Abstract
Petruševski and Škrekovski recently introduced the notion of an odd colouring of a graph: a proper vertex colouring of a graph $G$ is said to be odd if for each non-isolated vertex $x \in V(G)$ there exists a colour $c$ appearing an odd number of times in its neighbourhood $N(x)$. Petruševski and Škrekovski proved that for any planar graph $G$ there is an odd colouring using at most 9 colours and, together with Caro, showed that 8 colours are enough for a significant family of planar graphs. We show that 8 colours suffice for all planar graphs.

Keywords Planar graph · colouring · discharging method · odd colouring

1 Introduction

Let $G$ be a graph. An odd colouring of $G$ is a proper colouring $\varphi$ of $G$ such that for every non-isolated vertex $x \in V(G)$ there is a colour $c$ satisfying that $|\varphi^{-1}(c) \cap N(x)|$ is odd. The odd chromatic number of $G$, denoted $\chi_o(G)$, is the smallest number $k$ such that there exists an odd colouring using $k$ colours.

Odd colourings were recently introduced by Petruševski and Škrekovski [6]. Using the discharging method, they showed that $\chi_o(G) \leq 9$ holds for all planar graphs $G$. Furthermore, they made the following conjecture.

Conjecture 1.1 Every planar graph $G$ has odd chromatic number at most 5.

If true, this conjecture is clearly best possible as $\chi_o(C_5) = 5$. Since the introduction of the notion of odd colouring in [6], two papers have appeared on arXiv containing partial results about narrowing down the gap between the bounds of 5 and 9 colours for
planar graphs. Cranston [4] focused on the odd chromatic number of sparse graphs, obtaining for instance that $\chi_o(G) \leq 6$ for planar graphs $G$ of girth at least 6 and $\chi_o(G) \leq 5$ for planar graphs $G$ of girth at least 7. Caro, Petruševski and Škrekovski [3] studied various properties of the odd chromatic number, including the following important steps towards proving that 8 colours suffice for an odd colouring of any planar graphs.

**Lemma 1.1** If $G$ is a connected planar graph of even order, then $\chi_o(G) \leq 8$.

**Lemma 1.2** If $G$ is a connected planar graph of odd order which has a vertex of degree 2 or any odd degree, then $\chi_o(G) \leq 8$.

For the reader’s convenience, we include the proofs by Caro, Petruševski and Škrekovski of those lemmata at the beginning of the next section. Their proofs use the following theorem by Aashtab, Akbari, Ghanbari and Shidani [1], where a graph $H$ is said to be odd if for every vertex $v$ of $H$, its degree $d_H(v)$ is an odd number.

**Theorem 1.3** Let $G$ be a connected planar graph of even order. Then $V(G)$ can be partitioned into at most 4 sets such that each part induces an odd forest.

It is worth noting that the proof of Theorem 1.3 is based on the Four-Colour Theorem.

Since a minimal planar graph (with respect to the number of vertices) which does not admit an odd colouring with 8 colours must be connected, Lemma 1.1 and Lemma 1.2 obviously imply the following.

**Lemma 1.4** Let $G$ be a minimal planar graph which does not admit an odd colouring with 8 colours. Then $G$ has odd order and all degrees in $G$ are even and at least 4.

The goal of this paper is to prove that 8 colours are sufficient for an odd colouring of any planar graphs.

**Theorem 1.5** For every planar graph $G$ we have $\chi_o(G) \leq 8$.

As usual, we call a vertex of degree $d$ a $d$-vertex, and a vertex of degree at least $d$ a $d^+$-vertex. A face is of size $k$ if the enumeration of the vertices on its boundary has length $k$, counting with multiplicity a vertex that appears multiple times. A face of size $k$ is called a $k$-face and a face of size at least $k$ a $k^+$-face.

## 2 The Proof

We begin by giving the proofs of Lemma 1.1 and Lemma 1.2 due to Caro, Petruševski and Škrekovski.

**Proof of Lemma 1.1** By Theorem 1.3, we partition $V(G)$ into at most 4 sets such that each part induces an odd forest. Each odd forest $F$ obviously satisfies $\chi_0(F) = 2$. By using different colours for the different forests, we can find an odd colouring of $G$ with at most 8 colours. \( \square \)
Before moving onto the proof of Lemma 1.2, we introduce a convenient notation that will be used many times in the rest of the paper: suppose we remove a vertex \( v \) from a graph \( G \) and possibly add some edges between neighbours of \( v \) while keeping the resulting graph \( G' \) planar. Assume \( G' \) has an odd colouring \( c \) using at most 8 colours. To extend the colouring to \( G \), we need to check two things: 1) that there is a colour that appears an odd number of times in \( N(v) \), and 2) that there exists a colour for \( v \) such that the colouring remains proper and the neighbours of \( v \) still each have a colour that appears an odd number of times in their own neighbourhood. Note that for the second condition, every neighbour \( w \) of \( v \) prevents \( v \) from using at most 2 colours: the colour \( c(w) \) (since we are looking for a proper colouring of \( G \)), and at most one other in the case where \( w \) has exactly one colour that appears an odd number of times in \( N(w) \backslash \{v\} \). In the following, we will call the first of the colours forbidden at \( v \) by the proper colouring condition with respect to \( c \) and \( G \) and the second forbidden at \( v \) by the oddness condition with respect to \( c \) and \( G \). When \( c \) and \( G \) are clear from context, we will write simply forbidden at \( v \) by the proper (oddness) condition.

Given a graph \( G \), \( v \in V(G) \) and a colouring \( c \) of \( G \), we will say that \( v \) has its oddness condition satisfied if there exists a colour that appears an odd number of times in \( N(v) \).

**Proof of Lemma 1.2** If \( v \in V(G) \) is a vertex of odd degree, we attach to it a new leaf \( w \). The new graph has even order, and consequently admits an odd colouring with at most eight colours by Lemma 1.1. Deleting \( w \) gives an odd colouring of the original graph \( G \) since \( v \) is of odd degree and hence always has its oddness condition satisfied.

If \( v \in V(G) \) is a vertex with \( d(v) = 2 \), we remove \( v \) and add an edge between the neighbours of \( v \) (if they do not already have an edge between them). The new graph has even order, and consequently admits an odd colouring \( c \) with at most eight colours by Lemma 1.1. With respect to \( c \) and \( G \), \( v \) has now 2 colours forbidden by the proper colouring condition, and at most 2 colours forbidden by the oddness condition. Hence there is a colour that we can give to \( v \) to extend the colouring to an odd colouring of \( G \). \( \square \)

From now on, assume for a contradiction that there is a planar graph which does not admit an odd colouring with at most 8 colours, and let \( G = (V, E) \) be a minimal such graph. By Lemma 1.4, all the degrees in \( G \) are even and at least 4.

We notice that \( G \) cannot contain certain configurations of vertices:

**Claim 2.1** There do not exist two neighbouring vertices of degree 4 in \( G \), that share at least one common neighbour.

**Proof** Suppose \( u \) and \( v \) are two neighbouring 4-vertices with a common neighbour \( w \). Consider the graph \( G' \) obtained from \( G \) by removing \( u \) and \( v \). Let \( c \) be an odd colouring of \( G' \) using at most 8 colours.

First, suppose at least one of \( u \) and \( v \), say \( u \), has another neighbour of colour \( c(w) \). At most 7 colours are forbidden at \( v \): at most 3 colours are used for \( v \)'s neighbours (we have not coloured \( u \) yet), and at most 4 colours are forbidden by the oddness condition. We then colour \( v \) with any of the remaining colours. Now the neighbours of \( u \) forbid at most 7 colours, since the colour \( c(w) \) is forbidden by the two neighbours having
colour \( c(w) \), hence there is a choice of a colour for \( u \) that gives a valid odd colouring of \( G \) with 8 colours, which is a contradiction.

If neither \( u \) nor \( v \) has another neighbour of colour \( c(w) \), then we colour \( v \) in the same way as above. Now at \( u \), the colour \( c(w) \) is forbidden by two different sources: by \( w \) by the proper colouring condition, and by \( v \) by the oddness condition, leaving one colour available for \( u \), which again gives a valid odd colouring of \( G \) with 8 colours, which is a contradiction. \( \square \)

**Claim 2.2** There do not exist two neighbouring vertices \( v, x \) in \( G \) such that \( v \) is a 4-vertex, \( x \) is a 6-vertex and \( |N(v) \cap N(x)| \geq 2 \).

**Proof** Let \( y_1 \) and \( y_2 \) be two of the common neighbours of \( v \) and \( x \). Let \( w \) be the fourth neighbour of \( v \) and let \( z_1, z_2, z_3 \) be the other neighbours of \( x \). See Fig. 2 for illustration.

Set \( V' = V(G) \setminus \{v\} \) and \( E' = E \cup \{wv, vx, vy_2, vw\} \), and let \( c \) be an odd colouring of \( G' = (V', E') \) using at most 8 colours. For a vertex \( u \neq v \), let \( b(u) \) be the colour, if it exists, forbidden at \( u \) by the oddness condition, with respect to the colouring \( c \) and graph \( G \). Note that the oddness condition is already satisfied at \( v \) with \( c(x) \), hence \( c(y_1), c(x), c(y_2), c(w), b(y_1), b(x), b(y_2) \) and \( b(w) \) must all exist and be different, otherwise, there would be a colour for \( v \) that extends \( c \) to an odd colouring on \( G \). Note that since \( b(x) \neq c(y_1) \), there exists at least one neighbour of \( x \) with colour \( c(y_1) \) under the colouring \( c \), say \( z_1 \) by symmetry. Similarly, we may suppose \( c(z_2) = c(y_2) \). With those assumptions, it follows that \( c(z_3) = b(x) \neq c(w) \). To complete our proof of Claim 2.2 we will now construct an odd colouring \( c' \) of \( G \) agreeing with \( c \) on \( V \setminus \{v, x\} \), and so contradicting the definition of \( G \). Let \( c'(u) = c(u) \) for each vertex \( u \in V \setminus \{v, x\} \). Furthermore, let \( b'(z_1) \) be the unique colour, if it exists, appearing an odd number of times among \( c'(u), u \in N(z_1) \setminus \{x\} \). Similarly, we define \( b'(z_2) \) and \( b'(z_3) \).
If \( b(y_1) \) and \( b(y_2) \) both appear among \( b'(z_1), b'(z_2) \) and \( b'(z_3) \), then set \( c'(v) = c(x) \). Now, with respect to the graph \( G \) and odd colouring \( c' \), \( x \) has 4 forbidden colours by the proper colouring condition: \( c'(y_1) = c'(z_1) = c(z_1), c'(y_2) = c'(z_2) = c(y_2) = c(z_2), c'(z_3) = c(z_3) \) and \( c'(v) = c(x) \). But it has at most 3 forbidden colours by the oddness condition: \( b'(z_1), b'(z_2) \) and \( b'(z_3) \), as the oddness condition at \( v \) will always be satisfied since \( c'(w) = c(w) \), \( c'(y_1) = c(y_1) \) and \( c'(y_2) = c(y_2) \) are all different. Thus there is a colour that can be assigned to \( x \). It is clear that this colouring is proper. Moreover, \( w \) has its oddness condition satisfied, as \( c'(v) = c(x) \neq b(w) \). The oddness condition is also satisfied for all other vertices. Thus this colouring is an odd colouring.

If at least one of \( b(y_1) \) and \( b(y_2) \) does not appear among \( b'(z_1), b'(z_2) \) and \( b'(z_3) \), say by symmetry \( b(y_1) \), then we assign \( c'(x) = b(y_1) \) and \( c'(v) = b(y_2) \). It is straightforward to check that the resulting colouring \( c' \) is proper. Moreover, the vertex \( w \) has its oddness condition satisfied as \( c'(v) = b(y_2) \neq b(w) \). The vertex \( x \) has its oddness condition satisfied as \( c'(v) = b(y_2) \) is different than \( c'(y_1) = c(y_1) = c'(z_1) \), \( c'(y_2) = c(y_2) = c'(z_2) = c(z_2) \) and \( c'(z_3) = c(z_3) = b(x) \). The vertex \( y_1 \) has its oddness condition satisfied as \( c'(v) = b(y_2) \) and \( b(y_2) \neq b(y_1) \) and \( b(y_2) \neq c(x) \). The vertex \( y_2 \) has its oddness condition satisfied as \( c'(x) = b(y_1) \) and \( b(y_1) \neq b(y_2) \) and \( b(y_1) \neq c(x) \). Each of the vertices \( z_1, z_2, z_3 \) have their own oddness condition satisfied as \( c'(x) = b(y_1) \) is by assumption different than \( b'(z_1), b'(z_2) \) and \( b'(z_3) \). Thus the resulting colouring \( c' \) is an odd colouring of the whole \( G \), giving a contradiction.

On top of the previous two claims we will use the discharging method. The general idea of the discharging method is to obtain certain local configurations in any planar drawing of the given graph. This is done by assigning initial charge to the vertices and faces so that the sum of all the charges is negative, and then distributing the charge according to so-called discharging rules. Those typically make many final charges non-negative. Since the sum of all charges does not change, there has to exist a vertex or a face whose final charge is negative. The discharging rules thus give information about the surroundings of this vertex or face.

In our case, we assign to each vertex \( v \) initial charge \( c_0(v) = d(v) - 6 \) and to each face \( F \) initial charge \( c_0(F) = 2d(F) - 6 \) where by \( d(v) \) we mean the degree of \( v \) and by \( d(F) \) we mean the degree of the face \( F \).

The discharging rules we apply are:

(R1) Every \( 8^+ \)-vertex \( u \) sends charge \( 1/2 \) to every neighbouring \( 4 \)-vertex such that the next neighbour of \( u \) in the counter-clockwise order is a \( 6^+ \)-vertex.

(R2) Every \( 4^+ \)-face \( F \) sends charges to its neighbouring \( 4 \)-vertices according to the following rules:

(i.) If \( F \) is a \( 4 \)-face with all vertices having degree \( 4 \), then \( F \) sends charge \( 1/2 \) to each of its vertices.

(ii.) If \( F \) is a \( 4 \)-face with three incident \( 4 \)-vertices and one incident \( 6^+ \)-vertex \( u \), and suppose the counter-clockwise enumeration of the vertices on the boundary of \( F \) is \( uv_1wv_2 \), then \( F \) sends charge \( 3/4 \) to \( v_1 \) and \( v_2 \) and \( 1/2 \) to \( w \).
(iii.) If $F$ is a 5-face with all vertices having degree 4, then $F$ sends charge $\frac{3}{4}$ to all its incident 4-vertices.
(iv.) Otherwise, $F$ sends charge 1 to all its incident 4-vertices.

If a vertex appears multiple times on the boundary of a face $F$, then $F$ sends it charges with multiplicity.

Let $ch_1(v)$ and $ch_1(F)$ be the charges of a vertex $v$ and of a face $F$ after discharging. Let $F(G)$ be the set of faces of $G$. Applying Euler’s formula, we have
\[
\sum_{v \in V(G)} ch_1(v) + \sum_{F \in F(G)} ch_1(F) = \sum_{v \in V(G)} ch_0(v) + \sum_{F \in F(G)} ch_0(F) = -12.
\]

Using Claims 2.1 and 2.2, we will show that, after discharging, each vertex and each face of $G$ have non-negative charge, therefore contradicting the existence of $G$. Note that it is easy to already see that, after discharging, any 6$^+$-vertex and any 4$^+$-face have non-negative charges, as proved in the two following claims.

**Claim 2.3** After discharging, every 6$^+$-vertex carries a non-negative charge.

**Proof** Only (R1) affects the charges of 6$^+$-vertices. The charge of 6-vertices remains 0 after discharging. Let $x$ be an 8$^+$-vertex. It has at most $d(x) \leq 2$ neighbouring 4-vertices such that the previous neighbour in the counter-clockwise order is a 6$^+$-vertex. Therefore, we have:
\[
ch_1(x) \geq (d(x) - 6) - \frac{1}{2} \cdot \frac{1}{2} \cdot d(x) \geq 0.
\]
\[\square\]

**Claim 2.4** After discharging, each face carries a non-negative charge.

**Proof** Only (R2) can change the charge of a face. The charge of a 3-face remains 0 after discharging. Let $F$ be a 6$^+$-face. Then, by (R2)(iv.), $ch_1(F) \geq 2d(F) - 6 - d(F) \geq 0$.

If $F$ is a 5-face, there are two possibilities. If all its incident vertices are 4-vertices, then the charge $F$ sends is $5 \cdot \frac{3}{4}$ as by (R2)(iii.), and thus $ch_1(F) = 2 \cdot 5 - 6 - 15/4 > 0$. On the other hand, if there is at least one 6$^+$-vertex incident with $F$, then $F$ sends charge 1 to at most 4 vertices. Then, $ch_1(F) \geq 2 \cdot 5 - 6 - 4 \cdot 1 = 0$.

Finally, let $F$ be a 4-face. If there are at most two 4-vertices incident with it, $F$ sends a charge at most 2, and thus $ch_1(F) \geq 2 \cdot 4 - 6 - 2 = 0$. If there are exactly three 4-vertices incident with $F$, then $F$ sends charge 2 in accordance with (R2)(ii.).
The face $F$ also sends charge 2 if all four vertices incident with it are 4-vertices in accordance with (R2)(i.). Thus, in every case, $ch_1(F) \geq 0$. □

Finally, let $v$ be a 4-vertex with negative charge $ch_1(v) < 0$. Let the neighbours of $v$ in clockwise order be $x_1, x_2, x_3, x_4$.

First, we note that the four faces around $v$ whose boundaries respectively contain the edges $vx_1$ and $vx_2$, the edges $vx_2$ and $vx_3$, the edges $vx_3$ and $vx_4$, and the edges $vx_4$ and $vx_1$, are all different. Indeed, if any two of them had been the same, the face would have been a $6^+\text{-face}$, or a $5\text{-face}$ with $x_i x_i + 2 \in E$ for some $i$ – and by Claim 2.1 such $x_j$ would not be a 4-vertex. In both cases, the face would send charge at least 2 to $v$ by (R2)(iv).

Since $ch_1(v) < 0$ and $ch_0(v) = -2$, by (R2) at least one of these faces is a 3-face, call it $F$. We can assume without loss of generality that $x_1 x_2 \in E$. By Claim 2.1, both $x_1, x_2$ are $6^+\text{-vertices}$.

By (R2) again, there is at least one other 3-face around $v$ different than the face $(x_1 x_2 v)$, as otherwise the faces around $v$ containing $\{vx_1, vx_4\}$ and $\{vx_2, vx_3\}$ would each send charge at least $3/4$ to $v$ in addition to the charge at least $1/2$ sent to $v$ by the remaining non-triangular face.

We claim that one of $x_1 x_4$, $x_2 x_3$ is an edge. Indeed, if the above-mentioned 3-face is formed by $v, x_3, x_4$, then $x_3$ and $x_4$ are $6^+\text{-vertices}$ by Claim 2.2 which in turn means that one of the remaining two faces has to be a 3-face too, as otherwise in view of (R2) each of them would send charge 1 to $v$. Without loss of generality let $x_2 x_3 \in E$. By Claim 2.1, $x_3$ is a $6^+\text{-vertex}$. Applying Claim 2.2 again, either $x_3$ is an $8^+\text{-vertex}$ (in case $x_3 x_4 \in E$) or $x_1$ is an $8^+\text{-vertex}$ (if $x_4 x_1 \in E$).

Now, $v$ receives charge at least 1 due to (R1). Therefore, the remaining face around $v$ is a 3-face, as otherwise it would send charge 1 in accordance with (R2). This means that $x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_1$ are all edges and the respective faces around $v$ are all 3-faces. Applying Claim 2.1 and Claim 2.2, all of $x_1, x_2, x_3, x_4$ are $8^+\text{-vertices}$. But by (R1), $v$ receives charge 1/2 from all of them, contradicting $ch_1(v) < 0$.

Therefore, there is no planar graph with odd chromatic number greater than 8, which proves Theorem 1.5.

**Note Added in Proof**

For an alternative approach, see a paper of Fabrici et al. [5] posted on arXiv shortly after ours, proving that for each planar graph there exists a conflict-free colouring (that is, for each vertex there is a colour appearing exactly once in its neighbourhood) using at most eight colours. That result could also follow from adapting the proof of Theorem 28 by Bhyravarapu et al. [2].

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