Memory Limited, Streaming PCA

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Abstract

We consider streaming, one-pass principal component analysis (PCA), in the high-dimensional regime, with limited memory. Here, \( p \)-dimensional samples are presented sequentially, and the goal is to produce the \( k \)-dimensional subspace that best approximates these points. Standard algorithms require \( O(p^2) \) memory; meanwhile no algorithm can do better than \( O(kp) \) memory, since this is what the output itself requires. Memory (or storage) complexity is most meaningful when understood in the context of computational and sample complexity. Sample complexity for high-dimensional PCA is typically studied in the setting of the spiked covariance model, where \( p \)-dimensional points are generated from a population covariance equal to the identity (white noise) plus a low-dimensional perturbation (the spike) which is the signal to be recovered. It is now well-understood that the spike can be recovered when the number of samples, \( n \), scales proportionally with the dimension, \( p \). Yet, all algorithms that provably achieve this, have memory complexity \( O(p^2) \). Meanwhile, algorithms with memory-complexity \( O(kp) \) do not have provable bounds on sample complexity comparable to \( p \). We present an algorithm that achieves both: it uses \( O(kp) \) memory (meaning storage of any kind) and is able to compute the \( k \)-dimensional spike with \( O(p \log p) \) sample-complexity – the first algorithm of its kind. While our theoretical analysis focuses on the spiked covariance model, our simulations show that our algorithm is successful on much more general models for the data.

1 Introduction

Principal component analysis is a fundamental tool for dimensionality reduction, clustering, classification, and many more learning tasks. It is a basic preprocessing step for learning, recognition, and estimation procedures. The core computational element of PCA is performing a (partial) singular value decomposition, and much work over the last half century has
focused on efficient algorithms (e.g., \cite{7} and references therein) and hence on computational complexity.

The recent focus on understanding high-dimensional data, where the dimensionality of the data scales together with the number of available sample points, has led to an exploration of the sample complexity of covariance estimation. This direction was largely influenced by Johnstone’s spiked covariance model, where data samples are drawn from a distribution whose (population) covariance is a low-rank perturbation of the identity matrix \cite{11}. Work initiated there, and also work done in \cite{19} (and references therein) has explored the power of batch PCA in the $p$-dimensional setting with sub-Gaussian noise, and demonstrated that the singular value decomposition (SVD) of the empirical covariance matrix succeeds in recovering the principal components (extreme eigenvectors of the population covariance) with high probability, given $n = O(p)$ samples.

This paper brings the focus on another critical quantity: memory/storage. This is relevant in the so-called streaming data model, where the samples $x_i \in \mathbb{R}^p$ are collected sequentially, and unless we store them, they are irretrievably gone.\footnote{This is similar to what is sometimes referred to as the single pass model.} The only currently available algorithms with provable sample complexity guarantees either store all $n = O(p)$ samples (note that for more than a single pass over the data, the samples must all be stored) or explicitly form the empirical $p \times p$ (typically dense) covariance matrix. Either case requires at least $O(p^2)$ storage. Despite the availability of massive local and distributed storage systems, for high-dimensional applications (e.g., where data points are high resolution photographs, biometrics, video, etc.), $p$ could be on the order of $10^{10} - 10^{12}$, making $O(p^2)$ prohibitive, if not in fact impossible to manage. Indeed, at multiple computing scales, manipulating vectors of length $O(p)$ is possible, when storage of $O(p^2)$ is not. A typical desktop may have 10-20 GB of RAM, but will not have more than a few TB of total storage. A modern smart-phone may have as much as a GB of RAM, but has a few GB, not TB, of storage.

We consider the streaming data setting, where data points are generated sequentially, and are never stored. In the setting of the so-called spiked covariance model (and natural generalizations) we show that a simple algorithm requiring $O(kp)$ storage – the best possible – performs as well as batch algorithms (namely, SVD on the empirical covariance matrix), with sample complexity $O(p \log p)$. To the best of our knowledge, this is the only algorithm with both storage complexity and sample complexity guarantees. We discuss the connection to past work in more detail in Section 2. We introduce the model with all related details in Section 3 and present the solution to the rank 1 case, the rank $k$ case, and the perturbed-rank-$k$ case in Sections 4.1, 4.2 and 4.3, respectively. In Section 5 we provide simulations that not only confirm the theoretical results, but demonstrate that our algorithm works well outside the assumptions of our main theorems.

### 2 Related Work

Memory- and computation-efficient algorithms that operate on streaming data are plentiful in the literature and many seem to do well in practice. However, there is no algorithm that provably recovers the principal components in the same noise and sample-complexity regime as the batch PCA algorithm does and maintains a provably light memory footprint. Because
of the practical relevance, there has been renewed interest recently in this problem, and the fact that this is an important unresolved issue has been pointed out in numerous places, e.g., [21][1].

A large body of work has focused on the non-statistical data paradigm that deals with a fixed pool of samples. This includes work on online PCA and low-rank matrix approximation in the streaming scenario, including sketching and dimensionality-reduction based techniques.

Online-PCA for regret minimization has been considered in several papers, most recently in [21], where the multiplicative weights approach is adapted for this problem (now experts correspond to subspaces). The goal there is to control the regret, improving on the natural follow-the-leader algorithm that performs batch-PCA at each step. However, the algorithm can require $O(p^2)$ memory, in order to store the multiplicative weights. A memory-light variant described in [1] typically requires much less memory, but there are no guarantees for this, and moreover, for certain problem instances, its memory requirement is on the order of $p^2$.

Sub-sampling, dimensionality-reduction and sketching form another family of low-complexity and low-memory techniques, see, e.g., [5][13][8]. These save on memory and computation by performing SVD on the resulting smaller matrix. The results in this line of work provide worst-case guarantees over the pool of data, and typically require a rapidly decaying spectrum (which we do not have in our setting) to produce good bounds. More fundamentally, these approaches are not appropriate for data coming from a statistical model such as the spiked covariance model. It is clear that subsampling approaches, for instance, simply correspond to discarding most of the data, and for fundamental sample complexity reasons, cannot work. Sketching produces a similar effect: each column of the sketch is a random (+/-) sum of the data points. If the data points are, e.g., independent Gaussian vectors, then so will each element of the sketch, and thus this approach again runs against fundamental sample complexity constraints. Indeed, it is straightforward to check that the guarantees presented in ([5][8]) are not strong enough to guarantee recovery of the spike. This is not because the results are weak; it is because they geared towards worst-case bounds.

Algorithms focused on sequential SVD (e.g., [4][3][6] and more recently [2][9]) seek to have the best subspace estimate at every time (i.e., each time a new data sample arrives) but without performing full-blown SVD at each step. While these algorithms indeed reduce both the computational and memory burden of batch-PCA, there are no rigorous guarantees on the quality of the principal components or on the statistical performance of these methods.

In a Bayesian mindset, some researchers have come up with expectation maximization approaches [16][18], that can be used in an incremental fashion. The finite sample behavior is not known.

Stochastic-approximation-based algorithms along the lines of [15] are also quite popular, because of their low computational and memory complexity, and excellent performance in practice. They go under a variety of names, including Incremental PCA (though the term Incremental has been used in the online setting as well [10]), Hebbian learning, and stochastic power method [1]. The basic algorithms are some version of the following: upon receiving data point $x_t$ at time $t$, update the estimate of the top $k$ principal components via:

$$U^{(t+1)} = \text{Proj}(U^{(t)} + \eta_t x_t x_t^\top U^{(t)}), \quad (1)$$
where \( \text{Proj}(\cdot) \) denotes the “projection” that takes the SVD of the argument, and sets the top \( k \) singular values to 1 and the rest to zero (see [1] for further discussion).

While empirically these algorithms perform well, to the best of our knowledge - and efforts - there does not exist any rigorous finite sample guarantee for these algorithms. The analytical challenge seems to be the high variance at each step, which makes direct analysis difficult.

In summary, while much work has focused on memory-constrained PCA, there has as of yet been no work that simultaneously provides sample complexity guarantees competitive with batch algorithms, and also memory/storage complexity guarantees close to the minimal requirement of \( O(kp) \) – the memory required to store only the output. We present an algorithm that provably does both.

3 Problem Formulation and Notation

We consider a streaming model, where at each time step \( t \), we receive a point \( x_t \in \mathbb{R}^p \). Furthermore, any vector that is not explicitly stored can never be revisited. Now, our goal is to compute the top \( k \) principal components of the data: the \( k \)-dimensional subspace that offers the best squared-error estimate for the points. We assume a probabilistic generative model, from which the data is sampled at each step \( t \). Specifically, we assume,

\[
x_t = Az_t + w_t,
\]

where \( A \in \mathbb{R}^{p \times k} \) is a fixed matrix, \( z_t \in \mathbb{R}^{k \times 1} \) is a multivariate normal random variable, i.e.,

\[
z_t \sim \mathcal{N}(0_{k \times 1}, I_{k 
 k}),
\]

and vector \( w_t \in \mathbb{R}^{p \times 1} \) is the “noise” vector and is also sampled from a multivariate normal distribution, i.e.,

\[
w_t \sim \mathcal{N}(0_{p \times 1}, \sigma^2 I_{p \times p}).
\]

Furthermore, we assume that all \( 2n \) random vectors \( (z_t, w_t, \forall 1 \leq t \leq n) \) are mutually independent.

In this regime, it is well-known that batch-PCA is asymptotically consistent (hence recovering \( A \) up to unitary transformations) with number of samples scaling as \( n = O(p) \) [20]. It is interesting to note that in this high-dimensional regime, the signal-to-noise ratio quickly approaches zero, as the signal, or “elongation” of the major axis, \( \|Az\|_2 \), is \( O(1) \), while the noise magnitude, \( \|w\|_2 \), scales as \( O(\sqrt{p}) \). The central goal of this paper is to provide finite sample guarantees for a streaming algorithm that requires memory no more than \( O(kp) \) and matches the consistency results of batch PCA in the sampling regime \( n = O(p) \) (possibly with additional log factors, or factors depending on \( \sigma \) and \( k \)).

We denote matrices by capital letters (e.g. \( A \)) and vectors by lower-case bold-face letters \( (x) \). \( \|x\|_q \) denotes the \( \ell_q \) norm of \( x \); \( \|x\| \) denotes the \( \ell_2 \) norm of \( x \). \( \|A\| \) or \( \|A\|_2 \) denotes the spectral norm of \( A \) while \( \|A\|_F \) denotes the Frobenius norm of \( A \). Without loss of generality (WLOG), we assume that: \( \|A\|_2 = 1 \), where \( \|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 \) denotes the spectral norm of \( A \). Finally, we write \( \langle a, b \rangle = a^\top b \) for the inner product between \( a, b \). In proofs the constant \( C \) is used loosely and its value may vary from line to line.
Algorithm 1 Block-Stochastic Power Method

| Algorithm 1 Block-Stochastic Power Method | Block-Stochastic Orthogonal Iteration |
|------------------------------------------|--------------------------------------|
| **input** \( \{x_1, \ldots, x_n\} \), Block size: \( B \) | **input**  \( H^i \sim \mathcal{N}(0, I_{p \times p}), 1 \leq i \leq k \) (Initialization) |
| 1: \( q_0 \sim \mathcal{N}(0, I_{p \times p}) \) (Initialization) | 2: \( q_0 \sim \mathcal{N}(0, I_{p \times p}) \) (Initialization) |
| 3: for \( \tau = 0, \ldots, n/B - 1 \) do | 3: for \( \tau = 0, \ldots, n/B - 1 \) do |
| 4: \( s_{\tau+1} \leftarrow 0 \) | 4: \( s_{\tau+1} \leftarrow 0 \) |
| 5: for \( t = B\tau + 1, \ldots, B(\tau + 1) \) do | 5: for \( t = B\tau + 1, \ldots, B(\tau + 1) \) do |
| 6: \( s_{\tau+1} \leftarrow s_{\tau+1} + \frac{1}{B} q_{\tau} x_t x_t^\top \) | 6: \( s_{\tau+1} \leftarrow s_{\tau+1} + \frac{1}{B} x_t x_t^\top Q_{\tau} \) |
| 7: end for | 7: end for |
| 8: \( q_{\tau+1} \leftarrow s_{\tau+1} / \|s_{\tau+1}\|_2 \) | 8: \( q_{\tau+1} \leftarrow s_{\tau+1} / \|s_{\tau+1}\|_2 \) |
| 9: end for | 9: end for |
| **output** | **output** |

### 4 Algorithm and Guarantees

In this section, we present our proposed algorithm and its finite sample analysis. It is a block-wise stochastic variant of the classical power-method. Stochastic versions of the power method are already popular in the literature and are known to have good empirical performance; see [1] for a nice review of such methods. However, the main impediment to the analysis of such stochastic algorithms (as in [1]) is the potentially large variance of each step, due primarily to the high-dimensional regime we consider, and the vanishing SNR.

This motivated us to consider a modified stochastic power method algorithm, that has a variance reduction step built in. At a high level, our method updates only once in a "block" and within one block we average out noise to reduce the variance.

Below, we first illustrate the main ideas of our method as well as our sample complexity proof for the simpler rank-1 case. The rank-1 and rank-\( k \) algorithms are so similar, that we present them in the same panel. We provide the rank-\( k \) analysis in Section 4.2. We note that, while our algorithm describes \( \{x_1, \ldots, x_n\} \) as “input,” we mean this in the streaming sense: the data are no-where stored, and can never be revisited unless the algorithm explicitly stores them.

#### 4.1 Rank-One Case

We first consider the rank-1 case for which each sample \( x_t \) is generated using: \( x_t = uz_t + w_t \) where \( u \in \mathbb{R}^p \) is the principal component that we wish to recover. Our algorithm is a block-wise method where all the \( n \) samples are divided in \( n/B \) blocks (for simplicity we assume that \( n/B \) is an integer). In the \( (\tau + 1) \)-st block, we compute

\[
 s_{\tau+1} = \left( \frac{1}{B} \sum_{t=B\tau+1}^{B(\tau+1)} x_t x_t^\top \right) q_{\tau}. \tag{3}
\]

Then, the iterate \( q_{\tau} \) is updated using \( q_{\tau+1} = s_{\tau+1} / \|s_{\tau+1}\|_2 \). Note that, \( s_{\tau+1} \) can be easily computed in an online manner where \( O(p) \) operations are required per step. Furthermore, storage requirements are also linear in \( p \).
4.1.1 Analysis

We now present the sample complexity analysis of our proposed method (Algorithm 1). We show that, using \(O\left(\frac{\sigma^4 p \log(p/\epsilon^2)}{\epsilon^2}\right)\) samples, Algorithm 1 obtains a solution \(q_T\) of accuracy \(\epsilon\), i.e. \(\|q_T - u\|_2 \leq \epsilon\).

**Theorem 1.** Denote the data stream by \(x_1, \ldots, x_n\), where \(x_t \in \mathbb{R}^p, \forall t\) is generated by (2). Set the total number of iterations \(T = \Omega\left(\frac{\log(p/\epsilon)}{\log((\sigma^2 + .75)/\epsilon)}\right)\) and the block size \(B = \Omega\left(\frac{(1 + 3(\sigma + \sigma^2))\sqrt{p} \log(T)}{\epsilon^2\log((\sigma^2 + .75)/\epsilon)}\right)\). Then, with probability 0.99, \(\|q_T - u\|_2 \leq \epsilon\), where \(q_T\) is the \(T\)-th iterate of Algorithm 1. That is, Algorithm 1 obtains an \(\epsilon\)-accurate solution with number of samples \(n\) given by:

\[
n = \tilde{\Omega}\left(\frac{(1 + 3(\sigma + \sigma^2))\sqrt{p}}{\epsilon^2\log((\sigma^2 + .75)/\epsilon)}\right).
\]

Note that in the total sample complexity, we use the notation \(\tilde{\Omega}(\cdot)\) to suppress the extra \(\log(T)\) factor for clarity of exposition, as \(T\) already appears in the expression linearly.

**Proof.** The proof decomposes the current iterate into the component of the current iterate, \(q_T\), in the direction of the true principal component (the spike) \(u\), and the perpendicular component, showing that the former eventually dominates. Doing so hinges on three key components: (a) for large enough \(B\), the empirical covariance matrix \(F_{\tau+1} = \frac{1}{B} \sum_{t=B\tau+1}^{t=B(\tau+1)} x_t x_t^\top\) is close to the true covariance matrix \(M = uu^\top + \sigma^2 I\), i.e., \(\|F_{\tau+1} - M\|_2\) is small. In the process, we obtain “tighter” bounds for \(\|u^\top (F_{\tau+1} - M)u\|\) for fixed \(u\); (b) with probability 0.99 (or any other constant probability), the initial point \(q_0\) has a component of at least \(O\left(\frac{1}{\sqrt{p}}\right)\) magnitude along the true direction \(u\); (c) after \(\tau\) iterations, the error in estimation is at most \(O(\gamma^\tau)\) where \(\gamma < 1\) is a constant.

There are several results that we use repeatedly, which we collect here, and prove individually in the appendix.

**Lemmas 4, 5 and 6.** Let \(B, T\) and the data stream \(\{x_t\}\) be as defined in the theorem. Then:

- (Lemma 4): With probability \(1 - C/T\), for \(C\) a universal constant, we have:

\[
\left\|\frac{1}{B} \sum_t x_t x_t^\top - uu^\top - \sigma^2 I\right\|_2 \leq \epsilon.
\]

- (Lemma 5): With probability \(1 - C/T\), for \(C\) a universal constant, we have:

\[
u^\top s_{\tau+1} \geq u^\top q_T (1 + \sigma^2) \left(1 - \frac{\epsilon}{4(1 + \sigma^2)}\right),
\]

where \(s_t = \frac{1}{B} \sum_{B\tau < t \leq B(\tau+1)} x_t x_t^\top q_T\).

- (Lemma 6): Let \(q_0\) be the initial guess for \(u\), given by Steps 1 and 2 of Algorithm 1. Then, w.p. 0.99: \(|\langle q_0, u \rangle| \geq \frac{C_0}{\sqrt{p}}\), where \(C_0 > 0\) is a universal constant.
Step (a) is proved in Lemmas 4 and 5 while Lemma 6 provides the required result for the initial vector \( q_0 \). Using these lemmas, we next complete the proof of the theorem. We note that both (a) and (b) follow from well-known results; we provide them for completeness.

Let \( q_t = \sqrt{1 - \delta_t} u + \sqrt{\delta_t} g_t, 1 \leq \tau \leq n/B \), where \( g_t \) is the component of \( q_t \) that is perpendicular to \( u \) and \( \sqrt{1 - \delta_t} \) is the magnitude of the component of \( q_t \) along \( u \). Note that \( g_t \) may well change at each iteration; we only wish to show \( \delta_t \to 0 \).

Now, using Lemma 5, the following holds with probability at least \( 1 - C/T \):

\[
\begin{align*}
    u^T s_{\tau+1} & \geq \sqrt{1 - \delta_t (1 + \sigma^2)} \left( 1 - \frac{\epsilon}{4(1 + \sigma^2)} \right).
\end{align*}
\]  

Next, we consider the component of \( s_{\tau+1} \) that is perpendicular to \( u \):

\[
\begin{align*}
    g_{\tau+1}^T s_{\tau+1} & = g_{\tau+1}^T \left( \frac{1}{B} \sum_{t=B\tau+1}^{B(\tau+1)} x_t x_t^T \right) q_t
    & = g_{\tau+1}^T (M + E) q_t,
\end{align*}
\]

where \( M = uu^T + \sigma^2 I \) and \( E_t \) is the error matrix: \( E_t = M - \frac{1}{B} \sum_{t=B\tau+1}^{B(\tau+1)} x_t x_t^T \). Using Lemma 4, \( \|E_t\|_2 \leq \epsilon \) (w.p. \( \geq 1 - C/T \)). Hence, w.p. \( \geq 1 - C/T \):

\[
\begin{align*}
    g_{\tau+1}^T s_{\tau+1} & = \sigma^2 g_{\tau+1}^T q_t + \|g_{\tau+1}\|_2 \|E_t\|_2 \|q_t\|_2
    & \leq \sigma^2 \sqrt{\delta_t} + \epsilon.
\end{align*}
\]  

Now, since \( q_{\tau+1} = s_{\tau+1}/\|s_{\tau+1}\|_2 \),

\[
\begin{align*}
    \delta_{\tau+1} & = (g_{\tau+1}^T q_{\tau+1})^2
    & \leq \frac{(g_{\tau+1}^T s_{\tau+1})^2}{(u^T s_{\tau+1})^2 + (g_{\tau+1}^T s_{\tau+1})^2},
    & \leq \frac{(u^T s_{\tau+1})^2}{(u^T s_{\tau+1})^2 + (g_{\tau+1}^T s_{\tau+1})^2},
    & \leq \frac{(\sigma^2 \sqrt{\delta_t} + \epsilon)^2}{(\sigma^2 \sqrt{\delta_t} + \epsilon)^2 + (\sigma^2 \sqrt{\delta_t} + \epsilon)^2},
\end{align*}
\]  

where, (i) follows from (4) and (ii) follows from (5) along with the fact that \( \frac{x}{\epsilon + x} \) is an increasing function in \( x \) for \( c, x \geq 0 \).

Assuming \( \sqrt{\delta_t} \geq 2\epsilon \) and using (6) and bounding the failure probability with a union bound, we get (w.p. \( \geq 1 - t \cdot C/T \))

\[
\begin{align*}
    \delta_{\tau+1} & \leq \frac{\delta_t (\sigma^2 + 1/2)^2}{(1 - \delta_t)(\sigma^2 + 3/4)^2 + \delta_t (\sigma^2 + 1/2)^2}
    & \leq \frac{\gamma^{2\tau} \delta_0}{1 - (1 - \gamma^{2\tau}) \delta_0}
    & \leq C_1 \gamma^{2\tau} \rho,
\end{align*}
\]  

where \( \gamma = \frac{\sigma^2 + 1/2}{\sigma^2 + 3/4} \) and \( C_1 > 0 \) is a global constant. Inequality (ii) follows from Lemma 6 to prove (i), we need one final result: the following lemma shows that the recursion given by (7) decreases \( \delta_t \) at a fast rate. Interestingly, the rate of decrease in error \( \delta_t \) initially (for small \( \tau \)) might be sub-linear but for large enough \( \tau \) the rate turns out to be linear. We defer the proof to the appendix.
Lemma 2. If for any $\tau \geq 0$ and $0 < \gamma < 1$, we have $\delta_{\tau+1} \leq \frac{\gamma^2 \delta_0}{1 - \delta_\gamma \delta_\gamma}$, then,

$$
\delta_{\tau+1} \leq \frac{\gamma^{2 \tau+2} \delta_0}{1 - (1 - \gamma^{2 \tau+2}) \delta_0}.
$$

Hence, using the above equation after $T = O\left(\log(p/\varepsilon)/\log(1/\gamma)\right)$ updates, with probability at least $1 - C$, $\sqrt{\delta_T} \leq 2\sqrt{\frac{\sigma^2}{k} + 0.75 \lambda_k^2}$. The result now follows by noting that $\|u - q_T\|_2 \leq 2\sqrt{\frac{\sigma^2}{k} + 0.75 \lambda_k^2}$.

Remark: Note that in Theorem 1, the probability of accurate principal component recovery is a constant and does not decay with $p$. One can correct this by either paying a price of $O(\log p)$ in storage, or in sample complexity: for the former, we can run $O(\log p)$ instances of Algorithm 1 in parallel; alternatively, we can run Algorithm 1 $O(\log p)$ times on fresh data each time, using the next block of data to evaluate the old solutions, always keeping the best one. Either approach guarantees a success probability of at least $1 - \frac{1}{p^{O(1)}}$.

4.2 General Rank-$k$ Case

In this section, we consider the general rank-$k$ PCA problem where each sample is assumed to be generated using the model of equation (2), where $A \in \mathbb{R}^{p \times k}$ represents the $k$ principal components that need to be recovered. Let $A = UAV^\top$ be the SVD of $A$ where $U \in \mathbb{R}^{p \times k}$, $\Lambda, V \in \mathbb{R}^{k \times k}$. The matrices $U$ and $V$ are orthogonal, i.e., $U^\top U = I, V^\top V = I$, and $\Sigma$ is a diagonal matrix with diagonal elements $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_k$. The goal is to recover the space spanned by $A$, i.e., $\text{span}(U)$. Without loss of generality, we can assume that $\|A\|_2 = \lambda_1 = 1$.

Similar to the rank-1 problem, our algorithm for the rank-$k$ problem can be viewed as a streaming variant of the classical orthogonal iteration used for SVD. But unlike the rank-1 case, we require a more careful analysis as we need to bound spectral norms of various quantities in intermediate steps and simple, crude analysis can lead to significantly worse bounds. Interestingly, the analysis is entirely different from the standard analysis of the orthogonal iteration as there, the empirical estimate of the covariance matrix is fixed while in our case it varies with each block.

For the general rank-$k$ problem, we use the largest-principal-angle-based distance function between any two given subspaces:

$$
\text{dist} (\text{span}(U), \text{span}(V)) = \text{dist}(U, V) = \|U^\top V\|_2 = \|V^\top U\|_2,
$$

where $U_\perp$ and $V_\perp$ represent an orthogonal basis of the perpendicular subspace to $\text{span}(U)$ and $\text{span}(V)$, respectively. For the spiked covariance model, it is straightforward to see that this is equivalent to the usual PCA figure-of-merit, the expressed variance.

Theorem 3. Consider a data stream, where $\mathbf{x}_t \in \mathbb{R}^p$ for every $t$ is generated by (2), and the SVD of $A \in \mathbb{R}^{p \times k}$ is given by $A = UAV^\top$. Let, wlog, $\lambda_1 = 1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. Let,

$$
T = \Omega\left(\log(p/k\varepsilon)/\log\left(\frac{\sigma^2 + 0.75 \lambda_k^2}{\sigma^2 + 0.5 \lambda_k^2}\right)\right), \quad B = \Omega\left(\frac{(1 + \sigma)^2 \sqrt{k} + \sigma \sqrt{1 + \sigma^2} \sqrt{k} \sqrt{p}}{\lambda_k^2 \varepsilon^2} \log(T)\right).
$$
Then, after $T$ B-size-block-updates, w.p. $0.99$, $\text{dist}(U, Q_T) \leq \epsilon$. Hence, the sufficient number of samples for $\epsilon$-accurate recovery of all the top-$k$ principal components is:

$$n = \tilde{\Omega} \left( \frac{(1 + \sigma)^2 \sqrt{k + \sigma \sqrt{1 + \frac{\sigma^2 k}{\sqrt{p}}}}^2 \log(p/k\epsilon)}{\lambda_k^4 \epsilon^2 \log \left( \frac{\sigma^2 + 0.75\lambda_k^2}{\sigma^2 + 0.5\lambda_k^2} \right)} \right).$$

Again, we use $\tilde{\Omega}(\cdot)$ to suppress the extra $\log(T)$ factor.

The key part of the proof requires the following additional lemmas that bound the energy of the current iterate along the desired subspace and its perpendicular space (Lemmas 8 and 9), and Lemma 10, which controls the quality of the initialization.

**Lemmas 8, 9 and 10** Let the data stream, $A$, $B$, and $T$ be as defined in Theorem 3, $\sigma$ be the variance of noise, $F_{\tau+1} = \frac{1}{B} \sum_{B\tau < t \leq B(\tau+1)} x_t x_t^\top$ and $Q_\tau$ be the $\tau$-th iterate of Algorithm 1.

- **(Lemma 8):** $\forall \ v \in \mathbb{R}^k$ and $\|v\|_2 = 1$, w.p. $1 - 5C/T$ we have:

  $$\|U^T F_{\tau+1} Q_\tau v\|_2 \geq (\lambda_k^2 + \sigma^2 - \frac{\lambda_k^2 \epsilon}{4}) \sqrt{1 - \|U_Q^\top Q_\tau\|_2^2}.$$  

- **(Lemma 9):** With probability at least $1 - 4C/T$, $\|U^T F_{\tau+1} Q_\tau\|_2 \leq \sigma^2 \|U_Q^\top Q_\tau\|_2 + \lambda_k^2 \epsilon/2$.

- **(Lemma 10):** Let $Q_0 \in \mathbb{R}^{p \times k}$ be sampled uniformly at random as in Algorithm 1. Then, w.p. at least 0.99: $\sigma_k(U^T Q_0) \geq C \sqrt{\frac{1}{kp}}$.

We provide the proof of the lemmas and theorem in the appendix.

### 4.3 Perturbation-tolerant Subspace Recovery

While our results thus far assume $A$ has rank exactly $k$, and $k$ is known a priori, here we show that both these can be relaxed; hence our results hold in a quite broad setting.

Let $x_t = A z_t + w_t$ be the $t$-th step sample, with $A = U \Lambda V^T \in \mathbb{R}^{p \times r}$ and $U \in \mathbb{R}^{p \times r}$ where $r \geq k$ is the true rank of $A$ which is unknown. However, we run Algorithm 1 with rank $k$ and the goal is to recover a subspace $Q_T$, s.t., $Q_T$ is contained in $U$.

We first observe that the largest-principal angle based distance function that we use in the previous section can directly be used for our more general setting. That is, $\text{dist}(U, Q_T) = \|U_Q^\top Q_T\|_2$ measures the component of $Q_T$ “outside” the subspace $U$ and the goal is to show that component is $\leq \epsilon$.

Now, our analysis can be easily modified to handle this more general setting as crucially our distance function does not change. Naturally, now the number of samples we require increases according to $r$. In particular, if

$$n = \tilde{\Omega} \left( \frac{(1 + \sigma)^2 \sqrt{r + \sigma \sqrt{1 + \frac{\sigma^2 r \sqrt{p}}}}^2 \log(p/r\epsilon)}{\lambda_k^4 \epsilon^2 \log \left( \frac{\sigma^2 + 0.75\lambda_k^2}{\sigma^2 + 0.5\lambda_k^2} \right)} \right),$$


then dist($U, Q_T$) ≤ ε. Furthermore, if we assume $r \geq C \cdot k$ (or a large enough constant $C > 0$) then the initialization step provides us better distance, i.e., dist($U, Q_0$) ≤ $C'/\sqrt{p}$ rather than dist($U, Q_0$) ≤ $C'/\sqrt{k^2 p}$ bound if $r = k$. This initialization step enables us to give tighter sample complexity as the $r\sqrt{p}$ in the numerator above can be replaced by $\sqrt{r p}$.

### 5 Experiments

In this section, we show that, as predicted by our theoretical results, our algorithm performs close to the optimal batch SVD. We provide the results from simulating the spiked covariance model, and demonstrate the phase-transition in the probability of successful recovery that is inherent to the statistical problem. Then we stray from the analyzed model and performance metric and test our algorithm on real world–and some very big–datasets, using the metric of explained variance.

In the experiments for Figures 1 (a)-(b), we draw data from the generative model of
Our results are averaged over at least 200 independent runs. Algorithm 1 uses the block size prescribed in Theorem 3, with the empirically tuned constant of 0.2. As expected, our algorithm exhibits linear scaling with respect to the ambient dimension $p$ – the same as the batch SVD. The missing point on batch SVD’s curve (Figure 1(a)), corresponds to $p > 2.4 \cdot 10^4$. Performing SVD on a dense $p \times p$ matrix, either fails or takes a very long time on most modern desktop computers; in contrast, our streaming algorithm easily runs on this size problem. The phase transition plot in Figure 1(b) shows the empirical sample complexity on a large class of problems and corroborates the scaling with respect to the noise variance we obtain theoretically.

Figures 1(c)-(d) complement our complete treatment of the spiked covariance model, with some out-of-model experiments. We used three bag-of-words datasets from [14]. We evaluated our algorithm’s performance with respect to the fraction of explained variance metric: given the $p \times k$ matrix $V$ output from the algorithm, and all the provided samples in matrix $X$, the fraction of explained variance is defined as $\frac{\text{Tr}(V^T XX^T V)}{\text{Tr}(XX^T)}$. To be consistent with our theory, for a dataset of $n$ samples of dimension $p$, we set the number of blocks to be $T = \lceil \log(p) \rceil$ and the size of blocks to $B = \lfloor n/T \rfloor$ in our algorithm. The NIPS dataset is the smallest, with 1500 documents and 12K words and allowed us to compare our algorithm with the optimal, batch SVD. We had the two algorithms work on the document space ($p = 1500$) and report the results in Figure 1(c). The dashed line represents the optimal using $B$ samples. The figure is consistent with our theoretical result: our algorithm performs as well as the batch, with an added $\log(p)$ factor in the sample complexity.

Finally, in Figure 1(d), we show our algorithm’s ability to tackle very large problems. Both the NY Times and PubMed datasets are of prohibitive size for traditional batch methods – the latter including 8.2 million documents on a vocabulary of 141 thousand words – so we just report the performance of Algorithm 1. It was able to extract the top 7 components for each dataset in a few hours on a desktop computer. A second pass was made on the data to evaluate the results, and we saw 7-10 percent of the variance explained on spaces with $p > 10^4$. 

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References

[1] R. Arora, A. Cotter, K. Livescu, and N. Srebro. Stochastic optimization for PCA and PLS. In 50th Allerton Conference on Communication, Control, and Computing, Monticello, IL, 2012.

[2] L. Balzano, R. Nowak, and B. Recht. Online identification and tracking of subspaces from highly incomplete information. In Communication, Control, and Computing (Allerton), 2010 48th Annual Allerton Conference on, page 704–711, 2010.

[3] M. Brand. Fast low-rank modifications of the thin singular value decomposition. Linear algebra and its applications, 415(1):20–30, 2006.

[4] Matthew Brand. Incremental singular value decomposition of uncertain data with missing values. Computer Vision—ECCV 2002, page 707–720, 2002.

[5] Kenneth L. Clarkson and David P. Woodruff. Numerical linear algebra in the streaming model. In Proceedings of the 41st annual ACM symposium on Theory of computing, page 205–214, 2009.

[6] P. Comon and G. H. Golub. Tracking a few extreme singular values and vectors in signal processing. Proceedings of the IEEE, 78(8):1327–1343, 1990.

[7] Gene H. Golub and Charles F. Van Loan. Matrix computations, volume 3. JHUP, 2012.

[8] Nathan Halko, Per-Gunnar Martinsson, and Joel A. Tropp. Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. SIAM review, 53(2):217–288, 2011.

[9] J. He, L. Balzano, and J. Lui. Online robust subspace tracking from partial information. arXiv preprint arXiv:1109.3827, 2011.

[10] Mark Herbster and Manfred K. Warmuth. Tracking the best linear predictor. The Journal of Machine Learning Research, 1:281–309, 2001.

[11] Iain M. Johnstone. On the distribution of the largest eigenvalue in principal components analysis.(english. Ann. Statist, 29(2):295–327, 2001.

[12] Y. Li. On incremental and robust subspace learning. Pattern recognition, 37(7):1509–1518, 2004.

[13] Boaz Nadler. Finite sample approximation results for principal component analysis: a matrix perturbation approach. The Annals of Statistics, page 2791–2817, 2008.

[14] Ian Porteous, David Newman, Alexander Ihler, Arthur Asuncion, Padhraic Smyth, and Max Welling. Fast collapsed gibbs sampling for latent dirichlet allocation. In Proceedings of the 14th ACM SIGKDD international conference on Knowledge discovery and data mining, page 569–577, 2008.
[15] Herbert Robbins and Sutton Monro. A stochastic approximation method. *The Annals of Mathematical Statistics*, page 400–407, 1951.

[16] Sam Roweis. EM algorithms for PCA and SPCA. *Advances in neural information processing systems*, page 626–632, 1998.

[17] Mark Rudelson and Roman Vershynin. Smallest singular value of a random rectangular matrix. *Communications on Pure and Applied Mathematics*, 62(12):1707–1739, 2009.

[18] Michael E. Tipping and Christopher M. Bishop. Probabilistic principal component analysis. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 61(3):611–622, 1999.

[19] R. Vershynin. How close is the sample covariance matrix to the actual covariance matrix? *Journal of Theoretical Probability*, page 1–32, 2010.

[20] Roman Vershynin. Introduction to the non-asymptotic analysis of random matrices. *arXiv preprint arXiv:1011.3027*, 2010.

[21] Manfred K. Warmuth and Dima Kuzmin. Randomized online PCA algorithms with regret bounds that are logarithmic in the dimension. *Journal of Machine Learning Research*, 9:2287–2320, 2008.
A Lemmas from Section 4.1

We first give the statement of all the Lemmas whose proofs we omitted in the body of the paper. Then we provide some results from the literature – what we call Preliminaries – and then we prove Theorem 3 and the supporting lemmas.

Lemma 4. Let $B$, $T$ and the data stream $\{x_t\}$ be as defined in Theorem 1. Then, w.p. $1 - C/T$ we have:

$$\left\| \frac{1}{B} \sum_t x_t x_t^\top - uu^\top - \sigma^2 I \right\|_2 \leq \epsilon.$$

Lemma 5. Let $B$, $T$ and the data stream $\{x_t\}$ be as defined in Theorem 1. Then, w.p. $1 - C/T$ we have:

$$u^\top s_{\tau+1} \geq u^\top q_{\tau}(1 + \sigma^2) \left(1 - \frac{\epsilon}{4(1 + \sigma^2)}\right),$$

where $s_t = \frac{1}{B} \sum_{B\tau < t \leq B(\tau+1)} x_t x_t^\top q_t$.

Lemma 6. Let $q_0$ be the initial guess for $u$, given by Steps 1 and 2 of Algorithm 1. Then, w.p. 0.99: $|\langle q_0, u \rangle| \geq C_0 \sqrt{p}$, where $C_0 > 0$ is a universal constant.

Lemma 7. If for any $\tau \geq 0$ and $0 < \gamma < 1$, we have $\delta_{\tau+1} \leq \frac{\gamma^2 \delta_0}{1 - \delta_0 + \gamma^2 \delta_0}$, then,

$$\delta_{\tau+1} \leq \frac{\gamma^{2\tau+2} \delta_0}{1 - (1 - \gamma^{2\tau+2}) \delta_0}.$$

B Lemmas from Section 4.2

Lemma 8. Let $X, A, B, F_{\tau+1}$, and $T$ be as defined in Theorem 3. Also, let $\sigma$ be the variance of noise, $F_{\tau+1} = \frac{1}{B} \sum_{B\tau < t \leq B(\tau+1)} x_t x_t^\top$, and $Q_{\tau}$ be the $\tau$-th iterate of Algorithm 1. Then, $\forall v \in \mathbb{R}^k$ and $\|v\|_2 = 1$, w.p. $1 - 5C/T$ we have:

$$\|U^\top F_{\tau+1} Q_{\tau} v\|_2 \geq (\lambda_k^2 + \sigma^2 - \frac{\lambda_k^2 \epsilon}{4}) \sqrt{1 - \|U_\perp^\top Q_{\tau}\|_2^2}.$$

Lemma 9. Let $X, A, B, F_{\tau+1}$, $Q_\tau$ be as defined in Lemma 8. Then, w.p. $1 - 4C/T$, $\|U^\top F_{\tau+1} Q_{\tau}\|_2 \leq \sigma^2 \|U_\perp^\top Q_{\tau}\|_2 + \lambda_k^2 \epsilon/2$.

Lemma 10. Let $Q_0 \in \mathbb{R}^{p \times k}$ be sampled uniformly at random from the set of all $k$-dimensional subspaces (see Initialization Steps of Algorithm 1). Then, w.p. at least 0.99: $\sigma_k(U^\top Q_0) \geq C \sqrt{\frac{p}{kp}}$, where $C > 0$ is a global constant.
C Preliminaries

Lemma 11 (Lemma 5.4 of [20]). Let $A$ be a symmetric $k \times k$ matrix, and let $N_\epsilon$ be an $\epsilon$-net of $S^{k-1}$ for some $\epsilon \in [0, 1)$. Then,

$$\|A\|_2 \leq \frac{1}{(1 - 2\epsilon)} \sup_{x \in N_\epsilon} |\langle Ax, x \rangle|.$$

Lemma 12 (Proposition 2.1 of [19]). Consider independent random vectors $x_1, \ldots, x_n$ in $\mathbb{R}^p$, $n \geq p$, which have sub-Gaussian distribution with parameter 1. Then for every $\delta > 0$ with probability at least $1 - \delta$ one has,

$$\|\frac{1}{n} \sum_{i=1}^n x_i x_i^T - \mathbb{E}[x_i x_i^T]\|_2 \leq C \sqrt{\log(2/\delta)} \sqrt{\frac{p}{n}}.$$

Lemma 13 (Corollary 3.5 of [20]). Let $A$ be an $N \times n$ matrix whose entries are independent standard normal random variables. Then for every $t \geq 0$, with probability at least $1 - 2 \exp(-t^2/2)$ one has,

$$\sqrt{N} - \sqrt{n} - t \leq \sigma_k(A) \leq \sqrt{N} + \sqrt{n} + t.$$

Lemma 14 (Theorem 1.2 of [17]). Let $\zeta_1, \ldots, \zeta_n$ be independent centered real random variables with variances at least 1 and subgaussian moments bounded by $B$. Let $A$ be an $k \times k$ matrix whose rows are independent copies of the random vector $(\zeta_1, \ldots, \zeta_n)$. Then for every $\epsilon \geq 0$ one has

$$\Pr(\sigma_{\min}(A) \leq \epsilon/\sqrt{k}) \leq C \epsilon + c n,$$

where $C > 0$ and $c \in (0, 1)$ depend only on $B$. Note that $B = 1$ for the standard Gaussian variables.

Lemma 15. Let $x_i \in \mathbb{R}^m, 1 \leq i \leq B$ be i.i.d. standard multivariate normal variables. Also, $y_i \in \mathbb{R}^n$ are also i.i.d. normal variables and are independent of $x_i, \forall i$. Then, w.p. $1 - \delta$,

$$\left\|\frac{1}{B} \sum_{i=1}^B x_i y_i^T\right\|_2 \leq \sqrt{\frac{C \max(m, n) \log(2/\delta)}{B}}.$$

Proof. Let $M = \sum_i x_i y_i^T$ and let $m > n$. Then, the goal is to show that, the following holds w.p. $1 - \delta$:

$$\frac{1}{B} \|Mv\|_2 \leq \sqrt{\frac{Cm \log(2/\delta)}{B}} \text{ for all } v \in \mathbb{R}^n \text{ s.t. } \|v\|_2 = 1.$$

We prove the lemma by first showing that the above mentioned result holds for any fixed vector $v$ and then use standard epsilon-net argument to prove it for all $v$.

Let $N$ be the 1/4-net of $S^{n-1}$. Then, using Lemma 5.4 of [20] (see Lemma 11),

$$\left\|\frac{1}{Bm} M^T M\right\|_2 \leq 2 \max_{v \in N} \frac{1}{Bm} \|Mv\|_2^2. \quad (8)$$

Now, for any fixed $v$: $Mv = \sum_i x_i y_i^T v = \sum_i x_i c_i$, where $c_i = y_i^T v \sim N(0, 1)$. Hence,

$$\|Mv\|_2^2 = \sum_{\ell=1}^m \left(\sum_{i=1}^B x_{i\ell} c_i\right)^2.$$
Now, $\sum_{i=1}^B x_i c_i \sim N(0, \|c\|_2^2)$ where $c^T = [c_1 \ c_2 \cdots c_B]$. Hence, $\sum_{i=1}^B x_i c_i = \|c\|_2 h_\ell$ where $h_\ell \sim N(0,1)$.

Therefore, $\|Mv\|_2^2 = \|c\|_2^2 \|h\|_2^2$ where $h^T = [h_1 \ h_2 \cdots h_B]$. Now,

$$
Pr\left(\frac{\|c\|_2^2 \|h\|_2^2}{Bm} \geq 1 + \gamma\right) \leq Pr\left(\frac{\|c\|_2^2}{B} \geq \sqrt{1 + \gamma}\right) + Pr\left(\frac{\|h\|_2^2}{m} \geq \sqrt{1 + \gamma}\right)
$$

\[ \leq 2 \exp\left(-\frac{B \gamma^2}{32}\right) + 2 \exp\left(-\frac{m \gamma^2}{32}\right) \leq 4 \exp\left(-\frac{m \gamma^2}{32}\right), \quad (9) \]

where $0 < \gamma < 3$ and $\zeta_1$ follows from Lemma 13.

Using (8), (9), the following holds with probability $(1 - 9^{a+1} e^{-\frac{m \gamma^2}{32}})$:

$$
\frac{\|M\|_2^2}{Bm} \leq 1 + 2 \gamma.
$$

The result now follows by setting $\gamma$ appropriately and assuming $n < Cm$ for small enough $C$. \qed

### D Proof of Theorem 3

Recall that our algorithm proceeds in a blockwise manner; for each block of samples, we compute

$$S_{\tau+1} = \left(\frac{1}{B} \sum_{t=B(\tau+1)}^{B(\tau+1)+1} x_t x_t^\top\right) Q_\tau, \quad (11)$$

where $Q_\tau \in \mathbb{R}^{p \times k}$ is the $\tau$-th block iterate and is an orthogonal matrix, i.e., $Q_\tau^T Q_\tau = I_{k \times k}$. Given $S_{\tau+1}$, the next iterate, $Q_{\tau+1}$, is computed by the QR-decomposition of $S_{\tau+1}$. That is,

$$Q_{\tau+1} = Q_{\tau+1} R_{\tau+1}, \quad (12)$$

where $R_{\tau+1} \in \mathbb{R}^{k \times k}$ is an upper-triangular matrix.

**Proof.** By using update for $Q_{\tau+1}$ (see (11), (12)):

$$Q_{\tau+1} R_{\tau+1} = F_{\tau+1} Q_\tau, \quad (13)$$

where $F_{\tau+1} = \frac{1}{B} \sum_{B \tau < t \leq B(\tau+1)} x_t x_t^\top$. That is,

$$U_{\perp}^T Q_{\tau+1} R_{\tau+1} v = U_{\perp}^T F_{\tau+1} Q_\tau v, \quad \forall v \in \mathbb{R}^k, \quad (14)$$

where $U_{\perp}$ is an orthogonal basis of the subspace orthogonal to span$(U)$. Now, let $v_1$ be the singular vector corresponding to the largest singular value, then:

$$
\|U_{\perp}^T Q_{\tau+1} v_1\|_2^2 = \frac{\|U_{\perp}^T Q_{\tau+1} v_1\|_2^2}{\|v_1\|_2^2} = \frac{\|U_{\perp}^T Q_{\tau+1} R_{\tau+1} v_1\|_2^2}{\|R_{\tau+1} v_1\|_2^2}
$$

\[ \leq \frac{\|U_{\perp}^T Q_{\tau+1} R_{\tau+1} v_1\|_2^2}{\|U_{\perp}^T Q_{\tau+1} v_1\|_2^2 + \|U_{\perp}^T Q_{\tau+1} R_{\tau+1} v_1\|_2^2} \]

\[ \leq \frac{\|U_{\perp}^T F_{\tau+1} Q_\tau v_1\|_2^2}{\|U_{\perp}^T F_{\tau+1} Q_\tau v_1\|_2^2 + \|U_{\perp}^T F_{\tau+1} Q_\tau v_1\|_2^2} \quad (15) \]
where $\tilde{v}_1 = \frac{R_{\tau+1}^{-1}v_1}{\|R_{\tau+1}^{-1}v_1\|_2}$. (i) follows as $Q_{\tau+1}$ is an orthogonal matrix and $[U, U_\perp]$ form a complete orthogonal basis; (ii) follows by using $\|R_{\tau+1}^{-1}\|_2 \geq \|U^T Q_{\tau+1} R_{\tau+1}^{-1} \|_2 = \|U^T F_{\tau+1} Q_{\tau+1} \|_2 > 0$, where $\zeta_0$ is the singular vector of $R_{\tau+1}$ corresponding to its smallest singular value, $\sigma_k(R_{\tau+1})$.

Now, using (13) with Lemmas 8 and using the fact that $x/(x+c)$ is an increasing function of $x$, for all $x > 0$, we get (w.p. $\geq 1 - 2C/T$):

$$\|U^T_{\perp} Q_{\tau+1}\|_2^2 \leq \frac{(\sigma^2 \|U^T_{\perp} Q_{\tau}\|_2^2 + \lambda^2_k \epsilon/2)^2}{(\lambda_k^2 + \sigma^2 - \lambda_k^2 \epsilon)^2(1 - \|U^T_{\perp} Q_{\tau}\|_2^2) + (\sigma^2 \|U^T_{\perp} Q_{\tau}\|_2^2 + 0.5 \lambda^2_k \epsilon)^2}.$$

Now, assuming $\epsilon \leq \|U^T_{\perp} Q_{\tau}\|_2^2$, using the above equation and by using union bound, we get (w.p. $\geq 1 - 2C/T$):

$$\|U^T_{\perp} Q_{\tau+1}\|_2^2 \leq \frac{\gamma^2 \|U^T_{\perp} Q_{\tau}\|_2^2}{1 - \|U^T_{\perp} Q_{\tau}\|_2^2 + \gamma^2 \|U^T_{\perp} Q_{\tau}\|_2^2},$$

where $\gamma = \frac{\sigma^2 + \lambda^2_k \epsilon/2}{\sigma^2 + 3 \lambda^2_k \epsilon/4} < 1$ for $\lambda_k > 0$. Using Lemma 2 along with the above equation, we get (w.p. $\geq 1 - 2C/T$):

$$\|U^T_{\perp} Q_{\tau+1}\|_2^2 \leq \gamma^2 \frac{\|U^T_{\perp} Q_{\tau}\|_2^2}{1 - \|U^T_{\perp} Q_{\tau}\|_2^2}.$$

Now, using Lemma 10 we know that $\|U^T_{\perp} Q_{\tau}\|_2^2$ is at most $1/\Omega(1/(kp))$. Hence, for $T = O(\log(p/\epsilon)/\log(1/\gamma))$, we get: $\|U^T_{\perp} Q_{\tau}\|_2^2 \leq \epsilon$. Furthermore, we require $B$ (as mentioned in the Theorem) samples per block. Hence, the total sample complexity bound is given by $O(BT)$, concluding the proof.

**E  Proof of Lemma 4**

**Proof.** Note that,

$$\frac{1}{B} \sum_t x_t x_t^T - uu^T - \sigma^2 I = uu^T - \sigma^2 I + \frac{1}{B} \sum_t (z_t^2 - 1) + \frac{1}{B} \sum_t (w_t w_t^T - \sigma^2 I) + \frac{1}{B} \sum_t z_t w_t^T + \frac{1}{B} \sum_t z_t w_t^T. \quad (17)$$

We now individually bound each of the above given terms in the RHS. Using standard tail bounds for covariance estimation (see Lemma 12), we can bound the first two terms (w.p. $\geq 1 - 2C/T$):

$$\frac{1}{B} \left| \sum_t (z_t^2 - 1) \right| \leq \sqrt{\frac{C \log(T)}{B}},$$

$$\|\frac{1}{B} \sum_t (w_t w_t^T - \sigma^2 I)\|_2 \leq \sigma^2 \sqrt{\frac{C \log(T)}{B}}. \quad (18)$$

4
Similarly, using Lemma 15, we can bound the last two terms in (17) (w.p. 1 − 2C/T):

\[ \| \frac{1}{B} \sum_t z_t w_t u^\top \|_2 \leq \| \frac{1}{B} u \sum_t z_t w_t^\top \|_2 \leq \sigma \sqrt{\frac{C_1 p \log(T)}{B}}. \] (19)

The lemma now follows by using (17), (18), (19) along with B as given by Theorem 1.

F Proof of Lemma 5

Proof. Let \( q_\tau = \sqrt{1 - \delta_\tau} u + \sqrt{\delta_\tau} u^\perp_\tau \), where \( u^\perp_\tau \) is the component of \( q_\tau \) that is orthogonal to \( u \). Now,

\[
\begin{align*}
    u^\top s_{\tau+1} &= \frac{1}{B} \sum_t (u^\top x_t)(x^\top_t q_t) \\
    &= \frac{1}{B} \sum_t (z_t + u^\top w_t)(\sqrt{1 - \delta_\tau}(z_t + u^\top w_t) + \sqrt{\delta_\tau} w_t^\top u^\perp_\tau) \\
    &= \frac{\sqrt{1 - \delta_\tau}}{B} \sum_t (z_t + u^\top w_t)^2 + \frac{\sqrt{\delta_\tau}}{B} \sum_t (z_t + u^\top w_t) w_t^\top u^\perp_\tau.
\end{align*}
\] (20)

Now, the first term above is a summation of \( B \) i.i.d. chi-square variables and hence using standard results (see Lemma 13), w.p. \( 1 - 2C/T \):

\[
\begin{align*}
    \frac{1}{B} \sum_t (z_t + u^\top w_t)^2 \geq (1 + \sigma^2)(1 - \sqrt{\frac{C \log(2T)}{B}}).
\end{align*}
\] (21)

Also, \( w_t^\top u \) and \( w_t^\top u^\perp_\tau \) are independent random variables, as both \( w_t^\top u \), \( w_t^\top u^\perp_\tau \) are Gaussians and \( E[w_t^\top u^\perp_\tau u^\top w_t] = 0 \). Hence, using Lemma 13 the following holds with probability \( \geq 1 - 4C/T \):

\[
\begin{align*}
    \| \frac{1}{B} \sum_t (z_t + u^\top w_t) w_t^\top u^\perp_\tau \|_2 \leq \sigma \sqrt{1 + \sigma^2} \sqrt{\frac{C \log(T)}{B}} \leq \sigma \sqrt{1 + \sigma^2} \sqrt{\frac{C_1 p \log(T)}{B(1 - \delta_0)}} \sqrt{1 - \delta_\tau},
\end{align*}
\] (22)

where (i) follows by using inductive hypothesis (i.e., \( \sqrt{1 - \delta_\tau} > \sqrt{1 - \delta_{\tau-1}} \), induction step follows as we show that the error decreases at each step) and Lemma 6.

The lemma now follows by using (20), (21), (22) and by setting \( B, T \) appropriately.

G Proof of Lemma 6

Proof. Using standard tail bounds for Gaussians (see Lemma 13), \( \| q_0 \|_2 \leq 2\sqrt{p} \) with probability \( 1 - \exp(-C_1 p) \), where \( C_1 > 0 \) is a universal constant. Furthermore, \((\|q_0\|_2 q_0)^\top u \sim N(0, 1) \). Hence, there exists \( C_0 > 0 \), s.t., with probability 0.99, \( |(\|q_0\|_2 q_0)^\top u| \geq C_0 \). Hence, \( |q_0^\top u| \geq \frac{C_0}{2\sqrt{p}} \).
H Proof of Lemma 2

Proof. We prove the lemma using induction. The base case (for $\tau = 0$) follows trivially. Now, by the inductive hypothesis, $\delta_\tau \leq \frac{\gamma^{2t}\delta_0}{1-(1-\gamma^{2t})\delta_0}$. That is,

$$\frac{1}{\delta_\tau} \geq \frac{1 - (1 - \gamma^{2t})\delta_0}{\gamma^{2t}\delta_0}.$$ 

Finally, by assumption,

$$\delta_{\tau+1} \leq \frac{\gamma^2}{\delta_\tau} - (1 - \gamma^2) \leq \frac{\gamma^2}{1-(1-\gamma^{2t})\delta_0} - (1 - \gamma^2).$$

The lemma follows after simplification of the above given expression. \qed

I Proof of Lemma 8

Proof. Using the generative model (2), we get:

$$U^T F_{\tau+1} Q_\tau v = \Lambda \left( \frac{1}{B} \sum_t z_t z_t^T \right) \Lambda U^T Q_\tau v + \left( \frac{1}{B} \sum_t U^T w_t U^T U^T Q_\tau v \right) + \left( \frac{1}{B} \sum_t (\Lambda z_t + U^T w_t) w_t^T U^T Q_\tau v \right).$$

(23)

Note that in the equation and rest of the proof, $t$ varies from $B\tau < t \leq B(\tau + 1)$.

We now show that each of the five terms in the above given equation concentrate around their respective means. Also, let $y_t = U^T w_t$ and $y_t^\perp = U^T Q_\tau v$. Note that, $y_t \sim N(0, \sigma^2 I_{k \times k})$ and $y_t^\perp \sim N(0, \sigma^2 I_{(p-k) \times (p-k)})$.

(a): Consider the first term in (23). Using $\|Av\|_2 \leq \|A\|_2 \|v\|_2$ and the assumption that $\lambda_1 = 1$, we get: $\|\Lambda \left( \frac{1}{B} \sum_t z_t z_t^T \right) \Lambda U^T Q_\tau v\|_2 \leq \| \left( \frac{1}{B} \sum_t z_t z_t^T - I \right) \| U^T Q_\tau v \|_2$. Using Lemma 12 we get (w.p. $1 - C/T$):

$$\| \frac{1}{B} \sum_t z_t z_t^T - I \|_2 \leq \sqrt{\frac{C_1 k \log(T)}{B}}.$$ 

That is,

$$\|\Lambda \left( \frac{1}{B} \sum_t z_t z_t^T - I \right) \Lambda U^T Q_\tau v\|_2 \leq \sqrt{\frac{C_1 k \log(T)}{B}} \| U^T Q_\tau v \|_2.$$ 

(24)

(b): Similarly, the second term in (23) can be bounded as (w.p. $1 - C/T$):

$$\| \left( \frac{1}{B} \sum_t U^T w_t U - \sigma^2 I \right) \| U^T Q_\tau v \|_2 \leq \sigma^2 \sqrt{\frac{C_1 k \log(T)}{B}} \| U^T Q_\tau v \|_2.$$ 

(25)
(c): Now consider the third and the fourth term. Now $\mathbf{w}_t$ and $\mathbf{z}_t$ are independent 0-mean Gaussians, hence using Lemma 15, we get:  
$$ \left\lVert \frac{1}{B} \sum_t U^T \mathbf{w}_t \mathbf{z}_t^T \right\lVert_2 \leq \sigma \sqrt{\frac{C_1 k \log(T)}{B}}. $$
Hence, w.p. $1 - 2C/T$,
$$ \left\lVert \frac{1}{B} \sum_t \mathbf{z}_t^T \mathbf{w}_t U^T Q_r \mathbf{v} \right\lVert + \left\lVert \left( \frac{1}{B} \sum_t U^T \mathbf{w}_t \mathbf{z}_t \right) \Lambda U^T Q_r \mathbf{v} \right\lVert \leq 2\sigma \sqrt{\frac{C_1 k \log(T)}{B}} \left\lVert U^T Q_r \mathbf{v} \right\lVert_2. \tag{26} $$

(d): Finally, we consider the last term in (23). Note that, $(\Lambda \mathbf{z}_t + U^T \mathbf{w}_t) \sim N(0, D)$ where $D$ is a diagonal matrix with $D_{ii} = \lambda_i^2 + \sigma^2$. Also, $Q^T U_1 U_1^T \mathbf{w}_t \sim N(0, \sigma^2 I_{(p-k) \times (p-k)})$ and is independent of $(\Lambda \mathbf{z}_t + U^T \mathbf{w}_t)$ as $E[Q^T U_1 U_1^T \mathbf{w}_t \mathbf{w}_t^T U] = 0$; recall that for Gaussian RVs, covariance is zero iff RVs are independent. Hence, using Lemma 15, w.p. $1 - C/T$:
$$ \left\lVert \frac{1}{B} \sum_t (\Lambda \mathbf{z}_t + U^T \mathbf{w}_t) \mathbf{w}_t^T U_1 U_1^T Q_r \mathbf{v} \right\lVert_2 \leq \sqrt{1 + \sigma^2} \sqrt{\frac{C_1 k \log(T)}{B}}. \tag{27} $$

Now, using (23), (24), (25), (26) (w.p. $1 - 5C/T$)
$$ \left\lVert U^T F_{r+1} Q_r \mathbf{v} \right\lVert_2 \geq \left( \Lambda^2 + \sigma^2 I \right) U^T Q_r \mathbf{v} \right\lVert_2 - \sqrt{\frac{C_1 k \log(T)}{B} \left\lVert U^T Q_r \mathbf{v} \right\lVert_2 \left( 1 + \sigma^2 \right) \left\lVert U^T Q_r \mathbf{v} \right\lVert_2} \right). \tag{28} $$

Now, $\left\lVert U^T Q_r \mathbf{v} \right\lVert_2 \geq \sigma_k(U^T Q_r \mathbf{v})$. Next, by using the inductive hypothesis (i.e., $\sigma_k(U^T Q_r) \geq \sigma_k(U^T Q_{r-1})$), induction step follows as we show that the error decreases at each step) and Lemma 10, we have $\left\lVert U^T Q_r \mathbf{v} \right\lVert_2 \geq \sigma_k(U^T Q_0) \geq \sqrt{\frac{C_0}{N}}$ with probability $\geq 0.99$.

Also, $\left\lVert (\Lambda^2 + \sigma^2 I) U^T Q_r \mathbf{v} \right\lVert_2 \geq (\lambda_k^2 + \sigma^2) \left\lVert U^T Q_r \mathbf{v} \right\lVert_2$. Additionally, $\left\lVert U^T Q_r \mathbf{v} \right\lVert_2 \geq \sqrt{1 - \left\lVert U^T Q_r \mathbf{v} \right\lVert_2^2}$. Hence, lemma follows by using these facts with (28) and by selecting $B$ as given in Theorem 8.

\section*{J Proof of Lemma 9}

\textbf{Proof.} Similar to our proof for Lemma 8, we separate out the “error” or deviation terms in $\left\lVert U_1^T F_{r+1} Q_r \right\lVert_2$ and bound them using concentration bounds. Now,
$$ \left\lVert U_1^T F_{r+1} Q_r \mathbf{v} \right\lVert_2 = \left\lVert U_1^T (U \Lambda^2 U^T + \sigma^2 I + E_r) Q_r \mathbf{v} \right\lVert_2 \leq \left\lVert \sigma^2 U_1^T Q_r \mathbf{v} \right\lVert_2 + \left\lVert U_1^T E_r Q_r \mathbf{v} \right\lVert_2 \leq \sigma^2 \left\lVert U_1^T Q_r \mathbf{v} \right\lVert_2 + \left\lVert E_r \right\lVert_2, \tag{29} $$
where $E_r$ is the error matrix representing deviation of the estimate $F_{r+1}$ from its mean. That is,
$$ E = \frac{1}{B} \sum_t x_t x_t^T - U \Lambda^2 U^T - \sigma^2 I $$
$$ = U \Lambda \left( \frac{1}{B} \sum_t z_t z_t^T - I \right) \Lambda U^T + \left( \frac{1}{B} \sum_t w_t w_t^T - \sigma^2 I \right) $$
$$ + U \Lambda \frac{1}{B} \sum_t z_t w_t^T + \frac{1}{B} \sum_t w_t z_t^T A U. \tag{30} $$
Note that the above given four terms correspond to similar four terms in (23) and hence can be bounded in similar fashion. In particular, the following holds with probability $1 - 4C/T$:

$$\|E\|_2 \leq \sqrt{\frac{C_1 k \log(T)}{B}} + \sigma^2 \sqrt{\frac{C_1 p \log(T)}{B}} + 2\sigma \sqrt{\frac{C_1 p \log(T)}{B}} \leq \lambda_k^2 \epsilon/2, \quad (31)$$

where the second inequality follows by setting $B$ as required by Theorem 3. The lemma now follows using (29), (30), (31).

\[\square\]

**K Proof of Lemma 10**

*Proof.* Using Step 2 of Algorithm \(H = Q_0 R_0\). Let \(v_k\) be the singular vector of \(U^\top Q_0\) corresponding to the smallest singular value. Then,

$$\sigma_k(U^\top Q_0) = \frac{\|U^\top Q_0 R_0^{-1} v_k\|_2}{\|R_0^{-1} v_k\|_2} \geq \sigma_k(U^\top Q_0 R_0) \sigma_k(R_0^{-1}). \quad (32)$$

Now, \(\sigma_k(R_0^{-1}) = \frac{1}{\|R_0\|_2} = \frac{1}{\|Q_0 R_0\|_2} = \frac{1}{\|H\|_2}. \) Note that \(\|H\|_2\) is the spectral norm of a random matrix with i.i.d. Gaussian entries and hence can be easily bounded using standard results. In particular, using Lemma 13 we get: \(\|H\|_2 \leq C_1 \sqrt{p}\) w.p. \(\geq 1 - e^{-C_2 p}\), where \(C_1, C_2 > 0\) are global constants.

By Theorem 1.1 of [17] (see Lemma 14), w.p. \(\geq 0.99\), \(\sigma_k(U^\top Q_0 R_0) = \sigma_k(H) \geq C/\sqrt{k}\). The lemma now follows using the above two bounds with (32). \(\square\)