Some congruences involving powers of Delannoy polynomials

Victor J. W. Guo

Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200241, People’s Republic of China
jwguo@math.ecnu.edu.cn, http://math.ecnu.edu.cn/~jwguo

Abstract. The Delannoy polynomial $D_n(x)$ is defined by

$$D_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k.$$ 

We prove that, if $x$ is an integer and $p$ is a prime not dividing $x(x+1)$, then

$$\sum_{k=0}^{p-1} (2k+1)D_k(x)^3 \equiv \left(\frac{-4x-3}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} (2k+1)D_k(x)^4 \equiv p \pmod{p^2},$$

$$\sum_{k=0}^{p-1} (-1)^k(2k+1)D_k(x)^3 \equiv p \left(\frac{4x+1}{p}\right) \pmod{p^2},$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol. The first two congruences confirm a conjecture of Z.-W. Sun [Sci. China 57 (2014), 1375–1400]. The third congruence confirms a special case of another conjecture of Z.-W. Sun [J. Number Theory 132 (2012), 2673–2699]. We also prove that, for any integer $x$ and odd prime $p$, there holds

$$\sum_{k=0}^{p-1} (-1)^k(2k+1)D_k(x)^4 \equiv p \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k}^2 (x^2 + x)^k(2x + 1)^{2k} \pmod{p^2},$$

and conjecture that it holds modulo $p^3$.

Keywords: congruences; Delannoy polynomials; Clausen’s formula; Zeilberger’s algorithm; Fermat’s little theorem

MR Subject Classifications: 11A07, 11B65, 05A10
1 Introduction

The central Delannoy numbers (see [1, 9]) are defined by

\[ D_n = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}. \]

Z.-W. Sun [11–13], among other things, proved many interesting congruences on sums involving Delannoy numbers, such as

\[ p - 1 \sum_{k=0}^{p-1} (2k+1)D_k \equiv p + 2p(2^{p-1} - 1) - p(2^{p-1} - 1)^2 \pmod{p^4}, \]
\[ n - 1 \sum_{k=0}^{n-1} (2k+1)D_k^2 \equiv 0 \pmod{n^2}, \]

where \( p \) is a prime greater than 3. Z.-W. Sun [13] also introduced the Delannoy polynomial \( D_n(x) \) as follows:

\[ D_n(x) = \sum_{k=0}^{n} \binom{n+k}{k} \binom{n+k}{k} x^k, \]

i.e., \( D_n(x) = P_n(2x + 1) \), where \( P_n(x) \) is the Legendre polynomial of degree \( n \) (see, for example, [6, p. 1]). Then he raised the following conjecture.

**Conjecture 1.1** [13, Conjecture 5.1] Let \( x \) be an integer and let \( m \) and \( n \) be positive integers. Then

\[ n - 1 \sum_{k=0}^{n-1} (2k+1)D_k(x)^m \equiv 0 \pmod{n}. \] (1.1)

If \( p \) is a prime not dividing \( x(x+1) \), then

\[ p - 1 \sum_{k=0}^{p-1} (2k+1)D_k(x)^3 \equiv p \left( \frac{-4x - 3}{p} \right) \pmod{p^2}, \] (1.2)
\[ p - 1 \sum_{k=0}^{p-1} (2k+1)D_k(x)^4 \equiv p \pmod{p^2}, \] (1.3)

where \( \left( \frac{\cdot}{p} \right) \) denotes the Legendre symbol.

The congruence (1.1) in a more general form has been confirmed by Pan [7] recently. However, Pan [7] did not give an integer coefficient polynomial formula for

\[ \frac{1}{n} \sum_{k=0}^{n-1} (2k+1)D_k(x)^m. \]

In this paper we shall prove the following results.
Theorem 1.2 Let $n$ be a positive integer. Then
\[
\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1)D_k(x)^3
\]
\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{i} \binom{n}{j+k+1} \binom{n+j+k}{j+k} \binom{i+j}{i} \binom{j+k}{i-k} \binom{2i}{i} x^{i+j}(x+1)^i,
\]
(1.4)

\[
\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1)D_k(x)^4
\]
\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{i} \binom{n}{j+k+1} \binom{n+j+k}{j+k} \binom{i+j}{i} \binom{j+k}{i-k} \binom{2i}{i} \binom{2j}{j} (x^2 + x)^{i+j}.
\]
(1.5)

Theorem 1.3 The supercongruences (1.2) and (1.3) are true.

Theorem 1.4 Let $x$ be an integer and $p$ an odd prime. Then
\[
\sum_{k=0}^{p-1} (-1)^k (2k + 1)D_k(x)^3 \equiv p \left( \frac{4x+1}{p} \right) \pmod{p^2}, \text{ provided that } p \nmid x(x+1),
\]
(1.6)

\[
\sum_{k=0}^{p-1} (-1)^k (2k + 1)D_k(x)^4 \equiv p \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k}^2 (x^2 + x)^k (2x+1)^{2k} \pmod{p^2}.
\]
(1.7)

For any positive integer $n$ and $p$-adic integer $x$, Z.-W. Sun [12, (4.6)] conjectured that
\[
\nu_p \left( \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k (2k + 1)D_k(x)^3 \right) \geq \min \{ \nu_p(n), \nu_p(4x+1) \},
\]
(1.8)

where $\nu_p(x)$ denotes the $p$-adic valuation of $x$. It is clear that the congruence (1.6) confirms the $n = p$ case of (1.8).

2 Proof of Theorem 1.2

It is easy to see that (see [10, Lemma 3.2])
\[
D_n(x)^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \binom{2k}{k} x^k(x+1)^k,
\]
(2.1)
which can be deduced from Clausen’s formula [3] (with \( a = -\frac{n}{2}, b = \frac{n+1}{2} \) and \( x \to -4x(x+1) \)):

\[
2F_1 \left[ \begin{array}{c} a, b \\ a + b + \frac{1}{2} \end{array} ; x \right]^2 = 3F_2 \left[ \begin{array}{c} 2a, 2b, a + b \\ 2a + 2b, a + b + \frac{1}{2} \end{array} ; x \right], \quad |x| < 1, \quad (2.2)
\]

and the following quadratic transformation of Gauss hypergeometric function (see [6, p. 180]):

\[
2F_1 \left[ \begin{array}{c} a, b \\ a + b + \frac{1}{2} ; 4x(1-x) \right] = 3F_2 \left[ \begin{array}{c} 2a, 2b, a \\ a + b + 2, a + b + \frac{1}{2} ; x \right]. \quad (2.3)
\]

Writing \( D_\ell(x)^3 \) as \( D_\ell(x)^2 \cdot D_\ell(x) \) and applying (2.1), we have

\[
\frac{1}{n} \sum_{\ell=0}^{n-1} (2\ell + 1)D_\ell(x)^3 = \frac{1}{n} \sum_{\ell=0}^{n-1} (2\ell + 1) \sum_{i=0}^{\ell} \frac{\ell}{i} \binom{\ell + i}{i} \binom{2i}{i} x^i(x + 1)^i \sum_{j=0}^{\ell} \frac{\ell}{j} \binom{\ell + j}{j} x^j. \quad (2.4)
\]

Note that (see the proof of [5, Lemma 4.2])

\[
\binom{\ell}{i} \binom{\ell + i}{i} \binom{\ell + j}{j} = \sum_{k=0}^{i} \binom{i + j}{i - k} \binom{j + k}{j - k} \binom{\ell}{k} \binom{\ell + j + k}{j + k}. \quad (2.5)
\]

Moreover, by induction on \( n \), we can easily prove that

\[
\sum_{\ell=k}^{n-1} (2\ell + 1) \binom{\ell}{k} \binom{\ell + k}{k} = n \binom{n}{k+1} \binom{n+k}{k}. \quad (2.6)
\]

Substituting (2.5) into (2.4), exchanging the summation order, and then utilizing (2.6), we complete the proof of (1.4).

Similarly, writing \( D_\ell(x)^4 \) as \( D_\ell(x)^2 \cdot D_\ell(x)^2 \) and applying (2.1), we can prove (1.5).

### 3 Proof of Theorem 1.3

**Proof of (1.2).** Letting \( n = p \) be a prime in (1.4), and noticing that \( (\binom{p}{k}) \equiv 0 \pmod{p} \) for \( 1 \leq k \leq p-1 \) and \( (\frac{2p-1}{p}) \equiv 1 \pmod{p} \), we obtain

\[
\frac{1}{p} \sum_{k=0}^{p-1} (2k + 1)D_k(x)^3
= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{k=0}^{j + k} \binom{p}{j + k + 1} \binom{p + j + k}{j + k} \binom{i + j}{i} \binom{j + k}{j} \binom{2i}{i} x^{i+j}(x+1)^i
\equiv \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{i + j}{i} \binom{j}{p - i - 1} \binom{p - 1}{j} \binom{2i}{i} x^{i+j}(x+1)^i \pmod{p}. \quad (3.1)
\]
For \(0 \leq i, j \leq p - 1\), there holds
\[
\binom{i+j}{i} \binom{j}{p-i-1} = 0, \quad \text{if } i + j < p - 1, \\
\equiv 0 \pmod{p}, \quad \text{if } i + j \geq p. \tag{3.2}
\]
Therefore, the possible nonzero summands in (3.1) must satisfy \(i + j = p - 1\). In other words, the congruence (3.1) may be simplified as
\[
\frac{1}{p} \sum_{k=0}^{p-1} (2k + 1) D_k(x)^3 \equiv \sum_{i=0}^{p-1} \binom{p-1}{i} \binom{p-1}{p-i-1} \binom{2i}{i} x^{p-1}(x+1)^i \\
\equiv \sum_{i=0}^{p-1} \binom{2i}{i} (x+1)^i \pmod{p},
\]
where we used the fact \(\binom{p-1}{i} \equiv (-1)^i \pmod{p}\) and Fermat’s little theorem. The proof then follows from the congruence
\[
\sum_{k=0}^{p-1} \binom{2k}{k} x^k \equiv \left( \frac{1 - 4x}{p} \right) \pmod{p} \tag{3.3}
\]
due to Sun and Tauraso [14, Theorem 1.1] (see also [13, Lemma 2.1]). \(\Box\)

**Proof of (1.3).** Let \(n = p\) be a prime in (1.5). Similarly to the proof of (1.2), we have
\[
\frac{1}{p} \sum_{k=0}^{p-1} (2k + 1) D_k(x)^4 \\
\equiv \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{i+j}{i} \binom{j}{p-i-1} \binom{p-1}{j} \binom{2j}{j} (x^2 + x)^{i+j} \\
\equiv \sum_{i=0}^{p-1} \binom{2i}{i} \binom{2p - 2i - 2}{p-i-1} \pmod{p}, \tag{3.2}
\]
(by (3.2) and Fermat’s little theorem)
\[
\equiv 1 \pmod{p},
\]
where in the last step we used the following fact
\[
\binom{2i}{i} \equiv 0 \pmod{p} \quad \text{for} \quad \frac{p-1}{2} < i < p, \tag{3.4}
\]
and \(\binom{p-1}{i}^2 \equiv 1 \pmod{p}. \quad \Box\)
4 Proof of Theorem 1.4

We need the following two lemmas.

Lemma 4.1 Let \( n \) be a positive integer. Then

\[
\frac{1}{n} \sum_{k=0}^{n-1} (-1)^{n-k-1} (2k + 1) D_k(x)^3
\]

\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{i} \binom{n-1}{j+k} \binom{n+j+k}{i+k} \binom{j+k}{i-j-k} \binom{2i}{i} x^{i+j} (x+1)^i, \quad (4.1)
\]

\[
\frac{1}{n} \sum_{k=0}^{n-1} (-1)^{n-k-1} (2k + 1) D_k(x)^4
\]

\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{i} \binom{n-1}{j+k} \binom{n+j+k}{i+k} \binom{j+k}{i-j-k} \binom{2i}{i} \binom{2j}{j} (x^2 + x)^{i+j}. \quad (4.2)
\]

Proof. It is exactly similar to the proof of Theorem 1.2. The difference is that we need to replace (2.6) by the following identity:

\[
\sum_{\ell=k}^{n-1} (-1)^{n-\ell-1} (2\ell + 1) \binom{\ell}{k} \binom{\ell + k}{k} = n \binom{n-1}{k} \binom{n+k}{k},
\]

which can also be proved by induction on \( n \).

Lemma 4.2 Let \( n \) be a positive integer. Then

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{k}{n-k} (-4)^{n-k}. \quad (4.3)
\]

Proof. Applying Zeilberger’s algorithm (see [6,8]), we find that both sides of (4.3) satisfy the following recurrence relation:

\[
(n+2)^2 S(n+2) - 4(3n^2 + 9n + 7) S(n+1) + 32(n+1)^2 S(n) = 0.
\]

Noticing that they also have the same initial values \( S(0) = 1 \) and \( S(1) = 4 \), we complete the proof. \( \square \)
Proof of (1.6). Letting \( n = p \) be a prime not dividing \( x(x + 1) \) in (4.1), we have

\[
\frac{1}{p} \sum_{k=0}^{p-1} (-1)^k (2k + 1) D_k(x)^3
\]

where we used the fact that, for \( 0 \leq j, k \leq p - 1 \),

\[
\binom{p - 1}{j + k} \binom{p + j + k}{j + k} \equiv (-1)^{j+k} \binom{j + k}{k} \quad (\text{mod } p).
\]

By the Chu-Vandermonde summation formula, we get

\[
\sum_{k=0}^{\min\{p-1,m\}} (-1)^k \binom{j}{i-k} \binom{j+k}{k} = (-1)^i.
\]  
\[
\binom{p - 1}{j + k} \binom{p + j + k}{j + k} \equiv (-1)^{j+k} \binom{j + k}{k} \quad (\text{mod } p).
\]  

Substituting (4.5) into (4.4) and using the binomial theorem, we obtain

\[
\frac{1}{p} \sum_{k=0}^{p-1} (-1)^k (2k + 1) D_k(x)^3
\]

\[
= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (-1)^{i+j} \binom{i+j}{i} \binom{2i}{i} x^{i+j} (x+1)^i
\]

\[
= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} (-1)^{i+j+k} \binom{i+j}{i} \binom{2i}{i} \binom{k}{k} x^{i+j+k}
\]

\[
\equiv \sum_{m=0}^{3p-3} \min\{p-1,m\} \sum_{i=0}^{\min\{p-1,m-i\}} (-1)^i \binom{2i}{i} \sum_{j=0}^{\min\{p-1,m-i\}} (-1)^j \binom{i+j}{i} \binom{i}{m-i-j} \quad (\text{mod } p).
\]

Note that, if \( m - i \leq p - 1 \), then

\[
\sum_{j=0}^{\min\{p-1,m-i\}} (-1)^j \binom{i+j}{i} \binom{i}{m-i-j} = \sum_{j=0}^{m-i} (-1)^j \binom{i+j}{i} \binom{i}{m-i-j} = (-1)^{m-i};
\]

while if \( m - i \geq p \), then for \( 0 \leq i, j \leq p - 1 \), there holds \( \binom{i+j}{i} \binom{i}{m-i-j} \equiv 0 \quad (\text{mod } p) \). Hence, we may simplify (4.6) to

\[
\frac{1}{p} \sum_{k=0}^{p-1} (-1)^k (2k + 1) D_k(x)^3 \equiv \sum_{m=0}^{p-1} (-x)^m \sum_{i=0}^{m} \binom{2i}{i} + \sum_{m=p}^{2p-2} (-x)^m \sum_{i=m-p+1}^{p-1} \binom{2i}{i} \quad (\text{mod } p).
\]  
\[
(4.7)
\]
By (3.3) and Fermat’s little theorem, we have

\[
\sum_{m=0}^{p-1} (-x)^m \sum_{i=0}^{m} \binom{2i}{i} = \sum_{i=0}^{p-1} \binom{2i}{i} \sum_{m=i}^{p-1} (-x)^m \\
= \sum_{i=0}^{p-1} \binom{2i}{i} \frac{(-x)^i - (-x)^p}{1+x} \\
\equiv \frac{1}{1+x} \left( \frac{1+4x}{p} \right) + \frac{x}{1+x} \left( \frac{-3}{p} \right) \pmod{p}, \quad (4.8)
\]

and

\[
\sum_{m=p}^{2p-2} (-x)^m \sum_{i=m-p+1}^{m-1} \binom{2i}{i} = (-x)^p \sum_{m=0}^{p-2} (-x)^m \sum_{i=m+1}^{p-1} \binom{2i}{i} \\
= (-x)^p \sum_{i=1}^{p-1} \binom{2i}{i} \sum_{m=0}^{i-1} (-x)^m \\
= (-x)^p \sum_{i=0}^{p-1} \binom{2i}{i} \frac{1 - (-x)^i}{1+x} \\
\equiv -x \left( \frac{-3}{p} \right) + \frac{x}{1+x} \left( \frac{1+4x}{p} \right) \pmod{p}, \quad (4.9)
\]

Substituting (4.8) and (4.9) into (4.7), we complete the proof. \(\square\)

**Proof of (1.7).** Let \(n = p\) be a prime in (4.2). Then similarly to (4.4) we have

\[
\frac{1}{p} \sum_{k=0}^{p-1} (-1)^k (2k+1) D_k(x)^4 \equiv \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (-1)^{i+j} \binom{i+j}{i} \binom{2i}{i} \binom{2j}{j} (x^2 + x)^{i+j} \\
= \sum_{n=0}^{p-1} (-1)^n (x^2 + x)^n \sum_{i=0}^{n} \binom{n}{i} \binom{2i}{i} \binom{2n-2i}{n-i} \pmod{p}, \quad (4.10)
\]

where we used the fact that \(\binom{i+j}{i} \equiv 0 \pmod{p}\) for \(0 \leq i, j \leq p-1\) and \(i+j \geq p\).
By (4.3), the right-hand side of (4.10) is equal to

\[ \sum_{n=0}^{p-1} (-1)^n (x^2 + x)^n \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{k}{n-k} (-4)^{n-k} \]

\[ = \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k}^2 \sum_{n=k}^{p-1} \binom{k}{n-k} 4^{n-k} (x^2 + x)^n \]

\[ = \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k}^2 \sum_{n=k}^{2k} \binom{k}{n-k} 4^{n-k} (x^2 + x)^n \]

\[ = \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k}^2 (x^2 + x)^k (1 + 4x + 4x^2)^k \pmod{p}, \]

where we used the congruence (3.4) and the binomial theorem. This completes the proof. □

5 Two open problems

Motivated by (1.8), we raise the following conjecture:

**Conjecture 5.1** Let \( n \) be a positive integer and \( x \) a \( p \)-adic integer. Then

\[ \nu_p \left( \frac{1}{n} \sum_{k=0}^{n-1} (2k + 1) D_k(x)^3 \right) \geq \min\{\nu_p(n), \nu_p(4x + 3)\}. \] (5.1)

It is obvious that Theorem 1.3 means that the \( n = p \) case of (5.1) is true.

Finally, numerical calculation suggests the following refinement of (1.7).

**Conjecture 5.2** Let \( x \) be an integer and \( p \) an odd prime. Then

\[ \sum_{k=0}^{p-1} (-1)^k (2k + 1) D_k(x)^4 \equiv p \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k}^2 (x^2 + x)^k (2x + 1)^{2k} \pmod{p^3}. \]

Acknowledgments. This work was partially supported by the Fundamental Research Funds for the Central Universities and the National Natural Science Foundation of China (grant 11371144).

References

[1] J.S. Caughman, C.R. Haithcock and J.J.P. Veerman, A note on lattice chains and Delannoy numbers, Discrete Math. 308 (2008), 2623–2628.
[2] S. Chowla, J. Cowles and M. Cowles, Congruence properties of Apéry numbers, J. Number Theory 12 (1980), 188–190.

[3] T. Clausen, Ueber die Fälle, wenn die Reihe von der Form $y = 1 + x^{\alpha} \beta/1.\gamma + \cdots$ ein Quadrat von der Form $z = 1 + x^{\alpha'} \beta'/1.\delta' + \cdots$ hat, J. Reine Angew. Math. 3 (1828) 89–91.

[4] V.J.W. Guo and J. Zeng, New congruences for sums involving Apéry numbers or central Delannoy numbers, Int. J. Number Theory 8 (2012), 2003–2016.

[5] V.J.W. Guo and J. Zeng, Proof of some conjectures of Z.-W. Sun on congruences for Apéry polynomials, J. Number Theory, 132 (2012), 1731–1740.

[6] W. Koepf, Hypergeometric Summation, an Algorithmic Approach to Summation and Special Function Identities, Friedr. Vieweg & Sohn, Braunschweig, 1998.

[7] H. Pan, On divisibility of sums of Apéry polynomials, J. Number Theory 143 (2014), 214–223.

[8] M. Petkovšek, H. S. Wilf and D. Zeilberger, A = B, A K Peters, Ltd., Wellesley, MA, 1996.

[9] R.A. Sulanke, Objects counted by the central Delannoy numbers, J. Integer Seq. 6 (2003), Article 03.1.5.

[10] Z.-H. Sun, Congruences concerning Legendre polynomials II, J. Number Theory 133 (2013), 1950–1976.

[11] Z.-W. Sun, On Delannoy numbers and Schroder numbers, J. Number Theory 131 (2011), 2387–2397.

[12] Z.-W. Sun, On sums of Apéry polynomials and related congruences, J. Number Theory 132 (2012), 2673–2699.

[13] Z.-W. Sun, Congruences involving generalized trinomial coefficients, Sci. China 57 (2014), 1375–1400.

[14] Z.-W. Sun and R. Tauraso, New congruences for central binomial coefficients, Adv. Appl. Math. 45 (2010), 125–148.