Scalar Perturbations of Nonlinear Charged Lifshitz Black Branes with Hyperscaling Violation

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Abstract

We study scalar perturbations of nonlinear charged Lifshitz black branes with hyperscaling violating factor, and we find numerically the quasinormal modes for scalar fields. Then, we study the stability of these black branes under massive and massless scalar field perturbations. Also, we consider different values of the dynamical exponent, the nonlinear exponent and the hyperscaling violating exponent.
I. INTRODUCTION

Lifshitz spacetimes have received considerable attention from the condensed matter point of view due to the AdS/CFT correspondence, i.e., the searching for gravity duals of Lifshitz fixed points for condensed matter physics and quantum chromodynamics \[1\]. From the quantum field theory point of view, there are many invariant scale theories of interest when studying such critical points. Such theories exhibit the anisotropic scale invariance \(t \rightarrow \chi^z t, \quad x \rightarrow \chi x\), with \(z \neq 1\), where \(z\) is the relative scale dimension of time and space, and these are of particular interest in studies of critical exponent theory and phase transitions. Systems with such behavior appear, for instance, in the description of strongly correlated electrons. The importance of possessing a tool to study strongly correlated condensed matter systems is beyond question, and consequently much attention has focused on this area in recent years. In this sense, Lifshitz holographic superconductivity has been a topic of numerous studies and interesting properties are found when one generalizes the gauge/gravity duality to non-relativistic situations \[2–11\].

The Lifshitz spacetimes are described by the metrics

\[
    ds^2 = -\frac{r^{2z}}{\ell^{2z}} dt^2 + \frac{\ell^2}{r^2} dr^2 + \frac{r^2}{\ell^2} d\vec{x}^2 ,
\]  

(1)
where $\vec{x}$ represents a $D - 2$ dimensional spatial vector, $D$ is the spacetime dimension and $\ell$ denotes the length scale in this geometry. If $z = 1$, the spacetime is the usual anti-de Sitter metric in Poincaré coordinates. Furthermore, all scalar curvature invariants are constant and these spacetimes have a null curvature singularity at $r \to 0$ for $z \neq 1$, which can be seen by computing the tidal forces between infalling particles. This singularity is reached in finite proper time by infalling observers, so the spacetime is geodesically incomplete \cite{12}. The metrics of Lifshitz black holes asymptotically have the form (1); however, obtaining analytical solutions does not seem to be a trivial task, and therefore constructing finite temperature gravity duals requires the introduction of strange matter content with a theoretical motivation that is not clear. Another way of finding such a Lifshitz black hole solution is by considering carefully-tuned higher-curvature modifications to the Hilbert-Einstein action, as in new massive gravity (NMG) in 3-dimensions or $R^2$ corrections to general relativity. Some Lifshitz black holes solutions have been found in references \cite{13–19}. Thermodynamically, it is difficult to compute conserved quantities for Lifshitz black holes; however, progress was made on the computation of mass and related thermodynamic quantities by using the ADT method \cite{20, 21} and the off-shell extension of the ADT formalism \cite{22} as well as the Euclidean action approach \cite{23, 24}. Also, phase transitions between Lifshitz black holes and other configurations with different asymptotes have been studied in \cite{25}. However, due to their different asymptotes these phases transitions do not occur.

A generalization of the above metric is given by

$$ds^2 = r^{-\frac{2\theta}{D}} \left( -\frac{r^{2z}}{\ell^{2z}} dt^2 + \frac{\ell^2}{r^2} dr^2 + \frac{r^2}{\ell^2} d\vec{x}^2 \right), \quad (2)$$

which, besides having an anisotropic scaling as the Lifshitz metric, have an overall hyperscaling violating factor with hyperscaling exponent $\theta$, thus, this line element is conformally related to the Lifshitz metric. This space-time is important in the study of the dual field theories with hyperscaling violation \cite{26, 27}. Lifshitz black holes with hyperscaling violation have been found in \cite{28–30}.

In this work, we study scalar perturbations of nonlinear charged Lifshitz black branes with hyperscaling violation. The matter is parameterized by scalar fields minimally coupled to gravity. Then, we obtain numerically the quasinormal frequencies (QNFs) for scalar fields. We focus our study in the influence of the dynamical exponent, the nonlinear exponent and the hyperscaling exponent in the stability.
The study of the QNFs \[31–36\] gives information about the stability of black holes under matter fields that evolve perturbatively in their exterior region, without backreacting on the metric. In general, the oscillation frequencies are complex, where the real part represents the oscillation frequency and the imaginary part describes the rate at which this oscillation is damped, with the stability of the black hole being guaranteed if the imaginary part is negative. The QNFs are independent of the initial conditions and depend only on the parameters of the black hole (mass, charge and angular momentum) and the fundamental constants (Newton constant and cosmological constant) that describe a black hole, just like the parameters that define the test field. On the other hand, the QNFs determine how fast a thermal state in the boundary theory will reach thermal equilibrium according to the AdS/CFT correspondence \[37\], where the relaxation time of a thermal state is proportional to the inverse of the imaginary part of the QNFs of the dual gravity background, which was established due to the QNFs of the black hole being related to the poles of the retarded correlation function of the corresponding perturbations of the dual conformal field theory \[38\]. Fermions on a Lifshitz background were studied in \[39\] by using the fermionic Green’s function in 4-dimensional Lifshitz spacetime with \(z = 2\); the authors considered a non-relativistic (mixed) boundary condition for fermions and showed that the spectrum has a flat band. Also, the Dirac quasinormal modes (QNMs) for a 4-dimensional Lifshitz black hole were studied in \[40\]. Generally, the Lifshitz black holes are stable under scalar perturbations, quasinormal modes under scalar field perturbations have been studied in \[24, 41–48\] and electromagnetic quasinormal modes in \[49\]. Moreover, it was established that for \(d > z + 1\), at zero momenta, the modes are non-overdamped, whereas for \(d \leq z + 1\) the system is always overdamped \[50\]. The QNFs have been calculated by means of numerical and analytical techniques, some remarkably numerical methods are: the Mashhoon method, Chandrasekhar-Detweiler, WKB method, Frobenius method, method of continued fractions, Nollert, asymptotic iteration method (AIM) and improved AIM among others. In the context of black hole thermodynamics, QNMs allow the quantum area spectrum of the black hole horizon to be studied as well as the mass and the entropy spectrum.

The paper is organized as follows. In Sec. II we give a brief review of nonlinear charged Lifshitz black branes with hyperscaling violation that we will consider as background. In Sec. III we calculate the QNFs of scalar perturbations numerically by using the improved AIM. Finally, our conclusions are in Sec. IV.
II. NONLINEAR CHARGED LIFSHITZ BLACK BRANES WITH HYPERSONAL VIOLATION

The nonlinear charged Lifshitz black brane that we consider is a solution of the Einstein-dilaton gravity in the presence of a linear and a nonlinear electromagnetic field, this solution was found in [29]. The action is given by

\begin{equation}
S = \frac{1}{16\pi} \int_{\mathcal{M}} d^D x \sqrt{-g} \left( R - \frac{1}{2} (\nabla \phi)^2 + V(\phi) - \frac{1}{4} e^{\lambda_1 \phi} H_{\mu\nu} H^{\mu\nu} + \frac{1}{4} e^{\lambda_2 \phi} (-F)^s \right),
\end{equation}

where \( R \) is the Ricci scalar on the manifold \( \mathcal{M} \), \( \phi \) is the dilatonic field, \( \lambda_1 \) and \( \lambda_2 \) are free parameters of the model, \( F \) is the Maxwell invariant of the electromagnetic field \( F_{\mu\nu} = \partial_{[\mu} A_{\nu]} \), where \( A_\mu \) is the electromagnetic potential and \( H_{\mu\nu} = \partial_{[\mu} B_{\nu]} \) is a linear electromagnetic field, where \( B_\mu \) is the electromagnetic potential. The following metric is solution of the theory defined by the action (3), and it represents a black brane solution with hyperscaling violating factor

\begin{equation}
ds^2 = r^{2\alpha} \left( -r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \sum_{i=1}^{D-2} dx_i^2 \right),
\end{equation}

with

\begin{equation}
f(r) = 1 - \frac{M}{r^{z+D-2-\theta}} + \frac{Q^{2s}}{r^{\Gamma+D-2+z-\theta}},
\end{equation}

where \( \alpha = -\frac{\theta}{D-2} \) has been used, and \( \theta \) is the hyperscaling exponent, also

\begin{equation}
Q^{2s} = \frac{(2s-1) r_0^{2z-1}}{4(D-2-\theta)\Gamma}(2q_2^2)^s, \quad \Gamma = z - 2 + \frac{D - 2 - \theta}{2s - 1},
\end{equation}

where \( M \) is an integration constant related to the mass of the black brane and \( q_2 \) is an integration constant related to its electric charge. The solutions are not valid for \( \alpha = -1 \). To have \( f(r) \to 1 \) when \( r \to \infty \), the following inequalities must be satisfied

\begin{equation}
z + D - 2 - \theta > 0, \quad \Gamma + D - 2 + z - \theta > 0,
\end{equation}

however, it can be shown that \( \Gamma > 0 \) [29]. The gauge and dilatonic fields are given by

\begin{align*}
F_{rt} &= q_2 r_0^{2(\frac{\theta}{D-2} + z - 1)} r^{-(\Gamma + 1)}, \\
H_{rt} &= q_1 r_0^{2(\frac{\theta}{D-2} + 1 - \theta - 1)} r^{(D-2+z-1)}, \\
\phi(r) &= \ln \left( \frac{r}{r_0} \right) \sqrt{2(D-2-\theta)\left(\frac{\theta}{D-2} + z - 1\right)},
\end{align*}

\( \phi(r) \),
Thus, for a real dilatonic field we must have \((D - 2 - \theta) \left(\frac{\theta}{D-2} + z - 1\right) \geq 0\). Moreover, since \(z + D - 2 - \theta > 0\), \(z\) has to be larger than 1 too. It is worth to mention that the condition for having a black hole is

\[
\left(\frac{\Gamma M}{\Gamma + z + D - 2 - \theta}\right)^{\Gamma + z + D - 2 - \theta} \geq \left(\frac{\Gamma Q^{2s}}{z + D - 2 - \theta}\right)^{z + D - 2 - \theta}.
\]

The temperature and entropy of the solution are given by

\[
T = \frac{1}{4\pi} \left( (z + D - 2 - \theta)r_h^z - \Gamma Q^{2s}r_h^{-(\Gamma + D - 2 - \theta)} \right), \quad S = \frac{1}{4} r_h^{D - 2 - \theta},
\]

where \(r_h\) denotes the event horizon. The study of the thermodynamics was performed in detail in [29].

### III. Quasinormal Modes

The QNMs of scalar perturbations in the background of a \(D\)-dimensional nonlinear charged Lifshitz black brane are given by the scalar field solution of the Klein-Gordon equation with suitable boundary conditions for a black brane geometry. This means there are only ingoing waves on the event horizon and we consider that the scalar field vanishes at spatial infinity, known as Dirichlet boundary condition. The Klein-Gordon equation for a scalar field minimally coupled to curvature is

\[
\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \varphi \right) = m^2 \varphi \,
\]

(11)

where \(m\) is the mass of the scalar field \(\varphi\). Now, by means of the following ansatz

\[
\varphi = e^{-i\omega t} e^{i\vec{x} \cdot \vec{r}} R(r),
\]

(12)

where \(\vec{x}\) is a spatial vector in \(D - 2\) dimensions, and \(-\kappa^2\) is the eigenvalue of the Laplacian in the flat base submanifold. The Klein-Gordon equation reduces to

\[
\frac{1}{r^\beta} \frac{d}{dr} \left( r^{2+\beta-2\alpha} f(r) \frac{dR}{dr} \right) + \left( \frac{\omega^2}{r^{2\alpha+2z} f(r)} - \frac{\kappa^2}{r^{2\alpha+2}} - m^2 \right) R(r) = 0,
\]

(13)

where \(\beta = (\alpha + 1) D + z - 3\) has been defined. Now, defining \(R(r)\) as

\[
R(r) = \frac{F(r)}{r^n},
\]

(14)
where \( n = \frac{(D-2)(1+\alpha)}{2} \), and by using the tortoise coordinate \( x \) given by
\[
d x = \frac{d r}{r^{z+1} f(r)} ,
\]
the Klein-Gordon equation can be written as a one-dimensional Schrödinger equation
\[
\frac{d^2 F(x)}{dx^2} - V(r) F(x) = -\omega^2 F(x) ,
\]
with an effective potential \( V(r) \) given by
\[
V(r) = \frac{1}{4} r^{2z-2} f(r) \left( 4(m^2 r^{2+2\alpha} + \kappa^2) + (D-2)(1+\alpha) \left( ((D-2)(1+\alpha) + 2z) f(r) + 2r \frac{df}{dr} \right) r^2 \right) ,
\]
that diverges at spatial infinity. It is worth to mention that is not trivial to find analytical solutions to Eq. (13). In the next section, we will perform numerical studies by using the improved AIM [51], which is an improved version of the method proposed in references [52, 53] and it has been applied successful in the context of QNMs for different black holes geometries, see for instance [40, 51, 54–56].

1. Stability analysis

Following the argument used in [57], adapted to Lifshitz geometries with hyperscaling violation, we can verify when the imaginary part of the quasinormal frequency \( \omega \) is always negative. By using outgoing Eddington-Filkenstein coordinates \( v = t + x \), metric (11) can be transformed to
\[
\text{d} s^2 = r^{2\alpha} \left( r^{2z} f(r) \text{d}v^2 + 2r^{z-1} \text{d}v \text{d}r + r^{2} \sum_{i=1}^{D-2} \text{d}x_i^2 \right) ,
\]
Now, taking as ansatz
\[
\varphi = e^{-i\omega t} e^{i\arctan^2 \frac{\psi(r)}{r^n}} ,
\]
with \( n = \frac{(D-2)(1+\alpha)}{2} \), the Klein-Gordon equation yields
\[
\frac{d}{dr}(r^{1+z} f(r) \psi'(r)) - 2i\omega \psi'(r) - V(r) \psi(r) = 0 ,
\]
where
\[
V(r) = nr^z f'(r) + n(n+z)r^{z-1} f(r) + \kappa^2 r^{z-3} + m^2 r^{z-2\alpha-1} .
\]
Notice that $n > -\frac{5}{2}$ according to the inequalities \([7]\). Then, multiplying equation \([20]\) by $\psi^*$ and performing integrations by parts, and using Dirichlet boundary condition for the scalar field at spatial infinity, one can obtain

$$
\int_{r_h}^{\infty} dr \left( r^{4+z} f(r) \left| \frac{d\psi}{dr} \right|^2 + V(r) |\psi|^2 \right) = -\frac{\left| \omega \right|^2 |\psi(r = r_h)|^2}{\Im(\omega)},
$$

thus, the stability of the black brane under scalar field perturbations is guaranteed for a strictly positive potential $V(r)$ outside the horizon, because in this case, equation \([22]\) is satisfied only for $\Im(\omega) < 0$. Notice that the potential \([21]\) is positive for $n > 0$ or $\alpha > -1$, what guaranties the stability of the black brane solution. In this work we focus our attention to this case ($\alpha > -1$).

2. Numerical analysis

In order to implement the improved AIM we make the following change of variable $y = 1 - r_h/r$ to equation \([13]\), where $r_h$ denotes the location of the event horizon, thus, in this coordinate, the event horizon is located at $y = 0$ and the spatial infinity at $y = 1$. Then, the Klein-Gordon equation becomes

$$
\frac{d^2 R}{dy^2} + \left( \frac{-2 \alpha - \beta}{1 - y} + \frac{f'(y)}{f(y)} \right) \frac{dR}{dy} + \left( \frac{\omega^2 (1 - y)^{2z-2}}{r_h^{2z} f(y)^2} - \frac{r_h^2}{r_h f(y)} - \frac{m^2 r_h^{2\alpha}}{(1 - y)^{2\alpha+2} f(y)} \right) R = 0,
$$

in this equation $f(y)$ refers to the function $f(r)$ evaluated at $r = \frac{r_h}{1-y}$; that is

$$
f(y) = 1 - \frac{M(1 - y)^{z+D-2-\theta}}{r_h^{z+D-2-\theta}} + \frac{Q^2 s(1 - y)^{\Gamma+D-2+z-\theta}}{r_h^{\Gamma+D-2+z-\theta}},
$$

and $f'(y) = \frac{df(y)}{dy}$. Now, we must consider the behavior of the scalar field on the event horizon and at spatial infinity. First, notice that the tortoise coordinate is given in terms of the $y$ coordinate by

$$
dx = \frac{dr}{r^{z+1} f(r)} = \frac{(1 - y)^{-z-1} dy}{r_h^2 f(y)}.
$$

- Event horizon

In the limit $y \to 0$ ($r \to r_h$) the function $f(y)$ tends to $f(y) = f'(y)y + \mathcal{O}(y^2)$, where $\mathcal{O}(y^2)$ denotes terms of order $y^2$ and higher which can be neglected when $y \to 0$. 
Therefore, the tortoise coordinate is given explicitly by

$$x = \frac{\ln(y)}{f'(0)r_h^z},$$

where \( f'(0) = \frac{df(y)}{dy} |_{y=0} \) and the effective potential \( V(x) \) tends to zero in this limit; thus, equation (16) reduces to

$$\frac{d^2F(x)}{dx^2} = -\omega^2 F(x), \quad (27)$$

and its solution is

$$F(x) = C_1e^{-\omega x} + C_2e^{\omega x}. \quad (28)$$

Imposing as boundary condition that only ingoing waves exist on the event horizon, we must set \( C_2 = 0 \). Therefore, the solution near the horizon is given by

$$F(x) = C_1e^{-\omega x} \sim y^{-\frac{\omega}{r_h f'(0)}}. \quad (29)$$

**Spatial infinity**

On the other hand, when \( y \to 1 \ (r \to \infty) \), the tortoise coordinate is given by

$$x = \frac{(1 - y)^z}{z r_h^z}. \quad (30)$$

In the following we will consider two cases: the first case corresponds to \(-1 < \alpha < 0\), and the second case corresponds to \(\alpha > 0\) and \(m = 0\). In both cases the effective potential tends to:

$$V(x) = \frac{\delta}{z^2 x^2}, \quad (31)$$

where \( \delta = n(n + z) = \frac{1}{4}(1 + \alpha)(D - 2)(2z + (D - 2)(1 + \alpha)) \). Then, equation (16) for \( x \to 0 \) becomes

$$\frac{d^2F(x)}{dx^2} - \frac{\delta}{z^2 x^2} F(x) = 0, \quad (32)$$

whose solution is

$$F(x) = D_1x^\frac{1}{2}(1 - \sqrt{1 + \frac{4\delta}{z^2}}) + D_2x^\frac{1}{2}(1 + \sqrt{1 + \frac{4\delta}{z^2}}). \quad (33)$$

Notice that the effective potential asymptotically tends to \(+\infty\) or to \(-\infty\) depending if \( \delta \) is positive or negative, respectively. In this work we focus on \( \delta > 0 \), and imposing Dirichlet boundary condition, that is, to have a null field at spatial infinity, we must set \( D_1 = 0 \). Therefore the solution becomes

$$F(x) = D_2x^\frac{1}{2}(1 + \sqrt{1 + \frac{4\delta}{z^2}}) \sim (1 - y)^\frac{1}{2}(1 + \sqrt{1 + \frac{4\delta}{z^2}}). \quad (34)$$
Notice that, due to (7), \( n + z > z/2 \). So, requiring \( \delta = n(n + z) > 0 \) implies \( n > 0 \), or \( \alpha > -1 \).

Thus, taking into account these behaviors we define

\[
R(y) = y - \frac{\sqrt{r_h f'(0)}}{(1 - y)\sqrt{1 + 4\delta/z^2}} \chi(y),
\]

as ansatz. Then, by inserting these fields in Eq. (23) we obtain the homogeneous linear second-order differential equation for the function \( \chi(z) \)

\[
\chi'' = \lambda_0(y)\chi' + s_0(y)\chi,
\]

where

\[
\lambda_0(y) = \frac{1}{r_h f'(0)(1 - y) f(y)} \left( (r_h^2 f'(0) y (3 + 2\alpha - D(1 + \alpha) + z \sqrt{1 + 4\delta/z^2}) + 2i(1 - y) \omega f(y) - r_h^2 f'(0) y (1 - y)^2 f'(y) \right),
\]

\[
s_0(y) = -\frac{1}{2r_h^{2(z+1)} f'(0)^2 y^2 (1 - y)^{2(1 + \alpha)} f(y)^2} \left( 2r_h^2 f'(0)^2 \omega^2 y^2 (1 - y)^{2(z + \alpha)} - r_h^2 (1 - y)^{2\alpha} \right.
\]

\[
\left. \left( r_h^2 f'(0)^2 y^2 (-2\delta + (D - 2) z (1 + \alpha)(1 + \sqrt{1 + 4\delta/z^2})) - 2ir_h f'(0)(1 - y) \right)
\right)
\]

\[
\left. (1 - (D - 2)(1 + \alpha)y + z \sqrt{1 + 4\delta/z^2} \omega + 2(1 - y)^2 \omega^2) f(y)^2 - r_h^2 f'(0) y f(y) \right)
\]

\[
\left. (2r_h^2 f'(0) y (r_h^{2\alpha + 2} m^2 + \kappa^2 (1 - y)^{2+2\alpha}) + r_h^2 (1 - y)^{1+2\alpha} (r_h^2 f'(0) z (1 + \sqrt{1 + 4\delta/z^2}) y
\right)
\]

\[
\left. + 2i(1 - y) \omega f'(y)) \right).
\]

Then, in order to implement the improved AIM it is necessary to differentiate Eq. (36) \( n \) times with respect to \( y \), which yields the following equation:

\[
\chi^{n+2} = \lambda_n(y)\chi' + s_n(y)\chi,
\]

where

\[
\lambda_n(y) = \lambda_{n-1}'(y) + s_{n-1}(y) + \lambda_0(y) \lambda_{n-1}(y),
\]

\[
s_n(y) = s_{n-1}'(y) + s_0(y) \lambda_{n-1}(y).
\]

Then, by expanding the \( \lambda_n \) and \( s_n \) in a Taylor series around some point \( \eta \), at which the improved AIM is performed yields

\[
\lambda_n(\eta) = \sum_{i=0}^{\infty} c_i (y - \eta)^i,
\]
\[ s_n(\eta) = \sum_{i=0}^{\infty} d_i^n (y - \eta)^i, \tag{43} \]

where the \( c_i^n \) and \( d_i^n \) are the \( i^{th} \) Taylor coefficients of \( \lambda_n(\eta) \) and \( s_n(\eta) \), respectively, and by replacing the above expansions in Eqs. (40) and (41) the following set of recursion relations for the coefficients is obtained:

\[
\begin{align*}
    c_i^n &= (i + 1)c_{i-1}^{n-1} + d_{i-1}^{n-1} + \sum_{k=0}^{i} c_{i-k}^k c_{i-k}^{n-1}, \tag{44} \\
    d_i^n &= (i + 1)d_{i-1}^{n-1} + \sum_{k=0}^{i} d_{i-k}^k c_{i-k}^{n-1}. \tag{45}
\end{align*}
\]

In this manner, the authors of the improved AIM have avoided the derivatives that contain the AIM in [51, 54], and the quantization condition, which is equivalent to imposing a termination to the number of iterations, is given by

\[ d_0^n c_{n-1}^0 - d_0^{n-1} c_0^n = 0. \tag{46} \]

We solve this equation numerically and we choose different values for the parameters. Thus, without loss of generality, we choose the following values \( D = 4, M = 4, z = 3, r_0 = 1, m = 0 \) and \( \kappa = 0 \) in Tables I, II and III. In Table I, we show some lowest QNFs, for massless scalar field with \( s = 1.5, q_2 = 1 \) and different values of \( \theta \). Then, in Table II we show some lowest QNFs, for \( \theta = 1, q_2 = 1 \) and different values of \( s \) and in Table III we show QNFs with \( \theta = 1, s = 2 \) and different values of \( q_2 \). In Table IV we show QNFs for \( D = 4, M = 4, \theta = 1, s = 2, r_0 = 1, q_2 = 1, m = 0, \kappa = 0 \) and different values of \( z \). We observe that in all cases analyzed the QNFs have an imaginary part that is negative, which ensures the stability of nonlinear charged Lifshitz black branes with hyperscaling violation under massless scalar perturbations. Then, in Table V we show some lowest QNFs for massive scalar fields, for \( s = 2, \theta = 1, \kappa = 1, \) and different values of \( m \). Finally, in Table VI we show fundamental QNFs for massless scalar field for \( D = 4 \) and \( D = 5 \), with different values of the dynamical and hyperscaling violating exponents. Note that the Klein-Gordon equation depends on the combination \( D - \theta \), this is the reason why, for instance, the same QNFs are obtained for \( D = 4, \theta = 0.5 \) and \( D = 5, \theta = 1.5 \) in Table VI.
Table I. Some quasinormal frequencies for $D = 4, M = 4, s = 1.5, r_0 = 1, q_2 = 1, z = 3, m = 0, \kappa = 0$ and different values of $\theta$.

| $n$ | $\theta = -1$ | $\theta = -0.5$ | $\theta = 0$ | $\theta = 0.5$ | $\theta = 1$ | $\theta = 1.5$ |
|-----|---------------|-----------------|--------------|---------------|-------------|-------------|
| 0   | -10.71750i    | -8.77694i       | -7.57904i    | -6.57065i     | -5.45750i   | -2.85448i   |
| 1   | -12.82510i    | -13.22240i      | -12.62600i   | -11.65420i    | -10.15690i  | -5.56952i   |
| 2   | -21.81980i    | -19.50890i      | -18.14820i   | -16.81610i    | -14.83420i  | -8.26014i   |
| 3   | -25.13070i    | -24.87590i      | -23.57970i   | -21.99220i    | -19.51670i  | -10.94460i  |

Table II. Quasinormal frequencies for $D = 4, M = 4, s = 3, \theta = 1, r_0 = 1, q_2 = 1, m = 0, \kappa = 0$ and different values of $s$.

| $n$ | $s = 1.5$ | $s = 2$ | $s = 2.1$ |
|-----|---------|-------|---------|
| 0   | -5.45750i | -3.16787i | -2.22508i |
| 1   | -10.15690i | -6.05288i | -4.30905i |
| 2   | -14.83420i | -8.87462i | -6.34705i |
| 3   | -19.51670i | -11.68160i | -8.36704i |

IV. CONCLUDING COMMENTS

In this work we have calculated numerically the QNFs of scalar field perturbations of nonlinear charged Lifshitz black branes with hyperscaling violation by imposing suitable boundary conditions on the event horizon and at spatial infinity. The scalar field is considered as a mere test field, without backreaction over the spacetime itself. In this study we have considered the case $\alpha > -1$, additionally, for $\alpha > 0$ we have restricted to massless scalar field $m = 0$. Then, we have studied the stability of these black branes under massive and massless scalar field perturbations. In general, our results show that the QNFs have a negative imaginary part. Therefore, the black brane is stable under massive and massless scalar field perturbations. Also, we can see that there is a limit on the dynamical exponent $z$ above which the system is always overdamped for a given dimension, and the hyperscaling violating exponent shifts this limit. For instance, as we can observe in Table [VI] for $D = 5$ and $\theta = -1$ we have found that the system is non-overdamped for $z = 3.5$, but it is overdamped for $z = 4$. However, for $D = 5$ and $\theta = 0$ we have found that the system is non-
Table III. Quasinormal frequencies for $D = 4$, $M = 4$, $z = 3$, $s = 2$, $\theta = 1$, $r_0 = 1$ $m = 0$, $\kappa = 0$ and different values of $q_2$.

| $n$ | $q_2 = 0.1$ | $q_2 = 0.5$ | $q_2 = 1$ |
|-----|---|---|---|
| 0   | $-6.66991i$ | $-6.49942i$ | $-3.16787i$ |
| 1   | $-12.25930i$ | $-11.96360i$ | $-6.05288i$ |
| 2   | $-17.91470i$ | $-17.48060i$ | $-8.87462i$ |
| 3   | $-23.57020i$ | $-22.99920i$ | $-11.68160i$ |

Table IV. Quasinormal frequencies for $D = 4$, $M = 4$, $\theta = 1$, $s = 2$, $r_0 = 1$, $q_2 = 1$, $m = 0$, $\kappa = 0$ and different values of $z$.

| $n$ | $z = 3$ | $z = 5$ | $z = 8$ |
|-----|---|---|---|
| 0   | $-3.16787i$ | $-8.80358i$ | $-15.17150i$ |
| 1   | $-6.05288i$ | $-16.87330i$ | $-29.48100i$ |
| 2   | $-8.87462i$ | $-24.90450i$ | $-43.77070i$ |
| 3   | $-11.68160i$ | $-32.93630i$ | $-58.06180i$ |

overdamped for $z = 2.5$ and it is overdamped for $z = 3$, and for $D = 5$ and $\theta = 0.5$ we have found that the system is overdamped for $z = 2.5$. It is worth to mention that the shift also depends on the dimension. On the other hand, when we increase the hyperscaling violating exponent the relaxation time of the dual thermal states increases (due to the absolute value of the imaginary part of the QNFs decreases) and when we increase the nonlinear exponent the relaxation time increases too, when the system is overdamped.

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Table V. Quasinormal frequencies for $D = 4$, $M = 4$, $z = 3$, $\theta = 1$, $r_0 = 1$, $q_2 = 1$, $\kappa = 1$, $s = 2$ and different values of $m$.

| $n$ | $m = 1$      | $m = 2$      | $m = 3$      |
|-----|-------------|-------------|-------------|
| 0   | $-3.49027i$ | $-3.94694i$ | $-4.58214i$ |
| 1   | $-6.32653i$ | $-6.75628i$ | $-7.37113i$ |
| 2   | $-9.11635i$ | $-9.52451i$ | $-10.12130i$|
| 3   | $-11.89980i$| $-12.28760i$| $-12.86430i$|

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Table VI. Fundamental quasinormal frequencies for $M = 4, Q = 1, m = 0, \kappa = 0, s = 2, r_0 = 1, q_2 = 1$ and different values of $D, z$ and $\theta$. (In the empty rows no event horizon exist or $\Gamma < 0$).

| $z$ | $\theta = -1$ | $\theta = 0$ | $\theta = 0.5$ | $\theta = 1$ | $\theta = 1.5$ |
|-----|---------------|--------------|----------------|---------------|----------------|
| 1.5 | $2.86396 - 5.49737i$ | -- | -- | -- | -- |
| 2   | $2.71457 - 7.93528i$ | $-3.50142i$ | -- | -- | -- |
| 2.5 | $1.54271 - 9.78902i$ | $-5.52189i$ | $-3.64301i$ | -- | -- |
| 3   | $-9.75248i$ | $-6.70287i$ | $-5.34727i$ | $-3.16787i$ | -- |
| 3.5 | $-10.11190i$ | $-7.73738i$ | $-6.61238i$ | $-5.01956i$ | -- |
| 4   | $-10.76490i$ | $-8.72993i$ | $-7.74275i$ | $-6.41728i$ | $-2.33210i$ |
| 4.5 | $-11.52580i$ | $-9.70835i$ | $-8.81550i$ | $-7.65052i$ | $-4.61700i$ |

| $z$ | $\theta = -1$ | $\theta = 0$ | $\theta = 0.5$ | $\theta = 1$ | $\theta = 1.5$ |
|-----|---------------|--------------|----------------|---------------|----------------|
| 1.5 | $5.45477 - 5.84803i$ | $2.86396 - 5.49737i$ | -- | -- | -- |
| 2   | $5.38082 - 8.28982i$ | $2.71457 - 7.93528i$ | $0.25590 - 7.06029i$ | $-3.50142i$ | -- |
| 2.5 | $4.76150 - 10.52210i$ | $1.54271 - 9.78902i$ | $-7.28570i$ | $-5.52189i$ | $-3.64301i$ |
| 3   | $3.74054 - 12.47720i$ | $-9.75248i$ | $-8.01991i$ | $-6.70287i$ | $-5.34727i$ |
| 3.5 | $2.29788 - 14.15880i$ | $-10.11190i$ | $-8.83530i$ | $-7.73738i$ | $-6.61238i$ |
| 4   | $-14.10110i$ | $-10.76490i$ | $-9.69375i$ | $-8.72993i$ | $-7.74275i$ |
| 4.5 | $-14.01030i$ | $-11.52580i$ | $-10.57970i$ | $-9.70835i$ | $-8.81550i$ |

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