Interval estimation for linear discrete-time delay systems

International Online Seminar: Interval Methods in Control Engineering

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Why TDS are of interest?

Delays may induce:
- Poor performances
- Instability
- Oscillations
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Delays may induce:
- Poor performances
- Instability
- Oscillations
Time-delay effects

\[ \dot{x}(t) = -x(t - h) \]
Consider system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + A_h x(t - h) \\
x(t) &= \phi(t) \quad t \in [-h, 0]
\end{align*}
\]
Consider system

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\begin{align*}
\dot{x}(t) &= Ax(t) + A_h x(t - h) \\
x(t) &= \phi(t) \quad t \in [-h, 0]
\end{align*}
\]

Infinite dimensional system

Time-delay systems belong to the class of functional differential equations.
### State estimation of TDS

- **Luenberger-type observers** [Hewer et al., 1973] [Gressang, 1974] [Bhat et al., 1976]
- **Unknown input observers (UIO)** [Sename, 1997] [Fattouh et al., 1999] [fu et al., 2004] [Darouach, 2004] [Hassan et al., 2013] [Zheng et al., 2015] [Warrad et al., 2016]
- **$H_\infty$ observers** [Choi et al., 1996] [Fattouh et al., 1999] [Fridman et al., 2001] [Briat, 2008]
- **Sliding-mode observers** [Seuret et al, 2007] [Niu et al., 2004] [Hu et al., 2012]
State estimation of TDS

- Luenberger-type observers [Hewer et al., 1973] [Gressang, 1974] [Bhat et al., 1976]
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When there exist uncertainties

- Point estimation cannot converge to the real states.
- Interval estimation calculates lower and upper bounds enclosing all the feasible states in a guaranteed way.
Classification of interval estimation approaches

Interval estimation

- Interval observers
  - Zonotopes
  - Polytopes
  - Ellipsoids
  - Parallelotopes

- Set-membership estimation
Interval observers

Concept

- Provide upper and lower bounds of the state vector using the available information.

Limitation

- Design two point observers such that the estimation error dynamics are both cooperative and stable.
- Relaxation via coordinate transformation [Mazenc and Bernard, 2011] [Raïsi et al, 2012] $\implies$ additional conservatism [Chambon et al., 2016]
Zonotope-based interval estimation

- **Good trade-off** between estimation accuracy and computational complexity [Tang et al., 2019].
- Intuitive and independent of cooperativity constraint and coordinate transformation.
Zonotopes

Definition

- An $s$-zonotope $Z \subseteq \mathbb{R}^n$ is the affine image of a hypercube $\mathbb{B}^s = [-1, 1]^s$ in $\mathbb{R}^n$ and can be expressed as:

\[ Z = \langle p, H \rangle = p + H\mathbb{B}^s = \{ z \in \mathbb{R}^n : z = p + Hb \} \]
## Properties of zonotopes

### Minkowski Sum

**Property 1: [Combastel, 2003]** The Minkowski sum of two zonotopes \( Z_1 = \langle p_1, H_1 \rangle \) and \( Z_2 = \langle p_2, H_2 \rangle \) is also a zonotope

\[
Z = Z_1 \oplus Z_2 = \langle p_1 + p_2, [H_1 H_2] \rangle
\]

### Linear transformation

**Property 2: [Combastel, 2003]** The linear transformation of a given zonotope \( Z = \langle p, H \rangle \) is

\[
K \odot Z = \langle Kp, KH \rangle
\]

### Reduction order

**Property 3: [Combastel, 2005]** A high dimensional zonotope can be bounded by a lower one via a reduction operator

\[
Z = \langle p, H \rangle \subseteq \langle p, \downarrow_q (H) \rangle
\]
Interval state estimation methods applied for:

- Biological systems [Gouzé., 2000]
- Bioreactors [Moisan et al., 2007]
- Nonlinear systems control [Raïssi et al., 2012]
- LPV systems [Efimov et al., 2012]
- Switched systems [Ethabet et al., 2018] [Zammali et al., 2020]
- Descriptor systems [Wang et al., 2018] [Tang et al, 2020]
- Fault diagnosis [Wang et al., 2018]
State of art

Interval state estimation methods applied for:

- Biological systems [Gouzé., 2000]
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- Descriptor systems [Wang et al., 2018] [Tang et al, 2020]
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Interval state estimation for time-delay systems

- Interval observer design for descriptor linear and nonlinear systems with time-delay [Efimov et al., 2015] [Zheng et al., 2016]
- Reduced-order interval observer design for linear and nonlinear systems with time-delay [Efimov et al., 2013] [Huong et al., 2020]
- Interval observers for time-invariant exponentially stable linear systems [Mazenc et al., 2012]
Contributions

Motivation

- Existing solutions have been developed only for continuous-time delay systems.
- Zonotope-based interval estimation for time-delay systems has not been fully considered.
Motivation

- Existing solutions have been developed only for continuous-time delay systems
- Zonotope-based interval estimation for time-delay systems has not been fully considered

Contribution

- Interval observer design and zonotope-based methods for linear discrete-time systems with constant time-delay with external disturbances and measurement noises.
Outline

1. Interval observer design for linear discrete-time delay systems

2. Zonotope-based interval estimation for linear discrete-time delay systems

3. Simulations
   - Example 1: Case of Cooperative Error System
   - Example 2: Case of Non-cooperative Error System

4. Conclusions & Perspectives
Outline

1. Interval observer design for linear discrete-time delay systems

2. Zonotope-based interval estimation for linear discrete-time delay systems

3. Simulations
   - Example1 : Case of Cooperative Error System
   - Example2 : Case of Non-cooperative Error System

4. Conclusions & Perspectives
Consider the following linear discrete-time delay system:

\[
\begin{align*}
    x(k + 1) &= A_0 x(k) + A_1 x(k - h) + B u(k) + D w(k), \\
    y(k) &= C x(k) + F v(k), \\
    x(s) &= \phi(s), \quad s = -h, ..., 0
\end{align*}
\]

- \( x \in \mathbb{R}^{n_x} \) is the state vector
- \( u \in \mathbb{R}^{n_u} \) is the input vector
- \( y \in \mathbb{R}^{n_y} \) is the output vector
- \( w \in \mathbb{R}^{n_w} \) is the state disturbance
- \( v \in \mathbb{R}^{n_v} \) is the measurement noise
- \( h \) is a constant time-delay
- \( \phi \) is the initial function vector
Assumptions

- The pair \((A_0, C)\) is observable.
- The initial state vector \(x(k)\) satisfies \(x(k) \in [\underline{x}(k), \overline{x}(k)]\) for all \(k \in [-h, 0]\).
- The state disturbance and the measurement noise are unknown but bounded with well known bounds:

\[
\underline{w} \leq w(k) \leq \overline{w}; \quad -VE_{n_v} \leq v(k) \leq VE_{n_v}
\]
Goal

- Develop an interval observer which consists of two dynamical systems to estimate respectively the upper and lower bounds of the state.
Interval observer design for linear discrete-time delay systems

Interval Observer

Goal

- Develop an interval observer which consists of two dynamical systems to estimate respectively the upper and lower bounds of the state.

Interval Observer Structure

\[
\begin{align*}
\bar{x}(k+1) &= A_0\bar{x}(k) + A_1\bar{x}(k-h) + Bu(k) + L_0(y(k) - C\bar{x}(k)) \\
&\quad + L_1(y(k-h) - C\bar{x}(k-h)) + D^+\bar{w} - D^-\bar{w} \\
&\quad - (|L_0F| + |L_1F|)KE_{n_v}, \\
\bar{x}(k+1) &= A_0\bar{x}(k) + A_1\bar{x}(k-h) + Bu(k) + L_0(y(k) - C\bar{x}(k)) \\
&\quad + L_1(y(k-h) - C\bar{x}(k-h)) + D^+\bar{w} - D^-\bar{w} \\
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Interval Observer

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- Develop an interval observer which consists of two dynamical systems to estimate respectively the upper and lower bounds of the state.

**Interval Observer Structure**

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\begin{cases}
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+ L_1(y(k-h) - C\overline{x}(k-h)) + D^+w - D^-w \\
- (|L_0F| + |L_1F|)KE_{n_y},
\end{cases}
\]

\[
\begin{cases}
\underline{x}(k+1) = A_0 \underline{x}(k) + A_1 \underline{x}(k-h) + Bu(k) + L_0(y(k) - C\underline{x}(k)) \\
+ L_1(y(k-h) - C\underline{x}(k-h)) + D^+\overline{w} - D^-\overline{w} \\
+ (|L_0F| + |L_1F|)KE_{n_y},
\end{cases}
\]

⇒ Finding gain matrices $L_0$ and $L_1$ for ensuring:
- Cooperativity condition: $\overline{x}(k) \leq x(k) \leq \underline{x}(k)$, $\forall k \in \mathbb{Z}_+$
- Stability of $\varepsilon = x - \underline{x}$ and $\overline{\varepsilon} = \overline{x} - x$
Cooperative system [Haddad et al, 2004]

Consider a linear discrete-time dynamical system with a constant time-delay:

\[ x(k + 1) = A_0 x(k) + A_1 x(k - h) + w(k), \quad w : \mathbb{R}_+ \rightarrow \mathbb{R}^n_+ \]

\[ x(k) = \phi(k) \geq 0, \quad k \in [-h, 0] \]

where \( x(k) \in \mathbb{R}^n \) is the state vector, \( h \) is a constant time-delay and the matrices \( A_0 \) and \( A_1 \) have appropriate dimensions. This system is called cooperative or nonnegative for all \( h \in \mathbb{Z}_+ \) if the matrices \( A_0 \) and \( A_1 \) are nonnegative.
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**State Estimation Error Convergence**

**Stability Condition**

- Lyapunov-Krasovskii functional for the upper estimation error (similarly for the lower estimation error)

\[
V(\bar{e}(k)) = \bar{e}(k)^T P \bar{e}(k) + \sum_{j=1}^{h} \bar{e}(k - j)^T Q_d \bar{e}(k - j),
\]

\[
P^T = P \succ 0, \quad Q^T = Q \succ 0
\]
State Estimation Error Convergence

Stability Condition

- Lyapunov-Krasovskii functional for the upper estimation error (similarly for the lower estimation error)

\[ V(\bar{e}(k)) = \bar{e}(k)^T P \bar{e}(k) + \sum_{j=1}^{h} \bar{e}(k-j)^T Q_d \bar{e}(k-j), \]

\[ P^T = P \succ 0, \quad Q^T = Q \succ 0 \]

Uncertainties Attenuation Condition

- The robustness interval observer design against the unknown disturbances is defined:

\[ ||\bar{e}||_2 < \gamma^2 ||d||_2 \quad ||e||_2 < \gamma^2 ||d||_2 \]
LMI-based Conditions

In order to satisfy the above conditions, the following inequalities hold

\[
\begin{bmatrix}
-P + Q + I_{n_x} & * & * & * & * & * & * \\
0 & -Q & * & * & * & * & * \\
0 & 0 & -\gamma^2 I_{n_w} & * & * & * & * \\
0 & 0 & 0 & -\gamma^2 I_{n_v} & * & * & * \\
0 & 0 & 0 & 0 & -\gamma^2 I_{n_v} & * & * \\
PA_0 - K_0 C & PA_1 - K_1 C & P & K_0 F & K_1 F & -P \\
\end{bmatrix} \preceq 0,
\]

- \( P \succ 0 \),
- \( Q \succ 0 \),
- \( PA_0 - K_0 C \geq 0 \),
- \( PA_1 - K_1 C \geq 0 \),

- Positive scalar \( \gamma \)
- \( P \in \mathbb{R}^{n_x \times n_x} \) diagonal matrix, \( Q \in \mathbb{R}^{n_x \times n_x} \) symmetric matrix
- \( L_0 = P^{-1} K_0 \) and \( L_1 = P^{-1} K_1 \) are the observer gain matrices
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    y(k) &= C x(k) + F v(k), \\
    x(s) &= \phi(s), \quad s = -h, \ldots, 0
\end{align*}
\]

The initial system state vector \( x(k) \), disturbances vector \( w(k) \) and measurement noises vector \( v(k) \) are assumed to be unknown but bounded

\[
x(k) \in \langle p_k, H_k \rangle, \quad k = -h, \ldots, 0
\]

\[
w(k) \in \mathcal{W} = \langle 0, H_w \rangle, \quad v(k) \in \mathcal{V} = \langle 0, H_v \rangle
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    y(k) &= Cx(k) + Fv(k), \\
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    w(k) \in \mathcal{W} = \langle 0, H_w \rangle, \quad v(k) \in \mathcal{V} = \langle 0, H_v \rangle
\]

**Goal**

- Estimate the intervals of state vector as accurate as possible
Main idea

Proposed method

- Zonotope-based interval estimation method for linear discrete-time delay systems
### Main idea

#### Proposed method

- Zonotope-based interval estimation method for linear discrete-time delay systems
  - Pointwise observer design via $H_\infty$ formalism
Main idea

Proposed method

- Zonotope-based interval estimation method for linear discrete-time delay systems
  - Pointwise observer design via $H_\infty$ formalism
  - Zonotope-based state reachable set estimation
Main idea

Proposed method

- Zonotope-based interval estimation method for linear discrete-time delay systems
  1. Pointwise observer design via $H_\infty$ formalism
  2. Zonotope-based state reachable set estimation
  3. Interval estimation via the smallest outer box
Step 1: $H_\infty$ observer design

**Luenberger-type observer structure**

\[
\hat{x}(k+1) = A_0\hat{x}(k) + A_1\hat{x}(k-h) + Bu(k) + L_0(y(k) - C\hat{x}(k)) \\
+ L_1(y(k-h) - C\hat{x}(k-1))
\]
Step 1: $H_\infty$ observer design

Luenberger-type observer structure

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+ L_1(y(k-h) - C\hat{x}(k-1))
\]

Estimation error dynamics $e(k) = x(k) - \hat{x}(k)$

\[
e(k+1) = (A_0 - L_0C)e(k) + (A_1 - L_1C)e(k-h) + Ed(k)
\]

Delay-Independent Stability Condition

\[
V(e(k)) = e(k)^T Pe(k) + \sum_{j=1}^{h} e(k-j)^T Qe(k-j), \quad P^T = P \succ 0, \quad Q^T = Q \succ 0
\]

Uncertainties attenuation condition

\[
||e||_2 < \gamma^2 ||d||_2
\]
Step 1: $H_\infty$ observer design

LMI-based Optimization Problem

In order to satisfy the above conditions, the following LMI holds

$$\begin{bmatrix}
-P + Q + I_{nx} & * & * & * & * \\
0 & -Q & * & * & * \\
0 & 0 & -\gamma^2 I_{nw} & * & * \\
0 & 0 & 0 & -\gamma^2 I_{nv} & * \\
0 & 0 & 0 & 0 & -\gamma^2 I_{nv} \\
(PA_0 - R_0 C) & (PA_1 - R_1 C) & PD & -R_0 F & -R_1 F & -P
\end{bmatrix} \prec 0$$

- $\gamma > 0$ is a scalar
- $P, Q \in \mathbb{R}^{nx \times nx}$ symmetric and positive definite matrices
- $L_0 = P^{-1}R_0$ and $L_1 = P^{-1}R_1$ observer gain matrices
Step 2 : Zonotope-based state reachable set estimation

The reachable set estimation for the error dynamic system:

\[ e(k + 1) = (A_0 - L_0 C)e(k) + (A_1 - L_1 C)e(k - h) \]
\[ + Dw(k) - L_0 Fv(k) - L_1 Fv(k - h) \]

where

\[ e(k) \in \langle 0, H_k \rangle, \quad k = -h, ..., 0 \]
\[ w(k) \in \mathcal{W} = \langle 0, H_w \rangle, \quad v(k) \in \mathcal{V} = \langle 0, H_v \rangle \]
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\]

\[
w(k) \in \mathcal{W} = \langle 0, H_w \rangle, \quad v(k) \in \mathcal{V} = \langle 0, H_v \rangle
\]

- Set propagation equation:

\[
e(k + 1) \in \Omega_{k+1} = (A_0 - L_0 C) \circ \langle 0, \hat{H}_k \rangle \oplus (A_1 - L_1 C) \circ \langle 0, \hat{H}_{k-h} \rangle \]

\[
\oplus D \circ \langle 0, H_w \rangle \oplus (-L_0 F) \circ \langle 0, H_v \rangle \oplus (-L_1 F) \circ \langle 0, H_v \rangle
\]

\[
= \langle 0, \hat{H}_{k+1} \rangle
\]

- Estimate the set of the state vector:

\[
\begin{cases}
x(k + 1) = \hat{x}(k + 1) + e(k + 1) \\
e(k + 1) \in \langle 0, \hat{H}_{k+1} \rangle
\end{cases}
\]

\[
\Rightarrow x(k + 1) \in \langle \hat{x}(k + 1), \hat{H}_{k+1} \rangle
\]
Step 2 : Zonotope-based state reachable set estimation

- The reachable set estimation for the error dynamic system:

\[
e(k + 1) = (A_0 - L_0 C)e(k) + (A_1 - L_1 C)e(k - h) \\
+ Dw(k) - L_0 Fv(k) - L_1 Fv(k - h)
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where

\[
e(k) \in \langle 0, H_k \rangle, \quad k = -h, ..., 0 \\
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- Set propagation equation:

\[
e(k + 1) \in \Omega_{k+1} = (A_0 - L_0 C) \circ \langle 0, \hat{H}_k \rangle \oplus (A_1 - L_1 C) \circ \langle 0, \hat{H}_{k-h} \rangle \\
\oplus D \circ \langle 0, H_w \rangle \oplus (-L_0 F) \circ \langle 0, H_v \rangle \oplus (-L_1 F) \circ \langle 0, H_v \rangle \\
= \langle 0, \hat{H}_{k+1} \rangle
\]

- Estimate the set of the state vector:

\[
\begin{cases}
x(k + 1) = \hat{x}(k + 1) + e(k + 1) \\
e(k + 1) \in \langle 0, \hat{H}_{k+1} \rangle
\end{cases}
\]

\[\Rightarrow x(k + 1) \in \langle \hat{x}(k + 1), \hat{H}_{k+1} \rangle \quad \text{How to obtain } \hat{H}_{k+1} ?\]
Step 2 : Zonotope-based state reachable set estimation

- Applying Properties 1 and 2 of zonotopes to the set propagation equation:

\[
\begin{align*}
\hat{H}_{k+1} &= [(A_0 - L_0 C)\hat{H}_k (A_1 - L_1 C)\hat{H}_{k-h}] DH_w - L_0 F H_v - L_1 F H_v \\
\hat{H}_k &= H_k, \quad k \in [-h, 0]
\end{align*}
\]
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- Applying Properties 1 and 2 of zonotopes to the set propagation equation:
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    \hat{H}_k &= H_k, \quad k \in [-h, 0]
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  \]

- The column number of the generator matrix $\hat{H}_{k+1}$ will increase linearly which may cause the curse of dimensionality.
Step 2 : Zonotope-based state reachable set estimation

Applying Properties 1 and 2 of zonotopes to the set propagation equation:

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\hat{H}_{k+1} = \left[ (A_0 - L_0 C)\hat{H}_k (A_1 - L_1 C)\hat{H}_{k-h} \right] DH_w - L_0 F H_v - L_1 F H_v \\
\hat{H}_k = H_k, \quad k \in [-h, 0]
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The column number of the generator matrix \(\hat{H}_{k+1}\) will increase linearly which may cause the curse of dimensionality.

Applying Property 3 to the matrix \(\hat{H}_{k+1}\):

\[
\hat{H}_{k+1} = \left[ (A_0 - L_0 C) \downarrow_q (\hat{H}_k) (A_1 - L_1 C)\hat{H}_{k-h} \right] DH_w - L_0 F H_v - L_1 F H_v \\
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Step 2 : Zonotope-based state reachable set estimation

- Applying Properties 1 and 2 of zonotopes to the set propagation equation:
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  \hat{H}_{k+1} &= [(A_0 - L_0 C) \downarrow_q (\hat{H}_k) (A_1 - L_1 C)\hat{H}_{k-h} DH_w - L_0 F H_v - L_1 F H_v] \\
  \hat{H}_k &= H_k, \quad k \in [-h, 0]
  \end{align*}
  \]

- The estimation error is bounded by
  \[
  e(k) \in (0, \hat{H}_k) \Rightarrow e(k + 1) \in (0, \hat{H}_{k+1})
  \]
Step 2: Zonotope-based state reachable set estimation

- Applying Properties 1 and 2 of zonotopes to the set propagation equation:
  \[
  \begin{align*}
  \hat{H}_{k+1} &= \left[(A_0 - L_0C)\hat{H}_k (A_1 - L_1C)\hat{H}_{k-h} DH_w - L_0FH_v - L_1FH_v\right] \\
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  \[
  \begin{align*}
  \hat{H}_{k+1} &= \left[(A_0 - L_0C) \downarrow_q (\hat{H}_k) (A_1 - L_1C)\hat{H}_{k-h} DH_w - L_0FH_v - L_1FH_v\right] \\
  \hat{H}_k &= H_k, \quad k \in [-h, 0]
  \end{align*}
  \]

- The estimation error is bounded by
  \[
  e(k) \in \langle 0, \hat{H}_k \rangle \Rightarrow e(k + 1) \in \langle 0, \hat{H}_{k+1} \rangle
  \]

- From \(x(k) = \hat{x}(k) + e(k)\), the estimated set of the states are given:
  \[
  x(k) \in \langle \hat{x}(k), \hat{H}_k \rangle
  \]
Step 3: Interval estimation via the smallest outer box

Smallest outer box [Combastel, 2003]

**Definition.** For a zonotope \( \mathcal{Z} = \langle p, H \rangle \subset \mathbb{R}^n \), its smallest outer box \( \text{Box}(\mathcal{Z}) \) is the smallest interval vector containing it, which is denoted by:

\[
\mathcal{Z} \subset \text{Box}(\mathcal{Z}) = \{ z \in \mathbb{R}^n : z = p + \text{diag}\{ \sum_{j=1}^{m} |H_{1,j}| \cdots \sum_{j=1}^{m} |H_{n,j}| \} \mathbb{B}^s \}
\]

Interval State Estimation

- Smallest interval vector \([\underline{x}(k), \overline{x}(k)]\) that contains the real state:

\[
\begin{align*}
\underline{x}(i, k) &= \hat{x}(i, k) - \sum_{j=1}^{m} |\hat{H}_{i,j}|, \quad i = 1, \ldots, n \\
\overline{x}(i, k) &= \hat{x}(i, k) + \sum_{j=1}^{m} |\hat{H}_{i,j}|, \quad i = 1, \ldots, n
\end{align*}
\]
1. Interval observer design for linear discrete-time delay systems

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4. Conclusions & Perspectives
A linear discrete-time delay system is considered with following parameters:

$$A_0 = \begin{bmatrix} 0.5 & 0.3 \\ -0.8 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.11 & 0.03 \\ 0.17 & 0.11 \end{bmatrix}, \quad F = 0.1$$

$$B = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad D = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad h = 3.$$
Simulation Results

- A linear discrete-time delay system is considered with following parameters:

\[
A_0 = \begin{bmatrix}
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B = \begin{bmatrix}
0.1 \\
0.2 
\end{bmatrix}, \quad D = \begin{bmatrix}
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- \(u(k) = 0.1\) is the input.
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- \( u(k) = 0.1 \) is the input.

- \( w(k) \in \langle 0, H_w \rangle, \quad v(k) \in \langle 0, H_v \rangle \) are the unknown disturbance and measurement noises where \( H_w = 0.1 \) and \( H_v = 0.01 \).
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- The initial state vector is bounded by \( X = \langle 0, H_k \rangle \) where

\[
H_k : \begin{cases}
0, & k = -h, \ldots, -1 \\
0.5 \times I, & k = 0
\end{cases}
\]

\( H_k \):
In this case, the cooperativity condition is satisfied and the interval observer is designed with $\gamma = 3.8$ and

$$L_0 = \begin{bmatrix} 0.36 \\ -0.91 \end{bmatrix}, \quad L_1 = \begin{bmatrix} -0.25 \\ 0.04 \end{bmatrix},$$

$$A_0 - L_0 C = \begin{bmatrix} 0.13 & 0.3 \\ 0.11 & 0.10 \end{bmatrix}, \quad A_1 - L_1 C = \begin{bmatrix} 0.14 & 0.03 \\ 0.12 & 0.11 \end{bmatrix}.$$
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- Zonotope-based interval estimation method is achieved with $H_\infty$ index $\gamma = 1.94$ and the following matrices :

$$L_0 = \begin{bmatrix} 0.5 \\ -0.79 \end{bmatrix}, \quad L_1 = \begin{bmatrix} -0.10 \\ 0.16 \end{bmatrix}.$$
Simulation Results

**Figure:** $x_1$ and its interval estimation
Simulation Results

**FIGURE:** $x_2$ and its interval estimation
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4. Conclusions & Perspectives
A linear discrete-time delay system is considered from [Lam et al., 2015] with following parameters :

\[
A_0 = \begin{bmatrix} 0.5 & -0.3 \\ -0.8 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.11 & 0.03 \\ 0.17 & -0.11 \end{bmatrix}, \quad F = 0.1
\]

\[
B = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad D = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad h = 3.
\]

\( u(k) = 0.1 \) is the input.
\( w(k) \in \langle 0, H_w \rangle, \quad v(k) \in \langle 0, H_v \rangle \) are the unknown disturbance and measurement noises where \( H_w = 0.1 \) and \( H_v = 0.01 \).

The initial state vector is bounded by \( \chi = \langle 0, H_k \rangle \) where

\[
H_k : \begin{cases} 0, & k=-h,\ldots,-1 \\ 0.5 \times I, & k=0 \end{cases}
\]
In this case, the interval observer can not be designed since the cooperativity constraint isn’t verified.
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Independent of the cooperativity constraint, an interval estimation, based on the zonotope-based method, can be implemented by obtaining $\gamma = 1.94$. and

\[
L_0 = \begin{bmatrix} 0.5 \\ -0.8 \end{bmatrix}, \quad L_1 = \begin{bmatrix} -0.11 \\ 0.17 \end{bmatrix}.
\]
Simulation Results

**Figure:** $x_1$ and its interval estimation
Simulation Results

\textbf{Figure:} $x_2$ and its interval estimation
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Interval Estimation for Linear Discrete-Time Delay Systems

- An interval observer design and zonotope-based interval estimation methods are proposed for linear discrete-time systems with time-delay affected by bounded disturbances and measurement noises.
Interval Estimation for Linear Discrete-Time Delay Systems

- An interval observer design and zonotope-based interval estimation methods are proposed for linear discrete-time systems with time-delay affected by bounded disturbances and measurement noises.
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Interval Estimation for Linear Discrete-Time Delay Systems

- An interval observer design and zonotope-based interval estimation methods are proposed for linear discrete-time systems with time-delay affected by bounded disturbances and measurement noises.

- The interval observer is designed based on cooperativity conditions of estimation error dynamics.

- The zonotope-based interval estimation method is proposed by a robust observer design based on $H_\infty$ formalism and zonotopic analysis.
Conclusions

Interval Estimation for Linear Discrete-Time Delay Systems

- An interval observer design and zonotope-based interval estimation methods are proposed for linear discrete-time systems with time-delay affected by bounded disturbances and measurement noises.

- The interval observer is designed based on cooperativity conditions of estimation error dynamics.

- The zonotope-based interval estimation method is proposed by a robust observer design based on $H_\infty$ formalism and zonotopic analysis.

- Compared with interval observers, the proposed method gives less conservative estimation results and it is independent of cooperativity constraint and coordinate transformation.
An interval observer design and zonotope-based interval estimation methods are proposed for linear discrete-time systems with time-delay affected by bounded disturbances and measurement noises.

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The zonotope-based interval estimation method is proposed by a robust observer design based on $H_\infty$ formalism and zonotopic analysis.

Compared with interval observers, the proposed method gives less conservative estimation results and it is independent of cooperativity constraint and coordinate transformation.

The interval observer has less computational complexity than the zonotope-based method based on set operations.
Further works

- Extension to delay-dependent stability approach.
Further works

- Extension to delay-dependent stability approach.
- Robust fault diagnosis for discrete-time delay systems with time-varying delay.
Thank you for your attention
Questions?