The WDVV Equations in $N = 2$ Supersymmetric Yang-Mills Theory

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Abstract

We present a simple proof of the WDVV equations for the prepotential of four-dimensional $N = 2$ supersymmetric Yang-Mills theory with all $ADE$ gauge groups. According to our proof it is clearly seen that the WDVV equations in four dimensions have their origin in the associativity of the chiral ring in two-dimensional topological Landau-Ginzburg models. The WDVV equations for the $BC$ gauge groups are also studied in the Landau-Ginzburg framework. We speculate about the topological field theoretic interpretation of the Seiberg-Witten solution of $N = 2$ Yang-Mills theory.

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The exact low-energy effective action of $N = 2$ supersymmetric Yang-Mills theory is determined by a holomorphic prepotential, which is obtained from the period integrals of the meromorphic one-form, the Seiberg-Witten (SW) differential, on a Riemann surface. For a simple gauge group $G$, it has been recognized that the spectral curves of the periodic Toda lattice associated with the dual affine Lie algebra $(\hat{G})^\vee$ provide the exact solution consistent with the microscopic instanton calculations. In our previous papers, we obtained the Picard-Fuchs differential equations for the period integrals and studied their solutions in the weak-coupling region for the $ADE$ gauge groups. For our analysis, it is essential to regard the spectral curve as the fibration over $\mathbb{CP}^1$ whose fiber is the single-variable version of the superpotential for the $ADE$ topological Landau-Ginzburg (LG) models. This relation between the four-dimensional SW theory and two-dimensional topological LG model suggests a hidden topological field theoretical structure of the low-energy effective action. In fact, Marshakov et al. have shown that the prepotential obeys the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations, which play an important role in evaluating the free energy in two-dimensional topological field theory. This type of non-linear equations would provide a useful tool for determining the instanton corrections without relying on the explicit form of the underlying Riemann surfaces. They might also have an important application to investigating the Donaldson-Witten theory for four-dimensional manifolds.

The analysis of the WDVV equations in is based on the associative property of the one-forms over the hyperelliptic Riemann surface. In [9], on the other hand, Bonelli and Matone derive the WDVV equation for $SU(3)$ gauge group from the Picard-Fuchs equation. In this paper, we will obtain the WDVV equations for the $ADE$ gauge groups from the Gauss-Manin system that the SW period integrals obey. It will be observed that the WDVV equations arise from those of the corresponding topological LG model. In a similar vein the WDVV equations for $B_r$ as well as $C_r$ gauge groups will be obtained from the corresponding Gauss-Manin system.

We begin with introducing a spectral curve for a gauge group $G$ and summarizing some basic properties of two-dimensional topological LG models. The Toda spectral curve for
a simply laced gauge group $G = \{A_r, D_r, E_6, E_7, E_8\}$ of rank $r$ takes the form:

$$z + \frac{\mu^2}{z} = W_G(x; t_1, \ldots, t_r),$$

(1)

where $W_G(x; t_1, \ldots, t_r)$ is identified as the superpotential for the LG models of type $G = ADE$ with flat coordinates $t_i$ ($i = 1, \ldots, r$) \cite{11},\cite{4}. The overall degree of $W_G$ equals $h^\vee$, the dual Coxeter number of $G$, and $t_i$ has degree $q_i = e_i + 1$ where $e_i$ is the $i$-th exponent of $G$. In particular $q_2 = 2$ and $q_r = h$ with $h$ being the Coxeter number of $G$. For $ADE$ gauge groups we have $h^\vee = h$.

In topological LG models it is well-known \cite{12} that the primary fields $\phi_i = \partial_{t_i} W(x)$ where $\partial_{t_i} = \frac{\partial}{\partial t_i}$, generate the chiral ring

$$\phi_i(x) \phi_j(x) = \sum_{k=1}^r C_{ij}^{\ k}(t) \phi_k(x) + Q_{ij}(x) \partial_x W(x),$$

(2)

where $C_{ij}^{\ k}(t)$ are the structure constants. There is a distinguished coordinate $t_r$ satisfying $\phi_r = \partial_{t_r} W(x) = 1$. Note that the flatness condition implies the relation

$$\partial_x Q_{ij}(x) = \partial_{t_i} \partial_{t_j} W(x),$$

(3)

and the topological metric $\eta_{ij}$ is given by

$$\eta_{ij} \equiv \langle \phi_i \phi_j \phi_i \rangle = \delta_{e_i + e_j, h^\vee}.$$  

(4)

The associativity of the chiral ring $\phi_i(\phi_j \phi_k) = (\phi_i \phi_j) \phi_k$ implies the relation $C_{ij}^{\ l} C_{lk}^{\ m} = C_{jk}^{\ l} C_{li}^{\ m}$. Denoting by $C_i$ the matrix with the elements $(C_i)_j^k = C_{ij}^{\ k}$, we get

$$[C_i, C_j] = 0.$$  

(5)

In topological field theory there exists the free energy $F(t_1, \ldots, t_r)$ in such a way that a three-point function $\langle \phi_i \phi_j \phi_k \rangle$ is given by $F_{ijk} \equiv \partial_{t_i} \partial_{t_j} \partial_{t_k} F(t)$. From the relation $F_{ijk} = C_{ij}^{\ l} \eta_{lk}$ and \cite{3}, we obtain the WDVV equations \cite{7}

$$F_i \eta^{-1} F_j = F_j \eta^{-1} F_i,$$

(6)

where the matrix $F_i$ has the elements $(F_i)_{jk} = F_{ijk}$. 

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Let us turn to four-dimensional $N=2$ Yang-Mills theory with the dynamical scale $\Lambda$. The spectral curve (11) with $\mu^2 = \Lambda^2 h^{'}/4$ yields the Riemann surface which is needed to describe the non-perturbative low-energy behavior of the Coulomb branch. The flat directions of the Coulomb branch are parametrized by $t_i$. The SW differential

$$\lambda_{SW} = \frac{x}{2\pi i} \frac{dz}{z}$$

is used to define the period integrals

$$a_I = \oint_{A_I} \lambda_{SW}, \quad a_{DI} = \oint_{B_I} \lambda_{SW}, \quad I = 1, \cdots, r$$

where $A_I$ and $B_I$ are canonical one-cycles on the curve. The periods $\Pi = (a_I, a_{DI})$ obey a set of differential equations, called the Picard-Fuchs (PF) equations. They take the simple form in terms of the flat coordinates [4]:

$$L_0 \Pi \equiv \left( \sum_{i=1}^{r} q_i t_i \partial_{t_i} - 1 \right)^2 \Pi - 4\mu^2 (h^{'})^2 \partial_{\mu}^2 \Pi = 0,$$

$$L_{ij} \Pi \equiv \partial_{t_i} \partial_{t_j} \Pi - \sum_{k=1}^{r} C_{ij}^k(t) \partial_{t_k} \partial_{t_r} \Pi = 0.$$  \hspace{1cm} (9)

The first equation arises from the scaling property of $\lambda_{SW}$

$$\left( \sum_{i=1}^{r} q_i t_i \partial_{t_i} + h^{'\mu} \partial_{\mu} - 1 \right) \lambda_{SW} = \partial_x(*) dx.$$  \hspace{1cm} (10)

The second equations are nothing but the Gauss-Manin system for the $ADE$ singularity [13]. Note that the first order scaling equation (10) is valid for any gauge group.

Now we introduce the prepotential $F(a)$ of the low-energy effective theory by

$$a_{DI} = \frac{\partial F(a)}{\partial a_I}$$

Make a change of variables from the flat coordinates $\{t_i\}$ to the periods $\{a_I\}$, then, in terms of new variables, the Gauss-Manin system is expressed as

$$\left( \partial_{a_I} a_I \partial_{a_J} a_J - \sum_{k=1}^{r} C_{ij}^k(t) \partial_{t_k} a_I \partial_{t_r} a_J \right) \partial_{a_I} \partial_{a_J} \Pi + P_{ij}^l \partial_{a_I} \Pi = 0,$$  \hspace{1cm} (12)

where

$$P_{ij}^l = \partial_{t_i} \partial_{t_j} a_I - \sum_{k=1}^{r} C_{ij}^k(t) \partial_{t_k} \partial_{t_r} a_I.$$  \hspace{1cm} (13)
Since $a_I$ is the solution of the Gauss-Manin system, it is obvious that $P_{ij} = 0$ and $\Pi = a_I$ satisfies (12). We next put $\Pi = a_{DI}$ in (12). This gives rise to the third order differential equations for $\mathcal{F}(a)$ of the form:

$$\tilde{F}_{ijk} = \sum_{l=1}^{r} C_{ijl} \tilde{F}_{lkr},$$  \hspace{1cm} (14)

where

$$\tilde{F}_{ijk} = \partial_{t_i} a_I \partial_{t_j} a_J \partial_{t_k} a_K \mathcal{F}_{IJK}, \quad \mathcal{F}_{IJK} = \partial_{a_I} \partial_{a_J} \partial_{a_K} \mathcal{F}(a).$$  \hspace{1cm} (15)

Defining the metric by $G_{ij} = \tilde{F}_{ij}$, we have

$$\tilde{F}_i = C_i G,$$  \hspace{1cm} (16)

where $\tilde{F}_i$ is the matrix with entries $(\tilde{F}_i)_{jk} = \tilde{F}_{ijk}$. From commutativity of the matrices $C_i$ (3) it follows that

$$\tilde{F}_i G^{-1} \tilde{F}_j = \tilde{F}_j G^{-1} \tilde{F}_i.$$  \hspace{1cm} (17)

Thus $G^{-1} \tilde{F}_i$ commute with each other, and hence the matrices

$$\tilde{F}_k^{-1} \tilde{F}_i = (G^{-1} \tilde{F}_k)^{-1} G^{-1} \tilde{F}_i$$  \hspace{1cm} (18)

also commute for fixed $k$. Therefore we obtain

$$\tilde{F}_i \tilde{F}_k^{-1} \tilde{F}_j = \tilde{F}_j \tilde{F}_k^{-1} \tilde{F}_i.$$  \hspace{1cm} (19)

Removing the Jacobians $\partial_{t_i} a_I$ from (19) we find the WDVV equations for $\mathcal{F}(a)$

$$\mathcal{F}_I \mathcal{F}_K^{-1} \mathcal{F}_J = \mathcal{F}_J \mathcal{F}_K^{-1} \mathcal{F}_I,$$  \hspace{1cm} (20)

where $\mathcal{F}_I$ is the matrix with $(\mathcal{F}_I)_{JK} = \mathcal{F}_{IJK}$.

The WDVV equations in the form (20) have been introduced in [6] and proved for $N = 2$ Yang-Mills theory which admits the description of the low-energy behavior in terms of hyperelliptic curves. The associative algebra of one-differentials is employed in their proof.\footnote{This approach is generalized to non-hyperelliptic curves in the case of classical gauge groups with an adjoint matter. See the second reference of [4].} Our proof here applies to $N = 2$ $ADE$ Yang-Mills theory for which the relevant Riemann surfaces are not necessarily of hyperelliptic type. It is quite clear
that the WDVV equations in $N = 2$ $ADE$ Yang-Mills theory are a consequence of the associativity of the chiral ring in two-dimensional $ADE$ topological LG models.

Since the Gauss-Manin system for $ADE$ gauge groups does not contain the instanton terms and $[\mathcal{L}_{ij}, \partial_{t_r}] = 0$, it is easy to work out the weak-coupling solution around $\mu^2 = 0$:

$$a_I = \bar{a}_I + \sum_{k=1}^{\infty} \frac{\mu^{2k}}{(k!)^2} \partial^{2k}_{t_r} \bar{a}_I,$$

where each coefficient in $\mu^2$ obeys the Gauss-Manin system. This means that the one-loop contribution to the prepotential

$$F_{1-loop} = \frac{i}{4\pi} \sum_{\alpha \in \Delta_+} (\alpha, a)^2 \log \left( \frac{(\alpha, a)^2}{\Lambda^2} \right),$$

where $\Delta_+$ denotes the set of positive roots, also satisfy the WDVV equation (20). In this regard it is interesting to see that the one-loop WDVV equation admits the flat metric

$$K_{IJ} = (\alpha_I, \alpha_J) F_{1-loop I} K^{-1} F_{1-loop J} = F_{1-loop J} K^{-1} F_{1-loop I},$$

where $\alpha_I$ are the simple roots of $G$. This relation comes from

$$F_{1-loop IJK} = \frac{i}{\pi} \sum_{\alpha \in \Delta_+} \frac{(\alpha, \alpha_I)(\alpha, \alpha_J)(\alpha, \alpha_K)}{(\alpha, a)} (\alpha, a)$$

and

$$a_I F_{1-loop IJK} = \frac{i h^\vee}{\pi} K_{JK}.$$ 

For the $AD$ case one may check the one-loop WDVV equations (23) directly with the use of the explicit formula given in [5]. For $E_6$ and $E_7$ we have checked the above relation on the computer.

So far we have studied the WDVV equations for simply laced gauge groups. In order to proceed to the case of non-simply laced gauge groups, we wish to write down the PF equations [14] in terms of the flat coordinates for the LG models of non-simply laced type. According to [15], the non $ADE$ LG models are readily constructed as quotient of the $ADE$ models by a discrete symmetry $\Gamma$ of the Dynkin diagram. Then the $BC_r$ models are obtained from the $A_{2r-1}$ models ($\Gamma = \mathbb{Z}_2$), the $F_4$ model from the $E_6$ model ($\Gamma = \mathbb{Z}_2$) and

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\(^1\)It is also checked numerically that the one-loop WDVV equations hold for $F_4$.\]
the $G_2$ model from the $D_4$ model ($\Gamma = \mathbb{Z}_3$). The quotient procedure amounts to simply setting flat coordinates which are not invariant under the action of $\Gamma$ equal to zero. Since the WDVV equation is trivial for $G_2$, we shall concentrate on the $BC$-type gauge groups in the present work. Because of the complicated structure of the $F_4$ Toda spectral curve, a detailed analysis of the $F_4$ gauge group is left for future study.

Let us start with the gauge group $B_r$. The spectral curve reads

$$z + \frac{\mu^2}{z} = \tilde{W}_B(x; u_1, \cdots, u_r) \equiv \frac{W_{BC}(x; u_1, \cdots, u_r)}{x},$$

where the LG superpotential of type $BC$

$$W_{BC}(x; u_1, \cdots, u_r) = x^{2r} - \sum_{i=1}^r u_i x^{2r-2i}$$

is obtained from the $A_{2r-1}$ superpotential $W(x; \bar{u}_1, \cdots, \bar{u}_{2r-1})$ by the restriction $\bar{u}_{2k} = 0$ ($k = 1, \cdots, r-1$) and setting $u_k = \bar{u}_{2k-1}$ ($k = 1, \cdots, r$). Let $\Phi_a$ ($a = 1, \cdots, 2r-1$) be primary fields coupled to the flat coordinates $T_a$ of the $A_{2r-1}$ topological LG model. Then $t_i = T_{2i-1}|_{T_2=\cdots=T_{2r-2}=0}$ are the flat coordinates of the $BC_r$ LG model.

Now, for the SW differential (7), one may follow [4] to derive the differential equation

$$\partial_t \partial_t \lambda_{SW} = \sum_{k=1}^r C_{ij}^k(t) \partial_t \partial_t \lambda_{SW} + \frac{1}{2\pi i} \mu \partial_\mu \left( \frac{x^{-2} Q_{ij}}{(W_B^2 - 4\mu^2)^{1/2}} \right) dx + \partial_x(*) dx,$$

where the total derivative term in $x$ disappears after the integration over a closed one-cycle. The second term in the RHS of (28), which is absent for simply laced gauge groups, may be calculated as follows: From the $A_{2r-1}$ LG model one has

$$Q_{ij} = \begin{cases} 0 & \text{for } i + j > r, \\ \Phi_{2(i+j)-2} & \text{for } i + j \leq r. \end{cases}$$

Since $\Phi_{2(i+j)-2}(x)$ is an odd polynomial in $x$ and becomes divisible by $x$ after the restriction of the $A$ to $BC$ models we have

$$Q_{ij}(x) = x \sum_{k=1}^r D_{ij}^k(t) \phi_k(x),$$

where

$$D_{ij}^k(t) = \text{res}_\infty \left( \frac{Q_{ij} \phi_k^*}{x \partial_x W_{BC}} \right).$$
Here \( \text{res}_\infty \) means to take the coefficient of \( x^{-1} \) in the expansion at \( x = \infty \) and \( \phi_k^* \) is the Poincaré dual of \( \phi_k \), i.e. \( \langle \phi_k \phi_k^* \phi_r \rangle = 1 \). Therefore we get the PF equation for \( \Pi = \oint \lambda_{SW} \):

\[
\partial_i \partial_j \Pi - \sum_{k=1}^{r} C_{ij}^k(t) \partial_t_k \partial_r \Pi + \sum_{k=1}^{r} D_{ij}^k(t) \mu \partial_r \partial_{t_k} \Pi = 0. \tag{32}
\]

Note that the Gauss-Manin system is not identical with that for the topological \( B_r \) LG model, but is modified in the \( N = 2 \) \( B_r \) Yang-Mills theory.

As in the \( ADE \) case the remaining PF equation is obtained from the scaling property (10) with \( h^\vee = 2r - 1 \) for \( B_r \). Using the relation

\[
(\mu \partial_\mu)^2 \lambda_{SW} = 4\mu^2 \partial_{u_{r-1}} \partial_{u_r} \lambda_{SW} = -4\mu^2 \left( t_1 \partial^2_{t_r} - \partial_{t_1} \partial_{t_{r-1}} \right) \lambda_{SW}, \tag{33}
\]

we find the scaling PF equation

\[
\left( \sum_{i=1}^{r} q_i t_i \partial_{t_i} - 1 \right)^2 \Pi + 4\mu^2 (h^\vee)^2 \left( t_1 \partial^2_{t_r} - \partial_{t_1} \partial_{t_{r-1}} \right) \Pi = 0. \tag{34}
\]

Next we discuss the gauge group \( C_r \) for which the spectral curve is known to be \( \mathbb{CP}^1 \) with \( h^\vee = r + 1 \) and the relation

\[
z + \frac{\mu^2}{z} = \tilde{W}_C(x; u_1, \cdots, u_r) \equiv \left( x^2 W_{BC}(x; u_1, \cdots, u_r)^2 + 4\mu^2 \right)^{1/2}. \tag{35}
\]

A similar procedure to the \( B_r \) case yields the PF equation

\[
\partial_i \partial_j \Pi - \sum_{k=1}^{r} C_{ij}^k(t) \partial_{t_k} \partial_r \Pi - \sum_{k=1}^{r} D_{ij}^k(t) \mu \partial_r \partial_{t_k} \Pi = 0. \tag{36}
\]

Notice that the difference between the gauge groups \( B \) and \( C \) is only the sign of the additional term in the Gauss-Manin system. The scaling PF equation comes from (10) with \( h^\vee = r + 1 \) and the relation

\[
(\mu \partial_\mu)^2 \lambda_{SW} = 4\mu^2 \partial_{t_1} \partial_{t_r} \lambda_{SW}. \tag{37}
\]

The result reads

\[
\left( \sum_{i=1}^{r} q_i t_i \partial_{t_i} - 1 \right)^2 \Pi - 4\mu^2 (h^\vee)^2 \partial_{t_1} \partial_{t_r} \Pi = 0. \tag{38}
\]

Here we wish to comment on possible instanton terms in the PF equation. In (32) and (36), although the form of the Gauss-Manin system deviates from (9), the term carrying \( \mu \partial_\mu \) is rewritten in terms of \( t_i \) variables via the scaling relation (10), and is not
the instanton effect. On the other hand, there exist genuine instanton terms in the PF equation for the $G_2$ gauge group [13]. The spectral curve for $G_2$ is [3]

$$z + \frac{\mu^2}{z} = \frac{1}{6}(p_1 + \sqrt{p_1^2 + 12p_2}),$$

(39)

where

$$p_1 = 6x^4 - 2ux^2, \quad p_2 = x^8 - 2ux^6 + u^2x^4 - vx^2 + 12\mu^4.$$  

(40)

Let us introduce the flat coordinates $t_1 = u/3$ and $t_2 = v/6 - u^3/81$. These are obtained by the restriction of $D_4$ to $G_2$. The PF equation is shown to be

$$\left(\partial_{t_1}^2 - C_{11}^2 \partial_{t_2}^2\right)\Pi - \frac{1}{8}t_1\mu\partial_{t_2}\Pi - \frac{8}{9}\mu^2\partial_{t_2}^2\Pi = 0,$$

$$\left(2t_1\partial_{t_1} + 6t_2\partial_{t_2} - 1\right)^2\Pi - 128\mu^2(t_1^2\partial_{t_2} + \partial_{t_1})\partial_{t_2}\Pi = 0,$$

(41)

where $C_{11}^2 = t_1^4$. Hence in this case the Gauss-Manin system also receives instanton corrections.

We are now ready to consider the WDVV equations for the gauge groups $BC$. It is easy to show from (32), (36) with the aid of (10) that the $N = 2$ prepotential $F(a)$ obeys the third order differential equations

$$\tilde{F}_{ijk} - \sum_{l=1}^{r} C_{ij} \tilde{F}_{lrk} - \sum_{l,n=1}^{r} b_n D_{ij} \tilde{F}_{lnk} = 0,$$

(42)

where $b_n = \epsilon 2nt_n/h^\vee$ with $\epsilon = +(-)$ for $B_r (C_r)$. In the obvious matrix notation (12) is expressed as

$$C_i = \tilde{C}_i + \sum_{n=1}^{r} b_n D_i \tilde{C}_n,$$

(43)

where we have set $\tilde{C}_i = \tilde{F}_i \mathcal{G}^{-1}$ with the same notation for $\tilde{F}$ and $\mathcal{G}$ as in (16). Suppose here that

$$[\tilde{C}_i, \tilde{C}_j] = 0,$$

(44)

then the WDVV equations (20) follow immediately as seen before. To prove (14) the coupled matrix-valued equations (13) for $\tilde{C}_i$ are solved explicitly since they are linear in $\tilde{C}_i$. For instance, we find for the rank 3 case

$$\tilde{C}_1 = \begin{pmatrix}
0 & 0 & 1 \\
2t_1t_2 - b_3 & -b_2 - t_2 + t_1^2 & -b_1 \\
(C_1)_3 & (C_1)_3^2 & b_1^2 - b_1 t_1 - b_2
\end{pmatrix},$$
\[
\tilde{C}_2 = \begin{pmatrix}
0 & 1 & 0 \\
-t_2 + t_1^2 & -t_1 & 1 \\
2t_1t_2 - b_3 & -b_2 - t_2 + t_1^2 & -b_1
\end{pmatrix}, \quad \tilde{C}_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (45)
\]

where
\[
\begin{align*}
(\tilde{C}_1)_3^1 &= -2b_1t_1t_2 + b_1b_3 + t_2^2 + t_1^4 - b_3t_1 + b_2t_2 - b_2t_1^2, \\
(\tilde{C}_1)_3^2 &= b_1b_2 + b_1t_2 - b_1t_1^2 + 2t_1t_2 - b_3. \quad (46)
\end{align*}
\]

It is remarkable that these matrices indeed commute with each other. For \( B_4 \) and \( C_4 \) we have also verified (44) explicitly. Although the WDVV equations for the \( BC \) gauge groups have been proved in a somewhat different manner \([6]\) it will be interesting to complete the proof along the lines presented here in the LG framework.

Finally we wish to discuss possible physical implications of the WDVV equations in view of two-dimensional topological field theory. For this, let us first describe an interesting observation made in \([17]\) in which the WDVV equations (20) are extended by treating the scale parameter \( a_0 \equiv \Lambda \) on an equal footing with the periods \( a_i \). From the scaling equation for the prepotential
\[
\left( \sum_{I=1}^{r} a_I \partial a_I + a_0 \partial a_0 \right) \mathcal{F}(a) = 2 \mathcal{F}(a), \quad (47)
\]
one obtains
\[
\begin{align*}
\mathcal{F}_{0IJ}(a) &= - \sum_{K=1}^{r} a_0^{-1} a_K \mathcal{F}_{IJK}(a), \\
\mathcal{F}_{00I}(a) &= a_0^{-2} \sum_{J,K=1}^{r} a_Ja_K \mathcal{F}_{IJK}(a), \\
\mathcal{F}_{000}(a) &= -a_0^{-3} \sum_{I,J,K=1}^{r} a_Ia_Ja_K \mathcal{F}_{IJK}(a). \quad (48)
\end{align*}
\]

Define the \((r + 1) \times (r + 1)\) matrices \( \hat{\mathcal{F}}_\alpha \) \((\alpha = 0, 1, \cdots, r)\) as
\[
\hat{\mathcal{F}}_\alpha = \begin{pmatrix}
\mathcal{F}_{\alpha 00} & \mathcal{F}_{\alpha 0I} \\
\mathcal{F}_{\alpha I0} & \mathcal{F}_{\alpha II}
\end{pmatrix}, \quad (49)
\]
then it is easy to show that the inverse of \( \hat{\mathcal{F}}_\alpha \) turns out to be
\[
\hat{\mathcal{F}}^{-1}_\alpha = \begin{pmatrix}
A_\alpha & B_\alpha \\
B_\alpha & C_\alpha
\end{pmatrix}, \quad (50)
\]
where

\[ A_\alpha = \frac{1}{2a_0^2 \sum_{r,K=1}^r a_J J K a_K} \], \quad (B_\alpha)_J = \frac{a_0 a_J}{2 \sum_{K,L=1}^r a_K J K a_L}, \]

\[(C_\alpha)_{JK} = (F_\alpha^{-1})_{JK} + \frac{a_J a_K}{2 \sum_{L,M=1}^r a_L J L a_M}. \] (51)

In the last equation of (51), \( F_0 \) is the \( r \times r \) matrix given by

\[ F_0 = -\frac{1}{a_0} \sum_{I=1}^r a_I F I a^I. \]

For \( \hat{F}_\alpha \) one can now prove the extended WDVV equation [17]

\[ \hat{F}_\alpha \hat{F}_\gamma \hat{F}_\beta = \hat{F}_\beta \hat{F}_\gamma \hat{F}_\alpha. \] (52)

This result implies that \( F(a_0, a_1, \ldots, a_r) \) may be thought of as the free energy of certain topological field theory whose deformations are parametrized by the special coordinates \( a_\alpha \) of special geometry. Writing the spectral curve [11] in the form

\[ W^*_G(x, z; \mu^2, t_1, \ldots, t_r) = z + \frac{\mu^2}{z} - W_G(x; t_1, \ldots, t_r), \] (53)

we see, as is pointed out in [11], that \( W^*_G \) is the sum of the superpotentials for the LG descriptions of the topological \( \mathbb{CP}^1 \) model [18] and the \( ADE \) model. It is then natural to express \( \mu^2 = \mu_0^{2h^\vee} e^{t_0} / 4 \) where \( t_0 \) is the flat coordinate in the topological \( \mathbb{CP}^1 \) model [19]. The \( ADE \) superpotential originates from a part of the defining equation for the ALE space with complex structure deformations. On the other hand, \( t_0 \) is a moduli parameter for Kähler deformations of the topological \( \sigma \)-model on \( \mathbb{CP}^1 \) which is classified as the topological A-model. Note that the LG description of the \( \mathbb{CP}^1 \) model is now recognized as the mirror partner of the \( \mathbb{CP}^1 \) \( \sigma \)-model [20]. Therefore it is consistent to regard \( W^*_G \) as a superpotential for a topological B-model (denoted as \( X_B \) henceforth) which is a tensor product of the \( \mathbb{CP}^1 \) and \( ADE \) LG models. Then it is tempting to speculate that the SW prepotential \( F(a_0, a_1, \ldots, a_r) \) will be the free energy of a topological theory related to the model \( X_B \). Notice that in constructing \( F \) we have utilized a period map [8]

\[ \Pi : (t_0, t_1, \ldots, t_r) \rightarrow (a_0, a_1, \ldots, a_r) \] (54)

with \( a_0 = \Lambda = \mu_0 e^{t_0 / 2h^\vee} \). Note also that [14] is written as

\[ F_{IJK}(a) = \frac{\partial t_i(a)}{\partial a_I} \frac{\partial t_j(a)}{\partial a_J} \frac{\partial t_k(a)}{\partial a_K} G_{kl}(a(t)) C_{ij}^l(t(a)) \] (55)
which is reminiscent of the mirror map for the Yukawa couplings from a Calabi-Yau threefold to its mirror manifold. In the RHS of (55) the “B-model Yukawa couplings” $C_{ij}^l$ are free from the instanton corrections, while in the LHS the “A-model Yukawa couplings” $F_{IJK}$ receive full instanton corrections. This suggests a fascinating possibility, though speculative, that the map (54) is understood as a kind of mirror map under which the prepotential $F(a_0, a_1, \cdots, a_r)$ is obtained as the free energy of a topological A-model which is the mirror partner of the model $X_B$ defined in terms of the superpotential $W_G$. It may be worth pursuing further the idea discussed here toward uncovering the hidden topological field theoretic property of the SW solution of $N = 2$ Yang-Mills theory. We expect that the $M$-theory/Type II fivebrane interpretation of the SW solution will certainly play a role [21].

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