Justification announcements in discrete time. Part II: frame definability results

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Abstract. In Part I of this paper, we presented a Hilbert-style system $\Sigma_D$ axiomatizing of stit logic of justification announcements (JA-STIT) interpreted over models with discrete time structure. In this part, we prove three frame definability results for $\Sigma_D$ using three different definitions of a frame plus a yet another version of completeness result.

1 Introduction

The so-called stit logic of justification announcements (JA-STIT, for short) was introduced in [6] as an explicit fragment of the richer logic of $E$-notions introduced in [8]. The underlying idea was to interpret the proving activity as an activity that results in presenting (or, as it were, demonstrating) proofs to the community thus making them publicly available within this community. JA-STIT borrows the representation of proofs which get presented to the community in this way from justification logic by S. Artemov et al. [2], whereas the model for the agentive activities within the community is based on stit logic by N. Belnap et al. [4]. Both of these logics are imported into JA-STIT rather explicitly, which leads to the presence of a full set of their respective modalities in the JA-STIT language. In a similar fashion, the intended models for JA-STIT contain the full set of structural elements present in the models for both justification logic and stit logic.

The interaction between agents and proofs is then provided for by a common pool of proofs publicly announced within the community. This model for justification announcements is inspired by a rather common occurrence when a group of agents tries to produce a proof working on a shared whiteboard. In JA-STIT this situation is modelled in an idealized form, so that one abstracts away from (1) the other available media (like private notes, private messages, etc.), (2) the natural limitations of the actual whiteboard (like limited space and the necessity to erase old proofs), and (3) from the natural limitations of the agents’ communication capacities (like bad handwriting on the part of presenting agents or short-sightedness on the part of spectators).
The state of the common body of publicly presented proofs, or of the community whiteboard, as we will sometimes call it, is described in JA-STIT by modality $Et$, where $t$ is an arbitrary proof polynomial of justification logic. The informal interpretation of $Et$ is that the proof $t$ is presented to the community, or that $t$ is on the whiteboard. This reading also explains the choice of $E$ as notation for this modality, since it serves as a sort of existence predicate for the pool of proofs publicly announced within the community.

The axiomatization of JA-STIT w.r.t. the full class of its intended models was given in [6] in the form of Hilbert-style axiomatic system $\Sigma$. At the same time, Proposition 1 of [6] showed that, rather surprisingly, this axiomatization is sensitive to the temporal structure of the underlying models, even though neither stit logic, nor justification logic, nor else $Et$-modality seem to be relevant to temporal logic, and the standard temporal modalities prove undefinable within JA-STIT. Nevertheless, it turned out that once the class of underlying models is restricted to the models based on discrete time, the axiomatization is no longer complete. The first part of this paper focuses on finding a strongly complete axiomatization $\Sigma_D$ of JA-STIT over the subclass of its intended models which are based on discrete time. We also found a number of less restrictive classes of models in the process — which all induce the same set of validities as the models with discrete temporal substructure. This result shows that one cannot enforce a discrete temporal substructure onto a model by simply postulating an appropriate set of JA-STIT validities. A natural follow-up question then presents itself, namely, how much of a structure can be enforced on the underlying model by simply postulating the set of theorems of $\Sigma_D$. Given that JA-STIT is a variant of modal propositional logic, it is more productive to recast this question in terms of frame definability rather than model definability. In this way, we ask:

**Main question.** Assuming all the theorems of $\Sigma_D$ are valid over the class of models based on a given frame $F$, what can be said about $F$ itself?

The exact meaning of this question clearly depends on how we define the notion of a frame on which a given model is based. Indeed, if we are primarily interested in what our axiomatization has to say about temporal sub-structure of the underlying frame, we need to include at least the set of moments woven together by a temporal accessibility relation. This gives us what we call a temporal frame; but we will show below that the restriction on the class of frames induced by the set of $\Sigma_D$ theorems is not affected at all if we also add the choice function to the frame structure thus extending a temporal frame to a stit frame. By contrast, the situation changes dramatically if we further add to a stit frame the epistemic accessibility relations as these can interact with the stit substructure of the frame in most intricate and subtle ways. In this way we obtain a justification stit frame and another frame definability theorem which is very different from the similar results for temporal and stit frames.

The layout of the rest of the paper is as follows. In Section 2 we define the language and the semantics of the logic at hand. We then define the three versions of the frame notion mentioned in the previous paragraph and consider some natural subclasses in both frame types relevant to the main part of the paper (these will appear in the results presented in Section 3). Next, we recall the definition of $\Sigma_D$ and recapitulate, without a proof, some results from Part I to be used in this second part as well.

Section 3 proves the frame definability results for the three versions of a frame
notion. Additionally, we identify yet another class of models w.r.t. which our axiomatization is complete, this time using the notion of justification stit frame. Section 4 gives some conclusions and drafts directions for future work.

In what follows we will be assuming, due to space limitations, a basic acquaintance with both stit logic and justification logic. We refer the readers to [1] for a quick introduction into the basics of stit logic, and [5, Ch. 2] for the same w.r.t. justification logic.

2 Preliminaries

2.1 Basic definitions and notation

We fix some preliminaries. First we choose a finite set $A_g$ disjoint from all the other sets to be defined below. Individual agents from this set will be denoted by letters $i$ and $j$. Then we fix countably infinite sets $PVar$ of proof variables (denoted by $x, y, z$) and $PConst$ of proof constants (denoted by $c, d$). When needed, subscripts and superscripts will be used with the above notations or any other notations to be introduced in this paper. Set $Pol$ of proof polynomials is then defined by the following BNF:

$$t := x \mid c \mid s + t \mid s \times t \mid !t,$$

with $x \in PVar, c \in PConst, and s, t$ ranging over elements of $Pol$. In the above definition $+$ stands for the sum of proofs, $\times$ denotes application of its left argument to the right one, and $!$ denotes the so-called proof-checker, so that $!t$ checks the correctness of proof $t$.

In order to define the set $Form^{A_g}$ of formulas we fix a countably infinite set $Var$ of propositional variables to be denoted by letters $p, q$. Formulas themselves will be denoted by letters $A, B, C, D$, and the definition of $Form^{A_g}$ is supplied by the following BNF:

$$A := p \mid A \land B \mid \neg A \mid [j]A \mid \Box A \mid t : A \mid KA \mid Et,$$

with $p \in Var, j \in A_g and t \in Pol$.

It is clear from the above definition of $Form^{A_g}$ that we are considering a version of modal propositional language. As for the informal interpretations of modalities, $[j]A$ is the so-called cstit action modality and $\Box$ is the historical necessity modality, both modalities are borrowed from stit logic. The next two modalities, $KA$ and $t : A$, come from justification logic and the latter is interpreted as “t proves A”, whereas the former is the strong epistemic modality “A is known”.

We assume $\Diamond$ as notation for the dual modality of $\Box$. As usual, $\omega$ will denote the set of natural numbers including 0, ordered in the natural way.

For the language at hand, we assume the following semantics. A justification stit (jstit) model for $A_g$ is a structure

$$\mathcal{M} = \langle Tree, \preceq, Choice, Act, R, Re, E, V \rangle$$

such that:

1. $Tree$ is a non-empty set. Elements of $Tree$ are called moments.
2. $\leq$ is a partial order on $\text{Tree}$ for which a temporal interpretation is assumed. We will also freely use notations like $\succ$, $\prec$, and $\triangleright$ to denote the inverse relation and the irreflexive companions.

3. $\text{Hist}$ is a set of maximal chains in $\text{Tree}$ w.r.t. $\leq$. Since $\text{Hist}$ is completely determined by $\text{Tree}$ and $\leq$, it is not included into the structure of model as a separate component. Elements of $\text{Hist}$ are called histories. The set of histories containing a given moment $m$ will be denoted $H_m$. The following set:

$$MH(\mathcal{M}) = \{(m, h) \mid m \in \text{Tree}, h \in H_m\},$$

called the set of moment-history pairs, will be used to evaluate formulas of the above language.

4. $\text{Choice}$ is a function mapping $\text{Tree} \times Ag$ into $2^{\text{Hist}}$ in such a way that for any given $j \in Ag$ and $m \in \text{Tree}$ we have as $\text{Choice}(m, j)$ (to be denoted as $\text{Choice}_j^m$ below) a partition of $H_m$. For a given $h \in H_m$ we will denote by $\text{Choice}_j^m(h)$ the element of partition $\text{Choice}_j^m$ containing $h$.

5. $\text{Act}$ is a function mapping $MH(\mathcal{M})$ into $2^{\text{Pol}}$.

6. $R$ and $R_e$ are two pre-order on $\text{Tree}$ giving two versions of epistemic accessibility relation. They are assumed to be connected by the inclusion $R \subseteq R_e$.

7. $\mathcal{E}$ is a function mapping $\text{Tree} \times \text{Pol}$ into $2^{\text{Form}}$ called admissible evidence function.

8. $V$ is an evaluation function, mapping the set $\text{Var}$ into $2^{MH(\mathcal{M})}$.

Furthermore, a jstit model has to satisfy a number of additional constraints. In order to facilitate their exposition, we introduce a couple of useful notations first. For a given $m \in \text{Tree}$ and any given $h, g \in H_m$ we stipulate that:

$$\text{Act}_m := \bigcap_{h \in H_m} (\text{Act}(m, h)),$$

and:

$$h \approx_m g \iff (\exists m' \triangleright m)(h, g \in H_{m'}).$$

Whenever we have $h \approx_m g$, we say that $h$ and $g$ are undivided at $m$.

The list of constraints on jstit models then looks as follows:

1. Historical connection:

$$(\forall m, m_1 \in \text{Tree})(\exists m_2 \in \text{Tree})(m_2 \leq m \land m_2 \leq m_1).$$

2. No backward branching:

$$(\forall m, m_1, m_2 \in \text{Tree})(m_1 \leq m \land m_2 \leq m) \Rightarrow (m_1 \leq m_2 \lor m_2 \leq m_1)).$$

\footnote{A more common notation $\leq$ is not convenient for us since we also widely use $\leq$ in this paper to denote the natural order relation between elements of $\omega$.}
3. No choice between undivided histories:

\[(\forall m \in \text{Tree})(\forall h, h' \in H_m)(h \approx_m h' \Rightarrow \text{Choice}_j^m(h) = \text{Choice}_j^m(h'))\]

for every \(j \in A_g\).

4. Independence of agents:

\[(\forall m \in \text{Tree})(\forall f : A_g \rightarrow 2^{H_m})(\forall j \in A_g)(f(j) \in \text{Choice}_j^m) \Rightarrow \bigcap_{j \in A_g} f(j) \neq \emptyset)\].

5. Monotonicity of evidence:

\[(\forall t \in \text{Pol})(\forall m, m' \in \text{Tree})(R_e(m, m') \Rightarrow \mathcal{E}(m, t) \subseteq \mathcal{E}(m', t))\].

6. Evidence closure properties. For arbitrary \(m \in \text{Tree}, s, t \in \text{Pol}\) and \(A, B \in \text{Form}\) it is assumed that:

(a) \(A \rightarrow B \in \mathcal{E}(m, s) \land A \in \mathcal{E}(m, t) \Rightarrow B \in \mathcal{E}(m, s \times t)\);
(b) \(\mathcal{E}(m, s) \cup \mathcal{E}(m, t) \subseteq \mathcal{E}(m, s + t)\).
(c) \(A \in \mathcal{E}(m, t) \Rightarrow t : A \in \mathcal{E}(m, !t)\).

7. Expansion of presented proofs:

\[(\forall m, m' \in \text{Tree})(m' \triangleleft m \Rightarrow \forall h \in H_m(\text{Act}(m', h) \subseteq \text{Act}(m, h)))\].

8. No new proofs guaranteed:

\[(\forall m \in \text{Tree})(\text{Act}_m \subseteq \bigcup_{m' \triangleleft m, h \in H_m} \text{Act}(m', h))).\]

9. Presenting a new proof makes histories divide:

\[(\forall m \in \text{Tree})(\forall h, h' \in H_m)(h \approx_m h' \Rightarrow (\text{Act}(m, h) = \text{Act}(m, h'))).\]

10. Future always matters:

\(\triangleleft \subseteq R\).

11. Presented proofs are epistemically transparent:

\[(\forall m, m' \in \text{Tree})(R_e(m, m') \Rightarrow (\text{Act}_m \subseteq \text{Act}_{m'}))).\]

The components like \(\text{Tree}, \triangleleft, \text{Choice}\) and \(V\) are inherited from stit logic, whereas \(R, R_e\) and \(\mathcal{E}\) come from justification logic. The only new component is \(\text{Act}\) which represents the above-mentioned common pool of proofs demonstrated to the community or the state of the community whiteboard at any given moment under a given history. When interpreting \(\text{Act}\), we invoke the classical stit distinction between dynamic (agentive) and static (moment-determinate) entities, assuming that the presence of a given proof polynomial \(t\) on the community whiteboard only becomes an accomplished fact at \(m\).
when \( t \) is present in \( \text{Act}(m, h) \) for every \( h \in H_m \). On the other hand, if \( t \) is in \( \text{Act}(m, h) \) only for some \( h \in H_m \), this means that \( t \) is rather in a dynamic state of being presented, rather than being present, to the community.

Due to space limitations, we skip the explanation of the intuitions behind jstit models. The interested reader may find such an explanation either in Section 2 of Part I of this paper, or in [7, Section 3].

For the members of \( \text{Form}^{Ag} \), we will assume the following inductively defined satisfaction relation. For every jstit model \( M = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_e, E, V \rangle \) and for every \( (m, h) \in MH(M) \) we stipulate that:

\[
\begin{align*}
\mathcal{M}, m, h \models p & \iff (m, h) \in V(p); \\
\mathcal{M}, m, h \models [j]A & \iff (\forall h' \in \text{Choice}_j^p(h))(\mathcal{M}, m, h' \models A); \\
\mathcal{M}, m, h \models \Box A & \iff (\forall h' \in H_m)(\mathcal{M}, m, h' \models A); \\
\mathcal{M}, m, h \models KA & \iff \forall m' \forall h'(R(m, m') \& h' \in H_{m'} \Rightarrow \mathcal{M}, m', h' \models A); \\
\mathcal{M}, m, h \models t:A & \iff A \in E(m, t) \& (\forall m' \in \text{Tree})(R_e(m, m') \& h' \in H_{m'} \Rightarrow \mathcal{M}, m', h' \models A); \\
\mathcal{M}, m, h \models Et & \iff t \in \text{Act}(m, h).
\end{align*}
\]

In the above clauses we assume that \( p \in \text{Var} \); we also assume standard clauses for Boolean connectives.

### 2.2 Frames and their subclasses

In a modal propositional context, it is customary to consider frames alongside models, and frames are normally defined as reducts of the models to components not involving any linguistic entities. In this way, within the classical modal logic a frame is a model minus the evaluation for propositional variables. Thus, in stit logic, a frame will contain \( \text{Tree}, \preceq \) and \( \text{Choice} \) but omit \( V \). In pure justification logic the situation is slightly more complicated, since also the admissible evidence function \( E \) invokes polynomials and sets of formulas. Therefore, in [2] a justification frame is just a set of worlds pre-ordered by the two epistemic accessibility relations; it does not contain \( E \) which will rather be construed as a part of a model based on this frame. When we turn to jstit models, we find \( \text{Act} \) as a further language-dependent component. Even though one can argue that with \( \text{Act} \) we enter a sort of grey area as compared to \( E \), since \( \text{Act} \) invokes proof polynomials but not formulas, in the context of JA-STIT it is clear that \( \text{Act} \) must be outside of frame structure. Indeed, one of the traditional distinctions between frames and models would be that one can evaluate formulas of a given language in models but not in frames. An exception is made for 0-ary connectives like \( \bot \) and \( \top \) and the formulas built from these connectives. Now, within the context of JA-STIT one can evaluate every formula of the form \( Et \) for \( t \in \text{Pol} \) using \( \text{Act} \) alone, and it would be tough to argue that such formulas are just another example of 0-ary connectives.

Having these considerations in mind, we define our frame notions as follows. If \( \mathcal{M} = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_e, E, V \rangle \) is a jstit model for \( Ag \), then \( F = \langle \text{Tree}, \preceq, \text{Choice}, R, R_e \rangle \) is a justification stit (or jstit, for short) frame for \( Ag \), and \( \mathcal{M} \) is said to be based on \( F \). Similarly, we define that \( C = \langle \text{Tree}, \preceq, \text{Choice} \rangle \) (resp. \( T = \langle \text{Tree}, \preceq \rangle \)) is a stit frame (resp. temporal frame) for \( Ag \). In this case, both \( \mathcal{M} \) and \( F \) as defined above are said to be based on \( C \) (resp. \( T \)). Given a class \( \mathcal{F} \) of jstit (resp. stit, temporal) frames, we
will denote the class of jstit models based on the frames from $\mathcal{F}$ by $Mod(\mathcal{F})$. We also observe, that notations like $Hist(F)$ and $MH(F)$ still make perfect sense when $F$ is a jstit, stit, or temporal frame.

An important subclass of jstit frames is made up of what we will call unirelational jstit frames. These are the frames satisfying the additional constraint that $R_e \subseteq R$. It is known (see [2]) that switching from the full class of models to the unirelational models (that is to say, to the models based on unirelational frames) in the context of pure justification logic still leaves us with a class of models w.r.t. which the logic is complete. We have shown (in [6]) that the same holds for JA-STIT over the general class of models based on regular jstit frames. Whenever $\mathcal{F}$ is a class of jstit frames, we will denote $\{F \in \mathcal{F} \mid F$ is unirelational\} by $\mathcal{F} \downarrow$. Similarly, whenever $\mathcal{C}$ is a class of stit or temporal frames we will denote by $Mod^i(\mathcal{C})$ the class of unirelational jstit models based on frames from $\mathcal{C}$.

Before we move on, we need to introduce the notation for an immediate $\triangleleft$-successor of a given moment as it will play an important part in the frame restrictions to be considered below. So whenever $C = \langle Tree, \preceq, Choice \rangle$ is a stit frame and $m, m' \in Tree$, we set that:

$$Next(m, m') \iff (m \triangleleft m' \& (\forall m'' \prec m')(m'' \preceq m)).$$

We now remind the reader the definition of a mixed successor stit frame originally given in Part I:

**Definition 1.** Let $C = \langle Tree, \preceq, Choice \rangle$ be a stit frame. Then we say that $C$ is a mixed successor frame iff for all $m, m_1 \in Tree$ it is true that:

$$[m \triangleleft m_1 \Rightarrow (\exists m_2 \preceq m_1)(Next(m, m_2))] \lor [(\forall h, g \in H_m)(h \approx_m g)].$$

We will denote the class of mixed successor stit frames by $\mathcal{C}_{mixsucc}$, and the same class restricted to the stit frames for a given community $Ag$ by $\mathcal{C}_{mixsucc}^Ag$. Since condition (mixsucc) does not invoke Choice function of a stit frame, it makes perfect sense for temporal frames as well. Therefore, we will denote by $Tree_{mixsucc}$ the class of mixed successor temporal frames and by $Tree_{mixsucc}^Ag$ the class of such frames for $Ag$.

We now proceed to defining the restriction used in our jstit frame definability result. First we need one further technical notion:

**Definition 2.** Let $F = \langle Tree, \preceq, Choice, R, R_e \rangle$ be a jstit frame and let $m \in Tree$. We define $\Theta_m \subseteq 2^{Tree}$ setting that $S \subseteq Tree$ is in $\Theta_m$ iff all of the following conditions hold:

1. $m \in S$;
2. $(\forall m_1, m_2 \in Tree)((m_1 \in S \& R_e(m_1, m_2)) \Rightarrow m_2 \in S)$;
3. $(\forall m_1 \in Tree)((\forall h \in H_{m_1})(\exists m_2 \in h)(Next(m_1, m_2) \& m_2 \in S) \Rightarrow m_1 \in S)$;
4. $(\forall m_1 \in Tree)((m_1 \in S \& (\forall m_2 \triangleleft m_1) \exists m_3 (m_2 \triangleleft m_3 \triangleleft m_1) \Rightarrow (\exists m_4 \triangleleft m_1)(m_4 \in S))$.

We give one important consequence of the above definition as a lemma:
Lemma 1. Let $F = \langle \text{Tree}, \preceq, \text{Choice}, R, R_e \rangle$ be a jstit frame, let $m \in \text{Tree}$, and let $S \in \Theta_m$. Then:

$$(\forall m_0 \in \text{Tree})(m_0 \in S \Rightarrow (\exists m_1 \in \text{Tree})(m_1 \prec m_0)).$$

Proof. Assume, for contradiction, that $m_0 \in S$ but there is no moment $m_1$ such that $m_1 \prec m_0$. Then, by contraposition of Definition 2.4, we must have that:

$$\neg(\forall m_2 \prec m_0)(\exists m_3(m_2 \prec m_3 \prec m_0),$$

whence, pushing the negation inside, we get that:

$$(\exists m_2 \prec m_0)(\forall m_3(m_2 \prec m_3 \Rightarrow \neg m_3 \prec m_0).$$

In particular, for any such $m_2$ we will have $m_2 \prec m_0$ and thus we have got our contradiction in place.

Lemma 1 shows that for a given $m \in \text{Tree}$ the family $\Theta_m$ may end up being empty, for example, when we have $R_e(m, m')$ and $m'$ is the $\preceq$-least moment in $\text{Tree}$. On the other hand, in case when $\text{Tree}$ has no $\preceq$-least moment, $\Theta_m$ is always non-empty, since we will have $\text{Tree} \in \Theta_m$ for all moments $m$. However, within this paper we will be mostly interested in less trivial configurations of $\Theta_m$ families:

Definition 3. Let $F = \langle \text{Tree}, \preceq, \text{Choice}, R, R_e \rangle$ be a jstit frame. Then we say that $F$ is regular iff the following holds for all $m, m_1 \in \text{Tree}$:

$$(m \prec m_1 \& (\exists S \in \bigcap_{m \leq m_0 \leq m_1} \Theta_{m_0})(m \notin S \& (\exists h' \in H_m)((\forall g \in H_{m_1})(h' \neq g) \& (\forall m' \in h'[\text{Next}(m, m') \Rightarrow m' \notin S]))) \Rightarrow (\exists m_2 \preceq m_1)(\text{Next}(m, m_2)) \quad \text{(reg)}$$

Just as with the (mixsucc) restriction on stit frames, we introduce the notation $F_{\text{reg}}$ for the class of regular jstit frames and the notation $F_{\text{reg}}^A_g$ for the class of regular jstit frames for a given agent community $A_g$.

Before we move on to actually proving something, we mention a couple of technical lemmas which were proved in Part I and will be used here without a proof:

Lemma 2. Let $C = \langle \text{Tree}, \trianglelefteq, \text{Choice} \rangle$ be a stit frame. Then:

1. $(\forall m \in \text{Tree})(\forall h \in H_m)(\exists m'(m' \triangleright m) \Rightarrow (\exists m'' \triangleright m)(h \in H_{m''}))$;
2. $(\forall m, m' \in \text{Tree})(m \preceq m' \Rightarrow H_{m'} \subseteq H_m)$;
3. $\approx_m$ is an equivalence relation for every $m \in \text{Tree}$.

Lemma 3. Let $M = \langle \text{Tree}, \trianglelefteq, \text{Choice}, \text{Act}, R, R_e, E, V \rangle$ be a jstit model. Then:

$$(\forall m, m' \in \text{Tree})(\forall h \in H_{m'})(\forall t \in \text{Act})(m \prec m' \& t \in \text{Act}(m, h') \Rightarrow t \in \text{Act}_{m'}).$$

We now establish a connection between the above-defined classes of stit and jstit frames:
Lemma 4. Let $Ag$ be a community of agents, let $F = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_c \rangle$ be a jstit frame for $Ag$, and let $C$ be its reduct to stit frame. Then all of the following statements are true:

1. If $C \in C_{mixsuc}^{Ag}$, then $F \in F_{reg}^{Ag}$;

2. It is possible that $F \in F_{reg}^{Ag}$ but $C \notin C_{mixsuc}^{Ag}$.

Proof. (Part 1) Assume that $C \in C_{mixsuc}^{Ag}$; we show that $F \in F_{reg}^{Ag}$. Indeed, assume that $m, m_1 \in \text{Tree}$, $h' \in H_{m_1}$, and $S \subseteq \text{Tree}$ verify the antecedent of (reg). This implies, among other things that:

$m \preceq m_1 \& (\forall g \in H_{m_1})(h' \not\approx m g)$.

Now, choose any $h \in H_{m_1}$. By Lemma 2.2, we get that $h \in H_m$, and, by the second conjunct of (1), we obtain that $h \not\approx_m h'$ thus falsifying the second disjunct in the condition (mixsuc) for $m, m_1$. Therefore, the first disjunct of the same condition must hold, whereby, given that $m \preceq m_1$, we get that $(\exists m_2 \preceq m_1)(\text{Next}(m, m_2))$, as desired.

As for Part 2, consider a jstit frame $F = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_c \rangle$ such that $C$ is outside $C_{mixsuc}^{Ag}$ and $R = R_c = \text{Tree} \times \text{Tree}$. With these settings we will have:

$$\Theta_m = \{ \emptyset \text{ if there is a } \preceq \text{-least element in Tree; } \}
\{ \text{Tree} \}, \text{ otherwise.}$$

for all $m \in \text{Tree}$. By Definition 2.2, therefore, the second conjunct in the antecedent of (reg) will be trivially falsified. This means that $F \in F_{reg}^{Ag}$, as desired. \qed

Next we recall the definition of the Hilbert-style axiomatic system $\Sigma_D$ from Part I. We first fix an arbitrary agent community $Ag$ (and will keep it fixed till Section 4). The set of axiom schemes for $\Sigma_D$ then looks as follows:

A full set of axioms for classical propositional logic (A0)

$S5$ axioms for $\square$ and $[j]$ for every $j \in Ag$ (A1)

$\square A \rightarrow [j]A$ for every $j \in Ag$ (A2)

$(\Diamond [j]A_1 \& \ldots \& \Diamond [j_n]A_n) \rightarrow \Diamond ([j_1]A_1 \& \ldots \& [j_n]A_n)$ (A3)

$(s: (A \rightarrow B)) \rightarrow (t: A \rightarrow (s \times t):B)$ (A4)

$t: A \rightarrow (t: (t: A) \& KA)$ (A5)

$(s: A \lor t: A) \rightarrow (s + t): A$ (A6)

$S4$ axioms for $K$ (A7)

$KA \rightarrow \square K \square A$ (A8)

$\square Et \rightarrow K \square Et$ (A9)

The assumption is that in (A3) $j_1, \ldots, j_n$ are pairwise different.
The rules of inferences are then as follows:
From $A, A \rightarrow B$ infer $B$; (R1)
From $A$ infer $KA$; (R2)
From $KA \rightarrow (\neg \square Et_1 \lor \ldots \lor \neg \square Et_n \lor \square Es_1 \lor \ldots \lor \square Es_k)$
infer $\Rightarrow KA \rightarrow (\neg Et_1 \lor \ldots \lor \neg Et_n \lor Es_1 \lor \ldots \lor Es_k)$. (R_D)

$\Sigma_D$ is a minimal system in which we make no assumptions as to the properties of proof constants. One standard way to extend this minimal system (following a pattern established in the pure justification logic) is to add a number of assumptions about proof constants. More precisely, let us call a constant specification any set $CS$ such that:

- $CS \subseteq \{c_n: \ldots c_1: A | c_1, \ldots, c_n \in PConst, A \text{ an instance of } (A0) - (A9)\}$;
- Whenever $c_{n+1}: c_1: A \in CS$, then also $c_n: \ldots c_1: A \in CS$.

For a given constant specification, we can define the corresponding inference rule $R_{CS}$ as follows:

If $c_n: \ldots c_1: A \in CS$, infer $c_n: \ldots c_1: A$. (R_{CS})

We now define that $\Sigma_D(CS)$ is just $\Sigma_D$ extended with the rule $R_{CS}$. Since $\emptyset$ is clearly one example of constant specification, we have that $\Sigma_D(\emptyset) = \Sigma_D$ so that our initial axiomatic system is also in the class of systems of the form $\Sigma_D(CS)$. However, when $CS \neq \emptyset$, the corresponding system $\Sigma_D(CS)$ will prove some formulas which are not valid even if we restrict our attention to jstit models based on any class of frames defined in Section 2. We therefore have to describe the restriction on jstit models which comes with a commitment to a given $CS$. We say that a jstit model $M$ is $CS$-normal iff it is true that:

$$(\forall c \in PConst)(\forall m \in Tree)\{{A | c: A \in CS} \subseteq \mathcal{E}(m, c)\},$$

where $\mathcal{E}$ is the $M$’s admissible evidence function. Again, it is easy to see that the class of $\emptyset$-normal jstit models is just the whole class of jstit models so that the representation $\Sigma_D(\emptyset) = \Sigma_D$ does not place any additional restrictions on the class of intended models of $\Sigma_D$. Whenever $F$ is a class of frames, jstit or stit, we will denote the class of $CS$-normal jstit models based on the frames from $F$ by $Mod_{CS}(F)$.

We then define a proof in $\Sigma_D(CS)$ as a finite sequence of formulas such that every formula in it is either an axiom or is obtained from earlier elements of the sequence by one of inference rules. A proof is a proof of its last formula. If an $A \in Form^{Ag}$ is provable in $\Sigma_D(CS)$, we will write $\vdash_{CS} A$. We say that $\Gamma \subseteq Form^{Ag}$ is inconsistent in $\Sigma_D(CS)$ (or $CS$-inconsistent) iff for some $A_1, \ldots, A_n \in \Gamma$ we have $\vdash_{CS} (A_1 \land \ldots \land A_n) \rightarrow \bot$, and we say that $\Gamma$ is consistent in $\Sigma_D(CS)$ (or $CS$-consistent) iff it is not inconsistent in $\Sigma_D(CS)$.

Finally, we cite (in a somewhat weakened form) the main result of Part I, which we will use in this part without a proof:

**Theorem 1.** Let $\Gamma \subseteq Form^{Ag}$. Then $\Gamma$ is $CS$-consistent iff it is satisfiable in $Mod_{CS}(C^{Ag}_{mix succ})$ iff it is satisfiable in $Mod^{4}_{CS}(C^{Ag}_{mix succ})$. 

We can immediately state a similar completeness result for the temporal frames:

**Corollary 1.** Let $\Gamma \subseteq \text{Form}^{Ag}$. Then $\Gamma$ is $\mathcal{CS}$-consistent iff it is satisfiable in $\text{Mod}_{\mathcal{CS}}(\mathcal{M}^{mixsuc})$ iff it is satisfiable in $\text{Mod}_{\mathcal{CS}}(\mathcal{T}^{Ag}_{mixsuc})$.

**Proof.** Note that $\mathcal{M}$ is a (unirelational) jstit model based on a frame from $\mathcal{C}^{Ag}_{mixsuc}$ iff $\mathcal{M}$ is a (unirelational) jstit model based on a frame from $\mathcal{T}^{Ag}_{mixsuc}$.

### 3 Frame definability results

#### 3.1 Temporal and stit frames

We deal with the stit frames first. The principal lemma looks as follows:

**Lemma 5.** Let $\mathcal{CS}$ be a constant specification and let $C = \langle \text{Tree}, \leq, \text{Choice} \rangle$ be a stit frame outside $\mathcal{C}^{Ag}_{mixsuc}$. Then there is a $\mathcal{CS}$-normal jstit model $\mathcal{M} = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, R_e, \mathcal{E}, V \rangle$ based on $C$ such that for some $(m, h) \in MH(\mathcal{M})$ it is true that:

$$\mathcal{M}, m, h \not\models K(\Box Ex \lor \neg \Box Ey) \rightarrow (Ex \lor \neg Ey).$$

**Proof.** Assume that $C \notin \mathcal{C}^{Ag}_{mixsuc}$. Then we can choose $m_0, m_1 \in \text{Tree}$ and $h_0, h_1 \in H_{m_0}$ such that:

$$(h_0 \not\approx_{m_0} h_1) \land (m_0 \prec m_1) \land (\forall m \leq m_1)(\neg \text{Next}(m_0, m)).$$

We now extend $C$ to $\mathcal{M}$ setting $R = R_e = \leq, \mathcal{E}(m, t) = \text{Form}^{Ag}$ for all $m \in \text{Tree}$ and $t \in \text{Pol}$, and setting $V(p) = \emptyset$ for all $p \in \text{Var}$. As for $\text{Act}$, we set as follows. We first choose an arbitrary $h_2 \in H_{m_1}$. By Lemma 2.2 we know that also $h_2 \in H_{m_0}$. Now for an arbitrary $m \in \text{Tree}$ we define that:

$$\text{Act}(m, h) = \begin{cases} \{y\}, & \text{if } m = m_0 \text{ and } h \approx_{m_0} h_2; \\ \{x, y\}, & \text{if } m \succ m_0 \text{ and } h \approx_{m_0} h_2; \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is obvious that every semantical constraint on jstit models is satisfied, except possibly for the constraints involving $\text{Act}$, and it is also clear that such an $\mathcal{M}$ satisfies $\mathcal{CS}$-normality condition for every constant possible specification $\mathcal{CS}$.

As for $\text{Act}$ itself, we start by establishing the following:

**Claim.** Under the current settings for $\mathcal{M}$ we have, for an arbitrary $m \in \text{Tree}$:

$$\text{Act}_m = \begin{cases} \{x, y\}, & (\exists h \in H_m)(m \succ m_0 \land h \approx_{m_0} h_2); \\ \emptyset, & \text{otherwise.} \end{cases}$$

Indeed, assume that $m \in \text{Tree}$ and $h \in H_m$ are such that $m \succ m_0$ and $h \approx_{m_0} h_2$.

Now, if $g \in H_m$ is arbitrary, then, by Lemma 2.2, $g, h \in H_{m_0}$, so that $g \approx_{m_0} h$. By Lemma 2.3, we get then $g \approx_{m_0} h_2$ so that $\text{Act}(m, g) = \{x, y\}$. Since $g \in H_m$ was chosen arbitrarily, this means that also $\text{Act}_m = \{x, y\}$.

On the other hand, if either $m_0 \not\approx m$ or no history in $H_m$ is undivided from $h_2$ at $m_0$, then we obviously have $\text{Act}_m = \emptyset$. Assume then that $m = m_0$. Recall that $h_0, h_1 \in H_{m_0} = H_m$ are such that $h_0 \not\approx_{m_0} h_1$. Therefore, by Lemma 2.3, we must have
either \( h_0 \not\approx_{m_0} h_2 \) or \( h_1 \not\approx_{m_0} h_2 \), whence either \( \text{Act}(m_0, h_0) \) or \( \text{Act}(m_0, h_1) \) equals to \( \emptyset \). In any case, we will have \( \emptyset = \text{Act}_{m_0} = \text{Act}_m \).

We now look into the semantical constraints dependent on \( \text{Act} \) in some detail.

**Expansion of presented proofs.** Assume that \( m \vdash m' \) and that \( h \in H_m' \). Then also \( h \in H_m \) by Lemma 2.2. Now, if \( m_0 \not\subset m \), then \( \text{Act}(m, h) = \emptyset \) and the constraint is verified trivially. The same argument applies, if \( h \not\approx_{m_0} h_2 \). Further, if \( m = m_0 \) and \( h \approx_{m_0} h_2 \), then we must have \( \text{Act}(m, h) = \{ y \} \) and \( \text{Act}(m', h) = \{ x, y \} \), respectively, and the constraint is satisfied. Finally, if \( m \triangleright m_0 \) and \( h \approx_{m_0} h_2 \), then we must have \( \text{Act}(m, h) = \text{Act}(m', h) = \{ x, y \} \), and the constraint is again satisfied.

**Presenting a new proof makes histories divide.** Assume that \( m \in \text{Tree} \) and that \( h \approx_m g \), so that for some \( m' \triangleright m \) it is true that \( m' \in h \cap g \). Now, if \( m_0 \not\subset m \), then \( \text{Act}(m, h) = \text{Act}(m, g) = \emptyset \) and the constraint is verified. The same argument applies when \( h, g \not\subset_{m_0} h_2 \). Finally, if \( m_0 \leq m \) and \( h, g \approx_{m_0} h_2 \), then either \( \text{Act}(m, h) = \text{Act}(m, g) = \{ y \} \) or \( \text{Act}(m, h) = \text{Act}(m, g) = \{ x, y \} \) depending on whether \( m = m_0 \) or \( m \triangleright m_0 \).

**No new proofs guaranteed.** Assume that \( m \in \text{Tree} \). If \( \text{Act}_m = \emptyset \), then the constraint is trivially satisfied. On the other hand, if \( \text{Act}_m \neq \emptyset \), then, by the Claim above, we must have \( \text{Act}_m = \{ x, y \} \) and also that \( m \triangleright m_0 \) and \( h \approx_{m_0} h_2 \) for some \( h \in H_m \). Therefore, we can choose an \( m' \triangleright m_0 \) such that \( m' \in h_2 \cap h \). But then \( m' \) must be \( \preceq \)-comparable with \( m \) and we need to deal with the two cases:

*Case 1.* \( m \preceq m' \). Then, by Lemma 2.2, \( h_2 \in H_m \). Recall that, by its choice, \( h_2 \in H_{m_1} \), so that \( m_0 \) must be \( \preceq \)-comparable with \( m_1 \) as well. By the Claim above, we clearly have \( \text{Act}(m_1, h_2) = \{ x, y \} \), therefore, if \( m_1 \not\subset m \), then we are done. On the other hand, if \( m \preceq m_1 \), then, by Lemma 2.2, \( \neg \text{Next}(m_0, m) \). The latter means that we can choose an \( m'' \not\subset m \) such that \( m'' \not\subset m_0 \). Hence by the absence of backward branching, we will have \( m_0 \not\subset m'' \). Note that by Lemma 2.2 and \( m'' \not\subset m \), we will also have \( h_2 \in H_{m''} \) whence, by \( m_0 \not\subset m'' \), we get that \( \text{Act}(m''', h_2) = \{ x, y \} \) again satisfying the constraint.

*Case 2.* \( m' \not\subset m \). Then, since \( m' \triangleright m_0 \) and \( h \approx_{m_0} h_2 \), we must have \( \text{Act}(m', h) = \{ x, y \} \) thus satisfying the constraint.

**Presented proofs are epistemically transparent.** Assume that \( m, m' \in \text{Tree} \) are such that \( R_n(m, m') \). Then, by definition of \( R_n \), above, we will also have \( m \preceq m' \). Now, if \( \text{Act}_m \) is empty, then the constraint is trivially verified. Otherwise we will have \( \text{Act}_m = \{ x, y \} \) by the Claim above. Let \( h \in H_{m'} \) be arbitrary. By Lemma 2.2, \( h \in H_m \), therefore \( \text{Act}(m, h) = \{ x, y \} \). But then, by the expansion of presented proofs constraint (verified above) we must have \( \text{Act}(m', h) = \{ x, y \} \). Since \( h \in H_{m'} \) was chosen arbitrarily, this shows that \( \text{Act}_{m'} = \{ x, y \} \) and the constraint is satisfied.

Therefore, the above-defined \( \mathcal{M} \) is shown to be a just model for \( \text{Ag} \) and by the Claim above we obviously have that:

\[
\mathcal{M}, m_0, h_2 \not\models K(\Box E x \lor \neg \Box E y) \rightarrow (E x \lor \neg E y).
\]

Indeed, whenever \( m' \in \text{Tree} \), then, by the Claim above, we will either have \( \text{Act}_{m'} = \emptyset \) (and then \( \mathcal{M}, m', g \models \neg \Box E y \) for all \( g \in H_{m'} \)), or \( \text{Act}_{m'} = \{ x, y \} \) (and then \( \mathcal{M}, m', g \models \Box E x \) for all \( g \in H_{m'} \)). Therefore, it is clear that we have:

\[
\mathcal{M}, m_0, h_2 \models K(\Box E x \lor \neg \Box E y),
\]
and yet, on the other hand it is true that:

\[ \mathcal{M}, m_0, h_2 \models \neg Ex \land Ey. \]

The frame definability result for stit frames is now straightforward:

**Theorem 2.** Let \( C = (\text{Tree}, \leq, \text{Choice}) \) be a stit frame for \( Ag \). For any constant specification \( \mathcal{CS} \) it is true that:

\[
(\forall \mathcal{M} \in \text{Mod}_{\mathcal{CS}}(\{C\}))(\mathcal{M} \models \{ A \in \text{Form}^{Ag} \models_{\mathcal{CS}} A \}) \iff C \in C^{Ag}_{\text{mixsucc}}.
\]

**Proof.** The \((\leq)-part follows from Theorem \[\square\]. For the \((\Rightarrow)-part, note that for a given \( x, y \in PV ar \) we have

\[ \vdash_{CS} K(\Box Ex \lor \neg \Box Ey) \rightarrow (Ex \lor \neg Ey) \]

by one application of \((R_{PR})\) to an appropriate instance of \((A7)\). Given this fact, the \((\Rightarrow)-part follows from Lemma \[\square\].

An analogous result for temporal frames is an easy corollary of the facts established above. More precisely, we claim the following:

**Corollary 2.** Let \( \mathcal{CS} \) be a constant specification and let \( T = (\text{Tree}, \leq) \) be a temporal frame outside \( T^{Ag}_{\text{mixsucc}} \). Then there is a \( \mathcal{CS} \)-normal jstit model \( \mathcal{M} = (\text{Tree}, \leq, \text{Choice}, \text{Act}, R, R_e, \mathcal{E}, V) \) based on \( C \) such that for some \((m, h) \in MH(\mathcal{M})\) it is true that:

\[ \mathcal{M}, m, h \not\models K(\Box Ex \lor \neg \Box Ey) \rightarrow (Ex \lor \neg Ey). \]

**Proof.** Just repeat the proof of Lemma \[\square\] adding to the definition of \( \mathcal{M} \) that we set \( \text{Choice}_{j}^m = H_m \) for all \( m \in \text{Tree} \) and \( j \in Ag \).

Now we can establish the following theorem in the same way as Theorem \[\square\] using Corollaries \[\square\] and \[\square\] instead of Theorem \[\square\] and Lemma \[\square\] respectively:

**Theorem 3.** Let \( T = (\text{Tree}, \leq) \) be a temporal frame for \( Ag \). For any constant specification \( \mathcal{CS} \) it is true that:

\[
(\forall \mathcal{M} \in \text{Mod}_{\mathcal{CS}}(\{T\}))(\mathcal{M} \models \{ A \in \text{Form}^{Ag} \models_{\mathcal{CS}} A \}) \iff T \in T^{Ag}_{\text{mixsucc}}.
\]

### 3.2 Justification stit frames

We now turn to the much more complex case of jstit frames. First, we need to know how \( \Sigma_D(\mathcal{CS}) \) stands in relation to the \( \mathcal{CS} \)-normal models based on regular jstit frames, and we start answering this question by establishing a soundness claim. This claim mostly reduces to a routine check that every axiom is valid and that rules preserve validity. We treat the less obvious cases in some detail:

**Theorem 4.** Let \( \mathcal{CS} \) be an arbitrary constant specification. Then every instance of \((A0)\), \((A9)\) is valid over the class \( \text{Mod}_{\mathcal{CS}}(\mathcal{F}^{Ag}_{\text{reg}}) \), and every application of rules \((R1)\), \((R2)\), \((R3)\), and \((R_{CS})\) to formulas which are valid over \( \text{Mod}_{\mathcal{CS}}(\mathcal{F}^{Ag}_{\text{reg}}) \) yields a formula which is valid over the same class.
Proof. First, note that if \( \mathcal{M} = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_e, \mathcal{E}, V \rangle \) is a CS-normal stit model based on a stit frame from \( \mathcal{F}_{\text{reg}}^{\text{Ag}} \), then \( \langle \text{Tree}, \preceq, \text{Choice}, V \rangle \) is a model of stit logic. Therefore, axioms (A0)–(A3), which were copy-pasted from the standard axiomatization of stit logic, must be valid. Second, note that if \( \mathcal{M} = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_e, \mathcal{E}, V \rangle \) is a CS-normal stit model, then \( \mathcal{M} = \langle \text{Tree}, R, R_e, \mathcal{E}, V \rangle \) is what is called in [2 Section 6] a justification model with the form of constant specification given by CS\(^2\). This means that also all of the (A4)–(A7) must be valid, whereas (R1), (R2), and (R\(_{\text{CS}}\)) must preserve validity, given that all these parts of our axiomatic system were borrowed from the standard axiomatization of justification logic. The validity of other parts of \( \Sigma_{\text{DS}}(\text{CS}) \) will be motivated below in some detail. In what follows, \( \mathcal{M} = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_e, \mathcal{E}, V \rangle \) will always stand for an arbitrary stit model in \( \text{Mod}_{\text{CS}}(\mathcal{F}_{\text{reg}}^{\text{Ag}}) \), and \( (m, h) \) for an arbitrary element of \( MH(\mathcal{M}) \).

As for (A\(_{\text{R}}\)), assume for reductio that \( \mathcal{M}, m, h \models KA \land K \neg A \). Then \( \mathcal{M}, m, h \models KA \) and also \( \mathcal{M}, m, h' \models K \neg A \) for some \( h' \in H_m \). By reflexivity of \( R \), it follows that \( \neg A \) will be satisfied at \( (m, h) \) in \( \mathcal{M} \). The latter means that, for some \( h'' \in H_m \), \( A \) must fail at \( (m, h'') \) and therefore, again by reflexivity of \( R \), \( KA \) must fail at \( (m, h) \) in \( \mathcal{M} \), a contradiction.

We consider next (A\(_9\)). If \( \Box Et \) is true at \( (m, h) \) in \( \mathcal{M} \), then, by definition, \( t \in \text{Act}_m \). Now, if \( m' \in \text{Tree} \) is such that \( R(m, m') \), then, by \( R \subseteq R_e \), we will have \( R_e(m, m') \), and, by the epistemic transparency of presented proofs constraint, we must have \( t \in \text{Act}_{m'} \), so that for every \( g \in H_{m'} \) we will have \( \mathcal{M}, m', g \models \Box Et \). Therefore, we must have \( \mathcal{M}, m, h \models K \Box Et \) as well.

The hardest part is to show that (R\(_{\text{DS}}\)) preserves validity over stit models from \( \text{Mod}_{\text{CS}}(\mathcal{F}_{\text{reg}}^{\text{Ag}}) \). Assume that \( KA \rightarrow (\neg \Box Et_1 \lor \ldots \lor \neg \Box Et_n \lor \Box E_{s_1} \lor \ldots \lor \Box E_{s_k}) \) is valid over this class of stit models, and assume also that we have:

\[
\mathcal{M}, m, h \models KA \land Et_1 \land \ldots \land Et_n \land \neg E_{s_1} \land \ldots \land \neg E_{s_k}. \tag{3}
\]

By validity of (A\(_{\text{I}}\)), it follows that:

\[
\mathcal{M}, m, h \models KA \land \neg E_{s_1} \land \ldots \land \neg E_{s_k}.
\]

Whence, by the assumed validity we know that also:

\[
\mathcal{M}, m, h \models \neg \Box Et_1 \lor \ldots \lor \neg \Box Et_n,
\]

therefore, we can choose a natural \( u \) such that \( 1 \leq u \leq n \) and:

\[
\mathcal{M}, m, h \models \neg \Box Et_u.
\]

The latter, in turn, means that for some \( h' \in H_m \), we have that:

\[
\mathcal{M}, m, h' \models \neg Et_u. \tag{4}
\]

Comparison between (3) and (4) shows that \( \text{Act}(m, h) \neq \text{Act}(m, h') \), whence by the presenting a new proof makes histories divide constraint we get that \( h \neq m, h' \). Hence

\(^2\)See, e.g. [4 Ch. 17], although \( \Sigma \) uses a simpler format closer to that given in [3 Section 2.3].

\(^3\)The format for the variable assignment \( V \) is slightly different, but this is of no consequence for the present setting.
we know that \( m \) cannot be \( \preceq \)-maximal in \( \text{Tree} \). Using Lemma 2.1, we can choose in \( \text{Tree} \) some \( m_1 \succ m \) such that \( m_1 \in h \). We now establish the following claims:

Claim 1. \(( \forall g \in H_{m_1})(h' \nRightarrow m g)\).

The argument is the same as for \( h \): if \( g \in H_{m_1} \), then \( m_1 \in g \cap h \) so that, by \( m_1 \succ m \) we must have \( g \approx_m h \). But then, given Lemma 2.3 and \( h \nRightarrow_m h' \), we cannot have \( g \approx_m h' \).

Claim 2. \( S = \{ m'' \in \text{Tree} \mid t_1, \ldots, t_n \in \text{Act}_{m''} \} \in \Theta_{m''} \) for every \( m'' \) such that \( m \prec m'' \preceq m_1 \). Furthermore, \( m \notin S \).

The fact that \( m \notin S \) immediately follows from (4). Now, choose in \( \text{Tree} \) an arbitrary \( m' \) such that \( m \prec m' \preceq m_1 \). Since \( h \in H_{m_1} \), we know, by Lemma 2.2, that \( h \in H_{m'} \). Further, if \( g \in H_{m'} \) is arbitrary, then, by the same lemma, \( g \in H_m \). Therefore, \( g \approx_m h \) and, by the presenting a new proof makes histories divide constraint, \( \text{Act}(m, g) = \text{Act}(m, h) \supseteq \{ t_1, \ldots, t_n \} \). Whence, by \( m \prec m' \) and the expansion of presented proofs constraints we get that \( \{ t_1, \ldots, t_n \} \subseteq \text{Act}(m', g) \). Since \( g \in H_{m'} \) was chosen arbitrarily, this means that \( t_1, \ldots, t_n \in \text{Act}_{m'} \) and hence \( m' \in S \) thus verifying Definition 2.1.

Next, if \( m_2 \in S \) and \( R_e(m_2, m_3) \), then \( t_1, \ldots, t_n \in \text{Act}_{m_2} \), hence by the epistemic transparency of presented proofs \( t_1, \ldots, t_n \in \text{Act}_{m_3} \), which means that also \( m_3 \in S \) and Definition 2.2 is also verified.

Furthermore, assume that \( m_2 \in \text{Tree} \) is such that, for all \( g \in H_{m_2} \), there exists \( m_g \in g \) with the property \( N_{ext}(m_2, m_g) \& m_g \in S \). So choose an arbitrary \( g \in H_{m_2} \).

We have then \( t_1, \ldots, t_n \in \text{Act}_{m_g} \), whence, by the no new proofs guaranteed constraint we can choose \(( m^1_g, h^1_g), \ldots, ( m^n_g, h^n_g) \) such that:

\[
\begin{align*}
  h^1_g &\in H_{m_g} \& m^1_g \prec m_g \& t_1 \in \text{Act}(m^1_g, h^1_g); \\
  & \ldots \\
  h^n_g &\in H_{m_g} \& m^n_g \prec m_g \& t_n \in \text{Act}(m^n_g, h^n_g).
\end{align*}
\]

By \( N_{ext}(m_2, m_g) \) and (5) we get that:

\[
(6) m^1_g, \ldots, m^n_g \preceq m_2.
\]

From \( g, h^1_g, \ldots, h^n_g \in H_{m_g}, m_2 \prec m_g \) and Lemma 2.2 we get that:

\[
(7) h^1_g, \ldots, h^n_g \in H_{m_2}.
\]

and, further:

\[
(8) h^1_g \approx_{m_2} g, \ldots, h^n_g \approx_{m_2} g.
\]

By the presenting a new proof makes histories divide constraint, this further means that:

\[
(9) \text{Act}(m_2, h^1_g) = \ldots = \text{Act}(m_2, h^n_g) = \text{Act}(m_2, g).
\]

Next, by (8), (7), the expansion of presented proofs constraint, and (5) we get that:

\[
(10) t_1 \in \text{Act}(m_2, h^1_g), \ldots, t_n \in \text{Act}(m_2, h^n_g).
\]

It follows now from (3) and (10) that \( t_1, \ldots, t_n \in \text{Act}(m_2, g) \). Since \( g \in H_{m_2} \) was chosen arbitrarily, this further means that \( t_1, \ldots, t_n \in \text{Act}_{m_2} \) and thus \( m_2 \in S \), as desired. In this way, Definition 2.3 is verified.
Now, let \( m_2 \in S \) and assume that:

\[
(\forall m_3 < m_2) \exists m_4 (m_3 < m_4 < m_2).
\]  

(11)

By \( m_2 \in S \) we know that \( t_1, \ldots, t_n \in \text{Act}_{m_2} \). Again, by the no new proofs guaranteed constraint we can choose \((m^1, h^1), \ldots, (m^n, h^n)\) such that:

\[
h^1 \in H_{m_2} \land m^1 < m_2 \land t_1 \in \text{Act}(m^1, h^1);
\]

\[
\ldots
\]

\[
h^n \in H_{m_2} \land m^n < m_2 \land t_n \in \text{Act}(m^n, h^n).
\]

(12)

By \( \text{(12)} \) and the absence of backward branching it follows that all of \( m^1, \ldots, m^n \) are \( \leq \)-comparable, so we let \( m' \) be the \( \leq \)-greatest moment among \( m^1, \ldots, m^n \). By the choice of \( m' \) and \( \text{(12)} \), we have:

\[
m' < m_2.
\]

Therefore, by Lemma 2.2, we get \( H_{m_2} \subseteq H_{m'} \), whence:

\[
h^1, \ldots, h^n \in H_{m'}.
\]

(13)

It follows then from \( \text{(12)} \) and \( \text{(13)} \) that:

\[
h^1 \approx_{m'} \ldots \approx_{m'} h^n,
\]

(15)

which further means, by the presenting a new proof makes histories divide constraint that:

\[
\text{Act}(m', h^1) = \ldots = \text{Act}(m', h^n).
\]

(16)

Again by the choice of \( m' \) and the expansion of presented proofs constraint, we further get that:

\[
t_1 \in \text{Act}(m', h^1), \ldots, t_n \in \text{Act}(m', h^n).
\]

(17)

It follows then from \( \text{(17)} \) and \( \text{(16)} \) that \( t_1, \ldots, t_n \in \text{Act}(m', h^1) \). Now, by \( \text{(11)} \) and \( \text{(13)} \), we can choose an \( m'' \in \text{Tree} \) such that \( m' < m'' < m_2 \). By Lemma 2.2, we know that \( H_{m_2} \subseteq H_{m''} \), whence, by \( \text{(12)} \), \( h^1 \in H_{m''} \). It follows, by Lemma 3 that \( t_1, \ldots, t_n \in \text{Act}_{m''} \) and thus \( m'' \in S \), as desired. This ends both the verification of Definition 2.4 and the proof of Claim 2.

Claim 3. \((\forall m'_1 \in h')(\text{Next}(m, m'_1) \Rightarrow m'_1 \not\in S)\).

Indeed, assume the contrary, i.e. that for some \( m'_1 \in h' \) we have both \( \text{Next}(m, m'_1) \) and \( m'_1 \in S \). Then we will have \( t_1, \ldots, t_n \in \text{Act}_{m'_1} \). But then, by the no new proofs guaranteed constraint, we can choose a \( g \in H_{m'_1} \) and \( m'' < m'_1 \) such that \( t_u \in \text{Act}(m'', g) \). By \( \text{Next}(m, m'_1) \) we know that \( m'' \leq m \) and by Lemma 2.2 and \( m'_1 \triangleright m \) we know that \( g \in H_m \). Therefore, by the expansion of presented proofs, we get that \( t_u \in \text{Act}(m, g) \). Moreover, note that \( m'_1 \in h' \cap g \) so that \( h' \approx_{m} g \). Therefore, by the presenting a new proof makes histories divide constraint, we must have \( \text{Act}(m, h') = \text{Act}(m, g) \triangleright t_u \) which is in plain contradiction with \( \text{(4)} \).

In view of the Claims 1–3 above, we must be able to choose an \( m_2 \in \text{Tree} \) such that both \( m_2 \leq m_1 \) and \( \text{Next}(m, m_2) \). So we consider such an \( m_2 \). Given that \( m_1 \in h \), we know, by Lemma 2.2, that \( h \in H_{m_2} \). Therefore, it follows from \( \text{(3)} \) and Lemma 3 that \( t_1, \ldots, t_n \in \text{Act}_{m_2} \), or, equivalently:

\[
\mathcal{M}, m_2, h \models \Box Et_1 \land \ldots \land \Box Et_n.
\]

(18)
Furthermore, by the future always matters constraint we know that $R(m, m_2)$, whence it follows, again by (3), that:

$$\mathcal{M}, m_2, h \models KA.$$  \hfill (19)

Finally, choose an arbitrary $r$ between 1 and $k$. If $s_r \in Act_{m_2}$, then, by the no new proofs guaranteed constraint, there must be some $g \in H_{m_2}$ and some $m_0 < m$ such that $s_r \in Act(m_0, g)$. Then, by Lemma 22, $g \in H_m$, hence $h \approx_m g$. Therefore, by the presenting a new proof makes histories divide constraint, $Act(m, g) = Act(m, h)$. By Next$(m, m_2)$ we must have $m_0 \leq m$, therefore, by the expansion of presented proofs, $s_r \in Act(m, g)$, whence also $s_r \in Act(m, h)$. But this plainly contradicts (3). Since $1 \leq r \leq k$ was chosen arbitrarily, this means that all of $s_1, \ldots, s_k$ are outside $Act_{m_2}$ so that we have:

$$\mathcal{M}, m_2, h \models \neg \Box Es_1 \land \ldots \land \neg \Box Es_k.$$  \hfill (20)

Taken together, (18), (20) contradict the assumed validity of $KA \rightarrow (\neg \Box Et_1 \lor \ldots \lor \neg \Box Et_n \lor \Box Es_1 \lor \ldots \lor \Box Es_k)$. \hfill $\square$

It follows from Theorem 4 that we cannot have a result analogous to Theorem 2 w.r.t. jstit frames. Indeed, by Lemma 3, there exists a regular jstit frame $F$ for $Ag$, which violates condition mixsucc. However, by Theorem 4 every $CS$-normal jstit model based on $F$ will make every theorem of $\Sigma_D(CS)$ valid. Therefore, the frame definability result for jstit frames has to use a much more involved regularity condition in place of mixsucc.

Even though Theorem 4 is already sufficient to derive the frame definability theorem for jstit frames, we pause to observe that one can actually get a completeness theorem as well:

**Theorem 5.** Let $\Gamma \subseteq Form^{Ag}$ and let $F$ be a class of jstit frames such that $F^{Ag}_{reg} \downarrow \subseteq F \subseteq F^{Ag}_{reg}$. Then $\Gamma$ is $CS$-consistent iff it is satisfiable in $\text{Mod}_{CS}(F)$.

**Proof.** $(\Rightarrow)$. Let $\Gamma \subseteq Form^{Ag}$ be satisfiable in $\text{Mod}_{CS}(F)$ so that for some $\mathcal{M} \in \text{Mod}_{CS}(F)$ and some $(m, h) \in MH(\mathcal{M})$ we have $\mathcal{M}, m, h \models \Gamma$. Then we must have $\mathcal{M} \in \text{Mod}_{CS}(F^{Ag}_{reg})$. If $\Gamma$ were $CS$-inconsistent, this would mean that for some $A_1, \ldots, A_n \in \Gamma$ we would have $\vdash_{CS} (A_1 \land \ldots \land A_n) \rightarrow \bot$. By Theorem 4 this would mean that:

$$\mathcal{M}, m, h \models (A_1 \land \ldots \land A_n) \rightarrow \bot,$$

whence clearly $\mathcal{M}, m, h \models \bot$, which is impossible. Therefore, $\Gamma$ must be $CS$-consistent.

$(\Leftarrow)$. We can re-use the canonical model $\mathcal{M}^{Ag}_{CS}$ from Part I of this paper. In Part I, $\mathcal{M}^{Ag}_{CS}$ was shown to be $CS$-universal in that it satisfies every $CS$-consistent subset of $Form^{Ag}$. It was also shown that $\mathcal{M}^{Ag}_{CS} \in \text{Mod}_{CS}(C^{mixsucc}_{reg})$, whence, by Lemma 41, we get that $\mathcal{M}^{Ag}_{CS}$ is in $\text{Mod}_{CS}(F^{Ag}_{reg})$ and therefore in $F$. \hfill $\square$

Now for the frame definability for jstit frames:

**Lemma 6.** $CS$ be a constant specification and let $F = \langle Tree, \leq, Choice, R, R_e \rangle$ be a jstit frame outside $F^{Ag}_{reg}$. Then there is a $CS$-normal jstit model $\mathcal{M} = \langle Tree, \leq, Choice, Act, R, R_e, E, V \rangle$ based on $F$ such that for some $(m, h) \in MH(\mathcal{M})$ it is true that:

$$\mathcal{M}, m, h \not\models K(\Box Ex \lor \neg \Box Ey) \rightarrow (Ex \lor \neg Ey).$$
Proof. Assume that \( F \notin \mathcal{F}_{\text{reg}} \). Then we can choose \( m_0, m_1 \in \text{Tree} \), \( h' \in H_m \), and \( S \in \bigcap_{m_0 < m \leq m_1} \Theta_m \) such that:

\[
\begin{align*}
(m_0 < m_1) \land & m_0 \notin S \land (\forall g \in H_{m_1})(g \not\approx_m h') \land \\
& (\forall m' \in h')(\text{Next}(m_0, m') \Rightarrow m' \notin S) \land (\forall m \leq m_1)(\neg\text{Next}(m_0, m)). 
\end{align*}
\] (21)

We now extend \( F \) to \( M \) setting \( \mathcal{E}(m, t) = \text{Form}_{\text{Ag}} \) for all \( m \in \text{Tree} \) and \( t \in \text{Pol} \), and setting \( V(p) = \emptyset \) for all \( p \in \text{Var} \). As for \( \text{Act} \), we set as follows. We first choose an arbitrary \( h_2 \in H_{m_0} \). By Lemma 2 we know that also \( h_2 \in H_{m_0} \). Now for an arbitrary \( m \in \text{Tree} \) we define that:

\[
\text{Act}(m, h) = \begin{cases} 
\{ y \}, & \text{if } m = m_0 \text{ and } h \approx m_0 h_2; \\
\{ x, y \}, & \text{if } m \in S \lor \exists m'(m' \in h \land S \land \text{Next}(m, m')); \\
\emptyset, & \text{otherwise.}
\end{cases}
\]

It is obvious that every semantical constraint on \( \text{jsit} \) models is satisfied, except possibly for the constraints invoking \( \text{Act} \), and it is also clear that such an \( M \) satisfies \( \mathcal{CS} \)-normality condition for every possible constant specification \( \mathcal{CS} \).

As for \( \text{Act} \) itself, we start by establishing the following claims:

Claim 1. \( (\forall m \in \text{Tree})(m \in S \Leftrightarrow \text{Act}_m = \{ x, y \}) \).

Indeed, whenever \( m \in S \), we will have \( \text{Act}(m, h) = \{ x, y \} \) for every \( h \in H_m \) and hence \( \text{Act}_m = \{ x, y \} \). In the other direction, assume that for every \( h \in H_m \) it is true that \( \text{Act}(m, h) = \{ x, y \} \). If \( m \in S \), then we are done. If \( m \notin S \), then for every \( h \in H_m \) we must have \( m_h \in h \) such that both \( m_h \in S \) and \( \text{Next}(m, m_h) \). But then, by Definition 2, we must also have \( m \in S \) despite our initial assumption.

Claim 2. \( (\forall m \in \text{Tree})(\text{Act}_m = \emptyset \lor \text{Act}_m = \{ x, y \}) \).

Indeed, if \( m \neq m_0 \) then for every \( h \in H_m \) we will have either \( \text{Act}(m, h) = \emptyset \) or \( \text{Act}(m, h) = \{ x, y \} \) just by definition of \( \text{Act} \) so that the claim is obviously true. And if \( m = m_0 \), then we know that \( \text{Act}(m_0, h') = \emptyset \) so that we must have \( \text{Act}_{m_0} = \emptyset \).

Claim 3. Under the settings for \( M \) we have, for an arbitrary \( m \in \text{Tree} \):

\[
\text{Act}_m = \begin{cases} 
\{ x, y \}, & m \in S; \\
\emptyset, & \text{otherwise.}
\end{cases}
\]

Immediate from Claims 1 and 2.

Claim 4. \( (\forall m \in \text{Tree})(m \in h_2 \land \text{Act} > m_0 \Rightarrow m \in S) \).

Indeed, if \( m \in h_2 \), then \( m \) must be \( \preceq \)-comparable to \( m_1 \). Now, if \( m \leq m_1 \), then \( m_0 \prec m \leq m_1 \) so that \( m \in S \) by Definition 2.1. On the other hand, if \( m_1 \prec m \), then note that we clearly have \( m_1 \in S \) by Definition 2.1. By the future always matters constraint and \( R \subseteq R_e \) we further get \( R_e(m_1, m) \), whence by Definition 2 we again get \( m \in S \).

We now look into the semantical constraints dependent on \( \text{Act} \) in some detail.

Expansion of presented proofs. Assume that \( m \prec m' \) and \( h \in H_{m'} \). We have three cases to consider.

Case 1. \( \text{Act}(m, h) = \emptyset \). The constraint is verified trivially.

Case 2. \( \text{Act}(m, h) = \{ y \} \). Then \( m = m_0 \) and \( h \approx_{m_0} h_2 \). The latter means that we can choose an \( m'' > m = m_0 \) such that \( m'' \in h \cap h_2 \). By Claim 4, we get then that \( m'' \in S \). Now, since also \( m' \in h, m' \) and \( m'' \) must be \( \preceq \)-comparable. If \( m'' \prec m' \), then \( R_e(m'', m') \) by the future always matters constraint and \( R \subseteq R_e \), and, further, \( m' \in S \) by Definition 2. If \( m' \preceq m'' \) then by Lemma 2, we get that \( m' \in h_2 \) and since also
Adding this up with the above definition of $\text{Act}$ and the constraint is satisfied.

Case 3. $\text{Act}(m, h) = \{x, y\}$. If $m \in S$ then also $m' \in S$ by $R \subseteq R_c$, the future always matters constraint, and Definition 2.2. On the other hand, if there exists $m'' \in h$ such that $m'' \in S \& \text{Next}(m, m'')$, then $m'$ and $m''$ are both in $h$ and must be $\preceq$-comparable. We cannot have $m' \prec m''$ since by $\text{Next}(m, m'')$ this would mean that $m' \preceq m$, in contradiction with our assumptions. Therefore, we must have $m'' \preceq m'$, whence by $R \subseteq R_c$, the future always matters constraint, and Definition 2.2 we again get that $m'' \in S$. Thus we get $m' \in S$ anyway, which means that $\text{Act}(m', h) = \{x, y\}$ and the constraint is satisfied.

**Presenting a new proof makes histories divide.** Assume that $h, g \in H_m$ and that there exists an $m' > m$ such that $m' \in g \cap h$. We consider four cases according to the above definition of $\text{Act}$:

Case 1. $m \in S$. Then clearly $\text{Act}(m, h) = \text{Act}(m, g) = \{x, y\}$ and the constraint is satisfied.

Case 2. For some $m'' \in h$ it is true that $m'' \in S$ and $\text{Next}(m, m'')$. Then $\text{Act}(m, h) = \{x, y\}$, and also $m'$ and $m''$ must be $\preceq$-comparable. We cannot have $m' \prec m''$ since by $\text{Next}(m, m'')$ this would mean that $m' \preceq m$, in contradiction with our assumptions. Therefore, we must have $m'' \preceq m'$, whence by Lemma 2.2, we must have $m'' \in g$ so that we get $\text{Act}(m, g) = \{x, y\}$ as well, and the constraint is satisfied. A symmetrical (and similar) subcase would start from the assumption that $m'' \in g$.

Case 3. $m = m_0$ and $h \approx_m h_2$. Then $\text{Act}(m, h) = \{y\}$. By Lemma 2.3, we get that $g \approx_{m_0} h_2$ so that $\text{Act}(m, g) = \{y\}$ as well, and the constraint is satisfied. Again, a symmetrical (and similar) subcase would start from the assumption that $g \approx_{m_0} h_2$.

Case 4. None of the above cases applies either for $h$ or for $g$. Then $\text{Act}(m, h) = \text{Act}(m, g) = \emptyset$ and the constraint is satisfied.

**No new proofs guaranteed.** Let $m \in \text{Tree}$ be arbitrary. If $\text{Act}_m = \emptyset$, then the constraint is satisfied trivially. On the other hand, if $\text{Act}_m \neq \emptyset$, then, by Claim 3, we must have both $m \in S$ and $\text{Act}_m = \{x, y\}$. Then we have to consider two cases:

Case 1. There exists an $m' \prec m$ such that $m' \in S$. Then choose an arbitrary $h \in H_m$. We have $m' \in h$ by Lemma 2.2, and $\text{Act}(m', h) = \{x, y\}$ by the above definition of $\text{Act}$, so that the constraint is satisfied.

Case 2. For all $m' \prec m$ we have $m' \notin S$. Then, by Definition 2.4 we must have:

$$(\exists m_2 \prec m)(\forall m_3 \prec m)(\neg m_2 \prec m_3).$$

We choose such an $m_2$. Of course, whenever $m_3 \prec m_3$, $m_3$ must be $\preceq$-comparable to $m_2$ by the absence of backward branching, therefore, given that we never have $m_2 \prec m_3$, we must get that:

$$\forall m_3 \prec m)(m_3 \preceq m_2).$$

Adding this up with $m_2 \prec m$, we get that $\text{Next}(m_2, m)$. Now, choose an arbitrary $h \in H_m$. We have $m_2 \in h$ by Lemma 2.2 and $\text{Act}(m_2, h) = \{x, y\}$ by the fact that $m \in h \cap S \& \text{Next}(m_2, m)$ and the above definition of $\text{Act}$, so that the constraint is again satisfied.

**Presented proofs are epistemically transparent.** Assume that $m, m' \in \text{Tree}$ are such that $R_e(m, m')$. Then, if $\text{Act}_m = \emptyset$, the constraint is satisfied trivially. On the other hand, if $\text{Act}_m \neq \emptyset$, then, by Claim 3, we must have $m \in S$. But then, by
Definition 2.2, we will also have \( m' \in S \), and, by Claim 3, \( \text{Act}_{m'} = \text{Act}_{m} = \{x, y\} \) so that the constraint is again satisfied.

Therefore, the above-defined \( M \) is shown to be a jstit model for \( Ag \) and we obviously have that:

\[
M, m_0, h_2 \models K(\Box Ex \vee \neg \Box Ey) \rightarrow (Ex \vee \neg Ey).
\]

Indeed, whenever \( m' \in \text{Tree} \), then, by Claim 2 above, we will either have \( \text{Act}_{m'} = \emptyset \) (and then \( M, m', g \models \neg \Box Ey \) for all \( g \in H_{m'} \)), or \( \text{Act}_{m'} = \{x, y\} \) (and then \( M, m', g \models \Box Ex \) for all \( g \in H_{m'} \)). Therefore, it is clear that we have:

\[
M, m_0, h_2 \models K(\Box Ex \vee \neg \Box Ey),
\]

and yet, on the other hand it is true that:

\[
M, m_0, h_2 \models \neg Ex \land Ey.
\]

The frame definability result for jstit frames is now also straightforward:

**Theorem 6.** Let \( F = \langle \text{Tree}, \sqsubseteq, \text{Choice}, R, R_e \rangle \) be a jstit frame for \( Ag \). For any constant specification \( CS \) it is true that:

\[
(\forall M \in \text{Mod}_{CS}(\{F\}))(M \models \{A \in \text{Form}^{Ag} \models_{CS} A\}) \iff F \in F_{reg}^{Ag}.
\]

**Proof.** Same as for Theorem 2, using Theorem 5 and Lemma 6 in place of Theorem 1 and Lemma 5, respectively.

\[\square\]

### 4 Conclusions and further research

We have established that \( \Sigma_D \), our axiomatization of stit logic of justification announcements from Part I, has a reasonably clear-cut meaning (given by condition \text{mixsucc} in Definition 1 above) when it comes to restrictions induced by it on the temporal substructure of the underlying frame. The fact that \( \Sigma_D \), as it follows from the main result of Part I, cannot distinguish between mixed successor frames and a group of other stronger restrictions all the way up to discrete time structures underscores the limitations of expressive power of JA-STIT. We have also seen that once the epistemic accessibility relations enter the picture, the complexity of the restriction on frames imposed by \( \Sigma_D \) goes up significantly. One may even question the possible utility of such a complex defining condition as an insight into the nature of JA-STIT. We believe, however, that the notion of the family of sets \( \Theta_m \) for a given moment is interesting at least in that the claims we have established in the course of proofs of Theorems 4 and 6 given above apparently suggest that these families allow one to more or less characterize, for a given finite set of proof polynomials \( \sigma \), the set of moments \( m \) in a given jstit model for which we have \( \sigma \subseteq \text{Act}_m \) without mentioning \( \text{Act} \) at all. This by-product of the above results probably holds some potential for further research in this direction.

Additionally, the above results lay down a basis for similar enquiries into the expressive powers of some natural extensions of JA-STIT, like the logic of \( E \)-notions (see [8]) or the full basic jstit logic introduced in [7].
5 Acknowledgements

To be inserted.

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