FRAÎSSÉ STRUCTURES WITH SDAP\(^+\), PART I: INDIVISIBILITY

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ABSTRACT

This is Part I of a two-part series regarding Ramsey properties of Fraïssé structures satisfying a property called SDAP+, which strengthens the Disjoint Amalgamation Property. We prove that every Fraïssé structure in a finite relational language with relation symbols of any finite arity satisfying this property is indivisible. Novelties include a new formulation of coding trees in terms of 1-types over initial segments of the Fraïssé structure, and a direct proof of indivisibility which uses the method of forcing to conduct unbounded searches for finite sets. In Part II, we prove that every Fraïssé structure in a finite relational language with relation symbols of arity at most two having this property has finite big Ramsey degrees which have a simple characterization. It follows that any such Fraïssé structure admits a big Ramsey structure. Part II utilizes a theorem from Part I as a pigeonhole principle for induction arguments. This work offers a streamlined and unifying approach to Ramsey theory on some seemingly disparate classes of Fraïssé structures.

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1. Introduction

In recent years, the Ramsey theory of infinite structures has seen quite an expansion. This area seeks to understand which infinite structures satisfy some analogue of the infinite Ramsey theorem for the natural numbers.

**Theorem 1.1** (Ramsey, [29]): Given integers $k, r \geq 1$ and a coloring of the $k$-element subsets of the natural numbers into $r$ colors, there is an infinite set of natural numbers, $N$, such that all $k$-element subsets of $N$ have the same color.

For infinite structures, exact analogues of Ramsey’s theorem usually fail, even when the class of finite substructures has the Ramsey property. This is due to some unseen structure which persists in every infinite substructure isomorphic to the original, but which dissolves when considering Ramsey properties of classes of finite substructures. This was first seen in Sierpiński’s use of a well-ordering on the rationals to construct a coloring of unordered pairs of rationals with two colors such that both colors persist in any subcopy of the rationals (see [32]). The interplay between the well-ordering and the rational order forms additional structure which is in some sense essential, as it persists upon taking any subset forming another dense linear order without endpoints. The quest to characterize and quantify the often hidden but essential structure for infinite structures, more generally, is the area of big Ramsey degrees.

Given an infinite structure $M$, we say that $M$ has *finite big Ramsey degrees* if for each finite substructure $A$ of $M$, there is an integer $T$ such that the following holds: For any coloring of the copies of $A$ in $M$ into finitely many colors, there is a substructure $M'$ of $M$ such that $M'$ is isomorphic to $M$, and the copies of $A$ in $M'$ take no more than $T$ colors. When a $T$ having this property exists, the least such value is called the *big Ramsey degree* of $A$ in $M$, denoted $T(A, M)$. In particular, if the big Ramsey degree of $A$ in $M$ is one, then any finite coloring of the copies of $A$ in $M$ is constant on some subcopy of $M$.

While the area of big Ramsey degrees on infinite structures traces back to Sierpiński’s result that the big Ramsey degree for unordered pairs of rationals is at least two, and progress on the rationals and other binary relational structures was made in the decades since, the question of which infinite structures have finite big Ramsey degrees attracted extended interest due to the flurry of results in [23], [24], [27], and [31] in tandem with the publication of [19], in which Kechris, Pestov, and Todorcevic asked for an analogue of their correspondence
between the Ramsey property of Fraïssé classes and extreme amenability to the
setting of big Ramsey degrees for Fraïssé limits. This was addressed by Zucker in
[34], where he proved a connection between Fraïssé limits with finite big Ramsey
degrees and completion flows in topological dynamics. Zucker’s results apply
to big Ramsey structures, expansions of Fraïssé limits in which the big Ramsey
degrees of the Fraïssé limits can be exactly characterized using the additional
structure induced by the expanded language. This additional structure involves
a well-ordering, and characterizes the essential structure which persists in every
infinite subcopy of the Fraïssé limit. It is this essential structure we seek to
understand in the study of big Ramsey degrees.

We describe an amalgamation property, called the Substructure Disjoint
Amalgamation Property (SDAP), forming a strengthened version of disjoint
amalgamation. The Fraïssé limit of a Fraïssé class satisfying SDAP is said to
satisfy SDAP if it satisfies two additional properties, which we call the Diago-
nal Coding Tree Property and the Extension Property. The motivation behind
SDAP was to distill properties inherent in proofs of big Ramsey degrees which
have a simple characterization, and it has led to Theorems 1.2 and 1.3 below.

A particular case of Ramsey theory on infinite structures is when one colors
copies of a given substructure with universe of size one. A Fraïssé limit $K$
is called indivisible if every one-element substructure of $K$ has big Ramsey
degree equal to one. In Part I, we prove indivisibility for Fraïssé limits in
finite relational languages with relation symbols of any finite arity satisfying
SDAP. In the case when $K$ has exactly one substructure of size one (up to
isomorphism), as happens for instance when the language of $K$ has no unary
relation symbols and there are no “loops” in $K$, this definition reduces to the
usual one for indivisibility of structures like the Rado graph and the Henson
graphs (see [31], [21], and [11]).

**Theorem 1.2**: Suppose $K$ is a Fraïssé class in a finite relational language with
relation symbols in any arity such that its Fraïssé limit $K$ satisfies SDAP. Then $K$ is indivisible.

Theorem 1.2 provides new classes of examples of indivisible Fraïssé structures,
in particular for ordered structures, while recovering results in [11], [21], and
[12].
In Part II, we characterize the exact big Ramsey degrees for all Fraïssé limits in finite relational languages with relation symbols of arity at most two satisfying SDAP\(^+\), or a related property called LSDAP\(^+\). Our characterization, together with results of Zucker in \[34\], imply that such Fraïssé limits admit big Ramsey structures, and their automorphism groups have metrizable universal completion flows.

**Theorem 1.3:** Let \(\mathcal{K}\) be a Fraïssé class in a finite relational language with relation symbols of arity at most two such that the Fraïssé limit \(\mathcal{K}\) of \(\mathcal{K}\) has SDAP\(^+\) or LSDAP\(^+\). Then \(\mathcal{K}\) has finite big Ramsey degrees which have a simple characterization and, moreover, admits a big Ramsey structure. Hence, the topological group \(\text{Aut}(\text{Flim}(\mathcal{K}))\) has a metrizable universal completion flow, which is unique up to isomorphism.

Theorem 1.3 provides new classes of examples of big Ramsey structures while recovering results in \[5\], \[16\], \[23\], and \[24\] and extending special cases of the results in \[35\] to obtain exact big Ramsey degrees. Theorem 5.4 in this paper will serve as the starting point for proving Theorem 1.3 in Part II.

We now discuss several theorems which follow from Theorem 1.2 or Theorem 1.3 as well as new examples obtained from our results. A fuller description is provided in Section 5 of Part II (\[4\]).

We show in Part II that SDAP\(^+\) holds for disjoint amalgamation classes which are “unrestricted.” Particular instances of unrestricted classes include classes of structures with finitely many unary and binary relations such as graphs, directed graphs, tournaments, graphs with several edge relations, etc., as well as their ordered versions. Our examples encompass those unconstrained binary relational structures considered in \[24\] as well as their ordered expansions. We also show in Part II that SDAP\(^+\) holds for Fraïssé limits of free amalgamation classes which forbid 3-irreducible substructures, namely, substructures in which any three distinct elements appear in a tuple for which some relation holds, as well as for their ordered versions. Hence, Theorem 1.2 implies that all unrestricted classes, all free amalgamation classes which forbid 3-irreducible substructures, and their ordered expansions have Fraïssé limits which are indivisible. Theorem 1.3 implies that such classes with relation symbols of arity at most two have Fraïssé limits with big Ramsey degrees which have a simple characterization. See Propositions 5.2 and 5.4 and Theorem 5.5 of \[4\] for more details.
Our methods also apply to certain Fraïssé structures derived from the rational linear order. In Part II, we will show that the structure \( \mathbb{Q}_Q \), the dense linear order without endpoints with an equivalence relation such that all equivalence classes are convex copies of the rationals, satisfies a related property called LSDAP\(^+\). Theorem 1.3 (and hence also the conclusion of Theorem 1.2) holds for \( \mathbb{Q}_Q \), answering a question raised by Zucker at the 2018 Banff Workshop on *Unifying Themes in Ramsey Theory*. More generally, we show that members of a natural hierarchy of finitely many convexly ordered equivalence relations, where each successive equivalence relation coarsens the previous one, also admit big Ramsey structures with a simple characterization. Theorem 1.3 recovers known results including Devlin’s characterization of the big Ramsey degrees of the rationals [5] as well as results of Laflamme, Nguyen Van Thé, and Sauer in [23] characterizing the big Ramsey degrees of \( \mathbb{Q}_n \), the rational linear order with a partition into \( n \) dense pieces, as these structures satisfy SDAP\(^+\). See Theorem 5.12 of Part II for more details.

While many of the known big Ramsey degree results use sophisticated versions of Milliken’s Ramsey theorem for trees [26], and while proofs using the method of forcing to produce ZFC results have appeared in [6], [8], [9], and [35], there are three novelties to our approach which produce a clarity about indivisibility and more generally, about big Ramsey degrees. Given a Fraïssé class \( K \), we fix an enumerated Fraïssé limit of \( K \), which we denote by \( K \). By *enumerated Fraïssé limit*, we mean that the universe of \( K \) is ordered via the natural numbers. The first novelty is that we work with trees of quantifier-free 1-types (see Definition 3.1) and develop forcing arguments directly on them to prove the Level Set Ramsey Theorem (Theorem 5.4). It was suggested to the second author by Sauer during the 2018 BIRS Workshop, *Unifying Themes in Ramsey Theory*, to try moving the forcing methods from [8] and [9] to forcing directly on the structures. Using trees of quantifier-free 1-types seems to come as close as possible to fulfilling this request, as the 1-types allow one to see the essential hidden structure (the interplay of a well-ordering of the universe with first instances where 1-types disagree), whereas working only on the Fraïssé structures, with no reference to 1-types, obscures this central feature of big Ramsey degrees from view. We will be calling such trees *coding trees*, as there will be special nodes, called *coding nodes*, representing the vertices of \( K \): The \( n \)-th coding node will be the quantifier-free 1-type of the \( n \)-th vertex of \( K \) over
the substructure of $K$ induced on the first $n - 1$ vertices of $K$. (The 0-th coding node is the quantifier-free 1-type of the 0-th vertex over the empty set.)

The second novelty of our approach is that our Level Set Ramsey Theorem (Theorem 5.4) immediately yields indivisibility for all Fraïssé structures satisfying SDAP$^+$ with finitely many relations of any finite arity. This is a consequence of forcing on diagonal coding trees, developed in the second author’s work for big Ramsey degrees of Henson graphs in [9] and [8]. It is interesting to note that in general, indivisibility does not follow from forcing on widely branching coding trees; diagonal coding trees are necessary to obtain indivisibility directly.

The third novelty is that we find the exact big Ramsey degrees directly from the diagonal coding trees of 1-types, without appeal to the standard method of “envelopes”. This means that the upper bounds which we find via forcing arguments are shown to be exact.

Using trees of quantifier-free 1-types (partially ordered by inclusion) allows us to prove a characterization of big Ramsey degrees for Fraïssé limits with SDAP$^+$ which is a simple extension of the so-called “Devlin types” for the rationals in [5], and of the characterization of the big Ramsey degrees of the Rado graph achieved by Laflamme, Sauer, and Vuksanovic in [24]. Here, we present the characterization for structures without unary relations. The full characterization is given in Part II.

**Simple Characterization of big Ramsey degrees:** Let $\mathcal{L}$ be a language consisting of finitely many relation symbols, each of arity two. Suppose $\mathcal{K}$ is a Fraïssé class in $\mathcal{L}$ such that the Fraïssé limit $K$ of $\mathcal{K}$ satisfies SDAP$^+$ or LSDAP$^+$. Fix a structure $A \in K$. Let $(A, \prec)$ denote $A$ together with a fixed enumeration $\langle a_i : i < n \rangle$ of the universe of $A$. We say that a tree $T$ is a diagonal tree coding $(A, \prec)$ if the following hold:

1. $T$ is a finite tree with $n$ terminal nodes and branching degree two.
2. $T$ has at most one branching node in any given level, and no two distinct nodes from among the branching nodes and terminal nodes have the same length. Hence, $T$ has $2n - 1$ many levels.
3. Let $\langle d_i : i < n \rangle$ enumerate the terminal nodes in $T$ in order of increasing length. Let $D$ be the $\mathcal{L}$-structure induced on the set $\{d_i : i < n\}$ by the increasing bijection from $\langle a_i : i < n \rangle$ to $\langle d_i : i < n \rangle$, so that $D \cong A$. Let $\tau_i$ denote the quantifier-free 1-type of $d_i$ over $D$, the substructure of $D$ on vertices $\{d_m : m < i\}$. Given $i < j < k < n$, if $d_j$ and $d_k$ both
extend some node in $T$ that is at the same level as $d_i$, then $d_j$ and $d_k$
have the same quantifier-free 1-types over $D_i$. That is, $\tau_j \upharpoonright D_i = \tau_k \upharpoonright D_i$.

Let $D(A, <)$ denote the number of distinct diagonal trees coding $(A, <)$; let $O.A$ denote a set consisting of one representative from each isomorphism class of ordered copies of $A$. Then

$$T(A, K) = \sum_{(A, <) \in O.A} D(A, <)$$

If $\mathcal{L}$ also has unary relation symbols, in the case that $K$ is a free amalgamation class, the simple characterization above holds when modified to diagonal coding trees with the same number of roots as unary relations. In the case that $K$ contains a transitive relation, then the above characterization still holds. The full characterization will be given in Theorem 4.8 in Part II.

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2. Amalgamation Properties

The inspiration for the amalgamation property SDAP defined in this section comes from a strengthening of the free amalgamation property, which we call the Substructure Free Amalgamation Property (SFAP). We originally found that any binary relational Fraïssé structure with an age satisfying SFAP has finite big Ramsey degrees that are characterized in a manner similar to the characterizations, in [24], of big Ramsey degrees for the Rado graph and other unconstrained binary relational structures with disjoint amalgamation. SFAP is satisfied by the ages of all unconstrained relational structures having free amalgamation, as well as by Fraïssé classes with forbidden irreducible and 3-irreducible substructures. The Substructure Disjoint Amalgamation Property
(SDAP) is a natural extension of SFAP to a broader collection of Fraïssé classes with disjoint amalgamation.

In Subsection 2.1 we review the basics of Fraïssé theory, the Ramsey property, and indivisibility. Big Ramsey degrees and big Ramsey structures will be discussed in Section 2 of Part II. More general background on Fraïssé theory can be found in Fraïssé’s original paper [14], as well as [15]. The properties SFAP and SDAP are presented in Subsection 2.2. The presentation of SDAP will be given in Definition 4.18, after coding trees of 1-types and related notions are defined in Section 3.

2.1. Fraïssé theory and indivisibility. All relations in this paper will be finitary, and all languages will consist of finitely many relation symbols (and no constant or function symbols). We use the set-theoretic notation $\omega$ to denote the set of natural numbers, $\{0, 1, 2, \ldots\}$, and treat $n \in \omega$ as the set $\{i \in \omega : i < n\}$.

Let $\mathcal{L} = \{R_i : i < I\}$ be a finite language where each $R_i$ is a relation symbol with associated arity $n_i \in \omega$. An $\mathcal{L}$-structure is an object

$$M = \langle M, R_0^M, \ldots, R_{i-1}^M \rangle$$

where $M$ is a nonempty set, called the universe of $M$, and each $R_i^M \subseteq M^{n_i}$. Finite structures will typically be denoted by $A, B$, etc., and their universes by $A, B$, etc. Infinite structures will typically be denoted by $J, K$ and their universes by $J, K$. We will call the elements of the universe of a structure vertices.

An embedding between $\mathcal{L}$-structures $M$ and $N$ is an injection $\iota : M \rightarrow N$ such that for each $i < I$ and for all $a_0, \ldots, a_{n_i-1} \in M$,

$$R_i^M(a_0, \ldots, a_{n_i-1}) \iff R_i^N(\iota(a_0), \ldots, \iota(a_{n_i-1})).$$

A surjective embedding is an isomorphism, and an isomorphism from $M$ to $N$ is an automorphism. The set of embeddings of $M$ into $N$ is denoted $\text{Emb}(M, N)$, and the set of automorphisms of $M$ is denoted $\text{Aut}(M)$. When $M \subseteq N$ and the inclusion map is an embedding, we say $M$ is a substructure of $N$. When there exists an embedding $\iota$ from $M$ to $N$, the substructure of $N$ having universe $\iota[M]$ is called a copy of $M$ in $N$, and it is a subcopy of $N$ if $M$ is isomorphic to $N$. The age of $M$, written $\text{Age}(M)$, is the class of all finite $\mathcal{L}$-structures that embed into $M$. We write $M \leq N$ when there is an embedding of $M$ into $N$, and $M \cong N$ when there is an isomorphism from $M$ to $N$. 
A class $\mathcal{K}$ of finite structures in a finite relational language is called a *Fraïssé class* if it is nonempty, closed under isomorphisms, hereditary, and satisfies the joint embedding and amalgamation properties. The class $\mathcal{K}$ is *hereditary* if whenever $B \in \mathcal{K}$ and $A \subseteq B$, then also $A \in \mathcal{K}$. The class $\mathcal{K}$ satisfies the *joint embedding property* if for any $A, B \in \mathcal{K}$, there is a $C \in \mathcal{K}$ such that $A \subseteq C$ and $B \subseteq C$. The class $\mathcal{K}$ satisfies the *amalgamation property* if for any embeddings $f : A \to B$ and $g : A \to C$, with $A, B, C \in \mathcal{K}$, there is a $D \in \mathcal{K}$ and there are embeddings $r : B \to D$ and $s : C \to D$ such that $r \circ f = s \circ g$. Note that in a finite relational language, there are only countably many finite structures up to isomorphism.

An $L$-structure $\mathcal{K}$ is called *ultrahomogeneous* if every isomorphism between finite substructures of $\mathcal{K}$ can be extended to an automorphism of $\mathcal{K}$. We call a countably infinite, ultrahomogeneous structure a *Fraïssé structure*. Fraïssé showed [14] that the age of a Fraïssé structure is a Fraïssé class, and that conversely, given a Fraïssé class $\mathcal{K}$, there is, up to isomorphism, a unique Fraïssé structure whose age is $\mathcal{K}$. Such a Fraïssé structure is called the *Fraïssé limit* of $\mathcal{K}$ or the *generic structure* for $\mathcal{K}$.

Throughout this paper, $\mathcal{K}$ will denote the Fraïssé limit of a Fraïssé class $\mathcal{K}$. We will sometimes write $\text{Flim}(\mathcal{K})$ for $\mathcal{K}$. We will assume that $\mathcal{K}$ has universe $\omega$, and call such a structure an *enumerated Fraïssé structure*. For $m < \omega$, we let $\mathcal{K}_m$ denote the substructure of $\mathcal{K}$ with universe $m = \{0, 1, \ldots, m-1\}$.

The following amalgamation property will be assumed in this paper: A Fraïssé class $\mathcal{K}$ satisfies the *Disjoint Amalgamation Property* if, given embeddings $f : A \to B$ and $g : A \to C$, with $A, B, C \in \mathcal{K}$, there is an amalgam $D \in \mathcal{K}$ with embeddings $r : B \to D$ and $s : C \to D$ such that $r \circ f = s \circ g$ and moreover, $r[B] \cap s[C] = r \circ f[A] = s \circ g[A]$. The disjoint amalgamation property is also called the *strong amalgamation property*. It is equivalent to the *strong embedding property*, which requires that for any $A \in \mathcal{K}$, $v \in A$, and embedding $\varphi : (A - v) \to \mathcal{K}$, there are infinitely many different extensions of $\varphi$ to embeddings of $A$ into $\mathcal{K}$. (See [3].)

A Fraïssé class has the *Free Amalgamation Property* if it satisfies the Disjoint Amalgamation Property and moreover, the amalgam $D$ can be chosen so that no tuple satisfying a relation in $D$ includes elements of both $r[B] \setminus r \circ f[A]$ and $s[C] \setminus s \circ g[A]$; in other words, $D$ has no additional relations on its universe other than those inherited from $B$ and $C$. 
For languages $\mathcal{L}_0$ and $\mathcal{L}_1$ such that $\mathcal{L}_0 \cap \mathcal{L}_1 = \emptyset$, and given Fraïssé classes $K_0$ and $K_1$ in $\mathcal{L}_0$ and $\mathcal{L}_1$, respectively, the free superposition of $K_0$ and $K_1$ is the Fraïssé class consisting of all finite ($\mathcal{L}_0 \cup \mathcal{L}_1$)-structures $A$ such that the $\mathcal{L}_i$-reduct of $A$ is in $K_i$, for each $i < 2$. (See also [2] and [17].) Note that the free superposition of $K_0$ and $K_1$ has free amalgamation if and only if each $K_i$ has free amalgamation; and similarly for disjoint amalgamation.

Given a Fraïssé class $K$ and substructures $M, N$ of $K$ (finite or infinite) with $M \leq N$, we use $\binom{N}{M}$ to denote the set of all substructures of $N$ which are isomorphic to $M$. Given $M \leq N \leq O$, substructures of $K$, we write $O \rightarrow (N)_{\ell}^{M}$ to denote that for each coloring of $\binom{O}{M}$ into $\ell$ colors, there is an $N' \in \binom{O}{N}$ such that $\binom{N'}{M}$ is monochromatic, meaning that all members of $\binom{N'}{M}$ have the same color.

**Definition 2.1:** A Fraïssé structure $K$ is indivisible if for every singleton substructure $A$ of $K$, $K \rightarrow (K)^A_{\ell}$ for every positive integer $\ell$.

Note that when there is only one quantifier-free 1-type over the empty set satisfied by elements of $K$, so that $K$ has exactly one singleton substructure up to isomorphism, indivisibility amounts to saying that for any partition of the universe of $K$ into finitely many pieces, there is a subcopy of $K$ contained in one of the pieces. Indivisibility has been proved for many structures, including the triangle-free Henson graph in [21], the $k$-clique-free Henson graphs for all $k \geq 4$ in [11], more general binary relational free amalgamation structures in [30], and for $k$-uniform hypergraphs, $k \geq 3$, that omit finite substructures in which all unordered triples of vertices are contained in at least one $k$-edge in [12]. For a much broader discussion of Fraïssé structures and indivisibility, the reader is referred to Nguyen Van Thé’s Habilitation [28].

**2.2. Substructure Amalgamation Properties.** Recall that given a Fraïssé class $\mathcal{K}$ in a finite relational language $\mathcal{L}$, we let $K$ denote an enumerated Fraïssé limit of $\mathcal{K}$ with underlying set $\omega$. All results will hold regardless of which enumeration is chosen. We make the following conventions and assumptions, which will hold in the rest of this paper.

All types will be quantifier-free 1-types, over a finite parameter set, that are realizable in $K$. With one exception, all such types will be complete; the
exception is the case of “passing types”, defined in Section 3, which may be partial. Complete types will be denoted simply “tp”.

We will assume that for any relation symbol $R$ in $L$, $R^K(\bar{a})$ can hold only for tuples $\bar{a}$ of distinct elements of $\omega$. In particular, we assume our structures have no loops. We further assume that all relations in $K$ are non-trivial: This means that for each relation symbol $R$ in $L$, there exists a $k$-tuple $\bar{a}$ of (distinct) elements of $\omega$ such that $R^K(\bar{a})$ holds, and a $k$-tuple $\bar{b}$ of (distinct) elements of $\omega$ such that $\neg R^K(\bar{b})$ holds. Since $\mathcal{K}$ has disjoint amalgamation by assumption, non-triviality will imply that there are infinitely many $k$-tuples from $\omega$ that satisfy $R^K$, and infinitely many that do not. We will further hold to the following convention:

*Convention 2.2:* If $L$ has at least one unary relation symbol, then letting $R_0, \ldots, R_{n-1}$ list them, we have that $n \geq 2$ and for each $A \in \mathcal{K}$ and each $a \in A$, $R^A_i(a)$ holds for exactly one $i < n$.

By possibly adding new unary relation symbols to the language, any Fraïssé class with unary relations can be assumed to meet this convention.

Finally, we assume that there is at least one non-unary relation symbol in $\mathcal{L}$. This poses no real restriction, as whenever a finite language has only unary relation symbols, any disjoint amalgamation class in that language will have a Fraïssé limit that consists of finitely many disjoint copies of $\omega$, with vertices in a given copy all realizing the same quantifier-free 1-type over the empty set. In this case, finitely many applications of Ramsey’s Theorem will prove the existence of finite big Ramsey degrees.

We now present the Substructure Free Amalgamation Property. This property also provides the intuition behind the more general amalgamation property SDAP (Definition 2.5), laying the foundation for the main ideas of this paper.

*Definition 2.3 (SFAP):* A Fraïssé class $\mathcal{K}$ has the *Substructure Free Amalgamation Property (SFAP)* if $\mathcal{K}$ has free amalgamation, and given $A, B, C, D \in \mathcal{K}$, the following holds: Suppose

1. $A$ is a substructure of $C$, where $C$ extends $A$ by two vertices, say $C \setminus A = \{v, w\}$;
2. $A$ is a substructure of $B$ and $\sigma$ and $\tau$ are 1-types over $B$ with $\sigma \upharpoonright A = \text{tp}(v/A)$ and $\tau \upharpoonright A = \text{tp}(w/A)$; and
(3) $\mathbf{B}$ is a substructure of $\mathbf{D}$ which extends $\mathbf{B}$ by one vertex, say $v'$, such that $\text{tp}(v'/\mathbf{B}) = \sigma$.

Then there is an $\mathbf{E} \in \mathcal{K}$ extending $\mathbf{D}$ by one vertex, say $w'$, such that $\text{tp}(w'/\mathbf{B}) = \tau$, $\mathbf{E} \upharpoonright (\mathbf{A} \cup \{v', w'\}) \cong \mathbf{C}$, and $\mathbf{E}$ adds no other relations over $\mathbf{D}$.

The definition of SFAP can be stated using embeddings rather than substructures in the standard way. We remark that requiring $\mathbf{C}$ in (1) to have only two more vertices than $\mathbf{A}$ is sufficient for all our uses of the property in proofs of big Ramsey degrees, and hence we have not formulated the property for $\mathbf{C}$ of arbitrary finite size.

Remark 2.4: SFAP is equivalent to free amalgamation along with a model-theoretic property that may be termed free $3$-amalgamation, a special case of the disjoint $3$-amalgamation property defined in [22]: In the definition of disjoint $n$-amalgamation in Section 3 of [22], take $n = 3$ and impose the further condition that the “solution” or $3$-amalgam disallows any relations (in any realization of the solution) that were not already stipulated in the initial $3$-amalgamation “problem”. Kruckman shows in [22] that if the age of a Fraïssé limit $\mathbf{K}$ has disjoint amalgamation and disjoint $3$-amalgamation, then $\mathbf{K}$ exhibits a model-theoretic tameness property called simplicity.

SFAP ensures that a finite substructure of a given enumerated Fraïssé structure can be extended as desired without any requirements on its configuration inside the larger structure. SFAP precludes any need for the so-called “witnessing properties” which were necessary for the proofs of finite big Ramsey degrees for constrained binary free amalgamation classes, as in the $k$-clique-free Henson graphs in [9] and [8], and the recent more general extensions in [35]. Free amalgamation classes with forbidden $3$-irreducible substructures satisfy SFAP, as shown in Proposition 5.2 of Part II.

The next amalgamation property extends SFAP to disjoint amalgamation classes. In the definition, we again use substructures rather than embeddings.

**Definition 2.5 (SDAP):** A Fraïssé class $\mathcal{K}$ has the **Substructure Disjoint Amalgamation Property (SDAP)** if $\mathcal{K}$ has disjoint amalgamation, and the following holds: Given $\mathbf{A}, \mathbf{C} \in \mathcal{K}$, suppose that $\mathbf{A}$ is a substructure of $\mathbf{C}$, where $\mathbf{C}$ extends $\mathbf{A}$ by two vertices, say $v$ and $w$. Then there exist $\mathbf{A}', \mathbf{C}' \in \mathcal{K}$, where $\mathbf{A}'$ contains a copy of $\mathbf{A}$ as a substructure and $\mathbf{C}'$ is a disjoint amalgamation of $\mathbf{A}'$ and $\mathbf{C}$.
over $A$, such that letting $v', w'$ denote the two vertices in $C' \setminus A'$ and assuming (1) and (2), the conclusion holds:

(1) Suppose $B \in \mathcal{K}$ is any structure containing $A'$ as a substructure, and let $\sigma$ and $\tau$ be 1-types over $B$ satisfying $\sigma \upharpoonright A' = \text{tp}(v'/A')$ and $\tau \upharpoonright A' = \text{tp}(w'/A')$.

(2) Suppose $D \in \mathcal{K}$ extends $B$ by one vertex, say $v''$, such that $\text{tp}(v''/B) = \sigma$.

Then there is an $E \in \mathcal{K}$ extending $D$ by one vertex, say $w''$, such that $\text{tp}(w''/B) = \tau$ and $E \upharpoonright (A \cup \{v'', w''\}) \cong C$.

**Remark 2.6:** We note that SFAP implies SDAP, taking $A' = A$ and $C' = C$ and because disjoint amalgamation is implied by free amalgamation. Further, it follows from their definitions that SFAP and SDAP are each preserved under free superposition.

**Example 2.7:** The idea behind allowing for an extension $A'$ of $A$ in the definition of SDAP is most simply demonstrated for the Fraïssé class $\mathcal{LO}$ of finite linear orders. Given $A, C \in \mathcal{LO}$, suppose $v, w$ are the two vertices of $C \setminus A$ and suppose that $v < w$ holds in $C$. We can require $A'$ to be some extension of $A$ in $\mathcal{K}$ containing some vertex $u$ so that the formula $(x < u)$ is in $\text{tp}(v/A')$ and $(u < x)$ is in $\text{tp}(w/A')$, where $x$ is a variable. Then given any 1-types $\sigma, \tau$ extending $\text{tp}(v/A'), \text{tp}(w/A')$, respectively, over some structure $B$ containing $A'$ as a substructure, any two vertices $v', w'$ satisfying $\sigma, \tau$ will automatically satisfy $v' < w'$, thus producing a copy of $C$ extending $A$.

In the case of finitely many independent linear orders, we can similarly produce an $A'$ which ensures that any vertices $v', w'$ satisfying such $\sigma, \tau$ as above produce a copy of $C$ extending $A$. In more general cases, the use of $A'$ only ensures that there exist such vertices $v', w'$.

**Remark 2.8:** Ivanov [18] and independently, Kechris and Rosendal [20], have formulated a weakening of the amalgamation property which is called *almost amalgamation* in [18] and *weak amalgamation* in [20]. This property arises in the context of generic automorphisms of countable structures. In the presence of disjoint amalgamation, SDAP may be thought of as a ternary version of weak amalgamation (one of several possible such versions), and as a “weak” version of the disjoint 3-amalgamation property from [22] (again, one of several possible such weakenings).
Remark 2.9: We note that we could have used the definition of the free 3-amalgamation property from Remark 2.4 and of an appropriately formulated version of a “weak” disjoint 3-amalgamation property as in Remark 2.8. We have chosen to use Definitions 2.3 and 2.5 instead, as they are the forms used in the proof of Theorem 5.4.

In Section 3 onwards we will be working with the new notion of coding trees of 1-types, which represent subcopies of a given Fraïssé limit $K$. For Fraïssé structures in languages with relation symbols of arity greater than two, a priori, these trees may have unbounded branching. However, for all classes with SFAP and for all classes with SDAP which we have investigated, one can construct subtrees with bounded branching which still represent $K$. Accordingly, we will formulate the strengthened version SDAP$^+$ of SDAP in Subsection 4.3 which imposes conditions on the branching in a coding tree for $K$.

3. Coding trees of 1-types for Fraïssé structures

Fix throughout a Fraïssé class $K$ in a finite relational language $L$. Recall that $K$ denotes an enumerated Fraïssé limit for $K$, meaning that $K$ has universe $\omega$. In order to avoid confusion, we shall usually use $v_n$ instead of just $n$ to denote the $n$-th member of the universe of $K$, and we shall call this the $n$-th vertex of $K$. For $n < \omega$, we write $K_n$, and sometimes $K \upharpoonright n$, to denote the substructure of $K$ on the set of vertices $\{v_i : i < n\}$. We call $K_n$ an initial segment of $K$. Note that $K_0$ is the empty structure.

In Subsection 3.1 we present a general construction of trees of complete 1-types over initial segments of $K$, which we call coding trees. Graphics of coding trees are then presented for various prototypical Fraïssé classes which will be proved in Part II to satisfy SDAP$^+$. In Subsection 3.2 we define passing types, extending the notion of passing number due to Laflamme, Sauer, and Vuksanovic in [24], which has been central to all prior results on big Ramsey degrees for binary relational structures. Then we extend the notion from [24] of similarity type for binary relational structures to structures with relations of any arity. In Subsection 4.1 we introduce diagonal coding trees. These will be key to obtaining indivisibility directly from Theorem 5.4 as well as precise big Ramsey degree results without appeal to the method of envelopes in Part II. We define the Diagonal Coding Tree Property, one of the conditions for SDAP$^+$ to hold.
3.1. Coding Trees of 1-types. All types will be quantifier-free 1-types, with variable $x$, over some finite initial segment of $K$. For $n \geq 1$, a type over $K_n$ must contain the formula $\neg(x = v_i)$ for each $i < n$. Given a type $s$ over $K_n$, for any $i < n$, $s \upharpoonright K_i$ denotes the restriction of $s$ to parameters from $K_i$. Recall that the notation “tp” denotes a complete quantifier-free 1-type.

Definition 3.1 (The Coding Tree of 1-types, $S(K)$): The coding tree of 1-types $S(K)$ for an enumerated Fraïssé structure $K$ is the set of all complete 1-types over initial segments of $K$ along with a function $c : \omega \to S(K)$ such that $c(n)$ is the 1-type of $v_n$ over $K_n$. The tree-ordering is simply inclusion.

We shall usually simply write $S$, rather than $S(K)$. Note that we make no requirement at this point on $K$; an enumerated Fraïssé limit of any Fraïssé class (with no reference to its amalgamation or Ramsey properties) naturally induces a coding tree of 1-types as above. We say that $c(n)$ represents or codes the vertex $v_n$. Instead of writing $c(n)$, we shall usually write $c_n$ for the $n$-th coding node in $S$.

We let $S(n)$ denote the collection of all 1-types $\text{tp}(v_i/K_n)$, where $i \geq n$. Note that each $c(n)$ is a node in $S(n)$. The set $S(0)$ consists of the 1-types over the empty structure $K_0$. For $s \in S(n)$, the immediate successors of $s$ are exactly those $t \in S(n+1)$ such that $s \subseteq t$. For each $n < \omega$, the set $S(n)$ is finite, since the language $L$ consists of finitely many finitary relation symbols.

We say that each node $s \in S(n)$ has length $n + 1$, and denote the length of $s$ by $|s|$. Thus, all nodes in $S$ have length at least one. While it is slightly unconventional to consider the roots of $S$ as having length one, this approach lines up with the natural correspondence between nodes in $S$ and certain sequences of partial 1-types that we define in the next paragraph. The reader wishing for a tree starting with a node of length zero may consider adding the empty set to $S$, as this will have no effect on the results in this paper. A level set is a subset $X \subseteq S$ such that all nodes in $X$ have the same length.

Let $n < \omega$ and $s \in S(n)$ be given. We let $s(0)$ denote the set of formulas in $s$ involving no parameters; $s(0)$ is the unique member of $S(0)$ such that $s(0) \subseteq s$. For $1 \leq i \leq n$, we let $s(i)$ denote the set of those formulas in $s \upharpoonright K_i$ in which $v_{i-1}$ appears; in other words, the formulas in $s \upharpoonright K_i$ that are not in $s \upharpoonright K_{i-1}$. In this manner, each $s \in S$ determines a unique sequence $\langle s(i) : i < |s| \rangle$, where $\{s(i) : i < |s|\}$ forms a partition of $s$. For $j < |s|$, $\bigcup_{i \leq j} s(i)$ is the node in
\(S(j)\) such that \(\bigcup_{i \leq j} s(i) \subseteq s\). For \(\ell \leq |s|\), we shall usually write \(s \upharpoonright \ell\) to denote \(\bigcup_{i < \ell} s(i)\).

Given \(s, t \in S\), we define the meet of \(s\) and \(t\), denoted \(s \land t\), to be \(s \upharpoonright K\) for the maximum \(m \leq \min(|s|, |t|)\) such that \(s \upharpoonright K = t \upharpoonright K\). It can be useful to think of \(s \in S\) as the sequence \(\langle s(0), \ldots, s(|s| - 1)\rangle\); then \(s \land t\) can be interpreted in the usual way for trees of sequences.

It will be useful later to have specific notation for unary relations. We will let \(\Gamma\) denote \(S(0)\), the set of complete 1-types over the empty set that are realized in \(K\). For \(\gamma \in \Gamma\), we write “\(\gamma(v_n)\) holds in \(K\)” when \(\gamma\) is the 1-type of \(v_n\) over the empty set; in practice, it will be the unary relation symbols in \(\gamma\) (if there are any) that will be of interest to us.

**Remark 3.2:** Our definition of \(s(i)\) sets up for the definition of passing type in Subsection 3.2, which directly abstracts the notion of passing number used in [31] and [24], and in subsequent papers building on their ideas.

**Remark 3.3:** In the case where all relation symbols in the language \(L\) have arity at most two, the coding tree of 1-types \(S\) has bounded branching. If \(L\) has any relation symbol of arity three or greater, then \(S\) may have branching which increases as the levels increase. If such a Fraïssé class satisfies SDAP, sometimes more work still must be done in order to guarantee that its Fraïssé limit has SDAP\(^+\).

We now provide graphics for coding trees of 1-types which are prototypical for Fraïssé classes which have SDAP. That their Fraïssé limits satisfy SDAP\(^+\) will be proved in Section 5 of Part II. We start with the rational linear order, since its coding tree of 1-types is the simplest, and also because the rationals were the first Fraïssé structure for which big Ramsey degrees were characterized (Devlin, [5]).

**Example 3.4** (The coding tree of 1-types \(S(\mathbb{Q})\)): Figure 1 shows the coding tree of 1-types for \((\mathbb{Q}, <)\), the rationals as a linear order. This is the Fraïssé limit of \(LO\), the class of finite linear orders. We assume that the universe of \(\mathbb{Q}\) is linearly ordered in order-type \(\omega\) as \(\langle v_n : n < \omega \rangle\). For each \(n\), the coding node \(c_n\) is the 1-type of vertex \(v_n\) over the initial segment \(\{v_i : i < n\}\) of \(\mathbb{Q}\). (Recall that \(x\) is the variable in all of our 1-types.) Thus, the coding node \(c_0\) is the empty 1-type, and \(c_1\) is the 1-type \(\{v_0 < x\}\). Thus, the coding nodes \(\{c_0, c_1\}\) represent the linear order \(v_0 < v_1\). Likewise, the coding node \(c_2\) is the 1-type \(\{x < v_0, x < v_1\}\)
over the linear order $v_0 < v_1$. Hence, $c_2$ represents the vertex $v_2$ satisfying $v_2 < v_0 < v_1$. The coding node $c_3$ is the 1-type $\{v_0 < x, x < v_1, v_2 < x\}$, so $c_3$ represents the vertex $v_3$ satisfying $v_2 < v_0 < v_3 < v_1$. Below the tree, we picture the linear order on the vertices $v_0, \ldots, v_5$ induced by the coding nodes. As the tree grows in height, the linear order represented by the coding nodes grows into the countable dense linear order with no endpoints.

Notice that only the coding nodes branch. This is because of the rigidity of the rationals: Given a non-coding node $s$ on the same level as a coding node $c_n$ (say $n \geq 1$), $s$ is a 1-type which is satisfied by any vertex which lies in some interval determined by the vertices $\{v_i : i < n\}$, and $v_n$ is not in that interval. Thus, the order between $v_n$ and any vertex satisfying $s$ is predetermined, so $s$ does not split. Said another way, letting $m$ denote the length of the meet of $c_n$ and $s$, $c_n$ and $s$ must disagree on the formula $x < v_m$; hence, $x < v_m$ is in $c_n$ if and only if $v_m < x$ is in $s$. In the case that the formula $x < v_m$ is in $c_n$, then it follows that $v_n < v_m$. On the other hand, any realization $v_i$ of the 1-type $s$ must satisfy $v_m < v_i$. Hence every realization of $s$ by some vertex $v_i$ must satisfy $v_n < v_i$. Thus, there is only one immediate successor of $s$ in the tree of 1-types. The tree of 1-types for $\mathbb{Q}$ eradicates the extraneous structure which appears in the more traditional approach of using the full binary branching tree and Milliken’s Theorem to approach big Ramsey degrees of the rationals.

**Example 3.5** (The coding tree of 1-types $S(\mathbb{Q}_2)$): Next, we consider coding trees of 1-types for linear orders with equivalence relations with finitely many equivalence classes, each of which is dense in the linear order. Figure 2 provides a graphic for the coding tree of 1-types for the structure $\mathbb{Q}_2$, the rationals with an equivalence relation with two equivalence classes which are each dense in the linear order. We point out that $c_0$ is the 1-type $\{U_1(x)\}$, $c_1$ is the 1-type $\{U_0(x), x < v_0\}$, $c_2$ is the 1-type $\{U_0(x), v_0 < x, v_1 < x\}$, etc.

Note that $S(\mathbb{Q}_2)$ looks like two identical disjoint copies of a coding tree for $\mathbb{Q}$. This is because each of the two unary relations, representing the two equivalence classes, appears densely in the linear order. The ordered structure $\mathbb{Q}_2$ appears below the two trees as the vertices $v_0, v_1, \ldots$. Unlike Figure 1 for $\mathbb{Q}$, the vertices in $\mathbb{Q}_2$ do not line up below the coding nodes in the trees representing them, since $S(\mathbb{Q}_2)$ has two roots. However, if we modify our definition of coding tree of 1-types to have individual coding nodes $c_n$ represent the unary relations satisfied by $v_n$ (rather than $S$ having $|\Gamma|$ many roots), this has the effect of producing a
one-rooted tree with “γ-colored” coding nodes appearing cofinally in the tree, for each γ ∈ Γ. This approach then shows the linear order Q₂ lining up below the coding nodes, recovers the characterization of the big Ramsey degrees in [23], and will aid us in proving SDAP⁺ for P₂. (See Definition 4.1 for this variation of tree of 1-types, which reproduces the approach in [23].)

Similarly, for any n ≥ 2, S(Qₙ) will have n roots, and above each root, the n trees will be copies of each other.

Example 3.6 (The coding tree of 1-types S(Qₐ)): Next, we present the tree of 1-types for the Fraïssé structure Qₐ. Recall that this is the Fraïssé limit of the class CO in the language L = {<, E}, where < is a linear order and E is a convexly ordered equivalence relation, meaning that all equivalence classes are intervals.

Figure 3 shows the first six levels of a coding tree of 1-types, S(Qₐ). The formulas which are in the 1-types can be read from the graphic. For instance,
$c_0$ is the empty type. $c_1$ is the 1-type $\{v_0 < x, E(x, v_0)\}$, so since the vertex $v_1$ satisfies this 1-type, we have $v_0 < v_1$ and $E(v_0, v_1)$ holding. Similarly, $c_2$ is the 1-type $\{x < v_0, \neg E(x, v_0), x < v_1, \neg E(x, v_1)\}$, so $v_2$ satisfies $v_2 < v_0$, and $v_2$ is not equivalent to either of $v_0$ or $v_1$. $c_3$ is the 1-type $\{v_0 < x, E(x, v_0), v_1 < x, E(x, v_1), v_2 < x, \neg E(x, v_2)\}$, and hence, we see that $v_1 < v_3$ and $v_3$ is equivalent to $v_1$ and hence also to $v_0$. Note that only coding nodes branch. Moreover, $c_n$ has splitting degree two if $c_n$ represents a vertex $v_n$ which is equivalent to $v_i$ for some $i < n$; otherwise $c_n$ has splitting degree four.

For each non-coding node $s$ on the level of a coding node $c_n$, there is only one possible 1-type extending $s$ over the initial structure on the first $n + 1$ vertices of $\mathbb{Q}_Q$. We will show in Theorem 5.10 in Part II that $\mathbb{Q}_Q$ satisfies SDAP$^+$. 

In Figure 2 below the tree $S(\mathbb{Q}_Q)$ is the linear order on the vertices $v_0, \ldots, v_5$ represented by the coding nodes $c_0, \ldots, c_5$; the lines between the vertices represent that they are in the same equivalence class. Thus, $v_0, v_1, v_3$ are all in the same equivalence class, $v_2, v_4$ are in a different equivalence class, and $v_5$ is in yet another equivalence class.

Next, we present graphics for coding trees of 1-types for some free amalgamation classes. The tree of 1-types for the Rado graph is simply a binary tree in which the coding nodes are dense and every node $s$ at the level of the $n$-th coding node splits into two immediate successors, representing the two possible extensions of $s$ to the 1-types $s \cup \{E(x, v_n)\}$ and $s \cup \{\neg E(x, v_n)\}$. This follows
Figure 3. Coding tree of 1-types for $\mathbb{Q}$ and linear order with convex equivalence relations represented by its coding nodes.
immediately from the Extension Property for the Rado graph. As this is simple to visualize, and as a graphic has already appeared in [10], we move on to bipartite graphs.

Example 3.7 (The coding tree of 1-types for the generic bipartite graph): Figure 4 presents a coding tree of 1-types for the generic bipartite graph. The unary relations $U_0$ and $U_1$, which keep track of which partition each vertex is in, are represented by “red” and “blue”, respectively. We have chosen to enumerate this structure so that odd indexed vertices are in one of the partitions, and even indexed vertices are in the other, for purely aesthetic reasons. The edge relation is represented as extension to the right, and non-edge is represented by extending left. On the left is the bipartite graph being represented by the coding nodes in the two-rooted tree of 1-types. For instance, $c_0$ is the 1-type $\{U_0(x)\}$, so $v_0$ is a vertex in the collection of “red” vertices. For another example, $c_3$ is the 1-type $\{U_1(x), E(x, v_0), \neg E(x, v_1), E(x, v_2)\}$. Thus, $v_3$ is in the collection of “blue” vertices and has edges exactly with $v_0$ and $v_2$. It is straightforward to check that the classes of $n$-partite graphs satisfy SFAP.

Lastly, we consider free amalgamation classes with relations of higher arity. The prototypical example of this is the generic 3-uniform hypergraph, and discussing it should provide the reader with reasonable intuition about coding trees for higher arities.

Example 3.8 (The coding tree of 1-types for the generic 3-uniform hypergraph): Figure 5 presents the coding tree of 1-types for the generic 3-uniform hypergraph. This tree has the property that every node at the same level branches...
into the same number of immediate successors, as there are no forbidden substructures. On the left of Figure 5 is a picture of the hypergraph being built, where $v_n$ is the vertex satisfying the 1-type of the coding node $c_n$ over the initial segment of the structure restricted to $\{v_i : i < n\}$.

Since hyperedges involve three vertices, $c_0$ and $c_1$ are both the empty 1-types. Technically, these nodes are the same, but we draw them distinctly in Figure 4 to aid the drawing of the hypergraph on the left. Letting $R$ denote the 3-hyperedge relation, $c_1$ branches into two 1-types over $\{v_0, v_1\}$: \{\neg R(x, v_0, v_1)\} and \{R(x, v_0, v_1)\}. Since $c_2 = \{R(x, v_0, v_1)\}$, it follows that $R(v_0, v_1, v_2)$ holds in the hypergraph represented on the left of the tree; this hyperedge is represented by the oval containing these three vertices.

Both nodes on the level of $c_2$ branch into four immediate successors. This is because for each node $s$ at the level of $c_2$, the immediate successors of $s$ range over the possibilities of adding a new formula $R(x, \cdot, \cdot)$ or $\neg R(x, \cdot, \cdot)$ containing the parameter $v_2$ and a choice of either $v_0$ or $v_1$ as the second parameter. In particular, the immediate successors of $c_2$ are the 1-types consisting of $\{R(x, v_0, v_1)\}$ unioned with one of the following:

1. $\{\neg R(x, v_0, v_2), \neg R(x, v_1, v_2)\}$;
2. $\{\neg R(x, v_0, v_2), R(x, v_1, v_2)\}$;
3. $\{R(x, v_0, v_2), \neg R(x, v_1, v_2)\}$;
4. $\{R(x, v_0, v_2), R(x, v_1, v_2)\}$.

Likewise, the immediate successors of the other node $s = \{\neg R(x, v_0, v_1)\}$ in level two of the tree consists of the extensions of $s$ by one of the four above cases. In general, each node on the level of $c_n$ branches into $2^n$ many immediate successors. This is because the new formulas in any immediate successor have the choice of $R(x, p, v_n)$ or its negation, where $p \in \{v_i : i < n\}$. However, the Fraïssé class of finite 3-uniform hypergraphs satisfies SFAP (by Proposition 5.2 of Part II), and Theorem 4.10 will provide a skew subtree coding the generic 3-hypergraph in which the branching degree is two (that is, a diagonal subtree).

The coding node $c_3$ is the 1-type $\{\neg R(x, v_0, v_1), R(x, v_0, v_2), \neg R(x, v_1, v_2)\}$. Thus, the hypergraph being built on the left has the hyperedge $R(v_0, v_2, v_3)$. The coding node $c_4$ is the 1-type consisting of $R(x, v_0, v_1), R(x, v_1, v_2), R(x, v_2, v_3)$ along with $\neg R(x, p_0, p_1)$ where $p_0, p_1$ are parameters in $\{v_0, \ldots, v_3\}$. This codes the new hyperedges $R(v_0, v_1, v_4), R(v_1, v_2, v_4)$ and $R(v_2, v_3, v_4)$. 
Figure 5. Coding tree of 1-types for the generic 3-uniform hypergraph.

3.2. Passing types and similarity. As before, let $\mathbf{K}$ be an enumerated Fraïssé structure and $\mathbb{S} := \mathbb{S}(\mathbf{K})$ be the corresponding coding tree of 1-types. We begin by defining the notion of a subtree of $\mathbb{S}$. As is standard in Ramsey theory on infinite trees (see Chapter 6 of [33]), a subtree is not necessarily closed under initial segments, but rather it is closed under those portions of initial segments that have certain prescribed lengths.

**Definition 3.9 (Subtree):** Let $T$ be a subset of $\mathbb{S}$, and let $L$ be the set of lengths of coding nodes in $T$ and lengths of meets of two incomparable nodes (not necessarily coding nodes) in $T$. Then $T$ is a subtree of $\mathbb{S}$ if $T$ is closed under meets and closed under initial segments with lengths in $L$, by which we mean that whenever $\ell \in L$ and $t \in T$ with $\ell \leq |t|$, then $t | \ell$ is also a member of $T$.

We now describe the natural correspondence from subtrees of $\mathbb{S}$ to substructures of $\mathbf{K}$. The following notation will aid in the translation.

**Notation 3.10:** Given a subtree $A \subseteq \mathbb{S}$, let $\langle c_n^A : n < N \rangle$ denote the enumeration of the coding nodes of $A$ in order of increasing length, where $N \leq \omega$ is the number of coding nodes in $A$. Let

$$(3) \quad N^A := \{ i \in \omega : \exists m \ (c_i = c_m^A) \},$$

the set of indices $i$ such that $c_i$ is a coding node in $A$. For $n < N$, let

$$(4) \quad N_n^A := \{ i \in N^A : \exists m < n \ (c_i = c_m^A) \};$$
the set of indices of the first \( n \) coding nodes in \( A \). Recall that \( \omega \) is the set of vertices for \( K \), and that we often use \( v_i \) to denote \( i \), the \( i \)-th vertex of \( K \). Thus, \( N^A \) is precisely the set of vertices of \( K \) represented by the coding nodes in \( A \). Let \( K \upharpoonright A \) denote the substructure of \( K \) on universe \( N^A \). We call this the \textit{substructure of \( K \) represented by the coding nodes in \( A \)}, or simply the \textit{substructure represented by \( A \)}.

The next definition extends the notion of \textit{passing number} developed in \cite{24} and \cite{31} to code binary relations using pairs of nodes in regular splitting trees. Here, we extend this notion to relations of any arity.

Recall from the discussion after Definition 3.1 that for \( s \in \mathbb{S} \), \( s(0) \) denotes the set of formulas in \( s \) without parameters; and for \( 1 \leq i < |s| \), \( s(i) \) denotes the set of those formulas in \( s \upharpoonright K_i \) in which \( v_{i-1} \) appears.

\textbf{Definition 3.11 (Passing Type)}: Given \( s, t \in \mathbb{S} \) with \( |s| < |t| \), we call \( t(|s|) \) the \textit{passing type of \( t \) at \( s \)}. We also call \( t(|s|) \) the \textit{passing type of \( t \) at \( c_n \)}, where \( n + 1 = |s| \), as \( |c_n| = n + 1 \).

Let \( A \) be a subtree of \( \mathbb{S} \), \( t \) be a node in \( \mathbb{S} \), and \( c_n \) be a coding node in \( \mathbb{S} \) such that \( |c_n| < |t| \). We write \( t(c_n; A) \) to denote the set of those formulas in \( t(|c_n|) \) in which all parameters are from among \( \{v_i : i \in N^A_m \cup \{n\}\} \), where \( m \) is least such that \( |c_n^A| \geq |c_n| \). We call \( t(c_n; A) \) the \textit{passing type of \( t \) at \( c_n \) over \( A \)}.

Given a coding node \( c_n^A \) in \( A \), we write \( t(n; A) \) to denote \( t(c_n^A; A) \), and call this the \textit{passing type of \( t \) at \( n \) over \( A \)}.

Note that passing types are partial types which do not include any unary relation symbols. Thus, one can have realizations of the same passing type by elements which differ on the unary relations. Further, note that the passing type of \( t \) at \( s \) only takes into consideration the length of \( s \), not \( s \) itself. Writing the “passing type of \( t \) at \( s \)” rather than “passing type of \( t \) at \( |s| \)” continues the convention set forth in \cite{24}, \cite{31}, and continued in all papers following on these two.

\textbf{Remark 3.12}: In the case where the language \( L \) only has binary relation symbols, passing type reduces to the concept of passing number, first defined and used in \cite{24} and \cite{31} and later used in \cite{9}, \cite{8}, \cite{6}, \cite{35}. This is because for binary relational structures, the tree \( \mathbb{S} \) has a bounded degree of branching. In the special case of the Rado graph, where the language has exactly one binary relation, say \( E \), the tree \( \mathbb{S} \) is regular 2-branching and may be correlated with
the tree of finite sequences of 0’s and 1’s; then the passing number 0 of \( t \) at \( s \) corresponds to the passing type generated by \( \{ \neg R(x, v_{|s|}) \} \), and the passing number 1 of \( t \) at \( s \) corresponds to the passing type generated by \( \{ R(x, v_{|s|}) \} \).

In the case of the rationals, the coding tree of 1-types \( S \) for \( \mathbb{Q} \) provides a minimalistic way to view the work of Devlin in [5], as \( S \) branches exactly at coding nodes and nowhere else. In our set-up, any antichain of coding nodes is automatically a so-called diagonal antichain, as defined in Subsection 4.1. This differs from the previous approaches to big Ramsey degrees of \( \mathbb{Q} \) in [25] and [5] (see also [33]), which use the binary branching tree, Milliken’s theorem, and the method of envelopes.

We will need to be able to compare structures represented by different sets of coding nodes in \( S \). The next notion provides a way to do so.

Recall that \( x \) is the variable used in all 1-types in \( S \). Given subsets \( X \) and \( Y \) of \( \omega \) and map \( f : X \rightarrow Y \), let \( f^* : X \cup \{ x \} \rightarrow Y \cup \{ x \} \) be the extension of \( f \) given by \( f^*(x) = x \).

**Definition 3.13 (Similarity of Passing Types over Subsets):** Let \( A \) and \( B \) be subsets of \( S \), and let \( m, n \in \omega \) be such that \( N_A \cap m \) has the same number of elements as \( N_B \cap n \), say \( p \). Let \( f \) be the increasing bijection from \( N_A \cap m \) to \( N_B \cap n \). Suppose \( s, t \in S \) are such that \( |c_m| < |s| \) and \( |c_n| < |t| \). We write

\[
s(c_m; A) \sim t(c_n; B)
\]

when, given any relation symbol \( R \in \mathcal{L} \) of arity \( k \) and \( k \)-tuple \((z_0, \ldots, z_{k-1})\), where all \( z_i \) are from among \( \{ v_i : i \in N^A_p \} \cup \{ x \} \) and at least one \( z_i \) is the variable \( x \), we have that \( R(z_0, \ldots, z_{k-1}) \) is in \( s(c_m; A) \) if and only if \( R(f^*(z_0), \ldots, f^*(z_{k-1})) \) is in \( t(c_n; B) \). When \( s(c_m; A) \sim t(c_n; B) \) holds, we say that the passing type of \( s \) at \( c_m \) over \( A \) is similar to the passing type of \( t \) at \( c_n \) over \( B \).

If \( A \) and \( B \) each have at least \( n + 1 \) coding nodes, then for \( s, t \in S \) with \( |c_m^A| < |s| \) and \( |c_n^B| < |t| \), define

\[
s(n; A) \sim t(n; B)
\]

to mean that \( s(c_n^A; A) \sim t(c_n^B; B) \). When \( s(n; A) \sim t(n; B) \), we say that \( s \) over \( A \) and \( t \) over \( B \) have similar passing types at the \( n \)-th coding node, or that the passing type of \( s \) at \( n \) over \( A \) is similar to the passing type of \( t \) at \( n \) over \( B \).

It is clear that for fixed \( n, \sim \) is an equivalence relation on passing types over subsets of \( S \).
The following fact is the essence of why we are interested in similarity of passing types: They tell us exactly when two structures represented by coding nodes are isomorphic as substructures of the enumerated structure $K$; that is, when there exists an $L$-isomorphism between the structures that preserves the order relation on their underlying sets inherited from $\omega$.

**Fact 3.14:** Let $A$ and $B$ be subsets of $S$ and $n < \omega$ such that $A$ and $B$ each have $n + 1$ many coding nodes. Then the substructures $K \upharpoonright A$ and $K \upharpoonright B$ are isomorphic, as ordered substructures of $K$, if and only if

1. For each $i \leq n$, the 1-types $c_i^A$ and $c_i^B$ contain the same parameter-free formulas; and
2. For all $i < j \leq n$, $c_j^A(i; A) \sim c_j^B(i; B)$.

We now extend the similarity relation on passing types over subsets of $S$ to a relation on subtrees of $S$ that preserves tree structure. For this, we first define a (strict) linear order $\prec$ on $S$: We may assume there is a linear ordering on the relation symbols and negated relation symbols in $L$, with the convention that all the negated relation symbols appear in the linear order before the relation symbols. (We make this convention to support the intuition that “moving left” from a node in a tree indicates that a relation does not hold, while “moving right” suggests that it does; the convention is not necessary for our results.) Extend the usual linear order $<$ on $\omega$, the underlying set of $K$, to the set $\{x\} \cup \omega$ by setting $x < n$ for each $n \in \omega$. Let $(\{x\} \cup \omega)^{<\omega}$, the set of finite sequences from $\{x\} \cup \omega$, have the induced lexicographic order. Then the induced lexicographic order on the set

$$(\{R : R \in \mathcal{L}\} \cup \{\neg R : R \in \mathcal{L}\}) \times (\{x\} \cup \omega)^{<\omega}$$

is a linear order on the set of atomic and negated atomic formulas of $\mathcal{L}$ that have one free variable $x$ and parameters from $\omega$. Since any node of $S$ is completely determined by such atomic and negated atomic formulas, this lexicographic order gives rise to a linear order on $S$, which we denote $\prec$. Observe that by definition of the lexicographic ordering, we have: If $s \subset t$, then $s \prec t$; and for any incomparable $s, t \in S$, if $|s \wedge t| = n$, then $s \prec t$ if and only if $s \upharpoonright (n+1) \prec t \upharpoonright (n+1)$. This order $\prec$ generalizes the lexicographic order for the case of binary relational structures in [31], [24], [9], [8], and [35].
Definition 3.15 (Similarity Map): Let $S$ and $T$ be meet-closed subsets of $\mathbb{S}$. A function $f : S \rightarrow T$ is a similarity map of $S$ to $T$ if for all nodes $s, t \in S$, the following hold:

1. $f$ is a bijection which preserves $\prec$: $s \prec t$ if and only if $f(s) \prec f(t)$.
2. $f$ preserves meets, and hence splitting nodes: $f(s \wedge t) = f(s) \wedge f(t)$.
3. $f$ preserves relative lengths: $|s| < |t|$ if and only if $|f(s)| < |f(t)|$.
4. $f$ preserves initial segments: $s \subseteq t$ if and only if $f(s) \subseteq f(t)$.
5. $f$ preserves coding nodes and their parameter-free formulas: Given an $n$ a coding node $c_{S,n} \in S$, $f(c_{S,n}) = c_{T,n}$; moreover, for $\gamma \in \Gamma$, $\gamma(v_{S,n}^S)$ holds in $\mathbf{K}$ if and only if $\gamma(v_{T,n}^T)$ holds in $\mathbf{K}$, where $v_{S,n}^S$ and $v_{T,n}^T$ are the vertices of $\mathbf{K}$ represented by coding nodes $c_{S,n}^S$ and $c_{T,n}^T$, respectively.
6. $f$ preserves relative passing types at coding nodes: $s(n; S) \sim f(s)(n; T)$, for each $n$ such that $|c_{S,n}^S| < |s|$.

When there is a similarity map between $S$ and $T$, we say that $S$ and $T$ are similar and we write $S \sim T$. Given a subtree $S$ of $\mathbb{S}$, we let $\text{Sim}(S)$ denote the collection of all subtrees $T$ of $\mathbb{S}$ which are similar to $S$. If $T' \subseteq T$ and $f$ is a similarity map of $S$ to $T'$, then we say that $f$ is a similarity embedding of $S$ into $T$.

Remark 3.16: It follows from (2) that $s$ is a splitting node in $S$ if and only if $f(s)$ is a splitting node in $T$. Moreover, if $s$ is a splitting node in $S$, then $s$ has the same number of immediate successors in $S$ as $f(s)$ has in $T$. Similarity is an equivalence relation on the subtrees of $\mathbb{S}$, since the identity map is a similarity map, the inverse of a similarity map is a similarity map, and the composition of two similarity maps is a similarity map.

Our notion of similarity extends the notion of strong similarity in [31] and [24] for trees without coding nodes, and in [9] and [8] for trees with coding nodes. We drop the word strong to make the terminology more efficient, since there is only one notion of similarity used in this paper.

Given two substructures $F, G$ of $\mathbf{K}$, we write $F \cong_{\omega} G$ when there exists a $\mathcal{L}$-isomorphism between $F$ and $G$ that preserves the linear order on their universes inherited from $\omega$. Note that for any subtrees $S, T$ of $\mathbb{S}$, $S \sim T$ implies that $\mathbf{K} \rest S \cong_{\omega} \mathbf{K} \rest T$. 

4. Diagonal coding trees and SDAP$^+$

In this section, we introduce concepts utilized in the proof of indivisibility in this paper and in the simple characterization of big Ramsey degrees in [4].

4.1. DIAGONAL CODING TREES. Our approach to proving indivisibility and to finding exact big Ramsey degrees for structures with unary and binary relations starts with the kinds of trees that will actually produce indivisibility as well as the exact degrees, upon taking a subcopy of $K$ represented by an antichain of coding nodes in such trees. Namely, we will work with diagonal coding trees.

First, the following modification of Definition 3.1 of $S(K)$ will be useful especially for Fraïssé classes which have both non-trivial unary relations and a linear order or some similar relation, such as the betweenness relation. Recall that $\Gamma$ denotes the set of complete 1-types having only parameter-free formulas; in particular, the only relation symbols that can occur in any $\gamma \in \Gamma$ will be unary.

Definition 4.1 (The Unary-Colored Coding Tree of 1-Types, $U(K)$): Let $\mathcal{K}$ be a Fraïssé class in language $\mathcal{L}$ and $K$ an enumerated Fraïssé structure for $\mathcal{K}$. For $n < \omega$, let $c_n$ denote the 1-type of $v_n$ over $K_n$ (exactly as in the definition of $S(K)$). Let $\mathcal{L}^-$ denote the collection of all relation symbols in $\mathcal{L}$ of arity greater than one, and let $K^-$ denote the reduct of $K$ to $\mathcal{L}^-$ and $K^-_n$ the reduct of $K_n$ to $\mathcal{L}^-$. For $n < \omega$, define the $n$-th level, $U(n)$, to be the collection of all 1-types $s$ over $K^-_n$ in the language $\mathcal{L}^-$ such that for some $i \geq n$, $v_i$ satisfies $s$. Define $U$ to be $\bigcup_{n<\omega} U(n)$. The tree-ordering on $U$ is simply inclusion. The unary-colored coding tree of 1-types is the tree $U$ along with the function $c: \omega \rightarrow U$ such that $c(n) = c_n$. Thus, $c_n$ is the 1-type (in the language $\mathcal{L}^-$) of $v_n$ in $U(n)$ along with the additional “unary color” $\gamma \in \Gamma$ such that $\gamma(v_n)$ holds in $K$.

Note that $K^-$ is not necessarily a Fraïssé structure, as the collection of reducts of members of $\mathcal{K}$ to $\mathcal{L}^-$ need not be a Fraïssé class. This poses no problem to our uses of $U$ or to the results.

Remark 4.2: In the case that $\mathcal{K}$ has no unary relations, $U$ is the same as $S$. Otherwise, the difference between $U$ and $S$ is that all non-coding nodes in $U$ are complete 1-types over initial segments of $K^-$ in the language $\mathcal{L}^-$, while all nodes in $S$, coding or non-coding, are complete 1-types over initial segments of
\( K \) in the language \( L \). In particular, \( S(0) \) equals \( \Gamma \), while \( U(0) \) has exactly one node, \( c_0 \).

Definition 3.11 of passing type applies to \( U \), as the notion of passing type involves no unary relations. Definition 3.13 of similarity of passing types and Definition 3.15 of similarity maps both apply to \( U \), since the notion of coding nodes is the same in both \( S \) and \( U \). Working inside \( U \) instead of \( S \) makes the upper bound arguments for Fraïssé classes with both a linear order and unary relations simpler, lining up with the previous approach for big Ramsey degrees of \( \mathbb{Q}_n \) in [23]. This set-up will allow us to do one uniform forcing proof in the next section for all classes satisfying SDAP\(^+\). For classes with SFAP, the exact bound proofs will return to the \( S \) setting.

Lastly, we point out that the tree \( U \) extends the approach used by Zucker in [35] for certain free amalgamation classes with binary and unary relations.

The following definition of diagonal, motivated by Definition 3.2 in [24], can be found in [9] and [8].

**Definition 4.3 (Diagonal tree):** We call a subtree \( T \subseteq S \) or \( T \subseteq U \) diagonal if each level of \( T \) has at most one splitting node, each splitting node in \( T \) has degree two (exactly two immediate successors), and coding node levels in \( T \) have no splitting nodes.

**Notation 4.4:** Given a diagonal subtree \( T \) (of \( S \) or \( U \)) with coding nodes, we let \( \langle c^T_n : n < N \rangle \), where \( N \leq \omega \), denote the enumeration of the coding nodes in \( T \) in order of increasing length. Let \( \ell^T_n \) denote \( |c^T_n| \), the length of \( c^T_n \). We shall call a node in \( T \) a critical node if it is either a splitting node or a coding node in \( T \).

Let

\[(7) \quad \widehat{T} = \{ t \upharpoonright n : t \in T \text{ and } n \leq |t| \}. \]

Given \( s \in T \) that is not a splitting node in \( T \), we let \( s^+ \) denote the immediate successor of \( s \) in \( \widehat{T} \). Given any \( \ell \), we let \( T \upharpoonright \ell \) denote the set of those nodes in \( \widehat{T} \) with length \( \ell \), and we let \( T \downarrow \ell \) denote the union of the set of nodes in \( T \) of length less than \( \ell \) with the set \( T \upharpoonright \ell \).

Extending Notation 3.10 to subtrees \( T \) of either \( S \) or \( U \), we write \( K \upharpoonright T \) to denote the substructure of \( K \) on \( N^T \), the set of vertices of \( K \) represented by the coding nodes in \( T \).
Definition 4.5 (Diagonal Coding Subtree): A subtree $T \subseteq \mathbb{U}$ is called a *diagonal coding subtree* if $T$ is diagonal and satisfies the following properties:

1. $K \upharpoonright T \cong K$.
2. For each $n < \omega$, the collection of 1-types in $T \upharpoonright (\ell_n^T + 1)$ over $K \upharpoonright (T \downharpoonright \ell_n^T)$ is in one-to-one correspondence with the collection of 1-types in $\mathbb{U}(n + 1)$.
3. Given $m < n$ and letting $A := T \downharpoonright (\ell_m^T - 1)$, if $c_n^T \supseteq c_m^T$ then
   $$(c_n^T)^+(c_n^T; A) \sim (c_m^T)^+(c_m^T; A).$$

Likewise, a subtree $T \subseteq S$ is a *diagonal coding subtree* if the above hold with $\mathbb{U}$ replaced by $S$.

Remark 4.6: Requirement (3) aids in the proofs in the next section and can be met by the Fraïssé limit of any Fraïssé class satisfying SDAP. Note that if $T \subseteq \mathbb{U}$ (or $T \subseteq S$) satisfies (3), then any subtree $S$ of $T$ satisfying $S \sim T$ automatically satisfies (3).

Now we are prepared to define the Diagonal Coding Tree Property, which is an assumption in Definition 4.18 of SDAP$^+$. We say that a tree $T$ is *perfect* if $T$ has no terminal nodes, and each node in $T$ has at least two incomparable extensions in $T$.

Recall our assumption that any Fraïssé class $\mathcal{K}$ that we consider has at least one non-unary relation symbol in its language. We make this assumption because if $\mathcal{K}$ has only unary relation symbols in its language, then $S$ is a disjoint union of finitely many infinite branches. In this case, finitely many applications of Ramsey’s Theorem will yield finite big Ramsey degrees.

We point out that whenever $\mathcal{K}$ satisfies SFAP, the trees $S$ and $\mathbb{U}$ are perfect. However, there are Fraïssé classes in binary relational languages that satisfy SDAP, and yet for which the trees $S$ and $\mathbb{U}$ are not perfect; for example, certain Fraïssé classes of ultrametric spaces. In such cases, Theorem 5.4 does not apply, as the forcing posets used in its proof are atomic. Thus, one of the requirements for SDAP$^+$ is that there is a perfect subtree of $\mathbb{U}$ which codes a copy of $K$, whenever $\mathcal{L}$ has relation symbols of arity greater than one. This is an ingredient in the next property.

Definition 4.7 (Diagonal Coding Tree Property): A Fraïssé class $\mathcal{K}$ in language $\mathcal{L}$ satisfies the *Diagonal Coding Tree Property* if given any enumerated Fraïssé
structure \( K \) for \( \mathcal{K} \), there is a diagonal coding subtree \( T \) of either \( S \) or \( U \) such that \( T \) is perfect.

From here through most of Section 5, we will simply work in \( U \) to avoid duplicating arguments, noting that for Fraïssé classes with SFAP, or without SFAP but with Fraïssé limits having SDAP\(^+\) and in a language with no unary relation symbols, the following can all be done inside \( S \).

We now define the space of coding subtrees of \( U \) with which we shall be working.

**Definition 4.8 (The Space of Diagonal Coding Trees of 1-Types, \( \mathcal{T} \)):** Let \( K \) be any enumerated Fraïssé structure and let \( T \) be a fixed diagonal coding subtree of \( U \). Then the space of coding trees \( \mathcal{T}(T) \) consists of all subtrees \( T \) of \( T \) such that \( T \sim T \). Members of \( \mathcal{T}(T) \) are called simply coding trees, where diagonal is understood to be implied. We shall usually simply write \( T \) when \( T \) is clear from context. For \( T \in \mathcal{T} \), we write \( S \leq T \) to mean that \( S \) is a subtree of \( T \) and \( S \) is a member of \( \mathcal{T} \).

**Remark 4.9:** Given \( T \) satisfying (1)–(3) in Definition 4.5 if \( T \subseteq T \) satisfies \( T \sim T \), then \( T \) also satisfies (1)–(3). Any tree \( T \) satisfying (1) and (2) has no terminal nodes and has coding nodes dense in \( T \). Condition (2) implies that the Fraïssé structure \( J := K \upharpoonright T \) represented by \( T \) has the following property: For any \( i - 1 < j < k \) in \( J \) satisfying \( J \upharpoonright (i \cup \{ j \}) \cong J \upharpoonright (i \cup \{ k \}) \), it holds that \( \text{tp}(j/K_i) = \text{tp}(k/K_i) \); equivalently, that whenever two vertices in \( J \) are in the same orbit over \( J_i \) in \( J \), they are in the same orbit over \( K_i \) in \( K \).

The first use of diagonal subtrees of the infinite binary tree in characterizing exact big Ramsey degrees was for the rationals in [5]. Diagonal subtrees of the infinite binary tree turned out to be at the heart of characterizing the exact big Ramsey degrees of the Rado graph as well as of the generic directed graph and the generic tournament in [31] and [24]. More generally, diagonal subtrees of boundedly branching trees turned out to be central to the characterization of big Ramsey degrees of unconstrained structures with finitely many binary relations in [31] and [24]. More recently, characterizations of the big Ramsey degrees for triangle-free graphs were found to involve diagonal subtrees ([9],[7]), and similarly, for free amalgamation classes with finitely many binary relations and finitely many finite forbidden irreducible substructures on three or more vertices ([1],[8],[35]). However, in these cases, properties additional to being diagonal are
essential to characterizing their big Ramsey degrees; hence, their big Ramsey
degrees do not have a “simple” characterization solely in terms of similarity
types of antichains of coding nodes in diagonal coding trees. We will prove that,
similarly to the rationals and the Rado graph, all unary and binary relational
Fraïssé classes with Fraïssé structure satisfying SDAP have big Ramsey degrees
which are characterized simply by similarity types of antichains of coding nodes
in diagonal coding trees, along with the passing types of their coding nodes.

Recalling from Notation 4.4 that $t \in T$ is called a critical node if $t$ is either
a splitting node or a coding node in $T$, any two critical nodes in a diagonal
coding tree have different lengths, and thus, the levels of $T$ are designated
by the lengths of the critical nodes in $T$. (This follows from the definition of
diagonal.) If $(d^T_m : m < \omega)$ enumerates the critical nodes in $T$ in order of strictly
increasing length, then we let $T(m)$ denote the collection of those nodes in $T$
with length $|d^T_m|$, which we call the $m$-th level of $T$.

Given a substructure $J$ of $K$, we let $U \upharpoonright J$ denote the subtree of $U$
induced by the meet-closure of the coding nodes $\{c_n : n \in J\}$. We call $U \upharpoonright J$ the subtree of $U$ induced by $J$. If $J = K \upharpoonright T$ for some $T \in T$, then $U \upharpoonright J = T$, as $T$ being
diagonal ensures that the coding nodes in $U \upharpoonright J$ are exactly those in $T$.

The final work in this subsection is to prove that Fraïssé classes satisfying
SFAP, as well as their ordered expansions, have Fraïssé limits satisfying the
Diagonal Coding Tree Property. The following notation will be used in the rest
of this subsection. Given $j < \omega$, sets vertices $\{v_{m_i} : i < j\}$ and $\{v_{n_i} : i < j\}$,
and 1-types $s, t \in S$ such that $|s| > m_{j-1}$ and $|t| > n_{j-1}$, we will write

\[(8) \quad s \upharpoonright (K \upharpoonright \{v_{m_i} : i < j\}) \sim t \upharpoonright (K \upharpoonright \{v_{n_i} : i < j\})\]

exactly when, for each $i < j$, $s(c_{m_i} ; \{c_{m_k} : k < i\}) \sim t(c_{n_i} ; \{c_{n_k} : k < i\})$.

**Theorem 4.10**: SFAP implies the Diagonal Coding Tree Property.

**Proof.** Suppose $\mathcal{K}$ is a Fraïssé class satisfying SFAP. Let $K$ be any enumerated
Fraïssé structure for $\mathcal{K}$, and let $S$ be the coding tree of 1-types over finite initial
segments of $K$. Recall that $c_n$ denotes the $n$-th coding node of $S$, that is, the
1-type of the $n$-th vertex of $K$ over $K_n$. If there are any unary relations in
the language $L$ for $\mathcal{K}$, then $S(0)$ will have more than one node. Recall our
convention that the “leftmost” or $\lessdot$-least node in $S(n)$ is the 1-type over $K_n$ in
which no relations of arity greater than one are satisfied.
We start constructing a diagonal coding subtree $T$ by letting the minimal level of $T$ equal $S(0)$. Take a level set $X$ of $S$ satisfying (a) for each $t \in S(0)$, the number of nodes in $X$ extending $t$ is the same as the number of nodes in $S(1)$ extending $t$, and (b) the subtree $U_0$ generated by the meet-closure of $X$ is diagonal. We may assume, for convenience, that the $\prec$-order of the splitting nodes in $U_0$ is the same as the ordering by their lengths.

Let $x_*$ denote the $\prec$-least member of $X$ extending $c_0$. (If there are no unary relation symbols in the language, then $x_*$ is the "leftmost" or $\prec$-least node in $X$.) Let $c^T_0$ denote the coding node of least length extending $x_*$. Extend the rest of the nodes in $X$ to the length of $c^T_0$ and call this set of nodes, along with $c^T_0$, $Y$; define $T \upharpoonright |c^T_0| = Y$. Then take one immediate successor in $S$ of each member of $Y$ so that there is a one-to-one correspondence between the 1-types in $Y$ over $K \upharpoonright \{v^T_0\}$, where $v^T_0$ is the vertex in $K$ represented by $c^T_0$, and the 1-types in $S(1)$: Letting $p = |S(1)|$, list the nodes in $S(1)$ and $Y$ in $\prec$-increasing order as $\langle s_i : i < p \rangle$ and $\langle y_i : i < p \rangle$, respectively. Take $z_i$ to be an immediate successor of $y_i$ in $S$ such that $z_i \upharpoonright (K \upharpoonright \{v^T_0\}) \sim s_i$. Such $z_i$ exist by SFAP. Let $T \upharpoonright (|c^T_0| + 1) = \{z_i : i < p\}$. This constructs $T$ up to length $|c^T_0| + 1$.

The rest of $T$ is constructed similarly: Suppose $n \geq 1$ and $T$ has been constructed up to the immediate successors of its $(n-1)$-st coding node, $c^T_{n-1}$. Take $W$ to be the set of nodes in $T$ of length $|c^T_{n-1}| + 1$. This set $W$ has the same size as $S(n)$; let $\varphi : W \to S(n)$ be the $\prec$-preserving bijection. Take a level set $X$ of nodes in $S$ extending $W$ so that (a) for each $w \in W$, the number of nodes in $X$ extending $w$ is the same as the number of nodes in $S(n+1)$ extending $\varphi(w)$, and (b) the tree $U$ generated by the meet-closure of $X$ is diagonal. Again, we may assume that the splitting nodes in $U$ increase in length as their $\prec$-order increases.

Note that $X$ and $S(n+1)$ have the same cardinality. Let $p = |S(n+1)|$ and enumerate $X$ in $\prec$-increasing order as $\langle x_i : i < p \rangle$. Let $i_*$ be the index so that $x_{i_*}$ is the $\prec$-least member of $X$ extending $\varphi(c_n)$. Let $c^T_n$ denote the coding node in $S$ of shortest length extending $x_{i_*}$. For each $i \in p \setminus \{i_*\}$, take one $y_i \in S$ of length $|c^T_n|$ extending $x_i$. Let $y_{i_*} = c^T_n$, $Y = \{y_i : i < p\}$, and $T \upharpoonright |c^T_n| = Y$. Let $\langle s_i : i < p \rangle$ enumerate the nodes in $S(n+1)$ in $\prec$-increasing order. Then for each $i < p$, let $z_i$ be an immediate successor of $y_i$ in $S$ satisfying

$$z_i \upharpoonright (K \upharpoonright \{v^T_m : m \leq n\}) \sim s_i \upharpoonright K_{n+1},$$

(9)
where $v^T_m$ is the vertex of $K$ represented by the coding node $c^T_m$. Again, such $z_i$ exist by SFAP. Let $T \upharpoonright (|c^T_m| + 1) = \{z_i : i < p\}$.

In this manner, we construct a subtree $T$ of $S$. It is straightforward to check that this construction satisfies (1) and (2) of Definition 4.5 of diagonal coding tree. Using SFAP, we may construct $T$ so that property (3) holds. As long as the language for $K$ contains at least one relation symbol, $T$ will be a perfect tree. Thus, any Fraïssé limit for $K$ satisfies the Diagonal Coding Tree Property.

Next, we consider ordered SFAP classes.

**Lemma 4.11:** Suppose $K$ is a Fraïssé class satisfying SFAP and let $K^<$ denote the Fraïssé class of ordered expansions of members of $K$. Then the Fraïssé limit $K^<$ of $K^<$ satisfies the Diagonal Coding Tree Property.

**Proof.** Let $L$ denote the language for $K$, and let $L^*$ be the expansion $L \cup \{<\}$, the language of $K^<$. Let $K^<$ denote an enumerated structure for $K^<$, and let $K$ denote the reduct of $K^<$ to $L$; thus, $K$ is an enumerated Fraïssé structure for $K$. The universes of $K$ and $K^<$ are $\omega$, which we shall denote as $\langle v_n : n < \omega \rangle$. Let $U$ denote the coding tree of 1-types induced by $K$, and $U^<$ denote the coding tree of 1-types induced by $K^<$. (As in the case for the $n$-partite graphs, if $K$ has unary relations and these unary relations do not occur densely in $U$, then work in $S$ and $S^<$ instead.) As usual, we let $c_n$ denote the $n$-th coding node in $K$, and we will let $c^<_n$ denote the $n$-th coding node in $K^<$. (Normally, if $<$ is in the language of a Fraïssé class $K$, then we will simply write $K$ for its enumerated Fraïssé structure and $U$ for its induced coding tree of 1-types, but here it will aid the reader to consider the juxtaposition of $U$ and $U^<$.) Notice that $K^<$ satisfies SDAP: This holds because SFAP implies SDAP, $LO$ satisfies SDAP, and SDAP is preserved under free superposition. So it only remains to show that there is a diagonal coding tree for $K^<$.

Note that since $L$ has at least one non-unary relation symbol and since $K$ satisfies SFAP, the tree $U$ is perfect. The branching of $U$ and $U^<$ are related in the following way: Each node $t \in U(0)^<$ has twice as many immediate successors in $U(1)^<$ as its reduct to $L$ has in $U(1)$. In general, for $n \geq 1$, given a node $t \in U^<(n)$, let $s$ denote the collection of formulas in $t$ using only relation symbols in $L$ and note that $s \in U(n)$. The number of immediate successors of $t$ in $U^<(n + 1)$ is related to the number of immediate successors of $s$ in $U(n + 1)$ as follows: Let $(*)_n(t)$ denote the following property:
\((*)_n(t)\): \(\{m < n : (x < v_m) \in t\} = \{m < n : (x < v_m) \in c_n^<\}\)

If \((*)_n(t)\) holds, then \(t\) has twice as many immediate successors in \(U^<(n+1)\) as \(s\) has in \(U(n)\), owing to the fact that each 1-type in \(U(n+1)\) extending \(s\) can be augmented by either of \((x < v_n)\) or \((v_n < x)\) to form an extension of \(t\) in \(U^<(n+1)\). If \((*)_n(t)\) does not hold, then any vertex \(v_i\), \(i > n\), satisfying \(t\) lies in an interval of the \(\prec\)-linearly ordered set \(\{v_m : m < n\}\), where neither of the endpoints are \(v_n\). Thus, the order between \(v_i\) and \(v_n\) is already determined by \(t\); hence \(t\) has the same number of immediate successors in \(U^<(n+1)\) as \(s\) has in \(U(n)\).

A diagonal coding subtree \(T^<\) of \(U^<\) can be constructed similarly as in Theorem 4.10 with the following modifications: Suppose \(T^<\) has been constructed up to a level set \(W\), where either \(n = 0\) and \(W = U^<(0) := \{c_0^<\}\), or else \(n \geq 1\) and \(W\) is the set of immediate successors of the \((n-1)\)-st coding node of \(T^<\). This set \(W\) has the same size as \(U^<(n)\); let \(\varphi : W \to U^<(n)\) be the \(\prec\)-preserving bijection. As \(K\) is a free amalgamation class, we may assume that for any \(s \in U^<(n)\), if \(t, u\) in \(U^<(n+1)\) are immediate successors of \(s\) with \((x < v_n) \in t\) and \((v_n < x) \in u\), then \(t \prec u\). Note that the two \(\prec\)-least extensions of \(s\) either both contain \((x < v_n)\), or else both contain \((v_n < x)\). Moreover, we may assume that the \(\prec\)-least immediate successor of \(s\) contains negations of all relations in \(s\) with \(v_n\) as a parameter.

Take a level set \(X\) of nodes in \(U^<\) extending \(W\) so that the following hold: (a) for each \(w \in W\), the number of nodes in \(X\) extending \(w\) is the same as the number of nodes in \(U^<(n+1)\) extending \(\varphi(w)\), and (b) the tree \(U\) generated by the meet-closure of \(X\) is diagonal, where each splitting node in \(U\) is extended by its two \(\prec\)-least immediate successors in \(U^<\), and all non-splitting nodes are extended by the \(\prec\)-least extension in \(U^<\). As in Theorem 4.10 we may assume that the splitting nodes in \(U\) increase in length as their \(\prec\)-order increases, though this has no bearing on the theorems in the next section.

Let \(p := |U^<(n+1)|\) and index the nodes in \(U^<(n+1)\) in \(\prec\)-increasing order as \(\langle s_i : i < p\rangle\). Note that \(X\) has \(p\)-many nodes; index them in \(\prec\)-increasing order as \(\langle x_i : i < p\rangle\). Let \(x_{i^*}\) denote the \(\prec\)-least member of \(X\) extending \(\varphi(c_n^<=)\), and extend \(x_{i^*}\) to a coding node in \(U^<\) satisfying the same \(\gamma \in \Gamma\) as \(c_n^<=\); label it \(y_{i^*}\). This node \(y_{i^*}\) will be the \(n\)-th coding node, \(c_n^<=\), of the diagonal coding subtree \(T^<\) of \(U^<\) which we are constructing. For each \(i \in p \setminus \{i^*\}\), take one \(y_i \in U^<\) of length \(|c_n^<=|\) extending \(x_i\) so that \(y_i\) is the \(\prec\)-least extension of \(x_i\),
subject to the following: Let \( n^*_n = c^n < n^*_n \). For \( i < p \), if \((v_n < x)\) is in \( s_i \), then we take \( y_i \) so that for some \( m < n^*_n \) such that \( v_n < v_m \), \((v_m < x)\) is in \( y_i \). This has the effect that if \((v_n < x)\) is in \( s_i \), then any vertex \( v_j \) represented by a coding node extending \( s_i \) will satisfy \( v_m < v_j \); and since \( v_n < v_m \), it will follow that \( v_n < v_j \); hence \((v_n, x)\) is automatically in \( y_i \).

Likewise, if \((x < v_n)\) is in \( s_i \), then we take \( y_i \) so that for some \( m < n^*_n \) such that \( v_m < v_n \), \((x < v_m)\) is in \( y_i \).

Let \( Y = \{y_i : i < p\} \) and define the set of nodes in \( T^< \) at the level of \( c^n < n \) to be \( Y \). For each \( i < p \), let \( z_i \) be an immediate successor of \( y_i \) in \( U^< \) satisfying

\[
\text{(10)} \quad z_i \upharpoonright \left( (K \upharpoonright \{v^< j \mid j \leq n\}) \sim s_i \upharpoonright K_{n+1},
\]

where \( v^< j \) is the vertex represented by \( c^j < n \). This is possible by SFAP. For the linear order, this was taken care of by SDAP and our choice of \( y_i \). Let \( T^< \upharpoonright (|c^n T| + 1) = \{z_i : i < p\} \).

In this manner, we construct a coding subtree \( T^< \) of \( U^< \) which is diagonal, representing a substructure of \( K^< \) which is again isomorphic to \( K^< \). By extending coding nodes in \( T^< \) by their \( \prec \)-least extensions in \( U^< \), we satisfy (3) of the definition of diagonal coding tree.

Hence, \( K^< \) satisfies the Diagonal Coding Tree Property.

We will work in a diagonal coding subtree of \( S \) whenever such a subtree exists. This is always the case for Fraïssé classes satisfying SFAP. For Fraïssé limits with no unary relations satisfying SDAP, note that \( S = U \); so in this case, a diagonal coding subtree of \( U \) is the same as a diagonal coding subtree of \( S \). If \( K \) is a Fraïssé class with unary relations satisfying SDAP and there is a diagonal coding subtree of \( U \) but no diagonal coding subtree of \( S \), then there are subsets \( P_0, \ldots, P_j \) of the unary relation symbols of \( K \) and a diagonal coding subtree \( T \subseteq U \) such that at some level \( \ell \) below the first coding node of \( T \), the following hold: \( T \upharpoonright \ell \) has exactly \( j+1 \) nodes, say \( t_0, \ldots, t_j \), and for each \( i \leq j \), every coding node in the tree \( T \) restricted above \( t_i \) has unary relation in \( P_i \) and moreover, each of the unary relations in \( P_i \) occurs densely in \( T \) restricted above \( t_i \). By possibly adding unary relation symbols, we may assume that \( P_0, \ldots, P_j \) is a partition of the unary relation symbols. Thus, without loss of generality, we will hold to the following convention for the remainder of this article.
Convention 4.12: Let \( K \) be a Fraïssé class in a language \( \mathcal{L} \) and \( K \) a Fraïssé limit of \( K \). Either there is a diagonal coding subtree of \( S(K) \), or else there is a diagonal coding subtree of \( U(K) \) in which all unary relations occur densely.

4.2. THE EXTENSION PROPERTY. In this section we will define the Extension Property, which is the last of the requirements for SDAP\(^+\) to hold.

Let \( T \) be a diagonal coding tree for the Fraïssé limit \( K \) of some Fraïssé class \( K \). Recall that the tree ordering on \( T \) is simply inclusion. We recapitulate notation from Subsection 3.1: Each \( t \in T \) can be thought of as a sequence \( \langle t(i) : i < |t| \rangle \) where \( t(i) = (t \upharpoonright K_i) \setminus (t \upharpoonright K_{i-1}) \). For \( t \in T \) and \( \ell \leq |t| \), \( t \upharpoonright \ell \) denotes \( \bigcup_{i<\ell} t(i) \), which we can think of as the sequence \( \langle t(i) : i < \ell \rangle \), the initial segment of \( t \) with domain \( \ell \). Note that \( t \upharpoonright \ell \in S(\ell - 1) \) (or \( t \upharpoonright \ell \in U(\ell - 1) \)). (We let \( S(-1) = U(-1) \) denote the set containing the empty set, just so that we do not have to always write \( \ell \geq 1 \).)

The following extends Notation 4.4 to subsets of trees. For a finite subset \( A \subseteq T \), let

\[
(11) \quad \ell_A = \max\{|t| : t \in A\} \quad \text{and} \quad \max(A) = \{s \in A : |s| = \ell_A\}.
\]

For \( \ell \leq \ell_A \), let

\[
(12) \quad A \upharpoonright \ell = \{t \upharpoonright \ell : t \in A \text{ and } |t| \geq \ell\}
\]

and let

\[
(13) \quad A \downarrow \ell = \{t \in A : |t| < \ell\} \cup A \upharpoonright \ell.
\]

Thus, \( A \upharpoonright \ell \) is a level set, while \( A \downarrow \ell \) is the set of nodes in \( A \) with length less than \( \ell \) along with the truncation to \( \ell \) of the nodes in \( A \) of length at least \( \ell \). Notice that \( A \upharpoonright \ell = \emptyset \) for \( \ell > \ell_A \), and \( A \downarrow \ell = A \) for \( \ell \geq \ell_A \). Given \( A, B \subseteq T \), we say that \( B \) is an initial segment of \( A \) if \( B = A \downarrow \ell \) for some \( \ell \) equal to the length of some node in \( A \). In this case, we also say that \( A \) end-extends (or just extends) \( B \). If \( \ell \) is not the length of any node in \( A \), then \( A \downarrow \ell \) is not a subset of \( A \), but is a subset of \( \hat{A} \), where \( \hat{A} \) denotes \( \{t \upharpoonright n : t \in A \text{ and } n \leq |t|\} \).

Define \( \max(A)^+ \) to be the set of nodes \( t \) in \( T \upharpoonright (\ell_A + 1) \) such that \( t \) extends \( s \) for some \( s \in \max(A) \). Given a node \( t \in T \) at the level of a coding node in \( T \), \( t \) has exactly one immediate successor in \( \hat{T} \), which we recall from Notation 4.4 is denoted as \( t^+ \).
Definition 4.13 (+-Similarity): Let $T$ be a diagonal coding tree for the Fraïssé limit $K$ of a Fraïssé class $K$, and suppose $A$ and $B$ are finite subtrees of $T$. We write $A \overset{+}{\sim} B$ and say that $A$ and $B$ are $+\text{-similar}$ if and only if $A \sim B$ and one of the following two cases holds:

**Case 1.** If $\max(A)$ has a splitting node in $T$, then so does $\max(B)$, and the similarity map from $A$ to $B$ takes the splitting node in $\max(A)$ to the splitting node in $\max(B)$.

**Case 2.** If $\max(A)$ has a coding node, say $c^A_n$, and $f : A \to B$ is the similarity map, then $s^+(n; A) \sim f(s)^+(n; B)$ for each $s \in \max(A)$.

Note that $\overset{+}{\sim}$ is an equivalence relation, and $A \overset{+}{\sim} B$ implies $A \sim B$. When $A \sim B$ ($A \overset{+}{\sim} B$), we say that they have the same similarity type ($+\text{-similarity type}$).

**Remark 4.14:** For infinite trees $S$ and $T$ with no terminal nodes, $S \sim T$ implies that for each $n$, letting $d^S_n$ and $d^T_n$ denote the $n$-th critical nodes of $S$ and $T$, respectively, $S \upharpoonright |d^S_n| \overset{+}{\sim} T \upharpoonright |d^T_n|$.

We adopt the following notation from topological Ramsey space theory (see [33]). Given $k < \omega$, we define $r_k(T)$ to be the restriction of $T$ to the levels of the first $k$ critical nodes of $T$; that is,

$$(14) \quad r_k(T) = \bigcup_{m<k} T(m),$$

where $T(m)$ denotes the set of all nodes in $T$ with length equal to $|d^T_m|$. It follows from Remark 4.14 that for any $S, T \in \mathcal{T}$, $r_k(S) \overset{+}{\sim} r_k(T)$. Define $\mathcal{AT}_k$ to be the set of $k\text{-th approximations}$ to members of $\mathcal{T}$; that is,

$$(15) \quad \mathcal{AT}_k = \{r_k(T) : T \in \mathcal{T}\}.$$  

For $D \in \mathcal{AT}_k$ and $T \in \mathcal{T}$, define the set

$$(16) \quad [D, T] = \{S \in \mathcal{T} : r_k(S) = D \text{ and } S \leq T\}.$$  

Lastly, given $T \in \mathcal{T}$, $D = r_k(T)$, and $n > k$, define

$$(17) \quad r_n[D, T] = \{r_n(S) : S \in [D, T]\}.$$  

More generally, given any $A \subseteq T$, we use $r_k(A)$ to denote the first $k$ levels of the tree induced by the meet-closure of $A$. We now have the necessary ideas to define the Extension Property.
Recall from Convention 4.12 that \( T \) is either a fixed diagonal coding tree in \( S \) or else is a fixed diagonal coding tree in \( U \) such that all unary relations occur densely in \( T \), for an enumerated Fra"issé limit \( K \) of a Fra"issé class \( \mathcal{K} \).

**Definition 4.15 (Extension Property):** We say that \( K \) has the *Extension Property* when the following condition (EP) holds:

(EP) Suppose \( A \) is a finite or infinite subtree of some \( T \in \mathcal{T} \). Let \( k \) be given and suppose \( \max(r_{k+1}(A)) \) has a splitting node. Suppose that \( B \) is a \(+\)-similarity copy of \( r_k(A) \) in \( T \). Let \( u \) denote the splitting node in \( \max(r_{k+1}(A)) \), and let \( s \) denote the node in \( \max(B)^+ \) which must be extended to a splitting node in order to obtain a \(+\)-similarity copy of \( r_{k+1}(A) \). If \( s^* \) is a splitting node in \( T \) extending \( s \), then there are extensions of the rest of the nodes in \( \max(B)^+ \) to the same length as \( s^* \) resulting in a \(+\)-similarity copy of \( r_{k+1}(A) \) which can be extended to a copy of \( A \).

**Remark 4.16:** The Extension Property easily holds for Fra"issé limits of all Fra"issé classes satisfying SFAP, as we show below in Lemma 4.17; and similarly for their ordered expansions. The same is true for Fra"issé limits of all unrestricted Fra"issé classes and their ordered expansions. In these cases, all splitting nodes in \( T \) allow for the construction of a \(+\)-similarity copy of \( A \). The Fra"issé structures \( \mathbb{Q}_n \) also trivially have the Extension Property.

**Lemma 4.17:** SFAP implies the Extension Property. Similarly, the Fra"issé limit of any SFAP class with an ordered expansion satisfies the Extension Property.

**Proof.** We will actually prove a slightly stronger statement which implies the Extension Property. Let \( A \) be a subtree of some \( T \in \mathcal{T} \). Without loss of generality, we may assume that either \( A \) is infinite and has infinitely many coding nodes, or else \( A \) is finite and the node in \( A \) of maximal length is a coding node. Let \( m \) either be 0, or else let \( m \) be a positive integer such that \( \max(r_m(A)) \) has a coding node. Let \( n > m \) be least above \( m \) such that \( \max(r_n(A)) \) has a coding node; let \( c^A_i \) denote this coding node.

Now suppose that \( B \) is a \(+\)-similarity copy of \( r_m(A) \), and suppose \( C \) is an extension of \( B \) in \( T \) such that \( C \) is \(+\)-similar to \( r_{n-1}(A) \). (Such a \( C \) is easy to construct since \( S \) is a perfect tree whenever \( K \) has at least one non-trivial relation of arity greater than one.) Let \( X \) denote \( \max(r_{n-1}(A))^+ \), let \( Y \) denote
max$(C)^+$, and let $\varphi$ be the $+\text{-similarity}$ map from $X$ to $Y$. Let $t$ denote the node in $X$ which extends to the coding node in $\text{max}(r_n(A))$, and let $y$ denote $\varphi(t)$. Extend $y$ to some coding node $c_{i}^{T}$ in $T$ such that the substructure of $K$ represented by the coding nodes in $B$ along with $c_{i}^{T}$ is isomorphic to the substructure of $K$ represented by the coding nodes in $r_n(A)$.

Fix any $u \in X$ such that $u \neq t$, and let $z$ denote $\varphi(u)$. Let $c_{j}^{T}$ denote the least coding node in $A$ extending $u$. By SFAP, there is an extension of $z$ to some coding node $c_{j}^{T}$ representing a vertex $w'$ in $K$ such that the substructure of $K$ represented by the coding nodes in $B$ along with $c_{i}^{T}$ and $c_{j}^{T}$ is isomorphic to the substructure of $K$ represented by the coding nodes in $r_n(A)$ along with $c_{j}^{T}$. Let $u'$ denote the unique extension of $u$ in $\text{max}(r_n(A))$, and let $z'$ denote the truncation of $c_{j}^{T}$ to the length $|c_{i}^{T}| + 1$. Then $(z')^+(c_{i}^{T}; B) \sim (u')^+(c_{i}^{T}; r_m(A))$. Therefore, the union of $C$ along with $\{u': u \in Y \setminus \{y\}\} \cup \{c_{i}^{T}\}$ is $+\text{-similar}$ to $r_n(A)$. It follows that the Extension Property holds.

The proof for the ordered expansion of an SFAP class is similar.

4.3. Substructure Disjoint Amalgamation Property$^+$. We now have all the components needed to define the strengthened version of Substructure Disjoint Amalgamation Property central to our results.

**Definition 4.18 (SDAP$^+$):** A Fraïssé structure $K$ has the Substructure Disjoint Amalgamation Property$^+$ (SDAP$^+$) if its age $K$ satisfies SDAP, and $K$ has the Diagonal Coding Tree Property and the Extension Property.

We point out that while the Diagonal Coding Tree Property and Extension Property are defined in terms of an enumerated Fraïssé structure, they are independent of the chosen enumeration, and hence SDAP$^+$ is a property of a Fraïssé structure itself.

By previous lemmas, it follows that SFAP implies SDAP$^+$.

**Theorem 4.19:** Let $K$ be a Fraïssé class in a language with finitely many relation symbols of any finite arity satisfying SFAP. Then the Fraïssé limit $K$ of $K$ and the Fraïssé limit $K^<$ of the ordered expansion $K^<$ both satisfy SDAP$^+$.

**Proof.** It follows immediately from the definitions that if $K$ satisfies SFAP, then both $K$ and $K^<$ satisfy SDAP. By Lemmas 4.10, 4.11, and 4.17 their Fraïssé limits satisfy SDAP$^+$.  

The motivation behind SDAP$^+$ was to distill the essence of those Fraïssé classes for which the forcing arguments in Theorem 5.4 work. As such, it yields quick proofs via forcing of indivisibility (Part I) as well as efficient proofs of big Ramsey degrees which have simple characterizations, similar to those of the rationals and the Rado graph (Part II). It is known that SDAP$^+$, and even SDAP, are not necessary for obtaining finite big Ramsey degrees. For instance, generic $k$-clique-free graphs [8] and the generic partial order [17] have been shown to have finite big Ramsey degrees, and their ages do not have SDAP.

We now present a coding tree version of SDAP$^+$. This version is implied by Definition 4.18 and will be used in proofs.

**Definition 4.20 (SDAP$^+$, Coding Tree Version):** A Fraïssé class $\mathcal{K}$ satisfies the Coding Tree Version of SDAP$^+$ if and only if $\mathcal{K}$ satisfies the disjoint amalgamation property and, letting $\mathcal{K}$ be any enumerated Fraïssé limit of $\mathcal{K}$, $\mathcal{K}$ satisfies the Diagonal Coding Tree Property, the Extension Property, and the following condition:

Let $T$ be any diagonal coding subtree of $\mathcal{U}(\mathcal{K})$ (or of $\mathcal{S}(\mathcal{K})$), and let $\ell < \omega$ be given. Let $i, j$ be any distinct integers such that $\ell < \min(|c^T_i|, |c^T_j|)$, and let $C$ denote the substructure of $\mathcal{K}$ represented by the coding nodes in $T \upharpoonright \ell$ along with $\{c^T_i, c^T_j\}$. Then there are $m \geq \ell$ and $s', t' \in T \upharpoonright m$ such that $s' \supseteq s$ and $t' \supseteq t$ and, assuming (1) and (2), the conclusion holds:

1. Suppose $n \geq m$ and $s'', t'' \in T \upharpoonright n$ with $s'' \supseteq s'$ and $t'' \supseteq t'$.
2. Suppose $c^T_{i'} \in T$ is any coding node extending $s''$.

Then there is a coding node $c^T_{j'} \in T$, with $j' > i'$, such that $c_{j'} \supseteq t''$ and the substructure of $\mathcal{K}$ represented by the coding nodes in $T \upharpoonright \ell$ along with $\{c^T_{i'}, c^T_{j'}\}$ is isomorphic to $C$.

4.4. LSDAP$^+$. We now present the Labeled Substructure Disjoint Amalgamation Property$^+$ which is applicable to structures such as $\mathbb{Q}_Q$, the Fraïssé limit of the Fraïssé class in language $\mathcal{L} = \{<, E\}$ of equivalence relations where each equivalence class is convex. It should be thought of as a weakening of SDAP$^+$, for if we were to allow $q = 1$ in the following definitions, SDAP$^+$ would be recovered.

**Definition 4.21 (Labeled Diagonal Coding Tree):** We say that a diagonal coding tree $T$ is *labeled* if the following hold: There is some $2 \leq q < \omega$, and a function
$\psi$ defined on the set of splitting nodes in $T$ and having range $q$, such that the following holds:

(a) If $s \subseteq t$ are splitting nodes in $T$, then $\psi(s) \geq \psi(t)$.

(b) For each splitting node $s \in T$ and each $n > |s|$, there is a splitting node $t \supseteq s$ with $|t| \geq n$ such that $\psi(t) = \psi(s)$.

(c) The language for $K$ has relation symbols of arity at most two, and each $m < q$ corresponds to a pair of partial 1-types $(\sigma_m, \tau_m)$ involving only binary relation symbols over a 1-element structure such that whenever $s$ is a splitting node in $T$, $\psi(s) = m$ if and only if the following hold:

- whenever $c_j^T, c_k^T$ are coding nodes in $T$ with $c_j^T \wedge c_k^T = s$ and $c_j^T < c_k^T$, if $j < k$ then $c_j^T(|c_j^T|) \sim \tau_m$ and if $j > k$ then $c_j^T(|c_k^T|) \sim \sigma_m$.

(d) The maximal splitting node $s$ below a coding node in $T$ has $\psi(s) = 0$.

Given (a) and (b), the function $\psi$ can be extended to all nodes of $T$ as follows: For each non-splitting node $t \in T$, define $\psi(t)$ to equal $\psi(s)$, where $s$ is the maximal splitting node in $T$ such that $s \subseteq t$.

Notation 4.22: For a labeled diagonal coding tree $T$, for $S, T$ subtrees of $T$, write $S \sim L T$ to mean that $S \sim T$ and the similarity map $f : S \to T$ preserves $\psi$, meaning that for each $s \in S$, $\psi(s) = \psi(f(s))$.

Definition 4.23 (L+-Similarity): Let $T$ be a labeled diagonal coding tree with labeling function $\psi$ for the Fraïssé limit $K$ of a Fraïssé class $K$, and suppose $A$ and $B$ are finite subtrees of $T$. We write $A \overset{L+}{\sim} B$ and say that $A$ and $B$ are $L^+$-similar if and only if $A \overset{\sim}{\sim} B$ and $A \overset{L}{\sim} B$.

Definition 4.24 (Labeled Extension Property): We say that $K$ has the Labeled Extension Property when the following condition (LEP) holds:

(LEP) There is some $2 \leq q < \omega$ and a labeling function $\psi$ taking $T$ onto $q$ satisfying Definition 4.21 such that the following holds: Suppose $A$ is a finite or infinite subtree of some $T \in T$. Let $k$ be given and suppose $\max(r_{k+1}(A))$ has a splitting node. Suppose that $B$ is an $L^+$-similarity copy of $r_k(A)$ in $T$. Let $u$ denote the splitting node in $\max(r_{k+1}(A))$, and let $s$ denote the node in $\max(B)^+$ which must be extended to a splitting node in order to obtain a $+\,$similarity copy of $r_{k+1}(A)$, and note that $\psi(s) \geq \psi(u)$. Then for each $s' \supseteq s$ in $T$ with $\psi(s') \geq \psi(u)$, there exists a splitting node $s^* \in T$ extending $s'$ such that $\psi(s^*) = \psi(u)$. 
Moreover, given such an $s^*$, there are extensions of the rest of the nodes in $\max(B)^+$ to the same length as $s^*$ resulting in an L+-similarity copy of $r_{k+1}(A)$.

**Definition 4.25 (LSDAP+):** A Fraïssé structure $K$ has the Labeled Substructure Disjoint Amalgamation Property+ (LSDAP+) if its age $K$ satisfies SDAP, and $K$ has a labeled diagonal coding tree satisfying the Diagonal Coding Tree Property and the Labeled Extension Property.

**Definition 4.26 (The space of diagonal coding trees for LSDAP+ structures):** If $K$ satisfies LSDAP+, then given a diagonal coding tree $T$ for $K$ with labeling $\psi$, we let $T$ denote the set of all subtrees $T$ of $T$ such that $T \sim T$.

5. Indivisibility via forcing the Level Set Ramsey Theorem

In Theorem 5.4 we use the technique of forcing to essentially conduct an unbounded search for a finite object, achieving within ZFC one color per level set extension of a given finite tree. It is important to note that we never actually go to a generic extension. In fact, the forced generic object is very much not a coding tree and will not represent a Fraïssé limit. Rather, we use the forcing to do two things: (1) Find a good set of nodes from which we can start to build a subtree which can have the desired homogeneity properties; and (2) Use the forcing to guarantee the existence of a finite object with certain properties. Once found, this object, being finite, must exist in the ground model.

We take here a sort of amalgamation of techniques developed in [9], [8], and [6], making adjustments as necessary. The main differences from previous work are the following: The forcing poset is on diagonal coding trees of 1-types; as such, we work with the general notion of passing type, in place of passing number used in the papers [6], [8], [9], and [35] for binary relational structures. Moreover, Definitions 4.13 and 4.23 present stronger requirements than just similarity. These address both the fact that relations can be of any arity, and the fact that we consider Fraïssé classes which have disjoint, but not necessarily free, amalgamation.

We now set up notation, definitions, and assumptions for Theorem 5.4. Recall Convention 4.12.

By an **antichain** of coding nodes, we mean a set of coding nodes which is pairwise incomparable with respect to the tree partial order of inclusion.
Set-up for Theorem 5.4. Let $T$ be a diagonal coding tree in $\mathcal{T}$. Fix a finite antichain of coding nodes $\bar{C} \subseteq T$. We abuse notation and also write $\bar{C}$ to denote the tree that its meet-closure induces in $T$. Let $\bar{A}$ be a fixed proper initial segment of $\bar{C}$, allowing for $\bar{A}$ to be the empty set. Thus, $\bar{A} = \bar{C} \downarrow \ell$, where $\ell$ is the length of some splitting or coding node in $\bar{C}$ (let $\ell = 0$ if $\bar{A}$ is empty).

Let $\ell_\bar{A}$ denote this $\ell$, and note that any non-empty $\max(\bar{A})$ either has a coding node or a splitting node. Let $\bar{x}$ denote the shortest splitting or coding node in $\bar{C}$ with length greater than $\ell_\bar{A}$, and define $\bar{X} = \bar{C} \upharpoonright |\bar{x}|$. Then $\bar{A} \cup \bar{X}$ is an initial segment of $\bar{C}$; let $\ell_{\bar{X}}$ denote $|\bar{x}|$.

There are two cases:

Case (a). $\bar{X}$ has a splitting node.

Case (b). $\bar{X}$ has a coding node.

Let $d + 1$ be the number of nodes in $\bar{X}$ and index these nodes as $\bar{x}_i$, $i \leq d$, where $\bar{x}_d$ denotes the critical node (recall that critical node refers to a splitting or coding node). Let

$$\tilde{B} = \bar{C} \upharpoonright (\ell_\bar{A} + 1).$$

Then $\bar{X}$ is a level set equal to or end-extending the level set $\tilde{B}$. For each $i \leq d$, define

$$\tilde{b}_i = \bar{x}_i \upharpoonright \ell_{\tilde{B}}.$$

Note that we consider nodes in $\tilde{B}$ as simply nodes to be extended; it does not matter whether the nodes in $\tilde{B}$ are coding, splitting, or neither in $T$.

Definition 5.1 (Weak similarity): Given finite subtrees $S, T \in \mathcal{T}$ in which each coding node is terminal, we say that $S$ is weakly similar to $T$, and write $S \sim_T T$, if and only if $S \setminus \max(S) \downarrow T \setminus \max(T)$. We say that $S$ is $L$-weakly similar to $T$, and write $S \Lsim_T T$, if and only if $S \setminus \max(S) \Lsim_T T \setminus \max(T)$.

In the following, we put the technicalities for the LSDAP+ case in parentheses.

Definition 5.2 (Ext$_T(B; \bar{X})$): Let $T \in \mathcal{T}$ be fixed and let $D = r_n(T)$ for some $n < \omega$. Suppose $A$ is a subtree of $D$ such that $A \sim_T \bar{A}$ ($A \Lsim_T \bar{A}$) and $A$ is extendible to a similarity ($L$-similarity) copy of $\bar{C}$ in $T$. Let $B$ be a subset of the level set $\max(D)^+$ such that $B$ end-extends or equals $\max(A)^+$ and $A \cup B \Lsim_T \bar{A} \cup \bar{B}$ ($A \cup B \Lsim_T \bar{A} \cup \bar{B}$). Let $X^*$ be a level set end-extending $B$ such that $A \cup X^* \Lsim_T \bar{A} \cup \bar{X}$ ($A \cup X^* \Lsim_T \bar{A} \cup \bar{X}$). Let $U^* = T \downarrow (\ell_B - 1)$. Define Ext$_T(B; X^*)$ to be the collection of all level sets $X \subseteq T$ such that...
(1) $X$ end-extends $B$;
(2) $U^* \cup X \uplus U^* \cup X^*$ ($U^* \cup X \uplus U^* \cup X^*$);
(3) $A \cup X$ extends to a copy of $\tilde{C}$.

For Case (b), condition (3) follows from (2). For Case (a), the Extension Property (Labeled Extension Property) guarantees that for any level set $Y$ end-extending $B$, there is a level set $X$ end-extending $Y$ such that $A \cup X$ satisfies condition (3). In both cases, condition (2) implies that $A \cup X \uplus \sim \tilde{A} \cup \tilde{X}$ ($A \cup X \uplus \sim \tilde{A} \cup \tilde{X}$).

The following theorem of Erdős and Rado will provide the pigeonhole principle for the forcing proof.

**Theorem 5.3** (Erdős-Rado, [13]): For $r < \omega$ and $\mu$ an infinite cardinal,

$$\sum_{r}(\mu)^+ \rightarrow (\mu^+)_{r+1}.$$

We are now ready to prove the Ramsey theorem for level set extensions of a given finite tree.

**Theorem 5.4** (Level Set Ramsey Theorem): Suppose that $K$ has Fraïssé limit $K$ satisfying SDAP$^+$ (or LSDAP$^+$), and $T \in T$ is given. Let $\tilde{C}$ be a finite antichain of coding nodes in $T$, $\tilde{A}$ be an initial segment of $\tilde{C}$, and $\tilde{B}$ and $\tilde{X}$ be defined as above. Suppose $D = r_n(T)$ for some $n < \omega$, and $A \subseteq D$ and $B \subseteq \max(D^+)$ satisfy $A \cup B \uplus \tilde{A} \cup \tilde{B}$, $A \cup B \uplus \tilde{A} \cup \tilde{B}$, $A \cup X^* \uplus \tilde{A} \cup \tilde{X}$, $A \cup X^* \uplus \tilde{A} \cup \tilde{X}$). Then given any coloring $h : \text{Ext}_T(B; X^*) \rightarrow 2$, there is a coding tree $S \in [D, T]$ such that $h$ is monochromatic on $\text{Ext}_S(B; X^*)$.

**Proof.** Enumerate the nodes in $B$ as $s_0, \ldots, s_d$ so that for any $X \in \text{Ext}_T(B; X^*)$, the critical node in $X$ extends $s_d$. Let $M$ denote the collection of all $m \geq n$ for which there is a member of $\text{Ext}_T(B; X^*)$ with nodes in $T(m)$. Note that this set $M$ is the same for any $S \in T$. Let $L = \{|t| : \exists m \in M \ (t \in T(m))\}$, the collection of lengths of nodes in the levels $T(m)$ for $m \in M$.

For $i \leq d$, let $T_i = \{t \in T : t \supseteq s_i\}$. Let $\kappa$ be large enough, so that the partition relation $\kappa \rightarrow (\aleph_1)^2_{\aleph_0}$ holds. The following forcing notion $\mathbb{P}$ adds $\kappa$ many paths through each $T_i$, $i < d$, and one path through $T_d$. 
In both Cases (a) and (b), define $\mathbb{P}$ to be the set of finite partial functions $p$ such that

$$p : (d \times \bar{\delta}_p) \cup \{d\} \rightarrow T(m_p),$$

where

1. $m_p \in M$ and $\bar{\delta}_p$ is a finite subset of $\kappa$;
2. $\{p(i, \delta) : \delta \in \bar{\delta}_p\} \subseteq T_i(m_p)$ for each $i < d$;
3. $p(d)$ is the critical node in $T_d(m_p)$; and
4. For any choices of $\delta_i \in \bar{\delta}_p$, the level set $\{p(i, \delta_i) : i < d\} \cup \{p(d)\}$ is a member of $\text{Ext}_T(B; X^*)$.

Given $p \in \mathbb{P}$, the range of $p$ is defined as

$$\text{ran}(p) = \{p(i, \delta) : (i, \delta) \in d \times \bar{\delta}_p\} \cup \{p(d)\}.$$

Let $\ell_p$ denote the length of the nodes in $\text{ran}(p)$. If also $q \in \mathbb{P}$ and $\bar{\delta}_p \subseteq \bar{\delta}_q$, then we let $\text{ran}(q \upharpoonright \bar{\delta}_p)$ denote $\{q(i, \delta) : (i, \delta) \in d \times \bar{\delta}_p\} \cup \{q(d)\}$.

In Case (a), the partial ordering on $\mathbb{P}$ is defined by $q \leq p$ if and only if (1) and (2) hold:

1. $m_q \geq m_p$, $\bar{\delta}_q \supseteq \bar{\delta}_p$, $q(d) \supseteq p(d)$. (In the case of LSDAP$^+$, we also require that $\psi(q(d)) = \psi(p(d)).$
2. $q(i, \delta) \supseteq p(i, \delta)$ for each $(i, \delta) \in d \times \bar{\delta}_p$. (In the case of LSDAP$^+$, we also require that $\psi(q(i, \delta)) = \psi(p(i, \delta)).$)

In Case (b), we define $q \leq p$ if and only if (1) and (2) hold and additionally, the following third requirement holds:

3. Letting $U = T \upharpoonright (\ell_p - 1)$, $U \cup \text{ran}(p) \not\leq U \cup \text{ran}(q \upharpoonright \bar{\delta}_p)$.

(Requirement (3) is stronger than that which was used for the Rado graph in [6], because for relations of arity three or more, the extension $q$ must preserve information about 1-types over the fixed finite structure which we wish to extend.) Then $(\mathbb{P}, \leq)$ is a separative, atomless partial order.

The next part of the proof (up to and including Lemma 5.7) follows that of [6] almost verbatim. The key difference between the work here and in [6] is that here, information which makes the proof work for relations of any arity is embedded in the definition of $\text{Ext}_T(B, X^*)$. For $(i, \alpha) \in d \times \kappa$, let

$$(20) \quad \dot{b}_{i, \alpha} = \{\langle p(i, \alpha), p \rangle : p \in \mathbb{P} \text{ and } \alpha \in \bar{\delta}_p\},$$

a $\mathbb{P}$-name for the $\alpha$-th generic branch through $T_i$. Let

$$(21) \quad \dot{b}_d = \{\langle p(d), p \rangle : p \in \mathbb{P}\},$$
a \mathbb{P}\text{-}name for the generic branch through \( T_d \). Given a generic filter \( G \subseteq \mathbb{P} \), notice that \( \dot{b}_d^G = \{ p(d) : p \in G \} \), which is a cofinal path of critical nodes in \( T_d \). Let \( \dot{L}_d \) be a \( \mathbb{P}\text{-}name for the set of lengths of critical nodes in \( \dot{b}_d \), and note that \( \mathbb{P} \) forces that \( \dot{L}_d \subseteq L \). Let \( \dot{U} \) be a \( \mathbb{P}\text{-}name for a non-principal ultrafilter on \( \dot{L}_d \). Given \( p \in \mathbb{P} \), recall that \( \ell_p \) denotes the lengths of the nodes in \( \text{ran}(p) \), and notice that

\[
\tag{22}
 p \vDash \forall (i, \alpha) \in d \times \tilde{\delta}_p \left( \dot{b}_{i, \alpha} \upharpoonright \ell_p = p(i, \alpha) \right) \land \left( \dot{b}_d \upharpoonright \ell_p = p(d) \right).
\]

We will write sets \( \{ \alpha_i : i < d \} \) in \([k]^d\) as vectors \( \vec{\alpha} = \langle \alpha_0, \ldots, \alpha_{d-1} \rangle \) in strictly increasing order. For \( \vec{\alpha} \in [k]^d \), let

\[
\dot{b}_{\vec{\alpha}} = \langle \dot{b}_{0, \alpha_0}, \ldots, \dot{b}_{d-1, \alpha_{d-1}}, \dot{b}_d \rangle.
\]

For \( \ell < \omega \), let

\[
\dot{b}_{\vec{\alpha}} \upharpoonright \ell = \langle \dot{b}_{0, \alpha_0} \upharpoonright \ell, \ldots, \dot{b}_{d-1, \alpha_{d-1}} \upharpoonright \ell, \dot{b}_d \upharpoonright \ell \rangle.
\]

One sees that \( h \) is a coloring on level sets of the form \( \dot{b}_{\vec{\alpha}} \upharpoonright \ell \) whenever this is forced to be a member of \( \text{Ext}_T(B; X^*) \). Given \( \vec{\alpha} \in [k]^d \) and \( p \in \mathbb{P} \) with \( \vec{\alpha} \subseteq \tilde{\delta}_p \), let

\[
X(p, \vec{\alpha}) = \{ p(i, \alpha_i) : i < d \} \cup \{ p(d) \},
\]

recalling that this level set \( X(p, \vec{\alpha}) \) is a member of \( \text{Ext}_T(B; X^*) \).

For each \( \vec{\alpha} \in [k]^d \), choose a condition \( p_{\vec{\alpha}} \in \mathbb{P} \) satisfying the following:

1. \( \vec{\alpha} \subseteq \tilde{\delta}_{p_{\vec{\alpha}}} \).
2. There is an \( \varepsilon_{\vec{\alpha}} \in 2 \) such that \( p_{\vec{\alpha}} \vDash "h(\dot{b}_{\vec{\alpha}} \upharpoonright \ell) = \varepsilon_{\vec{\alpha}}" \) for \( \dot{U} \) many \( \ell \) in \( \dot{L}_d " \).
3. \( h(X(p_{\vec{\alpha}}, \vec{\alpha})) = \varepsilon_{\vec{\alpha}}. \)

Such conditions can be found as follows: Fix some \( X \in \text{Ext}_T(B; X^*) \) and let \( t_i \) denote the node in \( X \) extending \( s_i \), for each \( i \leq d \). For \( \vec{\alpha} \in [k]^d \), define

\[
p^0_{\vec{\alpha}} = \{ \langle (i, \delta), t_i \rangle : i < d, \delta \in \vec{\alpha} \} \cup \{ (d, t_d) \}.
\]

Then (1) will hold for all \( p \leq p^0_{\vec{\alpha}} \), since \( \tilde{\delta}_{p^0_{\vec{\alpha}}} = \vec{\alpha} \). Next, let \( p^1_{\vec{\alpha}} \) be a condition below \( p^0_{\vec{\alpha}} \) which forces \( h(\dot{b}_{\vec{\alpha}} \upharpoonright \ell) \) to be the same value for \( \dot{U} \) many \( \ell \in \dot{L}_d \). Extend this to some condition \( p^2_{\vec{\alpha}} \leq p^1_{\vec{\alpha}} \) which decides a value \( \varepsilon_{\vec{\alpha}} \in 2 \) so that \( p^2_{\vec{\alpha}} \) forces \( h(\dot{b}_{\vec{\alpha}} \upharpoonright \ell) = \varepsilon_{\vec{\alpha}} \) for \( \dot{U} \) many \( \ell \) in \( \dot{L}_d \). Then (2) holds for all \( p \leq p^2_{\vec{\alpha}} \). If \( p^2_{\vec{\alpha}} \) satisfies (3), then let \( p_{\vec{\alpha}} = p^2_{\vec{\alpha}} \). Otherwise, take some \( p^3_{\vec{\alpha}} \leq p^2_{\vec{\alpha}} \) which forces \( \dot{b}_{\vec{\alpha}} \upharpoonright \ell \in \text{Ext}_T(B; X^*) \) and \( h'(\dot{b}_{\vec{\alpha}} \upharpoonright \ell) = \varepsilon_{\vec{\alpha}} \) for some \( \ell \in \dot{L} \) with \( \ell_{p^3_{\vec{\alpha}}} < \ell \leq \ell_{p^3_{\vec{\alpha}}} \). Since \( p^3_{\vec{\alpha}} \) forces that \( \dot{b}_{\vec{\alpha}} \upharpoonright \ell \) equals \( \{ p^3_{\vec{\alpha}}(i, \alpha_i) \upharpoonright \ell : i < d \} \cup \{ p^3_{\vec{\alpha}}(d) \upharpoonright \ell \} \), which is
exactly $X(p^3_\alpha \upharpoonright \ell, \vec{a})$, and this level set is in the ground model, it follows that $h(X(p^3_\alpha \upharpoonright \ell, \vec{a})) = \varepsilon_\vec{a}$. Let $p_\vec{a}$ be $p^3_\alpha \upharpoonright \ell$. Then $p_\vec{a}$ satisfies (1)-(3).

Let $\mathcal{I}$ denote the collection of all functions $\iota: 2d \to 2d$ such that for each $i < d$, $\{\iota(2i), \iota(2i+1)\} \subseteq \{2i, 2i+1\}$. For $\vec{\theta} = \langle \theta_0, \ldots, \theta_{2d-1} \rangle \in [\kappa]^{2d}$, $\iota(\vec{\theta})$ determines the pair of sequences of ordinals $\langle \iota_e(\vec{\theta}), \iota_o(\vec{\theta}) \rangle$, where

\begin{align*}
\iota_e(\vec{\theta}) &= \langle \theta_{\iota(0)}, \theta_{\iota(2)}, \ldots, \theta_{\iota(2d-2)} \rangle \\
\iota_o(\vec{\theta}) &= \langle \theta_{\iota(1)}, \theta_{\iota(3)}, \ldots, \theta_{\iota(2d-1)} \rangle.
\end{align*}

(26)

We now proceed to define a coloring $f$ on $[\kappa]^{2d}$ into countably many colors. Let $\vec{\delta}_\vec{a}$ denote $\delta_{p_\vec{a}}$, $k_\vec{a}$ denote $|\vec{\delta}_\vec{a}|$, $\ell_\vec{a}$ denote $\ell_{p_\vec{a}}$, and let $\langle \delta_\vec{a}(j) : j < k_\vec{a} \rangle$ denote the enumeration of $\vec{\delta}_\vec{a}$ in increasing order. Given $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, to reduce subscripts let $\vec{\alpha}$ denote $\iota_e(\vec{\theta})$ and $\vec{\beta}$ denote $\iota_o(\vec{\theta})$, and define

\begin{align*}
f(\iota, \vec{\theta}) &= \langle \iota, \varepsilon_\vec{a}, k_\vec{a}, p_\vec{a}(d), \langle p_\vec{a}(i, \delta_\vec{a}(j)) : j < k_\vec{a} \rangle : i < d \rangle, \\
\langle \langle i, j \rangle : i < d, j < k_\vec{a}, \text{ and } \delta_\vec{a}(j) = \alpha_i \rangle, \\
\langle \langle j, k \rangle : j < k_\vec{a}, k < k_{\vec{\beta}}, \delta_\vec{a}(j) = \delta_{\vec{\beta}}(k) \rangle \rangle.
\end{align*}

(27)

Fix some ordering of $\mathcal{I}$ and define

\begin{equation}
(28)
f(\vec{\theta}) = \langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I} \rangle.
\end{equation}

By the Erdős-Rado Theorem there is a subset $K \subseteq \kappa$ of cardinality $\aleph_1$ which is homogeneous for $f$. Take $K' \subseteq K$ so that between each two members of $K'$ there is a member of $K$. Given sets of ordinals $I$ and $J$, we write $I < J$ to mean that every member of $I$ is less than every member of $J$. Take $K_i \subseteq K'$ be countably infinite subsets satisfying $K_0 < \cdots < K_{d-1}$.

Fix some $\vec{\gamma} \in \prod_{i<d} K_i$, and define

\begin{align*}
\varepsilon^* &= \varepsilon_{\vec{\gamma}}, \\
k^* &= k_{\vec{\gamma}}, \\
t_d &= p_{\vec{\gamma}}(d), \\
t_{i,j} &= p_{\vec{\gamma}}(i, \delta_{\vec{\gamma}}(j)) \text{ for } i < d, j < k^*.
\end{align*}

(29)

We show that the values in equation (29) are the same for any choice of $\vec{\gamma}$.

**Lemma 5.5:** For all $\vec{\alpha} \in \prod_{i<d} K_i$, $\varepsilon_{\vec{\alpha}} = \varepsilon^*$, $k_{\vec{\alpha}} = k^*$, $p_{\vec{\alpha}}(d) = t_d$, and $\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_\vec{a} \rangle = \langle t_{i,j} : j < k^* \rangle$ for each $i < d$.

**Proof.** Let $\vec{\alpha}$ be any member of $\prod_{i<d} K_i$, and let $\vec{\gamma}$ be the set of ordinals fixed above. Take $\iota \in \mathcal{I}$ to be the identity function on $2d$. Then there are $\vec{\theta}, \vec{\theta}' \in [K]^{2d}$ such that $\vec{\alpha} = \iota_e(\vec{\theta})$ and $\vec{\gamma} = \iota_e(\vec{\theta}')$. Since $f(\iota, \vec{\theta}) = f(\iota, \vec{\theta}')$, it follows that
\[ \varepsilon_{\bar{\alpha}} = \varepsilon_{\bar{\gamma}}, \ k_{\bar{\alpha}} = k_{\bar{\gamma}}, \ p_{\bar{\alpha}}(d) = p_{\bar{\gamma}}(d), \ \text{and} \ (p_{\bar{\alpha}}(i, \delta_{\bar{\alpha}}(j)) : j < k_{\bar{\alpha}}) = (p_{\bar{\gamma}}(i, \delta_{\bar{\gamma}}(j)) : j < k_{\bar{\gamma}}) : i < d). \]

Let \( l^* \) denote the length of the node \( t_d \), and notice that the node \( t_{i,j} \) also has length \( l^* \), for each \((i, j) \in d \times k^* \).

**Lemma 5.6:** Given any \( \bar{\alpha}, \bar{\beta} \in \prod_{i<d} K_i \), if \( j, k < k^* \) and \( \delta_{\bar{\alpha}}(j) = \delta_{\bar{\beta}}(k) \), then \( j = k \).

**Proof.** Let \( \bar{\alpha}, \bar{\beta} \) be members of \( \prod_{i<d} K_i \) and suppose that \( \delta_{\bar{\alpha}}(j) = \delta_{\bar{\beta}}(k) \) for some \( j, k < k^* \). For \( i < d \), let \( \rho_i \) be the relation from among \( \{<, = , >\} \) such that \( \alpha_i \rho_i \beta_i \). Let \( i \) be the member of \( \mathcal{I} \) such that for each \( \bar{\theta} \in [K]^{2d} \) and each \( i < d \), \( \theta_i(2i) \rho_i \theta_i(2i+1) \). Fix some \( \bar{\theta} \in [K']^{2d} \) such that \( \nu_e(\bar{\theta}) = \bar{\alpha} \) and \( \nu_o(\bar{\theta}) = \bar{\beta} \). Since between any two members of \( K' \) there is a member of \( K \), there is a \( \bar{\zeta} \in [K]^d \) such that for each \( i < d \), \( \alpha_i \rho_i \zeta_i \) and \( \zeta_i \rho_i \beta_i \). Let \( \bar{\mu}, \bar{\nu} \) be members of \( [K]^{2d} \) such that \( \nu_e(\bar{\mu}) = \bar{\alpha}, \ \nu_o(\bar{\mu}) = \bar{\zeta}, \ \nu_e(\bar{\nu}) = \bar{\zeta}, \) and \( \nu_o(\bar{\nu}) = \bar{\beta} \). Since \( \delta_{\bar{\alpha}}(j) = \delta_{\bar{\beta}}(k) \), the pair \( \langle j, k \rangle \) is in the last sequence in \( f(\nu, \bar{\theta}) \). Since \( f(\nu, \bar{\mu}) = f(\nu, \bar{\nu}) = f(\nu, \bar{\theta}) \), also \( \langle j, k \rangle \) is in the last sequence in \( f(\nu, \bar{\mu}) \) and \( f(\nu, \bar{\nu}) \). It follows that \( \delta_{\bar{\alpha}}(j) = \delta_{\bar{\zeta}}(k) \) and \( \delta_{\bar{\zeta}}(j) = \delta_{\bar{\beta}}(k) \). Hence, \( \delta_{\bar{\zeta}}(j) = \delta_{\bar{\beta}}(k) \), and therefore \( j \) must equal \( k \).

For each \( \bar{\alpha} \in \prod_{i<d} K_i \), given any \( \nu \in \mathcal{I} \), there is a \( \bar{\theta} \in [K]^{2d} \) such that \( \bar{\alpha} = \nu_o(\bar{\alpha}) \). By the second line of equation (27), there is a strictly increasing sequence \( \langle j_i : i < d \rangle \) of members of \( k^* \) such that \( \delta_{\bar{\alpha}}(j_i) = \alpha_i \). By homogeneity of \( f \), this sequence \( \langle j_i : i < d \rangle \) is the same for all members of \( \prod_{i<d} K_i \). Then letting \( t^*_i \) denote \( t_{i,j_i} \), one sees that

\[ p_{\bar{\alpha}}(i, \alpha_i) = p_{\bar{\alpha}}(i, \delta_{\bar{\alpha}}(j_i)) = t_{i,j_i} = t^*_i. \]

Let \( t^*_d \) denote \( t_d \).

**Lemma 5.7:** For any finite subset \( \bar{J} \subseteq \prod_{i<d} K_i \), \( p_{\bar{J}} := \bigcup \{ p_{\bar{\alpha}} : \bar{\alpha} \in \bar{J} \} \) is a member of \( \mathbb{P} \) which is below each \( p_{\bar{\alpha}}, \bar{\alpha} \in \bar{J} \).

**Proof.** Given \( \bar{\alpha}, \bar{\beta} \in \bar{J} \), if \( j, k < k^* \) and \( \delta_{\bar{\alpha}}(j) = \delta_{\bar{\beta}}(k) \), then \( j \) and \( k \) must be equal, by Lemma 5.6. Then Lemma 5.5 implies that for each \( i < d \),

\[ p_{\bar{\alpha}}(i, \delta_{\bar{\alpha}}(j)) = t_{i,j} = p_{\bar{\beta}}(i, \delta_{\bar{\beta}}(j)) = p_{\bar{\beta}}(i, \delta_{\bar{\beta}}(k)). \]

Hence, for all \( \delta \in \delta_{\bar{\alpha}} \cap \delta_{\bar{\beta}} \) and \( i < d \), \( p_{\bar{\alpha}}(i, \delta) = p_{\bar{\beta}}(i, \delta) \). Thus, \( p_{\bar{J}} := \bigcup \{ p_{\bar{\alpha}} : \bar{\alpha} \in \bar{J} \} \) is a function with domain \( \delta_{\bar{J}} \cup \{d\} \), where \( \delta_{\bar{J}} = \bigcup \{ \delta_{\bar{\alpha}} : \bar{\alpha} \in \bar{J} \} \); hence, \( p_{\bar{J}} \)
is a member of $\mathbb{P}$. Since for each $\vec{\alpha} \in \vec{J}$, \( \text{ran}(p_{\vec{J}} | \vec{J} \vec{\alpha}) = \text{ran}(p_{\vec{\alpha}}) \), it follows that $p_{\vec{J}} \leq p_{\vec{\alpha}}$ for each $\vec{\alpha} \in \vec{J}$. 

This ends the material drawn directly from [6].

We now proceed to build a (diagonal coding) tree $S \in [D, T]$ so that the coloring $h$ will be monochromatic on $\text{Ext}_S(B; X^*)$. Recall that $n$ is the integer such that $D = r_n(T)$. Let $\{m_j : j < \omega\}$ be the strictly increasing enumeration of $M$, noting that $m_0 \geq n$. For each $i \leq d$, extend the node $s_i \in B$ to the node $t_i^*$. Extend each node $u$ in $\text{max}(D)^+ \setminus B$ to some node $u^*$ in $T \setminus \ell^*$. If $X^*$ has a coding node and $m_0 = n$, require also that $(u^*)^+(u^*; D) \sim u^+(u; D)$; SDAP ensures that such $u^*$ exist. Set

\[(32) \quad U^* = \{t_i^* : i \leq d\} \cup \{u^* : u \in \text{max}(D)^+ \setminus B\}\]

and note that $U^*$ end-extends $\text{max}(D)^+$.

If $m_0 = n$, then $D \cup U^*$ is a member of $r_{m_0+1}[D, T]$. In this case, let $U_{m_0+1} = D \cup U^*$, and let $U_{m_1}$ be any member of $r_{m_1}[U_{m_0+1}, T]$. Note that $U^*$ is the only member of $\text{Ext}_{U_{m_1}}(B; X^*)$, and it has $h$-color $\varepsilon^*$. Otherwise, $m_0 > n$. In this case, take some $U_{m_0} \in r_{m_0}[D, T]$ such that $\text{max}(U_{m_0})$ end-extends $U^*$, and notice that $\text{Ext}_{U_{m_0}}(B; X^*)$ is empty.

Now assume that $j < \omega$ and we have constructed $U_{m_j} \in r_{m_j}[D, T]$ so that every member of $\text{Ext}_{U_{m_j}}(B; X^*)$ has $h$-color $\varepsilon^*$. Fix some $V \in r_{m_j+1}[U_{m_j}, T]$ and let $Y = \text{max}(V)$. We will extend the nodes in $Y$ to construct $U_{m_j+1} \in r_{m_j+1}[U_{m_j}, T]$ with the property that all members of $\text{Ext}_{U_{m_j+1}}(B; X^*)$ have the same $h$-value $\varepsilon^*$. This will be achieved by constructing the condition $q \in \mathbb{P}$, below, and then extending it to some condition $r \leq q$ which decides that all members of $\text{Ext}_T(B; X^*)$ coming from the nodes in $\text{ran}(r)$ have $h$-color $\varepsilon^*$.

Let $q(d)$ denote the splitting node or coding node in $Y$ and let $\ell_q = |q(d)|$. For each $i < d$, let $Y_i$ denote $Y \cap T_i$. For each $i < d$, take a set $J_i \subseteq K_i$ of size $\text{card}(Y_i)$ and label the members of $Y_i$ as $\{z_\alpha : \alpha \in J_i\}$. Let $\vec{J}$ denote $\prod_{i < d} J_i$. By Lemma [5.7] the set $\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is compatible, and $p_{\vec{J}} := \bigcup\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is a condition in $\mathbb{P}$.

Let $\delta_q = \bigcup\{\delta_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$. For $i < d$ and $\alpha \in J_i$, define $q(i, \alpha) = z_\alpha$. It follows that for each $\vec{\alpha} \in \vec{J}$ and $i < d$,

\[(33) \quad q(i, \alpha_i) \supseteq t_i^* = p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{J}}(i, \alpha_i),\]
and
\begin{equation}
q(d) \supseteq t^*_d = p_\bar{\alpha}(d) = p_{\bar{\alpha}}(d).
\end{equation}

For \( i < d \) and \( \delta \in \tilde{\delta}_q \setminus J_i \), we need to extend each node \( p_{\bar{\alpha}}(i, \delta) \) to some node of length \( \ell_q \) in order to construct a condition \( q \) extending \( p_{\bar{\alpha}} \). These nodes will not be a part of the construction of \( U_{m_j+1} \), however; they only are only a technicality allowing us to find some \( r \leq q \leq p_{\bar{\alpha}} \) from which we will build \( U_{m_j+1} \). In Case (a), let \( q(i, \delta) \) be any extension of \( p_{\bar{\alpha}}(i, \delta) \) in \( T \) of length \( \ell_q \). In Case (b), let \( q(i, \delta) \) be any extension of \( p_{\bar{\alpha}}(i, \delta) \) in \( T \) of length \( \ell_q \) with
\begin{equation}
q(i, \delta) p_{\bar{\alpha}}(i, \delta); T \upharpoonright (\ell^* - 1)) \sim p_{\bar{\alpha}}(i, \delta) p_{\bar{\alpha}}(i, \delta); T \upharpoonright (\ell^* - 1)).
\end{equation}
The SDAP guarantees the existence of such \( q(i, \delta) \). (In the case of LSDAP\(^+\), in addition to \( 35 \), we also require that \( \psi(q(i, \delta) = \psi(p(i, \delta)) \) for all \( i < d \) and \( \delta \in \tilde{\delta}_q \setminus J_i \). In this case, LSDAP\(^+\) guarantees the existence of such a \( q(i, \delta) \).)

Define
\begin{equation}
q = \{q(d)\} \cup \{(i, \delta), q(i, \delta) : i < d, \delta \in \tilde{\delta}_q\}.
\end{equation}
This \( q \) is a condition in \( \mathcal{P} \), and \( q \leq p_{\bar{\alpha}} \).

Now take an \( r \leq q \) in \( \mathcal{P} \) which decides some \( \ell_j \) in \( \hat{L}_d \) for which \( h(\hat{b}_\alpha \upharpoonright \ell_j) = \varepsilon^* \), for all \( \alpha \in \mathcal{J} \). This is possible since for all \( \alpha \in \mathcal{J} \), \( p_\alpha \) forces \( h(\hat{b}_\alpha \upharpoonright \ell) = \varepsilon^* \) for \( \hat{U} \) many \( \ell \in \hat{L}_d \). By the same argument as in creating the conditions \( p_\alpha \), we may assume that the nodes in the image of \( r \) have length \( \ell_j \). Since \( r \) forces \( \hat{b}_\alpha \upharpoonright \ell_j = X(r, \alpha) \) for each \( \alpha \in \mathcal{J} \), and since the coloring \( h \) is defined in the ground model, it follows that \( h(X(r, \alpha)) = \varepsilon^* \) for each \( \alpha \in \mathcal{J} \). Let
\begin{equation}
Y_0 = \{q(d)\} \cup \{(i, \alpha) : i < d, \alpha \in J_i\},
\end{equation}
and let
\begin{equation}
Z_0 = \{r(d)\} \cup \{(i, \alpha) : i < d, \alpha \in J_i\}.
\end{equation}

Now we consider the two cases separately. In Case (a), let \( Z \) be the level set consisting of the nodes in \( Z_0 \) along with a node \( z_y \) in \( T \upharpoonright \ell_j \) extending \( y \), for each \( y \in Y \setminus Y_0 \). Then \( Z \) end-extends \( Y \). By SDAP, it does not matter how the nodes \( z_y \) are chosen (except that in the case of LSDAP\(^+\), we also require that \( \psi(z_y) = \psi(y) \)). Letting \( U_{m_j+1} = U_{m_j} \cup Z \), we see that \( U_{m_j+1} \) is a member of \( r_{m_j+1}[U_{m_j}, T] \) such that \( h \) has value \( \varepsilon^* \) on \( \Ext_{U_{m_j+1}}(B; X^*) \).

In Case (b), \( r(d) \) is a coding node. Since \( r \leq q \), the nodes in \( \ran(r \upharpoonright \delta_q) \) have the same passing types over \( T \downarrow \ell_q \) as the nodes in \( \ran(q) \) have over \( T \downarrow \ell_q \). We
now need to extend all the other members of \( Y \setminus Y_0 \) to nodes with the required passing types at \( r(d) \). For each \( y \in Y \setminus Y_0 \), choose a member \( z_y \supseteq y \) in \( T_d \upharpoonright \ell_j \) so that
\[
(39) \quad z^+_y (r(d); U_{m_j}) \sim y^+(q(d); U_{m_j}).
\]
SDAP ensures the existence of such \( z_y \). (In the case of LSDAP\(^+\), in addition to (39), we also require that \( \psi(z_y) = \psi(y) \).) Let \( Z \) be the level set consisting of the nodes in \( Z_0 \) along with the nodes \( z_y \) for \( y \in Y \setminus Y_0 \). Then \( Z \) end-extends \( Y \) and moreover, \( U_{m_j} \cup Z \not\preceq V \). Letting \( U_{m_j+1} = U_{m_j} \cup Y \), we see that \( U_{m_j+1} \) is a member of \( r_{m_j+1}[U_{m_j}, T] \) and \( h \) has value \( \varepsilon^* \) on \( \text{Ext}_{U_{m_j+1}}(B; X^*) \).

Let \( U_{m_j+1} \) be any member of \( r_{m_j+1}[U_{m_j+1}, T] \). This completes the inductive construction. Let \( S = \bigcup_{j<\omega} U_{m_j} \). Then \( S \) is a member of \([D, T]\) and for each \( X \in \text{Ext}_S(B) \), \( h(X) = \varepsilon^* \). Thus, \( S \) satisfies the theorem.

\[ \square \]

**Remark 5.8:** By the construction in the previous proof, in Case (b) the coding nodes in any member \( X \in \text{Ext}_S(B; X^*) \) extend the coding node \( t^*_d \). It then follows from (3) in Definition 4.5 that for every level set \( X \subseteq S \) with \( A \cup X \sim \tilde{A} \cup X^* \), the coding node \( c \in X \) automatically satisfies \( c^+(c; A) \sim (t^*_d)^+(t^*_d; A) \sim \tilde{x}^+_d (\tilde{x}_d; \tilde{A}) \), where \( \tilde{x}_d \) denotes the coding node in \( X^* \). Thus, \( A \cup X \not\preceq \tilde{A} \cup X^* \) if and only if the non-coding nodes in \( X \) have immediate successors with similar passing types over \( A \cup \{c\} \) as their counterparts in \( X^* \) have over \( \tilde{A} \cup \{\tilde{x}_d\} \).

Moreover, for languages with only unary and binary relations, in Case (b) the set \( \text{Ext}_T(B; X^*) \) is exactly the set of all end-extensions \( X \) of \( B \) such that \( A \cup X \not\preceq \tilde{A} \cup \tilde{X} \) (\( A \cup X \not\preceq \tilde{A} \cup \tilde{X} \) in the case of LSDAP\(^+\)).

The main theorem of this paper follows immediately from the previous theorem.

**Theorem 1.2.** Suppose \( \mathcal{K} \) is a Fraïssé class in a finite relational language with relation symbols in any arity such that its Fraïssé limit \( K \) satisfies SDAP\(^+\).

Then \( \mathcal{K} \) is indivisible.

**Proof.** Let \( C \) be a singleton structure in \( \mathcal{K} \), and suppose \( h \) is a coloring of all copies of \( C \) inside \( K \) into two colors. Let \( T \) be a diagonal coding subtree of \( S \) representing \( K \), if one exists. Otherwise, we may without loss of generality assume that \( T \) is a diagonal coding subtree of \( U \) in which coding nodes representing \( C \) occur densely above any coding node. Let \( X^* \) be the coding node in \( T \) of least length representing a copy of \( C \). Let \( A = D = r_0(T) \) be the empty
sequence. In the case that $T$ is a subtree of $S$ and $\Gamma$ is of size at least two, let $B$ consist of $X^*$ along with one node $t_\gamma$ of the same length as $X^*$ extending $\gamma$, for each $\gamma \in \Gamma$. Otherwise, let $B$ be the initial segment of $X^*$ of length one. Then Theorem 5.4 provides us with a coding tree $S \in [D, T]$ such that $h$ is monochromatic on $\text{Ext}_S(B; X^*)$. Since $D = r_0(T)$, every coding node in $S$ representing a copy of $C$ is a member of $\text{Ext}_S(B; X^*)$. Thus, $K$ is indivisible.

Remark 5.9: The conclusion of Theorem 5.4 also holds for Fraïssé structures satisfying LSDAP$^+$ in languages with finitely many relation symbols of arity at most two, but more work is required for the proof. Indivisibility for such structures will follow from Theorem 3.6 in Part II.

6. Conclusion

The main theorem, Theorem 1.2, of this paper showing that all Fraïssé structures satisfying SDAP$^+$ with finitely many relations of any finite arities are indivisible followed from the Level Set Ramsey Theorem 5.4. In Part II, [4], we will start with Theorem 5.4 as the basis for an induction proof of upper bounds for the big Ramsey degrees of Fraïssé structures satisfying SDAP$^+$ with finitely many relations of arity at most two. Those upper bounds are given in terms of finite diagonal antichains of coding nodes representing a given finite structure. Such upper bounds will moreover be proved to be exact, leading to big Ramsey structures in the sense of Zucker [34] which have a simple presentation. Towards the end of [4], a catalogue of results on indivisibility and on big Ramsey degrees will be presented, showing which previous results are recovered by our methods and which results are new to our Parts I and II.
References

[1] M. Balko, D. Chodounský, N. Dobrinen, J. Hubička, M. Konečný, L. Vena, and A. Zucker, *Exact big Ramsey degrees for binary relational structures with forbidden irreducible substructures*, 2021, Submitted. arxiv:2110.08409, p. 97 pp.

[2] Manuel Bodirsky, *Ramsey classes: Examples and constructions*, London Mathematical Society Lecture Note Series, 424, Cambridge University Press, 2015.

[3] Peter Cameron, *Oligomorphic Permutation Groups*, Cambridge University Press, 1990.

[4] Rebecca Coulson, Natasha Dobrinen and Rehana Patel, *Fraïssé structures with SDAP+*, *Part II: Simply characterized big Ramsey structures*, 2022, p. 58 pp.

[5] Dennis Devlin, *Some partition theorems for ultrafilters on ω*, Ph.D. thesis, Dartmouth College, 1979.

[6] Natasha Dobrinen, *Borel sets of Rado graphs and Ramsey’s theorem*, To appear. arXiv:1904.00266v1, p. 29 pp.

[7] Natasha Dobrinen, *Ramsey theory of the universal homogeneous triangle-free graph, Part II: Exact big Ramsey degrees*, arXiv:2009.01985, p. 22pp.

[8] Natasha Dobrinen, *Ramsey theory of the universal homogeneous k-clique-free graph*, Journal of Mathematical Logic (2020), 75 pp.

[9] Natasha Dobrinen, *The Ramsey theory of the universal homogeneous triangle-free graph*, Journal of Mathematical Logic 20 (2020), no. 2, 2050012, 75 pp.

[10] Natasha Dobrinen, *Ramsey Theory on infinite structures and the method of strong coding trees*, Contemporary Logic and Computing (Adrian Rezus, ed.), College Publications, London, 2020, pp. 444–467.

[11] Mohamed El-Zahar and Norbert Sauer, *The indivisibility of the homogeneous Kn-free graphs*, Journal of Combinatorial Theory, Series B 47 (1989), no. 2, 162–170.

[12] Mohamed El-Zahar and Norbert Sauer, *On the divisibility of homogeneous hypergraphs*, Combinatorica 14 (1994), no. 2, 159–165.

[13] Paul Erdős and Richard Rado, *A partition calculus in set theory*, Bulletin of the American Mathematical Society 62 (1956), 427–489.

[14] Roland Fraïssé, *Sur l’extension aux relations de quelques propriétés des ordres*, Annales Scientifiques de l’Ecole Normale Supérieure 71 (1954), no. 3, 363–388.

[15] Wilfred Hodges, *A Shorter Model Theory*, Cambridge University Press, 1997.

[16] John Howe, *Big Ramsey degrees in homogeneous structures*, Ph.D. thesis, University of Leeds, Expected 2020.

[17] Jan Hubička, *Big Ramsey degrees using parameter spaces*, 2020, Preprint. arXiv:2009.00967, 19 pp.

[18] A. A. Ivanov, *Generic expansions of ω-categorical structures and semantics of generalized quantifiers*, The Journal of Symbolic Logic 64 (1999), no. 2, 775–789.

[19] Alexander Kechris, Vladimir Pestov and Stevo Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geometric and Functional Analysis 15 (2005), no. 1, 106–189.

[20] Alexander S. Kechris and Christian Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proceedings of the London Mathematical Society. Third Series 94 (2007), no. 2, 302–350.
[21] Péter Komjáth and Vojtěch Rödl, *Coloring of universal graphs*, Graphs and Combinatorics 2 (1986), no. 1, 55–60.

[22] Alex Kruckman, *Disjoint n-amalgamation and pseudofinite countably categorical theories*, Notre Dame Journal of Formal Logic 60 (2019), no. 1, 139–160.

[23] Claude Laflamme, Lionel Nguyen Van Thé and Norbert Sauer, *Partition properties of the dense local order and a colored version of Milliken’s theorem*, Combinatorica 30 (2010), no. 1, 83–104.

[24] Claude Laflamme, Norbert Sauer and Vojkan Vuksanovic, *Canonical partitions of universal structures*, Combinatorica 26 (2006), no. 2, 183–205.

[25] Richard Laver, unpublished.

[26] Keith R. Milliken, *A Ramsey theorem for trees*, Journal of Combinatorial Theory, Series A 26 (1979), 215–237.

[27] Lionel Nguyen Van Thé, *Big Ramsey degrees and divisibility in classes of ultrametric spaces*, Canadian Mathematical Bulletin 51 (2008), no. 3, 413–423.

[28] Lionel Nguyen Van Thé, *Structural Ramsey theory with the Kechris-Pestov-Todorcevic correspondence in mind*, Habilitation thesis, Université d’Aix-Marseille, 2013, p. 48 pp.

[29] Frank P. Ramsey, *On a problem of formal logic*, Proceedings of the London Mathematical Society 30 (1929), 264–296.

[30] Norbert Sauer, *Canonical vertex partitions*, Combinatorics, Probability, and Computing 12 (2003), no. 6, 671–704.

[31] Norbert Sauer, *Coloring subgraphs of the Rado graph*, Combinatorica 26 (2006), no. 2, 231–253.

[32] Sierpiński, *Sur une problème de lat théorie des relations*, Ann. Scuola Norm. Super. Pisa, Ser. 2 2 (1933), 239–242.

[33] Stevo Todorcevic, *Introduction to Ramsey Spaces*, Princeton University Press, 2010.

[34] Andy Zucker, *Big Ramsey degrees and topological dynamics*, Groups, Geometry and Dynamics 13 (2018), no. 1, 235–276.

[35] Andy Zucker, *A Note on Big Ramsey degrees*, 2020, Submitted. arXiv:2004.13162, 21 pp.