**Simplified $D = 11$ Pure Spinor $b$ Ghost**

Nathan Berkovits*, Max Guillen*

*ICTP South American Institute for Fundamental Research
Instituto de Física Teórica, UNESP-Universidade Estadual Paulista
R. Dr. Bento T. Ferraz 271, Bl. II, São Paulo 01140-070, SP, Brazil

E-mail: nberkovi@ift.unesp.br, luismax@ift.unesp.br

ABSTRACT: A $b$-ghost was constructed for the $D = 11$ non-minimal pure spinor superparticle by requiring that $\{Q, b\} = T$ where $Q = \Lambda^\alpha D_\alpha + R^\alpha \tilde{W}_\alpha$ is the usual non-minimal pure spinor BRST operator. As was done for the $D = 10$ $b$-ghost, we will show that the $D = 11$ $b$-ghost can be simplified by introducing an $SO(10,1)$ fermionic vector $\Sigma^i$ constructed out of the fermionic spinor $D_\alpha$ and pure spinor variables. This simplified version will be shown to satisfy $\{Q, b\} = T$ and $\{b, b\} = \text{BRST - trivial}$.

KEYWORDS: Supergravity, Superparticle, Pure spinors.
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1 Introduction

The \(D = 11\) pure spinor superparticle is a useful tool to describe \(D = 11\) linearized supergravity in a manifestly covariant way [1]. This formalism describes physical states as elements of the cohomology of a BRST operator defined by \(Q_{\text{min}} = \Lambda^\alpha D_\alpha\), where \(\Lambda^\alpha\)
is a $D = 11$ pure spinor\textsuperscript{1} satisfying the constraint $\Lambda \Gamma^a \Lambda = 0$, $a$ is an $SO(10,1)$ vector index, and $D_a$ are the first-class constraints of the $D = 11$ Brink-Schwarz-like superparticle [3]. The spectrum found by using this formalism coincides with that obtained via the BV quantization of $D = 11$ linearized supergravity and includes the graviton, gravitino, and 3-form at ghost-number 3, as well as their ghosts and antifields at other ghost number [1, 4], each one of them satisfying certain equations of motion and gauge invariances as dictated by the BV prescription.

Motivated by the non-minimal version of the pure spinor superstring [5], Cederwall formulated the $D = 11$ non-minimal pure spinor superparticle by introducing a new set of variables $\bar{\Lambda}_\alpha$, $R_\beta$ and their respective momenta $\bar{W}^\alpha$, $S^\beta$, where $\bar{\Lambda}_\alpha$ is a $D = 11$ bosonic spinor and $R_\beta$ is a $D = 11$ fermionic spinor satisfying the constraints $\Lambda \Gamma^a \bar{\Lambda} = 0$ and $\bar{\Lambda} \Gamma^a R = 0$ [6, 7]. In order for the new variables to not affect the physical spectrum, the BRST operator should be modified to $Q = \Lambda^a D_a + R_\alpha \bar{W}^\alpha$, as in the quartet argument of [8]. In the non-minimal pure spinor formalism of superstring, one can formulate a consistent prescription to compute scattering amplitudes by constructing a non-fundamental $b$-ghost satisfying $\{Q, b\} = T$. Therefore, it is important to know if a similar $b$ ghost can be constructed in the $D = 11$ superparticle case.

The $D = 11$ $b$-ghost was first constructed in [9] in terms of quantities which are not manifestly invariant under the gauge symmetries of $w_\alpha$ generated by $\Lambda \Gamma^a \Lambda = 0$. This $b$-ghost was later shown in [10] to be $Q$-equivalent to one written in terms of the gauge-invariant quantities $N_{ab}$ and $J$, and we will focus on this manifestly gauge-invariant version of the $b$-ghost.

The complicated form of the $b$-ghost in [10] makes it difficult to treat, so for instance its nilpotency property $\{b, b\}$ has not yet been analyzed. A similar complication exists in $D = 10$ dimensions, however, it was shown in [11] that the $D = 10$ $b$-ghost could be simplified by defining new fermionic vector variables. In this paper, a similar simplification involving fermionic vector variables will be found for the $D = 11$ $b$-ghost which will simplify the computations of $\{Q, b\} = T$ and $\{b, b\}$.

The paper is organized as follows: In section 2 we review the $D = 10$ non-minimal pure spinor superparticle, constructing the corresponding pure spinor $b$-ghost and its simplification. In section 3 we review the $D = 11$ pure spinor superparticle, constructing the manifestly gauge-invariant $b$-ghost and explaining how to translate the simplification of the $D = 10$ $b$-ghost to the $D = 11$ $b$-ghost by defining the $SO(10,1)$ composite fermionic vector $\Sigma^j$. Finally we construct the simplified $D = 11$ $b$-ghost and show that it satisfies the relations $\{Q, b\} = T$ and $\{b, b\} =$ BRST-trivial. Some comments are given at the end of the paper concerning the relation between the $b$-ghost found in [10] and this simplified $b$-ghost.

\textsuperscript{1}In this paper, a d=11 pure spinor $\Lambda^a$ will be defined to satisfy $\Lambda \Gamma^a \Lambda = 0$. A d=11 pure spinor is sometimes [2] defined to satisfy both $\Lambda \Gamma^a \Lambda = 0$ and $\Lambda \Gamma^{ab} \Lambda = 0$. 

\hspace{1cm} – 2 –
2 $D = 10$ non-minimal pure spinor superparticle

The $D = 10$ (minimal) pure spinor superparticle action is given by [12]:

$$S = \int d\tau (\dot{X}^m P_m + \dot{\theta}^\mu p_\mu - \frac{1}{2} P^m P_m + \dot{\lambda}^\mu w_\mu)$$

(2.1)

where $m$, $\mu$ are $SO(9,1)$ vector/spinor indices, $\theta^\mu$ is an $SO(9,1)$ Majorana-Weyl spinor, $p_\mu$ is its corresponding conjugate momentum and $P^m$ is the momentum. The variable $\lambda^\mu$ is a $D = 10$ pure spinor satisfying the constraint $\lambda^\mu \gamma^\mu = 0$ where $m$ is an $SO(9,1)$ vector index, and $w_\mu$ is its corresponding conjugate momentum. Because of the pure spinor constraint this $SO(9,1)$ antichiral spinor is defined up to the gauge transformation $\delta w_\mu = (\gamma^m \lambda)_\mu f_m$, where $f_m$ is an arbitrary vector. The $SO(9,1)$ gamma matrices denoted by $\gamma^m$ satisfy the Clifford algebra $(\gamma^m)_{\mu\nu}(\gamma^n)^{\nu\rho} + (\gamma^n)_{\mu\nu}(\gamma^m)^{\nu\rho} = 2\eta^{mn}\delta^\rho_\mu$. The physical states are defined as elements of the cohomology of the BRST operator $Q = \lambda^\mu d_\mu$, where $d_\mu = p_\mu - P_m(\gamma^m \theta)_\mu$ are the first-class constraints of the $D = 10$ Brink-Schwarz superparticle [13]. The spectrum turns out to describe the BV version of $D = 10$ (abelian) Super Yang-Mills [12, 14, 15].

In the non-minimal version of the pure spinor superparticle [5], one introduces a new pure anti-Weyl spinor $\bar{\lambda}_\mu$, and a fermionic field $r_\mu$ satisfying the constraint $\bar{\lambda}_\mu \gamma^{\mu r} = 0$, together with their respective conjugate momenta $\bar{\omega}^\mu$, $s^\mu$. In order to not affect the cohomology corresponding to $Q_{\text{min}}$, the non-minimal BRST operator is defined as $Q_{\text{non-min}} = \lambda^\mu d_\mu + \bar{\omega}^\mu r_\mu$. Thus the $D = 10$ non-minimal pure spinor superparticle is described by the action:

$$S = \int d\tau (\dot{X}^m P_m + \dot{\theta}^\mu p_\mu - \frac{1}{2} P^m P_m + \dot{\lambda}^\mu w_\mu + \bar{\omega}^\mu \bar{\lambda}_\mu + \dot{r}_\mu s_\mu)$$

(2.2)

and the BRST operator $Q = \lambda^\mu d_\mu + \bar{\omega}^\mu r_\mu$. By construction, the physical spectrum also describes BV $D = 10$ (abelian) Super Yang-Mills.

2.1 $D = 10$ b-ghost

As discussed in [16, 17] a consistent scattering amplitude prescription can be defined using a composite b-ghost satisfying $\{Q, b\} = T$, where $Q$ is the non-minimal BRST operator and $T = -\frac{1}{2} P^a P_a$ is the stress-energy tensor. This superparticle b-ghost is obtained by dropping the worldsheet non-zero modes in the superstring b ghost and is

$$b = \frac{1}{2} \frac{(\bar{\lambda} \gamma_m d) P^m}{\lambda \lambda} - \frac{1}{192} \frac{(\bar{\lambda} \gamma_{mnp} r)(d \gamma_{mnp} d) + 24 N_{mn} P_p}{(\lambda \lambda)^2} + \frac{1}{16} \frac{(r \gamma_{mnp}) (\bar{\lambda} \gamma^m d) N^{np}}{(\lambda \lambda)^3}$$

(2.3)

$$- \frac{1}{128} \frac{(r \gamma_{mnp}) (\bar{\lambda} \gamma^{mnp} r) N_{mn} N_{qr}}{(\lambda \lambda)}$$

where $N_{mn} = \frac{1}{2} \lambda \gamma_{mn} w$.

The complicated nature of this expression makes it difficult to prove nilpotence [18], however it was shown in [11] that the b-ghost can be simplified by introducing an $SO(9,1)$ composite fermionic vector $\Gamma_m$ satisfying the constraint $(\gamma_m \bar{\lambda})^\mu \Gamma^m = 0$. In the expression (2.3), the terms involving $d_\mu$ always appear in the combination

$$\Gamma^m = \frac{1}{2} \frac{(\bar{\lambda} \gamma^m d)}{(\lambda \lambda)} - \frac{1}{8} \frac{(\bar{\lambda} \gamma_{mnp} r) N_{np}}{(\lambda \lambda)^2},$$

(2.4)
and using this $\tilde{\Gamma}^m$, the $b$-ghost can be written in the simpler form:

$$b = P^m\tilde{\Gamma}_m - \frac{1}{4} \frac{(\lambda)^m n_r}{(\lambda\lambda)} \tilde{\Gamma}_m \tilde{\Gamma}_n$$  \hspace{1cm} (2.5)$$

This simplified $D = 10$ $b$-ghost was shown to satisfy the property $\{Q, b\} = T$ in [19], and as shown in Appendix I, the nilpotence property $\{b, b\} = 0$ easily follows from $\{\tilde{\Gamma}_m, \tilde{\Gamma}_n\} = 0$ and $[\tilde{\Gamma}_m, \lambda\lambda] = 0$.

3 $D = 11$ non-minimal pure spinor superparticle

The $D = 11$ non-minimal pure spinor superparticle action is given by [1]

$$S = \int d\tau (\bar{X}^a P_a + \bar{\Theta}^a P_a - \frac{1}{2} P^a P_a + \bar{\Lambda}^\alpha W_\alpha + \dot{\Lambda}_a \bar{W}^\alpha + \dot{R}_a S^\alpha)$$  \hspace{1cm} (3.1)$$

We use letters of the beginning of the Greek alphabet ($\alpha, \beta, \ldots$) to denote $SO(10,1)$ spinor indices and henceforth we will use Latin letters ($a, b, \ldots, l, m, \ldots$) to denote $SO(10,1)$ vector indices, unless otherwise stated. In (3.1) $\Theta^a$ is an $SO(10,1)$ Majorana spinor and $P_a$ is its corresponding conjugate momentum, and $P_a$ is the momentum for $X^a$. The variables $\Lambda^\alpha$, $\bar{\Lambda}_a$ are $D = 11$ pure spinors and $W_\alpha$, $\bar{W}^\alpha$ are their respective conjugate momenta, $R_a$ is an $SO(10,1)$ fermionic spinor satisfying $\Lambda^a R = 0$ and $S^\alpha$ is its corresponding conjugate momentum. The $SO(10,1)$ gamma matrices denoted by $\Gamma^a$ satisfy the Clifford algebra $(\Gamma^a)_{\alpha\beta}(\Gamma^b)_{\beta\gamma} + (\Gamma^b)_{\alpha\beta}(\Gamma^a)_{\beta\gamma} = 2\eta^{ab}\delta_\alpha^\gamma$. In $D = 11$ dimensions there exist an antisymmetric spinor metric $C_{\alpha\beta}$ (and its inverse $(C^{-1})^{\alpha\beta}$) which allows us to lower (and raise) spinor indices (e.g. $(\Gamma^a)^{\alpha\beta} = C^\alpha\sigma C^{\beta\delta}(\Gamma^a)_{\sigma\delta}$, $(\Gamma^a)_{\alpha\beta} = C^{\alpha\sigma}(\Gamma^a)_{\sigma\beta}$, etc).

The physical states described by this theory are defined as elements of the cohomology of the BRST operator $Q = \Lambda^a D_a + R_a \bar{W}^\alpha$ where $D_a = P_a - P_a(\Gamma^a\Theta)\alpha$ and describe $D = 11$ linearized supergravity.

3.1 $D = 11$ $b$-ghost and its simplification

As in the $D = 10$ case, a composite $D = 11$ $b$-ghost can be constructed satisfying the properties $\{Q, b\} = T$ where $T = -P^a P_a$, and was found in [9, 10, 20] to be:

$$b = \frac{1}{2} \eta^{-1}(\bar{\Lambda}_{ab}\Lambda)(\Lambda^{abcd} T^i D)_{P_i} + \eta^{-2} L^{(1)}_{abcd}(\Lambda^a D)(\Lambda^{bcd} D) + 2(\Lambda^{abc} i_j \Lambda)N^{di} P^j$$

$$+ \frac{2}{3} (\eta^b \eta^d q - \eta^b \eta q^d)(\Lambda^{apcij} \Lambda)N_{ij} P^q - \frac{1}{3} \eta^{-3} L^{(2)}_{abcd,e,f} (\Lambda^{abcij} \Lambda)(\Lambda^{def} D) N_{ij}$$

$$- 12 ((\Lambda^{abc}] \eta f j - \frac{2}{3} \eta f a (\Lambda^{bce} i j \Lambda)(\Lambda^d D) N_{ij})$$

$$+ \frac{4}{3} \eta^{-4} L^{(3)}_{abcd,e,f,gh}(\Lambda^{abcij} \Lambda)(\Lambda^{defgk} \Lambda) \eta^l \eta^h = \frac{2}{3} \eta^{-4} (\Lambda^{efg} \Lambda) \{N_{ij}, N_{kl}\}$$  \hspace{1cm} (3.2)$$

where

$$\eta = (\Lambda^{ab} \Lambda)(\bar{\Lambda}_{ab})$$  \hspace{1cm} (3.3)$$

$$L^{(n)}_{a_1 a_2 \ldots a_n b_1 \ldots b_n} = (\bar{\Lambda}_{[a_0 b_0} \Lambda)(\bar{\Lambda}_{a_1 b_1} R) \ldots (\bar{\Lambda}_{a_n b_n}) R$$  \hspace{1cm} (3.4)$$
and $[\ldots]$ means antisymmetrization between each pair of indices. The $D = 11$ ghost current is defined by $N_{ij} = \Lambda \Gamma_{ij} W$.

To simplify this complicated expression for the $D=11$ $b$-ghost, we shall mimic the procedure explained above for the $D=10$ $b$-ghost and look for a similar object to $\bar{\Gamma}_m$. A hint comes from looking at the quantity multiplying the momentum $P^i$ in the expression for the $D=11$ $b$-ghost:

$$b = P^i [\frac{1}{2} \eta^{-1} (\bar{\Lambda} \Gamma_{ab} \bar{\Lambda}) (\Lambda \Gamma_{ab} \Gamma_i D) + \eta^{-2} L^{(1)}_{ab,cd}(2(\Lambda \Gamma_{abc} \Lambda) N^{dk} + \frac{2}{3} (\eta^b p^d \eta^i - \eta^b d \eta_p) (\Lambda \Gamma_{aqc} \Lambda) N_{qj}) + \ldots]$$

Therefore our candidate to play the analog role to $\bar{\Gamma}_m$ is:

$$\bar{\Sigma}^i = \bar{\Sigma}_0^i + \frac{2}{\eta^2} \eta^{d} L^{(1)}_{ab,cd}(\Lambda \Gamma_{abc} \Lambda) N^d_k + \frac{2}{3\eta^2} \eta^{d} L^{(1)}_{ab,cd}(\Lambda \Gamma_{abc} \Lambda) N_{qj} - \frac{2}{3\eta^2} L^{(1)}_{ab,cd}(\Lambda \Gamma_{abc} \Lambda) N_{qj} \tag{3.5}$$

To show that the $b$-ghost of (3.8) satisfies $\{Q, b\} = T$, it will be convenient to first compute $\{Q, \Sigma^i\}$ where, using the identities (B.10), (B.13),

$$\Sigma^i = \Sigma_0^i + \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{ac} \Lambda)(\Lambda \Gamma_{i} \Lambda) \bar{\Sigma}^c + \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{ab} \Lambda)(\Lambda \Gamma_{i} \Lambda) \bar{\Sigma}_0 - \frac{2}{\eta^2} \eta^{d} L^{(1)}_{ab,cd}(\Lambda \Gamma_{abc} \Lambda) N_{qj} \tag{3.6}$$

Using equation (3.9) and the identities (B.20), (B.21), (B.22), (B.23):

$$\{Q, \Sigma^i\} = - P^i \frac{2}{\eta}(\bar{\Lambda} \Gamma_{a} \Lambda)(\Lambda \Gamma_{b} \Lambda)(\Lambda \Gamma_{i} \Lambda) P_m - \frac{2}{\eta}(\bar{\Lambda} \Gamma_{mn} \Lambda)(\Lambda \Gamma_{a} \Lambda) \Sigma^m_0 + \frac{4}{\eta}(\bar{\Lambda} \Gamma_{mn} \Lambda)(\Sigma^i_0 - \Sigma^i_0) - \frac{1}{\eta}(\bar{\Lambda} \Gamma_{ab} \Lambda)(\Lambda \Gamma_{i} \Lambda) D$$

$$- \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{ab} \Lambda)(\Lambda \Gamma_{i} \Lambda) (\Lambda \Gamma_{i} \Lambda) D - \frac{2}{\eta^2} \eta^d L^{(1)}_{ab,cd}(\Lambda \Gamma_{abc} \Lambda)(\Lambda \Gamma_{i} \Lambda) D$$

$$- \frac{2}{\eta^2} \eta^d L^{(1)}_{ab,cd}(\Lambda \Gamma_{abc} \Lambda)(\Lambda \Gamma_{i} \Lambda) D - \frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab} \Lambda)(\Lambda \Gamma_{cd} \Lambda)(\Lambda \Gamma_{i} \Lambda) N^{dk} \tag{3.7}$$

Furthermore, it will be shown in Appendix D that the $D_\alpha$’s appearing in $\bar{\Sigma}_0^i$ are the same as those appearing in the $b$-ghost. Therefore a plausible assumption for the simplification of the $b$-ghost would be $b = P^i \bar{\Sigma}_0 + O(\Sigma^2)$. As will now be shown, the simplified form of the $b$-ghost satisfying $\{Q, b\} = T$ is indeed

$$b = P^i \bar{\Sigma}_0 + \frac{2}{\eta} (\bar{\Lambda} \Gamma_{ac} \Lambda)(\Lambda \Gamma_{a} \Lambda) \bar{\Sigma}_0 - \frac{1}{\eta} (\bar{\Lambda} \Gamma_{i} \Lambda) \bar{\Sigma}_0 \bar{\Sigma}_0 \tag{3.8}$$

3.2 Computation of $\{Q, \Sigma^i\}$

To show that the $b$-ghost of (3.8) satisfies $\{Q, b\} = T$, it will be convenient to first compute $\{Q, \Sigma^i\}$ where, using the identities (B.10), (B.13),

$$\Sigma^i = \Sigma_0^i + \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{ac} \Lambda)(\Lambda \Gamma_{cd} \Lambda) N^d_k + \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ab} \Lambda)(\Lambda \Gamma_{cd} \Lambda) N_{qj}$$

$$- \frac{2}{\eta^2} \eta^{d} L^{(1)}_{ab,cd}(\Lambda \Gamma_{abc} \Lambda) N_{qj} \tag{3.9}$$

Using equation (3.9) and the identities (B.20), (B.21), (B.22), (B.23):

$$\{Q, \Sigma^i\} = - P^i \frac{2}{\eta}(\bar{\Lambda} \Gamma_{a} \Lambda)(\Lambda \Gamma_{b} \Lambda)(\Lambda \Gamma_{i} \Lambda) D - \frac{2}{\eta}(\bar{\Lambda} \Gamma_{i} \Lambda)(\Lambda \Gamma_{a} \Lambda) \Sigma^m_0 + \frac{2}{\eta}(\bar{\Lambda} \Gamma_{mn} \Lambda)(\Lambda \Gamma_{a} \Lambda) \Sigma^i_0 + \frac{4}{\eta}(\bar{\Lambda} \Gamma_{mn} \Lambda)(\Sigma^i_0 - \Sigma^i_0) - \frac{1}{\eta}(\bar{\Lambda} \Gamma_{ab} \Lambda)(\Lambda \Gamma_{i} \Lambda) D$$

$$- \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{ab} \Lambda)(\Lambda \Gamma_{i} \Lambda) \Lambda(\Lambda \Gamma_{i} \Lambda) D - \frac{2}{\eta^2} \eta^d L^{(1)}_{ab,cd}(\Lambda \Gamma_{abc} \Lambda)(\Lambda \Gamma_{i} \Lambda) D$$

$$- \frac{2}{\eta^2} \eta^d L^{(1)}_{ab,cd}(\Lambda \Gamma_{abc} \Lambda)(\Lambda \Gamma_{i} \Lambda) D - \frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab} \Lambda)(\Lambda \Gamma_{cd} \Lambda)(\Lambda \Gamma_{i} \Lambda) N^{dk} \tag{3.10}$$
\[-\frac{4}{3\eta^2} (\bar{\Lambda} \bar{\Gamma}_{ab} R)(\bar{\Lambda} \bar{\Gamma}_{cd} i R)(\bar{\Lambda} \bar{\Gamma}^{abcdk} L) N_{dk} \]  
\[- \frac{2}{3\eta^2} (\bar{\Lambda} \bar{\Gamma}_{ab} L)(R R)(\bar{\Lambda} \bar{\Gamma}^{iabcdk} L) N_{dk} \]  
(3.10)

As shown in Appendix F, this expression in invariant under the same gauge transformations under which $\Sigma_0^{ij}$ is invariant:

$$\delta D_\alpha = (\Gamma^{ij} \Lambda')_\alpha f_{ij}$$

(3.11)

where $(\Lambda')^\alpha = \frac{1}{\eta}(\bar{\Lambda} \bar{\Gamma}_{mn} \bar{\Lambda})(\bar{\Lambda} \bar{\Gamma}^{mn})^\alpha$ is a pure spinor, and $f_{ij}$ is an antisymmetric gauge parameter. Therefore we can write all $D_\alpha$’s in this object in terms of $\Sigma_0^{ij}$, and the result is (see Appendix F):

$$\{Q, \Sigma^i\} = -P_i - \frac{2}{\eta} (\bar{\Lambda} \bar{\Gamma}^{mb} \bar{\Lambda})(\bar{\Lambda} \bar{\Gamma}_{b}^i \bar{\Lambda}) \eta_{m} + \frac{4}{\eta^2} (\bar{\Lambda} \bar{\Gamma}_{mn} R)(\bar{\Lambda} \bar{\Gamma}^{mn} \bar{\Lambda})(\Sigma^i - \Sigma_0^{i})$$

$$- \frac{2}{\eta} (\bar{\Lambda} \bar{\Gamma}^{ci} R)(\bar{\Lambda} \bar{\Gamma}_{ck} \bar{\Lambda}) \Sigma_0^k + \frac{4}{\eta^2} (\bar{\Lambda} \bar{\Gamma}_{cd} R)(\bar{\Lambda} \bar{\Gamma}^{cd} \bar{\Lambda}) \Sigma_0^d + \frac{2}{\eta^2} (\bar{\Lambda} R)(\bar{\Lambda} \bar{\Gamma}^{ij} \bar{\Lambda}) \Sigma_0^k$$

$$- \frac{2}{\eta^2} (\bar{\Lambda} \bar{\Gamma}_{ab} R)(\bar{\Lambda} \bar{\Gamma}_{i}^j R)(\bar{\Lambda} \bar{\Gamma}^{abcdk} L) N_{dk} - \frac{4}{\eta^2} (\bar{\Lambda} \bar{\Gamma}_{ab} R)(\bar{\Lambda} R)(\bar{\Lambda} \bar{\Gamma}^{iabcdk} L) N_{dk}$$

$$- \frac{2}{3\eta^2} (\bar{\Lambda} \bar{\Gamma}_{ab} L)(R R)(\bar{\Lambda} \bar{\Gamma}^{iabcdk} L) N_{dk}$$

(3.12)

After plugging (3.9) into (3.12), all of the terms explicitly depending on $N_{ab}$ are cancelled and we get (see appendix G):

$$\{Q, \Sigma^i\} = -P_i - \frac{2}{\eta} (\bar{\Lambda} \bar{\Gamma}^{mb} \bar{\Lambda})(\bar{\Lambda} \bar{\Gamma}_{b}^i \bar{\Lambda}) \eta_{m} - \frac{2}{\eta}(\bar{\Lambda} \bar{\Gamma}^{ci} R)(\bar{\Lambda} \bar{\Gamma}_{ck} \bar{\Lambda}) \Sigma^k$$

$$+ \frac{4}{\eta^2} (\bar{\Lambda} \bar{\Gamma}_{cd} R)(\bar{\Lambda} \bar{\Gamma}^{cd} \bar{\Lambda}) \Sigma^d + \frac{2}{\eta^2} (\bar{\Lambda} R)(\bar{\Lambda} \bar{\Gamma}^{ij} \bar{\Lambda}) \Sigma_k - \frac{2}{\eta^2} (\bar{\Lambda} \bar{\Gamma}_{cd} R)(\bar{\Lambda} \bar{\Gamma}^{cd} \bar{\Lambda})(\bar{\Lambda} \bar{\Gamma}^{mn} \bar{\Lambda})(\bar{\Lambda} \bar{\Gamma}^{nk} \bar{\Lambda}) \Sigma_k$$

(3.13)

3.3 \{Q, b\} = T

Using (3.13) it is now straightforward to compute $\{Q, b\}$:

$$\{Q, b\} = P_i \{Q, \bar{\Sigma}_i\} - \frac{4}{\eta^2} (\bar{\Lambda} \bar{\Gamma}^{mn} \bar{\Lambda})(\bar{\Lambda} \bar{\Gamma}_{mn} R)(\bar{\Lambda} \bar{\Gamma}^{aj} R)(\bar{\Lambda} \bar{\Gamma}_{ak} \bar{\Lambda}) \Sigma^k \bar{\Sigma}_j$$

$$+ \frac{2}{\eta} (\bar{\Lambda} \bar{\Gamma}^{aj} R)(\bar{\Lambda} \bar{\Gamma}_{ak} \bar{\Lambda})(\{Q, \bar{\Sigma}^k\}) \bar{\Sigma}_j - \frac{2}{\eta} (\bar{\Lambda} \bar{\Gamma}^{aj} R)(\bar{\Lambda} \bar{\Gamma}_{ak} \bar{\Lambda}) \bar{\Sigma}_k \{Q, \bar{\Sigma}_j\}$$

$$- \frac{2}{\eta^2} (\bar{\Lambda} \bar{\Gamma}^{mn} \bar{\Lambda})(\bar{\Lambda} \bar{\Gamma}_{mn} R)(\bar{\Lambda} \bar{\Gamma}^{j} R)(\bar{\Lambda} \bar{\Gamma}_{jk} \bar{\Lambda}) \Sigma_j \bar{\Sigma}_k + \frac{4}{\eta}(R R)(\bar{\Lambda} \bar{\Gamma}^{ij} \bar{\Lambda}) \bar{\Sigma}_j \Sigma_k$$

$$+ \frac{4}{\eta}(\bar{\Lambda} R)(\bar{\Lambda} \bar{\Gamma}^{ij} \bar{\Lambda})(\{Q, \bar{\Sigma}_j\}) \bar{\Sigma}_k - \frac{1}{\eta}(\bar{\Lambda} R)(\bar{\Lambda} \bar{\Gamma}^{ij} \bar{\Lambda}) \bar{\Sigma}_j \{Q, \bar{\Sigma}_k\}$$

(3.14)

To make the computations transparent, each term in (3.14) involving $\{Q, \Sigma_i\}$ will be simplified separately:

$$M_1 = P_i \{Q, \bar{\Sigma}_i\}$$
Using (B.2), we get

\[ M_2 = \frac{2}{\eta} (\tilde{\Lambda} \Gamma^{aj} R)(\Lambda \Gamma_{ak} A)(\{Q, \tilde{\Sigma}^k\}) \tilde{\Sigma}_j \]

\[ = \frac{2}{\eta} (\tilde{\Lambda} \Gamma^{aj} R)(\Lambda \Gamma_{ak} A) - P^k + \frac{2}{\eta} (\tilde{\Lambda} \Gamma_{mb} A)(\tilde{\Lambda} \Gamma^{bk} \tilde{A}) P^m - \frac{2}{\eta} (\tilde{\Lambda} \Gamma_{ck} R)(\Lambda \Gamma_{cp} A) \tilde{\Sigma}^p \]

\[ - \frac{2}{\eta^2} (\tilde{\Lambda} \Gamma^{cd} R)(\Lambda \Gamma_{cd} A)(\tilde{\Lambda} \Gamma^{kn} \tilde{A})(\Lambda \Gamma_{np} A) \tilde{\Sigma}_j \]

\[ = -\frac{2}{\eta^2} (\tilde{\Lambda} \Gamma^{aj} R)(\Lambda \Gamma_{ak} A) P^k \tilde{\Sigma}_j + \frac{4}{\eta^2} (\tilde{\Lambda} \Gamma^{aj} R)(\Lambda \Gamma_{ak} A)(\tilde{\Lambda} \Gamma^{bk} \tilde{A}) P^m \tilde{\Sigma}_j \]

\[ - \frac{4}{\eta^3} (\tilde{\Lambda} \Gamma^{aj} R)(\Lambda \Gamma_{ak} A)(\tilde{\Lambda} \Gamma^{cd} R)(\Lambda \Gamma_{cd} A)(\tilde{\Lambda} \Gamma^{kn} \tilde{A})(\Lambda \Gamma_{np} A) \tilde{\Sigma}^p \tilde{\Sigma}_j \]

Using (B.2), we get

\[ M_3 = -\frac{2}{\eta} (\tilde{\Lambda} \Gamma^{aj} R)(\Lambda \Gamma_{ak} A) \tilde{\Sigma}^k (\{Q, \tilde{\Sigma}^k\}) \]

\[ = -\frac{2}{\eta} (\tilde{\Lambda} \Gamma^{aj} R)(\Lambda \Gamma_{ak} A) \tilde{\Sigma}^k \{1 - \frac{2}{\eta} [(\tilde{\Lambda} \Gamma_{jb} A)(\tilde{\Lambda} \Gamma_{bm} \tilde{A}) - (\tilde{\Lambda} \Gamma_{mb} A)(\tilde{\Lambda} \Gamma_{bj} \tilde{A})] P_m \]

\[ - \frac{2}{\eta^2} (\tilde{\Lambda} \Gamma_{cj} R)(\Lambda \Gamma_{cp} A) \tilde{\Sigma}_p + \frac{4}{\eta^2} (\tilde{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma_{cj} A) \tilde{\Sigma}_d + \frac{2}{\eta} (\tilde{\Lambda} R)(\Lambda \Gamma_{jp} A) \tilde{\Sigma}_p \]

\[ - \frac{2}{\eta^2} (\tilde{\Lambda} \Gamma^{cd} R)(\Lambda \Gamma_{cd} A)(\tilde{\Lambda} \Gamma^{np} \tilde{A})(\Lambda \Gamma_{np} A) \tilde{\Sigma}_p \]

\[ = -\frac{2}{\eta} (\tilde{\Lambda} \Gamma^{aj} R)(\Lambda \Gamma_{ak} A) \tilde{\Sigma}^k P_j + \frac{4}{\eta^2} (\tilde{\Lambda} \Gamma^{aj} R)(\Lambda \Gamma_{ak} A) \tilde{\Sigma}^k (\tilde{\Lambda} \Gamma_{jb} A)(\tilde{\Lambda} \Gamma_{bm} \tilde{A}) P_m \]

\[ - \frac{4}{\eta^2} (\tilde{\Lambda} \Gamma^{aj} R)(\Lambda \Gamma_{ak} A)(\tilde{\Lambda} \Gamma^{mb} A)(\Lambda \Gamma_{bj} A) P_m + \frac{4}{\eta^2} (\tilde{\Lambda} \Gamma^{aj} R)(\Lambda \Gamma_{ak} A) \tilde{\Sigma}^k (\tilde{\Lambda} \Gamma^{cj} R)(\Lambda \Gamma_{cp} A) \tilde{\Sigma}_p \]
\[
-M_3 = \frac{2}{\eta} (\bar{\Lambda} \Gamma^a R)(\Lambda \Gamma_{ak})(\Lambda \Gamma_{cd} R)\Sigma^c R (\bar{\Lambda} \Gamma^b \Lambda)\Sigma^d - \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^a R)(\Lambda \Gamma_{ak} R)\Sigma^c (\bar{\Lambda} R)(\Lambda \Gamma_{jp} \Lambda)\Sigma^p \\
+ \frac{4}{\eta^3} (\bar{\Lambda} \Gamma^a R)(\Lambda \Gamma_{ak} R)\Sigma^c (\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma_{jm} \Lambda)(\Lambda \Gamma_{np} \Lambda)\Sigma^p
\]

Using (B.2), (B.5), (B.19):

\[
M_3 = \frac{2}{\eta} (\bar{\Lambda} \Gamma^a R)(\Lambda \Gamma_{ak} R)\Sigma^c P_j + \frac{2}{\eta^2} (\bar{\Lambda} \Gamma^a R)(\Lambda \Gamma_{ak} R)(\Lambda \Gamma_{km} \Lambda)\Sigma^k P^m \\
+ \frac{2}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{mk} \Lambda) P_m \Sigma_k - \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{ac} \Lambda)(\bar{\Lambda} R)(\Lambda \Gamma_{kp} \Lambda)\Sigma_k \Sigma_p \\
- \frac{1}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{kp} \Lambda)\Sigma_k \Sigma_p + \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^a R)(\Lambda \Gamma_{ak} \Lambda)(\Lambda \Gamma_{cd} R)(\Lambda \Gamma_{ck} \Lambda)\Sigma^c \Sigma^d \\
+ \frac{2}{\eta^2} (\bar{\Lambda} \Gamma^a R)(\Lambda \Gamma_{aj} \Lambda)(\bar{\Lambda} R)(\Lambda \Gamma_{kp} \Lambda)\Sigma_k \Sigma_p - \frac{2}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{kp} \Lambda)(\Lambda \Gamma_{cd} \Lambda)\Sigma_k \Sigma_p \\
= \frac{2}{\eta} (\bar{\Lambda} \Gamma^a R)(\Lambda \Gamma_{ak} \Lambda)\Sigma^c P_j + \frac{2}{\eta^2} (\bar{\Lambda} \Gamma^a R)(\Lambda \Gamma_{ak} \Lambda)(\Lambda \Gamma_{km} \Lambda)\Sigma^k P^m \\
+ \frac{2}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{mk} \Lambda) P_m \Sigma_k - \frac{1}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{kp} \Lambda)\Sigma_k \Sigma_p + \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^a R)(\Lambda \Gamma_{ak} \Lambda)(\Lambda \Gamma_{cd} R)(\Lambda \Gamma_{ck} \Lambda)\Sigma^c \Sigma^d \\
- \frac{2}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{kp} \Lambda)(\Lambda \Gamma_{cd} \Lambda)\Sigma_k \Sigma_p
\]

\[
M_4 = \frac{1}{\eta} (\bar{\Lambda} R)(\Lambda \Gamma^j k \Lambda)\{Q, \Sigma_j\} \Sigma_k
\]

\[
\frac{1}{\eta} (\bar{\Lambda} R)(\Lambda \Gamma^j k \Lambda) P_j \Sigma_k + \frac{2}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma^j k \Lambda)(\Lambda \Gamma_{mj} \Lambda) P_m \Sigma_k \\
- \frac{1}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{mj} \Lambda) P_m \Sigma_k + \frac{1}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{km} \Lambda) P_m \Sigma_k \\
- \frac{1}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{mk} \Lambda) P_m \Sigma_k - \frac{1}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{jm} \Lambda) P_m \Sigma_k \\
= \frac{1}{\eta} (\bar{\Lambda} R)(\Lambda \Gamma^j k \Lambda) P_j \Sigma_k - \frac{1}{\eta} (\bar{\Lambda} R)(\Lambda \Gamma_{mj} \Lambda) P_m \Sigma_j \\
+ \frac{1}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{jm} \Lambda) P_m \Sigma_j + \frac{1}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{km} \Lambda) P_m \Sigma_j \\
= -\frac{2}{\eta} (\bar{\Lambda} R)(\Lambda \Gamma^j k \Lambda) P_j \Sigma_k
\]

\[
M_5 = \frac{1}{\eta} (\bar{\Lambda} R)(\Lambda \Gamma^j k \Lambda) \Sigma_j \{Q, \Sigma_k\}
\]

\[
\frac{1}{\eta} (\bar{\Lambda} R)(\Lambda \Gamma^j k \Lambda) \Sigma_j P_k - \frac{2}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma^j k \Lambda)(\Lambda \Gamma_{mj} \Lambda)(\bar{\Lambda} \Gamma_{bk} \Lambda) P_m \\
+ \frac{2}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma^j k \Lambda) \Sigma_j (\bar{\Lambda} \Gamma_{ck} \Lambda)(\Lambda \Gamma_{np} \Lambda) \Sigma_p + \frac{1}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma^j k \Lambda)(\Lambda \Gamma_{cd} \Lambda)(\bar{\Lambda} \Gamma_{kn} \Lambda)(\Lambda \Gamma_{np} \Lambda) \Sigma_p \\
= \frac{1}{\eta} (\bar{\Lambda} R)(\Lambda \Gamma^j k \Lambda) P_j \Sigma_k - \frac{1}{\eta} (\bar{\Lambda} R)(\Lambda \Gamma_{jm} \Lambda) P_m \Sigma_j \\
+ \frac{1}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{jm} \Lambda) P_m \Sigma_j + \frac{1}{\eta^2} (\bar{\Lambda} R)(\Lambda \Gamma_{kn} \Lambda) P_m \Sigma_j \\
= -\frac{2}{\eta} (\bar{\Lambda} R)(\Lambda \Gamma^j k \Lambda) P_j \Sigma_k
\]
Putting together all the terms in (3.14):

$$\{Q, b\} = \sum_{i=1}^{5} M_i - \frac{4}{\eta^2} (\Lambda \Gamma_{mn} \Lambda) (\bar{\Lambda} \Gamma_{mn} R) (\Lambda \Gamma_{ak} \Lambda) \Sigma^k \Sigma_j$$

$$- \frac{2}{\eta^2} (\Lambda \Gamma_{mn} \Lambda) (\bar{\Lambda} \Gamma_{mn} R) (\Lambda \Gamma_{jk} \Lambda) \Sigma_j \Sigma_k + \frac{1}{\eta^2} (RR) (\Lambda \Gamma_{jk} \Lambda) \Sigma_j \Sigma_k$$

$$= -P^2$$

(3.20)

Recalling that $T = -P^2$ is the stress-energy tensor, we have checked that $\{Q, b\} = T$.

### 3.4 $\{b, b\} = \text{BRST-trivial}$

In the D=10 case, the identity $\{\bar{\Gamma}^m, \bar{\Gamma}^n\} = 0$ was crucial for showing that $\{b, b\} = 0$. However, in the D=11 case, it is shown in Appendix H that $\{\Sigma^j, \bar{\Sigma}^k\}$ is non-zero and is proportional to $R_\alpha$. This implies that

$$\{b, b\} = R^\alpha G^\alpha (\Lambda, \bar{\Lambda}, R, W, D)$$

(3.21)

for some $G^\alpha (\Lambda, \bar{\Lambda}, R, W, D)$.

Note that $[Q, \{b, b\}] = 0$ since $[b, T] = 0$ where $T = -P_a P^a$. Since $Q = \Lambda^\alpha D_\alpha + R^a \bar{W}_a$, the quartet argument implies that the cohomology of $Q$ is independent of $R_\alpha$, which allows us to conclude that $\{b, b\} = \text{BRST-trivial}$. It would be interesting to investigate if this BRST-triviality of $\{b, b\}$ is enough for the scattering amplitude prescription using the $b$-ghost to be consistent.

### 4 Remarks

We have succeeded in finding a considerably simpler form in (3.8) for the D=11 $b$-ghost than that of equation (3.2) which was presented in [10]. Although this simplified version is not strictly nilpotent, it satisfies the relation $\{b, b\} = \text{BRST-trivial}$ which may be good enough for consistency.

It is natural to ask if the simplified $D = 11$ $b$-ghost (3.8) is the same as the $b$-ghost presented in (3.2). These two expressions are compared in Appendix I and we find that they coincide up to normal-ordering terms coming from the position of $N_{mn}$ in each expression. Note that the product of $N_{mn}$’s appears as an anticommutator in (3.2) whereas it appears as an simple ordinary product in (3.8). However, because we have ignored normal-ordering questions in our analysis, we will not attempt to address this issue.

### 5 Acknowledgments

MG acknowledges FAPESP grant 15/23732-2 for financial support and NB acknowledges FAPESP grants 2016/01343-7 and 2014/18634-9 and CNPq grant 300256/94-9 for partial financial support.
A  $D = 10$ gamma matrix identities

In $D = 10$ we have chiral and antichiral spinors which will be denoted by $\chi^\alpha$ and $\bar{\chi}_\alpha$ respectively. The product of two spinors can be decomposed into two forms depending on the chiralities of the spinors used:

$$\xi_\mu \chi^\nu = \frac{1}{16} \delta_\mu^\nu (\xi \chi) - \frac{1}{216} (\gamma_{mnpq})^\nu_\mu (\xi \gamma_{mnpq} \chi) + \frac{1}{416} (\gamma_{mnpq})^\nu_\mu (\xi \gamma_{mnpq} \chi)$$  \hspace{1cm} (A.1)

$$\xi^\mu \chi_\nu = \frac{1}{16} \gamma^\mu_\nu (\xi \gamma \chi) + \frac{1}{316} (\gamma_{mnpq})^\nu_\mu (\xi \gamma_{mnpq} \chi) + \frac{1}{5132} (\gamma_{mnpq})^\nu_\mu (\xi \gamma_{mnpq} \chi)$$  \hspace{1cm} (A.2)

The 1-form and 5-form are symmetric, and the 3-form is antisymmetric. Furthermore, it is true that

$$(\gamma_{mn})^\mu_\nu = - (\gamma_{nm})^\mu_\nu,$$

$$(\gamma_{mnpq})^\mu_\nu = (\gamma_{mnpq})^\mu_\nu.$$  \hspace{1cm} (A.4)

Two particularly useful identities are:

$$(\gamma^m)^\mu_\nu (\gamma_m)^\rho_\sigma = 0$$  \hspace{1cm} (A.3)

$$(\gamma^m)^\mu_\nu (\gamma_m)^\rho_\sigma = 4 (\gamma^m)^{\mu \rho} (\gamma_m)^{\nu \sigma} - 2 \delta_\nu^\mu \delta_\sigma^\rho - 8 \delta_\nu^\rho \delta_\sigma^\mu$$  \hspace{1cm} (A.4)

From A.4 we can deduce the following:

$$(\gamma_{mn})^\mu_\nu \gamma_{mnpq} = 2 (\gamma^m)^{\mu \rho} (\gamma_{pm})^\sigma_\nu + 6 (\gamma^m)^{\mu \rho} (\gamma_{pm})^\sigma_\nu - (\rho \leftrightarrow \sigma)$$  \hspace{1cm} (A.5)

$$(\gamma_{mn})^\mu_\nu \gamma_{mnpq} = -2 (\gamma_{pm})^\nu_\mu + 6 (\gamma_{pm})^\nu_\mu - (\rho \leftrightarrow \sigma)$$  \hspace{1cm} (A.6)

$$(\gamma_{mn})^\mu_\nu \gamma_{mnpq} = 12 [ (\gamma_{pm})^\nu_\mu - (\gamma_{nm})^\nu_\mu ]$$  \hspace{1cm} (A.7)

$$(\gamma_{mn})^\mu_\nu \gamma_{mnpq} = 48 [ (\gamma_{pm})^\nu_\mu - (\gamma_{nm})^\nu_\mu ]$$  \hspace{1cm} (A.8)

The Lorentz algebra satisfied by the ghost currents $N_{mn} = \frac{1}{2} (\lambda \gamma_{mn} w)$ is:

$$[N_{pq}, N_{rs}] = \eta_{qs} N_{pr} - \eta_{qr} N_{ps} - \eta_{ps} N_{qr} + \eta_{pr} N_{qs}$$  \hspace{1cm} (A.9)

B  $D = 11$ pure spinor identities

We list some pure spinor identities in eleven dimensions:

$$\bar{\Lambda} \Gamma^{ab} \bar{\Lambda} (\Gamma_b \bar{\Lambda})_\alpha = 0$$  \hspace{1cm} (B.1)

$$\bar{\Lambda} \Gamma^{ab} \bar{\Lambda} (\Gamma^{cd} \bar{\Lambda}) = 0$$  \hspace{1cm} (B.2)

$$\bar{\Lambda} \Gamma^{ab} \bar{\Lambda} (\Gamma^{cd} \bar{\Lambda}) = 0$$  \hspace{1cm} (B.3)

$$(\bar{\Lambda} \Gamma^{ab} \bar{\Lambda}) (\bar{\Lambda} \Gamma^{cd} \bar{\Lambda}) = 0$$  \hspace{1cm} (B.4)

$$(\bar{\Lambda} \Gamma_{ij} R) (\bar{\Lambda} \Gamma_{k} J R) = (\bar{\Lambda} \Gamma_{ik} \bar{\Lambda}) (\bar{\Lambda} R) + \frac{1}{2} (\bar{\Lambda} \Gamma_{ik} \bar{\Lambda}) (RR)$$  \hspace{1cm} (B.5)

$$(\bar{\Lambda} \Gamma_{ab} R) (\bar{\Lambda} \Gamma_{cd} R) g^{ac} g^{bd} = 0$$  \hspace{1cm} (B.6)

$$(\bar{\Lambda} \Gamma_{sk} \bar{\Lambda}) (\bar{\Lambda} \Gamma_{abcdk} \bar{\Lambda}) = 0$$  \hspace{1cm} (B.7)

$$(\Gamma_i \bar{\Lambda})_\alpha (\bar{\Lambda} \Gamma^{ab} \bar{\Lambda}) = 6 (\Gamma^{ab} \bar{\Lambda})_\alpha (\bar{\Lambda} \Gamma^{cd} \bar{\Lambda})$$  \hspace{1cm} (B.8)

$$(\Gamma_{ij} \bar{\Lambda})_\alpha (\bar{\Lambda} \Gamma^{abcd} \bar{\Lambda}) = -18 (\Gamma^{ab} \bar{\Lambda})_\alpha (\bar{\Lambda} \Gamma^{cd} \bar{\Lambda})$$  \hspace{1cm} (B.9)
where \( f^{ac}, g^{bd} \) are antisymmetric in \((a, c), (b, d)\) respectively. In addition, using (B.4) it can be shown that

\[
L_{ab,cd}^{(1)} f^{abc} = (\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R) f^{abc} \quad (B.10)
\]

\[
L_{ab,cd}^{(1)} f^{abc} = -(\bar{\Lambda} \Gamma_{cd} \bar{\Lambda})(\bar{\Lambda} \Gamma_{ab} R) f^{abc} \quad (B.11)
\]

\[
L_{ab,cd,ef}^{(2)} f^{abc} = \frac{1}{3} (\bar{\Lambda} \Gamma_{eb} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\bar{\Lambda} \Gamma_{ef} R) f^{abc} \quad (B.12)
\]

where \( f^{abc}, f^{abce} \) are antisymmetric in all of their indices.

Other useful identities:

\[
L_{ad,c}^{d} = (\bar{\Lambda} \Gamma_{ac} \bar{\Lambda})(\bar{\Lambda} R) \quad (B.13)
\]

\[
L_{ab,cd,e} = \frac{1}{3} [ (\bar{\Lambda} \Gamma_{eb} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R) - (\bar{\Lambda} \Gamma_{cd} \bar{\Lambda})(\bar{\Lambda} \Gamma_{ab} R) ] \quad (B.14)
\]

\[
L_{ab,cd,e} = \frac{1}{3} [ (\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R) - 2(\bar{\Lambda} \Gamma_{cd} \bar{\Lambda})(\bar{\Lambda} \Gamma_{ab} R) ] \quad (B.15)
\]

\[
(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda}) \Sigma^b = 0 \quad (B.16)
\]

Some useful commutation relations

\[
[\bar{\Sigma}^i, \eta] = 0 \quad (B.17)
\]

\[
[\bar{\Sigma}^j, (\Lambda \Gamma_{mn})] = \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{ef} \bar{\Lambda})(\bar{\Lambda} \Gamma_{gh} R)(\Lambda \Gamma_{mn}) \quad (B.18)
\]

\[
\{\bar{\Sigma}^j, (\bar{\Lambda} \Gamma_{mn} R)(\Lambda \Gamma_{mn})\} = 0 \quad (B.19)
\]

\[
[Q, \eta] = -2(\Lambda \Gamma_{mn} \bar{\Lambda})(\bar{\Lambda} \Gamma_{mn} R) \quad (B.20)
\]

\[
[Q, \Lambda \Gamma_{ab} \bar{\Lambda}] = -2(\Lambda \Gamma_{ab} R) \quad (B.21)
\]

\[
[Q, \Lambda \Gamma_{hi} \bar{\Lambda}] = (\Lambda \Gamma^{hi} D) \quad (B.22)
\]

\[
[Q, \Lambda \Gamma_{hi} \bar{\Lambda}] = -2(\Gamma_{mn} \bar{\Lambda}) \beta P_m \quad (B.23)
\]

\[
[N^{hi}, \eta] = -2(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})[-2\eta^a \Lambda(\Lambda \Gamma^{bh} \bar{\Lambda}) + 2\eta^b \Lambda(\Lambda \Gamma^{ah} \bar{\Lambda})] \quad (B.24)
\]

\[
[N^{hi}, (\Lambda \Gamma_{lmnp} \bar{\Lambda})] = -2\eta^q(\Lambda \Gamma^{chlmn} \bar{\Lambda}) + 2\eta^m(\Lambda \Gamma^{cchl} \bar{\Lambda}) - 2\eta^r(\Lambda \Gamma^{chln} \bar{\Lambda}) + 2\eta^l(\Lambda \Gamma^{chmn} \bar{\Lambda}) \quad (B.25)
\]

\[
[\Lambda \Gamma^{aW}, \Lambda \Gamma^{bW}] = -2 N^{ab} \quad (B.26)
\]

\[
[\Lambda \Gamma^{aW}, \Lambda \Gamma^{mn} W] = -2 \Gamma^{aW} W \quad (B.27)
\]

\[
[\Lambda \Gamma^{abcW}, \Lambda \Gamma^{mn} W] = 4\delta_{mp}^{bc} N^{am} - 4\delta_{mp}^{bc} N^{am} + 4\delta_{mn}^{bc} N^{ap} - 4\delta_{mp}^{ac} N^{bm} + 4\delta_{mp}^{ac} N^{bm} - 4\delta_{mn}^{ab} N^{bp} + 4\delta_{mp}^{ab} N^{cm} - 4\delta_{mp}^{ab} N^{cm} + 4\delta_{mn}^{ab} N^{cp} - 2 \Lambda \Gamma^{abe} W \quad (B.28)
\]
C Nilpotence of $D=10$ $b$-ghost

The nilpotency property satisfied by this object is not obvious to see so that we will check it in detail. The first step is to show that $\{\Gamma^m, \Gamma^n\} = 0$. This can be seen from the equation (2.5) and the use of the $U(5)$ decomposition of the pure spinor variables [14]. If we choose the only non-zero component of $\lambda_\mu$ to be $\bar{\lambda}_{----} \neq 0$ then $r_{++++}$, $r_{++--}$, $r_{++-+}$, $r_{+++--}$ all vanish as follows from the constraint $\lambda_\gamma^{m r} = 0$. This implies that the only components of $d_\mu$ and $N_{mn}$ appearing in (2.5) are $d_\nu$ with $\nu = \{(-++++), (+---), (+-+-), (++--), (++--), (++++), (++++)\}$ and $N_{pq}$ with $p, q = \{(1-2i), (3-4i), (5-6i), (7-8i), (9-10i)\}$. Because all the components of $\lambda_\mu$ with two plus signs are zero, the commutator $[N_{mn}, \bar{\lambda}_\mu \lambda^\mu]$ vanishes. Likewise the commutator $[N_{mn}, N_{pq}]$ vanishes for $(p, q, m, n)$ taking the values listed above because the metric components are zero for any combination of these values (see equation (A.9)). Thus we see that $\{\Gamma^m, \Gamma^n\} = 0$. The nilpotency property of the $b$-ghost can be seen as follows: Because the only contribution of $\lambda$ in $(\bar{\lambda} \lambda)$ is $\gamma^{++} \gamma^{++}$, the commutator $[\Gamma^m, (\bar{\lambda} \lambda)] = 0$. Furthermore, from the constraint $(\gamma_m \bar{\lambda})^\mu \Gamma^m = 0$ it is followed that the only non-zero components of $\Gamma^m$ are $\Gamma^m$ with $n = \{(1+2i), (3+4i), (5+6i), (7+8i), (9+10i)\}$. This implies that the term $(\lambda_\gamma^{m r} r)$ in (2.5) is non-zero only for the cases $m, n = \{(1-2i), (3-4i), (5-6i), (7-8i), (9-10i)\}$. Now, because $w_\mu$ can only appear with two or four plus signs in $N_{pq}$ and $\lambda^\mu$ appears in $(\lambda_\gamma^{m r} r)$ at least with two plus signs the only relevant situation is when $w_\mu$ has two plus signs and $\lambda^\mu$ has three plus signs, however when this occurs the only components of $r_\alpha$ which contribute are those with one minus sign making the whole expression vanish. Therefore the commutator $[\Gamma^m, (\lambda_\gamma^{pq} r)] \Gamma_p \Gamma_q = 0$. This implies immediately that $\{b, b\} = 0$.

This result can also be shown from the following covariant computation:

$$\{\Gamma^m, \Gamma^n\} = \frac{1}{2} \left(\frac{\lambda_\gamma^{m d}}{(\lambda \lambda)^2} - \frac{1}{8} \frac{(\lambda_\gamma^{mrs} r)_N_{rs}}{\lambda} - \frac{1}{8} \frac{(\lambda_\gamma^{npq} r)_N_{pq}}{(\lambda \lambda)^2}\right)$$

$$= -\frac{1}{4(\lambda \lambda)^2} (\lambda_\gamma^{m \gamma^k \gamma^r \lambda}) P_k + \frac{1}{32(\lambda \lambda)^4} (\lambda_\gamma^{m d} (\lambda_\gamma^{npq} r) (\lambda_\gamma^{npq} \lambda))$$

$$+ \frac{1}{32(\lambda \lambda)^4} (\lambda_\gamma^{m r} (\lambda_\gamma^{npq} r) (\lambda_\gamma^{npq} \lambda)) + \frac{1}{16(\lambda \lambda)^4} (\lambda_\gamma^{mrs} r) (\lambda_\gamma^{n p s} r) N_{pr}$$

$$+ \frac{1}{64(\lambda \lambda)^5} (\lambda_\gamma^{mrs} r) (\lambda_\gamma^{npq} r) (\lambda_\gamma^{npq} r) (\lambda_\gamma^{n p q} r) (\lambda_\gamma^{n p q} r)$$

where we have used that $\{d_\mu, d_\nu\} = -\gamma_\mu P_\mu$, $[N_{pq}, \bar{\lambda} \lambda] = -\frac{1}{2} (\lambda_\gamma^{npq} r)$ and the Lorentz algebra satisfied by $N_{pq}$ given in (A.9). The first term proportional to $P^m$ is zero because the pure spinor constraint and the bosonic nature of $\lambda_\alpha$. From the identity (A.5) we can show that $(\lambda_\gamma^{npq} r)(\lambda_\gamma^{npq} r) = 0$. Therefore the terms proportional to this expression vanish. So we are left with

$$\{\Gamma^m, \Gamma^n\} = \frac{1}{16} (\lambda_\gamma^{mrs} r) (\lambda_\gamma^{n p s} r) N_{pr} \quad \text{(C.1)}$$

The equation (A.2) allows putting this expression into the form

$$\{\Gamma^m, \Gamma^n\} = -\frac{1}{3!16^2} (\lambda_\gamma^{mrs} r) (\lambda_\gamma^{n p s} r) N_{pr} \quad \text{(C.2)}$$

- 12 -
Now we can use the GAMMA package [21] to do gamma matrix manipulations. The expansion of this expression, the use of the pure spinor constraint and the bosonic nature of $\bar{\lambda}_\mu$ give us the following result

$$
\{\bar{\Gamma}^m, \bar{\Gamma}^n\} = -\frac{1}{3!16^2}[\eta^{np}(\bar{\lambda}_tuvmr\bar{\lambda}) + \eta^{mp}(\bar{\lambda}_tuvnr\bar{\lambda}) - \eta^{mn}(\bar{\lambda}_tuvpr\bar{\lambda})]
$$
$$
+ 6\eta^p\bar{\lambda}_tuvmpr\bar{\lambda}] (\gamma_{tuvr}) N_{pr}
\quad (C.3)
$$

Using the reasons mentioned above we can write $\bar{\lambda}_tuvmr\bar{\lambda} = \bar{\lambda}_tupsym\bar{\lambda} = \bar{\lambda}_tuv\gamma^{mr}\bar{\lambda}$. Therefore after using the identity (A.7) and the constraint $\bar{\lambda}_t\gamma^{mr} = 0$ we obtain the result desired

$$
\{\bar{\Gamma}^m, \bar{\Gamma}^n\} = 0
\quad (C.4)
$$

Using this we can calculate $\{b, b\}$ directly:

$$
\{b, b\} = \{P^n\bar{\Gamma}_m - \frac{1}{4}(\bar{\lambda}\gamma^{mn})\bar{\Gamma}_m \bar{\Gamma}_n, P^p\bar{\Gamma}_p - \frac{1}{4}(\bar{\lambda}\lambda)\bar{\Gamma}_p \bar{\Gamma}_q\}
$$
$$
= 0
\quad (C.5)
$$

where we have used $[\bar{\Gamma}^m, \bar{\lambda}\lambda] = \frac{1}{16(\lambda\lambda)}(\bar{\lambda}\gamma^{mnp})(\lambda\gamma_{np}\lambda) = 0$ and $\{\bar{\Gamma}^m, \lambda\gamma^{rs}\bar{\Gamma}_s\} \bar{\Gamma}_s = \frac{1}{8(\lambda\lambda)}(\bar{\lambda}\gamma^{mnp})(\lambda\gamma_{np}\gamma^{rs}) \bar{\Gamma}_s = \frac{1}{8(\lambda\lambda)}[-(\bar{\lambda}\gamma^{mns})(\lambda\gamma_{ns}\lambda) + (\bar{\lambda}\gamma^{mnr})(\lambda\gamma_{nr}\lambda)] \bar{\Gamma}_s = 0$ because of the constraint $(\gamma_m\bar{\lambda})^m \bar{\Gamma}^m = 0$.

D The $b$-ghost and $\Sigma^j$ have the same $D_\alpha$’s

We should figure out which are the $D_\alpha$’s appearing in the expressions for $\Sigma^j$ and the $b$-ghost. For this we will decompose the eleven dimensional Lorentz group in the following way: $SO(10, 1) \rightarrow SO(3, 1) \times SO(7)$ and we will break the Lorentz invariance by choosing a special direction for $D_\alpha$:

$$
\bar{\Lambda}^0 = 0 \rightarrow \text{We choose the only non-zero component of $\bar{\Lambda}$ to be: $\bar{\Lambda}^{+0} \neq 0$} \quad (D.1)
\bar{\Lambda}^a R = 0 \rightarrow R^{+a} = R^{-0} = R^{+j} = 0 \quad (D.2)
$$

and from the pure spinor constraint $\Lambda^a\bar{\Lambda} = 0$ we have:

$$
\Lambda^{-0} = -\frac{\Lambda^{+j} + \Lambda^{-j}}{\Lambda^{-0}}
\quad (D.3)
\Lambda^{+0} = -\frac{\Lambda^{+j} - \Lambda^{-j}}{\Lambda^{+0}}
\quad (D.4)
\Lambda^{+j} = \frac{1}{\Lambda^{-0}}[\Lambda^{-j}\Lambda^{+0} - \Lambda^{-j}\Lambda^{+0} + \Lambda^{+j}\Lambda^{+0}]
\quad (D.5)
$$

where $j = 1, \ldots, 7$ and we have assumed that $\Lambda^{-0} \neq 0$. This allows us to expand the quadratic term in $D_\alpha$ in the $b$-ghost (I.1) in terms of these components:

$$
b_1 \propto \frac{(\Lambda^{+0}\Lambda^{+0})(\Lambda^{+0}R^{--})}{(\Lambda^{+0}\Lambda^{+0})^2}(\Lambda^{-0}\Lambda^{--} + \Lambda^{-k}\Lambda^{--})^2 \{\Lambda^{-0}D^{+0} + \Lambda^{--}D^{++} - \Lambda^{-k}D^{++} + \Lambda^{+0}D^{--} \}
$$
where $k,j$ contained in the expression for $\Lambda$. Therefore we see that the expression for $\Lambda$ contains the same combinations of $D_{\alpha}$'s as those present in the expression for $\Sigma^i_0$ (D.10), (D.11), (D.12), (D.13), (D.14)).

**E  $D_{\alpha}$ in terms of $\Sigma^i_0$**

Let us define the quantity:

$$H_{\alpha} = (\Lambda \Gamma_i)_{\alpha} \Sigma^i_0 = \frac{1}{2\eta} (\Gamma_i \Lambda)_{\alpha} (\Lambda \Gamma_{ab}) (\Lambda \Gamma^{ab} \Gamma^i D) \quad (E.1)$$
Now we will assume that there exist a matrix $(M^{-1})_{\alpha}^{\beta}$ such that:

$$D_\alpha = (M^{-1})_{\alpha}^{\beta} H_\beta$$

and let us make the following ansatz for $(M^{-1})_{\alpha}^{\beta}$:

$$(M^{-1})_{\alpha}^{\beta} = 2\delta_{\alpha}^{\beta} + \frac{2}{\eta}((\Lambda^m)_{\alpha}(\bar{\Lambda} \Gamma^m \Lambda)\Lambda_{\alpha}^{\beta})$$

Next we will check that this proposal for $(M^{-1})_{\alpha}^{\beta}$ is right:

$$H_\alpha = \frac{1}{2\eta}((\Gamma_i \Lambda)_{\alpha}(\bar{\Lambda} \Gamma_i \Lambda))(\Lambda \Gamma^c \Gamma^i M^{-1} H)$$

It can be shown that $\bar{\Sigma}_{0i}$ can be written in terms of $H_\alpha$:

$$H_\alpha = (\Lambda^c)_{\alpha} \bar{\Sigma}_{0c}$$

Therefore by using the constraint $(\bar{\Lambda} \Gamma_i \Lambda)\bar{\Sigma}_{0i} = 0$, we find

$$\bar{\Sigma}_{0c} = \frac{1}{\eta}((\bar{\Lambda} \Gamma_i \Lambda)(\Lambda \Gamma^c \Gamma^i H)$$

**F** The $D_\alpha$’s in $\{Q, \bar{\Sigma}^i\}$ are gauge invariant

We will show that the $D_\alpha$’s appearing in (3.10) are invariant under the gauge transformations (3.11). Therefore they are the same $D_\alpha$’s as those contained in the definition of $\bar{\Sigma}^i$.
of the heavy manipulation of gamma matrix identities which computations demanded. Let us simplify this object:

\[I^i = -\frac{1}{\eta} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[-\frac{2}{3\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[-\frac{2}{3\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[= -\frac{1}{\eta} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[+ \frac{8}{\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[+ \frac{8}{\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

The third term of this expression requires more careful manipulations, so we will do them in detail

\[I^{*i} = -\frac{2}{\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[+ \frac{8}{\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[+ \frac{8}{\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[= -\frac{4}{\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[+ \frac{4}{\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[+ \frac{4}{\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[= -\frac{2}{\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[+ \frac{8}{\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]

\[+ \frac{8}{\eta^2} (\Lambda_\alpha \Gamma_\beta) (\Lambda_\gamma \Gamma_\delta) \left( \frac{-18}{6} \right) \left[ 4(\Lambda_\alpha D)(\Lambda_\beta e\Lambda) + 2(\Lambda_\gamma D)(\Lambda_\delta a\Lambda) \right] \]
Furthermore, if we use (B.4) this result can be cast as

\[ I^{*i} = -\frac{8}{\eta^2} (\bar{\Lambda} R)(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha D) - \frac{2}{\eta^2} (\bar{\Lambda} \bar{\Lambda})(\bar{\Lambda} c_i D) + \frac{4}{\eta^2} (\bar{\Lambda} \bar{\Lambda})(\bar{\Lambda} d_i R) + \frac{1}{2} (\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha D) \]

\[ + \frac{4}{\eta^2} (\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha D) - \frac{1}{\eta^2} (\bar{\Lambda} R)(\Lambda^\alpha D) + \frac{1}{\eta^2} (\bar{\Lambda} \bar{\Lambda})(\bar{\Lambda} b_i R)(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha D) \]

\[ - \frac{2}{\eta} (\Lambda^\alpha c_i R)(\Lambda^\alpha D) - \frac{2}{\eta} (\bar{\Lambda} \bar{\Lambda})(\bar{\Lambda} d_i R)(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha D) \]

(F.3)

Plugging this result into (F.1), we find

\[ I^i = 4 \eta^2 (\bar{\Lambda} b_i \bar{\Lambda})(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha D) + 2 \eta^2 (\Lambda^\alpha b_i \Lambda)(\Lambda^\alpha D) \]

\[ + \frac{1}{\eta^2} (\bar{\Lambda} \bar{\Lambda})(\bar{\Lambda} b_i R)(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha D) \]

(F.4)

After applying the transformation (3.11) and using the identities (B.2), (B.3), (B.4) one can show that this expression is invariant as mentioned above.

Therefore we can replace the inverse relation (E.4) in (3.10). Let us do this for each term in (F.4):

\[ I^i_1 = \frac{8}{\eta^2} (\bar{\Lambda} \bar{\Lambda})(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha D)[(\Lambda^\alpha m \Lambda)\Sigma_{0m} + \frac{1}{\eta^2}(\Lambda^\alpha m \Lambda)(\bar{\Lambda} \bar{\Lambda})(\Lambda^\alpha k \Lambda)\Sigma_{0k}] \]

\[ = \frac{8}{\eta^2} (\bar{\Lambda} \bar{\Lambda})(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha D)[(\Lambda^\alpha m \Lambda)\Sigma_{0m} - \frac{1}{2}(\Lambda^\alpha k \Lambda)\Sigma_{0k}] \]

\[ = \frac{4}{\eta^2} (\bar{\Lambda} \bar{\Lambda})(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha D)\Sigma_{0m} \]

\[ = \frac{2}{\eta} (\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha k \Lambda)\Sigma_{0k} \]

(F.5)

\[ I^i_2 = \frac{4}{\eta^2} (\bar{\Lambda} \bar{\Lambda})(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha d_i R)[(\Lambda^\alpha d_i \Lambda)\Sigma_{0d} + \frac{1}{\eta^2}(\Lambda^\alpha d_i \Lambda)(\bar{\Lambda} \bar{\Lambda})(\Lambda^\alpha k \Lambda)\Sigma_{0k}] \]

\[ = \frac{4}{\eta^2} (\bar{\Lambda} \bar{\Lambda})(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha d_i R)[(\Lambda^\alpha d_i \Lambda)\Sigma_{0d} + \frac{1}{\eta^2}(\Lambda^\alpha d_i \Lambda)(\bar{\Lambda} \bar{\Lambda})(\Lambda^\alpha k \Lambda)\Sigma_{0k}] \]

\[ = \frac{4}{\eta}(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha d_i R)(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha k \Lambda)\Sigma_{0k} \]

(F.6)

\[ I^i_3 = \frac{1}{\eta^2} (\bar{\Lambda} \bar{\Lambda})(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha D)(\Lambda^\alpha k \Lambda) \]

\[ = \frac{2}{\eta^2} (\bar{\Lambda} \bar{\Lambda})(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha D)[(\Lambda^\alpha d_i \Lambda)\Sigma_{0d} + \frac{1}{\eta^2}(\Lambda^\alpha d_i \Lambda)(\bar{\Lambda} \bar{\Lambda})(\Lambda^\alpha k \Lambda)\Sigma_{0k}] \]

\[ = \frac{2}{\eta}(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha \bar{\Lambda})(\Lambda^\alpha k \Lambda)\Sigma_{0k} \]

(F.7)
Replacing these expressions in (F.4) and putting all together in (3.10) we obtain

\[
\{Q, \bar{\Sigma}^i\} = -\bar{P}^i - \frac{2}{\eta}[(\bar{\Lambda}^{mb} \bar{\Lambda})(\bar{\Lambda}^i_b \Lambda) - (\bar{\Lambda}^{ib} \bar{\Lambda})(\bar{\Lambda}^m_i \Lambda)]P_m + \frac{4}{\eta}(\bar{\Lambda}_{mn}R)(\bar{\Lambda}^{mn} \Lambda)\bar{\Sigma}^i
\]

\[
- \frac{2}{\eta} (\bar{\Lambda}_{mn}R)(\bar{\Lambda}^{mn} \Lambda)\bar{\Sigma}^i - \frac{\eta}{\bar{\Lambda}^{ci} \Lambda)(\Lambda^m_i \Lambda)^2 + \frac{4}{\eta}(\bar{\Lambda}_{cd}R)(\bar{\Lambda}^{ci} \Lambda)^2 \bar{\Sigma}^i
\]

\[
+ \frac{2}{\eta} (\bar{\Lambda}R)(\bar{\Lambda}^{ik} \Lambda)\bar{\Sigma}^i - \frac{2}{\eta} (\bar{\Lambda}^{cd}R)(\bar{\Lambda}^{ci} \Lambda)\bar{\Sigma}^i
\]

\[
+ \frac{4}{\eta^2}(\bar{\Lambda}_{ab}R)(\bar{\Lambda}^{ab} \Lambda)(\bar{\Lambda}^{cd}R)(\bar{\Lambda}^{cd} \Lambda)N_{qj}
\]

\[
\text{(F.8)}
\]

**G Cancellation of all of the } N_{ab} \text{ contributions in the equation (3.12)**

We will show this cancellation in two steps. First we will simplify the expression depending explicitly on } \bar{\Sigma}^i_0 \text{ and then simplify the expression depending explicitly on } N_{ab}. \text{ Finally we will see that these two expressions identically cancel out. We start with the following equation}

\[
J^i = \frac{4}{\eta}(\bar{\Lambda}^{mn}R)(\bar{\Lambda}^{mn} \Lambda)(\bar{\Lambda}^i - \bar{\Sigma}^i_0) - \frac{2}{\eta} (\bar{\Lambda}^{ci} \Lambda)(\Lambda^{m} \Lambda)\bar{\Sigma}^i_0 + \frac{4}{\eta}(\bar{\Lambda}_{cd}R)(\bar{\Lambda}^{ci} \Lambda)\bar{\Sigma}^i_0
\]

\[
+ \frac{2}{\eta} (\bar{\Lambda}R)(\bar{\Lambda}^{ik} \Lambda)\bar{\Sigma}^i_0 - \frac{2}{\eta^2}(\bar{\Lambda}^{cd}R)(\bar{\Lambda}^{ci} \Lambda)(\bar{\Lambda}^{m} \Lambda)\bar{\Sigma}^i_0
\]

\[
\text{(G.1)}
\]

Now let us focus on the contributions proportional to } \bar{\Lambda}R:\n
\[
J^i_1 = \frac{2}{\eta}(\bar{\Lambda}R)(\bar{\Lambda}^{ik} \Lambda)[-\frac{4}{3\eta^2}(\bar{\Lambda}_{ab} \bar{\Lambda})(\bar{\Lambda}^{ab} \Lambda)\bar{\Lambda}(\bar{\Lambda}^{abcd} \Lambda)N_{qj}]
\]

\[
+ \frac{4}{\eta}(\bar{\Lambda}^{mn}R)(\bar{\Lambda}^{mn} \Lambda)[\frac{2}{3\eta^2}(\bar{\Lambda}_{ab} \bar{\Lambda})(\bar{\Lambda}R)(\bar{\Lambda}^{abcd} \Lambda)N_{qj}]
\]

\[
+ \frac{4}{\eta}(\bar{\Lambda}_{cd}R)(\bar{\Lambda}^{ci} \Lambda)[-\frac{4}{3\eta^2}(\bar{\Lambda}_{ab} \bar{\Lambda})(\bar{\Lambda}R)(\bar{\Lambda}^{abcd} \Lambda)N_{qj}]
\]

\[
- \frac{4}{3\eta^2}(\bar{\Lambda}^{ik} \Lambda)(\bar{\Lambda}R)(\bar{\Lambda}^{ab} \Lambda)(\bar{\Lambda}^{cd} \Lambda)(\bar{\Lambda}R)(\bar{\Lambda}^{abcd} \Lambda)N_{qj}
\]

\[
+ \frac{8}{3\eta^2}(\bar{\Lambda}^{mn}R)(\bar{\Lambda}^{mn} \Lambda)(\bar{\Lambda}_{ab} \bar{\Lambda})(\bar{\Lambda}R)(\bar{\Lambda}^{abcd} \Lambda)N_{qj}
\]

\[
- \frac{16}{3\eta^3}(\bar{\Lambda}_{cd} \bar{\Lambda})(\bar{\Lambda}^{cd} \Lambda)(\bar{\Lambda}^{ab} \Lambda)(\bar{\Lambda}^{abcd} \Lambda)N_{qj}
\]

\[
\text{(G.2)}
\]

The last term can be written as

\[
- \frac{16}{3\eta^3}(\bar{\Lambda}_{cd} \bar{\Lambda})(\bar{\Lambda}^{cd} \Lambda)(\bar{\Lambda}_{ab} \bar{\Lambda})(\bar{\Lambda}^{abcd} \Lambda)N_{qj} = - \frac{16}{3\eta^3}(\bar{\Lambda}_{kc} \bar{\Lambda})(\bar{\Lambda}^{kij} \Lambda)(\bar{\Lambda}_{ab} \bar{\Lambda})(\bar{\Lambda}R)(\bar{\Lambda}^{abcd} \Lambda)N_{qj}
\]

\[
\text{(G.3)}
\]

Therefore,

\[
J^i_1 = \frac{12}{3\eta^3}(\bar{\Lambda}^{ik} \Lambda)(\bar{\Lambda}R)(\bar{\Lambda}_{ab} \bar{\Lambda})(\bar{\Lambda}^{ab} \Lambda)(\bar{\Lambda}^{abcd} \Lambda)N_{qj}
\]

\[
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\]
\[
J_i^1 = \frac{8}{\eta^3} (\bar{\Lambda}^m \Gamma_{mn} R)(\bar{\Lambda} \Gamma_{mn} \Lambda)(\bar{\Lambda} \Gamma_{ab} \bar{A})(\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{abqj} \Lambda)N_{qj}
\]
\[
= \frac{12}{3\eta^3} (\bar{\Lambda} \Gamma^{ik} \Lambda)(\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{ck} R)(\bar{\Lambda} R)\left[\frac{-18}{6}\right][4(\bar{\Lambda} \Gamma^{bc} \Lambda)(\bar{\Lambda} \Gamma^{a} W) + 2(\bar{\Lambda} \Gamma^{ab} \Lambda)(\bar{\Lambda} \Gamma^{c} W)]
\]
\[
+ \frac{8}{3\eta^3} (\bar{\Lambda} \Gamma^{mn} R)(\bar{\Lambda} \Gamma_{mn} \Lambda)(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{ab} \Lambda)(\bar{\Lambda} \Gamma^{c} W)
\]
\[
= -\frac{48}{\eta^3} (\bar{\Lambda} \Gamma^{ik} \Lambda)(\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{ck} R)(\bar{\Lambda} \Gamma^{ab} \Lambda)(\bar{\Lambda} \Gamma^{a} W)
\]
\[
- \frac{24}{\eta^3} (\bar{\Lambda} \Gamma^{ik} \Lambda)(\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{ck} R)(\bar{\Lambda} \Gamma^{c} W)
\]
\[
- \frac{32}{\eta^3} (\bar{\Lambda} \Gamma^{mn} R)(\bar{\Lambda} \Gamma_{mn} \Lambda)(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{ab} \Lambda)(\bar{\Lambda} \Gamma^{a} W)
\]
\[
- \frac{16}{\eta^2} (\bar{\Lambda} \Gamma^{mn} R)(\bar{\Lambda} \Gamma_{mn} \Lambda)(\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{i} W)
\]
\[
= \frac{24}{\eta^3} (\bar{\Lambda} \Gamma^{bi} \Lambda)(\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{ck} R)(\bar{\Lambda} \Gamma^{ck} \Lambda)(\bar{\Lambda} \Gamma^{a} W)
\]
\[
- \frac{24}{\eta^3} (\bar{\Lambda} \Gamma^{ik} \Lambda)(\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{ck} R)(\bar{\Lambda} \Gamma^{c} W)
\]
\[
- \frac{32}{\eta^3} (\bar{\Lambda} \Gamma^{mn} R)(\bar{\Lambda} \Gamma_{mn} \Lambda)(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{bi} \Lambda)(\bar{\Lambda} \Gamma^{a} W)
\]
\[
- \frac{16}{\eta^2} (\bar{\Lambda} \Gamma^{mn} R)(\bar{\Lambda} \Gamma_{mn} \Lambda)(\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{i} W)
\]
\[
(G.4)
\]
As a result, we get
\[
J_i^1 = \frac{8}{\eta^3} (\bar{\Lambda} \Gamma^{bi} \Lambda)(\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{ck} R)(\bar{\Lambda} \Gamma^{ck} \Lambda)(\bar{\Lambda} \Gamma^{a} W)
\]
\[
- \frac{24}{\eta^3} (\bar{\Lambda} \Gamma^{ik} \Lambda)(\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{ck} R)(\bar{\Lambda} \Gamma^{c} W)
\]
\[
- \frac{16}{\eta^2} (\bar{\Lambda} \Gamma^{mn} R)(\bar{\Lambda} \Gamma_{mn} \Lambda)(\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{i} W)
\]
\[
(G.5)
\]

Now let us focus on the term proportional to \((\bar{\Lambda} \Gamma^{a} R)\):
\[
J_i^2 = \frac{4}{\eta} (\bar{\Lambda} \Gamma^{mn} R)(\bar{\Lambda} \Gamma_{mn} \Lambda)[\frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma^{c i} R)(\bar{\Lambda} \Gamma^{abcqj} \Lambda)N_{qj}]
\]
\[
- \frac{2}{\eta^2} (\bar{\Lambda} \Gamma^{ci} R)(\bar{\Lambda} \Gamma_{ck} \Lambda)[-\frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma^{f k} R)(\bar{\Lambda} \Gamma^{abf qj} \Lambda)N_{qj}]
\]
\[
- \frac{2}{\eta^2} (\bar{\Lambda} \Gamma^{cd} R)(\bar{\Lambda} \Gamma_{cd} \Lambda)(\bar{\Lambda} \Gamma^{ka} \Lambda)(\bar{\Lambda} \Gamma_{nk} \Lambda)[-\frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma^{f k} R)(\bar{\Lambda} \Gamma^{abf qj} \Lambda)N_{qj}]
\]
\[
= \frac{8}{3\eta^3} (\bar{\Lambda} \Gamma^{mn} R)(\bar{\Lambda} \Gamma_{mn} \Lambda)(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma^{c i} R)(\bar{\Lambda} \Gamma^{abcqj} \Lambda)N_{qj}
\]
\[
+ \frac{4}{3\eta^3} (\bar{\Lambda} \Gamma^{ci} R)(\bar{\Lambda} \Gamma_{ck} \Lambda)(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma^{f k} R)(\bar{\Lambda} \Gamma^{abf qj} \Lambda)N_{qj}
\]
\[
+ \frac{4}{3\eta^3} (\bar{\Lambda} \Gamma^{cd} R)(\bar{\Lambda} \Gamma_{cd} \Lambda)(\bar{\Lambda} \Gamma^{ka} \Lambda)(\bar{\Lambda} \Gamma_{nk} \Lambda)(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma^{f k} R)(\bar{\Lambda} \Gamma^{abf qj} \Lambda)N_{qj}
\]
\[
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Now we use the identity (B.9):

\[
J_2 = \frac{6}{3\eta^3} (\bar{\Lambda}\Gamma^{mn}R)(\bar{\Lambda}\Gamma_{mn}\Lambda)(\bar{\Lambda}\Gamma_{ab}\bar{\Lambda})(\bar{\Lambda}\Gamma^i R)(\bar{\Lambda}\Gamma^{abqj}\Lambda) N_{lj}
\]

\[
+ \frac{4}{3\eta^3} (\bar{\Lambda}\Gamma^{ca} R)(\bar{\Lambda}\Gamma_{ck}\Lambda)(\bar{\Lambda}\Gamma_{ab}\bar{\Lambda})(\bar{\Lambda}\Gamma_f^k R)(\bar{\Lambda}\Gamma^{abqj}\Lambda) N_{lj}
\]

\[
+ \frac{2}{3\eta^3} (\bar{\Lambda}\Gamma^{cd} R)(\bar{\Lambda}\Gamma_{cd}\Lambda)(\bar{\Lambda}\Gamma_{ab}\bar{\Lambda})(\bar{\Lambda}\Gamma_{fi}^k R)(\bar{\Lambda}\Gamma^{abqj}\Lambda) N_{lj}
\]

\[
= \frac{6}{3\eta^3} (\bar{\Lambda}\Gamma^{mn} R)(\bar{\Lambda}\Gamma_{mn}\Lambda)(\bar{\Lambda}\Gamma_{ab}\bar{\Lambda})(\bar{\Lambda}\Gamma^i R)(\bar{\Lambda}\Gamma^{abqj}\Lambda) N_{lj}
\]

\[
+ \frac{4}{3\eta^3} (\bar{\Lambda}\Gamma^{ci} R)(\bar{\Lambda}\Gamma_{ck}\Lambda)(\bar{\Lambda}\Gamma_{ab}\bar{\Lambda})(\bar{\Lambda}\Gamma_f^k R)(\bar{\Lambda}\Gamma^{abqj}\Lambda) N_{lj}
\]

Therefore,

\[
J_2 = \frac{16}{\eta^3} (\bar{\Lambda}\Gamma^{mn} R)(\bar{\Lambda}\Gamma_{mn}\Lambda)(\bar{\Lambda}\Gamma_{ab}\bar{\Lambda})(\bar{\Lambda}\Gamma^i R)(\bar{\Lambda}\Gamma^{bc}\Lambda)(\bar{\Lambda}\Gamma^a W) - \frac{12}{\eta^2} (\bar{\Lambda}\Gamma^{mn} R)(\bar{\Lambda}\Gamma_{mn}\Lambda)(\bar{\Lambda}\Gamma_c^i R)(\bar{\Lambda}\Gamma^c W)
\]

\[
- \frac{8}{\eta^2} (\bar{\Lambda}\Gamma^{ci} R)(\bar{\Lambda}\Gamma_{ck}\Lambda)(\bar{\Lambda}\Gamma_{ab}\bar{\Lambda})(\bar{\Lambda}\Gamma_f^k R)(\bar{\Lambda}\Gamma^{fj}\Lambda)(\bar{\Lambda}\Gamma^j W)
\]

Now let us simplify the remaining terms in (G.1):

\[
J_3 = \frac{4}{\eta} (\bar{\Lambda}\Gamma^{mn} R)(\bar{\Lambda}\Gamma_{mn}\Lambda)(\bar{\Lambda}\Gamma_{cd} R)(\bar{\Lambda}\Gamma^{abckl}\Lambda) N^d_{\bar{k}l}
\]

\[
+ \frac{2}{\eta^2} (\bar{\Lambda}\Gamma_{cd} R)(\bar{\Lambda}\Gamma^{ci}\Lambda)[- \frac{2}{\eta^2} (\bar{\Lambda}\Gamma_{ab}\bar{\Lambda})(\bar{\Lambda}\Gamma_{ef} R)(\bar{\Lambda}\Gamma^{abkdl}\Lambda) N^f_k]
\]

\[
= \frac{8}{\eta^3} (\bar{\Lambda}\Gamma^{mn} R)(\bar{\Lambda}\Gamma_{mn}\Lambda)(\bar{\Lambda}\Gamma_{ab}\bar{\Lambda})(\bar{\Lambda}\Gamma_{cd} R)(\bar{\Lambda}\Gamma^{abckl}\Lambda) N^d_{\bar{k}l}
\]

\[
- \frac{8}{\eta^2} (\bar{\Lambda}\Gamma_{cd} R)(\bar{\Lambda}\Gamma^{ci}\Lambda)(\bar{\Lambda}\Gamma_{ab}\bar{\Lambda})(\bar{\Lambda}\Gamma_{ef} R)(\bar{\Lambda}\Gamma^{abkdl}\Lambda) N^f_k
\]

Now we apply the identity (B.8) to each term:

\[
J_3^{(1)} = \frac{8}{\eta^3} (\bar{\Lambda}\Gamma^{mn} R)(\bar{\Lambda}\Gamma_{mn}\Lambda)(\bar{\Lambda}\Gamma_{ab}\bar{\Lambda})(\bar{\Lambda}\Gamma_{cd} R) \left[ \frac{6}{24} [4(\bar{\Lambda}\Gamma^{ca}\Lambda)(\bar{\Lambda}\Gamma^{ab}\Gamma^{cd}\Lambda) - 4(\bar{\Lambda}\Gamma^{ca}\Lambda)(\bar{\Lambda}\Gamma^{ab}\Gamma^{cd}\Lambda) + 4(\bar{\Lambda}\Gamma^{cb}\Lambda)(\bar{\Lambda}\Gamma^{ca}\Gamma^{bd}\Lambda) - 4(\bar{\Lambda}\Gamma^{cb}\Lambda)(\bar{\Lambda}\Gamma^{ca}\Gamma^{bd}\Lambda) + 4(\bar{\Lambda}\Gamma^{cb}\Lambda)(\bar{\Lambda}\Gamma^{cb}\Gamma^{bd}\Lambda)]
\]
\[
J^{(2)}_3 = -\frac{8}{\eta^2}(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ac}\bar{A})(\bar{\Lambda}R)(\bar{\Lambda}G_{ei}\bar{A})\left(6\frac{6G_{cd}R}{24}\right)[4(\bar{\Lambda}G_{ed}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ab}\bar{A})] + 4(\bar{\Lambda}G_{ab}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ed}\Lambda)(\bar{\Lambda}G_{ab}\bar{A})] - (\bar{\Lambda}G_{ab}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ed}\Lambda)(\bar{\Lambda}G_{ab}\bar{A}) + 2(\bar{\Lambda}G_{ab}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ed}\Lambda)(\bar{\Lambda}G_{ab}\bar{A})] + 2(\bar{\Lambda}G_{ab}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ed}\Lambda)(\bar{\Lambda}G_{ab}\bar{A})]
\]

\[
= -\frac{8}{\eta^2}(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ac}\Lambda)(\bar{\Lambda}R)(\bar{\Lambda}G_{ei}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ab}\Lambda) + 4(\bar{\Lambda}G_{ab}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ed}\Lambda)(\bar{\Lambda}G_{ab}\bar{A})] - (\bar{\Lambda}G_{ab}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ed}\Lambda)(\bar{\Lambda}G_{ab}\bar{A}) + 2(\bar{\Lambda}G_{ab}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ed}\Lambda)(\bar{\Lambda}G_{ab}\bar{A})]
\]

\[
= -\frac{8}{\eta^2}(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ac}\Lambda)(\bar{\Lambda}R)(\bar{\Lambda}G_{ei}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ab}\Lambda) + 4(\bar{\Lambda}G_{ab}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ed}\Lambda)(\bar{\Lambda}G_{ab}\bar{A})] - (\bar{\Lambda}G_{ab}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ed}\Lambda)(\bar{\Lambda}G_{ab}\bar{A}) + 2(\bar{\Lambda}G_{ab}\Lambda)(\bar{\Lambda}G_{cd}R)(\bar{\Lambda}G_{ed}\Lambda)(\bar{\Lambda}G_{ab}\bar{A})]
\]
\[ J_3^i = \frac{8}{\eta^2}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) + \frac{8}{\eta^2}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) - \frac{8}{\eta^2}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) \] (G.10)

Therefore \( J_3^i \) takes the form

\[ J_3^i = \frac{8}{\eta^3}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) - \frac{4}{\eta^3}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) + \frac{16}{\eta^3}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) + \frac{16}{\eta^3}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) - \frac{16}{\eta^3}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) + \frac{8}{\eta^3}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) - \frac{8}{\eta^3}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) - \frac{24}{\eta^2}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) + \frac{16}{\eta^2}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) + \frac{8}{\eta^2}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) \] (G.11)

And we also have

\[ J_1^i + J_2^i = -\frac{16}{\eta^3}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) - \frac{12}{\eta^2}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) - \frac{8}{\eta^2}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) - \frac{8}{\eta^3}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) + \frac{24}{\eta^3}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) - \frac{16}{\eta^2}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) \] (G.12)

The sum of these quantities gives us the following result

\[ J^i = -\frac{4}{\eta^3}(\tilde{\Lambda} R)(\Lambda^e R)(\tilde{\Lambda} R)(\Lambda^i R)(\Lambda^f R)(\Lambda^g W) \]
For convenience let us focus first on the last three terms:

\[ + \frac{4}{\eta^2} (\bar{\Lambda}\Gamma_{mn} R)(\Lambda\Gamma_{mn} \Lambda)(\bar{\Lambda}\Gamma_{bd} R)(\Lambda\Gamma^{bd} i) \]

\[ - \frac{4}{\eta^2} (\bar{\Lambda}\Gamma_{mn} R)(\Lambda\Gamma_{mn} \Lambda)(\bar{\Lambda}\Gamma_{c} i R)(\Lambda\Gamma^{c} W) \]

\[ - \frac{8}{\eta^2} (\bar{\Lambda}\Gamma_{mn} R)(\Lambda\Gamma_{mn} \Lambda)(\bar{\Lambda}\Gamma_{cd} R)(\Lambda\Gamma^{cd} i) \]

\[ + \frac{16}{\eta^2} (\bar{\Lambda}\Gamma_{cd} R)(\Lambda\Gamma^{ci} \Lambda)(\bar{\Lambda}\Gamma_{ef} R)(\Lambda\Gamma^{ef} a \Lambda)(\Lambda\Gamma^{bd} i) \]

\[ + \frac{8}{\eta^2} (\bar{\Lambda}\Gamma_{cd} R)(\Lambda\Gamma^{ci} \Lambda)(\bar{\Lambda}\Gamma_{ef} R)(\Lambda\Gamma^{ef} d) \]

\[ - \frac{8}{\eta^2} (\bar{\Lambda}\Gamma_{c} i R)(\Lambda\Gamma^{ci} \Lambda)(\bar{\Lambda}\Gamma_{f k} R)(\Lambda\Gamma^{f} W) \]

where we have used that \((\bar{\Lambda}\Gamma_{mn} R)(\Lambda\Gamma_{mn} \Lambda)(\bar{\Lambda}\Gamma_{ab} R)(\Lambda\Gamma_{ab} \Lambda) = 0\). After using the identity (B.4) this expression simplifies to

\[ J^i = - \frac{4}{\eta^2} (\bar{\Lambda}\Gamma_{mn} R)(\Lambda\Gamma_{mn} \Lambda)(\bar{\Lambda}\Gamma_{bd} R)(\Lambda\Gamma^{bd} i) + \frac{4}{\eta^2} (\bar{\Lambda}\Gamma_{cd} R)(\Lambda\Gamma^{ci} \Lambda)(\bar{\Lambda}\Gamma_{ef} R)(\Lambda\Gamma^{ef} d) \]

\[ - \frac{4}{\eta^2} (\bar{\Lambda}\Gamma_{mn} R)(\Lambda\Gamma_{mn} \Lambda)(\bar{\Lambda}\Gamma_{c} i R)(\Lambda\Gamma^{c} W) - \frac{8}{\eta^2} (\bar{\Lambda}\Gamma_{c} i R)(\Lambda\Gamma^{ci} \Lambda)(\bar{\Lambda}\Gamma_{f k} R)(\Lambda\Gamma^{f} W) \]

\[ \text{(G.13)} \]

Now we will simplify the expressions containing \(N_{mn}\) explicitly:

\[ S^i = - \frac{4}{\eta^2} (\bar{\Lambda}\Gamma_{ab} R)(\bar{\Lambda}\Gamma_{cd} R)(\Lambda\Gamma^{abcdk} \Lambda) N_{d k} - \frac{4}{3\eta^2} (\bar{\Lambda}\Gamma_{ab} R)(\bar{\Lambda}\Gamma_{c} i R)(\Lambda\Gamma^{abcdk} \Lambda) N_{d k} \]

\[ - \frac{4}{3\eta^2} (\bar{\Lambda}\Gamma_{ab} R)(\bar{\Lambda} R)(\Lambda\Gamma^{abcdk} \Lambda) N_{d k} - \frac{2}{3\eta^2} (\bar{\Lambda}\Gamma_{ab} \Lambda)(\bar{\Lambda} R)(\Lambda\Gamma^{abcdk} \Lambda) N_{d k} \]

\[ \text{(G.14)} \]

For convenience let us focus first on the last three terms:

\[ S^i_2 = - \frac{4}{3\eta^2} (\bar{\Lambda}\Gamma_{ab} R)(\bar{\Lambda}\Gamma_{c} i R)(-\frac{18}{6})[4(\bar{\Lambda}\Gamma^{bc} \Lambda)(\bar{\Lambda} \Gamma^{a} W) + 2(\Lambda\Gamma^{ab} \Lambda)(\Lambda\Gamma^{c} W)] \]

\[ - \frac{4}{3\eta^2} (\bar{\Lambda}\Gamma_{ab} R)(\bar{\Lambda} R)(-\frac{18}{6})[2(\bar{\Lambda}\Gamma^{ab} \Lambda)(\bar{\Lambda} \Gamma^{i} W) + 4(\Lambda\Gamma^{bi} \Lambda)(\Lambda\Gamma^{a} W)] \]

\[ - \frac{2}{3\eta^2} (\bar{\Lambda}\Gamma_{ab} \Lambda)(\bar{\Lambda} R)(-\frac{18}{6})[2(\bar{\Lambda}\Gamma^{ab} \Lambda)(\bar{\Lambda} \Gamma^{i} W) + 4(\Lambda\Gamma^{bi} \Lambda)(\Lambda\Gamma^{a} W)] \]

\[ = \frac{16}{\eta^2} (\bar{\Lambda}\Gamma_{ab} R)(\bar{\Lambda}\Gamma_{c} i R)(\Lambda\Gamma^{bc} \Lambda)(\Lambda\Gamma^{a} W) + \frac{8}{\eta^2} (\bar{\Lambda}\Gamma_{ab} R)(\Lambda\Gamma^{bi} \Lambda)(\bar{\Lambda}\Gamma_{c} i R)(\Lambda\Gamma^{c} W) \]

\[ + \frac{8}{\eta^2} (\bar{\Lambda}\Gamma_{ab} R)(\Lambda\Gamma^{ab} \Lambda)(\bar{\Lambda} R)(\Lambda\Gamma^{i} W) + \frac{16}{\eta^2} (\bar{\Lambda}\Gamma_{ab} R)(\bar{\Lambda} R)(\Lambda\Gamma^{bi} \Lambda)(\Lambda\Gamma^{a} W) \]

\[ + \frac{4}{\eta^2} (\bar{\Lambda} R)(\Lambda\Gamma^{i} W) + \frac{8}{\eta^2} (\bar{\Lambda}\Gamma_{ab} \Lambda)(\bar{\Lambda} R)(\Lambda\Gamma^{bi} \Lambda)(\Lambda\Gamma^{a} W) \]

\[ \text{(G.16)} \]

The same manipulations for the first term of \(S^i\) give us

\[ S^i_1 = - \frac{4}{\eta^2} (\bar{\Lambda}\Gamma_{ab} R)(\bar{\Lambda}\Gamma_{cd} R)(\frac{6}{24})[4(\bar{\Lambda}\Gamma^{ci} \Lambda)(\Lambda\Gamma^{ab} T^{d} W) - 4(\Lambda\Gamma^{ca} \Lambda)(\Lambda\Gamma^{ib} T^{d} W) \]

\[ + 4(\Lambda\Gamma^{cb} \Lambda)(\Lambda\Gamma^{a} T^{d} W) + 4(\Lambda\Gamma^{ab} \Lambda)(\Lambda\Gamma^{ci} T^{d} W) - 4(\Lambda\Gamma^{bi} \Lambda)(\Lambda\Gamma^{ca} T^{d} W) + 4(\Lambda\Gamma^{ia} \Lambda)(\Lambda\Gamma^{ch} T^{d} W)] \]

\[ \text{(G.17)} \]
Therefore we get

\[- \frac{4}{\eta^2} (\bar{\Lambda} R)(\bar{\Lambda} R)(\bar{\Lambda} R)(\bar{\Lambda} R)(\bar{\Lambda} R)(\bar{\Lambda} R)(\bar{\Lambda} R)(\bar{\Lambda} R)\]

\[+ 2(\bar{\Lambda} \bar{\Lambda})\]

\[= \frac{4}{\eta^2} (\bar{\Lambda} b R)(\bar{\Lambda} a R)(\bar{\Lambda} a R)(\bar{\Lambda} a R)(\bar{\Lambda} a R)(\bar{\Lambda} a R)(\bar{\Lambda} a R)(\bar{\Lambda} a R)\]

\[+ 2(\bar{\Lambda} \bar{\Lambda})\]

\[\text{G.17}\]

Thus we have a full cancellation

\[\text{H Calculation of } \{\bar{\Sigma}^i, \bar{\Sigma}^j\}\]

The object \(\bar{\Sigma}^i\) has a part depending on \(D_\alpha\) and other part depending on \(N_{mn}\), as it can be seen in (3.9). The part depending on \(N_{mn}\) will be called \(\bar{\Sigma}_0^i\) and as before we use \(\bar{\Sigma}_0^i\) to denote the part depending on \(D_\alpha\). Therefore

\[\bar{\Sigma}^i = \bar{\Sigma}_0^i + \bar{\Sigma}_1^i\]

\[\text{H.1}\]

It is easy to see that \(\{\bar{\Sigma}_0^i, \bar{\Sigma}_0^j\} = 0\):

\[\{\bar{\Sigma}_0^i, \bar{\Sigma}_0^j\} = \left\{ \frac{1}{2\eta} (\bar{\Lambda} \bar{\Lambda}) (\bar{\Lambda} \bar{\Lambda} D), \frac{1}{2\eta} (\bar{\Lambda} \bar{\Lambda}) (\bar{\Lambda} \bar{\Lambda} D) \right\} \]

\[= \frac{1}{4\eta^2} (\bar{\Lambda} \bar{\Lambda}) (\bar{\Lambda} \bar{\Lambda}) (\bar{\Lambda} \bar{\Lambda} D) (\bar{\Lambda} \bar{\Lambda} D) \{D_\alpha, D_\beta\}\]
using the identity (B.3).

The next step is to compute the anticommutator \( \{ \bar{\Sigma}_0^j, \bar{\Sigma}_1^j \} \). To this end let us write \( \bar{\Sigma}_1^j \) explicitly:

\[
\bar{\Sigma}_1^j = \frac{2}{\eta^2} (\bar{\Lambda} \bar{\Gamma}^{ab} \bar{A})(\bar{\Lambda} \Gamma_{cd} R)(\bar{\Lambda} \Gamma^{abcd} \bar{A}) N^d_k + \frac{2}{3 \eta^2} (\bar{\Lambda} \bar{\Gamma}^{ab} \bar{A})(\bar{\Lambda} \Gamma^c R)(\bar{\Lambda} \Gamma^{abc} \bar{A}) N_k d \\
+ \frac{2}{3 \eta^2} (\bar{\Lambda} \bar{\Gamma}^{ab} \bar{A})(\bar{A} R)(\bar{\Lambda} \Gamma^{abcd} \bar{A}) N_d k
\]

and denote each term by \( \bar{\Sigma}_1^{(1)} \), \( \bar{\Sigma}_1^{(2)} \), \( \bar{\Sigma}_1^{(3)} \), respectively. It can be shown that \( \{ \bar{\Sigma}_0^j, \bar{\Sigma}_1^{(1)} \} = 0 \). Now we rewrite \( \bar{\Sigma}_1^{(2)} \), \( \bar{\Sigma}_1^{(3)} \) in a more convenient way:

\[
\bar{\Sigma}_1^{(2)} = -\frac{4}{\eta} (\bar{A} R)(\bar{\Lambda} \Gamma^{ij} \bar{A}) W - \frac{8}{\eta^2} (\bar{\Lambda} \bar{\Gamma}^{ab} \bar{A})(\bar{\Lambda} \Gamma^c R)(\bar{\Lambda} \Gamma^{ca} \bar{A}) (\bar{\Lambda} \Gamma^{b} W) \\
\bar{\Sigma}_1^{(3)} = -\frac{4}{\eta} (\bar{A} R)(\bar{\Lambda} \Gamma^{ij} \bar{A}) W - \frac{8}{\eta^2} (\bar{\Lambda} \bar{\Gamma}^{ab} \bar{A})(\bar{A} R)(\bar{\Lambda} \Gamma^{ja} \bar{A}) (\bar{\Lambda} \Gamma^{b} W)
\]

after using the identity (B.9). Therefore

\[
\{ \bar{\Sigma}_0^i, \bar{\Sigma}_1^{(2)} \} = \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{mn} \bar{A})(\bar{\Lambda} \Gamma^{ij} \bar{A}) R(\bar{\Lambda} \Gamma^{mn} \Gamma^i D) + \frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{mn} \bar{A})(\bar{\Lambda} \Gamma^{ab} \bar{A})(\bar{\Lambda} \Gamma^{ca} \bar{A}) (\bar{\Lambda} \Gamma^{b} \Gamma^{mn} \Gamma^i D) \\
= -\frac{8}{\eta^2} (\bar{\Lambda} \Gamma^{ab} \bar{A})(\bar{\Lambda} \Gamma^{ij} \bar{A}) (\bar{\Lambda} \Gamma^{ca} \bar{A}) (\bar{\Lambda} \Gamma^{b} \Gamma^{mn} \Gamma^i D) \\
+ \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{ab} \bar{A})(\bar{\Lambda} \Gamma^{ij} \bar{A}) (\bar{\Lambda} \Gamma^{ca} \bar{A}) (\bar{\Lambda} \Gamma^{mn} \Gamma^i D) \\
+ \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{ij} \bar{A})(\bar{A} R)(\bar{\Lambda} \Gamma^{mn} \bar{A}) (\bar{\Lambda} \Gamma^{ca} \bar{A}) (\bar{\Lambda} \Gamma^{b} \Gamma^{mn} \Gamma^i D) \\
+ \frac{2}{\eta^2} (\bar{\Lambda} \Gamma^{ij} \bar{A})(\bar{A} R)(\bar{\Lambda} \Gamma^{mn} \bar{A}) (\bar{\Lambda} \Gamma^{ca} \bar{A}) (\bar{\Lambda} \Gamma^{b} \Gamma^{mn} \Gamma^i D)
\]

and by doing the same for \( \bar{\Sigma}_1^{(3)} \) we obtain

\[
\{ \bar{\Sigma}_0^i, \bar{\Sigma}_1^{(3)} \} = \frac{2}{\eta^2} (\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{mn} \bar{A})(\bar{\Lambda} \Gamma^{ij} \Gamma^{mn} \Gamma^i D) + \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{ab} \bar{A})(\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{ja} \bar{A})(\bar{\Lambda} \Gamma^{ab} \bar{A})(\bar{\Lambda} \Gamma^{mn} \Gamma^i D) \\
= -\frac{8}{\eta^2} (\bar{\Lambda} \Gamma^{ab} \bar{A})(\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{ja} \bar{A})(\bar{\Lambda} \Gamma^{ab} \bar{A})(\bar{\Lambda} \Gamma^{mn} \Gamma^i D) \\
+ \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{ab} \bar{A})(\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{ja} \bar{A})(\bar{\Lambda} \Gamma^{ab} \bar{A})(\bar{\Lambda} \Gamma^{mn} \Gamma^i D)
\]
Hence we get

\[
\begin{align*}
\{\Sigma^i_0, \Sigma^j_1\} &= -\frac{4}{\eta^2} (\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{i j} \bar{\Lambda})(\Lambda D) - \frac{4}{\eta^2} (\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{i} \bar{\Lambda})(\Lambda^{i} \eta D) \\
&\quad - \frac{4}{\eta^2} (\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{i} \bar{\Lambda})(\Lambda^{i} \eta D) + \frac{4}{\eta^2} \eta^{ij} (\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{mn} \bar{\Lambda})(\Lambda^{i} \eta m D) \\
&\quad - \frac{4}{\eta^2} (\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{mn} \bar{\Lambda})(\Lambda^{i} \eta m D)
\end{align*}
\]  

(H.7)

Analogously we obtain

\[
\begin{align*}
\{\Sigma^i_1, \Sigma^j_0\} &= -\frac{4}{\eta^2} (\bar{\Lambda} R)(\bar{\Lambda} \Gamma^{i} \bar{\Lambda})(\Lambda^{i} \eta D) + \frac{2}{\eta^2} (\bar{\Lambda} \Gamma^{i} \bar{\Lambda})(\Lambda^{i} \eta D) \\
&\quad + \frac{2}{\eta^2} \eta^{ij} (\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{mn} \bar{\Lambda})(\Lambda^{i} \eta m D) - \frac{2}{\eta^2} (\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{mn} \bar{\Lambda})(\Lambda^{i} \eta m D)
\end{align*}
\]  

(H.8)

and thus the sum of (H.8) and (H.9) is

\[
\begin{align*}
\{\Sigma^i_0, \Sigma^j_1\} + \{\Sigma^i_1, \Sigma^j_0\} &= -\frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{i} \bar{\Lambda})(\Lambda^{i} \eta D) - \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{i} \bar{\Lambda})(\Lambda^{i} \eta D) \\
&\quad + \frac{2}{\eta^2} (\bar{\Lambda} \Gamma^{i} \bar{\Lambda})(\Lambda^{i} \eta D) + \frac{2}{\eta^2} (\bar{\Lambda} \Gamma^{i} \bar{\Lambda})(\Lambda^{i} \eta D) \\
&\quad - \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{i} \bar{\Lambda})(\Lambda^{i} \eta D) - \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{i} \bar{\Lambda})(\Lambda^{i} \eta D)
\end{align*}
\]  

(H.10)

Now we will simplify the expression corresponding to \(\Sigma^j_1\):

\[
\Sigma^j_1 = \frac{2}{\eta^2} (\bar{\Lambda} \Gamma^{abc} \bar{\Lambda})(\Lambda^{c} \eta d R)(\Lambda^{c} \eta d R)(\Lambda^{c} \eta d R)(\Lambda^{c} \eta d R)(\Lambda^{c} \eta d R)
\]

\[
- \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{c} \bar{\Lambda})(\Lambda^{c} \eta d R) - \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{c} \bar{\Lambda})(\Lambda^{c} \eta d R) - \frac{8}{\eta^2} (\bar{\Lambda} \Gamma^{c} \bar{\Lambda})(\Lambda^{c} \eta d R)(\Lambda^{c} \eta d R)
\]  

(H.11)

Let us call \(Y^j_1\) to the first term of this expression and expand it as follows

\[
Y^j_1 = \left(\frac{6}{24}\right)(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} \bar{\Lambda})(\Lambda^{abc} \eta d W) - 4(\Lambda^{abc} \eta d W) + 4(\Lambda^{abc} \eta d W)
\]

\[
+ 4(\Lambda^{abc} \eta d W) - 4(\Lambda^{abc} \eta d W) + 4(\Lambda^{abc} \eta d W)
\]

\[
= \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} \bar{\Lambda})(\Lambda^{abc} \eta d W) - 2(\Lambda^{abc} \eta d W) + (\Lambda^{abc} \eta d W)
\]

\[\text{(H.11)}\]
\[-2(\Lambda \Gamma^{jb} \Lambda)(\Lambda \Gamma^{ca} \Gamma^d \Lambda)\]

\[= \frac{2}{\eta^2}(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R) [2\eta^{bd}(\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{a} \Lambda) + (\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{abd} \Lambda) + 2\eta^{dj}(\Lambda \Gamma^{ca} \Lambda)(\Lambda \Gamma^{b} \Lambda) \]

\[-2\eta^{bd}(\Lambda \Gamma^{ca} \Lambda)(\Lambda \Gamma^{j} \Lambda) - 2(\Lambda \Gamma^{ca} \Lambda)(\Lambda \Gamma^{bdj} \Lambda) + \eta^{dj}(\Lambda \Gamma^{ab} \Lambda)(\Lambda \Gamma^{c} \Lambda) - \eta^{cd}(\Lambda \Gamma^{ab} \Lambda)(\Lambda \Gamma^{j} \Lambda) \]

\[-(\Lambda \Gamma^{ab} \Lambda)(\Lambda \Gamma^{bdj} \Lambda) + 2\eta^{cd}(\Lambda \Gamma^{jb} \Lambda)(\Lambda \Gamma^{c} \Lambda) - 2\eta^{cd}(\Lambda \Gamma^{jb} \Lambda)(\Lambda \Gamma^{c} \Lambda) + 2(\Lambda \Gamma^{jb} \Lambda)(\Lambda \Gamma^{a} \Lambda)]\]

\[= \frac{4}{\eta^2}(\bar{\Lambda} \Gamma_{ac} \bar{\Lambda})(\bar{\Lambda} R)(\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{a} \Lambda) + \frac{2}{\eta^2}(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{a} \Lambda) \]

\[+ \frac{4}{\eta^2}(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma^{cj} \Lambda)(\Lambda \Gamma^{a} \Lambda)(\Lambda \Gamma^{b} \Lambda) + \frac{4}{\eta^2}(\bar{\Lambda} \Gamma_{ac} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{a} \Lambda) \]

\[+ \frac{2}{\eta^2}(\bar{\Lambda} \Gamma_{cj} \bar{\L}(\Lambda \Gamma^{c} \Lambda) - \frac{2}{\eta^2}(\bar{\Lambda} \Gamma_{cd} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{a} \Lambda)(\Lambda \Gamma^{b} \Lambda) \]

\[= \frac{4}{\eta^2}(\bar{\Lambda} \Gamma_{ac} \bar{\Lambda})(\bar{\Lambda} R)(\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{a} \Lambda) + \frac{2}{\eta^2}(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{a} \Lambda) \]

\[+ \frac{2}{\eta^2}(\bar{\Lambda} \Gamma_{ac} \bar{\Lambda})(\bar{\Lambda} \Gamma^{cj} \Lambda)(\Lambda \Gamma^{a} \Lambda)(\Lambda \Gamma^{b} \Lambda) + \frac{4}{\eta^2}(\bar{\Lambda} \Gamma_{cd} \bar{\Lambda})(\bar{\Lambda} \Gamma^{cj} \Lambda)(\Lambda \Gamma^{a} \Lambda)(\Lambda \Gamma^{b} \Lambda) \]

\[+ \frac{4}{\eta^2}(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{bj} \Lambda)(\Lambda \Gamma^{a} \Lambda)(\Lambda \Gamma^{bd} \Lambda) \]

\[\tag{H.12} \]

Plugging this result into the equation (H.11)

\[\Sigma_{ij} = -\frac{2}{\eta^2}(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{bd} \Lambda) - \frac{1}{\eta}(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{cd} \Lambda) \]

\[+ \frac{2}{\eta^2}(\bar{\Lambda} \Gamma^{bj} \Lambda)(\bar{\Lambda} \Gamma^{ac} \Lambda)(\Lambda \Gamma^{a} \Lambda)(\Lambda \Gamma^{b} \Lambda) + \frac{4}{\eta^2}(\bar{\Lambda} \Gamma^{cj} \Lambda)(\bar{\Lambda} \Gamma^{bd} \Lambda)(\Lambda \Gamma^{a} \Lambda)(\Lambda \Gamma^{b} \Lambda) \]

\[- \frac{1}{\eta^2}(\bar{\Lambda} \Gamma^{bd} \Lambda)(\bar{\Lambda} \Gamma_{ac} \Lambda)(\Lambda \Gamma^{b} \Lambda)(\Lambda \Gamma^{c} \Lambda) - \frac{8}{\eta^2}(\bar{\Lambda} \Gamma_{ab} \Lambda)(\bar{\Lambda} \Gamma^{j} \Lambda)(\Lambda \Gamma^{bd} \Lambda)(\Lambda \Gamma^{a} \Lambda) \]

\[\frac{4}{\eta^2}(\bar{\Lambda} \Gamma^{b} \Lambda)(\Lambda \Gamma^{c} \Lambda) \]

\[= -\frac{2}{\eta^2}(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{bd} \Lambda) - \frac{1}{\eta}(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{cd} \Lambda) \]

\[+ \frac{2}{\eta^2}(\bar{\Lambda} \Gamma^{bj} \Lambda)(\bar{\Lambda} \Gamma^{ac} \Lambda)(\Lambda \Gamma^{a} \Lambda)(\Lambda \Gamma^{b} \Lambda) + \frac{4}{\eta^2}(\bar{\Lambda} \Gamma^{cj} \Lambda)(\bar{\Lambda} \Gamma^{bd} \Lambda)(\Lambda \Gamma^{a} \Lambda)(\Lambda \Gamma^{b} \Lambda) \]

\[- \frac{1}{\eta^2}(\bar{\Lambda} \Gamma^{bd} \Lambda)(\bar{\Lambda} \Gamma_{ac} \Lambda)(\Lambda \Gamma^{b} \Lambda)(\Lambda \Gamma^{c} \Lambda) - \frac{8}{\eta^2}(\bar{\Lambda} \Gamma_{ab} \Lambda)(\bar{\Lambda} \Gamma^{j} \Lambda)(\Lambda \Gamma^{bd} \Lambda)(\Lambda \Gamma^{a} \Lambda) \]

\[+ \frac{4}{\eta^2}(\bar{\Lambda} \Gamma^{b} \Lambda)(\Lambda \Gamma^{c} \Lambda) \]

\[= -\frac{2}{\eta^2}(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{bd} \Lambda) - \frac{1}{\eta}(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{cd} \Lambda) \]

\[+ \frac{2}{\eta^2}(\bar{\Lambda} \Gamma^{bj} \Lambda)(\bar{\Lambda} \Gamma^{ac} \Lambda)(\Lambda \Gamma^{a} \Lambda)(\Lambda \Gamma^{b} \Lambda) + \frac{4}{\eta^2}(\bar{\Lambda} \Gamma^{cj} \Lambda)(\bar{\Lambda} \Gamma^{bd} \Lambda)(\Lambda \Gamma^{a} \Lambda)(\Lambda \Gamma^{b} \Lambda) \]

\[- \frac{1}{\eta^2}(\bar{\Lambda} \Gamma^{bd} \Lambda)(\bar{\Lambda} \Gamma_{ac} \Lambda)(\Lambda \Gamma^{b} \Lambda)(\Lambda \Gamma^{c} \Lambda) - \frac{8}{\eta^2}(\bar{\Lambda} \Gamma_{ab} \Lambda)(\bar{\Lambda} \Gamma^{j} \Lambda)(\Lambda \Gamma^{bd} \Lambda)(\Lambda \Gamma^{a} \Lambda) \]

\[+ \frac{4}{\eta^2}(\bar{\Lambda} \Gamma^{b} \Lambda)(\Lambda \Gamma^{c} \Lambda) \]
\[
\frac{4}{\eta^2}(\bar{\Lambda} \Gamma^{mn} \bar{\Lambda})(\bar{\Lambda} \Gamma^{nr} R)(\Lambda \Gamma_{nr} \Lambda)(\Lambda \Gamma_m W) + \frac{8}{\eta^2}(\bar{\Lambda} \Gamma^{rj} \bar{\Lambda})(\bar{\Lambda} \Gamma^{mn} R)(\Lambda \Gamma_{nr} \Lambda)(\Lambda \Gamma_m W) - \frac{4}{\eta}(\bar{\Lambda} \Gamma^{cj} R)(\Lambda \Gamma_c W)
\]

This expression is invariant under the gauge symmetry generated by the pure spinor constraint, as it should. Now let us make the following definitions:

\[
W_{1j}^j = -\frac{2}{\eta^2}(\bar{\Lambda} \Gamma^{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma^{cd} R)(\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{ab} W)
\]

\[
W_{2j}^j = -\frac{1}{\eta}(\bar{\Lambda} \Gamma^{cd} R)(\Lambda \Gamma^{cdj} W)
\]

\[
W_{3j}^j = \frac{2}{\eta^2}(\bar{\Lambda} \Gamma^{bj} \bar{\Lambda})(\bar{\Lambda} \Gamma^{ca} R)(\Lambda \Gamma_{ca} \Lambda)(\Lambda \Gamma^{b} W)
\]

\[
W_{4j}^j = -\frac{4}{\eta^2}(\bar{\Lambda} \Gamma^{cj} \bar{\Lambda})(\bar{\Lambda} \Gamma^{ba} R)(\Lambda \Gamma_{ca} \Lambda)(\Lambda \Gamma^{b} W)
\]

\[
W_{5j}^j = \frac{1}{\eta^2}(\bar{\Lambda} \Gamma^{bd} \bar{\Lambda})(\bar{\Lambda} \Gamma^{ac} R)(\Lambda \Gamma^{ac} \Lambda)(\Lambda \Gamma^{bd} W)
\]

Consequently to compute \(\{\bar{\Sigma}_1^j, \bar{\Sigma}_2^j\}\) we should calculate the anticommutator between each pair of these \(W_{1,2,3,4,5}^j\) variables. Explicitly this computation works as follows

\[
\{W_{1j}^j, W_{1j}^j\} = \{-\frac{2}{\eta^2}(\bar{\Lambda} \Gamma^{mn} \bar{\Lambda})(\bar{\Lambda} \Gamma^{rs} R)(\Lambda \Gamma^{ri} \Lambda)(\Lambda \Gamma^{mns} W), -\frac{2}{\eta^2}(\bar{\Lambda} \Gamma^{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma^{cd} R)(\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{ab} W)\}
\]

\[
= \frac{4}{\eta^4}(\bar{\Lambda} \Gamma^{mn} \bar{\Lambda})(\bar{\Lambda} \Gamma^{rs} R)(\Lambda \Gamma^{ab} \Lambda)(\bar{\Lambda} \Gamma^{cd} R)[(\Lambda \Gamma^{ri} \Lambda)(\Lambda \Gamma^{mns} W), (\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{ab} W)]
\]

\[
= \frac{4}{\eta^4}(\bar{\Lambda} \Gamma^{mn} \bar{\Lambda})(\bar{\Lambda} \Gamma^{rs} R)(\Lambda \Gamma^{ab} \Lambda)(\bar{\Lambda} \Gamma^{cd} R)((\Lambda \Gamma^{ri} \Lambda)(\Lambda \Gamma^{mns} W))((\Lambda \Gamma^{cj} \Lambda)(\Lambda \Gamma^{ab} W))
\]

\[
+ (\Lambda \Gamma^{ri} \Lambda)((\Lambda \Gamma^{mns} W), (\Lambda \Gamma^{cj} \Lambda))(\Lambda \Gamma^{ab} W) + (\Lambda \Gamma^{cj} \Lambda)((\Lambda \Gamma^{ri} \Lambda), (\Lambda \Gamma^{ab} W))(\Lambda \Gamma^{mns} W))
\]

\[
= \frac{64}{\eta^4}(\bar{\Lambda} \Gamma^{mn} \bar{\Lambda})(\bar{\Lambda} \Gamma^{rs} R)(\Lambda \Gamma^{ab} \Lambda)(\bar{\Lambda} \Gamma^{cd} R)(\Lambda \Gamma^{ri} \Lambda)(\Lambda \Gamma^{cj} \Lambda)[\delta_{cd}^{mns} N^{nd}]
\]

\[
= \frac{32}{\eta^4}(\bar{\Lambda} \Gamma^{dm} \bar{\Lambda})(\bar{\Lambda} \Gamma^{rs} R)(\Lambda \Gamma^{ab} \Lambda)(\bar{\Lambda} \Gamma^{cd} R)(\Lambda \Gamma^{ri} \Lambda)(\Lambda \Gamma^{cj} \Lambda) N^{nd}
\]

\[
= 0
\]

because of the identities (B.28) and \((\bar{\Lambda} R)(\bar{\Lambda} R) = 0\).

\[
\{W_{2j}^j, W_{2j}^j\} = \{-\frac{1}{\eta}(\bar{\Lambda} \Gamma^{mn} R)(\Lambda \Gamma^{mni} W), -\frac{1}{\eta}(\bar{\Lambda} \Gamma^{cd} R)(\Lambda \Gamma^{cdj} W)\}
\]

\[
= \frac{1}{\eta^2}(\bar{\Lambda} \Gamma^{mn} R)(\bar{\Lambda} \Gamma^{cd} R)[(\Lambda \Gamma^{mni} W), (\Lambda \Gamma^{cdj} W)]
\]
\[
\begin{align*}
&= \frac{1}{\eta^2} (\bar{\Lambda} \Gamma_{mn} R)(\bar{\Lambda} \Gamma_{cd} R)(4\delta_{cd} N^{ij} - 8\delta_{c}^{mn} N^{id} - 8\delta_{d}^{mi} N^{nj} - 16\delta_{c}^{nj} N^{md}) \\
&= -\frac{8}{\eta^2} (\bar{\Lambda} \Gamma R)(\bar{\Lambda} \Gamma_{cd} R)N^{id} - \frac{8}{\eta^2} (\bar{\Lambda} \Gamma_{mn} R)(\bar{\Lambda} \Gamma_{mi} R)N^{nj} \\
&\quad - \frac{8}{\eta^2} (\bar{\Lambda} \Gamma_{c} R)(\bar{\Lambda} \Gamma_{cd} R)\eta^{ij} + \frac{8}{\eta^2} (\bar{\Lambda} \Gamma_{mn} R)(\bar{\Lambda} \Gamma^d R)N_{md} \\
&\quad (H.20)
\end{align*}
\]

The use of the identity (B.5) allows us to write
\[
\begin{align*}
\{W_2', W_2''\} &= -\frac{8}{\eta^2} (\bar{\Lambda} \Gamma^{jd} R)(\bar{\Lambda} R) + \frac{1}{2} (\bar{\Lambda} \Gamma^{jd} \bar{\Lambda})(RR))N^i_d \\
&\quad + \frac{8}{\eta^2} (\bar{\Lambda} \Gamma^{mi} R)(\bar{\Lambda} R) + \frac{1}{2} (\bar{\Lambda} \Gamma^{mi} \bar{\Lambda})(RR))N^j_n \\
&\quad + \frac{8}{\eta^2} \eta^{ij} (\bar{\Lambda} \Gamma^{md} R)(\bar{\Lambda} R) + \frac{1}{2} (\bar{\Lambda} \Gamma^{md} \bar{\Lambda})(RR))N_{md} \\
&\quad + \frac{8}{\eta^2} (\bar{\Lambda} \Gamma^{mj} R)(\bar{\Lambda} \Gamma^{id} R)N_{md} \\
&\quad (H.21)
\end{align*}
\]

\[
\begin{align*}
\{W_3', W_3''\} &= \left\{ \frac{2}{\eta^2} (\bar{\Lambda} \Gamma^{mi} \bar{\Lambda})(\bar{\Lambda} \Gamma^{rn} R)(\bar{\Lambda} \Gamma_{rn} \bar{\Lambda})(\bar{\Lambda} \Gamma_{rn} W), \frac{2}{\eta^2} (\bar{\Lambda} \Gamma^{bj} \bar{\Lambda})(\bar{\Lambda} \Gamma^{ca} R)(\bar{\Lambda} \Gamma_{ca} \bar{\Lambda})(\bar{\Lambda} \Gamma_{b} W) \right\} \\
&= -\frac{8}{\eta^2} (\bar{\Lambda} \Gamma^{mi} \bar{\Lambda})(\bar{\Lambda} \Gamma^{rn} R)(\bar{\Lambda} \Gamma^{bj} \bar{\Lambda})(\bar{\Lambda} \Gamma^{ca} R)(\bar{\Lambda} \Gamma_{ca} \bar{\Lambda})(\bar{\Lambda} \Gamma_{b} W)N_{nb} \\
&= 0 \\
&\quad (H.22)
\end{align*}
\]

because of the identities (B.26) and \((\bar{\Lambda} \Gamma_{mn} \bar{\Lambda})(\bar{\Lambda} \Gamma^{mn} R)(\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma^{ab} R) = 0\).

\[
\begin{align*}
\{W_4', W_4''\} &= \left\{ -\frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{ri} \bar{\Lambda})(\bar{\Lambda} \Gamma^{nm} R)(\bar{\Lambda} \Gamma_{rn} \bar{\Lambda})(\bar{\Lambda} \Gamma_{a} W), -\frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{cj} \bar{\Lambda})(\bar{\Lambda} \Gamma^{ba} R)(\bar{\Lambda} \Gamma_{ca} \bar{\Lambda})(\bar{\Lambda} \Gamma_{b} W) \right\} \\
&= -\frac{32}{\eta^2} (\bar{\Lambda} \Gamma^{ri} \bar{\Lambda})(\bar{\Lambda} \Gamma^{nm} R)(\bar{\Lambda} \Gamma^{cj} \bar{\Lambda})(\bar{\Lambda} \Gamma^{ba} R)(\bar{\Lambda} \Gamma_{ca} \bar{\Lambda})(\bar{\Lambda} \Gamma_{b} W)N_{nb} \\
&\quad (H.23)
\end{align*}
\]

which follows directly from the identity (B.26)

\[
\begin{align*}
\{W_5', W_5''\} &= \left\{ \frac{1}{\eta^2} (\bar{\Lambda} \Gamma_{ns} \bar{\Lambda})(\bar{\Lambda} \Gamma^{mr} R)(\bar{\Lambda} \Gamma_{mr} \bar{\Lambda})(\bar{\Lambda} \Gamma^{ns} \bar{\Lambda} W), \frac{1}{\eta^2} (\bar{\Lambda} \Gamma_{bd} \bar{\Lambda})(\bar{\Lambda} \Gamma^{ac} R)(\bar{\Lambda} \Gamma_{ac} \bar{\Lambda})(\bar{\Lambda} \Gamma^{bd} \bar{\Lambda} W) \right\} \\
&= \frac{1}{\eta^2} (\bar{\Lambda} \Gamma_{ns} \bar{\Lambda})(\bar{\Lambda} \Gamma_{bd} \bar{\Lambda})(\bar{\Lambda} \Gamma^{mr} R)(\bar{\Lambda} \Gamma^{ac} R)(\bar{\Lambda} \Gamma_{ac} \bar{\Lambda})(\bar{\Lambda} \Gamma^{bd} \bar{\Lambda} W) \\
&\quad [\bar{\Lambda} \Gamma^{ns} \bar{\Lambda} W], (\bar{\Lambda} \Gamma^{bd} \bar{\Lambda} W)] \\
&\quad (H.23)
\end{align*}
\]

\[
\begin{align*}
\{W_5', W_5''\} &= \frac{8}{\eta^2} (\bar{\Lambda} \Gamma^{mj} \bar{\Lambda})(\bar{\Lambda} \Gamma^{id} \bar{\Lambda})(\bar{\Lambda} \Gamma^{mr} R)(\bar{\Lambda} \Gamma^{ac} R)(\bar{\Lambda} \Gamma_{mr} \bar{\Lambda})(\bar{\Lambda} \Gamma_{ac} \bar{\Lambda})N_{md}
\end{align*}
\]
because of the identity $(\Lambda \Gamma_{mn}\Lambda)(\bar{\Lambda} \Gamma^{mn} R)(\Lambda \Gamma_{ab}\Lambda)(\bar{\Lambda} \Gamma^{ab} R) = 0$.

\[ \{W^1_1, W^1_2\} = \{- \frac{2}{\eta^2} (\Lambda \Gamma_{mn}\bar{\Lambda})(\bar{\Lambda} \Gamma_{rs} R)(\Lambda \Gamma^{ri}\Lambda)(\Lambda \Gamma^{mns} W), - \frac{1}{\eta} (\Lambda \Gamma_{cd} R)(\Lambda \Gamma^{cdj} W)\} \]

\[ = \frac{2}{\eta^3} (\Lambda \Gamma_{mn}\bar{\Lambda})(\bar{\Lambda} \Gamma_{rs} R)(\Lambda \Gamma_{cd} R)[(\Lambda \Gamma^{ri}\Lambda)(\Lambda \Gamma^{mns} W), (\Lambda \Gamma^{cdj} W)] \]

\[ + [(\Lambda \Gamma^{ri}\Lambda), (\Lambda \Gamma^{cdj} W)](\Lambda \Gamma^{mns} W) \]

\[ = \frac{2}{\eta^3} (\Lambda \Gamma_{mn}\bar{\Lambda})(\bar{\Lambda} \Gamma_{rs} R)(\Lambda \Gamma_{cd} R)[-8\delta_{c_m}^{m_s} + 8\delta_{c_d}^{m_s} - 16\delta_{c_j}^{m_s} N^{md}] + \frac{4}{\eta^3} (\Lambda \Gamma_{mn}\bar{\Lambda})(\bar{\Lambda} \Gamma_{rs} R)(\Lambda \Gamma_{cd} R)\Lambda \Gamma^{cdj\eta} W)(\Lambda \Gamma^{mns} W) \]

\[ = - \frac{16}{\eta^3} (\bar{\Lambda} \Gamma^{cdj\eta} W)(\Lambda \Gamma_{rs} R)(\bar{\Lambda} \Gamma^{ri}\Lambda) N^{sd} + \frac{16}{\eta^3} (\bar{\Lambda} \Gamma_{mn}\bar{\Lambda})(\bar{\Lambda} \Gamma_{rs} R)(\Lambda \Gamma^{cd} R)(\Lambda \Gamma^{ri}\Lambda) N^{mj} \]

\[ - \frac{16}{\eta^3} (\bar{\Lambda} \Gamma_{mn}\bar{\Lambda})(\bar{\Lambda} \Gamma_{rs} R)(\Lambda \Gamma^{cd} R)(\Lambda \Gamma^{ri}\Lambda) N^{mj} + \frac{4}{\eta^3} (\Lambda \Gamma_{mn}\bar{\Lambda})(\bar{\Lambda} \Gamma_{rs} R)(\Lambda \Gamma_{cd} R)\Lambda \Gamma^{cdj\eta} W)(\Lambda \Gamma^{mns} W) \]

\[ = \frac{32}{\eta^3} (\bar{\Lambda} \Gamma_{d}^{j\eta} W)(\Lambda \Gamma_{rs} R)(\bar{\Lambda} \Gamma^{ri}\Lambda) N^{sd} + \frac{16}{\eta^3} (\bar{\Lambda} \Gamma_{sd}^{j\eta} W)(\Lambda \Gamma^{ri}\Lambda) N^{sd} \]

\[ - \frac{16}{\eta^3} (\bar{\Lambda} \Gamma_{d}^{j\eta} W)(\Lambda \Gamma_{rs} R)(\Lambda \Gamma^{ri}\Lambda) N^{ds} + \frac{4}{\eta^3} (\Lambda \Gamma_{mn}\bar{\Lambda})(\bar{\Lambda} \Gamma_{rs} R)(\Lambda \Gamma_{cd} R)\Lambda \Gamma^{cdj\eta} W)(\Lambda \Gamma^{mns} W) \]

\[ (H.25) \]

\[ \{W^3_1, W^3_2\} = \{- \frac{2}{\eta^2} (\Lambda \Gamma_{mn}\bar{\Lambda})(\bar{\Lambda} \Gamma_{rs} R)(\Lambda \Gamma^{ri}\Lambda)(\Lambda \Gamma^{mns} W), 2 \frac{2}{\eta^2} (\Lambda \Gamma_{rj}^{cdj\eta} W)(\Lambda \Gamma^{co} R)(\Lambda \Gamma_{ca} \Lambda)(\Lambda \Gamma_{b} W)\} \]

\[ = \frac{8}{\eta^4} (\Lambda \Gamma_{mn}\bar{\Lambda})(\bar{\Lambda} \Gamma_{rs} R)(\bar{\Lambda} \Gamma^{co} R)(\Lambda \Gamma_{ca} \Lambda)(\Lambda \Gamma^{mns} W) \]

\[ = 0 \]

\[ (H.26) \]

where we have used the identities (B.2), (B.27).
\[-\frac{16}{\eta^3} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_r \Lambda) (\bar{\Lambda}_s R) (\bar{\Lambda}^b j) R (\bar{\Lambda}^{ri} \Lambda) (\Lambda^{mns} b W) \]
\[-\frac{16}{\eta^3} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_r \Lambda) (\bar{\Lambda}_s R) (\bar{\Lambda}^b j) R (\bar{\Lambda}^{ri} \Lambda) (\Lambda^{mns} b W) \]  

because of the identity (B.2).

\[
\{W_1^i, W_3^j\} = \left\{ -\frac{2}{\eta} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_r \Lambda) (\Lambda^{mns} W), \frac{1}{\eta^2} (\bar{\Lambda}_b j \bar{\Lambda}) (\bar{\Lambda}_c \Lambda) (\bar{\Lambda}^{ac} \Lambda) (\Lambda^{b dj} W) \right\} 
\]
\[
= -\frac{2}{\eta^2} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_r \Lambda) (\bar{\Lambda}_b \bar{\Lambda}) (\bar{\Lambda}_c \Lambda) [[(\Lambda^{r i} \Lambda) (\Lambda^{mns} W), (\Lambda^{ac} \Lambda) (\Lambda^{b dj} W)]]
\]
\[
= -\frac{2}{\eta^2} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_r \Lambda) (\bar{\Lambda}_b \bar{\Lambda}) (\bar{\Lambda}_c \Lambda) \{[(\Lambda^{r i} \Lambda) (\Lambda^{ac} \Lambda) [(\Lambda^{mns} W), (\Lambda^{b dj} W)]
\]
\[
+ [(\Lambda^{ac} \Lambda) [(\Lambda^{r i} \Lambda), (\Lambda^{b dj} W)]] (\Lambda^{mns} W) \}
\]
\[
= -\frac{4}{\eta^2} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_r \Lambda) (\bar{\Lambda}_b \bar{\Lambda}) (\bar{\Lambda}_c \Lambda) (\Lambda^{ac} \Lambda) (\Lambda^{b d r i} \Lambda) (\Lambda^{mns} W)
\]
\[
= -\frac{2}{\eta^2} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_r \Lambda) (\bar{\Lambda}_b \bar{\Lambda}) (\bar{\Lambda}_c \Lambda) (\Lambda^{ac} \Lambda) (\Lambda^{b r i} \Lambda) (\Lambda^{mns} W) [16 \delta_{b i} W]
\]
\[
= \frac{16}{\eta^2} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_r \Lambda) (\bar{\Lambda}_b \bar{\Lambda}) (\bar{\Lambda}_c \Lambda) (\Lambda^{ac} \Lambda) (\Lambda^{b r i} \Lambda) N_{nd}
\]
\[
= \frac{8}{\eta^2} (\bar{\Lambda}_r \Lambda) (\bar{\Lambda}^{nd} \Lambda) (\bar{\Lambda} R) (\bar{\Lambda}^{ac} \Lambda) (\Gamma^{b r i} \Lambda) (\Lambda^{mns} W) \]

\[
\{W_2^i, W_3^j\} = \left\{ -\frac{1}{\eta} (\bar{\Lambda}_m \bar{\Lambda}) (\Lambda^{m n i} W), \frac{2}{\eta^2} (\bar{\Lambda}_b j \bar{\Lambda}) (\bar{\Lambda}^{ca} \Lambda) (\Lambda^{b} W) \right\} 
\]
\[
= \left\{ -\frac{1}{\eta} (\bar{\Lambda}_m \bar{\Lambda}) (\Lambda^{m n i} W), -\frac{4}{\eta^2} (\bar{\Lambda}_b j \bar{\Lambda}) (\bar{\Lambda}^{ca} \Lambda) (\Lambda^{b} W) \right\} 
\]
\[
= \frac{4}{\eta^3} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_c j \bar{\Lambda}) (\Lambda^{b} W) [2 (\Lambda^{ca} \Lambda) (\Lambda^{mnib} W) - 2 (\Lambda^{mnica} \Lambda) (\Lambda^{a} W)]
\]
\[
= \frac{8}{\eta^3} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_c j \bar{\Lambda}) (\Lambda^{b} W) (\Lambda^{mnib} W)
\]
\[
= \frac{8}{\eta^3} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_c j \bar{\Lambda}) (\Lambda^{b} W) (\Lambda^{mnica} \Lambda) (\Lambda^{b} W) \]  

\[
\{W_2^i, W_3^j\} = \left\{ -\frac{1}{\eta} (\bar{\Lambda}_m \bar{\Lambda}) (\Lambda^{m n i} W), \frac{1}{\eta^2} (\bar{\Lambda}_b j \bar{\Lambda}) (\bar{\Lambda}^{ca} \Lambda) (\Lambda^{b} W) \right\} 
\]
\[
= \left\{ -\frac{1}{\eta} (\bar{\Lambda}_m \bar{\Lambda}) (\Lambda^{m n i} W), -\frac{4}{\eta^2} (\bar{\Lambda}_b j \bar{\Lambda}) (\bar{\Lambda}^{ca} \Lambda) (\Lambda^{b} W) \right\} 
\]
\[
= \left\{ -\frac{1}{\eta^3} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_c j \bar{\Lambda}) (\Lambda^{b} W) [\Lambda^{m n i} W), (\Lambda^{b} W) \right\} 
\]
\[
= \left\{ -\frac{1}{\eta^3} (\bar{\Lambda}_m \bar{\Lambda}) (\bar{\Lambda}_c j \bar{\Lambda}) (\Lambda^{b} W) [\Lambda^{m n i} W), (\Lambda^{b} W) \right\} 
\]
\[
= \frac{8}{\eta^3} (\bar{\Lambda}_r \Lambda) (\bar{\Lambda}^{nd} \Lambda) (\bar{\Lambda} R) (\bar{\Lambda}^{ac} \Lambda) (\Lambda^{b} W) N_{nd} \]  

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\[
\{ W_{3}, W_{4} \} = \left\{ \frac{2}{\eta} \bar{\Gamma}^{mn} \bar{\Lambda} \bar{\Gamma}^{rm} R (\bar{\Gamma}_{rm} \Lambda) (\bar{\Gamma}_{n} W), - \frac{4}{\eta} \bar{\Gamma}^{cqj} \bar{\Lambda} \bar{\Gamma}^{bca} R (\bar{\Gamma}_{ca} \Lambda) (\bar{\Gamma}_{b} W) \right\}
\]
\[
= \frac{16}{\eta} (\bar{\Gamma}^{mn} \bar{\Lambda} \bar{\Gamma}^{rm} R (\bar{\Gamma}_{rm} \Lambda) (\bar{\Gamma}^{cqj} \bar{\Lambda} \bar{\Gamma}^{bca} R (\bar{\Gamma}_{ca} \Lambda) N_{ab}) \quad (H.32)
\]

\[
\{ W_{3}, W_{5} \} = \left\{ \frac{2}{\eta} (\bar{\Gamma}^{mn} \bar{\Lambda} \bar{\Gamma}^{rm} R (\bar{\Gamma}_{rm} \Lambda) (\bar{\Gamma}_{n} W), \frac{1}{\eta} (\bar{\Gamma}_{bd} \bar{\Lambda} \bar{\Gamma}_{ac} R (\bar{\Gamma}^{ac} \Lambda) (\bar{\Gamma}^{bdj} W) \right\}
\]
\[
= 0 \quad (H.33)
\]

\[
\{ W_{4}, W_{5} \} = \left\{ - \frac{4}{\eta} (\bar{\Gamma}^{ri} \bar{\Lambda} \bar{\Gamma}^{mn} R (\bar{\Gamma}_{rm} \Lambda) (\bar{\Gamma}_{n} W), \frac{1}{\eta} (\bar{\Gamma}_{bd} \bar{\Lambda} \bar{\Gamma}_{ac} R (\bar{\Gamma}^{ac} \Lambda) (\bar{\Gamma}^{bdj} W) \right\}
\]
\[
= - \frac{8}{\eta} (\bar{\Gamma}^{ri} \bar{\Lambda} \bar{\Gamma}_{n} R (\bar{\Gamma}_{bd} \Lambda) (\bar{\Gamma}_{ac} R (\bar{\Gamma}^{ac} \Lambda) (\bar{\Gamma}^{bdj} W) \right\}
\]
\[
= - \frac{4}{\eta} (\bar{\Gamma}^{ri} \bar{\Lambda} R (\bar{\Gamma}_{bd} \Lambda) (\bar{\Gamma}_{ac} R (\bar{\Gamma}^{ac} \Lambda) (\bar{\Gamma}^{bdj} W) \right\}
\]
\[
= \frac{8}{\eta} \bar{\Lambda} (\bar{\Gamma}^{dij} R (\bar{\Gamma}_{bd} \Lambda) (\bar{\Gamma}_{ac} R (\bar{\Gamma}^{ac} \Lambda) (\bar{\Gamma}^{bdj} W) \right\}
\]
\[
= \frac{8}{\eta} \bar{\Lambda} (\bar{\Gamma}^{dij} R (\bar{\Gamma}_{bd} \Lambda) (\bar{\Gamma}_{ac} R (\bar{\Gamma}^{ac} \Lambda) (\bar{\Gamma}^{bdj} W) \right\}
\]
\[
= \frac{32}{\eta} \bar{\Lambda} (\bar{\Gamma}_{d} R (\bar{\Gamma}_{bd} \Lambda) (\bar{\Gamma}_{ac} R (\bar{\Gamma}^{ac} \Lambda) (\bar{\Gamma}_{b} W) \right\}
\]

Putting all together, the result is

\[
\{ \Sigma^{i}, \Sigma^{j} \} = - \frac{4}{\eta^2} (\bar{\Gamma}^{mn} \bar{\Lambda} \bar{\Gamma}^{mj} R (\bar{\Gamma}_{cn} D) + \frac{2}{\eta^2} (\bar{\Gamma}^{mn} \bar{\Lambda} \bar{\Gamma}_{c} R (\bar{\Gamma}^{cj} R (\bar{\Gamma}_{cm} D)\right)
\]
\[
+ \frac{2}{\eta^2} (\bar{\Gamma}^{mn} \bar{\Lambda} \bar{\Gamma}_{c} R (\bar{\Gamma}^{jmn} D) + \frac{2}{\eta^2} (\bar{\Gamma}^{mn} \bar{\Lambda} \bar{\Gamma}_{c} R (\bar{\Gamma}^{cjm} D)\right)
\]
\[
- \frac{4}{\eta^2} (\bar{\Gamma}^{n} \bar{\Lambda} \bar{\Gamma}_{n} R (\bar{\Gamma}^{jn} D) - \frac{4}{\eta^2} (\bar{\Gamma}^{j} \bar{\Lambda} \bar{\Gamma}_{n} R (\bar{\Gamma}^{jn} D) + \frac{4}{\eta^2} \bar{\Lambda} (\bar{\Gamma}^{mn} \bar{\Lambda} \bar{\Gamma}_{n} R (\bar{\Gamma}^{mn} \bar{\Lambda}) (\bar{\Gamma}^{jn} D)\right)
\]
\[
- \frac{8}{\eta^2} (\bar{\Gamma}^{dij} R (\bar{\Gamma}_{bd} \Lambda) (\bar{\Gamma}_{ac} R (\bar{\Gamma}^{ac} \Lambda) (\bar{\Gamma}^{bdj} W) \right\}
\]
\[
+ \frac{8}{\eta^2} \bar{\Lambda} (\bar{\Gamma}^{dij} R (\bar{\Gamma}_{bd} \Lambda) (\bar{\Gamma}_{ac} R (\bar{\Gamma}^{ac} \Lambda) (\bar{\Gamma}^{bdj} W) \right\}
\]
\[
+ \frac{32}{\eta^2} \bar{\Lambda} (\bar{\Gamma}_{d} R (\bar{\Gamma}_{bd} \Lambda) (\bar{\Gamma}_{ac} R (\bar{\Gamma}^{ac} \Lambda) (\bar{\Gamma}_{b} W) \right\}
\]

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One of the useful things that can be extracted from this result is the fact that \( \{ \tilde{\Sigma}^i, \tilde{\Sigma}'^j \} \) depends linearly and quadratically on \( R_\alpha \). This allows us to find the \( R_\alpha \)-dependence of \( \{ b, b \} \) which it turns out to be of the form:

\[
\{ b, b \} = R^\alpha f_\alpha^{(1)} + \ldots + R^\alpha R^\beta R^\rho R^\lambda \bar{f}_\alpha^{(6)} f_{\beta\delta\sigma\rho\lambda} \tag{H.36}
\]

where \( f_\alpha^{(i)} \), for \( i = 1, \ldots, 6 \) are functions of pure spinor variables \( \Lambda^\alpha, \bar{\Lambda}_\alpha, W_\alpha \) and the fermionic constraints \( D_\alpha \).
This can be used to check that \( \{b, b\} = Q\Omega \) where \( \Omega \) is an arbitrary function of pure spinor variables and the constraints \( D_\alpha \). To see this let us expand \( \Omega \) in terms of \( R^\alpha \):

\[
\Omega = \Omega^{(0)} + R^\alpha \Omega^{(1)}(\alpha) + R^{\alpha\beta} \Omega^{(2)}(\alpha\beta) + \ldots + R^{\alpha_1 \ldots \alpha_{23}} \Omega^{(23)}(\alpha_1 \ldots \alpha_{23}) \quad (H.37)
\]

Thus the action of the BRST operator \( Q = Q_0 + R^\alpha W_\alpha \) on \( \Omega \) gives us

\[
Q\Omega = Q_0 \Omega^{(0)} + R^\alpha (\frac{\partial}{\partial \Lambda^\alpha} \Omega^{(0)} + Q_0 \Omega^{(1)}(\alpha)) + R^{\alpha\beta} (\frac{\partial}{\partial \Lambda^\alpha} \Omega^{(1)}(\beta) + Q_0 \Omega^{(2)}(\alpha\beta)) + \ldots \quad (H.38)
\]

The comparison of this result with the equation \((H.36)\) determines the functions \( \Omega^{(k)} \) for \( k = 1, \ldots, 23 \):

\[
0 = Q_0 \Omega^{(0)} \quad (H.39)
\]

\[
f^{(1)}_{\alpha} = \frac{\partial}{\partial \Lambda^\alpha} \Omega^{(0)} + Q_0 \Omega^{(1)}(\alpha) \quad (H.40)
\]

\[
f^{(2)}_{\alpha\beta} = \frac{\partial}{\partial \Lambda^\alpha} \Omega^{(1)}(\beta) + Q_0 \Omega^{(2)}(\alpha\beta) \quad (H.41)
\]

\[
\vdots
\]

Therefore if we make the following definitions:

\[
\Omega^{(0)} = \bar{\Lambda}^\alpha f^{(1)}_{\alpha} \quad (H.42)
\]

\[
\Omega^{(1)}_{\beta} = \bar{\Lambda}^\alpha f^{(2)}_{\alpha\beta} \quad (H.43)
\]

\[
\Omega^{(2)}_{\beta\delta} = \bar{\Lambda}^\alpha f^{(3)}_{\alpha\beta\delta} \quad (H.44)
\]

\[
\vdots
\]

\[
\Omega^{(5)}_{\beta\delta\sigma\rho\lambda} = \bar{\Lambda}^\alpha f^{(6)}_{\alpha\beta\delta\sigma\rho\lambda} \quad (H.45)
\]

\[
\Omega^{(6)}_{\beta\delta\sigma\rho\lambda \gamma} = 0 \quad (H.46)
\]

\[
\vdots
\]

\[
\Omega^{(23)} = 0 \quad (H.47)
\]

the equations above are automatically solved.

**I Expanding the simplified \( D = 11 \) b-ghost**

In this Appendix we will reproduce the terms contained in \( O(\bar{\Sigma}^2) \) in the expression for the simplified \( D = 11 \) b-ghost. First we will reproduce the quadratic term in \( D_\alpha \) in the expression for the b-ghost \((3.2)\)

\[
\eta^{-2} L^{(1)}_{ab,cd}(\bar{\Lambda} \Gamma^a D)(\bar{\Lambda} \Gamma^{bcd} D) \quad (I.1)
\]

We will work with the expression

\[
b_{\text{simpl}} = R^i \bar{\Sigma}_i - \frac{4}{\eta^2} (\bar{\Lambda} \Gamma^{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\bar{\Lambda} \Gamma^{ad} \bar{\Lambda})(\bar{\Lambda} \Gamma^{bcd} \bar{\Lambda})(\bar{\Lambda} \Gamma^{c} \bar{\Lambda})(\bar{\Lambda} \Gamma_{sk} \bar{\Lambda})(\bar{\Lambda} \Gamma^{s} \bar{\Lambda}) \quad (I.2)
\]
It is useful to write $\Sigma_0^i$ in the convenient way:

$$\Sigma_0^i = \frac{1}{2\eta}[(\bar{\Lambda}_G^a \bar{\Lambda})(\Lambda^i D) + 4(\bar{\Lambda}_G^a \bar{\Lambda})(\Lambda^a D)] \tag{I.3}$$

Therefore we have

$$\langle \Lambda^i \Lambda \rangle \Sigma_0^j = \frac{2}{\eta} \langle \Lambda^i \Lambda \rangle (\bar{\Lambda}^a \bar{\Lambda}) \langle \Lambda^a \rangle \tag{I.4}$$

which is a direct consequence of the identity (B.1). Now we will expand $\Sigma^i$ as it was done in (3.9):

$$\Sigma^i = \Sigma_0^i + \frac{2}{\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{gh} \Lambda)(\Lambda^e f g i j \Lambda)N^d_k + \frac{2}{\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{ij} \Lambda)(\Lambda^e f g i j \Lambda)N_{qj} - \frac{2}{3\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{ij} \Lambda)(\Lambda^e f g i j \Lambda)N_{qj} \tag{I.5}$$

Using this equation we can write $\Sigma^j$, $\Sigma^c$, $\Sigma^k$ in the following way:

$$\Sigma^j = \Sigma_0^j + \frac{2}{\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{gh} \Lambda)(\Lambda^e f g i j \Lambda)N^h_i + \frac{2}{\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{ij} \Lambda)(\Lambda^e f g i h \Lambda)N_{hi} - \frac{2}{3\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{ij} \Lambda)(\Lambda^e f g i h \Lambda)N_{hi} \tag{I.6}$$

$$\Sigma^c = \Sigma_0^c + \frac{2}{\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{gh} \Lambda)(\Lambda^e f g i j \Lambda)N^p_q + \frac{2}{\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{ij} \Lambda)(\Lambda^e f g i h \Lambda)N_{pq} - \frac{2}{3\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{ij} \Lambda)(\Lambda^e f g i h \Lambda)N_{pq} \tag{I.7}$$

$$\Sigma^k = \Sigma_0^k + \frac{2}{\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{gh} \Lambda)(\Lambda^e f g i j \Lambda)N^w_i + \frac{2}{\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{ij} \Lambda)(\Lambda^e f g i h \Lambda)N_{iy} - \frac{2}{3\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{ij} \Lambda)(\Lambda^e f g i h \Lambda)N_{iy} \tag{I.8}$$

Replacing these expressions in (I.2) we have

$$b_{\text{simpl}} = p^i \Sigma_i - \frac{4}{\eta^2} (\bar{\Lambda} G_{ab} \bar{\Lambda})(\bar{\Lambda} G_{cd} \Lambda)(\Lambda^a \Lambda)\Sigma^j_0 + \frac{2}{\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{gh} \Lambda)(\Lambda^e f g i j \Lambda)N^q_i + \frac{2}{3\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{ij} \Lambda)(\Lambda^e f g i h \Lambda)N_{hi} + \frac{2}{3\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{ij} \Lambda)(\Lambda^e f g i h \Lambda)N_{hi} \times \{\Sigma_0^c + \frac{2}{\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{gh} \Lambda)(\Lambda^e f g i j \Lambda)N^p_q + \frac{2}{\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{ij} \Lambda)(\Lambda^e f g i h \Lambda)N_{pq} - \frac{2}{3\eta^2} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\bar{\Lambda} G_{ij} \Lambda)(\Lambda^e f g i h \Lambda)N_{pq} \}
\frac{1}{\eta} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\Lambda^e f g i j \Lambda)N^d_k + \frac{1}{\eta} (\bar{\Lambda} G_{ef} \bar{\Lambda})(\Lambda^e f g i j \Lambda)N^d_k \tag{I.9}$$

Hence the contributions proportional to $D^2$ are:

$$b_{\text{simpl}}^{(2)} = -\frac{4}{\eta^2} (\bar{\Lambda} G_{ab} \bar{\Lambda})(\bar{\Lambda} G_{cd} \Lambda)(\Lambda^a \Lambda)\Sigma_0^j + \frac{1}{\eta} (\bar{\Lambda} G^a \Lambda)(\bar{\Lambda} G_{cs} \Lambda)(\Lambda^a \Lambda)\Sigma_0^k.$$
and then on the part proportional to \( \eta \). This term can be calculated in two steps. First we focus on the part proportional to \( (\Lambda \Gamma^a) \) which will be called \( K_1 \) and then on the part proportional to \( (\Lambda \Gamma^{bd}) \) which will be called \( K_2 \). Thus

\[
K_1 = -\frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{cd} \Lambda)(\Lambda \Gamma^{a}) \Lambda \Sigma_{0j} \Sigma_{0k}^c \\
+ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{lm} \bar{\Lambda})(\bar{\Lambda} \Gamma_n \Lambda R)(\Lambda \Gamma^{lmnpq} \Lambda)N_{pq} - \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ln} \bar{\Lambda})(\bar{\Lambda} R)(\Lambda \Gamma^{lmpnq} \Lambda)N_{pq} \\
+ \frac{1}{\eta} (\bar{\Lambda} \Gamma_{s} \Lambda)(\Lambda \Gamma^{s} \Lambda)\left[ \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{rt} \bar{\Lambda})(\Lambda \Gamma_{uw} R)(\Lambda \Gamma^{rtyw} \Lambda) \right] N_{wy} \\
+ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{rt} \bar{\Lambda})(\Lambda \Gamma^{k} \Lambda R)(\Lambda \Gamma^{rtuw} \Lambda)N_{wy} - \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ru} \bar{\Lambda})(\bar{\Lambda} R)(\Lambda \Gamma^{rkuw} \Lambda)N_{wy} \right] 
\]

This term can be calculated in two steps. First we focus on the part proportional to \( (\Lambda \Gamma^a) \) which will be called \( K_1 \) and then on the part proportional to \( (\Lambda \Gamma^{bd}) \) which will be called \( K_2 \). Thus

\[
K_1 = -\frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{cd} \Lambda)(\Lambda \Gamma^{a}) \Lambda \Sigma_{0j} \Sigma_{0k}^c \\
+ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{lm} \bar{\Lambda})(\bar{\Lambda} \Gamma_n \Lambda R)(\Lambda \Gamma^{lmnpq} \Lambda)N_{pq} - \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ln} \bar{\Lambda})(\bar{\Lambda} R)(\Lambda \Gamma^{lmpnq} \Lambda)N_{pq} \\
+ \frac{1}{\eta} (\bar{\Lambda} \Gamma_{s} \Lambda)(\Lambda \Gamma^{s} \Lambda)\left[ \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{rt} \bar{\Lambda})(\Lambda \Gamma_{uw} R)(\Lambda \Gamma^{rtyw} \Lambda) \right] N_{wy} \\
+ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{rt} \bar{\Lambda})(\Lambda \Gamma^{k} \Lambda R)(\Lambda \Gamma^{rtuw} \Lambda)N_{wy} - \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ru} \bar{\Lambda})(\bar{\Lambda} R)(\Lambda \Gamma^{rkuw} \Lambda)N_{wy} \right] 
\]

We can use the identity (1.4) to simplify this expression:

\[
K_1 = -\frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab} \bar{\Lambda})(\bar{\Lambda} \Gamma_{cd} R)(\Lambda \Gamma^{cd} \Lambda)(\Lambda \Gamma^{a}) \Lambda \Sigma_{0j} \Sigma_{0k}^c \\
+ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{lm} \bar{\Lambda})(\bar{\Lambda} \Gamma_n \Lambda R)(\Lambda \Gamma^{lmnpq} \Lambda)N_{pq} - \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ln} \bar{\Lambda})(\bar{\Lambda} R)(\Lambda \Gamma^{lmpnq} \Lambda)N_{pq} \\
- \frac{1}{\eta} (\bar{\Lambda} \Gamma_{s} \Lambda)(\Lambda \Gamma^{s} \Lambda)\left[ \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{rt} \bar{\Lambda})(\Lambda \Gamma_{uw} R)(\Lambda \Gamma^{rtyw} \Lambda) \right] N_{wy} \\
+ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{rt} \bar{\Lambda})(\Lambda \Gamma^{k} \Lambda R)(\Lambda \Gamma^{rtuw} \Lambda)N_{wy} - \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ru} \bar{\Lambda})(\bar{\Lambda} R)(\Lambda \Gamma^{rkuw} \Lambda)N_{wy} \right] 
\]
\[
+ \frac{1}{\eta} (\bar{\Lambda} \Gamma_s \bar{\Lambda})(\Lambda \Gamma_s \bar{\Lambda}) \{ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{rt \bar{\Lambda}})(\bar{\Lambda} \Gamma_{u^k R})(\Lambda \Gamma^{rtau_{wp}} \bar{\Lambda})N_{wy} \} \\
= - \frac{8}{\eta^3} (\bar{\Lambda} \Gamma_{ab \bar{\Lambda}})(\Lambda \Gamma_{cd R})(\Lambda \Gamma^{bd \bar{\Lambda}})(\Lambda \Gamma^{xj \bar{\Lambda}})(\Lambda \Gamma^{x \bar{\Lambda}} D) \times \{ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ln \bar{\Lambda}})(\bar{\Lambda} \Gamma_{np R})(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})N_{pq} \\
+ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{lm \bar{\Lambda}})(\bar{\Lambda} \Gamma_{n c \bar{\Lambda}})(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})N_{pq} - \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ln \bar{\Lambda}})(\bar{\Lambda} \Gamma_{R})(\Lambda \Gamma^{incpq} \bar{\Lambda})N_{pq} \\
+ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma^{c \bar{\Lambda}})(\Lambda \Gamma^{x \bar{\Lambda}} D)(\Lambda \Gamma^{bd \bar{\Lambda}}) \times \{ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{lm \bar{\Lambda}})(\bar{\Lambda} \Gamma_{np R})(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})N_{pq} \\
+ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{lm \bar{\Lambda}})(\bar{\Lambda} \Gamma_{n c \bar{\Lambda}})(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})N_{pq} - \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ln \bar{\Lambda}})(\bar{\Lambda} \Gamma_{R})(\Lambda \Gamma^{incpq} \bar{\Lambda})N_{pq} \\
+ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma^{c \bar{\Lambda}})(\Lambda \Gamma^{x \bar{\Lambda}} D)(\Lambda \Gamma^{bd \bar{\Lambda}}) \} \ (I.11)
\]

Now it is useful to use the following identity which is followed from (B.4):
\[
(\bar{\Lambda} \Gamma_{x \bar{\Lambda}})(\Lambda \Gamma_{cd R})(\Lambda \Gamma^{bd \bar{\Lambda}}) = \frac{1}{2} (\bar{\Lambda} \Gamma_{xc \bar{\Lambda}})(\Lambda \Gamma_{bd \bar{\Lambda}}) + \frac{1}{2} (\bar{\Lambda} \Gamma_{bd \bar{\Lambda}})(\Lambda \Gamma_{xc \bar{\Lambda}}) + (\bar{\Lambda} \Gamma_{cd \bar{\Lambda}})(\Lambda \Gamma_{bx \bar{\Lambda}})(\Lambda \Gamma^{bd \bar{\Lambda}}) \ (I.12)
\]

With the additional use of (B.2) we obtain
\[
K_1 = - \frac{4}{\eta^3} (\bar{\Lambda} \Gamma_{xc R})(\Lambda \Gamma^{x \bar{\Lambda}} D)(\bar{\Lambda} \Gamma_{lm \bar{\Lambda}})(\Lambda \Gamma_{np R})(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})N_{pq} \\
- \frac{4}{3\eta^3} (\bar{\Lambda} \Gamma_{xc R})(\Lambda \Gamma^{x \bar{\Lambda}} D)(\bar{\Lambda} \Gamma_{lm \bar{\Lambda}})(\Lambda \Gamma_n ^c \bar{\Lambda})(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})N_{pq} \\
+ \frac{4}{3\eta^3} (\bar{\Lambda} \Gamma_{xc R})(\Lambda \Gamma^{x \bar{\Lambda}} D)(\bar{\Lambda} \Gamma_{lm \bar{\Lambda}})(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})(\Lambda \Gamma^{x \bar{\Lambda}} D)N_{pq} \\
- \frac{2}{3\eta^3} (\bar{\Lambda} \Gamma_{lm \bar{\Lambda}})(\bar{\Lambda} \Gamma_{xc R})(\Lambda \Gamma^{x \bar{\Lambda}} D)(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})(\Lambda \Gamma^{x \bar{\Lambda}} D)N_{pq} \\
- \frac{2}{3\eta^3} (\bar{\Lambda} \Gamma_{lm \bar{\Lambda}})(\bar{\Lambda} \Gamma_{xc R})(\Lambda \Gamma^{x \bar{\Lambda}} D)(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})(\Lambda \Gamma^{x \bar{\Lambda}} D)N_{pq} \\
= \frac{4}{\eta^3} (\bar{\Lambda} \Gamma_{lm \bar{\Lambda}})(\bar{\Lambda} \Gamma_{cx R})(\Lambda \Gamma_{np R})(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})(\Lambda \Gamma^{x \bar{\Lambda}} D)N_{pq} \\
- \frac{2}{\eta^3} (\bar{\Lambda} \Gamma_{lm \bar{\Lambda}})(\bar{\Lambda} \Gamma_{xm R})(\bar{\Lambda} \Gamma_{np R})(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})(\Lambda \Gamma^{x \bar{\Lambda}} D)N_{pq} \\
= \frac{4}{\eta^3} (\bar{\Lambda} \Gamma_{lm \bar{\Lambda}})(\bar{\Lambda} \Gamma_{cx R})(\Lambda \Gamma_{np R})(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})(\Lambda \Gamma^{x \bar{\Lambda}} D)N_{pq} \\
- \frac{2}{\eta^3} (\bar{\Lambda} \Gamma_{lm \bar{\Lambda}})(\bar{\Lambda} \Gamma_{nx R})(\bar{\Lambda} \Gamma_{np R})(\Lambda \Gamma^{lnmnpq} \bar{\Lambda})(\Lambda \Gamma^{x \bar{\Lambda}} D)N_{pq} \ (I.13)
\]
Now let us move on to compute the term proportional to \((\Lambda \Gamma^{abc} D)\), this term comes from the following contribution:

\[
K_2 = -\frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab} \bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd} R) (\bar{\Lambda} \Gamma^{bd} \Lambda) (\bar{\Lambda} \Gamma^{aj} \Lambda) \left[ \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{ef} \bar{\Lambda}) (\bar{\Lambda} \Gamma_{gh} R) (\bar{\Lambda} \Gamma^{efgij} \Lambda) \right] N^{hi}_i \\
+ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ef} \bar{\Lambda}) (\bar{\Lambda} \Gamma_{g} \bar{\Lambda}) (\bar{\Lambda} \Gamma^{efg} \Lambda) N_{hi} - \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{eg} \bar{\Lambda}) (\bar{\Lambda} R) (\bar{\Lambda} \Gamma^{e} \Lambda) N_{hi} \times \left[ \frac{1}{\eta} (\bar{\Lambda} \Gamma^{ca} \Lambda) (\bar{\Lambda} \Gamma^{sk} \Lambda) \Sigma_0 k \right] \\
- \frac{2}{\eta} (\bar{\Lambda} \Gamma_{ac} R) (\bar{\Lambda} \Gamma_{aj} \Lambda) \left[ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ef} \bar{\Lambda}) (\bar{\Lambda} \Gamma_{j} \Lambda) (\bar{\Lambda} \Gamma^{efg} \Lambda) N_{hi} \times \left[ \bar{\Sigma}_{c0} \right] \right] \\
= -\frac{2}{\eta} (\bar{\Lambda} \Gamma_{ac} R) (\bar{\Lambda} \Gamma_{aj} \Lambda) \left[ \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ef} \bar{\Lambda}) (\bar{\Lambda} \Gamma_{j} \Lambda) (\bar{\Lambda} \Gamma^{efg} \Lambda) N_{hi} \times \left[ \bar{\Lambda} \Gamma_{mn} \Lambda \right] \right] \\
+ \frac{1}{\eta} (\bar{\Lambda} \Gamma^{ca} \Lambda) (\bar{\Lambda} \Gamma^{sk} \Lambda) \Sigma_0 k \tag{1.14}
\]

where we have just used the identities (B.4) and (B.7). Therefore

\[
K_2 = -\frac{4}{3\eta^3} (\bar{\Lambda} \Gamma_{ac} R) (\bar{\Lambda} \Gamma^{a} \Lambda) [(\bar{\Lambda} \Gamma_{ef} \bar{\Lambda}) (\bar{\Lambda} \Gamma_{g} \Lambda) (\bar{\Lambda} \Gamma^{efgh} \Lambda) N_{hi} \times \left[ \frac{1}{2\eta} (\bar{\Lambda} \Gamma_{mn} \Lambda) (\bar{\Lambda} \Gamma^{mn} D) \right] \\
- \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ac} R) (\bar{\Lambda} \Gamma^{a} \Lambda) [(\bar{\Lambda} \Gamma_{ef} \bar{\Lambda}) (\bar{\Lambda} \Gamma_{g} \Lambda) (\bar{\Lambda} \Gamma^{efgh} \Lambda) N_{hi} \times (\bar{\Lambda} \Gamma_{mn} \Lambda) (\bar{\Lambda} \Gamma^{mn} D) \\
+ \frac{1}{3\eta^3} (\bar{\Lambda} \Gamma_{ac} \Lambda) (\bar{\Lambda} \Gamma_{ge} R) (\bar{\Lambda} \Gamma_{ef} R) (\bar{\Lambda} \Gamma^{efgh} \Lambda) N_{hi} (\bar{\Lambda} \Gamma^{mn} D) \\
- \frac{1}{3\eta^3} (\bar{\Lambda} \Gamma_{ac} \Lambda) (\bar{\Lambda} \Gamma_{mn} R) (\bar{\Lambda} \Gamma_{ef} R) (\bar{\Lambda} \Gamma^{efgh} \Lambda) N_{hi} (\bar{\Lambda} \Gamma^{mn} D) \\
+ \frac{1}{3\eta^3} (\bar{\Lambda} \Gamma_{ac} \Lambda) (\bar{\Lambda} \Gamma_{mn} R) (\bar{\Lambda} \Gamma_{ge} R) (\bar{\Lambda} \Gamma^{efgh} \Lambda) N_{hi} (\bar{\Lambda} \Gamma^{mn} D) \\
- \frac{1}{3\eta^3} (\bar{\Lambda} \Gamma_{ac} \Lambda) (\bar{\Lambda} \Gamma_{ge} R) (\bar{\Lambda} \Gamma_{mn} R) (\bar{\Lambda} \Gamma^{efgh} \Lambda) N_{hi} (\bar{\Lambda} \Gamma^{mn} D) \tag{1.15}
\]

Thus the term desired is

\[
b^{(3)}_{\text{simp}} = \frac{4}{\eta^3} (\bar{\Lambda} \Gamma_{lm} \Lambda) (\bar{\Lambda} \Gamma_{cm} R) (\bar{\Lambda} \Gamma_{n} \Lambda) (\bar{\Lambda} \Gamma^{lmncmq} \Lambda) (\bar{\Lambda} \Gamma^{z} D) N_{zp} \\
- \frac{2}{\eta^3} (\bar{\Lambda} \Gamma_{lm} \Lambda) (\bar{\Lambda} \Gamma_{n} R) (\bar{\Lambda} \Gamma^{lmnpq} \Lambda) (\bar{\Lambda} \Gamma^{z} D) N_{p} \\
- \frac{1}{3\eta^3} (\bar{\Lambda} \Gamma_{ef} \Lambda) (\bar{\Lambda} \Gamma_{ge} R) (\bar{\Lambda} \Gamma_{mn} R) (\bar{\Lambda} \Gamma^{efgh} \Lambda) N_{hi} (\bar{\Lambda} \Gamma^{mn} D) \tag{1.16}
\]

The last term to be calculated is that proportional to \(\eta^{-4}\). The relevant terms are (after using (B.7)):

\[
b^{(4)}_{\text{simp}} = -\frac{4}{\eta^3} (\bar{\Lambda} \Gamma_{ab} \Lambda) (\bar{\Lambda} \Gamma_{cd} R) (\bar{\Lambda} \Gamma^{bd} \Lambda) (\bar{\Lambda} \Gamma^{aj} \Lambda) \left( \frac{2}{3\eta^2} (\bar{\Lambda} \Gamma_{ef} \Lambda) (\bar{\Lambda} \Gamma_{g} \Lambda) (\bar{\Lambda} \Gamma^{efgh} \Lambda) N_{hi} \right) \frac{2}{\eta^2} (\bar{\Lambda} \Gamma_{lm} \Lambda) \times
\]
\[ (\bar{\Lambda}_{rn}R)(\bar{\Gamma}^{mnqcK}A)N_{pq} + \frac{2}{3\eta^2}(\bar{\Lambda}_{lm}A)(\bar{\Gamma}_n^cR)(\bar{\Gamma}^{lmnpq}A)N_{pq} + \frac{2}{3\eta^2}(\bar{\Lambda}_{lm}A)(\bar{\Gamma}^{lmnpq}A)N_{pq} + \frac{1}{3\eta^2}(\bar{\Lambda}_{nt}A)(\bar{\Gamma}^{ctupq}A)N_{pq} + \frac{2}{3\eta^2}(\bar{\Lambda}_{nt}A)(\bar{\Gamma}^{ctupq}A)N_{pq} \]

\[ = -\frac{4}{3\eta^2}(\bar{\Lambda}_{nt}A)(\bar{\Lambda}_{n}^cR)(\bar{\Gamma}^{effghi}A)N_{hi} + \frac{2}{3\eta^2}(\bar{\Lambda}_{lm}A)(\bar{\Gamma}^{lmnpq}A)N_{pq} + \frac{1}{3\eta^2}(\bar{\Lambda}_{lm}A)(\bar{\Gamma}^{lmnpq}A)N_{pq} - \frac{2}{3\eta^2}(\bar{\Lambda}_{nt}A)(\bar{\Gamma}^{lmnpq}A)N_{pq} + \frac{1}{3\eta^2}(\bar{\Lambda}_{nt}A)(\bar{\Gamma}^{ctupq}A)N_{pq} \]

\[ = \frac{2}{3\eta^2}(\bar{\Lambda}_{nt}A)(\bar{\Lambda}_{n}^cR)(\bar{\Gamma}^{effghi}A)N_{hi} - \frac{2}{3\eta^2}(\bar{\Lambda}_{lm}A)(\bar{\Gamma}^{lmnpq}A)N_{pq} + \frac{1}{3\eta^2}(\bar{\Lambda}_{lm}A)(\bar{\Gamma}^{lmnpq}A)N_{pq} - \frac{2}{3\eta^2}(\bar{\Lambda}_{nt}A)(\bar{\Gamma}^{lmnpq}A)N_{pq} + \frac{1}{3\eta^2}(\bar{\Lambda}_{nt}A)(\bar{\Gamma}^{ctupq}A)N_{pq} \]

\[ = \frac{2}{3\eta^2}(\bar{\Lambda}_{nt}A)(\bar{\Lambda}_{n}^cR)(\bar{\Gamma}^{effghi}A)N_{hi} - \frac{2}{3\eta^2}(\bar{\Lambda}_{lm}A)(\bar{\Gamma}^{lmnpq}A)N_{pq} + \frac{1}{3\eta^2}(\bar{\Lambda}_{lm}A)(\bar{\Gamma}^{lmnpq}A)N_{pq} - \frac{2}{3\eta^2}(\bar{\Lambda}_{nt}A)(\bar{\Gamma}^{lmnpq}A)N_{pq} + \frac{1}{3\eta^2}(\bar{\Lambda}_{nt}A)(\bar{\Gamma}^{ctupq}A)N_{pq} \]

So our simplified \( D = 11 \) b-ghost has the following expansion:

\[ b_{\text{simpl}} = \frac{1}{2\eta^2} - (\bar{\Lambda}_{ab}A)(\bar{\Gamma}^{ab}R_1D) + \eta^{-2}L^{(1)}_{ab,cd}[2(\bar{\Gamma}^{abcD}_{*KI}A)N^{dk} \]

\( - 39 - \)
The quadratic term in $D_\alpha$ is easy to obtain using the identity \eqref{B.10}

$$
 b^{(2)} = \frac{1}{\eta^2}(\bar{\Lambda} \Gamma_{ab} \Lambda)(\Lambda \Gamma^{abc} D) P_i + \eta^{-2} L_{ab,cd}^{(1)}(\Lambda \Gamma^{bcd} D) + 2(\Lambda \Gamma^{abc} D) N^{di} P^j
$$

We can compare this result with the expansion of the $b$-ghost in \eqref{3.2}

$$
 b = \frac{1}{2} \eta^{-1}(\bar{\Lambda} \Gamma_{ab} \Lambda)(\Lambda \Gamma^{abc} D) P_i + \eta^{-2} L_{ab,cd}^{(1)}(\Lambda \Gamma^{bcd} D) + 2(\Lambda \Gamma^{abc} D) N^{di} P^j
$$

Now we will find the term proportional to $\eta^{-3}$. Let us do this in two steps: First let us focus on the term proportional to $\Lambda \Gamma^{a} D$ (which will be called $K'_{1}$) and then on the term proportional to $\Lambda \Gamma^{abc} D$ (which will be called $K'_{2}$):

$$
 K'_{1} = \frac{4}{\eta^3} L_{ab,cd,ef}^{(2)} [(\Lambda \Gamma_{ae} \Lambda)(\Lambda \Gamma^{d} D) \eta^{f} N_{ij} - \frac{2}{3} \eta^{f[a} (\Lambda \Gamma^{bce} D)(\Lambda \Gamma^{d} D) N_{ij}]
$$

$$
 K'_{2} = \frac{4}{\eta^3} L_{ab,cd,ef}^{(2)} [(\Lambda \Gamma_{ae} \Lambda)(\Lambda \Gamma^{d} D) \eta^{f} N_{ij} - \frac{2}{3} \eta^{f[a} (\Lambda \Gamma^{bce} D)(\Lambda \Gamma^{d} D) N_{ij}]
$$

$$
 = \frac{4}{\eta^3} (\bar{\Lambda} \Gamma_{ab} \Lambda)(\bar{\Lambda} \Gamma_{ed} \Lambda)(\Lambda \Gamma^{d} D) N_{ij}
$$

$$
 - \frac{8}{3 \eta^3} L_{ab,cd,ef}^{(2)} [\eta^{a(b} (\Lambda \Gamma^{d} D)(\Lambda \Gamma^{d} D) N_{ij}]
$$

$$
 = \frac{4}{\eta^3} (\bar{\Lambda} \Gamma_{ab} \Lambda)(\bar{\Lambda} \Gamma_{ed} \Lambda)(\Lambda \Gamma^{d} D) N_{ij}
$$

$$
 - \frac{2}{3 \eta^3} L_{ab,cd,ef}^{(2)} [2 \eta^{a} (\Lambda \Gamma^{d} D)(\Lambda \Gamma^{d} D) N_{ij}]
$$

$$
 = \frac{4}{\eta^3} (\bar{\Lambda} \Gamma_{ab} \Lambda)(\bar{\Lambda} \Gamma_{ed} \Lambda)(\Lambda \Gamma^{d} D) N_{ij}
$$

$$
 - \frac{2}{3 \eta^3} [2 L_{ab,cd,ef}^{(2)} a (\Lambda \Gamma^{d} D)(\Lambda \Gamma^{d} D) N_{ij}]
$$

$$
 + L_{ab,cd,ef}^{(2)} c (\Lambda \Gamma^{d} D)(\Lambda \Gamma^{d} D) N_{ij}
$$

(1.18)

$$
 - \frac{1}{3 \eta^3} L_{ab,cd,ef}^{(2)} [2 \eta^{a} (\Lambda \Gamma^{d} D)(\Lambda \Gamma^{d} D) N_{ij}]
$$

$$
 = \frac{4}{\eta^3} (\bar{\Lambda} \Gamma_{ab} \Lambda)(\bar{\Lambda} \Gamma_{ed} \Lambda) N_{ij}
$$

$$
 - \frac{2}{3 \eta^3} L_{ab,cd,ef}^{(2)} [2 \eta^{a} (\Lambda \Gamma^{d} D)(\Lambda \Gamma^{d} D) N_{ij}]
$$

$$
 = \frac{4}{\eta^3} (\bar{\Lambda} \Gamma_{ab} \Lambda)(\bar{\Lambda} \Gamma_{ed} \Lambda) N_{ij}
$$

$$
 - \frac{2}{3 \eta^3} [2 L_{ab,cd,ef}^{(2)} a (\Lambda \Gamma^{d} D)(\Lambda \Gamma^{d} D) N_{ij}]
$$

$$
 + L_{ab,cd,ef}^{(2)} c (\Lambda \Gamma^{d} D)(\Lambda \Gamma^{d} D) N_{ij}
$$

(1.19)
where the identity (B.12) was used from the first to the second line. Now we make use of the identities (B.14) and (B.15):

\[
K_1' = \frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{e}^j R)(\Lambda \Gamma^{abc\Lambda})(\Lambda \Gamma^{dD}) N_{ij} \\
- \frac{2}{9\eta^2} \{ [4(\bar{\Lambda} \Gamma_{eb}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R) (\bar{\Lambda} R) - 2(\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{eb}\Lambda)](\Lambda \Gamma^{bceij} \Lambda) \\
+ [ (\bar{\Lambda} \Gamma_{ab}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{e}^c R) - 2(\bar{\Lambda} \Gamma_{ed}R)(\bar{\Lambda} \Gamma_{ab}\Lambda)](\Lambda \Gamma^{abcij} \Lambda) \} (\Lambda \Gamma^d D) N_{ij} \\
= \frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{e}^j R)(\Lambda \Gamma^{abc\Lambda})(\Lambda \Gamma^{dD}) N_{ij} \\
- \frac{2}{9\eta^2} \{ [6(\bar{\Lambda} \Gamma_{eb}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} R) - 2(\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{eb}\Lambda)](\Lambda \Gamma^{bceij} \Lambda) \\
- (\bar{\Lambda} \Gamma_{ed}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{e} R)(\bar{\Lambda} R)(\Lambda \Gamma^{bceij} \Lambda) - (\bar{\Lambda} \Gamma_{eb}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{ad}R)(\bar{\Lambda} \Gamma_{e} a R)(\Lambda \Gamma^{bceij} \Lambda) \} (\Lambda \Gamma^d D) N_{ij} \\
= \frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{e}^j R)(\Lambda \Gamma^{abc\Lambda})(\Lambda \Gamma^{dD}) N_{ij} \\
- \frac{2}{9\eta^2} \{ [6(\bar{\Lambda} \Gamma_{eb}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} R) - 2(\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{e} a R) \\
- (\bar{\Lambda} \Gamma_{ed}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{e} R)(\bar{\Lambda} R)](\Lambda \Gamma^{bceij} \Lambda) \} (\Lambda \Gamma^d D) N_{ij} \\
= \frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{e}^j R)(\Lambda \Gamma^{abc\Lambda})(\Lambda \Gamma^{dD}) N_{ij} \\
- \frac{2}{9\eta^2} \{ [6(\bar{\Lambda} \Gamma_{eb}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} R) - 2(\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{e} a R) \\
- (\bar{\Lambda} \Gamma_{ed}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{e} R)(\bar{\Lambda} R)](\Lambda \Gamma^{bceij} \Lambda) \} (\Lambda \Gamma^d D) N_{ij}
\]

where we have made use of the identity (B.4). By using the antisymmetry in \((b,c,e)\) we show that:

\[
K_1' = \frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{e}^j R)(\Lambda \Gamma^{abc\Lambda})(\Lambda \Gamma^{dD}) N_{ij} \\
- \frac{2}{9\eta^2} (\bar{\Lambda} \Gamma_{eb}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} R)(\Lambda \Gamma^{bceij} \Lambda) (\Lambda \Gamma^d D) N_{ij} \\
= \frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{e}^j R)(\Lambda \Gamma^{abc\Lambda})(\Lambda \Gamma^{dD}) N_{ij} \\
(1.21)
\]

Now let us focus on the term proportional to \((\Lambda \Gamma^{bed} D)\). This term appears in the expression (3.2) in the form (after using (B.12)):

\[
K_2' = -\frac{1}{3\eta^2} (\bar{\Lambda} \Gamma_{ab}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{e}^f R)(\Lambda \Gamma^{abcpq} \Lambda)(\Lambda \Gamma^{def} D) N_{pq} \\
(1.22)
\]

Putting these results together we obtain

\[
b^{(3)} = \frac{4}{\eta^2} (\bar{\Lambda} \Gamma_{ab}\bar{\Lambda}) (\bar{\Lambda} \Gamma_{cd}R)(\bar{\Lambda} \Gamma_{e}^j R)(\Lambda \Gamma^{abc\Lambda})(\Lambda \Gamma^{dD}) N_{ij}
\]
\[- \frac{2}{\eta^4} (\bar{\Lambda} \Gamma_{bc} \bar{A})(\bar{\Lambda} \Gamma_{cd} R)(\bar{\Lambda} R)(\bar{\Lambda} \Gamma_{ij} \Lambda)(\bar{\Lambda} \Gamma_{d} D) N_{ij} \]

\[- \frac{1}{3 \eta^2} (\bar{\Lambda} \Gamma_{ab} \bar{A})(\bar{\Lambda} \Gamma_{cd} R)(\bar{\Lambda} \Gamma_{ef} R)(\bar{\Lambda} \Gamma_{abc} \Lambda)(\bar{\Lambda} \Gamma_{def} D) N_{pq} \]

which it should be compared with the analog expression corresponding to $b_{simpl}$, equation (I.16).

The last term to be computed is that proportional to $\eta^{-4}$ in (3.2):

\[ b^{(4)} = 4 \frac{1}{3} \eta^{-4} L_{ab,cd,ef,gh}^{(3)} (\bar{\Gamma} \Gamma_{abcd} \Lambda)(\bar{\Gamma} \Gamma_{defg} \Lambda) \eta^{hl} \{ N_{ij}, N_{kl} \} \tag{1.24} \]

Let us use some identities in order to write this expression in a simpler way. It is more convenient to do this in two steps: First we will focus on the first term ($P_1$) and then on the second term ($P_2$):

\[
P_1 = 4 \frac{1}{3} \eta^{-4} L_{ab,cd,ef,gh}^{(3)} (\bar{\Gamma} \Gamma_{abcd} \Lambda)(\bar{\Gamma} \Gamma_{defg} \Lambda) \eta^{hl} \{ N_{ij}, N_{kl} \}
\]

\[
= 4 \frac{1}{3 \eta^4} \left( \frac{1}{4} (\bar{\Lambda} \Gamma_{ab} \bar{A})(\bar{\Lambda} \Gamma_{cd} R)(\bar{\Lambda} \Gamma_{ef} R)(\bar{\Lambda} \Gamma_{gh} R) - (\bar{\Lambda} \Gamma_{cd} \bar{A})(\bar{\Lambda} \Gamma_{ef} R)(\bar{\Lambda} \Gamma_{gh} R) + (\bar{\Lambda} \Gamma_{ef} \bar{A})(\bar{\Lambda} \Gamma_{ab} R)(\bar{\Lambda} \Gamma_{cd} R)(\bar{\Lambda} \Gamma_{gh} R) - (\bar{\Lambda} \Gamma_{gh} \bar{A})(\bar{\Lambda} \Gamma_{ab} R)(\bar{\Lambda} \Gamma_{cd} R)(\bar{\Lambda} \Gamma_{ef} R) \right) (\bar{\Gamma} \Gamma_{abcd} \Lambda) \times (\bar{\Gamma} \Gamma_{defg} \Lambda) \eta^{hl} \{ N_{ij}, N_{kl} \}
\]

\[
= 4 \frac{1}{3 \eta^4} (\bar{\Lambda} \Gamma_{ab} \bar{A})(\bar{\Lambda} \Gamma_{cd} R)(\bar{\Lambda} \Gamma_{ef} R)(\bar{\Lambda} \Gamma_{gh} R) (\bar{\Gamma} \Gamma_{abcd} \Lambda)(\bar{\Gamma} \Gamma_{defg} \Lambda) \eta^{hl} \{ N_{ij}, N_{kl} \} \tag{1.25}
\]

The simplifications made here are result of repeated uses of the identities (B.10) and (B.11). Now let us focus on $P_2$:

\[
P_2 = - \frac{8}{9 \eta^4} L_{ab,cd,ef,gh}^{(3)} (\bar{\Gamma} \Gamma_{abcd} \Lambda)(\bar{\Gamma} \Gamma_{defg} \Lambda) \eta^{hl} (\bar{\Gamma} \Gamma_{efg} \Lambda) \{ N_{ij}, N_{kl} \}
\]

\[
= - \frac{2}{9 \eta^4} L_{ab,cd,ef,gh}^{(3)} (\bar{\Gamma} \Gamma_{abcd} \Lambda) \eta^{hl} (\bar{\Gamma} \Gamma_{efg} \Lambda) - 2 \eta^{he} (\bar{\Gamma} \Gamma_{efg} \Lambda) \{ N_{ij}, N_{kl} \}
\]

\[
= - \frac{2}{9 \eta^4} [L_{ab,cd,ef,gh}^{(3)} \eta^{hl} (\bar{\Gamma} \Gamma_{abcd} \Lambda)(\bar{\Gamma} \Gamma_{defg} \Lambda) - 2 L_{ab,cd,ef,gh}^{(3)} \eta^{he} (\bar{\Gamma} \Gamma_{abcd} \Lambda)(\bar{\Gamma} \Gamma_{defg} \Lambda)] \{ N_{ij}, N_{kl} \} \tag{1.26}
\]

We can simplify each term separately:

\[
(P_2')_{ijkl} = L_{ab,cd,ef,gh}^{(3)} \eta^{hl} (\bar{\Gamma} \Gamma_{abcd} \Lambda)(\bar{\Gamma} \Gamma_{defg} \Lambda)
\]

\[
= \frac{1}{4} (\bar{\Lambda} \Gamma_{ab} \bar{A})(\bar{\Lambda} \Gamma_{cd} R)(\bar{\Lambda} \Gamma_{ef} R)(\bar{\Lambda} \Gamma_{d} R) - (\bar{\Lambda} \Gamma_{cd} \bar{A})(\bar{\Lambda} \Gamma_{ab} R)(\bar{\Lambda} \Gamma_{ef} R)(\bar{\Lambda} \Gamma_{d} R)
\]

\[- 42 \]
This result together with (I.25) gives us the following result for which it should be compared with the analog expression corresponding to (I.17).

\[
(P_2)_{ijkl} = -2\tilde{\Lambda}^{(3)}_{ab,cd,e,f,gh}(\Lambda^g_{e,f}A)(\Lambda^g_{a,b}A)\Lambda(R)(\Lambda^f_{e,f}A)\Lambda(R)(\Lambda^g_{a,b}A)\Lambda(R)(\Lambda^e_{f,g}A)
\]

This result together with (I.25) gives us the following result for \(b^{(4)}\):

\[
b^{(4)} = \frac{4}{3\eta^3}(\tilde{\Lambda}_{ab,cd}A)(\tilde{\Lambda}_{e,f}R)(\tilde{\Lambda}_{g,h}R)(\Lambda^{abcij}\Lambda)(\Lambda^{def,ijkl}\Lambda)\eta^{kl}\{N_{ij}, N_{kl}\}
\]

which it should be compared with the analog expression corresponding to \(b_{\text{simp}}\), equation (I.17).

Hence we can write the full expansion for the \(b\)-ghost given in (3.2)

\[
b = P^i \left[ \frac{1}{2} \eta^{-1}(\tilde{\Lambda}_{ab,cd}A)(\Lambda^{ab}T_i D) + \eta^{-2}L^{(1)}_{ab,cd}(2(\Lambda^a_{bc}D)(\Lambda^{bcd}D) + \frac{2}{3}(\eta^b_{p} \eta^d_{q} - \eta^b_{q} \eta^d_{p})(\Lambda^a_{pqij}D)N_{ij} \right] + \frac{1}{\eta^2}L^{(1)}_{ab,cd}(\Lambda^a_{bcd}D)(\Lambda^{bcd}D) + \frac{4}{\eta^3}(\tilde{\Lambda}_{ab,cd}A)(\tilde{\Lambda}_{e,f}R)(\Lambda^{abcij}\Lambda)(\Lambda^{d}D)N_{ij} - \frac{2}{\eta^3}(\tilde{\Lambda}_{ab,cd}A)(\tilde{\Lambda}_{e,f}R)(\Lambda^{abcij}\Lambda)(\Lambda^{d}D)N_{ij} - \frac{1}{3\eta^2}(\tilde{\Lambda}_{ab,cd}A)(\tilde{\Lambda}_{e,f}R)(\Lambda^{abcij}\Lambda)(\Lambda^{d}D)N_{pq} + \frac{4}{3\eta^3}(\tilde{\Lambda}_{ab,cd}A)(\tilde{\Lambda}_{e,f}R)(\tilde{\Lambda}_{gh}R)(\Lambda^{abcij}\Lambda)(\Lambda^{d}D)N_{pq} - \frac{2}{3\eta^3}(\tilde{\Lambda}_{ab,cd}A)(\tilde{\Lambda}_{e,f}R)(\tilde{\Lambda}_{gh}R)(\Lambda^{abcij}\Lambda)(\Lambda^{d}D)N_{pq}
\]
This expression differs from (I.18) in two points. First, the position of \( N_{hi} \) in the last term proportional to \( \eta^{-3} \) is not at the end of the expression as it is in (I.31). Second, in the terms proportional to \( \eta^{-4} \) we do not have the anticommutator of \( N_{ab} \)’s in (I.18) as we do in (I.31), and once again the position of \( N_{hi} \) is not at the end of the expressions in (I.18) as it is in (I.31).

In order to have a clearer idea on what is happening, we will move all of the \( N_{ab} \)’s at the end of the expressions mentioned above in (I.18). Let us start with the term proportional to \( \eta^{-3} \). We should put the ghost current \( N_{hi} \) to the right hand side of \((\Lambda \Gamma_{cmn} D)\). For this purpose we compute the commutator between \( N_{hi} \) and \((\Lambda \Gamma_{cmn} D)\) with the symmetry properties written in (I.15):

\[
[N_{hi}, (\Lambda \Gamma_{cmn} D)] = -2\eta_{hi}^{\Gamma_{mn}} (\Lambda \epsilon fghi \Lambda)(\Lambda \Gamma_{cmn} D)N_{hi} + (\Lambda \Gamma_{chimn} D)
\]

(I.32)

The use of the identities (B.4), (B.6) allows us to cancel out all of the terms except the last one, so

\[
K_2 = -\frac{1}{3\eta^3}(\Lambda \epsilon f \bar{\Lambda})(\Lambda \Gamma_{cmn} R)(\bar{\Lambda} \Gamma_{mn} R)(\Lambda \Gamma_{efghi} \Lambda)(\Lambda \Gamma_{cmn} D)N_{hi} - \frac{1}{3\eta^3}(\Lambda \epsilon f \bar{\Lambda})(\Lambda \Gamma_{cmn} R)(\bar{\Lambda} \Gamma_{mn} R)(\Lambda \Gamma_{efghi} \Lambda)(\Lambda \Gamma_{chimn} D)
\]

(I.33)

Therefore \( b^{(3)}_{\text{simp}} \) changes by the factor \(-\frac{1}{3\eta^3}(\Lambda \epsilon f \bar{\Lambda})(\Lambda \Gamma_{cmn} R)(\bar{\Lambda} \Gamma_{mn} R)(\Lambda \Gamma_{efghi} \Lambda)(\Lambda \Gamma_{chimn} D)\) when \( N_{mn} \) is placed at the end of the full expression.

Now we move on to the terms proportional to \( \eta^{-4} \). We should move \( N_{hi} \) to the right hand side of \((\Lambda \Gamma_{cmn} D)\):

\[
[N_{hi}, (\Lambda \Gamma_{cmn} D)] = 4\eta_{hi}\eta^q (\Lambda \Gamma_{cimnq} D) - 4\eta_{hi}\eta^m (\Lambda \Gamma_{cimnq} D) - 8\eta_{hi}\eta^l (\Lambda \Gamma_{cimnq} D) - 4\eta_{hi}\eta^m (\Lambda \Gamma_{cimnq} D)
\]

(I.34)

These are the relevant terms in (I.17) and it is a direct consequence of the equation (B.25) and the symmetry properties of the expression (I.17). The last two terms do not contribute as it can be seen after using (B.4), (B.6). So we are left with:

\[
Z_1 = \frac{4}{3\eta^4}(\Lambda \epsilon f \bar{\Lambda})(\Lambda \Gamma_{cmn} D)(\Lambda \Gamma_{mn} R)(\bar{\Lambda} \Gamma_{mn} R)(\Lambda \Gamma_{efghi} \Lambda)(\Lambda \Gamma_{cimnq} D)N_{hi} N_{qp} + \frac{16}{3\eta^4}(\Lambda \epsilon f \bar{\Lambda})(\Lambda \Gamma_{cmn} D)(\Lambda \Gamma_{mn} R)(\bar{\Lambda} \Gamma_{mn} R)(\Lambda \Gamma_{efghi} \Lambda)(\Lambda \Gamma_{cimnq} D)N_{hi} N_{qp}
\]

\[
= \frac{4}{3\eta^4}(\Lambda \epsilon f \bar{\Lambda})(\Lambda \Gamma_{cmn} D)(\Lambda \Gamma_{mn} R)(\bar{\Lambda} \Gamma_{mn} R)(\Lambda \Gamma_{efghi} \Lambda)(\Lambda \Gamma_{cimnq} D)N_{hi} N_{qp} + \frac{16}{3\eta^4}(\Lambda \epsilon f \bar{\Lambda})(\Lambda \Gamma_{cmn} D)(\Lambda \Gamma_{mn} R)(\bar{\Lambda} \Gamma_{mn} R)(\Lambda \Gamma_{efghi} \Lambda)(\Lambda \Gamma_{cimnq} D)
\]

\[
= \frac{4}{3\eta^4}(\Lambda \epsilon f \bar{\Lambda})(\Lambda \Gamma_{cmn} D)(\Lambda \Gamma_{mn} R)(\bar{\Lambda} \Gamma_{mn} R)(\Lambda \Gamma_{efghi} \Lambda)(\Lambda \Gamma_{cimnq} D)N_{hi} N_{qp}
\]

\[
= \frac{4}{3\eta^4}(\Lambda \epsilon f \bar{\Lambda})(\Lambda \Gamma_{cmn} D)(\bar{\Lambda} \Gamma_{cg} R)(\Lambda \Gamma_{mn} R)(\bar{\Lambda} \Gamma_{mn} R)(\Lambda \Gamma_{efghi} \Lambda)(\Lambda \Gamma_{cimnq} D)N_{hi} N_{qp}
\]

\[
= \frac{4}{3\eta^4}(\Lambda \epsilon f \bar{\Lambda})(\Lambda \Gamma_{cmn} D)(\bar{\Lambda} \Gamma_{cg} R)(\Lambda \Gamma_{mn} R)(\bar{\Lambda} \Gamma_{mn} R)(\Lambda \Gamma_{efghi} \Lambda)(\Lambda \Gamma_{cimnq} D)N_{hi} N_{qp}
\]
\[ + \frac{16}{3\eta^4} [(\tilde{\Lambda} \Gamma_{ef} \tilde{\Lambda}) (\tilde{\Lambda} \Gamma_{gm} \tilde{R})(\tilde{\Lambda} \Gamma_{ln} \tilde{R})(\tilde{\Lambda} \Gamma_{np} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmn} \Lambda)] \]

\[ - \frac{16}{3\eta^4} (\tilde{\Lambda} \Gamma_{gm} \tilde{\Lambda}) (\tilde{\Lambda} \Gamma_{ln} \tilde{R})(\tilde{\Lambda} \Gamma_{np} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmn} \Lambda)] N_{qp} \]

\[ = \frac{4}{3\eta^4} (\tilde{\Lambda} \Gamma_{ef} \tilde{\Lambda})(\tilde{\Lambda} \Gamma_{gm} \tilde{R})(\tilde{\Lambda} \Gamma_{ln} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmn} \Lambda)] N_{hi} N_{qp} \]

\[ + \frac{16}{3\eta^4} [(\tilde{\Lambda} \Gamma_{ef} \tilde{\Lambda})(\tilde{\Lambda} \Gamma_{gm} \tilde{R})(\tilde{\Lambda} \Gamma_{ln} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmn} \Lambda)] N_{qp} \]

\[ = \frac{4}{3\eta^4} (\tilde{\Lambda} \Gamma_{ef} \tilde{\Lambda})(\tilde{\Lambda} \Gamma_{gm} \tilde{R})(\tilde{\Lambda} \Gamma_{ln} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmn} \Lambda)] N_{hi} N_{qp} \quad \text{(I.35)} \]

Now let us make the same procedure with the remaining term. The relevant commutation relation is:

\[ [\eta^{hi}, (\tilde{\Lambda} \Gamma_{ilmnq} \Lambda)] = -8\eta^{hp} (\tilde{\Lambda} \Gamma_{ilmnq} \Lambda) - 4\eta^{hm} (\tilde{\Lambda} \Gamma_{ilmnq} \Lambda) + 8\eta^{hl} (\tilde{\Lambda} \Gamma_{ilmnq} \Lambda) \quad \text{(I.36)} \]

Once again, we have obtained this result by using the identity (B.25) and the symmetry properties of the corresponding expression in (I.17). When applying the equations (B.4), (B.6), the last two terms vanish and we obtain

\[ Z_2 = -\frac{2}{3\eta^4} (\tilde{\Lambda} \Gamma_{ef} \tilde{\Lambda})(\tilde{\Lambda} \Gamma_{gm} \tilde{R})(\tilde{\Lambda} \Gamma_{ln} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmnq} \Lambda)] N_{hi} N_{pq} \]

\[ + \frac{16}{3\eta^4} (\tilde{\Lambda} \Gamma_{ef} \tilde{\Lambda})(\tilde{\Lambda} \Gamma_{gm} \tilde{R})(\tilde{\Lambda} \Gamma_{ln} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmnq} \Lambda)] N_{pq} \quad \text{(I.37)} \]

Now we will show that the last term is zero. Let us see how this happens:

\[ M = \frac{16}{3\eta^4} (\tilde{\Lambda} \Gamma_{ef} \tilde{\Lambda})(\tilde{\Lambda} \Gamma_{gm} \tilde{R})(\tilde{\Lambda} \Gamma_{ln} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmnq} \Lambda)] N_{pq} \]

\[ = -\frac{16}{3\eta^4} (\tilde{\Lambda} \Gamma_{ln} \tilde{\Lambda})(\tilde{\Lambda} \Gamma_{mg} \tilde{R})(\tilde{\Lambda} \Gamma_{ef} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmnq} \Lambda)] N_{pq} \]

\[ = \frac{16}{3\eta^4} (\tilde{\Lambda} \Gamma_{ef} \tilde{\Lambda})(\tilde{\Lambda} \Gamma_{gm} \tilde{R})(\tilde{\Lambda} \Gamma_{ln} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmnq} \Lambda)] N_{pq} \]

\[ = \frac{16}{3\eta^4} (\tilde{\Lambda} \Gamma_{ef} \tilde{\Lambda})(\tilde{\Lambda} \Gamma_{gm} \tilde{R})(\tilde{\Lambda} \Gamma_{ln} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmnq} \Lambda)] N_{pq} \]

\[ = -M \quad \text{(I.38)} \]

Therefore,

\[ Z_2 = -\frac{2}{3\eta^4} (\tilde{\Lambda} \Gamma_{ef} \tilde{\Lambda})(\tilde{\Lambda} \Gamma_{gm} \tilde{R})(\tilde{\Lambda} \Gamma_{ln} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmnq} \Lambda)] N_{hi} N_{pq} \quad \text{(I.39)} \]

This means that \( \Gamma_{simp}^{(4)} \) does not change when we move \( N_{mn} \) to the end of the full expression.

We can summarize the result in the following expression for the simplified \( b \)-ghost:

\[ b_{simp} = P_i \frac{1}{2} \eta^{-1} (\tilde{\Lambda} \Gamma_{ab} \tilde{\Lambda})(\tilde{\Lambda} \Gamma_{abc} \tilde{\Gamma}_i \tilde{D}) + \eta^{-2} L_{ab,cd}^{(1)} [2(\tilde{\Lambda} \Gamma_{abc} \tilde{\Lambda}) \tilde{N}^{dk} \quad \text{(I.39)} \]

Therefore,

\[ Z_2 = -\frac{2}{3\eta^4} (\tilde{\Lambda} \Gamma_{ef} \tilde{\Lambda})(\tilde{\Lambda} \Gamma_{gm} \tilde{R})(\tilde{\Lambda} \Gamma_{ln} \tilde{R})(\tilde{\Lambda} \Gamma_{egfghi} \Lambda) (\tilde{\Lambda} \Gamma_{ilmnq} \Lambda)] N_{hi} N_{pq} \quad \text{(I.39)} \]

This means that \( \Gamma_{simp}^{(4)} \) does not change when we move \( N_{mn} \) to the end of the full expression.

We can summarize the result in the following expression for the simplified \( b \)-ghost:
+ \frac{2}{3}(\eta^{b} p^{d} l - \eta^{b} d \eta_{p q})(\Lambda^{a p c q j}(\Lambda)N_{a j}) + \frac{1}{\eta^{2}} L^{(1)}_{m c, r s}(\Lambda^{a m} D)(\Lambda^{c r s} D) + \\
+ \frac{4}{\eta^{3}}(\tilde{\Lambda}_{l m} A)(\tilde{\Lambda}_{c d R})(\tilde{\Lambda}_{n p} R)(\Lambda^{l m c n q}(\Lambda)(\Lambda^{c x} D) N_{q p} \\
- \frac{2}{\eta^{3}}(\tilde{\Lambda}_{l m} A)(\tilde{\Lambda}_{n x} R)(\tilde{\Lambda} R)(\Lambda^{c l m n p q}(\Lambda)(\Lambda^{c x} D) N_{p q} \\
- \frac{1}{3 \eta^{3}}(\tilde{\Lambda}_{e f A})(\tilde{\Lambda}_{g c R})(\tilde{\Lambda}_{m n} R)(\Lambda^{e f g h i}(\Lambda)(\Lambda^{c m n} D) N_{h i} \\
+ \frac{4}{3 \eta^{4}}(\tilde{\Lambda}_{e f A})(\tilde{\Lambda}_{g c R})(\tilde{\Lambda}_{l m} R)(\tilde{\Lambda}_{n p} R)(\Lambda^{e f g h i}(\Lambda)(\Lambda^{c l m n p q} N_{h i}, N_{q p} \\
- \frac{2}{3 \eta^{4}}(\tilde{\Lambda}_{e f A})(\tilde{\Lambda}_{g m} R)(\tilde{\Lambda}_{l n} R)(\tilde{\Lambda} R)(\Lambda^{e f g h i}(\Lambda)(\Lambda^{c l m n p q} N_{h i}, N_{p q} \\
- \frac{1}{3 \eta^{3}}(\tilde{\Lambda}_{e f A})(\tilde{\Lambda}_{g c R})(\tilde{\Lambda}_{m n} R)(\Lambda^{e f g h i}(\Lambda)(\Lambda^{c h i m n} D) (I.40)

We can write the anticommutator instead of the ordinary product of $N_{a b}$’s recalling the relation $2 N_{h i}, N_{q p} = [N_{h i}, N_{q p}] + \{N_{h i}, N_{q p}\}$. The commutator contribution vanishes because of the identity (B.7). Therefore the final form for the expansion of the simplified $b$-ghost is

$$
b_{simpl} = \frac{1}{2} \eta^{-1}(\Lambda_{a b} \Lambda)(\Lambda^{a b} \Gamma_{i} D) + \eta^{-2} L^{(1)}_{a b, c d}(2(\Lambda^{a b c d \eta_{k l}} \Lambda) N^{d k}
+ \frac{2}{3}(\eta^{b} p^{d} l - \eta^{b} d \eta_{p q})(\Lambda^{a p c q j}(\Lambda)N_{a j}) + \frac{1}{\eta^{2}} L^{(1)}_{m c, r s}(\Lambda^{a m} D)(\Lambda^{c r s} D) + \\
+ \frac{4}{\eta^{3}}(\tilde{\Lambda}_{l m} A)(\tilde{\Lambda}_{c d R})(\tilde{\Lambda}_{n p} R)(\Lambda^{l m c n q}(\Lambda)(\Lambda^{c x} D) N_{q p} \\
- \frac{2}{\eta^{3}}(\tilde{\Lambda}_{l m} A)(\tilde{\Lambda}_{n x} R)(\tilde{\Lambda} R)(\Lambda^{c l m n p q}(\Lambda)(\Lambda^{c x} D) N_{p q} \\
- \frac{1}{3 \eta^{3}}(\tilde{\Lambda}_{e f A})(\tilde{\Lambda}_{g c R})(\tilde{\Lambda}_{m n} R)(\Lambda^{e f g h i}(\Lambda)(\Lambda^{c m n} D) N_{h i} \\
+ \frac{4}{3 \eta^{4}}(\tilde{\Lambda}_{e f A})(\tilde{\Lambda}_{g c R})(\tilde{\Lambda}_{l m} R)(\tilde{\Lambda}_{n p} R)(\Lambda^{e f g h i}(\Lambda)(\Lambda^{c l m n p q} N_{h i}, N_{q p} \\
- \frac{2}{3 \eta^{4}}(\tilde{\Lambda}_{e f A})(\tilde{\Lambda}_{g m} R)(\tilde{\Lambda}_{l n} R)(\tilde{\Lambda} R)(\Lambda^{e f g h i}(\Lambda)(\Lambda^{c l m n p q} N_{h i}, N_{p q} \\
- \frac{1}{3 \eta^{3}}(\tilde{\Lambda}_{e f A})(\tilde{\Lambda}_{g c R})(\tilde{\Lambda}_{m n} R)(\Lambda^{e f g h i}(\Lambda)(\Lambda^{c h i m n} D) (I.41)

The differences between this equation and (I.31) is a factor of 2 in the coefficients proportional to $\eta^{-4}$ and the non-zero extra term proportional to $(\Lambda^{c h i m n} D)$. However this mismatch could be fixed in a possible (normal-ordered) quantum version of both expressions.

References

[1] N. Berkovits, “Towards covariant quantization of the supermembrane,” JHEP 09 (2002) 051, arXiv:hep-th/0201151 [hep-th].

[2] P. S. Howe, “Pure spinors, function superspaces and supergravity theories in ten-dimensions and eleven-dimensions,” Phys. Lett. B273 (1991) 90–94.
[3] M. B. Green, M. Gutperle, and H. H. Kwon, “Light cone quantum mechanics of the eleven-dimensional superparticle,” *JHEP* **08** (1999) 012, arXiv:hep-th/9907155 [hep-th].

[4] M. Cederwall, B. E. W. Nilsson, and D. Tsimpis, “Spinorial cohomology and maximally supersymmetric theories,” *JHEP* **02** (2002) 009, arXiv:hep-th/0110069 [hep-th].

[5] N. Berkovits, “Pure spinor formalism as an N=2 topological string,” *JHEP* **10** (2005) 089, arXiv:hep-th/0509120 [hep-th].

[6] M. Cederwall, “Towards a manifestly supersymmetric action for 11-dimensional supergravity,” *JHEP* **1001** (2010) 117, arXiv:0912.1814 [hep-th].

[7] M. Cederwall, “D=11 supergravity with manifest supersymmetry,” *Phys. Lett.* **A25**, (2010) 3201, arXiv:1001.0112 [hep-th].

[8] T. Kugo and I. Ojima, “Local Covariant Operator Formalism of Nonabelian Gauge Theories and Quark Confinement Problem,” *Prog. Theor. Phys. Suppl.* **66** (1979) 1–130.

[9] M. Cederwall and A. Karlsson, “Loop amplitudes in maximal supergravity with manifest supersymmetry,” *JHEP* **03** (2013) 114, arXiv:1212.5175 [hep-th].

[10] A. Karlsson, “Ultraviolet divergences in maximal supergravity from a pure spinor point of view,” *JHEP* **04** (2015) 165, arXiv:1412.5983 [hep-th].

[11] N. Berkovits, “Dynamical twisting and the b ghost in the pure spinor formalism,” *JHEP* **06** (2013) 091, arXiv:1305.0693 [hep-th].

[12] N. Berkovits, “Covariant quantization of the superparticle using pure spinors,” *JHEP* **09** (2001) 016, arXiv:hep-th/0105050 [hep-th].

[13] L. Brink and J. H. Schwarz, “Quantum Superspace,” *Phys. Lett.* **B100** (1981) 310–312.

[14] N. Berkovits, “ICTP lectures on covariant quantization of the superstring,” in *Superstrings and related matters. Proceedings, Spring School, Trieste, Italy, March 18-26, 2002*, pp. 57–107. 2002. arXiv:hep-th/0209059 [hep-th].

http://www.ictp.trieste.it/~pub_off/lectures/lns013/Berkovits/Berkovits.pdf.

[15] O. A. Bedoya and N. Berkovits, “GGI Lectures on the Pure Spinor Formalism of the Superstring,” in *New Perspectives in String Theory Workshop Arcetri, Florence, Italy, April 6-June 19, 2009*. 2009. arXiv:0910.2254 [hep-th].

https://inspirehep.net/record/833767/files/arXiv:0910.2254.pdf.

[16] J. Bjornsson and M. B. Green, “5 loops in 24/5 dimensions,” *JHEP* **08** (2010) 132, arXiv:1004.2692 [hep-th].

[17] J. Bjornsson, “Multi-loop amplitudes in maximally supersymmetric pure spinor field theory,” *JHEP* **01** (2011) 022, arXiv:0909.5906 [hep-th].

[18] R. Lipinski Jusinskas, “Nilpotency of the b ghost in the non-minimal pure spinor formalism,” *JHEP* **05** (2013) 048, arXiv:1303.3966 [hep-th].

[19] N. Berkovits and O. Chandia, “Simplified Pure Spinor b Ghost in a Curved Heterotic Superstring Background,” *JHEP* **06** (2014) 001, arXiv:1403.2429 [hep-th].

[20] A. Karlsson, “Pure spinor indications of ultraviolet finiteness in D=4 maximal supergravity,” arXiv:1506.07505 [hep-th].

[21] U. Gran, “GAMMA: A Mathematica package for performing gamma matrix algebra and Fierz transformations in arbitrary dimensions,” arXiv:hep-th/0105086 [hep-th].