Stability analysis of Hadamard and Caputo-Hadamard fractional nonlinear systems without and with delay

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Abstract
This paper handles with the Hadamard and the Caputo-Hadamard fractional derivative and stability of related systems without and with delay. Firstly, the derivative inequalities are obtained, which is indispensable in applying the theorems derived in this paper. Then, for systems without delay, we get the stability results by using the Lyapunov direct method and for systems with delay, we explore two useful inequalities to verify the stability. Examples are presented with numerical simulations to illustrate the effectiveness of our results.

Keywords Hadamard system · Caputo-Hadamard system · Stability · Fractional Lyapunov method · Fractional Halanay inequality

Mathematics Subject Classification 26A33 · 34A08 · 37B25 · 34K37

1 Introduction

Differential equations with fractional derivatives have achieved growing attentions in recent years and many monographs have appeared [1–7]. The most common
definitions of fractional calculus (integration and differentiation) are for the Riemann-Liouville and the Caputo derivatives. Compared with these two types of definitions, the Hadamard fractional calculus, which was first introduced in 1892 by Hadamard [8], did not get much attention. Due to the integral kernel of form $\ln x$, the Hadamard derivative is efficient in characterizing some ultra-slow diffusion processes, besides, the fractional power $(t^{\alpha})$ is well suited to the case of the half-axis and is invariant relative to dilation.

Recently, the studies concerning the Hadamard equations are mainly on the fundamental theoretical fields. For example, the existence of the solutions are studied in [9–11], where the strip conditions and fixed point theorems are used. In [12], the authors present a new fractional comparison principle, and explored the stability of Hadamard fractional systems. In [13], the Gronwall inequality and the dependence of solution on parameters in Hadamard case are investigated. In [14], the asymptotic of higher order Caputo-Hadamard fractional equations is investigated. The regional controllability and observability of the Caputo-Hadamard fractional systems are studied in [15, 16]. For more results on the Hadamard calculus and the properties, we refer to [17–20].

For the application, we know that fractional calculus is widely used in anomalous diffusion, see [21, 22]. For the Hadamard fractional calculus, it is convenient to analyze the functions describing by logarithmic operator. In [23, 24], the authors used logarithmic function to describe the ultraslow diffusion, which implies that the Hadamard derivative is more effective in characterizing ultraslow diffusion than the Caputo derivative or the Riemann-Liouville derivative. For example, the logarithmic creep law is used to characterize the ultraslow diffusion of sedimentary rocks [25] and igneous rocks [26, 27]. In [28], the authors studied the dynamic evolution of COVID-19 caused by the Omicron variant via the Caputo-Hadamard fractional SEIR model and the prediction of the spread is obtained. Base on the above applications, the researches on Hadamard fractional calculus not only have theoretical significance, but also can provide theoretical support for practical applications.

The Lyapunov direct method is a useful tool for analysing the stability of nonlinear systems. Its fractional version is first introduced by Li et al. in [29] with the Riemann-Liouville and the Caputo fractional derivative. In [30–32], the authors obtained very useful fractional derivative inequalities which makes the fractional Lyapunov method to handle the stability more implementable. Some results are obtained to deal with the stability of fractional order systems, like fractional system with noninstantaneous impulses [33], the fractional boundary value problems with order $2 < \alpha < 3$, [34].

Considering the systems with delay, which is a common phenomenon in practical application, Halanay [35] proposed a powerful tool to verify the stability, called Halanay inequality. Recently, this classical inequality was extended to fractional case, see [36, 37], then, some other consequences are obtained for different kinds of systems, for instance, the fractional difference Halanay inequality [38], the fractional Halanay inequality with distributed delays [39], the fractional Halanay inequality for stochastic systems [40]. These researches are mainly on the Caputo or the Riemann-Liouville fractional case. Very recently, the Caputo-Hadamard Halanay inequality was explored in [41].
Motivated mainly by [29, 30, 37, 41], this paper concentrates on the stability of the Hadamard and the Caputo-Hadamard systems. The rest of this article is organized as follows: Section 2 presents some fundamental conceptions and properties of the Hadamard and the Caputo-Hadamard fractional calculus. In Section 3, some useful derivative inequalities are given. Section 4 investigates the stability of fractional systems without and with delay. To support our theory, some numerical examples are shown in Section 5.

2 Hadamard and Caputo-Hadamard fractional calculus

Now let us review some notions and properties which are necessary for this present work. For more related works on fractional calculus, Hadamard and Caputo-Hadamard operator, we refer to [4–7].

2.1 Definition and properties of Hadamard fractional calculus

Definition 1 [1, Page 110] For \( a \geq 0 \) and \( t > a \), the Hadamard integral and derivative of order \( \alpha > 0 \) are defined as

\[
H_a D^{-\alpha}_t y(t) = \int_a^t \frac{1}{\Gamma(\alpha)} \left( \ln \frac{t}{\eta} \right)^{\alpha-1} \frac{y(\eta)}{\eta} d\eta
\]

and

\[
H_a D^\alpha_t y(t) = \left( t \frac{d}{dt} \right) \frac{1}{\Gamma(m - \alpha)} \int_a^t \left( \ln \frac{t}{\eta} \right)^{m-\alpha-1} \frac{y(\eta)}{\eta} d\eta
\]

respectively, where \( m = [\alpha] + 1 \).

Particularly, for \( 0 < \alpha < 1 \), the Hadamard derivative becomes

\[
H_a D^\alpha_t y(t) = t \frac{d}{dt} \frac{1}{\Gamma(1 - \alpha)} \int_a^t \left( \ln \frac{t}{\eta} \right)^{-\alpha} \frac{y(\eta)}{\eta} d\eta.
\]

Property 1 [1, Page 112, Property 2.24] The Hadamard derivative of a constant is generally not equal to zero. In fact, for any constant \( c \),

\[
H_a D^\alpha_t c = \frac{c}{\Gamma(1 - \alpha)} \left( \ln \frac{t}{a} \right)^{-\alpha}, \quad 0 < \alpha < 1.
\]

Property 2 [1, Page 113, Corollary 2.4] The equality \( H_a D^\alpha_t y(t) = 0 \) is valid if and only if

\[
y(t) = \sum_{j=1}^m c_j \left( \ln \frac{t}{a} \right)^{\alpha-j}, \quad \text{for any } c_i \in \mathbb{R}
\]
where \( m = [\alpha] + 1 \).

Particularly, \( H_a D_t^\alpha y(t) = 0 \) holds iff \( y(t) = c (\ln \frac{t}{a})^{\alpha-1} \) for any \( c \in \mathbb{R} \) and \( 0 < \alpha < 1 \).

**Property 5** [1, Page 114, Property 2.26] For \( \alpha > 0 \), \( \beta > 0 \), if \( y(t) \) is integrable, then

\[
H_a D_t^{-\alpha} H_a D_t^{-\beta} y(t) = H_a D_t^{-(\alpha + \beta)} y(t).
\]

**Property 4** [1, Page 115, Property 2.27] If \( \alpha > \beta > 0 \) and \( y(t) \) is integrable, then

\[
H_a D_t^\beta H_a D_t^{-\alpha} y(t) = H_a D_t^{-\alpha + \beta} y(t). \tag{2.1}
\]

**Property 5** [1, Page 116, Property 2.28] If \( \alpha > 0 \) and \( y(t) \) is integrable, then

\[
H_a D_t^\alpha H_a D_t^{-\alpha} y(t) = y(t). \tag{2.2}
\]

**Property 6** Assume that \( y(t) \) is continuous and the derivative \( H_a D_t^{\beta - \alpha} y(t) \) exists when \( \beta > \alpha > 0 \), then the formula

\[
H_a D_t^\beta H_a D_t^{-\alpha} y(t) = H_a D_t^{-\alpha + \beta} y(t) \tag{2.3}
\]

holds for \( \alpha, \beta > 0 \).

**Proof** If \( \alpha > \beta \), (2.3) holds directly according to Property 4.

If \( \beta > \alpha \), let \( p = [\beta] + 1 \) and \( q = [\beta - \alpha] + 1 \), it can be seen that \( q \leq p \) holds. Then, by the definition of Hadamard derivative and Property 3, we can obtain

\[
\begin{aligned}
H_a D_t^\beta H_a D_t^{-\alpha} y(t) &= \frac{d^p}{dt^p} \left[ H_a D_t^{-(p-\beta)} \left( H_a D_t^{-\alpha} y(t) \right) \right] = \frac{d^p}{dt^p} \left\{ H_a D_t^{\beta - \alpha - p} y(t) \right\} \\
&= \frac{d^q}{dt^q} \left[ H_a D_t^{\beta - \alpha - q} y(t) \right] = H_a D_t^{\beta - \alpha} y(t).
\end{aligned}
\]

\( \Box \)

**Property 7** [1, Page 116, Theorem 2.3] If \( y(t) \) is integrable and \( H_a D_t^{m-\alpha} y(t) \in AC^m[a, b] \), then

\[
H_a D_t^{-\alpha} H_a D_t^{\alpha} y(t) = y(t) - \sum_{k=1}^{m-1} \left( \frac{d}{dt} \right)^{m-k-1} \frac{H_a D_t^{\alpha-k-1} y(a)}{\Gamma(\alpha - k + 1)} \left( \ln \frac{t}{a} \right)^{\alpha-k-1},
\]

where \( m = [\alpha] + 1 \). In particular, it has

\[
H_a D_t^{-\alpha} H_a D_t^{\alpha} y(t) = y(t) - \frac{H_a D_t^{\alpha-1} y(a)}{\Gamma(\alpha)} \left( \ln \frac{t}{a} \right)^{\alpha-1} \text{ for } 0 < \alpha < 1.
\]
**Property 8** Assume that \( \alpha > 0, 0 < \beta < 1 \), then

\[
H_a D_t^{-\alpha} \left( H_a D_t^{\beta} y(t) \right) = \frac{H_a D_t^{-\alpha} y(t)}{\Gamma(\alpha)} - \frac{H_a D_t^{\beta-1} y(a)}{\Gamma(\beta)} \left( \ln \frac{t}{a} \right)^{\beta-1}. \tag{2.4}
\]

**Proof** To prove the equality (2.4), we use Property 3 (if \( \beta \leq \alpha \)) or Property 6 (if \( \alpha \leq \beta \)) and then Property 7, which gives

\[
H_a D_t^{-\alpha} \left( H_a D_t^{\beta} y(t) \right) = \frac{H_a D_t^{-\alpha} y(t)}{\Gamma(\beta)} - \frac{H_a D_t^{\beta-1} y(a)}{\Gamma(\beta)} \left( \ln \frac{t}{a} \right)^{\beta-1}, \tag{2.5}
\]

where the known Hadamard integral and the derivative of the logarithmic function [1, Page 112, Property 2.24]

\[
H_a D_t^{-\alpha} \left( \ln \frac{t}{a} \right)^{p-1} = \frac{\Gamma(p)}{\Gamma(p+\alpha)} \left( \ln \frac{t}{a} \right)^{p+\alpha-1},
\]

\[
H_a D_t^{\alpha} \left( \ln \frac{t}{a} \right)^{p-1} = \frac{\Gamma(p)}{\Gamma(p-\alpha)} \left( \ln \frac{t}{a} \right)^{p-\alpha-1},
\]

are used. \( \square \)

**Lemma 1** Let \( 0 < \alpha < 1 \) and \( y(t), z(t) \in \mathbb{R} \), if \( H_a D_t^{\alpha-1} y(a) \geq H_a D_t^{\alpha-1} z(a) \) and \( H_a D_t^{\alpha} y(t) \geq H_a D_t^{\alpha} z(t) \) hold. Then \( y(t) \geq z(t) \) for \( t > a \).

**Proof** Since \( H_a D_t^{\alpha} y(t) \geq H_a D_t^{\alpha} z(t) \), a nonnegative function \( R(t) \) exists satisfying

\[
H_a D_t^{\alpha} y(t) = R(t) + H_a D_t^{\alpha} z(t). \tag{2.6}
\]

Taking the Hadamard integral of order \( \alpha \) on both sides of (2.6) and using Property 7 yields

\[
y(t) - \frac{H_a D_t^{\alpha-1} y(a)}{\Gamma(\alpha)} \left( \ln \frac{t}{a} \right)^{\alpha-1} = H_a D_t^{-\alpha} R(t) + z(t) - \frac{H_a D_t^{\alpha-1} z(a)}{\Gamma(\alpha)} \left( \ln \frac{t}{a} \right)^{\alpha-1}. \]

That is

\[
y(t) = z(t) + \frac{1}{\Gamma(\alpha)} \left( \ln \frac{t}{a} \right)^{\alpha-1} \left( H_a D_t^{\alpha-1} y(a) - H_a D_t^{\alpha-1} z(a) \right) + H_a D_t^{-\alpha} R(t). \tag{2.7}
\]

By the assumption \( H_a D_t^{\alpha-1} y(a) \geq H_a D_t^{\alpha-1} z(a) \), then

\[
y(t) \geq z(t) + H_a D_t^{-\alpha} R(t).
\]
From the definition of Hadamard integral, \( R(t) \geq 0 \) implies \( H_a D_t^{-\alpha} R(t) \geq 0 \). Then \( y(t) \geq z(t) \) holds.

### 2.2 Definition and properties of Caputo-Hadamard fractional calculus

**Definition 2** [1, 20, 42] The Caputo-Hadamard derivative of order \( \alpha > 0 \) is defined by

\[
CH_a D_t^\alpha y(t) = H_a D_t^\alpha \left[ y(t) - \sum_{k=0}^{m-1} \frac{\delta^k y(a)}{k!} \left( \ln \frac{t}{a} \right)^k \right], \quad t \geq a,
\]

where \( \delta = t \frac{d}{dt} \) and \( m = [\alpha] + 1 \).

Particularly, for \( 0 < \alpha < 1 \),

\[
CH_a D_t^\alpha y(t) = H_a D_t^\alpha [y(t) - y(a)].
\]

**Lemma 2** [20, Lemma 2.1] [1, Equation 2.7.35] For \( y(t) \in AC^m[a, b] \), \( CH_a D_t^\alpha y(t) \) exist everywhere and

\[
CH_a D_t^\alpha y(t) = \frac{1}{\Gamma(m - \alpha)} \int_a^t \left( \ln \frac{t}{\eta} \right)^{m-\alpha-1} \left( \eta \frac{d}{d\eta} \right)^m \frac{y(\eta)}{\eta} \frac{d\eta}{\eta}, \text{ where } m = [\alpha] + 1.
\]

Particularly, for \( 0 < \alpha < 1 \),

\[
CH_a D_t^\alpha y(t) = \frac{1}{\Gamma(1 - \alpha)} \int_a^t \left( \ln \frac{t}{\eta} \right)^{-\alpha} y'(\eta) d\eta.
\]

**Lemma 3** [20, Theorem 2.2] For any continuous function \( y(t) \), \( CH_a D_t^\alpha y(t) \) is continuous and

\[
CH_a D_t^\alpha y(a) = 0, \text{ where } m = [\alpha] + 1.
\]

**Property 9** [20, Lemma 2.4] For \( \alpha > 0 \), it has

\[
CH_a D_t^\alpha \left( H_a D_t^{-\alpha} y \right)(t) = y(t).
\]

**Property 10** [20, Lemma 2.5] For \( y(t) \in AC[a, b] \) and \( 0 < \alpha < 1 \), it has

\[
H_a D_t^{-\alpha} \left( CH_a D_t^\alpha y \right)(t) = y(t) - y(a).
\]

**Lemma 4** For \( 0 < \alpha < 1 \), and \( y(t), z(t) \in \mathbb{R} \), if \( y(a) \geq z(a) \) and \( CH_a D_t^\alpha y(t) \geq CH_a D_t^\alpha z(t) \) hold. Then \( y(t) \geq z(t) \) for \( t \geq a \).
Proof Since $C^H_a D^\alpha_t y(t) \geq C^H_a D^\alpha_t z(t)$, a nonnegative function $R(t)$ exists satisfying

$$C^H_a D^\alpha_t y(t) = R(t) + C^H_a D^\alpha_t z(t). \tag{2.8}$$

Taking the Hadamard integral of order $\alpha$ on both sides of (2.8) and using Property 10, yields

$$y(t) - y(a) = \frac{H^D_{a} - \alpha t}{a} R(t) + z(t) - z(a).$$

By the assumption $y(a) \geq z(a)$, then

$$y(t) \geq \frac{H^D_{a} - \alpha t}{a} R(t) + z(t). \tag{2.9}$$

From the definition of Hadamard integral, $R(t) \geq 0$ implies $\frac{H^D_{a} - \alpha t}{a} R(t) \geq 0$. Then $y(t) \geq z(t)$ holds. $\square$

2.3 Other definitions and properties

Definition 3 [43, Definition 5.11] A continuous function $k(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is in class-$K$ if $k(0) = 0$ and it is strictly increasing.

Definition 4 The one parameter and two parameter Mittag-Leffler functions are defined as

$$E_p(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(pk + 1)} \text{ and } E_{p,q}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(pk + q)}$$

respectively, where $p > 0$, $q > 0$.

Lemma 5 Let $0 < \alpha < 1$ and $y(a)$ be a nonnegative constant, then

$$C^H_a D^\alpha_t y(t) \leq \frac{H^D_{a}}{a} D^\alpha_t y(t).$$

Proof By the definition of Caputo-Hadamard derivative, for $0 < \alpha < 1$, it has

$$C^H_a D^\alpha_t y(t) = \frac{H^D_{a}}{a} D^\alpha_t [y(t) - y(a)] = \frac{H^D_{a}}{a} D^\alpha_t y(t) - \frac{y(a)}{\Gamma(1 - \alpha)} \left( \ln \frac{t}{a} \right)^\alpha.$$

Since $y(a) \geq 0$, we get $C^H_a D^\alpha_t y(t) \leq \frac{H^D_{a}}{a} D^\alpha_t y(t)$. $\square$

Lemma 6 [1, Page 235, Example 4.13] The solution to the Cauchy problem with Hadamard derivative

$$\begin{cases} \frac{H^D_{a}}{a} D^\alpha_t y(t) = \lambda y(t) + f(t), & 0 < \alpha < 1, \\ \frac{H^D_{a}}{a} D^{(1-\alpha)}_t y(a) = b, & \lambda, b \in \mathbb{R}, \end{cases}$$

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can be expressed as
\[ y(t) = b\left(\ln \frac{t}{a}\right)^{\alpha - 1} E_{\alpha,a} \left[ \lambda \left(\ln \frac{t}{a}\right)^{\alpha} \right] + \int_{a}^{t} \left(\ln \frac{t}{\xi}\right)^{\alpha - 1} E_{\alpha,a} \left[ \lambda \left(\ln \frac{t}{\xi}\right)^{\alpha} \right] \frac{f(\xi)}{\xi} d\xi. \]

**Lemma 7** [41] The solution to the Cauchy problem with Caputo-Hadamard derivative
\[
\begin{cases}
C_{a}^{H} D_{t}^{\alpha} y(t) = \lambda y(t) + f(t), & 0 < \alpha < 1, \\
y(a) = b, & \lambda, b \in \mathbb{R},
\end{cases}
\]
can be expressed as
\[ y(t) = bE_{\alpha,a} \left[ \lambda \left(\ln \frac{t}{a}\right)^{\alpha} \right] + \int_{a}^{t} \left(\ln \frac{t}{\xi}\right)^{\alpha - 1} E_{\alpha,a} \left[ \lambda \left(\ln \frac{t}{\xi}\right)^{\alpha} \right] \frac{f(\xi)}{\xi} d\xi. \]

### 3 Several derivative inequaities for Hadamard and Caputo-Hadamard derivative

This section presents several key lemmas on the inequalities of the Hadamard and the Caputo-Hadamard derivative, which makes the Lyapunov method and the Halanay inequality applicable to the Hadamard and the Caputo-Hadamard fractional systems.

#### 3.1 Derivative inequaities for Hadamard fractional derivative

**Theorem 1** Let \( \alpha \in (0, 1) \) and \( y(t) \in \mathbb{R}^{n} \) be differentiable, \( V(y) : \mathbb{R}^{n} \to \mathbb{R} \) be continuous and differentiable, suppose that \( V(0) = 0 \) and \( V(y) \) is convex over \( \mathbb{R}^{n} \), then
\[ \frac{H_{a}^{\alpha} D_{t}^{\alpha} V(y(t))}{D_{t}^{\alpha} y(t)} \leq \left( \frac{H_{a}^{\alpha} D_{t}^{\alpha} y(t)}{D_{t}^{\alpha} y(t)} \right)^{\top} \frac{\partial V}{\partial y}. \]  
\[ (3.1) \]

**Proof** Clearly, inequality (3.1) is equivalent to
\[ \frac{H_{a}^{\alpha} D_{t}^{\alpha} V(y(t))}{D_{t}^{\alpha} y(t)} - \left( \frac{H_{a}^{\alpha} D_{t}^{\alpha} y(t)}{D_{t}^{\alpha} y(t)} \right)^{\top} \frac{\partial V}{\partial y} \leq 0. \]  
\[ (3.2) \]

It suffices to prove that (3.2) holds. Due to the Newton-Leibniz formula, \( y(t) \) can be expressed as
\[ y(t) = y(a^{+}) + \int_{a}^{t} \dot{y}(\xi) d\xi \]
\[ = y(a^{+}) + \int_{a}^{t} \frac{\xi \dot{y}(\xi)}{\xi} d\xi = y(a^{+}) + \frac{H_{a}^{\alpha} D_{t}^{-1}(t \dot{y}(t))}. \]  
\[ (3.3) \]

derivative on both sides of (3.3) and using (2.1), we have
Combing (3.4) and (3.5), we derive

\[
\frac{\text{d}^\alpha}{\text{d}t^\alpha} y(t) = \frac{\text{d}^\alpha}{\text{d}t^\alpha} y(a^+) + \frac{\text{d}^{\alpha-1}}{\text{d}t^{\alpha-1}}(t \dot{y}(t))
\]

\[
= \frac{y(a^+)}{\Gamma(1 - \alpha)} \left( \ln \frac{t}{a} \right)^{-\alpha} + \frac{1}{\Gamma(1 - \alpha)} \int_a^t \left( \ln \frac{t}{\xi} \right)^{-\alpha} \dot{y}(\xi) d\xi \]

\[
= \frac{1}{\Gamma(1 - \alpha)} \left[ y(a^+) \left( \ln \frac{t}{a} \right)^{-\alpha} + \int_a^t \left( \ln \frac{t}{\xi} \right)^{-\alpha} \dot{y}(\xi) d\xi \right].
\] (3.4)

Similar to (3.3) and (3.4), the Hadamard derivative of \( V(y(t)) \) is

\[
\frac{\text{d}^\alpha}{\text{d}t^\alpha} V(y(t)) = \frac{1}{\Gamma(1 - \alpha)} \left[ V(y(a^+)) \left( \ln \frac{t}{a} \right)^{-\alpha} + \int_a^t \left( \ln \frac{t}{\xi} \right)^{-\alpha} \dot{y}(\xi) \frac{\partial V(y(\xi))}{\partial y} d\xi \right].
\] (3.5)

Combining (3.4) and (3.5), we derive

\[
\frac{\text{d}^\alpha}{\text{d}t^\alpha} V(y(t)) - \left( \frac{\text{d}^\alpha}{\text{d}t^\alpha} y(t) \right)^T \frac{\partial V}{\partial y} \]

\[
= \frac{1}{\Gamma(1 - \alpha)} \left( \ln \frac{t}{a} \right)^{-\alpha} \left[ V(y(a^+)) \left( \ln \frac{t}{a} \right)^{-\alpha} - \left( y(a^+) \right)^T \frac{\partial V(y(t))}{\partial y} \right]
\]

\[
+ \int_a^t \frac{1}{\Gamma(1 - \alpha)} \left( \ln \frac{t}{\xi} \right)^{-\alpha} \dot{y}(\xi) \left[ \frac{\partial V(y(\xi))}{\partial y} - \frac{\partial V(y(t))}{\partial y} \right] d\xi.
\] (3.6)

Calculating the integral term of (3.6) leads to

\[
\int_a^t \left( \ln \frac{t}{\xi} \right)^{-\alpha} \dot{y}(\xi) \left[ \frac{\partial V(y(\xi))}{\partial y} - \frac{\partial V(y(t))}{\partial y} \right] d\xi
\]

\[
= \int_a^t \left( \ln \frac{t}{\xi} \right)^{-\alpha} \left[ \frac{\partial V(y(\xi))}{\partial y} - \frac{\partial V(y(t))}{\partial y} \right] d\xi
\]

\[
= \int_a^t \left( \ln \frac{t}{\xi} \right)^{-\alpha} \left[ - V(y(\xi)) \right] d\xi
\]

\[
= - \lim_{\xi \to t} \left( \ln \frac{t}{\xi} \right)^{-\alpha} \left[ - V(y(\xi)) \right] + \left( y(t) \right)^T \frac{\partial V(y(t))}{\partial y} - V(y(t))
\]
\[
\times V(y(\xi)) - V(y(t)) \right] - \left( \ln \frac{t}{a} \right)^{-\alpha} \left[ V(y(a)) - \left( y(a) \right)^\top \frac{\partial V(y(t))}{\partial y} + \left( y(t) \right)^\top \frac{\partial V(y(t))}{\partial y} \right] \\
+ \left( y(t) \right)^\top \frac{\partial V(y(t))}{\partial y} - V(y(t)) \right] - \alpha \int_a^t \frac{1}{\xi} \left( \ln \frac{t}{\xi} \right)^{-\alpha - 1} \\
\times \left[ \left( y(t) - y(\xi) \right)^\top \left( \frac{\partial V(y(t))}{\partial y} \right) - \left( V(y(t)) - V(y(\xi)) \right) \right] d\xi. \tag{3.7}
\]

By the L’Hôpital’s rule, one gets
\[
\lim_{\xi \to t^-} \left( \ln \frac{t}{\xi} \right)^{-\alpha} \left[ V(y(\xi)) - \left( y(\xi) \right)^\top \frac{\partial V(y(t))}{\partial y} + \left( y(t) \right)^\top \frac{\partial V(y(t))}{\partial y} \right] \\
- V(y(t)) = \lim_{\xi \to t^-} \left[ \left( \dot{y}(\xi) \right)^\top \frac{\partial V(y(t))}{\partial y} - \left( \dot{y}(\xi) \right)^\top \frac{\partial V(y(t))}{\partial y} \right] = 0. \tag{3.8}
\]

Combing (3.6), (3.7) and (3.8), it is seen that (3.2) can be reduced to
\[
\frac{H D^\alpha_t}{a} V(y(t)) - \left( \frac{H D^\alpha_t}{a} y(t) \right)^\top \frac{\partial V(y(t))}{\partial y} \\
= \frac{1}{\Gamma(1 - \alpha)} \left( \ln \frac{t}{a} \right)^{-\alpha} \left[ V(y(t)) - \left( y(t) \right)^\top \frac{\partial V(y(t))}{\partial y} \right] \\
- \frac{\alpha}{\Gamma(1 - \alpha)} \int_a^t \frac{1}{\xi} \left( \ln \frac{t}{\xi} \right)^{-\alpha - 1} \left[ \left( y(t) - y(\xi) \right)^\top \left( \frac{\partial V(y(t))}{\partial y} \right) \\
- \left( V(y(t)) - V(y(\xi)) \right) \right] d\xi \leq 0. \tag{3.9}
\]

Noting
\[
V(y(\xi)) = V(y(t)) + \left( y(\xi) - y(t) \right)^\top \frac{\partial V(y(t))}{\partial y} \\
+ \frac{1}{2} \left( y(\xi) - y(t) \right)^\top \frac{\partial^2 V(y(\xi))}{\partial y^2} \left( y(\xi) - y(t) \right)
\]

and
\[
\lim_{\xi \to t^-} \left( \ln \frac{t}{\xi} \right)^{-\alpha - 1} \left[ \left( y(t) - y(\xi) \right)^\top \frac{\partial V(y(t))}{\partial y} - \left( V(y(t)) - V(y(\xi)) \right) \right] \\
= \lim_{\xi \to t^-} \frac{1}{2} \left( \ln \frac{t}{\xi} \right)^{-\alpha - 1} \left( y(\xi) - y(t) \right)^\top \frac{\partial^2 V(y(\xi))}{\partial y^2} \left( y(\xi) - y(t) \right) \\
= 0,
\]

we know
\[
\int_a^t \frac{1}{\xi} \left( \ln \frac{t}{\xi} \right)^{-\alpha - 1} \left[ \left( y(t) - y(\xi) \right)^\top \left( \frac{\partial V(y(t))}{\partial y} \right) - \left( V(y(t)) - V(y(\xi)) \right) \right] d\xi
\]
makes sense. From the property of convex function and $V(y(t))$ being convex, it is seen that

$$\left( y(t) - y(\xi) \right)^{\top} \left( \frac{\partial V(y(t))}{\partial y} \right) - \left( V(y(t)) - V(y(\xi)) \right) \geq 0$$

holds. This, together with $V(0) = 0$, yields $V(y(t)) \leq \left( y(t) \right)^{\top} \frac{\partial V(y(t))}{\partial y}$, which implies that (3.9) is always true, hence $H_{a} D_{t}^{\alpha} V(y(t)) \leq \left( \frac{\partial V(y(t))}{\partial y} \right)^{\top} H_{a} D_{t}^{\alpha} y(t)$ holds. 

If we take $V(y) = y^{2}(t)$ for $y(t) \in \mathbb{R}$ or $V(y) = y^{\top}(t)y(t)$ for $y(t) \in \mathbb{R}^{n}$ in Theorem 1, it is clear that $V(y)$ is convex and $V(0) = 0$. Then we can obtain the following corollaries directly.

**Corollary 1** Let $0 < \alpha < 1$ and $y(t)$ be differentiable, then

$$\frac{1}{2}H_{a} D_{t}^{\alpha} y^{2}(t) \leq y(t)^{\top} H_{a} D_{t}^{\alpha} y(t), \quad \text{for } y(t) \in \mathbb{R},$$

and

$$\frac{1}{2}H_{a} D_{t}^{\alpha} y^{\top}(t)y(t) \leq \left( H_{a} D_{t}^{\alpha} y(t) \right)^{\top} y(t), \quad \text{for } y(t) \in \mathbb{R}^{n}$$

hold.

**Corollary 2** Let $0 < \alpha < 1$ and $y(t) \in \mathbb{R}^{n}$ be differentiable vector function, then for constant matrix $S \in \mathbb{R}^{n \times n}$, it has

$$\frac{1}{2}H_{a} D_{t}^{\alpha} (y^{\top}(t)Sy(t)) \leq \left( H_{a} D_{t}^{\alpha} y(t) \right)^{\top} Sy(t),$$

where $S$ is symmetric and positive definite.

**Proof** From the property of matrix $S$, one can see that $V(y) = y^{\top}(t)Sy(t)$ is convex and satisfies $V(0) = 0$. The result can be derived directly according to Theorem 1. 

### 3.2 Derivative inequalities for Caputo-Hadamard derivative

**Theorem 2** Let $y(t) \in \mathbb{R}^{n}$ be differentiable, $V(y) : \mathbb{R}^{n} \to \mathbb{R}$ be continuous and differentiable, suppose $V(y)$ is convex over $\mathbb{R}^{n}$, then for any $t \geq a$,

$$C_{a} H_{a} D_{t}^{\alpha} V(y(t)) \leq \left( C_{a} H_{a} D_{t}^{\alpha} y(t) \right)^{\top} \frac{\partial V}{\partial y}, \quad \alpha \in (0, 1). \quad (3.10)$$

**Proof** Clearly, inequality (3.10) means

$$C_{a} H_{a} D_{t}^{\alpha} V(y(t)) - \left( \frac{\partial V}{\partial y} \right)^{\top} C_{a} H_{a} D_{t}^{\alpha} y(t) \leq 0.$$
Due to the Newton-Leibniz formula, one has $y(t) = y(a) + \frac{H}{a} D_{t}^{-1}(t \dot{y}(t))$. Then by Lemma 3 and Property 6, we have

$$C_{t}^{H} D_{t}^{\alpha} y(t) = \frac{H}{a} D_{t}^{-(1-\alpha)}(t \dot{y}(t)) = \int_{a}^{t} \frac{1}{\Gamma(1-\alpha)} \left( \ln \frac{t}{\xi} \right)^{-\alpha} \dot{y}(\xi) d\xi. \quad (3.11)$$

Similarly, for $V(y(t))$, it has

$$V(y(t)) = V(y(a)) + \frac{H}{a} D_{t}^{-1} \left[ \left( \dot{y}(t) \right)^{\top} \left( t \frac{\partial V(y(t))}{\partial y} \right) \right], \quad (3.12)$$

and

$$C_{t}^{H} D_{t}^{\alpha} V(y(t)) = C_{t}^{H} D_{t}^{\alpha-1} \left[ \left( \dot{y}(t) \right)^{\top} \left( t \frac{\partial V(y(t))}{\partial y} \right) \right]$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} \left( \ln \frac{t}{\xi} \right)^{-\alpha} \left( \dot{y}(\xi) \right)^{\top} \left( \frac{\partial V(y(\xi))}{\partial y} \right) d\xi. \quad (3.13)$$

Combing (3.11) and (3.13), we have

$$C_{t}^{H} D_{t}^{\alpha} V(y(t)) = \left( C_{t}^{H} D_{t}^{\alpha} y(t) \right) ^{\top} \frac{\partial V}{\partial y}$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} \left( \ln \frac{t}{\xi} \right)^{-\alpha} \left( \dot{y}(\xi) \right)^{\top} \left[ \frac{\partial V(y(\xi))}{\partial y} - \frac{\partial V(y(t))}{\partial y} \right] d\xi$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} \left( \ln \frac{t}{\xi} \right)^{-\alpha} \left[ \frac{\partial V(y(\xi))}{\partial y} - \frac{\partial V(y(t))}{\partial y} \right]^{\top} d\xi$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} \left( \ln \frac{t}{\xi} \right)^{-\alpha} \left[ V(y(\xi)) - V(y(t)) \right]^{\top} d\xi$$

$$= \int_{a}^{t} \left( \ln \frac{t}{\xi} \right)^{-\alpha} \left[ \left( y(t) - y(a) \right) ^{\top} \frac{\partial V(y(t))}{\partial y} - \left( V(y(t)) - V(y(\xi)) \right) \right] d\xi$$

$$= -\frac{1}{\Gamma(1-\alpha)} \left( \ln \frac{t}{a} \right)^{-\alpha} \left[ \left( V(y(t)) - V(y(\xi)) \right) \left( \frac{\partial V(y(t))}{\partial y} \right) \right]$$

$$- \left( V(y(t)) - V(y(\xi)) \right) \left( \frac{\partial V(y(t))}{\partial y} \right)$$

$$\times \left( \ln \frac{t}{\xi} \right)^{-\alpha-1} \frac{a}{\Gamma(1-\alpha)} \int_{a}^{t} \left( \ln \frac{t}{\xi} \right)^{-\alpha-1} d\xi. \quad (3.15)$$
Since \( V(y(t)) \) is convex over \( \mathbb{R}^n \), we know from the property of convex function that
\[
\left( y(t) - y(a) \right)^\top \left( \frac{\partial V(y(t))}{\partial y} \right) - \left( V(y(t)) - V(y(a)) \right) \geq 0
\]
and
\[
\left( y(t) - y(\xi) \right)^\top \left( \frac{\partial V(y(t))}{\partial y} \right) - \left( V(y(t)) - V(y(\xi)) \right) \geq 0,
\]
which, jointly with (3.14) and (3.15), gives
\[
\frac{CH}{\alpha} a D_t^\alpha V(y(t)) - \left( \frac{CH}{\alpha} a D_t^\alpha y(t) \right)^\top \frac{\partial V(y(t))}{\partial y} \leq 0.
\]
The proof is completed. \( \square \)

Analogous to Corollary 1 and Corollary 2, we can obtain the following two corollaries.

**Corollary 3** Let \( 0 < \alpha < 1 \) and function \( y(t) \) be differentiable, then
\[
\frac{1}{2} \frac{CH}{\alpha} a D_t^\alpha y^2(t) \leq y(t) \frac{CH}{\alpha} a D_t^\alpha y(t), \quad \text{for } y(t) \in \mathbb{R},
\]
and
\[
\frac{1}{2} \frac{CH}{\alpha} a D_t^\alpha y^\top(t) y(t) \leq \left( \frac{CH}{\alpha} a D_t^\alpha y(t) \right)^\top y(t), \quad \text{for } y(t) \in \mathbb{R}^n
\]
hold.

**Corollary 4** Let \( 0 < \alpha < 1 \) and function \( y(t) \in \mathbb{R}^n \) be differentiable, then for constant matrix \( S \in \mathbb{R}^{n \times n} \), it has
\[
\frac{1}{2} \frac{CH}{\alpha} a D_t^\alpha (y^\top(t) S y(t)) \leq \left( \frac{CH}{\alpha} a D_t^\alpha y(t) \right)^\top S y(t),
\]
where \( S \) is symmetric and positive definite.

## 4 Fractional nonlinear systems and Hadamard Mittag-Leffler stability

Consider the following fractional nonlinear system
\[
a D_t^\alpha y(t) = f(t, y), \quad (4.1)
\]
where $0 < \alpha < 1$, $a D_t^\alpha$ represents the Hadamard derivative or the Caputo-Hadamard derivative. $y \in \mathbb{R}^n$, $f : (a, \infty \to (a, \infty)$ (for the Hadamard case while $[a, \infty \to [a, \infty$ for the Caputo-Hadamard case) $\times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function.

The initial condition of system (4.1) corresponding to the Hadamard derivative and the Caputo-Hadamard derivative are, respectively,

$$H_a D_t^{(1-\alpha)}y(a) = y_a,$$

and

$$y(a) = y_a.$$

**Definition 5** $y_0$ is said to be an equilibrium point of the Hadamard or the Caputo-Hadamard fractional system (4.1) if and only if $a D_t^\alpha y_0 = f(t, y_0)$.

For the initial value and equilibrium point of the Hadamard fractional systems and the Caputo-Hadamard fractional systems, one can see the explanation in [27].

In the follows, we suppose that $y = 0$ is the equilibrium point of system (4.1). Now let us introduce the Lyapunov stability of the zero solution to system (4.1).

**Definition 6** [27, Definition 2.8,2.9] The solution of system (4.1) is said to be:

1. stable, if for any initial value $y_a$, there exists an $\varepsilon > 0$ such that $\|y(t)\| < \varepsilon$ for all $t > a > 0$ (or $t \geq a > 0$ for Caputo-Hadamard case).
2. asymptotically stable, iff $\lim_{t \to +\infty} \|y(t)\| = 0$.

### 4.1 Lyapunov method for fractional nonlinear systems without delay

Now we present the Lyapunov method for the Hadamard and the Caputo-Hadamard fractional systems, combining the inequalities of the Hadamard and Caputo-Hadamard derivative, these theorems can be used to check the stability for some specific examples.

**Theorem 3** Let $V(y) : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function satisfying

$$c_1\|y\|^u \leq V(y) \leq c_2\|y\|^u, \quad \text{and}$$

$$CH_a D_t^{\alpha} V(y) \leq -c_3\|y\|^u,$$

where $\alpha \in (0, 1), c_1, c_2, c_3, u, v$ are arbitrary positive constants. Then system (4.1) with Caputo-Hadamard derivative is asymptotically stable.

**Proof** Combining (4.2) and (4.3), we have

$$CH_a D_t^{\alpha} V(y) \leq -\frac{c_3}{c_2} V(y),$$

which implies that a nonnegative function $R(t)$ exists satisfying

$$CH_a D_t^{\alpha} V(y) = -\frac{c_3}{c_2} V(y) - R(t).$$


By Lemma 7, the solution of system (4.4) with initial $V(y(a)) = V_0$ is

$$V(y) = V_0 E_\alpha \left( - \frac{c_3}{c_2} \left( \ln \frac{t}{a} \right)^\alpha \right)$$

$$- \int_a^t \left( \ln \frac{t}{\xi} \right)^{\alpha-1} E_{\alpha,\alpha} \left( - \frac{c_3}{c_2} \left( \ln \frac{t}{\xi} \right)^\alpha \right) R(\xi) \frac{d\xi}{\xi}. \quad (4.5)$$

Since $R(t) \geq 0$, the integral term in (4.5) is non-negative. Then,

$$V(y) \leq V_0 E_\alpha \left( - \frac{c_3}{c_2} \left( \ln \frac{t}{a} \right)^\alpha \right). \quad (4.6)$$

Substituting (4.6) into (4.2) yields

$$\|y(t)\| \leq \left[ \frac{V_0}{c_1} E_\alpha \left( - \frac{c_3}{c_2} \left( \ln \frac{t}{a} \right)^\alpha \right) \right]^{\frac{1}{u}}.$$ 

By the property of the Mittag-Leffler function, for $\alpha \in (0, 1)$, $E_\alpha(-t)$ tends to zero as $t \to \infty$, so $y(t) \to 0$ as $t \to \infty$. This implies the asymptotically stable of system (4.1). The proof is completed. $\square$

**Corollary 5** Let $V(y) : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function convex over $\mathbb{R}^n$ and satisfies

$$c_1 \|y\|^u \leq V(y) \leq c_2 \|y\|^u v,$$

and

$$\left( \frac{\partial V}{\partial y} \right)^T C \mathcal{H} D_t^\alpha y(t) \leq -c_3 \|y\|^u v,$$

where $\alpha \in (0, 1), c_1, c_2, c_3, u, v$ are arbitrary positive constants. Then system (4.1) with Caputo-Hadamard derivative is asymptotically stable.

**Proof** The result of this corollary can be obtained directly according to Theorem 2 and Theorem 3. $\square$

**Theorem 4** Let $V(y) : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function satisfying

$$c_1 \|y\|^u \leq V(y) \leq c_2 \|y\|^u v,$$ \quad (4.7)

$$\mathcal{H} D_t^\alpha V(y) \leq -c_3 \|y\|^u v,$$ \quad (4.8)

where $\alpha \in (0, 1), c_1, c_2, c_3, u, v$ are arbitrary positive constants. Then system (4.1) with Hadamard derivative is asymptotically stable.

**Proof** By Lemma 5 and the nonnegativeness of $V(y(t))$, we can obtain

$$C \mathcal{H} D_t^\alpha V(y) \leq \mathcal{H} D_t^\alpha V(t) \leq -c_3 \|y\|^u v,$$

$\square$
which means that condition (4.3) holds. Then by Theorem 3, we can get that

\[
\|y(t)\| \leq \left[ \frac{V_{a}}{c_{1}} E_{\alpha} \left( -\frac{c_{3}}{c_{2}} \left( \ln \frac{t}{a} \right)^{\alpha} \right) \right]^{\frac{1}{u}},
\]

By the property of the Mittag-Leffler function, for \( \alpha \in (0, 1) \), \( E_{\alpha}(-t) \) tends to zero as \( t \to \infty \), so \( y(t) \to 0 \) as \( t \to \infty \). This implies the system (4.1) with Hadamard derivative is asymptotically stable.

\[\square\]

Similar to Corollary 5, we have:

**Corollary 6** Let \( V(y) : \mathbb{R}^{n} \to \mathbb{R} \) be a continuously differentiable convex function and satisfy

\[
c_{1}\|y\|^{u} \leq V(y) \leq c_{2}\|y\|^{u}, \quad \text{and} \quad \left( \frac{\partial V}{\partial y} \right)^{T} D_{a}^{\alpha} y(t) \leq -c_{3}\|y\|^{u}
\]

where \( \alpha \in (0, 1) \), \( c_{1}, c_{2}, c_{3}, u \), \( v \) are arbitrary positive constants. Then system (4.1) with Hadamard derivative is asymptotically stable.

**Remark 1** In Theorem 3, Theorem 4, Corollary 5, and Corollary 6, we obtain the stability in the form of

\[
\|y(t)\| \leq \left\{ m(y_{a}) E_{\alpha} \left[ -d \left( \ln \frac{t}{a} \right)^{\alpha} \right] \right\}^{\frac{1}{u}},
\]

where \( c, d > 0 \), and \( m(y_{a}) \) is a function with respect to the initial condition. This indicates the state decays towards zero being a composition of the Mittag-Leffler function and \( \ln \frac{t}{a} \). From the property of the Mittag-Leffler function [44, Theorem 4], it is seen that the state decays towards zero like \( \frac{1}{\left[ 1 + (\ln \frac{t}{a})^{\alpha} \right]^{c}} \). In [29], the authors introduce a similar expression and call it Mittag-Leffler stable, which is slightly different from the asymptotical stable because the Mittag-Leffler stability implies the asymptotical stability and gives the decay expression while the asymptotical stability cannot generally tell us the speed of the decay or the decay expression. Following the manner of [29], here, we can call it Hadamard Mittag-Leffler stable.

**Theorem 5** Let \( V(y) \) be a Lyapunov function and let \( \gamma_{1}(\cdot) \) be a class-K function. If

\[
\gamma_{1}(\|y(t)\|) \leq V(y) \tag{4.9}
\]

and

\[
D_{a}^{\alpha} V(y) \leq 0. \tag{4.10}
\]

Then system (4.1) with Hadamard derivative is asymptotically stable.
Proof By (4.10), a nonnegative function \( R(t) \) exists satisfying \( H^a D_t^\alpha V(y) + R(t) = 0 \). Calculating the Hadamard integral of order \( \alpha \) on both side of this equation yields

\[
V(t) - \frac{V_a}{\Gamma(\alpha)} \left( \ln \frac{t}{a} \right)^{\alpha - 1} + \int_a^t \frac{1}{\Gamma(\alpha)} \left( \ln \frac{t}{\xi} \right)^{\alpha - 1} \frac{R(\xi)}{\xi} d\xi = 0,
\]

where \( V_a = a^\alpha D_t^{-\alpha} V(a) \). Since \( R(\xi) \geq 0 \), one has

\[
V(t) \leq \frac{V_a}{\Gamma(\alpha)} \left( \ln \frac{t}{a} \right)^{\alpha - 1}.
\]

Considering (4.9), we have

\[
\gamma_1(\|y\|) \leq V(y) \leq \frac{V_a}{\Gamma(\alpha)} \left( \ln \frac{t}{a} \right)^{\alpha - 1}.
\]

Then \( \|y\| \leq \gamma_1^{-1} \left( \frac{V_a}{\Gamma(\alpha)} \left( \ln \frac{t}{a} \right)^{\alpha - 1} \right) \). Hence, the asymptotical stability of system (4.1) follows from the fact that \( \gamma_1 \) is in class-\( K \) and the fact that as \( t \to \infty \), \( (\ln(t/a))^{\alpha - 1} \to 0 \). This ends the proof.

\[ \square \]

Theorem 6 Let \( V(y) \) be a Lyapunov function and let \( \gamma_1(\cdot) \) be a class-\( K \) function. If

\[ \gamma_1(\|y(t)\|) \leq V(y(t)) \quad (4.11) \]

and

\[ C^H D_t^\alpha V(y(t)) \leq 0. \quad (4.12) \]

Then system (4.1) with Caputo-Hadamard derivative is stable.

Proof By (4.12), we can see that a nonnegative function \( R(t) \geq 0 \) exists such that \( C^H D_t^\alpha V(y(t)) + R(t) = 0 \). Calculating the Hadamard integral of order \( \alpha \) on this equation, yields

\[
V(y(t)) = V(y(a)) - \int_a^t \frac{1}{\Gamma(\alpha)} \left( \ln \frac{t}{\xi} \right)^{\alpha - 1} \frac{R(\xi)}{\xi} d\xi,
\]

which, together with \( R(\xi) \geq 0 \) implies \( V(y(t)) \leq V(y(a)) \). This, combining with (4.11), gives \( \|y(t)\| \leq \gamma_1^{-1}(V(y)) \leq \gamma_1^{-1}(V(a)) \). Hence system (4.1) with Caputo-Hadamard derivative is stable.

\[ \square \]

Remark 2 For the Caputo fractional differential systems, the integer order differential systems and the Caputo-Hadamard fractional differential systems, corresponding conditions (4.11) and (4.12) can only obtain the stability results, see the Caputo fractional case in [31, Theorem 3], and the integer order case in [45, Page 62, Theorem 3.2],

\[ \square \]
but for the Hadamard fractional differential system, the asymptotical stability can be realised, the item \((\ln t)^{\alpha-1}\) makes the solution go to zero. This is a difference between the Caputo type system and the Hadamard system.

According to Theorem 5 and Theorem 6, we have the following corollaries.

**Corollary 7** For the Hadamard fractional system

\[
\frac{H}{a} D_t^\alpha y(t) = f(y(t)), \quad (4.13)
\]

if \(y(t) f(y(t)) \leq 0\) holds for \(\forall y(t) \in \mathbb{R}\), then system (4.13) is asymptotically stable.

**Proof** Choosing \(V(y(t)) = \frac{1}{2}y^2(t)\), one has \(\frac{H}{a} D_t^\alpha V(y(t)) \leq y(t) \frac{H}{a} D_t^\alpha y(t) = y(t) f(y(t)) \leq 0\), from which we see that \(V(y(t))\) satisfies (4.9) and (4.10). Hence, by Theorem 5, system (4.13) is asymptotically stable. \(\square\)

**Corollary 8** For the Caputo-Hadamard fractional system

\[
\frac{CH}{a} D_t^\alpha y(t) = f(y(t)), \quad (4.14)
\]

if \(y(t) f(y(t)) \leq 0\) holds for \(\forall y(t) \in \mathbb{R}\), then system (4.14) is stable.

**Proof** Taking \(V(y(t)) = \frac{1}{2}y^2(t)\), we get \(\frac{CH}{a} D_t^\alpha V(y(t)) \leq y(t) \frac{CH}{a} D_t^\alpha y(t) = y(t) f(y(t)) \leq 0\). It follows from Theorem 6 that system (4.14) is stable. \(\square\)

### 4.2 Fractional Halanay inequalities for Hadamard and Caputo-Hadamard systems with time delay

**Theorem 7** Suppose that \(W : (a, +\infty) \rightarrow \mathbb{R}^+\) and \(\tau : (a, +\infty) \rightarrow \mathbb{R}^+\) are continuous and \((\ln t)^{1-\alpha} W(t)\) is continuous, and there exists a monotonically increasing function \(g(t)\) satisfying \(\lim_{t \rightarrow +\infty} g(t) = +\infty\) such that \(t - \tau(t) \geq g(t) > a\) for all \(t > a\), and \(g(t)\) is integrable on \((a, T]\) for some \(T > a\). Let \(W(t)\) satisfy

\[
\frac{H}{a} D_t^\alpha W(t) \leq -\lambda W(t) + \mu \sup_{-\tau(t) \leq \sigma \leq 0} W(t + \sigma), \quad \alpha \in (0, 1) \quad (4.15)
\]

for some constants \(\lambda > \mu > 0\). Then \(\lim_{t \rightarrow +\infty} W(t) = 0\).

**Proof** According to (4.15), we know that there exist a nonnegative function \(R(t)\) such that

\[
\frac{H}{a} D_t^\alpha W(t) = -\lambda W(t) - R(t) + \mu \sup_{-\tau(t) \leq \sigma \leq 0} W(t + \sigma).
\]

This equation has solution
\[ W(t) = \frac{H_d}{a} D_t^{-(1-\alpha)} W(a) \left( \frac{t}{a} \right)^{\alpha-1} E_{\alpha,\alpha} \left[ -\lambda \left( \ln \frac{t}{a} \right)^\alpha \right] \]
\[ + \int_a^t \left( \frac{t}{s} \right)^{\alpha-1} E_{\alpha,\alpha} \left[ -\lambda \left( \ln \frac{t}{s} \right)^\alpha \right] ds \sup_{-\tau(s) \leq \sigma \leq 0} W(s+\sigma)s^{-\alpha} \, ds. \]

Considering the fact that \((\ln t)^{\alpha-1}, E_{\alpha,\alpha}[-\lambda(\ln t)^\alpha], R(t)\) are nonnegative, we have the following estimate

\[ W(t) \leq \frac{H_d}{a} D_t^{-(1-\alpha)} W(a) \left( \frac{t}{a} \right)^{\alpha-1} E_{\alpha,\alpha} \left[ -\lambda \left( \ln \frac{t}{a} \right)^\alpha \right] \]
\[ + \int_a^t \left( \frac{t}{s} \right)^{\alpha-1} E_{\alpha,\alpha} \left[ -\lambda \left( \ln \frac{t}{s} \right)^\alpha \right] \sup_{-\tau(s) \leq \sigma \leq 0} W(s+\sigma)s^{-\alpha} \, ds. \] (4.16)

Using the notations \(u(t)\) and \(K(h(t))\) with

\[ u(t) = \frac{H_d}{a} D_t^{-(1-\alpha)} W(a) \left( \frac{t}{a} \right)^{\alpha-1} E_{\alpha,\alpha} \left[ -\lambda \left( \ln \frac{t}{a} \right)^\alpha \right] \]
\[ and K(h(t)) = (h(t))^{\alpha-1} E_{\alpha,\alpha} \left[ -\lambda(h(t))^\alpha \right], \text{ then the inequality (4.16) can be rewritten as} \]

\[ W(t) \leq u(t) + \int_a^t K(h(t) - h(s)) \sup_{s-\tau(s) \leq \eta \leq s} W(\eta) dh(s), \]

where \(h(t) = \ln t\), it can be established that \(\lim_{t \to +\infty} u(t) = \lim_{t \to +\infty} K(h(t)) = 0\).

Now we prove \(\mu \|K\|_{L^1(\mathbb{C})} = \mu \int_0^{h(t)} K(h(s)) dh(s) < 1\). In fact, for the Mittag-Leffler function, it has

\[ \frac{d}{dt} [t^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha)] = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \]

if we take \(h(t)\) as an independent variable, then we have

\[ \frac{d}{dh(t)} [h^\alpha(t) E_{\alpha,\alpha+1}(-\lambda h^\alpha(t))] = h^{\alpha-1}(t) E_{\alpha,\alpha}(-\lambda h^\alpha(t)), \]

which implies

\[ \int_0^{h(t)} h^{\alpha-1}(s) E_{\alpha,\alpha}(-\lambda h^\alpha(s)) dh(s) = h^{\alpha}(t) E_{\alpha,\alpha+1}(-\lambda h^\alpha(t)), \]

together with

\[ E_{\alpha,\alpha+1}(-\lambda h^\alpha(t)) = \frac{1}{\lambda h^\alpha(t)} + O \left( \frac{1}{\lambda^2 h^{2\alpha}(t)} \right). \]
which yields

\[
\lim_{h(t) \to +\infty} h^\alpha(t) E_{\alpha, \alpha+1}(-\lambda h^\alpha(t)) = \frac{1}{\lambda}.
\]

By the fact that \( h(t) = \ln t \) is increasing with respect to \( t \) and the fact that \( \int_0^{h(t)} h^{\alpha-1}(s) E_{\alpha, \alpha}(-\lambda h^\alpha(s)) dh(s) \) is increasing with respect to \( h(t) \), we can see that \( \int_0^{h(t)} h^{\alpha-1}(s) E_{\alpha, \alpha}(-\lambda h^\alpha(s)) dh(s) \) is increasing with respect to \( t \), and so does for \( h^\alpha(t) E_{\alpha, \alpha+1}(-\lambda h^\alpha(t)) \). Thus

\[
\int_0^{h(t)} h^{\alpha-1}(s) E_{\alpha, \alpha}(-\lambda h^\alpha(s)) dh(s) \leq \frac{1}{\lambda}, \quad \forall t > a,
\]

combining \( \lambda > \mu \), which implies

\[
\mu \| K \|_{L^1(C)} = \mu \int_0^{h(t)} K(h(s)) d(h(s)) \leq \frac{\mu}{\lambda} < 1.
\]

By \([37, \text{Theorem } 2]\), we know that \( \lim_{t \to +\infty} W(t) = 0 \). \( \square \)

Similar to Theorem 7, we have integral inequality for Caputo-Hadamard fractional case.

**Theorem 8** Suppose that \( W, \tau : [a, +\infty) \to \mathbb{R}^+ \) are continuous. Let \( \tau \) satisfy \( t \geq a + \tau(t) \) for \( t \in [a, +\infty) \) and \( \lim_{t \to \infty} t - \tau(t) = \infty \). Let \( W(t) \) satisfy

\[
C^HD_t^\alpha W(t) \leq -\lambda W(t) + \mu \sup_{-\tau(t) \leq \sigma \leq 0} W(t + \sigma), \quad \alpha \in (0, 1), \quad (4.17)
\]

for some constants \( \lambda > \mu > 0 \). Then \( \lim_{t \to +\infty} W(t) = 0 \).

**Remark 3** Inequalities (4.15) and (4.17) can be considered as the generalization of the classic Halanay inequality \([35]\), which are useful in verifying the stability of delay system. Furthermore, the delay can be bounded, unbounded or distributed.

**Remark 4** From Theorem 3, Theorem 4, Corollary 5, and Corollary 6, it is seen that the state of system without delay under certain conditions can decay towards zero like \( \frac{1}{[1 + (\ln \frac{t}{a})^\alpha]^c} \). An interesting and nontrivial problem arises naturally: Can we get the decay speed of state of system with delay under under certain conditions? More precisely, we leave an open question here: Do (4.15) and (4.17) imply that \( W(t) \) decays towards zero like \( \frac{1}{[1 + (\ln \frac{t}{a})^\alpha]^c} \)?

### 5 Examples

In this section, some illustrative examples are provided to show the usefulness of the Lyapunov theorem and the Halanay inequality.
Example 1 Consider the following fractional order system with Hadamard derivative
\[
\begin{align*}
\frac{1}{1}D_t^{0.8} y_1(t) &= -y_1(t) - 2y_2(t) \cos t, \\
\frac{1}{1}D_t^{0.8} y_2(t) &= 2y_1(t) \cos t - y_2(t).
\end{align*}
\tag{5.1}
\]

Letting \( V(y) = \frac{1}{2}(y_1^2(t) + y_2^2(t)) \), we derive
\[
\begin{align*}
\frac{1}{1}D_t^{0.8} V(y) &\leq y_1 \frac{1}{1}D_t^{0.8} y_1(t) + y_2 \frac{1}{1}D_t^{0.8} y_2(t) \\
&= -y_1^2(t) - 2y_1(t)y_2(t) \cos t + 2y_1(t)y_2(t) \cos t - y_2^2(t) \\
&= -y_1^2(t) - y_2^2(t) \\
&\leq -2V(y).
\end{align*}
\]

By Theorem 3, system (5.1) is asymptotically stable.

Example 2 Consider the following fractional order system with Caputo-Hadamard derivative
\[
\begin{align*}
\frac{1}{1}D_t^{0.98} y_1(t) &= -4y_1(t) + y_2(t) + 2y_2(t)y_3(t), \\
\frac{1}{1}D_t^{0.98} y_2(t) &= -y_1(t) - 4y_2(t) + 5y_1(t)y_3(t), \\
\frac{1}{1}D_t^{0.98} y_3(t) &= -4y_3(t) - 7y_1(t)y_2(t),
\end{align*}
\tag{5.2}
\]

with initial value \( (y_1(1), y_2(1), y_3(1))^T = (0.8, 0.6, 0.4)^T \).

Letting \( V(y) = \frac{1}{2}(y_1^2(t) + y_2^2(t) + y_3^2(t)) \), it has
\[
\begin{align*}
\frac{1}{1}D_t^{0.98} V(y) &\leq y_1(t) \frac{1}{1}D_t^{0.98} y_1(t) + y_2(t) \frac{1}{1}D_t^{0.98} y_2(t) + y_3(t) \frac{1}{1}D_t^{0.98} y_3(t) \\
&= -4y_1^2(t) + y_1(t)y_2(t) + 2y_1(t)y_2(t)y_3(t) - y_1(t)y_2(t) - 4y_2^2(t) \\
&\quad + 5y_1(t)y_2(t)y_3(t) - 4y_3^2(t) - 7y_1(t)y_2(t)y_3(t) \\
&= -4(y_1^2(t) + y_2^2(t) + y_3^2(t)) = -8V(y).
\end{align*}
\]

By Theorem 3, system (5.2) is asymptotically stable. The evolution for the system can be seen in Fig. 1.

Example 3 Consider the the following Hadamard system with time-varying delay
\[
\begin{align*}
\frac{1}{1}D_t^{0.7} y_1(t) &= -4y_1(t) - 2y_1(t - \frac{1}{2}t) + y_1(t) \sin(y_2(t)), \\
\frac{1}{1}D_t^{0.7} y_2(t) &= -\frac{7}{2}y_2(t) - y_2(t - \frac{1}{2}t) + y_2(t) \sin(y_1(t - \frac{1}{2}t)).
\end{align*}
\tag{5.3}
\]

Letting \( W(t) = y_1^2(t) + y_2^2(t) \), we can verify that \( \frac{1}{1}D_t^{0.7} W(t) \leq -\lambda W(t) + \mu \sup_{\tau(t) \leq \eta \leq 0} W(t + \eta) \) holds for \( \lambda = 4, \mu = 1 \). In fact,
The initial value is $(y_1(1), y_2(1))^\top = (0.25, 0.2)^\top$.

To verify the stability of system (5.4), we check the conditions of Theorem 8. To this end, let $W(t) = y_1^2(t) + y_2^2(t)$. Since

\[ C_H D_t^{0.95} W(t) + \lambda W(t) - \mu \sup_{-\tau(t) \leq \eta \leq 0} W(t + \eta) \leq 2y_1(t)(-3y_1(t) + 2y_1(t) \sin(y_2(t - \frac{1}{2}))) + 2y_2(t)(-2y_2(t) + 2y_2(t) \cos(y_1(t - \frac{1}{2}))) + 2\lambda y_1^2(t) + 2\lambda y_2^2(t) - \mu \sup_{-\frac{1}{4} \leq \tau(t) \leq 0} (y_1^2(t + \eta) + y_2^2(t + \eta)) \]

By Theorem 7, system (5.3) is asymptotically stable.

**Example 4** Consider the following Caputo-Hadamard system with time-varying delay

\[ C_H D_t^{0.95} y_1(t) = -3y_1(t) + 2y_1(t) \sin(y_2(t - \frac{1}{4})), \]
\[ C_H D_t^{0.95} y_2(t) = -2y_2(t) + y_2(t) \cos(y_1(t - \frac{1}{4})). \]

(5.4)
To verify its stability, let the inequality

\[ CH_{D_t}^{0.95} W(t) \leq -\lambda W(t) + \mu \sup_{-\eta \leq\theta \leq 0} W(t + \eta) \]

holds for \( \lambda = 1, \mu = 0.5 \). Therefore, it is seen that the system (5.4) is asymptotically stable. The evolution of the system is shown in Fig. 2.

**Example 5** Consider the following fractional partial differential system with delay

\[
\begin{align*}
&\left\{ \begin{array}{l}
CH_{D_t}^{0.8} z(x, t) = z_{xx}(x, t) - 2z(x, t) + z(x, t - \sin^2 t), \ t > 1, \\
z(0, t) = z(1, t) = 0, t \geq 1, \\
z_0(x, t) = \sin 3\pi x - 3 \sin 2\pi x, 0 \leq x \leq 1, 0 \leq t \leq 1.
\end{array} \right.
\end{align*}
\]

(5.5)

To verify its stability, let \( W(t) = \int_0^1 z^2(x, t) dx \). Then

\[
\begin{align*}
&CH_{D_t}^{0.8} W(t) \\
&\leq 2 \int_0^1 z(x, t)^2 CH_{D_t}^{0.8} z(x, t) dx \\
&= 2 \int_0^1 z(x, t) \left\{ z_{xx}(x, t) - 2z(x, t) + z(x, t - \sin^2 t) \right\} dx \\
&= \int_0^1 \left\{ -2z_x^2(x, t) - 4z^2(x, t) + 2z(x, t)z_{xx}(x, t - \sin^2 t) \right\} dx \\
&= \int_0^1 \left\{ -2z_x^2(x, t) - 4z^2(x, t) + 2z(x, t)z(x, t - \sin^2 t) \right\} dx.
\end{align*}
\]

(5.6)

So we have

\[
\begin{align*}
&CH_{D_t}^{0.8} W(t) + \lambda W(t) - \mu \sup_{-\sin^2 t \leq\theta \leq 0} W(t + \eta) \\
&\leq \int_0^1 \left[ -2z_x^2(x, t) + (\lambda - 4)z^2(x, t) + 2z(x, t)z(x, t - \sin^2 t) \\
&- \mu z^2(x, t - \sin^2 t) \right] dx
\end{align*}
\]
\[ \leq \int_0^1 \left[ (-2\pi^2 + \lambda - 4)z^2(x, t) + 2z(x, t)z(x, t - \sin^2 t) - \mu z^2(x, t - \sin^2 t) \right] dx \]
\[ = \int_0^1 \left( z(x, t) z(x, t - \sin^2 t) \right) \left( -2\pi^2 - 4 + \lambda \frac{1}{1} - \mu \right) \left( z(x, t) z(x, t - \sin^2 t) \right) dx, \quad (5.7) \]

where the Wirtinger’s inequality \( \int_0^1 z_x^2(x) dx \geq \pi^2 \int_0^1 z^2(x) dx \) is used. It can be seen that when \( \mu = 1, \lambda = 2\pi^2 + 3.5, \) inequality \( ^{CH}D_{0^+}^{0.8} W(t) + \lambda W(t) - \mu \sup_{-\sin^2 t \leq n \leq 0} W(t + \eta) \leq 0 \) is always true. Hence, by Theorem 8, system (5.5) is asymptotically stable. The evolution of the system is shown in Fig. 3.

6 Concluding remarks

This paper focuses on the stability of the Hadamard and the Caputo-Hadamard fractional systems. We have derived several useful inequalities of the Hadamard fractional derivative and explored the Lyapunov method to cope with the stability of the Hadamard and the Caputo-Hadamard fractional nonlinear systems and the Halanay inequality to deal with the stability of delay systems. There are some other interesting problems to be studied for the Hadamard fractional calculus such as controllability, observability and detectability.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.
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