The Quantum Orthogonal Mystery

A. Sudbery

Department of Mathematics, University of York, Heslington, York, England

YO1 5DD

Based on a talk at the XXX Karpacz Winter School, February 1994

QUANTUM LIE ALGEBRAS

The familiar examples of quantum groups arise from semisimple Lie algebras, and it ought to be possible to consider them as similar sorts of objects \cite{3}. Usually the emphasis is on their resemblance to enveloping algebras, i.e. infinite-dimensional associative algebras, but there is always a finite-dimensional Lie algebra-like structure to be found in a quantised enveloping algebra. Let \( L \) be a simple finite-dimensional Lie algebra, and let \( U_q(L) \) be a quantisation of its enveloping algebra \( U(L) \). Then the representations of \( U_q(L) \) are deformations of the representations of \( U(L) \) (and therefore of those of \( L \)); for every representation \( \rho \) of \( U(L) \) on a finite-dimensional vector space \( V_\rho \) there is a representation \( \rho_q \) of \( U_q(L) \) on the same vector space. Moreover, \( U_q(L) \) itself under the adjoint action decomposes into irreducibles in essentially the same way as \( U(L) \) \cite{2}. In particular, it contains a copy of the deformation of the adjoint representation of \( L \); hence there is a subspace \( L_q \subset U_q(L) \) which is invariant under \( \text{ad}X \) for each \( X \in U_q(L) \). Thus we can define a bracket on \( L_q \) by

\[
[X, Y] = \text{ad}X(Y) = \sum X(1)Y\kappa(X(2))
\]

where, as usual, the coproduct in \( U_q(L) \) is denoted by \( \Delta(X) = \sum X(1) \otimes X(2) \), and \( \kappa \) denotes the antipode in \( U_q(L) \). For example, in the simplest case of \( L = \mathfrak{sl}(2) \) the quantised enveloping algebra has a three-dimensional subspace spanned by \( V_+, V_0 \) and \( V_- \) which is closed under the above bracket, as follows:

\[
\begin{align*}
[V_+, V_+] &= 0, & [V_+, V_0] &= -q^{-1}V_+, & [V_+, V_-] &= V_0, \\
[V_0, V_+] &= qV_+, & [V_0, V_0] &= (q - q^{-1})V_0, & [V_0, V_-] &= -q^{-1}V_-, \\
[V_-, V_+] &= -V_0, & [V_-, V_0] &= qV_-, & [V_-, V_-] &= 0.
\end{align*}
\]

In any Hopf algebra the bracket defined by (1) obeys two versions of the Jacobi identity:

\[
[X, [Y, Z]] = \sum [[X(1), Y], [X(2), Z]]
\]

and

\[
[[X, Y], Z] = \sum [X(1), [Y, \kappa(X(2)), Z]]
\]
The first of these says that each \( \text{ad}X \) is a generalised derivation of the non-associative algebra structure defined by the bracket; the second seems to be trying to say that the map \( X \mapsto \text{ad}X \) is, in some sense, a representation of this algebra structure. These identities provide good reason for thinking of the bracket (1) as defining a kind of Lie algebra. However, subspaces which are closed under the bracket (such as the three-dimensional subspace exhibited in (2)) cannot be regarded as instances of this kind of Lie algebra, since they are not usually subcoalgebras and so the coproduct (and therefore the bracket) cannot be defined in an intrinsic way. Moreover, there does not seem to be any version of the antisymmetry property of the Lie bracket in this context.

Woronowicz [13], working in a more geometrical framework, has produced a different set of axioms for a quantum Lie algebra; these arise from the properties of vector fields in non-commutative geometry. They have only one Jacobi identity (corresponding to the representation property of the adjoint map), but they also have an anti-commutativity axiom and a natural notion of representation, and they have a wealth of good finite-dimensional examples. They are obtained from the differential geometry of quantum groups in a similar way to the construction of classical Lie algebras from Lie groups, and at first sight Woronowicz’s theory promises to give genuine deformations of all classical Lie algebras; in particular, it looks as if these quantum Lie algebras should have the same dimensions as the classical ones. But this expectation is to be disappointed; it seems to be difficult, in the majority of cases, to construct non-commutative differential spaces whose tangent spaces have the same dimension as in the classical theory. Satisfactory deformations exist in the case of the general linear groups but not, it appears, in other cases.

In this talk I will examine the orthogonal groups to try to understand this difference between the classical and the non-commutative geometry. I will start by presenting Woronowicz’s theory in a formulation which emphasises its links with classical differential geometry.

NON-COMMUTATIVE DIFFERENTIATION

The differential geometry associated with a non-commutative space (= non-commutative algebra, regarded as an algebra of functions) in the theories of Woronowicz [13], Wess and Zumino [12] and their co-workers [9] is based on an algebra of differential forms on the space, defined by means of their commutation relations with the non-commuting coordinate functions and by the algebraic properties of an exterior derivative \( d \); apart from these algebraic properties, there is little connection with the intuitive ideas of differentiation. Classically, differential forms are defined more geometrically in terms of tangent vectors, which in turn are defined in terms of directional derivatives of functions. It is possible to follow this more geometrical line in the non-commutative case also, using ideas
of Majid [4] which show something like a true process of differentiation behind
the non-commutative differential algebra. Here I will show how these ideas lead
to the relations between coordinates and differentials which are a starting point
for Woronowicz and his followers.

Let \( A \) be a (non-commutative) algebra generated by \( x^1, \ldots, x^n \); we think
of these as coordinates on a vector space \( V \), and therefore use \( V^* \) to denote the
vector space spanned by the \( x^i \). The additive structure on a classical vector space
\( V \), which is a map from \( V \times V \) to \( V \), can be expressed in terms of the algebra \( A \) of
functions on the space by a \textit{coaddition} map \( \Delta : A \to A \otimes A \) [6, 3, 11]; classically,
\( \Delta(x^i) = x^i \otimes 1 + 1 \otimes x^i \) and \( \Delta \) is an algebra homomorphism from \( A \) to \( A \otimes A \)
with the usual algebra structure in the tensor product, so that \( x^i \otimes 1 \) commutes
with \( 1 \otimes x^i \). Now suppose that \( A \) is non-commutative, with relations between the
generators \( x^i \) given by an \( R \)-matrix in the form

\[
R^{ij}_{kl} x^k x^l = qx^i x^j \tag{5}
\]

where \( q \) is an eigenvalue of \( R \). Then the \( \Delta(x^i) \) must satisfy the same relations.
We keep the same formula for them, namely

\[
\Delta(x^i) = x^i \otimes 1 + 1 \otimes x^i, \tag{6}
\]

but change to a non-commutative algebra structure in \( A \otimes A \):

\[
(1 \otimes x^i)(x^j \otimes 1) = B^{ij}_{kl}(x^k \otimes 1)(1 \otimes x^l) = B^{ij}_{kl} x^k \otimes x^l \tag{7}
\]

where \( B \) satisfies the braid relation and commutes with \( R \).

Now let \( v \) be a vector in our non-commutative space \( V \) (formally, an element
of the dual vector space to the space spanned by the \( x^i \)). Then [4] we can
define \( D_v \), the directional derivative along \( v \), by following the classical idea that
the directional derivative of the function \( f \) at the point with coordinates \( x \) is
obtained from \( f(x + y) \) by picking out the first-order terms in \( y \) and evaluating
them with \( y \) equal to the coordinates of \( v \). Writing \( d_v f = D_v f(0) = f_1(v) \) where
\( f_1 \) is the first-order part of the function \( f \), this prescription for forming \( D_v f \) as a
function of \( x \) is

\[
D_v = (\text{id} \otimes d_v) \Delta(f) \tag{8}
\]

which is also a good definition of \( D_v : A \to A \) when \( A \) is a non-commutative
algebra of functions. Exactly as in commutative calculus, we define \( df \) to be the
map which associates to each vector \( v \in V \) the directional derivative \( D_v f \); this is
a linear map \( df : V \to A \) and can therefore be regarded as an element of \( A \otimes V^* \)
which is a subset of \( A \otimes A \) since \( V^* \) generates \( A \). Identifying the coordinates \( x^i \)
with \( x^i \otimes 1 \), we have both coordinates and differentials in the algebra \( A \otimes A \), with
the relations (8); the differentials \( dx^i \) are identified with \( 1 \otimes x^i \), so the relations
become

\[
dx^i x^j = B^{ij}_{kl} x^k dx^l. \tag{9}
\]
These relations are usually taken as a starting point for a non-commutative differential calculus.

**INvariant Vector Fields**

Now let us try to follow the classical construction of the Lie algebra of a Lie group as the set of left-invariant vector fields on the group manifold. Let $\mathcal{H}$ be a Hopf algebra, which we think of as analogous to the function space $C^\infty(G)$ of a Lie group $G$. There is a natural definition of a left-invariant operator $X : \mathcal{H} \to \mathcal{H}$ in terms of the coproduct $\Delta$ of $\mathcal{H}$: $X$ is left-invariant if

$$\Delta \circ X = (\text{id} \otimes X) \circ \Delta. \quad (10)$$

The set $L(H)$ of all such left-invariant operators is closed under operator multiplication, and is isomorphic as an algebra to the Hopf dual $H^*$ by the correspondence

$$L(H) \ni X \mapsto X_e = \epsilon \circ X \in H^* \quad (11)$$

$$H^* \ni X_e \mapsto X = (\text{id} \otimes X_e) \circ \Delta \in L(H) \quad (12)$$

where $\epsilon$ is the counit of $\mathcal{H}$. (Classically, if $X$ is a differential operator, $X_e$ maps $f \in \mathcal{H}$ to the number $Xf(e)$ where $e$ is the identity of $G$; in particular, if $X$ is a vector field then $X_e$ is the corresponding tangent vector at the identity.) If $X_e \in H^*$ has a coproduct $\mu^*(X_e) = \sum X_{(1)e} \otimes X_{(2)e}$, where $\mu$ is the multiplication in $\mathcal{H}$, then the corresponding $X \in L(H)$ satisfies

$$X(fg) = \sum (X_{(1)}f)(X_{(2)}g), \quad (13)$$

i.e. it is a generalised derivation of $\mathcal{H}$. Classically, being a derivation is the defining property of a vector field (i.e. a first-order differential operator); but the generalised property of derivation provided by Hopf algebra theory is too general for this definition, for it includes higher-order differential operators. To capture the notion of a first-order differential operator we need more structure than that of a Hopf algebra on the algebra of functions; this is just what is provided by Woronowicz’s idea of a differential calculus.

Rearranging Woronowicz’s theory a little, we can take a differential calculus on a Hopf algebra $\mathcal{H}$ to be a set $\mathcal{D}$ (of differential forms) which is an $\mathcal{H}$-bimodule and a differential bialgebra, i.e. $\mathcal{D}$ has the following structures:

1. An $\mathbb{N}$-grading $\mathcal{D} = D_0 \oplus D_1 \oplus \cdots$ with $D_0 = \mathcal{H}$. We can describe this grading by a map $\nu : \mathcal{D} \to \mathcal{D}$ such that $D_0 = \ker(\nu - n)$.

2. A $\mathbb{Z}_2$-grading $\sigma = (-1)^\nu$.

3. A differential $d : \mathcal{D} \to \mathcal{H}$ of $\nu$-degree 1, i.e. $d(D_n) \subseteq D_{n+1}$ and

$$d(\theta\phi) = (d\theta)\phi + \sigma(\theta)d\phi. \quad (14)$$
4. An algebra homomorphism \( \widetilde{\Delta} : D \to D \) with \( \widetilde{\Delta}|H = \Delta \) and
\[
\widetilde{\Delta} \circ d = (d \otimes 1 + \sigma \otimes d) \circ \widetilde{\Delta};
\] (15)

5. \( D_1 \) is generated by \( d(H) \) as a left \( H \)-module.

Thus differential forms can be multiplied by functions both on the left and on
the right, and there are relations giving a product of the form \( df \cdot g \) as a sum of
products of the form \( g' df' \).

We define a vector field on \( H \) (relative to the differential calculus \( D \)) to be an
operator \( X : H \to H \) which is of the form \( X = \iota_X \circ d \) where \( \iota_X : D_1 \to H \) is left
\( H \)-linear. Such an operator satisfies a deformed version of the usual derivation
property which defines a vector field classically:
\[
X(fg) = f(Xg) + \sum g'(Xf') \text{ where } df \cdot g = \sum g' df'.
\] (16)

It follows from this that the action of a vector field on elements of \( H \) is determined
by its action on the generators of \( H \). Left-invariant vector fields correspond to
elements of \( \mathcal{H}^* \), so if \( H \) has a finite number of generators then the set of left-
invariant vector fields is a finite-dimensional vector space.

THE LIE ALGEBRA OF A QUANTUM GROUP

To define a Lie bracket between left-invariant vector fields, as just defined, we
return to the general Hopf-algebra bracket \([\cdot, \cdot]\). Between elements \( X_e, Y_e \) of the
dual Hopf algebra \( \mathcal{H}^* \), this bracket can be written
\[
\langle [X_e, Y_e], f \rangle = \langle X_e \otimes Y_e, \text{ad}(f) \rangle
\] (17)
where the angle brackets denote the pairing between a vector space and its dual,
and \( \text{ad} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \) is the adjoint coaction:
\[
\text{ad}(f) = \sum f_1(1) \kappa(f_3(3)) \otimes f_2(2).
\] (18)

In the classical case, where \( f \) is a function of one variable in \( G \), \( \text{ad}(f) \) is the
function of two variables given by \( \text{ad}(f)(x, y) = f(yx^{-1}) \), and tangent vectors
at the identity like \( X_e \) are given as elements of \( \mathcal{H}^* \) by
\[
X_e f = \frac{d}{dt} f(e^{tX}) \bigg|_{t=0},
\] (19)
the definition \([17]\) corresponds to
\[
[X_e, Y_e] f = \frac{d}{ds} \frac{d}{dt} f(e^{sX} e^{tY} e^{-sX}) \bigg|_{s=t=0}
\] (20)
By the isomorphism between $\mathcal{H}^*$ and $L(\mathcal{H})$, this definition gives a Lie bracket between left-invariant differential operators. The classical fact that the set of tangent vectors is closed under this bracket carries over to the general case of a Hopf algebra with a differential calculus, and so do the familiar properties of the Lie bracket:

**Theorem** (Woronowicz [13]). Let $\mathcal{H}$ be a Hopf algebra with a differential calculus, and let $L$ be the set of left-invariant fields on $\mathcal{H}$. Then:

1. If $X$, $Y$ are left-invariant vector fields on $\mathcal{H}$, so is $[X, Y]$.
2. There is a braiding $\beta : L \otimes L \to L \otimes L$ such that
   
   (a) $[X, Y] = XY - \mu \circ \beta(X \otimes Y)$ (quommutator)
   
   (b) For $T \in L \otimes L$, $\beta(T) = T \implies [T] = 0$ (q-antisymmetry)
   
   (c) $\text{ad}_L[X, Y] = \text{ad}_L X \cdot \text{ad}_L Y - \mu \circ (\text{ad}_L \otimes \text{ad}_L) \circ \beta(X \otimes Y)$ (q-Jacobi)

where $\text{ad}_L X$ takes $Y \in L$ to $[X, Y] \in L$.

THE GOOD BEHAVIOUR OF GL$_q$(n)

When $\mathcal{H}$ is the quantised function algebra of GL($n$), this theory works as well as one could possibly hope. In this case $\mathcal{H}$ is generated by the elements of an $n \times n$ matrix $A = (a^i_j)$ with relations

$$R_{12} A_1 A_2 = A_1 A_2 R_{12}, \quad \text{i.e.} \quad R^{ij}_{kl} a^i_m a^j_n = a^k_m a^l_n R^{kl}_{mn}, \quad (21)$$

where the $n^2 \times n^2$ matrix $R$ is diagonalisable with two eigenvalues $\lambda = q$ and $\mu = -q^{-1}$, so that $(R - q)(R + q^{-1}) = 0$. Then a differential calculus on $\mathcal{H}$ is defined by the commutation relations

$$dA_1 A_2 = -\mu^{-1} R_{12} A_1 dA_2 R_{12} \quad (22)$$

(which are categorically determined [10] by a differential calculus on the associated quantum space). This differential calculus yields $n^2$ independent left-invariant vector fields and a matrix of $n^2$ independent left-invariant (Maurer-Cartan) forms $\Omega = A^{-1} dA$, and hence an $n^2$-dimensional Lie algebra with a basis $E^i_j$ and Lie brackets

$$[E^i_j, E^k_l] = E^i_j E^k_l - R^{k_2}_{j_2} R^{j_1}_{i_1} (R^{-1})^{j_1}_{k_1} (R^{-1})^{h_1}_{e_1} (R^{-1})^{d_1}_{c_1} E^a_b E^c_d \quad (23)$$

where $R^{-1}$ is the inverse of $R$ as an element of $\text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ (acting on contravariant tensors $t^{ij}$), and $R^{-1}$ is the inverse of the same matrix as an element of $\text{End}(\text{End}\mathbb{C}^n)$ (acting on mixed tensors $t^i_2$); i.e.

$$(R^{-1})^{ij}_{kl} R^{kl}_{mn} = \delta^i_m \delta^j_n \quad \text{and} \quad (R^{-1})^{ij}_{kl} R^{ln}_{jm} = \delta^i_n \delta^j_k. \quad (24)$$
This can be regarded as a bracket on the space of \( n \times n \) matrices \( X = x^j_i E_i^j \), with

\[
[X, Y]_i^j = x^k_i y^j_k - x^m_k y^l_m R^{kr}_{ns} R^{pm}_{qt} t^i_q (R^{-1})^q_i
\]

where \( t^i_j = (R^{-1})^k_i \) is the quantum trace. This looks less unpleasant in a graphical notation:

\[
\begin{array}{c}
\text{Y} \\
\text{X}
\end{array}
\]

where \( \begin{array}{c}
\text{i} \\
\text{j}
\end{array} \begin{array}{c}
\text{k} \\
\text{l}
\end{array} = R^{ij}_{kl}, \quad \begin{array}{c}
\text{i} \\
\text{j}
\end{array} \begin{array}{c}
\text{k} \\
\text{l}
\end{array} = (R^{-1})^{ij}_{kl} \quad \text{and} \quad \begin{array}{c}
\text{j} \\
\text{i}
\end{array} = t^i_j.
\]

\[\ldots\ \text{BUT O}_q(n) \text{ WON’T PLAY BALL}\]

Now let \( \mathcal{H} \) be the quantised function algebra of \( \text{GO}(n) \), the orthogonal group together with dilatations in \( n \) dimensions. Just like the case of \( \text{GL}(n) \), this \( \mathcal{H} \) has \( n^2 \) generators \( a^i_j \) forming a matrix \( A \), subject to relations \( R_{12} A_1 A_2 = A_1 A_2 R_{12} \), but now the \( n^2 \times n^2 \) matrix \( R \) has three eigenvalues:

\[
\lambda_+ = q; \quad \text{eigenspace } E_+ \quad \text{with dim} E_+ = \frac{1}{2} n(n + 1) - 1
\]
\[
\lambda_- = -q^{-1}; \quad \text{eigenspace } E_- \quad \text{with dim} E_- = \frac{1}{2} n(n - 1)
\]
\[
\lambda_0 = q^{1-n}; \quad \text{eigenspace } E_0 \quad \text{with dim} E_0 = 1
\]

The one-dimensional eigenspace \( E_0 \subset \mathbb{C}^n \otimes \mathbb{C}^n \) is spanned by the quantum metric \( g^{ij} \) (not a symmetric tensor). We have projectors \( \Pi_+ , \Pi_- , \Pi_0 \) onto the three eigenspaces, with

\[
(\Pi_0)^{ij}_{kl} = \alpha g^{ij} g_{kl}
\]

where \( \alpha \) is a scalar and \( g_{ij} \) is the inverse of \( g^{ij} \). In terms of these projectors, the relations can be written

\[
(\Pi_0 + \Pi_+) A_1 A_2 \Pi_- = \Pi_- A_1 A_2 (\Pi_0 + \Pi_+) = 0,
\]

\[\ldots\]
which is a $q$-deformation of the statement that the matrix elements of $A$ commute, and

$$\Pi_0 A_1 A_2 \Pi_+ = \Pi_+ A_1 A_2 \Pi_0 = 0, \quad (28)$$

which is a deformation of the orthogonality equation $g^{kl} a^i_k a^j_l = Q g^{ij}$. In the quantum case the scalar $Q$ is given by

$$\Pi_0 A_1 A_2 \Pi_0 = Q \Pi_0 \quad (29)$$

and commutes with all the matrix elements $a^i_j$.

Suppose that in an attempt to construct a differential calculus on the algebra $\mathcal{H}$, we assume the same commutation relations as in the case of $\text{GL}(n)$:

$$dA_1 A_2 = qR_{12} A_1 dA_2 R_{12} \quad (30)$$

Then applying the external derivative $d$ to the relations $RA_1 A_2 = A_1 A_2 R$ gives

$$[R, 1 \otimes \Omega + R(1 \otimes \Omega) R] = 0 \quad (31)$$

where $\Omega = A^{-1} dA$ is the matrix of left-invariant 1-forms, which we expect to be antisymmetric. But from the above equation we get both

$$\Pi_0 (1 \otimes \Omega) \Pi_+ = 0, \quad (32)$$

which says that the traceless part of $\Omega$ is $q$-antisymmetric, and

$$\Pi_0 (1 \otimes \Omega) \Pi_− = 0, \quad (33)$$

which says that $\Omega$ is $q$-symmetric. These conditions leave only one independent left-invariant 1-form in the matrix $\Omega$.

On the other hand, Carow-Watamura et al. [1] found a bicovariant differential calculus on $\mathcal{H}$ in which all $n^2$ 1-forms in the matrix $\Omega$ were independent.

Schmüdgen and Schüler [7, 8] have considered the general theory of a bicovariant differential calculus on the algebra $\mathcal{H}$. We have $n^2$ left-invariant 1-forms $\omega^i_j = \kappa(a^i_k) d a^k_j$; we can use the quantum metric to lower an index and obtain $\omega_{ij} = g_{ik} \omega^k_j$. It follows from the fact that the $a^i_j$ generate $\mathcal{H}$ that the $\omega_{ij}$ span the space of left-invariant 1-forms. The general theory of Woronowicz [13] then gives commutation relations between the $\omega^i_j$ and the matrix elements $a^i_j$ of the form

$$\omega_{ij} a^k_l = a^k_p T^{pnm}_{ijl} \omega_{mn} \quad (34)$$

where $T$ is a numerical tensor. The requirement of bicovariance is that there should be left and right coactions

$$\delta_L(da^i_j) = a^k_i \otimes da^k_j, \quad \delta_R(da^i_j) = da^k_i \otimes a^l_j \quad (35)$$

i.e.

$$\delta_L(\omega_{ij}) = 1 \otimes \omega_{ij}, \quad \delta_R(\omega_{ij}) = \omega_{kl} \otimes a^k_i a^l_j \quad (36)$$
which are $\mathcal{H}$-bimodule homomorphisms, i.e. are consistent with the coproducts of matrix elements and the commutation relations between differentials and matrix elements. This leads to

$$T_{123}A_1A_2A_3 = A_1A_2A_3T_{123}$$  \hspace{1cm} (37)

if we assume that the 1-forms $\omega_{ij}$ are all independent over $\mathcal{H}$. There are two further consistency conditions. By considering the product $\Omega_1A_2A_3$ and requiring consistency between the commutation relations (21) and (34) (where “consistency” means independence of the 1-forms $\omega_{ij}$ over $\mathcal{H}$), we are led to

$$R_{12}T_{234}T_{123} = T_{234}T_{123}R_{34}.$$  \hspace{1cm} (38)

Schmüdgen and Schüler show that this requires $T$ to be of the form

$$T_{123} = X_2^{-1}R_{12}^{-1}R_{23}X_{12}.$$  \hspace{1cm} (39)

Finally, differentiating $RA_1A_2 = A_1A_2R$ and again requiring that the $\omega_{ij}$ should be independent, they show that $X$ must be

$$X = R - \lambda_0^{-1}$$  \hspace{1cm} (40)

which gives precisely the differential calculus of al. et Weich [1].

But the assumptions in this argument are unreasonable; the $\omega_{ij}$ ought not to be independent. This is particularly evident in the last step: the relations $RA_1A_2 = A_1A_2R$ include a version of the orthogonality condition on $A$, and differentiating this ought to lead to a version of the antisymmetry of $\omega_{ij}$. However, making the assumptions more reasonable (in classical terms) leaves the conclusion the same:

**Theorem.** Let $t_{ij}^\alpha$ be a set of independent $q$-antisymmetric tensors, i.e. a basis for the image of $\Pi_-$, and suppose that the 1-forms $t_{ij}^\alpha\omega_{ij}$ are independent over $\mathcal{H}$. Then the matrix $X$ in (39) is $X = R - \lambda_0^{-1}$ and all the $\omega_{ij}$ are independent.

**THE QUANTUM SPHERE**

Another place where we might expect to find a set of vector fields whose Lie brackets give an orthogonal Lie algebra is on the quantum sphere, which is a homogeneous space for the quantum orthogonal group. On the sphere, unlike the group, a differential calculus can be defined which is a genuine deformation of the classical one.

Homogeneous spaces can be constructed for any Hopf algebra $\mathcal{H}$ with matrix comultiplication generated by matrix elements $a_j^i$ satisfying relations (21) with a diagonalisable $R$-matrix

$$R = \lambda_1\Pi_1 + \cdots + \lambda_r\Pi_r.$$  \hspace{1cm} (41)
where the $\Pi_i$ are projectors onto the eigenspaces $E_i$ of $R$. For each of these eigenspaces, or for any subset $K = \{k_1, \ldots, k_p\}$ of them, we can form a quantum space $S_K$ with generators $x^i$ and relations

$$\Pi_k x_1 x_2 = 0 \quad \text{for each } k \in K$$  \hspace{1cm} (42)

The space $S_K$ then admits a coaction of the bialgebra $\mathcal{H}$,

$$\delta_K : S_K \to H \otimes S_K \quad \text{with} \quad x^i \mapsto a^i_j \otimes x^j,$$  \hspace{1cm} (43)

that is to say the relations (42) are preserved by this map. We can also consider spaces defined by an equation like (42) but with a non-zero right-hand side in the form of a numerical tensor $t_k \in E_k \subset \mathbb{C}^n \otimes \mathbb{C}^n$. To obtain a coaction on such a space it will be necessary to add further relations to (41). In the case of the quantum orthogonal group we have

$$R = q\Pi_+ - q^{-1}\Pi_- + \lambda_0\Pi_0$$  \hspace{1cm} (44)

and the quantum sphere is given by the relations

$$\Pi_- x_1 x_2 = 0 \quad \text{and} \quad \Pi_0 x_1 x_2 = \rho g$$  \hspace{1cm} (45)

where $\rho$ is the radius of the sphere. The first equation, a deformation of the statement that the coordinates commute, defines a $q$-orthogonal flat space; the second is the equation of a sphere in this space. They can be combined in the single equation

$$R_{ij}^{kl} x^k x^l = q x^i x^j + \rho(\lambda_0 - q)g^{ij}$$  \hspace{1cm} (46)

If $\rho \neq 0$ the relations (41) must be supplemented by

$$g^{kl} a^i_k a^j_l = g^{ij}$$  \hspace{1cm} (47)

i.e. $Q = 1$ in (21). We will use $\mathcal{F}(S)$ to denote the algebra of functions on the sphere, i.e. the algebra generated by the $x^i$ with the relations (41). We define a differential calculus $\mathcal{D}(S)$ on the sphere by means of the commutation relations

$$dx_1 x_2 = q R_{12} x_1 dx_2$$  \hspace{1cm} (48)

and the relations between the $dx^i$ obtained by differentiating this. Differentiating the sphere equation (46) then yields

$$\Pi_0 x_1 dx_2 = 0, \quad \text{i.e.} \quad g_{ij} x^i dx^j = 0$$  \hspace{1cm} (49)

as we would expect classically. Thus the number of independent monomials in coordinates and differentials is the same as for the classical sphere.

We define a vector field on the sphere to be a left $\mathcal{F}(S)$-linear map $X : \mathcal{F}(S) \to \mathcal{F}(S)$ which is obtained from a map $i_X : \mathcal{D}(S) \to \mathcal{F}(S)$ by $X = i_X \circ d$. Then $X$ has a derivation property, and is determined by its action on the generators $x^i$ by

$$X(x_1 \cdots x_n) = (1 + qR_{n-1,n} + \cdots + q^{n-1} R_{12} R_{23} \cdots R_{n-1,n}) x_1 \cdots x_{n-1} X(x_{n-1}).$$  \hspace{1cm} (50)
INFINITESIMAL ROTATIONS

Even in the absence of a notion of a tangent vector at the identity of the orthogonal group, we can distinguish those vector fields on the quantum sphere which are transmitted from differential operators at the identity of the group by means of the coaction $\delta$ of (43). Such an infinitesimal rotation is a vector field whose action on $\mathcal{F}(S)$ is given by

$$Xf = (\xi \otimes \text{id})\delta(f) \quad \text{for some } \xi \in H^*.$$  

The linearity of the coaction then means that the action of $X$ on the coordinates must be linear:

$$X(x^i) = m^i_j x^j$$  

for some numerical matrix $M = (a^i_j)$. The derivation equation (50) together with the sphere equation (46) then imply that $M$ must satisfy

$$\Pi_0(1 \otimes M)\Pi_+ = 0$$  

which is a deformation of the statement that $M$ (with its trace removed) is antisymmetric with respect to the quantum metric $g$. This is what one would expect classically. However, in the quantum case there is an extra condition

$$\rho\Pi_0(1 \otimes M)\Pi_- = 0$$  

which, if $\rho \neq 0$, implies that $M$ is q-symmetric as well as q-antisymmetric, and therefore $M$ must be a multiple of the identity matrix. Thus infinitesimal rotations exist only on the null-sphere.

THE LIE BRACKET OF VECTOR FIELDS ON A QUANTUM HOMOGENEOUS SPACE

We have just seen that a homogeneous space of a quantum group may have a satisfactory notion of vector field which the group itself lacks. On the other hand, on a quantum space with a differential calculus there is no natural definition of the Lie bracket of vector fields; the coaction of the group can remedy this deficiency.

Let $\mathcal{F}$ be a (non-commutative) function algebra with a differential calculus $\mathcal{D}$, and let $\mathcal{H} = \mathcal{F}(G)$ be a Hopf algebra coacting on $\mathcal{F}$, so that there is an algebra homomorphism $\delta : \mathcal{F} \to \mathcal{H} \otimes \mathcal{F}$. As on the quantum sphere, we define an infinitesimal generator of the coaction to be a map $X : \mathcal{F} \to \mathcal{F}$ which is both a vector field, so that $X = \iota_X \circ d$ for some map $\iota_X : \mathcal{D} \to \mathcal{F}$, and is obtained from the coaction by

$$X = (\xi \otimes \text{id}) \circ \delta \quad \text{for some } \xi \in H^*.$$  

11
If $X$, $Y$ are two such infinitesimal generators obtained from $\xi, \eta \in H^*$, we can form the adjoint Lie bracket $[\xi, \eta]$ by (17) and hence define the Lie bracket of $X$ and $Y$ by

$$[X, Y] = ([\xi, \eta] \otimes \text{id}) \circ \delta$$  \hspace{1cm} (56)

It can be shown that $[X, Y]$ is also a vector field on $\mathcal{F}$. However, in general, a given infinitesimal generator $X$ is not obtained from a unique $\xi \in H^*$, and $[X, Y]$ will depend on the choice of $\xi$ and $\eta$.

**THE SLIPPERY PATH FROM SPHERE TO GROUP**

On the Sunday in the middle of the Karpacz school, some of us climbed Snieżka and discovered how slippery uphill paths can be. The attempt to lift vector fields from the quantum sphere to the orthogonal group is reminiscent of that walk.

Suppose $X$ is an infinitesimal rotation of the quantum sphere determined by $\xi \in H^*$. Then the action of $X$ on coordinates is given by a matrix $M$ according to (52), and this implies that the value of $\xi$ on the generators $a_{ij}$ of $H = \mathcal{F}(G)$ is also given by $M$ according to

$$\langle \xi, a_{ij} \rangle = m_{ik}a_{jk} \hspace{1cm} (57)$$

To define $\xi$ completely, we need to specify $\langle \xi, f \rangle$ for all polynomials $f \in \mathcal{F}(G)$.

If $\xi = \tilde{X}$, where $\tilde{X}$ is a vector field on $G$ with respect to some differential calculus on $\mathcal{F}(G)$, then

$$X(x_1x_2) = \iota_X(x_1dx_2 + dx_1x_2) = \langle \xi, \pi A_1 \pi A_2 \rangle x_1x_2 = \iota_{\tilde{X}}(A_1dA_2 + dA_1A_2)x_1x_2$$

But this forces the differential calculus on $\mathcal{F}(G)$ to be the impoverished one that we first considered, with relations (30):

**Theorem.** The only differential calculi on $\mathcal{F}(S)$ and $\mathcal{F}(G)$ which are compatible in the above sense are defined by the relations

$$x_1x_2 = qR_{12}x_1dx_2 \hspace{1cm} dA_1A_2 = qR_{12}A_1dA_2R_{12}$$

In terms of the general analysis of Schm"udgen and Sch"uler, this differential calculus on the group has $T_{123} = X_{23}^{-1}R_{12}^{-1}R_{23}X_{12}$ with $X = 1$.

**PRESS ON REGARDLESS**

Suppose we just ignore the unsatisfactory features of the differential calculus defined by (30), and develop its formal consequences. It gives a derivation property
for vector fields $\tilde{X}$ on the quantum orthogonal group which makes it possible to evaluate $\tilde{X}_e$ on products of matrix elements; for example,

$$\langle \tilde{X}_e, A_1A_2A_3 \rangle = \langle \tilde{X}_e, A_3 \rangle + R_{23}\langle \tilde{X}_e, A_3 \rangle R_{23} + R_{12}R_{23}\langle \tilde{X}_e, A_3 \rangle R_{23}R_{12}\quad (58)$$

This makes sense (when multiplied by $x_1x_2x_3$) even if the differential calculus doesn’t, and gives a Lie algebra between antisymmetric $n \times n$ matrices $X = (x^i_j)$ which is best presented graphically and in terms of $x^{ij} = g^{ik}x^j_k$:

$[X, Y] = X \bullet Y - \bullet \beta(X \otimes Y)\quad (60)$

where $(X \bullet Y)^{ij} = x^{ik}g_{kl}y^{lj}$, defining $\bullet : M \otimes M \to M$, and $\bullet \beta : M \otimes M \to M$ is the composition of $\beta$ and $\bullet$.

**Theorem.** Let $T$ be the set of covariant tensors $X = (x^{ij})$, and let $L$ be the subset satisfying $\Pi_+(X) = 0$. Define the Lie bracket $[X, Y]$ between two such tensors by the above diagram. Then we have

1. **Closure**

   $$X, Y \in L \implies [X, Y] \in L\quad (59)$$

2. **Braiding** There is an operator $\beta : M \otimes M \to M \otimes M$ satisfying the braid relation $\beta_{12}\beta_{23}\beta_{12} = \beta_{23}\beta_{12}\beta_{23}$, such that

   $$[X, Y] = X \bullet Y - \bullet \beta(X \otimes Y)$$

(classically, $[X, Y] = XY + Y^T X$). This has surprisingly good properties:
3. **Antisymmetry** If \( T = \sum X_i \otimes Y_i \) satisfies \( \beta(T) = 0 \), then \( \sum [X_i, Y_i] = 0 \).

However, the braiding \( \beta \) is not defined purely in terms of the putative Lie algebra \( L \) itself; and there is no Jacobi identity for this bracket. The braiding \( \beta \) is shown in the following diagram:

\[
\beta = -\lambda_0^{-1}
\]

since

\[
\lambda_0^{-1}
\]
References

[1] U.Carow-Watamura, M.Schlieker, S.Watamura and W.Weich, Bicovariant
differential calculus on quantum groups $SU_q(N)$ and $SO_q(N)$. *Commun.
Math. Phys.* (1991), 142, 605.

[2] A.Joseph and G.Letzter, Separation of variables for quantized enveloping
algebras. *Am. J. Math.* (1994) 116, 127.

[3] S.Majid, Quantum and braided linear algebra. *J. Math. Phys.* (1993) 34,
1176.

[4] S.Majid, Free braided differential calculus, braided binomial theorem and
the braided exponential map. *J. Math. Phys.* (1993) 34, 4843.

[5] S.Majid, Quantum and braided Lie algebras. *J. Geom. Phys.* (1994), 13,
207.

[6] S.Majid, On the addition of quantum matrices. *J. Math. Phys.* (1994), 35,
2617.

[7] K.Schm"udgen and A.Sch"uler, Covariant differential calculi on quantum
spaces and on quantum groups. *C. R. Acad. Sci. Paris* (1993), 316, 1155.

[8] K.Schm"udgen and A.Sch"uler, Classification of bicovariant differential calculi
on quantum groups of type $A$, $B$, $C$, and $D$. Leipzig preprint (1993).

[9] P.Schupp, P.Watts and B.Zumino, Bicovariant quantum algebras and quantum
Lie algebras. *Commun. Math. Phys.* (1993), 157, 305.

[10] A.Sudbery, The algebra of differential forms on a full matric bialgebra. *Math.
Proc. Camb. Phil. Soc.* (1993) 114, 111.

[11] A.A.Vladimirov, Coadditive differential complexes on quantum groups and quantum spaces. Dubna preprint E2-94-39 (1994).

[12] J.Wess and B.Zumino, Covariant differential calculus on the quantum hyperplane. *Nucl. Phys. B (Proc.,Suppl.)* 18, 302.

[13] S.L.Woronowicz, Differential calculus on compact matrix pseudogroups
(quantum groups). *Commun. Math. Phys.* (1989) 122, 125.