Genuinely multipartite entangled states in higher dimensions: a generalization of balancedness

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Abstract
I generalize the concept of balancedness to qudits with arbitrary dimension $d$. It is an extension of the concept of balancedness by Osterloh and Siewert (2010 New J. Phys. 12 075025). At first, I define maximally entangled states as being the stochastic states (with local reduced density matrices $I/d$ for a $d$-dimensional local Hilbert space) that are not product states and show that every so-defined maximal genuinely multi-qudit entangled state is balanced. Furthermore, all irreducibly balanced states are genuinely multi-qudit entangled and are locally equivalent with respect to $\text{SL}(d)$ transformations (i.e. the local filtering transformations) to a maximally entangled state. In particular the concept given here gives a condition that all maximal genuinely multi-qudit entangled states for general local Hilbert space dimension $d$ have to satisfy. A general genuinely multi-qudit entangled state is an element of the partly balanced $\text{SU}(d)$-orbits.

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1. Introduction

Entanglement is one of the cornerstones of quantum information theory and it gains an increasing importance in every branch of physics. In a work from 2000 [2] Vidal has listed the minimal requirements that a measure of entanglement has to satisfy, and called such a quantity an entanglement monotone. In particular, it has to be invariant under the local unitary group, and it must not increase under arbitrary local operations, considering also classical communications (LOCC). The transformations that entanglement has to be invariant with have recently been enlarged to the stochastic version of LOCC, the SLOCC, and the relevant
invariance group is the SL [3], or the complex representations of the local unitary group [4]. This has given rise to more detailed analysis of the SL invariant measures of entanglement [5–15]. Concerning the entangled states, it was observed immediately that they satisfy a ‘center of mass’ condition for pure states of two, and three qubits [5]. This condition paved the way towards the balancedness condition, a condition which roughly states that the number of times the local state $|0\rangle$ appears, has to be equal to that of the occurrence of a local $|1\rangle$ on each qubit. It has been demonstrated to be due to the underlying SL symmetry, and that every pure genuinely multi-qubit entangled state satisfies these conditions [1]. In the meantime a generalization of the method of local antilinear operators in [12, 14] to qudits of dimensions 3 and 4 has appeared in [16], and the question of how this generalization would be transported into some condition for the genuinely multi-particle entangled states was natural. I introduce here this generalization of the concept of balancedness to higher local dimensions. We will see that, roughly speaking, each of the numbers $d_0,...,2d_3$ which represent the $+S_2^2$ eigenstates of a local spin $S$, must occur the same number of times in order for a state being possibly maximally entangled. These are the states that are prescribed by the symmetry as genuinely many-qudit entangled.$^1$

The outline of the article is as follows. After giving a brief introduction into the balancedness condition for many qubits in section 2, a definition of maximally genuinely entanglement will be presented in section 3. Then, a generalization of the balancedness conditions is elaborated in section 4 for states of many qudits, which are the generalized balanced states in this case. The conclusions are drawn in section 5.

2. Balancedness for qubit states

It has been observed in [5] that for the three-tangle

$$\tau_3 = 4 \left| d_1 - 2d_2 + 4d_3 \right|$$

$$d_1 = \psi_{000}^2 \psi_{111}^2 + \psi_{001}^2 \psi_{110}^2 + \psi_{010}^2 \psi_{101}^2 + \psi_{100}^2 \psi_{011}^2$$

$$d_2 = \psi_{000} \psi_{111} \psi_{011} \psi_{000} + \psi_{000} \psi_{111} \psi_{101} \psi_{010} + \psi_{011} \psi_{110} \psi_{010} \psi_{010}$$

$$d_3 = \psi_{000} \psi_{110} \psi_{011} \psi_{010} + \psi_{111} \psi_{001} \psi_{010} \psi_{100}$$

(1)

the components of the $d_i$ correspond to a line ($d_1$), a rectangle ($d_2$), or a tetrahedron ($d_3$) with the property, that their ‘center of mass’ coincides with that of the underlying cube. For two-qubit Bell states the same is valid for a two-dimensional square. Here, $\psi_{ijk}$ are the coefficients of the three qubit wavefunction $\ket{\psi} = \sum \psi_{ijk} \ket{i,j,k}$. The generalization of this observation is called balancedness [1], and it stems from the local invariant operator, which is $\epsilon_{ij}$ in the case of qubits. It is also known as the spinor metric. In other words, every spin-singlet will be a candidate for a maximally genuinely many-qubit entangled state. Interestingly, the ground states of diverse spin models are singlet states [19–21]. The simplest SL(2) invariant two-qubit measure would then be the determinant of the state, when written in matrix form

$^1$ There are two independent notions for the word ‘genuinely multipartite entanglement’ in the literature. The first goes back to the work [17] and means not bipartite, whereas the notion used in this work follows [14] and means not bipartite and distinguishable by a non-zero SL(N)$^{\mathbb{R}}$-invariant. Non bipartite states of the null-cone, like the W-states, make the difference [18].
where we imposed Einstein sum convention. Key ingredients are the binary matrix $B$ or, equivalently, the alternating matrix $A$. For example, the state $\alpha_1 |111\rangle + \alpha_0 |000\rangle$ for $\alpha_i \neq 0$ has the binary matrix

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix},$$

and the corresponding alternating matrix is

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

The balancedness condition is then written as

$$\sum_{l=1}^{L} A_{il} n_l = 0,$$  \hspace{1cm} (2)

where $n_l \in \mathbb{N}_+$ and for all $i = 1, \ldots, q$, where $q$ is the number of qubits. The number $l$ is running from 1 to $L$, which is the length of the state. Here, $L$ is to be understood as the minimum length the state can have on its local SU(2) orbit, which is an invariant with respect to SL(2) as well, and therefore balancedness is also an SU(2) invariant concept. We refer to [1] for the details, but we want to notice here that every state whose form of minimal length is balanced is genuinely multi-qubit entangled or a product of genuinely entangled $q_i$ qubit states ($q_i \geq 2$) such that $\sum q_i = q$; every state whose form of minimal length is irreducibly balanced is genuinely multi-qubit entangled. Furthermore, every maximal genuinely multi-qubit entangled state is balanced. Also, every (irreducibly) balanced state is proven to be SL equivalent to a stochastic state. This means that there are different classes of genuinely entangled states, depending whether they have a form of minimal length that is (irreducibly) balanced, or which contains still an unbalanced part—the partly (irreducibly) balanced states. The difference of the two is that whereas the (irreducibly) balanced states are SL equivalent to a stochastic state, this is only true for the partly (irreducibly) balanced states after an infinite sequence of local filtering operations.

3. Maximally entangled states

At the beginning we want to define what we consider a maximally entangled $q$ qudit state.

**Definition 3.1.** (Maximal genuine multi-qudit entanglement)

We call a pure $q$-qudit state $|\psi_q\rangle$ maximal genuine multipartite qudit entangled, iff

1. The state is not a product, i.e. the minimal rank of any reduced density matrix of $|\psi_q\rangle$ is 2.
2. It is stochastic [9], that is, its reduced single qudit density matrices are all equal to $\mathbb{1}_{d}/d$ for a qudit of dimension $d$.

\[2\] For a precise definition of the binary and alternating matrix, see section 3.2 in [1].
This definition is a direct generalization of the one used in [1] and is widely accepted in the community. It follows the principle that a maximally entangled state has maximal local uncertainty.

An additional requirement has been put forward in [22], demanding also arbitrary reduced density matrices to be maximally mixed. That this is a too strong requirement has been proven in [23]. Thus, this requirement has been split up into a hierarchy [24–26] of \(k\)-uniform states. Here, we only deal with the first requirement in this hierarchy, the stochastic states, which are the 1-uniform states. It is clear that also the rank of many-site reduced density matrices is a discrete and hence non-polynomial invariant of the underlying group, but this subject remains an open question for future work on this subject.

We will demonstrate below that maximally and genuinely multi-qudit entanglement is furthermore directly connected to SL-invariance, as it turned out for qubits [1]. Please note that here, we do not consider entanglement of fewer than \(q\) particles, as is the case for \(W\) type of states for qubit systems, nor an entanglement that does not fill all the Hilbert space, in the sense that some local density matrix is \(1/d'\), with \(d' < d\). These types of entanglement only range over part of the available resources. These resources are already classifiable by some corresponding SL-invariant measures of fewer particles, as e.g. the \(W\) state in qubit systems is already classified by the concurrence. In the second case, the entanglement can be classified by SL-invariants of a smaller Hilbert space dimension \(d' < d\).

4. Maximally entangled states predicted by the symmetry

We have the determinant of the two-qudit state when written in matrix form, \(\ket{\psi} = (\psi^{(i)})_{ij}\), as the simplest polynomial SL-invariant measure [16, 27, 28], and all SL-invariant measures are constructed from its local invariant operators (see [16] for dimensions 3 and 4) that are connected to \(e_{ii}^{(i)}\). This determinant is a combination of elements of a \(d\)-dimensional square matrix, such that each row and column occurs precisely once. We want to stress here that this means that each element of the local Hilbert space, \(|0\rangle\) to \(|S\rangle\), is occurring once. This will define the generalization of the term ‘balancedness′ for qubits to an arbitrary dimension of the local Hilbert space dimension, hence arbitrary qudits. We want to add that the balancedness condition for qubits [1] also tells that each element, \(|0\rangle\) and \(|1\rangle\), is occurring with exactly the same multiplicity for each qubit. Hence, the corresponding logic applies for qudits as well as for qubits. We now have to cast this observation into an equation or a set of equations, which will be the generalized balancedness conditions each genuinely multi-qudit entangled state has to satisfy.

To this end, we consider a pure \(q\) qudit state of local Hilbert space dimension \((2S + 1)\)

\[
\ket{\psi} = \sum_{i_1, \ldots, i_q=1}^{d=2S+1} \psi_{i_1, \ldots, i_q} \ket{i_1, \ldots, i_q}.
\]

The state is represented in an orthogonal product basis. What we ask for, is first an analogue of the binary matrix \(B_{ik}\) from [1] where the weights are not written due to the underlying SL(\(d\)). We call the number of orthogonal product vectors the length \(L\) of this representation. E.g. for the single site state of a qutrit \((S = 1)\) \(\ket{\Psi} = \alpha_0 \ket{00} + \alpha_1 \ket{11} + \alpha_2 \ket{22}\) this would mean, \(B_{00} = (0, 1, 2)\), and length \(L = 3\). For a 2 qutrit state \(\ket{\Psi} = \alpha_0 \ket{00} + \alpha_1 \ket{11} + \alpha_2 \ket{22}\) it takes the form
which has also length \( L = 3 \).

For representing the condition that each basis state has to occur the same number of times, note that the \( n \)th roots of unity sum up to zero, \( \sum_{j=1}^{n} e^{i2\pi j/n} = 0 \). Therefore, one idea is to take a state in the local Hilbert space of spin \( S/2 \), and attribute to it precisely one number out of \( \{0,\ldots,2S\} \). If \( S/2 \) is prime, this works without parts of the sum yet being zero. This leads to the condition for spin \( S \) (\( S/2 + 1 \) prime, e.g. \( S = 1/2, 1, 2, 3, 5, 6, \ldots \))

\[
\exists \, n_1, \ldots, n_L \in \mathbb{N}_+ \ni \sum_{j=1}^{L} \sum_{k=1}^{q} e^{i2\pi j/n_k} = 0 \quad \forall \, j = 1, \ldots, q
\]

with \( q \) being the number of qudits and \( L \) being the length of the state. This condition defines generalized balancedness for \( S/2 + 1 \) being prime.

At first we briefly show that for spin-1/2, the above exponential gives precisely the alternating matrix. In this case the condition reads \( \sum_{j=1}^{L} e^{i2\pi 0.5 n_j} = \sum_{j=1}^{L} A_{jk} n_k = 0 \). The maximally entangled state for this case is [1]

\[
|\psi\rangle = \sum_{k=1}^{L} \sqrt{n_k} \prod_{j=1}^{q} \left| B_{jk} \right>,
\]

whose length is \( L = \sum_{k=1}^{q} n_k \). One could hence think of a matrix \( A_{jk} \) which is

\[
A_{jk} = e^{i2\pi j/n_k}.
\]

For \( S/2 + 1 \) not being prime, one still has the \( S/2 \) conditions that every two occurrences appear the same number of times. For these partial conditions let e.g. \( (A_{ij}^{(j+1)})_{ik} \) be the matrix whose entry is \(-1 \) (1) if the \( k \)th qubit of the \( i \)th basis state in \( |\psi\rangle \) is \( j \) (\( j+1 \)) and 0 if it has some other value. Thus,\n
\[
\exists \, n_k \in \mathbb{N}_+ \ni \sum_{k=1}^{L} n_k^{(j)} (A_{ij}^{(j+1)})_{ik} = 0 \quad \forall \, i = 1, \ldots, q; \quad \forall \, j = 0, \ldots, 2S.
\]

These are \( 2S \) conditions per qudit. We want to outline that the \( A_{ij}^{(j+1)} \) can be added up to yield \( A_{ij}^{(j+2)} \) in the following way

\[
A_{ij}^{(j+1)} + A_{ij}^{(j+1, j+2)} = A_{ij}^{(j+2)},
\]

\[
A_{ij}^{(j+1)} + A_{ij}^{(j+1, j+2)} = A_{ij}^{(j+2)}.
\]

To see what this means, we write down explicitly \( (A_{ij}^{(j+1)})_{ik} \) for the maximally entangled two qutrit state from equation (4)

\[
A_{ij}^{(0, 1)} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad A_{ij}^{(1, 2)} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.
\]

This leads to an \( n^{(i)} = (n_1^{(i)}, n_2^{(i)}, n_3^{(i)}) \) as follows

\[
n^{(0)} = (1, 1, 2) \quad \text{and} \quad n^{(1)} = (n, 1, 1).
\]

We therefore have to choose the free integers to be \( m = n = 1 \) and hence \( n^{(i)} = n = (1, 1, 1) \) for \( j = 0, 1 \). Notice that on every qudit one can choose to make a basis change. Permutations
are just examples for such a change of basis. Since the balancedness condition acts on every singlet qudit, this means that $n_k^{(i)} = n_k$ holds in general.

It must be stressed that an arbitrary solution for the $n_k$ with non-negative integer numbers can also be realized by writing the $k$th state with a multiplicity of $n_k$, i.e. $n_k$ number of times.

Thus, we finally have

$$\exists \ n_k \in \mathbb{N}_+ \ni \sum_{k=1}^{L} n_k \left( A_{\psi}^{(i,j+1)} \right)_k = 0 \quad \forall \ i = 1, \ldots, q; \ \forall \ j = 0, \ldots, 2S.$$  (13)

The corresponding genuinely multi-qudit entangled state again is

$$|\psi\rangle = \sum_{k=1}^{L} \sqrt{n_k} \prod_{i=1}^{q} |B_{ik}\rangle,$$  (14)

the $n_k$ are chosen to be relatively prime.

This leads to the following definitions, which are directly transcribed from [1].

**Definition 4.1. (Balancedness)**

1. A pure SU-orbit of a spin-$S$ state is called balanced, iff the conditions (13) are satisfied for every state in the orbit.
2. Let $|\psi\rangle$ be a balanced SU-orbit. It is called irreducibly balanced iff for a state in its form of minimal length it cannot be split into different balanced parts.
3. An SU-orbit is called partly balanced if some (but not all) of the $n_k$ are admitted to be zero. A partly balanced state is called reducible/irreducible iff its balanced part is reducible/irreducible.

We have to define also what we mean when calling an SU-orbit ‘unbalanced’.

**Definition 4.2. (Complete unbalancedness)**

We call a state out of an SU-orbit ‘unbalanced’ if it is locally unitarily equivalent to a state without balanced part.

The next theorem states the well-definedness of the concept.

**Theorem 4.1.** Product states are not irreducibly balanced.

In the proof for this theorem the only modification is to replace $mn/2$ in [1] by $mn/(2S + 1)$. We hence do not show it here, and refer to [1] instead.

Also the next theorem translates directly to the higher dimensional version of spin-$S$.

**Theorem 4.2.** Every stochastic state (local reduced density matrices equal to $1/(2S + 1)$ for spin $S$) is balanced.

**Proof.** Consider a $q$-qudit state $|\psi\rangle$ that is stochastic. We can write the state in the form

$$\begin{pmatrix}
  w_1 & \ldots & w_m & w_{m+1} & \ldots & w_{m+n+\ldots+1} & \ldots & w_{L-m-n-\ldots} \\
  0 & \ldots & 0 & 1 & \ldots & 1 & \ldots & 2S & \ldots & 2S \\
  \phi_1 & \ldots & \phi_m & \phi'_1 & \ldots & \phi'_{m-1} & \ldots & \phi_{L-m-n-\ldots}^{(2S)} & \ldots & \phi_{L-m-2S-n-\ldots}^{(2S)}
\end{pmatrix},$$  (15)
with weights $w_i \in \mathbb{C}$. Let some of the states out of $\Phi_i$, $\Phi'_i$, $\Phi^{(2S)}_i$ coincide and call their superposition $\Psi_i$, and their complements $\Phi^c_i$, $\Phi'^c_i$, $\Phi^{(2S)}_i$. The state is then written as $(|0\rangle + \alpha |1\rangle + \ldots + \alpha^{(2S)}|2S\rangle) \otimes |\Psi_i\rangle + |\beta |0\rangle \otimes |\Phi^c_i\rangle + \ldots + \beta^{(2S)}|2S\rangle \otimes |\Phi^{(2S)}_i\rangle$. Since the local density matrix is proportional to the identity matrix in every basis, we find that $\alpha = 0$, $\ldots$, $\alpha^{(2S)} = 0$. Hence, the $\Phi_i$, $\ldots$, $\Phi^{(2S)}_i$ are orthogonal in pairs. Defining $\rho_{ij} = w_i |i\rangle |j\rangle$, we find

$$\rho^{(1)} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} + \sum_{i \in I_i} p_i I_i$$

where $I_i$ are the column numbers such that the $j$th qudit takes value $k = 0, \ldots, 2S$. These sums of $p_i$ have to be equal in pairs. The conditions we get out of (16) are the same as those for balancedness.

For the maximal length of an irreducibly balanced state, we have

**Theorem 4.3.** Every balanced $q$-qudit state of spin-$S$ and length larger than $2Sq + 1$ is reducible.

**Proof.** Balancedness means the existence of integers $n_k$, $k = 1, \ldots, L$, such that (13) is satisfied. Here we must demonstrate that every state of length $L > 2Sq + 1$ must be reducible if balanced. Add a vertical cut $\kappa \cup \kappa' = I$ with $|\kappa|, |\kappa'| \geq \frac{2Sq + 1}{2}$. Define $\tilde{a}_f^K := (a_{i,1}^K, \ldots, a_{i,q}^K)$ such that $a_{i,1}^K = \sum_{k \in \kappa} n_k (A_{ik}^{(j+1)})_k$. Irreducibility means that $a_{i,1}^K \neq 0$ for all $i = 1, \ldots, q$ and some $j = 1, \ldots, 2S$. Now make a cut $\kappa$ and $\kappa'$ in $\kappa$ and $\kappa'$ and define $\kappa := (\kappa \setminus \kappa) \cup \kappa'$. Including arbitrary positive integers $m_k$, $k \in \kappa'$, and keeping $m_k = n_k$ for $k \in \kappa$, one finds

$$\sum_{k \in \kappa} m_k (A_{ik}^{(j+1)})_k = a_{i,1}^K - \sum_{k \in \kappa} m_k (A_{ik}^{(j+1)})_k + \sum_{k \in \kappa'} m_k (A_{ik}^{(j+1)})_k.$$

Irreducibility implies that for arbitrary such subsets $\kappa$ and $\kappa'$ no integer numbers $\tilde{m}_k \in \mathbb{Z}^{\kappa + |\kappa'|}$ do exist such that $\sum_{k \in \kappa \cup \kappa'} \tilde{m}_k (A_{ik}^{(j+1)})_k = a_{i,1}^K$ for all $i \in \{1, \ldots, q\}$ and all $j \in \{1, \ldots, 2S\}$. Without loss of generality $(A_{ik}^{(j+1)})_k \in \mathbb{C}$ has rank $q$ (due to a suitable choice of $\kappa$ and $\kappa'$). This implies that each of the at most $2Sq$ conditions can be satisfied, given that we can have up to $2Sq$ variables.

Finally, we have a one-to-one correspondence of irreducibly balanced states and stochastic states, which have all single-qudit reduced density matrices proportional to the identity.

**Theorem 4.4.** Every (irreducibly) balanced state of length $L \leq 2Sq + 1$ (or $q^2 + 1$ for $2S > q$) is equivalent under local filtering operations (LFO) $SL(2S + 1, \mathbb{C})^{\otimes q}$ to a stochastic state.

**Proof.** Let $a_j$, $j = 1, \ldots, L$ be the amplitudes of the product state written in the $j$th column of $B_q$. Let the LFO be
$$T_{\text{LFO}}^{(i)} = \text{diag}\left\{ \frac{t_{0,i}}{t_{0,i}}, \ldots, \frac{1}{t_{2S-1,i}} \right\}, \quad (17)$$

with $t_{k,i} := t^{z_{k,i}}$ for some positive $t$, $z_{k,i} \in \mathbb{C}$ and $\forall \ i = 1, \ldots, q$, and diag the diagonal matrix with given entries. It is to mention that the $k$th entry has exponent $z_{k,i} = z_{k-1,i}$, $z_{-1,i} := z_{2S} := 0$, for every $k$. We must demonstrate that after suitable such operations all the $a_j$ can be equalized. Let the $n_k = 1$ for all $k = 1, \ldots, L$ without loss of generality and $B_{i1} = 0$ for all $i = 1, \ldots, q$. Then, after the LFO we get

$$a_j \sum \left( z_{B_{0,i}} - z_{B_{0,i-1}} \right) = a_j \phi_j$$

and a following division by $a_0 \phi_0$ of the amplitudes, we get

$$\frac{a_j}{a_0} \sum_{i=1}^q \left( z_{B_{0,i}} - z_{B_{0,i-1}} \right) - \sum_{i=1}^q z_{n_i}.$$ 

Equality for all these quantities therefore means

$$\log \frac{a_0}{a_j} = \sum_{i=1}^q (z_{B_{0,i}} - z_{B_{0,i-1}} - z_{0,i}).$$

These are at most $2S\hat{q}$ independent variables due to the balancedness. Since we have $L - 1$ conditions for the amplitudes to be satisfied, this results in a total length of at most $L = 2S\hat{q} + 1$ amplitudes for irreducibly balanced states which can be equalized. For $2S > q$ we obtain at most $q^2$ independent variables and hence a maximal length of $L = q^2 + 1 < 2S\hat{q} + 1$. The resulting state is stochastic. $\square$

It is important to notice that the above proof works as well, if the state is only reducibly balanced, and of length shorter than, or equal to, $2S\hat{q} + 1$.

As the essence of this work, I will prove the following theorem, which states that every irreducibly balanced state belongs to the class of semi-stable states, hence it is detected by some SL-invariant. A representative of the irreducibly balanced state, namely its normal form in the sense of [9] whose reduced single-site density matrix is $1^{(2S+1)/(2S+1)}$, is stochastic and hence is maximally entangled, according to definition 3.1.

**Theorem 4.5.** Every (irreducibly) balanced state is stable, hence it is not in the zero-class with respect to SLOCC transformations. That means they are robust against infinitely many LFO’s in $\text{SL}(2S+1)^{\otimes q}$ and possess a finite normal form [9].

**Proof.** We have to show that no LFO can exist that reduces the whole state, hence each of the $L$ product states in the superposition. An LFO of this type, when applied infinitely many times, will eventually annihilate the state. Hence, it is sufficient to demonstrate that one of the product states in superposition always gains a factor larger than one. The state is assumed already in its normal form in which it is (irreducibly) balanced. Hence, the remaining class of LFO’s is given by equation (17).

Balancedness means that each state from $|0\rangle$ until $|2S\rangle$ is occurring exactly the same number of times $p_i$ for each qudit $i \in \{0, \ldots, q\}$. We assume that all the $n_k = 1$ in equation (13) at first. This leads to the product of the $L$ factors being equal to
distributed among the $L$ product states in the superposition as $r_j$ for $j \in \{1, \ldots, L\}$ such that $\prod_{j=1}^{L} r_j = 1$. This holds for every single qudit. Applied this for each qudit, one ends up with a product of the $L$ factors of

$$1 = \left[ \begin{array}{cccc} t_0 & t_1 & \ldots & t_{2S-1} \\ t_0 & t_0 & \ldots & t_{2S-2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{2S-1} & t_{2S-1} & \ldots & t_{2S-1} \end{array} \right]^{p_j}$$

distributed among the $L$ product states in the superposition just as mentioned above. Now we assume all factors $r_j$ in front of the $(L - 1)$ product states in the superposition to be smaller than one. Then, the factor in front of the $L$th product state will be

$$r_L = \left( \prod_{j=1}^{L-1} r_j \right)^{-1} > 1.$$ 

Since the all over product has to be equal to one.

Now we let the $n_k \in \mathbb{N}_{+}$, and relabel the product states in the superposition such that, without loss of generality, $n_k = \min \{n_k; k = 1, \ldots, L\}$. Then, the same line of thought applies also here, and we have that the $L$th product state is multiplied by a number $r_L$ which is larger than one.

It is to be mentioned of course that the above proof applies to an arbitrary balanced state, even if it is only partly balanced in its normal form. The maximally entangled states for qudits formally coincide with equation (6) for qubits.

This is the essential theorem of this work, because it directly connects (partly) balancedness with its relation to stochasticity, and the (semi)-stablness of the state with respect to $\text{SU}(d)^{\otimes q}$ trasformations. Thereby, every state which is partly balanced is connected to its balanced mother state, which is SL equivalent to a stochastic state. If the state is not a product, it contains hence genuine entanglement of dimension $d = 2S + 1$ and $q$ qudits due to definition 3.1, and it is detected by some $\text{SL}(d)^{\otimes q}$-invariant. Consequently, the phrases ‘A state contains some genuine entanglement of dimension $d$ and of $q$ qudits following definition 3.1’, and ‘A state is detected by some $\text{SL}(d)^{\otimes q}$-invariant and is not a product’, are equivalent.

5. Conclusions

The concept of generalized balancedness with respect to that for qubits in [1] is introduced in terms of equal occurrence of each qudit basis state. It is an SU invariant concept, as for qubits, in that a state is called (irreducibly) balanced if it has in its SU orbit a state of minimal length that is (irreducibly) balanced. The well-definedness of this concept is demonstrated in showing that no product state can be an irreducible balanced state.

This paves the way towards a classification of entanglement for arbitrary Hilbert-space dimension and number of constituents, defining as maximally entangled of genuinely $d$ dimensions and also of genuinely $q$ constituents the stochastic states [9] from definition 3.1.

Based on this stochasticity of maximally entangled states we manage to prove that stochasticity implies balancedness and that every (irreducible) balanced state with minimal length $L \leq 2Sq + 1$ is SL-equivalent to a stochastic state. Finally it is proven that all states that are (reducibly) balanced are robust against SLOCC transformations. It must be
highlighted that also those states which fall only partly into this scheme (hence having a (irreducibly) balanced and an unbalanced part) belong to the class of semi-stable states. They are all detected by some $\text{SL}(d)$ invariant measure of entanglement and are thus entangled of genuinely dimension $d$ and of genuinely $q$ qudits, in contrast to the unstable states that are in the $\text{SL}(d)^{\otimes q}$ null-cone. These states can have an entanglement of a lower dimension $d' < d$ and/or a lower number of qudits $q' < q$.

The generalization is given for arbitrary local dimension. The normal form of an arbitrary genuinely many-qudit entangled states with minimal length is (irreducibly) balanced plus eventually some unbalanced part, which applies to qubits as well.

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