Abstract. We prove that the spatial realization of a rational complete Lie algebra $L$, concentrated in degree 0, is isomorphic to the simplicial Bar construction on the group, obtained from the Baker-Campbell-Hausdorff product on $L$.

Introduction

In [1], we construct a cosimplicial differential graded complete Lie algebra (henceforth cdgl) $(\mathcal{L}_\bullet, d)$, in which $(\mathcal{L}_1, d)$ is the Lawrence-Sullivan model of the interval introduced in [5]. As in the work of Sullivan ([7]) for differential commutative graded algebras, the existence of this cosimplicial object gives an adjoint pair of functors between the category cdgl of cdgl’s and the category Sset of simplicial sets, see [1] or [3]. In this work, we focus on one of them, the spatial realization functor,

$$\langle - \rangle : \text{cdgl} \to \text{Sset},$$

defined by $\langle L \rangle = \text{Hom}_{\text{cdgl}}(\mathcal{L}_\bullet, L)$ for $L \in \text{cdgl}$. (Let us also notice that $\langle L \rangle$ is isomorphic to the nerve of $L$, a deformation retract of the Getzler-Hinich realization, see [2], [6].)

More precisely, we are interested in the realization $\langle L \rangle$ of a complete Lie algebra, $L$, concentrated in degree 0 and (thus) with the differential 0. In this case, a group structure can be defined on the set $L$ from the Baker-Campbell-Hausdorff formula. We denote by exp $L$ this group. The main result of this work is

**Main Theorem.** Let $L$ be a complete differential graded Lie algebra, concentrated in degree 0. Then, its spatial realization $\langle L \rangle$ is isomorphic to the simplicial Bar construction on exp $L$.

In Section 1, we recall basic background on cdgl and our construction $\mathcal{L}_\bullet$. Section 2 consists of the proof of the Main theorem.

1. Some Reminders

We first recall the construction of the cosimplicial cdgl $\mathcal{L}_\bullet$. Let $V$ be a finite dimensional graded vector space. The completion of the free graded Lie algebra on $V$, $\mathbb{L}(V)$,
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\[ \delta^i a_{i_0 \ldots i_p} = a_{j_{i_0} \ldots j_p} \quad \text{with} \quad j_k = \begin{cases} i_k + 1 & \text{if } i_k < i, \\ i_k & \text{if } i_k \geq i, \end{cases} \]  

(1.1)

and the codegeneracies \( \sigma^i \) of the cosimplicial cdgl \( \mathfrak{L}_\bullet \) are defined by

\[ \sigma^i a_{i_0 \ldots i_p} = \begin{cases} 0 & \text{if } \{i, i + 1\} \subset \{i_0, \ldots, i_p\}, \\ a_{j_{i_0} \ldots j_p} & \text{otherwise, with} \quad j_k = \begin{cases} i_k & \text{if } i_k \leq i, \\ i_k - 1 & \text{if } i_k > i. \end{cases} \end{cases} \]  

(1.2)

2. Proof of the main theorem

The simplicial bar construction \( B_\bullet G \) on a group \( G \), with rational coefficients, is the simplicial rational vector space with \( B_n G \) the \( \mathbb{Q} \)-vector space generated by the symbols \( [g_1 \ldots g_n], \ g_i \in G \). Its faces \( d_i \) and degeneracies \( s_i \) are defined as follows:

\[ d_0 [g_1 \ldots g_n] = [g_2 \ldots g_n], \]  

(2.1)

\[ d_i [g_1 \ldots g_n] = [g_1 \ldots g_{i-1}g_{i+1} \ldots g_n], \quad \text{for} \quad 0 < i < n, \]  

\[ d_n [g_1 \ldots g_n] = [g_1 \ldots g_{n-1}]. \]
The degeneracy \( s_i \) consists to insert the identity \( e \) in position \( i \).

Let \( L \in \text{cdgl} \) be generated in degree 0 and \( f: \mathcal{L}_n \to L \) a morphism in \( \text{cdgl} \). For degree reasons, we have \( f(a_{i_0 \ldots i_k}) = 0 \) if \( k \neq 1 \). Moreover, since \( f \) commutes the differential, from the definition of the differential in \( \mathcal{L}_2 \), we get

\[
0 = df(a_{0rs}) = f(a_{0r}) \ast f(a_{rs}) \ast f(a_{0s})^{-1}.
\]

Therefore, for any \( r, s > 0 \), we have

\[
f(a_{rs}) = f(a_{0r})^{-1} \ast f(a_{0s}).
\]

The map \( f \) being entirely defined by its values on the \( a_{qr} \), we have a bijection

\[
\Phi: \text{Hom}_{\text{cdgl}}(\mathcal{L}_n, L) \to L^n,
\]

defined by \( f \mapsto (f(a_{01}), f(a_{02}), \ldots, f(a_{0n})) \).

We now determine the image of the faces and degeneracies on \( \text{Hom}_{\text{cdgl}}(\mathcal{L}_n, L) \), induced from (1.1) and (1.2). For the face operators, as only the elements \( (a_{0r}) \) play a role, it suffices to consider,

\[
\delta^0(a_{0r}) = \begin{cases} a_{0r} & \text{if } r < i \\ a_{0r+1} & \text{if } r \geq i \end{cases} \quad \text{for } i > 0, \quad \text{and } \delta^0(a_{0r}) = a_{1,r+1}.
\]

Let \( f: \mathcal{L}_n \to L \) described by \( (f(a_{01}), \ldots, f(a_{0n})) \). Then \( d_0 f = f \circ \delta^0: \mathcal{L}_{n-1} \to L \) is described by

\[
d_0(f(a_{01}), \ldots, f(a_{0(n-1)})) = (f \circ \delta^0(a_{01}), \ldots, f \circ \delta^0(a_{0(n-1)})) = (f(a_{12}), \ldots, f(a_{1n})) = (f(a_{01})^{-1} \ast f(a_{02}), \ldots, f(a_{01})^{-1} \ast f(a_{0n})).
\]

Therefore, the face operator \( d_0 \) on \( L^\bullet \), induced from \( \Phi \), is

\[
d_0(x_1, \ldots, x_n) = (x_1^{-1} \ast x_2, \ldots, x_1^{-1} \ast x_n).
\]

Similar arguments give, for \( i > 0 \),

\[
d_i(x_1, \ldots, x_n) = (x_1, \ldots, \hat{x}_i, \ldots, x_n).
\]

As for the degeneracies, starting from

\[
\sigma^i(a_{0r}) = \begin{cases} a_{0r} & \text{if } r \leq i \\ a_{0,r-1} & \text{if } r > i \end{cases} \quad \text{for } i > 0,
\]

\[
\sigma^0(a_{0r}) = a_{0, r-1} \quad \text{if } r > 1, \quad \text{and } \sigma^0(a_{0,1}) = 0,
\]

we get

\[
s_0(a_1, \ldots, a_n) = (0, a_1, \ldots, a_n)
\]

and for \( i > 0 \),

\[
s_i(a_1, \ldots, a_n) = (a_1, \ldots, a_i, \ldots, a_n).
\]

Now a straightforward and easy computation shows that the morphism

\[
\Psi: \text{Hom}_{\text{cdgl}}(\mathcal{L}_n, L) \to B^\bullet(L)
\]

defined by

\[
\Psi(a_1, \ldots, a_n) = (a_1, a_1^{-1}a_2, a_2^{-1}a_3, \ldots, a_{n-1}^{-1}a_n)
\]

is an isomorphism of simplicial sets.
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