STANDARD EMBEDDINGS OF SMOOTH SCHUBERT VARIETIES IN RATIONAL HOMOGENEOUS MANIFOLDS OF PICARD NUMBER 1

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Abstract. Smooth Schubert varieties in rational homogeneous manifolds of Picard number 1 are horospherical varieties. We characterize standard embeddings of smooth Schubert varieties in rational homogeneous manifolds of Picard number 1 by means of varieties of minimal rational tangents. In particular, we mainly consider nonhomogeneous smooth Schubert varieties in symplectic Grassmannians and in the 20-dimensional $F_4$-homogeneous manifold associated to a short simple root.

1. Introduction

A rational homogeneous manifold is a homogeneous space $G/P$ for a complex simple Lie group $G$ and a parabolic subgroup $P \subset G$. Under the action of a Borel subgroup $B$ of $G$, the closure of a $B$-orbit in $G/P$ is called a Schubert variety of $G/P$. For details about the parabolic subgroups and the Schubert varieties of $G/P$, see Springer [28].

Most Schubert varieties are singular, and smooth Schubert varieties have been classified by using combinatorial and geometric methods (for the combinatorial smoothness criterion, see Billey-Postnikov [1]). Since conjugacy classes of parabolic subgroups of a simple Lie group are in one-to-one correspondence with subsets of the set of simple roots (equivalently, nodes of the corresponding Dynkin diagram), the Dynkin diagrams with a marked node correspond to rational homogeneous manifolds of Picard number 1. A marked subdiagram of the marked Dynkin diagram of $G/P$ defines a homogeneous submanifold $G_0/P_0$ of $G/P$, the $G_0$-orbit of the base point $eP \in G/P$, then it is a smooth Schubert variety (see Section 2 of Hong-Mok [9]). Lakshmibai-Weyman [20] and Brion-Polo [2] showed that when $G/P$ is a Hermitian symmetric space of compact type, any smooth Schubert variety in $G/P$ is a homogeneous submanifold associated to a subdiagram of the marked Dynkin diagram of $G/P$. More generally, when $G/P$ is associated to a long simple root, all smooth Schubert varieties are homogeneous submanifolds associated to subdiagrams of the marked Dynkin diagram (Proposition 3.7 of Hong-Mok [9]).
On the other hand, when $G/P$ is associated to a short simple root, there may exist a smooth Schubert variety which is not homogeneous. Recently, Hong [4] and Hong-Kwon [5] have classified all smooth Schubert varieties in this case.

A smooth Schubert variety $Z$ of $G/P$ is canonically embedded in $G/P$ by an equivariant embedding induced from the inclusion $B \subset G$. By a standard embedding of $Z$ into $G/P$, we will mean the composite of the canonical equivariant embedding and an element of the automorphism group of $G/P$. When $G/P$ is associated to a simple root and a homogeneous submanifold $G_0/P_0$ is not linear, we have a characterization of standard embeddings of $G_0/P_0$ into $G/P$ by means of varieties of minimal rational tangents as follows.

**Theorem 1.1** (Theorem 1.2 of Hong-Mok [8], Theorem 1.2 of Hong-Park [10]).
Let $X$ be a rational homogeneous manifold associated to a simple root and let $Z$ be a nonlinear rational homogeneous manifold associated to a subdiagram of the marked Dynkin diagram of $X$. If $f$ is a holomorphic embedding from a connected open subset $U$ of $Z$ into $X$ which respects varieties of minimal rational tangents for a general point $z \in U$, then $f$ extends to a standard embedding of $Z$ into $X$.

Given a uniruled projective variety $X$ equipped with a minimal rational component $\mathcal{K}$, the variety $\mathcal{C}_z(X) \subset \mathbb{P}(T_zX)$ of minimal rational tangents (VMRT) at a general point $x \in X$ is defined by the closure of the space of the tangent vectors of minimal rational curves belonging to $\mathcal{K}$ passing through $x$. If $X$ is a rational homogeneous manifold $G/P$ associated to a simple root, then there is a canonical choice of a minimal rational component, namely, the irreducible family of lines $\mathbb{P}^1$ which are contained in $X$ after we embed $X$ into $\mathbb{P}^N$ by the ample generator of the Picard group of $X$. Similarly, there is a canonical choice of a minimal rational component for a smooth Schubert variety. For a general reference on the theory of rational curves and varieties of minimal rational tangents, see Kollár [19], Hwang-Mok [14], Hwang [12] and Mok [24].

For a holomorphic embedding $f: U \rightarrow X$ from an open subset $U$ of a uniruled projective variety $Z$ with a minimal rational component $\mathcal{H}$, we say that $f$ respects varieties of minimal rational tangents if

$$df(\mathcal{C}_z(Z)) = df(\mathbb{P}(T_zZ)) \cap \mathcal{C}_{f(z)}(X)$$

for a general point $z \in U$, where $\mathcal{C}_z(Z)$ is the variety of minimal rational tangents of $Z$ at $z \in Z$ associated to $\mathcal{H}$ and $\mathcal{C}_{f(z)}(X)$ is the variety of minimal rational tangents of $X$ at $f(z)$ associated to $\mathcal{K}$.

If $Z$ is linear, then the condition that $f: U \rightarrow X$ respects varieties of minimal rational tangent is equivalent to the condition that $df(\mathbb{P}(T_zZ))$ should be contained in $\mathcal{C}_{f(z)}(X)$ for any $z \in U$. In other words, for each $f(z) \in f(U)$ there is a linear space in $X$ which is tangent to $f(U)$ at $f(z)$. In general, when $Z$ is linear, there is a non-standard embedding from an open subset $U$ of $Z$ into $X$ so that Theorem 1.1 does not hold. For example, there is an embedding of $Z = \mathbb{P}^1$ into $X$ with $df(\mathbb{P}(T_zZ)) \subset \mathcal{C}_{f(z)}(X)$ for any $z \in \mathbb{P}^1$ whose image is not contained in any linear space in $X$ (see Section 6 of Choe-Hong [3]). However, when $Z$ is a maximal linear space, Theorem 1.3 of Hong-Park [10] gives a related result with some exceptions involving counterexamples constructed by Choe-Hong [3].

In this paper, we will prove a generalization of Theorem 1.1 in the case that $Z$ is a smooth Schubert variety of $G/P$ by the arguments developed in Hong-Mok [8] and Hong-Park [10]. The fundamental tools for the proof are the non-equidimensional
Cartan-Fubini type extension theorem (Proposition 2.5) and the parallel transport of VMRTs along minimal rational curves (see Section 2.2).

**Theorem 1.2.** Let $X$ be a rational homogeneous manifold associated to a simple root and let $Z$ be a nonlinear smooth Schubert variety in $X$. If $f$ is a holomorphic embedding from a connected open subset $U$ of $Z$ into $X$ which respects varieties of minimal rational tangents for a general point $z \in U$, then $f$ extends to a standard embedding of $Z$ into $X$.

We denote the rational homogeneous manifold $G/P$ associated to a simple root $\alpha_i$ by $(G, \alpha_i)$. Among rational homogeneous manifolds associated to a short simple root, since $(B_4, \alpha_2) \cong (D_{4\ell+1}, \alpha_\ell)$, $(C_\ell, \alpha_1) \cong \mathbb{F}^{2\ell-1}$ and $(G_2, \alpha_1) \cong (B_4, \alpha_1) \cong \mathbb{Q}^5$ as complex manifolds, three cases can be regarded as rational homogeneous manifolds associated to a long simple root. Moreover, any smooth Schubert variety in the 15-dimensional $F_4$-homogeneous manifold $(F_4, \alpha_4)$ associated to the short simple root $\alpha_4$ is a linear space by Theorem 1.3 of Hong-Kwon [6]. Thus, it suffices to consider nonhomogeneous smooth Schubert varieties in the symplectic Grassmannians (Section 3) and in the 20-dimensional $F_4$-homogeneous manifold $(F_4, \alpha_3)$ associated to the short simple root $\alpha_3$ (Section 4). Because these Schubert varieties are smooth nonhomogeneous horospherical varieties of Picard number 1, we review notions and facts about horospherical varieties in Section 2.1.

2. Horospherical varieties and Cartan-Fubini extension

2.1. Spherical and horospherical varieties. For a complex reductive algebraic group $G$, a complex algebraic variety with an action of $G$ is called a $G$-variety. A $G$-spherical variety is a normal $G$-variety having an open orbit under the action of a Borel subgroup $B$ of $G$. A normal $G$-variety is horospherical if $G$ acts with an open orbit $G/H$ isomorphic to a torus bundle over a rational homogeneous manifold, or equivalently, if the isotropy subgroup $H$ of a general point contains the unipotent radical of a Borel subgroup $B$. The dimension of the torus fiber is called the rank of a horospherical variety.

The Bruhat decomposition of $G$ implies that horospherical varieties are spherical (see Section 5.3 of Perrin [27]). Toric varieties and rational homogeneous manifolds are the well-known examples of horospherical varieties. Furthermore, we know that all smooth Schubert varieties in rational homogeneous manifolds of Picard number 1 are horospherical varieties from the classification result of Hong-Mok [9], Hong [1] and Hong-Kwon [6].

Let \{$\alpha_1, \ldots, \alpha_n$\} be a system of simple roots of $G$ following the standard numbering (e.g. Humphreys [11]) and $P(\alpha_i)$ denote the maximal parabolic subgroup associated to a simple root $\alpha_i$. For the corresponding system \{$\omega_1, \ldots, \omega_n$\} of fundamental weights, $V(\omega_i)$ denotes the irreducible $G$-representation space with the $i$-th fundamental weight $\omega_i$ as a highest weight. When we take a highest weight vector $v_i$ of $V(\omega_i)$, the $G$-orbit of $[v_i]$ in $\mathbb{P}(V(\omega_i))$ is closed and isomorphic to the rational homogeneous manifold $G/P(\alpha_i)$ which is denoted by $(G, \alpha_i)$.

If $v_i$ and $v_j$ are highest weight vectors of $V(\omega_i)$ and $V(\omega_j)$ respectively, we will consider the closure of the $G$-orbit of the point $[v_i + v_j]$ in $\mathbb{P}(V(\omega_i) \oplus V(\omega_j))$. For any $i \neq j$, the open orbit $G.[v_i + v_j]$ is isomorphic to a $\mathbb{C}^*$-bundle over a rational homogeneous manifold $G/(P(\alpha_i) \cap P(\alpha_j))$. According to Proposition 2.1 of Hong [5], since the closure of $G.[v_i + v_j]$ in $\mathbb{P}(V(\omega_i) \oplus V(\omega_j))$ is a normal variety, $G.[v_i + v_j]$...
is a horospherical $G$-variety and we denote it by $(G, \alpha_i, \alpha_j)$. The smooth projective horospherical varieties of Picard number 1 are classified by Pasquier [26] using the fact that any nonlinear smooth horospherical variety of Picard number 1 is of the form $(G, \alpha_i, \alpha_j)$.

**Proposition 2.1** (Theorem 0.1 of Pasquier [26]). Let $G$ be a connected reductive algebraic group. A smooth projective horospherical $G$-variety $X$ of Picard number 1 is either homogeneous or horospherical of rank 1. In the nonhomogeneous case, its automorphism group $\text{Aut}(X)$ is a connected non-reductive linear algebraic group acting with exactly two orbits $X_0$ and $Z$; moreover, $X$ is uniquely determined by its two closed $G$-orbits $Y \subset X_0$ and $Z$, isomorphic to rational homogeneous manifolds $G/P_Y$ and $G/P_Z$, respectively, where $(G, P_Y, P_Z)$ is one of the triples in the following list:

1. $(B_n, P(\alpha_{n-1}), P(\alpha_n))$ with $n \geq 3$;
2. $(B_3, P(\alpha_1), P(\alpha_3))$;
3. $(C_n, P(\alpha_k), P(\alpha_{k-1}))$ with $n \geq k \geq 2$;
4. $(F_4, P(\alpha_2), P(\alpha_3))$;
5. $(G_2, P(\alpha_2), P(\alpha_1))$.

**Proposition 2.2** (Theorem 1.11 of Pasquier [26]). In the above cases (1) – (5), the automorphism group of $X$ is isomorphic to $(\text{SO}(2n+1) \times \mathbb{C}^*) \ltimes V(\omega_2)$, $(\text{SO}(7) \times \mathbb{C}^*) \ltimes V(\omega_3)$, $(\text{Sp}(2n) \times \mathbb{C}^*)/\{\pm 1\} \ltimes V(\omega_1)$, $(F_4 \times \mathbb{C}^*) \ltimes V(\omega_4)$ and $(G_2 \times \mathbb{C}^*) \ltimes V(\omega_1)$, respectively.

Recently, Hong [5] showed that a smooth horospherical variety $X$ of Picard number 1 can be embedded as a linear section into a rational homogeneous manifold of Picard number 1 except when $X$ is $(B_n, \alpha_{n-1}, \alpha_n)$ for $n \geq 7$. For a description of their tangent space based on weights and roots, see Proposition 2.6 of Kim [18].

**Example 2.3** (Odd symplectic Grassmannian $(C_n, \alpha_k, \alpha_{k-1})$). Let $V$ be a complex vector space endowed with a skew-symmetric bilinear form $\omega$ of maximal rank. We denote the variety of all $k$-dimensional isotropic subspaces in $V$ by $\text{Gr}_\omega(k, V) = \{W \subset V : \dim W = k, \omega|_W = 0\}$. When $\dim V$ is even, say, $2n$, the form $\omega$ is a nondegenerate symplectic form and this variety $\text{Gr}_\omega(k, 2n)$ is the usual symplectic Grassmannian, which is homogeneous under the action of the symplectic group $\text{Sp}(2n)$. But when $\dim V$ is odd, say, $2n + 1$, the skew-form $\omega$ has a one-dimensional kernel. The variety $\text{Gr}_\omega(k, 2n + 1)$, called the odd symplectic Grassmannian, is not homogeneous and has two orbits under the action of its automorphism group if $2 \leq k \leq n$ (cf. Mihai [22] and Proposition 1.12 of Pasquier [26]). If $k = 1$, then the isotropic condition holds trivially so that $\text{Gr}_\omega(1, V)$ is just the linear space $\mathbb{P}^{\dim V - 1}$. Next, for $k = n + 1$ the odd symplectic Grassmannian $\text{Gr}_\omega(n + 1, 2n + 1)$ is isomorphic to the symplectic Grassmannian $\text{Gr}_\omega(n, 2n)$ because any $(n + 1)$-dimensional isotropic subspace must contain the one-dimensional kernel of $\omega$. In what follows, we will assume that $2 \leq k \leq n$ when considering the odd symplectic Grassmannians.

Let $S$ be an odd symplectic Grassmannian $\text{Gr}_\omega(k, 2n + 1)$ for $2 \leq k \leq n$. Then $S$ is a smooth Fano manifold of Picard number 1 and the automorphism group $\text{Aut}(S)$ of $S$ is isomorphic to the semi-direct product $((\text{Sp}(2n) \times \mathbb{C}^*)/\{\pm 1\}) \ltimes \mathbb{C}^{2n}$. We know that $S$ has two orbits under its automorphism group. The closed orbit $\{W \in \text{Gr}_\omega(k, 2n + 1) : \text{Ker} \omega \subset W\}$ is isomorphic to the symplectic Grassmannian $\text{Gr}_\omega(k - 1, 2n)$ and the open orbit $\{W \in \text{Gr}_\omega(k, 2n + 1) : \text{Ker} \omega \not\subset W\}$ is isomorphic
to the dual tautological sub-bundle on the symplectic Grassmannian $\text{Gr}_n(k, 2n)$. In fact, choosing a supplementary subspace $V' \subset V$ so that $V = \text{Ker } \omega \oplus V'$, any $W \in \text{Gr}_n(k, V) = \text{Gr}_n(k, 2n + 1)$ containing $\text{Ker } \omega$ corresponds a point of $\text{Gr}_n(k - 1, V') = \text{Gr}_n(k - 1, 2n)$. And the projection coming from the above decomposition gives a map from the open orbit onto $\text{Gr}_n(k, 2n)$ of which the fiber at a point $E \in \text{Gr}_n(k, 2n)$ is $E^*$ (for details, see Proposition 4.3 of Mihai [22]). Consequently, the odd symplectic Grassmannian $\text{Gr}_n(k, 2n + 1)$ has three orbits under the semisimple part $\text{Sp}(2n)$ of its automorphism group. In particular, the $\text{Sp}(2n)$-closed orbit lying in the open orbit is isomorphic to a symplectic Grassmannian $\text{Gr}_n(k, 2n)$.

2.2. Second fundamental form and Cartan-Fubini extension. Let $V$ be a finite-dimensional vector space and let $A \subset \text{P}(V)$ be a complex-analytic subvariety. Denote by $\tilde{A} \subset V \setminus \{0\}$ the affine cone of $A$, i.e., the pre-image $\pi^{-1}(A)$ of the canonical projection $\pi: V \setminus \{0\} \to \text{P}(V)$. For a smooth point $\eta \in \tilde{A}$, the second fundamental form

$$\sigma_\eta: T_\eta \tilde{A} \times T_\eta \tilde{A} \to V/T_\eta \tilde{A}$$

of $\tilde{A} \subset V$ at $\eta \in \tilde{A}$ is defined by $\sigma_\eta(\xi, \zeta) = \nabla \xi \tilde{\zeta}$ mod $T_\eta \tilde{A}$ for any $\xi, \zeta \in T_\eta \tilde{A}$, where $\tilde{\zeta}$ is a local vector field with $\tilde{\zeta}(\eta) = \zeta$, and $\nabla$ is the Euclidean flat connection on the Euclidean space $V$. Another definition is given by the differential of the Gauss map

$$\Gamma: \tilde{A} \to \text{Gr}(d, V), \quad \beta \in \tilde{A} \mapsto [T_\beta \tilde{A}] \in \text{Gr}(d, V)$$

at $\eta$, where $d = \dim A + 1$. The differential of the Gauss map $\Gamma$ at $\eta \in \tilde{A}$ is a linear map

$$d\Gamma_\eta: T_\eta \tilde{A} \to T_{[T_\beta \tilde{A}]} \text{Gr}(d, V) \cong \text{Hom}(T_\eta \tilde{A}, V/T_\eta \tilde{A}).$$

The canonical isomorphism $\chi$ between the tangent space $T_{[W]} \text{Gr}(d, V)$ of a Grassmannian and $\text{Hom}(W, V/W)$ is given by $\xi \mapsto \chi_\xi$ with $\chi_\xi(w) := \rho_0(0) + W$, where $\rho: D \to V$ is a moving vector field with $\rho(0) = w$ along a holomorphic path $\xi$ from a connected open subset $D \subset \mathbb{C}$ into $\text{Gr}(d, V)$ such that $\xi(0) = [W]$ and $\xi'(0) = \xi$. Here, $\rho: D \to V$ is called a moving vector field along a holomorphic path $\xi$ if $\rho(t) \in \xi(t)$ for every $t \in D$. If we use this canonical isomorphism, the differential of the Gauss map is described as follows.

For $\xi, \zeta \in T_\eta \tilde{A}$ we choose the following gadgets:

- a holomorphic path $\alpha: D \to \tilde{A}$ with $\alpha(0) = \eta$ and $\alpha'(0) = \xi$;
- a vector field $\rho: D \to V$ along $\alpha$ with $\rho(0) = \zeta$, i.e., $\rho(t) \in T_{\alpha(t)} \tilde{A}$ for every $t \in D$.

If we set $\tilde{\xi} := \Gamma \circ \alpha: D \to \text{Gr}(d, V)$, then $\tilde{\xi}(0) = [T_\eta \tilde{A}]$ and $\tilde{\xi}'(0) = d\Gamma_\eta(\xi) \in \text{Hom}(T_\eta \tilde{A}, V/T_\eta \tilde{A})$ under the canonical isomorphism. Since $\rho$ is a moving vector field along $\tilde{\xi}$,

$$d\Gamma_\eta(\xi)(\zeta) = \tilde{\xi}'(0)(\zeta) = \rho'(0) + T_\eta \tilde{A} \in V/T_\eta \tilde{A}.$$

Restating the above construction, we have obtained a symmetric bilinear map, the second fundamental form, $\sigma_\eta(\alpha'(0), \rho(0)) = \rho'(0) + T_\eta \tilde{A}$ for every holomorphic path $\alpha$ in $\tilde{A}$ with $\alpha(0) = \eta$ and every moving vector field $\rho$ along $\alpha$.

For a subspace $E$ of $T_\eta \tilde{A}$ we define

$$\text{Ker } \sigma_\eta(\cdot, E) := \{\zeta \in T_\eta \tilde{A} : \sigma_\eta(\zeta, \xi) = 0, \forall \xi \in E\}.$$
From the fact that \( \tilde{A} \) is a cone with the vertex at 0, it follows that \( \sigma_\eta(\eta, \xi) = 0 \) for any \( \xi \in T_\eta \tilde{A} \). In particular, \( C_\eta \) is contained in \( \text{Ker} \sigma_\eta(\cdot, E) \) for any subspace \( E \) of \( T_\eta \tilde{A} \). At \( [\eta] = \pi(\eta) \in A \) the tangent space \( T_{[\eta]} A \) is given by \( (T_\eta \tilde{A} / C_\eta) \otimes (\mathbb{C}_\eta)^* \). Thus the second fundamental form \( \sigma_\eta : T_\eta \tilde{A} \times T_\eta \tilde{A} \to V/T_\eta \tilde{A} \) of \( \tilde{A} \) at \( \eta \) induces the projective second fundamental form \( \bar{\sigma}_{[\eta]} : T_{[\eta]} A \times T_{[\eta]} A \to T_{[\eta]} \mathbb{P}(V)/T_{[\eta]} A \) of \( A \) at \([\eta] \). From now on we will use the notation \( \sigma_{[\eta]} \) instead of \( \bar{\sigma}_{[\eta]} \) for the sake of convenience. For a subspace \( \mathcal{E} \) of \( T_{[\eta]} A \) we define \( \text{Ker} \sigma_{[\eta]}(\cdot, \mathcal{E}) \) by \( \{ \vec{\xi} \in T_{[\eta]} A : \sigma_{[\eta]}(\vec{\xi}, \xi) = 0, \forall \xi \in \mathcal{E} \} \).

**Definition 2.4.** Let \((X, K)\) and \((Z, H)\) be two polarized uniruled projective manifolds equipped with a minimal rational component. For a holomorphic immersion \( f : U \to X \) from an open subset \( U \) of \( Z \) in the analytic topology, we say that \( f \) is **nondegenerate with respect to** \((K, H)\) if

1. its image \( f(U) \) is not contained in the bad locus of \( K \), which means the smallest subvariety \( B \) of \( X \) such that for all \( x \in X \setminus B \), any minimal rational curve passing through \( x \) is free and a general minimal rational curve passing through \( x \) is standard, and
2. for a general point \( z \in U \) and a general smooth point \( \alpha \in \tilde{C}_z(Z) \), \( df(\alpha) \) is a smooth point of \( \tilde{C}_z(Z) \) and \( \text{Ker} \sigma_{df(\alpha)}(\cdot, T_{df(\alpha)}(df(\tilde{C}_z(Z)))) = \mathbb{C}df(\alpha) \),

where \( \sigma_{df(\alpha)} \) denotes the second fundamental form of the affine cone \( \tilde{C}_f(z)(X) \subset T_{f(z)}X \) at \( df(\alpha) \).

Now, as the main ingredient for the proof of Theorem 1.2, we state the non-equidimensional Cartan-Fubini type extension theorem, which says the rational extension of germs of holomorphic maps respecting varieties of minimal rational tangents. For an introductory exposition on an analytic continuation along minimal rational curves and Cartan-Fubini extension, we refer to Section 2 of Mok [25].

**Proposition 2.5** (Theorem 1.1 of Hong-Mok [8]). Let \((X, K)\) and \((Z, H)\) be two uniruled projective manifolds equipped with a minimal rational component. Assume that \( Z \) is of Picard number 1 and that \( C_z(Z) \) is positive-dimensional at a general point \( z \in Z \). Let \( f : U \to X \) be a holomorphic immersion defined on a connected open subset \( U \subset Z \). If \( f \) respects varieties of minimal rational tangents and is nondegenerate with respect to \((K, H)\), then \( f \) extends to a rational map \( F : Z \to X \).

To use this result, we need to check the second fundamental form of the variety of minimal rational tangents as subvariety in the projective tangent space and to check the nondegeneracy of the pair of varieties of minimal rational tangents (Proposition 4.5 and Proposition 4.5). Then we can apply the non-equidimensional Cartan-Fubini type extension theorem and get a rational extension \( F : Z \to X \) of \( f \). Up to the action of \( \text{Aut}(X) \), \( F(x_0) = x_0 \) and \( C_{x_0}(F(Z)) = C_{x_0}(Z) \) for a fixed general point \( x_0 \in U \subset Z \). Since \( f \) sends minimal rational curves in \( Z \) to minimal rational curves in \( X \) and the tangency property of the two VMRTs of \( Z \) and \( F(Z) \) at an intersection point does imply equality of these VMRTs in the case of smooth Schubert varieties in a rational homogeneous manifold of Picard number 1, as established in the next sections, we can extend the map inductively along minimal rational curves. Consequently, \( F \) is the identity map up to the action of \( \text{Aut}(X) \).
3. Smooth Schubert varieties in symplectic Grassmannians

Let $G$ be a connected simple Lie group of type $C_\ell$ and let $X$ be a rational homogeneous manifold $G/P$ associated to a simple root $\alpha_k$ $(1 \leq k \leq \ell)$. Then $X$ is the symplectic Grassmannian $\text{Gr}_\omega(k, 2\ell)$ of isotropic $k$-subspaces in $V = \mathbb{C}^{2\ell}$ with respect to a symplectic form $\omega$ on $\mathbb{C}^{2\ell}$, where the symplectic form means a nondegenerate skew-symmetric bilinear form on $V$. Take a basis $\{e_1, \ldots, e_{2\ell}\}$ of $V$ such that $\omega(e_{\ell-i}, e_{\ell+i+1}) = -\omega(e_{\ell+i+1}, e_{\ell-i}) = 1$ for $0 \leq i \leq \ell - 1$, and all other $\omega(e_i, e_j)$ are zero. Define $F_j \subset V$ as the subspace generated by $e_1, \ldots, e_j$ for $1 \leq j \leq 2\ell$ and set $F_0 = \{0\}$. Then $F_{\ell-i} = F_{\ell+i}$ for $0 \leq i \leq \ell$. The symplectic group $G = \text{Sp}(V)$ naturally acts on $\text{Gr}_\omega(k, 2\ell)$ and the parabolic subgroup $P$ is the isotropy subgroup of $G$ at $[F_k]$. If $k = 1$, then the isotropic condition holds trivially so that $\text{Gr}_\omega(1, 2\ell)$ is just the linear space $\mathbb{P}^{2\ell-1}$. On the other hand, if $k = \ell$, a rational homogeneous manifold associated to the long simple root $\alpha_\ell$ is the Lagrangian Grassmannian $\text{Gr}_{\ell}(\ell, 2\ell)$ of which any smooth Schubert variety is a homogeneous submanifold associated to a subdiagram of the marked Dynkin diagram of $\text{Gr}_{\ell}(\ell, 2\ell)$ by Lakshmibai-Weyman [20], Brion-Polo [2] and Hong-Mok [9]. In what follows, we will assume that $1 < k < \ell$.

Fix a $k$-dimensional isotropic subspace $E \subset V$. Since we can view $X$ as a subvariety of the Grassmannian $\text{Gr}(k, V)$ of $k$-dimensional subspaces in $V$ and the tangent space of $\text{Gr}(k, V)$ at $[E]$ is naturally isomorphic to $\text{Hom}(E, V/E)$, we have

$$T_{[E]}X = \{h \in \text{Hom}(E, V/E) : \omega(h(e_1), e_2) + \omega(e_1, h(e_2)) = 0, \forall e_1, e_2 \in E\}.$$

Putting $E^\perp := \{v \in V : \omega(v, e) = 0, \forall e \in E\}$, $E^\perp$ is a subspace of dimension $2\ell - k$ containing $E$ because $E$ is an isotropic subspace. From the nondegeneracy of $\omega$, the isomorphism $V/E^\perp \cong E^\perp$ is induced by the symplectic form $\omega$. Then, under the map $\psi : E^* \otimes V/E \to E^* \otimes V/E^\perp \cong E^* \otimes E^\perp$ which is composition of projection and the isomorphism $V/E^\perp \cong E^\perp$, the tangent space of the symplectic Grassmannian $\text{Gr}_\omega(k, 2\ell)$ at $[E]$ is the inverse image $\psi^{-1}(S^2 E^*)$ of the symmetric square $S^2 E^* \subset E^* \otimes E^*$ and can be identified with

$$T_{[E]} \text{Gr}_\omega(k, 2\ell) = (E^* \otimes (E^\perp/E)) \oplus S^2 E^*.$$

Minimal rational curves of $X$ are lines of $\text{Gr}(k, 2\ell)$ lying on $X$. Thus, the variety $\mathcal{C}_{[E]}(X)$ of minimal rational tangents of $X$ at a point $[E] \in X$ is the variety of decomposable tensors in $T_{[E]}X$. From now on, we take the standard inner product on $E^*$ associated with Lie group $\text{SO}(E^*)$, which gives the correspondence $e^* \mapsto e$ between $E^*$ and $E$. If a decomposable tensor $h = e^* \otimes v$ is contained in $T_{[E]}X \subset E^* \otimes V/E$, then

$$\omega(v, e') = \omega(h(e), e') = -\omega(e, h(e'))$$

$$= -\omega(e, (e^* \otimes v)e') = -\omega(e, e^*(e')v)$$

$$= -\omega(e, v)e^*(e') \quad \text{for all } e' \in E,$$

that is, $\omega(v, \cdot)|_E \in \mathbb{C}e^*$. Conversely, if $\omega(v, \cdot)|_E \in \mathbb{C}e^*$, then $e^* \otimes v$ is contained in $T_{[E]}X$. Therefore, the affine cone of $\mathcal{C}_{[E]}(X)$ is

$$\tilde{\mathcal{C}}_{[E]}(X) = \{e^* \otimes v \in E^* \otimes (V/E) : \omega(v, \cdot)|_E \in \mathbb{C}e^*\} \setminus \{0\}.$$

By Proposition 3.2.1 of Hwang-Mok [17] or Corollary 5.5 of Landsberg-Manivel [21], the variety $\mathcal{A}$ of minimal rational tangents of $\text{Gr}_\omega(k, 2\ell)$ at a point $[E] \in$
The second fundamental form \(\sigma\) tangent space \(T\)

where \(U = E^*\) and \(Q = E^\perp / E\). Under the projection \(\mathcal{A} \to \mathbb{P}(U) = \mathbb{P}^{k-1}\) defined by \(u \otimes q + cu^2 \mapsto u\), \(\mathcal{A}\) becomes a \(\mathbb{P}^{2m}\)-bundle over \(\mathbb{P}^{k-1}\), where \(m = \ell - k\).

For integers \(a, b\) with \(0 \leq a < k < b \leq 2\ell - a\), define

\[
\text{Gr}_\omega(k, 2\ell; F_a, F_b) := \{E \in \text{Gr}_\omega(k, 2\ell) : F_a \subset E \subset F_b\},
\]

where \(F_j \subset V\) is the subspace generated by \(e_1, \cdots, e_j\). Recently, Hong [4] has classified smooth Schubert varieties in the symplectic Grassmannian \(\text{Gr}_\omega(k, 2\ell)\). From this result, all smooth Schubert varieties are of this form satisfying certain condition:

**Lemma 3.1.** Smooth Schubert varieties of the symplectic Grassmannian \(\text{Gr}_\omega(k, 2\ell)\) are of the form \(\text{Gr}_\omega(k, 2\ell; F_a, F_b)\), where one of the following holds:

1. \(0 \leq a < k\) and \((k < b \leq \ell\) or \(b = 2\ell - a\)); a homogeneous submanifold associated to a subdiagram of the marked Dynkin diagram corresponding to the symplectic Grassmannian \(\text{Gr}_\omega(k, 2\ell)\).

2. \(0 \leq a < k\) and \(b = 2\ell - a - 1\); an odd symplectic Grassmannian \((C\ell - 1, \alpha_{k-a}, \alpha_{k-a-1})\), \(a = k - 1\) and \(\ell + 1 \leq b \leq 2\ell - k\); a linear space \(\mathbb{P}^{k-1}\).

**Proof.** Proposition 3.1 and Proposition 4.7 of Hong [4].

As we have seen in Example 2.2, the odd symplectic Grassmannian \(\text{Gr}_\omega(k, 2\ell; F_a, F_{2\ell-a-1})\) is not homogeneous but a smooth Schubert variety of the symplectic Grassmannian \(\text{Gr}_\omega(k, 2\ell)\). To prove Theorem [1.2] in the case that \(X\) is the symplectic Grassmannian \(\text{Gr}_\omega(k, 2\ell)\), it suffices to consider when \(Z\) is an odd symplectic Grassmannian. In the remaining of the section, we will prove Theorem [1.2] in this case.

**Lemma 3.2.** Let \(X\) be the symplectic Grassmannian \(\text{Gr}_\omega(k, 2\ell)\) with \(1 < k < \ell\) and \(A\) be the variety of minimal rational tangents of \(X\) at a point \([E]\) \(\in X\). The tangent space \(T_\beta\) of \(\tilde{A}\) at \(\beta \in \tilde{A}\) is given by

\[
T_\beta = \{u \otimes q' + u' \otimes q + 2u \circ u' : u', q' \in Q\} \text{ if } \beta = u \otimes q + u^2,
T_\beta = \{u \otimes q' + u' \otimes q + cu^2 : u', q' \in Q, c \in \mathbb{C}\} \text{ if } \beta = u \otimes q.
\]

The second fundamental form \(\sigma: T_\beta \times T_\beta \to (T_{[E]}X)/T_\beta\) of \(\tilde{A} \subset T_{[E]}X\) at \(\beta \in \tilde{A}\) is given as follows:

1. for \(\beta = u \otimes q + u^2\),

\[
\sigma(u' \otimes q + 2u \circ u', u \otimes q') = u' \otimes q',
\sigma(u' \otimes q + 2u \circ u', u'' \otimes q' + 2u \circ u'') = 2u' \circ u'',
\sigma(u \otimes q', u \otimes q'') = 0;
\]
(II) for $\beta = u \otimes q$,

$$
\begin{align*}
\sigma(u' \otimes q, u \otimes q') &= u' \otimes q' \\
\sigma(u' \otimes q, u^2) &= 2u \circ u' \\
\sigma(u' \otimes q, u'' \otimes q) &= 0 \\
\sigma(u \otimes q', u \otimes q'') &= 0 \\
\sigma(u \otimes q', u^2) &= 0 \\
\sigma(u^2, u^2) &= 0,
\end{align*}
$$

where $u', u'' \in U$ and $q', q'' \in Q$.

Remark 3.3. The second fundamental form $\sigma$ of $\widetilde{A}$ at $\beta \in \widetilde{A}$ has its image in the quotient space $(T_{[E]}X)/T_{[\beta]}$. For simplicity, here and henceforth we will use the same notation for an element $v \in T_{[E]}X$ and its image in the quotient $(T_{[E]}X)/T_{[\beta]}$. We will use the same convention for the second fundamental forms of other subvarieties.

Proof. This result is given in Lemma 3.2 of Hong-Park [10] without details. We give the details of the proof. First, to obtain the tangent space $T_{[\beta]}\widetilde{A}$, we consider the velocity vectors of curves in the affine cone $\widetilde{A}$. Let $\{u_t\} \subset U$ be a curve with $u_0 = u$ and $\{q_t\} \subset Q$ be a curve with $q_0 = q$. The curves $u_t \otimes q + u_t^2$, $u \otimes q_t + u^2$ in the affine cone $\widetilde{A}$ pass through a point $u \otimes q + u^2$ and their velocity vectors are $u' \otimes q + 2u \circ u'$ for some $u' \in U$ and $u \otimes q'$ for some $q' \in Q$, respectively. Because $\dim \widetilde{A} = k + 2m = \dim U + \dim Q$, the tangent space $T_{[\beta]}\widetilde{A}$ at a point $\beta = u \otimes q + u^2$ is spanned by the vectors $\{u' \otimes q + 2u \circ u' : u' \in U\}$ and $\{u \otimes q' : q' \in Q\}$. Similarly, the curves $u_t \otimes q$ and $u \otimes q_t$ pass through a point $u \otimes q$ when $t = 0$ so that their velocity vectors $\{u' \otimes q : u' \in U\}$ and $\{u \otimes q' : q' \in Q\}$ lie in $T_{[\beta]}\widetilde{A}$ at a point $\beta = u \otimes q$. But these vectors do not span the whole tangent space $T_{[\beta]}\widetilde{A}$ since $\{u' \otimes q : u' \in \mathbb{C}u\} = \{u \otimes q' : q' \in \mathbb{C}q\}$. Therefore, we additionally consider a curve $u \otimes q + cu^2$ such that $c_t \in \mathbb{C}$ and $c_0 = 0$, from which we obtain the tangent vectors of the form $cu^2$ for some $c \in \mathbb{C}$.

The second fundamental form $\sigma: T_{[\beta]} \times T_{[\beta]} \rightarrow (T_{[E]}X)/T_{[\beta]}$ is given by the differential of the Gauss map $\widetilde{A} \rightarrow \text{Gr}(d, T_{[E]}X)$, $\beta \mapsto [T_{[\beta]}\widetilde{A}]$, where $d = \dim \widetilde{A}$, as explained in Section 2.2.

Let $\{u_t\} \subset U$ be a curve with $u_0 = u$ and $\{q_t\} \subset Q$ be a curve with $q_0 = q$. Then the holomorphic curves $[T_{[\beta]}]$ in $\text{Gr}(d, T_{[E]}X)$ for $\{[\beta_t]\} \subset \widetilde{A}$ such that $\beta_0 = \beta$ are as follows:

1. for $\beta_t = u_t \otimes q + u_t^2$, $T_{[\beta]} = \{u_t \otimes q' + u' \otimes q + 2u_t \circ u' : u' \in U, q' \in Q\}$;
2. for $\beta_t = u \otimes q_t + u^2$, $T_{[\beta]} = \{u \otimes q' + u' \otimes q_t + 2u \circ u' : u' \in U, q' \in Q\}$;
3. for $\beta_t = u_t \otimes q + u^2$, $T_{[\beta]} = \{u_t \otimes q' + u' \otimes q + cu^2 : u' \in U, q' \in Q, c \in \mathbb{C}\}$;
4. for $\beta_t = u \otimes q_t$, $T_{[\beta]} = \{u \otimes q' + u' \otimes q_t + cu^2 : u' \in U, q' \in Q, c \in \mathbb{C}\}$;
5. for $\beta_t = u \otimes q + cu^2$, $T_{[\beta]} = \{u \otimes q' + u' \otimes q + cu_t u' + cu^2 : u' \in U, q' \in Q, c \in \mathbb{C}\}$, where $\{c_t\} \subset \mathbb{C}$ is a curve with $c_0 = 0$.

By differentiating the curve $[T_{[\beta]}]$ in $\text{Gr}(d, T_{[E]}X)$, we can compute the second fundamental form $\sigma$ of $\widetilde{A}$. To be specific, for any tangent vectors $\xi, \zeta \in T_{[\beta]}\widetilde{A}$ we choose

- a holomorphic curve $\beta_t$ into $\widetilde{A}$ such that $\beta_0 = \beta$ and $\frac{d}{dt}|_{t=0}\beta_t = \xi$, which gives the curve $[T_{[\beta]}]$ in $\text{Gr}(d, T_{[E]}X)$,
- a vector field $\rho_t$ along the above curve $\beta_t$ such that $\rho_0 = \zeta$ and $\rho_t \in T_{[\beta_t]}$ for every $t$. 

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Then we have \( \sigma(\xi, \zeta) = \sigma(\frac{d}{dt}|_{t=0}\beta, \rho_0) = \frac{d}{dt}|_{t=0}\rho_1. \)

(Case I : \( \beta = u \otimes q + u^2 \)). (i) First, to compute \( \sigma(u' \otimes q + 2u \circ u', u \otimes q') \), take a curve \( \beta_t = u_t \otimes q + u_t^2 \) as in (1) and assume that \( u_0 = u, \frac{d}{dt}|_{t=0}u_t = u' \). Then \( \beta_0 = u \otimes q + u^2 = \beta \) and \( \frac{d}{dt}|_{t=0}\beta_t = u' \otimes q + 2u \circ u' \). Since \( u_t \otimes q' \in T_{\beta_t} \) for any \( t \), the differential \( \frac{d}{dt}|_{t=0}[T_{\beta_t}] : T_{\beta} \rightarrow T_{[\xi]}X/T_{\beta} \) maps \( u \otimes q' \in T_{\beta} \) to \( \frac{d}{dt}|_{t=0}u_t \otimes q' = u' \otimes q' \). Thus we have \( \sigma(u' \otimes q + 2u \circ u', u \otimes q') = u' \otimes q' \).

(ii) Taking the same curve \( \beta_t = u_t \otimes q + u_t^2 \) as in (i), \( u'' \otimes q + 2u \circ u'' \in T_{\beta_t} \), for any \( t \). So \( \sigma(u' \otimes q + 2u \circ u', u'' \otimes q + 2u \circ u'') = \frac{d}{dt}|_{t=0}(u'' \otimes q + 2u_t \circ u'') = 2u' \circ u'' \).

(iii) For \( \beta_t = u \otimes q + u_1 \) with \( \frac{d}{dt}|_{t=0}u_1 = q' \) as in (2), \( u \otimes q'' \in T_{\beta_t} \), for any \( t \). So \( \sigma(u \otimes q', u \otimes q'') = \frac{d}{dt}|_{t=0}u \otimes q'' = 0 \).

(Case II : \( \beta = u \otimes q \)). (i) Similarly, we take \( \beta_t = u_t \otimes q \) as in (3) and assume that \( u_0 = u, \frac{d}{dt}|_{t=0}u_t = u' \). Then \( \beta_0 = u \otimes q = \beta, \frac{d}{dt}|_{t=0}\beta_t = u' \otimes q \) and \( u_t \otimes q' \in T_{\beta_t} \), for any \( t \), hence \( \sigma(u' \otimes q, u \otimes q') = \frac{d}{dt}|_{t=0}u_t \otimes q' = u' \otimes q' \).

(ii) For the above curve \( \beta_t = u_t \otimes q, u_t^2 \in T_{\beta_t} \), for any \( t \). So \( \sigma(u' \otimes q, u^2) = \frac{d}{dt}|_{t=0}u_t^2 = 2u \circ u' \).

(iii) For the above curve \( \beta_t = u \otimes q, u'' \otimes q \in T_{\beta_t} \), for any \( t \). So \( \sigma(u' \otimes q, u'' \otimes q) = \frac{d}{dt}|_{t=0}u'' \otimes q = 0 \).

(iv) Taking \( \beta_t = u \otimes q_t \) as in (4), \( \beta_0 = u \otimes q = \beta \) and \( \frac{d}{dt}|_{t=0}\beta_t = u \otimes q_t \). Since \( u \otimes q'' \in T_{\beta_t} \), for any \( t \), we obtain \( \sigma(u \otimes q', u \otimes q''') = \frac{d}{dt}|_{t=0}u \otimes q'' = 0 \).

(v) For the above curve \( \beta_t = u \otimes q_t, u^2 \in T_{\beta_t} \), for any \( t \). So \( \sigma(u \otimes q, u^2) = \frac{d}{dt}|_{t=0}u^2 = 0 \).

(vi) Finally, we take \( \beta_t = u \otimes q + c_t u^2 \) as in (5) and assume that \( \frac{d}{dt}|_{t=0}c_t = 1 \). Then \( \beta_0 = u \otimes q = \beta \) and \( \frac{d}{dt}|_{t=0}\beta_t = u^2 \). Since \( u^2 \in T_{\beta_t} \), for any \( t \), we obtain \( \sigma(u^2, u^2) = \frac{d}{dt}|_{t=0}u^2 = 0 \). \( \square \)

**Definition 3.4.** Let \( X \) be a polarized uniruled projective manifold equipped with a minimal rational component \( \mathcal{K} \). Assume that \( Z \) is an embedded submanifold in \( X \). Let \( \mathcal{A} := \mathcal{C}_x(X) \subset \mathcal{P}(T_x X) \) and \( \mathcal{B} := \mathcal{C}_x(Z) \subset \mathcal{P}(T_x Z) \subset \mathcal{P}(T_x X) \) be the varieties of minimal rational tangents at a common general point \( x \) of \( X \) and \( Z \), respectively. We say that the pair \( (\mathcal{A}, \mathcal{B}) \) is nondegenerate if

\[
\text{Ker} \sigma_{\beta} : \mathcal{A}(\beta, T_{\beta} \tilde{B}) = \mathbb{C}\beta
\]

for any \( \beta \in \tilde{B} \), where \( \sigma_{\beta} : T_{\beta} \tilde{A} \times T_{\beta} \tilde{A} \rightarrow (T_x X)/T_{\beta} \tilde{A} \) is the second fundamental form of the affine cone \( \tilde{A} \) in \( T_x X \) at \( \beta \).

**Proposition 3.5.** Let \( X \) be the symplectic Grassmannian \( \text{Gr}_\omega(k, 2\ell) \) with \( 1 < k < \ell \) and \( Z \) be an odd symplectic Grassmannian \( \text{Gr}_\omega(k, 2\ell; F_a, F_{2\ell-a-1}) \) with \( 0 < a < k \). Let \( \mathcal{A} \subset \mathcal{P}(T_x X) \) and \( \mathcal{B} \subset \mathcal{P}(T_x Z) \subset \mathcal{P}(T_x X) \) be the varieties of minimal rational tangents at a common general point \( x \) of \( X \) and \( Z \), respectively. Then the pair \( (\mathcal{A}, \mathcal{B}) \) is nondegenerate.

**Proof.** Now take a point \( [E] \in Z \) such that \( E \cap F_{a+1} = F_a \). Then the dimension of \( F_{2\ell-a-1} \cap E^\perp \) is \( 2\ell - k - 1 \) and so \( F_{2\ell-a-1}/(F_{2\ell-a-1} \cap E^\perp) \) is isomorphic to \( (E/F_a)^* \). From Lemma 4.2 (2) of Hong-Mok [9], the variety \( \mathcal{B} \) of minimal rational tangent of \( Z \) at a general point \( [E] \in Z \) is the projectivization of the affine cone

\[
\tilde{B} = \{ u \otimes q + cu^2 : u \in U_a, q \in Q_a, c \in \mathbb{C} \} \setminus \{0\} \subset (U_a \otimes Q_a) \oplus S^2U_a,
\]
where $U_a = (E/F_a)^*$ and $Q_a = (F_{2k- a-1} \cap E^\perp)/E$. Note that $Q_a$ is a codimension 1 subspace of $Q$. By this description of the varieties of minimal rational tangents of $Z$ and by the computation of the second fundamental form of $A$ (Lemma 3.22), we get the desired results.

1. The tangent space $T_\beta \mathcal{B}$ at $\beta = u \otimes q$ is given by $\{u \otimes q' + u' \otimes q + cu^2 : u' \in U_a, q' \in Q_a, c \in \mathbb{C}\}$. Then we have $\ker \sigma_\beta(\cdot, U_a \otimes q) = U \otimes q$, $\ker \sigma_\beta(\cdot, u \otimes Q_a) = \{u \otimes q' + cu^2 : q' \in Q, c \in \mathbb{C}\}$ and $\ker \sigma_\beta(\cdot, Cu^2) = \{u \otimes q + cu^2 : q' \in Q, c \in \mathbb{C}\}$. Therefore, $\ker \sigma_\beta(\cdot, T_\beta \mathcal{B}) = \mathbb{C}(u \otimes q) = \mathbb{C}\beta$.

2. The tangent space $T_\beta \mathcal{B}$ at $\beta = u \otimes q + u^2$ is given by $\{u \otimes q' + u' \otimes q + 2u \otimes u' : u' \in U_a, q' \in Q_a\}$. Then we have $\ker \sigma_\beta(\cdot, u \otimes Q_a) = \{u \otimes q' + cu^2 : q' \in Q, c \in \mathbb{C}\}$ and $\ker \sigma_\beta(\cdot, \{u' \otimes q + 2u \otimes u' : u' \in U_a\}) = \mathbb{C}(u \otimes q + u^2)$. Therefore, $\ker \sigma_\beta(\cdot, T_\beta \mathcal{B}) = \mathbb{C}(u \otimes q + u^2) = \mathbb{C}\beta$.

We will use the same notation for $g \in G$ and for the differential of the action $g \colon X \to X$ at $x \in X$ and its projectivization $\mathbb{P}(T_xX) \to \mathbb{P}(T_{gX}X)$, for simplicity.

**Proposition 3.6.** In the setting of Proposition 3.3, if $B' = A \cap \mathbb{P}(W')$ is another linear section of $A$ by a linear subspace $\mathbb{P}(W')$ of $\mathbb{P}(T_xX)$ such that $(B \subset \mathbb{P}(T_xZ))$ is projectively equivalent to $(B' \subset \mathbb{P}(W'))$, then there is an element $h$ in a Levi factor of the parabolic subgroup $P$ such that $B'' = hB$.

**Proof.** Since $B$ is a $\mathbb{P}^{2m-1}$-bundle on $\mathbb{P}(U_a)$, $B'$ is also a $\mathbb{P}^{2m-1}$-bundle on $\mathbb{P}^r$, where $r = \dim U_a - 1 = k - a - 1$. Let $B'_1$ be a codimension 1 linear section of $B'$ which is projectively equivalent to the codimension 1 linear section $B_1 := B \cap \mathbb{P}(U \otimes Q) \cong \mathbb{P}(U_a) \times \mathbb{P}(Q_a)$ of $B$.

Suppose that $B'_1$ is not contained in $\mathbb{P}(U \otimes Q)$. Take $b' = u \otimes q + u^2 \in B'_1 \cap (\mathcal{A}\setminus(U \otimes Q))$. Since $B'$ is a linear section $\mathcal{A} \cap \mathbb{P}(W')$ of $\mathcal{A}$, the tangent space $T_{b'} B'$ at $b'$ is contained in the intersection $W' \cap T_{b'} \mathcal{A}$ and the second fundamental form $\sigma_{b'}^{B'}_\mathcal{B} : T_{b'} B' \times T_{b'} \mathcal{B} \to W'/T_{b'} B'$ of $B'$ at $b'$, composed with the quotient map $W'/T_{b'} B' \to T_{x'X}/T_{b'} \mathcal{A}$, is the restriction of the second fundamental form $\sigma_{b'}^\mathcal{B}$ of $\mathcal{A}$ to $T_{b'} B' \times T_{b'} \mathcal{B}$. Hence $\{v \in T_{b'} \mathcal{B} : \sigma_{b'}^{B'}_\mathcal{B}(v, v) = 0\}$ is a linear subspace of $\{u \otimes q' : q' \in Q_a\}$ because $B'$ is a linear section of $\mathcal{A}$. Since $Z$ is not linear, for $b \in B_1$, $\{v \in T_b \mathcal{B} : \sigma_{b'}^{B'}_\mathcal{B}(v, v) = 0\}$ is the union of two subspaces $\{u' \otimes q : u' \in U_a\}$ and $\{u \otimes q' : q' \in Q_a\}$, while for $b' \in B'_1 \cap (\mathcal{A}\setminus(U \otimes Q))$, $\{v \in T_{b'} \mathcal{B} : \sigma_{b'}^{B'}_\mathcal{B}(v, v) = 0\}$ is only one linear subspace. Thus the second fundamental form $\sigma_{b'}^{B'}_\mathcal{B}$ is not isomorphic to $\sigma_{b'}^\mathcal{B}$ and hence $B \subset \mathbb{P}(T_xZ)$ cannot be projectively equivalent to $B' \subset \mathbb{P}(W')$.

Therefore, $B'_1$ is contained in $\mathbb{P}(U \otimes Q) \cap \mathcal{A} \cong \mathbb{P}(U) \times \mathbb{P}(Q)$. By Lemma 2 of Mok (23) about linear maps between nontrivial tensor product spaces, any linear section of $\mathbb{P}(U) \times \mathbb{P}(Q)$ which is projectively equivalent to $\mathbb{P}(U_a) \times \mathbb{P}(Q_a) \subset \mathbb{P}(T_xZ)$ is of the form $\mathbb{P}(U'_a) \times \mathbb{P}(Q'_a)$ for some subspaces $U'_a \subset U$ and $Q'_a \subset Q$ with $\dim U'_a = \dim U_a$, $\dim Q'_a = \dim Q_a$. To characterize the variety $B$ of minimal rational tangents of $Z$, we use the base locus $\{v \in T_{b'} \mathcal{B} : \sigma_{b'}(v, v) = 0\}$ of the second fundamental form $\sigma$ of $\mathcal{A} \subset T_xX$ at a generic point $b \in \mathcal{A}$. Let $B' = \mathcal{A} \cap \mathbb{P}(W')$ be a linear section of $\mathcal{A}$ which is projectively equivalent to $B \subset \mathbb{P}(T_xZ)$. Then for a general point $b' \in B'$ the second fundamental form $\sigma_{b'}^{B'}$ of $B'$ at $b'$ is isomorphic to the second fundamental form $\sigma_{b'}^\mathcal{B}$ of $\mathcal{B}$ at $b$. Hence $\{v \in T_{b'} \mathcal{B} : \sigma_{b'}^\mathcal{B}(v, v) = 0\}$ is isomorphic to $\{v \in T_{b'} \mathcal{B} : \sigma_{b'}^{B'}(v, v) = 0\}$. From the fact that $B'$ is a linear section of $\mathcal{A}$, it follows that $\{v \in T_{b'} \mathcal{B} : \sigma_{b'}^{B'}(v, v) = 0\}$ is...
0 \} \) is contained in \( \{ v \in T_{\tilde{\mathbb{A}}} : \sigma_b(v, v) = 0 \} \). For \( b \in B' \setminus B'_0 \subset A \setminus \mathbb{P}(U \otimes Q) \) the linear space \( \mathbb{P}^{2m-1} \) in \( B' \) passing through \( b \) is contained in the fiber of the projection \( A \to \mathbb{P}(U) \) containing \( b \), because \( \{ v \in T_{\tilde{\mathbb{A}}} : \sigma_b(v, v) = 0 \} \) is the tangent space to the fiber of \( \tilde{\mathbb{A}} \to U \). Thus \( B' \) is the restriction of the \( \mathbb{P}^{2m-1} \)-bundle on \( \mathbb{P}(U) \) to the subspace \( \mathbb{P}(U') \). Because any hyperplane in \( \mathbb{P}(Q) \) can be transformed another hyperplane in \( \mathbb{P}(Q) \) under the action of \( \text{Sp}(Q) \), \( B' = h\mathbb{B} \) for some \( h \in \text{SL}(U) \times \text{Sp}(Q) \) which is the semisimple part of \( P \).

**Proof of Theorem 1.2 in the case that** \( P \) **is the symplectic Grassmannian** \( \text{Gr}_{\omega}(k, 2\ell) \). From Lemma 3.1 and Theorem 1.1 it suffices to consider odd symplectic Grassmannians in the symplectic Grassmannian \( \text{Gr}_{\omega}(k, 2\ell) \) with \( 1 < k < \ell \). Let \( Z \) be an odd symplectic Grassmannian \( \text{Gr}_{\omega}(k, 2\ell; F_0, F_{2\ell-a-1}) \) with \( 0 \leq a < k \).

Let \( f : U \to X \) be a holomorphic embedding from a connected open subset \( U \) of \( Z \) into \( X \) which respects varieties of minimal rational tangents for a general point \( z \in U \). Then \( df(C_z(Z)) \) is the linear section \( C_{f(z)}(X) \cap df(\mathbb{P}(T_z(Z))) \) of \( C_{f(z)}(X) \) and \( df(C_z(Z)) \subset df(\mathbb{P}(T_z(Z))) \) is projectively equivalent to \( C_z(Z) \subset \mathbb{P}(T_z(Z)) \). By Proposition 3.6, for each general point \( z \in U \) there is an element \( h = h(z) \) in a Levi factor of the parabolic subgroup \( P \) such that \( df(C_z(Z)) = C_{f(z)}(hZ) \). Thus \( f \) is nondegenerate with respect to \( (K, \mathcal{H}) \) by Proposition 3.5. Then Proposition 2.1 of Hong-Mok [8] implies that \( f \) sends minimal rational curves in \( Z \) to minimal rational curves in \( X \) and we get a rational extension \( F : Z \to X \) of \( f \) by Proposition 2.5. Then the total transformation \( F(Z) \) of \( F \) is rationally saturated, i.e., for every smooth point \( x \in F(Z) \) and for any minimal rational curve \( C \) on \( X \) passing through \( x \), \( C \) must lie on \( F(Z) \) whenever \( C \) is tangent to \( F(Z) \) at \( x \). For a general point \( x \) in \( F(Z) \), the variety \( C_x(F(Z)) \) of minimal rational tangents of \( F(Z) \) is \( df(C_z(Z)) \) where \( x = F(z) \).

Fix a general point \( x_0 \in U \). From the homogeneity of \( X \) and Proposition 3.6, \( F(x_0) = g x_0 \) for some \( g \in G \) and \( C_{g x_0}(F(Z)) = h g C_{x_0}(Z) \) for an element \( h \in P \). Then \( F(\Sigma) = \Sigma \) up to the action of \( G \), where \( \Sigma \) denotes the subvariety of \( Z \) swept out by minimal rational curves in \( Z \) passing through \( x_0 \). Let \( C \) be a standard minimal rational curve in \( Z \) passing through \( x_0 \) and let \( y \in C \) be a smooth point different from \( x_0 \). Then the tangent direction \( [T_y C] \) is contained both in \( C_y(Z) \) and in \( C_y(F(Z)) \). By the deformation theory of minimal rational curves (Lemma 2.8 of Hong-Mok [8]), the tangent space \( T_y \Sigma \) of \( \Sigma \) at \( x \) can be identified with the tangent space of \( C_y(Z) \) at \( \alpha \in T_x C \). Note that by Proposition 4.3 of Hong-Mok [9], if \( h \) is an element in the isotropy subgroup \( P_{E} \) of \( G \) such that \( h \mathbb{B} = \mathbb{B} \) is tangent at a point of intersection, then \( h \mathbb{B} \) is equal to \( \mathbb{B} \). Since \( F(\Sigma) = \Sigma, C_y(Z) \) is tangent to \( C_y(F(Z)) \) at \( [T_y C] \), and thus we have \( C_y(F(Z)) = C_y(Z) \). Therefore, \( C_y(F(Z)) = C_y(Z) \) for a generic point \( y \in \Sigma \).

Since \( Z \) is an uniruled projective manifold of Picard number 1, there is a sequence of irreducible varieties \( U_0 = \{ x_0 \} \subset U_1 \subset \cdots \subset U_k \) with \( \dim U_k = \dim Z \) such that a general point in \( U_{k+1} \) can be connected to a point in \( U_k \) by a minimal rational curve in \( Z \) (see Section 4.3 of Hwang-Mok [13] or Section 3 of Mok [24]). Applying the same arguments as above inductively, we get that \( F(U_k) = U_k \) and thus we have \( F(Z) = Z \). From Proposition 2.2, we know that \( \text{Aut}(Z) \) is isomorphic to the projective odd symplectic group \( \text{PSp}(2\ell - 1) := (\text{Sp}(2\ell - 2) \times \mathbb{C}^*)/(\pm 1) \times \mathbb{C}^{2\ell-2} \). Consequently, Proposition 3.3 of Mihai [22] implies that there exists \( g' \in \text{Sp}(2\ell) \) such that \( g'|_Z = F \). Therefore, \( f \) is the restriction of the standard embedding of \( Z \) into \( X \). \( \square \)
4. Smooth Schubert varieties in $F_4$-homogeneous manifolds

Let us start with the facts about the complex simple Lie algebra $\mathfrak{g}$ of type $F_4$. We choose a system $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of simple roots such that $\alpha_3$ and $\alpha_4$ are short roots. Then the highest long root of $\mathfrak{g}$ is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, hence the grading on $\mathfrak{g}$ associated to $\alpha_3$ is of depth 4.

\[
(F_4) \begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
2 & 3 & 4 & 2
\end{array}
\]

Let $\mathfrak{p}$ be the maximal parabolic subalgebra of $\mathfrak{g}$ associated to the simple root $\alpha_3$. Given an integer $k$, $-4 \leq k \leq 4$, $\Phi_k$ denotes the set of all roots $\alpha = \sum_{q=1}^4 c_q \alpha_q$ with the third coefficient $c_3 = k$. Define

\[
\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_k = \bigoplus_{\alpha \in \Phi_k} \mathfrak{g}_\alpha, \quad k \neq 0.
\]

Then the parabolic subalgebra $\mathfrak{p}$ is decomposed as a graded Lie algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \mathfrak{g}_4$ with

\[
\dim \mathfrak{g}_0 = 12, \quad \dim \mathfrak{g}_1 = 6, \quad \dim \mathfrak{g}_2 = 9, \quad \dim \mathfrak{g}_3 = 2, \quad \dim \mathfrak{g}_4 = 3.
\]

Let $X$ be the rational homogeneous manifold $(F_4, \alpha_3)$ associated to the short root $\alpha_3$. Then $X$ is the closed $F_4$-orbit of the space of lines on the rational homogeneous manifold of type $(F_4, \alpha_4)$, which is a smooth hyperplane section of the (complex) Cayley plane $\mathbb{OP}^2 = (E_6, \alpha_1)$ (cf. Section 6 of Landsberg-Manivel [21]).

Since $\dim \mathfrak{g} = 52$ and $\dim \mathfrak{p} = 32$, the rational homogeneous manifold $X$ of type $(F_4, \alpha_3)$ is a projective variety of dimension 20. Let $o = eP$ be the base point of $X = G/P$. The tangent space $T_o(G/P)$ is canonically isomorphic to $\mathfrak{g}/\mathfrak{p}$ and so the first Chern number of the tangent bundle $T_{G/P}$ is computed by

\[
\sum_{\beta \in \Phi_1} \beta(H_{\alpha_3}) + \sum_{\beta \in \Phi_2} \beta(H_{\alpha_3}) + \sum_{\beta \in \Phi_3} \beta(H_{\alpha_3}) + \sum_{\beta \in \Phi_4} \beta(H_{\alpha_3}) = 1 + 3 + 1 + 2 = 7,
\]

the first Chern class of $X$ is $c_1(X) = 7L \in H^2(X, \mathbb{Z}) \cong H^1(X, \mathcal{O}_X) = \text{Pic}(X)$. Here $L$ is the ample generator with $\text{Pic}(G/P) = \mathbb{Z}L$ and gives an embedding $G/P \subset \mathbb{P}^{272} = \mathbb{P}(V(\omega_3))$, where $\omega_3$ is the third fundamental weight of $G$. Furthermore, $G/P$ is covered by lines of $\mathbb{P}^{272}$ and the Chow space $\mathcal{K}_o$ consists of all lines passing through $o$, which are contained in $G/P$. Hence the tangent map $\tau_o : \mathcal{K}_o \to C_o$ is an embedding and the variety $C_o$ of minimal rational tangents at $o$ is 5-dimensional, because $c_1(G/P) = 7L$.

Now we take a choice of a Levi factor $\mathfrak{k}$ of $\mathfrak{p}$. The semisimple part of $\mathfrak{k}$ is isomorphic to $\mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. So a reductive subgroup $K \subset P$ with Lie algebra $\mathfrak{k}$ is isogenous to $\mathbb{C}^* \times \text{SL}(3, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$ as a complex Lie group. Under the identification $T_o(G/P) = \mathfrak{g}/\mathfrak{p}$, $\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-4}$ is the graded decomposition into irreducible $K$-modules.

Let $E$ be a 3-dimensional complex vector space with dual $E^*$ with respect to the standard inner product on $E$ and $Q$ be a 2-dimensional complex vector space.
Then, we can check the following $K$-module isomorphisms:

$$g_{-1} = E^* \otimes Q, \quad g_{-2} = \wedge^2 E^* \otimes S^2 Q, \quad g_{-3} = Q, \quad g_{-4} = E^*.$$  

In particular, one can determine the highest weight variety $W_1 \subset \mathbb{P}g_{-1}$ consisting of highest weight vectors of the irreducible $K$-module $g_{-1}$. Because the highest weight variety $W_1 \subset \mathbb{P}g_{-1}$ of $X$ is a homogeneous manifold associated to the marked Dynkin diagram having markings corresponding to the simple roots $\alpha_2$ and $\alpha_4$ which are adjacent to $\alpha_3$ in the Dynkin diagram of the semisimple part of $P$, we have $W_1 = \mathbb{P}^2 \times \mathbb{P}^1 \subset \mathbb{P}^5$ embedded in the Segre embedding and its affine cone is equal to $\{e^* \otimes q \in E^* \otimes Q : e^* \in E^*, q \in Q\}\{0\}$.

$$(F_4, \alpha_3) \quad \xrightarrow{\text{identification}} \quad \mathbb{P}X \times \mathbb{P}W_1 \subset \mathbb{P}g_{-1}$$  

The variety $C_o(X)$ of minimal rational tangents at the base point $o \in X$ contains the highest weight variety $W_1$ in $\mathbb{P}g_{-1}$ but $C_o(X)$ is strictly bigger than $W_1$ since $\dim C_o(X) = 5$. Hence, we consider the highest weight variety $W_2 \subset \mathbb{P}g_{-2}$ with respect to the $K$-action, which we expect to be contained in $C_o(X)$. The affine cone of $W_2$ is equal to

$$\{(e_1^* \wedge e_2^*) \otimes q^2 \in \wedge^2 E^* \otimes S^2 Q : e_1^*, e_2^* \in E^*, q \in Q\}\{0\}.$$  

By Section 3 of Hwang-Mok [16] or Proposition 6.9 of Landsberg-Manivel [21], $C_o(X)$ is contained in $\mathbb{P}(g_{-1} \oplus g_{-2})$ and is the projectivization of the affine cone

$$\{e^* \otimes q + (e_1^* \wedge e_2^*) \otimes q^2 : e^* \wedge e_1^* \wedge e_2^* = 0, e^*, e_1^*, e_2^* \in E^*, q \in Q\}\{0\}$$  

in $(E^* \otimes Q) \oplus (\wedge^2 E^* \otimes S^2 Q)$. Since $\wedge^2 E^*$ is isomorphic to $E$ as $\text{SL}(E)$-modules, we will make a fixed choice of the identification. Now, we denote a subvariety $C_o(X) \subset \mathbb{P}((E^* \otimes Q) \oplus (E \otimes S^2 Q))$ by $\tilde{A}$. Then the affine cone $\tilde{A}$ over $A$ is given by

$$\tilde{A} = \{e^* \otimes q + f \otimes q^2 : \langle e^*, f \rangle = 0, e^* \in E^*, f \in E, q \in Q\}\{0\},$$  

where $\langle e^*, f \rangle$ denotes the evaluation of $e^*$ at $f$. Under the projection map $e^* \otimes q + f \otimes q^2 \mapsto q$, $\tilde{A}$ is a fiber bundle over $\mathbb{P}(Q) = \mathbb{P}^1$ with fibers which are isomorphic to a 4-dimensional quadric $Q^4$. In other words, $\tilde{A}$ is the Grassmannian bundle of 2-planes of the vector bundle $E^*$ on $\mathbb{P}^1$, where $E$ is a vector bundle of rank 4 which splits as $O(1)^3 \oplus O$. In fact, the Plücker line bundle $\xi$ on $\text{Gr}(2, E^*)$ defines an embedding of $\text{Gr}(2, E^*)$ into $\mathbb{P}H^0(\text{Gr}(2, E^*), \xi)$. Since $H^0(\text{Gr}(2, E^*), \xi) = H^0(\mathbb{P}^1, \wedge^2 E) = H^0(\mathbb{P}^1, O(1)^3 \oplus O(2)^3)$, under the identification $\mathbb{P}^1 = \mathbb{P}(Q^*)$, we have $H^0(\text{Gr}(2, E^*), \xi) = (E^* \otimes Q) \oplus (\wedge^2 E^* \otimes S^2 Q)$. For the detailed descriptions as a projective variety, see Section 2 of Hwang-Mok [16].

**Lemma 4.1.** Let $X$ be the rational homogeneous manifold of type $(F_4, \alpha_3)$ and $A$ be the variety of minimal rational tangents of $X$ at a point $x \in X$. The tangent space $T_\beta$ of $\tilde{A}$ at $\beta \in \tilde{A}$ is given by

$$T_\beta = \{e^* \otimes q' + e'^* \otimes q + f^* \otimes q^2 + f \otimes (2q \circ q') : \langle e'^*, f' \rangle = 0, e'^* \in E^*, f' \in E, q' \in Q\} \text{ if } \beta = e^* \otimes q + f \otimes q^2,$$

$$T_\beta = \{e^* \otimes q' + e'^* \otimes q + f^* \otimes q^2 : \langle e'^*, f' \rangle = 0, e'^* \in E^*, f' \in E, q' \in Q\} \text{ if } \beta = e^* \otimes q.$$
The second fundamental form $\sigma: T_\beta \times T_\beta \to (T_x X)/T_\beta$ of $\tilde{A} \subset T_x X$ at $\beta \in \tilde{A}$ is given as follows:

(I) for $\beta = e^* \otimes q + f \otimes q^2$,

\[
\begin{align*}
\sigma(e^* \otimes q' + f \otimes (2q \circ q'), e^* \otimes q'' + f \otimes (2q \circ q'')) &= f \otimes (2q' \circ q'') \\
\sigma(e^* \otimes q' + f \otimes (2q \circ q'), e^* \otimes q) &= e^* \otimes q' \\
\sigma(e^* \otimes q' + f \otimes (2q \circ q'), f' \otimes q^2) &= f' \otimes (2q \circ q') \\
\sigma(e^* \otimes q, e'' \otimes q) &= 0 \\
\sigma(e^* \otimes q, f' \otimes q^2) &= (-\langle e^*, f' \rangle e) \otimes q^2 \\
\sigma(f' \otimes q^2, f'' \otimes q^2) &= 0;
\end{align*}
\]

(II) for $\beta = e^* \otimes q$,

\[
\begin{align*}
\sigma(e^* \otimes q, e^* \otimes q') &= 0 \\
\sigma(e^* \otimes q, e^* \otimes q) &= e^* \otimes q' \\
\sigma(e^* \otimes q, f' \otimes q^2) &= f' \otimes (2q \circ q') \\
\sigma(e^* \otimes q, e'' \otimes q) &= 0 \\
\sigma(e^* \otimes q, f' \otimes q^2) &= (-\langle e^*, f' \rangle e) \otimes q^2 \\
\sigma(f' \otimes q^2, f'' \otimes q^2) &= 0,
\end{align*}
\]

where $e^*, e'' \in E^*, f', f'' \in E$ and $q', q'' \in Q$.

Proof. This is given in Lemma 4.2 of Hong-Park [11] without details. We give the details of the proof. First, to obtain the tangent space $T_{\beta(\tilde{A})}$, we consider the velocity vectors of curves in the affine cone $\tilde{A}$. Let $\{e_i^\beta\} \subset E^*$, $\{f_i\} \subset E$ and $\{q_i\} \subset Q$ be curves with $e_0^\beta = e^*$, $f_0 = f$ and $q_0 = q$, respectively. Assuming $\langle e^*, f \rangle = 0$, the curve $e^* \otimes q + f \otimes q^2$ lies in the affine cone $\tilde{A}$ and passes through a point $e^* \otimes q + f \otimes q^2$ in $\tilde{A}$ and its velocity vector is $e^* \otimes q + f' \otimes q^2$ for some $e^* \in E^*$, $f' \in E$ such that $\langle e^*, f \rangle = \langle e^*, f' \rangle = 0$. Next, for the curve $\beta_t = e^* \otimes q + f_t \otimes q^2$ with $f_0 = 0$ and $e^* \otimes f_t = 0$, $\beta_0 = e^* \otimes q$ and $\frac{d}{dt}|_{t=0} \beta_t = e^* \otimes (\frac{d}{dt}|_{t=0} f_t) \otimes q^2 = f_0 \otimes (\frac{d}{dt}|_{t=0} f_t^2) = e^* \otimes q + f' \otimes q^2$ for some $f' \in E$, $q' \in Q$ such that $\langle e^*, f' \rangle = 0$.

By a similar computation as in Lemma 3.2, we get the above results. Let $\{e_i^\beta\} \subset E^*$, $\{f_i\} \subset E$ and $\{q_i\} \subset Q$ be curves with $e_0^\beta = e^*$, $f_0 = f$ and $q_0 = q$, respectively. Then the holomorphic curves $[T_{\beta_i}]$ in $\text{Gr}(d, T_x X)$ for $\{\beta_i\} \subset A$ such that $\beta_0 = \beta$ are as follows:

1. for $\beta_t = e^* \otimes q_t + f \otimes q_t^2$, $T_{\beta_t} = \{e^* \otimes q' + e^* \otimes q_t + f' \otimes q_t^2 + f \otimes (2q_t \circ q') : \langle e^*, f \rangle + \langle e^*, f' \rangle = 0, e^* \in E^*, f' \in E, q' \in Q\}$;
2. for $\beta_t = e_t^\beta \otimes q + f \otimes q^2$, $T_{\beta_t} = \{e_t^\beta \otimes q' + e^* \otimes q + f' \otimes q^2 + f \otimes (2q \circ q') : \langle e_t^\beta, f \rangle + \langle e^*, f' \rangle = 0, e^* \in E^*, f' \in E, q' \in Q\}$;
3. for $\beta_t = e_t^\beta \otimes q + f_t \otimes q^2$ with $f_0 = f$, $\beta_t = \{e_t^\beta \otimes q' + e^* \otimes q + f' \otimes q^2 + f_t \otimes (2q \circ q') : \langle e_t^\beta, f \rangle + \langle e^*, f' \rangle = 0, e^* \in E^*, f' \in E, q' \in Q\}$;
4. for $\beta_t = e^* \otimes q_t + f' \otimes q_t^2$, $T_{\beta_t} = \{e^* \otimes q' + e^* \otimes q_t + f' \otimes q_t^2 : \langle e^*, f' \rangle = 0, e^* \in E^*, f' \in E, q' \in Q\}$;
5. for $\beta_t = e_t^\beta \otimes q_t + q' \otimes q^2$, $T_{\beta_t} = \{e_t^\beta \otimes q' + e^* \otimes q_t + f' \otimes q_t^2 : \langle e_t^\beta, f' \rangle = 0, e^* \in E^*, f' \in E, q' \in Q\}$;
(6) for \( \beta_t = e^* \otimes q + f_t \otimes q^2 \) with \( f_0 = 0, T_{\beta_t} = \{ e^* \otimes q' + e^* \otimes q + f' \otimes q^2 + f_t \otimes (2q \otimes q') : \langle e^*, f_t \rangle + \langle e^*, f' \rangle = 0 \}, e^* \subset E^*, f' \subset E, q' \subset Q \}.

As in Lemma 3.2, the second fundamental form is computed in the following manner: \( \sigma(\frac{d}{dt}|_{t=0} \beta_t, \rho_t) = \frac{d}{dt}|_{t=0} \rho_t \), where \( \rho_t \) is a vector field along the curve \( \beta_t \) such that \( \rho_t \subset T_{\beta_t} \) for every \( t \).

(Case I : \( \beta = e^* \otimes q + f \otimes q^2 \)). (i) We take a curve \( \beta_t = e^* \otimes q_t + f \otimes q_t^2 \) as in (1) and assume that \( \frac{d}{dt}|_{t=0} \beta_t = q' \). Then \( \beta_0 = e^* \otimes q + f \otimes q^2 = \beta \) and \( \frac{d}{dt}|_{t=0} \beta_t = e^* \otimes q' + f \otimes (2q \otimes q') \). Since \( e^* \otimes q' + f \otimes (2q \otimes q') \subset T_{\beta_t} \) for any \( t \), the differential \( \frac{d}{dt}|_{t=0}[T_{\beta_t}] : T_{\beta} \rightarrow V/T_{\beta} \) maps \( e^* \otimes q' + f \otimes (2q \otimes q') \subset T_t \) to \( \frac{d}{dt}|_{t=0}(e^* \otimes q'' + f \otimes (2q \otimes q')) = f \otimes (2\frac{d}{dt}|_{t=0} q_t) \otimes q'' = f \otimes (2q \otimes q''). \) Thus we have \( \sigma(e^* \otimes q' + f \otimes (2q \otimes q'), e^* \otimes q'' + f \otimes (2q \otimes q'')) = f \otimes (2q \otimes q'') \).

(ii) If \( e^* \otimes q \subset T_{\beta_t} \), then the relation \( \langle e^*, f \rangle = 0 \) holds. For the above curve \( \beta_t = e^* \otimes q_t + f \otimes q_t^2, e^* \otimes q_t \subset T_{\beta_t} \), for any \( t \) so \( \sigma(e^* \otimes q + f \otimes (2q \otimes q'), e^* \otimes q) = \frac{d}{dt}|_{t=0}(e^* \otimes q_t) = e^* \otimes q' \).

(iii) If \( f' \otimes q^2 \subset T_{\beta_t} \), then the relation \( \langle e^*, f' \rangle = 0 \) holds. For the above curve \( \beta_t = e^* \otimes q_t + f \otimes q_t^2, f' \otimes q_t^2 \subset T_{\beta_t} \), for any \( t \) so \( \sigma(e^* \otimes q + f \otimes (2q \otimes q'), f' \otimes q^2) = \frac{d}{dt}|_{t=0}(f' \otimes q_t^2) = f' \otimes (2q \otimes q') \).

(iv) Taking a curve \( \beta_t = e^*_t \otimes q + f \otimes q^2 \) as in (2) such that \( \frac{d}{dt}|_{t=0} e^*_t = e^*, \beta_0 = e^* \otimes q + f \otimes q^2 = \beta \) and \( \frac{d}{dt}|_{t=0} \beta_t = e^* \otimes q \). If \( e^* \otimes q \subset T_{\beta_t} \), then the relation \( \langle e^*, f \rangle = 0 \) holds. Since \( e^* \otimes q \subset S_{\beta_t} \), for any \( t \) we obtain \( \sigma(e^* \otimes q, e^* \otimes q) = \frac{d}{dt}|_{t=0} e^* \otimes q = 0 \).

(v) We take the above curve \( \beta_t = e^*_t \otimes q + f \otimes q^2 \) and a vector field \( f_t \otimes q^2 \) along \( \beta_t \) with \( f_0 = f' \). Then \( f_t \otimes q^2 \subset T_{\beta_t} \) whenever \( \langle e^*_t, f_t \rangle = 0 \) for any \( t \). Differentiating the equation \( \langle e^*_t, f_t \rangle = 0 \) at \( t = 0 \), we have \( \langle e^*, f' \rangle + \sigma(e^*, e^*) = \frac{d}{dt}|_{t=0}(e^* \otimes q_t) = 0 \). Hence \( \frac{d}{dt}|_{t=0} f_t = -(e^*, f')e \) and so \( \sigma(e^* \otimes q, f' \otimes q^2) = \frac{d}{dt}|_{t=0} f_t \otimes q^2 = -(e^*, f')e \otimes q^2. \)

(vi) Taking a curve \( \beta_t = e^* \otimes q + f_t \otimes q^2 \) as in (3) such that \( f_0 = f \) and \( \frac{d}{dt}|_{t=0} f_t = f' \), \( \beta_0 = e^* \otimes q + f \otimes q^2 = \beta \) and \( \frac{d}{dt}|_{t=0} \beta_t = f' \otimes q^2. \) Since \( f' \otimes q^2 \subset T_{\beta_t} \) for any \( t \), we obtain \( \sigma(f' \otimes q^2, f' \otimes q^2) = \frac{d}{dt}|_{t=0} f' \otimes q^2 = 0 \).

(Case II : \( \beta = e^* \otimes q \)). (i) Now take a curve \( \beta_t = e^* \otimes q_t \) as in (4) and assume that \( \frac{d}{dt}|_{t=0} q_t = q' \). Then \( \beta_0 = e^* \otimes q = \beta \) and \( \frac{d}{dt}|_{t=0} \beta_t = e^* \otimes q' \). Since \( e^* \otimes q' \subset T_{\beta_t} \), for any \( t \) we have \( \sigma(e^* \otimes q', e^* \otimes q'') = \frac{d}{dt}|_{t=0} e^* \otimes q'' = 0 \).

(ii) For the above curve \( \beta_t = e^*_t \otimes q_t + e^* \otimes q_t \subset T_{\beta_t} \) for any \( t \). So \( \sigma(e^* \otimes q', e^* \otimes q) = \frac{d}{dt}|_{t=0} e^* \otimes q = e^* \otimes q'. \)

(iii) If \( f' \otimes q^2 \subset T_{\beta_t} \), then the relation \( \langle e^*, f' \rangle = 0 \) holds. For the above curve \( \beta_t = e^*_t \otimes q_t, f' \otimes q_t^2 \subset T_{\beta_t} \) for any \( t \) so \( \sigma(e^* \otimes q', f' \otimes q^2) = \frac{d}{dt}|_{t=0} (f' \otimes q^2) = f' \otimes (2q \otimes q') \).

(iv) Taking a curve \( \beta_t = e^*_t \otimes q \) as in (5) such that \( \frac{d}{dt}|_{t=0} e^*_t = e^*, \beta_0 = e^* \otimes q = \beta \) and \( \frac{d}{dt}|_{t=0} \beta_t = e^* \otimes q. \) Since \( e^* \otimes q \subset T_{\beta_t} \) for any \( t \), we obtain \( \sigma(e^* \otimes q, e^* \otimes q) = \frac{d}{dt}|_{t=0} e^* \otimes q = 0 \).

(v) We take the above curve \( \beta_t = e^*_t \otimes q + f_t \otimes q^2 \) and a vector field \( f_t \otimes q^2 \) along \( \beta_t \) with \( f_0 = 0. \) Then \( f_t \otimes q^2 \subset T_{\beta_t} \) whenever \( \langle e^*_t, f_t \rangle = 0 \) for any \( t \). Differentiating this equation, we know \( \frac{d}{dt}|_{t=0} f_t = -(e^*, f')e \) as in (v) of Case I. Hence \( \sigma(e^* \otimes q, f' \otimes q^2) = \frac{d}{dt}|_{t=0} f_t \otimes q^2 = -(e^*, f')e \otimes q^2. \)

(vi) Taking a curve \( \beta_t = e^* \otimes q + f_t \otimes q^2 \) as in (6) such that \( f_0 = 0 \) and \( \frac{d}{dt}|_{t=0} f_t = f' \),
\[ \beta_0 = e^* \otimes q = \beta \text{ and } \frac{d}{dt}|_{t=0} \beta_t = f' \otimes q^2. \] Since \( f'' \otimes q^2 \in T_{\beta_t} \) for any \( t \), we obtain
\[ \sigma(f' \otimes q^2, f'' \otimes q^2) = \frac{d}{dt}|_{t=0} f'' \otimes q^2 = 0. \] \hfill \Box

Hong-Kwon \[6\] have classified smooth Schubert varieties in the \( F_4 \)-homogeneous manifold \((F_4, \alpha_3)\). Thus, for the proof of Theorem \[12\] it suffices to consider the only two cases for \( Z \):

**Lemma 4.2.** Let \( X \) be the rational homogeneous manifold of type \((F_4, \alpha_3)\). A nonhomogeneous smooth Schubert variety \( Z \) of \( X \) is one of the followings:

1. the horospherical variety \((B_3, \alpha_2, \alpha_3)\),
2. the horospherical variety \((C_2, \alpha_2, \alpha_1)\) which is isomorphic to a smooth Schubert variety \( \Gr_n(2,6;F_0, F_5) \) in the symplectic Grassmannian \((C_3, \alpha_2)\).

**Remark 4.3.** Recall that all nonlinear homogeneous submanifolds associated to subdiagrams of the marked Dynkin diagram of \((F_4, \alpha_3)\) are \((B_3, \alpha_3)\) and \((C_3, \alpha_2)\).

As considered in Section 3, the odd symplectic Grassmannian \((C_2, \alpha_2, \alpha_1)\) is a unique nonhomogeneous smooth Schubert variety of \((C_3, \alpha_2)\).

**Lemma 4.4.** Let \( Z \) be a nonhomogeneous smooth Schubert variety of the rational homogeneous manifold of type \((F_4, \alpha_3)\). Then the variety \( B \) of minimal rational tangents of \( Z \) at a general point \( z \in Z \) is

1. a \( \PP^2 \)-bundle \( \PP(O(-1) \oplus O(-2)^2) \) over \( \PP(Q) = \PP^1 \) if \( Z \) is of type \((B_3, \alpha_2, \alpha_3)\),
2. a \( \PP^1 \)-bundle \( \PP(O(-1) \oplus O(-2)) \) over \( \PP(Q) = \PP^1 \) if \( Z \) is of type \((C_2, \alpha_2, \alpha_1)\).

**Proof.** (1) Hong and Kim \[7\] showed that the variety of minimal rational tangents of the horospherical variety \((B_n, \alpha_{n-1}, \alpha_n)\) is \( \PP(O_{p^2}(-1) \oplus O_{p^2}(-2)^{n-1}) \) by calculating the Chern numbers based on a gradation on its tangent space.

(2) As already described in Section 3, the variety of minimal rational tangents of the odd symplectic Grassmannian \((C_n, \alpha_k, \alpha_{k-1})\) is \( \PP(O_{p^{k-1}}(-1)^{2n-2k+1} \oplus O_{p^{k-1}}(-2)) \). \hfill \Box

Let \( Z \) be a smooth Schubert variety of type \((B_3, \alpha_2, \alpha_3)\). The gradation on the tangent space of \( Z \) described in Proposition 25 of Kim \[18\] could be embedded in the gradation on the tangent space

\[ g_{-1} = E^* \otimes Q, \quad g_{-2} = E \otimes S^2Q, \quad g_{-3} = Q, \quad g_{-4} = E^* \]

as a linear section by Lemma \[4.2\] (1) after proper shifting of the gradation on the tangent space of \( Z \). Let \( g'_{-1} \oplus g'_{-2} \oplus g'_{-3} \) be the induced gradation on the tangent space of \( Z \) from \( X \). Then

\[ g'_{-1} = F^* \otimes Q, \quad g'_{-2} = F^* \otimes S^2Q, \quad g'_{-3} = \wedge^2 F^* \]

where \( F^* \subset E^* \) is a fixed subspace of dimension 1 and \( F^{\perp} = \{ f \in E : \langle e^*, f \rangle = 0, \forall e^* \in E^* \} \). Hence, the variety \( B \) of minimal rational tangents of \( Z \) at a general point \( x \) is

\[ B = \PP(\{ e^* \otimes q + f \otimes q^2 : e^* \in F^*, f \in F^{\perp}, q \in Q \}) \] as a linear section of

\[ A = \PP(\{ e^* \otimes q + f \otimes q^2 : \langle e^*, f \rangle = 0, e^* \in E^*, f \in E, q \in Q \}). \]
This $\mathcal{B}$ is a $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ over $\mathbb{P}(Q) = \mathbb{P}^1$. This result coincides with Lemma 4.4 (1).

Let $Z$ be a smooth Schubert variety of type $(C_2, \alpha_2, \alpha_3)$. By Lemma 4.4 (2) and Proposition 25 of Kim [18], after proper shifting of the gradation on the tangent space of $Z$, we let $g'_{-1} \oplus g'_{-2}$ be the induced gradation on the tangent space of $Z$ from $X$. Then

$$g'_{-1} = F^* \otimes Q, \quad g'_{-2} = F' \otimes S^2Q,$$

where $F^* \subset E^*$ is the above fixed subspace and $F' \subset F^{*\perp}$ is an 1-dimensional subspace. Hence, the variety $\mathcal{B}$ of minimal rational tangents of $Z$ at a general point $x$ is

$$\mathcal{B} = \mathbb{P}(\{e^* \otimes q + f \otimes q^2 : e^* \in F^*, f \in F', q \in Q\})$$

as a linear section of $A$. This $\mathcal{B}$ is a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ over $\mathbb{P}(Q) = \mathbb{P}^1$. This result coincides with Lemma 4.4 (2).

**Proposition 4.5.** Let $X$ be the rational homogeneous manifold of type $(F_4, \alpha_3)$ and $Z$ be a smooth Schubert variety of type $(B_3, \alpha_2, \alpha_3)$ or $(C_2, \alpha_2, \alpha_1)$. Let $A \subset \mathbb{P}(T_xX)$ and $B \subset \mathbb{P}(T_yZ) \subset \mathbb{P}(T_xX)$ be the varieties of minimal rational tangents at a common general point $x$ of $X$ and $Z$. Then the pair $(A, B)$ is nondegenerate.

**Proof.** By this description of the variety $\mathcal{B}$ of minimal rational tangents of $Z$ as a linear section of the variety $A$ of minimal rational tangents of $X$ and by the computation of the second fundamental form of $A$, we get the desired results.

(1) Let $Z$ be a smooth Schubert variety of type $(B_3, \alpha_2, \alpha_3)$. (i) The tangent space $T_\beta \mathcal{B}$ at $\beta = e^* \otimes q$ is given by $\{e^* \otimes q' + e^* \otimes q + f^* \otimes q^2 : e^* \in F^*, f' \in F^{*\perp}, q' \in Q\}$. Then we have $\text{Ker} \sigma_\beta(\cdot, e^* \otimes Q) = e^* \otimes Q$, $\text{Ker} \sigma_\beta(\cdot, F^* \otimes q) = \{e^* \otimes q + f^* \otimes q^2 : e^* \in F^*, f' \in F^{*\perp}\}$ and $\text{Ker} \sigma_\beta(\cdot, F^{*\perp} \otimes q^2) = \{e^* \otimes q + f^* \otimes q^2 : e^* \in F^*, f' \in E\}$. Therefore, $\text{Ker} \sigma_\beta(\cdot, T_\beta \mathcal{B}) = C(e^* \otimes q) = C\beta$. (ii) The tangent space $T_\beta \mathcal{B}$ at $\beta = e^* \otimes q + f \otimes q^2$ is given by $\{e^* \otimes q' + e^* \otimes q + f^* \otimes q^2 + f \otimes (2q \otimes q') : e^* \in F^*, f' \in F^{*\perp}, q' \in Q\}$. Then we have $\text{Ker} \sigma_\beta(\cdot, F^* \otimes q) \cap \text{Ker} \sigma_\beta(\cdot, F^{*\perp} \otimes q^2) = \{e^* \otimes q + f^* \otimes q^2 : e^* \in F^*, f' \in F^{*\perp}\}$ and $\text{Ker} \sigma_\beta(\cdot, F^* \otimes q + 2f \otimes q \otimes q' : q' \in Q) = C(e^* \otimes q + f \otimes q^2)$. Therefore, $\text{Ker} \sigma_\beta(\cdot, T_\beta \mathcal{B}) = C(e^* \otimes q + f \otimes q^2) = C\beta$.

(2) Let $Z$ be a smooth Schubert variety of type $(C_2, \alpha_2, \alpha_1)$. (i) The tangent space $T_\beta \mathcal{B}$ at $\beta = e^* \otimes q$ is given by $\{e^* \otimes q' + e^* \otimes q + f^* \otimes q^2 : e^* \in F^*, f' \in F', q' \in Q\}$. Then we have $\text{Ker} \sigma_\beta(\cdot, e^* \otimes Q) = e^* \otimes Q$, $\text{Ker} \sigma_\beta(\cdot, F^* \otimes q) = \{e^* \otimes q + f^* \otimes q^2 : e^* \in E^*, f' \in F'\}$ and $\text{Ker} \sigma_\beta(\cdot, F^{*\perp} \otimes q^2) = \{e^* \otimes q + f^* \otimes q^2 : e^* \in F^*, f' \in F'\}$. Therefore, $\text{Ker} \sigma_\beta(\cdot, T_\beta \mathcal{B}) = C(e^* \otimes q) = C\beta$. (ii) The tangent space $T_\beta \mathcal{B}$ at $\beta = e^* \otimes q + f \otimes q^2$ is given by $\{e^* \otimes q' + e^* \otimes q + f^* \otimes q^2 + f \otimes (2q \otimes q') : e^* \in F^*, f' \in F', q' \in Q\}$. Then we have $\text{Ker} \sigma_\beta(\cdot, F^* \otimes q) \cap \text{Ker} \sigma_\beta(\cdot, F^{*\perp} \otimes q^2) = \{e^* \otimes q + f^* \otimes q^2 : e^* \in F^*, f' \in F^{*\perp}\}$ and $\text{Ker} \sigma_\beta(\cdot, e^* \otimes q + 2f \otimes q \otimes q' : q' \in Q) = C(e^* \otimes q + f \otimes q^2)$. Therefore, $\text{Ker} \sigma_\beta(\cdot, T_\beta \mathcal{B}) = C(e^* \otimes q + f \otimes q^2) = C\beta$. □

**Proposition 4.6.** In the setting of Proposition 4.3, if $h$ is an element in the isotropy subgroup $P_x$ of $G$ at a general point $x$ in $Z$ such that $h\mathcal{B}$ and $\mathcal{B}$ are tangent at a general point of intersection, then $h\mathcal{B}$ is equal to $\mathcal{B}$.

**Proof.** We recall $K = C^* \times \text{SL}(E^*) \times \text{SL}(Q)$-module isomorphisms:

$$g_{-1} = E^* \otimes Q, \quad g_{-2} = \wedge^2 E^* \otimes S^2Q, \quad g_{-3} = Q, \quad g_{-4} = E^*.$$
under the identification $T_xX \cong g_{-1} \oplus g_{-2} \oplus g_{-3} \oplus g_{-4}$ at the base point $o \in X = G/P$.

(1) Let $Z$ be a smooth Schubert variety of type $(B_3, \alpha_2, \alpha_3)$. The variety of minimal rational tangents at a general point $x$ is

$$B = \mathbb{P}\{\langle e^* \otimes q + f \otimes q^2 : e^* \in F^*, f \in F^{*\perp}, q \in Q \rangle\},$$

where $F^* \subset E^*$ is a subspace of dimension 1 and $F^{*\perp} = \{f \in E : \langle e^*, f \rangle = 0, \forall e^* \in F^*\}$.

Let $h$ be an element in the isotropy subgroup $P_x$ of $G$, then $h = h_0h_1h_2h_3h_4$ where $dh_i \in \bigoplus_{j \in Z} \text{Hom}(g_j, g_{j+i}).$ The left-multiplication actions of $h_i$ at $e^* \otimes q + f \otimes q^2 \in B$ are

$$h_0(e^* \otimes q + f \otimes q^2) = h_0e^* \otimes h_0q + h_0f \otimes h_0q^2 \in (E^* \otimes Q) \oplus (\wedge^2 E^* \otimes S^2 Q),$$

$$h_1(e^* \otimes q + f \otimes q^2) = e^* \otimes q + f \otimes q^2 + dh_1(f \otimes q^2),$$

$$h_i(e^* \otimes q + f \otimes q^2) = e^* \otimes q + f \otimes q^2 \text{ for } i = 2, 3, 4.$$

Let $f = f_1^* \wedge f_2^*$ for $f_1^*, f_2^* \in E^*$. For the action $dh_1 \in \bigoplus_{j \in Z} \text{Hom}(g_j, g_{j+i})$, we consider the adjoint action of $e^* \otimes q + f \otimes q^2$ as follows:

$$(E \otimes Q^*) \times (\wedge^2 E^* \otimes \text{Sym}^2 Q) \rightarrow E^* \otimes Q$$

$$(e' \otimes q^*, f \otimes q^2) \mapsto cf(e') \otimes q$$

where $e' \in E$, $q^* \in Q^*$, $c$ is a scalar which is zero if $\langle q'^*, q \rangle = 0$, and $f(e') = (f_1^* \wedge f_2^*)(e') = (f_1^*, e')f_2^* - (f_2^*, e')f_1^*$. From now on, $h_1$ action on a subspace $F^{*\perp} \subset \wedge^2 E^*$ means $f(e')$, i.e., $f(e') \in h_1F^{*\perp}$ and $dh_1(f \otimes q^2) = h_1f \otimes cq$ for some scalar $c$.

Since $dh_1(f \otimes q^2) \in E^* \otimes Q$ and $h_0Q = Q$, it follows that

$$hB = \mathbb{P}\{\langle e^* \otimes q + f \otimes q^2 : e^* \in h_0F^* + h_0h_1(F^{*\perp}), f \in h_0(F^{*\perp}), q \in Q \rangle\}.$$

If $B$ and $hB$ intersect at a general point $\beta = e^* \otimes q + f \otimes q^2 \in B \cap (hB)$, then the tangent space $T_\beta(hB)$ is given by

$$\{e^* \otimes q' + e^* \otimes q + f' \otimes q^2 + f \otimes (2q \circ q') : e^* \in h_0F^* + h_0h_1(F^{*\perp}), f' \in h_0(F^{*\perp}), q' \in Q\}.$$

By assumption, $T_\beta(hB)$ coincide with $T_\beta(B)$, we see $h_0F^* + h_0h_1(F^{*\perp}) = F^*$ and $h_0(F^{*\perp}) = F^{*\perp}$. Hence, $hB = B$.

(2) Let $Z$ be a smooth Schubert variety of type $(C_2, \alpha_2, \alpha_1)$. The variety of minimal rational tangents at a general point $x$ is

$$B = \mathbb{P}\{\langle e^* \otimes q + f \otimes q^2 : e^* \in F^*, f \in F', q \in Q \rangle\},$$

where $F^* \subset E^*$ and $F' \subset F^{*\perp}$ are subspaces of dimension 1.

If $h$ is an element in the isotropy subgroup $P_x$ of $G$, then

$$hB = \mathbb{P}\{\langle e^* \otimes q + f \otimes q^2 : e^* \in h_0F^* + h_0h_1F', f \in h_0F', q \in Q \rangle\}.$$

If $B$ and $hB$ intersect at a general point $\beta = e^* \otimes q + f \otimes q^2 \in B \cap (hB)$, then the tangent space $T_\beta(hB)$ is given by

$$\{e^* \otimes q' + e^* \otimes q + f' \otimes q^2 + f \otimes (2q \circ q') : e^* \in h_0F^* + h_0h_1F', f' \in h_0F', q' \in Q\}.$$

By assumption, $T_\beta(hB)$ coincide with $T_\beta(B)$, we see $h_0F^* + h_0h_1F' = F^*$ and $h_0(F^{*\perp}) = F^{*\perp}$. Hence, $hB = B$. \qed
Remark 4.7. In the proof (1) of Proposition 4.6 suppose that \( h_0F^* \subset h_0h_1(F^*) \), the dimension of \( h_0h_1(F^*) \) is 2, and the dimension of \( h_0(F^*) \) is 1, then \( hB = \mathbb{P}(e^* \otimes q + f \otimes q^2 : e^* \in h_0h_1(F^*), f \in h_0(F^*), q \in Q) \) which is isomorphic to \( \mathbb{P}(O_{\mathbb{P}^1}(-1)^2 \otimes O_{\mathbb{P}^1}(-2)) \). In this case, two rank 3 vector bundles \( hB = \mathbb{P}(O_{\mathbb{P}^1}(-1)^2 \otimes O_{\mathbb{P}^1}(-2)) \) and \( B = \mathbb{P}(O_{\mathbb{P}^1}(-1) \otimes O_{\mathbb{P}^1}(-2)^2) \) are not tangent at a general point of intersection.

Proposition 4.8. In the setting of Proposition 4.6, if \( B' = A \cap \mathbb{P}(W') \) is another linear section of \( A \) by a linear subspace \( \mathbb{P}(W') \) of \( \mathbb{P}(T_xX) \) such that \( (B \subset \mathbb{P}(T_xZ)) \) is projectively equivalent to \( (B' \subset \mathbb{P}(W')) \), then there is an element \( h \) in a Levi factor of \( P \) such that \( B' = hB \).

Proof. Let \( Z \) be a smooth Schubert variety of type \((B_3, \alpha_2, \alpha_3)\). Since \( B \) is a \( \mathbb{P}^2 \)-bundle on \( \mathbb{P}(Q) = \mathbb{P}^1 \), \( B' \) is also a \( \mathbb{P}^2 \)-bundle on \( \mathbb{P}^1 \). Let \( B'_1 \) be a linear section of \( B' \) which is projectively equivalent to the linear section \( B_1 := B \cap \mathbb{P}(E^* \otimes Q) \simeq \mathbb{P}(F^*) \times \mathbb{P}(Q) \) of \( B \).

Suppose that \( B_1 \) is not contained in \( \mathbb{P}(E^* \otimes Q) \). Take \( b' = e^* \otimes q + f \otimes q^2 \in B'_1 \cap (\widetilde{A}(E^* \otimes Q))^\perp \). Since \( B' \) is a linear section \( A \cap \mathbb{P}(W') \) of \( A \), the tangent space \( T_{b'}B' \) at \( b' \) is contained in the intersection \( W' \cap T_{b'}A \) and the second fundamental form \( \sigma_{b'}^E : T_{b'}B' \times T_{b'}B' \to W'/T_{b'}B' \) of \( B' \) at \( b' \), composed with the quotient map \( W'/T_{b'}B' \to T_{b'}X/T_{b'}A \), is the restriction of the second fundamental form \( \sigma_{b'} \) of \( A \) to \( T_{b'}B' \). In particular, \( \{ v \in T_{b'}B' : \sigma_{b'}^E(v, v) = 0 \} \) is contained in

\[
\{ v \in T_{b'}A : \sigma_{b'}(v, v) = 0 \} = \{ e^* \otimes q + f' \otimes q^2 : e^*, f' \in E^*, f' \in E \}.
\]

Hence, \( \{ v \in T_{b'}B' : \sigma_{b'}^E(v, v) = 0 \} \) is a linear section of an affine cone of a hyperquadric \( Q^3 \) because \( B' \) is a linear section of \( A \). For \( b \in B_1 \), \( \{ v \in T_bB : \sigma_b^E(v, v) = 0 \} \) is the union of three subspaces \( F^* \otimes q, e^* \otimes Q \) and \( F^* \otimes q^2 \), while for \( b' \in B_1 \cap (\widetilde{A}(E^* \otimes Q)), \{ v \in T_{b'}B' : \sigma_{b'}^E(v, v) = 0 \} \) is as above. Thus the second fundamental form \( \sigma_{b'}^E \) is not isomorphic to \( \sigma_{b'} \) and hence \( B \subset \mathbb{P}(T_xZ) \) cannot be projectively equivalent to \( B' \subset \mathbb{P}(W') \), which is a contradiction. Therefore, \( B_1 \) is contained in \( \mathbb{P}(E^* \otimes Q) \cap A \simeq \mathbb{P}(E^*) \times \mathbb{P}(Q) \).

For \( b' = e^* \otimes q \in B_1 \), \( \{ v \in T_{b'}B' : \sigma_{b'}^E(v, v) = 0 \} \) should be the union of three subspaces \( R^* \otimes q, e^* \otimes Q \) and \( R^* \otimes q^2 \) for a subspaces \( R^* \subset E^* \) of \( \dim R^* = 1 \). Hence, \( B' \) has nonzero intersection \( B'_1 = \mathbb{P}(R^*) \times \mathbb{P}(Q) \) with \( \mathbb{P}(E^*) \times \mathbb{P}(Q) \).

Since \( B \) and \( B' \) are linear sections of \( A \) and second fundamental forms of \( B \) and \( B' \) are equivalent, we see that \( \{ v \in T_{b'}B' : \sigma_{b'}^E(v, v) = 0 \} \) is contained in \( \{ v \in T_{b'}A : \sigma_{b'}(v, v) = 0 \} \) as a linear section:

\[
\{ e^* \otimes q + f' \otimes q^2 : (e^*, f) + (e^*, f') = 0, e^* \in R^*, f' \in R^* \}.
\]

More precisely, for \( b' = e^* \otimes q + f \otimes q^2 \in B'_1 \cap (\widetilde{A}(E^* \otimes Q)), \) the space \( \{ v \in T_{b'}B' : \sigma_{b'}(v, v) = 0 \} \) contains \( \{ e^* \otimes q + f' \otimes q^2 : (e^*, f) + (e^*, f') = 0, e^* \in R^*, f' \in R^* \} \), meanwhile, for \( b = e^* \otimes q + f \otimes q^2 \in B_1 \cap (\widetilde{A}(E^* \otimes Q)), \) the space \( \{ v \in T_bB : \sigma_b(v, v) = 0 \} \) is \( \{ e^* \otimes q + f' \otimes q^2 : (e^*, f) + (e^*, f') = 0, e^* \in F^*, f' \in F^* \} \). Because second fundamental forms of \( B' \) and \( B \) are equivalent, the dimensions of base locus are same, the space \( \{ v \in T_{b'}B : \sigma_{b'}(v, v) = 0 \} \) should be \( \{ e^* \otimes q + f' \otimes q^2 : (e^*, f) + (e^*, f') = 0, e^* \in R^*, f' \in R^* \} \) which is tangent to the fiber of the projection \( A \to \mathbb{P}(Q) \). Hence, \( B' = \mathbb{P}(e^* \otimes q + f \otimes q^2 : q \in \mathbb{P}(Q)) \).
Recall that $F^*$ and $R^*$ are linear subspaces in $E^*$ with the dimensions $\dim F^* = \dim R^* = 1$. Thus, $B' = hB$ for some $h \in \text{SL}(E^*) \times \text{SL}(Q)$, which is contained in the semisimple part of $P$.

For a smooth Schubert variety of type $(C_2, \alpha_2, \alpha_1)$, we can prove this in the same way.

Proof of Theorem 1.2 in the case that $X$ is the $F_4$-homogeneous manifold associated to a short simple root. Since any smooth Schubert variety in the 15-dimensional $F_4$-homogeneous manifold $(F_4, \alpha_4)$ is a linear space by Theorem 1.3 of Hong-Kwon [6], it suffices to consider nonhomogeneous smooth Schubert varieties of type $(B_3, \alpha_2, \alpha_3)$ and $(C_2, \alpha_2, \alpha_1)$ in the 20-dimensional $F_4$-homogeneous manifold $(F_4, \alpha_3)$. Using Proposition 2.2, Proposition 4.5, Proposition 4.6 and Proposition 4.8, the same argument in the proof of Theorem 1.2 given in Section 3 completes the proof.

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