On Nonlinear Gauge Theory
from a Deformation Theory Perspective

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(Received October 21, 1999)

Nonlinear gauge theory is a gauge theory based on a nonlinear Lie algebra (finite W algebra) or a Poisson algebra, which yields a canonical star product for deformation quantization as a correlator on a disk. We pursue nontrivial deformation of topological gauge theory with conjugate scalars in two dimensions. This leads uniquely to a two-dimensional nonlinear gauge theory, which implies its essential uniqueness. We also consider a possible generalization to higher dimensions.

Nonabelian Chern-Simons gauge theory, a topological field theory (TFT) of the Schwarz type, provides an intriguing framework to deal with three-dimensional pure gravity on one hand and knot invariants on the other. A two-dimensional analogue of such a framework is given by nonlinear gauge theory, which is a gauge theory based on a nonlinear Lie algebra (finite W algebra) or a Poisson algebra. That is, nonlinear gauge theory provides a TFT framework to deal with two-dimensional pure gravity (dilaton gravity) on one hand and star products on the other.

Nonlinear gauge theory was originally constructed in part by inspection. In this paper, we consider a nontrivial deformation of abelian BF theory in two dimensions, which uniquely results in nonlinear gauge theory. A two-dimensional nonlinear gauge theory is unique in this sense, as is the case for Chern-Simons gauge theory in three dimensions.

We first recapitulate a few aspects of nonlinear gauge theory. The two-dimensional theory is given by an action functional

$$ S = \int_\Sigma h^a d\phi_a + \frac{1}{2} W_{ab}(\phi) h^a h^b, $$

where $\phi_a$ is a scalar field, $h^a$ is a one-form gauge field with an internal index $a$, and $d$ denotes the exterior derivative on a two-dimensional spacetime or worldsheet $\Sigma$. The structure constants of the algebra determine the couplings in the theory, as nonabelian gauge theory is a gauge theory based on a nonabelian Lie algebra, whose structure constants yield the couplings of the gauge interaction.

The metric field along with the dilaton yields no local physical degrees of freedom in two dimensions, as is the case for the metric field in three dimensions.

A quadratically nonlinear gauge theory was obtained in a somewhat related approach by Dayi.

This action is intrinsically two-dimensional, as the Chern-Simons action is peculiar to three dimensions.
structure function $W_{ab}$ is determined by a nonlinear Lie algebra (finite $W$ algebra $^7$) $[T_a, T_b] = W_{ab}(T)$ or a Poisson bracket $[\phi_a, \phi_b] = W_{ab}(\phi)$ of the coordinates $\phi_a$ and $\phi_b$ on a Poisson manifold $M$, as was pointed out by Schaller and Strobl. $^8$ This action is invariant under the nonlinear gauge transformation

$$
\delta \phi_a = W_{ba} \epsilon^b, \quad \delta h^a = d\epsilon^a + \frac{\partial W_{bc}}{\partial \phi_a} h^b \epsilon^c
$$

(2)
due to the Jacobi identity satisfied by $W_{ab}$. $^5$

With an ordinary Lie algebra, this theory reduces to nonabelian BF theory in two dimensions. When we adopt a nonlinear extension of the two-dimensional Poincaré algebra, we obtain the Palatini form of two-dimensional dilaton gravity, $^5$ as the Palatini form of three-dimensional gravity is obtained as Chern-Simons gauge theory of the three-dimensional Poincaré algebra. $^2$

When the Poisson bracket is nondegenerate to yield a symplectic form $\omega$ on $M$, we can formally integrate out the fields $h^a$ to obtain the action

$$
S = \int_{\partial \Sigma} d^{-1}\omega
$$

(3)
of particles with the phase space $M$ and a vanishing Hamiltonian, where we adopt a disk as the worldsheet $\Sigma$. Thus $\partial \Sigma$ is the boundary circle. Then the path integral

$$
\int D\phi \delta_x(\phi(\infty)) f(\phi(1)) g(\phi(0)) \exp i \frac{\bar{h}}{\hbar} S
$$

(4)
yields an estimation at a point $x \in M$ of the (time-)ordered product of functions $f$ and $g$ on $M$, where $\delta_x$ is a delta function centered at $x$, and the arguments 0, 1 and $\infty$ of $\phi$ denote three points on the circle $\partial \Sigma$, placed in counterclockwise order.

More generally, the canonical star product given explicitly by Kontsevich $^9$ for deformation quantization $^{13}$ is obtained perturbatively as a correlator:

$$
f \star g(x) = \int D\phi Dh \delta_x(\phi(\infty)) f(\phi(1)) g(\phi(0)) \exp i \frac{\bar{h}}{\hbar} S.
$$

(5)

Here $S$ is the action Eq. (1) of the nonlinear gauge theory on the disk $\Sigma$, as was elucidated by Cattaneo and Felder. $^{10}$ We have omitted the ghost part in the above functional integral.

Let us now consider nontrivial deformation of abelian BF theory $^*$ in two dimensions with the aid of the Barnich-Henneaux approach $^{11}$ based on the Batalin-Vilkovisky formalism $^{14}$ where a solution to the master equation provides a gauge transformation and an invariant action simultaneously.

The free action is given by

$$
S_0 = \int d^2 \xi \epsilon^{\mu \nu} h^a_\mu \partial_\nu \phi_a,
$$

(6)

$^*$ For the abelian theory, the star product Eq. (5) reduces to the ordinary product, so that deformation of the theory corresponds to that of the product.
which is invariant under the gauge transformation
\[ \delta_0 \phi_a = 0, \quad \delta_0 h^a_{\mu} = \partial_\mu \epsilon^a. \]
The minimal solution to the classical master equation reads
\[ \bar{S} = S_0 + \int d^2 \xi h^a_{\mu} \partial_\mu \epsilon^a, \]
and the BRST symmetry is then given by
\[ s = \partial_\mu \epsilon^a \partial^2_\mu + \epsilon^{\mu \nu} \partial_\nu \phi^a \partial^\alpha_\nu - \partial_\mu h^a_{\alpha} \partial \epsilon^\alpha_\mu. \]
A deformation \( \tilde{L} \) of the Lagrangian should obey
\[ s \tilde{L} + da_1 = 0, \]
with its descent equations
\[ sa_1 + da_0 = 0, \quad sa_0 = 0. \]
Thus
\[ a_0 = - \frac{1}{2} f_{ab}(\phi) c^a c^b, \]
where \( f_{ab}(\phi) \) is antisymmetric. This implies
\[ a_1 = \frac{1}{2} \partial f_{ab} h^a_{\alpha} c^b + f_{ab} h^a_{\alpha} c^b, \]
where \( h^a_{\alpha} = \epsilon_{\mu \nu} h^a_{\mu \nu} \), and leads to
\[ \tilde{L} = - \frac{1}{4} \partial^2 f_{ab} h^a_{\alpha} c^b c^c + \partial f_{ab} \left( \frac{1}{2} c^a c^b c^c - h^a h^b \right) - f_{ab} \left( \phi^{\alpha} c^b - \frac{1}{2} h^a_{\mu} h^b \right). \]
This deformation is consistent only when the functions \( f_{ab}(\phi) \) satisfy the Jacobi identity. Then the deformed action
\[ \bar{S} + \int \tilde{L} \]
satisfies the classical master equation. For the vanishing antifields, it reduces to the action Eq. (1) of nonlinear gauge theory with \( f_{ab}(\phi) = W_{ab}(\phi) \). Hence we uniquely obtain the nonlinear gauge theory as a deformation of the free theory in two dimensions.

We finally consider a possible generalization to higher dimensions. Let us try the BF-like Lagrangian
\[ \mathcal{L} = A^{ab} D_\mu \phi_a + \frac{1}{2} B^{\mu \nu} R^a_{\mu \nu}, \]
where $B_{\mu}^{\nu}$ is an antisymmetric tensor and

$$D_\mu \phi_a = \partial_\mu \phi_a + W_{ab} h_{\mu}^b, \quad R_{\mu}^a = \partial_\mu h_a^a - \partial_\nu h_a^\nu + \partial W_{be} h_{\mu}^b h_{\nu}^c. \quad (17)$$

This turns out to be invariant under the nonlinear gauge transformation Eq. (2) with

$$\delta A^{\mu a} = A^{bp} \frac{\partial W_{bc}}{\partial \phi_a} \varepsilon^c - B_{d}^{\mu} \frac{\partial^2 W_{bc}}{\partial \phi_d \partial \phi_a} h_{\nu}^b \varepsilon^c, \quad \delta B_{a}^{\mu} = B_{c}^{\mu} \frac{\partial W_{ba}}{\partial \phi_c} \varepsilon^b, \quad (18)$$

since

$$\delta (D_\mu \phi_a) = (D_\mu \phi_c) \frac{\partial W_{bc}}{\partial \phi_a} \varepsilon^b,$$

$$\delta R_{\mu}^a = R_{d}^{\mu} \frac{\partial W_{bc}}{\partial \phi_a} \varepsilon^c + \left\{ (D_\mu \phi_d) \frac{\partial^2 W_{bc}}{\partial \phi_d \partial \phi_a} h_{\nu}^b \varepsilon^c - (\mu \leftrightarrow \nu) \right\}. \quad (19)$$

Thus deformation theoretic analysis is also called for in higher-dimensional cases.

To conclude, we have obtained a nonlinear gauge theory uniquely as the nontrivial deformation of the abelian gauge theory with conjugate scalars in two dimensions. This may be regarded as a partial result of a systematic search for topological gauge field theories of the Schwarz type: Chern-Simons, BF and nonlinear gauge theories. They constitute the simplest class of gauge field theories, with no local physical degrees of freedom, whereas they have fertile mathematical content. This is not so surprising; quantum field theories are even expected to describe the whole nature effectively, which is apparently the most complicated in nature with huge mathematical structures.

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