The General Self-dual solution of the Einstein Equations

Sucheta Koshti† and Naresh Dadhich∗

Inter-University Center for Astronomy and Astrophysics,
Pune-411007, India.

Abstract

We obtain the most general explicit (anti)self-dual solution of the Einstein equations. We find that any (anti)self-dual solution can be characterised by three free functions of which one is harmonic. Any stationary (anti)self-dual solution can be characterised by a harmonic function. It turns out that the form of the Gibbons and Hawking multi-center metrics is the most general stationary (anti)self-dual solution. We further note that the stationary (anti)self-dual Einstein equations can be reinterpreted as the (anti)self-dual Maxwell equations on the Euclidean background metric.

∗ nkd@iucaa.ernet.in, † sucheta@iucaa.ernet.in
1 Introduction

Real (anti)self-dual solutions are the solutions of the Euclidean Einstein equations (EE) having (anti)self-dual[(A)SD] Riemann tensor. In Quantum gravity, (A)SD solutions are significant, for, they correspond to saddle points of the Einstein-Hilbert action, therefore giving large contributions to a path integral over Euclidean metrics [1]. They can be interpreted as “one particle states” in quantum gravity [2]. In four dimensions, the necessary condition for a metric to be hyperkähler is that, it must have either SD or ASD curvature tensor [3].

There have been several attempts made to tackle the problem of constructing the general (A)SD solution of the Einstein equation. Ashtekar [4] has simplified the self-dual Einstein equations (SDE) by using the new Hamiltonian variables for general relativity. In terms of Ashtekar’s new Hamiltonian variables, SDE may be rewritten as evolution equations for three divergence free vector fields given on a three dimensional surface with a fixed volume element. Penrose [2] has used twistor techniques and shown that, in principal, one can construct the general SD solution but the problem of constructing the Penrose’s deformed twistors is as difficult as solving the partial differential equations. Grant [5] has shown that any SD metric can be characterised by a function that satisfies a non-linear evolution equation, to which the general solution can be found iteratively. Grant’s general SD solution is formal and not explicit. In this paper our main motivation is to find explicitly, the most general (A)SD solution of the EE.

(Anti)Self-duality of the Riemann tensor automatically ensures the van-
ishing of the Ricci tensor, and so they are solutions of the vacuum EE with vanishing cosmological constant. In four dimensions, the Riemann tensor is (A)SD iff the curvature 2-form $R^a_{\ b}$ is (A)SD. The (A)SD 2-form $R^a_{\ b}$ can be considered to come from a (A)SD connection $\omega^a_{\ b}$ by choosing an appropriate gauge called “(A)SD gauge”. Therefore one way to find the general (A)SD solution of the EE is to write the most general form of the metric and demand for the (anti)self-duality of its O(4) Levi-Civita connection. This way the problem reduces to solving first order partial differential equations.

In this paper we find the general (A)SD solution of the vacuum EE by demanding the (anti)self-duality of the O(4) Levi-Civita connection of the most general metric.

Here we work in four dimensions and with the Euclidean metric signature, for, real, nontrivial (A)SD solutions exist only in the Euclidean signature. In section 2, we write the general form of the metric in terms of six unknown functions of four variables $(t, x, y, z)$ and show that any (A)SD solution can be characterised by three free functions of which one is harmonic. In section 3, we show that any stationary (A)SD solution can be characterised by one free harmonic function. It is also demonstrated that the stationary (A)SD solutions of the EE can be reinterpreted as the static (A)SD Maxwell solutions on the Euclidean background metric. Section 4 deals with some particular (A)SD solutions and gives a strategy to generate non-stationary (A)SD solutions from the stationary (A)SD solutions. Conclusions are summarised in section 6.
2 Construction of the general (A)SD solution of the vacuum EE.

In four dimensions, the general metric is characterised by ten functions, $g_{ij}$'s of the four coordinates. All the ten $g_{ij}$'s are not independent but they are subject to four coordinate conditions, the Bianchi identities. Therefore there are only six independent $g_{ij}$. By means of coordinate transformations, we can arbitrarily assign four of the ten $g_{ij}$, provided it does not lead to a reduction in the dimensions. Therefore without loss of generality we can assume the form of the most general metric as follows.

$$ds^2 = u^{-1}(dt + A \cdot dx)^2 + v_1dx^2 + v_2(dy^2 + dz^2).$$

where $u, v_1, v_2$ and $A_a, a = 1, 2, 3$ are functions of $t, x, y, z$. We chose the vierbein or the tetrad:

$$e^a_\mu = \{u^{-\frac{1}{2}}(dt + A \cdot dx), v_1^{\frac{1}{2}}dx, v_2^{\frac{1}{2}}dy, v_3^{\frac{1}{2}}dz\},$$

where $a$ and $\mu$ are the tetrad and spacetime indices respectively which run from 0 to 3. For the metric (1), the $O(4)$ Levi-Civita connection compatible with $e^a$ is given by,

$$\omega^0_1 = -\frac{1}{2} e^0[u^{-\frac{1}{2}}v_1^{-\frac{1}{2}}(u, x - \dot{u}A_1) + v_1^{-\frac{1}{2}}\dot{A}_1] - \frac{1}{2} e^1[u^{\frac{1}{2}}v_1^{-\frac{1}{2}}v_1]$$

$$+ \frac{1}{2} e^2[(uv_1v_2)^{-\frac{1}{2}}((\dot{A} \times A + \nabla \times A)_3)]$$

$$- \frac{1}{2} e^3[(uv_1v_2)^{-\frac{1}{2}}((\dot{A} \times A + \nabla \times A)_2)]$$
\[\omega^2_3 = -\frac{1}{2} e^0 [u^{-\frac{3}{2}} v_1^{-1} [(\dot{A} \times A + \nabla \times A)_1]] - \frac{1}{2} e^2 [v_2^{-\frac{3}{2}} (\dot{v}_3 A_3 - v_2 z)] - \frac{1}{2} e^3 [v_2^{-\frac{1}{2}} (-\dot{v}_3 A_{2} + v_{2,y})]
\]

\[\omega^0_2 = -\frac{1}{2} e^0 [u^{-1} v_2^{-\frac{3}{2}} (u_{,y} - \ddot{u} A_{2}) + v_2^{-\frac{1}{2}} \dot{A}_{2}] - \frac{1}{2} e^1 [(uv_1 v_2)^{-\frac{1}{2}} [(\dot{A} \times A + \nabla \times A)_0]] - \frac{1}{2} e^2 [u^{-\frac{3}{2}} v_2^{-1} \dot{v}_2] + \frac{1}{2} e^3 [u^{-\frac{1}{2}} v_2^{-1} [(\dot{A} \times A + \nabla \times A)_1]]
\]

\[\omega^3_1 = -\frac{1}{2} e^0 [(uv_2 v_1)^{-\frac{1}{2}} [(\dot{A} \times A + \nabla \times A)_2]] - \frac{1}{2} e^1 [v_1^{-1} v_2^{-\frac{1}{2}} (-\dot{v}_1 A_3 + v_1 z)] - \frac{1}{2} e^3 [v_1^{-\frac{3}{2}} (\dot{v}_3 A_1 - v_2 x)]
\]

\[\omega^0_3 = -\frac{1}{2} e^0 [u^{-1} v_2^{-\frac{3}{2}} (u_{,z} - \ddot{u} A_{3}) + v_2^{-\frac{1}{2}} \dot{A}_{3}] + \frac{1}{2} e^1 [(uv_2 v_1)^{-\frac{1}{2}} [(\dot{A} \times A + \nabla \times A)_1]] - \frac{1}{2} e^2 [u^{-\frac{3}{2}} v_2^{-1} [\dot{v}_3 A_3 - v_2 y]] - \frac{1}{2} e^3 [u^{-\frac{1}{2}} v_2^{-1} \dot{v}_2]
\]

\[\omega^1_2 = -\frac{1}{2} e^0 [(uv_1 v_2)^{-\frac{1}{2}} [(\dot{A} \times A + \nabla \times A)_0]] - \frac{1}{2} e^1 [v_1^{-1} v_2^{-\frac{1}{2}} (\dot{v}_3 A_2 - v_1 y)] - \frac{1}{2} e^2 [v_2^{-1} v_3^{-\frac{1}{2}} (-\dot{v}_3 A_1 + v_2 x)] - \frac{1}{2} e^3 [v_2^{-\frac{3}{2}} (\dot{v}_3 A_1 - v_2 y)]
\]

where over dot (\dot{\cdot}) is the partial differentiation with respect to \(t\).

In our conventions, the Riemann tensor is (A)SD iff it satisfies,

\[R_{abcd} = \mp * R_{abcd} \equiv \mp \frac{1}{2} \epsilon_{abfg} R^{fg}_{\ cd}, \quad (4)
\]
where $*$ is the Hodge-dual operator and $\epsilon_{a b f g}$ is the Levi-Civita tensor. The Riemann tensor satisfies the cyclic identity $R^{a}{}_{b [c d]} = 0$. Therefore the (anti)self-duality of the Riemann tensor automatically ensures the vanishing of the Ricci tensor, and so they are solutions of the vacuum Einstein equations with vanishing cosmological constant. In four dimensions, the Hodge duality takes 2-forms to 2-forms. The most important 2-form associated with a four dimensional metric is its curvature 2-form $R^{a}{}_{b}$. Demanding the (anti)self-duality of the Riemann tensor is equivalent to demanding the (anti)self-duality of $R^{a}{}_{b}$. Any (A)SD 2-form $R^{a}{}_{b}$ can be considered to come from a (A)SD connection $\omega^{a}{}_{b}$ by choosing an appropriate gauge called “(A)SD gauge”. Suppose $R^{a}{}_{b}$ is (A)SD but $\omega^{a}{}_{b}$ is not (A)SD. Then by decomposing $\omega^{a}{}_{b}$ into SD and ASD parts and using an $O(4)$ gauge transformation, one can always remove the (A)SD part leaving $\omega^{a}{}_{b}$ SD or ASD. The only change in $R^{a}{}_{b}$ under the gauge transformation is a rotation by an orthogonal matrix which does not change its duality. The metric (1) will be (A)SD if $\omega^{a}{}_{b}$ given by (3) is (A)SD. By demanding for the (anti)self-duality of the connection given by (3), we get,

\[
\dot{v}_1 = \dot{v}_2 = 0, \tag{5}
\]

and

\[
v_1 = e^{h(x)} v_2, \tag{6}
\]

where $h(x)$ is an integration function. The function $h(x)$ can be absorbed by redefining the coordinates and without loss of generality one can assume $v_1 = v_2 \equiv v(x, y, z)$. Then the metric (1) reduces to:
\[ ds^2 = u^{-1}(dt + A \cdot dx)^2 + v(dx^2 + dy^2 + dz^2). \] (7)

Substituting \( v_1 = v_2 \equiv v(x, y, z) \) in (3), (anti)self-duality conditions on the connection \( \omega^a_{\ b} \), further leads to,

\[ \dot{A} + \Phi A - \Psi = 0, \] (8)

and

\[ \dot{A} \times A + \nabla \times A = \mp 2u^{\frac{1}{2}}\nabla v^{\frac{1}{2}}, \] (9)

where \( \Phi \) and \( \Psi \) are given by,

\[ \Phi = \partial_t(\ln u^{-\frac{1}{2}}) \] (10)

and

\[ \Psi = \nabla(\ln \frac{v^{\frac{1}{2}}}{u^{\frac{1}{2}}}). \] (11)

Substituting the vector equation (8) in the vector equation (9) we get,

\[ \Psi \times A + \nabla \times A = \mp 2u^{\frac{1}{2}}\nabla v^{\frac{1}{2}}. \] (12)

Solving the equations (8) and (9) is equivalent to solving the equations (8) and (12). Therefore solving the equations (8) and (12) simultaneously, we get,

\[ A = u^{\frac{1}{2}}[\int_0^t (u^{-\frac{1}{2}}\Psi) \, dt + f(x, y, z)], \] (13)
where $f$ satisfies the equation:

$$\nabla \times (v^\frac{1}{2} f) = \mp \nabla v.$$  \hfill (14)

The above equation implies that $v$ is a harmonic function. Note that, given a harmonic function $v$, there are many solutions of the equation (14). Given $v$, if $f$ is a solution, then $f + v^{-\frac{1}{2}} \nabla \phi$ for an arbitrary $\phi$, is also a solution of the equation (14). This implies, given $u$ and $v$, if $A$ is a solution of the equations (8) and (12), so is $A + u^\frac{1}{2} v^{-\frac{1}{2}} \nabla \phi$, for an arbitrary function $\phi$. The solution of the equation (14) will be unique after fixing the boundary conditions and the gauge. Here, $\phi$ is a free function and the transformation, $A \rightarrow A + u^\frac{1}{2} v^{-\frac{1}{2}} \nabla \phi$ leads to different (A)SD metrics. This implies that every (A)SD metric is characterised by two free functions $u$ and $\phi$ and a harmonic function $v$. Given a harmonic function $v$, the solution (see Appendix A) of (14) is unique up to a gauge and is given by,

$$f_x = \mp v^{-\frac{1}{2}} \left[ \int_{z_0}^{z} v_{y} \, dz + \partial_x \phi \right],$$

$$f_y = \mp v^{-\frac{1}{2}} \left[ -\int_{z_0}^{z} v_{x} \, dz + \int_{x_0}^{x} v_{z} (x, y, z_0) \, dx + \partial_y \phi \right],$$

$$f_z = \mp v^{-\frac{1}{2}} [\partial_z \phi],$$  \hfill (15)

where $\phi$ is an arbitrary function.

Thus the metric:

$$ds^2 = u^{-1} (dt + A \cdot dx)^2 + v (dx^2 + dy^2 + dz^2),$$  \hfill (16)

is the general (A)SD solution of the EE where $u$ and $v$ are free functions of which $v$ is harmonic and $A$ is given by (13), (11) and (15). Thus any (A)SD metric can be written in the above form.
3 The general stationary (A)SD metric.

The metric (7) will be general stationary (A)SD metric if $u, v$ and $A$ are independent of $t$ and obey the relations (13) and (14). This implies, $\Psi = 0$ or $u = kv$ where $k$ is an arbitrary positive constant and $A$ satisfies the relation,

$$\nabla \times A = \mp \nabla v.$$  \hspace{1cm} (17)

(Note that, a method of solving the above equation is given in the Appendix A.) Without loss of generality, one can assume $u = v$, for, by redefining the coordinates, one can absorb the constant $k$. Conversely, if $u = v$ then by virtue of (11), (13) and (14), $\Psi = 0$ and $A = v^{\frac{1}{2}}f$, where, $f$ is given by the equation (15). Thus the general stationary (A)SD metric is given by

$$ds^2 = v^{-1}(dt + A \cdot dx)^2 + v(dx^2 + dy^2 + dz^2).$$  \hspace{1cm} (18)

where $A = v^{\frac{1}{2}}f$, $f$ is given by the equation (15) and $v$ is a free harmonic function. This implies that, any stationary (A)SD solution of the EE can be characterised by a harmonic function. Note that, here, the gauge freedom $A \rightarrow A + \nabla \phi$ does not generate a new stationary (A)SD metric but leads to a coordinate transformed (A)SD metric.

We notice that the equation (17) can be reinterpreted as the static (A)SD Maxwell equation on the Euclidean background metric by reinterpreting $A$ as the Maxwell vector potential and $v$ as the Maxwell scalar potential. Thus there is a one-to-one correspondence between stationary (A)SD solutions of the EE and the static (A)SD Maxwell solutions on the Euclidean background.
4 Some particular solutions.

The multi-center (A)SD metrics given by Gibbons and Hawking \[6\] are the stationary (A)SD metrics and is given by,

\[
ds^2 = v^{-1}(dt + A \cdot dx)^2 + v(dx^2 + dy^2 + dz^2).
\]  

(19)

where,

\[
v = l + 2m \sum_{i=1}^{k} \frac{1}{|x - x_i|}.
\]

(20)

Here, \(l\) and \(m\) are constants and \(A\) can be obtained (see Appendix A) from the equation,

\[
\nabla \times A = \mp \nabla v.
\]

(21)

Thus for arbitrary harmonic function \(v\), the form of the Gibbons and Hawking multi-center metric is the most general stationary (A)SD metric.

We now give a strategy to generate non-stationary (A)SD solutions from the stationary solutions.

**Strategy:** Start with any harmonic function \(v(x, y, z)\). Obtain \(A_s\) by solving (see Appendix A) the equation \(\nabla \times A_s = \nabla v\). This is a stationary (A)SD solution of the EE. Substitute \(A_s\) in

\[
A = u^{\frac{1}{2}}[\int_0^t (u^{-\frac{1}{2}} \Psi) dt + v^{-\frac{1}{2}}A_s],
\]

(22)

where \(u\) is a free function of four coordinates such that \(u \neq kv\) and \(\Psi = \nabla(\ln \frac{u}{u^2})\). Then the metric:

\[
ds^2 = u^{-1}(dt + A \cdot dx)^2 + v(dx^2 + dy^2 + dz^2).
\]

(23)
is a non-stationary (A)SD solution of the EE.

5 Conclusions

We have explicitly obtained the most general (A)SD solution of the EE. We have shown that any (A)SD solution of the vacuum EE can be characterised by three free functions $u, \phi$ and $v$ where $v$ is harmonic. The metric is stationary, iff $u = kv$. Any stationary (A)SD metric can be characterised by one free harmonic function. Therefore the problem of finding stationary (A)SD solutions is equivalent to solving the 3-d Laplace equation. We also point out that, any stationary (A)SD metric can be reinterpreted as a static (A)SD Maxwell’s solution. The form of the Gibbons and Hawking multi-center metric is the most general stationary (A)SD metric.

It should be emphasised that we have not imposed any kind of boundary conditions on our solutions at infinity. However, one can obtain large variety of gravitational instantons by imposing appropriate boundary conditions.

Acknowledgements

We would like to thank Tarun Souradeep and Ravi Kulkarni, for valuable discussions.

SK was supported during this work, by the National Board for Higher Mathematics, India, through the Post-doctoral fellowship.
A. The general solution of $\nabla \times \mathbf{A} = \nabla v$ for given $v$

Given $v$, the solution of $\nabla \times \mathbf{A} = \nabla v$ will be unique up to a gauge. Any two solutions $\mathbf{A}_1$ and $\mathbf{A}_2$ will be related by,

$$\mathbf{A}_1 = \mathbf{A}_2 + \nabla \phi,$$

(24)

where $\phi$ is an arbitrary function. Therefore it is sufficient to find a particular solution of $\nabla \times \mathbf{A} = \nabla v$ and all other solutions can be found by gauge transformation. Without loss of generality, let us assume that the coordinates have been chosen such that $\mathbf{A}$ is parallel to the xy plane. i.e.

$$A_z = 0.$$  

(25)

The equation $\nabla \times \mathbf{A} = \nabla v$ with $A_z = 0$ implies,

$$- A_{y,z} = v_{,x}$$

(26)

$$A_{x,z} = v_{,y}$$

(27)

$$A_{y,x} - A_{x,y} = v_{,z}.$$  

(28)

Integrating (26) and (27) with respect to $z$, we get,

$$A_x = \int_{z_0}^{z} v_{,y}dz + g_1(x, y)$$

(29)

and

$$A_y = -\int_{z_0}^{z} v_{,x}dz + g_2(x, y),$$

(30)
where \( g_1 \) and \( g_2 \) are the integration functions of \( x \) and \( y \). Substituting the above two equations in (28) and using the fact that \( v \) is harmonic, we get,

\[
v_{,z}(x, y, z) - v_{,z}(x, y, z_0) - g_{1,y}(x, y) + g_{2,x}(x, y) = v_{,z}(x, y, z).
\] (31)

This implies,

\[
-g_{1,y}(x, y) + g_{2,x}(x, y) = v_{,z}(x, y, z_0).
\] (32)

As a particular solution, we choose \( g_1(x, y) = 0 \). Then the equation (32) gives, \( g_2 = \int_{x_0}^{x} v_{,z}(x, y, z_0) \, dx \).

Thus given the harmonic function \( v \), the general solution of \( \nabla \times A = \nabla v \) is:

\[
A_x = \int_{z_0}^{z} v_{,y} \, dz + \partial_x \phi,
\]

\[
A_y = -\int_{z_0}^{z} v_{,x} \, dz + \int_{x_0}^{x} v_{,z}(x, y, z_0) \, dx + \partial_y \phi
\]

\[
A_z = \partial_z \phi,
\]

where \( \phi \) is an arbitrary function.
References

[1] S. W. Hawking, Phys. Letts. 60A, 81 (1977).

[2] R. Penrose, Gen. Rel. & Grav. 7, 31 (1976).

[3] M. F. Atiyah, N. J. Hitchin & I. M. Singer, Proc. Roy. Soc. Lond. A362, 425 (1978).

[4] A. Ashtekar, Phys. Rev. D36, 1587, (1987).

[5] J. D. E. Grant, Preprint DAMPT-R 92/47.

[6] G. W. Gibbons & S. W. Hawking, Phys. Letts. 78B, 430 (1978).