On Compact Perturbations of Laurent and Toeplitz Operators

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Abstract

Let \( f \) be a regular real-valued non-constant symbol defined on \( \mathbb{T}^d \). Denote respectively by \( \kappa \) and \( L \), its set of critical points and the associated Laurent operator on \( l^2(\mathbb{Z}^d) \). If \( V \) is a suitable compact perturbation, we prove that the operator \( L + V \) has finite point spectrum and no singular continuous component away from the set \( f(\kappa) \). Some propagating estimates are derived. We also show that a similar result can be established for compact perturbations of Toeplitz matrices.

Keywords: Spectrum, Commutator, Laurent operator, Toeplitz matrices, perturbation.

1 Introduction

It is known that the essential spectrum of a self-adjoint operator remains invariant under relatively compact perturbations (Weyl’s Theorem, e.g. [24]). Unfortunately, even if the spectrum of the unperturbed operator is well-known, this result tells nothing about the fine structure of the essential spectrum of the perturbed one. This issue requires the elaboration of complementary tools, which actually depend mostly on the nature of the unperturbed operator. The spectral theory of Laurent and Toeplitz operators has been extensively studied over the years, but to our knowledge, only partial answers have been provided regarding their compact perturbations: these limitations are expressed either in terms of the class of the symbols considered (polynomials mainly) [9] or in terms of the perturbations [19].

In the present paper, we develop a systematic approach based on the regular Mourre theory, which applies for a general class of symbol and perturbations. First, we extend the strategy of [9] to study the spectral effects of a compact and regular perturbation on Laurent operators associated to some regular real-valued symbol (Theorem 2.1). Proposition 2.1 gives some propagation estimates. Then, these results are applied to analyze the spectral properties of some compact perturbations of Toeplitz matrices (Theorem 2.2). Let us mention the existence of complementary methods and results in random settings [8], [11].

The manuscript is structured as follows. The main results are introduced in Section 2 and illustrated in Section 3. The proofs are presented in Sections 4, 5 and 6. Some auxiliary results are postponed to Section 7.

Notations: The resolvent set of a closed operator \( C \) defined on some Hilbert space \( \mathcal{H} \) is denoted by \( \rho(C) \) and its spectrum by: \( \sigma(C) = \mathbb{C} \setminus \rho(C) \). The unit circle and the one-dimensional torus are denoted by \( \partial \mathbb{D} \) and \( \mathbb{T} \) respectively. To any Borel function \( F \) on \( \partial \mathbb{D}^d \) is associated a unique function \( f \) defined on \( \mathbb{T}^d \) by: \( f(\theta) = F(e^{i\theta}) \), for all \( \theta \in \mathbb{T}^d \). If the function \( f \) belongs to \( C^1(\mathbb{T}^d) \), the derivatives of \( f \), \( (\partial_{\theta_j} f)_{j=1}^d \) (resp.
\( \nabla f \) are associated in a unique manner to a family of functions on \( \partial \mathbb{D}^d \) denoted by \( (\partial_j F)^d_{j=1} \) (resp. \( \nabla F \)), which can be expressed by the chain rule: \( \forall j \in \{1, \ldots, d\}, \forall \theta \in \mathbb{T}^d, \)

\[
(\partial_j F)(e^{i\theta}) := \partial \theta_j f(\theta) = i e^{i\theta_j} (\partial_{\theta_j} F)(e^{i\theta}) .
\]

If \( U \) is a unitary operator defined on \( \mathcal{H} \) and if its spectral family is denoted by \( (E(\Delta))_{\Delta \in \mathcal{B}(\mathbb{T})} \), where \( \mathcal{B}(\mathbb{T}) \) stands for the family of Borel sets of \( \mathbb{T} \), we will have that:

\[
G(U) = \int_{\mathbb{T}} g(\theta) dE(\theta) = \int_{\mathbb{T}} G(e^{i\theta}) dE(\theta) ,
\]

for any bounded Borel functions \( G \) (resp. \( g \)). If \( A \) is a self-adjoint operator defined on \( \mathcal{H} \), we define \( \langle A \rangle := \sqrt{1 + A^2} \). The Fourier transform between \( l^2(\mathbb{Z}^d) \) and \( L^2(\mathbb{T}^d) \) is denoted by \( \mathcal{F} \).

## 2 Hypotheses and Main Results

Our first main result, Theorem \[2.1\] deals with the spectral properties of self-adjoint operators of the form \( L_f + V \), where \( L_f \) is the Laurent operator on \( l^2(\mathbb{Z}^d) \) associated to the real-valued regular symbol \( f \) and \( V \) is a compact regular symmetric perturbation. A precise meaning is given to this sentence in the next paragraphs.

### 2.1 Regularity issues

**Regularity of the symbol.** Let us denote by \( n = (n_1, n_2, \ldots, n_d) \), \( n_i \in \mathbb{Z} \), the vectors of the \( d \)-dimensional lattice \( \mathbb{Z}^d \) and for \( i = 1, \ldots, d \), we define the shift operators \( T_i \) acting on the Hilbert space \( \mathcal{H} = l^2(\mathbb{Z}^d) \) as follows:

\[
T_i e_n = e_{S_i(n)}/
\]

where \( S_i(n) = (n_1, \ldots, n_{i-1}, n_i + 1, n_{i+1}, \ldots, n_d) \). The unitary operators \( (T_i)^d_{i=1} \) commute: \( [T_i, T_j] = 0 \) for all \( i, j = 1, \ldots, d \).

Let \( f \) be a real-valued continuous symbol. Using the functional calculus, we can define the (bounded symmetric) Laurent operator \( L_f \) by:

\[
L_f = F(T) := F(T_1, \ldots, T_d) .
\]

In particular, if the symbol \( f \) belongs to the Wiener algebra \( \mathcal{A}(\mathbb{T}^d) \) (this is the case if \( f \) belongs to \( C^2(\mathbb{T}, \mathbb{R}) \)): \( f(\theta) = \sum_{\alpha} a_{\alpha} e^{i\alpha \cdot \theta} \), where we write \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d \) and \( (a_{\alpha})_{\alpha \in \mathbb{Z}^d} \) belongs to \( l^1(\mathbb{Z}^d; \mathbb{C}) \) \[14, 20\]. Denoting \( T^\alpha = T_{1}^{\alpha_1} \ldots T_{d}^{\alpha_d} \), the operator \( L_f \) rewrites

\[
L_f = \sum_{\alpha \in \mathbb{Z}^d} a_{\alpha} T^\alpha ,
\]

where the RHS is norm convergent. Since \( f \) is real-valued, \( a_{-\alpha} = \overline{a_{\alpha}} \) for any \( \alpha \in \mathbb{Z}^d \).

The set of critical points associated to the symbol \( f \) is denoted as follows:

\[
\kappa_f = \{ \theta \in \mathbb{T}^d; f \text{ is not differentiable at } \theta \text{ or } \nabla f(\theta) = 0 \} .
\]

If \( f \) is of class \( C^1 \), the set \( \kappa_f \) is a closed subset of \( \mathbb{T}^d \), hence compact. By continuity, it follows that \( f(\kappa_f) \) is a compact subset of \( \sigma(L_f) = \text{Ran} f \).

**Remark:** The operator \( L_f \) is unitarily equivalent to the multiplication operator by \( f(\cdot) \) on \( L^2(\mathbb{T}^d) \):

\[
L_f = \mathcal{F}^* f(\cdot) \mathcal{F} .
\]

This means its spectral properties can be controlled according to the properties of the function \( f \). In particular, we have that:
• the spectrum of $L_f$ coincides with the essential range of $f$, which is is the closure of the range of $f$, $f(T^d)$ if $f$ is continuous. In this case, $\sigma(L_f)$ is a connected and compact subset of $\mathbb{R}$.

• $\lambda$ is an eigenvalue of $L_f$ if and only if $f^{-1}(\{\lambda\})$ has non zero Lebesgue measure

• $L_f$ has purely absolutely continuous spectrum in a subset $I$ of $\mathbb{R}$ if and only if for any Borel set $N \subset I$ of zero Lebesgue measure, $f^{-1}(N)$ is also of measure zero.

• $L_f$ has non-trivial singular continuous spectrum if and only if there exists a Borel set $N \subset \mathbb{R}$ of zero Lebesgue measure, such that $f^{-1}(N)$ has non-zero measure but $f^{-1}(\{\lambda\})$ is of zero measure for each $\lambda \in N$.

\textbf{Regularity of the perturbation.} The regularity of the perturbation $V$ is defined via suitable commutation conditions w.r.t. a fixed auxiliary self-adjoint operator. Generally speaking, if an operator $B$ belongs to $\mathcal{B}(\mathcal{H})$ and $A$ is a self-adjoint operator defined on $\mathcal{H}$ with domain $\mathcal{D}(A)$, we say that $B$ is of class $C^1$ with respect to $A$ ($B \in C^1(A)$), if there exists a dense linear subspace of $\mathcal{H}$, $\mathcal{S} \subset \mathcal{D}(A)$, such that the sesquilinear form $Q_1$, defined by

$$Q_1(\varphi, \psi) := \langle A\varphi, B\psi \rangle - \langle \varphi, BA\psi \rangle$$

for any $(\varphi, \psi) \in \mathcal{S} \times \mathcal{S}$, extends continuously to a bounded form on $\mathcal{H} \times \mathcal{H}$. The bounded linear operator associated to the extension of $Q_1$ is denoted by $\text{ad}_A(B) = [A, B]$. By extension, if $k \in \mathbb{N}$, the operator $B$ is of class $C^k$ with respect to $A$ ($B \in C^k(A)$), if $B \in C^{k-1}(A)$ and if there exists a dense linear subspace of $\mathcal{H}$, $\mathcal{S} \subset \mathcal{D}(A)$, such that the sesquilinear form $Q_k$, defined by

$$Q_k(\varphi, \psi) := \langle A\varphi, \text{ad}_A^{k-1}(B)\psi \rangle - \langle \varphi, \text{ad}_A^{k-1}(B)A\psi \rangle$$

for any $(\varphi, \psi) \in \mathcal{S} \times \mathcal{S}$, extends continuously to a bounded form on $\mathcal{H} \times \mathcal{H}$. The bounded linear operator associated to the extension of $Q_k$ is denoted by $\text{ad}_A(\text{ad}_A^{k-1}(B)) = \text{ad}_A^k(B)$. In other words, the operation $\text{ad}_A$ defines a derivation on $\mathcal{B}(\mathcal{H})$ (see Section 7.2 for details). Equivalently, $B \in C^k(A)$ if and only if the map defined by: $t \mapsto e^{-iAt}Be^{iAt}$ is strongly $C^k$ (see e.g. Section 7 or [4]).

**Remark:** Actually, the regularity scale given by the classes $C^k(A)$ can be refined with the introduction of interpolation spaces $\mathbb{I}$. Without going into details, let us just mention for later purposes that $B \in C^{1,1}(A)$ if:

$$\int_0^1 \|e^{iAt}Be^{-iAt} + e^{-iAt}Be^{iAt} - 2B\| \frac{d\tau}{|\tau|^2} < \infty .$$

In particular, $C^{1,1}(A)$ is a linear subspace of $\mathcal{B}(\mathcal{H})$ and the following inclusions hold: $C^2(A) \subset C^{1,1}(A) \subset C^1(A)$. The reader who would like to avoid excessive technicalities may replace $C^{1,1}(A)$ by $C^3(A)$ in the statements of our main results.

Let us introduce the class of auxiliary self-adjoint operators (or conjugate operators) $(A_g)_{g \in C^1(T^d, \mathbb{R}^d)}$ with respect to which the regularity of $V$ will be measured. First, we denote by $(A_j)_{j=1}^d$ the family of linear operators defined on the canonical orthonormal basis of $l^2(\mathbb{Z}^d)$ by: $A_j e_n = n_j e_n$. These operators $(A_j)_{j=1}^d$ are essentially self-adjoint on $\langle e_n; n \in \mathbb{Z}^d \rangle$ and their respective self-adjoint extensions are also denoted by $A_j$. Note that for all $(j, k) \in \{1, \ldots, d\}^2$, $[A_j, T_k] = \delta_{j,k} T_k$ and that $FA_jF^* = -i\partial_j$. We write:

$$\mathcal{D}_A := \{ \phi \in l^2(\mathbb{Z}^d); \sum_{n \in \mathbb{Z}^d} (1 + n_1^2 + \ldots + n_d^2) |\phi_n|^2 < \infty \} .$$

Given $g \in C^1(T^d, \mathbb{R}^d)$, we can define the operator $A_g$ on $\mathcal{D}_A$ by:

\begin{equation}
A_g := -\frac{i}{2} \mathcal{F}^* (g(\theta) \cdot \nabla + \nabla \cdot g(\theta)) \mathcal{F} = \frac{1}{2} \sum_{j=1}^d L_{g_j} A_j + A_j L_{g_j} = \frac{1}{2} (L_g \cdot A + A \cdot L_g) .
\end{equation}
The proof of the following result is exposed in Section 4.2:

**Lemma 2.1** Let $g$ in $C^1(T^d;\mathbb{R}^d)$. Then the operator $A_g$ defined by (1) is essentially self-adjoint on $\mathcal{D}_A$. Moreover, if $g \in C^k(T^d,\mathbb{R}^d)$ for some $k \geq 1$, then $A_g^s$ is relatively bounded with respect to $\langle A \rangle^s$, $\langle A \rangle := \sqrt{A_1^2 + \ldots + A_d^2 + I}$, for any $s \in \{1, \ldots, k\}$.

The self-adjoint extension of $A_g$ defined in Lemma 2.1 will be also denoted by $A_g$.

Let us state now our main results.

### 2.2 Laurent operators

Let $H$ be the self-adjoint operator defined on $l^2(\mathbb{Z}^d)$ by: $H = L_f + V = F(T) + V$ where $f$ is a continuous real-valued symbol on $\mathbb{T}^d$ and $V$ a compact symmetric perturbation. Using the unitary functional calculus and Weyl’s Theorem, we have that: $\sigma_{ess}(H) = \sigma_{ess}(L_f) = \operatorname{Ran} f$. A precise qualitative description of the spectral nature of $H$ can be obtained under some additional hypotheses:

**Theorem 2.1** Let $f \in C^2(\mathbb{T}^d)$ be a non-constant symbol defined on $\mathbb{T}^d$ and $V$ be a compact symmetric operator defined on $l^2(\mathbb{Z}^d)$ which belongs to $C^{1,1}(A^{\mathbb{V}_f})$. Assume in addition that the set $f(\kappa_f)$ has a finite number of accumulation points and denote: $H = L_f + V$. Then,

(a) $H$ has at most a finite number of eigenvalues in any compact subinterval of $\operatorname{Ran} f \setminus f(\kappa_f)$ and each of these eigenvalues has finite multiplicity.

(b) $H$ has no singular continuous spectrum.

The proof of Theorem 2.1 is a consequence of Lemma 4.1. A Limiting Absorption Principle also holds for $H$ on $\operatorname{Ran} f \setminus \sigma_{pp}(H) \cup f(\kappa_f)$ (w.r.t $A^{\mathbb{V}_f}$) (see Section 4 for details).

**Remark:** The fact that $f$ belongs to $C^2(\mathbb{T}^d, \mathbb{R})$ implies that $L_f \in C^2(A^{\mathbb{V}_f})$. The conclusions of Theorem 2.1 can be drawn under slightly weaker regularity hypotheses on the function $f$ (see [4] Paragraph 7.6).

**Remark:** The spectral results given by Theorem 2.1 are qualitative. In particular, nothing is said about the distribution of the point spectrum and what occurs at $f(\kappa_f)$. The eigenvalues (within the essential or the discrete spectrum of $H$) may accumulate at $f(\kappa_f)$.

**Remark:** In order to satisfy the hypotheses of Theorem 2.1 $V$ can be:

- a finite rank operator whose range is included in $\bigcap_j \mathcal{D}(A_j^2)$. In this case, $V \in C^2(A^{\mathbb{V}_f})$.
- a diagonal operator defined by: $V e_n = v_n e_n$, where $(v_n)_{n \in \mathbb{Z}^d}$ is a bounded real-valued sequence such that for some $0 < a < b$,

\[
\int_1^\infty \sup_{ar < |k| < br} |v_k| \, dr < \infty,
\]

(provided $f \in C^3(\mathbb{T}^d)$). In that case, the reader will prove that $V$ satisfies the required hypotheses, following mutatis mutandis the strategy described by [3] for the discrete Schrödinger operator. See in particular Lemma 2.1 [4] Theorem 7.5.8 or [9] Theorem 6.1 (with $\mathcal{G} = \mathcal{H} = \mathcal{G}^*$).

A formal computation shows that: $\text{id} \langle A \rangle L_f = A^{\mathbb{V}_f}$, where $\langle A \rangle^2 = 1 + A_1^2 + \ldots + A_d^2 = \mathcal{F}^*(1 - \Delta_0)\mathcal{F}$. This induces the idea that the evolutions generated by $L_f$ and $H$ satisfy some propagating and energy growth estimates. This is expressed more precisely by the following proposition:
Proposition 2.1 Let $f \in C^2(\mathbb{T}^d)$ be a non-constant symbol defined on $\mathbb{T}^d$ and $V$ be a compact symmetric operator defined on $l^2(\mathbb{Z}^d)$ which belongs to $C^{1,1}(A_{\varphi f})$. Let $H = L_f + V$ and $\lambda \in \text{Ran } f \setminus (\sigma_{pp}(H) \cup f(\kappa_f))$. Then, there exists an open interval containing $\lambda$, $I_\lambda$ such that for any $\Phi \in C_0^\infty(I_\lambda)$ and any $\psi \in \cap_{j=1}^d D(A_j)$ with $\Phi(H)\psi \neq 0$,

$$0 < \liminf_{t \to \pm \infty} |t|^{-1} |\langle e^{iHt}\Phi(H)\psi, A_{\varphi f}e^{iHt}\Phi(H)\psi \rangle| .$$

If in addition $V \in \cap_{j=1}^d (C^1(A_j) \cap C^1(A_j^2))$, then

$$0 < \liminf_{t \to \pm \infty} |t|^{-1} \|e^{-iHt}\psi\|_A ,$$

where $\|\psi\|_A = \sqrt{\|\psi\|^2 + \sum_{j=1}^d \|A_j\psi\|^2}$.

Remark: Following the proof of Proposition 2.1 the reader will convince himself that the regularity hypotheses also entail:

$$\limsup_{t \to \pm \infty} |t|^{-1} |\langle e^{iHt}\Phi(H)\psi, A_{\varphi f}e^{iHt}\Phi(H)\psi \rangle| < \infty \text{ and } \limsup_{t \to \pm \infty} |t|^{-1} \|e^{-iHt}\psi\|_A < \infty .$$

The proofs of Theorem 2.1 and Proposition 2.1 are respectively postponed to Sections 4 and 5.

2.3 From Laurent operators to Toeplitz matrices

Historically, the analysis of the spectral properties of Toeplitz matrices can be traced back in a series of papers [16], [28] which culminated with the proof of their absolute continuity [27]. For further generalizations and/or general references, see [7], [12], [13]. In this paragraph, we show that these properties remain stable under suitable compact perturbations in the one-dimensional case.

In the following, the Hilbert space $l^2(\mathbb{N})$ is considered as a closed linear subspace of $l^2(\mathbb{Z}) \simeq l^2(\mathbb{N}) \oplus l^2(\mathbb{Z}_-)$ and $Q$ denotes the orthogonal projection on this subspace. If $L = L_f = F(T)$ denotes the Laurent operator associated to a continuous symbol $f$ on $\mathbb{T}$, we define the Toeplitz operator $M$ associated to $f$ on the Hilbert space $l^2(\mathbb{N})$ by $M = QLQ$. Following [17], we have that: $\sigma(M) = \sigma_{ess}(M) = f$. Any bounded linear perturbation defined on $l^2(\mathbb{N})$ will be considered as the restriction of a bounded operator $W$ defined on $l^2(\mathbb{Z})$ in the sense that: $V = QWQ = W$. If $V$ is compact and symmetric, it follows from Weyl’s Theorem that: $\sigma_{ess}(M + V) = \sigma_{ess}(M)$. A precise qualitative description of the spectral nature of the matrix $M + V$ can be obtained under some additional hypotheses:

Theorem 2.2 Let $f \in C^1(\mathbb{T}, \mathbb{R})$ be a non-constant symbol and $V$ be a compact symmetric operator defined on $l^2(\mathbb{N})$ such that $W$ belongs to $C^{1,1}(A_{\varphi f})$. Assume in addition that $f(\kappa_f)$ has a finite number of accumulation points. Then,

(a) $M + V$ has at most a finite number of eigenvalues in any compact subinterval of $\text{Ran } f \setminus f(\kappa_f)$ and each of these eigenvalues has finite multiplicity.

(b) $M + V$ has no singular continuous spectrum.

The proof of Theorem 2.2 is postponed to Section 6.

3 An Illustration

We apply the former results to study perturbations of the discrete Schrödinger operator and show in particular, how [9] Theorem 2.1 can be easily recast as a special case of our framework.
Let \( L \) be self-adjoint operator defined on \( l^2(\mathbb{Z}^d) \) by:

\[
L = \sum_{j=1}^{d} T_j + T_j^*
\]

i.e. associated to the symbol: \( f(\theta) = 2 \sum_{j=1}^{d} \cos \theta_j \). In particular, \( \sigma(L) = [-2d, 2d] \). For all \( k \in \{1, \ldots, d\} \), we have that: \( [A_k, L] = T_k - T_k^* \). In other words, the operator \( i\text{ad}_{A_k} L \) is associated to the symbol \( (\partial_{\theta_k} f)(\theta) = -2 \sin \theta_k \). Following the former section, we have that for all \( \theta \in \mathbb{T}^d \),

\[
|\nabla f(\theta)|^2 = 4 \sum_{i=1}^{d} \sin^2 \theta_i.
\]

It follows clearly that the corresponding critical set is: \( \kappa = \{ \theta \in \mathbb{T}^d; \forall l \in \{1, \ldots, d\}, \theta_l = 0[\pi] \} \) and \( f(\kappa) = \{2(k-d); k \in \{0, \ldots, 2d\}\} \). Both are finite. Therefore, \[ \text{Theorem 2.1} \] is a corollary of \[ \text{Theorem 2.1} \]. However, the eigenvalues may accumulate at the thresholds in \( f(\kappa) \). Note that the existence of these critical points inside the spectrum of \( H \) was underlined in \[ \text{Theorem 2.1} \].

The reader will draw easily similar conclusions for compact perturbations of the discrete Schrödinger operator on the half-line by using \[ \text{Theorem 2.2} \].

**Remark:** These results are qualitative. The reader is referred to \[ \text{[3, 10, 18, 25]} \] for complementary results regarding the (point) spectrum of the discrete Schrödinger operator. Let us mention also that in dimension 1, it is difficult to produce absolutely continuous spectrum in this model \[ \text{[26]} \].

### 4 Proof of Theorem 2.1

Theorem 2.1 is actually a consequence of Lemma 4.1:

**Lemma 4.1** Let \( f \in C^0(\mathbb{T}^d) \) be a non-constant symbol defined on \( \mathbb{T}^d \). Let \( I \) be an open subinterval of \( \text{Ran } f \) such that \( \overline{f^{-1}(I)} \subset \mathbb{T}^d \setminus \kappa_f \) and assume there exists an open subset \( \Theta \subset \mathbb{T}^d \), which contains \( \overline{f^{-1}(I)} \) and on which \( f \) is of class \( C^2 \). Let \( \phi \) be any smooth function, compactly supported on \( \Theta \), which takes value 1 on \( \overline{f^{-1}(I)} \). Let \( V \) be a compact symmetric operator defined on \( l^2(\mathbb{Z}^d) \) which belongs to \( C^1(\mathbb{R}) \). Then, if \( H := L_f + V \),

(a) \( H \) has at most a finite number of eigenvalues in \( I \) and each of these eigenvalues has finite multiplicity.

(b) \( H \) has no singular continuous spectrum in \( I \).

(c) a LAP holds for \( H \) on \( I \setminus \sigma_{pp}(H) \) (w.r.t \( A \partial \nabla f \)).

The proof of Lemma 4.1 to which the remainder of the section is dedicated, relies on a suitable application of the regular Mourre theory. In particular, the meaning of statement (c) is clarified in the next paragraph.

#### 4.1 Regular Mourre Theory for self-adjoint operators

Throughout this paragraph, \( \mathcal{H} \), \( B(\mathcal{H}) \) and \( H \) denote respectively a fixed Hilbert space, the algebra of bounded operators on \( \mathcal{H} \) and a fixed self-adjoint operator in \( B(\mathcal{H}) \). The spectral measure of \( H \) is denoted by \( (E_{\Delta})_{\Delta \in B(\mathbb{R})} \). The concept of conjugacy are central in the developments of the Mourre theory:

**Definition 4.1** Assume that there exist a self-adjoint operator \( A \) with domain \( D(A) \subset \mathcal{H} \) such that \( H \in C^1(A) \). For a given \( \Lambda \in B(\mathbb{R}) \), we say that
• **H is weakly conjugate with respect to A** if \(i[A, H] > 0\) (i.e. non-negative and injective).

• **H is conjugate with respect to the observable A on \(\Lambda\)** if there exist \(c > 0\) and a compact operator \(K\) such that: 
  \[ E_\Lambda i[A, H] E_\Lambda \geq c E_\Lambda + K \]

• **H is strictly conjugate with respect to the observable A on \(\Lambda\)** if there exist \(c > 0\) such that: 
  \[ E_\Lambda i[A, H] E_\Lambda \geq c E_\Lambda \]

If \(H\) is strictly conjugate with respect to the observable \(A\) (on \(\mathbb{R}\)), it is of course weakly conjugate with respect to \(A\). We also write that the operator \(H\) is (strictly) conjugate for \(A\) at a point \(\theta\) in \(\mathbb{R}\), when there exists an open interval \(\Lambda_\theta\) containing \(\theta\) such that \(H\) is (strictly) conjugate for \(A\) on \(\Lambda_\theta\). This is equivalent to claim that there exist a smoothed characteristic function \(\Phi\) supported in \(\Lambda_\theta\), which takes value 1 on a neighborhood of \(\theta\) and a positive constant \(c\) such that:

\[
\Phi(H)i[A, H]\Phi(H) \geq c\Phi(H)^2.
\]

The development of the Mourre theory (or conjugate operator method) follows two steps. First, a control on the point spectrum is provided by the Virial Theorem (see [4] Proposition 7.2.10):

**Proposition 4.1** Assume that \(H\) belongs to \(C^1(A)\). Then, 
\[ E_{\{\theta\}} i[A, H] E_{\{\theta\}} = 0 \] for all \(\theta \in \mathbb{R}\).

As particular consequences, we have that:

**Corollary 4.1** Assume that \(H\) is weakly conjugate with respect to \(A\). Then, \(\sigma_{pp}(H) = \emptyset\).

**Corollary 4.2** Assume that \(H\) is conjugate with respect to \(A\) on the Borel subset \(\Lambda \subset \mathbb{R}\). Then, \(H\) has a finite number of eigenvalues in \(\Lambda\). Each of these eigenvalues has finite multiplicity.

Corollary 4.1 is straightforward while Corollary 4.2 is a reformulation of [4] Corollary 7.2.11.

Once controlled the point spectrum of \(H\), the Mourre theory proves the existence of a (local) Limiting Absorption Principle (LAP) away from the set of eigenvalues and therefore allows to rule out the existence of singular continuous spectrum. We say that a LAP holds for \(H\) on some Borel subset \(\Lambda \subset \mathbb{R}\) (w.r.t \(A\)) if:

• For any compact subset \(K \subset \Lambda\)
  \[
  \sup_{\Im z \neq 0, R z \in K} \|\langle A \rangle^{-1}(z - H)^{-1}\langle A \rangle^{-1}\| < \infty.
  \]

• If \(z\) tends to \(\lambda \in \Lambda\) (non-tangentially), then \(\langle A \rangle^{-1}(z - H)^{-1}\langle A \rangle^{-1}\) converges in norm to a bounded operator denoted \(F^+(\lambda)\) (resp. \(F^-(\lambda)\)) if \(\Im z > 0\) (resp. \(\Im z < 0\)). This convergence is uniform on any compact subset \(K \subset \Lambda\).

• The operator-valued functions defined by \(F^\pm\) are continuous on each connected component of \(\Lambda\), with respect to the norm topology on \(B(H)\).

**Remark:** If \(\Lambda \subset \rho(H)\), the properties described above are trivially satisfied. In this case, \(F^+(\lambda) = F^-(\lambda) = \langle A \rangle^{-1}(\lambda - H)^{-1}\langle A \rangle^{-1}\) for any \(\lambda \in \Lambda\).

We refer to [4] for a proof of the following result:

**Proposition 4.2** Let \(\Lambda\) be an open subset of \(\mathbb{R}\). Assume \(H\) is conjugate w.r.t \(A\) on \(\Lambda\) and that \(H \in C^{1,1}(A)\). Then,

• \(H\) has no singular continuous spectrum in \(\Lambda\).

• A LAP holds for \(H\) on \(\Lambda \setminus \sigma_{pp}(H)\) w.r.t \(A\).
Note that:

**Lemma 4.2** Let $\mathcal{H}$ be a Hilbert space and $A$ a self-adjoint operator defined on $\mathcal{H}$ with domain $\mathcal{D}(A)$. If $B$ is a compact operator on $\mathcal{H}$ which belongs to $C^{1,1}(\mathcal{A})$, then $\operatorname{ad}_A B$ is also compact.

The proof of Lemma 4.2 corresponds actually to the remark (ii) made in the proof of [4] Theorem 7.2.9. Due to the inclusions $(5.2.10)$ noted in [4], $\operatorname{ad}_A B$ can be expressed as the norm-limit when $\varepsilon$ tends to 0, of the family of compact operators $(-i\varepsilon^{-1}(e^{iA\varepsilon} B e^{-iA\varepsilon} - B))_{\varepsilon > 0}$.

Since Lemma 4.2 is essentially an application of the preceding results, it remains to recast it within this framework.

### 4.2 Regularity of the symbol revisited

The main purpose of this paragraph is to reinterpret the regularity hypotheses on the symbol $f$ in Theorem 7.11 in terms of regularity of $L_f = F(T)$ with respect to the commutation operation $\operatorname{ad}_{A_{\mathcal{F}_f}}$. Following the notations of paragraph 2.1, the reader will note that the shift operators $T_k$ belong to $C^{\infty}(A_j)$ for all $k \in \{1, \ldots, d\}$ and that for any nonnegative integral number $l$, $\operatorname{ad}_A(T_k) = \delta_{jk}T_k'$ and $\operatorname{ad}'_A(T_k) = (-1)^l\delta_{jk}T_k^{l'}$. More generally, we have that:

**Lemma 4.3** Let $h \in C^k(\mathbb{T}^d)$ for some $k$ in $\mathbb{N}$. Then, for all $j \in \{1, \ldots, d\}$, $L_h = H(T) \in C^k(A_j)$. In particular,

$$\operatorname{ad}_A H(T) = \operatorname{ad}_A L_h = L_{-i\partial_h} = T_j(\partial_{e^{i\theta}}, H)(T) = -i(\partial_j H)(T).$$

**Proof:** The result can be deduced directly by Fourier transform $\mathcal{F}$: if $h \in C^k(\mathbb{T}^d)$, then for all $\alpha \in \mathbb{Z}_+^d$ such that $0 < |\alpha| \leq k$, $\operatorname{ad}_{\mathcal{F}_f} h = (\partial^\alpha h)$. In particular, if $|\alpha| = 1$, the chain rule tells us that $-i(\partial_h) = e^{i\theta}, (\partial_{e^{i\theta}}, H)(e^{i\theta})$ for all $\theta \in \mathbb{T}^d$.

**Proof of Lemma 2.1** As $\mathcal{F}C^1(\mathbb{T}^d) \subset D_A$, the proof results from the fact that $i(g(\theta) \cdot \nabla + \nabla \cdot g(\theta))$ is (defined and) essentially self-adjoint on $C^1(\mathbb{T}^d)$. This statement follows from a suitable adaptation of the proof of [4] Proposition 7.6.3 (a) together with Nelson’s Lemma (see also [4] Chapter 2 and [2] Chapter 7). By Lemma 4.3, we have that for all $j \in \{1, \ldots, d\}$,

$$(L_{g_j}, A_j + A_jL_{g_j}) = 2L_{g_j}A_j - i\partial_{g_j}.$$ 

Since for all $j \in \{1, \ldots, d\}$, $A_j$ is relatively bounded with respect to $A$, the second assertion is proved in the case $s = 1$. For $1 < s \leq k$, the proof follows by induction. In particular, we can rewrite (on the domain of $A^s$):

$$A^s = \sum_{m=0}^s G_{s,m}(T)P_{s,m}(A),$$

where the functions $g_{s,m} \in C^{k-s+m}(\mathbb{T}^d)$ and $P_{s,m}(A) := P_{s,m}(A_1, \ldots, A_d)$ are homogeneous polynomials of degree $m$ in the variables $(A_1, \ldots, A_d)$. 

It follows that:

**Lemma 4.4** Let $f \in C^0(\mathbb{T}^d, \mathbb{R})$. Let $(k_1, k_2) \in \mathbb{N} \times \mathbb{N}$, $g \in C^{k_2}(\mathbb{T}^d, \mathbb{R}^d)$ and assume there exists an open set $\Theta \subset \mathbb{T}^d$ which contains the support of $g$, and on which $f$ is of class $C^{k_1}$. Then $L_f \in C^1(A_g)$ and $\operatorname{ad}_{A_g} L_f = L_{-i\nabla f}$. It follows that $\operatorname{ad}_A L_f \in C^{\min(k_1-1, k_2)}(A_g)$. In particular, if $f \in C^{k_1}(\mathbb{T}^d, \mathbb{R})$ then $L_f \in C^{k_1}(A_{\mathcal{F}_f})$.

**Proof:** Using Fourier transform $\mathcal{F}$, the first part is straightforward. The second follows by induction by noting that the symbol $g \cdot \nabla f$ belongs to $C^{\min(k_1-1, k_2)}(\mathbb{T}^d)$ and using again the Fourier transform. The
last part is obtained as a special case: \( g = \nabla f \).

The last step before the application of the Mourre theory consists in establishing the conjugacy properties satisfied by the operators \( L_f \) and \( H \):

**Proposition 4.3** Let \( f \in C^0(\mathbb{T}^d) \) be a non-constant symbol defined on \( \mathbb{T}^d \). Let \( I \) be an open subinterval of \( \text{Ran } f \) such that \( \overline{f^{-1}(I)} \subset \mathbb{T}^d \setminus \kappa_f \) and assume there exists an open subset \( \Theta \subset \mathbb{T}^d \), which contains \( \overline{f^{-1}(I)} \) and on which \( f \) is of class \( C^2 \). Let \( \phi \) be any smooth function, compactly supported on \( \Theta \), which takes value 1 on \( f^{-1}(I) \). Let \( V \) be a compact symmetric operator defined on \( l^2(\mathbb{Z}^d) \) which belongs to \( C^{1,1}(A_{\phi \nabla f}) \). Then, the operators \( L_f \) and \( H \) are respectively strictly conjugate and conjugate w.r.t. \( A_{\phi \nabla f} \) on \( I \).

**Proof:** It follows from Lemma 4.4 that \( L_f = F(T) \) belongs to \( C^1(A_{\phi \nabla f}) \) and: \( \text{id}_{A_{\phi \nabla f}} L_f = L_{\phi \| \nabla f \|^2} \).

Let \( \phi \in \mathcal{H} \setminus \{ 0 \} \). \( \overline{f^{-1}(I)} \) is a closed subset of \( \mathbb{T}^d \) hence compact. Since \( \theta \mapsto \| \nabla f(\theta) \| \) is continuous and does not vanish on \( f^{-1}(I) \), there exists \( c > 0 \) such that for all \( \theta \in f^{-1}(I) \), \( \| \nabla f(\theta) \| \geq c \). It follows that: \( \langle \phi, E_t(L_f)(\text{id}_{A_{\phi \nabla f}} L_f)E_t(L_f)\phi \rangle \geq c\| E_t(L_f)\phi \|^2 \). This proves the first part of the proposition. Since \( E_t \in C^{1,1}(A_{\phi \nabla f}) \subset C^{1}(A_{\phi \nabla f}) \), \( H \in C^{1,1}(A_{\phi \nabla f}) \subset C^{1}(A_{\phi \nabla f}) \). Due to Lemma 4.2, \( \text{id}_{A_{\phi \nabla f}} V \) is compact and \( \text{id}_{A_{\phi \nabla f}}/H = \text{id}_{A_{\phi \nabla f}} L_f + \text{id}_{A_{\phi \nabla f}} V \). It follows that for any \( \Phi \in C^\infty_c(I) \)

\[
\Phi(H)\text{id}_{A_{\phi \nabla f}} H \Phi(H) - \Phi(H)\text{id}_{A_{\phi \nabla f}} L_f \Phi(H)
\]

is compact. On the other hand,

\[
\Phi(H)\text{id}_{A_{\phi \nabla f}} L \Phi(H) = \Phi(L)\text{id}_{A_{\phi \nabla f}} L \Phi(L) + (\Phi(H) - \Phi(L))(\text{id}_{A_{\phi \nabla f}} L)\Phi(L) + \Phi(H)(\text{id}_{A_{\phi \nabla f}} L)(\Phi(H) - \Phi(L))
\]

Since \( H - L \) is compact, \( \Phi(H) - \Phi(L) \) is compact (a consequence of Stone-Weierstrass Theorem). Therefore, the operator \( (\Phi(H) - \Phi(L))(\text{id}_{A_{\phi \nabla f}} L)\Phi(L) + \Phi(H)(\text{id}_{A_{\phi \nabla f}} L)(\Phi(H) - \Phi(L)) \) is compact and so is \( \Phi(H)(\text{id}_{A_{\phi \nabla f}} H)\Phi(H) - \Phi(L)(\text{id}_{A_{\phi \nabla f}} L)\Phi(L) \). The last statement follows.

**Remark:** If \( f \in C^2(\mathbb{T}^d, \mathbb{R}) \), the set open \( \Theta \) and the function \( \phi \) can be taken respectively as: \( \Theta = \mathbb{T}^d \) and \( \phi \equiv 1 \).

### 4.3 Proof of Lemma 4.1

As already mentioned, \( \text{Ran } f = \sigma_{\text{ess}}(L_f) = \sigma_{\text{ess}}(H) \) due to Weyl’s Theorem. According to Lemma 4.4, \( L_f \in C^2(A_{\phi \nabla f}) \). It follows that \( H \in C^{1,1}(A_{\phi \nabla f}) \). The proof of Lemma 4.1 follows once combined Corollary 4.2, Propositions 4.2 and 4.3.

### 5 Proof of Proposition 2.1

Due to Lemma 4.1, \( H \) belongs to \( C^1(A_{\nabla f}) \). Therefore, for any \( t \in \mathbb{R} \) and any \( \Phi \in C^\infty_0(\mathbb{R}) \), \( e^{itH} \) and \( \Phi(H) \) also belong to \( C^1(A_{\nabla f}) \) (see e.g. Proposition 7.4 and Corollary 6.2.6 respectively). In particular, if \( \psi \in \cap_{j=1}^d \mathcal{D}(A_j) \), \( \psi \in \mathcal{D}(A_{\nabla f}) \) and \( e^{itH}\Phi(H)\psi \) also belongs to \( \mathcal{D}(A_{\nabla f}) \) (see e.g. Proposition 7.2).

Keep in mind that \( \text{id}_{A_{\nabla f}} V \) is compact according to Lemma 4.2. Combining it with Proposition 4.3 (and the remark which follows this proposition), we deduce that \( H \) is strictly conjugate w.r.t. \( A_{\nabla f} \) at each point of \( \text{Ran } f \setminus \sigma_{\text{pp}}(H) \cup \sigma_f(k_f) \). In other words, given \( \lambda \in \text{Ran } f \setminus \sigma_{\text{pp}}(H) \cup \sigma_f(k_f) \), there exists an open interval \( I_\lambda \) containing \( \lambda \) where \( H \) is strictly conjugate w.r.t. \( A_{\nabla f} \). From now, \( \Phi \) denotes a smooth function with compact support in \( I_\lambda \). For all \( t > 0 \),

\[
\langle \Phi(H)\psi, e^{-itH}A_{\nabla f}e^{itH}\Phi(H)\psi \rangle - \langle \Phi(H)\psi, A_{\nabla f}\Phi(H)\psi \rangle = \int_0^t \langle \Phi(H)\psi, e^{-isH}(\text{id}_{A_{\nabla f}} H)e^{isH}\Phi(H)\psi \rangle \, ds 
\]

\[
\geq ct\| \Phi(H)\psi \|^2,
\]

### (2)
for some \( c > 0 \). For \( t < 0 \), we proceed similarly. This proves the first part. Now, repeating the initial argument of this proof, we observe that since \( H \in \cap_{j=1}^d C^1(A_j) \), the vectors \( e^{iHT} \Phi(H) \psi \) and \( H e^{iHT} \Phi(H) \psi \) also belongs to \( \cap_{j=1}^d D(A_j) \) for all \( t \in \mathbb{R} \). Now, fix for the moment \( j \in \{1, \ldots, d\} \). Observe first that for all \( t \in \mathbb{R} \),

\[
\|A_j e^{iHT} \Phi(H) \psi\|^2 - \|A_j \Phi(H) \psi\|^2 = i \int_0^t \langle A_j e^{iHs} \Phi(H) \psi, A_j H e^{iHs} \Phi(H) \psi \rangle - \langle A_j H e^{iHs} \Phi(H) \psi, A_j e^{iHs} \Phi(H) \psi \rangle \, ds
\]

Splitting \( H = L_f + V = F(T) + V \) and observing that \( V \in C^1(A_2^2) \), we can rewrite:

\[
\|A_j e^{iHT} \Phi(H) \psi\|^2 - \|A_j \Phi(H) \psi\|^2 = \int_0^t \langle e^{iHs} \Phi(H) \psi, (L_{\partial \theta_j} f A_j + A_j L_{\theta_j} f) e^{iHs} \Phi(H) \psi \rangle \, ds
\]

Observe that the last integral grows at most linearly with \( t \). Now summing over \( j, j \in \{1, \ldots, d\} \) and using inequality (2) (or its counterpart for \( t < 0 \)) implies the last part.\( \square \)

### 6 Proof of Theorem 6.1

Theorem 2.2 is actually a consequence of Lemma 6.1.

**Lemma 6.1** Let \( f \in C^4(T, \mathbb{R}) \) be a non-constant symbol and \( V \) be a compact symmetric operator defined on \( l^2(\mathbb{N}) \) such that \( W \) belongs to \( C^{1,1}(A_{\sqrt{f}}) \). Let \( I \) be a subinterval of \( \text{Ran} f \) such that: \( \overline{f^{-1}(I)} \subset T^d \setminus \kappa_f \).

Then,

(a) \( M + V \) has at most a finite number of eigenvalues in \( I \) and each of these eigenvalues has finite multiplicity.

(b) \( M + V \) has no singular continuous spectrum in \( I \).

Section 6 is dedicated to the proof of Lemma 6.1. In order to simplify our notations, we write: \( L = L_f = F(T), L' = L_f' = (\partial F)(T), M = QLQ, V = QWQ, \overline{Q} = I - Q \) and

\[
D = M + \overline{Q}LQ
\]

\[
L = D + \overline{Q}LQ + QLQ
\]

Note that \( Q \) and \( \overline{Q} \) belong trivially to \( C^\infty(A) \) since \( [A, Q] = 0 = [A, \overline{Q}] \). We are interested by the spectral properties of the matrix \( M + V \). This will be done looking at the spectral properties of the operator \( D + QWQ \) and then reducing the problem to \( \text{Ran} Q \). The analysis concerning the perturbation \( V \) has been carried out in Section 4. So, our discussion will be focused on the off-diagonal operators \( \overline{Q}LQ \) and \( QLQ \).

#### 6.1 Preliminaries

The operators \( \overline{Q}LQ \) and \( QLQ \) are Hankel operators associated to the symbols \( \theta \mapsto f(\theta) \) and \( \theta \mapsto f(-\theta) \). Since these symbols are continuous by hypothesis, it follows from Hartman’s Theorem that \( \overline{Q}LQ \) and \( QLQ \) are compact [6, 17, 22, 23].

**Lemma 6.2** If \( f \) belongs to \( C^k(T) \) for some \( k \geq 1 \), then \( L, \overline{Q}LQ, QLQ \) and \( D \) belong to \( C^k(A) \). In addition, \( \text{ad}_A \overline{Q}LQ \) and \( \text{ad}_A QLQ \) are compact.
Proof: Since $L = F(T)$, the first part reinterprets Lemma 4.3. Since $f$ is of class $C^k$, the Hankel operators $QL\bar{Q}$ and $\bar{Q}LQ$ also belong to $C^k(A)$ because for all $j \in \{1, \ldots, k\}$,

$$
\text{ad}_A^j QL\bar{Q} = Q(\text{ad}_A^j L)\bar{Q} = (-i)^j QL_{f(j)}\bar{Q}$$
$$\text{ad}_A^j \bar{Q}LQ = \bar{Q}(\text{ad}_A^j L)Q = (-i)^j \bar{Q}L_{f(j)}Q.
$$

This also means that $\text{ad}_A^j QL\bar{Q}$ and $\text{ad}_A^j \bar{Q}LQ$ are Hankel operators associated to some continuous symbols, hence are compact. Since the class $C^k(A)$ is a linear subspace of $B(l^2(\mathbb{Z}))$, the last part follows. □

If $f \in C^2(\mathbb{T})$, then following Lemma 2.4 the symmetric operator

$$A_{f'} = \frac{1}{2}(L'A + AL')$$

is essentially self-adjoint on $\mathcal{D}(A)$.

Proposition 6.1 Let $s \in \mathbb{N}$ and $f \in C^k(\mathbb{T})$ for some $k > s + 1$. Then, the operators $QL\bar{Q}$ and $\bar{Q}LQ$ belong to $C^s(A_{f'})$.

Proof: For symmetry reasons, we restrict our discussion to the case $QL\bar{Q}$. We actually prove that $QL\bar{Q} \in C^{s,1}(A_{f'}) \subset C^s(A_{f'})$. Due to Lemma 2.1 and 9 Theorem 6.1, it is enough to check that:

$$\int_1^\infty r^{s-1}\|\chi(A/r)QL\bar{Q}\|dr < \infty$$

where $\chi$ is some compactly supported smooth function on $\mathbb{R}$ such that $\chi(x) > 0$ for $x \in (a,b) \subset (0,\infty)$. Since $f$ belongs to $C^k(\mathbb{T})$, $QL\bar{Q}$ belongs to $\mathcal{S}_k$ (see Section 7.1) and it follows from Lemma 7.3 that

$$\|\chi(A/r)QL\bar{Q}\| \leq Cr^{1-k}$$

for some $C > 0$. The above integral converges if $k > s + 1$. □

The proof of the following proposition can be greatly simplified if the symbol is a polynomial i.e. when the operator $L$ displays a band structure (as in the discrete Schrödinger model).

Proposition 6.2 Let $f \in C^k(\mathbb{T})$ for some $k > 3$. Then, $QL\bar{Q}$ and $\bar{Q}LQ$ belong to $C^1(A_{f'})$. Moreover, $\text{ad}_{A_{f'}} QL\bar{Q}$ and $\text{ad}_{A_{f'}} \bar{Q}LQ$ are compact.

Proof: The first part reinterprets Proposition 6.1. For symmetry reasons, we restrict our discussion to the case $QL\bar{Q}$. Using quadratic forms and the fact that $A$ and $Q$ commute, we have that:

$$2\text{ad}_{A_{f'}} QL\bar{Q} = (\text{ad}_{L'} QL\bar{Q})A + L'Q(\text{ad}_{A} L)\bar{Q} + A(\text{ad}_{L'} QL\bar{Q}) + Q(\text{ad}_{A} L)\bar{Q}L' \quad (3)$$

Since $\text{ad}_{A} L = -iL'$ and $f$ belongs to $C^3(\mathbb{T})$, it follows that $Q(\text{ad}_{A} L)\bar{Q}$ is bounded and compact. This means that the second and the fourth terms on the RHS are compact. Now, rewriting the commutator factors in the first and third terms of the RHS entails:

$$[L', QL\bar{Q}] = QL'Q\bar{Q} - QL\bar{Q} + \bar{Q}L'Q - QL'\bar{Q} \quad (4)$$
$$= -QL'\bar{Q}L - QL\bar{Q}L' + QL'\bar{Q}L - QL'\bar{Q}L' \quad (5)$$

Recall that the operators $QL\bar{Q}$, $\bar{Q}LQ$ belong to $\mathcal{S}_k$ and that $QLL'\bar{Q}$ belongs to $\mathcal{S}_{k-1}$ since $LL' = (\partial F)(T)F(T) = L_{f''}$ and $ff' \in C^{k-1}(\mathbb{T})$. Now, once identity (4) (resp. (5)) multiplied on the right (resp. on the left) by $A$, it follows from Lemma 7.2 that the first and third terms on the RHS of identity (3) can be expressed as linear combinations of products of bounded operators by operators in $\mathcal{S}_{k-2}$ (hence compact by Lemma 7.1). They are therefore compact, which implies the result. □
6.2 Proof of Lemma 6.1

By hypothesis, Lemma 4.4 and Proposition 6.1, \((D + W) \in C^{1,1}(A_f)\). Proposition 6.2 implies that the differences \((D + W) - L\) and \(\text{ad}_{A_f}(D + W) - \text{ad}_{A_f}L\) are compact. Therefore, \(\sigma_{\text{ess}}(D + W) = \sigma_{\text{ess}}(L) = \text{Ran } f\). In view of Proposition 6.3 and Lemma 4.1 we also know that: \(\sigma_{\text{sc}}(D + W) \cap I = \emptyset\), and \(\sigma_{pp}(D + W) \cap I\) is finite with finite multiplicity.

On the other hand, we know that: \(\sigma(M) = \sigma(L) = \text{Ran } f = \sigma_{\text{ess}}(M)\), since \(f\) is continuous. The perturbation \(V\) is compact so that: \(\sigma_{\text{ess}}(M + V) = \text{Ran } f\). Now, by reduction on \(\text{Ran } Q\), \(M + V = QDQ + QWQ\). The qualitative observations made on the point and singular continuous component of the spectrum of \(D + W\) in the interval \(I\) are also valid for \(M + V\). This allows us to conclude.

7 Auxiliary Results

7.1 Some estimates

Let \(p \in \mathbb{N}\) and \(B\) a closed operator defined on \(l^2(\mathbb{N})\). If it has a matrix representation \((B_{ij})\) on the canonical orthonormal basis of \(l^2(\mathbb{N})\) such that:

\[
\sup_{n,m} |(n + m)^p B_{nm}| < \infty
\]

we write that \(B\) belongs to the class \(S_p\). Naturally, if \(p \leq q\), \(S_q \subset S_p\). If \(p > 1\), the Schur-Holmgren criterion shows readily that \(S_p\) is a vector subspace of \(B(l^2(\mathbb{N}))\) [21]. Actually, we have that:

**Lemma 7.1** If \(p > 1\) and \(B \in S_p\), then \(B\) is compact.

**Proof:** Set \(\epsilon = (p - 1)/4\), \(q = (p + 1)/2\) and consider \(\Lambda\) the diagonal linear operator defined on the canonical orthonormal basis by: \(\Lambda e_n = p e_n\). For any \((n, m) \in \mathbb{N}^2\),

\[
|(n + m)^q (nm)^p B_{nm}| \leq (n + m)^{-2q} (nm)^p |(n + m)^p B_{nm}| \leq 2^{-2q} \sup_{n,m} |(n + m)^p B_{nm}| < \infty
\]

since \(B \in S_p\). Using again the Schur-Holmgren criterion, it follows that the linear operator \(\Lambda^* B \Lambda^*\) belongs to \(S_q\), hence is bounded. Since the operator \(\Lambda^*\) is compact and \(B = \Lambda^{-1} (\Lambda^* B \Lambda^*) \Lambda^{-1}\), the conclusion follows. \(\square\)

**Remark:** If the symbol \(f\) belongs to \(C^p(\mathbb{T})\), then the Hankel matrices \(QF(T)Q\) and \(\overline{QF(T)}Q\) belong to \(S_p\). Compared with Hartman’s Theorem, the hypothesis \(p > 1\) in Lemma 7.1 is not optimal. But it allows us to consider operators which are not Hankel as shown in the previous section.

The proof of the following result is immediate:

**Lemma 7.2** Let \(B \in S_p\) for some \(p > 1\). Let \(\Lambda\) the linear operator defined on the canonical orthonormal basis of \(l^2(\mathbb{N})\) by: \(\Lambda e_j = je^{i\theta_j} e_j\) for some \((\theta_j) \in \mathbb{T}^N\). Then, the operators \(\Lambda B\) and \(B \Lambda\) belong to \(S_{p-1}\).

**Lemma 7.3** Let \(B \in S_p\) for some \(p > 1\). For \((n, N) \in \mathbb{N}^2\), \(n < N\), define

\[
P_{[n,N]} = \sum_{k=n+1}^{N} |e_k\rangle\langle e_k|.
\]

Then, there exists \(C > 0\) such that for all \(N > n\),

\[
\|P_{[n,N]} B\| \leq C n^{1-p}
\]
\[
\|BP_{[n,N]}\| \leq C n^{1-p}
\]

**Proof:** Let us justify the first one. Using Schur-Holmgren criterion [21], \(\|P_{[n,N]} B\|\) is estimated by the maximum of \(\sup_{j \in \mathbb{N}} \sum_{i=n+1}^{N} |B_{ij}|\) and \(\sup_{j \in \{n + 1, \ldots, N\}} \sum_{j \in \mathbb{N}} |B_{ij}|\). Since there exists \(C > 0\) such that for all \((i, j) \in \mathbb{N}^2\), \(|B_{ij}| \leq C(i + j)^{-p}\), the result follows. The second case is obtained by symmetry. \(\square\)
7.2 Regularity classes for bounded operators

In this section, we have summed up some basic properties of the regularity classes \( C^k(A) \). For more details see [4] Chapter 5.

From now, \( A \) denotes a fixed self-adjoint operator, densely defined on a fixed Hilbert space \( \mathcal{H} \), with domain \( \mathcal{D}(A) \). The regularity of a bounded operator defined on \( \mathcal{H} \) w.r.t \( A \) is associated to the algebra of derivation on \( \mathcal{B}(\mathcal{H}) \) defined by the operation \( \text{ad}_A \). From a theoretical point of view, it is often more convenient to reformulate this concept of derivation in terms of the regularity of the strongly continuous function:

\[
\mathcal{W}_B : \mathbb{R} \rightarrow \mathcal{B}(\mathcal{H}) \quad t \mapsto e^{iAt}B e^{-iAt}.
\]

Most of the properties derived below can be deduced easily once established the following equivalence:

**Proposition 7.1** Let \( k \in \mathbb{N} \). The following assertions are equivalent:

- \( B \in C^k(A) \)
- The map \( \mathcal{W}_B \) is \( C^k \) with respect to the strong topology on \( \mathcal{B}(\mathcal{H}) \).
- The map \( \mathcal{W}_B \) is \( C^k \) with respect to the weak topology on \( \mathcal{B}(\mathcal{H}) \).

Moreover, \( \mathcal{W}_B^{(k)}(0) = i^k \text{ad}^kB \).

See [4] Lemma 6.2.9 and Theorem 6.2.10 in association with Lemma 6.2.1 and Definition 6.2.2 for a proof. According to Proposition 7.1, \( \mathcal{B}(\mathcal{H}) = C^0(A) \) and for any \( B \in \mathcal{B}(\mathcal{H}), \mathcal{W}_B(0) = B = \text{ad}^0_A B \). For all nonnegative integral number \( k \), \( C^{k+1}(A) \subset C^k(A) \).

**Proposition 7.2** If \( B \in C^1(A) \), then \( \mathcal{B}(\mathcal{D}(A)) \subset \mathcal{D}(A) \).

For any nonnegative integral number \( k \), \( C^k(A) \) is clearly a vector subspace of \( \mathcal{B}(\mathcal{H}) \). These classes also share the following natural algebraic properties:

**Proposition 7.3** Let \( k \in \mathbb{N} \) and \((B,C) \in C^k(A) \times C^k(A) \). then,

- \( B^* \in C^k(A) \) and for all \( j \in \{0, \ldots, k\} \), \( \text{ad}^j_A B^* = (-1)^j (\text{ad}^j_A B)^* \)
- \( BC \in C^k(A) \) and for all \( j \in \{1, \ldots, k\} \),

\[
\text{ad}^j_A BC = \sum_{i_1+i_2 = j} \frac{j!}{i_1! i_2!} (\text{ad}^{i_1}_A B)(\text{ad}^{i_2}_A C) .
\]

In particular, \( \text{ad}^j_A BC = (\text{ad}^j_A B)C + B(\text{ad}_A C) \)
- for all \( j \in \{0, \ldots, k\} \), \( \text{ad}^j_A B \in C^{k-j}(A) \).
- If \( B \) is invertible (i.e \( B^{-1} \in \mathcal{B}(\mathcal{H}) \)) and \( B \in C^1(A) \), then \( B^{-1} \in C^1(A) \); \( \text{ad}^j_A B^{-1} = -B^{-1}(\text{ad}^j_A B)B^{-1} \).

See [4] Propositions 5.1.2, 5.1.5, 5.1.6, 5.1.7 for a proof. Combining the last statements of Proposition 7.3, we deduce that if an invertible bounded operator \( B \) belongs to \( C^k(A) \), then its inverse \( B^{-1} \) also belongs to \( C^k(A) \). The relationships between these regularity classes and the self-adjoint functional calculus are deeply explored in [4] (Theorem 6.2.5 and Corollary 6.2.6) and [15] (using Helffer-Sjostrand formula). A proof of the next result can be found in [5] Section 2:

**Proposition 7.4** Let \( k \in \mathbb{N} \) and \((B_n)_{n \in \mathbb{N}} \subset C^k(A) \) be a sequence of operators such that for all \( j \in \{0, \ldots, k\} \), the sequence \((\text{ad}^j_A B_n)_{n \in \mathbb{N}} \) converges weakly to a bounded operator \( C_j \). Then, \( C_0 \in C^k(A) \) and \( \text{ad}^j_A C_0 = C_j \).

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