D4-branes wrapped on supersymmetric four-cycles

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Abstract

In F(4) gauged supergravity in six dimensions, we derive supersymmetry equations for D4-branes wrapped on two Riemann surfaces. We find a new AdS$_2$ solution. Using the AdS$_2$ solution, we calculate entropy of asymptotically AdS$_6$ black holes with AdS$_2 \times \Sigma_{g_1} \times \Sigma_{g_2}$ horizon. We find a match of the entropy with the recently calculated topologically twisted index of 5d USp$(2N)$ gauge theory on $\Sigma_{g_1} \times \Sigma_{g_2} \times S^1$ in the large $N$ limit. We also study D4-branes wrapped on Kähler four-cycles in Calabi-Yau fourfolds and on Cayley four-cycles in Spin(7) manifolds, and find new AdS$_2$ solutions.
1 Introduction

The AdS$_6$/CFT$_5$ correspondence remains as one of the less appreciated among its family [1]. In SU$(2) \times$ U$(1)$-gauged $\mathcal{N} = 4$ supergravity in six dimensions, commonly known as F(4) gauged supergravity named after the F(4) superalgebra in six dimensions, there is a unique supersymmetric fixed point [2]. This fixed point is known to be dual of 5d superconformal USp$(2N)$ gauge theory [3] which is one of the few 5d SCFTs known so far [4, 5]. In [6] it was shown that F(4) gauged supergravity is a consistent truncation of massive type IIA supergravity [7]. The fixed point uplifts to AdS$_6 \times S^4$ near-horizon geometry of the D4-D8 brane system [8].

In order to study RG flows from 5d SCFTs to lower dimensional ones via the AdS/CFT correspondence in the spirit of [12], twisted compactifications of F(4) gauged supergravity were studied. The supergravity solutions describe the near-horizon geometries of wrapped D4-branes on various supersymmetric cycles. D4-branes wrapped on two- and three-cycles were studied in [13, 14]. They found AdS$_4$ and AdS$_3$ fixed point solutions. See for more recent results on three-cycles in [15].

In this paper, we study D4-branes wrapped on supersymmetric four-cycles. We begin by deriving supersymmetry equations for D4-branes wrapped on two Riemann surfaces. We find a new AdS$_2$ solution. Using the AdS$_2$ solution, we calculate entropy of asymptotically AdS$_6$ black holes with AdS$_2 \times \Sigma_{g_1} \times \Sigma_{g_2}$ horizon. Analogous to the 3d gauge theory examples in

\footnote{It is also a consistent truncation of type IIB supergravity [9, 10, 11].}
[16, 17], this entropy would give the topologically twisted index of 5d $USp(2N)$ gauge theory on $\Sigma_{g_1} \times \Sigma_{g_2} \times S^1$ in the large $N$ limit. Indeed we find that the entropy match recent calculation of the topologically twisted index in [18].

Then, we consider D4-branes wrapped on Kähler four-cycles in Calabi-Yau fourfolds and on Cayley four-cycles in $Spin(7)$ manifolds. We believe these are all possible four-cycles on which D4-branes can wrap. Two Riemann surfaces falls into a special case of Kähler four-cycles in Calabi-Yau fourfolds. We derive the supersymmetry equations, and obtain new supersymmetric $AdS_2$ solutions for each case.

We comment on the comparison with the recent field theory results. The topologically twisted index can be written as the contour integral of meromorphic differential form in variables parametrizing the Cartan subgroup and subalgebra of the gauge group, summed over the lattice of gauge magnetic fluxes, $m$, on the internal manifold [16, 17]. Recently, topologically twisted index of 5d $USp(2N)$ gauge theory on $\Sigma_{g_1} \times \Sigma_{g_2} \times S^1$ was calculated at finite $N$ by two independent groups, [18] and [19], and their calculations agree. When considering the large $N$ limit, they both employed a conjecture to extremize the prepotential in order to get a saddle point distribution for dominant eigenvalue distribution. In [19], by considering the non-zero magnetic fluxes as a subleading contribution, the contribution to the twisted index from zero magnetic fluxes, $m = 0$, was evaluated. As we explain in detail below, it matches the gravitational entropy of the $AdS_2$ solution in [14] which is expected to be incorrect, and it counts the half of the full contribution to the twisted index. However, in [18], instead of setting the magnetic fluxes to zero, they extremized the twisted superpotential in order to get a saddle point distribution for the fluxes as well. In this paper, we will show that the full gravitational entropy of asymptotically $AdS_6$ black holes with $AdS_2 \times \Sigma_{g_1} \times \Sigma_{g_2}$ horizon matches the topologically twisted index in the large $N$ limit, only when the contributions from non-zero gauge theory magnetic fluxes are accounted as it was done in [18].

In section 2, we review $F(4)$ gauged supergravity. In section 3, we consider D4-branes wrapped on two Riemann surfaces. We derive the supersymmetry equations, find a new $AdS_2$ solution, and calculate entropy of the black holes. In section 4, we consider D4-branes wrapped on Kähler four-cycles in Calabi-Yau fourfolds and on Cayley four-cycles in $Spin(7)$ manifolds. In section 5, we conclude. The equations of motion of $F(4)$ gauged supergravity are presented in appendix A.

\footnote{These cases were previously studied in [14] as D4-branes wrapped on Kähler four-cycles in Calabi-Yau threefolds and on co-associative four-cycles in $G_2$ manifolds. However, as they have not turned on the two form gauge potential which is needed to have a consistent set of supersymmetry equations and to satisfy the equations of motion, we conclude that their equations and solutions are not correct.}
2 $F(4)$ gauged supergravity in six dimensions

We review $SU(2) \times U(1)$-gauged $\mathcal{N} = 4$ supergravity in six dimensions [2]. The bosonic field content consists of the metric, $g_{\mu\nu}$, a real scalar, $\phi$, an $SU(2)$ gauge field, $A_I^\mu$, $I = 1, 2, 3$, a $U(1)$ gauge field, $A_\mu$, and a two-form gauge potential, $B_{\mu\nu}$. The fermionic field content is four gravitinos, $\psi_{\mu i}$, and four gauginos, $\chi_i$, $i = 1, 2$. The field strengths are defined by

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
F_I^{\mu\nu} = \partial_\mu A_I^\nu - \partial_\nu A_I^\mu + g e^{IJK} A_J^\mu A^K_\nu ,
G_{\mu\nu\rho} = 3 \partial_\mu B_{\nu\rho}\]

and

\[
H_{\mu\nu} = F_{\mu\nu} + m B_{\mu\nu} .
\]

The bosonic Lagrangian is given by

\[
e^{-1} \mathcal{L} = - \frac{1}{4} R + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{8} \left( g^2 e^{\sqrt{2} \phi} + 4 g m e^{-\sqrt{2} \phi} - m^2 e^{-3\sqrt{2} \phi} \right)
- \frac{1}{4} e^{-\sqrt{2} \phi} \left( \mathcal{H}_{\mu\nu} \mathcal{H}^{\mu\nu} + F_I^{\mu\nu} F_I^{\mu\nu} \right) + \frac{1}{12} e^{2\sqrt{2} \phi} G_{\mu\nu\rho} G^{\mu\nu\rho}
- \frac{1}{8} e^{\mu\rho\sigma\tau\kappa} B_{\mu\nu} \left( F_{\rho\sigma} F_{\tau\kappa} + m B_{\rho\sigma} F_{\tau\kappa} + \frac{1}{3} m^2 B_{\rho\sigma} B_{\tau\kappa} + F_I^{\rho\sigma} F_I^{\tau\kappa} \right),
\]

where $g$ is the $SU(2)$ gauge coupling constant and $m$ is the mass of the two-form gauge potential. The supersymmetry transformations of the fermionic fields are

\[
\delta \psi_{\mu i} = \nabla_\mu \epsilon_i + g A_I^\mu (T^I)_i^j \epsilon_j - \frac{1}{8\sqrt{2}} \left( g e^{-\sqrt{2} \phi} + m e^{-3\sqrt{2} \phi} \right) \gamma_\mu \gamma_7 \epsilon_i
- \frac{1}{8\sqrt{2}} e^{-\sqrt{2} \phi} (\mathcal{F}_{\mu\lambda} + m B_{\mu\lambda}) \left( \gamma_{\mu}^{\nu\lambda} - 6 \delta_{\mu}^{\nu} \gamma_{\lambda} \right) \epsilon_i
- \frac{1}{4\sqrt{2}} e^{-\sqrt{2} \phi} F_{\mu\lambda} \left( \gamma_{\mu}^{\nu\lambda} - 6 \delta_{\mu}^{\nu} \gamma_{\lambda} \right) \gamma_7 (T^I)_i^j \epsilon_j
- \frac{1}{24} e^{\sqrt{2} \phi} G_{\mu\nu\rho} \gamma_7 \gamma_{\nu}\gamma_{\rho} \gamma_\mu \epsilon_i ,
\]

\[
\delta \chi_i = \frac{1}{\sqrt{2}} \gamma_\mu \partial_\mu \phi \epsilon_i + \frac{1}{4\sqrt{2}} \left( g e^{-\sqrt{2} \phi} - 3 m e^{-3\sqrt{2} \phi} \right) \gamma_7 \epsilon_i
+ \frac{1}{4\sqrt{2}} e^{-\sqrt{2} \phi} (\mathcal{F}_{\mu\nu} + m B_{\mu\nu}) \gamma^{\mu\nu} \epsilon_i
+ \frac{1}{2\sqrt{2}} e^{-\sqrt{2} \phi} F_{\mu\nu} \gamma^{\mu\nu} \gamma_7 (T^I)_i^j \epsilon_j
- \frac{1}{12} e^{\sqrt{2} \phi} G_{\mu\nu\lambda} \gamma_7 \gamma^{\mu\nu\lambda} \epsilon_i ,
\]

where $T^I$, $I = 1, 2, 3$, are the $SU(2)$ left-invariant one-forms,

\[
T^I = -\frac{i}{2} \sigma^I.
\]
Described by the above Lagrangian, there are five inequivalent theories: \( \mathcal{N} = 4^+ \) \((g > 0, m > 0)\), \( \mathcal{N} = 4^- \) \((g < 0, m > 0)\), \( \mathcal{N} = 4^g \) \((g > 0, m = 0)\), \( \mathcal{N} = 4^m \) \((g = 0, m > 0)\), \( \mathcal{N} = 4^0 \) \((g = 0, m = 0)\). The \( \mathcal{N} = 4^+ \) theory admits a supersymmetric AdS\(_6\) fixed point when \( g = 3m \).

3 D4-branes wrapped on two Riemann surfaces

3.1 The supersymmetry equations

We consider the metric,

\[
ds^2 = e^{2f(r)} \left( \, dt^2 - dr^2 \right) - e^{2g_1(r)} \left( d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 \right) - e^{2g_2(r)} \left( d\theta_2^2 + \sin^2 \theta_1 d\phi_2^2 \right),
\]

(3.1)

for the \( S^2 \times S^2 \) background, and

\[
ds^2 = e^{2f(r)} \left( \, dt^2 - dr^2 \right) - e^{2g_1(r)} \left( d\theta_1^2 + \sinh^2 \theta_1 d\phi_1^2 \right) - e^{2g_2(r)} \left( d\theta_2^2 + \sinh^2 \theta_1 d\phi_2^2 \right),
\]

(3.2)

for the \( H^2 \times H^2 \) background. The only non-vanishing component of the non-Abelian \( SU(2) \) gauge field, \( A^I_\mu \), \( I = 1, 2, 3 \), is given by

\[
A^3 = -a_1 \cos \theta_1 d\phi_1 - a_2 \cos \theta_2 d\phi_2,
\]

(3.3)

for the \( S^2 \times S^2 \) background, and

\[
A^3 = a_1 \cosh \theta_1 d\phi_1 + a_2 \cosh \theta_2 d\phi_2,
\]

(3.4)

for the \( H^2 \times H^2 \) background, where the magnetic charges, \( a_1 \) and \( a_2 \), are constant. In order to have equal signs for field strengths, we set opposite signs of the gauge fields for \( S^2 \times S^2 \) and \( H^2 \times H^2 \) backgrounds. We also have a non-trivial two-form gauge potential, \( B_{\mu \nu} \), and we will determine it later. We turn off the Abelian \( U(1) \) gauge field, \( A_\mu \).

The supersymmetry equations are obtained by setting the supersymmetry variations of the
fermionic fields to zero. From the supersymmetry variations, we obtain

\[
f'^{-f} \gamma^i \epsilon_i - \frac{1}{4\sqrt{2}} \left( ge^{\frac{\phi}{\sqrt{2}}} + me^{-\frac{3\phi}{\sqrt{2}}} \right) \gamma \epsilon_i,
\]

\[
- \frac{1}{2\sqrt{2}} e^{-\frac{\phi}{\sqrt{2}}} \left( a_1 e^{-2g_1 \gamma^i \phi^i} + a_2 e^{-2g_2 \gamma^i \phi^i} \right) \gamma_2 (T^3)_{i,j} \epsilon_j - \frac{3}{\sqrt{2}m} a_1 a_2 e^{\frac{\phi}{\sqrt{2}} - 2g_1 - 2g_2} \gamma^i \epsilon_i = 0,
\]

\[
(3.5)
\]

\[
g'^{-f} \gamma^i \epsilon_i - \frac{1}{4\sqrt{2}} \left( ge^{\frac{\phi}{\sqrt{2}}} + me^{-\frac{3\phi}{\sqrt{2}}} \right) \gamma \epsilon_i
\]

\[
+ \frac{1}{2\sqrt{2}} e^{-\frac{\phi}{\sqrt{2}}} \left( 3a_1 e^{-2g_1 \gamma^i \phi^i} - a_2 e^{-2g_2 \gamma^i \phi^i} \right) \gamma_2 (T^3)_{i,j} \epsilon_j + \frac{1}{\sqrt{2}m} a_1 a_2 e^{\frac{\phi}{\sqrt{2}} - 2g_1 - 2g_2} \gamma^i \epsilon_i = 0,
\]

\[
(3.6)
\]

\[
g'^{-f} \gamma^i \epsilon_i - \frac{1}{4\sqrt{2}} \left( ge^{\frac{\phi}{\sqrt{2}}} + me^{-\frac{3\phi}{\sqrt{2}}} \right) \gamma \epsilon_i
\]

\[
+ \frac{1}{2\sqrt{2}} e^{-\frac{\phi}{\sqrt{2}}} \left( 3a_1 e^{-2g_1 \gamma^i \phi^i} - a_2 e^{-2g_2 \gamma^i \phi^i} \right) \gamma_2 (T^3)_{i,j} \epsilon_j + \frac{1}{\sqrt{2}m} a_1 a_2 e^{\frac{\phi}{\sqrt{2}} - 2g_1 - 2g_2} \gamma^i \epsilon_i = 0,
\]

\[
(3.7)
\]

\[
\frac{1}{\sqrt{2}} \phi'^{-f} \gamma^i \epsilon_i + \frac{1}{4\sqrt{2}} \left( ge^{\frac{\phi}{\sqrt{2}}} - 3me^{-\frac{3\phi}{\sqrt{2}}} \right) \gamma \epsilon_i
\]

\[
+ \frac{1}{2\sqrt{2}} e^{-\frac{\phi}{\sqrt{2}}} \left( a_1 e^{-2g_1 \gamma^i \phi^i} + a_2 e^{-2g_2 \gamma^i \phi^i} \right) \gamma_2 (T^3)_{i,j} \epsilon_j - \frac{1}{\sqrt{2}m} a_1 a_2 e^{\frac{\phi}{\sqrt{2}} - 2g_1 - 2g_2} \gamma^i \epsilon_i = 0,
\]

\[
(3.8)
\]

where the hatted indices are the flat indices. The \(t\)-, \(\theta_1\)-, and \(\theta_2\)-components of the gravitino variations give (3.5), (3.6), (3.7), and the gaugino variation gives (3.8). The \(\phi_1\)-, \(\phi_2\)-components of the variations are identical to the \(\theta_1\)-, and \(\theta_2\)-components beside few more terms,

\[
\epsilon_i = -2ga_1 \gamma^i \phi^i (T^3)_{i} \epsilon_j,
\]

\[
\epsilon_i = -2ga_2 \gamma^i \phi^i (T^3)_{i} \epsilon_j.
\]

\[
(3.9)
\]

We employ the projection conditions,

\[
\gamma^i \gamma^i \epsilon_i = \epsilon_i,
\]

\[
\gamma^i \phi^i (T^3)_{i} \epsilon_j = \frac{\lambda}{2} \epsilon_i,
\]

\[
\gamma^i \phi^i (T^3)_{i} \epsilon_j = \frac{\lambda}{2} \epsilon_i,
\]

\[
(3.10)
\]

where \(\lambda = \pm 1\). Solutions with the projection conditions preserve 1/8 of the supersymmetries. By employing the projection conditions, we obtain the complete supersymmetry equations,

\[
f'^{-f} = - \frac{1}{4\sqrt{2}} \left( ge^{\frac{\phi}{\sqrt{2}}} + me^{-\frac{3\phi}{\sqrt{2}}} \right) - \frac{\lambda}{2\sqrt{2}} e^{-\frac{\phi}{\sqrt{2}}} \left( a_1 e^{-2g_1} + a_2 e^{-2g_2} \right) - \frac{3}{\sqrt{2}m} a_1 a_2 e^{\frac{\phi}{\sqrt{2}} - 2g_1 - 2g_2},
\]

\[
g'^{-f} = - \frac{1}{4\sqrt{2}} \left( ge^{\frac{\phi}{\sqrt{2}}} + me^{-\frac{3\phi}{\sqrt{2}}} \right) + \frac{\lambda}{2\sqrt{2}} e^{-\frac{\phi}{\sqrt{2}}} \left( 3a_1 e^{-2g_1} - a_2 e^{-2g_2} \right) + \frac{1}{\sqrt{2}m} a_1 a_2 e^{\frac{\phi}{\sqrt{2}} - 2g_1 - 2g_2},
\]

\[
g'^{-f} = - \frac{1}{4\sqrt{2}} \left( ge^{\frac{\phi}{\sqrt{2}}} + me^{-\frac{3\phi}{\sqrt{2}}} \right) + \frac{\lambda}{2\sqrt{2}} e^{-\frac{\phi}{\sqrt{2}}} \left( 3a_2 e^{-2g_2} - a_1 e^{-2g_1} \right) + \frac{1}{\sqrt{2}m} a_1 a_2 e^{\frac{\phi}{\sqrt{2}} - 2g_1 - 2g_2},
\]

\[
\frac{1}{\sqrt{2}} \phi'^{-f} = \frac{1}{4\sqrt{2}} \left( ge^{\frac{\phi}{\sqrt{2}}} - 3me^{-\frac{3\phi}{\sqrt{2}}} \right) + \frac{\lambda}{2\sqrt{2}} e^{-\frac{\phi}{\sqrt{2}}} \left( a_1 e^{-2g_1} + a_2 e^{-2g_2} \right) - \frac{1}{\sqrt{2}m} a_1 a_2 e^{\frac{\phi}{\sqrt{2}} - 2g_1 - 2g_2}.
\]

\[
(3.11)
\]
From (3.9) we also obtain the twist conditions on the magnetic charges,

\[ a_1 = -\frac{k}{\lambda g}, \quad a_2 = -\frac{k}{\lambda g}, \quad (3.12) \]

where \( k = +1 \) for the \( S^2 \times S^2 \) background and \( k = -1 \) for the \( H^2 \times H^2 \) background. \[^3\]

In the derivation of the supersymmetry equations, we determined the non-zero components of the two-form gauge potential, \( B_{\mu\nu} \). We determined the normalization by solving the equations of motion,

\[ B_{tr} = -\frac{2}{m^2} a_1 a_2 e^{\sqrt{2} \phi + 2f - 2g_1 - 2g_2}. \quad (3.13) \]

The three-form field strength of the two-form gauge potential, \( G_{\mu\nu\lambda} \), vanishes identically. The supersymmetry equations satisfy the equations of motion. We present the equations of motion in appendix A.

When we plug the twist conditions, (3.12), in the supersymmetry equations, we obtain

\[
\begin{align*}
    f' e^{-f} &= -\frac{1}{4\sqrt{2}} \left( ge^{\sqrt{2} \phi} + me^{-\sqrt{2} \phi} \right) - \frac{k}{2\sqrt{2}g} e^{-\sqrt{2} \phi} \left( e^{-2g_1} + e^{-2g_2} \right) - \frac{3}{\sqrt{2}g^2 m} e^{\sqrt{2} \phi - 2g_1 - 2g_2}, \\
    g'_e e^{-f} &= -\frac{1}{4\sqrt{2}} \left( ge^{\sqrt{2} \phi} + me^{-\sqrt{2} \phi} \right) - \frac{k}{2\sqrt{2}g} e^{-\sqrt{2} \phi} \left( 3e^{-2g_1} - e^{-2g_2} \right) + \frac{1}{\sqrt{2}g^2 m} e^{\sqrt{2} \phi - 2g_1 - 2g_2}, \\
    g'_e e^{-f} &= -\frac{1}{4\sqrt{2}} \left( ge^{\sqrt{2} \phi} + me^{-\sqrt{2} \phi} \right) - \frac{k}{2\sqrt{2}g} e^{-\sqrt{2} \phi} \left( e^{-2g_2} - e^{-2g_1} \right) + \frac{1}{\sqrt{2}g^2 m} e^{\sqrt{2} \phi - 2g_1 - 2g_2}, \\
    \frac{1}{\sqrt{2}} \phi' e^{-f} &= \frac{1}{4\sqrt{2}} \left( ge^{\sqrt{2} \phi} + 3me^{-\sqrt{2} \phi} \right) - \frac{k}{2\sqrt{2}g} e^{-\sqrt{2} \phi} \left( e^{-2g_1} + e^{-2g_2} \right) - \frac{1}{\sqrt{2}g^2 m} e^{\sqrt{2} \phi - 2g_1 - 2g_2},
\end{align*}
\]

where \( k = +1 \) for the \( S^2 \times S^2 \) background and \( k = -1 \) for the \( H^2 \times H^2 \) background. The supersymmetry equations in (3.14) are analogous to the equations for M5-branes wrapped on two Riemann surfaces in [20], and more recently generalized in [21].

### 3.2 The AdS\(_2\) solution and entropy of black holes

Now we will consider the \( N = 4^+ \) theory, \( g > 0, m > 0 \). We find a new AdS\(_2\) fixed point solution for the \( H^2 \times H^2 \) background with \( k = -1 \). \[^4\]

\[
\begin{align*}
    e^f &= \frac{2^{1/4}}{g^{3/4}m^{1/4}} \frac{1}{r}, \quad e^{g_1} = e^{g_2} = \frac{2^{3/4}}{g^{3/4}m^{1/4}}, \quad e^{\sqrt{2} \phi} = \frac{2^{1/4}m^{1/4}}{g^{1/4}}. \quad (3.16)
\end{align*}
\]

\[^3\]It is possible to have geometries like \( S^2 \times H^2 \) for \( k_1 = +1 \) and \( k_2 = -1 \), or vice versa. One can easily generalize our supersymmetry equations and the twist conditions to that case.

\[^4\]The solution can also be presented as

\[
\begin{align*}
    e^f &= \frac{\sqrt{2}}{g} e^{-\sqrt{2} \phi} \frac{1}{r}, \quad e^{g_1} = e^{g_2} = \frac{2}{g} e^{-\sqrt{2} \phi}, \quad e^{-2\sqrt{2} \phi} = \frac{g}{2m}. \quad (3.15)
\end{align*}
\]
When we consider the $S^2 \times S^2$ background with $k = +1$, $AdS_2$ fixed point does not exist.

By employing the uplift formulae in [6], the $AdS_2$ solution can be uplifted to a solution of massive type IIA supergravity. We only present the uplift formulae for the metric,

$$ds_{10}^2 = \sin^{1/12} \xi X^{1/8} \left[ \Delta^{3/8} ds_6^2 + \frac{2\Delta^{3/8} X^2}{g^2} d\xi^2 + \frac{\cos^2 \xi}{2g^2 \Delta^{5/8} X} \sum_{i=1}^3 (\sigma^i - gA^i)^2 \right], \quad (3.17)$$

and the dilaton field, $\Phi$,

$$e^\Phi = \frac{\Delta^{1/4}}{\sin^{5/6} \xi X^{5/4}}, \quad (3.18)$$

where we define

$$\Delta = X \cos^2 \xi + \frac{1}{X^3} \sin^2 \xi, \quad X = e^{-\frac{\phi}{\sqrt{2}}}.$$

The solution would describe $AdS_2 \times \Sigma_{g_1} \times \Sigma_{g_2}$ horizon of six-dimensional black holes. When we consider the $\mathcal{N} = 4^+$ theory with $g = 3m$, there is a supersymmetric fixed point which is known to be dual of 5d $USp(2N)$ gauge theory in the large $N$ limit. Analogous to the 3d gauge theory examples in [16, 17], entropy of the black hole could match the topologically twisted index of 5d $USp(2N)$ gauge theory on $\Sigma_{g_1} \times \Sigma_{g_2} \times S^1$ in the large $N$ limit. Recently, there has been development in calculating topologically twisted index of 5d $USp(2N)$ gauge theory on $M_4 \times S^1$ by two independent groups, [18] and [19]. See the introduction for more details on their calculations.

Now we calculate entropy of asymptotically $AdS_6$ black holes with $AdS_2 \times \Sigma_{g_1} \times \Sigma_{g_2}$ horizon. We would like to consider the $AdS_6$ fixed point in the $\mathcal{N} = 4^+$ theory by taking $g = 3m$. In order to have unit radius, $L_{AdS_6} = 1$, we set $m = \sqrt{2}$ $^5$ The Bekenstein-Hawking entropy of the black hole is given by

$$S_{BH} = \frac{L_{AdS_6}^{-2}}{4G_N^{(6)}} = \frac{1}{4G_N^{(6)}} = \frac{vol(M_4)}{4G_N^{(6)}} = \frac{e^{2g}vol(\Sigma_{g_1})e^{2g}vol(\Sigma_{g_2})}{4G_N^{(6)}} = \frac{8\pi^2(\mathfrak{g}_1 - 1)(\mathfrak{g}_2 - 1)}{27G_N^{(6)}}, \quad (3.20)$$

where $L$, $vol$, and $G_N$ are radius, volume, and the Newton’s gravitational constant for the corresponding spaces, respectively. We used that the volume of Riemann surfaces with genus, $g$, is

$$vol(\Sigma_g) = 4\pi (g - 1), \quad (3.21)$$

and, in the last equality, we used the value of the warp factor at our $AdS_2$ solution, $e^{4g} = 2/3^3$. We can relate the gravitational entropy to the free energy of 5d SCFTs on $S^5$ by using the universal formula, $^7$

$$F_{S^5} = -\frac{\pi^2 L_{AdS_6}^4}{3G_N^{(6)}}, \quad (3.22)$$

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$^5$See discussions around (4.6) in [23].

$^6$See, for example, (73) in [12].

$^7$See, for example, (4.5) in [23].
and then entropy of the black hole is
\[ S_{BH} = -\frac{8}{9}(g_1 - 1)(g_2 - 1)F_{S^5}. \] (3.23)

The free energy of 5d $USp(2N)$ gauge theory with $N_f$ flavors on $S^5$ in the large $N$ limit is given by [24],
\[ F_{S^5} = -\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8 - N_f}}. \] (3.24)

Therefore, entropy of the black hole can be written by
\[ S_{BH} = \frac{8\sqrt{2}\pi (1 - g_1)(1 - g_2)N^{5/2}}{5\sqrt{8 - N_f}}. \] (3.25)

This nicely matches the topologically twisted index of 5d $USp(2N)$ gauge theory on $\Sigma_{g_1} \times \Sigma_{g_2} \times S^1$ in the large $N$ limit calculated in [18]. As it was explained in detail in the introduction, the contribution from zero magnetic fluxes, $m = 0$, obtained in [19], counts the half of the gravitational entropy.

## 4 D4-branes wrapped on supersymmetric four-cycles

In this section, we consider D4-branes wrapped on Kähler four-cycles in Calabi-Yau fourfolds and on Cayley four-cycles in $Spin(7)$ manifolds. We believe these are all possible four-cycles on which D4-branes can wrap in $F(4)$ gauged supergravity. D4-branes on two Riemann surfaces in the previous section fall into a special case of D4-branes on Kähler four-cycles in Calabi-Yau fourfolds. The analogous results of M5-branes wrapped on supersymmetric four-cycles were studied in [22, 20].

### 4.1 Kähler four-cycles in Calabi-Yau fourfolds

We consider the metric,
\[ ds^2 = e^{2f(r)}(dt^2 - dr^2) - e^{2g(r)}ds_{M_4}^2, \] (4.1)
where $M_4$ is a Kähler four-cycle in Calabi-Yau fourfolds. The curved coordinates on the Kähler four-cycles will be denoted by $\{x_1, x_2, x_3, x_4\}$, and the hatted ones are the flat coordinates. For Kähler four-cycles in Calabi-Yau fourfolds, there are four directions transverse to D4-branes in the fourfolds. The normal bundle of the four-cycle has $U(2) \subset SO(4)$ structure group. We identify $U(1)$ part of the structure group with $U(1)$ gauge field from the non-Abelian $SU(2)$ gauge group, [22, 25]. The only non-vanishing component of the non-Abelian $SU(2)$ gauge field, $A^I_\mu$, $I = 1, 2, 3$, is given by
\[ F^3_{x_1 x_2} = a_1 e^{-2g}, \quad F^3_{x_3 x_4} = a_2 e^{-2g}, \] (4.2)
where the magnetic charges, \( a_1 \) and \( a_2 \), are constant. The only non-vanishing component of the two-form gauge potential is

\[
B_{tr} = -\frac{2}{m^2} a_1 a_2 e^{\sqrt{2}\phi + 2f - 4g}.
\]

We employ the projection conditions,

\[
\gamma^\ell \gamma^7 \epsilon_i = \epsilon_i, \quad \gamma^{\hat{x}_1 \hat{x}_2} (T^3)_i^j \epsilon_j = \frac{\lambda}{2} \epsilon_i, \quad \gamma^{\hat{x}_3 \hat{x}_4} (T^3)_i^j \epsilon_j = \frac{\lambda}{2} \epsilon_i,
\]

where \( \lambda = \pm 1 \). Solutions with the projection conditions preserve \( 1/8 \) of the supersymmetries. By employing the projection conditions, we obtain the complete supersymmetry equations,

\[
f' e^{-f} = -\frac{1}{4\sqrt{2}} \left( g e^{\frac{\phi}{\sqrt{2}}} + m e^{-\frac{3\phi}{\sqrt{2}}} \right) + \frac{k}{\sqrt{2g}} e^{-\frac{\phi}{\sqrt{2}}} - \frac{3}{\sqrt{2g^2 m}} e^{\frac{\phi}{\sqrt{2}}},
\]

\[
g' e^{-f} = -\frac{1}{4\sqrt{2}} \left( g e^{\frac{\phi}{\sqrt{2}}} + m e^{-\frac{3\phi}{\sqrt{2}}} \right) - \frac{k}{\sqrt{2g}} e^{-\frac{\phi}{\sqrt{2}}} + \frac{1}{\sqrt{2g^2 m}} e^{\frac{\phi}{\sqrt{2}}},
\]

\[
\frac{1}{\sqrt{2}} g' e^{-f} = \frac{1}{4\sqrt{2}} \left( g e^{\frac{\phi}{\sqrt{2}}} - 3m e^{-\frac{3\phi}{\sqrt{2}}} \right) - \frac{k}{\sqrt{2g}} e^{-\frac{\phi}{\sqrt{2}}} - \frac{1}{\sqrt{2g^2 m}} e^{\frac{\phi}{\sqrt{2}}},
\]

with the twist conditions,

\[
a_1 = -\frac{k}{\lambda g}, \quad a_2 = -\frac{k}{\lambda g},
\]

where \( k \) determines the curvature of the Kähler four-cycles in Calabi-Yau fourfolds.

Two Riemann surfaces considered in the previous section is a special case of Kähler four-cycles in Calabi-Yau fourfolds. When we identify \( g_1 = g_2 \) in the supersymmetry equations for D4-branes wrapped on two Riemann surfaces, \((3.14)\), we obtain the supersymmetry equations here, \((4.5)\). By solving the supersymmetry equations, we find an \( AdS_2 \) fixed point solution which is identical to the one obtained in the previous section, \((3.16)\).

### 4.2 Cayley four-cycles in \( Spin(7) \) manifolds

We consider the metric,

\[
ds^2 = e^{2f(r)} \left( dt^2 - dr^2 \right) - e^{2g(r)} ds_{M_4}^2,
\]

where \( M_4 \) is a Cayley four-cycle in manifolds with \( Spin(7) \) holonomy. The curved coordinates on the Cayley four-cycles will be denoted by \( \{ x_1, x_2, x_3, x_4 \} \), and and the hatted ones are the flat coordinates. In order to preserve supersymmetry for D4-branes wrapped on Cayley four-cycles in \( Spin(7) \) manifolds, we identify self-dual \( SU(2)_+ \) subgroup of the \( SO(4) \) isometry of the four-cycle,

\[
SO(4) \rightarrow SU(2)_+ \times SU(2)_- ,
\]

with the non-Abelian \( SU(2) \) gauge group, \([22, 25]\). The self-duality is defined by

\[
\gamma_{\mu\nu} = \pm \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \gamma^{\rho\sigma},
\]
and we denoted the self-duality and anti-self-duality by + and −, respectively. For the self-dual part, components are identified by

\[ \gamma^{\hat{x}_1\hat{x}_2} = \gamma^{\hat{x}_3\hat{x}_4}, \quad \gamma^{\hat{x}_1\hat{x}_3} = \gamma^{\hat{x}_4\hat{x}_2}, \quad \gamma^{\hat{x}_1\hat{x}_4} = \gamma^{\hat{x}_2\hat{x}_3}. \] (4.10)

The only non-vanishing components of the non-Abelian SU(2) gauge field, \( A^I_{\mu}, \ I = 1, 2, 3, \) are given by

\[ F^1_{\hat{x}_1\hat{x}_2} = F^1_{\hat{x}_3\hat{x}_4} = a_1 e^{-2g}, \]
\[ F^2_{\hat{x}_1\hat{x}_3} = F^2_{\hat{x}_4\hat{x}_2} = a_2 e^{-2g}, \]
\[ F^3_{\hat{x}_1\hat{x}_4} = F^3_{\hat{x}_2\hat{x}_3} = a_3 e^{-2g}, \] (4.11)

where the magnetic charges, \( a_1, a_2 \) and \( a_3, \) are constant. The only non-vanishing component of the two-form gauge potential is

\[ B_{tr} = -\frac{2}{m^2} \left( a_1^2 + a_2^2 + a_3^2 \right) e^{\sqrt{2}f} e^{-4g}. \] (4.12)

We employ the projection conditions,

\[ \gamma^i \gamma^j \epsilon_i = \epsilon_j, \]
\[ \gamma^{\hat{x}_1\hat{x}_2} (T^1)_i \epsilon_j = \gamma^{\hat{x}_3\hat{x}_4} (T^1)_i \epsilon_j = \lambda \frac{\epsilon_i}{2}, \]
\[ \gamma^{\hat{x}_1\hat{x}_3} (T^2)_i \epsilon_j = \gamma^{\hat{x}_4\hat{x}_2} (T^2)_i \epsilon_j = \lambda \frac{\epsilon_i}{2}, \]
\[ \gamma^{\hat{x}_1\hat{x}_4} (T^3)_i \epsilon_j = \gamma^{\hat{x}_2\hat{x}_3} (T^3)_i \epsilon_j = \lambda \frac{\epsilon_i}{2}, \] (4.13)

where \( \lambda = \pm 1. \) Solutions with the projection conditions preserve \( 1/16 \) of the supersymmetries. By employing the projection conditions, we obtain the complete supersymmetry equations,

\[ f' e^{-f} = -\frac{1}{4\sqrt{2}} \left( ge^{\frac{f}{2}} + me^{-\frac{3f}{2}} \right) + \frac{k}{\sqrt{2}g} e^{-\frac{f}{2} - 2g} - \frac{\lambda}{\sqrt{2}g^2 m} e^{\frac{f}{2} - 4g}, \]
\[ g'_I e^{-f} = -\frac{1}{4\sqrt{2}} \left( ge^{\frac{f}{2}} + me^{-\frac{3f}{2}} \right) - \frac{k}{\sqrt{2}g} e^{\frac{f}{2} - 2g} + \frac{\lambda}{3\sqrt{2}g^2 m} e^{\frac{f}{2} - 4g}, \]
\[ \frac{1}{\sqrt{2}} \phi' e^{-f} = \frac{1}{4\sqrt{2}} \left( ge^{\frac{f}{2}} - 3me^{-\frac{3f}{2}} \right) - \frac{k}{\sqrt{2}g} e^{\frac{f}{2} - 2g} - \frac{\lambda}{3\sqrt{2}g^2 m} e^{\frac{f}{2} - 4g}, \] (4.14)

with the twist conditions,

\[ a_1 = -\frac{k}{3\lambda g}, \quad a_2 = -\frac{k}{3\lambda g}, \quad a_3 = -\frac{k}{3\lambda g}, \] (4.15)

where \( k \) determines the curvature of the Cayley four-cycles in \( \text{Spin}(7) \) manifolds.
Now we will consider the $\mathcal{N} = 4^+$ theory, $g > 0$, $m > 0$. By solving the supersymmetry equations, we find a new $AdS_2$ fixed point solution for the negatively curved Cayley four-cycles with $k = -1$,
\[
e^f = \frac{3^{1/4}}{g^{3/4}m^{1/4}} \frac{1}{r}, \quad e^g = \frac{2}{3^{1/4}g^{3/4}m^{1/4}}, \quad e^{\phi_2} = \frac{2^{1/2}m^{1/4}}{3^{1/4}g^{1/4}}.
\] (4.16)

When we consider $k = +1$, $AdS_2$ fixed point does not exist. It will be interesting to have a field theory interpretation of this $AdS_2$ fixed point solution.

5 Conclusions

In this paper, we studied D4-branes wrapped on two Riemann surfaces. We obtained a new $AdS_2$ solution. With the $AdS_2$ solution, we calculated entropy of asymptotically $AdS_5$ black holes with $AdS_2 \times \Sigma_{g_1} \times \Sigma_{g_2}$ horizon. Our gravitational entropy nicely matches the topologically twisted index of 5d $USp(2N)$ gauge theory on $\Sigma_{g_1} \times \Sigma_{g_2} \times S^1$ in the large $N$ limit obtained in [18]. The contribution from zero magnetic fluxes, $m = 0$, in [19], counts the half of the gravitational entropy.

We also considered D4-branes wrapped on Kähler four-cycles in Calabi-Yau fourfolds and Cayley four-cycles in $Spin(7)$ manifolds. We obtained the supersymmetry equations and their $AdS_2$ solutions. We have shown that there should be non-zero two-form gauge potential, $B_{\mu\nu}$, in order to have a consistent set of supersymmetry equations, and satisfy the equations of motion. For this reason, as explained in the introduction, we concluded that the equations and solutions for the corresponding cases in [14] are not correct.

It would be also interesting to study supersymmetric four-cycles in matter coupled $F(4)$ gauged supergravity [26] along the line of [27], to obtain more examples of $AdS_2$ solutions [28].

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A The equations of motion

In this appendix, we present the equations of motion of $F(4)$ gauged supergravity,

$$
R_{\mu\nu} = 2\partial_\mu \phi \partial_\nu \phi + \frac{1}{8} g_{\mu\nu} \left( g^2 e^{\sqrt{2} \phi} + 4 g m e^{-\sqrt{2} \phi} - m^2 e^{-3\sqrt{2} \phi} \right) - 2 e^{-\sqrt{2} \phi} \left( H_{\mu\nu} H^{\mu\nu} - \frac{1}{8} g_{\mu\nu} H_{\rho\sigma} H^{\rho\sigma} \right) 
$$

$$
- 2 e^{-\sqrt{2} \phi} \left( F^{I\mu \rho} F^{I\nu \rho} - \frac{1}{8} g_{\mu\nu} F^{I\rho \sigma} F^{I\rho \sigma} \right) + e^{2\sqrt{2} \phi} \left( G_{\mu\rho \sigma} G^{\nu \rho \sigma} - \frac{1}{6} g_{\rho \sigma} G_{\mu \nu} G^{\rho \sigma \tau} \right),
\quad (A.1)
$$

$$
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) = \frac{1}{4\sqrt{2}} \left( g^2 e^{\sqrt{2} \phi} - 4 g m e^{-\sqrt{2} \phi} + 3 m^2 e^{-3\sqrt{2} \phi} \right) 
$$

$$
+ \frac{1}{2\sqrt{2}} e^{-\sqrt{2} \phi} \left( H_{\mu\nu} \mathcal{H}^{\mu\nu} + F^{I\mu \nu} F^{I\mu \nu} \right) + \frac{1}{3\sqrt{2}} e^{2\sqrt{2} \phi} G_{\mu \nu \rho} G^{\mu \nu \rho},
\quad (A.2)
$$

$$
\mathcal{D}_\nu \left( e^{\sqrt{2} \phi} H^{\mu \nu} \right) = \frac{1}{6} e e^{\mu \rho \sigma \tau \kappa} H_{\nu \rho} G_{\sigma \tau \kappa},
\quad (A.3)
$$

$$
\mathcal{D}_\nu \left( e^{\sqrt{2} \phi} F^{I \mu \rho} \right) = \frac{1}{6} e e^{\mu \rho \sigma \tau \kappa} F^{I}_{\nu \rho} G_{\sigma \tau \kappa},
\quad (A.4)
$$

$$
\mathcal{D}_\rho \left( e^{2\sqrt{2} \phi} G^{\mu \nu} \right) = -\frac{1}{4} e e^{\mu \rho \sigma \tau \kappa} \left( H_{\rho \sigma} H^{\tau \kappa} + F^{I}_{\rho \sigma} F^{I \tau \kappa} \right) - m e^{\sqrt{2} \phi} \mathcal{H}^{\mu \nu}.
\quad (A.5)
$$

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