AN EXAMPLE OF A TEICHMÜLLER DISK IN GENUS 4 WITH DEGENERATE KONTSEVICH-ZORICH SPECTRUM

GIOVANNI FORNI AND CARLOS MATHEUS

ABSTRACT. We construct an orientable holomorphic quadratic differential on a Riemann surface of genus 4 whose $SL(2, \mathbb{R})$-orbit is closed and has a highly degenerate Kontsevich-Zorich spectrum. This example is related to a previous similar construction in genus 3 by the first author.

1. INTRODUCTION

The goal of this note is the construction of an orientable quadratic differential $q$ on a Riemann surface $M$ of genus 4 such that the Kontsevich-Zorich cocycle along the $SL(2, \mathbb{R})$-orbit of $(M, q)$ has a totally degenerate Kontsevich-Zorich spectrum, in the sense that all non-trivial Lyapunov exponents of the Kontsevich-Zorich cocycle are equal to zero. A similar example in genus 3 appeared in [6]. Our example is given by a non-primitive Veech surface in the stratum $H(2, 2, 2)$ of abelian differentials with 3 double zeros. Unpublished work of M. Möller implies that there are very few examples of this kind and only for abelian differentials on surfaces of genus $g \leq 5$. It is likely that the Forni’s example in genus 3 [6] and the example presented here are the only examples of Veech surfaces with vanishing non-trivial Kontsevich-Zorich exponents [11].

In order to explain more precisely our results, let us recall the definition of the Teichmüller flow and the Kontsevich-Zorich cocycle.

Given a surface $M$, we denote by $\text{Diff}^+(M)$ the group of orientation-preserving diffeomorphisms of $M$ and $\text{Diff}_0^+(M)$ the connected component of the identity in $\text{Diff}^+(M)$ (i.e., $\text{Diff}_0^+(M)$ is the subset of diffeomorphisms in $\text{Diff}^+(M)$ which are isotopic to the identity).

Definition 1.1. Let $Q_g$ be the Teichmüller space of holomorphic quadratic differentials on a surface of genus $g \geq 1$:

$$Q_g = \{\text{holomorphic quadratic differentials}\} / \text{Diff}_0^+(M).$$

Also, let $\mathcal{M}_g$ be the moduli space of holomorphic quadratic differentials on a surface of genus $g \geq 1$:

$$\mathcal{M}_g = \{\text{holomorphic quadratic differentials}\} / \Gamma_g,$$

Date: September 30, 2008.
where $\Gamma_g$ is the mapping class group $\Gamma_g := \text{Diff}^+(M) / \text{Diff}_0^+(M)$.

Moreover, we note that the group $\text{SL}(2, \mathbb{R})$ acts naturally on $M_g$ by linear transformations on the pairs of real-valued 1-forms $(\text{Re}(q^{1/2}), \text{Im}(q^{1/2}))$. The Teichmüller (geodesic) flow $G_t$ is given by the action of the diagonal subgroup $\text{diag}(e^t, e^{-t})$ of $\text{SL}(2, \mathbb{R})$ on $M_g$.

For later reference, we recall some of the main structures of the Teichmüller space $Q_g$ and the moduli space $M_g$:

- $M_g$ and $Q_g$ are stratified into analytic spaces $M_\kappa$ and $Q_\kappa$ obtained by fixing the multiplicities $\kappa = (k_1, \ldots, k_\sigma)$ of the zeroes $\{p_1, \ldots, p_\sigma\}$ of the quadratic differentials (here $\sum k_i = 4g - 4$);
- the total area function $A : M_g \to \mathbb{R}^+$, $A(q) = \int_M |q|$ is $\text{SL}(2, \mathbb{R})$-invariant so that the unit bundle $M_g(1) := A^{-1}(1)$ and its strata $M_\kappa^{(1)} := M_g \cap M_\kappa$ are $\text{SL}(2, \mathbb{R})$-invariant (and, a fortiori, $G_t$-invariant);
- a stratum $M_\kappa$ of orientable quadratic differentials (that is, quadratic differentials obtained as squares of holomorphic 1-forms) has a locally affine structure modeled on the cohomology $H^1(M, \Sigma_\kappa, \mathbb{C})$, relative to the zero set $\Sigma_\kappa := \{p_1, \ldots, p_\sigma\}$, with local charts are given by the period map $q \mapsto [q^{1/2}] \in H^1(M, \Sigma_\kappa, \mathbb{C})$;
- the Lebesgue measure on the Euclidean space $H^1(M, \Sigma_\kappa, \mathbb{C})$, appropriately normalized, induces an absolutely continuous $\text{SL}(2, \mathbb{R})$-invariant measure $\mu_\kappa$ on $M_\kappa$ such that the conditional measure $\mu_\kappa^{(1)}$ induced on $M_\kappa^{(1)}$ is $\text{SL}(2, \mathbb{R})$-invariant (and hence $G_t$-invariant).

Once we get the existence of a good invariant measure $\mu_\kappa^{(1)}$ for the Teichmüller flow, it is natural to ask whether $\mu_\kappa^{(1)}$ has finite mass and/or $\mu_\kappa^{(1)}$ is ergodic with respect to the Teichmüller dynamics. In this direction, Veech [13] showed that the strata are not always connected. More recently, Kontsevich and Zorich [8] (in the orientable case) and Lanneau [9] (in the non-orientable case) gave a complete classification of the connected components of all strata of holomorphic quadratic differentials. Taking this into account, we have the following result:

**Theorem 1.2 (Masur [10], Veech [12]).** The total volume of $\mu_\kappa^{(1)}$ is finite and the Teichmüller flow $G_t = \text{diag}(e^t, e^{-t})$ is ergodic on each connected component of $M_\kappa$ with respect to $\mu_\kappa^{(1)}$.

In order to analyze the Lyapunov spectrum (i.e., the collection of the Lyapunov exponents) of the Teichmüller flow, Kontsevich and Zorich [7] introduced the following notion: the Kontsevich-Zorich cocycle $G_t^{KZ}$ is the quotient of the trivial cocycle $G_t \times \text{id} : Q_g \times H^1(M, \mathbb{R}) \to Q_g \times H^1(M, \mathbb{R})$ with respect to the action of the mapping class group $\Gamma_g$. It is known that
the cocycle $G^K_Z$ is symplectic, so that the Lyapunov spectrum of $G^K_Z$ with respect to any $G_t$-invariant ergodic probability $\mu$ is symmetric:

$$1 = \lambda_1^\mu \geq \ldots \lambda_g^\mu \geq 0 \geq -\lambda_g^\mu \geq \ldots \geq -\lambda_1^\mu = -1.$$  

It turns out that the $g$ numbers $1 = \lambda_1^\mu \geq \lambda_2^\mu \geq \ldots \geq \lambda_g^\mu$ appearing in the non-negative part of the Kontsevich-Zorich spectrum determines the Lyapunov spectrum of the Teichmüller flow (this is one of the motivation for introducing the Kontsevich-Zorich cocycle). Indeed, it is possible to show that, for any ergodic probability measure $\mu$ on $M_g$ supported on a stratum of orientable quadratic differentials with $\sigma \in \mathbb{N}$ distinct zeros, the Lyapunov spectrum of the Teichmüller flow is

$$2 = (1 + \lambda_1^\mu) \geq \ldots \geq \underbrace{(1 + \lambda_g^\mu)}_{\sigma-1} \geq 1 = \ldots = 1 \geq (1 - \lambda_g^\mu) \geq \ldots \geq (1 - \lambda_1^\mu) \geq 1.$$  

On the other hand, concerning the Lyapunov spectrum of the Kontsevich-Zorich cocycle, Zorich and Kontsevich conjectured that the Lyapunov exponents of $G^K_Z$ for the canonical absolutely continuous measure $\mu^{(1)}_\kappa$ on any stratum of orientable quadratic differentials are all non-zero (i.e., non-uniform hyperbolicity) and distinct (i.e., all Lyapunov exponents have multiplicity 1). After the fundamental works of G. Forni [5] (showing the non-uniform hyperbolicity of $G^K_Z$) and Avila, Viana [1] (proving the simplicity of the Lyapunov spectrum), it follows that Zorich-Kontsevich conjecture is true. In other words, the Lyapunov exponents of a $\mu^{(1)}_\kappa$-generic point are all non-zero and they have multiplicity 1. However, it remains to understand the dynamical behavior under $G^K_Z$ of the non-generic orbits (with respect to $\mu^{(1)}_\kappa$). In particular, one can follow Veech and ask how “degenerate” the Lyapunov spectrum of $G^K_Z$ can be along a non-typical orbit. This question was first answered by G. Forni [6] who exhibited an example of an orientable holomorphic quadratic differential $q$ on a Riemann surface $M$ of genus $g = 3$ such that the Kontsevich-Zorich spectrum of the $SL(2, \mathbb{R})$-invariant measure $\mu$ supported on the (closed) $SL(2, \mathbb{R})$-orbit of $(M, q)$ verifies $\lambda_2^\mu = \lambda_3^\mu = 0$.

At this point, we are able to state our main result:

**Theorem 1.3.** There exists an orientable holomorphic quadratic differential $q$ on a Riemann surface $M$ of genus 4 such that the Lyapunov exponents (with respect to $G^K_Z$) of the $SL(2, \mathbb{R})$-invariant probability $\mu$ supported on
the $SL(2, \mathbb{R})$-orbit of $(M, q)$ verifies
\[ \lambda_2^q = \lambda_3^q = \lambda_4^q = 0. \]

The organization of this note is the following: in §2, we will recall Forni’s version of the Kontsevich-Zorich formula for the sum of the Lyapunov exponents of $G_t^{KZ}$, in §3 we review Forni’s method [6] to exploit symmetries and, in section §4, we will complete the proof of our theorem.

Acknowledgements: We would like to thank J.-C. Yoccoz, who motivated our work by asking whether the example in [6] could be generalized. We are also very grateful to M. Möller who introduced us to the notion of a cyclic cover, directed us to the reference [2] and suggested the correct algebraic equation of our example.

2. THE KONTSEVICH-ZORICH FORMULA REVISITED

Let $q$ be a holomorphic quadratic differential on a Riemann surface $M$ of genus $g \geq 2$. Fix $z = x + iy$ a holomorphic local coordinate and write $q = \phi(z)dz^2$. It follows that the (degenerate) Riemannian metric $R_q$ and the area form $\omega_q$ induced by $q$ are $R_q = |\phi(z)|^{1/2}(dx^2 + dy^2)^{1/2}$, $\omega_q = |\phi(z)|dx \wedge dy$. Denote by $S = \partial/\partial x$ and $T = \partial/\partial y$ the horizontal and vertical directions. Define $L^2_q(M) := L^2(M, \omega_q)$ the space of complex-valued square-integrable functions and $H^1_q(M)$ the Sobolev space of functions $v \in L^2_q(M)$ such that $Sv, Tv \in L^2_q(M)$.

Lemma 2.1 (Forni [4], prop. 3.2). The Cauchy-Riemann operators $\partial^\pm := S\pm iT/2$ with (dense) domain $H^1_q(M) \subset L^2_q(M)$ are closed. Moreover, $\partial^\pm$ has closed range of finite codimension (equal to the genus $g$ of $M$). Furthermore, denoting by $M_q^\pm \subset L^2_q(M)$ the subspaces of meromorphic (resp. anti-meromorphic) functions, we have the orthogonal decompositions:

\[ L^2_q(M) = \text{Ran}(\partial^+_q) \oplus M_q^- = \text{Ran}(\partial^-_q) \oplus M_q^+. \]

Denote by $\pi^\pm_q : L^2_q(M) \to M_q^\pm$ the orthogonal projection. Let $H_q$ be the (non-negative definite) Hermitian form on $M_q^+ \subset L^2_q(M)$ given by

\[ H_q(m_1^+, m_2^+) := \langle \pi_q^-(m_1^+), \pi_q^-(m_2^+) \rangle_q := \int_M \pi_q^-(m_1^+) \cdot \overline{\pi_q^-(m_2^+)} \omega_q \]

for all $m_1^+, m_2^+ \in M_q$. Let $1 \equiv \Lambda_1(q) \geq \Lambda_2(q) \geq \cdots \geq \Lambda_g(q) \geq 0$ be the eigenvalues of $H_q$.

Next, we consider the sets
\[ R^{(1)}_g(k) := \{ q \in M^{(1)}_g : \Lambda_{k+1}(q) = \cdots = \Lambda_g(q) = 0 \}. \]
Definition 2.2. The set \( R_g^{(1)}(g - 1) \) is called the determinant locus.

The relevance of the natural filtration of sets \( R_g^{(1)}(1) \subset \cdots \subset R_g^{(1)}(g - 1) \) for the study of the Lyapunov spectrum of \( G^{KZ}_t \) becomes evident from the following version of a formula by Kontsevich and Zorich [7] for the sum of Lyapunov exponents:

**Theorem 2.3** (Forni [5], Corollary 5.3). Let \( \mu \) be a \( SL(2, \mathbb{R}) \)-invariant ergodic probability measure on \( \mathcal{M}_g^{(1)} \). Then, the Lyapunov exponents of \( G^{KZ}_t \) with respect to \( \mu \) satisfy the formula:

\[
\lambda_1^\mu + \cdots + \lambda_g^\mu = \int_{\mathcal{M}_g^{(1)}} (\Lambda_1(q) + \cdots + \Lambda_g(q)) d\mu(q).
\]

In particular, since \( \lambda_1^\mu = 1 \equiv \Lambda_1(q) \), we have

\[
\lambda_2^\mu + \cdots + \lambda_g^\mu = \int_{\mathcal{M}_g^{(1)}} (\Lambda_2(q) + \cdots + \Lambda_g(q)) d\mu(q).
\]

A direct consequence of this formula is:

**Corollary 2.4** (Forni [6], Corollary 7.1). Let \( \mu \) be any \( SL(2, \mathbb{R}) \)-invariant ergodic probability measure on \( \mathcal{M}_g^{(1)} \) supported on a stratum of orientable quadratic differentials. The measure \( \mu \) is supported on the locus \( R_g^{(1)}(1) \) if and only if the non-trivial Kontsevich-Zorich spectrum vanishes, that is,

\[
\lambda_2^\mu = \cdots = \lambda_g^\mu = 0.
\]

3. **Symmetries**

In this section we recall the simple method developed in [6] to derive bounds on the rank of the matrix \( H_q \) from symmetries of the orientable holomorphic quadratic differential \( q \). Let \( Aut(M) \) be the group of holomorphic automorphism of the Riemann surface \( M \). Let \( Aut(q) \subset Aut(M) \) is the subgroup formed by automorphisms \( a \in Aut(M) \) such that \( a^*(q) = q \).

Note that there is a natural unitary action of \( Aut(q) \) on \( \mathbb{M}_q^+ \) (by pull-back).

Given \( a \in Aut(q) \), fix \( \{m_1^+(a), \ldots, m_g^+(a)\} \) an orthonormal basis of eigenvectors and denote by \( \{u_1(a), \ldots, u_g(a)\} \) the associated eigenvalues. Let \( B^a(q) \) be the matrix of the operator \( \pi^{-}_q : \mathbb{M}_q^+ \rightarrow \mathbb{M}_q^- \) with respect to the basis \( \{m_1^+(a), \ldots, m_g^+(a)\} \subset \mathbb{M}_q^+ \) and \( \{m_1^+(a), \ldots, m_g^+(a)\} \subset \mathbb{M}_q^- \):

\[
B^a_{ij}(q) = \int_{M} m_i^+(a)m_j^+(a)\omega_q.
\]

For any \( I, J \subset \{1, \ldots, g\} \) with \( \#I = \#J \), denote by \( det B^a_{I,J} \) the minor of the matrix \( B^a(q) \) with entries \( B^a_{ij}(q) \), \( i \in I, j \in J \).
Lemma 3.1 (Forni [6], Lemma 7.2). The following holds:

$$\prod_{i \in I} \prod_{j \in J} u_i(a)u_j(a) \neq 1 \implies \det B^a_{IJ}(q) = 0.$$

Corollary 3.2. If $$\prod_{i \in I} \prod_{j \in J} u_i(a)u_j(a) \neq 1$$ for all $$I, J \subset \{1, \ldots, g\}$$ with $$\#I = \#J = k$$, then rank $$(H_q) \leq g - k$$, hence $$q \in \mathcal{R}_g^{(1)}(k)$$.

Proof. Since the matrix $$H(q)$$ of $$H_q$$ with respect to $${m_1^+(a), \ldots, m_g^+(a)}$$ satisfies $$H(q) = B^a(q)^*B^a(q) = B^a(q)B^a(q)$$ (see equation (44) of [6]), the statement follows immediately from the previous lemma. □

At this point, we are ready to present our genus 4 example and prove Theorem 1.3.

4. The Example

Given an integer $$N > 1$$ and a 4-tuple of integers $$a := (a_1, \ldots, a_4)$$ with $$0 < a_\mu < N$$, $$\gcd(N, a_1, \ldots, a_4) = 1$$ and $$\sum_{\mu=1}^4 a_\mu \equiv 0 \pmod{N}$$. Let $$x_1, x_2, x_3, x_4 \in \mathbb{C}$$ be 4 distinct points and let $$M := M_N(a)$$ be the connected, non-singular Riemann surface determined by the algebraic equation:

$$w^N = (z - x_1)^{a_1}(z - x_2)^{a_2}(z - x_3)^{a_3}(z - x_4)^{a_4}.$$ 

The surface $$M$$ is a cyclic cover of the Riemann sphere $$\mathbb{P}^1(\mathbb{C})$$ branched over the points $$x_1, \ldots, x_4 \in \mathbb{C} \equiv \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$$. In fact, every such cyclic cover (up to isomorphisms) can be written as above (see Example 4.2 in [2]). The surface $$M$$ has genus

$$g = N + 1 - \frac{1}{2} \sum_{\mu=1}^4 \gcd(a_\mu, N).$$

The surface $$M$$ is called a cyclic cover since its automorphism group is cyclic. In fact, it is generated by the automorphism $$T : M \to M$$ given by

$$T(z, w) = (z, \varepsilon^i w),$$

where $$\varepsilon^N = 1$$ is a primitive $$N$$th root of unity.

For $$i \in \{1, \ldots, N - 1\}$$, let $$L_i$$ denote the eigenspace with eigenvalue $$\varepsilon^i$$ for the action of $$T$$ on holomorphic abelian differentials:

$$L_i := \{\omega \in H^1_{\text{dr}}(M, \mathbb{C}) : T^*\omega = \varepsilon^i \cdot \omega\}.$$
Lemma 4.1 (Bouw [2], Lemma 4.3). The following formulas hold:

\[ \dim \mathcal{L}_i = \sum_{\mu=1}^{4} \langle i \cdot a_{\mu} \rangle - 1 \]

(here \(\langle \cdot \rangle\) denotes the fractional part).

We recall that the genus 3 example in [6] was given by the \(SL(2, \mathbb{R})\) orbit of orientable quadratic differentials (or, equivalently, of abelian differentials) constructed from cyclic covers of type \(M_4(1, 1, 1, 1)\).

Proof of Theorem 1.3. Our example is given by an orientable holomorphic quadratic differential constructed from cyclic covers of type \(M_6(1, 1, 1, 3)\).

Let \(M\) be any branched cover of type \(M_6(1, 1, 1, 3)\). Observe that \(M\) has genus 4. Lemma 4.1 yields:

\[ \dim \mathcal{L}_1 = \dim \mathcal{L}_2 = 0; \]
\[ \dim \mathcal{L}_3 = \dim \mathcal{L}_4 = 1; \]
\[ \dim \mathcal{L}_5 = 2. \]

Thus, we can fix \(\theta_1 \in L_3, \theta_2 \in L_4\) and \(\theta_3, \theta_4 \in L_5\) so that \(\{\theta_1, \ldots, \theta_4\}\) is a basis of the space of holomorphic differentials on \(M\) and the action of \(T^*\) on this basis is diagonal with eigenvalues \(-1 = \varepsilon^3\) (with multiplicity 1), \(\varepsilon^4\) (with multiplicity 1) and \(\varepsilon^5\) (with multiplicity 2). Let \(q = \theta_1^2\) be the orientable quadratic differential obtained as pull-back to \(M\) of the unique holomorphic quadratic differential on \(\mathbb{P}(\mathbb{C})\) with simple poles at the branching points \(\{x_1, \ldots, x_4\}\). It follows that the spectrum of the action of \(T \in \text{Aut}(q)\) on the space \(\mathcal{M}_q^+ \subset L^2_2(M)\) is

\[ u_1(T) = 1, \quad u_2(T) = -\varepsilon^4, \quad u_3(T) = -\varepsilon^5, \quad u_4(T) = -\varepsilon^5. \]

Consequently, \(q \in \mathcal{R}_4^{(1)}(1)\). In fact, a direct application of Corollary 3.2 shows that the form \(H_q\) has rank 1, so that \(q \in \mathcal{R}_4^{(1)}(1)\).

Let \(V\) be the set of all orientable quadratic differentials on a cyclic cover of type \(M_6(1, 1, 1, 3)\) branched over 4 points \(\{x_1, \ldots, x_4\} \in \mathbb{P}(\mathbb{C})\) obtained as pull-back of the unique (non-orientable) holomorphic quadratic differential on \(\mathbb{P}(\mathbb{C})\), of total area equal to 1, with simple poles at \(x_1, \ldots, x_4\). Since the stratum of all quadratic differentials on the Riemann sphere with 4 simple poles is \(SL(2, \mathbb{R})\)-invariant and consists of a single \(SL(2, \mathbb{R})\) orbits, the set \(V\) consists of a single \(SL(2, \mathbb{R})\) orbit in the stratum of orientable quadratic differentials with 3 distinct zeros of order 4 (equivalently, of the stratum of squares of abelian differentials with 3 distinct double zeros). The above computation shows that \(V \subset \mathcal{R}_4^{(1)}(1)\), hence (in view of Corollary 2.4) the proof of Theorem 1.3 is completed. \(\square\)
We conclude by the remark that our exploration of other cyclic covers has failed to provide additional example of totally degenerate Kontsevich-Zorich spectra. We recall that according to Möller [11], there are no more examples of totally degenerate Kontsevich-Zorich spectrum among Veech surfaces in any genus.

REFERENCES

[1] A. Avila and M. Viana, Simplicity of Lyapunov Spectra: Proof of the Zorich-Kontsevich conjecture, Acta Math., v. 146, no. 2, 295–344, 1997.
[2] I. Bouw, The $p$-rank of ramified covers of curves, Compositio Math., v. 126, 295–322, 2001.
[3] I. Bouw and M. Möller, Teichmüller curves, triangle groups and Lyapunov exponents, to appear in Ann. of Math.
[4] G. Forni, Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus, Ann. of Math., v. 146, no. 2, 295–344, 1997.
[5] ______, Deviation of ergodic averages for area-preserving flows on surfaces of higher genus, Annals of Math., v.155, no. 1, 1–103, 2002.
[6] ______, On the Lyapunov exponents of the Kontsevich-Zorich cocycle, Handbook of Dynamical Systems v. 1B, B. Hasselblatt and A. Katok, eds., Elsevier, 2006, 549-580.
[7] M. Kontsevich, Lyapunov exponents and Hodge theory, in ‘The mathematical beauty of physics’, Saclay, 1996. Adv. Ser. Math. Phys. v. 2, 318–332, World Scientific, River Edge, NJ, 1997.
[8] M. Kontsevich and A. Zorich, Connected components of the moduli spaces of Abelian differentials with prescribed singularities, Invent. Math. v. 153, no. 3, 631-678, 2003.
[9] E. Lanneau, Connected components of the strata of the moduli spaces of quadratic differentials, An. Sc. de l’ENS, v. 41, 1-56, 2008.
[10] H. Masur, Interval exchange transformations and measured foliations, Annals of Mathematics, v. 115, 169-200, 1982.
[11] M. Möller, personal communication, August 2008.
[12] W. Veech, Teichmüller Geodesic Flow, Annals of Mathematics, v. 124, 441-530, 1986.
[13] ______, Moduli spaces of quadratic differentials, J. Anal. Math., v. 55, 117–171, 1990.

GIOVANNI FORNI: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742-4015, USA
E-mail address: gforni@math.umd.edu.

CARLOS MATHEUS: COLLÈGE DE FRANCE, 3, RUE D’ULM, PARIS, CEDEX 05, FRANCE
E-mail address: matheus@impa.br.