Edges not covered by monochromatic bipartite graphs

Xiutao Zhu\textsuperscript{1,2}, Ervin Győri\textsuperscript{1}, Zhen He\textsuperscript{1,3}, Zequn Lv\textsuperscript{*1,3}, Nika Salia\textsuperscript{1,5}, Casey Tompkins\textsuperscript{1}, and Kitti Varga\textsuperscript{1,4}

1Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences.
2Department of Mathematics, Nanjing University.
3Department of Mathematical Sciences, Tsinghua University.
4Department of Computer Science and Information Theory, Budapest University of Technology and Economics.
5Extremal Combinatorics and Probability Group, Institute for Basic Science, Daejeon, South Korea.

Abstract

Let $f_k(n, H)$ denote the maximum number of edges not contained in any monochromatic copy of $H$ in a $k$-coloring of the edges of $K_n$, and let $\text{ex}(n, H)$ denote the Turán number of $H$. In place of $f_2(n, H)$ we simply write $f(n, H)$. In [5], Keevash and Sudakov proved that $f(n, H) = \text{ex}(n, H)$ if $H$ is an edge-critical graph or $C_4$ and asked if this equality holds for any graph $H$. All known exact values of this question require $H$ to contain at least one cycle. In this paper we focus on acyclic graphs and have the following results:

(1) We prove $f(n, H) = \text{ex}(n, H)$ when $H$ is a spider or a double broom.
(2) A tail in $H$ is a path $P_3 = v_0v_1v_2$ such that $v_2$ is only adjacent to $v_1$ and $v_1$ is only adjacent to $v_0, v_2$ in $H$. We obtain a tight upper bound for $f(n, H)$ when $H$ is a bipartite graph with a tail. This result provides the first bipartite graphs which answer the question of Keevash and Sudakov in the negative.
(3) Liu, Pikhurko and Sharifzadeh [6] asked if $f_k(n, T) = (k-1)\text{ex}(n, T)$ when $T$ is a tree. We provide an upper bound for $f_2(n, P_{2k})$ and show it is tight when $2k-1$ is prime. This provides a negative answer to their question.

1 Introduction

Given any graph $H$, the classical theorem of Ramsey asserts that there exists an integer $R(H, H)$ such that every 2-coloring of the edges of the complete graph $K_n$ with $n \geq R(H, H)$ contains a monochromatic copy of $H$. A natural extension of this problem is determining how many monochromatic copies of $H$ there are. For the case of $H = K_3$, this question was answered by Goodman [4] and the case of $H = K_4$ was settled by Thomason [10].

In a different direction, one can ask how many edges must be contained in some monochromatic copy of $H$ in every 2-coloring of the edges of $K_n$ (equivalently how many edges there can be in a 2-coloring which are not contained in any monochromatic copy of $H$). The first result about this topic is due to Erdős, Rousseau and Schelp [2]. They considered the maximum number of edges not contained in any monochromatic triangle in a 2-coloring of the edges of $K_n$. Erdős also wrote “many further related questions can be asked” in [2]. In this paper, we will consider problems of this type.

Let $c$ be a 2-coloring of the edges of $K_n$ and let $H$ be a graph. If an edge of $K_n$ is not contained in any monochromatic copy of $H$, then we say it is NIM-$H$. Let $E(c, H)$ denote the

\textsuperscript{*}Corresponding author. Email: lvzq19@mails.tsinghua.edu.cn
set of all NIM-$H$ edges in $K_n$ under the 2-edge-coloring $c$ and let

$$f(n, H) = \max \{|E(c, H)| : c \text{ is a 2-edge-coloring of } K_n\}.$$ 

Let $ex(n, H)$ be the Turán number of $H$. If one considers a 2-coloring of the edges of $K_n$ in which one of the colors yields an extremal graph for $H$, then it is easy to see

$$f(n, H) \geq ex(n, H).$$  \hspace{1cm} (1)

As observed by Alon, the result on $f(n, K_3)$ by Erdős, Rousseau and Schelp [2] can also be deduced from a result of Pyber [9] (see [5]). In [5], Keevash and Sudakov studied $f(n, H)$ systematically. They proved that if $H$ contains an edge $e$ such that $\chi(H - e) < \chi(H)$ or $H = C_4$, then equality holds in (1) for sufficiently large $n$. Furthermore, they asked if the equality holds for all $H$.

**Question 1** (Keevash, Sudakov [5]). Is it true that for any graph $H$ we have $f(n, H) = ex(n, H)$ when $n$ is sufficiently large?

In 2017, Ma [7] provided an affirmative answer to Question 1 for an infinite family of bipartite graphs $H$, including all even cycles and complete bipartite graphs $K_{s,t}$ for $t > s^2 - 3s + 3$ or $(s,t) \in \{(3,3),(4,7)\}$. In 2019, Liu, Pikhurko and Sharifzadeh [6] extended Ma’s result by providing a larger family of bipartite graphs for which $f(n, H) = ex(n, H)$ holds (however, the graphs they construct still contain a cycle). Surprisingly, Yuan [11] recently found an example showing that the assertion in Question 1 does not hold in general.

**Theorem 1** (Yuan [11]). Let $p \geq t + 1 \geq 4$ and $K_t^{p+1}$ denote the graph obtained from $K_t$ by replacing each edge of $K_t$ with a clique $K_{p+1}$. When $n$ is sufficiently large, then

$$f(n, K_t^{p+1}) = ex(n, K_t^{p+1}) + \binom{t-1}{2}.$$ 

Based on this result, he conjectured the following.

**Conjecture 1** (Yuan [11]). Let $H$ be any graph and $n$ be sufficiently large. Then there exists a constant $C = C(H)$ such that $f(n, H) = ex(n, H) + C$.

As mentioned earlier, the known results about the exact value of $f(n, H)$ require that $H$ contains a cycle. For acyclic graphs and some other bipartite graphs, the situation is less clear. Thus, in this paper, we will focus on this case. A spider is the graph consisting of $t$ paths with one common end vertex such that all other vertices are distinct. A double broom with parameters $t, s_1$ and $s_2$ is the graph consisting of a path with $t$ vertices with $s_1$ and $s_2$ distinct leaves appended to each of its respective end vertices.

**Theorem 2.** Let $H$ a spider or a double broom with $s_1 < s_2$ and $n$ be sufficiently large, we have

$$f(n, H) = ex(n, H).$$

A tail in a (not necessary acyclic) graph $H$ is a path $P_3 = v_0v_1v_2$ such that $v_2$ is only adjacent to $v_1$ and $v_1$ is only adjacent to $v_0$ and $v_2$.

**Theorem 3.** Let $H = (A, B, E)$ be a bipartite graph containing a tail and $|A| \leq |B|$. When $n$ is sufficiently large, we have

$$f(n, H) \leq ex(n, H) + \binom{|A| - 1}{2}. \hspace{1cm} (2)$$

Furthermore, the upper bound is tight.
Remark 1. In Theorem 3, there are many bipartite graphs $H$ such that $f(n, H)$ achieves an upper bound greater than $\text{ex}(n, H)$. This implies that even for the bipartite case, the answer to Question 1 can be negative. However, the graphs from Theorem 3 satisfy Conjecture 1.

We will also consider the case of edge colorings with 3 or more colors. Let $f_k(n, H)$ be the maximum number of edges not contained in any monochromatic copy of $H$ in a $k$-coloring of the edges of $K_n$. Thus, $f_k(n, H) = f(n, H)$. It appears likely that for $k \geq 3$, the function $f_k(n, H)$ has different behavior for bipartite graphs and non-bipartite graphs. For non-bipartite graphs, one can see that $f_k(n, H) \neq (k-1)\text{ex}(n, H)$ since $(k-1)\text{ex}(n, H) \geq \binom{k}{2}$.

For a tree $T$, Ma [7] constructed a lower bound by taking random overlays of $k-1$ copies of extremal $T$-free graphs, and the construction implies $f_k(n, T) \geq (k-1-o(1))\text{ex}(n, T)$. Liu, Pikhurko and Sharifzadeh [6] showed that this lower bound is asymptotically correct.

**Theorem 4** (Liu, Pikhurko, Sharifzadeh [6]). Let $T$ be a tree with $h$ vertices. Then there exists a constant $C(k, h)$ such that for all sufficiently large $n$, we have

$$|f_k(n, T) - (k-1)\text{ex}(n, T)| \leq C(k, h).$$

For more general bipartite graph $H$, Ma [7] wrote “it may be reasonable to ask if $f_k(n, H) = (k-1)\text{ex}(n, H)$ holds for sufficiently large $n$”. However, this is not true for disconnected bipartite graphs. Liu, Pikhurko and Sharifzadeh [6] gave an example and showed $f_k(n, 2K_2) = (k-1)\text{ex}(n, 2K_2) - \binom{k-1}{2}$. Based on this example, Liu, Pikhurko and Sharifzadeh [6] asked the following question.

**Question 2** (Liu, Pikhurko, Sharifzadeh [6]). Is it true that $f_k(n, T) = (k-1)\text{ex}(n, T)$ for any tree $T$ and sufficiently large $n$?

Our third result concerns the case when $T$ is a path with an even number of vertices and yields a negative answer to Question 2.

**Theorem 5.** Let $k \geq 1$ and $n \geq (2k)^{2k^2}$ be integers. We have

$$f_{2k}(n, P_{2k}) \leq (2k-1)\text{ex}(n, P_{2k}) + (k-1)\binom{2k-1}{2}.$$ 

Furthermore, equality holds when $2k-1$ is a prime and $n \in \{a(2k-1) + (k-1), a(2k-1) + k\}$.

**Notation and organization.** For a given graph $G$, we use $e(G)$ to denote the number of edges of $G$. For a subset of vertices $X$, let $G[X]$ denote the subgraph induced by $X$ and $G - X$ denote the subgraph induced by $V(G) \setminus X$. For two disjoint subset $X, Y$, let $G[X, Y]$ denote the bipartite subgraph of $G$ consisting of the edges of $G$ with one end vertex in $X$ and the other in $Y$. In a red-blue edge-colored complete graph $K_n$, we say that $u$ is a red (or blue) neighbor of $v$ if the edge $uv$ is red (or blue). For a set $X$ of vertices, let $N_r(v, X)$ and $N_b(v, X)$ denote the red and blue neighbors of $v$ in $X$, respectively. Let $d_r(v, X) = |N_r(v, X)|$ and $d_b(v, X) = |N_b(v, X)|$. If $X = V(K_n)$, then we simply write $d_r(v)$ and $d_b(v)$. For two graphs $G$ and $H$, we use $G \cup H$ to denote the disjoint union of $G$ and $H$. Let $G + H$ be the graph obtained from $G \cup H$ by adding all edges with one end vertex in $V(G)$ and one end vertex in $V(H)$.

The rest of the paper is organized as follows. In Sections 2 and 3, we study the function $f(n, H)$ and prove Theorems 2 and 3, respectively. In Section 4, we study the general function $f_k(n, H)$ and prove Theorem 5.

## 2 Proof of Theorem 2

Let $H$ be a spider or a double broom on $k$ vertices and $c$ be a red-blue edge-coloring of $K_n$ with $|E(c, H)|$ being maximum. If $E(c, H)$ contains no $H$, then

$$f(n, H) = |E(c, H)| \leq \text{ex}(n, H),$$
and we are done. Hence we may assume there is a non-monochromatic copy of $H$ in $E(c, H)$.

Since we can take $n$ to be larger than the Ramsey number $R(k^2, k^2)$, it follows, without loss of generality, that $K_n$ contains a blue clique $K$ of size at least $k^2$. We partition $V(K_n)$ into two parts $X$ and $Y$ such that $Y$ is maximal with the property that any vertex $v$ in $Y$ has $d_b(v, Y) \geq k$ and $X$ consists of the remaining vertices. Note that the large blue clique $K$ is contained in $Y$, and hence $|Y| \geq k^2$. Since each vertex in $Y$ has blue degree at least $k$ in $Y$, every blue edge in $Y$ or between $X$ and $Y$ can be extended to a blue copy of $H$. Hence, all blue NIM-$H$ edges are contained in $X$ and $|X| \geq 2$.

For each vertex $u$ in $X$, we have $d_b(u, Y) \leq (k - 1)$. Thus for each subset $X'$ of $X$, the subset $Y' = Y \setminus N_b(X', Y)$ is such that $K_n[X', Y']$ is a red complete bipartite graph and $|Y'| \geq |Y| - (k - 1)|X'|$. We call $Y'$ the corresponding subset of $X'$.

First assume $|X| \geq \left\lceil \frac{k}{2} \right\rceil + 1$. For each red edge $uv$ contained in $X$ or between $X$ and $Y$, we can find a subset $X' \subseteq X$ of size $\left\lfloor \frac{k}{2} \right\rfloor$ that contains exactly one of $u$ and $v$. Using the corresponding subset $Y'$ of $X'$, this red edge $uv$ can be extended to a red copy of $H$. Hence all red NIM-$H$ edges are contained in $Y$ and

$$|E(c, H)| \leq \text{ex}(|Y|, H) + \text{ex}(|X|, H) \leq \text{ex}(n, H).$$

Therefore, in the rest of the proof, we will assume $|X| \leq \left\lceil \frac{k}{2} \right\rceil$. Furthermore, each red edge in $Y$ is NIM-$H$, otherwise we replace the color of this edge by blue and since $E(c, H)$ is maximum, it has no changes.

Next we distinguish two cases based on whether $H$ is a spider or a double broom.

**The proof when $H$ is a spider.** Let $H$ be a spider consisting of $t$ paths with a common initial vertex $v_0$. We call each path starting from $v_0$ a branch, and we assume that the lengths of these $t$ branches are $\ell_1, \ldots, \ell_t$ such that $v(H) = k = 1 + \sum_{i=1}^t \ell_i$.

Now we choose a copy of $H$ from $E(c, H)$ and denote it by $H'$. Let $X' = X \cap V(H')$. Since $H'$ contains blue edges and all NIM-$H$ blue edges are contained in $X$, we have $X' \neq \emptyset$ and the corresponding subset $Y'$ is of size at least

$$|Y'| - (k - 1)|X'| \geq k.$$

For every branch of $H'$, we apply the following method to replace all blue edges with red edges. First, every branch consisting entirely of blue edges is replaced by a red path of the same length in $K_n[X', Y']$. This can be done since $K_n[X', Y']$ is a complete bipartite graph consisting of only red edges and $Y'$ is large enough. For any remaining branch $v_0v_1 \ldots v_{\ell_m}$, let $v_iv_{i+1}$ be the first red edge on this branch, i.e., every edge in the path $v_0v_1 \ldots v_i$ is blue. If $i$ is even, we replace the path $v_{2j}v_{2j+1}v_{2j+2}$ by a new red path $v_{2j}y_jv_{2j+2}$ with a distinct $y_j \in Y'$ for all $0 \leq j \leq \frac{k}{2} - 1$. If $i$ is odd, we replace the path $v_{2j}v_{2j+1}v_{2j+2}$ by a new red path $v_{2j}y_jv_{2j+2}$ with a distinct $y_j \in Y'$ for all $0 \leq j \leq \frac{k}{2} - \frac{1}{2} - 1$ and replace the single edge $v_{i-1}v_i$ by a new red path $v_iy'v_{i-1}$ with a distinct $y' \in Y'$. For all other blue edges after $v_iv_{i+1}$, we replace them by a new red path $P_3$ with the middle vertices in $Y'$. Again, this can be done since $K_n[X', Y']$ is a complete bipartite graph consisting of only red edges and $Y'$ is large enough.

After this, the original branch becomes a longer red path and we take the first segment of length $\ell_m$ as the new branch. Note that this new branch still contains the original red edge $v_iv_{i+1}$ unless $i$ is odd and $i + 1 = \ell_m$. Let $H''$ be the resulting copy of $H$.

If $H''$ still contains one of the original red edges, then we have a monochromatic copy of $H$, a contradiction since the original edges are NIM-$H$. Otherwise every branch of $H'$ is either entirely blue or has even length and is such that only the final edge is red. However, then we have $|X| \geq |X'| \geq \left\lceil \frac{k}{2} \right\rceil + 1$, a contradiction of our assumption that $|X| \leq \left\lceil \frac{k}{2} \right\rceil$ (recall that the blue edges are in $X'$). The proof is complete for spiders.

**The proof when $H$ is a double broom.** Let $H$ be a double broom with parameters $t$, $s_1$ and $s_2$ such that $k = t + s_1 + s_2$ and $s_1 < s_2$.
First, assume that $t$ is odd and $|X| \geq \lceil \frac{t}{2} \rceil + 1$. For a red edge $uv$ with $u \in X$, $v \in Y$, there is a subset $X' \subseteq X$ of size $\frac{t+1}{2}$ containing $u$. Let $Y'$ be the corresponding subset for $X'$. Then there is a path $P_1$ in $K_n[X', Y']$ which starts from $u$ and ends at another vertex, say $w$ in $X'$, and avoids $v$. Since $|Y'| \geq k^2 - (k-1)\frac{t+1}{2}$, we can select additional red edges incident to $u$ and $w$, which together with the edge $uv$ represent the set of edges incident to the leaves of $H$. It follows that $uv$ is not NIM-$H$. Hence all red NIM-$H$ edges are contained in $X$ and $Y$, and we have

$$|E(c, H)| \leq \text{ex}(n - |X|, H) + \left(\frac{|X|}{2}\right) \leq \text{ex}(n, H),$$

where the second inequality holds since $|X| \leq \frac{n}{2}$.

Now assume that $t$ is even and $|X| \geq \lceil \frac{t}{2} \rceil + 1$. Let $Y_1 = \{v \in Y : d_r(v, X) \geq 1\}$ and $Y_2 = Y \setminus Y_1$. Since each vertex in $X$ has at most $k - 1$ blue neighbors in $Y$, we have $|Y_2| \leq k - 1$.

Now we show that for each vertex $v \in Y_1$, there are at most $s_1 + \frac{t}{2} - 1$ NIM-$H$ edges incident to $v$. Suppose by way of contradiction that for a vertex $v \in Y_1$, there are at least $s_1 + \frac{t}{2}$ red NIM-$H$ edges incident to $v$. By the definition of $Y_1$, there is a red edge $vu$ with $u \in X$. Let $X' = X$ and let $Y' \subseteq Y$ be the corresponding subset of $X'$. We extend the red edge $vu$ to a red path $P_1$ in such a way that: (1) one of the end vertex is $v$ and the other end vertex $w$ is in $X'$, (2) every second vertex of the path is in $X'$ and the remaining vertices of the path are in $Y'$, (3) there remain at least $s_1$ red NIM-$H$ edges incident to $v$ which are not vertices of the path. These conditions can be satisfied since $|Y'|$ is sufficiently large. Now at least $s_1$ red NIM-$H$ edges incident to $v$ are not covered by the vertices of the path, which we can view as leaf edges of $H$ incident to $v$. Select another $t$ red (but not necessarily NIM-$H$) edges incident to $w$ and to some vertices which have not been used yet. Thus we found a red copy of $H$ containing at least one NIM-$H$ edge, a contradiction.

Therefore, for each vertex $v \in Y_1$, there are at most $s_1 + \frac{t}{2} - 1$ NIM-$H$ edges incident to $v$. All other NIM-$H$ edges are contained in $Y_2$ and $X$. Hence,

$$|E(c, H)| \leq |Y_1| \left(s_1 + \frac{t}{2} - 1\right) + \left(\frac{|Y_2|}{2}\right) + \left(\frac{|X|}{2}\right) \leq \text{ex}(n, H),$$

where the second inequality holds since the coefficient of $|Y_1|$ satisfies $s_1 + \frac{t}{2} - 1 < \frac{k-2}{2}$ and $|Y_2| \leq k - 1$, $|X| \leq \frac{n}{2}$. Thus, we are done in the case $|X| \geq \lceil \frac{t}{2} \rceil + 1$.

Finally, we consider the case when $|X| \leq \lceil \frac{t}{2} \rceil$. Since $|X| \geq 2$, we have $t \geq 4$. Let $Y_1 = \{v \in Y : d_r(v, X) \geq 2\}$ and $Y_2 = Y \setminus Y_1$. Now we show that there is no red path of length $t - 2|X| + 1$ in $Y_1$. Suppose by way of contradiction that $P$ is a red path of length $t - 2|X| + 1$ in $Y_1$. First, we extend $P$ to a red path of length $t - 1$ using vertices in $X$ and the corresponding subset of $X$ in $Y$ such that the two end vertices of this longer path, say $u$ and $v$, are contained in $X$. Since each vertex in $X$ has red degree at least $(|Y| - (k-1))$ in $Y$, we can find $s_1$ new red neighbors of $u$ and $s_2$ new red neighbors of $v$ in $Y$ and view them as the leaf-edges of $H$. That is, we extended the red path $P$ to a red copy of $H$. However, as we assumed all red edges in $Y$ are NIM-$H$, we have a contradiction.

Now we show $|Y_2| \leq s_1 - 1$. Suppose by way of contradiction that $|Y_2| \geq s_1$. If there are two vertices $v_1, v_2$ in $Y_2$ such that $N_b(v_1, X) \cup N_b(v_2, X) = X$, then for any blue edge $u_1u_2$ in $X$, we have that $v_1u_1u_2v_2$ or $v_1u_2u_1v_2$ is a blue path. Since $t \geq 4$ and all vertices in $Y$ have large blue degree in $Y$, this blue path can be extended to a blue copy of $H$. Hence there are no blue NIM-$H$ edges, a contradiction. Thus by the definition of $Y_2$, there exists a vertex $w \in X$ such that $N_b(v, X) = X \setminus \{v\}$. Let $uv'$ be a blue NIM-$H$ edge in $Y$ with $u \neq w$. Using $uv'$ and $s_1$ blue edges between $u$ and $Y_2$, we can find a blue star with $s_1 + 1$ leaves. By the definition of $Y$, we can extend this blue star to a blue copy of $H$ using other vertices in $Y$,
a contradiction. Hence we have $|Y_2| \leq s_1 - 1$. Furthermore, there are at most $|Y_2|$ red NIM-$H$
edges between $X$ and $Y_2$.

Therefore, we have

$$|E(c, H)| \leq \text{ex}(|Y_1|, P_{t-2|X|+2}) + |Y_1|(|Y_2| + |X|) + \left(\frac{|Y_2|}{2}\right) + \left(\frac{|X|}{2}\right) + |Y_2|$$

$$\leq \frac{t - 2|X|}{2}|Y_1| + |Y_1|(|Y_2| + |X|) + \left(\frac{|Y_2|}{2}\right) + \left(\frac{|X|}{2}\right) + |Y_2|$$

$$\leq \frac{t + 2(s_1 - 1)}{2}(n - (s_1 - 1) - |X|) + \left(\frac{s_1}{2}\right) + \left(\frac{|X|}{2}\right)$$

$$\leq \frac{t + 2s_1 - 2}{2}n \leq \text{ex}(n, H), \tag{2}$$

where the last inequality holds since $s_1 < s_2$. The proof is complete. \hfill \blacksquare

**Remark 2.** One may note that in inequality (1) and (2), we need the condition $s_1 < s_2$ to ensure that $\frac{t + 2s_1 - 2}{2}n \leq \text{ex}(n, H)$. For the case $s_1 = s_2$, these inequalities still show $f(n, H) \leq k/2n$ but this does not imply $f(n, H) \leq \text{ex}(n, H)$ for all $n$. With additional details, one could extend the proof to the case $s_1 = s_2$. But this would make our proof more complicated, so we omit it.

### 3 Proof of Theorem 3

We first construct some bipartite graphs which attain the upper bound in (2). Our idea comes from a theorem of Bushaw and Kettle [1]. Before we present the detailed constructions, we recall some results which we will require.

It is well-known that $\text{ex}(n, T) \leq \frac{v(T)-2}{2}n$ when $T$ is a path or star. For a general tree $T$, this is the celebrated Erdős–Sós Conjecture.

**Conjecture 2 (Erdős–Sós).** For a tree $T$, we have $\text{ex}(n, T) \leq \frac{v(T)-2}{2}n$.

In 2005, McLennan [8] proved that the Erdős–Sós Conjecture holds for trees of diameter at most four.

**Theorem 6 (McLennan [8]).** Let $T$ be a tree of diameter at most four, then $\text{ex}(n, T) \leq \frac{v(T)-2}{2}n$.

A tree is called balanced if it has the same number of vertices in each color class when the tree is viewed as a bipartite graph. A forest is called balanced if each of its components is a balanced tree. Bushaw and Kettle [1] proved the following theorem.

**Theorem 7 (Bushaw and Kettle [1]).** Let $H$ be a balanced forest on $2a$ vertices which comprises at least two trees. If the Erdős–Sós Conjecture holds for each component tree in $H$, then for any $n \geq 3a^2 + 32a^2\binom{2a}{a}$, we have

$$\text{ex}(n, H) = \begin{cases} \binom{a-1}{2} + (a - 1)(n - a + 1) & \text{if } H \text{ admits a perfect matching,} \\ (a - 1)(n - a + 1) & \text{otherwise.} \end{cases}$$

Now, making use of Theorems 6 and 7, we construct some bipartite graphs $H$ which are negative examples for Question 1. Let $\mathcal{H}_1$ be the family of all balanced trees on $2a$ vertices which admit no perfect matching and for which the Erdős–Sós Conjecture holds. One can see that $\mathcal{H}_1$ is not empty since a double star $S_{a-1, a-1}$ is a balanced tree on $2a$ vertices and the Erdős–Sós Conjecture holds for it by Theorem 6. Let $\mathcal{H}_2$ be the family of balanced trees on $2a$ vertices for which the Erdős–Sós Conjecture holds for sufficiently large $n$. Note that $\mathcal{H}_2$ is also nonempty, for example a path on $2a$ vertices belongs to $\mathcal{H}_2$. 

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Let $H_1 \in \mathcal{H}_1$, $H_2 \in \mathcal{H}_2$ and set $H = H_1 \cup H_2$. We know that $H$ is a balanced forest on $4a$ vertices. Since $H_1$ admits no perfect matching, $H$ admits no perfect matching either. The Erdős–Sós Conjecture holds for each component of $H$, hence by Theorem 7, when $n$ is sufficiently large, we have

$$ex(n, H) = (2a - 1)(n - 2a + 1).$$

On the other hand, consider a partition of the vertices of the complete graph $K_n$ into parts $X$ and $Y$ with $|X| = 2a - 1$ and $|Y| = n - 2a + 1$. We color all edges between $X$ and $Y$ red and the remaining edges blue. One can see that the red edges induce a complete bipartite graph $K_{2a - 1, n - 2a + 1}$ which contains no red copy of $H$. The blue edges induce a blue $(2a - 1)$-clique and a blue $(n - 2a + 1)$-clique which are disjoint with each other. Since each component of $H$ contains $2a$ vertices, all blue copies of $H$ are contained in the $(n - 2a + 1)$-clique. Therefore, all red edges and all the edges in the blue $(2a - 1)$-clique are NIM-$H$, that is,

$$f(n, H) \geq \left(\frac{2a - 1}{2}\right) + (2a - 1)(n - 2a + 1) = \left(\frac{2a - 1}{2}\right) + ex(n, H).$$

Therefore, such a bipartite graph $H$ attains the upper bound of the inequality (2).

Next we prove that if the bipartite graph $H$ contains a tail $v_0v_1v_2$, then $f(n, H) \leq ex(n, H) + \binom{|A| - 1}{2}$. Note that it is possible that $H$ is disconnected, hence let $H = H_1 \cup \cdots \cup H_q$, where $H_i$ are its components (if $H$ is connected, then $H = H_1$) and we say the tail $v_0v_1v_2$ is contained in $H_1$. Let $A_i, B_i$ be the two color classes of $H_i$ with $|A_i| \leq |B_i|$ for any $1 \leq i \leq q$, and let $A = \bigcup_{i=1}^q A_i$, $B = \bigcup_{i=1}^q B_i$. Set $a = |A|$.

Since we take $n$ to be sufficiently large, we may assume $n \geq R(K_{v(H)}, K_{v(H)})$. Let $c$ be a red-blue edge-coloring of $K_n$. Without loss of generality, there is a blue clique on at least $v(H)$ vertices in $K_n$. Let $K_t$ be a blue clique in $K_n$ such that $t$ is as large as possible. We have $t \geq v(H)$ and every other vertex has a red neighbor in $V(K_t)$. We partition $V(K_n) \setminus V(K_t)$ into two subsets $X, Y$ such that $Y$ consists of the vertices which have blue neighbors in $V(K_t)$ and $X$ consists of the remaining vertices. Hence all edges between $V(K_t)$ and $X$ are red.

The following claims will be used several times.

**Claim 1.** All blue NIM-$H$ edges are contained in $X$.

**Proof.** Obviously, the blue edges in $K_t$ and $K_0[V(K_t), Y]$ are not NIM-$H$. Let $xy$ be a blue edge with $y \in Y$ and $x \in X \cup Y$. By the definition of $Y$, the vertex $y$ has a blue neighbor, say $v$, in $V(K_t)$. If we embed $V(H) \setminus \{v_1, v_2\}$ into $V(K_t)$ and view $vyx$ as the tail of $H$, then we find a blue copy of $H$ containing $xy$. Thus $xy$ is not NIM-$H$. Therefore, all blue NIM-$H$ edges are contained in $X$. \hfill \square

**Claim 2.** If $|X| \geq a$, then the red edges between $X$ and $V(K_t) \cup Y$ are not NIM-$H$.

**Proof.** Since the red edges between $X$ and $V(K_t)$ induce a red complete bipartite graph and $|X| \geq a$ and $t \geq v(H)$, each such edge is contained in a red copy of $H$, thus these edges are not NIM-$H$. Let $xy$ be a red edge with $x \in X$, $y \in Y$. By the maximality of $K_t$, the vertex $y$ has a red neighbor, say $v$, in $V(K_t)$. Actually, $\{x, y, v\}$ induces a red triangle. If the tail $v_0v_1v_2$ of $H$ satisfies $\{v_0, v_2\} \subset B$ and $v_1 \in A$, then embed $B \setminus \{v_2\}$ into $V(K_t)$ so that $v_0$ is identified with $v$, embed $A \setminus \{v_1\}$ into $X \setminus \{x\}$ and view $vxy$ as the tail of $H$, thus we find a red copy of $H$ containing $xy$. So in this case, $xy$ is not NIM-$H$. If the tail $v_0v_1v_2$ of $H$ satisfies $\{v_0, v_2\} \subset A$ and $v_1 \in B$, then embed $B \setminus \{v_1\}$ into $V(K_t) \setminus \{v\}$, embed $A \setminus \{v_2\}$ into $X$ so that $v_0$ is identified with $x$. View $xyv$ as the tail, we find a red copy of $H$ containing $xy$. So in this case, $xy$ is not NIM-$H$ either. \hfill \square

We distinguish three cases based on the size of $X$. 
Case 1: $|X| \geq a + 1$. In this case, we first claim that the red edges in $X$ are also not NIM-$H$. Let $xx'$ be a red edge contained in $X$ and $v$ be a vertex in $K_1$. If the tail $v_0v_1v_2$ in $H$ satisfies \{v_0, v_2\} $\subset B$ and $v_1 \in A$, then since $|X \setminus \{x, x'\}| \geq a - 1 = |A \setminus \{v_1\}|$, we can embed $A \setminus \{v_1\}$ into $X \setminus \{x, x'\}$, embed $B \setminus \{v_2\}$ into $V(K_1)$ so that $v_0$ is identified with $v$ and view $vxx'$ as the tail $v_0v_1v_2$, thereby finding a red copy of $H$ containing $xx'$. So in this case, $xx'$ is not NIM-$H$.

If the tail $v_0v_1v_2$ in $H$ satisfies \{v_0, v_2\} $\subset A$ and $v_1 \in B$, then we embed $A \setminus \{v_2\}$ into $X \setminus \{x\}$ so that $v_0$ is identified with $x$, embed $B \setminus \{v_1\}$ into $V(K_1) \setminus \{v\}$ and view $xx'v$ as the tail, and again we can find a red copy of $H$ containing $xx'$. Therefore, $xx'$ is not NIM-$H$.

By Claim 2 and the above result, all red NIM-$H$ edges are contained in $V(K_1) \cup Y$. Note that the red NIM-$H$ edges contained in $V(K_1) \cup Y$ induce an $H_1$-free graph. Otherwise, such a red copy of $H_1$, together with a red copy of $H_2 \cup \cdots \cup H_q$ if $H$ is disconnected) contained in the complete bipartite graph $K_n[X, V(K_1)]$ yields a red copy of $H$ containing an NIM-$H$ edge, a contradiction. Analogously, the blue NIM-$H$ edges contained in $X$ induce a graph which is $H_1$-free. Hence,

$$|E(c, H)| \leq \text{ex}(|X|, H_1) + \text{ex}(n - |X|, H_1)$$

$$\leq \text{ex}(n, H_1) \leq \text{ex}(n, H),$$

where the second inequality holds since $H_1$ is connected. The proof is complete in this case.

Case 2: $|X| = a$. By Claim 2, the set of red NIM-$H$ edges can be partitioned into two parts: the ones contained in $V(K_1) \cup Y$ and the remaining ones which are contained in $X$. Since all blue NIM-$H$ edges are contained in $X$ by Claim 1, the sum of the total number of blue NIM-$H$ edges and the number of red NIM-$H$ edges contained in $X$ is at most $\binom{a}{2}$. The set of red NIM-$H$ edges contained in $V(K_1) \cup Y$ yields an $H_1$-free graph. Indeed, otherwise together with a red copy of $H_2 \cup \cdots \cup H_q$ (if $H$ is disconnected) in $K_n[X, V(K_1)]$, we could find a red copy of $H$ containing a red NIM-$H$ edge, a contradiction. Thus the number of red NIM-$H$ edges contained in $V(K_1) \cup Y$ is at most $\text{ex}(n - a, H_1)$.

Therefore, the total number of NIM-$H$ edges is at most $\text{ex}(n - a, H_1) + \binom{a}{2}$. Since $H_1$ is connected and contains a tail, it follows that the union of a star $S_{a-1}$ on $a$ vertices and an extremal graph for $\text{ex}(n - a, H_1)$ is still $H_1$-free. Hence,

$$\text{ex}(n - a, H_1) + (a - 1) \leq \text{ex}(n, H_1).$$

Thus, we have

$$|E(c, H)| \leq \text{ex}(n - a, H_1) + \binom{a}{2} \leq \text{ex}(n, H_1) + \binom{a - 1}{2}$$

$$\leq \text{ex}(n, H) + \binom{a - 1}{2},$$

and the proof of this case is complete.

Case 3: $|X| \leq a - 1$. By Claim 1, the number of blue NIM-$H$ edges is at most $\binom{a-1}{2}$, and the red NIM-$H$ edges yield an $H$-free graph. Hence

$$|E(c, H)| \leq \text{ex}(n, H) + \binom{a - 1}{2},$$

and the proof is complete.

\[ \blacksquare \]

Remark 3. In [12], the first author and Chen also give a family of examples such that $\chi(H) = 3$ and $f(n, H) > \text{ex}(n, H)$. 

8
4 Proof of Theorem 5

We first give a 2k-edge-coloring of $K_n$ with $(2k-1)ex(n, P_{2k})+(k-1)(2k-1)$ NIM-$P_{2k}$ edges when $2k-1$ is a prime and $n \in \{a(2k-1) + (k-1), a(2k-1) + k\}$. Before showing our construction, we need to recall the exact value of $ex(n, P_t)$.

**Theorem 8** (Faudree and Schelp [3]). Let $n = a(\ell - 1) + b$ with $0 \leq b \leq \ell - 2$. Then we have

$$ex(n, P_t) = a\left(\frac{\ell - 1}{2}\right) + \left\lceil \frac{b}{2} \right\rceil.$$

If $\ell$ is even and $b \in \{\ell/2, \ell/2-1\}$, then the extremal graphs are $tK_{\ell-1} \cup (K_{\ell/2-1} + \bar{K}_{n-t(\ell-1)-\ell/2+1})$ for any $0 \leq t \leq a$. Otherwise $aK_{\ell-1} \cup K_b$ is the unique extremal graph.

Therefore, by Theorem 8, when $n \in \{a(2k-1) + (k-1), a(2k-1) + k\}$, the extremal graphs for $ex(n, P_{2k})$ are $tK_{2k-1} \cup (K_{k-1} + \bar{K}_{n-t(2k-1)-(k-1)})$ for any $0 \leq t \leq a$.

Let $U$ be a subset of size $(2k-1)^2$ of $V(K_n)$ and label the vertices of $U$ by $[i, j]$ where $1 \leq i, j \leq 2k-1$. We divide $U$ into $2k-1$ subsets by setting

$$U_i = \{[i, 1], [i, 2], \ldots, [i, 2k-1]\}, \quad 1 \leq i \leq 2k-1.$$

When it is not confusing, we also let $U$ and $U_i$ denote the cliques induced by the vertices in them.

For any $1 \leq i, j \leq 2k-1$, let $\sigma_{ji}$ denote the clique induced by the vertices $[1, i], [2, i + j], \ldots, [2k-1, i + (2k-2)j]$, where the indices are taken modulo $2k-1$. For any $1 \leq j \leq 2k-1$, let

$$C_j = \{\sigma_{ji} : 1 \leq i \leq 2k-1\}.$$

Then $C_j$ is a set consisting of $2k-1$ disjoint $(2k-1)$-cliques.

Let $c : E(K_n) \to \{c_1, \ldots, c_{2k}\}$ be a 2k-edge-coloring defined as follows. Let $W = V(K_n) \setminus U$. For any $j \in [2k-1]$, we assign the color $c_j$ to the edges of each clique $\sigma_{ji}$ in $C_j$. Let $\sigma_{j1}^c$ denote the clique induced by the vertices $[k+1, 1 + kj], \ldots, [2k-1, 1 + (2k-2)j]$. Clearly, we have $\sigma_{j1}^c \subset \sigma_{ji}$. Now consider the sub-clique $\sigma_{j1} - \sigma_{j1}^c$ and replace the color $c_j$ by $c_{2k}$ inside it. With this, $\sigma_{j1}$ decomposes into a copy of $K_{k-1} + \bar{K}_k$ colored by $c_j$ and a copy of $K_k$ colored by $c_{2k}$. After this, we assign the color $c_{2k}$ to all the edges between $\sigma_{j1}^c$ and $V$. Figure 1 shows the subgraph induced by the edges colored by $c_{2k-1}$. Finally, we assign the color $c_{2k}$ to the edges which have not been colored yet.

![Figure 1: The subgraph induced by the edges of color $c_{2k-1}$.](image)

In the next two paragraphs, we show that this $2k$-edge-coloring is well-defined, namely, each edge is assigned exactly one color. Clearly, each edges is assigned at least one color and the edges inside $W$ or between $U_1 \cup \cdots \cup U_k$ and $W$ are assigned exactly one color.
Note that $U$ is a $(2k - 1)^2$-clique. Let $1 \leq i, \ell, s, t \leq 2k - 1$. Clearly, the edge $[i, s][i, t]$ is only covered by the clique $U_i$. If the edge $[i, s][\ell, t]$ with $i < \ell$ were covered by two cliques, say by one in $C_j$ and by another one in $C_j'$ for some $1 \leq j, j' \leq 2k - 1$, then

$$\begin{cases} t \equiv s + (\ell - i)j \pmod{2k - 1} \\
 t \equiv s + (\ell - i)j' \pmod{2k - 1}
\end{cases}$$

would hold, and since $2k - 1$ is prime, we would have $j = j'$, a contradiction. Thus, each edge inside $U$ is covered by at most one clique in $C_j$ or by the clique $U_i$. On the other hand, considering the number of edges in $U$ and the total number of edges of cliques in each $C_j$ and $U_i$ yields

$$e(U) = \sum_{i=1}^{2k-1} e(U_i) + \sum_{j=1}^{2k-1} \sum_{\sigma_j \in C_j} e(\sigma_j).$$

Therefore, the cliques in each $C_j$ together with the cliques $U_i$ for all $1 \leq i, j \leq 2k - 1$ form an edge-decomposition of the large clique $U$. Hence each edge in $U$ is assigned one color.

Now we show that for any $1 \leq j, j' \leq 2k - 1$ with $j \not= j'$, the sub-cliques $\sigma_j$ and $\sigma_j'$ are vertex-disjoint. Supposing that a vertex $[i, 1 + (i - 1)j] \in V(\sigma_j)$ is also contained in $\sigma_{j'}$ for some $1 \leq i, j, j' \leq 2k - 1$, we obtain

$$1 + (i - 1)j \equiv 1 + (i - 1)j' \pmod{2k - 1}.$$

Since $2k - 1$ is a prime number, we get $j = j'$, a contradiction. Thus the sub-cliques $\sigma_j$ for all $1 \leq j \leq 2k - 1$ form a vertex-decomposition of $U_{k+1} \cup \cdots \cup U_{2k-1}$. Hence, each edge between $U_{k+1} \cup \cdots \cup U_{2k-1}$ and $W$ is assigned one color in $\{c_1, \ldots, c_{2k-1}\}$. Therefore, our $2k$-edge-coloring $c$ is well-defined.

Note that for any $1 \leq j \leq 2k - 1$, the subgraph induced by the edges of color $c_j$ is a copy of $tK_{2k-1} \cup (K_{k-1} + \overline{K}_{n-(k-1)-t(2k-1)})$ with $t = 2k - 2$, and this graph is extremal for $\text{ex}(n, P_{2k})$ when $n \not\in \{a(2k - 1) + (k - 1), a(2k - 1) + k\}$. Now consider the edges colored by $c_{2k}$. They are in the cliques $U_i$ with $1 \leq i \leq 2k - 1$, inside $\sigma_{j1} - \sigma_{j1}'$ with $1 \leq j \leq 2k - 1$, inside $W$, and between $U_1 \cup \cdots \cup U_k$ and $W$. Note that for any $k + 1 \leq i \leq 2k - 1$, $U_i$ are independent $(2k - 1)$-cliques colored by $c_{2k}$, hence the edges in $U_i$ are also NIM-$P_{2k}$. For all other $c_{2k}$-edges, they construct a large connected component such that $W$ is a clique in the component. Hence none of these edges are NIM-$P_{2k}$.

Therefore,

$$|E(c, P_{2k})| = (2k - 1)\text{ex}(n, P_{2k}) + (k - 1)\binom{2k - 1}{2},$$

and we are done.

**Remark 4.** Note that $tK_{2k-1} \cup (K_{k-1} + \overline{K}_{n-(k-1)-t(2k-1)})$ is not extremal for $\text{ex}(n, P_{2k})$ when $n \not\in \{a(2k - 1) + (k - 1), a(2k - 1) + k\}$, but we still have

$$\text{ex}(n, P_{2k}) - e\left(tK_{2k-1} \cup (K_{k-1} + \overline{K}_{n-(k-1)-t(2k-1)})\right) < (k - 1)^2.$$

Hence in our construction, when $2k - 1$ is prime, the number of NIM-$P_{2k}$ edges is more than $(2k - 1)\text{ex}(n, P_{2k})$. That is to say, when $2k - 1$ is prime, we have $f_{2k}(n, P_{2k}) > (2k - 1)\text{ex}(n, P_{2k})$ for every sufficiently large $n$.

Next we prove the upper bound of $f_{2k}(n, P_{2k})$. Let $c : E(K_n) \to \{c_1, \ldots, c_{2k}\}$ be a $2k$-edge-coloring of $K_n$. We call an edge a $c_i$-edge if it is of color $c_i$ and we let $G_i$ denote the subgraph induced by all $c_i$-edges, for any $1 \leq i \leq 2k$. Without loss of generality, we can assume $e(G_{2k}) \geq \binom{n}{2}/2k$. By Theorem 8, there is a path $P$ of at least $\frac{n}{2k}$ vertices in $G_{2k}$. Let $G'_{2k}$ be
the component of $G_{2k}$ which contains the path $P$, and let $X = V(G'_{2k})$ and $Y = V(K_n) - X$. Then we have $|X| \geq \frac{n}{2k}$ and there is no $c_{2k}$-edge between $X$ and $Y$. Since the component $G'_{2k}$ contains a long path $P$, each edge of $G'_{2k}$ is contained in a monochromatic copy of $P_{2k}$. Hence, all NIM-$P_{2k}$ $c_{2k}$-edges are contained in $Y$.

For each $1 \leq i \leq 2k-1$, there are at most $\text{ex}(n, P_{2k})$ NIM-$P_{2k}$ $c_{i}$-edges. If $|Y| \leq (k-1)(2k-1)$, then there are at most $\text{ex}(|Y|, P_{2k}) \leq (k-1)(2k-1)$ NIM-$P_{2k}$ $c_{2k}$-edges. Hence, the total number of NIM-$P_{2k}$ edges is at most

$$(2k-1)\text{ex}(n, P_{2k}) + (k-1)\binom{2k-1}{2},$$

so we are done. Therefore, we may assume $|Y| \geq (k-1)(2k-1) + 1$.

Let us define a procedure to find pairs $(X_i, Y_i)$ satisfying the following conditions:

(i) $X_i \subseteq X$ with $|X_i| = 2k$ and $Y_i \subseteq Y$ with $|Y_i| = k$ for any $1 \leq i \leq 2k$;

(ii) $Y_i$ and $Y_j$ are disjoint for any $1 \leq i, j \leq 2k$ with $i \neq j$;

(iii) $K_n[X_i, Y_i]$ forms a monochromatic copy of complete bipartite graph for any $1 \leq i \leq 2k$.

Assume that for some $1 \leq i \leq 2k$, we have found $(X_1, Y_1), \ldots, (X_{i-1}, Y_{i-1})$ which satisfy the conditions. Let $s = (k-1)(2k-1) + 1$. If

$$|Y \setminus \bigcup_{j=1}^{i-1} Y_j| \leq s - 1,$$

then the procedure terminates. Otherwise we choose a subset $Y'_i$ of $Y \setminus \bigcup_{j=1}^{i-1} Y_j$ with $|Y'_i| = s$. Let $Y'_i = \{y_1, \ldots, y_s\}$. For each $x \in X$, we define a vector $\vec{e}(x, Y'_i) = (e_1, \ldots, e_s)$ as follows: for any $1 \leq j \leq s$, let $e_j = i$ if and only if the edge $xy_j$ is colored by $c_i$. Since no edge between $X$ and $Y$ is colored by $c_{2k}$, we have $\vec{e}(x, Y'_i) \in \{1, \ldots, 2k-1\}^s$ for any $x \in X$. For each $\vec{v} \in \{1, \ldots, 2k-1\}^s$, let $X_{\vec{v}}$ denote the set of vertices $x \in X$ for which $\vec{e}(x, Y'_i) = \vec{v}$. Hence, $X$ is divided into $(2k-1)^s$ subsets and clearly, at least one subset, say $X_{\vec{v}}$, contains at least $|X|/(2k-1)^s$ vertices. Observe that $K_n[X_{\vec{v}}, Y_i]$ is a monochromatic star for any $y_j \in Y'_i$. Since $|Y'_i| = (k-1)(2k-1) + 1$ and there are at most $2k-1$ different colors between $X_{\vec{v}}$ and $Y'_i$, by pigeonhole principle, there exists a subset $Y_i' \subseteq Y'_i$ such that $|Y_i'| = k$ and the edges between $X_i$ and $Y_i'$ are monochromatic. That is $K_n[X_{\vec{v}}, Y_i']$ is a monochromatic complete bipartite graph. Since $n \geq (2k)^{2k^2}$,

$$|X_{\vec{v}}| \geq \frac{|X|}{(2k-1)^s} \geq \frac{n}{(2k)^s} \geq 2k.$$

We can choose a subset $X_i$ from $X_{\vec{v}}$ with $|X_i| = 2k$, thereby finding the pair $(X_i, Y_i)$ as we wanted.

Note that since $Y$ is finite, the procedure terminates. Let $t$ denote the number of steps the algorithm took, and let $(X_1, Y_1), \ldots, (X_t, Y_t)$ be the pairs the algorithm found. Let $Y_0 = Y \setminus \bigcup_{i=1}^{t} Y_i$. Then we have $|Y_0| \leq (k-1)(2k-1)$. For any $1 \leq i \leq 2k-1$, let $t_i$ denote the number of the pairs $(X_j, Y_j)$ for which the edges of $K_n[X_j, Y_j]$ are of color $c_i$. Without loss of generality, we may assume that $t_1, \ldots, t_h > 0$ for some $1 \leq h \leq 2k-1$. Then $t = \sum_{i=1}^{h} t_i$. Let $1 \leq i \leq h$ and consider the $c_i$-edges. Without loss of generality, we can assume that $K_n[X_1, Y_1], \ldots, K_n[X_{t_i}, Y_{t_i}]$ are of color $c_i$. Then each NIM-$P_{2k}$ $c_i$-edge is contained in $V(K_n) \setminus \bigcup_{j=1}^{t_i} (X_j \cup Y_j)$. Since the sets $Y_1, \ldots, Y_{t_i}$ are pairwise disjoint and $X_1, \ldots, X_{t_i} \subseteq X$, we have

$$\left| \bigcup_{j=1}^{t_i} (X_j \cup Y_j) \right| \geq t_i k + 2k,$$
thus the number of NIM-$P_{2k}$ $c_i$-edges is at most $\operatorname{ex}(n - t_i k - 2 k, P_{2k})$. Now let $h + 1 \leq i \leq 2k - 1$ (if such an index exists). Since $t_i = 0$, the number of NIM-$P_{2k}$ $c_i$-edges is at most $\operatorname{ex}(n, P_{2k})$.

As we have proved, all NIM-$P_{2k}$ $c_2k$-edges are contained in $Y$ and $|Y| \leq (k - 1)(2k - 1) + tk$. Therefore, the total number of NIM-$P_{2k}$ edges is at most

$$\operatorname{ex}((k - 1)(2k - 1) + tk, P_{2k}) + \sum_{i=1}^{h} \operatorname{ex}(n - t_i k - 2 k, P_{2k}) + (2k - 1 - h) \operatorname{ex}(n, P_{2k}).$$

(3)

To prove the final result, we need the following lemma.

**Lemma 1.** Let $n_1$, $n_2$ and $c$ be constants. Then we have

$$\operatorname{ex}(n_1, P_t) + \operatorname{ex}(n_2, P_t) < \operatorname{ex}(n_1 - c, P_t) + \operatorname{ex}(n_2 + c + \ell, P_t).$$

**Proof.** Let $n_1 - c = a_1(\ell - 1) + b_1$ and $n_2 + c = a_2(\ell - 1) + b_2$, where $0 \leq b_1, b_2 \leq \ell - 2$. By Theorem 8, we have

$$\operatorname{ex}(n_1 - c, P_t) + \operatorname{ex}(n_2 + c, P_t) + \operatorname{ex}(\ell, P_t) \geq \operatorname{ex}(n_1 - c, P_t) + \operatorname{ex}(n_2 + c, P_t) + \operatorname{ex}(\ell, P_t) > (a_1 + a_2)\left(\frac{\ell - 1}{2}\right) + \left(\frac{b_1}{2}\right) + \left(\frac{b_2}{2}\right) + \left(\frac{\ell - 1}{2}\right)$$

and

$$\operatorname{ex}(n_1, P_t) + \operatorname{ex}(n_2, P_t) \leq \frac{\ell - 2}{2}(n_1 + n_2) = (a_1 + a_2)\left(\frac{\ell - 1}{2}\right) + (b_1 + b_2)\frac{\ell - 2}{2}.$$

Hence we have

$$\operatorname{ex}(n_1 - c, P_t) + \operatorname{ex}(n_2 + c, P_t) - (\operatorname{ex}(n_1, P_t) + \operatorname{ex}(n_2, P_t)) > \left(\frac{b_1}{2}\right) + \left(\frac{b_2}{2}\right) + \left(\frac{\ell - 1}{2}\right) - (b_1 + b_2)\frac{\ell - 2}{2} > 0.$$

we are done. \hfill \Box

When applying the above lemma to (3), we get

$$\operatorname{ex}((k - 1)(2k - 1) + tk, P_{2k}) + \sum_{i=1}^{h} \operatorname{ex}(n - t_i k - 2 k, P_{2k}) + (2k - 1 - s) \operatorname{ex}(n, P_{2k})$$

$$< (2k - 1) \operatorname{ex}(n, P_{2k}) + \operatorname{ex}((k - 1)(2k - 1), P_{2k})$$

$$= (2k - 1) \operatorname{ex}(n, P_{2k}) + (k - 1)\left(\frac{2k - 1}{2}\right).$$

Thus the proof is complete. \hfill \blacksquare

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