THE ROLE OF THE APPARENT HORIZON IN THE EVOLUTION OF
ROBINSON-TRAUTMAN EINSTEIN-MAXWELL SPACETIMES

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ABSTRACT
The ‘runaway solutions’ of the Lorentz-Dirac equation of a charged particle interacting
with its own field in classical electrodynamics are well-known. This type of self ac-
celerated phenomena also exists in the solutions of the Einstein-Maxwell equations in
general relativity. In particular, runaway solutions occur in a class of simple models
known as the ‘Asymptotically Flat Robinson-Trautman Einstein-Maxwell’ (AFRTEM)
spacetimes. Consequently these spacetimes cannot evolve to their unique regular steady
state, viz. a charged non-rotating black hole. This seems to contradict the established
results that charged non-rotating black holes are stable under first order perturbations.
We show that if an AFRTEM spacetime also possesses an apparent horizon, then it has
a Lyapunov functional. This suggests that the evolution equations with additional con-
straints arising from the apparent horizon would evolve stably to a charged non-rotating
black hole.

1. Introduction

The runaway solutions of the Lorentz-Dirac equation of a charged particle inter-
acting with its own field are considered to be unphysical. This type of self acceler-
ated motions has never been observed down to the length scale of a classical electron.
Its non-existence can be explained within the realm of classical electrodynamics.\(^1\)
It is not surprising that runaway solutions of the Einstein-Maxwell equations also
exist in general relativity.\(^2\) In particular, runaway solutions of the evolution equa-
tions occur in a class of simple models known as the *Asymptotically Flat Robinson-
Trautman Einstein-Maxwell* (AFRTEM) spacetimes.\(^3\) Consequently this class of
spacetimes cannot evolve to its unique regular steady state, viz. a charged non-
rotating black hole represented by the Reissner-Nordström solution. There are a
number of reasons why such solutions are thought to be unphysical. (1) It seems to
contradict the result that first order perturbations to a charged non-rotating black hole are stable.\(^4\) (2) The corresponding charge-free models, which are the non-static *Asymptotically Flat Robinson-Trautman* (AFRT) vacuum spacetimes, emit gravitational waves then settle down to the unique steady state as a non-rotating black hole represented by the Schwarzschild solution.\(^5\),\(^6\),\(^7\),\(^8\) (3) The runaway solutions suggest that the AFRTEM spacetimes possess some kind of compact sources which supply the release of an unlimited amount of gravitational and electromagnetic radiation. (4) Numerical simulations show that the Bondi-Sachs 4-momentum becomes infinite as the spacetimes evolve and this result can be interpreted as self accelerated motion of the associated source.

We show that if we assume that the AFRTEM spacetimes are dynamic black holes, then we can eliminate the runaway solutions of the evolution equations. Mathematically, the assumption gives rise to regularity conditions which serve as inner boundary conditions ‘protecting’ the past singularities. In technical terms, we assume the existence of a past apparent horizon in these spacetimes, which guarantees the presence of a particle horizon. As a consequence, we are able to prove that these restricted AFRTEM spacetimes possess a Lyapunov functional, which is the sum of the square of the Bondi mass and the surface area of the apparent horizon. This suggests that the AFRTEM evolution equations with additional constraint equations arising from the apparent horizon assumption would evolve stably to the Reissner-Nordström solution, provided solutions to the nonlinear system of equations exist. We have not proved the existence of solutions of this nonlinear system of three functions of three independent variables, consisting of three hypersurface equations, one hypersurface inequality and one two-surface equation. However in the conclusion we indicate that the linearised equations of these restricted spacetimes are stable about the Reissner-Nordström solution (details in preparation).

In Section 2 we briefly summarise the evolution equations of the AFRT EM spacetimes and then show that the linearisation of these equations about their steady-state solution are unstable. We also consider a number of conserved quantities arising from the evolution equations and prove that the Bondi mass of the spacetime is a monotonically decreasing quantity. For a more detailed discussion see Singleton.\(^9\) In Section 3 the constraint equations arising from the apparent horizon assumption are presented (see Chow and Lun\(^10\) for derivation). In Section 4, we prove that (1) the Bond mass of a restricted AFRTEM spacetime is positive,\(^10\) (2) the surface area of the past apparent horizon is monotonically decreasing, and (3) the sum of the square of the Bondi mass and the surface area of the apparent horizon is a Lyapunov functional, whose critical point is the Reissner Nordström solution.

2. The *Asymptotically Flat RTEM* Spacetimes

Physically the AFRTEM spacetimes can be thought of as electrically charged non-rotating compact sources emitting purely outgoing coupled gravitational and electromagnetic radiations. A more general mathematical formulation can be found in
Kramer et al.\textsuperscript{11} In this paper we restrict our discussion to the asymptotically flat subclass. The AFRTEM spacetimes can be described by the metric

$$ds^2 = \left( -2r(\ln P)_u + K - \frac{2M}{r} + \frac{Q_0^2}{r^2} \right) du^2 + 2dudr - \frac{2r^2d\zeta d\bar{\zeta}}{P^2}$$

in \((u,r,\zeta,\bar{\zeta})\) coordinates. We use \(\partial\) or \(,\) to denote partial derivatives. Here \(M = M(u,\zeta,\bar{\zeta})\) and \(P = P(u,\zeta,\bar{\zeta})\) are real-valued functions. \(Q_0\) is a real constant corresponding to the electric charge of the spacetime. \(K := \Delta (\ln P)\), where

\[
\Delta := 2P^2\partial_{\zeta}\bar{\zeta}
\]

is the Laplacian on the closed two-surface \(S_{u,r}\) which has the topology of a two-sphere \(S^2\). Geometrically \(K\) is the Gaussian curvature of \(S_{u,r}\). The spacetime is foliated by these two-surfaces as \(u\) and \(r\) vary. \(u\) is the retarded time and \(u = u_0 = \text{constant}\) is a future null cone whose generators are a congruence of diverging, shearfree null geodesics affinely parametrised by \(r\). The variables \(M\) and \(P\) can be interpreted as the mass-energy of the spacetime. Information about the rest mass and electromagnetic radiations of the spacetime is prescribed by \(M\) while information about gravitational radiations is given by \(P\).\textsuperscript{10} After partial integration of the Einstein-Maxwell equations, the residual field equations can be written as :

\[
\begin{align*}
(\ln P)_u & = -\frac{1}{4Q_0^2}\Delta M \\
M_{tu} & = -\frac{3M}{4Q_0^2}\Delta M + \frac{1}{4}\Delta K - \frac{P^2}{2Q_0^2}M_{,\zeta}M_{,\bar{\zeta}}
\end{align*}
\]

Since the evolution equations (2) and (3) are independent of \(r\), the whole spacetime is determined by the evolution of \(M\) and \(P\) with respect to the retarded time \(u\) on a \(r = r_0 = \text{constant}\) hypersurface. Without any loss of generality we take \(r_0 = 1\) in the following.

2.1. Linearised Evolution equations

In contrast to the AFRT vacuum spacetimes, the AFRTEM evolution equations (2) and (3) are linearly unstable about the Reissner Nordström solution, which is the unique regular steady state solution of these equations. The steady state solution is given by

\[
M = M_0 = \text{constant}, \quad P = P_0 := \frac{1}{\sqrt{2}}(a + \bar{b}\zeta + b\bar{\zeta} + c\zeta\bar{\zeta})
\]

where \(ac - \bar{b}\bar{b} = 1\) and \(M_0\) is the mass of the Reissner-Nordström solution. In terms of \(P_0\), the metric of the two-sphere \(S^2\) is \(g_0 = 2P_0^{-2}d\zeta d\bar{\zeta}\). The condition \(ac - \bar{b}\bar{b} = 1\) normalises the constant Gaussian curvature of \(S^2\) to \(K = K_0 = 1\).

Following Singleton,\textsuperscript{9} we linearise (2) and (3) about the steady state solution (4). Let \(P = P_0(1 + \epsilon\tilde{G})\) and \(M = M_0(1 + \epsilon\tilde{E})\), where the parameter \(\epsilon\) is a first order quantity. Physically \(\tilde{G}\) and \(\tilde{E}\) describe the purely outgoing gravitational and electromagnetic radiations, respectively. The linearised equations are
\[
\begin{align*}
\tilde{G}_{,u} &= -\frac{M_0}{4Q_0^2} \Delta_0 \tilde{E} \\
\tilde{E}_{,u} &= -\frac{3M_0}{4Q_0^2} \Delta_0 \tilde{E} + \frac{1}{4M_0} (\Delta_0 + 2) \Delta_0 \tilde{G}
\end{align*}
\]

where \(\Delta_0 = 2P_0^2 \partial_\zeta \zeta\) denotes the Laplacian on \(S^2\). Decomposition into spherical harmonics \(\tilde{G} = \sum G_{\ell,m} Y_{\ell,m}\) and \(\tilde{E} = \sum E_{\ell,m} Y_{\ell,m}\) reduces the linearised equations (5) and (6) to a linear autonomous system in \(G_{\ell,m}\) and \(E_{\ell,m}\):

\[
\frac{d}{du} \begin{pmatrix} G_{\ell,m} \\ E_{\ell,m} \end{pmatrix} = \begin{pmatrix} 0 & \frac{M_0}{4Q_0^2} \ell (\ell + 1) \\ \frac{1}{4M_0} (\ell + 2)(\ell + 1)\ell - 1 & \frac{3M_0}{4Q_0^2}\ell (\ell + 1) \end{pmatrix} \begin{pmatrix} G_{\ell,m} \\ E_{\ell,m} \end{pmatrix}
\]

The eigenvalues of the linear system are given by

\[
\lambda_{\ell,\pm} = \frac{3M_0}{8Q_0^2} \ell (\ell + 1) \left( 1 \pm \sqrt{1 + \frac{4Q_0^2(\ell + 2)(\ell - 1)}{9M_0^2}} \right)
\]

Thus all modes with \(\ell \geq 1\) of the linearised equations are unstable. The critical point of the system is a saddle. Therefore the nonlinear equations will also be unstable about the steady state solution. This would pose a rather serious problem for a physically meaningful interpretation of the AFRTEM spacetimes. Furthermore it apparently contradicts the established results affirming the stability of the Reissner-Nordström black hole against linear perturbations. We note, however, that for any mode \(\ell\), there is a stable submanifold given by a linear combination of the eigenvectors corresponding to \(\lambda_{\ell,-}\). For \(\ell = 1\) (i.e. the dipole modes), the eigenvalue \(\lambda_{1,+} = \frac{3M_0}{2Q_0^2}\) is the same as the exponent in the runaway solution of the radiation reaction equation of motion for a free particle in classical electrodynamics. Hence this justifies calling these unstable solutions of the AFRTEM evolution equations runaway solutions of the Einstein-Maxwell equations.

### 2.2. Conserved Quantities

From the evolution equations (2) and (3), one can derive a number of conserved quantities. It is more convenient to prove these and subsequent results if we ‘factor out’ the two-sphere geometry from the evolution equations by defining \(e^{-\Lambda} := P/P_0\). The metric of the two-surface \(S_{u,1}\) is conformal to the metric of \(S^2\) with the conformal factor \(e^{2\Lambda}\) so that

\[
\bar{g}_1 = \frac{2e^{2\Lambda} d\tilde{\zeta} d\tilde{\zeta}}{P_0^2} = e^{2\Lambda} g_0
\]

It is important to note that, unlike \(P\) which is singular at a point (at least one) on \(S_{u,1}\) due to the non-trivial topology of \(S^2\), the conformal factor \(e^\Lambda\) is regular (cf. the form of \(P_0\) in Eq. (4)). The surface elements of \(S^2\) and \(S_{u,1}\) are given, respectively, by
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\( \omega_0 = \frac{id\zeta \wedge d\zeta}{P_0^2}, \quad \omega_1 = e^{2\Lambda} \omega_0 \)

The evolution equations (2) and (3) are now expressed in terms of regular functions \( \Lambda \) and \( M \) on \( S^2 \) as:

\[
e^{2\Lambda} \Lambda_{,u} = \frac{1}{4Q_0^2} \Delta_0 M \quad (8)
\]

\[
e^{2\Lambda} M_{,u} = -\frac{3M}{4Q_0^2} \Delta_0 M + \frac{1}{4} \Delta_0 K - \frac{(P_0 M, \zeta)(P_0 M, \bar{\zeta})}{2Q_0^2} \quad (9)
\]

We define three quantities on the closed two-surface \( S_{u,1} \), its surface area \( A_{S_{u,1}} \), its Euler characteristic \( \chi \) and its ‘irreducible mass’ \( M \) as follows:

\[
A_{S_{u,1}} := \int_{S_{u,1}} \omega_1 = \int_{S^2} e^{2\Lambda} \omega_0 \quad (10)
\]

\[
\chi := \frac{1}{2\pi} \int_{S_{u,1}} K \omega_1 = \frac{1}{2\pi} \int_{S^2} Ke^{2\Lambda} \omega_0 \quad (11)
\]

\[
M := \int_{S_{u,1}} M \omega_1 = \int_{S^2} Me^{2\Lambda} \omega_0 \quad (12)
\]

The conservation of the surface area \( A_{S_{u,1}} \) (i.e. \( \frac{d}{du} A_{S_{u,1}} = 0 \)) follows from (8) by integrating it over \( S^2 \) and applying Stokes’ theorem to the right side. In terms of \( \Lambda \), the Gaussian curvature

\[
K = e^{-2\Lambda}(1 - \Delta_0 \Lambda) = e^{-2\Lambda} + e^{-\Lambda} \Delta_0 e^{-\Lambda} - 2(P_0 e^{-\Lambda})_\zeta (P_0 e^{-\Lambda})_{\bar{\zeta}} \quad (13)
\]

Substituting (13) into (11) gives the topological invariant

\[
\chi = \frac{1}{2\pi} \int_{S^2} (1 - \Delta_0 \Lambda) \omega_0 = \frac{1}{2\pi} \int_{S^2} \omega_0 = 2 \quad (14)
\]

This implies the conservation of the Euler characteristic \( \chi \) and that \( S_{u,1} \) is a topological \( S^2 \) for all retarded time \( u \). Alternately, from (8) and the definition of \( K \), one can show that

\[
K_{,u} = -\frac{1}{4Q_0^2}(\Delta + 2K) \Delta_0 M
\]

\[
(e^{2\Lambda} K)_{,u} = -\frac{1}{4Q_0^2} \Delta_0 (e^{-2\Lambda} \Delta_0 M)
\]

\( \frac{d}{du} \chi = 0 \) follows from the latter equation. Hence the result derived from the evolution equations is consistent with that from (14). Since \( \chi \) is an integer, a non-constant
\( \chi \) would indicate pathological and unphysical behaviour of the field equations. Combining (8) and (9) gives
\[
(e^{2\Lambda} M)_u = \frac{1}{4} \Delta_0 (K - \frac{M^2}{2Q_0^2})
\]
which implies the conservation of the irreducible mass \( M \). The proof is similar to that of (12).

### 2.3. Bondi Mass

In an asymptotically flat spacetime, the notion of the energy-momentum at a given retarded time exists, viz. the Bondi-Sachs 4-momentum. One can consider the spacetime geometry as one approaches null infinity \( I^+ \) on an asymptotically null hypersurface. This would enable one to discuss quantitatively the amount of energy and momentum carried away by coupled gravitational and electromagnetic radiation. Using Penrose’s conformal technique (see Penrose and Rindler [13]), the Bondi-Sachs 4-momentum of the AFRTEM spacetime can be shown to be [9,10]

\[
P_a = \frac{1}{4\pi} \int_{S_{u,\infty}} M e^\Lambda \xi_a \omega_1 = \frac{1}{4\pi} \int_{S^2} M e^{3\Lambda} \xi_a \omega_0 \tag{15}
\]
where

\[
\xi_a = (1, x, y, z), \quad x = \frac{\zeta + \bar{\zeta}}{\zeta + 1}, \quad y = -\frac{i(\zeta - \bar{\zeta})}{\zeta + 1}, \quad z = \frac{\zeta\bar{\zeta} - 1}{\zeta + 1}
\]
corresponds to the asymptotic symmetries generating the translations, and \( S_{u,\infty} \) is a two-sphere \( S^2 \) at \( I^+ \) on a \( u = \text{constant} \) future null cone (called a ‘cut’ of \( I^+ \)).

The time (first) component of \( P_a \) is the Bondi mass which is denoted by \( M_B \), i.e.

\[
P_t := M_B = \frac{1}{4\pi} \int_{S_{u,\infty}} M e^\Lambda \omega_1 = \frac{1}{4\pi} \int_{S^2} M e^{3\Lambda} \omega_0 \tag{16}
\]

The components of the Bondi-Sachs 4-momentum are written as \( M_B, P_x, P_y \) and \( P_z \). The Bondi mass is not a conserved quantity, indeed, it is monotonically decreasing.

**Proposition 1:** The Bondi mass of an AFRTEM spacetime is a monotonically decreasing function of the retarded time \( u \).

**Proof:** Firstly, we use (13) and the identity

\[
[\Delta_0(e^{-\Lambda})]_{\zeta} = [2P_0^2(e^{-\Lambda})_{\zeta\zeta}]_{\zeta} = 2 \left[ P_0^2 \partial_\zeta \left( \frac{1}{P_0^2} [P_0^2(e^{-\Lambda})_{\zeta}]_{\zeta} \right) - (e^{-\Lambda})_{\zeta} \right]
\]
to expand \( \Delta_0 K \) in terms of \( e^{-\Lambda} \):
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\[ \Delta_0 K = 4 \left\{ e^{-\Lambda} P_0^2 \partial_\zeta \left( \frac{1}{P_0^2} [P_0^2(e^{-\Lambda}), \zeta] \right) - [P_0^2(e^{-\Lambda}), \zeta] [P_0^2(e^{-\Lambda}), \zeta] \right\} \]  

(17)

Then (8) and (9) imply

\[ (e^{3\Lambda} M)_u = e^\Lambda \left( \frac{1}{4} \Delta_0 K - \frac{(P_0 M, \zeta)(P_0 M, \bar{\zeta})}{2Q_0^2} \right) \]  

(18)

Substitute (17) into (18), then integrate over $S^2$ and apply Stokes’ theorem to the right side. We obtain

\[ \frac{d}{du} M_B = -\frac{1}{4\pi} \int_{S^2} e^\Lambda \left( [P_0^2(e^{-\Lambda}), \zeta] [P_0^2(e^{-\Lambda}), \bar{\zeta}] + \frac{(P_0 M, \zeta)(P_0 M, \bar{\zeta})}{2Q_0^2} \right) \omega_0 \leq 0 \]  

(19)

since the integrand is non-negative. □.

The result is a special case of the positive mass loss theorem due to Bondi et al.\(^{14}\) and Sachs.\(^{15}\) Using Hölder’s inequality one can show that

\[ \left( \int_{S^2} \omega_0 \right)^{\frac{4}{3}} \left( \int_{S^2} e^{3\Lambda} \omega_0 \right)^{\frac{2}{3}} \geq \int_{S^2} e^{2\Lambda} \omega_0 \]  

(20)

In an AFRT vacuum spacetime, where $M = M_0$ is a positive constant, the conservation of surface area (20) implies that the Bondi mass of an AFRT vacuum spacetime is positive : $M_B \geq M_0$.\(^9\) However unlike the vacuum case, the expression (17) for the Bondi mass of an AFRTEM spacetime is not manifestly positive, since $M$ is no longer a positive constant. In view of the positive Bondi mass theorem of Gibbons et al.\(^{16}\) which assumes the presence of an apparent horizon when spacetime singularities exist, such as the $r = 0$ singularity in the AFRT and the AFRTEM spacetimes, the positivity of $M_B$ arising from (20) is a stronger result than the positive mass theorem. This is due to the special features that in an AFRT vacuum spacetime $M$ is constant and the inequality (20) holds. In the next section, we will investigate how the presence of an apparent horizon in an AFRTEM spacetime can lead to the positivity of the associated Bondi mass.

3. Apparent Horizon

In this section we assume that in an AFRTEM spacetime a past apparent horizon exists and we will derive the governing equations. We define an outer apparent horizon $H$ to be a non-timelike hypersurface $r = \Re(u, \zeta, \bar{\zeta})$ such that, on each $u = u_0$ null hypersurface, the spacelike two-surface $r = \Re(u_0, \zeta, \bar{\zeta})$ is a past marginally trapped surface denoted by $T_{u_0}$. These closed two-surfaces $T_u$ can be identified by the local conditions (1) that the ingoing future directed congruence of orthogonal
null geodesics must have vanishing divergence, and (2) that the outgoing future directed congruence of orthogonal null geodesics is diverging.17 Hence the closed two-surfaces \( \mathcal{T}_u \) foliate the hypersurface \( \mathcal{H} \). By the singularity theorems of Penrose 18 and Hawking 19, the existence of an outer past apparent horizon guarantees a ‘white hole’. By a ‘white hole’ we mean a past spacetime singularity ‘hidden’ behind a null hypersurface called the ‘particle horizon’, which is defined as the boundary of the region to which particles or photons from the past infinity can reach.

### 3.1. Newman-Penrose Quantities

In an AFRTEM spacetime, we can choose a null tetrad \( (l^n, n^a, m^a, \bar{m}^a) \) such that \( l^n \) is tangent to a congruence of outgoing future directed null geodesics which is diverging, shearfree, twistfree and affinely parametrised by \( r \). The contravariant form of the metric \( \{l\} \) is given by \( g^{ab} := l^a n^b + n^a \bar{m}^b - m^a \bar{m}^b \) with

\[
\begin{align*}
l^a &= \partial_r, \\
n^a &= \partial_u - \left( H - \frac{P^2 R_\zeta R_\bar{\zeta}}{r^2} \right) \partial_r + \frac{P^2 R_\zeta}{r^2} \partial_\zeta + \frac{P^2 R_\bar{\zeta}}{r^2} \partial_{\bar{\zeta}} \\
m^a &= \frac{P}{r} (\partial_\zeta + R_\zeta \partial_r) \\
\bar{m}^a &= \frac{P}{r} (\partial_{\bar{\zeta}} + R_{\bar{\zeta}} \partial_r)
\end{align*}
\]

Here \( H = -r (\ln P)_u + \frac{1}{2} K - \frac{M}{r} + \frac{Q_0^2}{2r^2} \) and \( R = R(u, \zeta, \bar{\zeta}) \) is a real-valued function whose presence does not affect the metric. The nonvanishing Newman-Penrose 20 quantities are:

\[
\begin{align*}
\rho &= -\frac{1}{r}, \quad \tau = \pi = \omega + \beta = -\frac{P R_\zeta}{r^2}, \quad \beta = -\frac{P_\zeta}{2r}, \\
\lambda &= (\partial_\zeta + R_\zeta \partial_r) \left( \frac{P^2 R_\zeta}{r^2} \right), \\
\mu &= -\frac{1}{2r} \left[ K - \frac{2}{r} (M + P^2 R_\zeta) + \frac{1}{r^2} (Q_0^2 + 2 P^2 R_\zeta R_{\bar{\zeta}}) \right], \\
\gamma &= -\frac{1}{2} \left[ (\ln P)_u - \frac{1}{r^2} (M + PP_\zeta R_\zeta - PP_{\bar{\zeta}} R_{\bar{\zeta}}) + \frac{1}{r^3} (Q_0^2 + 2 P^2 R_\zeta R_{\bar{\zeta}}) \right], \\
\nu &= \frac{P}{r} (\partial_\zeta + R_\zeta \partial_r) \left( R_{\zeta u} - r (\ln P)_u + \frac{K}{2} - \frac{M}{r} + \frac{Q_0^2}{2r^2} + \frac{P R_\zeta R_{\bar{\zeta}}}{r^2} \right), \\
\phi_1 &= \frac{Q_0}{2r^2}, \quad \phi_2 = \frac{P}{2Q_0 r} \left( M_{\zeta \zeta} + \frac{2 Q_0^3 R_\zeta}{r^2} \right), \quad \Psi_2 = -\frac{1}{r^3} \left( M - \frac{Q_0^2}{r} \right), \\
\Psi_3 &= -\frac{PK_{\zeta \zeta}}{2r^2} + \frac{3PM_{\zeta \zeta}}{2r^3} - \frac{3PM R_\zeta}{r^4} + \frac{3Q_0^2 P R_{\bar{\zeta}}}{r^5}.
\end{align*}
\]
From (21), the complex null vectors \( \tilde{T} \) is orthogonal to \( \tilde{T} \). Consider a hypersurface \( H \) which foliates the hypersurface \( H \). According to (21), the null vectors \( \tilde{l}^a \) and \( \tilde{n}^a \) are orthogonal to \( \tilde{T} \). It is straightforward to check that the vector

\[
\tilde{Z}^a = \tilde{n}^a + \left( \tilde{R}_a + \tilde{H} + \frac{P^2 \tilde{R}_\zeta \tilde{R}_\bar{\zeta}}{\tilde{R}^2} \right) \tilde{l}^a
\]

\[
= \partial_a + \left( \tilde{R}_a + \frac{2P^2 \tilde{R}_\zeta \tilde{R}_\bar{\zeta}}{\tilde{R}^2} \right) \partial_r + \frac{P^2 \tilde{R}_\zeta \tilde{\zeta}}{\tilde{R}^2} \partial_\zeta + \frac{P^2 \tilde{R}_\bar{\zeta} \bar{\zeta}}{\tilde{R}^2} \partial_{\bar{\zeta}} \tag{24}
\]

is orthogonal to \( \tilde{N}^a \) and therefore is tangent to the hypersurface \( H \). From (23) and (24), one obtains the ‘magnitude’ of \( \tilde{N}^a \) and of \( \tilde{Z}^a \)

\[
\tilde{N}_a \tilde{N}^a = -\tilde{Z}_a \tilde{Z}^a = -2 \left( \tilde{R}_a + \tilde{H} + \frac{P^2 \tilde{R}_\zeta \tilde{R}_\bar{\zeta}}{\tilde{R}^2} \right) \tag{25}
\]

It is more convenient to work with non-null normalised vectors. Thus when \( \tilde{R}_a + \tilde{H} + (P^2 \tilde{R}_\zeta \tilde{R}_\bar{\zeta})/\tilde{R}^2 \neq 0 \), we use

\[
\tilde{k}^a = \frac{\tilde{N}^a}{\sqrt{|\tilde{N}_b \tilde{N}^b|}}, \quad \tilde{z}^a = \frac{\tilde{Z}^a}{\sqrt{|\tilde{Z}_b \tilde{Z}^b|}}
\]

In this case the spacelike hypersurface \( H \) is spanned by \([\tilde{z}^a, \tilde{m}^a, \tilde{m}^a] \) with unit normal vector \( \tilde{k}^a \).

(21) and (22) imply that the null vector \( \tilde{l}^a \) is future directed, outward pointing, geodetic (\( \tilde{k} = 0 \)), diverging (\( \tilde{\rho} = -\frac{1}{\tilde{R}} \)), null and orthogonal to the two-surface \( T_u \).
The conditions for $\mathcal{H}$ to be an outer past apparent horizon in an AFRTEM spacetime reduce to (1) the null vector $\vec{n}^a$ has vanishing divergence, i.e. $\vec{\mu} + \vec{\Omega} = 0$, and (2) the apparent horizon $\mathcal{H}$ is non-timelike. The normal vector $\vec{N}^a$ is causal and hence $\vec{N}_a \vec{N}^a \geq 0$. From (22) and (25), these conditions become, respectively,

$$K - \frac{2M}{\mathcal{R}} + \frac{Q_0^2}{\mathcal{R}^2} - \Delta(\ln \mathcal{R}) = 0 \quad (26)$$

$$2\Re \partial_u \left( \ln \frac{\mathcal{R}}{P} \right) + K - \frac{2M}{\mathcal{R}} + \frac{Q_0^2}{\mathcal{R}^2} + \frac{2P^2\mathcal{R}_\xi \mathcal{R}_\xi}{\mathcal{R}^2} \leq 0 \quad (27)$$

Since $\mu$ is real, (26) implies $\vec{\mu} = 0$. Consequently the directional derivative of $\vec{\mu}$ along the non-timelike vector $\vec{Z}^a$ tangent to $\mathcal{H}$ must vanish. (24) and (25) then imply

$$\vec{Z}^a \nabla_a \vec{\mu} = \vec{n}^a \nabla_a \vec{\mu} - \frac{1}{2} (\vec{N}_h \vec{N}^h) \vec{Z}^a \nabla_a \vec{\mu} = 0 \quad (28)$$

Substituting (22) into the Newman-Penrose equations (4.11.12.a') and (4.11.12.f') in Penrose and Rindler gives, respectively,

$$\vec{n}^a \nabla_a \vec{\nu} = \vec{m}^a \nabla_a \vec{\nu} + \vec{\nu} \left( (\vec{\pi} + \vec{\alpha} + 3\vec{\beta}) + \vec{\pi} \phi_2 - 2\vec{\phi}_2 \phi_2 \right)$$

$$= -\partial_\zeta \left( \frac{P^2\mathcal{R}_\zeta}{\mathcal{R}^2} \right) \partial_\zeta \left( \frac{P^2\mathcal{R}_\zeta}{\mathcal{R}^2} \right) - \frac{P^2}{2Q_0^2} \mathcal{R}_\zeta \partial_\zeta \left( M - \frac{2Q_0^2}{\mathcal{R}} \right) \partial_\zeta \left( M - \frac{2Q_0^2}{\mathcal{R}} \right)$$

$$- \frac{P^2}{2\Re} \left[ \partial_\zeta \left( \frac{\vec{N}_a \vec{N}^a}{\mathcal{R}} \right) - (\vec{N}_a \vec{N}^a) \partial_\zeta \left( \frac{1}{\mathcal{R}} \right) \right] \quad (29)$$

$$\vec{m}^a \nabla_a \vec{\pi} + \vec{\pi} \left( (\vec{\pi} - \vec{\alpha} + \vec{\beta}) + \vec{\Psi}_2 \right)$$

$$= \frac{P^2}{\Re} \partial_\zeta \left( \frac{1}{\mathcal{R}} \right) - M \frac{Q_0^2}{\mathcal{R}^4} \quad (30)$$

In (23) we have used $\vec{\nu} = -\frac{\mu}{2\mathcal{R}} (\vec{N}_a \vec{N}^a) \zeta$. Combine (28), (29) and (30) to give

$$\frac{1}{\mathcal{R}} \Delta \left( \frac{\vec{N}_a \vec{N}^a}{\mathcal{R}} \right) - 2\vec{\Psi}_2 (\vec{N}_a \vec{N}^a) \phi_2 = 4(\vec{\lambda} + 2\vec{\phi}_2 \phi_2) \quad (31)$$

An AFRTEM spacetime with apparent horizon can be described then by the evolution equations (5) and (9) together with the constraint equations (26) and (31), and the constraint inequality (27).

### 4. Lyapunov Functional

In this section we will show that the existence of a past apparent horizon in an AFRTEM spacetime implies that the Bondi Mass of the spacetime will always be positive. This can be regarded as a special case of the general Bondi Mass positivity result for charged black holes. In fact, all that is required for this result to hold...
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on any given slice $u = u_0$ in an AFRTEM spacetime is that there be a marginally trapped surface on that slice; i.e. only (26) is used. The inequality (27) guarantees that the area of the apparent horizon monotonically decreases with increasing $u$. These results will be combined to show the existence of a Lyapunov functional for the spacetime. We also use $e^\Phi = \Re e^\Lambda = \Re \Omega$.

**Proposition 2:** If an AFRTEM spacetime possesses past marginally trapped surface $\mathcal{T}_0$ on a null slice $u$, then the Bondi mass $\mathcal{M}_B$ at $u$ is positive and satisfies $\mathcal{M}_B^2 \geq Q_0^2$.

**Proof:** We first show that the Bondi mass is non-negative. (26) gives

$$2Me^{3\Lambda} - Q_0^2e^{4\Lambda - \Phi} = e^\Phi (1 - \Delta_0 \Phi)$$

The non-negativity of the Bondi Mass follows from this expression:

$$\mathcal{M}_B = \frac{1}{4\pi} \int_{S^2} Me^{3\Lambda} \omega_0$$

$$\geq - \frac{1}{2} \int_{S^2} e^\Phi \Delta_0 \Phi \omega_0$$

$$\geq - \frac{1}{2} \int_{S^2} [\Delta_0 (e^\Phi) - 2e^\Phi (P_0 \Phi, \zeta) (P_0 \Phi, \bar{\zeta})] \omega_0$$

$$\geq \int_{S^2} e^\Phi (P_0 \Phi, \zeta) (P_0 \Phi, \bar{\zeta}) \omega_0$$

$$\Rightarrow \mathcal{M}_B \geq 0$$

Since the electric charge $Q_0$ is assumed to be nonvanishing, we can show by using the Schwarz inequality and $\int_{S^2} e^{2\Lambda} \omega_0 = 4\pi$ that

$$16\pi^2 \mathcal{M}_B^2 = \left( \int_{S^2} Me^{3\Lambda} \omega_0 \right)^2$$

$$= \frac{1}{4} \left( \int_{S^2} Q_0^2 e^{4\Lambda - \Phi} \omega_0 - \int_{S^2} e^\Phi (1 - \Delta_0 \Phi) \omega_0 \right)^2$$

$$+ \left( \int_{S^2} Q_0^2 e^{4\Lambda - \Phi} \omega_0 \right) \left( \int_{S^2} e^\Phi (1 - \Delta_0 \Phi) \omega_0 \right)$$

$$\geq \left( \int_{S^2} Q_0^2 e^{4\Lambda - \Phi} \omega_0 \right) \left( \int_{S^2} e^\Phi \omega_0 + \int_{S^2} e^\Phi (P_0 \Phi, \zeta) (P_0 \Phi, \bar{\zeta}) \omega_0 \right)$$

$$\geq \left[ \left( \int_{S^2} Q_0^2 e^{4\Lambda - \Phi} \omega_0 \right)^{1/2} \left( \int_{S^2} e^\Phi \omega_0 \right)^{1/2} \right]^2$$

$$\geq \left( \int_{S^2} Q_0^2 e^{2\Lambda} \omega_0 \right)^2$$

$$= 16\pi^2 Q_0^2$$

$$\Rightarrow \mathcal{M}_B^2 \geq Q_0^2 > 0 \quad \Box.$$
Proposition 2 is a special case of the positive mass result of Gibbons et al.\textsuperscript{16}

We can define the surface area of the marginally trapped surface $\mathcal{T}_u$ as

$$A_T = \int_{S_2} R^2 \omega_1 = \int_{S_2} e^{2\Phi} \omega_0$$  \hspace{1cm} (32)

**Proposition 3:** If an AFRTEM spacetime possesses an outer past apparent horizon $\mathcal{H}$, then the surface area $A_T$ of the outer marginally trapped surface $\mathcal{T}_u$ is a monotonically decreasing function of the retarded time $u$.

**Proof:** Apply (26) to the inequality (27). We get

$$\left( e^{2\Phi} \right)_u + \Delta_0 (e^{\Phi-L}) = -e^{\Phi+L} (\widetilde{N}_a \widetilde{N}^a) \leq 0$$ \hspace{1cm} (33)

Since $\widetilde{N}^a$ is causal, by Stokes’ Theorem

$$\frac{d}{du} A_T = \frac{d}{du} \int_{S_2} e^{2\Phi} \omega_0 = -\int_{S_2} e^{\Phi+L} (\widetilde{N}_a \widetilde{N}^a) \omega_0 \leq 0$$ \hspace{1cm} (34)

Proposition 3 is simply the “area decrease theorem” for apparent horizons.\textsuperscript{22} If a spacetime possesses a horizon, then as it evolves the area of its past apparent horizon will decrease, just as its future horizon will increase in area. We can now prove that the sum of the square of the Bondi mass and the area of the outer marginally trapped surface at a retarded time $u$ is a Lyapunov functional.

**Theorem:** $\mathcal{L}(M, \Lambda, \Phi) = 16\pi M_B^2 + A_T$ is a Lyapunov functional for a regular AFRTEM spacetime that possesses an outer past apparent horizon $\mathcal{H}$; i.e. $\mathcal{L}$ satisfies the following conditions:

(a) $\mathcal{L}(M, \Lambda, \Phi) > 0$,

(b) $\frac{d}{du} \mathcal{L}(M, \Lambda, \Phi) \leq 0$, and

(c) $\frac{d}{du} \mathcal{L}(M_0, \Lambda_0, \Phi_0) = 0$ if and only if $(M_0, \Lambda_0, \Phi_0)$ is an equilibrium solution of the system.

**Proof:**

(a) Proposition 2 gives $M_B^2 > 0$, and by definition $A_T > 0$. Hence $\mathcal{L} > 0$.

(b) Proposition 1 gives $\frac{d}{du} M_B \leq 0$, Proposition 3 gives $\frac{d}{du} A_T \leq 0$, and Proposition 2 gives $M_B > 0$. Hence

$$\frac{d}{du} \mathcal{L} = 32\pi M_B \frac{d}{du} M_B + \frac{d}{du} A_T \leq 0$$

(c) \textbf{(i)} \hspace{1cm} (\Rightarrow)

$$\frac{d}{du} M_B = 0 \Rightarrow M_\zeta = 0$$

and $[P_0^2 (e^{-\Lambda})_\zeta]_{\zeta} = 0$ (from (19) in Proposition 1),
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\[ \frac{d}{du} A_T = 0 \Rightarrow \tilde{N}_a \tilde{N}^a = 0 \] (from (34) in Proposition 3)
\[ \Rightarrow [P^2(\mathcal{R}^{-1})]_{\zeta} = 0, \text{ and } M_{\zeta} = 2Q_0^2(\mathcal{R}^{-1})_{\zeta} \] (from (31))

So, \[ \frac{d}{du} L = 0 \Rightarrow \frac{d}{du} M_B = 0, \text{ and } \frac{d}{du} A_T = 0 \]
\[ \Rightarrow M_{\zeta} = \mathcal{R}_{\zeta} = [P_0^2(e^{-\Lambda})]_{\zeta} = 0 \]

Now, \[ M_{\zeta} = 0 \Rightarrow \Delta_0 M = 0 \]
\[ \Rightarrow [P_0^2(e^{-\Lambda})]_{\zeta} = 0 \Rightarrow \Delta_0 K = 0 \] (from (17) in Proposition 1)
\[ \Rightarrow M_{,u} = \Lambda_{,u} = 0 \] (from (3) and (4)),
\[ \Rightarrow M = M_0 = \text{constant, and } \Lambda = \Lambda_0 = \ln P_0 - \ln(A + B\zeta + C\zeta^2 + C\zeta^2). \]

Also, \[ \mathcal{R}_{\zeta} = 0 \Rightarrow (e^{\Phi-\Lambda})_{\zeta} = 0 \Rightarrow \Phi_{,u} = 0 \] (from (33) in Proposition 3).

Thus equilibrium is reached when \( \mathcal{L} \) is stationary. The equilibrium value of \( \mathcal{R} \) can be obtained from (23) and (31); i.e. \( \mathcal{R}_0 = M_0 + \sqrt{M_0^2 - Q_0^2} \)
\[ \text{(ii) } (\leq) \]
\[ \Lambda = \Lambda_0, \quad M = M_0, \quad \mathcal{R} = \mathcal{R}_0 \]
\[ \Rightarrow \frac{d}{du} M_B = 0 \] (from (19)), \quad and \[ \frac{d}{du} A_T = 0 \] (from (33) and (34))
\[ \Rightarrow \frac{d}{du} L = 0 \] \( \square \).

5. Conclusion

Rendall \(^{23}\) pointed out, in relation to the AFRT evolution, that the presence of a Lyapunov functional does not guarantee the existence of a solution, as the phase space is infinite dimensional. Furthermore, due to the non-uniqueness of the equilibrium solution, arising from conformal motions of the sphere, the existence of a Lyapunov functional was not sufficient to guarantee convergence to the equilibrium. An assumption of antipodal symmetry would restrict the class of solutions such that there is a unique equilibrium.\(^6\) It has subsequently been suggested, in the vacuum case, that this extra freedom can be factored out, effectively fixing a gauge condition.\(^9\) It has been shown that the conformal motions that preserve the equilibrium solution are equivalent to transformations at \( I^+ \) between different accelerated frames. We will discuss this point further in a future paper.

We believe that the existence of a Lyapunov functional is very suggestive, and that the AFRTEM system, in the case where a past apparent horizon exists, evolves stably to the Reissner Nordström equilibrium. \cite{11} implies that \( \int_{S^2}(M - Q^2) \omega_0 \geq 0. \)

The condition \( M - Q^2 \geq 0 \), that is \( \Psi_2 \leq 0 \), if it holds pointwise on \( \mathcal{H} \), would be sufficient to rule out the exponentially growing modes in \( \Phi \). Note that near equilibrium \( M \geq Q^2 \), with equality holding in the extreme Reissner Nordström geometry. This is in contrast to the instability suggested by the linear analysis (see Section 2 above). We would infer from this that the subclass of AFRTEM spacetimes that possess an outer past apparent horizon are those that correspond to the stable submanifold of the linearized system. The requirement for the existence of an apparent horizon amounts to the imposition of a global condition selecting the physically meaningful solutions of the evolution equations. This would resolve the apparent contradiction between the instability of the linearized AFRTEM system and the stability of the perturbed Reissner Nordström black hole, since only that
subclass of the AFRTEM spacetimes containing apparent horizons correspond to perturbed black hole spacetimes.

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