Fractional diffusion-wave equation: hidden regularity for weak solutions

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August 10, 2021

1 Introduction

The equation
\[ \partial_t^\alpha u(t, x) = \Delta u(t, x) \] (1)
is obtained from the diffusion equation by replacing the first order time-derivative by the Caputo fractional derivative of order \( \alpha \), where \( 1 < \alpha < 2 \), that is
\[ \partial_t^\alpha f(t) = \frac{1}{\Gamma(2 - \alpha)} \int_0^t (t - \tau)^{1 - \alpha} \frac{d^2 f}{d\tau^2}(\tau) \, d\tau. \]

We will prove the following regularity results.

**Theorem 1.1** If \( u_0 \in H_0^1(\Omega) \) and \( u_1 \in L^2(\Omega) \), then the unique weak solution \( u \) of problem
\[
\begin{cases}
\partial_t^\alpha u(t, x) = \Delta u(t, x), & t \geq 0, \ x \in \Omega, \\
u(t, x) = 0 & t \geq 0, \ x \in \partial\Omega, \\
u(0, x) = u_0(x), & u_t(0, x) = u_1(x), \ x \in \Omega,
\end{cases}
\]
belongs to \( C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; D(A^{-\theta})) \), \( \theta \in (\frac{2-\alpha}{2\alpha}, \frac{1}{2}) \), and
\[
\lim_{t \to 0} \|u(t, \cdot) - u_0\|_{H_0^1(\Omega)} = \lim_{t \to 0} \|\partial_t u(t, \cdot) - u_1\|_{D(A^{-\theta})} = 0,
\]
\[
\|u\|_{C([0, T]; H_0^1(\Omega))} + \|\partial_t u\|_{C([0, T]; D(A^{-\theta}))} \leq C(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)}),
\] (3)

In addition, for any \( \theta \in (0, \frac{1}{2}) \) there exists a constant \( C > 0 \) such that
\[
\|\nabla u\|_{L^2(0, T; D(A^{-\theta}))} \leq C(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)}),
\] (4)

and for any \( \theta \in (\frac{\alpha-1}{2\alpha}, \frac{1}{2}) \) there exists a constant \( C > 0 \) such that
\[
\|\partial_t^\alpha u\|_{L^2(0, T; D(A^{-\theta}))} \leq C(\|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)}).
\] (5)

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‡ This paper is published (in revised form) in Fract. Calc. Appl. Anal. Vol. 24, No 4 (2021), pp. 1015-1034, DOI: 10.1515/fca-2021, and is available online at https://www.degruyter.com/journal/key/FCA/html
Theorem 1.2 Let \( u_0 \in H^1_0(\Omega) \) and \( u_1 \in L^2(\Omega) \). If \( u \) is the weak solution of (2) then, for any \( T > 0 \) there is a constant \( c_0 = c_0(T) \) such that, denoting by \( \partial_{\nu} u \) the normal derivative of \( u \), we have
\[
\int_0^T \int_{\partial \Omega} |\partial_{\nu} u|^2 d\sigma dt \leq c_0(\|\nabla u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2).
\] (6)

For previous results related to this problem see [18, 19, 27, 23, 24] and references therein.

2 Preliminaries

Let \( \Omega \subset \mathbb{R}^N, N \geq 1, \) be a bounded open set with \( C^2 \) boundary. We consider \( L^2(\Omega) \) endowed with the usual inner product and norm
\[
\langle u, v \rangle = \int_{\Omega} u(x)v(x) \, dx, \quad \|u\|_{L^2(\Omega)} = \left( \int_{\Omega} |u(x)|^2 \, dx \right)^{1/2} \quad u, v \in L^2(\Omega).
\]

Definition 2.1 For any \( f \in L^1(0, T) \) \( (T > 0) \) we define the Riemann–Liouville fractional integral \( I^\beta \) of order \( \beta \in \mathbb{R}, \beta > 0 \), by
\[
I^\beta(f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) \, d\tau, \quad \text{a.e. } t \in (0, T),
\] (7)
where \( \Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} \, dt \) is the Euler gamma function.

We note that
\[
I^1(f)(t) = \int_0^t f(\tau) \, d\tau.
\] (8)

For the sequel it is convenient to introduce the following function
\[
\Phi_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} \quad t > 0,
\] (9)
so
\[
I^\beta(f)(t) = (\Phi_\beta * f)(t), \quad \text{a.e. } t \in (0, T).
\] (10)

For \( f \in L^2(0, T) \) we have
\[
\|I^\beta(f)\|_{L^2(0,T)} \leq \|\Phi_\beta\|_{L^1(0,T)} \|f\|_{L^2(0,T)}.
\] (11)

If we take into account that
\[
\Phi_\beta * \Phi_\gamma(t) = \Phi_{\beta+\gamma}(t) \quad t > 0 \quad \beta, \gamma > 0,
\] (12)
we have
\[
I^\beta I^\gamma(f) = I^{\beta+\gamma}(f).
\] (13)

\[
\partial_t^\alpha f(t) = \begin{cases}
I^{1-\alpha}(\frac{d^2f}{dt^2})(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d^2f}{d\tau^2}(\tau) \, d\tau & 0 < \alpha < 1, \\
I^{2-\alpha}(\frac{d^2f}{dt^2})(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \frac{d^2f}{d\tau^2}(\tau) \, d\tau & 1 < \alpha < 2.
\end{cases}
\] (14)
We define the operator $A$ in $L^2(\Omega)$ by
\[
D(A) = H^2(\Omega) \cap H^1_0(\Omega) \\
(Au)(x) = -\triangle u(x), \quad x \in \Omega, \quad u \in D(A).
\]
The fractional powers $A^\theta$ are defined for $\theta > 0$, see e.g. [30] and [25, Example 4.34]. We recall that the spectrum of $A$ consists of a sequence of positive eigenvalues, each of them with finite dimensional eigenspace, and there exists an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $A$. We denote such a basis by $\{e_n\}_{n \in \mathbb{N}}$ and by $\lambda_n$ the eigenvalue with eigenfunction $e_n$, that is $Ae_n = \lambda_n e_n$. Then, for $\theta > 0$ the domain $D(A^\theta)$ of $A^\theta$ consists of those functions $u \in L^2(\Omega)$ such that
\[
\sum_{n=1}^{\infty} \lambda_n^{2\theta} |\langle u, e_n \rangle|^2 < +\infty
\]
and
\[
A^\theta u = \sum_{n=1}^{\infty} \lambda_n^\theta \langle u, e_n \rangle e_n, \quad u \in D(A^\theta).
\]
Moreover $D(A^\theta)$ is a Hilbert space with the norm
\[
\|u\|_{D(A^\theta)} = \|A^\theta u\|_{L^2(\Omega)} = \left( \sum_{n=1}^{\infty} \lambda_n^{2\theta} |\langle u, e_n \rangle|^2 \right)^{1/2}, \quad u \in D(A^\theta).
\] (15)
We have $D(A^\theta) \subset H^{2\theta}(\Omega)$. In particular, $D(A^{1/2}) = H^1_0(\Omega)$. If we identify the dual $(L^2(\Omega))'$ with $L^2(\Omega)$ itself, then we have $D(A^\theta) \subset L^2(\Omega) \subset (D(A^\theta))'$. From now on we set
\[
D(A^{-\theta}) := (D(A^\theta))',
\] (16)
whose elements are bounded linear functionals on $D(A^\theta)$. If $\varphi \in D(A^{-\theta})$ and $u \in D(A^\theta)$ the value of $\varphi$ applied to $u$ is denoted by
\[
\langle \varphi, u \rangle_{-\theta,\theta} := \varphi(u).
\] (17)
In addition, $D(A^{-\theta})$ is a Hilbert space with the norm
\[
\|\varphi\|_{D(A^{-\theta})} = \left( \sum_{n=1}^{\infty} \lambda_n^{-2\theta} |\langle \varphi, e_n \rangle_{-\theta,\theta}|^2 \right)^{1/2}, \quad \varphi \in D(A^{-\theta}).
\] (18)
We also recall that
\[
\langle \varphi, u \rangle_{-\theta,\theta} = \langle \varphi, u \rangle \quad \text{for} \quad \varphi \in L^2(\Omega), u \in D(A^\theta),
\] (19)
e.g. see [4, Chapitre V].
For $\alpha, \beta > 0$ arbitrary constants, we define the Mittag–Leffler functions by
\[
E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}.
\] (20)
By the power series, one can note that $E_{\alpha,\beta}(z)$ is an entire function of $z \in \mathbb{C}$. 

3
Lemma 2.2 Let $1 < \alpha < 2$ and $\beta > 0$ be. Then for any $\mu$ such that $\pi \alpha / 2 < \mu < \pi$ there exists a constant $C = C(\alpha, \beta, \mu) > 0$ such that

$$|E_{\alpha, \beta}(z)| \leq \frac{C}{1 + |z|}, \quad \mu \leq |\arg(z)| \leq \pi. \tag{21}$$

Lemma 2.3 For any $0 < \beta < 1$ the function $x \to \frac{x^\beta}{1 + x}$ gains its maximum on $[0, +\infty[$ at point $\frac{\beta}{1 - \beta}$ and the maximum value is given by

$$\max_{x \geq 0} \frac{x^\beta}{1 + x} = \beta^\beta (1 - \beta)^{1 - \beta}. \tag{22}$$

For a Hilbert space $H$ endowed with the norm $\| \cdot \|_H$ and $\beta \in (0,1)$, $H^\beta(0, T; H)$ is the space of all $u \in L^2(0, T; H)$ such that

$$[u]_{H^\beta(0, T; H)} := \left( \int_0^T \int_0^T \frac{|u(t) - u(\tau)|_H^2}{|t - \tau|^{1 + 2\beta}} \, dt \, d\tau \right)^{1/2} < +\infty,$$

that is $[u]_{H^\beta(0, T; H)}$ is the so-called Gagliardo semi-norm of $u$. $H^\beta(0, T; H)$ is endowed with the norm

$$\| \cdot \|_{H^\beta(0, T; H)} := \| \cdot \|_{L^2(0, T; H)} + [ \cdot ]_{H^\beta(0, T; H)}. \tag{23}$$

We will use later the following extension to the case of vector valued functions of a known result, see [12, Theorem 2.1].

Theorem 2.4 Let $H$ be a separable Hilbert space.

(i) The Riemann–Liouville operator $I^\beta : L^2(0, T; H) \to L^2(0, T; H)$, $0 < \beta \leq 1$, is injective and the range $\mathcal{R}(I^\beta)$ of $I^\beta$ is given by

$$\mathcal{R}(I^\beta) = \begin{cases} H^\beta(0, T; H), & 0 < \beta < \frac{1}{2}, \\
\left\{ v \in H^\beta(0, T; H) : \int_0^T t^{-1} |v(t)|^2 \, dt < \infty \right\}, & \beta = \frac{1}{2}, \\
oH^\beta(0, T; H), & \frac{1}{2} < \beta \leq 1, \end{cases} \tag{24}$$

where $\text{no}H^\beta(0, T) = \{ u \in H^\beta(0, T) : u(0) = 0 \}$.

(ii) For the Riemann–Liouville operator $I^\beta$ and its inverse operator $I^{-\beta}$ the norm equivalences

$$\| I^\beta(u) \|_{H^\beta(0, T; H)} \sim \| u \|_{L^2(0, T; H)}, \quad u \in L^2(0, T; H),$$

$$\| I^{-\beta}(v) \|_{L^2(0, T; H)} \sim \| v \|_{H^\beta(0, T; H)}, \quad v \in \mathcal{R}(I^\beta), \tag{25}$$

hold true.

For the sake of completeness, we recall the notion of a weak solution for fractional diffusion-wave equations, see [33, Definition 2.1].

Definition 2.5 Let $1 < \alpha < 2$. We define $u$ as a weak solution to problem

$$\begin{cases}
\partial_t^\alpha u(t, x) = \Delta u(t, x) & t \in (0, T), \ x \in \Omega, \\
u(t, x) = 0 & t \in (0, T), \ x \in \partial \Omega, \\
u(0, x) = u_0(x), \ u_1(0, x) = u_1(x) & x \in \Omega, \tag{26}
\end{cases}$$
if \( \partial_t^\alpha u(t, \cdot) = \Delta u(t, \cdot) \) holds in \( L^2(\Omega) \), \( u(t, \cdot) \in H^1_0(\Omega) \) for almost all \( t \in (0, T) \) and for some \( \theta > 0 \), depending on the initial data \( u_0, u_1 \), one has \( u, \partial_t u \in C([0, T]; D(A^{-\theta})) \) and

\[
\lim_{t \to 0} \| u(t, \cdot) - u_0 \|_{D(A^{-\theta})} = \lim_{t \to 0} \| \partial_t u(t, \cdot) - u_1 \|_{D(A^{-\theta})} = 0. \tag{27}
\]

We also need to recall some existence results given in [33, Theorem 2.3], that we have integrated with other essential regularity properties of the solution, see [28] below.

**Theorem 2.6**  
(i) Let \( u_0 \in L^2(\Omega) \) and \( u_1 \in D(A^{\frac{-1}{\alpha}}) \). Then there exists a unique weak solution \( u \in C([0, T]; L^2(\Omega)) \cap C((0, T]; H^2(\Omega) \cap H^1_0(\Omega)) \) to (26) with \( \partial_t^\alpha u \in C((0, T]; L^2(\Omega)) \) and satisfying

\[
\lim_{t \to 0} \| u(t, \cdot) - u_0 \|_{L^2(\Omega)} = 0, \quad \| u \|_{C([0, T]; L^2(\Omega))} \leq C \left( \| u_0 \|_{L^2(\Omega)} + \| u_1 \|_{D(A^{\frac{-1}{\alpha}})} \right),
\]

\[
\lim_{t \to 0} \| \partial_t u(t, \cdot) - u_1 \|_{D(A^{-\theta})} = 0, \quad \theta \in \left( \frac{1}{\alpha}, 1 \right),
\]

\[
\| \partial_t u \|_{C([0, T]; D(A^{-\theta}))} \leq C \left( \| u_0 \|_{L^2(\Omega)} + \| u_1 \|_{D(A^{\frac{-1}{\alpha}})} \right),
\]

for some constant \( C > 0 \). Moreover, if \( u_1 \in L^2(\Omega) \) we have

\[
u(t, x) = \sum_{n=1}^{\infty} \left[ \langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right] e_n(x), \tag{29}
\]

\[
\partial_t u(t, x) = \sum_{n=1}^{\infty} \left[ -\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) \right] e_n(x), \tag{30}
\]

\[
\partial_t^\alpha u(t, x) = \sum_{n=1}^{\infty} \left[ -\lambda_n \langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) - \lambda_n \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right] e_n(x), \tag{31}
\]

\[
\| \partial_t u(t, \cdot) \|_{L^2(\Omega)} \leq C \left( t^{-1} \| u_0 \|_{L^2(\Omega)} + \| u_1 \|_{L^2(\Omega)} \right) \quad (C > 0).
\]

(ii) If \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) and \( u_1 \in H^1_0(\Omega) \), then the unique weak solution \( u \) to (26) given by (29) belongs to \( C([0, T]; H^2(\Omega) \cap H^1_0(\Omega)) \cap C^1(0, T]; L^2(\Omega)) \) and \( \partial_t^\alpha u \in C([0, T]; L^2(\Omega)) \). In addition, there exists a constant \( C > 0 \) such that

\[
\| u \|_{C([0, T]; H^2(\Omega))} + \| u \|_{C^1([0, T]; L^2(\Omega))} + \| \partial_t^\alpha u \|_{C([0, T]; L^2(\Omega))} \leq C \left( \| u_0 \|_{H^2(\Omega)} + \| u_1 \|_{H^1(\Omega)} \right). \tag{32}
\]

**Proof.** We refer to [33, Theorem 2.3] for the proof of all statements, except for the proof of (28). We first observe that, since \( u_1 \in D(A^{\frac{-1}{\alpha}}) \), the expression (30) for \( \partial_t u \) has to be written in the form

\[
\partial_t u(t, x) = \sum_{n=1}^{\infty} \left[ -\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t \frac{E_{\alpha,1}(-\lambda_n t^\alpha)}{-\lambda_n} \right] e_n(x).
\]

For \( \theta \in (0, 1) \) to choose suitably later, we have

\[
\| \partial_t u(t, \cdot) - u_1 \|_{D(A^{-\theta})}^2 = \sum_{n=1}^{\infty} \lambda_n^{-2\theta} \left| -\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t \frac{E_{\alpha,1}(-\lambda_n t^\alpha)}{-\lambda_n} - \frac{1}{\alpha} \right|^2.
\]

\[
\leq 2t^{2\alpha-1} \sum_{n=1}^{\infty} \lambda_n^{2(1-\theta)} \left| \langle u_0, e_n \rangle E_{\alpha,\alpha}(-\lambda_n t^\alpha) \right|^2 + 2 \sum_{n=1}^{\infty} \lambda_n^{-2\theta} \left| \langle u_1, e_n \rangle - \frac{1}{\alpha} \right|^2 \left( E_{\alpha,1}(-\lambda_n t^\alpha) - 1 \right)^2. \tag{33}
\]
Moreover, if we assume $u$ while, regarding the second sum, we have
\[\lambda_n^{-2\theta} \left| \langle u_1, e_n \rangle - \frac{1}{n^\alpha} (E_{\alpha,1}(-\lambda_n t^\alpha) - 1) \right|^2 = \lambda_n^{-2(\theta - \frac{1}{\alpha})} \lambda_n^{-\frac{2}{\alpha}} \left| \langle u_1, e_n \rangle - \frac{1}{n^\alpha} \right|^2 \left| E_{\alpha,1}(-\lambda_n t^\alpha) - 1 \right|^2 .\]
Therefore, plugging the above two estimates into (33) we obtain
\[\| \partial_t u(t, \cdot) - u_1 \|^2_{D(A^{-\theta})} \leq C t^{2(\alpha - 1)} \| u_0 \|^2_{L^2(\Omega)} + 2 \sum_{n=1}^{\infty} \lambda_n^{-2(\theta - \frac{1}{\alpha})} \lambda_n^{-\frac{2}{\alpha}} \left| \langle u_1, e_n \rangle - \frac{1}{n^\alpha} \right|^2 \left| E_{\alpha,1}(-\lambda_n t^\alpha) - 1 \right|^2 ,\]
whence it follows that for $\theta > \frac{1}{\alpha}$ (28) holds true. \(\square\)

3 Regularity for \(u_0 \in H^1_0(\Omega)\) and \(u_1 \in L^2(\Omega)\)

We establish a result about the regularity of the weak solutions assuming on the data \(u_0, u_1\) a degree of regularity intermediate between those assumed in (i) and (ii) of Theorem 2.6.

**Theorem 3.1** If \(u_0 \in H^1_0(\Omega)\) and \(u_1 \in L^2(\Omega)\), then the unique weak solution \(u\) to (26) given by (29)–(31) belongs to \(C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; D(A^{-\theta}))\), \(\theta \in \left(\frac{2 - \alpha}{2\alpha}, \frac{1}{2}\right]\), and
\[\lim_{t \to 0} \| u(t, \cdot) - u_0 \|_{H^1_0(\Omega)} = \lim_{t \to 0} \| \partial_t u(t, \cdot) - u_1 \|_{D(A^{-\theta})} = 0 ,\]
\[\| u \|_{C([0, T]; H^1_0(\Omega))} + \| \partial_t u \|_{C([0, T]; D(A^{-\theta}))} \leq C \left( \| u_0 \|_{H^1_0(\Omega)} + \| u_1 \|_{L^2(\Omega)} \right).\] (34)
In addition, for any \(\theta \in (0, \frac{1}{2\alpha})\) there exists a constant \(C > 0\) such that
\[\| \nabla u \|_{L^2(0, T; D(A^{-\theta}))} \leq C \left( \| u_0 \|_{H^1_0(\Omega)} + \| u_1 \|_{L^2(\Omega)} \right),\] (35)
and for any \(\theta \in \left(\frac{2 - \alpha}{2\alpha}, \frac{1}{2}\right]\) there exists a constant \(C > 0\) such that
\[\| \partial_t^\alpha u \|_{L^2(0, T; D(A^{-\theta}))} \leq C \left( \| u_0 \|_{H^1_0(\Omega)} + \| u_1 \|_{L^2(\Omega)} \right).\] (36)
Moreover, if we assume \(u_0 \in D(A^{\frac{1}{2} + \varepsilon})\) with \(\varepsilon \in \left(\frac{2 - \alpha}{2\alpha}, \frac{1}{2}\right]\), then
\[\lim_{t \to 0} \| \partial_t u(t, \cdot) - u_1 \|_{L^2(\Omega)} = 0 ,\]
\[\| \partial_t u \|_{C([0, T]; L^2(\Omega))} \leq C \left( \| u_0 \|_{D(A^{\frac{1}{2} + \varepsilon})} + \| u_1 \|_{L^2(\Omega)} \right).\] (37)

**Proof.** In virtue of the expression (29) for the solution \(u\) we have
\[\| u(t, \cdot) - u_0 \|^2_{H^1_0(\Omega)} = \sum_{n=1}^{\infty} \lambda_n \left| \langle u_0, e_n \rangle \left( E_{\alpha,1}(-\lambda_n t^\alpha) - 1 \right) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right|^2 \leq 2 \sum_{n=1}^{\infty} \lambda_n \left| \langle u_0, e_n \rangle \right|^2 \left| E_{\alpha,1}(-\lambda_n t^\alpha) - 1 \right|^2 + t^{2\alpha} C^2 \sum_{n=1}^{\infty} \left| \langle u_1, e_n \rangle \right|^2 \left( \frac{\lambda_n t^\alpha \frac{1}{2}}{1 + \lambda_n t^\alpha} \right)^2 ,\] (38)
thanks also to (21). We observe that for any $n \in \mathbb{N}$ $\lim_{t \to 0} (E_{\alpha,1}(-\lambda_n t^\theta) - 1) = 0$. Moreover, again by (21), we get for $n \in \mathbb{N}$ and $0 \leq t \leq T$

$$\lambda_n \|\langle u_0, e_n \rangle\|^2 |E_{\alpha,1}(-\lambda_n t^\theta) - 1|^2 \leq 2\lambda_n \|\langle u_0, e_n \rangle\|^2 \left(\frac{C}{(1 + \lambda_n t^\theta)^2} + 1\right) \leq C\lambda_n \|\langle u_0, e_n \rangle\|^2,$$

hence by (33) we deduce $\lim_{t \to 0} \|u(t, \cdot) - u_0\|_{H^1_0(\Omega)} = 0$ and for any $t \in [0, T]$

$$\|u(t, \cdot)\|^2_{H^1_0(\Omega)} \leq C(\|u_0\|^2_{H^1_0(\Omega)} + \|u_1\|^2_{L^2(\Omega)}).$$

To complete the proof of (34), we fix $\theta \in \left(\frac{2-\alpha}{2\alpha}, \frac{1}{2}\right]$ and use formula (30) to note that

$$\|\partial_t u(t, \cdot) - u_1\|^2_{D(A^{1-\theta})} = \sum_{n=1}^{\infty} \lambda_n^{-2\theta} \left|\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\theta) + \langle u_1, e_n \rangle (E_{\alpha,1}(-\lambda_n t^\theta) - 1)\right|^2 \leq C t^{\alpha-2+2\alpha \theta} \|u_0\|^2_{H^1_0(\Omega)} + 2 \sum_{n=1}^{\infty} \left|\lambda_n \langle u_0, e_n \rangle t\right|^2 \|E_{\alpha,1}(-\lambda_n t^\theta) - 1\|^2,$$  

(39)

thanks also to (21). Since $0 < \frac{1-2\theta}{2} < 1$ we can apply (22) to have

$$\|\partial_t u(t, \cdot) - u_1\|^2_{D(A^{1-\theta})} \leq C t^{\alpha-2+2\alpha \theta} \|u_0\|^2_{H^1_0(\Omega)} + 2 \sum_{n=1}^{\infty} \left|\lambda_n \langle u_0, e_n \rangle t\right|^2 \|E_{\alpha,1}(-\lambda_n t^\theta) - 1\|^2.$$

Therefore, by analogous arguments to those done before, since $\alpha - 2 + 2\alpha \theta > 0$ we deduce $\lim_{t \to 0} \|\partial_t u(t, \cdot) - u_1\|_{D(A^{1-\theta})} = 0$ and for any $t \in [0, T]$

$$\|\partial_t u(t, \cdot)\|^2_{D(A^{1-\theta})} \leq C(\|u_0\|^2_{H^1_0(\Omega)} + \|u_1\|^2_{L^2(\Omega)}).$$

$$\|\nabla u(\cdot, t)\|^2_{D(A^{\theta})} = \sum_{n=1}^{\infty} \lambda_n^{1+2\theta} \left|\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\theta) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\theta)\right|^2 \leq C \sum_{n=1}^{\infty} \lambda_n \|\langle u_0, e_n \rangle\|^2 \frac{\lambda_{\alpha}^{2\theta}}{(1 + \lambda_n t^\theta)^2} + C \sum_{n=1}^{\infty} \left|\langle u_1, e_n \rangle\right|^2 \frac{\lambda_1^{1+2\theta} t^2}{(1 + \lambda_n t^\theta)^2}.$$  

(40)

Since

$$\frac{\lambda_{\alpha}^{2\theta}}{(1 + \lambda_n t^\theta)^2} = \left(\frac{\lambda_{\alpha} t^\theta}{1 + \lambda_n t^\theta}\right)^2 t^{-2\alpha \theta},$$

$$\frac{\lambda_1^{1+2\theta} t^2}{(1 + \lambda_n t^\theta)^2} = \left(\frac{(\lambda_{\alpha} t^\theta)^{1+2\theta}}{1 + \lambda_n t^\theta}\right)^2 t^{-\alpha(1+2\theta)},$$

for $0 < \theta < \frac{1}{2}$, we can apply (22) to have

$$\|\nabla u(\cdot, t)\|^2_{L^2(0,T;D(A^{\theta}))} \leq C t^{-2\alpha \theta} \|u_0\|^2_{H^1_0(\Omega)} + C t^{2-\alpha(1+2\theta)} \|u_1\|^2_{L^2(\Omega)}$$

Thanks to (18), (31) and (21) we get

$$\|\partial_t^2 u(\cdot, t)\|^2_{D(A^{1-\theta})} = \sum_{n=1}^{\infty} \lambda_n^{-2\theta} \left|\lambda_n \langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\theta) + \lambda_n \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\theta)\right|^2 \leq C \sum_{n=1}^{\infty} \lambda_n \|\langle u_0, e_n \rangle\|^2 \frac{\lambda_{\alpha}^{-2\theta}}{(1 + \lambda_n t^\theta)^2} + C \sum_{n=1}^{\infty} \left|\langle u_1, e_n \rangle\right|^2 \frac{\lambda_1^{2(1-\theta)} t^2}{(1 + \lambda_n t^\theta)^2}.$$  

(41)
\[
\frac{\lambda_{n}^{1-2\theta}}{(1 + \lambda_{n}t^{\alpha})^2} = \left(\frac{(\lambda_{n}t^{\alpha})^{\frac{1-2\theta}{2}}}{1 + \lambda_{n}t^{\alpha}}\right)^2 t^{(2\theta - 1)}
\]

\[
\frac{\lambda_{n}^{2(1-\theta)}t^{2}}{(1 + \lambda_{n}t^{\alpha})^2} = \left(\frac{(\lambda_{n}t^{\alpha})^{1-\theta}}{1 + \lambda_{n}t^{\alpha}}\right)^2 t^{2+2\alpha(\theta - 1)}
\]

\[
\|\partial_{\alpha}^{\ast} u(\cdot, t)\|_{D(A^{1/2})}^2 \leq C t^{1-\alpha} \|u_0\|_{H_{0}^{1/2}(\Omega)}^2 + C t^{3-2\alpha} \|u_1\|_{L^2(\Omega)}^2. \quad (42)
\]

By assuming, in addition, that \(u_0 \in D(A^{1/2+\varepsilon})\) with \(\varepsilon \in \left(\frac{2-\alpha}{2\alpha}, \frac{1}{2}\right)\) we have

\[
\|\partial_{t} u(t, \cdot) - u_1\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} \left| -\lambda_{n} \langle u_0, e_n \rangle t^{\alpha - 1} E_{\alpha, \alpha} (-\lambda_{n}t^{\alpha}) + \langle u_1, e_n \rangle (E_{\alpha, 1} (-\lambda_{n}t^{\alpha}) - 1) \right|^2
\]

\[
\leq C t^{\alpha - 2 + 2\alpha\varepsilon} \sum_{n=1}^{\infty} \lambda_{n}^{1+2\varepsilon} \left| \langle u_0, e_n \rangle \right|^2 \left(\frac{(\lambda_{n}t^{\alpha})^{1-2\varepsilon}}{1 + \lambda_{n}t^{\alpha}}\right)^2 + 2 \sum_{n=1}^{\infty} \left| \langle u_1, e_n \rangle \right|^2 \left| E_{\alpha, 1} (-\lambda_{n}t^{\alpha}) - 1 \right|^2. \quad (43)
\]

Thanks to (22) with \(\beta = \frac{1-2\varepsilon}{2}\) we obtain

\[
\|\partial_{t} u(t, \cdot) - u_1\|_{L^2(\Omega)}^2 \leq C t^{\alpha - 2 + 2\alpha\varepsilon} \|u_0\|_{D(A^{1/2+\varepsilon})} + 2 \sum_{n=1}^{\infty} \left| \langle u_1, e_n \rangle \right|^2 \left| E_{\alpha, 1} (-\lambda_{n}t^{\alpha}) - 1 \right|^2,
\]

hence, since \(\alpha - 2 + 2\alpha\varepsilon > 0\), we deduce (37). \(\square\)

**Remark 3.2** Comparing the regularity results given in Theorems 2.6 and 3.1, we have to observe that if \(\theta \in \left(\frac{2-\alpha}{2\alpha}, \frac{1}{2}\right)\) then \(D(A^{-\theta}) \subset D(A^{-\eta})\) for any \(\eta \in \left(\frac{1}{\alpha}, 1\right]\). Therefore Theorem 3.1 effectively improves the regularity of the weak solution.

Moreover, taking into account the argumentations used to get (39), we note that to secure a regularity of \(\partial_{t} u\) in \(L^2(\Omega)\) we have to assume the datum \(u_0\) more regular than \(u_0 \in H_{0}^{1/2}(\Omega) = D(A^{1/2})\), that is \(u_0 \in D(A^{1/2+\varepsilon})\) with \(\varepsilon \in \left(\frac{2-\alpha}{2\alpha}, \frac{1}{2}\right)\), see (37).

## 4 Hidden regularity results

To begin with we single out some technical results that we will use later in the main theorem.

**Lemma 4.1** For any \(w \in H^{2}(\Omega)\) one has

\[
2 \int_{\Omega} \nabla w \cdot \nabla w \; dx = \int_{\partial \Omega} \left[2 \partial_{w} w \cdot \nabla w - h \cdot \nabla w \right] \; d\sigma - 2 \sum_{i,j=1}^{N} \int_{\Omega} \partial_{i} h_{j} \partial_{i} w \partial_{j} w \; dx
\]

\[
\int_{\Omega} \sum_{j=1}^{N} \partial_{j} h_{j} |\nabla w|^{2} \; dx. \quad (44)
\]

**Proof.** We integrate by parts to get

\[
\int_{\Omega} \nabla w \cdot \nabla w \; dx = \int_{\partial \Omega} \partial_{w} w \cdot \nabla w \; d\sigma - \int_{\Omega} \nabla w \cdot \nabla (h \cdot \nabla w) \; dx. \quad (45)
\]
Since
\[ \int_\Omega \nabla w \cdot \nabla (h \cdot \nabla w) \, dx = \sum_{i,j=1}^N \int_\Omega \partial_i w \partial_i (h_j \partial_j w) \, dx = \sum_{i,j=1}^N \int_\Omega \partial^2_i w \partial_i h_j \partial_j w \, dx + \sum_{i,j=1}^N \int_\Omega h_j \partial_i w \partial_j (\partial_i w) \, dx, \]
we evaluate the last term on the right-hand side again by an integration by parts, so we obtain
\[ \sum_{i,j=1}^N \int_\Omega h_j \partial_i w \partial_j (\partial_i w) \, dx = \frac{1}{2} \sum_{j=1}^N \int h_j \partial_j \left( \sum_{i=1}^N (\partial_i w)^2 \right) \, dx \]
\[ = \frac{1}{2} \int h \cdot v |\nabla w|^2 \, d\sigma - \frac{1}{2} \int_\Omega \sum_{j=1}^N \partial_j h_j \cdot |\nabla w|^2 \, dx. \]
Therefore, if we merge the above two identities with (45), then we have (44). \( \square \)

**Lemma 4.2** Assume \( 1 < \alpha < 2 \) and the weak solution \( u \) of
\[ \partial^\alpha_t u(t, x) = \triangle u(t, x) \quad \text{in} \quad (0, \infty) \times \Omega \quad \text{(46)} \]
belonging to \( C([0, +\infty); H^2(\Omega) \cap H^1_0(\Omega)) \cap C^1([0, +\infty); L^2(\Omega)) \) with \( \partial^\alpha_t u \in C([0, +\infty); L^2(\Omega)) \). Then, for a vector field \( h : \overline{\Omega} \to \mathbb{R}^N \) of class \( C^1 \) and \( \beta, \theta \in (0, 1) \) the following identities hold true
\[ \int_{\partial\Omega} \left[ 2I^\beta(\partial_\nu u)(t) \cdot I^\beta(\nabla u)(t) - h \cdot \nu |I^\beta(\nabla u)(t)|^2 \right] \, d\sigma = 2\langle I^\beta(\partial_t^\alpha u)(t), h \cdot I^\beta(\nabla u)(t) \rangle_{-\theta, \theta} \]
\[ + 2 \sum_{i,j=1}^N \int_\Omega \partial_i h_j I^\beta(\partial_\nu u)(t)I^\beta(\partial_j u)(t) \, dx - \int_\Omega \sum_{j=1}^N \partial_j h_j \cdot |I^\beta(\nabla u)(t)|^2 \, dx, \quad t > 0, \quad \text{(47)} \]
\[ \int_{\partial\Omega} \left[ 2(I^\beta(\partial_\nu u)(t) - I^\beta(\partial_\nu u)(\tau)) \cdot h \cdot (I^\beta(\nabla u)(t) - I^\beta(\nabla u)(\tau)) - h \cdot \nu |I^\beta(\nabla u)(t) - I^\beta(\nabla u)(\tau)|^2 \right] \, d\sigma \]
\[ = 2\langle I^\beta(\partial_t^\alpha u)(t) - I^\beta(\partial_t^\alpha u)(\tau), h \cdot (I^\beta(\nabla u)(t) - I^\beta(\nabla u)(\tau)) \rangle_{-\theta, \theta} \]
\[ + 2 \sum_{i,j=1}^N \int_\Omega \partial_i h_j (I^\beta(\partial_\nu u)(t) - I^\beta(\partial_\nu u)(\tau)) (I^\beta(\partial_j u)(t) - I^\beta(\partial_j u)(\tau)) \, dx \]
\[ - \sum_{j=1}^N \int_\Omega \partial_j h_j \cdot |I^\beta(\nabla u)(t) - I^\beta(\nabla u)(\tau)|^2 \, dx, \quad t, \tau > 0. \quad \text{(48)} \]

**Proof.** First, we apply the operator \( I^\beta, \beta \in (0, 1) \), to equation (46):
\[ I^\beta(\partial_t^\alpha u)(t) = I^\beta(\triangle u)(t) \quad t > 0. \quad \text{(49)} \]
Fix \( \theta \in (0, 1) \), by means of the duality \( \langle \cdot, \cdot \rangle_{-\theta, \theta} \) introduced by (17) we multiply the terms of the previous equation by
\[ 2h \cdot \nabla I^\beta(u)(t), \]
that is
\[ 2\langle I^\beta(\partial_\nu^\beta u)(t), h \cdot \nabla I^\beta(u)(t)\rangle_{-\theta,\theta} = 2\langle \triangle I^\beta(u)(t), h \cdot \nabla I^\beta(u)(t)\rangle_{-\theta,\theta}. \]

Thanks to the regularity of data and (19), the term on the right-hand side of the previous equation can be written as a scalar product in \( L^2(\Omega) \), so we have
\[ 2\langle I^\beta(\partial_\nu^\beta u)(t), h \cdot \nabla I^\beta(u)(t)\rangle_{-\theta,\theta} = 2\int_\Omega \triangle I^\beta(u)(t) h \cdot \nabla I^\beta(u)(t) \, dx \tag{50} \]

To evaluate the term
\[ 2\int_\Omega \triangle I^\beta(u)(t) h \cdot \nabla I^\beta(u)(t) \, dx, \]
we apply Lemma 4.1 to the function \( w(t, x) = I^\beta(u)(t) \), so from (44) we deduce
\[ 2\int_\Omega \triangle I^\beta(u)(t) h \cdot \nabla I^\beta(u)(t) \, dx = \int_{\partial\Omega} \left[ 2I^\beta(\partial_\nu u)(t) h \cdot I^\beta(\nabla u)(t) - h \cdot \nu |I^\beta(\nabla u)(t)|^2 \right] d\sigma \]
\[ - 2 \sum_{i,j=1}^N \int_\Omega \partial_i h \partial_j I^\beta(\partial_i u)(t) I^\beta(\partial_j u)(t) \, dx + \int_\Omega \sum_{j=1}^N \partial_j h \frac{|I^\beta(\nabla u)(t)|^2}{2} \, dx. \]

In conclusion, plugging the above formula into (50), we obtain (47).

The proof of (48) is similar to that of (47). Indeed, starting from
\[ I^\beta(\partial_\nu^\beta u)(t) - I^\beta(\partial_\nu^\beta u)(\tau) = I^\beta(\triangle u)(t) - I^\beta(\triangle u)(\tau) \quad t, \tau > 0, \]
by means of the duality \( \langle \cdot, \cdot \rangle_{-\theta,\theta} \) one multiplies both terms by
\[ 2h \cdot \nabla (I^\beta(u)(t) - I^\beta(u)(\tau)). \]

Then applying Lemma 4.1 to the function \( w(t, \tau, x) = I^\beta(u)(t) - I^\beta(u)(\tau) \), one can get the identity (48). We omit the details. \( \square \)

**Theorem 4.3** Let \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \), \( u_1 \in H^1_0(\Omega) \) and \( u \) the weak solution of

\[
\begin{align*}
\partial_\nu^\beta u(t, x) &= \Delta u(t, x), \quad t \geq 0, \ x \in \Omega, \\
u(t, x) &= 0 \quad t \geq 0, \ x \in \partial \Omega, \\
u(0, x) &= u_0(x), \quad \nu(t, 0, x) = u_1(x), \quad x \in \Omega.
\end{align*}
\tag{51}
\]

Then, for any \( T > 0 \) there is a constant \( c_0 = c_0(T) \) such that \( u \) satisfies the inequality
\[ \int_0^T \int_{\partial\Omega} |\partial_\nu u|^2 \, d\sigma dt \leq c_0(\|\nabla u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2). \tag{52} \]

**Proof.** We will use Theorem 2.31 with \( H = L^2(\partial\Omega) \) and \( \beta \in (0, 1) \). Indeed, thanks to (25) we have
\[ \|\partial_\nu u\|_{L^2(0,T;L^2(\partial\Omega))} \sim \|I^\beta(\partial_\nu u)\|_{H^\beta(0,T;L^2(\partial\Omega))}, \tag{53} \]
so, taking also into account (23), the proof of (52) is equivalent to prove
\[ \|I^\beta(\partial_\nu u)\|_{L^2(0,T;L^2(\partial\Omega))}^2 + \|I^\beta(\partial_\nu u)\|_{H^\beta(0,T;L^2(\partial\Omega))}^2 \leq c_0(\|\nabla u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2). \tag{54} \]
To this end we will employ the two identities in Lemma 4.2 with a suitable choice of the vector field \( h \). Indeed, we take a vector field \( h \in C^1(\Omega; \mathbb{R}^N) \) satisfying the condition

\[
  h = \nu \quad \text{on} \quad \partial \Omega
\]

(see e.g. [16] for the existence of such vector field \( h \)) and first consider the identity (47). Since

\[
  \nabla u = (\partial_\nu u)\nu \quad \text{on} \quad (0, T) \times \partial \Omega,
\]

(see e.g. [27, Lemma 2.1] for a detailed proof) the left-hand side of (47) becomes

\[
  \int_{\partial \Omega} |I^3(\partial_\nu u)|^2 \, d\sigma.
\]

Thanks to that choice of \( h \), if we integrate (47) over \([0, T]\), then we obtain

\[
  \int_0^T \int_{\partial \Omega} |I^3(\partial_\nu u)|^2 \, d\sigma dt = 2 \int_0^T \langle I^3(\partial_t^\alpha u)(t), h \cdot I^3(\nabla u)(t) \rangle_{-\theta, \theta} \, dt
\]

\[
  + 2 \sum_{i,j=1}^N \int_0^T \int_{\Omega} \partial_\nu h_j I^3(\partial_t^\alpha u)(t)I^3(\partial_j u)(t) \, dx dt - \int_0^T \int_{\Omega} \sum_{j=1}^N \partial_\nu h_j |I^3(\nabla u)(t)|^2 \, dx dt.
\]

Thanks again to the condition (56) the left-hand side of (48) becomes

\[
  \int_{\partial \Omega} |I^3(\partial_\nu u)(t) - I^3(\partial_\nu u)(\tau)|^2 \, d\sigma.
\]

Therefore, if we multiple both terms of (48) by \( \frac{1}{|t-\tau|^{1+2\beta}} \) and then integrate over \([0, T] \times [0, T]\), we have

\[
  \left[I^3(\partial_\nu u)\right]_{H^\beta(0,T;L^2(\partial \Omega))}^2
\]

\[
  = 2 \int_0^T \int_0^T \frac{1}{|t-\tau|^{1+2\beta}} \langle I^3(\partial_t^\alpha u)(t) - I^3(\partial_t^\alpha u)(\tau), h \cdot (I^3(\nabla u)(t) - I^3(\nabla u)(\tau)) \rangle_{-\theta, \theta} \, dt d\tau
\]

\[
  + 2 \int_0^T \int_0^T \frac{1}{|t-\tau|^{1+2\beta}} \sum_{i,j=1}^N \int_{\Omega} \partial_\nu h_j (I^3(\partial_t^\alpha u)(t) - I^3(\partial_t^\alpha u)(\tau))(I^3(\partial_j u)(t) - I^3(\partial_j u)(\tau)) \, dx \, dt d\tau
\]

\[
  - \int_0^T \int_0^T \frac{1}{|t-\tau|^{1+2\beta}} \int_{\Omega} \sum_{j=1}^N \partial_\nu h_j |I^3(\nabla u)(t) - I^3(\nabla u)(\tau)|^2 \, dx \, dt d\tau.
\]

To estimate the first term on the right-hand side of the above identity, we note that

\[
  2 \int_0^T \int_0^T \frac{1}{|t-\tau|^{1+2\beta}} \langle I^3(\partial_t^\alpha u)(t) - I^3(\partial_t^\alpha u)(\tau), h \cdot (I^3(\nabla u)(t) - I^3(\nabla u)(\tau)) \rangle_{-\theta, \theta} \, dt d\tau
\]

\[
  \leq C \left[ I^3(\partial_t^\alpha u) \right]_{H^\beta(0,T;D(A^{-\theta}))}^2 + C \left[ I^3(\nabla u) \right]_{H^\beta(0,T;D(A^{\theta}))}^2.
\]

If we choose \( \theta \in \left( \frac{\alpha - 1}{2\alpha}, \frac{1}{2\alpha} \right) \), then we can apply Theorem 3.1 to get \( \partial_t^\alpha u \in L^2(0,T;D(A^{-\theta})) \) and \( \nabla u \in L^2(0,T;D(A^{\theta})) \). Therefore, thanks to Theorem 2.4 we have

\[
  \|I^3(\partial_t^\alpha u)\|_{H^\beta(0,T;D(A^{-\theta}))} \sim \|\partial_t^\alpha u\|_{L^2(0,T;D(A^{-\theta}))},
\]

\[
  \|I^3(\nabla u)\|_{H^\beta(0,T;D(A^{\theta}))} \sim \|\nabla u\|_{L^2(0,T;D(A^{\theta}))}.
\]

\( \square \)
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