THE CONVERGENCE ESTIMATES FOR GALERKIN-WAVELET SOLUTION OF PERIODIC PSEUDODIFFERENTIAL INITIAL VALUE PROBLEMS

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Using the discrete Fourier transform and Galerkin-Petrov scheme, we get some results on the solutions and the convergence estimates for periodic pseudodifferential initial value problems.

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1. Introduction. In recent years, wavelets have been developing intensively and have become a powerful tool to study mathematics and technology, for example, the theory of the singular integral, singular integro-differential equations, the areas such as sound analysis, image compression, and so on (see [9, 10] and references therein). In this paper, we use a scaling function and a multilevel approach to estimate the error of the problem

\[ \frac{\partial u(x,t)}{\partial t} = a \cdot Au(x,t), \quad x \in \mathbb{Z}^n, \quad t > 0, \quad a \in \mathbb{R}, \]

\[ u(x,0) = [u_0](x), \quad x \in \mathbb{Z}^n, \tag{1.1} \]

where \( A \) is a pseudodifferential operator (see [1, 2, 3, 4, 6, 8, 9, 12]) with a symbol \( \sigma \in C^\infty(\mathbb{R}^n) \), \( \sigma \) is positively homogeneous of degree \( r > 0 \) such that

\[ |D^\alpha \sigma(\xi)| \leq C_\alpha (1 + |\xi|)^{r-|\alpha|}, \quad \text{for all multi-index } \alpha \in \mathbb{N}^n, \tag{1.2} \]

\( \mathbb{Z}^n = \mathbb{R}^n / \mathbb{Z}^n \), and \([u_0](x) = \sum_{k \in \mathbb{Z}^n} u_0(x + k) \) is a periodic operator.

We discuss only problem (1.1) with the following condition:

\[ a \sigma(\xi) \leq 0, \quad \forall \xi \in \mathbb{Z}^n. \tag{1.3} \]

2. Preliminaries and notations. The continuous Fourier transform of the function \( f \in L_2(\mathbb{R}^n) \) is defined by

\[ \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \xi} f(x) dx, \quad \xi \in \mathbb{R}^n \tag{2.1} \]
with the inverse Fourier formula

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \xi} \hat{f}(\xi) d\xi, \quad \xi \in \mathbb{R}^n$$

(2.2)

(see [4, 8, 11]).

The discrete Fourier transform of the function $f \in L_2(\mathbb{J}^n)$ is

$$\mathcal{F}(f)(\xi) = \tilde{f}(\xi) := \int_{[0,1]^n} e^{-2\pi i x \xi} f(x) dx, \quad \xi \in \mathbb{Z}^n,$n

(2.3)

and the inverse Fourier transform is

$$f(x) := \sum_{\xi \in \mathbb{Z}^n} \tilde{f}(\xi) e^{2\pi i x \xi}$$

(2.4)

(see [6]).

Some simple properties of the discrete Fourier transform are

$$(f, g)_0 = \sum_{\xi \in \mathbb{Z}^n} \tilde{f}(\xi) \overline{\tilde{g}(\xi)},$$

(2.5)

where $(\cdot, \cdot)_0$ is the $L_2(\mathbb{J}^n)$-inner product,

$$\|f\|_0^2 = \sum_{\xi \in \mathbb{Z}^n} |\tilde{f}(\xi)|^2 = \|\tilde{f}\|_{l_2}^2,$$

(2.6)

where $\|\cdot\|_0$ is $L_2(\mathbb{J}^n)$-norm and $\|\cdot\|_{l_2}$ is $l_2$-norm.

Let $s \in \mathbb{R}$. Denote

$$H^s(\mathbb{J}^n) = \{ u \in D'(\mathbb{J}^n) \mid \langle D \rangle^s u \in L_2(\mathbb{J}^n) \},$$

(2.7)

where

$$\langle \xi \rangle = \begin{cases} 1 & \text{if } \xi = 0, \\ |\xi| & \text{if } \xi \neq 0, \end{cases}$$

(2.8)

then $H^s(\mathbb{J}^n)$ is the Sobolev space endowed with the norm

$$\|u\|_{s}^2 = \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2,$$

(2.9)

and the inner product

$$\langle u, v \rangle_s = \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2s} \hat{u}(\xi) \overline{\hat{v}(\xi)}.$$

(2.10)

Here, we also define the discrete Sobolev space $H^s_d(\mathbb{R}^n), s \in \mathbb{R}$, of the functions $f \in H^s(\mathbb{R}^n)$ such that the following norm is finite:

$$\|f\|_{s,d}^2 = \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2.$$

(2.11)
Denote
\[
\mathcal{L}_2 = \left\{ f \in L_2(\mathbb{R}^n) : \sum_{\xi \in \mathbb{Z}^n} |f(\cdot - \xi)| \in L_2([0,1]^n) \right\}.
\] (2.12)

It is clear that any function \( f \in L_2(\mathbb{R}^n) \), which has compact support, or any function, for which \( \int_{k+(0,1)^n} |f(x)|^2 dx \) decays exponentially as \(|k|\) tends to infinity, belongs to \( \mathcal{L}_2 \). The periodic operator \([u]\) is totally defined if \( u \in \mathcal{L}_2 \).

Here, we assume that \( u_0 \in \mathcal{L}_2 \).

**Remark 2.1.** (1) It follows from (2.1) and (2.3) that if \( u \in \mathcal{L}_2 \), then \( \mathcal{F}([u])(\xi) = \hat{u}(\xi) \), \( \xi \in \mathbb{Z}^n \).

(2) It is clear that if \( t \leq s \), \( s, t \in \mathbb{R} \), then \( H^t(\mathbb{Z}^n) \subset H^s(\mathbb{Z}^n) \).

Using the variable separate method and the discrete Fourier transform, the solution of problem (1.1) can be represented as
\[
u(x,t) = E(t) [u_0](x) = \sum_{\xi \in \mathbb{Z}^n} \exp(a \sigma(\xi) t) \mathcal{F}([u_0])(\xi)e^{2\pi i x \xi},
\] (2.13)
where \( E(t) \) is a differentiable function and \( E(0) = 1 \).

We recall that a multiresolution approximation (MRA) of \( L_2(\mathbb{R}^n) \) is, as a definition, an increasing sequence \( V_j \), \( j \in \mathbb{Z} \), of closed linear subspaces of \( L_2(\mathbb{R}^n) \) with the following properties:
\[
\bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \quad \bigcup_{j \in \mathbb{Z}} V_j = L_2(\mathbb{R}^n);
\] (2.14)
for all \( f \in L_2(\mathbb{R}^n) \) and all \( j \in \mathbb{Z} \),
\[
f(x) \in V_j \iff f(2x) \in V_{j+1};
\] (2.15)
for all \( f \in L_2(\mathbb{R}^n) \) and \( k \in \mathbb{Z}^n \),
\[
f(x) \in V_0 \iff f(x-k) \in V_0.
\] (2.16)

There exists a function, called the scaling function (SF) \( \phi(x) \in V_0 \), such that the sequence
\[
\{\phi(x-k), k \in \mathbb{Z}^n\}
\] (2.17)
is a Riesz basic of \( V_0 \) (see [5, 9]).

An SF \( \phi \) is called \( \mu \)-regular (\( \mu \in \mathbb{N} \)) if, for each \( m \in \mathbb{N} \), there exists \( c_m \) such that the following condition holds:
\[
|D^\alpha \phi(x)| \leq c_m (1 + |x|)^{-m}, \quad \forall \alpha, \ |\alpha| \leq \mu.
\] (2.18)
Remark 2.2.  (1) Denote $\phi_{jk}(x) = 2^{nj/2}\phi(2^j x - k)$, $k \in \mathbb{Z}^n$. It follows from (2.14), (2.15), (2.16), and (2.17) that $V_j = \text{span}\{\phi_{jk}(x), k \in \mathbb{Z}^n\}$, $j \in \mathbb{Z}$.

(2) For each $\mu \in \mathbb{N}$, there exists an SF $\phi(x)$ with compact support, and $\phi(x)$ is $\mu$-regular; so in what follows, we always assume that $\phi$ has compact support and is $\mu$-regular (see [9]).

Using the periodic operator and an MRA of $L_2(\mathbb{R}^n)$, we can build an MRA of $L_2(\mathbb{R}^n)$ with the SF $[\phi]$ as follows.

Denote

$$
\phi^j_k(x) = 2^{nj/2} \sum_{l \in \mathbb{Z}^n} \phi_{jk}(x + l) = 2^{nj/2} \sum_{l \in \mathbb{Z}^n} \phi(2^j(x + l) - k), \quad j \geq 0,
$$

(2.19)

where $\mathbb{Z}^{nj} = \mathbb{Z}^n/2^j \mathbb{Z}^n$.

Then, the sequence $[V_j]_{j \geq 0}$ satisfies

$$
[V_0] \subset [V_1] \subset \cdots, \quad \bigcup_{j \geq 0} [V_j] = L_2(\mathbb{R}^n).
$$

(2.20)

It is clear that $\text{dim}[V_j] = 2^{nj}$, and if $(\phi_{jk}, \phi_{jl}) = \delta_{kl}$, $k, l \in \mathbb{Z}^n$, then $(\phi^j_k, \phi^j_l) = \delta_{kl}$, $k, l \in \mathbb{Z}^{nj}$ (see [6]).

For each $j \geq 0$, let $P_j : L_2(\mathbb{R}^n) \rightarrow [V_j]$ be the orthogonal projection from $L_2(\mathbb{R}^n)$ on $[V_j]$, which has the following property.

Theorem 2.3 (see [6, page 600]). Let $-\mu - 1 \leq s \leq \mu$, $-\mu \leq q \leq \mu + 1$, and $s \leq q$, then

$$
\|u - P_j u\|_s \leq c 2^{j(s-q)} \|u\|_q
$$

(2.22)

for all $u \in H^q(\mathbb{R}^n)$, where $c$ is independent of $j$ and $u$.

Denoting $h = 2^{-j}$ and $V_h = [V_j]$, we can write (2.22) as

$$
\|u - P_j u\|_s \leq c h^{q-s} \|u\|_q.
$$

(2.23)

3. The Galerkin-wavelet solution. Fix a distribution with compact support $\eta \in H^{-s'}(\Gamma)$, where $s' \geq 0$ satisfying $AV_h \subset H^{s'}(\mathbb{R}^n)$ and where $\Gamma \subset \mathbb{R}^n$ is some fixed compact domain such as a hypercube. For $f \in H^{s'}(\mathbb{R}^n)$, define

$$
\eta^j_k(f) = 2^{-nj/2} \eta(f(2^{-j}(\cdot + k))).
$$

(3.1)

The space

$$
X^j := \text{span}\{\eta^j_k, k \in \mathbb{Z}^{nj}\}
$$

(3.2)
is contained in \((AV_h)'\), which is the dual of \(AV_h\). The corresponding Galerkin-Petrov-wavelet scheme is then given by

\[
\eta_k^j \left( \frac{\partial u_h}{\partial t} \right) = a \eta_k^j (A u_h), \quad k \in \mathbb{Z}^{nj},
\]

\[
u_h(x,0) = R_{h}[u_0](x),
\]

where \(R_h v\) is a linear approximation of \(v\) in \(V_h\) and \(u_h : [0, \infty) \rightarrow V_h\) is a differentiable operator.

Set

\[
u_h(x,t) = \sum_{k \in \mathbb{Z}^{nj}} c_k(t) \phi_j^k(x),
\]

\[
R_h[u_0](x) := [u_0]_{h}(x) := \sum_{k \in \mathbb{Z}^{nj}} c_k(0) \phi_j^k(x).
\]

Then the scheme (3.3) and (3.4) provides an algebra equation system and the solution can be solved by Fourier series.

**Lemma 3.1.** The following formulas hold true:

\[
\mathcal{F}(\phi_j^k)(\xi) = h^{n/2} \hat{\phi}(h\xi)e^{-2\pi ihk\xi},
\]

\[
\mathcal{F}(A\phi_j^k)(\xi) = h^{n/2} \sigma(\xi) \hat{\phi}(h\xi)e^{-2\pi ihk\xi}.
\]

**Proof.** (a) It follows from (2.3) and (2.19) that

\[
\mathcal{F}(\phi_j^l)(\xi) = h^{-n/2} \sum_{l \in \mathbb{Z}^{nj}} \int_{[0,1]^n} e^{-2\pi i l x \xi} \phi(2^j(x + l) - k) dx
\]

\[
= h^{n/2} \sum_{l \in \mathbb{Z}^{nj}} \int_{2^j[l+[0,1]^n]-k} e^{-2\pi i h x \xi} \phi(x) dx e^{-2\pi i h k \xi}
\]

\[
= h^{n/2} \int_{\mathbb{R}^n} e^{-2\pi i h x \xi} \phi(x) dx e^{-2\pi i h k \xi}
\]

\[
= h^{n/2} \hat{\phi}(h\xi)e^{-2\pi i h k \xi}.
\]

(b) We have

\[
\mathcal{F}(Au)(\xi) = \sigma(\xi) \hat{u}(\xi);
\]

consequently,

\[
\mathcal{F}(A\phi_j^k)(\xi) = \sigma(\xi) \mathcal{F}(\phi_j^k)(\xi) = h^{n/2} \sigma(\xi) \hat{\phi}(h\xi)e^{-2\pi ihk\xi}.
\]

The proof of the lemma is complete.
**Corollary 3.2.** The following formulas hold true:

\[
\eta^j_k(\phi^j_l) = h^n \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi) \overline{\eta(h\xi)} e^{-2\pi ih(l-k)\xi},
\]

\[
\eta^j_k(A\phi^j_l) = h^n \sum_{\xi \in \mathbb{Z}^n} \sigma(\xi) \hat{\phi}(h\xi) \overline{\eta(h\xi)} e^{-2\pi ih(l-k)\xi}.
\]

**(3.11)**

**Proof.** (a) Using (2.4), Lemma 3.1, and (3.1), we have

\[
\eta^j_k(\phi^j_l) = h^n \sum_{\xi \in \mathbb{Z}^n} \mathcal{F}(\phi^j_l)(\xi) e^{2\pi ix\xi}
\]

\[
= h^n \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi) e^{-2\pi ihl\xi} e^{2\pi ix\xi}
\]

\[
= h^n \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi) \overline{\eta(h\xi)} e^{-2\pi ih(l-k)\xi}.
\]

**(3.12)**

(b) Similarly, we can get the second assertion. □

The following lemma is extracted from [6].

**Lemma 3.3.** The following formula holds valid:

\[
\sum_{m \in \mathbb{Z}^{nj}} e^{-2\pi ihm(k-\xi)} = \begin{cases} 2^{nj} & \text{if } \xi = k + 2^j \theta, \theta \in \mathbb{Z}^n, \\ 0 & \text{otherwise}. \end{cases}
\]

**(3.13)**

Set

\[
\alpha(k) = \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi) \overline{\eta(h\xi)} e^{2\pi ihk\xi},
\]

**(3.14)**

\[
\delta(k) = \sum_{\xi \in \mathbb{Z}^n} \sigma(h\xi) \hat{\phi}(h\xi) \overline{\eta(h\xi)} e^{2\pi ihk\xi}, \quad k \in \mathbb{Z}^{nj}.
\]

**(3.15)**

The series

\[
\tilde{\alpha}(\zeta) = h^n \sum_{k \in \mathbb{Z}^{nj}} \alpha(k) e^{-2\pi ihk\zeta},
\]

**(3.16)**

\[
\tilde{\delta}(\zeta) = h^n \sum_{k \in \mathbb{Z}^{nj}} \delta(k) e^{-2\pi ihk\zeta},
\]

**(3.17)**

\[
\tilde{\epsilon}(\zeta, t) = h^n \sum_{k \in \mathbb{Z}^{nj}} c_k(t) e^{-2\pi ihk\zeta}, \quad \zeta \in \mathbb{Z}^n
\]

**(3.18)**

are called discrete Fourier series.
It follows from (3.3), (3.5), the positively homogeneous condition, and Corollary 3.2 that

\[
\sum_{k \in \mathbb{Z}^n} c_k'(t) \alpha(l - k) = ah^{-r} \sum_{k \in \mathbb{Z}^n} c_k(t) \delta(l - k), \quad l \in \mathbb{Z}^n.
\]

Thus

\[
\tilde{\alpha}'(\zeta, t) \tilde{\alpha}(\zeta) = ah^{-r} \tilde{\alpha}(\zeta, t) \tilde{\delta}(\zeta),
\]

\[
\tilde{\alpha}(\zeta, t) = \exp\left(\frac{at}{h^r} \frac{\tilde{\delta}(\zeta)}{\tilde{\alpha}(\zeta)}\right) \tilde{\alpha}(\zeta, 0).
\]

For each \(\tau = 0, 1\), set

\[
g_{\phi, \tau}(\zeta) = \sum_{k \in \mathbb{Z}^n} \sigma(h \zeta + k)^\tau \hat{\phi}(h \zeta + k) \hat{\eta}(h \zeta + k).
\]

**LEMMA 3.4.** If the series (3.22) converges absolutely, then

\[
\tilde{\alpha}(\zeta) = g_{\phi, 0}(\zeta), \quad \tilde{\delta}(\zeta) = g_{\phi, 1}(\zeta).
\]

**PROOF.** (a) From (3.14) and (3.16), it follows that

\[
\tilde{\alpha}(\zeta) = h^n \sum_{k \in \mathbb{Z}^n} \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h \xi) \hat{\eta}(h \xi) e^{-2\pi ihk (\zeta - \xi)}.
\]

By the hypothesis of the lemma, we can interchange the summation in the above double sum; then by using the variable change and Lemma 3.3, it is easy to see that

\[
\tilde{\alpha}(\zeta) = h^n \sum_{\xi \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} e^{-2\pi ihk (\zeta - \xi)}
\]

\[
= \sum_{\theta \in \mathbb{Z}^n} \hat{\phi}(h \zeta + \theta) \hat{\eta}(h \zeta + \theta) = g_{\phi, 0}(\zeta).
\]

(b) Similarly, the second assertion of the lemma will be checked.

From (3.5), (3.6), and (3.21), it follows that

\[
\hat{u}_h(\xi, t) = \exp\left(\frac{at}{h^r} \frac{\hat{\delta}(\xi)}{\hat{\alpha}(\xi)}\right) \mathcal{F}\left([u_0]_h\right)(\xi).
\]

Let \(F_h(t)\) be the operator defined by

\[
\mathcal{F}(F_h(t)v(\cdot))(\xi) = \exp\left(\frac{at}{h^r} \frac{\hat{\delta}(\xi)}{\hat{\alpha}(\xi)}\right) \hat{v}(\xi),
\]

then the approximation \(u_h(x)\) can be represented by

\[
u_h(x) = F_h(t)R_h[u_0](x).
\]
4. The error estimate of approximation solutions. Now to estimate the error, we need some restrictions on the $\sigma$, $\phi$, and $\eta$ used above. The triplet $(\sigma, \phi, \eta)$ is called admissible if the following properties hold:

(i) there exists $p \in \mathbb{N}$, $p \geq r$, such that the series

$$\sum_{k \in \mathbb{Z}^n} \sigma(h\xi + k) \hat{\phi}(h\xi + k) \hat{\eta}(h\xi + k)$$

converges absolutely and

$$\sum_{k \in \mathbb{Z}^n} \sigma(h\xi + k) \hat{\phi}(h\xi + k) \hat{\eta}(h\xi + k) = \sigma(h\xi) \hat{\phi}(h\xi) \hat{\eta}(h\xi) + o(|h\xi|^p)$$

as $|h\xi| \to 0$,

(ii) $\hat{\phi}(\xi) \hat{\eta}(\xi) \geq 0$, for all $\xi \in \mathbb{R}^n$, $\hat{\phi}(0) \hat{\eta}(0) \neq 0$,

(iii) the series

$$\sum_{k \in \mathbb{Z}^n} \hat{\phi}(h\xi + k) \hat{\eta}(h\xi + k)$$

converges and

$$\sum_{k \in \mathbb{Z}^n} \hat{\phi}(h\xi + k) \hat{\eta}(h\xi + k) = \hat{\phi}(h\xi) \hat{\eta}(h\xi) + o(|h\xi|^p)$$

as $|h\xi| \to 0$.

**Remark 4.1.** (1) If $\eta = \phi$ and $\sigma$ is a pseudodifferential operator with symbol $\sigma(\xi) = |\xi|^r$, $0 < r \leq \mu$, then the triplet $(\sigma, \phi, \phi)$ is automatically admissible at least for $p = \mu$, where $\mu \in \mathbb{N}$ is used in (2.18) (see [7] for detail).

(2) If $\eta = \phi$ and $\sigma$ is a pseudodifferential operator with symbol $\sigma(\xi) = \langle \xi \rangle^2$, then the triplet $(\langle \xi \rangle^2, \phi, \phi)$ is admissible for $p = \mu$ (see [6]).

Write

$$u - u_h = [u - F_h(t)[u_0]] + F_h(t)[[u_0] - R_h[u_0]].$$

We have

$$\mathcal{F}(F_h(t)[u_0](\cdot))(\xi) = \exp \left( \frac{at}{h^r} \tilde{\delta}(\xi) \right) \mathcal{F}([[u_0]])(\xi)$$

$$= \exp \left( \frac{at}{h^r} \tilde{\delta}(\xi) \right) \hat{u}_0(\xi), \ \xi \in \mathbb{Z}^n,$$

thus

$$\mathcal{F}(u - F_h(t)[u_0])(\xi)$$

$$= \left\{ \exp(at\sigma(\xi)) - \exp \left( \frac{at}{h^r} \tilde{\delta}(\xi) \right) \right\} \hat{u}_0(\xi), \ \xi \in \mathbb{Z}^n.$$
If the triplet \((\sigma, \phi, \eta)\) is admissible, then it follows from (3.22) and Lemma 3.4 that

\[
\frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)} = \sigma(h\xi) + O(|h\xi|^p) \quad \text{as } |h\xi| \to 0.
\]

(4.8)

**Theorem 4.2.** Suppose that \(r + s' \leq s \leq p\), \(0 \leq m \leq s\), and it is assumed that the triplet \((\sigma, \phi, \eta)\) is admissible. Then, for \(u_0 \in L_2 \cap H^m_{\text{ad}}(\mathbb{R}^n), 0 \leq t \leq T\), with \(h\) small enough, we get

\[
||u - F_h(t)[u_0]||_m \leq ch^{s-r}||u_0||_{s'+m,d},
\]

(4.9)

where \(c\) is independent of \(u, h,\) and \(u_0\).

**Proof.** It follows from (4.8) that

\[
\left| at\sigma(\xi) - \frac{at}{hr}\tilde{\delta}(\xi) \right| \leq ch^{p-r}|\xi|^p \quad \text{as } |h\xi| \leq 1.
\]

(4.10)

The equality

\[
e^{ta} - e^{tb} = t(a-b) \int_0^1 e^{sta+(1-s)tb} ds,
\]

(4.11)

(4.10), and (1.3) imply that, for \(r \leq s \leq p\) and \(0 \leq t \leq T\),

\[
\left| \exp(at\sigma(\xi)) - \left(\frac{at}{hr}\tilde{\alpha}(\xi)\right)^p \right| \leq ch^{s-r}|\xi|^s \quad \text{as } |h\xi| \leq 1.
\]

(4.12)

Hence, from (4.7) and (4.12), we obtain

\[
|\mathcal{F}(u(\cdot, t) - F_h(t)[u_0](\cdot))(\xi)| \leq ch^{s-r}|\xi|^s \hat{u}_0(\xi) \quad \text{as } |h\xi| \leq 1.
\]

(4.13)

By (1.3) and the admissibility of the triplet \((\sigma, \phi, \eta)\), inequality (4.13) is also valid for all \(\xi \in \mathbb{Z}^n\). Hence, for each \(0 \leq m \leq s, r + s' \leq s \leq p\), and \(0 \leq t \leq T\), we get

\[
||u - F_h(t)[u_0]||_m^2 = \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2m} |\mathcal{F}(u(\cdot, t) - F_h(t)[u_0](\cdot))(\xi)|^2
\leq ch^{2(s-r)} \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2(m+s)} |\hat{u}_0(\xi)|^2
\leq ch^{2(s-r)}||u_0||_{m+s,d}^2.
\]

(4.14)

The theorem is thus proved. \(\square\)
From the admissibility of the triplet \((\sigma, \phi, \eta)\) and (1.3), it follows that \(F_h(t) : H^m(\mathbb{R}^n) \to H^m(\mathbb{R}^n), 0 \leq m \leq s\), is a continuous linear operator. Consequently,

\[
\|F_h(t)([u_0] - R_h[u_0])\|_m \leq c\|[u_0] - R_h[u_0]\|_m. \quad (4.15)
\]

Therefore, if we assume that

\[
\|(I - R_h)[u_0]\|_m \leq ch^s\|[u_0]\|_{m+s}, \quad (4.16)
\]

then

\[
\|F_h(t)([u_0] - R_h[u_0])\|_m \leq ch^s\|[u_0]\|_{m+s}. \quad (4.17)
\]

**Remark 4.3.** It follows from (2.23) that the assumption (4.17) is satisfied, when \(R_h = P_j\) for \(0 \leq m, m + s \leq \mu + 1\).

Thus from (4.5), (4.9), and (4.17), we obtain the following theorem.

**Theorem 4.4.** If all the hypotheses of Theorem 4.2 and assumption (4.17) are satisfied, then

\[
\|u - u_h\|_m \leq ch^{s-r}\|[u_0]\|_{m+s,d} + ch^s\|[u_0]\|_{m+s}, \quad (4.18)
\]

where \(c\) is independent of \(u_0, h\).

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