Abstract
We determine the $L^2$-Betti numbers of all one-relator groups and all surface-plus-one-relation groups. We also obtain some information about the $L^2$-cohomology of left-orderable groups, and deduce the non-$L^2$ result that, in any left-orderable group of homological dimension one, all two-generator subgroups are free.

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1 Notation and background
Let $G$ be a (discrete) group, fixed throughout the article.

We use $\mathbb{R} \cup \{-\infty, \infty\}$ with the usual conventions; for example, $\frac{1}{\infty} = 0$, and $3 - \infty = -\infty$. Let $\mathbb{N}$ denote the set of finite cardinals, $\{0, 1, 2, \ldots\}$. We call $\mathbb{N} \cup \{\infty\}$ the set of vague cardinals, and, for each set $X$, we define its vague cardinal $|X| \in \mathbb{N} \cup \{\infty\}$ to be the cardinal of $X$ if $X$ is finite, and to be $\infty$ if $X$ is infinite.

Mappings of right modules will be written on the left of their arguments, and mappings of left modules will be written on the right of their arguments.

Let $\mathbb{C}[G]$ denote the set of all functions from $G$ to $\mathbb{C}$ expressed as formal sums, that is, a function $a : G \to \mathbb{C}$, $g \mapsto a(g)$, will be written as $\sum_{g \in G} a(g)g$. Then $\mathbb{C}[G]$ has a natural $CG$-bimodule structure, and contains a copy of $\mathbb{C}G$ as $\mathbb{C}G$-sub-bimodule. For each $a \in \mathbb{C}[G]$, we define $\|a\| := (\sum_{g \in G} |a(g)|^2)^{1/2} \in [0, \infty]$, and $\text{tr}(a) := a(1) \in \mathbb{C}$.

Define
$$ l^2(G) := \{a \in \mathbb{C}[G] : \|a\| < \infty\}. $$

We view $\mathbb{C} \subseteq \mathbb{C}G \subseteq l^2(G) \subseteq \mathbb{C}[G]$. There is a well-defined external multiplication map
$$ l^2(G) \times l^2(G) \to \mathbb{C}[G], \quad (a, b) \mapsto a \cdot b, $$
where, for each $g \in G$, $(a \cdot b)(g) := \sum_{h \in G} a(h)b(h^{-1}g)$; this sum converges in $\mathbb{C}$, and, moreover, $|(a \cdot b)(g)| \leq \|a\| \|b\|$, by the Cauchy-Schwarz inequality. The external multiplication extends the multiplication of $\mathbb{C}G$.

The group von Neumann algebra of $G$, denoted $\mathcal{N}(G)$, is the ring of bounded $\mathbb{C}G$-endomorphisms of the right $\mathbb{C}G$-module $l^2(G)$; see [19 §1.1]. Thus $l^2(G)$ is an $\mathcal{N}(G)$-$\mathbb{C}G$-bimodule. We view $\mathcal{N}(G)$ as a subset of $l^2(G)$ by the map $\alpha \mapsto \alpha(1)$, where $1$ denotes the identity element of $\mathbb{C}G \subseteq l^2(G)$. It can be shown that
$$ \mathcal{N}(G) = \{a \in l^2(G) \mid a \cdot l^2(G) \subseteq l^2(G)\}, $$
and
and that the action of $\mathcal{N}(G)$ on $l^2(G)$ is given by the external multiplication. Notice that $\mathcal{N}(G)$ contains $\mathbb{C}G$ as a subring and also that we have an induced ‘trace map’ $\text{tr}: \mathcal{N}(G) \to \mathbb{C}$. The elements of $\mathcal{N}(G)$ which are injective, as operators on $l^2(G)$, are precisely the (two-sided) non-zerodivisors in $\mathcal{N}(G)$, and they form a left and right Ore subset of $\mathcal{N}(G)$; see [19, Theorem 8.22(1)].

Let $\mathcal{U}(G)$ denote the ring of unbounded operators affiliated to $\mathcal{N}(G)$; see [19, §8.1]. It can be shown that $\mathcal{U}(G)$ is the left, and the right, Ore localization of $\mathcal{N}(G)$ at the set of its non-zerodivisors. For example, it is then clear that,

if $x$ is an element of $G$ of infinite order, then $x - 1$ is invertible in $\mathcal{U}(G)$. (1.0.1)

Moreover, $\mathcal{U}(G)$ is a von Neumann regular ring in which one-sided inverses are two-sided inverses, and, hence, one-sided zerodivisors are two-sided zerodivisors; see [19, §8.2].

There is a continuous, additive von Neumann dimension that assigns to every left $\mathcal{U}(G)$-module $M$ a value $\dim_{\mathcal{U}(G)} M \in [0, \infty]$; see Definition 8.28 and Theorem 8.29 of [19]. For example,

if $e$ is an idempotent element of $\mathcal{N}(G)$, then $\dim_{\mathcal{U}(G)} \mathcal{U}(G)e = \text{tr}(e)$; (1.0.2)

By Definition 6.50, Lemma 6.51 and Theorem 8.29 of [19], we can define, for each $n \in \mathbb{N}$, the $n$th $L^2$-Betti number of $G$ as

$$b_n^{(2)}(G) := \dim_{\mathcal{U}(G)} H_n(\mathcal{U}(G) \otimes_{\mathbb{Z}G} \mathcal{P}),$$

where $\mathcal{U}(G)$ is to be viewed as a $\mathcal{U}(G)$-$\mathbb{Z}G$-bimodule. Of course,

$$H_n(\mathcal{U}(G) \otimes_{\mathbb{Z}G} \mathcal{P}) = \text{Tor}^Z_n(\mathcal{U}(G), Z) \simeq \text{Tor}^Z_{\mathbb{Z}G}_n(\mathcal{U}(G), Z) = H_n(G; \mathcal{U}(G)),$$

where, for the purposes of this article, it will be convenient to understand that $H_n(G; -)$ applies to right $G$-modules. Thus the $L^2$-Betti numbers do not depend on the choice of $Z$, nor on the choice of $\mathcal{P}$.

1.1 Remark. If $G$ contains an element of infinite order, then [10, 11] implies that $\mathcal{U}(G) \otimes_{\mathbb{Z}G} Z = 0$, and $\mathcal{U}(G) \otimes_{\mathbb{Z}G} P_1 \to \mathcal{U}(G) \otimes_{\mathbb{Z}G} P_0 \to 0$ is exact, and $H_0(G; \mathcal{U}(G)) = 0$, and $b_0^{(2)}(G) = 0$. □

1.2 Remarks. In general, there is little relation between the $n$th $L^2$-Betti number, $b_n^{(2)}(G) = \dim_{\mathcal{U}(G)} H_n(G; \mathcal{U}(G)) \in [0, \infty]$, and the $n$th (ordinary) Betti number,

$b_n(G) := \dim_{\mathbb{Q}} H_n(G; \mathbb{Q}) \in [0, \infty]$. 
We say that \( G \) is of type FL if, for \( Z = \mathbb{Z} \), there exists a resolution such that all the \( P_n \) are finitely generated free left \( ZG \)-modules and all but finitely many of the \( P_n \) are 0.

If \( G \) is of type FL, then it is easy to see that the \( L^2 \)-Euler characteristic
\[
\chi^{(2)}(G) := \sum_{n \geq 0} (-1)^n b_n^{(2)}(G)
\]
is equal to the (ordinary) Euler characteristic
\[
\chi(G) := \sum_{n \geq 0} (-1)^n b_n(G).
\]

We say that \( G \) is of type VFL if \( G \) has a subgroup \( H \) of finite index such that \( H \) is of type FL. In this event, the (ordinary) Euler characteristic of \( G \) is defined as
\[
\chi(G) := \frac{1}{[G:H]} \chi(H);
\]
this is sometimes called the virtual Euler characteristic. Here again, \( \chi^{(2)}(G) = \chi(G) \); see [19, Remark 6.81].

## 2 Summary of results

In outline, the article has the following structure. More detailed definitions can be found in the appropriate sections.

In Section 3, we prove a useful technical result about \( \mathcal{U}(G) \) for special types of groups.

In Section 4, we calculate the \( L^2 \)-Betti numbers of one-relator groups. Let us describe the results.

For any element \( x \) of a group \( G \), we define the exponent of \( x \) in \( G \), denoted \( \exp_G(x) \), as the supremum in \( \mathbb{Z} \cup \{ \infty \} \) of the set of those integers \( m \) such that \( x \) equals the \( m \)th power of some element of \( G \). Then \( \exp_G(x) \) is a nonzero vague cardinal. We write \( G/\langle x \rangle \) to denote the quotient group of \( G \) modulo the normal subgroup of \( G \) generated by \( x \).

Suppose that \( G \) has a one-relator presentation \( \langle X \mid r \rangle \). Thus \( r \) is an element of the free group \( F \) on \( X \), and \( G = F/\langle r \rangle \).

Set \( d := |X| \in [0, \infty] \), \( m := \exp_F(r) \in [1, \infty] \), and \( \chi := 1 - d + \frac{1}{m} \in [-\infty, 1] \).

It is known that if \( d < \infty \) then \( G \) is of type VFL and \( \chi(G) = \chi \). If \( d = \infty \), then \( G \) is not finitely generated and \( \chi = -\infty \); here we define \( \chi(G) = -\infty \), which is non-standard, but it is reasonable.

In general, \( \max\{\chi(G), 0\} = \frac{1}{m} \).

In Theorem 4.2 we will show that,

\[
\begin{cases}
    \max\{\chi(G), 0\} & \text{if } n = 0, \\
    \max\{-\chi(G), 0\} & \text{if } n = 1, \\
    0 & \text{if } n \geq 2.
\end{cases}
\]

Lück [19, Example 7.19] gave some results and conjectures concerning the \( L^2 \)-Betti numbers of torsion-free one-relator groups, and (2.0.1) shows that the conjectured statements are true.

In Section 5, we calculate the \( L^2 \)-Betti numbers of an arbitrary surface-plus-one-relation group \( G = \pi_1(\Sigma)/\langle \alpha \rangle \). Here \( \Sigma \) is a connected orientable surface, and \( \alpha \) is
an element of the fundamental group, \( \pi_1(\Sigma) \). The surface-plus-one-relation groups were introduced and studied by Hempel [12], and further investigated by Howie [15]; these authors called the groups ‘one-relator surface groups’, but we are reluctant to adopt this terminology.

If \( \Sigma \) is not closed, then \( \pi_1(\Sigma) \) is a countable free group, see [20], and \( G \) is a countable one-relator group. In light of Theorem 4.2 we may assume that \( \Sigma \) is a closed surface.

Let \( g \) denote the genus of the closed surface \( \Sigma \), and let \( m = \exp_{\pi_1(\Sigma)}(\alpha) \). It is not difficult to deduce from known results that \( G \) is of type VFL and

\[
\chi(G) = \begin{cases} 
1 & \text{if } g = 0, \\
0 & \text{if } g = 1, \\
2 - 2g + \frac{1}{m} & \text{if } g \geq 2.
\end{cases}
\]

Then \( \chi(G) \in (-\infty, 1] \) and \( \max\{\chi(G), 0\} = \frac{1}{|G|} \). In Section 5 we will show that (2.0.1) is also valid for surface-plus-one-relation groups.

For any group \( G \), \( b_n^{(2)}(G) = \frac{1}{|G|} \); see [19] Theorem 6.54(8)(b)]. It is obvious that if \( G \) is finite then \( b_n^{(2)}(G) = 0 \) for all \( n \geq 1 \). Thus, in essence, the foregoing results assert that if \( G \) is an infinite one-relator group, or an infinite surface-plus-one-relation group, then

\[
b_n^{(2)}(G) = \begin{cases} 
-\chi(G) & \text{if } n = 1, \\
0 & \text{if } n \neq 1,
\end{cases}
\]

and we emphasize that, in this case, we understand that \( \chi(G) = -\infty \) if \( G \) is not finitely generated.

In Section 6 we consider a variety of situations where \( Z \) is a nonzero ring and there exists some positive integer \( n \) such that \( P_n = ZG^2 \) in a projective \( ZG \)-resolution (1.0.3) of \( ZG \). For example, this happens for two-generator groups and for two-relator groups.

Thus, in Corollary 6.8, we recover Lück’s result [19] Theorem 7.10] that all the \( L^2 \)-Betti numbers of Thompson’s group \( F \) vanish; see [6] for a detailed exposition of the definition and main properties of \( F \).

2.1 Definitions. Recall that \( G \) is left orderable if there exists a total order \( \leq \) of \( G \) which is left \( G \)-invariant, that is, whenever \( g, x, y \in G \) and \( x \leq y \), then \( gx \leq gy \). One then says that \( \leq \) is a left order of \( G \). The reverse order is also a left order. Since every group is isomorphic to its opposite through the inversion map, we see that ‘left-orderable’ is a short form for ‘one-sided-orderable’.

A group is said to be locally indicable if every finitely generated subgroup is either trivial or has an infinite cyclic quotient. Burns and Hale [5] showed that every locally indicable group is left orderable. This often provides a convenient way to prove that a given group is left orderable.

Recall that the cohomological dimension of \( G \) with respect to a ring \( Z \), denoted \( \text{cd}_Z G \), is the least \( n \in \mathbb{N} \) such that \( P_{n+1} = 0 \) in some projective \( ZG \)-resolution (1.0.3) of \( ZG \). The cohomological dimension of \( G \), denoted \( \text{cd}_Z G \), is \( \text{cd}_G G \). A classic result of Stallings and Swan says that the groups of cohomological dimension at most one are precisely the free groups.
Similarly, the homological dimension of $G$ with respect to a ring $Z$, denoted $\text{hd}_Z G$, is the least $n \in \mathbb{N}$ such that $P_{n+1} = 0$ in some flat $ZG$-resolution of $ZG$. The homological dimension of $G$, denoted $\text{hd}_Z G$.

We understand that Robert Bieri, in the 1970’s, first raised the question as to whether the groups of homological dimension at most one are precisely the locally free groups. Notice that a locally free group has homological dimension at most one, since the augmentation ideal of a locally free group is a directed union of free groups. Recently, in [16], it was proved that if the homological dimension of $G$ is at most one and $G$ satisfies the Atiyah conjecture (or, more generally, the group ring $ZG$ embeds in a one-sided Noetherian ring), then $G$ is locally free. In Corollary 6.12 we show that if $G$ is locally indicable, or, more generally, left orderable, and the homological dimension of $G$ is at most one, then every two-generator subgroup of $G$ is free.

Finally, in Proposition 6.13 we calculate the first three $L^2$-Betti numbers of an arbitrary left-orderable two-generator group of cohomological dimension at least three.

2.2 Notation. We will frequently consider maps between free modules over a ring $U$, and we will use the following format.

Let $X$ and $Y$ be sets.

By an $X \times Y$ row-finite matrix over $U$ we mean a function $(u_{x,y}) : X \times Y \to U$, $(x,y) \mapsto u_{x,y}$ such that, for each $x \in X$, $\{y \in Y \mid u_{x,y} \neq 0\}$ is finite.

We write $\oplus_X U$ to denote the direct sum of copies of $U$ indexed by $X$. If $n \in \mathbb{N}$, and $X = \{1,\ldots,n\}$, we identify $X = n$ and also write $\oplus_n U$ as $U^n$. An element of $\oplus_X U$ will be viewed as a $1 \times X$ row-finite matrix $(u_{1x})$ over $U$. Then $\oplus_X U$ is a left $U$-module in a natural way.

A map $\oplus_X U \to \oplus_Y U$ of left $U$-modules will be thought of as right multiplication by a row-finite $X \times Y$ matrix $(u_{x,y})$ in a natural way, and we will write $\oplus_X U \xrightarrow{(u_{x,y})} \oplus_Y U$.

3 Preliminary results about $U(G)$

For $a = \sum_{g \in G} a(g)g \in \mathbb{C}[[G]]$, we let $a^* = \sum_{g \in G} \overline{a(g^{-1})}g$ where $\overline{z}$ indicates the complex conjugate of $z$. This involution restricts to $\mathbb{C}(G)$ and $\mathcal{N}(G)$, and extends in a unique way to $U(G)$. Furthermore, if $a, b \in \mathcal{N}(G)$, then $(ab)^* = b^*a^*$ and $a^*a = 0$ if and only if $a = 0$.

In Sections 4 and 5, we shall see that the narrow hypotheses of the following result hold whenever $G$ is a one-relator group or a surface-plus-one-relation group.

3.1 Theorem. Suppose that $G$ has a normal subgroup $H$ such that $H$ is the semidirect product $F \rtimes C$ of a free subgroup $F$ by a finite subgroup $C$, and that $G/H$ is locally indicable, or, more generally, left orderable.

Let $m = |C|$, and let $e = \frac{1}{m} \sum_{c \in C} c \in \mathbb{C}G$.

Then the following hold.

(i) Each torsion subgroup of $G$ embeds in $C$.

(ii) Each nonzero element of $e \mathbb{C}Ge$ is invertible in $e U(G)e$.

(iii) For all $x \in U(G)e$ and $y \in e \mathbb{C}Ge$, if $xy = 0$ then $x = 0$ or $y = 0$. 
Proof. Each torsion subgroup of $G$ lies in $H$ and has trivial intersection with $F$, and therefore embeds in $C$.

Notice that $e$ is a projection, that is, $e$ is idempotent and $e^* = e$. Clearly, $\text{tr}(e) = \frac{1}{m}$. Also, $eU(G)e$ is a ring and $eCGe$ is a subring of $eU(G)e$. Moreover, in $eU(G)e$, one-sided inverses are two-sided inverses.

Let $a \in eCGe - \{0\}$. We want to show that $a$ is left invertible in $eU(G)e$.

Let $T$ be a transversal for the right (or left) $H$-action on $G$, and suppose that $T$ contains 1. Write $a = t_1a_1 + \cdots + t_na_n$ where the $t_i$ are distinct elements of $T$, and, for each $i$, $a_i \in \mathbb{C}(H)e - \{0\}$.

Let $x$ be a left order for $G/H$. We may assume that $t_1H \prec \cdots \prec t_nH$. To show that $a$ is left invertible in $eU(G)e$, it suffices to show that $(ea_1t_1^{-1}e)a$ is left invertible in $eU(G)e$. On replacing $a$ with $(ea_1t_1^{-1}e)a = a_1t_1^{-1}a$, we see that we may assume that $t_1 = 1$ and $a_1 \in eCGe - \{0\}$.

By (1), $m$ is the least common multiple of the orders of the finite subgroups of $H$. Now the strong Atiyah conjecture holds for $H$; see [18] or [19] Chapter 10. Hence $\dim_{U(H)}U(H)a_1 \geq \frac{1}{m} = \text{tr}(e)$. Of course, $U(H)a_1 \subseteq U(H)e$, and thus $\dim_{U(H)}U(H)a_1 \leq \dim_{U(H)}U(H)e = \text{tr}(e)$. Hence $\dim_{U(H)}U(H)a_1 = \text{tr}(e)$.

Also, $U(H)(a_1 + 1 - e) = U(H)a_1 \oplus U(H)(1 - e)$. Hence

$$\dim_{U(H)}U(H)(a_1 + 1 - e) = \dim_{U(H)}U(H)a_1 + \dim_{U(H)}U(H)(1 - e) = \text{tr}(e) + \text{tr}(1 - e) = 1.$$ 

This implies that $a_1 + 1 - e$ is invertible in $U(H)$. The $*$-dual of [17] Theorem 4 now implies that $a + 1 - e = 1(a_1 + 1 - e) + t_2a_2 + \cdots + t_na_n$ is invertible in $U(G)$.

It is then straightforward to show that $a$ is invertible in $eU(G)e$.

(1) Suppose that $y \neq 0$. Then $x^*xyy^* = 0$, $yy^* \in eCGe - \{0\}$ and $x^*x \in eU(G)e$. By (1), $yy^*$ is invertible in $eU(G)e$. Hence $x^*x = 0$ and $x = 0$. \qed

3.2 Remark. The above proof shows that the conclusions of Theorem 3.1 hold under the following hypotheses: $H$ is a normal subgroup of $G$; $G/H$ is left orderable; the strong Atiyah conjecture holds for $H$; and, $e$ is a nonzero projection in $\mathbb{C}H$ such that $\frac{1}{\text{tr}(e)}$ is the least common multiple of the orders of the finite subgroups of $H$. \qed

3.3 Theorem. If $G$ is locally indicable, or, more generally, left orderable, then every nonzero element of $\mathbb{C}G$ is invertible in $U(G)$. \qed

4 One-relator groups

We shall now calculate the $L^2$-Betti numbers of one-relator groups.

4.1 Notation. Suppose that $G$ is a one-relator group, and let $(X \mid r)$ be a one-relator presentation of $G$.

Here $r$ is an element of the free group $F$ on $X$ and $G = F/\langle r \rangle$.

Let $m = \text{exp}_F(r)$ and let $d = |X|$. These are vague cardinals. Here $m \neq 0$; moreover, $m = \infty$ if and only if $r = 1$, in which case $G = F$. 

If \( m < \infty \), then \( r = q^m \) for some \( q \in F \). Let \( c \) denote the image of \( q \) in \( G \), and let \( C = \langle c \rangle \leq G \). Then \( C \) has order \( m \). Let \( e = \frac{1}{m} \sum_{x \in C} x \in \mathbb{C}G \).

If \( m = \infty \), we define \( e = 0 \in \mathbb{C}G \).

In any event \( e \) is a projection and \( \text{tr}(e) = \frac{1}{m} \).

There is an exact sequence of left \( \mathbb{Z}G \)-modules

\[
0 \rightarrow \bigoplus_{x \in X} \mathbb{Z}G \rightarrow \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0 \quad \text{if } m = \infty,
\]

\[
0 \rightarrow \mathbb{Z}[G/C] \rightarrow \mathbb{Z} \rightarrow 0 \quad \text{if } d = 1 \text{ and } m < \infty,
\]

\[
0 \rightarrow \mathbb{Z}[G/C] \rightarrow \bigoplus_{x \in X} \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0 \quad \text{if } d \geq 2 \text{ and } m < \infty.
\]

see [7], specifically, Lemma 6.21 and \((*)\) on p. 167 in the proof of Theorem 6.22. In all cases, there is then an exact sequence of left \( \mathbb{C}G \)-modules

\[
0 \rightarrow \mathbb{C}Ge \xrightarrow{(a_1,x)} \bigoplus_{x \in X} \mathbb{C}G \xrightarrow{(b_{x,1})} \mathbb{C}G \rightarrow \mathbb{C} \rightarrow 0; \quad (4.1.1)
\]

for each \( x \in X \), \( b_{x,1} \) is the image of \( q \) in \( \mathbb{C}G \), and \( a_{1,x} \) is the left Fox derivative \( \frac{\partial q}{\partial x} \in e\mathbb{C}G \).

If \( d < \infty \), then \( G \) is of type VFL and

\[
\chi(G) = 1 - d + \frac{1}{m} \in (-\infty, 1]; \quad (4.1.2)
\]

see Theorem 6.22 and Corollary 6.15 of [7], for the cases where \( m < \infty \) and \( m = \infty \), respectively.

In the case where \( d = \infty \), that is, \( G \) is a non-finitely-generated one-relator group, we define \( \chi(G) := -\infty \). This is non-standard, but it extends (4.1.2).

It is easy to verify that \( \frac{1}{|G|} = \max \{\chi(G), 0\} \). In fact, by abelianizing \( G \), we see that \( G \) is finite if and only if either \( d = 1 \) and \( m < \infty \), or \( d = 0 \) (and hence \( m = \infty \)).

We shall now prove the following.

4.2 Theorem. If \( G \) is a one-relator group, then, for \( n \in \mathbb{N} \),

\[
b_n^{(2)}(G) = \begin{cases} 
\max\{\chi(G), 0\} \left(= \frac{1}{|G|}\right) & \text{if } n = 0, \\
\max\{-\chi(G), 0\} & \text{if } n = 1, \\
0 & \text{if } n \geq 2.
\end{cases} \quad (4.2.1)
\]

Proof. Suppose that Notation 4.1 holds.

Unaugmenting (4.1.1) and applying \( \mathcal{U}(G) \otimes_{\mathbb{Z}G} \) gives

\[
0 \rightarrow \mathcal{U}(G)e \xrightarrow{(a_1,x)} \bigoplus_{x \in X} \mathcal{U}(G) \xrightarrow{(b_{x,1})} \mathcal{U}(G) \rightarrow 0; \quad (4.2.2)
\]

the homology of (4.2.2) is then \( H_n(G; \mathcal{U}(G)) \).

We claim that

\[
\text{if } y \in \mathcal{U}(G)e - \{0\} \text{ and } a \in e\mathbb{C}G - \{0\}, \text{ then } ya \neq 0. \quad (4.2.3)
\]

This is vacuous if \( m = \infty \).

If \( m < \infty \), let \( H \) denote the normal subgroup of \( G \) generated by \( c \). Then \( G/H = \langle X \mid q \rangle \) is a torsion-free one-relator group. Hence \( G/H \) is locally indicable.
where \( x, y \)
\[
α \not\in S.
\]

### 5.1 Theorem

Following.

Let \( g \in G \) denote the genus of \( \Sigma \). Then \( g \in \mathbb{N} \)
\[
\pi_1(\Sigma) = \langle x_1, x_2, \ldots, x_{2g-1}, x_{2g} \mid [x_1, x_2][x_3, x_4] \cdots [x_{2g-1}, x_{2g}] \rangle,
\]
where \([x, y] \) denotes \( xyx^{-1}y^{-1} \). Since this is a one-relator presentation, we have \( α \neq 1 \). In particular, \( g \) is nonzero. The non-one-relator cases are included in the following.

### 5.1 Theorem

Let \( Σ \) be a closed orientable surface of genus at least one, let \( S = π_1(Σ) \), let \( α \) be a nontrivial element of \( S \), and let \( G = S/\langle α \rangle \).

Let \( g \) denote the genus of \( Σ \), let \( m = \exp_α(α) \), and let \( Q \) be a nonzero ring in which \( \frac{1}{m} \) is defined, that is, if \( m < \infty \) then \( mQ = Q \). Then the following hold.

(i) \( G \) is of type VFL and \( χ(G) = \min\{2 - 2g + \frac{1}{m}, 0\} \)
\[
\begin{align*}
0 & \quad \text{if } g = 1, \\
2 - 2g + \frac{1}{m} & \quad \text{if } g \geq 2.
\end{align*}
\]

(ii) \( cdQ(G) = \min\{2, g\} \)
\[
\begin{align*}
1 & \quad \text{if } g = 1, \\
2 & \quad \text{if } g \geq 2.
\end{align*}
\]

(iii) For \( n \in \mathbb{N} \), \( b_{2n}^0(G) = -δ_{n,1}χ(G) = \begin{cases} -χ(G) & \text{if } n = 1, \\
0 & \text{if } n \neq 1. \end{cases} \)
Proof. We break the proof up into a series of lemmas and summaries of notation.

5.2 Notation. As in [3] Examples I.3.5(v), the expression \( S_1 *_{S_0} \alpha \) will denote an HNN extension, where it is understood that \( S_1 \) is a group, \( S_0 \) is a subgroup of \( S_1 \) and \( \alpha \) is an injective group homomorphism \( \alpha: S_0 \to S_1, a \mapsto a^\alpha \). The image of this homomorphism is denoted \( S_0^\alpha \).

5.3 Lemma (Hempel). If \( g \geq 2 \), then there exists an HNN-decomposition \( S = S_1 *_{S_0} \alpha \) such that \( S_1 \) is a free group, \( \alpha \) lies in \( S_1 \), and the normal subgroup of \( S_1 \) generated by \( \alpha \) intersects both \( S_0 \) and \( S_0^\alpha \) trivially.

Hence, \( G = S/\langle \alpha \rangle \) has a matching HNN-decomposition \( S/\langle \alpha \rangle = S_1/\langle \alpha \rangle *_{S_0} S_0 \).

Proof. This was implicit in the proof of [12] Theorem 2.2, and was made explicit in [14] Proposition 2.1.

5.4 Lemma (Hempel). If \( m < \infty \), there exists \( \beta \in S \) such that \( \beta^m = \alpha \), and the image of \( \beta \) in \( G \) has order \( m \).

Proof. As this is obvious for \( g = 1 \), we may assume that \( g \geq 2 \). Thus we have matching HNN-decompositions \( S = S_1 *_{S_0} \alpha \) and \( G = S/\langle \alpha \rangle = S_1/\langle \alpha \rangle *_{S_0} S_0 \), as in Lemma 5.3.

Let \( m' = \exp S_1 \alpha \). Since \( \alpha \neq 1 \) and \( S_1 \) is free, we see that \( m' < \infty \). Choose \( \beta \in S_1 \) such that \( \beta^{m'} = \alpha \). Let \( c \) denote the image of \( \beta \) in \( G \), and let \( C = \langle c \rangle \leq G \). Then \( C \) has order \( m' \), and every torsion subgroup of \( S_1/\langle \alpha \rangle \) embeds in \( C \). From the HNN decomposition for \( G \), we see that any finite subgroup of \( G \) is conjugate to a subgroup of \( S_1/\langle \alpha \rangle \), and hence has order dividing \( m' \).

A similar argument shows that for any positive integer \( i \), \( S/\langle \alpha^i \rangle \) has a matching HNN decomposition, and therefore has a subgroup of order \( m'i \) and a subgroup of order \( i \). It follows that if \( \alpha = \gamma^j \) for some positive integer \( j \) then \( S/\langle \alpha \rangle \) has a subgroup of order \( j \), and hence \( j \) divides \( m' \). It now follows that \( m = m' < \infty \).

5.5 Notation. Let \( \beta \) denote an element of \( S \) such that \( \beta^m = \alpha \).

Let \( c \) denote the image of \( \beta \) in \( G \). Let \( C = \langle c \rangle \), a cyclic subgroup of \( G \) of order \( m \).

Let \( e = \frac{1}{m} \sum_{x \in C} x \), an idempotent element of \( CG \) with \( \text{tr}(e) = \frac{1}{m} \); we shall also view \( e \) as an idempotent element of \( QG \).

Let \( H \) denote the normal subgroup of \( G \) generated by \( c \); thus, \( G/H \simeq S/\langle \beta \rangle \).

5.6 Lemma. (i) \( H \) has a free subgroup \( F \) such that \( H = F \rtimes C \).

(ii) \( G/H \) is locally indicable.

(iii) Every torsion subgroup of \( G \) embeds in \( C \).

(iv) If \( x \in U(G) e - \{0\} \) and \( y \in eCG - \{0\} \), then \( xy \neq 0 \).

Proof. (i) As this is clear for \( g = 1 \), we may assume that \( g \geq 2 \).

By Lemma 5.3 with \( \beta \) in place of \( \alpha \), there exists an HNN-decomposition \( S = S_1 *_{S_0} \alpha \) where \( S_1 \) is a free group, \( \beta \) lies in \( S_1 \), and the normal subgroup of \( S_1 \) generated by \( \beta \) intersects both \( S_0 \) and \( S_0^\beta \) trivially. Hence \( \alpha \) lies in \( S_1 \), and the normal subgroup of \( S_1 \) generated by \( \alpha \) intersects both \( S_0 \) and \( S_0^\alpha \) trivially. It follows that we can make identifications:

\[
G = S/\langle \alpha \rangle = S_1/\langle \alpha \rangle *_{S_0} S_0 \quad \text{and} \quad G/H = S/\langle \beta \rangle = S_1/\langle \beta \rangle *_{S_0} S_0.
\]
Thus we have matching HNN-decompositions for \( S, G \) and \( G/H \).

Let us apply Bass-Serre theory, following, for example, [3] Chapter 1. Consider the action of \( H \) on the Bass-Serre tree for the above HNN-decomposition of \( G \). Then \( H \) acts freely on the edges. Let \( H_0 \) denote the normal subgroup of \( S_1/\langle \alpha \rangle \) generated by \( c \). Then \( H_0 \) is a vertex stabilizer for the \( H \)-action, and the other vertex stabilizers are \( G \)-conjugates of \( H_0 \). By Bass-Serre theory, or the Kurosh Subgroup Theorem, \( H \) is the free product of a free group and various \( G \)-conjugates of \( H_0 \).

By [11] Theorem 1, \( H_0 \) itself is a free product of certain \( S_1/\langle \alpha \rangle \)-conjugates of \( C \).

Thus \( H \) is the free product a free group and various \( G \)-conjugates of \( C \). If we map each of these \( G \)-conjugates of \( C \) isomorphically to \( C \), and map the free group to 1, we obtain an epimorphism \( H \rightarrow C \). Applying [3] Proposition I.4.6] to this epimorphism, we see that its kernel \( F \) is free. Clearly, \( H = F \times C \). This proves (i) and (v). Since \( G/H = S/\langle \beta \rangle \) and \( \beta \) is not a proper power in \( S, G/H \) is locally indicable by [12] Theorem 2.2).

Let us dispose of the case where \( g = 1 \), which is well known and included only for completeness.

5.7 Lemma. If \( g = 1 \), then the following hold.

(i) \( H = C \) and \( G/C \) is infinite cyclic generated by \( xC \) for some \( x \in G \).

(ii) \( 0 \rightarrow \mathbb{Z}[G/C] \xrightarrow{x-1} \mathbb{Z}[G/C] \rightarrow \mathbb{Z} \rightarrow 0 \) is an exact sequence of left \( \mathbb{Z}G \)-modules.

(iii) \( 0 \rightarrow QGe \xrightarrow{x-1} QGe \rightarrow Q \rightarrow 0 \) is an exact sequence of left \( QG \)-modules.

(iv) \( \langle x \rangle \) is an infinite cyclic subgroup of \( G \) of finite index, \( G \) is of type \( VFL \), \( \chi(G) = 0 \) and \( cdQG = 1 \).

(v) The homology of \( 0 \rightarrow U(G)e \xrightarrow{x-1} U(G)e \rightarrow 0 \) is \( H_n(G;U(G)) \).

(vi) For each \( n \in \mathbb{N} \), \( b^{(2)}_n(G) = 0 \).}

5.8 Remark. For \( g = 1 \), Lemma 5.7(ii) gives the augmented cellular chain complex of a one-dimensional \( E(G) \) which resembles the real line.

5.9 Notation. Henceforth we assume that \( g \geq 2 \).

Let \( X = \{x_1, x_2, \ldots, x_{2g-1}, x_{2g}\} \), let \( F \) be the free group on \( X \), and let \( r_1 = [x_1, x_2] \cdots [x_{2g-1}, x_{2g}] \in F \). Then \( S = \langle X \mid r_1 \rangle \).

Let \( g_2 \) be any element of \( F \) which maps to \( \beta \) in \( S \), and let \( r_2 = g_2^n \). Then \( G = \langle X \mid r_1, r_2 \rangle \).

For \( i \in \{1, 2\}, j \in \{1, \ldots, 2g\} \), we set \( a_{i,j} := \frac{\partial r_i}{\partial x_j} \in \mathbb{Z}G \), the left Fox derivatives, and \( b_{j,1} := x_j - 1 \in \mathbb{Z}G \).

Notice that \( me = \sum_{x \in C} x \in \mathbb{Z}G \) and \( a_{2,j} = \frac{\partial x_1}{\partial x_j} = (me) \frac{\partial x_1}{\partial x_j} \).

5.10 Lemma (Howie). The sequence of left \( \mathbb{Z}G \)-modules

\[
0 \rightarrow \mathbb{Z}G \oplus \mathbb{Z}[G/C] \xrightarrow{(a_{1,i})} \mathbb{Z}G^{2g} \xrightarrow{(b_{j,1})} \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0
\]

is exact.
Proof. Howie [15, Theorem 3.5] describes a \( K(G,1) \), and it is straightforward to give it a CW-structure as follows.

We take a \( K(S,1) \) with one zero-cell, \( 2g \) one-cells, and a two-cell which is a \( 2g \)-gon, and then the exact sequence of left \( ZS \)-modules arising from the augmented cellular chain complex of the universal cover of the \( K(S,1) \) is

\[
0 \longrightarrow ZS \overset{(a_{1,j})}{\longrightarrow} ZS^{2g} \overset{(b_{j,1})}{\longrightarrow} ZS \longrightarrow Z \longrightarrow 0,
\]

where we view the \( a_{1,j} \) and \( b_{j,1} \) as elements of \( ZS \).

We take a \( K(C,1) \) with one cell in each dimension such that the infinitely repeating exact sequence of left \( ZC \)-modules arising from the augmented cellular chain complex of the universal cover of the \( K(C,1) \) is

\[
\cdots \longrightarrow ZC \overset{mc}{\longrightarrow} ZC \overset{-1}{\longrightarrow} ZC \overset{mc}{\longrightarrow} ZC \overset{-1}{\longrightarrow} ZC \longrightarrow Z \longrightarrow 0.
\]

By [15, Theorem 3.5], we get a \( K(G,1) \) by melding the one-skeleton of our \( K(C,1) \) into the one-skeleton of our \( K(S,1) \) in the natural way; the attaching map of the two-cell at the homology level is then \( (a_{2,j}) \). The exact sequence of left \( ZG \)-modules arising from the augmented cellular chain complex of the three-skeleton of the universal cover of the \( K(G,1) \) is

\[
ZG \overset{(0,1-c)}{\longrightarrow} ZG^2 \overset{(a_{1,j})}{\longrightarrow} ZG^{2g} \overset{(b_{j,1})}{\longrightarrow} ZG \longrightarrow Z \longrightarrow 0.
\]

The lemma now follows easily. \( \square \)

We now imitate the proof of [11, Theorem 2].

5.11 Lemma. \( G \) is of type VFL and \( \chi(G) = 2 - 2g + \frac{1}{m} \).

Proof. Let \( p \) be a prime divisor of \( m \). It was shown in [11] that \( S \) is residually a finite \( p \)-group; see [10, Theorem B] for an alternative proof. Hence there exists a finite \( p \)-group \( P = P(p) \) and a homomorphism \( S \to P \) whose kernel does not contain \( \beta_m^P \), and we assume that \( P \) has smallest possible order. The centre \( Z(P) \) of \( P \) is nontrivial. By minimality of \( P \), \( \beta_m^P \) lies in the kernel of the composite \( S \overset{1}{\longrightarrow} P \overset{1}{\longrightarrow} P/Z(P) \). Thus \( \beta_m^P \), and \( \beta_m^P \), are mapped to \( Z(P) \). By minimality of \( P \), \( \beta_m^P \) is mapped to 1 in \( P \).

By considering the direct product of such \( P(p) \), one for each prime divisor \( p \) of \( m \), we find that there is a finite quotient of \( S \) in which the image of \( \beta \) has order exactly \( m \).

Hence there exists a normal subgroup \( N \) of \( G \) such that \( N \) has finite index in \( G \) and \( N \cap C = \{1\} \). It follows that \( N \) acts freely on \( G/C \). The number of orbits is

\[
|N\setminus(G/C)| = |N\setminus G/C| = |(N\setminus G)/C| = [G:N]/m,
\]

where the last equality holds since \( C \) acts freely on \( N\setminus G \), on the right.

Now [5.10.1] is a resolution of \( Z \) by free left \( ZN \)-modules. Thus \( N \) is of type FL, and, in particular, \( N \) is torsion-free. It is now a simple matter to calculate \( \chi(G) \) \( (= \frac{1}{[G:N] \chi(N)} \).

\( \square \)

Together Lemma 5.7(iv) and Lemma 5.11 give Theorem 5.1(i).
By Lemma 5.10, the following is clear.
5.12 Corollary. The sequence of left $QG$-modules

$$0 \rightarrow QG \oplus QGe \xrightarrow{(a_{i,j})} QG^{2g} \xrightarrow{(b_{j,i})} QG \rightarrow Q \rightarrow 0$$

is exact. □

5.13 Lemma. $\text{cd}_Q G = 2$.

Proof. By Corollary 5.12, $\text{cd}_Q G \leq 2$. It remains to show that $\text{cd}_Q G \geq 1$. Let us suppose that $\text{cd}_Q G \leq 1$ and derive a contradiction.

By Notation 5.6 and Lemma 5.9, $H$ is the (normal) subgroup of $G$ generated by the elements of finite order. By Dunwoody’s Theorem [S, Theorem IV.3.13], $G$ is the fundamental group of a graph of finite groups; by [S, Proposition I.7.11], $H$ is the normal subgroup of $G$ generated by the vertex groups. From the presentation of $G$ as in [S, Notation I.7.1], it can be seen that $G/H$ is a free group.

Since $G/H = S/\langle \beta \rangle$, the abelianization of $G/H$ has $\mathbb{Z}$-rank $2g$ or $2g - 1$. Thus the rank of the free group $G/H$ is $2g$ or $2g - 1$. Hence $\chi(S/\langle \beta \rangle)$ is $2 - 2g$ or $-2g$.

But $\chi(S/\langle \beta \rangle) = 3 - 2g$ by Lemma 5.11. This is a contradiction. □

Together Lemma 5.7 and Lemma 5.13 give Theorem 5.11. By Corollary 5.12 with $Q = \mathbb{C}$, the following is clear.

5.14 Corollary. The homology of

$$0 \rightarrow U(G) \oplus U(G)e \xrightarrow{(a_{i,j})} U(G)^{2g} \xrightarrow{(b_{j,i})} U(G) \rightarrow 0$$

is $H_\ast(G;U(G))$. □

We now come to the subtle part of the argument.

5.15 Lemma. $U(G) \oplus U(G)e \xrightarrow{(a_{i,j})} U(G)^{2g}$ is injective.

Proof. Let $(u, v)$ be an arbitrary element of the kernel. Thus, $(u, v) \in U(G) \oplus U(G)e$ and

for each $j \in \{1, \ldots, 2g\}$, $ua_{1,j} + va_{2,j} = 0$ in $U(G)$. (5.15.1)

Consider first the case where $u$ does not lie in $vCG$. We shall obtain a contradiction.

We form the right $CG$-module $W = U(G)/(vCG)$, and let $w = u + vCG \in W$. By (5.15.1),

for each $j \in \{1, \ldots, 2g\}$, $wa_{1,j} = 0$ in $W$. (5.15.2)

Let $K = \{x \in G \mid wx = w\}$. Clearly, $K$ is a subgroup of $G$.

We claim that $K = G$; it suffices to show that $\{x_1, \ldots, x_{2g}\} \subseteq K$.

We will show by induction that, if $j \in \{0, 1, \ldots, g\}$, then $\{x_1, \ldots, x_j\} \subseteq K$. This is clearly true for $j = 0$. Suppose that $j \in \{1, \ldots, g\}$ and that it is true for $j - 1$. We will show it is true for $j$. Let $k = [x_1, x_2] \cdots [x_{2j-3}, x_{2j-2}]$; then $k$ lies in $K$ by the induction hypothesis. Recall that $r_1 = [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$. By (5.15.2) and Notation 5.9

$$0 = wa_{1,j-1} = w \frac{\partial r_1}{\partial x_{2j-1}} = wk(1 - x_{2j-1} x_{2j} x_{2j-1}^{-1})$$
and

\[ 0 = wa_{1,j} = u \left( \frac{\partial r_1}{\partial x_{2j}} \right) = w k x_{2j-1} (1 - x_{2j} x_{2j-1}^{-1} x_{2j}^{-1}). \]

Since \( K = \{ x \in G \mid w(1 - x) = 0 \} \), we see that \( K \) contains

\[ k(x_{2j-1} x_{2j} x_{2j-1}^{-1}) k^{-1} \]

and \( (k x_{2j-1})(x_{2j} x_{2j-1}^{-1} x_{2j}^{-1})(k x_{2j-1})^{-1} \).

Thus \( K \) contains

\[ x_{2j-1} x_{2j} x_{2j-1}^{-1} \]

and \( x_{2j-1} (x_{2j} x_{2j-1}^{-1} x_{2j}^{-1}) x_{2j-1}^{-1} \),

and it follows easily that \( K \) contains \( x_{2j} x_{2j-1}^{-1} \), \( x_{2j} - 1 \), and \( x_{2j} \). This completes the proof by induction.

Hence, \( K = G \), and \( w \) is fixed under the right \( G \)-action on \( W \). Thus, the subset \( u + vCG \) of \( \mathcal{U}(G) \) is closed under the right \( G \)-action on \( \mathcal{U}(G) \). We denote the set \( u + vCG \) viewed as right \( G \)-set by \( (u + vCG)_G \). Notice that \( u + vCG \) does not contain 0.

By Lemma 5.6(iii), the surjective map \( eCG \to vCG \), \( x \mapsto vx \), is either injective or zero. In either event, \( vCG \) is a projective right \( C \)-module. By the left-right dual of \( \mathbb{F} \) Corollary 5.6 there exists a right \( G \)-tree with finite edge stabilizers and vertex set \( (u + vCG)_G \). It follows that there exists a (left) \( G \)-tree \( T \) with finite edge stabilizers and vertex set \( C(u + vCG) \). Alternatively, one can use \( T \) to prove that \( b_n^{(2)}(G) = 0 \) and deduce that \( (u, v) = (0, 0) \), which is also a contradiction.

Thus \( u \) lies in \( vCG \), and there exists \( y \in eCG \) such that \( u = vy \).

We consider first the case where \( v \neq 0 \). For each \( j \in \{1, \ldots, 2g\} \),

\[ v(y a_{1,j} + a_{2,j}) = u a_{1,j} + v a_{2,j} = 0 \]

by Lemma 5.10, and by Lemma 5.6(iv), \( 0 = y a_{1,j} + a_{2,j} = y a_{1,j} + e a_{2,j} \). Hence, \( (y, e) \) lies in the kernel of \( CG \oplus CG e \xrightarrow{(a_{i,j})} CG^{2g} \); since this map is injective by Corollary 5.12 we see \( e = 0 \), which is a contradiction.

Thus \( v = 0 \), and hence \( u = 0 \).

By Lemma 5.10 and Remark 1.7 it is straightforward to obtain the following.

5.16 Lemma. The \( \mathcal{U}(G) \)-dimensions of the kernel and the image of the map

\[ \mathcal{U}(G) \oplus (G)e \xrightarrow{(a_{i,j})} \mathcal{U}(G)^{2g} \]

are 0 and \( 1 + \frac{1}{m} \), respectively.

The \( \mathcal{U}(G) \)-dimensions of the image and the kernel of the map

\[ \mathcal{U}(G)^{2g} \xrightarrow{(b_{i,j})} \mathcal{U}(G) \]

are 1 and \( 2g - 1 \), respectively.

For \( n \in \mathbb{N} \), \( b_n^{(2)}(G) = \begin{cases} (2g - 1) - (1 + \frac{1}{m}) & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases} \]

Together Lemma 5.7[1] and Lemma 5.10 give Theorem 5.1[1]. This completes the proof of Theorem 5.1[1].
6 Left-orderable groups

Throughout this section we will frequently make the following assumption.

6.1 Hypotheses. There exist nonzero rings \( Z \) and \( U \) such that \( ZG \) is a subring of \( U \) and each nonzero element of \( ZG \) is invertible in \( U \).

This holds, for example, if \( G \) is locally indicable, or, more generally, left orderable, with \( Z \) being any subring of \( \mathbb{C} \), and \( U \) being \( U(G) \), by Theorem 6.2.

Notice that \( ZG \) has no nonzero zerodivisors, and hence \( G \) is torsion free.

6.2 Lemma. Let \( U \) be a ring, and let \( X \) and \( Y \) be sets.

Let \( A \) and \( B \) be nonzero row-finite matrices over \( U \) in which each nonzero entry is invertible, such that \( A \) is \( X \times 2 \), \( B \) is \( 2 \times Y \), and the product \( AB \) is the zero \( X \times Y \) matrix.

Then \( \oplus_X U \xrightarrow{A} U^2 \xrightarrow{B} \oplus_Y U \) is an exact sequence of free left \( U \)-modules.

Moreover, \( U^2 \) has a left \( U \)-basis \( v_1, v_2 \) such that \( \ker B = \im A = Uv_1 \) and \( B \) induces an isomorphism \( Uv_2 \simeq \im B \).

Proof. Write \( A = (a_{x,i}) \) and \( B = (b_{y,j}) \).

There exists \( x_0 \in X \) such that \( (a_{x_0,1}, a_{x_0,2}) \neq (0, 0) \). We take \( v_1 = (a_{x_0,1}, a_{x_0,2}) \).

Clearly \( Uv_1 \subseteq \im A \subseteq \ker B \). Without loss of generality, there exists \( y_0 \in Y \) such that \( b_{1,y_0} \) is invertible in \( U \). We take \( v_2 = (1, 0) \).

Since \( AB = 0 \), \( a_{x_0,1}b_{1,y_0} + a_{x_0,2}b_{2,y_0} = 0 \). Thus \( a_{x_0,1} = -a_{x_0,2}b_{2,y_0}b_{1,y_0}^{-1} \). Hence \( a_{x_0,2} \) cannot be zero, and is therefore invertible.

Hence \( v_1, v_2 \) is a basis of \( U^2 \), and \( b_{2,y_0}b_{1,y_0}^{-1} = -a_{x_0,2}a_{x_0,1} \).

Consider any \( (a_1, a_2) \in \ker B \). Then \( a_1b_{1,y_0} + a_2b_{2,y_0} = 0 \), and

\[
(a_1, a_2) = (-a_2b_{2,y_0}b_{1,y_0}^{-1}, a_2) = a_2(-b_{2,y_0}b_{1,y_0}^{-1})
\]

\[
= a_2(a_{x_0,2}^{-1}a_{x_0,1}, 1) = a_2a_{x_0,2}^{-1}(a_{x_0,1}, a_{x_0,2}) = a_2a_{x_0,2}^{-1}v_1 \in Uv_1,
\]

as desired. Finally, \( Uv_2 \simeq (Uv_1 + Uv_2)/Uv_1 = U^2/\ker B \simeq \im B \).

6.3 Remark. We see from the proof that the hypotheses that \( A \) and \( B \) are nonzero and every nonzero entry in \( A \) and \( B \) is invertible can be replaced with the hypotheses that some element of the first row of \( B \) is invertible, and some element of the second column of \( A \) is invertible.

There are other variations, but the stated form is most convenient for our purposes.

6.4 Proposition. Suppose that Hypotheses 6.1 hold, and suppose that there exists a positive integer \( n \) and a resolution of \( \mathbb{Z} \) by projective left \( ZG \)-modules such that \( P_n = ZG^2 \). Then either the map \( P_{n+1} \to P_n \) in is the zero map or \( H_n(G; U) = 0 \).

Proof. We may assume that \( P_{n+1} \to P_n \) is nonzero. Then we have an exact sequence

\[
P_{n+1} \to P_n \to P_{n-1},
\]
and we want to deduce that

\[
U \otimes_{ZG} P_{n+1} \to U \otimes_{ZG} P_n \to U \otimes_{ZG} P_{n-1}
\]
remains exact.

This is clear if \( P_n \to P_{n-1} \) is the zero map. Thus we may assume that the maps in (6.4.1) are nonzero.

By adding a suitable \( ZG \)-projective summand to \( P_{n+1} \) with a zero map to \( P_n \), we may assume that \( P_{n+1} \) is \( ZG \)-free without affecting the images. Similarly, we may assume that we have specified \( ZG \)-bases of \( P_{n+1} \), \( P_n \) and \( P_{n-1} \), and that the maps in (6.4.1) are represented by nonzero matrices over \( ZG \).

The maps in (6.4.2) are then represented by nonzero matrices over \( U \) with all coefficients lying in \( ZG \). Now we may apply Lemma 6.2 to deduce that (6.4.2) is exact, as desired.

\[ 6.5 \text{ Remark.} \] In Proposition 6.4 if we replace the hypothesis \( P_n = ZG^2 \) with the hypothesis \( P_n = ZG^1 \), then it is easy to see that at least one of the maps \( P_{n+1} \to P_n \), \( P_n \to P_{n-1} \) is necessarily the zero map.

Applying Proposition 6.4 with \( U = U(G) \), together with Theorem 3.3, we obtain the following two results.

\[ 6.6 \text{ Corollary.} \] Let \( G \) be a left-orderable group, and let \( Z \) be a subring of \( C \). Suppose that there exists a positive integer \( n \) and a resolution (1.0.3) of \( Z \) by projective left \( ZG \)-modules such that \( P_n = ZG^2 \). Then either \( \text{cd}_Z G \leq n \) or \( b^{(2)}_n(G) = 0 \).

\[ 6.7 \text{ Corollary.} \] If \( G \) is a left-orderable group, and there exists an exact \( CG \)-sequence of the form

\[ \cdots \to C \to CG^2 \to CG \to CG \to CG \to C 
\]

in which all the \( \partial_n \) are nonzero, then all the \( b^{(2)}_n(G) \) are zero.

\[ \text{Proof.} \] Since \( \partial_0 \) is nonzero, we see that \( G \) is nontrivial. Since \( G \) is torsion-free, \( b^{(2)}_0(G) = 0 \). For \( n \geq 1 \), \( b^{(2)}_n(G) = 0 \) by Proposition 6.4.

\[ 6.8 \text{ Corollary (Lück [19, Theorem 7.10])}. \] All the \( L^2 \)-Betti numbers of Thompson’s group \( F \) vanish.

\[ \text{Proof.} \] This follows from Corollary 6.7 since \( F \) is orderable, see [6], and has a resolution as in (6.7.1), see [4].

We now look at situations where we can deduce that a two-generator group is free.

\[ 6.9 \text{ Proposition.} \] Suppose that Hypotheses 6.1 hold. The following are equivalent.

\( a) \) \( G \) is a two-generator group, and \( H_1(G; U) \cong U \).

\( b) \) \( G \) is a two-generator group, and \( H_1(G; U) \neq 0 \).

\( c) \) \( G \) is free of rank two.

\[ \text{Proof.} \] \( (a) \Rightarrow (b) \) is obvious.

\( (b) \Rightarrow (c) \). Let \( \{x, y\} \) be a generating set of \( G \). Then we have an exact sequence of left \( ZG \)-modules

\[ \oplus_R ZG \to ZG^2 \to ZG \to Z \to 0, \]
where $R$ is the set of relators which have a nonzero left Fox derivative in $ZG$. By Proposition [6.4] with $n = 1$, we see that $R$ is empty, and that the augmentation ideal is left $ZG$-free on $x - 1$ and $y - 1$.

A result of Bass-Nakayama [21, Proposition 1.6] then says that $G$ is freely generated by $x$ and $y$. This can be seen geometrically, as follows. Let $\Gamma = \Gamma(G, \{x, y\})$ denote the Cayley graph of $G$ with respect to the subset $\{x, y\}$. The above exact sequence is precisely the augmented cellular $Z$-chain complex of $\Gamma$. It is then straightforward to show that $\Gamma$ is a tree, and that $G$ is freely generated by $x$ and $y$.

(c) $\Rightarrow$ (a) is straightforward.

6.10 Corollary. The following are equivalent.

(a) $G$ is a two-generator left-orderable group and $b_1^{(2)}(G) \neq 0$.
(b) $G$ is free of rank two.

6.11 Theorem. Suppose that Hypotheses 6.1 hold. If $\text{hd}_Z G \leq 1$ then every two-generator subgroup of $G$ is free.

Proof. Since the hypotheses pass to subgroups, we may assume that $G$ itself is generated by two elements, and it remains to show that $G$ is free.

We calculate $H_*(G; U)$ in the case where $G$ is not free.

By Hypotheses 6.4 $G$ is torsion free. As in Remark 1.1 if $H_0(G; U) \neq 0$, then $G$ is free of rank zero. Thus we may assume that $H_0(G; U) = 0$.

By Proposition 6.9 if $H_1(G, U) \neq 0$, then $G$ is free of rank two. Thus we may assume that $H_1(G; U) = 0$.

Since $H_n(G; U) = 0$, we have an exact sequence of projective left $U$-modules

$$0 \longrightarrow U \otimes ZG P \longrightarrow U^2 \longrightarrow U \longrightarrow 0.$$  

This sequence splits, and we see that $U(U \otimes ZG P)$ is finitely generated.

Hence $ZG P$ is finitely generated, by the following standard argument. Let $R$ be a set such that $P$ is a $ZG$-summand of $\oplus_R ZG$, that is, $P$ is a $ZG$-submodule of $\oplus_R ZG$ and we have a $ZG$-linear retraction of $\oplus_R ZG$ onto $P$. We may assume that $R$ is minimal, that is, for each $r \in R$, the image of $P$ under projection onto the $r$th coordinate is nonzero. Then $U \otimes ZG P$ is a $U$-submodule of $\oplus_R U$, and here also $R$ is minimal. Since $U(U \otimes ZG P)$ is finitely generated, $R$ is finite, as desired.

Now $ZG Z$ has a resolution by finitely generated projective left $ZG$-modules. By [2] Theorem 4.6(c), $\text{cd}_Z G \leq 1$: in essence, $ZG \omega$ is finitely related and flat, and is therefore projective. Since $G$ is torsion free, $G$ is free by Stallings’ Theorem; see Remark II.2.3(ii) (or Corollary IV.3.14) in [8].

6.12 Corollary. Suppose that $G$ is locally indicable, or, more generally, that $G$ is left orderable. If $\text{hd}_G \leq 1$ then every two-generator subgroup of $G$ is free.
We now turn from two-generator groups to two-relator groups.

6.13 Proposition. Suppose that $G$ is left orderable, that $G$ has a presentation $(X \mid R)$ with $|R| = 2$, and that $cd G \geq 3$.

Then $b_0^{(2)}(G) = 0$, $b_1^{(2)}(G) = |X| - 2$, and $b_2^{(2)}(G) = 0$.

Proof. The given presentation of $G$ yields an exact sequence of $\mathbb{Z}$-modules

$$\cdots \rightarrow \oplus Y \mathbb{Z}G \xrightarrow{A} \mathbb{Z}G^2 \xrightarrow{B} \oplus X \mathbb{Z}G \xrightarrow{C} \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0.$$  

Then $H_*(G, \mathcal{U}(G))$ is the homology of the sequence

$$\cdots \rightarrow \oplus Y \mathcal{U}(G) \xrightarrow{A} \mathcal{U}(G)^2 \xrightarrow{B} \oplus X \mathcal{U}(G) \xrightarrow{C} \mathcal{U}(G) \rightarrow 0. \tag{6.13.1}$$

Since $G$ is left orderable, $G$ is torsion free. Since $cd G \neq 0$, $G$ is non-trivial. Hence $G$ has an element of infinite order. By Remark 1.1 $b_0^{(2)}(G) = 0$ and the $\mathcal{U}(G)$-dimension of ker $C$ in (6.13.1) is $|X| - 1$.

Since $G$ is left orderable, all nonzero elements of $CG$ are invertible in $\mathcal{U}(G)$ by Theorem 3.3. Since $cd G \geq 3$, $b_2^{(2)}(G) = 0$ by Corollary 6.6. Moreover, by Lemma 6.2 the $\mathcal{U}(G)$-dimension of im $B$ in (6.13.1) is one.

Finally, $b_1^{(2)}$ is the difference between the $\mathcal{U}(G)$-dimensions of ker $C$ and im $B$ in (6.13.1), that is, $|X| - 2$. Of course, the hypotheses clearly imply that $|X| \geq 2$. \hfill $\square$

Suppose that $G$ is a left-orderable two-relator group. We know the first three $L^2$-Betti numbers of $G$ if $cd G \geq 3$ by Proposition 6.13. If $cd G \leq 1$, then $G$ is free, and again one knows the $L^2$-Betti numbers. There remains the case where $cd G = 2$; here all we know are the $L^2$-Betti numbers of torsion-free surface-plus-one-relation groups; these groups are left-orderable by [12, Theorem 2.2] and they are clearly two-relator groups.

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