WAVELET TRANSFORM AND DIFFUSION EQUATIONS: APPLICATIONS TO THE PROCESSING OF THE "CASSINI" SPACECRAFT OBSERVATIONS

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We show that continuous transform with the complex Morlet wavelet is easily performed if we replace the integration of the fast-oscillation function by the solution of the diffusion differential equations. The most important advantage of this approach is that the initial data can be represented by non-uniform sample of an arbitrary node number. We apply the proposed method to the processing of the image of the Saturn A-ring obtained from the Cassini spacecraft. Also we have got via the wavelet transform using PDE, the local correlation coefficient of the signal and the harmonic function.

1. Introduction

As is known, during last 20 years the wavelet transform is one of the contemporary methods for the signal processing. The main advantage of this approach is the high localization of the base functions in both, spatial and frequent regions (see, e.g., 1, 2). This allows us to analyze effectively non-stationary signals and signals with certain singularities. For our convenience, the continuous wavelet transform

\[ w(a, b) = \int_{-\infty}^{+\infty} f(t) \psi^* \left( \frac{t - b}{a} \right) \frac{dt}{a}, \]

(1)
(where the asterisk means the complex conjugation) with the amplitude norm
$$\int_{-\infty}^{\infty} \left| \psi \left( \frac{t-b}{a} \right) \right| \frac{dt}{a} = \text{const.}$$

is the most convenience.

The wavelet basis which is often used for the signal processing is the complex Morlet basis. Its exact form that satisfy the admissible condition
$$\int_{-\infty}^{\infty} \psi(\xi) d\xi = 0,$$

has the following form \(^2\):
$$\psi(\xi) = \frac{1}{\sqrt{2\pi}} \left( e^{-i\omega_0 \xi} - e^{-\frac{\xi^2}{2}} \right) e^{-\frac{\xi^2}{2}}.$$  \(^3\)

In this case, the corresponding wavelet-transform \(w(a,b)\) plays the role of the local spectrum distribution by period \(a\) in the neighborhood of the point \(b\). This feature allows, on the basis of the instant frequency notion \(^3\), to develop the method of the analysis of the solution for Hamiltonian systems. Another approach developed on the basis of the mean-square deviation of the investigated series from the model data (but without of the wavelets) has been proposed in \(^4\). In the present paper, we show that these methods are similar to each other.

It should be however noted, that the standard realization of the continuous wavelet transform based on the transition to the frequency domain and fast Fourier transform (FFT) has certain disadvantages. They follow directly from FFT: the initial data should be presented by the sample \(2^N\) equispaced nodes. Neglect of this condition leads to the algorithm complication and the lost of the accuracy. But the data sets obtained via dynamical systems and some astronomical observations can be presented by sufficiently non-equispaced sample.

Thus, we should rely upon an alternative method of the evaluation of the wavelet transform with the real-valued Gaussian family:
$$\psi_n(\xi) = \frac{1}{\sqrt{2\pi}} \frac{d^n}{d\xi^n} e^{-\frac{\xi^2}{2}}, \quad n \geq 1,$$

This approach presents the wavelet-transform as a derivative of the diffusively smooth initial function \(^5\):
$$w(a,b) = a^n \frac{\partial^n}{\partial b^n} \int_{-\infty}^{+\infty} f(t) \psi_0 \left( \frac{t-b}{a} \right) \frac{dt}{a}.$$
This function can be found as a solution of the diffusion equation with the iterative or finite-difference methods. In this case, there are no restrictions to the initial date representation. In addition, this approach can be adapted for the multidimensional systems. For example, in the analysis of 3D turbulent solutions.

The main purpose of the present paper is to reduce the continuous integral wavelet transform with complex wavelet to the solution of the partial differential equations (a Cauchy problem). It allows us to use the developed algorithm in the local frequency analysis. This approach (which cannot be realized by the standard FFT) additionally makes it possible almost arbitrary scale decompositions. This problem is very important now in application to small-scale structure of the planetary rings, in particular, the Saturn rings (see, e.g., a review). For the first time, analysis the Voyager-2 data by the wavelet transform has been undertaken in. The authors using the real wavelets, detect the structure of the rings in the noisy images. At the same time, the problem related to the local periodicity is still open, but it can be resolved by the complex wavelet transform. This is due to the fact that in the present the necessary data are available by the Huygens-Cassini mission.

2. Continuous transform with the Morlet wavelet

For applications, when $\omega_0$ is a large enough, in (3) one can neglect the second summand. Thus, in the present Part we will used the following simplification:

$$\psi(\xi) = \frac{\omega_0^2}{\sqrt{2\pi}} e^{-i\omega_0 \xi} e^{-\xi^2/2}.$$  \hspace{1cm} (4)

This expression corresponds to the normalization (2) with const = $\exp(\omega_0^2/2)$. Let us show that such a choice (owing to the violation of admissible conditions) makes it possible to use the transformed function as an initial condition for the equation

$$ \left( a \frac{\partial^2}{\partial b^2} - \frac{\partial}{\partial a} - i\omega_0 \frac{\partial}{\partial b} \right) w(a, b) = 0 \hspace{1cm} (5)$$

which is satisfied by the wavelet-image (1) with the Morlet wavelet (see ). Let us write the wavelet-image as a sum of the real and the imaginary parts

$$w(a, b) = u(a, b) + iv(a, b).$$
With respect to these variables Eqs. (5) may be presented by the following system:

\[
\frac{\partial u}{\partial a} = \frac{a}{\partial b^2} \frac{\partial^2 u}{\partial b^2} + \omega_0 \frac{\partial v}{\partial b}, \tag{6}
\]

\[
\frac{\partial v}{\partial a} = \frac{a}{\partial b^2} \frac{\partial^2 v}{\partial b^2} - \omega_0 \frac{\partial u}{\partial b}. \tag{7}
\]

To find the corresponding initial conditions let us rewrite the continuous transform (1) with the kernel (4) in the form:

\[
w(a,b) = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{2\pi a^2}} e^{-\frac{(t-b)^2 - i\omega_0}{2}} dt. \tag{8}
\]

As is known, the integral (8) does not depend on the imaginary subtrahend in the exponent. Also, in the limit \(a \to 0\) the transformation kernel is a delta-function. Therefore, we can get initial conditions

\[
u(0,b) = Re(f(b)),
\]

\[
v(0,b) = Im(f(b))
\]

for the system of differential equations (6)–(7).

Let us apply the described algorithm to the analysis of radial distributions of the matter density in the Saturn A-ring. Here we use the data obtained in July 2004 from the Cassini orbiter (image PIA 06091 from the collection NASA/JPL/Space Science Institute). This image consists of a stripe with the width of 29 pixels in the radial direction. The natural length of the image is about 248 kilometers, and the width is about 7.2 kilometers. In such an approximation we may neglect the curvature of the ring. The function \(f(t)\) obtained by the sample average is shown in Fig.1a.

As the signal is real-valued, we used the following pair of initial conditions: \(u(0,b) = f(b)\) and \(v(0,b) = 0\). Owing to the bound size, the Cauchy problem for the Eqs. (6)–(7) can be replaced by the boundary problem of the first kind. For the real component of the function we used the initial signal value. For the imaginary component zero boundary conditions has been taken. Because it is necessary to get the most good resolution for the present numeric sample, the based frequency has been chosen as \(\omega_0 = \pi\). By the reason that the inverse wavelet transform is not necessary, we neglect by the deviation of the wavelet for the admissible conditions.

The results of numerical analysis, the modulus and the phase of the
wavelet-transform determined as

$$|w(a,b)| = \sqrt{u^2(a,b) + v^2(a,b)}, \quad \phi(a,b) = \arctan \frac{u(a,b)}{v(a,b)},$$

are shown in Fig.1b,c. In this Figure, the larger values correspond more dark regions.

Let us consider the modulus of the wavelet-transform (Fig.1b) in detail. One can easily see that there are regions with monotonically changing instant periods. These regions belong to the intervals 0.05–0.2 and 0.5–0.85. In the non-regular region of the ring A periodic components do not revealed. Note that to avoid the boundary effect, we do not consider intervals closely related to the ends of the sample.

As is known, values and dynamics of the instant period are determined by the modulus maxima lines of the wavelet-transform. Let us extract them from the region with the regular image (see Fig.1b). The obtained lines are shown in Fig.2a,b. The form of these lines confirm the understanding of the fine stricture as the regions corresponding to the high-order resonances with Saturn’s moons. In particular, the analyzed image PIA 06091 from the collection NASA/JPL/Space Science Institute can be construed as the
region between the resonances $11:12$ Prometheus and $3:5$ Mimas. Using the third Kepler law we may obtain the distribution of the major semi-axes corresponding to the Farey sequence from this interval (Fig.2c,d). In this Figure, we clearly see the near resemblance of the found parts of the sequence with the line of the instant radial periods.

![Graphs showing the distribution of radial particle distribution in Saturn's A-ring and the Farey sequence.](image)

Figure 2. Instant periods of the radial particle distribution in the Saturn A-ring (a,b) and the Farey sequence (c,d).

### 3. The wavelet transform with the Morlet basis and the correlation coefficient

The rigorous definition of the Morlet basis (3) makes it possible to find quantitatively the correlation of the instant period with period of the monochromatic harmonics. Introduce a new denotation $\alpha = a/\omega_0$ and rewrite the corresponding integral convolution:

$$
e^{-\frac{1}{2}} \int_{-\infty}^{+\infty} f(t) e^{i(\alpha^{-1})t} e^{\frac{(t-b)^2}{2\alpha^2\omega_0^2}} \frac{1}{\sqrt{2\pi}} dt.
$$

(9)
At $f(t) = 1$ the integral (9) has a closed form solution $e^{i\frac{1}{2}}e^{-\frac{1}{2\alpha^2}}$. Thus, the following representation is admissible:

$$e^{-\frac{1}{2\alpha^2}} = e^{-i\frac{1}{2}} \int_{-\infty}^{+\infty} e^{i(\alpha-1)t} \frac{e^{-\frac{(t-b)^2}{2\alpha^2\omega_0^2}}}{\sqrt{2\pi\alpha^2\omega_0^2}} dt.$$  (10)

Therefore, the wavelet transform (1) with the Morlet basis (3) can be rewritten as:

$$w(a,b) = e^{-i\frac{b}{2}} \left[ \int_{-\infty}^{+\infty} f(t) e^{i(\alpha-1)t} \frac{e^{-\frac{(t-b)^2}{2\alpha^2\omega_0^2}}}{\sqrt{2\pi\alpha^2\omega_0^2}} dt - \int_{-\infty}^{+\infty} e^{i(\alpha-1)t} \frac{e^{-\frac{(t-b)^2}{2\alpha^2\omega_0^2}}}{\sqrt{2\pi\alpha^2\omega_0^2}} dt \right].$$

Let us compare here the expression in the square brackets with the correlation coefficient of two random variables $x$ and $y$:

$$R = \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\sqrt{D_x D_y}},$$

where $\langle \cdot \rangle$ means the averaging operation and $D_x = \langle \xi^2 \rangle - \langle \xi \rangle^2$ is a dispersion. We see that this wavelet transform is equivalent to

$$w(\alpha, b) = e^{-i\frac{b}{2}} \left[ \langle f(t) e^{i(\alpha-1)t} \rangle - \langle f(t) \rangle \langle e^{i(\alpha-1)t} \rangle \right],$$

and $t$ are normally distributed sample nodes. This distribution is centered in $b$.

Let us introduce dispersions:

$$D_f = \left[ \int_{-\infty}^{+\infty} f(t)^2 \frac{e^{-\frac{(t-b)^2}{2\alpha^2\omega_0^2}}}{\sqrt{2\pi\alpha^2\omega_0^2}} dt - \left( \int_{-\infty}^{+\infty} f(t) \frac{e^{-\frac{(t-b)^2}{2\alpha^2\omega_0^2}}}{\sqrt{2\pi\alpha^2\omega_0^2}} dt \right)^2 \right]^{1/2},$$

$$D_{exp} = \left[ \int_{-\infty}^{+\infty} e^{i(\alpha-1)t} \frac{e^{-\frac{(t-b)^2}{2\alpha^2\omega_0^2}}}{\sqrt{2\pi\alpha^2\omega_0^2}} dt - \left( \int_{-\infty}^{+\infty} e^{i(\alpha-1)t} \frac{e^{-\frac{(t-b)^2}{2\alpha^2\omega_0^2}}}{\sqrt{2\pi\alpha^2\omega_0^2}} dt \right)^2 \right]^{1/2}.$$
Now we are able to find quantitatively the local correlation coefficient of the signal $f(t)$ and the function $\exp(i(\alpha^{-1})t)$ in a neighborhood of the given point $t = b$:

$$R_\alpha(b) = \frac{|w(\alpha, b)|}{\sqrt{D_fD_{exp}}}. \quad (11)$$

All the integrals contained the analyzed function are the solution of the Cauchy problem for the simplest diffusion equation:

$$\frac{\partial u(\tau, b)}{\partial \tau} = \alpha^2 \frac{\partial^2 u(\tau, b)}{\partial b^2}.$$  

These solutions are taken in time $\tau = \omega_0^2/2$. As an initial condition we used $f(t) \exp(-(\alpha^{-1})t)$, $f(t)$ or $f(t)^2$. To solve this equation there are robust methods (see, e.g. 10) even on the non-uniform grid.

4. Conclusions

Thus, we showed that continuous transform with the complex Morlet wavelet is easily performed if we replace the integration of the fast-oscillation function by the solution of the differential equations. In spite of fact that this method requires more computer resources, it has certain advantages. The most important of them is that the initial data can be represented by non-uniform sample of an arbitrary number nodes. In addition, this algorithm is realized via the standard MATLAB. During the calculation, the time variable in the diffusion equation is associated with the scale variable of the wavelet transform. Therefore, the choice of a small enough step allows us to trace the evolution of the instant period in detail.

In the present article we apply the proposed method to the processing of the image of the Saturn A-ring PIA 06091. This analysis confirm the resonance origin in its structure.

More detail quantitative description is the subject of our following analysis. This analysis is possible due to the video material obtained from the Cassini spacecraft last half 2004 year. These quantitative estimations can be based on results described in the second part of the article, where we got the dependence of the local correlation coefficient of the signal and the harmonic function in neighborhoods of the given point. Also, the computation of this coefficient may be realized via the solution of diffusion equations.

The proposed algorithm may be additionally considered as a certain extension of the local time-series analysis. Namely, in 4, a set of sequence with the compact support has been advanced as test values. These sequences
are obtained by the periodization of sample's parts. In our analysis, as a test we propose to use the harmonics.

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