Exact quantization of the XXZ spin chain embedded in a topological manifold

Yi Qiao,1,2 Pei Sun,1 Junpeng Cao,1,2,3,4,5 Wen-Li Yang,1,5,6 Kangjie Shi,1 and Yupeng Wang2,3

1Institute of Modern Physics, Northwest University, Xian 710127, China
2Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
3School of Physical Sciences, University of Chinese Academy of Sciences, Beijing, China
4Songshan Lake Materials Laboratory, Dongguan, Guangdong 523808, China
5Peng Huanan Center for Fundamental Theory, Xian 710127, China
6Shaanxi Key Laboratory for Theoretical Physics Frontiers, Xian 710127, China

A novel Bethe Ansatz scheme is proposed to deal with topological quantum integrable systems. As an example, the anti-periodic XXZ spin chain, a typical correlated many-body system embedded in a topological manifold, is examined. Conserved “momentum” and “charge” operators are constructed despite the absence of translational invariance and $U(1)$ symmetry. The ground state energy and elementary excitations are derived exactly. It is found that two intrinsic fractional (one half) zero modes exist in most of the eigenstates. The elementary excitations show quite a different picture from that in the periodic boundary case. This method can be applied to other quantum integrable models with nontrivial topology or boundary conditions.

PACS numbers: 75.10.Pq, 03.65.Vf, 71.10.Pm

Exact quantization in topological manifold is an important issue in modern physics [1]. It is related to several interesting research fields such as quantum topological phases in correlated many-body systems [2], exact quantization in Calabi-Yau manifold in the string theory [3], and many others. Recently, a research focus in topological phases is to take correlations into account to seek for new physical effects. Sometimes exact quantization procedure is finally attributed to solve some quantum integrable models [6]. However, those quantum integrable models possess nontrivial topology structure. A formidable problem to fulfill exact quantization in nontrivial topological manifold is the absence of $U(1)$ symmetry, which makes us frustrated to work in a traditional particle-hole representation. We remark that though some methods have been developed to approach quantum integrable models without $U(1)$ symmetry [7–13], including the off-diagonal Bethe Ansatz [14, 15] proposed by some of the present authors, with which the formal solutions of the eigenvalues can be expressed in an inhomogeneous $t–Q$ relation, exact quantum numbers and elementary excitations for those models are still unclear because of complicated distribution of Bethe roots associated with inhomogeneous Bethe Ansatz equations.

In this letter, we propose a novel Bethe Ansatz scheme for obtaining exact quantized spectrum of topological quantum integrable models. By constructing an operator identity of the transfer matrix for arbitrary spectral parameter $u$, factorized Bethe Ansatz equations (BAEs) about the zero roots of the transfer matrix can be derived. It is found that the root distribution in the complex plane possesses a simple structure, which allows us to define quantum numbers associated with zero roots of the transfer matrix and to calculate physical quantities such as ground state energy and exact elementary excitations (quasi particles) in the thermodynamic limit. A counterpart of momentum operator and a conserved charge (correspondence of $U(1)$ charge in the periodic boundary case) in the topological manifold are also defined to classify the eigenstates and elementary excitations.

To clarify our procedure clearly, we study the anti-periodic XXZ spin chain [16, 17] as a concrete example. The model Hamiltonian reads

$$H = -\sum_{n=1}^{N} (\sigma_{n}^{x} \sigma_{n+1}^{x} + \sigma_{n}^{y} \sigma_{n+1}^{y} + \cosh \eta \sigma_{n}^{z} \sigma_{n+1}^{z}),$$

with the topological boundary condition

$$\sigma_{1+N}^{\alpha} = \sigma_{1}^{\alpha} \sigma_{0}^{\alpha} \sigma_{1}^{\alpha}, \quad \text{for} \quad \alpha = x, y, z,$$

where $\sigma^{x}, \sigma^{y}, \sigma^{z}$ are the usual Pauli matrices and $\eta$ is the coupling constant. This nontrivial boundary condition mixes the spin up and spin down states in the Hilbert space and makes the system forming a quantum Möbius strip. The $U(1)$ symmetry is thus broken and a discrete $Z_{2}$ invariance $[H, U] = 0$ is left with $U = \prod_{j=1}^{N} \sigma_{j}^{x}$ and $U^{2} = 1$. In the following text, we put $\eta = i\gamma$ as an imaginary constant. The real $\eta$ case can be studied straightforwardly. The integrability of the model is associated with the well-known six-vertex $R$-matrix

$$R_{0,j}(u) = \frac{\sinh(u + \eta) + \sinh u}{2\sinh \eta} + \frac{1}{2}(\sigma_{j}^{x} \sigma_{0}^{x} + \sigma_{j}^{y} \sigma_{0}^{y})$$

$$+ \frac{\sinh(u + \eta) - \sinh u}{2\sinh \eta} \sigma_{j}^{z} \sigma_{0}^{z},$$

which satisfies the Yang-Baxter equation [18, 19], where $u$ is the spectral parameter.

Let us introduce the monodromy matrix

$$T_{0}(u) = R_{0,N}(u) \cdots R_{0,1}(u).$$

exactquantization of the XXZ spin chain embedded in a topological manifold

Yi Qiao,1,2 Pei Sun,1 Junpeng Cao,1,2,3,4,5 Wen-Li Yang,1,5,6 Kangjie Shi,1 and Yupeng Wang2,3

1Institute of Modern Physics, Northwest University, Xian 710127, China
2Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
3School of Physical Sciences, University of Chinese Academy of Sciences, Beijing, China
4Songshan Lake Materials Laboratory, Dongguan, Guangdong 523808, China
5Peng Huanan Center for Fundamental Theory, Xian 710127, China
6Shaanxi Key Laboratory for Theoretical Physics Frontiers, Xian 710127, China

A novel Bethe Ansatz scheme is proposed to deal with topological quantum integrable systems. As an example, the anti-periodic XXZ spin chain, a typical correlated many-body system embedded in a topological manifold, is examined. Conserved “momentum” and “charge” operators are constructed despite the absence of translational invariance and $U(1)$ symmetry. The ground state energy and elementary excitations are derived exactly. It is found that two intrinsic fractional (one half) zero modes exist in most of the eigenstates. The elementary excitations show quite a different picture from that in the periodic boundary case. This method can be applied to other quantum integrable models with nontrivial topology or boundary conditions.

PACS numbers: 75.10.Pq, 03.65.Vf, 71.10.Pm

Exact quantization in topological manifold is an important issue in modern physics [1]. It is related to several interesting research fields such as quantum topological phases in correlated many-body systems [2], exact quantization in Calabi-Yau manifold in the string theory [3], and many others. Recently, a research focus in topological phases is to take correlations into account to seek for new physical effects. Sometimes exact quantization procedure is finally attributed to solve some quantum integrable models [6]. However, those quantum integrable models possess nontrivial topology structure. A formidable problem to fulfill exact quantization in nontrivial topological manifold is the absence of $U(1)$ symmetry, which makes us frustrated to work in a traditional particle-hole representation. We remark that though some methods have been developed to approach quantum integrable models without $U(1)$ symmetry [7–13], including the off-diagonal Bethe Ansatz [14, 15] proposed by some of the present authors, with which the formal solutions of the eigenvalues can be expressed in an inhomogeneous $t–Q$ relation, exact quantum numbers and elementary excitations for those models are still unclear because of complicated distribution of Bethe roots associated with inhomogeneous Bethe Ansatz equations.

In this letter, we propose a novel Bethe Ansatz scheme for obtaining exact quantized spectrum of topological quantum integrable models. By constructing an operator identity of the transfer matrix for arbitrary spectral parameter $u$, factorized Bethe Ansatz equations (BAEs) about the zero roots of the transfer matrix can be derived. It is found that the root distribution in the complex plane possesses a simple structure, which allows us to define quantum numbers associated with zero roots of the transfer matrix and to calculate physical quantities such as ground state energy and exact elementary excitations (quasi particles) in the thermodynamic limit. A counterpart of momentum operator and a conserved charge (correspondence of $U(1)$ charge in the periodic boundary case) in the topological manifold are also defined to classify the eigenstates and elementary excitations.

To clarify our procedure clearly, we study the anti-periodic XXZ spin chain [16, 17] as a concrete example. The model Hamiltonian reads

$$H = -\sum_{n=1}^{N} (\sigma_{n}^{x} \sigma_{n+1}^{x} + \sigma_{n}^{y} \sigma_{n+1}^{y} + \cosh \eta \sigma_{n}^{z} \sigma_{n+1}^{z}),$$

with the topological boundary condition

$$\sigma_{1+N}^{\alpha} = \sigma_{1}^{\alpha} \sigma_{0}^{\alpha} \sigma_{1}^{\alpha}, \quad \text{for} \quad \alpha = x, y, z,$$

where $\sigma^{x}, \sigma^{y}, \sigma^{z}$ are the usual Pauli matrices and $\eta$ is the coupling constant. This nontrivial boundary condition mixes the spin up and spin down states in the Hilbert space and makes the system forming a quantum Möbius strip. The $U(1)$ symmetry is thus broken and a discrete $Z_{2}$ invariance $[H, U] = 0$ is left with $U = \prod_{j=1}^{N} \sigma_{j}^{x}$ and $U^{2} = 1$. In the following text, we put $\eta = i\gamma$ as an imaginary constant. The real $\eta$ case can be studied straightforwardly. The integrability of the model is associated with the well-known six-vertex $R$-matrix

$$R_{0,j}(u) = \frac{\sinh(u + \eta) + \sinh u}{2\sinh \eta} + \frac{1}{2}(\sigma_{j}^{x} \sigma_{0}^{x} + \sigma_{j}^{y} \sigma_{0}^{y})$$

$$+ \frac{\sinh(u + \eta) - \sinh u}{2\sinh \eta} \sigma_{j}^{z} \sigma_{0}^{z},$$

which satisfies the Yang-Baxter equation [18, 19], where $u$ is the spectral parameter.

Let us introduce the monodromy matrix

$$T_{0}(u) = R_{0,N}(u) \cdots R_{0,1}(u).$$
The transfer matrix $t(u)$ is given by \[ t(u) = tr_0 \{ \sigma_0^u T_0(u) \}, \] where $tr_0$ denotes trace over the “auxiliary space” 0. From Eqs. (3)-(5) we know that $t(u + i\pi) = (-1)^{N-1}t(u)$ and $t(u)$ as a function of $u$, is an operator-valued trigonometric polynomial of degree $N - 1$. The Hamiltonian described by (1) and (2) is given by

$$H = -2 \sinh \eta \frac{\partial \ln t(u)}{\partial u}|_{u=0} + N \cosh \eta. \quad (6)$$

The commutativity of the transfer matrices with different spectral parameters ensured by the Yang-Baxter equation implies that they have common eigenstates. Let $|\Psi\rangle$ be an eigenstate of $t(u)$, which does not depend upon $u$, with the eigenvalue $\Lambda(u)$, i.e.,

$$t(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle. \quad (7)$$

Obviously, $L(u + i\pi) = (-1)^{N-1}L(u)$. $\Lambda(u)$ as a function of $u$, is a trigonometric polynomial of degree $N - 1$ and can be expressed in terms of its $N - 1$ zero roots $\{z_j - \eta/2| = 1, \cdots, N - 1\}$ and an overall coefficient $\Lambda_0$ as follows

$$\Lambda(u) = \Lambda_0 \prod_{j=1}^{N-1} \sinh(u - z_j + \frac{\eta}{2}). \quad (8)$$

The corresponding eigenvalue of the Hamiltonian given by (6) can be expressed as

$$E = 2 \sinh \eta \sum_{j=1}^{N-1} \coth(z_j - \frac{\eta}{2}) + N \cosh \eta. \quad (9)$$

**Bethe Ansatz**: The key point of the present Bethe Ansatz is to construct an operator identity for the transfer matrix, which allows us to derive self-consistent equations determining the zero roots of its eigenvalues. We note that $R_{1,2}(-\eta) = -2P_{1,2}^{(-)}$ and $P_{1,2}^{(\pm)} = (1 \pm P_{1,2})/2$, where $P_{1,2}^{(\pm)}$ and $P_{1,2}$ are the projection operators and permutation operator, respectively. With the fusion techniques $[21, 22]$ we have

$$t(u)t(u - \eta) = tr_1,2\{P_{1,2}^{(-)} \sigma_1^u \sigma_2^u T_2(u) T_1(u - \eta) P_{1,2}^{(-)} \} + tr_1,2\{P_{1,2}^{(\pm)} \sigma_1^u \sigma_2^u T_2(u) T_1(u - \eta) P_{1,2}^{(\pm)} \}, \quad (10)$$

we have the following $t - W$ relation

$$t(u)t(u - \eta) = -a(u)d(u - \eta) \times \text{id} + d(u)W(u), \quad (11)$$

where

$$a(u) = d(u + \eta) = \frac{\sinh^N(u + \eta)}{\sinh^N \eta}. \quad (12)$$

$W(u)$ is an operator-valued degree $N$ trigonometric polynomial of $u$ with $[W(u), t(u)] = 0$; and $\text{id}$ is the identity operator in the Hilbert space. Acting (10) on an eigenstate $|\Psi\rangle$ we have

$$\Lambda(u)\Lambda(u - \eta) = -a(u)d(u - \eta) + d(u)W(u), \quad (13)$$

where $W(u)$ is the eigenvalue of $W(u)$. Let

$$W(u) = W_0 \sinh^{-N} \eta \prod_{i=1}^{N} \sinh(u - w_i), \quad (14)$$

with $W_0$ a constant (depending on the roots). An important fact is that (13) is a degree $2N$ polynomial equation and thus gives $2N + 1$ independent equations for the coefficients, which determines the $N - 1$ $z_j$ roots, $N$ $w_i$ roots and the two constants $\Lambda_0$ and $W_0$ completely. Since $\Lambda(u)$ is a degree $N - 1$ trigonometric polynomial of $u$, the leading terms in the right hand side of (13) must be zero. Therefore, $W_0 e^{\pm \sum_{i=1}^{N} w_i} = 1$, or $W_0^2 = 1$ and $\sum_{i=1}^{N} w_i = 0 \mod(i\pi)$. Let $u = z - \eta/2$ in (13), we obtain

$$\sinh^N(z_j - \frac{3\eta}{2}) \sinh^N(z_j + \frac{\eta}{2}) = W_0 \sinh^N(z_j - \frac{\eta}{2}) \prod_{i=1}^{N} \sinh(z_j - w_i - \frac{\eta}{2}). \quad (15)$$

Let $u = w_i$ in (13) we obtain

$$\Lambda_0 \prod_{j=1}^{N-1} \sinh(w_i - z_j + \frac{\eta}{2}) \sinh(w_i - z_j - \frac{\eta}{2}) = - \sinh^{-2N} \eta \sinh^N(w_i + \eta) \sinh^N(w_i - \eta). \quad (16)$$

The coefficient $\Lambda_0$ can be determined by putting $u = 0$ in (13) as

$$\Lambda_0 \prod_{j=1}^{N-1} \sinh(z_j + \frac{\eta}{2}) \sinh(z_j - \frac{\eta}{2}) = (-1)^{N-1}. \quad (17)$$

From the intrinsic properties of the $R$-matrix, for imaginary $\eta$ we have

$$t^*(u) = (-1)^{N-1}t(u^* - \eta), \quad \Lambda^*(u) = (-1)^{N-1}\Lambda(u^* - \eta). \quad (18)$$

The above relation implies that if $z_j$ is a root, $z^*_j$ must also be a root! Therefore, $z_j$ can be classified into 3 sets: (1) real $z_j$; (2) $z_j = \mu_i - i\pi/2$, with $\mu_i$ real (this is because its conjugate shifted by $i\pi$ becomes itself); (3) complex conjugate pairs. Similarly, we have $W^*(u^*) = (-1)^N W(u)$, indicating that if $w_i$ is a root of $W(u)$, $w_i^*$ must also be a root! In fact, both the exact numerical solutions for finite $N$ and analytic analysis in the thermodynamic limit (as shown below) indicate
that the imaginary parts of a $z$-root conjugate pair are around $\pm n\eta/2$ with $n \geq 2$ a positive integer. For $n = 2$, the $z$-root conjugate pair is accompanied by a $w$-root conjugate pair with imaginary parts around $\pm 3\eta/2$. For $n > 2$, the $z$-root conjugate pair is accompanied by a $w$-root 4-string with imaginary parts $\pm(n - 1)\eta/2$ and $\pm(n + 1)\eta/2$. Such a simple structure of the roots is quite similar to the string structure appeared in the conventional Bethe Ansatz solvable models \[ \text{22} \] and allows us to calculate physical properties exactly in the thermodynamic limit. The exact diagonalization of the transfer matrix up to $N = 12$ was performed numerically and all the roots solved indeed exactly coincide with those by solving the BAEs (13)-(15). The numerical results for $N = 4$ and $\eta = 0.6i$ are shown in Table I.

**Table I:** $z$ roots calculated via exact numerical diagonalization of the transfer matrix for $N = 4$ and $\eta = 0.6i$. Each set of solutions is doubly degenerate due to the $Z_2$ symmetry.

| $z_1$ | $z_2$ | $z_3$ |
|-------|-------|-------|
| -0.2890 | 0.0000 | 0.2890 |
| -1.4697 | -0.0531 | 0.2266 |
| -0.2266 | 0.0531 | 1.4697 |
| -1.4190 | 0.0000 | -1.5708i |
| -0.7908 - 1.5708i | 0.0000 | 0.7908 - 1.5708i |
| -0.1652 | 0.1384 - 0.6102i | 0.1384 + 0.6102i |
| -0.1384 - 0.6102i | -0.1384 + 0.6102i | 0.1652 |
| 0.0000 - 1.5708i | 0.0000 - 0.6238i | 0.0000 + 0.6238i |

**Conserved quantities:** Due to the topological boundary, the model possesses neither translational invariance nor $U(1)$ symmetry. Nevertheless we find that

$$ t(0) = \sigma^z P_{1,N} P_{1,N-1} \cdots P_{1,2}, \quad (18) $$

is a conserved quantity and represents the shift operator in the topological manifold. A corresponding “momentum” operator can thus be defined as $P_q = -i \ln t(0)$. From the definition of the transfer matrix we have $t^{2N}(0) = 1$, indicating that the eigenvalues of $P_q$ take values of

$$ k = \frac{\pi l}{N} \text{mod} \{\pi\}, \quad (19) $$

with $l = \{-N, -N+1, \cdots, N-1\}$ denoting the quantum winding numbers. The topological momentum is related to the $z$-roots as

$$ k = -\frac{i}{2} \sum_{j=1}^{N-1} \ln \frac{\sinh(z_j + \frac{n}{2})}{\sinh(z_j - \frac{n}{2})} + (1 - (-1)^{N-1}) \frac{\pi}{4}. \quad (20) $$

Similarly, we have the following conserved charge operator

$$ M_q = \frac{1}{2} (I_q^+ + I_q^-) = \frac{1}{4} e^{-\frac{N+1}{2} \lim_{u \to \infty} (2 \sinh \eta e^{-u})^{N-1} \mathbf{t}(u)}, \quad (21) $$

where

$$ I_q^\pm = \frac{1}{2} \sum_{j=1}^{N} e^{\mp \frac{\eta}{2} \sum_{k=j+1}^{N} \sigma^z \sigma^z_j e^{\pm \frac{\eta}{2} \sum_{k=j+1}^{N-1} \sigma^z}}, \quad (22) $$

are two generators of the quantum group $\mathfrak{g}_2^{[2]}$ associated with the model. The corresponding eigenvalues of charge operator $M_q$ is given by

$$ M_q = \frac{1}{4} \sinh^{N-1} \eta \Lambda_0 e^{-\sum_{k=1}^{N-1} z_k}. \quad (23) $$

When $\eta \to 0$, the model tends to an isotropic spin chain and the $U(1)$ symmetry recovers with $M_q = \sum_{j=1}^{N} \sigma^z_j/2$, which is just the $U(1)$ charge. We note that $M_q$ may not take values of integers for generic $\eta$.

**Ground state:** For the ground state, all roots $z_j$ and $w_l$ take real values around zero symmetrically. Taking the logarithm of (14) and its complex conjugate we have

$$ 2\theta(z_j) - \theta(z_j) = 2\pi I_j - \frac{1}{N} \sum_{l=1}^{N} \theta(z_j - w_l), \quad (24) $$

and

$$ \ln |\Lambda_0 \sinh z_j - z_j/2| = \frac{1}{N} \sum_{l=1}^{N} \ln |\sinh(z_j - w_l - z_j/2)|, \quad (25) $$

where $I_j$ is the quantum numbers (integers or half odd integers depending on the parity of $N$) associated with the root $z_j$ and $\theta_n(x) = 2 \cot^{-1}(\coth(x) \tan \frac{\pi n}{2})$. The quantum numbers take values

$$ I_j = \left\{ -\frac{N-2}{2}, -\frac{N-4}{2}, \cdots, \frac{N-4}{2}, \frac{N-2}{2} \right\}. $$

In the thermodynamic limit $N \to \infty$, we define the density of $z$-roots and the density of $w$-roots per unit site as $\rho(z)$ and $\rho^b(z)$, the density of $w$-roots as $\sigma(w)$, respectively. Taking the continuum limits of (24) and (25) we have

$$ 2a_1(z) - a_3(z) = 2\rho(z) + 2\rho^b(z) - a_1 \ast \sigma(z), \quad (26) $$

$$ b_3(z) = b_1 \ast \sigma(z), \quad (27) $$

where $a_n(z) = \theta_n(z)/(2\pi)$, $b_n(z) = \ln|\sinh(z-n\eta/2)|/\pi$ and * indicates convolution. With Fourier transformation we readily have

$$ \rho(z) + \rho^b(z) = \frac{2 \cosh(\frac{\pi z}{\eta}) \sin(\frac{\pi z}{\eta})}{(\pi - \gamma)[\cosh(2z/\eta) + \cos(2z - 2\gamma/\eta)]}, $$

$\rho^b(z)$ is non-zero only in the range $|z| > D$ ($D \to \infty$ in the thermodynamic limit) with $N \int_{-D}^{D} \rho^b(z) dz = 1/2$ and $N \int_{-D}^{-D} \rho^b(z) dz = 1/2$. The hole density is introduced to ensure the total number of roots to be $N - 1$. Clearly, the holes contribute two fractional (half) modes and zero energy. These holes preserve Kitaev’s boundary Majorana
zero modes [24] and exist in most of the eigenstates, despite the present model is defined in a closed ring! The ground state energy density reads
\[ e_g = -\sin \gamma \int \frac{\cosh[(\pi-2\gamma)\tau]}{\sinh(\frac{\pi}{2})} \tan \left[ \frac{\pi}{2} - \frac{\gamma}{2} \right] d\tau + \cos \gamma, \]
which is the same to that of the periodic chain [25]. It seems that the anti-periodic boundary induces energy deviation only in order of \( O(1/N) \).

From the exact numerical diagonalization results we find that besides the ground state there exist several sets of real \( z_j \) solutions for small \( \gamma \) but their distributions are asymmetric around the origin. Correspondingly, a boundary conjugate pair \( \beta \pm m\eta/2 \) exists in the set of \( w \)-roots. A typical set of such solutions is shown in Fig.1. In the thermodynamic limit, \( \beta \to \pm \infty \) to keep the density functions to be convergent; and to ensure the associated energy to be real, \( m \) can only take values of odd integers \((\geq 3)\), coinciding with the numerical results. In this case, the energy is almost degenerate to that of the ground state but the Majorana-like zero modes disappear.

**Elementary excitations I:** The elementary excitations can be obtained by studying the root distribution away from real axis. The first kind of elementary excitations is that a single root locates in the axis \( \text{Im}z = -i\pi/2 \) and all the other roots remain in the real axis. Corresponding to such an excitation, a set of roots derived by exact numerical diagonalization for \( N = 10 \) is shown in Fig.2(a). Let us denote the single complex root as \( z = \alpha - i\pi/2 \), with \( \alpha \) a real number. Accordingly, two \( w \)-roots form a conjugate pair \( w_{\pm} = \beta \pm m\eta/2 \) with \( \beta \) and \( m \) two real numbers, and all the other \( w \)-roots keep real. In the thermodynamic limit, by taking the complex roots into account, we can also derive the density \( \rho(z) \) for real \( z_j \). Notice that the fractional zero modes still exist in this excited state. To ensure the convergence of the density function, the following constraints are needed
\[ m + 1 - \frac{\pi}{\gamma} = 0, \quad \beta = \alpha. \quad (28) \]

The above relations not only fix the relative position between the complex \( z \)-root and the \( w \) conjugate pair but also the imaginary parts of the \( w \) conjugate pair. The associated excitation energy reads
\[ \delta e_1 = \sin \gamma \int \frac{\cos(\tau\alpha) \tan \left[ \frac{\pi}{2} - \frac{\gamma}{2} \right] \cosh(\frac{\pi\tau}{2})}{\sinh(\frac{\pi}{2})} d\tau + \frac{2\sin^2 \gamma}{\cosh(2\alpha) + \cos \gamma}. \quad (29) \]

**Elementary excitations II:** When \( \eta \) is away from \( i\pi/2 \), conjugate pairs of \( z \)-roots can exist. Here we consider the solution of one single conjugate pair and all the other roots remain in the real axis. The simplest conjugate pair is given by \( z_{\pm} \sim \alpha \pm \eta \). The corresponding \( w \)-roots are formed by a \( w \) conjugate pair \( w_{\pm} \sim \alpha \pm 3\eta/2 \) and \( N - 2 \) real \( w_i \). Both the positions and the imaginary parts of the conjugate pairs are determined by convergence requirement of the density functions. A typical set of root solutions describing such an excitation for \( N = 10 \) is shown in Fig3(a).

The momentum associated with \( \alpha \) can be determined by [20]. For \( \eta = i\pi/2 \), all the possible zero roots \( z_j \) lie either in the real axis or the line \( \text{Im}z = -i\pi/2 \). In this case, this kind of excitations is the only possible one and resembles particle-hole type or spin-wave type in the periodic case.

**Elementary excitations III:** General conjugate pair excitation is given by a conjugate pair \( z_{\pm} \sim \alpha \pm n\eta/2 \) with \( n \geq 3 \), and all the other \( z \)-roots remain in the real axis. In this case, the corresponding \( w \)-roots are formed by a four-string \( \sim \alpha \pm (n+1)\eta/2 \), \( \alpha \pm (n-1)\eta/2 \) and \( N - 4 \) real \( w \)-roots. A set of roots corresponds to this kind of excitations for \( N = 10 \) is shown in Figs3(b,c).
In the thermodynamic limit, the excitation energy reads

$$\delta e_3 = 2 \sin \gamma \int \frac{\cos(\tau \alpha) \tanh(\frac{\tau}{2}) f(\tau) d\tau}{\sinh(\frac{\pi \tau}{2})}$$

$$+ \frac{2 \sin \gamma \sin((n-1)\gamma)}{\cosh(2\alpha) - \cos((n-1)\gamma)}$$

$$- \frac{2 \sin \gamma \sin((n+1)\gamma)}{\cosh(2\alpha) - \cos((n+1)\gamma)}, \quad (31)$$

where $f(\tau) = \cosh[(1 - \delta_{n-1} - \delta_{n+1})\pi \tau/2] \cosh[(\delta_{n-1} - \delta_{n+1})\pi \tau/2]$ and $\delta_m = m\gamma/(2\pi) - [m\gamma/(2\pi)].$

We remark that there is indeed intrinsic difference between elementary excitations in the topological boundary case and those in the periodic boundary case. In the present case, the dispersion of the excitation energy relies only on a single parameter $\alpha$ (or its corresponding quantum number $I_a/N$) besides the number $n$; while in the periodic boundary case, at least two parameters (in terms of Bethe roots, two holes in the real axis) appear in the energy dispersion relation accounting for the spinon excitations [26]. Though only single complex root (or conjugate pair) excitations are discussed, a number of excitations can exist simultaneously in a single eigenstate, indicating that the “quasi-particles” are countable even without $U(1)$ symmetry.

In conclusion, a novel Bethe Ansatz scheme is proposed to deal with the topological quantum integrable models. The factorized Bethe Ansatz equations allow us to perform exact quantization of the spectrum with associated quantum numbers and to derive physical quantities in the thermodynamic limit. For real $\eta$, after replacing $z_j$ and $w_l$ by $iz_j$ and $iw_l$, we find the roots have similar structures to those of the imaginary $\eta$ case but the excitations have a gap. This scheme can be applied to other Yang-Baxter quantum integrable systems.

The financial supports from National Program for Basic Research of MOST (Grant Nos. 2016 YFA0300600 and 2016YFA0302104), National Natural Science Foundation of China (Grant Nos. 11934015, 11975183, 11774397, 11775178, 11775177 and 11947301), Major Basic Research Program of Natural Science of Shaanxi Province (Grant No. 2017ZJJC-32), Australian Research Council (Grant No. DP 190101529), and Double First-Class University Construction Project of Northwest University are gratefully acknowledged. WL Yang would like to thank IoP/CAS for the hospitality during his visit.

* wlyang@nwu.edu.cn
† yupeng@iphy.ac.cn

[1] N.A. Nekrasov and S.L. Shatashvili, arXiv:0908.4052
[2] X.G. Wen, Rev. Mod. Phys. 89, 41004 (2017).
[3] X.Wang, G. Zhang, and M.X. Huang, Phys. Rev. Lett. 115, 121601 (2015).
[4] L. Fu and C.L. Kane, Phys. Rev. Lett. 100, 096407 (2008).
[5] R.M. Lutchyn, J.D. Sau, and S. Das Sarma, Phys. Rev. Lett. 105, 077001 (2010).
[6] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, and C. Vafa, Commun. Math. Phys. 261, 451 (2006).
[7] J. Cao, H.Q. Lin, K.J. Shi, and Y. Wang, Nucl. Phys. B 663, 487 (2003).
[8] R.I. Nepomechie, J. Phys. A: Math. Gen. 34, 9993 (2001); Nucl. Phys. B 662, 615 (2002).
[9] P. Baseilhac, Nucl. Phys. B 754, 309 (2006).
[10] W. Gallwey, Nucl. Phys. B 790, 524 (2008).
[11] E.K. Sklyanin, Lect. Notes Phys. 226, 196 (1985); J. Sov. Math. 31, 3417 (1985).
[12] E.K. Sklyanin, Prog. Theor. Phys. Suppl. 118, 35 (1995).
[13] S. Belliard and N. Crampé, SIGMA 9, 072 (2013).
[14] J. Cao, W.-L. Yang, K. Shi, and Y. Wang, Phys. Rev. Lett. 111, 137201 (2013).
[15] X. Zhang, Y.-Y. Li, J. Cao, W.-L. Yang, K. Shi and Y. Wang, J. Stat. Mech. 05, P05014 (2015).
[16] C.M. Yang and M.T. Batchelor, Nucl. Phys. B 446, 461 (1995).
[17] M.T. Batchelor, R.J. Baxter, M.J. O’Rourke, and C.M. Yang, J. Phys. A 28, 2759 (1995).
[18] C.N. Yang, Phys. Rev. Lett. 19, 1312 (1967); Phys. Rev. 168, 1920 (1968).
[19] R.J. Baxter, Ann. Phys. 70, 323 (1972).
[20] P.P. Kulish, N. Yu. Reshetikhin, and E.K. Sklyanin, Lett. Math. Phys. 5, 393 (1981).
[21] N. Yu Reshetikhin, Sov. Phys. JETP 57, 691 (1983).
[22] M. Takahashi, Thermodynamics of One-Dimensional Solvable Models (Cambridge University Press, Cambridge, 1999) and references therein.
[23] V. Chari and A. Pressley, A Guide to Quantum Groups (Cambridge University Press, Cambridge, 1999).
[24] A.Y. Kitaev, Phys. Usp. 44, 131 (2001).
[25] C.N. Yang and C.P. Yang, Phys. Rev. 150, 327 (1966).
[26] L.D. Faddeev and L.A. Takhtajan, Phy. Lett. A 85, 375 (1981).