Symmetric punctured intervals tile $\mathbb{Z}^3$

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Abstract

Extending the methods of Metrebian (2018), we prove that any symmetric punctured interval tiles $\mathbb{Z}^3$. This solves a question of Gruslys, Leader and Tan (2016).

1 Introduction

Given $n$, let $T$ be a tile in $\mathbb{Z}^n$, i.e. a finite subset of $\mathbb{Z}^n$. Recently, confirming a conjecture of Chalcraft that was posed on MathOverflow, Gruslys, Leader and Tan [2] showed that $T$ tiles $\mathbb{Z}^d$ for some $d$. This is an existence result and they wondered about better bounds in terms of the dimension $n$ and the size $|T|$. They conjectured the following for the case $n = 1$.

Conjecture 1.1 (Gruslys, Leader, Tan [2]). For any positive integer $t$ there is a number $d$ such that any tile $T$ in $\mathbb{Z}$ with $|T| = t$ tiles $\mathbb{Z}^d$.

Let us note that Adler and Holroyd [1] had earlier investigated which tiles in $\mathbb{Z}$ can tile $\mathbb{Z}$. In general, for any fixed $d$, there are one-dimensional tiles which cannot tile $\mathbb{Z}^d$, see Section 4. When dealing with one-dimensional tiles, we find it convenient to use the same notation as in [1]: a tile $T$ in $\mathbb{Z}$ which is the union of $n$ intervals $I_1$ up to $I_n$, such that the length of interval $I_i$ is $a_i$ and the gap between $I_i$ and $I_{i+1}$ is $b_i$, will be denoted by $a_1(b_1)a_2(b_2)a_3 \ldots (b_{n-1})a_n$. With this notation, we can state the concrete question that motivated this work [2, Qu. 21]. It asks for the optimal tiling dimension for symmetric punctured intervals.

Question 1.2 (Gruslys, Leader, Tan [2]). What is the least $d$ for which $T = k(1)k$ tiles $\mathbb{Z}^d$?

Very recently, Metrebian [4] showed that $d \leq 4$ suffices and that $d = 3$ is optimal when $k$ is odd or $k \equiv 4 \pmod{8}$. He noted that for $k \geq 3$ one has $d \geq 3$, while for $k \in \{1, 2\}$ the optimal $d$ equals $k$. We extend Metrebian’s methods to solve Question 1.2 in its entirety.

Theorem 1.3. The least $d$ for which $T = k(1)k$ tiles $\mathbb{Z}^d$ equals $\min\{k, 3\}$.

This will be a corollary of a slightly more general result, Proposition 3.4 below.

The organization of the paper is as follows. In Section 2 we prove a lemma implying that it is enough to find some structured partial tilings of $\mathbb{Z}^2$. We exhibit such constructions in Section 3. After having constructed upper bounds on the dimension $d$ such that $T$ tiles $\mathbb{Z}^d$, we then in Section 4 examine lower bounds that are related to Conjecture 1.1. We conclude in Section 5 with some speculative thoughts towards Conjecture 1.1.

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2 From partial to complete tilings

To tile $\mathbb{Z}^d$ with a one-dimensional tile $T$ for some minimal $d > 1$, one clearly needs tiles directed in each of the $d$ orthogonal directions. Choosing one particular direction and removing all tiles in the tiling of $\mathbb{Z}^d$ in this direction, one has partial tilings of spaces isomorphic to $\mathbb{Z}^{d-1}$ which are orthogonal to the chosen direction. Noting that $T$ has some gap, one can derive a contradiction assuming there are only two partial tilings.

In this section, we will prove that three different partial tilings can be enough when $T$ contains only one gap, i.e. when $T$ is the union of two intervals. This is done in Lemma 2.1 which is a generalization of Lemma 4 in \cite{4}.

**Lemma 2.1.** Let $T$ be the one-dimensional tile $k(m)\ell$. Suppose there are three disjoint subsets $A, B, C$ of $\mathbb{Z}^d$ with the same cardinality such that one can tile $\mathbb{Z}^d \setminus (A \cup B)$, $\mathbb{Z}^d \setminus (A \cup C)$ and $\mathbb{Z}^d \setminus (B \cup C)$ with $T$. Then $T$ tiles $\mathbb{Z}^{d+1}$.

**Proof.** First assume $m < \min\{k, \ell\}$. We construct a subset $Y \subset \mathbb{Z} \times \{0, 1, 2\}$ such that $|Y \cap \{(z) \times \{0, 1, 2\}\}| = 2$ for every $z \in \mathbb{Z}$ and such that $T$ tiles $Y$. Let $(x, i) \in Y$ for some $x \in \mathbb{Z}$ and $i \in \{0, 1, 2\}$ if and only if

$$x - i(k + \ell) \equiv 1, 2, \ldots, k; k + m + 1, k + m + 2, \ldots, k + m + \ell \pmod{3k + 3\ell} \text{ or }$$

$$\equiv 2k + \ell + 1, 2k + 2\ell; 2k + 2\ell + m + 1, \ldots, 3k + 2\ell + m \pmod{3k + 3\ell}.$$

The construction has been sketched in Figure 1 for $\{1, 2, \ldots, 3(k + \ell)\} \times \{0, 1, 2\}$. By gluing infinitely many copies of that picture together, one gets the full construction of $Y$.

\[
\begin{array}{c}
  0 \\
  1 \\
  2 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
  \pi_1 & & & & & & & \pi_3 \\
& \lVert & & & & & \rVert & \\
& \pi_1 & & & & & \pi_2 & \\
& & \lVert & & & & & \rVert & \\
& & & \pi_1 & & & & \pi_3 & \\
\end{array}
\]

**Figure 1:** Construction of $Y$.

Now we explain why this construction meets the conditions we need. Let $S_1 = \{1, 2, \ldots, k\}$, $S_2 = \{k + m + 1, k + m + 2, \ldots, k + m + \ell\}$, $S_3 = \{2k + \ell + 1, \ldots, 2k + 2\ell\}$ and $S_4 = \{2k + 2\ell + m + 1, \ldots, 3k + 2\ell + m\}$. Let $S_0 = S_1 + S_3$ and $S_e = S_2 + S_4$. Then both $S_0 \cup ((k + \ell) + S_0) \cup (2(k + \ell) + S_0)$ and $S_e \cup ((k + \ell) + S_e) \cup (2(k + \ell) + S_e)$ cover all elements in $\mathbb{Z}_{3(k+\ell)}$ exactly once, from which the result follows.

The elements of $A \cup B \cup C$ can be partitioned into triples $\{a_i, b_i, c_i\}$ since $A, B, C$ have the same cardinality. Every set $\mathbb{Z} \times \{a_i, b_i, c_i\}$ has a subset $Y_i \cong Y$ which can be tiled by $T$ in the same manner, i.e. there exists a partition $\{Z_1, Z_2, Z_3\}$ of $Z$ such that for every $i$ we have $Y_i \cap \{(z) \times \{a_i, b_i, c_i\}\} = \{a_i, b_i\}$ for every $z \in Z_1$, $Y_i \cap \{(z) \times \{a_i, b_i, c_i\}\} = \{a_i, c_i\}$ for every $z \in Z_2$ and $Y_i \cap \{(z) \times \{a_i, b_i, c_i\}\} = \{b_i, c_i\}$ for every $z \in Z_3$. Now $\mathbb{Z}^{d+1} \setminus (\cup_i Y_i)$ can be written as $Z_1 \times (\mathbb{Z}^d \setminus (A \cup B)) \cup Z_2 \times (\mathbb{Z}^d \setminus (A \cup C)) \cup Z_3 \times (\mathbb{Z}^d \setminus (B \cup C))$ and by the assumptions this can be tiled by $T$ as well, so $T$ tiles $\mathbb{Z}^{d+1}$. Looking at Figure 1 every hyperplane $\pi_i$ will be covered by the intersections with $\cup_i Y_i$ and a partial tiling isomorphic to one of $\mathbb{Z}^d \setminus (A \cup B), \mathbb{Z}^d \setminus (A \cup C)$ or $\mathbb{Z}^d \setminus (B \cup C)$.  

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When $m \geq \min\{k, \ell\}$, where we assume without loss of generality $k = \min\{k, \ell\}$, one can glue two copies $T_1, T_2$ of $T$ together to a tile $T'$ with $k' = \ell' = k + \ell$ and $m' = m - k$ by taking $T_1 = \{-k, -k + 1, \ldots, -1\} \cup \{m, m + 1, \ldots, m + \ell - 1\}$ and $T_2 = \{-k - \ell, -k - \ell + 1, \ldots, -k - 1\} \cup \{m - k, m - k + 1, \ldots, m - 1\}$. See Figure 2 for a depiction. When $m' \geq k'$, one can glue $[m'/k' + 1]$ copies of $T'$ together, which are translates of $T'$ with initial point at $0, k', \ldots, [m'/k'] k'$. Hence we have reduced this to the case which has been proven already.

![Figure 2: Gluing $T_1$ and $T_2$ and copies $T'$.](image)

### 3 Most punctured intervals tile $\mathbb{Z}^3$

Rather than focusing solely on Question 1.2, we consider a slightly more general setting. Throughout this section, we let $T$ be a punctured interval tile, which is the union of an interval of length $k$ and an interval of length $\ell$ with a gap of size 1. So $T = k(1)\ell$ equals a translate of $\{-k, -k + 1, \ldots, -1, 1, 2, \ldots, \ell\}$ as a subset of $\mathbb{Z}$. By applying Lemma 2.1, we prove that in most cases $T$ tiles $\mathbb{Z}^3$. When tiles do not tile $\mathbb{Z}^2$, the partial tilings cannot be only horizontal (similarly not only vertical). Hence it is natural to try to combine partial vertical tilings with partial horizontal tilings up to a set of the desired form.

As a warm up, we construct three partial tilings of the plane satisfying the conditions of Lemma 2.1 when $T$ is the symmetric punctured interval $k(1)k$ with $k \equiv 2 \pmod{4}$.

**Proposition 3.1.** If $k \equiv 2 \pmod{4}$, then $T = k(1)k$ tiles $\mathbb{Z}^3$.

**Proof.** Let $X$ be a set of diagonals which are a distance $k + 1$ apart, e.g. $X = \{(x, y) \in \mathbb{Z}^2 | x - y \equiv 0 \pmod{k + 1}\}$. Let $A = \{(x, y) \in X | x \equiv 0, 1 \pmod{4}\}$ and $B = \{(x, y) \in X | x \equiv 2, 3 \pmod{4}\}$. Furthermore, choose $C = B + (0, 1) = \{(x, y) \in \mathbb{Z}^2 | y - x \equiv 1 \pmod{k + 1}\}$, $x \equiv 2, 3 \pmod{4}$. The construction is shown in Figure 3. Then $\mathbb{Z}^2 \setminus (A \cup B) = \mathbb{Z}^2 \setminus X$ can be tiled by $T$ in many ways. Note that one can tile $\mathbb{Z}^2 \setminus (A \cup C)$ easily by vertical copies of $T$, i.e. for every $x \in \mathbb{Z}$ one can tile $(\{x\} \times \mathbb{Z}) \setminus (A \cup C)$ straightforwardly with copies of $T$.

One can also see that $\mathbb{Z}^2 \setminus (B \cup C)$ can be tiled, by placing copies of $T$ horizontally. One can check that for every $y \in \mathbb{Z}$ the set $(\mathbb{Z} \times \{y\}) \setminus (B \cup C)$ is periodic (with period $4k + 4$) and its period can be covered with two copies of $T$, which have one edge in common (i.e. which are translates of each other with distance $2k + 1$).

By Lemma 2.1 we know $T$ tiles $\mathbb{Z}^2$ as the conditions of the lemma are satisfied.

In the not-necessarily-symmetric case, we start with two constructions that work for certain punctured intervals and then proceed to Proposition 3.4 which implies the upperbound in Theorem 1.3.

**Proposition 3.2.** If $2 \nmid k + \ell$, then $T = k(1)\ell$ tiles $\mathbb{Z}^3$. 

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Proof. If \(k + \ell\) is odd, then \(K = k + \ell + 1\) is even. Take \(A = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv y \pmod{K}, x \equiv 0 \pmod{2}\}\), \(B = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv y \pmod{K}, x \equiv 1 \pmod{2}\}\) and \(C = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv y - 1 \pmod{K}, x \equiv 0 \pmod{2}\}\).

Now one can tile \(\mathbb{Z}^2 \setminus (A \cup B)\) and \(\mathbb{Z}^2 \setminus (A \cup C)\) by placing copies of \(T\) horizontal, i.e. by placing copies of \(T\) from \((x - k, y)\) to \((x + \ell, y)\) for any \((x, y) \in A \cup B\), resp. \(A \cup C\). Similarly one can tile \(\mathbb{Z}^2 \setminus (B \cup C)\) with vertical copies of \(T\) from \((x, y - k)\) to \((x, y + \ell)\) for any \((x, y) \in B \cup C\).

Hence the result follows from Lemma 2.1.

\(\square\)

**Proposition 3.3.** If \(2 \nmid k, \ell\), then \(T = k(1)\ell\) tiles \(\mathbb{Z}^3\).

**Proof.** Let \(K = k + \ell + 2\). Take \(A_1 = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv y \pmod{K}, x \equiv 0 \pmod{2}\}\) and \(A_2 = A_1 + (k + 1, 0) = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv y + k + 1 \pmod{K}, x \equiv 0 \pmod{2}\}\). Let \(B_i = (1, 1) + A_i\) and \(C_i = (0, 1) + A_i\) for every \(i \in \{1, 2\}\). Set \(A = A_1 \cup A_2, B = B_1 \cup B_2\) and \(C = C_1 \cup C_2\).

Now one can tile \(\mathbb{Z}^2 \setminus (A \cup B)\) and \(\mathbb{Z}^2 \setminus (A \cup C)\) by placing copies of \(T\) horizontal, i.e. by placing copies of \(T\) from \((x - k, y)\) to \((x + \ell, y)\) for any \((x, y) \in A_2 \cup B_2\), resp. \(A_2 \cup C_2\). Similarly one can tile \(\mathbb{Z}^2 \setminus (B \cup C)\) with vertical copies of \(T\) from \((x, y - k)\) to \((x, y + \ell)\) for any \((x, y) \in B_1 \cup C_1\).

Hence the result follows from Lemma 2.1.

\(\square\)

**Proposition 3.4.** If \(v_2(k) = v_2(\ell)\), then \(T = k(1)\ell\) tiles \(\mathbb{Z}^3\).

**Proof.** Let \(v_2(k) = v_2(\ell) = n\) and \(q = 2^n\). When \(n = 0\), the result follows from Proposition 3.3. So from now on, we assume \(n \geq 1\).

Let \(A \subset \mathbb{Z}^2\) be the sets containing the elements \((x, y)\) if and only if

\[
x - y \equiv [(k + \ell + 2)(q - 1) + 1] \pmod{2(\ell + k + 2)q}
\]

for some \(0 \leq i \leq q - 1\) and \(j \in \{0, k + 1\}\). Let \(B = (q(\ell + k + 2), 0) + A\). Let \(C \subset \mathbb{Z}^2\) be the sets containing the elements \((x, y)\) if and only if

\[
x - y \equiv [(k + \ell + 2)(q - 1) + 1] \pmod{2(\ell + k + 2)q}
\]

for some \(q \leq i \leq 2q - 1\) and \(j \in \{k, \ell + k + 1\}\). One can see a depiction of this in Figure 4 in the case \(q = 2, n = 1\).
Figure 4: Construction of partial planar tilings when $v_2(k) = v_2(\ell)$.

Now one can tile $\mathbb{Z}^2 \setminus (A \cup B)$ with $T$ as $A \cup B$ is the union of diagonals which are alternately distance $k + 1$ and $\ell + 1$ apart. One can tile $\mathbb{Z}^2 \setminus (A \cup C)$ horizontally. For this, it is enough to tile one horizontal line as every horizontal line is a translate of that one and due to periodicity in particular the set

$$\mathbb{Z}^2 \setminus (A \cup C) \cap (\{0, 1, \ldots, 2(\ell + k + 2)q - 1\} \times \{0\}).$$

For this, use translates of $T$ starting at $(1 + i(\ell + k + 2), 0)$ for $0 \leq i \leq q - 1$ and at $(i(\ell + k + 2), 0)$ for $q \leq i \leq 2q - 1$.

To finish, we note that we can tile $\mathbb{Z}^2 \setminus (B \cup C)$ vertically. For this, we only have to check $(\{0\} \times \mathbb{Z}) \setminus (B \cup C)$, since $\gcd\{2(\ell + k + 2)q, (k + \ell + 2)(q - 1) + 1\} = 1$ and hence every vertical line is up to some translation identical to every other vertical line. By noting that $B$ and $C$ are subsets of some diagonals on the plane, one checks that

$$(\{0\} \times \mathbb{Z}) \cap B = \{0\} \times \{y \mid y \equiv i(\ell + k + 2) + \text{mod } 2(\ell + k + 2)q, q \leq i \leq 2q - 1, j \in \{0, \ell + 1\}\}.$$
For this, note that
\[
0 - [i(\ell + k + 2) + j] \cdot [(k + \ell + 2)(q - 1) + 1] \\
\equiv -[i(k + \ell + 2)(q - 1) + i + j(q - 1)](k + \ell + 2) - j \pmod{2q(k + \ell + 2)} \\
\equiv (i - j(q - 1))(k + \ell + 2) - j \pmod{2q(k + \ell + 2)}
\]
since \(2q \mid l + k\). When \(j = 0\), we get \(i(k + \ell + 2) \pmod{2q(k + \ell + 2)}\) for \(q \leq i \leq 2q - 1\). When \(j = \ell + 1\), we get \((k + \ell + 2) + k + 1 \pmod{2q(k + \ell + 2)}\) for \(q \leq i \leq 2q - 1\), since \(\ell \equiv q \pmod{2q}\) and \(2 \not\mid q\), so \((\ell + 1)(q - 1) \equiv -1 \pmod{2q}\). Similarly one has
\[
(\{0\} \times \mathbb{Z}) \cap C = \{0\} \times \{y \mid y \equiv i(\ell + k + 2) + j \pmod{2(\ell + k + 2)q}, 0 \leq i \leq q - 1, j \in \{-k, 1\}\}.
\]
Hence one can tile \((\{0\} \times \mathbb{Z}) \setminus (B \cup C)\) by putting vertical tiles starting at \((0, i(\ell + k + 2) - k + 1)\) for every \(i \equiv 0, 1, \ldots, q - 1 \pmod{2q}\) and \((0, i(\ell + k + 2) - k)\) for every \(i \equiv q, q + 1, \ldots, 2q - 1 \pmod{2q}\). Hence the result follows from Lemma 2.1.

We can handle a good fraction more cases as follows, but for the remaining cases for \(T = k(1)\), we suspect that an additional idea may be needed.

**Proposition 3.5.** If \(\ell \equiv 2 \pmod{4}, 4 \mid \ell\) (or vice versa) then \(T = k(1)\ell\) tiles \(\mathbb{Z}^3\).

**Proof.** Let \(2K = k + \ell + 2\) and note that \(K\) is even. We will choose \(A, B, C\) again such that we can apply Lemma 2.1.

Let \(j = \frac{k}{g}, g = \gcd(j, K)\) and \(K' = \frac{K}{g}\). We will construct a permutation \(a_1, a_2, \ldots, a_K\) of \(\{1, 2, \ldots, K\}\) satisfying \(a_i - a_{i-1} \equiv j, j + 1 \pmod{K}\) for every \(1 \leq i \leq K\). Write \(i - 1 = q_i K' + r\), where \(q_i\) and \(0 \leq r \leq K' - 1\) are integers. For every \(1 \leq i \leq K\), choose \(1 \leq a_i \leq K\) such that \(a_i \equiv j(i - 1) + q_i \pmod{K}\). In particular, one can note that \(a_i - a_{i-1} \equiv j + 1 \pmod{K}\) exactly when \(i\) is a multiple of \(K'\) and otherwise \(a_i - a_{i-1} \equiv j \pmod{K}\). To check that this is a permutation, note that if \(a_i \equiv a_h \pmod{K}\) then \(q_i \equiv q_h \pmod{g}\). Since \(0 \leq q_i, q_h \leq g - 1\), this implies \(q_i = q_h\). Hence \(j i \equiv j h \pmod{K} \Rightarrow i \equiv h \pmod{K'}\) and so combining with \(q_i = q_h\) we conclude \(i = h\).

For every \(1 \leq i \leq K\), let
\[
U_i = \{(x, y) \in \mathbb{Z}^2 \mid (x, y) \equiv (2i, 2i), (2i + 1, 2i), (2i, 2i + k + 1), (2i + 1, 2i + k + 1) \pmod{2K}\}.
\]
For every \(1 \leq i \leq \frac{K}{2}\)
- if \(a_{2i} - a_{2i-1} \equiv j \pmod{K}\), let
\[
X_i = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv 2a_{2i-1}, 2a_{2i} + 1 \pmod{2K}, y \equiv y - k - 1 \equiv 2a_{2i-1}, 2a_{2i} \pmod{2K}\},
\]
- if \(a_{2i} - a_{2i-1} \equiv j + 1 \pmod{K}\), let
\[
X_i = \{(x, y) \in \mathbb{Z}^2 \mid x \equiv 2a_{2i-1} + 1, 2a_{2i} \pmod{2K}, y \equiv y - k - 1 \equiv 2a_{2i-1}, 2a_{2i} \pmod{2K}\}.
\]
Let
\[
A \cup B = U = \bigcup_{i=1}^{K} U_i, A \cup C = X = \bigcup_{i=1}^{\frac{K}{2}} X_i\text{ and } B \cup C = U \Delta X.
\]
This can be done by taking $A = U \cap X$, $B = U \setminus X$ and $C = X \setminus U$.

Now $Z^2 \setminus (A \cup B) = Z^2 \setminus U$ can be tiled vertically by copies of $T$, by placing those copies starting in all $(x, y)$ for $x \equiv 2i, 2i + 1 \pmod{2K}$, $y \equiv 2i + 1 \pmod{2K}$ for every $1 \leq i \leq K$.

The set $Z^2 \setminus (A \cup C) = Z^2 \setminus X$ can be tiled horizontally by copies of $T$. Put those copies starting at $(x, y)$ for some $x \equiv 2a_{2i-1} + 1, y \equiv 2a_{2i} + 2$ or $y \equiv k - 1 \equiv 2a_{2i-1}, 2a_{2i} \pmod{2K}$ when $a_{2i} - a_{2i-1} \equiv j \pmod{K}$ and otherwise for some $x \equiv 2a_{2i-1} + 2, y \equiv k - 1 \equiv 2a_{2i-1}, 2a_{2i} \pmod{2K}$.

For the last case, we will check that $Z^2 \setminus (B \cup C)$ can be tiled vertically by copies of $T$ again. For every $1 \leq i \leq K$, the set $(U_{a_{2i-1}} \cup U_{a_{2i}}) \triangle X_i$ equals

$\{(x, y) \in Z^2 | (x, y) \text{ or } (x, y - k - 1) \equiv v \pmod{2K} \text{, for some } v \in V\}$

where $V = \{(2a_{2i-1}, 2a_{2i}), (2a_{2i-1} + 1, 2a_{2i-1}), (2a_{2i}, 2a_{2i-1}), (2a_{2i} + 1, 2a_{2i-1})\}$

if $a_{2i} - a_{2i-1} \equiv j \pmod{K}$, or

$V = \{(2a_{2i-1}, 2a_{2i-1}), (2a_{2i-1} + 1, 2a_{2i}), (2a_{2i}, 2a_{2i-1}), (2a_{2i} + 1, 2a_{2i})\}$

if $a_{2i} - a_{2i-1} \equiv j + 1 \pmod{K}$.

So clearly $(U_{a_{2i-1}} \cup U_{a_{2i}}) \triangle X_i$ can be tiled vertically.

The conclusion now follows easily as

$$U \triangle X = \bigcup_{i=1}^{K/2} (U_{a_{2i-1}} \cup U_{a_{2i}}) \triangle X_i.$$ 

Hence the result follows from Lemma 2.1.

**Corollary 3.6.** More than 95% of the punctured intervals tile $Z^3$.

**Proof.** For this, we combine Proposition 3.2 (half of the cases), Proposition 3.4 (one third of the cases) and Proposition 3.5 (one eighth of the cases).

**4 Impossible tilings**

Conjecture 1.1 should be a substantially more difficult problem than Question 1.2. In this section, to give an indication of the subtleties, we collect two classes of (known) one-dimensional tiles that do not tile $Z^d$ for a given $d$.

Let $T_k$ be the tile $k \underbrace{(k-1)1(k-1)1 \ldots (k-1)1(k-1)}_{k \text{ times } (k-1)}$ as considered in [2].

Let $D_n$ be the tile $2(1)2(1)2 \ldots (1)2$ as considered in [3].

The following proposition shows that for every $d$, one can find $k$ and $n$ such that neither $D_n$ nor $T_k$ tiles $Z^d$. The reason behind this is slightly different for the two tiles. The first uses sparseness of tiles put in one direction. The other considers the intersection of the tiles with subdivisions of $Z^d$.

**Proposition 4.1.** $T_k$ does not tile $Z^d$ for $d < \frac{k^2 + 2k - 1}{3k - 1}$ and $D_n$ does not tile $Z^d$ for $n > 3^{d - 1}$.

**Proof.** In the case of $T_k$, one looks to the maximum volume covered by tiles in one of the $d$ orthogonal directions in a hypercube $[N]^d$. When $N \to \infty$, the ratio of the volume covered by
these tiles will have a limsup which is at most \( \frac{3^{k-1}}{k+2k-1} \) from which the result follows as the sum of the ratios over the \( d \) directions should sum to 1.

We assume \( D_n \) tiles \( \mathbb{Z}^d \) and look to the intersection of this fixed tiling with a hypercube \([N]^d\). Look to the \( 3^d \) possible partitions of \( \mathbb{Z}^d \) in hypercubes with side length 3. Call a nonempty intersection of \([N]^d\) with a hypercube of side length 3 for a given partition a subregion. We now count the total number \( \#D \) of intersections of a subregion of a partition and a \( D_n \) which are of size 2, in two different ways.

For each of the \( 3^d \) partitions, there are less than \( (\frac{N}{3} + 2)^d \) subregions. Each subregion will contain at most \( \frac{3^d - 1}{2} \) intersections with a \( D_n \) of size 2. Hence \( \#D < 3^d (\frac{N}{3} + 2)^d \frac{3^d - 1}{2} = \frac{3^d - 1}{2} (N + 6)^d \).

On the other hand, there are at least \( \frac{(N-6n)^d}{2n} \) \( D_n \)'s completely inside the hypercube. Every \( D_n \) of these, intersects \( n \) subregions in exactly 2 places for each of \( 2 \cdot 3^{d-1} \) partitions. For \( 3^{d-1} \) partitions, these \( D_n \) intersects \( n - 1 \) small hypercubes in exactly 2 places and 2 small hypercubes in exactly one place. This implies that \( \#D \geq \frac{(N-6n)^d}{2n} \cdot 3^{d-1} (3n - 1) \).

Hence \( \frac{3^d - 1}{2} (N + 6)^d > \frac{(N-6n)^d}{2n} \cdot 3^{d-1} (3n - 1) \) for all \( N \), in particular one finds that the leading coefficients satisfy \( \frac{3^d - 1}{2} \geq 3^{d-1} \cdot \frac{3n - 1}{2n} \Rightarrow n \leq 3^{d-1} \).

In the case of \( D_n \), this generalizes the ‘only if’ part of Proposition 1 in [3]. Let us remark that this also follows from a straightforward generalization of Theorem 1 in [3], which concerns ‘convolutions’ of tiles. In case it might be of use to others, we use the notation of [3] to state the generalization (and leave the proof to the reader). Theorem 1 in [3] is \( n = 2 \) and \( d = 2 \).

**Proposition 4.2 (3).** Suppose \( T \subset \mathbb{Z}^n \) is a tile. Suppose that \( S \subset \mathbb{Z}^d \) is a symmetric tile (i.e. no matter how the tile is oriented, it is a translate of itself). Then if for some \( m \in \mathbb{N} \) one has \( \|S \star m 1_T\|_1 < \|S\|_T \) or \( \|S \star m 1_T\|_\infty < \|1_T\|_1 \) and \( \|S\| \neq 0 \), then \( T \) does not tile \( \mathbb{Z}^d \).

5 Towards Conjecture 1.1

Since the examples we presented in Section 4 contain many gaps, it is natural to wonder if the following is true. If so, it would prove Conjecture 1.1.

**Question 5.1.** Does there exist a function \( f : \mathbb{N} \to \mathbb{N} \) such that any tile \( T \subset \mathbb{Z} \) with at most \( N \) gaps, i.e. \( T \) is the union of at most \( N + 1 \) intervals, \( \mathbb{Z}^f(\mathbb{N}) \)?

This question naturally leads to a number of subproblems, which if solved could lead to progress in Conjecture 1.1

- Does any punctured interval \( k(1)\ell \) tile \( \mathbb{Z}^3 \)?
- Does any one-dimensional tile \( k(m)\ell \) tile \( \mathbb{Z}^d \) for some small (uniform) choice of \( d \)?
- Find the smallest \( d \) such that \( D_n \) tiles \( \mathbb{Z}^d \).

Answering the first subquestion affirmatively would improve upon Corollary 3.6 and would confirm Question 11 in [4]. By the work in this paper, the remaining open cases are \( 2 \leq \nu_2(k) < \nu_2(\ell) \), the smallest case being the tile \( T = 4(1)8 \).

For the second subquestion, by the reduction used at the end of the proof of Lemma 2.1 one knows that it is enough to do this for \( m < k, \ell \). When \( d = \text{gcd}\{m, k, \ell\} > 1 \), one can form
the union of a $k \times d \times d$ and a $\ell \times d \times d$ cuboid with a $m \times d \times d$ gap. So one only needs to consider the case with $\gcd\{k, m, \ell\} = 1$. In the case $2m \mid k + \ell + m$, one can extend the construction in Proposition 3.2 by choosing $K = \frac{k + \ell + m}{m} + 1$ and replacing every element in the construction of $A$, $B$ and $C$ by squares of size $m \times m$. For this, it could be helpful to find some tiling where $m \nmid k + \ell$, e.g. $T = 3(2)4$ and then find a general construction for these cases.

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