Efficient Algorithms for One-Dimensional \textit{k}-Center Problems

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Abstract. We present an $O(n \log n)$ time algorithm for the (weighted) \textit{k}-center problem of \textit{n} points on a real line. We show that the problem has an $\Omega(n \log n)$ time lower bound, and thus our algorithm is optimal. We also show that the problem is solvable in $O(n)$ time in certain special cases. Our techniques involve developing efficient data structures for processing queries that find a lowest point in the common intersection of a certain subset of half-planes. This subproblem is interesting in its own right and our solution for it may find other applications.

1 Introduction

We study the weighted \textit{k}-center problem for a set of \textit{n} points on a real line. Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of \textit{n} points on a real line $L$. For each $i$ with $1 \leq i \leq n$, the point $p_i \in P$ has a weight $w(p_i) \geq 0$. For a point $p$ on $L$, denote by $L(p)$ the coordinate of $p$ on $L$, which we also refer to as the $L$-coordinate of $p$. For two points $p$ and $q$ on $L$, let $d(p, q) = |L(p) - L(q)|$ be the distance between $p$ and $q$. Further, for a set $F = \{f_1, f_2, \ldots, f_k\}$ of points and a point $q$ on $L$, define $d(q, F) = d(F, q) = \min_{1 \leq j \leq k} d(q, f_j)$. Given $P$ and an integer $k > 0$, the weighted one-dimensional \textit{k}-center problem seeks to determine a set $F = \{f_1, f_2, \ldots, f_k\}$ of \textit{k} points on $L$ such that the value $\psi(P, F) = \max_{p_i \in P} (w(p_i) \cdot d(p_i, F))$ is minimized. We use $1DkCenter$ to denote this problem. Also, the points in $F$ are called centers, and the points in $P$ are called demand points.

The unweighted version of $1DkCenter$ is the case where all points have the same weight. If $F \subseteq P$ is required, then the case is called the discrete version.

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Although many variants of the $k$-center problem are NP-hard [1,3,12,15,17], some special cases are solvable in polynomial time. Megiddo and Tamir [17] presented an $O(n \log^2 n \log \log n)$ time algorithm for the unweighted $k$-center problem on a tree of $n$ nodes, and the running time can be reduced to $O(n \log^2 n)$ by applying Cole’s parametric search [7]. Later, Frederickson [11] gave a linear time algorithm for the unweighted $k$-center problem on a tree. Jeger and Kariv [14] gave an $O(kn \log n)$ time algorithm for the weighted $k$-center problem on a tree. For the weighted $k$-center problem on a real line (i.e., the problem 1D$k$Center), Bhattacharya and Shi [2] recently proposed an algorithm with a time bound linear in $n$ but exponential in $k$. In addition, the discrete weighted $k$-center problem on a tree is solvable in $O(n \log^2 n)$ time [18] and the discrete unweighted $k$-center problem on a tree is solvable in $O(n \log n)$ time [11]. Tamir [19] indicated that the discrete weighted 1D$k$Center can be solved in $O(n \log n)$ time by using the procedure given in the last section of [18]. Below, unless otherwise stated, 1D$k$Center always refers to the weighted version.

In this paper, we solve the 1D$k$Center problem in $O(n \log n)$ time, which is faster than the $O(kn \log n)$ time result [14] when they are applied to 1D$k$Center. Our result is also better than the solution in [2] for large $k$ since the running time in [2] is exponential in $k$. In fact, we can show that the problem has an $\Omega(n \log n)$ time lower bound, and therefore, our $O(n \log n)$ time algorithm is optimal. Further, if all points in $P$ are given sorted on $L$ and their weights are also sorted, we solve 1D$k$Center in $O(n + k^2 \log^2 n \frac{n \log \log n}{k \log 2 n})$ time, which is in favor of small $k$. For example, if $k = O(n^{1/2-\varepsilon})$ for any $\varepsilon > 0$ (which is true in many applications), our algorithm runs in $O(n)$ time.

In addition, our techniques also yield an efficient data structure for processing queries for finding a lowest point in the common intersection of a certain subset of half-planes, which we call the 2-D sublist LP queries. Since the 2-D sublist LP query is a basic geometric problem, our data structure may be interesting in its own right.

1.1 An Overview of Our Approach

We model the 1D$k$Center problem as a problem of approximating a set of weighted points by a step function in the plane [5]. Two algorithms were given in [5] for this point approximation problem with the time bounds of $O(n \log^2 n)$ and $O(n \log n + k^2 \log^2 n \frac{n \log \log n}{k \log 2 n})$, respectively. Consequently, the 1D$k$Center problem can be solved in $O(\min\{n \log^2 n, n \log n + k^2 \log^2 n \})$ time.

However, the 1D$k$Center problem has some special properties that allow us to develop faster solutions. Specifically, after the geometric transformations, a key component to solving the problem is the following 2-D sublist LP query problem: Given a set of $n$ upper half-planes, $H = \{h_1, h_2, \ldots, h_n\}$, in the plane, for each query $q(i, j)$ ($1 \leq i \leq j \leq n$), compute a lowest point $p^*$ in the common intersection of all half-planes in $H_{ij} = \{h_t \mid i \leq t \leq j\}$. A data structure was proposed in [5] for this problem, which can be built in $O(n \log n)$ time and answers each query in $O(\log^2 n)$ time. On the 1D$k$Center problem, we observe
that the input half-plane set $H$ has a special property that the intersections between the $x$-axis and the bounding lines of the half-planes are ordered from left to right according to the half-plane indices in $H$. Exploiting this special property and using the compact interval trees [13], we design a new data structure for this special case of the 2-D sublist LP queries, which can be built in $O(n \log n)$ time and can answer each query in $O(\log n)$ time. This new data structure allows us to solve the $1DkCenter$ problem in $O(n \log n)$ time. Since the 2-D sublist LP query problem is a very basic problem, our new data structure may find other applications as well.

We should mention that very recently, by using Cole's parametric search [7], Fournier and Vigneron [10] gave an $O(n \log n)$ time algorithm for the above point approximation problem [5]. Thus, by combining our problem modeling and the algorithm in [10], the $1DkCenter$ problem can be solved in $O(n \log n)$ time. However, as pointed out in [10], the parametric search approach in [7] is quite complicated and involves large constants, and thus the algorithm in [10] is mainly of theoretical interest. In contrast, our approach is much simpler and more practical.

In the following, we present the high-level scheme of our algorithm in Section 2. In Section 3, we model our problem as the 2-D sublist LP queries and present our data structure. Section 4 concludes the paper and discusses the lower bound of the problem $1DkCenter$.

For simplicity of discussion, we make a general position assumption that no two points in $P$ are at the same position on $L$. We also assume the weight of each point in $P$ is positive and finite. These assumptions are only for ease of exposition and our algorithms can be easily extended to the general case.

2 The Algorithmic Scheme

In this section, we discuss the high-level framework of our algorithm. As pointed out in [2], it is possible that there is more than one optimal solution for the $1DkCenter$ problem. Our algorithm focuses on finding one optimal solution.

2.1 Preliminaries

For any two points $p$ and $q$ on $L$ with $L(p) \leq L(q)$ (recall that $L(p)$ is the coordinate of $p$ on $L$, and similarly for $L(q)$), denote by $[p,q]$ the (closed) interval of $L$ between $p$ and $q$.

We first sort all points of $P$ from left to right on $L$. Without loss of generality (WLOG), let $\{p_1, p_2, \ldots, p_n\}$ be the sorted order in increasing coordinates on $L$. For any two points $p_i, p_j \in P$ with $i \leq j$, denote by $I(i,j)$ the interval $[p_i, p_j]$. Let $\psi^*$ be the value of $\psi(P,F)$ for an optimal solution of $1DkCenter$. Suppose $F$ is the center set in an optimal solution; for a demand point $p \in P$ and a center $f \in F$, if $(w(p) \cdot d(f,p)) \leq \psi^*$, then we say that $p$ can be served by $f$. It is easy to see that there is an optimal solution $F$ such that each center of $F$ is in $[p_1, p_n]$. Further, as discussed in [2], there is an optimal solution $F$ such that the points
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referred to as the weighted step function min-

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size of $P$

2.2 The Reduction to the Planar Weighted Point Approximation Problem

point approximation problem \cite{5} in the next subsection.

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We first review the planar weighted point approximation problem and then show our problem reduction.

For any two integers $i$ and $j$ with $1 \leq i \leq j \leq n$, denote by $P_{ij}$ the subset of points of $P$ in the interval $I(i, j)$, i.e., $P_{ij} = \{p_i, p_{i+1}, \ldots, p_j\}$ ($P_{ij} = \{p_i\}$ for $i = j$). Consider the following weighted 1-center problem: Find a single center (i.e., a point) $f$ in the interval $I(i, j)$ such that the value of $\psi(P_{ij}, f) = \max_{p \in P_{ij}} (w(p) \cdot d(f, p))$ is minimized. Let $\alpha(i, j)$ denote the minimum value of $\psi(P_{ij}, f)$ for this weighted 1-center problem.

For solving the $IDkCenter$ problem, our strategy is to determine $k - 1$ integers $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{k-1} \leq n$ such that the value of $\max\{\alpha(1, i_1), \alpha(i_1 + 1, i_2), \ldots, \alpha(i_{k-1} + 1, n)\}$ is minimized and this minimized value is $\psi^*$. Note that in the above formulation, for each value $\alpha(i, j)$, exact one center is determined in the interval $I(i, j)$. To solve this problem, we reduce it to a planar weighted point approximation problem \cite{5} in the next subsection.

2.2 The Reduction to the Planar Weighted Point Approximation Problem

We first review the planar weighted point approximation problem and then show our problem reduction.

Let $P' = \{p'_1, p'_2, \ldots, p'_n\}$ be a point set in the plane with $p'_i = (x_i, y_i)$, and each point $p'_i$ be associated with a weight $w(p'_i) \geq 0$. Assume the points in $P'$ are ordered increasingly by their $x$-coordinates. Suppose $g$ is a step function (i.e., a piecewise constant function, e.g., see Fig. 1) which we use to approximate the points of $P'$ (in other words, we fit the step function $g$ to the point set $P'$). The weighted vertical distance between any point $p'_i \in P'$ and $g$ is defined as $d_w(p'_i, g) = w(p'_i) \cdot |y_i - g(x_i)|$ (see Fig. 1). The approximation error of $g$, denoted by $\epsilon(P', g)$, is defined as $\max_{p'_i \in P'} d_w(p'_i, g)$. The size of $g$ is the number of its horizontal line segments. Given an integer $k > 0$, the point approximation problem seeks a step function $g$ to approximate the points of $P'$ such that the size of $g$ is at most $k$ and the error $\epsilon(P, g)$ is minimized. In \cite{4}, this problem is referred to as the weighted step function min-$\epsilon$ problem, denoted by WSF. Here we also use WSF to denote this problem.
We now show that the 1D$k$Center problem can be reduced to WSF. Consequently, WSF algorithms can be used to solve 1D$k$Center. Indeed, consider the demand point set $P = \{p_1, p_2, \ldots, p_n\}$ for 1D$k$Center with the points ordered increasingly by their coordinates on $L$. For each demand point $p_i \in P$, $1 \leq i \leq n$, we create a point $p_i' = (i, L(p_i))$ in a 2-D Euclidean plane $\mathbb{R}^2$ (i.e., the $x$-coordinate of $p_i'$ in $\mathbb{R}^2$ is the index $i$ and its $y$-coordinate $y_i'$ in $\mathbb{R}^2$ is the coordinate of $p_i$ on $L$), and let the weight of $p_i'$ be that of $p_i$ (i.e., $w(p_i') = w(p_i)$). Let $P'$ be the set of $n$ weighted points thus created in $\mathbb{R}^2$. The next lemma states the relation between 1D$k$Center and the reduced instance of WSF.

**Lemma 1.** An optimal solution $OPT_{P'}$ for WSF on $P'$ in $\mathbb{R}^2$ corresponds to an optimal solution $OPT_P$ for 1D$k$Center on $P$. Further, once having $OPT_{P'}$, $OPT_P$ can be obtained in $O(n)$ time.

**Proof:** For any two integers $i$ and $j$ with $1 \leq i \leq j \leq n$, let $P'_{ij} = \{p_i', p_{i+1}', \ldots, p_j'\}$ ($P_{ij}' = \{p_i'\}$ for $i = j$). Consider the following problem: Find a value $Y$ for one single horizontal line segment with $Y$ as its $y$-coordinate such that the value of $d_w(P_{ij}', Y) = \max_{p_i' \in P_{ij}'} w(p_i') \cdot |y_i - Y|$ is minimized (where $y_i = L(p_i)$). Let $\alpha'(i, j)$ denote the minimized value of $d_w(P_{ij}', Y)$.

Let $\epsilon^*$ be the approximation error of an optimal solution for WSF on $P'$. It is easy to see that computing an optimal solution for WSF on $P'$ is equivalent to determining $k - 1$ integers $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{k-1} \leq n$ such that the value of $\max\{\alpha'(1, i_1), \alpha'(i_1 + 1, i_2), \ldots, \alpha'(i_{k-1} + 1, n)\}$ is minimized and this minimized value is $\epsilon^*$. According to the way that we create the point set $P'$ from the demand point set $P$, each value $\alpha'(i, j)$ is exactly equal to the value $\alpha(i, j)$, which is the minimized value of $\psi(P_{ij}, f)$ for the weighted 1-center problem on the demand point subset $P_{ij}$ by determining the value of $f$. Further, we have shown that to find an optimal solution for 1D$k$Center on $P$, it suffices to determine $k - 1$ integers $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{k-1} \leq n$ such that the value of $\max\{\alpha(1, i_1), \alpha(i_1 + 1, i_2), \ldots, \alpha(i_{k-1} + 1, n)\}$ is minimized and the minimized value is $\psi^*$.

The above discussion shows that to find an optimal solution for 1D$k$Center on $P$, it suffices to find an optimal solution for WSF on $P'$; further, $\psi^* = \epsilon^*$. Given an optimal solution $OPT_{P'}$ for WSF on $P'$, below we show how to obtain an optimal solution $OPT_P$ for 1D$k$Center on $P$ from $OPT_{P'}$ in linear time.

Note that $OPT_{P'}$ is a step function with $k$ steps (i.e., horizontal line segments). Let $i_0 = 0$ and $i_k = n$. For each $1 \leq j \leq k$, suppose the $j$-th step of $OPT_{P'}$ has a $y$-coordinate $y^j$ and covers the points of $P'$ from $p_{i_{j-1} + 1}'$ to $p_{i_j}'$, i.e., for each point $p_t' \in P'$ with $i_{j-1} + 1 \leq t \leq i_j$, the vertical line through $p_t'$ intersects the $j$-th horizontal segment of $OPT_{P'}$. We obtain $OPT_P$ for 1D$k$Center on $P$ as follows. For each $1 \leq j \leq k$, the $j$-th center $f_j$ is put at the position $y^j$ on $L$ (i.e., $L(f_j) = y^j$), which serves the demand points of $P$ from $p_{i_{j-1} + 1}$ to $p_{i_j}$.

Thus, once $OPT_{P'}$ is available, $OPT_P$ can be obtained in $O(n)$ time. □

Based on Lemma 1 to compute a set $F$ of $k$ centers for $P$ to minimize the value $\psi(P, F)$, it suffices to solve the corresponding WSF problem on $P'$ and $k$. Specifically, after an optimal step function $g$ for $P'$ is obtained, each horizontal
segment of \( g \) defines a center on \( L \) whose coordinate is equal to the \( y \)-coordinate of that horizontal segment of \( g \) in \( \mathbb{R}^2 \).

To apply the WSF algorithms to the 1DkCenter problem, we need a data structure for answering queries \( q(i, j) = \alpha(i, j) \) with \( 1 \leq i \leq j \leq n \). Suppose such a data structure can be built in \( O(\pi(n)) \) time and can answer each query \( \alpha(i, j) \) in \( O(q(n)) \) time; then we say the time bounds of the data structure are \( O(\pi(n), q(n)) \). The two lemmas below follow from the results in [5].

**Lemma 2.** [5] Suppose there is a data structure for the queries \( \alpha(i, j) \) with time bounds \( O(\pi(n), q(n)) \); then the 1DkCenter problem is solvable in \( O(\pi(n) + n \cdot q(n)) \) time.

**Lemma 3.** [5] Suppose there is a data structure for the queries \( \alpha(i, j) \) with time bounds \( O(\pi(n), q(n)) \); then the 1DkCenter problem is solvable in \( O(\pi(n) + q(n) \cdot k^2 \log^2 \frac{n}{k}) \) time.

Refer to [5] for the details of the algorithms in the above two lemmas. By the above two lemmas, Lemma 4 follows.

**Lemma 4.** Suppose there is a data structure for the queries \( \alpha(i, j) \) with time bounds \( O(\pi(n), q(n)) \); then the 1DkCenter problem can be solved in \( O(\min\{\pi(n) + n \cdot q(n), \pi(n) + q(n) \cdot k^2 \log^2 \frac{n}{k}\}) \) time.

A data structure based on fractional cascading [4] was given in [5] for answering the queries \( \alpha(i, j) \) with time bounds \( O(n \log n, \log^2 n) \). Consequently, by Lemma 4 the 1DkCenter problem is solvable in \( O(\min\{n \log^2 n, n \log n + k^2 \log^2 \frac{n}{k} \log n\}) \) time. In Section 3, we develop a data structure for processing the queries \( \alpha(i, j) \) with time bounds \( O(n \log n, \log n) \), which allows us to solve 1DkCenter in \( O(n \log n) \) time.

The reason why we can solve the 1DkCenter problem faster than simply applying the WSF algorithms [5] is that the WSF instance constructed above from the problem 1DkCenter has a special property: The \( y \)-coordinates of the points \( p_1', p_2', \ldots, p_n' \) are increasing. As shown in Section 3, this special property allows us to design a new data structure for the \( \alpha(i, j) \) queries with time bounds \( O(n \log n, \log n) \). Note that this special property does not hold for the general WSF problem studied in [5].

### 3 The Data Structure for Computing \( \alpha(i, j) \)

In this section, we present a data structure with time bounds \( O(n \log n, \log n) \) for answering the \( \alpha(i, j) \) queries. In the following, we first model the problem of computing \( \alpha(i, j) \) as the problem of finding a lowest point in the common intersection of a set of half-planes (i.e., the 2-D sublist LP query).
Fig. 2. Illustrating the common intersection of the half-planes defined by three points $p_1$, $p_2$, and $p_3$. The point $p^*$ is the lowest point in the common intersection.

3.1 The Problem Modeling

Consider a point subset $P_{ij} \subseteq P$ with $i \leq j$. Recall that $L(p_i) \leq L(p_{i+1}) \leq \cdots \leq L(p_j)$. To compute $\alpha(i, j)$, we need to find a point $f$ such that the value $\psi(P_{ij}, f) = \max_{i \leq t \leq j} (w(p_i) \cdot d(f, p_t)) = \max_{i \leq t \leq j} (w(p_i) \cdot |L(f) - L(p_t)|)$ is minimized and $\alpha(i, j)$ is the minimized value. Consider an arbitrary point $f'$ on $L$. Since $\psi(P_{ij}, f') = \max_{i \leq t \leq j} (w(p_i) \cdot |L(f') - L(p_t)|)$, each point $p_t \in P_{ij}$ defines two constraints: $w(p_i) \cdot (L(f') - L(p_t)) \leq \psi(P_{ij}, f')$ and $-w(p_i) \cdot (x - L(p_t)) \leq \psi(P_{ij}, f')$.

Consider a 2-D $xy$-coordinate system with $L$ as the $x$-axis. For each point $p_t \in P$, the inequality $w(p_i) \cdot |x - L(p_t)| \leq y$ defines two (upper) half-planes: $w(p_i) \cdot (x - L(p_t)) \leq y$ and $-w(p_i) \cdot (x - L(p_t)) \leq y$. Note that the two lines bounding the two half-planes intersect at the point $p_t$ on $L$.

Based on the above discussion, if $p^* = (x^*, y^*)$ is a lowest point in the common intersection of the $2(i - j + 1)$ (upper) half-planes defined by the points in $P_{ij}$, then $\alpha(i, j) = y^*$ and $L(f) = x^*$ is the coordinate of an optimal center $f$ on $L$ for $P_{ij}$. Figure 2 shows an example in which each “cone” is the intersection of the two upper half-planes defined by a point in $P_{ij}$. Clearly, this is an instance of the 2-D linear programming (LP) problem, which is solvable in $O(j - i + 1)$ time [9, 10]. However, we can make the computation faster by preprocessing. Let $H_P = \{h_1, h_2, \ldots, h_{2n}\}$ be the set of $2n$ (upper) half-planes defined by the $n$ points in $P$, such that for each $1 \leq i \leq n$, the demand point $p_i$ defines $h_{2i-1}$ and $h_{2i}$. Then to compute $\alpha(i, j)$, it suffices to find the lowest point $p^*$ in the common intersection of the half-planes defined by the points in $P_{ij}$, i.e., the half-planes in $H_{2i-1,2j} = \{h_{2i-1}, h_{2i}, h_{2i+1}, h_{2i+2}, \ldots, h_{2j-1}, h_{2j}\}$.

We actually consider a more general problem: Given in the plane a set of $n$ upper half-planes $H = \{h_1, h_2, \ldots, h_n\}$, each query $q(i, j)$ with $1 \leq i \leq j \leq n$ asks for a lowest point $p^*$ in the common intersection of all half-planes in $H_{ij} = \{h_t \mid i \leq t \leq j\} \subseteq H$. We call this problem the 2-D sublist LP query. Based on our discussion above, if we solve the 2-D sublist LP query problem, then the $\alpha(i, j)$ queries for the 1D$k$Center problem can be processed as well in the same time bound. A data structure for the 2-D sublist LP query problem with time bounds $O(n \log n, \log^* n)$ was given in [5].

Yet, the 2-D sublist LP query problem for 1D$k$Center is special in the following sense. For each half-plane $h \in H_P$ for 1D$k$Center, we call the $x$-coordinate of the intersection point between $L$ (i.e., the $x$-axis) and the line bounding $h$ the $x$-intercept of $h$ (or its bounding line). As discussed above, for each point $p_t \in P$,
the $x$-intercepts of both the half-planes $h_{2i-1}$ and $h_{2i}$ defined by $p_i$ are exactly the point $p_i$. Since all points of $P$ are ordered along $L$ from left to right by their indices, a special property of $H_P$ is that the $x$-intercepts of all half-planes in $H_P$ are ordered from left to right on $L$ by the indices of the half-planes. For a set $H$ of half-planes for a 2-D sublist LP query problem instance, if $H$ has the above special property, then we say that $H$ is $x$-intercept ordered.

Below, we show that if $H$ is $x$-intercept ordered, then there is a data structure for the specific 2-D sublist LP query problem with time bounds $O(n \log n, \log n)$. Henceforth, we assume that $H$ is an $x$-intercept ordered half-plane set. In the 1D$k$Center problem, since all the point weights for $P$ are positive finite values, the bounding line of each half-plane in $H_P$ is neither horizontal nor vertical. Thus, we also assume that no bounding line of any half-plane in $H$ is horizontal or vertical. Again, this assumption is only for simplicity of discussion.

In the next section, we solve the 2-D sublist LP queries by reducing it to computing the convex hull of a query sub-path of a given simple path [13]. Given a simple path in the plane, based on compact interval trees, data structures are proposed in [13] to compute (in logarithmic time) the convex hull of a query subpath that is specified by the indices of the beginning vertex and the end vertex of the subpath, and the convex hull is represented (by a compact interval tree) such that standard convex hull queries on it can be done in $O(\log n)$ time.

### 3.2 Answering 2-D Sublist LP Queries

For each half-plane $h_i \in H$, we denote by $l(h_i)$ the bounding line of $h_i$; let $l(H')$ be the set of the bounding lines of the half-planes in any subset $H' \subseteq H$.

Our problem reduction utilizes a duality transformation [8], which is a technique commonly used in computational geometry, as follows. Suppose we have a primary plane $\mathcal{P}$. For each point $(a, b) \in \mathcal{P}$, it corresponds to a line $y = ax - b$ in the dual plane $\mathcal{D}$; the line is also called the dual of the point and vice versa. Similarly, each line $y = a'x - b'$ in $\mathcal{P}$ corresponds to a point $(a', b')$ in $\mathcal{D}$.

Suppose all half-planes in $H$ are in the primary plane $\mathcal{P}$. By duality, we can obtain a set $H^*$ of points in the dual plane $\mathcal{D}$ corresponding to the lines in $l(H)$. For each query $q(i, j)$ on $H$, our goal is to locate the lowest point $p^*$ in the common intersection of all half-planes in $H_{ij}$ (note that due to our assumption that no line in $l(H)$ is horizontal, there is only one lowest point in the common intersection). Since all half-planes in $H$ are upper half-planes, an observation is that $p^*$ is also the lowest point of the upper envelope of the arrangement of the lines in $l(H_{ij})$. Denote by $U_{ij}$ the above upper envelope for $l(H_{ij})$. Denote by $H_{ij}^*$ the set of points (in the dual plane $\mathcal{D}$) dual to the lines in $l(H_{ij})$. Let $C_{ij}^*$ denote the lower hull of the convex hull of $H_{ij}^*$. It is commonly known [8] that the dual of $U_{ij}$ is exactly $C_{ij}^*$ (in the dual plane $\mathcal{D}$). Therefore, if we have a representation of $C_{ij}^*$ that can support standard binary-search-based queries, then we can compute the lowest point $p^*$ in logarithmic time accordingly.

One may attempt to design a data structure for querying the lower hull on any subset $H_{ij}^*$ of $H^*$. However, there are difficulties when doing so “directly”, which will be explained later. Instead, we use an “indirect” approach, as follows.
Recall that we have assumed the bounding line of any half-plane in \( H \) is not vertical or horizontal. We partition the half-plane set \( H \) into two subsets \( H_1 \) and \( H_2 \) such that a half-plane \( h_i \) of \( H \) is in \( H_1 \) (resp., \( H_2 \)) if and only if the slope of \( l(h_i) \) is negative (resp., positive). Accordingly, for each subset \( H_{ij} \), we also have \( H^1_{ij} \) and \( H^2_{ij} \), and we define the envelopes \( U^1_{ij} \) and \( U^2_{ij} \) accordingly. Since the bounding lines of all half-planes in \( H^1_{ij} \) have negative slopes, the upper envelope \( U^1_{ij} \) is monotone decreasing from left to right. Similarly, the upper envelope \( U^2_{ij} \) is monotone increasing from left to right. Hence, it is easy to see that the lowest point \( p^* \) is the single intersection of \( U^1_{ij} \) and \( U^2_{ij} \). Let \( H^1_{ij} \) and \( H^2_{ij} \) be the sets of points in the dual plane corresponding to the lines in \( l(H^1_{ij}) \) and \( l(H^2_{ij}) \), respectively. Denote by \( C^1_{ij} \) and \( C^2_{ij} \) the lower hulls of \( H^1_{ij} \) and \( H^2_{ij} \), respectively. By duality, the intersection of \( U^1_{ij} \) and \( U^2_{ij} \) corresponds exactly to the common tangent line of the two lower hulls \( C^1_{ij} \) and \( C^2_{ij} \) such that both hulls are above the tangent line. Note that since all lines in \( H^1_{ij} \) have negative slopes, by duality, all points in \( H^1_{ij} \) are to the left of the \( y \)-axis in the dual plane, and thus the lower hull \( C^1_{ij} \) is to the left of the \( y \)-axis. Similarly, the lower hull \( C^2_{ij} \) is to the right of the \( y \)-axis. Namely, the two lower hulls \( C^1_{ij} \) and \( C^2_{ij} \) are on different sides of the \( y \)-axis. This property can make our computation of their tangent line easier.

In summary, if we can represent both \( C^1_{ij} \) and \( C^2_{ij} \) in such a way that the common tangent line can be found efficiently, then \( p^* \) can be obtained immediately. Our remaining task is to derive a way to support convex hull (or lower hull) queries on any subset of consecutive points in \( H^1_{ij} \) (and similarly on \( H^2_{ij} \)). Our result is that a data structure can be built in \( O(n \log n) \) time and is represented in a way that supports binary-search-based queries (e.g., compute the common tangent of it and another lower hull, say \( C^2_{ij} \)). The details are given below.

Without loss of generality, we assume the bounding lines of the half-planes in \( H \) all have negative slopes (i.e., \( H = H_1 \)) and the other case can be handled analogously. Suppose the bounding line of each \( h_i \in H \) corresponds to the point \( h_i^* \in H^* \) in the dual plane \( D \). Let \( \gamma \) be the path by connecting all pairs of two consecutive points in \( H^* \) by line segments, i.e., connecting \( h_i^* \) to \( h_{i+1}^* \) for \( i = 1, 2, \ldots, n - 1 \). Consider the line segment connecting \( h_i^* \) and \( h_{i+1}^* \) and the line segment connecting \( h_j^* \) and \( h_{j+1}^* \); then the two segments are adjacent to each other if \( i + 1 = j \) or \( j + 1 = i \). The following Lemma 5 shows that the path \( \gamma \) is a simple path, that is, any two line segments of \( \gamma \) that are not adjacent do not intersect. As can be seen from the proof of Lemma 5, we note that the correctness of Lemma 5 heavily relies on two properties of \( H \): (1) \( H \) is \( x \)-intercept ordered; (2) the slopes of the bounding lines of the half-planes in \( H \) are all negative (or positive). Without either property above, the lemma would not hold, and the second property also explains why we need to partition the original set \( H \) into \( H_1 \) and \( H_2 \).

**Lemma 5.** The path \( \gamma \) is a simple path.
Fig. 3. Illustrating the two double edges $dw(s_i)$ and $dw(s_j)$ (the shaded regions) in the primary plane $P$ corresponding to the two segments $s_i$ and $s_j$ in the dual plane $D$. The two segments $x_i$ and $x_j$ are their intersections with the $x$-axis.

Proof: Consider two segments $s_i$ and $s_j$ where $s_i$ connects $h^*_i$ and $h^*_i+1$ and $s_j$ connects $h^*_j$ and $h^*_j+1$. Suppose $s_i$ and $s_j$ are not adjacent. To prove the lemma, it is sufficient to show $s_i$ and $s_j$ does not intersect. Since $s_i$ and $s_j$ are not adjacent, either $i+1 < j$ or $j+1 < i$. Without loss of generality, assume $i+1 < j$.

Note that $s_i$ and $s_j$ are in the dual plane $D$. It is commonly known [8] that the dual of $s_i$ in the primary plane $P$ is the double wedge bounded by the lines $l(h_i)$ and $l(h_{i+1})$ such that the double wedge does not contain a vertical line (e.g., see Fig. 3); we denote the double wedge by $dw(s_i)$. Similarly, the dual of $s_j$ is the double wedge $dw(s_j)$ bounded by $l(h_j)$ and $l(h_{j+1})$.

Assume to the contrary the two segments $s_i$ and $s_j$ intersect each other, say, at a point $p$. Then, $p$ corresponds to a line in the primary plane $P$, which is in the common intersection of the two double wedges $dw(s_i)$ and $dw(s_j)$. However, we claim that the common intersection of $dw(s_i)$ and $dw(s_j)$ does not contain any line, which incurs contradiction. Hence, $s_i$ cannot intersect $s_j$ and the lemma follows. Below, we prove the above claim.

Suppose to the contrary there is a line $l$ contained in $dw(s_i) \cap dw(s_j)$. Denote by $x_i$ (resp., $x_j$) the intersection of the $x$-axis and $dw(s_i)$ (resp., $dw(s_j)$), e.g., see Fig. 3. Hence, both $x_i$ and $x_j$ are line segments on the $x$-axis. Since the intersection of $l(h_j)$ and the $x$-axis is strictly to the right of the intersection of $l(h_{i+1})$ and the $x$-axis (since $H$ is $x$-intercept ordered), $x_i$ does not intersect $x_j$. Since the slopes of both $l(h_i)$ and $l(h_{i+1})$ are negative, a line contained in $dw(s_i)$ must intersect $x_i$, and thus the line $l$ intersects $x_i$ (due to $l \subseteq dw(s_i) \cap dw(s_j)$).

Similarly, $l$ also intersects $x_j$. Hence, we obtain that $l$ intersects both $x_i$ and $x_j$. Since both $x_i$ and $x_j$ lie in $x$-axis and $x_i$ does not intersect $x_j$, the line $l$ has to be the $x$-axis. However, since the slopes of both $l(h_i)$ and $l(h_{i+1})$ are negative, the double wedge $dw(s_i)$ cannot contain the $x$-axis and thus cannot contain $l$. Therefore, we obtain contradiction and the claim follows. □

In light of Lemma 5 we can utilize the results in [13]. Given a simple path in the plane, compact interval tree data structures are proposed in [13] to compute the convex hull of a query subpath that is specified by the indices of the beginning vertex and the end vertex of the subpath. If applied to $\gamma$ in our problem, then after spending $O(n \log n)$ time sorting the points in $H^*$ by their $x$-coordinates, we have the following results: A data structure can be constructed in $O(n \log \log n)$
time that can compute the lower hull $C_{ij}^*$ for any query $q(i, j)$ in $O(\log n)$ time and $C_{ij}^*$ is represented (by a compact interval tree) such that any standard convex hull queries on $C_{ij}^*$ can be done in $O(\log n)$ time, where the standard convex hull queries includes point-in-polygon tests, finding intersections with lines, finding tangents through query points, finding extreme vertices in a query direction, detecting intersections of two polygons, and finding common tangents of two convex hulls. Further, another data structure of construction time $O(n)$ and query time $O(\log n \log \log n)$ is also given in [13]; in addition, there is also a data structure of construction time $O(n \log^* n)$ and query time $O(\log n \log^* n)$ by making trade-off between the construction and query [13]. Both data structures are applicable to our problem.

By duality, the $x$-coordinate of each point $h_i^* \in H^*$ corresponds to the slope of the line $l(h_i)$. Thus, a sorted order of the points in $H^*$ by $x$-coordinate corresponds to a sorted order of the bounding lines of the half-planes in $H$ by slope.

The following lemma summarizes our discussions above.

**Lemma 6.** In $O(n \log n)$ time, we can build a data structure that can answer each 2-D sublist LP query in $O(\log n)$ time. Further, if the bounding lines of the half-planes in $H$ are sorted by their slopes, then there exist three data structures for the 2-D sublist LP queries, whose construction time complexities are $O(n)$, $O(n \log^* n)$, and $O(n \log \log n)$, respectively, and query time complexities are $O(\log n \log \log n)$, $O(\log n \log^* n)$, and $O(\log n)$, respectively.

According to the reduction procedure from the 1D$k$Center problem to the planar points approximation problem [6], the slopes of the bounding lines of the half-planes in $H$ correspond to the weights of the points in the input point set $P$. Therefore, we have the following corollary.

**Corollary 1.** There exists a data structure of time complexity $O(n \log n, \log n)$ for the $\alpha(i, j)$ queries. Further, if the points of $P$ are sorted on the line $L$ and the weights of the points in $P$ are also sorted, then we have three data structures for the $\alpha(i, j)$ queries of time complexities $O(n, \log n \log \log n)$, $O(n \log^* n, \log n \log^* n)$, and $O(n \log \log n, \log n)$, respectively.

By Lemma 6 we have the following result for the 1D$k$Center problem.

**Theorem 1.** The 1D$k$Center problem is solvable in $O(n \log n)$ time. Further, if the points of $P$ are sorted on $L$ and the weights of the points in $P$ are also sorted, then the 1D$k$Center is solvable in $O(\min\{n + k^2 \log^2 \frac{n}{k} \log n \log \log n, n \log^* n + k^2 \log^2 \frac{n}{k} \log n \log^* n, n \log \log n + k^2 \log^2 \frac{n}{k} \log n\})$ time.

Therefore, if the points of $P$ are sorted on the line $L$ and the weights of the points in $P$ are also sorted, for small $k$ (e.g., $k = O((\log^* n \log \log n)^{1/2})$, which is true in many applications), the 1D$k$Center problem is solvable in $O(n)$ time.
4 Conclusion

In this paper, we give an $O(n \log n)$ time algorithm for the k-center problem on a real line. In certain special cases, we can solve the problem in linear time.

As suggested by Tamir [19], when $k = n - 1$, the discrete unweighted 1DkCenter is equivalent to the Min Gap problem, i.e., finding the closest pair of neighbors in $P$. Hence, there is an $O(n \log n)$ time lower bound on the discrete version. In fact, by the reduction from the Min Gap problem, we can also show that the continuous unweighted 1DkCenter also has an $O(n \log n)$ lower bound on the running time. Indeed, in any optimal solution OPT for $k = n - 1$, there must be a center at the middle position of the closest pair of neighbors in $P$, and that center serves both neighbors; further, any other center in OPT serves one and only one demand point in $P$. Therefore, given an optimal solution OPT, since the demand points served by each center are known, the two demand points served by the same center are the closest neighbors in $P$. We thus obtain the $O(n \log n)$ time lower bound on the continuous unweighted 1DkCenter.

Note that the linear time algorithms for the unweighted continuous/discrete k-center problem on trees [11] do not violate the $O(n \log n)$ time lower bound discussed above because the tree structure already gives a partial order of the nodes in the tree. An open problem is whether the techniques given in this paper can be extended to the tree structure.

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