring spectra

$$i : \Sigma^\infty BGL_1(A)_+ \to K(A)$$
whose adjoint is given by the inclusion of \( A \)-lines into all cell \( A \)-modules (see \cite{ABG+08} for the construction on the level of \( \infty \)-categories; then \cite{EM06} gives the map of \( E_\infty \) ring spectra).

In \cite{SW11}, with Hisham Sati, we constructed an \( E_\infty \) map\(^2\) \( \varphi_n : K(\mathbb{Z}, n + 1) \to GL_1 E_n \). Delooping once and composing with \( i \), we obtain a map of \( E_\infty \) ring spectra

\[
i \circ B \varphi_n : \Sigma^\infty K(\mathbb{Z}, n + 2)_+ \to K(E_n).
\]

We may localise both the domain and range of this map with respect to \( K(n + 1) \); it is natural to ask about the behaviour of the composite map

\[
\beta_{n+1} = i \circ B \varphi_n \circ \rho_{n+1} : S(\det_\pm) \to L_{K(n+1)} K(E_n)
\]

(we note that the domain is the \( K(n+1) \)-local \( S(\det_\pm) \)).

Our methods are not suitable to construct such a map for the case \( n = 0 \) but it is instructive to consider this setting nonetheless. We interpret \( E_0 \) as singular cohomology with \( \mathbb{Q}_p \) coefficients; the resulting \( K \)-theory spectrum is the algebraic \( K \)-theory of \( \mathbb{Q}_p \). The map \( \beta_1 \), were it to exist, would be of the form \( S^2 \to L_{K(1)} K(\mathbb{Q}_p) \); it should be considered as the image in the \( K(1) \)-local category of the Bott element\(^3\) (considered in e.g., \cite{Tho85} and \cite{Mit97}) in the \( p \)-adic algebraic \( K \)-theory of \( \mathbb{Q}_p \).

**Theorem 1.5.** For \( p > 3 \), multiplication by \( \beta_{n+1} \in \pi_{S(\det_\pm)} L_{K(n+1)} K(E_n) \) is an equivalence. Therefore

\[
L_{K(n+1)} K(E_n) \simeq L_{K(n+1)} K(E_n)[\beta_{n+1}^{-1}],
\]

and so the map \( i \circ B \varphi_n \) makes \( L_{K(n+1)} K(E_n) \) an \( R_{n+1} \)-algebra spectrum.

This is proven as Corollary 6.3 below. It provides some evidence for the chromatic redshift conjecture, as enunciated in Conjecture 4.4 of \cite{AR06}. The spectrum of the theorem is the algebraic \( K \)-theory spectrum of a prominent \( K(n) \)-local spectrum. After localisation at \( K(n+1) \), we have shown it to support an algebra structure one chromatic level higher.

It is worth pointing out that this result is entirely \( K(n+1) \)-local. The corresponding statement for \( n = 0 \) – that multiplication by the Bott element as considered by Thomason is a \( K(1) \)-local equivalence – follows naturally from the fact that it descends to the familiar Bott class in the complex \( K \)-theory spectrum. A deeper statement in fact holds for \( n = 0 \): the Bott element exists prior to localisation, and multiplication by it is not nilpotent. Furthermore, inversion of this element defines the \( K(1) \)-localisation after smashing with an appropriate Moore spectrum (see \cite{Mit97}).

For \( n = 1 \), Ausoni has constructed in \cite{Aus10} a class that he calls the higher Bott element, \( b \in V(1)^{2p+2}(K(E_1)) \), and has verified that it, too, is not nilpotent using topological cyclic homology techniques. Again, this construction occurs at a stage prior to \( K(2) \)-localisation. It follows from his Theorem 1.1 that for \( p > 5 \), after localisation, \( b^{p-1} \) is a unit multiple of a \( V(1) \)-Hurewicz image of the element \( \beta_2^{p-1} \) in

\[
[S^2, L_{K(n)}(V(1) \wedge K(E_1))] \cong [S(\det_\pm)^{p-1}, L_{K(n)}(V(1) \wedge K(E_1))]
\]

(both this isomorphism follows from the same arguments as Proposition 6.1). In contrast to the intricate computational methods of \cite{Aus10}, our proof of the periodicity of \( \beta_{n+1} \) is achieved by detecting it

\(^2\)In the indicated paper, we constructed this map only at the prime 2. A mild generalisation of the argument is given for all primes in section 6.3 of this document. Indeed, \( \varphi_n \) is used in the equivalence given in Theorem 1.2.

\(^3\)This class doesn’t exist unless \( p \) is adjointed to \( \mathbb{Q}_p \); however, its \( p-1 \)st power does exist in \( K(\mathbb{Q}_p) \). Alternatively, \( \beta_1 \) exists after smashing with \( M\mathbb{Z}/p \). See section 6 for a similar phenomenon in our setting.
modulo $p$ as multiplication by an invertible element of the Picard-graded homotopy of the $K(n+1)$-local Moore spectrum.

However, an important caveat to Theorem 1.5 is that it does not actually show that the localisation $L_{K(n+1)}K(E_n)$ is nonzero. When $n = 0$, the fact that $L_{K(1)}K(\mathbb{Q}_p) \neq 0$ is visible in [HM03], where it is shown that $\mathcal{K}_s(\mathbb{Q}_p)$ contains the image of $J$ as a factor. When $n = 1$, the nonvanishing of $L_{K(2)}K(E_1)$ follows from Ausoni’s explicit computations of $V(1)_*K(E_1)$.

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2 Generalities on the $K(n)$-local category

2.1 Morava $K$ and $E$-theories

A central object of the chromatic program is the Honda formal group law $\Gamma_n$ of height $n$ over the finite field $\mathbb{F}_{p^n}$; the $p$-series of $\Gamma_n$ is $[p](x) = x^{p^n}$. We will write $\mathcal{K}$ or $\mathcal{K}_n$ for the 2-periodic Landweber-exact cohomology theory that supports this formal group law; thus the coefficients of $\mathcal{K}$ are $\pi_*\mathcal{K} = \mathbb{F}_{p^n}[u^\pm 1]$. This is closely related to the “standard” Morava $K$-theory, $K(n)$, whose homotopy groups are $\mathbb{F}_p[v_n^\pm 1]$, with $|v_n| = 2p^n - 2$.

The (Lubin-Tate) universal deformation of $\Gamma_n$ is defined over the ring $\mathbb{W}(\mathbb{F}_{p^n})[[t_1, \ldots, t_{n-1}]][u^\pm 1]$. There is a cohomology theory $E_n - Morava E$-theory — whose homotopy groups are given by that ring, and support this formal group law. The theorem of Goerss-Hopkins-Miller [GH04] equips $E_n$ with the structure of an $E_{\infty}$ ring spectrum.

If we write $m$ for the maximal ideal $m = (p, u_1, \ldots, u_{n-1}) \subseteq \pi_*(E_n)$, these cohomology theories are related via:

$$K^*(X) \cong (E_n/m)^*(X) \cong \mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} K(n)^*(X)[u]/(u^{p^n-1} - v_n),$$

and similarly in homology. Notice that the representing spectrum $K$ is equivalent to a bouquet of copies of $K(n)$. Thus its Bousfield class is the same as that of $K(n)$.

Recall that $\mathbb{F}_{p^n}$ consists of $p^n - 1$st roots of unity. Factoring $p^n - 1 = (p - 1)(1 + p + \cdots + p^{n-1})$, we note that $2(p - 1)$ divides $p^n - 1$ when $p$ is odd and $n$ is even. Choose a $p - 1$st root of $(-1)^{n-1}$, $\xi \in \mathbb{F}_{p^n}$, to be:

$$\xi := \begin{cases} 1, & n \text{ odd} \\ \text{a primitive } 2p - 2\text{nd root of } 1, & n \text{ even} \end{cases}$$

$^4$Here $\mathbb{W}(\mathbb{F}_{p^n})$ is the ring of Witt vectors over $\mathbb{F}_{p^n}$, alternatively defined by adjoining a primitive $p^n - 1$st root of unity to the $p$-adic integers, $\mathbb{Z}_p$. 

6
2.2 The Morava stabiliser group

We will briefly summarise some preliminaries regarding the Morava stabiliser group. We refer the reader to \cite{Mor85} and \cite{GHMR05} for more complete discussions of this material.

We will use the notation $\mathfrak{S}_n := \text{Aut}_{\mathfrak{F}_p^n}(\Gamma_n)$ to denote the group of automorphisms of $\Gamma_n$. Then $\mathfrak{G}_n$ is defined as the semidirect product

$$\mathfrak{G}_n := \mathfrak{S}_n \rtimes \text{Gal}(\mathfrak{F}_p^n/\mathfrak{F}_p).$$

The Goerss-Hopkins-Miller theorem lifts the defining action $\mathfrak{G}_n$ on $\Gamma_n$ to an action on the spectrum $E_n$ through $E_\infty$ maps. The work of Devinatz-Hopkins \cite{DH04}, Davis \cite{Dav06}, and Behrens-Davis \cite{BD10} defines homotopy fixed point spectra with respect to closed subgroups of $\mathfrak{G}_n$ and constructs associated descent spectral sequences.

We write $\mathcal{O}_n$ for the noncommutative ring

$$\mathcal{O}_n = \mathcal{W}(\mathfrak{F}_p^n)/(S^n - p, Sa = a^\sigma S)$$

where $\sigma$ denotes a lift of the Frobenius on $\mathfrak{F}_p^n$ to the ring $\mathcal{W}(\mathfrak{F}_p^n)$ of Witt vectors. Then $\mathcal{O}_n \cong \text{End}_{\mathfrak{F}_p^n}(\Gamma_n)$ is the ring of endomorphisms of $\Gamma_n$, and so $S_n \cong \mathcal{O}_n^\times$.

Now, $S_n$ naturally acts on $\mathcal{O}_n$ (by right multiplication) through left $\mathcal{W}(\mathfrak{F}_p^n)$-module homomorphisms, and so defines a map $S_n \to \text{GL}_n(\mathcal{W}(\mathfrak{F}_p^n))$, since $\mathcal{O}_n$ is free of rank $n$ over $\mathcal{W}(\mathfrak{F}_p^n)$, with basis $\{1, S, \ldots, S^{n-1}\}$. It turns out that the determinant of elements coming from $S_n$ actually lies in $\mathbb{Z}_p^\times$, instead of $\mathcal{W}(\mathfrak{F}_p^n)^\times$; this gives a homomorphism

$$\det : S_n \to \mathbb{Z}_p^\times.$$

As described in the introduction, we may extend this to $\det_{\pm} : \mathfrak{G}_n \to \mathbb{Z}_p^\times$ by sending a Frobenius generator of $\text{Gal}(\mathfrak{F}_p^n/\mathfrak{F}_p)$ to $(-1)^{n-1}$. We will write $SG_n^\pm$ for the kernel of this homomorphism.

We recall that there is an isomorphism $\mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p)^\times \cong \mu_{p-1} \times \mathbb{Z}_p$. Composing the determinant with the projection onto the second factor yields the reduced determinant $\widehat{\det}_{\pm} : \mathfrak{G}_n \to \mathbb{Z}_p$. If we write $\mathfrak{G}_n^1$ for the kernel of $\widehat{\det}_{\pm}$, then there is a split short exact sequence

$$1 \to S\mathfrak{G}_n^\pm \to \mathfrak{G}_n^1 \to \mu_{p-1} \to 1 \quad (2.2.1)$$

Furthermore, one may define a homomorphism $z : \mathbb{Z}_p^\times \to S_n$ by $a \mapsto a + 0 \cdot S + \cdots + 0 \cdot S^{n-1}$. The image is central, and the composite $\det_{\pm} \circ z : \mathbb{Z}_p^\times \to \mathbb{Z}_p$ is $a \mapsto a^n$. In particular, if $n$ is coprime to both $p$ and $p - 1$, this map is an isomorphism, yielding a splitting $\mathfrak{G}_n \cong S\mathfrak{G}_n^\pm \times \mathbb{Z}_p^\times$. Similarly, we get a splitting $\mathfrak{G}_n \cong \mathfrak{G}_n^1 \times \mathbb{Z}_p$.

2.3 The Picard group

We recall that the category of $K(n)$-local spectra is symmetric monoidal; the tensor product

$$A \otimes B := L_{K(n)}(A \wedge B)$$

is the $K(n)$-localisation of the smash product of $A$ and $B$. The unit is the $K(n)$-local sphere $S := L_{K(n)}(S^0)$.

\footnote{Since $\pm 1 \in \mu_{p-1}$, this is the same as the reduced determinant of, e.g., \cite{GHMR05}, so our $\mathfrak{G}_n^1$ is the same as in that article. However, the homomorphism to $\mu_{p-1}$ in equation (2.2.1) differs.}
An invertible spectrum $A$ is invertible with respect to this tensor product; that is, there exists another spectrum $B$ with $A \otimes B \simeq S$. The set of weak equivalence classes of invertible spectra forms an abelian group under tensor product, the Picard group of the $K(n)$-local category, denoted $\text{Pic}_n$.

It was shown in [HMS94] that $A$ is invertible if and only if $K(n)_*(A)$ is free of rank one over $K(n)_*$. Consequently there is a well-defined dimension $\dim(A) \in \mathbb{Z}/2(p^n - 1)$ which records the degree of a generator of $K(n)_*(A)$.

For spectra $A, B, F(A, B)$ will denote the function spectrum of maps from $A$ to $B$. If $A$ is an element of $\text{Pic}_n$, it is easily seen that the Spanier-Whitehead dual $F(A, S) = A^{-1}$ is the inverse to $A$: the evaluation map $A \otimes F(A, S) \to S$ is an equivalence.

### 2.4 $\text{Pic}_n$-graded invariants

In [HMS94] it was suggested that one should take a larger view of homotopy groups in the $K(n)$-local category, and grade homotopy by the Picard group. To invent some notation, we will write $\pi_A(X) := [A, X]$, for $K(n)$-local spectra $X$ and invertible $A$: then $\pi_{S^m}(X) = \pi_m(X)$.

We propose that the same approach be used to index\(^6\) the coefficients of more general (co)homology theories.

**Definition 2.1.** Let $X$ and $E$ be $K(n)$-local spectra. For an invertible spectrum $A$, define

$$E_A(X) := [A, X \wedge E], \quad E^A(X) := [A, X \otimes E], \quad \text{and} \quad E^A(X) := [X, A \otimes E].$$

These are the $\text{Pic}_n$-graded $E$-homology, completed $E$-homology, and $E$-cohomology.

We note that $E_A(X)$ need not be isomorphic to the $E^A(X)$, even for $A = S^m$. Indeed, when $E = E_n$ is Morava $E$-theory, the completed homology theory is the Morava module of $X$, which may differ from $(E_n)_m(X)$ when $X$ is an infinite complex. In this paper, we will largely be concerned with $E^A(X)$, rather than $E_A(X)$.

To be completely balanced, we should also consider the cohomological functor $[X, A \wedge E]$ (and perhaps decorate the functor $E^A(X)$ above with a $\vee$). However, $[X, A \wedge E] = [X, A \otimes E]$ when $A$ is a sphere (since $S^m \wedge E = \Sigma^m E$ is already $K(n)$-local). Furthermore, as in the homological setting, we will largely concern ourselves with $K(n)$-local constructions, making the non-local $A \wedge E$ awkward.

To distinguish from the standard indexing (by elements $m \in \mathbb{Z}$, corresponding to $S^m$), we will write

$$E^A_\bullet(X) := \bigoplus_{A \in \text{Pic}_n} E^A_m(X), \quad \text{but} \quad E^\vee_\bullet(X) := \bigoplus_{n \in \mathbb{Z}} E^A_n(X)$$

and similarly for cohomology.

**Example 2.2.** Since $K$ satisfies a strong Künneth theorem,

$$K^A(X) = [X, A \otimes K] = [X \otimes A^{-1}, K] = K^0(X \otimes A^{-1}) = (K^*(X) \otimes K^{-1}(A^{-1}))_0$$

If $A$ has dimension $d$, then $A^{-1}$ has dimension $-d$, so we conclude that $K^A(X) \cong K^d(X)$. So grading $K^\bullet$ by the full Picard group does not recover any more information than grading by the integers.

The same argument works if one replaces $K$ with $E_n$, since $E^*_n(A^{-1})$ is free of rank 1 over $E^*_n$, giving a collapsing Künneth spectral sequence. However, the result is far from true for all (co)homology theories, most notably stable homotopy.

\(^6\)The point is raised in [HMS94] that grading homotopy (and hence other such invariants) by the Picard group does not necessarily yield a well-behaved ring structure on the resulting direct sum. In practice, however, we will only ever consider indexing on integral powers of spheres and $S(\det_{+\mathbb{Z}})$, for which their Theorem 14.11 is more than sufficient.
Additionally, we note the following, whose proof is immediate:

**Proposition 2.3.** If $A$ is invertible and $E$ a $K(n)$-local ring spectrum, then $E^\wedge_n(A)$ and $E\wedge_n(A)$ are free $E\wedge_n := E\wedge_n(S)$-modules of rank one.

Note that, by the (collapsing) Künneth spectral sequence, this implies that for any spectrum $X$, the natural map

$$E\wedge_n(A) \otimes_{E\wedge_n} E\wedge_n(X) \to E\wedge_n(A \otimes X)$$

is an isomorphism (and similarly for the completed homology and cohomology).

### 2.5 Localising ring spectra

If $X$ is a $K(n)$-local ring spectrum with multiplication $\mu$, $A \in \text{Pic}_n$, and $f : A \to X$ an element of $\pi_A(X)$, one can localise $X$ away from $f$ in the same fashion as is usually done for homotopy elements.

**Definition 2.4.** Define $X[f^{-1}]$ to be the telescope (homotopy colimit) of the diagram

$$X \xrightarrow{m_f} A^{-1} \otimes X \xrightarrow{1 \otimes m_f} A^{-1} \otimes (A^{-1} \otimes X) \xrightarrow{1 \otimes 1 \otimes m_f} \ldots$$

Here $m_f$ is the composite

$$X = A^{-1} \otimes A \otimes X \xrightarrow{1 \otimes f \otimes 1} A^{-1} \otimes X \otimes X \xrightarrow{1 \otimes \mu} A^{-1} \otimes X$$

where $\mu$ is the multiplication in $X$.

As in section V.2 of [EKMM97], if $X$ is an $E\infty$-ring spectrum, then so too is $X[f^{-1}]$. Standard properties of homotopy colimits then give:

**Proposition 2.5.** The natural map $X \to X[f^{-1}]$ induces an isomorphism

$$\left(K\wedge_n X\right)[f^{-1}] \to K\wedge_n(X[f^{-1}]).$$

where $f$ is regarded as an element of $K_A(X) = [A, X \otimes K]$ by smashing with the unit of $K$. This induces an isomorphism

$$\left(K\wedge_n X\right)[f_*^{-1}] \to K\wedge_n(X[f_*^{-1}]).$$

where $f_* \in K_{\dim(A)}(X)$ is the image of a generator of $K\wedge_n(A)$ under $f$. The same holds with $K$ replaced by $E_n$.

Write $\eta$ for the unit in $X$. By construction, there is an equivalence $\mu \circ (f \otimes 1) : A \otimes X[f^{-1}] \to X[f^{-1}]$; let $g$ be the inverse equivalence. Define $f^{-1} : A^{-1} \to X[f^{-1}]$ as

$$A^{-1} = A^{-1} \otimes S \xrightarrow{1 \otimes \eta} A^{-1} \otimes X[f^{-1}] \xrightarrow{1 \otimes g} A^{-1} \otimes A \otimes X[f^{-1}] = X[f^{-1}]$$

By construction $\eta = f \cdot f^{-1} : S = A \otimes A^{-1} \to X[f^{-1}]$; thus $f$ is indeed invertible in $\pi\wedge_n(X[f^{-1}])$. 
2.6 Automorphism groups

For a spectrum \( A \), we will write \( \text{End}(A) := F(A, A) \) for the function spectrum of maps from \( A \) to itself. This is an associative ring spectrum under composition of functions, and so \( \pi_0 \text{End}(A, A) = [A, A] \) forms a ring.

**Definition 2.6.** Write \( \text{Aut}(A) \subseteq \Omega^\infty \text{End}(A) \) for the union of components whose image in the ring \( \pi_0 \text{End}(A, A) \) is invertible.

Note that the multiplication in \( \text{End}(A) \) equips \( \text{Aut}(A) \) with the structure of a grouplike \( A_\infty \) monoid. We also note that if \( A = S \), the adjoint of the identity on \( A \) defines an equivalence of ring spectra \( \text{End}(S) \simeq S \), and so \( \text{Aut}(S) \) is identified as the \( A_\infty \) monoid of units in the ring spectrum \( S \), \( \text{GL}_1(S) \):

\[
\text{Aut}(S) = \text{GL}_1(S) \subseteq \Omega^\infty S
\]

The following is immediate:

**Proposition 2.7.** If \( A \) is invertible and \( B \) any spectrum, tensoring with the identity of \( A \) gives an equivalence

\[
\text{id}_A \otimes - : \text{End}(B) \to \text{End}(A \otimes B).
\]

Passing to infinite loop spaces yields a natural equivalence of \( A_\infty \) monoids \( \text{Aut}(B) \to \text{Aut}(A \otimes B) \).

In particular, taking \( B = S \), we conclude

**Corollary 2.8.** If \( A \) is invertible, smashing with the identity of \( A \) defines an equivalence of \( A_\infty \) monoids

\[
\text{GL}_1(S) \to \text{Aut}(A).
\]

Loosely speaking, \( A \) becomes a \( \text{GL}_1(S) \)-spectrum by its action on the left tensor factor of \( S \otimes A = A \).

2.7 Thom spectra

Let \( X \) be a topological space, and \( \zeta : X \to B \text{GL}_1(S) \) a continuous map. Ando, et al. define the Thom spectrum \( X^\zeta \) in [ABG+08] as

\[
X^\zeta := \Sigma^\infty P_+ \wedge_{\Sigma^\infty \text{GL}_1(S)_+} S
\]

where \( P \) is the principal \( \text{GL}_1(S) \)-bundle over \( X \) defined by \( \zeta \). Using Corollary 2.8, we may modify and extend this definition in the \( K(n) \)-local category:

**Definition 2.9.** For an invertible spectrum \( A \), define the Thom spectrum \( X^A_\zeta \) as

\[
X^{A_\zeta} := \Sigma^\infty P_+ \otimes_{\Sigma^\infty \text{GL}_1(S)_+} A = L_{K(n)}(\Sigma^\infty P_+ \wedge_{\Sigma^\infty \text{GL}_1(S)_+} A).
\]

Note that even when \( A = S \), this differs slightly from the definition in [ABG+08] in that we have \( K(n) \)-localised the Thom spectrum. This extension of the definition of Thom spectra to have any invertible spectrum as “fibre” is convenient, but not very substantial; noticing that the action of \( \text{GL}_1(S) \) is on the \( S \) factor in \( A = S \otimes A \), one can easily show:

**Proposition 2.10.** \( X^{A_\zeta} = X^\zeta \otimes A \).
X \times X \xrightarrow{pr_1} X \xrightarrow{\zeta} BGL_1(S)

of \zeta with the projection onto the first factor defines a Thom spectrum over \(X \times X\), which may be identified with \(X^{A\zeta} \otimes X_+\). Furthermore, the diagonal map \(\Delta : X \to X \times X\) is covered by a map of Thom spectra

\[ D : X^{A\zeta} \to X^{A\zeta} \otimes X_+ \]

since \(pr_1 \circ \Delta = \text{id}_X\). The map \(D\) is the Thom diagonal, and for any \(K(n)\)-local ring cohomology theory \(E\), makes \(E^\bullet(X^{A\zeta})\) into a \(E^\bullet(X)\)-module.

Similarly, for each point \(x \in X\), the inclusion \(\{x\} \subseteq X\) is covered by a “fibre inclusion”

\[ i_x : A = X^{A\zeta}|_{\{x\}} \to X^{A\zeta}. \]

**Definition 2.11.** A class \(u \in E^A(X^{A\zeta})\) is a Thom class if, for every \(x \in X\) its restriction along \(i_x : A \to X^{A\zeta}\), \(i_x^*(u) \in E^A(A) \cong E_0\) is a unit.

**Proposition 2.12.** If \(X^{A\zeta}\) admits a Thom class \(u\) for \(E\), then \(E^\bullet(X^{A\zeta})\) is a free \(E^\bullet(X)\)-module of rank one, generated by \(u\).

When \(A = S\), this is classical, and is realised in homology by the map:

\[ E \otimes X^{A\zeta} \xrightarrow{1 \otimes D} E \otimes X^{A\zeta} \otimes X_+ \xrightarrow{1 \otimes i \otimes 1} E \otimes E \otimes A \otimes X \xrightarrow{\mu \otimes 1} E \otimes A \otimes X. \]

See, e.g., [MRS1] or [ABG+08], Prop 5.43. For general \(A\), this is obtained from that fact and the Pic\(_n\)-graded isomorphism

\[ E^\bullet(X^{A\zeta}) = E^\bullet(X^{A} \otimes A) \cong E^\bullet(X^{A\zeta}) \otimes_{E^\bullet} E^\bullet A. \]

### 3 Eilenberg-MacLane spaces

#### 3.1 Recollections from Ravenel-Wilson

Working stably, after \(p\)-completion, we note the equivalence\(^7\)

\[ \Sigma^\infty K(\mu_{p\infty}, n)_+ = \lim \Sigma^\infty K(\mathbb{Z}/p^j, n)_+ \cong \Sigma^\infty K(\mathbb{Z}, n + 1)_+ \cong \Sigma^\infty K(\mathbb{Z}, n + 1)_+ \]

(3.1.2)

(The first equivalence uses the Bockstein). This will be our main object of study:

**Definition 3.1.** Let \(X = X_n\) denote the \(K(n)\)-localisation of the unreduced suspension spectrum of \(K(\mathbb{Z}, n+1)\),

\[ X_n = L_{K(n)} \Sigma^\infty K(\mathbb{Z}, n + 1)_+. \]

We recall from Ravenel-Wilson [RW80] the Morava K-theory of this spectrum:

\[ K(n)^* K(\mathbb{Z}, n + 1) = K(n)_*[[x]], \quad K(n)_* K(\mathbb{Z}, n + 1) = \bigotimes_{k \geq 0} R(b_k). \]

Here \(|x| = 2g(n)|x| = 2g(n)\), where we define \(g(n) := \frac{p^n - 1}{p - 1}\). In the notation of section 12 of [RW80], \(x\) corresponds to the class \(x_S\) for \(S = (1,2,\ldots,n-1)\). Also, for each integer \(k \geq 0\), \(R(b_k)\) is the ring

\[ R(b_k) := K(n)_*[b_k]/(b_k^p - (-1)^{n-1}v_n^p b_k) = \mathbb{F}_p[x,v_n^\pm]/(b_k^p - (-1)^{n-1}v_n^p b_k), \]

\(^7\)Here \(\mu_{p\infty} \cong \lim \mathbb{Z}/p^j\) is the group of \(p^\text{th}\) power roots of unity in, e.g., \(\mathbb{C}\).
where the class $b_k$ has dimension $2p^k g(n)$ and is dual to $(-1)^{k(n-1)} x^{p^k}$. The notation $b_k$ is our abbreviation for Ravenel-Wilson’s $b_J$, with $J = (nk, 1, 2, \ldots, n - 1)$.

There are similar results for $K(\mathbb{Z}/p^j, n)$:

$$K(n)^* K(\mathbb{Z}/p^j, n) = K(n)_* [x]/x^{p^j}, \quad \text{and} \quad K(n)_* K(\mathbb{Z}/p^j, n) = \bigotimes_{k=0}^{j-1} R(b_k)$$

We have normalised these classes to be consistent across the limit in (3.1.2). This is not quite consistent with the notation of section 11 of [RW80]; there $K(n)_* K(\mathbb{Z}/p^j, n)$ is presented as being generated by classes $a_I$, with $I = (nk, n(j-1)+1, n(j-1)+2, \ldots, n(j-1)+n-1)$ with $0 \leq k < j$. This $a_I$ differs from $b_k$ by a power of $v_n$.

In the extension

$$K^*(X_n) = \mathbb{F}_p \otimes_{\mathbb{F}_p} K(n)^*(X_n)[u]/(u^{p^n-1} - v_n),$$

use the $2p - 2$th root of unity, $\xi$, to define a new (degree 0) coordinate $y := \xi x u^{g(n)}$; then

$$K^*(X_n) = K_*[[y]]$$

We may similarly normalise the $K$-homology; setting $c_k = (-1)^{k(n-1)} \xi^{-p^k} b_k u^{-p^k g(n)} = \xi^{-1} b_k u^{-p^k g(n)}$ we have

$$K_*(X_n) = K_*[c_0, c_1, \ldots]/(c_k^p - c_k).$$

and $\langle c_k, y^{p^n} \rangle = \delta_k^n$.

**Proposition 3.2.** The multiplication on $X_n$ equips $K^* X_n$ with the structure of a formal group over $K_*$ which is isomorphic to the formal multiplicative group, $\mathbb{G}_m$.

**Proof.** The computations described above equip $K^* X_n$ with a coordinate $y$, and the associative and unital multiplication on $X_n$ coming from the H-space structure on $K(\mathbb{Z}_p, n + 1)$ defines a formal group law on $K^* X_n = K_*[[y]]$.

We may compute the $p$-series of this formal group law using [RW80]. The relevant fact is that the Verschiebung, well-defined up to powers of $v_n$, satisfies $V(x) = (-1)^{n-1} x$. Thus

$$[p](x) = FV(x) = (-1)^{n-1} v_n^{-1} x^p$$

Consequently $[p](y) = y^p$. Such a $p$-series is only possible for the multiplicative group.

\[\square\]

We note that this computation allows one to formally define the $a$-series $[a](y) \in K_*[[y]]$ for any element $a \in \mathbb{Z}_p$.

**Definition 3.3.** Write $a$ in its $p$-adic expansion as $a = a_0 + a_1 p + a_2 p^2 + \ldots$, where $0 \leq a_i < p$. Then

$$[a](y) := [a_0](y) + F [a_1](y) + F [a_2](y) + \ldots$$

Here, $+F$ is addition according to the formal group law on $K_*[[y]]$.

The formula gives a well-defined series, since $\deg([p^n](y)) = p^n$ grows with $n$.\[\text{\textsuperscript{8}}\text{Because } \mathbb{F}_p = K(n)_*/(v_n - 1) \text{ is a perfect field, while } K(n)_* \text{ is not, } V \text{ is not naturally defined on } K(n)^*(X), \text{ but on its cyclically graded analogue } K(n)^*(X) = K(n)^*(X)/(v_n - 1).\]
3.2 Group actions

The $p$-adic integers $\mathbb{Z}_p$ are a topologically cyclic group with generator $1 \in \mathbb{Z}_p$; that is, the subgroup generated by $1$ (i.e., $\mathbb{Z}$) is dense in $\mathbb{Z}_p$. We will have occasion to write elements of $\mathbb{Z}_p$ in multiplicative notation; then we will write $h$ for the generator $1$.

Furthermore, for $p > 2$,

$$\mathbb{Z}_p^\times \cong \mu_{p-1} \oplus (1 + p\mathbb{Z}_p)^\times,$$

and the latter factor is isomorphic to $\mathbb{Z}_p$. Let $\zeta = \zeta_{p-1} \in \mu_{p-1}$ be a primitive $(p - 1)^{st}$ root of unity, i.e., a generator of $\mu_{p-1}$. A generator for $(1 + p\mathbb{Z}_p)^\times$ is $1 + p$. Then $\mathbb{Z}_p^\times$ is also topologically cyclic, with generator $g := (\zeta, 1 + p)$. We note that $g \mod p = \zeta \in \mathbb{F}_p^\times$.

The group $\mathbb{Z}_p^\times$ acts on $\mathbb{Z}_p$, and hence on $X_n = L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n + 1)_+$. For an element $a \in \mathbb{Z}_p^\times$, we will denote by $\psi_a$ the map $\psi_a : X_n \to X_n$ given by the action of $a$.

**Proposition 3.4.** The action of $\mathbb{Z}_p^\times$ on $K^*(X_n)$ is via ring homomorphisms, and is determined by $\psi_a(y) = [a](y)$.

**Proof.** This formula in fact holds for every $a \in \mathbb{Z}_p$. Note that this homotopy commutes, for any $m \in \mathbb{N}$:

$$K(\mathbb{Z}_p, n + 1) \xrightarrow{\Delta} K(\mathbb{Z}_p, n + 1)^{\times m} \xrightarrow{\psi^m} K(\mathbb{Z}_p, n + 1)$$

where $\Delta$ is the $m$-fold diagonal, and $\text{mult}$ is $m$-fold multiplication. The path along the upper right carries $y$ to $[m](y)$.

Being defined by space-level maps, the action of $\mathbb{Z}_p$ on $K^*(X_n)$ may be regarded as a continuous homomorphism $\mathbb{Z}_p \to \text{End}(K_*(\mathbb{Z}_p))$. Here, the topology on $\text{End}(K^*(X))$ is compact-open with respect to the natural topology on $K_*(Y)$ defined in section 11 of [HS99]. In that topology, the map $\mathbb{Z}_p \subseteq [X, X] \to \text{End}(K^*(X))$ is continuous. The argument above indicates that it agrees with the action $a \cdot y = [a](y)$ when $a \in \mathbb{N}$. Since the latter is also continuous, and $\mathbb{N}$ is dense in $\mathbb{Z}_p$, these two actions must agree.

The group $\mu_{p-1} \cong \mathbb{F}_p^\times$ acts via multiplication on $\mathbb{Z}/p = \mathbb{F}_p$, and therefore on $K(\mathbb{Z}/p, n)$.

**Proposition 3.5.** The action of $\mu_{p-1}$ on $K^*(\mathbb{Z}/p, n) \cong K_*[y]/y^n$ is given by $\psi_\zeta(y^m) = \zeta^m y^m$.

**Proof.** Since the action is space-level, it suffices to show that $\psi_\zeta(y) = \zeta y$. To see this, we note that the following diagram commutes:

$$\begin{array}{ccc}
K(\mathbb{Z}/p, 1) \times K(\mathbb{Z}/p, 1)^{\times n-1} & \xrightarrow{\psi^\xi} & K(\mathbb{Z}/p, n) \\
\psi^1 \downarrow & & \psi^\zeta \downarrow \\
K(\mathbb{Z}/p, 1) \times K(\mathbb{Z}/p, 1)^{\times n-1} & \xrightarrow{\psi^\xi} & K(\mathbb{Z}/p, n)
\end{array}$$

since both paths around the diagram represent $\zeta$ times the fundamental class in $H^n(K(\mathbb{Z}/p, 1)^{\times n}; \mathbb{F}_p)$. 

Thus in $K(n)_*$,

\[
\psi^\zeta(b_0) = \psi^\zeta(a_0) \circ a_1 \circ \cdots \circ a_{n-1}) = \psi^\zeta(a_0) \circ a_1 \circ \cdots \circ a_{n-1}) = \zeta a_0 \circ a_1 \circ \cdots \circ a_{n-1}) = \zeta b_0.
\]

The third equality uses the claim that $\psi^\zeta(a_0) = \zeta a_0 \in K(n)_2 K(Z/p, 1)$. This follows from the fact that both classes are carried injectively by the Bockstein to the unique class in $K(n)_2 K(Z_p, 2)$ which is $\zeta$ times the Hurewicz image of the fundamental class of $S^2 \to K(Z_p, 2)$. The claimed result then follows by duality.

\[
\square
\]

The commutativity of the diagram of exact sequences:

\[
\begin{array}{cccccc}
0 & \rightarrow & Z_p & \stackrel{p}{\rightarrow} & Z_p & \rightarrow Z/p & \rightarrow 0 \\
| & \downarrow{g} & | & \downarrow{g} & | & \downarrow{\zeta} & | \\
0 & \rightarrow & Z_p & \stackrel{p}{\rightarrow} & Z_p & \rightarrow Z/p & \rightarrow 0
\end{array}
\]

yields a commutative diagram of fibrations

\[
\begin{array}{cccccc}
K(Z/p, n) & \stackrel{\beta}{\rightarrow} & K(Z/p, n + 1) & \stackrel{\psi^p}{\rightarrow} & K(Z/p, n + 1) \\
\psi^\zeta & \downarrow & \psi^g & \downarrow & \psi^g \\
K(Z/p, n) & \stackrel{\beta}{\rightarrow} & K(Z/p, n + 1) & \stackrel{\psi^p}{\rightarrow} & K(Z/p, n + 1)
\end{array}
\]

(3.2.3)

Thus, the Bockstein

\[
\beta : L_{K(n)} \Sigma^\infty K(Z/p, n)_+ \rightarrow L_{K(n)} \Sigma^\infty K(Z/p, n + 1)_+ = X_n
\]

is $Z_p^\times$-equivariant where the action of $Z_p^\times$ on $L_{K(n)} \Sigma^\infty K(Z/p, n)_+$ is via the reduction $Z_p^\times \rightarrow \mu_{p-1}$. In particular, we conclude that

\[
\psi^\beta(y) \mod y^p = \zeta y
\]

### 3.3 Splitting $K(Z/p, n)$

For any $p$-complete spectrum $Y$, there is an action (distinct from the action described in the previous section) of $Z_p^\times$ on $Y$ where group elements acts in homotopy by multiplication; we will write the action of $\zeta$ simply as $\zeta$.

**Definition 3.6.** Define an endomorphism $\pi$ of $L_{K(n)} \Sigma^\infty K(Z/p, n)_+$ by

\[
\pi = \frac{1}{p - 1} \sum_{k=0}^{p-2} \zeta^{-k} (\psi^\zeta)^k = \frac{1}{p - 1} \sum_{k=0}^{p-2} \zeta^{-k} \psi^k \zeta
\]
We thank Tyler Lawson for this construction and the proof of the following proposition, which replaces an incorrect argument in a previous version of this paper.

**Proposition 3.7.** The map $\pi$ is a homotopy idempotent. It thus yields a splitting

$$L_{K(n)} \Sigma^\infty K(\mathbb{Z}/p, n)_{+} \simeq \mathbb{Z} \vee \mathbb{Z}^\perp$$

where the image of $Z$ in $K_{*} K(\mathbb{Z}/p, n)$ is the rank one $K_{*}$-subspace generated by $b_0$.

**Proof.** The idempotence follows from the fact that $\psi \cdot \zeta$ commutes with $\zeta$ and a brief computation. The splitting comes in the standard way, defining $Z$ as the spectrum representing the image of $\pi$ inside the functor $[-, L_{K(n)} \Sigma^\infty K(\mathbb{Z}/p, n)_{+}]$. Consequently, the image of $K_{*}(Z)$ is the image of $\pi_{*}$. However,

$$\pi_{*}(b_{0}^{\ell}) = \frac{1}{p-1} \sum_{k=0}^{p-2} \zeta^{-k} \psi^{k} \zeta(b_{0}^{\ell}) = \frac{1}{p-1} \sum_{k=0}^{p-2} \zeta^{-k} \zeta^{k \ell} b_{0}^{\ell} = \frac{1}{p-1} \sum_{k=0}^{p-2} \zeta^{(\ell-1)} b_{0}^{\ell}$$

which is precisely $b_{0}$ when $\ell = 1$, and 0 otherwise.

**Lemma 3.8.** $Z$ is an element of $\text{Pic}_{n}$. Furthermore,

$$L_{K(n)} \Sigma^\infty K(\mathbb{Z}/p, n)_{+} \simeq \bigvee_{k=0}^{p-1} \mathbb{Z}^\otimes k.$$ 

Lastly, $\mathbb{Z}^\otimes p^{-1} \simeq S$.

**Proof.** The first claim follows immediately from the previous Proposition. Then the map $\mathbb{Z}^\otimes k \to L_{K(n)} \Sigma^\infty K(\mathbb{Z}/p, n)_{+}$ induced by $k$-ary multiplication in $K(\mathbb{Z}/p, n)$ is an isomorphism in $K_{*}$ onto its image, which is the subspace generated by $b_{0}^{k}$. Thus $L_{K(n)} \Sigma^\infty K(\mathbb{Z}/p, n)_{+}$ decomposes as indicated into a wedge of tensor powers of $Z$.

The same argument, applied to $k = p$ (along with the fact that $b_{0}^{p} = (-1)^{n-1} v_{n} b_{0}$) yields $\mathbb{Z}^\otimes p \simeq Z$. The last result follows by cancelling a factor of $Z$.

Lastly, we note that $\pi \circ \psi^{\ell} = \zeta \cdot \pi$; thus the action of $\psi^{\ell}$ on $L_{K(n)} \Sigma^\infty K(\mathbb{Z}/p, n)_{+}$, when restricted to $Z$ is simply multiplication by $\zeta$.

### 3.4 An invariant description of $K_{*} K(\mathbb{Z}_{p}, n+1)$

Let $G$ be a profinite group and $R$ a topological ring. Following [Hov04], we define the completed group ring on $G$, $R[[G]]$ as the inverse limit of finite group rings

$$R[[G]] := \lim_{\leftarrow} U R[G/U]$$

where $U$ ranges over open subgroups of $G$. We recall that $R[[G]]$ admits the structure of an $R$-Hopf algebra where each $g \in G$ is grouplike: $\Delta(g) = g \otimes g$.

One may similarly define the $R$-Hopf algebra $C(G, R)$ to be the ring of continuous $R$-valued functions on $G$. The coproduct is the dual of multiplication in $G$. Theorem 5.4 of [Hov04] and the discussion that follows it give a proof that these two Hopf algebras are dual to each other over $R$:

$$R[[G]] \cong \text{Hom}_{R}(C(G, R), R).$$
Remark 3.9. The careful reader will note that in [Hov04], Hovey studies the twisted completed group ring $R[[G]]$ in the setting where $G$ acts continuously on $R$; the multiplication in $R[[G]]$ is deformed by this action. One similarly alters the coproduct (and right unit) in $C(G, R)$; the result is a Hopf algebroid. While of course essential to his goal of identifying $E_n^*E_*$ and its dual, for our purposes the untwisted analogues will suffice. We note in particular that the algebra structure on $C(G, R)$ (and coalgebra structure on $R[[G]]$) is the same in the twisted and untwisted setting.

When $G = \mathbb{Z}_p$ and $R = \mathbb{F}_p$, a natural family of functions is given as follows: if $m = m_0 + m_1 p + m_2 p^2 + \ldots \in \mathbb{Z}_p$ with $m_i \in \{0, \ldots, p-1\}$, define $f_k(m) = m$. We note that since the codomain of $f_k$ is $\mathbb{F}_p$, $f_k^p = f_k$. In fact, the $f_k$ form a set of generators for $C(\mathbb{Z}_p, \mathbb{F}_p)$, and so

$$C(\mathbb{Z}_p, \mathbb{F}_p) = \mathbb{F}_p[f_0, f_1, f_2, \ldots]/(f_k^p - f_k)$$

See, e.g., section 2 of [Rav76] or 3.3 of [Hov04]. Lastly, if $k$ is any finite extension of $\mathbb{F}_p$, then the natural map $C(\mathbb{Z}_p, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k \to C(\mathbb{Z}_p, k)$ is an isomorphism (this fact is perhaps more evident in the dual Hopf algebra).

Note that $\mathbb{Z}_p^\times$ acts on $\mathbb{Z}_p$, and hence on $R[[\mathbb{Z}_p]]$.

**Proposition 3.10.** There exists a $\mathbb{Z}_p^\times$-equivariant isomorphism of Hopf algebras

$$\phi : K_*[[\mathbb{Z}_p]] \to K_*^\times K(\mathbb{Z}_p, n+1) \cong K_*[y],$$

which carries a topological generator $h$ of $\mathbb{Z}_p$ to $1+y$.

**Proof.** The ring isomorphism $\phi$ is a basic fact about the Iwasawa algebra $\mathbb{Z}_p[[\mathbb{Z}_p]]$ (or, in this case, its reduction modulo $p$, and extension over $\mathbb{F}_p$). The dual map $\phi^*$ satisfies

$$\phi^*(c_k)(h^m) = \langle c_k, (1+y)^m \rangle = \binom{m}{p^k} \langle c_k, y^{p^k} \rangle = \binom{m}{p^k}$$

If we write $m$ in its $p$-adic expansion as $m = m_0 + m_1 p + m_2 p^2 + \ldots$, then by Lucas’ theorem, $\binom{m}{p^k} \mod p = m_k$. Therefore the map $\phi^*$ carries $c_k$ to $f_k \in C(\mathbb{Z}_p, K_*)$. It is evidently a ring isomorphism, and so $\phi$ is an isomorphism of Hopf algebras.

To see that the action of $\mathbb{Z}_p^\times$ is as claimed, we first note that $\mathbb{Z}_p^\times$ acts on $\mathbb{Z}_p$ through group homomorphisms; hence it acts on $K_*[[\mathbb{Z}_p]]$ through ring homomorphisms. Thus it suffices to show that for $\gamma \in \mathbb{Z}_p^\times$, $\phi(\gamma \cdot h) = \gamma \cdot \phi(h)$. The left-hand side is $\phi(h^\gamma) = \phi(h)^\gamma = (1+y)^\gamma$, whereas the right is $\gamma \cdot (1+y) = 1 + [\gamma](y)$. That these two are equal follows from the fact that $y$ is a coordinate on the formal multiplicative group.

3.5 Inverting roots of unity; the spectrum $R_n$

Define $\alpha = \beta \circ i : Z \to X$ as the composite of the Bockstein with the natural map $i : Z \to \Sigma^\infty K(\mathbb{Z}/p,n)_+$. Consider the localisation of $X$ at this element:

**Definition 3.11.** Write $R_n$ for the $E_\infty$ ring spectrum $R_n := L_{K(n)}X[\alpha^{-1}]$.

The $K(n)$-localisation in the formula above is necessary because it is not clear that $X[\alpha^{-1}]$, being a homotopy colimit of $K(n)$-local spectra, is itself local. Nonetheless, we will often suppress the $L_{K(n)}$ in the formula above, and consider the homotopy colimit as being performed in the category of $K(n)$-local spectra.
Lemma 3.13. There is a self map $\psi: X \rightarrow \psi \circ \psi$ action of $\mathbb{Z}_p^\infty$ on $X$ commuting with the natural map $\mathbb{Z}_p^\infty$. Passing to homotopy colimits, we get a well-defined map $\psi \circ \psi$.

Proof. Proposition 3.5 implies that $K_*(X[\alpha^{-1}]) = K_*(X)[\alpha^{-1}] = C(\mathbb{Z}_p^\infty, K_*)$ since $\alpha = \beta_0 = \xi_0 \psi^n(n)$. Note that $m \in \mathbb{Z}_p$ if and only if $f_0(m) = m \mod p$ is invertible. Therefore $C(\mathbb{Z}_p^\infty, K_*)[\alpha^{-1}]$ may be identified as $C(\mathbb{Z}_p^\infty, K_*)$.

We would like a $\mathbb{Z}_p^\infty$-equivariant version of this result. However, it is not immediately obvious that the action of $\mathbb{Z}_p^\infty$ on $X$ localises to an action on $X[\alpha^{-1}]$, as we have not been very careful with our point-set level construction of the localisation. Happily, we may avoid this issue by simply constructing a map which a topological generator $g$ of $\mathbb{Z}_p^\infty$ “should” act by, and verify that induces the expected action in $K_*$.

Diagram 3.2.3 naturally extends to the left along $i: Z \rightarrow \Sigma^\infty K(Z/p,n)_+$ to give

$$
\psi \circ \alpha \simeq \alpha \circ \zeta: Z \rightarrow X
$$

since $\zeta \simeq \psi^g$ on $Z$. Along with the fact that $\psi^g$ acts on $X$ through ring spectrum maps, we get a homotopy commuting diagram:

$$
\begin{array}{ccccccc}
X & \xrightarrow{=\psi^g} & Z^{-1} \otimes Z \otimes X \xrightarrow{1 \otimes \alpha \otimes 1} Z^{-1} \otimes X \otimes X \xrightarrow{1 \otimes \mu} Z^{-1} \otimes X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\zeta^{-1} \otimes \psi^g} & Z^{-1} \otimes Z \otimes X \xrightarrow{1 \otimes \alpha \otimes 1} Z^{-1} \otimes X \otimes X \xrightarrow{1 \otimes \mu} Z^{-1} \otimes X.
\end{array}
$$

In short, $m_\alpha \circ \psi^g \simeq (\zeta^{-1} \otimes \psi^g) \circ m_\alpha$.

Therefore, this diagram commutes:

$$
\begin{array}{ccccccc}
X & \xrightarrow{m_\alpha} & Z^{-1} \otimes X \xrightarrow{1 \otimes m_\alpha} Z^{-1} \otimes (Z^{-1} \otimes X) \xrightarrow{1 \otimes 1 \otimes m_\alpha} \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\psi^g} & Z^{-1} \otimes X \xrightarrow{1 \otimes m_\alpha} Z^{-1} \otimes (Z^{-1} \otimes X) \xrightarrow{1 \otimes 1 \otimes m_\alpha} \ldots
\end{array}
$$

Passing to homotopy colimits, we get a well-defined map $\psi^g: X[\alpha^{-1}] \rightarrow X[\alpha^{-1}]$. Since this commutes with the natural map $X \rightarrow X[\alpha^{-1}]$, we see that the action of $\psi^g$ on $K_*X[\alpha^{-1}]$ is the localisation of the action of $\psi^g$ on $K_*X$. Summarising, we have:

Lemma 3.13. There is a self map $\psi^g: X[\alpha^{-1}] \rightarrow X[\alpha^{-1}]$ whose induced map in $K_*X[\alpha^{-1}] = C(\mathbb{Z}_p^\infty, K_*)$ can be identified with translation of functions by $g$.

---

\(^9\)In section 3.3, we will show that $X[\alpha^{-1}]$ is homotopy equivalent to the spectrum $E_n^{\mathbb{A}, \mathbb{S}, \mathbb{G}, \mathbb{L}}$ which has an evident action of $\mathbb{Z}_p^\infty$ that rectifies the action that we are failing to construct here.
3.6 Invertible spectra as homotopy fibres

Proposition 3.14. Let \( \gamma \in \mathbb{Z}_p^\times \). The homotopy fiber \( F_\gamma \) of \( (\psi^g - \gamma) : R_n \to R_n \) is an invertible spectrum. When \( \gamma = 1 \), \( F_1 \) is equivalent to \( S \).

Proof. As in Proposition 3 of [Lur10], it is straightforward to see that \( \psi^g - \gamma \) is surjective in \( \mathbb{K}_\ast \), with kernel consisting of those functions \( f : \mathbb{Z} \times p \to \mathbb{K}_\ast \) which satisfy \( f(gx) = \gamma f(x) \). As such a function is determined by its value on 1, this has rank one over \( \mathbb{K}_\ast \). Thus \( \mathbb{K}_\ast(F_\gamma) = \ker((\psi^g - \gamma)_\ast) \) is rank one over \( \mathbb{K}_\ast \), and hence invertible.

When \( \gamma = 1 \), the kernel consists of those functions on \( \mathbb{Z}_p^\times \) which are invariant under translation by \( g \), namely the constant functions. As in Lemma 2.5 of [HMS94], the unit of the ring spectrum \( R_n \) – induced by the basepoint inclusion in \( K(\mathbb{Z}_p, n + 1) \) – lifts to \( F_1 \), and carries \( \mathbb{K}_\ast(S) \) onto these functions, yielding the desired isomorphism in \( \mathbb{K}_\ast \).

One should think of the second statement in this Proposition as saying that \( S \) is the homotopy fixed point spectrum for an action of \( \mathbb{Z}_p^\times \) on \( R_n \) that we have been too lazy to actually construct.

Definition 3.15. Denote by \( G \) the invertible spectrum \( F_g \).

We will show below that \( G = F_g \) is in fact the spectrum \( S(\det_+) \) of the introduction. With Theorem 1.2 in hand, this is largely an exercise in notation; until that result is proven, we must apologise for the multiplicity of different symbols for the same object.

Proposition 3.16. The assignment \( \gamma \mapsto F_\gamma \) defines a homomorphism \( \mathbb{Z}_p^\times \to \text{Pic}_n \).

Proof. We must show that \( F_{\gamma \sigma} \simeq F_\gamma \otimes F_\sigma \). Now \( \psi^g \) is a map of ring spectra, so this diagram homotopy commutes:

\[
\begin{array}{ccc}
F_\gamma \otimes F_\sigma & \xrightarrow{\gamma \otimes \sigma} & F_\gamma \otimes F_\sigma \\
\downarrow i & & \downarrow i \\
R_n \otimes R_n & \xrightarrow{\psi^g \otimes \psi^g} & R_n \otimes R_n \\
\downarrow \mu & & \downarrow \mu \\
R_n & \xrightarrow{\psi^g} & R_n
\end{array}
\]

Since \( \gamma \otimes \sigma = \gamma \sigma \), we see that \( \mu \circ i \) lifts to \( F_{\gamma \sigma} \); this map is easily seen to be an isomorphism in \( \mathbb{K}_\ast \).

Proposition 3.17. The canonical map \( \delta : G \to X[\alpha^{-1}] \) from the homotopy fibre \( G = F_g \) lifts to a map \( \rho : G \to X \):

\[
\begin{array}{ccc}
X & \xrightarrow{\psi^g - g} & X \\
\downarrow j & & \downarrow j \\
G & \xrightarrow{\delta} & X[\alpha^{-1}] \xrightarrow{\psi^g - g} X[\alpha^{-1}]
\end{array}
\]

Furthermore, there are equivalences of ring spectra

\[
X[\rho^{-1}] \xrightarrow{\sim} X[\alpha^{-1}][\delta^{-1}] \xrightarrow{\sim} X[\alpha^{-1}].
\]
Here \( j \) is the natural map from a ring spectrum to its localisation.

**Proof.** We note that since \( G \) is invertible it is a compact object, and so the map \( \delta \) must lift to one of the terms in the telescope. That is, there must then be a map \( \rho : G \to \mathbb{Z}^{-n} \otimes X \) which lifts \( \delta \). Composing with \( m_k \) an appropriate number of times if necessary, we may take \( n \) to be a multiple of \( p - 1 \), and get the desired map \( \rho : G \to X \).

The right equivalence is the localisation map at \( \delta \). By the proof of Proposition 3.14, the image of a generator for \( G \) under \( \delta \) is \( f_0(x) = x \mod p \). As this is clearly invertible in \( C(Z, K_\ast) = K_\ast(X[\alpha^{-1}]) \), \( m_\delta \) is an equivalence, and thus the directed system defining \( X[\alpha^{-1}][\delta^{-1}] \) is constant.

The left equivalence is induced by the map \( X \to X[\alpha^{-1}] \) after localisation at \( \rho \) (which maps to localisation at \( \delta \), since \( j \rho = \delta \)). By the same argument as above, the image of a generator for \( G \) under \( \rho \) is a function \( f : \mathbb{Z}_p \to K_\ast \) which restricts to \( f_0 \) along the inclusion \( \mathbb{Z}_p^\times \subseteq \mathbb{Z}_p \). Replacing \( \rho \) with \( m_k^{-1} \cdot \rho \) if necessary, we may assume that \( f \) vanishes on \( \mathbb{Z}_p \setminus \mathbb{Z}_p^\times \), and so \( f = f_0 \). Thus the localisation map \( X[\rho^{-1}] \to X[\alpha^{-1}][\delta^{-1}] \) is an \( K_\ast \)-isomorphism.

We note that since \( \text{im}(\rho_\ast) \) is generated by the class \( 10 \), we may conclude that \( \dim(G) = \dim(f_0) = 2g(n) \). We additionally record the following for later use:

**Proposition 3.18.** The spectrum \( Z \) is homotopy equivalent to \( F_\zeta \), via a map making the following diagram (whose bottom row is a fibre sequence) commute:

\[
\begin{array}{ccc}
Z & \xrightarrow{\alpha} & X \\
\downarrow{\psi^g - \zeta} & & \downarrow{\psi^g - \zeta} \\
F_\zeta & \xrightarrow{j} & X[\alpha^{-1}] \\
\end{array}
\]

**Proof.** Consider the composite

\[
L_{K(n)}K(Z/p,n) \xrightarrow{\pi} L_{K(n)}K(Z/p,n) \xrightarrow{\beta} X \xrightarrow{\psi^g} X
\]

Using the fact that \( \psi^g \circ \beta = \beta \circ \psi^\zeta \), we have

\[
\psi^g \circ \beta \circ \pi = \beta \circ \psi^\zeta \circ \left( \frac{1}{p - 1} \sum_{k=0}^{p-1} \zeta^{-k} (\psi^\zeta)^k \right)
\]

\[
= \beta \circ \zeta \cdot \pi
\]

\[
= \zeta \cdot \beta \circ \pi.
\]

Thus \( (\psi^g - \zeta) \circ j \circ \alpha = 0 \), and giving rise to the dashed arrow. The fact that it is a \( K_\ast \)-isomorphism (and hence equivalence) follows from inspection of the images of fundamental classes in \( K_\ast(X[\alpha^{-1}]) \); they are both generated by \( f_0 \).

\[\text{This class is detected by the primitive element } y \in K^\ast X. \text{ We thank Mike Hopkins, Jacob Lurie, and Eric Peterson for pointing out that this observation may be promoted to the claim that } G \text{ can be constructed as } \Sigma \text{ Cotor}_X(S,S) ; \text{ here } X \text{ is a coalgebra spectrum, and } \text{Cotor}_X(S,S) \text{ is the associated (reduced) cobar construction. See [Pet13] for details on this point of view and an algebro-geometric interpretation.}\]

---

10 This class is detected by the primitive element \( y \in K^\ast X \). We thank Mike Hopkins, Jacob Lurie, and Eric Peterson for pointing out that this observation may be promoted to the claim that \( G \) can be constructed as \( \Sigma \text{ Cotor}_X(S,S) \); here \( X \) is a coalgebra spectrum, and \( \text{Cotor}_X(S,S) \) is the associated (reduced) cobar construction. See [Pet13] for details on this point of view and an algebro-geometric interpretation.
3.7 Splitting $R_n$

There is an analogue for $R_n$ of the splitting of $p$-adic $K$-theory into a wedge of $p - 1$ Adams summands:

**Proposition 3.19.** There is an equivalence

$$\bigvee_{k=0}^{p-2} G^\otimes k \otimes R_n^{h\mu_p-1} \to R_n.$$ 

**Proof.** There is a natural forgetful map $R_n^{h\mu_p-1} \to R_n$. Additionally, one may produce a map $\sqrt[p-2]{G^\otimes k} \to R_n$ by wedging together powers of $\delta$; the product of these maps gives the desired equivalence.

To see that this map is an isomorphism in $K_*$ (and hence an equivalence), we note that since

$$K_*(R_n) = C(\mathbb{Z}_p^\times, F_{p^n}) = F_{p^n}[f_0, f_1, f_2, \ldots] / (f_0^{p-1} - 1, f_k^p - f_k),$$

then $K_* R_n^{h\mu_p-1}$ is the subspace generated by monomials in the $f_i$ whose total degree is a multiple of $p-1$, since $\psi^\vee(f_k) = \zeta f_k$. The whole space is a free module over this subalgebra, generated by the classes $\{1, f_0, f_0^2, \ldots, f_0^{p-2}\}$, which are the images of $G^\otimes k$ under $\delta^k$.

\[\square\]

3.8 The Morava module of $R_n$

**Proposition 3.20.** There are ring isomorphisms

$$C(\mathbb{Z}_p, E_{n*}) \to E_{n*}^\vee(X) \quad \text{and} \quad C(\mathbb{Z}_p^\times, E_{n*}) \to E_{n*}^\vee(R_n)$$

In this description, the action of $G_n$ on both of these Morava modules is induced by $\det : G_n \to \mathbb{Z}_p^\times$ and the natural action of $\mathbb{Z}_p^\times$ on $\mathbb{Z}_p$ and $\mathbb{Z}_p^\times$.

**Proof.** The proof of the first fact is based on the related computation of $E_{n*}^\vee, E_n$ given in [Hov04, section 2]. We note that since $K_{n*}(X)$ supports the multiplicative formal group (Prop. 3.2), and $E_n^*(X)$ is a deformation of it, $E_n^*(X)$ also supports the multiplicative group. Consequently, there is a coordinate $\bar{y} \in E_n^*(X)$ which reduces modulo $m$ to $y \in K^*(X)$.

Since $\bar{y}$ is a coordinate on the multiplicative group, the action of the Adams operations $\psi^k$ on $X$ gives a continuous, exponential map

$$a : \mathbb{Z}_p \to E_n^*(X) \quad \text{by} \quad k \mapsto \psi^k(1 + \bar{y})$$

This extends linearly over $E_{n*}$ to give a continuous ring homomorphism $a : E_{n*}[\mathbb{Z}_p] \to E_n^*(X)$. Writing $\mu : X \otimes X \to X$ for the multiplication in $X$, $\mu^k$ is the coproduct on $E_n^*(X)$. Then since $\psi^k : X \to X$ is a map of ring spectra and $\bar{y}$ is a coordinate on the multiplicative group,

$$\mu^*(\psi^k(1 + \bar{y})) = (\psi^k \otimes \psi^k)(\mu^*(1 + \bar{y})) = (\psi^k \otimes \psi^k)(1 \otimes 1 + \bar{y} \otimes 1 + 1 \otimes \bar{y} + \bar{y} \otimes \bar{y}) = \psi^k(1 + \bar{y}) \otimes \psi^k(1 + \bar{y}).$$

Therefore $a$ is a map of Hopf algebras. Applying the functor $\text{Hom}_{E_{n*}}^C(-, E_{n*})$ of continuous $E_{n*}$-module homomorphisms into $E_{n*}$ yields a Hopf algebra map

$$a^* : \text{Hom}_{E_{n*}}^C(E_n^*(X), E_{n*}) \to \text{Hom}_{E_{n*}}^C(E_{n*}[\mathbb{Z}_p], E_{n*}) \cong C(\mathbb{Z}_p, E_{n*}).$$
The identification of the codomain uses the fact that $E_{\infty,*}[\mathbb{Z}_p]$ is free. Similarly, the domain is

$$\text{Hom}^c_{E_{\infty,*}}(E^n_{\infty,*}(X), \lim_{I} \text{Hom}(E^n/I)_*) = \lim_{I} \text{Hom}((E^n/I)^*_*)(X), (E^n/I)_*) = \lim_{I} \text{Hom}(E^n/I)_*(X) = E^n_{\infty,*}(X)$$

where the limit is taken over the ideals $I = (p^n, u_1^{i_1}, \ldots, u_{n-1}^{i_{n-1}})$. We know that $E^n_{\infty,*}(X)$ is pro-free, concentrated in even dimensions, with reduction modulo $m$ isomorphic to $K_s(X) = C(\mathbb{Z}_p, \mathbb{F}_p[u^\pm 1])$.

The ring $C(\mathbb{Z}_p, E_{\infty,*})$ is also pro-free, and $a^* : E^n_{\infty,*}(X) \to C(\mathbb{Z}_p, E_{\infty,*})$ reduces modulo $m$ to the isomorphism $\phi^* : (E^n/I)_*(X) \to C(\mathbb{Z}_p, \mathbb{F}_p[u^\pm 1])$ of Prop. 3.10; it is then an isomorphism itself.

The same localisation technique as in the proof of Cor. 3.12 yields the corresponding result for $R_n$. Specifically, the image of the fundamental class of $Z$ under $\alpha$ is a function $B_0 : \mathbb{Z}_p \to E_{\infty,*}$ whose reduction modulo $m$ is $b_0 = \xi f_0 u^g(n)$. As in the residue field, an element $m \in \mathbb{Z}_p$ is invertible if and only if $B_0(m)$ is.

To see that the $G_n$ action is as claimed, we employ Peterson’s adaptation [Pet11] of Ravenel-Wilson’s results [RWS0] to identify the formal spectrum $\text{Spf} E^n_{\infty,*}K(\mathbb{Z}_p, n+1)$ in terms of the $n$th exterior power of the $p$-divisible group associated to $E_n$. Concretely, we recall that the action of $S_n$ on $E^n_{\infty,*}(K(\mathbb{Z}_p, 2)) \cong E^n_{\infty,*}(K(\mu_{p^n}, 1))$ is via the defining action of $O_n$ on $\Gamma_n$. Furthermore the Hopf ring circle product satisfies $a \circ b = -b \circ a$. Therefore, the action of $S_n$ on classes in $E^n_{\infty,*}(K(\mathbb{Z}_p, n+1)) \cong E^n_{\infty,*}(K(\mu_{p^n}, n))$ lying in the image of the iterated circle product

$$E^n_{\infty,*}(K(\mu_{p^n}, 1)^n) \to E^n_{\infty,*}(K(\mu_{p^n}, n))$$

is via the $n$th exterior power of the defining action. All classes in $K_s(\mu_{p^n}, n))$ may be obtained this way, and the above analysis lifts this statement to $E^n_{\infty,*}$. Therefore the action of $S_n$ must be via det. Examining the action of elements of $\text{Gal}(\mathbb{F}_p^\infty/\mathbb{F}_p)$, we note that if $\sigma$ is the Frobenius homomorphism, then

$$\sigma \cdot y = \sigma \cdot (\xi x u^g(n)) = \xi^p x u^g(n) = \xi^{p-1} y = (-1)^{n-1} y,$$

Here, $x u^g(n)$ is fixed by the action of Frobenius, since it descends to a class in $K(n)^*(X)$. This yields the result.

**Theorem 3.21.** The space of $E_{\infty,*}$ maps $R_n \to E_n$ is an infinite loop space with contractible components, the set of which is isomorphic to $\mathbb{Z}_{p}^\infty$. Furthermore, the action of $G_n \simeq \text{Aut}_{E_{\infty,*}}(E_n)$ on this set is via the homomorphism $\det_f : G_n \to \mathbb{Z}_{p}^\times$

**Proof.** The proof is essentially the same as the main technical result of [SW11], so we will be brief. The Goerss-Hopkins-Miller obstruction machinery shows that the higher homotopy groups of $\text{Map}_{E_{\infty,*}}(R_n, E_n)$ vanish if the cotangent complex for the map $\mathbb{F}_p \to (E^n_0)_0(R_n)/m$ is contractible. But the latter is

$$(E^n_0)_0(R_n)/m \cong C(\mathbb{Z}_p^\infty, \mathbb{F}_p^\infty) \cong \mathbb{F}_p^\infty[f_0, f_1, f_2, \ldots]/(f_0^{p-1} - 1, f_k^p - f_k).$$

The cotangent complex is contractible because the Frobenius on this ring is evidently an isomorphism.

The set of components of $\text{Map}_{E_{\infty,*}}(R_n, E_n)$ is then the set of continuous $E_{\infty,*}$-algebra homomorphisms $\text{Hom}_{E_{\infty,*}-alg}(E^n_{\infty,*}R_n, E_{\infty,*})$, which may be identified with $\mathbb{Z}_{p}^\infty = G_n/SG_{n}^\times$ by the previous result.

\[21\]
Reduction of a chosen generator of the group of such maps \( R_n \rightarrow E_n \) modulo \( m \) gives a natural transformation from \( R_n \) to \( K \) which will be useful in the following sections.

**Proposition 3.22.** The cohomology class \( 1 + y = \varphi \mod m \in K^*(X) = [X, K] \) extends over \( R_n \) to give a map of \( A_\infty \)-ring spectra \( t : R_n \rightarrow K \).

*Proof.* It was shown in [SW11] that the homotopy class \( 1 + y \) contains an \( A_\infty \) representative\(^{11}\) using Hochschild cohomology methods. Thus it suffices to show that \((1 + y) \circ \alpha : Z \rightarrow K\) is invertible. This follows, since the image of the fundamental class of \( Z \) under \( \alpha \) is \( b_0 \), and \((1 + y, b_0) = \xi \).

\[ \square \]

## 3.9 \( R_n \) as a homotopy fixed point spectrum

We now lift the results of Theorem 3.21 to an equivalence between \( R_n \) and \( E^{hSG}_{\mathbb{Z}} \). Some setup is required to study such homotopy fixed point spectra. Let \( H \) be a closed subgroup of \( \mathbb{G}_n \), and let \( A \) be a \( K(n) \)-local \( E_\infty \) ring spectrum, with the property that its Morava module is isomorphic to the ring of continuous functions on \( \mathbb{G}_n/H \):

\[ E_n^\mathbb{U}A \cong C(\mathbb{G}_n/H, E_\mathbb{U}). \]

Let \( K \leq \mathbb{G}_n \) be another closed subgroup. A consequence of [DH04] is that the homotopy fixed point spectrum \( E^{hK}_n \) admits a model which is an \( E_\infty \)-ring spectrum. When \( K = U \) is open, Devinatz-Hopkins construct a fibrant cosimplicial \( E_\infty \)-algebra whose totalisation is the spectrum \( E^{hU}_n \). This cosimplicial spectrum is a rectification of an \( h_\infty \) cosimplicial \( E_\mathbb{U} \)-Adams resolution of \( E^{hU}_n \), were it to exist (see also [BD10]).

Following their lead, but using their result that \( E^{hU}_n \) does exist, we will consider an actual cosimplicial \( K(n) \)-local \( E_n \) Adams resolution of \( E^{hU}_n \). Define

\[ B^s_U = E_n^{\otimes s+1} \otimes E^{hU}_n; \quad s \geq -1. \]

Taking \( s \geq 0 \), one may equip \( B^s_U \) with the structure of a cosimplicial spectrum in a familiar fashion, inserting units for coface maps, and applying multiplication for codegeneracies. We may freely replace \( B^s_U \) with a fibrant cosimplicial spectrum without changing the homotopy type of the terms \( B^s_U \). The natural coaugmentation \( E^{hU}_n = B^1_U \rightarrow \text{Tot}(B^s_U) \) is a weak equivalence, as can be seen from the associated Adams spectral sequence, and Theorem 1.(iv) of [DH04]. Lastly \( \text{Tot}(B^s_U) \), being the totalisation of a cosimplicial \( E_\infty \) algebra, is itself \( E_\infty \); the coaugmentation is an equivalence of \( E_\infty \) algebras.

**Lemma 3.23.** Let \( U \) be an open subgroup of \( \mathbb{G}_n \). The space of maps of \( E_\infty \) ring spectra from \( A \) to \( E^{hU}_n \) has contractible components. If \( U \) is a subgroup of \( H \), then there is a bijection from \( \pi_0(\text{Map}_{E_\infty}(A, E^{hU}_n)) \) to \( \mathbb{G}_n/H \); otherwise it is empty. Lastly, if \( U = H \), each of these maps is an equivalence.

*Proof.* The Morava module of \( B^s_U \) may be identified as the ring \( C(\mathbb{G}_n^{s+1} \times (\mathbb{G}_n/U), E_\mathbb{U}) \). By the same sort of argument as in the previous section, we see that the cotangent complex for the reduction of the degree 0 part of this algebra by \( m \) (that is, \( C(\mathbb{G}_n^{s+1} \times (\mathbb{G}_n/U), \mathbb{F}_p^\alpha) \)) is contractible. Thus

\[ \text{11} \]Strictly speaking, we showed this for \( p = 2 \), and with \( K(n) \) in place of \( K_n \). The same methods apply here.
the space \( \text{Map}_{E_n}(A, B_U^\infty) \) of maps of \( E_\infty \) ring spectra from \( A \) to \( B_U^\infty \) has contractible components which are in bijection with the set \( \text{Map}(G_n^{s+1} \times G_n \left/ (G_n/U), (G_n/H)\right.) \):

\[
\pi_0 \text{Map}_{E_n}(A, B_U^\infty) \cong \text{Hom}_{E_n, \text{alg}/E_n}^\delta(E_n^{s+1}(A), E_n^{s+1}(B_U^\infty))
\]

\[
= \text{Map}^\delta(G_n^{s+1} \times (G_n/U), (G_n/H))^G_n
\]

\[
= \text{Map}^\delta(G_n^{s+1} \times G_n \left/ (G_n/U), (G_n/H)\right.)
\]

Here, we are equipping all of these function spaces with the discrete topology; this is the meaning of the superscript \( \delta \). The discrete topology on \( \text{Map}(G_n^{s+1} \times G_n \left/ (G_n/U), (G_n/H)\right.) \) is not homeomorphic to the natural (compactly generated compact-open) one, which we will denote with a superscript \( co \). However, since \( G_n / H \) is totally disconnected, we note that for all \( s \), the evaluation at any point in \( \Delta^s \) gives a bijection

\[(*) \quad \text{Map}(\Delta^s, \text{Map}^{co}(G_n^{s+1} \times G_n \left/ (G_n/U), (G_n/H)\right.)) = \text{Map}^{co}(G_n^{s+1} \times G_n \left/ (G_n/U), (G_n/H)\right.).\]

Additionally, since \( \text{Map}^\delta \) is equipped with the discrete topology, the same evaluation yields a bijection

\[(***) \quad \text{Map}(\Delta^s, \text{Map}^\delta(G_n^{s+1} \times G_n \left/ (G_n/U), (G_n/H)\right.)) = \text{Map}^\delta(G_n^{s+1} \times G_n \left/ (G_n/U), (G_n/H)\right.).\]

We note that \( G_n^{s+1} \times G_n \left/ (G_n/U)\right. \) is the \( s \)-th term of a simplicial space, the Borel construction \( EG_n \times G_n \left/ (G_n/U)\right. = G_n^{s+1} \times G_n \left/ (G_n/U)\right. \). Therefore \( \text{Map}^{co}(G_n^{s+1} \times G_n \left/ (G_n/U), (G_n/H)\right.) \) is the \( s \)-th term of the cosimplicial space of continuous functions

\[\text{Map}^{co}(EG_n \times G_n \left/ (G_n/U), (G_n/H)\right.).\]

Being a function object from a simplicial space, this is a fibrant cosimplicial space.

In contrast, \( \text{Map}^\delta(G_n^{s+1} \times G_n \left/ (G_n/U), (G_n/H)\right.)) \) is the \( s \)-th term of the discretisation of this cosimplicial space, \( \text{Map}^\delta(EG_n \times G_n \left/ (G_n/U), (G_n/H)\right.)) \). It is also fibrant, being a cosimplicial set. The identity is a continuous map

\[\text{Tot}(\text{Map}^\delta(EG_n \times G_n \left/ (G_n/U), (G_n/H)\right.)) \to \text{Tot}(\text{Map}^{co}(EG_n \times G_n \left/ (G_n/U), (G_n/H)\right.)).\]

We have no expectation that this map is a homeomorphism. However, equations \((*)\) and \((***)\) imply that it is a bijection. In fact, this totalisation can be naturally computed as follows:

\[\text{Tot}(\text{Map}^{co}(EG_n \times G_n \left/ (G_n/U), (G_n/H)\right.)) \cong \text{Map}^{co}(EG_n \times G_n \left/ (G_n/U), (G_n/H)\right.)) \cong \text{Map}^{co}(EG_n \left/ (G_n/H)\right. U \cong (G_n/H)^U.\]

Here \( \cong \) indicates homeomorphism. The last homeomorphism is evaluation at the basepoint in \( EG_n \); it is a homeomorphism since \( EG_n \) is connected and \( G_n / H \) is totally disconnected. We may conclude that evaluation at the basepoint is a bijection

\[\text{Tot}(\text{Map}^\delta(EG_n \times G_n \left/ (G_n/U), (G_n/H)\right.)) = (G_n/H)^U.\]

If \( U \) is not a subgroup of \( H \), there is an element \( u \in U \) that acts nontrivially on \( G_n / H \), and so the set of fixed points is empty. Conversely, if \( U \leq H \), the action is trivial.

Now, since \( E_n^{hU} \simeq \text{Tot}(B_U^\infty) \) is (equivalent to) the totalisation of a cosimplicial \( E_\infty \)-algebra, we have

\[\text{Map}_{E_\infty}(A, E_n^{hU}) \simeq \text{Tot}(\text{Map}_{E_n}(A, B_U^\infty)).\]
Taking $B_U^\bullet$ to be a fibrant cosimplicial $E_\infty$-algebra, $\text{Map}_{E_\infty}(A, B_U^\bullet)$ is a fibrant cosimplicial space. Now, the projection to the set of components,

$$\text{Map}_{E_\infty}(A, B_U^\bullet) \rightarrow \pi_0 \text{Map}_{E_\infty}(A, B_U^\bullet) = \text{Map}^\delta(EG_n \bullet \times G_n / G_n / U, G_n / H)$$

is a levelwise equivalence of fibrant cosimplicial spaces; the fibrancy of domain and codomain yields an equivalence of totalisations. Therefore $\text{Map}_{E_\infty}(A, E_n^{hU})$ is empty if $U$ is not a subgroup of $H$ and has components in bijection with $G_n / H$ if it is.

Finally, if $H = U$, the Morava modules of both $A$ and $E_n^{hU}$ are both isomorphic to $C(G_n / U, E_n \ast)$, so every element of the $E_\infty$ mapping space implements an isomorphism of Morava modules, and hence an equivalence.

We cannot apply this result to get a map from $R_n$ to $E_n^{hSG_n^\pm}$, since $SG_n^\pm$ is not finite index, and so does not contain any open subgroups. However, we recall that for an arbitrary closed subgroup $K \leq G_n$ (such as $SG_n^\pm$), $E_n^{hK} = L_{K(n)}(\text{colim}_U E_n^{hU})$ where the colimit is taken over open subgroups $U$ of $G_n$ containing $K$.

Similarly, we may express $R_n$ as a homotopy colimit over ring spectra having an appropriate homotopy type. Specifically we may write $SG_n^\pm$ as an intersection $\bigcap_j U_j$ of open subgroups containing $SG_n^\pm$, where we may take the $U_j$ to be nested, and their projection onto $G_n / SG_n^\pm = \mathbb{Z}_p^\times$ to be of the form $(1 + p^{j+1} \mathbb{Z}_p)\times$. Writing $g$ for a topological generator of $\mathbb{Z}_p^\times$, this subgroup is generated by $g^{(p-1)p^j}$.

Define

$$R_n^j := \text{hofib}(1 - \psi g^{(p-1)p^j} : R_n \rightarrow R_n).$$

It is easy to verify that the Morava module of $R_n^j$ is

$$(E_n^\ast), R_n^j \cong C(G_n / U_j, E_n \ast) \cong C(\mathbb{Z}_p^\times / (1 + p^{j+1} \mathbb{Z}_p)\times, E_n \ast).$$

Note that there is a forgetful map $f_j : R_n^j \rightarrow R_n^{j+1}$ since $g^{(p-1)p^j+1} = (g^{(p-1)p^j})^p$. Since these maps factorize the natural forgetful map $R_n^j \rightarrow R_n$, we obtain a map

$$\text{hocolim}_{j \rightarrow \infty} R_n^j \rightarrow R_n$$

which gives an isomorphism of Morava modules, and thus an equivalence on $K(n)$-localisations. Notice that the $R_n^j$ are homotopy equalisers of a pair of $E_\infty$ maps, $1$ and $\psi g^{(p-1)p^j}$, and so are $E_\infty$ ring spectra. Since the category of $E_\infty$ ring spectra is cocomplete, so too is the indicated homotopy colimit. Thus $K(n)$-localisation of the forgetful map is an equivalence of $E_\infty$ ring spectra.

The previous lemma provides a family of $E_\infty$ equivalences $e_j : R_n^j \simeq E_n^{hU_j}$ indexed by $G_n / U_j$. Inductively choose these so that this diagram commutes

$$
\begin{array}{ccc}
R_n^j & \xrightarrow{f_j} & R_n^{j+1} \\
\downarrow{e_j} & & \downarrow{e_{j+1}} \\
E_n^{hU_j} & \xrightarrow{r_j} & E_n^{hU_{j+1}} \\
\end{array}
$$

where $r_j$ is induced by restriction along $U_{j+1} \leq U_j$. Consequently, we obtain an $E_\infty$ map

$$\text{hocolim}_{j \rightarrow \infty} R_n^j \rightarrow \text{colim}_{U_j} E_n^{hU_j}$$

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The latter colimit is cofinal in the one that Devinatz-Hopkins use to define $E_n^{hSG^\pm_n}$, so we get a zigzag of $\Omega_\infty$ equivalences:

$$R_n \leftarrow L_{K(n)}(\hocolim_{j \to \infty} R_n^j) \rightarrow L_{K(n)}(\colim_U E_n^{hU}) \rightarrow L_{K(n)}(\colim_U E_n^{hU}) = E_n^{hSG^\pm_n};$$

the composite of the rightward arrows is seen to be an equivalence since both sides have the same Morava modules.

**Corollary 3.24.** There is a weak equivalence of $\Omega_\infty$ ring spectra $R_n \simeq E_n^{hSG^\pm_n}$.

### 3.10 Gross-Hopkins duality

The previous sections imply that there is a homotopy commutative diagram

$$
\begin{array}{ccc}
R_n & \xrightarrow{\psi_g} & R_n \\
\cong & & \cong \\
E_n^{hSG^\pm_n} & \xrightarrow{\psi_g} & E_n^{hSG^\pm_n}
\end{array}
$$

where the action of $\psi_g \in \mathbb{Z}_p^\times$ on $E_n^{hSG^\pm_n}$ is from the residual $G_n/SG^\pm_n$ Morava stabiliser action. Consequently the homotopy fibres of each row are equivalent. If $n$ is odd (so $\det \pm = \det$), in [GHMR12], the lower homotopy fibre was named $S(\det)$, and its Morava module was shown to be $E_n^{\psi_g}(S(\det)) = E_n^{\psi_g}(\det)$. If $n$ is even we note that the squares of the characters agree: $\det \pm^2 = \det^2$.

In [HCG94] it was shown that this is the same Morava module as for $\Sigma^{n-n^2} I_n$, where $I_n$ is the Brown-Comenetz dual of $M_n S^0$, the $n$th monochromatic layer of the sphere spectrum. It was also shown that for $2p-2 \geq \max\{n^2, 2n+2\}$, an invertible spectrum is determined by its Morava module. We conclude:

**Corollary 3.25.** If $n$ is odd, there is an equivalence $G \simeq S(\det)$. When $2p-2 \geq \max\{n^2, 2n+2\}$, these may also be identified with $\Sigma^{n-n^2} I_n$. Also for large primes, if $n$ is even, then $G^{\otimes 2} \simeq S(\det)^{\otimes 2} \simeq \Sigma^{2n-n^2} I_n^{\otimes 2}$.

### 4 Thom spectra and characteristic classes

#### 4.1 Orientations and cannibalistic classes for $R_n$

We briefly review the theory of [May77] [ABG+08] for orientations of Thom spectra, especially from the point of view of [May09]. Recall that for an $E_\infty$ ring spectrum $R$, $\text{gl}_1 R$ is the spectrum whose infinite loop space is $\text{GL}_1 R$, the space of units in $\Omega_\infty R$. Further, $\text{gl}_1$ is functorial for maps of $\Omega_\infty$-ring spectra.

Let $\eta : S \to R_n$ be the unit of $R_n$, and define $b(S, R_n)$ to be the cofibre of $\text{gl}_1 \eta : \text{gl}_1 S \to \text{gl}_1 R_n$. Write $B(S, R_n) = \Omega_\infty b(S, R_n)$; then there is an equivalence

$$B(S, R_n) \simeq B(\ast, \text{GL}_1 S, \text{GL}_1 R_n)$$

between $B(S, R_n)$ and the bar construction for the action of $\text{GL}_1 S$ on $\text{GL}_1 R_n$ via $\text{GL}_1(\eta)$. The fibre sequence of infinite loop spaces associated to this construction is of the form

$$\text{GL}_1 S \xrightarrow{\eta} \text{GL}_1 R_n \xrightarrow{\gamma} B(S, R_n) \rightarrow B \text{GL}_1 S$$

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We recall that (homotopy classes of) maps of spaces $\zeta : Y \to B\text{GL}_1 S$ define Thom spectra with “fibre” $S$ – for such a map $\zeta$, the Thom spectrum $Y^\zeta$ is defined as

$$Y^\zeta = \Sigma^\infty P_+ \otimes_{\Sigma^\infty \text{GL}_1 S} S,$$

where $P$ is the homotopy fibre of $\zeta$. Furthermore (see section 3 of [May09]), the set of lifts of $\zeta$ over $\gamma$ to elements of $[Y, B(S, R_n)]$ are in bijection with $R_n$-orientations of $\zeta$, that is, Thom classes $u : Y^\zeta \to R$.

Consider a space $Y$ and a based map $f : Y \to B(S, R_n)$; write $\zeta = \gamma \circ f$. By the above, $f$ consists of the data of an orientation $u : Y^\zeta \to R_n$. Let $A \in \text{Pic}$, and note that a Thom class for $A\zeta$ is given by

$$1 \otimes u : A \otimes Y^\zeta = Y^{A\zeta} \to A \otimes R_n$$

If we insist that $A$ is a power of $G$, we may make employ the periodicity of $R_n$ to make the following definition:

**Definition 4.1.** The *normalised* Thom class $\overline{u} = \overline{u}(G^m \zeta) := \delta^m u \in R_n^0(Y)$ is defined as the composite

$$Y^G \zeta = G^m \otimes Y^\zeta \xrightarrow{1 \otimes u} G^m \otimes R_n \xrightarrow{\delta^m \otimes 1} R_n \otimes R_n \xrightarrow{\mu} R_n$$

Then for any $k \in \mathbb{Z}_+$, define $\theta_k(G^m \zeta) \in R_n^0(Y)$ by the equation

$$\psi^k(\overline{u}) = \theta_k(G^m \zeta) \cdot \overline{u}.$$  

Following Adams, we call $\theta_k(G^m \zeta)$ the (Bott) cannibalistic class of $G^m \zeta$.

**Proposition 4.2.** Let $\zeta$ and $\xi$ be Thom spectra associated to classes $f, h : Y \to B(S, R_n)$. Then

1. $\theta_k(0) = 1$.
2. $\theta_k(\zeta + \xi) = \theta_k(\zeta) \cdot \theta_k(\xi)$.
3. $\theta_k(G^m \zeta) = k^m \theta_k(\zeta)$.
4. $\theta_k(\zeta) = \psi^k(\theta_1) \cdot \theta_k$.

**Proof.** The proof that $\theta_k$ is exponential depends upon the fact that a product of Thom classes for $\zeta$ and $\xi$ is a Thom class for $\zeta + \xi$. Note that this implies that $\theta_k(\zeta)^{-1}$ exists, and is equal to $\theta_k(-\zeta)$. The third property uses the fact that $\psi^k(\delta^m \otimes u) = k^m \delta^m \otimes \psi^k(u)$. The last is an application of the equation $\psi^{kl} = \psi^k \psi^l$.

We note that this proposition implies that $\theta_k(\zeta)$ is invertible; $\theta_k(\zeta) \in R_n^0(Y)^\times$. Now for all $k$, $\psi^k : R_n \to R_n$ is a ring map, so there is a well defined map $\text{gl}_1 \psi^k : \text{gl}_1 R_n \to \text{gl}_1 R_n$. Write $\psi^k/1 : \text{gl}_1 R_n \to \text{gl}_1 R_n$ for the difference (using the *multiplicative* infinite loop space structure on $\text{GL}_1 R_n$) between this map and the identity. Since $\psi^k \circ \eta = \eta$, we see that $(\psi^k/1) \circ \text{gl}_1 \eta$ is nullhomotopic. Therefore there is a natural map $c(\psi^k) : b(S, R_n) \to \text{gl}_1 R_n$ making this diagram commute

$$\begin{array}{ccc}
\text{gl}_1 S & \xrightarrow{\text{gl}_1 \eta} & \text{gl}_1 R_n \\
\text{gl}_1 R_n & \xrightarrow{\gamma} & \Sigma \text{gl}_1 S \\
\psi^k/1 & \xrightarrow{c(\psi^k)} & \text{gl}_1 R_n
\end{array}$$
Then the cannibalistic classes may be computed via the operation $c(\psi^k)$:

$$\theta_k(\zeta) = c(\psi^k) \circ f;$$

see, e.g., Proposition 3.5 of [May09].

### 4.2 A model for $B(S, R_n)$

Let $r$ denote a topological generator for the $p$-adic units $\mathbb{Z}_p^\times$. At times it will be useful to take $r$ to be $g = \zeta(1 + p)$, where $\zeta$ is a primitive $p - 1$th root of unity. At others it will be beneficial to choose $r \in \mathbb{N}$; e.g., any positive integer whose reduction modulo $p^2$ generates $(\mathbb{Z}/p^2)^\times$. We note that for any choice, there is a cofibre sequence

$$S \xrightarrow{\eta} R_n \xrightarrow{\psi^r - 1} R_n$$

**Proposition 4.3.** There is an equivalence between

$$\Omega^\infty S = \Omega^\infty \text{hofib}(\psi^r - 1 : R_n \rightarrow R_n)$$

and the homotopy equaliser of the maps $\psi^r$ and $1 : \Omega^\infty R_n \rightarrow \Omega^\infty R_n$,

$$\text{hoeq}(\psi^r, 1 : \Omega^\infty R_n \rightarrow \Omega^\infty R_n).$$

**Proof.** An element of the homotopy fibre consists of a pair $(x, f)$, where $x \in \Omega^\infty R_n$, and $f$ is a path in $\Omega^\infty R_n$, starting at the basepoint (the additive unit 0), and ending at $\psi^r(x) - x$. An element of the homotopy equaliser is a path $h$ in $\Omega^\infty R_n$, starting at a point $x$, and ending at $\psi^r(x)$. An equivalence between these spaces is gotten by sending $(x, f)$ to the path $h$ gotten from $f$ by pointwise addition of $x$.

**Proposition 4.4.** There is an equivalence between $\text{hofib}(\psi^r / 1 : \text{GL}_1 R_n \rightarrow \text{GL}_1 R_n)$ and the homotopy equaliser of the maps $\psi^r$ and $1 : \text{GL}_1 R_n \rightarrow \text{GL}_1 R_n$,

$$\text{hoeq}(\psi^r, 1 : \text{GL}_1 R_n \rightarrow \text{GL}_1 R_n).$$

The proof is the same as for the previous result, replacing addition with multiplication and 0 with 1. We note that GL$_1$ S is the union of components of $\Omega^\infty S = \text{hoeq}(\psi^r, 1)$ lying over GL$_1$ R$_n$, and so conclude that there is an equivalence

$$\text{GL}_1 S \simeq \text{hofib}(\psi^r / 1 : \text{GL}_1 R_n \rightarrow \text{GL}_1 R_n).$$

**Corollary 4.5.** For any topological generator $r$, the infinite loop map $c(\psi^r) : B(S, R_n) \rightarrow \text{GL}_1 R_n$ is an equivalence on connected components.

**Proof.** The previous result allows us to identify the lower fibre sequence in the following commuting diagram of fibrations:

$$\begin{array}{c}
\Omega B(S, R_n) \xrightarrow{\gamma} \text{GL}_1 S \xrightarrow{\eta} \text{GL}_1 R_n \xrightarrow{c(\psi^r)} B(S, R_n) \\
\downarrow c(\psi^r) \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow c(\psi^r) \\
\Omega \text{GL}_1 R_n \xrightarrow{\gamma} \text{GL}_1 S \xrightarrow{\eta} \text{GL}_1 R_n \xrightarrow{\psi^r / 1} \text{GL}_1 R_n 
\end{array}$$
The leftmost column is an equivalence, since both spaces are homotopy fibres of the same map; this gives the result.

Perhaps surprisingly, this allows us to conclude that an $R_n$-orientation of a Thom spectrum $Y^\zeta$ over a connected space $Y$ is uniquely determined by its $r^{th}$ cannibalistic class $\theta_r(\zeta)$. More concretely, examining the exact sequences gotten from mapping a connected $Y$ into to the fibrations above, we have:

\[
\cdots \to [Y, GL_1(R_n)] \to [Y, B(S, R_n)] \to [Y, BGL_1(S)] \to \cdots
\]

Corollary 4.6. An $R_n$-oriented Thom spectrum $Y^\zeta$ over connected $Y$ is trivial (i.e., equivalent to $L_K(n)\Sigma\infty Y_+$) if and only if $\theta_r(\zeta) = \psi^r(s)/s$ for some unit $s \in R_0^*(Y)$.

For $n = 1$, this appears to be a strong form of a $K(1)$-local version of Corollary 5.8 of [Ada65].

4.3 An analogue of $MU$

Definition 4.7. For a space $Y$, write $Y(0)$ for the basepoint component of $Y$, and define

$$BX_n := B(S, R_n)(0)$$

Furthermore, we let $MX_n$ be the Thom spectrum

$$MX_n := L_{K(n)}BX_n^\gamma$$

associated to the composite $BX_n = B(S, R_n)(0) \xrightarrow{\gamma} B(S, R_n) \xrightarrow{\gamma} BGL_1 S$.

Remark 4.8. This notation is meant to evoke thoughts of the complex cobordism Thom spectrum $MU = BU^\gamma$ (where $\gamma$ is the tautological virtual bundle over $BU$). Indeed, this is precisely the case when $n = 1$ (at least $K(1)$-locally), since

$$R_1 = X_1[p^{-1}] = L_{K(1)}\Sigma\infty CP_+^\infty[\beta^{-1}] \simeq L_{K(1)}K \simeq L_{HZ/p}K,$$

by Snaith’s theorem. Thus $BX_1 \simeq GL_1(L_{HZ/p}K)(0)$ is the $p$-completion of $BU$, and so $MX_1 = L_{K(1)}MU$.

Additionally, we note that $\gamma$ is an infinite loop map. Using the methods of [LMSMS6], we conclude:

Proposition 4.9. The Thom spectrum $MX_n$ is an $E_\infty$-ring spectrum.

Note that by its very nature, $\gamma$ is $R_n$-oriented, yielding a Thom isomorphism

$$R_n^*(MX) \cong R_n^*BX$$

of $R_n^*BX$-modules.
4.4 Restricting to $K(\mathbb{Z}_p, n + 1)$

The following proposition, which is immediate from the connectivity of $Y$, is an analogue of the fact that a formal power series is invertible if and only if its constant term is.

**Proposition 4.10.** Let $R$ be an $A_\infty$ ring spectrum. If $Y$ is connected and $\alpha \in R^0(Y) = [Y, \Omega^\infty R]$, then $\alpha$ lies in $R^0(Y)^\times = [Y, \text{GL}_1 R]$ if and only if the restriction of $\alpha$ to a point in $Y$ is a unit; $\alpha|_\text{pt} \in R^\times_0$.

We will denote by $j$ the natural localisation map $j : X \to R_n = X[\rho^{-1}]$. We will often confuse $j$ with the corresponding element $j \in R_n^0(K(\mathbb{Z}_p, n + 1)) \cong R_n^0(X)$. Since $j|_\text{pt} = 1$ is the unit of $R_n$, the previous criterion shows that $j \in R_n^0(X)^\times$, while $j - 1$ is not a unit. Consequently, the following is not automatic:

**Proposition 4.11.** Let $r \in \mathbb{N}$ be a topological generator of $\mathbb{Z}_p^\times$. The element $\psi^r(j - 1) \in R_n^0(X)$ is divisible by $j - 1$, and the quotient $\frac{\psi^r(j - 1)}{j - 1}$ is a unit.

**Proof.** It follows from the proof of Proposition 3.4 that $\psi^r(j)$ is simply $j$ composed with the self map of $K(\mathbb{Z}_p, n + 1)$ which is multiplication by $r$ in $\pi_{n+1}$. Furthermore, this is implemented by the $r$-fold diagonal, followed by $r$-ary multiplication in $K(\mathbb{Z}_p, n + 1)$. Since $j$ is multiplicative, we conclude that $\psi^r(j) = j^r$. Thus $\psi^r(j - 1) = j^r - 1$ is evidently divisible by $j - 1$, with quotient

$$\frac{\psi^r(j - 1)}{j - 1} = 1 + j + \cdots + j^{r-1}$$

Restricting $j$ along the unit yields 1; thus the restriction of $\frac{\psi^r(j - 1)}{j - 1}$ along the unit gives $r \in R^\times_0$, so this class is invertible. \hfill \Box

**Definition 4.12.** Let $e : K(\mathbb{Z}_p, n + 1) \to BX_n = B(S, R_n)(0)$ be the unique map satisfying

$$c(\psi^r) \circ e = r^{-1} \frac{\psi^r(j - 1)}{j - 1} \in R_n^0(X)^\times.$$

Pulling $\gamma$ back over $e$ allows us to define a Thom spectrum $K(\mathbb{Z}_p, n + 1)^\gamma_{oe}$ which, for brevity, we will write as $X^\gamma$. Then $e$ is defined in such a way that the $r^{th}$ cannibalistic class of $\gamma$ is $\theta_r(\gamma) = r^{-1} \frac{\psi^r(j - 1)}{j - 1}$.

In fact, this definition of $e$ is independent of our choice of generator $r$. An induction using part 4 of Proposition 4.2 along with the continuity of the action of the Adams operations gives:

**Proposition 4.13.** For every $k \in \mathbb{Z}_p^\times$,

$$\theta_k(\gamma) = k^{-1} \frac{\psi^k(j - 1)}{j - 1}.$$

We will devote a lot of attention to the Thom spectrum $X^{G\gamma} = G \otimes X^\gamma$, the analogue in our setting of the Thom spectrum $\Sigma^2 CP^{\infty \gamma}$ (with Thom class in dimension 2). We note then that the cannibalistic class $\theta_\gamma(G\gamma)$ satisfies

$$(j - 1) \cdot \theta_\gamma(G\gamma) = \psi^\gamma(j - 1).$$
4.5 A zero section map

**Definition 4.14.** For an $R_n$-oriented Thom spectrum $Y^\zeta$ over $Y$, define $f_g(\zeta)$ as the composite

$$
R_n \otimes Y \xrightarrow{1 \otimes \Delta} R_n \otimes Y \otimes Y \xrightarrow{\psi^g \otimes \theta_g(\zeta)^{-1} \otimes 1} R_n \otimes R_n \otimes Y \xrightarrow{\mu \otimes 1} R_n \otimes Y
$$

**Proposition 4.15.** Let $Y^{G\zeta}$ be $R_n$-oriented with Thom class $u$ (and associated Thom isomorphism $T_u$). Then the following diagram commutes:

$$
\begin{array}{ccc}
R_n \otimes Y^{G\zeta} & \xrightarrow{T_u} & G \otimes R_n \otimes Y \\
\downarrow{\psi^g \otimes 1} & & \downarrow{g^{-1} \otimes 1} \\
R_n \otimes Y^{G\zeta} & \xrightarrow{1 \otimes f_g(\zeta)} & R_n \otimes Y
\end{array}
$$

**Proof.** It follows quickly from this definition and the fact that $\psi^g \circ \delta = g \cdot \delta$ that the right square commutes. Thus the result follows from the commutativity of the left square without the tensor factor of $G$. Here is an expanded version of that square, which commutes by inspection:

$$
\begin{array}{cccccc}
R_n \otimes Y^\zeta & \xrightarrow{1 \otimes \Delta} & R_n \otimes Y^\zeta & \xrightarrow{1 \otimes u \otimes 1} & R_n \otimes R_n \otimes Y & \xrightarrow{\mu \otimes 1} & R_n \otimes Y \\
\downarrow{\psi^g \otimes 1} & & \downarrow{1 \otimes 1 \otimes \Delta} & & \downarrow{1 \otimes \mu \otimes 1} & & \\
R_n \otimes Y^\zeta & \xrightarrow{1 \otimes 1 \otimes \Delta} & R_n \otimes Y^\zeta & \xrightarrow{1 \otimes u \otimes 1} & R_n \otimes R_n \otimes Y & \xrightarrow{\mu \otimes 1} & R_n \otimes Y \\
\downarrow{\psi^g \otimes 1 \otimes 1} & & \downarrow{\psi^g \otimes \theta_g(\zeta) \otimes u \otimes \theta_g(\zeta)^{-1} \otimes 1} & & \downarrow{1 \otimes \mu \otimes 1} & & \\
R_n \otimes Y^\zeta & \xrightarrow{1 \otimes 1 \otimes \Delta} & R_n \otimes Y^\zeta & \xrightarrow{1 \otimes u \otimes 1} & R_n \otimes R_n \otimes Y & \xrightarrow{\mu \otimes 1} & R_n \otimes Y \\
\end{array}
$$

Since the homotopy fibre of $\psi^g - 1 : R_n \to R_n$ is $S$, we see that $hofib((\psi^g - 1) \otimes 1) = Y^{G\zeta}$ in the diagram in the statement of Proposition 4.15. Since the horizontal maps are equivalences, we conclude:

**Corollary 4.16.** $hofib(g^{-1} f_g(\zeta) - 1) = hofib(f_g(\zeta) - g) = Y^{G\zeta}$.

We now focus on $X^{G\gamma}$. Define a map $\tau : X \to R_n \otimes X$ by $\tau = ((j - 1) \otimes 1) \circ \Delta$. Since $\psi^g (j - 1) = \theta_g (G\gamma) \cdot (j - 1) = g \theta_g (\gamma) \cdot (j - 1)$, we see that this commutes

$$
\begin{array}{cccccc}
X & \xrightarrow{\Delta} & X \otimes X & \xrightarrow{(j - 1) \otimes 1} & R_n \otimes X \\
\downarrow{g \tau} & & \downarrow{1 \otimes \Delta} & & \downarrow{1 \otimes \Delta} \\
X \otimes X & \xrightarrow{\Delta \otimes 1} & X \otimes X \otimes X & \xrightarrow{(j - 1) \otimes 1} & R_n \otimes X \otimes X \\
\downarrow{g \theta_g (j - 1) \otimes 1} & & \downarrow{g \theta_g (\gamma) \otimes (j - 1) \otimes 1} & & \downarrow{\psi^g \otimes \theta_g (\gamma)^{-1} \otimes 1} \\
R_n \otimes X & & R_n \otimes X & & R_n \otimes X
\end{array}
$$
Passage along the right side of this diagram gives $f_g(\gamma) \circ \varpi$, so $f_g(\gamma) \circ \varpi = g \varpi$. Thus $\varpi$ lifts to a map \[ z : X \to X^{G \gamma}. \]

which we will regard as the zero-section of the Thom spectrum.

The following result encourages us to regard $j - 1$ as the Euler class of $\gamma$ over $X$. Indeed, when $n = 1$, it is precisely the K-theoretic Euler class of the tautological line bundle over $\mathbb{C}P^\infty$.

We continue to write $e^*(u) \in R^G_n(X^{G \gamma})$ for the Thom class of $X^{G \gamma}$, and $e^*(\pi) \in R^G_0(X^{G \gamma})$ for its normalisation.

**Proposition 4.17.** $z^*(e^*(\pi)) = j - 1$.

**Proof.** We first observe that one may derive $e^*(\pi)$ from the fibre sequence for $f_g(\gamma) - g$; that is, this commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{z} & X^{G \gamma} \\
\varpi \downarrow & & \varpi \downarrow \\
R_n \otimes X & \xrightarrow{1 \otimes p} & R_n \otimes S = R_n \\
f_g(\gamma) - g \downarrow & & \\
R_n \otimes X & & \end{array}
\]

Here $p : X \to S$ is induced by the projection of $K(Z_p, n + 1)$ to a point. The commutativity of the upper right triangle follows from the diagram of fiber inclusions

\[
\begin{array}{ccc}
S \otimes X^{G \gamma} & = & X^{G \gamma} \\
\eta \otimes 1 \downarrow & & \downarrow \\
R_n \otimes X^{G \gamma} & \xrightarrow{1 \otimes \delta} & R_n \otimes R_n \otimes G \otimes X \\
1 \otimes e^*(u) \downarrow & & \downarrow \\
R_n \otimes G & \xrightarrow{\delta \otimes 1} & R_n \otimes X \\
1 \otimes p \downarrow & & \downarrow \\
R_n & & R_n,
\end{array}
\]

since passage along the lower left defines $\delta e^*(u) = e^*(\pi)$. Then

\[ z^*(e^*(\pi)) = (1 \otimes p) \circ \varpi = (1 \otimes p) \circ ((j - 1) \otimes 1) \circ \Delta = j - 1 \]

Recall that $K^*(X) = K_+[[y]]$, and the natural transformation $t : R_n \to K$ of Proposition 3.22. We note that $t_*(e^*(u))$ is the $K$-Thom class of $X^{G \gamma}$. It satisfies $z^*(t_*(e^*(u))) = y$, for

\[ z^*(t_*(e^*(u))) = t \circ (z^*(e^*(u))) = t \circ (j - 1) = (1 + y) - 1 = y. \]

We conclude:

**Proposition 4.18.** The composite of the Thom isomorphism for $X^{G \gamma}$ and $z^*$, as a map from $K^*(X) \cong K^{* + 2g(n)}(X^{G \gamma})$ to $K^{* + 2g(n)}X$ is multiplication by $y$.\footnote{The map $\varpi$ is, of course, not uniquely specified by this computation. However, any lift of $\varpi$ will serve for our purposes, as will be evident from the proof of Proposition 4.17.}
Note that the image of $z^*$ may be identified as the (split) subspace $yK_*[[y]] = \tilde{K}^*(\mathbb{Z}_p, n+1)$. Since $z^*$ is evidently injective, we obtain the main result of this section:

**Corollary 4.19.** The zero section restricts to a $K(n)$-local equivalence $z : L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1) \rightarrow X^{G\gamma}$.

The reader is likely familiar with the analogous (and much more easily proven) fact that the zero section of the Thom spectrum $\Sigma^2MU(1) = \Sigma^2\mathbb{C}P^{\infty\gamma}$ is an equivalence.

## 5 Higher orientations for chromatic homotopy theory

### 5.1 $n$-orientations and formal group laws

The map $\rho : G \rightarrow X$ allows us to extend the notion of a complex orientation of a cohomology theory to the $K(n)$-local category:

**Definition 5.1.** An $n$-orientation of a $K(n)$-local ring spectrum $E$ is a class $x \in E^G(X) = [X, G \otimes E]$ with the property that $\rho^*(x) \in E^G(G) = \pi_0E = E_0$ is a unit.

We note that since $\rho_1 : L_{K(1)}S^2 \rightarrow X$ is the $K(1)$-localisation of the inclusion $\mathbb{C}P^1 \subseteq \mathbb{C}P^{\infty}$, a $1$-orientation is precisely $K(1)$-local complex orientation.

**Examples 5.2.** The following are $n$-oriented spectra:

1. The spectrum $R_n = X[\rho^{-1}]$ is naturally $n$-oriented. Define $x \in R_n^G(X)$ by
   
   $x = \delta^{-1} \otimes (j-1) : X \rightarrow G \otimes X[\rho^{-1}]$.

   Then $\rho^*(x) = \delta^{-1} \cdot \rho^*(j-1) = \delta^{-1} \cdot \delta = 1$.

2. $K$ is $n$-oriented, via the conveniently named class $x \in K^{2g(n)}(X) = K^G(X)$. The image of $\rho$ in $K_* (X_n)$ is $b_0$, and so $\rho^*(x) = x(b_0) = 1$. More generally, any power series $f(x) \in K_*[[x]]$ which begins with a unit multiple of $x$ gives an $n$-orientation of $K$.

3. $E_n$ is $n$-oriented by a lift of the previous orientation. More carefully,
   
   $E_n^G(X) \cong E_n^{2g(n)}(X) = u^{g(n)} \cdot E_n^0(X) = u^{g(n)} \cdot W(\mathbb{F}_p \{[u_1, \ldots, u_{n-1}],[[z_p])$

   Then an orientation is given by the class of $u^{g(n)} \cdot g$, since the image of a fundamental class under $\rho$ in $E_n^*(X) \cong C(\mathbb{Z}_p, E_n^*)$ is a function which carries $g$ to a unit.

4. $MX_n$ is $n$-oriented via the map $x := (j^\gamma \otimes 1) \circ z$, as can be seen from the diagram:

   $X_n \xrightarrow{z} X^{G\gamma} \xrightarrow{=} X^\gamma \otimes G \xrightarrow{j^\gamma \otimes 1} MX_n \otimes G$

   which commutes by the proof of Proposition 4.17. Thus $\rho^*(x) = j^\gamma \circ \eta$ is the unit of the ring spectrum $MX_n$. 

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Theorem 5.3. An $n$-orientation $x$ of $E$ gives an isomorphism $E^\bullet(X) \cong E^\bullet[[x]]$, and the multiplication in $X$ equips this ring with a formal group law $F(x, y) \in E^\bullet[[x, y]]$.

Proof. We note that an $n$-orientation of $E$, $x \in E^G(X)$ defines, via the equivalence

$$z : L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n + 1) \to X^{G_{\gamma}},$$

a Thom class $u = (z^*)^{-1}(x) \in E^G(X^{G_{\gamma}})$, since restricting $u$ to each fibre gives the same class as restricting $x$ to $G$, namely, a unit. Furthermore, extending $z$ to all of $X = L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n + 1) \vee S$, we see that the augmentation ideal of $E^\bullet X$ is $E^\bullet K(\mathbb{Z}_p, n + 1) = z^*E^\bullet X^{G_{\gamma}}$. Moreover, by the Thom isomorphism, the latter is the cyclic ideal generated by $z^*u = x$.

We conclude two things: that the augmentation ideal of $E^\bullet X$ is generated by $x$, and that it is itself isomorphic to $E^\bullet X^{G_{\gamma}}$, and hence $E^\bullet X$ via the Thom isomorphism. We thereby inductively observe that the quotients of the filtration by powers of the augmentation ideal are free $E^\bullet$-modules of rank 1 generated by the powers of $x$, and so $E^\bullet X \cong E^\bullet[[x]]$.

The Künneth spectral sequence for $E^\bullet(X \otimes X)$ collapses to $E^\bullet(X_n) \otimes_{E^\bullet} E^\bullet(X_n) = E^\bullet[[x, y]]$, since each factor has free $E^\bullet$-cohomology. The properties of the formal group law all derive from the unital, associative multiplication on $X$.

\[ \Box \]

5.2 A remark on $(n - 1)$-gerbes

It is natural to ask what sort of object an $n$-orientation is orienting. We recall that a complex orientation of $E$ yields a theory of Chern classes for $E$ in which the orientation is the first Chern class of the tautological bundle. Furthermore, the formal group law of the cohomology theory encodes the Chern class of a tensor product of line bundles for the cohomology theory.

We will consider a $p$-adic $(n - 1)$-gerbe $V$ over a space $X$ to be a map $f : X \to K(\mathbb{Z}_p, n + 1)$. Here we are purposefully confusing a gerbe with its Dixmier-Douady type of characteristic class. For an $n$-oriented cohomology theory $E$, the orientation class $x$ defines a first Chern class for $V$ by the formula

$$c_1(V) := f_!(x) \in E^{E^{(\det \pm)}}(X).$$

The formal group law on $E^\bullet$ then allows one to compute $c_1(V \otimes W) = F(c_1(V), c_1(W))$.

It is not apparent to the author how to extend this notion to higher rank (i.e., non-abelian) $(n - 1)$-gerbes or higher Chern classes.

5.3 Multiplicative $n$-oriented spectra

Definition 5.4. An $n$-oriented spectrum $R$ is multiplicative if there is a unit $t \in R_\bullet$ such that the formal group law that $R$ supports on $R^\bullet(X_n)$ is given by the formula

$$F(x, y) = x + y + txy.$$

Theorem 5.5. The spectrum $R_n$ with its natural $n$-orientation is the universal (i.e., initial) multiplicative $n$-oriented spectrum.

Proof. This is argument closely follows that of Spitzweck-Østvær [SO09] for the motivic analogue of Snaith’s theorem.
First, $X_n[\rho^{-1}]$ is multiplicative. We note that $j$ is a map of ring spectra, as it is induced by the identity on $X_n$. Thus if $m$ denotes multiplication in $X_n$,

$$m^*(j) = j \otimes j \in R_n^\bullet(X_n \otimes X_n) = R_n^\bullet[[x,y]],$$

since the tensor product is multiplication in the cohomology of the smash product. Definitionally, $j \otimes 1 = 1 + \delta x$, and $1 \otimes j = 1 + \delta y$, so $j \otimes j = 1 + \delta(x + y + \delta xy)$. Thus

$$F(x, y) = m^*(x) = m^*(\delta^{-1}(j-1)) = \delta^{-1}(j \otimes j - 1) = x + y + \delta xy$$

Loosely, $R_n = X_n[\rho^{-1}]$ is universal because $j$ is initial amongst maps from $X_n$ to spectra in which $\rho$ is invertible. More carefully, let $E$ be an $n$-oriented, multiplicative spectrum, with orientation $v$, and whose formal group satisfies $F(v, w) = v + w + twv$ for some $t \in E_G$. Then there is a map of ring spectra $\phi : X_n \to E$ defined by $\phi = 1 + tv$. To check that $\phi$ is multiplicative, we need to see that $m^*(\phi) = \phi \otimes \phi \in E^\bullet(X_n \otimes X_n) = E^\bullet[[x,y]]$. But

$$m^*(\phi) = 1 + tm^*(v) = 1 + tF(v, w) = 1 + tv + tw + t^2 vw = (1 + tv)(1 + tw) = \phi \otimes \phi.$$ 

Similarly, $\phi$ is unital, since $t$ restricts to $0$ over $S^0$. We note that

$$\phi_*(\rho) = (1 + tv) \circ \rho = tv \circ \rho = t \cdot \rho^* v$$

is a product of units, so $\phi$ extends over $j$ to a map of ring spectra $\Phi : R_n = X_n[\rho^{-1}] \to E$.

Since $E$ is $n$-oriented, the map $\delta : G \to X_n$ defines a function

$$\delta^* : E^\bullet[[v]]_G = E^G(X_n) \to E^G(G) = E_0$$

which carries a power series in $v$ to the coefficient of $v$. Therefore $\phi \circ \delta = \delta^*(\phi) = t$. We see, then, that $\Phi$ is orientation preserving (i.e., $\Phi_*(x) = v$), since

$$v = t^{-1}(1 + tv - 1) = \Phi_*(\delta^{-1})(\phi - 1) = \Phi_*(\delta^{-1}(j - 1)) = \Phi_*(x).$$

Let $\Psi : X_n[\rho^{-1}] \to E$ is any other orientation-preserving map. Since it is orientation preserving, it must preserve the formal group law, so $\Psi_*(\delta) = t$. Consider the composite map $\psi := \Psi \circ j : X_n \to E$. Then

$$v = \Psi_*(x) = \Psi_*(\delta^{-1}(j - 1)) = t^{-1}(\psi - 1),$$

giving $\psi = 1 + tv = \phi$, and so $\Psi = \Phi$.

Therefore, for any multiplicative, $n$-oriented spectrum $E$, there exists a orientation-preserving map $\Psi : X_n[\rho^{-1}] \to E$, unique up to homotopy.

\[\square\]

### 5.4 Identifying the coefficients of $R_n$

There is an “integral lift” $\mathbb{W}(K)$ of the cohomology theory $K$ with homotopy groups

$$\mathbb{W}(K)_* = \mathbb{W}(F_{p^n})[u^{\pm 1}]$$

One may define $\mathbb{W}(K)$ as the $E_n$-algebra

$$\mathbb{W}(K) := E_n/(u_1, \ldots, u_{n-1}).$$

\[13\] We are not claiming that this map is highly structured, only that it preserves multiplication and units up to homotopy.

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Note that reduction modulo $p$ gives a natural transformation $\mathbb{W}(K) \to K$. Furthermore, the reduction map $E_n \to \mathbb{W}(K)$, being a ring homomorphism, carries an $n$-orientation of $E_n$ to an $n$-orientation for $\mathbb{W}(K)$.

**Proposition 5.6.** The formal group law on $\mathbb{W}(K)^{G_2}(X)$ is the universal multiplicative formal group law in the category of $\mathbb{W} (\mathbb{F}_{p^n})$-algebras.

**Proof.** $K$ supports the multiplicative formal group over $\mathbb{F}_{p^n}$, via Proposition 3.2. $\mathbb{W}(K)$ is the unique oriented map $\Phi : \mathbb{W}(\mathbb{F}_{p^n}) \to \mathbb{W}(\mathbb{F}_{p^n})$.

Recall that the Picard-graded homotopy of $E_n$ associated to powers of $G$ are

$$(E_n)_{G^r} = \mathbb{W} (\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]](s^\pm 1),$$

where $|s| = \dim(G) = 2g(n)$, following the discussion in Example 2.2. Thus $\mathbb{W}(K)_{G^r} = \mathbb{W} (\mathbb{F}_{p^n})[s^\pm 1]$.

We note that $\mathbb{Z}_p^\times$ acts on the multiplicative group $G_m$, and therefore on $\mathbb{W}(K)_{G^r}$, since the previous Proposition implies that $\mathbb{W}(K)_{G^r}$ co-represents $G_m$ in the category of $\mathbb{W}(\mathbb{F}_{p^n})$-algebras. The action (through $\mathbb{W}(\mathbb{F}_{p^n})$-algebra homomorphisms) is easily seen to be determined by the formula $\gamma \cdot s = \gamma s$.

For the next result, we recall that $R_n$ is also equipped with an action of $\mathbb{Z}_p^\times = G_n/S\mathbb{G}_n^\pm$:

**Corollary 5.7.** As a $\mathbb{Z}_p^\times$-representation, $\pi_{G^r} R_n$ contains $\mathbb{Z}_p[\rho^\pm 1]$ as a split summand.

**Proof.** We will write $\sigma$ for a lift of a primitive $p^n - 1$st root of unity in $\mathbb{F}_{p^n}$ to $\mathbb{W}(\mathbb{F}_{p^n})$, so that $\mathbb{W}(\mathbb{F}_{p^n}) = \mathbb{Z}_p[\sigma]$.

Since $R_n$ is the universal multiplicative $n$-oriented spectrum, the multiplicativity of the orientation of $\mathbb{W}(K)$ gives us a unique oriented map $\Phi : R_n \to \mathbb{W}(K)$, which induces $\Phi_{G^r} : \pi_{G^r} (R_n) \to \pi_{G^r} (\mathbb{W}(K))$. The last Proposition gives us a unique oriented map $\Xi : \pi_{G^r} (\mathbb{W}(K)) \to \pi_{G^r} (R_n)[\sigma]$. It must then be the case that $\Phi_{G^r} \circ \Xi = \text{id}$ once we extend $\Phi_{G^r}$ over $\mathbb{W}(\mathbb{F}_{p^n}) = \mathbb{Z}_p[\sigma]$. Thus $\pi_{G^r} (R_n)[\sigma]$ contains $\mathbb{W}(\mathbb{F}_{p^n})[\Xi(s)^{\pm 1}]$ as a split summand. As in the proof of Theorem 5.5, we must have $\Xi(s) = \rho$. As both maps were equivariant by universality properties, this splitting is equivariant.

We note that $\pi_0(S)$ does not contain any of the roots of unity in $\mu_{p^n-1}$ not lying in $\mu_{p-1}$. If it contained one such $\tau$, then for any $K(n)$-local ring spectrum $Y$, $\pi_0(Y)$ would be a $\mathbb{Z}_p[\tau]$-algebra. In particular, $\pi_0(K(n)) = \mathbb{F}_p[\tau]$ would be an $\mathbb{F}_p[\tau] = \mathbb{Z}_p[\tau]/p$-vector space, which is false if $\tau \notin \mu_{p-1} = \mathbb{F}_p^\times$.

Suppose now that $\tau \in \pi_0(R_n)$. Then the action of $\mathbb{Z}_p^\times$ on $\mathbb{Z}_p[\tau] \subseteq \pi_0(R_n)$ is trivial since it is trivial upon extending further to $\mathbb{Z}_p[\sigma] = \mathbb{W}(K)_0$. Examine the long exact sequence

$$\cdots \to \pi_0(S) \xrightarrow{\eta} \pi_0(R_n) \xrightarrow{\psi^{s-1}} \pi_0(R_n) \xrightarrow{} \cdots$$

Then $\mathbb{Z}_p[\tau] \subseteq \ker(\psi^{s-1}) \subseteq \pi_0(S)$, a contradiction.

Knowing that $\pi_{G^r} (R_n)[\sigma]$ contains $\mathbb{W}(\mathbb{F}_{p^n})[\rho^{\pm 1}]$ as a split summand, and that $\pi_0 R_n$ does not contain $\sigma$, we see that $\pi_{G^r} R_n$ contains $\mathbb{Z}_p[\rho^{\pm 1}]$ as a split summand.

\[35\]
5.5 $R_n$ for large primes

Let $q := g^{p-1} = \zeta^{p-1}(1+p)^{p-1} = (1+p)^{p-1}$; $q$ is a topological generator of $(1+p\mathbb{Z}_p)^\times$. Consequently, the equivalence $S = R_n^{h\mathbb{Z}_p}$ may be factorised (following [Dav09]) as

$$S = (R_n^{h\mu_{p-1}})^{h(1+p\mathbb{Z}_p)^\times} = \text{hofib}(q^g - 1 : R_n^{h\mu_{p-1}} \to R_n^{h\mu_{p-1}})$$

A variation on a standard sparseness result for the Adams-Novikov spectral sequence (see, e.g., [GHM12]) yields the following:

**Proposition 5.8.** If $n^2 < 2p - 3$, the long exact sequence in homotopy associated to

$S \xrightarrow{\eta} R_n^{h\mu_{p-1}} \xrightarrow{\psi^s - 1} R_n^{h\mu_{p-1}}$

splits in a range of degrees, giving

$$\mathbb{Z}_p = \pi_0(S) \cong \pi_0(R_n^{h\mu_{p-1}}), \text{ and } \pi_* (S) \cong \pi_* (R_n^{h\mu_{p-1}}) \oplus \pi_{s+1}(R_n^{h\mu_{p-1}})$$

when $-2p + 1 \leq s < 0$.

**Proof.** Equivalently, we may show these facts for $E_n^{hG_n^1} \cong R_n^{h\mu_{p-1}}$. The homotopy of this spectrum is computed via the spectral sequence

$$H^s(G_n^1, (E_n)_r) \Rightarrow \pi_{t-s}(E_n^{hG_n^1}).$$

Since $p$ is odd, our assumption implies that $(p - 1)$ and $p$ do not divide $n$, and thus $G_n$ is a $p$-adic analytic Lie group of dimension $n^2$ with no $p$-torsion; thus, its cohomological dimension is $n^2 + 1$ (see [Mor85]). Similarly, $G_n^1$ has cohomological dimension $n^2$. So the only contribution to $\pi_*$ comes from $H^s(G_n^1, (E_n)_{s+s})$ where $0 \leq s \leq n^2$.

However, one may compute this group cohomology as

$$H^s(G_n^1/\mu_{p-1}, (E_n)_{s+s}^{h\mu_{p-1}}) = H^s(SG_n^\pm, (E_n)_{s+s}^{h\mu_{p-1}}),$$

which vanishes when $s + *$ is not a multiple of $2(p - 1)$. Assuming that $n^2 < 2(p - 1)$ and that $-2(p - 1) < * \leq 0$ ensures that $-2(p - 1) < s + * \leq s < n^2 < 2(p - 1)$, so the only possible contribution is when $s = --$ (so $s + * = 0$). So for $* \leq 0$,

$$\pi_*(R_n^{h\mu_{p-1}}) = H^{-s}(G_n^1, (E_n)_0)$$

The same analysis holds (for $n^2 + 1 < 2(p - 1)$) to show that when $-2(p - 1) < * \leq 0$, $\pi_*(S) = H^{-s}(G_n, (E_n)_0)$.

Now, since $p \nmid n$, the reduced determinant is split, giving an isomorphism $G_n \cong G_n^1 \times \mathbb{Z}_p$, and the $\mathbb{Z}_p$ factor acts trivially on $(E_n)_0$ (and hence $H^*(G_n^1, (E_n)_0)$). Thus the (collapsing) Lyndon-Hochschild-Serre spectral sequence gives $H^0(G_n, (E_n)_0) \cong H^0(G_n^1, (E_n)_0)$, and if $* < 0$,

$$H^{-s}(G_n, (E_n)_0) \cong H^1(\mathbb{Z}_p, H^{-1-s}(G_n^1, (E_n)_0)) \oplus H^0(\mathbb{Z}_p, H^{-s}(G_n^1, (E_n)_0))$$

$$\cong H^{-1-s}(G_n^1, (E_n)_0) \oplus H^{-s}(G_n^1, (E_n)_0)$$

\hfill \Box

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This implies that for $0 \leq m \leq 2p-1$,

$$1 = \psi^q : \pi_{-m} R_n^{h\mu_{p-1}} \to \pi_{-m} R_n^{h\mu_{p-1}},$$

since the unit map $S \to R_n^{h\mu_{p-1}}$ is surjective in $\pi_{-m}$, and fixed by $\psi^q$. Now, the $G$-periodicity of $\pi_\bullet(R_n)$ yields a $G^{\otimes p-1}$-periodicity of $\pi_\bullet(R_n^{h\mu_{p-1}})$, so

$$[G^{\otimes k(p-1)}, \Sigma^m R_n^{h\mu_{p-1}}] \cong \pi_{-m}(R_n^{h\mu_{p-1}})$$

**Corollary 5.9.** If $n^2 < 2p - 3$, and $0 \leq m \leq 2p - 1$, the endomorphism $\psi^q$ of $[G^{\otimes k(p-1)}, \Sigma^m R_n^{h\mu_{p-1}}]$ is multiplication by $q^k$.

**Proof.** It suffices to observe that for $f \in \pi_{-m}(R_n^{h\mu_{p-1}})$ this commutes:

\[
\begin{array}{cccc}
\Sigma^{-m} G^{\otimes k(p-1)} = G^{\otimes k(p-1)} \otimes S^{-m} & \xrightarrow{1 \otimes f} & G^{\otimes k(p-1)} \otimes R_n^{h\mu_{p-1}} & \xrightarrow{\delta^k(p-1) \otimes \psi^q} & R_n^{h\mu_{p-1}} \\
q^k \otimes f & & \downarrow \psi^q & & \\
G^{\otimes k(p-1)} \otimes R_n^{h\mu_{p-1}} & & \xrightarrow{\delta^k(p-1) \otimes \psi^q} & & R_n^{h\mu_{p-1}}
\end{array}
\]

\[\]

5.6 Analogue of the image of J

**Corollary 5.10.** Let $k \in \mathbb{Z}$, and write $k = p^s m$, where $m$ is coprime to $p$. Then $[G^{\otimes k(p-1)}, S^1]$ contains a subgroup isomorphic to $\mathbb{Z}/p^{s+1}$. Furthermore, if $n^2 < 2p - 3$, there is an exact sequence

$$0 \to \mathbb{Z}/p^{s+1} \to [G^{\otimes k(p-1)}, S^1] \to N_{s+1} \to 0$$

where $N_{s+1} \leq \pi_{-1}(S)$ is the subgroup of $p^{s+1}$-torsion elements.

**Proof.** Without assumptions on $n$, Corollary 5.7 implies that $[G^{\otimes i}, R_n]$ has a split summand $Z_p$ upon which the action of $\psi^q$ is through multiplication by $g^i$. Taking $i = k(p-1)$, we conclude that the action of $\psi^q = \psi^g_{p^{-1}}$ on the corresponding summand $Z_p \leq [G^{\otimes k(p-1)}, R_n^{h\mu_{p-1}}] \cong \pi_0(R_n)$ is by multiplication by $q^k$.

Consider the long exact sequence obtained by applying $[G^{\otimes k(p-1)}, -]$ to the fibre sequence

\[
\cdots \longrightarrow R_n^{h\mu_{p-1}} \xrightarrow{\psi^q} R_n^{h\mu_{p-1}} \longrightarrow S^1 \longrightarrow \Sigma R_n^{h\mu_{p-1}} \xrightarrow{\psi^q} \Sigma R_n^{h\mu_{p-1}} \longrightarrow \cdots
\]

The first map contains a factor which is given by multiplication by $q^k - 1$ on $Z_p$. Since $q^k - 1$ generates the procyclical subgroup $p^{s+1}Z_p \subseteq Z_p$, we see that $[G^{\otimes k(p-1)}, S^1]$ contains the indicated kernel as a subgroup.

Now, if $n^2 < 2p - 3$, the results of the previous section give us a very precise computation of this sequence:

\[
Z_p \xrightarrow{\psi^q - 1} Z_p \longrightarrow [G^{\otimes k(p-1)}, S^1] \longrightarrow [G^{\otimes k(p-1)}, \Sigma R_n^{h\mu_{p-1}}] \xrightarrow{\psi^q - 1} [G^{\otimes k(p-1)}, \Sigma R_n^{h\mu_{p-1}}],
\]

and Corollary 5.9 implies that $\psi^q - 1$ is again multiplication by a unit multiple of $p^{s+1}$. Lastly, the identification $[G^{\otimes k(p-1)}, \Sigma R_n^{h\mu_{p-1}}] \cong \pi_{-1}(R_n^{h\mu_{p-1}})$ as a summand of $\pi_{-1}(S)$ (with complement $Z_p$) yields the desired short exact sequence.

\[\]
6 Redshift

We will write $M(p)$ for the $K(n)$-local mod $p$ Moore spectrum; i.e., the cofibre of the map $p : S \to S$. When $p > 2$, $M(p)$ is a ring spectrum, homotopy associative if $p > 3$; we will assume the latter throughout this section.

**Proposition 6.1.** There is a map $v : S(\det_{\pm}) \to Z \otimes M(p)$ which induces an isomorphism in $E_n$ after multiplying by the identity on $M(p)$:

$$v_* : E_{n*}(S(\det_{\pm}) \otimes M(p)) \cong E_{n*}(Z \otimes M(p)).$$

Thus $v$ induces an equivalence $S(\det_{\pm}) \otimes M(p) \simeq Z \otimes M(p)$.

**Proof.** Notice that after tensoring with $M(p)$, the element $g \in Z_p \subseteq [X_n, X_n]$ becomes homotopic to $\zeta \in \mathbb{F}_p \subseteq [X_n \otimes M(p), X_n \otimes M(p)]$. Thus the indicated lift in the following diagram exists:

$$
\begin{array}{ccc}
S(\det_{\pm}) & \cdots & Z \otimes M(p) \\
\delta & & \alpha \otimes 1 \\
R_n & \xrightarrow{\eta \otimes g} & R_n \otimes M(p) \\
\psi \otimes g & & (\psi \otimes \zeta) \otimes 1 \\
R_n & \xrightarrow{\eta \otimes g} & R_n \otimes M(p)
\end{array}
$$

To see that $v_*$ is an isomorphism, we note that the image in

$$E_{n*}(R_n \otimes M(p)) = C(Z_p, E_{n*}/p) = C(Z_p, \mathbb{F}_p[[u_1, \ldots, u_{n-1}]][u^{\pm 1}])$$

of $\alpha \otimes 1$ and $(1 \otimes \eta) \circ \delta$ are both generated by the function $f_0 : Z_p \to \mathbb{F}_p[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$ given by $f_0(x) = x \mod p$.

$$\square$$

This fact allows us to multiply any element of the Picard-graded homotopy of $Y \otimes M(p)$ with $v$, for any spectrum $Y$.

**Proposition 6.2.** The map $\beta^p_{n-1} : S(\det_{\pm}) \otimes p^{-1} \to L_{K(n)} K(E_{n-1})$ is, after tensoring with $M(p)$, homotopic to multiplication of the unit $\eta : S \to L_{K(n)} K(E_{n-1})$ by $\psi^{p-1}$.

**Proof.** The statement of the proposition amounts to the claim that the following diagram commutes:

$$
\begin{array}{ccc}
S(\det_{\pm}) \otimes p^{-1} & \xrightarrow{\psi \otimes p^{-1}} & X_n \xrightarrow{\psi \otimes \zeta \otimes 1} L_{K(n)} K(E_{n-1}) \\
& & \downarrow{\psi \otimes \zeta \otimes 1} \\
S \otimes M(p) & \xrightarrow{\eta \otimes 1} & X_n \otimes M(p) \xrightarrow{\psi \otimes \zeta \otimes 1} L_{K(n)} K(E_{n-1}) \otimes M(p),
\end{array}
$$

since the lower composite is the unit of $L_{K(n)} K(E_{n-1})$ tensored with the identity on $M(p)$. The first square commutes by construction, and the second is evident.

$$\square$$
Proposition, then multiplication by $\beta$ tion 6.1, smashing it with the identity of $L_{K(n)}K(E_{n-1})$ is also an equivalence. By the previous Proposition, then multiplication by $\beta_n^{p-1}$:

$$[\beta_n^{p-1}] = [v^{p-1}] : (E_n)^\bullet(K(E_{n-1}) \otimes M(p)) \to (E_n)^\bullet S_{\det^p}^{p-1}(K(E_{n-1}) \otimes M(p))$$

is an isomorphism. However, this implies that $\beta_n^{p-1}$ is an isomorphism in $K_*$ and hence an equivalence, since $K_n$ is a module spectrum for $E_n \otimes M(p)$, and hence in the same Bousfield class.

One may go slightly further. One of the chromatic redshift conjectures of [AR06] (specifically Conjecture 4.4) proposes an equivalence

$$L_{K(n)}K(\Omega_{n-1}) \simeq E_n,$$

where $\Omega_{n-1}$ is a suitable interpretation of the algebraic closure of the fraction field of $E_{n-1}$. In particular, this should restrict to a unital $E_\infty$ map $r_n : L_{K(n)}K(E_{n-1}) \to E_n$. When $n = 1$, this is a map $L_{K(1)}K(\mathbb{Q}_p) \to KU_1^\wedge$ corresponding to a choice of embedding $\mathbb{Q}_p \to \mathbb{C}$. Let us presume only the existence of such a map $r_n$. The previous results indicate the commutativity of the solid part of the following diagram:

$$\begin{array}{c}
L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n + 1)^+ \xrightarrow{B\varphi_{n-1}} L_{K(n)}\Sigma^\infty B\text{GL}_1(E_{n-1})^+ \xrightarrow{i} L_{K(n)}K(E_{n-1}) \\
\downarrow \simeq \downarrow \downarrow \downarrow \\
R_n \xrightarrow{\simeq} E_n \xrightarrow{\text{hSG}_n^+} E_n \\
\end{array}$$

Now, the path along the lower left is the map $\varphi_n$. The path along the upper right (including the conjectural redshift map $r_n$) is an $E_\infty$ map, and thus by Theorem 3.21, differs from $\varphi_n$ by at worst an Adams operation. So up to a suitable re-embedding of $E_n^{hSG}_n^+$ into $E_n$, the diagram must commute if $r_n$ is to exist. This provides a falsifiable condition upon any candidate redshift map $r_n$: it must give a factorisation of the homotopy fixed point inclusion $E_n^{hSG}_n^+ \to E_n$.

## 7 Questions

This section is purely speculative, and is intended as a collection of vaguely posed questions which the reader is invited to either clarify or (more likely) disprove.

Perhaps the most evident thing lacking in this paper is a geometric definition of the cohomology theory $R_n$. Despite all of the analogies with K-theory given above, we do not have a description of $R_n^*(X)$ akin to the Grothendieck group of (some generalisation of) vector bundles over $X$. The discussion of orientations on $p$-adic $(n-1)$-gerbes indicates a possible direction with this question, but it is not apparent to the author what the corresponding analogue of higher rank vector bundles should be (although the work of Michael Murray and others on bundle gerbe modules [Mur10, BCM+02] may point a way forward).
A related deficiency is the fact that we have described a $K(n)$-local analogue of the image of the J-homomorphism, but not the homomorphism itself. Classically, one can think of the J-homomorphism as built from either the functions $O(m) \to \Omega^m S^m$, or (in families, after delooping) the function that assigns to a vector bundle the associated sphere bundle. Neither of these has an obvious analogue in our construction. Such a description would be enlightening. In connection with the first question this raises the obvious question: does an $n$-bundle gerbe have a geometrically defined “sphere” bundle, where $S\langle \det \rangle$ is our replacement for the spherical fibre?

This, in turn, raises the question of whether there is a good unstable description of the map $\rho_n : S\langle \det \rangle \to L_{K(n)} \Sigma^\infty K(\mathbb{Z}_p, n+1)_+$ analogous to the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^\infty$. It follows from [Bou01] that $S\langle \det \rangle$ is the $K(n)$-localisation of a suspension spectrum, but a more geometric description of this spectrum and map is desirable.

Further, the definition of an $n$-oriented spectrum raises the question of whether it is possible to repeat the whole program of analysing complex-oriented spectra by their formal group laws, but in the $K(n)$-local category, with $K(\mathbb{Z}_p, n+1)$ replacing $\mathbb{C}P^\infty$. All of our examples, however, have resulted in essentially multiplicative formal groups. One standout is the spectrum $MX_n$ for which we haven’t even the beginnings of a computation of the associated formal group when $n > 1$. We would like to believe that it plays a universal role analogous to $MU$ in the complex-oriented case, but do not have any evidence to back this up.

Lastly, we have no examples in hand of $n$-oriented spectra whose associated formal group law is additive (i.e., the analogue of singular homology). Does such a theory exist? If so, and if it could be made part of an Atiyah-Hirzebruch spectral sequence for $K(n)$-local theories, the rather convoluted construction of the formal group law in Theorem 5.3 (requiring the entirety of section 4) could be replaced with a direct analogue of the argument familiar to most homotopy theorists for the proof that $h^*(\mathbb{C}P^\infty) \cong h_*[[x]]$ for complex-oriented theories $h$. However, when $n = 1$, this argument requires singular cohomology, a non-$K(1)$-local theory. So to mimic this argument, our definition of $n$-oriented spectrum would need to be extended beyond the $K(n)$-local setting to a larger class of spectra.

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14Of course, one may give such a homomorphism as the map $\gamma : B(S, R_n)/(0) \to B GL_1 S$; this then reiterates our desire for a more geometric definition of $R_n$. 

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