INVERSE MONOIDS AND IMMERSIONS OF CELL COMPLEXES

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ABSTRACT. An immersion \( f : D \to C \) between cell complexes is a local homeomorphism onto its image that commutes with the characteristic maps of the cell complexes. We study immersions between finite-dimensional connected \( \Delta \)-complexes by replacing the fundamental group of the base space by an appropriate inverse monoid. We show how conjugacy classes of the closed inverse submonoids of this inverse monoid may be used to classify connected immersions into the complex. This extends earlier results of Margolis and Meakin for immersions between graphs and of Meakin and Szakács on immersions into 2-dimensional \( CW \)-complexes.

1. INTRODUCTION

The notion of immersion arises from differential geometry: it is a differentiable function between differentiable manifolds whose derivative is everywhere injective. An immersion is essentially a local smooth embedding: a typical example is the immersion of the Klein bottle into 3-space — it is not an embedding, but it is a local embedding, which suffices for the purpose of visualization.

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In the absence of a differentiable structure, one can define a topological notion of immersion called a topological immersion, that is, a continuous map which is a local homeomorphism onto its image. Every immersion is a topological immersion. In our paper, we consider topological immersions between connected, finite-dimensional cell complexes (in particular, between CW-complexes and Δ-complexes), and in this context, we make the further assumption that the characteristic maps are respected. In the sequel, we call such maps between cell complexes immersions for short.

Covering maps are also immersions, and unlike immersions, they are very well understood by means of the fundamental group of the base space. Our aim is to generalize this characterization to immersions, where the fundamental group is replaced by an appropriate inverse monoid associated with the complex, which we call a loop monoid. (The term “fundamental inverse monoid” is reserved for something completely different in inverse semigroup theory, so we therefore refrain from using that terminology.) Loop monoids are defined in Section 4: the loop monoid of a complex \( C \) at a point \( v \) is denoted by \( L(C, v) \). Section 5 describes the one-to-one correspondence between closed inverse submonoids of loop monoids and immersions into the complex, closing with the following, main theorem of the paper, Theorem 5.5:

**Theorem.** Let \( C \) and \( D \) be connected \( \Delta \)-complexes labeled over a common \( \Delta \)-complex \( B(X, P) \), and suppose \( f : D \to C \) is an immersion that commutes with the labeling maps. If \( v \in D \), \( u \in C \), such that \( f(v) = u \), then \( f \) induces an embedding of \( L(D, v) \) into \( L(C, u) \). Conversely, let \( C \) be a \( \Delta \)-complex labeled over a \( \Delta \)-complex \( B(X, P) \), and let \( H \) be a closed inverse submonoid of the corresponding inverse monoid \( M(X, P) \) such that \( H \subseteq L(C, u) \) for some \( u \in C^0 \). Then there exists a \( \Delta \)-complex \( D \) labeled over the same \( B(X, P) \), and an immersion \( f : D \to C \) and a vertex \( v \in D^0 \) such that \( f(v) = u \) and \( L(D, v) = H \). Furthermore, \( D \) is unique, and \( f \) is unique. If \( H, K \) are two closed inverse submonoids of \( M(X, P) \) with \( H, K \subseteq L(C, u) \), then the corresponding complexes and immersions are equivalent if and only if \( H \) is conjugate to \( K \) in \( L(C, u) \).

An analogous theory has been developed for graphs in [5] and for 2-dimensional CW-complexes in [6]: this paper extends those results to finite dimensional \( \Delta \)-complexes. However, additional care is needed to define loop monoids and prove the topological lemmas needed to establish the one-to-one correspondence in the higher dimensional setting.

### 2. Preliminaries

In this section we introduce the basic notions of inverse monoids and immersions between cell complexes that will be used in the remainder of the paper.
2.1. **Inverse monoids.** An inverse monoid is a monoid $M$ with the property that for each $a \in M$ there is a unique element $a^{-1}$ (the inverse of $a$) in $M$ such that

$$a = aa^{-1}a \quad \text{and} \quad a^{-1} = a^{-1}aa^{-1}.$$ 

Inverse monoids arise naturally in the study of partial symmetry in mathematics in much the same way as groups arise in the study of symmetry. In fact the Wagner-Preston Theorem states that every inverse monoid embeds in an appropriate symmetric inverse monoid $\text{SIM}(Q)$, i.e. the monoid of all bijections between subsets of the set $Q$ under the usual composition of partial maps. For this and many additional properties of inverse monoids and their connections with other fields of mathematics, we refer the reader to the book of Lawson [3]. Some of the most basic properties of inverse monoids that we will need are listed in the following proposition.

**Proposition 2.1.** Let $M$ be an inverse monoid with set $E(M) = \{e \in M : e = e^2\}$ of idempotents. Then $E(M)$ is non-empty and

- The idempotents of $M$ commute. i.e. $ef = fe$ for all $e, f \in E(M)$. Thus the set $E(M)$ of idempotents forms a lower semilattice with respect to $e \wedge f = ef$. In particular, $g \leq ef$ if and only if $g \leq e$ and $g \leq f$ for any $e, f, g \in E(M)$.
- The relation defined on $M$ by $a \leq b$ iff $a = eb$ for some $e \in E(M)$ is a partial order on $M$, called the natural partial order on $M$. The natural partial order is compatible with the multiplication and inversion operations in $M$.
- The relation $\sigma_M$ defined on $M$ by $a \sigma_M b$ iff there exists $c \in M$ such that $c \leq a$ and $c \leq b$ is a congruence on $M$, called the minimum group congruence on $M$. The quotient $M/\sigma_M$ is a group, the maximum group homomorphic image of $M$.

Inverse monoids also arise naturally as transition monoids of *inverse automata*, which are automata whose underlying graphs are edge labeled over an alphabet $X \cup X^{-1}$ in the sense described below.

Let $X$ be a set and $X^{-1}$ a disjoint set in one-one correspondence with $X$ via a map $x \to x^{-1}$ and define $(x^{-1})^{-1} = x$. We extend this to a map on $(X \cup X^{-1})^*$ by defining $(x_1x_2\cdots x_n)^{-1} = x_n^{-1}\cdots x_2^{-1}x_1^{-1}$, giving $(X \cup X^{-1})^*$ the structure of the free monoid with involution on $X$. Throughout this paper by an $X$-graph (or just an edge-labeled graph if the labeling set $X$ is understood) we mean a strongly connected digraph $\Gamma$ with edges labeled over the set $X \cup X^{-1}$ such that the labeling is consistent with an involution: that is, there is an edge labeled $x \in X \cup X^{-1}$ from vertex $v_1$ to vertex $v_2$ if and only if there is an inverse edge labeled $x^{-1}$ from $v_2$ to $v_1$. The initial vertex of an edge $e$ will be denoted by $\alpha(e)$ and the terminal vertex by $\omega(e)$. If $X = \emptyset$, then we view $\Gamma$ as the graph with one vertex and no edges.

The label on an edge $e$ is denoted by $\ell(e) \in X \cup X^{-1}$. There is an evident notion of *path* in an $X$-graph. The initial (resp. terminal) vertex of a path $p$
will be denoted by $\alpha(p)$ (resp. $\omega(p)$). The label on the path $p = e_1e_2\ldots e_k$ is the word $\ell(p) = \ell(e_1)\ell(e_2)\ldots\ell(e_k) \in (X \cup X^{-1})^*$. 

$X$-graphs occur frequently in the literature. For example, the bouquet of $|X|$ circles is the $X$-graph $B_X$ with one vertex and one positively labeled edge labeled by $x$ for each $x \in X$. The Cayley graph $\Gamma(G,X)$ of a group $G$ relative to a set $X$ of generators is an $X$-graph: its vertices are the elements of $G$ and it has an edge labeled by $x$ from $g$ to $gx$ for each $x \in X \cup X^{-1}$.

If we designate an initial vertex (state) $\alpha$ and a terminal vertex (state) $\beta$ of $\Gamma$, then the birooted $X$-graph $\mathcal{A} = (\alpha, \Gamma, \beta)$ may be viewed as an automaton. See for example the book of Hopcroft and Ullman [2] for basic information about automata theory. The language accepted by this automaton is the subset $L(\mathcal{A})$ of $(X \cup X^{-1})^*$ consisting of the words in $(X \cup X^{-1})^*$ that label paths in $\Gamma$ starting at $\alpha$ and ending at $\beta$. This automaton is called an inverse automaton if it is deterministic (and hence co-deterministic), i.e. if for each vertex $v$ of $\Gamma$ there is at most one edge with a given label starting at $v$ or ending at $v$. This also implies that any path is uniquely determined by its initial vertex and its label. A deterministic $X$-graph can also be defined by a graph from which there is a label-preserving graph morphism to $B_X$ which is locally injective around the vertices. Such maps are called graph immersions in [11].

If $\Gamma$ is a deterministic $X$-graph, then each letter $x \in X \cup X^{-1}$ determines a partial injection of the set $V$ of vertices of $\Gamma$ that maps a vertex $v_1$ to a vertex $v_2$ if there is an edge labeled by $x$ from $v_1$ to $v_2$. The submonoid of $\text{SIM}(V)$ generated by these partial maps is an inverse monoid, called the transition monoid of the graph $\Gamma$.

For each subset $N$ of an inverse monoid $M$, we denote by $N^\omega$ the set of all elements $m \in M$ such that $m \geq n$ for some $n \in N$. The subset $N$ of $M$ is called closed if $N = N^\omega$.

Closed inverse submonoids of an inverse monoid $M$ arise naturally in the representation theory of $M$ by partial injections on a set [10]. An inverse monoid $M$ acts (on the right) by injective partial functions on a set $Q$ if there is a homomorphism from $M$ to $\text{SIM}(Q)$. Denote by $q.m$ the image of $q$ under the action of $m$ if $q$ is in the domain of the action by $m$. The following basic fact is well known (see [10]).

**Proposition 2.2.** If an inverse monoid $M$ acts on $Q$ by injective partial functions, then for every $q \in Q$, $\text{Stab}(q) = \{m \in M : qm = q\}$ is a closed inverse submonoid of $M$.

Conversely, given a closed inverse submonoid $H$ of $M$, we can construct a transitive representation of $M$ as follows. A subset of $M$ of the form $(Hm)^\omega$ where $mm^{-1} \in H$ is called a right $\omega$-coset of $H$. Let $X_H^\omega$ denote the set of right $\omega$-cosets of $H$. If $m \in M$, define an action on $X_H^\omega$ by $X_H^\omega$ if $(Ym)^\omega \in X_H^\omega$ and undefined otherwise. This defines a transitive action of $M$ on $X_H^\omega$. Conversely, if $M$ acts transitively on $Q$, then this action is
equivalent in the obvious sense to the action of $M$ on the right $\omega$-cosets of $\text{Stab}(q)$ in $M$ for any $q \in Q$. See [10] or [9] for details.

The $\omega$-coset graph $\Gamma_{(H,X)}$ (or just $\Gamma_H$ if $X$ is understood) of a closed inverse submonoid $H$ of an $X$-generated inverse monoid $M$ is constructed as follows. The set of vertices of $\Gamma_H$ is $X_H$ and there is an edge labeled by $x \in X \cup X^{-1}$ from $(Ha)^{\omega}$ to $(Hb)^{\omega}$ if $(Hb)^{\omega} = (Hax)^{\omega}$. Then $\Gamma_H$ is a deterministic $X$-graph. The birooted $X$-graph $(H,\Gamma_H, H)$ is called the $\omega$-coset automaton of $H$. The language accepted by this automaton is $H$ (or more precisely the set of words $w \in (X \cup X^{-1})^*$ whose natural image in $M$ is in $H$). Clearly, if $G$ is a group generated by $X$, then $\Gamma_H$ coincides with the coset graph of the subgroup $H$ of $G$.

We call two closed inverse submonoids $H_1, H_2$ of an inverse monoid $M$ conjugate if there exists $m \in M$ such that $mH_1m^{-1} \subseteq H_2$ and $m^{-1}H_2m \subseteq H_1$. It is clear that conjugacy is an equivalence relation on the set of closed inverse submonoids of $M$: however, conjugate closed inverse submonoids of an inverse monoid are not necessarily isomorphic. For example, the closed inverse submonoids $\{1, aa^{-1}, a^2a^{-2}\}$ and $\{1, aa^{-1}, a^{-1}a, aa^{-2}a\}$ of the free inverse monoid on the set $\{a\}$ are conjugate but not isomorphic.

Here we note that since inverse monoids form a variety of algebras (in the sense of universal algebra — i.e. an equationally defined class of algebras), free inverse monoids exist. We will denote the free inverse monoid on a set $X$ by $\text{FIM}(X)$. This is the quotient of $(X \cup X^{-1})^*$, the free monoid with involution, by the congruence that identifies $ww^{-1}w$ with $w$ and $ww^{-1}uw^{-1}$ with $wu^{-1}ww^{-1}$ for all words $u,w \in (X \cup X^{-1})^*$. See [9] or [3] for much information about $\text{FIM}(X)$. In particular, [9] and [8] provide an exposition of Munn’s solution [8] to the word problem for $\text{FIM}(X)$ via birooted edge-labeled trees called Munn trees.

In his thesis [12] and paper [13], Stephen initiated the theory of presentations of inverse monoids by extending Munn’s results about free inverse monoids to arbitrary presentations of inverse monoids. We refer the reader to [13] or our paper [6] for details of Stephen’s construction of Schützenberger graphs and Schützenberger automata and their use in the study of presentations of inverse monoids.

We recall the notion of an inverse category of paths on a graph. A category $C$ is called inverse if for every morphism $p$ in $C$ there is a unique inverse morphism $p^{-1}$ such that $p = pp^{-1}p$ and $p^{-1} = p^{-1}pp^{-1}$. The loop monoids $L(C,v)$ of an inverse category, that is, the set of all morphisms from $v$ to $v$, where $v$ is an arbitrary vertex, are inverse monoids. The free inverse category $\text{FIC}(\Gamma)$ on a graph $\Gamma$ is the free category on $\Gamma$ factored by the congruence induced by relations of the form $p = pp^{-1}p$, $p^{-1} = p^{-1}pp^{-1}$ and $pp^{-1}qq^{-1} = qq^{-1}pp^{-1}$ for all paths $p,q$ in $\Gamma$ with $\alpha(p) = \alpha(q)$. Its loop monoids are closed inverse submonoids of free inverse monoids - see [5].

2.2. Cell complexes. Recall the following definition [1] of a finite dimensional CW-complex $C$:
(1) Start with a discrete set $C^0$, the 0-cells of $C$.
(2) Inductively, form the $n$-skeleton $C^n$ from $C^{n-1}$ by attaching $n$-cells $C^n_\tau$ via attaching maps $\varphi_\tau: S^{n-1} \to C^{n-1}$. This means that $C^n$ is the quotient space of $C^{n-1} \cup_\tau B^n_\tau$ under the identifications $x \sim \varphi_\tau(x)$ for $x \in \partial B^n_\tau$. The cell $C^n_\tau$ is a homeomorphic image of $B^n_\tau - \partial B^n_\tau$ under the quotient map. (Here $B^n$ is the unit ball in $\mathbb{R}^n$ and $S^{n-1} = \partial B^n$ is its boundary).
(3) Stop the inductive process after a finite number of steps to obtain a finite dimensional CW-complex $C$.

The dimension of the complex is the largest dimension of one of its cells. Note that a 1-dimensional CW-complex is just an undirected graph, with the usual topology. We denote the set of $n$-cells of $C$ by $C^{(n)}$. We emphasize that each cell $C^n_\tau$ is open in the topology of the CW-complex $C$. A subset $A \subseteq C$ is open iff $A \cap C^n$ is open in $C^n$ for each $n$.

Each cell $C^n_\tau$ has a characteristic map $\sigma_\tau$, which is defined to be the composition $B^n_\tau \hookrightarrow C^{n-1} \cup_\tau B^n_\tau \to C^n \hookrightarrow C$. This is a continuous map whose restriction to the interior of $B^n_\tau$ is a homeomorphism onto $C^n_\tau$ and whose restriction to the boundary of $B^n_\tau$ is the corresponding attaching map $\varphi_\tau$. An alternative way to describe the topology on $C$ is to note that a subset $A \subseteq C$ is open iff $\sigma_\tau^{-1}(A)$ is open in $B^n_\tau$ for each characteristic map $\sigma_\tau$.

Our most general results apply to $\Delta$-complexes, which are CW-complexes with an additional restriction on the characteristic maps.

The standard $n$-simplex is the set
\[
\Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} : \Sigma_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}.
\]

We denote the $n + 1$ vertices of $\Delta^n$ by $v_i = (0, ..., 0, 1, 0, ..., 0)$ (1 in $i$th position). We order vertices by $v_i < v_j$ if $i < j$. The faces of the simplex are the subsimplices with vertices any non-empty subset of the $v_i$’s. There are $n + 1$ faces of dimension $n - 1$, namely the faces $\Delta_i^{n-1} = [v_0, ..., v_i-1, v_{i+1}, ..., v_n]$ for $i = 0, 1, ..., n$ spanned by omitting one vertex.

A $\Delta$-complex is a quotient space of a collection of disjoint simplices obtained by identifying certain of their faces via the canonical linear homeomorphisms that preserve the ordering of vertices.

Equivalently, a $\Delta$-complex is a CW-complex $X$ in which each $n$-cell $C^n_\alpha$ has a distinguished characteristic map $\sigma_\alpha : \Delta^n \to X$ such that the restriction of $\sigma_\alpha$ to each $(n-1)$-dimensional face of $\Delta^n$ is the distinguished characteristic map for an $(n-1)$-cell of $X$.

We refer the reader to Hatcher’s book [1] for more detail and many results about CW-complexes and $\Delta$-complexes.

The order on the vertices of the simplex makes it naturally possible to regard each $k$-cell $C^k_\tau$ of a $\Delta$-complex $C$ as a rooted cell, with distinguished root the image under the characteristic map $\sigma^k_\tau$ of the minimal 0-cell in the order on 0-cells in $\Delta^k$. We will denote the root of the cell $C$ by $\alpha(C)$. Thus we may regard the 1-skeleton as a digraph with each 1-cell (edge) $e$ directed...
from its initial vertex (the root of the cell) to its terminal vertex \( \omega(e) \) (the image of the maximal 0-cell of \( \Delta^1 \) under the characteristic map).

In the sequel, we will further assume all complexes to be connected.

### 2.3. Immersions between cell complexes

Let \( f \) be a map from the \( CW \)-complex \( D \) to the \( CW \)-complex \( C \) such that for each \( k \)-cell \( D^k_\tau \) of \( D \), \( f(D^k_\tau) \) is a \( k \)-cell of \( C \). Denote the corresponding distinguished characteristic maps of \( D^k_\tau \) and \( f(D^k_\tau) \) by \( \sigma^k_\tau : B^k \to D \) and \( \gamma^k_\tau : B^k \to C \) respectively. We say that \( f \) commutes with the characteristic maps of \( D \) and \( C \) if \( f \circ \sigma^k_\tau = \gamma^k_\tau \) for all \( k \)-cells \( D^k_\tau \) of \( D \). Note that if \( f \) commutes with the characteristic maps then it is a homeomorphism restricted to the (open) cells. In fact \( f \) must be a continuous map.

**Lemma 2.3.** If \( f \) is a map from the \( CW \)-complex \( D \) to the \( CW \)-complex \( C \) that commutes with the characteristic maps, then \( f \) is continuous.

**Proof.** Suppose \( f : D \to C \) is a map that commutes with the characteristic maps. Let \( U \) be an open subset of \( C \), let \( D \) be a \( k \)-cell of \( D \) and let \( C = f(D) \) be its image in \( C \). So \( C \) is a \( k \)-cell of \( C \). Let \( \sigma^C \) and \( \sigma^D \) be the corresponding characteristic maps. Then since \( U \) is open in the \( CW \)-complex \( C \), \( \sigma^C_1(U) \) is an open subset of the ball \( B^k \). But since \( f \) commutes with the characteristic maps, this implies that \( \sigma^D_1(f^{-1}(U)) = \sigma^C_1(U) \) is an open subset of \( B^k \), and hence \( f^{-1}(U) \) is an open set in the \( CW \)-complex \( D \). Hence \( f \) is continuous.

**Definition 2.4.** An immersion from a \( CW \)-complex \( D \) to a \( CW \)-complex \( C \) is a continuous map \( f : D \to C \) such that

(a) \( f \) is a local homeomorphism onto its image; that is, for each point \( x \in D \) there is an (open) neighborhood \( U \) of \( x \) such that \( f|_U \) is a homeomorphism from \( U \) onto \( f(U) \).

(b) \( f \) commutes with the characteristic maps of \( D \) and \( C \).

**Lemma 2.5.** A map \( f : D \to C \) between \( CW \)-complexes that commutes with the characteristic maps is an immersion if and only if it is locally injective at the vertices, that is, any 0-cell of \( \mathcal{D} \) has a neighborhood \( N \) such that \( f|_N \) is injective.

**Proof.** It is clear that an immersion between \( CW \)-complexes is locally injective at the vertices (0-cells) since it is a local homeomorphism onto its image. Conversely, suppose \( f : D \to C \) is a map that commutes with the characteristic maps and that is locally injective at vertices. Then by Lemma 2.3, \( f \) is continuous.

We need to show is that \( f \) is a local homeomorphism onto its image. We first show that \( f \) is locally injective around any point of \( D \). Indeed, let \( u \in D \) be any point. Indirectly, suppose that any neighborhood \( N^u \) of \( u \) contains distinct points \( w^1 \) and \( w^2 \) such that \( f(w^1) = f(w^2) \). Let \( S \) denote the set of
cells $D$ for which $u \in \overline{D}$. Take a neighborhood $N_u$ that is contained within $S$, and let the corresponding points be $w_1$ and $w_2$. For each $w_i$, there is exactly one cell $D_i$ such that $w_i \in D_i$. Since $f$ restricted to any cell of $D$ is a homeomorphism, $f(w_1) = f(w_2)$ implies $D_1 \neq D_2$, and $f(D_1) = f(D_2)$, in particular, $D_1$ and $D_2$ must be of the same dimension, say $k$. Let $D_u$ denote the unique cell containing $u$, note that then there are no other cells in $S$ with the same dimension as $D_u$, hence $D_u \neq D_1, D_2$. Therefore $u \in \overline{D}$ implies $u \in \partial D_1 \cap \partial D_2$, moreover, $D_u \subseteq \partial D_1 \cap \partial D_2$. Let $v$ be a 0-cell on the boundary of $D_u$. We will show that $f$ is not locally injective at $v$.

Let $N_v$ be an arbitrary neighborhood of $v$ in $D$. Denote the characteristic map of $D_i$ by $\sigma_{D_i}$, the common cell $f(D_1) = f(D_2)$ by $C$, the characteristic map of $C$ by $\sigma_C$, and take the set $U := \sigma_{D_1}^{-1}(N_v) \cap \sigma_{D_2}^{-1}(N_v) \subseteq B^k$. This is an open set in $B^k$: indeed $N_v$ is open in $D$, so is its preimages under the continuous characteristic maps, and the intersection of two open sets is also open. We claim that it is also nonempty. From $f \circ \sigma_{D_i} = \sigma_C$, one obtains $\sigma_{D_i}^{-1}(v) = \sigma_C^{-1}(f(v))$, in particular $\sigma_{D_1}^{-1}(v) = \sigma_{D_2}^{-1}(v)$, hence $\sigma_{D_i}^{-1}(v) \in U$. Take a point $x \in U \setminus \partial B^k$ – there certainly exists such a point, as $U$ is nonempty and open $\vdash$, and let $x_i = \sigma_{D_i}(x)$. Then as $x_i \in D_i$, we have $x_1 \neq x_2$, and $f \circ \sigma_{D_1} = f \circ \sigma_{D_2}$ implies $f(x_1) = f(x_2)$, a contradiction.

We have seen that for any point $u$ in $D$ there is a neighborhood $N_u$ of $u$ such that $f|_{N_u} : N_u \to f(N_u)$ is a continuous, bijective map. It suffices to show that if $V \subseteq N_u$ is an open set in $D$, then $f(V)$ is open in $f(D)$. So let $C$ be a $k$-cell in $f(D)$ and let $D$ be a $k$-cell in $D$ such that $f(D) = C$. Then $\sigma_{D_i}^{-1}(V) = \sigma_C^{-1}(f(V))$, so $\sigma_C^{-1}(f(V))$ is open in $B^k$ since $\sigma_{D_i}^{-1}(V)$ is open in $B^k$. Hence $f(V)$ is open in $C$ as required.

\[\square\]

3. Labeled $\Delta$-complexes

In this section, we introduce a way to assign labels to all cells of a $\Delta$-complex in a “deterministic” way: we expect paths to be uniquely determined by their initial vertex and their label. Our notion of paths is more restrictive than that of topological paths in $\Delta$-complexes. In particular, a path on the 1-skeleton is a path in the graph theoretic sense. As described in the previous section, we regard every 1-cell to be a directed edge from its root, but for the sake of allowing for paths traversing in the opposite direction, for each 1-cell (edge) $e$, we adjoin a distinct ghost edge denoted $e^{-1}$ with $\alpha(e^{-1}) = \omega(e)$ and $\omega(e^{-1}) = \alpha(e)$. We emphasize that a ghost edge is not present in the complex in the topological sense, but only serves to accurately describe paths.

So we will regard the 1-skeleton $C^0$ of a $CW$-complex to be a directed graph in the sense just described. Paths in this graph are of course sequences of directed edges $e_1 e_2 \dotsc e_n$ with $\omega(e_i) = \alpha(e_{i+1})$. We will also make use of generalized paths in a $CW$-complex $C$. By a generalized path in a $C$ we mean a sequence $e_1 e_2 \dotsc e_s$ where each $e_i$ is either a $k$-cell for $k \geq 2$ or a 1-cell or an
inverse of a 1-cell, and \( \omega(e_{i-1}) = \alpha(e_i) \) for \( i = 2, ..., s - 1 \). (Here \( \omega(e) = \alpha(e) \) if \( e \) is a \( k \)-cell with \( k \geq 2 \).)

**Lemma 3.1.** Every \( \Delta \)-complex \( C \) admits an immersion into a \( \Delta \)-complex with one 0-cell.

**Proof.** If we identify all 0-cells of \( C \), then the quotient cell complex is also a \( \Delta \)-complex \( B \) with one 0-cell. The corresponding map \( f : C \rightarrow B \) is an immersion since it is injective on all \( k \)-cells with \( k > 0 \).

\[ \square \]

Let \( B \) be a \( \Delta \)-complex of dimension \( n \) with one 0-cell. Let \( \{ e^k_{\rho} : \rho \in X \} \) be the set of 1-cells and \( \{ e^k_{\rho} : \rho \in P_k \} \) the set of \( k \)-cells of \( B \) for \( 2 \leq k \leq n \) and let \( \beta^k_{\rho} : \Delta^k \rightarrow B \) be the characteristic map of \( e^k_{\rho} \) for \( k \geq 1 \). Here we assume that the sets \( X, P_k \) are all mutually disjoint. We denote this \( \Delta \)-complex \( B \) by \( B(X, P_2, ..., P_n, \{ \beta^k_{\rho} \}) \), or more briefly by \( B(X, P) \) where \( P = P_2 \cup ... \cup P_n \). Then \( B = B_X \) if \( n = 1 \), and \( |X|, |P_2|, ..., |P_n| \) are all non-empty sets if \( n \geq 2 \) by definition of a \( \Delta \)-complex of dimension \( n \). We view \( X \) as a set of labels for the 1-cells of \( B(X, P) \) and \( P_k \) as a set of labels of the \( k \)-cells of \( B(X, P) \) for \( 2 \leq k \leq n \). That is, the label on the \( k \)-cell \( e^k_{\rho} \) is \( \ell(e^k_{\rho}) = \rho \).

The 1-skeleton of \( B(X, P) \) is \( B_X \). We regard this as an \( X \)-graph as usual; i.e. each edge labeled by \( x \in X \) is equipped with an inverse edge labeled by \( x^{-1} \). The labeling on the 1-cells of \( B(X, P) \) extends to a labeling on paths in the 1-skeleton of \( B(X, P) \) in the obvious way. The label on a path \( p \) in \( B(X, P) \) will be denoted by \( \ell(p) \); thus \( \ell(p) \in (X \cup X^{-1})^* \). More generally we may extend the labeling on cells to a labeling on generalized paths in the obvious way: if \( e_1 e_2 ... e_t \) is a generalized path, then \( \ell(e_1 e_2 ... e_t) = \ell(e_1) \ell(e_2) ... \ell(e_t) \in (X \cup X^{-1} \cup P)^* \). The label of the empty path is the empty word.

We say that a \( \Delta \)-complex \( C \) is labeled over a complex \( B(X, P) \) if it admits an immersion \( f : C \rightarrow B(X, P) \). In this case, the labeling on the \( k \)-cells of \( B(X, P) \) induces a labeling on the \( k \)-cells of \( C \) for \( k \geq 1 \): a \( k \)-cell \( C^k_{\tau} \) of \( C \) has label \( \ell(C^k_{\tau}) = \ell(f(C^k_{\tau})) \). So \( \ell(C^1_{\tau}) \in X \) and \( \ell(C^k_{\tau}) \in P_k \) if \( 2 \leq k \leq n \). Thus cells of \( C \) have the same label if and only if they have the same image under \( f \). If the underlying complex \( B(X, P) \) is understood, we just say that \( C \) is a labeled complex. By Lemma 3.1, every \( \Delta \)-complex admits some labeling. The immersion \( f \) constructed in the proof of that lemma assigns different labels to all cells of \( C \). Of course we would usually choose smaller sets \( X, P_k \) as sets of labels for the cells of \( C \) if possible.

According to the following lemma, the labeling introduced is a generalization of the deterministic labeling obtained by an immersion into \( B_X \) for graphs.

**Lemma 3.2.** If \( e_1 \) and \( e_2 \) are distinct 1-cells of a \( \Delta \)-complex with the same initial or terminal vertex, then \( \ell(e_1) \neq \ell(e_2) \). Furthermore, if \( C^k_{\gamma} \) and \( C^k_{\tau} \) are distinct cells of a labeled \( \Delta \)-complex \( C \) with the same root, then \( \ell(C^k_{\gamma}) \neq \ell(C^k_{\tau}) \).
Proof. This is obvious if \( k \neq s \) since the sets \( X, P_k \) are mutually disjoint. If \( k = s \), then it follows from the fact that the immersion \( C \to B \) is locally homeomorphic onto its image (hence injective) around the common root of the cells. \( \square \)

Definition 3.3. An immersion \( g : D \to C \) of \( \Delta \)-complexes is said to commute with the labeling if \( C \) and \( D \) are labeled over the same complex \( B(X, P) \) by immersions \( f_C : C \to B(X, P) \) and \( f_D : D \to B(X, P) \), and \( g \) commutes with these labeling maps, that is, \( f_D \circ g = f_C \).

Note that commuting with the labeling, in particular, implies \( g(\ell(D_k)) = \ell(g(D_k)) \) for any \( k \)-cell \( D_k \) of \( D \). Also note that if \( g : D \to C \) is an immersion of \( \Delta \)-complexes, then a labeling \( f_C \) of \( C \) over \( B(X, P) \) induces a labeling \( f_D \) of \( D \) over \( B(X, P) \) such that \( g \) respects the labeling by putting \( f_C \circ g = f_D \).

For the remainder of this paper, we will assume that all \( \Delta \)-complexes are labeled and that all immersions between \( \Delta \)-complexes respect the labeling, as described above.

4. The inverse monoid \( M(X, P) \)

In this section we construct an inverse monoid \( M(X, P) \) that enables us to study immersions into a cell complex \( B(X, P) \), and more generally to study immersions between cell complexes that are labeled over \( B(X, P) \). This inverse monoid will be given by generators and relations.

If \( \Delta^k = [v_0, v_1, \ldots, v_k] \), then \( \Delta^k \) has \( (k+1) \) faces of dimension \( k-1 \), namely the \( (k-1) \)-simplices \( \Delta^{[k-1]}_{i} = [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k] \) for \( i = 0, \ldots, k \). All of these faces except \( \Delta^{k-1}_0 \) contain the vertex \( v_0 \). The smallest vertex of \( \Delta^{k-1}_0 \) under the order on vertices is \( v_1 \).

If \( C \) is a \( \Delta \)-complex of dimension \( n \) and \( C^k \) is a \( k \)-cell of \( C \), there is a corresponding characteristic map \( \sigma^k : \Delta^k \to C \). The restriction of \( \sigma^k \) to \( \Delta^{k-1}_i \) is a characteristic map \( \sigma^{k-1}_i \) of some \( (k-1) \)-dimensional cell \( C^{k-1}_i \) of \( C \), by definition of a \( \Delta \)-complex. The root of \( C^k \) is \( \alpha(C^k) = \sigma^k(v_0) \) and the root of \( C^{k-1}_i \) is also \( \sigma^k(v_0) \) if \( i \neq 0 \) but the root of \( C^{k-1}_0 \) is \( \sigma^k(v_1) \). Thus the 1-cell \( \sigma^k([v_0, v_1]) \) is a directed edge in the 1-skeleton of \( C \) from the root of \( C^k \) to the root of \( C^{k-1}_0 \).

For a 2-cell \( C^2 \) of a \( \Delta \)-complex \( C \), we denote by \( bl(C^2) \) the boundary label of \( C^2 \). This is the label on the image in \( C \) of the path \( (v_0, v_1, v_2, v_0) \) in the 1-skeleton of \( \Delta^2 = [v_0, v_1, v_2] \) under the corresponding characteristic map from \( \Delta^2 \) to \( C \). For a \( k \)-cell \( C^k \) of \( C \), denote the image under \( \sigma^k \) of the 1-cell \( [v_0, v_1] \) of \( \Delta^k \) by \( e(C^k) \). For \( k \geq 3 \), denote by \( bl(C^k) \) the label on the generalized path \( C^{k-1}_k C^{k-1}_{k-1} \ldots C^{k-1}_1 e(C^k) C^{k-1}_0 (e(C^k))^{-1} \) and refer to this as the boundary label of the cell \( C^k \).

Boundary labels of cells are preserved by immersions.

Lemma 4.1. Let \( f : D \to C \) be an immersion of a \( \Delta \)-complexes that commutes with the labeling and let \( D^k \) be a \( k \)-cell of \( D \). Then
(a) \( f(\alpha(D^k)) = \alpha(f(D^k)) \) and 
(b) \( bl(D^k) = bl(f(D^k)) \).

In particular, if \( f \) is an immersion of \( D \) into \( B(X, P) \) that defines a labeling of \( D \), then \( bl(D^k) = bl(f(D^k)) \) for every \( k \)-cell \( D^k \) of \( D \).

**Proof.** The statement is immediate from the definition of immersion since immersions commute with characteristic maps. \( \square \)

It follows from Lemma 4.1 that if \( f \) is an immersion of \( C \) to \( B(X, P) \) that is used to define a labeling of \( C \), then any \( k \)-cells \( (k \geq 2) \) of \( C \) that have the same label in \( P_k \) have the same boundary label. Thus we may denote the boundary label of a \( k \)-cell \( C^k \) in \( C \) (or its image in \( B(X, P) \)) by \( bl(\rho) \) where \( \rho = \ell(C^k \rho) \).

**Definition 4.2.** We now define the inverse monoid \( M(X, P) \) as the inverse monoid with generators \( X \cup P \) and relations

- \( \rho^2 = \rho \) for each \( \rho \in P \) and
- \( \rho = \rho bl(\rho) \) for each \( \rho \in P \).

We remark that the conditions \( \rho = \rho^2 \) and \( \rho = \rho bl(\rho) \) are equivalent to \( \rho = \rho^2 \) and \( \rho \leq bl(\rho) \). It is sometimes more convenient to use this characterization of the defining relations for \( M(X, P) \). We will make use of this and the following fact about the monoid \( M(X, P) \) in the sequel.

**Lemma 4.3.** Let \( C \) be any \( k \)-cell with \( k \geq 2 \) in a \( \Delta \)-complex labeled by an immersion into \( B(X, P) \). If \( p \) is the image under the corresponding attaching map of any closed path around \( v_0 \) in the 1-skeleton of \( \Delta^k \), then \( \ell(C) \leq \ell(p) \) in \( M(X, P) \).

**Proof.** We prove the statement by induction on \( k \). Suppose first that \( k = 2 \) and let \( p = e_1 e_2 \ldots e_t \) where each \( e_i \) is a 1-cell or the inverse of a 1-cell on \( \partial(C) \). We proceed by induction on \( t \). If \( t = 0 \), then \( \ell(p) = 1 \), and since \( \ell(C) \) is an idempotent, the statement holds. We may assume without loss of generality that \( p \) contains no segment of the form \( qq^{-1}q \) where \( q = e_r e_{r+1} \ldots e_{r+m} \), since in that case we could omit \( qq^{-1} \) to obtain a shorter path with the same label in \( M(X, P) \).

If the word \( e_1 e_2 \ldots e_t \) is reduced as written, then \( \ell(p) = (bl(C))^s \) for some integer \( s \), and so it follows from the presentation that \( \ell(C) \leq \ell(p) \). If \( e_1 e_2 \ldots e_t \) is not group reduced as written, then the condition that \( p \) has no segment of the form \( qq^{-1}q \) forces that \( p = qq^{-1}p_1 \) or \( p = p_1 q^{-1}q \), where \( q \) is a reduced word that is an initial (terminal) segment of \( p \) and \( p_1 \) is a shorter path with the properties required by the lemma. By induction, \( \ell(C) \leq \ell(p_1) \). Also, \( q \) can be augmented so that it is an initial (terminal) segment of some reduced path \( p_2 \) with the properties in the lemma, hence \( \ell(C) \leq \ell(p_2) \), so \( \ell(C) \leq \ell(p_2 p_2^{-1}) \leq \ell(q q^{-1}) \). Then \( \ell(C) \leq \ell(q q^{-1}) \ell(p_1) = \ell(p) \), as desired. Hence the statement holds if \( k = 2 \).

Now suppose that the statement holds for cells of dimension up to \( k - 1 \) and that \( C \) is a \( k \)-cell of \( C \). Let \( q = f_1 \cdots f_l \) be a path on \( \Delta^k \) around \( v_0 \),
and let \( p = e_1e_2...e_t \) be its image under the attaching map. If \( t \leq k \) then 
\( p \) lies in one of the \( t \)-dimensional faces of \( \Delta^k \), say \( \Delta^k_{i-1} \), since \( q \) contains at 
most \( t + 1 \) distinct vertices of the 1-skeleton of \( \Delta^k \). Let \( C_i \) be the image 
of this face under the attaching map. Then, by the induction hypothesis, 
\( \ell(C_i) \leq \ell(p) \), and by the presentation, \( \ell(C) \leq \ell(C_i) \). So assume \( t > k \). Then 
the path \( f_1 \cdot \cdot \cdot f_k \) is a prefix of \( q \) and since it contains at most \( k + 1 \) distinct 
vertices, it is contained in one of the \( k \) dimensional faces \( \Delta^k \) of \( \Delta^t \) which 
contains \( v_0 \).

Let \( q_1 \) be a path of minimal length from \( \omega(f_k) \) to \( \alpha(f_1) \) in \( \Delta^k \) (that is, 
of length 0 or 1), and denote its image under the characterizing map of 
\( C \) by \( p_1 \). Consider the path \( e_1 \cdot \cdot \cdot e_k p_1(p_1^{-1}e_{k+1}...e_t) \) for which we have 
\( \ell(e_1 \cdot \cdot \cdot e_k p_1(p_1^{-1}e_{k+1}...e_t)) \leq \ell(e_1 \cdot \cdot \cdot e_t) \). As \( f_1 \cdot \cdot \cdot f_k q_1 \) is a path from \( v_0 \) to 
\( v_0 \) in \( \Delta^k \), by the induction hypothesis, like before, there exists a cell \( C_{i_1} \) with 
\( \ell(C) \leq \ell(C_{i_1}) \) and \( \ell(C_{i_1}) \ell(q_1^{-1}e_{k+1}...e_t) \leq \ell(e_1 \cdot \cdot \cdot e_k q_1) \ell(q_1^{-1}e_{k+1}...e_t) \). 
Applying the same argument to the shorter path \( q_1^{-1}e_{k+1}...e_t \) we obtain (formally by induction) that 
\( \ell(C_{i_1}) \ell(C_{i_2}) \cdot \cdot \cdot \ell(C_{i_m}) \leq \ell(e_1 \cdot \cdot \cdot e_t) = \ell(p) \) 
for some \( k \)-cells \( C_{i_1}, C_{i_2}, ..., C_{i_m} \) on the boundary of \( C \). Since the presentation implies that 
\( \ell(C) \leq \ell(C_{i_1}) \ell(C_{i_2}) \cdot \cdot \cdot \ell(C_{i_m}) \), we have \( \ell(C) \leq \ell(p) \) as 
required.

\[ \square \]

We extend this slightly to obtain a technical result about generalized 
paths that will be used later in the paper.

**Lemma 4.4.** Let \( C \) be any \( k \)-cell with \( k \geq 2 \) in a \( \Delta \)-complex labeled over 
\( B(X, P) \). If \( p \) is the image under the corresponding attaching map of any 
closed generalized path around \( v_0 \) on the boundary of \( \Delta^k \), then \( \ell(C) \leq \ell(p) \) 
in \( M(X, P) \).

**Proof.** Consider a generalized path \( q = q_1 \Delta_1 q_2 \Delta_2 ... q_t \Delta_t q_{t+1} \) on \( \partial \Delta^k \), 
where \( q_i \) is a path, and \( \Delta_i \) is a face of \( \Delta^k \) of at least 2-dimensions, and let 
\( p = p_1 D_1 p_2 D_2 ... p_{t} D_{t} p_{t+1} \) be its image under the attaching map. It follows 
inductively from the presentation for \( M(X, P) \) that \( \ell(C) \leq \ell(s_i) \ell(D_1) \ell(s_i)^{-1} \) 
for \( i = 1, ..., t \), where \( s_i \) is the image of a path on \( \partial \Delta^k \) from \( v_0 \) to \( \alpha(\Delta_i) \). 
By Lemma 4.3 we also have \( \ell(C) \leq \ell(p_1 s_1^{-1}) \ell(C) \leq \ell(s_1 p_2 s_2^{-1}) \cdot \cdot \cdot \ell(s_1 p_t s_{t+1}) \). Hence, by multiplying all of 
these inequalities, we obtain 
\( \ell(C) \leq \ell(p_1 s_1^{-1}) \ell(s_1) \ell(D_1) \ell(s_1^{-1}) \ell(s_1 p_2 s_2^{-1}) \ell(s_2) \ell(D_2) \ell(s_2^{-1}) \cdot \cdot \cdot \ell(s_1 p_t s_{t+1}) \ell(s_t) \ell(D_t) \ell(p_{t+1}) \leq \ell(p_1) \ell(D_1) \ell(p_2) \ell(D_2) \cdot \cdot \cdot \ell(D_t) \ell(p_{t+1}) = \ell(p) \), 
as required, since each \( \ell(s_i^{-1}) \ell(s_i) \) is an idempotent of \( M(X, P) \). \[ \square \]

**Proposition 4.5.**

(1) The inverse submonoid of \( M(X, P) \) generated by \( X \) is isomorphic to 
\( \text{FIM}(X) \).

(2) The maximum group image of \( M(X, P) \) is the fundamental group of 
the complex \( B(X, P) \).
Proof. For the first part of the theorem, note that $M(X, P)$ is obtained as a factor of the free monoid $X \cup X^{-1} \cup P$ by the defining relations of inverse monoids and those introduced in the presentation — these are all equations which, if they contain a letter in $P$, then they contain it on both sides. Therefore, to a word in $(X \cup X^{-1})^*$, one can only apply those relations which define the free inverse monoid on $X$.

The maximum group image of $M(X, P)$ has defining relations $bl(\rho) = 1$ for each 2-cell labeled by $\rho$, since the only idempotent of a group is the identity. But this is precisely a presentation of $\pi_1(B(X, P))$. \hfill \Box

Now let $C$ be a $\Delta$-complex labeled over $B(X, P)$. Then we may define a natural action of the inverse monoid $M(X, P)$ by partial one-to-one maps on the set $C^{(0)}$ of 0-cells of $C$ as follows. For $x \in X \cup X^{-1}$ and $v \in C^{(0)}$ define $v.x = w$ if there is an edge labeled by $x$ from $v$ to $w$ in the 1-skeleton of $C$, and $v.x$ is undefined if there is no such edge. For $\rho \in P_k$ with $k \geq 2$ and $v \in C^{(0)}$, define $v.\rho = v$ if $v = \alpha(C^k)$ for some $k$-cell $C^k$ with $\alpha(C^k) = v$, and $v.\rho$ is undefined otherwise.

Lemma 4.6. The action of the generators $X \cup P$ of $M(X, P)$ on $C^{(0)}$ extends to a well-defined action of $M(X, P)$ on $C^{(0)}$.

Proof. By Lemma 3.2 letters in $X \cup P$ act by partial one-to-one maps on $C^{(0)}$. Thus the action of the generators extends to a well-defined action of the free inverse monoid $\text{FIM}(X \cup P)$ on $C^{(0)}$. We need to show that the action respects the defining relations of $M(X, P)$. The partial actions of letters in $P$ are, by definition, identity maps on their domains, hence idempotent maps. Since the action by $\rho \in P_2$ is a restriction of the action by $bl(\rho)$, the action by $\rho$ is the same as the action by $\rho bl(\rho)$ for $\rho \in P_2$. Finally, for $\rho \in P_k$, $k \geq 3$, the action by $\rho$ is also a restriction of the action by $bl(\rho)$. This is because each of the elements $\ell(C^k_i)$ (for $i = 1, \ldots, k$) and $\ell(C^k_i)C^k_0^{-1}(C^k_i)$ that arise in the definition of the boundary label $bl(C^k)$ of a cell $C^k$ stabilize $\alpha(C^k)$. So again the action by $\rho$ is the same as the action by $\rho bl(\rho)$. Hence the action respects all defining relations in $M(X, P)$, as required. \hfill \Box

The stabilizer of a vertex in $C^{(0)}$ under the action by $M(X, P)$ is a closed inverse submonoid of $M(X, P)$, and stabilizers of different vertices in $C^{(0)}$ are conjugate closed inverse submonoids of $M(X, P)$. Hence the immersion $f : C \to B(X, P)$ that defines the labeling of $C$ gives rise to a conjugacy class of closed inverse submonoids of $M(X, P)$.

We may interpret the stabilizer of a vertex in $C^{(0)}$ under the action by $M(X, P)$ as the loop monoid of an appropriate inverse category based at that vertex. The construction of this inverse category closely follows the construction given in our paper [6].

To define this inverse category, it is convenient to first define a graph $\Gamma_C$ associated with the $\Delta$-complex $C$ as follows:
Proposition 5.1. Let conjugacy classes of closed inverse submonoids of loop monoids. The converse also holds: immersions are in one-to-one correspondence with \( M \) to a conjugacy class of closed inverse submonoid of its (unique) loop monoid. The proof follows from the fact that the fundamental groupoid \( \Gamma_C \) of a loop labeled by an idempotent in \( M \). If we abuse notation slightly by identifying the loop \( f_{C^k} \) in \( \Gamma_C \) with the \( k \)-cell \( C^k \), then paths in the graph \( \Gamma_C \) are just identified with generalized paths in the \( \Delta \)-complex \( C \).

Now let \( \sim \) be the congruence on the free category on \( \Gamma_C \) generated by the relations defining \( \text{fic}(\Gamma_C) \) and the ones of the form \( p^2 = p \) and \( p = pq \), where \( p, q \) are coterminal paths with \( \ell(p) \in P \) and \( \ell(q) = bl(\ell(p)) \). The inverse category \( \text{ic}(\mathcal{C}) \) corresponding to the \( \Delta \)-complex \( C \) is obtained by factoring the free category on \( \Gamma_C \) by \( \sim \). Note that if \( C = B(X, P) \), then \( \text{ic}(\mathcal{C}) = M(X, P) \).

Note that the relations of \( \sim \) are closely related to the ones defining \( M(X, P) \), that is, two coterminal paths \( p, q \) are in the same \( \sim \)-class if and only if \( \ell(p) = \ell(q) \) in \( M(X, P) \). The loop monoids \( L(\text{ic}(\mathcal{C}), v) \) consist of \( \sim \)-classes of \( (v, v) \)-paths, which can be identified with their (common) label in \( M(X, P) \). Using this identification, we have \( L(\text{ic}(\mathcal{C}), v) = \text{stab}(\mathcal{C}, v) \) for any vertex \( v \). Hence each loop monoid of \( \text{ic}(\mathcal{C}) \) is a closed inverse submonoid of \( M(X, P) \). In fact, if \( C = B(X, P) \), then the unique loop monoid of \( \text{ic}(\mathcal{C}) \) is \( M(X, P) \). The loop monoids play the role of the fundamental group, and \( \text{ic}(\mathcal{C}) \) plays the role of the fundamental groupoid in the classification of immersions. We will denote \( L(\text{ic}(\mathcal{C}), v) \) by \( L(\mathcal{C}, v) \) for brevity.

Proposition 4.7. For any vertex \( v \) in a connected \( \Delta \)-complex \( \mathcal{C} \), the greatest group homomorphic image of \( L(\mathcal{C}, v) \) is the fundamental group of \( \mathcal{C} \).

Proof. The proof follows from the fact that the fundamental groupoid of \( \mathcal{C} \) is \( \text{ic}(\mathcal{C}) \) factored by the congruence generated by relations of the form \( xx^{-1} = \text{id}_{\alpha(x)} \) for any morphism \( x \) (which implies \( bw(C) = \text{id}_{\alpha(C)} \) for any 2-cell \( C \)). Hence \( L(\mathcal{C}, v)/\sigma = \pi_1(\mathcal{C}) \).

5. Classification of Immersions

The previous section has shown how an immersion into \( B(X, P) \) gives rise to a conjugacy class of closed inverse submonoid of its (unique) loop monoid \( M(X, P) \). This section shows that this is true for any immersion, moreover, the converse also holds: immersions are in one-to-one correspondence with conjugacy classes of closed inverse submonoids of loop monoids.

Proposition 5.1. Let \( \mathcal{C} \) and \( \mathcal{D} \) be \( \Delta \)-complexes labeled over a common complex \( B(X, P) \), and suppose \( g: \mathcal{D} \to \mathcal{C} \) is an immersion, and let \( v \in \mathcal{D}^{(0)} \). Then \( L(\mathcal{D}, v) \) is a closed inverse submonoid of \( L(\mathcal{C}, f(v)) \).
Proof. There are immersions $f_C: C \to B(X, P)$ and $f_D: D \to B(X, P)$ defining the labeling, and $f_D \circ g = f_C$. Then $L(D, v) = \text{Stab}(D, v)$ and $L(C, f(v)) = \text{Stab}(C, f(v))$ are both closed inverse submonoids of the corresponding $M(X, P)$. Therefore it is enough to show that $\text{Stab}(D, v) \subseteq \text{Stab}(C, f(v))$. Indeed, suppose $w$ labels a closed path $p$ around $v$ in $D$. Then $g(p)$ is a closed path around $f(v)$ with the same label $w$, which proves the statement. \hfill \Box

We proceed to develop the theory of the converse part of the correspondence. Fix a $\Delta$-complex $C$ labeled over some $B(X, P)$ by an immersion $g: C \to B(X, P)$. Let $u \in C^{(0)}$. Given a closed inverse submonoid $H$ of $L(C, u)$, we construct a complex $C_H$ that immerses into $C$ with $H = L(C_H, v)$ for some $v \in C^{(0)}$.

The complex $C_H$ is defined with the help of the $\omega$-coset graph $\Gamma_H$ of $H$. The inverse monoid $M(XP)$ acts on the vertices of $\Gamma_H$. The idempotents, of course, all label loops.

We build a complex $C_H$ such that $\Gamma C_H = \Gamma_H$. This complex has the following sets of cells:

$$C_H^{(0)} = V(\Gamma_H),$$
$$C_H^{(1)} = \{e \in E(\Gamma_H) : \ell(e) \in X \cup X^{-1}\},$$
$$\text{if } k \geq 2, \quad C_H^{(k)} = \{C_e \in E(\Gamma_H) : \ell(e) \in P_k\}.$$ 

The attaching maps of 1-cells are the ones inherited from the graph $\Gamma_H$. Note that the 1-skeleton then immerses into $C$ via some map $f^{(1)}: C_H^1 \to C^1$ by the results of [5], as Proposition 4.5 ensures that our definitions and assumption reduce to those of [5] when applied to graphs. Moreover, $f(H) = u$.

We build the rest of the attaching maps of $C_H$ inductively. Let $\rho \in P_k$ be a label of a loop in $\Gamma_H$, based at $v$. Then $H$, and therefore $L(C, u)$ contain some conjugate of $\rho$, hence $\rho$ labels a $k$-cell $C^k$ in $C$. Denote its attaching map by $\varphi_k$. Suppose $C_H^S$ is a complex such that there exists an immersion $f^S: C_H^S \to C$, and $\Gamma C_H^S$ is a subgraph $S$ of $\Gamma_H$ which contains all edges labeled by $X \cup P_2 \cup \ldots \cup P_{k-1}$ (that is, the $k-1$-skeleton of $C_H^S$ is that of our desired complex), but does not contain a cell labeled by $\rho$ based at $v$.

**Lemma 5.2.** There exists a unique attaching map $\tilde{\varphi}_k : \partial \Delta_k \to C_H^S$ of a new $k$-cell $\tilde{C}_k$ based at $v$ such that $f^S \circ \tilde{\varphi}_k = \varphi_k$.

**Proof.** We can write the boundary of a $k$-simplex as a (non-disjoint) union of $k - 1$-dimensional simplices:

$$\partial \Delta^k = \bigcup_{i=1}^{k+1} \Delta^{k-1}_i.$$

By the definition of a $\Delta$-complex, $\varphi_k = \bigcup_{i=1}^{k+1} \sigma_i^{k-1}$ for the characteristic maps $\sigma_i^{k-1}: \Delta^{k-1}_i \to C$ of some cells $C_i^{k-1}$, and likewise, if such a $\tilde{\varphi}_k$ exists, it has to be the form of $\tilde{\varphi}_k = \bigcup_{i=1}^{k+1} \tilde{\sigma}_i^{k-1}$ for the characteristic maps
\[ \tilde{\sigma}_i^{k-1} : \Delta_i^{k-1} \to C^S_H \] of some cells \( \tilde{C}_i^{k-1} \). Since \( f \circ \tilde{\varphi}_k = \varphi_k \), \( \tilde{C}_i^{k-1} \) is some preimage of \( C_i^{k-1} \) under \( f^S \), in particular, \( \ell(C_i^{k-1}) = \ell(\tilde{C}_i^{k-1}) \).

For every \( i \) between 1 and \( k+1 \), let \( p_i \) be a path on the one-dimensional faces (edges) of \( \Delta^k \) from \( v_0 \) to the root of \( \Delta_i^{k-1} \). Put \( q_i = \varphi_k(p_i) \), these are paths on the one-cells of \( \mathcal{C} \). Since \( p_i \Delta_i^{k-1}(\Delta_i^{k-1})^{-1} p_i^{-1} \) is a closed generalized path on \( \partial \Delta^k \) around \( v_0 \), for the path \( \varphi_k(p_i \Delta_i^{k-1}(\Delta_i^{k-1})^{-1} p_i^{-1}) = q_i \tilde{C}_i^{k-1}(C_i^{k-1})^{-1} q_i^{-1} \) in \( \mathcal{C} \), by Lemma 4.4 we have

\[ \rho = \ell(C^k) \leq \ell(q_i) \ell(C_i^{k-1}) \ell(C_i^{k-1})^{-1} \ell(q_i)^{-1}, \]

therefore \( \ell(q_i) \ell(C_i^{k-1}) \ell(C_i^{k-1})^{-1} \ell(q_i)^{-1} \) labels a closed path in \( \Gamma_H \) around \( v \), and therefore in \( \tilde{C}_i^{k-1} \) also. The cell \( \tilde{C}_i^{k-1} \) must thus be the unique \( k-1 \)-cell with the label \( \ell(C_i^{k-1}) \) occurring in the previous path, and \( \tilde{\sigma}_i^{k-1} \) is the characteristic map corresponding to \( C_i^{k-1} \).

It remains to be shown that the map \( \tilde{\varphi}_k = \bigcup_{i=1}^{k+1} \tilde{\sigma}_i^{k-1} \), given as a union of maps on non-disjoint domains, is well-defined; that is, for any intersection \( \Delta_{i,j} = \Delta_i \cap \Delta_j \), we have \( \tilde{\sigma}_i^{k-1}|_{\Delta_{i,j}} = \tilde{\sigma}_j^{k-1}|_{\Delta_{i,j}} \). Since \( \Delta_{i,j} \) is a face of both \( \Delta_i \) and \( \Delta_j \), both maps \( \tilde{\sigma}_i^{k-1}|_{\Delta_{i,j}} \) and \( \tilde{\sigma}_j^{k-1}|_{\Delta_{i,j}} \) are characteristic maps \( \tilde{\sigma}_i^{k-2} \) and \( \tilde{\sigma}_j^{k-2} \) for some \( k-2 \)-cells \( \tilde{C}_i^{k-2} \) and \( \tilde{C}_j^{k-2} \). Of course, since \( \varphi_k = \bigcup_{i=1}^{k+1} \sigma_i^{k-1} \), we have \( \sigma_i^{k-1}|_{\Delta_{i,j}} = \sigma_j^{k-1}|_{\Delta_{i,j}} = \varphi_k|_{\Delta_{i,j}} \), hence

\[ f^S \circ \tilde{\sigma}_i^{k-2} = \sigma_i^{k-1}|_{\Delta_{i,j}} = \sigma_j^{k-1}|_{\Delta_{i,j}} = f^S \circ \tilde{\sigma}_j^{k-2}, \]

Now take a path \( s_i \) on the edges of \( \Delta_i \) from the root of \( \Delta_i \) to the root of \( \Delta_{i,j} \), and similarly a path \( s_j \) on \( \Delta_j \). Let \( t_i = \sigma_i^{k-1}(s_i) \), \( t_j = \sigma_j^{k-1}(s_j) \), and \( \bar{t}_i = \tilde{\sigma}_i^{k-1}(s_i) \), \( \bar{t}_j = \tilde{\sigma}_j^{k-1}(s_j) \). Since \( p_i s_i s_j^{-1} p_j^{-1} \) is a closed path on \( \Delta^k \) around \( v_0 \), for the path \( \varphi(p_i s_i s_j^{-1} p_j^{-1}) = q_i t_i t_j^{-1} q_j^{-1} \) in \( \mathcal{C} \), we have \( \ell(q_i t_i t_j^{-1} q_j^{-1}) \geq \rho \), and there is a closed path at \( v \) labeled by \( \ell(q_i t_i t_j^{-1} q_j^{-1}) \) in \( C^S_H \). But, the unique path from \( v \) labeled by \( \ell(q_i) \) ends in \( \alpha(\tilde{C}_i^{k-1}) \) — that is how \( \tilde{C}_i^{k-1} \) was defined —, and the unique path from \( \alpha(\tilde{C}_i^{k-1}) \) labeled by \( \ell(t_i) \) is \( \bar{t}_i \). The same can be said about \( \alpha(\tilde{C}_j^{k-1}) \) and \( \bar{t}_j \), which implies that \( \omega(\bar{t}_i) = \omega(\bar{t}_j) \), hence \( \alpha(\tilde{C}_i^{k-2}) = \alpha(\tilde{C}_j^{k-2}) \). Since \( f^S \) is an immersion which maps \( \tilde{C}_i^{n-2} \) and \( \tilde{C}_j^{n-2} \) to the same cell by (11), this immediately implies \( \tilde{C}_i^{k-2} = \tilde{C}_j^{k-2} \), and hence \( \tilde{\sigma}_i^{k-2} = \tilde{\sigma}_j^{k-2} \). That proves that \( \tilde{\varphi}_k \) is well-defined, and by nature of the construction, unique. Note that \( \tilde{\varphi}_k \) is continuous, since it is a union of continuous maps defined on closed sets, completing the proof.

\[ \square \]

**Lemma 5.3.** Let \( \tilde{C}_H^S \) denote the \( \Delta \)-complex obtained in the previous lemma. Then there exists a unique immersion \( f : \tilde{C}_H^S \to \mathcal{C} \) for which \( f|_{\tilde{C}_H^S} = f^S \).
**Proof.** Let \( \tilde{\sigma}^k \) denote the characteristic map of the cell \( \tilde{C}^k \) of \( \tilde{C}_H^S \), and \( \sigma^k \) denote the characteristic map of \( C^k \) of \( \mathcal{C} \). Let \( f : \tilde{C}_H^S \to \mathcal{C} \) be the map defined by

\[
\begin{align*}
  f|_{\tilde{C}_H^S} &= f^S \\
  f|_{C^k} &= \sigma_k|\text{int}\Delta^k \circ (\tilde{\sigma}^k|\text{int}\Delta^k)^{-1}
\end{align*}
\]

We show that \( f \) is an immersion: it suffices to show that \( f \) commutes with the characteristic maps, and is locally injective at the 0-cells.

Since \( f^S \) is an immersion, \( f \) clearly commutes with the characteristic map of any cell contained in \( C_H^S \). To see that \( f \) commutes with the characteristic map of \( \tilde{C}^k \), let \( x \in \Delta^k \) be an arbitrary point, and consider \( f \circ \tilde{\sigma}^k(x) \). If \( x \in \text{int}\Delta^k \), then \( f \circ \tilde{\sigma}^k(x) = \sigma^k(x) \) by (2). If \( x \in \partial\Delta^k \), then \( x \) lies in a simplex \( \Delta^j \) on \( \partial\Delta^k \), and \( \tilde{\sigma}^k|\Delta^j \) is a characteristic map \( f \) commutes with, therefore we again obtain \( f \circ \tilde{\sigma}^k(x) = \sigma^k(x) \), as desired.

To prove that \( f \) is locally injective at the 0-cells, let \( u \in \tilde{C}_H^S \). Suppose by contradiction that \( f \) is not locally injective at \( u \). Put \( C_1^k := \tilde{C}^k \). Since \( f^S \) is locally injective, there must a cell \( C_2^k \in \tilde{C}_H^S \) distinct from \( C_1^k \) such that any neighborhood of \( u \) contains points \( x_j \in C_2^k \) with \( f(x_1) = f(x_2) \), in particular, \( f(C_1^k) = f(C_2^k) = C^k \). Denote the respective characteristic maps of \( C_1^k \) and \( C_2^k \) by \( \sigma_1 \) and \( \sigma_2 \). Let \( N \) be a neighborhood of \( u \) such that \( \sigma_1^{-1}(N) \) (\( j = 1, 2 \)) is a disjoint union of open sets each containing one preimage of \( u \). One can further ensure that for such sets \( N_i^j \subseteq \sigma_j^{-1}(N) \) containing the vertices \( v_i \), (\( j = 1, 2 \)), the intersection \( N_1^1 \cap N_2^2 \) is empty if \( i_1 \neq i_2 \).

Choose points \( x_j \in C_1^k \cap N \) such that \( f(x_1) = f(x_2) \), denote the point \( \sigma_j^{-1}(x_j) \) by \( y_j \), and let \( N_i^j \) be the unique set in \( \sigma_j^{-1}(N) \) containing \( y_j \). Then \( f \circ \sigma_1(y_1) = f \circ \sigma_2(y_2) \), but since \( f(C_1^k) = f(C_2^k) \), we have \( f \circ \sigma_1 = f \circ \sigma_2 \), therefore \( y_1 = y_2 \). Hence the intersection \( N_1^1 \cap N_2^2 \) is non-empty, and \( i_1 = i_2 =: i \), then \( \sigma_1(v_i) = \sigma_2(v_i) = u \).

Let \( p_i \) be a path from \( v_0 \) to \( v_i \) on the 1-skeleton of \( \Delta^k \), and put \( q_1 = \sigma_1(p_i) \) and \( q_2 = \sigma_2(p_i) \) respectively. Then \( q_1^{-1}C_1^kq_1 \) and \( q_2^{-1}C_1^kq_2 \) both label closed generalized paths based at \( u \) in \( \tilde{C}_H^S \), therefore they label closed paths at \( u \) in \( \Gamma_H \), and \( f \circ \sigma_1 = f \circ \sigma_2 \) implies they have the same label. Since \( \Gamma_H \) is deterministic, this implies that they coincide, which contradicts \( C_1^k \neq C_2^k \). Therefore \( f \) is locally injective at any vertex \( u \), proving \( f \) is an immersion.

The uniqueness of \( f \) follows from the fact that any map satisfying the conditions of the lemma must satisfy (2).

We are ready to prove the following theorem:

**Theorem 5.4.** Let \( \mathcal{C} \) be a \( \Delta \)-complex labeled over some \( B(X, P) \), let \( u \in \mathcal{C}^{(0)} \), and let \( H \) be any closed inverse submonoid of \( L(\mathcal{C}, u) \). Then there exists a unique complex \( \mathcal{C}_H \) and a unique immersion \( f : \mathcal{C}_H \to \mathcal{C} \) with \( H = L(\mathcal{C}_H, v) \) for some vertex \( v \in \mathcal{C}_H \) with \( f(v) = u \).
The existence part of the theorem is clear from the previous construction. For uniqueness, note that \( H = L(C_H, v) \) dictates that \( \Gamma_{C_H} \) is isomorphic to the \( \omega \)-coset graph of \( \Gamma_H \), with \( v \) corresponding to \( H \). The uniqueness of the attaching map of 1-cells is clear from the uniqueness part of Theorem 4.5 of [6]. Then the uniqueness parts of Lemmas 5.2 and 5.3 yield the uniqueness of \( C_H \).

The following main theorem summarizes the previous results and characterizes immersions between finite dimensional \( \Delta \)-complexes.

**Theorem 5.5.** Let \( C \) and \( D \) be connected \( \Delta \)-complexes labeled over a common \( \Delta \)-complex \( B(X, P) \), and suppose \( f: D \to C \) is an immersion that commutes with the labeling maps. If \( v \in D \), \( u \in C \), such that \( f(v) = u \), then \( f \) induces an embedding of \( L(D, v) \) into \( L(C, u) \). Conversely, let \( C \) be a \( \Delta \)-complex labeled over a \( \Delta \)-complex \( B(X, P) \), and let \( H \) be a closed inverse submonoid of the corresponding \( M(X, P) \) such that \( H \subseteq L(C, u) \) for some \( u \in C^0 \). Then there exists a \( \Delta \)-complex \( D \) labeled over the same \( B(X, P) \), and an immersion \( f: D \to C \) and a vertex \( v \in D^0 \) such that \( f(v) = u \) and \( L(D, v) = H \). Furthermore, \( D \) is unique, and \( f \) is unique. If \( H, K \) are two closed inverse submonoids of \( M(X, P) \) with \( H, K \subseteq L(C, u) \), then the corresponding complexes and immersions are equivalent if and only if \( H \) is conjugate to \( K \) in \( L(C, u) \).

**Proof.** The first two statements follow from Proposition 5.1 and Theorem 5.4 respectively: the statement left to prove is the one regarding conjugacy.

First, suppose \( H \) and \( K \) are such that the unique complex and immersion corresponding to \( H \) and \( K \) are the same, that is, \( H = L(D, v_1) \) and \( K = L(D, v_2) \) for some complex \( D \). Let \( m \) label a path from \( v_1 \) to \( v_2 \) in \( D \), then \( m \) labels a path from \( f(v_1) = u \) to \( f(v_2) = u \) in \( C \), hence \( m \in L(C, u) \). It is clear then that \( mL(D, v_2)m^{-1} \subseteq L(D, v_1) \), and \( m^{-1}L(D, v_1)m \subseteq L(D, v_2) \), hence \( H \approx K \) in \( L(C, u) \) indeed.

For the converse, suppose \( H = L(D, v_1) \) for some complex \( D \), and suppose \( K \approx L(C, v_1) \). Then there exists some \( m \in L(C, u) \) such that \( m^{-1}L(D, v_1)m = K \) and \( mKm^{-1} = L(D, v_1) \), in particular, \( mm^{-1} \in L(D, v_1) \). Therefore \( m \) labels a (generalized) path from \( v_1 \) to some 0-cell \( v_2 \) in \( D \). If \( k \in K \), then \( mkm^{-1} \) labels a \( (v_1, v_1) \)-path, hence \( k \) labels a path from \( v_2 \) to \( v_2 \). Therefore \( K \subseteq L(D, v_2) \). On the other hand, if \( n \in L(D, v_2) \), then \( mmnm^{-1} \in L(D, v_1) \), and \( m^{-1}mmn^{-1}m \subseteq K \). Since \( K \) is closed and \( m^{-1}mmn^{-1}m \leq n \), this yields \( n \in K \), therefore \( K = L(D, v_2) \). This proves the statement.

\[ \square \]

6. Closing remarks

We remark that the constructions of the inverse monoid \( M(X, P) \) and of the complex associated with a closed inverse submonoid of \( M(X, P) \) are
effective. The proof of the following theorem makes use of Stephen’s construction of Schützenberger graphs \[13\] and an extension of this developed in \[6\]. We note that the result is somewhat surprising in view of the fact that the maximal group image of \(M(X, P)\) is the fundamental group of \(B(X, P)\), which may have undecidable word problem. However, the fact that \(M(X, P)\) is not \(E\)-unitary enables \(M(X, P)\) to have decidable word problem while its maximal group image may not necessarily have decidable word problem. The proof follows closely along the lines of the proof of Theorem 5.7 of \[6\], so we will omit it.

**Theorem 6.1.** (a) If \(X\) and \(P\) are finite sets, then the word problem for \(M(X, P)\) is decidable.

(b) If \(X\) and \(P\) are finite sets and \(H\) is a finitely generated closed inverse submonoid of \(M(X, P)\), then the associated \(\Delta\)-complex \(C_H\) is finite and effectively constructible.

Similarly, one may obtain the following characterization of the covering maps. Again the proof closely follows the proof of Theorem 6.1 of \[6\].

**Theorem 6.2.** Let \(C, D\) be \(\Delta\)-complexes labeled by an immersion over some complex \(B(X, P)\), let \(f : C \to D\) be an immersion that respects the labeling, and let \(v \in C^0\) be an arbitrary 0-cell. Then \(f\) is a covering map if and only if \(L(C, v)\) is a full closed inverse submonoid of \(L(D, f(v))\), that is, it contains all idempotents of \(L(D, f(v))\).

We conclude by raising the question as to whether an extension of some of the ideas contained in this paper may be developed to provide a classification of immersions between more general topological spaces (for example for arbitrary CW-complexes). It would also be of interest to provide a “presentation-free” characterization of the inverse category \(\text{IC}(\mathcal{C})\) that serves the role of the fundamental groupoid in covering space theory.

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