The Gauge String Solution of the 
\( D \geq 3 \) Yang-Mills Loop Equations.

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Abstract

I adapt the Gauge String [7], representing the strong coupling expansion in the continuous \( D \geq 3 \) Yang-Mills theory \((YM_D)\) with a sufficiently large bare coupling constant \( \lambda > \lambda_{cr} \) and a fixed ultraviolet cut off \( \Lambda \), to the analysis of the regularized Wilson’s loop-averages. When generalized to describe the fat (rather than infinitely thin) flux-tubes, the pattern of thus modified \( U(N) \) Gauge String is proved to be consistent with the chain of the judiciously regularized \( U(N) \) Loop equations. In particular, we reveal the dimensional reduction \( YM_D \rightarrow YM_2 \), taking place in the extreme \( SC \) limit \( \lambda \rightarrow \infty \), and compare it with the implications of the \( AdS/CFT \) correspondence conjecture. On the other hand, for the loop-averages associated to the sufficiently large minimal areas, the proposed stringy pattern is supposed to be in the one \textit{infrared} universality class (provided the loops are without zig-zag backtrackings) with the novel implementation of the noncritical \( D \)-dimensional Nambu-Goto string. The peculiarity is due to the nonstandard \( \Lambda^2 \)-scaling, \( \Lambda^2 = O(\sigma_{ph}) \), of the physical string tension \( \sigma_{ph} \). Being well-motivated from the viewpoint of the standard \( YM_4 \) theory with \( \lambda \rightarrow 0 \), this scaling is argued to entail that the considered modification of the Nambu-Goto system is in the stringy (rather than in the branched polymer) regime. In sum, the confinement in the continuous \( D \geq 3 \) \( U(N) \) (and, similarly, \( SU(N) \)) gauge theory is justified, for the first time, at least when both \( N \) and \( \lambda \) are sufficiently large. As a by-product, when continued to \( N = 1 \), the Gauge String is shown to describe the continuous \( U(1) \) gauge theory enriched with the \textit{monopoles} in the phase where the latter are supposed to be condensed.

Keywords: Yang-Mills, Loop equation, Duality, String, Strong-coupling expansion

PACS codes 11.15.Pg; 11.15.Me; 12.38.Aw; 12.38.Lg
1 Introduction.

An exact stringy reformulation of the continuous four-dimensional Yang-Mills theory \((YM_4)\) is one of the fundamental problems in theoretical physics, resolution of which is supposed to entail wide applications in the realm of phenomenology. Although a number of deep and impressive ideas have been invested (see e.g. \([1]-[4],[23, 24]\)), the situation in the subject remains far from being settled. As a step towards the rigorous reformulation, the author has recently proposed the novel Gauge String representation \([7]\) of certain \(D \geq 3\) dimensional lattice gauge systems in the large \(N\) strong coupling (SC) phase. Combining the nonabelian duality transformation \([6]\) with the Gross-Taylor stringy representation of the continuous \(YM_2\) theory on a 2d manifold, we find the exact reformulation of the considered \(D \geq 3\) \(YM_D\) systems in terms of the infinitely thin vortices of the colour-electric \(YM\)-flux.

In sharp contrast to the previous proposals in this direction, the lattice construction \([7]\) can be directly employed\(^1\) to define the variety of the continuous \(D \geq 3\) Gauge String models of smooth \(YM\) vortices. In turn, one can choose such a model that can be unambiguously associated to the continuous \(D \geq 3\) \(YM_D\) theory with the action

\[
S = \frac{1}{4g^2} \int d^D x \ tr (F_{\mu\nu}(x)F_{\mu\nu}(x)) ,
\]

where \(F_{\mu\nu} \equiv F_{\mu\nu}^a T^a_{ij}\), and \(T^a_{ij}\) are the properly normalized generators in the fundamental representation of a given Lie group to be fixed as \(U(N)\) unless otherwise specified. The considered explicit correspondence with the gauge theory \([11]\) is possible owing to the remarkable duality, to be formalized by eq. \((4.9)\). Certain conglomerates of the Feynman diagrams, comprising the weak-coupling (WC) series in a given continuous gauge system, are shown to be in the exact \(WC/SC\) correspondence with the judiciously associated variety of the appropriately weighted (piecewise) smooth flux-worldsheets. In this specific way, certain option of the smooth Gauge String allows, after the proper regularization (that will introduce a nonzero width of the \(YM\) vortex), to implement the \(1/N\) strong-coupling expansion of the regularized gauge invariant quantities in the theory \((1.1)\).

In the present paper, we investigate the regularized pattern of the proposed \(1/N\) \(SC\) (as opposed to the conventional \(WC\)) series further and directly justify the asserted \(YM_D/\text{String}\) duality employing the power of the Loop equation \([3, 10]\) associated to the regularized \(U(N)\) gauge theory \([11]\). To be more explicit, let the system \([1.1]\), defined in the \(D \geq 3\) Euclidean base-space \(\mathbb{R}^D\), be regularized at some ultraviolet (UV) scale \(\Lambda \sim N^0\). Also, the dimensionless and \(N\)-independent bare coupling constant

\[
\lambda = (g^2 N) \Lambda^{D-4} > \lambda_{cr}(D) \quad (1.2)
\]

is constrained to assume values larger than certain critical one \(\lambda_{cr}(D)\) (which is expected to be of order of unity) presumably associated to the large \(N\) phase transition. Once the inequality \((1.2)\) is fulfilled, the \(YM_D/\text{String}\) duality \([7]\) claims then that the free energy and the Wilson’s loop-averages in the \(YM_D\) theory \((1.1)\) can be reformulated in terms of the dual microscopic degrees of freedom: respectively the closed and the open sectors of the proposed smooth Gauge String.

\(^1\)Formally, this is possible due to the fact that the derived in \([7]\) lattice worldsheets’s weight \((1.3)\) is invariant, despite the discretization, under certain continuous (rather than discrete) group of the transformations (homeomorphisms) of the surface \(\tilde{M}_X\). The extension to the continuous space-time implies to trade the homeomorphisms for the corresponding diffeomorphisms characteristic for the theories of smooth strings.
Prior to the $UV$ regularization, the latter is defined through the 'bare' worldsheet’s weight

$$w[\tilde{M}_\chi] = N^\chi \exp\left( -\frac{\tilde{\lambda} \tilde{\Lambda}^2}{2} A[\tilde{M}_\chi] \right) J[\tilde{M}_\chi;\tilde{\lambda}], \tag{1.3}$$

where $A[\tilde{M}_\chi]$ denotes the total area of a given worldsheet $\tilde{M}_\chi$, $\tilde{\lambda} = (g^2 N) \tilde{\Lambda}^{D-4}$, while $N^\chi$ is the well-known ’t Hooft topological factor (with $\chi$ being the Euler character of $\tilde{M}_\chi$). As for $\tilde{\lambda}$, it is some auxiliary parameter which, tending to infinity before the regularization, will be related (see eq. (1.3)) to the $UV$ cut off $\Lambda$ on a later stage.

In contradistinction to the conventional Nambu-Goto pattern (given by eq. (1.6)), in eq. (1.3) there is one more factor $J[\tilde{M}_\chi;\tilde{\lambda}]$ which is crucial for the manifest correspondence with the weak-coupling series in the $YM_D$ theory (1.1). Being equal to unity for any nonselfintersecting surface $\tilde{M}_\chi$, this $\tilde{\lambda}$-dependent factor is in addition sensitive only to the topology, but not to the geometry, of selfintersections of $\tilde{M}_\chi$. Another distinction with the Nambu-Goto Ansatz is that the worldsheet’s selfintersections are allowed to possess certain topological singularities absent in the measure of the latter Ansatz. It is noteworthy, see [7], that the pattern of both the $J[\cdots]$-factor and the admissible worldsheet’s singularities does depend on the choice of the gauge group and the $YM_D$ lagrangian. In Section 4, it will be shown that, through this somewhat peculiar representation, one reproduces the contact interactions between the (self)intersecting (infinitely thin) vortices of the unit $YM$-flux. These interactions are to be distinguished from the selfenergy contribution which refers to the sheer Nambu-Goto pattern.

Next, by construction, eq. (1.3) refers to the dual gauge theory (1.1) considered prior to the $UV$ regularization that, in fact, is reflected by the vanishing width of the associated $YM$ vortices. On the other hand, in order to make contact with the technology of the loop equations [9, 10], one is to operate with the regularized $YM_D$ theory (1.1) where, therefore, some quasi-locality has to be introduced. In consequence, it turns out that we should deal with the fat flux-tubes of a nonzero width $\sqrt{<\mathbf{r}^2>} \sim \Lambda^{-1}$ described through certain quasi-local weight $w_r[M]$. The latter weight is supposed to be deduced via, roughly speaking, a kind of regularization-dependent smearing of the $\sqrt{<\mathbf{r}^2>} \rightarrow 0$ pattern (1.3) local on the worldsheet $M$. As a result, for the economic justification of the asserted matching (between the smeared variant of the pattern (1.3) and the regularized Loop equation), it is vital to reduce the regularization-dependence of the $YM_D$ loop-average $\langle W_C \rangle$ as much as possible.

Upon a reflection, there are two limiting regimes (formalized by eqs. (1.4) and (1.8) below) where the dependence of $\langle W_C \rangle$ on the choice of the regularization indeed can be reduced to the dependence of a few relevant coupling constants, entering the corresponding implementation of the Gauge String representation, on the bare coupling (1.2). In both cases, we are dealing with the dominance of the infrared phenomena: the contours $C$, being constrained to possess the radius of curvature $R(s) >> \Lambda^{-1}$ (for $\forall s$), should be associated to sufficiently large values of the minimal area $A_{\min}(C)$ of the saddle-point worldsheet $\tilde{M}_{\min}(C)$. Given the latter conditions, the required reduction is shown to be maintained when the characteristic amplitude $\sqrt{<h^2>}$ of the worldsheet’s fluctuations is either much larger or much smaller than the flux-tube’s width $\sqrt{<\mathbf{r}^2>} \sim \Lambda^{-1}$.

The smearing should comply with the natural requirement that, modulo admissible rescaling $\lambda \rightarrow \lambda'$ of the coupling constant (1.2), eq. (1.3) represents the universal limit of the variety of the differently regularized weights $w_r[M]$. The latter limit is achieved, possibly with the exception for some $\text{measure zero}$ subset of the worldsheet’s configurations, when the width $\sqrt{<\mathbf{r}^2>}$ is thinned back to zero.
As for the first of the regimes, the precise form of the defining necessary conditions reads

\[
\frac{\langle h^2 \rangle}{\langle r^2 \rangle} \sim \frac{D-2}{\lambda} \cdot \ln[A_{\text{min}}(C)\Lambda^2] \rightarrow \infty, \quad R(s)\Lambda \rightarrow \infty, \quad (1.4)
\]

which (being strengthened by the complementary constraint formulated prior to eq. (5.5)) entails that the characteristic YM vortices, spanned by \( C \), behave almost as if they are infinitely thin. In this case, taking advantage of certain hidden simplification inherent in the pattern (1.3), one can use a particularly simple regularization prescription for the Gauge String weight. For this purpose, let us consider the contours with at most point-like (self)intersections that, in particular, excludes zig-zag backtraclings of the loops. Then, the preferred prescription results in the stringy sum provides, in the regime (1.4), with choice (3.10) of \( w_2(M_\chi) \) which, in turn, is routed in the abelian Kalb-Ramond pattern [21]. Given the specific choice (1.5) can be viewed as a specific implementation of the general confining string Ansatz [32, 11] (see Appendix B (3.10)). We prove that thus defined stringy sum provides, in the regime (1.4), with the infrared universality

\[
\chi(M_\chi) = \exp \left( -\frac{\lambda \Lambda^2}{4} \int_{M_\chi} \int_{\tilde{M}_\chi} d\sigma_{\mu\nu}(x) d\sigma_{\mu\nu}(y) \right) \Lambda^2 G(\Lambda^2(x-y)^2), \quad (1.5)
\]

where \( d\sigma_{\mu\nu}(x) \) is the standard infinitesimal area-element (2.14) associated to the surface \( \tilde{M}_\chi \) of the Euler character \( \chi \). As for \( \Lambda^2 G(\Lambda^2 z^2) \), being introduced (see eq. (2.5)) to smear the \( D \)-dimensional \( \delta_D(x) \)-function, it is further constrained by the two conditions (given by eqs. (5.7) and (5.8)). We prove that thus defined stringy sum provides, in the regime (1.4), with the confining solution of the Dyson-Schwinger chain of the judiciously regularized \( D \geq 3 \) YM loops equations (derived for the system (1.4)) to all orders in \( 1/N \).

The deep reason, behind the relation between the stringy sums based respectively on the weights (1.3) and (1.5), is the following infrared universality taking place in the regime (1.4) as far as contours without zig-zag backtraclings are concerned. (The prescription, to implement the mandatory backtracing invariance of the YM loop-averages \( < W_C > \), will be discussed in Section 4.) To begin with, in this case, the above solution of the Loop equation is supposed to be in the one infrared universality class with the unconventional (owing to eq. (1.7)) implementation of the Nambu-Goto string which, in turn, is supposed to be reformulated in the spirit of the ‘low-energy’ noncritical Polyakov’s theory [4]. Thus associated Nambu-Goto theory, presumed to possess the same UV cut off \( \Lambda \), is endowed with the weight

\[
w_1(\tilde{M}_\chi) = N \exp \left( -\frac{\bar{\lambda}(\lambda) \Lambda^2}{2} \right), \quad (1.6)
\]

where \( L(\tilde{M}) \) is the length of the boundary \( \partial \tilde{M} \) of \( \tilde{M} \), while \( \bar{\lambda}(\lambda) \) and \( m_0(\lambda) \) are certain functions of \( \lambda \) which depend on the choice of both the flux-tube’s transverse profile and the prescription for the regularization of the string fluctuations. Then, the remaining step is to take into account that, being considered as resulting in the zero-width limit \( \lambda \Lambda^2 \rightarrow \infty \), the

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3In fact, the pattern (1.3) is reminiscent of (but, as it is discussed in the footnote after eq. (2.14), not equivalent to) the ad hoc smearing [3, 13] of the \( m_0 = 0 \) option of the Nambu-Goto weight (1.6). Complementary, the weight (1.3) can be viewed as a specific implementation of the general confining string Ansatz [32, 11] (see Appendix B for the explicit comparison) which, in turn, is routed in the abelian Kalb-Ramond pattern [21]. Given the specific choice (3.10) of \( G(\cdot) \), the \( D = 4 \) pattern (1.3) can be reduced (modulo certain boundary terms) to the rigid string but in the phase, see eq. (1.7) below, which is different from the one considered in [28, 29].

4Actually, as it is clear from eq. (4.12), the simple \( L(M) \)-dependence of the subleading boundary-contribution is valid only provided that all the loop’s self-intersections, if present, are point-like from the low-energy viewpoint.
associated to the weight (1.3) stringy sum is equivalent (see Appendix C) to the Nambu-Goto theory. The latter is endowed with the option of the weight (1.6) given by the identifications: \(m_0 = 0\), \(\lambda(\lambda)A^2 = \bar{\lambda}A^2\). In particular, the important consequence of the infrared unobservability of the selfintersection factor \(J[..]\), entering eq. (1.3) but missing in eq. (1.6), is that the pattern (1.6) works equally well for a large variety of the actions of the dual \(YM_D\) theories.

Next, despite its familiar appearance, the system (1.6) is not entirely conventional at least for sufficiently large \(N \geq 2\). The reason is that, within the \(1/N\) expansion, the consistent solution of the Loop equation mandatory results in the physical string tension \(\sigma_{ph}\) which is of order of (or, when \(\lambda \to \infty\), much larger than) the UV cut off \(\Lambda\) squared,

\[
\Lambda^2 = O(\sigma_{ph}) \quad ; \quad \sigma_{ph}^{(sc)} = \left(\frac{\bar{\lambda}(\lambda)}{2} - \zeta_D\right)\Lambda^2, \quad (1.7)
\]

where, in the \(N \to \infty\) semiclassical approximation \(\sigma_{ph}^{(sc)}\) for \(\sigma_{ph}\), the \(\lambda\)-independent constant \(\zeta_D\) (given by eqs. (5.6),(5.9)) encodes the \(D \geq 3\) entropy contribution \(\delta\sigma_{ent} = -\zeta_D\Lambda^2\) due to the regularized transverse string fluctuations. (In particular, eq. (1.7) implies that the coupling \(\bar{\lambda}(\lambda)\) is constrained to be sufficiently large.) In other words, the full-fledged quantum analysis of the stringy sum, endowed with the weight (1.6), requires to reformulate the latter as a somewhat unconventional stringy theory. Possessing the UV cut off \(\bar{\Lambda} >> \sqrt{\sigma_{ph}} \sim \Lambda\), this theory should not exhibit propagating degrees of freedom at the 'short distance' scales << \(1/\sqrt{\sigma_{ph}}\), while approaching the proposed Nambu-Goto pattern in the infrared domain at the scales larger than \(1/\sqrt{\sigma_{ph}}\).

According to the arguments of Section 7, the \(\Lambda^2\)-scaling (1.7) is sufficient to avoid the outgrowth of the microscopic baby-universes so that the Gauge String avoids the branched polymer phase. On the other hand, it is this scaling that hinders the direct application of the considered construction to the weak-coupling phase \(\lambda \to 0\). Actually, the somewhat odd dependence of \(\sigma_{ph}\) on the UV cut off \(\bar{\Lambda}\) becomes conceivable if one employs the proper reinterpretation [7] of the strongly coupled \(D = 4\) \(YM_D\) system (1.1). In view of the asymptotic freedom, the latter system can be viewed as a local prototype (or mathematical idealization) of the effective low-energy theory of the weakly coupled gauge theory (1.1). In this perspective, the cut off \(\bar{\Lambda}\) is to be identified with the confinement-scale which (in the \(D = 4\) system (1.1) with \(\lambda \to 0\) and the new UV cut off \(\bar{\Lambda} >> \Lambda\)) is supposed to be of order of the lowest glueball mass. From the results obtained, the considered confinement-scale can be viewed as the scale where the logarithmic renormgroup flow (of the running coupling constant) stops. Moreover, given the latter identification, in the \(N \to \infty\) limit the proposed implementation of the Nambu-Goto theory (endowed with the weight (1.6) is presumed to correctly describe the low-energy dynamics of the \(U(N)\) or \(SU(N)\) \(YM_D\) theory (1.1) in the WC phase with \(\lambda \to 0\). This statement is supported by the observation of Section 5.2 that the large \(N\) limit of the considered Nambu-Goto sum is supposed to be common for the infrared description of any \(D \geq 3\) \(U(\infty)\) or \(SU(\infty)\) pure gauge system with an arbitrary polynomial (in terms of \(F_{\mu\nu}\)) lagrangian providing with the \(N^2\)-scaling of the free energy.

In view of the latter universality, it is desirable to reconcile the reduction of the regularization-dependence with the observability of the particular choice of the associated \(U(N)\) action (1.1). On the side of the dual string theory, it is tantamount to the fact that the deviation of the selfintersection \(J[..]\)-factor (of eq. (1.3)) from unity is observable. This is indeed possible to achieve in the regime when the extreme SC limit \(\lambda \to \infty\) is performed before the large \([A_{\min}(C)\Lambda^2]\) limit (and, possibly, even before the large \(N\) limit):

\[
\frac{\ln[A_{\min}(C)\Lambda^2]}{\lambda} \to 0 \quad ; \quad [A_{\min}(C)\Lambda^2] \to \infty \quad , \quad \mathcal{R}(s)\Lambda \to \infty, \quad (1.8)
\]
while the radius of curvature \( R(s) \) is also kept to be sufficiently large (and the condition stated prior to eq. (1.3) is also assumed). The major simplification of the regime (1.8) is that the characteristic amplitude \( \sqrt{< \hat{h}^2 >} \) of the fluctuations is much smaller than the width \( \sqrt{< r^2 >} \sim \Lambda^{-1} \) of \( YM \) vortex. Furthermore, the leading asymptotics of the average \( < W_C > \) is represented by the contribution of the saddle-point vortex corresponding to the minimal area worldsheet(s) \( \tilde{M}_{\text{min}}(C) \) with the characteristic radius \( R(\gamma) \) of curvature being (at any point \( \gamma \equiv (\gamma_1, \gamma_2) \) of the surface) much larger than the flux-tube width:

\[
R(\gamma) \Lambda \longrightarrow \infty . \tag{1.9}
\]

Then, the landmark of the regime (1.8) is that the 'minimal area' worldsheet(s) are correctly described directly by the weight (1.3) so that the specifics of the \( YM_D \) action (1.1) is indeed observable.

In compliance with the prediction of [7], this can be formalized by the following concise prescription\(^5\) (to be compared with the implications of the AdS/CFT correspondence conjecture in Section 9). For simplicity, let the saddle-point (minimal area) worldsheets \( \tilde{M}_{\text{min}}(C) \) possess the support \( T_{\text{min}} = T_{\text{min}}(C) \) which, for simplicity, is presumed to be unique for a given \( C \). Additionally, keeping the conditions (1.3) fulfilled, let both the coupling constant \( \lambda \) and \( (R(s)\Lambda)^2 \) be much larger than \( N^2 \). Then, summing up the leading (with respect to the \( 1/\lambda \) expansion) subseries in \( 1/N^2 \), one obtains that the pattern of the \( D \geq 3 \) averages \( < W_C > |_{YM_D} \) is reduced,

\[
< W_C > |_{YM_D} \longrightarrow < W_C > |_{YM_2(T_{\text{min}})} \quad \text{if} \quad \lambda, (R(s)\Lambda)^2 \gg N^2 , \tag{1.10}
\]

to the one \( < W_C > |_{YM_2(T_{\text{min}})} \) in the continuous \( D = 2 \) \( YM_2 \) theory (1.1). Provided \( T_{\text{min}}(C) \) can be embedded into a \( 2d \) manifold, the latter \( YM_2 \) system is conventionally defined on \( T_{\text{min}}(C) \) as on the base-space; otherwise a slightly more subtle construction [7] is to be utilized. (When both \( \lambda \) and \( (R(s)\Lambda)^2 \) are large but \( \leq N^2 \), in the r.h. side of eq. (1.10) one is to retain only the leading \( O(N) \) order of the \( 1/N \) expansion of the \( YM_2(T_{\text{min}}) \) average.) The \( YM_2 \) coupling constant \( g_{YM_2} \) is related to the original \( D \geq 3 \) \( YM_D \) constant \( g_{YM_D} \) via the rescaling

\[
g_{YM_2}^2 = \xi \, g_{YM_D}^2 \, \Lambda^{D-2} = \xi \frac{\lambda}{N} \Lambda^2 \tag{1.11}
\]

that can be viewed (modulo the auxiliary \( \xi \)-factor\(^6\) which can be fixed to be unity) to the dimensional reduction \( YM_D \rightarrow YM_2 \). In particular, eq. (1.10) implies that the leading asymptotics of the physical string tension \( \sigma_{ph}(R) \) (associated to the average \( < W_C^R > \) of the Wilson loop in any \( \text{(anti)chiral representation} \ R \) of the \( U(N) \) group) is proportional, see eq. (1.1), to the eigenvalue \( C_2(R) \) of the second \( U(N) \) Casimir operator. As a result, the averaged force between the collinear elementary \( YM \) vortices is repulsion. Borrowing the classification from the pattern of the (dual) abelian superconductor, the (associated to the gauge theory (1.1)) Gauge String corresponds therefore to the type-II superconductor.

Finally, the considered in our paper stringy solution (1.3) of the Loop equation can be formally continued into the weak-coupling \( \lambda \rightarrow +0 \) phase as well. Yet, the scaling (1.7) is evidently in conflict with the implications of the standard perturbative computations valid for \( D = 3 \) and 4 in the limit \( \lambda \rightarrow +0 \). Therefore, in the large \( N \) weak-coupling limit, this solution in \( D = 3, 4 \) is presumed to describe the metastable pattern of the microscopic excitations which are

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\(^5\) The prescription (1.10) will be deduced, directly from the loop equations, in a separate paper. Here, we restrict our analysis only to a preliminary justification given by eq. (2.18).

\(^6\) In eq. (1.4), the parameter \( \xi \) will be related to the smearing function (2.3) reflecting the ambiguity of the \( UV \) regularization.
less energetically favourable than the ordinary perturbative gluons. Nevertheless, provided all the phase transitions (taking place in the process of decreasing $\lambda$ from infinity to zero) are not of deconfining nature, the analysed $SC$ dynamics foreshadows confinement in the $WC$ phase of the theory (1.1) when $D = 3, 4$.

1.1 Organization of the paper.

In Section 2, we first show that the Loop equation (2.2) reduces, for the contours without nontrivial selfintersections, to the linear equation (2.7) (which, for a subclass of the solutions, can be further reduced to the transparent form of eq. (2.11)) formulated directly for the worldsheet weight $w_2[\overline{M}(C)]$. Being restricted to the canonical space of the area-functionals, the general solution $w_2[\overline{M}(C)]$ of eq. (2.7) is given by the product of the particular solution (1.5) and the generic admissible zero mode $w_2^{(0)}[\overline{M}(C)]$ of the the Loop operator $\hat{L}_\nu$. Also, we briefly sketch how the $U(N)$ gauge theory’s correlators can be reconstructed on the stringy side and make a brief detour to the regime (1.8). Complementary, it is emphasized that the $N = 1$ option of the weight (1.5) constitutes, for arbitrary selfintersecting contours, the general $SC$ solution of the full Dyson-Schwinger chain of the loop equations associated to the $D \geq 2$ $U(1)$ gauge theory (1.1) which is regularized according to the prescription (2.5). In Section 3, on the example of 'short distance' expansion of the exponent of the weight (1.5), it is explained how the limit of the infinitely thin flux-tube can be used for the analysis of the Gauge String dynamics (of the fat $YM$ vortices) in the two distinguished regimes (1.4) and (1.8). In addition, we comment on the existence of the metastable stringy phase in the $WC$ regime $\lambda \to 0$.

Next, to properly utilize the solution of eq. (2.7) in the context of the $1/N$ $SC$ expansion for $N \geq 2$, in Section 4 the stringy sum endowed with the weight (1.5) is compared with the representation corresponding to the weight (1.3). It is shown that the difference between the ‘zero vortex width’ limit (1.2) of the weight (1.3) and the weight (1.5) reduces to the one associated to certain contact interactions (of the elementary $YM$ vortices) which are totally unobservable within the corresponding stringy sums. On the other hand, the quasi-contact interactions, built into the quasi-local pattern (1.5), are found to be of the abelian nature that has important implications for the regime (1.8). Also, the two complementary prescription, to implement the invariance (absent in the conventional Nambu-Goto string) of the averages $< W_C >$ under the backtrackings of the contour $C$, are introduced.

In Section 5, it is proved that, in the regime (1.4)/(1.7), both the stringy sum endowed with the weight (1.5) and the Gauge String representation based on the (smeared option of the) weight (1.3) are in the infrared universality class of the Nambu-Goto theory with the weight (1.6). In Section 6, the extreme $SC$ limit (1.8) is briefly discussed and an interesting infrared reduction of the number of the independent low-energy stringy excitations (corresponding to the nonelementary fat $YM$ vortex) is revealed in this regime. In particular, the $U(N)$ Casimir $C_2(R)$-scaling of the $SC$ asymptotics of the string tension $\sigma_{ph}(R)$ is deduced for $\forall R \in Y^{(N)}$.

In the dynamical regime (1.4), the novel implications of the $\Lambda^2$-scaling (1.7) of $\sigma_{ph}$ are clarified in Section 7. Section 8 contains the proof that the conditions (1.4)/(1.7) ensure that the sum over the $YM$ vortices assigned with the weight (1.5) indeed provides with the solution of the large $N$ Loop equation (2.2) (and, more generally, of the full Dyson-Schwinger chain of the loop equations). We demonstrate that the unconventional scaling (1.7) of $\sigma_{ph}$ implies the inapplicability of the arguments [8] concerning certain mismatch between the structure of the r.h. side of the Loop

\footnote{In $D \geq 3$, the latter abelian theory is enriched with the monopoles (to be treated similarly to the Wu-Yang formalism [12]) which are supposed to be condensed.}
equation \( \text{(2.2)} \) and the pattern of the Nambu-Goto/Polyakov’s string. Also, we adapt the gauge invariant regularization \( \text{(3)} \) of eq. \( \text{(2.2)} \) in such a way that makes it consistent with the pattern \( \text{(1.5)} \) of the solution of the reduced Loop equation \( \text{(2.7)} \). The conclusion provides with the summary of our results and emphasizes the central direction of the further research. A brief comment on the interrelation with the approach, based on the so-called AdS/CFT correspondence, is given as well. Finally, the Appendices serve to decrease the amount of technical details in the main text.

2 The solutions of the Loop equation on \( \Upsilon_0 \).

It is reasonable to expect that the (regularized implementation of the) \( SC/WC \) correspondence \( \text{(4)} \), discussed in the beginning of the introduction, is extended to the entire (regularized) \( YMD \) functional integral. In other words, the latter is supposed to be reformulated as the one over the smooth worldsheets with the string action predetermined by a smeared counterpart of the weight \( \text{(1.3)} \). To justify this expectation, we start with the representation of the continuous \( YMD \) theory in the space of the correlators of the Wilson loops

\[
W_C = \frac{1}{N} Tr \left[ \mathcal{P} \exp \left( i \oint_C dx_\mu A_\mu(x) \right) \right] , \quad A_\mu(x) \equiv A^a_\mu(x) T^a ,
\]

parametrized by the contours \( C \) which are represented by the trajectory \( x(s) \equiv x_\mu(s) \) in the Euclidean base-space \( \mathbb{R}^D \). The averages of the (products of the) loops \( \text{(2.1)} \) are known to satisfy the set of the Dyson-Schwinger equations which, in the limit \( N \rightarrow \infty \) (so that \( < W_C >_\infty \equiv < W_C >_{N \rightarrow \infty} \)), can be reduced to the single Loop equation \( \text{(1.10)} \)

\[
\hat{\mathcal{L}}_\nu(x(s)) < W_C >_\infty = \tilde{g}^2 \oint_C dy_\nu(s') \delta_D(y(s') - x(s)) < W_{C_{xy}} >_\infty < W_{C_{yx}} >_\infty ,
\]

where \( \tilde{g}^2 = g^2 N = \lambda A^{4-D} \), and \( \hat{\mathcal{L}}_\nu(x) \) is the Loop operator specified by eq. \( \text{(2.9)} \) below. As for the loop \( C \), owing to the constraint imposed by the \( D \)-dimensional \( \delta_D(.) \)-function, the r.h. side of eq. \( \text{(2.2)} \) vanishes unless \( C \) has a selfintersection at a point \( x(s) = y(s') \). In other words, \( C \equiv C_{xx} = C_{xy} C_{yx} \) is decomposable into the two subloops \( C_{xy} \) and \( C_{yx} \), with \( C_{xy} \) (or \( C_{yx} \)) collapsing to a point in the case of the trivial selfintersection.

The eq. \( \text{(2.2)} \) is notorious for being generically nonlinear, and the 'the devil' resides in the effects due to the nontrivial selfintersections of the loop \( C \). Furthermore, a priori, one may expect that in eq. \( \text{(2.2)} \) not any 'natural' (from the weak-coupling phase viewpoint) regularization of the \( \delta_D(x-y) \)-function results in a tractable regularization of the presumed stringy solution of the Loop equation in the \( SC \) phase. On the contrary, in the sector \( \Upsilon_0 \) (of the full loop space \( \Upsilon \)) comprised of the contours \( C \) which do not (nontrivially) selfintersect, eq. \( \text{(2.2)} \) considerably simplifies. In particular, there exists a simple gauge invariant regularization consistent with the transparent pattern of the regularized stringy solution on \( \Upsilon_0 \). To take advantage of the latter simplification, the idea is to decompose the solution of the Loop equation \( \text{(2.2)} \) into the two steps. First, one is to find a subclass of the regularized solutions of the considered \( \Upsilon_0 \)-reduction of the Loop equation that is the subject of the present section. Then, one is to analyse under what circumstances thus obtained solutions correctly reproduce the loop-averages for nontrivially selfintersecting contours. In other words, under what conditions there is such a judicious regularization of the full eq. \( \text{(2.2)} \) that makes the latter equation consistent with the considered \( \Upsilon_0 \)-solutions.\footnote{This can be compared to the qualitative analysis of \( \text{(1.3)} \) where it is argued that the area-law asymptotics of \( < W_C > \) is not in conflict with the Loop equation \( \text{(2.3)} \) in the \( WC \) limit \( \lambda \rightarrow 0 \).}
For any nonintersecting loop $C \in \Upsilon_0$, the r.h. side of eq. (2.2) receives a nonzero contribution only from the trivial selfintersection point $x(s) = y(s')$ with $s' = s$ so that one can put $< W_{C_{xy}} >= < W_{C_{yx}} >$ while $< W_{C_{yx}} >= 1$. Actually, the same simplification takes place for the finite $N$ extension (given by the $n = 1$ option of eq. (D.1) in Appendix D) of eq. (2.2) provided by the substitution

$$\left( \lim_{N \to \infty} < W_{C_{xy}} >= < W_{C_{yx}} > \right) \rightarrow < W_{C_{xy}} W_{C_{yx}} > . \tag{2.3}$$

It reduces the latter finite $N$ equation to the linear one

$$\hat{L}_\nu(x(s)) \mathcal{C} = g^2 < W_C > \int_D dy(s') \delta_D(y(s') - x(s)) , \quad C \in \Upsilon_0 , \tag{2.4}$$

where $g^2 = g^2 N \sim N^0$, and one can implement (consistently with the manifest gauge invariance) the smearing prescription

$$\delta_D(x - y) \rightarrow \Lambda^2 \mathcal{G}(\Lambda^2(x - y)^2) ; \quad \int d^2z \mathcal{G}(z^2) = 1 . \tag{2.5}$$

In eq. (2.4), the sufficiently smooth function $\mathcal{G}(z^2)$ (so that all its $n \geq 0$ moments (3.7) are presumed to be well-defined quantities) satisfies the natural normalization-condition that will be complemented by the two additional constraints (8.7) and (5.8).

Next, a priori, one can search for the solution $< W_C >$ of the reduced eq. (2.4) in the form of the sum $\sum w_2[\tilde{M}(C)]$ over the worldsheets $\tilde{M}(C)$, immersed into $\mathbb{R}^D$, which are weighted by some factor $w_2[\tilde{M}(C)]$. (More precise and formal definition of the measure in the latter sum will be given in Section 4.) To handle eq. (2.4), the first step is to plunge the stringy Ansatz and then perform the interchange of the relative order of the differentiation by $\hat{L}_\nu$ and summation over the worldsheets $M$ that amounts to

$$\sum_M^{(r)} \left( \hat{L}_\nu(x(s)) w_2[\tilde{M}(C)] - g^2 w_2[\tilde{M}(C)] \int_D dy(s') \delta_D(y(s') - x(s)) \right) = 0 , \tag{2.6}$$

where the sum’s superscript $(r)$ recalls that the transverse string’s fluctuations are to be regularized at the scale $\Lambda$. The point is to search for such solution $w_2[\tilde{M}(C)]$ that the expression in the big round parentheses of eq. (2.0) does vanish for any $\tilde{M}(C)$ involved into the sum. To this aim, let us first take into account that the Loop operator $\hat{L}_\nu$ is, in fact, the first (rather than second) order operator [3] complying therefore with the Leibnitz rule. As a result, eq. (2.4) can be transformed into the linear equation (to be regularized according to eq. (2.5))

$$\hat{L}_\nu(x(s)) \ln \left( w_2[\tilde{M}(C)] \right) = g^2 \int_D dy(s') \delta_D(y(s') - x(s)) \tag{2.7}$$

which, operating directly with the worldsheet’s weight $w_2[\tilde{M}(C)]$, is suitable for our further analysis. Remark also that, in accord with the general structure of the Dyson-Schwinger equations, eq. (2.7) merely determines the discontinuity that, being represented by its r.h. side, is exhibited by the total YM-flux (‘localized’ within the $YM$ vortex) which terminates at the Wilson loop source.

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9When $N = 1$, the average $< W_C >$ satisfies eq. (2.4) for an arbitrary selfintersecting contour.

10 As it will be clear, the solution (4.1)/(1.5) of the regularized eq. (2.13) is indeed consistent with the conditions selecting the latter type of the $C$-functionals.
2.1 The stringy form of Loop Equation for \( C \in Y_0 \).

Due to the first-order nature of \( \hat{L}_\nu \), the general solution \( w_2[\tilde{M}(C)] \) of eq. (2.7) assumes the form
\[
\begin{aligned}
  w_2[\tilde{M}(C)] &= \tilde{w}_2[\tilde{M}(C)] \quad ; \\
  \hat{L}_\nu(x(s)) ln(w_2^{(0)}[\tilde{M}(C)]) &= 0, 
\end{aligned}
\]
where \( \tilde{w}_2[..] \) is any particular solution of eq. (2.7) (to be identified with the one of eq. (1.5)), while \( w_2^{(0)}[\tilde{M}(C)] \) is formally allowed to be an arbitrary \( N \)-independent zero mode\(^1\) of the Loop operator. The latter can be defined \([3]\) as the combination
\[
\hat{L}_\nu(x(s)) = \partial^{(s)}_\mu \frac{\partial}{\partial \sigma_{\mu \nu}(x(s))}
\]
where \( \partial/\partial \sigma_{\mu \nu}(x(s)) \) and \( \partial^{(s)}_\mu \) denote respectively the Mandelstam area-derivative and the path-derivative (see Appendix B which recalls their definitions).

To proceed further, it is helpful to rewrite the r.h. side of eq. (2.7) as the surface integral over \( \tilde{M}(C) \). This can be done making use of the simple identity
\[
\int_{C=\partial \tilde{M}(C)} d\gamma y \ K((y-x)^2) = -\partial^{(s)}_\mu \int_{\tilde{M}(C)} d\sigma_{\mu \nu}(y) K((y-x)^2), \tag{2.10}
\]
where \( K((y-x)^2) \) is a generic smooth function which regularizes, according to eq. (2.5), the \( \delta D(\cdot) \)-function of eq. (2.7). (In the derivation of eq. (2.10), one is to employ that in this case the path-derivative \( \partial^{(s)}_\mu \equiv \partial/\partial x_\mu \) can be substituted by the ordinary derivative \( \partial/\partial x_\mu \) provided \( |\partial x_\mu(s)| \) is infinitesimal compared to the decay-length of \( K((y-x)^2) \). Also, the abelian Stokes theorem should be used.) Comparing the r.h. side of eq. (2.10) with the pattern (2.9) of \( \hat{L}_\nu \), one concludes that a particular solution \( \tilde{w}_2[..] \) of eq. (2.7) may be found from the equation
\[
\delta \ln \left( \tilde{w}_2[\tilde{M}(C)] \right) = -\frac{\tilde{g}}{2} \left( \delta_{\mu \alpha} \delta_{\nu \beta} - \delta_{\mu \beta} \delta_{\nu \alpha} \right) \int_{\tilde{M}(C)} d\sigma_{\alpha \beta}(z) \delta_{D}(x-z). \tag{2.11}
\]

Therefore, applying the area-derivative \( \delta/\delta \sigma_{\mu \alpha}(y(s')) \) to both sides of eq. (2.11), we arrive at the relatively simple equation which, prior to the regularization (2.5) of its r.h. side, reads formally
\[
\frac{2 \delta^2 \ln \left( \tilde{w}_2[\tilde{M}(C)] \right) \delta \sigma_{\mu \nu}(x(s)) \delta \sigma_{\rho \chi}(y(s'))}{\delta \sigma_{\rho \chi}(x(s)) \delta \sigma_{\mu \nu}(y(s'))} = -\left( \delta_{\mu \rho} \delta_{\nu \chi} - \delta_{\mu \chi} \delta_{\nu \rho} \right) \tilde{g}^2 \delta_{D}(x-y). \tag{2.12}
\]

Next, judging from the pattern of eq. (2.12), it is suggestive that this equation would be further simplified if the Mandelstam area-derivatives were traded for the ordinary functional area-derivatives (preliminary restricted to the boundary \( C \))
\[
\frac{\delta}{\delta \sigma_{\mu \nu}(x(s))} \rightarrow \frac{\delta f}{\delta p_{\mu \nu}(x(\gamma))} \bigg|_{x(\gamma) = x(s) \in C} \tag{2.13}
\]
with respect to the standard infinitesimal area-element
\[
d\sigma_{\mu \nu}(x(\gamma)) = p_{\mu \nu}(\gamma) d^2 \gamma, \quad p_{\mu \nu}(\gamma) = \varepsilon^{ab} \frac{\partial x_\mu(\gamma)}{\partial \gamma^a} \frac{\partial x_\nu(\gamma)}{\partial \gamma^b}, \tag{2.14}
\]

\(^1\) More precisely, \( \ln(w_2^{(0)}[..]) \) is presumed to support the proper cluster decomposition (to be introduced in Appendix B) similar to the one discussed in \([3]\).
where the coordinates \( x_\mu(\gamma) \equiv x_\mu(\gamma_1, \gamma_2) \) define the position of a given worldsheet \( \tilde{M} \) in the base-space \( \mathbb{R}^D \). To justify the substitution (2.13), the simplest and natural option is to constrain that the solution \(-\ln(w_2/N^x)\) of eq. (2.12) belongs to the space \( \Psi \) of the area-functionals\(^{12}\)

\[
\Psi : \sum_{n \geq 2} \int_{\tilde{M}} \cdots \int_{\tilde{M}} d\sigma_{\mu_1 \nu_1}(\gamma^{(1)}) \cdots d\sigma_{\mu_n \nu_n}(\gamma^{(n)}) \mathcal{S}^{(n)}_{\{\mu_\nu\}}(\{x(\gamma^{(i)}) - x(\gamma^{(j)})\}) , \tag{2.15}
\]

where \( \mathcal{S}^{(n)}_{\{\mu_\nu\}}(\cdot) \equiv \mathcal{S}^{(n)}_{\mu_\nu \nu_\mu}(\cdot) \) is an arbitrary translationally invariant \( 2n \)-tensor (with respect to the \( O(D) \) group of the Euclidean rotations) depending on the relative coordinates \( x(\gamma^{(i)}) - x(\gamma^{(j)}) \) which are allowed to take values in the full base-space \( \mathbb{R}^D \) rather than only on \( \tilde{M} \). (Eq. (2.13) implies that the area-derivative’s variation \( |\delta \sigma_{\mu \nu}(x(s))| \to 0 \) is infinitesimal compared to the decay-length of any of the kernels \( \mathcal{S}^{(n)}_{\{\mu_\nu\}}(\cdot) \).

Furthermore, on the space (2.13), in eq. (2.13) the restriction of the functional derivatives to the boundary \( C \) can be safely omitted. In sum, we arrive at the following reduction of eq. (2.12):\(^{13}\)

\[
\frac{2 \delta^2 \ln \left( \bar{w}_2[\tilde{M}(C)] \right)}{\delta p_{\mu \nu}(x(\gamma)) \delta p_{\rho \sigma}(y(\gamma'))} = -(\delta_{\mu \rho} \delta_{\nu \sigma} - \delta_{\mu \sigma} \delta_{\nu \rho}) \lambda A^4 G(A^2(x - y)^2) , \tag{2.16}
\]

where \( \lambda \) is defined in eq. (1.2), and we have implemented the smearing (2.3) of the \( \delta_D \)-function. Then, modulo the overall constant (excluded, in fact, through the matching with the full eq. (2.2)), the pattern\(^{14}\) (1.3) is the most general solution of eq. (2.16) when we presume one more constraint motivated by the \( Y \) \( M \)/\( S \) String duality \([7]\). Namely, we require the \( \exp[a\chi] \)-dependence of the weight on the Euler characteristics \( \chi \) of the worldsheet \( \tilde{M} \). As it is predetermined by the relations \( D.2 \) of Appendix D, in the regime (1.4)/ (1.7), the consistency of the pattern (1.3) with the full Loop equation (2.2) requires to identify \( a = \ln(N) \), where \( N \) parametrizes the associated \( U(N) \) group.

### 2.2 A brief digression to the extreme \( SC \) asymptotics (1.8).

At this step it is appropriate to remark that, when the first of the conditions (1.4) is violated, our previous analysis can be utilized at least in the following two situations. Namely, \( \text{either} \) in the \( N = 1 \) case of the \( U(1) \) pure gauge theory (with monopoles, see eq. (2.21), in \( D \geq 3 \)) \( \text{or} \), for \( N \geq 2 \), in the regime (1.8) when the saddle-point flux-tube is further \( \text{restricted} \) to be represented by an elementary \( Y \) \( M \) vortex with \( \text{unobservable} \) (modulo exponentially suppressed contribution going beyond the short-distance expansion (5.0)) selfoverlapping. In particular, in the extreme \( SC \) limit (1.8), the \( \Psi \)-space (2.13) can be employed to deduce the leading asymptotics (entering e.g. eq. (1.10)) of the average \( < W_{C}^{R} > \) in any given (anti)chiral representation \( R \in Y_n^{(N)} \) of \( U(N) \). The short-cut way to recover this asymptotics is to consider the analogue of the reduced Loop equation (2.2) but formulated for \( < W_{C}^{R} > \) rather than for \( < W_{C} > \equiv < W_{C}^{'} > \). Repeating the previous steps, one arrives at the modification of eq. (2.12) provided by the substitution

\[
g^2 N \longrightarrow g^2 C_2(R) \quad , \quad \lambda \longrightarrow \lambda C_2(R)/N \quad ; \quad R \in Y_n^{(N)} , \tag{2.17}
\]

\(^{12}\)The further analysis of the full \( U(N) \) Loop equation (2.2) reveals that the restriction to the \( \Psi \)-space (2.13) is indeed adequate at least in the regime (1.4)/ (1.7).

\(^{13}\)In (1), the \( \chi = 1 \) option of the pattern (1.3) was discussed, with \( \tilde{M} \) being \( \text{fixed} \) as the minimal area-worldsheet, as the ansatz for the solution \( < W_{C} > \) of the loop-equation considered on a limited \( \text{subspace} \) of the loop space. In (1) (1.11), a ‘cousin’ of the pattern (1.3) was considered: the normalization (2.3) \( \text{required} \) for the consistency with the regularized option of eq. (2.4) of the \( (D-2) \) th moment (1.7) of \( G(z^2) \) is \( \text{absent} \) while the alternative constraint, corresponding to \( \xi = 1 \) in eq. (3.1), is imposed on the \( \text{zeroth} \) moment.
where $C_2(R)$ is the (associated to $R$) eigenvalue \[^{\[2]\}\] of the second $U(N)$ Casimir operator. Thus modified equation evidently admits the solution similar to eq. \((1.4)\) that can be utilized to write down the following asymptotics of the average

$$
<\frac{W^R_C}{\text{dim} R}> \rightarrow \exp\left(-\frac{\lambda A^2 C_2(R)}{4N} \int_{T_{\text{min}}} \int_{T_{\text{min}}} d\sigma_{\mu\nu}(x) d\sigma_{\mu\nu}(y) \Lambda^2 G(\Lambda^2(x-y)^2)\right), \quad (2.18)
$$

where $\text{dim} R$ is the dimension of the (anti)chiral $U(N)$ representation $R$ in question, and $T_{\text{min}} \equiv T_{\text{min}}(C')$ is the support (of the relevant conglomerate, see eq. \((4.9)\)) of the minimal area worldsheets. A more refined analysis, to be presented in a separate paper, reveals that in the extreme $SC$ limit \((1.8)\) the asymptotics \((2.18)\) is indeed consistent with the full chain of the loop equations, at least provided one more condition is imposed. The support $T_{\text{min}}(C')$ should be further constrained to possess topology of such a smooth disc (which is, by construction, devoid of selfintersections) that the short-distance expansion \((3.6)\) of the exponent of eq. \((2.18)\) disregards\[^{14}\] in the latter equation only some exponentially suppressed 'nonperturbative' contribution irrelevant in the considered $SC$ limit.

### 2.3 Zero modes of the Loop operator $\hat{\mathcal{L}}_{\nu}$.

Given the particular solution \((1.4)\) of eq. \((2.7)\), the construction of the general solution \((2.8)\) requires to find the admissible class of the zero modes of the Loop operator \((2.9)\). Being as previously constrained to belong to the $\Psi$-space \((2.15)\), the general $N$-independent solution $w_2^{(0)}[\hat{M}(C)]$ of the zero-mode equation \((2.8)\) is obtained in Appendix B. There, we also make contact with the confining string Ansatz \([11]\) employing the Kalb-Ramond representation \([21]\) of the area-functionals \((2.15)\). As for the present simplest subsection, we merely present the simplest variety of $w_2^{(0)}[\hat{M}(C)]$ and comment on its interpretation.

The selected variety can be deduced from the general pattern \((2.15)\) retaining only the bilocal $n = 2$ contribution where $S_{\{\mu_k\nu_k\}}^{(2)}(x - y)$ is reduced to $\Lambda^4 \mathcal{M}_{\mu\nu;\rho\sigma}(\Lambda(x - y))$ with

$$
\mathcal{M}_{\mu\nu;\rho\sigma}(z) = [\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}] \partial_\lambda \partial_\lambda V(z^2) - [\partial_\mu \partial_\rho \delta_{\nu\sigma} - \partial_\nu \partial_\sigma \delta_{\mu\rho}] \text{ of the tensor indices. The defining property of the tensor (2.19) reads as } \partial_{\mu} \mathcal{M}_{\mu\nu;\rho\sigma}(z) = 0 \text{ which, as it is shown in Appendix B, can be easily generalized to construct the more general zero-mode solutions corresponding to the } n \geq 3 \text{ terms of the area-functional (2.14).}
$$

Finally, it is straightforward to verify that the modification \((2.8)\) of the particular solution \((1.4)\), due to the inclusion of the admissible zero-modes, does not alter the general conclusions of Sections 3 and 5 concerning the stringy dynamics in the regime \((1.4) / (1.7)\). (In particular, the general pattern of the short-distance expansion \((3.6)\) arises for any area-functional \((2.15)\).) To be more specific, let us consider the bilocal mode $w_2^{(0)}[\hat{M}(C)]$ corresponding to eq. \((2.19)\). On the one hand, the contribution associated to the first term of the latter equation can be viewed as the additive 'renormalization', $\mathcal{G} \rightarrow \mathcal{G} + 8\Lambda^4 \partial^\lambda V(z^2)/\lambda$, of the smearing function in eq. \((1.3)\). Furthermore, the normalization \((2.3)\) of the $(D - 2)$ th moment \((3.7)\) of $\mathcal{G}$ is not changed by this

\[^{14}\]More explicitly, at any point of $T_{\text{min}}(C')$, the line in the normal direction either does not have the second intersection with $T_{\text{min}}(C')$ or the second intersection takes place at a distance $>> \Lambda^{-1}$.}
modification, while the remaining moments of $G$ a priori are not fixed anyway. On the other hand, the double surface-integral of the second term ($\sim V(z^2)$) of eq. (2.19) can be transformed, via the abelian Stokes theorem, into the double contour-integral over the boundary $C$. In consequence, this subleading perimeter-type contribution is of secondary importance for the identification of the infrared universality class (predetermined, in the leading order, by the dynamics of the worldsheet’s interior) to which the analysed stringy system (1.5) belongs.

2.4 The $N = 1$ versus $N \geq 2$ cases.

Judging from the previous discussion, there still remains uncertainty if the solution (2.8)/(1.5) of eq. (2.7) can be employed to reproduce the $1/N$ expansion in the SC phase of the $U(N)$ gauge theory (1.1). The problem is foreshadowed by the fact that the purported ’t Hooft topological weight $N^x$ can not be determined from the reduced eq. (2.4) merely because the latter $\Upsilon_0$-equation, being $N$-independent (once $g^2$ is fixed independently of $N$), evidently decouples into the set of equations each associated to a given Euler character $\chi$ of the worldsheets $\tilde{M}_\chi$ involved into the relevant stringy sum (2.6) to be formalized by eq. (4.1). Upon a reflection, the derivation of the $N^x$-factor requires to check the consistency of the pattern (4.1)/(1.5) with the whole Dyson-Schwinger chain of the $U(N)$ loop equations considered within the framework of the 1/N series. In particular, this analysis will reveal that, even for $N \geq 2$, the Ansatz (4.1)/(1.5) (and its generalization (2.8)) is indeed consistent with the latter chain in the regime (1.4)/(1.7), provided the additional constraint (8.7) on the normalized smearing function (2.5) is satisfied. On the contrary, in the extreme SC limit (1.8), the considered $N \geq 2$ Anzať works only on a limited sector of the loop space (including a subclass of nonselfintersecting loops).

Before we begin to discuss the $N \geq 2$ issues, it is instructive to make a brief detour to the $N = 1$ case where the situation is substantially different. The reason is that, when $N = 1$, the reduced Loop equation (2.4) is valid for an arbitrary (selfintersecting) loop $C$. Furthermore, for a generic contour, the $N = 1$ option of the Ansatz (1.1)/(1.3) is consistent with the chain of the $U(1)$ loop equations (and, in fact, with the abelian Bianchi identities $\epsilon_{\mu\nu\rho\sigma} \partial_{\rho} \delta_{\sigma\mu\nu}(x) \delta_{\rho\sigma}(y) < W_C >= 0$ as it can be shown employing, e.g., the arguments from [11]). On the other hand, the subtlety arises because in $D \geq 3$ the considered variety (2.8) of the solutions is associated, see Appendix B, to the abelian gauge theory enriched with the monopoles which are supposed to be condensed. Actually, the presence of the Dirac-like monopoles (so that the magnetic current has the support on the union of the $(D - 3)$-dimensional hypersurfaces) could be anticipated in advance. Indeed, the $N = 1$ pattern (1.1)/(1.3) can be viewed as the continuum counterpart of the representation [20] (arising after the so-called abelian duality transformation) in the lattice pure $U(1)$ gauge theory with the Heat-Kernal action.

Similarly to the latter lattice theory, it is reasonable to expect that the $N = 1$ variant of the Gauge String provides with the representation of the SC series in the following compactified variant of the ordinary continuous $U(1)$ system (1.1). Selecting for simplicity the $D = 4$ case, in the $N = 1$ action (1.1) the field-strength tensor $F_{\mu\nu} = \partial_{\mu} \wedge A_{\nu}(x)$ is to be ‘minimally’ extended,

$$ F_{\mu\nu}(x) \longrightarrow \tilde{B}_{\mu\nu}(x) = F_{\mu\nu}(x) + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial_{\rho} \wedge \int d^4y < x| \frac{1}{\partial_{\lambda} \partial_{\lambda}} | y > k_{\sigma}(y) ,$$

(2.20)

to incorporate the contribution generated by the magnetic current $k_{\sigma}(y)$, to be averaged over, corresponding to the elementary magnetic charge $2\pi$. (The singular Dirac string is supposed to

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be eliminated through the Wu-Yang construction \(^{[12]}\). In the sector of \(n\) monopole’s loops \(C_q\), the current conventionally reads: \(k_\sigma(y) = \sum_{q=1}^{n} 2\pi \int_{C_q} dz_\sigma(s_q) \delta_4(z(s_q) - y)\), where one is sum over all immersions \(\tilde{C}_q \rightarrow C_q\) into \(\mathbb{R}^4\). Then, according to Appendices B and D, the \(N = 1\) Ansatz \((1.1)/(1.3)\) is supposed to be the general solution (of the set of the abelian loop equations) which reproduces the \(SC\) expansion of the arbitrary loop-observables in the regularized \(D \geq 3\) \(U(1)\) system\(^{[14]}\) with the action

\[
\sum_{n=0}^{\infty} \sum_{\{C_q[n]\}} \frac{1}{n!} \exp \left[ -\frac{1}{4g^2} \int d^4x \, d^4y \left( \tilde{B}_{\mu\nu}(x) \Lambda^D < x | 1/\tilde{G}(\partial_{\mu}\partial^\mu/\Lambda^2) | y > \tilde{B}_{\mu\nu}(y) \right) \right], \tag{2.21}
\]

(where \(\tilde{G}(p_{\mu}p^\mu)\) is the four-dimensional Fourier image of the smearing function \(G(z_\mu z^\mu)\) corresponding to the compactified (via the substitution \((2.20)\)) counterpart of the \(U(1)\) action \((1.1)\) regularized in compliance with eq. \((2.2)\). (Note that, akin the lattice case, one is to perform the canonical averaging over the \(n\)-loop ensembles \(\{C_q[n]\}\) with \(n = 1, 2, \ldots\).) On the other hand, apart from the regime \((1.4)\) where the infrared universality takes place, the nontrivial zero-modes \(w^{(0)}_2[\tilde{M}(C)] \neq 1\) (being constrained to meet the requirement of the proper cluster decomposition, see Appendix B) refer in \(D \geq 3\) to the continuous \(U(1)\) gauge theories other than the one of eq. \((2.2)\): the ‘compactification’ is represented by the combination of the minimal extension \((2.21)\) with some further \(F_{\mu\nu}\)-independent modification (see e.g. the expression in the large round brackets of eq. \((3.3)\) in Appendix B) of the abelian action.

2.5 The gauge theory’s correlators on the stringy side.

Let us presume that the smearing is consistent with the Bianchi identities (which indeed can be manifestly justified at least in the regimes \((1.4)\) and \((1.8)\)) reformulated, according to eq. \((2.22)\) below, in terms of the loop-operators \([3]\). Then, one can build up the regularized \(YM_D\) correlators from the properly associated states of the open\(^{[14]}\) Gauge String visualized in terms of the fat rather than infinitely thin flux-tubes. For this purpose, we are to employ the canonical reinterpretation \([3]\) of the area- and path-derivatives, combined into the Loop operator \((2.2)\), in terms of the corresponding \(YM_D\) structures (see eq. \((2.22)\) below). To this aim, recall first that, provided some (gauge invariant) regularization of the \(\delta_D\)-function, the representation \((2.2)\) of the Loop equation refers to the regularized, at some \(UV\) scale \(\Lambda\), but \(not\) to the renormalized \(YM_D\) theory \((1.1)\). The consistency of the regularized eq. \((2.2)\) requires that \(\Lambda^2 \cdot |\delta \sigma_{\mu\nu}(x(s))| \rightarrow 0, \Lambda \cdot |\delta x_{\mu}(s)| \rightarrow 0\), where the variations \(\delta \sigma_{\mu\nu}(x(s))\) and \(\delta x_{\mu}(s)\) refer, see Appendix B, to the corresponding derivatives in the Loop operator.

The key-point is that, employing thus introduced derivatives, from the multiloop averages \(< \prod_p W_{C_p} >\) one can reconstruct, at least in principle, the stringy representation for (the infrared asymptotics of) a generic field-strength correlator, with some path-ordered exponents of the gauge field possibly included, in the regularized \(U(N)\) gauge theory \((1.1)\). This can be done because, when the path-derivatives act on \(< \prod_p W_{C_p} >\) after the area-derivatives, they altogether can insert into the trace of each ordered exponent \((2.1)\) respectively the field-strength tensor and the conventional covariant derivative,

\[
\frac{\delta}{\delta \sigma_{\mu\nu}(x(s))} \quad \rightarrow \quad i F^a_{\mu\nu}(x(s)) T^a_{kj}, \quad \frac{\delta}{\delta x_\mu(s)} \quad \rightarrow \quad \hat{D}^a_{\mu}(A(x(s))) T^a_{kj}, \tag{2.22}
\]

\(^{16}\)For \(D = 2\), the result \((1.5)\) can be alternatively obtained applying the standard Feynman diagrammatics to the reduced variant of the \(U(1)\) system \((2.21)\) \(without\) monopoles.

\(^{17}\)As usual, the correspondence with the closed string states is maintained when the ’holes’, associated to the boundary loops, collapse to the points.
where the colour $k.j$-indices are to be contracted with the corresponding indices of the path-ordered exponents (originally combined into the set of the loops $W_{C_p}$) 'ingoing' and 'outgoing' from a particular point $x(s_p)$ of a given boundary $C_p$. Contracting all the involved loops $C_p$ to the points, in this way one is supposed to refine the amplitudes which originally appeared in the framework of the so-called dual resonance models stepping out at dawn of the hadron’s physics.

3 The fat vs. infinitely thin $YM$ vortices.

Prior to the analysis of the fat flux-tubes’ dynamics, it is helpful to consider first the mathematical idealization (e.g., of the solution (3.5)) corresponding to the infinitely thin $YM$ vortex described, therefore, through the weight which is local (like the one of eq. (1.3) or (3.2) below) on the worldsheet. The reason is twofold. On the one hand, in the extreme $SC$ limit (1.8), the condition (1.9) will be shown to ensure that the considered idealization provides with the adequate description of the (conglomerate of the) saddle-point worldsheets entering the r.h. side of the asymptotics (1.10) of $< W_C >$. On the other hand, in the regime (1.4)/(1.7), this local deformation of the Gauge String weight is expected to provide with the short-cut way to the identification of the universality class (stated in the Introduction) corresponding to the original fat $YM$ vortex.

As for the latter identification, according to Section 5, one can take advantage of the following evident freedom in the definition of the smearing of the Nambu-Goto ($NG$) weight (1.6). In the considered regime, any two stringy systems (formulated in terms of possibly fat $YM$ vortices) are supposed to belong to the same infrared universality class if and only if their local deformations are equivalent, modulo possible rescaling of the coupling constants, to the Nambu-Goto theory. The stated selection-rule is aimed to take into account that there are infinitely many (see Appendix C for a general discussion) seemingly different but in fact equivalent formulations of the considered $NG$ theory which are all admissible as the starting point for the subsequent smearing of the weight. The reformulations in question are generated merely by arbitrary nonsingular modifications (retaining the locality on the worldsheet) of the original $NG$ weight (1.6) on any measure zero subspace of the space spanned by relevant worldsheet’s configurations. In this perspective, the Nambu-Goto Ansatz can be reinterpreted as the simplest implementation of the stringy theory which alternatively can be defined, in the apparently more complicated way, either through the weight (1.3) or via the local deformation (given by eq. (3.2)) of the weight (1.5).

3.1 The 'brute force' local limit of the quasi-local weight (1.5).

In order to obtain the required local deformation of the pattern (1.5), we have to omit temporarily the normalization condition (2.7) and allow that the smearing function is analytically continued,

$$
\Lambda^2 g(\Lambda^2(x - y)^2) \longrightarrow \xi \delta^w_2(x(\gamma) - y(\gamma')) = \xi \frac{\delta_2(\gamma - \gamma')}{\sqrt{p_{\mu\nu}(\gamma)/2}},
$$

(3.1)

to merge with the 2-dimensional delta-function $\delta^w_2(x - y)$ on the worldsheet $\tilde{M}$ so that $x, y \in \tilde{M}$. (The area-element $p_{\mu\nu}(\gamma)$, entering eq. (1.1), has been introduced in eq. (2.14).) In this case, the quasi-local weight (1.5) reduces to its local limit

$$
w_3[\tilde{M}_\chi] = N^n \exp \left( -\frac{\bar{\lambda}\Lambda^2}{2} \sum_q (n_q)^2 \bar{A}_q \right) ; \quad \bar{\lambda} = \xi \lambda ,
$$

(3.2)
where the integer number $n_q$ is to be identified with the number of the (oriented) sheets covering the corresponding $q$th elementary domain $T_q$ (of the area $A[T_q] \equiv \tilde{A}_q = \int d^2 \gamma_q \sqrt{\epsilon_{\mu\nu}(\gamma_q)}/2$) of the support $T$ of $\tilde{M}_\chi$ so that $\sum_q |n_q| \tilde{A}_q = A[\tilde{M}_\chi]$.

Upon a reflection, the pattern (3.2) implies that the net effect of the contact interactions (to be discussed in Section 4 in more details) between the associated elementary $YM$ vortices is represented by the ratio

$$q[\tilde{M}_\chi] = \frac{w_3[\tilde{M}_\chi]}{w_1[\tilde{M}_\chi]_{m_0=0}} = \exp \left( -\frac{\bar{\lambda} \lambda^2}{2} \sum_q \left[ (n_q^2 - |n_q|) \tilde{A}_q \right] \right)$$

(3.3)
of the pattern (3.2) and the $m_0 = 0, \bar{\lambda}(\lambda) = \xi \lambda$ option of the Nambu-Goto weight (1.3), both assigned to a given worldsheet $\tilde{M}_\chi$ corresponding to a particular $\{n_q\}$-covering. Therefore, for $\prod_q n_q \neq 0, \varrho[\tilde{M}_\chi]$ is not equal to unity only for the multi-sheet covering of $T$, i.e. there must be at least one $q$ so that $|n_q| \geq 2$ which can be visualized as the corresponding 2-dimensional selfintersection (with the support on $T_q$) of the worldsheet $\tilde{M}_\chi$. In consequence, for any worldsheet $\tilde{M}_\chi$ without selfintersections on some 2d submanifolds, the pattern (3.2) reduces to the sheer $m_0 = 0$ option of the Nambu-Goto weight (1.4) corresponding to the bare string tension

$$\sigma_0 = \frac{\xi \lambda^2}{2} ; \quad \xi = \int d^2 z \mathcal{G}(z^2) \sim 1 .$$

(3.4)
furthermore, one can argue that (at least when $\lambda \sim 1$) the factor $\xi$, reflecting the $D \geq 3$ regularization ambiguity, is supposed to be of order of unity once the solution (1.5) describes stable, rather than metastable, stringy excitations. (As we will see, the same ambiguity is built into the mechanism of the $YM_D/String$ duality formalized by eq. (4.9) of Section 4.)

Next, to appropriately utilize the Ansatz (1.5), it is helpful first to compare its local limit (3.2) with the pattern corresponding to the worldsheet’s weight (1.3). Postponing the full analysis till the next section, here we state the two $D \geq 3$ equivalence-relations proved in Appendix C. In fact, these relations present a particular example of certain general freedom in the definition of a given variety of stringy theories (describing infinitely thin flux-tubes) that provide with one and the same set of the physical observables. The statement is that both the stringy sum (see eq. (4.7) of Section 4) endowed with the local weight (3.3) and the idealized implementation (formalized by eq. (4.1) of Section 4) of the Gauge String, corresponding to the direct application of the 'bare' weight (1.3), in $D \geq 3$ represent one and the same simpler system in disguise. Irrespective of the value of $\sigma_{ph}/\Lambda^2 > 0$, the latter system is the $m_0 = 0$ reduction of the good old Nambu-Goto Ansatz associated to eq. (1.6).

Then, in the regime (1.4)/(1.7), the latter equivalence-relations allow to design the economic prescription for the smearing of the 'bare' Gauge String weight (1.3). Namely, one is allowed to perform the smearing regularization$^{18}$ of the worldsheet’s weight after the selfintersection factor $J[..]$ in eq. (1.3) is omitted (and neglecting all the irregular topologies absent in the Nambu-Goto measure). In this way, the pattern (1.3) is traded for the one of eq. (1.5). As for the the physical interpretation of the latter two theorems, it is transparent: when the corresponding elementary $YM$ vortices are infinitely thin, their contact interactions (responsible for the discrepancy with the Nambu-Goto pattern) are completely unobservable within the associated $D \geq 3$ stringy sums.

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$^{18}$The first of the relations was implicitly assumed in the context of the heuristic 'regularization' $\mathcal{A}[\tilde{M}]$ of the area-functional $A[\tilde{M}]$.

$^{19}$Owing to the normalization (2.5) (to be reinterpreted from the viewpoint of the field-theoretic regularization of the gauge theory), the latter expansion has to be formulated in terms of the fat, rather than infinitely thin, $YM$ vortices of the width which is $\sim \Lambda^{-1}$ provided the condition (3.4) is fulfilled.
In particular, the first theorem merely formalizes the fact that the deviation (3.3) between the
$m_0 = 0$ option of the Nambu-Goto weight (1.6) and the corresponding pattern (3.2) takes place
only for those worldsheet’s configurations which are of measure zero in the $D \geq 3$ immersion space $\mathcal{I}(M, \mathbb{R}^D)$ relevant for both of the stringy representations.

3.2 The short-distance expansion of the quasi-local weight.

For the analysis of the next two sections, we will need the description of the quasilocal weight
(given either by eq. (1.5) or by a smeared counterpart of eq. (1.3)) through certain short-
distance expansion which starts with the leading term corresponding to the previously discussed
local deformation of the weight in question. But before we handle this issue, it is appropriate to
illuminate the two related points. The first is the physical reason why the regularization of the
$YM_D$ theory (1.1) mandatory requires to work with the smeared worldsheet’s weight assigned to
the fat flux-tube. In consequence, the second question arises: what is the precise interpretation
of the weight (1.3), formally associated to the infinitely thin $YM$ vortex, in the context of the
stringy representation of the regularized gauge theory (1.1).

For this purpose, we begin with the observation that the choice of the flux-tube profile (defined,
in the case of the bilocal pattern (1.3), through the kernel $G(z^2)$) can be reinterpreted as the
particular prescription for the regularization of the worldsheet’s weight. The subtlety is that not
every regularization of the weight can be mapped onto the corresponding regularization of the dual
gauge theory (1.1). The point is that, within the Loop equation (2.2), the kernel $G(z^2)$ satisfies
the constraint (2.5) which is not necessary from the viewpoint of the stringy representation alone.
In particular, presuming that $\xi$ meets the condition (3.4), the $D \geq 3$ analytical continuation
(3.1) disagrees with the normalization (2.5) of the $(D-2)$ th moment of $G(z^2)$. In turn, it matches
with the fact that the Nambu-Goto pattern (1.6) by itself is in conflict (see e.g. [3, 23]) with the
definition of Stoks functional [3] which makes the action of the Loop operator $\hat{L}_\nu$ ill-defined.

Taking into account the above discussion, one may expect (and this will be confirmed in Section
4) that the weight (1.3) refers to the formal implementation of the $1/N$ SC series in the ‘bare’
$YM_D$ system (1.1) prior to the full-fledged $UV$ regularization of the gauge theory. This point
can be most transparently justified at the example of the $U(1)$ gauge theory (2.21). When the
smearing function (2.3) approaches the original $\delta_D(x-y)$-function (that can be translated into
the infinite thinning of the $YM$ vortices), the corresponding $N = 1$ option of the weight (1.5)
evidently becomes divergent. The point is that the leading divergency is evidently associated to
the $(N = 1)$ variant of the) local limit (3.2) of the quasi-local pattern (1.5) so that in eq. (3.2),
one is to identify
$$\tilde{\Lambda}^{D-2} = \xi \Lambda^{D-2} \longrightarrow \infty .$$
(3.5)
Similarly, presuming the identification (3.5), the weight (1.3) is supposed to refer to the ‘leading
divergency’ (of some quasi-local weight $w, [M]$) exhibited in the considered local limit when the
‘bare’ $YM_D$ theory is formally recaptured.

On the other hand, the ‘bare’ weight (given, e.g., by eq. (3.2) or by the $\tilde{\Lambda}^{D-2} = \xi \Lambda^{D-2}$ option
of eq. (1.3) can be properly utilized for the description of the fat $YM$-vortex as well. Indeed, it can
be viewed as the ‘low-energy’ limit (taking place when the characteristic worldsheet’s curvature
satisfies the constraint (1.3) of the corresponding variety of the quasi-local weights assigned to

20On the field-theoretic side, the meaning of the latter normalization is transparent. Taking the $N = 1$ case
(2.21) as the simplest example, one immediately concludes: the considered condition (2.7) ensures that the residue,
of the $1/(p_\mu)^2$-pole in the regularized gauge field propagator (defined within the framework of the standard weak-
coupling expanation), is conventionally fixed to be unity modulo the standard prefactor.
the flux-tubes of nonzero width. More generally, the 'bare' weight can be viewed as the leading term in the 'short-distance' expansion of the quasi-local interaction between the surface elements \( V(D) \) in the area-functional akin to the one of eq. (2.13). To be more specific, we consider the latter expansion on the example of the bilocal exponent of the weight \( W(D) \) that reduces to the pattern \( (1.2) \) in the 'brute force' local limit \( (3.1) \). The expansion runs (see e.g. \([32, 11], [33, 36]\) and Appendix A) in terms of the scalar operators which, roughly speaking, refer to an effective theory of the two-dimensional gravity defined on the worldsheet \( \tilde{M} \) as on the base-space.

The simplest situation arises in the case of the worldsheets \( \tilde{M} \) without selfintersections\(^{21}\) so that the formal series start with the Nambu-Goto pattern \( (1.6) \). In particular, the contribution, associated to the interior of \( \tilde{M} \), can be symbolically written in the form

\[
\frac{\lambda \Lambda^2}{2} \sum_{p \geq 0} K_{2p}[G] \sum_{k=1}^{l(2p)} \frac{H_{2p}^{(k)}}{\Lambda^{2p}} \int_{\tilde{M}} d^2 \gamma \; Q_{2p}^{(k)}(\gamma) ,
\]

(3.6)

where the operators \( Q_{2p}^{(k)}(\gamma) \) are composed of the local tensors of the (intrinsic and extrinsic) curvature and torsion (and their derivatives) associated to a given point \( \gamma \) of \( \tilde{M} \). As for \( H_{2p}^{(k)} \), it is some \( \tilde{M} \)- and \( \mathcal{G} \)-independent numerical constant so that \( l(0) = 1 \) and \( H_{0}^{(1)} = 1 \). Finally, we have to pay for the microscopic nature of the (fat) \( Y \tilde{M} \) vortex by the explicit regularization-dependence of its bare 'effective action' \( (3.6) \). Namely, the overall coefficient, in front of the operators of a given dimension \( 2p \), depends on the transverse profile of the flux-tube through the \( 2p \)th moment

\[
K_n[G] = \int_{\mathcal{P}} d^2 z \; (z^2)^n \; \mathcal{G}(z^2) , \quad n \geq 0 ,
\]

(3.7)

of the smearing function \( (2.3) \) (and the integration in eq. (3.7) runs over the infinite \( 2d \) plane \( \mathcal{P} \), with \( z^2 \) being already measured in dimensionless units.)

Returning to the substitution \( (3.1) \), for \( \xi \sim 1 \) it becomes exact only when the smearing function \( (2.3) \) is such that

\[
K_0 = \xi \quad ; \quad K_n \rightarrow 0 \quad if \quad n \geq 1 .
\]

(3.8)

Similarly to the prescription of \([7]\), it corresponds to the infinitely small flux-tube width \( < r^2 > \) which is to be estimated as

\[
< r^2 > \sim \Lambda^{-1} \cdot \supr \left( \sqrt{< (z^2)^{\frac{n}{2}} >} \right) \quad ; \quad < (z^2)^{\frac{n}{2}} > = K_n/K_0 ,
\]

(3.9)

where \( \supr(\cdot) \) denotes the supremum with respect to \( n \geq 1 \). Then, the closest to eq. (3.8) \( D \geq 3 \) smearing, consistent with the normalization condition \( (2.5) \), reads

\[
K_0 = \xi , \; \quad K_{D-2} = \frac{2V_2}{D \cdot V_D} \quad ; \quad K_n \rightarrow 0 , \; n \neq 0 , (D-2) ,
\]

(3.10)

where \( V_D \) is the volume (e.g. \( V_2 = \pi \)) of the \( D \)-dimensional ball possessing the unit radius. Therefore, when \( D \) is even, the prescription \( (3.10) \) turns to zero the coefficients \( (3.6) \) in front of

\(^{21}\)The lines/points of selfintersections of \( \tilde{M} \) are assigned with some extra factors. Also, in addition to the bulk contribution \( (3.6) \), the expansion of the exponent of eq. \( (1.3) \) includes the boundary contribution which, in the leading order, is reduced to the perimeter-term \( (1.12) \) generalizing the one in eq. \( (1.6) \).

\(^{22}\)The operators \( Q_{2p}^{(k)} \) are characterized by their canonical dimension \( |Q_{2p}^{(k)}| = p \) which is evaluated postulating that \( [x_\mu] = [\gamma] = -1 \). The extra label \( k \) is introduced to distinguish, for \( p \geq 1 \), between \( l(p) \geq 1 \) different operators of the same dimension \( p \).
all the operators $Q_{2p}^{(k)}$ except for the ones of the dimension $2p = 0$ (i.e. the area term $A[\tilde{M}]$) and of the dimension $2p = (D - 2)$. Complementary, it implies that, when the dimension $D$ is odd, the additional (to $A[\tilde{M}]$) unsuppressed operators are associated to the boundary of the surface $\tilde{M}$ . Consequently, once the conditions (1.4) are valid for such $D$, in the interior of $\tilde{M}$ the Gauge String weight can be traded (after the identification (3.5)) for the sheer $m_0 = 0$ option of the Nambu-Goto pattern (1.3) with $\tilde{\lambda}(\lambda) = \xi \lambda$.

As for $D = 4$, apart from the full-derivative term localized at the boundary, the only nontrivial operator of dimension $(D - 2) = 2$ yields the well-known extrinsic curvature term so that the prescription (3.10) converts the exponent of eq. (1.3) into the pattern reminiscent, modulo certain boundary terms, of the so-called rigid string [28, 29]. (As it is clear from Appendix A, the parameter $\xi$ in eq. (3.10) is constrained to be larger than certain critical value when $D = 4$ case.) The important difference with the latter proposal is that the required $\Lambda^2$-scaling (1.7) of $\sigma_{ph}$ entails the dynamical regime distinct from the conventional regime presumed in [28]. To say the least, eq. (1.7) can be shown to imply the absence of the logarithmic flow of the running coupling constant in front of the extrinsic curvature term.

### 3.3 Existence of the metastable stringy phase in the $\lambda \to 0$ limit.

So far, we have never explicitly used that the coupling constant (1.2) is sufficiently large. The necessary condition, constraining $\lambda$, stems from the natural requirement that the physical string tension $\sigma_{ph} \sim \Lambda^2$ (given, in the semiclassical approximation, by eq. (1.7)) is positive. When the condition (3.4) on the auxiliary parameter $\xi$ (entering the bare string tension $\sigma_0$) is fulfilled, the constraint $\sigma_{ph} > 0$ indeed implies that $\lambda$ can not approach zero. On the other hand, the freedom in the choice of the $YM_D$ regularization can be employed to formally continue the considered stringy sum, endowed with the weight (2.8), into the weak-coupling regime $\lambda \to 0$. To explain this continuation understand its status in the simplest setting, we restrict our attention to the regime (1.4) presuming that the weight (1.3) is regularized according to the prescription (3.11).

The key-observation is that the ambiguity, in the choice of $\xi$, allows to keep $\tilde{\lambda} = \xi \lambda$ sufficiently large even when $\lambda \to +0$ at the expense of the scaling $\xi \sim 1/\lambda \to \infty$. Taking into account the estimate (5.6) of the entropy constant $\xi_D$, the latter observation can be employed to ensure that the physical string tension (1.7) (remaining positive even when $\lambda \to 0$) still complies with the $\Lambda^2$-scaling (1.7) that is mandatory for the consistency of the solution (1.3) with the full Loop equation (2.2). Unfortunately, being in the apparent conflict with the implications of the standard weak-coupling perturbative analysis (valid for $D = 3, 4$), thus constructed stringy representation provides with the metastable weak-coupling solution at least in $D = 3$ and $D = 4$. In turn, it matches with the conjecture of [9] that certain large $N$ phase transition at $\lambda = \lambda_{cr}(D) \geq \tilde{\lambda}^{-1}(2\zeta_D)$ may exclude the neighborhood of $\tilde{\lambda}(\lambda) \to 2\zeta_D$ (where eq. (1.7) is violated) from the domain of validity of the considered $YM_D/String$ duality. Let us also remark that, in the abelian case as well, one may expect that certain finite $N$ phase transition (or crossover) at some $g^2 = g_{cr}^2$ makes the $N = 1$ strong-coupling expansion (1.5)/(1.1) the unfaithful representation of the $U(1)$ gauge theory (2.21).

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23Furthermore, owing to eqs. (1.6) and (3.10), in this case the quasilocal weight (1.3) approaches (in the interior of $\tilde{M}$) the sheer Nambu-Goto pattern (1.6) with $\tilde{\lambda}(\lambda) = \xi \lambda$.

24The precise values of both $\lambda_{cr}(D)$ and $\zeta_D$ depend on the details of the regularization.
4   Gauge vs. Nambu-Goto string.

In the previous section, we have shown that the bridge, between the stringy sums based respectively on the weight (1.3) and on the smeared counterpart of the weight (1.3), is formed by the two equivalence-theorems. The theorems state that the deformations of the latter sums, corresponding to the 'brute force' thinning of the flux-tube’s width to zero, are both equivalent to the Nambu-Goto theory provided the appropriate identification of the relevant coupling constants. The aim of the present Section is to provide with the necessary details on the second of the above stringy sums. The central role in our discussion is played by the concept of the YM duality formalized by the identity (4.9) which can be used as the generating function for the 'bare' Gauge String’s weight (1.3) associated to the infinitely thin YM vortex. Complementary, the duality-relation (4.9) will be employed in Section 6 in order to deduce the SC asymptotics (1.10) implementing the dimensional reduction \( YM_D \rightarrow YM_2 \).

4.1   A formal definition of the generic stringy Ansatz.

To begin with, taking as an example the stringy sum endowed with the weight (1.0) or (2.8)/(1.3), let us introduce a more precise and abstract formulation of the latter systems which is presumably applicable to the generic stringy Ansatz. As for the sum associated to the weight (2.8)/(1.3), it can be symbolically written in the following form

\[
N^b < \prod_{k=1}^b W_{C_k} > = \left. \int_{\bar{\vartheta} \in \mathcal{I}(M, \mathbb{R}^D)} D\bar{\vartheta} \ w_2[\bar{M}(\{C_k\})] \right|_{\bar{M}=\bar{\vartheta}(M)}.
\]

As the average (4.1) is not irreducible for \( b \geq 2 \), one is to sum over the surfaces \( \bar{M}(\{C_k\}) \) possessing \( 1 \leq p \leq b \) open connected components with the boundaries associated to an arbitrary (ordered) partition \( \{b_j\} \) of the set of the \( b = \sum_{j=1}^b b_j \) contours \( C_k \) into nonempty subsets. As for the functional measure \( \sum_{\bar{M}} \rightarrow \int D\bar{\vartheta} \), it refers to the standard representation (4.2) of a given worldsheet \( \bar{M}(\{C_k\}) = \bar{M} \) as the image \( \bar{M} = \bar{\vartheta}(M) \) of the corresponding map

\[
\bar{\vartheta} : M(\{\tilde{C}_k\}) \rightarrow \mathbb{R}^D ; \quad \bar{\vartheta}(\tilde{C}_k) = C_k ,
\]

of an oriented (not necessarily connected) 2d manifold \( M(\{\tilde{C}_k\}) \equiv M \), with a given number \( b \) of the boundaries \( \tilde{C}_k \) (or without them at all), into the Euclidean base-space \( \mathbb{R}^D \). As a result, prior to the regularization, the generic measure is determined by the specification of the space \( \mathcal{X} \) of the admissible mappings (4.2) (i.e. of the admissible topologies of \( \bar{M} \)) while the area \( A[M] \) of \( \bar{M} \) is postulated to be averaged over. (As before, the frequencies of the transverse fluctuations of the worldsheet are to be somehow regularized at the UV scale \( \Lambda \) that trades \( D\bar{\vartheta} \) for its regularized counterpart \( D_r\bar{\vartheta} \). To simplify the notations the subscript \( r \) will be skipped.)

Next, returning to the case of the Ansatz (1.0) or (2.8)/(1.3), let us temporarily presume that the contours \( C_k \) are without backtracking. Then, once the conditions (4.4) are satisfied, the relevant set \( \mathcal{X} \) of the maps (4.2) can be consistently reduced to the space \( \mathcal{I}(M, \mathbb{R}^D) \) of the immersions \( \bar{\vartheta} \) of the manifold \( \bar{M} \) into \( \mathbb{R}^D \). The latter type of the mappings is characterized by the fact that any worldsheet \( \bar{M} \in \mathcal{I}(M, \mathbb{R}^D) \) locally looks like a domain of the 2-dimensional

\[\text{average} < W_C > \text{ briefly sketched in the end of this section.} \]

\[\text{The somewhat unconventional description of the measure is helpful to make contact with the formalisms of } [\frac{1}{2}, \frac{3}{2}] \text{. Not less important, it will allow for the economic implementation of the backtracking invariance of the average } < W_C > \text{ briefly sketched in the end of this section.} \]
Euclidean space $\mathbf{R}^2$. Therefore, having fixed a partition \( \{ b_j; j = 1, ..., p \} \), a given topology of an open surface $\tilde{M} \in \mathcal{I}(M, \mathbf{R}^D)$ in eq. (1.1) is specified by the $p$ positive integers $h_j$ (with $h = \sum_{j=1}^{p} h_j$) standing for the number of handles of $j$th open connected component of the worldsheet $\tilde{M}_\chi$. In turn, the total Euler character $\chi$ of $\tilde{M}_\chi$ (or, equally, of $M_\chi$) is related to the set of the quantum numbers $\{ h_j, b_j \}$ through the standard identity

$$\chi = \left( \sum_{j=1}^{p} 2 - 2h_j - b_j \right) = 2p - 2h - b \quad ; \quad b = \sum_{j=1}^{p} b_j \quad , \quad 1 \leq p \leq b \quad , \quad (4.3)$$

which ensures that $\chi = \sum_{j=1}^{p} \chi_j$ is the sum of the Euler characters $\chi_j$ associated to the connected components of $\tilde{M}_\chi$.

Finally, to obtain the conventional representation [1] of the stringy sum (1.1)/(1.6), one is to present the precise for-

4.2 The basics of the $YM_D$/String duality.

4.2.1 The idealized Gauge String sum in the limit of the zero vortex width.

Before we introduce the basic duality-relation (4.9), it is appropriate to present the precise formulation of the idealized Gauge String sum, endowed with the $\bar{\lambda}A^2 = \lambda A^2$ option (resulting after the identification (3.7)) of the 'bare' weight (1.3). The representation in question, arising when the formal limit

$$< r^2 > \quad \longrightarrow \quad 0 \quad (4.6)$$

of the zero flux-tube’s width (3.9) is performed prior to any other relevant limit, reads

$$N^b < \prod_{k=1}^{b} W_{C_k} > = \sum_{\chi} N^\chi \int_{\Delta(\bar{\chi}, M, \mathbf{R}^D)} \mathcal{D} \vartheta_\chi J[\vartheta_\chi | \bar{\lambda}] \exp \left( -\frac{\bar{\lambda}A^2}{2} A[\vartheta_\chi] \right) \quad , \quad (4.7)$$

where the worldsheets $\tilde{M}_\chi = \vartheta_\chi(M) \equiv \vartheta_\chi$, possessing a given total Euler character $\chi$, result from the mappings (1.2) comprised into certain space $\Delta(M, \mathbf{R}^D) = \cup_{\chi} \Delta_{\chi}(M, \mathbf{R}^D)$. (The functional integral $\int \mathcal{D} \vartheta_\chi = \sum_{\{ h_j, b_j \}} \int \mathcal{D} \vartheta_{\{ h_j, b_j \}}$ implies the discrete sum over the relevant set $\{ h_j, b_j \}$,

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26 The dimensional reduction (1.10), that takes place in the extreme $SC$ regime when the limits $\lambda, (\mathcal{R}(s)) \to \infty$ are performed before any other limits, can be formally deduced retaining from eq. (4.7) only the saddle-point contribution associated to the conglomerate of the minimal-area worldsheets.
constrained by eq. (1.3), which enumerates different topological sectors.) Without loss of the physical data, all the contours \( C_k \) are presumed to be devoid of backtrackings: the prescription to implement the backtracking invariance of \( \langle W_C \rangle \) is discussed in Appendix E. Also, as previously, an effective \( UV \) cut off \( \Lambda \) should be somehow imposed on the frequencies of the transverse fluctuations of the worldsheets \( \tilde{M}_\chi \).

According to Appendix C, the stringy sum \( \sum_{C} \) is equivalent\(^2\), in \( D \geq 3 \), to the \( m_0 = 0 \), \( \lambda(\lambda) = \xi \lambda \) option of the Nambu-Goto Ansatz \( (1.1)/(1.2) \) which, in turn, is indistinguishable from the local deformation \( (4.1)/(4.2) \) of the system \( (1.1)/(1.3) \). Let us briefly explain why it is the case. To begin with, the conventional immersion space \( \mathcal{I}(M, \mathbb{R}^D) \) (entering the sum \( (1.1) \)) can be viewed as the dense subspace of the \( \Delta(M, \mathbb{R}^D) \)-space relevant for eq. \( (4.7) \). Furthermore, the selfintersection-factor is considerably reduced when the worldsheet \( \tilde{M} \) belongs to the latter immersion-subspace:

\[
J[\vartheta(M) | \bar{\lambda}] = \frac{1}{|C_\vartheta|} \quad \text{if} \quad \tilde{M} = \vartheta(M) \in \mathcal{I}(M, \mathbb{R}^D),
\]

(4.8)

where \( |C_\vartheta| \) is the number of discrete transformations leaving \( \tilde{M} \) invariant, i.e. the number of the topologically inequivalent automorphisms \( \kappa \) of the immersion-map \( (4.2) \) which represents \( \tilde{M} : \vartheta \circ \kappa = \vartheta \). Then, the equivalence-theorem (stated in the beginning of this section) merely reflects the fact that, for \( D \geq 3 \), there is a dense subspace of \( \mathcal{I}(M, \mathbb{R}^D) \) where \( |C_\vartheta| = 1 \). Summarizing, in \( D \geq 3 \), the unobservable difference between the \( N \geq 2 \) stringy sums \( (4.7) \) and \( (4.1)/(3.2) \) refers exclusively to the distinction in the contact interactions between the elementary \( Y \) vortices of zero width.

4.2.2 The \( SC/WC \) correspondence.

Now we are in a position to formulate the duality-relation which allows to make contact between the representation \( (4.7) \) and the \( 1/N \) weak-coupling series in the associated gauge theory \( (1.1) \). Also, in Section 6, this relation will be crucial when we deduce the \( SC \) asymptotics \( (1.10) \) of \( \langle W_C \rangle \).

For this purpose, let us consider the \( T \)-restriction \( \Delta_T(M, \mathbb{R}^D) \) of \( \Delta(M, \mathbb{R}^D) \) that consists of all the maps \( \vartheta_T \) which result in the (piecewise) smooth worldsheets \( \tilde{M}(\{C_k\}) \) with the support \( Sup[\vartheta_T] \) (in \( \mathbb{R}^D \)) belonging to a given \( 2d \) cell-complex \( T(\{C_k\}) \) so that \( \vartheta_T(\{C_k\}) \in \cup_k C_k \). Also, let \( w[\vartheta(M)| \bar{\lambda}] \) denote the modification of the weight \( (1.3) \) arising after the identification \( (3.4) \). Then, the central implication of the \( Y \mathcal{M}_D/String \) duality is that the corresponding \( T \)-restriction of the idealized \( D \geq 3 \) Gauge String sum \( (4.7) \) represents the \( 1/N \) \( SC \) expansion of the (multi)loop average

\[
N^b \left< \prod_k W_{C_k} \right> \bigg|_{Y \mathcal{M}_D(T)} = \int_{\vartheta_T \in \Delta_T(M, \mathbb{R}^D)} d\vartheta_T w[\vartheta_T| \bar{\lambda}] + O(e^{-|a| N}),
\]

(4.9)

\(^{27}\)The advantage of the representation \( (4.7) \) is that, contrary to the other two reformulations, it allows to make contact with the \( 1/N \) weak-coupling expansion in the \( Y \mathcal{M}_D \) theory \( (1.1) \).

\(^{28}\)Due to possible selfintersections of \( \tilde{M}(\{C_k\}) \), the support \( T(\{C_k\}) \) may have topology of the \( 2d \)-skeleton of a \( D \)-dimensional lattice (embedded into \( \mathbb{R}^D \)) rather than simply of a \( 2d \) manifold. (Also, owing to the irrelevance \( \bar{\lambda} \) of the worldsheet backtrackings, \( T(\{C_k\}) \) is restricted to contain no backtracking \( 2d \) submanifolds.) On any given \( T(\{C_k\}) \), one can choose the graph \( \bar{T}_T \) where more than two outgoing \( 2d \) submanifolds of \( T(\{C_k\}) \) merge. Cutting along \( \bar{T}_T \), we make \( T(\{C_k\}) \) into a disjoint union of the elementary domains \( T_p(\{C_k\}) \) (entering e.g. eq. \( (3.2) \)) which possess topology of \( 2d \) manifolds with a boundary. By construction of \( \bar{\lambda} \), the support of a particular worldsheet \( \vartheta_T \) must be a nonempty union of some of the domains \( T_p(\{C_k\}) \).
evaluated in the two- (rather than $D$-) dimensional $YM_2$ system (1.1) of the curved target-space $T(\{C_k\}) \equiv T$ as on the base-space. (In the r.h. side of eq. (4.9), the exponentially suppressed residual term refers to the contribution which is essentially 'nonperturbative' with respect to the considered $1/N$ SC series. For example, in the $SU(N)$ case, the latter contribution includes the effects associated to the $SU(N)$ string-junctions.)

By construction, the relation (4.9) can be used as the generating function which formalizes the algorithm [7] to determine, for a generic nonabelian group, the pattern both of the relevant space $\Delta(M, R^D)$ of the mappings (1.2) and of the explicit form of the bare worldsheet’s weight $w[\hat{\vartheta}(M)]$. On the other hand, eq. (4.9) maintains the required correspondence between the two distinct $1/N$ expansions in the $YM_D$ theory (1.1). To explain the point, one is observe first the weaker variant of this correspondence between the amplitudes, associated to the conglomerates of the Feynman diagrams of the $1/N$ WC series (written for the $YM_2$ averages in the l.h. side of eq. (4.3)), and the amplitudes constituted by the $1/N$ SC series organized (in the r.h. side of eq. (4.9)) as the string-like representation. The final step is to notice that the latter WC conglomerates can be reinterpreted as certain subclasses of the Feynman diagrams in the bare $D$-dimensional $YM_D$ theory (1.1) prior to the UV regularization of the latter. More precisely, one is to rewrite first the perturbative propagators as the path integrals over the gluonic trajectories assigned with the colour indices. (To circumvent gauge-fixing, tricky to explicitly match between the $T$-restriction’s and standard formulations, one is to introduce an infinitesimally small mass term for the gauge field and then perform the considered comparison.) For a given support $T$, the relevant subclass of the diagrams includes those fishnet configurations which comply with the twofold constraint. The gluonic trajectories should be localized to $T$. while each colour $a$-component of the associated gluonic field-strength tensor $F^a_{\mu\nu}(z)$ (for any given $z \in T$) as the Euclidean $O(D)$-tensor should belong to the tangent space of $T$ at $z$.

In particular, the $\xi \Lambda^{D-2} = \tilde{\Lambda}^{D-2}$ implementation of the relation (1.11), between the $D$- and 2-dimensional coupling constants, emerges in the following way. The considered WC/SC resummation implies that, to each factor $g_{YM_D}^2$ of the amplitude associated to any given WC Feynman diagram, one is to assign the auxiliary factor $\tilde{\Lambda}^{D-2}$ which is to account for the different dimensionality of the microscopic excitations entering respectively the WC and SC series in the $YM_D$ system (1.1). Indeed, $(D-2)$ is the dimensionality of the subspace orthogonal to the tangent space associated to the support $T$ (spanned by the selected dual variety of the gluonic trajectories) of the corresponding conglomerate of the 2d flux-worldsheets. Summarizing, thus introduced SC/WC correspondence yields the refinement of the heuristic ’t Hooft arguments [2] (see also [4, 5]) concerning possible relation of the topological $1/N$ resummation (of the WC expansion) to the existence of the string-like representation for the matrix theories like (1.1).

The subtle point is that, as the WC series refer to the $YM_D$ theory before the UV regularization, the (apparently arbitrary) auxiliary parameter $\Lambda$ can be initially interpreted in the sense of the formal limiting procedure (3.5). Complementary, the representation (1.7) is to viewed as the analytical continuation (1.6) of the stringy sum based on certain quasi-local smearing $w_r[M]$ of the weight (1.3). This continuation is supposed to be performed extending the procedure which has been used to show that, in the local limit (3.1) of the zero flux-tube’s width, the Ansatz (4.1) / (3.2) is reduced to the system (4.1) / (3.3) describing the infinitely thin vortices. Actually, to deduce the smearing (consistent with the regularization (2.3) of the reduced Loop equation (2.4) of the worldsheet weight (1.3), one is to trade the average in the l.h. side of eq. (1.3) for its ‘regularized’ counterpart. The simple example of this prescription, implementing the proper smearing of the pattern (4.11) below, has been introduced in eq. (2.18). In particular, it provides with the complementary explanation of the reason why the 2d coupling constant $g_{YM_2}$ (entering, e.g., the SC asymptotics (1.10) of $< W_C >$) is related to the original $D$-dimensional one $g_{YM_D}$
At this step, it is appropriate to comment on the precise relation between the sum over the worldsheets \( \bar{M} \) in the r.h. side of eq. (4.9) and the superposition of the configurations visualized as (infinitely thin) flux-tubes. To identify the latter configurations, we recall first the following symbolic representation

\[
Z(\{C_k\}) < \prod_k W_{C_k} > \bigg|_{YM_2(T)} = \sum_{\{R_q\}} F(\{R_q\}) \exp \left( -\frac{\bar{\lambda} A^2}{2} \sum_q C_2(R_q) \bar{A}_q \right) \quad (4.10)
\]

The corresponding term in the r.h. side of eq. (4.9) is related to the character/flux-tube expansion (4.10) via a nontrivial resummation. The latter resummation, which serves to represent the contact interactions between the elementary YM vortices in stringy terms, is the clue for the proper understanding of the unconventional features of the \( \bar{Y} \)M vortices as seen within the 1/\( N \) SC expansion.

In a given representation \( R \) of the \( U(N) \) group, and the support \( T \) (with \( \partial T = C' \)) is constrained to be a curved disc of the area \( \Lambda T \) and with \( h \) handles so that \( \chi = 1 - 2h \).

Upon a reflection, a given flux-tube configuration is to be parametrized (as it is implied in the interpretation of the ratio (1.3)) by a single admissible \( \{R_q\} \)-assignment that refers to the corresponding term in the r.h. side of eq. (4.10). In consequence, once the l.h. side of eq. (4.9) does not comply with the shear Nambu-Goto pattern, the expansion in the r.h. side of eq. (4.9) is related to the character/flux-tube expansion (4.10) via a nontrivial resummation. The latter resummation, which serves to represent the contact interactions between the elementary YM vortices in stringy terms, is the clue for the proper understanding of the unconventional features of the (idealized) Gauge String pattern (1.7). One observes that, apart from the simplest situations (which, roughly speaking, refer to surfaces \( \bar{M} \) selfintersecting at most on a set of isolated points), there is no a direct identification of the weight (1.3) with the amplitude assigned to a particular flux-tube (of zero width). To say the least, a given YM vortex may contribute to all orders of the 1/\( N \) series. More precisely, the duality-relation (4.3) implies that, generically, certain superpositions of the flux-tubes (of regular topology) can be effectively reproduced through the corresponding stringy sums over the surfaces (including the ones of singular topology, not included into the immersion space \( \mathcal{I}(M, R^D) \)) endowed with the weight (1.3).

### 4.3 The case of the \( D \geq 3 \) \( U(1) \) gauge theory with monopoles.

The so far revealed matching, between the \( N \geq 1 \) patterns (4.1)/(3.2) and (1.7), is rooted in the unobservability (within the corresponding \( D \geq 3 \) stringy sums) of the difference in the associated

\footnote{This partition function is to be evaluated with the free boundary conditions for the \( YM_2(T) \) gauge fields on the boundary \( \partial T(\{C_k\}) \) of the base-space \( T \equiv T(\{C_k\}) \).}
contact interactions between the infinitely thin (elementary) flux-tubes. Actually, the matching becomes substantially more transparent in the \( N = 1 \) case. It may be expected from the fact that, irrespective of whether the conditions (1.3)/(1.7) are satisfied, the \( N = 1 \) option of the Ansatz (1.4)/(1.5) provides with the solution of the \( U(1) \) loop equations for generic Wilson loops.

For a preliminary orientation, observe first that the l.h. side (4.11) of the \( U(1) \) relation (1.9) is reminiscent of the \( N = 1 \) pattern (3.2). (Indeed, the quantity \( n^2 \) is the second Casimir eigenvalue \( C_2(R) \) associated to the \( U(1) \) irreducible representation \( R \) labelled by the single integer \( n \).) More precisely, the two corresponding patterns of the abelian contact interactions are equivalent, modulo certain minor details irrelevant for the associated \( D \geq 3 \) stringy sums.

To be more explicit, it is sufficient to restrict our attention to the simplest situation when the support \( T(\{C_k\}) \) (playing the role of the base-space for the \( YM_2(T) \) theory inherent in the l.h. side of eq. (1.3)) does not contain closed 2d submanifolds, i.e. closed 2-cycles. In this case, the partition function \( Z(\{C_k\}) \) in the l.h. side of eq. (1.4) is equal to unity, while the sum in the r.h. side of eq. (4.10) can be shown to reduce to a single term for which the fusion-rule function \( F(\{R_q\}) \) is equal to unity as well. On the other hand, recall that, in the local limit (3.8), the worldsheet’s weight (1.3) assumes the form of eq. (3.2) where one is to reidentify \( n^2 = C_2(n) \). Furthermore, any admissible \( \{n_q\} \)-assignment, induced by a particular immersion-map (4.2), satisfies the same fusion-rule constraints that are built into the \( U(1) \) character-expansion (4.10) with \( \{R_q\} \equiv \{n_q\} \). Altogether, for the considered topology of the supports \( T(\{C_k\}) \), one concludes that in the abelian case there exists the one-to-one correspondence between the l.h side of eq. (4.9) and the local limit (3.8) of the associated \( T \)-restriction of the \( N = 1 \) stringy sum (4.1)/(1.5).

Finally, presuming the validity of the condition (1.9), let us evaluate the leading boundary effects encoded in the weight (1.3) for generic smearing function (2.5). To simplify the consideration, we concentrate on the simplest option of the \( N = 1 \) Ansatz (1.4)/(1.5) for a \( D = 2 \) average \(< W_C >\). For a given worldsheet \( \tilde{M} \) corresponding to a particular \( \{n_q\} \)-assignment, the support \( C' \) (in \( \mathbb{R}^2 \)) of the contour \( C \) can be devided into the segments \( C'_p \) (of the length \( \tilde{L}_p \)) which separate the 2d domains of the worldsheet’s support \( T(C) \) covered respectively \( n^+_p \) and \( n^-_p \) times. Then, a computation yields that the quasi-locality of the weight (1.3) results in the boundary effects which, in the leading order of the short-distance expansion (akin to the one of eq. (3.9)), are described by the term

\[
\exp \left( -m_0(\lambda) \sum_p \left( n^+_p + n^-_p \right)^2 \tilde{L}_p \right),
\]

where \( m_0(\lambda) \) is given by eq. (A.13) of Appendix A.

One observes that \( (n^+_p + n^-_p)^2 \) is the second Casimir eigenvalue\(^{30}\) of the \( U(1) \) representation corresponding to the direct product of the associated representations labelled by \( n^+_p \) and \( n^-_p \). On the other hand, in the case of the sheer Nambu-Goto pattern (1.6), the exponent of eq. (4.12) would be traded for \( \sum_p |m_p| \tilde{L}_p \), where the integer \( |m_p| \) counts how many times the contour \( C \) covers the \( p \)th segment \( C'_p \) of the support \( C' \) of \( C \). Therefore, unless \( |n^+_p + n^-_p| = |m_p| \) for \( \forall p \), the boundary effects (encoded in the \( D = 2 \) pattern (1.3)) are sensitive to the quasi-contact interactions (responsible also for the deviation of the factor (3.3) from unity) which crucially depend on the choice of both the gauge group and the \( YM_D \) action.

\(^{30}\)In other words, \( T(\{C_k\}) \) can be embedded into a 2d manifold. This restriction is always fulfilled, e.g., for the minimal area worldsheets entering the \( SC' \) asymptotics (1.10) of \(< W_C >\).

\(^{31}\)Remark that it matches with the \( C_2(R) \)-scaling of the perimeter term in the pattern (2.18) of the \( U(N) \) average \(< W^R_C >\) in the extreme \( SC \) limit (1.8).
explains why in the regime \( \frac{1}{N} \), the necessary condition, for the Ansatz \( \frac{1}{N} \), to describe properly the (leading) boundary effects within the \( N \geq 2 \) series, requires that the involved loops possess at most point-like (self)intersections.

### 4.4 The backtracking invariance of \( < W_C > \).

In conclusion, we clarify the two alternative prescriptions to implement the invariance of the loop-averages \( < W_C > \) with respect to zig-zag backtracings of the loop \( C \). The short-cut prescription is provided by the following excessively simple rule (formalized by eq. (E.3) of Appendix E) that is predetermined by the basic duality-relation (4.9). Namely, the images \( \tilde{M}(C) \) of the \( \vartheta \)-mappings (4.4), (composed into the measure of the stringy representation (4.7) of \( < W_C > \), being kept intact in their interior, should be coherently deformed on their boundary \( \partial \tilde{M}(C) = C \) so that any particular data of backtracings is introduced at a given nonbacktracking reference-contour \( C \in \Upsilon_{nbt} \).

Although the above prescription is fully consistent, it looks somewhat 'external' to the \( m_0 = 0 \) option of the Nambu-Goto (NG) theory \( \frac{1}{L} \) to which the idealized Gauge String representation \( \frac{1}{L} \) is equivalent for nonbacktracking loops. Indeed, being extended from the subspace \( \Upsilon_{nbt} \) of nonbacktracking contours to the entire loop space \( \Upsilon \), the latter NG theory per se does not refer to the considered invariance which is, actually, absent (see Appendix E) in the conventional NG framework. Therefore, it is appropriate to discuss how the backtracking symmetry of the average \( < W_C > \) can be introduced as resulting from certain symmetry of the judiciously reformulated Gauge String action. To explain it in the simplest setting, let the constraints (1.4)/(1.7) be satisfied so that, for contours without backtracings, the Nambu-Goto pattern \( \frac{1}{L} \) correctly describes the low-energy dynamics of the \( D \geq 3 \) YM system \( \frac{1}{L} \). The key-observation is that, as we already discussed, the \( m_0 = 0 \) option of the NG theory \( \frac{1}{L} \) is equivalent to the local limit (3.2) of the stringy Ansatz \( \frac{1}{L} \) which, in turn, can be viewed as the specific member of the general confining string's family \( \frac{1}{L} \).

As a result, one can take advantage of the fact that (compared to the Nambu-Goto action of eq. (1.6)) the action in the exponent of eq. (1.5) is invariant under the larger group \( \frac{1}{L} \) of the reparametrizations which are now allowed to generate arbitrary worldsheets’ foldings. The central consequence of this extended worldsheets’ symmetry is that, being 'projected' onto the group of the reparametrizations of the boundary contour \( C \), it results in the invariance of \( < W_C > \) with respect to the contours’ foldings. To elevate this classical symmetry into the full-fledged quantum invariance, we observe that the l.h. side of the Ansatz \( \frac{1}{L} \) is invariant under certain deformation of the measure (in its r.h. side) paired up with the backtracking invariance of the weight \( \frac{1}{L} \). As we explain in Appendix E, this deformation merely trades any given worldsheets' measure \( M = \psi(M) \in \mathcal{I}(M, \mathbb{R}^p) \) for the whole equivalence class (i.e. orbit) of the surfaces. The latter orbit is generated through attachments to \( M \) all possible ‘collapsed’ (to bound a zero 3-volume) baby-universes which are visualized as the corresponding worldsheets’ foldings. In sum, there exists the twofold modification of the Nambu-Goto system \( \frac{1}{L} \) (i.e. when the weight \( \frac{1}{L} \) is substituted by the one of eq. (3.2), while the measure is deformed as described above) that allows for the 'natural' implementation of the required backtracking symmetry of \( < W_C > \).

In culmination, we formulate the general selection-rule following from the previous discussion. Consider a generic observable \( O_j \) in (open or closed string’s sector of) the Nambu-Goto theory \( \frac{1}{L} \) with \( m_0 = 0 \). The necessary condition, for such observable \( O_j \) to be associated to certain observable in the corresponding dual gauge theory \( \frac{1}{L} \), is that the above twofold modification of the pattern \( \frac{1}{L} \), being possible, leaves the quantity \( O_j \) unchanged. In this perspective,
the backtracking invariance of \(< W_C >\) together with the invariance of the properly introduced Gauge String action under the worldsheet’s backtracings can be viewed as a sort of 'pseudogauge' symmetry. On the other hand, the stringy action associated to either eq. (1.3) or eq. (1.6) can be viewed as arising after fixing of this 'pseudogauge' symmetry.

5 The infrared equivalence with the Nambu-Goto Ansatz.

It is time to put all the pieces together and demonstrate that, in the \(D \geq 3\) regime (1.4)/(1.7), the \(N \geq 1\) Nambu-Goto Ansatz (1.1)/(1.0) appears as the reference-model that 'marks' the universality class which includes the relevant Gauge String representation of the regularized \(U(N)\) loop-averages. Therefore, given a particular prescription to implement the \(UV\) cut off, the remaining regularization-dependence of the solution \(< W_C >\) (of the Loop equation) is localized within the pattern of the coupling constants \(\bar{\lambda}(\lambda)\) and \(m_0(\lambda)\) of the associated Nambu-Goto weight (1.0).

For this purpose, we are to employ that both the analytical continuation (4.7) (describing the infinitely thin \(YM\) vortices) of the presumable Gauge String representation and the stringy sum (1.1)/(3.2) (where the weight (3.2) is the local deformation (3.1) of the quasi-local pattern (1.5)) are exactly equivalent to the \(m_0 = 0\) option of the \(NG\) theory (4.1)/(1.6). In other words, with the exclusion of the measure zero subset of the surfaces, the characteristic worldsheet’s configurations within the two former sums are assigned with the \(NG\) weight (1.0). In consequence, when the conditions (1.4) are fulfilled, the smearing (1.5) of the local weight (3.2) simultaneously provides with the admissible smearing of the 'bare' Gauge String weight (1.3). The reason is that, in the regime (1.4), the weight (1.5) can be traded for the curvature expansion (3.6) which, for the dense variety of the worldsheets \(M\), starts with the leading term in the form of the \(m_0 = 0\) Nambu-Goto pattern (1.6) rather than of the more general pattern (3.2). In turn, it implies that the data of the quasi-contact interactions (which encodes the choice of the gauge group and the lagrangian for the dual \(YM_D\) theory) is unobservable within \(< W_C >\) in the considered \(D \geq 3\) regime (when the loop \(C\) is without 1d selfintersections).

To implement this idea, we first derive the estimate (1.4) for the characteristic amplitude \(\sqrt{< h^2 >}\) of the fluctuations of the extended flux-tube. Then, we employ certain general implications of the Wilsonian renormgroup in order to justify the following. There is the pertinent variety (including the particular choice (3.10)) of the smearing functions (2.5) so that, in the regime (1.4)/(1.7), the stringy systems (4.1)/(1.3) and (1.1)/(1.0) possess one and the same pattern of the low-energy theory.

5.1 The estimate of the characteristic amplitude of the fluctuations.

The required estimate (1.4) of the amplitude \(\sqrt{< h^2 >}\) is fairly straightforward in the case when the amplitude is either much larger or much smaller than the width of the vortex. As we will see,
in both cases\footnote{For a given $\bar{\lambda} = \xi \lambda$, the peculiar logarithmic scaling \eqref{eq:1} is observable (as a profile of the energy distribution) only in the regime when the height is sufficiently larger than the width of the flux-tube.} a simple computation (in the spirit of \cite{17}) provides with the equation

$$\frac{\langle h^2(\gamma) \rangle}{(D - 2)} \sim \frac{ln[A_{\text{min}} \Lambda^2]}{\xi \lambda \Lambda^2} \quad \Rightarrow \quad D_H \rightarrow +\infty,$$

where $A_{\text{min}} \equiv A[\tilde{M}_{\text{min}}(C)]$ is the minimal area corresponding to the saddle-point worldsheet $\tilde{M}_{\text{min}}(C)$ while $\lambda$ and $\xi$ are defined in eqs. \eqref{eq:1} and \eqref{eq:3} respectively. (According to the next section, in the regime \eqref{eq:3}, the estimate \eqref{eq:4} is modified through the substitution \eqref{eq:11}.) As for $h^2(\gamma)$, it is to be viewed as the squared distance between the particular point $\gamma = (\gamma_1, \gamma_2)$ of a given worldsheet $\tilde{M}(C)$ and the surface $\tilde{M}_{\text{min}}(C)$ (i.e. the relative height of $\tilde{M}$ measured, at the point $\gamma$ sufficiently far from the boundary $C$, in the direction normal to $\tilde{M}_{\text{min}}$) both spanned by the loop $C$ in question. Then, taking into account that the width $\sqrt{\langle r^2 \rangle}$ of the stable flux-tube (associated to the $SC$ solution of the Loop equation \eqref{eq:2}) is supposed to be $\sim \Lambda^{-1}$, we reproduce the first of the previously asserted conditions \eqref{eq:5}. Note also that eq. \eqref{eq:1} evidently implies that, in the large $N \text{ SC}$ regime \eqref{eq:4}, the Gauge String worldsheet is characterized by the infinite Hausdorff dimension $D_H$.

Let us now return to the derivation of eq. \eqref{eq:1} that, as a by-product, will allow to present the semiclassical evaluation the physical string tension $\sigma_{ph}(R)$ (see eqs. \eqref{eq:7} and \eqref{eq:9}) in the regime \eqref{eq:4}/\eqref{eq:7}. To this aim, we first consider the simplest situation when the profile of the smearing function $G(z^2)$ is deformed in accordance with the local limit \eqref{eq:8} when the flux-tube becomes infinitely thin. (The generalization of the analysis to the case of the fat $YM$ vortex will be given in the end of this subsection.) In the considered limit, the equivalence theorem (stated in the beginning of the section) ensures that the stringy system \eqref{eq:1}/\eqref{eq:2} is dynamically reduced to the Nambu-Goto one \eqref{eq:3}/\eqref{eq:4} with $m_0 = 0$ and $\lambda(\lambda) = \xi \lambda$. To make the required estimates for sufficiently large $\tilde{\lambda} = \xi \lambda$, one can employ the reparametrization symmetry \eqref{eq:5} and introduce the \lq\lq semirelativistic\rq\rq parametrization of the characteristic worldsheets

$$x_{\mu}(\gamma) = y_{\mu} \quad \text{if} \quad \mu = 1, 2 \quad ; \quad x_{j+2}(\gamma) = h_{j}(y) \quad \text{if} \quad j = 1, ..., D - 2,$$

where the coordinates $y_{a}$, $a = 1, 2$, parametrize the minimal area surface $\tilde{M}_{\text{min}}(C)$ presumed to have disc’s topology. The position of $\tilde{M}_{\text{min}}(C)$ in $R^D$ is encoded in the metric $\hat{g}^{ab}(y)$ to be normalized by the condition $det[\hat{g}(x)] = 1$ so that $A[\tilde{M}_{\text{min}}(C)] = \int_{\tilde{M}_{\text{min}}(C)} d^2 y$. Therefore, within the semiclassical approach one can employ the reduced pattern\footnote{The pattern \eqref{eq:2} is not applicable to those configurations of $\bar{M}(C)$ which, in the Hamiltonian framework, correspond to the creation of closed ‘sea’-strings from the vacuum. At least for $\tilde{\lambda}$ large enough, the latter creation processes are infrared irrelevant.} of the area-functional

$$\sigma_0 A[\tilde{M}(C)] \rightarrow \frac{\tilde{\lambda} \Lambda^2}{2} \int_{\tilde{M}_{\text{min}}(C)} d^2 y \sqrt{1 + \delta^{ij} \hat{g}^{ab}(y) \partial_a h_i(y) \partial_b h_j(y)} \quad \text{,}$$

where the height $h(y) = \{h_j(y), j = 1, ..., D - 2\}$ has been introduced in eq. \eqref{eq:1}, and $\tilde{M}_{\text{min}}(C)$ in what follows is constrained to be flat: $\hat{g}^{ab}(y) \rightarrow \delta^{ab}$.

Evidently, the action \eqref{eq:4} defines (together with the ghost-determinants due to the fixing of the symmetry \eqref{eq:3}) the \lq\lq two-dimensional\rq\rq effective field theory in the finite box of the 2-volume $A_{\text{min}}(C)$ and with the $UV$ cut off $\Lambda$. In the quasi-classical limit $\tilde{\lambda} \gg 1$, one can safely expand the square root in eq. \eqref{eq:4} that makes manifest the asymptotics

$$\tilde{\lambda} \rightarrow \infty : \quad \sqrt{\langle (\partial_a h)^{2n} \rangle} \sim \frac{1}{\tilde{\lambda}} \quad \Rightarrow \quad 1 + (\partial_a h)^2 \rightarrow 1 + \frac{(\partial_a h)^2}{2} \quad \text{,}$$

\footnote{For a given $\tilde{\lambda} = \xi \lambda$, the peculiar logarithmic scaling \eqref{eq:1} is observable (as a profile of the energy distribution) only in the regime when the height is sufficiently larger than the width of the flux-tube.}
which, in the leading order (when the contribution of the ghost-determinants can be neglected), reduces the system \((3.3)\) to the free 2d theory of the vector field \(h(y)\) with the \((D-2)\) components \(h_j\). This result can be immediately utilized to justify that, in the regime \((1.7)\) (i.e. for sufficiently large \(\bar{\lambda}\)), the estimate \((3.1)\) is indeed valid. The only subtlety is that one is to consider the variety of the large contours \(C\) so that the corresponding minimal area \(A_{\text{min}}(C)\) and the radius of curvature \(R\) comply with the conditions \((3.4)\). Additionally, we have to require that the considered contours \(C\) are macroscopic which can be formalized by the following twofold constraint. Firstly, except for the \(1/\Lambda\)-vicinity of the nontrivial (point-like) selfintersections of the boundary \(C\), the distance \(|x(s) - x(s')|\) is much larger than the flux-tube’s width \(\sqrt{<r^2>} \sim \Lambda^{-1}\), provided that the length of the corresponding segment of the boundary \(C = \partial \tilde{M}\) is much larger than \(\Lambda^{-1}\):

\[
\int_{s'}^s dt \sqrt{x''_\mu(t)} \gg \Lambda^{-1} \implies |x(s) - x(s')| \gg \Lambda^{-1} . \quad (5.5)
\]

Secondly, we always presume that, once \(C_{xx} = C_{xy}C_{yx}\) (where \(x(s) = y(s')\), \(s \neq s'\)) satisfies the above condition, then both \(C_{xy}\) and \(C_{yx}\) comply with this condition as well.

Next, employing the standard proper-time regularization of the determinant of the 2d Laplacian (see eq. \((5.9)\) below), the semiclassical evaluation of the functional integral (associated to the action \((3.3)\)) yields

\[
\zeta_D = \frac{D-2}{4\pi} \quad (5.6)
\]

for the entropy constant \(\zeta_D\). The latter enters the corresponding approximation for the physical string tension \((1.7)\) deduced, in the regime \((1.4)/(1.7)\), from the loop-average \(< W_C >\) in the (anti)fundamental representation of the \(U(N)\) group in question. More generally, consider the physical string tension \(\sigma_{ph}(R)\) corresponding to the average \(< W_R >\) in a given (anti)chiral \(U(N)\) representation \(R\). Taking into account the condition \((1.4)\), in the leading order of the \(1/N\) expansion, one readily obtains within the same semiclassical approach

\[
\sigma_{ph}^{(sc)}(R) = |n(R)| \left( \frac{\bar{\lambda}}{2} - \zeta_D \right) \Lambda^2 ; \quad R \in Y_n^{(N)} , \quad (5.7)
\]

where \(n(R) \equiv n \sim N^0\) is the total number (see eq. \((5.2)\)) of the elementary \(YM\)-fluxes associated to \(R\). The pattern \((5.7)\) is to be compared with the asymptotical \(C_2(R)\)-scaling \((6.1)\) of \(\sigma_{ph}^{(se)}(R)\) taking place in the complementary regime \((1.8)\).

Finally, let us show that, in the \(SC\) regime \((1.7)\), the eq. \((5.1)\) remains valid for a generic profile of the smearing function \(G(x^2)\) satisfying eqs. \((2.7)\) and \((3.4)\). Recall first that, once \(\xi = O(1)\), the normalization \((2.5)\) ensures that the width \((3.9)\) of the \(YM\) vortex is of order of \(\Lambda^{-1}\). Next, we presume that the width is much smaller than the amplitude of the fluctuations provided the minimal area \(A_{\text{min}}(C)\) is sufficiently large. To justify this presumption, observe that the scaling \((5.4)\) of \(\sqrt{<h^2>}\) evidently remains valid for generic worldsheet weight \((1.3)\) constrained to support the complementary scaling \((1.4)\) of \(\sigma_{ph}\). Employing the same parametrization\(^36\) of \(\tilde{M}\) as in eq. \((5.3)\), the leading nontrivial order of the operator expansion \((3.8)\) is reduced in the limit \(\bar{\lambda} \to \infty\) to the gaussian pattern

\[
\frac{\lambda \Lambda^2}{2} \int d^2y \, h_j(y) \, \hat{O}(\square) \, h^j(y) ; \quad \hat{O}(t^2) = \left( t^2 \hat{\mathcal{G}}_2(t^2 - \hat{\mathcal{O}}_2(t^2) + \hat{\mathcal{O}}_2(0)) \right) \bigg|_{t^2 > 0} > 0 , \quad (5.8)
\]

\(^36\)In this way, one has to neglect all the selfintersections of \(\tilde{M}(C)\) which, strictly speaking, are assigned in the short-distance expansion of eq. \((1.3)\) with the weights additional to the ones associated to eq. \((3.8)\). In the \(D = 3\) regime \((1.7)\), employing the technique of Appendix A, it can be shown that the latter extra weights, being infrared irrelevant, do not alter the estimate \((3.1)\).
where \( \Box = -\partial_b \partial^b \) (with \( \partial_b \equiv \partial/\partial y_b \)), while \( \tilde{G}_2(q_a q^a) = \int d^2 z e^{i q_a z^a} G(z_a z^a) \) and \( \tilde{O}_2(q_a q^a) \) are the 2d Fourier images of respectively the smearing function \( G(z_a z^a) \) and its derivative \( \frac{d \tilde{G}(r)}{dr} \bigg|_{r=|z_a z^a|} \), so that (in accordance with eq. (5.4)) \( \tilde{O}(t^2) \to \xi t^2/2 \) when \( t^2 \to 0 \). Upon a reflection, in the SC phase (1.7), the Gauge String is supposed to be stable with respect to the considered small fluctuations that implies the positivity-constraint: \( \tilde{O}(t^2) = (\xi t^2/2 + \sum_{n \geq 2} b_n (t^2/\Lambda^2)^n) > 0 \) for \( \forall t^2 > 0 \). In turn, combining the latter stability with the implications of the previous analysis, one justifies the required estimate (5.3) since, in the expansion of \( \tilde{O}(t^2) \), the higher order \( n \geq 2 \) terms in this case are irrelevant for the considered logarithmic asymptotics of \( < h^2 > \). Also, in the regime (1.4)/(1.7), eq. (5.8) can be straightforwardly used to generalize the estimate (5.6):

\[
\zeta_D = \frac{D-2}{4\pi} \int_1^\infty \frac{dt}{t} \int_0^{\infty} dt^2 \exp \left( -\frac{2t}{\xi} \tilde{O}(t^2) \right), \quad \tilde{O}(t^2) \bigg|_{t^2=0} = \frac{\xi}{2}, \tag{5.9}
\]

where the normalization of the exponent is fixed to match (in the limit \( \tilde{O}(t^2) = \xi t^2/2 \)) with the result which can be obtained employing the Pauli-Villars regularization used in the original computation of [7].

5.2 Identification of the proper universality class.

Finally, let us demonstrate that, for any macroscopic contour \( C \), the conditions (1.4)/(1.7) ensure the following important universality of the low-energy results. Under the latter conditions, for a large variety of the smearing functions \( G(x^2) \), the \( (b = 1) \) stringy sum (4.1)/(4.3) is in the same universality class\(^{37}\) with the properly associated Nambu-Goto system (1.1)/(1.6). To support the infrared equivalence with the Nambu-Goto theory, observe first that in the regime (1.4), the expansion (3.6) can be utilized to consistently define (building on the stringy representation (4.1)/(4.5)) the effective 2-dimensional gravity on the worldsheet \( \hat{M} \). In the infrared domain, the \( p = 0, p = 1 \) and \( p \geq 2 \) operators \( Q_{2p}^{(k)} \) in eq. (3.6) refer respectively to the relevant, marginal and irrelevant operators associated to the former operators via certain trick (4.28) sketched below. In turn, it allows to apply the general arguments of the Wilsonian renormgroup to the effective theory (3.6)/(4.1) in order to deduce that, at least in the SC phase (1.4)/(1.7), the pattern of the leading \( p = 0 \) Nambu-Goto term correctly singles out the universality class in question.

To sketch the purported arguments, recall first that the above mentioned trick (4.28) is aimed to circumvent the nonpolynomial dependence (on the derivatives \( \partial x_\mu(\gamma)/\partial \gamma_a \equiv \partial_a \) with respect to the coordinates \( x_\mu(\gamma) \) maintaining the parametrization (1.4) of the surface \( \hat{M} \) ) of the determinant \( det[\tilde{g}(\gamma)] = p^\mu_\mu/2 \) of the induced metric \( \tilde{g}_{ab}(\gamma) \) entering, see Appendix A, the operators \( Q_{2p}^{(k)}(\gamma) \). For this purpose, one is to introduce the auxiliary metric-tensor \( h_{ab}(\gamma) \) to be identified with \( g_{ab}(\gamma) \) through the Lagrange multipliers \( \varphi^{ab} \). Employing the conformal gauge \( h_{ab}(\gamma) = \rho(\gamma) \delta_{ab} \), it allows to trade \( (det[\tilde{g}(\gamma)])^{1/2} \) in \( Q_{2p}^{(k)}(\gamma) \) for the dimensionless auxiliary scalar field \( \rho(\gamma) \). Next, consider the SC phase (1.7), where \( \varphi^{ab} \) is supposed to exhibit a nontrivial condensate: \( < \varphi^{\alpha\beta} > = \varphi^{ab} \delta^{ab} \) with \( \varphi \neq 0 \). As a result, for the computation of the loop-averages in the large-area regime (1.4), the expansion (5.8) can be traded (generalizing the analysis of [28]) for the \( \int_M d^2 \gamma \)-integral of certain polynomial (with the \( \rho(\gamma) \)-dependent coefficients) expressed in terms of the derivatives \( \partial_{a_1}...\partial_{a_k} x_\mu(\gamma) \). In particular, provided that \( N, \lambda \to \infty \) while the conditions (1.4),(1.7) are fulfilled, for any macroscopic contour \( C \) the leading asymptotics of the \( b = 1 \) stringy sum (4.1)/(1.3) is

\(^{37}\)Furthermore, judging from the analysis of Section 8, it is reasonable to suppose that this is the only universality class suitable to provide with the \( \Lambda^2 \)-scaling (1.7) (of \( \sigma_{\rho h} \)) predetermined, in the large \( N \) SC phase, by the pattern (2.3) of the Loop equation.
given by the shear $m_0 = 0$, $\bar{\lambda}(\lambda) = \xi \lambda$ Nambu-Goto weight (1.6) assigned to the minimal-area surface $\bar{M}(C)$ with $\chi = 1$.

More generally, given any macroscopic contour $C$, the pattern of the renormgroup flow implies the following. For sufficiently large $\lambda$, there is such a variety of the smearing functions $G(x^2)$ (including those for which all the moments $K_n[G] = O(1)$) that the stringy systems (1.1), based respectively on the weights (1.3) and (1.6), merge in the regime (1.4)/(1.7) provided the loop-averages reassures that (modulo the subleading numerical factor depending on the number of the point-like (self)intersections of the loops) the loop-averages $< W_C >$ in both of the systems can be computed with the help of one and the same low-energy theory with the new $UV$ cut off $\Lambda'$; $R^{-1} \ll \Lambda' \ll \Lambda$. As previously, we presume that the boundary contour $C$ is devoid of any 1d selfintersections (including zig-zag backtracking).

The above renormgroup arguments are most straightforward to apply in the $D \geq 5$ case. Indeed, as the conditions (1.4) ensure that, for any macroscopic contour $C$, the $YM$ vortex behaves (in its interior) as if it is infinitely thin, one can neglect all the selfintersections in the interior of the flux-tube worldsheets. This is justified by the theorem [18] that the selfintersecting 2d surfaces, entering the measure of eq. (1.1), span the subspace which in $D \geq 5$ is of measure zero in the relevant space $\mathcal{I}(M, R^D)$ generated by the smooth immersions (1.2). In $D = 3$ and 4, one is to account for the extra contribution to the short-distance expansion of the worldsheet weight (1.3) which is associated respectively to 1d and point-like stable selfintersections. A more careful analysis shows then that the dynamical effect of the latter selfintersections in $D \geq 3$ is infrared-irrelevant at least as far as the constraints (1.4)/(5.5) are satisfied. Altogether, it substantiates the applicability of the arguments based on the notion of the effective 2

5.2.1 Infrared reduction of the $N$-dependence of the loop-averages.

It is noteworthy that, compared to the $SC$ asymptotics (1.10) where the $N$-dependent pattern of the quasi-contact interactions is observable, in the (multiloop generalization of the) regime (1.4) the $N$-dependence of the $U(N)$ loop-averages $N^b \ll \prod_{k=1}^{2} W_{C_k}$ is substantially reduced.[38] Once the macroscopic contours $C_k$ do not possess 1d (self)intersections, in the considered regime this dependence is reproduced by the purely topological ’t Hooft $N^x$-factor built into the weight (1.6) of the infrared equivalent Nambu-Goto theory. (As for the $SU(N)$ case, there appears the additional source of the $1/N$-dependence which is formalized via the substitution (5.3).)

Actually, the latter reduction can be viewed as a particular consequence of the well-known information-loss that generically takes place when the physical data is restricted to its low-energy sector. Namely, the remaining block of eq. (1.6) is universal: modulo the particular functional dependence of the parameters $\bar{\lambda}(\lambda)$ and $m_0(\lambda)$ on $\lambda$, this block is common for a large variety of the actions of the dual $U(N)$ gauge theory. (Complementary, there is a large variety of the quasi-local weights, given by the substitution of the bilocal pattern in the exponent of eq. (1.3) by a generic area-functional (2.15), that result in the stringy systems belonging to the same universality class ’labelled’ by the Nambu-Goto theory (1.4).) Furthermore, combining the previous analysis with the conclusions of [1], we expect the following large $N$ universality of the infrared properties

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[38] This reduction can be confronted with the hypothesis [11] that all gauge theories are described by the same universal string theory, with the dependence on the gauge group being reproduced through the factor (equal to $N^x$ in the $U(N)$ case) sensitive, in contradistinction to the $\lambda$-dependence of the $J[..]$-factor (1.3), only to the topology of the surface $\bar{M}$. 

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of the \(D \geq 3\) \(YM_D\) systems. Namely, the \(N \to \infty\) limit of the proposed implementation (1.7) of the Nambu-Goto theory (1.1)/(1.6) is presumed to provide with the low-energy description which is universal (modulo the particular dependence of \(\lambda(\lambda)\) and \(m_0(\lambda)\) on \(\lambda\)) for any \(D \geq 3\) \(U(\infty)\) or \(SU(\infty)\) pure gauge system (endowed with an arbitrary polynomial, in terms of \(F_{\mu\nu}\), lagrangian) once the free energy of the latter exhibits the \(N^2\)-scaling.

6 The loop-averages in the SC regime (1.8).

In the extreme SC limit \(\lambda \gg 1\), the area-parameter \(\ln[A_{min}\Lambda^2]\) can be used to vary the \(D \geq 3\) pattern of the average \(<W_{C'}^R>\) trading the regime (1.4) for the opposite regime (1.8) where the specifics of the particular gauge theory’s action (1.1) becomes most observable. In order to substantiate the dimensional reduction (1.10), we first derive the modification (2.17) of the estimate (5.1). For simplicity, we restrict the derivation to the simplest case of the \(U(1)\) gauge theory (described through the \(SU(1)\) expansion (1.1)/(1.6)), presuming that the appropriately formulated general conclusions are common for all \(N \geq 1\).

6.1 The collective dynamics of the fat composed \(YM\) vortex.

Let us consider the \(U(1)\) average \(<W_{C'}^R>\) in the representation \(R\) labelled by an integer number \(n \in \mathbb{Z}\). The associated \(N = 1\) stringy representation (1.1)/(1.6) of \(<W_{C'}^R>\) implies that the worldsheet \(\tilde{M}(C)\) has the boundary \(C = \cup_{k=1}^n C'_k\) (where \(C'_k\) is the \(k\)th copy of \(C'\)) resulting from the \(n\)-times winding around the base loop \(C'\). Therefore, \(\tilde{M}(C)\) can be visualized as the union \(\cup_{k=1}^n \tilde{M}_k(C'_k)\) of the \(k\) 'elementary' worldsheets \(\tilde{M}_k(C'_k)\) (each corresponding to the unit \(YM\) flux) spanned by the corresponding \(C'_k\). Altogether, we arrive at the picture of \(n\) independently fluctuating fat flux-tubes of the width \(\tilde{h}\) \(<\hbar^2>\sim \Lambda^{-1}\) which is much larger than the characteristic amplitude \(\sqrt{<\hbar^2>}\) of their fluctuations, once the conditions (1.8) are fulfilled. In consequence, the quasi-contact interactions between the fat elementary \(YM\) vortices can not be neglected even for \([A_{min}\Lambda^2]\) \(\gg 1\). The purpose of this subsection is to show that, from the infrared viewpoint (i.e. at the distance-scales \(\gg \Lambda^{-1}\)), these interactions ensure the remarkable reduction of the number of the independent low-energy stringy excitations. Namely, the considered conglomerate of the \(n\) fat elementary \(YM\) vortices behaves, in the infrared, as a single composed flux-tube which, fluctuating with the amplitude given by the modification (2.17) of eq. (5.1), supports the physical string tension (6.1).

To this aim, one is to introduce first the \(n\) different heights \(h_k(y) = \{h_{j,k}(y)\} j = 1,..,D-2; k = 1,..,n\) associated, according to the parametrization (5.2), to each of the previously defined worldsheets \(\tilde{M}_k(C'_k)\). Then, the simple algebraic manipulations vary that, for small fluctuations, the pattern of the \(N = 1\) weight (1.3) is reduced to the quadratic (with respect to \(h_k(y)\)) form given by the twofold modification of eq. (5.8). In addition to the \(N = 1\) option of the substitution (2.17), the single variable \(h(y)\) is superseded by the collective coordinate \((\sum_{k=1}^n h_k(y))/n\). The latter can be interpreted as the coordinate which, defining the auxiliary surface \(\tilde{M}_k(C'_k)\), refers to the 'center of mass' of the \(n\) worldsheets \(\tilde{M}_k(C'_k)\). The key-point is that, due to the \(S(n)\)-invariance with respect to the arbitrary permutations \(h_k(y) \to h_{\sigma(k)}(y), \sigma \in S(n)\), the considered quadratic form is independent of all the \(n-1\) 'relative' coordinates \(h_k(y) - h_{k-1}(y)\).

\footnote{It is the value of \(\sqrt{<\hbar^2>}\), rather than the averaged height (5.1), that determines the characteristic \(D \geq 3\) distribution of the energy density of the \(YM\) vortex in the regime (1.8). As a result, the logarithmic scaling (5.1) is essentially unobservable in this case.}
This observation can be strengthened further to reveal that, within the short-distance expansion (1.3), the relative fields $h_k(y) - h_q(y)$ are supposed to be massive owing to the presence of the potential-like terms $(h_k(y) - h_q(y))^m$, $m \geq 2$. Therefore, these degrees of freedom are supposed to be irrelevant at the distances $\gg \Lambda^{-1}$. Consequently, as long as the constraints (1.8) are satisfied, the residual infrared dynamics of the 'center of mass' effective excitation $\tilde{M}_+ (C')$ in the leading order reduces to the one described by the $N = 1$ variant of the quasi-local weight (2.18) where the support $T_{\min}$ should be changed for the collective coordinate $M_+ (C')$ to be integrated over as in eq. (4.1). To justify the consistency of the arguments, one is to observe that the fat flux-tube, assigned with the weight (2.17), indeed fluctuates with the amplitude given by the modification (2.17) of the estimate (5.1). In the limit $\ln[A_{\min} \Lambda^2] \gg 1$, this amplitude is supposed to be much larger than the characteristic amplitude of the relative fluctuations $|h_k(y) - h_{k-1}(y)|$ which is expected to be $\sim 1/\lambda$.

### 6.2 The $YM_D \to YM_2 (T_{\min})$ dimensional reduction.

Now we are in a position to deduce the prescription (1.10) for the leading asymptotics of the average $< W_C >$ in the $D$-dimensional $YM_D$ theory (1.1) evaluated in the extreme SC limit (1.8) accompanied by the auxiliary condition (1.9). To begin with, from the above discussion it is clear that in the considered limit, the $D \geq 3$ sum of the $YM$ vortices is localized on the (moduli space of the) saddle-point worldsheet(s) $\tilde{M}_{\min} (C)$. The latter, implementing the absolute minimum of the area-functional $A[M(C)]$, are presumed to possess a common support $T = T_{\min} (C)$. Let us further constrain that, at any point of $T_{\min} (C)$ which does not belong to the $1/\Lambda$-vicinity of possible 0- or 1-dimensional selfintersections of $T_{\min} (C)$, the line in the normal direction either does not have the second intersection with $T_{\min} (C)$ or the second intersection takes place at a distance $\gg \Lambda^{-1}$. Then, combining the condition (1.9) with the general pattern (3.6) of the short-distance expansion, one arrives at the following important universality. Namely, the smearing of the local pattern (4.11) associated to the support $T_{\min} (C)$ becomes essentially unobservable modulo the subleading contribution of the perimeter type akin to the one of eq. (1.12). (Furthermore, the latter perimeter-dependent terms can be eliminated, up to a constant factor depending on the number of point-like selfintersections of the loop, by the proper choice of the regularization of the Loop equation which will ensure $m_0 (\lambda) = 0$.) In other words, the flux-tube in its interior can be treated as infinitely thin which facilitates the regularization-independence of the leading asymptotics of the $D \geq 3$ average $< W_C >$ modulo the choice of auxiliary parameter $\xi$ in the expression (1.11) for $g_{YM_2}$.

As a result, the corresponding leading contribution of the fat $YM$ flux-tube(s) is correctly reproduced by the r.h. side of eq. (4.3) associated to the support $T_{\min} (C)$ (of the minimal area worldsheet(s) $\tilde{M}_{\min} (C)$) which, for simplicity, is presumed to be unique for a given $C$. In turn, it is supposed to justify the dimensionally reduced SC asymptotics (1.10) of $< W_C >$, once $\lambda$, $(\mathcal{R} (s) \Lambda)^2 \gg N^2$ so that one can retain only the leading order of both the $1/\lambda$-expansion and the curvature expansion (1.6) yet summing up the relevant $1/N^2$-subseries. (In the regime (1.8) with $\lambda$, $(\mathcal{R} (s) \Lambda)^2 \leq N^2$, one is to retain in the r.h. side of eq. (1.10) only the leading $O(N)$ asymptotics of the average $< W_C > |_{YM_2 (T_{\min})}$.) In particular, when the general regularization

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40Remark that, through the data of the singularities (excluded from the immersion-space $\mathcal{I}(M, R^D)$) of the map (1.2), a given minimal area worldsheet $\tilde{M}_{\min} (C)$ may have additional 'quantum numbers' apart from its area, genus and the support in $R^D$.

41For example, eq. (2.18) presents the smearing regularization of the local pattern (1.11).

42A somewhat similar dimensional reduction $YM_D \to YM_2$ is argued to take place in the heuristic model (based on certain picture of the stochastic vacuum) proposed to approximate the description of the $YM_D$ theory (1.1) but in the standard phase when $\lambda \to 0$. 

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prescription (2.3) is fixed, the parameter $\xi$ (entering e.g. eq. (1.11)) is to be identified with the zeroth moment (3.4) of the properly introduced smearing function $G(\ldots)$.

On the one hand, the prescription (1.10) implies that, even in the regime (1.8), there is a subspace of the full loop space where the $SC$ asymptotics (1.8) of the average $< W_C >_{YM_D}$ is still reproduced by the ($m_0 = 0$ option of the) Nambu-Goto pattern (1.6) applied to the minimal area worldsheet. In turn, the simple pattern (1.5) (applied to $\tilde{M}_{\text{min}}(C)$) remains applicable when the saddle-point flux-tube is given by the elementary $YM$ vortex with unobservable (within the short-distance expansion (3.6), provided the latter is convergent at least asymptotically) selfoverlapping. Then, the dimensional reduction (1.10) equates (modulo the $\xi$-rescaling) the $SC$ limit (3.4) of the physical string tension in the $YM_D$ system (1.1) with the string tension (13, 14) in the $D = 2$ $YM_2$ theory (1.1).

On the other hand, to implement in the simplest setting the deviation of the prescription (1.10) from the $m_0 = 0$ Nambu-Goto pattern (1.6), one is to consider the $D \geq 3$ average $< W_C^{R} >$ in a generic, not necessarily (anti)fundamental representation $R$ belonging the $U(N)$ (anti)chiral sector. Conventionally, we define the physical string tension $\sigma_{ph}(R)$ from the large area asymptotics (to be considered after the one of eq. (1.8)) of $ln[< W_C^{R} >]$. Evaluating $< W_C^{R} >_{YM_2(T)}$ on a nonselfintersecting disc bounded by $C$, one concludes from the pattern (4.11) that in the regime (1.8) the tension
\[ \sigma_{ph}^{(sc)}(R) = \left( \frac{\chi C_2(R)}{2N} - \zeta_D \right) \Lambda^2, \quad R \in Y_n^{(N)}, \quad (6.1) \]
asymptotically scales with respect to $R$ as the corresponding eigenvalue $C_2(R)$ of the second Casimir operator of the $U(N)$ group in question:
\[ C_2(R) = \sum_{i=1}^{N} n_i \cdot (n_i + N - 2i + 1) \quad ; \quad n(R) = \sum_{i=1}^{N} n_i. \quad (6.2) \]
As for the $N$ integers $n_i$, parametrizing a generic $U(N)$ representation $R$, they satisfy the condition: $n_1 \geq n_2 \geq \ldots \geq n_N$ so that $n \equiv |n(R)|$ measures the total amount of the elementary $U(N)$ fluxes, and $C_2(R) \sim N$ once $|n(R)| \sim N^0$. Note also that, for completeness, in the semiclassical result (6.1) (to be compared with eq. (5.7)) we have also included the next-to-leading term associated to the entropy of the flux-tube, parametrized by $R$, composed of $|n(R)|$ elementary vortices of unit flux.

Finally, in the $SC$ limit (1.8), the interactions between the latter elementary vortices is predetermined by the previously considered amplitude $< W_C^{R} >_{YM_2(T)}$. To reveal the type of these interactions, one is to take into account that (owing to the constraint $n_k \geq n_{k+1}$) the pattern (5.2) ensures the inequality
\[ C_2(R) \geq |n(R)|, \quad \forall R, \quad (6.3) \]
where $n(R)$ is given by eq. (5.2) and, for the subset of chiral $U(N)$ representations, the lowest bound $C_2(R) = n(R)$ is achieved when $n_i = 1$ for $\forall i = 1, 2, \ldots, N$. Therefore, apart from the latter exceptional configuration (and its antichiral analogue with $n_i = -1$ for $\forall i$), the averaged force between the elementary flux-tubes is the repulsion that reminds of the picture associated to the type-II (dual) abelian superconductor.

43 Recall that, in the latter sector, the $U(N)$ character $\chi_R(V)$ is composed of the products (10) of the traces $tr[(V)^k]$ only. The traces $tr[(V^{-1})^k]$, associated to the holonomies winding in the opposite direction, are not involved. The choice of a chiral or antichiral irrep $R \in Y_n^{(N)}$ implies that, in eq. (6.2), either $n_i \geq 0$ or $n_i \leq 0$ for $\forall i$. Consequently, all the $|n(R)|$ elementary fluxes, composed into the fat collective $YM$ vortex associated to $< W_C^{R} >$, possess one and the same orientation.
7 Peculiarity of the dynamical regime (1.4).

In the regime (1.4)/(1.7), the $D \geq 3$ Nambu-Goto representation (1.1)/(1.6) is presumed to be further reformulated in compliance with the general spirit of the noncritical Polyakov’s string [4]. Therefore, for macroscopic loops (satisfying, apart from the $1/\Lambda$-vicinity of the selfintersections, eq. (5.5)), the ‘microscopic’ $D \geq 3$ Nambu-Goto system (1.1)/(1.6) is to be traded for the ‘macroscopic’ low-energy theory with certain auxiliary worldsheet’s fields including the Liouville one. The important novelty is that the unconventional scaling (1.7) of $\sigma_{ph}$ implies certain features of the Gauge String dynamics which, in $D \geq 3$, are expected to be essentially different compared to the standard Polyakov’s noncritical string theory [4].

7.1 Implications of the unconventional scaling (1.7).

To begin with, the straightforward consequence of equation (1.7) is that, generically, $\sigma_{ph}$ is of order of $\Lambda^2$ unless $\bar{\lambda}(\lambda) \to 2\zeta_D$. The latter condition is in sharp contrast with the standard scaling

$$\sigma_{ph}/\Lambda^2 \to +0,$$

(7.1)

when the physical string tension is tuned to be much less than $\Lambda^2$. As we already stated in the Introduction, the seemingly bizarre scaling (1.7) could be expected in advance, being foreshadowed by the interpretation of $\Lambda$ as the confinement-scale in the effective low-energy theory of the $D = 4$ weakly-coupled gauge theory (1.1). Complementary, in view of eq. (1.7), the dynamics of the stringy system (1.1)/(1.6) is supposed to be devoid of the notorious branched-polymer ‘pathology’ inherent in the ‘fundamental’ $D \geq 3$ Nambu-Goto/Polyakov string associated to the regime (7.1).

To see how the scaling (1.7) improves the situation, let us first recall the expected source of the above ‘pathology’ in the conventional case (7.1). To begin with, choose an intermediate scale $\Lambda'$ (with $\sqrt{\sigma_{ph}} << \Lambda' << \Lambda$) and consider the effective low-energy theory (of the conventional Nambu-Goto string) with $\Lambda'$ as the new UV cut off. Then, at the momentum-scales less than $\Lambda'$, the dominant contribution to the worldsheet’s entropy is received from the graph-like surfaces visualized through the outgrowth of the baby-universes reminiscent of fat graphs with the characteristic transverse size of order of $1/\Lambda'$. Therefore, the considered effective theory is presumed to fit in the pattern of the branched polymer graphs filling the base-space $\mathbb{R}^D$.

On the other hand, it is easy to demonstrate that the $\Lambda^2$-scaling (1.7) of $\sigma_{ph}$ implies the severe suppression of the considered outgrowth of the baby-universes. To this aim, let us first note that (due to the UV cut off $\Lambda$) a tree-like worldsheet’s configuration of length $L$ costs the area at least of order of $L\Lambda^{-1}$ where the well-developed tree-structure implies $L \gg \Lambda^{-1}$. In the branched polymer phase, the characteristic worldsheet $\tilde{M}(C)$ would look like the minimal area surface $\tilde{M}_{min}(C)$ with the attached multiple tree-like tentacles. For large $A_{min}(C)\Lambda^2$, the area-density of these attachments is supposed to be of order of $1/\Lambda'$. Therefore, the considered effective theory is presumed to fit in the pattern of the branched polymer graphs filling the base-space $\mathbb{R}^D$.

To reveal the mismatch, one can employ the pattern (5.3) that yields

$$\left( A[\tilde{M}(C)] - A[\tilde{M}_{min}(C)] > A[\tilde{M}_{min}(C)] \right) \sim 1/\bar{\lambda},$$

(7.2)

which can not be much larger than unity in the $SC$ phase (1.7) that qualitatively justifies the lack of the ’branched polymer’ outgrowth in question.

\footnote{It is noteworthy that eq. (7.2), being consistent with the the estimate (5.1), is not in conflict with the infinite Hausdorff dimension of the characteristic worldsheets.}
At this step, it is appropriate to remark that the considered mechanism of the suppression (of the outgrowth) is not in conflict with the fact that the $3 \leq D \leq 26$ regime (1.4)/(1.7) of the Nambu-Goto theory (1.1)/(1.6) is as much nontrivial as the conventional regime (7.1) in the critical dimension $D = 26$. Furthermore, having presumably avoided the branched polymer phase, we can expect the absence of the tachion for $3 \leq D \leq 26$. Consequently, it implies that the mass $m_{gl}^{(0)}$ of the lowest glueball state, being positive at least when $3 \leq D \leq 25$, is of order (or even larger than) the inverse flux-tube’s width $\sim \Lambda$. Altogether, apart from the extreme SC limit $\lambda \to \infty$ (where additional peculiarities may be anticipated), the physical spectrum associated to the regime (1.7) of the stringy system (4.1)/(1.6) is expected to fit in the pattern of the Regge trajectories similar to the one in glueball sector of the realistic QCD. In particular, the trajectories are supposed to contain the glueball states of arbitrary large masses $m_{gl}^2 >> \sigma_{ph} \sim \Lambda^2$. Actually, owing to the weight pattern (1.6), the latter condition requires only that the characteristic ‘size’ $l_{gl}$ of the considered states is much larger than the inverse UV cut-off: $l_{gl} \Lambda >> 1$. Therefore, the asserted growth of the masses $m_{gl}^2$ is tantamount to the constraint $l_{gl} \sqrt{\sigma_{ph}} >> 1$ that is generically expected for the highly excited string-like bound states.

Finally, there is no difficulty to implement the scaling (1.7) at the level of the semiclassical evaluation (see e.g. [27]) of the Nambu-Goto averages (4.1)/(1.6) which is the adequate framework for obtaining a good approximation for $< W_C >$ in the considered regime (1.7). In particular, this computation reproduces the value (5.6) of the entropy constant $\zeta_D$ entering eqs. (1.7) and (5.7). To implement the regime (1.7) on the full-fledged quantum level, the problem arises because any direct ‘brute force’ UV cut off $\Lambda$ (for the transverse string fluctuations) by itself breaks the continuous symmetry under the worldsheet’s reparametrizations (4.4). In consequence, thus introduced cut off would produce spurious violation of the residual worldsheet’s conformal invariance (associated to the famous conformal anomaly [4]).

Upon a reflection, the known practical prescriptions [30, 31], to circumvent an explicit introduction of the UV cut off, work only in the conventional regime (7.1). Formally, the way out could be to introduce the proper counterterms to restore the conformal invariance. As this is rather unwieldy to perform explicitly, the better alternative is to make up for an effective UV cut off defined in a conformally invariant manner. (In particular, the manifest conformal symmetry of the regularized theory is crucial for the determination of the coefficient, proportional to the central charge associated to the conformal group, in front of the so-called Luscher term [17].) To this aim, the general idea is suggested by the concept of the Pauli-Villars regularization. We are to design such a renormalizable string theory, endowed with the new UV cut off $\bar{\Lambda}$ (to be sent to infinity), that has the following property. Approaching the Nambu-Goto system (4.1)/(1.6) at the momentum scales less than a fixed one $\Lambda = O(\sqrt{\sigma_{ph}})$, the theory in question should not support propagating stringy degrees of freedom at the momentum scales much larger than $\Lambda$. As for the scale $\Lambda^2$, being fixed compared to the physical string tension $\sigma_{ph}$, it is constrained to vanish (i.e. $\sigma_{ph}/\Lambda^2 \to 0$) in the units of $\Lambda^2$.

8 Including nontrivial selfintersections of the contour.

Now, we are in a position to explicitly justify one of the major results of the paper. To formulate the latter result, consider the subspace $\Upsilon'$ (of the full loop space $\Upsilon$) which is comprised of macroscopic loops (defined prior to eq. (5.5)) with at most point-like selfintersections, i.e. without selfintersections (e.g. backtracks) on 1d submanifolds. There is such a regularization of the
Loop equation (2.2) that yields, on the subspace $\mathcal{Y}'$, the solution in the form of the Ansatz (4.1)/(1.5), provided the conditions (1.4)/(1.7) and certain additional constraint (8.7) (on the smearing function (2.5)) are satisfied.

The proof of the latter statement is built on the following idea. To begin with, it is clear that the Ansatz (4.1)/(1.5) satisfies the (regularized variant of) the linear loop equation (2.4) irrespectively whether or not the contour $C$ selfintersects. Thus, the problem is to find such a regularization of the r.h. side of the nonlinear Loop equation (2.2) so that, for a generic point-like selfintersection, the latter can be reduced in the regime (1.4)/(1.7) to the regularized r.h. side of (the $N \to \infty$ limit of) eq. (2.4) on the entire subspace $\mathcal{Y}'$. This is indeed possible to accomplish owing to certain useful identity (8.3).

### 8.1 The problem to circumvent.

Before we implement this strategy, it is instructive to reveal certain important problem which resolution will require to impose the constraint (1.7) on $\sigma_{ph}$. For this purpose, it is sufficient to simplify the discussion and restrict our attention to the simple point-like selfintersections of $C$ where only two line-segments of the boundary intersect. Then, take a loop $C \in \mathcal{Y}'$ with a simple nontrivial selfintersection at $\mathbf{x}(s) = \mathbf{y}(s')$, $s' \neq s$. In this case, the r.h. side of eq. (2.2) obtains two different nonvanishing contributions both associated to $\mathbf{x}(s)$. The first one, $\mathbf{x}(s) = \mathbf{x}(\bar{s})$, $\bar{s} \to s$, as previously corresponds to the trivial selfintersection that properly ‘goes through’ the Loop equation (2.2). The apparent mismatch, between the considered Ansatz and the unregularized pattern of eq. (2.2), arises from the second term due to the additional contribution of the nontrivial selfintersection at $\mathbf{x}(s) = \mathbf{y}(s')$, $s' \neq s$. For this term, the ratio of the right and the left hand sides of eq. (2.2) is given by the factor

$$
\tau(C_{xx}|y) = \lim_{N \to \infty} \frac{\langle W_{C_{xy}} \rangle < W_{C_{yx}} \rangle}{\langle W_{C_{xx}} \rangle},
$$

(8.1)

where the averages are to be evaluated through the Ansatz (4.1)/(1.5). The problem, which in the regime (1.4)/(1.7) will be circumvented by the appropriate regularization of eq. (2.2), lies in the fact that the $D \geq 3$ factor (8.1) is generically not equal to unity. The reason is twofold. First, even in the strictly local limit (3.8) (when the stringy sum (4.1) is invariant under the substitution of the weight (1.5) by the $m_0 = 0$, $\lambda(\lambda) = \xi\lambda$ option of the Nambu-Goto weight (1.6)), the constraint $\tau(C_{xx}|y) = 1$ can not be generically satisfied if $D \geq 3$. Additionally, if the flux-tube width (3.9) is not infinitesimal compared to the amplitude of the fluctuations, the previously discussed quasi-contact interactions (of the elementary $YM$ vortices) also contribute, in the abelian manner, into the deviation of $\tau(C_{xx}|y)$ from unity for $D \geq 2$. Altogether, the Ansatz (1.1)/(1.5) runs in conflict with the factorized structure of the r.h. side of the $D \geq 3$ Loop equation (2.2), prior to the regularization of the latter.

In the standard regime (7.1), within the $1/N$ expansion, the considered mismatch can be shown to persist even after a generic regularization of the Loop equation (2.2) so that the considered Ansatz is not consistent with the latter equation. The situation improves drastically when the unconventional scaling (1.7) of $\sigma_{ph}$ is combined with the conditions (1.4) which, in particular,
ensure the essential unobservability of the quasi-contact interactions. Then, there exists an admissible regularization of eq. (2.2) that reconciles the latter equation with the residual deviation of the factor (8.1) from unity.

8.2 The proper regularization of the Loop equation.

Let us proceed with a few generalities which are relevant for the construction of such gauge-invariant regularization of (the r.h. side of) the Loop equation (2.2) that is consistent with the Ansatz (1.5)/(1.7) in the regime (1.4)/(1.7). To begin with, in the presence of a nontrivial selfintersection of a given contour \( C \), the smearing-trick (2.5) cannot be directly applied because of the two reasons: due to the deviation of the factor (8.1) from unity and the nonlinearity of eq. (2.2). Complementary, the standard prescription, to split the point of the nontrivial selfintersection, results formally in the Wilson averages of eq. (2.2). To properly combine the smearing (2.3) with the point-splitting, the general idea is rather straightforward (see e.g. (3)). In addition to the ordered exponents along the open paths \( C_{xy}, C_{yx} \), the proper regularization of eq. (2.2) should include the ordered exponents

\[
B(\Gamma_{xy}) = \mathcal{P} \ e^{i \oint_{\Gamma_{xy}} dz^\mu A_\mu(z) T^a}; \quad B(\Gamma_{yx}) = (B(\Gamma_{xy}))^{-1}, \quad (8.2)
\]

along two auxiliary (nonselfintersecting) paths \( \Gamma_{yx}, \Gamma_{xy} = \Gamma_{yx}^{-1} \) connecting \( x(s) \) and \( y(s') \). In consequence, the r.h. side of the regularized eq. (2.2) can be expressed as some linear functional (to be identified with r.h. side of eq. (8.3)) of the product \( < W_{C_{xy}} \Gamma_{xy} > < W_{C_{yx}} \Gamma_{yx} > \) that supersedes \( < W_{C_{xy}} > < W_{C_{yx}} > \) in the numerator of the ratio (8.1).

Next, one of the advantages of the regime (1.4)/(1.7) is that, dealing with any particular macroscopic contour \( C \), we can consistently distinguish between the trivial and nontrivial selfintersections of the latter. The point is that, in the considered regime, it is instructive first to design the two somewhat distinct regularization prescriptions for the latter two types of the selfintersections and then formulate certain condition (see eq. (8.7)) that ensures the proper matching between the prescriptions. As for the former type, it is convenient to retain the simple regularization (2.4)-(2.7). To handle nontrivial selfintersections, our strategy is to make use of the ambiguity in the introduction of the additional ordered exponents (8.2): one is to fix both the particular domain of integration over the positions of the (open) contours \( \Gamma_{xy} \) and the specific weight of the integration over \( \Gamma_{xy} \).

In the SC phase (1.7), to facilitate the required linearization of the regularized Loop equation (2.2), this ambiguity is to be fixed employing the following property of the \( b = 1, \chi = 1 \) functional integral (1.1)/(1.3). Let the point of a nontrivial selfintersection (at \( x(s) = y(s') \)) of a macroscopic contour \( C_{xx} = C_{xy}C_{yx} \) be resolved so that \( |x(s) - y(s')| = O(\Lambda^{-1}) \). The auxiliary identity asserts that, for any such \( C_{xx} = C_{xy}C_{yx} \) satisfying the constraints (1.4), there is such effective action \( S(\Gamma_{xy}|\mathcal{G}) \) that

\[
< W_{C_{xx}} > \int_{\Gamma_{xy} \in \mathbf{X}^{D-1}} \mathcal{D} z^{\mu}(t) e^{-S(\Gamma_{xy}|\mathcal{G})} < W_{C_{xy} \Gamma_{yx} > < W_{C_{yx}} \Gamma_{xy} > } \quad (8.3)
\]

where the functional integral runs over the (open, when \( x(s) \neq y(s') \)) paths \( \Gamma_{xy} = \Gamma_{yx}^{-1} \). Presumed to be nonselfintersecting in its interior, each path is parametrized,

\[
\Gamma_{xy} \in \mathbf{X}^{D-1} : \{ \ z^{\mu}(t) \ | \ z^{\mu}(0) = x^{\mu}(s), \ z^{\mu}(1) = y^{\mu}(s') \ \} , \quad (8.4)
\]

46The definition of macroscopic contours is given prior to eq. (7.4).
by the trajectory \( z_\mu(t) \) embedded into some \( D-1 \) dimensional subspace\(^47\) \( \mathbb{X}^{D-1} \) (diffeomorphic to \( \mathbb{R}^{D-1} \)) of the base-space \( \mathbb{R}^D \).

To explain the meaning of the relation (8.3), let us first explain the pattern of \( z_\mu(t) \). For this purpose, one is to isolate the cross-section \( \tilde{M}(C) \cap \mathbb{X}^{D-1} \) of a particular worldsheet \( \tilde{M}(C) \) (entering the measure (4.1)) with the subspace \( \mathbb{X}^{D-1} \) containing any two given points \( \mathbf{x}(s) \) and \( \mathbf{y}(s') \) of the macroscopic boundary loop \( C \equiv C_{xx} \). Then, for the dense subvariety of \( \tilde{M}(C) \in \mathcal{I}(\tilde{M}, \mathbb{R}^D) \), one can uniquely separate a nonselfintersecting contour\(^47\) to be identified with \( z_\mu(t) \), from the connected component of the resulting cross-section. As a result, in compliance with the pattern of the relation (8.3), in the regime (1.4) we are to identify \( \mathcal{Y}[\mathcal{G}] \) with the partial integration over all the worldsheets \( \tilde{M}(C_{xy} \Gamma_{yx}), \tilde{M}(C_{yx} \Gamma_{xy}) \) spanned by the fixed (for any given \( \Gamma_{xy} \)) loops \( C_{xy} \Gamma_{yx} \) and \( C_{yx} \Gamma_{xy} \).

Let us now utilize the identity (8.3) as a building block of the appropriate regularization scheme for the r.h. side of the Loop equation (2.2). First of all, as it is clear from the structure of the Ansatz (4.1)/(1.5), the following consequence of the unconventional \( \Lambda^2 \)-scaling \(^1\) of \( \sigma_{ph} \) is crucial for our purposes. Namely, once \( |\mathbf{x}(s) - \mathbf{y}(s')| \) is microscopic, the subspace \( \mathbb{X}^{D-1} \) can be chosen in such a way that the characteristic length \( \Lambda^{-1} \) of thus defined contour \( z_\mu(t) \) is microscopic as well:

\[
|\mathbf{x}(s) - \mathbf{y}(s')| \leq \Lambda^{-1} \quad \Rightarrow \quad < \mathcal{L}[\Gamma_{xy}] > = < \int_{0}^{1} dt \sqrt{2 \mu(t)} > \sim \Lambda^{-1} .
\]  

(8.5)

Turning to the construction of the regularization (of eq. (2.2)) consistent with the Ansatz (4.1)/(1.5), we start by acting on the average \( < \mathcal{W}_{C_{xx}} >_\infty \) by the Loop operator \( \hat{L}_\nu \). The result can be represented as the linear functional of \( < \mathcal{W}_{C_{xx}} >_\infty \) given by the (\( N \to \infty \) limit of the) r.h. side of eq. (2.3) regularized according to eq. (2.3). Next, we consider the simplest case of a contour \( C_{xx} \) with a single\(^47\) simple point-like selfintersection at \( \mathbf{x}(s) \). Then, for any given pair \( \mathbf{x}(s), \mathbf{y}(s') \) of the points entering the regularized r.h. side of eq. (2.4), one is to transform \( < \mathcal{W}_{C_{xx}} >_\infty \) in compliance with the identity (8.3). In sum, the Ansatz (4.1)/(1.5) satisfies (for macroscopic loops) the regularized variant of the Loop equation so that the r.h. side of eq. (2.2) is traded for

\[
\tilde{g}^2 \int_{C_{xx}} d\mu(s') \Lambda^D \mathcal{G}(\Lambda^2(\mathbf{x}(s) - \mathbf{y}(s'))^2) \int_{\Gamma_{xy}} \mathcal{D}z_\mu \ e^{-\mathcal{S}(\Gamma_{xy}|\mathcal{G})} < \mathcal{W}_{C_{xy} \Gamma_{yx}} >_\infty < \mathcal{W}_{C_{yx} \Gamma_{xy}} >_\infty
\]  

(8.6)

which synthesizes eqs. (8.3) and (2.3). Note also that, as it is clear from the smearing pattern of eq. (8.6), the contours with (point-like) selfintersection at \( \mathbf{x}(s) = \mathbf{y}(s') \) have to be considered on equal footing together with nonselfintersecting contours \( C_{xx} = C_{xy} C_{yx} \) with \( |\mathbf{x}(s) - \mathbf{y}(s')| \leq \Lambda^{-1} \).

Next, the matching between the prescriptions (2.3) and (8.6) (applied respectively to the trivial and nontrivial selfintersections of \( C \)) requires to ensure that the integrand in eq. (8.6) respects certain normalization condition inherited from the normalization of the \( \delta_D \)-function in the r.h. side of eq. (2.2). As a result, complementary to eq. (2.5), one is to impose the constraint

\[
\mathcal{Y}[\mathcal{G}] = \int d^D z \frac{\delta_{ab}}{N^2} < 0 \mid \hat{E}_{ab}[0|\mathcal{G}] \mid z > = 1,
\]  

(8.7)

\(^{47}\)The precise choice of \( \mathbb{X}^{D-1} \), that fixes a part of the regularization freedom, will not be necessary for our present purposes.

\(^{48}\)Moreover, the interior of the path \( \Gamma_{xy} \) generically does not intersect the contours \( C_{xy} \) and \( C_{yx} \).

\(^{49}\)Multiple simple point-like selfintersections can be treated through the straightforward generalization of the considered algorithm. As for nonsimple point-like selfintersections (i.e. when more than two line-segments intersect at a given point), they can be treated as the limiting case of the simple ones.
where the operator $\hat{E}^{ab}[0|G]$, which regularizes $\delta^{ab}1$, is to be introduced (see eq. (D.4) of Appendix D) rewriting eq. (8.6) in the form [3, 33] conventional for the weak-coupling analysis.

Finally, so far we have been able to derive that, in the $SC$ phase (1.4)/(1.7), the solution of the regularized Loop equation is consistent with the stringy representation (1.1)/(1.3) for the $b = 1$, $h = 0$ option of the ’t Hooft factor $N^2 - 2h - b$. From the pattern of the latter factor, it is manifest that, to justify it for $\forall h$ and $\forall b$, one is to consider the entire Dyson-Schwinger chain of the loop equations within the the $1/N$ expansion. This is done$^{[50]}$ in Appendix D where the validity of the $N^\chi$-factor is shown to be predetermined by the topological relations (D.2) which formalize the coupling between different worldsheet’s topologies taking place in the presence of a nontrivial (self)intersection of the loops. In particular, the consistency of the $N^\chi$-dependence of the weight (1.5) with the chain of the regularized $U(N)$ loop equations (D.1) requires to impose the following restriction (which strengthens the selection of the macroscopic contours $C$ without 1d selfintersections that is required in the case of the $N \to \infty$ eq. (2.2)). For $b \geq 2$ and large $N \neq \infty$, both the characteristic size (understood in the sense of eq. (5.5)) of each loop, presumed to be devoid of 1d selfintersections, and the minimal distance between any two different contours (entering eq. (D.1)) should be of order of $N^\alpha$ with some $\alpha > 0$. This restriction is aimed to suppress as $e^{-\beta N}$ (with some $\beta > 0$) all kinds of the quasi-local interactions (between the elementary flux-tubes) which abelian pattern, encoded in eq. (1.5), is in conflict with the $N \geq 2$ $U(N)$ Dyson-Schwinger chain.

Let us also note that, in view of the discussion in [7], the $SU(N)$ counterpart of the weight (1.3) is deduced through the substitution

$$U(N) \rightarrow SU(N) \quad \Rightarrow \quad \lambda_{U(N)} \rightarrow (1 - 1/N^2) \cdot \lambda_{SU(N)}$$

(8.8)

(where $\lambda_{SU(N)} = (g_{SU(N)})^2 N \Lambda^{D-4}$) that can be justified directly from the $SU(N)$ loop equations.

9 Conclusions.

To match with the regularization of the $YMD$ theory (1.1), we introduce the smearing regularizations of the 'bare' Gauge String weight (1.3) that allows to work with the fat $YM$ vortices of a nonzero width $\sim \Lambda^{-1}$, where $\Lambda$ is the $UV$ cut off. Given the appropriate smearing, the proposed in [7] stringy construction is shown to provide with the confining solution of the Dyson-Schwinger chain of the loop equations in the large $N$ strong coupling phase (1.2) of the regularized$^{[51]}$ $D \geq 3$ $U(N)$ gauge theory (1.1). Also, the concise prescription (formalized by eq. (E.3)) to implement the backtracking invariance of the average $<W_C>$ is found. Complementary, with the help of the vocabulary [3] that relates the gauge theory’s and the loop-space’s operators, the proposal of [7] can be strengthened further to present (at least at the formal level) the algorithm to reconstruct the ‘image’ of the field-theoretic $F_{\mu\nu}(x)$-correlators (in the regularized $YMD$ theory) on the stringy side.

$^{50}$The conclusions of Appendix D are to be compared with the general expectation (see e.g. [9]) that if a free (i.e. of genus zero) stringy ansatz satisfies the Loop equation (2.2) then the same interacting string complies, within the $1/N$ expansion, with the full Dyson-Schwinger chain of the loop equations.

$^{51}$In particular, we resolve the long standing problem of finding such class of the regularizations of the loop equations that can be explicitly translated into the tractable regularizations of the stringy solution of the latter equations. It is noteworthy that, due to the constraint (2.3), there is a variety of the regularizations (including the ‘most natural’ one (1.8) resulting in the infinitely thin $YM$ vortex) of the Gauge String weight which do not correspond to any regularization of the dual $YMD$ theory (1.1).
The land-mark of the considered $SC$ phase is that the infrared stable effective excitations, implementing the $1/N$ $SC$ expansion of the regularized gauge system, are the fat microscopic flux-tubes rather than point-like gluonic excitations. As the semiclassical analysis of the string fluctuations (leading, in particular, to the estimate of the entropy constant $\xi_D$) is supposed to correctly reproduce the major qualitative features of the dynamics, the mechanism of confinement in certain sense 'trivializes' within the proposed framework. (More precisely, as the gluonic excitations are not adequate to describe the large distance physics, the nontriviality is hidden in the formal resummation relating the $1/N$ $WC$ and the $1/N$ $SC$ series in the $YM_D$ theory.) Indeed, within the $SC$ series (avoiding the spurious infrared divergences characteristic for the individual Feynman graphs of the $WC$ expansion), the area-law asymptotics of the average $<W_C>$ emerges as naturally as the ordinary Coulomb interaction (between two heavy electric charges) appears in the standard $QED$.

Next, the duality-correspondence (4.9) implies that, in addition to the selfenergy (described, in the infrared domain, by the Nambu-Goto pattern), the world sheet's weight encodes certain (quasi-)contact interactions between the (self)overlapping elementary vortices assigned with the unit $YM$-flux. In the leading order (reproduced by eq. (1.3)) of the derivative expansion (3.6), these short-range interactions depend only on the choice of the gauge group and the lagrangian of the dual gauge theory, while in the subleading orders the dependence on the details of the regularization takes place. Nevertheless, we demonstrate that there are two dynamical regimes where the pattern of the Wilson loop-averages depends on the regularization prescription only through the dependence of a few relevant coupling constants on the bare $YM_D$ coupling.

The characteristic feature of the first regime (1.4)/(1.7) is that the flux-tube width is much smaller than the amplitude of the string fluctuations. In consequence, for the loops without $1d$ (self)intersections, it entails the infrared unobservability of the quasi-contact interactions so that (for a large variety of the actions of the dual $U(N)$ gauge theory) the corresponding implementation of the $U(N)$ Gauge String is in the same universality class with the Nambu-Goto theory. (Furthermore, we sort out those observables of the Nambu-Goto system which can be mapped onto the properly associated observables in the dual gauge theory.) The important novel feature arises due to the $\Lambda^2$-scaling of $\sigma_{ph}$ which, in the $SC$ phase, is mandatory at least for sufficiently large $N$. As a result, the stringy system (1.1)/(1.6) is supposed to avoid the collapse into the branched polymer phase. As for the formalism itself, in the considered regime the Loop equation (2.2) possesses, for nonselfintersecting loops, the particular solution (4.1)/(1.5) which satisfies the simple eq. (2.12). On the $\Psi$-space of the area-functionals (2.15), the general solution (4.1)/(2.8) of eq. (2.4) is also found and reinterpreted as a specific member of the 'confining strings' family.

On the contrary, in the extreme $SC$ limit (1.8), the vortex width is much larger than the amplitude of the string fluctuations. Therefore, the full set of the loop-averages does encode the data of the quasi-contact interactions and, consequently, reflects the choice of the particular action for the associated $U(N)$ gauge theory in question. Due to the relevant condition (1.9), the details of the regularization are irrelevant, and the results are represented via the universal local weight (1.3) assigned to the minimal area worldsheets. It is observable, for example, in the asymptotic Casimir scaling (6.1) of $\sigma_{ph}(R)$ for any given (anti)chiral $U(N)$ representation $R$. More generally, in the leading order of the short-distance expansion (3.9), the $SC$ asymptotics of the loop-averages in the $D \geq 3$ $YM_D$ theory (1.1) is reduced to the one in the l.h. side of eq. (1.9) evaluated in the corresponding dimensionally reduced $D = 2$ $YM_2$ system (1.1)/(1.11)\footnote{This phase can be reinterpreted as the continuum counterpart (maintaining the manifest $O(D)$ invariance of the Euclidean space $\mathbb{R}^D$) of the well-known $SC$ regime in the lattice gauge theories.}.
defined on the support $T$ of the minimal area surfaces. It is noteworthy that, starting with the regime (1.8) and then further increasing $A_{\min}(C)\Lambda^2$ (or, alternatively, decreasing the value of $\lambda$), the asymptotic $C_2(R)$-scaling (1.1) of $\sigma_{ph}(R)$ is superseded by the $|n(R)|$-scaling (5.7).

Actually, the previously discussed solution (2.2)/(1.8) has the twofold utilization in the regime (1.8) as well. For $N = 1$, the considered stringy sum presents the general solution of the loop equations in the $SC$ phase of the regularized $D \geq 2$ $U(1)$ gauge theory (2.21) (enriched with the monopoles in $D \geq 3$). In this way, this is valid irrespectively of the ratio between the vortex width and the amplitude of the fluctuations. As for $N \geq 2$, the pattern (1.1)/(2.8) properly reproduces, (being applied to the saddle-point worldsheet $M_{\min}((C_k))$, the extreme $SC$ asymptotics (1.8) of $< \prod_k W_{C_k} >$ when the minimal-area flux-tube is represented by the elementary $YM$ vortices (of unit flux) with unobservable\(^{54}\) (self)overlapping.

Finally, the ultimate goal of our project is to develop an approach which would make accessible the analysis of the low-energy phenomena in the $D = 4$ $YM_4$ theory (1.1) considered in the standard $WC$ phase with the vanishing (according to the perturbative renormgroup flow) bare coupling constant $\lambda \to 0$. Unfortunately, the direct continuation of the proposed stringy solution (of eq. (2.2)) into the $WC$ phase, being available, describes the metastable microscopic degrees of freedom, as it is predetermined by the mandatory $\Lambda^2$-scaling (1.7) of $\sigma_{ph}$. Nevertheless, more subtle utilization of the results obtained is expected to be possible in the $WC$ regime as well: the derived stringy pattern should be interpreted as describing the prototypes of the macroscopic flux-tubes\(^{46}\) which are supposed to ensure that confinement is inherited by the $WC$ phase from the $SC$ one. In other words, the extended $YM$ vortices are presumed to enter not directly the microscopic $\lambda \to 0$ gauge system (1.1) itself but rather its effective low-energy theory considered at the presumed confinement-scale. The latter is to be qualitatively identified with the mass of the lowest glueball excitation (that motivates the $\Lambda^2$-scaling (1.7) of $\sigma_{ph}$). At this scale, the effective theory is expected to be in the $SC$ phase which, in particular, manifests itself by the absence of the essentially logarithmic (with respect to the $UV$ cut off $\Lambda$) Wilsonian renormgroup flow of the running coupling constant $\lambda(\Lambda)$ in front of the $F_{\mu\nu}$-operator. (Roughly speaking, it can be reinterpreted as a kind of freezing of $\lambda(\Lambda)$.) Thus, the major problem is to find a technique which, being able to circumvent the 'brute force' Wilsonian renormgroup reduction of the $UV$ cut off, should incorporate the proposed formalism as the building block in the solution of the Loop equation in the $\lambda \to 0$ phase of the $YM_D$ theory (1.1). The work in this direction is in progress.

In conclusion, let us also remark that the regime (1.8) is reminiscent of the one which independently appeared in another recent framework (see also [25]). Being based on the conjecture about the so-called $AdS/CFT$ correspondence, this approach follows the somewhat alternative route towards certain stringy description of the pure gauge theories. It is interesting that, in the extreme $D = 4$ $SC$ regime $\lambda >> 1$, the purported $\lambda \Lambda^2$-scaling (24) of the physical string tension is consistent with the prediction (1.7) of our formalism. Accidentally, the large $N$ asymptotics of the loop-averages (entering the r.h. side of eq. (1.10)) in the $D = 2$ $YM_2$ theory (1.1) is discussed in [24] as providing with the tentative pattern of the $D = 4$ averages which may arise from the $AdS/CFT$ duality. Also, let us note the preliminary results (26) aimed at the justification of the latter correspondence directly from a variant of the Loop equation\(^{25}\) arising in the $N = 4$ SUSY $YM_4$ theory. One might expect that a deeper understanding of the interrelation between the two

\(^{54}\)For our present purposes, it is sufficient to require the unobservability within the short-distance expansion (3.3) constrained to be at least asymptotically convergent.

\(^{55}\)In particular, we conjecture that in the $WC$ phase the physical string tension $\sigma_{ph}(R)$ might exhibit a 'crossover', when $A_{\min}(C)\Lambda^2$ is varied, similar to the one interpolating between the two patterns (6.1) and (5.7).

\(^{50}\)For completeness, we remark that the formalism of the loop equations has been recently applied, see [34], in the context of the noncommutative gauge theories.
approaches could pay back to both of them.

Acknowledgements.

The author is grateful to Yu. Makeenko for a discussion at the final stage of the work. This project is partially supported by CRDF grant RP1-2108.

A: The short-distance expansion of the weight (1.5).

The required operator expansion (3.6) of the (exponent of the worldsheet’s weight (1.5)) is to be performed similarly to the more conventional multipole (or derivative) expansions. First, we introduce the convenient parametrization in terms of

\[
\gamma^- = (\gamma - \gamma') \quad , \quad \gamma^+ = y\gamma + (1 - y)\gamma' \quad ; \quad d^2\gamma d^2\gamma' = d^2\gamma_+ d^2\gamma_- ,
\]

where \(\gamma, \gamma'\) parametrize respectively the worldsheet’s coordinates \(x(\gamma)\) and \(y(\gamma') \equiv x(\gamma')\) (entering eq. (1.5)), while the auxiliary parameter \(0 \leq y \leq 1\) reflects the ambiguity in the choice of the ‘center of mass’ coordinate \(\gamma_+\). Then, for any given \(\gamma_+\), one is to arrange for the Taylor expansion (of all the structures involved into the bilocal exponent of eq. (1.5)) in terms of the relative coordinate \(\gamma^-\). Finally, integrating out \(\gamma^-\), the resulting ‘effective action’ is formulated as the expansion (3.6) in terms of the integrals of the local ‘scalar’ operators \(Q^{(k)}_{2\mu}(\gamma_+)\) which are composed of the \(\gamma_+\)-dependent tensors of the worldsheet’s curvature (and torsion).

Before we briefly discuss the general pattern (3.6), let us first apply this algorithm to derive the leading and the relevant next-to-leading terms of the considered expansion. Concerning the leading term (yielding the local limit (3.2) of (1.5)) given by eq. (3.2), it is deduced as follows. Keeping the leading \((\gamma^-)^0\)-contribution in

\[
p_{\mu\nu}(\gamma) = p_{\mu\nu}(\gamma_+) + (1 - y) \cdot (\partial_\alpha p_{\mu\nu}(\gamma_+))\gamma_+^\alpha + \frac{(1 - y)^2}{2} (\partial_\alpha \partial_\beta p_{\mu\nu}(\gamma_+))\gamma_+^\alpha \gamma_+^\beta + .. ,
\]

\[
p_{\mu\nu}(\gamma') = p_{\mu\nu}(\gamma_+) - y \cdot (\partial_\alpha p_{\mu\nu}(\gamma_+))\gamma_-^\alpha + \frac{y^2}{2} (\partial_\alpha \partial_\beta p_{\mu\nu}(\gamma_+))\gamma_-^\alpha \gamma_-^\beta + .. ,
\]

(where \(p_{\mu\nu}(\gamma)\) is defined by eq. (2.14)), in the argument of the smearing function

\[
(x(\gamma) - x(\gamma'))^2 = \left(\partial_\alpha x(\gamma_+)\gamma_+^\alpha + \frac{(1 - 2y)}{2} \partial_\alpha \partial_\beta x(\gamma_+)\gamma_+^\alpha \gamma_+^\beta + O((\gamma_-)^3)\right)^2
\]

one is to neglect all the \(O((\gamma_-)^3)\)-terms retaining the quadratic form

\[
\mathcal{G}(\Lambda^2(x(\gamma) - x(\gamma'))^2) \to \mathcal{G}(\Lambda^2 \hat{g}_{\alpha\beta}(\gamma_+)\gamma_+^\alpha \gamma_+^\beta)
\]

which depends on the induced worldsheet’s metric

\[
\hat{g}_{\alpha\beta}(\gamma) = \frac{\partial x_\mu}{\partial \gamma_\alpha} \frac{\partial x_\mu}{\partial \gamma_\beta} .
\]

In consequence, one recovers the pattern (3.2), with the bare string tension (3.4) being given by the product of \(\Lambda^2/2\) times the (curved space representation of the) zeroth moment (3.7)

\[
K_0 = \int_\mathcal{P} d^2z \sqrt{\hat{g}(\gamma_+)} \mathcal{G}(\hat{g}_{\alpha\beta}(\gamma_+)z^\alpha z^\beta) = \xi ,
\]
where \( \hat{g}(\gamma_+) \) is the determinant of the worldsheet metric \((A.6)\), and \( \bar{\mathcal{P}} \) is an arbitrary manifold diffeomorphic to the \( 2d \) plane. In compliance with the l.h. side of eq. \((3.1)\), the integral \((A.7)\) (being, by construction, independent of the metric \( \hat{g}_{\alpha\beta} \)) is constrained to be equal to \( \xi \).

Turning to the next-to-leading terms\(^{56}\) of the operator expansion, from the dimensional power-counting, they are given by a combination \([28, 29]\)

\[
\frac{\lambda}{2} \left( H^{(1)}_2 \int d^2 \gamma_+ \sqrt{\hat{g}(\gamma_+)} K^\alpha_{\beta(\gamma_+)} + H^{(2)}_2 \int d^2 \gamma_+ \sqrt{\hat{g}(\gamma_+)} R(\gamma_+) \right), \tag{A.8}
\]

where \( R(\gamma_+) \) is the intrinsic scalar curvature (with respect to the metric \( \hat{g}_{\alpha\beta}(\gamma_+) \)), while \( K^\alpha_{\beta(\gamma_+)} \) is the second fundamental form introduced via the standard geometrical relation

\[
\partial_\alpha \partial_\beta \mathbf{x} = \Gamma^\sigma_{\alpha\beta} \partial_\sigma \mathbf{x} + K^i_{\alpha\beta} \mathbf{n}_i ; \quad (\mathbf{n}_i, \mathbf{n}_j) = \delta_{ij} \quad , \quad (\mathbf{n}_i, \partial_\alpha \mathbf{x}) = 0. \tag{A.9}
\]

In eq. \((A.3)\), \( \Gamma^\sigma_{\alpha\beta}(\gamma) \) is the usual Christoffel symbol associated to \( \hat{g}_{\alpha\beta}(\gamma) \), and \( \mathbf{n}_i \equiv \mathbf{n}_i(\gamma), \quad i = 1, ..., D - 2 \), are the vectors normal (at the point \( \gamma \)) to the worldsheet parametrized by eq. \((A.4)\).

As the second term in eq. \((A.8)\) is the full derivative yielding vanishing contribution in the case of the closed string, we concentrate on the determination of the coefficient \( H^{(1)}_2 \) in front of the remaining extrinsic curvature term. There are the two sources of the contribution in question. First, keeping only the leading low-energy limit \((A.5)\) of the smearing function, one is to pick up those terms of the expansions \((A.2), (A.3)\) which result in the contribution quadratic in \( \gamma_- \).

Employing eq. \((A.9)\) and the identity

\[
\int_{\mathcal{P}} d^2 \hat{z} \sqrt{\hat{g}} \, \hat{g}(\hat{g}_{\alpha\beta} \hat{z}^\alpha \hat{z}^\beta) \hat{z}^a \hat{z}^b = \frac{K_2}{2} \hat{g}^{ab}, \tag{A.10}
\]

one concludes that the corresponding part of \( H^{(1)}_2 \) is given by \(-K_2/8\), where \( K_2 \) is defined in eq. \((B.7)\). (In the process of the derivation, we have traded the pair of cartesian antisymmetric tensors \( \epsilon^{ab} \) (with \( \epsilon^{12} = 1 \)) for their covariant counterparts: \( E^{ab} = \epsilon^{ab}/\sqrt{\hat{g}} \), so that \( E^{ab} E^{cd} g_{ac} = \hat{g}_{bd} \).)

As for the second type of the contribution, retaining the leading \((\gamma_-)^0\)-terms in eqs. \((A.2)\) and \((A.3)\), from the expansion of the smearing function one is to pick the term

\[
\mathcal{G}'(\Lambda^2 \hat{g}_{\alpha\beta}(\gamma_+)(\gamma_-)^{\alpha \beta}) \cdot \left( \frac{1 - 2y}{2} \partial_\alpha \partial_\beta \mathbf{x}(\gamma_+)(\gamma_-)^{\alpha \beta} \right)^2, \tag{A.11}
\]

and employ the identity (where \( \mathcal{G}'(t^2) \equiv \partial \mathcal{G}(t^2)/\partial t^2 \))

\[
\int_{\mathcal{P}} d^2 \hat{z} \sqrt{\hat{g}} \, \mathcal{G}'(\hat{g}_{\alpha\beta} \hat{z}^\alpha \hat{z}^\beta) \hat{z}^a \hat{z}^b \hat{z}^c \hat{z}^d = -\frac{K_2}{4} (\hat{g}^{ab} \hat{g}^{cd} + \hat{g}^{ac} \hat{g}^{bd} + \hat{g}^{ad} \hat{g}^{bc}) \tag{A.12}
\]

together with the standard relation \((K^\alpha_{\beta})^2 = R + K^\alpha_{\gamma\delta} K^\gamma_{\alpha \delta} \) \cite{256}. In sum, collecting both of the contributions, one obtains

\[
\frac{\lambda}{2} \frac{1}{2} H^{(1)}_2 = -\lambda \frac{K_2}{16} \left( 1 + 3(1 - 2y)^2 \right) < 0, \tag{A.13}
\]

where, in the \( D = 4 \) case, \( K_2 \) is fixed as in eq. \((3.10)\). In turn, it implies that, given the regularization prescription \((3.10)\), the positivity of the expansion \((3.0)\) in \( D = 4 \) requires that the factor \( \xi \) should be larger than certain critical value \( \xi_{cr} \).

Next, the above computations demonstrate that the generic area-functional \((2.15)\) can be expanded according to the pattern of eq. \((3.6)\). In particular, for macroscopic contours, the

\[\text{Similiar in spirit computations (employing somewhat different techniques) can found e.g. in [26, 27].}\]
domain of the integration in the relevant $n = 2p$ integrals (3.7) (defining the even moments $K_{2p}$, $p \geq 0$) can be extended up to the infinite $2d$ plane for the points $\gamma_+$ in the interior of the worldsheet $\tilde{M}$ (i.e. sufficiently far, in the units of $\Lambda^{-1}$, from the boundary of $\tilde{M}$). Note that, in contradistinction to the previous approaches [36, 37], the moments $K_{2p}$ are originally represented by the expressions like eq. (A.1) or (A.12) which are manifestly $2d$ generally-covariant. By construction, the latter integrals do not depend on the choice of the metric $\tilde{g}_{\alpha\beta}$ that, in eq. (3.7), is restricted to be flat: $\tilde{g}_{\alpha\beta} \to \delta_{\alpha\beta}$.

### A.1 The leading boundary term.

Finally, the part of the short-distance expansion of eq. (1.3), associated to the points $x(\gamma)$ in the vicinity of the boundary of the worldsheet $\tilde{M}$, deserves a special treatment. Here, the domain of the integration in the moments $K_n$ can be extended at best up to a $2d$ half-plane rather than to the full $2d$ plane. In the leading nontrivial order (of the expansion), this discrepancy results in the boundary contribution (1.12) that is reduced to the second term of the exponent of the Nambu-Goto weight (1.6) when the boundary contours are devoid of the $1d$ selfintersections. To obtain the expression (see eq. (A.13) below) for the effective mass $m_0(\lambda)$ entering eq. (1.12), the simplest option is to consider $M(C)$ in the form of a flat rectangle of the size $2l_1 \times 2l_2$. Then, the exponent of the weight (1.5) can be easily evaluated in the form

$$
\lambda \frac{1}{2} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{dp_1 dp_2}{(2\pi)^2} \tilde{G}_2(p_1^2 + p_2^2) \left(\frac{2\sin(p_1 \bar{l}_1)}{p_1}\right)^2 \left(\frac{2\sin(p_2 \bar{l}_2)}{p_2}\right)^2,
$$

(A.14)

where $\bar{l}_1 = l_1 \Lambda$, $\bar{l}_2 = l_2 \Lambda$, and $\tilde{G}_2(p_\alpha p^\alpha)$ is the two-dimensional Fourier image (on a $2d$ plane) of $G(z_\alpha z^\alpha)$ which is presumed to be a meromorphic function of $t^2 = p_\alpha p^\alpha$. Assuming that $\tilde{G}_2(t^2) = O(1/t^4)$ when $|t|^2 \to \infty$, the straightforward computation yields

$$m_0(\lambda) = -\lambda \Lambda \oint_{\Gamma_+} \frac{dt}{2\pi} \frac{\tilde{G}_2(t^2)}{(t + i\epsilon)^2},$$

(A.15)

where $\epsilon \to 0^+$, and the contour integral runs along the path $\Gamma_+$ which encircles the upper semiplane of the complex-valued variable $t$.

In particular, in the local limit (3.8), we obtain $\tilde{G}(0) = \xi$, while $\partial^n \tilde{G}_2(t^2)/\partial t^{2q} \to 0$ for $q \geq 1$ so that one indeed recovers: $m_0(\lambda) \to 0$. Let us also note that, being combined with the natural positivity constraint $\tilde{G}_2(t^2) > 0$ (for $\forall t^2 \geq 0$), eq. (A.13) implies that $m_0(\lambda) < 0$.

### B: The zero modes of $\hat{L}_\nu$ and the KR representation.

Before we analyse the zero-mode equation (2.8), let us first recall a few relevant details concerning the loop calculus. Recall first the meaning of the area- and the path-derivatives composed into the Loop operator (2.9). As for the area-derivative $\delta/\delta \sigma_{\alpha\nu}(x(s))$, it is associated to the variation of the base-loop $C \equiv C_{xx} \to C_{xx} \partial C_{xx}$ with respect to creation of the infinitesimal auxiliary

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57 For macroscopic loops, the next-to-leading term (4.12) is negligible compared to the leading area-term in the exponent of eq. (1.2).

58 In the process of the derivation, we have consistently neglected the exponentially suppressed (for $l_1, l_2 >> \Lambda^{-1}$) terms $\sim e^{-\mu l_1}, e^{-\mu l_2}$, where $\pm i\mu$ (with $-\mu^2 = \mu^2 \sim \Lambda^2$) are the positions of the two nearest, to the real axis, poles of $\tilde{G}_2(t^2)$. 

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loop $\tilde{C}_{xx}$ attached to $C_{xx}$ at the point $x(s)$. By definition, the loop $\tilde{C}_{xx}$, being projected onto the $\mu\nu$-plane, spans the surface-element of the (minimal) area $\Omega_{C_{xx}}$. As for the path-derivative $\frac{\partial f(x)}{\partial x_\mu(s)}$, it is associated to the variation $\delta f(x)$ of the functional (resulting after the application of $\delta_\mu W_C(x)$ to $W_C >$) with respect to the infinitesimal deformation $C_{xx} \to C_{xx} x^{-1}$ of the contour $C \equiv C_{xx}$. Here, the deformed loop $\Gamma_{xx} x^{-1}$ is obtained from $C_{xx}$ creating, at $x(s)$, a backtracking appendix to $z$ along the path $\Gamma_{xx}$ so that $z_\mu - x_\mu(s) \equiv \delta x_\mu(s) \to 0$ for $\forall \mu = 1, ..., D$.

In sum, on the $\Psi$-space (2.13), one easily deduces (employing, in particular, eq. (2.13)) that the action of the Loop operator $\mathcal{L}_\nu(x(s))$ considerably simplifies and can be represented by the corresponding differential operator

$$\mathcal{L}_\nu(x(s)) \ln \left( w_2[\tilde{M}(C)] \right) = \frac{\partial}{\partial x_\mu(s)} \frac{\delta f}{\delta p_\mu(x(s))} \ln \left( w_2[\tilde{M}(C)] \right).$$

where $p_\mu(x)$ is the area-element $\Omega_{C_{xx}}$, and the prescription (E.3) (introduced in Appendix E to implement the backtracking invariance of $< W_C >$) is implied.

Given eq. (B.1), the zero-mode equation (2.8) can be reformulated on the $\Psi$-space as the set of the linear equations yielding the corresponding constraints separately for each $n$th order tensor $S_{(\mu_1, ... , \mu_n)}^{(n)}(\ldots)$ entering $-\ln(w_2^{(0)}[\ldots])$. In the $n = 2$ case, one thus obtains the condition stated (after eq. (2.19)) in Section 2.3. To properly reinterpret the full set of the constraints, it is helpful to view (in the spirit of [11]) the set of the tensor-functions $S_{(\mu_1, ... , \mu_n)}^{(n)}(\ldots)$ as the connected correlators,

$$S_{(\mu_1, ... , \mu_n)}^{(n)}(\{ x_i - x_j \}) = \langle \langle B_{\mu_1, \nu_1}(x_1)B_{\mu_2, \nu_2}(x_2)\ldots B_{\mu_n, \nu_n}(x_n) \rangle \rangle,$$

in some theory of the tensor Kalb-Ramond (KR) field $B_{\mu\nu}(x)$ defined in the base-space $\mathbb{R}^D$. Then, taking into account eq. (B.1), the zero-mode equation (2.8) implies that

$$\partial_\mu B_{\mu\nu}(x) = 0.$$

To visualize this constraint, let us take for simplicity the $D = 4$ case when the Kalb-Ramond tensor can be canonically decomposed into the so-called exact and co-exact 2-forms $F_{\mu\nu}$ and $H_{\mu\nu}$,

$$B_{\mu\nu} = F_{\mu\nu} + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} ; \quad F_{\mu\nu}(x) = \partial_\mu \wedge A_\nu(x) , \quad H_{\mu\nu}(x) = \partial_\mu \wedge C_\nu(x),$$

which are conventionally expressed in terms of the normal and dual gauge potentials $A_\nu(x)$ and $C_\nu(x)$ respectively so that the ordinary $U(1)$ gauge symmetry is enlarged to the product $U(1) \otimes U_m(1)$ of the 'electric' and 'magnetic' factors. Then, the condition (B.3) (being $C_\rho$-independent) is transformed into the constraint that $\partial_\mu \wedge A_\nu(x)$ vanishes for $\forall \mu, \nu$, i.e. $A_\nu(x) = \partial_\mu f(x)$ with any $f(x)$. Finally, to make contact with [11], we note the following. On the $\Psi$-space of the area-functional (2.13), the general $D = 4$ solution of eq. (2.7) can be reproduced (for a given Euler character $\chi$ of $M_\chi(C)$) in the abelian $\text{KR}$ theory with the action

$$\int_{\mathbb{R}^4} d^4 x \left( \frac{1}{4g^2} B_{\mu\nu}^2(x) + W(\partial_\mu \ast B_{\mu\nu}(x)) \right) + \frac{i}{2} \int_{M_\chi} d\sigma_{\mu\nu}(y) B_{\mu\nu}(y),$$

This requirement ensures the appropriate cluster decomposition (akin to the one of [3]) that is necessary to maintain the selfconsistency of the gauge theory represented through the set of the Wilson loop-averages. This condition can be shown to exclude a variety of the spurious solutions of the zero-mode equation (2.8).
where \( d\sigma_{\mu\nu}(x) \) is given by eq. (2.14), while \( W(\ldots) \) is an arbitrary scalar function (of the combination \( \partial_{\mu} B_{\mu\nu} \)) presumed to be consistent with the existence of the corresponding functional integral over \( B_{\mu\nu}(x) \). In particular, the case of \( W(\ldots) = 0 \) reproduces the \((N = 1\) option of the) weight \((1.3)\).

Next, to make contact between eq. (B.5) and the action (2.21) of the abelian gauge theory enriched with the monopoles, let us consider the \( SC \) series representing the partition function \( Z_{U(1)} \) of the compactified \( U(1) \) gauge theory associated to eq. (B.5). To begin with, the logarithm of \( Z_{U(1)} \) is proportional to the partition function of the \( closed \) string. Therefore, in eq. (B.3) one is to sum over all \( \bar{\vartheta} \)-immersions \((3.1)\) into \( \mathbb{R}^4 \) (spanning the space \( \mathcal{I}(M, \mathbb{R}^4) \)), where \( M = \bar{\vartheta}(M) \equiv \bar{\vartheta} \) is not necessarily connected worldsheet (of an arbitrary Euler character \((1.3)\)) without boundaries. Then, the short-cut way between eqs. (B.5) and (2.21) is provided\(^{60}\) by the formal resummation prescription (reminiscent of the one in \([11]\))

\[
\int_{\mathcal{I}(M, \mathbb{R}^4)} D\bar{\vartheta} e^{-\frac{1}{\beta} \int d\sigma_{\mu\nu}(y) B_{\mu\nu}(y)} \leftrightarrow \int Dk_{\mu}(x) D\tilde{C}_{\mu}(x) e^{-i \int d^2x \tilde{C}_{\nu}(x)(\partial_{\mu} B_{\mu\nu}(x) - k_{\nu}(x))},
\]

(B.6)

to be substantiated in the end of this Appendix. In eq. (B.6), the functional integral over the monopole’s current \( k_{\mu}(x) \) symbolizes the averaging over the canonical multiloop ensembles akin to the one in eq. (2.21). As for the auxiliary dual potential \( \tilde{C}_{\nu}(x) \), it plays the role of the Lagrange multiplier that enforces the Bianchi identities modified in the presence of the magnetic charges quantized according to the Dirac rule.

Upon a reflection, the resummation-prescription (B.6) implies that the \( B_{\mu\nu} \)-pattern of the \( g^2 \)-dependent part of the lagrangian (2.21) is to be identified, when one reintroductes the \( \delta_D(x - y) \)-kernel removing the smearing (2.5), with the sum of the two terms (in the large round brackets) of the lagrangian (B.5). Therefore, it requires to put \( W(\ldots) = 0 \) in the latter equation. In turn, the fact, that the system (2.21) supports the existence of the flux-tubes of a \( finite \) width, implies (in compliance with the well-known dual superconductivity hypothesis) that the monopoles are in certain sense condensed. More precisely, the properly introduced dual gauge field \( C_{\nu}(x) \) is associated to the \( massive \) rather than massless degrees of freedom.

Finally, the somewhat heuristic derivation of the prescription (B.6) (where both sides have to be coherently regularized at some \( UV \) scale \( \Lambda \)) is based on the following observation. In the spirit of the arguments of \([1]\), the formal summation in the l.h. side of eq. (B.6) singles out only those configurations of the tensor Kalb-Ramond field \( B_{\mu\nu}(y) \) for which the exponent (in the l.h side) does \( not \) depend on the choice of the worldsheet \( M = \bar{\vartheta}(M) \) at all. The latter configurations are precisely those that comply with the modified Bianchi identities implemented by the r.h. side of eq. (B.6). Actually, the considered resummation can be understood as an analogue of the more familiar Borel resummation. While the l.h. side of eq. (B.6) yields the representation pertinent for the \( SC \) phase (i.e. for sufficiently large value of \( \lambda = g^2 \)), the r.h. side of this equation provides with pattern adequate in the \( WC \) regime for sufficiently small \( \lambda \).

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\[^{60}\]To interprete this ambiguity, one is to observe that, in any \( U_e(1) \otimes U_m(1) \) gauge theory with the monopoles, the full set of the gauge-invariant observables is \( larger \) than the set of the Wilson loop averages \((1.3)\). Additionally, we should fix the averages including the dual Wilson loops that measure the interaction between the monopoles. The latter loops are to be deduced from (the \( N = 1 \) option of) eq. (2.1) trading \( A_\rho(x) \) for its dual counterpart \( C_\rho(x) \) introduced through the resummation (B.6); in the r.h. side of eq. (B.6) one is to resolve the Bianchi identities via the decomposition (B.4), i.e. \( \partial_{\mu}(\partial_{\nu} C_{\mu\nu}(x)) = k_{\nu}(x) \).

\[^{61}\]As \( *B_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} B_{\rho\sigma}/2 \), the vector \( \partial_{\mu} B_{\mu\nu} \) is expressed in terms of the dual potential \( C_{\rho} \) only.

\[^{62}\]In the simplest setting, eq. (B.6) is to be applied to the local limit (1.8) of \( Z_{U(1)} \) (when the action \( W(\ldots) \) can be \( locally \) expressed in terms of the co-exact \( H_{\mu\nu} \)-form (B.4) as long as the gauge \( \partial_{\mu} C_{\mu}(x) = 0 \) is fixed) that allows to reproduce the limit (1.1)/(1.3) of the zero flux-tube’s width of the stringy representation (1.1)/(1.3).
C: The reduction to take advantage of.

The generic pattern of the type (1.1) is defined by fixing the configuration space $\mathcal{X}$ of the mappings (1.2) together with the weight $w[M]$, $M = \vartheta(M)$. The latter, being presumed finite for any map $\vartheta \in \mathcal{X}$, is further constrained to be local on the worldsheet $\tilde{M}$. Let the two $D \geq 3$ stringy representations (1.1) be determined respectively by $\{\mathcal{X}_a, w_a[M]\}$ and a presumably simpler pair $\{\mathcal{X}_b, w_b[M]\}$ so that $\mathcal{X}_b$ is dense in $\mathcal{X}_a$ (while $w_b[M]$ can be formally continued onto the entire space $\mathcal{X}_a$). By construction of the functional measure, the equivalence of the latter two representations (1.1) takes place if the two weight-patterns are distinct only on a measure zero subspace of $\mathcal{X}_a$ (where the relative complexity of $w_a[M]$ is supposed to show up.)

To present the rigorous formalization of this equivalence, assume first that there exists a subspace $\mathcal{X}_c \in \mathcal{X}_b$ which is dense in $\mathcal{X}_b$ (and, as a result, in $\mathcal{X}_a$), i.e.

$$V[\mathcal{X}_c] / V[\mathcal{X}_a] = V[\mathcal{X}_c] / V[\mathcal{X}_b] = 1 ,$$

(C.1)

where $V[\mathcal{X}]$ is the properly normalized volume of the space $\mathcal{X}$. In addition, we postulate that the patterns of the weights $w_a[M]$ and $w_b[M]$ coincide if $\tilde{M} = \vartheta(M)$ does belong to $\mathcal{X}_c$:

$$\vartheta(M) \in \mathcal{X}_c \implies w_a[\vartheta(M)] = w_b[\vartheta(M)] .$$

(C.2)

Then, the theorem is that the two functional integrals, fixed respectively by the pairs $\{\mathcal{X}_a, w_a[\tilde{M}]\}$ and $\{\mathcal{X}_b, w_b[\tilde{M}]\}$, coincide

$$\int_{\mathcal{X}_a} d\vartheta w_a[\vartheta(M)] = \int_{\mathcal{X}_b} d\vartheta w_b[\vartheta(M)] .$$

(C.3)

When $\partial\vartheta = \bigcup_k C_k$, the above sums provide with the two equivalent representations of the loop-averages (1.1). In the case of the worldsheets without boundaries (i.e. when $\partial\vartheta = 0$), the identity (C.3) equates the two partition functions associated to the corresponding averages (1.1).

C.1 Application to the Ansatz (4.1)/(3.2).

Let us apply the developed technology in order to prove that the $D \geq 3$ representation (4.1) is invariant under the substitution of the weight (3.2) by the $m_0 = 0$ option of the Nambu-Goto pattern (1.6). For this purpose, one is to identify $w_a[M]$ and $w_b[M]$ respectively with eq. (3.2) and the $m_0 = 0$ reduction of eq. (1.6), while

$$\mathcal{X}_a = \mathcal{X}_b = \mathcal{I}(M, \mathbb{R}^D), \quad \mathcal{X}_c = \mathcal{I}_1(M, \mathbb{R}^D) ,$$

(C.4)

where $\mathcal{I}(M, \mathbb{R}^D)$ stands for the space of the smooth immersions (4.2) (of a $2d$ manifold $M$ into $\mathbb{R}^D$) entering the $D \geq 3$ functional integral (4.1). As for $\mathcal{I}_d(M, \mathbb{R}^D)$, it denotes the subspace of $\mathcal{I}(M, \mathbb{R}^D)$ where the corresponding worldsheet $\tilde{M} = \vartheta(M)$, in its interior, is either a nonselfintersecting $2d$ manifold or a smooth surface with selfintersections on submanifolds of dimension less or equal to $d = 0, 1, 2$. As a result, according to eq. (3.3), on $\mathcal{I}_1(M, \mathbb{R}^D)$ the $m_0 = 0$ option of the pattern (1.6) coincides with the one of eq. (3.2).

Finally, in order to apply eq. (C.3), as in (1.1) one is to make use of the basic theorem [18] from the theory of immersions. For $d \geq 4 - D$, the subspace $\mathcal{I}_d(M, \mathbb{R}^D)$ is dense in $\mathcal{I}(M, \mathbb{R}^D)$:

$$V[\mathcal{I}_d(M, \mathbb{R}^D)] / V[\mathcal{I}(M, \mathbb{R}^D)] = 1 \quad if \quad d \geq 4 - D .$$

(C.5)

For example, the pattern (3.2) vs. the $m_0 = 0$ option of eq. (1.6); or eq. (1.3) vs. the same $m_0 = 0$ option of eq. (1.6).
and, in particular, the subspace $\mathcal{I}_1(M, \mathbb{R}^D)$ is indeed dense in $\mathcal{I}(M, \mathbb{R}^D)$ when $D \geq 3$. It completes the proof.

C.2 Application to the Ansatz (4.7).

The same general formula (C.3) implies the equivalence of the idealized Gauge String representation (4.7) to the $m_0 = 0$ option of the Nambu-Goto Ansatz (1.1)/(1.6). To apply eq. (C.3), recall first that any differentiable immersion is given [18] by a mapping (4.2) which is constrained to be locally one-to-one and in both sides $k \geq 1$ times differentiable. The $YM_D$/String duality [7] prescribes that $\Delta(M, \mathbb{R}^D)$ presents a specific extension of the space $\mathcal{I}(M, \mathbb{R}^D)$ of the smooth (i.e., infinitely differentiable) immersions. We are to take advantage that the space $\mathcal{I}(M, \mathbb{R}^D)$ is dense in the option of $\Delta(M, \mathbb{R}^D)$ associated to the $D \geq 3$ $U(N)$ gauge theory (1.1):

$$ U(N) : \quad \frac{V[\mathcal{I}(M, \mathbb{R}^D)]}{V[\Delta(M, \mathbb{R}^D)]} = 1 \quad \text{if} \quad D \geq 3 , \quad (C.6) $$

provided the boundary contours are either devoid of backtracking pieces or absent at all. (The $SU(N)$ counterpart of eq. (C.6) is obtained trading $\Delta(M, \mathbb{R}^D)$ for its reduction $\tilde{\Delta}(M, \mathbb{R}^D)$ which is to be induced, see [7], eliminating certain type of the singularities via the redefinition (8.8) of the bare string tension.) In the absence of the boundaries, the proof of the $D \geq 4$ variant of eq. (C.6) is given in [7]. Its extension to the case, when the boundary loops are without backtrackings, is straightforward. As for the $D = 3$ case, it takes a little more effort (to be reported elsewhere) to adapt the arguments of [7] in order to justify the required identity.

Next, in consequence of eq. (C.6), one is to identify $w_a[\bar{M}]$ and $w_b[\bar{M}]$ with respectively eq. (1.3) and the $m_0 = 0$ reduction of eq. (1.5) so that

$$ \Delta(M, \mathbb{R}^D) = \mathcal{X}_a , \quad \mathcal{I}(M, \mathbb{R}^D) = \mathcal{X}_b , \quad \mathcal{I}_1(M, \mathbb{R}^D) = \mathcal{X}_c , \quad (C.7) $$

where the subspace $\mathcal{I}_1(M, \mathbb{R}^D)$ of $\mathcal{I}(M, \mathbb{R}^D)$ is specified after eq. (C.4). To support the relevance of the latter identification, we first note that the selfintersection factor $J[.]$ (entering eq. (1.3)) greatly simplifies on $\mathcal{I}(M, \mathbb{R}^D)$ as it is formalized by eq. (1.8). Then, the key-observation is that, by construction of $|C_0|$, the dense (see eq. (C.3)) $D \geq 3$ subspace $\mathcal{I}_1(M, \mathbb{R}^D)$ of $\mathcal{I}(M, \mathbb{R}^D)$ is comprised of the worldsheets assigned with the $J[.] = 1$ factor:

$$ |C_0| = 1 \quad \text{if} \quad \vartheta(M) \in \mathcal{I}_1(M, \mathbb{R}^D) , \quad (C.8) $$

so that the weight (1.3) is reduced to the $m_0 = 0$ option of the weight (1.6) which, in turn, allows to apply the basic equivalence-identity (C.3). In sum, combining all the pieces together, we conclude that, in $D \geq 3$, the $m_0 = 0$ variant of the Nambu-Goto Ansatz (1.1)/(1.6) and the idealized Gauge String pattern (4.7) are indeed equivalent for nonbacktracking boundary contours. By the same token, the equivalence is maintained between the closed string’s sector of the latter Ansatz and the partition function of the idealized closed Gauge String (corresponding to the analytical continuation of the free energy of the $U(N)$ gauge theory (1.1)).

D: Justifying the ’t Hooft factor.

To justify the ’t Hooft factor $N^\chi$ in eq. (1.5), our strategy is as follows. First, we demonstrate that the $N = 1$ reduction of the Ansatz (1.1)/(1.5) is consistent with the entire Dyson-Schwinger

\footnote{The extension is performed allowing for certain singularities which correspond to the violation of the latter constraints on the mapping (4.2). Strictly speaking, the detailed pattern of $\Delta(M, \mathbb{R}^D)$ (and of the $J[.]$-factor in eq. (1.3)) does depend on the choice of the gauge group and the $YM_D$ lagrangian.}
chain of the regularized abelian loop equations. Then, certain topological relations are revealed which play the crucial role in the proof of the following matching. When the considered Ansatz is restricted to the (multiloop generalization of the) regime $[L.4]/[L.7]$, the $N^\infty$-factor allows to match (within each order of the $1/N$ expansion) the overall degree of $1/N$ in both sides of any given $U(N)$ loop equation $[D.1]$. Combining the two previous steps, we then sketch the proof that, under the above conditions, the Ansatz $[L.1]/[L.3]$ indeed ‘goes through’ the chain of the regularized $U(N)$ loop equations.

Before we proceed with the implementation of this program, let us briefly recall that a generic $n$th order loop equation in the $U(N)$ chain reads

$$\tilde{g}^{-2} \cdot \hat{\mathcal{L}}_\nu (x(s_q)) < W_{C^{(1)}} W_{C^{(2)}} ... W_{C^{(q)}} ... W_{C^{(n)}} >=$$

$$= \oint_{C^{(q)}} dy'_q (s'_q) \delta_D (y(s'_q) - x(s_q)) < W_{C^{(1)}} W_{C^{(2)}} ... W_{C^{(q)}} W_{C^{(q)}_{xy}} ... W_{C^{(n)}} >=$$

$$+ \sum_{k \neq q}^{n} \frac{1}{N^2} \oint_{C^{(k)}} dy_k (s'_k) \delta_D (y(s'_k) - x(s_q)) < W_{C^{(1)}} ... W_{C^{(q)}_{xy}} ... W_{C^{(k-1)}_{xy}} W_{C^{(k+1)}_{xy}} ... W_{C^{(n)}} >=$$

relating $n$-loop average with the $(n+1)$- and $(n-1)$-loop ones. Here, for any given $q = 1, ..., n$, the Loop operator $[L.1]$ is applied to the $q$th contour $C^{(q)} = C^{(q)}_{xx}$ which is decomposable (see the second line of eq. $[D.1]$) as $C^{(q)}_{xx} = C^{(q)}_{yy}$ in the same way as in eq. $[L.2]$. Complementary, in the last line, the $k$th contour $C^{(k)}$ (with $k \neq q$) is combined, if the $\delta_D (y - x)$-function does allow, with the contour $C^{(q)}$ gluing the two loops at the point $x$ into the single loop $C^{(q)}_{xx} C^{(k)}_{xx}$.

### D.1 The full $U(1)$ solution.

To handle the abelian variant of eqs. $[D.1]$, consider first the simplest case when all the involved contours, being nonselfintersecting, do not mutually intersect. For such loops, one easily observes that the $N = 1$ option of the Ansatz $[L.1]/[L.3]$ goes through the entire $U(1)$ chain of the regularized eqs. $[D.1]$ for any $n \geq 1$. This can be proved by a straightforward generalization of the steps resulting in eq. $[L.16]$. In particular, on the considered subspace of the multiloop space, both the $U(1)$ and the $U(N)$ chain reduce to the set of the linear loop equations. The latter can be deduced trading in eq. $[L.16]$ the surface $\hat{M}(C)$ with a single connected boundary for the worldsheet $\hat{M}([C^{(k)}])$ with an arbitrary number of connected boundary components so that both of the functional derivatives act on any given loop $C^{(q)}$.

Finally, for a generic set of the contours, the analysis$[^65]$ can be reduced to the previous case due to the well-known property of the $U(1)$ Wilson loops. Namely, if any two contours $C^{(1)}, C^{(2)}$ mutually intersect at $x$ then $W_{C^{(1)}} W_{C^{(2)}} = W_{C^{(1)}_{xx} C^{(2)}_{xx}}$, where the new single contour $C^{(1)}_{xx} C^{(2)}_{xx}$ is composed as in the third line of eq. $[D.1]$.

### D.2 The coupling between the different worldsheet’s topologies.

Let us now turn to the topological relations that necessitate the ’t Hooft $N^\infty$-factor of the weight $[L.3]$. The guiding idea (underlying the construction of the $D \geq 3$ low-energy solution $[L.1]/[L.3]$)

[^65]: The generalization of the conditions $[L.4]/[L.7]$ has been formulated in the very end of Section 8.

[^66]: Being maintained without imposing the constraints $[L.7], [L.3]$ (or $[L.8]$), the formal consistency of the $N = 1$ Ansatz $[L.1]/[L.3]$ with the $U(1)$ eq. $[D.1]$ does not account for possible phase transitions. The latter are expected to invalidate the considered $SC$ expansion as the faithful representation of the corresponding $D \geq 3$ abelian gauge theory (with the monopoles), defined by the action $[P.21]$ in the $D = 4$ case, in the $WC$ phase $g^2 \to 0$. 

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is that, in any given order of the $1/N$-expansion, the correspondence between the worldsheet’s topologies (associated, see eq. (D.2), to each of the three lines of eq. (D.1)) is $N$-independent, being thus the same as in the abelian case. In this perspective, we note first that a part of the reason behind the simplicity of the reduced $U(N)$ loop equation (2.4) lies in the fact that the latter is diagonal with respect to the genus $(1−\chi)/2$ of each particular worldsheet $\tilde{M}_\chi(C)$ entering the measure of the $b = 1$ stringy sum (4.1). On the other hand, the decoupling is no more valid when we consider the contribution (into the r.h. side of eq. (D.1)) either from a nontrivial selfintersection of a given contour or from an intersection of any two different loops.

Then, as it is shown in the end of this subappendix, in each particular order of the $1/N$ expansion, the following matching between the corresponding total Euler characters (4.3) takes place irrespectively whether or not the proper regularization of the loop equations is performed. Namely, the variety of the (not necessarily connected) worldsheets with the total Euler character $\chi$ in the first line of eq. (D.1) must be associated, through the action of $\hat{\mathcal{L}}_\nu(x(s))$, to the corresponding variety of the worldsheets in the second line with the total Euler character $\bar{\chi}$,

$$\bar{\chi} - \chi = \bar{b} - b = 1 ; \quad \chi - \hat{\chi} = b - \hat{b} = 1 ,$$

which is one unit larger than $\chi$ so that the difference $\bar{\chi} - \chi$ is equal to the one between the numbers $\bar{b}$ and $b$ of the connected boundary components in the second and in the first lines respectively. By the same token, the Euler character $\hat{\chi}$, associated to the third line, is one unit less than the character $\chi$ associated to the first line that matches with the difference between the numbers $\hat{b}$ and $b$ of the boundaries respectively in the third and in the first lines of eq. (D.1).

Upon a reflection, the relations (D.2) imply certain restriction which is necessary to impose, for the entire $1/N$ expansion (4.1)/(1.3) to 'go through' the regularized loop equations (D.1). One is to deal only with those multiloop sets $\{ C^{(k)} \}$ for which the (regularized) third line of eq. (D.1) vanishes in any order of the $1/N$ series. To justify this statement, we are to combine the relations (D.2) with the representation of the averages $\langle \prod_k W_{C^{(k)}} \rangle$ (of the Wilson loops (2.1)) in the form of the $1/N$-expansion starting with the $N^0$-term. Altogether, it suggests to search for the $N$-dependence of the low-energy weight $w_2(\tilde{M}_\chi(\{ C^{(k)} \}))$ in the form of the Ansatz $[f(N)]^x$. Then, it is straightforward to verify that the chain of the loop-equations (D.1) unambiguously require to put $f(N) = N$. Indeed, in this case, the overall powers $N^{\bar{b}−b}$ and $N^{\chi−\hat{\chi}}$ in the first and the second lines of eq. (D.1) are precisely equal for the considered (in the context of the first of the relations (D.2)) paired varieties of the worldsheets.

Next, when the third line of the regularized eq. (D.1) can not be discarded, the second of the relations (D.2) implies that the the overall powers $N^{\chi−b}$ and $N^{\chi−\hat{b}}/N^2$ in the first and the third lines do not match. To clarify when this harmful contribution can be consistently discarded, observe first that prior to the regularization the considered third line identically vanishes once the involved contours $C^{(k)}$ do not mutually intersect. Therefore, after the regularization, one is to require that both the characteristic size of each (macroscopic) loop $C^{(k)}$ and the minimal distance between any two different loops are of order of $N^\alpha$, $\alpha > 0$. Indeed, due to the latter condition, all sorts of the quasi-local interactions (between the elementary flux-tubes) are suppress as $e^{-\beta N}$ with some $\beta > 0$. This is necessary to accomplish because, if not suppressed, the abelian

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67 The decoupling of different worldsheet’s topologies immediately follows from the possibility to trade eq. (2.4) for eq. (2.7) formulated directly for any particular single surface $M_\chi(C)$.

68 Both of the relations (D.2) can be reformulated as the conservation of the composed topological number $(p−h−b)$, where $p$, $h$, and $b$ are introduced in eq. (4.3).

69 In the $N \to \infty$ limit, it is sufficient to weaken the $N^\alpha$-scaling of all the distances to the $|\phi|$-scaling with some $N$-independent $|\phi| \to \infty$. In the case of a single loop $C$, it is tantamount to the requirement (formulated prior to eq. (5.5)) that the contour $C$ is macroscopic.
pattern of these interactions is in conflict with the $N \geq 2$ $U(N)$ loop equations (D.4).

Returning to the derivation of the relations (D.2), (from the previous abelian analysis) it is clear that the second of the relations is topologically equivalent to the first one which we now focus on. Furthermore, for simplicity, we concentrate on the $n = 1$ eq. (D.4) that (in the $U(N)$ case) can be deduced from the large $N$ eq. (2.2) through the substitution (2.3). The generalization of the analysis to the $n \geq 2$ case of (nonintersecting) loops is straightforward.

As for the $n = 1$ option, it is instructive to prove the corresponding relation (D.2) for the regularized variant of the associated loop equation (D.1) so that we first sketch how the large $N$ identity (8.3) can be adapted to the case at hand. For this purpose, take any particular connected worldsheet $\tilde{M}_\chi(C) \in \mathcal{I}(M_\chi, \mathbb{R}^D)$ (entering eq. (1.1)) of the Euler character $\chi$. Let a nontrivial selfintersection point (at $x(s) = y(s')$ with $s \neq s'$) of a macroscopic loop $C_{xx} = C_{xy}C_{yx}$ be resolved according to the preliminary smearing (2.3). To properly generalize eq. (8.3), as previously one is to consider the intersection of $\tilde{M}_\chi(C)$ with some $(D - 1)$-dimensional domain $X^{D-1}$ including the points $x(s)$ and $y(s')$. The considered cross-section generically allows to select the nonselfintersecting contour (8.4). Then, in the generalized variant of the identity (8.3) written for generic values of $\chi$, the surface $\tilde{M}_\chi(C_{xx})$ (in its l.h. side) with the single boundary $C_{xx}$ is associated to the worldsheets $\tilde{M}_\chi(C_{xy} \Gamma_{yx}; C_{yx} \Gamma_{xy})$ (in the r.h. side). The latter, being endowed with the two boundaries $C_{xy} \Gamma_{yx}$ and $C_{yx} \Gamma_{xy}$, has the same area $A[\tilde{M}_\chi(C_{xx})] = A[\tilde{M}_\chi(C_{xy} \Gamma_{yx}; C_{yx} \Gamma_{xy})]$ but possesses a different value of the Euler character $\tilde{\chi}$ (to be determined below). Moreover, for any macroscopic contour $C_{xx}$, the effective action $S_\chi(\Gamma_{xy}(G))$ (introduced akin to eq. (8.3)) is evidently independent of $\chi$ in the relevant limit (1.4) of the large minimal areas associated to the loops $C_{xy} \Gamma_{yx}$ and $C_{yx} \Gamma_{xy}$ in question.

To substantiate eq. (D.2), there remains to prove that the total Euler character $\tilde{\chi}$ of the worldsheets $\tilde{M}_\chi(C_{xy} \Gamma_{yx}; C_{yx} \Gamma_{xy})$ is one unit larger than the one $\chi$ of $\tilde{M}_\chi(C_{xx})$. This fact immediately follows from the observation that, when being discretized, the former (not necessarily connected) worldsheets can be obtained from the latter by adding four extra cells: the two sites (0-cell), the two links (1-cells) and a single plaquette (2-cell). According to the definition of $\chi$, a given $k$-cell contributes to the total Euler character as $(-1)^k$, $k = 0, 1, 2$. Altogether, it reproduces the difference (D.2) between $\tilde{\chi}$ and $\chi$. To have an example of how it works, take the option when the surface $\tilde{M}_\chi(C_{xy} \Gamma_{yx}; C_{yx} \Gamma_{xy})$ remains to be connected. It implies that the trajectory (8.4) is supposed to cut exactly one (equivalence class of the) uncontractible closed path of $\tilde{M}_\chi(C_{xx})$. In turn, it ensures that the genus $\tilde{h} = -\tilde{\chi}/2$ is one unit smaller than $h = (1 - \chi)/2$ which is in exact agreement with eq. (D.2).

Finally, synthesizing the previous arguments with the analysis of Section 8, one can show the following. Let the regularization prescription (8.6) be appropriately adapted (for nontrivial point-like selfintersection of the loops) to the higher genus cases. Then, provided the $\Lambda^2$-scaling (1.7) of $\sigma_{ph}$ is fulfilled, the Ansatz (1.1)/(1.5) is the genuine $1/N$ expansion of the $U(N)$ solution for the entire Dyson-Schwinger chain of eqs. (D.1) restricted to the subspace of the macroscopic contours satisfying the conditions formulated in the very end of Section 8.

### D.3 The derivation of the constraint (8.7).

In order to derive the constraint (8.7), let us recast the prescription (8.6) into the form matching with the gauge-invariant regularization discussed in [1, 33]. Recall, that the latter is associated

\footnote{To say the least, the above interactions would produce spurious violation of the large $N$ factorization of the multiloop averages $\langle \prod_k W_{\ell(x)} \rangle$.}
(see [33]) to the so-called second order Dyson-Schwinger equation

\[
\left\langle \int d^D x \left( \hat{D}_{\mu}^{ab}(A) F_{\mu\nu}^b(x) \right) \frac{\delta}{\delta A^a_\nu(x)} W_C[A] \right\rangle =
\]

\[
= g^2 \left\langle \int d^D x \ d^D y \ < y | \hat{E}^{ab}[A] | x > \frac{\delta}{\delta A^a_\nu(y)} \frac{\delta}{\delta A^b_\nu(x)} W_C[A] \right\rangle _{YM_D},
\] (D.3)

where \( W_C[A] \equiv W_C \) is the Wilson loop (2.1), and \( < y | \hat{E}^{ab}[A] | x > \) is the matrix element which is to regularize \( \delta^{ab} \delta_D(x - y) \). To make contact with eq. (8.6), we first decompose

\[
< y | \hat{E}^{ab}[A] | G \rangle \left| x > = \Lambda^D G(\Lambda^2(x - y)^2) \langle y | \hat{E}^{ab}[A] | G \rangle \left| x > ,
\] (D.4)

where the gauge field \( A^a_\nu \) is supposed to enter the operator \( \hat{E}^{ab}[A] \) through the covariant derivative \( \hat{D}^a_\mu(A) \) (in the adjoint representation), and we have introduced an explicit functional dependence on the smearing \( G \)-function (2.5). Secondly, one is to identify that the path-integral representation of the matrix element in the r.h. side of eq. (D.3) is given by

\[
< y | \hat{E}_1^{ab}[A] | G \rangle \left| x > = \int_{\Gamma_{xy} \in \mathbb{X}^{D-1}} Dz_\mu(t) \ e^{-S(\Gamma_{xy}[G])} \ tr \left( T^a B(\Gamma_{xy}) T^b B(\Gamma_{yx}) \right),
\] (D.5)

where \( B(\Gamma_{xy}) \) is the path-ordered exponent (8.2) (with \( T^a \equiv T^a_0 \) standing for the \( U(N) \) generator in the fundamental representation with \( tr(T^a T^b) = \delta^{ab} \) so that one recognizes the building block of eq. (8.3) in the r.h. side of eq. (D.3).

Finally, for a generic smearing function, the normalization of the r.h. side of eq. (D.4) does not allow to interpret \( \hat{E}^{ab}[0] \) as the smearing of \( \hat{E}^{ab} \) (where \( \hat{1} \) acts in the coordinate space) as it is implied in the conventional gauge-invariant regularization [3, 33]. Therefore, in order to match between the prescriptions (8.7) and (2.3), one is to impose the constraint (8.7) complementary to the normalization (2.3).

In conclusion, let us make a few comments concerning the relation (8.3) underlying the considered above regularization. For a preliminary orientation, consider first the zero vortex-width limit (3.8) when the weight (1.5) can be traded, within the stringy sum (1.4), for the Nambu-Goto weight (1.6) with \( m_0 = 0, \lambda(\lambda) = \xi \lambda \). In this case, the interpretation of \( S(\Gamma_{xy}[G]) \) (in the exponent of eq. (8.3)) as the effective action for the cross-section paths \( \Gamma_{xy} \) is justified by the following observation. The consistency of the latter interpretation requires that \( S(\Gamma_{xy}[G]) \) has a finite limit when the minimal area of both \( M_{min}(C_{xy} \Gamma_{yx}) \) and \( M_{min}(C_{yx} \Gamma_{xy}) \) is sent to infinity (in the units of \( \Lambda^{-1} \)). In turn, the required finiteness is predetermined by the fact that the correlators of the normals to the worldsheet \( M(C_{xy}C_{yx}) \) are supposed to be short-ranged in the considered regime (1.4). As a result, even when \( |x - y| \leq \Lambda^{-1} \), the implicit dependence of the action \( S(\Gamma_{xy}[G]) \) on the geometry of the contour \( C = C_{xy}C_{yx} \) is reduced to the one on the geometry in the \( 1/\Lambda \) vicinity of \( x \) and \( y \).

As for the case, when the width of the \( YM \) vortex is \( \sim \Lambda^{-1} \), the only essential difference with the previous consideration is due to the presence of the quasi-contact interactions (of the abelian nature) between the elementary flux-tubes. The consistency is then maintained owing to the fact that, for any macroscopic contour \( C \) in the regime (1.4), the contribution of the latter interactions is unobservable everywhere except for the \( 1/\Lambda \) vicinity of the (point-like) selfintersections of \( C \). In consequence, by the same token as previously, \( S(\Gamma_{xy}[G]) \) remains finite when \( A[M_{min}(C_{xy} \Gamma_{yx})], A[M_{min}(C_{yx} \Gamma_{xy})] \rightarrow \infty \).

\[\text{Eq. (D.3)}\] regularizes the modified representation (1.1) of the Loop equation (2.2) which can be obtained from eq. (2.2) integrating both sides along the contour \( C \), i.e. performing the integral \( \oint_C dx_\nu \).
E: The backtracking invariance of $< W_C >$.

The basic symmetry of $< W_C >$, encoded in the duality-relation (4.9), is the invariance with respect to the creation of arbitrary zig-zag backtrackings (bounding zero 1-volume) of the contour $C$. Our aim is to discuss the two alternative prescriptions to implement the backtracking invariance of $< W_C >$ starting from any particular contour $C \in \Upsilon_{nbt}$ devoid of backtrackings.

Before we deduce from eq. (E.3) the first prescription (introduced in Section 4.4), let us first present the strict formalization of what we are going to obtain. Actually, the required prescription amounts to the following simple modification (given by eq. (E.3)) of the measure in eq. (4.7) (or (4.1)) while the action associated to the weight (1.3) (or, respectively, (1.6)) is kept intact. To begin with, it is helpful to reinterpret the loop space $\Upsilon$ as the space of the orbits $\mathcal{O}(C)$, each being generated via the appropriate attachment of the backtracking segments from a particular nonbacktracking contour $C \in \Upsilon_{nbt}$. To introduce the orbit structure, one is to associate to a given smooth immersion-map $\tilde{\vartheta}_b : \tilde{C} \to C$, resulting in the boundary contour $C \equiv C[\hat{1}]$, the smooth composed (i.e. inner-product) mapping

$$C[\hat{1}] = \tilde{\vartheta}_b(\tilde{C}) \longrightarrow \vartheta_b(\tilde{C}) \circ v(\tilde{C}) \equiv C[v] ,$$ (E.1)

where the map $v(\tilde{C})$ induces on a given reference-loop $C[\hat{1}] \in \Upsilon_{nbt}$ a particular data of backtrackings. Provided the backtracking segments possess the support on $C[\hat{1}] \in \Upsilon_{nbt}$, the deformation (E.1) can be evidently represented as the reparametrization of the contour $C[\hat{1}]$,

$$x_\mu(s) \to x_\mu(f(s)) \ ; \ \{ f(0) = 0 , \ f(1) = 1 \mid df(s)/ds > -\infty \} ,$$ (E.2)

originally introduced through the trajectory $x_\mu(s)$, $s \in [0,1]$. Then, the foldings are associated to those points $s_k$ where the derivative $df(s)/ds$ changes its sign (while the composite boundary-mapping (E.1) ceases to be an immersion).

Now we are ready to formalize the required prescription. The Gauge String representation of $< W_{C[v]} >$ can be deduced from the one of $< W_{C[\hat{1}]} >$ trading, in the measure of eq. (4.7) (or (4.1)), the $\vartheta(M(\tilde{C}))$- (or, respectively, $\tilde{\vartheta}(M(\tilde{C}))$-) maps for the modified mappings

$$\vartheta(M(\tilde{C})) \longrightarrow \vartheta(M(\tilde{C})) \circ v(\tilde{C}) \equiv M(C[v])$$ (E.3)

which, leaving the interior $\vartheta(M(\tilde{C}))/C$ of the surface intact

$$\left( \vartheta(M(\tilde{C})) \circ v(\tilde{C}) \right) /C[v] = \vartheta(M(\tilde{C}))/C[\hat{1}] ,$$ (E.4)

create a given data of the foldings (of the boundary contour $C \in \Upsilon_{nbt}$) in accordance with the $v$-pattern (E.1). The backtracking invariance of $< W_C >$ follows then from the fact that all the ingredients of the ‘action’ in eq. (4.7) (i.e. the area $A[M]$, the Euler character $\chi$, and the selfintersection factor $J[\cdot \cdot]$ deduced from eq. (4.9)) are invariant under the transformation (E.3).

Next, to obtain eq. (E.3) from the duality-relation (4.3), one is to observe that the prescription (E.3) in $D \geq 3$ effectively implements the following tree-irreducibility constraint (to be deduced below), lacking in the conventional Nambu-Goto paradigm. The considered irreducibility implies that, being cut along an arbitrary tree, any admissible worldsheet $\tilde{M}([C_k])$ does not split into a union of disjoint components. In order to reveal why the considered constraint is crucial for the

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72 Thus, the $\Upsilon_{nbt}$ subspace (comprised of all nonbacktracking contours) represents the proper section of $\Upsilon$ so that $\Upsilon = \mathcal{O}(\Upsilon_{nbt})$.

73 Recall that a tree is a graph without any closed 1-cycles.
backtracking invariance of $< W_C >$, one notes first that for $v \neq \hat{1}$ the support (in $\mathbb{R}^D$) of the complement $C[v]/C[\hat{1}]$ has topology of a (not necessarily connected) tree graph. Then, in addition to the surfaces $M(C[v])$ complying with eq. (E.3), there is another relevant contribution within the conventional Nambu-Goto measure: along $C[v]/C[\hat{1}]$, one can glue to $\tilde{\vartheta}(M(\hat{C}))$ arbitrary closed 'baby-universes'. The latter contribution, being of nonzero measure in $D \geq 3$, evidently violates both the tree-irreducibility and the backtracking invariance. It is the tree-irreducibility which in $D \geq 3$ makes the Gauge String representation rid of such 'baby-universes'.

Finally, it is straightforward to derive the considered irreducibility from the basic relation (4.9). Indeed, the functional integral of the $YM$ theory, defined (via, see [7], the Heat-Kernal lattice gauge system) on a taget-space $T$ irreducible with respect to the cut along a particular tree $T$, factorizes. The latter follows from the possibility [10] to choose such a gauge that all the link-variables (in the corresponding Heat-Kernal system on $T$), associated to the tree $T$ in question, are constrained to unity. As a result, the weight of the (conglomerates of the) tree-reducible worldsheets properly factorizes as well so that the latter worldsheets are to be interpreted as the disconnected contribution properly composed of the tree-irreducible connected components. Therefore, the required irreducibility constraint follows from the fact that, by construction, the stringy representation of the average in the r.h. side of eq. (4.9) involves only connected worldsheets.

E.1 Employing the larger symmetry of the string action (1.5).

Let us now turn to the complementary prescription to implement the backtracking invariance of $< W_C >$, for simplicity restricting our attention to the regime (1.4)/(1.7) where the stringy pattern (1.1)/(1.3) applies. To begin with, the logarithm of eq. (1.3) (akin to a generic polynomial functional (2.13) of the area-element $d\sigma_{\mu}(x)$) is evidently invariant under the extention [11] of the group of the worldsheet’s diffeomorphisms (4.4). The extension allows for a nonpositive Jacobian (4.3), and zeros of the Jacobian $J(\gamma)$ can possess support on closed curves which bound (on the worldsheet $\tilde{M}$ in question) connected $2d$ domains where $J(\gamma) < 0$.

As a result, the extended reparametrizations (E.2) of the boundary loop (allowing for negative $df(s)/ds$) can be alternatively implemented as the boundary restriction of the smooth reparametrization (1.4) (violating the $J(\gamma) > 0$ constraint) which involves the worldsheet’s interior as opposed to eq. (E.4). The latter reparametrization associates to a given strict immersion $\tilde{\vartheta}(M(\hat{C}))$ the whole orbit

$$\tilde{\vartheta}(M(\hat{C})) \longrightarrow \tilde{\vartheta}(M(\hat{C})) \circ \omega(M(\hat{C})) \equiv \phi(M(\hat{C})),$$

(E.5)
generated by the auxiliary maps $\omega(M(\hat{C}))$. Each particular $\omega(M(\hat{C}))$, geometrically, is visualized as an attachment of a given pattern of the worldsheet’s $2d$ foldings with the support on the surface $\tilde{M}(C)$ represented by the original immersion-map $\tilde{\vartheta}(M(\hat{C}))$. (In fact, $ln(w_2[M(C)])$ is invariant under the generalization of (E.3) when the foldings are substituted by arbitrary backtracking $2d$ segments (i.e. closed baby-universes bounding zero 3-volume). The latter segments, assumed to possess at least a 1-dimensional domain of intersection with $\tilde{M}(C)$, do not necessarily have the entire support on $\tilde{M}(C)$.)

To make contact with the extended group of the reparametrizations (E.2), eq. (E.3) should be compared with eq. (E.3). One concludes that the boundary restriction $\omega_b(\hat{C})$ of $\omega(M(\hat{C}))$ is to be identified with the corresponding mappings $\nu(\hat{C})$. The smoothness of the reparametrization

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74 This is in contrast to the previous prescription formalized via rather singular reparametrization (1.4) corresponding to the mapping (E.3).
functions $f_\alpha(\gamma)$ is now achieved because the loop’s backtrackings are not an isolated 1-dimensional ‘defect’ of the transformation (4.4) but a part of the 2-dimensional worldsheet’s folding associated to the domain of negative Jacobian. It is also noteworthy that, for $v \neq \hat{1}$, the topology of $M(C[v]) = \phi(M(\hat{C}))$ does violate the tree-irreducibility condition exactly on the tree where the worldsheet’s backtrackings (encoded in $\omega(M(\hat{C}))$) are attached.

Finally, in order to implement the backtracking invariance of $<W_C>$ (i.e. extended symmetry (E.2)) on the quantum level, the measure in the Ansatz (4.1) is to be modified according to the pattern (E.5):

$$\int D\bar{\theta}(M(\hat{C})) \rightarrow \int D\bar{\theta}(M(\hat{C})) \frac{d\omega(M(\hat{C}))}{K_{\bar{\theta}}} \equiv \int d\phi(M(\hat{C})),$$

(E.6)

where $K_{\bar{\theta}}$ stands for the proper normalization-factor. As the string action (1.5) is independent of $\omega(M(\hat{C}))$ (while the Wilson loops are invariant under the reparametrizations (E.2)), the original representation (4.1)/(1.5) is invariant under the extension (E.6) of the measure.

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