A SHORT EXPOSITION OF S. PARSA’S THEOREMS ON INTRINSIC LINKING AND NON-REALIZABILITY

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Abstract. We present a short exposition of the following results by S. Parsa.

Let \( L \) be a graph such that the join \( L \ast \{1, 2, 3\} \) (i.e. the union of three cones over \( L \) along their common bases) piecewise linearly (PL) embeds into \( \mathbb{R}^4 \). Then \( L \) admits a PL embedding into \( \mathbb{R}^3 \) such that any two disjoint cycles have zero linking number.

There is \( C \) such that every 2-dimensional simplicial complex having \( n \) vertices and embeddable into \( \mathbb{R}^4 \) contains less than \( Cn^{8/3} \) simplices of dimension 2.

We also present the analogue of the second result for intrinsic linking.

This paper provides short proofs of Theorems 1, 4 and 5 below. Let \([k] := \{1, \ldots, k\}\).

Theorem 1 (see Remark 3). Let \( L \) be a graph such that the join \( L \ast [3] \) (i.e. the union of three cones over \( L \) along their common bases) piecewise linearly (PL) embeds into \( \mathbb{R}^4 \). Then \( L \) admits a PL embedding into \( \mathbb{R}^3 \) such that any two disjoint cycles have zero linking number.

Proof. Consider \( L \ast [3] \) as a subcomplex of some triangulation of \( \mathbb{R}^4 \). Then there is a small general position 4-dimensional PL ball \( \Delta^4 \) containing the point \( \emptyset \ast 1 \in \mathbb{R}^4 \). Hence the intersection \( \partial \Delta^4 \cap (L \ast [3]) \) is PL homeomorphic to \( L \). Let us prove that this very embedding of \( L \) into the 3-dimensional sphere \( \partial \Delta^4 \) satisfies the required property.

Take any two disjoint oriented closed polygonal lines \( \beta, \gamma \subset \partial \Delta^4 \cap (L \ast [3]) \cong L \). Then \( (\beta \ast \{1, 2\}) - \text{Int} \Delta^4 \) and \( (\gamma \ast \{1, 3\}) - \text{Int} \Delta^4 \) are two disjoint 2-dimensional PL disks in \( \mathbb{R}^4 - \Delta^4 \) whose boundaries are \( \beta \) and \( \gamma \). Hence \( \beta \) and \( \gamma \) have zero linking number in the 3-dimensional sphere \( \partial \Delta^4 \) (by the following well-known lemma applied to the 4-dimensional ball \( S^4 - \text{Int} \Delta^4 \)).

Lemma 2. If two disjoint oriented closed polygonal lines in the 3-dimensional sphere \( \partial D^4 \) bound two disjoint 2-dimensional PL disks in the 4-dimensional ball \( D^4 \), then the polygonal lines have zero integer linking number in \( \partial D^4 \).

Proof. Denote by \( \beta, \gamma \subset D^4 \) the two disjoint oriented disks bounded by the polygonal lines \( \beta, \gamma \subset \partial D^4 \). Denote by \( \beta' \subset \mathbb{R}^4 - D^4 \) an oriented disk (e.g. a cone) bounded by \( \beta \). Denote by \( \gamma' \subset \partial D^4 \) a general position oriented singular disk (e.g. singular cone) bounded by \( \gamma \). We have \( \beta \cap \gamma' = (\beta \cup \beta') \cap (\gamma \cup \gamma') \). Denote by the same
letters integer chains carried by $\beta, \gamma, B, \Gamma$. Denote by $\cdot_M$ the algebraic intersection of integer chains in $M$. Then by general position the linking number of $\beta$ and $\gamma$ is $\beta \cdot_{D^4} \Gamma' = (B - B') \cdot_{R^4} (\Gamma - \Gamma') = 0$. \hfill $\Box$

**Remark 3.**
(a) Theorem 1 trivially generalizes to a $d$-dimensional finite simplicial complex $L$ and embeddings $L \ast [3] \to \mathbb{R}^{2d+2}$, $L \to \mathbb{R}^{2d+1}$. For $d \neq 2$ and a $d$-complex $L$ embeddability of $L \ast [3]$ into $\mathbb{R}^{2d+2}$ even implies embeddability of $L$ into $\mathbb{R}^{2d}$. For the case $d = 1$ considered in Theorem 1 this improvement follows from a theorem of Grünbaum [Gr69] (whose proof is more complicated). For the case $d \geq 3$ this improvement is proved in [MS06, (iv) $\Rightarrow$ (i) of Corollary 4.4], [Pa20a], [PS20].

(b) Theorem 1 is formally a corollary of [Pa15, Theorem 1] but is essentially a restatement of [Pa15, Theorem 1] accessible to non-specialists. In spite of being much shorter, the above proof of Theorem 1 is not an alternative proof comparatively to [Pa15, §3] but is just a different exposition avoiding sophisticated language. The above proof of Theorem 1 is analogous to [Sk03, Example 2] where a relation between intrinsic linking in 3-space and non-realizability in 4-space was found and used. Although the proof is simple, it easily generalizes to non-trivial results like a simple solution of the Menger 1929 conjecture and its generalizations [Sk03, Example 2, Lemmas 2 and 1'], see survey [Sk14].

The following Theorem 4.a is a higher-dimensional generalization of upper estimation on the number of edges in a planar graph.

An embedding of a simplicial complex into $\mathbb{R}^{2d+1}$ is called **linkless** if the images of any two $d$-dimensional spheres have zero linking number.

**Theorem 4.** (S. Parsa) (a) For every $d$ there is $C$ such that for every $n$ every $d$-dimensional simplicial complex having $n$ vertices and embeddable into $\mathbb{R}^{2d}$ contains less than $Cn^{d+1-3^{d-3}}$ simplices of dimension $d$.

(b) For every $d$ there is $C$ such that for every $n$ every $d$-dimensional simplicial complex having $n$ vertices and linklessly embeddable into $\mathbb{R}^{2d+1}$ contains less than $Cn^{d+1-4^{d-4}}$ simplices of dimension $d$.

The result (a) improves analogous result with $Cn^{d+1-3^{d-3}}$ [De93] and is covered by the Grünbaum-Kalai-Sarkaria conjecture (whose proof is announced in [Ad18]; I did not check that proof). See [Pa15, Theorems 3 and 4] and [Pa20].

The Flores 1934 Theorem states that the $d+1$ join power $[3]^{(d+1)} = [3] \ast \ldots \ast [3]$ (d+1 copies of [3]) is not (PL or topologically) embeddable into $\mathbb{R}^{2d}$ [Ma03, §5]. (We have $[3]^{*2} = K_{3,3}$, so the case $d = 1$ is even more classical.) The $d+1$ join power $[4]^{(d+1)}$ is not linklessly embeddable into $\mathbb{R}^{2d+1}$ [Sk03, Lemma 1]. (We have $[4]^{*2} = K_{4,4}$, so the case $d = 1$ is due to Sachs.) Theorem 4 is implied by these results and the following theorem.

**Theorem 5.** For every $d, r$ there is $C$ such that for every $n$ every $d$-dimensional simplicial complex having $n$ vertices and not containing a subcomplex homeomorphic to $[r]^{*(d+1)}$ contains less than $Cn^{d+1-r^{d-1}}$ simplices of dimension $d$.

Proof of Theorem 5 is based on the following lemma similar to the estimation of the number of edges in a graph not containing $K_{s+1,a}$ (Kovari-Sós-Turán Theorem).
Lemma 6. For every integers \( r, m, a, s \) and subsets \( S_1, \ldots, S_m \subset [a] \) every whose \( r \)-tuple intersection contains at most \( s \) elements we have

\[
|S_1| + \ldots + |S_m| \leq r(ma^{1-1/r}s^{1/r} + a).
\]

The case \( r = 3 \) of Theorem 5 and of Lemma 6 is essentially proved in \[Pa15\] §3, see [OC]. The case of arbitrary \( r \) is analogous.

Proof of Lemma 6. Denote by \( d_q \) the number of subsets among \( S_1, \ldots, S_m \) containing element \( q \in [a] \). We may assume that there is \( \nu \leq a \) such that \( d_q \geq r \) when \( q \leq \nu \) and \( d_q < r \) when \( q > \nu \). Then the required inequality follows by

\[
\sum_{j=1}^m |S_j| = \sum_{q=1}^a d_q + \sum_{q=1}^\nu d_q \quad \text{and}
\]

\[
\left( \sum_{q=1}^\nu d_q \right)^r \leq \nu^{r-1} \sum_{q=1}^\nu d_q^{r-1} \leq r^r \nu^{r-1} \sum_{q=1}^\nu \left( \frac{d_q}{r} \right)^r \leq r^r \nu^{r-1} \sum_{q=1}^a \left( \frac{d_q}{r} \right)^r \equiv (3)
\]

\[
\equiv r^r a^{r-1} \sum_{\{j_1, \ldots, j_r\}} |S_{j_1} \cap \ldots \cap S_{j_r}| \leq r^r a^{r-1} s \left( \frac{m}{r} \right) < r^r m^r a^{r-1} s.
\]

Here

- the inequality (1) is the inequality between the arithmetic mean and the degree \( r \) mean;
- since \( d_q \geq r \) when \( q \leq \nu \), the inequality (2) follows by

\[
r^r \left( \frac{d_q}{r} \right)^r = \frac{r^rd_q^r}{r!} \left( 1 - \frac{1}{d_q} \right) \left( 1 - \frac{2}{d_q} \right) \ldots \left( 1 - \frac{r-1}{d_q} \right) \geq \frac{r^rd_q^r}{r!} \frac{r-1}{r} \frac{r-2}{r} \ldots \frac{1}{r} = d_q^r;
\]

- the (in)equalities (3) and (4) are obtained by double counting the number of pairs \( (\{j_1, \ldots, j_r\}, q) \) of an \( r \)-element subset of \([m]\) and \( q \in S_{j_1} \cap \ldots \cap S_{j_r} \). \( \square \)

Proof of Theorem 5. Induction on \( d \). The base \( d = 1 \) follows because if a graph on \( n \) vertices does not contain a subgraph homeomorphic to \( K_{r,r} \), then the graph does not contain a subgraph homeomorphic to \( K_{2r}^d \) and hence has \( O(n) \) vertices \[BT98\] (this was apparently proved in the paper \[Ma68\] which is not easily available to me).

Let us prove the inductive step. If a \( d \)-dimensional simplicial complex \( K \) having \( n \) vertices does not contain a subcomplex homeomorphic to \([r]^{s(d+1)}\), then any \( r \)-tuple intersection of the links of vertices from \( K \) does not contain a subcomplex homeomorphic to \([r]^{3d} \). Apply Lemma 6 to the set of \( a = \binom{n}{d} < n^d \) simplices of \( K \) having dimension \( d - 1 \), and to \( m = n \) subsets defined by links of the vertices. By the inductive hypothesis \( s \leq Cn^{d-\nu^2-d} \). Hence the number of \( d \)-simplices of \( K \) does not exceed \( rmn^{(r-1)d/r} \left( Cn^{d-\nu^2-d} \right)^{1/r} = C'n^{d+1-r^{1-d}} \). \( \square \)

The following version of Lemma 6 is also possibly known. It was (re)invented by I. Mitrofanov and the author in discussions of the \( r \)-fold Khintchine recurrence theorem, see [OC] Problem 5.
Lemma 7. For every integers \( r, m, a \) and subsets \( S_1, \ldots, S_m \subset [a] \) we have

\[
a^{r-1} \sum_{j_1, \ldots, j_r=1}^m |S_{j_1} \cap \ldots \cap S_{j_r}| \geq \left( \sum_{j=1}^m |S_j| \right)^r.
\]

Proof. Consider the decomposition of \([a]\) by the sets \( S_j \) and their complements. The sets of this decomposition correspond to subsets of \([m]\). Denote by \( \mu_A \) the number of elements in the set of this decomposition corresponding to a subset \( A \subset [m] \).

To every pair \((A, j)\) of a subset \( A \subset [m] \) and a number \( j \in [m] \) assign 0 if \( j \notin A \) and assign \( \mu_A \) if \( j \in A \). Let us double count the sum \( \Sigma \) of the obtained \( 2^m \cdot m \) numbers. We obtain

\[
\sum_{j=1}^m |S_j| = \Sigma = \sum_{A \subset [m]} |A| \mu_A.
\]

To every pair \((A, (j_1, \ldots, j_r))\) of a subset \( A \subset [m] \) and a vector \((j_1, \ldots, j_r) \in [m]^r\) assign 0 if \( \{j_1, \ldots, j_r\} \notin A \) and assign \( \mu_A \) if \( \{j_1, \ldots, j_r\} \subset A \). Let us double count the sum \( \Sigma_r \) of the obtained \( 2^m \cdot m^r \) numbers. We obtain

\[
\sum_{j_1, \ldots, j_r=1}^m |S_{j_1} \cap \ldots \cap S_{j_r}| = \Sigma_r = \sum_{A \subset [m]} |A|^r \mu_A.
\]

Hence by the inequality between the weighted arithmetic mean and the weighted degree \( r \) mean, and using \( \sum_{A \subset [m]} \mu_A = a \), we obtain

\[
a^{r-1} \Sigma_r \geq \left( \sum_{A \subset [m]} |A| \mu_A \right)^r = \left( \sum_{j=1}^n |S_j| \right)^r.
\]

□

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