Some new inequalities of the Hermite–Hadamard type for extended \(((s_1,m_1)-(s_2,m_2))\)-convex functions on co-ordinates

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Some new inequalities of the Hermite–Hadamard type for extended \((s_1, m_1)-(s_2, m_2)\)-convex functions on co-ordinates

Bo-Yan Xi\(^*\), Chun-Ying He\(^1\) and Feng Qi\(^{2,3}\)

**Abstract:** In the paper, the authors introduce a new concept “extended \((s_1, m_1)-(s_2, m_2)\)-convex function on co-ordinates” and establish some new inequalities of the Hermite–Hadamard type for extended \((s_1, m_1)-(s_2, m_2)\)-convex functions of 2 variables on co-ordinates.

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**Keywords:** extended \((s_1, m_1)-(s_2, m_2)\)-convex function; co-ordinates; Hermite–Hadamard inequality

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1. Introduction

In Toader (1985) the concept of \(m\)-convex functions was introduced as follows.

**Definition 1.1** (Toader, 1985) For \(f: [0, b] \rightarrow \mathbb{R} \) and \(m \in (0, 1]\), if

\[
f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)
\]

is valid for all \(x, y \in [0, b]\) and \(t \in [0, 1]\), then we say that \(f\) is an \(m\)-convex function on \([0, b]\).

In Miheşan (1993) the concept of \((\alpha, m)\)-convex functions below was innovated.

**Definition 1.2** (Miheşan, 1993) For \(f: [0, b] \rightarrow \mathbb{R} \) and \((\alpha, m) \in (0, 1] \times (0, 1]\), if

\[
f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)
\]

is valid for all \(x, y \in [0, b]\) and \(t \in [0, 1]\), then we say that \(f\) is an \((\alpha, m)\)-convex function on \([0, b]\).
\[ f(tx + m(1 - t)y) \leq t^*f(x) + m(1 - t^*)f(y) \]

is valid for all \( x, y \in [0, b] \) and \( t \in [0, 1] \), then we say that \( f \) is a \((\alpha, m)\)-convex function on \([0, b]\).

In Dragomi (2002), the following the Hermite–Hadamard type inequality for \(m\)-convex functions was proved.

**Theorem 1.1** \((\text{Dragomi, 2002})\) Let \( f : \mathbb{R} \to \mathbb{R} \) be an \( m\)-convex with \( m \in (0, 1) \) and \( 0 \leq a < b \). If \( f \in L_1((a, b)) \), then

\[
\begin{align*}
\left( \frac{a + b}{2} \right) & \leq 1 \int_a^b f(x) + mf \left( \frac{x}{m} \right) \frac{dx}{2} \leq \frac{m + 1}{4} \left[ \frac{f(a) + f(b)}{2} + m \frac{f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right)}{2} \right].
\end{align*}
\]

**Definition 1.3** \((\text{Hudzik \& Maligranda, 1994})\) Let \( s \in (0, 1) \). A function \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is said to be \(s\)-convex (in the second sense) if

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]

holds for all \( x, y \in I \) and \( \lambda \in (0, 1) \).

In Drago (1999), Drago and Fitzpatrick proved a variant of Hadamard’s inequality for \(s\)-convex functions in the second sense.

**Theorem 1.2** \((\text{Dragomi, 1999})\) Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is an \( s\)-convex function in the second sense, where \( s \in (0, 1) \), and let \( a, b \in [0, \infty) \) with \( a < b \). If \( f \in L_1((a, b)) \), then

\[
2^{s-1}f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{s + 1}.
\]

**Theorem 1.3** \((\text{Kirmaci, Klaričić Bakula, Özdemir, \& Pečarić, 2007})\) Let \( f, g : [a, b] \to \mathbb{R} \) and \( f, g \in L_1((a, b)) \) with \( 0 \leq a < b < \infty \). If \( f \) is convex and nonnegative on \([a, b]\) and \( g \) is \(s\)-convex on \([a, b]\) for some fixed \( s \in (0, 1) \) then

\[
\frac{1}{b - a} \int_a^b f(x)g(x) \, dx \leq \frac{1}{s + 2} M(a, b) + \frac{1}{(s + 1)(s + 2)} N(a, b),
\]

where \( M(a, b) = f(a)g(a) + f(b)g(b) \) and \( N(a, b) = f(a)g(b) + f(b)g(a) \).

**Definition 1.4** \((\text{Dragomi, Pečarić, \& Persson, 1995})\) A map \( f : I \subseteq \mathbb{R} \to \mathbb{R}_0 \) said to belong to the class \( Q(I) \) if it is nonnegative and satisfies

\[
f(\lambda x + (1 - \lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1 - \lambda}
\]

for all \( x, y \in I \) and \( \lambda \in (0, 1) \).

**Theorem 1.4** \((\text{Dragomi, Pečarić, \& Persson, 1995, Theorem 2.1})\) Let \( f \in Q(I) \), \( a, b \in I \) with \( a < b \), and \( f \in L_1((a, b)) \). Then

\[
\begin{align*}
\left( \frac{a + b}{2} \right) & \leq \frac{4}{b - a} \int_a^b f(x) \, dx \quad \text{and} \quad \frac{1}{b - a} \int_a^b p(x)f(x) \, dx \leq \frac{f(a) + f(b)}{2},
\end{align*}
\]
where \( p(x) = \frac{(b - x)(x - a)}{(b - a)^2} \) for \( x \in I \).

**Definition 1.5** (Park, 2011) For \((s, m) \in (0, 1]^2\), a function \( f: [0, b] \to \mathbb{R} \) is said to be \((s, m)\)-convex if

\[
f(\lambda x + m(1 - \lambda)y) \leq \lambda^s f(x) + m(1 - \lambda)^s f(y)
\]

holds for all \( x, y \in I \) and \( \lambda \in [0, 1] \).

**Definition 1.6** (Xi & Qi, 2015) For some \( s \in [-1, 1] \), a function \( f: I \subseteq \mathbb{R} \to \mathbb{R} \) is said to be extended \( s\)-convex if

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)
\]

is valid for all \( x, y \in I \) and \( \lambda \in (0, 1) \).

Let us now consider a bi-dimensional interval \( \Delta = [a, b] \times [c, d] \) in \( R^2 \) with \( a < b \) and \( c < d \).

In Dragomir (2002), Dragomir and Pearce (2000) considered the convexity on the co-ordinates.

**Definition 1.7** (Dragomir, 2002; Dragomir & Pearce, 2000) A function \( f: \Delta \to \mathbb{R} \) is said to be convex on the co-ordinates with \( \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \) with \( a < b \) and \( c < d \) if the partial functions

\[
f_y: [a, b] \to \mathbb{R}, \quad f_y(u) = f(u, y) \quad \text{and} \quad f_x: [c, d] \to \mathbb{R}, \quad f_x(v) = f(x, v)
\]

are convex for all \( x \in (a, b) \) and \( y \in (c, d) \).

A formal definition for co-ordinated convex functions may be stated as follows.

**Definition 1.8** (Dragomir 2002; Dragomir & Pearce, 2000) A function \( f: \Delta \to \mathbb{R} \) is said to be convex on the co-ordinates with \( \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \) with \( a < b \) and \( c < d \) if and only if

\[
f(tx + (1 - t)y, tz + (1 - t)w) \leq tf(x, y) + t(1 - \lambda)f(x, w) + (1 - t)\lambda f(z, y) + (1 - t)(1 - \lambda)f(z, w)
\]

for all \( t, \lambda \in [0, 1] \) and \( (x, y), (z, w) \in \Delta \).

A formal definition for co-ordinated \( s\)-convex mappings may be stated as follows.

**Definition 1.9** (Latif & Alomari, 2005; Park, 2012) For \( s \in (0, 1) \), a mapping \( f: \Delta \to \mathbb{R} \) is called \( s\)-convex on the co-ordinates on \( \Delta \) if the inequality

\[
f(tx + (1 - t)y, tz + (1 - t)w) \leq t^s \lambda^s f(x, y) + t^s(1 - \lambda)^s f(x, w) + \lambda^s(1 - t)^s f(z, y) + (1 - t)^s(1 - \lambda)^s f(z, w)
\]

holds for all \( (x, y), (z, w), (x, w), (z, y) \in \Delta \) and \( t, \lambda \in [0, 1] \).

**Definition 1.10** (Xi, Hua, & Qi, 2014) For some \( s \in [-1, 1] \), a function \( f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R} \) is said to be extended \( s\)-convex on the co-ordinates with \( a < b \) and \( c < d \), if

\[
f(tx + (1 - t)y, tz + (1 - t)w) \leq t^s \lambda^s f(x, y) + t^s(1 - \lambda)^s f(x, w) + (1 - t)^s \lambda^s f(z, y) + (1 - t)^s(1 - \lambda)^s f(z, w)
\]

holds for all \( t, \lambda \in (0, 1) \) and \( (x, y), (z, w) \in \Delta \).

In Dragomir (2002), Dragomir and Pearce (2000), Dragomir established the following theorem.
The main aim of this paper is to introduce the concept “co-ordinated extended \((s_1, m_1)(s_2, m_2)\)-convex function” and to establish the Hermite–Hadamard inequalities for extended \((s_1, m_1)(s_2, m_2)\)-convex functions on the co-ordinates on a rectangle \(\Delta\) from the plane \(\mathbb{R}^2\).

2. Some integral inequalities of the Hermite–Hadamard type

Now we first state a new notion “co-ordinated extended \((s_1, m_1)(s_2, m_2)\)-convex function” as follows.

**Definition 2.1** For \((s_1, m_1), (s_2, m_2) \in [-1, 1] \times (0, 1]\), a function \(f:[0, b] \times [0, d] \to \mathbb{R}\) is said to be extended \((s_1, m_1)(s_2, m_2)\)-convex on the co-ordinates on \([0, b] \times [0, d]\) if and only if

\[
\begin{align*}
&f(tx + m_1(1-t)x, ty + m_2(1-t)y) \\
&\quad \quad \quad \quad \quad + m_1 t^s_1(1-t)^{s_1} f(x, y) + m_2 t^s_2(1-t)^{s_2} f(z, w) + m_1 m_2 (1-t)^{s_1+s_2} f(z, w)
\end{align*}
\]

holds for all \(t, \lambda \in (0, 1)\) and \((x, y), (z, w) \in [0, b] \times [0, d]\).

Now we start off to establish some integral inequalities of the Hermite–Hadamard type for the above-introduced co-ordinated extended \((s_1, m_1)(s_2, m_2)\)-convex functions.

**Theorem 2.1** Let \(f:[0, b]\times[0, d]\rightarrow \mathbb{R}\) be integrable on \([0, b]\times[0, d]\) with \(0 \leq a < b, 0 \leq c < d\), and some \(m_1, m_2 \in (0, 1)\). If \(f\) is co-ordinated extended \((s_1, m_1)(s_2, m_2)\)-convex on \([0, b]\times[0, d]\) for some \(s_1, s_2 \in [-1, 1]\), then

\[
2^{2(s_1+s_2)-4} f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{1}{2^{2-s_1-s_2}} \int_a^b \int_c^d \left[ F(x;c, d) \, dx + \frac{1}{d-c} \int_c^d G(a, by) \, dy \right]
\]

\[
\leq \frac{1}{16(b-a)(d-c)} \int_c^d \left\{ F(x;y, y) + m_1 F \left( \frac{x}{m_1}; \frac{y}{m_1}, \frac{y}{m_1} \right) + m_2 F \left( \frac{x}{m_2}; \frac{y}{m_2}, \frac{y}{m_2} \right) \right\} \, dy
\]

\[
+ m_1 F \left( \frac{x}{m_1}, y \right) + m_2 F \left( \frac{x}{m_2}, y \right)
\]

\[(2.2)\]
where
\[
\begin{align*}
F(x, y, z) &= f(x, y) + m_2f(x, m_1^{-1}z) + m_1f(m_1^{-1}x, y) + m_1m_2f(m_1^{-1}x, m_1^{-1}z) \\
\text{and} \\
G(x, y; z) &= f(x, z) + m_2f(x, m_1^{-1}z) + m_1f(m_1^{-1}y, z) + m_1m_2f(m_1^{-1}y, m_1^{-1}z)
\end{align*}
\]
(2.3)
for \(x, y, z \geq 0\).

Proof Letting \(y = \lambda c + (1 - \lambda)d, 0 \leq \lambda \leq 1\) and employing the co-ordinated extended \((s_1, m_1)\)\((s_2, m_2)\)-convexity of \(f\) (with \(t = \frac{1}{2}\) in (2.1)), we have
\[
f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) = \frac{1}{2} \int_{0}^{1} \left\{ f\left(\frac{a + b}{2}, \lambda c + (1 - \lambda)d\right) + m_1f\left(\frac{a + b}{2}, \frac{(1 - \lambda)c + \lambda d}{m_2}\right) \\
+ m_1f\left(\frac{a + b}{2m_1}, \lambda c + (1 - \lambda)d\right) + m_1m_2f\left(\frac{a + b}{2m_1}, \frac{(1 - \lambda)c + \lambda d}{m_2}\right)\right\} d\lambda
\]
(2.5)
and with \(t = \frac{1}{2}\) in (2.1), we obtain
\[
f\left(\frac{a + b}{2}, y\right) = \frac{1}{2} \int_{0}^{1} \left\{ f\left(\frac{ta + (1 - t)b}{2} + (1 - t)a + tb, y\right) + m_1f\left(\frac{ta + (1 - t)b}{m_2}, y\right) \\
+ m_1f\left(\frac{(1 - t)a + tb}{m_1}, y\right) + m_1m_2f\left(\frac{(1 - t)a + tb}{m_1}, \frac{y}{m_2}\right)\right\} dt
\]
(2.6)
and
\[
f\left(\frac{a + b}{2}, \frac{y}{m_2}\right) \leq \frac{1}{2} \int_{a}^{b} \left\{ F(x, y) + m_1f\left(x, \frac{y}{m_2}\right) + m_1f\left(\frac{x}{m_1}, y\right) + m_1m_2f\left(\frac{x}{m_1}, \frac{y}{m_2}\right)\right\} dx
\]
(2.7)
for \(x, y \geq 0\).
Substituting the above inequalities into the inequality (2.5) gives

\[
f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{2^{n+1}s_2(d - c)} \int_c^d G(a, b; y) \, dy
\]

\[
\leq \frac{1}{2^{n+1}s_2(b - a)(d - c)} \int_c^d \left\{ F(x; y, y) + m_1F \left( \frac{y}{m_1}, \frac{y}{m_1}, \frac{y}{m_1} \right) \right. \\
+ m_1F \left( \frac{x}{m_1}, y, y \right) + m_1m_2F \left( \frac{x}{m_1}, \frac{y}{m_1}, \frac{y}{m_1} \right) \right\} \, dx \, dy.
\]

(2.6)

By same argument as in the proof of inequality (2.6), we obtain

\[
f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{2^{n+1}s_2(b - a)} \int_a^b F(x; c, d) \, dx
\]

\[
\leq \frac{1}{2^{n+1}s_2(b - a)(d - c)} \int_c^d \left\{ F(x; y, y) + m_1F \left( \frac{y}{m_1}, \frac{y}{m_1}, \frac{y}{m_1} \right) \right. \\
+ m_1F \left( \frac{x}{m_1}, y, y \right) + m_1m_2F \left( \frac{x}{m_1}, \frac{y}{m_1}, \frac{y}{m_1} \right) \right\} \, dx \, dy.
\]

(2.7)

By adding inequalities (2.6) and (2.7) and dividing both sides by \(2\), we obtain the inequality (2.2). The proof of Theorem 2.1 is complete.

□

If putting \(m_1 = m_2 = m\) in Theorem 2.1, we can obtain the following corollary.

**COROLLARY 2.1.1** Under the conditions of Theorem 2.1 with \(m_1 = m_2 = m\), we have

\[
2^{2s_1+2s_2-2}f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{2^{n+1}s_1} \int_a^b \frac{1}{b - a} \int_a^b \frac{1}{d - c} \int_c^d G(a, b; y) \, dy \, dx \, dy
\]

\[
\leq \frac{1}{16(b - a)(d - c)} \int_c^d \left\{ F(x; y, y) + mF \left( \frac{y}{n}, \frac{y}{n}, \frac{y}{n} \right) \right. \\
+ mF \left( \frac{x}{n}, y, y \right) + m^2F \left( \frac{x}{n}, \frac{y}{n}, \frac{y}{n} \right) \right\} \, dx \, dy.
\]

**COROLLARY 2.1.2** In Theorem 2.1, if \(m_1 = m_2 = m = 1\), then

\[
2^{2s_1+2s_2-2}f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{2^{n+1}s_1} \int_a^b \frac{1}{b - a} \int_a^b \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) \, dy \, dx \, dy
\]

\[
\leq \frac{1}{(b - a)(d - c)} \int_c^d \int_a^b f(x, y) \, dx \, dy.
\]

Furthermore, when \(s_1 = s_2 = s\), one has

\[
2^{2s}f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{2^{n+2s}} \int_a^b \frac{1}{b - a} \int_a^b \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) \, dy \, dx \, dy
\]

\[
\leq \frac{1}{(b - a)(d - c)} \int_c^d \int_a^b f(x, y) \, dx \, dy.
\]
THEOREM 2.2 Let \( f : [0, \frac{b}{m_1^*}] \times [0, \frac{a}{m_2^*}] \rightarrow \mathbb{R} \) be integrable on \([0, \frac{b}{m_1^*}] \times [0, \frac{a}{m_2^*}]\) with \( 0 \leq a, 0 \leq c < d, \)

and \( m_1, m_2 \in (0, 1] \). If \( f \) is co-ordinated extended \((s_1, m_1) \cdot (s_2, m_2)\)-convex on \([0, \frac{b}{m_1^*}] \times [0, \frac{a}{m_2^*}]\) for some \( s_1, s_2 \in (-1, 1] \), then

\[
\frac{1}{(b - a)(d - c)} \int_c^d \int_a^b f(x, y) \, dx \, dy
\]

\[
\leq \frac{1}{2^{s_1+1}(s_1 + 1)(b - a)} \int_c^d F(x; c, d) \, dx + \frac{1}{2^{s_1+1}(s_1 + 1)(d - c)} \int_c^d G(a, b; y) \, dy
\]

\[
\leq \frac{1}{2^{s_1+1}(s_1 + 1)(s_2 + 1)} \left[ F(a; c, d) + m_1 F \left( a; \frac{c}{m_1}, \frac{d}{m_2} \right) + m_2 F \left( \frac{b}{m_1}, c, d \right) + m_1 m_2 F \left( \frac{b}{m_1}, \frac{c}{m_2}, \frac{d}{m_2} \right) \right],
\]

where \( F(x; y, z) \) and \( G(x; y, z) \) are given by (2.3) and (2.4), respectively.

Proof. Using the co-ordinated extended \((s_1, m_1) \cdot (s_1, m_2)\)-convexity of \( f \) with \( 0 < t < 1, \lambda = \frac{1}{2}, x = a, w = \frac{x}{y}, \) and \( z = \frac{b}{m_1} \) in (2.1), we have

\[
f(ta + (1 - t)b, y) \leq \frac{1}{2^s} \left\{ t^s f(a, y) + m_1 t^s f \left( a, \frac{y}{m_2} \right) + m_1 (1 - t)^s f \left( \frac{b}{m_1}, y \right) + m_1 m_2 f \left( \frac{b}{m_1}, \frac{y}{m_2} \right) \right\}
\]

(2.9)

for all \( y \in [0, d] \). Integrating the inequality (2.9) over \( t \) yields

\[
\int_0^1 f(ta + (1 - t)b, y) \, dt \leq \frac{1}{2^{s_1+1}(s_1 + 1)} \left\{ f(a, y) + m_1 f \left( a, \frac{y}{m_2} \right) + m_1 m_2 f \left( \frac{b}{m_1}, \frac{y}{m_2} \right) \right\}
\]

\[
= \frac{1}{2^{s_1+1}(s_1 + 1)} G(a, b; y).
\]

Therefore, we have

\[
\frac{1}{(b - a)(d - c)} \int_c^d \int_a^b f(x, y) \, dx \, dy = \frac{1}{d - c} \int_c^d \int_0^1 f(ta + (1 - t)b, y) \, dt \, dy
\]

\[
\leq \frac{1}{2^{s_1+1}(s_1 + 1)(d - c)} \int_c^d G(a, b; y) \, dy.
\]

(2.10)

Similarly, by replacing \( z = \frac{b}{m_1^*}, w = \frac{b}{m_2^*}, t = \frac{1}{2}, x = a, \) and \( y = c \) in (2.1), we obtain
By (2.11) and (2.12), we obtain the inequality (2.8). This completes the proof.

Applying the above inequalities to (2.10), we obtain

\[
\frac{1}{d-c} \int_c^d f(a, y) \, dy \leq \frac{1}{2^{s_1} (s_2 + 1)} \left\{ f(a, c) + m_2 (1 - \xi)^{\frac{m}{m_2}} f \left( a, \frac{d}{m_2} \right) 
\right. \\
+ m_1 \xi^{\frac{m}{m_1}} \left[ m_2 (1 - \xi)^{\frac{m}{m_2}} f \left( a, \frac{d}{m_2} \right) + m_1 \xi^{\frac{m}{m_1}} \left( f \left( a, \frac{d}{m_2} \right) + f \left( a, \frac{d}{m_2} \right) \right) \right] \right\} \\
= \frac{1}{2^{s_1} (s_2 + 1)} \left[ f(a, c) + m_1 f \left( a, \frac{d}{m_2} \right) + m_2 f \left( a, \frac{d}{m_2} \right) \right] \\
= \frac{1}{2^{s_1} (s_2 + 1)} \left[ F(a; c, d) \right] \frac{1}{d-c} \int_c^d f \left( a, \frac{x}{m_2} \right) \, dy \\
\leq \frac{1}{2^{s_1} (s_2 + 1)} F \left( a; \frac{c}{m_2}, \frac{d}{m_2} \right); \\
\frac{1}{d-c} \int_c^d f \left( \frac{b}{m_1}, y \right) \, dy \leq \frac{1}{2^{s_1} (s_2 + 1)} F \left( \frac{b}{m_1}, \frac{c}{m_2}, d \right); \\
\frac{1}{d-c} \int_c^d f \left( \frac{b}{m_1}, \frac{y}{m_2} \right) \, dy \leq \frac{1}{2^{s_1} (s_2 + 1)} F \left( \frac{b}{m_1}, \frac{c}{m_2}, \frac{d}{m_2} \right). 
\]

Applying the above inequalities to (2.10), we obtain

\[
\frac{1}{d-c} \int_c^d G(a; b, y) \, dy \leq \frac{1}{2^{s_1} (s_2 + 1)} F(a; c, d) \\
+ m_2 F \left( a; \frac{c}{m_2}, \frac{d}{m_2} \right) + m_1 F \left( \frac{b}{m_1}; c, d \right) + m_2 m_1 F \left( \frac{b}{m_1}, \frac{c}{m_2}, \frac{d}{m_2} \right). 
\] (2.11)

By similar argument as in the proof of the inequality (2.11), we obtain

\[
\frac{1}{b-a} \int_a^b F(x; c, d) \, dx \leq \frac{1}{2^{s_1} (s_2 + 1)} F(a; c, d) \\
+ m_2 F \left( a; \frac{c}{m_2}, \frac{d}{m_2} \right) + m_1 F \left( \frac{b}{m_1}; c, d \right) + m_2 m_1 F \left( \frac{b}{m_1}, \frac{c}{m_2}, \frac{d}{m_2} \right). 
\] (2.12)

By (2.11) and (2.12), we obtain the inequality (2.8). This completes the proof.

\[\square\]

**COROLLARY 2.2.1**  Under the conditions of Theorem 2.2, if taking \( m_1 = m_2 = m \), then

\[
\frac{1}{b-a} \int_a^b \int_c^d f(x, y) \, dx \, dy \\
\leq \frac{1}{2^{s_1+1} (s_2 + 1)} \int_a^b \int_c^d f(x; c, d) \, dx \, dy + \frac{1}{2^{s_1+1} (s_2 + 1)} \int_c^d G(a; b; y) \, dy \\
\leq \frac{1}{2^{s_1+1} (s_2 + 1)} \left[ F(a; c, d) + m_2 \left( a; \frac{c}{m_2}, \frac{d}{m} \right) \right. \\
+ m_1 \left( \frac{b}{m_1}; c, \frac{d}{m} \right) + m_2 m_1 \left( \frac{b}{m_1}, \frac{c}{m_2}, \frac{d}{m} \right) \right]. 
\]

If letting \( s_1 = s_2 = s \) or \( m_1 = m_2 = 1 \) in Theorem 2.2, we can obtain the following corollaries.
COROLLARY 2.2.2 Under the assumptions of Theorem 2.2, if \( s_1 = s_2 = s \), then

\[
\frac{1}{(b - a)(d - c)} \int_c^d b_a f(x, y) \, dx \, dy 
\leq \frac{1}{2^{s+1}(s + 1)} \left[ \frac{1}{b - a} \int_a^b F(x; c, d) \, dx + \frac{1}{d - c} \int_c^d G(a, b; y) \, dy \right] 
\leq \frac{1}{2^{s}(s + 1)^2} \left[ F(a; c, d) + m_2 \left( \frac{a + c}{m_2} \right) + m_1 \left( \frac{b - c}{m_1} \right) + m_1 m_2 \left( \frac{b - c}{m_1 m_2} \right) \right].
\]

COROLLARY 2.2.3 In Corollaries 2.1.1 and 2.2.1, if \( m_1 = m_2 = 1 \) and \(-1 < s_1, s_2 \leq 1\), then

\[
2^{2(s_1 + s_2) - 4} f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{2^{3-s_1-s_2}} \left[ \frac{1}{b - a} \int_a^b f \left( x, \frac{c + d}{2} \right) \, dx + \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) \, dy \right] 
\leq \frac{1}{(b - a)(d - c)} \int_c^d f(x, y) \, dx \, dy 
\leq \frac{1}{2^s(s_2 + 1)(b - a)} \left[ \int_a^b [f(x, c) + f(x, d)] \, dx \right] 
+ \frac{1}{2^s(s_1 + 1)(d - c)} \left[ \int_c^d [f(a, y) + f(b, y)] \, dy \right] 
\leq \frac{4}{2^{s_1 + s_2}(s_1 + 1)(s_2 + 1)} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right].
\]

In particular, when \( s_1 = s_2 = s \), one has

\[
2^{ks - 4} f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) 
\leq \frac{1}{2^{3-s_k}} \left[ \frac{1}{b - a} \int_a^b f \left( x, \frac{c + d}{2} \right) \, dx + \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \, dy \right] 
\leq \frac{1}{(b - a)(d - c)} \int_c^d f(x, y) \, dx \, dy 
\leq \frac{1}{2^s(s + 1)} \left[ \int_a^b [f(x, c) + f(x, d)] \, dx \right] 
+ \frac{1}{d - c} \int_c^d [f(a, y) + f(b, y)] \, dy 
\leq \frac{4}{2^{2s}(s + 1)^2} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right].
\]

Remark 2.1 The inequality (1.1) can be deduced from Corollary 2.2.3 applied to \( s_1 = s_2 = 1 \).

THEOREM 2.3 Let \( f : \left[ 0, \frac{b}{m_1} \right] \times \left[ 0, \frac{d}{m_2} \right] \rightarrow \mathbb{R} \) be integrable on \( \left[ 0, \frac{b}{m_1} \right] \times \left[ 0, \frac{d}{m_2} \right] \) with \( 0 \leq a < b, 0 \leq c < d \), and \( m_1, m_2 \in (0, 1) \). If \( f \) is co-ordinated extended \((-1, m_1)\)\((s_1, m_2)\)-convex on \( \left[ 0, \frac{b}{m_1} \right] \times \left[ 0, \frac{d}{m_2} \right] \) for \( s_1 = -1 \) and \( s_2 \in (-1, 1) \) then
\[
\frac{1}{(b - a)(d - c)} \int_c^b S_1(x)f(x, y) \, dx \, dy
\]

\[
\leq \frac{1}{(s_1 + 1)(b - a)} \int_c^b S_1(x)F(x; c, d) \, dx + \frac{1}{2^{s_1+1}(d - c)} \int_c^d G(a, b, y) \, dy
\]

(2.13)

where \( F(x; y, z) \) and \( G(x; y, z) \) are defined by (2.3) and (2.4), respectively, and

\[
S_1(x) = \frac{(b - x)(x - a)}{(b - a)^2}, \quad x \in [a, b].
\]

(2.14)

Proof Letting \( x = ta + (1 - t)b \) for \( 0 \leq t \leq 1 \), using the co-ordinated extended \((-1, m_1)\)-\((s_2, m_2)\)-convexity of \( f \) (with \( 0 < t < 1 \) and \( \lambda = \frac{1}{2} \) in (2.1)), and employing the inequality (2.11), we obtain

\[
\frac{1}{(b - a)(d - c)} \int_c^b S_1(x)f(x, y) \, dx \, dy
\]

\[
= \frac{1}{d - c} \int_0^1 t(1 - t)f(ta + (1 - t)b, y) \, dt \, dy
\]

\[
\leq \frac{1}{2^{s_1}(d - c)} \int_0^1 \left\{ (1 - t)f(a, y) + m_2(1 - t)f \left( a, \frac{y}{m_2} \right) \right. \\
+ m_1 t f \left( \frac{b}{m_1}, y \right) + m_1 m_2 f \left( \frac{b}{m_1}, \frac{y}{m_2} \right) \left. \right\} \, dt \, dy
\]

(2.15)

\[
= \frac{1}{2^{s_1+1}(d - c)} \int_c^d G(a, b, y) \, dy
\]

\[
\leq \frac{1}{2^{s_1}(s_2 + 1)} \left[ F(a; c, d) + m_2 F \left( a, \frac{c}{m_2}, \frac{d}{m_2} \right) \\
+ m_1 F \left( \frac{b}{m_1}; c, d \right) + m_1 m_2 F \left( \frac{b}{m_1}, \frac{c}{m_2}, \frac{d}{m_2} \right) \right].
\]

Similarly, using the inequality (2.12), we have

\[
\frac{1}{(b - a)(d - c)} \int_c^b S_2(x)f(x, y) \, dx \, dy = \frac{1}{b - a} \int_a^b S_2(x)f(x, \lambda c + (1 - \lambda)d) \, dx \, d\lambda
\]

\[
\leq \frac{1}{2^{s_1}(b - a)} \int_0^b \left\{ S_2(x) \left\{ \lambda^s f \left( x \right) + m_2(1 - \lambda)^s f \left( x, \frac{d}{m_2} \right) \right. \\
+ m_1 \lambda^s f \left( \frac{x}{m_1}, c \right) + m_1 m_2(1 - \lambda)^s f \left( \frac{x}{m_1}, \frac{d}{m_2} \right) \left. \right\} \, dx \, d\lambda
\]

(2.16)

\[
= \frac{1}{2^{s_1}(s_2 + 1)(b - a)} \int_0^b S_2(x)f(x; c, d) \, dx
\]

\[
\leq \frac{1}{2^{s_1}(s_2 + 1)} \left[ F(a; c, d) + m_2 F \left( a, \frac{c}{m_2}, \frac{d}{m_2} \right) \\
+ m_1 F \left( \frac{b}{m_1}; c, d \right) + m_1 m_2 F \left( \frac{b}{m_1}, \frac{c}{m_2}, \frac{d}{m_2} \right) \right].
\]
By (2.15) and (2.16), we obtain the inequality 2.3. The proof of Theorem 2.3 is complete. □

If taking \( m_1 = m_2 = m \) in Theorem 2.3, we can derive the following corollary.

**Corollary 2.3.1** Under the conditions of Theorem 2.3 with \( m_1 = m_2 = m \), we have

\[
\frac{1}{(b - a)(d - c)} \int_c^b S_1(x) f(x,y) \, dx \, dy
\]

\[
\leq \frac{1}{(s_2 + 1)(b - a)} \int_c^b S_1(x) F(x;c,d) \, dx + \frac{1}{2^{s_2}(d - c)} \int_c^d G(a,b;y) \, dy
\]

\[
\leq \frac{1}{2^{s_2}(s_2 + 1)} \left[ F(a;c,d) + mF \left( \frac{a - c}{m}, \frac{d}{m} \right) + mF \left( \frac{b - c}{m}, \frac{d}{m} \right) + m^2 F \left( \frac{b - c}{m}, \frac{d}{m} \right) \right].
\]

In particular, when \( m_1 = m_2 = 1 \) and \( s_2 = 1 \), one has

\[
\frac{1}{(b - a)(d - c)} \int_c^b S_1(x) f(x,y) \, dx \, dy
\]

\[
\leq \frac{1}{b - a} \int_c^b S_1(x) [f(x,c) + f(x,d)] \, dx + \frac{1}{4(d - c)} \int_c^d [f(a,y) + f(b,y)] \, dy
\]

\[
\leq f(a,c) + f(a,d) + f(b,c) + f(b,d).
\]

**Theorem 2.4** Let \( f: \left[ 0, \frac{b}{m_1} \right] \times \left[ 0, \frac{d}{m_2} \right] \to \mathbb{R} \) be integrable on \( \left[ 0, \frac{b}{m_1} \right] \times \left[ 0, \frac{d}{m_2} \right] \) with \( 0 \leq a < b, 0 \leq c < d \), and \( m_1, m_2 \in (0, 1) \). If \( f \) is co-ordinated extended \((s_1, m_1)\)-(−1, 1)-convex on \( \left[ 0, \frac{b}{m_1} \right] \times \left[ 0, \frac{d}{m_2} \right] \) for \( s_1 \in (-1, 1) \) and \( s_2 = -1 \), then

\[
\frac{1}{(b - a)(d - c)} \int_c^b S_1(y) f(x,y) \, dx \, dy
\]

\[
\leq \frac{1}{2^{s_2}(s_2 + 1)(b - a)} \int_c^b F(x;c,d) \, dx + \frac{1}{(s_2 + 1)(d - c)} \int_c^d S_1(y) G(a,b;y) \, dy
\]

\[
\leq \frac{1}{2^{s_2}(s_2 + 1)} \left[ F(a;c,d) + m_1 F \left( \frac{a - c}{m_2}, \frac{d}{m_2} \right) 
\right.
\]

\[
+ m_1 F \left( \frac{b - c}{m_2}, \frac{d}{m_2} \right) \left. + m_2 F \left( \frac{b - c}{m_2}, \frac{d}{m_2} \right) \right].
\]

where \( F(x; y, z) \) and \( G(x; y, z) \) are defined by (2.3) and (2.4), respectively, and

\[
S_1(y) = \frac{(d - y)(y - c)}{(d - c)^2}, \quad y \in [c, d]. \tag{2.17}
\]

If taking \( m_1 = m_2 = m \) in Theorem 2.4, we can obtain the following corollary.

**Corollary 2.4.1** Under the conditions of Theorem 2.4 with \( m_1 = m_2 = m \), we have
\[
\frac{1}{(b - a)(d - c)} \int_a^b \int_c^d S_2(y)f(x, y) \, dx \, dy \\
\leq \frac{1}{2^{s_1 + 2}(b - a)} \int_a^b F(x, c, d) \, dx + \frac{1}{(s_1 + 1)(d - c)} \int_c^d S_2(y)G(a, b, y) \, dy \\
\leq \frac{1}{2^s(s_1 + 1)} \left[ F(a, c, d) + mF \left( \frac{b - c}{m_1}; \frac{d}{m} \right) + m_2F \left( \frac{b}{m_1}; \frac{c}{m_2}; \frac{d}{m} \right) \right].
\]

In particular, when \( m_1 = m_2 = 1 \) and \( s_1 = 1 \), one has

\[
\frac{1}{(b - a)(d - c)} \int_a^b \int_c^d S_2(y)f(x, y) \, dx \, dy \\
\leq \frac{1}{4(b - a)} \int_a^b \left[ f(x, c) + f(x, d) \right] \, dx + \frac{1}{d - c} \int_c^d S_2(y) \left[ f(a, y) + f(b, y) \right] \, dy \\
\leq f(a, c) + f(a, d) + f(b, c) + f(b, d).
\]

**Theorem 2.5** Let \( f : \left[ 0, \frac{b}{m_1} \right] \times \left[ 0, \frac{d}{m_2} \right] \to \mathbb{R} \) be integrable on \( \left[ 0, \frac{b}{m_1} \right] \times \left[ 0, \frac{d}{m_2} \right] \) with \( 0 \leq a < b, 0 \leq c < d, \) and \( m_1, m_2 \in (0, 1) \). If \( f \) is co-ordinated extended \((−1, m_1)(−1, m_2)\)-convex on \( \left[ 0, \frac{b}{m_1} \right] \times \left[ 0, \frac{d}{m_2} \right] \) for \( s_1 = s_2 = -1 \), then

\[
\frac{1}{(b - a)(d - c)} \int_a^b \int_c^d S_1(x)S_2(y)f(x, y) \, dx \, dy \\
\leq \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b S_1(x)F(x; c, d) \, dx + \frac{1}{d - c} \int_c^d S_2(y)G(a, b, y) \, dy \right]
\]

\[
\leq F(a, c, d) + m_2F \left( \frac{b - c}{m_1}; \frac{d}{m_2} \right) + m_1F \left( \frac{b}{m_1}; \frac{c}{m_2}; \frac{d}{m} \right) + m_1m_2F \left( \frac{b}{m_1}; \frac{c}{m_2}; \frac{d}{m} \right),
\]

where \( F(x, y, z), G(x, y, z), S_1(x), \) and \( S_2(y) \) are, respectively, defined by (2.3), (2.4), (2.14), and (2.17).

**Proof** Using the co-ordinated extended \((−1, m_1)(−1, m_2)\)-convexity of \( f \) and utilizing inequalities (2.15) and (2.16), we have

\[
\frac{1}{(b - a)(d - c)} \int_a^b \int_c^d S_2(y)S_1(x)f(x, y) \, dx \, dy \\
= \frac{1}{d - c} \int_c^d \left[ \int_0^1 t(1 - t)S_2(y)f(ta + (1 - t)b, y) \, dt \right] \, dy
\]

\[
\leq \frac{1}{d - c} \int_c^d S_2(y)G(a, b, y) \, dy \\
\leq F(a, c, d) + m_2F \left( \frac{b - c}{m_1}; \frac{d}{m_2} \right) + m_1F \left( \frac{b}{m_1}; \frac{c}{m_2}; \frac{d}{m} \right) + m_1m_2F \left( \frac{b}{m_1}; \frac{c}{m_2}; \frac{d}{m} \right)
\]

and
\[
\frac{1}{(b - a)(d - c)} \int_{a}^{b} \int_{c}^{d} S_1(x)S_2(y)f(x,y) \, dx \, dy
\]
\[
\leq \frac{1}{b - a} \int_{a}^{b} S_1(x)f(x;c,d) \, dx
\]
\[
\leq F(\alpha;c,d) + m_1F \left( \frac{b}{m_1};\frac{c}{m_1}, \frac{d}{m_1} \right) + m_2F \left( \frac{b}{m_2};\frac{c}{m_2}, \frac{d}{m_2} \right).
\]

By (2.19) and (2.20), we obtain the inequality (2.18). The proof of Theorem 2.2 is complete.

If taking \( m_1 = m_2 = m \) in Theorem 2.5, we obtain the following corollary.

**Corollary 2.5.1** Under the conditions of Theorem 2.5 with \( m_1 = m_2 = m \), we have

\[
\frac{1}{(b - a)(d - c)} \int_{a}^{b} \int_{c}^{d} S_1(x)S_2(y)f(x,y) \, dx \, dy
\]
\[
\leq \frac{1}{2} \left[ \frac{1}{b - a} \int_{a}^{b} S_1(x)f(x;c,d) \, dx + \frac{1}{d - c} \int_{c}^{d} S_2(y)G(a,b;y) \, dy \right]
\]
\[
\leq F(\alpha;c,d) + mF \left( \frac{b}{m};\frac{c}{m}, \frac{d}{m} \right) + m^2F \left( \frac{b}{m};\frac{c}{m}, \frac{d}{m} \right).
\]

In particular, when \( m_1 = m_2 = m = 1 \), one has

\[
\frac{1}{(b - a)(d - c)} \int_{a}^{b} \int_{c}^{d} S_1(x)S_2(y)f(x,y) \, dx \, dy
\]
\[
\leq \frac{1}{2} \left[ \frac{1}{b - a} \int_{a}^{b} S_1(x)f(x;c,d) \, dx + \frac{1}{d - c} \int_{c}^{d} S_2(y)G(a,b;y) \, dy \right]
\]
\[
\leq 4 \left| f(\alpha,c) + f(\alpha,d) + f(b,c) + f(b,d) \right|.
\]

### 3. Conclusions

In this paper, we introduce a new notion “extended \((S_1, m_1)-(S_2, m_2)\)-convex function on co-ordinates” and establish some new inequalities of the Hermite–Hadamard type for extended \((S_1, m_1)-(S_2, m_2)\)-convex functions of two variables on co-ordinates.

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**Author’s contributions**

All authors contributed equally to the manuscript and read and approved the final manuscript.

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