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APPROXIMATION OF ROUGH PATHS OF FRACTIONAL BROWNIAN MOTION

ANNE MILLET∗† AND MARTA SANZ-SOLÉ∗

ABSTRACT. We consider a geometric rough path associated with a fractional Brownian motion with Hurst parameter \( H \in \left[ \frac{1}{4}, \frac{1}{2} \right] \). We give an approximation result in a modulus type distance, up to the second order, by means of a sequence of rough paths lying above elements of the reproducing kernel Hilbert space.

1. INTRODUCTION

Consider a \( d \)-dimensional fractional Brownian motion \( W^H \) with Hurst parameter \( H \in \left[ \frac{1}{4}, \frac{1}{2} \right] \cup \left[ \frac{1}{2}, 1 \right] \) and integral representation

\[
W^H_t = \int_0^1 K^H(t,s) dB_s, \tag{1.1}
\]

where \( K^H(t,s) = 0 \), if \( s \geq t \) and for \( 0 < s < t \),

\[
K^H(t,s) = c_H \left( t - s \right)^{H - \frac{1}{2}} + s^{H - \frac{1}{2}} F_1 \left( \frac{t}{s} \right) \tag{1.2}
\]

with

\[
F_1(z) = c_H \left( \frac{1}{2} - H \right) \int_0^{z-1} u^{H - \frac{3}{2}} \left( 1 - (u + 1)^{H - \frac{1}{2}} \right) du, \tag{1.3}
\]

for \( z > 1 \) (see for instance [1], equation (42)). In (1.1), \( B \) denotes a standard \( d \)-dimensional Brownian motion and in (1.2), (1.3), \( c_H \) denotes a positive real constant depending on \( H \).

Let \( p \in ]1, 4[ \) be such that \( pH > 1 \). In [2], it is proved that the sequence of smooth rough paths based on linear interpolations of \( W^H \) converges in the \( p \)-variation distance. The limit defines a geometric rough path with roughness \( p \) lying above \( W^H \). We will call this object the enhanced fractional Brownian motion.

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In the recent papers [3], [4], the $p$-variation distance on rough paths is replaced by a strictly stronger, modulus type distance defined as follows:

$$
\tilde{d}_p(x, y) = \sup_{0 \leq s < t \leq 1} \left( \sum_{i=1}^{[p]} \frac{|x_{s,i} - y_{s,i}|}{(t-s)^p} \right).
$$

In [3], it is proved that the enhanced fractional Brownian motion can actually be obtained by means of the $\tilde{d}_p$ distance and also that linear interpolations of $W^H$ define stochastic processes with values in $\mathcal{H}^H$, the reproducing kernel Hilbert space associated with $W^H$ (see Theorem 3.3 in [4] for a description of this space). Then, the authors state a characterization of the topological support of the enhanced fractional Brownian motion among other results.

Our aim in this work is to give a new approximation of the enhanced fractional Brownian motion by means of a sequence of geometric rough paths which, unlike those based on linear interpolations, are not smooth, but also belong to $\mathcal{H}^H$. For the sake of simplicity, we restrict to $[p] = 2$. We are pretty confident that our results extend to $[p] = 3$; however, dealing with higher generality would most likely produce a very technical paper. Our result, as is stated in Theorem 2.1, provides in particular a new approximation of the Lévy area of the fractional Brownian motion.

For any $m \in \mathbb{N}$, we consider the dyadic grid $(t^m_k = k2^{-m}, k = 0, 1, \ldots, 2^m)$ and set $\Delta^m_k = [t^m_{k-1}, t^m_k]$ and $\Delta^m_k B = B_{t^m_k} - B_{t^m_{k-1}}$. Define $\dot{B}(m)_0 = 0$ and for $t \in \Delta^m_k$, $\dot{B}(m)_t = B_{t^m_{k-1}} + 2^m(t - t^m_{k-1})\Delta^m_k B$. Our approximation sequence is defined by

$$
W(m)^H_t = \int_0^t K^H(t, s) \dot{B}(m)_s \, ds,
$$

(1.4)

$m \in \mathbb{N}$, where $\dot{B}(m)_s$ denotes the derivative with respect to $s$ of the path $s \mapsto B(m)_s$. Notice that $W(m)^H \in \mathcal{H}^H$.

Let $K^H_m(t, s)$ be the orthogonal projection of $K^H(t, \cdot)$ on the $\sigma$-field generated by $(\Delta^m_k, k = 1, \ldots, m)$. That is, for any $0 < s < t \leq 1$,

$$
K^H_m(t, s) = \sum_{k=1}^{2^m} 2^m \left( \int_{\Delta^m_{k} \cap [0, t]} K^H(t, u) \, du \right) 1_{\Delta^m_{k}}(s).
$$

(1.5)

We clearly have

$$
W(m)^H_t = \int_0^1 K^H_m(t, s) \, dB_s.
$$

(1.6)

For $H \in [\frac{1}{4}, 1[$, we set $W = (W_{s,t} = (W_{s,t}^{(1)}, 0 \leq s \leq t \leq 1), W(m) = (W(m)_{s,t} = (W(m)_{s,t}^{(1)}, 0 \leq s \leq t \leq 1)$, while for $H \in [\frac{1}{4}, \frac{1}{2}]$ we set $W = (W_{s,t} = (W_{s,t}^{(1)}, W_{s,t}^{(2)}, 0 \leq s \leq t \leq 1)$ and $W(m) = (W(m)_{s,t} = (W(m)_{s,t}^{(1)}, W(m)_{s,t}^{(2)}, 0 \leq s \leq t \leq 1), m \geq 1$. 


The main result of the paper states the convergence of $W(m)$ to $W$ in the $d_p$– metric for $p \in ]1, 3]$. For $p \in ]1, 2[$, the result is an almost trivial consequence of Lemma 3.2, which establishes Hölder continuity in the $L^2[0, 1]$ norm of the kernels $K^H$, $K^H_m$, respectively, and a control of the quadratic mean error in the approximation of $K^H$ by $K^H_m$. For $p \in [2, 3]$, the approximation of the Lévy area relies on representation formulas for the second order multiple integrals by means of the operator $K^*$ given in (2.3) and introduced in [1] (see also [2]). There are two fundamental ingredients. Firstly, Proposition 2.3, giving the rate of convergence of the approximation at the second order level in the $L^q(\Omega)$–modulus norm; secondly, Lemma 3.5, an extension of the Garsia–Rademich–Rumsey Lemma for geometric rough paths of any roughness $p$. Other technical results used in the proofs, mainly on the kernels $K^H$ and $K^H_m$, are given in the Appendix.

For simplicity, in general we shall not write explicitly the dependence on $H$; thus $W$ stands for $W^H$, $K(t, s)$ for $K^H(t, s)$, etc. For any $q \in [1, +\infty]$, we denote by $\| \cdot \|_q$ the $L^q(\Omega)$–norm. We make the convention $\sum_{k=a}^b x_k = 0$ if $b < a$ and denote by $C$ positive constants with possibly different values. For additional notions and notation on rough paths, we refer the reader to [4].

2. Approximation result

For $p \in ]1, +\infty]$ we set $\tilde{d}_p = \tilde{d}_{p \wedge 2}$, that is
\[
\tilde{d}_p(x, y) = \sup_{0 \leq s < t \leq 1} \left(\sum_{i=1}^{\lfloor p\rfloor} \frac{|x_s, i - y_s, i|}{(t - s)^{\frac{i}{p}}}\right)\,.
\]

The purpose of this section is to prove the following approximation result.

**Theorem 2.1.** Let $H \in ]1, \frac{1}{2}]$, $p \in ]2, 4]$ (resp. $H \in ]\frac{1}{2}, 1]$, $p \in ]1, 2]$, be such that $pH > 1$ and $q \in [1, +\infty]$. The sequence $(\tilde{d}_p(W(m), W), m \geq 1)$, converges to 0 in $L^q(\Omega)$ and a.s. Thus for $H \in ]\frac{1}{2}, 1]$ and $p \in ]1, 2]$, if $\mathcal{G}_p$ denotes the set of dyadic geometric rough paths endowed with the norm $d_p(0, \cdot)$ and $P^H$ denotes the law of the fractional Brownian motion $W^H$, then the triple $(X, \mathcal{H}^H, P^H)$ is an abstract Wiener space.

The next Proposition provides the auxiliary result to state the approximation of the first component of the enhanced fractional Brownian motion.

**Proposition 2.2.** Let $0 \leq s < t \leq 1$, $q \in [1, +\infty]$.

(i) For any $H \in ]0, \frac{1}{2}]$, $\lambda \in [0, H[$,
\[
\left\| W^{(1)}_{s, t} - W(m)^{(1)}_{s, t} \right\|_q \leq C2^{-m\lambda}|t - s|^{H - \lambda}.
\] (2.1)

(ii) For any $H \in ]\frac{1}{2}, 1]$, $\varepsilon \in [0, H[$, $\mu \in ]0, \frac{\varepsilon}{H(2H + 1)}[$,
\[
\left\| W^{(1)}_{s, t} - W(m)^{(1)}_{s, t} \right\|_q \leq C2^{-m\mu}|t - s|^{H - \varepsilon}.
\] (2.2)
Proof. By the hypercontractivity inequality, it suffices to prove the results for $q = 2$. In this case, it is an easy consequence of the identity

$$E \left( \left| W_{s,t}^{(1)} - W(m)_{s,t}^{(1)} \right|^2 \right) = \int_0^1 \left| (K(t,u) - K(s,u)) - (K_m(t,u) - K_m(s,u)) \right|^2 du$$

and of Lemma 3.2. Indeed, by (3.14), we have

$$E \left( \left| W_{s,t}^{(1)} - W(m)_{s,t}^{(1)} \right|^2 \right) \leq C |t - s|^{2H}.$$ 

Hence, if $t - s < 2^{-m}$, we easily obtain (2.1) and (2.2).

Assume now $H \in (0, \frac{1}{2}]$ and $t - s \geq 2^{-m}$. By (3.13), for $\epsilon \in [0, H]$, we obtain (2.2) with $\mu = \lambda \frac{2H}{1-2H}$, and (3.14) follows.

Let $H \in [\frac{1}{3}, 1]$ and $t - s \geq 2^{-m}$. Let $\alpha \in [0, 1]$; then (3.14) and (3.16) imply

$$\left\| W_{s,t}^{(1)} - W(m)_{s,t}^{(1)} \right\|_{q} \leq C |t - s|^{H(1-\alpha)} 2^{-m\alpha},$$

with $\lambda \in [0, \frac{1}{2H + 1}]$. By taking $\alpha = \frac{\epsilon}{2H}$, we obtain (2.2) with $\mu = \lambda \frac{2H}{1-2H}$.

Throughout the rest of this section, $H \in [\frac{1}{3}, \frac{1}{2}]$. Following [1], let $\mathcal{H}_K$ denote the set of functions $\varphi : [0, 1] \to \mathbb{R}$ such that

$$\| \varphi \|_{K}^2 = \int_0^1 \varphi(s)^2 K(1, s)^2 ds + \int_0^1 ds \left( \int_s^1 \varphi(t) - \varphi(s) \right) |K| dt, s \right) \leq +\infty.$$ 

For any $\varphi \in \mathcal{H}_K$, $0 < s < t$, set

$$K^* \left( 1_{[s,t]}(\cdot) (\varphi - \varphi_s) \right)(u) = 1_{[0,s]}(u) \int_s^t (\varphi_r - \varphi_s) K(dr, u)$$

$$+ 1_{[s,t]}(u) \left( K(t, u) (\varphi_u - \varphi_s) + \int_u^t (\varphi_r - \varphi_u) K(dr, u) \right).$$

(2.3)

Following [3],

$$W_{s,t}^{(2)} = \int_0^1 K^* \left( 1_{[s,t]}(\cdot) (W. - W_s) \right)(u) dB_u + \frac{1}{2} |t - s|^{2H}.$$ 

(2.4)

Moreover, by Theorem 9 in [4], for $W(m)$ defined in (1.4), we have

$$W(m)_{s,t}^{(2)} = \int_0^1 K^* \left( 1_{[s,t]}(\cdot) (W(m). - W(m)_s) \right) (u) \dot{B}(m)_u du.$$ 

(2.5)

Proposition 2.3. For each $m \in \mathbb{N}$, $0 < s < t \leq 1$, $q \in [1, \infty]$

$$\| W_{s,t}^{(2)} - W(m)_{s,t}^{(2)} \|_q \leq C 2^{-m\mu} |t - s|^{2H - \epsilon},$$

(2.6)

for some positive constants $C$ and any $\epsilon \in [0, 2H - \frac{1}{2}]$ and $\mu \in [0, \frac{1}{2H}]$. 


Before proving this proposition, we give an equivalent expression for $W(m)_{s,t}^2$, as follows. The integration by parts formula of Malliavin calculus (see e.g. [3], Equation (1.49)) and (1.6) yield $W(m)_{s,t}^{(2)} = A_{s,t}^1(m) + A_{s,t}^2(m)$, with

$$A_{s,t}^1(m) = \sum_{k=1}^{2m} \int_0^1 du \, 1_{\Delta_k^n}(u) \, 2^m K^* \left( 1_{[s,t]}(\cdot) \int_{\Delta_k^n} dB_r (W(m) - W(m)) \right)(u), \quad (2.7)$$

$$A_{s,t}^2(m) = \sum_{k=1}^{2m} \int_0^1 du \, 1_{\Delta_k^n}(u) \, 2^m K^* \left( 1_{[s,t]}(\cdot) \int_{\Delta_k^n} dr \, (K_m(\cdot, r) - K_m(s, r)) \right)(u). \quad (2.8)$$

By definition, for $r \in \Delta_k^n$, $K_m(t, r) = 2^m \int_{\Delta_k^n \cap [0,t]} K(t, u) du = 2^m K(1_{\Delta_k^n})(t)$.

Since $h := K(1_{\Delta_k^n}) \in H_K$, the duality relation given in [3], equation (58) and Lemma 3.3 yield

$$A_{s,t}^2(m) = \frac{1}{2} \int_0^1 dr \, |K_m(t, r) - K_m(s, r)|^2 = \frac{1}{2} ||W(m)_{s,t}^{(1)}||_2^2.$$

Thus, since $E|W_t - W_s|^2 = |t - s|^{2H}$, Schwarz’s inequality, (3.14), (3.15) imply

$$\left| A_{s,t}^2(m) - \frac{1}{2} |t - s|^{2H} \right| \leq C 2^{-m\varepsilon} |t - s|^{2H - \varepsilon}, \quad (2.9)$$

for some positive constant $C$ and $\varepsilon \in ]0, H[$.

Hence, in order to establish (2.6) it suffices to prove that for any small parameter $\varepsilon \in ]0, 4H - 1[$ and $\mu \in ]0, \varepsilon[$,

$$E \left( \left| \int_0^1 K^* \left( 1_{[s,t]}(\cdot) (W_r - W_s) \right)(u) dB_u - A_{s,t}^1(m) \right|^2 \right) \leq C 2^{-m\mu} |t - s|^{4H - \varepsilon}. \quad (2.10)$$

for all $m \geq 1$. We devote the next lemmas to the proof of this convergence, using the expression of the operator $K^*$ given in (2.8).

**Lemma 2.4.** For any $0 \leq s < t \leq 1$, $m \geq 1$, we set

$$T_1(s, t) = \int_0^s dB_u \left( \int_s^t (W_r - W_s) K(dr, u) \right),$$
Then for any $\epsilon \in ]0,2H[$ and $\mu \in ]0,\epsilon[$, there exists $C > 0$ such that
\[
E \left( |T_1(s,t,m) - T_1(s,t)|^2 \right) \leq C 2^{-m\mu} |t - s|^{4H - \epsilon}.
\]  

(2.11)

Proof. Assume $s \in \Delta^m_I$, $I \geq 1$; we consider the decomposition
\[
E \left( |T_1(s,t,m) - T_1(s,t)|^2 \right) \leq C \sum_{j=1}^3 \tau_{1,j}(s,t,m),
\]
with
\[
\tau_{1,1}(s,t,m) = \sum_{k \in \{1,1,1\}} E \left( \left| \int_{\Delta^m_k} dB_r \ 2^m \left( \int_{\Delta^m_k \cap [0,s]} du \left( \int_s^t (W(m)_v - W(m)_s) K(dv,u) \right) \right)^2 \right| \right),
\]  

(2.12)

\[
\tau_{1,2}(s,t,m) = \sum_{k \in \{1,1,1\}} E \left( \left| \int_{\Delta^m_k \cap [0,s]} dB_r \left( \int_s^t (W_v - W_s) K(dv,r) \right)^2 \right| \right),
\]  

(2.13)

\[
\tau_{1,3}(s,t,m) = E \left( \left| \sum_{k=2}^{I-2} \int_{\Delta^m_k} dB_r \int_{\Delta^m_k} du \left( \int_s^t (W(m)_v - W(m)_s) K(dv,u) \right)^2 \right| \right).
\]  

(2.14)

By Lemma 3.4, 3.4, Schwarz’s inequality and 3.14, any term in the right hand-side of (2.12) is bounded as follows. Let $\epsilon \in ]0,2H[$, $\lambda \in \left] \frac{1 - (2H - \epsilon)}{2}, \frac{1}{2} \right[$, then $2H - 3 + 2\lambda < -1, 1 - 2\lambda - (2H - \epsilon) < 0$ and
\[
E \left( \left| \int_{\Delta^m_k} dB_r \ 2^m \left( \int_{\Delta^m_k \cap [0,s]} du \left( \int_s^t (W(m)_v - W(m)_s) K(dv,u) \right) \right)^2 \right| \right)
\leq C \int_{\Delta^m_k} dr \int_0^1 d\rho \left( 2^m \int_{\Delta^m_k \cap [0,s]} du \left( \int_s^t (K_m(v,\rho) - K_m(s,\rho)) K(dv,u) \right)^2 \right)
\leq C \int_{\Delta^m_k} dr \int_0^1 d\rho 2^m \int_{\Delta^m_k \cap [0,s]} du \left( \int_s^t dv |v - u|^{2H - 3 + 2\lambda} \right) 
\times \left( \int_s^t dv K_m(v,\rho) - K_m(s,\rho)^2 |v - u|^{-2\lambda} \right)
\[ \leq C \int_{\Delta_m^m \cap [0, s]} du \left( (s - u)^{2H - 2 + 2\lambda} |t - s|^{2H} \left( |t - u|^{1 - 2\lambda} - |s - u|^{1 - 2\lambda} \right) \right) \]
\[ \leq C \int_{\Delta_m^m \cap [0, s]} du \left( (s - u)^{2H - 2 + 2\lambda} |t - s|^{2H} |t - s|^{2H - \varepsilon} |s - u|^{1 - 2\lambda - (2H - \varepsilon)} \right) \]
\[ \leq C |t - s|^{4H - \varepsilon} \int_{\Delta_m^m \cap [0, s]} du \left( |s - u|^{\varepsilon - 1} \leq C 2^{-m\varepsilon} |t - s|^{4H - \varepsilon}. \right) \]

Each term of the right hand-side of (2.13) can be studied using a similar strategy. Thus we obtain for \( \varepsilon \in [0, 2H] \):
\[ \tau_{1,1}(s, t, m) + \tau_{1,2}(s, t, m) \leq C 2^{-m\varepsilon} |t - s|^{4H - \varepsilon}. \] (2.15)

Set for \( s \geq 3 \cdot 2^{-m} \), and hence \( I \geq 4 \),
\[ X_r = \sum_{k=2}^{I-2} 1_{\Delta_k^m} \int_{\Delta_k^m} du \left( \int_s^t (W(m)_u - W(m)_s) K(dv, u) \right. \]
\[ \left. - \int_s^t (W_v - W_s) K(dv, r) \right). \]

Notice that \( X_r = \int_0^1 g(r, \rho) dB_r \), with
\[ g(r, \rho) = \sum_{k=2}^{I-2} 1_{\Delta_k^m} \int_{\Delta_k^m} du \left( \int_s^t K(dv, u) (K_m(v, \rho) - K_m(s, \rho)) \right. \]
\[ \left. - \int_s^t K(dv, r) (K(v, \rho) - K(s, \rho)) \right). \]

Hence, by Lemma 3.4 and Schwarz’s inequality, \( \tau_{1,3}(s, t, m) \leq C (\tau_{1,3,1}(s, t, m) + \tau_{1,3,2}(s, t, m)) \), with
\[ \tau_{1,3,1}(s, t, m) = \sum_{k=2}^{I-2} 2^m \int_{\Delta_k^m} dr \int_{\Delta_k^m} du \int_0^1 d\rho \]
\[ \times \left| \int_s^t (K_m(v, \rho) - K_m(s, \rho)) (K(dv, u) - K(dv, r)) \right|^2, \]
\[ \tau_{1,3,2}(s, t, m) = \sum_{k=2}^{I-2} 2^m \int_{\Delta_k^m} dr \int_0^1 d\rho \left| \int_s^t K(dv, r) \right. \]
\[ \left. \times (K_m(v, \rho) - K_m(s, \rho) - K(v, \rho) + K(s, \rho)) \right|^2. \]

Owing to (3.4), (3.7), we have for \( \lambda \in [0, 1] \), \( u, r \in \Delta_k^m \),
\[ \left| \frac{\partial K}{\partial v}(v, u) - \frac{\partial K}{\partial v}(v, r) \right| \]
\[ \leq C \left| \frac{\partial K}{\partial v}(v, u) - \frac{\partial K}{\partial v}(v, r) \right|^\lambda \left( \left| \frac{\partial K}{\partial v}(v, u) \right|^{1-\lambda} + \left| \frac{\partial K}{\partial v}(v, r) \right|^{1-\lambda} \right) \]
\[ \leq C 2^{-m \lambda} |v - (u \lor r)|^{H - \frac{1}{2}} \left[ (u \land r)^{-1} + |v - (u \lor r)|^{-1}\right]^{\lambda}. \quad (2.16) \]

Thus, taking \( \lambda := H \) yields \( \tau_{1,3,1}(s, t, m) \leq C 2^{-mH} \sum_{j=1}^{2} \tau_{1,3,1,j}(s, t, m) \), with
\[
\tau_{1,3,1,1}(s, t, m) = \sum_{k=2}^{l-2} 2^m \int_{\Delta_k^m} dr \int_{\Delta_k^m} du \int_0^1 d\rho \left( \int_s^t dv \left| K_m(v, \rho) - K_m(s, \rho) \right| \right) \times \left| v - (u \lor r)|^{H - \frac{3}{2}} (u \land r)\right| \left( u \land r \right)^2,
\]
\[
\tau_{1,3,1,2}(s, t, m) = \sum_{k=2}^{l-2} 2^m \int_{\Delta_k^m} dr \int_{\Delta_k^m} du \int_0^1 d\rho \left( \int_s^t dv \left| K_m(v, \rho) - K_m(s, \rho) \right| \right) \times \left| v - (u \lor r)|^{-\frac{3}{2}} \right|^2.
\]

Let \( a = 2 - \epsilon \), with \( \epsilon \in ]0, 2H[ \). Schwarz’s inequality along with (3.14) yield
\[
\tau_{1,3,1,1}(s, t, m) \leq C \sum_{k=2}^{l-2} 2^m \int_{\Delta_k^m} dr \int_{\Delta_k^m} du \left( \int_s^t dv \left| v - (u \lor r)\right|^{-a} \right) \times \left( \int_s^t dv \left| v - s \right|^{4H - 3 + \epsilon} |u \land r|^{-2H} \right)
\leq C |t - s|^{4H - \epsilon} \int_{t_1^m}^{t_2^m} du (s - \overline{u}_m)^{\epsilon - 1} (\overline{u}_m)^{-2H}
\leq C |t - s|^{4H - \epsilon} s^{-2H} \leq |t - s|^{4H - \epsilon} 2^{-m(\epsilon - 2H)}. \quad (2.17)
\]

Indeed, \( \int_s^t dv \left| v - (u \lor r)\right|^{-2+\epsilon} \leq C (s - \overline{u}_m)^{\epsilon - 1} \) for \( \overline{u}_m \) defined by (3.13). Let \( \epsilon \in ]0, 2H[ \) using Schwarz’s inequality and (3.14), we obtain
\[
\tau_{1,3,1,2}(s, t, m) \leq C \sum_{k=2}^{l-2} 2^m \int_{\Delta_k^m} dr \int_{\Delta_k^m} du \left( \int_s^t dv \left| v - (u \lor r)\right|^{-2 - 2H + \epsilon} \right) \times \left( \int_s^t dv \left| v - (u \lor r)\right|^{2H - \epsilon - 1} |v - s|^{2H} \right)
\leq C |t - s|^{4H - \epsilon} \int_{t_1^m}^{t_2^m} du \int_s^t dv (v - \overline{u}_m)^{-2 - 2H + \epsilon}
\leq C |t - s|^{4H - \epsilon} \int_{t_1^m}^{t_2^m} du (s - \overline{u}_m)^{-1 - 2H + \epsilon}
\leq C |t - s|^{4H - \epsilon} 2^{-m(\epsilon - 2H)} . \quad (2.18)
\]

From (2.17), (2.18) we deduce that for \( \epsilon \in ]0, 2H[ \),
\[
\tau_{1,3,1}(s, t, m) \leq C |t - s|^{4H - \epsilon} 2^{-m\epsilon}. \quad (2.19)
\]

Let \( \delta \in ]0, 2H[ \), \( \alpha \in ]0, 2H[ \), \( \lambda \in ]0, 1[ \) and \( \mu \in ]\frac{1}{2}, 1 - H[ \). Notice that for these choices, \(-2\mu + 1 - 2H + \delta < 0\). Hölder’s inequality together with (3.14) and
yield for any $\lambda \in ]0, 1]$, 
\[
\tau_{1,3,2}(s, t, m) \leq C \tau_{1,3,2,1}(s, t, m)^{\lambda} \tau_{1,3,2,2}(s, t, m)^{1-\lambda},
\]
where
\[
\tau_{1,3,2,1}(s, t, m) = \int_{t_{m-2}}^{t_{m-1}} dr \left( \int_{s}^{t} dv (v - r)^{2H-3+2\mu} \right) \left( \int_{s}^{t} dv (v - r)^{-2\mu_2 (v - s)^{2H}} \right),
\]
\[
\tau_{1,3,2,2}(s, t, m) = \int_{t_{m-2}}^{t_{m-1}} dr \left( \int_{s}^{t} dv (v - r)^{2H-3+2\mu} \right) \left( \int_{s}^{t} dv (v - r)^{-2\mu_2 m} \right).
\]
For the first term we have
\[
\tau_{1,3,2,1}(s, t, m) \leq C |t - s|^{4H-\delta} \int_{t_{m-2}}^{t_{m-1}} dr (s - r)^{2H-2+2\mu} (s - r)^{-2\mu_1+1-2H+\delta} \leq C |t - s|^{4H-\delta},
\]
while for the second one, we obtain
\[
\tau_{1,3,2,2}(s, t, m) \leq C 2^{-2mH} |t - s|^{2H-\alpha} \int_{t_{m-2}}^{t_{m-1}} dr (s - r)^{2H-2+2\mu} (s - r)^{-2\mu_1+1-2H+\alpha}.
\]
Consequently, 
\[
\tau_{1,3,2}(s, t, m) \leq C |t - s|^{(4H-\delta)\lambda + (2H-\alpha)(1-\lambda)} 2^{-2mH(1-\lambda)}.
\]
Take $\alpha, \delta$ arbitrarily small and $1 - \lambda = \frac{\epsilon - H\delta}{2H+\alpha}$. Then for $\beta < \epsilon < 2H$, we have proved that 
\[
\tau_{1,3,2}(s, t, m) \leq C |t - s|^{4H-\epsilon} 2^{-m\beta}.
\]
This inequality, together with (2.15) and (2.19) yields (2.11). \qed

**Lemma 2.5.** For any $0 \leq s < t \leq 1$, set
\[
T_2(s, t) = \int_{s}^{t} dB_u K(t, u)(W_u - W_s),
\]
\[
T_2(s, t, m) = \sum_{k=1}^{m} \int_{\Delta_k} dB_r 2^m \left( \int_{\Delta_k[s,t]} du K(t, u) (W(m)_u - W(m)_s) \right).
\]
Then, for $b \in [0, 2H[$, there exists a constant $C > 0$ such that for each $m \geq 1$
\[
E \left( |T_2(s, t, m) - T_2(s, t)|^2 \right) \leq C 2^{-mb} |t - s|^{4H-b}. \tag{2.20}
\]
**Proof.** Let $s \in \Delta_I^m$, $t \in \Delta_J^m$. We have
\[
E \left( |T_2(s, t, m) - T_2(s, t)|^2 \right) \leq C \sum_{j=1}^{3} T_{2,j}(s, t, m),
\]
with for $I = \{I, I + 1, J - 2, J - 1\}$
\[
T_{2,1}(s, t, m) = \sum_{k \in I} E \left( \int_{\Delta_k} dB_r 2^m \int_{\Delta_k[s,t]} du K(t, u) (W(m)_u - W(m)_s) \right)^2,
\]
and Schwarz's inequality, we have for any \( k \in \mathbb{I} \),

\[
T_{2,3}(s, t, m) = E \left( \left| \sum_{k=I}^{J-3} \int_{\Delta_k^m \cap [s, t]} dB_r [2^m \int_{\Delta_k^m \cap [s, t]} duK(t, u) (W(m)_u - W(m)_s) - K(t, r)(W_r - W_s)]^2 \right| \right).
\]

Owing to Lemma 3.4 applied to the Gaussian process

\[
X_r := 1_{\Delta_k^m}(r) \int_0^1 dB \int_{\Delta_k^m \cap [s, t]} duK(t, u) (K_m(u, \rho) - K_m(s, \rho))
\]

and Schwarz's inequality, we have for any \( k = 1, \ldots , 2^m \),

\[
T(s, t, m, k) := E \left( \left| \int_{\Delta_k^m \cap [s, t]} dB \int_{\Delta_k^m \cap [s, t]} duK(t, u) (W(m)_u - W(m)_s) \right|^2 \right) \leq C 2^{2m} \int_{\Delta_k^m} dr \int_0^1 d\rho \left( \int_{\Delta_k^m \cap [s, t]} duK^2(t, u) \right) \times \left( \int_{\Delta_k^m \cap [s, t]} du|K_m(u, \rho) - K_m(s, \rho)|^2 \right).
\]

Let \( k = I, I + 1 \); since \( \int_{\Delta_k^m \cap [s, t]} duK^2(t, u) \leq \int_{[s, t]} duK^2(t, u) \leq C |t - s|^{2H} \), we have for any \( b \in [0, 2H[ \),

\[
T(s, t, m, k) \leq C 2^m |t - s|^{2H} \int_{\Delta_k^m \cap [s, t]} du|u - s|^{2H} \leq C 2^m |t - s|^{4H-b} \int_{\Delta_k^m \cap [s, t]} du|u - s|^b \leq C 2^{-mb} |t - s|^{4H-b}.
\]

Let \( k = J - 2, J - 1, J \) with \( J - 2 > I + 1 \) then for \( u \in \Delta_k^m \), (3.5) implies \( |K(t, u)|^2 \leq C |t - u|^{2H-1} \) and \( |t - u| \leq C 2^{-m} \); we obtain for \( b \in [0, 2H[ \),

\[
T(s, t, m, k) \leq C 2^m \left( \int_{\Delta_k^m \cap [s, t]} du |t - u|^{2H-1-b} \right) \left( \int_{\Delta_k^m \cap [s, t]} du |u - s|^{2H} \right) \leq C |t - s|^{4H-b} 2^{-mb}.
\]

We therefore have proved that for \( b \in [0, 2H[ \),

\[
T_{2,1}(s, t, m) \leq C 2^{-bm} |t - s|^{4H-b}. \tag{2.21}
\]
The analysis of the term $T_{2.2}(s, t, m)$ is easier. Indeed, the isometry property of the stochastic integral yields for any $k = 1, \ldots, 2^m$,

$$E \left( \left| \int_{\Delta^m_k \cap [s,t]} dB_r K(t, r)(W_r - W_s) \right|^2 \right) = C \int_{\Delta^m_k \cap [s,t]} dr K^2(t, r)|r - s|^{2H}. \quad (2.22)$$

For the particular values of $k \in I$, the right hand-side of (2.22) can be analyzed following similar ideas as for $T_{2.1}(s, t, m)$, which yields for $b \in [0, 2H]$

$$T_{2.2}(s, t, m) \leq C 2^{-mb}|t - s|^{4H - b}. \quad (2.23)$$

We now study $T_{2.3}(s, t, m)$ and note that $T_{2.3}(s, t, m) = 0$ if $|t - s| \leq 2^{-m}$. Thus, we may assume that $t - s \geq 2^{-m}$. First, we apply Lemma 3.4 and obtain

$$T_{2.3}(s, t, m) \leq C(T_{2.3,1}(s, t, m) + T_{2.3,2}(s, t, m)),$$

where

$$T_{2.3,1}(s, t, m) = \int_{\mathbb{R}_m}^{\mathbb{R}_m - 2^{-m}} dr \int_0^1 du \left| 2^m \int_{\mathbb{R}_m}^\mathbb{R}_m du \left( K(t, u) - K(t, r) \right) \right|^2,$$

$$T_{2.3,2}(s, t, m) = \int_{\mathbb{R}_m}^{\mathbb{R}_m - 2^{-m}} dr \int_0^1 du \left| 2^m \int_{\mathbb{R}_m}^\mathbb{R}_m du K(t, r) \right|^2,$$

By Schwarz’s inequality and (3.14), for $b \in [0, 2H]$, 

$$T_{2.3,1}(s, t, m) \leq \int_{\mathbb{R}_m}^{\mathbb{R}_m - 2^{-m}} dr 2^m \int_{\mathbb{R}_m}^{\mathbb{R}_m} du |K(t, u) - K(t, r)|^2|u - s|^{2H},$$

$$\leq C|t - s|^{2H} \int_{\mathbb{R}_m}^{\mathbb{R}_m - 2^{-m}} dr 2^m \int_{\mathbb{R}_m}^{\mathbb{R}_m} du |K(t, u) - K(t, r)|^2,$$

$$\leq C 2^{-2mH}|t - s|^{2H} \leq C 2^{-mb}|t - s|^{4H - b}$$

where the last inequalities follow from (3.19) and $|t - s| \geq 2^{-m}$.

Owing to (3.19), we have for $u \in [\mathbb{R}_m, \mathbb{R}_m]$ 

$$\int_0^1 du |K_m(s, u) - K(s, u)|^2 \leq C 2^{-2mH},$$

$$\int_0^1 du |K_m(u, r) - K(r, r)|^2 \leq C \int_0^1 du \left( |K_m(u, r) - K(u, r)|^2 + |K(u, r) - K(r, r)|^2 \right) \leq C 2^{-2mH}. $$
Schwarz’s inequality, along with (3.3) and the above estimates yield

\[ T_{2,3,2}(s, t, m) \leq C \int_{\mathbb{R}^n} \int_{0}^{s} \int_{0}^{t} dr2^{-mH} \left( |r|^{2H-1} + |t-r|^{2H-1} \right) \]
\[ \leq C2^{-mH} \left( t^{2H} - s^{2H} + |t-s|^{2H} + 2^{-2mH} \right) \]
\[ \leq C2^{-mH} |t-s|^{2H} \leq C2^{-mb}|t-s|^{4H-b} \]

for \( b \in [0, 2H[ \). Indeed, for each \( H \in ]0, \frac{1}{2}[, \) and \( s < t, t^{2H} - s^{2H} \leq (t-s)^{2H} \) and we are assuming that \( 2^{-m} < |t-s| \). Thus, (2.21) is proved. \( \square \)

**Lemma 2.6.** For any \( 0 \leq s < t \leq 1 \), set

\[ T_3(s, t) = \int_{s}^{t} dB_u \int_{u}^{t} K(dr, u)(W_r - W_u) \]
\[ T_3(s, t, m) = \sum_{k=1}^{2m} 2^m \int_{\Delta_k^m} dB_r \int_{\Delta_k^m \cap [s, t]} du \int_{u}^{t} K(dv, u) (W(m)_v - W(m)_u). \]

There exists a positive constant \( C \) such that, for any \( \epsilon \in ]0, 4H - 1[ \)

\[ E \left( \left| T_3(s, t, m) - T_3(s, t) \right|^2 \right) \leq C2^{-me}|t-s|^{4H-\epsilon}, \tag{2.24} \]

for each \( m \geq 1 \).

**Proof.** Assume \( s \in \Delta_I^m, t \in \Delta_J^m \); we consider the upper bound

\[ E \left( \left| T_3(s, t, m) - T_3(s, t) \right|^2 \right) \leq C \sum_{j=1}^{3} T_{3,j}(s, t, m), \]

where for \( J = \{I, I + 1, J - 1, J\} \)

\[ T_{3,1}(s, t, m) = \sum_{k \in J} E \left( \left| 2^m \int_{\Delta_k^m} dB_r \int_{\Delta_k^m \cap [s, t]} du \int_{u}^{t} K(dv, u) \right. \right. \]
\[ \times (W(m)_v - W(m)_u)^2 \right) \tag{2.25}, \]
\[ T_{3,2}(s, t, m) = \sum_{k \in J} E \left( \left| \int_{\Delta_k^m \cap [s, t]} dB_r \int_{r}^{t} K(dv, r)(W_v - W_r) \right|^{2} \right), \tag{2.26} \]
\[ T_{3,3}(s, t, m) = E \left( \left| \sum_{k=i+2}^{J-2} 2^m \int_{\Delta_k^m} dB_r \int_{\Delta_k^m \cap [s, t]} du \right. \right. \]
\[ \times \left( \left| \int_{u}^{t} K(dv, u)(W(m)_v - W(m)_u) - \int_{r}^{t} K(dv, r)(W_v - W_r) \right|^{2} \right). \]

Lemma 3.3 along with Schwarz’s inequality yield for each term of the sum in the right hand side of (2.25) the upper bound

\[ C \int_{\Delta_k^m} dr \int_{0}^{1} d\rho 2^m \int_{\Delta_k^m \cap [s, t]} du \left( \int_{u}^{t} K(dv, u)(K_m(v, \rho) - K_m(u, \rho)) \right)^2. \]
Fix $a \in [2 - 4H, 1]$. From Schwarz’s inequality, (3.4) and (3.14) we deduce the following estimates for this integral:

$$C \int_{\Delta^m_k} d\nu^m \int_{\Delta^m_k(s,t]} du \left( \int_u^{\nu^r} dv |v - u|^{-a} \right) \left( \int_u^{\nu^r} |v - u|^{4H - 3 + a} \right)$$

$$\leq C \left( 2^{-m} \wedge |t - s| \right) |t - s|^{4H - 1}.$$  

A similar analysis yields the same result for each term in the right hand-side of (2.26). Consequently,

$$T_{3,1}(s, t, m) + T_{3,2}(s, t, m) \leq C \left( 2^{-m} \wedge |t - s| \right) |t - s|^{4H - 1}. \quad (2.27)$$

If $|t - s| \leq 2^{-m}$ then $T_{3,3}(s, t, m) = 0$. Hence, let us assume that $t - s \geq 2^{-m}$; in this case $T_{3,3}(s, t, m)$ is equal to $E \left( \int_0^{1} dB_r X_r \right)^2$, with $X_r = \int_0^{1} dB_r g(r, \rho)$, and

$$g(r, \rho) = \sum_{k = I + 2}^{J - 2} 1_{\Delta^m_k}(r) 2^m \int_{\Delta^m_k} du \left[ \int_u^{\nu^r} K(dv, u)(K_m(v, \rho) - K_m(u, \rho)) \right.$$

$$\left. - \int_r^{\nu^r} K(dv, r)(K(v, \rho) - K(r, \rho)) \right].$$

We at first study the contribution to $T_{3,3}(s, t, m)$ of the integrands

$$g_1(r, \rho) = \sum_{k = I + 2}^{J - 2} 1_{\Delta^m_k}(r) 2^m \int_{\Delta^m_k} du \int_u^{\nu^r} K(dv, u)(K_m(v, \rho) - K_m(u, \rho)),$$

$$g_2(r, \rho) = \sum_{k = I + 2}^{J - 2} 1_{\Delta^m_k}(r) 2^m \int_{\Delta^m_k} du \int_r^{\nu^r} K(dv, r)(K(v, \rho) - K(r, \rho)),$$

which we denote by $T_{3,3,j}(s, t, m)$, $j = 1, 2$. Actually, both are similar and therefore we only study the first one. Lemma (3.4), (3.14) and Schwarz’s inequality imply, for each $a \in [2 - 4H, 1]$,

$$T_{3,3,1}(s, t, m) \leq C \sum_{k = I + 2}^{J - 2} 2^m \int_{\Delta^m_k} dr \int_{\Delta^m_k} du \int_u^{\nu^r} dv |v - u|^{-a} \int_u^{\nu^r} dv |v - u|^{4H - 3 + a}$$

$$\leq C 2^{-m(4H - 1)} |t - s|. \quad (2.28)$$

We end the analysis of the term $T_{3,3}(s, t, m)$ by studying the contribution of $T_{3,3,3}(s, t, m)$ defined in terms of the integrand

$$g_3(r, \rho) = \sum_{k = I + 2}^{J - 2} \int_{\Delta^m_k} dr 2^m \int_{\Delta^m_k} du \int_u^{\nu^r} \left[ K(dv, u)(K_m(v, \rho) - K_m(u, \rho)) \right.$$

$$\left. - K(dv, r)(K(v, \rho) - K(r, \rho)) \right].$$

Notice that $g_3(r, \rho)$ is the sum of two analogous terms where the set $\Delta^m_k$ of the integral with respect to the variable $u$ is replaced by $[s, t], [r, \nu^{\nu^r}]$. 

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respectively. Again, the contribution of both terms is similar, so that we concentrate on the first one. That is, we consider

\[ T_{3,3,3}^{\lambda}(s,t,m) := E \left( \left| \sum_{k=1+2}^{J-2} 2^m \int_{\Delta_k^m} dB_r \int_{\mathbb{L}_{n,r}} du \int_0^t \left[ K(dv,u) \times (W(m)v - W(m)u) - K(dv,r)(W_v - W_r) \right]^2 \right| \right). \]

As before, all the arguments rely on Lemma 3.4, (3.4), (3.14), a suitable factorization of the integrands along with Schwarz’s inequality. In order to deal with the singularity at \( v = r \), we first replace the integral with respect to the variable \( v \) by \( \int_{r_m+2^{-m}}^r \). Given \( a \in [2 - 4H, 1] \), the corresponding contribution to \( T_{3,3,3}^{\lambda}(s,t,m) \) is bounded by

\[
C \sum_{k=1+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{\mathbb{L}_{n,r}} du \int_0^1 dp \left( \int_{r_m+2^{-m}}^r K(dv,u) \times (K_m(v,\rho) - K_m(u,\rho)) \right)^2 + \left( \int_{r_m+2^{-m}}^r K(dv,r)(K(v,\rho) - K(r,\rho)) \right)^2 \leq C \sum_{k=1+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{\mathbb{L}_{n,r}} du \int_r^{r_m+2^{-m}} dv |v-r|^{-a} \int_r^{r_m+2^{-m}} dv |v-r|^{4H-3+a} \leq C 2^{-m(4H-1)} |t-s|. \tag{2.29}
\]

Let us finally consider the range \( [r_m+2^{-m}, t[ \) for the variable \( v \). We have to study two terms:

\[
M_1(s,t,m) = \sum_{k=1+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{\mathbb{L}_{n,r}} du \int_0^1 dp \left( \int_{r_m+2^{-m}}^r dv \times |K_m(v,\rho) - K_m(u,\rho)||\partial K / \partial v(v,u) - \partial K / \partial v(v,r)| \right)^2,
\]

\[
M_2(s,t,m) = \sum_{k=1+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{\mathbb{L}_{n,r}} du \int_0^1 dp \left( \int_{r_m+2^{-m}}^r dv \left| \partial K / \partial v(v,r) \right| \times \left( (K_m(v,\rho) - K_m(u,\rho)) - (K(v,\rho) - K(r,\rho)) \right) \right)^2.
\]

For \( M_1(s,t,m) \), we proceed in a similar way as for the term \( \tau_{1,3,1}(s,t,m) \) in Lemma 2.4, as follows. By means of (2.16) we obtain for \( \lambda \in [0,1] \)

\[
M_{1,1}(s,t,m) = \sum_{k=1+2}^{J-2} 2^m \int_{\Delta_k^m} dr \int_{\mathbb{L}_{n,r}} du u^{-2\lambda} \int_0^1 \rho \left( \int_{r_m+2^{-m}}^r dv |K_m(v,\rho) - K_m(u,\rho)||v-r|^{H-3/2} \right)^2,
\]
\[ M_{1,2}(s, t, m) = \sum_{k=1+2}^{J-2} 2^m \int_{\Gamma_k} dr \int_{[\Gamma_m, r]} du \int_0^1 dp (\int_{t_m+2-m}^t dv) \|K_m(v, \rho) - K_m(u, \rho)\|v - r\|^{H-\frac{3}{2}-\lambda})^2. \]

Let \( a \in ]2 - 4H, 1], \lambda \in ]0, \frac{1}{2}[, \) since \( t - s \geq 2^{-m} \), for \( u \in [\Gamma_m, r] \), we have
\[ \int_{t_m+2-m}^t dv|v - r|^{2H-3+a}\|v - u\|^{2H} \leq C|t - r|^{4H+a-2}. \]
Consequently, since \( r \geq u \geq \frac{t}{2} \), implies \( u \geq \frac{t}{2} \),
\[ M_{1,1}(s, t, m) \leq C \sum_{k=1+2}^{J-2} 2^m \int_{\Gamma_k} dr \int_{[\Gamma_m, r]} du u^{-2\lambda} (\int_{t_m+2-m}^t dv|v - r|^{-a}) \times (\int_{t_m+2-m}^t dv|v - r|^{2H-3+a}\|v - u\|^{2H}) \leq C \int_s^t r^{-2a}|t - s|^{4H-1} dr \leq C|t - s|^{4H-2\lambda}. \quad (2.30) \]
Analogously, for \( b \in ]2 + 2\lambda - 4H, 1], \lambda \in ]0, 2H - \frac{1}{2}[, \) and \( |t - s| \geq 2^{-m} \),
\[ M_{1,2}(s, t, m) \leq C \sum_{k=1+2}^{J-2} 2^m \int_{\Gamma_k} dr \int_{[\Gamma_m, r]} du (\int_{t_m+2-m}^t dv|v - r|^{-a}) \times (\int_{t_m+2-m}^t dv|v - r|^{2H-3+a}\|v - u\|^{2H}) \leq C \int_s^t |t - r|^{4H-1-2\lambda} dr = C|t - s|^{4H-2\lambda}. \quad (2.31) \]
Finally, if we additionally use (3.13), we obtain for \( a \in ]2 - 4H, 1[ \)
\[ M_2(s, t, m) \leq C \sum_{k=1+2}^{J-2} 2^m \int_{\Gamma_k} dr \int_{[\Gamma_m, r]} du (\int_r^t dv|v - r|^{-a}) \times (\int_{t_m+2-m}^t dv|v - r|^{2H-3+a}\|v - u\|^{2H}) \leq C \int_s^t |t - r|^{4H-2\lambda - 2mH} dr \leq C2^{-mb}|t - s|^{4H-b} \quad (2.32) \]
for \( b \in ]0, 4H - 1[ \). We easily check that (2.24) follows from (2.27)–(2.32).

**Proof of Proposition 2.3**: We remark that Lemmas 2.4 to 2.6 yield the upper bound (2.10). Therefore, for \( q = 2, (2.6) \) follows from (2.9) and (2.11). The hypercontractivity inequality yields the validity of the same inequality for any \( q \in ]2, \infty[ \).

**Proof of Theorem 2.1**: 
Let \( H \in ]\frac{1}{2}, 1[ \) and \( p \in ]\frac{1}{M}, 2[ \). The convergence of \( \hat{d}_p(W(m), W) \) to zero in \( L^q(\Omega) \) is a consequence of (2.2) and the usual version of the Garsia-Rademich-Rumsey lemma (see e.g. [3], Theorem 2.1.3).

Consider the metric space \((G_p, d_p)\). The canonical embedding \( \mathcal{H}^H \hookrightarrow G_p \) is continuous. Indeed, let \( \hat{h}_i, i = 1, 2, \) belong to \( L^2([0, 1]) \). Then for \( h_i(.) = \int_0^1 K(\cdot, r) \hat{h}_i(r) dr \) and \( 0 \leq s < t \leq 1, \)

\[
|h_1(s,t) - (h_2(s,t)| \leq |t - s|^\frac{1}{A}\|\hat{h}_1 - \hat{h}_2\|_{2H} \leq |t - s|^\frac{1}{2}\|h_1 - h_2\|_{2H}.
\]

Consequently, the preceding convergence shows that \((G_p, \mathcal{H}^H, P^H)\) is an abstract Wiener space.

Let now \( H \in ]\frac{1}{2}, \frac{1}{2}[ \). We follow the outline of the proof of Lemma 3 in [3], but refer to the extension of the Garsia-Rademich-Rumsey lemma stated in the Lemma 3.5.

Fix \( p \in [2, 4[ \) such that \( pH > 1 \). We shall prove that there exists \( \theta > 0 \) such that for every \( q \in [1, \infty[ \),

\[
E\left( \left| d_p(W, W(m)) \right|^q \right) \leq C_2 q^{2-\theta pq}.
\]

Indeed, for a fixed \( q \in [1, \infty[ \), let \( M > q \) and \( N = 2M \) satisfy \( N > \frac{p^2}{2(Hp - 1)} \).

Let \( \alpha, \beta > 0 \) defined by \( \alpha = \frac{2}{p} + \frac{1}{M}, \beta = \frac{1}{p} + \frac{1}{N} \).

By virtue of (2.1) and (2.4), we easily check that the random variables

\[
A_1(m) := \int_0^1 \int_0^1 dsdt 1_{s<t} \frac{|W_{s,t}^{(1)} - W(m)_{s,t}^{(1)}|^{2N}}{|t - s|^{2N\beta}},
\]

\[
A_2(m) := \int_0^1 \int_0^1 dsdt 1_{s<t} \frac{|W_{s,t}^{(2)} - W(m)_{s,t}^{(2)}|^{2M}}{|t - s|^{2M\alpha}},
\]

satisfy

\[
E(A_1(m)) \leq C 2^{-\mu 2N}, \quad E(A_2(m)) \leq C 2^{-\mu 2M},
\]

for some \( \mu > 0 \).

Furthermore, the hypercontractivity property and the inequality (3.14) imply that for \( 0 \leq s < t \leq 1 \) and \( q \in [1, \infty[ \),

\[
\sup_m \left( \|W_{s,t}^{(1)}\|_q + \|W(m)_{s,t}^{(1)}\|_q \right) \leq C |t - s|^H.
\]

This yields

\[
\sup_m E(\eta(m)) \leq C,
\]

where

\[
\eta(m) := \int_0^1 \int_0^1 dsdt 1_{s<t} \frac{|W_{s,t}^{(1)}|^{2N} + |W(m)_{s,t}^{(1)}|^{2N}}{|t - s|^{2N\beta}}.
\]

By Lemma 3.5, we deduce that for any \( 0 \leq s < t \leq 1, \)

\[
|W_{s,t}^{(1)} - W(m)_{s,t}^{(1)}| \leq C A_1(m) \frac{1}{N} |t - s|^{\frac{1}{\beta}},
\]

\[
|W_{s,t}^{(2)} - W(m)_{s,t}^{(2)}| \leq C \left[ A_2(m) \frac{1}{M} + A_1(m) \frac{1}{N} \eta(m) \frac{1}{N} \right] |t - s|^{\frac{2}{\beta}}.
\]
Finally, Schwarz’s and Hölder’s inequalities together with (2.34)-(2.37) conclude the proof of the theorem.

3. Appendix

Let $W^H = (W^H_t, t \in [0, 1])$ be a $d$-dimensional fractional Brownian motion with Hurst parameter $H \in [0, 1]$ and integral representation given in (1.2).

Assume $H \in [\frac{1}{2}, 1[$; by computing the integral of the right hand-side of (1.3), we obtain the following expression for the kernel $K^H$ defined in (1.2):

$$K^H(t, s) = c_H \left( H - \frac{1}{2} \right) s^{H - \frac{1}{2}} F_2 \left( \frac{t}{s} \right),$$  

(3.1)

where for $z > 1$,

$$F_2(z) = \int_0^{z-1} u^{H - \frac{3}{2}} (u + 1)^{H - \frac{3}{2}} du.$$  

(3.2)

From (1.2), it follows that

$$\frac{\partial K^H}{\partial t}(t, s) = c_H \left( H - \frac{1}{2} \right) \left( \frac{s}{t} \right)^{H - \frac{3}{2}} (t - s)^{H - \frac{3}{2}}.$$  

(3.3)

holds for any $H \in [0, 1 \cup \frac{1}{2}, 1[ \text{ and } 0 < s < t < 1$. Consequently, for $H \in [0, \frac{1}{2}]$,

$$\left| \frac{\partial K^H}{\partial t}(t, s) \right| \leq C |t - s|^{H - \frac{3}{2}}.$$  

(3.4)

The next Lemma collects some technical estimates on the kernel $K^H(t, s)$.

**Lemma 3.1.** Let $0 < s < t < 1$.

1. Assume $H \in [0, \frac{1}{2}]$. Then,

$$|K^H(t, s)| \leq C \left( s^{H - \frac{1}{2}} 1_{[t, 1]}(s) + (t - s)^{H - \frac{1}{2}} 1_{[\frac{t}{2}, t]}(s) \right),$$  

(3.5)

$$\left| \frac{\partial K^H}{\partial s}(t, s) \right| \leq C \left( s^{H - \frac{1}{2}} 1_{[\frac{1}{2}, t]}(s) + (t - s)^{H - \frac{1}{2}} 1_{[\frac{1}{2}, \frac{t}{2}]}(s) \right),$$  

(3.6)

$$\left| \frac{\partial^2 K^H}{\partial t \partial s}(t, s) \right| \leq C (t - s)^{H - \frac{1}{2}} \left( s^{-1} 1_{[t, 1]}(s) + (t - s)^{-1} 1_{[\frac{t}{2}, t]}(s) \right).$$  

(3.7)

2. For $H \in [\frac{1}{2}, 1[$,

$$|K^H(t, s)| \leq C \left( (t - s)^{H - \frac{1}{2}} 1_{[\frac{1}{2}, \frac{1}{2}]}(s) + s^{H - \frac{1}{2}} 1_{[0, \frac{1}{2}]}(s) \right),$$  

(3.8)

$$\left| \frac{\partial K^H}{\partial s}(t, s) \right| \leq C (t - s)^{2H - 1} \left( s^{-(H + \frac{1}{2})} 1_{[0, \frac{1}{2}]}(s) + (t - s)^{-(H + \frac{1}{2})} 1_{[\frac{1}{2}, t]}(s) \right).$$  

(3.9)
Proof. Assume first $H \in [0, \frac{1}{2}]$. It is easy to check that, for any $u > 0$,

$$0 < 1 - (u + 1)^{H - \frac{3}{2}} \leq \left( \frac{1}{2} - H \right) u \wedge 1.$$ 

Hence, for $0 < s < t$, $0 < u < \frac{t}{s} - 1$,

$$u^{H - \frac{3}{2}} \left( 1 - (u + 1)^{H - \frac{1}{2}} \right) \leq C u^{H - \frac{3}{2}} 1_{[0, \frac{1}{2}] \cup \left( \frac{1}{2}, 1 \right]}(u) + C u^{H - \frac{3}{2}} 1_{\left( \frac{1}{2}, 1 \right]}(u).$$

(3.10)

Thus, from (1.3), (3.10), it follows that

$$\left| F_1 \left( \frac{t}{s} \right) \right| \leq C \int_0^{\frac{t}{s} - 1} u^{H - \frac{3}{2}} du \leq C,$$

for $\frac{t}{2} \leq s < t$, while for $0 < s < \frac{t}{2}$,

$$\left| F_1 \left( \frac{t}{s} \right) \right| \leq C \int_0^{1} u^{H - \frac{3}{2}} du + C \int_1^{\infty} u^{H - \frac{3}{2}} du \leq C.$$

Consequently

$$\sup_{0 \leq s < t} \left| F_1 \left( \frac{t}{s} \right) \right| \leq C$$

(3.11)

and the identity (1.2) yields (3.5).

By differentiating with respect to the variable $s$ in (1.2) and using (3.11), we obtain

$$\left| \frac{\partial K^H}{\partial s}(t, s) \right| \leq C \left( |t - s|^{H - \frac{3}{2}} + s^{H - \frac{3}{2}} + s^{-1} |t - s|^{H - \frac{3}{2}} \right),$$

which yields (3.6). The inequality (3.7) follows by differentiating with respect to the variable $s$ in (3.3).

Suppose now $H \in [\frac{1}{2}, 1]$. Consider the function $F_2$ given in (1.2). Clearly, if $\frac{t}{s} - 1 \leq 1$, that is, if $\frac{t}{2} \leq s < t$,

$$\left| F_2 \left( \frac{t}{s} \right) \right| \leq C.$$

Assume $\frac{t}{s} - 1 > 1$. For any $u \in [1, \frac{t}{s} - 1]$, $(1 + u)^{H - \frac{1}{2}} \leq C u^{H - \frac{1}{2}}$. Consequently,

$$\left| F_2 \left( \frac{t}{s} \right) \right| \leq C \left( \int_0^{1} u^{H - \frac{3}{2}} du + \int_1^{\frac{t}{s} - 1} u^{2H - 2} du \right) \leq C \left( \frac{t}{s} \right)^{2H - 1}.$$

The previous upper bounds, together with the representation of the kernel $K^H$ given in (3.1), imply

$$|K^H(t, s)| \leq C \left( s^{H - \frac{1}{2}} \left( \frac{t}{s} \right)^{2H - 1} 1_{[0, \frac{1}{2}]}(s) + s^{H - \frac{3}{2}} 1_{\left( \frac{1}{2}, 1 \right]}(s) \right) \leq \left( s^{H - \frac{3}{2}} 1_{[0, \frac{1}{2}]}(s) + s^{-H + \frac{1}{2}} (t - s)^{2H - 1} 1_{[0, \frac{1}{2}]}(s) + s^{H - \frac{3}{2}} 1_{\left( \frac{1}{2}, 1 \right]}(s) \right)$$

(3.12)
and (3.8) follows.

Differentiating with respect to the variable $s$ in (3.11) yields

\[
\left| \frac{\partial K^H}{\partial s}(t, s) \right| \leq C \left( s^{H-2} F_2 \left( \frac{t}{s} \right) + s^{H-\frac{1}{2}} \frac{t}{s^2} \left( \frac{t}{s} - 1 \right)^{H-\frac{1}{2}} \right) \\
\leq C \left( s^{H-2} \left( \frac{t}{s} \right)^{2H-1} 1_{[0, \frac{1}{2}]}(s) + s^{-(H+\frac{1}{2})} t^{H+\frac{1}{2}} (t-s)^{H-\frac{3}{2}} \right) \\
+ s^{H-\frac{3}{2}} 1_{[\frac{1}{2}, t]}(s),
\]

where in the last inequality we have applied the upper bounds for $F_2$ obtained before. Replacing in the last expression $t^{2H-1}$ by $C(s^{2H-1} + (t-s)^{2H-1})$ and $t^{H+\frac{1}{2}}$ by $C(s^{H+\frac{1}{2}} + (t-s)^{H+\frac{1}{2}})$, respectively, yields

\[
\left| \frac{\partial K^H}{\partial s}(t, s) \right| \leq C \left( s^{H-2} + (t-s)^{H-\frac{3}{2}} + s^{-(H+\frac{1}{2})} (t-s)^{2H-1} \right).
\]  

(3.12)

If $0 < s < \frac{t}{2}$ then, $s < t-s$ and $(t-s)^{H-\frac{3}{2}} < s^{H-\frac{3}{2}} < s^{-(H+\frac{1}{2})} (t-s)^{2H-1}$, while for $\frac{t}{2} \leq s < t$, the previous inequalities are reversed accordingly. Hence (3.9) clearly follows from (3.12). \(\square\)

We introduce the notation

\[
L_m = [2^m t] 2^{-m} \quad \text{and} \quad \tilde{t}_m = L_m + 2^{-m},
\]  

(3.13)

for any $m \in \mathbb{N}$. Notice that, $K^H_m$ given in (1.5) satisfies $K^H_m(t, s) = 0$ if $s \geq \tilde{t}_m$.

In the next result, we give a bound for the approximation in quadratic mean of the kernel $K^H$ by its projection $K^H_m$.

**Lemma 3.2.** (1) Let $H \in ]0, \frac{1}{2}[, 1]$. There exists a positive constant $C$ such that for any $0 < s < t \leq 1$,

\[
\sup_{m \geq 1} \int_0^1 \left( |K^H_m(t, u) - K^H_m(s, u)|^2 + |K^H(t, u) - K^H(s, u)|^2 \right) du \leq C|t-s|^{2H}.
\]  

(3.14)

(2) For $H \in ]0, \frac{1}{2}[$,

\[
\int_0^1 |K^H(t, u) - K^H_m(t, u)|^2 du \leq C (t \wedge 2^{-m})^{2H}.
\]  

(3.15)

(3) For $H \in ]\frac{1}{2}, 1[$ and any $\lambda \in ]0, \frac{1}{2H+1}[$,

\[
\int_0^1 |K^H(t, u) - K^H_m(t, u)|^2 du \leq C 2^{-2m\lambda} 2^{(H-\lambda)}.
\]  

(3.16)

**Proof.** The operator $\pi_m$ is a contraction on $L^2[0, 1]$. Thus,

\[
\sup_{m \geq 1} \int_0^1 \left( |K^H_m(t, u) - K^H_m(s, u)|^2 + |K^H(t, u) - K^H(s, u)|^2 \right) du
\]
proving (3.14).

By the same argument,
\[ \int_0^1 |K^H(t, u) - K^H_m(t, u)|^2 \, du \leq 4 \int_0^1 |K^H(t, u)|^2 \, du = 4t^{2H}. \]  
(3.17)

Therefore (3.15) holds for \( t \leq C 2^{-m} \).

Fix \( t \in \Delta^m_I \) with \( i > 7 \). We assume first \( H \in ]0, \frac{1}{2}[ \). Consider the decomposition
\[ \int_0^1 |K^H(t, u) - K^H_m(t, u)|^2 \, du \leq C \sum_{i=1}^5 T_i(t), \]  
(3.18)

with
\[
T_1(t) = \int_0^{t_0} |K^H(t, u) - K^H_m(t, u)|^2 \, du,
\]
\[
T_2(t) = \int_{t_0}^{t_1} |K^H(t, u) - K^H_m(t, u)|^2 \, du,
\]
\[
T_3(t) = \sum_{k=3}^{[2^{m-1}t]} \int_{\Delta^m_k} |K^H(t, u) - K^H_m(t, u)|^2 \, du,
\]
\[
T_4(t) = \sum_{k=[2^{m-1}t]+2}^{I-3} \int_{\Delta^m_k} |K^H(t, u) - K^H_m(t, u)|^2 \, du,
\]
\[
T_5(t) = \int_{\Delta^m_{[2^{m-1}t]+1}} |K^H(t, u) - K^H_m(t, u)|^2 \, du.
\]

Schwarz’s inequality and (3.3) imply
\[ T_1(t) \leq 4 \int_0^{t_0} |K^H(t, u)|^2 \, du \leq C \int_0^{t_0} u^{2H-1} \, du = C 2^{-mH}. \]

Similarly,
\[ T_2(t) \leq 4 \int_{t_0}^{t_1} |K^H(t, u)|^2 \, du \leq C \int_{t_0}^{t_1} |t - u|^{2H-1} \, du = C 2^{-mH}. \]

Let \( \lambda \in ]H, 1[ \) and \( k = 3, \ldots, [2^{m-1}t] \), which implies \( \Delta^m_k \subset ]0, \frac{t}{2}[ \). By Schwarz’s inequality, the mean value theorem and (3.5), (3.6), we obtain
\[
\int_{\Delta^m_k} |K^H(t, u) - K^H_m(t, u)|^2 \, du \leq 2^m \int_{\Delta^m_k} dv \int_{\Delta^m_k} dv |K^H(t, u) - K^H(t, v)|^2 \leq 2^m \int_{\Delta^m_k} dv \int_{\Delta^m_k} dv |K^H(t, u) - K^H(t, v)|^{2\lambda} |K^H(t, u)| + |K^H(t, v)|^{2(1-\lambda)}
\]
\[ \leq C 2^{-m(2\lambda - 1)} \int_{\Delta_k^m} du \int_{\Delta_k^m} dv \left((u \wedge v)^{2H - 1 - 2\lambda}\right). \]

For \(u, v \in \Delta_k^m\), \(u \wedge v \geq u - 2^{-m}\); thus,

\[ T_3(t) \leq C 2^{-2m\lambda} \int_{t - 2^{-m}}^{t} du \left(2^{-m}\right)^{2H - 1 - 2\lambda} \leq C 2^{-2mH}. \]

Fix now \(k = \lceil 2^{-m} \rceil + 2, \ldots, I - 3\), so that \(\Delta_k^m \subset \left[\frac{t}{2}, t\right]\). In this case

\[ \int_{\Delta_k^m} \left|K^H(t, u) - K^H(t, v)\right|^2 du \leq C 2^{-m(2\lambda - 1)} \]

\[ \times \int_{\Delta_k^m} du \int_{\Delta_k^m} dv \left(t - (u \vee v)\right)^{2H - 1 - 2\lambda}. \]

Since for \(u, v \in \Delta_k^m\), \(t - (u \vee v) \geq t - u - 2^{-m} \geq t_{l-2} - u\), the previous estimate implies

\[ T_4(t) \leq C 2^{-2m\lambda} \int_{t - 2^{-m}}^{t_{l-2}} du \left(2^{-m}\right)^{2H - 1 - 2\lambda} \leq C 2^{-2mH}. \]

We study the term \(T_5(t)\) using the same method as for \(T_3(t), T_4(t)\), as follows:

\[ T_5(t) \leq 2^m \int_{\Delta_{[2^{m-1}t] + 1}^m} du \int_{\Delta_{[2^{m-1}t] + 1}^m} dv \left|K^H(t, u) - K^H(t, v)\right|^2 \]

\[ \leq C 2^{-m(2\lambda - 1)} \int_{\Delta_{[2^{m-1}t] + 1}^m} du \int_{\Delta_{[2^{m-1}t] + 1}^m} dv \left((u \wedge v)^{H - \frac{3}{2}} + (t - (u \vee v)^{H - \frac{3}{2}}\right)^{2\lambda} \]

\[ \times \left((u \wedge v)^{H - \frac{3}{2}} + (t - (u \vee v)^{H - \frac{3}{2}}\right)^{2(1 - \lambda)}. \]

For \(u, v \in \Delta_{[2^{m-1}t] + 1}^m\), \(u \wedge v > \frac{t}{2} - 2^{-m}\), \(u \vee v < \frac{t}{2} + 2^{-m}\) and \(t - (u \vee v) > \frac{t}{2} - 2^{-m}\). Thus, the last integral is bounded by

\[ \int_{\Delta_{[2^{m-1}t] + 1}^m} du \int_{\Delta_{[2^{m-1}t] + 1}^m} dv \left(\frac{t}{2} - 2^{-m}\right)^{2H - 1 - 2\lambda}. \]

Moreover, since we are assuming that \(t \in \Delta_I^m\), with \(I > 7\), \(\frac{t}{2} - 2^{-m} \geq 2^{-m+1}\). Thus, we finally obtain for \(\lambda = \frac{1}{2}\),

\[ T_5(t) \leq C 2^{-2mH}. \]

Then (3.13) follows from the upper bounds obtained so far for \(T_i(t), i = 1, \ldots, 5\).

Notice that we have also proved that for \(H \in [0, \frac{1}{2}]\),

\[ \sum_{k=3}^{I-3} 2^m \int_{\Delta_k^m} du \int_{\Delta_k^m} dv |K^H(t, u) - K^H(t, v)|^2 \leq C 2^{-2mH}. \] (3.19)
Assume now $H \in ]\frac{1}{2}, 1]$ and fix $\lambda \in ]0, \frac{1}{2H+1}[$, so that $H - \lambda > 0$. Since the inequality (3.17) holds for any $H \in ]0, \frac{1}{2}[$, (3.18) holds for any $t \leq C 2^{-m}$. Let now $t \in \Delta^m_k$, with $I > 7$. We apply a similar method as we used in the case $H \in ]0, \frac{1}{2}[$, using the decomposition (3.18). In fact, owing to (3.8),

$$T_1(t) \leq C \int_{0}^{t} (t - u)^{2H-1} du \leq C 2^{-m} t^{2H-1},$$

$$T_2(t) \leq C \int_{t_{I-3}^m}^{t} u^{2H-1} du \leq C 2^{-m} t^{2H-1}.$$ Fix $k = 3, \ldots, [2^{m-1}t]$. Schwarz’s inequality, along with the mean value theorem and (3.8), (3.9), imply

$$\int_{\Delta_k^m} \left| K^H(t, u) - K^m(t, u) \right|^2 du \leq 2^m \int_{\Delta_k^m} du \int_{\Delta_k^m} dv \left| K^H(t, u) - K^H(t, v) \right|^{2\lambda} \times \left| \left| K^H(t, u) \right| + \left| K^H(t, v) \right| \right|^{2(1-\lambda)} \leq C 2^{-m(2\lambda-1)} \int_{\Delta_k^m} du \int_{\Delta_k^m} dv \left( (t - (u \wedge v))^{(\lambda+1)(2H-1)} (u \wedge v)^{-\lambda(2H+1)} \right) \leq C 2^{-2m\lambda} t^{(\lambda+1)(2H-1)} \int_{\Delta_k^m} du (u - 2^{-m})^{-\lambda(2H+1)}.$$ Since $\lambda < \frac{1}{2H+1}$, we have

$$T_3(t) \leq C 2^{-2m\lambda} t^{2(H-\lambda)}.$$ Let now $k = [2^{m-1}t] + 2, \ldots, I - 3$. With similar arguments as before, we deduce

$$\int_{\Delta_k^m} \left| K^H(t, u) - K^m(t, u) \right|^2 du \leq 2^m \int_{\Delta_k^m} du \int_{\Delta_k^m} dv \left| K^H(t, u) - K^H(t, v) \right|^{2\lambda} \times \left| \left| K^H(t, u) \right| + \left| K^H(t, v) \right| \right|^{2(1-\lambda)} \leq C 2^{-m(2\lambda-1)} \int_{\Delta_k^m} du \int_{\Delta_k^m} dv (t - (u \vee v))^{(\lambda+3)(2H-3)} (u \vee v)^{(1-\lambda)(2H-1)} \leq C 2^{-2m\lambda} t^{(1-\lambda)(2H-1)} \int_{\Delta_k^m} du (t - u - 2^{-m})^{\lambda(2H-3)}.$$ For $\lambda < \frac{1}{2H+1}$, $\lambda(2H - 3) + 1 > 0$. Hence,

$$T_4(t) \leq C 2^{-2m\lambda} t^{(1-\lambda)(2H-1)} \int_{\frac{t}{2}}^{t_{I-3}^m} (t - u - 2^{-m})^{\lambda(2H-3)} \leq C 2^{-2m\lambda} t^{2(H-\lambda)}.$$ Finally, we study the contribution of $T_5(t)$ as follows.

$$T_5(t) \leq 2^m \int_{\Delta_{[2^{m-1}t]+1}^m} du \int_{\Delta_{[2^{m-1}t]+1}^m} dv \left| K^H(t, u) - K^H(t, v) \right|^2$$
\[
\int_{m^{-1}t}^{m^{-1}t+1} du \int_{m^{-1}t}^{m^{-1}t+1} dv \left( (t - (u \land v))^{2H-1} \times \left( (u \land v)^{-\left(H + \frac{1}{2}\right)} + (t - (u \lor v))^{\left(H + \frac{1}{2}\right)} \right)^{2\lambda} \times \left( (t - (u \land v))^{H-\frac{1}{2}} + (u \lor v)^{H-\frac{1}{2}} \right)^{2(1-\lambda)} \right)
\]

For \( u, v \in \Delta_{m^{-1}t+1}^m \), \( u \land v > C_1 t \), \( u \lor v < C_2 t \), \( t - (u \land v) < C_3 t \) and \( t - (u \lor v) > C_4 t \). Thus,

\[ T_5(t) \leq C 2^{-m(2\lambda-1)2-2m \lambda^2(H-\lambda)-1} \leq C 2^{-2m \lambda^2(H-\lambda)} \]

The estimates obtained so far imply (3.16). □

In the next Lemma we prove a simple extension of a well-known integration formula for bounded variation functions.

**Lemma 3.3.** For any \( h \in \mathcal{H}, t \geq 0 \),

\[
\int_0^t h(u)h(du) = \frac{h^2(t)}{2},
\]

where the integral is understood in the sense of Proposition 5 in [7].

**Proof.** Let \( n \geq 1 \) and let \( h(n) \) be the function obtained by linear interpolation on the \( n \)-th dyadic grid of \( h \). We have proved in [7], Theorem 9 that

\[
\lim_{n \to \infty} \int_0^t h(n)(u)h(n)(du) = \int_0^t h(u)h(du),
\]

for any \( t \geq 0 \). Since (3.20) is true with \( h \) replaced by \( h(n) \), the result follows. □

The following result gives an upper bound for the \( L^2 \) norm of a Skorohod integral of a Gaussian process.

**Lemma 3.4.** Let \( X_t = \int_0^1 g(t,s)dB_s, t \in [0,1] \), with \( g \) a deterministic function belonging to \( L^2([0,1]^2) \). Then, the Skorohod integral \( \int_0^1 X_sdB_s \) satisfies

\[
E\left(\int_0^1 X_sdB_s\right)^2 \leq C \int_0^1 ds \int_0^1 dr |g(s,r)|^2.
\]

**Proof.** The isometry property of the Skorohod integral ([8], Equation (1.48)) yields

\[
E\left(\int_0^1 X_sdB_s\right)^2 \leq C \int_0^1 E(X_s)^2 ds + \int_0^1 ds \int_0^1 dr E(|D_r X_s|^2).
\]

Since \( E(X_s)^2 = \int_0^1 |g(s,r)|^2 dr \) and the Malliavin derivative \( D_r X_s \) is equal to \( g(s,r) \), (3.21) follows. □
We conclude this section by proving an extension of the Garsia-Rademich-Rumsey lemma used to estimate \( d_p(X, Y) \) when \( X \) and \( Y \) are geometric rough paths with roughness \( p \in [2, \infty[ \) (see [3], Definition 3.3.3).

**Lemma 3.5.** Let \( X \) and \( Y \) be geometric rough paths with the same roughness \( p \in [2, +\infty[. \) Set \( k = [p] \). For \( i = 1, \ldots, k \), let \( M_i \geq 1 \), \( \alpha_i = \frac{1}{p} + \frac{1}{M_i} \). Suppose that

\[
\int_0^1 \int_0^1 dsdt1_{\{s \leq t\}} \frac{|X_{s,t}^{(i)}|^{2M_i} + |Y_{s,t}^{(i)}|^{2M_i}}{|t - s|^{2M_i\alpha_i}} \leq A_i, \quad 1 \leq i \leq k - 1, \quad (3.22)
\]

\[
\int_0^1 \int_0^1 dsdt1_{\{s \leq t\}} \frac{|X_{s,t}^{(i)} - Y_{s,t}^{(i)}|^{2M_i}}{|t - s|^{2M_i\alpha_i}} \leq B_i, \quad 1 \leq i \leq k. \quad (3.23)
\]

Then, there exists a constant \( C > 0 \) such that for any \( 0 \leq s < t \leq 1 \),

\[
|X_{s,t}^{(i)}| + |Y_{s,t}^{(i)}| \leq CF_i |t - s|^{\frac{1}{p}}, \quad 1 \leq i \leq k - 1, \quad (3.24)
\]

\[
|X_{s,t}^{(i)} - Y_{s,t}^{(i)}| \leq CG_i |t - s|^{\frac{1}{p}}, \quad 1 \leq i \leq k. \quad (3.25)
\]

where \( F_i \) and \( G_i \) are defined recursively by

\[
F_i = A_i^{\frac{1}{2M_i}} + \sum_{j=1}^{i-1} F_j F_{i-j}, \quad 1 \leq i \leq k - 1, \quad (3.26)
\]

\[
G_i = B_i^{\frac{1}{2M_i}} + \sum_{j=1}^{i-1} G_j G_{i-j}, \quad 1 \leq i \leq k. \quad (3.27)
\]

**Remark:** For rough paths \( X, Y \) of roughness \( p \in [1, \infty[ \), \( X_{s,t} = X_{s,t}^{(1)} - X_{s,t}^{(1)} = (X - Y)_{s,t}^{(1)} \). The usual version of the Garsia-Rademich-Rumsey lemma yields the following. If

\[
\int_0^1 \int_0^1 dsdt1_{\{s \leq t\}} \frac{|X_{s,t}^{(1)} - Y_{s,t}^{(1)}|^{2M_1}}{|t - s|^{2M_1\alpha_1}} \leq B_1,
\]

then \( |X_{s,t}^{(1)} - Y_{s,t}^{(1)}| \leq C B_1^{\frac{1}{2M_1}} |t - s|^{\frac{1}{p}} \). Similarly, if

\[
\int_0^1 \int_0^1 dsdt1_{\{s \leq t\}} \frac{|X_{s,t}^{(1)}|^{2M_1} + |Y_{s,t}^{(1)}|^{2M_1}}{|t - s|^{2M_1\alpha_1}} \leq A_1,
\]

then \( |X_{s,t}^{(1)}| + |Y_{s,t}^{(1)}| \leq C A_1^{\frac{1}{2M_1}} |t - s|^{\frac{1}{p}} \).

**Proof of Lemma 3.3:** Throughout the proof, the constants \( F_i, 1 \leq i \leq k - 1 \) and \( G_i, 1 \leq i \leq k \) are defined by (3.26), (3.27), respectively. We introduce the following assumption:

(H_i)

\[
\int_0^1 \int_0^1 dsdt1_{\{s \leq t\}} \frac{|X_{s,t}^{(i)} - Y_{s,t}^{(i)}|^{2M_i}}{|t - s|^{2M_i\alpha_i}} \leq B_i,
\]
\[ |X_{s,t}^{(j)}| + |Y_{s,t}^{(j)}| \leq C F_j |t-s|^{\frac{j}{p}}, \quad 1 \leq j \leq i-1, \]
\[ |X_{s,t}^{(j)} - Y_{s,t}^{(j)}| \leq C G_j |t-s|^{\frac{j}{p}}, \quad 1 \leq j \leq i-1, \]
\( i \in \{2, \ldots, k\} \), and we prove that (H_i) implies
\[ |X_{s,t}^{(i)} - Y_{s,t}^{(i)}| \leq C G_i |t-s|^{\frac{i}{p}}. \tag{3.28} \]

For this, we use an argument similar to the proof of Theorem 2.1.3 in [3].

Indeed, for every \( t \in [0,1] \), set
\[ I(t) = \int_0^t \frac{|X_{s,t}^{(i)} - Y_{s,t}^{(i)}|^{2M_i}}{|t-s|^{2M_i \alpha_i}} \, ds, \quad J(t) = \int_0^1 \frac{|X_{s,t}^{(i)} - Y_{s,t}^{(i)}|^{2M_i}}{|u-t|^{2M_i \alpha_i}} \, du. \]

Then \( \int_0^1 I(t) \, dt = \int_0^1 J(t) \, dt \leq B_i \) and there exists \( t_0 > 0 \) such that \( I(t_0) + J(t_0) \leq 2 A_i \). We construct by induction a decreasing sequence \((t_n, n \geq 0)\) such that \( \lim_{n \to \infty} t_n = 0 \) and an increasing sequence \((s_n, n \geq 0)\) such that \( s_0 = t_0 \), \( \lim_{n \to \infty} s_n = 1 \) and such that there exists \( C > 0 \) such that for every \( n \geq 1 \),
\[ |X_{s_0,t_0}^{(i)} - Y_{s_0,t_0}^{(i)}| \leq C \int_0^1 |B_i|^{\frac{1}{2M_i}} u^{\frac{i}{p}-1} \, du + C \sum_{j=1}^{i-1} F_j G_{i-j}, \tag{3.29} \]
\[ |X_{s_t,t_0}^{(i)} - Y_{s_t,t_0}^{(i)}| \leq C \int_0^1 |B_i|^{\frac{1}{2M_i}} u^{\frac{i}{p}-1} \, du + C \sum_{j=1}^{i-1} F_j G_{i-j}. \tag{3.30} \]

Then Chen’s identity implies as \( n \to +\infty \),
\[ |X_{0,1}^{(i)} - Y_{0,1}^{(i)}| \leq |X_{0,t_0}^{(i)} - Y_{0,t_0}^{(i)}| + |X_{t_0,0}^{(i)} - Y_{t_0,0}^{(i)}| \]
\[ + \sum_{j=1}^{i-1} \left( |X_{0,t_0}^{(j)} - Y_{0,t_0}^{(j)}||X_{t_0,t_0}^{(i-j)}| + |Y_{0,t_0}^{(j)}||X_{0,t_0}^{(i-j)} - Y_{t_0,t_0}^{(i-j)}| \right). \tag{3.31} \]

With the hypothesis (H_i), we obtain \([3,28]\) with \( s = 0 \) and \( t = 1 \).

To construct \((t_n)\), we suppose that \( t_{n-1} \) has been chosen. Let \( d_{n-1} \) be defined by \( d_{n-1}^\alpha = \frac{1}{2} d_{n-1}^\alpha \). Then there exists \( t_n \leq 0, d_{n-1} \) such that
\[ I(t_n) \leq \frac{4 B_i}{d_{n-1}} \quad \text{and} \quad \frac{|X_{t_n,t_{n-1}}^{(i)} - Y_{t_n,t_{n-1}}^{(i)}|^{2M_i}}{|t_n-t_{n-1}|^{2M_i \alpha_i}} \leq \frac{2 I(t_{n-1})}{d_{n-1}}. \]

Indeed, the sets where each one of these inequalities may fail has Lebesgue measure less than \( \frac{d_{n-1}}{2} \). Furthermore, for every \( n \geq 0 \), \( 2 d_{n+1}^\alpha = t_{n+1}^\alpha \leq d_{n}^\alpha = \frac{1}{2} d_{n}^\alpha \) and \( |t_n-t_{n+1}|^\alpha \leq t_{n+1}^\alpha = 2 d_{n}^\alpha \leq 4 (d_{n}^\alpha - d_{n+1}^\alpha) \). Hence there exists \( a \in [0,1] \) such that \( t_{n+1} \leq a t_n \), so that \( \lim_{n} t_n = 0 \) and more precisely,
\[ t_n \leq a^n t_0, \tag{3.32} \]
while for any \( n \geq 1 \),
\[ |X_{t_{n+1},t_n}^{(i)} - Y_{t_{n+1},t_n}^{(i)}| \leq |2 I(t_n)|^{\frac{1}{2M_i}} d_{n}^{-\frac{1}{2M_i}} |t_n-t_{n+1}|^{\alpha_i}. \]
implies that for any $n \geq 1,$
\[
\left| X_{t_{n+1}, t_0}^{(i)} - Y_{t_{n+1}, t_0}^{(i)} \right| \leq \left| X_{t_{n+1}, t_0}^{(i)} - Y_{t_{n+1}, t_0}^{(i)} \right| + \left| X_{t_{n+1}, t_0}^{(i)} - Y_{t_{n+1}, t_0}^{(i)} \right|
\]
\[
\leq \left| X_{t_{n+1}, t_0}^{(i)} - Y_{t_{n+1}, t_0}^{(i)} \right| + C \int_{d_{n+1}}^{d_n} |8B_i| \frac{1}{2M_i} u^{-\frac{1}{p}+\alpha_i-1} \, du
\]

Thus if $b = a^{\frac{1}{2}} < 1,$ Chen’s identity and (3.33) imply that for any $n \geq 1,$
\[
\left| X_{t_{n+1}, t_0}^{(i)} - Y_{t_{n+1}, t_0}^{(i)} \right| \leq 4 \alpha_i \int_{d_{n+1}}^{d_n} |8B_i| \frac{1}{2M_i} u^{-\frac{1}{p}+\alpha_i-1} \, du.
\]
To deduce (3.28), for any $s, t \in [0, 1]$ with $s < t$, define $\bar{X}_u = X_{s + (t-s)u}$, $\bar{Y}_u = Y_{s + (t-s)u}$ for $u \in [0, 1]$. Then $\bar{X}$ and $\bar{Y}$ are geometric rough paths with the same roughness $p$. Moreover, for $0 \leq u < v \leq 1$, $j = 1, \cdots, k$, $\bar{X}_u^{(j)} = X_{s + (t-s)u, s + (t-s)v}^{(j)}$. In fact, by a change of variables, we see that this identity is obvious for smooth rough paths and therefore it is trivially extended to geometric rough paths.

Furthermore,

$$\int_0^1 \int_0^1 dudv 1_{\{u < v\}} \frac{|\bar{X}_{u,v}^{(i)} - \bar{Y}_{u,v}^{(i)}|^{2M_i}}{|v - u|^{2M_i \alpha_i}}$$

$$= (t - s)^{-2 + 2\alpha_i M_i} \int_s^t \int_s^t dudv 1_{\{u < v\}} \frac{|X_{u,v}^{(i)} - Y_{u,v}^{(i)}|^{2M_i}}{|v - u|^{2M_i \alpha_i}}$$

$$\leq (t - s)^{-2 + 2\alpha_i M_i} B_i = (t - s)^{2M_i} B_i.$$

Hence, if the pair $(X, Y)$ satisfies $(H_i)$ then $(\bar{X}, \bar{Y})$ satisfies a similar property with constants $\tilde{A}_j = (t - s)^{2M_j} A_j$, $\tilde{F}_j = |t - s|^{\frac{1}{p}} F_j, 1 \leq j \leq i - 1$, $\tilde{B}_j = (t - s)^{2M_j} B_j$, $\tilde{G}_j = |t - s|^{\frac{1}{p}}, 1 \leq j \leq i$. This finishes the proof of (3.28).

Taking in the preceding arguments first $X \equiv 0$ and then $Y \equiv 0$, we see recursively that (3.22) implies $(H_i)$ for any $i = 1, \ldots, k - 1$, with $B_i = A_i$. Hence we obtain (3.24). Moreover, we also see that $(H_i)$ holds true for any $i = 1, \ldots, k$, whenever (3.22), (3.23) are satisfied. This concludes the proof.

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