Switched Max-Plus Linear-Dual Inequalities: Application in Scheduling of Multi-Product Processing Networks

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Abstract: P-time event graphs are discrete event systems suitable for modeling processes in which tasks must be executed in predefined time windows. Their dynamics can be represented by systems of linear dynamical inequalities in the max-plus algebra and its dual, the min-plus algebra, referred to as max-plus linear-dual inequalities (LDIs). We define a new class of models called switched LDIs (SLDIs), which allow to switch between different modes of operations, each corresponding to an LDI, according to an infinite sequence of modes called schedule. In this paper, we focus on the analysis of SLDIs when the schedule is fixed and periodic. We show that SLDIs can model single-robot multi-product processing networks, in which every product has different processing requirements and corresponds to a specific mode of operation. Based on the analysis of SLDIs, we propose an algorithm to compute minimum and maximum cycle times for these processes that improves the time complexity of other existing approaches.

Keywords: Petri nets, P-time event graphs, scheduling, switched systems

1. INTRODUCTION

P-time event graphs (P-TEGs) are event graphs in which tokens are forced to sojourn in places in predefined time windows. They have been applied to solve scheduling problems for several processing networks, including electroplating lines and cluster tools, cf. Becha et al. [2013], Kim et al. [2003]. A common feature of these processing networks is that operations need to be executed in specified time intervals in order to obtain the desired quality of the final product, and P-TEGs are the ideal tools for modeling such constraints.

In this paper, we introduce a new class of systems called switched max-plus linear-dual inequalities (SLDIs). They extend the modeling power of P-TEGs by allowing to switch among different modes of operations, each consisting in a system of inequalities describing the dynamics of a P-TEG. We first highlight the equivalence between bounded consistency, an important property extended to SLDIs from P-TEGs, and the existence of periodic trajectories. SLDIs are then applied to model single-robot multi-product processing networks, namely, processing networks in which the type of products to be processed can change over time, each type requires to visit different processing stations, and products are transported by a single robot (see Kats et al. [2008] for a formal definition). In this case, each mode is associated with a certain product type.

When the sequence of modes is fixed and periodic with period \(|v| \in \mathbb{N}\), the minimum and maximum cycle times of such systems can be computed in strongly polynomial time \(O(|v|^4n^4)\) (in the worst case) using an algorithm presented in Kats et al. [2008], where \(n\) corresponds to the total number of processing stations in the network. We provide other two algorithms based on SLDIs that solve instances of the same problem. The first one is derived from an existing procedure that computes the cycle times of P-TEGs, and achieves time complexity \(O(|v|^4n^4)\). The second one, of time complexity \(O(|v||n|^3+n^4)\), improves the first one by using tools from automata theory to exploit the sparsity of a certain matrix in the max-plus algebra.

Tests are performed on an example of single-robot multi-product processing network to show the advantages of the proposed methods.

Notation

The set of positive, respectively non-negative, integers is denoted by \(\mathbb{N}\), respectively \(\mathbb{N}_0\). The set of non-negative real numbers is denoted by \(\mathbb{R}_{\geq 0}\). Moreover, \(\mathbb{R}_{\max} := \mathbb{R} \cup \{-\infty\}\), \(\mathbb{R}_{\min} := \mathbb{R} \cup \{\infty\}\), and \(\mathbb{R} := \mathbb{R}_{\max} \cup \{\infty\} = \mathbb{R}_{\min} \cup \{-\infty\}\). If \(A \in \mathbb{R}^{m \times n}\), we will use notation \(A^T\) to indicate \(-A^T\).
2. PRELIMINARIES

In the following subsections, some preliminary notions on idempotent semirings, precedence graphs, and multi-precedence graphs are recalled. For a more detailed discussion on the first two topics, we refer to Baccelli et al. [1992] and Hardouin et al. [2018]; multi-precedence graphs have been introduced in Zorzenon et al. [2022a].

2.1 Formal languages and the max-plus algebra

A dioid (or idempotent semiring) \((D, \oplus, \odot)\) is a set \(D\) endowed with two operations: \(\oplus\) (addition), and \(\odot\) (multiplication). Operation \(\oplus\) and \(\odot\) are associative and have a neutral element indicated, respectively, by \(e\) and \(\epsilon\); \(\oplus\) is commutative and idempotent \((a \oplus a = a)\), \(\odot\) distributes over \(\oplus\), and \(e\) is absorbing for \(\odot\). For the sake of brevity, we will often omit symbol \(\odot\). The order relation \(\leq\) is induced by \(\oplus\): \(a \leq b \iff a \oplus b = a\). A dioid is complete if it is closed for infinite sums and \(\odot\) distributes over infinite sums. In complete dioids, \(\top\) denotes the greatest element of \(D\), the Kleene star of an element \(a \in D\) is defined by \(a^* = \bigoplus_{k \in \mathbb{N}} a^k\), where \(a^k = a \cdot a^{k-1}\). An example of a dioid that will be used in this paper is the max-plus algebra \((D, \oplus, \odot)\), i.e., a complete dioid in which every element \(a \in D\) admits a multiplicative inverse \(a^{-1}\), i.e., \(a \odot a^{-1} = a^{-1} \odot a = e\). Then, \(\odot\) is defined as \(a \odot b = a \odot b\) if \(a, b \in D\) \(\{\top\}\), \(a \odot b = \top\) if \(a = \top\) or \(b = \top\) is a dioid for \((D, \odot, \oplus)\) (see Zorzenon et al. [2022b]).

The tensor (or Kronecker) product \(\otimes\) between two matrices \(A \in D^{m \times n}, B \in D^{p \times q}\) is defined as the matrix
\[
A \otimes B = \begin{bmatrix}
A_{11} B & \cdots & A_{1n} B \\
\vdots & \ddots & \vdots \\
A_{m1} B & \cdots & A_{mn} B
\end{bmatrix} \in D^{mp \times nq}.
\]

We recall the following properties of \(\otimes\), the first of which holds in commutative dioids, i.e., dioids in which \(\oplus\) is commutative.

**Proposition 1.** (Horn and Johnson [1991]). Let \((D, \odot, \oplus)\) be a commutative dioid, \(A \in D^{m \times n}, B \in D^{p \times q}, C \in D^{n \times k}, D \in D^{q \times k}\). Then \((A \odot B) \otimes (C \otimes D) = (A \otimes C) \otimes (B \otimes D)\).

**Proposition 2.** Let \((D, \odot, \oplus)\) be a dioid, \(A \in D^{m \times m}, B \in D^{p \times p}\). Then, \(tr(A \otimes B) = tr(A) \odot tr(B)\), where \(tr(M) = \bigoplus_{i=1}^m M_{ii}\) indicates the trace of matrix \(M\) in \(D^{q \times q}\).

**Proof.**
\[
tr(A \otimes B) = \bigoplus_{k=1}^m tr(A_{kk} \otimes B) = \bigoplus_{k=1}^m A_{kk} \odot tr(B) = \bigoplus_{k=1}^m \bigoplus_{i=1}^n A_{kk} \otimes (B_{ii}) = \bigoplus_{i=1}^n \bigoplus_{k=1}^m A_{kk} \otimes tr(B) = tr(A) \odot tr(B).\]

The max-plus algebra is the complete and commutative idempotent semifield \((R, \oplus, \odot)\), i.e., the set of extended real numbers endowed with the standard maximum operation \(\oplus\), and the standard addition \(\odot\). In the max-plus algebra, \(\epsilon = -\infty, e = 0, \top = \infty, \otimes\) is the standard minimum operation, \(\leq\) coincides with \(\leq\). The dual product \(\otimes\) is such that \(a \otimes b = a \odot b\) if \(a, b = \infty\), and \(a \otimes b = a\) if \(a = \infty\) or \(b = \infty\). The extension of the max-plus algebra to square matrices \((R, \otimes)\) is a complete dioid; in the rest of the paper, symbol \(\otimes\) will be reserved to compare matrices with elements from \(R\), i.e., \(\forall A, B \in R^{m \times n}, A \otimes B \iff A_{ij} \leq B_{ij} \forall i, j\). The product between a scalar \(\lambda \in \mathbb{R}\) and a matrix \(A \in R^{m \times n}\), \(\lambda \odot A = \lambda \otimes A\), will simply be indicated by \(\lambda A\). Note that, with the notation above, \((R, \oplus, \otimes)\) forms a dual dioid called the min-plus algebra.

2.2 Precedence graphs and multi-precedence graphs

A directed graph is a pair \((N, E)\) where \(N\) is a finite set of nodes and \(E \subseteq N \times N\) is the set of arcs. A weighted directed graph is a triplet \((N, E, w)\), where \((N, E)\) is a directed
The precedence graph associated with a matrix $A \in \mathbb{R}^{n \times n}$ is the weighted directed graph $G(A) = (N, E, w)$, where $N = \{1, \ldots, n\}$, and $E$ and $w$ are defined in the following (non-standard) way: there is an arc $(i, j) \in E$ from node $i$ to node $j$ if and only if $A_{ij} \neq -\infty$, and $w$ is such that $w((i, j)) = A_{ij}$. We adopt this non-standard convention of associating the weight of an arc $(i, j)$ instead of $(j, i)$, as this will simplify the interpretation of the label of a path in multi-precedence graphs. When elements of $A$ are functions of some real parameters, $A = A(\lambda_1, \ldots, \lambda_p)$, $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$, we say that $G(A)$ is a parametric precedence graph. A sequence of $r + 1$ nodes $\rho = (i_1, i_2, \ldots, i_{r+1})$, $r \geq 1$, such that $(i_j, i_{j+1}) \in E$ for all $j \in \{1, \ldots, r\}$ is a path of length $r$; a path $\rho$ such that $i_1 = i_{r+1}$ is called a circuit. The weight of a path is the sum (in conventional algebra) of the weights of the arcs composing it. Elements of the max-plus power of a matrix $A$ have a clear meaning with respect to precedence graph $G(A)$; indeed, $(A')^j$ corresponds to the maximum weight of all paths in $G(A)$ of length $r$ from node $i$ to node $j$. The maximum circuit mean of a precedence graph $G(A)$ with $n$ nodes can be computed in the max-plus algebra as $\text{mcm}(A) = \bigoplus_{k=1}^{n} \text{tr}(A^k)^{\frac{1}{k}}$, where $a^\frac{1}{k}$ is the $k$th max-plus root of $a \in \mathbb{R}_{\text{max}}$ and corresponds to $\frac{1}{k}$ in standard algebra. We recall that a precedence graph $G(A)$ does not contain circuits with positive weight if and only if $\text{tr}(A^r) = 0$; otherwise, if there is at least one circuit with positive weight in $G(A)$, then $\text{tr}(A^r) = \infty$.

In this paper, we will make use of another class of graphs, called multi-precedence graphs, which will allow us to analyze parametric precedence graphs using tools from formal languages and automata theory. The reader familiar with max-plus automata will notice their similarity to multi-precedence graphs. The multi-precedence graph associated with matrices $A_1, \ldots, A_l \in \mathbb{R}^{n_{\text{max}}}$ is the weighted multi-directed graph $G(A_1, \ldots, A_l) = (N, \Sigma, E, \mu, W)$, where $N = \{1, \ldots, n\}$ is the set of nodes, $\Sigma = \{a_1, \ldots, a_r\}$ is the alphabet of symbols $a_1, \ldots, a_r$, $\mu : \Sigma \to \mathbb{R}_{\text{max}}$ is the morphism defined by $\mu(a_i) = A_i$ for all $i \in \{1, \ldots, r\}$, and $E \subseteq N \times \Sigma \times N$ is the set of edges defined such that there is an arc $(i, z, j) \in E$ from node $i$ to node $j$ labeled $z$ with weight $\mu(z)$ if and only if $\mu(z) \neq -\infty$. A path in a multi-precedence graph $G(A_1, \ldots, A_l)$ is a sequence of alternating nodes and labels of the form $\sigma = (i_1, z_1, i_2, z_2, \ldots, z_r, i_{r+1})$, $r \geq 1$, such that $(i_j, z_j, i_{j+1}) \in E$ for all $j = 1, \ldots, r$; we will say that path $\sigma$ is labeled $\nu = z_1 z_2 \cdots z_r$.

It is convenient to extend morphism $\mu$ to $\mu : 2^\Sigma \to \mathbb{R}^{n_{\text{max}}}$ as follows: for all $z \in \Sigma$, $L_1, L_2 \subseteq \Sigma^*$, $\mu(\{e\}) = E_{\emptyset}$, $\mu(z) = \mu(z)$, $\mu(L_1 \cup L_2) = \mu(L_1) \oplus \mu(L_2)$, and $\mu(L_1 L_2) = \mu(L_1) \odot \mu(L_2)$. In this way, given a language $\Sigma \subseteq \Sigma^*$, $\mu(L_1)^{\frac{1}{k}} \supseteq \bigoplus_{k \in \mathbb{R}} \mathbb{S}(z_{i_j})$ corresponds to the supremum, for all strings $z \in \Sigma$, of the weights of all paths labeled $z$ in $G(A_1, \ldots, A_l)$ from node $i$ to node $j$; in particular, $\text{tr}(\mu(L)^{\frac{1}{k}}) = 0$ if and only if no circuits with positive weight exist in $G(A_1, \ldots, A_l)$ among those with label $z$. Moreover, the following properties hold: for all $L_1, L_2 \subseteq \Sigma^*$, $L_1 \subseteq L_2 \Rightarrow \mu(L_1) \leq \mu(L_2)$, and $\mu(L^*) = \mu(L^*)$. We will indicate by $\Gamma$, respectively, $\Gamma_M$, the set of all precedence graphs, respectively, multi-precedence graphs, that do not contain circuits with positive weight. The following proposition allows us to study the sign of circuit weights in some precedence graphs using multi-precedence graphs.

**Proposition 3.** (Zorzenon et al. [2022a]). Let $A_1, \ldots, A_l \in \mathbb{R}^{n_{\text{max}}}$. There exists a circuit with positive weight visiting node $i \in \{1, \ldots, n\}$ in multi-precedence graph $G(A_1, \ldots, A_l)$ if and only if there exists a circuit with positive weight visiting node $i$ in precedence graph $G(A_1 \oplus \ldots \oplus A_l)$.

Given a precedence graph $G(A)$, where $A = A(\lambda_1, \ldots, \lambda_l)$, the non-positive circuit weight problem (NCP) consists in characterizing the set $\Lambda_{\text{ncp}}(A) = \{\lambda_1, \ldots, \lambda_l \in \mathbb{R} \mid G(A) \in \Gamma\}$ of all values of parameter $(\lambda_1, \ldots, \lambda_l)$ for which $G(A)$ does not contain circuits with positive weight. When matrix $A$ has the form $A(\lambda) = \lambda P \otimes \lambda^{-1} I \oplus C$ for arbitrary matrices $P, I, C \in \mathbb{R}^{n_{\text{max}}}$ (called proportional, inverse, and constant matrix, respectively), then $\Lambda_{\text{ncp}}(A) = \{\lambda \mid \lambda \in \mathbb{R} \text{ is an interval; moreover, its extremes can be found either in weakly polynomial time using linear programming solvers such as the interior-point method, or in strongly polynomial time } O(n^4) \text{ using Algorithm 1, see Zorzenon et al. [2022a].}\}

**Algorithm 1: Solve_NCP($P, I, C$)**

**Input:** $P, I, C \in \mathbb{R}^{n_{\text{max}}}$

**Output:** $\Lambda_{\text{ncp}}(\lambda P \otimes \lambda^{-1} I \oplus C)$

1. If $G(C) \notin \Gamma$ then return $\emptyset$
2. $P \leftarrow C^* P C^*$, $I \leftarrow C^* I C^*$, $S \leftarrow E_{\emptyset}$
3. For $k = 1$ to $\left\lceil \frac{1}{\epsilon} \right\rceil$ do
   4. $S \leftarrow S \oplus I S P S \oplus E_{\emptyset}$
5. If $G(S) \notin \Gamma$ then return $\emptyset$
6. Return $\{mc(m(S^*)), (mc(m(S^*)))^{-1}\} \cap \mathbb{R}$

3. **P-TIME EVENT GRAPHS**

**Definition 4.** (From Calvez et al. [1997]). An unweighted P-time Petri net (P-TPN) is a 5-tuple $(P, T, E, m_0)\), where $(P \cup T, E)$ is a directed graph in which the set of places is partitioned into the set of places, $P$, and the set of transitions, $T$, the set of arcs $E$ is such that $E \subseteq (P \times T) \cup (T \times P)$, $m : P \to \mathbb{N}$ is a map such that $m(p)$ represents the number of tokens initially residing in place $p \in P$ (also called initial marking of $p$), and $\lambda : P \to \{\tau^-, \tau^+\}$ is a map that associates to every place $p \in P$ a time interval $\tau(p) = [\tau^-_p, \tau^+_p]$.

The dynamics of a P-TPN net is briefly described as follows. A transition $t$ is enabled when either it has no upstream place or each upstream place $p$ of $t$ contains at least one token which has resided in $p$ for a time between $\tau^-_p$ and $\tau^+_p$ (extremes included). When transition $t$ is enabled, it may fire; its firing causes one token to be removed instantaneously from each of the upstream places of $t$, and one token to be added, again instantaneously, to each of the downstream places of $t$. If a token sojourns more than $\tau^+_p$ time instants in a place $p$, then said token is dead, as it is forced to remain in $p$ forever.

A P-time event graph (P-TEG) is a P-TPN in which every place has exactly one upstream and one downstream
transitions. Without loss of generality (see Špaček and Komenda [2017]), we will suppose that the initial marking $m(p)$ is less than or equal to 1 for each place $p \in P$ of a P-TEG. This allows to rephrase the dynamics of a P-TEG with $|T| = n$ transitions as a max-plus linear-dual inequality system (LDI), i.e., a system of dynamical ($\otimes$, $\ominus$)- and ($\oplus$, $\ominus$)-linear inequalities of the form

$$\forall k \in \mathbb{N}_0, \quad \begin{cases} A^0 \otimes x(k) \leq x(k) \leq B^0 \otimes x(k) \\
 A^i \otimes x(k) \leq x(k + 1) \leq B^i \otimes x(k), \end{cases}$$

(1)

where $x : \mathbb{N}_0 \to \mathbb{R}^n$ is called dater function, $A^0, A^i \in \mathbb{R}^{n \times n}$, $B^0, B^i \in \mathbb{R}^{n \times n}$ are called characteristic matrices of the P-TEG, and are defined as follows. If there exists a place $p$ with initial marking $\mu \in \{0, 1\}$, upstream transition $t_j$ and downstream transition $t_i$, then $A^0_{ij} = \tau_p^j$ and $B^0_{ij} = \tau_p^i$; otherwise, $A^0_{ij} = -\infty$ and $B^0_{ij} = \infty$. By convention, element $x(k)$ of the dater function represents the time at which transition $t_i$ fires for the $(k + 1)^{st}$ time. Since the $(k + 2)^{nd}$ firing of any transition cannot occur before the $(k + 1)^{st}$, we require the dater to be a non-decreasing function, i.e., $\forall i \in \{1, \ldots, n\}, x_i(k + 1) \geq x_i(k)$.

If a non-decreasing dater trajectory $\{x(k)\}_{k \in \mathbb{N}_0}$ satisfying (1) exists, then the trajectory is said to be consistent for the P-TEG, as it does not cause the death of any token, and the P-TEG is said to be consistent. A trajectory $\{x(k)\}_{k \in \mathbb{N}_0}$ is 1-periodic with period $\lambda \in \mathbb{R}_{\geq 0}$, if it has the form $\{\lambda^k x(0)\}_{k \in \mathbb{N}_0}$, in the max-plus algebra sense; in standard algebra, this corresponds to a dater trajectory such that, for all $i \in \{1, \ldots, n\}$, $x_i(k + \lambda) = x_i(k)$. Moreover, we indicate by $\Lambda_{\mathsf{P-TEG}}(A^0, A^i, B^0, B^i) \subseteq \mathbb{R}$ the set of $\lambda \geq 0$ for which there exists a 1-periodic trajectory of period $\lambda$ that is consistent for the P-TEG characterized by matrices $A^0, A^i, B^0, B^i$; such periods are called cycle times. We say that a trajectory $\{x(k)\}_{k \in \mathbb{N}_0}$ is delay-bounded if there exists a positive real number $M$ such that, for all $i, j \in \{1, \ldots, n\}$ and for all $k \in \mathbb{N}_0$, $x_i(k) - x_j(k) < M$; a P-TEG admitting a consistent delay-bounded trajectory of the dater function is said to be boundedly consistent. To our knowledge, no algorithm that checks whether a P-TEG is consistent has been found until now; on the other hand, there exists an algorithm that checks bounded consistency of P-TEGs in time $O(n^4)$, which comes directly from the following result.

**Theorem 5.** (Zorzenon et al. [2020, 2022b]). A P-TEG is boundedly consistent if and only if it admits a consistent 1-periodic trajectory, i.e., if and only if set $\Lambda_{\mathsf{P-TEG}}(A^0, A^i, B^0, B^i)$ is non-empty. Moreover, $\Lambda_{\mathsf{P-TEG}}(A^0, A^i, B^0, B^i)$ coincides with

$$\Lambda_{\mathsf{SCP}}(\lambda B^1 \ominus \lambda^{-1} A^i \oplus (A^0 \ominus B^0)) \cap [0, \infty[,$$

**Example 6.** Consider the P-TEG represented in Figure 1, in which time windows are parametrized with respect to label $z$; in Table 1, values of time windows are given for $z \in \{a, b, c\}$. The matrices characterizing the P-TEG labeled $z$ are:

$$A^0_z = \begin{bmatrix} -\infty & -\infty \\ 0 & -\infty \end{bmatrix}, \quad A^i_z = \begin{bmatrix} \alpha_z & -\infty \\ -\infty & \beta_z \end{bmatrix},$$

$$B^0_z = \begin{bmatrix} \infty & \infty \\ \infty & \infty \end{bmatrix}, \quad B^1_z = \begin{bmatrix} \alpha_z & \beta_z \\ \beta_z & \beta_z \end{bmatrix}.$$

Since lower and upper bounds for the sojourn times of the two places with an initial token coincide, once dater $x_2(0)$ is chosen (such that the first inequality in (1) is satisfied for $k = 0$, i.e., $x_2(0) \geq x_1(0)$), the only trajectory $\{x(k)\}_{k \in \mathbb{N}_0}$ that is a candidate to be consistent for the P-TEG labeled $z$ is deterministically given by

$$\forall k \in \mathbb{N}_0, \quad x_2(k + 1) = \begin{cases} \alpha_z + x_2(k) \\ \beta_z + x_2(k) \end{cases}.$$

Conversely, it is easy to see that, for any valid choice of the initial dater, candidate trajectory $\{x(k)\}_{k \in \mathbb{N}_0}$ is not consistent (as for a sufficiently large $k$, $x_2(k) < x_1(k)$, and $\{x(k)\}_{k \in \mathbb{N}_0}$, despite being consistent, is not delay-bounded and results in the infinite accumulation of tokens in the place between $t_1$ and $t_2$ for $k \to \infty$. On the other hand, $\{x(z)\}_{k \in \mathbb{N}_0}$ is consistent and delay-bounded (in fact, it is 1-periodic with period 1); thus we can conclude that the P-TEG labeled $z$ is not consistent, the one labeled $b$ is consistent but not boundedly consistent, and the one labeled $c$ is boundedly consistent. Of course, we would have reached the same conclusions regarding delay-boundedness by using Theorem 5. In particular, applying Algorithm 1, we get

$$\Lambda_{\mathsf{P-TEG}}(A^0_0, A^1_0, B^0_0, B^1_0) = \Lambda_{\mathsf{P-TEG}}(A^0_0, A^1_0, B^0_0, B^1_0) = \emptyset,$$

$$\Lambda_{\mathsf{P-TEG}}(A^0_0, A^1_0, B^0_0, B^1_0) = [1, 1] = \{1\}.$$

### 4. SWITCHED MAX-PLUS LINEAR-DUAL INEQUALITIES

#### 4.1 General description

We start by defining a switched LDI (SLDI) as the natural extension of the dynamical inequalities of P-TEGs, in which the mode of operation can switch. Each mode is associated with a set of $n$ events that have to satisfy certain time window constraints. An SLDI is a 5-tuple $\mathcal{S} = (\Sigma, A^0, A^i, B^0, B^i)$, where $\Sigma = \{a_1, \ldots, a_m\}$ is a finite alphabet whose symbols are called modes, and $A^0, A^i : \Sigma \to \mathbb{R}_{\max}, B^0, B^i : \Sigma \to \mathbb{R}_{\min}$ are functions that associate a matrix to each mode of $\Sigma$; for sake of simplicity, given a mode $z \in \Sigma$, we will write $A^0_z, A^i_z, B^0_z, B^i_z$ in place of $A^0(z), A^i(z), B^0(z), B^i(z)$, respectively; A schedule $w \in \Sigma^*$ is an infinite concatenation of modes.

The dynamics of an SLDI $\mathcal{S}$ under schedule $w \in \Sigma^*$ is expressed by the following system of inequalities: for all $k \in \mathbb{N}_0$,

$$\begin{cases}
A^0_{w_{k+1}} \otimes x(w_k) \leq x(w_k) \leq B^0_{w_{k+1}} \otimes x(w_k) \\
A^i_{w_{k+1}} \otimes x(w_k) \leq x(w_{k+1}) \leq B^i_{w_{k+1}} \otimes x(w_k)
\end{cases},$$

(2)

where function $x : \mathsf{Pre}(w) \to \mathbb{R}^n$ is called dater of $\mathcal{S}$ associated with schedule $w$. Term $x(w_k)$ represents the

![Figure 1. Example of P-TEG.](image-url)
time of the occurrence of event $i$ associated with mode $w_{k+1}$.

When schedule $w$ is fixed, we can extend the definition of some properties of P-TEGs to SLDIs in a natural way. For instance, if there exists a trajectory of the dater $\{x(w_k)\}_{k \in N_0}$ that satisfies (2) for all $k \in N_0$, then the trajectory is consistent for the SLDI under schedule $w$, and we say that the SLDI is consistent under schedule $w$. The definitions of delay-bounded trajectory and bounded consistency are generalized to SLDIs under schedule $w$ in a similar way.

The interpretation of bounded consistency of an SLDI under a fixed schedule $w$ is analogous to the one of P-TEGs (see Zorzon et al. [2020]). When a process consisting of several tasks (each represented by an event) is modeled by an SLDI that is not boundedly consistent under a schedule $w$, then the execution of every possible sequence of tasks following $w$ will either lead to the violation of some time window constraints (if the SLDI is not consistent under $w$), or to the infinite accumulation of delay between the execution of some tasks (if the only consistent trajectories are not delay-bounded).

4.2 Analysis of fixed periodic schedules

In this subsection, we analyze bounded consistency and cycle times of an SLDI when schedule $w$ is periodic, i.e., when it can be written as $w = v^x$, where $v \in \Sigma^*$ is a finite subschedule. Similarly to P-TEGs, it is natural to assume the following non-decreasingness condition for the dater of an SLDI: for all $k \in N_0$, $h \in \{0, \ldots, |v| - 1\}$, $x(v^{k+1}v_1v_2\ldots v_h) \geq x(v^k v_1 v_2 \ldots v_h)$. The meaning is that events occurring during the $(k+2)^{th}$ repetition of mode $v_1$, at the $h^{th}$ position in subschedule $v$, cannot occur earlier than those taking place during the $(k+1)^{th}$ one.

We define $v$-periodic trajectories of period $\lambda \in \mathbb{R}_{\geq 0}$ for SLDIs under schedule $w = v^x$ as those dater trajectories that, for all $k \in N_0$, $h \in \{0, \ldots, |v| - 1\}$, satisfy $x(v^{k+1}v_1v_2\ldots v_h) = \lambda x(v^k v_1 v_2 \ldots v_h)$; $A_{SLDI}^\Sigma(S)$ denotes the set of all periods $\lambda$, called cycle times, for which there exists a consistent $v$-periodic trajectory. Their relationship with 1-periodic trajectories in P-TEGs is made clear by the following example.

Example 7. Let us analyze the SLDI $S$, with $\Sigma = \{a, b, c\}$, and $A_0^a, A_1^a, B_0^a, B_1^a$ defined as in Example 6; now label $z \in \Sigma$ is to be interpreted as a mode. Thus, for each event $k$, the dynamics of the SLDI may switch among those specified by the P-TEGs labeled $a$, $b$, and $c$. We consider periodic schedules (ac)$^v$ and (ab)$^v$; observe that for $w = v^x$, with $v \in \{ac, ab\}$ (i.e., $v_1 = a$ and $v_2 = c$ or $v_2 = b$) the SLDI following $w$ can be written as: for all $k \in N_0$,

$$
\begin{align*}
A_0^a \otimes x(v^k) &\preceq x(v^k) \preceq B_0^c \otimes x(v^k) \\
A_1^a \otimes x(v_1^k) &\preceq x(v^k v_1) \preceq B_1^c \otimes x(v^k v_1) \\
A_0^b \otimes x(v^k v_1) &\preceq x(v^k v_1 v_2) \preceq B_0^b \otimes x(v^k v_1 v_2) \\
A_1^b \otimes x(v^k v_1 v_2) &\preceq x(v^k v_1 v_2 v_3) \preceq B_1^b \otimes x(v^k v_1 v_2 v_3).
\end{align*}
$$

By defining $\bar{x}(k) = [x(v^k), x(v^k v_1), \ldots]$ and the above set of inequalities can be rewritten as an LDI: for all $k \in N_0$, where

$$
A_0^v = \begin{bmatrix} A_0^a & E \\ A_1^a & A_0^c \end{bmatrix}, \quad A_1^v = \begin{bmatrix} E & A_1^a \\ E & E \end{bmatrix}, \\
B_0^v = \begin{bmatrix} B_0^a & T \\ B_1^a & B_0^c \end{bmatrix}, \quad B_1^v = \begin{bmatrix} T & B_1^a \\ T & T \end{bmatrix}.
$$

To see the equivalence of (3) and (4), observe that the second block of (4a) reads $A_0^a \otimes x(v^k) \otimes A_1^c \otimes x(v^k v_1) \preceq x(v^k v_1) \preceq B_1^b \otimes x(v^k v_1) \preceq B_0^b \otimes x(v^k v_1)$. From this transformation, we can easily conclude that $S$ is boundedly consistent under $v^x$ if and only if the LDI with characteristic matrices $A_0^v, A_1^v, B_0^v, B_1^v$ is boundedly consistent, and that all consistent $v$-periodic trajectories of $S$ coincide with consistent 1-periodic trajectories of the LDI; hence,

$$
A_{SLDI}^v(S) = A_{P-TEG}(A_0^a, A_1^a, B_0^c, B_1^c) = \emptyset,
$$

$$
A_{SLDI}^v(S) = A_{P-TEG}(A_0^b, A_1^b, B_0^b, B_1^b) = [3, 3].
$$

It is worth noting that, although P-TEGs labeled $a$ and $b$ are not boundedly consistent, the SLDI under schedule $(ab)^v$ is. Thus, in general it is not possible to infer bounded consistency of an SLDI under a fixed schedule $w$ solely based on the analysis of each mode appearing in $w$.

By generalizing the procedure shown in Example 7, we can derive the following proposition through some simple algebraic manipulations (to set up an equivalent LDI) and applying Theorem 5.

Proposition 8. An SLDI $S$ is boundedly consistent under schedule $w = v^x$ if and only if it admits a $v$-periodic trajectory. Moreover, set $A_{SLDI}^v(S)$ coincides with $A_{NCP}(A^cP^a + \lambda^{-1}I_v + C_v)$, where

$$
P_v = Y_{|v|, 1} \otimes P_{v_{|v|}}, \quad I_v = Y_{1, |v|} \otimes I_{v_{|v|}},
$$

$$
|v| - 1
$$

$$
C_v = \bigoplus_{r=1}^{|v|} (Y_{r, r+1} \otimes P_{v_r} \otimes Y_{r+1, r} \otimes I_{v_r} \otimes Y_{1, |v|} \otimes C_{v|v|}),
$$

where, for all $i, j \in \{1, \ldots, |v|\}$, $Y_{i,j} \in \mathbb{R}^{|v| \times |v|}$, with $Y_{i,j, k} = 0$ if $h = i$ and $j = k$, $(Y_{i,j})_{hk} = -\infty$ else, for all $r \in \{1, \ldots, |v|\}$, $P_{v_r} = B_{v_r}^2$, $I_{v_r} = A_{v_r}^a$, and $C_{v_r} = A_{v_r}^b + P_{v_r}$.

For instance, when $|v| = 5$, matrix $\lambda P_v + \lambda^{-1}I_v + C_v$ has the form

$$
\begin{bmatrix}
C_{v_5} & P_{v_5} & E & E & \lambda^{-1}I_{v_5} \\
I_{v_5} & C_{v_4} & P_{v_4} & E & E \\
E & I_{v_4} & C_{v_3} & P_{v_3} & E \\
E & E & I_{v_3} & C_{v_2} & P_{v_2} \\
\lambda P_{v_3} & E & E & I_{v_2} & C_{v_1}
\end{bmatrix},
$$

which can be rewritten, using the tensor product, as

$$
Y_{1,1} \otimes C_{v_1} \otimes Y_{1,2} \otimes P_{v_1} \otimes \lambda^{-1}Y_{1,5} \otimes I_{v_5} \otimes Y_{2,1} \otimes I_{v_1} \otimes \ldots
$$

Proposition 8 directly provides an algorithm to compute the minimum and maximum cycle times of an SLDI under a fixed periodic schedule. Indeed, these values come from solving the NCP for parametric precedence graph $G(\lambda P_v + \lambda^{-1}I_v + C_v)$. However, this approach results in a slow (although strongly polynomial time) algorithm when the length of subschedule $v$ is large; indeed, its time complexity is $O((|v|n)^4) = O(|v|^4n^4)$, as the considered precedence graph has $|v|n$ nodes. In the next subsection, we show how to exploit the sparsity of $\lambda P_v + \lambda^{-1}I_v + C_v$, illustrated for $|v| = 5$ in (5), to develop an algorithm of linear complexity in the subschedule length.
Figure 2. Lumped-node representation of $G_v$ when $|v| = 5$.

Labels colored in blue, red, and black correspond to arcs whose weight depends proportionally, depends inversely, and does not depend on $\lambda$, respectively.

### 4.3 Improved algorithm

Let us start by defining the multi-precedence graph $G_v$ associated with parametric precedence graph $G(\lambda P_v \oplus \lambda^{-1} I_v \oplus C_v)$: $G_v = G(Y_{1,1} \otimes^t C_v, Y_{1,2} \otimes^t P_v, Y_{1,4} \otimes^t \lambda^{-1} I_v + Y_{1,3}) = (N, \Sigma, \mu, E)$ is such that $N = \{1, \ldots, |v| n\}$, $\Sigma = \{p_1, \ldots, p_n, i_1, \ldots, i_n, c_1, \ldots, c_n\}$, for all $r \in \{1, \ldots, |v| - 1\}$, $\mu(p_r) = Y_{r+1} \otimes^t P_v$, $\mu(i_r) = Y_{r+1} \otimes^t I_v$, $\mu(c_r) = Y_{r+1} \otimes^t C_v$. $\mu(|v|) = Y_{|v|,|v|} \otimes^t C_v$. The multi-precedence graph $G_v$ is schematized by the lumped-node representation of Figure 2 in the case $|v| = 5$. In this representation, $j$ indicates the set of nodes $\{j - 1|n + 1, \ldots, jn\}$ of $G_v$, and an arc from $i$ to $j$ with label $z$ indicates that, in $G_v$, every arc from a node in $i$ to a node in $j$ is labeled $z$.

Let $L_1$ be the (regular) language containing the labels of all circuits in $G_v$ from any node in $I = \{1, \ldots, n\}$. With the visual aid of the lumped-node representation, we can use automata-theory techniques to determine an expression for $L_1$: reinterpret the lumped-node representation of $G_v$ as a deterministic finite automaton with states $j$ for all $j \in \{1, \ldots, |v|\}$; then, $L_1$ is the language recognized by the automaton when $j$ is both initial and final state. Once we get $L_1$, values of $\lambda$ such that $\text{tr}(\mu(L_1)) = 0$ will correspond to those for which there are no circuits in $G_v$, visiting at least one node from $1$, with positive weight: we will see later how to derive from this observation a low-complexity algorithm that finds all $\lambda$'s such that $G_v \in \Gamma_M$ and, due to Proposition 3 these values correspond to the cycle inversely, and does not depend on $\lambda$, respectively.

For Example, in the case $|v| = 5$, we get

$$L_5 = \{\mu \in \mathbb{F}_v \mid \text{tr}(\mu(L_1)) = 0\}$$

Moreover, since $L_1 \cap \{0\} = \emptyset$, we can use Algorithm 1.

To find all $\lambda$'s for which $G_v \in \Gamma_M$, we still need to verify that there are no circuits with positive weight among those visiting only nodes that are not in 1 (if this is not true, then $\Lambda_{\text{SLDI}}(S) = \emptyset$). This can be done by checking that, for all $r \in \{1, \ldots, |v|\}$, $G(\mu(L_1)) \in \mathbb{F}_v$, and $\mu(L_1) = Y_{1,1} \otimes^t \lambda^{-1} I_v + Y_{1,4} \otimes^t C_v$. Finally, from Proposition 2 and $\text{tr}(Y_{1,1}) = 0$, we get $\text{tr}(\mu(L_1)) = 0$ and if only if $G(\lambda P_v \oplus \lambda^{-1} I_v + L_v \oplus C_v) \in \Gamma$. Observe that we obtained an NCP that can be solved in $O(n^4)$ using Algorithm 1.

The discussed procedure to compute the minimum and maximum cycle times of an SLDI $S$ under schedule $\nu^w$ is summarized in Algorithm 2. Note that the time complexity to run lines 1–11 is $O(|v| n^w)$, as the three for-loops perform...
Initially, the robot is positioned at station \( \iota \). We suppose that initially station \( \iota \) is not carrying any part, and it is not carrying any part, and \( \iota \) must be within the interval \( [77, 192] \), respectively, from \( S_1 \) to \( S_4 \) each after switching of mode from \( a \) to \( b \) (respectively, from \( b \) to \( a \)). we define \( (A'_{h})_{h=0}^{43} = 1 \), respectively, \( (A'_{h})_{h=0}^{43} = 134 \). The other elements of \( A'_{w} \), \( A'_{\ast} \) are taken from the characteristic matrices of P-TEG\(_{a} \), for \( z \in \{ a, b \} \). The modeling effort required to define \( S \) is repaid by the possibility to use Algorithm 2 for computing the minimum and maximum cycle times corresponding to a schedule \( w = w' \). For instance, we get \( \Lambda_{b}^{a}(S) = [77, 192] \). This means that, using schedule \( (a')_{h=0}^{43} \), we can obtain one final product of each type every at least 77 and at most 192 time units.

To appreciate the advantage of using Algorithm 2, in Figure 4 we show the computational time to get \( \Lambda_{b}^{a}(S) \) with increasing subschedule length \( |v| \), using different methods: Algorithm 2, the algorithm derived from Proposition 8 directly, the algorithm developed in Kats et al. [2008], and a linear programming solver. The first three algorithms were implemented on Matlab R2019a, for solving the linear programs we used CPLEX’s dual simplex method; the tests were executed on a PC with an Intel i7 processor at 2.20Ghz. From the results, we can see that the most time-consuming approach is the one using Proposition 8 directly, while Algorithm 2 achieves the fastest computation. This shows how critical the exploitation of the sparsity of matrix \( \lambda P_{v} \oplus \lambda^{-1} I_{v} \oplus C_{v} \) is for decreasing computation time.

6. FINAL REMARKS

We have shown that SLDIs can model plants such as multi-product processing networks, and provided an inexpensive method to compute minimum and maximum cycle times when they follow a fixed and periodic schedule. We remark that the complexity reduction achieved by exploiting the sparsity of matrix \( \lambda P_{v} \oplus \lambda^{-1} I_{v} \oplus C_{v} \) through techniques from automata theory could be generalized to solve NCPs on matrices with different distributions of non-\( e \) elements; practical applications in a variety of scheduling problems are expected. Regarding SLDIs, plenty of problems of theoretical and practical relevance remain open, such as the complexity of verifying the existence of a schedule \( w \) under which the SLDI is boundedly consistent. Finally, we argue that, as implicit switching max-plus linear systems generalize the dynamics of max-plus automata (cf. Van Den Boom and De Schutter [2006]), SLDIs generalize the
Figure 4. Time to compute $\Lambda^v_{SLDI}(\mathcal{S})$ for increasing values of $|v|$ using different methods.

dynamics of interval weighted automata. This would imply that SLDIs can be used to represent and solve scheduling problems for systems modeled by safe P-time Petri nets (cf. Komenda et al. [2020]).

REFERENCES

Baccelli, F., Cohen, G., Olse, G.J., and Quadrat, J.P. (1992). *Synchronization and linearity: an algebra for discrete event systems*. John Wiley & Sons Ltd.

Becha, T., Kara, R., Dutilleul, S.C., and Loiseau, J.J. (2013). Modelling, analysis and control of electropotential line modelled by P-time event graphs. *IFAC Proceedings Volumes*, 46(24), 311–316.

Calvez, S., Ayyalinc, P., and Khansa, W. (1997). P-time Petri nets for manufacturing systems with staying time constraints. *IFAC Proceedings Volumes*, 30(6), 1487–1492.

Hardouin, L., Cottenceau, B., Shang, Y., and Raisch, J. (2018). Control and state estimation for max-plus linear systems. *Foundations and Trends® in Systems and Control*, 6(1), 1–116.

Horn, R.A. and Johnson, C.R. (1991). *Topics in matrix analysis*. Cambridge University Press.

Kats, V., Lei, L., and Levner, E. (2008). Minimizing the cycle time of multiple-product processing networks with a fixed operation sequence, setups, and time-window constraints. *European Journal of Operational Research*, 187(3), 1196–1211. doi: https://doi.org/10.1016/j.ejor.2006.07.030.

Kim, J.H., Lee, T.E., Lee, H.Y., and Park, D.B. (2003). Scheduling analysis of time-constrained dual-armed cluster tools. *IEEE Transactions on Semiconductor Manufacturing*, 16(3), 521–534.

Komenda, J., Lai, A., Soto, J.G., Lahaye, S., and Boimond, J. (2020). Modeling of safe time Petri nets by interval weighted automata. *IFAC-PapersOnLine*, 53(4), 187–192. doi: https://doi.org/10.1016/j.ifacol.2021.04.018.

15th IFAC Workshop on Discrete Event Systems (WODES).

Špaček, P. and Komenda, J. (2017). Analysis of cycle time in interval P-time event graphs in dioid algebras. *IFAC-PapersOnLine*, 50(1), 13461–13467.

Van Den Boom, T.J. and De Schutter, B. (2006). MPC of implicit switching max-plus-linear discrete event systems - timing aspects. In *8th IFAC International Workshop on Discrete Event Systems (WODES)*, 457–462. doi:10.1109/WODES.2006.382516.

Zorzenon, D., Komenda, J., and Raisch, J. (2020). Bounded consistency of P-time event graphs. In 2020 59th IEEE Conference on Decision and Control (CDC), 79–85. doi:10.1109/CDC42340.2020.9304309.

Zorzenon, D., Komenda, J., and Raisch, J. (2022a). The non-positive circuit weight problem in parametric graphs: a solution based on dioid theory. Published on arXiv, available at https://arxiv.org/abs/2102.12264.

Zorzenon, D., Komenda, J., and Raisch, J. (2022b). Periodic trajectories in P-time event graphs and the non-positive circuit weight problem. *IEEE Control Systems Letters*, 6, 686–691. doi:10.1109/LCSYS.2021.3085521.