Replica-deformation of the $SU(2)$-invariant Thirring model via solutions of the qKZ equation

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Dedicated to the memory of Moshé Flato

Abstract

The response of an integrable QFT under variation of the Unruh temperature has recently been shown to be computable from an $S$-matrix preserving (“replica”) deformation of the form factor approach. We show that replica-deformed form factors of the $SU(2)$-invariant Thirring model can be found among the solutions of the rational $\mathfrak{s}l_2$-type quantum Knizhnik-Zamolodchikov equation at generic level. We show that modulo conserved charge solutions the deformed form factors are in one-to-one correspondence to the ones at level zero and use this to conjecture the deformed form factors of the Noether current in our model.
1 Introduction

The form factor approach to massive integrable quantum field theories (QFTs) \[9, 20\] provides a powerful tool to compute exact matrix elements of local operators from the knowledge of the (factorized) S-matrix. To be precise, a form factor in this context is the matrix element of some local operator between the physical vacuum and a multi-particle scattering state. These form factors contain all the information about the QFT as the Wightman functions can in principle be (re-)constructed in terms of them by saturating with a complete set of scattering states. Alternatively each local operator can be set into correspondence to a sequence of form factors; a feature that has been used to classify the full content of local operators in certain models, see e.g. \[22, 19\].

For an integrable QFT the form factors can be characterized axiomatically as tensor-valued meromorphic functions obeying a recursive set of functional equations, the so-called form factor equations \[20\].

As this will be of importance later we anticipate that one of these equations (the cyclic form factor equation), see equation (8) below, is closely related to the Bisignano-Wichmann-Unruh thermalization phenomenon \[1, 26\]. In particular the cyclicity parameter \(\beta = 2\pi\) is both physically and mathematically linked to the Unruh temperature \(T_U = 1/2\pi\) (in natural units), seemingly of a very different origin. This connection has been used to formulate the cyclic form factor equation without reference to bootstrap analyticity properties and to generalize it to non-integrable QFTs \[16\].

Among the most prominent non-scalar massive integrable models which have been solved by the form factor method at \(T_U = 1/2\pi\) are the Sine-Gordon model and the \(SU(2)\)-invariant Thirring model \[20, 21\]. We will refer to the latter, which will concern us in this paper, simply as the Thirring model (even though this clashes a bit with standard usage).

On the other hand, in an a priori independent direction of research it was shown in a pioneering work \[4\] that the classical Knizhnik-Zamolodchikov differential equation (for a survey see \[3\]) which arises in conformal field theory admits a quantum deformation. This deformation amounts in a consistent system of holonomic difference equation for a tensor valued meromorphic function, which is referred to as deformed or quantum Knizhnik-Zamolodchikov (qKZ) equation. The authors of \[4\] already realised that a subset of the standard form factor equations \[20\] provide a very specific example of a qKZ equation. Much work has been done during recent years, to elaborate this connection, see \[7, 13, 11\] and references therein.
In the seminal paper [24] the complete set of solutions of the qKZ equation for a rational sl$_2$-type R-matrix has been found. In this case one considers a function $\Psi(z_1, \ldots, z_n)$ taking values in a tensor product of sl$_2$-modules $V_1 \otimes \ldots \otimes V_n$, and subject to a system of difference equations with shift parameter $p$

$$\begin{align*}
\Psi(z_1, \ldots, z_m + p, \ldots, z_n) &= R_{m,m-1}(z_m - z_{m-1} + p) \cdots R_{m,m-1}(z_m - z_1 + p) \times \\
&\times R_{m,m-1}(z_m - z_n) \cdots R_{m,m+1}(z_m - z_1) \Psi(z_1, \ldots, z_m, \ldots, z_n).
\end{align*}
$$

(1)

Here, $R_{i,j}(z) \in \text{End}(V_i \otimes V_j)$ is the rational sl$_2$ R-matrix. The more general case of this equation involving an extra parameter $\kappa$ as studied in [24] will not be of importance in this paper.

In a nutshell the solutions of (1) are found as generalised hypergeometric integrals. Subsequently, the construction of [24] has been applied to irreducible representations of sl$_2$ in [12]. It is this case which is relevant here and which will provide our main mathematical tool.

Returning to form factors, it has been shown in [13] that Smirnov’s solutions of the form factor equations in the Thirring model [20, 21] can indeed be extracted from the general solutions of (1). In particular, Smirnov’s results follow if we take only two-dimensional fundamental representations $V$ of sl$_2$ (obviously corresponding to the physical states in the model) and set $p = -2\pi i$. This particular choice is referred to as level zero solution of the qKZ equation. In other words, the form factors of the Thirring model can be identified with specific solutions of the qKZ equation at level zero.

The physics problem we wish to solve is to construct the “replica-deformed” form factors in the sense of [14] for the SU(2)-invariant massive Thirring model.

We shall review some aspects of the replica-deformation in section 2. Here it may be sufficient to remark that in view of the aforementioned relation to the Bisignano-Wichmann-Unruh thermalization, taking $p \neq 2\pi i$ in the qKZ equations amounts to studying the “continuation” of the original QFT to some “off-critical” Unruh temperature. The quest for such a continuation naturally arises in the broader context of quantum gravity, see e.g. [2, 4], in particular one is interested in the response of a QFT under an infinitesimal variation of the Unruh temperature. Of course the problem in the first place consists in defining the “continuation” of the QFT in question. For the partition function Callan and Wilczek [2] proposed a “replica” prescription to define its continuation. In [14] more generally the invariance of the S-matrix has been proposed as the defining criterion. Specifically for an integrable QFT this criterion turned out to uniquely determine the proper deformation of the form factor equations. In particular there is still a cyclic equation whose
cyclicity parameter $p \neq 2\pi i$ physically corresponds to an off-critical Unruh temperature and which is equivalent to the qKZ equation at generic (non-zero) level. The main difference to the ordinary form factor equations lies in the residue equations which prescribe how solutions with different number of variables are arranged into sequences. For models with a diagonal S-matrix the qKZ is easy to implement and some replica-deformed form factors have been computed in [14, 18]. However, the full set has not yet been found for any model.

As it was mentioned above the standard form factors of the Thirring model can be found among the solutions of (1) at level zero. It will be the key point of this paper to show that the physical problem of construction the “replica” relatives of them can be solved by means of the solutions of (1) at generic level. To be a bit more precise we have to arrange solutions of (1) in sequences such that functions $\Psi(z_1, \ldots, z_n)$ with different numbers of arguments are linked by the form factor equations to be described in section 2.

In particular if the local operator whose form factors are being considered has isospin $j$ then in group theoretical terms its $n$-particle form factor is an intertwiner $V^\otimes_n \to V_j$, where $V_j$ is the irreducible $sl_2$-module of isospin $j$. In other words one has a one-to-one correspondence

\[
\text{local operator of isospin } j \quad \longleftrightarrow \quad \text{sequence of level zero qKZ solutions intertwinning } V^\otimes_n \to V_j, \quad n \geq n_0
\]  

(2)

Here $n_0$ is the starting member of a sequence and the non-vanishing members of a sequence have all either $n$ odd or $n$ even. For simplicity we shall refer to such a sequence as an isospin $j$ sequence. The aim in the bulk of the paper is to achieve something similar for the replica-deformed form factors. Since the replica-deformed system is no longer an ordinary Poincaré invariant QFT, the bootstrap viewpoint becomes even more important. That is the collection of deformed form factors is supposed to define the replica deformed system.

Our main result can then be summarized as follows: The right hand side of the correspondence (2) admits a unique replica-deformation, provided on both sides the equivalence classes modulo pointwise multiplication with a conserved charge eigenvalue (i.e. a regular solution of the form factor equations with trivial S-matrix) are being considered. Symbolically we can write

\[
\text{sequence of level zero qKZ solutions modulo conserved charge eigenvalue } \quad \longleftrightarrow \quad \text{sequence of level } p - 2\pi i \neq 0 \text{ qKZ solutions modulo conserved charge eigenvalue}
\]  

(3)
The structure of the conserved charge eigenvalues in both cases is very different and so is the kinematical arena [14]. However, in view of (3) the structure of the state space in both theories is very similar as will be shown in section 5, much in the spirit of the replica idea [2, 14].

The outline of the paper is as follows. In the next section we briefly review the replica-deformation of massive integrable field theories and describe the deformed form factor equations. In section 3 we summarise the essentials of the bootstrap description of the Thirring model and prepare the deformed minimal form factors [18]. In section 4 we rewrite the solutions of the rational $sl_2$-type qKZ-equation of [24, 12] in a way that facilitates the comparison with the (undeformed) form factors of the Thirring model. It turns out that the solutions can be parameterised by a polynomial similar to Smirnov’s form factors. We proceed by generalising the completeness proof of the level zero solutions [23] to generic level. It is essentially this completeness result that underlies the correspondence (3). In section 6 we give the result for the deformed form factors and first show that they satisfy a subset of the modified form factor equations of [14]. Here we take advantage of the fact that the replica deformation leaves the bootstrap $S$-matrix, and hence the structure of the algebraic Bethe vectors, unaffected. Then we derive conditions on the polynomial entering the qKZ-solutions under which the corresponding form factors in addition satisfy the modified kinematical residue equations. Finally, using (2) and (3) we give explicit expressions of some form factors which we conjecture to be the replica-deformed form factors of the Noether current in the SU(2) Thirring model. The last section is left to a discussion of the results.

2 Replica deformation of the form factor approach

In this section we review the results of [14, 15, 16] and state the modified form factor equations. The starting point is the Bisignano-Wichmann-Unruh thermalisation phenomenon stating that the vacuum of a Minkowski space QFT looks like a thermal state of inverse temperature $\beta = 2\pi$ with respect to the Killing time of the Rindler wedge [1, 26]. This fact can be encoded symbolically in the following formula, where $O_i(x_i), i = 1, \ldots, n$ denotes some local operator with support inside the Rindler wedge $W$

$$\langle 0|O_1(x_1)\cdots O_n(x_n)|0\rangle = \text{tr} \left[ \exp(2\pi K)O_1(x_1)\cdots O_n(x_n) \right].$$

(4)
Here \( K \) stands for the generator of Lorentz boosts in \( W \). Note that this trace can – in contrast to lattice models – never exist in a continuum QFT due to the noncompactness of \( K \) as described in [14, 16]. We therefore take (4) only as a mnemonic.

Suppose now that we want to study “the same” QFT with the parameter \( \beta \) shifted away from \( 2\pi \), a problem which naturally arises in the broader context of quantum gravity, see e.g. [2, 5]. Formally this amounts to replace
\[
\exp(2\pi K) \rightarrow (\exp(2\pi K))^{\beta/2\pi} = \exp(\beta K),
\]
in (4) while keeping everything else fixed. In the case of an integrable QFT we can require specifically that the factorized S-matrix of the model is unchanged upon (5). This can be viewed as a concrete realisation of the “replica” idea of Callan and Wilczek [2]. In particular, if one could make sense out of the deformed correlator (4), (5), its response under an infinitesimal variation of the Unruh temperature could be obtained simply by differentiation
\[
\beta \frac{\partial}{\partial \beta} \text{tr} \left[ \exp(2\pi K) O_1(x_1) \cdots O_n(x_n) \right] \bigg|_{\beta=2\pi}.
\]
The response (6) again has a meaning within the context of QFTs, while the deformed correlators themselves no longer correspond to a relativistic QFT.

The crux of the problem of course is to define the deformed correlators consistent with the replica idea (5) and in a way that avoids mathematical ambiguities. For example naively regularizing the trace in (4) would most likely lead to enormous technical problems with renormalizability since the translation invariance of the original QFT is lost for \( \beta \neq 2\pi \).

The solution proposed in [14] is to implement the replica idea on the level of form factors. A form factor in this context is the matrix element of a local operator \( O(x) \) between the vacuum and an \( n \)-particle scattering state. The scattering states depend on rapidities \( z_i \) and are created from the vacuum by means of Faddev-Zamolodchikov-type operators \( A_{a_i}(z_i) \), where the index \( a_i \) refers to the charge of a state. The form factor is then
\[
F_{a_1 \ldots a_n}(z_1, \ldots, z_n) = \langle 0 | O(x) | z_1, \ldots, z_n \rangle_{a_1 \ldots a_n} \sim \text{tr} \left[ e^{2\pi K} O(x) A_{a_1}(z_1) \cdots A_{a_n}(z_n) \right],
\]
where we are again using our trace-mnemonic (4). In essence (7) says that a form factor always looks like a trace, a fact that has been derived from general quantum field theoretical principles and linked to the thermalisation phenomenon (4) in [16]. Evidently the replica idea (5) can be applied to (7) as well, and here it can be made operational. This is because for the replica deformed form factors a modified system of form factor
equations exists. It is uniquely determined by (3) and the requirement that the S-matrix in unaffected by the deformation. For fixed $n$ the first two equations are as follows [14].

$$F_{a_1...a_n}(z_1 + i\beta, z_2, \ldots, z_n) = \eta F_{a_2...a_n,a_1}(z_2, \ldots, z_n, z_1),$$
$$F_{a_1...a_n,a_{n-1}}(z_1, \ldots, z_n, z_{n-1}) = S^n_{ab} F_{a_1...a_{n-1}}(z_1, \ldots, z_{n-1}, z_n). \quad (8)$$

Here $S^{cd}(z)$ is the factorised scattering operator. If this scattering operator has something to do with an R-matrix, it is not difficult to see that these two equations are equivalent to a qKZ equation [1] at generic level if we identify $p$ and $\beta$ in an obvious way. For $\beta = 2\pi$ we recover the level zero situation or the standard Watson equations for form factors [20]. The parameter $\eta$ is a phase, which may be different from $\eta = 1$ for certain local operators of the model, see [20, 21].

We need to supply the equations (8) with one more set of conditions. These conditions serve to single out those solutions of the qKZ equation which are of physical importance and provide a recursive relation between form factors involving different numbers of particles in the scattering state. We will refer to these additional conditions as kinematical residue equations.

We note that due to the first two requirements on the form factor (8) we may formulate without loss of generality the recursive relations just in the variables $z_{n-1}$ and $z_n$.

The particular solutions of (8) searched for are according to [14] supposed to have simple poles at $z_n = z_{n-1} = \pm i\pi \mod i\beta$. The residues at these poles are determined by the following equations which link solutions of (8) with $n$ and $n-2$ variables.

$$\text{res}_{z_n = z_{n-1} + i\pi} F_{a_1...a_{n-1}}(z_1, \ldots, z_{n-1}, z_n) = -C_{ab} F_{a_1...a_{n-2}}(z_1, \ldots, z_{n-2}).$$
$$\text{res}_{z_n = z_{n-1} - i\pi} C^{a_{n-1}}_{a_n} F_{a_1...a_{n-1}}(z_1, \ldots, z_{n-1}, z_n) = -F_{a_1...a_{n-2}}(z_1, \ldots, z_{n-2}). \quad (9)$$

Here $C_{ab}$ is the charge conjugation matrix and $\lambda$ is a numerical constant to be specified later. Note that the residue equation for $z_n = z_{n-1} + i\pi$ is similar to the condition satisfied by traces of vertex operators in lattice models [2, 8]. Without the second equation (9) however the recursion $n - 2 \rightarrow n$ would be highly ambiguous. Taking advantage of the qKZ equation to implement the analytic continuation $z_n \rightarrow z_n + i\beta$ the second equation (9) can be rewritten as follows

$$\text{res}_{z_n = z_{n-1} - i\pi + i\beta} F_{a_1...a_n}^{(n)}(z_1, \ldots, z_{n-1}, z_n) = \eta F_{b_1...b_{n-2}}^{(n-2)}(z_1, \ldots, z_{n-2}) \times C^{a_{n-1}}_{b_{n-1}} S^{b_1}_{a_1}(z_{n-1} - z_1) \cdots S^{b_{n-2}}_{a_{n-2}}(z_{n-1} - z_{n-2}). \quad (10)$$
This form turns out to be more convenient later. Observe that \( (10) \) looks like the second term in the ordinary \( \beta = 2\pi \) residue equation. For \( \beta \to 2\pi \) the poles at \( z_n = z_{n-1} + i\pi \) and \( z_n = z_{n-1} - i\pi + i\beta \) merge and the residues add up, so that the ordinary residue equation of \( (10) \) is recovered.

Equations (8) and (9) are the replica deformed form factor equations. As explained in the introduction for a given S-matrix we take their solutions to define the replica deformed system. The variation of of \( \beta \) away from \( 2\pi \) then has wide ranging physical consequences. First of all the standard translation invariance is broken while Lorentz symmetry is maintained. Quite generally the kinematical properties of the \( \beta \neq 2\pi \) models are drastically changed. For example it has been shown in [14] that the mass eigenvalues of asymptotic states are altered in a way such that it costs more energy to boost two particles relative to each other than in the \( \beta = 2\pi \) case. Moreover, under certain circumstances not the entire momentum phase space is accessible to the particles. This phenomenon is close in spirit to ‘t Hooft’s picture of scattering states subject to quantum gravitational transmutation [6].

### 3 The SU(2)-invariant Thirring model

The aim in the rest of the paper is to study the replica-deformation of the 1+1-dimensional SU(2)-Thirring model in terms of its form factors. In this section we prepare the basic ingredients for its bootstrap description. For the sake of orientation however let us also display the classical Lagrangian. It is given in terms of a two-component spinor \( \psi \), valued in the fundamental representation of SU(2).

\[
\mathcal{L} = i\bar{\psi} \gamma^\mu \partial_\mu \psi - g(\bar{\psi} \sigma^a \gamma^\mu \psi)(\bar{\psi} \sigma_a \gamma^\mu \psi).
\]  

Here \( \sigma^a \) are the Pauli matrices. The model exhibits dynamical mass generation and is believed to be asymptotically free. The physical particles of the theory are an SU(2) doublet of massive kinks. The index \( \varepsilon \in \{ \pm \} \) refers to either of these kinks. Moreover it is known that the model is integrable, and the factorised S-matrix is given as a function of the kink-rapidity \( z \) by the following expression.

\[
S_{\varepsilon_1,\varepsilon_2}(z) = S_0(z) \frac{z^{\varepsilon_1^{\varepsilon_2^{\varepsilon_1}}} + h}{z + h}, \quad h = -i\pi.
\]
We use the abbreviation $h = -i\pi$ as indicated in (12) from now on whenever suitable in accordance with [13]. The scalar part $S_0$ of the S-matrix is given by

$$S_0(z) = \frac{\Gamma\left(\frac{1}{2} - \frac{z}{2h}\right)\Gamma\left(\frac{z}{2h}\right)}{\Gamma\left(\frac{1}{2} + \frac{z}{2h}\right)\Gamma\left(-\frac{z}{2h}\right)}.$$  \hspace{1cm} (13)

Without $S_0$ the expression in (12) coincides with the rational $sl_2$-R-matrix in the defining representation [24, 12].

We also note that the charge conjugation matrix which enters the form factor equations (9) and (10) is given as follows.

$$C = i\sigma^2. \hspace{1cm} (14)$$

The ordinary form factors associated with this S-matrix can be found in [20]. They have two features one expects to be present also in the deformed case. First the tensor structure and second the appearance of a product of so-called minimal form factors in the solutions. Concerning the tensor structure we recall that an n-particle form factor (7) can be indexed by the kinks and anti-kinks of the scattering state and hence takes values in an n-fold tensor product $V^\otimes n$ of fundamental representations of $SU(2)$. Using the Pauli matrices $\Sigma^a = \sum_{i=1}^n \sigma_i^a$ we can then define the weight subspaces

$$(V^\otimes n)_l = \{ v \in V^\otimes n | \Sigma^3 v = (n - 2l) v \}. \hspace{1cm} (15)$$

We restrict our attention to solutions $\psi(z_1, \ldots, z_n)$ of the qKZ equation valued in the space of singular vectors in $(V^\otimes n)_l^{sing}$, i.e. to those satisfying $\Sigma^+ \psi(z_1, \ldots, z_n) = 0$. This is related to an unbroken $SU(2)$-symmetry. Then $2l \leq n$ and all other components in an irreducible $SU(2)$-multiplet of isospin $j = n/2 - l$ can be obtained by acting with $\Sigma^-$ on the singular vector.

To make the notation comparable to [13, 12] we map the indices $a_1, \ldots, a_n$ of the form factor (7) to a set $M$ as follows. Let $M = \{m_1 < m_2 < \ldots < m_l\} \subseteq \{1, 2, \ldots, n\}$, such that the number of elements of $M$ is $\#M = l$. In other words this set labels the positions at which the form factor (7) has kinks of type “−” in the scattering state, i.e. $M = \{i | \varepsilon_i = -\}$. Hence, we can identify vectors in $V^\otimes n$ as

$$v_M := v_{\varepsilon_1} \otimes \ldots \otimes v_{\varepsilon_n}. \hspace{1cm} (16)$$

Naturally we shall employ the weight decomposition (15) and the labellings (16) also in the deformed case.
Further we prepare here the deformed counterpart of the minimal form factor. This is (up to factors coming from representation theory and the pole structure) the two particle form factor as explained in [18]. The deformed minimal form factor \( f(z) \) is a solution of the functional equations

\[
f(z) = S_0(z)f(-z), \quad f(z + i\beta) = f(-z).
\]

(17)

The solution to these equations were found in [18] in terms of Barnes’ Digamma functions. Let complex numbers \( \omega_1, \omega_2 \) be such that \( \text{Re} \, \omega_i > 0 \) for \( i = 1, 2 \). We define the Digamma function [8, 18] by the Hankel contour integral as

\[
\log \Gamma_2(x|\omega_1, \omega_2) = \frac{1}{2\pi i} \int \frac{\exp(-xt)(\log(-t) + \gamma)}{t} \prod_{i=1}^{2} \frac{1 - \exp(-\omega_i t)}{t} \, dt,
\]

(18)

where \( \gamma \) denotes the Euler constant. Note that the periods \( \omega_1, \omega_2 \) enter symmetrically in (18). The Barnes’ periodicity of this function is due to the relation

\[
\frac{\Gamma_2(x + \omega_1|\omega_1, \omega_2)}{\Gamma_2(x|\omega_1, \omega_2)} = \frac{1}{\Gamma_1(x|\omega_2)},
\]

(19)

where \( \Gamma_1 \) can be identified with the usual \( \Gamma \) function as

\[
\Gamma_1(x|\omega_2) = \omega_2^{-1/2+x/\omega_2} \Gamma(x/\omega_2)/\sqrt{2\pi}.
\]

(20)

The relevant solutions of (17) are then given by

\[
f(z) = \frac{\Gamma_2(\pi - iz|2\pi, \beta)\Gamma_2(\pi + \beta + iz|2\pi, \beta)}{\Gamma_2(-iz|2\pi, \beta)\Gamma_2(\beta + iz|2\pi, \beta)}.
\]

(21)

Note that apart from the higher periodicity the structure of (21) is similar to the one of the scalar S-matrix (13). This feature was observed to be common for a large class of integrable field theories in [18]. The minimal form factor (21) was given an interpretation in terms of central extensions of Yangian doubles in [10]. As remarked before we can identify the “inverse Unruh temperature” \( \beta \) with the shift parameter \( p \) of the qKZ equation (1) via

\[
p = -i\beta.
\]

(22)

We find it convenient to use either of the variables throughout the paper, keeping the above identification in mind. For \( \beta = 2\pi \) one recovers using (13) from (21) up to trivial factors the minimal form factor \( \zeta(z) \) used by Smirnov in [20, 21].
For later use we conclude this section with the following identities among the minimal form factors

\[
    f(z)f(z-h) = \frac{1}{\Gamma_1(-iz|\beta)\Gamma_1(\beta - \pi + iz|\beta)}
\]

\[
    f(z)f(z+h+i\beta) = \frac{S_0(z)}{\Gamma_1(iz|\beta)\Gamma_1(\beta - \pi - iz|\beta)}.
\]

(23)

4 Solutions of the rational qKZ equation

It was mentioned in the previous section that – up to the scalar factor – the S-matrix of the Thirring model is just the rational \(sl_2\)-R-matrix in the defining representation. This means that the first two functional equations (8) to be satisfied by the form factor (7) can, – up to scalar factors which turn out to be the minimal form factors – be identified with the solutions of the qKZ equations with rational \(R\)-matrix as given in [24, 12]. Obviously our case is particularly simple in this respect because we have to work with only one type of representation of \(sl_2\).

Below we first recall the solutions of [24, 12] at arbitrary level and then proceed by rewriting them in a form suitable for our purposes. The new form readily allows one to make contact to the level zero solutions [20, 21, 13] in the limit \(p \to -2\pi i\), and secondly allows for a transparent procedure to single out those qKZ-solutions which in addition satisfy the modified kinematical residue equations (9) and (10).

We start by defining the rational hypergeometric space of [24] which will carry the information on the number and position of the kinks and anti-kinks of the form factor (7). This space consists of rational functions with at most simple poles and a certain asymptotic behaviour [24]. We define

\[
    \hat{g}_M(t_1, \ldots, t_l) = \prod_{a=1}^{l} \left( \frac{1}{t_a - z_{m_a} - \Lambda} \prod_{1 \leq k < m_a} \frac{t_a - z_k + \Lambda}{t_a - z_k - \Lambda} \right).
\]

(24)

Let \( f = f(t_1, \ldots, t_l) \) be a function. Let \( \sigma \) be an element of the symmetric group \( S^l \). Following [24, 12] we define a special symmetrisation in that for a simple transposition \((a, a + 1) \in S^l\) we set
\[ [f]_{(a,a+1)}(t_1, \ldots, t_a, t_{a+1}, \ldots, t_l) := f(t_1, \ldots, t_{a+1}, t_a, \ldots, t_l) \frac{t_a - t_{a+1} + h}{t_a - t_{a+1} - h}. \]  

(25)

Using this symmetrisation one can now introduce the following basis in the rational hypergeometric space.

\[ \hat{w}_M(t_1, \ldots, t_l) = \sum_{\sigma \in S^l} [\hat{g}_M(t_1, \ldots, t_l)]_{\sigma}. \]  

(26)

It is clear that in order to make our results comparable with \[20, 21\] we have to get rid of the unusual symmetrisation occuring in (26).

An important object in the theory qKZ equations is the phase function. This object provides the link between the analytic structure of the solutions and the geometry of local systems \[24, 3\]. In the conventions of \[24\] the phase function is given by:

\[ \hat{\Phi}(t_1, \ldots, t_l; z_1, \ldots, z_n) = \prod_{a=1}^{l} \prod_{j=1}^{n} \hat{\phi}(t_a - z_j; \Lambda) \prod_{1 \leq a < b \leq l} \hat{\phi}(t_a - t_b; h), \]  

(27)

where

\[ \hat{\phi}(x; \alpha) := \frac{\Gamma((x + \alpha)/p)}{\Gamma((x - \alpha)/p)}. \]  

(28)

We refrain from describing the connection coefficients arising from (27), details can be found in \[24\].

The third and last entity to describe the solution spaces of the rational qKZ-equation is the trigonometric hypergeometric space \( F_q \). The functions in this space are of the form

\[ P(\xi_1, \ldots, \xi_l; \zeta_1, \ldots, \zeta_n) \prod_{a=1}^{l} \prod_{j=1}^{n} \frac{\exp(i\pi(z_j - t_a)/p)}{\sin(\pi(t_a - z_j - \Lambda)/p)} \prod_{1 \leq a < b \leq l} \frac{\sin(\pi(t_a - t_b)/p)}{\sin(\pi(t_a - t_b - h)/p)}. \]  

(29)

Here we have set \( \xi_a = \exp(2\pi it_a/p) \) and \( \zeta_j = \exp(2\pi iz_j/p) \). \( P \) is a polynomial. It will be shown in the next section that the completeness of the solutions is encoded into the properties of \( P \). Moreover, the local operators in the model can be classified in terms of this object.

We can now write down the space of solutions of the qKZ equation for a function with values in \( V \otimes^n \) in terms of an integral representation. Let \( W = W_P \) be a function of the space \( F_q \) defined in (29). The integrand is given by
\[
I(\hat{w}_M, W_P) = \hat{w}_M(t_1, \ldots, t_l) \hat{\Phi}(t_1, \ldots, t_l) W_P(t_1, \ldots, t_l),
\]
where we omitted the dependence of \(I(\hat{w}_M, W_P)\) on \(z_1, \ldots, z_n\). The solution of the equation (30) with values in representations indicated in the previous section is then given by:

\[
\Psi_W(z_1, \ldots, z_n) = \sum_{#M=l} \int_{C_1} \cdots \int_{C_l} dt_1 \cdots dt_l I(\hat{w}_M, W_P) v_M.
\]

We specify the integration cycles \(C_l\) below after having rewritten this solution in a more convenient form. It has been shown in [24, 12] that \(\Psi_W\) is a solution to the qKZ equation. We will indicate how to check this after having introduced a basis in the solution space which is more appropriate to the aim of this paper. We stress again that the information on the space of representations of \(sl_2\) is contained only in the function \(\hat{w}_M\). Physically speaking this means that once we have fixed the number of kinks and anti-kinks in the candidate form factor, the only freedom of choice resides in the polynomial \(P\).

Eventually we are interested in constructing the form factors of the replica-deformed \(SU(2)\)-invariant Thirring model. To this end it is useful to rewrite the solution spaces (31) in a form which allows a direct comparison with the standard level zero form factors as they have been given in [20, 21, 13]. In particular we need to get rid of the non-standard symmetrising operation (25) for the rational weight function (26). However, the alterations to be performed do not change (up to an overall normalisation) the solution (31).

We replace some objects appearing in (31) as follows:

\[
t_a - z_j \to t_a - z_j + h, \quad \forall a, j; \quad \Lambda \to -h/2.
\]

This replacement will be in effect from now on.

Next, we replace a part of the phase function (27) which does depend only on differences of the variables \(t\) as follows.

\[
\frac{1}{\Gamma\left(\frac{t_a-t_b-h}{p}\right)} \sim \Gamma\left(\frac{h - (t_a-t_b)}{p}\right) (t_a - t_b - h) \sin \frac{\pi}{p} (t_a - t_b - h).
\]

As it is implicit already in [13] we can use this to replace the non-standard symmetrisation (23) by a standard one.

\[
w_M(t_1, \ldots, t_l) := \hat{w}_M(t_1, \ldots, t_l) \prod_{1 \leq a < b \leq l} (t_a - t_b - h) = \text{Asym} \left( g_M(t_1, \ldots, t_l) \right),
\]

\(12\)
where
\[ g_M(t_1, \ldots, t_l) = \hat{g}_M(t_1, \ldots, t_l) \prod_{1 \leq a < b \leq l} (t_a - t_b - h), \tag{35} \]
and Asym denotes the standard antisymmetrisation with respect to the variables \( t_1, \ldots, t_l. \)
This means that upon performing the steps outlined above, we can work with the same rational weight function as in the level zero case \([13]\). The same feature has been used in \([11]\) for determinant formulas in the case of the trigonometric qKZ equation.
As will be explained below, the natural consequence of this is that the (algebraic) Bethe vectors in the \( \beta \)-deformed case will be the same as in the standard level zero case.
In addition we can also cancel the pure \( t \)-dependent part in the denominator of (29). This has in particular the remarkable consequence that the space \( \mathcal{F}_q \) introduced in (29) almost coincides with the corresponding space introduced in \([13, 23]\) in the level zero case. We shall exploit this fact below.
Next we reorganise the remaining parts of the integrand in (31). This will be done in a way to point out the similarity between (31) and trace formulas of vertex operators as developed in \([7, 8, 10]\). To this end we first define the analogue of the phase function used by Smirnov \([20, 21]\).
\[
\Phi(y) := \Gamma \left( \frac{h/2 + y}{p} \right) \Gamma \left( \frac{p - 3h/2 - y}{p} \right) \exp \left( -\frac{\pi i}{p} y \right). \tag{36} \]
Note that here we put in the exponential factor from (29) just as a matter of convenience.
Then we define an odd function necessary to guarantee the complete symmetry of the integrand with respect to \( t. \)
\[
\psi(y) := -\frac{1}{\pi^3} \Gamma \left( \frac{h + y}{p} \right) \Gamma \left( \frac{h - y}{p} \right) \sin \left( \frac{\pi}{p} y \right). \tag{37} \]
For later use let us note the identity
\[
\Phi(y) \Phi(y + h) = \frac{\exp(-2\pi i(y + h/2)/p)}{\psi(y + 3h/2)(y + h/2) \sin \frac{\pi}{p} (y + h/2) \sin \frac{\pi}{p} (y + 5h/2)}. \tag{38} \]
Observe that this expression (up the the exponential factor) tends to the corresponding identity for the phase functions in \([20, 21]\) in the case \( \beta \rightarrow 2\pi. \)
Having now collected all the data we can write down the solution of the qKZ equation in a new form, which is, however, up to a normalisation identical with (31). Since now the
only degrees of freedom of the solution reside in the polynomial $P$ of (29) we prefer to label the solution by this object.

$$\Psi_P(z_1, \ldots, z_n) = \sum_{\#M=l} \int_{C_1} \cdots \int_{C_l} w_M(t_1, \ldots, t_l) P(\xi_1, \ldots, \xi_l; \zeta_1, \ldots, \zeta_n) \times$$

$$\times \prod_{a=1}^l \prod_{j=1}^n \Phi(t_a - z_j) \prod_{1 \leq a < b \leq l} \psi(t_a - t_b) \, dt_1 \cdots dt_l \, v_M. \tag{39}$$

The singular hyperplanes of the integrand in (39) are

$$t_a = z_j - h/2 + Z_{\leq 0} p,$$
$$t_a = z_j + p - 3h/2 + Z_{\geq 0} p,$$
$$t_a = z_j - 3h/2, \text{ depending on } M,$$
$$t_a = t_b - h + Z_{\leq 0} p,$$
$$t_a = t_b + h + Z_{\geq 0} p. \tag{40}$$

Note that the comment in the third line means that this pole may occur only in the basis function $w_M$ and hence its presence depends on the indexing set $M$.

This now allows us to specify the integration contours as follows. For any $a = 1, \ldots, l$ we let the integration contour $C_a$ separate

i. the hyperplanes of the first line from the ones of the second and third line in (40),

ii. the hyperplanes of the fourth line from the ones in the fifth line of (40).

This choice is convenient for our purposes, but as it has been indicated in [24, 12] the solution space is by analytical continuation actually independent of the particular choice of the integration contour.

We are now going to check that (39) is a solution of the rational qKZ-equation. Of course this is clear by construction [24]. However, along the way we introduce some new objects which will be of importance later in the paper. The use of these objects is nicely illustrated by verifying the qKZ equations.

A fundamental role in [24] is played by discrete derivates with respect to the variables $t$ and $z$ respectively. We describe these derivates within the modified form of the solution space (39). Therefore we introduce shift operators:

$$Z_j f(t_1, \ldots, t_l; z_1, \ldots, z_n) := \phi_{t+j}(t, z) f(t_1, \ldots, t_l; z_1, \ldots, z_j + p, \ldots, z_n),$$
$$Q_a f(t_1, \ldots, t_l; z_1, \ldots, z_n) := \phi_{t}(t, z) f(t_1, \ldots, t_l + p, \ldots, t_l; z_1, \ldots, z_n). \tag{41}$$
where

\[ \phi_{l+j}(t, z) = \prod_{a=1}^{l} \frac{t_a - z_j + 3h/2 - p}{t_a - z_j + h/2 - p}, \]

\[ \phi_a(t, z) = \prod_{b=1}^{l} \frac{t_a - t_b + h}{t_a - t_b + h - p} \prod_{j=1}^{n} \frac{t_a - z_j + 3h/2}{t_a - z_j + h/2}, \]

are the connection coefficients of [24, 12] adjusted to the choice of the solution (39).

With these objects we can define discrete partial derivatives, using the convention that

\[ D_a \text{ for } a = 1, \ldots, l \text{ acts on variable } t_a \text{ while } D_{l+j} \text{ for } j = 1, \ldots, n \text{ acts on variable } z_j. \]

\[ D_a = 1 - Q_a, \quad D_{l+j} = 1 - Z_j. \]

The proof that (39) is a solution of the rational qKZ equation is then parallel to the one given in [13] for the level zero case. We state the necessary steps here because we will need this result later for the solution of the first two form factor equations (8).

Following (12) and (13) let

\[ R(\varepsilon_1' \varepsilon_2') = S(\varepsilon_1' \varepsilon_2') S_0(z)^{-1}, \]

and \( f^{(a)}_{\varepsilon_1', \ldots, \varepsilon_n} \) for \( a = 1, \ldots, l \) be a family of functions which can be expressed in terms of the rational hypergeometric basis functions (34). The qKZ equation for functions of the form (39) valued in \( V^\otimes n \) is then equivalent to the following equations on the rational hypergeometric basis functions (34), taking into account the definition of the set \( M \) at the end of section 3.

We state the first one in \( z_1 \) only, but its generality is obvious.

\[ Z_1 w_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}(\cdot; z_1, z_2, \ldots, z_n) - w_{\varepsilon_2, \ldots, \varepsilon_n, \varepsilon_1}(\cdot; z_2, \ldots, z_n, z_1) = \sum_{a=1}^{l} D_a f^{(a)}_{\varepsilon_1', \ldots, \varepsilon_n}. \]

(44)

The second equation refers to the exchange of two \( z \) variables

\[ w_{\varepsilon_1, \ldots, \varepsilon_{i+1}, \varepsilon_i, \ldots, \varepsilon_n}(\cdot; z_1, \ldots, z_{i+1}, z_i, \ldots, z_n) = \sum_{\varepsilon'_1, \varepsilon'_2 = \pm} R^{\varepsilon'_1, \varepsilon'_2}_{\varepsilon_i \varepsilon_{i+1}}(z_i - z_{i+1}) w_{\varepsilon_1, \ldots, \varepsilon'_i, \varepsilon'_{i+1}, \ldots, \varepsilon_n}(\cdot; z_1, \ldots, z_i, z_{i+1}, \ldots, z_n). \]

(45)

We then have the following

**Theorem 1:** The function (39) valued in the tensor product \( V^\otimes n \) is a solution of the rational \( sl_2 \) qKZ-equation.
PROOF: For the proof it is sufficient to recall that the hypergeometric integral of \( [24, 12] \) as defined in (39) has exact hypergeometric forms (cf. [24] Lemma 2.21), which are essentially given by the derivatives \( D_a \) defined in (13).

5 Completeness of the solutions

Here we prove the completeness of the solutions (39) of the qKZ equation. On the basis of form factors in the Sine-Gordon model the completeness of the solutions was shown by Smirnov [22] in establishing a bilinear relation among the solutions of the form factor space. Later it was shown by Tarasov [23] that (at least for the rational qKZ equation) similar relations can be extracted from the solution spaces of the qKZ equation directly. In [23] the fact was used that for the form of the qKZ equation (1) studied in this paper, there exist zero solutions. In our language this means that there exist polynomials \( P \) in (39) such that the integral vanishes automatically. This guarantees that the dimension of the solution space matches the dimension of \( (V^\otimes n)_{\text{sing}} \) rather than that of \( V^\otimes n \). For a detailed presentation of this topic, see [24].

In this section we are going to derive a completeness relation for the solutions of the qKZ-equation at generic level, similar to the one in the level zero case [22, 23]. Since we modelled our solution parallel to the level zero case [13, 23] we can expect that the proofs here will be quite similar to the ones given in [13, 23]. The main difference is due to the more complicated pole structure of the integrand (40).

We begin by noting the dimension of the space \( F_q \) introduced in (29); it is finite dimensional with \( \dim F_q = \dim (V^\otimes n)_l = \binom{n}{l} \).

Let us first consider the case \( l = 1 \) in the integral (39), which means in particular that there are no \( \psi \)-functions present. Insert for the polynomial \( P \) either of the two choices

\[
P_{s1}(t_1) = e^{-2\pi i h n/p} \prod_{j=1}^{n} \exp \frac{\pi}{p} (t_1 - z_j + 3h/2), \quad \text{or} \quad P_{s2}(t_1) = \prod_{j=1}^{n} \exp \frac{\pi}{p} (t_1 - z_j + h/2).
\]

(46)

If we ignore for the moment that the partial degree in \( \exp(2\pi i t_1/p) \) is not as it may have been assumed in (39) it is not difficult to check that the first expression cancels all possible poles appearing in the second line of (10). Closing the contour \( C_1 \) in (39) such that it encircles the poles in question, we realise that the integral vanishes.
The same kind of argument applies to the second choice in (46). Here we close the contour $C_1$ such that it contains all possible poles appearing in the first line of (40).

Hence we have established that for $l = 1$

$$\Psi_{P_{s_1}} = 0 = \Psi_{P_{s_2}}.$$  

(47)

This result is the generalisation of Lemma 5.3 in [13] to the generic level case.

Let us proceed to the case of $l = 2$. According to [23] we set

$$\Xi^{(1)}(t) = P_{s_1}(t) - P_{s_2}(t).$$  

(48)

From the partial degree point of view this is a proper candidate for the polynomial $P$. It is clear from (47) that we also have

$$\Psi_{\Xi^{(1)}} = 0.$$  

(49)

Let us give an interpretation of this object. According to [24] it is possible to construct basis functions similar to (34) for the trigonometric hypergeometric space as defined on (29). If we take the definition (2.26) of [24] and rewrite the basis according to our form of the solutions (39) of the qKZ-equation, we find that these basis functions for $l = 1$ and variables $z_1, \ldots, z_n$ can be recast as a polynomial

$$P_\mu(t) = \exp \left( \frac{4\pi i\Lambda(n - \mu)}{p} \right) \prod_{1 \leq j < \mu} \left( e^{2\pi i(t - z_j + \Lambda)/p} - 1 \right) \prod_{\mu < j \leq n} \left( e^{2\pi i(t - z_j - \Lambda)/p} - 1 \right).$$  

(50)

Note that we still keep the convention (32). We can formally extend the definition (50) also to the case $\mu = 0$.

Then it is straightforward to check that

$$\prod_{j=1}^{n} \left( e^{2\pi i(t - z_j + \Lambda)/p} - 1 \right) = P_0(t) + (e^{4\pi i\Lambda/p} - 1) \sum_{j=1}^{n} P_j(t).$$  

(51)

The trigonometric hypergeometric basis functions for $l = 2$ in their polynomial form (appropriate to the conventions laid out in the previous section) can for $\mu < \nu$ be constructed from (50) as

$$P_{\mu\nu}(t_1, t_2) = \left( \sin \frac{\pi}{p}(t_1 - t_2) \right)^{-1} \text{Asym} \left( P_\mu(t_1) P_\nu(t_2) \sin \frac{\pi}{p}(t_1 - t_2 - h) \right).$$  

(52)
Having seen the structure of the basis functions, we can now write down a polynomial \( \Xi^{(2)} \)

\[
\Xi^{(2)}(t_1, t_2) = \Xi^{(1)}(t_1)\Xi^{(1)}(t_2) \left( \sin \frac{\pi}{p}(t_1 - t_2) \right)^{-1} \left( \sin \frac{\pi}{p}(t_1 - t_2 - h) - \sin \frac{\pi}{p}(t_2 - t_1 - h) \right).
\]

(53)

Taking into account the pole structure (40) it is possible along the lines in the \( l = 1 \) case to check that for \( l = 2 \) we have

\[
\Psi_{\Xi^{(2)}} = 0.
\]

(54)

The functions \( \Xi^{(1)} \) and \( \Xi^{(2)} \) are the analogs of the kernel functions at level zero as introduced by Tarasov in [23]. Therefore the construction of the graded spaces in section 7 of [23] can be applied to the case of generic level as well.

This means that we have now found the necessary set of kernel polynomials for the solutions of the qKZ-equation. The function \( \Xi^{(2)} \) is the analog of what was found in [22] as a consequence of the deformed Riemann bilinear identity at level zero.

We say that the arguments \( z_1, \ldots, z_n \) are in generic position, if the singular hyperlanes in (40) do not coincide. For the case when they do, see section 6.

The main point now is that the kernel solutions (49) and (54) reduce the actual dimension of the parameter space from \( \dim(V^\otimes n)_l \) to \( \dim(V^\otimes n)^{\text{sing}}_l \). This can be established using arguments absolutely parallel to the ones employed in the proof of Theorem 4.3 and 4.4 in [23]. This then gives rise to the following

**Theorem 2:** For generic values of the arguments \( z_1, \ldots, z_n \), the solutions (39) span the space \( (V^\otimes n)^{\text{sing}}_l \).

**Remark:** This result is crucial for the correspondence (3) between the ordinary and the deformed form factors. We shall return to it after Theorem 4. But let us indicate here that it means that the number of physical states in the replica-deformed model is identical to the one in the standard case. A fact which, however, could have been anticipated.

## 6 The replica-deformed form factors

In this section we define the replica-deformed form factors and check under which conditions on the objects \( P \) they satisfy the modified form factor equations of section 2.
According to what has been said in section 3 we can encode the index structure of the form factor (55) as

$$F_{\varepsilon_1, \ldots, \varepsilon_n}(z_1, \ldots, z_n) = F_M(z_1, \ldots, z_n).$$

(55)

We denote by $I_{P M}(z_1, \ldots, z_n)$ the summand corresponding to the basis element $w_M$ in the expression for the solution $\Psi_P$ of the qKZ-equation, as stated in (39).

Recall that the form factors of the standard Thirring model [20, 21] are (cf. [13]) special solutions of the rational $sl_2$ qKZ equation at level zero multiplied by a scalar function.

We will now show that the form factors in the replica-deformed Thirring model in the sense of [14] naturally arise from the solutions of the rational $sl_2$ qKZ equation at generic level. Specifically we will show that for a suitable choice the objects $P$ the functions

$$F_M(z_1, \ldots, z_n) = \frac{c_l}{(2\pi i)^l} \prod_{n \geq i > j \geq 1} f(z_i - z_j) \times I_{P M}(z_1, \ldots, z_n),$$

(56)

are solutions of the deformed form factor equations (8)--(10). Note that we take $2l \leq n$, later we shall see that $j = n/2 - l$ can be identified with the isospin of the underlying local operator. For the constant $c_l$ we take:

$$c_l = \left(2\pi h \Phi(-3h/2)\Gamma(-h/p)(\beta)^{\pi/\beta}e^{i\pi h/(2p)}\right)^{-l}.$$ 

(57)

The fact that (56) is a solution to the first two form factor equations (8) can be proved without problems. Due to its importance we state this fact as the following

**Theorem 3:** The function $F_M(z_1, \ldots, z_n)$ as defined in (56) satisfies the form factor equations (8).

**Proof:** Take the identities (17) for the minimal form factor $f(z)$ to realize that the factor $\prod_{n \geq i > j \geq 1} f(z_i - z_j)$ saturates the scalar part of the $S$-matrix (12) in the form factor equations (8). The Theorem in section 4 together with (44) and (45) establishes the result.

**Remark:** If we stick to the definition of the form factor given in (56), which means that we keep the the requirements set up earlier on the polynomial $P$ in (29), then $F_M$ satisfies (8) with $\eta = 1$. There are, however, situations where we have to weaken the requirements on $P$ depending on the local operator we are going to describe with this object. For example in the next section we will see an example, where we relax the $p$-periodicity in the variables $z$ of $P$. In this case $\eta$ may be different from unity but still satisfies $|\eta| = 1$. 

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We now come to the part in the form factor business going beyond the qKZ-equation. Namely, we have to single out those solutions described in Theorem 3, which do satisfy the equations (9) or (10).

Our strategy to prove this will be similar to the one used by Smirnov [20, 21]. In our case this means the following. In [20, 21] the source for the poles \( z_i = z_j - h \) were taken to reside entirely in the algebraic Bethe vectors \( \omega_M \). Additional contributions arising from the residue evaluation of the integrals (39) at level zero, the origin of which is the pinching of integration contours, are removed by a trick.

The algebraic Bethe vectors \( \omega_M \) were shown in [13] (Lemma 6.4) to descend directly from the rational basis functions \( w_M \). Now we have seen that at generic level by (34) we can work with the same basis functions as in the level zero case. This essentially reflects the replica-understanding of [2] in varying the Unruh temperature in an integrable quantum field theory as sketched in section 2. There we have seen that this thermalisation changes many aspect of the physics but preserves the \( S \)-matrix of the zero temperature model. And as the \( S \)-matrix is unchanged, the algebraic Bethe ansatz [21] remains, of course, unchanged. We could now outline along the lines of [13] how to construct the Bethe vectors from the the solution of the qKZ-equation (39). However, this is quite technical and in addition we would like to show that the kinematical residue equations (9) and (10) can be solved using the basis functions \( w_M \) directly and using the singular hyperpalnes of the hypergeometric integral (40) only (mainly because the other way is by Smirnov’s work very well understood).

For that purpose we simply state the now obvious but physically important

**Lemma 1:** The replica deformation of the \( SU(2) \)-invariant Thirring model has no effect on the algebraic Bethe ansatz.

We can now proceed to find solutions of the kinematical equations (9) and (10). We first collect some results on the residues of the functions \( w_M(t_1, \ldots, t_l; z_1, \ldots, z_n) \) as defined in (34) as they will be needed to evaluate some of the integrals \( I_{P}^{M} \) below. For the proofs to be given below it is sufficient to consider only poles in the variables \( z_{n-1} \) and \( z_n \). Therefore we prefer not to write down most of the formulas in the sequel in full generality, but rather stick to the sufficient case.

For both \( n - 1, n \notin M \) or \( n - 1, n \in M \) we find that

\[
\text{res}_{t_l = z_n - 3h/2} w_M(t_1, \ldots, t_l) = 0 = \text{res}_{t_l = z_{n-1} - 3h/2} w_M(t_1, \ldots, t_l).
\] (58)

The next case is when \( n \in M \) but \( n - 1 \notin M \). We set \( N = M \setminus \{n\} \).
Lemma 2: The form factor \( [54] \) has a KR pole \( z_n = z_{n-1} - h \) if \( n - 1 \in M \) and \( n \notin M \) and arises from the singular hyperplanes \( t_a = z_{n-1} - 3h/2 \).

The form factor \( [54] \) has a KR pole \( z_n = z_{n-1} + h \) if \( n - 1 \notin M \) and \( n \in M \) and arises from the singular hyperplanes \( t_a = z_{n-1} - 3h/2 \).

There are no KR poles if both \( n - 1, n \in M \) or \( n - 1, n \notin M \).

Proof: In \([40]\) we have collected the singular hyperplanes of the integral \([39]\). We could think of evaluating the integrals by means of the Leray residue method. Having done this one realises that the source for poles of the kind \( z_n = z_{n-1} - h \) or \( z_n = z_{n-1} + h \) can only reside in expressions arising from the phase function \([39]\). Now a kinematical pole of the first type appears if we take \( t_a = z_{n-1} - 3h/2 \). This singularity shows up iff it is present.

\[
\begin{align*}
\text{res}_{t_i = z_{n-1} - 3h/2} & w_M(t_1, \ldots, t_l; z_1, \ldots, z_n)|_{z_n = z_{n-1} - h} = 2 w_N(t_1, \ldots, t_{l-1}; z_1, \ldots, z_{n-2}) \\
& \times \prod_{k=1}^{n-2} \frac{z_{n-1} - z_k - 2h}{z_{n-1} - z_k - h} \prod_{a=1}^{l-1} (t_a - z_{n-1} + 3h/2), \quad (59)
\end{align*}
\]

If we take the residue with respect to \( z_{n-1} \) rather than \( z_n \), we get:

\[
\begin{align*}
\text{res}_{t_i = z_{n-1} - 3h/2} & w_M(t_1, \ldots, t_l; z_1, \ldots, z_n)|_{z_n = z_{n-1} - h} = -w_N(t_1, \ldots, t_{l-1}; z_1, \ldots, z_{n-2}) \\
& \times \prod_{k=1}^{n-2} \frac{z_{n-1} - z_k - h}{z_{n-1} - z_k} \prod_{a=1}^{l-1} (t_a - z_{n-1} + h/2),
\end{align*}
\]

\[
\begin{align*}
\text{res}_{t_i = z_{n-1} - 3h/2} & w_M(t_1, \ldots, t_l; z_1, \ldots, z_n)|_{z_n = z_{n-1} + h} = \\
& = - \text{res}_{t_i = z_{n-1} - 3h/2} w_M(t_1, \ldots, t_l; z_1, \ldots, z_n)|_{z_n = z_{n-1} - h}. \quad (60)
\end{align*}
\]

At last we treat the case of \( n \notin M \) but \( n - 1 \in M \). Set \( N' = M \setminus \{n - 1\} \). Here we find that \( \text{res}_{t_i = z_{n-1} - 3h/2} w_M(t_1, \ldots, t_l) = 0 \), while

\[
\begin{align*}
\text{res}_{t_i = z_{n-1} - 3h/2} w_M(t_1, \ldots, t_l; z_1, \ldots, z_n) &= w_N'(t_1, \ldots, t_{l-1}; z_1, \ldots, z_{n-2}) \\
& \times \prod_{k=1}^{n-2} \frac{z_{n-1} - z_k - h}{z_{n-1} - z_k} \prod_{a=1}^{l-1} (t_a - z_{n-1} + h/2), \quad (61)
\end{align*}
\]

which is obviously independent of the pole structure.
in $w_M$ and hence if $n - 1 \in M$. We have to check that if this is the case the presence of $n$ in $M$ leads to a vanishing contribution. This is easy to see because e.g.

$$\text{res}_{t_i = z_{n-1} - 3h/2} F_M |_{z_n = z_{n-1} - h} = 0.$$  \hspace{1cm} (62)

The proof of the second statement is similar. The last statement is then a consequence of (58) and the first two statements in Lemma 2.

We can now check under which conditions of $P$ the form factor satisfies the kinematical residue equations (9) and (10) in the thermalised case.

First of all it is clear from the Proposition that if both $n - 1, n \in M$ or $n - 1, n \notin M$ the residue of $F_M$ at the points in question vanishes. This means that in this case $F_M$ satisfies the equations (9) and (10) since the charge conjugation matrix $C$ vanishes here.

Next, we mention that due to (40) for the other cases discussed in the Proposition, i.e. $(\varepsilon_{n-1}, \varepsilon_n) = (-, +)$ or $(+, -)$ we do not encounter the pinching of integration contours which occurs in the case of standard form factors in the model [21].

The verification of the residue equation (9) for $z_n = z_{n-1} - h$ is more or less straightforward. We use the proposition of this section and evaluate the multiple integral $I_P M$ at $t_l = z_{n-1} - 3h/2$. This pole is present for $(\varepsilon_{n-1}, \varepsilon_n) = (-, +)$ and for $(\varepsilon_{n-1}, \varepsilon_n) = (+, -)$. Then we take the residue of the remaining expression at the point in question and use the identities (38) and (23).

Let us state that we set $\delta = 0$ if $(\varepsilon_{n-1}, \varepsilon_n) = (-, +)$ and $\delta = 1$ if $(\varepsilon_{n-1}, \varepsilon_n) = (+, -)$, and that we abbreviate the parameter $P$ in (56) by $P_{l,n}$. The result of the procedure outlined before is then

$$\text{res}_{z_n = z_{n-1} - h} F_M(z_1, \ldots, z_n) = (-1)^\delta \prod_{n-2 \geq i > j \geq 1} f(z_i - z_j) \frac{c_{l-1}}{(2\pi i)^{l-1}} \times$$

$$\times \int_{C_1} dt_1 \cdots \int_{C_{l-1}} dt_{l-1} w_N(t_1, \ldots, t_{l-1}; z_1, \ldots, z_{n-2}) P_{l,n} |_{t_l = z_{n-1} - 3h/2}$$

$$\times \prod_{a=1}^{l-1} \prod_{j=1}^{n-2} \Phi(t_a - z_j) \prod_{a < b}^{l-1} \prod_{j=1}^{n-2} e^{-i\pi(z_{n-1} - z_j - 3h/2)/p} \times$$

$$\times \prod_{a=1}^{l-1} \sin \frac{2\pi}{p}(t_a - z_{n-1} + h/2) \sin \frac{2\pi}{p}(t_a - z_{n-1} + 5h/2).$$  \hspace{1cm} (63)

This in turn means that $F_M$ satisfies the first kind of kinematical residue equations if
\[
P_{l,n} \prod_{a=1}^{l-1} \frac{e^{-2\pi i(t_a - z_{n-1} + h/2)/p}}{\sin \frac{\pi}{p}(t_a - z_{n-1} + h/2)} \prod_{j=1}^{n-2} e^{-i\pi(z_{n-1} - z_j - 3h/2)/p} \sim P_{l-1,n-2}.
\]

The “\(\sim\)” means that this is an equality up to a phase, which is permitted by the requirements on the form factors.

To check the second type of kinematical poles at \(z_n = z_{n-1} + h\), which is at least not explicitly necessary in the level zero case, we take a different strategy. Rather than to take the residue directly at this point we perform an analytic continuation and evaluate the residue at \(z_n = z_{n-1} + p - h\). The fact that the corresponding residue equations are equivalent is shown in cite [14]. In addition it may be nice to see how the \(S\)-matrix factors in (10) come about.

We use the property (44) which was used in the construction of solutions of the qKZ equations in section 3. Up to total derivatives we have

\[
w_{\varepsilon_1, \ldots, \varepsilon_{n-1}, \varepsilon_n}(\cdot; z_1, \ldots, z_{n-1}, z_n) \sim Z_1 w_{\varepsilon_n, \varepsilon_1, \ldots, \varepsilon_{n-1}}(\cdot; z_n, z_1, \ldots, z_{n-1}).
\]

(65)

Here \(Z_1\) acts on the first index \(\varepsilon_n\) associated with the variable \(z_n\).

Now we move \(z_n\) back to the place \(n - 1\) using the \(R\)-matrix property of the rational hypergeometric basis functions according to (45). Hence we get up to total derivatives

\[
w_{\varepsilon_1, \ldots, \varepsilon_{n-1}, \varepsilon_n}(\cdot; z_1, \ldots, z_{n-1}, z_n) \sim \phi(t, z) R_{\varepsilon_1 \varepsilon_n}^{\varepsilon_1 \alpha_1}(z_1 - z_n - p) R_{\varepsilon_2 \alpha_1}^{\varepsilon_2 \alpha_2}(z_2 - z_n - p) \cdots
\]

\[
\cdots R_{\varepsilon_{n-1} \alpha_{n-2}}^{\varepsilon_{n-1} \alpha_{n-2}}(z_{n-2} - z_n - p) w_{\varepsilon_n, \varepsilon_1, \ldots, \varepsilon_{n-2}, \alpha_{n-2}, \varepsilon_{n-1}}(\cdot; z_1, z_2, \ldots, z_{n-1}, z_n + p, z_{n-1}).
\]

(66)

Hence we can now evaluate the integral \(\mathcal{I}_M^P\) with this replacement in mind at \(t_l = z_n + p - 3h/2\) much in the same way as above.

The result then is
\[
\text{res}_{z_n=z_{n-1}+h-p} F_M(z_1, \ldots, z_n) = (-1)^{d+1} \prod_{n-2 \geq j \geq 1} f(z_i - z_j) \frac{c_{l-1}}{(2\pi i)^l} \times \prod_{i=1}^{n-2} S(z_{n-1} - z_i) \\
\times \int_{C_1} \cdots \int_{C_{l-1}} \Pi_{l-1} \psi(t_1, \ldots, t_{l-1}; z_1, \ldots, z_{n-1}) P_{l,n} |_{t_l=z_n+p-3h/2} \times \prod_{a=1}^{l-1} \prod_{j=1}^{l-1} \Phi(t_a - z_j) \prod_{a<b}^{l-1} \psi(t_a - t_b) \prod_{j=1}^{n-2} e^{-i\pi(z_{n-1} - z_j - h/2)/p} \times \prod_{a=1}^{l-1} \frac{e^{-2\pi i(t_a - z_{n-1} - h/2)/p}}{\sin \pi p(t_a - z_{n-1} - h/2/2)} \sin \frac{\pi}{p}(t_a - z_{n-1} + 3h/2).}
\]  

(67)

Hence the kinematical residue equation (11) is satisfied if

\[
P_{l,n} l-1 \prod_{a=1}^{l-1} \frac{e^{-2\pi i(t_a - z_{n-1} - h/2)/p}}{\sin \pi p(t_a - z_{n-1} - h/2/2)} \prod_{j=1}^{n-2} e^{-i\pi(z_{n-1} - z_j - h/2)/p} \sim P_{l-1,n-2}.  
\]  

(68)

We are going to discuss some important solutions of the equations (64) and (68) in the following section. Obviously, the value of \(n/2 - l\) an invariant under the kinematical residue equations. This invariant can be interpreted as the isospin of form factor sequence. This means that by (64) and (68) we get sequences of solutions of all form factor equations for each value of the isospin.

Let us collect the main results in a Theorem.

**Theorem 4:** There exist infinite sequences

\[
F_M(z_1, \ldots, z_n) = \frac{c_l}{(2\pi i)^l} \prod_{n-2 \geq j \geq 1} f(z_i - z_j) \times T_M^{P_{l,n}}(z_1, \ldots, z_n),
\]  

(69)

of solutions of the qKZ-equation with the R-matrix being the S-matrix (13) of the Thirring model. Each sequence is labelled by the invariant \(n/2 - l\). If in addition \(P_{l,n}\) satisfies the recursion relations (64) and (68) then the members of each sequence are linked by (9) and hence provide replica-deformed form factors of local operators in the Thirring model with isospin \(n/2 - l\).

Of course every sequence (69) can be multiplied pointwise with a sequence \(Q(z_1, \ldots, z_n)\) without changing the properties described, provided \(Q(z_1, \ldots, z_n)\) is completely symmetric and \(i\beta\)-periodic in all arguments, and satisfies

\[
24
\]
\[ Q(z_1, \ldots, z_n) \bigg|_{z_n = z_{n-1} \pm i\pi} = Q(z_1, \ldots, z_{n-2}). \quad (70) \]

Alternatively such sequences arise as ambiguities in the solution of the recursive equations for the polynomials \( P_{l,n} \). Physically the solutions of \( (70) \) correspond to eigenvalues of a local conserved charge. As shown in [14] the structure of these eigenvalues for \( \beta \neq 2\pi \) is quite different from that in the undeformed case. Also the eigenvalues in both cases are not automatically in one-to-one correspondence, though they can be made so by imposing suitable minimality conditions. In any case, modulo their respective ambiguities, the completeness result for section 5 (together with its undeformed counterpart) now implies the announced correspondence \( (3) \): The form factor sequences in the replica-deformed model are essentially in one-to-one correspondence to their undeformed QFT counterparts. In particular this allows one to identify the replica copy of the form factors of a local operators. As an illustration we present the replica-deformed form factors of the Noether current in the next section.

7 Deformed form factors of the Noether current

In this section we present solutions to the equations \( (64) \) and \( (68) \). As it was mentioned in the introduction, the form factor approach enables to classify the full space of local operators of a given model. This arises from the fact that there exist kernel solutions to the equations \( (64) \) and \( (68) \). This feature is illustrated in [20] where the local fields in the sine-Gordon model have been “counted”, and in [19] where the full space of local operators in the Sinh-Gordon model has been found. In the latter model the local operators in one superselection sector were identified with the integer powers of the elementary field appearing in the Lagrangian together with an exponential field.

It should be possible to mimic the analysis of the local fields of [22] in the present case as well. However, we would like to present in this section solutions to \( (64) \) and \( (68) \) which in the limit \( \beta = 2\pi \) turn into the form factors corresponding to the SU(2) Noether current in the Thirring model. These form factors are well understood in the standard case and it has been shown in [13] how these form factors arise from the qKZ-solutions at level zero and how they come about from the trace formulae of vertex operators in the Yangian double of [10].
Let \( j^a_\mu \) be the Noether current associated with the Lagrangian (11); \( \mu \) is the Lorentz index and \( a \) the \( SU(2) \)-index as introduced in section 3. It is convenient to switch to lightcone coordinates both in Minkowski space and in internal space, i.e.

\[
j_\pm = j^0_0 \pm j^1_0, \quad j_{\sigma}^\pm = j_{\sigma}^1 \pm i j_{\sigma}^2, \quad \sigma = \pm.
\]

Then we can introduce an index \( \tau = \pm, 3 \) labelling the components of the lightcone currents as \( j_{\tau}^\sigma \). A solution of the equations (64) and (68) is given by

\[
P_{l,n}^\sigma = d_{l,n}^\sigma e^{-i\pi(2l-n+2\sigma)} \sum_{j=1}^{n} \frac{z_j}{p} \prod_{a=1}^{l} e^{2\pi i (l+\sigma) t_a/p} \prod_{1 \leq a < b \leq l} \left( \sin \frac{\pi}{p} (t_a - t_b - h) \sin \frac{\pi}{p} (t_a - t_b + h) \right),
\]

where the constant \( d_{l,n}^\sigma \) is

\[
d_{l,n}^\sigma = \exp \left( -\frac{i\pi}{p} (n - 2l - 2(\sigma + 1)) l \right).
\]

Comparing this with the level zero counterpart [21, 13] we find that this choice of \( P_{l,n}^\sigma \) corresponds to the form factors of the current \( j^-_{\sigma} \). The form factors of the other components of \( j_{\tau}^\sigma \) can obviously be obtained by applying the operator \( \Sigma^+ \) once or twice to the corresponding form factor.

A closer look on the solution (72) shows that the corresponding form factor does satisfy the first two of the form factor equations (8) with a factor \( \eta \neq 1 \). But still we have \( |\eta| = 1 \) as it should be. The reasons why this feature appears is discussed in detail in [21].

Of course, in order to establish that these solutions do really correspond to the replica-deformed form factors of the currents we have to prove several conditions like in the level zero case, see section 6 of [21]. The most important of these conditions is the current conservation condition \( \partial_\mu j^a_\mu = 0 \).

To verify this in the deformed framework involves the knowledge of the deformed eigenvalues of the lightcone momenta. We remind that the kinematics in the deformed model is quite different from the usual model as was outlined in section 2. Even though these eigenvalues have been characterised and computed for some low values of the particle number \( n \) in [14], a general formula is not known at present. This certainly deserves a further study also in the light of the characterisation of other operators in the replica-copy of the Thirring model as well.

Therefore, we can at present only conjecture that the form factors characterised by the solutions (72) do correspond to the Noether-current operators in the replica-copy of the
Thirring model. Nevertheless we can give an argument to support our conjecture. Let $P_\pm$ be the eigenvalues of the lightcone momenta on an $n$-particle state in the $\beta \neq 2\pi$ copy of the Thirring model according to [14]. We denote by $\Psi_\sigma(z_1, \ldots, z_n)$ the functions which we conjecture to be the form factors of the currents $j_\sigma$.

The explicit difference of $\Psi_+$ and $\Psi_-$ resides first in the purely $z$-dependent part of (72). This bit is not affected by the integration. The second $\sigma$-dependent part comes from the first product term in (72). This term is trivial in the sense that it responsible neither for poles nor for zeros of the integrand. In addition it is $p$-periodic. Therefore it is clear that there is a scalar function $R(z_1, \ldots, z_n)$ such that for any particle number we have an identity $\Psi_+ = R\Psi_-$. According to what was said at the end of the previous section we can now redefine for any number of arguments $n$ the form factors of the currents by setting

$$J_+ := P_+ R^{-1/2} \Psi_+, \quad J_- := -P_- R^{1/2} \Psi_-.$$  

(74)

Then we can write the current conservation as follows:

$$P_+ J_- + P_- J_+ = P_+ P_- R^{-1/2} (\Psi_+ - R\Psi_-),$$  

(75)

which is zero due to the aforementioned identity.

### 8 Discussion

In this paper we have been trying to illustrate the natural connection between rational solutions of the qKZ equation and the replica-deformed form factors in the $SU(2)$-invariant Thirring model. Here we took up two independent directions of recent research. The first one is Niedermaier’s formalism showing that every integrable massive quantum field theory in $1+1$-dimensions admits an $S$-matrix preserving deformation, which can be interpreted as providing a “replica-copy” of the original QFT with an off-critical Unruh temperature. It was shown in [14] that this replica-copy can be described exactly in terms of its form factors, the latter being solutions of a a modification of the standard form factor equations [9, 20]. The present paper is the first in which these equations have been solved for a particular model in full generality.

As a technical tool we took advantage of another recent development, namely the study of the solutions of the rational $sl_2$ qKZ equation [24, 12]. As one could suspect by looking
at the first set of modified form factor equations (8), their solution in the Thirring model case should somehow reside in the space of solutions of the qKZ-equation for a particular representation of $sl_2$. The question then is whether among these solutions we can find some satisfying the deformed kinematical residue equations (9) as well.

The main purpose of this work was to show that this is indeed possible. In outline we rewrote the qKZ-solutions of [24, 12] at generic level in a way facilitating the comparison with the standard level zero situation. We argued that this form of the solution naturally shares the property that the algebraic Bethe vectors remain unchanged under the replica thermalisation. This fact could have been expected since the modified form factor equations use the usual scattering matrix. We then supplemented our qKZ-solutions with the minimal form factor to define an object (56) having only the polynomial $P$ unspecified. Using the pole structure of the qKZ-solutions we derived conditions on $P$ from the modified kinematical residue equations and solved these conditions to conjecture the deformed form factors of the Noether-current in the $SU(2)$ Thirring model.

To summarise we have shown that the Unruh-thermalised form factors arise naturally from qKZ-solutions at generic level as the standard form factors do at level zero.

We may also anticipate that the $\beta \to 0$ limit of our construction bears an unexpected relation to dimensionally reduced quantum gravity [17].

It should now be possible to derive integral representations of the form factors for the thermalised sine-Gordon model using the results on trigonometric qKZ-solutions as derived in [25]. It would be particularly interesting to construct the thermalised form factors of the breather sectors. This would involve also the construction of modified fusing relations for the form factors. The latter haven’t been supplied in [14]. In addition solving this problem would also naturally lead to an integral representation of the thermalised form factors in the Sinh-Gordon model, a problem which has only partially been solved in [14].

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