A GENERAL BEURLING-HELSON-LOWDENSLAGER THEOREM
ON THE DISK

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ABSTRACT. We give a simple proof of the Beurling-Helson-Lowdenslager in-
vARIANT subspace theorem for a very general class of norms on $L^\infty(\mathbb{T})$.

1. INTRODUCTION

Why should we care about invariant subspaces? In finite dimensions all of the
structure theorems for operators can be expressed in terms of invariant subspaces.
For example the statement that every $n \times n$ complex matrix $T$ is unitarily equivalent
to an upper triangular matrix is equivalent to the existence of a chain $M_0 \subset M_1 \subset
\cdots \subset M_n$ of $T$-invariant linear subspaces with $\dim M_k = k$ for $0 \leq k \leq n$.
Since every upper triangular normal matrix is diagonal, the preceding result yields the
spectral theorem. A matrix is similar to a single Jordan block if and only if its set
of invariant subspaces is linearly ordered by inclusion, so the Jordan canonical form
can be completely described in terms of invariant subspaces. In [3] L. Brickman and
P.A. Fillmore describe the lattice of all invariant subspaces of an arbitrary matrix.

In infinite dimensions, where we consider closed subspaces and bounded op-
erators, even the existence of one nontrivial invariant subspace remains an open
problem for Hilbert spaces. If $T$ is a normal operator with a $*$-cyclic vector, then,
by the spectral theorem, $T$ is unitarily equivalent to the multiplication operator
$M_z$ on $L^2(\sigma(T), \mu)$, i.e.,

$$(M_z f)(z) = z f(z),$$

where $\mu$ is a probability Borel measure on the spectrum $\sigma(T)$ of $T$. It was proved
by J. Bram [2] in 1955 that a normal operator with a $*$-cyclic vector has a cyclic
vector, which means that we can choose $\mu$ so that $L^2(\mu)$ equals $H^2(\mu)$ (analogous
to $H^2$) the closure of the polynomials in $z$. In this case von Neumann proved
that if a subspace $W$ that is invariant for $M_z$ and for $M^*_z = M_{\bar{z}}$, then the projection
$P$ onto $W$ is in the commutant of $M_z$, which is the maximal abelian algebra
$\{M_\varphi : \varphi \in L^\infty(\mu)\}$. Hence there is a Borel subset $E$ of $\sigma(T)$ such that $P = M_{\chi_E}$,
which implies $W = \chi_E L^2(\mu)$. It follows that if $T$ is a reductive normal operator, i.e.,
every invariant subspace for $T$ is invariant for $T^*$, then all invariant subspaces of $T$
have the form $\chi_E L^2(\mu)$. In [8] D. Sarason characterized the $(M_z, \mu)$ that are reduc-
tive; in particular, when $T = M_z$ is unitary (i.e., $\sigma(T) = T = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$),
then $M_z$ is reductive if and only if Haar measure $m$ on $\mathbb{T}$ is not absolutely continuous
with respect to $\mu$.

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The first important characterization of all the invariant subspaces of a non-normal operator, the unilateral shift, was due to A. Beurling [1] in 1949. His result was extended by H. Helson and D. Lowdenslager [6] to the bilateral shift operator, which is a non-reductive unitary operator.

Throughout m denotes Haar measure (i.e., normalized arc length) on the unit circle \( T = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \). Since \( \{ z^n : n \in \mathbb{Z} \} \) is an orthonormal basis for \( L^2 \), we see that \( M_z \) is a bilateral shift operator. The subspace \( H^2 \) which is the closed span of \( \{ z^n : n \geq 0 \} \) is invariant for \( M_z \) and the restriction of \( M_z \) to \( H^2 \) is a unilateral shift operator. A closed linear subspace \( W \) of \( L^2 \) is doubly invariant if \( zW \subseteq W \) and \( \bar{z} W \subseteq W \). Since \( \bar{z} = 1 \) on \( T \), \( W \) is doubly invariant if and only if \( zW = W \). Since the set of polynomials in \( z \) and \( \bar{z} \) is weak*-dense in \( L^\infty = L^\infty (T) \), and since the weak* topology on \( L^\infty \) coincides with the weak operator topology on \( L^\infty \) (acting as multiplication operators on \( L^2 \)), \( W \) is doubly invariant if and only if \( L^\infty \cdot W \subseteq W \). A subspace \( W \) is simply invariant if \( zW \not\subseteq W \), which means \( H^\infty \cdot W \subseteq W \), but \( \bar{z} W \not\subseteq W \).

The classical Beurling-Helson-Lowdenslager theorem for a closed subspace \( W \) of \( L^2 \) follows. A very short elegant proof is given in [7]. We give a short proof to make this paper self-contained.

**Theorem 1.1. (Beurling-Helson-Lowdenslager)** Suppose \( W \) is a closed linear subspace of \( L^2 \) and \( zW \subseteq W \). Then

1. if \( W \) is doubly invariant, then \( W = \chi_E L^2 \) for some Borel subset \( E \) of \( T \),
2. if \( W \) is simply invariant, then \( W = \varphi H^2 \) for some \( \varphi \in L^\infty \) with \( |\varphi| = 1 \) a.e. \((m)\), and
3. if \( 0 \neq W \subseteq H^2 \), then \( W = \varphi H^2 \) with \( \varphi \) an inner function (i.e., \( \varphi \in H^\infty \) and \( |\varphi| = 1 \) a.e. \((m)\)).

**Proof.** 1. This follows from von Neumann’s result discussed above.

2. If \( W \) is simply invariant, then \( M_z |W \) is a nonunitary isometry, which, by the Halmos-Wold-Kolmogorov decomposition must be a direct sum of at least one unilateral shift and an isometry. Thus \( W = W_1 \oplus W_2 \) and there is a unit vector \( \varphi \in W_1 \) with \( \{ z^n \varphi : n \geq 0 \} \) an orthonormal basis for \( W_1 \). Since \( \varphi \perp z^n \varphi \) for \( n \geq 1 \), we have

\[
\int_T |\varphi|^2 z^n \, dm = 0
\]

for \( n \geq 1 \), and taking conjugates, we also get the same for \( n \leq -1 \). Thus \( |\varphi|^2 \) is constant, and thus must be \( |\varphi|^2 = 1 \) a.e. \((m)\). Clearly,

\[
W_1 = \varphi \cdot \mathbb{P}(\{ z^n : n \geq 0 \}) = \varphi H^2.
\]

If \( g \) is a unit vector in \( W_2 \), then we have \( z^n \varphi \perp g \) and \( \varphi \perp z^n g \) for \( n \geq 0 \), which implies

\[
\int_T z^n \varphi g \, dm = 0
\]

for all \( n \in \mathbb{Z} \), which implies \( |g| = |\varphi g| = 0 \). Hence \( W = W_1 = \varphi H^2 \).

3. Clearly no nonzero subspace \( W \supseteq zW \) of \( H^2 \) can have the form \( L^2 \chi_E \), so part 2 applies and \( \varphi \in \varphi H^2 = W \subseteq H^2 \), which implies \( \varphi \in H^2 \) is inner. \( \square \)

These results are also true when \( \| \cdot \|_p \) is replaced with \( \| \cdot \|_{p \bar{\mu}} \) for \( 1 \leq p \leq \infty \), with the additional assumption that \( W \) is weak*-closed when \( p = \infty \) (see [9], [10], [11]). Many of the proofs for the \( \| \cdot \|_{p \bar{\mu}} \) case use the \( L^2 \) result and take cases when \( p \leq 2 \).
and 2 < p. In [4] the author proved versions of parts (1) and (3) for an even larger class of norms, called rotationally invariant norms. In this more general setting the cases p ≤ 2 and 2 < p have no analogue.

In this paper we extend the Beurling-Helson-Lowdenslager theorem to an even larger class of norms, with a proof that is simple even in the $L^p$ case.

2. Preliminaries

**Definition 2.1.** A norm $\alpha$ on $L^\infty$ is called a $|||_1$-dominating normalized gauge norm if

1. $\alpha(1) = 1$,
2. $\alpha(|f|) = |\alpha(f)|$ for every $f \in L^\infty$,
3. $\alpha(f) \geq \|f\|_1$ for every $f \in L^\infty$.

We say that a $|||_1$-dominating normalized gauge norm is continuous if

$$\lim_{m(E) \to 0^+} \alpha(\chi_E) = 0.$$ 

We let $\mathcal{N}$ denote the set of all $|||_1$-dominating normalized gauge norms, and we let $\mathcal{N}_c$ denote the set of continuous ones. Although a $|||_1$-dominating normalized gauge norm $\alpha$ is defined only on $L^\infty(T)$, we can define $\alpha$ for all measurable functions $f$ on $T$ by

$$\alpha(f) = \sup \{ \alpha(s) : s \text{ is a simple function}, |s| \leq |f| \}.$$ 

It is clear that $\alpha(f) = \alpha(|f|)$ still holds.

We define $L^\alpha(T) = \{ f : \alpha(f) < \infty \}$, and define $L^\alpha(T)$ to be the $\alpha$-closure of $L^\infty(T)$ in $L^\alpha$. The following elementary result was proved in [4, Proposition 2.1].

**Lemma 2.2.** Suppose $\alpha \in \mathcal{N}$ and $f, g : T \to \mathbb{C}$ are measurable. The following statements are true:

1. $|f| \leq |g| \implies \alpha(f) \leq \alpha(g)$;
2. $\alpha(fg) \leq \alpha(f)\|g\|_\infty$;
3. $\alpha(f) \leq \|f\|_\infty$;
4. $(L^\alpha, \alpha)$ is a Banach space and $L^\infty \subset L^\alpha \subset L^\alpha \subset L^1$.

It is clear that $\mathcal{N}$ contains $|||_p$ ($1 \leq p \leq \infty$) and is convex and compact in the topology of pointwise convergence on $L^\infty$. So if $1 \leq p_n < \infty$, then $\alpha = \sum_{n=1}^{\infty} \frac{1}{p_n} \|\|_p \in \mathcal{N}_c$ and $\alpha$ is not equivalent to some $|||_p$ if $p_n \to \infty$.

**Definition 2.3.** Suppose $\alpha \in \mathcal{N}$. We define the dual norm $\alpha'$ on $L^\infty$ by

$$\alpha'(f) = \sup \left\{ \left| \int_T fh dm \right| : h \in L^\infty, \alpha(h) \leq 1 \right\} = \sup \left\{ \int_T |fh| dm : h \in L^\infty, \alpha(h) \leq 1 \right\}.$$ 

**Lemma 2.4.** If $\alpha \in \mathcal{N}$, then $\alpha' \in \mathcal{N}$. 
Proof. Suppose \( f \in L^\infty \). If \( h \in L^\infty \) with \( \alpha(h) \leq 1 \), then
\[
\int_T |fh| \, dm \leq \|f\|_\infty \|h\|_1 \leq \|f\|_\infty \alpha(h) \leq \|f\|_\infty,
\]
thus \( \alpha'(f) \leq \|f\|_\infty \). On the other hand, since \( \alpha(1) = 1 \), we have
\[
\alpha'(f) \geq \int_T |f| 1 \, dm = \|f\|_1,
\]
it follows that \( \alpha' \) is a norm with \( \alpha'(|f|) = \alpha'(f) \) for every \( f \in L^\infty \). Since \( \|1\|_1 \leq \alpha'(1) \leq \|1\|_\infty \), we see that \( \alpha'(1) = 1 \).

When \( \alpha \) is continuous, we can compute the normed dual \( (L^\alpha)^{\#} \) of \( L^\alpha \).

**Proposition 2.5.** Suppose \( \alpha \in \mathcal{N}_c \). Then \( (L^\alpha)^{\#} = L^{\alpha'} \), i.e., for every \( \phi \in (L^\alpha)^{\#} \), there is an \( h \in L^{\alpha'} \) such that
\[
\phi(f) = \int_T fh \, dm
\]
for all \( f \in L^\alpha \) and with \( \|\phi\| = \alpha'(h) \).

**Proof.** If \( \{E_n\} \) is a disjoint sequence of Borel subsets of \( T \), it follows that
\[
\lim_{N \to \infty} m\left( \bigcup_{n=N+1}^\infty E_n \right) = 0,
\]
and the continuity of \( \alpha \) implies
\[
\lim_{N \to \infty} \left\| \phi\left( \sum_{n=N+1}^\infty \chi_{E_n} \right) \right\| \leq \lim_{N \to \infty} \|\phi\| \alpha\left( \sum_{n=N+1}^\infty \chi_{E_n} \right) = 0.
\]
Hence
\[
\phi\left( \sum_{n=1}^\infty \chi_{E_n} \right) = \sum_{n=1}^\infty \phi(\chi_{E_n}).
\]
It follows that the restriction of \( \phi \) to \( L^\infty \) is weak*-continuous, which implies there is an \( h \in L^1 \) such that, for every \( f \in L^\infty \),
\[
\phi(f) = \int_T fh \, dm.
\]
The definitions of \( \alpha' \) and \( \|\phi\| \) imply that \( \alpha' = \|\phi\| \). Since \( L^\infty \) is dense in \( L^\alpha \), it follows that
\[
\phi(f) = \int_T fh \, dm
\]
holds for all \( f \in L^\alpha \).

We let \( \mathbb{B} = \{ f \in L^\infty : \|f\|_\infty \leq 1 \} \) denote the closed unit ball in \( L^\infty \).

**Lemma 2.6.** Suppose \( \alpha \in \mathcal{N}_c \). The \( \alpha \)-topology and the \( \|\|_2 \)-topology coincide on \( \mathbb{B} \).

**Proof.** Since \( \alpha \) is \( \|\|_1 \)-dominating, \( \alpha \)-convergence implies \( \|\|_1 \)-convergence implies convergence in measure. Suppose \( \{f_n\} \) is a sequence in \( \mathbb{B} \) and \( f_n \to f \) in measure. If \( E_{n,k} = \{ z \in T : |f(z) - f_n(z)| \geq \frac{1}{k} \} \), then \( \lim_{n \to \infty} m(E_{n,k}) = 0 \), which implies \( \lim_{n \to \infty} \alpha\left( \chi_{E_{n,k}} \right) = 0 \). Then, for \( k \geq 1 \),
\[
\alpha(f_n - f) \leq \alpha\left( \chi_{E_{n,k}} \right) \|\chi_{E_{n,k}}(f_n - f)\|_\infty + \alpha\left( \chi_{T \setminus E_{n,k}} \right) \|\chi_{T \setminus E_{n,k}}(f_n - f)\|_\infty
\]

Hence \( \alpha (f_n - f) \to 0 \). Hence \( \alpha \)-convergence is equivalent to convergence in measure on \( \mathbb{B} \). Since \( \alpha \) was arbitrary, the same holds for \( \| \cdot \|_2 \)-convergence.

3. The Main Result

In this section we prove our generalization of the classical Beurling-Helson-Lowdenslager theorem. Suppose \( \alpha \in \mathcal{N}_c \). We define \( H^\alpha \) to be the \( \alpha \)-closure of \( H^\infty \), i.e.,

\[
H^\alpha = \left[ H^\infty \right]^{-\alpha}.
\]

Since the polynomials in \( z \) are weak*-dense in \( H^\infty \), we know from Lemma 2.2 that \( H^\alpha \) is the \( \alpha \)-closure of the set of polynomials. We need another characterization of \( H^\alpha \).

**Lemma 3.1.** Suppose \( \alpha \in \mathcal{N}_c \). Then \( H^\alpha = H^1 \cap L^\alpha \).

**Proof.** It is clear that \( H^\alpha \subset H^1 \cap L^\alpha \). Suppose \( f \in H^1 \cap L^\alpha \) and \( \varphi \in (L^\alpha)^\# \) and \( \varphi \| H^\alpha = 0 \). It follows from Proposition 2.5 that there is an \( h \in L^\alpha \) such that \( \varphi (v) = \int_T vhdm \). We know that \( h \in L^\alpha \subset L^1 \), so we can write \( h (z) = \sum_{n=\infty} c_n q^n \). Since \( \varphi \| H^\alpha = 0 \), we have

\[
c_{-n} = \int_T h z^n dm = 0
\]

for \( n \geq 0 \). Thus \( h \) is analytic and \( h (0) = 0 \). Since \( h \in L^\alpha \) and \( f \in L^\alpha \cap H^1 \) we see that \( f \) is analytic and \( fh \in L^1 \), which means \( f \in H^1 \). Hence

\[
\varphi (f) = \int_T fhdm = f (0) h (0) = 0.
\]

It follows from the Hahn Banach theorem that \( f \in H^\alpha \). Hence \( H^1 \cap L^\alpha \subset H^\alpha \). \( \Box \)

A key ingredient is based on the following result that uses the Herglotz kernel [3].

**Lemma 3.2.** \( \{ |g| : 0 \neq g \in H^1 \} = \{ \varphi \in L^1 : \varphi \geq 0 \text{ and } \log \varphi \in L^1 \} \). In fact, if \( \varphi \geq 0 \) and \( \varphi, \log \varphi \in L^1 \), then

\[
g (z) = \exp \int_T \frac{w + z}{w - z} \log \varphi (w) dm (w)
\]

defines an outer function \( h \) on \( \mathbb{D} \) and \( |g| = \varphi \) on \( \mathbb{T} \).

**Proposition 3.3.** Suppose \( \alpha \in \mathcal{N}_c \). If \( k \in L^\infty \) and \( k^{-1} \in L^\alpha \), then there is an unimodular function \( w \in L^\infty \) and an outer function \( h \in H^\infty \) such that \( k = wh \) and \( h^{-1} \in H^\alpha \).

**Proof.** Recall that an outer function is uniquely determined by its absolute boundary values, which are necessarily absolutely log integrable. Suppose \( k \in L^\infty \) with \( k^{-1} \in L^\alpha \). Observe that on the unit circle,

\[
-|k| \leq - \log |k| = \log |k^{-1}| \leq |k^{-1}|,
\]

it follows from \( k \in L^\infty \) and \( k^{-1} \in L^\alpha \subset L^1 \) that

\[
-\infty < - \int_T |k| dm \leq \int_T \log |k^{-1}| dm \leq \int_T |k^{-1}| dm < \infty,
\]
and hence $|k^{-1}|$ is log integrable. Then by Lemma 3.2, there is an outer function $g \in H^1$ such that $|g| = |k^{-1}|$ on $\mathbb{T}$. If we let $h = g^{-1}$, then $w = kg$, we see that $h \in H^\infty$, $w$ is unimodular, $k = wh$ and $h^{-1} = g = wk^{-1} \in L^\alpha \cap H^1 = H^\alpha$. □

**Corollary 3.4.** Suppose $\alpha \in \mathcal{N}_c$, and $W$ is an $\alpha$-closed linear subspace of $L^\alpha$, and $M$ is a weak*-closed linear subspace of $L^\infty$ such that $z M \subseteq M$ and $z W \subseteq W$. Then

1. $M = M^{-\alpha} \cap L^\infty$, and
2. $W = (W \cap L^\infty)^{-\alpha}$.

**Proof.** 1. Clearly, $M \subseteq M^{-\alpha} \cap L^\infty$. Assume, via contradiction, that $w \in M^{-\alpha} \cap L^\infty$ and $w \notin M$. Since $M$ is weak*-closed, there is an $F \in L^1$ such that $\int_F W \, dm \neq 0$ but $\int g F \, dm = 0$ for every $g \in M$. Since $k = \frac{1}{\overline{F} |F|} \in L^\infty$ and $k^{-1} \in L^1$, it follows from Lemma 3.3 that there is an $h \in H^\infty$, $1/h \in H^1$ and a unimodular function $u$ such that $k = uh$. Hence $h F = uh |F| \in L^\infty$. Choose a sequence $\{h_n\}$ in $H^\infty$ such that $\|h_n - 1/h\|_1 \to 0$. Thus

$$\|h_n h F - F\|_1 \leq \left\|h_n - \frac{1}{h}\right\|_1 \|h F\|_\infty \to 0.$$ 

For each $n \in \mathbb{N}$ and every $g \in M$ we know that $h_n g \in H^\infty M \subset M$. Hence

$$\int_T g F h_n \, dm = 0$$

for every $g \in M$. Since, $h_n h F \in L^\infty$,

$$\int_T g F h_n \, dm = 0$$

for every $g \in M^{-\alpha}$. In particular,

$$\int_T F h_n w \, dm = 0$$

for every $m \in \mathbb{N}$. Hence,

$$0 \neq \left| \int_T F W \, dm \right| \leq \lim_{n \to \infty} \|h_n F - F\|_1 \|w\|_\infty + \left| \int_T F h_n w \, dm \right| = 0,$$

a contradiction. Hence $M = M^{-\alpha} \cap L^\infty$.

2. It is clear that $W \cap [W \cap L^\infty]^{-\alpha}$. Suppose $f \in W$ and let $k = \frac{1}{|f| |f|}$. It follows from Lemma 3.3 that there is an $h \in H^\infty$, $1/h \in H^\alpha$ and a unimodular function $u$ such that $k = uh$, so $h f = uh \in L^\infty$. There is a sequence $\{h_m\}$ in $H^\infty$ such that $\alpha (h_m - \frac{1}{h}) \to 0$. Thus $\{h_m h f\}$ is a sequence in $L^\infty \cap W$ and

$$\alpha (h_m h f - f) \leq \alpha \left( h_m - \frac{1}{h} \right) \|h f\|_\infty \to 0.$$ 

Thus $f \in [L^\infty \cap M]^{-\alpha}$.

□

**Theorem 3.5.** Suppose $\alpha \in \mathcal{N}_c$ and $W$ is a closed subspace of $L^\alpha$. Then $z W \subseteq W$ if and only if either $W = \phi H^\alpha$ for some unimodular function $\phi$ or $W = \chi_E L^\alpha$ for some Borel set $E \subset \mathbb{T}$. If $0 \neq W \subset H^\alpha$, then $W = \varphi H^\alpha$ for some inner function $\varphi$. 


Proof. The "if" part is obvious. Suppose \( zW \subset W \) and let \( M = W \cap L^\infty \). We claim that \( M \) is weak*-closed in \( L^\infty \). Using the Krein-Smulian theorem, we need only show that \( M \cap B \) is weak*-closed. However, we know that \( M \cap B \) is \( \alpha \)-closed, which, by Lemma 2.6, means that \( M \cap B \) is \( \|\|_2 \)-closed. Since \( M \cap B \) is convex, \( M \) is closed in the weak topology on \( L^2 \). Since \( \{ f\bar{g} : g, f \in L^2 \} = L^1 \), the \( L^2 \)-weak topology on \( L^\infty \) coincides with the weak*-topology. Hence \( M \cap B \) is weak*-closed in \( L^\infty \), which verifies our claim. It is clear that \( zM \subset M \), since \( zW \subset W \).

Thus \( M^-\|_2 \) is an \( H^\infty \)-invariant closed subspace of \( L^2 \), so \( M^-\|_2 = \varphi H^2 \) for some unimodular function \( \varphi \) or \( M^-\|_2 = \chi_E L^2 \) for some Borel subset \( E \) of \( \mathbb{T} \). It follows that \( M = M^-\|_2 \cap L^\infty \) is either \( \varphi H^\infty \) or \( \chi_E L^\infty \). Thus \( W = M^-\alpha \) must be either \( \varphi H^\alpha \) or \( \chi_E L^\alpha \). If \( 0 \neq W \subset H^\alpha \), we must have \( W = \varphi H^\alpha \), so \( \varphi \in H^\alpha \subset H^1 \) must be an inner function. \( \square \)

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