Quantum noise reduction using a cavity with a Bose–Einstein condensate

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Abstract
We study an opto-mechanical system in which the collective density excitations (Bogoliubov modes) of a Bose–Einstein condensate (BEC) is coupled to a cavity field. We show that the optical force changes the frequency and the damping constant of the collective density excitations of the BEC. We further analyse the occurrence of normal mode splitting (NMS) due to mixing of the fluctuations of the cavity field and the fluctuations of the condensate with finite atomic two-body interaction. The NMS is found to vanish for small values of the two-body interaction. We further show that the density excitations of the condensate can be used to squeeze the output quantum fluctuations of the light beam. This system may serve as an opto-mechanical control of quantum fluctuations using a BEC.

1. Introduction
In recent years mechanical and optical degrees of freedom have become entangled experimentally by the underlying mechanism of radiation pressure forces. This field, known as cavity opto-mechanics, has played a vital role in the conceptual exploration of the boundaries between classical and quantum mechanical systems. The coupling of mechanical and optical degrees of freedom via radiation pressure has been a subject of early research in the context of laser cooling [1–3] and gravitational-wave detectors [4]. Recently there has been a great surge of interest in the application of radiation forces to manipulate the centre-of-mass motion of mechanical oscillators covering a huge range of scales from macroscopic mirrors in the Laser Interferometer Gravitational Wave Observatory (LIGO) project [5, 6] to nano-mechanical cantilevers [7–12], vibrating microtoroids [13, 14] and membranes [15]. The central accomplishment of the field of cavity opto-mechanics is the investigation of radiation pressure forces which allow one to manipulate the motional state of micromechanical oscillators. In particular, it has become possible to substantially cool the thermal excitation of a single mechanical mode, down to a few tens of remaining phonons [16]. With these developments, micro- and nano-mechanical resonators now represent an important model system with the prospect of demonstrating quantum effects on a macroscopic scale. Theoretical work has also proposed to use the radiation-pressure coupling for quantum non-demolition measurements of the light field [17].

Interfacing ultracold atomic systems with mechanical resonators could both profit from and contribute to these developments. The intriguing question is raised of whether the sophisticated toolbox for coherent manipulation of the quantum state of atoms could be employed to read out, cool and coherently manipulate mechanical oscillators. New possibilities for cavity opto-mechanics may emerge by combining the tools of cavity quantum electrodynamics (QED) with those of ultracold gases [18–23]. These studies show that sufficiently strong and coherent coupling would enable studies of atom-oscillator entanglement, quantum state transfer and quantum control of mechanical force sensors. Due to coupling between the condensate wavefunction and the cantilever, mediated by the cavity photons, the cantilever displacement is expected to strongly influence the superfluid properties of the condensate [20]. Recently, coupled dynamics of a movable mirror and atoms trapped in the standing wave light field of a cavity were studied [24]. It was shown that the dipole potential in which the atoms move is modified due to the back-action of the atoms and that the position of the atoms can become bistable.

Experimental implementation of a combination of cold atoms and cavity QED has made significant progress [25–29]. Placing an ensemble of atoms inside a high-finesse cavity enhances the atom–light interaction because the atoms collectively couple to the same light mode. The motional
degrees of freedom of ultracold atomic gases represent a new source of long-lived coherence affecting the light–atom interaction. Nonlinear optics arising from this long-lived coherent motion of ultracold atoms trapped within a high-finesse Fabry–Perot cavity was reported recently [19]. Strong optical nonlinearities were observed even at the low mean photon number of 0.05. This nonlinearity also gives rise to bistability in the transmitted probe light through the cavity. In the cavity the influence of atomic back-action and the external driving pump becomes important and modify the optical potential [30]. Due to the coupling between the condensate wavefunction and the cavity modes, the cavity light field develops a band structure [31]. Theoretically there have been some interesting works on the correlated atom-field dynamics in a cavity. It has been shown that the strong coupling of the condensed atoms to the cavity mode changes the resonance frequency of the cavity [32]. Finite cavity response times lead to damping of the coupled atom-field excitations [33]. The driving field in the cavity can significantly enhance the localization and the cooling properties of the system [34, 35].

It is well known that by using a resonant optical cavity around a Kerr medium, the quantum fluctuations of an incoming light beam can be reduced below the standard quantum-noise limit for a given quadrature component. The quantum optical properties of a mirror coupled via radiation pressure to a cavity field show interesting similarities to an intracavity Kerr-like interaction [36, 37] and can be used to reduce quantum noise of the light field reflected by such a cavity.

In this work, we study another kind of Kerr-type medium, namely the collective motion of a trapped macroscopic ensemble of an ultracold gas coupled to the intensity of the light field inside a cavity which serves as a mechanical oscillator (analogous to a movable mirror). First we show how the optical force modifies the frequency and damping constant of the collective density excitations of the Bose–Einstein condensate (BEC) and further show the occurrence of normal mode splitting (NMS) due to mixing of the optical mode and the Bogoliubov mode. Finally we demonstrate for the first time that the density excitations of the condensate can be used to reduce the quantum noise of the cavity field reflected by the BEC. The role of the two-body interaction on the dynamics of the coupled system is also explored. This system may serve as an opto-mechanical control of quantum fluctuations using a BEC.

2. Quantum Langevin equations for the system

We consider an elongated cigar-shaped BEC of $N$ two-level $^{87}$Rb atoms in the $| F = 1 \rangle$ state with mass $m$ and frequency $\omega_0$ of the $| F = 1 \rangle \rightarrow | F' = 2 \rangle$ transition of the $D_2$ line of $^{87}$Rb, interacting with a quantized single standing wave cavity mode of frequency $\omega_c$. The standing wave that forms in the cavity results in a one-dimensional optical lattice potential. The cavity field is also coupled to external fields incident from the one-side mirror. It is well known that high-$Q$ optical cavities can significantly isolate the system from its environment, thus strongly reducing decoherence and ensuring that the light field remains quantum-mechanical for the duration of the experiment. We also assume that the induced resonance frequency shift of the cavity is much smaller than the longitudinal mode spacing, so that we restrict the model to a single longitudinal mode. In order to create an elongated BEC, the frequency of the harmonic trap along the transverse direction should be much larger than one in the axial (along the direction of the optical lattice) direction. The system is also coherently driven by a laser field with frequency $\omega_p$ through the cavity mirror with amplitude $\eta$. This system is modelled by the opto-mechanical Hamiltonian $(H_{\text{om}})$ in a rotating wave (in the reference frame oscillating at the frequency $\omega_p$ of the pump) and the dipole approximation (according to dipole approximation, we can neglect the effects of the position of the electron when we are dealing with the atom-field interaction):

$$H_{\text{om}} = \frac{p^2}{2m} - \hbar \Delta_0 \sigma^+ \sigma^- - \hbar \Delta_c \hat{a}^\dagger \hat{a} - i\hbar g(x) \times [\sigma^+ \hat{a} - \sigma^- \hat{a}^\dagger] - i\eta (\hat{a} - \hat{a}^\dagger), \quad (1)$$

where $\Delta_0 = \omega_p - \omega_\text{om}$ and $\Delta_c = \omega_p - \omega_c$ are the large atom-pump and cavity-pump detuning, respectively. A large atom-pump detuning is essential to suppress the spontaneous emission which otherwise would lead to momentum diffusion and hence heating of the atomic sample. Here $\sigma^+$, $\sigma^-$ are the Pauli matrices. In the presence of the standing wave, the atom-field coupling is written as $g(x) = g_0 \cos(kx)$ with the wavenumber $k = \omega_p/c$. Here $g_0$ is the vacuum Rabi frequency and $\eta$ is the annihilation operator for a cavity photon. The input laser field populates the intracavity mode which couples to the atoms through the dipole interaction. The field in turn is modified by the back-action of the atoms. The system we are considering is intrinsically open as the cavity field is damped by the photon leakage through the massive coupling mirror. Since the detuning $\Delta_c$ is large, spontaneous emission is negligible and we can adiabatically eliminate the excited state using the Heisenberg equation of motion $\sigma^- = \frac{1}{\hbar}[H_{\text{om}}, \sigma^+]$. This yields the single particle Hamiltonian

$$H_0 = \frac{\hbar^2 \omega_c}{2m} - \hbar \Delta_c \hat{a}^\dagger \hat{a} + \cos^2(kx)[V_\text{cl} + \hbar U_0(\hat{a}^\dagger \hat{a})] - i\eta (\hat{a} - \hat{a}^\dagger), \quad (2)$$

The parameter $U_0 = \frac{g_0^2}{\omega_c}$ is the optical lattice barrier height per photon and represents the atomic back-action on the field [30]. $V_\text{cl}(r)$ is the external classical potential. Here we will always take $U_0 > 0$. In this case the condensate is attracted to the nodes of the light field and hence the lowest bound state is localized at these positions which leads to a reduced coupling of the condensate to the cavity compared to that for $U_0 < 0$. Along $x$, the cavity field forms an optical lattice potential of period $\lambda/2$ and depth $\hbar U_0(\hat{a}^\dagger \hat{a}) + V_\text{cl}$. We now write the Hamiltonian in a second quantized form including the two-body interaction term:

$$H = \int d^3x \Psi^\dagger(\vec{r}) H_0 \Psi(\vec{r}) + \frac{1}{2} \frac{4\pi \hbar^2}{m} \int d^3x \Psi^\dagger(\vec{r}) \Psi^\dagger(\vec{r}) \Psi(\vec{r}) \Psi(\vec{r}), \quad (3)$$

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where \( \Psi(\vec{r}) \) is the field operator for the atoms. Here \( l_s \) is the two-body s-wave scattering length. The corresponding opto-mechanical Bose–Hubbard (OMBH) Hamiltonian can be derived by writing \( \Psi(\vec{r}) = \sum_j \hat{b}_j w(\vec{r} - \vec{r}_j) \), where \( w(\vec{r} - \vec{r}_j) \) is the Wannier function and \( \hat{b}_j \) is the corresponding annihilation operator for the bosonic atom at the \( j \)th site. Note that this substitution is valid only for the lowest band of the potential. Retaining only the lowest band with the nearest-neighbour interaction, we have

\[
H = E_0 \sum_j \hat{b}_j^\dagger \hat{b}_j + E \sum_j (\hat{b}_j^\dagger \hat{b}_{j+1} + \hat{b}_{j+1}^\dagger \hat{b}_j) + (\hbar U_0 \hat{a}^\dagger \hat{a} + V_\text{cl}) \left\{ J_0 \sum_j \hat{b}_j^\dagger \hat{b}_j + J \sum_j (\hat{b}_{j+1}^\dagger \hat{b}_j + \hat{b}_j^\dagger \hat{b}_{j+1}) \right\} + \frac{U}{2} \sum_j \hat{b}_j^\dagger \hat{b}_j \hat{b}_j + \hbar \Delta_a \hat{a}^\dagger \hat{a} - \hbar \eta (\hat{a} - \hat{a}^\dagger),
\]

where

\[
U = \frac{4\pi l_s \hbar^2}{m} \int d^3 x |w(\vec{r})|^4,
\]

\[
E_0 = \int d^3 x w(\vec{r} - \vec{r}_j) \left\{ \frac{-\hbar^2 \nabla^2}{2m} \right\} w(\vec{r} - \vec{r}_j),
\]

\[
E = \int d^3 x w(\vec{r} - \vec{r}_j) \left\{ \frac{-\hbar^2 \nabla^2}{2m} \right\} w(\vec{r} - \vec{r}_{j\pm 1}),
\]

\[
J_0 = \int d^3 x w(\vec{r} - \vec{r}_j) \cos^2(kx) w(\vec{r} - \vec{r}_j),
\]

\[
J = \int d^3 x w(\vec{r} - \vec{r}_j) \cos^2(kx) w(\vec{r} - \vec{r}_{j\pm 1}).
\]

The OMBH Hamiltonian derived above is valid only for weak atom-field nonlinearity \([38]\). In the above OMBH Hamiltonian, we have assumed a constant scattering length \( l_s \). Strictly speaking this is not true. The strong atom-field coupling will induce significant density fluctuations which will make the scattering length time dependent. The parameter \( U_0 = g_0^2 / \Delta_a \) signifies the effective coupling of the atoms with the cavity field. A moderate value of \( U_0 \) can be achieved by tuning \( \Delta_a \) so as to keep the density fluctuations small. In this situation, we could safely take the scattering length to be constant. It has been shown \([24, 39]\) that the intracavity field intensity is bistable and leads to a bistable optical lattice potential. The position of the individual lattice wells is bistable as well since a mirror displacement \( l_m \) displaces each optical lattice well by \( l_m \) in the same direction. However, we consider a regime when \( l_m/(\pi/k) \ll 1 \), and thus we ignore this effect on the Wannier function used above. The nearest-neighbour nonlinear interaction terms are usually very small compared to the onsite interaction and are neglected as usual. We now write down the Heisenberg–Langevin equation of motion for the bosonic field operator \( \hat{b}_j \) and the internal cavity mode \( \hat{a} \) as

\[
\dot{\hat{b}}_j = -i \left( U_0 \hat{a}^\dagger \hat{a} + \frac{V_\text{cl}}{\hbar} \right) \{ J_0 \hat{b}_j + J (\hat{b}_{j+1}^\dagger \hat{b}_j + \hat{b}_j^\dagger \hat{b}_{j+1}) \} - \frac{iE}{\hbar} \hat{b}_{j+1}^\dagger \hat{b}_j - \frac{iE_0}{\hbar} \hat{b}_j - \frac{\Gamma_b}{2} \hat{b}_j + \sqrt{\Gamma_b} \xi_b(t) + \Gamma_b \xi_b(t).
\]

\[
\dot{\hat{a}} = -iU_0 \left\{ J_0 \sum_j (\hat{b}_{j+1}^\dagger \hat{b}_j + \hat{b}_j^\dagger \hat{b}_{j+1}) + J \sum_j (\hat{b}_{j+1}^\dagger \hat{b}_j + \hat{b}_j^\dagger \hat{b}_{j+1}) \right\} \hat{a} + \eta + i \left\{ \Delta \frac{k}{2} \right\} \hat{a} + \sqrt{\kappa} \xi_b(t).
\]

Here \( \kappa \) and \( \Gamma_b \) characterizes the dissipation of the cavity field and collective density excitations of the BEC, respectively; \( \Gamma_b \) depends on the chemical potential, Rabi frequency and atomic detuning \([21]\). Here, we follow a semi-classical theory by considering noncommuting noise operators for the input field, i.e. \( \langle \xi_b(t) \rangle = 0, \langle \xi_b(t') \xi_b(t) \rangle = n_p (\delta (t' - t)) \). The thermal noise input for the BEC is provided by the thermal cloud of atoms and can be considered classical when \( k_B T > \hbar \omega_m \). Here \( T \) is the temperature of the thermal reservoir. The quantities \( n_b \) and \( n_p \) are the equilibrium occupation numbers for the mechanical BEC and optical oscillators, respectively. We consider a deep lattice formed by a strong classical potential \( V_\text{cl}(r) \), so that the overlap between Wannier functions is small. Thus, we can neglect the contribution of tunnelling by putting \( E = 0 \) and \( J = 0 \). Under this approximation, the matter-wave dynamics is not essential for light scattering. In experiments, such a situation can be realized because the time scale of light measurements can be much faster than the time scale of atomic tunnelling. One of the well-known advantages of optical lattices is their extremely high tunability. Thus, tuning the lattice potential, tunnelling can be made very slow \([40]\).

## 3. Dynamics of small fluctuations: normal mode splitting

Here we show that the coupling of the cavity field fluctuations and the condensate fluctuations (Bogoliubov mode) leads to the splitting of the normal mode into two modes (NMS). The opto-mechanical however involves driving two parametrically coupled nondegenerate modes out of equilibrium. The NMS does not appear in the steady-state spectra but rather manifests itself in the fluctuation spectra of the mirror displacement. To this end, we will shift the canonical variables to their steady-state values (i.e. \( \hat{a} \rightarrow a_s + \hat{a}, \hat{b}_j \rightarrow (\sqrt{N}/M + \hat{b}) \)) and linearize to obtain the following Heisenberg–Langevin equations such that the fluctuations have zero mean:

\[
\dot{\hat{b}} = -i[v + 2U_\text{eff}] \hat{b} - iU_\text{eff} \hat{b}_j^\dagger \hat{b}_j - \Gamma_b \hat{b} + \sqrt{\Gamma_b} \xi_b(t) + \Gamma_b \xi_b(t)
\]

\[
\dot{\hat{a}} = \left( i\Delta_d - \frac{k}{2} \right) \hat{a} - ig_c (\hat{b}^\dagger \hat{b} + \sqrt{\kappa} \xi_b(t))
\]

Here, \( U_\text{eff} = \frac{U_0}{\hbar} \), \( v = U_0 J_0 \sqrt{N} a_s^2 \), \( \xi_b = U_0 J_0 |a_s|^2 + \frac{V_\text{cl}a_s^2}{2 \hbar} + \Gamma_b^2 \), \( \Delta_a = \Delta_a - U_0 N J_0 \) is the detuning with respect to the renormalized resonance. Note that the linearized equations are valid only when the atom–photon coupling is not so strong. In deriving the above equation, we have assumed \( a_s \), the steady-state value of \( \hat{a} \), to be real. This can be achieved by an appropriate choice of the laser phase. \( N \) is
the total number of atoms in $M$ sites and $n_0 = N/M$. As before, we assume negligible tunnelling ($J = E = 0$) and hence we drop the site index $j$ from the atomic operators. We will always assume $\Gamma_b \ll \kappa$. We transform to the quadratures: $X_p = \hat{a} + \hat{a}^\dagger$, $P_p = i(\hat{a}^\dagger - \hat{a})$, $X_b = \hat{b} + \hat{b}^\dagger$, $P_b = i(\hat{b}^\dagger - \hat{b})$. Note that the steady-state values can be obtained by putting $\hat{a} = 0$ and $\hat{b} = 0$ in equations (6) and (7) and solving for $\alpha_j$. This yields a cubic equation in $\alpha_j$ which has three real solutions for certain values of parameters. Out of these three real solutions, two are stable which represents bistability. The steady-state values of $\hat{a}$ and $\hat{b}$ represent points far from the turning points in the bistable systems. The system reaches a steady state only if it is stable and the condition of stability can be obtained by applying the Routh–Hurwitz criterion to equations (8) and (9). In the following we will always be in the stable regime. The displacement spectrum

$$S_\omega(\omega) = \frac{\beta_1^2}{|d(\omega)|^2} \left[ 4\Gamma_b n_b + \frac{8\kappa g_2^\ast \kappa (\Delta_d^2 + \omega^2 + \kappa^2/4)}{(\Delta_d^2 - \omega^2 + \kappa^2/4)^2 + \omega^2 \kappa^2} \right],$$

(10)

where

$$|d(\omega)|^2 = \left[ \Omega_{\text{eff}}^2 - \omega^2 \right]^2 + \omega^2 \Gamma_{\text{eff}}^2. \tag{11}$$

and the effective Bogoliubov mechanical frequency ($\Omega_{\text{eff}}$) and the effective Bogoliubov mechanical damping ($\Gamma_{\text{eff}}$)

$$\Omega_{\text{eff}}^2 = \beta_1 \beta_2 + \frac{4\Delta_d g_2^\ast \beta_1 (\Delta_d^2 - \omega^2 + \kappa^2/4)}{(\Delta_d^2 - \omega^2 + \kappa^2/4)^2 + \omega^2 \kappa^2} \tag{12}$$

and

$$\Gamma_{\text{eff}} = \Gamma_b - \frac{4\Delta_d g_2^\ast \beta_1 \kappa}{(\Delta_d^2 - \omega^2 + \kappa^2/4)^2 + \omega^2 \kappa^2}. \tag{13}$$

Here $\beta_1 = \nu + U_{\text{eff}}$ and $\beta_2 = \nu + 3U_{\text{eff}}$. This spectrum is characterized by a mechanical susceptibility $\chi(\omega) = 1/d(\omega)$ of the condensate that is driven by thermal noise ($\propto n_b$) and by the quantum fluctuations of the radiation pressure (quantum back-action).

The modification of the frequency of the Bogoliubov excitations of the condensate due to the radiation pressure shown by equation (12) is equivalent to the ‘optical spring effect’ in cavity opto-mechanical systems with the movable mirror. This effect leads to significant frequency shifts in the case of low-frequency oscillations. In figure 1, we show the plot of the normalized effective Bogoliubov mechanical frequency ($\Omega_{\text{eff}}/\omega_m$, $\omega_m = \sqrt{\beta_1 \beta_2}$) of the BEC versus normalized frequency ($\omega/\omega_m$). The deviation of the Bogoliubov frequency of the condensate from its bare Bogoliubov frequency $\omega_m$ increases as the strength of the interaction with the cavity field increases. The right plot shows the normalized effective Bogoliubov mechanical frequency of the BEC versus normalized frequency for two values of the effective two-body interaction. A higher two-body interaction makes the condensate more robust and the Bogoliubov frequency of the condensate does not significantly deviates from $\omega_m.$ Figure 2 displays a plot of the normalized effective Bogoliubov mechanical damping ($\Gamma_{\text{eff}}/\omega_m$, $\omega_m = \sqrt{\beta_1 \beta_2}$) of the BEC versus normalized frequency ($\omega/\omega_m$). A stronger coupling with the cavity photons induces a higher atom loss and hence a higher value of the effective damping. This light-induced back-action heating and consequent loss of atoms was observed in [19]. They found that the atom loss rate was enhanced near resonance. The right plot shows the normalized effective Bogoliubov mechanical damping of the BEC versus normalized frequency for two values of the effective two-body interaction. The other parameters are the same. The larger the two-body interaction, the higher the damping of the Bogoliubov modes of the BEC. Cooling of the Bogoliubov mode of the BEC by the radiation pressure can be understood in thermodynamical sense. Radiation pressure couples the BEC to the optical cavity mode, which behaves as an effective additional reservoir for the BEC oscillator. As a consequence, the effective temperature of the Bogoliubov mode of the BEC will be intermediate between the initial thermal reservoir temperature and that of the optical reservoir.
Figure 2. Plot of the normalized effective Bogoliubov mechanical damping ($\Gamma_{\text{eff}}/\omega_m$, $\omega_m = \sqrt{\beta_{1} \beta_{2}}$) of the BEC versus normalized frequency ($\omega/\omega_m$). Parameter values are (left plot): $\Gamma_b = 0.025\omega_m$, $\kappa = 32.5\omega_m$, $\Delta_\lambda = -40\omega_m$, $U_{\text{eff}} = 1000\omega_m$, $\nu = \omega_m$ and two values of the atom–photon interaction parameter, $g_c = 6.0\omega_m$ (thin line) and $g_c = 10\omega_m$ (thick line). The right plot shows the normalized effective Bogoliubov mechanical damping of the BEC versus normalized frequency for two values of the effective two-body interaction, $U_{\text{eff}} = 1000\omega_m$ (thin line), $U_{\text{eff}} = 1000\omega_m$ (thick line) and $g_c = 10\omega_m$. The other parameters are the same.

Figure 3. Plot of the displacement spectrum $S_x(\omega)$ of the BEC versus normalized frequency and normalized effective detuning ($\Delta_\lambda$) for two values of the atomic two-body interaction. $U_{\text{eff}} = 150 \times 10^7$ Hz (left plot), $U_{\text{eff}} = 150 \times 10^5$ Hz (right plot), $\nu = 4 \times 10^4$ Hz, $\Gamma_b = 735$ Hz, $n_b = 10^4$ and $\nu = \kappa = 7.35 \times 10^4$ Hz. Clearly, we see an NMS when $U_{\text{eff}} = 150 \times 10^7$ Hz. As the effective interaction decreases, the NMS vanishes.

which is practically zero due to the condition $n_p = 0$. Therefore one can approach the mechanical ground state of the BEC when the atom–photon coupling rate $g_c$ is much larger than the damping rate $\Gamma_b$. This explains why significant mechanical cooling of the Bogoliubov mode is obtained when radiation pressure coupling is strong.

Figure 3 shows the plot of the displacement spectrum $S_x(\omega)$ of the BEC versus normalized frequency and normalized effective detuning ($\Delta_\lambda$) for two values of the atomic two-body interaction. In the presence of larger interactions, we observe the usual normal mode splitting into two modes and we find that if the atom–atom interaction is significantly less, the normal mode splits vanish (right plot of figure 3). The NMS is associated with the mixing between the fluctuation of the cavity field around the steady state and the fluctuations of the condensate (Bogoliubov mode) around the mean field. The origin of the fluctuations of the cavity field is the beat of the pump photons with the photons scattered from the condensate atoms. The frequency of the Bogoliubov mode in the low momentum limit is $\approx \sqrt{U_{\text{eff}}}$. Hence in the absence of interactions, the Bogoliubov mode is absent and as a result NMS vanishes. In the presence of finite atom–atom interaction, the photon mode and the Bogoliubov mode form a system of two coupled oscillators. An important point to note is that in order to observe the NMS, the energy exchange between the two modes should take place on a time scale faster than the decoherence of each mode. Experimentally, Normal mode splitting of a system of large number of atoms coupled to the cavity field has been achieved recently [41]. It was observed that the NMS was observed only if the coupling between the atoms and the cavity was strong enough (strong cooperative coupling regime). This regime was achieved by increasing the atom numbers. One experimental limitation
could be spontaneous emission which leads to momentum diffusion and hence heating of the atomic sample [19].

4. Output intensity squeezing

From equations (8) and (9), we see the affect of the coupling between the cavity mode and the Bogoliubov mode of the BEC. As the intensity of the cavity field is \( |a_s|^2 \), there will be an intensity-dependent shift of the BEC frequency since \( X_b \) depends linearly on \( a_s \), thus introducing a coupling between the fluctuations \( \hat{a} \) and its conjugate \( \hat{a}^\dagger \). Thus, as a consequence of the dependence of \( \hat{a} \) on \( X_b \), the fluctuation of the internal cavity field will be squeezed. However, \( X_b \) also depends on \( \hat{a} \), so a further dynamical phase shift and damping is introduced by the coupling of the cavity mode to the BEC.

In frequency space, from the Heisenberg–Langevin equations (8) and (9), we can trivially find

\[
\frac{\kappa}{2 + i(-\Delta_d - \omega)} - i\eta(\omega)K\hat{a}(\omega) = \sqrt{\kappa}\xi_p(\omega) + i\frac{g_r}{\sqrt{\Gamma_b}}\chi(\omega)\xi_m(\omega),
\]

where \( |K| = 2\beta|a_s|^2/\omega_m^2 \) and here we have introduced the dimensionless dynamical response factor of the BEC:

\[
\chi(\omega) = \frac{(\omega_m^2 - \omega^2) - i\Gamma_b\omega}{\omega_m^2}\chi_1(\omega) + i\chi_2(\omega),
\]

with \( \chi^*(\omega) = -\chi(-\omega) \).

The input–output theory gives the following relation among the incoming field (\( \xi(\hat{a}(\omega)) \)), internal field (\( \hat{a} \)) and output field (\( \hat{a}_{\text{out}} \)) as a consequence of the boundary condition at the fixed mirror surface:

\[
\hat{a}_{\text{in}}(\omega) + \xi_p(\omega) = \sqrt{\kappa}\hat{a}(\omega).
\]

Using equation (14) and its conjugate, we write

\[
\hat{a}_{\text{out}}(\omega) = \zeta(\omega)\xi_p(\omega) + \eta(\omega)\xi_m(\omega) + \zeta(\omega)\xi_m(\omega),
\]

and \( \hat{a}^\dagger_{\text{out}}(\omega) = [\hat{a}_{\text{out}}(\omega)]^\dagger \), where

\[
\zeta(\omega) = -\frac{\omega^2 + i\epsilon + i\Delta_d}[\kappa/2 - 2\chi_2(\omega)|K| + i(2\chi_1(\omega)|K| + \Delta_d)]\Delta(\omega).
\]

\[
\eta(\omega) = \frac{i\kappa\chi(\omega)|K|}{\Delta(\omega)} = -\eta^*(-\omega),
\]

\[
\zeta(\omega) = \frac{i\kappa\chi(\omega)|K|}{\Delta(\omega)} = \frac{\sqrt{\Gamma_b}}{\omega_m^2}\Delta(\omega).
\]

\[
\Delta(\omega) = \left[ \frac{k^2}{4} + \frac{\Delta_d^2 - \omega^2 + 2\Delta_d\chi(\omega)|K|}{4} - i\kappa\omega - 2\Delta_d\chi_2(\omega)|K| \right],
\]

with \( \Delta(-\omega) = \Delta^*(\omega) \).

The output intensity spectrum \( S_I(\omega) \) is defined as

\[
S_I(\omega) = \frac{1}{\omega_{\text{out}}^2} \int d\omega'\langle \delta I_{\text{out}}(\omega)\delta I_{\text{out}}(\omega') \rangle
\]

and

\[
\delta I_{\text{out}}(\omega) = \alpha_{\text{out}}^\text{in}\hat{a}_{\text{out}}(\omega) + \alpha_{\text{out}}^\text{in}\hat{a}_{\text{out}}(\omega),
\]

where \( \alpha_{\text{out}} = \langle \hat{a}_{\text{out}}(\omega) \rangle \) and \( \alpha_{\text{in}} = \langle \hat{a}_s(\omega) \rangle \). As a consequence of the boundary conditions at the fixed mirrors, \( \alpha_{\text{out}} = \sqrt{\kappa}\alpha_{\text{in}} - \alpha_{\text{in}} \). This yields the output intensity spectrum for \( n_p = 0 \) as

\[
S_I(\omega) = 1 + \frac{4\kappa\omega\Delta_d K}{(\kappa^2/4 + \Delta_d^2)|\Delta(\omega)|^2}
\times \left\{ \frac{\omega\Delta_d|\chi(\omega)|^2n_b}{Q} + \chi_2(\omega)[\kappa^2/4 + \Delta_d^2 - \omega\Delta_d] + 1/2\omega\kappa\chi_1(\omega) \right\}.
\]

Here, \( Q = \omega_m/\Gamma_b \) is the mechanical quality factor of the condensate. We see that the thermal contribution destroys the squeezing.

Figure 4 shows the intensity squeezing spectrum versus normalized detuning \( \Delta_d/\kappa \) and normalized frequency \( \omega/\kappa \). In the \( (\omega, \Delta_d) \) parameter space, squeezing of the output light intensity \( S_I(\omega) < 1 \) is indicated by dark regions. The higher the squeezing, the darker the region. Clearly, we find that the squeezing region in the parameter space \( (\omega, \Delta_d) \) decreases with decreasing two-body atom–atom interaction. This is evident as we go from the left plot to the right plot. Significant squeezing is observed for \( \Delta_d < 0 \) and \( \omega < \kappa \). Note that we did not observe any squeezing for \( \Delta_d > 0 \). These results can be easily explained from equation (24). As mentioned before thermal contribution destroys the squeezing. In order to ensure that the thermal noise contribution does not influence the squeezing \( S_I(\omega) \), the mechanical quality factor of the condensate has to be large \( (\omega_m \gg \Gamma_b) \). Thus, for the left plot of figure 4, the mechanical quality factor \( Q = 1.06 \times 10^4 \) and while for the right plot \( Q = 3.54 \times 10^2 \). This implies that the thermal noise contribution is enhanced and the mechanical quality factor of the condensate is too low for \( \Delta_m \). Further, in order to observe enhanced squeezing, the Bogoliubov mechanical bosonization of the BEC should be larger than the cavity linewidth \( \kappa \). This can be achieved by increasing the atom–atom interaction \( U_{\text{eff}} \). If \( \omega_m < \kappa \), the cavity photons do not see any coherently variable collective position of the BEC (large amplitude of the Bogoliubov mode) but only small fluctuations of the collective position. This can also be interpreted in terms of the coherence length (the coherence length is inversely proportional to \( U_{\text{eff}} \)). There are two length scales, the coherence length and the spatial scale of variation of the density. At low values of the interaction, the coherence length of the condensate is large and the quantum pressure term dominates the usual pressure term and hence the spatial variations of the density occur on a length scale less than the coherence length. Under this circumstance, atoms behave as almost free particles. On the other hand, when the interactions are large, the coherence length decreases and the spatial scale of variation of the density become large compared to the coherence length and hence the atoms move collectively. We will then require \( \omega_m > \kappa \). This implies that when \( U_{\text{eff}} \)
The decay rate of the intracavity field by using high-κ cavities is reduced, which leads to a reduction in squeezing in the (ω, K) parameter space. The characteristic time-scales of coherent dynamics are significantly faster than those over the losses. It is important that the characteristic time-scales of coherent dynamics are significantly faster than those associated with losses.

5. Conclusions

In summary, we have analysed a novel scheme of cavity optomechanics with ultracold atoms. We showed that due to the optical force experienced by the BEC in the cavity, the damping rate and the frequency of the Bogoliubov mode of the condensate changes. In the presence of atom–atom interactions, the cavity field fluctuations and the condensate fluctuations (Bogoliubov mode) lead to the splitting of the normal mode into two modes (normal mode splitting). The system described here shows a complex interplay between two distinct systems, namely, an optical micro-cavity and a gas of ultracold atoms. We found that using a BEC with high mechanical quality factor, squeezing of the intensity of the output light field is obtained at low frequency. This scheme may lead to a possible realization of a quantum device to tailor quantum fluctuations of output light using cavity QED and BEC technology.

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