Branching Random Walks with Immigration.
Lyapunov Stability

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1 Introduction

The initial version of this article was published in proceedings of the international scientific conference ACMPT 2017 which was dedicated to the 90th birth anniversary of outstanding mathematician and expert of applied mathematics Professor A.D. Solov’ev. The extended version was prepared at that time. But it was not published due to some technical difficulties. The present article is the final version of the initial one.

Nowadays the branching random walks (BRWs) are an appropriate tool to describe and explore the evolution processes with birth, death and migration [13]. In practice such models may be used in biology [2] and demography [8].

Models presented in the paper give a reasonably good description for the demographic situation associated with immigration in different European countries. Population dynamics research the statistical equilibrium of the process, so-called steady state. One of the examples of such stochastic processes is a continuous-time critical Galton-Watson branching process where the rates of birth and annihilation are equal. This process with a random walk of particles and their generation at any point on the lattice was considered in [5]. Such BRW has steady state under some conditions. In [5] authors assume that the underlying random walk associated with the process is transient. Unfortunately, this population model is not stable with respect to (even small) random perturbation and the steady state destroys, see [7]. It is not difficult to understand the fact: in case when death rate $\mu$ and birth rate $\beta$ in model of binary splitting are equal and the perturbation has the form $\beta' = \beta + \varepsilon$, $\mu' = \mu$ (where $\beta'$ and $\mu'$ new rates of birth and death), then the process becomes supercritical for any small $\varepsilon > 0$. If the perturbation has the form $\beta' = \beta - \varepsilon$, then the process becomes subcritical and the population degenerates.
Typically, immigration was introduced in models of branching processes. One of the first models was introduced by B. Sevastyanov in [10]. He considered the model when each particle can produce an arbitrary number of offsprings and any number of particles could appear from the outside in the system. We are going to consider immigration in more complex BRW model, in which also contains a random walk in space.

The arising problem of absence the steady state can be solved by adding immigration. The presence of immigration can stablish the process and stop extinction when the birth rate is less than the death rate. Apparently, this approach was suggested by Han, Molchanov and Whitmeyer in [5], but only for the case of binary splitting, thus is each particle can produce only one offspring. For such BRW it is possible to have slightly different interpretation: each particle produce two offsprings which start their evolution processes independently and the parental particle die.

The structure of the paper is following. In section 2 we describe the model of the BRW with immigration the main feature of which is arbitrary splitting of particles. Moreover, we present some conditional expectations. In section 3 we study the first moment of the particle field, derive the differential equation and obtain the asymptotic behaviour of the solution. In section 4 we consider the second moment of the particle field. Similarly to the previous section, we get the asymptotic behaviour of the second moment, which can be obtained from the derived differential equation. Unlike the first moment, derivation of the equation of the second moment needs the usage of more complicated tools. Besides, we should research two cases: they depend on the equality of points on the lattice which we consider. In section 5 there are the differential equations of the higher moments with their derivations. In section 6 we consider the generating function of the process. Typically, this is a useful tool in studying the process, because it gives an opportunity to get all the moments of the random variable. However in our case is not a good method for research due to the fact that the usage of generating function does not allow to obtain the equations in all necessary cases. In section 7 we assume that the intensities of the birth, death and immigration are functions which depend on the points on the lattice. Under these assumptions we consider the Lyapunov stability. This is more realistic in applications [8]. For example, people prefer to live in areas where there are more resources required for living, rather than in areas with uncomfortable climate.

2 BRW model on the Multidimensional Lattice

In this section we consider countinuous-time symmetric BRW on the lattice $\mathbb{Z}^d$. The subject of the study is the particle field $n(t, x)$, where $t \geq 0$, $x \in \mathbb{Z}^d$. We assume that at the initial moment random variables $n(0, x)$ are independent and identically distributed with finite moments. For example, it can be said the $n(0, x) \equiv 1$ for any $x \in \mathbb{Z}^d$. In fact, we will show in the future that for $t \to \infty$ the contribution of $n(0, x)$ into $n(t, x)$ is exponentionally neglectable. The evolution of the particle field includes several options.

Each particle at the moment $t > 0$ in the point $x \in \mathbb{Z}^d$ stays at this point random time $\tau$ up to the first transformation. The process we consider is the Markov process, so the random
variable $\tau$ has the exponential distribution. Therefore, at the moment $t + \tau + 0$ there can be the following transformations:

1. Firstly, it can be the jump from the point $x$ to the point $x + z$ with the probability $a(z)$. We assume that $a(z) = a(-z)$, $\sum_{z \neq 0} a(z) = 1$ and $a(0) = -1$. The intensity of jump, also called the diffusion coefficient, is denoted by $\kappa > 0$. Thus, the probability to jump from the point $x$ to the point $x + z$ during the small time $dt$ is $\kappa a(z) dt$. The generator of underlying random form has the form:

$$\mathcal{L}\psi(x) = \kappa \sum_{z \neq 0} \left[ \psi(x + z) - \psi(x) \right] a(z).$$

Moreover, we assume that our random walk is irreducible, thus is $\text{Span}\{z : a(z) > 0\} = \mathbb{Z}^d$.

The operator $\mathcal{L}$ generates the Markov semigroup

$$P_t = \exp\{t\mathcal{L}\}.$$

with kernel (transition probability) $p(t, x, y)$. Here

$$\frac{\partial p(t, x, y)}{\partial t} = \mathcal{L}_x p(t, x, y) = \mathcal{L}_y p(t, x, y).$$

Each particle carries out random walk with the generator $\mathcal{L}$ up to the moment of the first transformation, which consists of either annihilation or splitting.

2. Secondly, particle can die with the probability $\mu dt$, where $\mu$ is the mortality rate.

3. Thirdly, each particle (independent on others) can produce $n$ offsprings (or we can say that it produces $n - 1$ particles and still stays at the same point on the lattice). Let $b_n, n \neq 1$ is the intensity of the transformation for the single parental particle into $n$ particles. Besides,

$$\mu + \sum_{n \geq 2} b_n = -b_1 > 0.$$

So we obtain the infinitesimal generating function:

$$F(z) = \mu + \sum_{n=1}^{\infty} b_n z^n = \mu - (\mu + \sum_{n \geq 2} b_n) z + \sum_{n \geq 2} b_n z^n.$$

We also assume that $F(z)$ is the analytical function in the circle $|z| < 1 + \delta$, $\delta > 0$, thus is the intensities $b_n$ as the functions of $n$ are exponentially decreasing. Lastly, new particles start their evolution independently on others at the points where they appear.
4. Finally, the new property of the process is the presence of immigration. It means that a new particle can appear at the time interval \((t, t + dt)\) at any point \(x \in \mathbb{Z}^d\) on the lattice with the probability \(kd\), where \(k\) is the immigration rate.

Further we assume that the intensities \(b_n, \mu\) and \(k\) are constant.

The study of the BRW is usually based on backward Kolmogorov equations:

\[
P_t = AP,
\]

where \(A\) is the transition probability matrix.

However in our model, because of the presence of immigration and random distribution of the initial particle field, we have to use forward Kolmogorov equations:

\[
P_t = PA .
\]

The derivation of the forward Kolmogorov equation in our model is based on the following representation:

\[
n(t + dt, x) = n(t, x) + \xi(dt, x) ,
\]

where \(\xi(dt, x)\) is the discrete random variable with the distribution

\[
\xi(dt, x) = \begin{cases} 
n - 1, & \text{with probability } b_n n(t, x) dt, \quad n \geq 3, \\
1, & \text{with probability } b_2 n(t, x) dt + k dt \\
-1, & \text{with probability } \mu n(t, x) dt + \zeta n(t, x) dt, \\
0, & \text{with probability } 1 - \sum_{n \geq 3} b_n n(t, x) dt \\
- (b_2 + \mu + \zeta) n(t, x) dt - k dt \\
- \sum_{z \neq 0} a(-z) n(t, x + z) dt.
\end{cases}
\]

One of the targets of our research is to find the asymptotic behaviour of the moments of the random variable \(n(t, x)\) (definition of moments will be in the next sections). In this case we need the technique of conditional expectations \[11, \text{гл. 2}\].

Notice that random variable \(\xi(dt, x)\) and the \(\sigma\)-algebra \(\mathcal{F}_{\leq t}\), where \(\mathcal{F}_{\leq t}\) is the \(\sigma\)-algebra of events before and including \(t\), are independent.

In the end of the section we consider some useful relations.

1. Firstly, we have the conditional expectation of the variable \(\xi(dt, x)\).

\[
\mathbb{E}[\xi(dt, x)|\mathcal{F}_{\leq t}] = \sum_{n=2}^{\infty} (n - 1) b_n n(t, x) dt + k dt + \sum_{z \neq 0} \zeta a(-z) n(t, x + z) dt \\
-(\mu + \zeta) n(t, x) dt; \quad (1)
\]
2. The expectation of \( \xi^2(dt, x) \) is

\[
E[\xi^2(dt, x)|F_{<t}] = \sum_{n=2}^{\infty} (n-1)^2 b_n n(t, x)dt + kdt + \sum_{z \neq 0} \kappa a(z)n(t, x + z)dt + (\mu + \kappa)n(t, x)dt;
\]  

(2)

3. The correlation of \( \xi(dt, x) \) and \( \xi(dt, y) \). Here we assume that \( dt^2 = 0 \).

\[
E[\xi(dt, x)\xi(dt, y)|F_{<t}] = -\kappa(a(y - x)n(t, x)dt + a(x - y)n(t, y)dt), \ x \neq y;
\]  

(3)

4. For three different points \( x, y, z \in \mathbb{Z}^d \) (using \( dt^2 = 0 \)) we get

\[
E[\xi(dt, x)\xi(dt, y)\xi(dt, z)|F_{<t}] = 0, \ x \neq y, \ x \neq z, \ x \neq z;
\]  

(4)

5. For the higher moments of the particle field we have the expectation of \( \xi^p(dt, x) \), \( p \geq 2 \).

\[
E[\xi^p(dt, x)|F_{<t}] = \sum_{n=2}^{\infty} (n-1)^p b_n n(t, x)dt + kdt + \kappa \sum_{y \neq 0} a(y)n(t, x + y)dt + (-1)^p[\mu + \kappa]n(t, x)dt, \ p>0;
\]  

(5)

6. The next relation is also useful for getting the equations for the higher moments.

\[
E[\xi^p(dt, x)\xi^q(dt, y)|F_{<t}] = \kappa[(-1)^p a(y - x)n(t, x)dt + (-1)^q a(x - y)n(t, y)dt], \ x \neq y, \ p, q > 0;
\]  

(6)

7. Finally, we present the relation which will be used in method of generationg fuction.

\[
E[e^{-z\xi(dt, x)}|F_{<t}] = \sum_{n \geq 2} e^{-z(n-1)} b_n n(t, x)dt \\
+ e^{-z}(kdt + \kappa \sum_{y \neq 0} a(y)n(t, x + y)dt) + e^z(\mu + \kappa)n(t, x)dt \\
+ (1 - \sum_{n \geq 2} b_n n(t, x)dt - kdt - \kappa \sum_{y \neq 0} a(y)n(t, x + y)dt \\
- \mu n(t, x)dt - \kappa n(t, x)dt), \ z \geq 0.
\]  

(7)

3 The First Moment

In the future study there will be used some properties of the conditional expectations, see \[11\] for details:

If \( \xi - \mathcal{G} - \text{measurable}, \) then \( E(\xi \eta|\mathcal{G}) = \xi E(\eta|\mathcal{G}) \), \( (8) \)
\[ E(\xi|\mathcal{G}) = E\xi . \]

Define the first moment for the particle field \( n(t,x) \) as following:

\[ m_1(t,x) = E n(t,x) . \]

Now we are going to get the differential equation for the first moment. Consider the first moment at the time \( t + dt \) to obtain the equation:

\[ En(t + dt, x) = E[E[n(t + dt, x)|\mathcal{F}_{\leq t}]] = E[E[n(t, x) + \xi(dt, x)|\mathcal{F}_{\leq t}] . \]

Using properties (8) and (9) and equality (11) we have

\[ En(t + dt, x) = E[E[n(t + dt, x)|\mathcal{F}_{\leq t}]] = E[E[n(t, x) + \xi(dt, x)|\mathcal{F}_{\leq t}] = m_1(t, x) + \sum_{n=2}^{\infty} (n - 1) b_n m_1(t, x) dt + k dt + \sum_{z \neq 0} \kappa a(z)(m_1(t, x + z) dt - m_1(t, x) dt - \mu m_1(t, x) dt . \]

Let

\[ \mathcal{L}_a f(t, x) := \sum_{z \neq 0} a(z) \left( f(t, x + z) - f(t, x) \right) . \]

Combining all above results, we get the differential equation for the first moment. Moreover, due to the space homogeneity, \( \mathcal{L}_a m_1(t, x) = 0 . \)

\[ \begin{cases} \frac{\partial m_1(t, x)}{\partial t} = \left( \sum_{n=2}^{\infty} (n - 1) b_n - \mu \right) m_1(t, x) + k , \\ m_1(0, x) = E n(0, x) . \end{cases} \]

Let us define the coefficient \( \beta \) — the birth rate as

\[ \beta := \sum_{n \geq 2} (n - 1) b_n . \]

Equation (10) is similar to equation which covers the case when \( \beta = \beta(x) \), \( \mu = \mu(x) \), \( k = k(x) \) are bounded functions on the lattice \( \mathbb{Z}^d \). This case will be considered later.

In case of constant coefficients the equation can be solved as an ordinary differential equation. The solution has the form

\[ m_1(t, x) = \frac{k}{\beta - \mu} (e^{(\beta - \mu) t} - 1) + e^{(\beta - \mu) t} E n(0, x) . \]
Remark 3.1 If $\beta > \mu$ (in this case we say that the process is supercritical) then the population grows exponentially, despite the presence of immigration.

If $\beta = \mu$ (such case is critical) and $k > 0$ then the population grows with linear speed.

We are interested in the last case (subcritical case) when $\mu > \beta$. Here, if $k = 0$ then the population vanishes. But if $k > 0$ then

$$m_1(t, x) \to \frac{k}{\mu - \beta}, \quad t \to \infty.$$  \hspace{1cm} (11)

Further we are going to consider the case when $\mu > \beta$ and $k > 0$.

In case of non-constant coefficients we obtain the Lyapunov stability for the first moment. See the section 7 for details.

Theorem 3.1 Let $b_n(x), n \geq 2$, $\mu(x), k(x), x \in \mathbb{Z}^d$ are bounded and $\mu(x) - \beta(x) \geq \delta_1 > 0$, $k(x) \geq \delta_2 > 0$. Then for the bounded initial conditions there exists the limit

$$m_1(\infty, x) = \lim_{t \to \infty} m_1(t, x).$$

4 The Second Moment

Let us denote the second moment:

$$m_2(t, x, y) = E[n(t, x)n(t, y)].$$

To find the asymptotic behaviour of the second moment we calculate the differential equations in two cases ($x = y$ and $x \neq y$) and combine them into one equation, using the properties of the BRW and relations (11), (8) and (9) obtained for the first moment.

4.1 Case 1. $x = y$

Here there is used the same technique as in section 3 when the equation for the first moment was received. It means that we consider the second moment at the time $t + dt$ and use properties (8) and (9) and equalities (1), (2), (3). To simplify the recording, we introduce the designation

$$\mathcal{L}_ax f(t, x, y) := \sum_{z \neq 0} a(z)\left(f(t, x + z, y) - f(t, x, y)\right).$$

Then

$$m_2(t + dt, x, x) = E[n^2(t + dt, x)] = E[E[n^2(t + dt, x)|\mathcal{F}_{\leq t}]]$$

$$= E[E[(n(t, x) + \xi(dt, x))^2|\mathcal{F}_{\leq t}]]$$

$$= 2(\beta - \mu)m_2(t, x, x)dt + 2\mathcal{L}_ax m_2(t, x, x)dt$$

$$+ 2km_1(t, x)dt + kdt + \sum_{n=2}^{\infty} (n - 1)^2 b_n m_1(t, x)dt$$

$$+ \mathcal{L}_a m_1(t, x)dt + 2\mathcal{L}_a m_1(t, x)dt + \mu m_1(t, x)dt.$$
From this and (11) it is easy to get the differential equation in case 1:

\[
\begin{aligned}
\frac{\partial m_2(t, x, x)}{\partial t} &= 2m_2(t, x, x)[\sum_{n=2}^{\infty}(n-1)b_n - \mu] \\
&\quad + \frac{k(2k+2\epsilon+2\mu+\sum_{n=2}^{\infty}(n-1)(n-2)b_n)}{\mu-\sum_{n=2}^{\infty}(n-1)b_n} \\
&\quad + 2\kappa L_{ax}m_2(t, x, x), \\
m_2(0, x, x) &= E_n^2(0, x).
\end{aligned}
\] (12)

4.2 Case 2. \(x \neq y\)

As in 4.1 explore the second moment at the time \(t + dt\). Let

\[
L_{ay}f(t, x, y) := \sum_{z \neq 0} a(z) \left( f(t, x, y + z) - f(t, x, y) \right).
\]

In case \(x \neq y\) the following representation is true

\[
m_2(t + dt, x, y) = E[E[n(t + dt, x)n(t + dt, y)|F_{t+dt}] = E[E[(n(t, x) + \xi(dt, x))(n(t, y) + \xi(dt, y))|F_{t+dt}] = m_2(t, x, y) + m_2(t, y, x)(2\beta - 2\mu)dt + \kappa L_{ax}m_2(t, x, y) \\
&\quad + \kappa L_{ay}m_2(t, x, y) + k\left(m_1(t, y) + m_1(t, x)\right)dt \\
&\quad - \kappa\left(a(y - x)m_1(t, x) + a(x - y)m_1(t, y)\right)dt.
\]

So the differential equation for the second case has the form:

\[
\begin{aligned}
\frac{\partial m_2(t, x, y)}{\partial t} &= m_2(t, x, y)(2\beta - 2\mu) + \kappa L_{ax}m_2(t, x, y) \\
&\quad + \kappa L_{ay}m_2(t, x, y) + k\left(m_1(t, x) + m_1(t, y)\right) \\
&\quad - \kappa\left(a(y - x)m_1(t, x) + a(x - y)m_1(t, y)\right), \\
m_2(0, x, y) &= (E_n(0, x))^2.
\end{aligned}
\] (13)

4.3 Differential Equation for the Second Moment

To obtain the differential equation for the second moment we should combine equations in 4.1 and 4.2. Note that for fixed \(t\) the number of particles \(n(t, x)\) homogeneous in space, therefore, it is possible to write

\[
m_2(t, x, y) = m_2(t, x - y) = m_2(t, u).
\]
Thus, the equation which combines (12) and (13) is
\[
\begin{aligned}
\frac{\partial m_2(t,u)}{\partial t} &= 2m_2(t,u)(\beta - \mu) + 2\kappa \mathcal{L}_{au}m_2(t,u) + 2\kappa a(u)\Phi(m_1) \\
&\quad + \delta_0(u)\Psi(m_1) , \\
m_2(0,u) &= (En(0,u))^2(1 - \delta_0(u)) + \delta_0(u)En^2(0,u) .
\end{aligned}
\]  

Here \(x - y = u\), functions \(\Phi(x)\) and \(\Psi(x)\) are known functions which depend linearly on \(x\).

The result obtained in (11) (the relation for the first moment of the particle field) allows to write the final differential equation for the second moment:
\[
\begin{aligned}
\frac{\partial m_2(t,u)}{\partial t} &= 2m_2(t,u)(\beta - \mu) + 2\kappa \mathcal{L}_{au}m_2(t,u) + \frac{2k^2}{\mu - \beta} - \frac{2\kappa a(u)}{\mu - \beta} \\
&\quad + \delta_0(u)k(2\mu + \sum_{n \geq 2} (n-1)(n-2)b_n) \\
m_2(0,u) &= (En(0,u))^2(1 - \delta_0(u)) + \delta_0(u)En^2(0,u) .
\end{aligned}
\]  

(14)

4.4 Asymptotic behaviour of the Second Moment

The next goal is to solve the equation (14) and find the asymptotic behaviour of the second moment when \(t \to \infty\). One way to resolve this problem is to divide (14) into three equations and solve them separately and sum the obtained solutions, so the common solution will be found.

\[
\begin{aligned}
\frac{\partial m_2(t,u)}{\partial t} &= 2m_2(t,u)(\beta - \mu) , \\
m_2(0,u) &= (En(0,u))^2(1 - \delta_0(u)) + \delta_0(u)En^2(0,u) ;
\end{aligned}
\]  

(15)

\[
\begin{aligned}
\frac{\partial m_2(t,u)}{\partial t} &= 2m_2(t,u)(\beta - \mu) + \frac{2k^2}{\mu - \beta} , \\
m_2(0,u) &= 0 ;
\end{aligned}
\]  

(16)

\[
\begin{aligned}
\frac{\partial m_2(t,u)}{\partial t} &= 2m_2(t,u)(\beta - \mu) + 2\kappa \mathcal{L}_{au}m_2(t,u) - \frac{2\kappa a(u)}{\mu - \beta} \\
&\quad + \delta_0(u)k(2\mu + \sum_{n \geq 2} (n-1)(n-2)b_n) \\
m_2(0,u) &= 0 .
\end{aligned}
\]  

(17)

At the beginning we solve the first equation (15). To reach this aim we are going to use Feinman-Kac formula (see [9]). Then the solution \(m_{2,1}(t,u)\) of (15) is
\[
m_{2,1}(t,u) = E[e^{-\int_0^t -2(\beta - \mu)ds}((En(0,u))^2(1 - \delta_0(u)) + \delta_0(u)En^2(0,u))] = \\
e^{2(\beta - \mu)t}E[(En(0,u))^2(1 - \delta_0(u)) + \delta_0(u)En^2(0,u)] .
\]  

(18)

Note that this solution tends to zero as \(t \to \infty\).
The second equation (16) is the ordinary differential equation and the solution \( m_{2,2}(t, u) \) can be found:

\[
m_{2,2}(t, u) = \frac{k^2}{(\mu - \beta)^2} (1 - e^{2(\beta - \mu)t}) .
\]

As \( t \to \infty \),

\[
m_{2,2}(t, u) \to \frac{k^2}{(\mu - \beta)^2} .
\]

Finally, find the solution of the last equation (17) which we denote by \( m_{2,3}(t, u) \). Here we will use the discrete Fourier transform defined as

\[
\hat{f}(\theta) = \sum_{u \in \mathbb{Z}^d} e^{i(\theta, u)} f(\theta), \quad \theta \in [-\pi, \pi]^d .
\]

(20)

In the last equation there is a term

\[
L_{au}m_{2,3}(t, u) = \sum_{z \neq 0} a(z)\left(m_{2,3}(t, u + z) - m_{2,3}(t, u)\right)
\]

\[
= \sum_{z \neq 0} a(z)m_{2,3}(t, u) - m_{2,3}(t, u) .
\]

The first term is the convolution of the functions \( a(z) \) and \( m_{2,3}(t, u) \). So applying the discrete Fourier transform (20) to this term shows that

\[
\widehat{L_{au}m_{2,3}}(t, \theta) = \hat{a}(\theta)\hat{m}_{2,3}(t, \theta) - \hat{m}_{2,3}(t, \theta) .
\]

Turn into discrete Fourier transform (20) in the third differential equation

\[
\left\{ \begin{array}{l}
\frac{\partial \hat{m}_{2,3}(t, \theta)}{\partial t} = 2\hat{m}_{2,3}(t, \theta)[\beta - \mu] + 2\kappa(\hat{a}(\theta) - 1)\hat{m}_{2,3}(t, \theta) \\
-2\kappa k\hat{a}(\theta) + \frac{k(2\mu + \sum_{n \geq 2}(n-1)(n-2)b_n)}{\mu - \beta} , \\
\hat{m}_{2,3}(\theta, 0) = 0 .
\end{array} \right.
\]

The solution of this equation has the form

\[
\hat{m}_{2,3}(t, \theta) = \frac{-2\kappa k\hat{a}(\theta) + \frac{k(2\mu + \sum_{n \geq 2}(n-1)(n-2)b_n)}{\mu - \beta}}{2(\beta - \mu) + 2\kappa(\hat{a}(\theta) - 1)} (e^{(\beta - \mu) + 2\kappa(\hat{a}(\theta) - 1) t} - 1) .
\]

As \( t \to \infty \):

\[
\hat{m}_{2,3}(t, \theta) \to \hat{m}_{2,3}(\theta) = \frac{-2\kappa k\hat{a}(\theta) + \frac{k(2\mu + \sum_{n \geq 2}(n-1)(n-2)b_n)}{\mu - \beta}}{2(\beta - \mu) + 2\kappa(\hat{a}(\theta) - 1)} .
\]
Then
\[ -\hat{m}_{2,3}(\theta) = -\frac{C_1\hat{a}(\theta) + C_2}{C_3 - C_4\hat{a}(\theta)}, \] 
(21)
where \( C_1 = \frac{k\mu - \beta}{\mu - \beta} \), \( C_2 = \frac{-k(\mu + \sum_{n \geq 2} (n-1)(n-2)b_n)}{C_3 - C_4\hat{a}(\theta)} \), \( C_3 = \mu - \beta + \kappa \), \( C_4 = \kappa \).

From (21) we receive
\[ -\hat{m}_{2,3}(\theta) = \frac{C_1}{C_4} + \left( \frac{C_1C_3 + C_2C_4}{C_4} \right) \frac{1}{C_4 - \hat{a}(\theta)}. \]

**Remark 4.1** The following properties are true

1. \( \hat{\delta}_0(\theta) = 1; \)

2. \( \frac{1}{\hat{C}_4 - \hat{a}(\theta)} = \frac{C_4}{C_3} \left( 1 + \sum_{n=1}^{\infty} \left( \frac{C_4}{C_3} \right)^n \hat{a}(\theta)^n \right), \) if \( |\frac{C_4}{C_3}\hat{a}(\theta)| < 1. \)

Then
\[ -m_{2,3}(u) = -\frac{C_1}{C_4}\delta_0(u) + \left( \frac{C_1}{C_4} + \frac{C_2}{C_3} \right) \left( \delta_0(u) + \sum_{n=1}^{\infty} \left( \frac{C_4}{C_3} \right)^n a^{*n}(u) \right). \]

Here \( a^{*n}(u) \) - \( n \)-convolution of the function \( a(z) \), thus is \( a^{*n}(u) = (a \ast \cdots \ast a)(u). \)

After substituting the values of coefficients \( C_1, C_2, C_3, C_4 \) we get the following result:
\[ m_{2,3}(u) = \frac{k}{\mu - \beta}\delta_0(u) + \frac{k}{(\mu - \beta)(\mu - \beta + \kappa)} \left( \sum_{n=2}^{\infty} \left( \begin{array}{c} n \\ 2 \end{array} \right) b_n - \kappa \right) \]
\[ \times \left( \delta_0(u) + \sum_{n=1}^{\infty} \left( \frac{\kappa}{\mu - \beta + \kappa} \right)^n a^{*n}(u) \right). \]
(22)

Gather results obtained in (18), (19), (22)
\[ m_2(t, u) = m_{2,1}(t, u) + m_{2,2}(t, u) + m_{2,3}(t, u). \]

Then
\[ m_2(t, u) \to \frac{k^2}{(\mu - \beta)^2} + \frac{k}{\mu - \beta}\delta_0(u) + \frac{k}{(\mu - \beta)(\mu - \beta + \kappa)} \left( \sum_{n=2}^{\infty} \left( \begin{array}{c} n \\ 2 \end{array} \right) b_n - \kappa \right) \]
\[ \times \left( \delta_0(u) + \sum_{n=1}^{\infty} \left( \frac{\kappa}{\mu - \beta + \kappa} \right)^n a^{*n}(u) \right), \quad t \to \infty. \]
5 The Higher Moments

Give the definition of the \( n^{th} \) moment:

\[
m_n(t, x_1, ..., x_n) = E \left[ \prod_{i=1}^{n} n(t, x_i) \right].
\]

Let \( 1_A \) is the indicator of the set \( A \), thus is

\[
1_A = \begin{cases} 
1, & \text{if } A \text{ is true;} \\
0, & \text{otherwise}.
\end{cases}
\]

Using the same methods of calculations (and relations (1)-(6), (8) and (9)) it is possible to receive the differential equations for higher moments.

\[
m_n(t + dt, x_1, ..., x_n) = E \left[ \prod_{i=1}^{n} n(t + dt, x_i) \right] = E \left[ E \left[ \prod_{i=1}^{n} n(t + dt, x_i) \right] | \mathcal{F}_{\leq t} \right]
\]

\[= E \left[ \prod_{i=1}^{n} (n(t, x_i) + \xi(dt, x_i)) \right] | \mathcal{F}_{\leq t} = m_n(t, x_1, ..., x_n)
\]

\[+ \sum_{i=1}^{n} E \left[ \prod_{j=1, j \neq i}^{n} n(t, x_j) E[[\xi(dt, x_i)]|\mathcal{F}_{\leq t})] \right]
\]

\[+ \sum_{i=1}^{n} \sum_{p=2}^{\infty} E \left[ \prod_{x_j \neq x_i}^{n} n(t, x_j) E[[(\xi(dt, x_i))^p]|\mathcal{F}_{\leq t})] \right] 1_A(p, x_i)
\]

\[+ \sum_{i,j=1, p,q>0}^{n} E \left[ \prod_{x_k:x_k \neq x_i, x_j} \left[ n(t, x_k) E[[(\xi(dt, x_i))^p(\xi(dt, x_j))^q]|\mathcal{F}_{\leq t})] \right] \right] 1_A(p, x_i) 1_A(q, x_j).
\]
To continue calculations we use equalities (1)–(6):

\[
m_n(t + dt, x_1, ..., x_n) = m_n(t, x_1, ..., x_n) + \sum_{i=1}^{n} E[\prod_{j=1, j\neq i}^{n} n(t, x_j)\sum_{r=2}^{\infty} (r - 1)b_r n(t, x_i)dt + kdt
+ \sum_{z \neq 0} a(z)n(t, x_i + z)dt - (\mu + \kappa)n(t, x_i)dt)] + \sum_{i=1}^{n} \sum_{p=2}^{n} E[\prod_{j=1, j \neq i}^{n} n(t, x_j)]
\times [\sum_{r=2}^{\infty} (r - 1)^p b_r n(t, x_i)dt + kdt + \kappa \sum_{z \neq 0} a(z)n(t, x_i + z)dt + (-1)^p(\mu + \kappa)
\times n(t, x_i)dt]1_{A(p, x_i)} + \sum_{i, j=1, p, q > 0}^{n} \sum_{x_i \neq x_j}^{q \leq p \leq q \leq n} E[\prod_{k=1}^{n} n(t, x_k)\sum_{r=2}^{\infty} (r - 1)^p b_r + (-1)^p\mu]dt
+ km_{n-p}(t, \tilde{x}_1, ..., \tilde{x}_{n-p})dt + \kappa L_{ax} m_{n-p+1}(t, x_i, \tilde{x}_1, ..., \tilde{x}_{n-p})dt + (1 + (-1)^p)\kappa
\times m_{n-p+1}(t, x_i, \tilde{x}_1, ..., \tilde{x}_{n-p})dt1_{A(p, x_i)} + \sum_{x_i \neq x_j}^{q \leq p \leq q \leq n} [m_{n-(p+q)+1}(t, x_i, \tilde{x}_1, ..., \tilde{x}_{n-(p+q)})
\times (-1)^p\kappa a(x_i - x_j)dt + m_{n-(p+q)+1}(t, x_j, \tilde{x}_1, ..., \tilde{x}_{n-(p+q)})(-1)^q\kappa a(x_i - x_j)dt]
\times 1_{A(p, x_i)}1_{A(q, x_j)}
\]

where \( A(p, x_i) \) is the following event:

\[ A(p, x_i) = \{ \text{sequence } \{x_1, ..., x_n\} \text{ contains exactly } p \text{ identical elements } x_i \}. \]

Hence, from the representation above, we get the differential equation

\[
\begin{align*}
\frac{\partial m_n(t, x_1, ..., x_n)}{\partial t} &= n[\beta - \mu]m_n(t, x_1, ..., x_n) + k \sum_{i=1}^{n} m_{n-1}(t, x_1, ..., \hat{x}_i, ..., x_n) + \kappa \sum_{i=1}^{n} L_{ax} m_{n}(t, x_1, ..., x_n) + \sum_{i=1}^{n} \sum_{p=2}^{n} [m_{n-p+1}(t, x_i, \tilde{x}_1, ..., \tilde{x}_{n-p})\sum_{r=2}^{\infty} (r - 1)^p b_r + (-1)^p\mu] + km_{n-p}(t, \tilde{x}_1, ..., \tilde{x}_{n-p})dt + (1 + (-1)^p)\kappa m_{n-p+1}(t, x_i, \tilde{x}_1, ..., \tilde{x}_{n-p})dt1_{A(p, x_i)} + \sum_{x_i \neq x_j}^{q \leq p \leq q \leq n} [m_{n-(p+q)+1}(t, x_i, \tilde{x}_1, ..., \tilde{x}_{n-(p+q)})
\times (-1)^p\kappa a(x_i - x_j)dt + m_{n-(p+q)+1}(t, x_j, \tilde{x}_1, ..., \tilde{x}_{n-(p+q)})(-1)^q\kappa a(x_i - x_j)dt]
\times 1_{A(p, x_i)}1_{A(q, x_j)}
\end{align*}
\]
Here record \( m_{n-p+1}(t, x_i, \tilde{x}_1, ..., \tilde{x}_{n-p}) \) means that when we sum by \( i \) there are no points \( x_i \) in the set \( \{\tilde{x}_1, ..., \tilde{x}_{n-p}\} \).

6 Generating Function

In this section we consider one of the most popular tools in BRWs studies. This is generating function.

6.1 Definition and the Differential Equation of the Generating Function.

Our target is to obtain the differential equation of the generating function to simplify calculations in the moments’ research.

6.1.1 Definition.

Define the generating function \( F_\infty(z, t, x) \) of the particle field \( n(t, x) \) as

\[
\begin{align*}
F_\infty(z, t, x) &= E e^{-zn(t,x)} \\
F_\infty(z, 0, x) &= E e^{-zn(0,x)}, \quad z \geq 0.
\end{align*}
\]

Generating function is used in calculating moments (to do it, we should take the derivative of the variable \( z \) and substitute \( z = 0 \)). Here we want to derive the differential equation for the generating function and compare with the results above.

6.1.2 Differential Equation of \( F_\infty(z, t, x) \).

Derivating of the equation for the function \( F_\infty(z, t, x) \) is based on the considering this function at the time \( t + dt \). Then using (7)

\[
\begin{align*}
F_\infty(z, t + dt, x) &= E e^{-zn(t+dt,x)} = E e^{-zn(t,x)} e^{-z(\xi)} = (2) E\left(E(e^{-zn(t,x)} e^{-z(\xi)} F_{\leq t})\right) \\
&= E\left(e^{-zn(t,x)} (E(e^{-z(\xi)} | F_{\leq t})\right) \\
&= E\left(e^{-zn(t,x)} \left(E(e^{-z(\xi)} | F_{\leq t})\right)\right).
\end{align*}
\]
Remark 6.1 Note that

\[
\frac{F_\infty(z, t + dt, x) - F_\infty(z, t, x)}{dt} = E(e^{-zn(t,x)}(\sum_{n \geq 2} e^{-z(n-1)}b_n n(t, x)dt + e^{-z}(kdt + \kappa \sum_{y \neq 0} a(y) n(t, x + y)dt)}
+ e^z(\mu + \kappa) n(t, x)dt + (1 - \sum_{n \geq 2} b_n n(t, x)dt - kdt - \kappa \sum_{y \neq 0} a(y) n(t, x + y)dt)
- \mu n(t, x)dt - \kappa n(t, x)dt)) = F_\infty(z, t, x) + dt E(e^{-zn(t,x)}(\sum_{n \geq 2} b_n n(t, x))
+ ke^{-z} + \kappa e^{-z} \sum_{y \neq 0} a(y) n(t, x + y) + e^z(\mu + \kappa) n(t, x) - \sum_{n \geq 2} b_n n(t, x)
- k - \kappa \sum_{y \neq 0} a(y) n(t, x + y) - \mu n(t, x) - \kappa n(t, x)\).
\]

Consequently,

\[
\frac{F_\infty(z, t + dt, x) - F_\infty(z, t, x)}{dt} = E(e^{-zn(t,x)}(\sum_{n \geq 2} (e^{-z(n-1)} - 1)b_n n(t, x) + (e^{-z} - 1)k
+ (e^z - 1)(\mu + \kappa) n(t, x) + (e^{-z} - 1) \sum_{y \neq 0} a(y) n(t, x + y))\).
\]

6.2 Usage of the generating function

As mentioned above, generating functions help to obtain the equations for the moments of random variables.

Concretely,

\[
m_1(t, x) = -\frac{\partial F_\infty(z, t, x)}{\partial t} \bigg|_{z=0},
\]

\[
m_2(t, x) = \frac{\partial^2 F_\infty(t, x, y)}{\partial t^2} \bigg|_{z=0}.
\]
Now check results obtained for differential equations of the first two moments in sections 3 and 4.

6.2.1 The First Moment

The equation for the first moment can be derived from

\[
\frac{\partial^2 F_\infty(z, t, x)}{\partial z \partial t} \bigg|_{z=0} = - \frac{\partial m_1(t, x)}{\partial t} .
\]

From this and 6.1.2 we get the left part

\[
\frac{\partial F_\infty(z, t, x)}{\partial z} \bigg|_{z=0} \left( \sum_{n \geq 2} (n - 1)b_n - \mu - \kappa \right) + kF_\infty(0, t, x)(-1) + \kappa E \sum_{y \neq 0} a(y)n(t, x + y)(-1) = -m_1(t, x)(\beta - \mu) + \kappa m_1(t, x) - k - \kappa \sum_{y \neq 0} a(y)m_1(t, x + y).
\]

The initial condition is

\[m_1(0, x) = E n(0, x).\]

Hence, combining the above results, there is Cauchy problem for the first moment

\[
\begin{cases}
\frac{\partial m_1(t, x)}{\partial t} = (\beta - \mu)m_1(t, x) + \kappa L_\alpha m_1(t, x) + k, \\
m_1(0, x) = E n(0, x).
\end{cases}
\]

6.2.2 The Second Moment

Identically do for the second moment:

\[
\frac{\partial^3 F_\infty(z, t, x)}{\partial z^2 \partial t} \bigg|_{z=0} = m_2(t, x, x).
\]
So, in the left part we have

\[
2 \frac{\partial^2 F_\infty(z, t, x)}{\partial z^2} \bigg|_{z=0} \left( \sum_{n \geq 2} (n-1)b_n - \mu - \kappa \right) + \frac{\partial F_\infty(z, t, x)}{\partial z} \bigg|_{z=0} \left( - \sum_{n \geq 2} (n-1)^2 b_n \right) - \mu - \kappa + 2k \frac{\partial F_\infty(z, t, x)}{\partial z} \bigg|_{z=0} (-1) + kF_\infty(0, t, x) + \kappa \mathbb{E} \left( \sum_{y \neq 0} a(y)n(t, x + y) \right) \times (2e^{-z(t,x)}n(t,x) + e^{-z(t,x)}) \bigg|_{z=0} = 2(\beta - \mu)m_2(t, x, x) - 2\kappa m_2(t, x, x) + m_1(t, x) \left( \sum_{n \geq 2} (n-1)^2 b_n + \mu + \kappa \right) + 2k \sum_{y \neq 0} a(y)m_2(t, x, x + y) + \kappa \sum_{y \neq 0} a(y)m_1(t, x + y) = 2(\beta - \mu)m_2(t, x, x) + 2\kappa \mathcal{L}_axm_2(t, x, x) + k + m_1(t, x) \times \left( \sum_{n \geq 2} (n-1)^2 b_n + \mu + 2\kappa + 2k \right) + \kappa \mathcal{L}_am_1(t, x) .
\]

The initial condition:

\[
m_2(t, x, x) = E_n^2(0, x) .
\]

Thus,

\[
\begin{cases}
\frac{\partial m_2(t, x, x)}{\partial t} = 2(\beta - \mu)m_2(t, x, x) + 2\kappa \mathcal{L}_axm_2(t, x, x) + k \\
m_2(t, x, x) = E_n^2(0, x) .
\end{cases}
\]

### 6.3 Summary

As we see in 3, 4.3 and 6.2.1, 6.2.2 obtained equations are identical. Therefore, using the generating function makes the derivating of the equations much easier. However it is not possible to receive equation for high-order moments; because taking the derivative of the variable \( z \) and substituting \( z = 0 \) gives the correlation functions only in case when points (which used as arguments) are identical. But in the right side of the equations there are various configurations of points. So these equations cannot be solved.

### 7 Lyapunov Stability

In sections 3 and 4 there were obtained asymptotics for the first two moments in case of constant rates \( \beta, \mu, k \). In this part we are going to explore moments when the intensities are functions which depend on the point on the lattice, thus is \( \beta = \beta(x), \mu = \mu(x), k = k(x) \). \( x \in \mathbb{Z}^d \). For the next study there will be used the following
Definition 7.1 [1, ch. 2, § 10] It is said that two systems of differential equations

\[
\frac{dx}{dt} = f(t, x) \quad \text{and} \quad \frac{dy}{dt} = g(t, y)
\]

are **asymptotically equivalent** systems if a one-to-one correspondence can be established between their solutions \(x(t), y(t)\) respectively:

\[
\lim_{t \to \infty} [x(t) - y(t)] = 0.
\]

7.1 Parabolic Problem

To study the Lyapunov stability we are going to use the following

**Lemma 7.1** Consider the parabolic problem

\[
\begin{align*}
\frac{\partial u(t, x)}{\partial t} &= \mathcal{L}_a u(t, x) - v(x)u(t, x) + f(t, x), \\
u(0, x) &= u_0(x), \quad x \in \mathbb{Z}^d, \quad t \in [0, T].
\end{align*}
\]

Let \(v(x), u_0(x), f(t, x)\) are bounded (the first two - on \(\mathbb{Z}^d\), \(f(t, x)\) - on \([0, T] \times \mathbb{Z}^d\)), \(x(t)\) is a symmetric random walk on \(\mathbb{Z}^d\) with the generator \(\mathcal{L}_a\). Then

\[
u(t, x) = E_x e^{-\int_0^t v(x(s))ds} u_0(x(t)) + E_x \int_0^t f(t - s, x(s))e^{-\int_0^s v(x(\tau))d\tau} ds.
\]

The proof of the Lemma [7,1] can be found in [3].

7.2 Assumptions

For the next research we make some assumptions. Let \(k_0, \beta_0, \mu_0, \tilde{u}_0 \in \mathbb{R}\) and for fixed \(\varepsilon > 0\) say

\[
\begin{align*}
\mu_0 - \beta_0 &= v_0 > 0; \quad (23a) \\
k_0 &> 0; \quad (23b) \\
u_0 &> 0; \quad (23c) \\
\tilde{u}_0 &> 0; \quad (23d) \\
v_0 - \varepsilon &\leq v(x) \leq v_0 + \varepsilon; \quad (23e) \\
k_0 - \varepsilon &\leq k(x) \leq k_0 + \varepsilon; \quad (23f) \\
u_0 - \varepsilon &\leq u_0(x) \leq u_0 + \varepsilon; \quad (23g) \\
\tilde{u}_0 - \varepsilon &\leq u_0(x, y) \leq \tilde{u}_0 + \varepsilon. \quad (23h)
\end{align*}
\]
7.3 The First Moment

Consider the equation for the first moment in case of non-constant rates:

\[
\begin{align*}
\frac{\partial m_1(t, x)}{\partial t} &= \mathcal{L}_a m_1(t, x) - v(x)m_1(t, x) + k(x), \\
m_1(0, x) &= u_0(x).
\end{align*}
\]

**Theorem 7.1** Under the assumptions in \ref{7.2} and for any \( t \geq 0 \)

\[
C_1^- e + C_0^- e^{-(v_0 + \varepsilon)t} \leq m_1(t, x) - \frac{k_0}{v_0} \leq C_1^+ e + C_0^+ e^{-(v_0 - \varepsilon)t}.
\]

Constants \( C_0^\pm \), \( C_1^\pm \) only depend on \( k_0 \), \( v_0 \), \( u_0 \) if \( \varepsilon \leq \frac{\min(k_0, v_0)}{2} \).

**Remark 7.1** \( \frac{k_0}{v_0} = \frac{k_0}{\mu_0 - \beta_0} \) is the limit value of the first moment in case of constant intensities when \( t \to \infty \).

**Proof:**

1. For the upper estimate we use assumptions \ref{23e}, \ref{23f} and \ref{23g}:

\[
m_1(t, x) \leq (u_0 + \varepsilon)e^{-(v_0 - \varepsilon)t} + \int_0^t (k_0 + \varepsilon)e^{-(v_0 - \varepsilon)s}ds
\]

\[
= (u_0 + \varepsilon)e^{-(v_0 - \varepsilon)t} + \frac{k_0 + \varepsilon}{v_0 - \varepsilon}(1 - e^{-(v_0 - \varepsilon)t})
\]

\[
= \frac{k_0 + \varepsilon}{v_0 - \varepsilon} + e^{-(v_0 - \varepsilon)t}(u_0 - \frac{k_0 + \varepsilon}{v_0 - \varepsilon})
\]

\[
= \frac{k_0}{v_0} + O(\varepsilon) + e^{-(v_0 - \varepsilon)t}(u_0 - \frac{k_0}{v_0} + O(\varepsilon)).
\]

2. Use the same technique the lower estimate:

\[
m_1(t, x) \geq (u_0 - \varepsilon)e^{-(v_0 + \varepsilon)t} + \int_0^t (k_0 - \varepsilon)e^{-(v_0 + \varepsilon)s}ds
\]

\[
= (u_0 - \varepsilon)e^{-(v_0 + \varepsilon)t} + \frac{k_0 - \varepsilon}{v_0 + \varepsilon}(1 - e^{-(v_0 + \varepsilon)t})
\]

\[
= \frac{k_0 - \varepsilon}{v_0 + \varepsilon} + e^{-(v_0 + \varepsilon)t}(u_0 - \frac{k_0 - \varepsilon}{v_0 + \varepsilon})
\]

\[
= \frac{k_0}{v_0} + O(\varepsilon) + e^{-(v_0 + \varepsilon)t}(u_0 - \frac{k_0}{v_0} + O(\varepsilon)).
\]

Now there is the assertion of the theorem. \( \blacksquare \)

From this theorem we get that the solutions for the equations in cases of constant and non-constant rates are asymptotically equivalent.
7.4 The Second Moment

In the equation of the second moment there are two operators \( L_{ax}, L_{ay} \), and the equation has the form:

\[
\begin{align*}
\frac{\partial m_2(t, x, y)}{\partial t} &= L_{ax} m_2(t, x, y) + L_{ay} m_2(t, x, y) - (v(x) + v(y)) m_2(t, x, y) \\
&+ k(x) m_1(t, x) + k(y) m_1(t, y) - \kappa(a(x - y)) m_1(t, x) \\
&+ a(y - x) m_1(t, y) + \delta_x(y) (m_1(t, x) (\mu(x) + \sum_{n \geq 2} (n - 1)^2 b_n(x))) \\
&+ k(x) + \kappa L_a m_1(t, x), \\

m_2(0, x, y) &= u_0(x, y).
\end{align*}
\]

Let

\[
F(t, x) = m_1(t, x) (\mu(x) + \sum_{n = 2}^{\infty} (n - 1)^2 b_n(x))) + k(x) + \kappa L_a m_1(t, x) ;
\]

\[
f(t, x, y) = k(x) m_1(t, y) + k(y) m_1(t, x) + \delta_x(y) F(t, x) \\
- \kappa a(x - y) (m_1(t, y) + m_1(t, x)) ;
\]

\[
V(x, y) = v(x) + v(y) .
\]

Then the equation can be rewritten:

\[
\begin{align*}
\frac{\partial m_2(t, x, y)}{\partial t} &= (L_{ax} + L_{ay}) m_2(t, x, y) - V(x, y) m_2(t, x, y) + f(t, x, y) , \\
m_2(0, x, y) &= u_0(x, y) .
\end{align*}
\]

For this equation we also want to apply lemma 7.1, so we reckon couple of two independent random walks on \( \mathbb{Z}^d \): \( x(t) \) and \( y(t) \) with generators \( L_{ax} \) and \( L_{ay} \) respectively. Thus, now we can use lemma 7.1 for the pair \( (x(t), y(t)) \).

From this proposition we receive

\[
m_2(t, x, y) = E_{(x,y)} e^{- \int_0^t V(x(s), y(s))ds} u_0(x(t), y(t)) \\
+ E_{(x,y)} \int_0^t f(t - s, x(s), y(s)) e^{- \int_0^s V(x(\tau), y(\tau))d\tau} d\tau .
\]

Remark 7.2 From (23a) and (26)

\[
2(v_0 - \varepsilon) \leq V(x, y) \leq 2(v_0 + \varepsilon) .
\]
Let
\[ G(t, x, y) := E_{(x,y)} e^{-\int_0^t V(x(s), y(s)) \, ds} u_0(x(t), y(t)) ; \]  

(27)
\[ H(t, x, y) := E_{(x,y)} \int_0^t f(t - s, x(s), y(s)) e^{-\int_s^t V(x(\tau), y(\tau)) \, d\tau} \, ds \]  

(28)
So
\[ m_2(t, x, y) = G(t, x, y) + H(t, x, y) \]  

(29)

1. Estimate function \( G(t, x, y) \) (defined in (27)) using (23h). Then
   - (a) \( G(t, x, y) \leq E_{(x,y)} e^{-\int_0^t 2(v_0 - \varepsilon) \, ds} (u_0 + \varepsilon) \)
   - (b) \( G(t, x, y) \geq E_{(x,y)} e^{-\int_0^t 2(v_0 + \varepsilon) \, ds} (u_0 - \varepsilon) \)

2. Estimate \( H(t, x, y) \) (from (28)).
   - (a) Firstly, find the boundaries for function \( f(t, x, y) \) (definition see in (25)).
     \[ a(x - y) = \begin{cases} 
     -1, & \text{if } x = y \\
     \in [0, 1], & \text{if } x \neq y 
     \end{cases} \]
     \[ -a(x - y) = \begin{cases} 
     1, & \text{if } x = y \\
     \in [-1, 0], & \text{if } x \neq y 
     \end{cases} \]
     i. Using assumptions (23e) and (23f) gives the upper boundary
     \[ f(t, x, y) \leq (k_0 + \varepsilon + \kappa) (m_1(t, x) + m_1(t, y)) + \delta_x(y) F(t, x) \]
     \[ = 2(k_0 + \varepsilon + \kappa) \left( \frac{k_0}{v_0} + C_1^+ \varepsilon + C_0^+ e^{-(v_0 - \varepsilon)t} \right) + \delta_x(y) F(t, x) ; \]
     ii. and the lower boundary
     \[ f(t, x, y) \geq (k_0 - \varepsilon - \kappa) (m_1(t, x) + m_1(t, y)) + \delta_x(y) F(t, x) = \]
     \[ = 2(k_0 - \varepsilon) \left( \frac{k_0}{v_0} + C_1^- \varepsilon + C_0^- e^{-(v_0 + \varepsilon)t} \right) \]
     \[ - 2\kappa \left( \frac{k_0}{v_0} + C_1^+ \varepsilon + C_0^+ e^{-(v_0 - \varepsilon)t} \right) + \delta_x(y) F(t, x) . \]
   - (b) Estimate \( H(t, x, y) \)
i. For upper estimate use i from 2.(a).

\[ H(t, x, y) \leq \int_0^t \delta_x(y) F(t - s, x) e^{-\int_0^t V(x(\tau), y(\tau))d\tau} ds \]

\[ + 2(k_0 + \varepsilon + \varkappa) \int_0^t \left( \frac{k_0}{v_0} + C_1^+ \varepsilon + C_0^- e^{-(v_0-\varepsilon)(t-s)} \right) e^{-2(v_0-\varepsilon)s} ds . \]

Let

\[ L(t, x, y) = \int_0^t \delta_x(y) F(t - s, x) e^{-\int_0^t V(x(\tau), y(\tau))d\tau} ds . \]

Then

\[ H(t, x, y) \leq L(t, x, y) + \frac{(k_0 + \varepsilon + \varkappa)(k_0 + \varepsilon + C_1^+ \varepsilon)}{v_0 - \varepsilon} (1 - e^{-2(v_0-\varepsilon)t}) \]

\[ + C_0^- e^{-(v_0-\varepsilon)t} 2(k_0 + \varepsilon + \varkappa) \frac{1}{v_0 - \varepsilon} (1 - e^{-(v_0-\varepsilon)t}) \]

\[ = L(t, x, y) + \frac{k_0(k_0 + \varkappa)}{v_0} + O(\varepsilon) + \frac{2C_0^- (k_0 + \varepsilon + \varkappa)}{v_0} e^{-(v_0-\varepsilon)t} \]

\[ - e^{-2(v_0-\varepsilon)t} \left( \frac{k_0(k_0 + \varkappa)}{v_0^2} + O(\varepsilon) + \frac{2C_0^+ (k_0 + \varkappa)}{v_0} \right) . \]

ii. For the lower estimate use ii from 2.(a).

\[ H(t, x, y) \geq L(t, x, y) + 2(k_0 - \varepsilon) \int_0^t \left( \frac{k_0}{v_0} + C_1^- \varepsilon + C_0^- e^{-(v_0+\varepsilon)(t-s)} \right) e^{-2(v_0+\varepsilon)s} ds \]

\[ - 2\varkappa \int_0^t \left( \frac{k_0}{v_0} + C_1^+ \varepsilon + C_0^- e^{-(v_0-\varepsilon)(t-s)} \right) e^{-2(v_0-\varepsilon)s} ds \]

\[ = L(t, x, y) + \frac{k_0^2}{v_0^2} + O(\varepsilon) - e^{-2(v_0+\varepsilon)t} \left( \frac{k_0^2}{v_0^2} + O(\varepsilon) \right) \]

\[ + \frac{2C_0^- k_0 e^{-(v_0+\varepsilon)t}}{v_0} - \left( \frac{2C_0^- k_0}{v_0} + O(\varepsilon) \right) e^{-2(v_0+\varepsilon)t} \]

\[ - \frac{\varkappa k_0}{v_0^2} + \left( \frac{\varkappa k_0}{v_0^2} + O(\varepsilon) \right) e^{-2(v_0-\varepsilon)t} - \frac{2\varkappa C_0^+}{v_0} e^{-(v_0-\varepsilon)t} \]

\[ + \left( \frac{2\varkappa C_0^+}{v_0} + O(\varepsilon) \right) e^{-2(v_0-\varepsilon)t} . \]
3. From (29) and 1. и 2. there is an estimate for the second moment in case of non-constant intensities:

(a) Upper limit

\[ m_2(t, x, y) \leq e^{-2(v_0 - \varepsilon)t}(u_0 + \varepsilon) + L(t, x, y) + \frac{k_0(k_0 + \kappa)}{v_0^2} \]
\[ + O(\varepsilon) + \frac{2C_0^+(k_0 + \varepsilon + \kappa)}{v_0} e^{-(v_0 - \varepsilon)t} \]
\[ - e^{-2(v_0 - \varepsilon)t}\left(\frac{k_0(k_0 + \kappa)}{v_0^2} + O(\varepsilon) + \frac{2C_0^+(k_0 + \varepsilon)}{v_0}\right) \]
\[ = L(t, x, y) + \frac{k_0^2}{v_0^2} + \frac{k_0\kappa}{v_0^2} + C_2^+ e^{-(v_0 - \varepsilon)t} \]
\[ + \left(u_0 - \frac{k_0 + \kappa}{v_0}\left(2C_0^+ + \frac{k_0}{v_0}\right) + O(\varepsilon)\right) e^{-2(v_0 - \varepsilon)t} + O(\varepsilon) . \]

(b) Lower limit

\[ m_2(t, x, y) \geq e^{-2(v_0 + \varepsilon)t}(u_0 - \varepsilon) + L(t, x, y) + \frac{k_0^2}{v_0^2} \]
\[ + O(\varepsilon) - e^{-2(v_0 + \varepsilon)t}\left(\frac{k_0^2}{v_0^2} + O(\varepsilon)\right) + \frac{2C_0^-k_0}{v_0} e^{-(v_0 + \varepsilon)t} \]
\[ - \left(\frac{2C_0^-k_0}{v_0} + O(\varepsilon)\right) e^{-2(v_0 + \varepsilon)t} - \frac{\kappa k_0}{v_0} e^{-2(\varepsilon)} + \frac{\kappa k_0}{v_0} e^{-2(\varepsilon)} \]
\[ - \frac{2\kappa C_0^+}{v_0} e^{-(v_0 - \varepsilon)t} + \frac{2\kappa C_0^+}{v_0} e^{-2(\varepsilon)} . \]

Finally, we obtain the following boundaries for the second moment:

\[ A \leq m_2(t, x, y) - L(t, x, y) - \frac{k_0^2}{v_0^2} \leq B , \]

where

\[ A = C_2^- e^{-(v_0 + \varepsilon)t} + C_3^- e^{-2(v_0 + \varepsilon)t} + C_4^- \varepsilon - \frac{\kappa k_0}{v_0^2} (1 - e^{-2(\varepsilon)}) ; \]

\[ B = C_2^+ e^{-(v_0 - \varepsilon)t} + C_3^+ e^{-2(v_0 - \varepsilon)t} + C_4^+ \varepsilon + \frac{\kappa k_0}{v_0^2} (1 - e^{-2(\varepsilon)}) . \]

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