NONCOMMUTATIVE RESOLUTIONS AND INTERSECTION COHOMOLOGY FOR QUOTIENT SINGULARITIES

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Abstract. For a large class of good moduli spaces $X$ of symmetric stacks $\mathcal{X}$, we define noncommutative motives $D^{nc}(X)$ which can be regarded as categorifications of the intersection cohomology of $X$. These motives are summands of noncommutative resolutions of singularities $\mathcal{D}(X) \subset D^b(\mathcal{X})$ of $X$. The category $\mathcal{D}(X)$ is a global analogue of the noncommutative resolutions of singularities of $V//G$ for $V$ a symmetric representation of a reductive group $G$ constructed by Špenko–Van den Bergh.

1. Introduction

1.1. Intersection cohomology. Intersection cohomology $IH^*(X, \mathbb{Q})$ and the BBDG Decomposition Theorem [5] are important tools in the study of the topology of algebraic varieties. They have many applications in representation theory, see [32], or more recently in construction of representation of $\mathcal{W}$-algebras due to Braverman–Finkelberg–Nakajima [8] and in the study of (Kontsevich–Soibelman) Cohomological Hall algebras [28], [15]. It is an important problem to define a K-theoretic version of intersection cohomology and it is expected that such a theory will have applications in representation theory. More generally, one can try to construct categorifications of intersection cohomology.

1.2. Noncommutative resolutions. A natural place to look for such categorifications is inside noncommutative resolutions (NCRs) of $X$. There are more NCRs than usual resolution of singularities. A conjecture of Bondal–Orlov [7, Section 5] says that there exists a minimal NCR, i.e. a category admissible inside any other NCR of $X$.

We look for natural candidates of minimal NCRs $\mathcal{D}$ of good moduli spaces $X$ of Artin stacks $\mathcal{X}$. Inspired by the Decomposition Theorem, we expect the periodic cyclic homology $HP_*(\mathcal{D})$ to have $\bigoplus_{i \in \mathbb{Z}} IH^{+2i}(X, \mathbb{C})$ as a direct summand. We then try to construct noncommutative motives $D^{nc}$ inside $\mathcal{D}$ whose periodic cyclic homology is $\bigoplus_{i \in \mathbb{Z}} IH^{+2i}(X, \mathbb{C})$.

1.3. NCRs for good moduli spaces. We make three possible assumptions on a stack $\mathcal{X}$:

A. $\mathcal{X}$ is an algebraic stack locally of finite type over $\mathbb{C}$ with a good moduli space $\pi: \mathcal{X} \to X$ (as defined by Alper [1]) such that $\pi$ has affine diagonal.

B. In addition to the above, assume that $\dim X = \dim \mathcal{X}$.

C. In addition to the above, assume that $\pi$ is generically an isomorphism.
Let $\mathcal{X}$ a smooth stack which satisfies Assumption A and let $p \in X(\mathbb{C})$. By work of Alper–Hall–Rydh [2, Theorem 1.2], there exists a smooth affine scheme $A$ with an action of a reductive group $G$ such that the following diagram is cartesian

$$
\begin{array}{ccc}
\mathcal{X} & \xleftarrow{e} & A/G \\
\pi & \downarrow & \\
X & \xleftarrow{e} & A / G,
\end{array}
$$

and $e$ is an étale map which contains $p$ in its image. The NCRs we are using are a global version of the NCRs constructed by Špenko–Van den Bergh [38] for quotients $V/G$ for $V$ a representation of $G$.

**Theorem 1.1.** Let $\mathcal{X}$ be a symmetric stack satisfying Assumption B with good moduli space $X$. Then there exists a subcategory $\mathbb{D}(\mathcal{X})$ admissible in $D^b(\mathcal{X})$ which is an NCR of $X$. Its complement is generated by complexes supported on attracting stacks/$\Theta$-stacks of $X$.

Assume that $\mathcal{X} = V/G$ where $V$ is a symmetric $G$-representation, i.e. the weights $\beta$ and $-\beta$ of $G$ have the same multiplicity in $V$. Let $M$ be the weight lattice of $G$, let $M_K = M \otimes \mathbb{R}$, and let $W$ be the Weyl group of $G$. For $\lambda : \mathbb{C}^* \to G$ a cocharacter, define

$$n_\lambda := \langle \lambda, \det L^* \rangle,$$

where $L^*$ is the cotangent complex of $\mathcal{X}$. Let $\delta \in M_K^W$. The NCR constructed by Špenko–Van den Bergh [38] is as follows: $\mathbb{D}_\delta(\mathcal{X})$ is the full subcategory of $D^b(\mathcal{X})$ generated by complexes $F$ such that for any cocharacter $\lambda : \mathbb{C}^* \to G$:

$$- \frac{n_\lambda}{2} + \langle \lambda, \delta \rangle \leq \langle \lambda, F|_0 \rangle \leq \frac{n_\lambda}{2} + \langle \lambda, \delta \rangle.$$

Next, let $\mathcal{X}$ be as in Assumption B. Recall the description (1) for a point $p \in X$. Locally analytically near a point $x \in X(\mathbb{C})$, the stack $\mathcal{X}$ is isomorphic to an open analytic subset of $N/G$, where $G$ is the stabilizer of $x$ and $N$ is its normal bundle in $\mathcal{X}$. We assume that all such representations are symmetric, and call stacks $\mathcal{X}$ with this property symmetric. Note that symmetric stacks are smooth. Let $\ell \in \text{Pic}(\mathcal{X})_\mathbb{R}$. For a morphism $\lambda : B\mathbb{C}^* \to \mathcal{X}$ with image in $\mathcal{X}(\mathbb{C})$, define

$$n_\lambda := \text{wt } \lambda^*(\det L^* \rangle,$$

where $L^*$ is the cotangent complex of $\mathcal{X}$. The category $\mathbb{D}_\ell(\mathcal{X})$ is the full subcategory of $D^b(\mathcal{X})$ such that for any map $\lambda : B\mathbb{C}^* \to \mathcal{X}$ with image in $\mathcal{X}(\mathbb{C})$, we have

$$- \frac{n_\lambda}{2} + \text{wt } \lambda^* \ell \leq \text{wt } \lambda^*(F) \leq \frac{n_\lambda}{2} + \text{wt } \lambda^* \ell.$$

To show $\mathbb{D}_\ell(\mathcal{X})$ is a smooth and proper category over $X$, we show in Subsection 3.3 that $\mathbb{D}_\ell(\mathcal{X})$ is admissible inside the Kirwan resolution of $\mathcal{X}$, a DM stack constructed by Edidin–Rydh [17] which recovers the Kirwan resolution in the local case [26]. The analogous result for quotients $Y/G$ was proved by Špenko–Van den Bergh [40]. Our approach is different from theirs and uses the variation of GIT techniques from [4], [20], [19]. In order for $\pi^* : D^b(\mathcal{X}) \to D^b(\mathcal{X})$ to have image in $\mathbb{D}_\ell(\mathcal{X})$, we choose $\ell$ to be trivial and drop it from the notation of $\mathbb{D}(\mathcal{X})$. 
1.4. **Intersection cohomology of symmetric good moduli spaces.** Let $\mathcal{X}$ be a symmetric stack satisfying Assumption B from Subsection 1.3. Define

$$B := \text{image} \left( \bigoplus_S H(\mathcal{Z}) \xrightarrow{p^*q^*} H(\mathcal{X}) \right),$$

over all attracting stacks $S$ (different from $\mathcal{X}$) with fixed stack and associated maps $\mathcal{Z} \xrightarrow{\mathcal{S}} \mathcal{X}$ and the singular cohomology is with $\mathbb{Q}$ coefficients. We denote by $P_{\leq 0} H(\mathcal{X})$ the zeroth piece of the perverse filtration on $H(\mathcal{X})$ induced by the map $\pi : \mathcal{X} \rightarrow X$. Observe that if $\mathcal{X}$ satisfies Assumption C, then $P_{\leq 0} H(\mathcal{X}) \cong IH(\mathcal{X})$.

In Section 4, we show that:

**Theorem 1.2.** For $\mathcal{X}$ a symmetric stack satisfying Assumption B, there is a decomposition

$$H(\mathcal{X}) = P_{\leq 0} H(\mathcal{X}) \oplus B.$$

The above result follows from a version of the BBDG Decomposition Theorem for stacks, see Proposition 4.2. The exceptional loci are covered by the attracting stacks, and by the symmetric assumption on $\mathcal{X}$, the images of classes from attracting stacks are in positive perverse degree.

1.5. **Noncommutative motives for symmetric good moduli spaces.** Let $\mathcal{X}$ be a symmetric stack satisfying Assumption B from Subsection 1.3. In Section 5.3 we define a noncommutative motive with rational coefficients $D^{nc}(\mathcal{X}) = (D(\mathcal{X}), e)$, where $e$ is an idempotent in $K_0(\text{rep}(D(\mathcal{X})))_{\mathbb{Q}}$, see Subsection 2.5 for a brief discussion of noncommutative motives. Define

$$BK(\mathcal{X}) := \text{image} \left( \bigoplus_S K(\mathcal{Z}) \xrightarrow{p^*q^*} K(\mathcal{X}) \right),$$

where the sum is an in (3) and $K$ is rational K-theory. In Section 5 we show an analogue on Theorem 1.2 in K-theory:

**Theorem 1.3.** For $\mathcal{X}$ a symmetric stack satisfying Assumption B, there is a decompositions

$$K(\mathcal{X}) = K(\mathcal{D}^{nc}(\mathcal{X})) \oplus BK(\mathcal{X}).$$

The category $D(\mathcal{X})$ contains, in general, complexes on attracting stacks, but it may be indecomposable as a triangulated category. For $\lambda : B\mathbb{C}^* \rightarrow \mathcal{X}$ with associated fixed stack $\mathcal{Z}$, let $D(\mathcal{Z})_b$ be the subcategory of $D(\mathcal{X})$ of complexes on which $\lambda$ acts with weight $b = \frac{n_\lambda}{2}$. The motive $D^{bc}(\mathcal{X})$ is a complement of the images

$$m_\lambda := p^*q^* : D(\mathcal{Z})_b \rightarrow D(\mathcal{X})$$

from fixed substacks $\mathcal{Z}$. By a result of Thomason [42, Corollary 2.17], one can compute rational K-theory using an étale cover, so it suffices to check the statement in the local case $\mathcal{X} = A/G$, where $A$ is a smooth affine scheme. The main tool in understanding these images for different fixed substacks is a product-coproduct type compatibility that we briefly explain. Let $\lambda$ and $\mu$ be two dominant cocharacters,
let $S$ be a set of cocharacters $\nu$ refining $\mu$ and $\lambda$, see Subsection 5.1.5. Then the following diagram commutes:

$$K_*(\mathbb{D}(X_{\lambda})_b) \xrightarrow{m_{\lambda}} K_*(\mathbb{D}(X))$$

$$\bigoplus_S \Delta_{\nu} \downarrow \quad \bigoplus_S \bigoplus \tilde{m}_{\nu} \downarrow \Delta_{\mu}$$

$$\bigoplus_S K_*(\mathbb{D}(X_{\nu})_b) \xrightarrow{\bigoplus_S \tilde{m}_{\nu}} K_*(\mathbb{D}(X_{\mu})_b).$$

Here, $\Delta$ are restriction maps and $\tilde{m}$ denotes a twist of the multiplication. We check that the diagram commutes by a direct computation using shuffle formulas for $m$ and $\Delta$.

1.6. Intersection K-theory. Let $X$ be a symmetric stack satisfying Assumption C and let $\overline{X}$ be its good moduli space. Then $P \leq \overline{X}$ and $H \cdot (\overline{X}) \cong IH \cdot (\overline{X}).$ Given Theorems 1.2 and 1.3 we propose to call $K_*(\mathbb{D}^{nc}(X)) := IK_*(\overline{X})$ the intersection K-theory of $X$. There is a Chern character map

$$\text{ch} : IK_*(\overline{X}) \to IH_*(\overline{X}).$$

Further, from the construction of $IK_*(\overline{X})$, we obtain a natural surjection

$$K_*(X) \twoheadrightarrow IK_*(\overline{X}).$$

The analogous statement in cohomology was proved by Kirwan [27]. Further, in Subsection 5.6.2 we show that if the Kirwan resolution is a scheme, then

$$HP_i(\mathbb{D}^{nc}(X)) \cong \bigoplus_{j \in \mathbb{Z}} IH^{i+2j}(X, \mathbb{C}).$$

We use the construction of K-theory of quiver varieties to prove a version of a Poincaré-Birkhoff-Witt Theorem for K-theoretic Hall algebras of quivers with potential [35, Theorem 1.2]. For example, for a quiver $Q$, the K-theoretic Hall algebra $\text{KHA}(Q,0)$ for zero potential is generated by spaces which are twisted versions of $IK_*(\overline{X}(d))$, where $X(d)$ is the coarse space of representation of $Q$ of dimension $d$.

1.7. Previous work on intersection K-theory. There are other approaches of defining intersection K-theory in particular cases. Cautis [11], Cautis–Kamnitzer [12] have an approach for categorification of intersection sheaves for certain subvarieties of the affine Grassmannian. For varieties with a cellular stratification, Eberhardt proposed a definition in [16].

A related problem is defining intersection Chow groups. Corte–Hanamura [13], [14] proposed two approaches towards intersection Chow groups for general varieties $X$, one which proves a version of the decomposition theorem under some conjectures. In [35], we propose a definition of intersection (graded) gr K-theory of any singular variety $X$ which is a summand of gr $K_*(Y)$ for any resolution of singularities $Y \to X$; the associated graded is taken with respect to the codimension filtration, so its zeroth level is a version of intersection Chow groups. De Cataldo–Migliorini [9] proposed a definition of intersection Chow motive for singularities with a semismall resolution.
1.8. **Future directions.** The definitions of $D_{nc}(X)$ and $\text{IK}(X)$ are for symmetric stacks $X$. It is worth trying to find analogues of these constructions beyond symmetric stacks. One idea is to look for symmetric substacks $X' \subset X$ whose complement is a union of $\Theta$-strata and thus have good moduli spaces $X'$. It is true that $\text{IH}(X)$ is a direct summand of $\text{IH}(X')$, but we do not know how to characterize the difference between them.

The discussion in Subsection 1.2 serves as motivation for our work, but we do not make progress towards the Bondal–Orlov conjecture. In general, $D(X)$ and $D_{nc}(X')$ are different. However, in the cases in which they are equal, for example for quotients $V/T$ where $T$ is a torus and $V$ is a $T$-representation, it is natural to guess that $D(V/T)$ is minimal in the sense of Bondal–Orlov. The category $D(V/T)$ is indecomposable [39, Appendix A], so it should be minimal if the Bondal–Orlov conjecture is true for this particular class of singularities.

We plan to return to these questions in future work.

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2. **Preliminaries**

2.1. **Conventions and notations.** All spaces considered are over $\mathbb{C}$. All schemes considered are of finite type over $\mathbb{C}$. All points considered are $\mathbb{C}$-points unless otherwise stated.

For a scheme with an action of a reductive group $G$, we denote the quotient stack by $A/G$ and its coarse space by $A/\text{sslash}G$.

For $X$ a scheme or stack, denote by $\text{QCoh}(X)$ the category of (unbounded) complexes of quasicoherent sheaves on $X$, by $D^b\text{Coh}(X)$ its subcategory of compact objects, i.e. the derived category of bounded complexes of coherent sheaves, and by $\text{Perf}(X)$ the subcategory of $D^b\text{Coh}(X)$ of perfect complexes. The functors used in the paper are derived; we sometimes drop $R$ or $L$ from notation, for example we write $f_*$ instead of $Rf_*$. Denote by $D_{\text{shvs}}(X)$ the category of complexes of constructible sheaves on a space $X$.

For $G$ a reductive group, fix a maximal torus and a Borel subgroup $T \subset B \subset G$. We denote by $M$ the character lattice, by $N$ the cocharacter lattice, by $W$ the Weyl group, by $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, and by

$$\langle , \rangle : N \times M \to \mathbb{Z}$$

the natural pairing; it induces a pairing between $N$ and $M_{\mathbb{R}}$. We assume that the weights in the Lie algebra of $B$ are negative roots. In particular, $B$ induces a choice of a dominant chamber $M_{\mathbb{R}}^+ \subset M_{\mathbb{R}}$. Denote by $\rho$ half the sum of positive roots of $G$. For $\chi$ a dominant weight of $G$, denote by $\Gamma(\chi)$ the irreducible representation of $G$ of highest weight $\chi$. By abuse of notation, for $V$ a representation of $G$, write

$$\langle \lambda, V \rangle := \langle \lambda, \det V \rangle = \left\langle \lambda, \sum_{\beta \text{ wt of } V} \beta \right\rangle.$$
We denote by \( w \cdot \chi \) the usual \( W \)-action and by \( w \ast \chi = w(\chi + \rho) - \rho \) the shifted \( W \)-action. We denote by \( \mathfrak{g} \) the Lie algebra of \( G \).

For a stack \( \mathcal{X} \), we denote by \( H(\mathcal{X}) \) the singular cohomology with \( \mathbb{Q} \) coefficients and by \( K(\mathcal{X}) \) the rational K-theory of \( \mathcal{X} \).

2.2. Semi-orthogonal decompositions and noncommutative resolutions.

In this Subsection, we recall some basic notions related to derived categories. References for this material are [29], [30].

2.2.1. Let \( T \) be a triangulated category, and let \( A_1, \cdots, A_n \subset T \) be triangulated subcategories. We say that there is a semi-orthogonal decomposition \( T = \langle A_1, \cdots, A_n \rangle \) if the following two conditions are satisfied:

(i) \( \text{Hom}(A_i, A_j) = 0 \) for all \( A_i \in A_i, A_j \in A_j \) and \( 1 \leq j < i \leq n \).

(ii) the smallest triangulated subcategory of \( T \) containing all \( A_i \)'s is \( T \).

2.2.2. Let \( T \) be a triangulated category, and let \( A \) be a subcategory. \( A \) is called right admissible in \( T \) if there exists a semi-orthogonal decomposition \( T = \langle - , A \rangle \). Equivalently, the inclusion functor \( A \hookrightarrow T \) has a right adjoint.

2.2.3. In this paper, we say that a triangulated category \( D \) is smooth if it is admissible inside \( D^b(Y) \) for a smooth DM stack \( Y \). We say that \( D \) is proper over a given variety \( S \) if it is admissible in \( D^b(T) \) for \( T \) a proper DM stack over \( S \).

2.2.4. Let \( X \) be a variety. We say that a smooth and proper over \( X \) triangulated category \( D \) is a noncommutative resolution of singularities (NCR) if there exists an adjoint pair of functors

\[
F : D \to D^b(X),
\]

\[
G : \text{Perf} (X) \to D
\]

such that \( FG = \text{id}_{\text{Perf}(X)} \). The definition is slightly more general than the definition in [30] Definition 3.1 and the paragraph after it.

**Example.** Let \( X \) be a variety with rational singularities and let \( f : Y \to X \) be a resolution of singularities. Then \( D^b(Y) \) is an NCR of \( X \) where the corresponding functors are \( f_* : D^b(Y) \to D^b(X) \) and \( f^* : \text{Perf} (X) \to D^b(Y) \).

2.3. Good moduli spaces.

2.3.1. Let \( \mathcal{X} \) be a stack. An algebraic space \( X \) with a morphism \( \pi : \mathcal{X} \to X \) is called a good moduli space if

(i) \( \pi_* : \text{QCoh} (\mathcal{X}) \to \text{QCoh} (X) \) is exact,

(ii) the induced morphism \( \mathcal{O}_X \to \pi_* \mathcal{O}_X \) is an isomorphism.

**Examples.**

(1) For \( A \) an affine variety and \( G \) a reductive group acting linearly on \( A \),

\[
\pi : A/G \to A \parallel G = \text{Spec} (\mathcal{O}_A^G)
\]

is a good moduli space.
(2) Let \( Y \) be a smooth projective variety, let \( \beta \in H^*(Y) \), and let \( \mathcal{L} \) be an ample line bundle on \( Y \). The moduli stack \( \mathcal{M}_\beta^{ss} \) of \( \mathcal{L} \)-Gieseker semistable sheaves with Chern character \( \beta \) has a good moduli space \( \mathcal{M}_\beta^{ss} \).

For properties and further examples of good moduli space, see Alper [1]. We assume that \( X \) is a scheme in this paper. The following is [2, Theorem 4.12, Theorem 1.2]:

**Theorem 2.1.** Let \( X \) be a stack satisfying Assumption A. Let \( x \in \mathcal{X}(\mathbb{C}) \) with stabilizer group \( G_x \) and normal bundle \( N_x \). Then there exists an affine scheme \( A \) with a linearizable action of \( G_x \), a point \( a \in A(\mathbb{C}) \) with stabilizer group \( G_x \), and étale maps \( e \) and \( f \) such that the following squares are cartesian:

\[
\begin{array}{ccc}
(N_x/G_x, 0) & \xleftarrow{f} & (A/G_x, a) & \xrightarrow{e} & (\mathcal{X}, x) \\
\downarrow & & \downarrow & & \downarrow \\
N_x \sslash G_x & \xleftarrow{f} & A \sslash G_x & \xrightarrow{e} & X.
\end{array}
\]

We will be using the following corollary:

**Corollary 2.2.** Let \( \pi : \mathcal{X} \to X \) be a stack satisfying Assumption A. Let \( p \in X(\mathbb{C}) \). There exists a quotient stack \( p : \mathcal{Y} := V/G \to Y := V \sslash G \) with \( G \) a reductive group, \( V \) a \( G \)-representation, and analytic open sets \( p \in U \subset X \) and \( 0 \in \mathcal{U} \subset Y \) such that the diagram

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xleftarrow{\sim} & p^{-1}(\mathcal{U}) \\
\downarrow & & \downarrow p \\
U & \xleftarrow{\sim} & \mathcal{U}.
\end{array}
\]

commutes. In particular, the diagram

\[
\begin{array}{ccc}
\widehat{\mathcal{X}}_p & \xleftarrow{\sim} & \widehat{\mathcal{Y}}_0 \\
\downarrow & & \downarrow p \\
\widehat{\mathcal{X}}_p & \xleftarrow{\sim} & \widehat{\mathcal{Y}}_0,
\end{array}
\]

where \( \widehat{\mathcal{X}}_p \) is the formal completion of \( \mathcal{X} \) along \( \pi^{-1}(p) \), \( \widehat{\mathcal{X}}_p \) is formal completion of \( \mathcal{X} \) at \( p \) etc.

2.4. **Theta-stratifications.** References for this Section are [20, Section 1], [19, Section 2.1], [3].

2.4.1. Let \( \mathcal{X} \) be an algebraic stack of finite type over \( \mathbb{C} \). Let \( \Theta = \mathbb{A}^1/G_m \). The stacks \( \text{Map}(B\mathbb{G}_m, \mathcal{X}) \) and \( \text{Map}(\Theta, \mathcal{X}) \) are algebraic stacks with natural (evaluation) maps to \( \mathcal{X} \). Their connected components are called fixed stacks and \( \Theta \)-stacks or attracting stacks, respectively. There is also a natural map

\[
\text{Map}(\Theta, \mathcal{X}) \to \text{Map}(B\mathbb{G}_m, \mathcal{X}).
\]
A $\Theta$-stack $S$ has an associated fixed stack $Z$, and fits in a diagram

$$
\begin{array}{ccc}
S & \xrightarrow{p} & X \\
\downarrow q & & \downarrow \\
Z & & 
\end{array}
$$

where $p$ is proper and $q$ is an affine bundle map. If $p$ is an immersion, we say that $S$ is a $\Theta$-stratum.

When $X = V/G$ for $V$ an representation a reductive group $G$, the fixed and $\Theta$-stacks are of the form

$$
\begin{array}{ccc}
V^\lambda_{\geq 0}/G^\lambda_{\geq 0} & \xrightarrow{p} & V/G \\
\downarrow q & & \\
V^\lambda/G^\lambda & & 
\end{array}
$$

where $\lambda : \mathbb{C}^* \to G$ is a cocharacter, $G^\lambda$ is the Levi group associated to $\lambda$, $G^\lambda_{\geq 0}$ is the parabolic group associated to $\lambda$, $V^\lambda \subset V$ is the $\lambda$-fixed locus and $V^\lambda_{\geq 0} \subset V$ is the $\lambda$-attracting locus. Such a $\Theta$-stack is a $\Theta$-stratum if the map $p$ is a closed immersion, so if it is a Kempf–Ness locus in the terminology of [19, Section 2.1].

2.5. Noncommutative motives. We briefly explain the definition of noncommutative motives. A general reference is [41]. Denote by $\text{dgcat}$ the category of (small) dg categories over $\mathbb{C}$. It has a Quillen model structure whose weak equivalences are derived Morita equivalences. Denote by $\text{Hmo}$ the corresponding homotopy category. The universal category through which all additive invariants factor (examples include cyclic homology, $K$-theory, and related constructions) is a smaller (additive) category $\text{Hmo}_0$ with objects dg categories and morphisms

$$
\text{Hom}_{\text{Hmo}_0}(A, B) = K_0(\text{rep}(A, B))
$$

where $A$ and $B$ are dg categories and $\text{rep}(A, B) \subset D^b(A^{op} \otimes B)$ is the full subcategory of bimodules $X$ such that $X(a, -) \in \text{Perf}(B)$ for any object $a \in A$. Consider the functor $U : \text{dgcat} \to \text{Hmo}_0$.

We consider the category $\text{Hmo}_{0; \mathbb{Q}}^\sharp$, the idempotent completion of the $\mathbb{Q}$-linearization of $\text{Hmo}_0$. We call its elements (by a slight abuse) noncommutative motives; the original definition considers the subcategory of $\text{Hmo}_{0; \mathbb{Q}}^\sharp$ generated by proper and smooth dg categories, but in our case we need to allow proper and smooth categories over (not necessarily proper) $X$.

2.6. A preliminary result. The following type of result is used by Špenko–Van den Bergh in their construction of NCRs.

Let $A$ be a smooth affine variety with an action of a reductive group $G$ and let $\mathcal{X} = A/G$. For a locally free sheaf $F$ on $\mathcal{X}$, its stalk at the origin is a representation $\Gamma$ of $G$. We call $\Gamma$ the associated representation of $F$.

We state the following result for future reference, the proof is same as [21, Section 3.2] and it uses an explicit Koszul resolution for pushforward along the map $A^\lambda_{\geq 0}/G^\lambda_{\geq 0} \hookrightarrow A/G^\lambda_{\geq 0}$ and the Borel-Bott-Weil Theorem for the map $A/G^\lambda_{\geq 0} \to A/G$. 

Proposition 2.3. Let \( \lambda \) be a cocharacter of \( G \) and consider the diagram of attracting loci

\[
\begin{array}{c}
A^\lambda/G^\lambda & \overset{q}{\leftarrow} & A^{\lambda>0}/G^{\lambda>0} & \overset{p}{\rightarrow} & A/G.
\end{array}
\]

Let \( \mathcal{F} \) be a locally free sheaf on \( A^{\lambda}/G^\lambda \) with associated representation \( \Gamma(\chi) \) where \( \chi \) is a dominant weight of \( G \). Then there is a complex

\[
\left( \bigoplus_I \mathcal{F}_I[I - \ell(w)], d \right) \rightarrow p_*q^* \mathcal{F},
\]

where the terms of the complex correspond to subsets \( I \subset \{ \beta | \langle \lambda, \beta \rangle < 0 \} \), \( \mathcal{F}_I \) is a locally free sheaf with associated representation

\[
\Gamma \left( \left( \chi - \sigma_I \right)^+ \right),
\]

where \( \sigma_I = \sum_{\beta \in I} \beta, (\chi - \sigma_I)^+ \) is the dominant Weyl-shifted conjugate of \( \chi - \sigma_I \) if it exists, and zero otherwise, and \( w \) is the element of the Weyl group such that \( w^* (\chi - \sigma_I) \) is dominant or zero of length \( \ell(w) \).

3. Noncommutative resolutions of quotient singularities

In this Section, we prove Theorem 1.1. The definition of the categories \( D_{\ell}(X) \) are for symmetric stacks satisfying Assumption A. In order to obtain NCRs, we need to assume that \( X \) satisfies Assumption C. Recall the construction of category \( D(X) \) and the strategy of proof discussed in Subsection 2.3.

3.1. Local case. Let \( X = V/G \) where \( V \) is a symmetric \( G \)-representation. Denote by \( X = V \sslash G \) and by \( p : X \rightarrow X \). We will use the notations from Subsection 2.1.

For a cocharacter \( \lambda : \mathbb{C}^* \rightarrow G \), recall the diagram of attracting loci \( (6) \) and define

\[
n_{\lambda} := \langle \lambda, V^{\lambda>0} \rangle - \langle \lambda, g^{\lambda>0} \rangle = \langle \lambda, \det L_p \rangle = \langle \lambda, \det L_{\lambda>0} \rangle.
\]

Let \( \delta \in M^W \), and let \( \mathbb{D}_{\delta}(X) \) be the full subcategory of \( D^b(X) \) generated by complexes \( \mathcal{F} \) such that for any cocharacter \( \lambda : \mathbb{C}^* \rightarrow G \):

\[
-\frac{n_{\lambda}}{2} + \langle \lambda, \delta \rangle \leq \langle \lambda, \mathcal{F} \rangle_0 \leq \frac{n_{\lambda}}{2} + \langle \lambda, \delta \rangle.
\]

Let \( W \subset \mathbb{R} \) be the polytope

\[
W := \text{sum } [0, \beta] \subset \mathbb{R},
\]

where the Minkowski sum is taken over all weights \( \beta \) of \( V \). The category \( \mathbb{D}_{\delta}(X) \) can be also described as the full subcategory of \( D^b(X) \) generated by vector bundles \( \mathcal{O}_X \otimes \Gamma(\chi) \) where \( \chi \) is a dominant weight of \( G \) such that

\[
\chi + \rho + \delta \in \frac{1}{2} W,
\]

where the sum is taken over all weights \( \beta \) of \( V \).

Define \( \mathcal{A}_{\delta} \) as the subcategory of \( D^b(X) \) generated by complexes \( p_{\lambda} q_{\lambda}^*(\mathcal{E}) \) where \( \mathcal{E} \) is a complex in \( D^b(X^\lambda) \) with

\[
\langle \lambda, \mathcal{E} \rangle_0 < -\frac{n_{\lambda}}{2} + \langle \lambda, \delta \rangle.
\]
The following was proved by Špenko–Van den Bergh \[38\] Proposition 8.4 (the semi-orthogonal decomposition in loc. cit. holds for quotient stacks satisfying Assumption A, the condition that $X$ has a $T$-fixed stable point in loc. cit. is necessary to identify the summands with NCRs):

**Theorem 3.1.** There exists a semi-orthogonal decomposition

$$D^b(X) = \langle A_\delta, D_\delta \rangle.$$ 

The semi-orthogonal decomposition holds relative to $X$ in the following sense: if $A \in A_\delta$ and $D \in D_\delta$, then

$$R\pi_* (R\text{Hom}_X(D, A)) = 0.$$ 

3.2. **Global case.** Assume that $\pi : X \to X$ is a symmetric stack satisfying Assumption B. Let $\ell \in \text{Pic}(X)_R$. Recall the definition of $D_\ell(X)$ from Subsection 1.3. As in Step 1, define $A_\ell$ as the subcategory of $D^b(X)$ generated by complexes of sheaves $p^* q^*(E)$, where $S$ is a $\Theta$-stack, $Z$ is its associated fixed stack with maps $Z \xleftarrow{\sim} S \xrightarrow{p} X$, and $E$ satisfies

$$\text{wt } \lambda^*(E) < -\frac{n_\lambda}{2} + \langle \lambda, \delta \rangle.$$ 

**Theorem 3.2.** There is a semi-orthogonal decomposition

$$D^b(X) = \langle A_\ell, D_\ell \rangle.$$ 

The semi-orthogonal decomposition holds relative to $X$ in the following sense: if $A \in A_\ell$ and $D \in D_\ell$, then

$$R\pi_* (R\text{Hom}_X(D, A)) = 0.$$ 

For $p \in X$, let $D^b_o(\hat{X}_p)$ be the split category generated by the restrictions of complexes in $D^b(X)$. For $U \subset X$ as in Corollary 2.2 let $X_U := \pi^{-1}(U)$ and define $D^b_o(X_U)$ as the split category of coherent analytic sheaves generate by restrictions of complexes in $D^b(X)$. For $Y = V/G$, the category $D^b(Y)$ is generated by $O_Y \otimes \Gamma(\chi)$ for $\chi$ a dominant weight of $G$, and thus $D^b(\hat{Y}_0)$ is generated by $O_{\hat{G}} \otimes \Gamma(\chi)$. By Theorem 2.1, the categories $D^b_o(X_U)$ and $D^b_o(\hat{X}_p)$ are also generated by these vector bundles.

Further, let $\hat{D}_{\ell, p} \subset D^b_o(\hat{X}_p)$ be the category generated by restrictions of sheaves in $D^b(X)$; we define $A_{\ell, p}$, $D_{\ell, U}$ etc. similarly. For the stack $Y = V/G$ from Corollary 2.2 let $\delta \in M(G)_{\mathbb{R}}$ be the restriction of $\ell$ and we denote by $A_\delta$ and $D_\delta$ the categories from Theorem 3.1. By Corollary 2.2 and using the notation from there, we have that

$$D_{\ell, U} \cong D^f_{\delta, U},$$

$$A_{\ell, U} \cong A^f_{\delta, U}$$

and the analogues equivalences for formal completions.
Proof of Theorem 3.2. We continue with the notations from the above. Let $A \in \mathfrak{A}_\ell$ and $D \in \mathfrak{D}_\ell$. To show (9), it suffices to prove the statement after restriction to $\hat{\mathfrak{X}}_p$ for all $p \in X$. Then $A \in \mathfrak{A}^\prime_{\delta,0}$ and $D \in \mathfrak{D}^\prime_{\delta,0}$ and thus the claim follows from Theorem 3.1. To show that $\mathfrak{A}_\ell$ and $\mathfrak{D}_\ell$ generate $D^b(X_U)$, observe that they generate the local categories $D^b(X_U)$. The claim follows as in [38, Proposition 3.5.8]. □

3.3. NCR. We show that $D_\ell(X)$ is a smooth and proper over $X$ category. More precisely, $D(X)$ is an admissible subcategory of the Kirwan resolution of $X$ as constructed by Edidin–Rydh [17]. In loc. cit., the authors do not use the language of $\Theta$-strata, but their construction is natural and applied to quotient stacks recovers Kirwan’s resolution of singularities [26, pages 475-476]. For a stack $X$ satisfying Assumption B, the Edidin–Rydh construction provides a sequence of stacks

$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow Y$,

with the following properties:

(i) the stacks $X_i$ have good moduli spaces $X_i$,
(ii) $X^ss_i \subset X_i$ is an open subset, complement to $\Theta$-strata,
(iii) étale locally on $X_k$, either $X_k \leftarrow X_{k+1}$ are isomorphisms, or there are neighborhoods as in Theorem 2.1,
(iv) The stack $Y := X^{ss}_{n+1}$ is a smooth Deligne–Mumford stack with a proper map $Y \to X$ of relative dimension zero.

The class $\ell \in \text{Pic}(X)^R$ induces classes which we also denote by $\ell \in \text{Pic}(X_k)^R$ for $1 \leq k \leq n + 1$.

Proposition 3.3. Let $0 \leq k \leq n$. Assume that $X_k$ is a symmetric stack. Then $X^ss_{k+1}$ is also a symmetric stack.

Proof. It suffices to check the statement in the local case

$V/G \leftarrow (\text{Bl}_0 V)/G \leftarrow (\text{Bl}_0 V)^{ss}/G$,

where $G$ is a reductive group, $V$ is a symmetric $G$-representation with 0 the only $G$-fixed point in $V$, and the linearization is given by the tautological line bundle $O(1)$.

We claim that the unstable loci of $\text{Bl}_0 V \cong \text{Tot}_{\mathcal{P}(V)}(O(-1))$ are determined by pairs

$$(\lambda, \mathbb{P}(V_0))$$
where $\lambda: \mathbb{C}^* \to G$, $V_0 \subset V$ is the subspace on which $\lambda$ acts with weight $a$, and $a < 0$. The GIT algorithm, see [19, Section 2.1], eliminates pairs $(\lambda, Z)$ for $\lambda: \mathbb{C}^* \to G$, $Z$ is a $\lambda$-fixed component on $\text{Tot}_{\mathbb{P}(V)} (\mathcal{O}(-1))$, and

$$\text{wt}_\lambda \mathcal{O}(1)|_Z > 0.$$  

The fixed loci $Z$ are $\mathbb{P}(V_a)$ for $a \neq 0$ and $\text{Tot}_{\mathbb{P}(V_0)} (\mathcal{O}(-1))$. Further, we compute

$$\text{wt}_\lambda \mathcal{O}(1)|_{\mathbb{P}(V_a)} = \sum_{n \in \mathbb{Z}} (n - a) \dim V_n > 0.$$  

The representation $V$ is symmetric, so $\dim V_i = \dim V_{-i}$ for $i \in \mathbb{Z}$. We thus have that

$$0 < \sum_{n \in \mathbb{Z}} n \cdot \dim V_n - a \cdot \dim V = -a \cdot \dim V,$$

so indeed $a < 0$.

Finally, let $\lambda: \mathbb{C}^* \to G$ be a cocharacter which fixes a point $v$ in $\text{Tot}_{\mathbb{P}(V)} (\mathcal{O}(1))$. If it lies on a $\lambda$-fixed component $\mathbb{P}(V_a)$ for $a \neq 0$, it is part of an unstable locus for either $\lambda$ or $\lambda^{-1}$. Thus it lies on $\text{Tot}_{\mathbb{P}(V_0)} (\mathcal{O}(1))$; the normal bundle is $\bigoplus_{i \in \mathbb{Z} \setminus \{0\}} V_i$, which is symmetric.

**Proposition 3.4.** Let $0 \leq k \leq n$. The category $\mathbb{D}(X_k)$ is admissible in $\mathbb{D}(X_{k+1})$.

**Proof.** To simplify the notation, let $X := X_k$, $Y := X_{k+1}$. We will see that the unstable loci are indexed by $(\lambda, a)$, where $\lambda: BG_m \to X$ and $a \in \mathbb{Z}$. For each $\Theta$-stratum $S_{\lambda, a}$ with associated fixed stack $Z_{\lambda, a}$, choose a real number $w_{\lambda, a} \notin \mathbb{Z}$. By [20, Theorem 3.9], there is an admissible subcategory $G \hookrightarrow D^b(Y)$ with objects complexes $F$ such that for any $\Theta$-stratum $S_{\lambda, a}$:

$$-\frac{n_{\lambda, a}}{2} + \lambda^* (\ell) + w_{\lambda, a} \leq \lambda^* F \leq \frac{n_{\lambda, a}}{2} + \lambda^* (\ell) + w_{\lambda, a}$$

with the property that

$$\text{res} : G \cong D^b(Y^{ss}).$$

We next characterize $\Theta$-strata and compute $n_{\lambda, a}$. We can assume that we are in the local case

$$V/G \leftarrow \text{Bl}_0 V/G \cong \text{Tot}_{\mathbb{P}(V)} (\mathcal{O}(-1)) \leftarrow (\text{Bl}_0 V)^{ss}/G,$$

where $G$ is a reductive group, $V$ is a symmetric $G$-representation with 0 the only $G$-fixed point in $V$, and the linearization is given by the tautological line bundle $\mathcal{O}(1)$. By the argument of Proposition 3.3, the unstable loci are determined by pairs $(\lambda, \mathbb{P}(V_a))$

where $V_a$ is the $\lambda$-weight $a$ subspace of $V$ and $a < 0$. The $\lambda$-positive part of the normal bundle is

$$N_{\mathbb{P}(V_a)/\mathbb{P}(V)}^{\lambda>0} \cong \bigoplus_{i > a} V_i.$$
and $\lambda$ acts with weight $i + a$ on $V_i$. The length of the window $n_{\lambda,a}$ is thus:

\begin{align*}
\langle \lambda, N^\lambda >_0 \rangle & - \langle \lambda, g^\lambda >_0 \rangle = \\
\sum_{i > a} (i + a) \dim V_i & - \langle \lambda, V^\lambda >_0 \rangle > - \langle \lambda, g^\lambda >_0 \rangle.
\end{align*}

The category $\mathcal{D}_\ell(\mathcal{X}) \subset D^b(\mathcal{X})$ is defined by the conditions

\begin{align*}
- \frac{m_{\lambda}}{2} + \lambda^*(\ell) & \leq \text{wt} \lambda^* \mathcal{F} & \leq \frac{m_{\lambda}}{2} + \lambda^*(\ell),
\end{align*}

where, in the local model $\mathcal{X} = V/G$ from above, $m_{\lambda} = \langle \lambda, V^\lambda >_0 \rangle - \langle \lambda, g^\lambda >_0 \rangle$.

Similarly, the category $\mathcal{D}_\ell(\mathcal{Y}^{ss})$ of $D^b(\mathcal{Y}^{ss})$ is defined by the conditions

\begin{align*}
- \frac{m_{\lambda}}{2} + \lambda^*(\ell) & \leq \text{wt} \lambda^* \mathcal{F} & \leq \frac{m_{\lambda}}{2} + \lambda^*(\ell).
\end{align*}

Indeed, a fixed substack $Z$ associated to a map $\lambda : B\mathbb{G}_m \rightarrow \mathcal{X}$ is isomorphic, in the local model, to $\text{Tot} \mathcal{P}(V_0)(\mathcal{O}(-1))$, where $V_0 \subset V$ is the $\lambda$-fixed locus. This follows from the analysis at the end of the proof of Proposition 3.3.

The $\lambda$-fixed loci of $\mathcal{Y}^{ss}$ are all above $\lambda$-fixed loci of $\mathcal{X}$. Using (10), (11), (12), we can thus choose weights $w_{\lambda,a} \in \mathbb{Z}$ such that $\pi^* : D^b(\mathcal{X}) \rightarrow D^b(\mathcal{Y})$

\[\mathcal{D}_\ell(\mathcal{X}) \subset \mathcal{D}_\ell(\mathcal{Y}^{ss}).\]

Let $\Phi : D^b(\mathcal{X}) \rightarrow \mathcal{D}_\ell(\mathcal{X})$ be the right adjoint of $\mathcal{D}_\ell(\mathcal{X}) \hookrightarrow D^b(\mathcal{X})$ which exists by Theorem 3.2, see Subsection 2.2.2. Consider the functor $\Phi \pi_* : \mathcal{D}_\ell(\mathcal{Y}^{ss}) \rightarrow \mathcal{D}_\ell(\mathcal{X})$, where recall that $\pi_*$ is the derived functor. We claim that $\pi^*$ and $\Phi \pi_*$ are adjoint. Let $A \in \mathcal{D}_\ell(\mathcal{X})$ and $B \in \mathcal{D}_\ell(\mathcal{Y}^{ss})$. Then

\[\text{RHom}_\mathcal{Y}(\pi^* A, B) \cong \text{RHom}_\mathcal{X}(A, \pi_* B) \cong \text{RHom}_\mathcal{X}(A, \Phi \pi_* B).\]

Finally, the functor $\pi^*$ is fully faithful. By the projection formula, it suffices to show that $\pi_* \mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$, which follows from a direct computation in the local case. Thus $\mathcal{D}_\ell(\mathcal{X})$ is admissible in $\mathcal{D}_\ell(\mathcal{Y}^{ss})$.

We thus obtain that:

**Corollary 3.5.** The category $\mathcal{D}_\ell(\mathcal{X})$ is admissible in $D^b(\mathcal{Y})$.

We finally show that $\mathcal{D}(\mathcal{X}) := \mathcal{D}_0(\mathcal{X})$ is an NCR of $X$.

**Proposition 3.6.** Let $\ell \in \text{Pic}(\mathcal{X})_\mathbb{R}$ be such that for any cocharacter $\lambda$,

\[-\frac{n_{\lambda}}{2} + \lambda^*(\ell) \leq 0 \leq \frac{n_{\lambda}}{2} + \lambda^*(\ell).\]

Then $\mathcal{D}_\ell(\mathcal{X})$ is an NCR of $X$. 

Proof. Consider the inclusion functor and its natural adjoint obtained by Theorem 3.2 and the discussion in Subsection 2.2.2:

\[ \iota : D^b(\mathcal{X}) \hookrightarrow D^b(\mathcal{X}) \]

\[ \Phi : D^b(\mathcal{X}) \rightarrow D^b(\mathcal{X}). \]

Consider the functors

\[ \pi_* \iota : D^b(\mathcal{X}) \rightarrow D^b(\mathcal{X}) \]

\[ \Phi \pi^* : \text{Perf}(X) \rightarrow D^b(\mathcal{X}). \]

We need to show that for \( F \in \text{Perf}(X) \), we have that \( \pi_* \iota \Phi \pi^*(F) = F \).

The complexes \( \pi^*F \) have weight zero for any cocharacter \( \lambda \), and thus they are in \( D^b(\mathcal{X}) \). This means that \( \iota \Phi \pi^*F = \pi^*F \). We have that

\[ \pi_* \iota \Phi \pi^*(F) = \pi_* \pi^*(F) = F \otimes \pi_* O_X = F. \]

The last equality follows from \( \pi_* O_X = O_X \), see Subsection 2.3. Finally, the category \( D^b(\mathcal{X}) \) is smooth and proper over \( X \) by Corollary 3.5, and thus it is an NCR of \( X \).

\[ \square \]

4. Intersection cohomology for quotient singularities

In this Section, \( \mathcal{X} \) is a symmetric stack satisfying Assumption B. Let

\[ \pi : \mathcal{X} \rightarrow X \]

be the good moduli space morphism. Denote by \( I \) the set of connected components of \( \text{Map}(BG_m, \mathcal{X}) \), by \( o \) the connected component \( \mathcal{X} \), and let \( J := I - o \). Further, let \( I' \) be the set of connected components of \( \text{Map}(\Theta, \mathcal{X}) \), \( o \) the connected component corresponding to \( \mathcal{X} \), and let \( J' := I' - o \). For an attracting stack \( S \) in \( J' \) with associated fixed stack \( Z \), consider the map

\[ p_{S*} q^*_S : H^i(Z) \rightarrow H^i(\mathcal{X}). \]

Define

\[ B := \text{image} \left( \bigoplus_{J'} p_{S*} q^*_S : H^i(Z) \rightarrow H^i(\mathcal{X}) \right). \]

Recall that \( D_{\text{shvs}}(X) \) is the category of complexes of constructible sheaves on a space \( X \). Let \( (\tau^{\leq i}, \tau^{> i}) \) be the functors corresponding to the usual t-structure on \( \mathcal{D}_{\text{shvs}}(X) \) and let \( D^\leq_{\text{shvs}}(X) \) the subcategory of sheaves with \( \tau^{> i} = 0 \). Let \( (p^{\tau^{\leq i}}, p^{\tau^{> i}}) \) be the functors associated to the perverse t-structure on \( D_{\text{shvs}}(X) \) and denote by \( pD^\leq_{\text{shvs}}(X) \) the subcategory of sheaves with \( p^{\tau^{> i}} = 0 \). The map \( \pi \) induces a perverse filtration:

\[ \mathbf{P}^{\leq i} H^i(\mathcal{X}) := \text{image}(H^i(X, p^{\tau^{\leq i}} R\pi_* IC_{\mathcal{X}}) \rightarrow H^i(\mathcal{X}, R\pi_* IC_{\mathcal{X}})) \subset H^i(\mathcal{X}). \]

If \( \mathcal{X} \) satisfies Assumption C, then \( \mathbf{P}^{\leq 0} H^i(\mathcal{X}) \cong \mathcal{H}^i(X) \) by Proposition 4.6. The main result we prove in this Section is:
Theorem 4.1. Let $X$ be a symmetric stack satisfying Assumption B. Then
\[ H^i(X) = P^\leq 0 H^i(X) \oplus B. \]

We first explain that the BBDG Decomposition Theorem \cite{5} implies a Decomposition Theorem for the map $\pi$.

Proposition 4.2. There is a decomposition
\[ R\pi_*IC_X \cong \bigoplus_{i \geq 0} pH^i(R\pi_*IC_X)[-i] \]
and each sheaf $pH^i(R\pi_*IC_X)$ is a direct sum of sheaves $IC_Z(L)$ for $Z \subset X$ and $L$ a local system on an open smooth subset of $Z$.

An explicit computation of $R\pi_*IC_X$ for $X$ a stack of representation of a quiver appears in \cite[Theorem 4.6]{33}, \cite[Proof of Theorem 4.10]{15}.

Proof. The idea, inspired by Totaro's approximations of quotient stacks, is to approximate the stack $X$ with smooth varieties proper over $X$ and apply the BBDG Decomposition Theorem \cite{5}. By Corollary \ref{2.2} it suffices to treat the local case $X = V/G$.

For $n \geq 1$, consider the stacks
\[ X_n := (V \oplus V^{\oplus n}) / G \times \mathbb{C}^*, \]
where $\mathbb{C}^*$ acts with weight zero on the first copy of $V$ and with weight 1 on the summand $V^{\oplus n}$, and $G$ acts naturally on all copies of $V$. Consider the linearization
\[ G \times \mathbb{C}^* \to \mathbb{C}^*. \]
Define the schemes
\[ S_n := (V \oplus V^{\oplus n})^{st} \sslash G \times \mathbb{C}^*, \]
\[ S_n^o := (V \oplus V^{\oplus n})^{st, nf} \sslash G \times \mathbb{C}^*. \]
The superscript $nf$ means that we take the open subset of the stable locus $(V \oplus V^{\oplus n})^{st}$ not fixed by any elements of $G$. Consider the natural maps:
\[ \begin{array}{ccc}
X_n & \xrightarrow{i_n} & S_n^o \\
\downarrow t_n & & \downarrow t_n \\
X & \xrightarrow{\pi} & S_n \\
\downarrow q_n & & \\
X.
\end{array} \]
Denote by $\pi_n := \pi t_n : X_n \to X$. We discuss two preliminary results.

Proposition 4.3. We have that $(V \oplus V^{\oplus n})^{st} = (V \oplus V^{\oplus n})^{ss}$.

Proof. Let $\mu$ be a cocharacter of $G \times \mathbb{C}^*$. It is enough to show that there are no $\mu$-fixed points on $(V \oplus V^{\oplus n})^{ss}$. Write $\mu = \lambda \cdot z^b$ for $b \in \mathbb{Z}$, $\lambda$ a cocharacter of $G$, and $z$ the identity cocharacter of $\mathbb{C}^*$. 

If \( b > 0 \), then \((V \oplus V^\oplus)^{\mu}\) is unstable. If \( b < 0 \), then \((V \oplus V^\oplus)^{\mu-1}\) is unstable. If \( b = 0 \), then 
\[
(V \oplus V^\oplus)^{\lambda} \subset (V \oplus V^\oplus)^{\lambda+1},
\]
so \((V \oplus V^\oplus)^{\lambda}\) is contained in an unstable locus. \(\square\)

**Proposition 4.4.** Fix \( a \in \mathbb{Z} \). For \( n \) large enough, we have that
\[
p_{\tau}^{-\leq a} R_{\pi_n S_n} Q_{X_n} \cong p_{\tau}^{-\leq a} R_{\pi_n S_n} i_{n\ast} Q_{S_n} \]
\[
p_{\tau}^{-\leq a} R_{\eta_n S_n} Q_{S_n} \cong p_{\tau}^{-\leq a} R_{\pi_n S_n} i_{n\ast} Q_{S_n}.
\]

**Proof.** We only explain the first equality; the second one is similar. All complexes consider have cohomology of finite dimension and are bounded on the left. For stacks, functoriality of such complexes is discussed by Laszlo–Olsson [24, 25]. Consider the substacks
\[
j_n \colon Z_n \hookrightarrow X_n \hookrightarrow S_n^0 : i_n.
\]
There is a distinguished triangle in \( D_{\text{shvs}}(Y) \):
\[
j_n j_n^1 Q_{X_n} = j_n i_{n\ast} \omega_{Z_n}[-2 \dim X_n] \to Q_{X_n} \to i_{n\ast} Q_{S_n} \xrightarrow{[1]}.
\]
The complex \( \omega_{Z_n} \) is in \( D_{\text{shvs}}^{-2 \dim Z_n}(Z_n) \), see [23, Section V.2]. Let \( c_n \) be the codimension of \( Z_n \) in \( X_n \). Then \( \omega_{Z_n}[-2 \dim X_n] \in D_{\text{shvs}}^{\geq 2 c_n}(Z_n) \). Pushforward preserves the categories \( D^\geq \), so
\[
R_{\pi_n S_n} i_{n\ast} \omega_{Z_n}[-2 \dim X_n] \in D_{\text{shvs}}^{\geq 2 c_n}(X).
\]
There is a constant \( b \) only depending on \( X \) such that
\[
R_{\pi_n S_n} i_{n\ast} \omega_{Z_n}[-2 \dim X_n] \in D_{\text{shvs}}^{b+2 c_n}(X),
\]
and this implies the desired conclusion for large enough \( n \). \(\square\)

Now we continue the proof of Proposition 4.2. By Proposition 4.3, the variety \( S_n \) has finite quotient singularities, thus \( IC_{S_n} \cong Q_{S_n}[\dim S_n] \). By Proposition 4.4, we have that for \( m > n \) large enough,
\[
p_{\tau}^{-\leq a} R_{\pi_n S_n} Q_{X_n} \cong p_{\tau}^{-\leq a} R_{\eta_n S_n} Q_{S_n} \cong p_{\tau}^{-\leq a} R_{\pi_m S_n} Q_{S_n}.
\]

Fix \( a \in \mathbb{Z} \). The complex \( R_{\pi_n S_n} Q_X \) is a direct summand of \( R_{\pi_n S_n} Q_{X_n} \), so \( p_{\tau}^{-\leq a} R_{\pi_n S_n} Q_X \) is a direct summand of \( p_{\tau}^{-\leq a} R_{\pi_n S_n} Q_{X_n} \). The Decomposition Theorem for the maps \( q_n \) implies the desired conclusion. \(\square\)

For a cocharacter \( \lambda \), define \( c_\lambda := \dim X - \dim X^{\lambda>0} \).

**Proposition 4.5.** Assume that \( X = V/G \). Let \( \lambda \) be a cocharacter of \( G \) and let \( B \) be a semisimple summand of \( \mathcal{H}^i(R_{\pi_n S_n} IC_X) \) with support contained in the image of \( X^\lambda \to X \). Then
\[
B \subset \text{image } (p_{\tau}^{\mathcal{H}^i(R_{\pi_n S_n} IC_X)[-c_\lambda]}) \to p_{\tau}^{\mathcal{H}^i(R_{\pi_n S_n} IC_X)}.
\]
Proof. We use the same notations as in the proof of Proposition \ref{prop:IC}. First, for any \( n \), there is an isomorphism
\begin{equation}
Rt_{n*}Q_{X_n} \cong Q_X \otimes Q[h],
\end{equation}
where \( Q[h] \) is the polynomial ring in a generator \( h \) of cohomological degree 2. It suffices to show the statement for summands \( B \) of \( R\pi_{n*}Q_{X_n} \). Choose \( n \) such that \( B \) is a summand of \( pH_i (Rq_{n*}Q_{S_n}[\dim X]) \). Define
\begin{align*}
Y_n := (V^{\lambda \geq 0} \oplus V^{\oplus n}) / G \times \mathbb{C}^*, \\
W_n := (V^{\lambda \geq 0} \oplus V^{\oplus n}) \text{st} / G \times \mathbb{C}^*, \\
W_n^o := (V^{\lambda \geq 0} \oplus V^{\oplus n}) \text{st,nf} / G \times \mathbb{C}^*.
\end{align*}
It suffices to show that:
\begin{equation}
B \subset \text{image} \left( (p^H (R\pi_{n*}p_*IC_{Y_n}[-c_\lambda]) \to p^H (R\pi_{n*}IC_{X_n})) \right).
\end{equation}
Consider the diagram
\begin{equation*}
\begin{array}{ccc}
S_n & \leftarrow & W_n \\
q_n \downarrow & & \downarrow p_n \\
X & \leftarrow & X^\lambda.
\end{array}
\end{equation*}
By \cite{14} Proposition 1.5, \( B \) appears in the image of
\begin{equation*}
p^H (Rr_{n*}IC_{W_n}[-c_\lambda]) \to p^H (Rq_{n*}IC_{S_n}).
\end{equation*}
Using an argument similar to Proposition \ref{prop:IC} for \( Y_n, W_n, \) and \( W_n^o \), the claim in (14) follows.

\begin{proposition}
Let \( X = V/G \) be a symmetric stack satisfying Assumption B. Let \( \lambda \) be a non-zero cocharacter of \( G \). Then
\begin{equation*}
\text{image} \left( R\pi_*p_*IC_{X^{\lambda \geq 0}[-c_\lambda]} \to R\pi_*IC_X \right) \subset pD_{sh}^\geq 1(X).
\end{equation*}
In particular, \( p_\tau \leq 0 R\pi_*IC_X \) is a direct sum of IC sheaves with full support.
\end{proposition}

Proof. We show the first statement. We use induction on \( \dim G \). Consider the diagram
\begin{equation*}
\begin{array}{ccc}
X^{\lambda \geq 0} & \to & X \\
q \downarrow & & \downarrow p \\
X^\lambda & \to & X. \\
p^\lambda \downarrow & & \downarrow \pi \\
X^\lambda & \to & X.
\end{array}
\end{equation*}
There are natural maps
\begin{equation*}
p_*q^*IC_{X^\lambda} \to p_*IC_{X^{\lambda \geq 0}[-c_\lambda]} \to IC_X.
\end{equation*}
The map \( q \) is an affine bundle map, so \( q_* \mathcal{Q}_{X^\lambda} = \mathcal{Q}_{X^\lambda} \). The stack \( X \) is symmetric, so \( \text{reldim} \ q = c_\lambda \). We thus need to show that

\[
\text{image} \left( R\pi_* \mathcal{I}_C \to R\pi_* \mathcal{I}_C \right) \subset P_{D_{\text{shvs}}}^1(X).
\]

Let \( \tilde{G}^\lambda \) be the quotient of \( G^\lambda \) by the torus which acts trivially on \( V^\lambda \). Then

\[
\tilde{\pi}^\lambda : \tilde{X}^\lambda := V^\lambda / \tilde{G}^\lambda \to X^\lambda
\]

is a good moduli space and \( \tilde{X}^\lambda \) is a symmetric stack satisfying Assumption B. We thus have that

\[
R\tilde{\pi}_* \mathcal{I}_{\tilde{X}^\lambda} \in P_{D_{\text{shvs}}}^0(X^\lambda)
\]

\[
R\pi_* \mathcal{I}_{X^\lambda} \in P_{D_{\text{shvs}}}^1(X^\lambda).
\]

The second statement follows from Proposition 4.5 and the fact that \( \pi \) is generically finite. \( \square \)

We next discuss some preliminary computations regarding the maps \( m_S \). For \( X = A/G \) for \( A \) an affine scheme (but one can assume for simplicity that \( A \) is an affine space by Corollary 2.2) and \( \lambda : \mathbb{C}^* \to G \), denote by

\[
m_{\lambda} := p_{\lambda-1} q_{\lambda-1}^*: H(A^\lambda) \to H(X).
\]

Let \( N := N_{a/A} \) be the normal bundle, and denote by \( A \) for the set of weights in \( N \), \( A_\lambda \), \( g_\lambda \) for the set of weights (counted with multiplicities) of \( N^\lambda > 0 \), \( g^\lambda > 0 \) etc. For \( \beta \) a weight of \( G \), denote by \( h_\beta \in H^2(T) \). The following computation are standard, for similar computations see [28, Theorem 2.2], [43, Proposition 1.2]:

**Proposition 4.7.** Let \( \lambda \) be a cocharacter of \( G \). Let \( x \in H(A^\lambda) \). Then

\[
m_{\lambda}(x) = \sum_{w \in W/W^\lambda} w \left( (-1)^{|A_\lambda| - |g_\lambda|} \prod_{\beta} h_\beta \prod_{\alpha} \frac{h_\beta}{g_\alpha} \right).
\]

**Proposition 4.8.** Let \( \lambda \) and \( \mu \) be cocharacters of \( G \) with the same associated Levi group \( L \). Then

\[
\text{image} \left( H(A^\lambda) \xrightarrow{m_{\lambda}} H(X) \right) = \text{image} \left( H(A^\lambda) \xrightarrow{m_{\mu}} H(X) \right).
\]

**Proof.** Let \( y \in H(A^\lambda) \). By Proposition 4.7 we have that:

\[
m_{\lambda}(y) = \pm \sum_{w \in W/W^\lambda} w \left( \prod_{\beta} h_\beta \prod_{\alpha} \frac{h_\beta}{g_\alpha} \right).
\]

\[
m_{\mu}(y) = \pm \sum_{w \in W/W^\lambda} w \left( \prod_{\beta} h_\beta \prod_{\alpha} \frac{h_\beta}{g_\alpha} \right).
\]

The representation \( N \) is symmetric, so

\[
\{ \pm h_\beta | \beta \in N^\lambda > 0 \} = \{ \pm h_\beta | \beta \in N^\mu > 0 \} = \{ h_\beta | \beta \in N/N^L \},
\]

and thus \( m_{\lambda}(y) = \pm m_{\mu}(y) \). \( \square \)
Let $\lambda$ be a cocharacter of $G$. Let $T^\lambda \subset G^\lambda$ be the torus which acts trivially on $A^\lambda$, and let $\widetilde{G}^\lambda := g^\lambda/T^\lambda$. Consider the map
\[
\widetilde{\pi}^\lambda : \widetilde{X}^\lambda := A^\lambda/\widetilde{G}^\lambda \to X^\lambda.
\]
Define
\[
H'(X^\lambda)' := H^\left(p_{p<0} R_{\pi^\lambda}^* \operatorname{IC}_{X^\lambda}\right) \otimes \mathbb{Q}[t^\lambda] \to H(X^\lambda),
\]
where the generators of $t^\lambda$ have cohomological degree 2. For each Levi group $L$, choose a cocharacter $\lambda_L$ such that $L \cong G^\lambda$.  

**Proposition 4.9.** We have that
\[
B \cong \text{image} \left( \bigoplus_L H\left(\chi^{\lambda_L}\right) \to H(X) \right)
\]
\[
B \cong \text{image} \left( \bigoplus_L H\left(\chi^{\lambda_L}\right)' \to H(X) \right).
\]

**Proof.** The first equality follows from Proposition 4.8 and the second follows by induction and the Decomposition Theorem. \qed

**Proof of Theorem 4.1.** Let $p \in X$ and recall the setting from Corollary 2.2. The restriction of fixed and $\Theta$-stacks to $p^{-1}(U)$ and $p^{-1}(U)$ are in a natural bijection. It suffices to prove the statement in the local case $X = V/G$.

Choose a splitting in the decomposition theorem for $\pi : X \to X$:
\[
H(X) \cong \bigoplus_i pH^i(X),
\]
where $pH^i(X) := H^i(pH^i(R\pi_* \operatorname{IC}_X))$. For $L$ a Levi subgroup, denote by $H(X)_{X^L}$ the cohomology of the summands in $\bigoplus_i pH^i(R\pi_* \operatorname{IC}_X)$ with support $X^L \to X$. These are all the supports that appear in the Decomposition Theorem for the map $\pi : X \to X$. Further, by Propositions 4.5, 4.6, and 4.8,
\[
\text{image} \left( H\left(\chi^{\lambda_L}\right)' \to H(X) \right) = H(X)_{X^L} \cong \bigoplus_{i \geq 1} pH^i(X)_{X^L}.
\]
The conclusion follows from Proposition 4.6. \qed

5. **Categorification of IH for quotient singularities**

In this Section, assume that $\mathcal{X}$ satisfies Assumption B and is symmetric. We will use the category $\mathbb{D}(\mathcal{X})$ for $\ell$ zero. For $\mathcal{X} = A/G$ as in Theorem 2.1 denote by $N = N_{a/A}$ the normal bundle. We write $A_\lambda$, $g_\lambda$, $n$, $n^\mu$ etc. for the sets of weights (counted with multiplicities) of $N^{\lambda>0}$, $g^{\lambda>0}$, $n$, $n^\mu$ etc. We (abuse notation and) denote by $N^{\lambda>0}$, $g^{\lambda>0}$ etc. the sum of weights in $A_\lambda$, $g_\lambda$ etc. Recall that $K$ denotes rational K-theory.

5.1. **Notations and definitions.** We begin with some preliminary constructions and definitions.
5.1.1. There is a natural isomorphism

\[ K(A/G) \cong K(A/T)^W. \]

5.1.2. Let \( \chi \) be a weight of \( T \). Denote by \( K(A/T)_{\chi} \) the subspace of \( K(A/T) \) which is the image in K-theory of the inclusion \( D^b(A/T)_{\chi} \subset D^b(A/T) \) of the subcategory generated by locally free \( T \)-equivariant sheaves with \( T \)-weight \( \chi \).

5.1.3. Let \( \lambda \) be a cocharacter. For \( I \) a subset of \( A_\lambda \), denote by

\[ \sigma_I := \sum_{\beta \in I} \beta. \]

5.1.4. For two cocharacters \( \lambda \) and \( \mu \), let \( I^\lambda_\mu \) be the set of weights \( \beta \) of \( A_\lambda \) such that \( \langle \mu, \beta \rangle < 0 \); define similarly \( J^\lambda_\mu \) for the adjoint representation. We will use the notations:

\[ d^\lambda_\mu = |I^\lambda_\mu|, \]
\[ e^\lambda_\mu = |J^\lambda_\mu|, \]
\[ c^\lambda_\mu = |I^\lambda_\mu| - |J^\lambda_\mu|, \]
\[ N^\lambda_\mu = \sum_{I^\lambda_\mu} \beta, \]
\[ g^\lambda_\mu = \sum_{J^\lambda_\mu} \beta, \]
\[ N^{\lambda\mu}_\mu = N^\lambda_\mu - g^\lambda_\mu. \]

5.1.5. Let \( \lambda \) and \( \mu \) be cocharacters of \( G \) with associated Levi and Weyl groups \( G^\lambda, W^\lambda, G^\mu, W^\mu \). Consider the set \( S := W^\lambda W/W^\mu \). Let \( V \) be the set of simple weights. For \( s \in S \), consider partitions of \( V \) induced by \( \lambda \) and \( w\mu \):

\[ V = \bigsqcup_{i \in I^\lambda} V_i, \quad V = \bigsqcup_{j \in I^{w\mu}} V_j. \]

The sets \( I^\lambda \) and \( I^{w\mu} \) are the sets of eigenvalues of \( \lambda \) and \( w\mu \) on \( \mathfrak{h} = \mathbb{C}V \), respectively. They are ordered by the natural ordering of \( \mathbb{Z} \). Define \( V^w_{ij} := V_i \cap V_j \) and use the lexicographic order on the set \( (I^\lambda, I^{w\mu}) \). Consider the decomposition

\[ V = \bigsqcup_{(i,j) \in (I^\lambda, I^{w\mu})} V^w_{ij}. \]

Similarly, consider the partitions of \( V \) induced by \( \mu \) and \( w^{-1}\lambda \):

\[ V = \bigsqcup_{i \in I^\mu} V_i, \quad V = \bigsqcup_{j \in I^{w^{-1}\lambda}} V_j. \]
Define $V_{ij}^{w} := V_i \cap V_j$. Use the lexicographic order on the set $(I^\mu, I^{w-1}_s)$ and consider the decomposition

$$V = \bigsqcup_{(i,j) \in (I^\mu, I^{w-1}_s)} V_{ij}^{w}.$$ 

Assume next that $\lambda$ and $\mu$ are dominant cocharacters. Let $w \in W^\lambda sW^\mu$ be an element such that the lexicographic order on $(I^\lambda, I^{w_1}_s)$ induces a dominant cocharacter $\nu$; $w$ can be chosen such that it does not permute elements of $V_i$ for $i \in I^\lambda$ among themselves, and similarly for $V_j$ for $j \in I^\mu$. We can choose $w$ to be the element of minimal length $w_s \in W^\lambda sW^\mu$. Let $\nu$ be the dominant cocharacter induced by $(I^\lambda, I^{w_s}_s)$. Further, $(I^\mu, I^{w^{-1}_s}_s)$ induces a dominant cocharacter $\nu'$.

Further, for any $w \in W^\lambda w_s W^\mu$, there exists $w' \in W^\lambda$ such that 

$$\{w'V_{ij}^{w} | i \in I^\lambda, j \in I^{w_s}_s\} = \{V_{ij}^{w_s} | i \in I^\lambda, j \in I^{w_s}_s\}.$$ 

5.1.6. Example. We discuss an example of the construction from the previous Subsection. Let $G = GL(n)$ and assume that $\lambda$ and $\mu$ are dominant cocharacters with parabolic groups

$$G^{\lambda > 0} = GL(a, b)$$

$$G^{\mu > 0} = GL(c, d).$$

We identify the set $V$ with $\{1, \cdots, n\}$. The partition of $V$ corresponding to $\lambda$ is $\{1, \cdots, a\} \sqcup \{a + 1, \cdots, n\}$, and the partition of $V$ corresponding to $\mu$ is $\{1, \cdots, c\} \sqcup \{c + 1, \cdots, n\}$. The set $S = \mathfrak{S}_a \times \mathfrak{S}_b \backslash \mathfrak{S}_n / \mathfrak{S}_c \times \mathfrak{S}_d$ parametrizes quadruplets $(e_1, e_2, e_3, e_4)$ such that

$$e_1 + e_2 = a,$$
$$e_3 + e_4 = b,$$
$$e_1 + e_3 = c,$$
$$e_2 + e_4 = d.$$ 

The decomposition (15) corresponds to a partition of $V$ in four sets $V_i$ of cardinal $e_i$ for $1 \leq i \leq 4$ such that $V_1 \sqcup V_2 = \{1, \cdots, a\}$; the decomposition (16) corresponds to a partition of $V$ in four sets $V'_i$ of cardinals $e_1$, $e_3$, $e_2$, and $e_4$ respectively, such that $V'_1 \sqcup V'_2 = \{1, \cdots, c\}$. The decomposition corresponding to $\nu$ is

$$V_1 = \{1, \cdots, e_1\}, \cdots, V_4 = \{e_1 + e_2 + e_3 + 1, \cdots, n\}.$$ 

The dominant cocharacters $\nu$ and $\nu'$ correspond to the parabolic groups

$$G^{\nu > 0} = GL(e_1, e_2, e_3, e_4)$$

$$G^{\nu' > 0} = GL(e_1, e_3, e_2, e_4).$$

The permutation $w_s \in \mathfrak{S}_n$ sends

$$i \mapsto i + e_2$$
$$i \mapsto i - e_2$$
for $e_1 + 1 \leq i \leq e_1 + e_2$,

$$i \mapsto i$$
for $e_1 + e_2 + 1 \leq i \leq e_1 + e_2 + e_3$. 


5.1.7. Recall the definition of $\mathbb{W}$ from [3]. Let $\lambda$ a cocharacter of $G$. Assume that $n_\lambda$ is even and let $b_\lambda = n_\lambda/2$. Denote by $\mathcal{F}(\lambda)$ the set of weights $\chi$ such that
$$\langle \lambda, \chi \rangle = b_\lambda.$$ 
Let $\chi$ be a weight satisfying the inequalities in (2) and such that $\chi \in \mathcal{F}(\lambda)$. Then there exists $\psi$ and $w \in W$ such that $w\psi$ is dominant, $w\psi + \rho_L \in \frac{1}{2} \mathbb{W}(\mathcal{X}^L)$, and
$$\chi = \frac{1}{2} N^{\lambda>0} - \frac{1}{2} g^{\lambda>0} + \psi.$$ 
Here $\rho_L$ is half the sum of positive roots of $\mathfrak{l} = g^\lambda$.

Indeed, the condition that $\chi$ satisfies the inequalities in (2) means that there exists $w \in W$ such that $w\chi$ is dominant and $w\chi + \rho \in \frac{1}{2} \mathbb{W}$.

This also implies that $w\lambda$ is dominant. By [21, Lemma 3.12], [35, Corollary 2.4], there exists a weight $\tau \in \frac{1}{2} \mathbb{W}(\mathcal{X}^L)$ such that
$$w\chi + \rho = \frac{1}{2} N^{w\lambda>0} + \tau.$$ 
Write $\rho = \frac{1}{2} g^{w\lambda>0} + \rho_L$. Then there exists $\omega$ dominant such that $\omega + \rho_L \in \frac{1}{2} \mathbb{W}(\mathcal{X}^L)$ and
$$w\chi = \frac{1}{2} N^{w\lambda>0} - \frac{1}{2} g^{w\lambda>0} + \omega.$$ 
For $\psi = w^{-1}\omega$ we obtain the desired conclusion.

5.1.8. Recall from the discussion in Subsection [12] that the categories $\mathbb{D}(\mathcal{X})$ may contain complexes supported on attracting stacks. We discuss how to characterize these complexes.

Assume that $\mathcal{X} = A/G$. Let $\lambda$ be a cocharacter. Recall that $b_\lambda = n_\lambda/2$. Denote by $\mathbb{D}(\mathcal{X}^\lambda)_b$ the subcategory of $\mathbb{D}(\mathcal{X}^\lambda)$ generated by sheaves of weights $\chi$ such that $\langle \lambda, \chi \rangle = b_\lambda$, see (17) for their description. A cocharacter $\lambda$ determines a map
$$m_\lambda := \frac{1}{|W\lambda|} p_{\lambda-1, q_{\lambda-1}^*} : K \left( \mathbb{D}(\mathcal{X}^\lambda)_b \right) \rightarrow K \left( \mathbb{D}(\mathcal{X}) \right).$$ 
To see that the image lies in $\mathbb{D}(\mathcal{X})$, the sheaves in Proposition [22] all have $r$-invariant $\leq 1/2$ by the argument in [21, Proposition 3.12]. A cocharacter $\nu : \mathbb{C}^* \rightarrow G$ with image in $G^\lambda$ determines a cocharacter of $G^\lambda$ and thus a map
$$m_\nu := \frac{1}{|W\nu|} p_{\nu-1, q_{\nu-1}^*} : K \left( \mathbb{D}(\mathcal{X}^\nu)_b \right) \rightarrow K \left( \mathbb{D}(\mathcal{X}^\lambda)_b \right).$$ 
There are similarly defined maps in the global case.

A dominant cocharacter $\lambda$ of $G$ determines a restriction map:
$$\Delta_\lambda := \beta_{\geq b_\lambda} p_\lambda^* : K \left( \mathbb{D}(\mathcal{X}) \right) \rightarrow K \left( \mathbb{D}(\mathcal{X}^\lambda)_b \right).$$ 
Here, $\beta_{\geq b_\lambda}$ is the functor which considers the top $\lambda$-weight component:
$$\beta_{\geq b_\lambda} : D^b(\mathcal{X}^{\lambda>0}) \rightarrow D^b(\mathcal{X}^{\lambda>0})_{b_\lambda} \cong D^b(\mathcal{X}^\lambda)_{b_\lambda},$$
see [19, Lemma 3.4, Corollary 3.17] for a definition of the functor \( \beta \). The equivalence is induced by the functor \( q_\lambda^* \) [19, Amplification 3.18]. It has the formula from Proposition 5.7. For dominant \( \nu \) as above, there is a restriction map:

\[
\Delta_\nu : K. \left( \mathbb{D}(A^{\lambda})_b \right) \rightarrow K. \left( \mathbb{D}(A^{\nu})_b \right).
\]

5.1.9. Assume that \( \mathcal{X} = A/G \). We use the notations and settings of Subsections 5.1.4 and 5.1.5. Consider dominant \( \lambda \) and \( \mu \) inducing dominant \( \nu \) and \( \nu' \). Then

\[
\{ \beta \in N \mid \langle \lambda, \beta \rangle > 0, \langle w_{\lambda \mu}, \beta \rangle < 0 \} = \{ \beta \in N \mid \langle \nu, \beta \rangle > 0, \langle \nu', \beta \rangle < 0 \},
\]

and so \( c_{w_{\lambda \mu}}^{\lambda} = c_{w_{\lambda \mu}}^{\nu'}, N_{w_{\lambda \mu}}^{\lambda} = N_{w_{\lambda \mu}}^{\nu'} \). A weight \( \beta \) of \( T \) determines \( q_\beta \in K_0(BT) \). Define

\[
\tilde{s}_w^s : K. (\mathbb{D}(A^{\nu})_b) \rightarrow K. (\mathbb{D}(A^{\nu'})_b)
\]

\[
y \mapsto (-1)^{c_{w_{\lambda \mu}}^{\nu'}} w^{-1}_s \left( y q^{-N_{w_{\lambda \mu}}^{\nu'}} \right) \]

5.2. Computations in K-theory. Recall the notations \( I, J, I', J' \) from the beginning of Section 4. For \( S \) an attracting stack in \( J \) with fixed locus \( Z \), consider the map

\[
m_S := p_s q^* : K.(Z) \rightarrow K.(\mathcal{X})
\]

and the subspace of \( K.(\mathcal{X}) \):

\[
B K.(\mathcal{X}) := \text{image} \left( \bigoplus_{J'} K.(Z) \rightarrow K.(\mathcal{X}) \right).
\]

We define \( PK.(\mathcal{X}) \) in Subsection 5.3. The main result we prove in this Section is:

**Theorem 5.1.** There is a decomposition

\[
K.(\mathcal{X}) = PK.(\mathcal{X}) \oplus BK.(\mathcal{X}).
\]

The definition of \( PK.(\mathcal{X}) \) is based on the following result. We restrict to the local case \( \mathcal{X} = A/G \). The following is the main ingredient in the proof of Theorem 5.1.

**Theorem 5.2.** Let \( \lambda \) and \( \mu \) be two dominant cocharacters, let \( S \) be the set defined in Subsection 5.1.5, and let \( \nu \) be dominant cocharacters as constructed in Subsection 5.1.5. The following diagram commutes:

\[
\begin{array}{ccc}
K. \left( \mathbb{D}(A^{\lambda})_b \right) & \xrightarrow{m_\lambda} & K. \left( \mathbb{D}(A^{\mu}) \right) \\
\bigoplus_S \Delta^{\lambda} & \downarrow & \bigoplus_S \Delta^{\mu} \\
\bigoplus_S K. \left( \mathbb{D}(A^{\nu})_b \right) & \xrightarrow{\bigoplus_S \tilde{m}_s^{\lambda}} & K. \left( \mathbb{D}(A^{\nu})_b \right)
\end{array}
\]

Before the start of the above result, we list some preliminary computations. The first two are standard, for example, for the first one see [43, Proposition 1.2], [36, Propositions 3.1 and 3.2]:

**Proposition 5.3.** Consider the maps \( A/T \xleftarrow{L} A/B \xrightarrow{S} A/G \). Then the map \( s_t^*: K.(A/T) \rightarrow K.(A/G) \) has the formula

\[
s_t^*(y) = \sum_{w \in W} w \left( \frac{y}{\prod_{\beta \in n} (1 - q^{-\beta})} \right).
\]
Proposition 5.4. The map \( m_\lambda : K.(\mathcal{X}^\lambda) \to K.(\mathcal{X}) \) has the formula
\[
m_\lambda(x) = \frac{1}{|W\lambda|} \sum_{w \in W/W\lambda} w \left( \frac{\prod_{\beta \in A_\lambda}(1 - q^{-\beta})}{\prod_{\beta \in g_\lambda}(1 - q^{-\beta})} \right).
\]

Proof. Consider the natural maps
\[
r : A \hookrightarrow A^{\lambda \geq 0} \times A^{\lambda \leq 0} \to A^\lambda,
\]
\[
v : A/G^{\lambda \geq 0} \to A/G.
\]
In the statement of Proposition 2.3, the sheaves \( F_I \) are \( v_*(r_*(F)(-\sigma_I)) \). The claim thus follows from Proposition 2.3, see also [36, Propositions 3.1] for a similar computation.

Proposition 5.5. Let \( \lambda \) and \( \mu \) be cocharacters with the same associated Levi group \( L \subset G \). For \( y \in K.(\mathcal{X}^L) \), let \( y' := (-1)^{c_\lambda \mu} yq^{-N_\mu} \in K.(\mathcal{X}^L) \). Then
\[
m_\lambda(y) = m_\mu(y').
\]

Proof. Using Proposition 5.4, we have that:
\[
m_\lambda(y) = \frac{1}{|W\lambda|} \sum_{W/W\lambda} w \left( \frac{\prod_{\beta \in A_\lambda}(1 - q^{-\beta})}{\prod_{\beta \in g_\lambda}(1 - q^{-\beta})} \right)
\]
\[
m_\mu(y) = \frac{1}{|W\lambda|} \sum_{W/W\lambda} w \left( \frac{\prod_{\beta \in A_\mu}(1 - q^{-\beta})}{\prod_{\beta \in g_\mu}(1 - q^{-\beta})} \right).
\]

Further, we have that
\[
\frac{\prod_{A_\mu}(1 - q^{-\beta})}{\prod_{g_\mu}(1 - q^{-\beta})} = \frac{\prod_{A_\lambda}(1 - q^{-\beta})}{\prod_{g_\lambda}(1 - q^{-\beta})} (-1)^{c_\lambda \mu} q^{N_\lambda},
\]
which implies the desired conclusion.

Corollary 5.6. Under the hypothesis of Proposition 5.5
\[
image \left( K.\left( \mathbb{D}(\mathcal{X}^\lambda)_b \right) \xrightarrow{m_\lambda} K.\left( \mathbb{D}(\mathcal{X}) \right) \right) = image \left( K.\left( \mathbb{D}(\mathcal{X}^\mu)_b \right) \xrightarrow{m_\mu} K.\left( \mathbb{D}(\mathcal{X}) \right) \right).
\]

Proof. Assume that \( y \in K.\left( \mathbb{D}(\mathcal{X}^\lambda)_b \right) \). Let \( \chi \) and \( \chi' \) be the weights of \( y \) and \( y' \), respectively. Then
\[
\chi' = \chi + N_\mu^\lambda.
\]
By the discussion in Subsection 5.1.7 and the decomposition in (17), there is a weight \( \psi \) with the properties mentioned there such that
\[
\chi = \frac{1}{2} N^\lambda > 0 - \frac{1}{2} g^\lambda > 0 + \psi
\]
\[
\chi' = \frac{1}{2} N^\mu > 0 - \frac{1}{2} g^\mu > 0 + \psi,
\]
and thus \( y' = (-1)^{c_\lambda \mu} yq^{N_\lambda^\mu} \in K.\left( \mathbb{D}(\mathcal{X}^\mu)_b \right) \).
Proposition 5.7. Let $\mu$ be a dominant character and let $y \in K(A/T)_\chi$ with $\chi$ on the face $F(\mu)$. Then the restriction map $\Delta_\mu : K(\mathcal{D}(\chi)) \to K(\mathcal{D}(\chi^\mu)_{\mu})$ sends
\[
\Delta_\mu \left( \sum_{w \in W} w \left( \frac{y}{\prod_{\beta \in n}(1 - q^{-\beta})} \right) \right) = \sum_{w \in W^\mu} w \left( \frac{y}{\prod_{\beta \in n^\mu}(1 - q^{-\beta})} \right).
\]
Let $y \in K(A/T)_\chi$ with $w \ast \chi$ not on $F(\mu)$ for any $w \in W$, then
\[
\Delta_\mu \left( \sum_{w \in W} w \left( \frac{y}{\prod_{\beta \in n}(1 - q^{-\beta})} \right) \right) = 0.
\]

Proof. The weight $\chi$ is dominant up to multiplication by an element of $W^\mu$, so we can assume it is dominant. For $F \in \mathcal{D}(\chi)$, we have that $\Delta_\mu(F) = \beta_{b_\mu} \varphi^*(\chi)$. For the second part, by Proposition 5.5 we can replace $\chi$ with any $w \ast \chi$ for $w \in W$, so we can assume that $\chi$ is dominant. Then $\chi$ is not on $F(\mu)$, so $\langle \mu, p_\mu^*(F) \rangle < b_\mu$, and the second part thus follows. \qed

Proposition 5.8. Consider $\lambda, \tau$ two dominant cocharacters and $\chi$ a dominant weight with $\chi \in F(\lambda)$. Assume there is a partial sum $\sigma$ of weights in $A_\lambda$ and an element $w \in W$ such that $w \ast (\chi - \sigma) \in F(\tau)$. Let $\mu = w^{-1} \tau$. Then
\[
\sigma = N_\mu^\lambda + \sigma',
\]
where $\sigma'$ is a partial sum of weights in $(N^\mu)_{\lambda > 0}$. Conversely, any such partial sum $\sigma$ has the property that $w \ast (\chi - \sigma) \in F(\tau)$.

Proof. The weight
\[
(\chi - \sigma)^+ + \rho \in \frac{1}{2} W
\]
by the same argument as in [34, Proposition 3.6], see also [21, Proof of Theorem 3.2]. Using the description in Subsection 5.1.7, write
\[
\chi + \rho = \frac{1}{2} N_\chi^\lambda + \psi
\]
\[
w \ast (\chi - \sigma) + \rho = \frac{1}{2} N_{\tau}^\tau + \phi',
\]
and so
\[
\chi - \sigma + \rho = \frac{1}{2} N_{\mu}^\mu + \phi,
\]
where $\psi$ is a sum of weights of $N_\chi^\lambda$ and $\phi$ is a sum of weights of $N_{\mu}^\mu$. For the first two relations above we use that $\lambda$ and $\tau$ are dominant. Then
\[
\sigma = N_\mu^\lambda + \psi - \phi.
\]
Write
\[
\psi = \psi_{\mu}^{\lambda_0} + \psi_{\mu_0}^{\lambda_0} + \psi_{\mu}^{\lambda_0},
\]
\[
\phi = \phi_{\mu}^{\lambda_0} + \phi_{\mu_0}^{\lambda_0} + \phi_{\mu}^{\lambda_0},
\]

NCRS AND IH FOR QUOTIENT SINGULARITIES 25
where $\psi^\lambda_\nu$ is a sum of weights in $(N^\lambda)^{\mu>0}$ etc. Then the decomposition (19) implies that

$$\psi = \psi^\lambda_\nu, \quad \phi = \phi^\lambda_\nu, \quad \phi^\lambda_\nu = \psi^\lambda_\nu.$$

This implies that $\sigma = N^\lambda - \phi^\lambda_\nu$, where $-\phi^\lambda_\nu$ is a partial sum of weights in $(N^\mu)^{\lambda>0}$.

Conversely, if $\sigma = N^\lambda + \sigma'$, the argument above shows that $\chi - \sigma + \rho = \frac{1}{2} N^{\mu>0} + \phi$ and by (18) we have that $w(\chi - \sigma) \in \mathbb{F}(\tau)$.  

\textbf{Proof of Theorem 5.2.} Let $x \in K(\mathbb{D}(\lambda^\lambda)_h)$. By Proposition 5.3 let $y \in K(A/T)$ such that

$$x = \sum_{v \in W^\lambda} v \left( \frac{y}{\prod_{\beta \in n^\lambda} (1 - q^{-\beta})} \right).$$

We may assume that $y$ is of dominant weight $\chi$. By Proposition 5.4, we have that

$$p^\lambda_{1} y_{\nu}^{\lambda-1} \sum_{v \in W^\lambda} v \left( \frac{y}{\prod_{\beta \in n^\lambda} (1 - q^{-\beta})} \right) = \sum_{u \in W/W^\lambda} u \sum_{I \subset A_{\lambda}} \sum_{v \in W^\lambda} v \left( \frac{(-1)^{|I|} y_{q^{-\sigma_I}}}{\prod_{\beta \in n} (1 - q^{-\beta})} \right).$$

The weight of $y_{q^{-\sigma_I}}$ is $\chi - \sigma_I$. By Propositions 5.7 and 5.8 such an element has non-zero $\mu$-restriction if and only if there exists $w \in W$ such that

$$\sigma_I = N^\lambda_{w\mu} + \sigma_I', \quad (20)$$

where $I'$ is a subset of $A_{\lambda}^{1\nu}$, the set of weights $\beta$ in $A_{\lambda}$ such that $\langle w\mu, \beta \rangle = 0$.

Fix $w$. Let $\nu$ and $\nu'$ be the cocharacters constructed as in Subsection 5.1.5. The weight $N^\lambda_{w\mu}$ and the set $A_{\lambda}^{w\mu}$ depend on the coset $W/W^{w\mu}$; if they have associated $\nu$ as above, then they depend on $W^\lambda/W^{\nu'} \subset W/W^{w\mu}$. Recall the element $w'$ as the end of Subsection 5.1.5. The element $y_{q^{-\sigma_I}}$ has weight $\chi - \sigma_I$. By (17), the weight $(\chi - \sigma_I)^+$ is on $\mathbb{F}(\nu)$. Then

$$\sum_{v \in W^\lambda} v \left( \frac{y}{\prod_{\beta \in n^\lambda} (1 - q^{-\beta})} \right) q^{-N^\lambda_{w\mu} - \sigma_I'} = \sum_{v \in W^\lambda} v \left( \frac{y}{\prod_{\beta \in n^\lambda} (1 - q^{-\beta})} \right) w' \left( q^{-N^\lambda_{w\mu} - \sigma_J} \right)$$

$$= \sum_{v \in W^\lambda} \left( \frac{y_{q^{-N^\nu_{w\mu} - \sigma_J}}}{\prod_{\beta \in n^\lambda} (1 - q^{-\beta})} \right),$$

where $J$ is a subset of $A_{\lambda}^{w\mu}$. Define

$$m_{\lambda\nu}(x) := \frac{1}{|W^\nu|} \sum_{u \in W/W^\lambda} u \sum_{I \subset A_{\lambda}^{w\mu}} \sum_{v \in W^\lambda} v \left( \frac{(-1)^{|I|} y_{q^{-N^\nu_{w\mu} - \sigma_J}}}{\prod_{\beta \in n} (1 - q^{-\beta})} \right).$$
Then $m_\lambda = \sum_\nu m_{\lambda \nu}$. It suffices to show that the following diagram commutes

$$
\begin{array}{ccc}
K.(\mathcal{D}(\mathcal{X}^\lambda)_b) & \xrightarrow{m_{\lambda \nu}} & K.(\mathcal{D}(\mathcal{X})) \\
\downarrow \Delta^\lambda_b & & \downarrow \Delta_\mu \\
K.(\mathcal{D}(\mathcal{X}^\mu)_b) & \xrightarrow{m_{\mu \nu}} & K.(\mathcal{D}(\mathcal{X}^\mu)_b).
\end{array}
$$

Let $\tau$ be the sum of weights $\beta$ in $\mathfrak{n}$ such that $w_s^{-1}\beta$ is not in $\mathfrak{n}$. Then $-w_s\tau = g^\lambda_{w_s \mu}$ because the two sides are sums over the weights in the following two sets

$$\{\beta \in \mathfrak{n} \text{ such that } -\beta \in w_s \mathfrak{n}\} = \{\beta \in g^\lambda \text{ such that } -\beta \in g_{w_s \mu}\}.$$

Recall the setting of Subsection 5.1.9. We have that

$$w_s^{-1} \left( \frac{(-1)^{d^\nu} w_s^{-1} (yq^{-N^\nu})}{\prod_{\beta \in \mathfrak{n}} (1-q^{-\beta})} \right) = \frac{(-1)^{d^\nu} w_s^{-1} (yq^{-N^\nu})}{\prod_{\beta \in \mathfrak{n}} (1-\frac{1}{q})} = \frac{(-1)^{d^\nu} w_s^{-1} (yq^{-N^\nu})}{\prod_{\beta \in \mathfrak{n}} (1-q^{-\beta})}.$$

We can thus rewrite

$$m_{\lambda \nu}(x) := \frac{1}{|W^\nu|} \sum_{J' \subset A^{w_s^{-1}}_\lambda} \sum_{v \in W^\nu} v \left( \frac{(-1)^{d^\nu} w_s^{-1} (yq^{-N^\nu})}{\prod_{\beta \in \mathfrak{n}} (1-q^{-\beta})} \right),$$

and thus

$$\Delta_\mu m_{\lambda \nu}(x) = \frac{1}{|W^\nu|} \sum_{J' \subset A^{w_s^{-1}}_\lambda} \sum_{v \in W^\nu} v \left( \frac{(-1)^{d^\nu} w_s^{-1} (yq^{-N^\nu})}{\prod_{\beta \in \mathfrak{n}} (1-q^{-\beta})} \right).$$

This is the same as the composition of left-bottom maps:

$$\tilde{m}_{\mu \nu} \Delta_\lambda^\lambda(x) = \tilde{m}_{\mu \nu} \Delta_\lambda^\lambda \left( \sum_{v \in W^\lambda} v \left( \frac{y}{\prod_{\beta \in \mathfrak{n}^\lambda (1-q^{-\beta})} \right) \right)$$

$$= \tilde{m}_{\mu \nu} \sum_{v \in W^\nu} v \left( \frac{y}{\prod_{\beta \in \mathfrak{n}^\nu (1-q^{-\beta})} \right)$$

$$= m_{\mu \nu} \sum_{v \in W^\nu} \left( (-1)^{d^\nu} w_s^{-1} (yq^{-N^\nu}) \right) \prod_{\beta \in \mathfrak{n}^\nu (1-q^{-\beta})}$$

$$= \frac{1}{|W^\nu|} \sum_{J' \subset A^{w_s^{-1}}_\lambda} \sum_{v \in W^\nu} v \left( \frac{(-1)^{d^\nu} w_s^{-1} (yq^{-N^\nu})}{\prod_{\beta \in \mathfrak{n}^\nu (1-q^{-\beta})} \right).$$
where the second equality follows from Proposition 5.7 and the last one by Proposition 5.4.

5.3. Primitive K-theory: the local case. Recall the setting and notations from the beginning of Subsection 5.2. In this Subsection, we define $\mathbf{P}K(\mathcal{X}) \subset K(\mathcal{D}(\mathcal{X}))$ which appears in Theorem 5.1. Assume that $\mathcal{X} = A/G$ is a local stack.

**Proposition 5.9.** Let $\lambda$ be a dominant cocharacter and let $\mathfrak{S}_\lambda \subset W$ be the set of elements $w_s$ for $s \in W^\lambda \backslash W/W^\lambda$ such that the weight $\nu$ constructed in Subsection 5.1.3 is equal to $\lambda$. Then $\mathfrak{S}_\lambda$ is a group and it acts on $K(\mathcal{D}(\mathcal{X}^\lambda)_b)$ via $\tilde{w}$. 

**Proof.** The elements $w_s$ above are the elements of $W$ that induce permutations of $I^\lambda$ and which do not permute elements in $V_i$ for $i \in I^\lambda$ among themselves. These elements are clearly closed under multiplication and taking inverses.

For the second part, consider elements $w_1, w_2 \in \mathfrak{S}_\lambda$ and let $y \in K(\mathcal{D}(\mathcal{X}^\lambda)_b)$. Let $\tilde{w}_1, \tilde{w}_2$ be the swap maps for the elements $w_1, w_2$, and $w_2w_1$, respectively. We need to show that $\tilde{w}_1\tilde{w}_2 = \tilde{w}_2\tilde{w}_1$:

$$\begin{align*}
(-1)^{c_{w_1}^\lambda w_1^{-1}} (-1)^{c_{w_2}^\lambda w_2^{-1}} (yq^{-N_{w_2}^\lambda} q^{-N_{w_1}^\lambda}) = \\
(-1)^{c_{w_1}^\lambda w_1^{-1} + c_{w_2}^\lambda w_2^{-1}} (yq^{-N_{w_2}^\lambda} - N_{w_2}^{w_1 \lambda}) = \\
(-1)^{c_{w_2 w_1}^\lambda(w_2 w_1)^{-1}} (yq^{-N_{w_2 w_1}^\lambda}).
\end{align*}$$

In the example from Subsection 5.1.6, $\mathfrak{S}_\lambda$ is trivial unless $a = b$, case in which it is the symmetric group $\mathfrak{S}_2$. More generally, for $\lambda$ a cocharacter of $GL(n)$ with corresponding decomposition in distinct parts $d_1, \ldots, d_k$ with multiplicities $m_1, \ldots, m_k$, the group $\mathfrak{S}_\lambda$ is the product of symmetric groups $\times_{i=1}^k \mathfrak{S}_{m_i}$. In the framework of the above Proposition, denote by

$$\text{Sym}_\lambda : K(\mathcal{D}(\mathcal{X}^\lambda)_b) \to K(\mathcal{D}(\mathcal{X}^\lambda)_b)$$

$$x \mapsto \sum_{\sigma \in \mathfrak{S}_\lambda} \sigma(x).$$

Define

$$\text{BK.} \left(\mathcal{D}(\mathcal{X}^\lambda)_b\right) = \text{image} \left(\bigoplus_{\nu} m_{\nu} : K. (\mathcal{D}(\mathcal{X}^\nu)_b) \to K. (\mathcal{D}(\mathcal{X}^\lambda)_b)\right),$$

where the sum is over all non-trivial cocharacters $\nu$ of $G^\lambda$. We define $\mathbf{P}K(\mathcal{D}(\mathcal{X}^\lambda)_b)$ inductively on $\dim G^\lambda$ such that

$$K. (\mathcal{D}(\mathcal{X}^\lambda)_b) = \mathbf{P}K(\mathcal{D}(\mathcal{X}^\lambda)_b) \oplus \text{BK.} \left(\mathcal{D}(\mathcal{X}^\lambda)_b\right).$$

There are then natural surjections $\pi_\lambda : K. (\mathcal{D}(\mathcal{X}^\lambda)_b) \to \mathbf{P}K(\mathcal{D}(\mathcal{X}^\lambda)_b)$.

When $\dim G^\lambda = 0$, then $\mathbf{P}K(\mathcal{D}(\mathcal{X}^\lambda)_b) = K. (\mathcal{D}(\mathcal{X}^\lambda)_b)$. Assume that $\dim G > 0$. For any Levi $L < G$, choose a dominant cocharacter $\lambda_L$ such that $G^{\lambda_L} = L$. Denote
by $m_H = m_{\lambda H}$ etc. Define

$$\text{PK} \left( \mathbb{D}(\chi^L)_{b} \right) = \bigcap_{H < L} \left( \text{Sym}_H \pi_H \Delta_H : K, \left( \mathbb{D}(\chi^L)_{b} \right) \to \text{PK} \left( \mathbb{D}(\chi^L)_{b} \right) \right).$$

Denote by $\iota : \text{PK} \left( \mathbb{D}(\chi^L)_{b} \right) \hookrightarrow K, \left( \mathbb{D}(\chi^L)_{b} \right)$ the natural map, and denote by $\Phi_H := \text{Sym}_H \pi_H \Delta_H$.

**Proposition 5.10.** The composition

$$\text{PK} \left( \mathbb{D}(\chi^L)_{b} \right) \xrightarrow{m_L} K, \left( \mathbb{D}(\chi) \right) \xrightarrow{\pi_L \Delta_L} \text{PK} \left( \mathbb{D}(\chi^L)_{b} \right)$$

is $\pi_L \Delta_L m(x) = \frac{1}{|W_\lambda|} \sum_{\sigma \in S_\lambda} \sigma(x)$.

**Proof.** By Theorem 5.2, we have that $\Delta_L \pi_L \Delta_L m(x) = \sum S \Delta \pi_L \Delta_L m(x)$.

For $\nu'$ different from $\lambda$, the element $m_{\nu'} \Delta \nu(x)$ is in $B K, \left( \mathbb{D}(\chi^L)_{b} \right)$. Then

$$\pi_L \Delta_L m_L(x) = \sum_{\sigma \in S_\lambda} \sigma(x).$$

□

**Proposition 5.11.** Let $L$ and $E$ be proper Levi groups of $G$ such that $E \nsubseteq L$. Let $\lambda$ and $\mu$ be the associated cocharacters to these Levi groups. The composition

$$\text{PK} \left( \mathbb{D}(\chi^L)_{b} \right) \xrightarrow{m_L} K, \left( \mathbb{D}(\chi) \right) \xrightarrow{\Delta_E} K, \left( \mathbb{D}(\chi^\mu)_{b} \right) \xrightarrow{\pi_E} \text{PK} \left( \mathbb{D}(\chi^\mu)_{b} \right)$$

is zero.

**Proof.** By Theorem 5.2, we have that $\Delta_E m_L(x) = \sum S \Delta E m_L(x)$.

If $E \nsubseteq L$, there is no $s \in S$ such that $\nu' = \mu$, and so the right hand side is in $B K, \left( \mathbb{D}(\chi^\mu)_{b} \right)$.

□

We now assume the statements in (21) for $L < G$.

**Proposition 5.12.** There is a surjection

$$\bigoplus_{L < G} \text{PK} \left( \mathbb{D}(\chi^L)_{b} \right) \xrightarrow{\Theta_L} B K, \left( \mathbb{D}(\chi) \right).$$

**Proof.** The image of $m_L : \text{PK} \left( \mathbb{D}(\chi^L)_{b} \right) \to K, \left( \mathbb{D}(\chi) \right)$ factors through the symmetrization map

$$\text{PK} \left( \mathbb{D}(\chi^L)_{b} \right) \xrightarrow{\text{Sym}_L} \text{PK} \left( \mathbb{D}(\chi^L)_{b} \right) \xrightarrow{\Theta_L} m_L K, \left( \mathbb{D}(\chi) \right)$$

by Proposition 5.5. The statement follows using (21) for $L < G$ and Proposition 5.6. □
Theorem 5.13. There is a decomposition
\[ \left( \iota, \bigoplus_{L < G} m_{\lambda L} \right): PK.(\mathbb{D}(\mathcal{X})) \oplus \bigoplus_{L < G} PK.\left(\mathbb{D}(\mathcal{X}^{\lambda L})_b\right)^{\Theta_L} \sim K.(\mathbb{D}(\mathcal{X})). \]

Proof. Using Propositions 5.10 and 5.11 we see that the map is an injection. To see it is a surjection, let \( x \in K.(\mathbb{D}(\mathcal{X})) \) and assume that \( \Phi_H(x) = 0 \) for all \( L < H < G \) and \( \pi_L(x) \neq 0 \). By Propositions 5.10 and 5.11 there is a constant \( c \) such that
\[ y := x - c \cdot m_{\lambda L} \Phi_L(x) \in K.(\mathbb{D}(\mathcal{X})) \]
satisfies \( \Phi_H(y) = 0 \) for \( L \leq H < G \). Repeating this process, we see that the map is indeed surjective. \( \square \)

We now prove Theorem 5.1 in the local case.

Corollary 5.14. There is a decomposition \( K.(\mathcal{X}) = PK.(\mathcal{X}) \oplus BK.(\mathcal{X}) \).

Proof. This follows from Theorem 3.2, Proposition 5.5, and Theorem 5.13. \( \square \)

5.4. Compatibility of decompositions along étale maps. Let \( e: \mathcal{X}' \to \mathcal{X} \) be an étale map. Let \( S \) be a \( \Theta \)-stack of \( \mathcal{X} \) with associated fixed stack \( Z \). Let \( S' \) be a \( \Theta \)-stack in \( \mathcal{X}' \) contained in \( e^{-1}(S) \), and let \( Z' \) be its associated fixed stack. Finally, let \( w \in Z \). Then
\[ e^*: D^b(Z)_w \to D^b(Z')_w, \quad e^*: D^b(S) \to D^b(S'), \]
\[ e_*: D^b(Z')_w \to D^b(Z)_w, \quad e_*: D^b(S') \to D^b(S). \]

By the construction of the categories \( \mathbb{D} \) from Section 3 we obtain functors
\[ e^*: \mathbb{D}(\mathcal{X}) \to \mathbb{D}(\mathcal{Y}), \]
\[ e_*: \mathbb{D}(\mathcal{Y}) \to \mathbb{D}(\mathcal{X}). \]

By the construction of the spaces \( PK \) and \( BK \), we see that \( e_* \) and \( e^* \) respect these spaces for \( \mathcal{X} \) and \( \mathcal{Y} \) quotient stacks of smooth affine varieties by reductive groups.

5.5. Primitive K-theory: the global case. Let \( \mathcal{X} \) be a symmetric stack satisfying Assumption B and let \( X \) be its good moduli space. Consider a direct system of étale covers \( \mathcal{A} \) containing étale maps \( \mathcal{Y} \to \mathcal{X} \) as in Theorem 2.1. Then, by [42, Corollary 2.17]:
\[ \check{H}^p(\mathcal{A}, K_q(-)) \Rightarrow K_{q-p}(\mathcal{X}), \]
where \( \check{H} \) denotes Čech cohomology and the spectral sequence converges strongly. It is essential that we use rational K-theory to obtain this statement. Any \( \lambda: BG_m \to \mathcal{X} \) induces a cocharacter \( \lambda \) in local charts \( \mathcal{Y} \). Further, any local attracting locus corresponds to a map \( \lambda: BG_m \to \mathcal{X} \) and thus determines an attracting stack. Denote by \( \mathcal{X}^{\lambda} \) the corresponding fixed stack. For \( \lambda: BG_m \to \mathcal{X} \), define \( K^{\lambda}(\mathcal{Y}) := K.(\mathcal{Y}^{\lambda}) \). Then
\[ \check{H}^p(\mathcal{A}, K^{\lambda}_q(-)) \Rightarrow K_{q-p}(\mathcal{X}^{\lambda}). \]

Thus the following spectral sequence converges strongly:
\[ \check{H}^p(\mathcal{A}, BK_q(-)) \Rightarrow BK_{q-p}(\mathcal{X}). \]
By Theorem 5.14, (22), and (23), we thus obtain that the following spectral sequence converges strongly and define $PK(\mathcal{X})$ such that
\[
\tilde{H}^p(A, PK_q(-)) \Rightarrow PK_{q-p}(\mathcal{X}).
\]

Proof of Theorem 5.7. The decomposition claimed in Theorem 5.1 follows from the construction of $PK(\mathcal{X})$ and $BK(\mathcal{X})$ and by Theorem 5.14. □

5.6. Categorification of intersection cohomology.

5.6.1. The categories $D^b(\mathcal{X})$ have natural dg enhancements and the admissible subcategories $D(\mathcal{X})$ also have natural dg enhancements [Section 4.1]. Recall the definitions from Subsection 2.5. The splitting
\[
K_0(D(\mathcal{X})) \to PK_0(D(\mathcal{X})) \to K_0(D(\mathcal{X}))
\]
induces an idempotent of $e_\mathcal{X} \in \text{Hom}_{\text{Hmo}_{0;Q}}(D(\mathcal{X}), D(\mathcal{X}))$. Indeed, all the functors used to construct the above splitting are constructed from the functors for attracting and fixed stacks
\[
p_* : D^b(S) \to D^b(\mathcal{X})
\]
\[
p^* : D^b(\mathcal{X}) \to D^b(S)
\]
\[
q_w^* : D^b(\mathcal{Z})_w \sim D^b(S)_w,
\]
see [19, Amplification 3.18] for the last functor. Thus the functors used are induced by Fourier–Mukai transforms in $\text{rep}(D^b(S), D^b(\mathcal{X}))$ and $\text{rep}(D^b(\mathcal{Z}), D^b(\mathcal{X}))$, respectively. We thus obtain a noncommutative motive:
\[
\mathbb{D}^{nc}(\mathcal{X}) := (D(\mathcal{X}), e_\mathcal{X}) \in \text{Hmo}_{0;Q}^3.
\]

Consider the Chern character
\[
\text{ch} : K(\mathcal{X}) \to \widehat{H}(\mathcal{X}) := \prod_{j \in \mathbb{Z}} H^{i+2j}(\mathcal{X}).
\]

We write $\text{gr} K(\mathcal{X})$ for the associated graded with respect to the codimension filtration [18, Definition 3.7, Section 5.4]. By Theorems 4.11 and 5.1, we obtain:

Corollary 5.15. There is an inclusion $\text{gr} K(\mathbb{D}^{nc}(\mathcal{X})) \subset P^{\leq 0}H(\mathcal{X})$.

If $\mathcal{X}$ is symmetric and satisfies Assumption C, then $P^{\leq 0}H(\mathcal{X}) \cong IH(\mathcal{X})$. In this situation, we define the intersection $K$-theory of $\mathcal{X}$:
\[
IK(\mathcal{X}) := K(\mathbb{D}^{nc}(\mathcal{X})).
\]

There is a natural map
\[
\text{ch} : IK(\mathcal{X}) \to IH(\mathcal{X})
\]
obtained using the splittings from Theorem 4.11 and Theorem 5.1:
\[
IK(\mathcal{X}) \leftrightarrow K(\mathcal{X}) \xrightarrow{\text{ch}} \widehat{H}(\mathcal{X}) \to IH(\mathcal{X}).
\]

Directly from the definition of $\mathbb{D}^{nc}(\mathcal{X})$, intersection $K$-theory satisfies a version of Kirwan surjectivity [26, Theorem 2.5]:
\[
K(\mathcal{X})_Q \to IK(\mathcal{X}).
\]
5.6.2. Denote by $K_{\text{top}}$ the Blanc topological K-theory of a dg category $[6]$. Recall that for $\mathcal{X}$ a smooth stack, $K_{\text{top}}(D^b(\mathcal{X}))$ recovers the Atiyah-Segal equivariant topological K-theory $[22, \text{Theorem } 3.9]$. For $j \in \{0,1\}$, denote by $\text{gr}^i K_{\text{top}}^j(\mathcal{X})$ the associated graded with respect to filtration $F^{>i}$ induced by $H^{>j+2i}(\mathcal{X})$ via

$$\text{ch} : K_{\text{top}}^j(\mathcal{X}) \to \prod_{i \in \mathbb{Z}} H^{j+2i}(\mathcal{X}).$$

The idempotent $e$ induces a well-defined direct summand $K_{\text{top}}^j(D_{nc}(\mathcal{X}))$ of $K_{\text{top}}(D(\mathcal{X}))$.

For $j \in \{0,1\}$, we have that

$$\text{gr}^i K_{\text{top}}^j(\mathcal{X}) \cong H^{j+2i}(\mathcal{X}, \mathbb{Q}).$$

Indeed, it suffices to show the statement for quotient stacks, case in which both sides can be computed using Totaro’s approximations $S_n$ of $\mathcal{X}$ from the proof of Proposition $[4,1]$. The isomorphism for a scheme $S_n$ holds by the Atiyah-Hirzebruch theorem. By Theorems $[4,1]$ and $[5,1]$ we have that $\text{gr}^i K_{\text{top}}^j(D_{nc}(\mathcal{X})) \cong IH^{j+2i}(X, \mathbb{Q})$ for $j \in \{0,1\}$. If the natural map

$$K_{\text{top}}^j(D_{nc}(\mathcal{X})) \to \text{gr}^i K_{\text{top}}^j(D_{nc}(\mathcal{X})), \quad (24)$$

is an isomorphism, then

$$K_{\text{top}}^j(D_{nc}(\mathcal{X})) \cong IH^{j+2i}(X, \mathbb{Q}) \quad (25)$$

for $j \in \{0,1\}$. We claim that if the Kirwan resolution $Y \to X$ is a scheme, then $(24)$ holds. Indeed, it suffices to check $(24)$ for $D(\mathcal{X})$ instead of $D_{nc}(\mathcal{X})$. It suffices to check that the Chern character map for $D(\mathcal{X})$ is injective. By Corollary $[3,5]$ it suffices to check that the Chern character map for $Y$ is injective. For a scheme $Y$, the Chern character is an isomorphism by the Atiyah-Hirzebruch theorem.

Denote by $HP$ the periodic cyclic homology of a dg category. The idempotent $e$ induces a well-defined direct summand $HP(D_{nc}(\mathcal{X}))$ of $HP(D(\mathcal{X}))$. By $[22, \text{Theorem A}]$ and $(25)$, we also obtain that if the Kirwan resolution $Y \to X$ is a scheme, then

$$HP^j(D_{nc}(\mathcal{X})) \cong \bigoplus_{j \in \mathbb{Z}} IH^{j+2j}(X, \mathbb{C}).$$

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