On the Solution of a Painlevé III Equation

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I.

In a 1977 paper of McCoy, Tracy and Wu there appeared for the first time the solution of a Painlevé equation in terms of Fredholm determinants of integral operators. Specifically, it was shown that a one-parameter family of solutions of the equation

\[ \psi''(t) + \frac{1}{t} \psi'(t) = \frac{1}{2} \sinh 2\psi + 2\alpha \frac{1}{t} \sinh \psi, \]

(1)
a special case of the Painlevé III equation, is given by

\[ \psi(t) = \sum_{n=0}^{\infty} \frac{2}{2n+1} \lambda^{2n+1} \int_1^\infty \cdots \int_1^\infty \prod_{j=1}^{2n+1} \frac{e^{-tu_j}}{u_j + u_j + 1} \left( \prod_{j=1}^{2n+1} \left( \frac{u_j - 1}{u_j + 1} \right)^{\alpha + \frac{1}{2}} + \prod_{j=1}^{2n+1} \left( \frac{u_j - 1}{u_j + 1} \right)^{-\alpha - \frac{1}{2}} \right) \prod_{j=1}^{2n+1} dy_j \cdots dy_{2n+1}. \]

This is clearly expressible in terms of the Fredholm determinants of the kernels

\[ \frac{e^{-tu}}{u + v \left( \frac{u - 1}{u + 1} \right)^{\alpha + \frac{1}{2}}} \]

acting on \( L^2(1, \infty) \). The proof in [2] is complicated, and the purpose of this note is to give a more straightforward one.

First we give an equivalent formulation of the solution in terms of the kernel

\[ K(x, y) = \frac{e^{-t(x+y^{-1})/2}}{x + y} \left| \frac{x - 1}{x + 1} \right|^{2\alpha} \]

acting on \( L^2(0, \infty) \). This is the representation

\[ \psi = \log \det (I + \frac{\lambda}{2} K) - \log \det (I - \frac{\lambda}{2} K), \]

where \( K \) is the operator with kernel \( K(x, y) \). (This is very likely known but seems not to have been written down in the literature before.) To derive this second representation of

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ψ we make the changes of variable \( u_j = (x_j + x_j^{-1})/2 \) in the multiple integral in the first representation. Then

\[
\sqrt{\frac{u_j + 1}{u_j - 1}} = \frac{x_j + 1}{x_j - 1}, \quad du_j = \frac{1}{2} (x_j^2 - 1) \frac{dx_j}{x_j^2},
\]

and the integral becomes

\[
\frac{1}{2^{2n+1}} \int_1^\infty \cdots \int_1^\infty \prod_{j=1}^{2n+1} \frac{e^{-t(x_j+x_j^{-1})/2}}{(x_j + x_{j+1})(x_j x_{j+1} + 1)} \left[ \prod_{j=1}^{2n+1} (x_j + 1)^2 \right.
\]
\[
+ \left. \prod_{j=1}^{2n+1} (x_j - 1)^2 \right] \left( \frac{x_j - 1}{x_j + 1} \right)^{2\alpha} dx_1 \cdots dx_{2n+1}.
\]

Let \( f \) be an eigenfunction for \( K \) with eigenvalue \( \lambda \),

\[
\int_0^\infty \frac{e^{-t(x+x^{-1})/2}}{x+y} \left| \frac{x - 1}{x + 1} \right|^{2\alpha} f(y) dy = \lambda f(x).
\]

Then the substitutions \( x \to x^{-1}, \ y \to y^{-1} \) show that \( x^{-1} f(x^{-1}) \) is also an eigenfunction corresponding to the same eigenvalue. Hence any eigenfunction can be written uniquely as the sum of an “even” eigenfunction \( f_+ \) satisfying \( x^{-1} f_+(x^{-1}) = f_+(x) \) and an “odd” eigenfunction \( f_- \) satisfying \( x^{-1} f_-(x^{-1}) = -f_-(x) \). The change of variable \( y \to y^{-1} \) and the relations \( y^{-1} f_\pm(y^{-1}) = \pm f_\pm(y) \) show that

\[
\int_0^1 \frac{e^{-t(x+x^{-1})/2}}{x+y} \left( \frac{1-x}{1+x} \right)^{2\alpha} f_\pm(y) dy = \pm \int_0^\infty \frac{e^{-t(x+x^{-1})/2}}{xy+1} \left( \frac{x-1}{x+1} \right)^{2\alpha} f_\pm(y) dy.
\]

We deduce that \( f_\pm \) are eigenfunctions corresponding to the eigenvalue \( \lambda \) for the operators \( K_\pm \) on \( L^2(1, \infty) \) with kernels

\[
K_\pm(x,y) = e^{-t(x+x^{-1})/2} \left[ \frac{1}{x+y} \pm \frac{1}{xy+1} \right] \left( \frac{x-1}{x+1} \right)^{2\alpha} = \frac{e^{-t(x+x^{-1})/2}}{(x+y)(xy+1)} (x\pm1) (y\pm1) \left( \frac{x-1}{x+1} \right)^{2\alpha}.
\]

Hence the last displayed multiple integral equals

\[
\text{tr} \ K_{2n+1}^2 + \text{tr} \ K_{2n+1}^{-2},
\]

and it follows that

\[
\psi = 2 \sum_{n=0}^{\infty} \frac{(\frac{1}{2} \lambda)^{2n+1}}{2n+1} (\text{tr} \ K_{2n+1}^{2n+1} + \text{tr} \ K_{2n+1}^{-2n+1}) = 2 \sum_{n=0}^{\infty} \frac{(\frac{1}{2} \lambda)^{2n+1}}{2n+1} \text{tr} \ K_{2n+1}^{2n+1}
\]
\[
= \log \det \left( I + \frac{\lambda}{2} K \right) - \log \det \left( I - \frac{\lambda}{2} K \right),
\]

as claimed.
II.

Direct proofs of the fact that this function $\psi$ satisfies the Painlevé equation when $\alpha = 0$ have already been given \[1, 3\]. We shall make use of some of the results of \[3\] here and therefore follow that paper’s notation, more or less.

First, we introduce parameters $r$ and $s$, define

$$E(x) = \sqrt{\frac{\lambda}{2}} e^{(rx+sx^{-1})/2} \left| \frac{x-1}{x+1} \right|^\alpha, \quad K(x, y) = \frac{E(x) E(y)}{x+y},$$

and let $K$ be the operator with this kernel $K(x, y)$. (In the notation of \[3\], $r = t_1, s = t_{-1}$. The formulas we quote from there will be in terms of our parameters $r$ and $s$.) Define

$$\varphi(r, s) := \log \det (I + K) - \log \det (I - K).$$

Then $\psi(t) = \varphi(-t/2, -t/2)$. We know from \[3\] that $\varphi$ satisfies the sinh-Gordon equation

$$\frac{\partial^2 \varphi}{\partial r \partial s} = \frac{1}{2} \sinh 2\varphi. \tag{2}$$

In order to deduce \[1\] from this we must first find a connection between the $r$ and $s$ derivatives of $\varphi$. (When $\alpha = 0$ the determinants, and so also $\varphi$, depend only on the product $rs$ and \[1\] in this case is almost immediate.) To this end we observe that the determinants are unchanged if $K(x, y)$ is replaced by $\tilde{K}(x, y) := sK(sx, sy)$. This is the same as replacing $E(x)$ by

$$\tilde{E}(x) = \sqrt{\frac{\lambda}{2}} e^{(rsx+sx^{-1})} \left| \frac{sx-1}{sx+1} \right|^\alpha.$$

Now

$$\partial_s \tilde{E}(x) = \left( rx + 2\alpha \frac{x}{s^2x^2 - 1} \right) \tilde{E}(x),$$

which gives

$$\partial_s \tilde{K}(x, y) = r \tilde{E}(x) \tilde{E}(y) + 2\alpha \frac{s^2xy - 1}{(s^2x^2 - 1)(s^2y^2 - 1)} \tilde{E}(x) \tilde{E}(y).$$

(The denominator $x + y$ in $\tilde{K}(x, y)$ was cancelled by its occurrence also as a factor in both summands.) Hence

$$\partial_s \log \det (I + K) = \partial_s \log \det (I + \tilde{K}) = \text{tr} (I + \tilde{K})^{-1} \left[ r \tilde{E}(x) \tilde{E}(y) + 2\alpha \frac{s^2xy - 1}{(s^2x^2 - 1)(s^2y^2 - 1)} \tilde{E}(x) \tilde{E}(y) \right].$$

We abused notation here by writing in the bracket the kernel of the operator that is meant.) Now we undo the variable change we made, which means we replace $x$ by $x/s$ and $y$ by $y/s$ and divide by $s$ in the expressions for the kernels, and we obtain

$$\partial_s \log \det (I + K) = \text{tr} (I + K)^{-1} \left[ \frac{r}{s} E(x) E(y) + \frac{2\alpha}{s} \frac{xy - 1}{(x^2 - 1)(y^2 - 1)} E(x) E(y) \right].$$
If we had differentiated with respect to \( r \) without making a preliminary variable change we would have obtained
\[
\partial_r \log \det (I + K) = \text{tr} \ (I + K)^{-1} E(x) E(y).
\]

Hence we have shown that
\[
s \partial_s \log \det (I + K) - r \partial_r \log \det (I + K) = 2\alpha \text{tr} \ (I + K)^{-1} \left[ \frac{xy - 1}{(x^2 - 1)(y^2 - 1)} E(x) E(y) \right].
\]
Replacing \( K \) by \(-K\) and subtracting gives the relation (we use subscript notation for derivatives)
\[
r \varphi_r - s \varphi_s = 4\alpha \text{tr} \ (I - K^2)^{-1} \left[ \frac{xy - 1}{(x^2 - 1)(y^2 - 1)} E(x) E(y) \right].
\] (3)
This is the desired connection between the \( r \) and \( s \) derivatives of \( \varphi \).

### III.

We want to show that \( \psi(t) = \varphi(-t/2, -t/2) \) satisfies (1), but because of those awkward factors \(-1/2\) we prefer to derive the equivalent equation
\[
\frac{d^2}{dt^2} \varphi(t, t) + t^{-1} \frac{d}{dt} \varphi(t, t) = 2 \sinh 2 \varphi(t, t) - 4\alpha t^{-1} \sinh \varphi(t, t).
\] (4)
We are going to use
\[
\frac{d^2}{dt^2} \varphi(t, t) = 2 \varphi_{rs}(t, t) + \varphi_{rr}(t, t) + \varphi_{ss}(t, t), \quad \frac{d}{dt} \varphi(t, t) = \varphi_r(t, t) + \varphi_s(t, t).
\] (5)
Now we know that \( \varphi(r, s) \) satisfies the sinh-Gordon equation (2) so let us see what identity we have to derive. Set
\[
T = \text{tr} \ (I - K^2)^{-1} \left[ \frac{xy - 1}{(x^2 - 1)(y^2 - 1)} E(x) E(y) \right].
\]
Differentiating (3) with respect to \( r \) and \( s \) gives
\[
r \varphi_{rr} + \varphi_r - s \varphi_{rs} = 4\alpha T_r, \quad r \varphi_{rs} - s \varphi_{ss} - \varphi_s = 4\alpha T_s.
\]
Therefore
\[
-(r + s) \varphi_{rs} + r \varphi_{rr} + s \varphi_{ss} + \varphi_r + \varphi_s = 4\alpha (T_r - T_s),
\]
\[
(r + s) \varphi_{rs} + r \varphi_{rr} + s \varphi_{ss} + \varphi_r + \varphi_s = 2(r + s) \varphi_{rs} + 4\alpha (T_r - T_s).
\]
Setting \( r = s = t \) and using (5) we get
\[
t \frac{d^2}{dt^2} \varphi(t, t) + \frac{d}{dt} \varphi(t, t) = 4t \varphi_{rs}(t, t) + 4\alpha (T_r - T_s)(t, t).
\]
Hence, by (2),
\[
\frac{d^2}{dt^2} \varphi(t, t) + t^{-1} \frac{d}{dt} \varphi(t, t) = 2 \sinh 2 \varphi(t, t) + 4\alpha t^{-1} (T_r - T_s)(t, t).
\]
It follows that (4) is equivalent to
\[
(T_r - T_s)(t, t) = -\sinh \varphi(t, t).
\]
The functions
\[ E_i(x) = x^i E(x), \quad F_i(x) = \frac{E_i(x)}{x^2 - 1}, \quad Q_i = (I - K^2)^{-1} E_i, \quad P_i = (I - K^2)^{-1} K E_i \]
will arise in the computations leading to this identity. We have
\[ T_r - T_s = \text{tr} \left( I - K^2 \right)^{-1} \left[ \frac{xy - 1}{(x^2 - 1)(y^2 - 1)} (\partial_r - \partial_s) E(x) E(y) \right] \]
\[ + \text{tr} (\partial_r - \partial_s) \left( I - K^2 \right)^{-1} \left[ \frac{xy - 1}{(x^2 - 1)(y^2 - 1)} E(x) E(y) \right]. \]  
(6)
Now
\[ (\partial_r - \partial_s) E(x) E(y) = \frac{1}{2} (x - x^{-1} + y - y^{-1}) E(x) E(y), \]
which gives
\[ \frac{xy - 1}{(x^2 - 1)(y^2 - 1)} (\partial_r - \partial_s) E(x) E(y) = \frac{1}{2} \left( \frac{y - x^{-1}}{y^2 - 1} + \frac{x - y^{-1}}{x^2 - 1} \right) E(x) E(y). \]
Hence the first summand in (6) equals
\[ (Q_0, F_1) - (Q_{-1}, F_0). \]  
(7)
Next, using the notation \( a \otimes b \) for the operator with kernel \( a(x) b(y) \), we have (3, p. 4)
\[ (\partial_r - \partial_s) (I - K^2)^{-1} = \frac{1}{2} (P_0 \otimes Q_0 + Q_0 \otimes P_0 - P_{-1} \otimes Q_{-1} + Q_{-1} \otimes P_{-1}), \]
and it follows that the second summand in (6) equals
\[ (Q_0, F_1) (P_0, F_1) - (Q_0, F_0) (P_0, F_0) - (Q_{-1}, F_1) (P_{-1}, F_1) + (Q_{-1}, F_0) (P_{-1}, F_0). \]  
(8)
We introduce notations for the various inner products:
\[ U_{i,j} = (Q_i, F_j), \quad V_{i,j} = (P_i, F_j). \]
These are analogous to the inner products
\[ u_{i,j} = (Q_i, E_j), \quad v_{i,j} = (P_i, E_j) \]
which play a crucial role in (3) and will here, also. We have shown that
\[ T_r - T_s = U_{0,1} - U_{-1,0} + U_{0,1} U_{0,1} - U_{0,0} V_{0,0} - U_{-1,1} V_{-1,1} + U_{-1,0} V_{-1,0}. \]  
(9)
There are relations among the various quantities appearing here. If we set
\[ u_i = u_{0,i}, \quad v_i = v_{0,i}, \quad U_i = U_{0,i}, \quad V_i = V_{0,i} \]
then we have the recursion formulas
\[ x Q_i(x) - Q_{i+1}(x) = v_i Q_0(x) - u_i P_0(x), \]
\[ x P_i(x) + P_{i+1}(x) = u_i Q_0(x) - v_i P_0(x). \]
(The first is formula (9) of [3], the second is obtained similarly.) Taking inner products with \( E_j \) gives the formulas
\[ u_{i,j+1} - u_{i+1,j} = v_i u_j - u_i v_j, \]
\[ v_{i,j+1} + v_{i+1,j} = u_i u_j - v_i v_j \] of [3] and taking inner products with \( F_j \) gives the analogous formulas
\[ U_{i,j+1} - U_{i+1,j} = v_i U_j - u_i V_j, \]
\[ V_{i,j+1} + V_{i+1,j} = u_i U_j - v_i V_j. \]
Observe the special case \( i = j = -1 \) of the second part of (10):
\[ u_{-1}^2 + 1 = (1 + v_{-1})^2. \]
(The \( u_{i,j} \) are symmetric in \( i \) and \( j \).) In fact ([3], p. 8),
\[ u_{-1} = \sinh \varphi, \quad 1 + v_{-1} = \cosh \varphi. \]

We see from the above formulas that all the \( U_{i,j} \) and \( V_{i,j} \) may be expressed in terms of the \( U_i \) and \( V_i \) (with coefficients involving the \( u_i \) and \( v_i \)). But notice that \( F_{i+2} - F_i = E_i \) (here we use the form of \( F_i \) for the first time). This gives \( U_{i+2} - U_i = u_i \), \( V_{i+2} - V_i = v_i \) and using this also it is clear that everything can be expressed in terms of the four unknown quantities \( U_0, V_0, U_1 \) and \( V_1 \) (and the \( u_i \) and \( v_i \)). Using (12) also we compute that (9) equals
\[ -v_{-1} U_1 + u_{-1} V_1 + u_{-1} \]
\[ + U_1 V_1 - U_0 V_0 - ((1 + v_{-1}) U_0 - u_{-1} V_0) (u_{-1} U_0 - (1 + v_{-1}) V_0) \]
\[ + ((1 + v_{-1}) U_1 - u_{-1} V_1 - u_{-1}) (u_{-1} U_1 - (1 + v_{-1}) V_1 - v_{-1}). \]

Now we are going to use, as we did before, the fact that conjugation by the unitary operator \( f(x) \to x^{-1} f(x^{-1}) \) has the effect on \( K \) of interchanging \( r \) and \( s \). Thus \( K \) is invariant under this conjugation when \( r = s \).

Since \( E_i \) is sent to \( E_{-i-1} \) and \( F_i \) to \( -F_{-i+1} \) we find that when \( r = s \)
\[ U_0 = - (Q_{-1}, F_{-1}) = -(1 + v_{-1}) U_0 + u_{-1} V_0, \]
\[ U_1 = -(Q_{-1}, F_0) = -(1 + v_{-1}) U_{-1} + u_{-1} V_{-1} = -(1 + v_{-1})(U_1 - u_{-1}) + u_{-1}(V_1 - v_{-1}). \]

From these we deduce that
\[ V_0 = \frac{u_{-1}}{v_{-1}} U_0, \quad V_1 = \frac{u_{-1}}{v_{-1}} U_1 - 1. \]

Using these we find that when \( r = s \) ([13]) simplifies to
\[ 2 \frac{u_{-1}}{v_{-1}} (U_0^2 - U_1^2). \]

Since \( u_{-1} = \sinh \varphi \) we have reduced the problem to showing that
\[ U_1(t, t)^2 - U_0(t, t)^2 = \frac{1}{2} v_{-1}(t, t). \]
V.

Let us compute \( d/dt \) of both sides of the desired identity. Of course, \( d/dt U_0(t, t) = (\partial_r + \partial_s) U_0(t, t) \), etc., so we begin by writing down these derivatives. We have

\[
2 (\partial_r + \partial_s) E_i = E_{i+1} + E_{i-1}, \quad 2 (\partial_r + \partial_s) F_i = F_{i+1} + F_{i-1},
\]

\[
2 (\partial_r + \partial_s) (I - K^2)^{-1} = \frac{1}{2} (P_0 \otimes Q_0 + Q_0 \otimes P_0 + P_{-1} \otimes Q_{-1} + Q_{-1} \otimes P_{-1}).
\]

(For the last, see [3], p. 4.) Using these we compute

\[
2 (\partial_r + \partial_s) U_i = 2 U_{i+1} + 2 u_0 V_i + (1 + u_{-1}^2 + (1 + v_{-1})^2) U_{i-1} - 2 u_{-1} (1 + v_{-1})^2 V_{i-1}.
\]

Taking \( i = 0 \) and \( 1 \) and using (14) and (12) we find that when \( r = s \)

\[
2 (\partial_r + \partial_s) U_0 = 2 u_0 \frac{u_{-1}}{v_{-1}} U_0 - 2 v_{-1} U_1,
\]

\[
2 (\partial_r + \partial_s) U_1 = 2 u_0 \frac{u_{-1}}{v_{-1}} U_1 - 2 v_{-1} U_0.
\]

Hence

\[
(\partial_r + \partial_s) (U_1^2 - U_0^2) = 2 u_0 \frac{u_{-1}}{v_{-1}} (U_1^2 - U_0^2). \quad (16)
\]

Now we compute in a similar way (cf. [3], p. 5)

\[
2 (\partial_r + \partial_s) v_i = u_{-1} u_0 + v_{-1} v_0 + v_0 + v_{-1,1} + u_{-1,1} u_{-1} + v_{-1,1,1} v_{-1} + v_{-2} + v_{-1,1}.
\]

Again applying the operator \( f(x) \to x^{-1} f(x^{-1}) \) we find that when \( r = s \) we have

\[
u_{-1,1} = 0, \quad v_{-1,1} = v_0, \quad v_{-2} = v_{-1,1},
\]

so the above is

\[
2 (u_{-1} u_0 + v_{-1} v_0 + v_0 + v_{-1,1}).
\]

Applying the second part of (14) with \( i = -1, \ j = 0 \) gives \( v_0 + v_{-1,1} = u_{-1} u_0 - v_{-1} v_0 \), and so we have shown that when \( r = s \)

\[
(\partial_r + \partial_s) v_{-1} = 2 u_{-1} u_0 = 2 u_0 \frac{u_{-1}}{v_{-1}} v_{-1}.
\]

This relation and (13) show that \( U_1(t, t)^2 - U_0(t, t)^2 \) and \( v_{-1}(t, t) \) are equal up to a constant factor, and to deduce (13) it remains only to compute this factor. We do this by determining the asymptotics of both quantities as \( t \to -\infty \). For convenience we evaluate everything at \( r = s = -t \) and let \( t \to +\infty \).

We have

\[
v_{-1} = \left( (I - K^2)^{-1} K E, E_{-1} \right).
\]
If we were to replace \((I - K^2)^{-1}\) by \(I\) we would be left with

\[
(KE, E_{-1}) = \int_0^\infty \int_0^\infty \frac{E(x)^2 E(y)^2}{x + y} y^{-1} dy \, dx \sim \frac{1}{2} \left( \int_0^\infty E(x)^2 \, dx \right)^2
\]
since the main contributions to the integrals come from neighborhoods of \(x = y = 1\). It is an easy exercise to show that

\[
\int_0^\infty e^{-t(x+x^{-1})} \left| \frac{x-1}{x+1} \right|^{2\alpha} dx \sim \Gamma(\alpha + \frac{1}{2}) 2^{-2\alpha} t^{-\frac{3}{4}} e^{-2t},
\]

and so

\[
(KE, E_{-1}) \sim \frac{\lambda^2}{4} \Gamma(\alpha + \frac{1}{2})^2 2^{-4\alpha - 1} t^{-2\alpha - 1} e^{-4t}.
\]
The error caused by our replacement of \((I - K^2)^{-1}\) by \(I\) is of smaller order of magnitude. This follows from the fact that the square of the \(L^2\) norm of \(E\) is \(O(t^{-\alpha - \frac{1}{2}} e^{-2t})\), as shown above, and hence so is the operator norm of \(K\). Thus the error, which equals \(((I - K^2)^{-1}K^2 E, E_{-1})\), is \(O(t^{-4\alpha - 2} e^{-8t})\). Therefore we have shown

\[
v_{-1} \sim \frac{\lambda^2}{4} \Gamma(\alpha + \frac{1}{2})^2 2^{-4\alpha - 1} t^{-2\alpha - 1} e^{-4t}.
\]

Next,

\[
U_1 - U_0 = \left( (I - K^2)^{-1} E, (x + 1)^{-1} E \right), \quad U_1 + U_0 = \left( (I - K^2)^{-1} E, (x - 1)^{-1} E \right)
\]
and \((U_1^2 - U_0^2)\) is the product of these. As before, replacing \((I - K^2)^{-1}\) by \(I\) in each factor will not affect the first-order asymptotics of the product. After this replacement the first inner product becomes \(\lambda/2\) times the integral in (17) but with an extra factor \(x + 1\) in the denominator. Thus

\[
U_1 - U_0 \sim \frac{\lambda}{2} \Gamma(\alpha + \frac{1}{2}) 2^{-2\alpha - 1} t^{-\alpha - \frac{1}{2}} e^{-2t}.
\]

After the replacement the second inner product becomes \(\lambda/2\) times

\[
\int_0^\infty e^{-t(x+x^{-1})} \left| \frac{x-1}{x+1} \right|^{2\alpha} dx = \frac{\lambda}{2} \Gamma(\alpha + \frac{1}{2})^2 2^{-4\alpha - 2} t^{-2\alpha - 1} e^{-4t}.
\]

This is a little trickier since when we make the variable change \(x = 1 + y\) to compute the asymptotics we must use the second-order approximations \(x + x^{-1} \to 2 + y^2 - y^3\) and \((x + 1)^{-2\alpha} \to 2^{-2\alpha}(1 - \alpha y)\). But it is still straightforward and we find that

\[
U_1 + U_0 \sim \frac{\lambda}{2} \Gamma(\alpha + \frac{1}{2})^2 2^{-2\alpha - 1} t^{-\alpha - \frac{1}{2}} e^{-2t}.
\]

Thus

\[
U_1^2 - U_0^2 \sim \frac{\lambda^2}{4} \Gamma(\alpha + \frac{1}{2})^2 2^{-4\alpha - 2} t^{-2\alpha - 1} e^{-4t} \sim \frac{1}{2} v_{-1}.
\]

We knew that \((U_1^2 - U_0^2)/v_{-1}\) is a constant and now we see that the constant equals 1/2. This establishes (13) and concludes the proof.
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