On the Structure of Bethe Vectors

J. Fuksa

a Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, 141980 Russia
b Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, Prague, Czech Republic

Abstract—The structure of Bethe vectors for generalised models associated with the rational and trigonometric R-matrix is investigated. The Bethe vectors in terms of two-component and multi-component models are described. Consequently, their structure in terms of local variables and operators is provided. This, as a consequence, proves the equivalence of coordinate and algebraic Bethe ansatzes for the Heisenberg spin chains. Hermitian conjugation of the elements of the monodromy matrix for the spin chains is studied.

Keywords: quantum integrable systems, Bethe ansatz, spin chains

DOI: 10.1134/S1547477117040094

1. INTRODUCTION

The quantum inverse scattering method (QISM) was formulated by Faddeev, Sklyanin, and Takhtadjan in [6, 14]. Many physically interesting models were solved by this method such as the one-dimensional Bose gas, the Heisenberg spin chains, the sine-Gordon model [6, 7, 11, 12, 14], etc.

To fix the notation the basic features of the QISM are described in section 2. In particular, we recall the notion of the Bethe vectors which are eigenvectors of a family of mutually commuting operators including Hamiltonian of the solved system. In the literature the on-shell and off-shell Bethe vectors are distinguished. The on-shell Bethe vectors are eigenvectors of the Hamiltonian and thus their spectral parameters solve the Bethe equations (hence the notion on-shell), whereas the off-shell Bethe vectors have the same structure but their spectral parameters do not solve the Bethe equations.

The aim of this article is to describe the structure of both the on-shell and off-shell Bethe vectors. For this purpose we describe the generalised two-component and multi-component models introduced in [8] and [9], respectively. The results are formulated as three propositions contained in section 3.

In proposition 1 the Bethe vectors in terms of the two-component model are constructed. They were obtained in [8] for models with the rational R-matrix, including the XXX spin chains and the one-dimensional Bose gas. After minor modifications similar representation can be established also for models with the trigonometric R-matrix, i.e., also for the XXZ spin chains, sine-Gordon model, and other models. For the proof see [13].

Proposition 2 describes the Bethe vectors in terms of the multi-component model. The form of these vectors was firstly published in [9] (see also [10]) for models with the rational R-matrix. It is valid for models associated with the trigonometric R-matrix as well. We formulate proposition 2 for the generalised models associated with the both types of R-matrix and provide its proof because, as we believe, it is missing in the literature.

Proposition 3 describes the local structure of the Bethe vectors for the generalised inhomogeneous models with the R-matrix of both the rational and trigonometric form. We give also its proof. A particular form of proposition 3 for homogeneous XXX spin chain was obtained in [9]. It clarified the relation between the QISM and the former coordinate Bethe ansatz [1]. The proof for the inhomogeneous XXX spin chain can be found in [4].

Moreover, in section 1 the hermitian conjugation of the elements of the monodromy matrix is studied. The matrix elements $A(\lambda)$ and $B(\lambda)$ are related to $D(\lambda)$ and $C(\lambda)$ under the hermitian conjugation, respectively.

2. QUANTUM INVERSE SCATTERING METHOD

We adopt the notation often used in the context of the QISM. Let $A$ be an operator acting in a vector space $V$, and let $B$ be an operator acting in a tensor product $V \otimes V$. Let us have a tensor product $V^{\otimes N} = V \otimes V \otimes \ldots \otimes V$ of $N$ spaces $V$. We denote as

---

1 The article is published in the original.
$A_i$, the operator acting as $A$ in the $i$-th copy of $V$ and trivially in the rest of the tensor product, and as $B_{ij}$ the operator acting as $B$ in the tensor product of the $k$-th and $l$-th vector space and trivially in the rest of the tensor product.

We briefly introduce in this section the basic notions of the the QISM. For more details see [5] or [10] and references therein.

Let us suppose that there is a chain of $N$ sites, each site being endowed with a local Hilbert space $h_j$ and special parameter $\xi_j$. Such a chain is called inhomogeneous. If all parameters $\xi_j$ are equal, it is called homogeneous. The total Hilbert space is $\mathcal{H} = h_1 \otimes h_2 \otimes \ldots \otimes h_N$. To the $j$-th site there corresponds an L-operator $L_j(\mu, \xi_j) = L_j(\mu - \xi_j)$. The L-operator is a matrix acting in an auxiliary space $V$ and the local Hilbert space $h_j$ of the $j$-th site (quantum space); it depends on two spectral parameters $\mu$ and $\xi_j$. Depending on a physical model the auxiliary and quantum space vary. Particular forms of the L-operators for particular models can be found in the literature, e.g., [5, 10]. The monodromy matrix $T(\lambda)$ is a product of the L-operators along the chain and acts, therefore, in $V \otimes \mathcal{H}$. We restrict ourselves to particular models with an R-matrix of the form (2.3) below. For such models, the monodromy matrix is of the form

$$T(\lambda) = \prod_{j=1}^{N} L_j(\lambda, \xi_j) = \prod_{j=1}^{N} \begin{pmatrix} (L_j(\lambda, \xi_j))_{11} & (L_j(\lambda, \xi_j))_{12} \\ (L_j(\lambda, \xi_j))_{21} & (L_j(\lambda, \xi_j))_{22} \end{pmatrix} = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{2.1}$$

Its matrix elements $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ are operators on $\mathcal{H}$. Although we do not indicate it explicitly in our notation, the monodromy matrix $T(\lambda)$ as well as its matrix elements depends on the inhomogeneity parameters $\xi_1, \ldots, \xi_N$.

The monodromy matrix $T(\lambda)$ satisfies the following bilinear equation (called the RTT-relation) in $V_1 \otimes V_2 \otimes \mathcal{H}$:

$$R_{12}(\lambda, \mu)T(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R_{12}(\lambda, \mu). \tag{2.2}$$

The matrix $R_{12}(\lambda, \mu)$ is called the R-matrix. We suppose here that it is of the form

$$R(\lambda, \mu) = \begin{pmatrix} f(\lambda, \mu) & 0 & 0 & 0 \\ 0 & 1 & g(\lambda, \mu) & 0 \\ 0 & g(\lambda, \mu) & 1 & 0 \\ 0 & 0 & 0 & f(\lambda, \mu) \end{pmatrix}. \tag{2.3}$$

There are two R-matrices of the form (2.3). The first one is rational with the matrix elements

$$f(\lambda, \mu) = \frac{\lambda - \mu + 1}{\lambda - \mu}, \quad g(\lambda, \mu) = \frac{1}{\lambda - \mu} \tag{2.4}$$

and corresponds, e.g., to the XXX spin chain and the one-dimensional Bose gas. The second one is trigonometric with the matrix elements

$$f(\lambda, \mu) = \frac{\sinh(\lambda - \mu + \eta)}{\sinh(\lambda - \mu)}, \quad g(\lambda, \mu) = \frac{\sinh \eta}{\sinh(\lambda - \mu)} \tag{2.5}$$

and corresponds, e.g., to the XXZ spin chain and the sine-Gordon model. Here, $\eta \in \mathbb{C}$ is a parameter deforming the symmetry algebra of models with the trigonometric R-matrix to $\mathcal{U}_q(s\ell_2)$ with $q = e^{i\eta}$, whereas the symmetry algebra of models with the rational R-matrix is $sl_2$.

The R-matrix (2.3) satisfies the famous Yang–Baxter equation in the tensor product of three auxiliary spaces $V_1 \otimes V_2 \otimes V_3$:

$$R_{12}(\lambda_1, \lambda_2)R_{13}(\lambda_1, \lambda_3)R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3)R_{13}(\lambda_1, \lambda_3)R_{12}(\lambda_1, \lambda_2). \tag{2.6}$$

Let us mention that the L-operator $L(\lambda, \xi)$ satisfies the same relation (2.2), as can be easily seen if we restrict ourselves in (2.1) to the chain of the length one. In fact, relation (2.2) for the monodromy matrix is a consequence of the same relation for the L-operators, see [5].

Relation (2.2) determines an algebra with bilinear commutation relations. The R-matrix (2.3) is an analogue of its structure constants and the Yang–Baxter equation (2.6) is an analogue of the Jacobi identity. Let us list some of the relations here:

$[T_{jk}(\lambda), T_{jk}(\mu)] = 0, \quad j, k = 1, 2, \tag{2.7}$

$A(\mu)B(\lambda) = f(\lambda, \mu)B(\lambda)A(\mu) + g(\mu, \lambda)B(\lambda)A(\lambda), \tag{2.8}$

$B(\mu)A(\lambda) = f(\lambda, \mu)A(\lambda)B(\mu) + g(\mu, \lambda)A(\lambda)B(\lambda), \tag{2.9}$

$D(\mu)B(\lambda) = f(\mu, \lambda)B(\lambda)D(\mu) + g(\mu, \lambda)B(\mu)D(\lambda), \tag{2.10}$

and another seven relations \tag{2.11}

which can be found, e.g., in review papers [5, 13].

Relation (2.2) is the starting point of the quantum inverse scattering method. What follows is a general procedure which is related not only to the R-matrix of the form (2.3).

The R-matrix (2.3) is invertible almost everywhere. From this fact one can immediately prove the commutation relation

$$[\widetilde{T}(\lambda), \widetilde{T}(\mu)] = 0 \tag{2.12}$$

for the transfer matrix

$$\widetilde{T}(\lambda) = \text{Tr}(T(\lambda)) = A(\lambda) + D(\lambda). \tag{2.13}$$

Indeed, we can write (2.2) as

$$R_{12}(\lambda, \mu)T_1(\lambda)T_2(\mu)R_{12}^{-1}(\lambda, \mu) = T_2(\mu)T_1(\lambda)$$

and taking traces in the spaces $V_1$ and $V_2$ we get (2.12).
The transfer matrix (2.13) constitutes a generating function for a class of \(N-1\) commuting operators. The QISM diagonalizes all these commuting operators simultaneously, as it diagonalizes their generating function \(\mathcal{T}(\lambda)\). We mention here only the result. For details see [5] (see also [3]). For diagonalisation of \(\mathcal{T}(\lambda)\), it is necessary to suppose that the Hilbert space \(\mathcal{H}\) has the structure of a Fock space with a cyclic vector \(|0\rangle\) called pseudo vacuum. Let the pseudo vacuum \(|0\rangle\) be an eigenstate of both the operators \(A(\lambda)\) and \(D(\lambda)\) and be annihilated by the operator \(C(\lambda)\):

\[
A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \quad C(\lambda)|0\rangle = 0.
\]

Such a model with unspecified functions \(a(\lambda)\) and \(d(\lambda)\) is called generalised.

Eigenvectors of the transfer matrix \(\mathcal{T}(\mu)\) are of the form

\[
|\{\lambda\}\rangle = \prod_{j=1}^{M} B(\lambda_{j})|0\rangle,
\]

if the parameters \(\lambda_{k}\) are pairwise distinct: \(\lambda_{j} \neq \lambda_{k}\) for \(j \neq k\) and satisfy Bethe equations

\[
\mathcal{Y}(\lambda_{k}|\{\lambda\}) = 0
\]

for all \(k = 1, \ldots, M\). Here, \(M\) is a number of excitations. The corresponding eigenvalue is

\[
\tau(\mu|\{\lambda\}) = a(\mu)\prod_{a=1}^{M} f(\lambda_{a}, \mu) + d(\mu)\prod_{a=1}^{M} f(\mu, \lambda_{a})
\]

and the function \(\mathcal{Y}(\mu|\{\lambda\})\) appearing in the Bethe equation is of the form:

\[
\mathcal{Y}(\mu|\{\lambda\}) = \tau(\mu|\{\lambda\})\prod_{a=1}^{M} g^{-1}(\lambda_{a}, \mu),
\]

where the functions \(f(\lambda_{a}, \mu)\) and \(g(\lambda_{a}, \mu)\) are the matrix elements of the R-matrix (2.3).

The vectors (2.15) with the spectral parameters \(\{\lambda_{1}, \ldots, \lambda_{M}\}\), which satisfy (2.16), are called the on-shell Bethe vectors. The rest of this text takes into account not only the on-shell Bethe vectors but all vectors of the form (2.15), which are in general called the Bethe vectors.

2.1. Spin Chains

It is impossible to not mention the spin chains here as the most famous representatives of the quantum integrable systems. The L-operator is of the form

\[
L(\lambda, \xi) = \begin{pmatrix}
\lambda - \xi + S^{+} & S^{-} \\
S^{+} & \lambda - \xi - S^{3}
\end{pmatrix},
\]

for the XXX spin chain and

\[
L(\lambda, \xi) = \frac{1}{\sinh \eta} \begin{pmatrix}
\sinh(\lambda - \xi + \eta S^{3}) & \sinh(\eta) S^{-} \\
\sinh(\eta) S^{+} & \sinh(\lambda - \xi - S^{3})
\end{pmatrix},
\]

for the XXZ spin chain, respectively. The operators \(S^{3}, S^{+}, S^{-}\) are generators of the Lie algebra \(s\ell_{2}\).

Let the representation of \(S^{3}, S^{\pm}\) on the local Hilbert spaces \(h_{j}, j = 1, \ldots, N\), is of the following property:

\[
(S^{3})^\dagger = S^{3}, \quad (S^{\pm})^\dagger = S^{\mp}.
\]

One can prove by induction on the length of the chain \(N\) the following relations amongst the matrix elements of the monodromy matrix:

\[
B_{i}(\lambda) = (-1)^{N-1} C(-\lambda), \quad A_{i}(\lambda) = (-1)^{N} D(-\lambda)
\]

for the XXX spin chain and

\[
B_{i}(\lambda) = (-1)^{N-1} C_{\eta}(-\lambda), \quad A_{i}(\lambda) = (-1)^{N} D_{\eta}(-\lambda)
\]

for the XXZ spin chain, where the deformation parameter \(\eta\) transfers to its complex conjugate \(\bar{\eta}\). The spectral parameters \(\lambda\) and \(\xi_{j}, j = 1, \ldots, N\), transfer also to their complex conjugates \(\bar{\lambda}\) and \(\bar{\xi}_{j}\), respectively.

3. Structure of Bethe Vectors

The aim of this section is to describe the detailed structure of the Bethe vectors, i.e., of vectors of the form (2.15). For models associated with the rational R-matrix, a lot was known from the origin of the QISM [8, 9]. It helped with the identification of the QISM with the former coordinate Bethe ansatz method [1] for the homogeneous XXX spin chains. We intend to generalise known results and to provide the structure of the Bethe vectors for generalised inhomogeneous models associated with both the rational and trigonometric R-matrix. We suppose, as in section 2, that the length of the chain is \(N\).

3.1. Multi-Component Model and Bethe Vectors

The two-component model was developed for calculations of correlation functions of a local operator sitting at a site \(x\). The chain is split into two subchains of the length \(x\) and \(N-x\) and we define correspondingly a monodromy matrix for each subchain. For details see [8]. This can be obviously generalised to an arbitrary number \(K \leq N\) of subchains [9]:

\[
T(\lambda) = T_{1}(\lambda)T_{2}(\lambda)\cdots T_{K}(\lambda).
\]

The total Hilbert space \(\mathcal{H}\) is divided into its \(K\) subspaces \(\mathcal{H} = \mathcal{H}_{1} \otimes \mathcal{H}_{2} \otimes \cdots \otimes \mathcal{H}_{K}\). Consequently, the
pseudo vacuum is split into \( |0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_K \), \( |0\rangle_j \in \mathcal{H}_j \), where \( j = 1, 2, \ldots, K \). The partial monodromy matrix

\[
T_j(\lambda) = \begin{pmatrix} A_j(\lambda) & B_j(\lambda) \\ C_j(\lambda) & D_j(\lambda) \end{pmatrix},
\]

\( j = 1, 2, \ldots, K \), satisfies the same RTT-condition (2.2) as the undivided monodromy matrix \( T(\lambda) \) with the same R-matrix (2.3). Its matrix elements act non-trivially only on the Hilbert subspace \( \mathcal{H}_j \) and due to (2.2) satisfy the same set of bilinear relations (2.7)–(2.11) as the matrix elements of the full model. The matrix elements of different partial monodromy matrices \( T_k(\lambda) \) and \( T_j(\lambda) \) mutually commute.

We suppose that the operators \( A_j(\lambda), D_j(\lambda), C_j(\lambda) \) for \( j = 1, \ldots, K \) act on the partial pseudo vacuum \( |0\rangle_j \in \mathcal{H}_j \) as

\[
A_j(\lambda)|0\rangle_j = a_j(\lambda)|0\rangle_j, \quad D_j(\lambda)|0\rangle_j = d_j(\lambda)|0\rangle_j, \quad C_j(\lambda)|0\rangle_j = 0.
\]

\[
|\{\lambda\}\rangle = \sum_{\mathcal{P}(M)} \prod_{k \in \mathcal{P}} \prod_{j \in k} f(\lambda_{k_j}, \lambda_{k_j}) d_j(\lambda_{k_j}) a_j(\lambda_{k_j}) B_j(\lambda_{k_j}) B_j(\lambda_{k_j}) |0\rangle_j,
\]

where \( f(\lambda, \mu) \) is the matrix element of the R-matrix (2.3) and the summation is performed over all sets \( \mathcal{P} \) from the power set \( \mathcal{P}(M) \) of the set \( \{1, \ldots, M\} \). \( \mathcal{F} \) is the complement of \( \mathcal{F} \) in \( \{1, \ldots, M\} \): \( \mathcal{F} \cap \mathcal{F} = \{1, \ldots, M\}, \mathcal{F} \cap \mathcal{F} = \emptyset \).

For proof see [13]. The result of proposition 1 can be generalised for an arbitrary number of components \( K \leq N \), as stated in [9].

\[
|\{\lambda\}\rangle = \sum_{\mathcal{P}(M)} \prod_{k \in \mathcal{P}} \prod_{j \in k} \prod_{j' \in k} a_j(\lambda_{k_{j'}}) d_j(\lambda_{k_j}) f(\lambda_{k_j}, \lambda_{k_j}) B_j(\lambda_{k_j}) B_j(\lambda_{k_j}) |0\rangle_j,
\]

where the summation is performed over the set \( \mathcal{P}(M) \) of all divisions of the set \( \{1, \ldots, M\} \) into its \( K \) subsets \( \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_K \) such that: \( \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_K = \{1, \ldots, M\} \) and \( \mathcal{F}_1 \cap \mathcal{F}_j = \emptyset \) for \( j \neq k \).

The proof is missing in the literature. We give it here.

**Proof.** We perform the proof by induction on the number of components \( K \). For \( K = 2 \), Eq. (3.6) coincides with (3.5).

Let us suppose that (3.6) is valid for \( K - 1 < N \). The chain of the length \( N \) is divided into \( K - 1 \) subchains. Consequently, the Hilbert space is divided into \( K - 1 \) subspaces \( \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{K-1} \).

Let us mention that we do not specify the explicit form of the division into \( K - 1 \) subchains. We suppose that the induction hypothesis holds for all possible divisions into \( K - 1 \) subchains. It is obvious that all possible divisions into \( K \) subchains are obtained by dividing the last subchain of the divisions into \( K - 1 \) subchains, if possible, into its two subchains.

To the division of the \((K-1)\)-st subchain into its two subchains there corresponds a division of the Hilbert space \( \mathcal{H}_{K-1} = \mathcal{H}_{K-1}^0 \otimes \mathcal{H}_{K}^0 \) into its two subspaces \( \mathcal{H}_{K-1}^0 \) and \( \mathcal{H}_{K}^0 \). Consequently, the pseudo vacuum \( |0\rangle_{K-1} = |0\rangle_{K-1}^0 \otimes |0\rangle_{K}^0 \) is divided into two pseudovacua \( |0\rangle_{K-1}^0 \) and \( |0\rangle_{K}^0 \). The eigenvalues of \( A_{K-1}(\lambda) \), resp., \( D_{K-1}(\lambda) \).
are divided into parts as in (3.4): $a_{k-1}(\lambda) = a'_{k-1}(\lambda) a''_{k}(\lambda)$, resp., $d_{k-1}(\lambda) = d'_{k-1}(\lambda) d''_{k}(\lambda)$. Using this and proposition 1, we get
\[
\prod_{k_{-1} \in J_{k-1}} B_{k-1}(\lambda_{k_{-1}}) |0\rangle_{k-1}
\]
\[\sum_{j_{k} \in J_{k}} \prod_{j_{k} \in J_{k}} f(\lambda_{j_{k}}, \lambda_{j_{k}}, d''_{j_{k}}(\lambda_{j_{k}}), a''_{j_{k}}(\lambda_{j_{k}})) (3.7)
\]
\[
\times B_{k-1}(\lambda_{i_{k}}) |0\rangle_{k-1} \otimes B'_{k}(\lambda_{i_{k}}) |0\rangle_{k},
\]
where the sum goes over all divisions of $J_{k-1}$ into its two disjoint subsets $J'_{k-1}$ and $J''_{k}$ such that $J'_{k-1} \cup J''_{k} = J_{k-1}$. The operators $B_{k-1}(\lambda)$ and $B'_{k}(\lambda)$ act on the new sub chains with the Hilbert spaces $H'_{k-1}$ and $H''_{k}$, respectively. Using this in the induction hypothesis for $K - 1$ we prove (3.6).

3.2. Local Structure of Bethe Vectors

We investigate in this subsection the local structure of the Bethe vectors. The approach developed in this subsection is the application of proposition 2 for subchains of the length one (1-subchains). We divide the chain of the length $N$ into its $N$ 1-subchains. The local Hilbert space corresponding to the $j$-th 1-subchain is $h_{j}$. The monodromy matrix $T_{j}(\lambda)$ is identical with the L-operator $L_{j}(\lambda, \xi_{j})$, as can be seen from (2.1).

Let the local quantum space $h_{j}$ contain a pseudo vacuum $|0\rangle_{j}$. Let $|0\rangle_{j}$ be a common eigenvector of the diagonal elements of $L_{j}(\lambda, \xi_{j})$ and be annihilated by the subdiagonal element:
\[
(L_{j}(\lambda, \xi_{j}))_{i1} |0\rangle_{j} = \alpha(\lambda, \xi_{j}) |0\rangle_{j},
\]
\[
(L_{j}(\lambda, \xi_{j}))_{22} |0\rangle_{j} = \delta(\lambda, \xi_{j}) |0\rangle_{j},
\]
(3.8)
\[
(L_{j}(\lambda, \xi_{j}))_{21} |0\rangle_{j} = 0.
\]

Assumptions (3.8) for the L-operators $L_{j}(\lambda, \xi_{j})$ ensure the validity of assumptions (3.3) of multi-component models for arbitrary $K$, $K \leq N$, and, consequently, the validity of propositions 1 and 2. Particularly, (3.8) ensures the validity of assumptions (2.14) necessary for the diagonalisation procedure.

The complete pseudo vacuum $|0\rangle \in \mathcal{H}$ is of the form of an $N$-fold tensor product
\[
|0\rangle = |0\rangle_{1} \otimes |0\rangle_{2} \otimes \ldots \otimes |0\rangle_{N}
\]
and the eigenvalues of $A(\lambda)$ and $D(\lambda)$ in (2.14) are:
\[
a(\lambda) = \prod_{j=1}^{N} \alpha(\lambda, \xi_{j}),
\]
\[
d(\lambda) = \prod_{j=1}^{N} \delta(\lambda, \xi_{j}).
\]
(3.10)

The creation-like operator $B_{j}(\lambda)$ of the $j$-th 1-subchain is identical with the over diagonal element of $L_{j}(\lambda, \xi_{j})$
\[
B_{j}(\lambda) = (L_{j}(\lambda, \xi_{j}))_{12}.
\]
(3.11)

We restrict our consideration to a very specific representation in which
\[
B_{j}(\lambda)B_{j}(\mu)|0\rangle = 0.
\]
(3.12)

This condition is satisfied for both the XXX and XXZ spin chains, see (2.19) and (2.21), and indicates the fermionic type behavior of the operators $B_{j}(\lambda)$.

**Proposition 3.** Let assumptions (3.8) be satisfied. Let the creation-like operators (3.11) satisfy condition (3.12). Then the Bethe vector (2.15) can be represented in the form
\[
|\{\lambda\}\rangle = \sum_{1 \leq \lambda_{1} < \ldots < \lambda_{M} \leq N} \sum_{\sigma_{2} \in S_{M}} \sum_{\delta_{M}} \sigma_{2} \delta_{M} |B_{\lambda_{1}}(\lambda_{1}) B_{\lambda_{2}}(\lambda_{2}) \ldots B_{\lambda_{M}}(\lambda_{M}) |0\rangle
\]
\[
\times \prod_{l=1}^{M} \prod_{i=1}^{\lambda_{i}-1} \alpha(\lambda_{i}, \xi_{j}) \prod_{j=\lambda_{i}+1}^{\lambda_{i+1}} \delta(\lambda_{j}, \xi_{j}) \prod_{r=\lambda_{i}+1}^{M} \prod_{j=\lambda_{i}+1}^{\lambda_{i+1}} f(\lambda_{i}, \lambda_{r}),
\]
(3.13)

where $S_{M}$ is the permutation group and its elements $\sigma_{2}$ permute the spectral parameters $\{\lambda_{1}, \ldots, \lambda_{M}\}$.

**Proof.** We realised above that assumptions (3.8) ensure the validity of proposition 2 for $K = N$. Condition (3.12) provides that the sets $J_{k}^{i}$, $k = 1, \ldots, N$, from proposition 2 for $K = N$ are of the cardinality maximally one: $|J_{k}^{i}| \leq 1$. Moreover, only $M$ of these sets are nonempty, let us denote them $J_{n_{1}}, J_{n_{2}}, \ldots, J_{n_{M}}$. The summation over all distributions $P_{N}(M)$ in (3.13) is then equivalent to the summation over all $n_{1}, n_{2}, \ldots, n_{M}$ such that $1 \leq n_{1} < n_{2} < \ldots < n_{M} \leq N$, and over all permutations $\sigma_{2} \in S_{M}$ of the set of spectral-parameters $\{\lambda_{1}, \ldots, \lambda_{M}\}$ provided that $k \in J_{n_{k}}$. This observation drastically simplifies the subsequent considerations.
Let us study what happens with the coefficient

$$\mathcal{C}_a = \prod_{k \in j_1} \cdots \prod_{k \in j_N} \prod_{1 \leq j < j' \leq N} a_i(\lambda_{k_{j'}})$$ (3.14)

appearing in (3.6) for $K = N$. We know with respect to the above considerations that only $j_1, j_2, \ldots, j_N$ from $\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_N$ are nonempty and, moreover, contain only one element. Therefore,

$$\mathcal{C}_a = \prod_{l=1}^{M} \prod_{i=1}^{n-1} a_i(\lambda_{k_{n_i}}).$$ (3.15)

Moreover, we have supposed that $k_{n_i} = l$ and $a_i(\lambda) = \alpha(\lambda, \xi_i)$. Hence,

$$\mathcal{C}_a = \prod_{l=1}^{M} \prod_{i=1}^{n-1} \alpha(\lambda_l, \xi_i).$$ (3.16)

Similarly, we obtain that

$$\mathcal{C}_f = \prod_{k \in j_1} \cdots \prod_{k \in j_N} \prod_{M < j < j' < N} d_j(\lambda_{k_{j'}})$$ (3.17)

and

$$\mathcal{C}_f = \prod_{l=1}^{M} \prod_{i=1}^{n-1} \delta(\lambda_{l_i}, \xi_i).$$ (3.18)

The product of the partial Bethe vectors is

$$\prod_{k \in j_1} \cdots \prod_{k \in j_N} B_1(\lambda_{k_{j_1}})B_2(\lambda_{k_{j_2}}) \cdots B_N(\lambda_{k_{j_N}})$$ (3.19)

$$= B_n(\lambda_1)B_n(\lambda_2) \cdots B_n(\lambda_M).$$

Gluing these results together we prove (3.13).

A simple consequence of this proposition is that the Bethe vectors vanish for $M > N$.

For models, where $B(\lambda)$ are parameter independent, expression (3.13) can be further simplified. This holds particularly for the XXX and XXZ spin chains, see (2.19) and (2.20). Moreover, for their homogeneous version we obtain the following representation of the Bethe vectors:

$$|\{\lambda\} = \sum_{I \leq n_1 < \cdots < n_M \leq N} B_{n_1}B_{n_2} \cdots B_{n_M} \prod_{j=1}^{M} \sum_{\lambda_{n_j}} \frac{\delta N(\lambda_{n_j}, \xi_{n_j})}{\alpha(\lambda_{n_j}, \xi_{n_j})}$$ (3.20)

$$\times \sum_{\sigma_\lambda \in S_M} \prod_{1 \leq i < j \leq M} f(\lambda_i, \lambda_j) \prod_{k=1}^{M} \left( \frac{\alpha(\lambda_{k_{j'}}, \xi_{k_{j'}})}{\delta(\lambda_{k_{j'}}, \xi_{k_{j'}})} \right).$$

The representation of the Bethe vectors (3.20) was published in [9] for the homogeneous XXX spin chain. The special form of proposition 3 for the inhomogeneous XXX spin chain is given in [4]. Proposition 3 provides the generalisation of the known results for the generalised inhomogeneous models associated with the R-matrix of both the rational and trigonometric form.

4. CONCLUSIONS

The results of section 3 are valid for the generalised inhomogeneous models associated with the rational and trigonometric R-matrix.

The representation (3.20) proves the equivalence of the QISM to the coordinate Bethe ansatz for the homogeneous XXX and XXZ spin chains, as was noted in [9] for the XXX spin chain (see also [3]).

Formula (3.13) is the most general formula giving an explicit expression for the Bethe vectors of all models satisfying condition (3.12), like the inhomogeneous XXX spin chain, the inhomogeneous XXZ chain, or the fermionic realization of the corresponding homogeneous chains (see [2, 3] and references therein).

In section 1 we have established relations (2.23) and (2.24) amongst the elements of the monodromy matrix under the hermitian conjugation.

ACKNOWLEDGMENT

The author would like to express his gratitude to his scientific leaders A. Isaev and Č. Burdík for their guidance in his research and to N. Slavnov for the advice to study two-component models in the context of the Bethe vectors.

The work of the author was supported by the Grant Agency of the Czech Technical University in Prague, grant no. SGS15/215/OHK4/3T/14 and by the Grant of the Plenipotentiary of the Czech Republic at JINR, Dubna.

REFERENCES

1. H. Bethe, “Zur theorie der metalle,” Zeitschr. Phys. 71, 205–226 (1931).
2. Č. Burdík, J. Fuksa, and A. Isaev, “Bethe vectors for xxx-spin chain,” J. Phys.: Conf. Ser. 563, 012011 (2014).
3. Č. Burdík, J. Fuksa, A. P. Isaev, S. O. Krivonos, and O. Navrátil, “Remarks towards the spectrum of the Heisenberg spin chain type models,” Phys. Part. Nucl. 46, 277–309 (2015); arXiv:1412.3999v2 [math-ph].
4. F. Essler, H. Frahm, F. Gohmann, A. Klumper, and V. E. Korepin, The One-Dimensional Hubbard Model (Cambridge Univ. Press, Cambridge, 2005).
5. L. D. Faddeev, “How algebraic Bethe Ansatz works for integrable model,” arXiv:hep-th/9605187v1.
6. L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan, “The quantum inverse problem method. 1,” Theor. Math. Phys. 40, 688 (1980).
7. A. Izergin and V. Korepin, “The pauli principle for one-dimensional bosons and the algebraic Bethe Ansatz,” J. Sov. Math. 34, 1933–1937 (1986).
8. A. G. Izergin and V. E. Korepin, “The quantum inverse scattering method approach to correlation functions,” Comm. Math. Phys. 94, 67–92 (1984).
9. A. G. Izergin, V. E. Korepin, and N. Y. Reshetikhin, “Correlation functions in a one-dimensional Bose gas,” J. Phys. A: Math. Gen. 20, 4799–4822 (1987).
10. V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, Quantum Inverse Scattering Method and Correlation Functions (Cambridge Univ. Press, Cambridge, 1993).
11. E. K. Sklyanin, “Method of the inverse scattering problem and the nonlinear quantum Schrödinger equation,” Sov. Phys. Dokl. 24, 107 (1979).
12. E. K. Sklyanin and L. D. Faddeev, “Quantum mechanical approach to completely integrable field theory models,” Sov. Phys. Dokl. 23, 978 (1978).
13. N. A. Slavnov, “The algebraic Bethe Ansatz and quantum integrable systems,” Russ. Math. Surv. 62, 727 (2007).
14. L. A. Takhtajan and L. D. Faddeev, “The quantum method of the inverse problem and the Heisenberg XYZ model,” Russ. Math. Surv. 34, 11 (1979).