An Effective Field Theory for Acoustic and Pseudo-Acoustic Phonons in Solids

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We present a relativistic effective field theory for the interaction between acoustic and gapped phonons, in the limit of small gap. While the former are the Goldstone modes associated to the spontaneous breaking of spacetime symmetries, the latter are pseudo-Goldstones, associated to some (small) explicit breaking. We hence dub them “pseudo-acoustic” phonons. In this first investigation, we build our effective theory for the cases of one and two spatial dimensions, two atomic species, and assuming large distance isotropy. As an illustrative example, we show how the theory can be applied to compute the total lifetime of both acoustic and pseudo-acoustic phonons. This construction can find applications that range from the physics of bi-layer graphene to sub-GeV dark matter detectors.

Keywords: Effective Theory, Acoustic Phonon, Pseudo-Acoustic Phonon, 2D Material

I. INTRODUCTION

Many properties of solids are dictated by the dynamics of their simplest collective excitations: the phonons. These are localized vibrational modes that, when characterized by wavelengths much larger than the atomic spacing, can be described in terms of quasi-particles. In a solid with a single atom per unit cell the phonons’ dispersion relation is gapless, i.e. when its wavevector vanishes so does its frequency. In this case one talks about “acoustic” phonons. However, for more complicated (and common) solids, some phonons can be gapped, with a frequency that tends to a finite positive value at zero wavevector. When the gap is large enough (at least comparable to the maximum frequency of the acoustic phonons), these modes are typically called “optical” phonons.

It is well known that the acoustic phonons are nothing but the Goldstone bosons associated to the spontaneous breaking of spatial translations induced by the solid background. Taking this idea as a starting point, recent years have witnessed the development of effective field theory (EFT) techniques, based on symmetry breaking and its consequences, applied to the study of collective excitations in different states of matter (see [3] and references therein). An EFT description of the phonon degrees of freedom, organized in a low energy/long wavelength expansion, has the advantage of being universal, i.e. not to rely on the often complicated microscopic physics, up to a finite number of effective coefficients. The latter must be obtained from experiment or determined in other ways, as for example Density Functional Theory calculations—see e.g. [6]-[10]. Such an EFT approach has already proven to be useful to a number of phenomenologically relevant problems, covering a wide range of fields, from the physics of \(^4\)He to cosmology (see e.g. [11]-[21]).

In this paper we develop a new EFT for the description of acoustic and gapped phonons and their interactions in a solid, in the regime where the gap is small compared to the typical frequency characterizing the microscopic system. For reasons that will be clear soon, we dub these collective excitations as “pseudo-acoustic” phonons. To the best of our knowledge, no bottom-up effective description of pseudo-acoustic phonons has been presented so far.

The construction we present here can be applied to any number of spatial dimensions and any number of atomic species within the solid. However, as we will show, we expect a small gap for the pseudo-acoustic phonons to be achieved when the different species are weakly coupled to each other. Although this might be hard to envision in three-dimensional materials, it does find examples in two dimensional ones, most notably bi-layer graphene [14]-[19], where the two sheets are coupled via van der Waals forces.

In this first study, we focus on solids that are both homogeneous and isotropic at large distances. While for large enough wavelengths the first property is always true, the second one is a simplifying assumption. The extension of our EFT to the case of solids which preserve only discrete rotations at large distance is straightforward, but tedious.

In constructing the EFT for acoustic and pseudo-acoustic phonons we impose relativistic Lorentz invariance. Although this might not be common in solid state physics, there are reasons why this approach is worth exploring. First of all, it is technically easier to impose the Lorentz symmetry rather than the nonrelativistic Galilei one, by simply contracting covariant indices. Moreover, given that the Lorentz group is more fundamental, one is always free to require invariance under it, and hence allow our EFT to

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1 This does not necessarily require an underlying discrete lattice. For example, a perfectly homogeneous jelly in statistical equilibrium would still break spatial translations.

2 See [22]-[23] for a proposal on how to describe optical phonons in a quantum field theory language.
Conventions: We work in natural units, and transport many of these considerations to the two-dimensional case, which instead constitutes the most relevant part of our work. In Figure 4 we report a schematic representation of the different phonon modes.

A. 1D Case

Consider for the moment a monoatomic solid 4. Its volume elements can be labeled by a single scalar field, $\phi(x)$—the comoving coordinate—which at equilibrium can always be taken to be proportional to the physical spatial coordinate, $\langle \phi(x) \rangle = \alpha x$, where $\alpha$ is a constant determining the degree of compression/dilation of the solid 21. From an EFT viewpoint this vacuum expectation value (vev) breaks boosts and spatial translations. Since all solids are homogeneous at large enough distances, one also postulates an internal $U(1)$ shift symmetry, $\phi \rightarrow \phi + c$, which is broken together with part of the Poincaré group 1 down to time translations and a diagonal $U(1)$, i.e. $ISO(1, 1) \times U(1) \rightarrow \mathbb{R} \times U(1)$. It is this last unbroken $U(1)$ that one uses to define large distance homogeneity.

The fluctuation of the comoving coordinate around its equilibrium configuration, $\phi(x) = \alpha x + \pi(x)$, is the Goldstone boson associated to the broken symmetries, and corresponds to the phonon of the solid. Since the symmetries are spontaneously broken, the dynamics of the phonon can be described via a Lagrangian that is invariant under the full unbroken group. In the long wavelength limit (i.e. at lowest order in a derivative expansion), the only quantity that is invariant under both the Poincaré group and the internal shift symmetry is $X = \partial_\mu \phi \partial^\mu \phi$, and the most general Lagrangian is $F(X)$, with $F$ an a priori generic function. Upon inspecting the stress-energy tensor of the theory, one finds that $F$ is nothing but minus the energy density 3. For a strongly coupled system its analytical expression is hard (or even impossible) to compute, and one must obtain it from experimental or numerical data.

Expanding the action in small fluctuations one obtains all possible interactions for the acoustic phonon which, being a Goldstone boson, is gapless—see e.g. 4 for details.

Let us now consider a second atomic species in our solid. One can introduce two comoving coordinates, $\phi_{A,B}(x)$, one for each species, featuring two independent shift symmetries. At equilibrium both of them are proportional to space, $\langle \phi_A(x) \rangle = \alpha A x$ and $\langle \phi_B(x) \rangle = \beta x$, and the symmetry breaking pattern is then $ISO(1, 1) \times U_A(1) \times U_B(1) \rightarrow \mathbb{R} \times U(1)$. Despite the number of broken generators, the above symmetry breaking pattern leads to only two Gold-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Schematic representation of the energy of different phonon modes as a function of their momentum. The shaded region indicates where the EFT description given here is valid.}
\end{figure}

3 The rate of dark matter scattering by pseudo-acoustic phonons is also subject to the coupling-to-mass suppression that was observed in the scattering by optical phonons 22.

4 More precisely, Lorentz boosts, spatial translations and spatial rotations are broken by a solid. In the one-dimensional case, the latter are clearly absent.
stones, as dictated by the inverse Higgs constraints. In the following we will set $\alpha = \beta = 1$ for simplicity.

Analogously to the monoatomic case, at lowest derivative order one can build three quantities that are invariant under the Poincaré group and the two internal $U(1)$’s, i.e.

$$
X_1 = \partial_\mu \phi_A \partial^\mu \phi_A , \quad X_2 = \partial_\mu \phi_B \partial^\mu \phi_B ,
X_3 = \partial_\mu \phi_A \partial^\mu \phi_B .
$$

In the ideal case of two non-interacting solids the effective Lagrangian is simply $F_1(X_1) + F_2(X_2)$. In the most general case, however, the two solids couple to each other and the action will depend on all the invariants in Eq. (1), describing two interacting acoustic phonons, both gapless.

This is, however, not the end of the story. In fact, it is now possible to build one more quantity, $\Delta = (\phi_A - \phi_B)^2$, which explicitly breaks the initial $U_A(1) \times U_B(1)$ but preserves their final diagonal combination. This operator generates a gap for one of the two degrees of freedom, which then becomes a pseudo-Goldstone boson, hence the name pseudo-acoustic phonon. If such a gap is smaller than the UV cutoff, the pseudo-acoustic phonon can still be treated perturbatively within the EFT. This means that, while the $A$ and $B$ solids can be separately arbitrarily strongly coupled at the microscopic level, we expect this regime to be achieved when they are weakly coupled to each other. A priori, the most general Lagrangian that incorporates a small explicit breaking (and hence a small gap) can be written as

$$
F(X_i, \Delta) = f(X_i) + \delta f(X_i, \Delta) ,
$$

with $\delta f \ll f$. However, for the most common systems we expect that, if the two solids are weakly coupled to each other, all interactions between the phonons of the two sectors will be weak, i.e.

$$
F_{\text{weak}}(X_i, \Delta) = f_A(X_1) + f_B(X_2) + \delta f(X_i, \Delta) ,
$$

where $f_{A,B}$ are (minus) the energy densities of the two solids in the limit where they are exactly decoupled.

Working with the most general action $S = \int d^2x F(X_i, \Delta)$, and expanding in small fluctuations, up to cubic order one gets

$$
S \supset \int d^2x \left[ - g_{(a \beta)} \pi_a \pi_\beta + g'_{(a \beta)} \partial_x \pi_a \partial_x \pi_\beta + g_{\Delta} (\pi_A - \pi_B)^2 + y_{(a \beta \gamma)} \partial_x \pi_a \partial_x \pi_\beta \partial_x \pi_\gamma \right] + \delta f(X_i, \Delta).
$$

where $\alpha, \beta, \gamma = A, B$, and (...) represents symmetric indices. The effective couplings, $g$ and $y$, are given in terms of the derivatives of $F$ in Appendix A. In general, the coefficients of the quadratic terms depends on derivatives up to the second, those of the cubic ones up to the third, and so on. The spectrum of the theory is obtained by looking at the eigenmodes of the quadratic part of the action. For small momenta one gets the following dispersion relations for acoustic and pseudo-acoustic phonons:

$$
\omega^2_{\text{ac}} = c^2_s q^2 , \quad \omega^2_{\text{ps}} = \delta^2 + \gamma q^2 ,
$$

where the gap is given by

$$
\delta = \sqrt{g_{AA} g_{BB} + 2g_{AB} g_{\Delta}} .
$$

The expression for $c_s$ and $\gamma$ are also reported in Appendix A. The gap of the pseudo-acoustic phonon indeed goes to zero with vanishing $g_{\Delta}$, this being the only parameter encoding the dependence of the theory on explicit breaking at quadratic order.

To make contact with physical systems, let us now show how the most general theory (4) can describe different instances of a linear diatomic chain. To do that, we will look at the following two example, and focus for now on the spectrum of the theory.

- **Two non-interacting monoatomic chains with different atomic masses:** A system of this kind is described, as already mentioned, by a Lagrangian as in Eq. (3) in the $\delta f \to 0$ limit. At the level of the quadratic action this implies $g_{AA} = g'_{AA} = g_{\Delta} = 0$, and one obtains two gapless modes with two generically different sound speeds, $c^2_s = g'_{AA}/g_{BB}$ and $c^2_{ps} = g_{BB}/g_{AB}$. Note that since the $q \to 0$ and $\delta \to 0$ limits do not commute, one cannot obtain the above sound speed as a limit of the dispersion relations.

- **Two identical chains with weak coupling between them:** Being the two separate chain identical, the system is obtained from the action (3) imposing symmetry under $X_1 \leftrightarrow X_2$, which implies $g_{AA}^{(l)} = g_{BB}^{(l)} = g_{\Delta}^{(l)} = 0$. In this case the small gap survives, but $\gamma \simeq c^2_s$.

Before proceeding, let us briefly discuss how the effective couplings of the theory can be determined from the static properties of the solid, e.g. by experiment or numerical simulations. It is clear the the structure of the action (4) could have been found also by simply writing down all possible interactions compatible with the unbroken symmetries. Nevertheless, to express the couplings in terms of derivatives of the Lagrangian with respect to the invariants allows to determine them in terms of the nonlinear stress-strain curve of the solid [18]. Imagine for example, statically stretching or compressing only one

\footnote{Our definition of $\Delta$ is such that the theory is invariant under $\phi_A, B \to -\phi_A, B$ while being analytic around $\pi_A, B = 0$.}

\footnote{Note that, despite the presence of a gap (albeit small), these phonons are usually still called “acoustic” in the literature. \cite{28}. We however stress the conceptual difference between the two: while standard acoustic phonons are Goldstone bosons, pseudo-acoustic phonons are not.}

\footnote{This is analogous to what happens, for example, for pions in QCD, whose mass is smaller than $4\pi f_{\pi} \sim 1$ GeV, the scale at which the chiral EFT breaks down (see e.g. \cite{29}).}
of the solids, say solid $A$, while keeping the other at its equilibrium configuration. This corresponds to exciting a time-independent mode $\pi_A(x)$ while keeping $\pi_B = 0$, i.e. a deformation of solid $A$. Clearly, the mode $\pi_A(x)$ must have a nontrivial spatial gradient, otherwise it would simply correspond to a global $U_A(1)$ transformation, which does not affect the system. At linear order in the deformation $\pi_A$, this induces a variation in $X_1$, $X_3$, while $X_2$ remains unchanged. Exciting a mode $\pi_B(x)$ has the same effect but with $X_1 \leftrightarrow X_2$. If instead we statically deform the two solids in opposite directions, $\pi_A(x) = -\pi_B(x)$, this will generate a variation in $X_1$, $X_2$, but not in $X_3$.

Recalling that the Lagrangian, $F(X_i, \Delta)$, is minus the energy density of the solid, one deduces that the effective couplings can be obtained by studying the nonlinear response of the energy density under the mechanical deformations described above. For example, by measuring the linear change in the free energy following the deformations described above one can determine the first derivatives of the Lagrangian with respect to the $X_i$ invariants. To obtain higher derivatives of the Lagrangian with respect to $X_i$, as well as the dependence on $\Delta$, one can study the the nonlinear response.

### B. 2D Case

Building on the results of the previous section, we now describe the case of a two-dimensional diatomic solid. This presents no conceptual novelty with respect to the previous section, but it does involve some technical aspects worth addressing.

In two spatial dimensions the comoving coordinates are described by two scalar fields for each species of solid, $\phi^I_\alpha(x)$ with $I = 1, 2$ and $\alpha = A, B$. At equilibrium they can be aligned to the physical coordinates, i.e. they acquire the vev $\langle \phi^I_\alpha(x) \rangle = x^I$. This again breaks boosts and spatial translations, but also spatial rotations. If, beside homogeneity, one also restricts oneself to solids that are isotropic at large distances, then it is necessary to impose an internal $ISO(2)$ symmetry [4]. Under this, the comoving coordinates transform as $\phi^I_\alpha \rightarrow O^I_\alpha \phi^I_\alpha + c^I_\alpha$, where $c^I_\alpha$ is a constant vector and $O^I_\alpha$ a constant $SO(2)$ matrix. This internal Euclidean group is again spontaneously broken, but a diagonal combination of it with the spacetime Euclidean group is preserved, $ISO(2, 1) \times ISO_A(2) \times ISO_B(2) \rightarrow \mathbb{R} \times ISO(2)$.

Let us now build all the independent operators that are invariant under the symmetry group. Imposing first Poincaré and shift invariance one obtains the following matrices

\[
B^{IJ} = \frac{1}{2}(\partial \phi^I \partial \phi^J - \partial \phi^J \partial \phi^I), \quad B^{AB} = \partial \phi^A \partial \phi^B, \quad B^{IJ} = \partial \phi^I \partial \phi^J, \quad \Sigma^{IJ} = (\phi^I - \phi^J)(\phi^I - \phi^J), \quad (7)
\]

The $B$ matrices transform linearly under the initial $SO_A(2) \times SO_B(2)$ group—i.e. $B_{\alpha \beta} \rightarrow O_\alpha \cdot B_{\alpha \beta} \cdot O_\beta^T$, where $O_\alpha$ is an orthogonal matrix belonging to $SO_\alpha(2)$. The matrix $\Sigma$, instead, only transforms linearly under the full broken $SO(2)$, and is only invariant under the broken $U(1)$.

We now need to build operators that are also invariant under internal rotations. There is a total of eleven independent invariants. The following eight are invariant under the full $SO_A(2) \times SO_B(2)$ group, and hence are compatible with its spontaneous breaking:

\[
X_1 = [B_{AA}], \quad X_2 = [B_{BB}], \quad X_3 = [B_{AB} B_{BA}^T], \quad X_4 = [B_{AA}^2], \quad X_5 = [B_{BB}^2], \quad X_6 = ([B_{AB} B_{BA}^T]^2), \quad (8)
X_7 = [B_{AB} B_{BA}^T B_{AA}], \quad X_8 = [B_{AB} B_{BA} B_{BB}].
\]

Here with $[\ldots]$ we represent the trace. The remaining three operators are only invariant under the unbroken group, and hence explicitly break the initial symmetry:

\[
\Delta_1 = [B_{AA} B_{BB}], \quad \Delta_2 = [\Sigma], \quad \Delta_3 = \frac{[\Sigma B_{AA}] + [\Sigma B_{BB}]}{2} - [\Sigma]. \quad (9)
\]

The expression for $\Delta_3$ has been chosen so that it does not contribute to the quadratic action.

Again, the phonons will be the fluctuations of the comoving coordinates around their equilibrium configuration, $\phi^I_\alpha(x) = x^I + \pi^I_\alpha(x)$, and the action for their dynamics is $S = \int d^3x F(X_i, \Delta_i)$, which can be now determined by the nonlinear response of the system to shear and stress, as discussed in the previous section. We can now expand the action up to cubic order in small fluctuations. Moreover, we perform a field redefinition, $\pi_\alpha = O_{\alpha \beta} S_{\beta \gamma} \chi_\gamma$, where $S$ is a matrix that brings the temporal kinetic term to its canonical form, and $O$ an orthogonal matrix that diagonalizes the mass term. The result is

---

8 Indeed the matrices in Eq. (7) have a total of twelve independent components. However, we can always perform an unbroken $SO(2)$ rotation to bring the number down to eleven.
\[ S \supset \int d^3x \left[ \frac{1}{2} \dot{X}^2 - \frac{1}{2} K^{(1)}_{(a\beta)} \nabla \cdot \chi_\alpha \nabla \cdot \chi_\beta - \frac{1}{2} K^{(2)}_{(a\beta)} \nabla^i \chi^\alpha_i \nabla^\beta_j - \frac{1}{2} M^2_{(a\beta)} \chi_\alpha \cdot \chi_\beta + \lambda^{(1)}_{(a\beta)} \chi_\alpha \cdot \dot{\chi}_\beta \right. \\
+ \lambda^{(2)}_{(a\beta)} \dot{\chi}^i_\alpha \dot{\chi}^j_\beta \nabla^i \chi^\gamma_j + \lambda^{(3)}_{(a\beta\gamma)} \nabla \cdot \chi_\alpha \nabla \cdot \chi_\beta \nabla \cdot \chi_\gamma + \frac{1}{2} M^2_{(a\beta\gamma)} \chi_\alpha \cdot \nabla \chi_\beta \chi_\gamma + \frac{1}{2} M^2_{(a\beta\gamma)} \nabla^j \chi^\alpha_j \nabla^i \chi^\beta_i \nabla \cdot \chi_\gamma \\
\left. + \lambda^{(4)}_{(a\beta\gamma)} \nabla^i \chi^\alpha_i \nabla^j \chi^\beta_j \chi^\gamma_k + \lambda^{(5)}_{(a\beta\gamma)} \nabla^j \chi^\alpha_j \nabla^i \chi^\beta_i \chi^\gamma_k + \lambda^{(6)}_{(a\beta\gamma)} \nabla^i \chi^\alpha_i \nabla^j \chi^\beta_j \nabla^k \chi^\gamma_k \\
\right] + \left( \omega^2_{L,ac} c^2_q q^2, \omega^2_{T,ac} c^2_T q^2 \right) \] (10)
Two comments are in order about the previous expression. First of all, one immediately sees that, when the gap grows, the third term in parenthesis can be neglected, and the rates becomes what one would obtain from an EFT for a single acoustic phonon [4], in agreement with the idea that the pseudo-acoustic phonon can be integrated out at large gap. Secondly, because of the considerations made above, the decay width of acoustic phonons in two spatial dimension is less suppressed at small momenta than what one would expect from naive scaling which, instead, would suggest a $\sim q^4$ behavior. This is a consequence of the well-known extra infrared divergences arising in low-dimensional systems.

Note also that our analysis applies to an ideal two-dimensional system, since it only involves in-plane phonons. Out-of-plane modes in two spatial dimensions have been shown to have peculiar properties in absence of external strain, and to contribute sensibly to the decay rate for acoustic phonons [41, 42]. Their dispersion relation quickly approaches a linear one, at small momenta from $q^0$ to $q^2$. Remarkably, the quadratic action becomes diagonal, i.e. $K^{(4)}_{\lambda_5} = 0$, and one also finds $c_5^2 = \gamma_5 + O(\delta f)$—in agreement, for example, with [29]. Moreover, the gap (squared) and all the couplings with mixed $A,B$ indices must arise from the coupling between the two solids, and are therefore of order $O(\delta f)$. Using the Feynman rules in Appendix [3] one obtains the following decay rate for a pseudo-acoustic phonon at rest:

$$\Gamma_{ps} = \left(\frac{2g_{ABA}^{(8)} - g_{ABA}^{(9)}}{16c_\lambda c_\tau (c_\lambda + c_\tau)^2} \right)^2 + \left(\frac{g_{ABA}^{(9)}}{64c_\lambda^4} \right)^2 + \frac{\left(2g_{ABA}^{(8)} - g_{ABA}^{(9)} - g_{BA}^{(9)}\right)^2}{64c_\lambda^4} + O(\delta f^3).$$

Note the interesting fact that, to this order in small explicit breaking, the decay rate is independent on the gap itself.

IV. CONCLUSIONS

In this work we have presented a relativistic effective field theory for the description of the low-energy degrees of freedom of a solid made of two species, weakly coupled to each other. In this regime the system features two distinct types of excitations: acoustic and pseudo-acoustic phonons. The first are the Goldstone bosons associated to the spontaneous breaking of spacetime symmetries and observed [42].

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We can now move to the spin-averaged decay rate for the pseudo-acoustic phonon. Given the large number of effective couplings, let us focus on the case of a 2D solid in which the two species, $A$ and $B$, are identical, physically relevant to a description of bi-layer graphene. In analogy to what we explained in Section II A the Lagrangian for such a system can be written as

$$F(X_i, \Delta_i) = f(X_1, X_4) + f(X_2, X_5) + \delta f(X_i, \Delta_i),$$

where we used the fact that the free energies of the two solids must be the same in absence of coupling. Moreover, since the system must respond in the same way to modes where either $\pi_A$ or $\pi_B$ is excited, the Lagrangian must be symmetric under the exchanges $X_1 \leftrightarrow X_2$, $X_4 \leftrightarrow X_5$ and $X_7 \leftrightarrow X_8$. It is simple to show that this implies that the effective couplings are symmetric under the exchange $A \leftrightarrow B$, as one might have guessed from the beginning. Remarkably, the quadratic action becomes diagonal, i.e. $K^{(4)}_{\lambda_5} = 0$, and one also finds $c_5^2 = \gamma_5 + O(\delta f)$—in agreement, for example, with [29]. Moreover, the gap (squared) and all the couplings with mixed $A,B$ indices must arise from the coupling between the two solids, and are therefore of order $O(\delta f)$. Using the Feynman rules in Appendix [3] one obtains the following decay rate for a pseudo-acoustic phonon at rest:

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are consequently gapless. The second are instead pseudo-Goldstone bosons, and are characterized by a small gap arising from a perturbative explicit breaking operator.

An EFT formulation of the problem has the important advantage of putting on a firm ground several properties of these collective excitations by connecting them to those universal features of the system that only depend on low-energy/large distance physics. It also allows analytical control over the observables, which can be computed for a large class of solids, only via symmetry arguments.

From this viewpoint there are several open questions of both conceptual and phenomenological relevance. One of them is to understand the nature of out-of-plane modes from a low-energy perspective. These modes are gapless
but feature a quadratic dispersion relation and, as already
commented, contribute to an important fraction of the
total decay rate of acoustic phonons [42] and, in turns,
to the thermal conductivity of two-dimensional mate-
rials [39]. In a similar direction, it would be interesting
to understand what is the contribution to the latter due
to pseudo-acoustic phonons. The contribution of optical
phonons is typically negligible because of their large gap.
However, pseudo-acoustic phonons having a perturbative
gap, they could play a relevant role. It would also be
interesting to understand if our action (10) captures any
of the features of true optical phonons, despite their large
gap. Finally, the generalization of the present construc-
tion to systems which only preserve a discrete subgroup of
rotations at large distances is clearly phenomenologically
relevant. We leave these and other interesting questions
for future work.

ACKNOWLEDGMENTS

We are grateful to G. Cuomo, A. Khmelnitsky, A. Nico-
is, R. Penco and R. Rattazzi for important discussions.
We are especially thankful to G. Chiriacò for several
enlightening conversations and comments on the draft.
A.E. is supported by the Swiss National Science Foun-
dation under contract 200020-169696 and through the
National Center of Competence in Research SwissMAP.
E.G. acknowledges support from the International Max
Planck Research School for Precision Tests of Funda-
mental Symmetries in Particle Physics, Nuclear Physics,
Atomic Physics and Astroparticle Physics. T.M. is sup-
ported by the World Premier International Research Cen-
ter Initiative (WPI) MEXT, Japan, and by JSPS KAK-
ENHI grants JP18K13533, JP19H05810, JP20H01896 and
JP20H00153.

Appendix A: Effective couplings for the 1D solid

Here we report the explicit expressions for the cou-
plings and parameters of the EFT written in terms of
the derivatives of the Lagrangian evaluated on the back-
ground configuration, \( \langle \phi \rangle = x \). The subscripts indicate
derivatives with respect to a given invariant. The effective
couplings appearing in the action (4) are

\[
\begin{align*}
g_{AA} &= F_{x_1} ; \\
g'_{AA} &= F_{x_1} + 2F_{x_1x_1} + \frac{1}{2}F_{x_1x_3} + 2F_{x_1x_3} ; \\
g_{BB} &= F_{x_2} ; \\
g'_{BB} &= F_{x_2} + 2F_{x_2x_2} + \frac{1}{2}F_{x_2x_3} + 2F_{x_2x_3} ; \\
g_{AB} &= \frac{1}{2}F_{x_3} + \frac{1}{2}F_{x_3x_3} + 2F_{x_3x_3} + F_{x_3x_3} + F_{x_2x_3} ; \\
g_{\Delta} &= F_{\Delta} ,
\end{align*}
\]

for the quadratic ones, and

\[
\begin{align*}
y_{AAA} &= 2F_{x_1x_1} + F_{x_1x_3} + \frac{4}{3}F_{x_1x_1x_1} \\
&+ 2F_{x_1x_1x_3} + F_{x_1x_3x_3} + \frac{1}{6}F_{x_3x_3x_3} , \\
y_{AAB} &= \frac{2}{3}F_{x_1x_2} + F_{x_1x_3} + \frac{1}{3}F_{x_3x_3} \\
&+ \frac{4}{3}F_{x_1x_1x_2} + \frac{2}{3}F_{x_1x_3x_3} + \frac{4}{3}F_{x_3x_1x_3} \\
&+ \frac{2}{3}F_{x_3x_1x_3} + \frac{1}{3}F_{x_3x_3x_3} + \frac{1}{6}F_{x_3x_3x_3} , \\
y_{ABB} &= y_{AAB} \text{ with } X_1 \leftrightarrow X_2 , \\
y_{BBB} &= y_{AAA} \text{ with } X_1 \leftrightarrow X_2 , \\
y'_{AAA} &= 2F_{x_1x_1} + F_{x_1x_3} , \\
y'_{AAB} &= 2F_{x_1x_2} + F_{x_1x_3} , \\
y'_{ABA} &= F_{x_1x_1} + \frac{1}{2}F_{x_3x_3} , \\
y'_{ABB} &= y'_{ABA} \text{ with } X_1 \leftrightarrow X_2 , \\
y'_{BBB} &= y'_{AAA} \text{ with } X_1 \leftrightarrow X_2 , \\
y'_{A} &= 2F_{x_1x_1} + F_{x_3x_3} , \\
y'_{B} &= y'_{A} \text{ with } X_1 \leftrightarrow X_2 .
\end{align*}
\]

\[\text{(A2)}\]

for the cubic ones.

The parameters appearing in the dispersion relation for
the acoustic and optical phonons in one spatial dimension,
Eqs. (5), are instead

\[
\begin{align*}
\epsilon_s^2 &= \frac{g'_{AA} + g'_{BB} + 2g'_{AB}}{g_{AA} + g_{BB} + 2g_{AB}} ; \\
\gamma &= \frac{1}{(g_{AA} + g_{BB} + 2g_{AB})(g_{AA}g_{BB} - g_{AB}^2)} \times \\
&\times \left[ g_{AA}g'_{BB} + g_{BB}g'_{AA} - 2g_{AA}g_{BB}g'_{AB} \right. \\
&\left. + g_{AB} (g'_{AA} + g'_{BB} - 2g'_{AB}) \right] \\
&+ 2g_{AA}g_{AB}(g'_{BB} - g'_{AB}) \\
&+ 2g_{BB}g_{AB}(g'_{AA} - g'_{AB}) .
\end{align*}
\]

\[\text{(A3)}\]

Appendix B: Canonical quantization and Feynman
rules for the 2D solid

In this appendix we perform the canonical quantization
for the fields \( \chi_a \) appearing in the action (10). Follow-
ning the standard canonical quantization procedure, we
expand them in creation and annihilation operators, and
require that they satisfy the equations of motion. Since
the quadratic action is in general non-diagonal, \( \chi_A \) and
\( \chi_B \) obey a set of coupled linear differential equation and,
therefore, both of them contain creation/annihilation op-
erators for the acoustic and pseudo-acoustic phonons. We
thus write
\[ \chi_\alpha(x) = \sum_{\lambda,j} \int \frac{d^2q}{(2\pi)^2} \epsilon^q_i C^\alpha_\lambda(q) a^\lambda_j e^{iqj \cdot x} + \text{h.c.,} \]
where \( \lambda \) is the phonon’s polarization (longitudinal/transverse) and \( f \) its ‘flavor’ (acoustic/pseudo-acoustic), and \( a^\lambda_j \) is the annihilation operator, normalized so that:
\[ \left[a^\lambda_j, (a^\lambda_j')^\dagger\right] = 2\omega_j (2\pi)^2 \delta^{(2)}(q - q') \delta^{\lambda\lambda'} \delta^{ji}. \] (B1)

Moreover, \( \epsilon^q_i \) is a polarization vector, which for longitudinal and transverse phonons is given respectively by \( \epsilon^q_i = q^j \) and \( \epsilon^q_i = \epsilon^{ji} q^j \), such that they satisfy the completeness relation, \( \sum_{\lambda,j} \epsilon^q_i \epsilon_{q,j} = \delta^{ij} \). To determine the overlap functions, \( C^\alpha_\lambda(q) \), we first require, for the fields to satisfy the equal-time commutation relations, \([\chi^\alpha_\lambda(t,x), \chi^\alpha_\lambda(t',y)] \propto i \delta^{(2)}(x - y) \delta^{ji} \), leading to
\[ \sum_f |C^\alpha_f|^2 = \sum_f |C^\alpha_f|^2 = 1. \] (B2)

We then impose the linear equations of motion and obtain, up to order \( O(q^3) \),
\[ C^{\alpha,ac}_A = C^{\alpha,ac}_B = 1, \quad C^{\alpha,ac}_C = -K_{1B}^{(1)} + K_{1B}^{(2)} q^2, \]
\[ C^{\alpha,ac}_C = -K_{1B}^{(2)} q^2, \quad C^{\alpha,ac}_C = |C^{(1)}_{B} + K_{1B}^{(2)} q^2, \]
\[ C^{\alpha,ac}_C = |C^{(2)}_{B} + K_{1B}^{(2)} q^2, \quad C^{\alpha,ac}_C = \text{sgn} \left( K_{1B}^{(1)} + K_{1B}^{(2)} q^2 \right), \quad C^{\alpha,ac}_C = \text{sgn} \left( K_{1B}^{(2)} q^2 \right). \] (B3)

Note that, in the case in which the two solids are the same—see Eq. (14)—one has \( K_{1B}^{(1)} = 0 \), and the two fields interpolate a single mode each, i.e. the acoustic phonon for \( \chi_A \) and the pseudo-acoustic one for \( \chi_B \).

Let us now present the Feynman rules. From the canonical expression for the \( \chi_\alpha \) field we can deduce the propagator, \( G_{\alpha\beta}(x) = \langle 0 | T \chi^\alpha_\lambda(x) \chi^\beta_\lambda(0) | 0 \rangle \), where \( | 0 \rangle \) is the solid vacuum, and \( T \) enforces the time-ordered product. With standard quantum field theory techniques we deduce the following rules:

\[ \begin{align*}
\langle \chi^\alpha_\lambda(t,x) \rangle & = \frac{i e^q_i e^{i q j \cdot x} C^\alpha_\lambda(q)}{\omega^2 - \omega^2_j(q) + i\varepsilon}, \\
\langle \chi^\beta_\lambda(t',y) \rangle & = \frac{i e^q_i e^{i q j \cdot y} C^\beta_\lambda(q)}{\omega^2 - \omega^2_j(q) + i\varepsilon}.
\end{align*} \]

where we recall that the indices \( i, j, k \) run over spatial components, \( \lambda, \sigma, \rho \) over longitudinal/transverse polarizations, \( \alpha, \beta, \gamma \) over the solid labels \( A \) and \( B \), and finally \( f, g, h \) over the acoustic/pseudo-acoustic flavors. Moreover, by “permutings” we mean the possible combinations of the collective indices \( \{q_1, i, \lambda, \alpha, f\} \), \( \{q_2, j, \sigma, \beta, g\} \) and \( \{q_3, k, \rho, \gamma, h\} \). One can check that the propagator is indeed the inverse of the kinetic matrix of the action (10).
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