Analysis of a family of HDG methods for second order elliptic problems *

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Abstract

In this paper, we propose and analyze a new family of hybridizable discontinuous Galerkin (HDG) methods for second order elliptic problems in two and three dimensions. We use piecewise polynomials of degree $k \geq 0$ for both the flux and numerical trace, and piecewise polynomials of degree $k + 1$ for the potential. We show the convergence orders of the flux and the potential in $L^2$-norm are $k + 1$ and $k + 2$, respectively. What’s more, we construct a local postprocessing for the flux, which produces a numerical flux with better conservation. We also study the conditioning of the HDG method. We finally do numerical experiments in two-space dimensions to verify our theoretical results.

Keywords. hybridizable discontinuous Galerkin method, convergence, postprocessing, conditioning

1 Introduction

The pioneering works on hybrid (also called mixed-hybrid) finite element methods are due to Pian [31] and Fraejis de Veubeke [26] for the numerical solution of linear elasticity problems. Here the term “hybrid”, as stated in [5, 33], means ”the constraints of displacement continuity and/or traction reciprocity at the inter-element boundaries are relaxed a priori” in the hybrid finite element model. One may refer to [36, 32, 35, 39, 37, 38, 34, 41, 42, 53, 49, 52] and to [47, 45, 50, 28] respectively for some developments of hybrid stress (also called assumed stress) methods and hybrid strain (also called enhanced assumed strain) methods based on generalized variational principles, such as Hellinger-Reissner principle and Hu-Washizu principle. In [7, 54, 8, 51], stability and convergence were analyzed for several 4-node hybrid stress/strain quadrilateral/rectangular

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elements. We refer to [9, 11, 10] for the analysis of hybrid methods for 4th order elliptic problems, and to [6, 30, 43, 44] for the analysis for second-order elliptic boundary-value problems. One may see [14, 10, 46, 34] for more references therein on the hybrid methods.

Due to the relaxation of function continuity at the inter-element boundaries, the hybrid finite element model allows for piecewise-independent approximation to the displacement/potential or stress/flux solution, thus leading to a sparse, symmetric and positive definite discrete system through local elimination of unknowns defined in the interior of the elements. This is one main advantage of the hybrid methods. The process of local elimination is also called "static condensation" in engineering literature. In the discrete system, the unknowns are only the globally coupled degrees of freedom of the approximation trace of the "displacement" or "traction" defined only on the boundaries of the elements.

In [18] Cockburn et. al. introduced a unifying framework for hybridization of finite element methods for the second order elliptic problem: Find the potential $u$ and the flux $\sigma$ such that

\begin{align*}
  c\sigma - \nabla u &= 0, \quad \text{in } \Omega \\
  -\text{div}\sigma &= f, \quad \text{in } \Omega \\
  u &= g, \quad \text{on } \Omega
\end{align*}

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) with Lipschitz boundary $\partial\Omega$, $c(x) \in R^{d \times d}$ is a matrix valued function that is symmetric and uniformly positive definite on $\Omega$. $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Here hybridization denotes the process to rewrite a finite element method as a hybrid version. The unifying framework includes as particular cases hybridized versions of mixed methods [3, 12, 15], the continuous Galerkin (CG) method [21], and a wide class of hybridizable discontinuous Galerkin (HDG) methods. In particular, three new kinds of HDG, or more precisely LDG-H (Local DG-hybridizable), methods were presented in [18]. We refer to [13, 17, 19] for the convergence analysis of several HDG methods. We note that the error analysis in [19] for one of the three HDG methods by [18] is based on the use of a projection operator inspired by the form of the numerical traces of the methods. Following the same idea as in [19], a unifying framework was proposed [20] to analyze a large class of methods including the hybridized versions of some mixed methods as well as several HDG methods.

As we know, the condition number of a discrete system is of importance, due to its roles in estimating the convergence of iterative algorithms and the sensitivity of the discrete solution to perturbations. In [27] a new technique was developed to study conditioning of the hybridized versions of Raviart-Thomas [3] and Brezzi-Douglas-Marini [12] mixed methods. Following [27], conditioning of a HDG method was presented in [24].

In this paper, we develop a new family of HDG methods for problem (1.1). We use piecewise polynomials of degree $k$ for both the flux approximation, $\sigma_h$, and the numerical trace, $\lambda_h$, of $u$, and use piecewise polynomials of degree $k + 1$ for the potential approximation, $u_h$. We show the
convergence orders of $\sigma_h$ and $u_h$ in $L^2$-norm are $k+1$ and $k+2$, respectively. Besides, we construct a local postprocessing for the flux, which produces a numerical flux $\sigma_h^*$ with better conservation. We also study the conditioning of the proposed HDG method.

We note that the analysis technique used in [19, 20] doesn’t apply to our method. We make a simple comparison in Table 1 between the methods in [18, 19] and ours. Since the only globally coupled degrees of freedom in a HDG method are those describing $\lambda_h$, our method and the third LDG-H method in Table 1 are of the same computation size. We can see that our convergence rate for $u_h$ is one order higher than the third method. We mention that in [20] it has been shown that there exists a postprocessing, in element-by-element fashion, for the third method to produce a new potential $u_h^*$ that converges to $u$ at order $k+2$ for $k \geq 1$.

Table 1: Several methods fitting in the framework ($k \geq 0$)

| Method         | potential $u_h$ | trace $\lambda_h$ | flux $\sigma_h$ | $\|u - u_h\|$ | $\|\sigma - \sigma_h\|$ |
|----------------|-----------------|-------------------|-----------------|---------------|-----------------|
| LDG-H [18]     | $P_k(T)$        | $P_{k+1}(F)$      | $[P_{k+1}(T)]^d$ |               |                 |
| LDG-H [18]     | $P_{k+1}(T)$    | $P_{k+1}(F)$      | $[P_k(T)]^d$    |               |                 |
| LDG-H [18, 19] | $P_k(T)$        | $P_k(F)$          | $[P_k(T)]^d$    | $k+1$         | $k+1$           |
| our method     | $P_{k+1}(T)$    | $P_k(F)$          | $[P_k(T)]^d$    | $k+2$         | $k+1$           |

The rest of this paper is organized as follows. In Section 2.2 we follow the general framework in [18] to give our HDG scheme. Section 3 is devoted to the convergence analysis of the proposed HDG method. Section 4 presents a simple postprocessing. We study in Section 5 the conditioning of the method. Finally Section 6 provides numerical results.

2 HDG method

2.1 Preliminaries and Notations

We use the standard definitions of Sobolev spaces and their norms([1]), namely, for an arbitrary open set, $D \subset \mathbb{R}^d$ and any nonnegative integer $s$,

$$H^s(D) := \{ v \in L^2(D) : \partial^\alpha v \in L^2(D), \forall |\alpha| \leq s \},$$

$$\|v\|_{s,D} = (\sum_{|\alpha| \leq s} \int_D |\partial^\alpha v|^2)^{1/2}.$$  

We denote by $(\cdot, \cdot)_D$ and $(\cdot, \cdot)_{\partial D}$ the $L^2$ inner-products on $L^2(D)$ and $L^2(\partial D)$ respectively, and by $\|\cdot\|_D$ and $\|\cdot\|_{\partial D}$ the corresponding $L^2$-norms. In particular, $(\cdot, \cdot)$ and $\|\cdot\|$ abbreviate, when $D = \Omega$, $(\cdot, \cdot)_{\Omega}$ and $\|\cdot\|_{\Omega}$ respectively. For an integer $k \geq 0$, we use $P_k(D)$ to denote the set of polynomials of degree no greater than $k$ on $D$. 


Assume \( \Omega \subset \mathbb{R}^d \) is a polygonal. Let \( \mathcal{T}_h \) be a conforming shape-regular triangulation of \( \Omega \) and \( \mathcal{F}_h \) be the set of all faces of \( \mathcal{T}_h \). For any \( T \in \mathcal{T}_h \), we denote by \( h_T \) the diameter of \( T \) and set \( h := \max_{T \in \mathcal{T}_h} h_T \). For \( T \in \mathcal{T}_h \) and \( F \in \mathcal{F}_h \), let \( V(T) \), \( M(F) \) and \( W(T) \) be local spaces of finite dimensions. Then we define

\[
V_h := \{ v \in L^2(\Omega) : v|_T \in V(T), \forall T \in \mathcal{T}_h \},
\]

\[
M_h := \{ \mu \in L^2(\mathcal{F}_h) : \mu|_F \in M(F), \forall F \in \mathcal{F}_h \},
\]

\[
W_h := \{ \tau \in [L^2(\Omega)]^d : \tau|_T \in W(T), \forall T \in \mathcal{T}_h \},
\]

\[
M(\partial T) := \{ \mu \in L^2(\partial T) : \mu|_F \in M(F), \forall \text{ face } F \text{ of } T \}.
\]

For any \( g \in L^2(\partial \Omega) \), set

\[
M_h(g) := \{ \mu \in M_h : (\mu, \eta)_{\partial \Omega} = (g, \eta)_{\partial \Omega}, \forall \eta \in M_h \},
\]

and denote \( M^0_h := M_h(0) \).

Let \( P_T^k : L^2(T) \to P_k(T) \) be the standard \( L^2 \) orthogonal projection operator. Then it holds the following well-known approximation result: For any \( v \in H^s(T) \) \((s \geq 1)\) and \( T \in \mathcal{T}_h \),

\[
\|v - P_T^k v\|_T + h_T^{\frac{s}{2}} \|v - P_T^k v\|_{\partial T} \lesssim h_T^{\min\{k+1,s\}} \|v\|_{s,T}.
\]  

(2.1)

Here and in what follows, \( x \lesssim y \) (or \( x \gtrsim y \)) denotes that there exists a positive constant \( C \) such that \( x \leq Cy \) (or \( x \geq Cy \)), where \( C \) only depends on \( c \), \( k \), the regularity of \( \mathcal{T}_h \), or \( \Omega \). The notation \( x \sim y \) abbreviates \( x \lesssim y \lesssim x \).

## 2.2 Formulations of HDG method

We first introduce a local projection \( P_T^0 : H^1(T) \to M(\partial T) \) defined by

\[
\langle P_T^0 v, \mu \rangle_{\partial T} = \langle v, \mu \rangle_{\partial T}, \forall v \in H^1(T), \forall \mu \in M(\partial T).
\]

It is easy to see \( P_T^0 v \mid_F \) is the standard \( L^2 \)-orthogonal projection of \( v \mid_F \) on the local space \( M(F) \) for any face \( F \) of \( T \).

Following [18], the general framework of HDG methods is as follows: Seek \( (u_h, \lambda_h, \sigma_h) \in V_h \times M_h(g) \times W_h \), such that

\[
\begin{align*}
(\sigma_h, \mathcal{H}h)_{\mathcal{H}} + (u_h, \text{div} u_h)_{\mathcal{H}} - \sum_{T \in \mathcal{T}_h} \langle \lambda_h, \mathcal{H}h \cdot n \rangle_{\partial T} &= 0, \\
- (v_h, \text{div} \sigma_h) + \sum_{T \in \mathcal{T}_h} \langle \alpha_T (P_T^0 u_h - \lambda_h), v_h \rangle_{\partial T} &= (f, v_h), \\
\sum_{T \in \mathcal{T}_h} \langle \sigma_h \cdot n - \alpha_T (P_T^0 u_h - \lambda_h), \mu_h \rangle_{\partial T} &= 0
\end{align*}
\]

(2.2a)

(2.2b)

(2.2c)

hold for all \( (v_h, \mu_h, \tau_h) \in V_h \times M_h \times W_h \), where

\[
\langle \sigma, \tau \rangle_{\mathcal{H}} := \langle \sigma, \tau \rangle \forall \sigma, \tau \in [L^2(\Omega)]^d,
\]

(4)
the broken $\text{div}$ operator, $\text{div}_h$, is defined as

$$\text{div}_h \tau_h |_T := \text{div}(\tau_h |_T), \quad \forall \tau_h \in W_h, T \in T_h,$$

and $\alpha_T$ denotes a nonnegative penalty function defined on $\partial T$.

In this paper we choose the local spaces $V(T)$, $M(F)$, $W(T)$ and the penalty parameter $\alpha_T$ as following:

$$V(T) = P_{k+1}(T), \quad M(F) = P_k(F), \quad W(T) = [P_k(T)]^d, \quad (2.3)$$

$$\alpha_T |_F = h_F^{-1}, \quad \forall \text{ face } F \text{ of } T, \quad (2.4)$$

where $h_F$ denotes the diameter of $F$.

**Remark 2.1.** Different choices of $V(T)$, $M(F)$, $W(T)$ and $\alpha_T$ lead to different methods. In [18], three choices were provided (cf. Table 1), one of which, namely $V(T) = P_k(T), M(F) = P_k(F), W(T) = [P_k(T)]^d$, was analyzed in [19], where $\alpha_T = O(1)$.

**Remark 2.2.** When $\alpha_T = 0$, then the RT-H and BDM-H methods proposed in [18] coincide with the classical hybridized versions of RT element method [3] and BDM element method [12], respectively.

It is easy to show the following existence and uniqueness result.

**Lemma 2.1.** The HDG scheme (2.2) with the choices (2.3)-(2.4) admits a unique solution $(u_h, \lambda_h, \sigma_h) \in V_h \times M_h(g) \times W_h$.

## 3 Error analysis

### 3.1 Estimation for flux approximation $\sigma_h$

Let $P_V : L^2(\Omega) \to V_h$, $P_M : L^2(F_h) \to M_h$ and $P_W : [L^2(\Omega)]^d \to W_h$ be the standard $L^2$-orthogonal projection operators. For any given $(\tau, v) \in [L^2(\Omega)]^d \times H^1(\Omega)$, we define

$$L_{\tau, v}(\psi) := \sum_{T \in T_h} \langle (P_W \tau - \tau) \cdot n - \alpha_T (P_T^a P_V v - P_M v), \psi \rangle_{\partial T}, \quad \forall \psi \in H^1(\Omega) \bigcup M_h \bigcup V_h. \quad (3.1)$$

Here $v$ in $P_M v$ is understood as its trace on $F_h$ for $v \in H^1(\Omega)$.

Denote

$$e_h^n := \sigma_h - P_W \sigma, \quad e_h^u := u_h - P_V u, \quad e_h^\lambda := \lambda_h - P_M u, \quad (3.2)$$

then we have the following lemma.
Lemma 3.1. For all \((v_h, \mu_h, \tau_h) \in V_h \times M_h^0 \times W_h\) it holds
\[
\begin{align*}
(e_h^\sigma, \tau_h)_c + (e_h^\mu, \text{div}_h \tau_h) - \sum_{T \in T_h} \langle e^\lambda \cdot n, \tau_h \cdot n \rangle_{\partial T} &= (\sigma - P_h \sigma, \tau_h)_c, \\
-(v_h, \text{div}_h e_h^\sigma) + \sum_{T \in T_h} \langle \alpha_T (P_T e_h^u - e_h^\lambda), v_h \rangle_{\partial T} &= L_{\sigma,u}(v_h), \\
\sum_{T \in T_h} \langle e_h^\mu \cdot n - \alpha_T (P_T e_h^u - e_h^\lambda), \mu_h \rangle_{\partial T} &= -L_{\sigma,u}(\mu_h).
\end{align*}
\] (3.3a, 3.3b, 3.3c)

Proof. It’s straightforward to show the relations
\[
\begin{align*}
(P_h \sigma, \tau_h)_c + (P_h u, \text{div}_h \tau_h) - \sum_{T \in T_h} \langle P_T^0 u, \tau_h \cdot n \rangle_{\partial T} &= (P_h \sigma - \sigma, \tau_h)_c, \\
-(v_h, \text{div}_h (P_h \sigma)) + \sum_{T \in T_h} \langle (P_h \sigma - \sigma) \cdot n, v_h \rangle_{\partial T} &= (f, v_h)
\end{align*}
\]
hold for all \((\tau_h, v_h) \in W_h \times V_h\). Subtracting (2.2a) and (2.2b) from the above two equations respectively, we obtain (3.3a) and (3.3b). The relation (3.3c) follows from (2.2c) and the fact
\[
\sum_{T \in T_h} \langle \sigma \cdot n, \mu_h \rangle_{\partial T} = 0, \text{ for } \mu_h \in M_h^0.
\]

We introduce a semi-norm \(\| \cdot \| : V_h \times M_h^0 \times W_h \to \mathbb{R}\) with
\[
\|(v_h, \mu_h, \tau_h)\| := \|\tau_h\|_c^2 + \sum_{T \in T_h} \left\| \alpha_T^\frac{1}{2} (P_T^0 v_h - \mu_h) \right\|^2_{\partial T}, \forall (v_h, \mu_h, \tau_h) \in V_h \times M_h^0 \times W_h,
\]
where
\[
\|\tau\|_c := (c \tau, \tau)^\frac{1}{2}, \forall \tau \in [L^2(\Omega)]^d.
\]

Lemma 3.2. It holds
\[
\| (e_h^\mu, e_h^\lambda, e_h^\sigma) \|^2 = I_1 + I_2 + I_3,
\] (3.4)

where
\[
\begin{align*}
I_1 &= (\sigma - P_h \sigma, e_h^\sigma)_c, \\
I_2 &= \sum_{T \in T_h} \langle \alpha_T (P_T u - u), e_h^\lambda - P_T^0 e_h^u \rangle_{\partial T}, \\
I_3 &= \sum_{T \in T_h} \langle (\sigma - P_h \sigma) \cdot n, e_h^\lambda - e_h^u \rangle_{\partial T}.
\end{align*}
\]

Proof. Taking \(\tau_h = e_h^\sigma\) in (3.3a), \(v_h = e_h^\mu\) in (3.3b) and adding the resultant two equations, we obtain
\[
I_1 + L_{\sigma,u}(e_h^\mu) = \| e_h^\mu \|_c^2 + \sum_{T \in T_h} \langle \alpha_T (P_P^0 e_h^u - e_h^\lambda), e_h^\mu \rangle_{\partial T} - \sum_{T \in T_h} \langle (e_h^\mu, e_h^\sigma) \cdot n \rangle_{\partial T}
\]
\[
= \| (e_h^\mu, e_h^\lambda, e_h^\sigma) \|^2 - \sum_{T \in T_h} \langle (e_h^\mu, e_h^\sigma) \cdot n - \alpha_T (P_T^0 e_h^u - e_h^\lambda), e_h^\lambda \rangle_{\partial T},
\]
which, together with (3.3c) and the fact \(e_h^\lambda \in M_h^0\), yields
\[
I_1 + L_{\sigma,u}(e_h^\mu) = \| (e_h^\mu, e_h^\lambda, e_h^\sigma) \|^2 + L_{\sigma,u}(e_h^\lambda).
\]
Thus it follows
\[
\| (e_h^u, e_h^\lambda, e_h^\sigma) \|^2 = I_1 + L_{\sigma,u}(e_h^u) - L_{\sigma,u}(e_h^\lambda)
\]
\[
= I_1 + \sum_{T \in \mathcal{T}_h} (\alpha_T (P_T^0 P_T V u - P_M u), e_h^\lambda - e_h^u)_{\partial T} + \sum_{T \in \mathcal{T}_h} ((\sigma - P_W \sigma) \cdot n, e_h^\lambda - e_h^u)_{\partial T}
\]
\[
= I_1 + \sum_{T \in \mathcal{T}_h} (\alpha_T (P_T V u - u), e_h^\lambda - P_T^0 e_h^u)_{\partial T} + \sum_{T \in \mathcal{T}_h} ((\sigma - P_W \sigma) \cdot n, e_h^\lambda - e_h^u)_{\partial T}
\]
\[
= I_1 + I_2 + I_3.
\]

In view of Cauchy-Schwarz inequality, we easily derive the following estimates for $I_1$ and $I_2$.

**Lemma 3.3.** It holds

\[
I_1 \lesssim \| \sigma - P_W \sigma \| \| (e_h^u, e_h^\lambda, e_h^\sigma) \|.
\] (3.5)

\[
I_2 \lesssim \left( \sum_{T \in \mathcal{T}_h} \left\| \alpha_T^2 (u - P_T V u) \right\|_{\partial T}^2 \right)^{1/2} \| (e_h^u, e_h^\lambda, e_h^\sigma) \|.
\] (3.6)

To estimate the term $I_3$, we need a key result as follows.

**Lemma 3.4.**

\[
\left( \sum_{T \in \mathcal{T}_h} \| \nabla e_h^u \|^2_T \right)^{1/2} \lesssim \| (e_h^u, e_h^\lambda, e_h^\sigma) \| + \| \sigma - P_W \sigma \|.
\] (3.7)

**Proof.** For any $T \in \mathcal{T}_h$, from (3.3a) and integration by parts, it follows

\[
(\nabla e_h^u, \tau_h)_T = (ce_h^\sigma, \tau_h)_T + (P_T^0 e_h^u - e_h^\lambda, \tau_h \cdot n)_{\partial T} - (c(\sigma - P_W \sigma), \tau_h)_T, \forall \tau_h \in W_h,
\]

which, by taking $\tau_h = \nabla e_h^u$, leads to

\[
\| \nabla e_h^u \|^2_T = (ce_h^\sigma, \nabla e_h^u)_T + (P_T^0 e_h^u - e_h^\lambda, \nabla e_h^u \cdot n)_{\partial T} - (c(\sigma - P_W \sigma), \nabla e_h^u)_T
\]
\[
\lesssim \| e_h^\sigma \|_T \| \nabla e_h^u \|_T + \| P_T^0 e_h^u - e_h^\lambda \|_{\partial T} \| \nabla e_h^u \|_{\partial T} + \| \sigma - P_W \sigma \|_T \| \nabla e_h^u \|_T.
\]

Since trace inequality and inverse estimate yields

\[
\| \nabla e_h^u \|_{\partial T} \lesssim h_T^{-\frac{1}{2}} \| \nabla e_h^u \|_T,
\]

we finally have

\[
\| \nabla e_h^u \|_T \lesssim \| e_h^\sigma \|_T + h_T^{-\frac{1}{2}} \| P_T^0 e_h^u - e_h^\lambda \|_{\partial T} + \| \sigma - P_W \sigma \|_T,
\]

which implies (3.7). □

**Lemma 3.5.** It holds

\[
I_3 \lesssim \left( \sum_{T \in \mathcal{T}_h} \left\| \alpha_T^{-\frac{1}{2}} (\sigma - P_W \sigma) \right\|_{\partial T}^2 \right)^{1/2} \left( \| (e_h^u, e_h^\lambda, e_h^\sigma) \| + \| \sigma - P_W \sigma \| \right).
\] (3.8)
Proof. By the property (2.1) for the projection operator \(P_T^h\), it holds
\[
h_T^{-\frac{1}{2}} \left\| P_T^h e_h^u - e_h^u \right\|_{\partial T} \lesssim \left\| \nabla e_h^u \right\|_{\partial T},
\]
which, together with (3.7), implies
\[
\sum_{T \in T_h} \left\| \alpha_T^\frac{1}{2} (P_T^h e_h^u - e_h^u) \right\|^2 \lesssim \left\| \left( e_h^u, e_h^e_\lambda, e_h^\sigma \right) \right\|^2 + \| \sigma - P_W \sigma \|^2.
\]
(3.9)
Therefore, we have
\[
I_3 = \sum_{T \in T_h} \left\langle (\sigma - P_W \sigma) \cdot n, e_h^e - e_h^e \right\rangle_{\partial T}
\]
\[
= \sum_{T \in T_h} \left\langle (\sigma - P_W \sigma) \cdot n, e_h^e - P_T^h e_h^e \right\rangle_{\partial T} + \sum_{T \in T_h} \left\langle (\sigma - P_W \sigma) \cdot n, \right. e_h^e - P_T^h e_h^e \left\rangle_{\partial T}
\]
\[
\leq \left( \sum_{T \in T_h} \left\| \alpha_T^{-\frac{1}{2}} (\sigma - P_W \sigma) \right\|_{\partial T} \right)^\frac{1}{2} \left( \left\| (e_h^u, e_h^e_\lambda, e_h^\sigma) \right\| + \left( \sum_{T \in T_h} \left\| \alpha_T^{-\frac{1}{2}} (P_T^h e_h^e - e_h^e) \right\|_{\partial T} \right)^2 \right)^\frac{1}{2}
\]
\[
\lesssim \left( \sum_{T \in T_h} \left\| \alpha_T^{-\frac{1}{2}} (\sigma - P_W \sigma) \right\|_{\partial T} \right)^\frac{1}{2} \left( \left\| (e_h^u, e_h^e_\lambda, e_h^\sigma) \right\| + \| \sigma - P_W \sigma \| \right), \quad \text{(by (3.9))}
\]
which completes the proof. \(\square\)

In light of Lemmas 3.2, 3.3 and 3.5, using Young’s inequality \(ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2\) for \(\epsilon > 0\), we easily derive the Lemma below.

Lemma 3.6. It holds
\[
\left\| (e_h^u, e_h^e_\lambda, e_h^\sigma) \right\| \lesssim \| \sigma - P_W \sigma \| + \left( \sum_{T \in T_h} \left\| \alpha_T^\frac{1}{2} (u - P_T^h u) \right\|_{\partial T} \right)^\frac{1}{2} + \left( \sum_{T \in T_h} \left\| \alpha_T^{-\frac{1}{2}} (\sigma - P_W \sigma) \right\|_{\partial T} \right)^\frac{1}{2}.
\]
(3.10)

Now we introduce a "broken" Sobolev space
\[
H^s(\mathcal{T}_h) := \{ v \in L^2(\Omega) : v|_T \in H^s(T), \ \forall T \in \mathcal{T}_h \}
\]
with norm \(\| \cdot \|_{s, \mathcal{T}_h}\) defined by
\[
\| v \|_{s, \mathcal{T}_h} := \left( \sum_{T \in T_h} \| v \|_{s, T}^2 \right)^\frac{1}{2}, \quad \forall v \in H^s(\mathcal{T}_h).
\]
Assume \((u, \sigma) \in H^{l_\alpha}(\mathcal{T}_h) \times [H^{l_\alpha}(\mathcal{T}_h)]^d\), then, for the projection operators \(P_V\) and \(P_W\) and any \(T \in \mathcal{T}_h\), it holds the following standard estimates:
\[
\| \sigma - P_W \sigma \|_T + h_T^{-\frac{1}{2}} \| \sigma - P_W \sigma \|_{\partial T} \lesssim h_T^{-\frac{5}{2}} \| \sigma \|_{l_\alpha, T}.
\]
Thus, from Lemma 3.6 it follows
\[
\left\| (e_h^u, e_h^e_\lambda, e_h^\sigma) \right\| \lesssim h_T^{\frac{5}{2}} \| u \|_{l_\alpha, T} + h_T^{-\frac{5}{2}} \| \sigma \|_{l_\alpha, T}.
\]
(3.11)
As a result, the estimate (3.11), together with triangle inequality, yields the following convergence results for the flux approximation.
Theorem 3.1. Assume \((u, \sigma) \in H^1_u(T_h) \times [H^1 \sigma(T_h)]^d\), then it holds

\[
\|\sigma - \sigma_h\| \lesssim h^\min\{k+1, l_u - 1\} \|u\|_{l_u, T_h} + h^\min\{k+1, l_\sigma\} \|\sigma\|_{l_\sigma, T_h}.
\]  

(3.12)

In particular, if \(l_u = k + 2, l_\sigma = k + 1\), then it holds

\[
\|\sigma - \sigma_h\| \lesssim h^{k+1} \left(\|u\|_{k+2, T_h} + \|\sigma\|_{k+1, T_h}\right).
\]  

(3.13)

3.2 Estimation for potential approximation \(u_h\)

Similarly to [19], we shall use Aubin-Nitsche’s technique of duality argument to derive the error estimation for the potential approximation \(u_h\). To this end we introduce the dual problem

\[
\begin{cases}
  c\Phi - \nabla \phi = 0 & \text{in } \Omega, \\
  \text{div} \Phi = -e_h^u & \text{in } \Omega, \\
  \phi = 0 & \text{on } \Omega,
\end{cases}
\]  

(3.14)

where, as defined in (3.2), \(e_h^u = u_h - P_V u\). In addition, we assume the following regularity property holds:

\[
\|\Phi\|_{1,\Omega} + \|\phi\|_{2,\Omega} \lesssim \|e_h^u\|.
\]  

(3.15)

We first present a basic equality as follows.

Lemma 3.7. It holds

\[
\|e_h^u\|^2 = \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5,
\]  

(3.16)

where

\[
\begin{align*}
\Pi_1 &= (e_h^\sigma, P_W \Phi - \Phi)_e, \\
\Pi_2 &= (P_W \sigma - \sigma, P_W \Phi)_e, \\
\Pi_3 &= \sum_{T \in T_h} \langle e_h^u - e_h^\lambda, (P_W \Phi - \Phi) \cdot n \rangle_{\partial T}, \\
\Pi_4 &= \sum_{T \in T_h} \langle \alpha_T (P_T^2 e_h^u - e_h^\lambda), \phi - P_V \phi \rangle_{\partial T}, \\
\Pi_5 &= L_{\sigma, u} (P_V \phi - \phi).
\end{align*}
\]  

(3.17)-(3.21)

Proof. From (3.14) we have

\[
\|e_h^u\|^2_T = -(e_h^u, \text{div} \Phi)_T \\
= (\nabla e_h^u, P_W \Phi)_T - \langle e_h^u, \Phi \cdot n \rangle_{\partial T} \\
= -(e_h^u, \text{div} P_W \Phi)_T + \langle e_h^u, (P_W \Phi - \Phi) \cdot n \rangle_{\partial T},
\]  

which yields

\[
\|e_h^u\|^2 = -(e_h^u, \text{div} P_W \Phi) + \sum_{T \in T_h} \langle e_h^u, (P_W \Phi - \Phi) \cdot n \rangle_{\partial T}.
\]  

(3.22)
By the continuity of $\Phi \cdot n$ and the fact that $e^\lambda_h|_{\partial T} = 0$, we obtain

$$
\|e^\mu_h\|^2 = -(e^\mu_h, \text{div}_h P_W \Phi) + \sum_{T \in T_h} \langle e^\lambda_h, P_W \Phi \cdot n \rangle_{\partial T} + \sum_{T \in T_h} \langle e^\mu_h - e^\lambda_h, (P_W \Phi - \Phi) \cdot n \rangle_{\partial T}
$$

$$
= (e^\sigma_h, P_W \Phi) e - (\sigma - P_W \sigma, P_W \Phi) e + \sum_{T \in T_h} \langle e^\mu_h - e^\lambda_h, (P_W \Phi - \Phi) \cdot n \rangle_{\partial T} \quad \text{(by (3.3a))}
$$

$$
= (e^\sigma_h, P_W \Phi) e + \Pi_2 + \Pi_3. \quad (3.23)
$$

Taking $v_h = P_V \phi$ in (3.3b), we get

$$
-(\phi, \text{div}_h e^\sigma_h) + \sum_{T \in T_h} \langle \alpha_T (P^d_T e^\mu_h - e^\lambda_h), P_V \phi \rangle_{\partial T} = L_{\sigma_\alpha}(P_V \phi). \quad (3.24)
$$

Taking $\mu_h = P_M \phi$ in (2.2c), we arrive at

$$
\sum_{T \in T_h} \langle \sigma_h \cdot n - \alpha_T (P^d_T u_h - \lambda_h), \phi \rangle_{\partial T} = \sum_{T \in T_h} \langle \sigma_h \cdot n - \alpha_T (P^d_T u_h - \lambda_h), P_M \phi \rangle_{\partial T} = 0,
$$

which, together with the definitions (3.1)-(3.2), indicates

$$
\sum_{T \in T_h} \langle e^\mu_h \cdot n - \alpha_T (P^d_T e^\mu_h - e^\lambda_h), \phi \rangle_{\partial T} = -L_{\sigma_\alpha}(\phi). \quad (3.25)
$$

In light of (3.14), (3.24) and (3.25), it holds

$$
(e^\sigma_h, \Phi) e = (\nabla \phi, e^\sigma_h)
$$

$$
= -(\phi, \text{div}_h e^\sigma_h) + \sum_{T \in T_h} \langle e^\sigma_h \cdot n, \phi \rangle_{\partial T}
$$

$$
= \sum_{T \in T_h} \langle \alpha_T (P^d_T e^\mu_h - e^\lambda_h), \phi - P_V \phi \rangle_{\partial T} + L_{\sigma_\alpha}(P_V \phi - \phi)
$$

$$
= \Pi_4 + \Pi_5. \quad (3.26)
$$

Finally, from (3.23) and (3.26) it follows

$$
\|e^\mu_h\|^2 = (e^\sigma_h, P_W \Phi) e + \Pi_2 + \Pi_3 - (e^\sigma_h, \Phi) e + \Pi_4 + \Pi_5
$$

$$
= (e^\sigma_h, P_W \Phi - \Phi) e + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5
$$

$$
= \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5.
$$

The following Lemma shows estimates for the terms $\Pi_j$ for $j = 1, 2, \cdots, 5$.

**Lemma 3.8.** It holds

$$
|\Pi_1| \lesssim h \|e^\sigma_h\|_e \|e^\mu_h\|, \quad (3.27)
$$

$$
|\Pi_2| \lesssim h \|\sigma - P_W \sigma\| \|e^\mu_h\|, \quad (3.28)
$$

$$
|\Pi_3| \lesssim h(||(e^\mu_h, e^\lambda_h)\|| + ||\sigma - P_W \sigma\|) \|e^\mu_h\|, \quad (3.29)
$$
|Π₄| ≲ h \left\| (\epsilon_h^\alpha, \epsilon_h^\gamma, \epsilon_h^\eta) \right\| \|\epsilon_h^\eta\|, \quad (3.30)

|Π₅| ≲ h \left\{ \sum_{T \in T_h} \left\| \alpha_T^{-\frac{1}{2}} (P_W \sigma - \sigma) \right\|_{\partial T}^{\frac{1}{2}} + \left( \sum_{T \in T_h} \left\| \alpha_T^{-\frac{1}{2}} (u - P_V u) \right\|_{\partial T}^{\frac{1}{2}} \right) ^{\frac{1}{2}} \right\} \|\epsilon_h^\eta\|. \quad (3.31)

**Proof.** By noticing

$$\Pi_1 = (\epsilon_h^\gamma, P_W \Phi - \Phi)_c$$

and

$$\Pi_2 = (\sigma - P_W \sigma, \Phi - P_W \Phi)_c - (\sigma - P_W \sigma, \Phi)_c$$

$$= (\sigma - P_W \sigma, \Phi - P_W \Phi)_c - (\sigma - P_W \sigma, \nabla \phi) \quad \text{(by (3.14))}$$

$$= (\sigma - P_W \sigma, \Phi - P_W \Phi)_c - (\sigma - P_W \sigma, \nabla \phi - P_W \nabla \phi),$$

the estimates (3.27)-(3.28) follow from Cauchy-Schwarz inequality, the projection approximation result (2.1) for $P_W$ and the regularity assumption (3.15).

Next we go to estimate $\Pi_3$. We have

$$|\Pi_3| = \left| \sum_{T \in T_h} (\epsilon_T^\alpha - e_h^\lambda, (P_W \Phi - \Phi) \cdot n)_{\partial T} \right|$$

$$\leq \left| \sum_{T \in T_h} (P_T^\theta \epsilon_T^\alpha - e_h^\lambda, (P_W \Phi - \Phi) \cdot n)_{\partial T} \right| + \left| \sum_{T \in T_h} (\epsilon_T^\alpha - P_T^\theta \epsilon_T^\alpha, (P_W \Phi - \Phi) \cdot n)_{\partial T} \right|. \quad (3.32)$$

Using Cauchy-Schwarz inequality, the projection approximation result (2.1) for $P_W$ and $P_T^\theta$, and the regularity assumption (3.15) once again, we obtain

$$\left| \sum_{T \in T_h} (P_T^\theta \epsilon_T^\alpha - e_h^\lambda, (P_W \Phi - \Phi) \cdot n)_{\partial T} \right| \leq \sum_{T \in T_h} \left\| P_T^\theta \epsilon_T^\alpha - e_h^\lambda \right\|_{\partial T} \| P_W \Phi - \Phi \|_{\partial T}$$

$$\leq h^\frac{1}{2} \left( \sum_{T \in T_h} \left\| \alpha_T^{-\frac{1}{2}} (P_T^\theta \epsilon_T^\alpha - e_h^\lambda) \right\|_{\partial T}^{\frac{1}{2}} \right) \left( \sum_{T \in T_h} \| P_W \Phi - \Phi \|_{\partial T}^{\frac{1}{2}} \right)$$

$$\lesssim h \left\| (\epsilon_h^\alpha, e_h^\lambda, \epsilon_h^\gamma) \right\| \|\epsilon_h^\eta\|,$n

$$\left| \sum_{T \in T_h} (\epsilon_T^\alpha - P_T^\theta \epsilon_T^\alpha, (P_W \Phi - \Phi) \cdot n)_{\partial T} \right| \leq \left( \sum_{T \in T_h} \left\| e_T^\alpha - P_T^\theta \epsilon_T^\alpha \right\|_{\partial T}^{\frac{1}{2}} \right) \left( \sum_{T \in T_h} \| P_W \Phi - \Phi \|_{\partial T}^{\frac{1}{2}} \right)$$

$$\lesssim h \|\epsilon_h^\alpha\| \left( \sum_{T \in T_h} \| \nabla \epsilon_T^\alpha \|_{\partial T}^{\frac{1}{2}} \right)$$

$$\lesssim h \|\epsilon_h^\alpha\| \left( \left\| (\epsilon_h^\alpha, e_h^\lambda, \epsilon_h^\gamma) \right\| + \|\sigma - P_W \sigma\| \right). \quad \text{(by (3.7))}$$

These two inequalities, together with (3.32), imply the desired estimate (3.29).

Similarly, we can easily derive the estimates (3.30)-(3.31) by virtue of the properties of the projection operators $P_V, P_M, P_T^\theta$ and the regularity (3.15).

A combination of Lemmas 3.7-3.8 yields the following conclusion.

**Lemma 3.9.** It holds

$$\|\epsilon_h^\alpha\| \lesssim h \left\{ \left\| (\epsilon_h^\alpha, e_h^\lambda, \epsilon_h^\gamma) \right\| + \|\sigma - P_W \sigma\| + \left( \sum_{T \in T_h} \left\| \alpha_T^{-\frac{1}{2}} (\sigma - P_W \sigma) \right\|_{\partial T}^{\frac{1}{2}} \right)^{\frac{1}{2}} + \left( \sum_{T \in T_h} \left\| \alpha_T^{-\frac{1}{2}} (u - P_V u) \right\|_{\partial T}^{\frac{1}{2}} \right)^{\frac{1}{2}} \right\}. \quad (3.33)$$
From Theorem 3.1 and Lemma 3.9, we easily derive the following convergence theorem for the potential approximation $u_h$.

**Theorem 3.2.** Assume $(u, \sigma) \in H^l_u(T_h) \times [H^l_\sigma(T_h)]^d$, then it holds

$$
\|u - u_h\| \lesssim h^{\min\{k+2,l_u\}} \|u\|_{l_u,T_h} + h^{\min\{k+2,l_\sigma+1\}} \|\sigma\|_{l_\sigma,T_h}.
$$

(3.34)

In particular, if $l_u = k + 2, l_\sigma = k + 1$, then it holds

$$
\|u - u_h\| \lesssim h^{k+2} \left(\|u\|_{k+2,T_h} + \|\sigma\|_{k+1,T_h}\right).
$$

(3.35)

**Remark 3.1.** If we set

$$
V(T) = P_k(T),
$$

$$
M(F) = P_k(F),
$$

$$
W(T) = [P_k(T)]^d,
$$

and choose $\alpha_T = 1$. According to the theory proposed in [19], for sufficient smooth solution and $k > 0$, although, the order of convergence for $u_h$ is $k + 1$, there exists an element by element postprocessing to produce a new approximation $u^*_h$, the order of convergence of which is $k + 2$.

However, for $k = 0$, such postprocessing doesn’t exist.

4 Flux postprocessing

In this section we follow the idea in [22, 19] to construct a local postprocessing so as to obtain a new flux approximation $\sigma^*_h \in H(div;\Omega)$. We shall show that $\sigma^*_h$ converges at the same order as $\sigma_h$, while its divergence converges at one higher order than $\sigma_h$.

We define

$$
\sigma^*_h := \sigma_h - \tilde{\sigma}_h,
$$

(4.1)

where, for any $T \in T_h$,

$$
\tilde{\sigma}_h|_T \in RT_{k+1}(T) := \{\tau : \tau = p + qx, \forall p \in [P_{k+1}(T)]^d, \forall q \in P_{k+1}(T)\}
$$

satisfies

$$
(\tilde{\sigma}_h, q)_T = 0, \quad \forall q \in [P_k(T)]^d,
$$

(4.2a)

$$
(\tilde{\sigma}_h \cdot n, \mu)_F = \langle \alpha_T(P_T^0 u_h - \lambda_h), \mu \rangle_F, \quad \forall \mu \in P_{k+1}(F), \forall \text{ face } F \text{ of } T,
$$

(4.2b)

We note that the existence and uniqueness of $\tilde{\sigma}_h$ follow from the property of the RT elements [40].

Moreover, we have the following theorem.
Theorem 4.1. It holds

$$\sigma^*_h \in H(\text{div}; \Omega) \quad \text{with} \quad \text{div}\sigma^*_h = P_V \text{div}\sigma.$$  \hspace{1cm} (4.3)

In addition, it holds

$$\|\sigma - \sigma^*_h\| \lesssim h^{k+1} \left( \|u\|_{k+2,T_h} + \|\sigma\|_{k+1,T_h} \right)$$  \hspace{1cm} (4.4)

if $$(u, \sigma) \in H^{k+2}(T_h) \times [H^{k+1}(\Omega)]^d$$, and it holds

$$\|\text{div}\((\sigma - \sigma^*_h)\)\| \lesssim h^{k+2} \|\text{div}\sigma\|_{k+2,T_h}$$  \hspace{1cm} (4.5)

if $\text{div}\sigma \in H^{k+2}(T_h)$.

Proof. First, by (2.2c) and (4.2b) it’s easy to verify that $\sigma^*_h \cdot n = 0$ on any $F \in \mathcal{F}_h$, which means $\sigma^*_h \in H(\text{div}; \Omega)$. From (2.2b) it follows

$$-(v_h, \text{div}_h\sigma_h)_T + \langle \alpha_T(P_T^\rho u_h - \lambda_h), v_h \rangle_{\partial T} = (f, v_h)_T, \ \forall v_h \in P_{k+1}(T),$$

which, together with integration by parts, yields

$$(\nabla v_h, \sigma_h)_T - (\sigma_h \cdot n - \alpha_T(P_T^\rho u_h - \lambda_h), v_h)_{\partial T} = (f, v_h)_T, \ \forall v_h \in P_{k+1}(T).$$

Then, in view of (4.2a)-(4.2b), it holds

$$(\nabla v_h, \sigma_h - \bar{\sigma}_h)_T - ((\sigma_h - \bar{\sigma}_h) \cdot n, v_h)_{\partial T} = (f, v_h)_T, \ \forall v_h \in P_{k+1}(T),$$

which implies

$$-(v_h, \text{div}_h\sigma^*_h)_T = (f, v_h)_T, \ \forall v_h \in P_{k+1}(T),$$

or equivalently,

$$\text{div}_h\sigma^*_h = P_V \text{div}\sigma.$$

The thing left is to prove the estimates (4.4)-(4.5). In light of (4.1), Lemma 3.6 and Theorem 3.1, it suffices to show

$$\|\bar{\sigma}_h\| \lesssim \|((e_h^u, e_h^\lambda, e_h^\sigma))\| + \left( \sum_{T \in \mathcal{T}_h} \|\alpha_T^{\frac{1}{2}}(u - P_V u)\|_{\partial T} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (4.6)

By (4.2a)-(4.2b), a simple scaling argument yields the estimate

$$\|\bar{\sigma}_h\|_T \lesssim \|\alpha_T^{\frac{1}{2}}(P_T^\rho u_h - \lambda_h)\|_{\partial T}, \quad \forall T \in \mathcal{T}_h.$$ 

Then, from $e_h^u = u_h - P_V u$ and $e_h^\lambda = \lambda_h - P_M u$ it follows

$$\|\bar{\sigma}_h\| \lesssim \left( \sum_{T \in \mathcal{T}_h} \|\alpha_T^{\frac{1}{2}}(P_T^\rho u_h - \lambda_h)\|_{\partial T}^2 \right)^{\frac{1}{2}}$$

$$\lesssim \left( \sum_{T \in \mathcal{T}_h} \|\alpha_T^{\frac{1}{2}}(P_T^\rho e_h^u - e_h^\lambda)\|_{\partial T}^2 \right)^{\frac{1}{2}} + \left( \sum_{T \in \mathcal{T}_h} \|\alpha_T^{\frac{1}{2}}(P_T^\rho P_V u - P_M u)\|_{\partial T}^2 \right)^{\frac{1}{2}}$$

$$\lesssim \|((e_h^u, e_h^\lambda, e_h^\sigma))\| + \left( \sum_{T \in \mathcal{T}_h} \|\alpha_T^{\frac{1}{2}}(P_T^\rho P_V u - P_M u)\|_{\partial T}^2 \right)^{\frac{1}{2}},$$
which, together with the fact
\[ \| \alpha_T^2 (P^T_P V u - P_M u) \|_{\partial T} = \| \alpha_T^2 P^T_P (P_V u - u) \|_{\partial T} \leq \| \alpha_T^2 (u - P_V u) \|_{\partial T}, \]
indicates the desired estimate (4.6). This completes the proof. \( \square \)

5 Conditioning of HDG method

In this section, we shall first consider the static condensation of the system (2.2) by following the idea in [15, 18]. We turn the system (2.2) into a SPD (symmetric positive definite) system that only involves the trace approximation \( \lambda_h \). Secondly, we shall study the conditioning of the SPD system, which is significant in the construction of efficient solvers.

In what follows we assume, without loss of generality, \( g = 0 \) for the sake of simplicity.

5.1 "Static condensation" for HDG method

As mentioned before, one main advantage of a hybrid finite element method lies in that the method results in a sparse, symmetric and positive definite discrete system through "static condensation". This "static condensation" is essentially a process of locally eliminating unknowns defined in the interior of the elements. In what follows we shall show the process of "static condensation" for the HDG method (2.2) by applying the framework developed in [18].

For any \( T \in T_h \), we introduce two local problems as follows.

Local problem 1: For any given \( \lambda \in M(\partial T) \), seek \( (u_\lambda, \sigma_\lambda) \in V(T) \times W(T) \) such that
\[
\begin{align*}
(c \sigma, \tau)_T + (u_\lambda, \text{div} \tau)_T &= \langle \lambda, \tau \cdot n \rangle_{\partial T}, \quad (5.1a) \\
-(v, \text{div} \sigma_\lambda)_T + \langle \alpha_T P^2_T u_\lambda, v \rangle_{\partial T} &= \langle \alpha_T \lambda, v \rangle_{\partial T} \quad (5.1b)
\end{align*}
\]
hold for all \( (v, \tau) \in V(T) \times W(T) \).

Local problem 2: For any given \( f \in L^2(T) \), seek \( (u_f, \sigma_f) \in V(T) \times W(T) \) such that
\[
\begin{align*}
(c \sigma_f, \tau)_T + (u_f, \text{div} \tau)_T &= 0, \quad (5.2a) \\
-(v, \text{div} \sigma_f)_T + \langle \alpha_T P^2_T u_f, v \rangle_{\partial T} &= (f, v)_T \quad (5.2b)
\end{align*}
\]
hold for all \( (v, \tau) \in V(T) \times W(T) \).

Lemma 5.1. For \( f \in L^2(T) \) and \( \lambda \in M(\partial T) \), the local problems (5.1) and (5.2) are well-posed.

Proof. It suffices to show that the homogeneous linear system
\[
\begin{align*}
(c \sigma_h, \tau)_T + (u_h, \text{div} \tau)_T &= 0, \quad \forall \tau \in W(T), \quad (5.3a) \\
-(v, \text{div} \sigma_h)_T + \langle \alpha_T P^2_T u_h, v \rangle_{\partial T} &= 0, \quad \forall v \in V(T) \quad (5.3b)
\end{align*}
\]
admits a unique solution \((u_h, \sigma_h) = (0, 0)\) in \(V(T) \times W(T)\). In fact, taking \(\tau = \sigma_h\) in (5.3a), \(v = u_h\) in (5.3b), and adding the two resultant equations, we have

\[
\left\| c^2 \sigma_h \right\|_T^2 + \left\| \alpha_T P_T^0 u_h \right\|_{\partial T}^2 = 0,
\]

which implies

\(\sigma_h = 0\) and \(P_T^0 u_h = 0\).

Then, from (5.3a) and the fact that \(\tau \cdot n \in M(\partial T)\) due to (2.3), it follows

\[
0 = (u_h, \nabla \cdot \tau)_T = -(\nabla u_h, \tau)_T + (u_h, \tau \cdot n)_{\partial T}
= -(\nabla u_h, \tau)_T + (P_T^0 u_h, \tau \cdot n)_{\partial T}
= -(\nabla u_h, \tau)_T, \quad \forall \tau \in W(T),
\]

which, together with \(\nabla V(T) \subset W(T)\), yields \(\nabla u_h = 0\), or equivalently, \(u_h = \text{constant}\). As a result, \(u_h = 0\) follows from \(P_T^0 u_h = 0\). This completes the proof.

This lemma means that we can eliminate the unknowns of \((u_h, \sigma_h)\) in terms of \(\lambda_h\) at the element level. The following theorem shows that HDG method (2.2) results in a SPD system.

**Theorem 5.1.** Suppose \((u_h, \lambda_h, \sigma_h) \in V_h \times M_h^0 \times W_h\) to be the solution of the system (2.2), and suppose, for any \(T \in T_h\), \((u_{\lambda h}, \sigma_{\lambda h}) \in V(T) \times W(T)\) and \((u_f, \sigma_f) \in V(T) \times W(T)\) to be the solutions of the local problems (5.1) and (5.2), respectively. Then it holds

\[
\sigma_h|_T = \sigma_{\lambda h} + \sigma_f, \quad (5.4)
\]

\[
u_h|_T = u_{\lambda h} + u_f, \quad (5.5)
\]

and \(\lambda_h \in M_h^0\) is the solution of the system

\[
d_h(\lambda_h, \mu_h) = b_h(\mu_h), \quad \forall \mu_h \in M_h^0, \quad (5.6)
\]

where

\[
d_h(\lambda_h, \mu_h) := (\sigma_{\lambda h}, \sigma_{\mu h})_c + \sum_{T \in T_h} (\alpha_T (P_T^0 u_{\lambda h} - \lambda_h), P_T^0 u_{\mu h} - \mu_h)_{\partial T}, \quad (5.7)
\]

\[
b_h(\mu_h) := (f, u_{\mu h}). \quad (5.8)
\]

We omit the proof of this theorem here, since it is just a trivial modification of the proof of Theorem 2.1 in [18].

**Remark 5.1.** We note that the system (5.6) leads to a linear system with sparse, symmetric and positive definite matrix, in which only unknowns of \(\lambda_h\) are remained. In this sense, the computational size is reduced, and one may use a conjugate gradient method or preconditioned conjugate gradient method to solve the resultant linear system.

**Remark 5.2.** Once \(\lambda_h\) is obtained, we can compute \(u_h\) and \(\sigma_h\) in an element-by-element fashion.
5.2 Main results for conditioning

At first, we introduce some mesh-dependent norms and seminorms as follows. For $\forall \mu \in L^2(\mathcal{F}_h)$, define
\[ \|\mu\|_h := \left( \sum_{T \in \mathcal{T}_h} \|\mu\|^2_{h,\partial T} \right)^{\frac{1}{2}}, \|\mu\|_{h,\partial T} := h_T^{\frac{1}{2}} \|\mu\|_{\partial T}, \]
\[ |\mu|_h := \left( \sum_{T \in \mathcal{T}_h} |\mu|^2_{h,\partial T} \right)^{\frac{1}{2}}, |\mu|_{h,\partial T} := h_T^{-\frac{1}{2}} \|\mu - m_T(\mu)\|_{\partial T}, \]
\[ m_T(\mu) := \frac{1}{|\partial T|} \int_{\partial T} \mu. \]

We state in Theorems 5.2-5.4 some main results for the conditioning of the system (5.6), and put the proof of Theorem 5.2 in Subsection 5.3.

**Theorem 5.2.** For any $\mu_h \in M_h$, it holds
\[ d_h(\mu_h, \mu_h) \simeq |\mu_h|^2_h. \quad (5.9) \]

**Theorem 5.3.** Suppose $\mathcal{T}_h$ to be quasi-uniform, then it holds
\[ \|\mu_h\|^2_h \lesssim d_h(\mu_h, \mu_h) \lesssim h^{-2} \|\mu_h\|^2_h, \forall \mu_h \in M_h^0. \quad (5.10) \]

**Proof.** The conclusion follows from the same routine as in the proof of Theorem 2.3 in [27]. \qed

**Theorem 5.4.** Suppose $\mathcal{T}_h$ to be quasi-uniform. If $h$ is sufficiently small, then it holds
\[ \sup_{\mu_h \in M_h^0} \frac{d_h(\mu_h, \mu_h)}{\|\mu_h\|^2_h} \gtrsim h^{-2}; \quad (5.11) \]
\[ \inf_{\mu_h \in M_h^0} \frac{d_h(\mu_h, \mu_h)}{\|\mu_h\|^2_h} \lesssim 1. \quad (5.12) \]

**Proof.** Introduce the standard $H^1$-conforming finite element space
\[ \mathcal{L}^1_1(\mathcal{T}_h) := \{ v_h \in H^1_0(\Omega) : v_h|_T \in P_1(T), \forall T \in \mathcal{T}_h \}. \]
Through standard scaling arguments, we have the estimates
\[ \|P_M v_h\|_h \simeq \|v_h\|, \forall v_h \in \mathcal{L}^1_1(\mathcal{T}_h), \quad (5.13) \]
\[ |P_M v_h|_h \simeq \|\nabla v_h\|, \forall v_h \in \mathcal{L}^1_1(\mathcal{T}_h). \quad (5.14) \]
On the other hand, it is well-known that the following estimates holds:
\[ \sup_{v_h \in \mathcal{L}^1_1(\mathcal{T}_h)} \frac{\|\nabla v_h\|^2}{\|v_h\|^2} \gtrsim h^{-2}; \quad (5.15) \]
\[ \inf_{v_h \in \mathcal{L}^1_1(\mathcal{T}_h)} \frac{\|\nabla v_h\|^2}{\|v_h\|^2} \lesssim 1. \quad (5.16) \]
Therefore, in light of Theorem 5.2 and (5.13)-(5.16), we obtain
\[
\sup_{\mu_h \in M_h^0} \frac{d_h(\mu_h, \mu_h)}{\|\mu_h\|_h^2} \gtrsim \sup_{v_h \in L^2(T_h)} \frac{\|P_M v_h\|^2_h}{\|v_h\|^2_h} \simeq \sup_{v_h \in L^2(T_h)} \frac{\|\nabla v_h\|^2}{\|v_h\|^2} \gtrsim h^{-2},
\]
\[
\inf_{\mu_h \in M_h^0} \frac{d_h(\mu_h, \mu_h)}{\|\mu_h\|_h^2} \lesssim \inf_{v_h \in L^2(T_h)} \frac{\|P_M v_h\|^2_h}{\|v_h\|^2_h} \simeq \inf_{v_h \in L^2(T_h)} \frac{\|\nabla v_h\|^2}{\|v_h\|^2} \lesssim 1.
\]
The theorem is proven. \[\square\]

**Remark 5.3.** Suppose \(\{\eta_1, \eta_2, \ldots, \eta_M\}\) to be the set of nodal basis functions for \(M_h^0\). For each \(\mu_h \in M_h^0\), we use \(\tilde{\mu}_h \in \mathbb{R}^M\) to denote the vector of coefficients of \(\mu_h\) in the \(\{\eta_i\}\)-basis. Then there exists a SPD matrix \(\mathcal{D}_h \in \mathbb{R}^{M \times M}\) such that
\[
\tilde{\lambda}_h^T \mathcal{D}_h \tilde{\mu}_h = d_h(\lambda_h, \mu_h), \ \forall \lambda_h, \mu_h \in M_h^0.
\]
By scaling arguments we have
\[
\|\mu_h\|_h \simeq h^{d/2} \|\tilde{\mu}_h\|, \ \forall \mu_h \in M_h^0,
\]
where \(\|\cdot\| : \mathbb{R}^M \to \mathbb{R}\) denotes the standard Euclidean norm. Then, from Theorem 5.3 and Theorem 5.4 we easily know that the condition number, \(\kappa(\mathcal{D}_h)\), of \(\mathcal{D}_h\) is of the estimate
\[
\kappa(\mathcal{D}_h) \simeq h^{-2}.
\]

### 5.3 Proof of Theorem 5.2

It is easy to see that the local problem (5.1) is equivalent to the following problem: For any given \(\lambda \in M(\partial T)\), seek \((u_\lambda, \sigma_\lambda) \in V(T) \times W(T)\) such that
\[
(c \sigma_\lambda, \tau)_T + (u_\lambda - m_T(\lambda), \text{div} \tau)_T = \langle \lambda - m_T(\lambda), \tau \cdot n \rangle_{\partial T}, \quad (5.17a)
\]
\[-(v, \text{div} \sigma_\lambda)_T + \langle \alpha_T(\partial_T^2 u_\lambda - m_T(\lambda)), v \rangle_{\partial T} = \langle \alpha_T(\lambda - m_T(\lambda)), v \rangle_{\partial T}, \quad (5.17b)
\]
hold for all \((v, \tau) \in V(T) \times W(T)\).

Before going on, we list below two well-known estimates which follow from trace inequality and inverse inequality:
\[
\|\tau\|_{\partial T} \lesssim h_T^{-\frac{1}{2}} \|\tau\|_T, \ \forall \tau \in W(T), \quad (5.18)
\]
\[
\|v\|_{\partial T} \lesssim h_T^{-\frac{1}{2}} \|v\|_T, \ \forall v \in V(T). \quad (5.19)
\]
For the sake of concision, we will use the above estimates without notification in what follows.

**Lemma 5.2.** For any given \(\lambda \in M(\partial T)\), it holds
\[
\|\nabla u_\lambda\|_T \lesssim \|\sigma_\lambda\|_T + \|\alpha_T(\partial_T^2 u_\lambda - \lambda)\|_{\partial T}. \quad (5.20)
\]
Proof. By integrating by parts, it follows from (5.1a) that

$$(\nabla u_\lambda, \tau)_T = (c\sigma, \tau)_T + (P_T^0u_\lambda - \lambda, \tau \cdot n)_{\partial T}, \quad \forall \tau \in W(T).$$

Thus we have

$$\|\nabla u_\lambda\|_T^2 = (c\sigma, \nabla u_\lambda)_T + \langle P_T^0u_\lambda - \lambda, \nabla u_\lambda \cdot n \rangle_{\partial T} \lesssim \|\sigma\|_T \|\nabla u_\lambda\|_T + h_T^{-\frac{1}{2}} \|P_T^0u_\lambda - \lambda\|_{\partial T} \|\nabla u_\lambda\|_T$$

which implies (5.20).

Further more, we have the following lemma.

**Lemma 5.3.** For any given $\lambda \in M(\partial T)$, it holds

$$\|\sigma\|_T + \left\| \alpha_T^\lambda (P_T^0u_\lambda - \lambda) \right\|_{\partial T} \simeq |\lambda|_{h, \partial T} . \quad (5.21)$$

**Proof.** We first show

$$|\lambda|_{h, \partial T} \lesssim \|\sigma\|_T + \left\| \alpha_T^\lambda (P_T^0u_\lambda - \lambda) \right\|_{\partial T} . \quad (5.22)$$

In fact, denote $\tilde{\lambda} := \lambda - m_T(\lambda)$ and $\bar{u}_\lambda := \frac{1}{|T|} \int_T u_\lambda$, then we obtain

$$\langle \tilde{\lambda}, u_\lambda \rangle_{\partial T} = \langle \tilde{\lambda}, u_\lambda - \bar{u}_\lambda \rangle_{\partial T} \leq \|\tilde{\lambda}\|_{\partial T} \|u_\lambda - \bar{u}_\lambda\|_{\partial T} \lesssim h_T^{\frac{1}{2}} \|\tilde{\lambda}\|_{\partial T} \|\nabla u_\lambda\|_T \lesssim h_T^{\frac{1}{2}} \|\tilde{\lambda}\|_{\partial T} \left( \|\sigma\|_T + \left\| \alpha_T^\lambda (P_T^0u_\lambda - \lambda) \right\|_{\partial T} \right) . \quad \text{by (5.20)}$$

This estimate, together with

$$\langle \tilde{\lambda}, \lambda - P_T^0u_\lambda \rangle_{\partial T} \leq \|\tilde{\lambda}\|_{\partial T} \|\lambda - P_T^0u_\lambda\|_{\partial T} \lesssim h_T^{\frac{1}{2}} \|\tilde{\lambda}\|_{\partial T} \left\| \alpha_T^\lambda (\lambda - P_T^0u_\lambda) \right\|_{\partial T} ,$$

yields

$$\|\tilde{\lambda}\|_{\partial T}^2 = \langle \tilde{\lambda}, \tilde{\lambda} - P_T^0u_\lambda \rangle_{\partial T} + \langle \tilde{\lambda}, P_T^0u_\lambda \rangle_{\partial T} = \langle \tilde{\lambda}, \lambda - P_T^0u_\lambda \rangle_{\partial T} + \langle \tilde{\lambda}, u_\lambda \rangle_{\partial T} \lesssim h_T^{\frac{1}{2}} \|\tilde{\lambda}\|_{\partial T} \left( \|\sigma\|_T + \left\| \alpha_T^\lambda (P_T^0u_\lambda - \lambda) \right\|_{\partial T} \right) ,$$

which implies

$$|\lambda|_{h, \partial T} = h_T^{-\frac{1}{2}} \|\tilde{\lambda}\|_{\partial T} \lesssim \|\sigma\|_T + \left\| \alpha_T^\lambda (P_T^0u_\lambda - \lambda) \right\|_{\partial T} ,$$

i.e. (5.22) holds.
Second we need to prove
\[
\|\sigma\|_T + \|\alpha_T^\frac{1}{2}(P^0_T u_\lambda - \lambda)\|_{\partial T} \lesssim |\lambda|_{h,\partial T}.
\] (5.23)
Taking \( \tau = \sigma \lambda \) in (5.17a), \( v = u_\lambda - m_T(\lambda) \) in (5.17b), and adding the two resultant equations, we have
\[
\left\| \frac{c^\frac{1}{2}}{\tau} \right\|_T^2 + \left\| \frac{\alpha^\frac{1}{2}}{\partial T}(P^0_T u_\lambda - m_T(\lambda)) \right\|_{\partial T}^2 \\
= (\lambda - m_T(\lambda), \sigma \lambda \cdot n)_{\partial T} + (\alpha_T(\lambda - m_T(\lambda)), P^0_T u_\lambda - m_T(\lambda))_{\partial T} \\
\leq \|\lambda - m_T(\lambda)\|_{\partial T} \|\sigma\|_T + \|\alpha^\frac{1}{2}_T(\lambda - m_T(\lambda))\|_{\partial T} \|\alpha^\frac{1}{2}_T(P^0_T u_\lambda - m_T(\lambda))\|_{\partial T} \\
\lesssim h_{-\frac{1}{2}}^\frac{1}{2} \|\lambda - m_T(\lambda)\|_{\partial T} \|\sigma\|_T + \|\alpha^\frac{1}{2}_T(\lambda - m_T(\lambda))\|_{\partial T} \|\alpha^\frac{1}{2}_T(P^0_T u_\lambda - m_T(\lambda))\|_{\partial T} \\
\lesssim |\lambda|_{h,\partial T} \left( \|\sigma\|_T + \|\alpha^\frac{1}{2}_T(P^0_T u_\lambda - m_T(\lambda))\|_{\partial T} \right),
\] which leads to
\[
\|\sigma\|_T + \|\alpha^\frac{1}{2}_T(P^0_T U\lambda - m_T(\lambda))\|_{\partial T} \lesssim |\lambda|_{h,\partial T}.
\] (5.24)
By noticing that the above estimate also indicates
\[
\|\alpha^\frac{1}{2}_T(P^0_T u_\lambda - \lambda)\|_{\partial T} \leq \|\alpha^\frac{1}{2}_T(P^0_T u_\lambda - m_T(\lambda))\|_{\partial T} + \|\alpha^\frac{1}{2}_T(\lambda - m_T(\lambda))\|_{\partial T} \\
\lesssim |\lambda|_{h,\partial T},
\] the estimate (5.23) follows immediately.

As a result, the desired equivalence (5.21) follows from (5.22) and (5.23).

Now we are ready to prove Theorem 5.2.

**Proof of Theorem 5.2.** In view of Lemma 5.3, it holds
\[
d_h(\mu_h, \mu_h) = (\sigma_{\mu_h}, \sigma_{\mu_h})_e + \sum_{T \in T_h} (\alpha_T(P^0_T u_{\mu_h} - \mu_h), P^0_T u_{\mu_h} - \mu_h)_{\partial T} \\
\simeq \sum_{T \in T_h} \left( \|\sigma_{\mu_h}\|_T^2 + \|\alpha^\frac{1}{2}_T(P^0_T u_{\mu_h} - \mu_h)\|_{\partial T}^2 \right) \\
\simeq \sum_{T \in T_h} |\mu_h|_{h,\partial T}^2 \\
\simeq |\mu_h|_{h}^2.
\]

### 6 Numerical experiments

This section provides numerical experiments in two-space dimensions to verify our theoretical results. We consider the problem (1.1) with \( \Omega = (0, 1) \times (0, 1) \) and
\[
c(x, y) = \begin{pmatrix}
1 + x^2 y^2 & 0 \\
0 & 1 + x^2 y^2
\end{pmatrix},
\]
and we set \( u(x, y) = \sin(\pi x) \sin(\pi y) \) to be the analytic solution.
We start with an initial mesh shown in Figure 1 with $h^{-1} = 2$ and obtain a sequence of refined meshes by bisection. Numerical results are presented in Tables 2-3 for the proposed HDG method (2.2) with $k = 0, 1$.

Table 2 shows the history of convergence for the potential approximation $u_h$ and the flux approximation $\sigma_h$. We can see that for $k = 0$, which corresponds to the lowest order HDG method, the potential error $\|u - u_h\|$ is of second-order accuracy, and the flux error $\|\sigma - \sigma_h\|$ is first-order accuracy, while for $k = 1$, $\|u - u_h\|$ is of third-order accuracy and $\|\sigma - \sigma_h\|$ is of second-order accuracy. These numerical results are conformable to the error estimates in Theorems 3.1-3.2.

Table 3 shows the history of convergence for the postprocessed flux approximation $\sigma^*_h$. We can see that for $k = 0$, $\|\sigma - \sigma^*_h\|$ is of first-order accuracy, and $\|\text{div}\sigma - \text{div}\sigma^*_h\|$ is second-order accuracy, while for $k = 1$, $\|\sigma - \sigma^*_h\|$ is of second-order accuracy and $\|\text{div}\sigma - \text{div}\sigma^*_h\|$ is of third-order accuracy. These numerical results are conformable to the error estimates in Theorem 4.1.

Figure 1: Initial mesh with $h^{-1} = 2$
| Degree $k$ | Mesh $h^{-1}$ | $\|u - u_h\|$ Error | Order | $\|\sigma - \sigma_h\|$ Error | Order |
|-----------|---------------|---------------------|-------|-------------------------------|-------|
| 0         | 2             | 3.052e-1            | -     | 1.230                         | -     |
| 4         | 7.828e-2      | 1.963               | 0.933 | 6.443e-1                      | 0.933 |
| 8         | 1.968e-2      | 1.992               | 0.987 | 3.250e-1                      | 0.987 |
| 16        | 4.927e-3      | 1.998               | 0.997 | 1.629e-1                      | 0.997 |
| 32        | 1.232e-3      | 1.999               | 0.999 | 8.147e-2                      | 0.999 |
| 1         | 2             | 3.431e-2            | -     | 2.524e-1                      | -     |
| 4         | 4.376e-3      | 2.971               | 2.023 | 6.211e-2                      | 2.023 |
| 8         | 5.510e-4      | 2.990               | 2.000 | 1.552e-2                      | 2.000 |
| 16        | 6.900e-5      | 2.997               | 2.000 | 3.882e-3                      | 2.000 |

Table 2: History of convergence for $u_h$ and $\sigma_h$

| Degree $k$ | Mesh $h^{-1}$ | $\|\sigma - \sigma_h^*\|$ Error | Order | $\|\text{div}\sigma - \text{div}\sigma_h^*\|$ Error | Order |
|-----------|---------------|----------------------------------|-------|-------------------------------------------------|-------|
| 0         | 2             | 1.080                            | -     | 0.7003                                         | -     |
| 4         | 5.616e-1      | 0.944                            | 1.912 | 0.1861                                         | 1.912 |
| 8         | 2.826e-1      | 0.991                            | 1.985 | 0.0470                                         | 1.985 |
| 16        | 1.415e-1      | 0.998                            | 1.996 | 0.0118                                         | 1.996 |
| 32        | 7.078e-2      | 0.999                            | 1.999 | 0.0029                                         | 1.999 |
| 1         | 2             | 2.278e-1                         | -     | 0.0114                                         | -     |
| 4         | 5.514e-2      | 2.046                            | 3.015 | 0.0014                                         | 3.015 |
| 8         | 1.373e-2      | 2.006                            | 2.997 | 1.7919e-4                                     | 2.997 |
| 16        | 3.429e-3      | 2.001                            | 2.999 | 2.2405e-5                                     | 2.999 |

Table 3: History of convergence for $\sigma_h^*$

References

[1] R. A. ADAMS, J. J. F. FOURNIER, Sobolev Spaces, Academic Press, 2nd ed., 2003.

[2] A. ALONSO, Error estimator for a mixed method, Numer. Math., 74 (1996), 385-395.

[3] D. N. ARNOLD, F. BREZZI, Mixed and non-conforming finite element methods: implementation, post-processing and error estimates, Modél. Math. Anal. Numér., 19 (1985), 7-35.
[4] D. N. ARNOLD, F. BREZZI, B. COCKBURN, L. D. MARINI, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal., 39 (2002), 1749-1779.

[5] S. N. ATLURI, H. MURAKAWA, On hybrid finite element models in nonlinear solid mechanics, Finite elements in nonlinear mechanics, 1 (1977), 3-41.

[6] I. BABUSKA, J. ODEN, J. LEE, Mixed-hybrid finite element approximations of second-order elliptic boundary-value problems, Comput. Methods Appl. Mech. Engrg., 11 (1977), 175-206.

[7] D. BRAESS, Enhanced assumed strain elements and locking in membrane problems, Comput. Methods Appl. Mech. Engrg., 165 (1998), 155-174.

[8] D. BRAESS, C. CARSTENSEN, B.D. REDDY, Uniform convergence and a posteriori error estimators for the enhanced strain finite element method, Numer. Math., 96 (2004), 461-479.

[9] F. BREZZI, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, R.A.I.R.O. Anal Numer., 8 (1974), 129-151.

[10] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.

[11] F. BREZZI, L. D. MARINI, On the numerical solution of plate bending problems by hybrid methods, R.A.I.R.O. Anal Numer., 9 (1975), 5-50.

[12] F. BREZZI, J. DOUGLAS, JR., L. D. MARINI, Two families of mixed finite elements for second order elliptic problems, Numer. Math., 47 (1985), 217-235.

[13] P. CASTILLO, B. COCKBURN, I. PERUGIA, D. SCHÖTZAU, An a priori error analysis of the local discontinuous Galerkin method for elliptic problems, SIAM J. Numer. Anal., 38 (2000), 1676-1706.

[14] P. CIARLET, The finite element method for elliptic problems, North-Holland, Amsterdam, 1978.

[15] B. COCKBURN, J. GOPALAKRISHNAN, A characterization of hybridized mixed methods for second order elliptic problems, SIAM J. Numer. Anal., 42 (2004), 283-301.

[16] B. COCKBURN, B. DONG, An analysis of the minimal dissipation local discontinuous Galerkin method for convection-diffusion problems, J. Sci. Comput., 32 (2007), 233-262.

[17] B. COCKBURN, B. DONG, J. GUZMÁN, A superconvergent LDG-hybridizable Galerkin method for second-order elliptic problems, Math. Comp., 77 (2008), 1887-1916.

[18] B. COCKBURN, J. GOPALAKRISHNAN, R. LAZAROV, Unified hybridization of discontinuous Galerkin, mixed, and conforming Galerkin methods for second order elliptic problems, SIAM J. Numer. Anal., 47 (2009), 1319-1365.

[19] B. COCKBURN, J. GOPALAKRISHNAN, F. J. SAYAS, A projection-based error analysis of HDG methods, Math. Comp., 79 (2010), 1351-1367.
[20] B. COCKBURN, W. QIU, K. SHI, Conditions for superconvergence of HDG methods for second-order elliptic problems, Math. Comp., 81 (2012), 1327-1353.

[21] B. COCKBURN, J. GOPALAKRISHNAN, AND H. WANG, Locally conservative fluxes for the continuous Galerkin method, SIAM J. Numer. Anal., 45 (2007), 1742-1776.

[22] B. COCKBURN, J. GUZMÁN, H. WANG, Superconvergent discontinuous Galerkin methods for second-order elliptic problems, Math. Comp., 78 (2009), 1-24.

[23] B. COCKBURN, J. GUZMÁN, S. C. SOON, H. K. STOLARSKI, An analysis of the embedded discontinuous Galerkin method for second order elliptic problems. SIAM J. Numer. Anal., 2009, 2686-2707.

[24] B. COCKBURN, O. DUBOIS, J. GOPALAKRISHNAN, Multigrid for an HDG Method. IMA J. Numer. Anal., 2013, doi: 10.1093/imanum/drt024.

[25] M. FARHLOUL, M. FORTIN, Dual hybrid methods for the elasticity and the Stokes problems: a unified approach, Numer. Math. 76 (1997), 419-440.

[26] B. FRAEJIS DE VEUBEKE, Displacement and equilibrium models in the finite element method, in Stress Analysis, O. C. Zienkiewicz and G. Holister, eds., Wiley, New York, 1965.

[27] J. GOPALAKRISHNAN, A Schwarz preconditioner for a hybridized mixed method, Comput. Meth. Appl. Math., 3 (2003), 116-134.

[28] E. P. KASPER, R. L. TAYLOR, A mixed-enhanced strain method - part I: Geometrically linear problems, Comp. Struct., 75 (2000), 237-250.

[29] J. ODEN, J. LEE, Theory of mixed and hybrid finite-element approximations in linear elasticity, Appl. Methods Funct. Anal. Problems Mech. (1976) 90-109.

[30] J. ODEN, J. LEE, Dual-Mixed Hybrid finite element method for second-order elliptic problems, Lecture Notes in Mathematics 606 (1977), 275-291.

[31] T. H. H. PIAN, Derivation of element stiffness matrices by assumed stress distributions, AIAA Journal, 2 (1964), 1333-1336.

[32] T. H. H. PIAN, Finite element formulation by variational principles with relaxed continuity requirements, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, Part II (A. K. Aziz, Editor), Academic Press, New York, 1972, pp. 671-687. MR 49 #11824.

[33] T. H. H. PIAN, A historical note about 'hybrid elements', Internat. J. Numer. Methods Engrg., 5 (1978), 891-892

[34] T. H. H. PIAN, State-of-the-art development of hybrid/mixed finite element method, Finite Elements in Analysis and Design, 21 (1995), 5-20.
[35] T. H. H. PIAN, K. SUMIHARA, Rational approach for assumed stress finite elements. Internat. J. Numer. Methods Engrg., 20 (1984), 1685-1695.

[36] T. H. H. PIAN, P. TONG, Basis of finite element methods for solid continua, Internat. J. Numer. Methods Engrg., 1 (1969), 3-28.

[37] T. H. H. PIAN, P. TONG, Relations between incompatible displacement model and hybrid stress model, Internat. J. Numer. Methods Engrg., 22 (1986), 173-181.

[38] T. H. H. PIAN, C. C. WU, A rational approach for choosing stress terms for hybrid finite element formulations, Internat. J. Numer. Methods Engrg., 26 (1988), 2331-2343.

[39] E. F. PUNCH, S. N. ATLURI, Development and testing of stable, invariant, isoparametric curvilinear 2- and 3-D hybrid-stress elements, Comput. Methods Appl. Mech. Eng., 47 (1984), 331-356.

[40] P.-A. RAVIART, J. M. THOMAS, A mixed finite element method for second order elliptic problems, Mathematical Aspects of Finite Element Method (I. Galligani and E. Magenes, eds.), Lecture Notes in Math. 606, Springer-Verlag, New York, 1977, 292-315.

[41] K. Y. SZE, Efficient formulation of robust hybrid elements using orthogonal stress/strain interpolants and admissible matrix formulation, Internat. J. Numer. Methods Engng., 35 (1992), 1-20.

[42] K. Y. SZE, Hybrid hexahedral element for solids, plates, shells and beams by selective scaling, Internat. J. Numer. Methods Engng., 36 (1993), 1519-1540.

[43] P. A. RAVIART, J. M. THOMAS, Primal hybrid finite element methods for 2nd order elliptic equations[J]. Mathematics of computation, 31 (1977), 391-413.

[44] P. A. RAVIART, J. M. THOMAS, Dual finite element models for second order elliptic problems[J]. Energy methods in finite element analysis. (A 79-53076 24-39) Chichester, Sussex, England, Wiley-Interscience, 1979, 175-191.

[45] B. REDDY, J. SIMO, Stability and convergence of a class of enhanced strain methods, SIAM J. Numer. Anal. 32 (1995), 1705-1728.

[46] J. E. ROBERTS, J. M. THOMAS, Mixed and hybrid methods, in Handbook of Numerical Analysis, II, Handb. Numer. Anal., II, North-Holland, Amsterdam, 1991, 523-639.

[47] J. C. SIMO, M. S. RIFAI, A class of mixed assumed strain methods and the method of incompatible modes, Internat. J. Numer. Methods Engrg., 29 (1990), 1505-1638.

[48] J. WANG, X. YE, A weak Galerkin finite element method for the Stokes equations, arXiv:1302.2707.

[49] X. P. XIE, T. X. ZHOU, Optimization of stress modes by energy compatibility for 4-node hybrid quadrilaterals, Internat. J. Numer. Methods Engrg., 59 (2004), 293-313.

[50] S. T. YEO, B. C. LEE, Equivalence between enhanced assumed strain method and assumed stress hybrid method based on the Hellinger-Reissner principle, Internat. J. Numer. Methods Engrg., 39 (1996), 3083-3099.
[51] G. Z. YU, X. P. XIE, C. CARSTENSEN, Uniform convergence and a posteriori error estimation for assumed stress hybrid finite element methods, Comput. Methods Appl. Mech. Engrg., 200 (2011), 2421-2433.

[52] S. Q. ZHANG, X. P. XIE, Accurate 8-node hybrid hexahedral elements with energy-compatible stress modes, Adv. Appl. Math. Mech., 2 (2010), 333-354.

[53] T. X. ZHOU, Y. F. NIE, Combined hybrid approach to finite element schemes of high performance. Internat. J. Numer. Methods Engrg., 51 (2001), 181-202.

[54] T. X. ZHOU, X. P. XIE, A unified analysis for stress/strain hybrid methods of high performance, Comput. Methods Appl. Mech. Engrg., 191 (2002), 4619-4640.