A CYCLE DECOMPOSITION AND ENTROPY PRODUCTION FOR CIRCULANT QUANTUM MARKOV SEMIGROUPS

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May 3, 2014

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Submitted to:
“Infinite Dimensional Analysis, Quantum Probability and Related Topics”

Abstract

We propose a definition of cycle representation for Quantum Markov Semigroups (qms) and Quantum Entropy Production Rate (QEPR) in terms of the ρ-adjoint. We introduce the class of circulant qms, which admit non-equilibrium steady states but exhibit symmetries that allow us to compute explicitly the QEPR, gain a deeper insight into the notion of cycle decomposition and prove that quantum detailed balance holds if and only if the QEPR equals zero.

Keywords: Non-equilibrium steady state, circulant quantum Markov semigroup, quantum cycle representation, entropy production rate, weighted detailed balance.

AMS Subject Classification: 46L55, 82C10, 60J27

1 Introduction

The notion of equilibrium state of physical systems is well understood and there exist several conditions that characterize such states, detailed balance and zero entropy
production among them. For classical Markov chains the equivalence of these two equilibrium criteria has been proved by Qian et al.\cite{12} using Kalpazidou’s cycle representation for Markov chains\cite{8}. Non-equilibrium state is a much more subtle notion, since there are a huge variety of behaviors involved in it.

This work is aimed at contributing to the program outlined in Reference\cite{1}, namely, to look for some interesting Gorini-Kossakowski-Sudarshan and Lindblad (GKSL) generators with properties that are rich enough to go beyond the equilibrium situation, but concrete enough to allow explicit study and, in some cases, explicit solutions. We define Quantum Entropy Production Rate (QEPR) for qms in terms of the \( \rho \)-adjoint and discuss its connection with Fagnola-Rebolledo’s\cite{6} definition. We propose a definition of cycle representation for Completely Positive (CP) maps and GKSL generators, discussing its connections with the QEPR. To test and illustrate the above notions, we introduce the class of circulant qms that admit non-equilibrium steady states but exhibit pretty symmetries which allow us explicit computation of the QEPR. The symmetry properties of our semigroups arise from an abelian group structure on the state space of the associated classical Markov chain.

Section 2 is a brief review of quantum detailed balance and its extensions. Our QEPR definition along with some basic properties are discussed in Section 3. A brief review of cycles and passage matrices is made in Section 4. In Section 5 we show how the transition probability matrix of a Markov chain on a finite abelian group is a circulant matrix, which is the leading concept of this article. Section 6 offers a quantum generalization of the former, named circulant operator, and we use it to define circulant qms; also in this section we propose a definition of quantum cycle representation for CP maps and GKSL generators. In Section 7 and 8 both QEPR and classical EPR are explicitly computed and compared for a circulant qms and its classical restriction using a diagonal invariant state. The remaining invariant states and its QEPR are studied in Section 9.

2 Preliminaries

2.1 Quantum detailed balance

For uniformly continuous qms on \( B(h) \), with \( h \) a separable Hilbert space, a notion of detailed balance was introduced first by Alicki\cite{3} and Frigerio-Gorini-Kossakowski-Verri\cite{9}. Indeed, a qms with GKSL generator \( \mathcal{L} \) satisfies a quantum detailed balance condition in the sense of Ref.\cite{3,9} with respect to a stationary state \( \rho \) (i.e., \( tr(\rho L(x)) = 0, \forall x \in B(h) \)), if there exists an operator \( \hat{L} \) on \( B(h) \) and a self-adjoint operator \( K \) on \( h \) such that for all \( x, y \in B(h) \) the following relations hold:

\[
tr(\rho \hat{L}(x)y) = tr(\rho x \mathcal{L}(y)), \\
\hat{L}(\cdot) - \mathcal{L}(\cdot) = 2i[K, \cdot].
\] (1)
The operator $\tilde{L}$, is called the $\rho$-adjoint of $L$. For a wide class of GKSL generators, including those deduced from the stochastic limit of quantum theory, the $\rho$-adjoint coincides with the time-reversed generator if quantum detailed balance holds. Therefore, $\tilde{L}$ can be considered as an extension of the time-reversed GKSL generator to the non-equilibrium situation and we expect that simple non-equilibrium situations should appear when studying the difference between $L$ and $\tilde{L}$, see Accardi-Fagnola-Quezada [1] and the references therein.

Other notions of quantum detailed balance have been introduced by Fagnola and Umanità [4, 5]. The main idea is to separate the invariant state $\rho$ into two pieces or, equivalently, define the $\rho$-adjoint using the inner product $\langle a, b \rangle_s = \text{tr}(\rho^{1-s}a^*\rho^s b)$ for $0 \leq s \leq \frac{1}{2}$, and replace relations (1) by

$$\text{tr}(\rho^{1-s}\tilde{L}(x)\rho^s y) = \text{tr}(\rho^{1-s}x\rho^s \tilde{L}(y)),$$

$$\tilde{L}(\cdot) - L(\cdot) = 2i[K, \cdot].$$

Due to the non-commutativity, these two definitions are not equivalent in general. Clearly, detailed balance in the sense of (1) corresponds with the case $s = 0$ in (2).

Notice that if $\rho$ is a stationary state for $L$, hence with $y = 1$ in (1) and using that $L(1) = 0$ we get

$$0 = \text{tr}(\rho x L(1)) = \text{tr}(\rho \tilde{L}(x)), \quad \forall x \in \mathcal{B}(h).$$

Therefore, $\rho$ is a stationary state also for $\tilde{L}$.

### 2.2 The $\rho$-adjoint and special representations

The $\rho$-adjoint (with $s = 0$) $\tilde{L}$ of a GKSL generator $L$ is a GKSL generator if and only if the last one commutes with the modular automorphism of $\rho$, i.e., $L \circ \sigma_{-i} = \sigma_{-i} \circ L$, where $\sigma_{-i}(a) = \rho a \rho^{-1}$, see Theorem 8 in Reference [5].

The Markov generators can be written in the standard Gorini-Kossakowski-Sudarshan and Lindblad (GKSL) representation

$$L(x) = i[H, x] - \frac{1}{2} \sum_{k \geq 1} (L^*_k L_k x - 2L^*_k x L_k + x L^*_k L_k),$$  \hspace{1cm} (2)

where $H, L_k \in \mathcal{B}(\mathcal{H})$ with $H = H^*$ and the series $\sum_{k \geq 1} L^*_k L_k$ is strongly convergent.

Given a normal state $\rho$ on $\mathcal{B}(\mathcal{H})$, a GKSL representation (2) of $L$ by a bounded self-adjoint operator $H$ and a finite or infinite sequence $(L_k)_{k \geq 1}$ of elements of $\mathcal{B}(\mathcal{H})$ such that:

(i) $\text{tr}(\rho L_k) = 0$ for each $k \geq 1$,

(ii) $\sum_{k \geq 1} L^*_k L_k$ is a strongly convergent sum,
(iii) if $\sum_{k \geq 0} |c_k|^2 < \infty$ and $c_0 + \sum_{k \geq 1} c_k L_k = 0$ for complex scalars $(c_k)_{k \geq 0}$, then $c_k = 0$ for every $k \geq 0$.

is called special. See Theorem 30.16 in Parthasarathy’s book[10] for a proof of the existence of these class of representations. Special representations are unique up to unitary transformations.

2.3 Weighted detailed balance

The notion of weighted detailed balance introduced in Reference[1], was aimed at characterizing a class of GKSL generators with properties rich enough to go beyond the equilibrium situation but concrete enough to allow explicit study. In terms of special representations, weighted detailed balance is stated as follows.

A uniformly continuous quantum Markov semigroup $(T_t)_{t \geq 0}$ satisfies a weighted detailed balance condition with respect to a faithful invariant state $\rho$, if its generator $L$ has a special GKSL representation by means of operators $H, L_k$, such that here exists a sequence of positive weights $q := (q_k)_k$ and operators $H', L_k'$ of a (possibly another) special representation of $L$ such that the difference $\tilde{L}_\rho - L$ has the structure

$$\tilde{L}_\rho - L = -2i[K, \cdot] + \Pi, \quad (3)$$

where $K = K^*$ is bounded and

$$\Pi(x) = \sum_k (q_k - 1)L_k^{*}xL_k'. \quad (4)$$

Quantum detailed balance holds if and only if $q_k = 1$ for all $k$.

3 Quantum Entropy Production Rate for quantum Markov semigroups

In this section we introduce a notion of Quantum Entropy Production based on the concept of $\rho$-adjoint. As well as detailed balance, our definition depends on which $\rho$-adjoint is used. Our definition is slightly different from the one introduced by Fagnola and Rebolledo[]. Both definitions coincide in the class of circulant quantum Markov semigroups introduced in Section 6 below.

Assume that $L$ and its $\rho$-adjoint $\tilde{L}$, are GKSL generators of strongly continuous qms $T$ and $\tilde{T}$, respectively, with an invariant state $\rho$. Let $T_{st}$ and $\tilde{T}_{st}$ denote the corresponding pre-dual semigroups.

**Definition 1** For every $t \geq 0$, let $\Omega_t$ and $\tilde{\Omega}_t$ be the states (density matrices) on $B(h \otimes h)$, with $h$ a separable Hilbert space, given by

$$\Omega_t = (\mathbb{1} \otimes T_{st})(|\Omega_\rho\rangle\langle\Omega_\rho|)$$


and
\[ \hat{\Omega}_t = \left( \mathbb{1} \otimes \tilde{T}_t \right) (|\Omega_\rho\rangle\langle\Omega_\rho|), \]
where \( \Omega_\rho = \sum \rho_i^\frac{1}{2} (e_i \otimes e_i) \in (h \otimes h) \), with \( (e_i)_{1 \leq i \leq p-1} \) the orthonormal basis of \( \rho \) in \( h \). The Quantum Entropy Production Rate of the uniformly continuous qms \( T_\ast \), with respect to the invariant state \( \rho \), is given by
\[ e_p(T_\ast, \rho) = \frac{d}{dt} S(\Omega_t, \hat{\Omega}_t) \bigg|_{t=0}, \]
where the relative entropy of the states \( \eta \) and \( \rho \) is defined as
\[ S(\eta, \rho) = tr \left( \eta \log \eta - \eta \log \rho \right) \]
if the nullspace of \( \eta \) contains the nullspace of \( \rho \) and \( \infty \) otherwise.

As a consequence of Klein’s Inequality, see the work of B. Ruskai [11], the relative entropy of every pair of states \( \rho, \eta \) is non-negative
\[ S(\eta, \rho) \geq 0. \]
Moreover, equality holds if and only if \( \eta = \rho \).

In the remaining sections we compute explicitly the Quantum Entropy Production Rate for circulant qms.

Remark 2  
(i) In the finite dimensional case \( \Omega_t \) (resp. \( \hat{\Omega}_t \)) is the so called Jamiołkowski [7], or Choi-Jamiołkowski, transform of the CP map \( T_\ast \circ T_\rho \) (resp. \( \tilde{T}_t \circ T_\rho \)), with \( T_\rho(x) = \rho^{\frac{1}{2}} x \rho^{\frac{1}{2}} \).

(ii) A simple computation shows that \( tr \left( |\Omega_\rho\rangle\langle\Omega_\rho| \right) = tr(\rho) = 1 \), hence \( |\Omega_\rho\rangle\langle\Omega_\rho| \) is a state on \( \mathcal{B}(h \otimes h) \) and \( \Omega_t \) is well defined.

(iii) In comparison to Fagnola-Rebolledo’s definition of entropy production rate, we remark that in our definition, the Jamiołkowski transform is not modified by an anti-unitary operator. Moreover, instead of forward and backward two-point states we use as forward dynamics the time-dependent state generated by Jamiołkowski transform of the semigroup \( (T_\ast)_t \geq 0 \) and as a backward dynamics the one associated with its \( \rho \)-adjoint \( (\tilde{T}_t)_t \geq 0 \).
4 Cycles and passage functions

Let $S$ be a numerable set and $c$ a periodic function from $\mathbb{Z}$ into the set $S$. Following the notations of Qian et al. [12], we call the values $c(n)$ of $c$ vertices (or nodes) of $c$, while the pairs $(c(n), c(n + 1))$ are called edges (directed edges or directed arcs) of $c$. The period of $c$ is the smallest integer $p$ such that $c(n + p) = c(n)$ for all $n \in \mathbb{Z}$. Two periodic functions $c$ and $c'$ are equivalent if one is a translation of the other, i.e., there exists $i \in \mathbb{Z}$ such that $c'(n) = c(n + i)$. The above is an equivalence relation and clearly two equivalent periodic functions have the same vertices and period. A directed circuit is an equivalence class of the above defined equivalence relation. Any directed circuit is determined either by its period $p$ and any $(p + 1)$-tuple $(i_0, i_1, \cdots, i_p)$ with $i_p = i_0$; or by its period $p$ and $p$ ordered pairs $(i_0, i_1), (i_1, i_2), \cdots, (i_{p-1}, i_p)$ with $i_p = i_0$, where $i_t = c(n + t - 1)$, $0 \leq i_t \leq p - 1$ for some $n \in \mathbb{Z}$.

Definition 3 The cycle (or directed cycle) associated with a given directed circuit $c = (i_0, i_1, \cdots, i_{p-1}, i_1)$, $p \geq 1$, with distinct vertices $i_0, i_1, \cdots, i_{p-1}$, is the ordered sequence $\hat{c} = (i_0, i_1, \cdots, i_{p-1})$.

Every cycle is invariant under cyclic permutation of its vertices. We also use the notation $\hat{c} = (c(0), c(1), \cdots, c(p - 1))$ for the cycle associated with the directed circuit $c = (c(0), c(1), \cdots, c(p - 1), c(0))$ of period $p$, and use the symbol $c$ for both the directed circuit and the cycle when no confusion is possible. For every directed circuit $c = (i_0, i_1, \cdots, i_{p-1}, i_0)$, the reverse circuit $c_\sim$ is defined as $c_\sim = (i_0, i_{p-1}, \cdots, i_1, i_0)$.

When all points of $c$ are distinct except for the extremes, then

$$J_c(i, j) = \begin{cases} 1 & \text{if } (i, j) \text{ is an edge of } c; \\ 0 & \text{otherwise.} \end{cases}$$

Hence, every cycle $c$ has associated an unique matrix $J_c = (J_c(i, j))$, in some complex matrix space, called passage matrix of $c$.

Example. If $c_0 = (0123)$ and $c_1 = (0312)$, then

$$J_{c_0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_{c_1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The passage matrix $J_{c_0}$ of the full length cycle $c_0 = (0, 1, 2, \ldots, p - 1)$ is often called the primary permutation matrix and the cycle $c_0$ the primary cycle. From now on $J_p$ will denote the primary permutation matrix in a $p \times p$ complex matrix space. Notice that given any cycle $c = (c(0), c(1), \cdots, c(p - 1))$, its passage matrix can be written in
terms of the canonical basis \( \{|e_i\rangle\langle e_j| : 0 \leq i, j \leq p - 1\} \) of the \( p \times p \) complex matrix space as

\[
J_c = \sum_{i=0}^{p-1} |e_{c(i)}\rangle\langle e_{c(i)+1}|,
\]

where \( \{e_j\}_{0 \leq j \leq p-1} \) is the canonical basis of \( \mathbb{C}^p \). Notice that \( J_c \) moves the canonical basis of \( \mathbb{C}^p \) according to the cycle \( c \), i.e., \( J_c e_{c(i)} = e_{c(i+1)} \) for all \( i \). So, the primary permutation matrix \( J_p \) is, in fact, the left shift operator for the canonical basis in \( \mathbb{C}^p \).

5 Circulant matrices

5.1 Markov chains on finite groups

Let \((G, \circ)\) be a finite group. Unless otherwise specified, we let \( p = |G| \) and denote by \( hg \) the product \( h \circ g \), \( h, g \in G \). Given a probability distribution \( \mu \) on \( G \), the transition probabilities

\[
p(g, hg) = \mu(\{h\}),
\]
define a discrete time Markov chain on \( G \).

**Example 1.** Consider the cyclic group \( \mathbb{Z}_p = \{0, 1, \cdots, p-1\} \) and any distribution probability \( \alpha = \{\alpha_0, \alpha_1, \cdots, \alpha_{p-1}\} \) on \( \mathbb{Z}_p \). Then the transition probability matrix is the circulant matrix

\[
A = \text{circ}(\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{p-1}) = \begin{pmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{p-1} \\
\alpha_{p-1} & \alpha_0 & \alpha_1 & \cdots & \alpha_{p-2} \\
\alpha_{p-2} & \alpha_{p-1} & \alpha_0 & \cdots & \alpha_{p-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_0
\end{pmatrix}.
\]

Notice that \( A \) is a convex linear combination of powers of the primary permutation matrix \( J_p = \sum_{j=0}^{p-1} |e_j\rangle\langle e_{j+1}| \); indeed,

\[
A = \sum_{j=0}^{p-1} \alpha_j J_p^j.
\]

**Example 2.** Let \( G \) be the abelian group \( G = \mathbb{Z}_p \times \mathbb{Z}_q \) where the symbol \( \times \) denotes direct product, with \( p, q \geq 2 \). We set the lexicographic order in \( \mathbb{Z}_p \times \mathbb{Z}_q \) and take \( \alpha = \{\alpha(0, 0), \cdots, \alpha(0, q-1), \alpha(1, 0), \cdots, \alpha(1, q-1), \cdots, \alpha(p-1, 0), \cdots, \alpha(p-1, q-1)\} \) any probability distribution on \( G \). One can easily see that the corresponding transition probability matrix is the block circulant matrix

\[
R = \text{circ}(R_0, R_1, \cdots, R_{p-1}),
\]
with circulant blocks

\[
R_i = \begin{pmatrix}
\alpha(i, 0) & \alpha(i, 1) & \alpha(i, 2) & \cdots & \alpha(i, q-1) \\
\alpha(i, q-1) & \alpha(i, 0) & \alpha(i, 1) & \cdots & \alpha(i, q-2) \\
\alpha(i, q-2) & \alpha(i, q-1) & \alpha(i, 0) & \cdots & \alpha(i, q-3) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha(i, 1) & \alpha(i, 2) & \alpha(i, 3) & \cdots & \alpha(i, 0)
\end{pmatrix}, \quad i = 0, 1, \ldots, p-1.
\]

The above matrix \( R \) is a convex linear combination of tensor products of powers of the primary permutation matrices \( J_p = \sum_{i=0}^{p-1} |e_i\rangle \langle e_{i+1}| \) and \( J_q = \sum_{j=0}^{q-1} |e_j\rangle \langle e_{j+1}| \); indeed,

\[
R = \sum_{0 \leq i \leq p-1, 0 \leq j \leq q-1} \alpha(i, j) J_p^i \otimes J_q^j. \quad (6)
\]

**Remark 4** Due to Birkhoff’s Theorem, every bi-stochastic matrix is a convex linear combination of permutation matrices. Notice that (3) and (4) are Birkhoff’s representations of bi-stochastic circulant and block-circulant matrices, respectively.

### 5.2 Diagonalization of circulant matrices

The discrete (or quantum) Fourier transform on \( \mathbb{C}^p \) is the unitary operator defined by means of

\[
F_p = \frac{1}{\sqrt{p}} \sum_{0 \leq j, k \leq p-1} \omega_p^{jk} |e_k\rangle \langle e_j|,
\]

where \( \omega_p \) is a primitive \( p \)-th root of unity and \( \{e_j\}_{0 \leq j \leq p-1} \) is the canonical basis of \( \mathbb{C}^p \). Before proving the discrete Fourier transform diagonalizes \( J_p \) we will need the next orthogonality relation between the \( p \)-th roots of unity.

**Proposition 5** For every pair \( i, k \in \{0, 1, \ldots, p-1\} \)

\[
\sum_{l=0}^{p-1} \omega_p^{(k-i)l} = p \delta_{ik}
\]

**Proof.** Fix \( i, k \in \{0, 1, \ldots, p-1\} \). Any primitive \( p \)-th root of unity \( \omega_p \) satisfies

\[
\omega_p^{(k-i)} \frac{1}{p} \sum_l \omega_p^{lk} \omega_p^{-lt} = \frac{1}{p} \sum_l \omega_p^{(l+1)k} \omega_p^{-((l+1)t)} = \frac{1}{p} \sum_l \omega_p^{lk} \omega_p^{-lt},
\]

therefore \( (\omega_p^{(k-i)} - 1) \frac{1}{p} \sum_l \omega_p^{(k-i)l} = 0 \). Since \( \omega_p^{(k-i)} - 1 = 1 - \delta_{ik} \) the conclusion follows. \( \square \)
Lemma 6 Let $Z_p = \text{diag}(1, \omega_p, \omega_p^2, \ldots, \omega_p^{p-1})$, then

(i) $F_p J_p F_p^* = Z_p$, 

(ii) $(F_p \otimes F_q)(J_p \otimes J_q)(F_p \otimes F_q)^* = Z_p \otimes Z_q$.

Proof. Direct computations show that

$$F_p J_p = \frac{1}{\sqrt{p}} \sum_{k,l} \omega_p^{kl} |e_k \rangle \langle e_{l+1}|,$$

$$F_p J_p F_p^* = \frac{1}{p} \left( \sum_{k,l} \omega_p^{kl} |e_k \rangle \langle e_{l+1}| \right) \left( \sum_{i,j} \omega_p^{ij} |e_j \rangle \langle e_i| \right) = \frac{1}{p} \sum_{i,k,l} \omega_p^{kl-(l+1)i} |e_k \rangle \langle e_i|$$

$$= \frac{1}{p} \sum_{i,k} \left( \sum_l \omega_p^{(k-i)l} \right) \omega_p^{-i} |e_k \rangle \langle e_i| = \sum_k \omega_p^k |e_k \rangle \langle e_k| = Z_p.$$

This proves (i). Item (ii) follows directly from (i).\hfill\square

Since each circulant matrix can be expressed in terms of the primary permutation matrix $J_p$, it follows that the discrete Fourier transform diagonalizes every circulant matrix as well as block circulant matrices with circulant blocks.

Theorem 7 If $A = \sum_i \alpha(i) J_p^i$ and $B = \sum_{i,j} \alpha(i, j) J_p^i \otimes J_q^j$, then

(i) $F_p A F_p^* = \sum_k \lambda_k |e_k \rangle \langle e_k|,$

with $\lambda_k = \sum_i \alpha(i) \omega_p^{ki}$, and

(ii) $(F_p \otimes F_q) B (F_p \otimes F_q)^* = \sum_{k,l} \lambda_{k,l} |e_k \otimes e_l \rangle \langle e_k \otimes e_l|,$

with $\lambda_{kl} = \sum_{i,j} \alpha(i, j) \omega_p^{ki} \omega_q^{lj}$.

Proof. Using the above Lemma 6, a direct computation shows that

$$F_p A F_p^* = \sum_i \alpha(i) F_p J_p^i F_p^* = \sum_i \alpha(i) \sum_k \omega_p^{ki} |e_k \rangle \langle e_k| = \sum_k \left( \sum_i \alpha(i) \omega_p^{ki} \right) |e_k \rangle \langle e_k|.$$

This proves (i).

Now observe that
\begin{align*}
(F_p \otimes F_q) B (F_p \otimes F_q)^* &= \sum_{i,j} \alpha(i,j) (F_p J^i_p F^*_{p}) \otimes (F_q J^j_q F^*_{q}) \\
&= \sum_{i,j} \alpha(i,j) \left( \sum_k \omega^k_p |e_k\rangle \langle e_k| \right) \otimes \left( \sum_l \omega^l_q |e_l\rangle \langle e_l| \right) \\
&= \sum_{k,l} \left( \sum_{i,j} \alpha(i,j) \omega^k_p \omega^l_q \right) |e_k \otimes e_l\rangle \langle e_k \otimes e_l|.
\end{align*}

This finishes the proof. \hfill \square

**Corollary 8** With the notations in the above theorem we have

(i) \hspace{1cm} e^{tA} = \frac{1}{p} \sum_{j,l} \Phi_{l-j}(t) |e_j\rangle \langle e_l|,

with \( \Phi_{m}(t) = \sum_{k} \omega^{mk}_p e^{t\lambda_k} \), and

(ii) \hspace{1cm} e^{tB} = \frac{1}{pq} \sum_{i,j,m,n} \Phi_{m-i,n-j}(t) |e_i \otimes e_j\rangle \langle e_m \otimes e_n|,

with \( \Phi_{i,j}(t) = \sum_{k,l} \omega^k_p \omega^l_q e^{t\lambda_{kl}} \).

**Proof.** The result of the above theorem and a direct computation show that

\[ e^{tA} = F^*_p \text{diag}(e^{t\lambda_k}) F_p = \frac{1}{p} \sum_{j,l} \left( \sum_{k} \omega^{(l-j)k}_p e^{t\lambda_k} \right) |e_j\rangle \langle e_l| = \frac{1}{p} \sum_{j,l} \Phi_{l-j}(t) |e_j\rangle \langle e_l|. \]

This proves (i).

In a similar way we see that

\[ e^{tB} = (F_p \otimes F_q)^* \text{diag}(e^{t\lambda_{kl}}) (F_p \otimes F_q) = \frac{1}{pq} \sum_{i,j,k,l,m,n} \omega^k_p \omega^l_q e^{t\lambda_{kl} \omega^{km}_p \omega^{ln}_q} |e_i \otimes e_j\rangle \langle e_m \otimes e_n|. \]

This proves (ii). \hfill \square
6 Circulant quantum Markov semigroups

6.1 Circulant completely positive maps

We first recall that a $p \times p$ complex matrix $A$ is called reducible if there exists a permutation matrix $P$ such that

$$PAP^* = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where $B$ and $D$ are square matrices of order at least 1. A matrix is called irreducible if it is not reducible. It is well known, see for instance Theorem 5.18 in Zhang’s book [13], that every irreducible $p \times p$ permutation matrix $A$ is permutation similar to the primary permutation matrix $J_p$, i.e., there exists a permutation matrix $P$ such that $A = PJ_pP^{-1}$.

**Lemma 9** For every irreducible permutation matrix $J \in M_p(\mathbb{C})$ there exists a unique cycle $c$ of maximal length such that $J = J_c$ is the passage matrix of $c$.

**Proof.** Being an irreducible permutation matrix, $J$ is permutation similar to $J_p$, i.e., there exists a permutation matrix $P$ such that $J = PJ_pP^{-1}$. Therefore for any element of the canonical basis $\{e_i\}$ of $\mathbb{C}^n$ we have

$$JPe_i = PJ_pe_i = Pe_{i-1}.$$ 

Define a unique cycle $c$ by means of the permutation $P$ taking $e_{c(i)} = Pe_{i-1}$. Clearly $J = J_c$ since $Je_{c(i)} = e_{c(i-1)}$. □

**Lemma 10** Let $B_0, B_1, \ldots, B_{p-1}$ be $p$-dimensional subspaces mutually orthogonal with respect to the Hilbert-Schmidt inner product in $M_p(\mathbb{C})$, with $B_0$ the subspace of all diagonal matrices. If $J_c$ is the passage matrix of any cycle $c$ of maximal length, then

$$J_cB_l = B_{l+1}, \forall 0 \leq l \leq p-1 \iff B_l = \text{span}\{\langle e_{c(k)}|e_{c(k+l)}\rangle |k = 0, 1, \ldots, n - 1\} \forall 0 \leq l \leq p-1,$$

where the sums in the indices $k, l$ is modulus $p$.

**Proof.** Clearly condition on the left hand side of (7) is sufficient for $J_cB_l = B_{l+1}$ for all $0 \leq l \leq p-1$. Let us proof the necessity by induction on $p \geq 1$. For $p = 1$ the condition on the left hand side clearly holds. Now, assuming that the condition holds for any $1 \leq p$ and let us proof that it holds for $p + 1$. We have $J_c|e_{c(i)}\rangle\langle e_{c(i+l)}| = |e_{c(i-1)}\rangle\langle e_{c(i+l)}| \in B_{l+1}, \forall 0 \leq i \leq p, 0 \leq l \leq p$ by assumption, where the sums in $c(i-1), c(i+l)$ is modulus $p + 1$. Hence we have that $B_l = \text{span}\{|e_{c(i)}\rangle\langle e_{c(i+l)}| : 0 \leq i \leq p\}, \forall 0 \leq l \leq p$ since these $p + 1$ vectors are linearly independent and $B_l$ is $(p + 1)$-dimensional. □
Definition 11 A linear operator $\Phi : \mathcal{M}_p(\mathbb{C}) \rightarrow \mathcal{M}_p(\mathbb{C})$ is called circulant map (or circulant quantum channel) if there exist $p$-dimensional subspaces $B_l$, $l = 0, \ldots, p - 1$, mutually orthogonal with respect to the Hilbert-Schmidt inner product with $B_0 = \text{span}\{ |e_k\rangle \langle e_k| : k = 0, \ldots, p - 1 \}$, invariant under the action of $\Phi$, such that

(i) $\mathcal{M}_p(\mathbb{C}) = \bigoplus_{l=0}^{p-1} B_l$.

(ii) there exists an irreducible permutation matrix $J \in \mathcal{M}_p(\mathbb{C})$ such that $JB_l = B_{l+1}$ sum modulus $p$.

(iii) If $c$ is the cycle associate with $J$ by Lemma [9], then under the isomorphism from $B_l$ into $\mathbb{C}^n$ defined by $|e_{c(k)}\rangle \langle e_{c(k+l)}| \mapsto e_{c(k)}$ we have that

$$\Phi(|e_{c(k)}\rangle \langle e_{c(k+l)}|) \mapsto e_{c(k)}Q$$

where $Q$ is a circulant $p \times p$ matrix.

Example. Let $c$ be any cycle of maximal length in $\{0, 1, \ldots, p - 1\}$, then the CP linear map defined by

$$\Phi_c(x) = \sum_{k=0}^{p-1} \gamma(p-k)J_c^k x J_c^{*k},$$

for some $\gamma(j) \geq 0$ is a circulant CP map. Let us define the subspaces $B_l = \text{span}\{ |e_{c(k)}\rangle \langle e_{c(k+l)}| : k = 0, 1, \ldots, p - 1 \}$ for $l = 0, \ldots, p - 1$, clearly condition (i) in the above definition holds. With $J = J_c$ in the above definition, let us prove the invariance of the subspaces $B_l$’s. For any $x^{(l)} = \sum_{j=0}^{p-1} x_{e_{c(j)}, e_{c(j+l)}} |e_{c(j)}\rangle \langle e_{c(j+l)}| \in B_l$ we have that

$$\Phi_c(x^{(l)}) = \sum_{k=0}^{p-1} \sum_{j=0}^{p-1} \gamma(p-k)x_{e_{c(j)}, e_{c(j+k-l)}} |e_{c(j)}\rangle \langle e_{c(j+k-l)}|$$

$$= \sum_{j=0}^{p-1} \left( \sum_{k=0}^{p-1} \gamma(p-k)x_{e_{c(j+k-l)}, e_{c(j)}} \right) |e_{c(j)}\rangle \langle e_{c(j+l)}| \in B_l.$$

Moreover, using the isomorphism induced by the cycle $c$ we get

$$\Phi_c(|e_{c(j)}\rangle \langle e_{c(j+l)}|) = \sum_{k=0}^{p-1} \gamma(p-k)|e_{c(j-k)}\rangle \langle e_{c(j-k+l)}| \mapsto$$

$$\sum_{k=0}^{p-1} \gamma(p-k)e_{c(j-k)} = e_{c(j)}$$

$$\begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \cdots & \gamma(p-1) \\ \gamma(p-1) & \gamma(0) & \gamma(1) & \cdots & \gamma(p-2) \\ \gamma(p-2) & \gamma(p-1) & \gamma(0) & \cdots & \gamma(p-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma(1) & \gamma(2) & \gamma(3) & \cdots & \gamma(0) \end{pmatrix}.$$
**Theorem 12** For every circulant CP map \( \Phi \) on \( \mathcal{M}_p(\mathbb{C}) \) there exists a cycle \( c \) of length \( p \) such that \( \Phi = \Phi_c \).

**Proof.** Assume that \( \Phi \) is a CP circulant map. By Lemma (11) and condition (iii) we have that \( \Phi(\langle e_{c(j)} \rangle \langle e_{c(j+l)} \rangle) = \sum_{k=0}^{p-1} \beta(k) \langle e_{c(j+k)} \rangle \langle e_{c(j+k+l)} \rangle \) with some \( \beta(k) \)'s independent of \( l \) and positive. On the other side, if \( \Phi_c(x) = \sum_k \beta(p-k)J^k_xJ^k_c^* \) we have that \( \Phi_c(\langle e_{c(j)} \rangle \langle e_{c(j+l)} \rangle) = \sum_{k=0}^{p-1} \beta(p-k) \langle e_{c(j-k)} \rangle \langle e_{c(j+k+l)} \rangle = \sum_k \beta(k) \langle e_{c(j+k)} \rangle \langle e_{c(j+k+l)} \rangle \), sums modulus \( p \), therefore \( \Phi = \Phi_c \) on \( B_l \) for every \( 0 \leq l \leq p-1 \). Hence by condition (ii) we can conclude that \( \Phi(x) = \Phi_c(x) \) for all \( x \in \mathcal{M}_p(\mathbb{C}) \) and this finishes the proof. \( \square \)

Consider the CP map on \( \mathcal{M}_p(\mathbb{C}) \otimes \mathcal{M}_q(\mathbb{C}) \) defined by

\[
\Phi_*(x) = \sum_{i,j} \alpha(p-i,q-j)(J^i_p \otimes J^j_q)x(J^i_p \otimes J^j_q)^*,
\]

with \( \alpha(i,j) \geq 0 \) for all \( 0 \leq i \leq p-1, 0 \leq j \leq q-1 \) and \( J_s, \ s = p, q, \) the left shift operator.

Motivated by the above discussion, maps of the class (8) will be called block circulant CP maps. More generally, we call block circulant CP map to any CP linear combination of tensor products of powers of passage matrices. Restriction of block circulant CP maps to invariant subspaces coincide with block circulant matrices with circulant blocks.

**Theorem 13** For every \( (k,l) \in \mathbb{Z}_p \otimes \mathbb{Z}_q \) let \( B_{kl} \) be the subspace of \( \mathcal{M}_p(\mathbb{C}) \otimes \mathcal{M}_q(\mathbb{C}) \) defined by

\[
B_{kl} = \text{span}\{|e_i\rangle\langle e_{i+k}| \otimes |e_j\rangle\langle e_{j+l}| : 0 \leq i \leq p-1, 0 \leq j \leq q-1\}.
\]

Then,

(i) the \( pq \)-dimensional subspaces \( B_{kl} \) are mutually orthogonal with the Hilbert-Schmidt product, invariant under the action of \( \Phi_* \) given by (3), \( \bigoplus_{kl} B_{kl} = \mathcal{M}_p(\mathbb{C}) \otimes \mathcal{M}_q(\mathbb{C}) \) and moreover,

(ii) the restriction of \( \Phi_* \) to any subspace \( B_{kl} \) reduces to the action the block circulant matrix \( Q = \sum_{i,j} \alpha(i,j)(J^i_p \otimes J^j_q) \), through the isomorphism from \( B_{kl} \) onto \( \mathbb{C}^p \otimes \mathbb{C}^q \) defined by \( |e_i \otimes e_j\rangle \langle e_{i+k} \otimes e_{j+l}| \mapsto e_i \otimes e_j \). More precisely,

\[
\Phi_*(|e_{i_0}\rangle\langle e_{i_0+k}| \otimes |e_{j_0}\rangle\langle e_{j_0+l}|) \\
= \sum_{i,j} \alpha\left(p-(i_0-i),q-(j_0-j)\right)|e_i \otimes e_j\rangle\langle e_{i+k} \otimes e_{j+l}| \\
= \sum_{i,j} \alpha\left(p-(i_0-i),q-(j_0-j)\right)e_i \otimes e_j = (e_{i_0} \otimes e_{j_0})Q.
\]
Where $Q$ is the block circulant matrix $Q = \text{circ}(Q_0, Q_1, \cdots, Q_{p-1})$ with circulant blocks

$$Q_i = \begin{pmatrix}
\alpha(i, 0) & \alpha(i, 1) & \alpha(i, 2) & \cdots & \alpha(i, q-1) \\
\alpha(i, q-1) & \alpha(i, 0) & \alpha(i, 1) & \cdots & \alpha(i, q-2) \\
\alpha(i, q-2) & \alpha(i, q-1) & \alpha(i, 0) & \cdots & \alpha(i, q-3) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\alpha(i, 1) & \alpha(i, 2) & \alpha(i, 3) & \cdots & \alpha(i, 0)
\end{pmatrix}, \quad 0 \leq i \leq p-1. \quad (9)$$

Proof. For every fixed $(i_0, J_0) \in \mathbb{Z}_p \times \mathbb{Z}_q$ we have that

$$\Phi_\ast(\langle e_{i_0} \rangle \langle e_{i_0+k} \rangle \otimes \langle e_{j_0} \rangle \langle e_{j_0+l} \rangle)$$

$$= \sum_{i,j} \alpha(p-i, q-j) \langle J_p^i \otimes J_q^j \rangle \langle e_{i_0} \otimes e_{j_0} \rangle \langle e_{i_0+k} \otimes e_{j_0+l} \rangle \langle J_p^i \otimes J_q^j \rangle^*$$

$$= \sum_{i,j} \alpha(p-i, q-j) \langle e_{i_0-i} \otimes e_{j_0-j} \rangle \langle e_{i_0+k} \otimes e_{j_0+l} \rangle \hookrightarrow$$

$$\sum_{i,j} \alpha(p-i, q-j) \langle e_{i_0-i} \otimes e_{j_0-j} \rangle = (e_{i_0} \otimes e_{j_0})Q.$$

This proves that every subspace $B_{kl}$ is invariant. They are mutually orthogonal, since

$$tr\left(\langle e_{i+k} \otimes e_{j+l} \rangle \langle e_i \otimes e_j \rangle \langle e_{i+k'} \otimes e_{j+l'} \rangle\right) = \delta_{kl,k'l}.$$

This proves the Theorem. \qed

6.2 Circulant quantum Markov semigroups

Consider the discrete time Markov chain on the abelian group $\mathbb{Z}_p \times \mathbb{Z}_q$ associated with a given probability distribution $\alpha : \mathbb{Z}_p \times \mathbb{Z}_q \mapsto [0, 1]$ with $\sum_{i,j} \alpha(i, j) = 1$. If we set $\alpha(0, 0) = 0$, then the corresponding bi-stochastic circulant transition probabilities matrix

$$\Pi = \sum_{i,j} \alpha(i, j)(J_p^i \otimes J_q^j),$$

can be considered as the transition probability matrix of the embedded Markov chain of the continuous time Markov chain with infinitesimal generator (or Q-matrix) $Q = \Pi - \mathbb{1}$, where $\mathbb{1}$ denotes the identity matrix in $\mathcal{M}_p(\mathbb{C}) \otimes \mathcal{M}_q(\mathbb{C}) \cong \mathcal{M}_n(\mathbb{C})$. Clearly $Q$ is a block circulant matrix with circulant blocks, we shall consider the quantum extensions, in pre-dual representation,

$$\Phi_\ast(x) = \sum_{(i,j) \in \mathbb{Z}_p \times \mathbb{Z}_q} \alpha(p-i, q-j)(J_p^i \otimes J_q^j)x(J_p^i \otimes J_q^j)^*. \quad (10)$$
and
\[ \mathcal{L}_s(x) = \Phi_s(x) - x. \] (11)
of \Pi and \ Q, respectively, with \( x \in \mathcal{M}_p(\mathbb{C}) \otimes \mathcal{M}_q(\mathbb{C}) \). Clearly \( \Phi_s \) is a circulant CP map (embedded quantum Markov chain). We call \( \mathcal{L}_s \) a circulant GKSL generator and circulant qms the semigroup generated by \( \mathcal{L}_s \).

Instead of the matrices \( J_p \) and \( J_q \), we can choose any pair of passage matrices \( J_{c_p}, J_{c_q} \) of cycles of maximal length in \( \mathbb{Z}_p \) and \( \mathbb{Z}_q \) respectively, having \( G = \mathbb{Z}_p \times \mathbb{Z}_q \). Even more, any finite number \( n \) of maximal length cycles \( c_k \) can be chosen in \( \mathbb{Z}_{pk} \) respectively, and follow the computation along the same lines with \( G = \times_{k=0}^{n-1}\mathbb{Z}_{pk} \). Moreover, if \( \mathbb{Z}_p \) has a prime order, then every power \( J_c^k \) of a passage matrix \( J_c \) is the passage matrix of some cycle \( c_k \neq c \) if \( 0 \neq k \neq 1 \mod p \).

Having this in mind and Kalpaziduo’s cycle representation\(^8\) of an irreducible Markov chain with uniform stationary measure \( \pi = \{\frac{1}{p}\} \) and circulant generator \( Q \),
\[ \frac{1}{p}Q = \sum_{c \in \mathcal{C}_\infty} w_c J_c - \frac{1}{p}I, \]
we can regard equations (10) and (11) as a quantum cycle representation of the circulant GKSL generator \( \mathcal{L}_s \) with cycle weights \( (\alpha(i, j))_{(i, j) \in \mathbb{Z}_p \times \mathbb{Z}_q} \). This motivates the following.

**Definition 14** Given a bounded GKSL generator of the form (2) with a discrete spectrum Hamiltonian, we call cycle representation of its embedded quantum Markov chain
\[ \Phi(x) = \sum_k L_k^* x L_k, \]
to a GKSL representation of \( \Phi \) of the form
\[ \Phi(x) = \sum_l \alpha_l U_l^* x U_l, \]
where for each \( l \), \( \alpha_l > 0 \) and \( U_l \) is a passage matrix.

Clearly, any cycle decomposition of the embedded chain \( \Phi \) induces a cycle representation of \( \mathcal{L} \).

**Remark 15** Tensor product like \( J_p^i \otimes J_q^j \) are irreducible matrices. Hence by Lemma 9, they are passage matrices of a cycle. This shows that our definition includes representations of the form (10) and its extensions involving any finite number of cycles or higher order tensor products.
By Theorem 13 each subspace $B_{kl}$ is invariant under $\Phi^*$, $L^*$ and, consequently, also under the action of the semigroups $T_\ast = (T_{st})_{t \geq 0}$ generated by $L$. The state $\rho = \frac{1}{pq} (\mathbb{1} \otimes \mathbb{1})$ is clearly invariant for $T_\ast$ since

$$L_\ast(\rho) = \sum_{(i,j) \neq (0,0)} \alpha(p-i,q-j) \mathbb{1} \otimes \mathbb{1} - \mathbb{1} \otimes \mathbb{1} = 0,$$

because $\sum_{(i,j) \neq (0,0)} \alpha(p-i,q-j) = 1$. The $\rho$-adjoint $\tilde{L}$ has the GKSL representation

$$\tilde{L}(x) = \sum_{i,j} L_{ij}^* x L_{ij},$$

with $L_{ij} = L_{ij}^0$ and $L_{ij} = \alpha(p-i,q-j)^{\frac{1}{2}} (J_p^i \otimes J_q^j)$. Hence $\tilde{L}_{ij} = \alpha(i,j)^{\frac{1}{2}} (J_p^i \otimes J_q^j)$. One can write in direct representation

$$\tilde{L}(x) = \sum_{(i,j) \neq (0,0)} \alpha(i,j)(J_p^i \otimes J_q^j)^* x (J_p^i \otimes J_q^j) - x.$$

Hence the difference between $L$ and its $\rho$-adjoint (reverse) operator looks like

$$\tilde{L}(x) - L(x) = \sum_{(i,j) \neq (0,0)} (\alpha(i,j) - \alpha(p-i,q-j)) (J_p^i \otimes J_q^j)^* x (J_p^i \otimes J_q^j)$$

$$= \sum_{(i,j) \neq (0,0)} (q(i,j) - 1) L_{ij}^* x L_{ij},$$

with $q(i,j) = \alpha(i,j) \alpha(p-i,q-j)^{-1}$ and the $L_{ij}$’s as above. Therefore, the semigroup $T_\ast$ satisfies a weighted detailed balance condition in the sense of Accardi-Fagnola-Quezada[1] with weights $q = (q_{ij} = \alpha(i,j) \alpha(p-i,q-j)^{-1})$, see equation (3) above. Consequently, by Corollary 2 in Ref.[1], detailed balance holds if and only if

$$\alpha(p-i,q-j) = \alpha(i,j), \forall (0,0) \neq (i,j) \in \mathbb{Z}_p \times \mathbb{Z}_q. \quad (12)$$

7 Quantum Entropy Production Rate for circulant qms

Let us compute the Quantum Entropy Production Rate (QEPR) for the circulant semigroup $T_\ast$ in the previous section. For simplicity we consider first the invariant state $\rho = \frac{1}{pq} \mathbb{1}$, other invariant states are studied in Section 9. We know that every subspace $B_{kl}$ of $M_p \otimes M_q$ is invariant under the action of the elements of $T_\ast$. This implies that the states $\Omega_t$ and $\tilde{\Omega}_t$ are diagonal with respect to the canonical basis.
Lemma 16  With the notations in Section 6 and Subsection 3 the following hold:

(i) for every \((i, j), (i', j') \in \mathbb{Z}_p \times \mathbb{Z}_q\), using the isomorphism induced by lemma 9, we have

\[ T^*_t (|e_i \otimes e_j\rangle\langle e_{i'} \otimes e_{j'}|) \mapsto (e_i \otimes e_j)e^{tQ} = \frac{1}{pq} \sum_{m,n} \Phi_{m-i,n-j}(t) (e_m \otimes e_n) \]

\[ \mapsto \frac{1}{pq} \sum_{m,n} \Phi_{m,n}(t) |e_{m+i} \otimes e_{n+j}\rangle \langle e_{m+i'} \otimes e_{n+j'}|. \]

where

\[ \Phi_{m,n}(t) = \sum_{k,l} \omega_p^{mk} \omega_q^{nl} e^{t \lambda_{kl}}, \quad \lambda_{kl} = \sum_{i,j} \alpha(i, j) \omega_p^{ik} \omega_q^{jl}. \]

We recall that in this case \(\alpha(0, 0) = -1\) and \(\sum_{(i, j) \neq (0, 0)} \alpha(i, j) = 1\). Moreover, the functions \(\Phi_{m,n}(t)\) are real-valued, since \(Q\) and hence \(e^{tQ}\) are real matrices.

(ii)

\[ \Omega_t = \frac{1}{pq} \sum_{m,n} \Phi_{m,n}(t) |u_{mn}\rangle\langle u_{mn}|, \]

where \(u_{mn} = \sum_{ij} \rho_{ij}^t (e_i \otimes e_j) \otimes (e_{m+i} \otimes e_{n+j}).\)

Proof. Item (i) is an immediate consequence of Theorem 7 and Corollary 8. Now a direct computation using (i) shows that

\[ \Omega_t = \frac{1}{pq} \sum_{m,n} \Phi_{m,n}(t) \rho_{ij}^t (e_i \otimes e_j) \otimes (e_{m+i} \otimes e_{n+j}) \langle \sum_{r,s} \rho_{rs}^t e_r \otimes e_s \otimes e_{m+r} \otimes e_{n+s} \] (15)

This finishes the proof. □

The subspaces \(B_{kl}\) are invariant also for the reverse semigroup \(\tilde{T}_t\). Moreover, similar computations yield the following.

Lemma 17  For the \(\rho\)-adjoint (reverse) semigroup we have:
\( T_{\varepsilon t}(\langle e_i \otimes e_j | e_i' \otimes e_j' \rangle) \mapsto (e_i \otimes e_j) e^{tQ^*} = \frac{1}{pq} \sum_{m,n} \tilde{\Phi}_{m-i,n-j}(t)(e_m \otimes e_n) \)

\( \mapsto \frac{1}{pq} \sum_{m,n} \tilde{\Phi}_{m,n}(t)(e_{m+i} \otimes e_{n+j})(e_{m+i'} \otimes e_{n+j'}) \),

where \( Q^* \) is the transpose of \( Q \) and \( \tilde{\Phi}_{m,n} = \Phi_{p-m,q-n} \).

\( \tilde{\Omega}_t = \frac{1}{pq} \sum_{m,n} \Phi_{p-m,q-n}(t)|u_{mn}\rangle\langle u_{mn}|, \quad (17) \)

**Theorem 18** Let \( L^* \) be a circulant GKSL generator of the form \( (11) \), then the Quantum Entropy Production Rate of the corresponding qms is given by

\[ e_p(T^*, \rho) = \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}_p \times \mathbb{Z}_q} \left( \alpha(m,n) - \alpha(p-m, q-n) \right) \log \frac{\alpha(m,n)}{\alpha(p-m, q-n)}. \]

**Proof.** From the above lemmata it follows that the relative entropy has the explicit expression,

\[ S(\tilde{\Omega}_t, \Omega_t) = \frac{1}{pq} \sum_{m,n} \Phi_{m,n}(t) \log \frac{\Phi_{m,n}(t)}{\Phi_{p-m,q-n}(t)} \]

\[ = \frac{1}{2} \sum_{m,n} \left( \Phi_{m,n}(t) - \Phi_{p-m,q-n}(t) \right) \log \frac{\Phi_{m,n}(t)}{\Phi_{p-m,q-n}(t)}. \]

For the Quantum Entropy Production Rate we have,

\[ e_p(T^*, \rho) = \lim_{t \to 0^+} \frac{S(\tilde{\Omega}_t, \Omega_t)}{t} \]

\[ = \frac{1}{2} \sum_{m,n} \left( \lim_{t \to 0^+} \frac{\Phi_{m,n}(t) - \Phi_{p-m,q-n}(t)}{t} \right) \lim_{t \to 0^+} \log \frac{\Phi_{m,n}(t)}{\Phi_{p-m,q-n}(t)}. \]

But a simple computation shows that for every \( m, n \),

\[ \lim_{t \to 0^+} \frac{\Phi_{m,n}(t)}{t} = \lim_{t \to 0^+} \left( \frac{e_0 \otimes e_0}{t} e^{tQ} - I \right) e^\varepsilon e_0 = \langle e_0 \otimes e_0, Q(e_m \otimes e_n) \rangle = \alpha(m,n). \]
Therefore
\[
e_p(T_\star, \rho) = \frac{1}{2} \frac{1}{pq} \sum_{m,n} \left( \alpha(m, n) - \alpha(p - m, q - n) \right) \log \frac{\alpha(m, n)}{\alpha(p - m, q - n)}.
\]

This finishes the proof. \(\square\)

8 Comparison to Classical Entropy Production Rate

The Quantum Entropy Production Rate (1) aims at generalizing the classical one, hence it is natural to expect that some relation can be found between them. In this section we compute explicitly the (classical) Entropy Production Rate for the restriction of Circulant Quantum Markov Semigroups to the diagonal commutative sub-algebra, namely \(B_{00}\), and show it actually coincides with its quantum counterpart.

According to Qian et al. [12], the Classical Entropy Production Rate of an irreducible Markov chain with intensity matrix \(Q = (q_{ij})_{i,j \in S}\) and stationary measure \(\pi = (\pi_i)_{i \in S}\), over a finite state space \(S\) is given by

\[
e_p = \frac{1}{2} \sum_{i,j \in S} (\pi_i q_{ij} - \pi_j q_{ji}) \log \frac{\pi_i q_{ij}}{\pi_j q_{ji}}.
\]

By Theorem 13, the restriction of \(\mathcal{L}_\star\) to \(B_{00}\) reduces to the action of the block circulant matrix \(Q = \text{circ}(Q_0, Q_1, \ldots, Q_{p-1})\), with circulant blocks of the form (9) and \(\alpha(0,0) = -1\). In terms of the distribution \(\alpha\), each matrix element of \(Q\) is given by

\[q_{ij} = \alpha((l - k)p, (r_j - r_i)q),\]

where for every pair \(0 \leq i, j \leq pq - 1\) we write \(i = qk + r_i, j = ql + r_j, 0 \leq k, l \leq p - 1, 0 \leq r_i, r_j \leq q - 1\), and for every \(-(s - 1) \leq x \leq s - 1\), \(s = p, q\), we define

\[(x)_s = \begin{cases} x & \text{if } x \geq 0 \\ s + x & \text{if } x < 0. \end{cases}\]

Clearly the relation \((-x)_s = s - (x)_s\) holds true.

**Corollary 19** The Quantum Entropy Production Rate of a Circulant qms equals the Classical Entropy Production Rate of its diagonal-restricted Markov chain, i.e.,

\[e_p(T_\star, \rho) = e_p.\]

**Proof.** An application of the above formula (19), re-ordering the sum according with the order of blocks and the change of variables \(m = (l - k)_p, n = (r_j - r_i)_q\) yields,
\[ e_p = \frac{1}{2pq} \sum_{k,l=0}^{p-1} \sum_{r_i,r_j=0}^{q-1} \left( \alpha((l-k)p, (r_j-r_i)q) - \alpha((k-l)p, (r_i-r_j)q) \right) \times \log \frac{\alpha((l-k)p, (r_j-r_i)q)}{\alpha((k-l)p, (r_i-r_j)q)} \]
\[ = \frac{1}{2pq} \sum_{(m,n) \in \mathbb{Z}_p \times \mathbb{Z}_q} \left( \alpha(m,n) - \alpha(p-m, q-n) \right) \log \frac{\alpha(m,n)}{\alpha(p-m, q-n)} \]
\[ = e_p(T_\ast, \rho). \]

This proves the corollary. \( \square \)

9 \quad \textbf{QEPR with respect to other invariant states}

To close the paper, in this section we compute the QEPR in any invariant state of the semigroup \( T_\ast \).

**Proposition 20** Every invariant state of \( L_\ast \) has the form
\[ \rho = \frac{1}{pq} \mathbb{1}_p \otimes \mathbb{1}_q + \sum_{ij} \rho_{ij} J_p^i \otimes J_q^j, \tag{20} \]
where \( \rho_{ij} \) are complex numbers constrained by the positiveness of \( \rho \).

**Proof.** We decompose \( \rho \) into its mutually orthogonal components in the subspaces \( B_{kl} \), namely \( \rho = \sum_{kl} \hat{\rho}_{kl} \). Clearly \( L_\ast(\rho) = 0 \) if and only if \( L(\hat{\rho}_{kl}) = 0 \) for every \( (k, l) \in \mathbb{Z}_p \times \mathbb{Z}_q \). As a consequence of Theorem 13 using the isomorphism defined there, each of the above conditions becomes a linear system of equations of the form \( \hat{\rho}_{kl} Q = 0 \), where \( Q \) is the same circulant matrix for all systems. Any solution to these systems is a multiple of the identity vector, which yields the solution (20). Although every choice of complex constants \( \rho_{kl} \) give a solution of \( L_\ast(\rho) = 0 \), not all of them give back a state \( \rho \). In fact, \( \rho_{00} = \frac{1}{pq} \) so that \( tr\rho = 1 \) while the remaining \( \rho_{kl} \)s are constrained by the positiveness of \( \rho \). Conversely, if \( \rho \) has the form (20) then \( L_\ast(\rho) = \rho \left( \sum_{ij \neq 0} \alpha(p-i, q-j) \mathbb{1}_p \otimes \mathbb{1}_q - \mathbb{1}_p \otimes \mathbb{1}_q \right) = 0. \) \( \square \)

By Lemma 6 any invariant state \( \rho \) can be diagonalized by the discrete Fourier Transform, indeed,
\[ \rho = \sum_{lk} \tilde{\rho}_{kl} | \hat{e}_l \otimes \hat{e}_k \rangle \langle \hat{e}_l \otimes \hat{e}_k |, \]
where \( \tilde{\rho}_{lk} = \frac{1}{pq} + \sum_{ij} \rho_{ij} \omega_{p}^{i} \omega_{q}^{j} \) and \( \tilde{e}_{l} = F_{p} e_{l}, \tilde{e}_{k} = F_{q} e_{k} \).

In the next computations it is understood that sums over the first coordinate of the tensor product go from 0 to \( p - 1 \) and sums over the second coordinate go from 0 to \( q - 1 \). We use the results and notations in Lemma 16.

Let us compute the state associated with \( T_{\ast} \) using the basis of \( \rho, \{ \tilde{e}_{i} \otimes \tilde{e}_{j} \}_{(i,j) \in \mathbb{Z}_{p} \times \mathbb{Z}_{q}} \),

\[
\Omega_{t} = \sum_{ii'jj'} |\tilde{e}_{i} \otimes \tilde{e}_{i'}\rangle \langle \tilde{e}_{j} \otimes \tilde{e}_{j'}| \otimes \mathcal{T}_{st}(\rho_{\frac{t}{2}}|\tilde{e}_{i} \otimes \tilde{e}_{i'}\rangle \langle \tilde{e}_{j} \otimes \tilde{e}_{j'}| \rho_{\frac{t}{2}})
\]

\[
= \frac{1}{pq} \sum_{ii'jj'} \sum_{mm''rr'} \rho_{ii'}^{\frac{1}{2}} \rho_{jj'}^{\frac{1}{2}} \omega_{p}^{i} \omega_{q}^{j} \omega_{p}^{i'} \omega_{q}^{j'} |e_{n} \otimes e_{n'}\rangle \langle e_{r} \otimes e_{r'}| \otimes \mathcal{T}_{st}(\rho_{mm''rr''} |\tilde{e}_{i} \otimes \tilde{e}_{i'}\rangle \langle \tilde{e}_{j} \otimes \tilde{e}_{j'}|)
\]

\[
= \frac{1}{(pq)^{2}} \sum_{ii'jj'} \sum_{mm''rr''} \rho_{ii'}^{\frac{1}{2}} \rho_{jj'}^{\frac{1}{2}} \omega_{p}^{(n+N)n'} \omega_{q}^{(r+R)r'} |e_{n} \otimes e_{n'}\rangle \langle e_{r} \otimes e_{r'}| \otimes \mathcal{T}_{st}(|e_{N} \otimes e_{N'}\rangle \langle e_{R} \otimes e_{R'}|)
\]

where \( u_{mm'} = \frac{1}{\sqrt{pq}} \sum_{ii'jj'} \beta(l, L, l', L') |e_{l} \otimes e_{l'}\rangle \otimes (e_{L+m} \otimes e_{L'+m'}) \), and \( \beta(l, L, l', L') = \frac{1}{\sqrt{pq}} \sum_{ii'} \tilde{\rho}_{ii'}^{\frac{1}{2}} \omega_{p}^{(l+L)l'} \omega_{q}^{(L'+L')l'} \).

Direct computations show that the \( \rho \)-adjoint semigroup, with respect to any \( \rho \) of the form (20), coincide with \( \tilde{T}_{\ast} \) given by Lemma 17. In a similar way we get

\[
\tilde{\Omega}_{t} = \frac{1}{pq} \sum_{mm'} \Phi_{p-m,q-m'}(t) |u_{mm'}\rangle \langle u_{mm'}|.
\]

It follows that the Quantum Entropy Production Rate in any invariant state \( \rho \) of the form (20) coincides with the one given by Theorem 18.

Theorem 21 Let \( T_{\ast} \) a circulant qms with GKSL generator \( L_{\ast} \) of the form (11), then the following are equivalent:

(i) \( T_{\ast} \) satisfies a quantum detailed balance condition with respect to any invariant state \( \rho \) of the form (20).
(ii) $\alpha(m, n) = \alpha(p - m, q - n)$ for all $(m, n) \in \mathbb{Z}_p \times \mathbb{Z}_q$.

(iii) the Quantum Entropy Production Rate of $T_*$ with respect to any stationary state $\rho$ of the form (20) equals zero, i.e., $e_p(T_*, \rho) = 0$.

Proof. The equivalence of (i) and (ii) follows from Corollary 2 in Ref. [1], see (12). And the equivalence of (ii) with (iii) follows from Theorem 18. □

Remark 22

(i) Theorems 18 and 21 have a direct generalization to the case of any finite number of cycles (or cyclic factors in the abelian group $G$).

(ii) We remark that in the case of a separable probability distribution $\alpha(i, j) = \alpha_p(i)\alpha_q(j)$, a direct computation using Lemmata 16, 17 shows that the states $\Omega_t$, $\tilde{\Omega}_t$ are separable. Indeed, $\Omega_t = \Omega_p(t) \otimes \Omega_q(t)$, with

$$\Omega_s(t) = \frac{1}{s} \sum_i \Phi_s(i, t)|u_s(i)\rangle\langle u_s(i)|,$$

where $\Phi_s(i, t) = \sum_j \omega^{ij}e^{\lambda_j(\alpha_s)}$, $\lambda_j(\alpha_s) = \sum_l \alpha_s(i)\overline{\omega}^{lj}$, and $u_s(k) = \frac{1}{\sqrt{s}} \sum_n |e_n\rangle\langle e_{n+k}|$, $s = p, q$.

Acknowledgement

The financial support from CONACYT-Mexico and Ministero degli Affari Esteri-Italy, through the joint research project “Dinámica Estocástica con Aplicaciones en Física y Finanzas”, is gratefully acknowledged.

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