On Self-Dual Gravity

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Abstract

We study the Ashtekar-Jacobson-Smolin equations that characterise four dimensional complex metrics with self-dual Riemann tensor. We find that we can characterise any self-dual metric by a function that satisfies a non-linear evolution equation, to which the general solution can be found iteratively. This formal solution depends on two arbitrary functions of three coordinates. We construct explicitly some families of solutions that depend on two free functions of two coordinates, included in which are the multi-centre metrics of Gibbons and Hawking.

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1 Introduction

In four dimensions the Hodge duality operation takes two forms to two forms. Given a four dimensional metric, the most important two form associated with it is the curvature two form \( R^a_{\ b} \). It is therefore natural to be interested in four dimensional metrics whose curvature form obeys the self-duality relation

\[
R^a_{\ b} = *R^a_{\ b},
\]

where * is the Hodge duality operator. We will refer to such metrics as “self-dual”. Such metrics automatically have vanishing Ricci tensor, and so satisfy the vacuum Einstein equations with vanishing cosmological constant. Unfortunately, the only real Lorentzian self-dual metric is flat Minkowski space, so we choose instead to work with metrics with four complex dimensions.

Physically these metrics may be of interest in attempts to quantise gravity, since they correspond to saddle points of the Einstein-Hilbert action, therefore giving large contributions to a path integral over euclidean metrics \([1]\). Alternatively, it may be possible to interpret them as “one particle states” in a quantised gravity theory \([2]\).

From a purely mathematical point of view these metrics are interesting since they are “hyperkähler”. Hyperkähler manifolds are \(4n\) dimensional manifolds (for \(n\) a positive integer) that admit a non-singular euclidean metric, \(g\), with respect to which there exist three automorphisms, \(J^i\), of the tangent bundle which obey the quaternion algebra \([3]\). In other words

\[
\nabla J^i = 0, \quad J^i J^j = -\delta_{ij} + \epsilon_{ijk} J^k,
\]

where \(\nabla\) is the covariant derivative with respect to the metric \(g\). In four
dimensions, it turns out that for a metric, $g$, to be hyperkähler it must have either self-dual, or anti-self-dual, curvature tensor \([4]\).

The problem of constructing metrics with self-dual curvature tensor has been tackled in several ways. The most direct approach is to formulate the problem in terms of partial differential equations \([5, 6]\). A more constructive approach is Penrose’s ‘Non-Linear Graviton’ technique \([2]\). Here, the task of solving partial differential equations is replaced by that of constructing deformed twistor spaces, and holomorphic lines on them. In practice this turns out to be just as difficult as solving partial differential equations, but in principle one can construct the general self-dual metric in this way.

Here we concentrate on partial differential equations. We find a formulation which is similar to Plebański’s First Heavenly equation \([5]\), but which can be viewed as simply an evolution equation. This means that the free functions in our solution are just a field and its time derivative on some initial hypersurface i.e. two free functions of three coordinates. We construct, in a somewhat formal manner, the general solution to this equation. We also construct explicitly some infinite dimensional families of solutions to these equations. In the appendices, we show how this formulation is equivalent to Plebański’s.

## 2 Construction of Self-Duality Condition

In \([7]\) the equations for complex self-dual metrics were reformulated in terms of the new Hamiltonian variables for General Relativity introduced in \([8]\). By fixing the four manifold to be of the form $\mathcal{M} = \Sigma \times \mathbb{R}$ and using the
coordinate $T$ to foliate the manifold, they reduced the problem of finding self-dual metrics to that of finding a triad of complex vectors $\{V_i : i = 1, 2, 3\}$ that satisfy the equations

$$\text{Div} V_i = 0, \quad (3)$$

$$\frac{\partial}{\partial T} V_i = \frac{1}{2} \epsilon_{ijk} [V_j, V_k]. \quad (4)$$

Defining the densitised inverse three metric

$$\hat{q}^{ab} = V_i^a V_j^b \delta_{ij}, \quad (5)$$

we recover the undensitised inverse three metric $q^{ab}$ by the relation $q^{ab} = \hat{q} \hat{q}^{ab}$, where $\hat{q} = \det \hat{q}_{ab} = (\det \hat{q}^{ab})^{-1}$. If we now define the lapse function $N$ by $N = (\det q_{ab})^{1/2}$ then we find that the metric defined by the line element

$$ds^2 = N^2 dT^2 + q_{ab} dx^a dx^b \quad (6)$$
is self-dual.

Later, it was found that this triad of vectors could be related to the complex structures $J^i$ that hyperkähler metrics admit [9]. Given a self-dual metric, we choose local coordinates $(T, x^a)$ to put the line element in the form of equation (6). If we define the triad of vectors $V_i = -J^i(\ast, \partial_T)$, then these vectors will satisfy (3) and (4).

Here we will concentrate on the problem of finding local solutions to equations (3) and (4). We thus introduce a local coordinate chart $(X, Y, Z)$ on the three surface, $\Sigma$, with its natural flat metric and connection. Thus (3) becomes just

$$\frac{\partial}{\partial x^a} V_i^a = 0. \quad (7)$$
The crucial step is to realise that we can write equation \((4)\) as
\[
\left[ \frac{\partial}{\partial T}, V_i \right] = \frac{1}{2} \epsilon_{ijk} [V_j, V_k].
\] (8)

If we consider only euclidean metrics, then we take the \(V_i\) to be real. In this case we define two complex vectors \(A, B\) by
\[
A = \frac{\partial}{\partial T} + i V_1, \quad \quad \quad \quad (9)
\]
\[
B = V_2 - i V_3. \quad \quad \quad \quad (10)
\]
which, by virtue of \((8)\), obey the Lie bracket algebra
\[
[A, B] = 0, \quad [\bar{A}, \bar{B}] = 0, \quad [A, \bar{A}] + [B, \bar{B}] = 0, \quad (11)
\]
where \(-\) denotes complex conjugate. We can generalise these equations by considering four complex vectors \(U, V, W\) and \(X\) that satisfy the relations
\[
[U, V] = 0, \quad \quad \quad \quad (12)
\]
\[
[W, X] = 0, \quad \quad \quad \quad (13)
\]
\[
[U, W] + [V, X] = 0. \quad \quad \quad \quad (14)
\]
Here we are thinking of \(W\) and \(X\) as “generalised complex conjugates” of \(U\) and \(V\) respectively. By Frobenius’ theorem, we can use \((12)\) to define a set of coordinates \((t, x)\) on the 2 (complex) dimensional surface defined by vectors \(U\) and \(V\), and take \(U\) and \(V\) to be
\[
U = \frac{\partial}{\partial t}, \quad V = \frac{\partial}{\partial x}. \quad \quad \quad \quad (15)
\]
We can now foliate our whole space using the coordinates \((t, x, y, z)\). The equation \((14)\) then becomes \(\partial_t W + \partial_x X = 0\). This means there exists a
vector field $Y$ such that $W = \partial_z Y, X = -\partial_t Y$. Thus we are only left with the problem of solving for vectors $Y$ that satisfy $[\partial_t Y, \partial_x Y] = 0$. We expand $W$ and $X$ as

$$W = \partial_t + f_x \partial_y + g_x \partial_z, \quad (16)$$

$$X = -f_t \partial_y - g_t \partial_z. \quad (17)$$

(The reason for the $\partial_t$ term in $W$ is, as alluded to above, we are thinking of $W$ as a sort of complex conjugate of $U = \partial_t$. Although this argument only seems sensible for $t$ a real coordinate, we are still perfectly at liberty to expand $W$ in this way if $t$ is complex.) If, by analogy with (3), we impose

$$\frac{\partial}{\partial x^a} W^a = \frac{\partial}{\partial x^a} X^a = 0,$$

then we find that there exists a function $h(t, x, y, z)$ such that $f = h_z, g = -h_y$. Imposing (13), we find that there exists a function $\alpha(t, x)$ such that

$$h_{tt} + h_{xz} h_{ty} - h_{xy} h_{tz} = \alpha(t, x). \quad (18)$$

We can absorb the arbitrary function $\alpha$ into the function $h$, and conclude that we can form a self-dual metric for any function $h$ that satisfies

$$h_{tt} + h_{xz} h_{ty} - h_{xy} h_{tz} = 0. \quad (19)$$

This is just an evolution equation. Thus we can arbitrarily specify data $h$ and $h_t$ on a $t = \text{constant}$ hypersurface and propagate it throughout the space according to (19) to get a solution. For example, if we expand $h$ around the

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2It was only after this work was completed that I learned of [10] where the ideas developed so far were found independently. From here onwards, however, our treatments are different.
\[ t = 0 \text{ hypersurface, and insist that } h \text{ is regular on this surface, then } h \text{ is of the form} \]

\[ h = a_0(x, y, z) + a_1(x, y, z) t + a_2(x, y, z) \frac{t^2}{2!} + a_3(x, y, z) \frac{t^3}{3!} + \ldots \tag{20} \]

Substituting this into (19) shows that \( a_0 \) and \( a_1 \) are arbitrary functions of \( x, y \) and \( z \). \( a_2, a_3 \ldots \) are then completely determined for chosen \( a_0 \) and \( a_1 \) by

\[ a_2 = a_{0xy} a_{1z} - a_{0xz} a_{1y}, \tag{21} \]

\[ a_3 = a_{0xy} a_{2z} - a_{0xz} a_{2y} + a_{1xy} a_{1z} - a_{1xz} a_{1y}, \tag{22} \]

and so on. Thus, in principle, we have a solution that depends on two arbitrary functions of three coordinates. It is interesting to compare our equation (19) with Plebański's First Heavenly equation

\[ \Omega_{\tilde{p}\tilde{q}} \Omega_{\tilde{p}\tilde{q}} - \Omega_{\tilde{p}\tilde{p}} \Omega_{\tilde{q}\tilde{q}} = 1. \tag{23} \]

Here it is not so obvious what our free functions are, and an expansion along the lines of (20) doesn’t work. (It is shown in the appendix how to get equation (23) from our equation, showing that the two approaches are equivalent. Thus for any self-dual metric there will exist a corresponding function \( h \) that satisfies (19).)

From the work of [11] we know that the vectors \( U, V, W, X \) are proportional to a null tetrad that determines a self-dual metric. Indeed the tetrad is given by \( \sigma_a = f^{-1}V_a \), where \( V_a = (U, V, W, X) \) for \( a = 0, 1, 2, 3 \) and \( f^2 = \epsilon(U, V, W, X) \), for \( \epsilon \) the four dimensional volume form \( dt \wedge dx \wedge dy \wedge dz \). In our case, \( f^2 = -h_{tt} \) and our line element is

\[ ds^2 = dt (h_{ty} dy + h_{tz} dz) + dx (h_{xy} dy + h_{xz} dz) + \frac{1}{h_{tt}} (h_{ty} dy + h_{tz} dz)^2. \tag{24} \]
3 The Formal Solution.

We now construct, at least formally, the general solution to (19). Instead of working with this equation directly, it is helpful to define two functions $A = h_t$, $B = h_x$, and rewrite (19) in the equivalent form

$$A_t + A_y B_z - A_z B_y = 0,$$  \hspace{1cm} (25)

$$A_x = B_t.$$  \hspace{1cm} (26)

If we just viewing $B$ as some arbitrary function, then the solution to (25) is

$$A(t, x, y, z) = \exp \left[ \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \left[ (B_y(t_1) \partial_z - B_z(t_1) \partial_y) \ldots (B_y(t_n) \partial_z - B_z(t_n) \partial_y) \right] a_1(x, y, z) \right],$$  \hspace{1cm} (27)

where $a_1(x, y, z)$ is the value of $A$ at $t = 0$ as in (20). The exponential here is defined by its power series with the $n$'th term in this series being

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \left[ (B_y(t_1) \partial_z - B_z(t_1) \partial_y) \ldots (B_y(t_n) \partial_z - B_z(t_n) \partial_y) \right] a_1(x, y, z).$$  \hspace{1cm} (28)

We now must impose (26) as a consistency condition on this solution. This gives us

$$A(t, x, y, z) = \exp \left[ \int_0^t dt_1 \int_0^{t_1} dt_2 (A_{xy}(t_2, x, y, z) \partial_z - A_{xz}(t_2, x, y, z) \partial_y) \right] a_1(x, y, z).$$  \hspace{1cm} (29)

Formally, this equation can now be solved iteratively. We can make successive approximations

$$A^{(0)} = a_1(x, y, z),$$  \hspace{1cm} (30)
\[
A^{(1)} = \exp\left[\left( ta_{0xy} + \frac{t^2}{2}a_{1xy}\right) \partial_z - \left( ta_{0xz} + \frac{t^2}{2}a_{1xz}\right) \partial_y\right] a_1(x, y, z), \quad (31)
\]
\[
A^{(n+1)} = \exp\left[\int_0^t \int_0^{t_1} dt_1 dt_2 \left(A^{(n)}_{xy} \partial_z - A^{(n)}_{xz} \partial_y\right)\right] a_1(x, y, z), \quad (32)
\]
for \( n \geq 1 \). Then defining \( A = \lim_{n \to \infty} A^{(n)} \) gives the formal solution for \( A \). Integrating \( A \) with respect to \( t \) and imposing \( h(t = 0) = a_0(x, y, z) \) then gives us a solution of (13).

Finally, we note that (25) means that the quantity \( A(t, x, \tilde{y}, \tilde{z}) \) is \( t \) independent, where \( \tilde{y} \) and \( \tilde{z} \) are defined implicitly by
\[
\tilde{y}(t) = y + \int_0^t dt_1 B_z(t_1, x, \tilde{y}(t_1), \tilde{z}(t_1)), \quad (33)
\]
\[
\tilde{z}(t) = z - \int_0^t dt_1 B_y(t_1, x, \tilde{y}(t_1), \tilde{z}(t_1)). \quad (34)
\]
This may be important if we were to look for action angle variables for the system. It also implies that \( A(t, x, y, z) = a_1(x, y', z') \), where the coordinates \( y', z' \) are defined by \( \tilde{y}(t, x, y', z') = y, \tilde{z}(t, x, y', z') = z \). Thus the dynamics are characterised by a coordinate transformation in the \( y, z \) plane\(^4\).

4 **Group Methods**

Several powerful techniques have been developed for the study of partial differential equations \([12]\). One of the most powerful is that of group analysis \([13, 14]\). By studying the Lie algebra under which a given system of partial

\(^3\)It is beyond the scope of this paper to show that the \( A^{(n)} \) actually do converge to a well defined limit.

\(^4\)An earlier version of this paper incorrectly stated that this transformation was area preserving.
differential equations is invariant, we can hopefully find new solutions to these equations. One method of doing so is to look for similarity solutions which are left invariant by the action of some sub-algebra of this symmetry algebra. This will reduce the number of independent variables present in the equation, possibly reducing a partial differential equation to an ordinary differential equation. However, such similarity solutions, by construction, will have some symmetries imposed upon them, so this method is not very useful if one is looking for the general solution to a system of equations.

A more powerful method is to exponentiate the infinitesimal action of the Lie algebra into a group action, which takes one solution of the equation to another. However, even if this is possible, it is unlikely that the group action can be used to find the general solution to the equation from any given solution.

Instead of attempting to find the symmetry algebra of (19), it is easier to work with the equivalent system (25) and (26). We find that (25) and (26) admit a symmetry group defined by the infinitesimal generators

\[ \xi_1 = f_A \partial_t - f_x \partial_B, \]  
\[ \xi_2 = (t g_x + B g_A) \partial_t + g_A \partial_x - g_x \partial_A - (tg_x + B g_A)_x \partial_B, \]  
\[ \xi_3 = k t \partial_t + k x \partial_x + k y \partial_y, \]  
\[ \xi_4 = l_z \partial_y - l_y \partial_z, \]

where \( f \) and \( g \) arbitrary functions of \( x \) and \( A \), \( l \) is an arbitrary function of \( y \) and \( z \), and \( k \) is an arbitrary constant. \( \xi_3 \) just generates dilations, whereas \( \xi_4 \) generates area preserving diffeomorphisms in the \( y - z \) plane. Although
\[ \xi_4 \text{ gives a representation of } W_\infty \text{ (modulo cocycle terms) } [15], \text{ they are really only coordinate transformations, so are not too interesting. However, we have two interesting symmetries, generated by } \xi_1 \text{ and } \xi_2. \]

It is possible to exponentiate the action of \( \xi_1 \) directly for an arbitrary function \( f \). We find that if \( A(t, x, y, z) \) and \( B(t, x, y, z) \) are a solution of the system (25) and (26) then we can implicitly define a new solution, \( \tilde{A} \) and \( \tilde{B} \), by

\[
\tilde{A} = A(t + f_A(x, \tilde{A}), x, y, z), \quad \tilde{B} = B(t + f_A(x, \tilde{A}), x, y, z) + f_x(x, \tilde{A}),
\]

for any function \( f(x, A) \). Using this implicit form we can solve iteratively for the functions \( \tilde{A} \) and \( \tilde{B} \) given functions \( A, B \) and \( f \). This means that given one solution of (19), we can form an infinite dimensional family of solutions depending on that solution. For a given function \( g \) we can also exponentiate the action of \( \xi_2 \), although its action cannot be exponentiated directly for a general function \( g \).

Although both (37) and (38) give rise to infinite dimensional families of solutions from any given solution, they are not enough to derive a solution with arbitrary initial data from any given solution.

If we compute the commutators of generators \( \xi_1(f_i) \) and \( \xi_2(g_j) \) for arbitrary functions \( f_i \) and \( g_j \) we find that they obey the algebra

\[
[\xi_1(f_1), \xi_1(f_2)] = 0, \quad (40)
\]

\[
[\xi_1(f), \xi_2(g)] = \xi_1(f_A g_x - f_x g_A), \quad (41)
\]

\[
[\xi_2(g_1), \xi_2(g_2)] = \xi_2(g_{1A} g_{2x} - g_{1x} g_{2A}). \quad (42)
\]
If we define a basis for transformations

\[ (\alpha)T^m_i = \xi_\alpha(x^{i+1} A^{m+1}) \] (43)

for \( \alpha = 1, 2 \), where \( m \) and \( i \) are integers, then the above algebra becomes

\[ [^{(1)}T^m_i , ^{(1)}T^n_j] = 0, \] (44)

\[ [^{(1)}T^m_i , ^{(2)}T^n_j] = ((m + 1)(j + 1) - (n + 1)(i + 1))^{(1)}T^{m+n}_{i+j}, \] (45)

\[ [^{(2)}T^m_i , ^{(2)}T^n_j] = ((m + 1)(j + 1) - (n + 1)(i + 1))^{(2)}T^{m+n}_{i+j}. \] (46)

The algebra (44) - (46) is the algebra of locally area preserving diffeomorphisms which, modulo cocycle terms, is just the extended conformal algebra \( W_\infty \). Thus (44) - (46) represent some generalisation of \( W_\infty \). These are similar results to those found in [16].

We now note that equation (19) can be derived from the Lagrangian

\[ S = \int d^4x \left\{ \frac{1}{2} h_t^2 + \frac{1}{3} h_t (h_y h_{xz} - h_z h_{xy}) \right\}. \] (47)

The Hamiltonian is then

\[ H = \frac{1}{2} \int \Sigma d^3x (\pi - \frac{1}{3} (h_y h_{xz} - h_z h_{xy}))^2. \] (48)

where \( \pi = h_t + \frac{1}{3} (h_y h_{xz} - h_z h_{xy}) \) is the momentum canonically conjugate to \( h \). We now define the Poisson Bracket of functionals of \( h \) and \( \pi \) by

\[ \{\alpha, \beta\} = \int \Sigma d^3x \left( \frac{\delta \alpha}{\delta h} \frac{\delta \beta}{\delta \pi} - \frac{\delta \alpha}{\delta \pi} \frac{\delta \beta}{\delta h} \right), \] (49)

The algebra (44) - (46) now reflects the fact that we have two infinite families of conserved quantities\(^5\) of the form

\[ I(f(x, A)) = \int \Sigma f(x, A) d^3x \] (50)

\(^5\text{We are assuming that we can ignore surface terms.}\)
\[ I_2(g(x, A)) = \int_{\Sigma} (tg_x + B g_A) d^3x. \] (51)

These quantities all have vanishing Poisson brackets, i.e. they are in involution. (The fact that these quantities are time independent comes from the conservation equations

\[ \partial_t (f) + \partial_y (f B_z) - \partial_z (f B_y) = 0, \] (52)

and

\[ \partial_t (t g_x + B g_A) + \partial_x (g) - \partial_y (t g_x A B_z + g_A B B_z)
+ \partial_z (t g_x A B_y + g_A B B_y) = 0, \] (53)

which follow from (25) and (26).)

5 Solutions

We begin by looking for solutions that admit a *triholomorphic* Killing vector, \( \xi \). This means the three complex structures, \( J^i \), are invariant under the action of \( \xi \), i.e. \( \mathcal{L}_\xi J^i = 0 \), where \( \mathcal{L} \) is the Lie derivative. Using the relationship between the complex structures and the vectors \( V_i \) given in Section 2 and the fact that \( \xi \) is a Killing vector, we see that we require \( \mathcal{L}_\xi V_i = 0 \).

If \( \partial_x \) is a triholomorphic Killing vector, this means that \( \partial_x X = \partial_x W = 0 \), where \( X \) and \( W \) are as in (16) and (17). This means \( h \) is of the form \( a(t, y, z) + xb(y, z) \) for some functions \( a \) and \( b \). In terms of functions \( A = h_t \) and \( B = h_x \) this means that \( A = A(t, y, z) \), \( B = B(y, z) \), so (20) is automatically satisfied. If we take \( A(t = 0) = a_1(y, z) \) and \( B(t = 0) = \ldots \)
\( \phi(y, z) \), it is straightforward to show that the solution to (25) is then

\[
A(t, y, z) = \exp\left\{ t (\phi_y \partial_z - \phi_z \partial_y) \right\} a_1(y, z),
\]

\[
B(y, z) = \phi(y, z).
\]

(54)

For given functions \( \phi \) and \( a_1 \) it is straightforward to do the exponentiation, giving \( A \) explicitly. Using the exponentiated form of (33) and (34), we could now use these solutions to generate new solutions which had some restricted \( x \)-dependent initial data as well.

We can also consider metrics with a triholomorphic Killing vector, \( \partial_z \). This means we require \( \partial_z V_i = 0 \). In this case, we take \( h = -tz + g(t, x, y) \). We then recover the result [17] that \( g \) must satisfy the three dimensional Laplace equation \( g_{tt} + g_{xy} = 0 \). The general solution to this is known, and can be written in terms of two arbitrary functions \( a_0(x, y) \) and \( a_1(x, y) \). An almost identical reduction occurs if we take \( \partial_y \) as a triholomorphic Killing vector. Again, using the symmetries (33) and (34), we can generate infinite dimensional families of new solutions, that in general have no Killing vectors.

We note in passing that the solution corresponding to the multi-centre Eguchi-Hansen metric [17] is

\[
A = -z + \alpha \sum_{i=1}^{s} \sinh^{-1} \left( \frac{t - t_i}{2\sqrt{(x - x_i)(y - y_i)}} \right),
\]

(55)

\[
B = -\alpha \sum_{i=1}^{s} \frac{\sqrt{(t - t_i)^2 + 4(x - x_i)(y - y_i)}}{(x - x_i)}.
\]

(56)

where \( \alpha \) is a constant. This is the only metric with a triholomorphic Killing vector that has a non-singular real (euclidean) section [18].

At present, work is underway to complete a study of holomorphic Killing vectors and to see how the problem relates to the known results on such
metrics $[19, 20]$. 

6 Conclusion

We have shown how, at least formally, to construct the general complex metric with self-dual Riemann tensor. We have also studied the symmetry algebra of the system and found two infinite dimensional families of conserved quantities that have vanishing Poisson brackets.

It should be emphasised that all the considerations here have been inherently local in nature, and we have imposed no sorts of boundary conditions on our solutions at infinity. If we were to look for metrics that are well defined globally, this would lead us to cohomological problems $[21]$, which appear to be best tackled using the twistor formalism $[2]$. 

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A  The First Heavenly Equation

Starting with (25) and (26), instead of looking on $A$ as a function of $t, x, y$ and $z$ we take $A$ as a coordinate and look on $f \equiv t$ and $g \equiv B$ as functions of $p \equiv A, q \equiv x, r \equiv y, s \equiv z$. This transformation is well defined as long as $A_t \neq 0$. Inverting (25) and (26) gives

$$f_q = -g_p, \quad (57)$$

$$f_r g_s - f_s g_r = 1. \quad (58)$$

(57) means we can introduce a function $\Omega(p, q, r, s)$ such that $f = -\Omega_p, g = \Omega_q$. (58) then means that $\Omega$ must satisfy $\Omega_{ps} \Omega_{qr} - \Omega_{pr} \Omega_{qs} = 1$. Carrying out the same transformation on the line element (24), we find it becomes

$$ds^2 = \Omega_{pr} dp dr + \Omega_{ps} dp ds + \Omega_{qr} dq dr + \Omega_{qs} dq ds.$$ 

Thus we have recovered the Plebański formalism.

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