On $L(2, 1)$-labelings of some products of oriented cycles

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Abstract

We refine two results of Jiang, Shao and Vesel on the $L(2, 1)$-labeling number $\lambda$ of the Cartesian and the strong product of two oriented cycles. For the Cartesian product, we compute the exact value of $\lambda(\vec{C}_m \Box \vec{C}_n)$ for $m, n \geq 40$; in the case of strong product, we either compute the exact value or establish a gap of size one for $\lambda(\vec{C}_m \bowtie \vec{C}_n)$ for $m, n \geq 48$.

1 Introduction

A $L(p, q)$-labeling, or $L(p, q)$-coloring, of a graph $G$ is a function $f : V(G) \to \{0, \ldots, k\}$ such that $|f(u) - f(v)| \geq p$, if $uv \in E(G)$; and $|f(u) - f(v)| \geq q$, if there is a path of length two in $G$ joining $u$ and $v$. To take into account the number of colors used, we say that $f$ is a $k$-$L(p, q)$-labeling of $G$ (note that, for historical reasons, the colorings are assumed to start with the label 0). The minimum value of $k$ such that $G$ admits a $k$-$L(p, q)$-labeling is denoted by $\lambda_{p, q}(G)$, and it is called the $L(p, q)$-labeling number of $G$.

The particular case of $L(p, q)$-labelings that attracted the most attention is $p = 2$ and $q = 1$, the $L(2, 1)$-labeling. It was introduced by Yeh [6], and it traces back to the frequency assignment problem of wireless networks introduced by Hale [3]. In this case, we write $\lambda(G)$ instead of $\lambda_{2, 1}(G)$ for short.

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The definitions above can be extended to oriented graphs (a directed graph whose underlying graph is simple), namely: if \( G \) is an oriented graph, a \( L(p, q) \)-labeling of \( G \) is a function \( f : V(G) \to \{0, \ldots, k\} \) such that \( |f(u) - f(v)| \geq p \), if \( uv \in E(G) \); and \( |f(u) - f(v)| \geq q \), if there is a directed path of length two in \( G \) joining \( u \) and \( v \). The corresponding \( L(p, q) \)-labeling number is again denoted by \( \lambda_{p, q}(G) \) (in some papers, the notation \( \lambda_{p, q}^{-\rightarrow}(G) \) is used instead). The \( L(p, q) \)-labelings of oriented graphs were first studied by Chang and Liaw [2], and the \( L(p, q) \)-labeling problem has been extensively studied since then in both undirected and directed versions. We refer the interested reader to the excellent surveys of Calamoneri [1] and Yeh [7].

In this paper, we study the \( L(2, 1) \)-labeling number of the Cartesian and the strong product of two oriented cycles, improving results of Jiang, Shao and Vesel [4]. In the case of Cartesian product, we compute the exact value of \( \lambda(C_m \square C_n) \) for \( m, n \geq 40 \); in the case of strong product, we either compute the exact value or establish a gap of size one for \( \lambda(C_m \boxast C_n) \) for \( m, n \geq 48 \).

2 Cartesian product

The Cartesian product of two graphs (resp. digraphs) \( G \) and \( H \) is the graph (resp. digraph) \( G \square H \) such that \( V(G \square H) = V(G) \times V(H) \), and where there is an edge joining \((a, x)\) and \((b, y)\) if \( ab \in E(G) \) and \( x = y \), or if \( a = b \) and \( xy \in E(H) \) (resp. there is an edge pointing from \((a, x)\) to \((b, y)\) if \( ab \in E(G) \) and \( x = y \), or if \( a = b \) and \( xy \in E(H) \)).

Let \( S(m, n) = \{am + bn : a, b \geq 0 \text{ integers not both zero}\} \). A classical result of Sylvester [5] states that \( t \in S(m, n) \) for all integers \( t \geq (m - 1)(n - 1) \) that are divisible by \( \gcd(m, n) \), the greatest common divisor of \( m \) and \( n \).

In [4], Jiang, Shao and Vesel prove the following theorem:

**Theorem 1 ([4]).** For all \( m, n \in S(5, 11) \), \( 4 \leq \lambda(C_m \square C_n) \leq 5 \). In particular, the result holds for every \( m, n \geq 40 \).

Our result in this section determines the exact value of \( \lambda \) in the range above. We start with a lemma which is a slightly stronger version of Lemma 5 from [4] that can be obtained with the same proof:

**Lemma 1.** For every \( m, n \geq 3 \) and every \( 4 \)-\( L(2, 1) \)-labeling \( f \) of \( C_m \square C_n \), the following periodicity condition holds:

\[
    f(i, j) = f(i + 1 \mod m, j - 1 \mod n) \quad \text{for all } i \in [m], j \in [n].
\]

We call labelings with this property diagonal.

The following lemma from [4] combined with the result of Sylvester will also help us:
Theorem 2. Let \( \lambda \) and \( \text{Lemma 2. (Lemmas 2 and 3 in [3])} \) Let \( a \) all \( \text{Proof.} \) For \( L \) \( f \) \( n \) that \( d \) Finally, if \( 3 \), we label \( 2 \), it is enough to prove that \( \lambda \) \( G \) and the directed edges of \( m \) \( n \) \( i,j \) \( g \) to together with the fact that \( f \) \( 1 \) as well. \( g \) \( f \) \( | \) \( n \) \( f \) \( 1 \) \( j \) \( m \) and \( n \) \( \) and \( \lambda(\overrightarrow{C_m \square C_n}) \leq k \) and \( \lambda(\overrightarrow{C_n \square C_n}) \leq k \), then \( \lambda(\overrightarrow{C_{m+n} \square C_n}) \leq k \).

In particular, if \( m \) and \( n \) are such that \( \lambda(\overrightarrow{C_m \square C_n}) \leq k \), \( \lambda(\overrightarrow{C_m \square C_m}) \leq k \) and \( \lambda(\overrightarrow{C_n \square C_n}) \leq k \), then \( \lambda(\overrightarrow{C_{a \square C_b}}) \leq k \) for all \( a, b \in S(m,n) \), and hence for all \( a, b \geq (m - 1)(n - 1) \) divisible by \( \gcd(m,n) \).

Theorem 2. Let \( m, n \geq 40 \). Then:

\[
\lambda(\overrightarrow{C_{m \square C_n}}) = \begin{cases} 
4, & \text{if } \gcd(m,n) \geq 3; \\
5, & \text{otherwise.} 
\end{cases}
\]

Proof. For \( m, n \geq 3 \), let \( G \) denote the graph \( \overrightarrow{C_m \square C_n} \), i.e., \( V(G) = [m] \times [n] \) and the directed edges of \( G \) point from \( (i,j) \) to \( (i + 1 \mod m, j) \) and to \( (i, j + 1 \mod n) \), for every \( i \in [m], j \in [n] \). For a labeling \( f \), we write \( f(i,j) \) instead of \( f((i,j)) \) for short.

Let \( d = \gcd(m,n) \) and assume first that \( d \notin \{1,2\} \). According to Lemma 2 it is enough to prove that \( \lambda(\overrightarrow{C_d \square C_d}) = 4 \). Any \( 4-L(2,1) \)-labeling \( f \) of \( \overrightarrow{C_d} \) can be extended to a \( 4-L(2,1) \)-labeling \( f' \) of \( \overrightarrow{C_{d \square C_d}} \) by setting \( f'(i,j) = f(i + j \mod d) \). It suffices to show, then, that \( \lambda(\overrightarrow{C_d}) = 4 \).

If \( d \equiv 0 \) (mod 3), then we can label \( \overrightarrow{C_d} \) with \( d/3 \) blocks 024. If \( d \equiv 1 \) (mod 3), we label \( \overrightarrow{C_d} \) with \( (d - 4)/3 \) consecutive blocks 024 and then one block 0314. Finally, if \( d \equiv 2 \) (mod 3), then we label \( \overrightarrow{C_d} \) with \( (d - 2)/3 \) consecutive blocks 024 and then a block 13.

On the other hand, assume for the sake of contradiction that \( d \in \{1,2\} \) and there is a \( 4-L(2,1) \)-labeling \( f \) of \( \overrightarrow{C_m \square C_n} \). In particular, \( m \neq n \), so let us assume that \( m > n \).

It is easy to check that, if \( m \geq n + 3 \), \( f \) induces a valid \( 4-L(2,1) \)-labeling of \( \overrightarrow{C_{m-n} \square C_n} \). In fact, let \( g(i,j) = f(i,j) \) for all \( 1 \leq i \leq m-n \) and \( 1 \leq j \leq n \). We claim that \( g \) is a \( 4-L(2,1) \)-labeling of \( \overrightarrow{C_{m-n} \square C_n} \), which, in particular, satisfies \([1]\) as well.

Indeed, all we have to check is that the following conditions hold for \( g \), since the other restrictions are inherited by \( f \): |\( g(m-n-1,j) - g(1,j) \)\| \( \geq 1 \) \( , |g(m-n,j) - g(2,j)\| \geq 1 \), \( |g(m-n,j) - g(1,j+1 \mod n)\| \geq 2 \), for every \( j \in [n] \). All these conditions follow from \( g(m-n-1,j) = f(m-n-1,j) = f(m-1,j+n \mod n) = f(m-1,j) \) and \( g(m-n,j) = f(m-n,j) = f(m,j+n \mod n) = f(m,j) \), which result from the application of \([1]\) \( n \) times, together with the fact that \( f \) is a \( L(2,1) \)-labeling of \( \overrightarrow{C_m \square C_n} \).

Applying this argument consecutively, using the fact that \( d = \gcd(m,n) \) and by the symmetry of the factors of the product, we conclude that \( f \) induces a \( 4-L(2,1) \)-labeling \( c \) of either \( \overrightarrow{C_{k+1} \square C_k} \) or \( \overrightarrow{C_{k+2} \square C_k} \), for some \( k \geq 3 \). This is a
contradiction, since in this case we would have \( c(1, 1) = c(2, k) = \cdots = c(k+1, 1) \) and \((k + 1, 1)\) and \((1, 1)\) are joined by an edge or by a directed path of length two, respectively.

\( \square \)

### 3 Strong product

The strong product of two graphs (resp. digraphs) \( G \) and \( H \) is the graph (resp. digraph) \( G \boxtimes H \) such that \( V(G \boxtimes H) = V(G) \times V(H) \), and where there is an edge joining \((a, x)\) and \((b, y)\) if either \( ab \in E(G) \) and \( x = y \), or if \( a = b \) and \( xy \in E(H) \), or if \( ab \in E(G) \) and \( xy \in E(H) \).

In the same paper, Jiang, Shao and Vesel prove the following result for the strong product of two directed cycles:

**Theorem 3** ([4]). If \( m, n \geq 48 \), then \( 6 \leq \lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) \leq 8 \).

In this section, we refine this theorem in the following way:

**Theorem 4.** If \( m, n \geq 48 \), then

\[
\lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) = \begin{cases} 
6, & \text{if } m \equiv n \equiv 0 \pmod{7}; \\
7 \text{ or } 8, & \text{otherwise.}
\end{cases}
\]

The key lemma of in the proof of Theorem 4 is analogous to Lemma 1:

**Lemma 3.** Let \( m, n \geq 4 \) be integers. Any 6-\( L(2, 1) \)-labeling \( f \) of \( \overrightarrow{C_m} \boxtimes \overrightarrow{C_n} \) is diagonal, i.e., the following condition holds:

\[
f(i, j) = f(i + 1 \mod m, j - 1 \mod n) \text{ for all } i \in [m], j \in [n].
\]  

(2)

**Proof of Lemma 3.** Let \( G \) be the graph \( \overrightarrow{C_m} \boxtimes \overrightarrow{C_n} \). For every vertex \((i, j)\) of \( G \), there is a \( \overrightarrow{P_4} \boxtimes \overrightarrow{P_4} \) subgraph of \( G \) as in Figure 1 such that \( v_{22} \) is the vertex \((i, j)\). It suffices to show that \( f(v_{22}) = f(v_{13}) \) for every 6-\( L(2, 1) \)-labeling of \( G \).

Let \( f \) be a 6-\( L(2, 1) \)-labeling of \( G \). By the fact that \( 6 - f \) is also a 6-\( L(2, 1) \)-labeling of \( G \), we may assume that \( f(v_{22}) \in \{0, 1, 2, 3\} \). We will divide the rest of the proof in cases according to the value of \( f(v_{22}) \). In each case, we will assume that \( f(v_{13}) \neq f(v_{22}) \) and reach a contradiction by applying the rules of \( L(2, 1) \)-labeling and finding a vertex for which there is no available color. We will use the notation \( \mathcal{v}! \) to mean that there is no color available for the vertex \( v \), and the notation \( \{u, v\} \in \mathcal{S}! \) to mean that \( u \) and \( v \) cannot be colored using the colors in \( S \), where \( S \) is the set of possible colors for \( u \) and \( v \) based on the colors of the previous vertices and the rules of \( L(2, 1) \)-labeling. For instance, if \( u \) and \( v \) are joined by an edge, then \( \{u, v\} \in \{0, 1\}! \).
Case 1: \( f(v_{22}) = 3 \)
- \( f(v_{13}) \in \{5, 6\} \Rightarrow \{v_{12}, v_{23}\} \in \{0, 1\} !. \)
- \( f(v_{13}) = 4 \Rightarrow f(v_{12}) = 6 \) and \( f(v_{23}) \in \{0, 1\} \) (wlog) \( \Rightarrow f(v_{33}) = 5 \) \( \Rightarrow f(v_{34}) = 2 \Rightarrow f(v_{23}) = 0 \Rightarrow v_{24} !. \)
- \( f(v_{13}) \in \{0, 1\} \Rightarrow \{v_{12}, v_{23}\} \in \{5, 6\} !. \)
- \( f(v_{13}) = 2 \Rightarrow f(v_{12}) = 0 \) and \( f(v_{23}) \in \{5, 6\} \) (wlog) \( \Rightarrow f(v_{33}) = 1 \) \( \Rightarrow f(v_{34}) = 4 \Rightarrow f(v_{23}) = 6 \Rightarrow v_{24} !. \)

Case 2: \( f(v_{22}) = 0 \)
- \( f(v_{13}) = 3 \Rightarrow \{v_{12}, v_{23}\} \in \{5, 6\} !. \)
- \( f(v_{13}) = 5 \Rightarrow \{v_{12}, v_{23}\} \in \{2, 3\} !. \)
- \( f(v_{13}) = 2 \Rightarrow f(v_{12}) = 6 \) and \( f(v_{23}) = 4 \) (wlog) \( \Rightarrow v_{33} !. \)
- \( f(v_{13}) = 4 \Rightarrow f(v_{12}) = 6 \) and \( f(v_{23}) = 2 \) (wlog) \( \Rightarrow v_{24} !. \)
- \( f(v_{13}) = 6 \Rightarrow f(v_{12}) = 2 \) and \( f(v_{23}) = 4 \) (wlog) \( \Rightarrow v_{33} !. \)
- \( f(v_{13}) = 1 \Rightarrow f(v_{12}) = 3 \) and \( f(v_{23}) = 5 \) or \( f(v_{12}) = 6 \) and \( f(v_{23}) = 4 \)
or \( f(v_{12}) = 6 \) and \( f(v_{23}) = 3 \) (wlog) \( \Rightarrow v_{24} ! \) (in the first two cases) or \( f(v_{24}) = 5 \) (in the third case) \( \Rightarrow v_{24} ! \) or \( v_{34} !. \)

Case 3: \( f(v_{22}) = 1 \)
- \( f(v_{13}) = 6 \Rightarrow \{v_{12}, v_{23}\} \in \{3, 4\} !. \)
- \( f(v_{13}) = 5 \Rightarrow \{v_{12}, v_{23}\} \in \{3\} !. \)
- \( f(v_{13}) = 4 \Rightarrow \{v_{12}, v_{23}\} \in \{6\} !. \)
- \( f(v_{13}) = 2 \Rightarrow f(v_{12}) = 4 \) and \( f(v_{23}) = 6 \) (wlog) \( \Rightarrow f(v_{33}) = 3 \) and \( f(v_{24}) = 0 \Rightarrow v_{34} !. \)

Case 4: \( f(v_{22}) = 2 \)
- \( f(v_{13}) = 5 \Rightarrow \{v_{12}, v_{23}\} \in \{0\} !. \)
- \( f(v_{13}) = 6 \Rightarrow f(v_{12}) = 0 \) and \( f(v_{23}) = 4 \) (wlog) \( \Rightarrow v_{33} !. \)
- \( f(v_{13}) = 4 \Rightarrow f(v_{12}) = 0 \) and \( f(v_{23}) = 6 \) (wlog) \( \Rightarrow v_{33} !. \)

\( \Box \)
Proof of Theorem 4. By Theorem 3, it is enough to prove that $G = \lambda(\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}) = 6$ if and only if $7$ divides $m$ and $n$.

If both $m$ and $n$ are divisible by $7$, the following periodic labeling is easily checked to be a $L(2,1)$-labeling of $\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}$: the pattern $0246135$ is repeated along the cycles. More explicitly, $f(i, j) = 0, 2, 4, 6, 1, 3, 5$ if $i + j \equiv 2, 3, 4, 5, 6, 0, 1 \pmod{7}$, respectively.

On the other hand, assume that $G$ admits a $6-L(2,1)$-labeling $f$. By Lemma 3, $f$ is diagonal. Similarly as in the proof of Theorem 2, it is simple to check that, if $m \geq n + 3$, then $f$ induces a $6-L(2,1)$-labeling of $\overrightarrow{C_m-n} \boxtimes \overrightarrow{C_n}$. Again, applying this argument consecutively, we are either left with a $\overrightarrow{C_{d+1}} \boxtimes \overrightarrow{C_{d}}$, where $d = \gcd(m, n)$, if $d \geq 3$, or a $\overrightarrow{C_{k+2}} \boxtimes \overrightarrow{C_{k}}$, or a $\overrightarrow{C_{k+2}} \boxtimes \overrightarrow{C_{k}}$.

In the last two cases, the fact that $f$ is diagonal immediately implies that there are either two consecutive vertices or two vertices within distance two which receive the same color, which is a contradiction.

We are done, then, if we prove that $\lambda(\overrightarrow{C_d} \boxtimes \overrightarrow{C_d}) \geq 7$ if $d$ is not a multiple of $7$. Indeed, assume that there is a $6-L(2,1)$-labeling of $\overrightarrow{C_d} \boxtimes \overrightarrow{C_d}$. Again, by Lemma 3, it should be diagonal. In particular, it means that it corresponds to a labeling of the cycle $C_d$ with the following property: every pair of vertices with distance at most two must receive colors two apart, and every pair of vertices with distance three or four must receive distinct colors: indeed, the vertex $(i + 1, j)$ is a neighbor of $(i, j)$, and the vertex $(i + 2, j)$ has the same color as $(i + 1, j + 1)$, which is adjacent to $(i, j)$ in $\overrightarrow{C_d \boxtimes C_d}$, so they must receive colors two apart from the color of $(i, j)$; similarly we prove that $(i + 3, j)$ and $(i + 4, j)$ must receive distinct colors from $(i, j)$. This coloring is denoted in Figure 1: A $\overrightarrow{P_4} \boxtimes \overrightarrow{P_4}$ subgraph of $\overrightarrow{C_m} \boxtimes \overrightarrow{C_n}$.
the literature by $L(2, 2, 1, 1)$-labeling. Note that, in particular, this implies the statement for $d = 3$ and $d = 4$. Let us assume in what follows that $d \geq 5$.

If $C_d$ has such a coloring, it is readily checked that it must use the color 0 or 6. Indeed, otherwise all available colors are 12345, and it is impossible to color a $P_5$ subgraph of $C_d$ with only these colors.

By symmetry, we may assume that 0 is used. Let $c$ be the coloring of $C_d$, and let the integers modulo $d$ represent its vertices. Let us assume that $c(0) = 0$.

If $c(1) = 3$, then 2 and 3 must receive colors from 5, 6. If $c(2) = 5$, then $c(3) = 1$, and then $c(-1) = 6$, $c(-2) = 2$, $c(-3) = 4$, and finally there is no available color for -4. If $c(2) = 6$, then $c(3) = 1$, and then $c(-1) = 5$, $c(-2) = 2$, and there is no available color for -3.

If $c(1) = 4$, then $c(2)$ is either 2 or 6. In the first case, $c(-1) = 6$ and there is no available color for 3. In the second, $c(-1) = 2$, which implies $c(-2) = 5$ and then there is no available color for -3.

If $c(1) = 6$, then $c(2) \in \{2, 3, 4\}$. The first implies $c(3) = 4$ and there is no color for 4. The second implies $c(3) = 1$, and then $c(4) = 5$ and there is no color for 5. Finally, the third implies that either $c(3) = 1$ or $c(3) = 2$, both of which makes impossible to find a color for 4.

The argument above implies that the neighbors of 0 must have colors 2 and 5. Without loss of generality, we may assume that $c(1) = 5$ and $c(-1) = 2$. This implies that $c(2) = 3$, which in turn implies $c(3) = 1$, then $c(4) = 6$ and $c(5) = 4$. It follows that $c(6) \in \{0, 2\}$, but by the paragraphs above, 4 cannot be a neighbor of 0, so $c(6) = 2$. Then $c(7)$ is either 0 or 5. If it is 5, it implies that $c(8) = 0$, and by the argument above, $c(9)$ must be 2, with contradicts $c(6) = 2$. This means that $c(7) = 0$, and the block 2053164 of size 7 is repeated. The only way the coloring can be completed along the cycle is, then, if 7 divides $d$.

4 Final remarks

The natural next step would be to close the gap left from Theorem 4, deciding for which $m$ and $n$ we have $\lambda(C_m \boxtimes C_n) = 7$.

In the proof of Theorem 4, we gave a periodic 6-labeling of $\lambda(C_7 \boxtimes C_7)$, namely that one in which the pattern 0246135 is repeated along the cycles diagonally. In a similar fashion, the following periodic 7-coloring works for $\lambda(C_8 \boxtimes C_8)$: 02461357. Concatenating these two patterns, one can show that $\lambda(C_m \boxtimes C_m) = 7$ for every sufficiently large $m$ (namely, for every $m \in S(7, 8)$; in particular for $m \geq 42$), and consequently $\lambda(C_m \boxtimes C_n) = 7$ for every $m, n$ such that 7 does not divide both $m$ and $n$ and $\gcd(m, n) \geq 42$.

Finally, we remark that it is simple to check that the proof of Lemma 2 works in the setting of strong product of cycles as well. As we know from the
paragraph above that $\lambda(\overrightarrow{C}_m \boxtimes \overrightarrow{C}_m) = 7$ for every $m$ in $S(7, 8)$ (and, in particular for every $m \geq 42$), to prove that $\lambda(\overrightarrow{C}_m \boxtimes \overrightarrow{C}_n) = 7$ for all sufficiently large $m$ and $n$, it would be enough to find a pair of coprime integers $a, b \in S(7, 8)$ such that $\lambda(\overrightarrow{C}_a \boxtimes \overrightarrow{C}_b) = 7$.

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