Hamiltonian paths, closed complexes, and determinantal facet ideals

Bruno Benedetti *
Dept. of Mathematics
University of Miami
bruno@math.miami.edu

Lisa Seccia **
Dip. di Matematica
Univ. degli Studi di Genova
seccia@dima.unige.it

Matteo Varbaro **
Dip. di Matematica
Univ. degli Studi di Genova
varbaro@dima.unige.it

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Abstract

We study $d$-dimensional generalizations of three mutually related topics in graph theory: Hamiltonian paths, (unit) interval graphs, and binomial edge ideals. We provide partial high-dimensional generalizations of Ore and Pósa’s sufficient conditions for a graph to be Hamiltonian. We introduce a hierarchy of combinatorial properties for simplicial complexes that generalize unit-interval, interval, and co-comparability graphs. We connect these properties to the already existing notions of determinantal facet ideals and (tight and weak) Hamiltonian paths in simplicial complexes. Some important consequences of our work are:

1. Every almost-closed strongly-connected $d$-dimensional simplicial complex is traceable. (This extends the well-known result “unit-interval connected graphs are traceable”.)

2. Every almost-closed $d$-complex that remains strongly connected after the deletion of $d$ or less vertices, is Hamiltonian. (This extends the fact that “unit-interval 2-connected graphs are Hamiltonian”.)

3. The minors defining the determinantal facet ideal of any almost-closed complex form a lex-Grobner basis. (This revises a recent theorem by Ene et al., and extends a result by Herzog and others.)

4. The determinantal facet ideals of all under-closed and semi-closed complexes have a square-free initial ideal with respect to any diagonal monomial order. In positive characteristic, they are even Frobenius split. (This provides the largest known class of determinantal facet ideals that are radical.)

Introduction

The first Combinatorics paper in History is probably Leonhard Euler’s 1735 solution of the Königsberg bridge problem. In that article, Euler introduced the notion of graph, and studied cycles (now called ‘Eulerian’) that touch all edges exactly once. Euler proved that the graphs admitting them are exactly those graphs with all vertices of even degree. Hamiltonian cycles are instead cycles that touch all vertices exactly once; they are named after sir William Rowan Hamilton, who in 1857 invented a puzzle game which asked to find one such cycle in the icosahedron. Unlike for the Eulerian case, figuring out if a graph admits a Hamiltonian cycle or not is a hard problem, now known to be NP-complete [Kar72].

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Even if simple characterizations are off the table, in the 1960s, Dirac, Ore, Pósa and others were able to obtain simple conditions on the vertex degrees (in analogy with Euler’s work) that are sufficient for a graph to admit Hamiltonian cycles. Ore’s theorem, for example, says, “Any graph with \( n \) vertices such that \( \deg u + \deg v \geq n \) for all non-adjacent vertices \( u, v \), admits a Hamiltonian cycle”. Ore’s condition is far from being necessary: In any cycle, for example, one has \( \deg u + \deg v = 4 \) for all \( u, v \).

About at the same time of Ore’s results, the two papers [Lb62] and [GH64] initiated the study of unit-interval graphs. This very famous class consists, as the name suggests, of all intersection graphs of a bunch of length-one open intervals on the real line. (That is, we place a node in the middle of each interval, and we connect two nodes with an arc if and only if the corresponding intervals overlap). If they are connected, such graphs always admit Hamiltonian paths, i.e. paths that touch all vertices once [Ber83], but not necessarily Hamiltonian cycles. However, Chen–Chang–Chang’s theorem states that “2-connected unit-interval graphs admit Hamiltonian cycles” [CCC97]. For these results, the length-one request can be weakened to “pairwise not-nested”, but it cannot be dismissed: Within the larger world of interval graphs, one encounters connected graphs such as \( K_{1,3} \) that do not admit Hamiltonian paths, and also 2-connected graphs like the \( G_5 \) of Remark 45 that do not admit Hamiltonian cycles.

In the 1970s, the work of Stanley and Reisner established a fundamental bridge between Combinatorics and Commutative Algebra, namely, a natural bijection between labeled simplicial complexes on \( n \) vertices and radical monomial ideals in a polynomial ring with \( n \) variables. This correspondence lead Stanley to prove the famous Upper Bound Theorem for triangulated spheres [St14]. After this success, many authors have investigated ways to encode graphs into monomial ideals. In 2010, Herzog et al. [H&10] first considered a natural way to encode graphs into binomial ideals, the so-called binomial edge ideals. The catch is that all such binomial edge ideals are radical [H&10]. In the process, Herzog et al. re-discovered unit-interval graphs, characterizing them as the graphs whose binomial edge ideals have quadratic Gröbner bases with respect to \( \text{lex} \) [H&10, Theorem 1.1].

So far, we sketched three important graph theory topics from the last three centuries: Hamiltonian paths, (unit) interval graphs, binomial edge ideals. In the last 20 years, there has been an increasing interest in expanding these three notions to higher dimensions. Specifically:

- Katona–Kierstead [KK99] and many others [HS10, K&a10, RSR08] have studied “tight Hamiltonian paths” and “loose Hamiltonian paths” in \( d \)-dimensional simplicial complexes; both notions for \( d = 1 \) boil down to ordinary Hamiltonian paths. The good news is that extremal combinatorics provides a non-trivial way to extend Dirac’s theorem for \( d \)-complexes with a very large number of vertices that satisfy certain ridge-degree conditions. The bad news is that already Ore and Pósa’s theorems seem very hard to extend.
- Ene et al. [E&13] introduced “determinantal facet ideals”, which directly generalize binomial edge ideals, and “closed \( d \)-complexes”, which generalize ‘unit-interval graphs’. The good news is that the definitions are rather natural. The bad news is that determinantal facet ideals are not radical in general (see Example 71), and they are hard to manipulate; alas, the two main results of the paper [E&13] are incorrect, cf. Remark 77.

In the present paper we take a new, unified look at these approaches. In Chapter 1, we introduce a notion of ‘weakly-Hamiltonian paths’ for \( d \)-dimensional simplicial complexes that for \( d = 1 \) also boils down to ordinary Hamiltonian paths. This weaker notion enables us to obtain a first, partial extension of Dirac, Ore and Pósa’s theorem to higher dimensions:

**Main Theorem I** (Higher-dimensional Ore and Dirac, cf. Proposition 18 and Corollary 20). Let \( \Delta \) be any traceable \( d \)-complex on \( n > 2d \) vertices. If in some labeling that makes \( \Delta \) traceable the two \((d-1)\)-faces \( \sigma \) and \( \tau \) formed by the first \( d \) and the last \( d \) vertices, respectively, have facet
degrees summing up to at least \( n \), then \( \Delta \) admits a weakly-Hamiltonian cycle.

In particular, if in a traceable pure \( d \)-complex with \( n \) vertices, every \((d-1)\)-face belongs to at least \( \frac{n}{2} \) facets, then the complex admits a weakly-Hamiltonian cycle.

**Main Theorem II** (Higher-dimensional Pósa, cf. Proposition 23). Let \( \Delta \) be any traceable pure \( d \)-complex on \( n \) vertices, \( n > 2d \). Suppose that with any labeling in which \( \Delta \) has a weakly-Hamiltonian path, \( \Delta \) is traceable. Let \( \sigma_1, \sigma_2, \ldots, \sigma_s \) be the \((d-1)\)-faces of \( \Delta^* \), ordered so that \( d_1 \leq d_2 \leq \ldots \leq d_s \), where \( d_i \) is the number of \( d \)-faces containing \( \sigma_i \). If for every \( d \leq k < \frac{n}{2} \) one has \( d_{k-d+1} > k \), then \( \Delta \) admits a weakly-Hamiltonian cycle.

As you can see these results are conditional: ‘Traceability’, i.e. the existence of a tight Hamiltonian path, must be known a priori, in order to infer the existence of a weakly-Hamiltonian cycle. This sounds like a bad deal, but in the one-dimensional case our results above still immediately imply the original theorems by Ore and Pósa for graphs. Moreover, since no extremal combinatorics is used in the proof, there is an advantage: Main Theorems I and II do not require the number of vertices to be extremely large. On the contrary: In the two-dimensional case, they already apply to complexes with five vertices.

In Chapter 2, we introduce a hierarchy of four natural properties that progressively weaken (for strongly-connected complexes) the notion of “closed \( d \)-complexes”, as originally proposed in [E&13]. We introduce “almost-closed”, “under-closed”, and “weakly-closed” complexes, as natural combinatorial higher-dimensional generalizations of unit-interval graphs, of interval graphs, and of co-comparability graphs, respectively. The forth property, called “semi-closed”, is intermediate between “under-closed” and “weakly-closed”; it is also defined very naturally, but it seems to be new already for graphs. We will see its algebraic consequence in Main Theorem VI below.

The main goal of Chapter 2 is to connect this hierarchy to the notions of Chapter 1:

**Main Theorem III** (Higher-dimensional Bertossi, Theorem 54). Every almost-closed strongly-connected \( d \)-dimensional simplicial complex is traceable.

**Main Theorem IV** (Higher-dimensional Chen–Chan–Chang, Theorem 58). Every almost-closed \( d \)-dimensional simplicial complex that remains strongly connected after the deletion of \( d \) or less vertices, however chosen, is Hamiltonian.

Finally, Chapter 3 is dedicated to the connection with commutative algebra. Building on the very recent work of the second author [Se21], we are able to resurrect one of the results claimed (with incomplete proof) in Ene et al [E&13], and to extend it from closed to almost-closed complexes:

**Main Theorem V** (Theorems 76). The minors defining the determinantal facet ideal of any almost-closed complex form a Gröbner basis with respect to any diagonal term order.

For a homogeneous ideal of polynomials, having a square-free Gröbner degeneration is a strong and desirable property. In 2020, Conca and Varbaro proved Herzog’s conjecture that if a homogeneous ideal \( I \) has square-free initial ideal \( \text{in}(I) \), then the extremal Betti numbers of \( I \) and \( \text{in}(I) \) are the same [CV20]. In particular, this allows us to infer the depth, the Castelnuovo–Mumford regularity, and many other invariants of the ideal \( I \), simply by computing these invariants on its initial ideal. The latter task is much simpler, because the aforementioned Stanley–Reisner correspondence enables us to use techniques from combinatorial topology.

We conclude our work with a result that provides a broad class of determinantal facet ideals that are radical:
Main Theorem VI (Theorem 74). The determinantal facet ideals of all semi-closed complexes are radical. Indeed, they have a square-free initial ideal with respect to any diagonal monomial order. Moreover, in prime characteristic \( p > 0 \), the quotients by these ideals are all \( F \)-pure.

The proof relies once again on the recent work by the second author [Se21]. Since all shifted complexes are under-closed, and in particular semi-closed, Theorem 74 immediately implies that the determinantal facet ideals of shifted complexes admit a square-free Gröbner degeneration and, in positive characteristic, define \( F \)-pure rings.

**Notation**

Throughout \( d, n \) are positive integers, with \( d < n \). We denote by \( \Sigma^d \) the \( d \)-simplex, and by \( \Sigma^d_n \) the \( d \)-skeleton of \( \Sigma^{n-1} \). We write each face of \( \Sigma^d_n \) by listing its vertices in increasing order. We describe simplicial complexes by listing their facets in any order, e.g. \( \Delta = 123 \), \( 124 \), \( 235 \). For any \( d \)-face \( F = a_0a_1\cdots a_d \) of \( \Sigma^d_n \), we call \( \text{gap} \) of \( F \) the integer \( \text{gap}(F) = a_d - a_0 - d \), which counts the integers \( i \) strictly between \( a_0 \) and \( a_d \) that are not present in \( F \).

For \( 1 \leq i \leq n - d \) let us call \( H_i \) the \( d \)-face of \( \Sigma^d_n \) with vertices \( i, i+1, \ldots, i+d \). Clearly, \( H_1, H_2, \ldots, H_{n-d} \) are exactly those faces of \( \Sigma^d_n \) that have gap zero. With abuse of notation, we extend the definition of \( H_i \) also to \( i \in \{n - d + 1, \ldots, n\} \) by using the “congruence modulo \( n \)” convention. In other words, by “\( n + 1 \)” we mean vertex 1, by “\( n + 2 \)” we mean vertex 2, and so on. So \( H_n \) will be the \( d \)-face adjacent to \( H_1 \) and of vertices \( \{n, 1, 2, 3, \ldots, d\} \), which we are supposed to write down in increasing order, so \( H_n = 123\cdots d \). Note that \( \text{gap}(H_i) > 0 \) when \( i > n - d \).

**Definition 1** (traceable, Hamiltonian). A complex \( \Delta \) is (tight-) traceable if it has a labeling such that \( H_1, \ldots, H_{n-d} \) are in \( \Delta \). It is (tight-) Hamiltonian if it has a labeling such that all of \( H_1, \ldots, H_n \) are in \( \Delta \).

Clearly, Hamiltonian implies traceable. For \( d = 1 \), Definition 1 boils down to the classical notions of traceable and Hamiltonian graphs, that is, graphs that admits a Hamiltonian path and a Hamiltonian cycle, respectively. In fact, nobody prevents us from relabeling the vertices in the order in which we encounter them along such path (or cycle).

Recall that two facets of a pure simplicial \( d \)-complex are adjacent if their intersection has cardinality \( d \), or equivalently, dimension \( d - 1 \). For example, each \( H_i \) is adjacent to \( H_{i+1} \). The dual graph of a pure simplicial \( d \)-complex \( \Delta \) has nodes corresponding to the facets of \( \Delta \), and arcs according to the following rule: Two nodes are connected by an arc if and only if the corresponding facets of \( \Delta \) are adjacent. A pure simplicial \( d \)-complex \( \Delta \) is called strongly-connected if its dual graph is connected. For \( d \geq 1 \), every strongly-connected \( d \)-complex is connected, and when \( d = 1 \) the two notions coincide. Note that, according to our convention, all strongly-connected simplicial complexes are pure.

**Remark 2.** The statement “the dual graph of any Hamiltonian \( d \)-complex is Hamiltonian” holds true only for \( d = 1 \): For example, the Hamiltonian simplicial complex

\[
\Delta_1 = 123, 234, 345, 456, 567, 678, 789, 189, 129, 147
\]

is not even strongly connected: The dual graph has an isolated vertex, corresponding to 147. Note also that the deletion of vertex 1 from \( \Delta_1 \) yields a simplicial complex that is not even pure.
1 Weakly-traceable/Hamiltonian complexes and ridge degrees

In this section, we introduce two weaker notions of traceability and Hamiltonicity that first appeared in [K&a10], and we study their nontrivial relationship with the “ridge degree”, i.e. how many $d$-faces contain any given $(d-1)$-face. This relationship has a long history, beginning in 1952 with one of the most classical results in graph theory, due to Gabriel Dirac [Dir52], the son of Nobel Prize physicist Paul Dirac:

**Theorem 3** (Dirac [Dir52]). Let $G$ be a graph with $n$ vertices. If $\deg v \geq \frac{n}{2}$ for every vertex $v$, then $G$ is Hamiltonian.

Later Øystein Ore [Ore60] improved Dirac’s result and extended it to traceable graphs:

**Theorem 4** (Ore [Ore60]). Let $G$ be a graph with $n$ vertices.

(A) If $\deg u + \deg v \geq n$ for all non-adjacent vertices $u, v$, the graph $G$ is Hamiltonian.

(B) If $\deg u + \deg v \geq n - 1$ for all non-adjacent vertices $u, v$, the graph $G$ is traceable.

Two years later Pósa extended Ore’s condition (A) much further:

**Theorem 5** (Pósa [Pó62]). Let $G$ be a graph with $n$ vertices. Order the vertices $v_1, \ldots, v_n$ so that the respective degrees are weakly increasing, $d_1 \leq d_2 \leq \ldots \leq d_n$.

(C) If for every $k < \frac{n}{2}$ one has $d_k > k$, the graph $G$ is Hamiltonian.

These theorems have been generalized in five main directions, over the course of more than a hundred papers (see also Li [Li13] for a survey with a different perspective than ours):

1. Bondy and Chvátal [Bon69, Bon71a, Chv84, BC71] weakened the antecedent in the implication (C) of Pósa’s theorem (see [Far99] for an application to self-complementary graphs);
2. Bondy [Bon71b] strengthened the conclusion of Ore’s theorem, from Hamiltonian to pancyclic (=containing cycles of length $\ell$ for any $3 \leq \ell \leq n$); later Schmeichel–Hakimi [SH74] showed that Pósa and Chvátal’s theorems can be strengthened in the same direction;
3. Fan [Fan84] showed that for 2-connected graphs, it suffices to check Ore’s condition for vertices $u$ and $v$ at distance 2; and even more generally, it suffices to check that for any two vertices at distance two, at least one of them has degree $\geq \frac{n}{2}$. With these weaker assumptions he was still able to achieve a pancyclicity conclusion. See [BCS93], [LLF07], [CSZ14] for recent extensions of Fan’s work.

4. A forth line of generalizations of Ore’s theorem involved requiring certain vertex sets to have large neighborhood unions, rather than large degrees: Compare Broersma–van den Heuvel–Veldman [BHV93] and Chen–Schelp [CS92].

Here we are interested in the fifth main direction, namely, the generalization to higher dimensions. This is historically a rather difficult task: As of today, no straightforward extension of Ore’s theorem or of Pósa’s theorem is known. However, some elegant positive results were obtained in 1999 by Katona and Kierstead [KK99], who applied extremal graph theory to generalize Dirac’s theorem to simplicial complexes with a huge number of vertices. Building on the work by Katona and Kierstead [KK99], Rödl, Szemerédi, and Ruciński [RSR08] were able in 2008 to prove the following ‘extremal’ version of Dirac’s theorem:

**Theorem 6** (Rödl–Szemerédi–Ruciński [RSR08]). For all integers $d \geq 2$ and for every $\varepsilon > 0$ there exists a (very large) integer $n_0$ such that every $d$-dimensional simplicial complex $\Delta$ with more than $n_0$ vertices, and such that every $(d-1)$-face of $\Delta$ is in at least $n(\frac{1}{2} + \varepsilon)$ facets, is Hamiltonian.

Now we are ready to introduce the main definition of the present section. Recall that two facets of a pure simplicial $d$-complex are *incident* if their intersection is nonempty.
Definition 7 (weakly-traceable, weakly-Hamiltonian). A $d$-dimensional simplicial complex $\Delta$ is weakly-traceable if it has a labeling such that $\Delta$ contains a subset $H_{i_1}, \ldots, H_{i_k}$ of $\{H_1, \ldots, H_{n-d}\}$ that altogether cover all vertices, and such that $H_{i_j}$ is incident to $H_{i_{j+1}}$ for each $j \in \{1, \ldots, k-1\}$. In this case, we call $H_{i_1}, \ldots, H_{i_k}$ a weakly-Hamiltonian path.

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Remark 8. These notions are not new. For what we called “weakly-Hamiltonian”, Keevash et al. [K&a10] use the term “generic Hamiltonian”. Their paper [K&a10] focuses however on the stronger notion of “loose-Hamiltonian” complexes, which are weakly-Hamiltonian complexes where all of the intersections $H_{i_j} \cap H_{i_{j+1}}$ consist of a single point (with possibly one exception). By definition, all Hamiltonian complexes are loose-Hamiltonian, and all loose-Hamiltonian complexes are weakly-Hamiltonian. For $d = 1$ all these different notions converge: “Weakly-Hamiltonian 1-complexes” are simply “graphs with a Hamiltonian cycle”, and “weakly-traceable 1-complexes” are “graphs with a Hamiltonian path”. In 2010 Han–Schacht [HS10] and independently Keevash et al. [K&a10] proved the following extension of Theorem 6 above:

Theorem 9 (Han–Schacht [HS10], Keevash et al. [K&a10]). For all integers $d \geq 2$ and for every $\varepsilon > 0$ there exists a (very large) integer $n_0$ such that every $d$-dimensional simplicial complex $\Delta$ with more than $n_0$ vertices, and such that every $(d-1)$-face of $\Delta$ is in at least $n(\frac{1}{2d} + \varepsilon)$ facets, is loose-Hamiltonian, and in particular weakly-Hamiltonian.

Remark 10. In Definition 7, note that if $\Delta$ is weakly-traceable, necessarily $i_1 = 1$ and $i_k = n-d$, because otherwise 1 and $n$ would not be covered. So equivalently, in Def. 7 we could demand

$$\{i_2, \ldots, i_{k-1}\} \subset \{2, \ldots, n-d-1\}.$$ 

Note also that if a labeling $v_1, \ldots, v_n$ makes $\Delta$ (weakly-) traceable, so does the “reverse labeling” $v_n, \ldots, v_1$. As for Hamiltonian complexes: If a labeling $v_1, \ldots, v_n$ makes $\Delta$ weakly-Hamiltonian, so does its reverse, and also $v_i, \ldots, v_n$, where $(i_1, \ldots, i_n)$ is any cyclic permutation of $(1, \ldots, n)$. So we may assume that $i_1 = 1$. Or we may assume that $i_k = n-d$. But as the next remark shows, we cannot assume both.

Remark 11. When $d > 1$, not all weakly-Hamiltonian $d$-complexes are weakly-traceable. For $d = 2$, a simple counterexample is given by

$$\Delta_0 = 123, 156, 345.$$ 

The weakly-Hamiltonian cycle is of course $H_1, H_3, H_5$. Any labeling that makes $\Delta_0$ weakly-Hamiltonian is either the reverse or a cyclic shift (or both) of the labeling above. For parity reasons, in any labeling that makes $\Delta_0$ weakly-Hamiltonian, only one of $H_1$ and $H_4$ is in $\Delta_0$.

Remark 12. Weakly-traceable complexes are obviously connected. Weakly-Hamiltonian complexes are even 2-connected, in the sense that the deletion of any vertex leaves them connected. The converses are well-known to be false already for $d = 1$. In fact, let $n \geq 4$. Let $A_{n-2}$ be the edge-less graph on $n - 2$ vertices. Let $x, y$ be two new vertices. The “suspension”

$$\text{susp}(A_{n-2}) \overset{\text{def}}{=} A_{n-2} \cup \{x * v : v \in A_{n-2}\} \cup \{y * v : v \in A_{n-2}\}$$

is a 2-connected graph on $n$ vertices that is not Hamiltonian for $n \geq 5$, and not even traceable for $n \geq 6$. In higher dimensions, the $\Delta^d_3$ of Lemma 44 is $d$-connected, but neither weakly-traceable nor weakly-Hamiltonian.
We start with a few Lemmas that are easy, and possibly already known; we include nonetheless a proof for the sake of completeness. For the following lemma, a subword of a word is a subsequence formed by consecutive letters of a word: So for us “word” is a subword of “subword”, whereas “sword” is not.

**Lemma 13.** Let \( d \geq 2 \). If a \( d \)-complex \( \Delta \) is weakly-Hamiltonian (resp. weakly traceable), then for any \( k \in \{1, \ldots, d\} \) the \( k \)-skeleton of \( \Delta \) is weakly-Hamiltonian (resp. weakly traceable).

**Proof.** Given a weakly-Hamiltonian path/cycle, replace any \( d \)-face \( H_1 \) with its \((k+1)\)-letter subwords, ordered lexicographically. The result, up to canceling possible redundancies, will be a weakly-Hamiltonian path/cycle for the \( k \)-skeleton.

For example: if \( d = 3 \) and \( k = 1 \), suppose that a 3-complex \( \Delta \) on 8 vertices admits the Hamiltonian path \( 1234, 2345, 5678 \).

Then the 1-skeleton of \( \Delta \) admits the Hamiltonian path \( 12, 23, 34, 23, 34, 45, 56, 67, 78 \).

The next Lemma is an analog to the fact that Hamiltonian complexes are traceable.

**Lemma 14.** Let \( \Delta \) be a \( d \)-dimensional complex that has a weakly-Hamiltonian cycle \( H_{i_1}, \ldots, H_{i_k} \), with \( k \geq 3 \). For any \( j \) in \( \{1, \ldots, k\} \), let \( m_j \) be the number of vertices of \( H_{i_j} \) that are neither contained in \( H_{i_{j-1}} \) nor in \( H_{i_{j+1}} \) (where by convention \( i_{k+1} \equiv i_1 \)).

- If \( m_j > 0 \), the deletion of those \( m_j \) vertices from \( \Delta \) yields a weakly-traceable complex.
- If \( m_j = 0 \), and in addition \( H_{i_{j-1}} \) and \( H_{i_{j+1}} \) are disjoint, then \( \Delta \) itself is weakly-traceable.

**Proof.** Fix \( j \) in \( \{1, \ldots, k\} \). If \( m_j > 0 \), the \( m_j \) vertices that belong to \( H_{i_j} \) and to no other facet of the cycle are labeled consecutively. So up to relabeling the vertices cyclically, we can assume that they are the vertices \( n - m_j + 1, n - m_j + 2, \ldots, n - 1, n \). Thus the facet in the cycle they all belong to is the last one, \( H_{i_k} \). Now let \( D \) be the complex obtained from \( \Delta \) by deleting these \( m_j \) vertices. It is easy to see that

\[
H_1 = H_{i_1}, H_{i_2}, \ldots, H_{i_{k-1}}
\]

is a weakly-Hamiltonian path for \( D \).

The case \( m_j = 0 \) is similar: Up to relabeling the vertices cyclically, \( i_{j+1} = 1 \) and thus \( j = k \). By assumption \( H_{i_{k-1}} \) and \( H_1 \) are disjoint. But since \( m_k = 0 \), and vertex \( n \) does not belong to \( H_1 \), it must belong to \( H_{i_{k-1}} \). Therefore \( H_{i_{k-1}} = H_{n-d} \). So

\[
H_1 = H_{i_1}, H_{i_2}, \ldots, H_{i_{k-1}}
\]

is a weakly-Hamiltonian path for \( \Delta \) itself.

The next Lemma can be viewed as a \( d \)-dimensional extension of the fact that the cone over the vertex set of a graph \( G \) is a Hamiltonian graph if and only if the starting graph \( G \) is traceable.

**Lemma 15.** Let \( \Delta \) be any \( d \)-complex on \( n \) vertices. Let \( \Sigma^{d-1} \) be the \((d-1)\)-simplex. Let \( \Gamma \) be the \( d \)-complex obtained by adding to \( \Delta \) a \( d \)-face \( v \ast \Sigma^{d-1} \) for every vertex \( v \) in \( \Delta \). Then

\( \Delta \) is weakly-traceable \( \iff \Gamma \) is weakly-Hamiltonian.
Proof. \( \Rightarrow \): If \( H_1, \ldots, H_k \) is a list of facets proving that \( \Delta \) is weakly-traceable, then the list \( H_1, \ldots, H_k, H_n, H_{n+1} \) shows that \( \Gamma \) is weakly-Hamiltonian.

\( \Leftarrow \): Pick a labeling that makes \( \Gamma \) weakly-Hamiltonian. By how the complex \( \Gamma \) is constructed, the vertices of \( \Sigma^{d-1} \) must be labeled consecutively; so without loss, we may assume that they are \( n + 1, \ldots, n + d \). Take a weakly-Hamiltonian cycle for \( \Gamma \) and delete from the list all the \( d \)-faces containing any vertex whose label exceeds \( n \).

\( \square \)

Remark 16. The following statements are valid only for \( d = 1 \).

(i) “\( \Delta \) is weakly-traceable \( \iff \) \( \Delta \cup w \ast (d-1)\text{-skel}(\Delta) \) is weakly-Hamiltonian.”
(ii) “Deleting a single vertex from a weakly-Hamiltonian \( d \)-complex yields a weakly-traceable complex.”
(iii) “Deleting (the interior of) any of the \( H_i \)'s from a weakly-Hamiltonian \( d \)-complex yields a weakly-traceable complex.”

Simple counterexamples in higher dimensions are:

(i) \( \Delta_1 = 126,234,456,489,678 \) is not weakly-traceable, yet \( \Delta_2 \equiv \Delta_1 \cup (10 \ast 2\text{-skel}(\Delta_1)) \) admits the weakly-Hamiltonian cycle 234, 456, 678, 8910, 1210. This is a counterexample to \( \Leftarrow \). In contrast, the direction \( \Rightarrow \) holds in all dimensions.
(ii) If from the \( \Delta_2 \) above we delete vertex 10, we get back to \( \Delta_1 \), not weakly-traceable.
(iii) \( \Delta_3 = 1234,2345,5678,16710,18910 \) is weakly-Hamiltonian, but the deletion of (the interior of) 5678 yields a complex that is not weakly-traceable.

Our first non-trivial result is an “Ore-type result”: We shall see later that in some sense it extends ‘most’ of the proof of Ore’s theorem 4, part (A), to all dimensions.

Definition 17. Let \( \Delta \) be a pure \( d \)-dimensional simplicial complex, and let \( \sigma \) be any \( (d-1) \)-face of \( \Delta \). The degree \( d_\sigma \) of \( \sigma \) is the number of \( d \)-faces of \( \Delta \) containing \( \sigma \).

Proposition 18. Let \( \Delta \) be a traceable \( d \)-dimensional simplicial complex on \( n \) vertices, \( n > 2d \). If in some labeling that makes \( \Delta \) traceable the two \( (d-1) \)-faces \( \sigma \) and \( \tau \) formed by the first \( d \) and the last \( d \) vertices, respectively, satisfy \( d_\sigma + d_\tau \geq n \), then \( \Delta \) is weakly-Hamiltonian.

Proof. Since \( n > 2d \), the two faces \( \sigma \) and \( \tau \) are disjoint. Let \( J \equiv \{d + 2, d + 3, \ldots, n - d\} \). For every \( i \) in \( J \), which has cardinality \( n - 2d - 1 \), consider the two \( d \)-faces of \( \Sigma^d_n \)

\[
S_i \equiv \sigma \ast i \quad \text{and} \quad T_i \equiv (i - 1) \ast \tau.
\]

Now there are two cases, both of which will result in a weakly-Hamiltonian cycle:

Case 1: For some \( i \), both \( S_i, T_i \) are in \( \Delta \). We are going to introduce a new vertex labeling \( \ell_1, \ldots, \ell_n \). The “consecutive facets of the new labeling” will be called \( L_1 = (\ell_1, \ldots, \ell_d, \ell_{d+1}), L_2 = (\ell_2, \ldots, \ell_{d+2}) \), and so on. The following describes a weakly-Hamiltonian cycle:

- Start with the first \( i - 1 \) vertices in the same order: That is, set \( \ell_1 \equiv 1, \ldots, \ell_{i-1} \equiv i - 1 \). Hence \( L_1 = H_1, L_2 = H_2, \ldots, \) up until \( L_{i-d-1} = H_{i-d-1} \), which (since \( \Delta \) is traceable) is the first of the \( H_i \)'s that contains the vertex \( i - 1 \).
- Then set \( L_i \equiv T_i \). The vertices of \( \tau \) are to be relabeled by \( \ell_i, \ell_{i+1}, \ldots, \ell_{i+d} \). Specifically, label by \( \ell_i \) the vertex that is in \( H_{n-d} \) but not in \( H_{n-d-1} \), by \( \ell_{i+2} \) the vertex in \( H_{n-d-1} \) but not in \( H_{n-d-2} \), and so on. Facet-wise, we are traveling in reverse order across the last facets of the original labeling. Stop until you get to relabel vertex \( i \) by \( \ell_n \). (Or equivalently, if you prefer to think about facets, stop once you reach facet \( H_n \).)
- The weakly-Hamiltonian cycle gets then concluded with \( S_i \), which is adjacent to \( L_1 = H_1 \) via \( \sigma \). The facets previously called \( H_{i-d}, H_{i-d+1}, \ldots, H_{i-1} \) are not part of the new weakly-Hamiltonian cycle.
Figure 1: LEFT: The dashed triangles are $S_5$ and $T_5$. Were they both in $\Delta$, then one could relabel the vertices and create a weakly-Hamiltonian cycle (RIGHT).

**Case 2:** For all $i$, at most one of $S_i$, $T_i$ is in $\Delta$. Since the two sets \( \{ i \in J : \sigma \ast i \in \Delta \} \) and \( \{ i \in J : (i - 1) \ast \tau \in \Delta \} \) are disjoint, the sum of their cardinalities is the cardinality of their union, which is contained in $J$. So

\[
|\{ i \in J : \sigma \ast i \in \Delta \}| + |\{ i \in J : (i - 1) \ast \tau \in \Delta \}| \leq |J| = n - 2d + 1. \tag{1}
\]

Now, we claim that either $\sigma \ast n$ or $1 \ast \tau$ is a face of $\Delta$. From the claim the conclusion follows immediately, as such face creates a weakly-Hamiltonian cycle. We prove the claim by contradiction. Suppose $\Delta$ contains neither $\sigma \ast n$ nor $1 \ast \tau$. Every $d$-face containing $\sigma$ is of the form $\sigma \ast v$, where $v$ is either in $J$ or in the set \( \{ d, n - d + 1, n - d + 2, \ldots, n - 1 \} \) (which has size $d$). So

\[
d_\sigma \leq |\{ i \in J : \sigma \ast i \in \Delta \}| + d. \tag{2}
\]

Symmetrically, the $d$-faces containing $\tau$ are of the form $w \ast \tau$, with $w$ either in $J$ or in the size-$d$ set \( \{ 2, 3, \ldots, d, n - d + 1 \} \). So

\[
d_\tau \leq |\{ i \in J : (i - 1) \ast \tau \in \Delta \}| + d. \tag{3}
\]

Putting together inequalities 1, 2 and 3, we reach a contradiction:

\[
d_\sigma + d_\tau \leq (n - 2d - 1) + d + d = n - 1. \]

**Corollary 19.** Let $\Delta$ be a traceable $d$-dimensional simplicial complex on $n$ vertices, $n > 2d$. If for any two disjoint $(d-1)$-faces $\sigma$ and $\tau$ one has $d_\sigma + d_\tau \geq n$, then $\Delta$ is weakly-Hamiltonian.

**Corollary 20.** Let $\Delta$ be a traceable $d$-dimensional simplicial complex on $n$ vertices, $n > 2d$. If every $(d-1)$-face of $\Delta$ belongs to at least $\frac{n}{2}$ facets of $\Delta$, then $\Delta$ is weakly-Hamiltonian.

**Example 21.** Let $n > 2d$. Let $\Delta$ be the simplicial complex on $n$ vertices obtained from $\Sigma^d_n$ by removing the interior of the $d$-faces $H_{n-d+1}, H_{n-d+2}, \ldots, H_n$. By construction $\Delta$ is traceable, but the given labeling (as well as any labeling obtained from it by reversing or cyclic shifting) fails to prove that $\Delta$ is weakly-Hamiltonian. Now, in $\Delta$, the $(d-1)$-faces $\mu_i = H_i \cap H_{i+1}$, with $i \in \{ n - d + 1, n - d + 2, \ldots, n - 1 \}$, have degree $n - d - 2$. All other $(d-1)$-faces $\nu_j$ contained in one of $H_{n-d+1}, H_{n-d+2}, \ldots, H_n$ have degree $n - d - 1$. Finally, all $(d-1)$-faces not contained in any of $H_{n-d+1}, H_{n-d+2}, \ldots, H_n$ have degree $n - d$. Therefore:
• If \( n \geq 2d + 4 \), Corollary 20 tells us that \( \Delta \) is weakly-Hamiltonian, because \( n - d - 2 \geq \frac{n}{2} \).

• If \( n = 2d+3 \) or \( n = 2d+2 \), any two of the \( \mu_i \)'s are incident, and any \( \nu_j \) is incident to all of the \( \mu_i \)'s. Hence, for any two disjoint \((d-1)\)-faces \( \sigma \) and \( \tau \), we do have \( d_\sigma + d_\tau \geq 2n - 2d - 2 \geq n \). So we can still conclude that \( \Delta \) is weakly-Hamiltonian via Corollary 19.

• If \( n = 2d+1 \), then the assumptions of Corollaries 20 and 19 are not met, but Proposition 18 is still applicable. In fact, for the facets \( \sigma \) resp. \( \tau \) formed by the first resp. the last vertices of the given labeling, one has \( d_\sigma + d_\tau = (n-d) + (n-d-1) = 2n - (2d + 1) = n \).

So in all cases, \( \Delta \) is weakly-Hamiltonian. The proof of Proposition 18 also suggests a relabeling that works: \( \ell_1 \equiv 1, \ell_2 \equiv 2, \ldots, \ell_{d-1} \equiv d - 1, \ell_{d+2} = n - 1, \ldots, \ell_n \equiv d + 2 \).

To see in what sense Proposition 18 is a higher-dimensional version of Ore’s theorem 4, part (A), the best is to give a proof of the latter using the former:

**Proof of Ore’s theorem 4, part (A).** By contradiction, let \( G \) be a non-Hamiltonian graph satisfying \( \deg u + \deg v \geq n \) for all non-adjacent vertices \( u, v \). Add edges to it until you reach a maximal non-Hamiltonian graph \( G^* \). Since any further edge between the existing vertices would create a Hamiltonian cycle, \( G^* \) is traceable, and obviously it still satisfies \( \deg u + \deg v \geq n \). By Proposition 18 \( G^* \) is (weakly-)Hamiltonian, a contradiction. \( \Box \)

It is possible that the bound of Proposition 18 can be improved. But in any case, the possible improvement could only be small, as the following construction shows.

**Non-Example 22.** Let \( d < m \) be positive integers. Take the disjoint union of two copies \( A', A'' \) of \( \Sigma^d_m \). Let \( \mu \) be any facet of \( \Sigma^d_m \) and let \( \mu', \mu'' \) be its copies in \( A' \) and \( A'' \), respectively. Glue to \( A' \cup A'' \) a triangulation without interior vertices of the prism \( \mu \times [0,1] \), so that the lower face \( \mu \times \{0\} \) is identified with \( \mu' \), and the upper face \( \mu \times \{1\} \) is identified with \( \mu'' \). Let \( \Delta \) be the resulting \( d \)-complex on \( n = 2m \) vertices. This \( \Delta \) is traceable: the added prism, triangulated as a path of \( d \)-faces, serves as “bridge” to move between the two copies of \( \Sigma^d_m \). However, this bridge can only be traveled once, so \( \Delta \) is not weakly-Hamiltonian. For the labeling that makes it traceable, \( d_\sigma + d_\tau = (m-d) + (m-d) = n - 2d \).

Our next result is a “Pósa–type” result, in the sense that it extends most of Nash–Williams’ proof [Nas66] of Pósa’s theorem [Pós62] to all dimensions. We focus on complexes \( \Delta \) with the property that any labeling that makes them weakly-traceable, makes them also traceable. Such class is nonempty: for example, it contains all 1-dimensional complexes and all trees of \( d \)-simplices (i.e. all triangulations of the \( d \)-ball whose dual graph is a tree).

**Proposition 23.** Let \( \Delta \) be any traceable pure \( d \)-complex on \( n \) vertices, \( n > 2d \). Suppose that any labeling that makes \( \Delta \) weakly-traceable makes it also traceable. Let \( \sigma_1, \sigma_2, \ldots, \sigma_s \) be an ordering of the \((d-1)\)-faces of \( \Delta^* \), such that the respective degrees \( d_{\sigma_i} \equiv d_{\sigma_i} \) are weakly-increasing, \( d_1 \leq d_2 \leq \ldots \leq d_s \). If for every \( d \leq k < \frac{n}{2} \) one has \( d_{k-d+1} > k \), then \( \Delta \) is weakly-Hamiltonian.

**Proof.** Among all possible labelings that make \( \Delta \) weakly-traceable (and thus traceable, by assumption), choose one that maximizes \( d_\sigma + d_\tau \), where \( \sigma \) is the \((d-1)\)-face of \( H_1 \) spanned by the first \( d \) vertices (that is, \( 1, 2, \ldots, d \)) and \( \tau \) is the \((d-1)\)-face of \( H_{n-d} \) spanned by the last \( d \) vertices (that is, \( n-d+1, \ldots, n \)). Since \( n > 2d \), the faces \( \sigma \) and \( \tau \) are disjoint. If \( d_\sigma + d_\tau \geq n \), using the proof of Proposition 18 we get that \( \Delta \) is weakly-Hamiltonian, and we are done. If not, then one of \( \sigma \) or \( \tau \) has degree less than \( \frac{n}{2} \). Up to reversing the labeling, which would swap \( \sigma \) and \( \tau \), we can assume that \( d_\sigma < \frac{n}{2} \). Now let \( J \equiv \{d+2, d+3, \ldots, n-d\} \). For every \( i \) in \( J \), which has cardinality \( n - 2d - 1 \), consider the two \( d \)-faces of \( \Sigma^d_n \)

\[
S_i \equiv \sigma * i \quad \text{and} \quad T_i \equiv (i-1) * \tau.
\]
We may assume that at most one of these two faces is in $\Delta$, otherwise a weakly-Hamiltonian cycle arises, exactly as in the proof of Proposition 18. Now for each $i$ in $J_1 \overset{\text{def}}{=} \{i \in J : \sigma \ast i \in \Delta\}$, consider the $(d-1)$-face $\rho_i$ with vertices $\{i - d, i - d + 1, \ldots, i - 1\}$.

If for some $i$ in $J_1$ the $d$-face $\rho_i \ast w$ is in $\Delta$, then there is a new relabeling $\ell_1, \ldots, \ell_n$ of the vertices for which we have a weakly-Hamiltonian cycle: see Figure 2 above. (The proof is essentially identical to that of Proposition 18, up to replacing $T_i$ with $T_i^\prime \overset{\text{def}}{=} \rho_i \ast w$, reversing the order, and permuting it cyclically, so that $\rho_i$ is the first face.) Also in this case, we are done.

It remains to discuss the case in which for all $i$ in $J_1 \overset{\text{def}}{=} \{i \in J : \sigma \ast i \in \Delta\}$, the $d$-face $\rho_i \ast w$ is not in $\Delta$. In this case the relabeling $\ell_1, \ldots, \ell_n$ introduced above makes $\Delta$ weakly-traceable, and thus traceable by assumption. For such relabeling, the $(d-1)$-faces spanned by the first and the last $d$ vertices are $\rho_i$ and $\tau$, respectively. So by the way our original labeling was chosen, $d_{\rho_i} + d_{\tau} \leq d_\sigma + d_{\tau}$, and in particular

$$d_{\rho_i} \leq d_\sigma < \frac{n}{2}.$$ 

Now, any $d$-face containing $\sigma$ is of the form $\sigma \ast v$, where $v$ is either in the set $J_1$ or in the set $Z \overset{\text{def}}{=} \{d + 1, n - d + 1, \ldots, n - 1\}$, which has cardinality $d$. So $d_\sigma \leq |J_1 \cup Z|$. Since $J$ and $Z$ are disjoint, and $J_1 \subset J$, the sets $J_1$ and $Z$ are also disjoint and we have

$$d_\sigma - d = d_\sigma - |Z| \leq |J_1 \cup Z| - |Z| = |J_1| + |Z| - |Z| = |J_1|.$$ 

So the set $\{\rho_i : i \in J_1\}$ contains at least $d_\sigma - d$ faces of dimension $d - 1$ and degree $\leq d_\sigma$. If we count also $\sigma$, we have in $\Delta$ at least $d_\sigma - d + 1$ faces of dimension $d - 1$ and degree $\leq d_\sigma$. But then, setting $k \overset{\text{def}}{=} d_\sigma$, we obtain

$$d_{k-d+1} \leq k < \frac{n}{2},$$

which contradicts the assumption. \(\square\)

Again, to see in what sense Proposition 23 is a higher-dimensional version of Pósa’s Theorem 5, perhaps the best is to see how easily the latter follows from the former:

**Proof of Pósa’s theorem 5.** By contradiction, if $G$ is not Hamiltonian, we can add edges to it until we reach a maximal non-Hamiltonian graph $G^*$, which still satisfies the degree conditions and is traceable. By Proposition 23, $G^*$ is (weakly-)Hamiltonian, a contradiction. \(\square\)
A natural question is whether one can generalize to higher dimensions also part (B) of Ore’s theorem 4. The answer is positive, although some extra work is required. In fact, for graphs part (B) of Ore’s theorem can be quickly derived from part (A) by means of a coning trick. This trick however does not extend to higher dimensions, as we explained in Remark 16, so we’ll have to take a long detour, which makes the proof three times as long. The bored reader may skip directly to the next section.

**Definition 24.** A \(d\)-dimensional complex \(\Delta\) is quasi-traceable if there exists a vertex labeling for which \(\Delta \cup H_j\) is weakly-traceable, and moreover, with respect to the same labeling,

(a) if \(j = 1\), then \(\Delta\) contains all of \(H_2, \ldots, H_{n-d}\) (i.e., \(\Delta \cup H_1\) is traceable);
(b) if \(j \in \{2, \ldots, n-2d\}\), then \(\Delta\) already contains all of \(H_1, \ldots, H_{j-1}\) and \(H_{j+d}, \ldots, H_{n-d}\) (i.e., \(\Delta \cup H_j \cup \ldots \cup H_{j+d-1}\) is traceable);
(c) if \(j \in \{n-2d+1, \ldots, n-d-1\}\), then \(\Delta\) contains all of \(H_1, \ldots, H_{j-1}\) and also \(H_{n-d}\) (i.e., \(\Delta \cup H_j \cup \ldots \cup H_{n-d-1}\) is traceable);
(d) if \(j = n-d\), then \(\Delta\) already contains all of \(H_1, \ldots, H_{n-d-1}\) (i.e., \(\Delta \cup H_{n-d}\) is traceable).

**Example 25.** The complex \(\Delta = 123\) is quasi-traceable, although not weakly-traceable. In fact, \(\Delta\) becomes weakly-traceable if we add one of the facets 345 and 456, and it becomes even traceable if we add both.

Definition 24 allows the “added faces” to be already present in \(\Delta\). In particular, all traceable complexes are quasi-traceable. Here comes our high-dimensional version of Theorem 4, part (B):

**Proposition 26.** Let \(\Delta\) be a quasi-traceable \(d\)-dimensional simplicial complex on \(n\) vertices, \(n > 2d\). If in some labeling that makes \(\Delta\) quasi-traceable the two \((d-1)\)-faces \(\sigma\) and \(\tau\) formed by the first \(d\) and the last \(d\) vertices satisfy \(d_\sigma + d_\tau \geq n-1\), then \(\Delta\) is weakly-traceable.

**Proof.** By contradiction, suppose \(\Delta\) is not weakly-traceable; we treat the four cases of Definition 24 separately.

- **Case (a)** is symmetric to Case (d), so we will leave it to the reader.
- **Case (b)** is the main case. Since \(j \in \{2, \ldots, n-2d\}\), by definition \(\Delta\) contains all of \(H_1, \ldots, H_{j-1}\) and also \(H_{j+d}, \ldots, H_{n-d}\). Since \(\Delta\) is not weakly-traceable, it does not contain \(H_j\). Moreover, \(\sigma \ast (d + j)\) cannot be a facet of \(\Delta\), otherwise the two “halfpaths” above would be connected into a weakly-Hamiltonian path. For the same reason, since \((d + j - 1) \in H_{j-1}\), the \(d\)-face \((d + j - 1) \ast \tau\) cannot be in \(\Delta\). So let \(J' \eqdef \{d + 2, d + 3, \ldots, n-d\} \setminus \{d + j\}\). For every \(i\) in \(J'\), which has cardinality \(n-2d-2\), consider the two \(d\)-faces of \(\Sigma^d_i\)

\[
S_i \eqdef \sigma \ast i \quad \text{and} \quad T_i \eqdef (i-1) \ast \tau.
\]

Now there are two subcases: Either there exists an \(i\) such that \(S_i, T_i\) are both in \(\Delta\), or not.

- **CASE (b.1):** For some \(i\), both \(S_i\) and \(T_i\) are in \(\Delta\). There are two subsubcases, according to whether \(i\) is “before the gap” or “after the gap”.

  - **Case (b.1.1):** \(i < d + j\). A weakly-Hamiltonian path arises from a relabeling as follows: We start at the beginning of the second halfpath, with the facets previously called \(H_{j+d}, H_{j+d+1}, \ldots\), until we reach \(H_{n-d}\). Then we use \(T_i\) to get back to the vertex previously labeled by \(i-1\). Next, we use in reverse order the facets previously called \(H_{i-d-1}, H_{i-d-2}, \ldots, H_2, H_1\). Finally use \(S_i\) to jump forward to the vertex previously called \(i\), and conclude the path with the facets previously called \(H_i, H_{i+1}, \ldots, H_{j-1}\).
Case (b.2): For all \( i \), at most one of \( S_i \) and \( T_i \) is in \( \Delta \). Since the two sets \( \{ i \in J' : \sigma * i \in \Delta \} \) and \( \{ i \in J' : (i - 1) * \tau \in \Delta \} \) are disjoint, we obtain a numerical contradiction:

\[
d_{\sigma} + d_{\tau} \leq d + |\{ i \in J' : \sigma * i \in \Delta \}| + d + |\{ i \in J' : (i - 1) * \tau \in \Delta \}| = 2d + |\{ i \in J' : \sigma * i \in \Delta \} \cup \{ i \in J' : (i - 1) * \tau \in \Delta \}| \leq 2d + |J'| = 2d + n - 2d - 2 = n - 2.
\]

Case (c) is the easiest. If \( j \in \{ n - 2d + 1, \ldots, n - d - 1 \} \), then \( H_{j-1} \) intersects \( H_{n-d} \). Since \( \Delta \) contains \( H_1, \ldots, H_{j-1} \) and also \( H_{n-d} \), it is weakly-traceable, a contradiction.

Case (d) is the last one. So, assume \( j = 1 \) and set \( J'' \triangleq \{ d + 2, d + 3, \ldots, n - d \} \). For every \( i \in J'' \), which has cardinality \( n - 2d - 1 \), consider the two \( d \)-faces of \( \Sigma_n^d \)

\[
S_i \triangleq \sigma * i \quad \text{and} \quad T_i \triangleq (i - 1) * \tau.
\]

Now there are two subcases: Either there exists an \( i \) such that \( S_i, T_i \) are both in \( \Delta \), or not.

Case (d.1): For some \( i \), both \( S_i \) and \( T_i \) are in \( \Delta \). Then we obtain a weakly-Hamiltonian path as follows: Starting with \( \ell_1 = 1 \), first we use the face \( \sigma * i \), then \( H_2, \ldots, H_{l_1 - d - 1} \) in their order, and then we use \((i - 1) * n\) to jump forward, and then we come back with \( H_{n-d} \).

Case (d.2): For all \( i \), at most one of \( S_i \) and \( T_i \) is in \( \Delta \). We know by that \( \sigma * d = H_1 \) is not in \( \Delta \) because we are treating the case \( j = 1 \), and we know that \( \sigma * n \) is not in \( \Delta \) otherwise we would have a weakly-Hamiltonian path. Thus any \( d \)-face containing \( \sigma \) is of the form \( \sigma * v \), where \( v \) is either in \( J'' \) or in the disjoint set \( \{ n - d + 1, \ldots, n - 1 \} \), which has cardinality \( d - 1 \). In contrast, any \( d \)-face containing \( \tau \) is of the form \((i - 1) * \tau \), where \( i \) is either in \( J'' \) or in the set \( \{ 2, \ldots, d + 1 \} \), which has cardinality \( d \). Since the two sets \( \{ i \in J'' : \sigma * i \in \Delta \} \) and \( \{ i \in J'' : (i - 1) * \tau \in \Delta \} \) are disjoint, the sum of their cardinality is equal to the cardinality of their union, which is a subset of \( J'' \). So also in this case we obtain a contradiction

\[
d_{\sigma} + d_{\tau} \leq d - 1 + |\{ i \in J'' : \sigma * i \in \Delta \}| + d + |\{ i \in J'' : (i - 1) * \tau \in \Delta \}| = 2d - 1 + |\{ i \in J'' : \sigma * i \in \Delta \} \cup \{ i \in J'' : (i - 1) * \tau \in \Delta \}| \leq 2d - 1 + |J''| = 2d - 1 + n - 2d - 1 = n - 2.
\]

Example 27. Let \( \Delta \) be the simplicial complex on 5 vertices obtained from \( \Sigma_5^2 \) by removing the interior of the two triangles 123 and 124. Clearly \( \Delta \) is quasi-traceable with \( j = 1 \), because \( \Delta \cup H_1 \) is traceable. Since \( d_{12} + d_{15} = 4 = n - 1 \), by Proposition 26 \( \Delta \) is weakly-traceable. (In fact, the reader may verify that \( \Delta \) is even Hamiltonian, with the relabeling \( \ell_1 = 1 \), \( \ell_2 = 2 \), \( \ell_3 = 5 \), \( \ell_4 = 3 \), \( \ell_5 = 4 \).)

For completeness, we conclude this section by showing how Proposition 26 implies part (B) of Ore’s theorem 4:

Proof of Ore’s theorem 4, part (B). By contradiction, let \( G \) be a non-traceable graph satisfying \( \deg u + \deg v \geq n - 1 \) for all non-adjacent vertices \( u, v \). Add edges to it until we reach a maximal non-traceable graph \( G^* \). This \( G^* \) is quasi-traceable and still satisfies \( \deg u + \deg v \geq n - 1 \). By Proposition 26 \( G^* \) is (weakly-)traceable, a contradiction.
2 Interval graphs and semiclosed complexes

In the present section,
(1) we introduce “weakly-closed $d$-complexes”, generalizing co-comparability graphs;
(2) we create a hierarchy of properties between closed and weakly-closed complexes, among which a $d$-dimensional generalization of interval graphs; and
(3) we connect such hierarchy to traceability and chordality.

2.1 A foreword on interval graphs and related graph classes

**Interval graphs** are the intersection graphs of intervals of $\mathbb{R}$. They have long been studied in combinatorics, since the pioneering papers by Lekkerkerker–Boland [Lb62] and Gilmore–Hoffman [GH64], and have a tremendous amount of applications; see e.g. [Gol80, Ch. 8, Sec. 4] for a survey. **Unit-interval graphs**, also known as “indifference graphs” [Rob69] or “proper interval graphs”, are the intersection graphs of unit intervals, or equivalently, the intersection graphs of sets of intervals no two of which are nested. The claw $K_{1,3}$ is the classical example of a graph that can be realized as intersection of four intervals, three of which contained in the forth; but it cannot be realized as intersection of unit intervals.

Bertossi noticed in 1983 that connected unit-interval graphs are traceable [Ber83], whereas connected interval graphs in general are not: The claw strikes. All 2-connected unit-interval graphs are Hamiltonian [CCC97][PD03]; again, this does not extend to 2-connected interval graphs. That said, for interval graphs (and even co-comparability graphs, see below for the definition) the Hamiltonian Path Problem and the Longest Path Problem can be solved in polynomial time [DS52] [MC12]; whereas for arbitrary graphs both problems are well known to be NP-complete, cf. [Kar72].

Given a finite set of intervals in the horizontal real line, we can swipe them “left-to-right”, and thus order them by increasing left endpoint. This so-called “canonical labeling” of the vertices of an interval graph obviously satisfies the following property: for all $a < b < c$,

\[ ac \in G \implies ab \in G. \]  

(4)

This “under-closure” is a characterization: It is easy to prove by induction that any graph with $n$ vertices labeled so that (4) holds can be realized as the intersection graph of $n$ intervals. This result was first discovered by Olario, cf. [LO93, Proposition 4].

There is a “geometrically dual argument” to the one above: Given a finite set of intervals in $\mathbb{R}$, we could also swipe them right-to-left, thereby ordering the intervals by decreasing right endpoint. This yields a vertex labeling that again satisfies (4), for the same geometric reasons. In general, since some of the intervals may be nested, this “dual labeling” bears no relation with the canonical one. But if we start with a finite set of unit intervals, then the dual labeling is simply the reverse of the canonical labeling. Thus in unit-interval graphs, not only the canonical labeling is under-closed, but also its reverse is. Or equivalently, in unit-interval graphs, the canonical labeling is closed ‘both below and above’: in mathematical terms, for all $a < b < c$,

\[ ac \in G \implies ab, bc \in G. \]  

(5)

Again, it is not difficult to prove by induction that any graph with $n$ vertices, labeled so that (5) holds, can be realized as the intersection graph of $n$ unit intervals [LO93, Theorem 1]; see Gardi [Gar07] for a computationally-efficient construction.

Recently Herzog et al. [H&10, E&13] rediscovered unit-interval graphs from an algebraic perspective, which will be discussed in the next chapter. They called them closed graphs and expanded the notion to higher dimensions as well ("closed $d$-complexes"). Later Matsuda [Mat18]...
extended this algebraic approach to the broader class of “co-comparability graphs” (or “weakly-closed graphs”), that we shall now describe in terms of their complement.

Any graph can be given an acyclic orientation by choosing a vertex labeling and then by directing all edges from the smaller to the larger endpoint. Every acyclic orientation can be induced this way. (This is not a bijection: different labelings may induce the same orientation). The drawings of posets, also called comparability graphs, admit also transitive orientations, namely, orientations such that if \( \vec{ab} \) and \( \vec{bc} \) are present, so is \( \vec{ac} \). Let us rephrase this in terms of a vertex labeling, which happens to be the same as a choice of a linear extension of the poset: Comparability graphs are those graphs \( G \) that admit a labeling such that, for all \( a < b < c \),

\[
ab \in G \text{ and } bc \in G \implies ac \in G.
\]

Not all graphs admit transitive orientations: The pentagon, for example, does not.

Co-comparability graphs are by definition the complements of comparability graphs. So they have a labeling that satisfies the contrapositive of the property above: Namely, for all \( a < b < c \),

\[
ac \in G \implies ab \in G \text{ or } bc \in G.
\] (6)

By comparing (4) and (6), it is clear that all interval graphs are co-comparability.

We should mention other two famous properties that all interval graphs enjoy. A graph is perfect if its chromatic number equals the size of the maximum clique. For example, even cycles are perfect, but odd cycles are not, because they have chromatic number 3 and maximal cliques of size 2. Note that in poset drawings, a clique (resp. an independent set) is just a chain (resp. an antichain) in the poset, whereas a coloring represents a partition of the poset into antichains. Thus Dilworth’s theorem (“for every partially ordered set, the maximum size of an antichain equals the minimum number of chains into which the poset can be partitioned” [Dil50] – see Fulkerson [Ful56] for an easy proof) can be equivalently stated as “every co-comparability graph is perfect”. Not all perfect graphs are co-comparability, as shown by large even cycles.

Last property: A graph is chordal if it has no induced subcycles of length \( \geq 4 \). One can characterize chordality in the same spirit of (4), (5) and (6): Namely, a graph is chordal if and only if it admits a labeling such that, for all \( a < b < c \),

\[
ac, bc \in G \implies ab \in G.
\] (7)

In fact, if a graph \( G \) has a labeling that satisfies (7), then \( G \) is obviously chordal, because if \( c \) is the highest-labeled vertex in any induced cycle, then its neighbors \( a \) and \( b \) in the cycle must be connected by a chord by (7). The converse, first noticed by Fulkerson–Gross [FG65], follows recursively from Dirac’s Lemma that every chordal graph has a “simplicial vertex”, i.e. a vertex whose neighbors form a clique (cf. [Gol80, p. 83] for a proof). In fact, let us pick any simplicial vertex and label it by \( n \). Then, in the (chordal!) subgraph induced on the unlabeled vertices, let us pick another simplicial vertex and label it by \( n - 1 \); and so on. The result is a labeling that satisfies (7). See [Gol80, pp. 84–87] for two algorithmic implementations.

Now, if the same labeling satisfies (6) & (7), then it trivially satisfies (4); and conversely, if (4) holds, then also (6) & (7) trivially hold. Thus it is natural to conjecture that interval graphs are the same as the co-comparability chordal graphs. The conjecture is true, although the ‘obvious’ proof does not work: Some labelings on chordal graphs satisfy (6) but not (4), like 13, 23, 24 on the three-edge path. However, Gilmore–Hoffman proved that any labeling that satisfies (6) on a chordal graph (or more generally, on a graph that lacks induced 4-cycles) can be modified in a way that ‘linearly orders’ all maximal cliques [Gol80, Theorem 8.1] and thus satisfies (4). For more characterizations, and a proof that all chordal graphs are perfect, see Golumbic [Gol80, Chapter 4].
2.2 Higher-dimensional analogs and a hierarchy

A $d$-dimensional extension* of Characterization (7) of chordality was provided in 2010 by Emtander [Emt10], and is equivalent to the following:

**Definition 28** (chordal). Let $\Delta$ be a pure $d$-dimensional simplicial complex with $n$ vertices. $\Delta$ is called chordal if there exists a labeling $1, \ldots, n$ of its vertices (called a “PEO” or “Perfect Elimination Ordering”) such that for any two facets $F = a_0a_1\cdots a_d$ and $G = b_0\cdots b_d$ of $\Delta$ with $a_d = b_d$, the complex $\Delta$ contains the full $d$-skeleton of the simplex on the vertex set $F \cup G$.

In 2013, Characterization (5) of unit-interval graphs was generalized as well:

**Definition 29** (closed [E&13]). Let $\Delta$ be a pure $d$-dimensional simplicial complex with $n$ vertices. The complex $\Delta$ is called closed if there exists a labeling $1, \ldots, n$ of its vertices such that for any two facets $F = a_0a_1\cdots a_d$ and $G = b_0\cdots b_d$ of $\Delta$ with $a_i = b_i$ for some $i$, the complex $\Delta$ contains the full $d$-skeleton of the simplex on the vertex set $F \cup G$.

Obviously, closed implies chordal. We now present four notions that in the strongly connected case are progressive weakenings of the closed property (see Theorem 48 and Proposition 52 for the proofs); the first property still implies chordality, whereas the last three do not. In Section 2.3, we connect all these notions to traceability (Theorem 61). One of these properties is “new” even for $d = 1$: We will see its importance in Chapter 3.

**Definition 30** (almost-closed). Let $\Delta$ be a pure $d$-dimensional simplicial complex with $n$ vertices. The complex $\Delta$ is called almost-closed if there exists a labeling $1, \ldots, n$ of its vertices such that for any $d$-face $F = a_0a_1\cdots a_d$ of $\Delta$, the complex $\Delta$ contains the whole $d$-skeleton of the simplex with vertex set $\{a_0, a_0 + 1, a_0 + 2, \ldots, a_d\}$.

**Definition 31** (under-closed). Let $\Delta$ be a pure $d$-dimensional simplicial complex with $n$ vertices. The complex $\Delta$ is called under-closed if there exists a labeling $1, \ldots, n$ of its vertices such that for any $d$-face $F = a_0a_1\cdots a_d$ of $\Delta$ the following condition holds:

- all faces $a_{0i_1}a_{i_2}\cdots a_{i_d}$ of $\Sigma^d_n$ with $i_1 \leq a_1, i_2 \leq a_2, \ldots, i_d \leq a_d$, are in $\Delta$.

**Definition 32** (semi-closed). Let $\Delta$ be a pure $d$-dimensional simplicial complex with $n$ vertices. The complex $\Delta$ is called semi-closed if there exists a labeling of its vertices such that for any $d$-face $F = a_0a_1\cdots a_d$ of $\Delta$, at least one of the two following conditions holds:

- either all faces $a_{0i_1}a_{i_2}\cdots a_{i_d}$ of $\Sigma^d_n$ with $i_1 \leq a_1, i_2 \leq a_2, \ldots, i_d \leq a_d$, are in $\Delta$;
- or all faces $a_{i_0}a_{i_1}\cdots a_{i_d}$ of $\Sigma^d_n$ with $i_0 \geq a_0, i_1 \geq a_1, \ldots, i_{d-1} \geq a_{d-1}$ are in $\Delta$.

**Definition 33** (weakly-closed). Let $\Delta$ be a pure $d$-dimensional simplicial complex with $n$ vertices. The complex $\Delta$ is called weakly-closed if there exists a labeling of its vertices such that for each $d$-face $F = a_0a_1\cdots a_d \in \Delta$, for every integer $g \notin F$ with $a_0 < g < a_d$, there exists a $d$-face $G = b_0b_1\cdots b_d$ in $\Delta$ such that $G$ contains $g$, $G$ is adjacent to $F$, and at least one of the following two conditions hold:

- either $b_d \neq a_d$;
- or $b_0 \neq a_0$.

**Remark 34.** For $d = 1$, and assuming connectedness:

- “closed 1-complexes” and “almost closed 1-complexes” are the same as the unit interval graphs; compare Looges–Olario [LO93, Theorem 1] and Matsuda [Mat18, Prop. 1.3].

*Several different $d$-dimensional generalizations of chordality exist in the literature, e.g. toric chordality [ANS16] or ridge-chordality, cf. e.g. [BB21]. Emtander chose the name “$d$-chordal” for what here we call “chordal".
• “under-closed 1-complexes” are the same as the interval graphs, cf. [LO93, Proposition 4].
• “weakly-closed 1-complexes” are the same as the co-comparability graphs; this is clear from the definition we gave, but a proof is also in Matsuda [Mat18, Theorem 1.9].

We will see that “semi-closed 1-complexes” are an intermediate class between the previous two. For example, such class contains the 4-cycle but not the complement of long even cycles, as we will prove in Theorem 48.

Remark 35 (“Almost-closed” vs. “chordal”). Suppose $F$ and $G$ are two faces of a complex $\Delta$ with $\min F = \min G$. Then any of the two conditions “$\Delta$ is closed”, “$\Delta$ is almost-closed” forces $\Delta$ to contain the full $d$-skeleton of the simplex on the vertex set $F \cup G$. (Instead, the condition “$\Delta$ is under-closed” does not suffice: See Remark 36 below). Symmetrically, if $F$ and $G$ are $d$-faces of $\Delta$ with $\max F = \max G$, and $\Delta$ is either closed or almost-closed, then $\Delta$ must contains the full $d$-skeleton of the simplex on the vertex set $F \cup G$. For this reason, all almost-closed $d$-dimensional complexes are chordal.

Remark 36 (“Under-closed” vs. “chordal”). Not all chordal complexes are under-closed: Already for $d = 1$, the chordal graph $G = 12, 13, 14, 23, 25, 36$, known as “3-sun” or “net graph”, is neither interval nor co-comparability. However, while all interval graphs are chordal (and co-comparability), the statement “all under-closed $d$-complexes are chordal” is false for $d > 1$. In fact, we leave it to the reader to verify that the smallest counterexample is the 2-complex $\Delta \equiv 123, 124, 234, 235$.

The other direction in Gillmore–Hoffman’s theorem (namely, “all chordal co-comparability graphs are interval graphs”) does not extend to $d > 1$ either, as the next Proposition shows.

Proposition 37. (i) Some chordal simplicial complexes are semi-closed, but not under-closed.
(ii) If a simplicial complex is chordal and semi-closed with respect to the same labeling, then with respect to that labeling the complex is also under-closed.

Proof. (i) The example we found is the complex

$$\Sigma = 123, 124, 134, 135, 167, 234, 246.$$ 

The labeling above is a PEO, so $\Sigma$ is chordal. A convenient relabeling (we leave it to the reader to figure out the bijection from the vertex degrees) allows us to rewrite it as

$$\Sigma = 123, 256, 345, 346, 347, 356, 456.$$ 

With this new labeling we see that $\Sigma$ is weakly- and semi-closed. However, with the help of a software designed by Marta Pavelka, we verified that $\Sigma$ is not under-closed.

(ii) Let $\Delta$ be a simplicial complex with a labeling that is a PEO and makes $\Delta$ semi-closed. Let $F = a_0 \cdots a_d$ be a face of $\Delta$ with gap $F > 0$. Let $G = a_0 b_1 \cdots b_d$ be a different $d$-face of $\Sigma_n^d$ such that $G \leq F$ (componentwise) and $\min G = \min F$. We claim that for any $b_i$ not in $F$, there exists a $d$-face $A_i$ of $\Sigma_n^d$ that contains $b_i$, such that $A_i \geq F$ (componentwise) and $\max A_i = \max F$. In fact, by construction $a_0 < b_i \leq b_d \leq a_d$. Since $b_i$ is not in $F$, there exists a unique $j \in \{0, \ldots, d-1\}$ such that $a_j < b_i < a_{j+1}$. Thus if we set

$$A_i \equiv a_0 \cdots a_{j-1} b_i a_{j+1} \cdots a_d$$

the claim is proven. Now, either $F$ satisfies condition (i) of the semi-closed definition, and then $G \in \Delta$; or $F$ satisfies condition (ii), in which case all $A_i$’s are in $\Delta$. But
by construction, the maximum of all these $A_i$’s is $a_d$, the same maximum of $F$. So by chordality, $\Delta$ must contain all the $d$-faces of $\Sigma^d_n$ with vertex set contained in

$$F \cup \bigcup_{i \text{ s.t. } b_i \notin F} A_i = \{a_0, a_1, \ldots, a_d\} \cup \{b_1, \ldots, b_d\} = F \cup G.$$ 

So also in this case $G \in \Delta$. \hfill \Box

**Remark 38.** Part (ii) of Proposition 37 is false if one replaces the assumption “semi-closed” with “weakly-closed”: The subcomplex $\Sigma' = 123, 124, 134, 135, 234$ of $\Sigma$ is weakly-closed and chordal with respect to this labeling, but to prove it under-closed, we need to change labeling.

**Remark 39** ("Under-closed" vs. "Shifted"). Recall that a simplicial complex $\Delta$ on $n$ vertices is called shifted if for every face $F$ of $\Delta$, and for every face $G$ of the simplex on $n$ vertices, if $\dim F = \dim G$ and $F \leq G$ componentwise, then also $G \in \Delta$. Shifted complexes are obviously under-closed. The converse is false, as shown by the graph 12, 23, 34.

**Remark 40.** Being shifted is maintained under taking cones, by assigning label 1 to the new vertex. In contrast, $G = 12, 13, 23$ is closed and chordal, but the cone over it is neither closed nor chordal. In fact, none of the five “closure properties” above (closed, almost-closed, under-closed, semi-closed, weakly-closed) is maintained under taking cones. A counterexample for all is the unit-interval graph $G = 12, 34, 56, 78$. The cone over $G$ is the $U^2_3$ of Lemma 43 below.

Let us start exploring the relations between all the new properties with some Lemmas.

**Lemma 41.** Let $d \geq k \geq 1$ be integers. If a pure $d$-dimensional simplicial complex is almost-closed (resp. under-closed, resp. semi-closed, resp. weakly-closed), then its $k$-skeleton is also almost-closed (resp. under-closed, resp. semi-closed, resp. weakly-closed).

**Proof.** It suffices to prove the claim for $k = d - 1$; the general claim follows then by iterating. We prove only the weakly-closed case; the others are easier. Let $\Delta$ be a pure weakly-closed $d$-complex. Let $\sigma = a_0 \cdots a_{d - 1}$ be a $(d - 1)$-face of $\Delta$. Let $g \notin \sigma$ be an integer such that $a_0 < g < a_{d - 1}$. Since $\Delta$ is pure, there exists a $d$-face $F$ of $\Delta$ that contains $\sigma$. Let $v$ be the vertex of $F$ not in $\sigma$. If $v = g$, i.e. if $F = \{g\} \cup \sigma$, then all the $d$ facets of $\Delta$ different than $\sigma$ are adjacent to $\sigma$ and contain $g$; if we choose one of these $d$ facets that has either different minimum or different maximum than $\sigma$, we are done. So let us assume that $v \neq g$, or equivalently, that $F$ does not contain $g$. By the weakly-closed assumption, there exists a $d$-face $G$ in $\Delta$ such that $G$ contains $g$, $G$ is adjacent to $F$, and $G$ and $F$ do not have same minimum and maximum. If $G$ contains the entire face $\sigma$, i.e. $G = \sigma \cup g$, then again we could conclude as above, choosing some facet of $G$ different than $\sigma$. So we can assume that $G$ does not contain the whole of $\sigma$, or in other words, that the vertex $v$ is present in $G$. Let $\tau$ be the unique face of $G$ that does not contain $v$. By construction, $\sigma$ and $\tau$ are adjacent, and $g \in \tau$. If $\sigma$ and $\tau$ had same minimum and maximum, then also $F$ and $G$ would, because $F$ and $G$ are obtained by adding to $\sigma$ and $\tau$, respectively, the same element $v$. Hence, the $(d - 1)$-skeleton of $\Delta$ is weakly-closed. \hfill \Box

**Lemma 42.** Let $d \geq 2$. Let $C^{d+1}$ be the $(d + 1)$-dimensional simplicial complex with facets $H_1$ and $H_2$. The boundary $S^d$ of $C^{d+1}$ is strongly-connected, semi-closed, but not under-closed. The $d$-skeleton $B^d$ of $C^{d+1}$ is traceable, strongly-connected, almost-closed, but not closed. In particular, the $k$-skeleton of a closed complex need not be closed.

**Proof.** Note that $S^d$ is $B^d$ minus a $d$-face, so since $d \geq 2$ the 1-skeleta of $B^d$ and of $S^d$ coincide. The vertices of $B^d$ (respectively, of $S^d$) can be partitioned with respect to the number of edges
In fact:

- If the label 1 is assigned to a basepoint, then any other vertex is contained in a facet that contains also 1. The same is true if $d + 3$ is assigned to a basepoint. So either way, there is a face $H$ containing both 1 and $d + 3$. Thus gap $H = 2$. But then if the labeling is under-closed, the complex must contain all three facets $12 \cdots dj$, with $j \in \{d + 1, d + 2, d + 3\}$. So we found in $S^d$ three different facets containing the $(d − 1)$-face $\sigma := 12 \cdots d$. This is a contradiction because $S^d$ is topologically a sphere: Every $(d − 1)$-face in it lies in exactly two $d$-faces.

Thus the two claims are proven. So up to a rotation that does not affect the list of facets, both for $B^d$ and $S^d$ we may focus on the labeling that we introduced from the start. With respect to that labeling, $S^d$ is clearly semi-closed, but it is not under-closed, because the $d$-face with vertices $2, 3, \ldots, d + 1, d + 2$ is missing. Similarly, with respect to that labeling, $B^d$ is traceable and almost-closed, but it is not closed for the following reason. Let $F$ (resp. $G$) be the face of vertices $1, 3, 4, \ldots, d + 1, d + 2$ (resp. $2, 3, 4, \ldots, d + 1, d + 3$). Since $F$ (resp. $G$) is contained in the facet $H_1$ (resp. $H_2$) of $C^{d+1}$, it is in $B^d$. Yet vertex 3 appears in second position in both $F$ and $G$. However, the face $H_3$ of vertices $1, 3, 4, \ldots, d + 1, d + 3$ contains the edge connecting the two apices, so $H_3$ is not in $B^d$.

---

**Figure 3:** (i) A 2-complex $B^2 = 123, 124, 134, 234, 235, 245, 246$ that is almost-closed, but not closed; if we remove the triangle 234, we get a 2-complex $S^2$ that is semi-closed, not under-closed, cf. Lemma 42.

(ii) A 2-complex $U_3^2 = 124, 345, 467$ that is closed, but not weakly-closed, cf. Lemma 43.

(iii) A 2-complex $\Delta^2_4 = 123, 124, 125$ that is under-closed, but not almost-closed, cf. Lemma 44.

(iv) A 2-complex $Q^2 = 123, 125, 234, 245$ that is weakly-closed, but not semi-closed, cf. Lemma 46.
Lemma 43. Let $d$ and $k$ be positive integers. Let $U^d_k$ be a one-point union of $k$ copies of $\Sigma^d$. Then $U^d_k$ is closed if and only if $k \leq d + 1$, and it is weakly-closed if and only if $k \leq 2$. In particular, for all $d \geq 2$, the $d$-complex $U^d_{d+1}$ is closed, but not weakly-closed.

Proof. Let $v$ be the vertex common to all facets. When $k > d + 1$, by the pigeonhole principle there are two facets in which $v$ appears in the same position; were $U^d_k$ closed, its dual graph would have to contain a clique, which is not the case. When $k \leq d + 1$, we force the closed property by giving $v$ a label so that $v$ appears in a different position in all facets. We show an algorithm to do this in case $k = d + 1$, leaving the case $k < d + 1$ to the reader. We label $v$ by $f_d = \binom{d+1}{2} + 1$. We label the vertices of the first facet by $123 \ldots d f_d$: so in the first facet, $v$ comes last. Then for all $i = 2, 3, \ldots, k = d + 1$, we label the $i$-th facet by using the next available $d - i$ integers below $f_d$, then $f_d$, then the first $i - 1$ available integers after $f_d$. This way in the $i$-th facet, $v$ comes “$i$-th last”. For example, the labeling we construct for $U^3_4$, since $f_3 = \binom{4}{2} + 1 = 7$, is $U^3_4 = 1237, 4578, 67910, 7111213$. Finally, suppose that $U^d_k$ is weakly-closed. No face of $U^d_k$ has an adjacent facet. Hence, the labeling satisfying the weakly-closed condition must consist only of gap-0 faces. But labeling all facets with consecutive vertices is possible if and only if $k \leq 2$. \hfill \qed

Lemma 44. Let $k \geq 1$ and $d \geq 2$ be integers. Let $\Delta^d_k$ be the $d$-dimensional complex on $d + k$ vertices obtained by joining the $(d-1)$-simplex $\Sigma^{d-1}$ to a $0$-complex consisting of $k$ points. Then

(a) $\Delta^d_k$ is under-closed for all $k$.

(b) $\Delta^d_k$ is (almost) closed, if and only if it is (weakly) traceable, if and only if $k \leq 2$.

Proof. Let us label the vertices of $\Sigma^{d-1}$ by $1, 2, \ldots, d$. This labeling immediately shows that $\Delta^d_k$ is under-closed. Moreover, the $d$-complex $\Delta^d_k$ is strongly-connected. It has exactly $d + k$ vertices and $k$ facets. When $k \leq 2$ its dual graph is a path, so clearly the obvious, consecutive labeling makes $\Delta^d_k$ a closed, almost closed, and traceable complex. But when $k \geq 3$, the “path of $k$ $d$-simplices” is not a subcomplex of $\Delta_d$. Hence, for $k \geq 3$ the complex $\Delta^d_k$ is not traceable, not weakly-traceable, and not weakly-Hamiltonian. The fact that $\Delta^d_k$ is neither almost-closed nor closed can be verified either directly, or using Proposition 52 and Theorem 54 below. \hfill \qed

Remark 45. The $1$-skeleton of $\Delta^3_3 = 123, 124, 125$ (cf. Figure 3) is the graph

$$G_5 = 12, 13, 14, 15, 23, 24, 25$$

which is under-closed by Lemma 41. It is not difficult to see that $G_5$ is the smallest 2-connected interval graph that is not Hamiltonian.

Lemma 46. Let $d \geq 2$ be an integer. Let $Q^d$ be the $d$-dimensional complex on $d + 3$ vertices obtained by taking $d - 1$ consecutive cones over the square. Then $Q^d$ is weakly-closed, but not semi-closed.

Proof. Both $Q^2 = 123, 125, 234, 245$ and $Q^3 = 1236, 1256, 2346, 2456$ are weakly-closed. If we label further coning vertices using consecutive labels after 6, we claim that the weakly-closed property is maintained. (This is not obvious, as the weakly-closed property is not maintained under arbitrary coning, cf. Remark 40.) In fact, since every face $F$ of $Q^3$ contains 6, the gap of $F$ equals the gap of $F \cup \{7\}$, and the missing integers are the same. So as we check whether $Q^4 = 12367, 12567, 23467, 24567$ is weakly-closed, for each face $F \cup \{7\}$ of $Q^4$ we may choose $G \cup \{7\}$ as adjacent face, where $G$ was the face of $Q^3$ adjacent to $F$ that is choosable to show that $Q^3$ is weakly-closed. For the same reasons, were $Q^4$ semi-closed, so would be $Q^3$. So it remains to see that $Q^2$ and $Q^3$ are not semi-closed. This can be done with the help of a computer. \hfill \qed
Lemma 47. Let $\Delta$ be a pure $d$-complex where every vertex is in at most $k$ facets.

(1) In any labeling that makes $\Delta$ weakly-closed, every facet has gap $\leq 2k - 2$.

(2) In any labeling that makes $\Delta$ semi-closed, every facet has gap $\leq k - 1$.

If in addition $d = 1$ and $\Delta$ is a $k$-regular graph, then in any labeling that makes $\Delta$ semi-closed, the $k$ edges of the type $1j$, with $2 \leq j \leq k + 1$, are all in $\Delta$; and so are all the $k$ edges of the type $in$, with $n - k \leq i \leq n - 1$.

(3) In any labeling that makes $\Delta$ almost-closed, every facet has gap $\leq g$, where $g$ is the largest integer such that $(\frac{g + d}{d}) \leq k$; in particular, every facet has gap $\leq \sqrt{kd} - 1$.

Proof. For any vertex $v$ of $\Delta$, let $\deg v$ be the number of facets of $\Delta$ containing it. For any facet $F$ of $\Delta$, let $S_F$ be the set of integers $i \notin F$ such that $\min F < i < \max F$. By definition, $S_F$ has cardinality equal to $\text{gap } F$. For brevity, set $a \equiv \min F$ and $b \equiv \max F$.

(1) For every $i$ in $S_F$, there is a face $G_i$ adjacent to $F$ that contains the vertex $i$ and exactly $d$ vertices of $F$, among which exactly one of $a$, $b$. Clearly as $i$ ranges over $S_F$, the $G_i$’s are all different. So $\deg a + \deg b \geq \text{gap } F + 2$. (The summand 2 is due to the fact that we should count also $F$ itself, once contributing to $\deg a$ and once to $\deg b$). Since $2k \geq \deg a + \deg b$ by assumption, we conclude that $\text{gap } F \leq 2k - 2$.

(2) For every $i$ in $S_F$, either $\Delta$ contains the $n_a \geq \text{gap } F + 1$ facets (including $F$ itself) with minimum $a$ that are componentwise $\leq F$, or $\Delta$ contains the $n_b \geq \text{gap } F + 1$ facets (including $F$ itself) with maximum $b$ that are componentwise $\geq F$. Either way, there is a vertex $v$ (either $a$ or $b$) with $\deg v \geq \text{gap } F + 1$. Since $\deg v \leq k$ by assumption, we conclude that $\text{gap } F \leq k - 1$. So the first claim is settled. From this applied to $d = 1$, it follows that

$$\{ \text{edges of } \Delta \text{ containing } 1 \} \subseteq \{ 1j \text{ such that } 2 \leq j \leq k + 1 \}.$$

The two sets above have size $\deg 1$ and $k$, respectively. If $\Delta$ is $k$-regular, the two quantities are equal, hence the sets coincide. The same argument applies to the edges containing $n$.

(3) For every $i$ in $S_F$, by definition of almost-closed, $\Delta$ contains the $(\text{gap } F + d) = \frac{\text{gap } F + d}{d}$ faces that contain vertex $i$ and have vertices in $\{a, a + 1, \ldots, b\}$. So we must have $(\text{gap } F + d) \leq k$. In particular, since $(\frac{g + d}{d}) \geq (\frac{(g+1)d}{d})$ for all positive integers $d$, we cannot have $(\frac{\text{gap } F + 1}{d}) > k$. \qed

Theorem 48. For each $d \geq 1$, for (pure) simplicial $d$-complexes, one has the hierarchy

$$\{ \text{almost-closed} \} \subset \{ \text{under-closed} \} \subset \{ \text{semi-closed} \} \subset \{ \text{weakly-closed} \} \subset \{ \text{all} \}.$$

Proof. All inclusions are obvious except perhaps the third one. Let $F = a_0a_1\ldots a_d$ be a face of $\Delta$. If $F$ satisfies condition (i) in the definition of semi-closed, and there is a $g$ such that $a_i < g < a_{i+1}$, then $G' \overset{\text{def}}{=} a_0a_1\ldots a_i g a_{i+1}\ldots a_d$ is componentwise $\leq F$ and thus belongs to $\Delta$; moreover, since $\max G' < \max F$, the face $G'$ satisfies condition (i) in the definition of weakly-closed. If instead $F$ satisfies condition (ii) in the definition of semi-closed, and $a_i < g < a_{i+1}$ for some $g$, then $G'' \overset{\text{def}}{=} a_1\ldots a_i g a_{i+1}\ldots a_d$ is componentwise $\geq F$, so $G''$ is in $\Delta$; and since $\min G'' > \min F$, this $G''$ satisfies condition (ii) in the definition of weakly-closed.

Next, we discuss the strictness of the inclusions, which is the interesting part of the theorem.

(i) For $d = 1$, the claw graph 12, 13, 14 is under-closed only with this labeling, which is not almost-closed because for example 23 is missing.

For $d \geq 2$, strictness follows by Lemma 44.

(ii) For $d = 1$, the 4-cycle is semi-closed with the labeling 12, 13, 24, 34. By Lemma 47, part (2), only this labeling makes the 4-cycle semi-closed. This labeling is not under-closed, because 24 is an edge, but 23 is not. More generally, for any $n \geq 4$, one can show that
the graph $\text{sus}(\mathcal{X})$ of Remark 12 is semi-closed (with the suspension apices labeled by $1$ and $n$), but not under-closed.

For $d \geq 2$, the strictness of the inclusion follows by Lemma 42.

(iii) For $d = 1$: Since $C_{2k}$ is a comparability graph (it is the nonempty-face poset of the $k$-gon), $\overline{C_{2k}}$ is co-comparability. We claim that $\overline{C_{2k}}$ is not semi-closed for any $k \geq 3$. For notational simplicity, we give the proof for $k = 3$; the case of arbitrary $k$ has a completely analogous proof. Suppose by contradiction that $\overline{C_{6}}$ has a semi-closed labeling. Since $C_{6}$ is 2-regular, its complement is $(6 - 1 - 2)$-regular, i.e. 3-regular. By Lemma 47, part (2), all of 12, 13, 14 and 36, 46, 56 are edges. In contrast, 15, 16 and 26 are not edges, again by Lemma 47. But then 25 must be an edge of $\overline{C_{6}}$, for otherwise 15, 16, 26 and 25 would form a 4-cycle inside the complement, which is $C_{6}$. We claim that this edge 25 cannot satisfy the semi-closed condition. In fact, if all of 23, 24, 25 were edges, together with 12 we would have 4 edges containing vertex 2, contradicting 3-regularity; and similarly, if all of 25, 35, 45 were edges, counting also 56 we would have 4 edges containing vertex 5. This shows strictness of the inclusion for $d = 1$; the case $d \geq 2$ is settled by Lemma 46.

(iv) For $d = 1$: Cycles of length five or more are well known not to be co-comparability. For completeness, we sketch a simple proof taken from Matsuda [Mat18]. Matsuda shows something stronger, namely, that co-comparability graphs have no induced subcycles of length $n \geq 5$. By contradiction, let $a_{1}, \ldots, a_{n}$ be the vertices of one such induced subcycle, labeled clockwise and such that $a_{1}$ is the smallest of the $a_{i}$'s. Since $n \geq 5$, the labels $a_{n-1}, a_{1}, a_{2}, a_{3}$ are distinct and there is no edge between $a_{3}$ and $a_{n}$. Were $a_{3} < a_{n}$, we would have a contradiction with the weakly-closed assumption: $a_{1}a_{n}$ is in $G$, but neither $a_{1}a_{3}$ nor $a_{3}a_{n}$ is. So $a_{n} < a_{3}$. Similarly, were $a_{n-1} < a_{2}$, we would have a contradiction: $a_{1}a_{2}$ is in $G$, but neither $a_{1}a_{n-1}$ nor $a_{n-1}a_{2}$ is. So $a_{2} < a_{n-1}$. But then were $a_{n} < a_{2} < a_{n-1}$, we would again have a contradiction, because the edge $a_{n}a_{n-1}$ is in $G$, but neither $a_{n}a_{2}$ nor $a_{2}a_{n-1}$ is. So $a_{2} < a_{n}$. Summing up, we have concluded that $a_{2} < a_{n-1}$ and $a_{2} < a_{n} < a_{3}$. Which also results in a contradiction: $a_{2}a_{3}$ is in $G$, but neither $a_{2}a_{n}$ nor $a_{n}a_{3}$ is. This shows strictness of the inclusion for $d = 1$.

For $d \geq 2$, we rely on Lemma 41: The $d$-dimensional annulus

$$A_{n}^{d} \overset{\text{def}}{=} H_{1}, H_{2}, \ldots, H_{n}$$

has a shortest non-contractible cycle in its 1-skeleton of length $\approx \frac{n}{d}$, and such cycle is induced. So by Matsuda’s argument, for $n$ larger than $5d$ the 1-skeleton of $A_{n}^{d}$ is not weakly-closed. By Lemma 41, $A_{n}^{d}$ is not weakly-closed either. (The bound $n > 5d$ is not tight for the conclusion we seek: Already $A_{6}^{2}$, for example, is not weakly-closed, although its 1-skeleton is weakly- and even semi-closed.)

Figure 4: One-dimensional simplicial complexes that are: (i) Not almost-closed, but under-closed. (ii) Not under-closed, but semi-closed. (iii) Not semi-closed, but weakly-closed. (iv) Not even weakly-closed.
2.3 Shortest dual paths and relation with traceability

As we saw in Lemma 43, there exist complexes like $U_3^2 = 124, 345, 467$ that are closed but not weakly-closed. So at this point we owe the reader some explanation: Why did we choose notation like “almost-closed” or “weakly-closed” for properties not implied by “closed”? Here is the reason. We are going to show that all strongly-connected closed complexes are almost-closed (Proposition 52). We will then prove that all such complexes are traceable (Theorem 54), which can be viewed as a higher-dimensional generalization of the graph-theoretical results by Bertossi [Ber83] and Herzog et al’s [H&10, Proposition 1.4]. The key to our generalization is to focus on shortest paths in the dual graph.

**Definition 49.** Let $F$ be a facet a pure $d$-dimensional simplicial complex $\Delta$. Let $v$ be a vertex of $\Delta$. A shortest path between $F$ and $v$ is a path in the dual graph of $\Delta$ of minimal length from $F$ to some facet containing $v$. The distance between $F$ and $v$ is the length of a shortest path, if any exists, or $+\infty$, otherwise.

**Definition 50.** Let $\Delta$ be a pure $d$-dimensional simplicial complex, with vertices labeled from 1 to $n$. A path $F_0, F_1, \ldots, F_\ell$ in the dual graph of $\Delta$ is called ascending, if each $F_i$ is obtained from $F_{i-1}$ by replacing the smallest vertex of $F_{i-1}$, with a vertex greater than all remaining vertices of $F_{i-1}$. A path is called descending, if the reverse path is ascending.

For example, suppose that a 2-complex $\Delta$ contains the facets $124$, $245$, $456$, and $356$. The dual path they form is not ascending – or better, it is ascending, except for the last step. Such dual path demonstrates that the vertex $v = 3$ is at distance $\leq 3$ from $124$. Now suppose that we know in advance that $\Delta$ is closed: Then from $356$, $456 \in \Delta$, we immediately derive that $\Delta$ must contain the whole 2-skeleton of the simplex $3456$. Note that the same conclusion could be reached also if we knew in advance that $\Delta$ is almost-closed, rather than closed. Either way: $\Delta$ contains the facet $G = 345$ which contains 3 and is adjacent to 245. So $124, 245, 345$ yields a “shortcut” to the original path, thereby proving that $v = 3$ is actually at distance $\leq 2$ from 124. And it gets even better: Since 245 and 345 are in $\Delta$, by the closed assumption (or the almost-closed assumption) on $\Delta$, we may conclude that $\Delta$ contains the whole 2-skeleton of the simplex 2345. So also 234 is in $\Delta$, which means that $v = 3$ is at distance 1 from 124.

This example generalizes as follows, in what can be viewed as a higher-dimensional version of Cox–Erskine’s narrowness property [CE15]:

**Lemma 51.** Let $\Delta$ be a pure $d$-dimensional simplicial complex, with a labeling that makes it either closed or almost-closed. Let $F = a_0a_1\cdots a_d$ be a facet of $\Delta$. Let $v$ be a vertex. If the distance between $F$ and $v$ is a finite number $\ell \geq 2$, then

- either there is a shortest path from $F$ to $v$ that is ascending (and thus $v > a_d$),
- or there is a shortest path from $F$ to $v$ that is descending (and thus $v < a_0$).

If instead $a_0 < v < a_d$, and some facet containing $v$ is in the same strongly-connected component of $F$, then the distance between $F$ and $v$ is at most one, and $\Delta$ contains the whole $d$-skeleton of the simplex on the vertex set $F \cup \{v\}$.

**Proof.** Let $F = F_0, \ldots, F_{i-1}, F_i, F_{i+1}$ be a shortest path from $F$ to a vertex $v \in F_{i+1}$. Suppose the path is ascending until $F_i$, but it stops being ascending when passing from $F_i$ to $F_{i+1}$. This means that $\max F_i = \max F_{i+1}$. By Remark 35, $\Delta$ contains the whole $d$-skeleton of the simplex with vertex set $F_i \cup F_{i+1}$. In
particular, if we set $\gamma \overset{\text{def}}{=} F_{i-1} \cap F_i$, the complex $\Delta$ contains $G \overset{\text{def}}{=} \gamma \cup v$. But since $G$ is a $d$-face that contains $v$ and is already adjacent to $F_{i-1}$,

$$F = F_0, \ldots, F_{i-1}, G$$

is a shorter path from $F$ to $v$ than the one we started with, a contradiction. The same argument applies to descending paths. If instead $a_0 < v < a_d$, clearly there cannot be any ascending or descending path from $F$ to $v$. So either $v \in F$, in which case the distance from $F$ to $v$ is 0 and there is nothing to prove, or $v \notin F$, in which case the distance is 1. In the latter case, $F$ and the adjacent face $G$ containing $v$ have same maximum, so again by Remark 35 the complex $\Delta$ contains the $d$-skeleton of the simplex on $F \cup G = F \cup \{v\}$.

\[ \square \]

**Proposition 52.** All strongly-connected closed simplicial complexes are almost-closed.

**Proof.** Let $\Delta$ be a strongly-connected $d$-dimensional simplicial complex that is closed with respect to some-labeling. Let $F = a_0a_1 \cdots a_d \in \Delta$. We claim the following:

(*) If there exist $m \in \{1, \ldots, d\}$ and $g_1, \ldots, g_m$ not in $F$, with $a_0 < g_1 < g_2 < \ldots < g_m < a_d$, then $\Delta$ contains the $d$-skeleton of the simplex with vertex set $\{a_0, a_1, g_1, \ldots, g_m\}$.

If $\text{gap}(F) = 0$, then the implication is trivially true, because the antecedent is never verified. So suppose $\text{gap}(F) > 0$, and let us proceed by induction on $m$.

For $m = 1$: Pick a vertex $g$ of $\Delta$ not in $F$, with $a_0 < g < a_d$. Since $\Delta$ is strongly connected, by the second part of Lemma 51 the complex $\Delta$ has a facet $G$ that contains $g$ and is adjacent to $F$. Had $G$ neither same minimum nor same maximum of $F$, then either $G = a_1a_2 \cdots a_dg$ or $G = ga_0a_1 \cdots a_{d-1}$. But both cases contradict the assumption $a_0 < g < a_d$. Hence, $F$ and $G$ have either same minimum or same maximum (or both), so they share at least one vertex in the same position. Since $\Delta$ is closed, $\Delta$ contains the $d$-skeleton of the simplex on $F \cup G = F \cup \{g\}$.

For $m > 1$: let $H$ be a subset of $\{a_0, \ldots, a_d, g_1, \ldots, g_m\}$ of cardinality $d + 1$. If $H$ contains at most $m - 1$ elements of $\{g_1, \ldots, g_m\}$, then we know that $H \in \Delta$ by the inductive assumption.

If $g_1, \ldots, g_m$ are all vertices of $H$, let us consider a new face $H'$ with exactly the same vertices of $H$, except for one replacement, to be decided as follows:

- If $\min H = a_0$ and $\max H = a_d$, we shall replace $g_1$ with any vertex $v$ of $F$ that is not in $H$. This way, since $a_0 \leq v \leq a_d$, we have that as real intervals

  $$(\min H, \max H) = (a_0, a_d) = (\min H', \max H').$$

- If $\min H = g_1$, or if $\min H = a_i$ for some $i > 0$, we shall replace $g_1$ with $a_0$. This way

  $$(\min H, \max H) \subseteq (a_0, \max H) = (\min H', \max H').$$

- If $\max H = g_m$, or $\max H = a_i$ for some $i < d$, we shall replace $g_m$ with $a_d$. This way

  $$(\min H, \max H) \subseteq (\min H, a_d) = (\min H', \max H').$$

In all three cases, if $w$ is the only element that belongs to $H$ but not to $H'$, then $w$ is either $g_1$ or $g_m$, and we have

$$\min H' < w < \max H'.$$

Moreover, $H'$ contains at most $m - 1$ elements of $\{g_1, \ldots, g_m\}$, so by the inductive assumption $H'$ is in $\Delta$. But since $\min H' < w < \max H'$, by the second part of Lemma 51 we conclude that also $H$ is in $\Delta$. By the genericity of $H$, this proves Claim (*). From the Claim the conclusion follows immediately, by choosing $m$ maximal. \[ \square \]
Remark 53. The converse is false: The complex with $k$ disjoint $d$-simplices is obviously not strongly-connected, yet it is almost-closed with the natural labeling below:

$$
\Delta = H_1, H_{d+2}, H_{2d+3}, \ldots, H_{(k-1)d+k}.
$$

For connected graphs, it is obvious that “closed” and “almost-closed” are the same: This is noticed also in Matsuda [Mat18, Proposition 1.3] and Crupi–Rinaldo [CR14]. However, as we saw in Lemma 42, higher-dimensional complexes that are both strongly-connected and almost-closed might not be closed.

We have arrived to the main result of this section, the generalization of Bertossi’s theorem:

**Theorem 54 (Higher-dimensional Bertossi).** Let $\Delta$ be a pure $d$-dimensional simplicial complex that is either closed or almost-closed. Then

$$
\Delta \text{ is strongly-connected } \iff \Delta \text{ is traceable}.
$$

*Proof.*

$\Leftarrow$: Let $F$ be a $d$-face of $\Delta$. We want to find a walk from $\Delta$ to $H_1$ in the dual graph. If gap $F = 0$, then $F = H_j$ for some $j$, and $H_1, H_2, \ldots, H_j$ is the desired path. If gap $F > 0$, let $i \overset{df}{=} \min F$. Since $F$ and $H_j$ have same minimum, by Remark 35 $\Delta$ contains the whole $d$-skeleton of the simplex on $F \cup H_i$. But the $d$-skeleton of a higher-dimensional simplex is strongly-connected, which means that in the dual graph of $\Delta$ we can walk from $F$ to $H_1$. And since $H_j$ has gap 0, we can walk from it to $H_1$.

$\Rightarrow$: Fix a labeling for which $\Delta$ is (almost-)closed. We are going to show by induction on $j$ that with the *same* labeling, every $H_j$ is in $\Delta$. For $j = 1$, since $\Delta$ is pure, it contains a face $F = a_0a_1\cdots a_d$ with $a_0 = 1$, and then it is easy to derive (either directly, or using that the labeling satisfies the under-closed condition by Theorem 48) that $H_1$ is in $\Delta$. Now suppose that $\Delta$ contains $H_j$ and let us show that $\Delta$ contains $H_{j+1}$. By Lemma 51, $\Delta$ has a $d$-face $H'$ that contains $d + j + 1$ and is adjacent to $H_j$. Such $H'$ has the same vertices of $H_j$, with the exception of a single vertex $i$ that was replaced by $d + j + 1$. Now either $i = j$, in which case $H' = H_{j+1}$ and we are done; or $i > j$. If $i > j$, then $j$ was not replaced, so it is still present in $H'$. Hence $H'$ and $H_j$ are adjacent faces with the same minimum, namely, $j$. By Remark 35, this implies that $H_{j+1}$ is in $\Delta$.

Remark 55. If the “almost-closed” assumption is weakened to “under-closed”, then the direction “$\Rightarrow$” of Theorem 54 no longer holds, with $K_{1,3}$ playing the usual role of the counterexample. The direction “$\Leftarrow$” instead is still valid. We claim in fact that *all weakly-closed traceable complexes are strongly-connected*. To see this, it suffices to show that from any $d$-face $F$ of positive gap we can walk in the dual graph to some gap-0 face. But the weakly-closed definition tells us how to move in the dual graph from $F$ to a face $F'$ of smaller gap than $F$. So if we iterate this, eventually we get from $F$ to a gap-0 face. (The same type of argument is carried out in details in the proof of Theorem 61, item (5), below.) That said, the “weakly-closed” assumption is needed for “$\Leftarrow$”. In fact, for any $d \geq 2$, if $G_d \overset{df}{=} \{1, d+2, 2d+3, \ldots, (k-1)d+k, kd+(k+1), \ldots, d^2+d+1\}$, then the traceable $d$-complex with $d^2 + d + 1$ vertices $\Delta = H_1, H_2, \ldots, H_d, H_{d+1}, G_d$ is not strongly-connected. Its dual graph is a path of length $d^2 + 1$ plus an isolated vertex.

Generalizing a result by Chen, Chang, and Chang [CCC97, Theorem 2], we can push Theorem 54 a bit further. If $D$ is a simplicial complex obtained from $\Delta$ by deleting some vertices $v_1, \ldots, v_k$, then any labeling of $\Delta$ naturally induces a *compressed labeling* for $D$, just by ordering the vertices of $D$ in the same way as they are ordered inside $\Delta$. For example, if $\Delta = 123, 134, 345$, the compressed labeling for $D = \text{del}(2, \Delta)$ is 123, 234. A priori, this $D$ need not be pure.
Lemma 56. Let \( \Delta' \) be a \( d \)-dimensional simplicial complex obtained by deleting some vertices from a \( d \)-dimensional simplicial complex \( \Delta \). If \( \Delta \) is almost-closed (resp. under-closed, resp. semi-closed), then so is \( \Delta' \).

Proof. If the original labeling satisfied the almost-closed (resp. under-closed, resp. semi-closed) condition, so does the compressed labeling. \( \square \)

Lemma 57. Let \( \Delta \) be a \( d \)-dimensional strongly-connected simplicial complex, with a labeling that makes it almost-closed. The following are equivalent:

(a) The deletion of \( d \) or less vertices, however chosen, yields a \( d \)-complex that is strongly connected.

(b) The deletion of \( d \) or less vertices, however chosen, yields a pure \( d \)-complex that with the compressed labeling is traceable.

(c) \( \Delta \) contains all faces of gap \( \leq d \).

Proof. (a) \( \Leftrightarrow \) (b): By Lemma 56 the compressed labeling satisfies the almost closed condition. Via Theorem 54, we conclude.

(b) \( \Rightarrow \) (c): By deleting zero vertices we notice that \( \Delta \) is itself traceable. Let \( F = a_0 \cdots a_d \) be any \( d \)-face of \( \Sigma_n^d \) that has gap \( \leq d \). If \( \text{gap}(F) = 0 \), then \( F \) is one of \( H_1, \ldots, H_{n-d} \), so \( F \) is in \( \Delta \) by definition of traceable. Otherwise, set \( S_F = \{ j \notin F \text{ such that } a_0 < j < a_d \} \). Let \( \Delta' \) be the complex obtained from \( \Delta \) by deleting the vertices in \( S_F \), which are at most \( d \).

By assumption, \( \Delta' \) is traceable with the “compressed labeling”. So \( \Delta' \) contains a gap-0 face of minimum \( a_0 \). But by how the compressed labeling is defined, this face has exactly the vertices that in the original labeling for \( \Delta \) were called \( a_0, a_1, \ldots, a_d \). So \( F \) is in \( \Delta \).

(c) \( \Rightarrow \) (b): Delete from \( \Delta \) the vertices in \( S_F \), defined as above, and call \( \Delta' \) the resulting complex.

With the compressed labeling, \( \Delta' \) is traceable, because any gap-0 \( d \)-face of \( \Delta' \) with the compressed labeling, is a \( d \)-face of \( \Delta \) that had gap \( \leq d \) in the original labeling. It remains to see that \( \Delta' \) is pure. We prove that \( \Delta' \) has no facets of dimension \( d - 1 \), leaving the case of facets of even lower dimensions to the reader. We claim that every \( (d-1) \)-face \( \sigma \) of \( \Delta \) lies in at least \( d + 1 \) distinct \( d \)-faces of \( \Delta \). From the claim the conclusion follows via the pigeonhole principle: If we delete \( d \) vertices, however chosen, then at least one of the \( d \)-faces containing \( \sigma \) will survive the deletion, which implies that \( \sigma \) is not a facet in \( \Delta' \).

So let us prove the claim. Let \( \sigma = b_0 \cdots b_{d-1} \). If \( b_{d-1} = b_0 - d + 1 \) \( \text{def} \text{ gap}(\sigma) \leq d \), then \( b_{d-1} + 1 \leq b_0 + 2d \). So for each \( i \) in the \( (d+1) \)-element set

\[
T_\sigma = \{ b_0, b_0 + 1, \ldots, b_{d-1}, b_{d-1} + 1, \ldots, b_0 + 2d \} \setminus \{ b_0, b_1, \ldots, b_{d-1} \}
\]

the \( d \)-face \( \sigma \cup \{ i \} \) has gap \( \leq d \), and thus is in \( \Delta \) by assumption. If instead \( \text{gap}(\sigma) \geq d + 1 \), we use the almost-closed assumption: for every \( i \) in \( S_{\sigma} \) \( \text{def} \{ i \notin \sigma \text{ such that } b_0 < i < b_{d-1} \} \), the \( d \)-face \( \sigma \cup \{ i \} \) is in \( \Delta \). So either way the claim is proven. \( \square \)

Theorem 58 (Higher-dimensional Chen–Chang–Chang). Let \( \Delta \) be a pure \( d \)-dimensional simplicial complex.

- If \( \Delta \) is almost-closed and the deletion of \( \leq d \) vertices, however chosen, yields a strongly-connected \( d \)-complex, then \( \Delta \) is Hamiltonian.
- If \( \Delta \) is weakly-closed and Hamiltonian, the deletion of \( \leq 1 \) vertices, however chosen, yields a strongly-connected \( d \)-complex.

Proof. For the second claim: Up to a cyclic reshuffling, the vertex we wish to delete is \( n \). The argument of Remark 55 yields a dual path in \( \Delta \) from each \( d \)-face \( F \) to \( H_1 \). If \( F \) does not contain
n, none of the $d$-faces in such dual path does, so the path belongs to the dual graph of the deletion of $n$ from $\Delta$.

Now we prove the first claim. By Lemma 57, $\Delta$ contains all $d$-faces of gap $\leq d$. In particular:

- for any odd $i$ such that $1 \leq i \leq n - 2d$, $\Delta$ contains the gap-$d$ face $O_i$ formed by $i$ and by the first $d$ consecutive odd integers after $i$;
- for any even $j$ such that $2 \leq j \leq n - 2d$, $\Delta$ contains the gap-$d$ face $E_j$ formed by $j$ and by the first $d$ consecutive even integers after $j$;
- $\Delta$ contains the gap-$(d - 1)$ face $F = 1, 2, 4, \ldots, 2d$ formed by $1$ and by the $d$ smallest even natural numbers;
- $\Delta$ contains the gap-$(d - 1)$ face $G$ formed by the largest even integer $\leq n$ and by the $d$ largest odd integers $\leq n$.

Now consider the following sequence $C$ of $d$-faces in $\Delta$: First all $O_i$’s in increasing order, then $G$, then all $E_j$’s in decreasing order, then $F$. Note that any two $O_i$’s are adjacent, and the last of them is adjacent to $G$; symmetrically, any two $E_j$’s are adjacent, and $F$ is adjacent to $E_2$.

We claim that this sequence would form a weakly-Hamiltonian cycle if we relabeled the vertices of $\Delta$ first by listing the odd ones increasingly, and then the even ones decreasingly.

Formally, if $n$ is odd, we introduce the new labeling

$$
\ell_1 \overset{\text{def}}{=} 1, \ell_2 \overset{\text{def}}{=} 3, \ell_3 \overset{\text{def}}{=} 5, \ldots, \ell_{n+1} \overset{\text{def}}{=} n, \ell_{n+1+1} \overset{\text{def}}{=} n - 1, \ell_{n+1+2} \overset{\text{def}}{=} n - 3, \ldots, \ell_{n-1} \overset{\text{def}}{=} 4, \ell_n \overset{\text{def}}{=} 2.
$$

And if instead $n$ is even, we introduce the new labeling

$$
\ell_1 \overset{\text{def}}{=} 1, \ell_2 \overset{\text{def}}{=} 3, \ell_3 \overset{\text{def}}{=} 5, \ldots, \ell_{n} \overset{\text{def}}{=} n, \ell_{n+1} \overset{\text{def}}{=} n - 1, \ell_{n+1+2} \overset{\text{def}}{=} n - 2, \ldots, \ell_{n-1} \overset{\text{def}}{=} 4, \ell_n \overset{\text{def}}{=} 2.
$$

Let us set $L_1 \overset{\text{def}}{=} \ell_1 \ell_2 \cdots \ell_{d+1}$, $L_2 \overset{\text{def}}{=} \ell_2 \ell_3 \cdots \ell_{d+2}$, and so on. Then the sequence $C$ described above is equal (whether $n$ is even or odd) to

$$L_1, L_2, \ldots, L_{\lfloor \frac{n+1}{2}\rfloor-(d-1)}, L_{\lfloor \frac{n+1}{2}\rfloor+1}, L_{\lfloor \frac{n+1}{2}\rfloor+2}, \ldots, L_{n-d}, L_{n-(d-1)}.$$

This shows that with the new labeling $\Delta$ is weakly-Hamiltonian. It remains to show for $d \geq 2$ that our weakly-Hamiltonian cycle can indeed be ‘completed’ to a Hamiltonian cycle, in the sense that the $L_i$’s that were not mentioned in $C$ are anyway contained in $\Delta$. First of all, note that $\Delta$ with the original labeling contained all the $d$-faces of gap $\leq d$, so in particular it contained all $d$-faces containing $1$ and with vertex set contained in $F \cup O_1$. This shows that with the new labeling, $L_{n-(d-2)}, \ldots, L_n$ are all in $\Delta$. So it remains to consider the missing $L_i$’s from the ‘center’ of the sequence $C$. For the “$n$ odd” case (the case for $n$ even is analogous), we have to see whether $\Delta$ contains also the $d - 1$ facets

$$L_{\frac{n+1}{2}-d+2}, L_{\frac{n+1}{2}-d+3}, \ldots, L_{\frac{n+1}{2}}.$$

When we translate these $d$-faces back into the old labeling, it is easy to see that the face with the largest gap is the last one, which has gap $d - 1$. So all these faces are in $\Delta$ by assumption. □

**Example 59.** Let $\Delta$ be an almost-closed 3-complex on $n = 9$ vertices that contains all tetrahedra with gap $\leq 3$. With the notation of Theorem 58 the complex $\Delta$ contains the sequence $C$ below:

$$O_1 = 1357, O_2 = 3579, G = 5789, E_2 = 2468, F = 1246.$$

If we relabel the vertices as in the proof of Theorem 58, the list above becomes

$$L_1, L_2, L_3, L_6, L_7.$$

Thus $\Delta$ is weakly-Hamiltonian. To prove that it is Hamiltonian, we need to check that $L_4, L_5$ and $L_8, L_9$ are in $\Delta$. Translated into the original labeling, this means checking that $6789, 4689$ and $1234, 1235$ are in $\Delta$, which is clearly the case because they all have gap $\leq 2$. 27
Remark 60. For $d = 1$, Theorem 58 boils down to Chen–Chang–Chang’s result that “unit interval graphs are Hamiltonian if and only if they are 2-connected” [CCC97, Theorem 2]. The $G_5$ of Remark 45 is 2-connected and not Hamiltonian; hence the “almost-closed” assumption in the first claim of Theorem 58 is necessary. As for the second claim, the “weakly-closed” assumption is necessary for $d > 1$, because we saw in Remark 2 that some Hamiltonian $d$-complexes are not strongly-connected.

We may condense most of the results of this chapter in the following summary:

Theorem 61. Let $\Delta$ be a $d$-dimensional simplicial complex.

1. If $\Delta$ is (almost)-closed and strongly connected, then $\Delta$ is traceable.
2. If $\Delta$ is (almost)-closed, and the deletion of $d$ or less vertices, however chosen, yields a strongly connected complex, then $\Delta$ is Hamiltonian.
3. If $\Delta$ is under-closed, it contains $H_1$. If in addition $\Delta$ has a face of minimum $i$ for each $i \in \{2, \ldots, n - d\}$, then $\Delta$ is traceable.
4. If $\Delta$ is semi-closed, then for every face $F = a_0 \cdots a_d$ of $\Delta$ either $H_{a_0}$ or $H_{a_d - a}$ in $\Delta$.
5. If $\Delta$ is weakly-closed, then $\Delta$ contains at least one of the $H_i$’s.

Proof. (1) This is given by Proposition 52 and Theorem 54 above.

(2) This is given by Proposition 52 and Theorem 58 above.

(3) By definition of under-closed, if $\Delta$ has a face of minimum $i$, then $\Delta$ contains $H_i$. The fact that $\Delta$ has a face of minimum 1 follows from the assumption that $\Delta$ is pure.

(4) This is straightforward from the definition of semi-closed.

(5) Let $F = a_0a_1 \cdots a_d$ be any facet of $\Delta$ with gap$(F) > 0$. Let $g \notin F$ such that $a_0 < g < a_d$. By definition of “weakly-closed”, some face $G = b_0b_1 \cdots b_d$ of $\Delta$ contains $g$, is adjacent to $F$, and has either $b_0 \neq a_0$ or $b_d \neq a_d$. Thus gap$G < \text{gap } F$. Iterating the process, eventually we find in $\Delta$ a gap-0 face, which has to be one of $H_{a_0}, H_{a_0+1}, \ldots, H_{a_d - d}$.

As for the second claim: By assumption, $\Delta$ contains $H_1$. Also, $\Delta$ contains $H_{a_d - d}$, because no other face has minimum $n - d$. Now let $H' = 2a_1 \cdots a_d$ be a face of $\Delta$ with minimum 2 and gap $\leq d - 1$. By the argument above, we know that $\Delta$ must contain at least one of $H_2, H_3, \ldots, H_{a_d - d}$.

Let us call this face $H_{i_2}$. By how $H'$ was chosen,

$$2 \leq i_2 \leq a_d - d = \text{gap}(H') + 2 \leq d + 1.$$ 

But since $H_1$ contains all vertices from 1 to $d + 1$, in particular it contains $i_2$. So $H_{i_2}$ is incident with $H_1$. Now let $H'' = a_0a_1 \cdots a_d$ be a face of $\Delta$ with gap smaller than $d$, and minimum $a_0 = i_2 + 1$. Repeating the argument above, $\Delta$ contains one of $H_{i_2 + 1}, H_{i_2 + 2}, \ldots, H_{a_d - d}$.

Call this facet $H_{i_3}$; as above, it must intersect $H_{i_2}$. And so on. Eventually, we obtain a list $H_1 = H_{i_1}, H_{i_2}, \ldots, H_{i_{k-1}}, H_{i_k} = H_{n-d}$ of facets of $\Delta$ that makes it weakly-traceable. \qed

Remark 62. In the previous theorem, a relabeling was necessary only to prove item (2). For all other items, the original labeling was already suitable for the desired conclusion. So for item (1) we proved a slightly stronger statement: “If $\Delta$ is strongly-connected, then any labeling that makes $\Delta$ almost-closed automatically makes $\Delta$ traceable”. Same for items (3), (4), (5).
3 Algebraic interpretation

In this section, we review Ene et al’s definition of determinantal facet ideals [E&13]. We find out a large class of them that are radical. In fact, we prove the following:

• if a simplicial complex is semi-closed, then its determinantal facet ideal has a square-free Gröbner degeneration (and in particular is radical), and the quotient by such ideal in positive characteristic is \( F \)-pure (Theorem 74);
• if the complex is also almost-closed, then the generators of its determinantal facet ideal even form a Gröbner basis with respect to a diagonal monomial order (Theorem 76).

3.1 A foreword on \( F \)-pure rings, \( F \)-split rings, and Knutson ideals

Let \( p \) be a prime number. Let \( R \) be a ring of characteristic \( p \). Recall that the Frobenius map is the ring homomorphism from \( R \) to itself that maps an element \( r \in R \) to \( r^p \). We denote by \( F^* R \) the \( R \)-module defined as follows: \( F^* R \) as additive group, and \( r \cdot x \) def = \( r^p x \) for all \( r \in R \) and \( x \in F^* R \). This allows us to view the Frobenius map as a map of \( R \)-modules, \( F : R \to F^* R \).

The ring \( R \) is reduced if and only if \( F \) is injective. So the following definitions are natural:

Definition 63. \( R \) is \( F \)-pure if \( F \otimes 1_M : M \to F_* R \otimes_R M \) is injective for any \( R \)-module \( M \).

Definition 64. \( R \) is \( F \)-split if there exists a homomorphism \( \theta : F_* R \to R \) of \( R \)-modules such that \( \theta \circ F = 1_R \). Such a \( \theta \) is called an \( F \)-splitting of \( R \).

If a ring is \( F \)-split, it is clearly \( F \)-pure. The converse does not hold in general. However, the two concepts are equivalent in a number of cases, for example:

Lemma 65. Let \( R = \bigoplus_{i \in \mathbb{Z}} R_i \) be a Noetherian graded ring of characteristic \( p \) having a unique homogeneous ideal \( \mathfrak{m} \) that is maximal with respect to inclusion. Furthermore, assume that the Noetherian local ring \( R_0 \) is complete. Then the following are equivalent:

(a) \( R \) is \( F \)-split.
(b) \( R \) is \( F \)-pure.
(c) \( F \otimes 1_E : E \to F_* R \otimes_R E \) is injective, where \( E \) is the injective hull of \( R/\mathfrak{m} \).

Proof. (a) \( \implies \) (b) \( \implies \) (c) are obvious implications. To see (c) \( \implies \) (a): the map

\[
F \otimes 1_E : E \to F_* R \otimes_R E
\]

is injective if and only if the corresponding map

\[
\text{Hom}_R(F_* R, \text{Hom}_R(E, E)) \cong \text{Hom}_R(F_* R \otimes_R E, E) \to \text{Hom}_R(E, E)
\]

is surjective. Hence, by [BH93, Corollary 3.6.7, Proposition 3.6.16, Theorem 3.6.17], the corresponding map \( \alpha : \text{Hom}_R(F_* R, R) \to R \) is surjective. So there exists \( \theta \in \text{Hom}_R(F_* R, R) \) such that \( \alpha(\theta) = 1 \). On the other hand, by construction \( \alpha(\theta) = \theta(F(1)) \), hence we have \( \theta \circ F = 1_R \), i.e. \( \theta \) is an \( F \)-splitting of \( R \).

Since we want to study homogeneous quotients of a polynomial rings over a field, by Lemma 65 we may as well regard the \( F \)-split notion and the \( F \)-pure notion as equivalent.

In the following the concept of Knutson ideal will be fundamental. The name arises from the work of Knutson [Kn09], later systematically investigated by the second author [Se20], who extended several properties from \( \mathbb{Z}/p\mathbb{Z} \) to any field. The main result from [Se20] that we shall need is the following:
The elements of this polynomial ring in $(d,n)$ are called... Let $C_g$ be the smallest set of ideals of $S$ containing $(g)$ and such that:

1. $I \in C_g \implies I : h \in C_g$ whenever $h \in S$,
2. $I, J \in C_g \implies I + J \in C_g$, $I \cap J \in C_g$.

If $I \in C_g$, then $\text{in}_{<}(I)$, and therefore $I$, is radical. Furthermore, if $I, J \in C_g$, then $\text{in}_{<}(I + J) = \text{in}_{<}(I) + \text{in}_{<}(J)$ and $\text{in}_{<}(I \cap J) = \text{in}_{<}(I) \cap \text{in}_{<}(J)$. Finally, if $K$ has positive characteristic, $S/I$ is $F$-split whenever $I \in C_g$.

**Example 67.** If $g = x_1x_2 \cdots x_n$, it is simple to check that $C_g$ is the set of squarefree monomial ideals.

### 3.2 Determinantal facet ideals

Let $d, n$ be positive integers with $d + 1 \leq n$. Let $S \overset{\text{def}}{=} K[x_{ij} : i = 1, \ldots, n, j = 0, \ldots, d]$ be a polynomial ring in $(d + 1)n$ variables over some field $K$. Set

$$X = \begin{bmatrix} x_{01} & x_{02} & \cdots & x_{0n} \\ x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{d1} & x_{d2} & \cdots & x_{dn} \end{bmatrix}.$$ 

Given $1 \leq r \leq d$, and integers $0 \leq a_0 < a_1 < \ldots < a_r \leq d$ and $1 \leq b_0 < \ldots < b_r \leq n$, an $(r + 1)$-minor of $X$ is any element of the form

$$[a_0a_1 \ldots a_r | b_0b_1 \ldots b_r] \overset{\text{def}}{=} \det \begin{bmatrix} x_{a_0b_0} & x_{a_0b_1} & \cdots & x_{a_0b_r} \\ x_{a_1b_0} & x_{a_1b_1} & \cdots & x_{a_1b_r} \\ \vdots & \vdots & \ddots & \vdots \\ x_{a_rb_0} & x_{a_rb_1} & \cdots & x_{a_rb_r} \end{bmatrix}.$$ 

If $r = d$, the row indices are forced to be $a_0 = 0, a_1 = 1, \ldots, a_d = d$. For this reason we denote $[01 \ldots d | b_0b_1 \ldots b_d]$ simply by $[b_0b_1 \ldots b_d]$. The ideal of $S$ generated by the $r + 1$-minors of $X$ is denoted by $I_{r+1}(X)$. This ideal defines the variety of $(d + 1)n$ matrices with entries in $K$ and with rank at most $r$.

The set $\Pi$ of all the minors of $X$ can be partially ordered by the relation

$$[a_0a_1 \ldots a_r | b_0b_1 \ldots b_r] \leq [c_0c_1 \ldots c_s | d_0d_1 \ldots d_s] \iff r \geq s, \ a_i \leq c_i \text{ and } b_i \leq d_i \forall i = 0, \ldots, s.$$ 

In particular, for maximal minors the previous definition restricts to

$$[a_0a_1 \ldots a_d] \leq [b_0b_1 \ldots b_d] \overset{\text{def}}{=} a_0 \leq b_0, a_1 \leq b_1, \ldots, a_d \leq b_d.$$ 

It is not our intent to review the theory of *Algebras with Straightening Law* here, as the interested reader can learn it directly from the standard source [BV88]. However, we wish to introduce a few concepts for the sake of clarity. The starting observation is that the polynomial ring $S$ is generated by $\Pi$ as a $K$-algebra. In fact, it turns out that a basis of $S$ as $K$-vector space is given by

$$\{ \pi_1 \cdots \pi_m : m \in \mathbb{N}, \ \pi_i \in \Pi, \ \pi_1 \leq \pi_2 \leq \ldots \leq \pi_m \}.$$ 

The elements of this $K$-basis are called *standard monomials*. It may happen that the product of two standard monomials is not a standard monomial. However, such product will be uniquely
writable as $K$-linear combination of standard monomials, which is in some sense compatible with the poset structure on $\Pi$. This is what is known as ‘Straightening Law’; compare [BV88, Theorem 4.11]. What we wish to outline is that the ideals of $S$ generated by poset ideals of $\Pi$ (i.e. subsets $\Omega \subset \Pi$ such that for all $\omega \in \Omega$, $\pi \in \Pi$, $\pi \leq \omega \implies \pi \in \Omega$) are particularly nice. We make an example to gain a bit of confidence.

**Example 68.** For any $r \leq d$, the ideal $I_{r+1}(X)$ is generated by the poset ideal $\Omega_{\geq r+1}$ consisting of all $t$-minors of $X$ with $t \geq r + 1$. Also, $\Omega_{\geq r+1}$ has a unique maximal element, namely $[d-r \ldots d][n-r \ldots n]$.

Some new notation: if $1 \leq i < j \leq n$, by $X_{[i,j]}$ we mean the matrix

$$X_{[i,j]} = \begin{bmatrix}
x_{0i} & x_{0,i+1} & \cdots & x_{0j} \\
x_{1i} & x_{1,i+1} & \cdots & x_{1j} \\
\vdots & \vdots & \ddots & \vdots \\
x_{di} & x_{d,i+1} & \cdots & x_{dj}
\end{bmatrix},$$

so $I_{r+1}(X_{[i,j]})$ is the ideal of $S$ generated by the $r+1$-minors of $X_{[i,j]}$, whenever $r \leq \min\{d, j-i\}$.

Eventually, we say that a monomial order $<$ on $S$ is a diagonal monomial order if, for all $1 \leq r \leq d$ and integers $0 \leq a_0 < a_1 < \ldots < a_r \leq d$ and $1 \leq b_0 < \ldots < b_r \leq n$, $\text{in}_<(\{a_0 a_1 \ldots a_r|b_0 b_1 \ldots b_r\}) = x_{a_0 b_0} x_{a_1 b_1} \cdots x_{a_r b_r}$. For example, the lexicographic monomial order on $S$ extending the linear order of the variables given by $x_{ij} > x_{ik}$ if and only if $i < h$ or $i = h$ and $j < k$ is a diagonal monomial order. We will use the following result from [St90]:

**Theorem 69 (Sturmfels [St90]).** If $<$ is a diagonal monomial order, $1 \leq i < j \leq n$ and $r \leq \min\{d, j-i\}$, then $\{|a_0 a_1 \ldots a_r| b_0 b_1 \ldots b_r\} : 0 \leq a_0 < a_1 < \ldots < a_r \leq d$ and $i \leq b_0 < \ldots < b_r \leq j$ is a Gröbner basis of the $I_{r+1}(X_{[i,j]})$.

**Definition 70.** Let $\Delta$ be a pure $d$-dimensional simplicial complex on $n$ vertices. Let $K$ be any field. Let $S = K[x_{ij} : i = 1, \ldots, n, j = 0, \ldots, d]$. The determinantal facet ideal of $\Delta$ is the ideal

$$J_\Delta := \langle [a_0 a_1 \ldots a_d] : a_0 a_1 \ldots a_d \in \Delta \rangle \subset S.$$

When $d = 1$, then $\Delta$ is a graph, and $J_\Delta$ is the binomial edge ideal of $\Delta$. Binomial edge ideals have been intensively studied in the recent literature: Among the many papers on this topic, see for example [H&10], [Oht11], [MM13], [Mat18]. Unlike binomial edge ideals, determinantal facet ideals are not always radical – not even if the complex is weakly-closed, as the following example shows:

**Example 71.** Consider the weakly-closed 2-dimensional simplicial complex on five vertices $\Delta = 124, 145, 234, 345$. It can be checked using Macaulay 2 that $J_\Delta$ is not radical.

Let us warm up by showing the algebraic effects of the traceability of $\Delta$ on the determinantal facet ideal $J_\Delta$:

**Proposition 72.** Let $\Delta$ be a traceable $d$-dimensional simplicial complex on $n$ vertices. Then $\text{height}(J_\Delta) = n - d$. Furthermore, if $J_\Delta$ is radical and unmixed, then it admits a square-free initial ideal. If, additionally, $K$ has positive characteristic, then $S/J_\Delta$ is $F$-split.
Proof. Let us fix a labeling for which $\Delta$ is traceable. Set 
\[ C \overset{\text{def}}{=} ([1\ldots d+1],[2\ldots d+2],\ldots,[n-d\ldots n]) \subset J_\Delta. \]

Let us fix a diagonal monomial order $< \text{ on } S$. Note that 
\[ \text{in}_<(\{i\ldots i+d\}) = x_0x_{11+i} \cdots x_{dd+i} \quad \text{and} \quad \text{in}_<(\{j\ldots j+d\}) = x_0x_{11+j} \cdots x_{dd+j} \]
are coprime if $i \neq j$. So $\{[1\ldots d+1],[2\ldots d+2],\ldots,[n-d\ldots n]\}$ is a Gröbner basis of $C$ and 
\[ \text{in}_<(C) = (x_0x_{12} \cdots x_{dd+1}, x_0x_{13} \cdots x_{dd+2}, \ldots, x_0x_{n-d}x_{11+n-d} \cdots x_{dn}) \]
is a complete intersection of height $c$. Hence $C$ is a complete intersection of height $c$ inside $J_\Delta$, which implies height($J_\Delta$) $\geq n-d$. On the other hand height($J_\Delta$) $\leq n-d$ because $J_\Delta$ is contained in $I_{d+1}(X)$, which has height equal to $n-d$.

For the second part of the statement, set $g = [1\ldots d+1]\cdots[n-d\ldots n]$. Notice that in$_<(g)$ is square-free. Obviously, we also have $C \subset C_g$. But if $J_\Delta$ is radical and unmixed, since height($J_\Delta$) = height($C$) by the previous part, then $J_\Delta$ must be of the form $C : h$ for some $h \in S$. Thus $J_\Delta \subset C_g$ and we conclude via Theorem 66. \(\square\)

The next lemma will help us identify a large class of complexes whose determinantal facet ideal is radical.

Lemma 73. Let $1 \leq a_0 < a_1 < \ldots < a_d \leq n$, and $\Gamma_a$ the simplicial complex generated by the facets $a_0i_1 \ldots i_d$ with $i_j \leq a_j$ for all $j = 1, \ldots, d$. Then 
\[ J_{\Gamma_a} = I_{d+1}(X_{[a_0,a_d]}) \cap I_d(X_{[a_0,a_{d-1}]}) \cap I_{d-1}(X_{[a_0,a_{d-2}])} \cap \ldots \cap I_1(X_{[a_0,a_1]}). \]

Analogously, if $\Gamma^a$ is the simplicial complex generated by the facets $i_0i_1 \ldots a_d$ with $i_j \geq a_j$ for all $j = 0, \ldots, d-1$, then 
\[ J_{\Gamma^a} = I_{d+1}(X_{[a_0,a_d]}) \cap I_d(X_{[a_1,a_d]}) \cap I_{d-1}(X_{[a_2,a_d])} \cap \ldots \cap I_1(X_{[a_{d-1},a_d]}). \]

Proof. Since the two identities are symmetric, we prove only the first one. The containment '$\subseteq$' is obvious; so we shall focus on proving '$\supseteq$'. To make the notation lighter, we will make the harmless assumption that $a_0 = 1$. Note that $J_{\Gamma_a}$ is generated by a poset ideal, namely by 
\[ \Omega = \{ \pi \in \Pi : \pi \leq [a_0 \ldots a_d] \}. \]
Similarly, for all $j = 0, \ldots, d$, the ideal $I_{j+1}(X_{[1,a_j]}$) is generated by the poset ideal 
\[ \Omega_j = \{ \pi \in \Pi : \pi \leq [d-j \ldots d|a_j - j | a_j] \}. \]
Since is easy to check that $\Omega = \cap_{j=0}^d \Omega_j$, via [BV88, Proposition (5.2)] we obtain 
\[ J_{\Gamma_a} = I_{d+1}(X_{[1,a_d]}) \cap I_d(X_{[1,a_{d-1}])} \cap I_{d-1}(X_{[1,a_{d-2}]} \cap \ldots \cap I_1(X_{[1,1]}). \]

Now, let $G \in S$ be the product of the minors whose main diagonals are illustrated in the $7 \times 13$ matrix below.

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More precisely,
\[ G = [d|1][d-1, d|1, 2] \cdots [1, 2, \ldots, d-1, d|1, 2, \ldots, d-1, d|d]. \]
\[ [1, 2, \ldots, d, d+1] \cdots [n-d, n-d+1, \ldots, n-1, n]. \]
\[ [n-d+1, n-d+2, \ldots, n-1, n|0, 1, \ldots, d-2, d-1] \cdots [n-1, n|0, 1|n|0]. \]

The reason we brought it up is that if \( < \) is a diagonal monomial order, we have
\[ \text{in}_<(G) = \prod_{i=0}^{d} \prod_{j=1}^{n} x_{ij}. \]

We are now ready to prove our main results of this Chapter.

**Theorem 74.** Let \( \Delta \) be a pure \( d \)-dimensional semi-closed simplicial complex on \( n \) vertices. Then \( J_\Delta \) is a radical ideal. Moreover:

1. For any diagonal monomial order (compatible with the labeling which makes \( \Delta \) semi-closed),
   \[ \text{in}(J_\Delta) \] is a squarefree monomial ideal.
2. If the field \( K \) has positive characteristic, \( S/J_\Delta \) is \( F \)-pure.

**Proof.** We will prove that if \( \Delta \) is semi-closed with respect to the given labeling then \( J(\Delta) \in C_G \), whence both claims follow by Theorem 66.

Let \( 1 \leq a_0 < a_1 < \ldots < a_d \leq n \). Using the notation of Lemma 73, since \( \Delta \) is semi-closed, either \( \Gamma_a \) or \( \Gamma^a \) is contained in \( \Delta \) whenever \( a_0 a_1 \cdots a_d \in \Delta \). For any \( a_0 a_1 \cdots a_d \in \Delta \), set \( \Delta_a = \Gamma_a \) if \( \Gamma_a \subseteq \Delta \), and \( \Delta_a = \Gamma^a \) otherwise. Then
\[ \Delta = \bigcup_{a_0 a_1 \cdots a_d \in \Delta} \Delta_a. \]

In particular,
\[ J(\Delta) = \sum_{a_0 a_1 \cdots a_d \in \Delta} J(\Delta_a). \]

Since \( C_G \) is closed under sums, in order to show that \( J(\Delta) \in C_G \) we only need to check that each \( J(\Delta_a) \in C_G \). To verify this, we use a result in [Se21]: The ideal \( I_{r+1}(X_{ij}) \in C_G \) whenever \( 1 \leq i < j \leq n \) and \( 0 \leq r \leq \min\{d, j-i\} \). Since \( C_G \) is closed under intersections, Lemma 73 guarantees that \( J(\Delta_a) \in C_G \), as desired.

**Remark 75.** The assumption “semi-closed” is best possible: if we replace it with “weakly-closed”, the theorem no longer holds, cf. Example 71. That said, the converse of Theorem 74 is false: There are non-semi-closed simplicial complexes for which \( J_\Delta \) is radical, \( \text{in}(J_\Delta) \) is squarefree, and \( S/J_\Delta \) is \( F \)-pure, cf. Remark 78 below.
Theorem 76. Let $\Delta$ be a pure $d$-dimensional simplicial complex, with a labeling that makes it almost-closed. Then the set $\{[a_0 \ldots a_d] : a_0 \ldots a_d \in \Delta\}$ is a Gröbner basis of $J_\Delta$ with respect to any diagonal monomial order.

Proof. We will prove that if $\Delta$ is almost-closed, then $J_\Delta$ is a sum of determinantal ideals belonging to $C_G$. From this via Theorem 66 it follows that its Gröbner basis is the union of the Gröbner bases of these determinantal ideals.

By definition of almost-closed, $\Delta$ can be written as the union of $d$-skeleta of simplices on consecutive vertices. We can choose these $d$-skeleta to be maximal with respect to inclusion. Formally, this yields a decomposition

$$\Delta = \bigcup_{i_1,j_1} \Sigma^d_{[i_1,j_1]} \cup \bigcup_{i_2,j_2} \Sigma^d_{[i_2,j_2]} \cup \ldots \cup \bigcup_{i_l,j_l} \Sigma^d_{[i_l,j_l]},$$

where $\Sigma^d_{[i_k,j_k]}$ denotes the $d$-skeleton of the simplex on vertices $i_k, i_k + 1, i_k + 2, \ldots, j_k$. Therefore

$$J_\Delta = I_{d+1}(X_{[i_1,j_1]}) + I_{d+1}(X_{[i_2,j_2]}) + \ldots + I_{d+1}(X_{[i_l,j_l]}).$$

But in [Se21] it is proved that $I_{r+1}(X_{[ij]}) \in C_G$ whenever $1 \leq i < j \leq n$ and $0 \leq r \leq \min\{d, j-i\}$. This implies that $J_\Delta \in C_G$; so by Theorem 66

$$\text{in}_<(J_\Delta) = \text{in}_<(I_{d+1}(X_{[i_1,j_1]})) + \text{in}_<(I_{d+1}(X_{[i_2,j_2]})) + \ldots + \text{in}_<(I_{d+1}(X_{[i_l,j_l]})).$$

Thus we get from Theorem 69 that $\{[a_0, \ldots, a_d] : a_0 \ldots a_d \in \Delta\}$ is a Gröbner basis for $J_\Delta$. \qed

Remark 77. Two of the results of [E&13] are incorrect because of the following counterexamples. The graph

$$G = 12, 13, 23, 24, 34$$

is closed, but one can verify that $S/J_G$ is not Cohen-Macaulay. Thus [E&13, Corollary 1.3] is incorrect already for $d = 1$. Moreover, the complex

$$B^2 = 123, 124, 134, 234, 235, 245$$

of Figure 3 is not closed, but the set of all the minors $[abc]$, where $abc$ ranges over all facets of $B^2$, is a Gröbner basis of $J_{B^2}$ for any diagonal monomial order by Theorem 76. The same holds for any $d$-dimensional complex $B_d$ of Lemma 42. Thus one direction of [E&13, Theorem 1.1] is incorrect for all $d > 1$. The other direction is correct, and we will use it for the next Remark.

Remark 78. The assumption “almost-closed” in Theorem 76 is best possible: if we replace it with “under-closed”, the theorem is false already for $d = 1$. In fact, by the work of Herzog et al. [H&10, Theorem 1.1], a graph is unit-interval if and only if the natural generators of its binomial edge ideal form a Gröbner basis with respect to any diagonal monomial order. That said, the converse of Theorem 76 does not hold. In fact, the simplicial complex $U^2_3 = 124, 345, 467$ of Figure 3 is not almost-closed and not even weakly-closed. However $124, 345, 467$ form a Gröbner basis of $J_{C_3^3}$ for any diagonal monomial order, because the initial monomials are pairwise coprime, in accordance with the ‘correct direction’ of [E&13, Theorem 1.1].
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