A COMPOSITIONAL COALGEBRAIC SEMANTICS FOR STRATEGIC GAMES

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ABSTRACT. We provide a compositional coalgebraic semantics for strategic games. In our framework, like in the semantics of functional programming languages, coalgebras represent the observable behaviour of systems derived from the behaviour of the parts over an unobservable state space. We use coalgebras to describe and program stage games, finitely and potentially infinitely repeated hierarchical or parallel games with imperfect and incomplete information based on deterministic, non-deterministic or probabilistic decisions of learning agents in possibly endogenous networks. Our framework is compositional in that arbitrarily complex network of games can be composed. The coalgebraic approach allows to represent self-referential or reflexive structures like institutional dynamics, strategic network formation from within the network, belief formation, learning agents or other self-referential phenomena that characterise complex social systems of cognitive agents. And finally our games represent directly runnable code in functional programming languages that can also be analysed by sophisticated verification and logical tools of software engineering.

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1. Introduction

We provide a coalgebraic semantic for strategic games based on categorical methods \[15, 11, 21, 9\]. The first formulation of games in terms of coalgebraic semantics appeared in \[19\], later work comprise \[8, 13, 3\]. In addition to these approaches we provide operations in order to compose arbitrarily complicated games from more basic ones. We allow the players to decide based on deterministic, non-deterministic or probabilistic algorithms given their epistemic state that arises from observations, i.e. we feature econometric or learning agents.

The coalgebraic approach to semantics has evolved for programming languages that are modelled as abstract unobservable state transition systems \[22, 20, 21, 9\]. Being build on this mathematical framework our games are directly implementable for example in Haskell. The key idea that we want to exploit in this paper is an analogy to bialgebraic theories \[10\] covering the behaviour of programming languages. The first usage of bialgebras for a semantic of cellular automatons for multi-agent systems was developed in \[24\]. We extend these ideas to game theory with the operations in the cellular automatons being players and the cellular grid being a network of players. The semantics of cellular automatons is here then a semantics of the overall game played by the coalgebraic players.

The coalgebraic constructions allow not only for infinite horizons or repetitions of games but also for infinite reflexive structures like beliefs of beliefs and so on that arise in economic game theory as Harsanyi type spaces \[4, 5, 6, 16, 7\]. This structure has been formulated coalgebraically in computer science \[18, 17\]. Reflexive structures may also arise as games within networks that are played over the very structure of the network itself. Reflexivity naturally arise in systems of cognitive agent who reason about the system they are part of, see for a sociological account thereof in \[14\]. The resulting mathematical paradoxes are discussed in \[12, 25, 2\].

Finally, the coalgebraic approach interfaces to the program verification tools or specification languages like modal logics \[11\] that can be used to analyse the strategic games in our framework.

2. Framework

In order to keep the category theoretical overhead to a minimum, we will introduce all category theoretical notions only in the special case of the category of sets where the objects are sets and the arrows are total functions. Other examples of categories contain sets and relations, measurable spaces and measurable functions, or topological spaces and continuous functions.

The main concepts we will need are those of a category, functor and natural transformation.

**Definition 2.1.** A category \( \mathcal{C} \) consists of a class \( \text{C}^{\text{obj}} \) of objects, a class \( \text{C}^{\text{arr}} \) of arrows, and a composition operation \( \circ \) on arrows. Each arrow \( f \in \text{C}^{\text{arr}} \) has a domain \( X \in \text{C}^{\text{obj}} \) and a codomain \( Y \in \text{C}^{\text{obj}} \). We write \( f : X \to Y \) to indicate that \( f \) is an arrow with domain \( X \) and codomain \( Y \). The composition operation is assumed to satisfy the following two conditions:

1. The composition \( f \circ g \) of two arrows is defined if, and only if, the domain of \( f \) is equal to the codomain of \( g \).
The composition operation is associative, i.e., for all arrows \( f : X \to Y, \)
\( g : Y \to Z, \) \( h : Z \to W, \)
\[
(h \circ g) \circ f = h \circ (g \circ f).
\]

(3) For each object \( X, \) there is an identity arrow \( \text{id}_X : X \to X \) such that
\[
f \circ \text{id}_X = f \quad \text{and} \quad \text{id}_X \circ g = g,
\]
for all arrows \( f : X \to Y \) and \( g : Z \to X. \)

**Definition 2.2.** A functor \( F \) (from the category of sets to itself) is an operation assigning
- to each set \( X \) a new set \( F(X) \) and
- to each function \( g : X \to Y \) a function \( F(g) : F(X) \to F(Y) \)
such that
\[
F(\text{id}_X) = \text{id}_{F(X)} \quad \text{and} \quad F(f \circ g) = F(f) \circ F(g),
\]
for all sets \( X \) and all functions \( f : Y \to Z \) and \( g : X \to Y. \)

As an example, let us introduce three functors that will be used below.

(1) The identity functor \( \text{id} \) maps every set \( X \) and every function \( f : X \to Y \)
to itself.

(2) The finite power-set functor \( \mathcal{P}_{\text{fin}} \) maps every set \( X \) to the set \( \mathcal{P}_{\text{fin}}(X) \) of its finite subsets, and it maps a function \( f : X \to Y \) to the function
\[
\mathcal{P}_{\text{fin}}(f) : \mathcal{P}_{\text{fin}}(X) \to \mathcal{P}_{\text{fin}}(Y) : S \mapsto \{ f(s) \mid s \in S \}.
\]

(3) The finite probability functor \( \mathbb{D}_{\text{fin}} \) maps a set \( X \) to the set of all finite probability distributions on \( X, \) i.e., all maps \( d : X \to [0,1] \) such that only finitely many elements of \( X \) are mapped to non-zero values. For a function \( f : X \to Y, \) it returns the function
\[
\mathbb{D}_{\text{fin}}(f) : \mathbb{D}_{\text{fin}}(X) \to \mathbb{D}_{\text{fin}}(Y) : d \mapsto df,
\]
where
\[
df(y) := \sum_{x \in f^{-1}(y)} d(x). \]

Beside the notion of a functor, we also need those of a natural transformation and a distributive law.

**Definition 2.3.** (a) A natural transformation \( \eta : F \Rightarrow G \) from a functor \( F \) to a functor \( G \) is a family \( \eta = (\eta_X)_X \) of functions
\[
\eta_X : F(X) \to G(X),
\]
indexed by sets \( X, \) satisfying
\[
\eta_Y \circ F(f) = G(f) \circ \eta_X, \quad \text{for every function} \ f : X \to Y.
\]
(b) A **distributive law** between two functors $\mathbb{F}$ and $\mathbb{G}$ is a natural transformation $\eta : \mathbb{F} \circ \mathbb{G} \Rightarrow \mathbb{G} \circ \mathbb{F}$.

Examples of natural transformations will appear in Section 2.2 below.

2.1. **Processes.** Before introducing games, let us define the simpler notion of a process, which corresponds to a game with a single player. Processes will provide the technical machinery our framework is based on.

A process is a state based system transforming an input sequence into an output sequence. In each step it receives an input value and, depending on its current state, it produces an output value and changes its state. Alternatively, a process can decide to terminate. Formally, a **process** is given by

- a set $S$ of **states**,
- a set $I$ of **inputs**,
- a set $O$ of **outputs**,
- a set $R$ of **results**, and
- a function $\pi : S \times I \rightarrow (C(R + S \times O))^I$, for some functor $C$.

The function $\pi$ describes one step of the process. When in state $s \in S$ and given the input $i \in I$, the process chooses a possible continuation that consists in either terminating with a result $r \in R$, or in continuing in a state $s' \in S$ and producing an output value $c \in O$.

In the above definition, the **choice functor** $C$ determines which kind of process we are dealing with. Important examples for choice functors are the following ones.

1. The **deterministic choice functor** $C_{\text{det}} = \text{id}$ is the identity functor. It can be used if the input uniquely determines what happens next.
2. The **non-deterministic choice functor** $C_{\text{ndet}} = \mathcal{P}_{\text{fin}}$ is the finite power-set functor. It can be used if, for a given input, there might be several possible continuations of the process.
3. The **probabilistic choice functor** $C_{\text{prob}} = \mathbb{D}_{\text{fin}}$ is the finite probability functor. It can be used if the continuation of the process is random.

To apply the category theoretical machinery it will be convenient to write the function $\pi$ in the form

$$\pi : S \rightarrow (C(R + S \times O))^I.$$  

In category theoretical terms, such functions can be seen as so-called coalgebras.

**Definition 2.4.** Let $\mathbb{F}$ be a functor. An **$\mathbb{F}$-coalgebra** is a function $h : X \rightarrow \mathbb{F}(X)$, for some set $X$.

Hence, a process $\pi$ becomes a $\Pi_0$-coalgebra $\pi : S \rightarrow \Pi_0(S)$, where $\Pi_0$ is the **process functor**

$$\Pi_0(X) := C(R + X \times O)^I.$$  

We denote by

$$\Pi(S; I, O, R) := \Pi_0(S)^S$$
the set of all processes with states $S$, inputs $I$, outputs $O$, and results $R$.

2.2. **Transformations of choice functors.** In this section we present several natural transformations between choice functors that will be needed in the next section.

1. For two choice functors $C_1$ and $C_2$, we define a natural transformation
   \[ \mu_{1,2} : C_1 \circ C_2 \Rightarrow C_{1,2} \]
   that combines a choice of $C_1$ followed by a choice of $C_2$ into a single choice with respect to a combined functor $C_{1,2}$.

2. For a choice functor $C$ and fixed sets $A, B$, we define a distributive law
   \[ \delta : A + B \times C(X) \Rightarrow C(A + B \times X) . \]

3. For two choice functors $C_1$ and $C_2$, we define a natural transformation
   \[ \lambda_{1,2} : C_1(X) \times C_2(Y) \Rightarrow C_{1,2}(X \times Y) . \]

The definitions of all three natural transformations are the ones you would expect from looking at the respective types. We encourage the reader to skip the formal definitions below, which are only included for the sake of completeness.

1. For $C_2 = C_{\text{det}}$, we can use $C_{1,2} := C_1$ and the identity function
   \[ \mu_{1,\text{det}} : C_1(X) \Rightarrow C_1(X) . \]
   Analogously, we can define $\mu_{1,2}$ for $C_1 = C_{\text{det}}$. For $C_1 = C_2 = C_{\text{ndet}}$, we use $C_{1,2} := C_{\text{ndet}}$ and the functions
   \[ \mu_{\text{ndet},\text{ndet}} : \mathcal{P}_{\text{fin}}(\mathcal{P}_{\text{fin}}(X)) \Rightarrow \mathcal{P}_{\text{fin}}(X) : U \mapsto \bigcup_{Z \in U} Z \]
   mapping a set $U \subseteq \mathcal{P}_{\text{fin}}(X)$ to its union. For $C_1 = C_2 = C_{\text{prob}}$, we use $C_{1,2} := C_{\text{prob}}$ and the functions
   \[ \mu_{\text{prob},\text{prob}} : \mathcal{D}_{\text{fin}}(\mathcal{D}_{\text{fin}}(X)) \Rightarrow \mathcal{D}_{\text{fin}}(X) \]
   mapping a distribution $d$ over $\mathcal{D}_{\text{fin}}(X)$ to the distribution
   \[ x \mapsto \sum_{d' \in \mathcal{D}(X)} d(d') \cdot d'(x) . \]

The case where one of $C_1$ and $C_2$ equals $C_{\text{ndet}}$ and the other one equals $C_{\text{prob}}$ is more involved. We omit the definitions.

2. We define $\delta$ as follows. If $C = C_{\text{det}}$, we can use the identity map
   \[ \delta_{\text{det}} : A + B \times X \Rightarrow A + B \times X . \]
   If $C = C_{\text{ndet}}$, we use the map
   \[ \delta_{\text{ndet}} : A + B \times \mathcal{P}_{\text{fin}}(X) \Rightarrow \mathcal{P}_{\text{fin}}(A + B \times X) , \]
   defined by
   \[ \delta_{\text{ndet}}(x) := \begin{cases} \{x\} & \text{for } x \in A , \\ \{(b, u) \mid u \in U\} & \text{for } x = (b, U) \in B \times \mathcal{P}_{\text{fin}}(X) . \end{cases} \]
   If $C = C_{\text{prob}}$, we use the map
   \[ \delta_{\text{prob}} : A + B \times \mathcal{D}_{\text{fin}}(X) \Rightarrow \mathcal{D}_{\text{fin}}(A + B \times X) , \]
where defined by
\[
\delta_{\text{prob}}(x) := \begin{cases} 
  d_x & \text{for } x \in A , \\
  d_{b,c} & \text{for } (b,c) \in B \times \mathcal{D}_{\text{fin}}(X)
\end{cases},
\]
where
\[
d_x(y) := \begin{cases} 
  1 & \text{for } y = x , \\
  0 & \text{otherwise .}
\end{cases}
\] and \(d_{b,c}(y) := \begin{cases} 
  e(c) & \text{for } y = (b,c) , \\
  0 & \text{otherwise .}
\end{cases}\)

(3.) For \(C_1 = C_{\text{det}}\), we can use for
\[
\lambda_{\text{det},\text{det}} : X \times C_2(Y) \Rightarrow C_2(X \times Y)
\]
the distributive law \(\delta\) from (2.) (setting \(A : = \emptyset\) and \(B : = X\)). The case where \(C_2 = C_{\text{det}}\) is handled symmetrically. For \(C_1 = C_2 = C_{\text{ndet}}\), we define
\[
\lambda_{\text{ndet},\text{ndet}} : \mathcal{P}_{\text{fin}}(X) \times \mathcal{P}_{\text{fin}}(Y) \rightarrow \mathcal{P}_{\text{fin}}(X \times Y) : (U, V) \mapsto U \times V.
\]
For \(C_1 = C_2 = C_{\text{prob}}\), we define
\[
\lambda_{\text{prob},\text{prob}} : \mathcal{D}_{\text{fin}}(X) \times \mathcal{D}_{\text{fin}}(Y) \rightarrow \mathcal{D}_{\text{fin}}(X \times Y) : (d, d') \mapsto e_{d,d'}
\]
where
\[
e_{d,d'}(x, y) := d(x) \cdot d'(y).
\]
Again, we omit the cases mixing \(C_{\text{ndet}}\) and \(C_{\text{prob}}\).

2.3. Operations on processes. Before introducing games, let us present several operations intended to construct processes from simpler ones. We start with sums and products of processes.

(a) The sum of two processes is a process where, depending on the state, either the first process takes a step, or the second one does. We only support the case where both processes use the same choice functor. Formally, the sum \(+\) is the operation
\[
+ : \Pi(S_0; I, O_0, R_0) + \Pi(S_1; I, O_1, R_1) \rightarrow \Pi(S_0 + S_1; I, O_0 + O_1, R_0 + R_1)
\]
defined by
\[
(\pi_0 + \pi_1)(s) := \begin{cases} 
  \pi_0(s) & \text{if } s \in S_0 , \\
  \pi_1(s) & \text{if } s \in S_1.
\end{cases}
\]

(b) The product \(\pi_1 \times \pi_2\) of two processes is a process where both components take steps simultaneously. We support the case where \(\pi_1\) and \(\pi_2\) use different choice functors. Suppose that \(\pi_1\) uses \(C_1\), while \(\pi_2\) uses \(C_2\). Formally, the product \(\times\) is the operation
\[
\times : \Pi(S_0; I_0, O_0, R_0) + \Pi(S_1; I_1, O_1, R_1) \rightarrow \\
\Pi(S_0 \times S_1; I_0 \times I_1, O_0 \times O_1, R_0 \times R_1 + R_0 + R_1)
\]
defined by
\[
(\pi_0 \times \pi_1)(s_0, s_1)(i_0, i_1) := (C_1,2(f) \circ \lambda_{1,2})(\pi_0(s_0)(i_0), \pi_1(s_1)(i_1)),
\]
where \(\lambda\) is the natural transformation from Section 2.2 and
\[
f : (R_0 + S_0 \times O_0) \times (R_1 + S_1 \times O_1) \rightarrow \\
(R_0 \times R_1 + R_0 + R_1 + S_0 \times S_1 \times O_0 \times O_1)
is the function

\[
f(x_0, x_1) := \begin{cases} 
(x_0, x_1) & \text{if } x_0 \in R_0 \text{ and } x_1 \in R_1, \\
x_0 & \text{if } x_0 \in R_0 \text{ and } x_1 \notin R_1, \\
x_1 & \text{if } x_0 \notin R_0 \text{ and } x_1 \in R_1, \\
(s_0, s_1, c_0, c_1) & \text{if } x_0 = (s_0, c_0) \text{ and } x_1 = (s_1, c_1),
\end{cases}
\]

(c) We also introduce two operations to modify the inputs and outputs. Given a process \( \pi \) and a function \( f \), we define new processes \( \pi \triangleright f \) and \( f \triangleright \pi \) as follows.

For a function \( f : S \times O \to S \times O' \) and a process \( \pi \in \Pi(S; I, O, R) \), the process \( \pi \triangleright f \) applies, after each step, the function \( f \) to the returned state-output pair. Formally, we define \( \pi \triangleright f \in \Pi(S; I, O', R) \) by

\[
(\pi \triangleright f)(s)(i) := C(id + f)(\pi(s)(i)).
\]

For a function \( f : I' \to I \) and a process \( \pi \in \Pi(S; I, O, R) \), the process \( f \triangleright \pi \) applies, before each step, the function \( f \) to the given input value. Formally, we define \( f \triangleright \pi \in \Pi(S; I', O, R) \) by

\[
(f \triangleright \pi)(s)(i) := \pi(s)(f(i)).
\]

(d) Finally, we introduce two more complicated operations on processes. The feedback operation takes a process \( \pi \) and feeds back its output as an additional input. That is, given a process \( \pi \in \Pi(S; I \times O, O, R) \) we construct a new process \( \pi^\circ \in \Pi(S \times O; I, O, R) \) which, at each step, calls the process \( \pi \) with its current input value and the output of the previous turn. We define

\[
\pi^\circ(s, c)(i) := C(id_R + f)(\pi(s)(i, c)),
\]

where

\[
f : S \times O \to (S \times O) \times O : (s, c) \mapsto ((s, c), c).
\]

(e) The cascading operation takes two processes \( \pi \) and \( \varrho \), runs them in parallel, and uses the outputs of the first process as inputs of the second one. We support the case where \( \pi \) and \( \varrho \) use different choice functors. Suppose that \( \varrho \) uses \( C_1 \), while \( \pi \) uses \( C_2 \). Given \( \pi \in \Pi(S; I, M, P) \) and \( \varrho \in \Pi(T; M, O, R) \), we define \( \pi \triangleright \varrho \in \Pi(S \times T; I, O, P + R) \) as follows. Let

\[
\varrho' : T \times M \to C_2(R + T \times O) : (t, m) \mapsto \varrho(t)(m),
\]

\[
\pi' : S \times T \times I \to C_1(P + S \times T \times M) : (s, t, i) \mapsto C_1(id_P + f_t)(\pi(s)(i)),
\]

where

\[
f_t : S \times M \to S \times T \times M : (s, m) \mapsto (s, t, m).
\]

We set

\[
(\pi \triangleright \varrho)(s, t)(i) := (C_{1,2}(g) \circ \mu \circ \delta \circ C_1(id_P + id_S \times \varrho'))(\pi'(s, t, i)),
\]

where \( \mu \) and \( \delta \) are the natural transformations from Section 2.2 and

\[
g : P + S \times (R + T \times O) \to P + R + S \times T \times O
\]

is the function

\[
g(x) := \begin{cases} 
x & \text{if } x \in P, \\
r & \text{if } x = (s, r) \in S \times R, \\
(s, t, c) & \text{if } x = (s, (t, c)) \in S \times T \times O.
\end{cases}
\]
2.4. Games. We consider games between several players that can consist of finitely many or infinitely many rounds. The game starts in a certain state and, in each round, every player chooses an action to perform. These actions determine the state the game enters next. To determine the outcome of a game, we assume that it produces an output value with each turn and that, at the end of the game, it returns some result. Together, the produced sequence of output values and the final result will determine the outcome. Formally, a game is therefore given by

- a set \( N \) of players,
- for each player \( p \in N \), a set \( A_p \) of actions for player \( p \),
- a set \( S \) of states of the game,
- a set \( R \) of results,
- a set \( O \) of output values, and
- a function

\[
\gamma : S \times \prod_{p \in N} A_p \rightarrow \mathbb{C}(R + S \times O).
\]

Thus, a game is a process where the input has the special form \( \prod_{p \in N} A_p \). In particular, a game \( \gamma \) is a \( \Gamma \)-coalgebra

\[
\gamma : S \rightarrow \Gamma(S),
\]

where \( \Gamma \) is the game functor

\[
\Gamma(S) := \mathbb{C}(R + S \times O)\prod_{p \in N} A_p.
\]

Example 2.5. To formalize the Prisoner’s Dilemma in our framework we use two players \( N := \{1, 2\} \), each with two actions \( A_p := \{c, d\} \) (‘confess’ and ‘deny’). The game needs only one state \( S := \{\ast\} \), no outputs \( O := \emptyset \), and results \( R := \mathbb{R} \times \mathbb{R} \). The deterministic game function \( \gamma : S \rightarrow R^{A_1 \times A_2} \) is defined by

\[
\begin{align*}
\gamma(\ast)(a_1, a_2) := & \begin{cases} 
(1, 1) & \text{if } (a_1, a_2) = (c, c), \\
(2, -1) & \text{if } (a_1, a_2) = (d, c), \\
(-1, 2) & \text{if } (a_1, a_2) = (c, d), \\
(0, 0) & \text{if } (a_1, a_2) = (d, d).
\end{cases}
\end{align*}
\]

Example 2.6. Let us also formalize the Repeated Prisoner’s Dilemma. Again, there are two players \( N := \{1, 2\} \) with two actions \( A_p := \{c, d\} \) each. We still have only one state \( S := \{\ast\} \), but now use outputs \( O := \mathbb{R} \times \mathbb{R} \) and no results \( R := \emptyset \). The deterministic game function \( \gamma : S \rightarrow (S \times O)^{A_1 \times A_2} \) is defined by

\[
\begin{align*}
\gamma(\ast)(a_1, a_2) := & \begin{cases} 
(\ast, (1, 1)) & \text{if } (a_1, a_2) = (c, c), \\
(\ast, (2, -1)) & \text{if } (a_1, a_2) = (d, c), \\
(\ast, (-1, 2)) & \text{if } (a_1, a_2) = (c, d), \\
(\ast, (0, 0)) & \text{if } (a_1, a_2) = (d, d).
\end{cases}
\end{align*}
\]

Example 2.7. For a more involved example, we consider a social game using endogenous networks. Given a group \( N \) of players, we model their social interactions as a graph \( \langle N, E \rangle \) where the edge relation \( E \) connects two players if they are friends. In each turn of the game, new friendships may form and old ones may end. Thus, the graph changes in the course of the game. We can model this game in our framework by using as set of states \( S \) the set of all possible edge relations \( E \). Each player \( p \) has two possible actions: he can befriend another player \( q \), or he can end an existing
friendship with some player. The game function \( \gamma : S \to \Gamma(S) \) takes the current network \( E \) as an input and modifies it according to the actions of all players.

2.5. Players and strategies. Let \( \gamma : S \to \Gamma(S) \) be a game. A strategy for a player \( p \in N \) is a function telling him which action to choose in a given turn of the game. The player has access to his current observations and his knowledge of the play so far. Thus, formally a strategy is a function

\[
\sigma : E_p \times B_p \to C(E_p \times A_p),
\]

where \( E_p \) is the epistemic state of player \( p \) and \( B_p \) is the set of possible observations. Again, we write \( \sigma \) as a coalgebra

\[
\sigma : E_p \to C(E_p \times A_p)^{B_p},
\]

that is, a process with inputs \( B_p \), outputs \( A_p \), and results \( R = \emptyset \).

The observations of a player depend on the current input, the output of the previous turn, and the actions of all players during the previous turn. To specify what exactly player \( p \) can observe, we use a function \( \beta_p : O \times \prod_{p \in N} A_p \to B_p \), which we assume to be a part of the description of the game.

Example 2.8. Suppose we are playing the Repeated Prisoner’s Dilemma. A probabilistic strategy for player 1 would be to copy the previous action of the other player with probability \( \frac{2}{3} \), and to choose the other action with probability \( \frac{1}{3} \). We use only one state \( E_1 := \{\ast\} \) and the observations \( B_1 := \{c, d\} \) are the previous actions of player 2.

\[
\sigma_1 : \{\ast\} \to \mathbb{D}_{\text{fin}}(\{\ast\} \times \{c, d\})^{\{c,d\}} : \ast \mapsto d
\]

where

\[
d(x)(\ast, y) := \begin{cases} 
\frac{2}{3} & \text{if } x = y, \\
\frac{1}{3} & \text{if } x \neq y.
\end{cases}
\]

If, in a game \( \gamma \), we fix strategies \( (\sigma_p)_{p \in N_0} \) for a subset \( N_0 \subseteq N \) of the players, we obtain a new game with players \( N \setminus N_0 \). We denote this game by \( \gamma[\sigma_p]_{p \in N_0} \). The formal definition is as follows. For players \( p \in N \setminus N_0 \) where no strategy is provided, we introduce a non-deterministic dummy strategy that, independently of the input, always tells the player to play some action from \( A_p \) without restricting his choice. This strategy uses only one state. Its formal definition is

\[
\sigma_p : 1 \to \mathcal{P}_{\text{fin}}(1 \times A_p)^{B_p} : i \mapsto A_p.
\]

With the help of these dummy strategies, we can define the desired game as

\[
\gamma[\sigma_p]_{p \in N_0} := \left[ \left[ f \triangleright \prod_{p \in N} \sigma_p \right] \circ \gamma \right] \in \Pi(S \times \prod_{p \in N} E_p \times O \times \prod_{p \in N} A_p, \emptyset, O, R),
\]

where the function

\[
f : O \times \prod_{p \in N} A_p \to \prod_{p \in N} B_p : (c, \bar{a}) \mapsto (\beta_p(c, \bar{a}))_{p \in N},
\]
computes the observations of each player. The states of this new game are tuples 
\[(s, \vec{e}, c, \vec{a}) \in S \times \prod_{p \in \mathbb{N}} E_p \times O \times \prod_{p \in \mathbb{N}} A_p\]
consisting of a state \(s\) of the old game, the epistemic states \(\vec{e}\) of the players, the
output \(c\) of the last turn, and the actions \(\vec{a}\) the players chose last turn.

2.6. Game trees. Given a game \(\gamma\) and strategies \(\sigma_p\) for each player, we would
like to compute the result of the game if each player follows her strategy. Besides
the techniques from the previous section, we need one more definition: that of a
game tree. Informally, a game tree is a tree containing all possible sequences of
events allowed in the game. The formal definition is based on the notion of a final
coalgebra.

**Definition 2.9.** Let \(\mathcal{F}\) be a functor. An \(\mathcal{F}\)-coalgebra \(\omega : \Omega \to \mathcal{F}(\Omega)\) is final if, for
every \(\mathcal{F}\)-coalgebra \(h : X \to \mathcal{F}(X)\), there exists a unique morphism \(\varphi : X \to \Omega\) such
that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & \Omega \\
h \downarrow & & \downarrow \omega \\
\mathcal{F}(X) & \xrightarrow{\mathcal{F}(\varphi)} & \mathcal{F}(\Omega)
\end{array}
\]

commutes.

For the process functors
\[\Pi^0_0(X) = (\mathbb{C}(R + X \times O))^f\]
there exist final \(\Pi^0_0\)-coalgebras \(\omega : \Omega \to \Pi^0_0(\Omega)\), provided that the choice functor \(\mathbb{C}\) is
sufficiently well-behaved. In particular, this is the case for the three choice functors
\(\mathbb{C}_{\text{det}}, \mathbb{C}_{\text{ndet}},\) and \(\mathbb{C}_{\text{prob}}\).

Let us describe the final \(\Pi^0_0\)-coalgebras for the choice functors \(\mathbb{C}\) introduced
above. The elements of these final coalgebras are trees, which are directed acyclic
graphs such that there exists one vertex, the root of the tree, with the property
that every other vertex can be reached by a unique path from the root. A tree is
\((A, B, C)\)-labelled if it has more than one vertex and

- the root is unlabelled,
- every other inner vertex is labelled by an element of \(B\),
- every leaf is labelled by an element of \(A \cup B\),
- every edge is labelled by an element of \(C\).

If there is an edge with label \(c\) from a vertex \(x\) to a vertex \(y\), we call \(y\) the \(c\)-successor
of \(x\).

(a) We start with the functor
\[\Pi^0_0(X) := (A + X \times B)^C\]
for \(\mathbb{C} = \mathbb{C}_{\text{det}}\). In this case the final \(\Pi^0_0\)-coalgebra \(\omega : \Omega \to \Pi^0_0(\Omega)\) takes the following
form. The set \(\Omega\) consists of all \((A, B, C)\)-labelled trees that are deterministic, that
is, such that every leaf has a label in \(A\) and every inner vertex has exactly one
\(c\)-successor, for each \(c \in C\). The function \(\omega\) is defined as follows: given a tree \(T\) and
a value \(c \in C\), we distinguish two cases depending on the label of the \(c\)-successor \(x\)
of the root. If \(x\) is labelled by an element \(a \in A\), we set \(\omega(T)(c) := a\). If \(x\) is
labelled by an element \( b \in B \), we set \( \omega(T)(c) := \langle T', b \rangle \) where \( T' \) is the subtree of \( T \) rooted at \( x \).

To see that this is indeed the final \( \Pi_0 \)-coalgebra, consider an arbitrary \( \Pi_0 \)-coalgebra \( \pi : S \rightarrow \Pi_0(S) \). The required unique function \( \varphi : S \rightarrow \Omega \) is given by

\[
\varphi(s) := T_s, \quad \text{for } s \in S,
\]

where the tree \( T_s \) is defined as follows: we first construct a graph \( \langle V, E \rangle \) with set of vertices \( V := A + S \times B \) and the following edges. For every \( \langle s, b \rangle \in S \times B \), there is a \( c \)-labelled edge from \( \langle s, b \rangle \) to \( \pi(s)(c) \). The elements of \( A \) have no outgoing edges. The vertex labelling is the natural one: a vertex \( a \in A \) gets the label \( a \) and a vertex \( \langle s, b \rangle \) gets the label \( b \in B \).

The tree \( T_s \) is now obtained from the unravelling of this graph starting at a vertex \( \langle s, b \rangle \), for an arbitrary \( b \in B \), by forgetting the label \( b \) of the root. Formally, the 

unravelling

of a graph \( \langle V, E \rangle \) starting at a vertex \( s \) is defined as the tree consisting of all finite paths through the graph that start at \( s \). There is an edge between two such paths if the second one is obtained from the first one by appending a single edge. This edge also determines the label of the edge label. The vertex labelling of the tree is obtained by labelling each path with the label of its end-vertex.

**Example 2.10.** Let \( \pi \in \Pi((s_a, s_b); \{0, 1\}, \{a, b\}, \emptyset) \) be the deterministic process defined by

\[
\pi(s_x)(y) := \begin{cases} 
(s_a, x) & \text{if } y = 0, \\
(s_b, x) & \text{if } y = 1.
\end{cases}
\]

The (top of the) tree \( \varphi(s_a) \) has the following form:

```
  0 __ 1
 / \
\ a / \ a
/ \ / \ / \
0 / 1 0 / 1
\ a / \ a / \ b / \ b
\ : / \ : / \ : / \ : 
```

To see that the function \( \varphi \) defined in this way has the required property we need to check that

\[
\Pi_0(\varphi) \circ \pi = \omega \circ \varphi.
\]

For \( s \in S \) and \( c \in C \), suppose that

\[
\pi(s)(c) = \langle s', b \rangle \in S \times B.
\]

Let \( T := \varphi(s) \) and \( T' := \varphi(s') \). Note that \( T' \) is equal to the subtree of \( T \) rooted at the \( c \)-successor of the root and that this \( c \)-successor is labelled by \( b \). Hence,

\[
(\Pi_0(\varphi) \circ \pi)(s)(c) = \Pi_0(\varphi)(\langle s', b \rangle) = \langle T', b \rangle = \omega(T)(c) = (\omega \circ \varphi)(s)(c).
\]

In the case where \( \pi(s)(c) = a \in A \) we argue similarly.

(b) Consider the functor

\[
\Pi_0(X) := \mathcal{P}_{\text{fin}}(A + X \times B)^C
\]
for non-deterministic games. In this case the final $\Pi_0$-coalgebra $\omega : \Omega \to \Pi_0(\Omega)$ takes the following form. The set $\Omega$ consists of all $(A, B, C)$-labelled trees where each vertex has only finitely many $c$-successors, for every $c \in C$. The function $\omega$ is defined as follows. Given a tree $T$ and a value $c \in C$, let $S$ be the set of all $c$-successors of the root of $T$. Then $\omega(T)(c)$ returns the set
\[
\{ a \in A | \text{some } x \in S \text{ has label } a \} \cup \{ (T_x, b) | x \in S \text{ has label } b \in B \},
\]
where $T_x$ is the subtree of $T$ rooted at $x$.

Given an arbitrary $\Pi_0$-coalgebra $\pi : S \to \Pi_0(S)$, the required unique function $\varphi : S \to \Omega$ is defined similarly as in (a). We set $\varphi(s) := T_s$, for $s \in S$, where $T_s$ is the unravelling of the following graph $\langle V, E \rangle$. Again the set of vertices is $V := A + S \times B$ and the vertex labelling is the natural one. For each $(s, b) \in S \times B$, there is a $c$-labelled edge from $(s, b)$ to $x$, for every $x \in \pi(s)(c)$.

As above, a straightforward calculation shows that the function $\varphi$ defined in this way has the required properties.

**Example 2.11.** Let $\pi \in \Pi(\{\ast\}; \{0, 1\}, \{a, b\}, \emptyset)$ be the non-deterministic process defined by
\[
\pi(\ast)(x) := \{(\ast, a), (\ast, b)\}.
\]
The tree $\varphi(\ast)$ has the following form:
\[
\begin{array}{cccccc}
a & 0 & 0 & 1 & 1 & 1 \\
\hline
b & a & a & b & a & b
\end{array}
\]

(c) Finally, consider the functor
\[
\Pi_0(X) := D_{fin}(A + X \times B)^C
\]
for probabilistic games. In this case the final $\Pi_0$-coalgebra $\omega : \Omega \to \Pi_0(\Omega)$ takes the following form: the set $\Omega$ consists of all $(A, B, C \times [0, 1])$-labelled trees where, for every $c \in C$ and every vertex $v$,
\begin{itemize}
  \item $v$ has only finitely many outgoing edges labelled $\langle c, p \rangle$, for some $p \in [0, 1]$,  
  \item the sum of all values $p$ such that there is an outgoing edge with label $\langle c, p \rangle$ equals 1, and  
  \item $v$ does not have two outgoing edges with labels $\langle c, p \rangle$ and $\langle c, p' \rangle$ where $p, p' \in [0, 1]$ and such that the subtrees rooted at the corresponding successors are isomorphic.
\end{itemize}
The function $\omega$ is defined as follows: given a tree $T$, a value $c \in C$, $a \in A$, and $\langle T', b \rangle \in \Omega \times B$, we set
\[
\omega(T)(c)(a) := p
\]
if the root of $T$ has an outgoing edge with label $\langle c, p \rangle$ that leads to a leaf with label $a$, and we set
\[
\omega(T)(c)((T', b)) := p
\]
if the root of $T$ has an outgoing edge with label $\langle c, p \rangle$ that leads to an inner vertex $x$ with label $b$ such that the subtree of $T$ rooted at $x$ is equal to $T'$. In all other cases, we set

$$\omega(T)(c)(x) := 0.$$ 

Given an arbitrary $\Pi_0$-coalgebra $\pi : S \to \Pi_0(S)$, the required unique function $\varphi : S \to \Omega$ is defined similarly as in (a). We set

$$\varphi(s) := T_s,$$

where $T_s$ is the unravelling of the following graph $\langle V, E \rangle$: again the set of vertices is $V := A + S \times B$ and the vertex labelling is the natural one. For each $(s, b) \in S \times B$ and every $x \in A + S \times B$, there is a $\langle c, \pi(s)(c)(x) \rangle$-labelled edge from $(s, b)$ to $x$.

As above, a straightforward calculation shows that the function $\varphi$ defined in this way has the required properties.

**Example 2.12.** Let $\pi \in \Pi(\{\ast\}; \{0, 1\}, \{a, b\}, \emptyset)$ be the probabilistic process defined by

$$\pi(\ast)(x)(s, c) := 1/2.$$

The tree $\varphi(\ast)$ has the following form:

```
0, 1/2 1/2 1/2
a b a b a b a b a b a b a b a b
1, 1/2 1/2 1/2
b a b a b a b a b a b a b a b
```

(Due to space considerations we have omitted some edge labels.)

We have seen that the final coalgebras consist of trees describing all possible sequences in the game. Given a game $\gamma : S \to \Gamma(S)$ and the unique morphism $\varphi : S \to \Omega$ into the final $\Gamma$-coalgebra, we call the tree $\varphi(s)$ the game tree of $\gamma$ when starting in state $s \in S$.

2.7. The outcome of a game. After these preparations we can determine the outcome of a game. Given a game $\gamma : S \to \Gamma(S)$ and strategies $\sigma_p$ for each player, we can compute a game $\gamma[\sigma_p]_p$ without players and determine its game tree $T$. Hence, it remains to define how to read off the outcome from a game tree.

Let $\gamma : S \to \Gamma(S)$ be a game without players and let $\omega : \Omega \to \Gamma(\Omega)$ be the final $\Gamma$-coalgebra. To define the outcome $\gamma$ we specify a set $U$ of outcomes and two functions $\varrho : \Omega \to U$ and $\tau : \Gamma(U) \to U$ such that

$$\Omega \xrightarrow{\omega} \Gamma(\Omega)$$

$$\begin{array}{ccc}
\varrho & \quad \Gamma(\varrho) \\
\downarrow & \quad \downarrow \\
U & \xleftarrow{\tau} & \Gamma(U)
\end{array}$$

Intuitively, $\varrho$ maps a game tree to its outcome, while $\tau$ computes the outcome of a game from the outcomes of its subgames. Hence, $\tau$ performs a local computation, while $\varrho$ is needed to compute the limit of an infinite sequence of turns. Ideally, the function $\tau$ uniquely determines $\varrho$. This is the case, for instance, for discounted pay-off games, where the value of a game mostly depends on an initial segment of the game tree.
Example 2.13. Consider a deterministic two player game with $R = \mathbb{R} \times \mathbb{R}$ and $O = \mathbb{R} \times \mathbb{R}$. Fixing deterministic strategies for both players, we obtain a deterministic zero-player game, the game tree of which is either an infinite sequence over $O$ or a finite sequence where the last element is from $R$ and the remaining ones are from $O$.

Choosing a discount factor $\lambda \in (0, 1)$, we can define the outcome by the functions

$$\tau : R + U \times O \to U$$

and

$$\varrho : O \to U,$$

where $U := \mathbb{R} \times \mathbb{R}$ and

$$\tau((x, y), (u, v)) := (\lambda x + u, \lambda y + v), \quad \text{for } ((x, y), (u, v)) \in U \times O.$$

The function $\varrho$ is uniquely determined by $\tau$. An explicit definition is

$$\varrho(x_n, y_n)_{n<\alpha} := \left(\sum_{n<\alpha} \lambda^n x_n, \sum_{n<\alpha} \lambda^n y_n\right).$$

2.8. Nash equilibria. Having defined the outcome of a game, we can introduce equilibria. Consider a game $\gamma$ with set of players $N$ and set of outcomes $U := \mathbb{R}^N$. We fix an output value $\hat{c} \in O$ and actions $\hat{a} \in \prod_{p} A_p$ that will serve as imaginary outcome of the ‘first game turn’. Let $\tau : \Gamma(U) \to U$ and $\varrho : \Omega \to U$ be the functions to compute the outcome of $\gamma$. For a tuple $\bar{\sigma} = (\sigma_p)_{p \in N}$ of strategies, we denote by $\varphi[\bar{\sigma}] : S \to \Omega$ the function from the reduced game $\gamma[\bar{\sigma}]$ to the final coalgebra.

Given strategies $\sigma_p$, for each $p \in N$, initial states $s_0 \in S$ and $e_p \in E_p$, for $p \in N$, and a player $q \in N$, we say that $\langle \sigma_q, e_q \rangle$ is a best response to $\langle \sigma_p, e_p \rangle_{p \in N \setminus \{q\}}$ in the game $\langle \gamma, s_0 \rangle$ if

$$\varrho(\varphi[\bar{\sigma}](s_0, \hat{e}, \hat{c}, \hat{a})) \geq \varrho(\varphi[\bar{\sigma}'](s_0, \hat{e}', \hat{c}, \hat{a})),$$

for all tuples $\bar{\sigma}'$ and $\bar{\sigma}'$ that differ from, respectively, $\bar{\sigma}$ and $\bar{\sigma}$ only in the $p$-th component.

We say that $\langle \bar{\sigma}, \bar{e} \rangle$ is a Nash equilibrium of $\langle \gamma, s_0 \rangle$ if, for every player $q \in N$, $\langle \sigma_q, e_q \rangle$ is a best response to $\langle \sigma_p, e_p \rangle_{p \in N \setminus \{q\}}$ in $\langle \gamma, s_0 \rangle$.

There exist especially well-behaved Nash equilibria called subgame perfect equilibria. In order to define them, we need the notion of an $n$-modification of a strategy $\sigma : E \to \mathbb{C}(E \times A)^B$. Intuitively, an $n$-modification of $\sigma$ is a new strategy that coincides with $\sigma$, except for the first $n$ turns of the game. Formally, we define it as a strategy

$$\sigma' : E + [n] \to \mathbb{C}((E + [n]) \times A)^B$$

(where $[n] := \{0, \ldots, n - 1\}$) that satisfies the following conditions:

$$\sigma'(e) = \sigma(e), \quad \text{for } e \in E,$$

$$\sigma'(k) \in \mathbb{C}\{\{k + 1\} \times A\}^B \quad \text{for } k \in [n], \ k < n - 1,$$

$$\sigma'(k) \in \mathbb{C}(E \times A)^B \quad \text{for } k = n - 1.$$

We say that a Nash equilibrium $\langle \bar{\sigma}, \bar{e} \rangle$ is subgame perfect if, for every $n \in \mathbb{N}$, all $n$-modifications $\bar{\sigma}'$ of $\bar{\sigma}$, and every player $q \in N$, $\langle \sigma'_p, 0 \rangle$ is a best response to $\langle \sigma'_p, 0 \rangle_{p \in N \setminus \{q\}}$ in $\langle \gamma, s_0 \rangle$, where we restrict the notion of a best response only to consider strategies coinciding with the given one in the first $n$ turns.

Example 2.14. Consider the Repeated Prisoner’s Dilemma introduced in Example 2.6. To define the outcome of the game we use a discounted sum with discount factor $\lambda < 1$. 


We take a look at two strategies: (i) a simple strategy $\sigma$ that always denies and (ii) a ‘tit-for-tat’ strategy $\sigma'$ which mirrors the last move of the opponent. We can define these two strategies as follows. Both strategies use no epistemic states $E_p := \{\ast\}$ and as observations $B_p := \{c, d\}$ the last action of the opponent.

$$\sigma(\ast)(x) := d \quad \text{and} \quad \sigma'(\ast)(x) := x.$$ 

The pair $\langle \sigma, \sigma \rangle$ of simple strategies is a Nash equilibrium with outcome $\langle 0, 0 \rangle$, since every change of one strategy results in a negative outcome for that player. The equilibrium is subgame perfect, as both the strategies and the game do not depend on the history of the play.

The pair $\langle \sigma', \sigma' \rangle$ of ‘tit-for-tat’ strategies is also a Nash equilibrium. Its outcome is $\langle 2/(1-\lambda), 2/(1-\lambda) \rangle$. This time the equilibrium is not subgame perfect. Consider the 1-modification $\langle \sigma'_c, \sigma'_d \rangle$ of $\langle \sigma', \sigma' \rangle$ where in the first turn, player 1 plays $c$ while player 2 plays $d$. This leads to the play $(c,d)(d,c)(c,d)(d,c)\ldots$ with outcome $\frac{2-\lambda}{1-\lambda}$ (for the first player). If, instead of $\sigma'_d$, the second player chooses the strategy of always playing $d$, we obtain the play $(c,d)(d,d)(d,d)(d,d)\ldots$ with outcome $2$. Since $0 < \lambda < 1$, this is larger than $\frac{2-\lambda}{1-\lambda}$.

2.9. **Summary.** Summing up the preceding sections, we have seen that we can specify a game by the following data:

- a set $N$ of players,
- for each $p \in N$, a set $A_p$ of actions for player $p$,
- for each $p \in N$, a set $B_p$ of observations for player $p$,
- a set $S$ of states of the game,
- a set $R$ of results,
- a set $O$ of output values,
- a set $U$ of outcomes,
- a function $\gamma : S \to \mathbb{C}(R + S \times O)^{\prod_{p \in N} A_p}$ computing a single step of the game,
- for each $p \in N$, a function $\beta_p : O \times \prod_{p \in N} A_p \to B_p$ computing the observations of player $p$, and
- two functions $\varrho : \Omega \to U$ and $\tau : \Gamma(U) \to U$ that satisfy
  $$\varrho = \tau \circ \Gamma(\varrho) \circ \omega$$
  and thus compute the outcome of a play.

3. **Examples**

In this section we present the formulations of two basic games in our framework. The first game is one with imperfect information as imperfect monitoring. The second game is the one with incomplete information [23].

The usual approach in game theory is to reduce incomplete information to imperfect one. Incomplete information denotes situations where the type of agents is not known while imperfect information denotes situations where the state of the game is not known. In our framework this differentiation is not important since both kinds of information deficiencies are captured by unobservable state spaces.
3.1. Imperfect Public Monitoring. The imperfect information game with imperfect monitoring and a noisy signal considers games where the agents’ actions may not be directly observable. The state of the game is driven by a probabilistic state transition and may be either “good” or “bad”. This information is publicly observed by the agents, i.e. all players observe the same signal. The payoff is a function of this public outcome.

Again we have two players with two actions each: $N = \{1, 2\}$ and $A_p = \{c, d\}$. There are no results $R = \emptyset$ since the game never ends. The output values are $O = \mathbb{R} \times \mathbb{R} \times Y$ with $Y := \{G, B\}$ that encode the payoffs in the stage games and the public signal. The game has a single state $S = \{\ast\}$.

The game for the probabilistic functor $C_{\gamma} = C_{\text{prob}}$, is the function

$$\gamma : S \rightarrow C_{\text{prob}}(S \times O)^{A_1 \times A_2}$$

$$(*, a_1, a_2) \mapsto (*, (r_1, r_2, y))$$

where

$$r_p = \begin{cases} 1 + \frac{2 - 2k}{k - m} & \text{if } (a_p, y) = (c, G) \\ 1 - \frac{2k}{k - m} & \text{if } (a_p, y) = (c, B) \\ \frac{2m - 2n}{m - n} & \text{if } (a_p, y) = (d, G) \\ -\frac{2n}{m - n} & \text{if } (a_p, y) = (d, B) \end{cases}$$

$$y = \begin{cases} G \text{ with probability } k & \text{if } (a_1, a_2) = (c, c) \\ G \text{ with probability } m & \text{if } (a_1, a_2) = (c, d) \lor (a_1, a_2) = (d, c) \\ G \text{ with probability } n & \text{if } (a_1, a_2) = (d, d) \\ B \text{ with probability } 1 - k & \text{if } (a_1, a_2) = (c, c) \\ B \text{ with probability } 1 - m & \text{if } (a_1, a_2) = (c, d) \lor (a_1, a_2) = (d, c) \\ B \text{ with probability } 1 - n & \text{if } (a_1, a_2) = (d, d) \end{cases}$$

The probabilities of the state transition are characterized by the parameters $k > m > n$. The parameters are chosen so that the expected value of the payoffs is given by the prisoner’s dilemma matrix:

|     | $c$   | $d$   |
|-----|-------|-------|
| $c$ | (1, 1)| (-1, 2)|
| $d$ | (2, -1)| (0, 0)|

The game has imperfect information so that the epistemic state of the players is not the state of the game. Each player knows the history of his actions and the history of the public signals

$$E_p = (A_p \times Y)^*.$$ 

The observation function is

$$\beta_p : O \times A_1 \times A_2 \rightarrow B_p$$

$$((r_1, r_2, y), a_1, a_2) \mapsto (r_p, y, a_p).$$
We consider deterministic strategies with choice functor \( C_\sigma = C_{\text{det}} \). An example of always (unconditionally) playing \( d \) for a player \( p \) is given by

\[
\sigma_1 : E_1 \to (E_1 \times A_1)^{B_1}, \\
(h, (r_1, y, a_1)) \mapsto (h(a_1, y), d)
\]

where \( h(a_p, y) \) denotes that the history \( h \) of the epistemic state is extended by the action \( a_p \) and the public signal \( y \).

### 3.2. Incomplete Information.

In a Bayesian game of incomplete information types of agents are not common knowledge. In the simplest case we take types of agents to be represented as different payoff functions of the game and each agent knows his own type but not the one of his opponent.

In the following example we define that the game is played only once. However, our framework is rich enough to easily extend the game to be played finitely, infinitely or potentially infinitely often. The types of agents can be drawn repeatedly or as in the following example only once. The agents can use Bayesian updating or in fact any kind of a learning rule.

We define a Bayesian game of four \( 2 \times 2 \) games: Matching Pennies (MP), Prisoner’s Dilemma (PD), Coordination Game (CG) and Battle of the Sexes (BS) with equivalence classes \( I_{i,j} \) for players \( i \) and types \( j \). Player 1, if of type 1, knows that the payoff is either MP or PD and if of type 2, that the payoff is either CG or BS. Player 2, if of type 1, knows that the payoff is either MP or CG and if of type 2, that the payoff is either PD or BS. The probabilities are given by \( p_{\text{MP}} = 0.3, p_{\text{PD}} = 0.1, p_{\text{CG}} = 0.2 \) and \( p_{\text{BS}} = 0.4 \).

The state space of the game is \( S = \{\ast, \text{MP, PD, CG, BS}\} \). The output space of the game is \( O = S \) and the result space is \( R = \mathbb{R} \times \mathbb{R} \). The action spaces are \( A_1 = \{U, D\} \) and \( A_2 = \{L, R\} \) and the epistemic state spaces have a single state \( E_1 = E_2 = \{\ast\} \). We formalize this game in two rounds.
(1) In the first round the game is in state $*$ and nature realizes the types of the players, the actions of the players are irrelevant.

$$\gamma : S \rightarrow \mathbb{C}_{\text{prob}}(R + S \times O)^{A_1 \times A_2}$$

$$(*, a_1, a_2) \mapsto \begin{cases} (MP, MP) & \text{with } p_{MP} = 0.3 \\ (PD, PD) & \text{with } p_{PD} = 0.1 \\ (CG, CG) & \text{with } p_{CG} = 0.2 \\ (BS, BS) & \text{with } p_{BS} = 0.4 \end{cases}$$

In the second round the game yields a result.

$$(MP, a_1, a_2) \mapsto \begin{cases} (2, 0) & \text{if } a_1 = U, a_2 = L \\ (0, 2) & \text{if } a_1 = U, a_2 = R \\ (0, 2) & \text{if } a_1 = D, a_2 = L \\ (2, 0) & \text{if } a_1 = D, a_2 = R \end{cases}$$

An analogous definition has to be given for the other type realizations $PD, CG, BS$.

(2) The observation function of the first player is

$$\beta_1 : O \times A_1 \times A_2 \rightarrow B_1$$

$$(o, a_1, a_2) \mapsto \begin{cases} * & \text{if } o = * \\ \{MP, PD\} & \text{if } o = MP \lor o = PD \\ \{CG, BS\} & \text{if } o = CG \lor o = BS \end{cases}$$

The observation function of player 2 is defined analogously.

(3) The strategies for both players are given by

$$\sigma_p : \{*\} \rightarrow \mathbb{C}(\{*\} \times A_p)^{B_p}$$

For example, the strategy of player 1 who plays $U$ if he is of type 1 and $D$ if he is of type 2 is given by

$$(*, b_1) \mapsto \begin{cases} U & \text{if } b_1 = \{MP, PD\} \\ D & \text{if } b_1 = \{CG, BS\} \end{cases}$$

In the first round when nature chooses the types, the strategy of the players is irrelevant since it does not matter in the game function.

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