THE MASS-SHIFT METHOD AND THE SELF-ACTION OF CLASSICAL CHARGE

S.L. Lebedev, 1

1 Chuvash state pedagogical university, 428000 Cheboksary

The complex non-local action functional is used in classical electrodynamics to describe the back-reaction effects for the charge moving in the constant homogeneous electromagnetic field. We apply the mass-shift method to obtain the higher order radiation effects in the non-relativistic cyclotron motion and generalize the method to the case of Bargmann-Michel-Telegdi particle.

1 Introduction

In 1978, considering the mass operator of the electron in the constant homogeneous EM field, V. Ritus [1] found the field-dependent correction to the mass of the electron, which in the weak-field limit $\beta \ll 1$ ($\beta \equiv e\hbar\varepsilon/m^2c^3$, $\varepsilon$ - the electric field) takes the form

$$\Delta m = \frac{\alpha m}{2\pi} \left[ -\pi \beta - i \beta \left( 2 \ln \frac{2\beta m}{\gamma_E \mu_{ph}} - 1 \right) + \ldots \right]. \quad (1)$$

Here $\ln \gamma_E = 0.577 \ldots$ and we have omitted the quantum corrections of orders $\beta^2 \ln \beta$ and $\beta^2$. The leading terms in the real and imaginary parts of the mass shift (MS), as it can be deduced from (1), are purely classical. Subsequently, Ritus [2] suggested a method of calculation of the MS (1) which relies, entirely, on a classical quantity

$$\Delta W = \frac{1}{2} e^2 \int d\tau \int d\tau' \hat{x}_\alpha(\tau) \hat{x}_\alpha(\tau') \Delta_e(x - x'; \mu_{ph}) \bigg|_0^F. \quad (2)$$

$\Delta W$ is the change in the self-action of the charge caused by the external field. The overdots in (2) denote the derivatives of the world line $x_\alpha(\tau)$ w.r.t. the proper time.

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1With some obvious exceptions we use the system of units where $c = 1$, $\hbar = 1$. Fine structure constant $\alpha = e^2/4\pi\hbar c$.

2The IR parameter $\mu_{ph}$ ("photon mass") should be interpreted as a minimal wave number $k_{\perp\text{min}} = \mu_{ph} c/\hbar$ of radiating quanta [3].
In the constant homogeneous field the self-action (2) determines the MS according to
\[ \Delta W = -\Delta m T , \] (3)
where the proper time interval \( T \) of the motion (in external field) is much greater than the formation time of \( \Delta m \).

The Green’s function \( \Delta_c \) in (2) is the causal one and, consequently, \( \Delta W \) is in general the complex quantity. The next important point is the presence of the subtraction \(|F_0^J|\) which manifests the fact that all self-action effects taken at zero external field should be accounted for by the definition of the observable mass of the charge. This means that no UV divergences should appear in (2).

To interpret \( \Im \Delta W \) in the context of classical electrodynamics [2] (CED) one ought to remind the connection between the latter and the spectral function of radiation,
\[ d\mathcal{E}_k = |j_\mu(k)|^2 \frac{d^3k}{16\pi^3}, \frac{1}{\hbar} \Im \Delta W = \int \frac{d\mathcal{E}_k}{\hbar\omega} , \] (4)
produced by the classical source (see e.g. [3])
\[ j_\mu(x) = e \int d\tau \dot{x}_\mu(\tau)\delta^{(4)}(x - x(\tau)) . \] (5)

The quantity \( \exp \left( \frac{i}{\hbar} \Delta W \right) \) appears in QED as well in the form of an amplitude of the preservation of the photon vacuum in the presence of the classical source (5), so that the imaginary part of \( \Delta W \) determines the corresponding probability [3]:
\[ \exp(-2\Im \Delta W) = |<0^+|0^->_j|^2 . \] (6)
This probability is less than unity if the particle radiates.

The consideration of the radiation reaction effects through the non-local reparametrization invariant functional (2) could bring the information which is supplementary to one obtained from Abraham-Lorentz-Dirac (ALD) equation. Let us list some known applications of the formula (2) (this applications, certainly, are not restricted with the case of constant electric field):
– the self-energy of the charged particle rest in a proximity to the charged black hole. MS determines a repulsive force acting on this particle [4];
– the real part of \( \Delta m \) in (1) determines (at \( \beta \ll 1 \)) the correction to the classical action in the exponential of barrier factor. This results [3] in 2-loop radiative correction to Schwinger pair creation rate;
– the classical radiative corrections to the cyclotron motion of the electron near a boundary [6]. Those were in the focus of attention in connection with the
apparatus-dependent effects in $g - 2$ experiments of Washington group [7]. It should be stressed that in the second example the result could not be deduced from ALD equation in view of infrared origin of the real part of the MS [2].

2 Uniformly accelerated charge

Here we consider the MS of the uniformly accelerated (UA) charge radiating vector (that is EM) or scalar field (electric or scalar charge for brevity) and moving in the space-time of arbitrary dimensions $D$. For $D = 4$ this type of motion is distinguished by zero radiation reaction force, so that the world line $x(\tau)$ of UA charge is an exact solution of the ALD equation. At that time one finds (see [8]) that the value $D = 4$ is in a sense singular for the existence of the ALD-type equation. We find the corresponding peculiarities for the MS as well.

To begin with we remind one remarkable similarity ($D = 4$) between the mass shifts of electric charge considering at two different situations. a) The self-field is massive (see (1)-(3)),

$$\Delta m_{el} = \frac{\alpha w_0}{2\pi} S_{el}(\Lambda), \quad \Lambda = \mu_{ph}/w_0 \equiv k_{\perp min}/w_0$$

($\Re S_{el}$ and $\Im S_{el}$ are expressed through the bilinear combinations of Bessel functions, $w_0$ is an acceleration in the rest frame of the charge and parameter $\beta$ from (1) is $\beta = w_0/m$); b) the self-field is massless but the charge is moving parallel to and a distance $L$ apart from the perfect mirror,

$$\Delta m_{el} = \frac{\alpha w_0}{2\pi} V_{el}(\bar{\Lambda}), \quad \bar{\Lambda} = (\gamma_E L w_0)^{-1}$$

(Euler’s constant is used here to fix the correspondence between $S$ and $V$ in the IR limit). The function $V_{el}$ is an elementary one:

$$V_{el} = \pi \left(- \coth \theta + \frac{1}{2} \csch \frac{\theta}{2}\right) + i (1 - \theta \coth \theta), \quad \cosh \theta = 1 + 2(\bar{\Lambda} \gamma_E)^{-2}. \quad (9)$$

The correspondence mentioned above,

$$\Lambda \leftrightarrow \bar{\Lambda} \simeq (L w_0)^{-1}, \quad (10)$$

is fulfilled (as it can be seen from the Fig.1) in the wide interval

$$0 < \Lambda, \bar{\Lambda} < 10, \quad (11)$$
despite that only in the IR region $\Lambda, \bar{\Lambda} \ll 1$ the asymptotics of $S_{el}$ and $V_{el}$ are expected to be in common (for real parts, $\Re S_{el}$ and $\Re V_{el}$, this similarity extends over the whole positive axis).

For the scalar UA source we have the same correspondence (10) between $S_{sc}(\Lambda)$ and $V_{sc}(\bar{\Lambda})$ (Fig.1b). The IR limit of those functions is distinct from (1) [2, 4, 9]:

$$\Delta m_{sc} = -i\frac{\alpha w_0}{2\pi} + \ldots (\Lambda, \bar{\Lambda} \ll 1).$$  \hspace{1cm} (12)

Now, in addition to abovementioned motivation it is interesting to consider the MS $\Delta m_{el}$ and $\Delta m_{sc}$ as a functions of another IR regulator (the latter being now $D - 4$).

We need usual dimensional generalization of $\Delta_c$ in the self-action (2) for $D \neq 4$ and introduce mass scale $\mu$. Now the source (5) has the charge ($e-$dimensionless constant)

$$e' = e\mu^\frac{D-4}{2}$$  \hspace{1cm} (13)

and $D$-dimensional $\delta$-function is implied there. The simple calculations along
the lines of the work \[2\] result in the expression:

\[
\Delta m_{el} = -\frac{e^2 w_0}{2(2\pi)^{D/2}} \left( \frac{w_0}{\mu} \right)^{D-4} \exp \left( \frac{-i\pi D}{4} \right) \int_0^\infty dz \, z^{D-4} e^{i\pi} K_1(iz). \quad (14)
\]

The corresponding formula for the \(\Delta m_{sc}\) could be obtained from (14) by the substitution \((-K_0(iz))\) in place of \(K_1(iz))\) \((K_0\)'s are the McDonald functions).

The integral in (14) is transformed with the help of 2.16.6(5) in [10], that gives the final expressions:

\[
\Delta m_{el} = \frac{e^2 w_0}{(4\pi)^{D/2}} \left( \frac{w_0}{\mu} \right)^{D-4} \frac{i}{\sqrt{\pi}} \exp \left( \frac{-i\pi D}{2} \right) \Gamma \left( \frac{D-2}{2} \right) \Gamma \left( \frac{3-D}{2} \right), \quad (15)
\]

\[
\Delta m_{sc} = \frac{e^2 w_0}{(4\pi)^{D/2}} \left( \frac{w_0}{\mu} \right)^{D-4} \frac{i}{\sqrt{\pi}} \exp \left( \frac{-i\pi D}{2} \right) \Gamma \left( \frac{D-2}{2} \right) \Gamma \left( \frac{3-D}{2} \right). \quad (16)
\]

At \(2\epsilon \equiv D - 4 \to +0\) one finds

\[
\Delta m_{el} = \frac{\alpha w_0}{2\pi} \left[ -\pi - i \left( \frac{1}{\epsilon} + \ln \frac{w_0^2}{\pi \mu^2 \gamma E} - 1 \right) \right], \quad (17)
\]

and \(\Delta m_{sc}\) in the same limit is precisely (12). With the correspondence \(\mu^2_{ph} = 4\pi \mu^2 \gamma_E^{-1} \exp (-1/\epsilon)\) expression (17) can be easily reconciled with (1).

The following properties are characteristic of the MS (15) and (16): i) the real part of \(\Delta m_{el}\) in (17) appears in tandem with the IR singularity \(\epsilon^{-1}\) present in (15), precisely as it were for the MS (7) and (8); ii) the real parts \(\Re \Delta m_{el}\) and \(\Re \Delta m_{sc}\) do not appear for even dimensions \(D \geq 6\); iii) for odd \(D \geq 3\) the real parts \(\Re \Delta m_{el}\) and \(\Re \Delta m_{sc}\) are divergent, while imaginary ones are finite. This should be juxtaposed with the absence of the Huygence principle in odd-dimensional space-times.

### 3 Cyclotron motion

For the non-relativistic cyclotron motion in magnetic field \((D = 4\) below\) one knows an exact solution of the ALD equation,

\[
\frac{d\xi}{dt} - \frac{1}{b} \frac{d^2\xi}{dt^2} = -i\omega_c \xi, \quad (18)
\]

where the following notations have been introduced:

\[
\xi = v_x + iv_y, \quad \omega_c = \frac{eH}{m}, \quad \frac{1}{b} = \frac{2}{3} \frac{e^2}{4\pi m}. \quad (19)
\]
We shall discuss only the plane motion of the charge. The non-relativistic cyclotron frequency $\omega_c$ describes unperturbed motion without radiation damping, present as the second term in the l.h.s. of (18). Physically acceptable (and exact) solution of (18) was found in [11] and looks like

$$\xi = A \exp(-i\Omega t)$$

(20)

$$\Re\Omega = \frac{b}{2} \left( -\frac{1}{2} + \frac{1}{2} \sqrt{1 + 16\frac{\omega_c^2}{b^2}} \right)^{1/2} = \omega_c \left( 1 - 2\frac{\omega_c^2}{b^2} + \ldots \right),$$

(21)

$$\Im\Omega = \frac{b}{2} \left[ 1 - \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + 16\frac{\omega_c^2}{b^2}} \right)^{1/2} \right] = -\frac{\omega_c^2}{b} + \ldots.$$  

(22)

Note, that real and imaginary parts of $\Omega$ are represented here as an infinite series in radiative constant $(\omega_c/b)^2$ and that $\Im\Omega$ is negative (this property corresponds to the dissipation of the energy and to the shrinkage of the cyclotron orbit).

MS for the plane cyclotron motion was found in the aforementioned Ritus’s work [2] and is purely imaginary quantity [3]:

$$\Delta m = -i\frac{\alpha \omega_c}{2\pi} \int_0^\infty dx \left( \frac{1 - v_x^2 \cos 2x}{x^2 - v_x^2 \sin^2 x} - \frac{1}{x^2} \right) \equiv -i\frac{\alpha \omega_c}{2\pi} I(v_x^2).$$

(23)

Now we shall exploit the dependence of $\Delta m$ on the integral $v_x^2 \equiv v_x^2 + v_y^2$ of the unperturbed motion. The first order correction to Lagrange function in laboratory system, $\Delta L = -\frac{\Delta m}{\gamma_2}$, in non relativistic regime takes the form:

$$\Delta L = \frac{1}{2} \delta m^{(1)} v_\perp^2,$$

(24)

where the first-order (in $e^2$) radiative correction to the mass,

$$\delta m^{(1)} = \frac{2i\alpha}{3m} eH$$

(25)

($\delta m^{(1)}$ should not be confused with $\Delta m$: having different physical meaning, they differ in sign). We postulate that the correction up to the second order must be determined by the same principle, but with $m + \delta m^{(1)}$ substituted.

$^3$MS (23) is not an exact result in the sense that $\Delta W$ (2) is computed for the unperturbed cyclotron world line.
in place of $m$ in (25). Application of this procedure \textit{ad infinitum} gives us the equation for the complete radiative mass-correction

$$
\delta m = \frac{2i\alpha eH}{3(m + \delta m)}.
$$

(26)

It could be easily checked that the ”new” cyclotron frequency $eH/(m + \delta m)$ is just the $\Omega$ from (21), (22):

$$
\frac{\delta m}{m} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{i\omega_c}{b}}, \quad \Omega = \frac{eH}{m + \delta m}.
$$

(27)

Qualitatively, this coincidence is not surprising: radiation effects, when properly accounted, must entail in dynamics. Really unexpected is quantitative correspondence between formulas (27) and (21), (22). So, one should ask about the foundation of such ”empirical” perturbative approach. As it seems now the main points are the following: i) the structure of the radiative correction (24) to Lagrange function (or energy) is just the one of the ”bare” Lagrangian. $\delta m^{(1)}$ does not contain dynamical variables, so that the relation (25) is independent of the concrete conditions of the cyclotron motion; ii) the constancy of the field which makes going from non-local $\Delta W$ to the MS possible; iii) two dimensional character of the unperturbed motion is not destroyed by radiation; the latter does give the better conditions for the applicability of the non-relativistic approximation. Could one hope to use this renormalisation method as a general tool for solving ALD equation? It does not seem so because there is no universal dimensionless parameter in CED which would govern the radiation effects. QED does have such a parameter which is the fine structure constant.

4 The probe of radiation reaction effects for spinning particles

Several reasons make this topic interesting: i) considerable recent attention to the pseudoclassical models which stems from the close relations between those models and string theory; ii) with rare exception, the problem of back-reaction effects have not been considering in those models; iii) it is desirable to endow the spinning particles models with a quasiclassical status -in the same sense as one uses the BMT equation (which does not include back reaction). For example, there is still a need in adequate quasiclassical interpretation of the radiation polarization phenomenon \[12\].
The self field of the charged particle possessing the magnetic moment \( \mu \) 
\( (= \frac{2}{3} \mu_B, \ \mu_B \equiv e\hbar/2mc) \) satisfies the equation

\[
-\partial^2 A_\beta(x) = j_\beta(x) + \partial_\gamma M_{\beta\gamma}(x),
\]

where \( j_\beta(x) \) is the source (5), and

\[
M_{\beta\gamma}(x) = \int d\tau \mu_{\beta\gamma}(\tau) \delta^4(x - x(\tau))
\]

is the polarization density. The dependence of \( \mu_{\beta\gamma} = i\mu\varepsilon_{\beta\gamma\delta} \dot{x}_\delta S_\delta \)
on \( \tau \) is determined from the Lorentz and BMT equations [3]:

\[
\dddot{x}_\alpha = 2\mu_B F_{\alpha\beta} \dot{x}_\beta, \quad \ddot{S}_\alpha = 2\mu_B F_{\alpha\beta} S_\beta
\]

(we put \( g=2 \) for simplicity). The self-action obtained from (2) by substitution

\( j_\beta \rightarrow j_\beta + \partial_\gamma M_{\beta\gamma} \), can be decomposed into the following terms:

\[
\Delta W = \Delta W_{or} + \Delta W_{s-o} + \Delta W_{s-s},
\]

where ”orbit” part is precisely \( \Delta W \) in (2), and ”spin-orbit” and ”spin-spin” terms are:

\[
\Delta W_{s-o} = -e \int d\tau \int d\tau' \delta_\beta(\tau) \mu_{\beta\alpha}(\tau') \dot{x}_\alpha(x - x')^{\mathcal{F}}_0, \\
\Delta W_{s-s} = \frac{1}{2} \int d\tau \int d\tau' \mu_{\alpha\beta} \mu'_{\alpha\gamma} \dot{x}_\beta \dot{x}_\gamma \delta_{\beta\gamma}(x - x')^{\mathcal{F}}_0.
\]

For the constant homogeneous magnetic field the MS is expressed through the geometrical invariants of the world line, the latter being curvature \( k \) and the first torsion \( \tau_1 \) (the second one is equal to zero for the plane motion):

\[
k = 2\mu_B H v_\perp \gamma_\perp, \quad \tau_1 = 2\mu_B H \gamma_\perp.
\]

Without going into details, we give the final expression for the \( \Delta W_{s-o} \) term in
(33) regarding the latter as a major contribution between two terms (33) and
(34) in the decomposition (32):

\[
\Delta m_{s-o} = -i \frac{e\mu}{2\pi^2} \zeta \omega_c^2 f(v_\perp).
\]
Here $\zeta$ is $z$-component of spin (in $\hbar/2$ units), and the formfactor $f(v_\perp)$ includes relativistic retardation effects. It is obtained from (33) (where IR regulator $\mu_{ph}$ could be omitted):

$$f(v_\perp) = 2v_\perp^2 \gamma_\perp^{-1} \int_0^\infty dx \frac{4 \sin^2(x/2) - x \sin x}{[x^2 - 4v_\perp^2 \sin^2(x/2)]^2}.$$  (37)

Following asymptotic expressions are obtainable from the representation (37):

$$f(v_\perp) = \begin{cases} 
\frac{\pi}{6} v_\perp^2, & v_\perp \ll 1, \\
\frac{\pi}{4\sqrt{3}} \gamma_\perp, & \gamma_\perp \gg 1.
\end{cases}$$  (38)

Comparing it with the ultrarelativistic behaviour of function $I(v_\perp^2)$ introduced in (23), $I(v_\perp^2) = \frac{5\pi}{2\sqrt{3}} \gamma_\perp$, we can conclude about the notable growth of the spin effects relative to orbit ones in the ultrarelativistic region $\gamma_\perp \gg 1$:

$$\frac{\Delta m_{s-o}}{\Delta m_{o-r}} \simeq \frac{1}{5} \frac{H}{H_c \gamma_\perp}.$$  (39)

Here $H_c = m^2 c^3/\hbar$ and we put $\zeta \sim 1$ (see (23) and (36)).

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