GEODESICALLY CONVEX ENERGIES AND CONFINEMENT OF SOLUTIONS FOR A MULTI-COMPONENT SYSTEM OF NONLOCAL INTERACTION EQUATIONS

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ABSTRACT. We consider a system of $n$ nonlocal interaction evolution equations on $\mathbb{R}^d$ with a differentiable matrix-valued interaction potential $W$. Under suitable conditions on convexity, symmetry and growth of $W$, we prove $\lambda$-geodesic convexity for some $\lambda \in \mathbb{R}$ of the associated interaction energy with respect to a weighted compound distance of Wasserstein type. In particular, this implies existence and uniqueness of solutions to the evolution system. In one spatial dimension, we further analyse the qualitative properties of this solution in the non-uniformly convex case. We obtain, if the interaction potential is sufficiently convex far away from the origin, that the support of the solution is uniformly bounded. Under a suitable Lipschitz condition for the potential, we can exclude finite-time blow-up and give a partial characterization of the long-time behaviour.

1. INTRODUCTION

In this work, we analyse the following system of $n \in \mathbb{N}$ nonlocal interaction evolution equations

$$
\partial_t \mu_1 = \text{div}(m_1 \mu_1 \nabla (W_{11} * \mu_1 + W_{12} * \mu_2 + \ldots + W_{1n} * \mu_n)),
$$

$$
\partial_t \mu_2 = \text{div}(m_2 \mu_2 \nabla (W_{21} * \mu_1 + W_{22} * \mu_2 + \ldots + W_{2n} * \mu_n)),
$$

$$
\vdots
$$

$$
\partial_t \mu_n = \text{div}(m_n \mu_n \nabla (W_{n1} * \mu_1 + W_{n2} * \mu_2 + \ldots + W_{nn} * \mu_n)).
$$

The sought-for $n$-vector-valued solution $\mu = (\mu_1, \ldots, \mu_n)(t)$ describes the distribution or concentration of $n$ different populations or agents on $\mathbb{R}^d$ at time $t \geq 0$, $d \in \mathbb{N}$ denoting the spatial dimension. Apart from the constant mobility magnitudes $W_{ij}$, $i, j \in \{1, \ldots, n\}$, system (1) is mainly governed by the matrix-valued interaction potential $W : \mathbb{R}^d \to \mathbb{R}^{n \times n}$ satisfying the following main requirements:

(W1) $W(z)$ is a symmetric matrix for each $z \in \mathbb{R}^d$.

(W2) $W_{ij} \in C^1(\mathbb{R}^d; \mathbb{R})$ for all $i, j \in \{1, \ldots, n\}$.

(W3) $W(z) = W(-z)$ for all $z \in \mathbb{R}^d$.

(W4) There exists a matrix $W \in \mathbb{R}^{n \times n}$ such that for each $i, j \in \{1, \ldots, n\}$ and all $z \in \mathbb{R}^d$:

$$
|W_{ij}(z)| \leq W_{ij}(1 + |z|^2).
$$

(W5) There exists a matrix $\kappa \in \mathbb{R}^{n \times n}$ such that, for each $i, j \in \{1, \ldots, n\}$, $W_{ij}$ is $\kappa_{ij}$-(semi-)convex, i.e. the map $z \mapsto W_{ij}(z) - \frac{1}{2} \kappa_{ij} |z|^2$ is convex.

System (1) possesses a formal gradient flow structure: On the subspace $\mathcal{P}$ of those $n$-vector Borel measures on $\mathbb{R}^d$ with fixed total masses $\mu_j(\mathbb{R}^d) = p_j > 0$, fixed (joint, weighted) center of mass $\sum_{j=1}^n \frac{1}{m_j} \int_{\mathbb{R}^d} x \, d\mu_j(x) = E \in \mathbb{R}^d$ and finite second moments $\ell_2(\mu_j) := \int_{\mathbb{R}^d} |x|^2 \, d\mu_j(x)$, the multi-component interaction energy functional

$$
W(\mu) := \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_{ij}(x-y) \, d\mu_i(x) \, d\mu_j(y)
$$

induces (1) as its gradient flow w.r.t. the following compound metric of Wasserstein-type distances for each of the components of the vector measures $\mu^0, \mu^1 \in \mathcal{P}$:

$$
W_M(\mu^0, \mu^1) = \left[ \sum_{j=1}^n \frac{1}{m_j} \min \left\{ \int_{\mathbb{R}^d} |t_j(x) - x|^2 \, d\mu_j^0(x) \mid t_j : \mathbb{R}^d \to \mathbb{R}^d \text{ measurable, } \mu_j^0 = t_j \# \mu_j^0 \right\} \right]^{1/2},
$$

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where $t_j \# \mu_0^j$ denotes the push-forward of the measure $\mu_0^j$ under the mapping $t_j$. It easily follows from the properties of the usual Wasserstein distance for probability measures with finite second moment that $W^*_M$ defines a distance on the (geodesic) space $\mathcal{P}$ (see for instance [1], [2] for more details on optimal transport and gradient flows). Note that (W1)-(W5) imply (at least formally) that $\mathcal{P}$ is a positively invariant set along the flow of the evolution (1). In this work, we give a rigorous proof for these formal arguments.

We obtain in the case of genuine irreducible systems a novel sufficient condition on the model parameters such that the interaction energy functional $\mathcal{W}$ becomes $\lambda$-geodesically convex on $\mathcal{P}$ with respect to the distance $W^*_M$ for some $\lambda \in \mathbb{R}$. That is, for every pair $\mu^0, \mu^1 \in \mathcal{P}$, there exists a constant-speed geodesic curve $(\mu^s)_{s \in [0,1]}$ in $\mathcal{P}$ connecting $\mu^0$ and $\mu^1$ for which

$$\mathcal{W}(\mu^s) \leq (1-s)\mathcal{W}(\mu^0) + s\mathcal{W}(\mu^1) - \frac{1}{2}s(1-s)\lambda W^*_M(\mu^0, \mu^1)^2, \quad \forall \ s \in [0,1].$$

(4)

We call the energy uniformly geodesically convex if (4) admits $\lambda > 0$. Define, for each $i \in \{1, \ldots, n\}$, the numbers $\eta_i := \min_{j \neq i} \mu_{ij} m_j$. We prove in Section 2 the $\lambda$-geodesic convexity of $\mathcal{W}$ for all

$$\lambda \leq \min_{i \in \{1, \ldots, n\}} \left[ p_i \min(0, m_i \kappa_{ii} - \eta_i) + \frac{1}{2} \sum_{j=1}^n p_j \left( \eta_j + \eta_i \frac{m_i}{m_j} \right) \right].$$

(5)

Even if some of the $W_{ij}$ are not uniformly convex (i.e. $\kappa_{ij} \leq 0$), we might still obtain a convexity modulus $\lambda > 0$, if attraction dominates repulsion as required in $\mathcal{E}$. Using this new condition, we are in position to invoke the theory on ($\lambda$-contractive) gradient flows in metric spaces by Ambrosio, Gigli and Savaré [2] to obtain (cf. Section 2.2) the existence of a gradient flow solution $\mu \in AC^2_{loc}((0, \infty); (\mathcal{P}, W^*_M))$ to (1), endowed with an initial datum $\mu^0 \in \mathcal{P}$. Moreover, uniqueness of solutions follows from the contraction estimate for all $t > 0$: $W^*_M(\mu(t), \nu(t)) \leq e^{-t} W^*_M(\mu^0, \nu^0)$.

If the modulus of geodesic convexity is strictly positive, the measure $\mu^\infty := (p_1, \ldots, p_n)^T \delta_{x^\infty}$, with $x^\infty := E \left[ \sum_{j=1}^n p_j \frac{1}{m_j} \right]^{-1} \in \mathbb{R}^d$, is the unique minimizer – the ground state – of $\mathcal{W}$ and the unique stationary state of (1) on $\mathcal{P}$. It is globally asymptotically stable since gradient flow solutions $\mu(t)$ converge exponentially fast in $(\mathcal{P}, W^*_M)$ with rate $\lambda$ to $\mu^\infty$.

In contrast, if $\mathcal{W}$ is $\lambda$-geodesically convex with only $\lambda \leq 0$, the dynamics of system (1) are more involved.

There, we restrict to one spatial dimension ($d = 1$) and rewrite the system in terms of inverse distribution functions: Given the (scaled) cumulative distribution functions

$$F_i(t, x) = \int_{-\infty}^x \frac{1}{p_i} \ d\mu_i(t, y) \quad \in [0,1],$$

(6)

let $u_i$ be their corresponding pseudo-inverse, i.e.

$$u_i(t, z) = \inf \{ x \in \mathbb{R} : F_i(t, x) > z \} \quad \text{for} \ z \in [0,1).$$

(7)

Then, system (1) transforms into (cf. Section 3)

$$\partial_t u_i(t, z) = m_i \sum_{j=1}^n p_j \int_0^1 W'_{ij}(u_j(t, \xi) - u_i(t, z)) \ d\xi \quad (i = 1, \ldots, n).$$

(8)

In terms of system (3), if $\mu \in AC^2_{loc}((0, \infty); (\mathcal{P}, W^*_M))$, one has $u \in AC^2_{loc}((0, \infty); L^2([0,1]; \mathbb{R}^n))$ and for all $t \geq 0$ and all $i \in \{1, \ldots, n\}$, $u_i(t, \cdot)$ is a non-decreasing càdlàg function on $[0,1)$. Section 3 is devoted to the analysis of the qualitative behaviour of the solution $\mu$ to (1) by means of investigation of the corresponding solution $u$ to (3). In that part, our main result is a confinement property of the solution: For admissible interaction potentials satisfying (W1)-(W5) only, we prove (cf. Proposition 3.3) that

$$\sup \mu_i(t) \subset [-K(T, \mu^0)i, K(T, \mu^0)] \quad \forall t \in [0, T] \quad \forall \ i \in \{1, \ldots, n\}.$$  

(9)

for some finite constant $K > 0$ depending on the (compactly supported) initial datum $\mu_0^i$ and the (finite) time horizon $T > 0$. Due to repulsion effects in this general setting, $K(T, \mu^0) \rightarrow \infty$ may occur as $T \rightarrow \infty$. We propose a confining condition on the interaction potential $W$ (cf. Definition 3.2) such that the above property extends to $T = \infty$ (cf. Theorem 3.4): In a nutshell, we require $W$ to behave – outside a compact set – like a potential inducing a uniform geodesically convex energy functional $\mathcal{W}$, in the sense of our criterion (5). Note that we do not require all $W_{ij}$ to be uniformly convex far away from the origin.

Thus, in many cases, mass cannot escape to infinity. In contrast, is it possible to have concentration in finite time, i.e. can it occur that absolutely continuous solutions collapse to measures with nonvanishing singular part in finite
Section 3.3 is devoted to the study of the long-time behaviour of the solution to (1). We first prove (cf. Theorem 3.6) that if the solution is a priori confined to a compact set, the $\omega$-limit set of the system only contains steady states of (1). More specifically, assume that (19) is true for some $K > 0$ and $T = \infty$ and assume that all $W_{ij}$ are Lipschitz-continuous on the interval $[-2K, 2K]$. Then,

$$\lim_{t \to \infty} \left( \frac{d}{dt} W(\mu(t)) \right) = 0.$$  

Moreover, for each sequence $t_k \to \infty$, there exists a subsequence and a steady state $\mu \in \mathcal{P}$ of (11) such that on the subsequence

$$\lim_{k \to \infty} W_i(\mu_i(t_k), \mu_i) = 0 \quad \forall i \in \{1, \ldots, n\}.$$

This holds even for initial data with compact support. In contrast to convergence w.r.t. the $\mathcal{W}_2$-Wasserstein distance between finite measures. However, this large-time limit $\mu$ is not unique since it depends both on the sequence $(t_k)_{k \in \mathbb{N}}$ chosen and the extracted subsequence.

Even if the interaction potential does neither yield uniform geodesic convexity of the energy nor is confining and Lipschitz, we may observe a $\delta$-separation phenomenon: If the initial datum has compact support and the model parameters admit $\sum_{j=1}^n \kappa_{ij} p_j > 0$ for all $i$, the diameter of the support of the solution shrinks exponentially fast over time (cf. Proposition 3.8). Still, the solution does in general not converge to a fixed steady state. However, in the uniformly geodesically convex regime ($\lambda > 0$ in (13)), we obtain convergence even w.r.t. the stronger topology of the $L^\infty$-Wasserstein distance $W_\infty$,

$$\lim_{t \to \infty} W_\infty(\mu_i(t), \mu_i^\infty) = 0 \quad \forall i \in \{1, \ldots, n\},$$

for initial data with compact support. In contrast to convergence w.r.t. $W_\infty$ (cf. Corollary 2.11), we do not obtain a specific rate of convergence.

System (11) is a natural generalization of the scalar nonlocal evolution equation

$$\partial_t \mu = \text{div} [\mu \nabla W(\mu)],$$

to multiple components. For the corresponding interaction energy functional

$$W(\mu) = \frac{1}{2} \int_{\mathbb{R}^d} (W * \mu) \, d\mu,$$

McCann provided in his seminal paper [39] a criterion for $\lambda$-geodesic convexity with respect to the $L^2$-Wasserstein distance (see also [51, Thm. 5.15(c)]). In a nutshell, if $W$ is $\kappa$-convex in the Euclidean sense on $\mathbb{R}^d$ for some $\kappa \in \mathbb{R}$, then $W$ is $\min(0, \kappa)$-geodesically convex on the space of probability measures endowed with the $L^2$-Wasserstein distance. On the subspace of those measures having fixed center of mass, $W$ is $\kappa$-geodesically convex (i.e. uniform convexity is retained in the metric framework). It was proven by Ambrosio, Gigli and Savaré in [2] that geodesic convexity essentially leads to existence and uniqueness of weak solutions for the associated gradient flow evolution equation and to contractivity of the associated flow map. Geodesic convexity also yields useful error estimates, e.g. for the semi-discrete JKO scheme [33, 42] often used to construct weak solutions to equations with gradient flow structure. An immediate consequence of $\lambda$-geodesic convexity of functionals – for strictly positive $\lambda \in \mathbb{R}$ – is existence and uniqueness of minimizers (for recent results without using convexity, see e.g. [17, 15]). For more general genuine systems of equations with gradient flow structure, geodesic convexity has been studied in [53].

Model equations of the form (11) have arisen in the study of population dynamics in many cases (e.g. [10, 11, 13, 20, 21, 24, 25, 31, 32, 34, 35, 40, 41, 17]):
- In the parabolic-elliptic Keller-Segel model for chemotaxis in two spatial dimensions, the interaction potential is given by (the negative of) the Newtonian potential, i.e. $W(z) = \frac{1}{r} \log(|z|)$, which is singular at $z = 0$ and attractive.
- Typical mathematical models of swarming processes include so-called attractive-repulsive potentials of the form $W(z) = -C_\alpha e^{-|z|^2/\lambda} + C_\tau e^{-|z|^2/\tau}$, a special case of which is the attractive Morse potential $W(z) = -e^{-|z|^2}$.
- Also, Gaussian-type attractive-repulsive potentials $W(z) = -C_\alpha e^{-|z|^2/\lambda} + C_\tau e^{-|z|^2/\tau}$ are conceivable.

Nonlocal interaction potentials also appear in several models of physical applications such as models for granular media ([4, 22, 23, 37, 43, 25]), opinion formation [16] or interactions between particles (e.g. in crystals [43] or fluids [52]) with a broad range of reasonable interaction potentials. One can e.g. consider:
- convex and $C^1$-regular potentials, e.g. $W(z) = |z|^q$ for $q > 1$.
- non-convex, but regular potentials such as the double-well potential $W(z) = |z|^4 - |z|^2$. 

These, and related, potentials can be used in the context of the model equations of this paper.
• non-convex and singular potentials, e.g. the Lennard-Jones potential.

In the case of a radially symmetric potential \( W(z) = w(|z|) \), the effect of the interaction potential is reflected by the sign of \( w' \): If \( w' \) is positive, the individuals of the population attract each other, whereas in the case of negative \( w' \) the dynamics are repulsive. The force generated by the potential \( W \) points towards or away from the origin for positive or negative \( w' \), respectively. Radially symmetric potentials describe interactions only depending on the distance of the particles. With the sum of convolutions appearing in the flux on the r.h.s. of system (11), we take into account that every species generates a – probably long-range – force on every other species.

Naturally, aggregation processes modelled by nonlocal interaction potentials are often combined with diffusive processes yielding (nonlinear) drift-diffusion equations as mathematical models. The question of global existence of solutions to equations of these forms has been addressed in various publications. Using the theory of gradient flows, global existence of measure-valued solutions was proven in [18, 21], also for non-smooth potentials, in generalization of [22, 28]. Methods from optimal transportation theory were also useful for proving uniqueness, see e.g. [24, 27]. Well-posedness in the measure-valued sense was also studied in [16], and in [28] for a similar system for two species (see below).

A second field of study is the analysis of the qualitative behaviour of solutions to equations like (11), such as the speed of propagation, finite- and infinite-time blow-up of solutions and possible attractors, also with focus on self-similarity of solutions. It is not surprising that (11) exhibits blow-ups if the potential is sufficiently attractive. The aforementioned properties were investigated e.g. in [3, 5, 6, 7, 8, 9, 19, 36, 44]. One specific object of study is equation (11) considered in one spatial dimension. For instance, in [43, 29, 30] by Raoul and Fellner, rewriting the speed of propagation, finite- and infinite-time blow-up of solutions and possible attractors, also with focus on self-similarity of solutions. It is not surprising that (11) exhibits blow-ups if the potential is sufficiently attractive.

In this section, we derive a sufficient condition for geodesic convexity of the interaction energy \( W \) (cf. formula (2)) and conclude existence and uniqueness of solutions to (11). Throughout the paper, the assumptions (W1)-(W5) above shall be fulfilled.

2. Geodesic convexity and existence of solutions

In this section, we derive a sufficient condition for geodesic convexity of the interaction energy \( W \) (cf. formula (2)) and conclude existence and uniqueness of solutions to (11). Throughout the paper, the assumptions (W1)-(W5) above shall be fulfilled.

2.1. Geodesic convexity of the multi-component interaction energy. We first recall some basic properties of the interaction energy \( W \).

Lemma 2.1 (Proper domain and lower semicontinuity). The following statements hold:

(a) For all \( \mu \in \mathcal{P} \), one has \( |W(\mu)| < \infty \).

(b) \( W \) is lower semicontinuous on the metric space \( (\mathcal{P}, W_M) \).

Proof. Part (a) easily follows from the growth condition (W4) using the fact that all second moments \( \ell_2(\mu_i) := \int_{\mathbb{R}^d} |x|^2 d\mu_i(x) \) are finite on \( \mathcal{P} \). Part (b) is a consequence of (W4) and Lemma 5.1.7 observing that the distance \( W_M \) is the canonical metric on a (weighted) product of Wasserstein spaces.

Lemma 2.2 (Growth control on the gradient). There exists a matrix \( \overline{C} \in \mathbb{R}^{n \times n} \) such that for all \( z \in \mathbb{R}^d \) and all \( i, j \in \{1, \ldots, n\} \):

\[
|\nabla W_{ij}(z)| \leq \overline{C}_{ij}(|z| + 1). \tag{13}
\]

Proof. We give a short proof for the sake of completeness and omit the indices \( i, j \) in the following. From (W2) and (W5), it easily follows for all \( x, y \in \mathbb{R}^d \) that

\[
W(y) - W(x) - \frac{\kappa}{2} |y - x|^2 \geq \nabla W(x)^T (y - x).
\]

Putting \( \alpha := (\max(4W, 2|\kappa|) - \kappa)^{-1} > 0 \) and \( y := x + \alpha \nabla W(x) \), we end up – using (W4) – with

\[
\alpha |\nabla W(x)|^2 \leq W(2 + 3|x|^2) + \frac{1}{2} \alpha |\nabla W(x)|^2.
\]
The claim immediately follows, since
\[ |\nabla W(x)| \leq \left( \frac{2W(2+3|x|^2)}{\max(4W,2|\kappa|)} \right)^{1/2} \]
implies an estimate of the form [13].

Remark 2.3 (Invariants). Along the flow of system [1], the set \( \mathcal{P} \) is positively invariant. We give a formal indication of this fact: Let an initial datum \( \mu^0 \in \mathcal{P} \) be given. Since [1] is in divergence form, we immediately obtain conservation of mass:
\[ \frac{d}{dt} \int_{\mathbb{R}^d} d\mu(t, x) = 0. \]

Furthermore, by formal integration by parts, one has
\[ \frac{d}{dt} \sum_{i=1}^{n} \ell_2(\mu_i(t)) = - \sum_{i=1}^{n} 2m_i \int_{\mathbb{R}^d} x^T \left( \sum_{j=1}^{n} \nabla W_{ij} \ast \mu_j(t) \right)(x) \, d\mu_i(t, x), \]
from which it is possible to derive using the Young and Jensen inequalities and Lemma 2.3 the estimate
\[ \frac{d}{dt} \sum_{i=1}^{n} \ell_2(\mu_i(t)) \leq A \sum_{i=1}^{n} \ell_2(\mu_i(t)) + B, \]
for suitable \( A, B \in \mathbb{R} \). Gronwall’s lemma yields finiteness of second moments at a fixed time \( t \geq 0 \). Finally,
\[ \frac{d}{dt} \sum_{i=1}^{n} \frac{1}{m_i} \int_{\mathbb{R}^d} x \, d\mu_i(t, x) = - \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla W_{ij}(x-y) \, d\mu_j(t, y) \, d\mu_i(t, x). \]
Using assumptions (W1)&(W3) in combination with Fubini’s theorem, we observe that the r.h.s. above is in fact equal to 0.

Definition 2.4 (Irreducible systems). We call a system of the form [1] irreducible, if the graph \( G = (V_G, E_G) \) with nodes \( V_G = \{1, \ldots, n\} \) and edges \( E_G = \{(i, j) \in V_G \times V_G : \nabla W_{ij} \neq 0 \text{ on } \mathbb{R}^d\} \) is connected. That is, irreducible systems cannot be split up into independent subsystems.

The main result of this section is about geodesic convexity of the interaction energy \( W \):

Theorem 2.5 (Criterion for geodesic convexity). Let \( n > 1 \) and let [1] be irreducible. Define for \( i \in \{1, \ldots, n\} \) the quantity \( \eta_i := \min_{j \neq i} \kappa_{ij} m_j \in \mathbb{R} \). Then, \( W \) is \( \lambda \)-geodesically convex on \( \mathcal{P} \) w.r.t. \( W_M \) for all \( \lambda \leq \lambda_0 \) with
\[ \lambda_0 := \min_{i \in \{1, \ldots, n\}} \left[ p_i \min(0,m_i \kappa_{ii} - \eta_i) + \frac{1}{2} \sum_{j=1}^{n} p_j \left( \eta_j + \eta_i \frac{m_i}{m_j} \right) \right]. \quad (14) \]

Proof. Let \( \mu^0, \mu^1 \in \mathcal{P} \) and let \( t_1, \ldots, t_n : \mathbb{R}^d \to \mathbb{R}^d \) be measurable and such that
\[ W_M^2(\mu^0, \mu^1) = \sum_{i=1}^{n} \frac{1}{m_i} \int_{\mathbb{R}^d} |t_i(x) - x|^2 \, d\mu_0^0(x), \quad (15) \]
i.e. \( t = (t_1, \ldots, t_n) \) is an optimal transport map for \( \mu^0, \mu^1 \). For more details on existence of these maps, see for instance [51]. Define for \( s \in [0, 1] \) the associated constant-speed geodesic curve \( \mu^s \in \mathcal{P} \), i.e. \( \mu^s := [(1-s) \text{id} + s t_i] \# \mu_i \). We obtain using the transformation theorem (write \( \mu := \mu^0 \) for simplicity):
\[ W(\mu^s) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_{ij}(x-y+s[t_j(x) - t_i(y) - (x-y)]) \, d\mu_j(x) \, d\mu_i(y). \]
Assumption (W5) now yields
\[ W(\mu^s) \leq (1-s)W(\mu^0) + sW(\mu^1) - \frac{1}{2} s(1-s) \cdot \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \kappa_{ij} |t_j(x) - t_i(y) - (x-y)|^2 \, d\mu_j(x) \, d\mu_i(y). \]
In view of [14], we have to verify that
\[ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \kappa_{ij} |t_j(x) - t_i(y) - (x-y)|^2 \, d\mu_j(x) \, d\mu_i(y) \geq \lambda_0 \sum_{i=1}^{n} \frac{1}{m_i} \int_{\mathbb{R}^d} |t_i(x) - x|^2 \, d\mu_i(x). \quad (16) \]
We first split up the l.h.s. of (16) into its diagonal and off-diagonal part and perform an estimate on the latter introducing the numbers $\eta_i = \min_{j \neq i} \kappa_{ij} m_j$:

$$
\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \eta_i \kappa_{ij} |t_j(x) - t_i(y) - (x - y)|^2 \, d\mu_j(x) \, d\mu_i(y)
$$

$$
\geq \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \eta_i (|t_j(x) - x| - (t_i(y) - y)|^2 \, d\mu_j(x) \, d\mu_i(y)
$$

$$
+ \frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \kappa_{ii} (|t_i(x) - x| - (t_i(y) - y)|^2 \, d\mu_i(x) \, d\mu_i(y).
$$

Expanding the squares yields

$$
\frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\eta_i}{m_j} (|t_j(x) - x| - (t_i(y) - y)|^2 d\mu_j(x) \, d\mu_i(y)
$$

$$
+ \frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \kappa_{ii} (|t_i(x) - x| - (t_i(y) - y)|^2 \, d\mu_i(x) \, d\mu_i(y)
$$

$$
= \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} \left( \int_{\mathbb{R}^d} \frac{\eta_i}{m_j} |t_j(x) - x|^2 \, d\mu_j(x) + \int_{\mathbb{R}^d} \frac{\eta_i}{m_j} |t_i(x) - x|^2 \, d\mu_i(x) \right)
$$

$$
- \sum_{i=1}^{n} \left( \sum_{j \neq i} \int_{\mathbb{R}^d} \frac{1}{m_j} (t_j(x) - x) \, d\mu_j(x) \right)^T \left( \int_{\mathbb{R}^d} \eta_i (t_i(x) - x) \, d\mu_i(x) \right)
$$

$$
+ \sum_{i=1}^{n} \kappa_{ii} \left( \int_{\mathbb{R}^d} p_i |t_i(x) - x|^2 \, d\mu_i(x) \right) - \sum_{i=1}^{n} \kappa_{ii} \left| \int_{\mathbb{R}^d} (t_i(x) - x) \, d\mu_i(x) \right|^2.
$$

Now, the special structure of $\mathcal{P}$ comes into play: Since the weighted center of mass $E$ is fixed on $\mathcal{P}$, one has

$$
E = \sum_{j=1}^{n} \frac{1}{m_j} \int_{\mathbb{R}^d} x \, d(t_j \# \mu_j),
$$

and consequently

$$
\sum_{j \neq i} \int_{\mathbb{R}^d} \frac{1}{m_j} (t_j(x) - x) \, d\mu_j(x) = - \int_{\mathbb{R}^d} \frac{1}{m_i} (t_i(x) - x) \, d\mu_i(x).
$$

We exploit this fact in order to simplify the second term on the r.h.s. above:

$$
\text{r.h.s.} = \sum_{i=1}^{n} \left\{ \int_{\mathbb{R}^d} (t_i(x) - x) \, d\left( \frac{1}{p_i} \mu_i \right)(x) \right\} \left( \frac{\eta_i}{m_i} - \kappa_{ii} \right) + \int_{\mathbb{R}^d} |t_i(x) - x|^2 \, d\left( \frac{1}{p_i} \mu_i \right)(x) p_i \left( \kappa_{ii} - \frac{\eta_i}{m_i} \right)
$$

$$
+ \frac{1}{m_i} \int_{\mathbb{R}^d} |t_i(x) - x|^2 \, d\mu_i(x) \cdot \frac{1}{2} \sum_{j} p_j \left( \eta_j + \eta_i \frac{m_i}{m_j} \right) \right\} = \sum_{i} S_i.
$$

We analyse each $S_i$ separately.

If $\frac{\eta_i}{m_i} - \kappa_{ii} \geq 0$, the first term in $S_i$ is nonnegative, so

$$
S_i \geq \frac{1}{m_i} \int_{\mathbb{R}^d} |t_i(x) - x|^2 \, d\mu_i(x) \cdot \left[ p_i (m_i \kappa_{ii} - \eta_i) + \frac{1}{2} \sum_{j} p_j \left( \eta_j + \eta_i \frac{m_i}{m_j} \right) \right].
$$

If $\frac{\eta_i}{m_i} - \kappa_{ii} < 0$, the sum of the first two terms in $S_i$ is nonnegative thanks to Jensen’s inequality. Hence,

$$
S_i \geq \frac{1}{m_i} \int_{\mathbb{R}^d} |t_i(x) - x|^2 \, d\mu_i(x) \cdot \frac{1}{2} \sum_{j} p_j \left( \eta_j + \eta_i \frac{m_i}{m_j} \right).
$$

Defining $\lambda_0$ as in (14) clearly leads to (16), completing the proof. \qed

Remark 2.6 (Non-irreducible systems). If system (14) is governed by an interaction potential satisfying (W1)-(W5), but is not irreducible, there exists an I-vector partition ($I \in \mathbb{N}$) of $n \in \mathbb{N}$ into $n_1 + n_2 + \ldots + n_I = n$ such that (14) decomposes into I independent irreducible subsystems having the same structure as (14), but with $n$ replaced by $n_1, \ldots, n_I$, respectively. The modulus of geodesic convexity of the interaction energy $W$ can now be computed as the
minimum of the respective convexity moduli of each subsystem. If \( n_k > 1 \) for some \( k \in \{1, \ldots, I\} \), formula (14) applies; if \( n_k = 1 \), McCann’s criterion [39] applies (and yields convexity modulus \( m_{ikp} \) for the respective \( m, \kappa, p \) of the scalar subsystem \( k \) in our framework).

**Remark 2.7** (Necessary condition for \( \lambda_0 > 0 \)). If \( \lambda_0 > 0 \) in (14), then for all \( i \in \{1, \ldots, n\} \):

\[
\sum_{j=1}^{n} \kappa_{ij} p_j > 0.
\]

This condition is not sufficient (cf. Example 2.8 below).

**Proof.** Fix \( i \in \{1, \ldots, n\} \). The following holds:

\[
m_i \sum_j \kappa_{ij} p_j = p_i (m_i \kappa_{ii} - \eta_i) + p_i \eta_i + \frac{1}{2} \sum_{j \neq i} p_j \kappa_{ij} m_i + \frac{1}{2} \sum_{j \neq i} p_j \kappa_{ij} m_j m_i / m_j,
\]

where we used the symmetry of \( \kappa \) (W1). Now, we estimate using the definition of \( \eta_i, \eta_j \):

\[
m_i \sum_j \kappa_{ij} p_j \geq p_i \min(0, m_i \kappa_{ii} - \eta_i) + \frac{1}{2} p_i \eta_i + \frac{1}{2} \sum_{j \neq i} p_j \eta_j + \frac{1}{2} \sum_{j \neq i} p_j \eta_j m_i / m_j \geq \lambda_0 > 0.
\]

We conclude this section with several examples for our convexity condition (14).

**Example 2.8.** Set, for simplicity, \( m_j = 1 \) and \( p_j = 1 \) for all \( j \in \{1, \ldots, n\} \). Then, (14) simplifies to

\[
\lambda_0 = \min_{i \in \{1, \ldots, n\}} \left[ \min(0, \kappa_{ii} - \eta_i) + \frac{1}{2} \sum_{j=1}^{n} \eta_j + \frac{n}{2} \eta_i \right] \quad (n > 1)
\]

and \( \eta_i = \min_{j \neq i} \kappa_{ij} \). In the even more specific setting of two species \( n = 2 \), one has \( \eta_1 = \kappa_{12} = \eta_2 \) and

\[
\lambda_0 = \min \{ \kappa_{11}, \kappa_{12}, \kappa_{22} \} + \kappa_{12} \quad (n = 2).
\]

Given the matrix \( \kappa \) from (W5), we obtain the following moduli for geodesic convexity \( \lambda_0 \), respectively:

\[
\kappa = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \implies \lambda_0 = 2, \quad \kappa = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \implies \lambda_0 = 1,
\]

\[
\kappa = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \implies \lambda_0 = 0, \quad \kappa = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \implies \lambda_0 = -2,
\]

\[
\kappa = \begin{pmatrix} -1 & -1 & a \\ -1 & -1 & b \\ a & b & -1 \end{pmatrix} \implies \lambda_0 = \frac{b - 5}{2} \quad (a \geq b \geq -1).
\]

2.2. Existence and uniqueness of gradient flow solutions. With the results of Lemma 2.1 and Theorem 2.5 at hand, the following statement follows thanks to [2, Chapter 11]:

**Theorem 2.9** (Existence and uniqueness). Consider (1) endowed with an initial datum \( \mu^0 \in \mathcal{P} \). Then, there exists a gradient flow solution \( \mu \in AC^2_{\text{loc}}([0, \infty); (\mathcal{P}, W_M)) \) to this initial-value problem: System (1) holds in the sense of distributions and one has \( \mu(0) = \mu^0 \). Moreover, with \( \lambda_0 \) from (14), the evolution variational estimate holds for almost every \( t > 0 \) and all \( \nu \in \mathcal{P} \):

\[
\frac{1}{2} \frac{d^+}{dt} W_M(\mu(t), \nu)^2 + \frac{\lambda_0}{2} W_M(\mu(t), \nu)^2 \leq W(\nu) - W(\mu(t)).
\]

Given another initial datum \( \nu^0 \in \mathcal{P} \) and the respective gradient flow solution \( \nu \in AC^2_{\text{loc}}([0, \infty); (\mathcal{P}, W_M)) \), the following contraction estimate holds for all \( t \geq 0 \):

\[
W_M(\mu(t), \nu(t)) \leq e^{-\lambda_0 t} W_M(\mu^0, \nu^0),
\]

which implies in particular the uniqueness of solutions.
Corollary 2.10 (The uniformly convex case). If \( \mu \) yields \( \lambda_0 > 0 \), the measure
\[
\mu^\infty := (p_1, \ldots, p_n)^T \delta_{x^\infty}, \quad \text{with} \quad x^\infty := E \left[ \sum_{j=1}^n \frac{p_j}{m_j} \right]^{-1} \in \mathbb{R}^d,
\]
is the unique minimizer of \( W \) and the unique stationary state of \( \mu \) on \( \mathcal{P} \). It is globally asymptotically stable: The solution from Theorem 2.9 converges exponentially fast in \((\mathcal{P}, W_M)\) at rate \( \lambda_0 \) to \( \mu^\infty \).

As for scalar equations of the form \( \mu \), system \( \mu \) can be viewed as a continuum limit of a multi-particle system. To this end, we introduce the concept of particle solutions as a conclusion to this section.

Remark 2.11 (Particle solutions). Assume that the initial datum is discrete, i.e. each component \( \mu_i^0 \) is a finite linear combination of Dirac measures:
\[
\mu_i^0 = \sum_{k=1}^{N_i} p_{i,k}^0 \delta_{x_{i,k}^0} \quad (i = 1, \ldots, n).
\]
There, the \( N_i \in \mathbb{N} \) particles of species \( i \) have mass \( p_{i,k}^0 > 0 \) and are at initial position \( x_{i,k}^0 \in \mathbb{R}^d \), for \( k = 1, \ldots, N_i \), respectively. Let \( N := \sum_{i=1}^n N_i \) and let a family \( x = (x_i^k) \) \((k = 1, \ldots, N_i; i = 1, \ldots, n)\) of \( L^2 \)-absolutely continuous curves \( x_i^k : [0, \infty) \to \mathbb{R}^d \) be given, such that the following initial-value problem for a system of \( N \) ordinary differential equations on \( \mathbb{R}^d \) is globally solved:
\[
\frac{dx_i^k(t)}{dt} = -m_i \sum_{j=1}^n \sum_{l=1}^{N_j} p_{i,l}^k \nabla W_{ij}(x_i^k(t) - x_j^l(t)), \quad x_i^k(0) = x_{i,k}^0, \quad (k = 1, \ldots, N_i; i = 1, \ldots, n). \tag{18}
\]
Then it is easy to verify that the particle solution
\[
\mu_i(t) = \sum_{k=1}^{N_i} p_{i,k}^t \delta_{x_{i,k}^t} \quad (i = 1, \ldots, n) \tag{19}
\]
is the unique gradient flow solution to system \( \mu \) with initial datum \( \mu_i^0 \) given above. However, it is a non-trivial question if such \( x \) exist, since (W1)-(W5) do not imply global Lipschitz-continuity of the r.h.s. in \( \mu \). Nevertheless, \( \mu \) admits \( L^2 \)-absolutely continuous solutions since this system possesses an underlying (discrete) gradient flow structure: Define the finite-dimensional space
\[
\mathcal{P}_d := \left\{ x \in \prod_{i=1}^n \prod_{k=1}^{N_i} \mathbb{R}^d \approx \mathbb{R}^{Nd} : p_i = \sum_{k=1}^{N_i} p_{i,k}^0 (i = 1, \ldots, n); E = \sum_{i=1}^n \frac{1}{m_i} \sum_{k=1}^{N_i} p_{i,k}^0 x_{i,k}^0 \right\},
\]
endowed with the (weighted Euclidean) distance
\[
d(x, y) = \left[ \sum_{i=1}^n \frac{1}{m_i} \sum_{k=1}^{N_i} \sum_{j=1}^{N_j} p_{i,k}^0 x_{i,k}^0 - y_{i,j}^0 \right]^{1/2},
\]
and define the discrete interaction energy \( W_\mu \) on \( \mathcal{P}_d \) as
\[
W_\mu(x) := \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^{N_i} \sum_{l=1}^{N_j} p_{i,k}^0 p_{j,l}^0 W_{ij}(x_{i,k}^0 - x_{j,l}^0).
\]
Applying the same method of proof as for Theorem 2.9 mutatis mutandis for the discrete framework, one can show that \( W_\mu \) is \( \lambda_0 \)-geodesically convex on \((\mathcal{P}_d, d)\) with the same modulus of convexity \( \lambda_0 \) as in the continuous case \( \mu \).

We can again invoke \( \mu \) to obtain existence and uniqueness of a solution curve \( x \in AC_{\text{loc}}(0, \infty); (\mathcal{P}_d, d) \) to the particle system \( \mu \). Conversely, thanks to the uniqueness of solutions to both \( \mu \) and \( \mu \), a gradient flow solution \( \mu \) to \( \mu \) can be represented by a solution \( x \) to \( \mu \).

3. Qualitative properties in one spatial dimension

In this section, we analyse the qualitative behaviour of the solution from Theorem 2.9 in the general scenario, i.e. the criterion for geodesic convexity may only yield \( \lambda_0 \leq 0 \). In this case, the contraction estimate \( \mu \) does not allow for conclusions on the long-time behaviour of the solution.

From now on, consider \( \mu \) in one spatial dimension \( d = 1 \); and let \( \mu \) be the solution to \( \mu \) with initial datum \( \mu^0 \in \mathcal{P} \), as given in Theorem 2.9. First, we rewrite system \( \mu \) in terms of the inverse distribution functions \( u = (u_1, \ldots, u_n) \); recall their definition from \( 6 \) & \( 7 \).
For all $z \in [0, 1)$, one has $z = F_i(t, u_i(t, z))$. Differentiation w.r.t. $t$ yields

$$0 = \partial_t F_i(t, u_i(t, z)) + \partial_x F_i(t, u_i(t, z)) \partial_t u_i(t, z)$$

$$= \int_{-\infty}^{u_i(t, z)} \frac{1}{p_i} \partial_y \left( \sum_{j=1}^{n} m_j \mu_j(t)(y) (W_{ij}'(y)) \mu_j(y) \right) dy + \frac{1}{p_i} \mu_i(t, u_i(t, z)) \partial_t u_i(t, z)$$

$$= \frac{n}{p_i} \sum_{j=1}^{n} \mu_i(t, u_i(t, z)) \int_{\mathbb{R}} W_{ij}'(u_i(t, z) - y) \mu_j(y) dy + \frac{1}{p_i} \mu_i(t, u_i(t, z)) \partial_t u_i(t, z).$$

Rearranging yields – with the help of (W3) and the transformation $\xi := F_j(t, y)$:

$$\partial_t u_i(t, z) = m_i \sum_{j=1}^{n} p_j \int_{0}^{1} W_{ij}'(u_j(t, \xi) - u_i(t, z)) d\xi \quad (i = 1, \ldots, n).$$

(20)

It is a consequence of Theorem 2.9 that given a gradient flow solution $\mu$ to (11), the corresponding curve of pseudo-inverse distribution functions $u \in AC_{sc}^2([0, \infty); L^2([0, 1]; \mathbb{R}^n))$ solves (20). Furthermore, since $\mu(t) \in \mathcal{P}$ for all $t \geq 0$, $u_i(t, \cdot)$ is a non-decreasing càdlàg function on $(0, 1)$. Conservation of the weighted center of mass $E$ over time is reflected in terms of $u$ by the identity

$$E = \sum_{j=1}^{n} \frac{p_j}{m_j} \int_{0}^{1} u_j(t, z) dz \quad \forall t \geq 0.$$ (21)

The concept of inverse distribution functions substantially simplifies the analysis of solutions to (11) since there does not appear any spatial derivative on the right-hand side of (20) anymore. However, this approach can be employed in one spatial dimension $d = 1$ only.

3.1. The purely quadratic case. This paragraph is devoted to another specific example for system (11), namely the case where all entries in $W$ are purely quadratic functions, i.e. $W_{ij}(z) = \frac{1}{2} \kappa_{ij} z^2$. There, it is possible to solve system (20) analytically: We obtain by elementary calculations – involving the usage of (21) – that

$$\partial_t u_i(t, z) = -m_i \left( \sum_{j} \kappa_{ij} p_j \right) u_i(t, z) + m_i^2 \kappa_{ii} E + \sum_{j \neq i} m_i \left( \kappa_{ij} - \kappa_{ii} \frac{m_i}{m_j} \right) \int_{0}^{1} p_j u_j(t, \xi) d\xi.$$ (22)

Define $v_i(t) := \int_{0}^{1} u_i(t, z) dz$ and integrate (22) w.r.t. $z \in (0, 1)$:

$$\frac{d}{dt} v_i(t) = -m_i \left( \sum_{j} \kappa_{ij} p_j \right) v_i(t) + m_i^2 \kappa_{ii} E + \sum_{j \neq i} m_i p_j \left( \kappa_{ij} - \kappa_{ii} \frac{m_i}{m_j} \right) v_j(t).$$ (23)

Consequently, with the definitions

$$A_{ii} := -m_i \sum_{j=1}^{n} \kappa_{ij} p_j, \quad A_{ij} := m_i p_j \left( \kappa_{ij} - \kappa_{ii} \frac{m_i}{m_j} \right) \quad (i \neq j), \quad b_i := m_i^2 \kappa_{ii},$$

for $i, j \in \{1, \ldots, n\}$, the following holds for $v = (v_1, \ldots, v_n)$:

$$\frac{d}{dt} v(t) = A v(t) + E b, \quad \partial_t (u(t, z) - v(t)) = \text{diag}(A)(u(t, z) - v(t)).$$ (24)

There, $\text{diag}(A)$ is meant to be the diagonal matrix with the same diagonal as $A$. The linear systems in (21) can easily be solved:

$$u(t, z) = \exp(\text{diag}(A)t) \left( u(0, z) - \int_{0}^{1} u(0, \xi) d\xi \right) + \exp(At) \int_{0}^{1} u(0, \xi) d\xi$$

$$+ \left( \sum_{j=1}^{n} \int_{0}^{1} \frac{p_j}{m_j} u_j(0, \xi) d\xi \right) \int_{0}^{t} \exp(A(t-s)) b ds, \quad \forall z \in [0, 1).$$

We expect exponential convergence of $u$ to the spatially constant equilibrium $(x^\infty, \ldots, x^\infty) \in \mathbb{R}^n$ as $t \to \infty$ if both $A$ and $\text{diag}(A)$ possess eigenvalues with negative real parts only. Clearly, our result on geodesic convexity (cf. Section 2) shows that if $\lambda_0 > 0$ in (14), these conditions are bound to hold. The necessary condition from Remark 2.7 implies that $\text{diag}(A)$ is negative definite.
The specific case of two species \((n = 2)\) deserves a closer look. Thanks to the invariant \((21)\), the two equations for \(u_1\) and \(u_2\) in \((20)\) can be separated completely:

\[
\begin{align*}
\partial_t u_1(t, z) &= -m_1(\kappa_1 p_1 + \kappa_2 p_2) u_1(t, z) + p_1(m_1 \kappa_1 - m_2 \kappa_1) v_1(t) + m_1 m_2 \kappa_1 E, \\
\partial_t u_2(t, z) &= -m_2(\kappa_2 p_1 + \kappa_2 p_2) u_2(t, z) + p_2(m_2 \kappa_2 - m_1 \kappa_1) v_2(t) + m_1 m_2 \kappa_2 E,
\end{align*}
\]

\[
\frac{d}{dt} v_i(t) = -v_i(t)(m_1 p_2 + m_2 p_1) \kappa_i + m_1 m_2 \kappa_i E \quad \text{(for both } i \in \{1, 2\}).
\]

Comparing with our criterion for geodesic convexity \((14)\) shows that the solution to the system above – for generic initial data – is unbounded in time if \(\lambda_0 < 0\) in \((13)\). For example, if \(m_1 = 1 = m_2\) and \(p_1 = 1 = p_2\), one easily sees that \(\kappa_1 > 0, \kappa_1 + \kappa_2 \geq 0\) and \(\kappa_2 + \kappa_2 \geq 0\) are necessary for a bounded solution. These conditions are equivalent to \(\lambda_0 \geq 0\) in the case of an irreducible system of two components.

### 3.2. Speed of propagation and confinement

In this section, we investigate the rate of propagation of the solution to \((11)\) in space over time, given an initial datum with compact support. We first obtain – for arbitrary potentials satisfying \((W1)-(W5)\) – boundedness of the support of \(\mu(t)\) for fixed time \(t > 0\), and second – under more restrictive requirements on the potential \(W\) – \(t\)-uniform boundedness of \(\text{supp } \mu(t)\).

**Proposition 3.1** (Finite speed of propagation). Let an initial datum \(\mu^0\) with compact support and \(T > 0\) be given. Then, there exists a constant \(K = K(T, \mu^0) > 0\) such that for all \(t \in [0, T]\),

\[
\text{supp } \mu(t) \subset [-K, K].
\]

**Proof.** For \(t \in [0, T]\) and \(i \in \{1, \ldots, n\}\), denote

\[
u_i(t, 1^-) := \lim_{\varepsilon \searrow 0} u_i(t, 1 - \varepsilon) \in \mathbb{R} \cup \{+\infty\} \quad \text{and} \quad \nu_i(t, 0^+) := \lim_{\varepsilon \nearrow 0} u_i(t, \varepsilon) \in \mathbb{R} \cup \{-\infty\}.
\]

The assertion will follow from finiteness of those limits. Let \(\varepsilon > 0\). Then,

\[
\partial_t (u_i(t, 1 - \varepsilon)^2) \leq 2|u_i(t, 1 - \varepsilon)| \sum_{j=1}^n m_i p_j \int_0^1 |W_{ij}(u_j(t, \xi) - u_i(t, 1 - \varepsilon))| d\xi.
\]

Lemma \([22]\) Hölder’s and Young’s inequality eventually lead to

\[
2|u_i(t, 1 - \varepsilon)| \sum_{j=1}^n m_i p_j \int_0^1 |W_{ij}(u_j(t, \xi) - u_i(t, 1 - \varepsilon))| d\xi \leq 2|u_i(t, 1 - \varepsilon)| \sum_{j=1}^n c_{ij} m_i p_j \left( \int_0^1 |u_j(t, \xi)| d\xi + |u_i(t, 1 - \varepsilon)| + 1 \right)
\]

\[
\leq 2 \left[ \sum_{j=1}^n c_{ij} m_i p_j + 1 \right] u_i(t, 1 - \varepsilon)^2 + \left( \sum_{j=1}^n c_{ij} m_i p_j \right)^2 + 2 \max_j \left( c_{ij}^2 m_i^2 m_j p_j \right) \sum_{j=1}^n \int_0^1 \frac{p_j}{m_j} u_j(t, \xi)^2 d\xi.
\]

With the transformation \(\xi := F_j(t, x)\), we observe that the sum in the last term on the r.h.s. of \((24)\) can be expressed in terms of the second moments \(\mathcal{L}_i(\mu_j(t))\) and of \(W_M(\mu(t), \delta_0 \varepsilon)\), where \(\varepsilon = (1, 1, \ldots, 1)^T \in \mathbb{R}^n\):

\[
\sum_{j=1}^n \int_0^1 \frac{p_j}{m_j} u_j(t, \xi)^2 d\xi = \sum_{j=1}^n \frac{1}{m_j} \mathcal{L}_2(\mu_j(t)) = W_M^2(\mu(t), \delta_0 \varepsilon).
\]

Since \(\mu \in AC^2([0, T]; (\mathcal{P}, W_M))\), there exists \(\varphi \in L^2([0, T])\) such that

\[
W_M(\mu(t), \mu^0) \leq \int_0^t \varphi(s) ds, \quad \forall \ t \in [0, T].
\]

We obtain

\[
\sum_{j=1}^n \int_0^1 \frac{p_j}{m_j} u_j(t, \xi)^2 d\xi \leq 2W_M^2(\mu(t), \mu^0) + 2W_M^2(\mu^0, \delta_0 \varepsilon) \leq 2 \left( \int_0^t \varphi(s) ds \right)^2 + 2W_M^2(\mu^0, \delta_0 \varepsilon),
\]

\[
\leq 2T \varphi^2_{L^2([0, T])} + 2 \sum_{j=1}^n \frac{1}{m_j} \mathcal{L}_2(\mu_j^0),
\]

which is a constant depending on \(T\) and \(\mu^0\). Inserting into \((25)\), we observe

\[
\partial_t (u_i(t, 1 - \varepsilon)^2) \leq Au_i(t, 1 - \varepsilon)^2 + B(T, \mu^0),
\]
for suitable constants $A, B > 0$. We apply Gronwall’s lemma, let $\varepsilon \searrow 0$ and use that – since $\mu^0$ has compact support by assumption – the limit $u_i(0, 1^-)$ exists in $\mathbb{R}$:

$$u_i(t, 1^-)^2 \leq \left[ u_i(0, 1^-)^2 + \frac{B}{A} \right] \exp(AT) \quad \forall t \in [0, T].$$

Thus, $u_i(t, 1^-)$ is a finite value, at each $t \in [0, T]$. Along the same lines, it can be shown that

$$u_i(t, 0^+)^2 \leq \left[ u_i(0, 0^+)^2 + \frac{B}{A} \right] \exp(AT) \quad \forall t \in [0, T].$$

Since by construction $\text{supp} \mu_i(t) \subset [u_i(t, 0^+), u_i(t, 1^-)]$, the assertion is proven. □

The statement of Proposition 3.1 shows that at fixed $t \geq 0$, the limits

$$u_i(t, 1^-) := \lim_{\varepsilon \searrow 0} u_i(t, 1 - \varepsilon) \quad \text{and} \quad u_i(t, 0^+) := \lim_{\varepsilon \searrow 0} u_i(t, \varepsilon)$$

exist (in $\mathbb{R}$). In order to prove uniform confinement of the solution, we show uniform boundedness of those limits. We first introduce a requirement on the potential by the following

**Definition 3.2 (Confining potentials).** We call an interaction potential $W$ satisfying (W1)-(W5) confining if there exists $R > 0$ such that:

(i) System (1) is irreducible at large distance, that is the graph $G' = (V_{G'}, E_{G'})$ with nodes $V_{G'} = \{1, \ldots, n\}$ and edges $E_{G'} = \{(i, j) \in V_{G'} \times V_{G'} : W_{ij} \neq 0 \text{ on } (R, \infty)\}$ is connected.

(ii) There exists a matrix $C \in \mathbb{R}^{n \times n}$ such that for each $i, j \in \{1, \ldots, n\}$, the map $W_{ij}$ is $C_{ij}$-(semi-)convex on the interval $(R, \infty)$ and the following holds:

If $n = 1$, then $C > 0$. If $n > 1$, with $\hat{\eta}_i := \min_{j \neq i} C_{ij} p_j$ for all $i \in \{1, \ldots, n\}$,

$$\hat{\lambda}_0 := \min_{i \in \{1, \ldots, n\}} \left[ p_i \min(0, m_i C_{ii} - \hat{\eta}_i) + \frac{1}{2} \sum_{j=1}^{n} p_j \left( \hat{\eta}_j + \hat{\eta}_i \frac{m_i}{m_j} \right) \right] > 0. \quad (26)$$

**Remark 3.3 (Geodesic convexity and confinement).** In the scalar case $n = 1$, uniform geodesic convexity of the interaction energy $\mathcal{W}$ is equivalent to $\kappa$-convexity of $W$ with $\kappa > 0$ [29]. So, the potential is confining. Also for genuine systems, if $\lambda_0 > 0$ in (14), the definition $\hat{\lambda}_0 = \lambda_0 > 0$. Hence, our criterion for uniform geodesic convexity of $\mathcal{W}$ necessarily implies that $W$ is a confining potential. Naturally, if the system is not irreducible at large distance, the independent irreducible subsystems should be considered separately.

**Theorem 3.4 (Confinement).** Assume that $W$ is confining and let $\mu^0$ have compact support. Then, there exists a constant $K = K(\mu^0) > 0$ independent of $t$ such that for all $t \geq 0$:

$$\text{supp} \mu(t) \subset [-K, K]. \quad (27)$$

**Proof.** We prove the assertion in the case of genuine systems $n > 1$.

Step 1: $L^2$ estimate. Let $\varepsilon > 0$ be sufficiently small such that replacing $C_{ij}$ by $C_{ij}' := C_{ij} - \varepsilon$ in (26) still yields a number $\hat{\lambda}_0' > 0$, possibly with $\hat{\lambda}_0' < \hat{\lambda}_0$. From the $C_{ij}$-convexity of $W_{ij}$ on $(R, \infty)$ and with the help of Young’s inequality, we get for all $z > R$:

$$W_{ij}(z) \geq W_{ij}(R) + W_{ij}'(R)(z - R) + \frac{1}{2} C_{ij}(z - R)^2 \geq \frac{1}{2} C_{ij}' z^2 - D_{ij},$$

for appropriate constants $D_{ij} > 0$. Thanks to (W2)&(W3), enlarging the constants, there exists $D > 0$ such that for all $i, j \in \{1, \ldots, n\}$ and for all $z \in \mathbb{R}$:

$$W_{ij}(z) \geq \frac{1}{2} C_{ij}' z^2 - D. \quad (28)$$

We now use boundedness of the energy $\mathcal{W}$ along the (gradient flow) solution to obtain with (28) for all $t \geq 0$:

$$2W(\mu^0) \geq 2W(\mu(t)) \geq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\mathbb{R}} \int_{\mathbb{R}} C_{ij}'(x - y)^2 \text{d}\mu_j(x) \text{d}\mu_i(y) - D \left( \sum_{j=1}^{n} p_j \right)^2.$$
The first term on the r.h.s. has precisely the same structure as the l.h.s. in (10), for \( t_i = t_j = 0 \). Arguing exactly as in the proof of Theorem 2.5 we obtain

\[
2W(\mu^0) \geq \tilde{\lambda}_0 \sum_{j=1}^{n} \frac{1}{m_j} \ell_2(\mu_j(t)) - D \left( \sum_{j=1}^{n} p_j \right)^2.
\]

All in all, we have proven uniform boundedness of the second moments: There exists \( C_2 > 0 \) such that for all \( t \geq 0 \) and all \( i \in \{1, \ldots, n\} \), one has \( \ell_2(\mu_i(t)) \leq C_2 \).

Step 2: \( L^\infty \) estimate.

We first prove an upper bound. For each \( t \geq 0 \), we consider those indices \( i \in \{1, \ldots, n\} \), where

\[
u_i(t, 1^-) \geq \max_{j \in \{1, \ldots, n\}} u_j(t, 1^-) - R.
\]

That is, for all \( \xi \in [0, 1) \) and all \( j \in \{1, \ldots, n\} \): \( u_j(t, 1^-) \geq u_j(t, \xi) - R \). We thus have a partition of \([0, 1)\) into two sets \( A_1 \) and \( A_2 \), where

\[
A_1 := \{ \xi \in [0, 1) : u_j(t, \xi) - u_i(t, 1^-) < -R \}, \quad A_2 := \{ \xi \in [0, 1) : |u_j(t, \xi) - u_i(t, 1^-)| \leq R \}.
\]

Since \( W'_{ij} \) is continuous thanks to (W2), it is bounded on the interval \([-R, R] \). The \( C_{ij} \)-convexity of \( W \) on \((-\infty, -R) \) yields using (W3):

\[
W'_{ij}(z) \leq C_{ij} z + C_{ij} R - W'_{ij}(R), \quad \forall z < -R.
\]

Hence, we obtain

\[
\partial_t \nu_i(t, 1^-) \leq \sum_{j=1}^{n} \int_{A_1} m_i C_{ij} p_j (u_j(t, \xi) - u_i(t, 1^-)) \, d\xi + C_0,
\]

for some constant \( C_0 > 0 \). Then, with the help of Hölder’s inequality,

\[
\sum_{j=1}^{n} \int_{A_1} m_i C_{ij} p_j (u_j(t, \xi) - u_i(t, 1^-)) \, d\xi
\]

\[
\leq m_i \sum_{j=1}^{n} C_{ij} p_j \left( \int_0^1 (u_j(t, \xi) - u_i(t, 1^-)) \, d\xi - \int_{A_2} (u_j(t, \xi) - u_i(t, 1^-)) \, d\xi \right)
\]

\[
\leq -m_i \sum_{j=1}^{n} C_{ij} p_j u_i(t, 1^-) + C' \sum_{j=1}^{n} \int_0^1 p_j u_j(t, \xi)^2 \, d\xi + C_1 = -m_i \sum_{j=1}^{n} C_{ij} p_j u_i(t, 1^-) + C' \sum_{j=1}^{n} \ell_2(\mu_j(t)) + C_1,
\]

for some constants \( C', C_1 > 0 \). We now employ step 1 and observe that, as in Remark 2.7, we have \( \sum_{j=1}^{n} C_{ij} p_j \geq \tilde{\lambda}_0 > 0 \):

\[
\partial_t \nu_i(t, 1^-) \leq -m_i \tilde{\lambda}_0 u_i(t, 1^-) + C'',
\]

for \( C'' > 0 \). Gronwall’s lemma yields – thanks to \( u_i(0, 1^-) < \infty \) – the existence of a constant \( K > 0 \) such that \( \max_{j \in \{1, \ldots, n\}} u_j(t, 1^-) \leq K \) for all \( t \geq 0 \).

In analogy, we now consider those \( i \in \{1, \ldots, n\} \) such that

\[
u_i(t, 0^+) \leq \min_{j \in \{1, \ldots, n\}} u_j(t, 0^+) + R,
\]

yielding a partition \([0, 1) = B_1 \cup B_2 \) with

\[
B_1 := \{ \xi \in [0, 1) : u_j(t, \xi) - u_i(t, 0^+) > R \}, \quad B_2 := \{ \xi \in [0, 1) : |u_j(t, \xi) - u_i(t, 0^+)| \leq R \}.
\]

Similar to step 1, using the symmetry property (W3), we get

\[
-\partial_t \nu_i(t, 0^+) \leq -m_i \sum_{j=1}^{n} C_{ij} p_j (-u_i(t, 0^+)) + C' \sum_{j=1}^{n} \ell_2(\mu_j(t)) + C_1 \leq -m_i \tilde{\lambda}_0 (-u_i(t, 0^+)) + C'',
\]

allowing to proceed as before.

Putting the bounds together finishes the proof: \( \sup_{t \geq 0} \max_{j \in \{1, \ldots, n\}} \|u_j(t, \cdot)\|_{L^\infty([0, 1])} \leq K. \)

We thus know, given a confining potential, that the solution lives on a fixed compact interval. It is now a natural question to ask if, for absolutely continuous initial conditions, partial or total collapse of the support can occur in finite time. This question is addressed in the following
Proposition 3.5 (Exclusion of finite-time blow-up). Let \( i \in \{1, \ldots, n\} \) be fixed, but arbitrary. Assume that for all \( j \in \{1, \ldots, n\} \) the maps \( W^i_{i,j} \) are Lipschitz-continuous. Suppose moreover that \( \text{supp} \mu_i^0 \) is a (possibly unbounded) interval and \( \mu_i^0 \) is absolutely continuous w.r.t. the Lebesgue measure. Assume that its Lebesgue density is continuous on the interior of \( \text{supp} \mu_i^0 \) and globally bounded. Then, \( \mu_i(t) \) is absolutely continuous for all \( t \geq 0 \).

Proof. Our method of proof is an adaptation of the proof of [14, Thm. 2.9] to the situation at hand. We show that for all \( t \geq 0 \), there exists \( \gamma(t) > 0 \) such that for all \( z \in [0, 1) \) and all \( h > 0 \) with \( z + h < 1 \):

\[
\frac{1}{h} (u_i(t, z + h) - u_i(t, z)) \geq \gamma(t) > 0.
\]

That is, \( u_i(t, \cdot) \) is strictly increasing at each \( t \geq 0 \). The assumptions on the initial datum above ensure that \( \gamma(t) > 0 \) is true at \( t = 0 \) with some \( \gamma(0) > 0 \). If \( (29) \) holds at a given \( t_0 \), the cumulative distribution function \( F_i(t_0, \cdot) \) is Lipschitz-continuous, which implies absolute continuity of \( \mu_i(t_0) \).

From \( (20) \), we get

\[
\partial_t (u_i(t, z + h) - u_i(t, z)) = m_i \sum_{j=1}^{n} p_j \int_0^1 \left[ W^i_{ij}(u_j(t, \xi) - u_i(t, z + h)) - W^i_{ij}(u_j(t, \xi) - u_i(t, z)) \right] \, d\xi.
\]

Denote by \( L_{ij} > 0 \) the Lipschitz constant of \( W^i_{ij} \). From the monotonicity \( u_i(t, z + h) - u_i(t, z) \geq 0 \), it follows that

\[
\partial_t (u_i(t, z + h) - u_i(t, z)) \geq -m_i \sum_{j=1}^{n} L_{ij} p_j (u_i(t, z + h) - u_i(t, z)).
\]

We subsequently obtain for \( \tilde{C}_i := m_i \sum_{j=1}^{n} L_{ij} p_j \) that \( \partial_t [(u_i(t, z + h) - u_i(t, z)) e^{\tilde{C}_i t}] \geq 0 \), and hence

\[
\frac{1}{h} (u_i(t, z + h) - u_i(t, z)) \geq \frac{1}{h} e^{-\tilde{C}_i t} (u_i(0, z + h) - u_i(0, z)) \geq e^{-\tilde{C}_i t} \gamma(0) > 0.
\]

Letting \( \gamma(t) := e^{-\tilde{C}_i t} \gamma(0) \), \( (29) \) follows. \( \square \)

Naturally, the above result does not extend to \( t \to \infty \) since e.g. in the uniformly geodesically convex case, the solution collapses to a Dirac measure in the large-time limit.

3.3 Long-time behaviour. We now analyse the long-time behaviour of the solution to \( (1) \) in the non-uniformly convex case.

Theorem 3.6 (Long-time behaviour). Assume that the solution \( \mu \) to \( (1) \) is uniformly confined, i.e. there exists \( K > 0 \) such that \( \text{supp} \mu_i(t) \subset [-K, K] \) holds for all \( t \geq 0 \) and all \( i \in \{1, \ldots, n\} \) as in \( (27) \). Moreover, suppose that the maps \( W^i_{ij} \) are Lipschitz-continuous on the interval \([\ell, 2K] \) for all \( i, j \in \{1, \ldots, n\} \). Set, for \( t \geq 0 \), \( W^i := W(\mu(t)) \). The following holds:

(a) There exists \( W^\infty \in \mathbb{R} \) such that

\[
\lim_{t \to \infty} W^i(t) = W^\infty,
\]

and

\[
\lim_{t \to \infty} \left( \frac{d}{dt} W^i(t) \right) = 0.
\]

(b) For each sequence \( (t_k)_{k \in \mathbb{N}} \) in \((0, \infty) \) with \( t_k \to \infty \) as \( k \to \infty \), there exists a subsequence \( (t_{k_l})_{l \in \mathbb{N}} \) and a steady state \( \pi_i \in \mathcal{P} \) of \( (1) \) such that for all \( i \in \{1, \ldots, n\} \):

\[
\lim_{l \to \infty} W_1(\mu_i(t_{k_l}), \pi_i) = 0.
\]

Thus, the \( \omega \)-limit set of the dynamical system associated to \( (1) \) can only contain steady states of \( (1) \).

Proof. We proceed similarly to the proof of [43, Prop. 1] and observe that along the solution \( \mu \), the dissipation of \( W \) reads

\[
\frac{d}{dt} W^i = -\sum_{i=1}^{n} m_i \left( \sum_{j=1}^{n} W^i_{ij} * \mu_j(t) \right)^2 \, d\mu_i(t)
\]

\[
= -\sum_{i=1}^{n} m_i p_i \int_0^1 \left( \sum_{j=1}^{n} p_j \int_0^1 W^i_{ij}(u_i(t, z) - u_j(t, \xi)) \, d\xi \right)^2 \, dz.
\]
which is non-positive. By \([27]\), all \(u_i\) are bounded in time and space by the constant \(K\). Since \(W'_{ij}\) is Lipschitz-continuous, it is differentiable almost everywhere on \([-2K,2K]\). So, another differentiation of the dissipation w.r.t. \(t\) shows

\[
\frac{d^2}{dt^2} W^t = -2 \sum_{i=1}^n m_i p_i \int_0^1 \left( \sum_{j=1}^n \int_0^1 p_j W'_{ij}(u_i(t,z) - u_j(t,\zeta)) \,d\zeta \right) \,dz.
\]

By elementary estimates, using in particular that \(|C_i|\) for some \(C\)

Elementary calculations – using e.g. Lemma 2.2 – show

for part (b), let a sequence of time points \(\tau_k\) nondecreasing functions is uniformly bounded in \(BV([0,1])\). Thus, there exist a subsequence \(\tau_k\) such that \(u_i(t_k,\cdot)\) converges to \(\overline{u}_i\) in \(L^1([0,1])\) and almost everywhere on \([0,1]\), as \(l \to \infty\) (for details, see e.g. [1]). The corresponding measure \(\overline{\mu}\) belongs to \(\mathcal{P}\) thanks to the dominated convergence theorem. It remains to show that \(\overline{\mu}\) is a steady state of system \([1]\). Define

\[
\omega := - \sum_{i=1}^n m_i p_i \int_0^1 \left( \sum_{j=1}^n \int_0^1 W'_{ij}(\overline{u}_i(z) - \overline{u}_j(\xi)) \,d\xi \right) \,dz.
\]

Elementary calculations – using e.g. Lemma 2.2, show

for a suitable constant \(C_0 > 0\). The Lipschitz-continuity of the \(W'_{ij}\) on \([-2K,2K]\), the triangle inequality and then imply

Hence, because of \([34]\), \(\omega = 0\). Specifically, this means that for each \(i \in \{1,\ldots,n\}\) and almost every \(x \in \text{supp}\, \overline{\mu}_i\), the following holds:

So, \(\overline{\mu}\) is a solution to \([1]\) and the proof is complete. \(\Box\)
Remark 3.7. The result of Theorem 3.8 does neither yield uniqueness of steady states of (11) nor convergence of the entire curve $\mu$ to some specific object as $t \to \infty$. If there only exists the trivial steady state $\mu^\infty$ from Corollary 2.10 in the set of those elements from $\mathcal{P}$ with support contained in $[-K, K]$, then Theorem 3.8 implies
\[
\lim_{t \to \infty} W_1(\mu_i(t), \mu_i^\infty) = 0 \quad \forall i \in \{1, \ldots, n\},
\]
without obtaining any specific rate of convergence.

If the potential is not confining, convergence may not occur. However, we might observe a $\delta$-separation phenomenon: The support of each component $\mu_i$ collapses to a single (but not necessarily fixed) point as $t \to \infty$.

Proposition 3.8 ($\delta$-separation). Let $i \in \{1, \ldots, n\}$ be fixed, but arbitrary. Assume that the support of $\mu_i^0$ is compact and that $S_i := \sum_{j=1}^n \kappa_{ij} p_j > 0$ holds. Then,
\[
\text{diam} \text{ supp } \mu_i(t) \leq e^{-m_i S_i t} \text{diam} \text{ supp } \mu_i^0.
\]
That is, the support of $\mu_i$ contracts at exponential speed.

Proof. Recall that $\text{diam} \text{ supp } \mu_i(t) = u_i(t, 1^-) - u_i(t, 0^+)$. We have
\[
\begin{aligned}
\partial_t (u_i(t, 1^-) - u_i(t, 0^+)) &= \sum_{j=1}^n m_i p_j \int_0^1 \left[ W_{ij}'(u_j(t, \xi) - u_i(t, 1^-)) - W_{ij}'(u_j(t, \xi) - u_i(t, 0^+)) \right] d\xi \\
&\leq \sum_{j=1}^n m_i p_j \kappa_{ij} \int_0^1 \left[ (u_j(t, \xi) - u_i(t, 1^-)) - (u_j(t, \xi) - u_i(t, 0^+)) \right] d\xi = -m_i S_i (u_i(t, 1^-) - u_i(t, 0^+)),
\end{aligned}
\]
the second-to-last step being a consequence of $\kappa_{ij}$-convexity (W5). Applying Gronwall’s lemma completes the proof. \qed

In the regime where Proposition 3.8 is applicable for all $i \in \{1, \ldots, n\}$, system (11) behaves asymptotically as $t \to \infty$ like the particle system (18) in the case of only one (heavy) particle for each component ($N_i = 1$ for all $i$). Obviously, by Remark 2.7, the condition $S_i > 0$ above is met in the scenario with uniformly geodesically convex energy. This enables us to improve the convergence result from Section 2.4 in one spatial dimension for compactly supported initial data:

Proposition 3.9 (The uniformly convex case in one spatial dimension). Assume that the criterion for geodesic convexity (14) yields $\lambda_0 > 0$ and suppose that $\mu^0$ has compact support. Then, for each $i \in \{1, \ldots, n\}$,
\[
\lim_{t \to \infty} W_\infty(\mu_i(t), \mu_i^\infty) = 0.
\]

In view of Corollary 2.10, we obtain convergence w.r.t. the stronger topology of the $L^\infty$-Wasserstein distance, but lose the exponential rate of convergence.

Proof. Fix $i \in \{1, \ldots, n\}$. Since $\lambda_0 > 0$ and $\text{supp } \mu_i^0$ is compact, we know from Corollary 2.10 Theorem 3.4 and Proposition 3.8 that $\|u_i(t, \cdot) - x^\infty\|_{L^\infty([0, 1])} \to 0$ as $t \to \infty$, $\|u_i(t, \cdot)\|_{L^\infty([0, 1])} \leq K$ for all $t \geq 0$ and $u_i(t, 1^-) - u_i(t, 0^+) \to 0$ as $t \to \infty$. Obviously, if $\lim_{t \to \infty} u_i(t, 1^-) = x^\infty$ holds, the desired result follows immediately from
\[
\|u_i(t, \cdot) - x^\infty\|_{L^\infty([0, 1])} = \max(|u_i(t, 1^-) - x^\infty|, |u_i(t, 0^+) - x^\infty|),
\]
since then also $\lim_{t \to \infty} u_i(t, 0^+) = x^\infty$ holds. So, assume that $u_i(t, 1^-)$ does not converge to $x^\infty$ as $t \to \infty$. Then, there exists $\varepsilon > 0$ and a sequence $t_k \to \infty$ such that
\[
|u_i(t_k, 1^-) - x^\infty| \geq \varepsilon \quad \forall k \in \mathbb{N}.
\tag{36}
\]
Thanks to the observations above, there exist a subsequence $(t_{k_i})_{i \in \mathbb{N}}$ and $\omega \in \mathbb{R}$ such that
\[
\lim_{l \to \infty} (u_i(t_{k_i}, 1^-) - x^\infty) = \omega, \quad \text{and } \lim_{l \to \infty} u_i(t_{k_i}, z) = x^\infty \text{ for a.e. } z \in (0, 1).
\]
Immediately, it follows that $\lim_{l \to \infty} u_i(t_{k_i}, 0^+) = x^\infty + \omega$ and consequently $\omega = 0$ by monotonicity. But $\omega = 0$ is a contradiction to (36). \qed

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