A Differential Harnack Inequality for the Newell-Whitehead-Segel Equation

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\textbf{Abstract.} This paper will develop a Li-Yau-Hamilton type differential Harnack estimate for positive solutions to the Newell-Whitehead-Segel equation on $\mathbb{R}^n$. We then use our LYH-differential Harnack inequality to prove several properties about positive solutions to the equation, including deriving a classical Harnack inequality and characterizing standing solutions and traveling wave solutions.

\textbf{Key Words:} Newell-Whitehead-Segel equation, Harnack estimate, Harnack inequality, wave solutions.

\textbf{AMS Subject Classifications:} 35C07, 35K10, 35K55

1 Introduction

Consider any positive solution $f: \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ to the Newell-Whitehead-Segel Equation,

$$f_t = \Delta f + af - bf^3,$$

here, we assume $a > 0$, $b > 0$. This equation was first introduced by A. C. Newell and J. A. Whitehead in 1969 [6] and shortly after was studied by L. Segel [9]. Exact solutions to the equation were computed using the Homotopy Perturbation method by S.
Nourazar, M. Soori and A. Nazari-Golshan in 2011 [8], while some approximate solutions were computed in 2015 by J. Patade and S. Bhalekar [7]. The equation is an example of a reaction-diffusion equation, as it is used to model the change of concentration of a substance, given any chemical reactions that the substance may be undergoing (modeled by the $af - bf^3$ term) and any diffusion causing the chemical to spread throughout the medium (modeled by the $\Delta f$ term). More specifically, the Newell-Whitehead-Segel equation models Rayleigh-Bénard convection, a reaction-diffusion phenomenon that occurs when a fluid is heated from below.

In this paper, we are just concerned with positive solutions on $\mathbb{R}^n$. For further discussion about working with functions on closed manifolds or complete non-compact manifolds, see [3]. Our main theorem, Theorem 1.1, will outline a Li-Yau-Hamilton type differential Harnack estimate (2) that we will prove based on computing time-evolutions of the relevant quantities, see Hamilton [4]. In the following, Harnack inequality or Harnack estimate refers to an LYH-type differential Harnack inequality. As an application, we will integrate our estimate (2) along a space time curve to obtain a classical Harnack inequality (16), see Corollary 4.1. Then we will use our Harnack estimate to characterize both traveling wave solutions and standing solutions to the Newell-Whitehead-Segel equation.

**Theorem 1.1.** With $f > 0$ a solution to (1.1), define $l = \log f$. Then:

\[
H = \alpha \Delta l + \beta |\nabla l|^2 + \gamma e^{2l} + \varphi(t) \geq 0,
\]

(1.2)

provided the following three inequalities hold:

(a) $\alpha > \beta \geq 0$,

(b) $\gamma \leq -\frac{nba^2(2a + \beta)}{3n\alpha^2 - 2(\alpha - \beta)\beta} < 0$,

(c) $4\gamma(\alpha - \beta) + na^2b < 0$,

with

\[
\varphi(t) = \left(\frac{a\alpha}{1 - e^{2at}}\right) \left(\frac{\gamma e^{2at} - \alpha \gamma n}{4\gamma(\alpha - \beta) + a^2bn}\right).
\]

If, instead of inequality (c), we have:

(d) $4\gamma(\alpha - \beta) + na^2b \geq 0$,

then:

\[
H = \alpha \Delta l + \beta |\nabla l|^2 + \gamma e^{2l} + \psi(t) \geq 0,
\]

(1.3)

for:

\[
\psi(t) = \begin{cases} 
\frac{na^2}{2(\alpha - \beta)t} & t \leq T := \frac{na^2}{2(\alpha - \beta)(-a\gamma)} \left(2\left(\frac{\alpha - \beta}{na^2}\right)\gamma + b\right), \\
-ana^2\gamma \left(e^{2at} - 1\right) & t > T.
\end{cases}
\]