Heat kernel asymptotics for real powers of Laplacians

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Abstract. We describe the small-time heat kernel asymptotics of real powers $\Delta^r$, $r \in (0,1)$ of a non-negative self-adjoint generalized Laplacian $\Delta$ acting on the sections of a Hermitian vector bundle $E$ over a closed oriented manifold $M$. First, we treat separately the asymptotic on the diagonal of $M \times M$ and in a compact set away from it. Logarithmic terms appear only if $n$ is odd and $r$ is rational with even denominator. We prove the non-triviality of the coefficients appearing in the diagonal asymptotics, and also the non-locality of some of the coefficients. In the special case $r = 1/2$, we give a simultaneous formula by proving that the heat kernel of $\Delta^{1/2}$ is a polyhomogeneous conormal section in $E \boxtimes E^*$ on the standard blow-up space $M_{\text{heat}}$ of the diagonal at time $t = 0$ inside $[0, \infty) \times M \times M$.

1 Introduction

Let $\Delta$ be a self-adjoint generalized Laplacian acting on the sections of a Hermitian vector bundle $E$ over an oriented, compact Riemannian manifold $M$ of dimension $n$. Denote by $p_t$ the heat kernel of $\Delta$, i.e., the Schwartz kernel of the operator $e^{-t\Delta}$. It is known since Minakshisundaram–Pleijel [21] that $p_t(x, y)$ has an asymptotic expansion as $t \searrow 0$ near the diagonal

$$p_t(x, y) \sim t^{-n/2} e^{-d(x,y)^2/(4t)} \sum_{j=0}^{\infty} t^j \Psi_j(x, y),$$

where $d(x, y)$ is the geodesic distance between $x$ and $y$, and the $\Psi_j$’s are recursively defined as solutions of certain ODE’s along geodesics (see, e.g., [4, 5]). This asymptotic expansion applied to $D^* D$, where $D$ is a twisted Dirac operator, plays a leading role in the heat kernel proofs of the Atiyah–Singer index theorem (see [6, 7, 12]).

Bär and Moroianu [2] studied the short-time asymptotic behavior of the heat kernel of $\Delta^{1/m}$, $m \in \mathbb{N}^*$, for a strictly positive self-adjoint generalized Laplacian $\Delta$. They give explicit asymptotic formulæ separately in the case when $t \searrow 0$ along the diagonal $\text{Diag} \subset M \times M$, and when $t$ goes to 0 in a compact set away from the diagonal. The asymptotic behavior depends on the parity of the dimension $n$ and of the root $m$.

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More precisely, logarithmic terms appear when \( n \) is odd and \( m \) is even. They use the Legendre duplication formula, and the more general Gauss multiplication formula for the \( \Gamma \) function (see, e.g., [22]). Another crucial argument in [2] is to use integration by parts in order to show that the Schwartz kernel \( q_\Delta \), of the pseudodifferential operator \( \Delta^{-s}, s \in \mathbb{C} \), defines a meromorphic function when restricted to the diagonal in \( M \times M \).

1.1 Small-time heat asymptotic for real powers of \( \Delta \)

The purpose of this paper is to study the short-time asymptotic of the Schwartz kernel \( h_t \) of the operator \( e^{-t \Delta} \), where \( r \in (0,1) \) and \( \Delta \) is a non-negative self-adjoint generalized Laplacian, like, for instance, \( \Delta = D^* D \) for a Dirac operator \( D \). We give separate formulæ as \( t \) goes to 0 in \( [0, \infty) \times \text{Diag} \), and when \( t \searrow 0 \) in \( [0, \infty) \times K \), where \( K \subset M \times M \) is a compact set disjoint from the diagonal. In Theorem 6.1, we obtain that \( h_t|_{(0, \infty) \times K} \in t^{\circ} \mathcal{C}_0^\infty \left( [0, \infty) \times K \right) \) is a smooth function vanishing at least to order 1 at \( \{ t = 0 \} \). The asymptotic along the diagonal depends on the parity of \( n \) (like in [2]) and on the rationality of \( r \). In Theorem 7.1, the most interesting case occurs when logarithmic terms appear. This happens only if \( n \) is odd, \( r = \frac{\alpha}{\beta} \) is rational, and the denominator \( \beta \) is even. In that case,

\[
\begin{aligned}
   h_t|_{\text{Diag}} \sim t^{\gamma} \sum_{j=0}^{(n-1)/2} \frac{t^{-n-2j}}{\frac{\alpha}{2}} \cdot A_{n-2j} + \sum_{j=1}^{\infty} \frac{t^{2j+1}}{\frac{\alpha}{2} + j} \cdot A_{2j+1} \\
   + \sum_{j=1}^{\infty} t^j \cdot A_j + \sum_{\beta \mid j \text{ odd}}^{\infty} \alpha t^{l \frac{j}{\beta}} \log t \cdot B_l.
\end{aligned}
\]

Similar expansions are proved in Theorem 7.1 in all the other cases. Furthermore, we prove the non-triviality of the coefficients appearing in the diagonal asymptotics (Theorem 1.1), and also the non-locality of some of them (Theorem 1.3).

In the special case \( r = 1/2 \), Bär and Moroianu [2] described the small-time asymptotic behavior of \( h_t \) on the diagonal and away from it separately. In Theorem 1.4, we give an uniform description of the transition between the on- and off-diagonal behavior by proving that the heat kernel of \( \Delta^{1/2} \) is a polyhomogeneous conormal section in \( \mathcal{E} \boxtimes \mathcal{E}^* \) on the standard blow-up space \( [[0, \infty) \times M \times M, \{ t = 0 \} \times \text{Diag}] \).

1.2 Comparison to previous results

Fahrenwaldt [11] studied the off-diagonal short-time asymptotics of the heat kernel of \( e^{-t f(P)} \), where \( f : [0, \infty) \to [0, \infty) \) is a smooth function with certain properties, and \( P \) is a positive self-adjoint generalized Laplacian. The function \( f(x) = x^r, r \in (0,1) \) does not satisfy the third condition in [11, Hypothesis 3.3], which seems to be crucial for the arguments and statements in that paper, so the results of [11] do not seem to apply here.

Duistermaat and Guillemin [10] give the asymptotic expansion of the heat kernel of \( e^{-t P} \), where \( P \) is a scalar positive elliptic self-adjoint pseudodifferential operator. The order of \( P \) in [10] seems to be a positive integer. It is claimed in [1] that this asymptotic holds true in the context of fiber bundles. Furthermore, Grubb [16, Theorem 4.2.2]
studied the heat asymptotics for $e^{-tP}$ in the context of fiber bundles when the order of $P$ is positive, not necessary an integer. In Theorem 7.1, we obtain the vanishing of some terms appearing in [16, Corollary 4.2.7] in our particular case when $P = \Delta'$ is a real power of a self-adjoint non-negative generalized Laplacian $\Delta$, $r \in (0,1)$. We also show that the remaining terms do not vanish in general.

**Theorem 1.1** For each $r \in (0,1)$, none of the coefficients in the small-time asymptotic expansion of $h_1$ appearing in Theorem 7.1 vanishes identically for every generalized Laplacian $\Delta$.

The logarithmic coefficients $B_l$ and the coefficients $A_j$ for $j \notin \mathbb{Z}$ can be computed in terms of the heat coefficients for $e^{-t\Delta}$ appearing in (1.1). It is well known that the heat coefficients of a generalized Laplacian are locally computable in terms of the curvature of the connection on $E$, the Riemannian metric of $M$ and their derivatives (see, e.g., [5]). This is no longer the case for the coefficients of positive integer powers of $t$ from Theorem 7.1 as we shall see now.

By applying Theorem 7.1 for $r \in (0,1)$ and a set of geometric data, namely a hermitic vector bundle $E$ over an oriented, compact Riemannian manifold $(M, g)$, a metric connection $\nabla$ and an endomorphism $F \in \text{End} E$, $F^* = F$, we produce an endomorphism $A_l (M, g, E, h_E, \nabla, F) \in \mathcal{C}^\infty (M, \text{End} E)$ for each index $l$ appearing in (1.2).

**Definition 1.1** (i) We say that a function $A$ which associates to any set of geometric data $(M, g, E, h_E, \nabla, F)$ a section in $\mathcal{C}^\infty (M, \text{End} E)$ is locally computable if for any two sets of geometric data $(M, g, E, h_E, \nabla, F), (M', g', E', h_{E'}, \nabla', F')$ which agree on an open set (i.e., there exist an isometry $a : U \rightarrow U'$ between two open sets $U \subset M, U' \subset M'$, and a metric isomorphism $\beta : E_{|U} \rightarrow E'_{|U'}$, which preserves the connection and $\beta_x \circ F_x \circ \beta_a^{-1}(s) = F'_a(x)$, we have

$$\beta_x \circ A_x \circ \beta_a^{-1}(s) = A_a(x),$$

for any $x \in U$.

(ii) A scalar function $a$ defined on the set of all geometric data $(M, g, E, h_E, \nabla, F)$ with values in $\mathbb{C}$ is called locally computable if there exists a locally computable function $C$ as in (i) above such that $a = \int_M \text{Tr} C \, d\text{vol}_g$ for any $(M, g, E, h_E, \nabla, F)$.

(iii) A function $A$ as in (i) is called cohomologically locally computable if there exists a locally computable function $C$ as in (i) such that for any $(M, g, E, h_E, \nabla, F)$,

$$[\text{Tr} A \, d\text{vol}_g] = [\text{Tr} C \, d\text{vol}_g] \in H^n_{dR} (M).$$

**Remark 1.2** (i) If a function $A$ is locally computable, then the integral $a := \int_M \text{Tr} A \, d\text{vol}_g$ is locally computable.

(ii) A function $A$ is cohomologically locally computable if and only if $a := \int_M \text{Tr} A \, d\text{vol}_g$ is locally computable.

**Theorem 1.3** If $r$ is irrational, the heat coefficients $A_j$ in Theorem 7.1 (and in particular in (1.2)) are not locally computable for integer $j \geq 1$. If $r = \frac{a}{b}$ is rational, then $A_j$ are not locally computable for $j \in \mathbb{N} \setminus \{1 \beta : l \in \mathbb{N}\}$. All the other coefficients can be written in terms of the heat coefficients of $e^{-t\Delta}$, hence they are locally computable.
Consider the asymptotic expansion in [10, Corollary 2.2] for a scalar admissible operator, i.e., an elliptic, self-adjoint, positive pseudodifferential operator $P$ of positive integer order $d$:

$$e^{-tP} \sim \sum_{l=0}^{\infty} A_l(P) t^{(l-n)/d} + \sum_{k=1}^{\infty} B_k(P) t^k \log t.$$  

Gilkey and Grubb [14, Theorem 1.4] proved that the coefficients $a_l(P)$ for $l \geq 0$ and $b_k(P)$ for $k \geq 1$ from the corresponding small-time heat trace expansion

\begin{equation}
\text{Tr} e^{-tP} \sim \sum_{l=0}^{\infty} a_l(P) t^{(l-n)/d} + \sum_{k=1}^{\infty} b_k(P) t^k \log t
\end{equation}

are generically non-zero in the above class of admissible operators. In Theorem 1.1, we prove the same type of statement. However, in our case, the order of the operator $\Delta^r$ is $2r$; thus, it is integer only for $r = 1/2$. Even in this case, the non-vanishing result in Theorem 1.1 is not a consequence of [14, Theorem 1.4] since, in our case, we do not consider the whole class of admissible operators of fixed integer order $d$ in the sense of Gilkey and Grubb [14], but the smaller class of square roots of generalized Laplacians.

Furthermore, in [14, Theorem 1.7], it is proved that the coefficients $a_l(P)$ in (1.3) corresponding to $t^{(l-n)/d}$, for $(l-n)/d \in \mathbb{N}$, are not locally computable. Remark that the meaning of “locally computable” in [14] is different from Definition 1.1. More precisely, in the definition of Gilkey and Grubb, a locally computable function $A$ has to be a smooth function in the jets of the homogeneous components of the total symbol of the operator. A locally computable coefficient in the sense of Gilkey and Grubb [14] is clearly locally computable in the sense of Definition 1.1(ii).

For $r = 1/2$, Bär and Moroianu [2] remark that for odd $k = 1, 3, \ldots$, the coefficients $A_k$ in (1.2) corresponding to $t^k$ appear to be non-local. In Section 9, we clarify this remark by proving that they are indeed non-local in the sense of Definition 1.1 (i) (Theorem 1.3). In fact, we prove that the $A_k$’s are not cohomologically local. By Remark 1.2 (ii), it also follows that the integrals $a_k := \int_M \text{Tr} A_k \text{dvol}_g$ are not locally computable in the sense of Definition 1.1 (ii). Therefore, the $a_k$’s for odd $k$ are also not locally computable in the sense of Gilkey and Grubb [14].

For $d = 1$, the non-local coefficients in the heat expansion (1.3) in [14] are $a_{n+1}, a_{n+2}, \ldots$, whereas in our case corresponding to $r = d/2 = 1/2$, the non-local coefficients are $a_1, a_3, \ldots$. Despite some formal resemblances, it appears therefore that the results of the present paper are quite different from those of [14].

1.3 The heat kernel as a conormal section

Recall that a smooth function $f$ on the interior of a manifold with corners is said to be polyhomogeneous conormal if for any boundary hypersurface given by a boundary defining function $\theta$, $f$ has an expansion with terms of the form $\theta^l \log^k \theta$ toward $\{\theta = 0\}$ (only natural powers $l$ are allowed). In [19], Melrose introduced the heat space $M^2_H$ by performing a parabolic blow-up of the diagonal in $M \times M$ at time $t = 0$. The new space is a manifold with corners with boundary hypersurfaces given by the boundary defining functions $\rho$ and $\omega_0$. Then the heat kernel $p_t$ has the form $\rho^{-n} C^\infty(M^2_H)$, and it vanishes rapidly at $\{\omega_0 = 0\}$ (see [19, Theorem 7.12]).
In the special case $r = 1/2$, we are able to give a simultaneous formula for the asymptotic behavior of $h_t$ as $t$ goes to zero both on the diagonal and away from it. We can understand better the heat operator $e^{-t \Delta^{1/2}}$ on a homogeneous (rather than parabolic) blow-up heat space $M_{\text{heat}}$, the usual blow-up of $\{0\} \times \text{Diag in } [0, \infty) \times M \times M$. The new added face is called the front face and we denote it $ff$, whereas the lift of the old boundary is the lateral boundary, denoted $lb$.

**Theorem 1.4** If $n$ is even, then the Schwartz kernel $h_t$ of the operator $e^{-t \Delta^{1/2}}$ belongs to $\rho^{-n} \omega_0 \cdot \mathcal{C}^\infty(M_{\text{heat}})$, while if $n$ is odd, $h_t \in \rho^{-n} \omega_0 \cdot \mathcal{C}^\infty(M_{\text{heat}}) + \rho \log \rho \cdot \omega_0 \cdot \mathcal{C}^\infty(M_{\text{heat}})$.

Theorem 1.4 improves the results of [2] twofold. First, it holds true for non-negative generalized Laplacians. Second, while Bär–Moroianu describe the asymptotic behavior of $h_t$ on the diagonal and away from it separately, this theorem also gives a precise, uniform description of the transition between these two regions by showing that $h_t$ is a polyhomogeneous conormal section on $M_{\text{heat}}$ with values in $\mathcal{E} \otimes \mathcal{E}^*$. 

Note that throughout the paper, integral kernels act on sections by integration with respect to the fixed Riemannian density from $M$ in the second variable, so $h_t$ does not contain a density factor. We feel that in the present context this exhibits more clearly the asymptotic behavior.

Based on the study of the case $r = 1/2$ and on the separate asymptotic expansions of the heat kernel $h_t$ of $\Delta^r$, $r \in (0, 1)$ as $t$ goes to 0 given in Theorems 6.1 and 7.1, we can conjecture that the heat kernel $h_t$ is a polyhomogeneous conormal function for all $r \in (0, 1)$ on a “transcendental” heat blow-up space $M^r_{\text{heat}}$, depending on $r$. We leave this as a future project.

## 2 The heat kernel of a generalized Laplacian

Let $\mathcal{E}$ be a Hermitian vector bundle over a compact Riemannian manifold $M$ of dimension $n$. Consider $\Delta$ to be a generalized Laplacian, i.e., a second-order differential operator which satisfies

$$\sigma_2(\Delta)(x, \xi) = |\xi|^2 \cdot \text{id}_\mathcal{E}.$$ 

For example, if $\nabla$ is a connection on $\mathcal{E}$ and $F \in \Gamma(\text{End } \mathcal{E})$, $F^* = F$, then $\nabla^* \nabla + F$ is a symmetric generalized Laplacian on $\mathcal{E}$.

Suppose that $\Delta$ is self-adjoint. Since $M$ is compact, the spectrum of $\Delta$ is discrete and $L^2(M, \mathcal{E})$ splits as an orthogonal Hilbert direct sum

$$L^2(M, \mathcal{E}) = \bigoplus_{\lambda \in \text{Spec } \Delta} E_\lambda,$$ 

where $E_\lambda$ is the eigenspace corresponding to the eigenvalue $\lambda$ of $\Delta$. Moreover, $\dim E_\lambda < \infty$ and by elliptic regularity, the eigensections are smooth (see, e.g., [8]). Let $e^{-t \Delta}$ be the heat operator defined as

$$e^{-t \Delta} \Phi = e^{-t \lambda} \Phi,$$ 

for any $\Phi \in E_\lambda, \lambda \in \text{Spec } \Delta$. 
Definition 2.1 The heat kernel of a self-adjoint elliptic pseudodifferential operator $P$ acting on the sections of $\mathcal{E}$ is the Schwartz kernel of the operator $e^{-tP}$.

If we denote by $\{\Phi_j\}$ an orthonormal Hilbert basis of $\Delta$-eigensections, then the heat kernel $p_t(x, y)$ satisfies

$$p_t(x, y) = \sum_j e^{-\lambda_j t} \Phi_j(x) \otimes \Phi^*_j(y)$$

in $C^\infty((0, \infty) \times M \times M)$.

Recall that the $L^2$-product of two sections $s_1, s_2 \in \Gamma(\mathcal{E})$ is given by

$$\langle s_1, s_2 \rangle_{L^2(\mathcal{E})} = \int_M h_\mathcal{E}(s_1, s_2) \, d\text{vol}_g,$$

where $g$ is the metric on $M$ and $h_\mathcal{E}$ is the Hermitian product on $\mathcal{E}$.

Let $y \in M$ be a fixed point. We work in geodesic normal coordinates defined by the exponential map

$$\exp_y : T_y M \rightarrow M.$$ 

Since $M$ is compact, there exists a global injectivity radius $\varepsilon$. For $x$ close enough to $y$ ($d(x, y) \leq \varepsilon$), take $x \in T_y M$ the unique tangent vector of length smaller than $\varepsilon$ such that $x = \exp_y x$. Let

$$j(x) = \frac{\exp_y^* dx}{dx},$$

namely the pull-back of the volume form $dx$ on $M$ through the exponential map $\exp_y$ is equal with $j(x)dx$. More precisely,

$$j(x) = |\det (d_x \exp_{x_0})| = \det^{1/2} (g_{ij}(x)).$$

Denote by $\tau^y_x : \mathcal{E}_x \rightarrow \mathcal{E}_y$ the parallel transport along the unique minimal geodesic $x_s = \exp_y(s x)$, where $s \in [0, 1]$, which connects the points $x$ and $y$. The heat kernel $p_t(x, y)$ belongs to the space $C^\infty((0, \infty) \times M \times M, \mathcal{E}_x \otimes \mathcal{E}^*_y)$ and $p_t(x, y)$ satisfies the heat equation

$$(\partial_t + \Delta_x) p_t(x, y) = 0.$$ 

Furthermore, $\lim_{t \to 0} P_t s = s$, in $\|\cdot\|_0$, for any smooth section $s \in \Gamma(M, \mathcal{E})$, where

$$(P_t s)(x) = \int_M p_t(x, y)s(y) \, dg(y),$$

where $dg(y)$ is the Riemannian density of the metric $g$. The next theorem is due to Minakshisundaram and Pleijel (see, for instance, [4, 21]).

Theorem 2.1 The heat kernel $p_t$ has the following asymptotic expansion near the diagonal:

$$p_t(x, y) \overset{t \to 0}{\sim} (4\pi t)^{-n/2} e^{-d(x, y)^2/4t} \sum_{i=0}^{\infty} t^i \Psi_i(x, y),$$

where $\Psi_i(x, y)$ are certain functions.
where \( \Psi_i : \mathcal{E}_x \rightarrow \mathcal{E}_x \) are \( \mathcal{C}^\infty \) sections defined near the diagonal. Moreover, the \( \Psi_i \)'s are given by the following explicit formulae:

\[
\Psi_0(x, y) = j^{-1/2}(x) \tau_y^x,
\]

\[
\tau_x^y \Psi_i(x, y) = -j^{-1/2}(x) \int_0^1 s^i j^{-1/2}(x_s) \Delta_x \Psi_{i-1}(x_s, y) ds.
\]

The asymptotic sum in Theorem 2.1 can be understood using truncation and bounds of derivatives as in [5]. We prefer the interpretation given in [19], where the heat kernel \( p_t \) is shown to belong to \( \rho^{-n}\mathcal{C}^\infty(M_H^2) \) on the parabolic blow-up space \( M_H^2 \) and to vanish rapidly at the temporal boundary face \( \{ \omega_0 = 0 \} \) (see Section 10).

**Example 2.2** Let \( \mathbb{T}^n = \left(S^1\right)^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n \) be the \( n \)-dimensional torus with the standard product metric \( g = d\theta_1^2 \otimes \cdots \otimes d\theta_n^2 \). Consider the trivial bundle \( \mathcal{E} = \mathbb{C} \) over \( \mathbb{T}^n \) with the standard metric \( h_\mathcal{E} \), the trivial connection \( \nabla = d \), and the zero endomorphism \( F \). Let \( \Delta_t \) be the Laplacian on \( \mathbb{T}^n \) given by the metric \( g \). The eigenvalues of \( \Delta_t \) are \( \{ k_1^2 + \cdots + k_n^2 : k_1, \ldots, k_n \in \mathbb{Z} \} \). Let \( \varphi_1(\xi) = \frac{1}{\sqrt{2\pi}} e^{i\xi t} \) be the standard orthonormal basis of eigenfunctions of each \( \Delta_{S^1} \). Then, for \( \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n \), the heat kernel \( p_t \) of \( \Delta_1 \) is the following:

\[
p_t(\theta, \theta) = \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n} e^{-t(k_1^2 + \cdots + k_n^2)} \varphi_{k_1}(\theta_1) \varphi_{k_2}(\theta_2) \cdots \varphi_{k_n}(\theta_n)
\]

Since \( \varphi_1(\xi) \overline{\varphi_1(\xi)} = \frac{1}{2\pi} \), for any \( \xi \in S^1 \), we get

\[
p_t(\theta, \theta) = \frac{1}{(2\pi)^n} \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n} e^{-t(k_1^2 + \cdots + k_n^2)}
\]

Remark that the Fourier transform of the function \( f_t : \mathbb{R}^n \rightarrow \mathbb{R} \), \( f_t(x) = e^{-t|x|^2} \) is given by

\[
\hat{f}_t(\xi) = \frac{n^{n/2}}{\pi^n} e^{-\frac{|\xi|^2}{4t}}
\]

Using the multidimensional Poisson formula (see, for instance, [3]), we obtain that

\[
p_t(\theta, \theta) = \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}^n} \hat{f}_t(2\pi k) = \frac{n^{n/2}}{(2\pi)^n} t^{-n/2} + \frac{n^{n/2}}{(2\pi)^n} t^{-n/2} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} e^{-\frac{\pi^2|k|^2}{4t}}
\]

Since the last sum is of order \( \Theta \left( e^{-\frac{1}{t}} \right) \) as \( t \rightarrow 0 \), it follows that the first coefficient in the asymptotic expansion at small-time \( t \) of \( p_t \) is \( \frac{n^{n/2}}{(2\pi)^n} t^{-n/2} \) and all the others vanish.

From now on, suppose that \( \Delta \) is non-negative (i.e., \( h_\mathcal{E} (\Delta f, f) \geq 0 \), for any \( f \in \mathcal{C}^\infty(M, \mathcal{E}) \)). For \( s \in \mathbb{C} \), we define the complex powers \( \Delta^{-s} \in \mathcal{V}^{-2s}(M, \mathcal{E}) \) of \( \Delta \) as

\[
\Delta^{-s} \Phi = \begin{cases}
\lambda^{-s} \Phi, & \text{if } \Phi \in E_\lambda, \ \lambda \neq 0, \\
0, & \text{if } \Phi \in \text{Ker } \Delta.
\end{cases}
\]

Remark that \( (\Delta^s)_{s \in \mathbb{C}} \) is a holomorphic family of pseudodifferential operators. Let \( r \in (0, 1) \). We denote by \( h_t \) the heat kernel of \( \Delta^r \), namely the Schwartz kernel of the
operator $e^{-t\Delta}$. We have seen that

$$(2.1) \quad p_t(x, x) \overset{t \to 0}{\sim} t^{-n/2} \sum_{j=0}^{\infty} t^j a_j(x, x),$$

with smooth sections $a_j(x, x) \in E_x \otimes E^*_x$.

3 The link between the heat kernel and complex powers of the Laplacian

**Proposition 1** (Mellin Formula)  With the notations above, for $\Re s > 0$, we have

$$\Delta^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( e^{-t\Delta} - P_{\text{Ker} \Delta} \right) dt,$$

where $P_{\text{Ker} \Delta}$ is the orthogonal projection onto the kernel of $\Delta$.

**Proof**  It is straightforward to check that both sides coincide on eigensections $\Phi \in E_\lambda$, $\lambda \in \text{Spec} \Delta$. Since $\{\Phi_j\}_j$ is a Hilbert basis, the result follows.

We will write $P_{\text{Ker} \Delta}(x, y)$ for the Schwartz kernel $\sum_k \varphi_k(x) \otimes \varphi_k^*(y)$, where $\{\varphi_k\}$ is an orthonormal basis in $\text{Ker} \Delta$. Denote by $q_{-s}$ the Schwartz kernel of the operator $\Delta^{-s}$. Let us first study the poles and the zeros of $q_{-s}$ away from the diagonal.

**Proposition 2**  Let $K$ be a compact in $M \times M \setminus \text{Diag}$. Then, for $(x, y) \in K$, the function $s \mapsto q_{-s}|_K \in \mathcal{C}^\infty(K, E \otimes E^*)$ is entire. Moreover, $q_{-s}|_K$ vanishes at each negative integer $s$.

**Proof**  For $\Re s > 0$, let $f_{x,y}(s) = \int_0^\infty t^{s-1} (p_t(x, y) - P_{\text{Ker} \Delta}(x, y)) dt$. Remark that

$$f_{x,y}(s) = \int_0^\infty t^{s-1} (p_t(x, y) - P_{\text{Ker} \Delta}(x, y)) dt$$

$$= \int_1^\infty t^{s-1} (p_t(x, y) - P_{\text{Ker} \Delta}(x, y)) dt$$

$$+ \int_0^1 t^{s-1} p_t(x, y) dt - P_{\text{Ker} \Delta}(x, y) \int_0^1 t^{s-1} dt.$$ 

Since $p_t(x, y) - P_{\text{Ker} \Delta}(x, y)$ decays exponentially fast as $t$ goes to $\infty$, the first integral is absolutely convergent in $C^k$ norms. The heat kernel $p_t$ vanishes with all of its derivatives as $t \to 0$ in the compact $K$, thus the second integral is also absolutely convergent. The last integral term is well-defined for $\Re s > 0$, and it extends to a meromorphic function on $\mathbb{C}$ with a simple pole in $s = 0$. Therefore, $s \mapsto f_{x,y}(s)$ extends to a meromorphic function on $\mathbb{C}$. By Proposition 1 and the identity theorem, the equality of meromorphic functions

$$\Gamma(s) q_{-s}(x, y) = f_{x,y}(s)$$

holds for any $s \in \mathbb{C}$. In particular, we obtain $q_{0}(x, y) = -P_{\text{Ker} \Delta}(x, y)$. Furthermore, $q_{-s}|_K$ is an entire function and vanishes in $s = -1, -2, \ldots$.

**Remark 3.1**  The fact that $q_{-s}|_K$ vanishes for negative integers $s$ also follows from the fact that then $\Delta^{-s}$ is a differential operator.
Now we check the behavior of $q_{-s}$ along the diagonal. It is no longer holomorphic there, and the coefficients $a_j(x, x)$ from (2.1) appear as residues of $q_{-s}(x, x)$.

**Proposition 3** Let $x \in M$. Then the function $s \mapsto \Gamma(s)q_{-s}(x, x)$ has a meromorphic extension from the set $\{ s \in \mathbb{C} : \Re s > \frac{n}{2} \}$ to $\mathbb{C}$ with simple poles in $s \in \{ 0 \} \cup \{ \frac{n}{2} - j : j \in \mathbb{N} \}$. The residue of $\Gamma(s)q_{-s}(x, x)$ in $s = \frac{n}{2} - j$, $j \neq \frac{n}{2}$, is $a_j(x, x)$. If $n$ is even, then the residue of $\Gamma(s)q_{-s}(x, x)$ in $s = 0$ is $a_{\frac{n}{2}}(x, x) - P_{\Ker \Delta}(x, x)$. If $n$ is odd, the residue in $s = 0$ is $- P_{\Ker \Delta}(x, x)$ and the meromorphic extension of $q_{-s}(x, x)$ vanishes at $s \in \{-1, -2, \ldots \}$.

**Proof** Consider the function $f_{x, x}(s) = \int_0^\infty t^{s-1} (p_t(x, x) - P_{\Ker \Delta}(x, x)) \, dt$ for $\Re s > \frac{n}{2}$. We have

$$f_{x, x}(s) = \int_0^\infty t^{s-1} (p_t(x, x) - P_{\Ker \Delta}(x, x)) \, dt$$

$$= \int_1^\infty t^{s-1} (p_t(x, x) - P_{\Ker \Delta}(x, x)) \, dt$$

$$+ \int_0^1 t^{s-1} p_t(x, x) \, dt - P_{\Ker \Delta}(x, x) \cdot \int_0^1 t^{s-1} \, dt.$$

The first integral is absolutely convergent, as seen in the proof of Proposition 2. The last integral term is meromorphic with a simple pole at $s = 0$ with residue $- P_{\Ker \Delta}(x, x)$.

Let us analyze the behavior of the second term $A_x(s) = \int_0^1 t^{s-1} p_t(x, x) \, dt$.

Using (2.1), we have that for $N \geq 0$,

$$t^{n/2} p_t(x, x) = \sum_{j=0}^N t^j a_j(x, x) + R_{N+1}(t, x),$$

where $R_{N+1}$ is of order $O(t^{N+1})$ as $t \to 0$. Furthermore, we obtain

$$A_x(s) = \int_0^1 t^{s-\frac{n}{2}-1} t^{\frac{n}{2}} p_t(x, x) \, dt = \sum_{j=0}^N \int_0^1 t^{s-\frac{n}{2}-1} t^j a_j(x, x) \, dt + \int_0^1 t^{s-\frac{n}{2}-1} R_{N+1}(t, x) \, dt$$

$$= \sum_{j=0}^N a_j(x, x) \frac{1}{s - \frac{n}{2} + j} + \int_0^1 t^{s-\frac{n}{2}-1} R_{N+1}(t, x) \, dt.$$

Thus $s \mapsto A_x(s)$ extends to a meromorphic function on $\mathbb{C}$ with simple poles in $\{ \frac{n}{2} - j : j = 0, N+1 \}$. Using again Proposition 1 and the identity theorem, we deduce the equality

$$\Gamma(s)q_{-s}(x, x) = f_{x, x}(s),$$

for any $s \in \mathbb{C}$. It follows that $\Gamma(s)q_{-s}(x, x)$ is meromorphic on $\mathbb{C}$ with simple poles in $s \in \{ 0 \} \cup \{ \frac{n}{2} - j : j \in \mathbb{N} \}$. Moreover, the residue of $\Gamma(s)q_{-s}(x, x)$ in a pole $\frac{n}{2} - j$ is $a_j(x, x)$, and the conclusion follows. ■

For $p \in \mathbb{C}$ and $\varepsilon > 0$, let $B_{\varepsilon}(p)$ be the open disk centered in $p$ of radius $\varepsilon$. We need the following technical result.
Proposition 4  Consider $\alpha < \beta$, and let $\varepsilon > 0$, $l \in \mathbb{N}$.

- If $K$ is a compact set disjoint from the diagonal, then the function $s \mapsto \Gamma(s)q_{-\varepsilon|_{K}}$ is uniformly bounded in $\{ s \in \mathbb{C} : \alpha \leq \Re s \leq \beta \} \setminus B_{\varepsilon}(0)$ in the $\mathcal{C}^{l}$ norm on $K$.
- The function $s \mapsto \Gamma(s)q_{-\varepsilon|_{\text{Diag}}}$ defined on $\{ s \in \mathbb{C} : \alpha \leq \Re s \leq \beta \} \setminus \bigcup_{j \in \mathbb{N} \cup \{ \frac{1}{2} \}} B_{\varepsilon}\left(\frac{n}{2} - j\right) \rightarrow \mathcal{C}^{l}(\text{Diag}, \mathcal{E} \otimes \mathcal{E}^{*})$ is uniformly bounded.

Proof  With the same argument as in the proof of Proposition 2, the restriction of the $\mathcal{C}^{l}$ norm on $K$ of the function $s \mapsto f_{x,\varepsilon}(s)$ is absolutely convergent in $\{ s \in \mathbb{C} : \alpha \leq \Re s \leq \beta \} \setminus B_{\varepsilon}(0)$, hence it is uniformly bounded.

As in the proof of Proposition 3, the $\mathcal{C}^{l}$ norm along Diag of $s \mapsto f_{x,\varepsilon}(s)$ converges absolutely in $\{ s \in \mathbb{C} : \alpha \leq \Re s \leq \beta \} \setminus \bigcup_{j \in \mathbb{N} \cup \{ \frac{1}{2} \}} B_{\varepsilon}\left(\frac{n}{2} - j\right)$, thus the conclusion follows.

4  The behavior of quotients of Gamma functions along vertical lines

A fundamental result used in [2] is the Legendre duplication formula

$$\frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)} = \frac{1}{\sqrt{2\pi}} 2^{s-\frac{3}{2}} \Gamma\left(\frac{s+1}{2}\right),$$

together with the rapid decay of the Gamma function in vertical lines $\Re s = \tau$ (see, e.g., [22]). These results are replaced in our case by the following estimate.

Proposition 5  The function $s \mapsto \frac{\Gamma(s)}{\Gamma(rs)}$ decreases in vertical lines faster than $|s|^{-k}$, for any $k \geq 0$, uniformly in each strip $\{ s \in \mathbb{C} : \alpha \leq \Re s \leq \beta \}$, for any $\alpha, \beta \in \mathbb{R}$.

Proof  For $z \in \mathbb{C}\setminus\mathbb{R}$, recall the Stirling formula (see, for instance, [23])

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - \frac{1}{2} \log(2\pi) + \Omega(z),$$

where $\log$ is defined on its principal branch, and $\Omega$ is an analytic function of $z$. For $|\arg z| < \pi$ and $|z| \to \infty$, $\Omega$ can be written as

$$\Omega(z) = \sum_{j=1}^{N-1} \frac{B_{2j}}{2j(2j-1)z^{2j-1}} + R_{N}(z),$$

where $B_{2j}$ are the Bernoulli numbers ($B_{2} = \frac{1}{6}$, $B_{4} = -\frac{1}{30}$, $B_{6} = \frac{1}{42}$, etc.). Moreover, the error term satisfies

$$|R_{N}(z)| \leq \frac{|B_{2N}|}{2N(2N-1)} \frac{\sec^{2N}(\frac{\arg z}{2})}{|z|^{2N-1}};$$

thus, $R_{N}(z)$ is of order $O\left(|z|^{-2N+1}\right)$ as $|z| \to \infty$ (see, for instance, [22, equation (2.1.6)]). For $s \notin (-\infty, 0)$, it follows that

$$\frac{\Gamma(s)}{\Gamma(rs)} = s^{r-1}e^{s\arg z} e^{\frac{1}{2} - rs} \Omega(s) - \Omega(rs).$$
Let \( s = a + ib, \ a \in \mathbb{R} \) fixed. As \(|b| \to \infty\), the difference \(|\Omega(s) - \Omega(rs)| \to 0\); thus, \(|e^{\Omega(s)} - e^{\Omega(rs)}| \to 1\). Note that \(|r^{\frac{2}{r} - rs}| = |r^{\frac{2}{r} - r}|\) and \(e^{(r-1)s} = e^{(r-1)a}\), so these terms are bounded. We show in Lemma 4.1 that for any \( k \geq 0 \), \(|s|^{k}|s^{r}|\) goes to 0 as \( R s = a \) is fixed and \(|\text{Im}\ s|\) tends to \( \infty\). It follows that the quotient \( \frac{\Gamma(s)}{\Gamma(rs)}\) indeed decreases in vertical lines faster than \(|s|^{-k}\), for any \( k \geq 0\), uniformly in vertical strips. \hfill \blacksquare

**Lemma 4.1** Let \( k \geq 0 \). If \( a \in \mathbb{R} \) is fixed and \(|b| \to \infty\), then \(|(a + ib)^{k+a+ib}|\) tends to zero.

**Proof** Let \( s = a + ib \notin (-\infty, 0) \) and set \( \log(a + ib) = x + iy \). Then \( x = \log \sqrt{a^2 + b^2}\), \( y = \text{arg}\ s \in (-\pi, \pi)\); hence,

\[
|s^{k}| = |e^{(k+a+ib)\log(a+ib)}| = e^{(k+a)x - by} = e^{(k+a)\log \sqrt{a^2 + b^2} - b\text{arg}\ s}.
\]

Since \( b = \tan \text{arg} s \cdot a\), the exponent is equal to

\[
(4.1) \quad (k + a) \log \sqrt{a^2 + b^2} - b\text{arg}\ s = (k + a) \log a + \frac{k + a}{2} \log (1 + \tan^2 \text{arg} s) - a\tan \text{arg} s \cdot \text{arg} s.
\]

If \( a > 0\), then \( \text{arg} s \neq \frac{\pi}{2} \) or \( \text{arg} s \neq -\frac{\pi}{2} \), and in both cases \( t := \tan \text{arg} s \) tends to \( \infty\). The exponent \((4.1)\) behaves as the function \( t \to \log(1 + t^2) - t\); therefore, as \( t \to \infty\), the exponent goes to \(-\infty\) and the statement of the claim follows.

If \( a < 0\), then \( \text{arg} s \neq \frac{\pi}{2} \) or \( \text{arg} s \neq -\frac{\pi}{2} \). In the first case when \( \text{arg} s \neq \frac{\pi}{2} \), it follows that \( t = \tan \text{arg} s \to -\infty\). The exponent \((4.1)\) behaves as \( \pm \log(1 + t^2) + t\); hence, the conclusion follows. While if \( \text{arg} s \neq -\frac{\pi}{2} \), then \( t \to \infty\), and the exponent \((4.1)\) behaves as \( \pm \log(1 + t^2) - t\); thus, the exponent tends again to \(-\infty\). Therefore, \(|s^{k+s}|\) goes to zero, which ends the proof. \hfill \blacksquare

5 **Link between the complex powers of \( \Delta \) and the heat kernel of \( \Delta^r \)**

**Proposition 6** (Inverse Mellin Formula) For \( R \tau > 0 \), the operators \( e^{-t\Delta^r} \) and \( \Delta^{-s} \) are related by the following formula:

\[
e^{-t\Delta^r} - \text{P}_{\text{Ker} \Delta} = \frac{1}{2\pi i} \int_{\text{Re} s = \tau} t^{-s} \Gamma(s) \Delta^{-rs} \, ds.
\]

**Proof** The equality holds on each eigensection \( \Phi_j \) corresponding to an eigenvalue \( \lambda_j \in \text{Spec} \Delta \). Since \( \{\Phi_j\}_j \) is a Hilbert basis, the result follows. \hfill \blacksquare

Set \( \tau > \frac{n}{2r} \). Then the Schwartz kernel \( q_{-rs} \) of \( \Delta^{-rs} \) is continuous and by the inverse Mellin formula, we get an identity which relates the Schwartz kernels \( h_t \) and \( q_{-rs} \):

\[
h_t(x, y) - \text{P}_{\text{Ker} \Delta}(x, y) = \frac{1}{2\pi i} \int_{\text{Re} s = \tau} t^{-s} \Gamma(s) q_{-rs}(x, y) \, ds
\]

\[
= \frac{1}{2\pi i} \int_{\text{Re} s = \tau} t^{-s} \frac{\Gamma(s)}{\Gamma(rs)} \cdot \Gamma(rs) q_{-rs}(x, y) \, ds.
\]

Now let \( k > 0 \). By changing \( \tau \) to \( \tau + \varepsilon \) (for a small \( \varepsilon > 0 \)) if needed, we can assume that \( \tau - k \notin \left\{ \frac{n}{2} - j : j \in \mathbb{N} \right\} \cup \{0\} \). Using Propositions 4 and 5, we can apply the residue
formula and move the line of integration to the left:

\[
h_t(x, y) = \frac{1}{2\pi i} \int_{\mathbb{C}} t^{-s} \frac{\Gamma(s)}{\Gamma(rs)} \cdot \Gamma(rs) q_{rs}(x, y) ds + \sum_{s \in -\mathbb{N} \cup \{ \frac{n-2j}{2r} : j \in \mathbb{N} \}} \text{Res}_{s} \left( t^{-s} \Gamma(s) q_{rs}(x, y) \right) + P_{\text{Ker} \Delta}(x, y).
\]

(5.1)

Notice that \(-\mathbb{N} \cup \{ \frac{n-2j}{2r} : j \in \mathbb{N} \}\) is the set of all possible poles of \(s \mapsto \Gamma(s) q_{rs}(x, y)\), but some of them might actually be regular points. We will study the sum (5.1) in detail in Theorems 6.1 and 7.1.

Let \(K\) be a compact set in \(M \times M \setminus \text{Diag}\) and \(l \in \mathbb{N}\). Remark that the integral term in (5.1) is of order \(O\left(t^{k-\epsilon}\right)\) in \(C^l(K, \mathcal{E} \otimes E^*)\). Indeed,

\[
\left\| \int_{\mathbb{C}} t^{-s} \frac{\Gamma(s)}{\Gamma(rs)} q_{rs} ds \right\| _{l} \leq t^{-\tau+k} \cdot \int_{s=\tau-k}^{\tau} \left\| \frac{\Gamma(s)}{\Gamma(rs)} \cdot \Gamma(rs) q_{rs}\right\| _{l} ds,
\]

and using again Propositions 4 and 5, the claim follows. Furthermore, when \(k\) goes to \(\infty\), we get

\[
h_{l}|_{k} \sim_{t} 0 \sum_{a=0}^{\infty} t^{a} \cdot \text{Res}_{s=-a} \left( \Gamma(s) q_{rs}\right) + t^{0} \cdot P_{\text{Ker} \Delta}|_{k},
\]

(5.2)

The meaning of the asymptotic sign in (5.2) is that if we set \(h_{l}^{N}|_{k}\) to be the right-hand side in (5.2) restricted to \(a \leq N\), then the difference \(|\partial_{l}^{i} (h_{l}|_{k} - h_{l}^{N}|_{k})|\) is of order \(O(t^{N+1-j})\) in \(C^l(K, \mathcal{E} \otimes E^*)\), for any \(N, j \in \mathbb{N}\).

Remark that using again Propositions 4 and 5, the integral term in (5.1) is of order \(O\left(t^{k-\epsilon}\right)\) in \(C^l(\text{Diag}, \mathcal{E} \otimes E^*)\). Therefore when \(k\) tends to \(\infty\), we obtain

\[
h_{l}|_{\text{Diag}} \sim_{t} 0 \sum_{a \in (-N) \cup \{ \frac{n-2j}{2r} : j \in \mathbb{N} \}} t^{-a} \cdot \text{Res}_{s=a} \left( \Gamma(s) q_{rs}\right) + t^{0} \cdot P_{\text{Ker} \Delta}|_{\text{Diag}},
\]

(5.3)

in the sense of the following:

**Definition 5.1** Consider \(l \in \mathbb{N}\) and let \(A, B \subset \mathbb{R}\). We say that \(h_{l}|_{\text{Diag}} \sim t^{0} \sum_{a \in A} t^{a} c_{a} + \sum_{\beta \in B} t^{\beta} \log t \cdot c_{\beta}\) if for any \(k, N \in \mathbb{N}\), the difference

\[
\partial_{l}^{i} \left( h_{l}|_{\text{Diag}} - \sum_{a \leq N} t^{a} c_{a} - \sum_{\beta \leq N} t^{\beta} \log t \cdot c_{\beta}\right)
\]

is of order \(O(t^{N+1-j} \log t)\) in \(C^l(\text{Diag}, \mathcal{E} \otimes E^*)\).

### 6 The asymptotic expansion of \(h_t\) away from the diagonal

**Theorem 6.1** The Schwartz kernel \(h_t\) of the operator \(e^{-t \Delta'}\) is \(C^\infty\) on \([0, \infty) \times (M \times M \setminus \text{Diag})\). Furthermore, let \(K \subset M \times M \setminus \text{Diag}\) be a compact set. Then the Taylor series of \(h_t|_K\) as \(t \searrow 0\) is the following:

\[
h_{t}|_{K} \sim_{t} 0 \sum_{j=1}^{\infty} t^{j} q_{rj}|_{K} (-1)^{j} \frac{j!}{j!}.
\]
Moreover, if $r = \frac{\alpha}{\beta}$ is rational with $\alpha, \beta$ coprime, then the coefficient of $t^j$ vanishes for $j \in \beta \mathbb{N}^*$.

**Proof** Let $j \in \mathbb{N}$. Using Propositions 4 and 5, $(-s)(-s-1)\ldots(-s-j+1)t^{-s-j} \frac{\Gamma(s)}{\Gamma(rs)} \Gamma(rs) q_{-rs|k}$ is $L^1$ integrable on $\Re s = \tau - k$ in $C^1(K, E \otimes E^*)$, for sufficiently large $k$ and for any $l \in \mathbb{N}$. It follows that $h_l$ is $C^\infty$ on $(0, \infty) \times (M \times M \setminus \text{Diag})$. By Proposition 2, the function $s \mapsto q_{-rs}(x, y)$ is entire for any $(x, y) \in K$. Since $\text{Res}_{s=-j} \Gamma(s) = \frac{(-1)^j}{j!}$, using (5.2) we get

$$h_{t|k} \sim 0 \sum_{j=0}^\infty t^j q_{jr|k} \frac{(-1)^j}{j!} + p_{\text{Ker} \Delta|k}.$$  

We obtained in the proof of Proposition 2 that $q_{0|k} = -p_{\text{Ker} \Delta|k}$; thus,

$$h_{t|k} \sim 0 \sum_{j=1}^\infty t^j q_{jr|k} \frac{(-1)^j}{j!},$$

and therefore $h_{t|k}$ is $C^\infty$ also at $t = 0$, and vanishes at order 1. Moreover, using again Proposition 2, if $r = \frac{\alpha}{\beta}$ is rational and $j$ is a non-zero multiple of $\beta$, then $q_{jr|k} \equiv 0$ and the conclusion follows. 

### 7 The asymptotic expansion of $h_t$ along the diagonal

To obtain the coefficients in the asymptotic of $h_t$ along the diagonal as $t \searrow 0$, we need to compute the residues from (5.3). Some of them are related to the heat coefficients $a_j$’s of $p_t$ due to Proposition 3. We will distinguish three cases. If $n$ is even, $\Gamma(s) q_{-rs}(x)$ has simple poles in $\left\{ \frac{n}{2r}, \frac{n-2}{2r}, \ldots, \frac{2}{2r} \right\} \cup \left\{ 0, -1, \ldots \right\}$ and the residues will give rise to real powers of $t$. If $n$ is odd and either $r$ is irrational or $r$ is rational with odd denominator, $\Gamma(s) q_{-rs}(x)$ has simple poles in $\left\{ 0, -1, \ldots \right\} \cup \left\{ \frac{n-2j}{2r} : j = 0, 1, \ldots \right\}$. Otherwise, if $n$ is odd and $r$ is rational with even denominator, then there exist some double poles which give rise to logarithmic terms in the asymptotic expansion of $h_t$.

**Theorem 7.1** Let $a_j(x, x)$ be the coefficients in (2.1) of the heat kernel $p_t$ of the non-negative self-adjoint generalized Laplacian $\Delta$. The asymptotic expansion of the Schwartz kernel $h_t$ of the operator $e^{-t \Delta^*}$, $r \in (0, 1)$ along the diagonal when $t \searrow 0$ is the following:

1. If $n$ is even, then

$$h_{t|\text{Diag}} \sim 0 \sum_{j=0}^{n/2-1} t^{\frac{n-2j}{2r}} \cdot A_{-\frac{n-2j}{2r}} + a_{n/2} + \sum_{j=1}^\infty t^j \cdot A_j.$$  

If $r = \frac{\alpha}{\beta}$ is rational, for $j = l\beta$, $l \in \mathbb{N}^*$, we obtain that $q_{jr}(x, x) = (-1)^l \cdot j! \cdot A_{\frac{n}{2} + l\alpha}(x, x)$, and the coefficient of $t^{l\beta}$ can be described more precisely as

$$A_{l\beta} = a_{\frac{n}{2} + l\alpha}.$$
(2) If $r \in \mathbb{R} \setminus \mathbb{Q}$ or the denominator of $r$ is odd, then

$$h_{\mathbb{R}_{|\text{Diag}}} \sim \sum_{j=0}^{(n-1)/2} t^{n-2j \cdot \frac{1}{2r}} \cdot A_{n-2j \cdot \frac{1}{2r}} + \sum_{j=1}^{\infty} t^{j \cdot \frac{1}{2r}} \cdot A_{j \cdot \frac{1}{2r}} + \sum_{j=1}^{\infty} t^{2j+1 \cdot \frac{1}{2r}} \cdot A_{2j+1 \cdot \frac{1}{2r}}.$$ 

Moreover, if $r = \frac{a}{b}$ is rational and $\beta$ is odd, then $A_{1\beta} = 0$ for any $l \in \mathbb{N}^*$. 

(3) If $n$ is odd, $r = \frac{a}{b}$ is rational and its denominator $\beta$ is even, then

$$h_{\mathbb{R}_{|\text{Diag}}} \sim \sum_{j=0}^{(n-1)/2} t^{n-2j \cdot \frac{1}{2r}} \cdot A_{n-2j \cdot \frac{1}{2r}} + \sum_{j=1}^{\infty} t^{j \cdot \frac{1}{2r}} \cdot A_{j \cdot \frac{1}{2r}} + \sum_{j=1}^{\infty} t^{2j+1 \cdot \frac{1}{2r}} \cdot A_{2j+1 \cdot \frac{1}{2r}} + \sum_{l=1}^{\infty} t^{l \cdot \frac{1}{2r}} \cdot A_{l \cdot \frac{1}{2r}} + \sum_{l=1}^{\infty} t^{l \cdot \frac{1}{2r}} \cdot \log t \cdot B_{l \cdot \frac{1}{2r}}.$$ 

In all these cases, the coefficients are

$$A_{n-2j \cdot \frac{1}{2r}}(x) = \frac{\Gamma\left(\frac{n-2j}{2r}\right)}{\Gamma\left(\frac{n-2j}{2r} \cdot \frac{1}{2r}\right)} \cdot \frac{1}{r} \cdot a_{j}(x, x), \quad A_{j \cdot \frac{1}{2r}}(x) = \frac{(-1)^j}{j!} \cdot q_{rj}(x, x),$$

$$A_{2j+1 \cdot \frac{1}{2r}}(x) = \frac{\Gamma\left(\frac{2j+1}{2r}\right)}{\Gamma\left(\frac{2j+1}{2r} \cdot \frac{1}{2r}\right)} \cdot \frac{1}{r} \cdot a_{n-2j \cdot \frac{1}{2r}}(x, x), \quad B_{l \cdot \frac{1}{2r}}(x) = \frac{(-1)^l}{l!} \cdot \frac{1}{\Gamma\left(-\frac{l}{2r}\right)} \cdot a_{n+2j \cdot \frac{1}{2r}}(x, x),$$

$$A_{l \cdot \frac{1}{2r}}(x) = \frac{(-1)^l}{l! \cdot \Gamma\left(-\frac{l}{2r}\right)} \cdot \text{FP}_{s=-l \cdot \frac{1}{2r}} \left(\Gamma(rs) q_{rs}(x, x)\right) + \text{FP}_{s=-l \cdot \frac{1}{2r}} \left(\frac{\Gamma(s)}{\Gamma(rs)}\right) \cdot \frac{a_{n+2l \cdot \frac{1}{2r}}(x, x)}{r}.$$ 

Proof We compute the coefficients from (5.3) by using Proposition 3. 

7.1 The case when $n$ is even

For $j \in \{0, 1, \ldots, n/2 - 1\}$, we have

$$\text{Res}_{s = -\frac{n}{2r}} \left( t^{-s} \frac{\Gamma(s)}{\Gamma(rs)} \Gamma(rs) q_{rs}(x, x) \right) = t^{-\frac{n}{2r}} \cdot \frac{\Gamma(\frac{n-2j}{2r})}{\Gamma(\frac{n-2j}{2r} \cdot \frac{1}{2r})} \cdot \frac{a_{j}(x, x)}{r}.$$ 

The residue in $s = 0$ is given by

$$\text{Res}_{s=0} \left( t^{-s} \frac{\Gamma(s)}{\Gamma(rs)} \Gamma(rs) q_{rs}(x, x) \right) = \text{Res}_{s=0} \left( t^{-s} \frac{\Gamma(s)}{\Gamma(rs)} \Gamma(rs) q_{rs}(x, x) \right) = r \cdot \frac{1}{r} \left( a_{\frac{n}{2r}}(x, x) - P_{\text{Ker} \Delta}(x, x) \right) = a_{\frac{n}{2r}}(x, x) - P_{\text{Ker} \Delta}(x, x),$$

thus the coefficient of $t^0$ in the asymptotic expansion (5.3) is $a_{\frac{n}{2r}}(x, x)$.
7.1.1 The case when \( n \) is even and \( r \) is irrational

Let \( j \in \mathbb{N}^+ \). Then

\[
\text{Res}_{s=-j} (t^{-s} \Gamma(s) q_{-rs}(x, x)) = t^j \frac{(-1)^j}{j!} \cdot q_{rf}(x, x).
\]

Therefore, in this case, the asymptotic expansion of \( h_t \) is the following:

\[
h_t(x, x) \sim \sum_{j=0}^{n/2-1} t^{-n-2j} \frac{\Gamma\left(\frac{n-2j}{2r}\right) a_j(x, x)}{\Gamma\left(\frac{n-2j}{2}\right)} + a_s(x, x) + \sum_{j=1}^{\infty} t^j \frac{(-1)^j}{j!} q_{rf}(x, x).
\]

7.1.2 The case when \( n \) is even and \( r = \frac{a}{b} \) is rational with \( (\alpha, \beta) = 1 \)

Some of the coefficients \( q_{rf}(x, x) \) from (7.2) can be expressed in terms of the \( a_k \)'s from (2.1). Remark that \( \frac{\Gamma(s)}{\Gamma(rs)} \) has simple poles in \( \{-1, -2, \ldots\} \setminus \left\{ \frac{-1}{r}, \frac{-2}{r}, \ldots \right\} \). For \( j \in \mathbb{N}^+, s := -\frac{l}{r} \in \{-1, -2, \ldots\} \) if and only if \( j \) is a multiple of \( a \), which is equivalent to \( s = -\frac{la}{r} = -l\beta \) for some \( l \in \mathbb{N}^+ \). In this case, we obtain

\[
\text{Res}_{s=-l\beta} (t^{-s} \Gamma(s) q_{-rs}(x, x)) = \text{Res}_{s=-l\beta} \left( t^{-s} \frac{\Gamma(s)}{\Gamma(rs)} \cdot q_{-rs}(x, x) \right) = t^{l\beta} \cdot \frac{1}{r} a_{\frac{s}{r} + l\alpha}(x, x) = t^{l\beta} a_{\frac{s}{r} + l\alpha}(x, x).
\]

Hence, for rational \( r = \frac{a}{b} \), if \( j = l\beta, l \in \mathbb{N}^+ \), we conclude that

\[
q_{rf}(x, x) = (-1)^j \cdot j! \cdot a_{\frac{s}{r} + l\alpha}(x, x),
\]

and \( h_t(x, x) \) has the following asymptotic expansion as \( t \searrow 0 \):

\[
\sum_{j=0}^{n/2-1} t^{-n-2j} \frac{\Gamma\left(\frac{n-2j}{2r}\right) a_j(x, x)}{\Gamma\left(\frac{n-2j}{2}\right)} + a_s(x, x) + \sum_{j=1}^{\infty} t^j \frac{(-1)^j}{j!} q_{rf}(x, x) + \sum_{l=1}^{\infty} t^{l\beta} a_{\frac{s}{r} + l\alpha}(x, x).
\]

7.2 The case when \( n \) is odd

For \( j \in \{0, 1, \ldots, (n-1)/2\} \), the coefficient of \( t^{-n-2j} \) is computed as in (7.1). Furthermore, in \( s = 0 \),

\[
\text{Res}_{s=0} (t^{-s} \Gamma(s) q_{-rs}(x, x)) = \text{Res}_{s=0} \left( t^{-s} \frac{\Gamma(s)}{\Gamma(rs)} \cdot \Gamma(rs) q_{-rs}(x, x) \right) = r \cdot \frac{-1}{r} \cdot P_{\ker \Delta}(x, x) = -P_{\ker \Delta}(x, x);
\]

hence, there is no free term in the asymptotic expansion of \( h_t \) as \( t \) goes to zero.

Now we have to compute the residues of the function \( t^{-s} \Gamma(s) q_{-rs}(x, x) \) in \( s \in \{-1, -2, \ldots\} \) and \( s \in \left\{ \frac{-1}{2r}, \frac{-3}{2r}, \ldots \right\} \).
7.2.1 The case when $n$ is odd and $r$ is irrational

Then these sets are disjoint; thus, all poles of the function $\Gamma(s)q_{-rs}(x)$ are simple. For $j \in \mathbb{N}^*$, the coefficient of $t^j$ is obtained as in (7.2). Furthermore, for $j \in \mathbb{N}$, we get

\[
\text{Res}_{s=-2j+1 \over 2r} \left( t^s \frac{\Gamma(s)}{\Gamma(rs)} \cdot \Gamma(rs)q_{-rs}(x,x) \right) = t^{2j+1 \over 2r} \cdot \frac{\Gamma(-2j+1 \over 2r)}{\Gamma(-2j+1 \over r)} \cdot \frac{a_{n+2j+1}(x,x)}{r}.
\]

Therefore, the small-time asymptotic expansion of $h_t$ is the following:

\[
h_t(x,x) \sim \sum_{j=0}^{n/2-1} t^{n+2j \over 2r} \cdot \frac{\Gamma(n-2j \over 2r)}{\Gamma(n-2j \over 2)} \cdot \frac{a_j(x,x)}{r} + \sum_{j=1}^{\infty} t^j \cdot \frac{(-1)^j}{j!} q_{rj}(x,x)
\]

\[
+ \sum_{j=0}^{\infty} t^{2j+1 \over 2r} \cdot \frac{\Gamma(2j+1 \over 2r)}{\Gamma(2j+1 \over 2)} \cdot \frac{a_{n+2j+1}(x,x)}{r}.
\]

7.2.2 The case when $n$ is odd and $r = \frac{a}{\beta}$ is rational

Consider the sets

\[A := \{-1, -2, \ldots\}, \quad B := \{-\frac{1}{2r}, -\frac{2}{2r}, \ldots\}, \quad C := \{\frac{1}{r}, \frac{2}{r}, \ldots\} \]

Remark that $A$ is the set of negative poles of $s \mapsto t^{-s}\Gamma(s)q_{-rs}(x,x)$, and $A \setminus C$ is the set of poles of the function $s \mapsto \frac{\Gamma(s)}{\Gamma(rs)}$. Clearly $B$ and $C$ are disjoint. Moreover, $A \cap C = \{-l\beta: \ l \in \mathbb{N}^*\}$. Furthermore, if $\beta$ is odd, then $A \cap B = \emptyset$, and otherwise if $\beta$ is even, then $A \cap B = \{-l \beta \over 2: \ l \in 2\mathbb{N} + 1\}$. Such an $s = -2j+1 \over 2r = l \beta \over 2 \in A \cap B$ is a double pole for $\Gamma(s)q_{rs}(x)$.

7.2.3 Suppose that $\beta$ is odd

Then $A$ and $B$ are disjoint. Thus, for $s = -2j+1 \over 2r \in B$, $j \in \mathbb{N}$, the residue of $t^{-s}\Gamma(s)q_{-rs}(x,x)$ is the one computed in (7.4).

For $s = -j \in A \setminus C$ (which means that $j \in \mathbb{N}^*, \beta \neq j$), the residue of $t^{-s}\Gamma(s)q_{-rs}(x,x)$ in $s$ is the one computed in (7.2).

If $s = -l\beta = -l \beta \over r \in A \cap C$ for some $l \in \mathbb{N}^*$, then $\Gamma(s)$ has a simple pole in $s$ and by Proposition 3, (the meromorphic extension of) $q_{-rs}(x,x)$ vanishes at $s = -l\beta$. Hence, the product $t^{-s}\Gamma(s)q_{-rs}(x,x)$ is holomorphic in $s = -l\beta$ and $t^{l\beta}$, $l \in \mathbb{N}^*$, does not appear in the asymptotic expansion.

Therefore, if $r = \frac{a}{\beta}$ is rational and $\beta$ is odd, we obtain

\[
h_t(x,x) \sim \sum_{j=0}^{n/2-1} t^{n+2j \over 2r} \cdot \frac{\Gamma(n-2j \over 2r)}{\Gamma(n-2j \over 2)} \cdot \frac{a_j(x,x)}{r} + \sum_{j=1}^{\infty} t^j \cdot \frac{(-1)^j}{j!} q_{rj}(x,x)
\]

\[
+ \sum_{j=1}^{\infty} t^{2j+1 \over 2r} \cdot \frac{\Gamma(2j+1 \over 2r)}{\Gamma(2j+1 \over 2)} \cdot \frac{a_{n+2j+1}(x,x)}{r}.
\]
**7.2.4 Assume now that \( \beta \) is even**

For \( s = -\frac{2j+1}{2r} \), the residue is computed as in (7.4). For \( s = -j \in \mathbb{A}(B \cup C) \) (namely \( j \in \mathbb{N}^* \), \( \frac{\beta}{2} + j \)), the residue is computed as in (7.2).

For \( s \in C \cap A \) (namely \( s = -l\beta, l \in \mathbb{N}^* \)), the residue is again 0. Indeed, \( \Gamma(s) \) has a simple pole in \( -l\beta \) and by Proposition 3, (the meromorphic extension of) \( q_{-rs}(x, x) \) vanishes in \( -l\beta \), thus \( t^{l\beta} \) does not appear in the asymptotic expansion of \( h_t \).

Finally, if \( s = -\frac{1a}{2r} = -\frac{l^2}{r} \in A \cap B \), \( l \in 2\mathbb{N} + 1 \), then \( s \) is a double pole for \( \Gamma(s) )q_{-rs}(x, x) \). We write the Laurent expansions of the functions \( t^{-s} \), \( \Gamma(s) \), and \( \Gamma(rs) \) respectively, in \( s = -\frac{1a}{2r} = -\frac{l^2}{r} =: -k:

\[
\begin{align*}
    t^{-s} &= t^k - t^k \log t + \mathcal{O}(s + k)^2, \\
    \frac{\Gamma(s)}{\Gamma(rs)} &= \frac{(-1)^k}{k!} \cdot \frac{1}{\Gamma(-kr)} (s + k)^{-1} + \cdots, \\
    \Gamma(rs)(q_{-rs}(x, x)) &= \frac{1}{r} a_{\frac{n+1}{2}}(x, x) (s + k)^{-1} + \cdots.
\end{align*}
\]

Thus, we finally obtain that

\[
\text{Res}_{s = -k} \left( t^{-s} \cdot \frac{\Gamma(s)}{\Gamma(rs)} \cdot \Gamma(rs) q_{-rs}(x, x) \right) = t^k \cdot \frac{(-1)^k}{k!} \cdot \frac{\Gamma(s)}{\Gamma(rs)} \cdot \text{FP}_{s = -k} \left( \frac{\Gamma(rs) q_{-rs}(x, x)}{r} \right)
\]

\[
+ t^k \cdot \text{FP}_{s = -k} \left( \frac{\Gamma(s)}{\Gamma(rs)} \right) \cdot \frac{a_{\frac{n+1}{2}}(x, x)}{r}
\]

\[
+ t^k \log t \cdot \frac{(-1)^k}{k!} \cdot \frac{a_{\frac{n+1}{2}}(x, x)}{r}.
\]

**8 Non-triviality of the coefficients**

Let us prove Theorem 1.1. Recall the definition of the zeta function of a non-negative self-adjoint generalized Laplacian \( \Delta \):

\[
\zeta_\Delta(s) := \sum_{\lambda \in \text{Spec } \Delta \setminus \{0\}} \lambda^{-s} = \int_M q_{-s}(x, x) dg(x).
\]

This series is absolutely convergent for \( \Re s > \frac{n}{2} \) and extends meromorphically to \( \mathbb{C} \) with possible simple poles in the set

\[
\left\{ \frac{n}{2} - j : j \in \mathbb{N} \setminus \left\{ \frac{n}{2} \right\} \right\}
\]

(see, for instance, [13]).

Consider the trivial bundle \( \mathbb{C} \) over a compact Riemannian manifold \( M \). As in [17], let \( (\Delta + \xi)_{\xi > 0} \) be a family of generalized Laplacians indexed by \( \xi > 0 \), and denote by \( q_{-s}^\xi \) the Schwartz kernels of the operators \( (\Delta + \xi)^{-s} \). Note that for \( \Re s > \frac{n}{2} \),

\[
\int_M q_{-s}^\xi(x, x) dx = \text{Tr } (\Delta + \xi)^{-s} = \zeta_{\Delta + \xi}(s) = \sum_{\lambda_j \in \text{Spec } \Delta} (\lambda_j + \xi)^{-s}.
\]


Since for $\mathfrak{R}s > \frac{n}{2}$ the sum is absolutely convergent, we obtain

$$
\frac{d}{d\xi} \zeta_{\Delta + \xi}(s) = -s \cdot \sum_{\lambda \in \text{Spec } \Delta} (\lambda_j + \xi)^{-s-1} = -s \cdot \zeta_{\Delta + \xi}(s + 1).
$$

By induction, it follows that for $\mathfrak{R}s > \frac{n}{2}$,

$$(8.2) \quad \frac{d}{d\xi} \zeta_{\Delta + \xi}(s) = (-1)^k s(s + 1) \ldots (s + k - 1) \cdot \zeta_{\Delta + \xi}(s + k).$$

Using the identity theorem, (8.2) holds true on $\mathbb{C}$ as an equality of meromorphic functions. Consider $s \in \mathbb{R} \setminus (-\infty)$ and $k \in \mathbb{N}$ large enough such that $s + k > \frac{n}{2}$. Since $\zeta_{\Delta + \xi}(s + k)$ is a convergent sum of strictly positive numbers, the right-hand side is non-zero. Thus, for any fixed $s \in \mathbb{R} \setminus (-\infty)$, the function $\xi \mapsto \zeta_{\Delta + \xi}(s)$ is not identically zero on $\mathbb{R}$, and by (8.1), $q_{s}(x, x)$ cannot be constant zero on $M$. Hence, for $s = -rj \notin \mathbb{N}$, there exist $\xi_0 \in (0, \infty)$ and $x_0 \in M$ such that the coefficient $q_{s}(x_0, x_0)$ of the asymptotic expansion of the Schwartz kernel $h_t$ of $e^{-t(\Delta + \xi_0)^{r}}$ is non-zero.

Now suppose that $rj \in \mathbb{N}$. Then $r = \frac{n}{2}$ is rational and $j$ is a multiple of $\beta$, $j := l\beta$. If $n$ is odd, we already proved in Theorem 7.1 that $t^{1/\beta}$ does not appear in the asymptotic expansion of $h_t$ as $t \to 0$. Furthermore, if $n$ is even, by (7.3), $q_{s}(x, x)$ is a non-zero multiple of the coefficient $a_{s + 1\alpha}(x, x)$ in the asymptotic expansion (2.1) of the heat kernel $p_t$. It is well known that the heat coefficients in (2.1) are non-trivial (see, for instance, [13]). It follows that all coefficients obtained in Theorem 7.1 indeed appear in the asymptotic expansion, proving Theorem 1.1.

### 9 Non-locality of the coefficients $A_j(x)$ in the asymptotic expansions

Let us prove Theorem 1.3. We give an example of an $n$-dimensional manifold and a Laplacian for which the coefficients $A_j(x) = \frac{(-1)^j}{j!} q_{rj}(x, x)$, $j \in \mathbb{N}^*$, $rj \notin \mathbb{N}$ appearing in Theorem 7.1 are not locally computable in the sense of Definition 1.1 (i). Let $\mathbb{T}^n = \mathbb{R}^n / (2\pi \mathbb{Z})^n$ be the $n$-dimensional torus from Example 2.2. Let $\Delta_g$ be the Laplacian on $\mathbb{T}^n$ given by the metric $g = d\theta_1^2 + \ldots + d\theta_n^2$.

Remark that the eigenvalues of $\Delta_g$ are $\{k_1^2 + \ldots + k_n^2 : k_1, \ldots, k_n \in \mathbb{Z}\}$. Let $\varphi_l(t) = \frac{1}{\sqrt{2\pi}} e^{itn}$ be the standard orthonormal basis of eigenfunctions of each $\Delta_g$. Then, for $\mathfrak{R}s > \frac{n}{2}$ and $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n$, the Schwartz kernel of $\Delta_g^{-s}$ is given by

$$
q_{-s}(\theta, \theta) = \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n \setminus \{0\}} (k_1^2 + \ldots + k_n^2)^{-s} \varphi_{k_1}(\theta_1) \varphi_{k_1}(\theta_1) \ldots \varphi_{k_n}(\theta_n) \varphi_{k_n}(\theta_n).
$$

Consider the $n$-dimensional zeta function

$$
\zeta_n(s) := \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n \setminus \{0\}} (k_1^2 + \ldots + k_n^2)^{-s} = \sum_{k \in \mathbb{N}^*} k^{-s} R_n(k),
$$

where $R_n(k) = \sum_{d \mid k} \mu(d) \sqrt{d}$.
where \( R_n(k) \) is the number of representations of \( k \) as a sum of \( n \) squares. Since 
\[
\varphi(t) \varphi(t) = \frac{1}{2\pi} \quad \text{for all } t \in S^1,
\]
it follows that 
\[
\tag{9.1} q_{-s}^\Delta (\theta, \theta) = \frac{1}{(2\pi)^n} \zeta_n(s),
\]
for any \( \Re s > \frac{n}{2} \), and clearly \( q_{-s}^\Delta \) is independent of \( \theta \).

Now let us change the metric locally on each component \( S^1 \). Let \( U \) be an open interval in \( S^1 \), and \( \psi : S^1 \rightarrow [0, \infty) \) a smooth function with \( \supp \psi \subset U \). Consider the new metric \( (1 + \psi(\theta)) \, d\theta^2 \) on each \( S^1 \). Then there exist \( p > 0 \) and an isometry \( \Phi : (S^1, (1 + \psi(\theta)) \, d\theta^2) \rightarrow (S^1, p^2 \, d\theta^2) \). Remark that the Laplacian on \( S^1 \) given by the metric \( p^2 \, d\theta^2 \) corresponds under this isometry to \( p^{-2} \) times the Laplacian for the metric \( d\theta^2 \). Let
\[
\tilde{g} = \sum_{j=1}^n (1 + \psi(\theta_j)) \, d\theta_j^2 
\]
and 
\[
g_p = \sum_{j=1}^n p^2 \, d\theta_j^2 = p^2 \tilde{g}.
\]
Then clearly \( \Phi \times \cdots \times \Phi : (\mathbb{T}^n, \tilde{g}) \rightarrow (\mathbb{T}^n, g_p) \) is an isometry, and let \( \tilde{\Delta}, \Delta_p \) be the corresponding Laplacians on \( \mathbb{T}^n \). Denote by \( q_{-s}^\Delta \) and \( q_{-s}^{\Delta_p} \) the Schwartz kernels of the complex powers \( \tilde{\Delta}^{-s} \) and \( \Delta_p^{-s} \). We have for \( \Re s > \frac{n}{2} \),
\[
\tag{9.2} q_{-s}^{\Delta_p} (\theta, \theta) = \frac{1}{(2\pi)^n} \sum_{k=(k_1,\ldots,k_n) \in \mathbb{Z}^n \setminus \{0\}} \left( p^{-2}k_1^2 + \cdots + p^{-2}k_n^2 \right)^{-s} = \frac{p^{2s}}{(2\pi)^n} \zeta_n(s).
\]
Remark that 
\[
q_{-s}^{\Delta} (\theta, \theta) = q_{-s}^{\Delta_p} (\Phi(\theta), \Phi(\theta)),
\]
and both of them are independent of \( \theta \). By (9.2), for \( \Re s > \frac{n}{2} \), we obtain
\[
\tag{9.3} q_{-s}^{\Delta_p} (\theta, \theta) = \frac{p^{2s-n}}{(2\pi)^n} \zeta_n(s).
\]
Now we prove that \( \zeta_n(s) \) has a meromorphic extension on \( \mathbb{C} \) with so-called trivial zeros at \( s = -1, -2, \ldots \). By Proposition 1, for \( \Re s > \frac{n}{2} \), we have
\[
\zeta_n(s) \Gamma(s) = \int_0^\infty t^{s-1} \sum_{k=(k_1,\ldots,k_n) \in \mathbb{Z}^n \setminus \{0\}} e^{-t(k_1^2 + \cdots + k_n^2)} \, dt = \int_0^\infty t^{s-1} F(t) \, dt,
\]
where \( F(t) := \sum_{k=(k_1,\ldots,k_n) \in \mathbb{Z}^n \setminus \{0\}} e^{-t(k_1^2 + \cdots + k_n^2)} \). Using the multidimensional Poisson formula (see, for instance, [3]), it follows that
\[
1 + F(t) = \sum_{k \in \mathbb{Z}^n} f_t(k) = \sum_{k \in \mathbb{Z}^n} \tilde{f}_t(2\pi k) = \pi^{n/2} t^{-n/2} \left( 1 + F\left( \frac{\pi^2}{t} \right) \right),
\]
and therefore
\[
F(t) = -1 + \pi^{n/2} t^{-n/2} + \pi^{n/2} t^{-n/2} F\left( \frac{\pi^2}{t} \right).
\]
Since $F(t)$ goes to 0 rapidly as $t \to \infty$, the function $A(s) = \int_0^\infty t^{s-1} F(\pi t)\, dt$ is entire. Remark that

\[
\zeta_n(s)\Gamma(s) = \int_0^n t^{-s} F(t)\, dt + \int_n^\infty t^{s-1} F(t)\, dt = \pi^s \left( \frac{1}{s} + \frac{1}{s-n} + A \left( \frac{n}{2} - s \right) + A(s) \right),
\]

so

\[
\pi^{-s} \zeta_n(s)\Gamma(s) = -\frac{1}{s} + \frac{1}{s-n} + A \left( \frac{n}{2} - s \right) + A(s).
\]

(9.4)

Therefore, $\zeta_n$ extends meromorphically to $\mathbb{C}$ with a simple pole in $s = \frac{n}{2}$ and zeros at $s = -1, -2, \ldots$. Furthermore, since the RHS is invariant through the involution $s \mapsto \frac{n}{2} - s$, it follows that $\zeta_n(s)$ does not have any other zeros for $s \in (-\infty, 0)$. We obtain the well-known functional equation of the Epstein zeta function

\[
\pi^{-s} \zeta_n(s)\Gamma(s) = \pi^{-s-n/2} \zeta_n \left( \frac{n}{2} - s \right) \Gamma \left( \frac{n}{2} - s \right)
\]

(see, for instance, [9, equation (63)]). Remark that for $r \in (0, 1)$ and $j \in \mathbb{N}^+$ with $rj \notin \mathbb{N}$, $\zeta_n(-rj)$ is not zero.

Using the identity theorem, it follows that (9.1) and (9.3) hold true as an equality of meromorphic functions on $\mathbb{C}$, and furthermore, we get

\[
q_{rj}^\Delta (\theta, \theta) = q_{rj}^\Delta (\theta, \theta),
\]

for $rj \notin \mathbb{N}$. Since we modified the metric locally in $U^n \subset \mathbb{T}^n$ and the corresponding kernel $q_{rj}^\Delta$ changed its behavior globally, it follows that it is not locally computable in the sense of Definition 1.1 (i).

Furthermore, let us see that the heat coefficients $A_j(x) = \frac{(-1)^j}{r!} q_{rj}^\Delta(x, x)$ for $j = \mathbb{N}^+$, $rj \notin \mathbb{N}$ are not cohomologically local in the sense of Definition 1.1 (iii). We argue by contradiction. Let $j$ be fixed. Suppose that there exists a function $C$, locally computable in the sense of Definition 1.1 (i), such that

\[
\int_{\mathbb{T}^n} q_{rj}^\Delta \, d\text{vol}_g = \int_{\mathbb{T}^n} C(g) \, d\text{vol}_g, \quad \int_{\mathbb{T}^n} q_{rj}^\Delta \, d\text{vol}_{\tilde{g}} = \int_{\mathbb{T}^n} C(\tilde{g}) \, d\text{vol}_{\tilde{g}}.
\]

Using (9.1) and (9.3), it follows that

\[
(2\pi)^n \zeta_n(-rj) = \int_{\mathbb{T}^n} C(g) \, d\text{vol}_g, \quad (2\pi p)^n p^{-2rj} \zeta_n(-rj) = \int_{\mathbb{T}^n} C(\tilde{g}) \, d\text{vol}_{\tilde{g}}.
\]

Remark that in the case of the trivial bundle with the trivial connection over a locally homogeneous Riemannian manifold $(M, h)$ (i.e., such that every two points have isometric neighborhoods), the function $C(M, h) \in C^\infty(M)$ is constant on $M$. This follows directly from Definition 1.1 (i). Therefore, $C(g), C(\tilde{g})$, and $C(g_p)$ are constant functions.

Since $(\mathbb{T}^n, \tilde{g})$ is (globally) isometric to $(\mathbb{T}^n, g_p)$, it follows that $C(\tilde{g}) = C(g_p)$. Furthermore, since $(\mathbb{T}^n, g_p)$ is locally isometric to $(\mathbb{T}^n, g)$ and $C(g_p), C(g)$ are constant functions, it also follows that they are equal: $C(g_p) = C(g)$. Hence we
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conclude that \( C(\tilde{g}) = C(g_p) = C(g) =: C \), for some \( C \in \mathbb{C} \), and thus we have

\[ \int_{T^n} C \, d\text{vol}_{\tilde{g}} = \int_{T^n} C \, d\text{vol}_{g_p}. \tag{9.6} \]

Since \( g_p = p^2 g \), we obtain that

\[ \int_{T^n} C \, d\text{vol}_{g_p} = p^n \int_{T^n} C \, d\text{vol}_g, \tag{9.7} \]

and then using (9.5)–(9.7), we get

\[ (2\pi p)^n p^{-2rj} \zeta_n(-rj) = p^n \cdot (2\pi)^n \zeta_n(-rj). \]

But, we proved above that \( \zeta_n(-rj) \) does not vanish for \( rj \not\in \mathbb{N} \). We obtain a contradiction because \( p^{-2rj} \neq 1 \) for \( r \in (0, 1), j = 1, 2, \ldots \).

10 Interpretation of \( h_t \) on the heat space for \( r = 1/2 \)

In Theorems 6.1 and 7.1, we studied the asymptotic behavior of the heat kernel \( h_t \) of \( \Delta', r \in (0, 1) \) for small-time \( t \) in two distinct cases: when we approach \( t = 0 \) along the diagonal in \( M \times M \), and when we approach a compact set away from the diagonal. We now give a simultaneous asymptotic expansion formula for both cases when \( r = 1/2 \). Furthermore, in order to understand the asymptotic behavior as \( t \) goes to zero in any direction (not just the case when \( t \) goes to 0 in the vertical one), we will pull-back the formula on a certain linear heat space \( M_{\text{heat}} \).

In [19], Melrose used his blow-up techniques to give a conceptual interpretation for the asymptotic of the heat kernel \( p_t \). Recall that the heat space \( M_{H}^2 \) is obtained by performing a parabolic blow-up of \( \{ t = 0 \} \times \text{Diag} \) in \( [0, \infty) \times M \times M \). The heat space \( M_{H}^2 \) is a manifold with corners with boundary hypersurfaces given by the boundary defining functions \( \rho \) and \( \omega_0 \). The heat kernel \( p_t \) belongs to \( \rho^{-n} C^\infty(M_{H}^2) \), and vanishes rapidly at the boundary hypersurface \( \{ \omega_0 = 0 \} \) (see [19, Theorem 7.12]).

In order to study the Schwartz kernel \( h_t \) of \( e^{-t\Delta'} \), we introduce the linear heat space \( M_{\text{heat}} \), which is just the standard blow-up of \( \{ 0 \} \times \text{Diag} \) in \( [0, \infty) \times M \times M \) (see [20] for details regarding the blow-up of a submanifold). Let \( \text{ff} \) be the front face, i.e., the newly added face, and denote by \( \text{lb} \) the lateral boundary which is the lift of the old boundary \( \{ 0 \} \times M \times M \). The blow down map is given locally by

\[ \beta_H : M_{\text{heat}} \longrightarrow [0, \infty) \times M \times M \quad \beta_H(\rho, \omega, x') = (\rho \omega_0, \rho \omega' + x', x'), \]

where

\[ \omega \in S_{H}'' = \{ \omega = (\omega_0, \omega') \in \mathbb{R}^{n+1} : \omega_0 \geq 0, \omega_0^2 + |\omega'|^2 = 1 \}. \]

Proof of Theorem 1.4 We want to show that \( h_t \in \rho^{-n} \omega_0 \cdot C^\infty(M_{\text{heat}}) + \rho \log \rho \cdot \omega_0 \cdot C^\infty(M_{\text{heat}}) \), and in fact, the second (logarithmic) term does not occur when \( n \) is even. First, we deduce the unified formula for \( h_t \) as \( t \searrow 0 \) both on the diagonal and away from it. By Mellin formula 1 and inverse Mellin formula 6, for \( \tau > n \), we get
\[ h_t(x, y) - P_{\text{Ker}\Delta}(x, y) = \frac{1}{2\pi i} \int_{\mathcal{C}_{x,y}} t^{-s} \frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)} q_{-s/2}(x, y) ds \]

\[ = \frac{1}{2\pi i} \int_{\mathcal{C}_{x,y}} t^{-s} \frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty T^{s-1} (P_T(x, y) - P_{\text{Ker}\Delta}(x, y)) dT ds. \]

We use the Legendre duplication formula as in [2] (see, for instance, [22]):

\[ \frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)} = \frac{1}{\sqrt{\pi}} 2^{s-1} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}, \]

obtaining that \( h_t(x, y) - P_{\text{Ker}\Delta}(x, y) \) is equal to

\[ \frac{1}{4\pi^2} \frac{1}{2\pi i} \int_{\mathcal{C}_{x,y}} \int_0^\infty \left(\frac{2\sqrt{T}}{t}\right)^s \Gamma\left(\frac{s+1}{2}\right) (p_T(x, y) - P_{\text{Ker}\Delta}(x, y)) dT ds. \]

Set \( X := \frac{2\sqrt{T}}{t} \). Using Propositions 4, 5, and Fubini, we first compute the integral in \( s \).

Changing the variable \( S = \frac{s+1}{2} \) and applying the residue theorem, we get

\[ \frac{1}{2\pi i} \int_{\mathcal{C}_{x,y}} X^s \Gamma\left(\frac{s+1}{2}\right) ds = \frac{2}{\sqrt{T}} \int_{\mathcal{C}_{x,y}} X^{2s-1} \Gamma(S) dS = 2 \sum_{k=0}^\infty \frac{(-1)^k}{k!} X^{-2k-1} \]

\[ = 2X^{-1}e^{-X^{-2}} = \frac{t}{\sqrt{T}} e^{-t^2/4\pi}. \]

Thus, we obtain

\[ (10.1) \quad h_t(x, y) - P_{\text{Ker}\Delta}(x, y) = \frac{t}{2\sqrt{\pi}} \int_0^\infty T^{-3/2} e^{-t^2/4\pi} (p_T(x, y) - P_{\text{Ker}\Delta}(x, y)) dT. \]

Since \( p_T(x, y) - P_{\text{Ker}\Delta}(x, y) \) decays exponentially as \( T \) goes to infinity, it follows that the integral from 1 to \( \infty \) in the right-hand side of equation (10.1) is of the form \( \mathcal{O}(T^{-3/2}) \). Furthermore, by the change of variable \( u = \frac{t}{2\sqrt{T}} \), we have

\[ -\frac{t}{2\sqrt{\pi}} \int_0^1 T^{-3/2} e^{-u^2} dT \cdot P_{\text{Ker}\Delta}(x, y) = -\frac{2}{\sqrt{T}} \int_{t/2}^\infty e^{-u^2} du \cdot P_{\text{Ker}\Delta}(x, y). \]

Since \( \int_{t/2}^\infty e^{-u^2} du \) tends to \( \frac{\sqrt{\pi}}{2} \) as \( t \to 0 \), the term \( -\frac{2}{\sqrt{T}} \int_0^1 T^{-3/2} e^{-u^2} dT P_{\text{Ker}\Delta}(x, y) \) will cancel in the limit as \( t \to 0 \) with \( -P_{\text{Ker}\Delta}(x, y) \) from the left-hand side of (10.1).

Let us study the remaining integral term \( \frac{t}{2\sqrt{\pi}} \int_0^1 T^{-3/2} e^{-t^2/4\pi} p_T(x, y) dT \). By Theorem 2.1,

\[ p_T(x, y) = T^{-n/2} e^{-\frac{d(x,y)^2}{4T}} \sum_{j=0}^N T^j a_j(x, y) + R_{N+1}(T, x, y), \]

where the remainder \( R_{N+1}(T, x, y) \) is of order \( \mathcal{O}(T^{N+1}) \); therefore,

\[ \frac{t}{2\sqrt{\pi}} \int_0^1 T^{-3/2} e^{-t^2/4\pi} p_T(x, y) dT = \frac{t}{2\sqrt{\pi}} \int_0^1 T^{-3/2} e^{-t^2/4\pi} R_{N+1}(T, x, y) dT \]

\[ + \frac{t}{2\sqrt{\pi}} \int_0^1 T^{-3/2} e^{-t^2/4\pi} T^{-n/2} e^{-\frac{d(x,y)^2}{4T}} \sum_{j=0}^N T^j a_j(x, y) dT. \]
Since $R_{N+1}(T, x, y)$ is of order $O(T^{N+1})$, the first integral is again of type $t \cdot C_{t,x,y}^{\infty}$. By changing the variable $u = \frac{t^2 + d(x,y)^2}{4T}$ in the second integral, we get

$$
\frac{t}{2\sqrt{\pi}} \sum_{j=0}^{N} a_j(x, y) \int_{0}^{\sqrt{\frac{4T}{t^2 + d(x,y)^2}}} T^{-\frac{n+1}{2} + j} e^{-\frac{t^2 + d(x,y)^2}{4T}} dT
$$

$$
= \frac{t}{2\sqrt{\pi}} \sum_{j=0}^{N} a_j(x, y) \left( \int_{t^2 + d(x,y)^2}^{\infty} u^{-\frac{n+1}{2} + j} e^{-u} du \right)
$$

$$
= \frac{t}{2\sqrt{\pi}} \sum_{j=0}^{N} a_j(x, y) \Gamma \left( \frac{n+1}{2} - j, \frac{t^2 + d(x,y)^2}{4} \right) \left( \frac{t^2 + d(x,y)^2}{4} \right)^{-\frac{n+1}{2} + j},
$$

where $\Gamma(z, \xi) := \int_{\xi}^{\infty} u^{-z} e^{-u} du$ is the upper incomplete Gamma function. We conclude that $h_t(x, y)$ is equal to (10.2)

$$
t \cdot C_{t,x,y}^{\infty} + \frac{t}{2\sqrt{\pi}} \sum_{j=0}^{N} a_j(x, y) \Gamma \left( \frac{n+1}{2} - j, \frac{t^2 + d(x,y)^2}{4} \right) \left( \frac{t^2 + d(x,y)^2}{4} \right)^{-\frac{n+1}{2} + j}.
$$

**10.1 The case when $n$ is even**

If $z > 0$, then one can easily check that $\Gamma(z, \xi) \in \ell^{\infty} C_{\xi}^{\infty}[0, \epsilon] + \Gamma(z)$, for some $\epsilon > 0$. Furthermore, for $z \in (-\infty, 0] \{0, -1, -2, \ldots \}$,

$$
\Gamma(z, \xi) = -\frac{1}{z} \ell^{\infty} \epsilon^{-\xi} + \frac{1}{z} \Gamma(z + 1, \xi)
$$

$$
= \ell^{\infty} \epsilon^{-\xi} \sum_{k=0}^{a-1} \frac{1}{z(z+1) \ldots (z+k)} \ell^{k} + \frac{1}{z(z+1) \ldots (z+a)} \Gamma(z + a, \xi)
$$

$$
= \ell^{\infty} C_{\xi}^{\infty}[0, \epsilon] + \frac{1}{z(z+1) \ldots (z+a-1)} \Gamma(z + a, \xi),
$$

where $a$ is a positive integer such that $z + a > 0$. Thus, for a non-integer $z < 0$, we have

$$
\Gamma(z, \xi) = \ell^{\infty} C_{\xi}^{\infty}[0, \epsilon] + \frac{1}{z(z+1) \ldots (z+a-1)} \Gamma(z + a).
$$

We want to interpret equation (10.2) on the heat space $M_{heat}$; thus, we pull back (10.2) through $\beta_{H}^{\star}$:

$$
\beta_{H}^{\star} h = \rho \omega_{0} \beta_{H}^{\star} C_{t,x,y}^{\infty} + \frac{1}{\sqrt{2\pi}} \rho \omega_{0} \sum_{j=0}^{n} \left( \frac{a^2}{4} \right)^{-\frac{n+1}{2} + j} \beta_{H}^{\star} a_j(x, y) \Gamma \left( \frac{n+1}{2} - j, \frac{\rho^2}{4} \right)
$$

$$
= \rho \omega_{0} \beta_{H}^{\star} C_{t,x,y}^{\infty} + \frac{1}{\sqrt{2\pi}} \rho^{-n} \omega_{0} \sum_{j=0}^{n/2} \rho^{2j} \beta_{H}^{\star} a_j(x, y) \Gamma \left( \frac{n+1}{2} - j \right)
$$

$$
+ \frac{1}{2\sqrt{\pi}} \rho \omega_{0} \sum_{j=0}^{n/2} \beta_{H}^{\star} a_j(x, y) C_{\rho}^{\infty}[0, \epsilon] + \frac{1}{\sqrt{2\pi}} \rho \omega_{0} \sum_{j=n/2+1}^{N} \beta_{H}^{\star} a_j(x, y) C_{\rho}^{\infty}[0, \epsilon]$$
\[ + \frac{1}{2\sqrt{\pi}} \rho^{-n} \omega_0 \sum_{j=n/2+1}^{N} \rho^{2j} 2^{n+1-2j} \beta_{H}^{*} a_j(x, y) \frac{2^{-n/2+j}}{(n + 1 - 2j)(n + 3 - 2j) \ldots (-1) \Gamma \left( \frac{1}{2} \right)}. \]

Since \( \Gamma \left( \frac{n+1}{2} - j \right) = \frac{\sqrt{\pi(n-2j-1)!}}{2^j(n-2j-1)!} \) for \( j \in \{0,1, \ldots, n/2\} \), it follows that

\[ (10.3) \]

\[ \beta_{H}^{*} h = \rho \omega_0 \beta_{H}^{*} C_{t, x, y}^{\infty} + \omega_0 \rho \beta_{H}^{*} C_{t, x, y}^{\infty} [0, \epsilon] + \rho^{-n} \omega_0 \sum_{j=0}^{n/2} \rho^{2j} 2^{n/2-j} (n - 2j - 1)!! \beta_{H}^{*} a_j(x, y) \]

\[ + \rho^{-n} \omega_0 \sum_{j=n/2+1}^{N} \rho^{2j} (2j - n - 1)!! \beta_{H}^{*} a_j(x, y). \]

The case \( \rho \neq 0 \) and \( \omega_0 \to 0 \) corresponds to \( x \neq y \) and \( t \to 0 \) before the pull-back. We obtain that \( \beta_{H}^{*} h \) is in \( C^{\infty}(M_{\text{heat}}) \) and it vanishes at first order on \( \text{lb} \), which is compatible with Theorem 6.1.

If \( \rho \to 0 \) and \( \omega_0 = 1 \), which corresponds to \( x = y \) and \( t \to 0 \), then \( \beta_{H}^{*} h = \rho^{-n} \omega_0 \sum_{j=0}^{N} \rho^{2j} A_j(x) \), where we denoted by \( A_j(x) \) the coefficients appearing in (10.3). Again, this result is compatible with Theorem 7.1, and moreover, the coefficients are precisely the ones from [2, Theorem 3.1].

Remark that formula (10.3) is stronger than Theorems 6.1 and 7.1. If both \( \rho \) and \( \omega_0 \) tend to 0 (with different speeds), it describes the behavior of \( h_t \) as \( t \) goes to zero from any positive direction (not only the vertical one).

### 10.2 The case when \( n \) is odd

Remark that for small \( \xi \), we have

\[ \Gamma(0, \xi) = \int_{\xi}^{\infty} t^{-1} e^{-t} dt = \int_{\xi}^{1} t^{-1} e^{-t} dt + \int_{1}^{\infty} t^{-1} e^{-t} dt = - \log \xi + C_{\xi}^{\infty}[0, \epsilon). \]

Furthermore, if \( p \) is a negative integer, inductively we obtain

\[ \Gamma(-p, \xi) = \frac{e^{-\xi} \xi^{\frac{-p}{p+1}}}{p+1} \sum_{k=0}^{p-1} (-1)^k (p - k - 1)! \xi^k + \frac{(-1)^p}{p!} \Gamma(0, \xi) \]

\[ = \xi^{-p} C_{\xi}^{\infty}[0, \epsilon] - \frac{(-1)^p}{p!} \log \xi + C_{\xi}^{\infty}[0, \epsilon). \]

We pull-back equation (10.2) on the heat space \( M_{\text{heat}} \):

\[ \beta_{H}^{*} h = \rho \omega_0 \beta_{H}^{*} C_{t, x, y}^{\infty} + \frac{1}{2\sqrt{\pi}} \rho \omega_0 \sum_{j=0}^{N} \left( \rho^{2j} \right)^{-\frac{n+1}{2}} \beta_{H}^{*} a_j(x, y) \frac{2^{j+n}-j}{(n + 1 - 2j)(n + 3 - 2j) \ldots (-1) \Gamma \left( \frac{1}{2} \right)}. \]

\[ = \rho \omega_0 \beta_{H}^{*} a_j(x, y) + \frac{1}{2\sqrt{\pi}} \rho \omega_0 \sum_{j=(n-1)/2}^{(n-1)/2} \beta_{H}^{*} a_j(x, y) C_{\rho}^{\infty}[0, \epsilon) \]

\[ + \frac{1}{\sqrt{\pi}} \rho^{-n} \omega_0 \sum_{j=0}^{(n-1)/2} \rho^{2j} \beta_{H}^{*} a_j(x, y) 2^{n-2j} \Gamma \left( \frac{n + 1}{2} - j \right). \]
11 The heat kernel as a polyhomogeneous conormal section

Let us recall the notions of index family and polyhomogeneous conormal functions on a manifold with corners with two boundary hypersurfaces. (For an accessible introduction, see [15], and for full details of the theory, see [18].) A discrete subset $F \subset \mathbb{C} \times \mathbb{N}$ is called an index set if the following conditions are satisfied:

1) For any $N \in \mathbb{R}$, the set $F \cap \{ (z, p) : \Re z < N \}$ is finite.
2) If $p > p_0$ and $(z, p) \in F$, then $(z, p_0) \in F$.

If $X$ is a manifold with corners with two boundary hypersurfaces $B_1$ and $B_2$ given by the boundary defining functions $x$ and $y$, a smooth function $f$ on $\hat{X}$ is said to be polyhomogeneous conormal with index sets $E$ and $F$, respectively, if in a small neighborhood $[0, \varepsilon) \times B_1$, $f$ has the asymptotic expansion

$$f(x, y) \asymp_{\varepsilon \to 0} \sum_{(z, p) \in F} a_{z, p}(y) \cdot x^z \log^p x,$$
where $a_{z,p}$ are smooth coefficients on $B_2$, and for each $a_{z,p}$ there exists a sequence of real numbers $b_{w,q}$, such that

$$a_{z,p}(y) \overset{y\to 0}{\sim} \sum_{(w,q)\in E} b_{w,q} \cdot y^w \log^q y.$$ 

One can prove that $f$ is a polyhomogeneous conormal function on $X$ with index sets $F_p = \{(k,0): k \in \mathbb{Z}, k \geq -p\}$ and $F_0 = \{(n,0): n \in \mathbb{N}\}$ if and only if $f \in y^{-p} C^\infty(X)$. Furthermore, $f$ is a polyhomogeneous conormal function on $X$ with index sets $F' = \{(n,1): n \in \mathbb{N}^*\}$ and $F_0$ if and only if $f \in C^\infty(X) + \log y \cdot C^\infty(X)$. Therefore, we can restate Theorem 1.4 as follows:

**Theorem 11.1** For $r = \frac{1}{2}$, the heat kernel $h_t$ of the operator $e^{-t \Delta^{1/2}}$ is a polyhomogeneous conormal section on the linear heat space $M_{heat}$ with values in $E \otimes E^*$. The index set for the lateral boundary is

$$F_{lb} = \{(k,0): k \in \mathbb{N}^*\}.$$ 

If $n$ is even, the index set of the front face is

$$F_{ff} = \{(-n+k,0): k \in \mathbb{N}\},$$

whereas for $n$ odd, the index set toward $ff$ is given by

$$F_{ff} = \{(-n+k,0): k \in \mathbb{N}\} \cup \{(k,1): k \in \mathbb{N}^*\}.$$ 

It seems reasonable to expect that the Schwartz kernel $h_t$ of the operator $e^{-t \Delta}$ for $r \in (0,1)$ can be lifted to a polyhomogeneous conormal section in a certain “transcendental” heat space $M_{Heat}^r$ depending on $r$ with values in $E \otimes E^*$. However, already in the case $r = 1/3$, our method leads to complicated computations involving Bessel modified functions. We therefore leave this investigation open for a future project.

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