The Schmidt number as a universal entanglement measure

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Received 3 February 2011
Accepted for publication 3 February 2011
Published 8 March 2011
Online at stacks.iop.org/PhysScr/83/045002

Abstract

The class of local invertible operations is defined, and the invariance of entanglement under such operations is established. For the quantification of entanglement, universal entanglement measures are defined, which are invariant under local invertible transformations. They quantify entanglement in a very general sense. It is shown that the Schmidt number is a universal entanglement measure, which is most important for the general amount of entanglement. For special applications, pseudo-measures are defined to quantify the entanglement useful for a certain quantum task. The entanglement quantification is further specified by operational measures, which include the observables accessible by a given experimental setup.

PACS numbers: 03.67.Mn, 03.65.Ud, 42.50.Dv

1. Introduction

Entanglement is a key resource in the vast fields of quantum information processing, quantum computation and quantum technology; for an introduction, see e.g. [1, 2]. For example, applications of entangled states include those in quantum key distribution [3], quantum dense coding [4] and quantum teleportation [5]. Thus both the identification and quantification of entanglement have a major role in future applications [6].

The phenomenon of entanglement is closely related to the superposition principle of quantum mechanics. A pure separable state is represented by the product of states for both systems. A general pure state is a superposition of factorizable states. The number of superpositions of factorizable states is given by the Schmidt rank [1]. A separable mixed quantum state is a convex combination of pure factorizable quantum states [7]. The generalization of the Schmidt rank to mixed quantum states delivers the Schmidt number. This generalization and introduction of Schmidt number witnesses are given in [8–10]. The Schmidt number of a mixed quantum state fulfils the axioms of an entanglement measure, cf [11–13]. More precisely, it is a convex roof measure as defined in [14, 15].

Since the amount of entanglement cannot increase under local operations and classical communication, all entanglement measures must satisfy the local operations and classical communication (LOCC) paradigm. But, in general, different entanglement measures do not deliver the same ordering of entangled quantum states [16–18]. It is important to note that for a given quantum task, an adequate definition of the corresponding LOCC plays a crucial role in entanglement quantification.

Maximally entangled states are usually considered to have the highest amount of entanglement. It turns out that for different quantum tasks different entangled quantum states are beneficial [19]. The experimentally demonstrated possibility of noise-free linear amplification makes a larger class of separable operations accessible [20]. Recently, it has been shown that these separable operations can increase entanglement with respect to certain measures [21]. A subclass of separable operations—the so-called local filter operations—play a crucial role in the universal entanglement distillation protocol [22]. This necessitates a careful consideration of local invertible transformations in the context of entanglement measures. It is also important that all entangled quantum states have some usable amount of entanglement, nonlocality and potential applications for quantum processing [23–25]. The inclusion of all these aspects requires a critical study of the entanglement quantification.

In this paper, we study entanglement measures and in particular the Schmidt number. We discuss the notion of maximally entangled states in connection with an arbitrary entanglement measure. We conclude that in the most general sense the available amount of entanglement depends on
the Schmidt number. We also define pseudo-measures and operational measures, which quantify the entanglement for a specific quantum task and for a special experimental setup, respectively.

The paper is structured as follows. In section 2, we discuss general entanglement measures and their properties in relation to one special entanglement measure—the Schmidt number. The definition of entanglement pseudo-measures and their application to an arbitrary experimental situation are given in section 3. A summary and some conclusions are given in section 4.

2. Entanglement measures

In this section, we consider special properties of entanglement measures in connection with LOCC. Starting with the given definitions of entanglement measures, we obtain some properties that indicate a fundamental role of the Schmidt number. Here and in the following, we assume finite but arbitrary dimensional Hilbert spaces \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). The generalization for continuous variable entanglement directly follows from the method of finite spaces as presented in [26].

2.1. The LOCC paradigm and entanglement measures

Entanglement measures are usually defined by using LOCC. Therefore, an accurate definition of these operations is essential for the understanding of entanglement measures. The general idea of a LOCC is a map that cannot create entanglement. Therefore, let us write the most general form of such a map \( \Lambda \), cf [2, 13, 27], the separable operations \( \Lambda \in \mathcal{C}_{\text{sep}} \):

\[
\Lambda(\rho) = \sum_i [A_i \otimes B_i]_i \rho [A_i \otimes B_i]^\dagger.
\]  

(1)

The quantum state will be normalized by

\[
\rho \mapsto \rho' = \frac{\Lambda(\rho)}{\text{tr} \Lambda(\rho)}.
\]  

(2)

The operations in the set \( \mathcal{C}_{\text{sep}} \) are also called stochastic separable operations. An important subset is \( \mathcal{C}_{\text{LI}} \), which denotes all local unitaries \([U_1 \otimes U_2]_i \rho[U_1 \otimes U_2]^\dagger\).

A more general substructure \( \mathcal{C}_\chi \) of \( \mathcal{C}_{\text{sep}} \) is defined in the following way. We have a set of operations \( \chi \subseteq \mathcal{C}_{\text{sep}} \). The substructure is defined by all \( \Lambda \in \chi \) and all compositions of elements of \( \chi \):

\[
\Lambda_1, \Lambda_2 \in \chi \Rightarrow \Lambda_1 \circ \Lambda_2 \in \chi.
\]  

(3)

This is the algebraic structure of a semi-group. It is usually considered that this substructure must at least include local changes of the basis by local unitaries \( \mathcal{C}_{\text{LU}} \). Here and in the following, we will call such a substructure, LOCC \( \mathcal{C}_\chi \). These particular LOCC are the applied operations for a special quantum task, for example, quantum key distribution [3] or quantum dense coding [4].

The usually considered class of LOCC operations is given by two-way classical communications, cf [2], for quantum teleportation [5]. These operations define a substructure of stochastic separable operations as given in equation (3). However, for other processes such as distillation protocols the whole set of stochastic separable operations are applied [22, 27].

For further studies, we may consider operations of the form \( A \otimes B \), the so-called local filtering operations [2, 22], with

\[
\Lambda(a|a) \otimes |b\rangle\langle b| = \Lambda_A(a)a \otimes \Lambda_B(|b\rangle\langle b|) = A(a|a)A^\dagger|B\rangle\langle B|B^\dagger. 
\]  

(4)

Note that the problem of normalization, \( A^\dagger A \otimes B^\dagger B \leq I \otimes I \), is discussed in [18]. In addition to such substructures, we define the following LOCC:

\[
\mathcal{C}_{\text{LI}}, \text{local invariables.} \text{ Operations of the form}
\]

\[
\rho \mapsto [T_1 \otimes T_2]_i \rho [T_1 \otimes T_2]^\dagger,
\]  

(5)

with \( T_1 \) and \( T_2 \) invertible, are called local ininvertible. Note that for any operation there exists an inverse operation given by \( T_1^{-1} \) and \( T_2^{-1} \). Further on, the LOCC \( \mathcal{C}_{\text{LI}} \) include the identity, \( I_1 \otimes I_2 \). Thus, \( \mathcal{C}_{\text{LI}} \) is a group. A subgroup of these operations is \( \mathcal{C}_{\text{LU}} \).

\[
\mathcal{C}_{\text{LP}}, \text{local projections.} \text{ In addition to all local unitaries this set contains local projections given by}
\]

\[
\rho \mapsto [P_1 \otimes P_2]_i \rho [P_1 \otimes P_2]^\dagger,
\]  

(6)

with \( P_1 \) and \( P_2 \) being projection operators. Recently, we used these operators to show that continuous variable entanglement can always be identified in finite dimensions [26].

Now we can define a general entanglement measure \( E \), which must fulfill the following definition; see [6].

**Definition 1.** \( E \) is an entanglement measure, if

(i) \( \sigma \) separable \( \Leftrightarrow E(\sigma) = 0 \),

(ii) \( \forall \Lambda \in \chi: E(\rho) \geq E \left( \frac{\Lambda(\rho)}{\text{tr} \Lambda(\rho)} \right) \).

Usually, a third condition is that an entanglement measure must be invariant under local unitaries. However, condition (ii), implies that the additional invariance under local unitaries is superfluous, see the appendix. Note that, instead of condition (ii) often a non-increasing behavior on average of the entanglement measure is considered, cf our related comments in section 2.6.

Last but not least, definition 1 depends on the chosen LOCC. Thus the precise mathematical definition of the LOCC \( \mathcal{C}_\chi \)—as given above—is crucial for the measure itself. The physical interpretation of LOCC is given in section 3. We discussed above that different quantum tasks use different sets of LOCC. For example, quantum teleportation uses deterministic two-way classical communication, and distillation uses all stochastic separable operations.

2.2. The Schmidt number

A well-known example of an entanglement measure is the Schmidt number \( r_S \). Let us consider the pure state \(|\Psi\rangle\) with
a Schmidt decomposition [11]

\[ |\Psi\rangle = \sum_{n=1}^{r(\Psi)} \lambda_n |e_n, f_n\rangle, \tag{7} \]

with the Schmidt rank \( r(\Psi) \), the Schmidt coefficients \( \lambda_k > 0 \) and \( \{|e_k\}_{k=1,...,r}, \{|f_k\}_{k=1,...,r} \) being orthonormal in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively. Any mixed quantum state \( \rho \) is a convex combination of pure states:

\[ \rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|, \tag{8} \]

and each vector \( |\psi_k\rangle \) of this decomposition has an individual Schmidt rank. For this distinct decomposition the Schmidt rank of \( \rho \) is given by the maximal Schmidt rank of all vectors. The Schmidt number of the mixed quantum state is given by the minimal Schmidt rank of \( \rho \) for all possible decompositions:

\[ rs(\rho) = \inf \left\{ r_{\max}: \rho = \sum_k p_k |\psi_k\rangle \langle \psi_k| \right\} \]

\[ \text{and } r_{\max} = \sup r(\psi_k) \]. \tag{9} \]

An overview of the Schmidt number as an entanglement measure can be found in [2].

For a pure state, the Schmidt rank is identical to the Schmidt number, and it counts the minimal number of superpositions of factorizable states needed for generating the state under study. Therefore, the interpretation in quantum physics is the quantum superposition principle. In compound Hilbert spaces, this was shown to be an unexpected property in relation to classical physics. In the following, we will throughout use the notion Schmidt number. The definition of the Schmidt number implies that the suitable LOCC in definition 1 is the complete set \( \mathcal{C}_{\text{sep}} \). For separable quantum states, the Schmidt number is equal to 1. The simple shift, \( rs \rightarrow rs - 1 \), delivers condition (i) in the definition for entanglement measures, with \( E = rs - 1 \).

In the following, let us consider some basic properties of arbitrary entanglement measures and their connection with the Schmidt number. It will become clear that the Schmidt coefficients \( \lambda_n \) as given in equation (7) play a minor role compared with the Schmidt number in the quantification of entanglement. The Schmidt number delivers a discrete and monotonic quantification of quantum states with respect to entanglement.

2.3. Entanglement invariance under local invertibles

Before quantifying entanglement, it is important to detect the entanglement of a given quantum state. This can be done by optimized entanglement witnesses [28–30]. These are Hermitian operators \( W \) with a non-negative expectation value for separable quantum states. However, entangled states can have negative expectation values.

The importance of the above-defined LOCC subgroup \( \mathcal{C}_{LI} \) of local invertible operations follows from its influence on the property of entanglement itself. More precisely, a general LOCC operation can eliminate the entanglement of a quantum state. However, a general mixed quantum state is entangled, iff it is entangled under local invertibles.

**Theorem 1. Entanglement under \( \mathcal{C}_{LI} \).**

(i) If the operator \( W \) is an optimized entanglement witness, then \( (T_1 \otimes T_2)^\dagger W(T_1 \otimes T_2) \) is an optimized entanglement witness, with \( T_1 \otimes T_2 \) invertible.

(ii) A quantum state

\[ \rho' = \frac{(T_1 \otimes T_2)^\dagger \rho (T_1 \otimes T_2)}{\text{tr}[(T_1 \otimes T_2)^\dagger \rho (T_1 \otimes T_2)]]} \]

is entangled, iff \( \rho \) is entangled, with \( T_1 \otimes T_2 \) invertible.

The proof is given in the appendix.

These findings of theorem 1 are known [31, 32], but in the following we consider these results in the frame of entanglement measures. This theorem proves that the entanglement of a quantum state is preserved under local operations that are invertible, \( \mathcal{C}_{LI} \). It also delivers a method to construct an equivalent class of entanglement witnesses detecting the entanglement of arbitrary quantum states. This can be done in a form that two quantum states, \( \rho \) and \( \rho' \), share the same kind of (e.g. bound or free) entanglement, \( \rho \cong \rho' \) [18], if they can be transformed into each other by a local invertible transformation

\[ \rho' = \frac{\Lambda_{LI}(\rho)}{\text{tr} \Lambda_{LI}(\rho)} \]. \tag{10} \]

with \( \Lambda_{LI} \in \mathcal{C}_{LI} \). From theorem 1 it follows that we should carefully study the quantification of entanglement with respect to \( \mathcal{C}_{LI} \), since this subgroup is of fundamental importance for the property of entanglement itself.

2.4. Maximally entangled states and universal entanglement measures

Let \( E \) be an arbitrary entanglement measure, and \( \rho_{\text{max}} \) is maximally entangled:

\[ \forall \rho: E(\rho_{\text{max}}) \geq E(\rho). \tag{11} \]

Now let us define a new entanglement measure

\[ E'(\rho) \overset{\text{def}}{=} E \left( \frac{\Lambda_{LI}(\rho)}{\text{tr} \Lambda_{LI}(\rho)} \right) \]. \tag{12} \]

with a local invertible operation \( \Lambda_{LI} \). If \( E \) is defined by the LOCC \( \mathcal{C}_X \), then \( E' \) is defined by \( \mathcal{C}_X' \). An element \( \Lambda' \) of \( \mathcal{C}_X' \) has the form

\[ \Lambda' = \Lambda_{LI}^{-1} \circ \Lambda \circ \Lambda_{LI}, \quad \text{with } \Lambda \in \mathcal{C}_X. \tag{13} \]

Obviously \( E' \) is an entanglement measure for \( \mathcal{C}_X' \); for details see the appendix. The state \( \rho'_{\text{max}} \) satisfies equation (11) for the measure \( E' \):

\[ \rho'_{\text{max}} = \frac{\Lambda_{LI}^{-1}(\rho_{\text{max}})}{\text{tr} \Lambda_{LI}^{-1}(\rho_{\text{max}})} \]. \tag{14} \]

However, the initial state \( \rho_{\text{max}} \) is not necessarily maximally entangled for \( E' \).
2.6 If the LOCC delivers a pure entangled state with the Schmidt number \( r \), then the entanglement measure is a universal entanglement measure. However, the resulting measure \( E \) does not depend on the Schmidt coefficients.

It follows that all similar states \( \rho = \rho' \), cf equation (10), have the same amount of entanglement. From the physical point of view it is clear that the number of nonlocal superpositions (the Schmidt number) is a good characterization of entanglement. Since local invertible transformations conserve the Schmidt number, the latter itself may serve as a universal entanglement measure.

In [18], it has been shown for distance-type measures that the idea, Measure = Distance = Amount of entanglement, can be misleading. Quantification of the amount of entanglement strongly depends on the choice of the distance. Moreover, the general importance of the local invertible operations becomes clear from theorem 1, proving that the property entanglement of a quantum state is unchanged under local invertibles.

2.5 Schmidt rank monotones

Let us consider entanglement measures defined by the LOCC \( C_{\text{LO}} \), which includes local projections. This is a weak requirement, since many LOCC sub-semi-groups fulfill this requirement [2]. The projection \( P \otimes I_2 \),

\[
P \otimes I_2 = \left( \sum_{k=1}^{r-1} |k\rangle \langle k| \right) \otimes I_2,
\]

maps the state \( |\phi_1\rangle \) to \( |\phi_{r-1}\rangle \) (neglecting the normalization). The LOCC condition (ii) in definition 1 delivers \( E(|\phi_1\rangle |\phi_1\rangle) \geq E(|\phi_{r-1}\rangle |\phi_{r-1}\rangle) \). In general, a pure state with a given Schmidt number contains less entanglement than a pure state with a higher Schmidt number. The next section implies the same result for mixed states.

**Observation 3.** An entanglement measure, being defined by LOCC \( C_{\text{LO}} \) including local projections, is a monotone of the Schmidt number.

This means that a sequence of quantum states with decreasing number of nonlocal superpositions cannot increase its entanglement with respect to \( E \). Thus, the entanglement of Schmidt number states, for a given \( r \), delivers upper boundaries for measure with projections and arbitrary quantum states with a Schmidt number less than \( r \).

Note that we can also use a deterministic (trace-preserving) operation instead of \( P \otimes I_2 \), see equation (19). Such an operation \( \Lambda \) can be given as

\[
\Lambda(\rho) = \left( P \otimes I_2 \right)^\dagger \left( P \otimes I_2 \right) \left( r |r\rangle \langle r| \otimes I_2 \right) \rho \left( r |r\rangle \langle r| \otimes I_2 \right).
\]

Obviously it follows that \( \Lambda(|\phi_1\rangle |\phi_1\rangle) \) is a Schmidt number \( r - 1 \) state:

\[
\Lambda(|\phi_1\rangle |\phi_1\rangle) = \frac{r-1}{r} |\phi_{r-1}\rangle \langle \phi_{r-1}| + \frac{1}{r} |r\rangle \langle r|, r \rangle.
\]

In the following, we generalize our observations to mixed quantum states.

2.6. Mixed quantum states and measuring entanglement measures

So far, we have seen properties that are defined by LOCC operations that include local invertible or local projections.
In the following, we study entanglement measures including both. Let us consider an arbitrary mixed quantum state \( \sigma_r = \sum_k p_k |\tilde{\psi}_k\rangle\langle \tilde{\psi}_k| \) with a Schmidt number \( r_k(\sigma_r) = r \). This state can be generated by an arbitrary pure state \( |\tilde{\psi}_r\rangle \)—with the Schmidt number \( r \)—and a LOCC operation \( \Lambda \in \mathcal{C}_{\text{sep}} \).

\[
\sigma_r = \Lambda(|\tilde{\psi}_r\rangle) = \sum_k (A_k \otimes B_k) |\tilde{\psi}_r\rangle (A_k \otimes B_k)^\dagger,
\]

with

\[
A_k \otimes B_k |\tilde{\psi}_r\rangle = \sqrt{p_k}|\psi_k\rangle
\]
as explained above by local invertibles and local projections together with classical mixing. Let us call the pure state \( |\tilde{\psi}_r\rangle \) the generator of \( \sigma_r \). From the property (ii) of an entanglement measure, we can conclude that for each generator \( |\tilde{\psi}_r\rangle \) of the state \( \sigma_r \), the following holds:

\[
E(|\tilde{\psi}_r\rangle \langle \tilde{\psi}_r|) \geq E(\sigma_r).
\]

Thus, we can formulate the following observation.

**Observation 4.** The generator of any mixed quantum state has an equal or a larger amount of entanglement than the generated mixed quantum state. Under all states \( \rho_{\text{max}} \) satisfying equation (11) must exist in a pure state.

This statement generalizes observations 2 and 3 to mixed quantum states.

2.7. Entanglement on average

Sometimes condition (ii) in definition 1 is replaced by a stronger condition

\[
(ii') \quad \forall \Lambda \in \mathcal{C}_X : \rho \xrightarrow{\Lambda} \sum_k p_k \rho_k \Rightarrow E(\rho) \geq \sum_k p_k E(\rho_k).
\]

This condition postulates that entanglement cannot increase on average [2, 6]. It is straightforward to show that our main observations 2–4 remain valid also for the stronger condition (ii').

First, let us consider operations of the form \( \Lambda(\rho) = [A \otimes B] \rho [A \otimes B]^\dagger \). Obviously, these operations \( \Lambda \) do not create an additional mixture:

\[
|\chi\rangle \langle \chi| = [A \otimes B]|\psi\rangle\langle \psi| [A \otimes B]^\dagger,
\]

\[
E(|\psi\rangle\langle \psi|) \geq E\left( \frac{\Lambda(|\psi\rangle\langle \psi|)}{\text{tr} \Lambda(|\psi\rangle\langle \psi|)} \right).
\]

In this case, (ii) and (ii') are equivalent. Thus, the local invertible operation \( T \) delivers the same results of ordering of quantum states as concluded in section 2.4. The universal measures are independent of the Schmidt coefficients, see observation 2 as derived from definition 2. If \( \Lambda \) has the form of a local projection \( P \), then the monotonic behavior with respect to the Schmidt number in observation 3 follows immediately as shown in section 2.5. Thus, observations 2 and 3 are also true for (ii').

Starting from the generator \( |\tilde{\psi}_r\rangle \) of the quantum state \( \sigma_r \), we obtain an additional mixture, see equation (22),

\[
\sigma_r = \sum_k p_k |\psi_k\rangle\langle \psi_k|.
\]

It follows from condition (ii) that the operation defined in equation (22) delivers

\[
E(|\tilde{\psi}_r\rangle \langle \tilde{\psi}_r|) \geq \sum_k p_k E(|\psi_k\rangle\langle \psi_k|).
\]

Therefore, the results of observation 4—the generator \( |\tilde{\psi}_r\rangle \) of the quantum state \( \sigma_r \) delivers an upper boundary of the amount of entanglement—remain valid if we assume (ii') instead of (ii). In all of our considered cases, condition (ii') delivers the same results as condition (ii).

2.8. Preliminary results

So far, we have seen that the term ‘maximally entangled’ state can be used only with respect to a given measure, which led us to definition 2 of a universal measure. We have also shown that entanglement measures there have a monotonic behavior with respect to the Schmidt number, see observation 3, and under all maximally entangled states for a given measure must exist a pure state, see observation 4. At this point the role of the Schmidt number for arbitrary mixed states as a universal entanglement measure becomes clear.

In general, entanglement measures are defined by a mathematical background. Not all used LOCC \( C_X \) operations can be performed in an experiment. Thus, the value \( E \) is not given by a direct experimental measurement. The other way round, a general experimental setup cannot use every kind of entanglement; for example, bound entangled states for a distillation protocol. It turns out that the state \( |\phi_r\rangle \)—with equally distributed Schmidt coefficients—is not the best one to perform quantum computation, see [19]. In contrast, \( |\phi_r\rangle \) is the best state for quantum teleportation of an \( r \)-qudit. An entanglement measure optimized for a special experimental setup needs to be found, and we need to find the suitable set of LOCC \( C_X \) for this specific experiment. This measure should quantify the usable amount of entanglement of a quantum state.

Examples of such tasks are the distillation protocols. In this context, a state that cannot be distilled has the same usable amount of entanglement as a separable quantum state. This observation leads us to a generalization of the concept of entanglement measures, which will be discussed in the following.

3. Pseudo-measures and operational measures

The above discussions of entanglement measures in the context of a certain task lead to a generalization of entanglement measures. Condition (i) in definition 1 can be relaxed to define pseudo-measures.

**Definition 3.** The non-negative function \( E \) is a pseudo-measure of the LOCC \( C_X \), if

\[
(i') \quad \sigma \text{ separable } \Rightarrow E(\sigma) = 0,
\]

\[
(ii) \quad \forall \Lambda \in C_X : E(\rho) \geq E\left( \frac{\Lambda(\rho)}{\text{tr} \Lambda(\rho)} \right).
\]
The new condition (i’) implies that even an entangled state can have a vanishing amount of usable entanglement, for a given quantum task to be specified by $C_X$. Now let us consider the application and usefulness of pseudo-measures.

3.1. Example: partial transposition (PT) entanglement

For example, let us consider the Peres criterion for the partial transposition (PT) [33]. A state is entangled, if it does not remain a quantum state under PT, $\rho^{PT} \neq 0$. Entangled states $\rho_{BE}$ with a positive PT are bound entangled states. These states cannot be used for distillation protocols. Thus, we need to define an entanglement pseudo-measure $E_{PT}$, with

$$\rho^{PT} \geq 0 \iff E_{PT}(\rho) = 0,$$

$$\forall \Lambda \in C_{distill} : E_{PT}(\rho) \geq E_P \left( \frac{\Lambda(\rho)}{\text{tr} \Lambda(\rho)} \right).$$

The LOCC $C_{distill}$ are the allowed operations for a distillation protocol. Since all separable quantum states $\sigma$ remain non-negative under PT, it follows that $E_{PT}(\sigma) = 0$. However, a state with a non-negative PT is in general not separable, but $E_{PT}(\rho_{BE}) = 0$. We conclude that equations (29) and (30) define the pseudo-measure $E_{PT}$ with respect to definition 3. But it is not an entanglement measure, see definition 1.

One possible way to construct such measures is given by measures based on entanglement witnesses, see [34, 35].

$$E_W(\rho) = -\inf_W [\text{tr}(\rho W)] = \sup_W [-\text{tr}(\rho W)],$$

with entanglement witnesses $W$ of a given form. A quantum state $\rho$ has a negative PT if and only if there exists a positive operator, $C = |\psi\rangle\langle \psi|$, with

$$\text{tr}(\rho^{PT} C) = \text{tr}(\rho C^{PT}) < 0.$$ (32)

As we have seen above, any $|\psi\rangle$ can be generated by a $|\phi_i\rangle$ and local invertibles and local projections. Thus $C^{PT}$ can be generated by a certain $\Lambda \in C_{sep}$ and $C^{PT} = \Lambda(V)$, with

$$V = (|\phi_i\rangle\langle \phi_i|)^{PT} = \sum_{k,l}^{r} |k,l\rangle\langle l,k|.$$ (33)

We may define the following entanglement pseudo-measure:

$$E_{PT}(\rho) = \sup_{\Lambda \in C_{sep}} \left( -\frac{\text{tr}[\Lambda(\rho)V]}{\text{tr} \Lambda(\rho)} \right).$$ (34)

This pseudo-measure fulfills equation (29), because a PPT entangled state cannot be distilled. And it fulfills equation (30), since $C_{distill}$ is a subset of $C_{sep}$.

3.2. Operational entanglement measures

Let us generalize this situation. We consider an experimental measurement given by the Hermitian operator $M$. Now we use the entanglement condition: a quantum state $\rho$ is entangled, iff $\text{tr} \rho M > f_{AB}(M)$ [28], where $f_{AB}(M)$ denotes the maximal expectation value of $M$ for separable states. We also use the maximal expectation value $f(M) = \sup\{\psi|M|\psi : \langle \psi|\psi \rangle = 1\}$ for all states to define an operational measure $E_M$.

**Definition 4.** An operational measure $E_M$ is a pseudo-measure defined by

$$E_M(\rho) = \sup_{\Lambda \in C_X} \left( \frac{\Lambda(\rho)}{\text{tr} \Lambda(\rho)} M - f_{12}(M) \right).$$

This definition is analogous to the definition of an operational measure for non-classicality, see [36].

From the experimental point of view, we have different devices. The set $X = \{A_1, \ldots, A_k\}$ denotes the action $A_k$ of the $k$th devices onto the quantum state. Thus, $C_X$ defines arbitrary combinations of the used devices. The definition of the operational measure $E_M$ obviously fulfills definition 3 for the LOCC given by $C_X$. For $M = -V$ and $C_X = C_{sep}$, we obtain the operational measure for $E_{PT}$, with $f_{AB}(-V) = 0$.

Some measurements do not use any kind of entanglement, e.g. $M = M_A \otimes M_B$. In this case, $f(M) = f_{AB}(M)$, we define $E_M \equiv 0$. In general, if the operational entanglement vanishes, $E_M(\rho) = 0$, then the state is either separable or the setup cannot use the specific kind of entanglement. The maximally entangled states are states $\rho_{max}$ together with an operation $\Lambda \in C_X$, such that

$$\text{tr} \left( \frac{\Lambda(\rho_{max})}{\text{tr} \Lambda(\rho_{max})} M \right) = f(M).$$ (35)

The amount of operational entanglement of this state is $E_M(\rho_{max}) = 1$. This is equivalent to the statement that the state $\rho_{max}$ under the transformation $\Lambda$ is within the range of the maximal eigenvalue of $M$. Let us summarize these statements in the following proposition.

**Proposition 1.** The operational entanglement $E_M$ has the following properties:

(i) the operational entanglement is a value between zero and one;

(ii) the operational entanglement vanishes, $E_M(\rho) = 0$, if and only if the state is separable or $(M, C_X)$ cannot use the specific kind of entanglement of the state $\rho$;

(iii) the operational entanglement is maximal if and only if the state $\rho_{max}$ under the transformation $\Lambda \in C_X$ is within the range of the maximal eigenvalue of the observable $M$.

4. Summary and conclusions

In conclusion, we have studied the role of local invertible transformations in the context of entanglement. We have reconsidered the invariance of entanglement under local invertibles, showing that these transformations are intrinsically related to the property entanglement. Hence the consideration of local invertibles in the context of entanglement quantification is an important issue.

We have proved that the Schmidt number of a pure state has a larger influence on the amount of entanglement than its Schmidt coefficients. In particular, the Schmidt number yields a discontinuous entanglement quantification that preserves the requirements of an entanglement measure.
under the most general class of stochastic separable operations. To account for this fact, we have defined universal entanglement measures, which are invariant under local invertible transformations and hence independent of the Schmidt coefficients. Further on, we have shown that a general class of entanglement measures has a monotonic behavior with respect to the Schmidt number. Since the Schmidt number represents the superposition principle of quantum physics, its general importance for the amount of entanglement has a clear physical background, which is deeply related to the main differences between quantum and classical physics.

For many applications of entangled states it is important to quantify the usable amount of entanglement. For this purpose, we have considered entanglement pseudo-measures for a certain quantum task, which is specified by the used set of LOCC. The pseudo-measure is zero whenever the entanglement of a given quantum state is not useful for the application under consideration. In terms of pseudo-measures such states are as useful as separable states, even though they may be entangled.

The entanglement pseudo-measures can be further specified as operational measures. In this case, the observables of the experimental setup are included in the definition. It is important that our findings for entanglement measures are also true for operational measures. Thus, the Schmidt number gives universal limits for all types of quantum tasks using entanglement as a resource.

Acknowledgment

This work was supported by the Deutsche Forschungsgemeinschaft through SFB 652.

Appendix. Local transformations

First of all let us prove theorem 1 in the presented form.

Proof of theorem 1. (i) An optimized entanglement witness \( W \) is given by \((a, b| W| a, b) ≥ 0 \) and \((a_0, b_0| W| a_0, b_0) = 0 \) for some \( |a_0, b_0⟩ = |a_0⟩ ⊗ |b_0⟩ \). Obviously the following hold,

\[
(T_1|a⟩ ⊗ T_2|b⟩)^\dagger W (T_1|a⟩ ⊗ T_2|b⟩) ≥ 0
\]

and

\[
(T_1|a'_0⟩ ⊗ T_2|b'_0⟩)^\dagger W (T_1|a'_0⟩ ⊗ T_2|b'_0⟩) = 0,
\]

with \( |a'_0⟩ = T_1^\dagger|a_0⟩ \) and \( |b'_0⟩ = T_2^\dagger|b_0⟩ \). Note that the normalization of the states does not effect relations with zero. 

(ii) Let \( ρ \) be entangled, and \( W \) be an optimized witness detecting the entanglement of the state, \( tr ρ W < 0 \). It follows for the witness \( W' = (T_1^\dagger ⊗ T_2^\dagger)W (T_1^\dagger ⊗ T_2^\dagger) \) that

\[
tr \left[ W' (T_1 ⊗ T_2)^\dagger ρ (T_1 ⊗ T_2) \right] = \frac{tr W [ρ (T_1 ⊗ T_2)]}{tr (T_1 ⊗ T_2) ρ (T_1 ⊗ T_2)} < 0.
\]

From (i) it follows that \( W' \) is an optimized witness. Thus \( ρ' \) is entangled. The opposite direction follows analogously using the inverse local operation.

The set of local unitaries \( C_{LU} \) is a LOCC operation of any \( C_λ \). Condition (ii) delivers for an arbitrary quantum state \( ρ \) and an arbitrary local unitary transformation \( ρ' = [U_1 ⊗ U_2]|ρ|U_1 ⊗ U_2⟩ \) (together with \( ρ = [U_1 ⊗ U_2]^{\dagger} ρ' [U_1 ⊗ U_2] \)):

\[
E(ρ) ≥ E(ρ') \quad \text{and} \quad E(ρ') ≥ E(ρ).
\]  

In equation (11), we consider the maximally entangled state \( ρ_{max} \), with \( E(ρ_{max}) = \) maximal. Now let us prove that the state \( ρ'_{max} \) has the same property for the measure \( E' \):

\[
E'(ρ'_{max}) = E \left( \frac{Λ_{LI}(ρ'_{max})}{tr Λ_{LI}(ρ'_{max})} \right)
\]

\[
= E \left( \frac{Λ_{LI}LI(Λ_{LI}^{-1}(ρ_{max}))}{tr Λ_{LI}LI(Λ_{LI}^{-1}(ρ_{max}))} \right)
\]

\[
= E \left( \frac{Λ_{LI}LI(Λ_{LI}^{-1}(ρ_{max}))}{tr Λ_{LI}LI(ρ_{max})} \right) = E(ρ_{max}) = \text{maximal}.
\]  

In addition, let us now prove that the set \( C_{λ'} \) satisfies condition (ii) in definition 1. It is obvious that

\[
Λ_{LI} \circ Λ' = Λ_{LI} \circ (Λ_{LI}^{-1} \circ Λ \circ Λ_{LI}) = Λ \circ Λ_{LI}.
\]

Now we can obtain

\[
E' \left( \frac{Λ'(ρ)}{tr Λ'(ρ)} \right) = E \left( \frac{Λ_{LI} LI(Λ'(ρ))}{tr Λ_{LI} LI(Λ'(ρ))} \right)
\]

\[
= E \left( \frac{Λ(Λ'(ρ))}{tr Λ(Λ'(ρ))} \right)
\]

\[
= E \left( \frac{Λ_{LI}(ρ)}{tr Λ_{LI}(ρ)} \right) = E' (ρ).
\]

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