Thick spectral walls in solitonic collisions

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Abstract: We study how the transition of a bound mode through the mass threshold of a scalar field theory in 1+1 dimensions affects the soliton dynamics in a generic (non-self-dual process) process, i.e., when a static intersoliton force shows up. We show that the thin, precisely localised spectral wall which forms in the self-dual limit (no static force), broadens in a well-defined manner in the non-self-dual case, giving rise to what we will call a thick spectral wall. This phenomenon just requires that a discrete mode crosses into the continuum at some intermediate stage of the dynamics and, therefore, should be observable in many soliton-antisoliton collisions. As an example, we consider soliton-antisoliton scattering in a one-parameter version of the $\phi^4$ model.
1 Motivation

Scattering of topological solitons is a complicated process revealing many nontrivial phenomena, even in (1+1) dimensional theories [1–3]. Typically, solitons interact in three different ways, i.e., by a static force [1, 4], with normal (and even quasi-normal) modes [5–7] which may store and release portions of the energy, and with radiation [8, 9]. Of course, during a soliton-antisoliton (SAS) collision all these types of interactions have a nontrivial impact on the dynamics, which leads to quite an involved pattern of behaviour. As a consequence, soliton collisions are still rather poorly understood. As the best example may serve the $\phi^4$ theory where, after 40 year of struggle, there is basically no explanation of the observed phenomena occurring in SAS scattering, e.g., the fractal structure [5]-[15]. In fact, this problem concerns various solitonic processes, even in (1+1) dimensions, especially if solitons possesses internal modes.

One reason for this situation is that we do not have any well defined analytical technique allowing for a rigorous mathematical investigation of SAS scattering. The usual unstable manifold construction [16], which, in the case of non-self-dual dynamics (e.g. soliton-soliton scattering), replaces the moduli space of self-dual processes, cannot be applied. The reason is that in a SAS process the final state, i.e., the vacuum, has much lower energy than the initial state (the infinitely separated SAS pair). In addition, any bound mode which exists on a free soliton, must necessarily disappear during the annihilation. In non-annihilating collisions (e.g., soliton-soliton or soliton-localized impurity) although the bound modes do not have to necessary cross the mass threshold, the frequency of the modes change significantly [17]. These facts were ignored in all effective model approaches, where
the full field theory dynamics was replaced by a small number of (the lightest) degrees of freedom.

Recently, a novel approach based on the so-called self-dual background field provided a framework [18–20] in which this issue can be treated in a systematic and mathematically rigorous fashion. Furthermore, the unique advantage of this approach is that it allows to disentangle the role played by internal modes in the soliton dynamics. Specifically, for a given solitonic process in a given (1+1) dimensional field theory $L[\phi]$, there is a background field $\sigma$ (an impurity) which transforms this process into a self-dual one, by a certain deformation of the initial Lagrangian

$$ L[\phi] \rightarrow L[\phi, \sigma] $$

Physically, e.g., in the case of SAS scattering, this means that there is no static force between the colliding soliton and antisoliton [20]. Hence, they can be placed at any distance from each other. Thus, there is a whole family of infinitely many static SAS solutions with the same energy, which gives rise to the appearance of a moduli space. As the solitons change their position (which corresponds to a flow on the moduli space), their spectral structure also changes. This allowed us to analytically study the effect of the transition of a bound mode to the continuum. Surprisingly, such a crossing triggers the so-called spectral wall (SW) phenomenon [21, 22]. A SW is a well-defined spatial point (e.g., the distance between the soliton and the antisoliton) where the solitons reveal a nontrivial behaviour. Its position is given by the point in moduli space at which the corresponding mode crosses the mass threshold. As the moduli space coordinate can, to some extent, be translated into the SAS distance, it is effectively observed as a barrier in real (physical) space. Specifically, a spectral wall acts as a filter which does not allow kinks to pass through if the pertinent mode is exited too much. Therefore, spectral walls are the leading factor governing the dynamics beyond geodesic flow in a self-dual process.

Of course, a generic SAS scattering in (1+1) dimensional scalar field theories like, for example, in the $\phi^4$ model, is not a self-dual process. It means that, even in the limit of a very slow collision, it does not happen via a sequence of energetically equivalent (self-dual) states. In other words, solitons acts with a static force on each other which influences the outcome and complexity of the scattering process in a highly nontrivial manner. Thus the natural question arises whether spectral walls exist also for generic (non self-dual) solitonic collisions. An affirmative answer definitely matters for their importance for realistic processes.

It is the aim of this work to study this issue in a broken self-dual regime. This is achieved by adding a self-duality breaking term to the original Lagrangian in which a particular SAS scattering occurs in a self-dual manner [22]. The resulting model enjoys the robust qualitative features of a generic SAS scattering, that is, the existence of a non-vanishing static force. However, being a (small) deformation of the self-dual theory, it provides a solid mathematical ground where all emerging phenomena may be given a rigorous, quantitative explanation. This allows us to assume that the obtained results are robust and apply basically to any soliton-antisoliton scattering in (1+1) dimensions.
As an example, we choose the $\phi^4$ theory as a representative model with topological kinks. However, we want to emphasize that similar results can be obtained for any other (1+1) dimensional solitonic field theory.

2 Breaking of self-duality

2.1 The prescription

The unique, special feature of the self-dual background field deformation of a solitonic process in a given field theoretical model is that there is no static force between the solitons. To relate the process in question to a realistic, non-selfdual counterpart, one has to switch on such forces. Basically, from a qualitative point of view, any breaking of the self-duality is equivalent, i.e., it introduces an intersoliton static force. As a consequence, the resulting theory should qualitatively reproduce the main features of the considered SAS process in the original, $L[\phi]$ theory, regardless of the particular way of breaking of the self-duality.

Among the infinitely many possible ways to break the self-duality of a SAS process in the $L[\phi, \sigma]$ model, there is one particularly simple one. Namely,

$$L = (1 - \epsilon^2) L[\phi, \sigma] + \epsilon^2 L[\phi] \quad (2.1)$$

where $\epsilon \in [0, 1]$ is a self-duality breaking parameter, allowing to interpolate between the self-dual background field model $L[\phi, \sigma]$ ($\epsilon = 0$) and the original theory ($\epsilon = 1$). Specifically, for SAS collisions in the $\phi^4$ model we have

$$L[\phi, \sigma] = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} \phi_x^2 + \frac{1}{2} (\phi_x + \sigma(1 - \phi^2))^2 \right] \quad (2.2)$$

and

$$L[\phi] = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[ \phi_t^2 + \phi_x^2 + (1 - \phi^2)^2 \right] \quad (2.3)$$

while the relevant impurity takes the form $\sigma = \tanh x$. Note that this model has the same mass threshold, $\omega^2 = 4$, for any $\epsilon$. Furthermore, since for $x \to \pm \infty$ the impurity tends to $\pm 1$, we recover the usual $\phi^4$ model. Hence, asymptotic states are exactly the $\phi^4$ kink and antikink. Also the kinetic term is independent of $\epsilon$.

2.2 The self-dual limit and the moduli space

It has been recently established that in the limit when $\epsilon = 0$ the model possesses a self-dual (SD) sector consisting of topologically trivial SAS solutions [20], [23]

$$\phi(x; \phi_0) = \frac{1 + \phi_0 - (1 - \phi_0) \cosh^2 x}{1 + \phi_0 + (1 - \phi_0) \cosh^2 x}, \quad (2.4)$$

where the value of the field at the origin, $\phi_0 \in (-\infty, 1)$, is a moduli space coordinate. For $\phi_0 \to 1$ it describes an infinitely separated pair of a $\phi^4$ kink and antikink. As $\phi_0$ decreases, the solitons approach each other, losing their identity for $\phi_0 < 0$. For $\phi_0 = -1$ we get the vacuum $\phi = -1$. Then, for $\phi_0 < -1$ the solution reveals a negative bump which gets
deeper for decreasing $\phi_0$ and whose bottom, i.e., $\phi_0$, may approach $-\infty$. This negative
bump is in fact a welcome property of the moduli space flow, because it is observed in
SAS annihilation in the pure $\phi^4$ model. Of course, the difference is that the singularity
$\phi_0 \to -\infty$ is not attainable in the latter case.

The moduli space metric is (see Fig. 1 left)

$$M(\phi_0) = \int_{-\infty}^{\infty} dx \left( \frac{d}{d\phi_0} \phi(x; \phi_0) \right)^2$$

$$= \frac{1 + a}{24a} \left( -3 + 4a(4 + a) + \frac{3(1 + 6a)}{(1 + a)\sqrt{1 + a}} \right)$$

where for compactness we introduce $a \equiv (1 + \phi_0)/(1 - \phi_0)$.

2.3 The near self-dual sector and the unstable manifold

Here we assume that $\epsilon$ is a small number. Therefore, the moduli space can be replaced by
the unstable manifold where the self-dual breaking part modifies the geodesic flow by the
appearance of a drag force due to an effective potential

$$L = \frac{1}{2} M(\phi_0) \dot{\phi}_0^2 - \epsilon^2 V(\phi_0)$$

where

$$V(\phi_0) = \frac{1}{3(1 + a)^2} \left( 3 + 4a(1 + 2a(3 + a)) + 3 \frac{1 - 2a + 4a^2}{\sqrt{a(1 + a)}} \arctanh \sqrt{\frac{a}{1 + a}} \right)$$

The SD breaking term strongly modifies the geodesic dynamics for $\phi_0 < -1$. Indeed, the
effective potential grows rapidly. This provides a mechanism which prevents the self-dual
solution from developing too large a negative bump. In other words, a repulsive core
emerges. Its position $\phi_0^*$ can be obtained from the first integral

$$\frac{1}{2} M(\phi_0) \dot{\phi}_0^2 + \epsilon^2 V(\phi_0) = E$$
where $E$ is the initial energy. Indeed, it obeys the relation

$$
\frac{1}{2} M (\phi_0^{\text{in}}) (\dot{\phi}_0^{\text{in}})^2 + \epsilon^2 V(\phi_0^{\text{in}}) = \epsilon^2 V(\phi_0^*)
$$

(2.10)

where $\phi_0^{\text{in}}$ and $\dot{\phi}_0^{\text{in}}$ is the initial position on the moduli and its velocity. Assuming that the initial state consists of the infinitely separated kink and antikink with velocity $v$ we find that $\phi_0^{\text{in}} \to 1$ and $\dot{\phi}_0^{\text{in}} = 2v\gamma (1 - \phi_0^{\text{in}})$ (here $\gamma = 1/\sqrt{1-v^2}$). Thus, the core can be read off from the algebraic equation

$$
V(\phi_0^*) = \frac{4}{3} \left( \frac{v^2\gamma^2}{\epsilon^2} + 2 \right),
$$

(2.11)

where the l.h.s. does not depend on $\epsilon$. As $\epsilon$ tends to 0 or $v$ increases we can climb higher on $V$, that is, go further in the moduli space towards more negative $\phi_0$.

Note that the effective potential has a local maximum at $\phi_0^* = 0.89167$. It means that kinks in the initially infinitely separated SAS pair repel each other. After crossing this little barrier, they start to strongly attract. The barrier is a very small one as $V(\phi_0^*) - V(\phi_0 = 1) = (2.67029 - 8/3) = 0.00363$. Therefore, at least for weak self-duality breaking (small $\epsilon^2$), it is very easy for boosted solitons to climb over the potential barrier.
3 Spectral walls in the near self-dual sector

3.1 Thin spectral walls in the self-dual limit

To analyse spectral walls, we excite the asymmetric superposition of the shape modes of the asymptotically free kink and antikink of the usual $\phi^4$ model. This means that the initial configuration is

$$\phi_{in}(x, t) = -\tanh(\gamma(x - x_0 - vt)) + \tanh(\gamma(x - x_0 - vt)) - 1$$

$$-A \frac{\sinh(\gamma(x - x_0 - vt))}{\cosh^2(\gamma(x - x_0 - vt))} \cos \omega \gamma(t - vx)$$

$$-A \frac{\sinh(\gamma(x + x_0 + vt))}{\cosh^2(\gamma(x + x_0 + vt))} \cos \omega \gamma(t + vx)$$ (3.1)

where the frequency $\omega^2 = 3$ and $\gamma = 1/\sqrt{1 - v^2}$. In the self-dual limit, such a superposition combines into a bound mode which changes its frequency as we flow on the moduli space i.e., the solitons approach each other. Importantly it crosses the mass threshold at $\phi_0 = -0.013$, see the green line in Fig. 1, right panel. Here the corresponding spectral wall occurs.

Typically, for a weakly excited self-dual SAS solution, the dynamics follows the geodesic flow and the kink and antikink go through the wall and annihilate, forming the negative bump of arbitrary depth. There is a critical amplitude for which the incoming pair freezes at the wall, i.e., the solitons do not change their mutual distance although they slightly oscillate. The value of $\phi_0$ (moduli space coordinate) at which such a stationary (saddle point) solution exists, coincides with the point on the moduli space at which the pertinent mode enters the continuum. Hence, the position of the stationary solution identifies the wall. For even higher amplitudes the SAS pair is reflected back.

Note that the position of the SW does not depend on the velocity of the incoming solitons. It is a stiff infinitely thin wall [21].

However, in our example the spectral wall exists for a value of the moduli coordinate at which the constituents lose their identity, i.e., below $\phi_0 < 0$. Importantly, it is close to the position of the vacuum wall (VW), $\phi_0 = -1$, another important factor in the self-dual dynamics [22]. The mechanism behind the existence of the vacuum wall is completely different, not related to a mode transition to the continuum. Therefore, it is not a selective phenomenon. On the contrary, it is triggered by any sufficiently excited self-dual solution. It was also observed that a spectral wall can be significantly affected if it is located too close to the vacuum wall. In particular, the solitons may not form a clearly visible stationary state [22]. This is exactly what happens in our case. The trajectory is slightly flattened at the position of the spectral wall but then it is repelled due to the interaction with the vacuum wall. One can say that the spectral wall is hidden in the shadow of the vacuum wall. This behavior is shown in Fig. 2 (left panel).
3.2 The near self-dual regime

Now we switch on the self-duality breaking term. We begin with a very small velocity $v = 0.005$. The first effect is that the vacuum wall is less important as $\epsilon$ increases and very quickly such a wall disappears completely. The reason for that is the following. The vacuum wall exists due to the zero mode. When a self-dual solution passes through the vacuum, an arbitrary small perturbation may result in the formation of a SAS pair. After the breaking of the self-duality, the vacuum is still a solution. However, for a sufficiently big difference of the potential energy at $\phi_0 = -1$ and at the SAS final state, small perturbations do not grow arbitrarily (forming solitons), but either are localized as oscillations around the vacuum or travel deep into the negative values of $\phi_0$ where they meet the repulsive core. Thus, the force induced by the non-SD part takes the trajectory through $\phi_0 = -1$. This occurs, in fact, very quickly. The VW exists for $\epsilon = 0.001$ (Fig. 2 right panel) but already for $\epsilon = 0.002$ it completely disappears (Fig. 3 left panel).

Secondly, the reduced importance of the vacuum wall and, as a consequence, its shadow, makes the stationary trajectory very well visible, see for example Fig. 2 (right panel) where the dynamics for $\epsilon = 0.001$ is presented. The vacuum wall is still there, but the stationary trajectory is clearly visible.

Importantly, the position of the stationary trajectory, which we call a barrier, rapidly moves towards bigger values of $\phi_0$ (see Fig. 3 right panel). Observe that, due to the weak self-duality breaking, $\phi_0$ may still be used as a relevant coordinate on the unstable manifold, giving a good insight into the physics of the system. In fact, even for very small values of the parameter $\epsilon$, the barrier varies significantly, e.g., from $\phi_0 = -0.013$ for $\epsilon = 0$ to $\phi_0 = 0.424$ for $\epsilon = 0.008$, see Fig. 4 (left panel, yellow dots). Now the SW forms for well defined solitons, that is, before they lose their identity.

For higher values of $\epsilon$, the repulsive core is also well visible, Fig. 3 (right panel), and its position is captured by our geodesics analysis with very good accuracy, see eq. (2.11). Note that, for this case, also bouncing solutions are present. They describe a SAS pair trapped between the core and the barrier where the stationary trajectory is formed.
Figure 5. Thick spectral wall for $\epsilon = 0.01$. Dynamics of $\phi_0$ for different initial velocities of the incoming solitons (different colors) with the stationary solution (the flat region of the trajectory) frozen at the barrier.

4 Thick spectral wall

Surprisingly, a closer look on the barrier in the non-SD regime reveals even more fascinating findings.

In contrast to the SD limit, the position of the stationary trajectory (the wall) significantly depends on the velocity of the incoming solitons. Indeed, it goes toward smaller $\phi_0$ as $v$ increases, see Fig. 4 right panel. In means that the barrier which was the spectral wall (at $\epsilon = 0$) behaves (for $\epsilon \neq 0$) rather as a thick spectral wall of a certain stiffness which can be penetrated by the SAS solution. When the initial velocity grows the solution can compress the wall more strongly. However, the bottom of the thick spectral wall, which is located in the vicinity of the original thin SW, apparently does not move. Hence, at some point even a large change of $v$ changes the position of the stationary trajectory very weakly, see Fig. 5. The stiffness of the thick spectral wall increases with the SD breaking parameter $\epsilon$.

As we know, in the SD limit the thin spectral wall occurs when a mode enters the continuum. However, when we depart from this extreme SD regime, we find that the barrier happens before the relevant mode enters the continuum, Fig. 6. It means that the mechanism behind the think spectral wall phenomenon must be a modification of the thin SW mechanism.

An explanation of the appearance of the thick SW phenomenon can be achieved in terms of the perturbation theory around the SD solution. The main idea is that the relevant bound mode cancels the acceleration due to the effective potential (2.8), leading to a stationary solution. Therefore, we search for an approximated solution in the form

$$\phi \approx \Phi + A_0\eta_0 + A_1\phi^{(1)} + A^{(2)}_1\phi^{(2)}$$

(4.1)

where to the SD solution $\Phi = \phi(x; \phi_0)$, we add the mode $\eta_0$, inducing the flow on the unstable manifold and related to the force due to the effective potential (2.8), as well as
Figure 6. Solution frozen on the barrier in the thick spectral wall for $\epsilon = 0.02$. Upper left: dynamics of $\phi_0$. Lower left: strength of the excited modes at $x = 0$. Upper right: radiation at $x = 50$. Lower right: excited modes at $x = 50$.

the excited bound mode $\phi_1 \equiv \phi^{(1)}$. $A_0, A_1$ are the amplitudes and $\phi^{(2)}$ is a second order correction. Note that the flow on the unstable manifold (or on the moduli space for $\epsilon = 0$) is generated by the change of $A_0(t)$. If this amplitude is constant, the SAS solution (or $\phi_0$) is frozen on a certain point on the unstable manifold. We neglect higher order terms in (4.1).

The equation in $O(A_0^0)$ takes the form

$$\phi^{(0)}_{tt} - \phi^{(0)}_{xx} + U'(\phi^{(0)}) = 0$$

where a prime denotes the differentiation w.r.t. $\phi$. In the non-SD model, there exist only three static solutions describing the vacuum, a pair of infinitely separated kinks, or an unstable solution. Other solutions are not static and accelerate due to the effective potential. For small $\epsilon$, this acceleration is also small and we can assume that $\phi^{(0)} = \Phi + A_0 \eta_0(x)$. The zeroth order equation is

$$\ddot{A}_0 \eta_0 + A_0 H \eta_0 = \Phi_{xx} - U'$$

where $H \equiv -d^2/dx^2 + U''$ while $U' \equiv U'(\Phi)$ and $U'' \equiv U''(\Phi)$. Using the fact that $\eta_0$ is the unstable eigen-mode with $\omega_0^2 < 0$ and projecting the equation on $\eta_0$ we find

$$\ddot{A}_0 + \omega_0^2 A_0 = \int dx \eta_0(\Phi_{xx} + U')$$

For the SD case, the r.h.s. vanishes and $\omega_0 = 0$.

In the first order, the bound modes oscillate independently

$$\phi^{(1)} = \phi_1 = \frac{1}{2} \eta_1(x) e^{i\omega_1 t} + c.c.$$

$\eta_0, \eta_1$ are the eigen-modes of $H$. The equation in $O(A_0^1)$ takes the form

$$\phi^{(1)}_{tt} - \phi^{(1)}_{xx} + U'(\phi^{(0)}) = 0$$

where $U'(\phi^{(0)}) \equiv U'(\Phi + A_0 \eta_0)$ and $\phi^{(1)} = \phi^{(1)}(x) e^{i\omega t} + c.c.$.
where the profiles and frequencies are obtained from the eigenvalue equation $H\eta_i = \omega_i^2 \eta_i$.

Finally, the second order leads to an inhomogeneous equation

$$\phi_i^{(2)} - H\phi^{(2)} = -\frac{1}{2}U'''(\phi^{(1)})^2.$$  \hspace{1cm} (4.6)

Taking all this information into account, the full equation up to the second order projected onto the translational mode reads

$$\ddot{A}_0 + \omega_0^2 A_0 = \int dx\eta_0(\Phi_{xx} - U')$$
$$- \frac{1}{4} A_1^2 (1 + \cos 2\omega_1 t) \int dx\eta_0 U'''\eta_1^2$$  \hspace{1cm} (4.7)

Thus the amplitude of the oscillational mode $A_1$ which cancels the acceleration generated by the effective potential is

$$A_1 = 2\sqrt{\frac{\int dx\eta_0(\Phi_{xx} - U')}{\int dx\eta_0 U'''\eta_1^2}}$$  \hspace{1cm} (4.8)

There is also a non-vanishing non-homogeneous part. However, it does not destabilize the solution but generates small oscillations with frequency $\omega_1$. In fact, such small oscillations of $\phi_0$ are clearly visible for the stationary trajectory frozen on the thick SW. This equation gives, for fixed $\epsilon$ and $v$, a relation between the amplitude of the exited mode which leads to the stationary solution, i.e., the appearance of the barrier in the thick SW, and its position.

To test our predictions, we have calculated the amplitude $A_1$ obtained from eq. (4.8) as a function of $\phi_0$. Note that in order to calculate the integrals, first we had to find the eigenmodes $\eta_0$, $\eta_1$ and the appropriate eigenvalues $\omega_i^2$ numerically, all of which depend on $\phi_0$ and $\epsilon$. We solved the eigenvalue problem using a shooting method, integrating the linearized equation and matching with the exponential tail. Next, we calculated the integrals approximating the eigenfunctions using the Hermite interpolation method.

**Figure 7.** Critical amplitude of the excited mode vs. the position of the stationary solution for $\epsilon = 0.01$ and 0.02.
The eigenfunctions for a given position on the moduli space, $\phi_0$, were also used to prepare initial conditions for the evolution of the full Euler-Lagrange equation. For a small amplitude of the excitation, the solitons attract each other and collide. Too large amplitudes lead to repulsion. Using the shooting method, we were able to find the amplitude which allows for the stationary (saddle point) solution, which remains at the same position for a long time (of the order of $t = 200–1000$ depending on $\epsilon$ and $\phi_0$).

In Fig. 7, we compare the theoretical prediction with the full numerical computation for $\epsilon = 0.01$ and 0.02. We plot the amplitude providing the stationary solution and its position $\phi_0$, which corresponds to different initial velocities $v$. The agreement is striking.

In addition, there is a simple scaling relation between the $A_1$ vs $\phi_0$ curves for different $\epsilon$. Indeed, after multiplication by $\epsilon$ the curves coincide. This follows from eq. (4.8), where the first non-zero contribution in the nominator is of the order $\epsilon^2$ (the order $\epsilon^0$ vanishes due to the SD property of $\Phi$) while the denominator has a nontrivial part of the order $\epsilon^0$. This linear scaling may receive some correction for bigger $\epsilon$.

## 5 Summary

The main result of this letter is the observation that, once we allow for a non-zero static intersoliton force, which is a generic feature of all realistic soliton-antisoliton processes, thin spectral walls transmute into thick spectral walls with a sponge-like behaviour. This means that, if we move from a self-dual to non-self-dual SAS process, instead of a sharp selective barrier, whose position in principle does not change with the velocity of the scattered solitons, we find a sort of sponge which has a non-zero thickness and a certain stiffness. This implies that for larger velocities the barrier is located closer to the original thin SW. Further, the stiffness of the thick SW increases as we move away from the self-dual regime. For $\epsilon \to 0$ the thick SW basically disappears and we see only its bottom i.e., the thin SW. As $\epsilon$ grows, the thick SW behaves more resistant, that is, for a given velocity the barrier (location of the stationary trajectory) is located further away from the original thin SW. Hence, if $\epsilon$ increases, the kink and the antikink feel the barrier sooner, while they are still further apart and their identity is more pronounced.

On the other hand, the thickness of the thick SW does not change with $\epsilon$. Its bottom is always the thin SW (which is probably a model independent feature) while it ends at the position of the local maximum of the effective potential $\phi_0^b$. This latter position is certainly a particular property of our SD breaking set-up. One can imagine a situation where the thick SW extends to $\phi_0 \to 1$, if no maximum exists. In fact, we found some evidence that for $\epsilon \to 1$, as a result of some nonlinearities, this small maximum is moved to $\phi_0 \to 1$.

The appearance of the thick SW can be viewed as an enhancement of the original thin SW phenomenon, which is a welcome effect, as most of the solitonic interactions (especially SAS scatterings) are strongly non-SD processes. Furthermore, we also observe a weakening of the vacuum wall due to the non-SD part of the model (the effective potential). As a consequence, the barrier is not any longer distorted by the vicinity of the vacuum wall (see the thin SW in the self-dual limit) and now it is much better visible.
The existence of the thin and thick spectral walls is based on different mechanisms. While the former is due to the mode transition to the continuum spectrum, the later is based on the appearance of a stationary saddle point solution by a compensation of the acceleration of the solitons (as dictated by the effective potential on the unstable manifold) by the sufficiently excited mode. Thus, now, this barrier happens before the mode passes the mass threshold. This is a nice feature, as it leads to a well defined effective theory, allowing for a quantitative description of the effect, see eq. (4.8).

It is worth underlining that the self-duality breaking (appearance of intersolitonic forces) acts differently on thin spectral walls and vacuum walls. In the self-dual limit, the vacuum wall is the leading factor of the dynamics beyond the geodesic approximation. It concerns all perturbations and can strongly affect thin spectral walls located in too close neighborhood. When self-duality is broken, a thin spectral wall expands to a thick spectral wall which is much more pronounced. This especially concerns thin spectral walls which were hidden in the shadow of the vacuum wall. Furthermore, the vacuum wall is very quickly removed. So, in practice, it disappears from non-self-dual processes. Hence we can say that, in contrast to thin spectral walls, the VW is a pure SD effect.

Although we considered a particular model, with a particular mode excited, it must be underlined that the results are much more general. The existence of a static force between a soliton and an antisoliton is a generic feature shared by any realistic model supporting topological solitons. Furthermore, thick spectral walls will exists in near-self-dual SAS processes in any (1+1) dimensional theory, also for a model which supports (anti)kinks without oscillating modes. This is due to the fact that the initial configuration, i.e., an infinitely separated pair of kink and antikink, has two zero modes. Then, in the self-dual background field limit only one zero mode survives, namely the symmetric superposition of the zero modes of the asymptotical states. The asymmetric superposition becomes a massive mode which necessarily crosses into the continuum during the collision. Hence the corresponding thin spectral wall shows up. In a weakly broken self-dual process, this thin spectral wall will again be enhanced to a thick spectral wall. As a result, we expect that thick spectral walls will exist in various processes in many (1+1) dimensional field theories [24]-[27].

Looking from a more general perspective, we extended the self-dual background field framework, which previously allowed to understand solitonic processes in given models in the limit of no static force, to a case where the self-dual property is weakly broken. This means that a weak static intersoliton force appears. This regime is still mathematically well-defined and, as a consequence, allows for a rigorous mathematical analysis and a good understanding of the dynamics.

In the final step, the limit $\epsilon \to 1$, which reproduces the original theory we would like to investigate, should be considered. That could, for example, prove the existence of the spectral walls in the original model. We will investigate this problem in a forthcoming paper.
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