Electromagnetic Extraction of Energy from Kerr Black Holes

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Abstract

The energy extraction process from Kerr black holes is elucidated compared with that from pulsars in force-free degenerate electrodynamics (FFDE). It is argued for the force-free magnetosphere of a Kerr black hole in a steady axisymmetric state that the function of unipolar inductors in the presence of global magnetic fluxes threading the horizon is equipped by a frame-dragging effect in the Kerr spacetime metric to the upper null surface, \( S_N \), with the axial distance, \( \sigma_N \), where the local value of the angular velocity of the frame-dragging, \( \omega \), is equal to the rotational frequency of each field line, i.e., \( \omega(\sigma_N) = \Omega_F \), and the rotational velocity of each field line, \( \nu_F \), measured by “zero-angular-momentum-observers” changes in sign. There must be a pair-creation gap at \( S_N \), from which the paired winds, ingoing and outgoing, are initiated. The poloidal electric current, \( I \), for each wind is determined by solving the eigenvalue problem with the “criticality condition” imposed at the fast surfaces nearly at the horizon and at infinity, and the “current-closure condition” at \( S_N \) determines the ultimate eigenvalues of \( \Omega_F \) and \( I \) in terms of the hole’s angular velocity, \( \Omega_H = \omega(r_H) \), and the distribution of the magnetic flux at the horizon. It is not in the ordinary ergosphere, but in the effective ergosphere in the region of \( \Omega_H > \omega > \Omega_F \) and \( \nu_F < 0 \) that general-relativistic effects play the most significant role for electromagnetic extraction of the hole’s energy. The flow there must be regarded as an ingoing magneto-centrifugal wind, rather than an accretion flow. In a kind of extreme state that “force-free” field lines be frozen in “massless” particles in FFDE, the “membrane paradigm” combined with the DC circuit model is useful in black hole as well as pulsar magnetospheres.

Key words: black hole physics — magnetic fields — magnetohydrodynamics: MHD — stars: winds, outflows

1. Introduction

1.1. Brief Review

Since the discovery of quasars in 1963, black holes as an energizer have been attracting much attention in high-energy astrophysics. This is because there would be nothing more practically useful and powerful than the rotational energy of a rapidly rotating (Kerr) black hole, apart from the processes related to the surrounding accretion disk. This energy source could certainly be enormous (up to the 27% of the total mass energy), if and only if an efficient process is developed for extracting it without violating any physical law, such as the causality principle. The principle of extraction was soon clarified, and is now well known as the Penrose process, which of course makes use of a purely general-relativistic effect, that is, the existence of negative-energy orbits of infalling particles (as seen from distant observers) in the (ordinary) ergosphere defined by the static-limit surface. It is known that the process may be a useful way as a major source of energy supply in a highly-civilized future city constructed outside the ergosphere around the hole (Misner et al. 1973), but it was soon realized that this “mechanical” means is insufficient for astrophysical purposes (Bardeen et al. 1972; Wald 1974; Phinney 1983b).

As for pulsars, discovered several years later after quasars, it did not take long before a viable mechanism was developed (Goldreich, Julian 1969, Michel 1969). The spin-down energy of rapidly rotating, highly magnetized neutron stars is soon recognized as serving to accelerate the plasma particles as relativistic winds. The foundation of pulsar electrodynamics in the force-free limit (FFDE; see the next subsection) has thus been established since then (Goldreich, Julian 1969; Michel 1973a, b; Okamoto 1974; Mestel 1999), and also pulsar magnetohydrodynamics (MHD) was constructed in parallel, with particle inertiafully taken into account (Michel 1969; Goldreich, Julian 1970; Okamoto 1978; Kennel et al. 1983; see, e.g., Mestel 1999 for review). It seemed at that time that pulsar MHD/FFDE was promising for a complete understanding of the magnetosphere-wind system of neutron stars.

With success regarding the second most compact objects before their eyes, it was natural for people to consider next that a similar mechanism would also be at work in the most compact objects, i.e., rapidly rotating black holes, although these are so compact that the new general-relativistic effects must be directly and properly included. Blandford and Znajek (1977) tried to marry (pulsar) electrodynamics with general relativity, after a prescription of the “boundary conditions at the horizon” was made by Znajek (1977). The union of both was soon to be celebrated by the 3 + 1 formalism of general relativity by Macdonald and Thorne (1982), by its extension to the MHD formalism by Phinney (1983a, b) and then by the Membrane Paradigm by Thorne, Price, and Macdonald (1986). Thus this union seems to have been perfectly accomplished. In this BZ process, it has been expected that a massive rotating black hole plays a major role in powering observed relativistic
jets associated with double radio sources and extended radio lobes (see, e.g., Begelman et al. 1982). It has recently been suggested that not only microquasars, but also gamma-ray bursters (GRB) may be powered by a hole’s rotational energy through the BZ process (see, e.g., Mirabel, Rodríguez 1999; Mészáros 2002 for review).

1.2. Basic Assumptions and Relations

We use the 3 + 1 formalism with the Boyer–Lindquist coordinates formulated by Macdonald and Thorne (1982) for general relativity. Then, essential general-relativistic effects for this class of problem are condensed into two metric functions: \( \alpha \) is the redshift factor or lapse function, and \( \omega \) is the angular velocity of dragging the inertial frames, i.e.,

\[
\alpha = \frac{\rho \sqrt{\Delta}}{\Sigma}, \quad \omega = \frac{2aGM\rho}{c^2 \Delta},
\]

where \( \rho = J/Mc \), and

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - \frac{2GMr}{c^2} + a^2, \quad \Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad \varpi = \frac{\Sigma}{\rho} \sin \theta,
\]

in usual notation. The horizon radius is given by \( \alpha = \Delta = 0 \), i.e.,

\[
r_H = \frac{GM}{c^2} + \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2}.
\]

The differential operator in the Boyer–Lindquist coordinates, \( \nabla \), becomes

\[
\nabla = \left(\frac{\sqrt{\Delta}}{\rho} \frac{\partial}{\partial r}, \frac{1}{\rho \varpi} \frac{\partial}{\partial \theta}, \frac{1}{\varpi} \frac{\partial}{\partial \phi}\right).
\]

The electromagnetic quantities, such as the magnetic field \( B \), electric field \( E \), etc., are measured by zero-angular-momentum-observers (ZAMOs) with the proper time \( \tau \), and if they use global time \( t \), the transformation yields such quantities as \( a\alpha B \), \( a\alpha E \), etc., where \( d\tau/dt = \alpha \) (Macdonald, Thorne 1982).

The most important premise is that the MHD approximation is robust and useful even in the general-relativistic regime. We postulate the presence of global magnetic fluxes threading the horizon, to extend to infinity in the steady, axisymmetric state. Using the induction equation, we consider why and where such a unipolar inducer(s) should exist in the black hole magnetosphere, as is usually supposed to exist on the neutron star surface.

As usual, we denote the “stream” and “current functions” by \( \Psi \) and \( I \) (see, e.g., Okamoto 2003), and then we have for the poloidal and toroidal components of the magnetic field:

\[
B_p = -\frac{t \times \nabla \Psi}{2\pi \varpi}, \quad B_t = -\frac{2I t}{\alpha \varpi c},
\]

where \( t \) is the unit toroidal vector. The poloidal and toroidal components of the electric current become

\[
j_p = -\frac{t \times \nabla I}{2\pi \varpi}, \quad j_t = \frac{2\varpi \Psi}{\alpha c^2}.
\]

\[\frac{j_t}{c} = -\frac{1}{2\pi \varpi} \nabla \ln B_p - \frac{B_p}{R} = -\frac{1}{2\pi \varpi} \left( \nabla^2 \Psi - \nabla \Psi \cdot \nabla \ln \varpi^2 \right), \tag{7}\]

where \( R \) is the curvature radius of each field line, given by

\[
\frac{1}{R} = \frac{\nabla^2 \Psi - \nabla \Psi \cdot \nabla \ln \varpi}{|\nabla \Psi|} \tag{8}\]

(see Okamoto, Sigalo 2006 for derivation). The components of \( j_p \), normal and parallel to \( B_p \), are given by

\[
j_p = \frac{1}{2\pi \varpi} \frac{\partial I}{\partial \ell}, \tag{9}\]

\[
j_t = -\frac{1}{2\pi \varpi} \frac{\partial I}{\partial n}, \tag{10}\]

where \( (\ell, n) \) is the curve-linear orthogonal coordinate system with \( \ell \) being measured along each poloidal field line and \( n \) being perpendicular to it; also, \( (\partial/\partial n) = -|\nabla \Psi|(\partial/\partial \Psi) \). Denoting the angle between the poloidal field vector \( B_p \) and the \( \varpi \)-axis by \( \psi \), then one has \( 1/R = (\partial \Psi/\partial \ell)_\Psi \), and \( 1/R > 0 \) indicates the collimation of field lines toward the rotation axis, and \( 1/R < 0 \) indicates decollimation toward the equatorial plane. On the other hand, the toroidal current, \( j_t \), is indispensable to sustain the field structure, although \( E_t \equiv 0 \) in the axisymmetric state.

We assume perfect electrical conductivity of the magnetospheric plasma. The frozen-in condition in the frame comoving with the plasma motion with velocity \( v \) requires the electric field \( E' \) to vanish, i.e., \( E' = [E + (v/c) \times B][1 - (v/c)^2]^{-1/2} = 0 \), and hence the frozen-in field becomes \( E = -v/c \times B \) [see equation (2.17d) in Macdonald, Thorne 1982], where \( v \) is measured by the ZAMOs. Then, the “induction equation” reduces in the steady state to \( \nabla \times \alpha E = (\varpi/c)(B \cdot \nabla) \omega \), which, with \( E_t = 0 \) due to axisymmetry, leads to \( v_p \times B_p = 0 \) and Ferraro’s isorotational law, i.e., \( \Omega_F = \Omega_F(\Psi) \), and thus we have the kinematic relations

\[
v_p = \kappa B_p, \quad v_t = \kappa B_t + \frac{(\Omega_F - \omega) \varpi}{\alpha}, \quad v = \kappa B + v_F, \tag{11}\]

where \( \kappa \) is a scalar function, and the rotational velocity of each field line, \( v_F \), is measured by the ZAMOS,

\[
v_F = \frac{(\Omega_F - \omega) \varpi}{\alpha} t. \tag{12}\]

Then we obtain for \( E_p \) in terms of \( v_F \) and \( \nabla \Psi \)

\[
E_p = -\frac{v}{c} \times B = -\frac{v_F}{c} \times B_p = \frac{\Omega_F - \omega}{2\pi \alpha c} \nabla \Psi. \tag{13}\]

Equation (13) means that the “frozen-in” field is also equal to the “corotational” field measured in the frame corotating with the field line rotational velocity, \( v_F \). Note that \( v_F \) can be interpreted as indicating the magnetic slingshot effect in the general-relativistic setting (Okamoto 2004).

It will be worth while remarking here that the “corotational” electric field in equation (14) is obtained through the induction
equation in ideal MHD under the Kerr background. It must therefore be this $E_p$ that gives rise to not only a pulsar unipolar inductor [by putting $\alpha = 1$ and $\omega = 0$; see equation (19)], but also a black hole unipolar inductor(s) or a black hole battery (correctly, dual), which has often been referred to in the literature (e.g., Blandford, Znajek 1977; Znajek 1978; Blandford 1979), but whose existence has not so far been confirmed. It is indeed a correct interpretation of relations (12), (13), and (14) that leads to a correct understanding of a viable extraction mechanism and a consequent solution of the “causality question”. It is in the domain created by $\omega$ and $\omega$ appearing in equations (12) and (14), i.e., effective ergosphere that purely general-relativistic effects really show up (Okamoto 1992; see figure 1).

In the following the role of “current lines” is crucially important, as opposed to “field lines” in the DC circuit model in MHD/FFDE. It goes without saying that in the $3 + 1$ formalism for MHD/FFDE there appear vector quantities, such as the mean flow velocity $\mathbf{v}$, the current density $\mathbf{j}$, the magnetic field $\mathbf{B}$, and the electric field $\mathbf{E}$, other than the scalar quantities, such as the mass density $\rho$ and the charge density $\rho_e$. In the steady state, $\mathbf{v}_p = \lambda \mathbf{B}_p$, and hence one can define the field-streamlines denoted with $\Psi = \text{constant}$ (simply “field lines”), along which the plasma particles flow. On the other hand, under the frozen-in condition the electric current $\mathbf{j}$ is given by the curl of magnetic field vector $\mathbf{B}$, and one can introduce the “current lines” $I = \text{constant}$, along which the poloidal current flows. The crucial difference between the former and the latter is that whereas some of field lines may be “open” to infinity in the wind zone, every current line must be “close” in the steady state. That is, as mentioned in subsection 1.3, the “current-closure condition” must hold for every current line in the steady state. Furthermore, the concept of a DC circuit is helpful even in MHD, and the “wires” in the MHD DC circuit must not be “field lines”, but “current lines” emanating from one terminal of a “battery” and returning to the other terminal. On the other hand, in the FFDE DC circuit the “current lines” may coincide with respective “field lines” everywhere at finite distances, but must be closed by the surface current on the sphere-at-infinity, crossing field lines threading there (see subsection 2.7 for example).

1.3. Pulsar/Black Hole MHD/FFDE Global Conditions

We are revisiting the electromagnetic extraction of rotational energy from Kerr black holes through their “force-free” magnetospheres, referring to the case of rotating neutron stars. Force-freeness requires that the electromagnetic energy density be much larger than the rest-mass energy density, or equivalently that the Lorentz force be much stronger than inertial forces. This means that the role of plasma particles is just to carry positive/negative charges, treated as if they were effectively massless or inertia-free. When the “force-free condition” is then used together with the “frozen-in condition” in ideal MHD, such an extreme physical state takes place such that “force-free” field lines are frozen in and dragged around by “massless” particles. The electrodynamics that describes such an extreme system is recently often referred to as “force-free degenerate electrodynamics” (FFDE).

Let us at first classify the global conditions necessary to elucidate the MHD/FFDE field-flow properties in the steady axisymmetric state. To do so, we use the following terminology:

(i) The “boundary condition” at the stellar surface, $S_S$, or some magnetospheric base, $S_B$, which specifies the sources of magnetic fluxes with the field angular velocity, $\Omega_B$, as well as the mass flux per unit flux tube, $\eta$, and the energy per unit mass, $\mu_\delta$ (one of the Bernoulli integrals) in MHD. It is needless to say that the current source must also be specified by the boundary condition. Under the frozen-in condition, $\Omega_F$ may be regarded as given by the rotation of the stellar surface or magnetospheric base, i.e., $\Omega_F \equiv \Omega_S$ or $\Omega_B$. This must imply the existence of a kind of unipolar battery there, to drive the electric current.

(ii) The “criticality condition”, which requires the flow-field structure to be nonsingular or nondivergent at the “critical” surfaces, such as the Alfvénic surface, $S_A$, and the fast surface, $S_F$. The criticality condition at $S_F$ imposes a kind of eigenvalue problem for the fourth of MHD constants of motion (the angular momentum flux per unit flux tube) $\beta$, or the current function $I$ in FFDE.

(iii) The “regularity condition” at the sphere-at-infinity, $S_\infty$, which requires the solution to be well behaved and/or indicates no source of not only mass, energy, and angular momentum, but also magnetic flux.

(iv) The “plasma condition”. In FFDE, particle inertia is regarded as being negligible everywhere at finite distances. This means that there are no plasma particles of finite inertia on which the Lorentz force is at work, thereby to transfer electromagnetic energy/angular momentum at finite distances. The “plasma condition” here is the condition under which particle inertia is restored, not at finite distances from the source in FFDE, but near the horizon surface as well as near
the sphere-at-infinity. In MHD, there is of course no need for introduction of such a condition.

The “current-closure condition”, which one must impose as one of the global conditions, i.e., the condition that no net loss or gain of charges takes place at any closed surface threaded by current lines in the steady state. We include no snapping of wires (i.e., “current lines”) in DC circuit theory to this condition. It is shown that this is indispensable for the DC circuit model to be applicable to the pulsar/black hole MHD/FFDE.

1.4. Degeneracies and EMF’s

In FFDE, the force-free condition is naturally in use with the frozen-in condition, and hence a kind of extreme state emerges, in which some important degeneracies arise in related equations and conditions. This in turn produces both simplicity and complexity, that is, FFDE makes the mathematical treatment of the problem much easier than MHD, but any physical interpretation of the result much more difficult, or even confusing, particularly when unified with general relativity, of which the interpretation of the result much more difficult, or even confusing, particularly when unified with general relativity, of which the interpretation of the result much more difficult, or even confusing, particularly when unified with general relativity, of which the interpretation of the result much more difficult, or even confusing, particularly when unified with general relativity.

One of appearances of complexity is that FFDE does not close by itself in the steady state. For example, one of the force-free constants of motion, i.e., the “current function”, $I$, cannot be determined without a breakdown of the force-freeness as well as the frozen-inness per se. One must formally restore the inertia of plasma particles with finite electrical resistivity in some way or another, and this is referred above to as the “plasma condition”. Also, one must be most careful about the other force-free constants of motion, i.e., $\Omega_F$, because it seemed so far, as if there were no such evident boundary surface definable by $\Omega_F$ in the black hole magnetosphere as the neutron-star surface.

The terminology degenerate or degeneracy is often so far used in the field of MHD/FFDE (see, e.g., Carter 1979; Macdonald, Thorne 1982). Indeed, FFDE itself contains the word degenerate. In ideal MHD with perfect conductivity, in the particle-rest frame, the electric field $E'$ vanishes, which leads to no direct “linear” acceleration along field lines in the inertial frame because $E_\parallel = 0$. We refer here to this as the first degeneracy. Through the induction equation, the frozen-in condition instead produces a large potential or voltage drops between the poles and the equator across the field lines in the inertial frame, as can be seen in equation (14). The voltage drops on the source surface(s) play the role of an unipolar battery or producing electro-motive force (EMF) between a pair of field lines appropriately chosen.

It is usually thought that there is such a unipolar battery on the neutron star surface, as given in equation (19) later, and then if one current line, $I(\ell, \Psi) = \text{constant}$, emanates from one terminal of it at $\Psi = \Psi_2$ and returns to the other terminal at $\Psi = \Psi_1$ without snapping on the way, the current line crosses the field lines between $\Psi_1 < \Psi < \Psi_2$, performing MHD working by $j_\perp > 0$ [see equation (27)], and thereby transferring the Poynting flux to the kinetic flux. In the neutron-star surface there is surely perfectly conducting matter into which field lines are anchored by freezing, and hence a unipolar inductor existing on the surface will be at work to drive electric currents in the pulsar magnetosphere in the steady state (Goldreich, Julian 1969).

For the transition from MHD to FFDE, a necessary procedure seems to be simply to take $\rho \to 0$ or $\eta \to 0$ in the equations and conditions of MHD. This means that no transfer of energy from the field to the “massless” particles is allowed to take place anywhere at finite distances from the source. Then, in addition to the first degeneracy of $E_\parallel = 0$, due to the frozen-inness in ideal MHD, the combination of force-freeness with frozen-inness in ideal FFDE produces the second degeneracy of $j_\perp = 0$ as the result of a disappearance of the dependence of $I$ upon $\ell$, i.e., $I = I(\Psi)$ and $j_\perp = -(dI/d\Psi)B_\perp$ [see equation (9)]. However, every current line should not be snapped on the way of a circuit in the steady state, but there is no load impedance at finite distances where the volume current flows. Therefore, any current line must be connected by the surface current on the sphere-at-infinity (the “force-free sphere-at-infinity” $S_{\text{ff}}$), where one must introduce a kind of artificial resistivity by breaking down both the force-freeness and the frozen-inness, and equation (14) for the voltage drop must instead lead to Ohm’s law for the surface current. This means that the main MHD acceleration domain of $S_F \lesssim S < S_\infty$ is made compressed onto $S_{\text{ff}}$ as a substitute of it.

The critical solutions in MHD must satisfy the “criticality conditions” at the Alfvénic surface, $S_A$, and the fast surface, $S_F$, to pass these surfaces smoothly. As a result of the second degeneracy, these surfaces are made reduced to the light surface $S_L$ and $S_{\text{ff}}$, followed by changes of the “criticality conditions” per se. That is, as $S_F \to S_{\text{ff}}$, for $\rho \to 0$, the “criticality condition” at $S_F$ coincides with the “regularity condition” as well as the “plasma condition”, and all of these conditions yield the same eigenvalue for $\beta$ or $I$, i.e., $I = (1/2) \Omega_F (B_p \sigma^2)^2_{\text{ff}}$ in the pulsar-FFDE (abbreviated to P-FFDE) (see subsection 2.4).

These degeneracies naturally take place in the black hole FFDE (BH-FFDE) magnetosphere. But the situation is much more complicated with a necessity to include general-relativistic effects. As pointed out by Punsly and Coroniti (1989), the expression for $I$, i.e., $I = (1/2)(\Omega_H - \Omega_F)(B_p \sigma^2)_H$ in FFDE cannot be given as the “boundary condition at the horizon” (cf. Blandford, Znajek 1977). This must be derived by the “criticality/plasma/regularity condition” for the ingoing critical solution (see subsection 3.4). Equation (14) for $E_p$, which is obtained from the “induction equation” under the condition of frozen-inness, with general-relativistic effects properly taken into account, should lead to a black hole battery, and it turns out that yet two batteries with a pair-creation gap between them are needed for the outgoing and ingoing winds (see subsection 3.3).

Thus, the second degeneracy due to force-freeness makes the “whole” magnetosphere at finite distances a forbidden region of any acceleration, by pushing away the load impedances formally to $S_{\text{ff}}$ and $S_{\text{ff}}$. As a result, one can, and must, presume the existence of a kind of membrane with finite resistivity, not only on $S_{\text{ff}}$, but on the “force-free horizon surface”, $S_{\text{ff}}$ (see Thorne et al. 1986), where the surface currents flow to cross the field lines threading there, to satisfy the “current-closure condition”. The most important thing in the present context is, however, the degeneration of a pair-creation gap with the dual unipolar inductors at the inner and outer surfaces of it into a (apparently) mere surface at the (upper) null surface,
S_N (Okamoto 1992; see subsections 3.2 and 3.3). To reach the energy-extraction process in FFDE/MHD, free from the criticism of causality violation, one must first of all elucidate the extreme state of complete degeneration brought about by the combination of frozen-in-ness and force-freeness, and dig out the pair-creation gap with the dual unipolar inductors, i.e., the "black hole dynamo" mechanism from under the upper null surface, S_N (see subsection 3.3).

1.5. The Plan of This Paper

The purpose of this paper is to present a correct electromagnetic process for extracting a Kerr black hole's rotational energy, and thereby to solve a long-standing formidable task, i.e., the "question of causality" in the BH-FFDE. We get rid of any ambiguities having remained in the pulsar MHD/FFDE theory, to correctly match P-FFDE with general relativity, and to show where the unipolar inductor(s) should be located in the black hole magnetosphere.

For that purpose we first summarize the basic properties of MHD pulsar wind theory succinctly in subsection 2.1, and next of those of FFDE pulsar theory in subsection 2.2. Based on the pulsar experiences, we elucidate the FFDE black hole magnetospheres in section 3. Figure 1 schematically shows double-structure of the magnetosphere around a Kerr black hole. In section 4, the causes why the causality question has taken place in the BH-FFDE are analysed in some detail. A summary is presented in section 5, and conclusions are given in the last section (see Okamoto 2005 for a brief summary of this paper in some senses).

2. The Pulsar MHD/FFDE Magnetospheres

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2. The Pulsar MHD/FFDE Magnetospheres

The equation of motion in the cold-wind approximation is given by

\[ \rho \frac{1}{c} j \times B = \rho (v \cdot \nabla) v \]  
\[ \approx 0 \]  
\[ \text{in MHD}, \]  
\[ \text{in FFDE} \]  
\[ (Michel 1969, 1973a, b; Goldreich, Julian 1969, 1970; Okamoto 1974, 1978; Okamoto, Sigalo 2006). \]

The "frozen-in" condition combines with the induction equation in the pulsar case, to yield the kinematic relations with \( \alpha = 1 \) and \( \omega = 0 \) everywhere in equations (11)–(14), i.e.,

\[ v_p = \kappa B_p, \quad v_i = \Omega_p \sigma + \kappa B_i \]  
\[ (17) \]  
and

\[ E_p = -\frac{v_F}{c} \times B = -\frac{\Omega_p}{2\pi c} \nabla \Psi, \quad v_F = \Omega_p \sigma t. \]  
\[ (18) \]

The electromotive force driving the poloidal current through the pulsar magnetosphere is the potential difference between a pair of field lines (say, \( \Psi_1 \) and \( \Psi_2 \)), generated by rotation of the neutron star as a magnetized conductor; at \( S_S \) with \( \ell = \ell_S \) from equation (18) one obtains

\[ \text{EMF} = -\frac{1}{2\pi c} \int \Psi_1 \Omega_p(\Psi) d\Psi. \]  
\[ (19) \]

Then, the "current line" \( I_{\Psi_1} \) is specified by \( I(\Psi_1, \ell) = I(\Psi_2, \ell_S) = I(\Psi_1, \ell_S) \equiv I_{\Psi_1} \), and crosses every field line in the range of \( \Psi_1 < \Psi < \Psi_2 \) only. In other words, each current line starts at \( S_S \) at \( \ell = \ell_S \) with \( I = I(\Psi_1, \ell_S) \) and returns at \( S_S \) with \( I = I(\Psi_2, \ell_S) \), as in usual DC circuits. In the steady state, let us impose the condition of no net increase or decrease of charges in the central star, which is referred to as the "current-closure condition" in subsection 1.3, and hence on an arbitrary closed surface, \( S_\ell \), normally threaded by \( B_p \),

\[ 0 = \int_0^{\Psi} j_p \cdot dA = -\int_0^{\Psi} I(\Psi, \ell) \frac{\partial I}{\partial \Psi} d\Psi, \]  
\[ (20) \]

where \( dA = (B_p/B_p') d\Psi \). Hence,

\[ I(0, \ell) = I(\Psi, \ell) = 0, \]  
\[ (21) \]

which requires the existence of some "critical" field line, \( \Psi_c \), where \( \partial I/\partial \Psi = 0 \) with the extremum of \( I \), and which indicates \( j_p < 0 \) in the range of \( 0 < \Psi < \Psi_c \) and \( j_p > 0 \) in the range of \( \Psi_c < \Psi < \Psi \), and \( \Psi \) is some limiting field line of the wind zone (see figure 2).

Needless to say, the charge conservation law, \( \nabla \cdot j = 0 \), as a local condition is automatically fulfilled, but the "current-closure" condition in equations (20) and (21) is indispensable as one of the global conditions to avoid one-way charging up or down in the central star, and to ensure a DC circuit model to hold.

2.1. MHD Magnetospheres

As is well known, there appear four constants of motion in the form of integral constants in the MHD wind, i.e., \( \Omega, \eta, \beta, \) and \( \mu_3 \) or \( \mu_r \). The integrated forms of the MHD equations of motion containing these constants are as follows:
\[
\rho \kappa = \frac{\rho v_p}{B_p} = \eta ,
\]

\[
- \frac{\sigma B_i}{4\pi} + \rho \kappa \gamma \sigma v_i = - \beta \frac{\rho \kappa}{4\pi},
\]

\[
\gamma \left( 1 - \frac{\Omega_F \sigma v_i}{c^2} \right) = \mu_4 ,
\]
or

\[
\gamma = \mu_e + \frac{\Omega_F \sigma B_i}{4\pi \eta c^2} = \mu_e - \frac{\Omega_F I}{2\pi \eta c^2}
\]

where \( \mu_e \) is related to \( \mu_4 \) as

\[
\mu_4 = \mu_e + \frac{\Omega_F \beta}{4\pi \eta c^2}
\]

and, as stated in subsection 1.3, \( \eta \) is the mass flux per unit flux tube, \( \beta \) is the angular momentum flux per unit flux tube, and \( \mu_4 \) and \( \mu_e \) are the Bernoulli integral constants. All of them are functions of the stream function, \( \Psi \).

The expression for the change of \( \gamma \) along each field line, which leads to equation (25), can be written as

\[
\frac{\partial \gamma}{\partial \ell} = \frac{\mathbf{j} \cdot \mathbf{E}}{\rho c^2 v_p} = \frac{\Omega_F \sigma \gamma j_z B_z}{c} - \frac{\Omega_F}{\rho c^2 v_p} = -\frac{\Omega_F}{\rho c^2} \frac{\partial I}{\partial \ell}.
\]

For MHD acceleration to take place, i.e., for \( \gamma \) to increase along a field line, the condition \( j_z > 0 \) must hold; that is, \( I \) must be a decreasing function of \( \ell \), because \( \gamma \) is also a linearly decreasing function of \( I \), as can be seen in equation (25).

The (cold) wind flow must pass smoothly through the two critical surfaces, \( S_\Lambda \) and \( S_F \), without becoming singular. The “criticality condition” at \( S_\Lambda \) relates \( \beta \) to the Alfvénic distance \( \sigma \Lambda \), \( \Omega_F \), and \( \mu_4 \) as

\[
\beta = -4\pi \eta \mu_4 \sigma \Lambda \Omega_F^2
\]

and also

\[
\frac{\Omega_F^2 \sigma \Lambda^2}{c^2} - 1 - M^2 = 1 - \mu_4 \mu_4^2
\]

where \( M^2 = (4\pi \eta \gamma^2 \Lambda / \rho) \) is the Alfvénic Mach number and \( M^2_\Lambda \) is the value at \( S_\Lambda \). Then, \( \mu_e \) becomes by equations (26), (28), and (29)

\[
\mu_e = \frac{\mu_4}{1 - \frac{\Omega_F^2 \sigma \Lambda^2}{c^2}} = \frac{\mu_4}{M^2_\Lambda}.
\]

When the distribution of magnetic fluxes is given on the stellar surface, \( S_S \), or the appropriate magnetospheric base surface, \( S_B \), the three constants of motion among the four, i.e., \( \Omega_F, \eta, \) and \( \mu_4 \), must be specified by the “boundary condition” at \( S_S \) or \( S_B \). The remaining constants (\( \beta, \sigma \Lambda, \) and \( \mu_e \)) must be determined as the eigenvalues due to the “criticality condition” at the fast surface, \( S_F \) (Michel 1969; Goldreich, Julian 1970; Okamoto 1978, 2003; Kenneel et al. 1983; Okamoto, Sigalo 2006). The condition at \( S_F \) for a cold MHD wind is given by

\[
1 - M^2 - \frac{\Omega_F^2 \sigma \Lambda^2}{c^2} + \frac{B_i^2}{B_p^2} = 0,
\]

which leads to determinations of the eigenvalues of \( \beta \) and \( \mu_e \) by equations (28) and (30):

\[
\beta = -4\pi \eta \mu_4 \sigma \Lambda \Omega_F^2 \left( 1 + \left( \frac{\Omega_F^2 \Phi_F}{4\pi \eta c^3} \right)^{2/3} \right)^{2/3} - \mu_4
\]

\[
\mu_e = \mu_4 \Omega_F^2 \left( 1 + \left( \frac{\Omega_F^2 \Phi_F}{4\pi \eta c^3} \right)^{2/3} \right)^{2/3}
\]

where \( \Phi_F = B_i \sigma \Lambda^2 \) stands for the magnetic flux and \( \Phi_F \) is the value of \( \Phi \) at \( S_F \). The expressions for \( \sigma_\Lambda^2 \) are given by equations (29) and (33). The expression for the location of \( S_F \) becomes

\[
\frac{\Omega_F^2 \sigma \Lambda^2}{c^2} = \left[ 1 - \gamma_F^2 \left( 1 - \frac{\Omega_F^2 \sigma \Lambda^2}{c^2} \right) \right] \\
\times \left( \gamma_F^2 + 2 - \frac{\Omega_F^2 \sigma \Lambda^2}{c^2} \right) \frac{\partial \sigma}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma} = \frac{\Omega_F^2 \Phi}{4\pi \eta c^3}.
\]

In deriving equation (34), use is made of the facts that all quantities in the asymptotic domain of \( S_\Lambda \approx S_F \lesssim S < S_\infty \) depend on the coordinates \( \sigma \) and \( z \) only through \( \Phi \) or \( \sigma \), and \( S_F \) must be situated at the innermost parts of the asymptotic domain where \( \sigma_\Lambda^2 \gg \sigma_\Lambda^2 \). If one denotes rather vaguely the surface at the innermost distances of the asymptotic domain by \( S_a \), it is then seen that \( S_F \approx S_a \) (see Okamoto, Sigalo 2006 for details).

Here, we are implicitly discussing an idealized situation in which the pulsar wind will reach \( S_\infty \) without coming across interstellar media. As mentioned in subsection 1.3, we refer to the condition describing the “well-behaved” state of the flowfield structure at \( S_\infty \) as the “regularity condition”. Without violating the “regularity condition”, the critical solution determined by the “criticality condition” at \( S_F \) will asymptotically reach the state \( I \rightarrow 0 \) as \( \Phi \rightarrow 0 \) and \( \sigma \rightarrow 0 \), and hence using equations (23), (25), and (28) one obtains

\[
\gamma_\infty = \mu_e, \quad (\sigma v_i)_\infty = \Omega_F \sigma \Lambda^2.
\]

Thus, the critical solution satisfies the “regularity condition” at \( S_\infty \) automatically. Since the value of \( \gamma \) at \( S_F \) is given by \( \gamma_F = \mu_e^1/3 \), this implies that \( \gamma_\infty \gg \gamma_F \) for a relativistic wind and the domain of \( S_F \approx S_a < S < S_\infty \) is the domain of the main MHD acceleration. The full acceleration up to the extremum value, \( \gamma_\infty = \mu_e = \gamma_F^3 \) at \( S_\infty \), can thus be achieved in the superfast domain of \( S > S_F \), with no Poynting flux reaching there.

The above fact must of course be consistent with the results on the basis of the transfield equation, which can be transformed in the asymptotic domain of \( S \gg S_\Lambda \) to

\[
\frac{\sigma R}{\Phi} = \frac{\varrho_\kappa E_p + (1/c) j_{\parallel} B_i}{\rho \gamma v_p^2 \sigma \Lambda} = \frac{2\pi}{\Omega_F \gamma} \left( 1 - \frac{\gamma}{\mu_e} \right) \frac{\partial}{\partial \Psi} \left[ \Omega_F \left( \frac{\mu_e}{\gamma} - 1 \right) \right]
\]
The transfield component of equation (16), which is often the stream equation (42), the energy/angular momentum fluxes with field line, \( j_\perp > 0 \) brings about a straightening of the field lines crossing the current lines. In the FFDE treatment, however, the argument concerning collimation/decollimation as well as acceleration of the flow loses its physical significance, because the superfast domain of \( S_F < S < S_{\infty} \) is made to be compressed onto a “thin” membrane at the force-free sphere-at-infinity \( S_{F\infty} \) as a result of the second degeneracy, as discussed in the following.

### 2.2. FFDE Magnetospheres

At first, the basic properties of a FFDE pulsar magnetosphere are clarified. The elimination of \( E_p \) in equations (16) and (18) yields \((j - \rho_\psi v_\psi) \times B = 0\), and hence \((\rho_\psi v_\psi - j) = KB\) with \( K \) being a scalar constant, or \( j_\parallel = -K B p, \ j_\perp = \rho_\psi v_\psi - K B, \) and obviously \( j_\perp = 0 \). Comparing these with equation (10) yields

\[
I = I(\Psi), \quad K = \frac{dI}{d\Psi},
\]

\[
j_\parallel = -\frac{dI}{d\Psi} B_p, \quad j_\perp = \rho_\psi v_\psi + \frac{1}{c} \frac{dI^2}{d\Psi},
\]

The transfield component of equation (16), which is often called the stream equation, can be obtained by equating equation (7) to (41) and inserting the divergence of \( E_p \) in equation (18) for \( \rho_c \), i.e.,

\[
\nabla \cdot \left[ \frac{1}{\sigma^2} \left( 1 - \frac{\Omega_\psi^2}{c^2} \right) \nabla \Psi \right] + \frac{\Omega_\psi}{c^2} \frac{d\Omega_\psi}{d\Psi} \left( \nabla \Psi \right)^2 + \frac{8\pi^2 I^2}{c^2\sigma^2} \frac{dI^2}{d\Psi} = 0,
\]

(Michel 1973a, b; Scharlemann, Wagoner 1974; Okamoto 1974), or equivalently in terms of the curvature radius of each field line, \( \mathcal{R} \), from equation (8),

\[
\frac{\sigma}{\mathcal{R}} \left( 1 - \frac{\Omega_\psi^2}{c^2} \right) + \frac{8\pi^2 \sigma^2}{c^2 \Phi} \frac{\partial}{\partial \Psi} \left( I^2 - \frac{\Omega_\psi^2}{2 \sqrt{2}} \right) + 4\pi^2 \sigma^2 \frac{\partial}{\partial \Psi} \left( \frac{\Phi}{\sigma^2} \right) = 0,
\]

where \( (1 - (\nabla \Psi)^2) \nabla \Psi \cdot \nabla \Psi = \partial^2 / \partial \Psi \).

In the force-free domain, whose structure is described by the stream equation (42), the energy/angular momentum fluxes become from equations (5) and (18)

\[
S_E = \frac{\Omega_\psi I}{2\pi c} B_p + \frac{\Omega_\psi \sigma}{4\pi} B_p^2 I, \quad S_I = \frac{I}{2\pi c} B_p.
\]

It turns out that \( S_E^0 = \Omega_\psi I^0 \) and \( S_E^0 / S_E^0 = c / (\Omega_\psi \sigma) \ll 1 \) for \( \sigma \gg mR = c / \alpha \). Then, the total energy/angular momentum fluxes passing any closed surface, \( S_c \), become

\[
\oint S_E \cdot dA = \frac{1}{2\pi c} \oint \Omega_\psi(\Psi) I(\Psi) d\Psi, \quad (46)
\]

\[
\oint S_I \cdot dA = \frac{1}{2\pi c} \oint I(\Psi) d\Psi, \quad (47)
\]

where \( dA = (1/B_p) d\Psi \) is the areal element on \( S_c \).

Any magnetospheric theory based on FFDE or a DC circuit model will be incomplete unless it elucidates where and how the force-free domain begins with a “battery” and terminates with an “impedance”. The force-free approximation will exactly be valid in the immediate neighborhood only on the outside of a rapidly rotating, strongly magnetized neutron star with finite surface \( S_B \) but this approximation is often extended to the “whole” space, that is, from the origin to infinity. As a result, the stream equation contains no scale length explicitly, except for the light cylinder, \( r_L = c/\Omega_\psi \), and there are apparently no surfaces bounding the force-free domain at finite distances. Then, the origin in FFDE is not the physical origin, and we denote this by \( S_{\infty 0} \). Similarly, we denote the force-free sphere-at-infinity by \( S_{\infty \infty} \), which obviously does not coincide with the physical sphere-at-infinity, \( S_{\infty} \).

To solve the stream equations (42) or (43) in the “whole” space in \( S_{\infty 0} < S < S_{\infty \infty} \), one must specify the “boundary condition” at \( S = S_{\infty 0} \) and the “criticality/plasma/regularity condition” at \( S = S_{\infty \infty} \), as shown in subsections 2.3 and 2.4. Also, the “whole” magnetosphere can be described by DC circuit theory (see subsection 2.5). The force-freeness requires a replacement of the acceleration domain of \( S_F \lesssim S \lesssim S_{\infty} \) with some sort of dissipative membrane, \( S_{\infty \infty} \) (see subsection 2.6).

### 2.3. The “Boundary Conditions” at \( S = S_{\infty 0} \)

In the P-FFDE, the neutron star may be regarded as being compressed onto the origin, \( S_{\infty 0} \), but field lines have usually been presumed to emanate from the stellar surface, \( S_S \), rotating with the same angular velocity of the star, because the field lines are frozen in the stellar matter filling up under \( S_S \), i.e., \( \Omega_\psi \approx \Omega_S \). Then the EMF, which drives the volume current in the domain of \( S_{\infty 0} < S < S_{\infty \infty} \) and the surface current on the “force-free infinity surface”, \( S_{\infty \infty} \), is given by equation (19) between a pair of two field lines, \( \Psi_1 \) and \( \Psi_2 \), which are related to \( I(\Psi_1) = I(\Psi_2) = I_{BC} \). In this case in FFDE, the current line, \( I_{BC} \), coincides with the field lines, \( \Psi_1 \) and \( \Psi_2 \), at finite distances with \( j_\perp < 0 \), respectively, in the force-free domain, but must deviate from lines \( \Psi_1 \) and \( \Psi_2 \), to connect them on \( S_{\infty \infty} \), crossing field lines in the range of \( \Psi_1 < \Psi < \Psi_2 \) (see subsection 2.5 and figure 3).

In FFDE, the mass flux per unit flux tube, \( \eta = \rho_\psi v_\psi / B_p \), may be taken as being infinitely small, while one of the Bernoulli constants, \( \mu_\psi \), may take some non-zero value. It must, however, be assumed that charged, though massless, particles of both signs flow out freely (at least in principle) from \( S = S_{\infty 0} \), by the EMF.

### 2.4. The “Criticality/Plasma/Regularity Condition” at \( S = S_{\infty \infty} \)

We treat these three conditions at \( S = S_{\infty \infty} \) separately in the following.
2.4.1. The “criticality condition”

In order to move from MHD to FFDE, one may take just \( \rho \to 0 \) or \( \eta \to 0 \) at finite distances, as done in equation (16). From equations (5) and (23) one obtains

\[
I(\Psi) = -\frac{c \sigma B_t}{2} = -\frac{c \beta}{2},
\]

(48)

which implies a coincidence of “current lines” and “field lines” in the force-free domain. Also, putting \( \eta \to 0 \) in equations (32) and (33) for \( \beta \) and \( \mu_e \) and in equation (29) for \( \sigma^2 \), one obtains

\[
\beta = -\frac{\Omega_F \Phi_{f\infty}}{c},
\]

(49)

\[
\mu_e \approx \frac{\Omega_F^2 \Phi_{f\infty}}{4 \pi \eta c^3} \to \infty,
\]

(50)

\[
\sigma^2 \to \sigma^2_\infty, \quad c \sigma^2 \to \infty,
\]

(51)

where \( \Phi_{f\infty} \) is the value of \( \Phi = B_p \sigma^2 \) at \( S_{f\infty} \). Hence, by using equations (48) and (49) one obtains the eigenvalue for \( I \) in FFDE.

\[
I(\Psi) = \frac{1}{2} \Omega_F \Phi_{f\infty}.
\]

(52)

One can obtain the same result directly from equation (31), since \( M^2 = 0 \) in FFDE, and for \( \sigma_F \to \infty \), the last two terms must cancel each other; that is, \( |B_1| \approx B_p (\Omega_F \sigma / c) \), which reduces to equation (52).

Because the Alfvénic surface, \( S_A \), reduces to \( S_L \), the connection between the two eigenvalues \( \beta \) and \( \sigma^2 \) disappears, and \( \beta \) must be related to \( I \) after having lost dependence on \( \ell \), i.e., \( I = I(\Psi) \) as shown in equation (48). Thus in FFDE the current function itself must be determined as the eigenvalue by the criticality condition (see subsubsections 2.4.2 and 2.4.3 later).

2.4.2. The “plasma condition”

As can be seen in equations (30) and (50), \( \gamma_{f\infty} = \mu_e \to \infty \) with \( \sigma_F \to \infty \) for finite \( \mu_F \), which requires us to take inertia into account, even in FFDE, but in the neighborhood of \( S_{f\infty} \). We referred to this as the “plasma condition” in subsection 1.3.

Needing to say, \( \gamma_{f\infty} = \gamma_{\infty} = \gamma_{\infty} \) at \( S_{f\infty} \) does not imply \( \gamma_{\infty} = \gamma_{\infty} \) at \( S_{\infty} \). This is the cost of making use of FFDE, and \( \gamma_{f\infty} \to \infty \) just indicates the necessity of restoring the inertia, to terminate the force-free domain (see subsection 2.6 later). Let us clarify what this “plasma condition” implies mathematically and physically.

The kinematic motion of “massless” charged particles in FFDE is still described by equation (17), from which one obtains

\[
v_1 = \Omega_F \sigma v_p B_t B_p = \Omega_F \sigma \left[ 1 - \frac{v_p}{c} \frac{2I}{\Omega_F \Phi} \right].
\]

(53)

Substituting \( v_1 \) into the equality relation \( v^2 = v_p^2 + v_1^2 \) yields a quadratic equation for \( v_p \) in terms of \( v \) and \( 2I/\Omega_F \Phi \),

\[
1 + \left( \frac{\Omega_F^2 \sigma^2}{c^2} \right) \left( \frac{2I}{\Omega_F \Phi} \right)^2 \left( \frac{v_p}{c} \right)^2 - 2 \left( \frac{\Omega_F^2 \sigma^2}{c^2} \right) \left( \frac{2I}{\Omega_F \Phi} \right) \left( \frac{v_p}{c} \right) + \left( \frac{\Omega_F^2 \sigma^2}{c^2} \right) - \frac{v^2}{c^2} = 0.
\]

(54)

When the total velocity is specified in some way or another, equations (53) and (54) indicate how to divide it into the poloidal and toroidal components. The solutions become

\[
\frac{v_p}{c} = \left( \frac{\Omega_F \Phi}{2I} \right) \sqrt{1 \pm \left( \frac{c}{\Omega_F \Phi} \right) \left( \frac{\Omega_F^2 \Phi}{2I} \right) \left( \frac{v^2}{c^2} - \frac{2I}{\Omega_F \Phi} \right) - 1 + \left( \frac{\Omega_F^2 \sigma^2}{c^2} \right) ^2}.
\]

(55)

Then, for \( \sigma^2 \gg \sigma^2_\infty \), one has for \( v_p \) and \( v_1 \):

\[
\frac{v_p}{c} \approx \left( \frac{\Omega_F \Phi}{2I} \right) \sqrt{1 + \left( \frac{c}{\Omega_F \Phi} \right) \left( \frac{\Omega_F^2 \Phi}{2I} \right) \left( \frac{v^2}{c^2} - \frac{2I}{\Omega_F \Phi} \right) - 1}.
\]

(56)

The solutions with the upper/lower sign in equations (55) and (56) correspond to the physical/unphysical ones. Let us consider the “plasma condition” for “massless” particles to restore their inertia. For \( \gamma_{f\infty} \to \infty \) or \( v \to c \) for \( S \to S_{f\infty} \) with \( \sigma \to \infty \), equations (55) and (56) reduce to

\[
\frac{v_p}{c} \approx \left( \frac{\Omega_F \Phi}{2I} \right) ,
\]

(57)

\[
\frac{v_1}{c} \approx \left( \frac{\Omega_F \Phi}{2I} \right) \sqrt{\left( \frac{2I}{\Omega_F \Phi} \right)^2 - 1}.
\]

(58)

By the fact that \( v_p \gg v_1 \), we must require the square root in equation (58) to vanish for \( S \to S_{f\infty} \), and hence we have the same eigen expression for \( I \) as already driven in equation (52)

\[
\frac{v_p}{c} \to 1, \quad \frac{v_1}{c} \to 0.
\]

(59)

It can also be seen by equation (18) and \( |B_1| \gg B_p \) that

\[
\frac{E}{|B|} \approx \frac{\Omega_F \sigma}{c} \frac{|B_2|}{|B|} = \frac{\Omega_F \Phi}{2I} \to 1.
\]

(60)

It thus turns out that the “plasma condition” resulting from \( \gamma_{f\infty} = \mu_e \to \infty \) mathematically leads to the requirement that the two solutions for \( v_1/c \), and similarly those for \( v_p/c \), cross each other like \( X \) at \( S = S_{f\infty} \); this requirement is nothing but the “criticality condition” for \( v_1/c \) at \( S \approx S_{f\infty} \) in FFDE.

In FFDE the volume Lorentz force vanishes everywhere at finite distances, and the “force-free” constants of motion, i.e., \( \Omega_F \) and \( I \), remain undetermined within the force-free domain; \( \Omega_F \) must be provided by the “boundary condition” at the particle source surface from which “field lines” emanate together with “current lines”, and it is done by imposing the “plasma condition” at \( S_p \approx S_{f\infty} \) that determines the “current function”, \( I \), in terms of \( \Omega_F \). As shown in subsection 2.4.3,
the eigenvalue \( I \) combines with the frozen-in condition, to require us to define the surface current on \( S_{\Omega \infty} \), which makes the surface Lorentz force nonvanishing on \( S_{\Omega \infty} \), similarly on \( S_{\Omega \infty} \) (see subsection 3.6). That is to say, the “plasma condition” physically leads to the surface Lorentz force at work onto \( S_{\Omega \infty} \), which is a substitute of the MHD acceleration domain of \( S_{\Omega} \leq S \leq S_{\Omega \infty} \) (see also subsection 2.6).

It seems natural that the result from the “plasma condition” due to breaking down force-freeness in FFDE should be coincident with the force-free limit of that in MHD. Needless to say, it is not necessary to impose such a “plasma condition” in MHD.

2.4.3. The “regularity condition”

The solution for the field-flow structure in FFDE must of course be “well-behaved” toward \( S_{\Omega \infty} \), and we impose what is here called the ‘regularity conditions’ at \( S_{\Omega \infty} \), quite similarly at \( S_{\Omega \infty} \) (cf. subsection 2.3 in Macdonald, Thorne 1982\(^1\); subsection 8.2 in Okamoto 1992; later subsubsection 3.4.3):

(i) The surface current \( \mathcal{I}_{\Omega \infty} \), i.e., charge crossing a unit length perpendicular to \( \mathcal{I}_{\Omega \infty} \) per unit time, completes the circuit of all volume electric currents that reach \( S_{\Omega \infty} \) along “current lines” \( I \) = constant.

(ii) Also, \( \mathcal{I}_{\Omega \infty} \) terminates all toroidal magnetic fields there, i.e.,

\[
B_t = -\frac{2I(\Psi)}{c \sigma} - \frac{4\pi}{c} \mathcal{I}_{\Omega \infty} \times p,
\]
or

\[
\mathcal{I}_{\Omega \infty} = \frac{I(\Psi)}{2\pi c} n_{\Omega \infty},
\]  

where \( p = B_p/B_t \) and \( n = -\nabla \Psi/|\nabla \Psi| \) constitute a system of unit orthogonal vectors together with \( t \).

(iii) The force-free infinity surface \( S_{\Omega \infty} \) as well as the force-free horizon surface \( S_{\Omega \infty} \) has the same resistivity of free space (Znajek 1978) as given by

\[
R_{\Omega \infty} = R_{\Omega \infty} = \frac{4\pi}{c} = 377 \, \text{ohm}
\]  

in the sense that Ohm’s law holds between \( E_{\Omega \infty} = \left[ \left( \Omega \Psi / \sigma \right) / \mathcal{I}_{\Omega \infty} \right]_{\Omega \infty} \) from the frozen-in condition in equation (18) and \( \mathcal{I}_{\Omega \infty} \), i.e.,

\[
E_{\Omega \infty} = R_{\Omega \infty} \mathcal{I}_{\Omega \infty}.
\]  

Then, by substituting equations (61) and (62) into equation (63), we reach the same eigenvalue for \( I \) as already given in equation (52).

The force-free domain in \( S_{\Omega \infty} < S < S_{\Omega \infty} \) is of course “force-free” everywhere, but not at the interface on \( S_{\Omega \infty} \); that is, there is a nonvanishing surface Lorentz force at work on the non-force-free domain beyond. For example, the surface torque upon \( S_{\Omega \infty} \) is given by

\[
\int \left[ (\mathcal{I}_{\Omega \infty} / c) \times B_p \right] \cdot \omega_t \, dA = \frac{1}{2\pi c} \int \frac{I(\Psi)}{c \sigma} \, d\Psi.
\]  

Also, the power transferred from the force-free domain to the domain beyond is

\[
\mathcal{P} = \frac{c}{4\pi} \int (E \times B) \cdot d\mathcal{A} = \frac{1}{2\pi c} \int \Omega_{\Psi}(\Psi) I(\Psi) \, d\Psi
\]

(see section 8 in Okamoto 1992).

The eigenvalue for \( I \) in equation (52) indicates that magnetic fluxes thread \( S_{\Omega \infty} \), i.e., \( \Phi_{\Omega \infty} \neq 0 \), and this means by equations (46) and (47), or (64) and (65) that energy/angular momentum fluxes reach \( S_{\Omega \infty} \). It can also be seen by equation (65) that “the tangential components of all electromagnetic fields look like outgoing electromagnetic waves”; that is, the “radiative condition” is satisfied on \( S_{\Omega \infty} \). We refer to this as the “regularity condition” rather than the “boundary condition” at \( S_{\Omega \infty} \) (cf. e.g., pp. 23–24 in Thorne et al. 1986; also see subsubsection 3.4.3 later).

The asymptotic behavior of the stream equation (43) reduces for \( \sigma \gg \sigma L \) to

\[
\frac{\sigma}{R} \approx \frac{8\pi^2}{\Omega_{\Psi}^2} \frac{\partial}{\partial \Psi} \left[ I^2 - \frac{1}{2} \Omega_{\Psi}^2 \right]^{1/2}.
\]  

Since \( \sigma / R \to 0 \) for \( S \to S_{\Omega \infty} \), one has \( I(1/2 \Omega_{\Psi} \Phi) \to 0 \), or vice versa. Thus, the “regularity condition” on the stream equation indicates the eigenvalue in equation (52).

2.5. The DC Circuit Model and Current-Closure Condition

In the steady state, each continual “current line” must be established to connect the battery at \( S_{\Omega \infty} \) through the force-free domain to the impedance at \( S_{\Omega \infty} \). Then, similarly in the MHD case, the “current-closure condition” must be satisfied, but everywhere independently of \( \ell \) at finite distances in FFDE, since \( I = I(\Psi) \) and \( j_\perp = 0 \) (cf. figure 2). Therefore, one has \( I(0) = I(\Psi) = 0 \) from equation (21) on an arbitrary closed surface \( \Omega \) normally threaded by open field lines, where \( \Psi \) is some field line limiting the range of the open field lines. If there is an equatorial outgoing current sheet at \( \Psi = \Psi_c \), then \( I(\Psi) \neq 0 \), and hence there must be an ingoing line current at the pole, i.e., \( I(0) \neq 0 \). There must exist some critical field line, \( \Psi_c \), where \( dI/d\Psi = 0 \) and \( j_p > 0 \) in the range of \( 0 < \Psi < \Psi_c \), while \( j_p > 0 \) in the range of \( \Psi_c < \Psi < \Psi \). This means that one can choose such a pair of field lines, \( \Psi_1 \) and \( \Psi_2 \), that \( I(\Psi_1) = I(\Psi_2) \) in the range of \( 0 < \Psi_1 < \Psi_c < \Psi_2 < \Psi \). And it is the surface current on \( S_{\Omega \infty} \) that connects the two lines \( I(\Psi_1) \) and \( I(\Psi_2) \) (see figure 3).

It can thus be seen that the “criticality/plasma/regularity condition” combines with the “frozen-in condition” on \( S_{\Omega \infty} \) to yield in the eigen-state,

\[
(\sigma / |B|)_{\Omega \infty} = \frac{2I}{c} \frac{\Omega_{\Psi}}{\Phi_{\Omega \infty}} = (\sigma / E)_{\Omega \infty},
\]  

which indicates Ohm’s law as well as the radiative condition on \( S_{\Omega \infty} \). Then, in a DC circuit model for the P-FFDE, the EMF is given by equation (19), the “current function” defining the current line is given by equation (52), and the resistor is defined by equation (62). The current line in the force-free domain is described by a pair of solutions of condition \( I(\Psi) = c \) constant (say \( I_{\Psi_1} \) and \( I_{\Psi_2} \), \( \Psi_1 < \Psi < \Psi_2 < \Psi \)), and it is the surface current \( \mathcal{I}_{\Omega \infty} = \left[ I_{\Psi_1}(2\pi c \sigma) \right]_{\Omega \infty} \) on \( S_{\Omega \infty} \) that connects the current lines on \( I(\Psi_1) \) and \( I(\Psi_2) \) (see e.g., subsection 2.7).

It can thus be seen that the force-free magnetosphere consists

\(^1\) They used terminology “boundary conditions at the horizon”, but here we use the “regularity conditions”, to avoid confusion.
of an infinite number of those DC circuits in the range of \( I(0) = I(\Psi_1) \leq I(\Psi) \leq I(\Psi_2) \).

2.6. Pulsars: The Membrane Paradigm

In the P-FFDE, the “whole” space is (mathematically) occupied by the force-free domain, i.e., \( S_{\text{ff}0} < S < S_{\text{ff}\infty} \). The star with the unipolar battery on \( S_0 \) is made to shrink to the origin \( S_{\text{ff}0} \), i.e., \( S_0 < S < S_3 \rightarrow S_{\text{ff}0} \), and the acceleration domain is dispersed to shrink infinity, i.e., \( S_{\text{ff}} \lesssim S \lesssim S_\infty \rightarrow S_{\text{ff}\infty} \). Hence, the volume currents in the acceleration domain of \( S_{\text{ff}} \lesssim S \lesssim S_\infty \) in MHD must be transformed to the surface currents on \( S_{\text{ff}\infty} \), which must then be allowed to cross the field lines threading \( S_{\text{ff}\infty} \). In ideal MHD we may of course assume perfect conductivity of the plasma, and we need not take into account the physical resistivity anywhere at finite distances, because the \((1/c)(j \times B)\) force is at work on the plasma to transform the Poynting flux to the kinetic flux. But in FFDE we must take into account a kind of artificial resistivity for the surface current to cross each field line, thereby allowing the hypothetical Joule dissipation of the surface current, which may be regarded as a substitute of the MHD acceleration in the domain of \( S_{\text{ff}} \approx S_4 \lesssim S \lesssim S_\infty \). That is, on the surface \( S_{\text{ff}\infty} \) bounding the force-free domain, not only the “force-free condition”, but also the “frozen-in condition”, must be broken down, to allow the cross-field surface current to flow there and Ohm’s law to hold formally. Thus, the FFDE treatment of the pulsar magnetosphere is equivalent to replacing the central star with the point source at \( S_{\text{ff}0} \) and the acceleration domain with a dissipative “membrane” endowed electrical resistivity on the sphere-at-infinity, \( S_{\text{ff}\infty} \) (Thorne et al. 1986; see subsection 3.7 later). In other words, for FFDE to hold its physical meaning, it is indispensable to introduce a kind of membrane on which the force-free condition is broken down, to dissolve the second degeneracy.

Then, equation (27) for the change of \( \gamma \) along each field line should not lose its physical significance on \( S_{\text{ff}\infty} \). Since \( S_{\text{ff}\infty} \) is a “thin” surface onto which the “vast” domain of MHD acceleration is forcibly compressed, one can integrate equation (27) over an “infinitesimal” length across \( S_{\text{ff}\infty} \) (from \( \ell_{\text{ff}\infty1} \) to \( \ell_{\text{ff}\infty2} \)), and then

\[
\int_{\ell_{\text{ff}\infty1}}^{\ell_{\text{ff}\infty2}} \left( \rho c^2 \gamma \frac{d\gamma}{d\ell} \right) d\ell = \int_{\ell_{\text{ff}\infty1}}^{\ell_{\text{ff}\infty2}} \frac{\Omega_{\text{ff}}}{c} B_p \left( \frac{1}{2 \pi c} \frac{dI}{d\ell} \right) \frac{d\ell}{\pi} \approx \frac{\rho c^2}{\Omega_{\text{ff}} B_p} \left( \frac{1}{2 \pi c} \frac{dI}{d\ell} \right) d\ell. \tag{68}
\]

If one takes \( \ell_{\text{ff}\infty1} \) just inside the surface \( S_{\text{ff}\infty} \), where \( \rho \approx 0 \) and \( \partial I/\partial \ell \approx 0 \), then this integral may be approximated by

\[
(\rho c^2 \gamma)_{\text{ff}\infty} \approx |E_{\text{ff}\infty} \cdot I_{\text{ff}\infty}| = \frac{R_{\text{ff}\infty} |I_{\text{ff}\infty}|^2}{E_{\text{ff}\infty}|^2} = R_{\text{ff}\infty} |I_{\text{ff}\infty}|^2 / E_{\text{ff}\infty}, \tag{69}
\]

which clarifies the role of the hypothetical membrane, \( S_{\text{ff}\infty} \), stated above. Of course, the integration of \( (\rho c^2 \gamma)_{\text{ff}\infty} \) over \( S_{\text{ff}\infty} \) coincides with the power \( P \) in equation (65).

2.7. Exact Monopole Magnetosphere

It will be instructive to show an illustrative example for a DC circuit/membrane model, making use of the exact monopole solution of the stream equation (42) (Michel 1973b; Okamoto 1974). The monopole in the P-FFDE is expressible by the point source of magnetic fluxes at \( S_{\text{ff}0} \) uniformly rotating given by

---

Fig. 3. Schematic picture showing the DC circuit model for a pulsar magnetosphere in FFDE. The rotation-induced battery produces a potential difference between a pair of field lines, \( \Psi_1 \) and \( \Psi_2 \). The volume current line along one field line, \( \Psi_1 \), given by \( I(\Psi_1) = -I(\Psi_2) \) in the polar range of \( 0 < \Psi_1 \leq \Psi_2 \), is connected to the line along another field line, \( \Psi_2 \), in the equatorial range of \( \Psi_1 \leq \Psi_2 < \Psi_3 \), by the surface current \( \mathcal{I} = -\int_{S_{\text{ff}}} (\mathbf{A}(\Psi)) \cdot d\mathbf{S} \) on \( S_{\text{ff}} \), where \( I(\Psi_1) = I(\Psi_2) = 0 \) (also see figures 2 and 4).
\[ \Psi = \Psi_c (1 - \cos \theta), \]  
(70)

\[ \Phi \equiv B_\rho \sigma^2 = \Psi \left( 2 - \frac{\Psi}{\Psi_c} \right), \]  
(71)

\[ I(\Psi) = \frac{1}{2} \Omega_F \Phi = \frac{1}{2} \Omega_F \Psi_c \sin^2 \theta, \]  
(72)

with \( \Omega_F = \text{constant} \) everywhere. Note that \( \Psi = 0 \) at the pole, \( \Psi = \Psi_c \) at the equator and \( \Psi = 2 \Psi_c \) at the antipole. The field structure is radial everywhere with \( v_r = c \), and \( I(0) = I(2 \Psi_c) = 0 \), and \( I(\Psi_c) = (1/2) \Omega_F \Psi_c \) is the extremum with \( dI/d\Psi = 0 \) at \( \theta = \pi/2 \), and hence the “current-closure condition” (52) is satisfied. It can also be seen in equation (72) that the “plasma/criticality/regularity condition” (52) is fulfilled everywhere. The total energy/angular momentum fluxes become by equations (46) and (47)

\[ \oint S_e \cdot dA = \Omega_F \oint S_i \cdot dA = \frac{\Omega_F}{4\pi c} \int_{-\Psi_c}^{2\Psi_c} \Psi \left( 2 - \frac{\Psi}{\Psi_c} \right) d\Psi = \frac{\Omega_F^2 \Psi_c^2}{3\pi c}, \]  
(73)

which indicates that the constant total flux passes through an arbitrary surfaces, \( S_e \).

The charge density and the electric current become

\[ \rho_e = -\frac{\Omega_F \Psi_c}{4\pi c} \left( 1 - \frac{\Psi}{\Psi_c} \right), \]  
(74)

\[ j_i = j_{\perp} = 0, \quad j_\parallel = -\frac{\Omega_F}{2\pi \sigma^2} \Psi \left( 1 - \frac{\Psi}{\Psi_c} \right) \left( 2 - \frac{\Psi}{\Psi_c} \right). \]  
(75)

Thus, \( \rho_e < 0, j_\parallel < 0 \) in the upper hemisphere with \( 0 < \Psi < \Psi_c \), and \( \rho_e > 0, j_\parallel > 0 \) in the lower hemisphere with \( \Psi_c < \Psi < 2 \Psi_c \).

If one picks up one current line, say \( I = \text{constant} = I_{12} \), then the field lines coinciding with \( I = I_{12} \) in the force-free domain of \( S_{10} < S < S_{\infty} \) are given by the solutions of \( I(\Psi) = I_{12} \), i.e., from equations (71) and (72),

\[ \Psi_{1,2} = \Psi_c \left( 1 \pm \sqrt{1 - \frac{2I_{12}}{\Omega_F \Psi_c}} \right). \]  
(76)

It can be seen that \( 0 < \Psi_1 < \Psi_c < \Psi_2 < 2 \Psi_c \) and \( j_\parallel(\Psi_1) < 0 < j_\parallel(\Psi_2) \), i.e.,

\[ j_\parallel(\Psi_{1,2}) = \pm \frac{\Omega_F}{\pi \sigma^2} \sqrt{1 - \frac{2I_{12}}{\Omega_F \Psi_c}}. \]  
(77)

The surface current on \( S_{\infty} \), i.e., \( J_{\infty} = -\{I_{12}/(2\pi \sigma r)\}_{\infty} \), flows poleward, crossing field lines threading \( S_{\infty} \) in the range of \( \Psi_1 < \Psi < \Psi_2 \) (see figure 4). The EMF driving the current along the current line \( I = I_{12} \) becomes from equations (19) and (76)

\[ \text{EMF} = \frac{\Omega_F \Psi_c}{\pi c} \sqrt{1 - \frac{2I_{12}}{\Omega_F \Psi_c}}. \]  
(78)

Note in this exact monopole solution that there is no current sheet present at the equator with \( \Psi = \Psi_c \), because the current function \( I \) is a continuous function of \( \Psi \) across the equator, as can be seen in equation (72).

2.8. The “Single” Eigenvalue Problem in Pulsar Magnetospheres

The force-freeness has produced the second degeneracy and, as a result, one has \( S_S \rightarrow S_{10}, S_\Lambda \rightarrow S_1, \) and \( S_F \rightarrow S_{\infty} \); it is the two constants of motion that survive among the four in MHD, i.e. \( \Omega_F \) and \( \beta = -2I/c \). Then one must only specify \( \Omega_F(\Psi) \) as the “boundary condition” at \( S_{10} \) and determine only one eigenvalue, \( \beta(\Psi) = [-2I/\Omega_F]/c \) for the angular momentum flux per unit flux tube, as the eigenvalue problem due to the criticality condition at \( S_{\infty} \).

Conversely, the inertia to be recovered in the MHD description makes \( S_\Lambda \) split from \( S_1 \), thereby restoring the role of the Alfvénic distances, \( \sigma_\Lambda \), as one of the critical points, and relating \( \sigma_\Lambda \) to \( \beta(\Psi) \) as the angular-momentum flux per unit flux tube, rather than the current, i.e., \( \beta = -4\pi \eta \mu_3 \Omega_F \sigma_\Lambda^2 \). Simultaneously, the splitting of \( \sigma_\Lambda \) from \( \sigma_1 \) restores the finite value of \( \mu_3 \) from infinity, to be determined as one of the eigenvalues. Also, the inertia makes \( S_{\infty} \) split to the two surfaces, \( S_F \) and \( S_{\infty} \), to restore the original role of \( \sigma_\infty \) at finite distances, where \( \beta \) should be determined as one of the eigenvalues, and hence separating the “criticality condition” from the “regularity condition”, with the “plasma condition” finishing its role. This means that the coincidence of “current lines” with “field lines” has melted away as well, allowing both to cross each other at finite distances, mainly in the domain of \( S_F \) to \( S_{\infty} \); that is, \( I = I(\Psi, I) \) and \( j_\parallel > 0 \), for the MHD acceleration to take place.

Correspondingly, the quantities to be specified as the boundary conditions at \( S_S \) or \( S_{10} \) increase in number; that is, \( \eta(P) = \rho v \sigma_\infty/B_\rho \) is the mass flux per unit flux tube, given by the integral of the continuity equation, and \( \mu_3(P) \) is one of the Bernoulli integrals indicating the energy per unit particle mass at \( S_S \). Then, the eigenvalues to be determined in terms of \( \Omega_F, \eta, \) and \( \mu_3 \) are \( \sigma_\Lambda, \mu_3, \) and \( \beta \), by the criticality condition at \( S_F \) at finite distances.

It may be necessary here to explain the meaning of the “single” eigenvalue problem. The pulsar magnetosphere naturally consists of a seamless “single” magnetosphere, and
the eigenvalues are determined by the criticality condition at \( S_F \approx S_s \) at finite distances in MHD or at \( S_F \approx S_{ff, \infty} \) in FFDE. On the other hand, it is shown in the following that the “double” eigenvalue problem is the case in the black hole magnetospheres (see figure 1), because the black hole magnetosphere consists of the two, i.e., inner and outer magnetospheres, each of which possesses a respective “single” eigenvalue problem due to the “criticality condition” at respective fast surfaces. Also, one must fit the two eigenvalues by the “boundary condition” at the interface, i.e., at the (outer) null surface \( S_N \), to determine the final eigenvalue of the angular velocity of field lines \( \Omega_f \). That is, one must solve the “double” eigenvalue problem in the BH-FFDE (see subsection 3.5 later).

3. The Black Hole FFDE

In the P-FFDE it is due to force-freeness that the stellar surface is made to shrink to the origin, i.e., \( S_s \to S_{p0} \) together with the pulsar unipolar inductor. Furthermore, the coincidence of “field lines” and “current lines” and no dissipation at finite distances enforce, rather than allow, us to introduce a dissipative “membrane” on \( S_{ff, \infty} \). When force-freeness is combined with general relativity in the BH-FFDE, an important query is why and where appears, with general relativity in the BH-FFDE, an important query is "field lines” and “current lines” and no dissipation at distances enforce, rather than allow, us to introduce a dissipation “membrane” at the interface, i.e., at the (outer) null surface \( S_N \), to determine the final eigenvalue of the angular velocity of field lines \( \Omega_f \). That is, one must solve the “double” eigenvalue problem in the BH-FFDE (see subsection 3.5 later).

3.1. The Force-Free Domains

The field structure of the force-free domain is formally described by the transfield component of equation (16), if one uses the 3 + 1 formalism in the Boyer–Lindquist coordinates in equations (1)–(4). One then has

\[
\nabla \cdot \left( \frac{\alpha}{\sigma^2} \left[ 1 - \frac{\sigma^2(\Omega_E - \omega)^2}{\alpha^2 c^2} \right] \nabla \Psi \right) + \frac{(\Omega_E - \omega)}{\alpha c^2} \frac{d\Omega_E}{d\Psi} \nabla \Psi + \frac{8\pi^2}{\alpha \sigma^2 c^2} \frac{dI^2}{d\Psi} = 0
\]

(79)

(Blandford, Znajek 1977; Macdonald, Thorne 1982; Okamoto 1992). Equivalently, in terms of the curvature radius of each field line, \( \mathcal{R} \),

\[
\frac{\sigma}{\mathcal{R}} \left[ 1 - \frac{\sigma^2(\Omega_E - \omega)^2}{\alpha^2 c^2} \right] + \frac{4\pi \sigma^2}{\alpha^2 c^2} \frac{\partial}{\partial \Psi} \left\{ I^2 - \left( \frac{1}{2}(\Omega_E - \omega)\Phi \right)^2 \right\} + \frac{2\pi \sigma^2}{\alpha} \frac{\partial}{\partial \Psi} \left( \frac{\alpha \Phi}{\sigma^2} \right) + \frac{4}{2\pi \alpha^2 c^2 \Phi} (\nabla \Psi \cdot \nabla) \ln \alpha + \frac{\sigma^2(\Omega_E - \omega)}{2\pi \alpha^2 c^2 \Phi} (\nabla \Psi \cdot \nabla) \omega = 0.
\]

(80)

The formula for \( \mathcal{R} \) given in equation (8) is still useful in the Kerr metric with \( \nabla \) given in equation (4). This stream equation, (79) or (80), obviously reduces to stream equation (42) or (43) in the classical limit of \( \alpha \to 1 \) and \( \omega \to 0 \).

The fluxes of redshifted energy/angular momentum are given from equations (12), (13), and (5) by

\[
S_E = \frac{\Omega_E I \Phi}{2\pi \alpha c \sigma^2} \frac{B_p}{B_p} + \frac{t}{4\pi} \left( \frac{\Omega_E - \omega}{\sigma} \right) \frac{B^2}{I^2},
\]

(81)

\[
S_J = \frac{I \Phi}{2\pi \alpha c \sigma^2} \frac{B_p}{B_p} + \frac{t}{8\pi} \left\{ -1 + \left( \frac{(\Omega_E - \omega)\Phi}{2I} \right)^2 + \left( \frac{\alpha c \Phi}{2I \sigma} \right)^2 \right\} \frac{B^2}{I^2},
\]

(82)

whose poloidal components possess the following properties:

\[
S_E^p = \frac{\alpha c}{4\pi} (E \times B)_{p} + \omega S_J^p = \frac{\Omega_E I}{2\pi \alpha c} B_p,
\]

(83)

\[
S_J^p = \frac{1}{2\pi \alpha c} \frac{B_p}{B_p}
\]

(84)

[see Macdonald, Thorne 1982; equations (2.18a, b) in Okamoto 1992]. The total fluxes are

\[
\oint a S_E dA = \frac{1}{2\pi c} \oint \Omega_E(\Psi) I(\Psi) d\Psi,
\]

(85)

\[
\oint a S_J dA = \frac{1}{2\pi c} \oint I(\Psi) d\Psi
\]

(86)

[see equations (46) and (47)]. As in the P-FFDE, \( \Omega_E \) and \( I \) are functions of \( \Psi \) in the force-free domain(s), and the above question reduces to how and where these two “constants of motion” should be determined in the BH-FFDE.

3.2. Existence of a Kind of Critical Surface

What is given to us as the background in the steady state is the spacetime metric around a rotating black hole with mass \( M \) and angular momentum \( J \) as two hairs. Also, the most significant role in the BH-FFDE is played by the two scalar functions, i.e., the redshift factor \( \alpha \) and the frame-dragging angular frequency \( \omega \) (Macdonald, Thorne 1982).
The “asymptotic infinity”, $S_{f I \infty}$, for the outflow in the BH-FFDE is given by $\alpha = 1$, $\omega = 0$, and there must certainly be the “classical” (as opposed to “general-relativistic”) domain, where $\alpha$ is not so small and $\omega$ is not large, but as one comes near the horizon with $\alpha = 0$, the frame-dragging effect becomes important with $\omega \to \Omega_H$, and indeed it is this frame-dragging that makes the angular velocity of field lines measured in the dragged inertial frame, $\Psi_1$, in equation (12) negative toward the horizon, $\Psi < 0$, and it is by $\alpha = 0$ that produces “another asymptotic infinity”, $S_{f H I}$, for the inflow, in the sense that both $S_{f H \infty}$ and $S_{f H I}$ are negative acceptors of information, material, and positive/negative form of energy and angular momentum (Punsly, Coroniti 1989). Just as $S_{f H \infty}$ is discriminated from the physical sphere-at-infinity, $S_{\infty}$, in the BH-FFDE as well as the P-FFDE, we discriminate the force-free sphere-at-horizon, $S_{f H I}$, from $S_{f H I}$ in the following.

The gravity is given by $g = -c^2 \nabla \ln \alpha$ throughout the magnetosphere (Macdonald, Thorne 1982), but the enormously strong gravity of the hole toward the horizon is already embodied by the presence of the horizon, itself, in the BH-FFDE. Then, “massless” particles in the “zero-inertia” limit do not feel this strong gravity at all, even near the horizon under assumptions (cf. Punsly 2003). It is due not to gravity, but to the “plasma condition” $v \to c$, that particles restore inertia toward $S_{f H I}$ in the BH-FFDE, similarly toward $S_{f H \infty}$ (see subsubsections 2.4.2 and 3.4.2).

As far as the stream equation (79) is concerned, it appears that the force-free domain, $S_{f H I} < S < S_{f H \infty}$, may be one seamless region in which there cannot be any existing source of electric currents, nor surfaces bounding the force-free domain at finite distances from $S_{f H I}$ to $S_{f H \infty}$. One cannot find any other characteristic length except for the horizon radius, $r = r_H$, which is defined by $\alpha = 0$, and except for the inner and outer light cylinders, $\sigma_{iL}$ and $\sigma_{oL}$, which are defined by $\nu_F = \pm c t$ in equation (12), i.e.,

$$\sigma_{iL} = \frac{\alpha c}{\omega - \Omega_F} \bigg|_{iL}, \quad \sigma_{oL} = \frac{\alpha c}{\Omega_F - \omega} \bigg|_{oL}. \tag{87}$$

In what follows, the surfaces defined by these lengths are denoted by $S_{iL}$ and $S_{oL}$. However, just as $\nu_F = \Omega_F$ in equation (18) for the P-FFDE indicates the magnetic slingshot toward $S_{\infty}$ or $S_{f H \infty}$, $\nu_F$ in equation (12) for the BH-FFDE should have the same physical meanings in the black hole magnetosphere. It can be seen from equation (12) that $\nu_F = \pm c t$ at $S = S_{oL}$ and $\nu_F \to \infty$ for $\sigma \to \infty$ or $S \to S_{f H \infty}$, whereas $\nu_F = -c t$ at $S = S_{iL}$ and $\nu_F \to - \infty$ for $\alpha \to - \infty$ or $S \to S_{f H I}$; hence, there must be such a surface, a kind of “critical” surface, at a distance $\ell = \ell_N$ along each field line where $\nu_F$, and hence $E_p$, change sign, i.e.,

$$\nu_F = E_p = 0, \quad \omega(\ell_N, \Psi) = \Omega_F(\Psi). \tag{88}$$

The location of the surface with $\omega = \Omega_F(\Psi)$ corresponds just to the location where Thorne, Price, and Macdonald’s (1986) equations (4.32) and (4.33) for $\nu_F$ and $E$ change sign, or where $E$ changes direction in their figure 38; it was already referred to as the (upper) null surface $S_N$ in Okamoto (1992) and Horiiuchi et al. (1995).

It turns out now that the force-free region is not jointless, but is divided by $S_N$ into two parts with crucially important differences, i.e., inner and outer magnetospheres. The outer
magnetosphere with \( v_F > 0 \) is that through which a pulsar-type outflow takes place toward infinity, \( S_{II\infty} \), passing through \( S_{II} \), while the inner magnetosphere with \( v_F < 0 \) is that through which an anti-pulsar-type inflow appears, passing \( S_I \) toward the horizon, \( S_{IH} \). The former is a classical (i.e., non-general-relativistic) domain, while the latter is a purely general-relativistic domain, created by the \( \alpha \omega \)-mechanism in the presence of global magnetic fluxes. In the FFDE treatment, the two parts of the magnetosphere seem to be separated by just a thin surface (see figure 1), but in reality there must be a gap with some important fine structure hidden under the surface, \( S_N \). Just as in the P-FFDE, there must be a point source presumed at the origin, \( S_{IO} \), for charged particles and unipolar battery, so in the BH-FFDE there must be not only the surface source for charged, though massless, particles maintaining electric current, but also dual unipolar batteries existing behind \( S_N \), as argued in the following (see figure 5).

If one wants to explain the ingoing wind by gravity-pulled accretion flow, one cannot physically determine the unique location of its divider with the outgoing wind due to the magnetocentrifugal wind (cf. Komissarov 2004). The stream equation (79) is discussed for a flux tube \( \Psi \), with the battery on the horizon surface (cf. Phinney 1983a; Thorne et al. 1986; section 9 in Okamoto 1992; Blandford 1993). Also, the image of a pair-creation gap so far was as follows: if the fountain there dries up occasionally and the magnetic field lines threading the horizon are devoid of electric charges, there would be induced a quadrupole-like electric field around the midsurface (just like in the pair-creation zone of the pulsar magnetosphere), which would be followed by discharges due to spark gaps and photon-photon collisions (Blandford, Znajek 1977). Otherwise the particles would be supplied by pair creation due to collisions of photons from the accretion disk (Phinney 1983a; see Beskin et al. 1992; Hirotan, Okamoto 1998, 2002).

In the P-FFDE the “boundary condition” must be specified at \( S_{IO} \), while in the BH-FFDE the “boundary condition” must be given at \( S_N \). This means that charged particles of both signs must be created and charge-separated by the potential difference between the two gap surfaces, and then supplied as current-carriers to the respective, outer and inner, force-free magnetospheres. Thus, in this sense the force-free condition must be broken down inside the gap, but it will be allowable to assume for the sake of simplicity that the (force-free) magnetic field lines be continuous across the (non-force-free) gap, because the gap may be replaced with the infinitely thin surface \( S_N \) in the BH-FFDE. The stream equation (79)

\[
\frac{\partial \Psi}{\partial \ell} = \frac{1}{2\pi c} \int_{\Psi_1}^{\Psi_2} \frac{\partial \omega}{\partial \ell} d\Psi
\]

(93)

(given in figure 5). It must be noted here that a pair of field lines under consideration must be chosen in such a way that these constitute one “current line” in respective, inner and outer, magnetospheres, that is, \( I_{\text{in}}(\Psi_1) = I_{\text{in}}(\Psi_2) \) and \( I_{\text{out}}(\Psi_1) = I_{\text{out}}(\Psi_2) \), and yet \( j_\ell < 0 \) along \( \Psi = \Psi_1 \) and \( j_\ell > 0 \) along \( \Psi = \Psi_2 \), where \( I_{\text{in}} \) and \( I_{\text{out}} \) are the “current functions”, which are to be determined as described in the next subsection. The outer magnetosphere with \( \alpha \approx 1 \) and \( \omega \approx 0 \) will be well approximated by a classical pulsar magnetosphere, so that one may use equation (52) for \( I_{\text{out}} \).

It is implicitly assumed here that every open field line threads the gap under the null surface, \( S_N \), without being damaged. The presence of the EMF’s between the pair of field lines automatically implies that there are such potential drops across the gap along each field line, \( \Psi_1 \) and \( \Psi_2 \), in the steady state such that \( \text{EMF}_{\text{out}} + \text{EMF}_{\text{in}} \approx 2 \text{EMF}_{\text{out}} \), and the accompanying electric field produced across \( S_N \) is given by

\[
\frac{\text{EMF}_{\text{out}}}{\Delta \ell} = \pm \frac{1}{2\pi c} \int_{\Psi_1}^{\Psi_2} \frac{\partial \omega}{\partial \ell} d\Psi
\]

\[
\approx \pm \frac{(\Psi_2 - \Psi_1)}{2\pi c} \left| \frac{\partial \omega}{\partial \ell} \right|, \quad (94)
\]

where the sign corresponds to the field lines \( \Psi = \Psi_1, 2 \). The potential drops across the gap must be capable of creating pair particles by, e.g., steady continuous discharges. Note that \( (E_1)_N \) is almost independent of \( \Delta \ell \), and hence will survive in the limit \( \Delta \ell \to 0 \) and ensure the current-closure condition across \( S_N \) in FFDE (Okamoto 2005).
or (80) itself must be regarded as applicable everywhere at finite distances outside of it.

In general, the net current at any point in the wind zone is

\[ j = e(n^+ v^+ - n^- v^-) \]

where \( n^+ \) and \( v^+ \) are the number densities and velocities of electrons and positrons, and one can get a local balance of outgoing and ingoing current with the outflow of both electrons and positrons, e.g., by having an excess of electron flow from the polar caps and an excess of positrons from the surrounding collars, but note here that we are using a simplified picture of complete charge separation (see section 6 in Okamoto 1992). That is, the pair plasma of electron/positron will be created not only in the gap, but also will be charge-separated by the same steady potential difference across the gap \( \sim 2|\text{EMF}|_{\text{in}} \). In the outer magnetosphere, electrons flow out along one of the pair of field lines, i.e., \( \Psi_1 (j_P < 0) \), and positrons flow out along the other of the pair of field lines, i.e., \( \Psi_2 (j_P > 0) \), just like in the P-FFDE. On the other hand, in the inner magnetosphere, positrons flow in along one of the pair of field lines, i.e., \( \Psi_1 (j_P > 0) \), and electrons flow in along the other of the pair of field lines, i.e., \( \Psi_2 (j_P < 0) \). Naturally, the pair of field lines \( \Psi_1 \) and \( \Psi_2 \) must make up one current line in the respective domains, i.e., \( I_{\text{out}}(\Psi_1) = I_{\text{out}}(\Psi_2) \) and \( I_{\text{in}}(\Psi_1) = I_{\text{in}}(\Psi_2) \). The current line in the outer magnetosphere, given by \( I_{\text{out}} \) (say \( I_{\text{FFDE}} \)), must emanate from and come back to the EMF in, given by equation (92), after consuming the electric power on the dissipative membrane at \( S_{\text{Hf}} \). Similarly, the current line in the inner magnetosphere, given by \( I_{\text{in}} \), must emanate from and come back to the EMF in, given by equation (93), Joule-dissipating the Poynting flux into an increase of the hole’s irreducible mass, or surface area (see figure 5 and subsection 3.6 later).

It must be emphasized that, whereas \( \Omega_p \) appears directly in equation (19) for the voltage drop in the classical case, one has a difference in the angular velocity of the frame-dragging \( \omega \) between the center and the surfaces of the gap for the voltage difference between a pair of field lines. This is because whereas one need not require the potential gap locally along each field line on the pulsar surface to supply charges in the steady state (except in a particle source model with a gap far above the surface), in the general-relativistic case one necessitates a steady voltage difference, \( \sim 2|\text{EMF}|_{\text{in}} \), across the gap with width \( 2\Delta \ell \) along the field lines \( \Psi_1 \) and \( \Psi_2 \), which must be responsible for not only providing the dual EMF’s for the circuit currents but also for creating charges steadily inside the gap to maintain the currents, themselves. Then, the two EMS’s at the battery surfaces are almost the same in strength and opposite in sign. This situation explains why in the classical limit, equation (92) does not reduce to equation (19), because we have already intruded too deeply into the realm of general relativity. To restore equation (19), one must come back to equation (18), by taking \( \alpha = 1 \) and \( \omega = 0 \).

3.4. The Criticality/Plasma/Regularity Condition

As shown in subsections 3.2 and 3.3, a pair plasma of electron/positron is created, charge-separated and driven by the potential difference/dual unipolar inductors from the gap at \( S_{\text{N}} \), to flow both toward \( S_{\text{Hf}} \) and \( S_{\text{H}} \). Importantly, the inflow as well as the outflow is the magnetocentrifugal wind. This means that just as the eigenvalue of \( I \) for the outgoing wind in the P-FFDE is done in subsection 2.4, so the eigenvalue of \( I \) for the ingoing wind must be determined by the similar criticality/plasma/regularity condition.

3.4.1. The criticality condition

Full MHD theory for the ingoing magneto-centrifugal wind has unfortunately not yet been constructed, with the inner fast surface, \( S_{\text{F}} \), probably still near the horizon surface, \( S_{\text{H}} \), and with the eigenvalue for \( \beta \) determined by the criticality condition at \( S_{\text{F}} \). That is, no analytic formula is available for the eigenvalue \( \beta \) by taking the force-free limit of which one can reach the expression for the eigenvalue of \( I \) in the BH-FFDE, similar to equation (52) in the P-FFDE. But it seems to be plausible to presume that one obtains the same eigenvalue for \( I \) as given in equation (103) later, if one can derive an analytic expression of \( \beta \) in the BH-MHD and take the limit \( \eta \to 0 \).

3.4.2. The plasma condition

Ingoing “massless” particles along each magnetic flux do not feel any gravity under assumptions, but are under control of the magnetic slingshot effect expressed in equation (12). The kinetic equation is still given by

\[ v_{\text{in}} = \frac{(\omega - \Omega_p) \sigma}{\alpha} + v_p \frac{B_t}{B_p} \]  

(95)

in the general-relativistic domain of \( S < S_{\text{N}} \) [cf. equation (53)]. Substituting this into the equality relation \( v^2 = v_p^2 + v_t^2 \) yields

\[ v_p^2 + 2v_p B_t B_p (\omega - \Omega_p) \sigma \frac{1}{\alpha c} \left( \frac{E_p^2}{B^2} \right) \]

(96)

of which the solutions, physical/unphysical, become

\[ \frac{v_{\text{in}}}{c} = \frac{B_p}{B} \left( \frac{B_t (\omega - \Omega_p) \sigma}{\alpha c} \pm \sqrt{\frac{v_t^2 - E_p^2}{B_t^2}} \right) \]  

(97)

and also

\[ \frac{v_{\text{in}}}{c} = -\frac{B_p}{B} \left( \frac{B_t (\omega - \Omega_p) \sigma}{\alpha c} \pm \sqrt{\frac{v_t^2 - E_p^2}{B_t^2}} \right) \]  

(98)

where \( E_p \) is the “corotational” electric field determined by the frozen-in condition equation (13), i.e.,

\[ E_p = \frac{(\omega - \Omega_p) \sigma}{\alpha c} B_p. \]  

(99)

Also, the factor \( E_p/B \) in equations (97) and (98) becomes

\[ \frac{E_p}{B} \approx \frac{E_p}{|B|} \frac{(\omega - \Omega_p)(B_p \sigma)^2}{2I} \]

(100)

for \( \alpha \to 0 \), because \( B \approx |B| \gg B_p \). The first term of equation (98) reduces to

\[ -\frac{B_p^2 (\omega - \Omega_p) \sigma}{\alpha c} \pm \frac{E_p^2}{B_t^2 (\omega - \Omega_p) \sigma} \to 0, \]  

(101)

and hence equation (98) becomes for \( S \to S_{\text{H}} \)
\[ \frac{v_i}{c} = \mp \sqrt{\frac{v^2}{c^2} - \frac{E^2_p}{B^2}}. \]  

(102)

If one considers the force-free limit of the MHD ingoing wind in the same way as that of the MHD outgoing wind, one would have \( \gamma_{H^i} = \mu_r \rightarrow \infty \) or \( v \rightarrow c \) in the BH-FFDE as well, and hence one should impose the “plasma condition”, and with the fact \( |v_i| \ll |v_p| \) taken into account, one may require the square root to vanish for \( a \rightarrow 0 \) and \( v \rightarrow c \), to yield \( |E_p/B| \rightarrow 1 \); that is, one obtains the eigenvalue for \( I_m \),

\[ I_m = \frac{1}{2} (\Omega_H - \Omega_E) (B_p \sigma^2 \gamma_{H^i})_{H^i} \]  

(103)

[see equation (6.7) in Macdonald, Thorne 1982, equations (4.7a, b) and (5.6a, b) in Okamoto 1992], and

\[ \frac{v_p}{c} \approx \frac{B_p}{\alpha |B_i|} \frac{(\Omega_H - \Omega_E) \sigma}{c} \approx -1. \]  

(104)

It thus turns out that the “plasma condition” \( v \rightarrow c \) does not imply the “physical” velocity exactly reaching the speed of light. In the BH-FFDE as well as the P-FFDE, one of the force-free constants, \( I \), cannot be determined within the framework of force-freeness, and this must be done by terminating the force-free domains on \( S_{H^i} \), similarly on \( S_{H^\infty} \) in the P-FFDE, and \( v \rightarrow c \) is formally necessary for “massless” particles to feel inertia. It must also be stressed that for “massless” particles to reach \( v \rightarrow c \) even formally, to feel inertia, this must be accomplished not gravitationally, but electromagnetically, because massless particles possess charge, but no inertial to feel gravity, by assumption. In FFDE, whether ingoing or outgoing, the winds that are equivalent to the electric currents driven by the EMF’s must be regarded as the magneto-centrifugal winds boosted by the magnetic slingshot effect.

3.4.3. The regularity condition

Just as the solution in the outer magnetosphere should be so toward \( S_{H^\infty} \), so the solution in the inner magnetosphere must be “well behaved” toward \( S_{H^i} \). The solution must satisfy the “regularity condition”, instead of the “boundary condition”. Let us imagine one current line given by the equation \( I_m(\Psi) = I_{12} \), the solutions of which should be given by (say) \( \Psi_1 \) and \( \Psi_2 \), with \( j_p < 0 \) along the line \( \Psi_1 \) and \( j_p > 0 \) along the line \( \Psi_2 \). Then the regularity condition indicates as follows:

(i) The surface current \( I_{H^i} \), i.e., charge crossing a unit length perpendicular to \( \mathbf{I}_{H^i} \) per unit time,

\[ I_{H^i} = \frac{I_{12}}{2 \pi \alpha \sigma} \mathbf{n} \bigg|_{H^i}, \]  

(105)

completes the circuit of \( I_m = I_{12} \) that reaches \( S_{H^\infty} \), by flowing on \( S_{H^i} \) from \( \Psi_2 \) to \( \Psi_1 \).

(ii) \( I_{H^\infty} \) terminates the toroidal magnetic field there, i.e.,

\[ B_\perp = \frac{2 I_{12}}{\alpha \sigma} \frac{4 \pi c}{\mathbf{I}_{H^\infty} \times \mathbf{p}}, \]  

(106)

where \( \mathbf{p} = B_p/|B_p|, \mathbf{n} = -\nabla \Psi/|\nabla \Psi|, \) and \( \mathbf{p} = t \times \mathbf{n} \).

(iii) The force-free horizon surface, \( S_{H^H} \), has the same resistivity of free space as \( S_{H^\infty} \) (Znajek 1978), as already given in equation (62), in the sense that Ohm’s law holds between \( \sigma E_{H^H} = (1/\Omega_H - \Omega_E)(\sigma/c) B_p n_{H^H} \) from the frozen-in condition in equation (14) or (99) and \( \alpha \mathbf{I}_{H^i} \), i.e.,

\[ R_{H^H} \mathbf{I}_{H^H} = E_{H^H}. \]  

(107)

Then substituting equations (105) and (62) into equation (107), we reach the same eigenvalue for \( I \) as already given in equation (103). Ohm’s law must hold for all of the current lines emanating from the EMF’s given by equation (93), and coming back there, after flowing as the surface current on \( S_{H^i} \).

The stream equation (80) for \( S \gg S_{H^i} \), or \( S \ll S_{H^i} \), i.e., \( |\sigma (\Omega_E - \omega)/\alpha c| \gg 1 \) also reduces to

\[ \frac{\sigma}{R} = \frac{4 \pi}{(\Omega_E - \omega)^2 \Phi} \partial \Psi \left\{ f^2 - \left[ 1 + \frac{2}{(\Omega_E - \omega)} \Phi \right]^2 \right\} \]  

\[ + \frac{2 \pi \alpha c^2}{(\Omega_E - \omega)^2 \Phi^2} \partial \Psi \left[ \frac{(\alpha \Phi)}{\sigma^2} + \frac{\sigma^2}{2 \pi \Phi} (\nabla \Psi \cdot \nabla) \ln \alpha \right] \]  

\[ + \frac{\sigma^2}{2 \pi (\Omega_E - \omega) \Phi} (\nabla \Psi \cdot \nabla) \omega. \]  

(108)

For \( S \rightarrow S_{H^i}, \alpha \rightarrow 0, \) and \( \omega \rightarrow \Omega_E \), the last three terms become negligible, and the term \( \sigma/\Phi \) reduces asymptotically to null, and hence one obtains equation (103).

It can thus be seen that the “criticality/plasma/regularity condition” combines with the “frozen-in condition” on \( S_{H^i} \) to yield the following relations on \( S_{H^\infty} \):

\[ (\alpha \sigma |B_i|)_{H^i} = 2 I/c = \frac{(\Omega_H - \Omega_E)}{c} \Phi_H - \Phi_{H^H} = (\sigma \alpha E)_{H^H}, \]  

(109)

which indicates that magnetic fluxes thread \( S_{H^i} \), and thereby that the “radiative condition” as well as Ohm’s law holds on \( S_{H^i} \) (Thorne et al. 1986; Okamoto 1992) [cf. equation (67)].

3.5. The “Double” Eigenvalue Problem in Black Hole Magnetospheres

In the P-FFDE, the “criticality/plasma/regularity condition” determines the eigenvalue for \( I \) in terms of \( \Omega_E \) and \( \Omega_E \) itself must be specified as the “boundary condition” at the point source with \( \Omega_{S^i} \) or \( \Omega_{B} \). In the BH-FFDE, on the other hand, the classical pulsar-type outer magnetosphere is connected at \( S_{N\perp} \) to the purely general-relativistic, inner magnetosphere, which is \( \alpha \omega \)-generated and anti-pulsar-type. Each magnetosphere has its own eigenvalue, \( I_m/I_{out} \), determined by the “criticality/plasma/regularity conditions” at \( S_{H^i}/S_{H^\infty} \). The black hole magnetosphere is not a single, seamless one, and hence the last task to be solved is how smoothly to make both magnetospheres joint at \( S_{N\perp} \).

By digging out from under the surface \( S_{N\perp} \), we have already found that the potential difference creates and charge-separates pair plasma, and that the dual EMF’s drive the electric current in both magnetospheres, but the FFDE treatment does not necessarily need such a “fine” structure explicitly. That is to say, in the BH-FFDE the null surface, \( S_{N\perp} \), looks just like one of surfaces through which the fluxes of redshifted energy/angular momentum, \( S_E^H/S^H \), given in equations (83) and (84), must flow.
smoothly and continuously. After having determined eigenvalues $I_{\text{in}}/I_{\text{out}}$ at $S_{\text{ffH}}$, what is apparently needed is just to impose the continuity of $\Phi_2^c$ and $\Phi_2^I$ as the “boundary condition” at $S_N$ and to determine the final eigenvalue for $\Omega_F$ in terms of $S_{\text{ffH}}$. We have already presumed the magnetic field lines to be continuous across $S_N$. Then, $I_{\text{out}} = I_{\text{in}}$ and equations (52) and (103) yield

$$\Omega_F(\Psi) = \frac{\Phi_{\text{ffH}}}{\Phi_{\text{ffH}} + \Phi_{\text{ffI}}} \Omega_H, \quad \quad (110)$$

and

$$I_{\text{out}}(\Psi) = I_{\text{in}}(\Psi) = \frac{\Phi_{\text{ffH}} \Phi_{\text{ffI}}}{\Phi_{\text{ffH}} + \Phi_{\text{ffI}}} \Omega_H \quad \quad (111)$$

for the eigenvalues of $\Omega_F$ and $I_{\text{in}} = I_{\text{out}}$. Then, the location of $S_N$, i.e., $\ell = \ell_N$ is determined by solving the equation

$$\omega(\ell_N, \Psi) = \Omega_F(\Psi) = \frac{\Phi_{\text{ffH}}}{\Phi_{\text{ffH}} + \Phi_{\text{ffI}}} \Omega_H \quad \quad (112)$$

for each field line with $\Psi$ threading $S_N$. Thus, if the task in the P-FFDE is to solve the “single” eigenvalue problem, that in the BH-FFDE will be to solve the “double” eigenvalue problem.

We may practically need to consider the problem in two steps. In the first step, the current functions $I_{\text{out}}$ and $I_{\text{in}}$ in the outer and inner magnetospheres are determined as the eigenvalues by the plasma/criticality/regularity condition at $S_{\text{ffH}}$ as given in equations (52) and (103), in terms of $\Omega_F$ chosen appropriately beforehand. In the next step, since we have already presumed the continuity of magnetic fluxes across the gap, then as the “boundary condition” at $S_N$ we impose the “current-closure condition” which may in this case be regarded as a substitute of the energy/angular momentum conservation law, and therefore what is really needed is just to equate the two current functions determined as the eigenvalue in each magnetosphere, classical and general-relativistic, i.e., $I_{\text{out}} = I_{\text{in}}$, because by equations (83) and (84) the fluxes of the Poynting and angular momentum must be continuous across $S_N$ in the BH-FFDE.

Thus, it turns out that what should finally be determined in the BH-FFDE is the angular velocity of each field line, $\Omega_F$, as the final eigenvalue of the “double-eigenvalue problem”. It must be remarked here that we are assuming that the induction equation is applicable so that $\Omega_F$ can be defined inside the gap, and a plenty of plasma particles pair-created are rotating with the frame-dragging angular frequency, $\omega(\ell_N)$, and are frozen in by magnetic fluxes with $\Phi_F = \omega(\ell_N)$. However, in FFDE the half-width, $\Delta \ell$, does not appear explicitly, nor the dual unipolar inductors, with everything covered by $S_N$. It is nevertheless of crucial importance to recall the existence of the $\left( \frac{dJ}{dt} \right)_B$ at $S_N$ given in equation (94), which is the remains of the dual unipolar inductors at both gap surfaces in the force-free limit.

### 3.6. Increase of the Hole’s Entropy

Similarly on $S_{\text{ffH}}$, the “criticality/plasma/regularity condition” in the BH-FFDE is equivalent to the requirement of breaking-down of the force-freeness, which combines with another breaking-down of frozen-inness on $S_{\text{ffH}}$, to allow the surface Lorentz force on $S_{\text{ffH}}$, though the volume Lorentz force vanishes anywhere else. It is in reality through the torque exerted by the surface Lorentz force on $S_{\text{ffH}}$ that enables one to extract the hole’s rotational energy. The change of the hole’s angular momentum is given by integrating the torque on $S_{\text{ffH}}$.

$$\frac{dJ}{dt} = -\oint \left[ (\mathbf{I}_{\text{ffH}}/c) \times \mathbf{B}_F \right] \cdot \mathbf{\sigma}_t \mathbf{d}A \quad \quad (113)$$

[see equation (86)], and the Joule dissipation on $S_{\text{ffH}}$ produces an increase of the hole’s entropy, $T \frac{dS}{dt} = \oint E_{\text{ffH}} \cdot \mathbf{I}_{\text{ffH}} dA = \oint R_{\text{ffH}} |\mathbf{I}_{\text{ffH}}|^2 dA \quad \quad (114)$

where $J$, $S$, and $T$ are the hole’s angular momentum, entropy and surface temperature, respectively (see Thorne et al. 1986; Okamoto, Kaburaki 1990, 1991; section 8 in Okamoto 1992). From the first law of black hole thermodynamics, $c^2 \frac{dM}{dt} = T dS + \Omega_H dJ$, one obtains the output power, $P = -c^2 \frac{dM}{dt} = -\Omega_H \frac{dJ}{dt} - T \frac{dS}{dt}$

$$= \frac{1}{2\pi c} \oint \Phi_F(\Psi) I_{\text{in}}(\Psi) d\Psi \quad \quad (115)$$

[see equation (85)].

Substitution of the equations (110) and (111) into equations (114) and (115) yields

$$T \frac{dS}{dt} = \frac{\Omega_H^2}{2\pi c} \oint \frac{\zeta}{(1 + \zeta^2)^2} \Phi^H d\Psi \quad \quad (116)$$

and

$$P = \frac{\Omega_H^2}{2\pi c} \oint \frac{\zeta}{(1 + \zeta^2)^2} \Phi^H d\Psi \quad \quad (117)$$

where $\zeta = \Phi_{\text{ffH}}/\Phi_H$. As easily confirmed, $\zeta = 1$, i.e., $\Phi_{\text{ffH}} = \Phi_H$ yields $\Omega_F = (1/2) \Omega_H$, which corresponds to the condition for the maximum power production, and one obtains

$$P = T \frac{dS}{dt} = \frac{\Omega_H}{8\pi c} \oint \Phi^H d\Psi \quad \quad (118)$$

However, if $T dS = -\Omega_H dJ$, i.e., $dM = 0$, then all of the hole’s reducible mass will be dissipated on the resistive membrane on $S_{\text{ffH}}$ with $P = 0$, to contribute only to the entropy increase. If $T dS = 0$, this means that without entropy production on $S_{\text{ffH}}$, all of the hole’s rotational energy may be usable as the output power on $S_{\text{ffH}}$, however the time scale may be infinitely long. The former $M = \text{constant}$ process corresponds to $\zeta = 0$, i.e., $\Omega_F = \Omega_H$, with zero size of the effective ergosphere, and the latter adiabatic process corresponds to $\zeta = \infty$, i.e., $\Omega_F = 0$ with an infinitely large effective ergosphere (see Okamoto 1992). It is shown here that the eigen-effective ergosphere has an eigenvalue of $\Omega_F = (1/2) \Omega_H$ for $\zeta = 1$, by solving the “double” eigenvalue problem.
3.7. Black Holes: The Membrane Paradigm

The membrane paradigm for black holes was developed by Thorne, Price, and Macdonald (1986), who showed that by “stretching” the horizon by an arbitrarily small quantity, $\alpha$, one can deal with the diverging infall time, as measured by the clock for a universal time coordinate $t$, for a particle to cross the horizon, replacing the null horizon with a time-like physical membrane endowed with electrical, mechanical, and thermodynamical properties. However, Thorne, Price, and Macdonald have in addition endowed the stretched horizon with even a function of dynamo generating the current system by the “boundary condition”. For this, Punsly and Coroniti (1989) have shown that the horizon behaves like an imperfect conductor only when it accepts information passively, and the horizon behaves like an asymptotic infinity to accreting plasma and electromagnetic waves. Although one may be able to surely regard the stretched horizon as possessing electrical, mechanical, and thermodynamical properties, one cannot endow it with any dynamo that can generate a global current system in the whole magnetosphere.

It thus seems that, combined with the claims of Znajek (1978), Blandford and Znajek (1977), and Macdonald and Thorne (1982) for the “boundary conditions at the horizon surface” for $J$, Thorne, Price, and Macdonald’s oversight of the shift of the dynamo layer due to the $\omega\alpha$ effect to $S_N$ was the start of a later longstanding confusion. It was Punsly and Coroniti (1989) who first pointed out the acausality of the Blandford–Znajek process, but they unfortunately overstated this mere mistake/oversight as if it were a physical question, and thereby made it much more difficult to find the correct solution. It must be said that the “causality question” has thus been created by one mistake after another (see subsection 4.1).

In FFDE, plasma particles are treated as massless charges, but it does not make sense without including particle inertia eventually. For the P-FFDE, this means that the force-free domain must be attached to the non-force-free surface, $S_{ff\infty}$, which is exerted by the non-vanishing surface Lorentz force. The surface $S_{ff\infty}$ must thus be regarded as a sort of membrane that possesses passive, dissipative properties (see figure 3 and subsection 2.6), and thereby as a “thin” membrane comprising the superfast, MHD-accelerating “vast” domain from $S_F$ to $S_\infty$. The condition at work on $S_{ff\infty}$ cannot be described in terms of the “boundary condition”, but in terms of the “criticality/regularity/plasma condition”. For the BH-FFDE, one must also introduce the non-force-free dissipative membranes, not only on the hole’s event horizon $S_{FFH}$, but also on the force-free infinity surface, $S_{ff\infty}$, upon which the non-force-free surface Lorentz forces are at work to transfer energy/angular momentum fluxes (positive or negative) to the non-force-free domains. We have already referred to these as the Horizon membrane and the Load-membrane (see section 8 in Okamoto 1992).

After all, it is the two degeneracies due to frozen-inness and force-freeness that compel one to introduce the dissipative membranes, to terminate the force-free, frozen-in domain by invalidating the assumption, itself. The force-free domain in the P-FFDE occupies the “whole” space (mathematically), and hence the L-membrane must be set up at $S_{ff\infty}$, while in the BH-FFDE, the L- as well as H-membranes must be introduced at $S_{ff\infty}$ and $S_{FFH}$, to terminate respective, inner and outer, force-free domains, and the role of the membranes is, owing to their non-force-free and non-frozen-in properties, to make it possible to flow the surface currents, to close the DC circuits, and to “Joule-dissipate” on the membranes. Anyway, as long as one does not endow such a non-passive property as a battery to these membranes, the membrane paradigm, in combination with the $\omega\alpha$ mechanism, is a useful tool for describing the extraction process of a hole’s rotational energy, if one does not mind “sweeping the dirt under the carpet” (Mestel 1994).

4. Discussion

The two issues are discussed in this section; the first is on the “causality question” related to the energy extraction process, and the second is on the connection between the analytic analysis of the steady state theory and numerical simulations of dynamical states.

4.1. The Causality Question

It is when the author was asked to comment on Punsly and Coroniti’s critiques by the referee of the paper (Okamoto 1992) that he has first met with the “causality question”. A naive physical intuition then was “The force-free condition is a mere physical approximation useful in the solar atmospheric phenomena and pulsar magnetospheres. Such an approximation must not give rise to such a serious problem as the causality violation in physics, even when it is applied to a general-relativistic object”. This is because it is not known that gravity and electromagnetism are incompatible with each other. Then, if the force-free approximation is applicable to a compact object, like a neutron star, it must also be so even to an extreme gravitating object, like a black hole, if general-relativistic effects are properly taken into account. It was shown then that the ingoing wind is produced by the $\omega\alpha$ mechanism with the pair plasma source situated at the null surface, $S_{\infty}$.

It seems that the direct cause giving birth to the causality question is the two misinterpretations of the “boundary conditions at the horizon” and the “battery in the horizon surface”, in the context of how to determine the two constants of motion, $I$ and $\Omega_F$. The former seems to come from the ambiguities of FFDE/MHD wind theory at that time (see subsection 4.1.1), and the latter seems to be due to the shortage of a complete understanding of the general-relativistic effects (subsection 4.1.2). And both combined to lead to a mismatching of FFDE/MHD with general relativity. It seems for example that the wording, such as “ergosphere” and “accretion”, which have quite often appeared in the literature in the context of extracting the hole’s rotational energy, was not necessarily appropriate. It was thus indispensable to fully understand the various properties of FFDE/MHD wind theory and to replace the ambiguous/wrong phrases or terminology with correct ones when matching.

\footnote{It seemed that a careful check of the manuscript did not detect any sign of yielding such a pathological feature. It must however now be admitted that it was still tacitly thought that the battery surface was on the horizon.}
Also, the issues accompanying the causality question are the questions of stability of the FFDE magnetospheres (subsubsection 4.1.3) and information communication due to the force-free waves (subsubsection 4.1.4).

4.1.1. The ambiguities of MHD/FFDE wind theory

For definiteness, we have defined the “regularity condition” as the terminology of describing the constraints at the horizon and infinity surfaces, discriminated from the “boundary condition” at the plasma source and the “criticality condition” at the inner and outer fast surfaces, $S_{FFD}$ and $S_{DF}$. For example, Znajek’s “boundary condition” must be rephrased by the “regularity condition” at $S_{FH}$, which expresses the requirement that “the electromagnetic fields be nonsingular when measured in an infalling frame”, that is, the “radiative condition”. Confusingly enough, because of the degeneracy brought about by force-freeness, this condition, however, coincides simultaneously with the other two conditions: the “criticality condition” at the inner fast surface near the horizon, which requires the critical solution to pass through $S_{DF}$ without becoming singular, and the “plasma condition” for restoring plasma inertia for $v \to c$. The “criticality/plasma/regularity condition” thus yields the eigenvalue for $I_{in}$, which combines with the “frozen-in condition”, to yield Ohm’s law on the dissipative membrane, $S_{FH}$, thereby leading to a nonvanishing surface torque on $S_{FH}$.

In relation to the “boundary condition” and the “criticality condition”, it may be helpful to introduce two words, external and internal, the former of which is related to the expression of “external physical injected into the wind” or “outside the wind zone” used by Punsly (1996), and the latter is the opposite to the former. Then, the “boundary condition” is external and the “criticality condition” is internal. As already described, there are two constants of motion in FFDE, i.e., $\Omega_f$ and $I$ (or $\beta$), and four in MHD, i.e., $\Omega_f$, $\eta$, $\mu_\phi$, and $\beta$. It could be said that the ambiguities with respect to how to determine these constants of motion in FFDE/MHD wind theory have given rise to one misinterpretation after another. As clarified here, one of the two in FFDE, $\Omega_f$, and three of the four in MHD ($\Omega_f$, $\eta$, and $\mu_\phi$) should be regarded as external to the wind zone, whereas the remaining one, $\beta$ (and $\mu_\phi$, $\sigma^2_\Omega$ together in MHD), must be internally solved as the eigenvalue in terms of other constants of motion by the “criticality condition” at the fast surface. This procedure has already been followed by Michel (1969), Goldreich and Julian (1970), and Kennel, Fijimura, and Okamoto (1983).

One of the causes resulting in the confusion in the BH-FFDE is that $I$ (or $\beta$) has, mistakenly probably owing to the ambiguities at that time combined with the horizon properties yet unclarified (see subsubsection 4.1.2), been regarded as external given as the “boundary condition at the hole’s horizon” by Znajek (1977), Blandford and Znajek (1977) and Thorne, Price, and Macdonald (1986) (see also Okamoto 1997 for the pulsar case). Their claim gave rise to the next misinterpretation of $I_{in}$ by Punsly (1998a, b), who was led to regarding the minimum-torque solution of Michel (1969) as being closely related to the “minimum-entropy principle” as the external “boundary condition” in the stellar surface. Needless to say, Michel’s minimum-torque solution is nothing but the eigensolution satisfying the internal “criticality condition” at the fast surface existing at infinity under the prescribed split-monopolar structure (Goldreich, Julian 1970). This misinterpretation has further led to his thinking of the BZ process as being intrinsically acausal, and then of the ideal MHD wind models as being acausal and useless as well, eventually pursuing a dissipative non-ideal MHD model of extracting a hole’s rotational energy, that is, “gravitomagnetohydrodynamic dynamo” (see, e.g., chap. 6 in Punsly 2001). Moreover, this was followed recently by Komissarov (2004), who proposes “resistive electrodynamics” beyond the force-free approximation, to handle the ergospheric dissipative current sheets (see subsubsection 4.2.1). Thus, the confusion has been further deepened by one misinterpretation after another.

It must also be pointed out as another cause that the concept of a DC circuit and the “current-closure condition” has not yet been established in a “complete” form, i.e., in connection to the EMF related to $\Omega_f$, the “current lines” and the load impedances (MHD working/Joule dissipation”). This is partly responsible for the incorrect belief concerning implausibility of MHD acceleration and the general tendency of global collimation of MHD outflows (see Okamoto 2002, 2003; Okamoto, Sigalo 2006), as well as for the present causality question.

4.1.2. Shortage of understanding for the $\omega_0$ mechanism

It will be evident that $S_{FH}$ as well as $S_{F\infty}$ cannot possess any battery function. Then, if the statements on the boundary conditions at the horizon in Znajek (1977), Blandford and Znajek (1977), Macdonald and Thorne (1982), and Thorne, Price, and Macdonald (1986) and the phrases on a battery in the hole’s horizon in Phinney (1983a), Thorne, Price, and Macdonald (1986), and Okamoto (1992) were not mere misinterpretations and misstatements based on inaccuracies in MHD wind theory at that time, the matching of general relativity with electrodynamics would certainly not have been accomplished without violating causality as a principal premise. Indeed, Punsly and Coroniti (1990) state in their discussion that if the external unipolar generator, which drives current toward the horizon and asymptotic infinity, produces a toroidal field, such that “the Znajek horizon boundary condition” corresponds to $\Omega_f \sim (1/2) \Omega_\Phi$, the resulting winds will correspond to Phinney’s solution, Phinney’s wind probably does represent the minimally dissipative, maximum electromagnetic energy extraction solution. This positive statement toward a correct understanding was however followed by a negative statement that Our arguments show only that this solution is neither unique nor guaranteed by the MHD electrodynamics, because the black hole alone cannot enforce the global unipolar current system. One of the purposes of this paper is to show that not the latter statement but the former is in reality the case, although “the Znajek horizon boundary condition” must be replaced by the “criticality/plasma/regularity condition”. It was already shown in subsection 3.3 that there are indeed two external EMF's with opposite signs, existing back to back at the null surface, $S_N$, with the pair-creation gap there, which are responsible for driving current toward both the horizon and infinity.

On the other hand, the ergosphere and the Penrose process therein have so far been often mentioned in the context of energy extraction and the causality question (recently see, e.g., Punsly 2001; Blandford 2002; Koide 2003; Komissarov 2004). As is well known, one of the most important quantities in black
hole “mechanics” (BH-M, as opposed to BH-FFDE/MHD) is
\[ g_\text{h} = -\left(\Delta - a^2 \sin^2 \theta \right)/\rho^2 \]
in the usual notation, because the condition \( g_\text{h} = 0 \) defines the static-limit surface dividing the space around the hole into two, i.e., \textit{mechanically} separating the purely general-relativistic from classical domains, and the former is here referred to as the \textit{ordinary} ergosphere; it has been expected that the Penrose process will be at work therein.

It is however already shown that the Penrose process, itself, is inefficient for powering the relativistic jets from AGN (e.g., Phinne 1983b). In this context it will be relevant to point out a well-known fact that no “mechanical” process is capable of explaining the slow rotation of late-type main-sequence stars, including the Sun. It is generally believed that the loss of angular momenta must be due to magnetic braking by stellar winds (see, e.g., Wever, Davis 1967; Mestel 1968). It will then be reasonable to extend this classical fact to special- and

waves (see, e.g., Wever, Davis 1967; Mestel 1968). It will then be reasonable to extend this classical fact to special- and even general-relativistic objects, and to say that no “mechanical” process will be viable in “braking” the rotation of neutron stars and black holes within relevant timescales. The \textit{ordinary} ergosphere is defined “mechanically”, and it does not seem to have been shown that the \textit{ordinary} ergosphere may possess any putative dynamo process as well as a particle source in the area inside as one of the intrinsic properties in the steady state. Indeed, the condition \( g_\text{h} \lesssim 0 \) never formally appears as an essential one in the BH-FFDE/MHD.

It is the condition \( v_F \leq 0 \), but not the condition \( g_\text{h} \leq 0 \), that creates the purely general-relativistic domain relevant to provide the means efficient enough to “brake” the rotation of a black hole by its own wind. Coupled with the magnetic fluxes of the angular velocity, \( \Omega_F \), present around the hole, the dragging of inertial frames by the hole’s rotation creates an entirely new, purely general-relativistic region in \( \Omega_F \leq \omega \leq \Omega_H \), which is significantly different from the \textit{ordinary} ergosphere, although \( \Omega_F \textit{ per se} \) must be determined as the eigenvalue of the “double” eigenvalue problem. It may however be physically meaningful that for the \textit{eigen}-magnetosphere with the eigenvalue \( \Omega_F = \omega(\ell) \approx (1/2)\Omega_H \), the pair-creation gap with dual unipolar inductors at both surfaces of it is present within the \textit{ordinary} ergosphere. It may well be said that a general-relativistic domain consisting of a Kerr hole and the \textit{effective} ergosphere with \( v_F \leq 0 \) as a whole behaves like a “classical” rapidly rotating, strongly magnetized object with the particle source and unipolar inducer at its “surface”, \( S_N \), and with \( v_F > 0 \) in its “magnetosphere” outside (see figure 1).

4.1.3. Questions of stability and causality

The lack of a proper unipolar inductor(s) in the BZ solution and consequent reliance of the solution on Znajek’s boundary condition have led Punsly and Coroniti to the correct conclusion that the BZ process (as it is) is acausal. But these have unfortunately led them to an extra uncertain conjecture that since the solution is nonphysical, though it is a proper mathematical solution, its nonphysical nature must give itself away via instability (Punsly, Coroniti 1990; Punsly 2001). This conjecture has given rise to another confusion of mixing the question of causality with that of stability. The use of the condition \( \Omega_F = (1/2)\Omega_H \), irrespective of how it is derived, leads to a mathematically proper solution, and it will be implausible that this FFDE solution becomes unstable by itself, just as shown by numerical simulations (Komissarov 2001, 2004). But this asymptotic stability does not imply automatically that the BZ solution is not acausal, because the BZ solution is reliant on Znajek’s “boundary condition at the horizon”.

The conjecture by Punsly and Coroniti has thus induced the incorrect statement by Komissarov that “the BZ solution is asymptotically stable, and hence causal”. The question of causality originally had nothing to do with that of stability. If the result of \( \Omega_F = (1/2)\Omega_H \) were due to the consequence of the double-eigenvalue problem, that is, the two criticality conditions at \( S_F \) and \( S_{\text{of}} \) and the boundary condition at \( S_N \), the BZ solution would then have given the proper, causal, stable solution.

4.1.4. Characteristic wave analysis

In order to refute Punsly and Coroniti’s criticisms against the causality violation, Blandford (1989, 1993, 2002) introduced the method of the characteristic analysis of the force-free waves [see Uchida (1997a, b) for a general theory of force-free waves in a black hole magnetosphere], and continued attempts to justify the “boundary condition at the horizon” for \( I \) given in equation (103), by making use of the information communication due to the \textit{force-free} waves to the source region and the outgoing wind beyond. Against that, Punsly developed extensive analyses of information communication by the \textit{force-free} waves (Punsly 2001, 2003; Punsly, Bini 2004; see also Komissarov 2002a). To refute Blandford’s (2002) recent claim, Punsly (2003), for example, elucidated the nature of information that can be transported along the characteristics for each of the two MHD waves, the Alfvén and fast modes, in the short-wavelength limit, and concluded that the information on charge and current perturbations can be transported along the Alfvén wave mode characteristics, but not along the fast wave characteristics in the \textit{force-free} magnetosphere. He also showed that long-wavelength plasma modes are extremely ineffective at transporting magnetic stresses, field-aligned current or charge outward from near the horizon in the \textit{force-free} magnetosphere (chap. 5 in Punsly 2001). He then interpreted these wave characteristics as indicating that “the results of the characteristic analysis elucidate the nature of the causality violation in the \textit{force-free} black hole magnetospheres”.

In a general case of the \textit{purely} force-free limit, \textit{before} fixing the eigenvalue, \( I(\Psi) \), by breaking down force-freeness/frozen-inness, there will be no information determined to be conveyed by the \textit{purely} force-free waves along each field line. It then seems to be of no physical meaning in the causality question to apply wave communication theory to the \textit{purely} force-free magnetosphere, because there is no information to communicate. On the other hand, for the \textit{eigen}-magnetosphere, one may, if necessary, think that \textit{quasi}-force-free waves will communicate the current information determined by the criticality/regularity/plasma condition along the “\textit{eigen} current lines” established at \( S_{\text{ff}} \) and \( S_{\infty} \) toward \( S_N \), to fulfill the current-closure condition and thereby to fix the final eigenvalue of \( \Omega_F \) in the \textit{eigen}-circuit. In the \textit{eigen}-magnetosphere, fast magnetosonic disturbances do naturally affect the global electric current flowing in the system, and hence \( S_F \) and \( S_{\text{of}} \) are causal wind boundaries in the \textit{eigen}-magnetosphere.

Strictly speaking, the \textit{force-free} magnetosphere cannot be \textit{force-free everywhere}, but must be \textit{non-force-free} on \( S_{\text{ff}} \) and
$S_{\Omega_2 \infty}$, because the eigenvalues for $I_{\infty}$ and $I_{\Omega_2 \infty}$ must be determined by breaking down force-freeness at $S_{\Omega_2 H}$ and $S_{\Omega_2 \infty}$ and also, to extract the hole’s rotational energy, the nonvanishing surface Lorentz force must be at work there (cf. chap. 6 in Punsly 2001 and Punsly, Bini 2004 for an opposite interpretation of break-down of the force-free condition near the horizon).

4.2. Numerical Simulations

 Needless to say, analytic treatments may basically be restricted to more or less steady state analyses, whereas numerical simulations are more suited for pursuing time-evolution of the system by making use of full time-dependent MHD/FFDE equations under appropriate initial/boundary conditions in the general-relativistic setting. It seems, however, in the simulations so far carried out that the black hole magnetosphere has been treated as “seamless”, without separating the pure general-relativistic domain (i.e., effective ergosphere) from the classical domain. And more or less uniform plasma distributions, with or without an accretion disk, are initially assumed under the presence of global magnetic fluxes, with resulting importance of the ordinary ergosphere emphasized. It will then be important to point out the necessity of including the pair-creation gap with the dual unipolar inductors at $S_N$ as the initial/boundary condition in simulations, because this “microphysics” will not spontaneously show up in the simulation process of solving the “macroscopic” set of MHD/FFDE equations, and the noninclusion may have produced something that looks like physical phenomena. In the following some comments are given on the results of recent numerical simulations on FFDE/MHD magnetospheres.

4.2.1. Force-free simulations

As Punsly has persistently been stating, the BZ solution with $\Omega_{\Omega_2} \approx (1/2)\Omega_H$ in use is certainly mathematically a proper solution, but not physically so as it is, because Znajek’s “boundary condition” at the horizon surface was used for $I$ or otherwise the impedance matching between $S_{\Omega_2 \infty}$ and $S_{\Omega_2 H}$ had to be assumed for $\Omega_{\Omega_2}$. It seems that almost the same situation can be seen in recent FFDE numerical simulations by Komissarov (2001, 2002a, 2004). The disagreements with the present analytic analysis are briefly summarized in the following:

1) The time-dependent simulations by Komissarov will be significant, if these have reached the “steady state” with $\Omega_{\Omega_2} \approx (1/2)\Omega_H$, independent of, or free from, the initial states. This is because the steady state solutions may have reproduced numerically a “proper” solution corresponding to the analytic BZ solution, and because the traces of unphysical waves and currents on the way of computations (see Punsly, Bini 2004) may probably have been erased away in the steady state. It is claimed by Komissarov (2001) that in the inner region, the solution for an initial monopole field configuration quickly settles to a steady state with an outgoing Poynting flux, with its maximum power output given by $\Omega_{\Omega_2} \approx (1/2)\Omega_H$ in all cases. The probable reproduction of the analytic BZ solution by the simulations, however, does not necessarily mean that the numerical solutions are not “acausal”, because the simulations have not clarified physically why and where the electric current surely flowing through the magnetosphere was generated and how it was maintained in the steady state with $\Omega_{\Omega_2} \approx (1/2)\Omega_H$, just as in the BZ analytic solution. It was stated by Komissarov (2002b), himself, that the “driving source” for the Blandford–Znajek mechanism must be located between the inner and outer Alfvén surfaces of a black hole magnetosphere and, thus, lay well outside of the horizon, as cited by Punsly and Bini (2004). That is, the numerical results as well have not yet succeeded to show the presence of a “black hole dynamo”, nor explain physically “why $\Omega_{\Omega_2} \approx (1/2)\Omega_H$?”.

2) In spite of Komissarov’s (2004) conclusion, it seems to be implausible that the key role in the Blandford–Znajek mechanism, itself, is played by the (ordinary) ergosphere, where according to his scenario the poloidal currents are driven by the gravitationally induced electric field which cannot be screened by any static distribution of electric charge of locally created pair plasma. The reason is as follows. The FFDE stream equation (79), which Blandford and Znajek (1977) solved in the slow-rotation limit in the steady state, has been obtained by requiring the general-relativistic Lorentz force to vanish. Then the equation naturally contains the $\alpha_0$-modified, rotationally induced electric field given by equation (14) and its divergence for the electric part and the toroidal magnetic field multiplied by the poloidal electric current for the magnetic part. What is exactly needed by the BZ solution is not a current sheet with radiative resistivity at the equator for the source of energy/angular momentum at infinity to the surrounding force-free magnetosphere, but a pair of unipolar inductors with the pair-plasma source between.

3) Znajek’s “boundary condition” is reinterpreted by Komissarov (2004) as the “regularity condition” at the fast surface (which seems to be equivalent to the “criticality condition” here), but it does not seem that what is physically meant by this reinterpretation has been sufficiently clarified. The “criticality condition” determines the current function in the eigen state, which defines the current lines along which the poloidal electric current flows, and the current lines are coincident with the relevant field lines in only the force-free domain. It must be the poloidal electric current that connects the unipolar inductors to the impedances at $S_{\Omega_2 \infty}$ and $S_{\Omega_2 H}$. Thus, the poloidal current implied by the “criticality condition” for $I_{\Omega_2}$ is obviously different from that driven by the gravitationally induced electric field, which Komissarov (2004) considers cannot be screened within the ergosphere by any static distribution of the electric charge of locally created pair plasma. On the other hand, the BZ solution is obtained basically for the screened electric field, i.e., rotationally induced, as given in equation (14).

4) What the criticality condition at $S_{\Omega_2}$ means for the ingoing wind must be the same as what the criticality condition at $S_{\Omega_2}$ means for the outgoing wind. That is, if the outgoing wind is a magnetocentrifugal wind, the ingoing wind must also be so, although both winds are oppositely directed, while being allowed to change in sign at $S_{\Omega_2}$ of the magnetic slingshot effect at work [see equation (12)]. These criticality conditions are meaningful for the field lines threading both $S_{\Omega_2 \infty}$ and $S_{\Omega_2 H}$ where the surface currents are Joule-dissipated, but will be meaningless for the field lines threading the ergosphere (cf. Komissarov 2004). It must also be emphasized that these conditions are obtainable at both ends of the force-free domain,
by breaking down the force-free condition, e.g., by taking $v \to c$ formally at $S_{R\infty}$ and $S_{H\infty}$ (see subsection 3.4).

4.2.2. MHD simulations

As a topic supporting his process for “gravitomagnetohydrodynamic dynamo”, Punsly (2003) pointed out the results of some numerical simulations on the extraction process of a hole’s rotational energy (e.g., Koide et al. 2002; also see Koide 2003).

As the initial condition, Koide et al. utilized a simple system of a large-scale uniform field, thin uniform plasma initially at rest and a Kerr black hole with no accretion disk, and followed the time evolution of the system. The domain of their “MHD Penrose process” seems to be divided into three regions: (i) the “Alfvén wave region”, where the Alfvén wave propagation carries the outward flow of electromagnetic energy against the plasma infall, (ii) the “negative energy-falling region”, where for material at the foot point of the Alfvén region, the energy-at-infinity becomes negative, causing the plasma there to fall rapidly into the black hole, and (iii) the “frame-dragging dynamo region”, which is situated between the Alfvén wave region and the negative energy-falling region. Here the kinetic energy of the plasma is converted into the electromagnetic energy by the frame-dragging dynamo effect.

It seems, however, that their “MHD Penrose process” remains transient at present, which suggests that the simulations were still under the influence of rather “unrealistic” initial conditions, so it may not be directly possible to compare these nonstationary results with the properties elucidated concerning the stationary magnetosphere, e.g., the presence of a kind of critical surface, $S_N$, with the dual unpolar inductors/pair-creation source behind, but it may be helpful for future studies to point out the following:

1) The appearance of the “Alfvén wave region” would be indicative of a non-stationary transient phenomenon, and the conversion of kinetic energy of infalling plasma to the Poynting flux in the “frame-dragging dynamo region” seems to be in common with Punsly’s mechanism. But it may be mainly the gravitational potential energy released by the infalling plasma that the Kerr hole’s rotational energy that may be carried outward by the Alfvén wave in this transient stage. It seems that this suggests possible significance of Punsly’s mechanism in a transient stage, where the infalling matter has still significant effects as the energy source.

2) The initial conditions used in the simulations may not well reflect the “realistic” environments around the Kerr hole, to simulate the extraction process in the steady state. In this context, one may need to remember the formation process of a black hole and the origin of the hole’s magnetosphere. The “no-hair theorem” indicates that the magnetic flux does not conserve during the gravitational collapse, probably dissipated into the hole’s entropy increase, and the fluxes now threading the horizon must have been supplied $a$ posteriori together with the frozen-in matter from, e.g., the accretion disk (see Thorne et al. 1986; cf. Punsly 1998a for a charged hole). On the other hand, the initial condition of a large-scale field will be quite helpful in collimating the outflow into a jet-like flow along the field direction, but it seems implausible that the uniform field immersed in a uniform plasma can be so strong that the magnetic field-dominated state is realized globally from near the central hole to infinity.

3) The black hole magnetosphere in the steady state will not be seamless, but possess a kind of critical surface, $S_N$, dividing it into the two, inner and outer, with the magnetic slingshot effect changing sign in direction there, and there must exist the dual unpolar inductors back to back with the pair-creation gap between them. However, in FFDE simulations this “microphysics” may be hidden beneath $S_N$, and the “current lines” as well as the “field lines” may seemingly be treated as if they were continuous across the gap at $S_N$, so that the transient phenomena may soon have been erased. It may, on the other hand, be probable that unless the “microphysics” in the neighborhood of $S_N$ is explicitly taken into consideration as the “boundary condition” in MHD simulations, $S_N$ may behave like a kind of singular surface, and hence the lack of the “microphysics” may have manifested itself as a transient phenomena apparently, like such frame-dragging dynamo and generation of the Alfvén waves, as suggested by “gravito-magnetohydrodynamic dynamo” (Punsly 2003), probably due to, e.g., difficulties of maintaining continuous “current lines” across $S_N$.

4) It seems that the MHD numerical simulations carried out so far nowhere took into account the plasma injection process in the magnetosphere explicitly as the “boundary conditions”, whereas every theory for stationary magnetospheres presumes the presence of the plasma injection or pair creation gap somewhere between the two light surfaces (e.g., Blandford, Znajek 1977; Phinney 1983b), and we are insisting that the location should be at the upper null surface $S_N$ in this paper as well (Okamoto 1992). Then if plasma particles are injected in the most appropriate region as the “boundary conditions”, after the effects of preexisting uniform plasma disappear, the solution may settle down quickly to the steady state. Conversely, unless the “microphysics” for pair-creation/unipolar inductors is explicitly inputted to the system as the “boundary condition” at the upper null surface, $S_N$, at $\omega = \Omega_{P} \approx (1/2) \Omega_{H}$, it seems improbable that such “microphysics” will emerge spontaneously in the time evolution of the system with the use of the “macroscopic” MHD equations, and reach the exact steady state [see also 3) in subsection 4.2.3].

4.2.3. FFDE/MHD simulations

To solve a weakly magnetized thick disk around a Kerr black hole, McKinney and Gammie (2004) carried out numerical simulations of general-relativistic MHD, making references to (i) the BZ process and (ii) the Gammie model for a stationary, axisymmetric equatorial MHD inflow.

1) It is stated that the “funnel” region near the polar axis of the black hole is consistent with the BZ model, more or less like Komissarov (2001). Although in the magnetically dominated state the necessity of explicit inclusion of the pair-creation gap with the unipolar batteries at $S_{N}$ may not be large, this does not mean that the process is free from causality violation, because the hole’s unipolar inductor(s) driving the current is not equipped in the original BZ solution, as pointed out by Punsly and Coroniti (1989).

2) It is presumed in the turned-inside-out version of the solar wind model by Weber and Davis (1967) that the equatorial MHD inflow takes place in the “plunging” region between the innermost stable circular orbit and the event horizon.
If this inflow is a gravitationally pulled accretion flow, it is not clear how much the inflow contributes to extracting the hole’s rotational energy. If the inflow is a magneto-centrifugal wind inward-directed, then just as the solar wind necessitates the driving source (i.e., a kind of unipolar battery) at the solar surface, the inflow in the “plunging” region also necessitates the same driving source, but in the opposite direction somewhere near SN, where \( \omega = \Omega_F \).

3) In common with almost all MHD simulations, it is not clear how and where the poloidal electric currents start and terminate in the black hole magnetosphere. The concept of a “current line”, defined by \( I(\Psi, \ell) = \text{constant} \), is as important as the concept of a “field line”, but with significant differences. Field lines are regarded as extending from the horizon to infinity, passing through the pair-creation gap with the dual unipolar inductors in FFDE/MHD, while current lines may be allowed to cross the gap in FFDE with \( \Delta \ell \approx 0 \), but will not in MHD with finite \( \Delta \ell \). If the whole domain of simulation was treated as if being seamless, then a kind of critical surface \( S_N \) may have manifested itself as a kind of singular region, like a “rapidly fluctuating plunging region”. It will thus be conjectured that unless the gap structure with the microphysics is properly taken into account as the “boundary condition” at \( S_N \), MHD simulations may naturally show some nonphysical behaviors due to the lack of what should externally be inputted [see also 4) in subsubsection 4.2.2].

5. Summary

The important properties of Kerr black hole magnetospheres clarified in this paper are summarized as follows (see also Okamoto 2005):

1) When a neutron star (hypothetically) collapses to produce a Kerr black hole, the \( \alpha \omega \) mechanism, coupled with magnetic fluxes threading the horizon, creates an entirely new domain of magnetosphere between the horizon surface and the classical domain (the effective ergosphere; see figure 1 and subsubsection 4.1.2). The “stellar surface”, under which matter was tightly filled in the neutron star preexisting, must shift together with its EMF to \( S_N \) with \( \alpha(\ell_S) = \Omega_B \), where there must be situated a pair-creation gap with the two EMF’s at the inner and outer surfaces, EMF_{in} and EMF_{out}, for the paired, in- and out-going, winds, respectively. The gradient of the frame-dragging angular velocity, \( \omega \), at \( S_N \), multiplied by the half-width of the gap, \( \Delta \omega \approx |\partial \omega / \partial \ell|_S \Delta \ell \), is responsible for both the inner and outer EMF’s, which are almost equal to each other and are opposite in direction. It is indeed the corotational, frozen-in electric field \( E_p \) per se, perpendicular everywhere to field lines rotating with \( \Omega_F \), that generates the EMF’s at the particle source surfaces, to drive the current along each current line.

2) The “microphysics” in the gap depends upon \( \Delta \omega \) across the gap, and the \( \alpha \omega \)-induced potential difference is \( \sim E_p \parallel \approx |\text{EMF}_{\alpha \omega} / |\Delta \ell| \), and will be strong enough to pair-create, to charge-separate inside the gap, and to provide the currents for both the inner and outer circuits. One current line, \( I(\Psi) \), in each circuit consists of a pair of field lines, with \( j_p > 0 \) along one field line \( \Psi_2 \) in the equatorial side, and \( j_p < 0 \) along the other field line \( \Psi_1 \) in the polar side, where \( \Psi_{1,2} \) are the two solutions of equation \( I(\Psi) = I(\Psi) \). The surface current connecting the volume currents along the field lines \( \Psi_1 \) and \( \Psi_2 \) crosses the field lines in \( \Psi_1 < \Psi < \Psi_2 \), following Ohm’s law on \( S_{\text{ff} \infty} \) and \( S_{\text{ff} \text{H}} \) endowed with finite resistivity. The electromagnetic energy Joule-dissipated on \( S_{\text{ff} \infty} \) expresses the MHD acceleration of the mean flow due to the volume Lorentz force in the asymptotic, superfast domain in \( S \lesssim S \lesssim S_{\infty} \), whereas that on \( S_{\text{ff} \text{H}} \) means the Joule-heating leading to the increase of the hole’s irreducible mass (see figure 5).

3) One has the “single” eigenvalue problem for \( I \) in the P-FFDE in terms of \( \Omega_F \), whereas one has to solve the “double” eigenvalue problem in the BH-FFDE. This is because after determining the two eigenvalues \( I \)'s for the classical and general-relativistic domains by the “criticality conditions” at \( S_{\text{ff} \text{H}} \) and \( S_{\text{ff} \infty} \), one must determine the final eigenvalue for \( \Omega_F \) by the “boundary condition” at \( S_N \), which expresses the continuity of the energy/angular momentum fluxes across the gap, or the current-closure condition, i.e., \( I_{\text{in}}(\Psi) = I_{\text{out}}(\Psi) \). One important premise here is the existence of magnetic fluxes that possess the same \( \Omega_F(\Psi) \) throughout the classical and general-relativistic domains, connecting \( S_{\text{ff} \text{H}} \) with \( S_{\text{ff} \infty} \) through \( S_N \). This will imply that both terminals are communicated beyond the pair-creation gap with the dual unipolar inductors by FFDE/MHD waves, such as Alfvenic and fast and/or their hybrid ones in the steady eigen state with the “current lines” established. In the final eigen-state in the BH-FFDE, the eigenvalue \( \Omega_F \), is given in terms of \( \Omega_H \) and the ratio of the terminal fluxes of the poloidal field, i.e., \( \xi = \Psi_{\text{ff} \infty} / \Psi_{\text{ff} \text{H}} \). Importantly, \( E_p \parallel \) remains finite at \( S_N \) even in the limit of \( \Delta \ell \rightarrow 0 \). For \( \xi \approx 1 \), one obtains the optimal value of \( \Omega_F \approx (1/2)\Omega_H \).

4) When the force-free approximation is in use together with the frozen-in approximation, that is, when force-free field lines are regarded as frozen in massless particles, a kind of extreme physical state emerges, in which the two degeneracies take place in equations and conditions. These degeneracies produce simplicity in mathematics, which conversely brings about complexity in physics. In fact, unless the processes of degeneracies taking place are clarified completely, “how the physics works?” cannot be elucidated. For example, the gap which the “microphysics” should be incorporated into as the “boundary condition” for the “macroscopic FFDE equation” is made closed to simply a surface \( S_N \), by \( \Delta \ell \rightarrow 0 \). The physical significance of this \( S_N \), itself, has really so far been overlooked since the first suggestion (Okamoto 1992).

5) The necessity of the particle source being somewhere above the horizon is always mentioned (e.g., Znajek 1977; Blandford, Znajek 1977). Instead of “a battery in the horizon surface” (Phinney 1983a; Thorne et al. 1986; Okamoto 1992), the existence of a dynamo process or the driving force of the BZ process has recently been suggested again to exist somewhere between the inner and outer Alfvenic surfaces, well outside of the horizon (e.g., Takahashi et al. 1990; Beskin, Kuznetsova 2000; Punsly 2001; Komissarov 2002b; Blandford 2002). However, unfortunately, the presence of the two unipolar inductors existing back to back at \( S_{\text{ff}} \), as described in this paper, has so far been overlooked in the MHD/FFDE models. It is only massless charged particles carrying electric currents, and thereby the Poynting flux that are apparently needed in FFDE, and the strengths of the current could be fixed just by imposing the load impedances by the
“boundary condition” and \( \Omega_F \approx (1/2) \Omega_H \) could be fixed by impedance matching at both terminals (see, e.g., Thorne et al. 1986). It might thus be difficult to detect the double EMP’s associated with the particle source.

6. Conclusion

It will be worth while reiterating that the key relations leading to a correct understanding of energy extraction from a Kerr black hole are given by equations (12) and (14). For example, the expression for \( v_F \), which indicates the magnetic slingshot effect (Okamoto 2005), exactly shows how the general-relativistic effects intervene in FFDE/MHD magnetospheres, that is, how the redshift factor \( \alpha \) and the frame-dragging angular frequency \( \omega \) couple with the large-scale fields existing there. Although implicit in the pioneering paper by Blandford and Znajek (1977), these relations were derived long ago by Macdonald and Thorne (1982) and Thorne, Price, and Macdonald (1986) in the 3+1 formalism, and are often cited in the literature. Nevertheless, probably because these seem to have been simple or quite simple, it may have been rather difficult to detect its deep physical implications, that is, to find that this simple expression for \( v_F \) helps us to realize the existence of a new general-relativistic domain separated by \( S_N \) from the classical domain, i.e., the effective ergosphere produced by the \( \alpha \omega \) mechanism, quite different in nature from the ordinary ergosphere (Okamoto 1992). It will also be worth while stressing that, as far as extraction of the hole’s rotational energy is concerned, the ingoing wind in the effective ergosphere is an inward-directed magneto-centrifugal wind blowing through the critical surfaces toward the horizon, fundamentally different from an accretion flow in the ordinary ergosphere. The battery mechanism (the \( \alpha \omega \) dynamo) as well as the particle creation process must be located in the interface \( S_N \) for the paired wind system. Among the wind parameters (the constants of motion), the angular velocity of each field-streamline must be deduced from the induction equation in the gap, the location of which is itself defined by the condition \( \omega(\sigma_N) = \Omega_F \), and the poloidal current \( I ' \)’s are given as the eigenvalues in terms of the input parameters in the particle creation zone by the “criticality conditions” at the inner and outer fast surfaces. The current-closure condition at the interface, \( S_N \), as the “boundary condition” determines \( \Omega_F \) and \( I ' \)'s in the steady state as the ultimate eigenvalues in the “double” eigenvalue problem.

Basically, gravity and electromagnetism are not uncongenial rivals to each other, like the Montagues and the Capulets, so we can avoid such a tragedy in marrying general relativity and FFDE/MHD as Romeo and Juliet. It can be concluded that the “causality question” is fortunately not of physical origin. Unless misused or overlooked, FFDE/MHD are probably the most useful tools for describing the interactions among the field, rotation and motion, and consequent extraction of energy by “braking magnetically”, from not only non-relativistic and special-relativistic, but also general-relativistic objects. Hereafter, one will be able to enjoy applying the BZ process and the membrane paradigm with the “microphysics” incorporated at \( S_N \) for a variety of phenomena in black hole astrophysics, without worrying about causality violation. One will, for example, be capable of making use of the modified BZ process as a viable process of extracting the rotational energy of a Kerr black hole, stellar-sized or super-massive, not only for modeling quasars, microquasars, and GRBs, but also for the black hole Carnot cycle in constructing a cosmic power station at least in the stage of a thought experiment (Kaburaki, Okamoto 1991).

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