Anisotropic Raviart–Thomas interpolation error estimates using a new geometric parameter

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Abstract
We present precise Raviart–Thomas interpolation error estimates on anisotropic meshes. The novel aspect of our theory is the introduction of a new geometric parameter of simplices. It is possible to obtain new anisotropic Raviart–Thomas error estimates using the parameter. We also include corrections to an error in “General theory of interpolation error estimates on anisotropic meshes” (Japan Journal of Industrial and Applied Mathematics, 38 (2021) 163-191), in which Theorem 3 was incorrect.

Keywords Raviart–Thomas finite element · Interpolation error estimates · Anisotropic meshes

Mathematics Subject Classification 65D05 · 65N30

1 Introduction

Interpolation error analysis is an essential theme in finite element analysis. It is well known that the shape-regular condition for mesh partitions leads to optimal interpolation error estimates, which results in the convergence of finite element approximation for partial differential equations. However, the geometric condition can be relaxed to the maximum-angle condition, which makes it possible to use anisotropic meshes; see [5] for two-dimensional cases and [19] for three-dimensional cases. Anisotropic meshes are effective for, for example, problems in which the solution has anisotropic behaviour in some direction of the domain.

This paper considers Raviart–Thomas interpolation error estimates on anisotropic meshes. The Raviart–Thomas finite element space was proposed in [20]. The space is used for mixed finite element methods for second-order elliptic and incompressible
flow problems. Anisotropic Raviart–Thomas interpolation theory was developed [1, 2]. The key idea is to derive the component-wise stabilities of the Raviart–Thomas interpolation on reference elements (Lemmata 6 and 7).

In contrast, this paper proposes anisotropic Raviart–Thomas interpolation error estimates using a new parameter \( H_{T_0} \) of \( d \)-simplices, \( d \in \{2, 3\} \), (see Definition 3) proposed in a recent paper [14] under an assumption and using the component-wise stabilities, for example, we derive the following anisotropic error estimate (Theorem 2):

\[
\| I_{T_0}^{RT_k} v_0 - v_0 \|_{L^p(T_0)^d} \leq c \left( \frac{H_{T_0}}{h_{T_0}} \sum_{|\varepsilon| = \ell + 1} \mathcal{H}^\varepsilon \| \partial_x \psi_{T_0}^{-1}(v_0) \|_{L^p(\Phi_{T_0}^{-1}(T_0)^d)} \right. \\
+ h_{T_0} \sum_{|\beta| = \ell} \mathcal{H}_\beta \| \partial_x \psi_{T_0}^{-1}(v_0) \|_{L^p(\Phi_{T_0}^{-1}(T_0)^d)}),
\]

also see Theorem 3, where \( I_{T_0}^{RT_k} : W^{1,1}(T_0)^d \to RT^k(T_0) \) is the Raviart–Thomas interpolation on \( T_0 \) defined by (2.15) and (2.16), \( \Phi_{T_0} \) and \( \psi_{T_0} \) are respectively the Affine mapping and Piola transformation defined in Sects. 2.3 and 2.4, \( \varepsilon := (\varepsilon_1, \ldots, \varepsilon_d) \in \mathbb{N}_0^d \) (\( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \)) is a multi-index, and \( \mathcal{H}_\varepsilon \) is defined in Sect. 2.5. We remark that the heart of the proof of Theorems 2 and 3 is the scaling argument described in Sect. 3. We are naturally able to consider the following geometric condition, which is equivalent to the maximum-angle condition, as being sufficient to obtain optimal order estimates: there exists \( \gamma_0 > 0 \) such that

\[
\frac{H_{T_0}}{h_{T_0}} \leq \gamma_0 \quad \forall T_h \in \{ T_h \}, \quad \forall T_0 \in T_h,
\]

where \( T_h = \{ T_0 \} \) is a simplicial mesh of a domain in \( \mathbb{R}^d \). The new geometric condition appears to be simpler than the maximum-angle condition. Furthermore, the quantity \( \frac{H_{T_0}}{h_{T_0}} \) can be easily calculated in the numerical process of finite element methods. Therefore, the new condition may be useful. We expect that the Raviart–Thomas interpolation error estimates that include the new mesh condition will be effective for a posteriori error estimates. We call the element \( T_0 \) “good” when there exists a constant \( \gamma_0 > 0 \) that satisfies the new geometric condition (see Remark 1). Arguments about “good elements” can be found in [13, 17].

We describe the previous paper [14, 15]. The authors derived Raviart–Thomas interpolations on \( d \)-simplices. However, the proof of Theorem 3 in [14] included a mistake, and we need to modify its statement to correct this error (Sect. 4.2). The Babuška and Aziz technique is generally not applicable on anisotropic meshes in the proof of Theorem 3 in [14]. Although we argued in [14, 15] that we do not impose either the shape-regular or maximum-angle condition during mesh partitioning, we realised that the new geometric condition (1.1) is necessary for the stability results of the global Raviart–Thomas interpolation.
When there is no ambiguity, we use the notation and definitions given in [14]. Throughout this paper, $c$ denotes a constant independent of $h$ (defined later) unless specified otherwise. These values may change in each context. Let $\mathbb{R}_+$ be the set of positive real numbers. For $k \in \mathbb{N}_0$, $P^k(T)$ is spanned by the restriction to $T$ of polynomials in $P^k$, where $P^k$ denotes the space of polynomials with the degree at most $k$. Furthermore, we often use the following inequality (see [11, Exercise 1.20]). Let $0 < r \leq s$ and $a_i \geq 0$, $i = 1, 2, \ldots, n$ ($n \in \mathbb{N}$), be real numbers. We then have

$$
\left( \sum_{i=1}^{n} a_i^s \right)^{1/s} \leq \left( \sum_{i=1}^{n} a_i^r \right)^{1/r}.
$$

(1.2)

For matrix $A \in \mathbb{R}^{d \times d}$, we denote by $[A]_{ij}$ the $(i, j)$-component of $A$. We set $\|A\|_{\text{max}} := \max_{1 \leq i, j \leq d} |[A]_{ij}|$. We also use the inequality

$$
\| A \|_{\text{max}} \leq \| A \|_2.
$$

(1.3)

2 Settings for the analysis of anisotropic interpolation theory

Throughout this paper, let $d \in \{2, 3\}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain. Let $\mathbb{T}_h = \{T_0\}$ be a simplicial mesh of $\overline{\Omega}$ made up of closed $d$-simplices, such as

$$
\overline{\Omega} = \bigcup_{T_0 \in \mathbb{T}_h} T_0,
$$

with $h := \max_{T_0 \in \mathbb{T}_h} h_{T_0}$, where $h_{T_0} := \text{diam}(T_0)$. For simplicity, we assume that $\mathbb{T}_h$ is conformal: that is, $\mathbb{T}_h$ is a simplicial mesh of $\overline{\Omega}$ without hanging nodes.

2.1 Motivation

Let $T_0 \subset \mathbb{R}^d$ and $\hat{T} \subset \mathbb{R}^d$ be an element on a mesh $\mathbb{T}_h$ and a reference element defined in Sect. 2.2. Let these two elements be affine equivalent. As the usual manner, the transformation $\Phi_0$ takes the form

$$
\Phi_0 : \hat{T} \ni \hat{x} \mapsto \Phi_0(\hat{x}) := B_0 \hat{x} + b_0 \in T_0,
$$

where $B_0 \in \mathbb{R}^{d \times d}$ is an invertible matrix and $b_0 \in \mathbb{R}^d$. According to the classical theory (e.g., see [11, Theorem 1.114]), it holds that

$$
\| v - I^{RT}_{T_0} v \|_{L^p(T_0)^d} \leq c \left( \| B_0 \|_2 \| B_0^{-1} \|_2 \right) \| B_0 \|_2 |v|_{W^{1,p}(T_0)^d} \quad \forall \ v \in W^{1,p}(T_0)^d.
$$
Here, the quantity $\|B_0\|_2\|B_0^{-1}\|_2$ is called the Euclidean condition number of $B_0$. By standard estimates (e.g., see [11, Lemma 1.100]), we have
\[
\|B_0\|_2\|B_0^{-1}\|_2 \leq c \frac{h_{T_0}}{\rho_{T_0}}, \quad \|B_0\|_2 \leq c h_{T_0},
\]
where $\rho_{T_0}$ is the diameter of the largest ball that can be inscribed in $T_0$. It thus holds that
\[
\|v - I_{T_0}^R T_0^0 v\|_{L^p(T_0)^d} \leq c \frac{h_{T_0}}{\rho_{T_0}} h_{T_0}|v|_{W^{1,p}(T_0)^d}.
\]

As geometric conditions to obtain global interpolation error estimates and to prove that this estimate converges to zero as $h \to 0$, the shape-regularity condition is widely used and well known. If the shape-regularity condition is violated, that is, the simplex becomes too flat as $h_{T_0} \to 0$, the quantity $\frac{h_{T_0}}{\rho_{T_0}} h_{T_0}$ may diverge.

To overcome the difficulty, we considered a new strategy ([14, Sect. 3] and [17, Sect. 2]) to use anisotropic mesh partitions. Using an intermediate simplex $T \subset \mathbb{R}^d$ imposing Condition 1 or 2 described in Sect. 2.3, we construct two affine mappings $\Phi_T : \hat{T} \to T$ and $\Phi_{T_0} : T \to T_0$. We first define the affine mapping $\Phi_T : \hat{T} \ni \hat{x} \mapsto x := A_T \hat{x} \in T$,
\[
\Phi_T : \hat{T} \ni \hat{x} \mapsto x := A_T \hat{x} \in T,
\]
where $A_T \in \mathbb{R}^{d \times d}$ is an invertible matrix. Details are given in Sect. 2.3.1. We next define an affine mapping $\Phi_{T_0} : T \to T_0$ as
\[
\Phi_{T_0} : T \ni x \mapsto x^{(0)} := \Phi_{T_0}(x) := A_{T_0} x + b_{T_0} \in T_0,
\]
where $b_{T_0} \in \mathbb{R}^d$ and $A_{T_0} \in \mathbb{R}^{d \times d}$ is an invertible matrix. Details are given in Sect. 2.3.2. We define an affine mapping $\Phi : \hat{T} \to T_0$ as
\[
\Phi := \Phi_{T_0} \circ \Phi_T : \hat{T} \ni \hat{x} \mapsto x^{(0)} := \Phi(x) = (\Phi_{T_0} \circ \Phi_T)(\hat{x}) = A \hat{x} + b_{T_0} \in T_0,
\]
where $A := A_{T_0} A_T \in \mathbb{R}^{d \times d}$. Our strategy uses the affine mapping $\Phi$ instead of the mapping $\Phi_0$.

2.2 Reference elements

We define reference elements $\hat{T} \subset \mathbb{R}^d$.

Two-dimensional case

Let $\hat{T} \subset \mathbb{R}^2$ be a reference triangle with vertices $\hat{x}_1 := (0, 0)^T$, $\hat{x}_2 := (1, 0)^T$, and $\hat{x}_3 := (0, 1)^T$. 

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Three-dimensional case

In the three-dimensional case, we need to consider the following two cases (i) and (ii); also see Condition 2.

Let $\hat{T}_1$ and $\hat{T}_2$ be reference tetrahedra with the following vertices:

(i) $\hat{T}_1$ has the vertices $\hat{x}_1 := (0, 0, 0)^T$, $\hat{x}_2 := (1, 0, 0)^T$, $\hat{x}_3 := (0, 1, 0)^T$, and $\hat{x}_4 := (0, 0, 1)^T$;
(ii) $\hat{T}_2$ has the vertices $\hat{x}_1 := (0, 0, 0)^T$, $\hat{x}_2 := (1, 0, 0)^T$, $\hat{x}_3 := (0, 1, 0)^T$, and $\hat{x}_4 := (0, 0, 1)^T$.

We thus set $\hat{T} \in \{\hat{T}_1, \hat{T}_2\}$.

2.3 Affine mappings

2.3.1 Construct of the mapping $\Phi_T : \hat{T} \rightarrow T$

We define the affine mapping $\Phi_T : \hat{T} \rightarrow T$ as

$$\Phi_T : \hat{T} \ni \hat{x} \mapsto x := \Phi_T(\hat{x}) := A_T \hat{x} \in T$$

with the element $T$ satisfying Condition 1 when $d = 2$ or Condition 2 when $d = 3$, which are described later. We define the matrix $A_T \in \mathbb{R}^{d \times d}$ as follows. We first define the diagonal matrix;

$$\hat{A} := \text{diag}(\alpha_1, \ldots, \alpha_d), \quad \alpha_i \in \mathbb{R}_+ \quad \forall i.$$  \hspace{1cm} (2.2)

When $d = 2$, we define the regular matrix $\tilde{A} \in \mathbb{R}^{2 \times 2}$ as

$$\tilde{A} := \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix},$$

with parameters

$$s^2 + t^2 = 1, \quad t > 0.$$  \hspace{1cm} (2.3)

For the reference element $\hat{T}$, let $\hat{T}^{(2)}$ be the family of triangles

$$T = \Phi_T(\hat{T}) = A_T(\hat{T}), \quad A_T := \hat{A} \hat{A}$$

with vertices $x_1 := (0, 0)^T$, $x_2 := (\alpha_1, 0)^T$, and $x_3 := (\alpha_2 s, \alpha_2 t)^T$. Then, $\alpha_1 = |x_1 - x_2| > 0$ and $\alpha_2 = |x_1 - x_3| > 0$.

When $d = 3$, we define the regular matrices $\tilde{A}_1, \tilde{A}_2 \in \mathbb{R}^{3 \times 3}$ as

$$\tilde{A}_1 := \begin{pmatrix} 1 & s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad \tilde{A}_2 := \begin{pmatrix} 1 - s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}$$

\hspace{1cm} (2.4)
with parameters

\[
\begin{cases}
  s_1^2 + t_1^2 = 1, \ s_1 > 0, \ t_1 > 0, \ \alpha_2 s_1 \leq \alpha_1/2, \\
  s_2^2 + t_2^2 = 1, \ t_2 > 0, \ \alpha_3 s_2 \leq \alpha_1/2.
\end{cases}
\]

We thus set \( \tilde{A} \in [\tilde{A}_1, \tilde{A}_2] \). For the reference elements \( \hat{T}_i, i = 1, 2 \), let \( \mathcal{F}_i^{(3)}, i = 1, 2 \), be the family of tetrahedra

\[
T_i = \Phi_{T_i}(\hat{T}_i) = A_T(\hat{T}_i), \quad A_T := \tilde{A}_i \tilde{A}, \quad i = 1, 2,
\]

with vertices

\[
\begin{align*}
x_1 &:= (0, 0, 0)^T, \quad x_2 := (\alpha_1, 0, 0)^T, \quad x_4 := (\alpha_3 s_2, \alpha_3 s_2, \alpha_3 t_2)^T, \\
x_3 &:= (\alpha_2 s_1, \alpha_2 t_1, 0)^T \quad \text{for case (i)}, \\
x_3 &:= (\alpha_1 - \alpha_2 s_1, \alpha_2 t_1, 0)^T \quad \text{for case (ii)}.
\end{align*}
\]

Then, \( \alpha_1 = |x_1 - x_2| > 0, \alpha_3 = |x_1 - x_4| > 0, \) and

\[
\alpha_2 = \begin{cases}
  |x_1 - x_3| > 0 & \text{for case (i)}, \\
  |x_2 - x_3| > 0 & \text{for case (ii)}.
\end{cases}
\]

In the following, we impose conditions on \( T \in \mathcal{F}^{(2)} \) in the two-dimensional case and \( T \in \mathcal{F}_1^{(3)} \cup \mathcal{F}_2^{(3)} := \mathcal{F}^{(3)} \) in the three-dimensional case.

**Condition 1** (Case in which \( d = 2 \)) Let \( T \in \mathcal{F}^{(2)} \) with the vertices \( x_i \ (i = 1, \ldots, 3) \) introduced in this section. We assume that \( x_3 x_5 \) is the longest edge of \( T \); i.e., \( h_T := |x_3 - x_5| \). Recall that \( \alpha_1 = |x_1 - x_2| \) and \( \alpha_2 = |x_1 - x_3| \). We then assume that \( \alpha_2 \leq \alpha_1 \). Note that \( \alpha_1 = O(h_T) \).

**Condition 2** (Case in which \( d = 3 \)) Let \( T \in \mathcal{F}^{(3)} \) with the vertices \( x_i \ (i = 1, \ldots, 4) \) introduced in this section. Let \( L_i \ (1 \leq i \leq 6) \) be the edges of \( T \). We denote by \( L_{\text{min}} \) the edge of \( T \) that has the minimum length; i.e., \( |L_{\text{min}}| = \min_{1 \leq i \leq 6} |L_i| \). We set \( \alpha_2 := |L_{\text{min}}| \) and assume that

the endpoints of \( L_{\text{min}} \) are either \( \{x_1, x_3\} \) or \( \{x_2, x_3\} \).

Among the four edges that share an endpoint with \( L_{\text{min}} \), we take the longest edge \( L_{\text{max}}^{(\text{min})} \). Let \( x_1 \) and \( x_2 \) be the endpoints of edge \( L_{\text{max}}^{(\text{min})} \). We thus have that

\[
\alpha_1 = |L_{\text{max}}^{(\text{min})}| = |x_1 - x_2|.
\]

Consider cutting \( \mathbb{R}^3 \) with the plane that contains the midpoint of edge \( L_{\text{max}}^{(\text{min})} \) and is perpendicular to the vector \( x_1 - x_2 \). We then have two cases:

(Type i) \( x_3 \) and \( x_4 \) belong to the same half-space;

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(Type ii) $x_3$ and $x_4$ belong to different half-spaces.

In each case, we set

(Type i) $x_1$ and $x_3$ as the endpoints of $L_{\min}$, that is, $\alpha_2 = |x_1 - x_3|$;

(Type ii) $x_2$ and $x_3$ as the endpoints of $L_{\min}$, that is, $\alpha_2 = |x_2 - x_3|$.

Finally, we set $\alpha_3 = |x_1 - x_4|$. Note that we implicitly assume that $x_1$ and $x_4$ belong to the same half-space. In addition, note that $\alpha_1 = O(h_T)$.

### 2.3.2 Construct of the mapping $\Phi_{T_0} : T \to T_0$

Let $\Phi_{T_0}$ be the affine mapping defined as

$$
\Phi_{T_0} : T \ni x \mapsto x^{(0)} := \Phi_{T_0}(x) := A_T x + b_{T_0} \in T_0,
$$

where $b_{T_0} \in \mathbb{R}^d$ and $A_{T_0} \in O(d)$ is a rotation and mirror imaging matrix. Note that none of the lengths of a simplex’s edges or the simplex’s measure is changed by the transformation.

**Example 1** As examples, we define the matrices $A_{T_0}$ as

$$
A_{T_0} := 
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix},
A_{T_0} := 
\begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

where $\theta$ is an angle.

### 2.4 Piola transformations

The Piola transformation $\Psi := \Psi_{T_0} \circ \Psi_T : C(\hat{T})^d \to C(T_0)^d$ is defined as

$$
\Psi : C(\hat{T})^d \to C(T_0)^d
$$

$$
\hat{v} \mapsto v_0(x) := \Psi_{T_0} \circ \Psi_T(\hat{v})(x) = \frac{1}{\det(A)} A_T \hat{v}(\hat{x}),
A = A_{T_0} A_T
$$

with two Piola transformations:

$$
\Psi_T : C(\hat{T})^d \to C(T)^d
$$

$$
\hat{v} \mapsto v(x) := \Psi_T(\hat{v})(x) := \frac{1}{\det(A_T)} A_T \hat{v}(\hat{x}),
$$

$$
\Psi_{T_0} : C(T)^d \to C(T_0)^d
$$

$$
v \mapsto v_0(x^{(0)}) := \Psi_{T_0}(v)(x^{(0)}) := \frac{1}{\det(A_{T_0})} A_{T_0} v(x).
$$
2.5 Additional notation and assumption

For convenience, we introduce two definitions (Fig. 1, 2).

**Definition 1** We define a parameter \( \mathcal{H}_i, \ i = 1, \ldots, d, \) as

\[
\begin{cases}
\mathcal{H}_1 := \alpha_1, & \mathcal{H}_2 := \alpha_2 t & \text{if } d = 2, \\
\mathcal{H}_1 := \alpha_1, & \mathcal{H}_2 := \alpha_2 t_1, & \mathcal{H}_3 := \alpha_3 t_2 & \text{if } d = 3.
\end{cases}
\]

For a multi-index \( \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d, \) we use the following notation:

\[
\mathcal{H}^\beta := \mathcal{H}_{\beta_1} \cdots \mathcal{H}_{\beta_d}, \quad \mathcal{H}^{\beta^\vphantom{\beta}} := \mathcal{H}_{\beta_1}^{\vphantom{\beta}} \cdots \mathcal{H}_{\beta_d}^{\vphantom{\beta}}.
\]

We also define \( \alpha^\beta := \alpha_{\beta_1} \cdots \alpha_{\beta_d} \) and \( \alpha^{\beta^\vphantom{\beta}} := \alpha_{\beta_1}^{\vphantom{\beta}} \cdots \alpha_{\beta_d}^{\vphantom{\beta}}. \)

**Definition 2** We define vectors \( r_n \in \mathbb{R}^d, \ n = 1, \ldots, d, \) as follows. If \( d = 2, \)

\[
r_1 := (1, 0)^T, \quad r_2 := (s, t)^T,
\]

and if \( d = 3, \)

\[
r_1 := (1, 0, 0)^T, \quad r_2 := (s_{21}, s_{22}, t_2)^T,
\]

\[
\begin{cases}
r_2 := (s_1, t_1, 0)^T & \text{for case (i)}, \\
r_2 := (-s_1, t_1, 0)^T & \text{for case (ii)}.
\end{cases}
\]

For a sufficiently smooth function \( \varphi_0 \) and vector function \( v_0 := (v_{0,1}, \ldots, v_{0,d})^T, \) we define the directional derivative as, for \( i \in \{1 : d\}, \)

\[
\frac{\partial \varphi_0}{\partial r_i(0)} := [(A_{T_0} r_i) \cdot \nabla x(0)] \varphi_0 = \sum_{i_0=1}^d (A_{T_0} r_i)_{i_0} \frac{\partial \varphi_0}{\partial x_{i_0}} = \sum_{i_0,j_0=1}^d [A_{T_0}]_{i_0,j_0} (r_i)_{i_0} \frac{\partial \varphi_0}{\partial x_{j_0}},
\]

**Fig. 1** New parameters \( \mathcal{H}_i, \)

\( i = 1, 2, 3 \)
\[ \frac{\partial v}{\partial r_i^{(0)}} := \left( \frac{\partial v_{0,1}}{\partial r_i^{(0)}}, \ldots, \frac{\partial v_{0,d}}{\partial r_i^{(0)}} \right)^T = \left( [(A_{T_0}r_i) \cdot \nabla x]v_{0,1}, \ldots, [(A_{T_0}r_i) \cdot \nabla x]v_{0,d} \right)^T, \]

where \( A_{T_0} \in O(d) \) is the orthogonal matrix defined in (2.5). For a multi-index \( \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d \), we use the notation
\[ \frac{\partial^{\beta} \varphi_0}{\partial r_1^{(0)} \ldots \partial r_d^{(0)}} := \frac{\partial^{|\beta|} \varphi_0}{\partial r_1^{(0)} \beta_1 \ldots \partial r_d^{(0)} \beta_d}. \]

For \( \varphi = \varphi_0 \circ \Phi_{T_0} \) and \( v = \Psi_{T_0}^{-1} v_0 \), we define the directional derivative as, for \( i \in \{1 : d\} \),
\[ \frac{\partial \varphi}{\partial r_i} := [r_i \cdot \nabla x] \varphi = \sum_{i_0=1}^{d} (r_i)_{i_0} \frac{\partial \varphi}{\partial x_{i_0}}, \]
\[ \frac{\partial v}{\partial r_i} := \left( \frac{\partial v_1}{\partial r_i}, \ldots, \frac{\partial v_d}{\partial r_i} \right)^T := \left( [r_i \cdot \nabla x]v_1, \ldots, [r_i \cdot \nabla x]v_d \right)^T. \]

For a multi-index \( \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d \), we use the notation
\[ \frac{\partial^{\beta} \varphi}{\partial r_1^{\beta_1} \ldots \partial r_d^{\beta_d}} := \frac{\partial^{|\beta|} \varphi}{\partial r_1^{\beta_1} \ldots \partial r_d^{\beta_d}}. \]

**Assumption 1** In anisotropic interpolation error analysis, we may impose a geometric condition for the simplex \( T \):

1. If \( d = 2 \), there are no additional conditions;
2. If \( d = 3 \), there exists a positive constant \( M \) independent of \( h_T \) such that \( |s_{22}| \leq M \frac{\alpha_2}{\alpha_3} \). Note that if \( s_{22} \neq 0 \), this condition means that the order with respect to \( h_T \) of \( \alpha_3 \) coincides with the order of \( \alpha_2 \), and if \( s_{22} = 0 \), the order of \( \alpha_3 \) may be different from that of \( \alpha_2 \).
Note 1 Recall that

\[ |s| \leq 1, \quad \alpha_2 \leq \alpha_1 \quad \text{if } d = 2, \]
\[ |s_1| \leq 1, \quad |s_2| \leq 1, \quad \alpha_2 \leq \alpha_3 \leq \alpha_1 \quad \text{if } d = 3. \]

When \( d = 3 \), if Assumption 1 is imposed, there exists a positive constant \( M \) independent of \( h_T \) such that \( |s_{22}| \leq M \frac{\alpha_1}{\alpha_3} \). We thus have, if \( d = 2 \),

\[ \alpha_1|\langle \tilde{A} \rangle_{j1}| \leq \mathcal{H}_j, \quad \alpha_2|\langle \tilde{A} \rangle_{j2}| \leq \mathcal{H}_j, \quad j = 1, 2, \]

and, if \( d = 3 \), for \( \tilde{A} \in \{ \tilde{A}_1, \tilde{A}_2 \} \),

\[ \alpha_1|\langle \tilde{A} \rangle_{j1}| \leq \mathcal{H}_j, \quad \alpha_2|\langle \tilde{A} \rangle_{j2}| \leq \mathcal{H}_j, \quad \alpha_3|\langle \tilde{A} \rangle_{j3}| \leq \max\{1, M\} \mathcal{H}_j, \quad j = 1, 2, 3. \]

2.6 New parameters

We proposed two geometric parameters, \( H_T \) and \( H_{T_0} \), in [14].

Definition 3 The parameter \( H_T \) is defined as

\[ H_T := \prod_{i=1}^{d} \alpha_i |\mathcal{T}| h_T, \]

and the parameter \( H_{T_0} \) is defined as

\[ H_{T_0} := \frac{h_{T_0}^2}{|\mathcal{T}_0|} \min_{1 \leq i \leq 3} |L_i| \quad \text{if } d = 2, \quad H_{T_0} := \frac{h_{T_0}^2}{|\mathcal{T}_0|} \min_{1 \leq i, j \leq 6, i \neq j} |L_i| |L_j| \quad \text{if } d = 3 \]

where \( L_i \) denotes the edges of the simplex \( T_0 \subset \mathbb{R}^d \).

The following lemma shows the equivalence between \( H_T \) and \( H_{T_0} \).

Lemma 1 It holds that

\[ \frac{1}{2} H_{T_0} < H_T < 2 H_{T_0}. \]

Furthermore, in the two-dimensional case, \( H_{T_0} \) is equivalent to the circumradius \( R_2 \) of \( T_0 \).

Proof The proof is found in [14, Lemma 3]. \( \square \)

We introduce the geometric condition proposed in [14], which is equivalent to the maximum-angle condition [16].
Assumption 2 A family of meshes $\{T_h\}$ has a semi-regular property if there exists $\gamma_0 > 0$ such that \(1.1\). Equivalently, there exists $\gamma_1 > 0$ such that

$$\frac{H_T}{h_T} \leq \gamma_1 \quad \forall T_h \in \{T_h\}, \quad \forall T_0 \in T_h, \quad T = \Phi^{-1}_{T_0}(T_0). \quad (2.6)$$

Remark 1 In [17], we considered good elements on meshes. On anisotropic meshes, the good elements may satisfy conditions such as

\((d = 2)\) $\alpha_2 \approx \alpha_2 t = \mathcal{H}_2$;  
\((d = 3)\) $\alpha_2 \approx \alpha_2 t_1 = \mathcal{H}_2$ and $\alpha_3 \approx \alpha_3 t_2 = \mathcal{H}_3$.

We have the following theorem concerning the new condition.

Theorem 1 Condition \(1.1\) holds if and only if there exist $0 < \gamma_2, \gamma_3 < \pi$ such that

\[d = 2: \quad \theta_{T_0, \text{max}} \leq \gamma_2 \quad \forall T_h \in \{T_h\}, \quad \forall T_0 \in T_h, \quad (2.7)\]

where $\theta_{T_0, \text{max}}$ is the maximum angle of $T_0$, and

\[d = 3: \quad \theta_{T_0, \text{max}} \leq \gamma_3, \quad \psi_{T_0, \text{max}} \leq \gamma_3 \quad \forall T_h \in \{T_h\}, \quad \forall T_0 \in T_h, \quad (2.8)\]

where $\theta_{T_0, \text{max}}$ is the maximum angle of all triangular faces of the tetrahedron $T_0$ and $\psi_{T_0, \text{max}}$ is the maximum dihedral angle of $T_0$. Conditions \(2.7\) and \(2.8\) are called the maximum-angle condition.

Proof In the case that $d = 2$, we use the previous result presented in [18]; i.e., there exists a constant $\gamma_3 > 0$ such that

$$\frac{R_2}{h_{T_0}} \leq \gamma_3 \quad \forall T_h \in \{T_h\}, \quad \forall T_0 \in T_h,$$

if and only if condition \(2.7\) is satisfied. Combining this result with the fact that $H_{T_0}$ is equivalent to the circumradius $R_2$ of $T_0$, we have the desired conclusion. In the case that $d = 3$, the proof can be found in the recent paper [16]. \( \square \)

Lemma 2 (Euclidean condition number) It holds that

\[
\|\hat{A}\|_2 \leq h_T, \quad \|\hat{A}\|_2 \|\hat{A}^{-1}\|_2 = \frac{\max\{\alpha_1, \ldots, \alpha_d\}}{\min\{\alpha_1, \ldots, \alpha_d\}}, \quad (2.9a)
\]

\[
\|\tilde{A}\|_2 \leq \begin{cases} \sqrt{2} & \text{if } d = 2, \\ 2 & \text{if } d = 3 \end{cases}, \quad \|\tilde{A}\|_2 \|\tilde{A}^{-1}\|_2 \leq \begin{cases} \frac{\alpha_1 \alpha_2}{|T|} = \frac{H_T}{h_T} & \text{if } d = 2, \\ \frac{2}{3} \frac{\alpha_1 \alpha_2 \alpha_3}{|T|} = \frac{H_T}{3 h_T} & \text{if } d = 3 \end{cases}, \quad (2.9b)
\]

\[
\|A_{T_0}\|_2 = 1, \quad \|A_{T_0}^{-1}\|_2 = 1. \quad (2.9c)
\]

Furthermore, we have

\[
|\det(A_T)| = |\det(\tilde{A})| |\det(\hat{A})| = d! |T|, \quad |\det(A_{T_0})| = 1. \quad (2.10)
\]
The proof of (2.9b) is found in [14, (4.4), (4.5), (4.6) and (4.7)]. The inequality (2.9a) is easily proved. Because $A_{T_0} \in O(d)$, one can easily have $A_{T_0}^{-1} \in O(d)$ and (2.9c). The proof of equality (2.10) is standard. \hfill \Box

2.7 Raviart–Thomas finite element generation

We follow the procedure described in [14, Sect. 3.5] and [11, Sects. 1.4.1 and 1.2.1].

For a simplex $T_0 \subset \mathbb{R}^d$, we define the local Raviart–Thomas polynomial space of order $k \in \mathbb{N}_0$ as follows:

$$RT^k(T_0) := \mathcal{P}^k(T_0)^d + x\mathcal{P}^k(T_0), \quad x \in \mathbb{R}^d.$$ (2.11)

Let $\hat{T} \subset \mathbb{R}^d$ be the reference element defined in Sections 2.2. The Raviart–Thomas finite element on the reference element is defined by the triple $\{\hat{T}, \hat{P}, \hat{\Sigma}\}$ as follows:

1. $\hat{P} := RT^k(\hat{T})$;
2. setting $N^{(RT)} := \dim RT^k$, $\hat{\Sigma}$ is a set $\{\hat{\chi}_i\}_{1 \leq i \leq N^{(RT)}}$ of $N^{(RT)}$ linear forms with its components such that, for any $\hat{q} \in \hat{P}$,

$$\int_{\hat{F}} \hat{q} \cdot \hat{n}_{\hat{F}} \hat{p}_k d\hat{s}, \quad \forall \hat{p}_k \in \mathcal{P}^k(\hat{F}), \quad \hat{F} \subset \partial \hat{T},$$ (2.12)

$$\int_{\hat{F}} \hat{q} \cdot \hat{p}_{k-1} d\hat{x}, \quad \forall \hat{p}_{k-1} \in \mathcal{P}^{k-1}(\hat{T})^d,$$ (2.13)

where $\hat{n}_{\hat{F}}$ denotes the outer unit normal vector of $\hat{T}$ on the face $\hat{F}$. Note that for $k = 0$, the local degrees of freedom of type (2.13) are violated.

For the simplicial Raviart–Thomas element in $\mathbb{R}^d$, it holds that

$$\dim RT^k(\hat{T}) = \begin{cases} (k + 1)(k + 3) & \text{if } d = 2, \\ \frac{1}{2}(k + 1)(k + 2)(k + 4) & \text{if } d = 3. \end{cases}$$ (2.14)

The Raviart–Thomas finite element with the local degrees of freedom of (2.12) and (2.13) is unisolvent; for example, see [6, Proposition 2.3.4].

We set the domain of the local Raviart–Thomas interpolation to $V(\hat{T}) := \mathcal{W}^{1,1}(\hat{T})^d$; for example, see [12, p. 188]. The local Raviart–Thomas interpolation $I_{\hat{T}}^{RT^k} : V(\hat{T}) \rightarrow \hat{P}$ is then defined as follows: For any $\hat{v} \in V(\hat{T})$,

$$\int_{\hat{F}} (I_{\hat{T}}^{RT^k} \hat{v} - \hat{v}) \cdot \hat{n}_{\hat{F}} \hat{p}_k d\hat{s} = 0 \quad \forall \hat{p}_k \in \mathcal{P}^k(\hat{F}), \quad \hat{F} \subset \partial \hat{T},$$ (2.15)

and if $k \geq 1$,

$$\int_{\hat{T}} (I_{\hat{T}}^{RT^k} \hat{v} - \hat{v}) \cdot \hat{q}_{k-1} d\hat{x} = 0 \quad \forall \hat{q}_{k-1} \in \mathcal{P}^{k-1}(\hat{T})^d.$$ (2.16)
The triples \( \{T, P, \Sigma\} \) and \( \{T_0, P_0, \Sigma_0\} \) are defined as

\[
\begin{align*}
T &= \Phi_T(T); \\
P &= \{\Psi_T(p); \ p \in P\}; \\
\Sigma &= \{\{\chi_i\}_{1 \leq i \leq N(RT)}; \chi_i = \chi_i(\Psi_T^{-1}(p)), \forall p \in P, \chi_i \in \hat{\Sigma}\},
\end{align*}
\]

and

\[
\begin{align*}
T_0 &= \Phi_{T_0}(T); \\
P_0 &= \{\Psi_{T_0}(p); \ p \in P\}; \\
\Sigma_0 &= \{\{\chi_{T_0,i}\}_{1 \leq i \leq N(RT)}; \chi_{T_0,i} = \chi_i(\Psi_{T_0}^{-1}(p_0)), \forall p_0 \in P_0, \chi_i \in \Sigma\}.
\end{align*}
\]

The triple \( \{T, P, \Sigma\} \) and \( \{T_0, P_0, \Sigma_0\} \) are then the Raviart–Thomas finite elements. Furthermore, let

\[
I_{RT}^{T_k}: V(T) \rightarrow P, \quad I_{T_0}^{RT_k}: V(T_0) \rightarrow P_0
\]

be the associated local Raviart–Thomas interpolations defined in (2.15) and (2.16).

### 3 Scaling argument

We present estimates related to the scaling argument corresponding to [11, Lemma 1.113]. The estimates play significant roles in our analysis.

**Lemma 3** Let \( p \in [1, \infty) \). Let \( T \in \mathbb{S}^{(d)} \) satisfy Condition 1 or Condition 2. Let \( T_0 \subset \mathbb{R}^d \) be a simplex such that \( T = \Phi_{T_0}^{-1}(T_0) \). It holds that, for any \( \hat{v} = (\hat{v}_1, \ldots, \hat{v}_d)^T \in L^p(T) \) with \( v = (v_1, \ldots, v_d)^T := \Psi_T \hat{v} \) and \( v_0 = (v_{0,1}, \ldots, v_{0,d})^T := \Psi_{T_0} v \),

\[
\|v_0\|_{L^p(T_0)^d} \leq c|\det(A_T)|^{\frac{1-p}{p}}\|\tilde{A}\|^2_2 \left( \sum_{j=1}^d \alpha_j^p \|\tilde{v}_j\|_{L^p(T)}^p \right)^{1/p}.
\]  

where \( \tilde{A} = A \) and \( \alpha_j^p = 1 \) if \( p = 1 \).

Let \( \ell, m \in \mathbb{N}_0 \) and \( k \in \mathbb{N} \) with \( 1 \leq k \leq d \). Let \( \beta := (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d \) and \( \gamma := (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}_0^d \) be multi-indices with \( |\beta| = \ell \) and \( |\gamma| = m \), respectively. It then holds that, for any \( \hat{v} = (\hat{v}_1, \ldots, \hat{v}_d)^T \in W^{1,|\beta|+|\gamma|,p}(T) \) with \( v = (v_1, \ldots, v_d)^T := \Psi_T \hat{v} \) and \( v_0 = (v_{0,1}, \ldots, v_{0,d})^T := \Psi_{T_0} v \),

\[
\left\| \partial_x^{\beta} \partial_{\hat{x}}^\gamma \hat{v}_k \right\|_{L^p(T)} \leq c|\det(A_T)|^{\frac{p-1}{p}}\alpha_k^{-1}\|\tilde{A}^{-1}\|^2_2 \sum_{|\epsilon| = |\beta| + |\gamma|} \alpha^\epsilon \left\| \partial_{\hat{x}}^\epsilon v_0 \right\|_{L^p(T_0)^d}.
\]  

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If Assumption 1 is imposed, it holds that, for any \( \hat{v} = (\hat{v}_1, \ldots, \hat{v}_d)^T \in W^{[\beta]+|\gamma|,p}(\widehat{T})^d \) with \( v = (v_1, \ldots, v_d)^T := \Psi_T \hat{v} \) and \( v_0 = (v_{0,1}, \ldots, v_{0,d})^T := \Psi_{T_0} v \),

\[
\left\| \partial^\beta \partial^\gamma \hat{v}_k \right\|_{L^p(\widehat{T})} \leq c \left( \det(A_T) \right)^{\frac{p-1}{p}} \| \hat{A}^{-1} \|_2 \sum_{\ell \mid \ell = |\beta|+|\gamma|} H^\ell \left\| \partial^\ell \left( \Psi_{T_0}^{-1} v_0 \right) \right\|_{L^p(\Phi_{T_0}^{-1}(T_0))^d}.
\]

(3.3)

**Proof** Because the space \( C(\widehat{T})^d \) is dense in the space \( L^p(\widehat{T})^d \), we show (3.1) for \( \hat{v} \in C(\widehat{T})^d \) with \( v = \Psi_T \hat{v} \) and \( v_0 = \Psi_{T_0} v \). The following inequality is found in [11, Lemma 1.113]. There exists a positive constant \( c \) such that

\[
\| v_0 \|_{L^p(T_0)^d} \leq c \| A_{T_0} \|_2 |\det(A_{T_0})|^{-\frac{1}{p}} \| v \|_{L^p(T)^d},
\]

which leads to, using (2.9c) and (2.10),

\[
\| v_0 \|_{L^p(T_0)^d} \leq c \| v \|_{L^p(T)^d},
\]

(3.4)

where \( p' \) is a real number such that \( \frac{1}{p} + \frac{1}{p'} = 1 \). From the definition of the Piola transformation, for \( i = 1, \ldots, d \),

\[
v_i(x) = \frac{1}{\det(A_T)} \sum_{j=1}^d \tilde{A}_{ij} \alpha_j \hat{v}_j(\hat{x}).
\]

If \( 1 \leq p < \infty \), for \( i = 1, \ldots, d \),

\[
\| v \|_{L^p(T)^d}^p = \sum_{i=1}^d \| v_i \|_{L^p(T)}^p \leq c \left( \det(A_T) \right)^{1-p} \| A \|_2 \sum_{j=1}^d \alpha_j^p \| \hat{v}_j \|_{L^p(T)}^p,
\]

which leads to (3.1) together with (3.4).

Because the space \( C^{\ell+m}(\widehat{T})^d \) is dense in the space \( W^{\ell+m,p}(\widehat{T})^d \), we prove (3.2) and (3.3) for \( \hat{v} \in C^{\ell+m}(\widehat{T})^d \) with \( v = \Psi_T \hat{v} \) and \( v_0 = \Psi_{T_0} v \). Using (1.3), (2.9c) and (2.10), through a simple calculation, we have, for \( 1 \leq k \leq d \),

\[
\left| \partial^\beta \partial^\gamma \hat{v}_k \right| = \left| \frac{\partial^{|\beta|+|\gamma|}}{\partial \hat{x}^\beta_1 \cdots \partial \hat{x}^\beta_d \partial \hat{x}^\gamma_1 \cdots \partial \hat{x}^\gamma_d} \hat{v}_k \right| \leq c \left( \det(A_T) \right) \| \hat{A}^{-1} \|_2 \left( \sum_{v=1}^d \alpha_v^\beta \alpha_v^\gamma \right)^{-\frac{1}{d}} \sum_{i_1 = 1}^{d} \sum_{i_1 = 0}^{d} [A_{T_0}]_{i_1}^{(0,1)} [A_{T_0}]_{i_1}^{(1)} (r_1)_{i_1}^{(1)} \cdots \left[ \sum_{j_1 = 1}^{\beta_1 \times \text{times}} \sum_{j_1 = 0}^{d} [A_{T_0}]_{j_1}^{(0,1)} [A_{T_0}]_{j_1}^{(1)} (r_1)_{j_1}^{(1)} \cdots \right]
\]

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\[
\sum_{\beta_1}^{d} \sum_{\gamma_1}^{d} \left[ A_{T_0} \right]_{j_1(0,1)}^{\beta_1(d)} (r_d)_{j_1(0,1)} \left( r_d \right)_{j_1(0,1)} \cdots \sum_{\gamma_1}^{d} \left[ A_{T_0} \right]_{j_1(0,1)}^{\beta_1(d)} (r_d)_{j_1(0,1)} \left( r_d \right)_{j_1(0,1)} \cdots
\]
\[
\sum_{\beta_1}^{d} \sum_{\gamma_1}^{d} \left[ A_{T_0} \right]_{j_1(0,1)}^{\beta_1(d)} (r_d)_{j_1(0,1)} \left( r_d \right)_{j_1(0,1)} \cdots \sum_{\gamma_1}^{d} \left[ A_{T_0} \right]_{j_1(0,1)}^{\beta_1(d)} (r_d)_{j_1(0,1)} \left( r_d \right)_{j_1(0,1)} \cdots
\]
\[
\sum_{\beta_d}^{d} \sum_{\gamma_d}^{d} \left[ A_{T_0} \right]_{j_1(0,1)}^{\beta_d(d)} (r_d)_{j_1(0,1)} \left( r_d \right)_{j_1(0,1)} \cdots \sum_{\gamma_d}^{d} \left[ A_{T_0} \right]_{j_1(0,1)}^{\beta_d(d)} (r_d)_{j_1(0,1)} \left( r_d \right)_{j_1(0,1)} \cdots
\]
\[
\frac{\partial x^{(0)}_{j_1(0,1)}}{\partial x^{(0)}_{j_1(0,1)}} \cdots \frac{\partial x^{(0)}_{j_1(0,1)}}{\partial x^{(0)}_{j_1(0,1)}} \left( \frac{\partial x^{(0)}_{j_1(0,1)}}{\partial x^{(0)}_{j_1(0,1)}} \right) \cdots \left( \frac{\partial x^{(0)}_{j_1(0,1)}}{\partial x^{(0)}_{j_1(0,1)}} \right) \cdots \left( \frac{\partial x^{(0)}_{j_1(0,1)}}{\partial x^{(0)}_{j_1(0,1)}} \right) \cdots
\]
\[
\frac{\partial x^{(0)}_{j_1(0,1)}}{\partial x^{(0)}_{j_1(0,1)}} \cdots \frac{\partial x^{(0)}_{j_1(0,1)}}{\partial x^{(0)}_{j_1(0,1)}} \left( \frac{\partial x^{(0)}_{j_1(0,1)}}{\partial x^{(0)}_{j_1(0,1)}} \right) \cdots \left( \frac{\partial x^{(0)}_{j_1(0,1)}}{\partial x^{(0)}_{j_1(0,1)}} \right) \cdots \left( \frac{\partial x^{(0)}_{j_1(0,1)}}{\partial x^{(0)}_{j_1(0,1)}} \right) \cdots
\]
\[
\leq c \left| \det (A_T) \right| \| \tilde{A}^{-1} \|_{2} \alpha_k^{-1} \sum_{v=1}^{d} \sum_{|\varepsilon|=|\beta|+|\gamma|} \alpha^p |\partial \varepsilon^p v|.
\]

Because \(1 \leq p < \infty\), it holds that, for \(1 \leq k \leq d\),
\[
\left\| \partial_x^{\beta} \partial_x^{\gamma} \hat{v}_k \right\|_{L^p(T)}^p \leq c \left| \det (A_T) \right|^{p-1} \| \tilde{A}^{-1} \|_2 \alpha_k^{-p} \sum_{|\varepsilon|=|\beta|+|\gamma|} \alpha^p \int_{T_0} |\partial \varepsilon^p v_0|^p dx^{(0)},
\]

which leads to (3.2) together with (1.2).

Using (1.3) and Note 1, through a simple calculation, we have, for \(1 \leq k \leq d\),
\[
|\partial_x^{\beta+\gamma} \hat{v}_k| = \left| \frac{\partial^{\beta+\gamma}}{\partial \hat{x}_1^{\beta_1} \cdots \partial \hat{x}_d^{\beta_d} \partial \hat{\gamma}_1 \cdots \partial \hat{\gamma}_d} \hat{v}_k \right|
\[
\leq c \left| \det (A_T) \right| \alpha_k^{-1} \sum_{\eta=1}^{d} |[\tilde{A}^{-1}]_{\eta \eta}|.
\]
Because $1 \leq p < \infty$, it holds that, for $1 \leq k \leq d$,

$$\left\| \frac{\partial^\beta}{\partial x} \frac{\partial^\gamma}{\partial \eta} \hat{\psi}_k \right\|_{L^p(T)}^p \leq c \mid \det(A_T) \mid \| \hat{A}^{-1} \|_2^p \alpha_k^{-p} \sum_{|\epsilon|=|\beta|+|\gamma|} \mathcal{H}_E^p \int_T |\hat{\psi}_k^\epsilon|^p dx,$$

which leads to (3.3) together with (1.2), $T = \Phi_T^{-1}(T_0)$ and $v = \Psi_T^{-1} v_0$. 

Remark 2 In inequality (3.3), it is possible to obtain the estimates in $T_0$ by explicitly determining the matrix $A_T$. 

\[ \square \]
Let \( \hat{v} \in C^1(\hat{T})^d \) with \( v = \Psi_T \hat{v} \) and \( v_0 = \Psi_{T_0} v \). Using (1.3), (2.9c), (2.10) and the definition of Piola transformations, we have, for \( 1 \leq i, k \leq d \),

\[
\left| \frac{\partial \hat{v}_k}{\partial \hat{x}_i} \right| \leq c | \det(A_T)| \| \hat{A}^{-1} \|_2 \alpha_k^{-1} \sum_{\nu=1}^{d} \sum_{i_1^{(1)}, \ldots, i_1^{(1)} = 1} \mathcal{H}_1 \left( A_{T_0} \right)_{i_1^{(1)}, \ldots, i_1^{(1)}} \left| \frac{\partial v_{0,v}}{\partial x_{i_1^{(0)}}} \right| .
\]

Let \( d = 3 \). We define the matrix \( A_{T_0} \) as

\[
A_{T_0} := \begin{pmatrix}
\cos \frac{\pi}{3} - \sin \frac{\pi}{3} & 0 \\
\sin \frac{\pi}{3} \cos \frac{\pi}{3} & 0 \\
0 & 1
\end{pmatrix}.
\]

We then have

\[
\left| \frac{\partial \hat{v}_k}{\partial \hat{x}_i} \right| \leq c | \det(A_T)| \| \hat{A}^{-1} \|_2 \alpha_k^{-3} \sum_{\nu=1}^{3} \left( \mathcal{H}_1 \left| \frac{\partial v_{0,v}}{\partial x_2^{(0)}} \right| + \mathcal{H}_2 \left| \frac{\partial v_{0,v}}{\partial x_1^{(0)}} \right| + \mathcal{H}_3 \left| \frac{\partial v_{0,v}}{\partial x_3^{(0)}} \right| \right) .
\]

Because \( 1 < p < \infty \), it holds that, for \( 1 \leq i, k \leq 3 \),

\[
\left\| \frac{\partial \hat{v}_k}{\partial \hat{x}_i} \right\|_{L^p(\hat{T})}^p \leq c | \det(A_T)|^{p-1} \| \hat{A}^{-1} \|_2 \alpha_k^{-p} \left( \mathcal{H}_1 \left\| \frac{\partial v_{0}}{\partial x_2^{(0)}} \right\|_{L^p(T)}^p + \mathcal{H}_2 \left\| \frac{\partial v_{0}}{\partial x_1^{(0)}} \right\|_{L^p(T)}^p + \mathcal{H}_3 \left\| \frac{\partial v_{0}}{\partial x_3^{(0)}} \right\|_{L^p(T)}^p \right) .
\]

The following two lemmata are divided into the element on \( \Sigma^{(2)} \) or \( \Sigma_1^{(3)} \) and the element on \( \Sigma_2^{(3)} \).

**Lemma 4** Let \( T \in \Sigma^{(2)} \) or \( T \in \Sigma_1^{(3)} \) satisfy Condition 1 or Condition 2, respectively. Let \( T_0 \subset \mathbb{R}^d \) be a simplex such that \( T = \Phi^{-1}_{T_0}(T_0) \). Let \( \beta := (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d \) be a multi-index with \( |\beta| = \ell \). Let \( p \in [1, \infty) \). It then holds that, for any \( \hat{v} = (\hat{v}_1, \ldots, \hat{v}_d)^T \in W^{\ell+1,p}(\hat{T})^d \) with \( v = (v_1, \ldots, v_d)^T := \Psi_T \hat{v} \) and \( v_0 = (v_{0,1}, \ldots, v_{0,d})^T := \Psi_{T_0} v \),

\[
\left\| \frac{\partial^\ell \nabla \hat{v}}{\partial \hat{x}^\ell} \cdot \hat{v} \right\|_{L^p(\hat{T})} \leq c | \det(A_T)|^{\frac{p-1}{p}} \sum_{|\epsilon| = \ell} \alpha^\epsilon \left\| \frac{\partial^\ell \nabla v}{\partial x^\ell} \cdot v_0 \right\|_{L^p(T_0)} . \tag{3.5}
\]

If Assumption 1 is imposed, it holds that

\[
\left\| \frac{\partial^\ell \nabla \hat{v}}{\partial \hat{x}^\ell} \cdot \hat{v} \right\|_{L^p(\hat{T})} \leq c | \det(A_T)|^{\frac{p-1}{p}} \sum_{|\epsilon| = \ell} \mathcal{H}^\epsilon \left\| \frac{\partial^\ell \nabla x \cdot (\Psi_{T_0}^{-1} v_0)}{\partial x^\ell} \right\|_{L^p(\Phi_{T_0}^{-1}(T_0))} . \tag{3.6}
\]
\textbf{Proof} Because the space $C^{\ell+1}(\hat{T})^d$ is dense in the space $W^{\ell+1,p}(\hat{T})^d$, we show (3.5) and (3.6) for $\hat{v} \in C^{\ell+1}(\hat{T})^d$ with $v = \Psi_{\hat{T}} \hat{v}$ and $v_0 = \Psi_{T_0}v$.

For a general derivative $\partial^\beta \nabla_{\hat{x}} \cdot \hat{v}$ with order $|\beta| = \ell$, we obtain, using (2.10),

$$
|\partial^\beta \nabla_{\hat{x}} \cdot \hat{v}| = \left| \frac{\partial |\beta|}{\partial \hat{x}^{\beta_1} \cdots \partial \hat{x}^{\beta_d}} \nabla_{\hat{x}} \cdot \hat{v} \right| \\
\leq |\det(A_T)||\det(A_{T_0})|
$$

$$
\sum_{i_1^{(0,1)} = 1}^d \alpha_1 [A_{T_0}]_{i_1^{(0,1)}(r_1)} \cdots \sum_{i_1^{(0,1)} = 1}^d \alpha_1 [A_{T_0}]_{i_1^{(0,1)}(r_1)} \\
\sum_{i_1^{(0,d)} = 1}^d \alpha_d [A_{T_0}]_{i_1^{(0,d)}(r_d)} \cdots \sum_{i_1^{(0,d)} = 1}^d \alpha_d [A_{T_0}]_{i_1^{(0,d)}(r_d)}
$$

$$
= \frac{\partial |\beta|}{\partial x^{(0)}_{i_1^{(0,1)}}} \cdots \frac{\partial |\beta|}{\partial x^{(0)}_{i_1^{(0,1)}}} \nabla_{x^{(0)}} \cdot v_0 \\
\beta_1\text{times} \quad \beta_1\text{times}
$$

$$
\leq |\det(A_T)| \sum_{|\epsilon| = \ell} \alpha^\epsilon |\partial^\epsilon \nabla_{x^{(0)}} \cdot v_0|
$$

It then holds that, using (1.2),

$$
\|\partial^\beta \nabla_{\hat{x}} \cdot \hat{v}\|_{L^p(\hat{T})} \leq c |\det(A_T)|^{p-1} \sum_{|\epsilon| = \ell} \alpha^\epsilon \|\partial^\epsilon \nabla_{x^{(0)}} \cdot v_0\|_{L^p(T_0)},
$$

which is the desired inequality.

By an analogous argument, if Assumption 1 is imposed, for a general derivative $\partial^\beta \nabla_{\hat{x}} \cdot \hat{v}$ with order $|\beta| = \ell$, we have, using Note 1,

$$
|\partial^\beta \nabla_{\hat{x}} \cdot \hat{v}| = \left| \frac{\partial |\beta|}{\partial \hat{x}^{\beta_1} \cdots \partial \hat{x}^{\beta_d}} \nabla_{\hat{x}} \cdot \hat{v} \right| \\
\leq c |\det(A_T)|
$$

$$
\sum_{i_1^{(1)} = 1}^d \alpha_1 [A]_{i_1^{(1)}} \cdots \sum_{i_1^{(1)} = 1}^d \alpha_1 [A]_{i_1^{(1)}} \\
\sum_{i_1^{(d)} = 1}^d \alpha_d [A]_{i_1^{(d)}} \cdots \sum_{i_1^{(d)} = 1}^d \alpha_d [A]_{i_1^{(d)}}
$$

$$
= \frac{\partial |\beta|}{\partial x^{(0)}_{i_1^{(0,1)}}} \cdots \frac{\partial |\beta|}{\partial x^{(0)}_{i_1^{(0,1)}}} \nabla_{x^{(0)}} \cdot v_0 \\
\beta_1\text{times} \quad \beta_1\text{times}
$$

$$
\leq c |\det(A_T)| \sum_{|\epsilon| = \ell} \alpha^\epsilon \|\partial^\epsilon \nabla_{x^{(0)}} \cdot v_0\|_{L^p(T_0)},
$$

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\[
\begin{align*}
\left| \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}} \nabla_x \cdot v \right|_{\beta_1 \text{times}} & \cdot \left| \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}} \nabla_x \cdot v \right|_{\beta_d \text{times}} \\
\leq & \; c |\det(A_T)| \sum_{|\epsilon| = \ell} \mathcal{H}^e |\partial^{\epsilon} \nabla_x \cdot v|.
\end{align*}
\]

It then holds that, using (2.10) and (1.2),
\[
\| \partial^{\beta} \nabla_x \cdot \hat{v}\|_{L^p(\hat{T})} \leq c |\det(A_T)| \frac{p-1}{p} \sum_{|\epsilon| = \ell} \mathcal{H}^e \|\partial^{\epsilon} \nabla_x \cdot v\|_{L^p(T)},
\]
which leads to (3.6) together with \( T = \Phi^{-1}_{T_0}(T_0) \) and \( v = \Psi^{-1}_{T_0}v_0 \).

Lemma 5 Let \( d = 3 \). Let \( T \in \mathbb{R}^3 \) satisfy Condition 2. Let \( T_0 \in \mathbb{R}^3 \) be a simplex such that \( T = \Phi^{-1}_{T_0}(T_0) \). Let \( \ell \in \mathbb{N}_0 \) and \( k \in \mathbb{N}_0 \) with \( 1 \leq k \leq 3 \). Let \( \beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}_0^3 \) be a multi-index with \( |\beta| = \ell \). Let \( p \in [1, \infty) \). It holds that, for any \( \hat{v} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)^T \in W^{\ell+1,p}(\hat{T})^3 \) with \( v = (v_1, v_2, v_3)^T := \Psi T \hat{v} \) and \( v_0 = (v_{0,1}, v_{0,2}, v_{0,3})^T := \Psi_{T_0} v \),
\[
\left\| \partial^{\beta} \nabla_x \cdot \hat{v} \right\|_{L^p(\hat{T})} \leq c |\det(A_T)| \frac{p-1}{p} \|\sim^{-1}\|_2 \sum_{|\epsilon| = \ell} \mathcal{H}^e \left\| \partial^{\epsilon} \frac{\partial v_0}{\partial r_0} \right\|_{L^p(\phi_{T_0}^{-1}(T_0))^3}. \tag{3.7}
\]
If Assumption 1 is imposed, it holds that
\[
\left\| \partial^{\beta} \nabla_x \cdot \hat{v} \right\|_{L^p(\hat{T})} \leq c |\det(A_T)| \frac{p-1}{p} \|\sim^{-1}\|_2 \sum_{|\epsilon| = \ell} \mathcal{H}^e \left\| \frac{\partial (\Psi_{T_0}^{-1} v)}{\partial r_k} \right\|_{L^p(\phi_{T_0}^{-1}(T_0))^3}. \tag{3.8}
\]

Proof Because the space \( C^{\ell+1}(\hat{T})^3 \) is dense in the space \( W^{\ell+1,p}(\hat{T})^3 \), we show (3.5) and (3.6) for \( \hat{v} \in C^{\ell+1}(\hat{T})^3 \) with \( v = \Psi T \hat{v} \) and \( v_0 = \Psi_{T_0} v \).

For a general derivative \( \partial^{\beta} \hat{v}_k / \partial x_k \) (1 \leq k \leq 3) with order \( |\beta| = \ell \), we obtain, using (1.3), (2.9c) and (2.10),
\[
\left| \partial^{\beta} \hat{v}_k / \partial x_k \right| = \left| \frac{\partial^{\beta}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta}}{\partial x_d^{\beta_d}} \frac{\partial \hat{v}_k}{\partial x_k} \right| \\
\leq |\det(A_T)| |\det(A_{T_0})| \sum_{\eta,\nu=1}^3 \left| [\sim^{-1}]_{k\eta} [A_{T_0}^{-1}]_{\eta\nu} \right|.
\]
\[
\sum_{i_1^{(1)}, j_1^{(1)}=1}^{3} \alpha_1 [A_{T_0}]_{i_1^{(1)}, j_1^{(1)}} (r_1)_{i_1^{(1)}} \cdots \sum_{i_1^{(1)}, j_1^{(1)}=1}^{3} \alpha_1 [A_{T_0}]_{i_1^{(1)}, j_1^{(1)}} (r_1)_{i_1^{(1)}} \leq \beta_1 \text{ times}
\]

\[
\sum_{i_1^{(2)}, j_1^{(2)}=1}^{3} \alpha_2 [A_{T_0}]_{i_1^{(2)}, j_1^{(2)}} (r_2)_{i_1^{(2)}} \cdots \sum_{i_1^{(2)}, j_1^{(2)}=1}^{3} \alpha_2 [A_{T_0}]_{i_1^{(2)}, j_1^{(2)}} (r_2)_{i_1^{(2)}} \leq \beta_2 \text{ times}
\]

\[
\sum_{i_1^{(3)}, j_1^{(3)}=1}^{3} \alpha_3 [A_{T_0}]_{i_1^{(3)}, j_1^{(3)}} (r_3)_{i_1^{(3)}} \cdots \sum_{i_1^{(3)}, j_1^{(3)}=1}^{3} \alpha_3 [A_{T_0}]_{i_1^{(3)}, j_1^{(3)}} (r_3)_{i_1^{(3)}} \leq \beta_3 \text{ times}
\]

\[
\frac{\partial \hat{\beta}_1}{\partial x^{(0)}} \cdots \frac{\partial \hat{\beta}_1}{\partial x^{(0)}} \frac{\partial \hat{\beta}_2}{\partial x^{(0)}} \cdots \frac{\partial \hat{\beta}_2}{\partial x^{(0)}} \frac{\partial \hat{\beta}_3}{\partial x^{(0)}} \cdots \frac{\partial \hat{\beta}_3}{\partial x^{(0)}} \frac{\partial v_{0,v}}{\partial r^{(0)}} \frac{\partial v_{0,v}}{\partial r^{(0)}} \frac{\partial v_{0,v}}{\partial r^{(0)}} \leq c \left| \det (A_T) \right| \| \tilde{A}^{-1} \| L_2 \sum_{\mathcal{V} = 1}^{3} \sum_{|\varepsilon| = \beta}^3 \alpha^\varepsilon \left| \frac{\partial \hat{v}_0}{\partial r^{(0)}} \right| \frac{\partial v_{0,v}}{\partial r^{(0)}} \right| \| \tilde{A}^{-1} \|_{L_2},
\]

which is the desired inequality.

It then holds that, using (1.2),

\[
\left\| \frac{\partial^\beta \hat{v}_k}{\partial x^{(0)}} \right\|_{L^p(T)} \leq c \left| \det (A_T) \right| \left\| \tilde{A}^{-1} \right\|_{L_2} \sum_{\mathcal{V} = 1}^{3} \sum_{|\varepsilon| = \beta} \alpha^\varepsilon \left| \frac{\partial \hat{v}_0}{\partial r^{(0)}} \right| \frac{\partial v_{0,v}}{\partial r^{(0)}} \right| \| \tilde{A}^{-1} \|_{L_2},
\]

By an analogous argument, if Assumption 1 is imposed, for a general derivative \( \frac{\partial^\beta \hat{v}_k}{\partial x^{(0)}} (1 \leq k \leq 3) \) with order \( |\beta| = \ell \), we have, using Note 1 and (1.3),

\[
\left\| \frac{\partial^\beta \hat{v}_k}{\partial x^{(0)}} \right\|_{L^p(T)} \leq c \left| \det (A_T) \right| \sum_{\mathcal{V} = 1}^{3} \left| \tilde{A}^{-1} \right| \sum_{\ell} \left| \hat{v}_\ell \right| \sum_{\mathcal{V} = 1}^{3} \sum_{|\varepsilon| = \beta}^3 \alpha^\varepsilon \left| \frac{\partial \hat{v}_0}{\partial r^{(0)}} \right| \frac{\partial v_{0,v}}{\partial r^{(0)}} \right| \| \tilde{A}^{-1} \|_{L_2},
\]

which is the desired inequality.
\[
\sum_{i_1^{(3)}=1}^{3} \alpha_3 ||\tilde{A}_{i_1^{(3)}}||_3 \cdots \sum_{i_{\beta_3}^{(3)}=1}^{3} \alpha_3 ||\tilde{A}_{i_{\beta_3}^{(3)}}||_3 
\]

\[
\frac{\partial \beta_1}{\partial x_{i_1^{(1)}}} \cdots \frac{\partial \beta_1}{\partial x_{i_{\beta_1}^{(1)}}} \frac{\partial \beta_2}{\partial x_{i_1^{(2)}}} \cdots \frac{\partial \beta_2}{\partial x_{i_{\beta_2}^{(2)}}} \frac{\partial \beta_3}{\partial x_{i_1^{(3)}}} \cdots \frac{\partial \beta_3}{\partial x_{i_{\beta_3}^{(3)}}} \frac{\partial v_\eta}{\partial r_k}
\]

\[
\leq c |\text{det}(A_T)||\tilde{A}^{-1}_\varepsilon||_2 \sum_{\eta=1}^{3} \sum_{|\varepsilon|=|\beta|} \mathcal{H}^{\varepsilon} \left| \frac{\partial v_\eta}{\partial r_k} \right|
\]

It then holds that, using (1.2),

\[
\left\| \frac{\partial \beta}{\partial x} \frac{\partial \tilde{\nu}_k}{\partial x} \right\|_{L^p(\hat{T})} \leq c |\text{det}(A_T)||\tilde{A}^{-1}_\varepsilon||_2 \sum_{|\varepsilon|=|\beta|} \mathcal{H}^{\varepsilon} \left| \frac{\partial v}{\partial r_k} \right|_{L^p(\hat{T})}
\]

which leads to (3.8) together with \( T = \Phi_{T_0}^{-1}(T_0) \) and \( v = \Psi_{T_0}^{-1}v_0 \).

\[\square\]

### 4 Local Raviart–Thomas interpolation error estimates

We introduce component-wise stability for the Raviart–Thomas interpolation proposed in [2]; see also [1].

We first introduce component-wise stability estimates in the reference element \( \hat{T}_1 = \text{conv}\{0, e_1, \ldots, e_d\} \), where \( e_1, \ldots, e_d \in \mathbb{R}^d \) are the canonical basis.

**Lemma 6** For \( k \in \mathbb{N}_0 \), there exists a constant \( C_1^{(i)}(k), i = 1, \ldots, d \) such that for all \( \hat{u} = (\hat{u}_1, \ldots, \hat{u}_d)^T \in W^{1,p}(\hat{T}_1)^d \),

\[
\| (I_{\hat{T}_1}^{RT}) \hat{u} \|_{L^p(\hat{T})} \leq C_1^{(i)}(k) \left( \| \hat{u} \|_{W^{1,p}(\hat{T}_1)} + \| \nabla \hat{u} \cdot \hat{u} \|_{L^p(\hat{T}_1)} \right), \quad i = 1, \ldots, d.
\]

**Proof** The proof is provided in [2, Lemma 3.3] for the case \( d = 3 \). The estimate in the case \( d = 2 \) can be proved analogously. \[\square\]

We next provide component-wise stability estimates in the reference element \( \hat{T}_2 = \text{conv}\{0, e_1, e_1 + e_2, e_3\} \).

**Lemma 7** For \( k \in \mathbb{N}_0 \), there exists a constant \( C_2^{(i)}(k), i = 1, 2, 3 \) such that, for all \( \hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)^T \in W^{1,p}(\hat{T}_2)^3 \),

\[
\| (I_{\hat{T}_2}^{RT}) \hat{u} \|_{L^p(\hat{T})} \leq C_2^{(i)}(k) \left( \| \hat{u} \|_{W^{1,p}(\hat{T}_2)} + \sum_{j=1, j \neq i}^{3} \| \frac{\partial \hat{u}_j}{\partial \hat{x}_j} \|_{L^p(\hat{T}_2)} \right), \quad i = 1, 2, 3.
\]

\[\square\]
Proof The proof is provided in [2, Lemma 4.3]. In this case, we remark that our reference element is different from that in [2, Lemma 4.3]. However, the lemma can be proved using an analogous argument.

4.1 Stability of the local Raviart–Thomas interpolation

Lemma 8 Let \( p \in [1, \infty) \). Let \( T \in \mathcal{T}^{(2)} \) or \( T \in \mathcal{T}^{(3)}_1 \) satisfy Condition 1 or Condition 2, respectively. Let \( T_0 \subset \mathbb{R}^d \) be a simplex such that \( T = \Phi_{T_0}^{-1}(T_0) \). It holds that, for any \( \hat{v} = (\hat{v}_1, \ldots, \hat{v}_d)^T \in W^{1,p}(\hat{T})^d \) with \( v = (v_1, \ldots, v_d)^T := \Psi_T \hat{v} \) and \( v_0 = (v_{0,1}, \ldots, v_{0,d})^T := \Psi_{T_0} v \),

\[
\| I_{T_0}^{RT} v_0 \|_{L^p(T_0)^d} \leq c \left[ \frac{H_{T_0}}{h_{T_0}} \left( \| v_0 \|_{L^p(T_0)^d} + \sum_{|\kappa|=1} \alpha^\kappa \| \partial_{\Gamma(0)} \|_{L^p(T_0)^d} \right) + h_{T_0} \| \nabla v_0 \|_{L^p(T_0)^d} \right].
\]

Let \( d = 3 \). Let \( T \in \mathcal{T}_2^{(3)} \) satisfy Condition 2. Let \( T_0 \subset \mathbb{R}^3 \) be a simplex such that \( T = \Phi_{T_0}^{-1}(T_0) \). It holds that, for any \( \hat{v} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)^T \in W^{1,p}(\hat{T})^3 \) with \( v = (v_1, v_2, v_3)^T := \Psi_T \hat{v} \) and \( v_0 = (v_{0,1}, v_{0,2}, v_{0,3})^T := \Psi_{T_0} v \),

\[
\| I_{T_0}^{RT} v_0 \|_{L^p(T_0)^3} \leq c \frac{H_{T_0}}{h_{T_0}} \left[ \| v_0 \|_{L^p(T_0)^3} + h_{T_0} \sum_{k=1}^3 \| \partial v_0 \|_{L^p(T_0)^3} \right].
\]

Proof The inequality (4.3) follows from (3.1), the component-wise stability (4.1), (4.2) with \( \ell = 0 \) and \( m \in \{0, 1\} \), and (3.5) with \( \ell = 0 \). The inequality (4.4) follows from (3.1), the component-wise stability (4.2), (4.3) with \( \ell = 0 \) and \( m \in \{0, 1\} \), and (3.7) with \( \ell = 0 \).

4.2 Remarks on anisotropic interpolation error analysis

In the proof of Theorem 3 in [14], we used the following estimate in [14, Lemmas 6 and 7]: For any \( v \in H^1(T)^d \),

\[
\frac{\| I_T^{RT} v - v \|_{L^2(T)^d}}{\|v\|_{H^1(T)^d}} \leq C_{p,d} H_T \frac{h_T}{h_T} \left( \sum_{i=1}^d \alpha_i^2 \| (I_T^{RT} \hat{v})_i - \hat{v}_i \|_{L^2(\hat{T})}^2 \right)^{1/2}.
\]

If the component-wise stability of the Raviart–Thomas interpolation on the reference element \( \hat{T} \)

\[
\| (I_T^{RT} \hat{v})_i - \hat{v}_i \|_{L^2(\hat{T})} \leq C \| \hat{v}_i \|_{H^1(\hat{T})}, \quad i = 1, \ldots, d
\]
holds, then the target estimate
\[ \| I_{RT} v - v \|_{L^2(T)^d} \leq c \frac{H_T}{h_T} h_T |v|_{H^1(T)^d} \]
holds. However, the estimate (4.5) generally does not hold (see [2, Introduction]): that is, we cannot apply the Babuška and Aziz technique [5]. We provide a counterexample of [2, Introduction].

We consider the simplex \( \hat{T} \subset \mathbb{R}^2 \) with vertices \( \hat{P}_1 := (0, 0)^T, \hat{P}_2 := (1, 0)^T, \) and \( \hat{P}_3 := (0, 1)^T \). For \( 1 \leq i \leq 3 \), let \( \hat{F}_i \) be the face of \( \hat{T} \) opposite to \( \hat{P}_i \). The Raviart–Thomas interpolation of \( \hat{v} \) is defined as
\[
I_{RT} \hat{v} = \sum_{i=1}^{3} \left( \int_{\hat{F}_i} \hat{v} \cdot \hat{n}_i d\hat{s} \right) \hat{\theta}_i \in RT^0,
\]
where
\[
\hat{\theta}_i := \frac{1}{2|\hat{T}|} (\hat{x} - \hat{P}_i), \quad \hat{x} = (\hat{x}_1, \hat{x}_2)^T.
\]
Setting \( \hat{v} := (0, \hat{x}_2^2)^T \) yields \( I_{RT} \hat{v} = \frac{1}{3}(\hat{x}_1, \hat{x}_2)^T \). This implies that \( (I_{RT} \hat{v})_1 - \hat{v}_1 \neq 0 \) for any \( \hat{x} \in \mathbb{R}^2 \).

### 4.3 Main theorems

The Bramble–Hilbert–type lemma (e.g., see [8, 10]) plays a significant role in interpolation error analysis. We use the following estimates on anisotropic meshes proposed in [3, Lemma 2.1].

**Lemma 9** Let \( D \subset \mathbb{R}^d \) with \( d \in \{2, 3\} \) be a connected open set that is star-shaped with respect to balls \( B \). Let \( \gamma \) be a multi-index with \( m := |\gamma| \) and \( \varphi \in L^1(D) \) be a function with \( \partial^\gamma \varphi \in W^{\ell-m,p}(D) \), where \( \ell \in \mathbb{N}, m \in \mathbb{N}_0, 0 \leq m \leq \ell, p \in [1, \infty] \). It then holds that
\[
\| \partial^\gamma (\varphi - Q^{(\ell)} \varphi) \|_{W^{\ell-m,p}(D)} \leq C^{BH} |\partial^\gamma \varphi|_{W^{\ell-m,p}(D)},
\]
where \( C^{BH} \) depends only on \( d, \ell, \text{diam} D, \) and \( \text{diam} B \), and \( Q^{(\ell)} \varphi \) is defined as
\[
(Q^{(\ell)} \varphi)(x) := \sum_{|\delta| \leq \ell-1} \int_B \eta(y) (\partial^\delta \varphi)(y) \frac{(x - y)^\delta}{\delta!} dy \in \mathcal{P}^{\ell-1},
\]
where \( \eta \in C_0^\infty(B) \) is a given function with \( \int_B \eta dx = 1 \).

The following two theorems are divided into the element on \( T^{(2)} \) or \( T_1^{(3)} \) and the element on \( T_2^{(3)} \).
Theorem 2 Let \( p \in [1, \infty) \). Let \( T \in \Sigma^{(2)} \) or \( T \in \Sigma^{(3)} \) satisfy Condition 1 or Condition 2, respectively. Let \( T_0 \subset \mathbb{R}^d \) be a simplex such that \( T = \Phi_{T_0}^{-1}(T_0) \). For \( k \in \mathbb{N}_0 \), let \( \{ T_0, RT^k(T_0), \Sigma_0 \} \) be the Raviart–Thomas finite element and \( I_{T_0}^{RT^k} \) the local interpolation operator defined in (2.17). Let \( \ell \) be such that \( 0 \leq \ell \leq k \). For any \( \hat{v} \in W^{\ell+1,p}(\hat{T})^d \) with \( v = (v_1, \ldots, v_d)^T := \Psi T \hat{v} \) and \( v_0 = (v_{0,1}, \ldots, v_{0,d})^T := \Psi_{T_0} v \), it holds that
\[
\| I_{T_0}^{RT^k} v_0 - v_0 \|_{L^p(T_0)^d} \leq c \left( \frac{H_{T_0}}{h_{T_0}} \sum_{|\beta| = \ell} \alpha^\beta \| \partial_x^\beta \Psi_{T_0}^{-1} v_0 \|_{L^p(T_0)^d} + h_{T_0} \sum_{|\beta| = \ell} \alpha^\beta \| \partial_x^\beta (\Psi_{T_0}^{-1} v_0) \|_{L^p(T_0)^d} \right).
\]

(4.8)

If Assumption 1 is imposed, it holds that
\[
\| I_{T_0}^{RT^k} v_0 - v_0 \|_{L^p(T_0)^d} \leq c \left( \frac{H_{T_0}}{h_{T_0}} \sum_{|\beta| = \ell} \mathcal{H}^\beta \| \partial_x^\beta (\Psi_{T_0}^{-1} v_0) \|_{L^p(T_0)^d} + h_{T_0} \sum_{|\beta| = \ell} \mathcal{H}^\beta \| \partial_x^\beta (\Psi_{T_0}^{-1} v_0) \|_{L^p(T_0)^d} \right).
\]

(4.9)

Proof Let \( \hat{v} \in W^{\ell+1,p}(\hat{T})^d \). Let \( I_{\hat{T}}^{RT^k} \) be the local interpolation operators on \( \hat{T} \) defined by (2.15) and (2.16). If \( q \in \mathcal{P}^\ell(T_0)^d \subset RT^k(T_0) \), then \( I_{T_0}^{RT^k} q = q \).

We set \( \Omega^{(\ell+1)} v_0 := (Q^{(\ell+1)} v_{0,1}, \ldots, Q^{(\ell+1)} v_{0,d})^T \in \mathcal{P}^\ell(T_0)^d \), where \( Q^{(\ell+1)} v_{0,j} \) is defined by (4.7) for any \( j \). We then obtain
\[
\| I_{T_0}^{RT^k} v_0 - v_0 \|_{L^p(T_0)^d} \leq \| I_{T_0}^{RT^k} (v_0 - \Omega^{(\ell+1)} v_0) \|_{L^p(T_0)^d} + \| \Omega^{(\ell+1)} v_0 - v_0 \|_{L^p(T_0)^d}.
\]

(4.10)

The inequality (3.1) for the first term on the right-hand side of (4.10) yield
\[
\| I_{T_0}^{RT^k} (v_0 - \Omega^{(\ell+1)} v_0) \|_{L^p(T_0)^d} \leq c |\det(A_T)|^{\frac{1-p}{p}} \| \hat{\tilde{A}} \|_2 \left( \sum_{j=1}^d \alpha_j^p \| I_{\hat{T}}^{RT^k} (\hat{\tilde{v}} - \hat{\tilde{\Omega}}^{(\ell+1)} \hat{\tilde{v}}) \|_{L^p(\hat{T})} \right)^{1/p}.
\]

(4.11)

The component-wise stability (4.1) and (1.2) yields
\[
\left( \sum_{j=1}^d \alpha_j^p \| I_{\hat{T}}^{RT^k} (\hat{\tilde{v}} - \hat{\tilde{\Omega}}^{(\ell+1)} \hat{\tilde{v}}) \|_{L^p(\hat{T})} \right)^{1/p}.
\]

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\[
\leq \sum_{j=1}^{d} \alpha_j \| I_{T}^{\mathcal{T}^k} (\hat{v} - \hat{\mathcal{Q}}^{(\ell+1)} \hat{v}) \|_{L^p(\hat{T})} j \]
\[
\leq c \sum_{j=1}^{d} \alpha_j \left( \| \hat{v}_j - \hat{\mathcal{Q}}^{(\ell+1)} \hat{v}_j \|_{W^{1,p}(\hat{T})} + \| \nabla_{\hat{x}} \cdot (\hat{v} - \hat{\mathcal{Q}}^{(\ell+1)} \hat{v}) \|_{L^p(\hat{T})} \right). \quad (4.12)
\]

The inequalities (3.1) for the second term on the right-hand side of (4.10) and (1.2) yield
\[
\| \mathcal{Q}^{(\ell+1)} v_0 - v_0 \|_{L^p(T_0)^d} \leq c | \det (A_T) |^{1-p} \| \tilde{A} \|_2 \sum_{j=1}^{d} \alpha_j \left( \| \hat{v}_j - \hat{\mathcal{Q}}^{(\ell+1)} \hat{v}_j \|_{W^{1,p}(\hat{T})} + \| \nabla_{\hat{x}} \cdot (\hat{v} - \hat{\mathcal{Q}}^{(\ell+1)} \hat{v}) \|_{L^p(\hat{T})} \right) \quad (4.13)
\]

The inequality (4.10) together with (4.11), (4.12) and (4.13) leads to
\[
\| I_{T_0}^{\mathcal{T}^k} v_0 - v_0 \|_{L^p(T_0)^d} \leq | \det (A_T) |^{1-p} \| \tilde{A} \|_2 \sum_{j=1}^{d} \alpha_j \left( \| \hat{v}_j - \hat{\mathcal{Q}}^{(\ell+1)} \hat{v}_j \|_{W^{1,p}(\hat{T})} + \| \nabla_{\hat{x}} \cdot (\hat{v} - \hat{\mathcal{Q}}^{(\ell+1)} \hat{v}) \|_{L^p(\hat{T})} \right). \quad (4.14)
\]

The Bramble–Hilbert-type lemma (Lemma 9) and (3.2),
\[
\| \hat{v}_j - \hat{\mathcal{Q}}^{(\ell+1)} \hat{v}_j \|_{W^{1,p}(\hat{T})} \]
\[
= \| \hat{v}_j - \hat{\mathcal{Q}}^{(\ell+1)} \hat{v}_j \|_{L^p(\hat{T})} \sum_{k=1}^{d} \left\| \frac{\partial}{\partial \hat{x}_k} \right( \hat{v}_j - \hat{\mathcal{Q}}^{(\ell+1)} \hat{v}_j \right) \|_{L^p(\hat{T})} \]
\[
\leq c \left( \sum_{|\gamma|=\ell+1} \| \hat{v}_j \|_{L^p(\hat{T})} + \sum_{k=1}^{d} \sum_{|\beta|=\ell} \left\| \frac{\partial^\beta}{\partial \hat{x}_k} \hat{v}_j \right\|_{L^p(\hat{T})} \right) \]
\[
\leq c | \det (A_T) |^{p-1} \alpha_j^{-p} \| \tilde{A}^{-1} \|_2 \left( \sum_{|\xi|=\ell+1} \| \xi^\gamma \|_{L^p(T_0)^d} \right)^p \quad (4.15)
\]

Because from [8, Proposition 4.1.17] it holds that
\[
\nabla_{\hat{x}} \cdot (\hat{\mathcal{Q}}^{(\ell+1)} \hat{v}) = \hat{Q}^\ell (\nabla_{\hat{x}} \cdot \hat{v}), \quad (4.16)
\]
from Lemma 9 and (3.5),

\[
\| \nabla \hat{x} \cdot (\hat{\upsilon} - \hat{\Omega}^{(\ell+1)} \hat{\upsilon}) \|_{L^p(T)}^p \\
\leq \| \nabla \hat{x} \cdot \hat{\upsilon} - \hat{\Omega}^{\ell} (\nabla \hat{x} \cdot \hat{\upsilon}) \|_{L^p(T)}^p \\
\leq c |\nabla \hat{x} \cdot \hat{\upsilon}|_{W^{l,p}(T)}^p = c \sum_{|\beta| = \ell} \| \partial^\beta \nabla \hat{x} \cdot \hat{\upsilon} \|_{L^p(T)}^p \\
\leq c |\det(A_T)|^{p-1} \left( \sum_{|\epsilon| = \ell} \alpha^\epsilon \| \partial^\epsilon \nabla x \cdot v_0 \|_{L^p(T_0)} \right)^p. 
\]

(4.17)

Combining (4.14), (4.15), and (4.17) with (2.9b) yields the target result (4.8).

If Assumption 1 is imposed, we use instead (3.3) of (3.2) and (3.6) instead of (3.5).

The inequality (4.9) then follows from (4.14), (3.3) and (3.6) with (2.9b).

\[\Box\]

**Theorem 3** Let \(d = 3\) and \(p \in [1, \infty)\). Let \(T \in T_2^{(3)}\) satisfy Condition 2. Let \(T_0 \subset \mathbb{R}^3\) be a simplex such that \(T = \Phi^{-1}_T(T_0)\). For \(k \in \mathbb{N}_0\), let \(\{T_0, RT^k(T_0), \Sigma_0\}\) be the Raviart–Thomas finite element and \(I_{T_0}^{RT^k}\) the local interpolation operator defined in (2.17). Let \(\ell\) be such that \(0 \leq \ell \leq k\). For any \(\hat{\upsilon} \in W^{\ell+1,p}(T)\) with \(v = (v_1, v_2, v_3)^T := \Psi_T \hat{\upsilon}\) and \(v_0 = (v_{0,1}, v_{0,2}, v_{0,3})^T := \Psi_{T_0} v\), it holds that

\[
\| I_{T_0}^{RT^k} v_0 - v_0 \|_{L^p(T_0)^3} \leq c \frac{H_{T_0}}{h_{T_0}} \left( h_{T_0} \sum_{k=1}^3 \sum_{|\epsilon| = \ell} \alpha^\epsilon \left\| \partial^\epsilon \nabla x \cdot v_0 \right\|_{L^p(T_0)^3} \right). 
\]

(4.18)

If Assumption 1 is imposed, it holds that

\[
\| I_{T_0}^{RT^k} v_0 - v_0 \|_{L^p(T_0)^3} \leq c \frac{H_{T_0}}{h_{T_0}} \left( \sum_{|\epsilon| = \ell+1} \mathcal{H}^\epsilon \| \partial^\epsilon (\Psi_{T_0}^{-1} v_0) \|_{L^p(\Phi_{T_0}^{-1}(T_0))}\right)^3 \]

\[
+ h_{T_0} \sum_{k=1}^3 \sum_{|\epsilon| = \ell} \mathcal{H}^\epsilon \left\| \partial^\epsilon (\Psi_{T_0}^{-1} v_0) \right\|_{L^p(\Phi_{T_0}^{-1}(T_0))}\right). 
\]

(4.19)

**Proof** An analogous proof of Theorem 2 yields the desired results (4.18) and (4.19), where we use the component-wise stability (4.2) and Lemma 5 instead of Lemma 4.

\[\Box\]

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