A NONLINEAR EVOLUTION EQUATION WITH 2 + 1
DIMENSIONS RELATED TO NONSTATIONARY DIRAC-TYPE
SYSTEM

MANSUR I ISMAILOV

Abstract. In this paper the inverse scattering problem for the nonstationary
Dirac-type system on the whole plane was considered. A nonlinear evolution
ystem of equation related to nonstationary Dirac-type system is introduced
and the solviblity of this system using the IST method is studied.

1. Introduction

As is known, every linear scattering problem determines a class of nonlinear
evolution equations for which the inverse scattering transform (IST) method is
suitable to integrate these equations. The investigation of the nonlinear evolution
equations by the IST method has been stated in [1, 2]. The essence of this method
is to represent this nonlinear equation as a Lax equation

\[
\frac{\partial L}{\partial t} = [L, A]
\]

by using the Lax pair \( L \) and \( A \). The equation in the form of (0.1) allows to
investigate evolution of scattering data instead of the evolution of the operator \( L \),
if the inverse scattering problem is investigated for the operator \( L \).

The IST method for the nonlinear evolution equations with 1+1 dimensions
(one space and one time dimensions) have been described in a variety reviews and
monographs. The generalization of the IST method to nonlinear evolution equations
with 2+1 dimensions (two space and one time dimension ) has been developed in
monograph [3]. In this direction one can also notice the papers [4 - 7], in which
the inverse scattering problem for the first order systems is studied and this inverse
problem was applied to integration of \( N \)- wave interactions, Davey-Stewartson and
Kodomtsev-Petviashvili equations.

Let \( L = \frac{\partial}{\partial y} - M \), where \( M \) is an ordinary differential operator in \( x \). Then
equation (0.1) is rewritten as (see [2])

\[
\left[ \frac{\partial}{\partial y} - M, \frac{\partial}{\partial t} - A \right] = 0
\]

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and it is a relation for ordinary differential operators. Some nonlinear evolution equations in 2+1 dimension are presented in [8] by using commutativity condition (0.2) when $M$ is an scalar coefficients ordinary differential operator in $x$.

This article is organized as follows: In Section 2, we consider the inverse scattering problems for the $L = \frac{\partial}{\partial y} - M$, with $M = \sigma \frac{\partial}{\partial x} + Q(x,y)$ in the whole plane. We introduce the scattering data for the operator $L$ in the plane (the minimal information for a unique restoration of matrix-function $Q(x,y)$). All the results in Section 2 are the matrix generalization of the L. P. Nizhnik's results for two component nonstationary Dirac system on the whole plane in [9] (see [4] also). Taking into account that this generalization is not difficult, so we do not give the proofs. In Section 3, we introduce some system of nonlinear evolution equations with 2+1 dimensions related to nonstationary Dirac-type systems and we study the solvability of this system using the IST method. In [10], the IST method is applied to integrate the two component nonlinear Schrödinger equation, using the inverse scattering problem for stationary Dirac-type system in whole line.

2. Inverse scattering problem for the Dirac-type system in the plane

Let us consider a system of first-order partial differential equations (PDEs)

$$L(\psi) \equiv \frac{\partial \psi}{\partial y} - \sigma \frac{\partial \psi}{\partial x} - Q(x,y) \psi = 0, \quad -\infty < x, y < +\infty,$$

where

$$\sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & q_1 \\ 0 & 0 & q_2 \\ q_3 & q_4 & 0 \end{bmatrix},$$

with the measurable complex-valued functions $q_i$ ($i = 1, \ldots, 4$).

We give fundamental results on the inverse scattering problem for the system (1.1) in the case where the coefficients $q_i$ ($i = 1, \ldots, 4$) decrease quite fast with respect to variables $x$ and $y$ at infinity. Each bounded solution of the system (1.1) has the asymptotic

$$\psi(x,y) = F_x a_-(y) + o(1), \quad y \to -\infty,$$

$$\psi(x,t) = F_x a_+(t) + o(1), \quad y \to +\infty.$$

where $F_x$ denotes the diagonal matrix shift operator, such that for a vector function $a(y) = \text{col} [a_1(y), a_2(y), a_3(y)]$, $F_x a(y) = \text{col} [a_1(t+x), a_2(t+x), a_3(t-x)]$

The scattering operator $S$ is defined by the equality

$$a_+(y) = Sa_- (y).$$

This operator has the inverse $S^{-1}$ and $S = I + F$ and $S^{-1} = I + G$, where $F = [F_{ij}]_{i,j=1}^3$ and $G = [G_{ij}]_{i,j=1}^3$ are matrix integral operators. To regenerate the system (1.1), i.e., the coefficients $q_i$ ($i = 1, \ldots, 4$) it is sufficient to know $F_{13}, F_{23}, G_{31}$ and $G_{32}$. The collection $\{F_{13}, F_{23}, G_{31}, G_{32}\}$ is called the scattering data for the system (1.1) in the plane. The solution of the inverse problem reduces to the integral equation
A non-linear evolution equation related to nonstationary Dirac-type systems

\[ A(x, y, \tau) - \int_{-\infty}^{y} \left[ \int_{y}^{+\infty} A(x, y, z) \tilde{G}(z - x, s - x) \, dz \right] \tilde{F}(s + x, \tau - x) \, ds = \tilde{F}(y + x, \tau - x), \]

(1.3)

\[ B(x, y, \tau) - \int_{y}^{+\infty} \left[ \int_{-\infty}^{y} B(x, y, z) \tilde{F}(z + x, s - x) \, dz \right] \tilde{G}(s + x, \tau - x) \, ds = \tilde{G}(y - x, \tau + x), \]

where \( \tilde{F} = \begin{bmatrix} F_{13} \\ F_{23} \end{bmatrix} \) and \( \tilde{G} = \begin{bmatrix} G_{31} & G_{32} \end{bmatrix} \).

The coefficients \( q_i \ (i = 1, \ldots, 4) \) of the system (1.1) are expressed in terms of the solution of (1.3) by means of the equations

\[ \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}(x, y) = -2A(x, y, y), \quad \begin{bmatrix} q_3 & q_4 \end{bmatrix}(x, y) = -2B(x, y, y). \]

3. Nonlinear evolution equation related to nonstationary Dirac-type systems

Let \( M \) and \( A \) be first order matrix operators in (0.2):

\[ M = \sigma \frac{\partial}{\partial x} + Q, \quad A = \tau \frac{\partial}{\partial x} + P. \]

Then the equation (0.2) becomes to the form

\[ \left( \frac{\partial}{\partial y} - \sigma \frac{\partial}{\partial x} - Q, \frac{\partial}{\partial t} - \tau \frac{\partial}{\partial x} - P \right) = 0. \]

(2.1)

Here \( \tau \) and \( P \) are third order square matrices. Let the matrix \( \tau \) be real and diagonal: \( \tau = \text{diag}(b_1, b_2, b_3), \) \( b_1 > b_2 > b_3 \) and the matrices \( Q \) and \( P \) obey the relation: \( [\sigma, P] = [\tau, Q] \).

Take \( P = \begin{bmatrix} 0 & v_{12} & \frac{b_1 - b_2}{2} q_1 \\ 0 & 0 & \frac{b_2 - b_3}{2} q_2 \\ \frac{b_1 - b_3}{2} q_3 & \frac{b_2 - b_3}{2} q_4 & 0 \end{bmatrix} \), where the functions \( v_{12} \) and \( v_{21} \) are solutions of the equations

\[ \frac{\partial}{\partial y} v_{12} - \frac{\partial}{\partial x} v_{12} = -\frac{b_1 - b_2}{2} q_1 q_4, \]
\[ \frac{\partial}{\partial y} v_{21} - \frac{\partial}{\partial x} v_{21} = -\frac{b_2 - b_1}{2} q_2 q_3, \]

respectively.

Then, the conditions (2.1) are reduced to the system of equations

\[ \begin{align*}
\frac{\partial}{\partial t} q_1 + k_1 \frac{\partial}{\partial y} q_1 + k_2 \frac{\partial}{\partial x} q_1 &= v_{12} q_2, \\
\frac{\partial}{\partial t} q_2 + k_1 \frac{\partial}{\partial y} q_2 + k_2 \frac{\partial}{\partial x} q_2 &= v_{21} q_1,
\end{align*} \]

(2.2)

\[ \begin{align*}
\frac{\partial}{\partial t} q_3 + k_3 \frac{\partial}{\partial y} q_3 + k_4 \frac{\partial}{\partial x} q_3 &= -v_{21} q_4, \\
\frac{\partial}{\partial t} q_4 + k_3 \frac{\partial}{\partial y} q_4 + k_4 \frac{\partial}{\partial x} q_4 &= -v_{12} q_3,
\end{align*} \]
\[ k_1 = -\frac{b_1 - b_2}{2}, \quad k_2 = -\frac{b_1 + b_3}{2}, \quad k_3 = -\frac{b_2 - b_3}{2}, \quad k_4 = -\frac{b_2 + b_3}{2}. \]

Let us denote \( P = \frac{\partial}{\partial t} - \tau \frac{\partial}{\partial y} - P \). The Lax form (2.1) of the system of equations (2.2) enables us to apply IST method for the integration.

**Lemma 1.** Let \( \psi \) be a solution of the Dirac-type system (1.1), whose the coefficients \( q_i, i = 1, \ldots, 4 \) satisfy Eqs. (2.2). Then the function \( \varphi = P\psi \) also satisfy the system (1.1).

**Proof.** Let us apply the operator equation to \( \psi \):

\[
(LP - PL)\psi = L(P\psi) - P(L\psi) = 0.
\]

Taking into account that \( L\psi = 0 \), then \( L(P\psi) = 0 \) follows from the last equation. It means that \( P\psi \) is solution of system (1.1). \( \Box \)

The next theorem is true for the evolution of the scattering data.

**Lemma 2.** Let the coefficients \( q_i, i = 1, \ldots, 4 \) of the Dirac-type system (1.1) depend on \( t \) as a parameter and satisfy the system of equation (2.2). Besides that

\[ v_{12}(x, \pm\infty) = 0, \quad v_{21}(x, \pm\infty) = 0. \]

Then the kernels \( F_{13}(y, \tau; t), F_{23}(y, \tau; t), G_{31}(y, \tau; t), G_{32}(y, \tau; t) \) of the integral operators \( F_{13}, F_{23}, G_{31}, G_{32} \) corresponding to the scattering operator \( S \) for the system (1.1) in the plane satisfy the system of equations

\[
\frac{\partial}{\partial t} F_{13}(y, \tau; t) - b_1 \frac{\partial}{\partial y} F_{13}(y, \tau; t) + b_3 \frac{\partial}{\partial \tau} F_{13}(y, \tau; t) = 0,
\]

(2.4)

\[
\frac{\partial}{\partial t} F_{23}(y, \tau; t) - b_1 \frac{\partial}{\partial y} F_{23}(y, \tau; t) + b_3 \frac{\partial}{\partial \tau} F_{23}(y, \tau; t) = 0
\]

and

\[
\frac{\partial}{\partial t} G_{31}(y, \tau; t) + b_1 \frac{\partial}{\partial y} G_{31}(y, \tau; t) - b_1 \frac{\partial}{\partial \tau} G_{31}(y, \tau; t) = 0,
\]

(2.5)

\[
\frac{\partial}{\partial t} G_{32}(y, \tau; t) + b_1 \frac{\partial}{\partial y} G_{32}(y, \tau; t) - b_2 \frac{\partial}{\partial \tau} G_{32}(y, \tau; t) = 0.
\]

**Proof.** By virtue of definition of the scattering operator \( S \), from Lemma 1 we get

\[ \varphi_+ = S\varphi_- \]

where \( \varphi_+ = P a_+ \), \( P = \frac{\partial}{\partial t} - \tau \sigma \frac{\partial}{\partial y} \).

Since \( a_+ = Sa_- \) (see (1.2)), the equality

\[ PS = SP \]

(2.7)

follows from (2.6).

Analogously,

\[ PS^{-1} = S^{-1}P. \]

(2.8)

From the matrix operator equation (2.7) it follows that the kernels of the integral operators \( F_{13} \) and \( F_{23} \) satisfy system of equation (2.4). The similarly system of
equation (2.5) for the kernels of the integral operators $G_{31}$ and $G_{32}$ follows from the matrix operator equation (2.8).

According to Chapter 2 and Lemma 2, let us give a procedure for the solution of the system (2.2) by IST method.

**Theorem 1.** The system of equations (2.2) admits integration in the class of decreasing functions that satisfy the condition (2.3) by the IST method. The solution $q_i, i = 1, \ldots, 4$ of the system (2.2) is determined by formulae

$$
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}
(x,y,t) = -2A(x,y,y,t), \quad \begin{bmatrix}
q_3 \\
q_4
\end{bmatrix}
(x,y,t) = -2B(x,y,y,t),
$$

where the vector functions $A$ and $B$ are the solutions of the integral equations

$$
A(x,y,\tau;t) - \int_{-\infty}^{y} \left[ \int_{-\infty}^{+\infty} A(x,y,z;\tau) \tilde{G}(z-x,s+x;\tau)ds \right] \tilde{F}(s+x,\tau-x;t)ds = \tilde{F}(y+x,\tau-x;t),
$$

$$
B(x,y,\tau;t) - \int_{y}^{+\infty} \left[ \int_{-\infty}^{+\infty} B(x,y,z;\tau) \tilde{G}(z-x,s-x;\tau)ds \right] \tilde{G}(s+x,\tau-x;t)ds = \tilde{G}(y-x,\tau+x;t),
$$

where $\tilde{F} = \begin{bmatrix} F_{13} \\ F_{23} \end{bmatrix}$ and $\tilde{G} = \begin{bmatrix} G_{31} & G_{32} \end{bmatrix}$ are satisfied the evolution equations (2.4) and (2.5).

Thus the system of nonlinear evolution equation (2.2) admits the integration by IST method. Let us denote by $\Pi$ the operator transforming the coefficients $q(t) = \text{col}(q_{1}(x,y,t), q_{2}(x,y,t), q_{3}(x,y,t), q_{4}(x,y,t))$ of the equation (1.1) to scattering data $T(t) = \text{col}(F_{13}(y,\tau;t), F_{23}(y,\tau;t), G_{31}(y,\tau;t), G_{32}(y,\tau;t))$ as follows:

$$
\Pi : q(t) \longrightarrow T(t).
$$

Then the solution of the system (2.2) can be represent as

$$
q(t) = \Pi^{-1} e^{iMT}
$$

where $T = T(0)$ i.e. $T = \text{col}(F_{13}(y,\tau), F_{23}(y,\tau), G_{31}(y,\tau), G_{32}(y,\tau))$ and

$$
M = \begin{bmatrix}
b_{1} \frac{\partial}{\partial y} - b_{3} \frac{\partial}{\partial \tau} & 0 & 0 & 0 \\
0 & b_{2} \frac{\partial}{\partial y} - b_{3} \frac{\partial}{\partial \tau} & 0 & 0 \\
0 & 0 & b_{1} \frac{\partial}{\partial y} - b_{3} \frac{\partial}{\partial \tau} & 0 \\
0 & 0 & 0 & b_{2} \frac{\partial}{\partial \tau} - b_{3} \frac{\partial}{\partial y}
\end{bmatrix}.
$$

The formula (2.9) is a specified form of solution $q(t)$ of (2.2).

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Department of Mathematics, Gebze Institute of Technology, Gebze-Kocaeli 41400, Turkey

E-mail address: mismailov@gyte.edu.tr