QUASI-PERIODIC SOLUTIONS FOR NONLINEAR WAVE EQUATION WITH LIOUVILLEAN FREQUENCY

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Abstract. In this paper, one dimensional nonlinear wave equation
\[ u_{tt} - u_{xx} + \varepsilon f(\omega t, x, u; \xi) = 0 \]
with Dirichlet boundary condition is considered, where \( \varepsilon \) is small positive parameter, \( \omega = \xi \hat{\omega} \), \( \hat{\omega} \) is weak Liouvillean frequency. It is proved that there are many quasi-periodic solutions with Liouvillean frequency for the above equation. The proof is based on an infinite dimensional KAM Theorem.

1. Introduction. Both the CWB (Craig-Wayne-Bourgain) method [7, 8, 9, 10, 11, 13] and the infinite dimensional KAM theory [20, 26, 29] are important tools to study the existence of finite dimensional tori for Hamiltonian PDEs. The CWB method is a generalization of the Lyapunov-Schmidt reduction and Newton’s method. And the infinite dimensional KAM theory is the extension of classical KAM theory. Compared to the CWB method, the KAM method has its own advantage. Besides obtaining the existence of quasi-periodic solutions, it also provides more information of the dynamics. With these two methods, the quasi-periodic solutions for Hamiltonian PDEs (such as NLS, NLW, BEAM, KDV, GBQ, BO and etc.) are constructed. For more details, one may refer to [4, 15, 16, 19, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31] and the references therein.

The study of quasi-periodic solutions with frequency varying in a line was first proposed by Bourgain [12] and Eliasson [14]. In the infinite dimensional Hamiltonian setting, this was first proved by Geng-Ren [17] for 1D wave equation and then Berti-Biasco [5] for 1D NLS, Berti-Bolle [6] for higher dimensional NLS.

The frequencies of the quasi-periodic solutions mentioned above must satisfy some Diophantine condition. A vector \( \omega \in \mathbb{R}^d \) is said to be Diophantine, if there

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exists $\tau > d, \chi > 0$, such that
\[
|\langle k, \omega \rangle| \geq \frac{\chi}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}.
\]
Later, it is found that the Diophantine condition on the frequencies can be weakened to Brjuno condition, that is
\[
B(\omega) = \sum_{n \geq 0} \frac{1}{2^n \max_{0 < |k| \leq 2^n, k \in \mathbb{Z}^d} \ln |\langle k, \omega \rangle|} < \infty.
\]
If $\omega$ doesn’t satisfy Diophantine or Brjuno condition, we call it Liouvillean.

For $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2)$ with $\bar{\omega}_1 = (\alpha, 1), \alpha \in \mathbb{R} \setminus \mathbb{Q}, \bar{\omega}_2 \in \mathbb{R}^d$, we say that the frequency $\bar{\omega}$ is weak Liouvillean, if there exist $\chi > 0$ and $\tau > d + 6$, such that
\[
\beta(\alpha) = \limsup_{n \to 0} \frac{\ln q_{n+1}}{\ln q_n} < \infty
\]
and
\[
|\langle k, \bar{\omega}_1 \rangle + \langle l, \bar{\omega}_2 \rangle| \geq \frac{\chi}{(|k| + |l|)^\tau}, \quad \forall k \in \mathbb{Z}^d, l \in \mathbb{Z}^d \setminus \{0\},
\]
where $\{q_n\}_{n \in \mathbb{N}}$ are the best rational approximations to $\alpha$ (see the definition in section 2). Denote by $WL(\chi, \tau, \beta)$ the set of such frequencies and by $WL$ the union
\[
WL = \bigcup_{\chi > 0, \tau > d + 6, 0 \leq \beta < \infty} WL(\chi, \tau, \beta).
\]
It is obvious that $WL$ is of full Lebesgue measure.

We should mention Avila-Fayad-Krikorian [3] and Hou-You [18], where they obtained the rotation reducibility of quasi-periodic $SL(2, \mathbb{R})$ cocycles with Liouvillean frequency. A quasi-periodic $SL(2, \mathbb{R})$ cocycle $(\alpha, A) \in \mathbb{R} \setminus \mathbb{Q} \times C^\omega(T, SL(2, \mathbb{R}))$ is a linear skew product system:
\[
(\alpha, A) : \quad T^1 \times \mathbb{R}^2 \to T^1 \times \mathbb{R}^2
\]
\[
(\theta, v) \mapsto (\theta + \alpha, A(\theta) \cdot v).
\]
The dynamics of quasi-periodic $SL(2, \mathbb{R})$ cocycles is closely related to the spectral theory of 1D quasi-periodic Schrödinger operators.

Reducibility theory of quasi-periodic $SL(2, \mathbb{R})$ cocycles with Liouvillean frequency [1, 2, 3, 18] plays a quite important role in the spectral theory of quasi-periodic Schrödinger operators. Based on the reducibility results developed in quasi-periodic cocycles, Xu-You-Zhou [32] make the breakthrough to prove the existence of quasi-periodic solutions with Liouvillean frequency for Hamiltonian partial differential equation. They consider 1D forced nonlinear Schrödinger equation
\[
iu_t - u_{xx} + v(x)u + \varepsilon f(\omega t, x, u, \bar{u}; \xi) = 0 \quad (2)
\]
with the Dirichlet boundary condition:
\[
u(t, 0) = u(t, \pi) = 0, \quad -\infty < t < \infty.
\]
They proved the equation (2) has a $C^\infty$ smooth quasi-periodic solution with Liouvillean frequency.

Motivated by the result in [32], in this paper we consider one dimensional forced nonlinear wave equation
\[
u_{tt} - u_{xx} + mu + \varepsilon f(\omega t, x, u; \xi) = 0 \quad (4)
\]
with the Dirichlet boundary condition.
Our main result is in the following:

**Theorem 1.1.** Let \( \hat{\omega} \in W L, \xi \in \mathcal{O} = [\frac{1}{2}, \frac{3}{2}], |m| < \frac{1}{2}, f(\Theta, x, u; \xi) \) is real analytic in \( \Theta, x, u, \) Lipschitz in \( \xi \) and satisfies \( f(\Theta, 0, u; \xi) = f(\Theta, \pi, u; \xi) = 0. \) Then for any small \( \gamma > 0, \) there exists \( \varepsilon_0 > 0 \) and \( \mathcal{O}_\gamma \subset \mathcal{O} \) with \( \text{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma), \) such that if \( \varepsilon < \varepsilon_0, \) then the equation (4) has a \( C^\infty \) smooth quasi-periodic solution with frequency \( \omega = \xi \hat{\omega} \) for any \( \xi \in \mathcal{O}_\gamma. \)

2. Preliminaries. In this section, for an irrational number \( \alpha \in \mathbb{R} \setminus \mathbb{Q}, \) we give the definitions of the best approximation for \( \alpha \) and the CD bridge.

Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) be irrational. We first set \( a_0 = 0, \alpha_0 = \alpha \) and then we define inductively for \( n \geq 1 \)

\[
a_n = [\alpha_{n-1}^{-1}], \quad \alpha_n = \alpha_{n-1}^{-1} - a_n = \left\{ \frac{1}{\alpha_{n-1}} \right\}.
\]

We define

\[
p_0 = 0, \quad p_1 = 1, \quad q_0 = 1, \quad q_1 = a_1
\]

and

\[
p_n = a_np_{n-1} + p_{n-2}, \quad q_n = a_nq_{n-1} + q_{n-2}.
\]

Then \( (q_n)_{n \in \mathbb{N}} \) is the sequence of denominators of the best rational approximations for \( \alpha \in \mathbb{R} \setminus \mathbb{Q}. \) To be more precise, for any \( 1 \leq k < q_n, \) one has

\[
\|k\alpha\| \geq \|q_{n-1}\alpha\|, \quad \frac{1}{q_{n+1} + q_n} \leq \|q_n\alpha\| \leq \frac{1}{q_{n+1}},
\]

where \( \|x\| = \inf_{p \in \mathbb{Z}} |x - p|. \)

For any \( \alpha \in \mathbb{R} \setminus \mathbb{Q}, \) let \( (q_n) \) be the sequence of denominators of best rational approximation for \( \alpha. \) We choose two particular subsequences of \( (q_n): \) the first is \( (q_k) \) for simple, the second is \( (q_{n_k+1}) \) which we denote by \( (Q_k). \) The properties required from our choice of the subsequence \( (Q_k) \) are summarized below.

**Definition 2.1.** [3] Let \( 0 < A \leq B \leq C, \) we say the pair of denominators \( (q_l, q_n) \) form a CD-bridge if:

1. \( q_{l+1} \leq q_{l+1}^A, \quad i = l, \ldots, n-1; \)
2. \( q_l^B \leq q_n \leq q_l^C. \)

**Lemma 2.2.** [3] For any \( A > 0, \) there exists a subsequence \( (Q_k) \), such that \( Q_0 = 1 \) and for each \( k \geq 0, \) \( Q_{k+1} \leq Q_k^A \), and either \( Q_k \geq Q_k^A \) or the pairs \( (Q_{k-1}, Q_k) \) and \( (Q_k, Q_{k+1}) \) are both \( CD(A, A, A^3) \) bridges.

Similar to [32], we assume \( A \geq 16, \) and \( (Q_n) \) is the selected subsequence in Lemma 2.2 accordingly. Then one has the following Lemma:

**Lemma 2.3.** [22] We have \( Q_n \geq Q_{n-1}^A \) for every \( n \geq 1. \) Furthermore, if \( \beta(\alpha) < \infty, \) then there exists \( U = U(\alpha) > 0, \) such that

\[
\sup_{n>0} \frac{\ln \ln Q_{n+1}}{\ln Q_n} \leq U(\alpha).
\]
An infinite dimensional KAM theorem. In this section, we state an infinite dimensional KAM theorem, by which we could prove Theorem 1.1.

First we introduce some useful notations: for any fixed \( a > 0 \) and \( \rho > 0 \), let \( P^{a,\rho} \) be the Hilbert space of all complex sequence \( w = (w_1, w_2, \cdots) \) with
\[
|w|_{a,\rho}^2 = \sum_{p \geq 1} |w_p|^2 p^{2\rho} e^{2\rho p} < \infty.
\]
Fix \( \bar{\omega} \in WL \), and assume that the tangential frequency \( \omega = (\omega_1, \omega_2) = \xi (\bar{\omega}_1, \bar{\omega}_2) \) with \( \xi \in \mathcal{O} = [\frac{1}{2}, \frac{3}{2}] \), and the normal frequencies \( \Omega_p(\xi) \) are real functions depending on \( \xi \in \mathcal{O} \). Then we consider a small perturbed Hamiltonian
\[
H = N + P
\]
\[
= (\omega_1, I) + (\omega_2, J) + \sum_{p \geq 1} \Omega_p(\xi)|z_p|^2 + P(\theta, \varphi, z, \bar{z}; \xi).
\]

The Hamiltonian is defined on the phase space \( P^{a,\rho} = \mathbb{T}^{2+d} \times \mathbb{R}^{2+d} \times P^{a,\rho} \) with symplectic structure \( dI \wedge d\theta + dJ \wedge d\varphi + i \sum_{p \geq 1} dz_p \wedge d\bar{z}_p \). Denote by \( I = (I, J) \) with \( I \in \mathbb{R}^2 \), \( J \in \mathbb{R}^d \) and \( \varphi \in \mathbb{T}^d \).

The perturbation \( P = P(\Theta, z, \bar{z}; \xi) \) is independent of \( I \), real analytic in \( \Theta, z, \bar{z} \) and Lipschitz in the parameter \( \xi \). For each \( \xi \in \mathcal{O} \), the Hamiltonian vector field \( X_p = (-P_\Theta, 0, iP_z, -iP_{\bar{z}}) \) defines a real analytic map from \( P^{a,\rho} \) to itself. To make this precise, we introduce the complex neighborhoods of \( \mathcal{T}^{2+d} = \mathbb{T}^{2+d} \times \{0,0\} \):
\[
D(r,s) = \{(\Theta, \mathcal{I}, z, \bar{z}) : |\text{Im} \Theta| < r, |\mathcal{I}| < s^2, |z|_{a,\rho} < s, |ar{z}|_{a,\rho} < s \},
\]
where \( |.| \) denotes the sup-norm of complex vectors, and the weighted phase space norm for \( W = (X,Y,U,V) \) is defined by
\[
|W|_s = |W|_{a,s,\rho} = |X| + \frac{1}{s^2} |Y| + \frac{1}{s} |U|_{a,\rho} + \frac{1}{s} |V|_{a,\rho}.
\]

We assume that \( X_p \) is real analytic on \( D(r,s) \) for \( s, r > 0 \) with finite norm
\[
\|X_p\|_{s,D(r,s)} = \sup_{D(r,s) \times \mathcal{O}} |X_p|_s
\]
and with finite Lipschitz semi-norm
\[
\|X_p\|_{s,D(r,s)}^\mathcal{L} = \sup_{\xi, \eta \in \mathcal{O}, \xi \neq \eta} \sup_{D(r,s)} \frac{|\Delta_{\xi,\eta} X_p|_s}{|\xi - \eta|},
\]
where \( \Delta_{\xi,\eta} X_p = X_p(\xi) - X_p(\eta) \). For simplicity, we usually write
\[
\|X_p\|_{s,D(r,s)}^\mathcal{L} = \|X_p\|_{s,D(r,s)} + \|X_p\|_{s,D(r,s)}^\mathcal{L}.
\]
The semi-norm of any function \( f(\xi) \) on \( \xi \in \mathcal{O} \) is defined as
\[
|f|_\mathcal{O} = |f|_\mathcal{O} + |f|_\mathcal{L},
\]
where the Lipschitz semi-norm is defined analogously to \( \|X_p\|_{s,D(r,s)}^\mathcal{L} \).

Now we state the following infinite dimensional KAM theorem:

**Theorem 3.1.** Let \( \bar{\omega} \in WL(\chi, \tau, \beta) \) with \( 0 \leq \beta < \infty, \tau > d + 6, \chi > 0 \). Suppose the Hamiltonian (7) satisfies the assumptions:

(A1) The tangential frequency \( \omega = \xi \bar{\omega} \) with \( \xi \in \mathcal{O} = [\frac{1}{2}, \frac{3}{2}] \).
Remark 1. The choice of the value \( \frac{1}{\sqrt{2} + 1} \) in assumption (10) is motivated by the application to the wave equation, see (24).

Similar to [32], Theorem 3.1 is proved by a modified KAM scheme which involves an infinite sequence of transformation of variables. What makes this KAM scheme complicated is that \( \omega \) is Liouvillean, then some \( \theta \) dependent terms of the perturbation cannot be solved and then have to be included in the new normal form. Consequently, we need to consider the Hamiltonian of the form

\[
H_n = e_n(\theta; \xi) + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} (\Omega_p^0(\xi) + B_p^0(\theta; \xi))|z_p|^2 + P_n(\theta, \varphi, z, \bar{z}; \xi),
\]

where \( \langle \omega_1, \omega_2 \rangle = \xi(\bar{\omega}_1, \bar{\omega}_2) \), \( B_p^0(\theta; \xi) \) is of initial size \( \varepsilon_0 \), and the perturbation \( P_n(\theta, \varphi, z, \bar{z}; \xi) \) is of size \( \varepsilon_n(\varepsilon_n \ll \varepsilon_0) \). As usual, we will construct a symplectic transformation \( \Phi \), which is close to the identity, such that \( \Phi_n \) transforms (12) into

\[
H_{n+1} = e_{n+1}(\theta; \xi) + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} (\Omega_p^{n+1}(\xi) + B_p^{n+1}(\theta; \xi))|z_p|^2 + P_{n+1}(\theta, \varphi, z, \bar{z}; \xi),
\]

where \( B_p^{n+1}(\theta; \xi) \) is still of size \( \varepsilon_0 \), and the perturbation \( P_{n+1}(\theta, \varphi, z, \bar{z}; \xi) \) is of size \( \varepsilon_{n+1}(\varepsilon_{n+1} \ll \varepsilon_0) \).

Similar to [32], we introduce the global iteration. At first we introduce some useful notations. Let \( D(r) = \{ (\theta, \varphi) \in \mathbb{C}^2 \times \mathbb{C}^d : |\text{Im } \theta| + |\text{Im } \varphi| < r \} \) be the complex neighborhood of \( T^2 \times T^d \). Denote by \( \mathcal{B}_r \) the set of functions \( g(\theta, \varphi) \) that are real analytic on \( D(r) \), and \( \mathcal{B}_r(\mathcal{O}) \) the set of functions \( g(\theta, \varphi; \xi) \) that are real analytic on \( D(r) \) and Lipschitz on \( \mathcal{O} \) such that

\[
|g|_{r, \mathcal{O}}^* = \sum_{(k,l) \in \mathbb{Z}^2 \times \mathbb{Z}^d} |\hat{g}_{(k,l)}(\xi)|_{\mathcal{O}}^* e^{||k||+|l||}r < \infty,
\]

where \( g(\theta, \varphi; \xi) = \sum_{(k,l) \in \mathbb{Z}^2 \times \mathbb{Z}^d} \hat{g}_{(k,l)}(\xi)e^{i((k,l) \cdot (\theta, \varphi))} \). The average of the function \( g(\theta, \varphi) \) is defined by

\[
[g(\theta, \varphi)] = \int_{T^d} g(\theta, \varphi) \, d\varphi, \quad [g(\theta, \varphi)] = \int_{T^{2d}} g(\theta, \varphi; \xi) \, d\theta d\varphi.
\]
The operator norm $| \cdot |_{s, \tilde{s}}$ is defined as

$$|L|_{s, \tilde{s}} = \sup_{W \in D(r, s), W \neq 0} \frac{|LW|_{s, D(r, s)}}{|W|_{s, D(r, s)}}.$$ 

For $r, s, M_1, M_2, M_3 > 0$ and $\mathcal{O} \subset \mathbb{R}$, let $\mathcal{F}_{r, s, \mathcal{O}}(M_1, M_2, M_3)$ be the space of analytic function $(e(\theta; \xi), \hat{\Omega}_p(\xi), B_p(\theta; \xi), P(\theta, \varphi, z, \tilde{z}; \xi))$ satisfying

$$\begin{cases} |e(\theta; \xi)|_{r, \mathcal{O}} \leq M_1, & \sup_{p \geq 1} |\hat{\Omega}_p(\xi)|_{\mathcal{O}} \leq \frac{1}{2\sqrt{2}} + M_1, \\ |B_p(\theta; \xi)|_{r, \mathcal{O}} \leq M_2, & \|X_p\|_{s, D(r, s)} \leq M_3. \end{cases}$$

For simplicity, we just denote them as

$$e(\theta; \xi) + \sum_{p=1}^{\infty} (\hat{\Omega}_p(\xi) + B_p(\theta; \xi))|z_p|^2 + P(\theta, \varphi, z, \tilde{z}; \xi) \in \mathcal{F}_{r, s, \mathcal{O}}(M_1, M_2, M_3).$$

We introduce the iteration sequence as following. Let $r_0, s_0 > 0$ and $\bar{\omega} \in W\tilde{L}(\chi, \tau, \beta)$ with $0 \leq \beta < \infty, \chi > 0$ and $\tau > d + 6$. For any $\gamma > 0$, let $\gamma_0 = \gamma$ and $\varepsilon_0$ is supposed to be small enough such that

$$\varepsilon_0 \leq \min\left\{ \frac{(r_0s_0\gamma_0)^{12\tau + 36}}{Q_{1, r_0}^2}, e^{-2\varepsilon_0U} \right\} \text{ and } \ln\varepsilon_0^{-1} \leq \varepsilon_0^{-\tau \varepsilon_0 - \varepsilon_0},$$

where $U = U(\alpha)$ is defined in Lemma 2.3, $c > \frac{18\tau + 27}{2\varepsilon_0}$ is a global constant. We define some sequences for $n \geq 1$:

$$s_n = s_{n-1}^{(n+1)}e_{n-1}^{(n+1)} - 1, \quad b_{n+1} = 2 + \frac{2^{n+1}\tau U \ln Q_{n+1}}{C^2 + 9},$$

$$r_n = \frac{r_0}{4Q_{1, r_0}}, \quad \varepsilon_n = \varepsilon_{n-1} \cdot Q_{n+1}^{n+1} \tau U,$$

$$\mathcal{E}_n = \sum_{m=0}^{n-1} \frac{1}{2} \varepsilon_n, \quad \gamma_n = \gamma_0 - 3 \sum_{m=0}^{n-1} \varepsilon_n, \quad K_n = 40r_{n+1} \ln \varepsilon_n^{-1}, \quad D_n = D(r_n, s_n)$$

with $\mathcal{E}_0 = 0$. Now we state the following proposition:

**Proposition 1.** Let $\bar{\omega} \in W\tilde{L}(\chi, \tau, \beta)$ with $0 \leq \beta < \infty, \chi > 0, \tau > d + 6, (\omega_1, \omega_2) = \xi(\bar{\omega}_1, \bar{\omega}_2)$. Then the following holds for all $n \geq 0$: suppose that the Hamiltonian

$$H_n = e_n(\theta; \xi) + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{p \geq 1} \left( \Omega_p^0(\xi) + B_p^0(\theta; \xi) \right) \|z_p\|^2 + P_n(\theta, \varphi, z, \tilde{z}; \xi)$$

satisfies

1. For any $\xi \in \mathcal{O}_n$, if $(k, l) \in \mathbb{Z}^2 \times \mathbb{Z}^d$ with $|k| + |l| \leq K_n$ and $m \in \mathbb{Z}^N$ with $1 \leq |m| \leq 2$, one has

$$\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \langle m, \Omega_n^\alpha \rangle \geq \frac{\langle m \rangle \gamma_n}{(|k| + |l| + 1)^r}$$

where $\langle m \rangle = |\sum_{p \geq 1} m_p \cdot p|$ and $|m| = \sum_{p \geq 1} |m_p|$.

2. The functions $e_n(\theta; \xi), B_p^0(\theta; \xi) \in B_{r_n}(\mathcal{O}_n)$ with $|B_p^0(\theta; \xi)| = 0$. 
Lemma 3.2. Suppose that the tangential frequency estimate. \[ e_n(\theta; \xi) + \sum_{p=1}^{n} |\hat{\Omega}_p(\xi) + B_p(\theta; \xi)| |z_p|^2 + P_n(\theta, \varphi, z, \xi) \in F_{r, s, n, \Omega, \xi}(E_n, E_n, \varepsilon_n). \]

Then there exists \( O_{n+1} \subseteq O_n \) with
\[
\text{meas}(O_n \setminus O_{n+1}) \leq \frac{c_{n+1}}{K_{n+1}},
\]
and a real analytic symplectic transformation \( \Phi_n : D_{n+1} \times O_{n+1} \to D_n \) such that for \( H_{n+1} = H_n \circ \Phi_n \), the same assumptions as above are satisfied with \( n+1 \) in place of \( n \). Furthermore, \( \Phi_n \) satisfies
\[
\| \Phi_n - \text{id} \|_{s, D_{n+1}} \leq \frac{\varepsilon}{n},
\]
\[
\| D(\Phi_n - \text{id}) \|_{s, s, D_{n+1}} \leq \frac{\varepsilon}{n}.
\]

The proof of proposition 1 is very similar to that of proposition 3.1 in [32]. To prove proposition 3.1 in [32], Xu-You-Zhou introduce a global KAM iteration. One may refer to [32] for details. In the following we present a key lemma on measure estimate.

Lemma 3.2. Suppose that the tangential frequency \( \omega = (\omega_1, \omega_2) = \xi(\tilde{\omega}, \tilde{\omega}_2) \) with \( \xi \in \mathcal{O} \), the normal frequency \( \Omega \) are defined on \( \mathcal{O} \) with
\[
\sup_{p \geq 1} |\Omega_p - p|_{\mathcal{O}} < \frac{1}{8}.
\]

For any \( (k, l) \in \mathbb{Z}^2 \times \mathbb{Z}^d, m \in \mathbb{Z}^N, 1 \leq |m| \leq 2, \tilde{\gamma} > 0 \), we define the resonant set
\[
\mathcal{R}_{klm}(\tilde{\gamma}) = \{ \xi \in \mathcal{O} : |(k, \omega_1) + (l, \omega_2) + (m, \Omega)| < \frac{\tilde{\gamma}(m)}{(|k| + |l| + 1)^{\tilde{\gamma}}} \}
\]
with \( |m| = \sum_{p \geq 1} m_p \cdot p \). Then we have
\[
\text{meas}(\mathcal{R}_{kl}(\tilde{\gamma})) = \text{meas}(\bigcup_{1 \leq |m| \leq 2} \mathcal{R}_{klm}(\tilde{\gamma})) \leq \frac{c_{\tilde{\gamma}}}{(|k| + |l| + 1)^{\tilde{\gamma} - 2}},
\]
where \(|m| = \sum_{p \geq 1} |m_p|\).

Proof. Note that one can rewrite the resonant set \( \mathcal{R}_{kl}(\tilde{\gamma}) \) as
\[
\mathcal{R}_{kl}(\tilde{\gamma}) = \bigcup_{p \geq 1} \mathcal{R}_{klp}(\tilde{\gamma}) \cup \mathcal{R}_{klpq}(\tilde{\gamma}) \bigcup_{p \geq 1} \mathcal{R}_{klpq}(\tilde{\gamma})
\]
with
\[
\mathcal{R}_{klp}(\tilde{\gamma}) = \{ \xi \in \mathcal{O} : |(k, \omega_1) + (l, \omega_2) + \Omega_p| < \frac{p^\gamma}{(|k| + |l| + 1)^{\tilde{\gamma}}} \},
\]
\[
\mathcal{R}_{klpq}(\tilde{\gamma}) = \{ \xi \in \mathcal{O} : |(k, \omega_1) + (l, \omega_2) + \Omega_p + \Omega_q| < \frac{p^\gamma q^\gamma}{(|k| + |l| + 1)^{\tilde{\gamma}}} \},
\]
\[
\mathcal{R}_{klpq}(\tilde{\gamma}) = \{ \xi \in \mathcal{O} : |(k, \omega_1) + (l, \omega_2) \pm (\Omega_p + \Omega_q)| < \frac{p^\gamma q^\gamma}{(|k| + |l| + 1)^{\tilde{\gamma}}} \}.
\]
To prove the desired result, we first need the following lemma:
Lemma 3.3. If \( \max\{p, q\} > |p - q| > c(|k| + |l|) \), we have \( \mathcal{R}_{k lp}(\tilde{\gamma}) = \mathcal{R}_{k lp q}(\tilde{\gamma}) = \emptyset \). If \( \max\{p, q\} > |p - q| > c(|k| + |l|), c \geq 6|\tilde{\omega}| \) and \( p \neq q \), we have \( \mathcal{R}_{k lp q}(\tilde{\gamma}) = \emptyset \).

Proof. We only consider the resonant set \( \mathcal{R}_{k lp q}(\tilde{\gamma}) \) with \( p \neq q \). If \( \max\{p, q\} > |p - q| > c(|k| + |l|) \), we have

\[
|\langle k, \omega_1 \rangle + \langle l, \omega_2 \rangle + \Omega_p - \Omega_q|
\]

\[
> |p - q| - |\Omega_p - p| - |\Omega_q - q| - |\xi(|k| + |l|)|\bar{\omega}|
\]

\[
\geq |p - q| - \frac{1}{4} - \frac{3}{4}|\bar{\omega}||k| + |l| > \frac{|p - q|}{2},
\]

which follows that \( \mathcal{R}_{k lp q}(\tilde{\gamma}) = \emptyset \).

By lemma 3.3, we only need to consider the case \( \max\{p, q\} \leq c(|k| + |l|) \). In this case, since \( \xi \in [\frac{4}{3}, \frac{5}{3}] \), it follows that

\[
\mathcal{R}_{k lp q}^{11} \subset \mathcal{Q}_{k lp q}^{11} = \{ \xi \in \mathcal{O} : |\langle k, \bar{\omega}_1 \rangle + \langle l, \bar{\omega}_2 \rangle + \Omega_p - \Omega_q| < \frac{2|p - q|\bar{\gamma}}{|k| + |l| + 1} \}.
\]

By (17), it is easy to see that

\[
|\Delta_{\xi}(\langle k, \bar{\omega}_1 \rangle + \langle l, \bar{\omega}_2 \rangle + \Omega_p - \Omega_q)| = |\frac{\Omega_p(\xi) - \Omega_q(\xi)}{\xi} - \frac{\Omega_p(\eta) - \Omega_q(\eta)}{\eta}| \geq \frac{1}{\bar{\gamma}}|p - q||\xi - \eta|
\]

and then

\[
\text{meas}(\mathcal{R}_{k lp q}^{11}) \leq \text{meas}(\mathcal{Q}_{k lp q}) \leq \frac{36\bar{\gamma}}{|k| + |l| + 1}.
\]

The other resonant sets can be handled similarly.

As a conclusion, we have

\[
\text{meas}(\bigcup_{1 \leq |m| \leq 2} \mathcal{R}_{k l m}(\tilde{\gamma}))
\]

\[
\leq \text{meas}(\bigcup_{p \leq c(|k| + |l|)} \mathcal{R}_{k lp}(\tilde{\gamma})) + \text{meas}(\bigcup_{p \neq q,\ \max\{p, q\} \leq c(|k| + |l|)} \mathcal{R}_{k lp q}^{11}(\tilde{\gamma})
\]

\[
+ \text{meas}(\bigcup_{\max\{p, q\} \leq c(|k| + |l|)} \mathcal{R}_{k lp q}^2(\tilde{\gamma}))
\]

\[
\leq \frac{36\bar{\gamma}}{|k| + |l| + 1} \left( 1 + \sum_{1 \leq p \leq c(|k| + |l|)} 1 + \sum_{1 \leq p \leq c(|k| + |l|)} 1 \right) \frac{c}{|k| + |l| + 1}.
\]

\[
4. \ \text{Proof of theorem 1.1 and theorem 3.1.} \ \text{The proof of Theorem 3.1 is similar to that of Theorem 2 in [32]. In the following, we will prove Theorem 1.1 by using Theorem 3.1.}
\]

Introducing \( v = u_t \) and \( B = -\partial_{xx} + m \), then (4) reads
The Hamiltonian equations we obtain a real analytic Hamiltonian with the symplectic structure

\[ H = \frac{1}{2} \langle v, v \rangle + \frac{1}{2} \langle Bu, u \rangle + \varepsilon \int_0^\pi F(\omega t, x, u; \xi) \, dx \]

where

\[ F_u(\omega t, x, u; \xi) = f(\omega t, x, u; \xi), \]

for real functions \( g, h \), we define \( \langle g, h \rangle = \int_0^\pi g(x) h(x) \, dx \).

Let

\[ u(t, x) = S_1 q = \sum_{j \geq 1} \lambda_j^{1/2} q_j(t) \phi_j(x), \quad v(t, x) = S_2 p = \sum_{j \geq 1} \lambda_j^{1/2} p_j(t) \phi_j(x), \]

where \( q(t) = (q_1(t), q_2(t), \cdots), p(t) = (p_1(t), p_2(t), \cdots) \) are real valued sequences in \( l^{\alpha, \beta} \), \( \phi_j(x) = \sqrt{\frac{2}{\pi}} \sin j x \), for \( j = 1, 2, \cdots \) are the Dirichlet eigenfunctions of the operator \( B \) with eigenvalues \( \lambda_j = j^2 + m \), setting \( \mu_j = \sqrt{\lambda_j} \). Then, associated with the symplectic structure \( \sum_{j \geq 1} dq_j \wedge dp_j \) on \( l^{\alpha, \beta} \times l^{\alpha, \beta} \), we have the following Hamiltonian equations

\[ \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad j \geq 1, \]

\[ H = \Lambda + G, \]

\[ \Lambda = \frac{1}{2} \sum_{j \geq 1} \mu_j (q_j^2 + p_j^2), \]

\[ G = \varepsilon \int_0^\pi F(\omega t, x, S_1 \xi) \, dx. \]

By introducing the complex coordinates

\[ z_j = \frac{1}{\sqrt{2}}(q_j + i p_j), \quad \bar{z}_j = \frac{1}{\sqrt{2}}(q_j - i p_j), \]

we obtain a real analytic Hamiltonian

\[ H = \sum_{j \geq 1} \mu_j |z_j|^2 + \varepsilon \int_0^\pi F(\omega t, x, S_1(\frac{z + \bar{z}}{\sqrt{2}})) \, dx \]

on the now complex Hilbert space \( l^{\alpha, \beta} \) with the symplectic structure \( i \sum_{j \geq 1} dz_j \wedge d\bar{z}_j \), where \( \omega = \xi(\bar{\omega}_1, \bar{\omega}_2) \) with \( \xi \in [\frac{1}{2}, \frac{3}{2}] \) and \( (\bar{\omega}_1, \bar{\omega}_2) \in WL(\chi, \tau, \beta) \). Let

\[ P(\omega t, z, \bar{z}; \xi) = \varepsilon \int_0^\pi F(\omega t, x, S_1(\frac{z + \bar{z}}{\sqrt{2}})) \, dx \]

with \( (z, \bar{z}) \in l^{\alpha, \beta} \times l^{\alpha, \beta} \), then the corresponding system is

\[
\begin{align*}
\dot{z}_j &= -i\mu_j z_j - i \partial_z P(\omega t, z, \bar{z}; \xi), \quad j \geq 1, \\
\dot{\bar{z}}_j &= i\mu_j \bar{z}_j + i \partial_{\bar{z}} P(\omega t, z, \bar{z}; \xi), \quad j \geq 1.
\end{align*}
\]

We then rewrite (21) as an autonomous system
\[
\begin{align*}
\dot{\theta} & = \omega_1, \\
\dot{\phi} & = \omega_2, \\
\dot{I} & = -\partial_\theta P(\theta, \phi, z, \bar{z}; \xi), \\
\dot{J} & = -\partial_\phi P(\theta, \phi, z, \bar{z}; \xi), \\
\dot{z}_j & = -i\mu_j z_j - i\partial_{z_j} P(\theta, \phi, z, \bar{z}; \xi), \quad j \geq 1, \\
\dot{\bar{z}}_j & = i\mu_j \bar{z}_j + i\partial_{\bar{z}_j} P(\theta, \phi, z, \bar{z}; \xi), \quad j \geq 1.
\end{align*}
\] (22)

To make the KAM iteration work fluently, we name \((\theta, \phi)\) as angle variable and equip \((\theta, \phi)\) with new action variable \((I, J)\). The perturbation is dependent on \(\theta, \phi, z, \bar{z}\) and independent on the action \(I\) and \(J\). Thus it is reasonable to consider an autonomous Hamiltonian system in the extended phase space \(\mathcal{P}^{a, \rho} = \mathbb{Z}^{2+d} \times \mathbb{R}^{2+d} \times l^{a, \rho} \times l^{a, \rho}\),

\[
\begin{align*}
\dot{\theta} & = \omega_1, \\
\dot{\phi} & = \omega_2, \\
\dot{I} & = -\partial_\theta P(\theta, \phi, z, \bar{z}; \xi), \\
\dot{J} & = -\partial_\phi P(\theta, \phi, z, \bar{z}; \xi), \\
\dot{z}_j & = -i\mu_j z_j - i\partial_{z_j} P(\theta, \phi, z, \bar{z}; \xi), \quad j \geq 1, \\
\dot{\bar{z}}_j & = i\mu_j \bar{z}_j + i\partial_{\bar{z}_j} P(\theta, \phi, z, \bar{z}; \xi), \quad j \geq 1.
\end{align*}
\] (23)

This system can be treated as a perturbed Hamiltonian

\[H = N + P = \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{j \geq 1} \mu_j |z_j|^2 + \varepsilon \int_0^\pi F(\theta, \phi, x, S_1(\frac{z + \bar{z}}{\sqrt{2}}); \xi) \, dx,\]

which is defined on \(D(r, s) \times \mathcal{O}\) with symplectic structure \(dI \wedge d\theta + dJ \wedge d\phi + i \sum_{j \geq 1} dz_j \wedge d\bar{z}_j\). We now verify that it satisfies all the assumptions of Theorem 3.1.

Recalling \(\mu_j = \sqrt{j^2 + m}\) and \(|m| < \frac{1}{2}\), one has

\[|\mu_j - j| = \frac{|m|}{\sqrt{j^2 + m} + j} < \frac{1}{2 + \sqrt{2}}\] (24)

The regularity of the perturbation is obtained by the following:

\[\partial_{z_j} P(\omega t, z, \bar{z}; \xi) = \frac{\varepsilon}{\sqrt{2\mu_j}} \int_0^\pi f(\omega t, x, \sum_{j \geq 1} \frac{z_j + \bar{z}_j}{\sqrt{2\mu_j}} \phi_j(x); \xi) \phi_j(x) \, dx,\]

it follows that \(\|X_P\|_{C_r,D_r(s)} \leq \varepsilon\).

Since \((\bar{\omega}_1, \bar{\omega}_2) \in \mathcal{W}\) by the assumption of Theorem 1.1, Theorem 3.1 is applicable if \(\varepsilon\) is small enough. Thus there exists a symplectic transformation \(\Phi\) such that the Hamiltonian (20) is transformed into

\[H^* = e^*(\theta; \xi) + \langle \omega_1, I \rangle + \langle \omega_2, J \rangle + \sum_{j \geq 1} (\Omega_j^* + B_j^*(\theta; \xi)) |z_j|^2 + P^*(\theta, \phi, z, \bar{z}; \xi),\]

with \(P^*(\theta, \phi, z, \bar{z}; \xi) = \sum_{|\alpha + \beta| \geq 3} P_{\alpha \beta}^*(\theta, \phi; \xi) z^\alpha \bar{z}^\beta\) and the corresponding system is
\[
\begin{align*}
\dot{\theta} &= \omega_1, \\
\dot{\phi} &= \omega_2, \\
\dot{j} &= -\partial_\theta (e^* (\theta; \xi) + \sum_{j \geq 1} B_j^*(\theta; \xi) |z_j|^2 + P^*(\theta, \varphi, z, \bar{z}; \xi)), \\
\dot{\bar{j}} &= -\partial_\varphi P^*(\theta, \varphi, z, \bar{z}; \xi), \\
\dot{z}_j &= -i \Omega_j^* z_j - i \partial_{\bar{z}_j} P^*(\theta, \varphi, z, \bar{z}; \xi), \quad j \geq 1, \\
\dot{\bar{z}_j} &= i \Omega_j^* \bar{z}_j + i \partial_z P^*(\theta, \varphi, z, \bar{z}; \xi), \quad j \geq 1.
\end{align*}
\]

(25)

By similar discussion as page 15 in [32], we obtain the system (21) has a quasi-periodic solution \((z(t), \bar{z}(t)) = (\Phi_z(\omega_1 t, \omega_2 t, 0, 0; \xi), \Phi_{\bar{z}}(\omega_1 t, \omega_2 t, 0, 0; \xi))\) for every \(\xi \in \mathcal{O}_\gamma\), and the equation (4) has a quasi-periodic solution

\[u(t, x) = \sum_{j \geq 1} \frac{1}{\sqrt{2\mu_j}} \left[ \Phi_{z_j}(\omega_1 t, \omega_2 t, 0, 0; \xi) + \Phi_{\bar{z}_j}(\omega_1 t, \omega_2 t, 0, 0; \xi) \right] \phi_j(x).
\]

The solution is \(C^\infty\) since the transformation \(\Phi\) is \(C^\infty\) in \((\theta, \varphi)\) and analytic in \(z, \bar{z}\). \(\Box\)

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