ASYMPTOTIC SYMMETRY GROUPS OF LONG-RANGED GAUGE CONFIGURATIONS

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Abstract

We make some general remarks on long-ranged configurations in gauge or diffeomorphism invariant theories where the fields are allowed to assume some non vanishing values at spatial infinity. In this case the Gauss constraint only eliminates those gauge degrees of freedom which lie in the connected component of asymptotically trivial gauge transformations. This implies that proper physical symmetries arise either from gauge transformations that reach to infinity or those that are asymptotically trivial but do not lie in the connected component of transformations within that class. The latter transformations form a discrete subgroup of all symmetries whose position in the ambient group has proven to have interesting implications. We explain this for the dyon configuration in the $SO(3)$ Yang-Mills-Higgs theory, where we prove that the asymptotic symmetry group is $\mathbb{Z}_m \times \mathbb{R}$ where $m$ is the monopole number. We also discuss the application of the general setting to general relativity and show that here the only implication of discrete symmetries for the continuous part is a possible extension of the rotation group $SO(3)$ to $SU(2)$.

Introduction

In theories with gauge or diffeomorphism invariance some of the canonical variables do not really label physically existent degrees of freedom. Rather, they are labels on a phase space, $\Gamma$, whose points represent physical states in a redundant fashion. We leave the question aside as to whether the employment of redundant labelings is purely a matter of convenience or points towards some deeper underlying necessity. In any case, the Hamiltonian formulations of such theories display that fact by presenting constraints – usually called Gauss constraints for gauge theories or

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diffeomorphism constraints for general relativity* – which define the constraint surface \( \tilde{\Gamma} \subset \Gamma \) (their zero level set) and whose Hamiltonian flows connect points which label the \textit{same} physical state. Through each point of \( \tilde{\Gamma} \) passes exactly one of the orbits generated by the Gauss constraints. The program of Hamiltonian reduction (see e.g. \cite{1}) now advises to construct the so-called reduced phase space, \( \hat{\Gamma} \), which is just the space of gauge orbits in \( \tilde{\Gamma} \). It would then furnish a faithful phase space where any two different points label different physical states. This can be interpreted as saying that there are sufficiently many physical observables to separate any two points in \( \hat{\Gamma} \). Clearly this cannot be the case on \( \Gamma \) or \( \tilde{\Gamma} \) if physical observables are required to (Poisson) commute with the Gauss constraints. Unfortunately the construction of \( \hat{\Gamma} \) is forbiddingly difficult in many cases of physical interest. This makes it a working necessity to employ the redundant label space \( \Gamma \) with unsolved constraints.

In many cases, including those mentioned above, the redundant phase space has the structure of an \( \mathcal{R} \)-principal bundle, where \( \mathcal{R} \) now denotes the group that is generated by the Gauss constraints. We call it the \textit{redundancy group}. A crucial remark is that this group is generally only a normal proper subgroup of the group of all admissible gauge transformations which we call the \textit{invariance group}, denoted by \( \mathcal{I} \). Both of these groups have the property of mapping solution curves to the Hamiltonian equations on \( \Gamma \) onto solution curves. However, neither of these deserves to be called a symmetry. To make this clear, we pretend for the moment that we succeeded in the construction of \( \hat{\Gamma} \). Then the quotient \( \mathcal{S} := \mathcal{I}/\mathcal{R} \) would act on \( \hat{\Gamma} \) thereby mapping solution curves to solution curves. But since \( \hat{\Gamma} \) is a faithful label space for physical states, we may say that \( \mathcal{S} \) maps solution curves to new, \textit{physically different} solution curves. It is this feature of being a \textit{physically} active transformation group that, in our opinion, distinguishes a symmetry transformation from a mere redundancy. We thus call \( \mathcal{S} \) the \textit{symmetry group}. The relation of the three groups introduced is compactly displayed by the following sequence of groups and homomorphisms, where at each step the image of the ‘arriving’ map equals the kernel of the ‘departing’ one. One says the sequence is exact. (The unit 1 denotes the trivial group.):

\[
1 \rightarrow \mathcal{R} \xrightarrow{i} \mathcal{I} \xrightarrow{p} \mathcal{S} \rightarrow 1
\]  

* In order to avoid too many repetitions we shall in this section adopt the convention that gauge transformations and Gauss constraints are collective names also including diffeomorphism and the diffeomorphism constraints respectively.
Here the map \( i \) denotes the injective inclusion map, and \( p \) the surjective quotient map from \( I \) onto \( S := I/R \). In group theoretic terms, \( I \) is an \( R \)-extension of \( S \).

However, we here want to consider the situation where one works with \( \Gamma \) rather than \( \hat{\Gamma} \). In this case we are primarily given the invariance group \( I \) and the normal subgroup \( R \) acting on \( \Gamma \) (or \( \hat{\Gamma} \)). The symmetry group \( S \) then arises only as a quotient and cannot generally be expected to also act on \( \Gamma \). This is precisely what happens in gauge theories or general relativity. In these cases the invariance group is given the group of all gauge transformations on a Cauchy hypersurface. If this hypersurface is open with one asymptotic region, as we assume, then \( I \) has to leave invariant the boundary conditions in the asymptotic region. Those boundary conditions may well include those for which the fields assume nonzero and possibly non constant values in the asymptotic region. The group \( R \) is then given by the identity component of asymptotically trivial gauge transformations, where the condition of asymptotic triviality refers to certain falloff condition that must be met in order for them to be generated by the Gauss constraint. In the following sections we will specify the groups \( R \) and \( I \) for long ranged configurations in Yang-Mills-Higgs theory and determine the symmetry group \( S \). The last section deals with general relativity. These parts of the present work may be considered as an elaboration on some aspects presented in less detail in [2] and [3].

**Symmetries of SO(3)-Dyon Configurations**

We consider the standard \( SO(3) \) Yang-Mills-Higgs model, with the Higgs field \( \phi \) in the adjoint representation and symmetry braking potential \( V(\phi) \). As spacetime we take \( \mathbb{R}^4 \). The \( SO(3) \) bundle is topologically trivial so that gauge transformations can be identified with \( SO(3) \)-valued functions.

We identify the \( SO(3) \) Lie algebra with \( \mathbb{R}^3 \) in the standard fashion: \( so(3) \ni \{L_{ij}\} \mapsto \{\frac{1}{2}\varepsilon^{aij}L_{ij}\} =: \{L^a\} =: \vec{L} \). The connection is called \( \vec{A}_\mu \) and the adcovariant derivative reads \( D_\mu = \partial_\mu + \vec{A}_\mu \times \), where the vector product is defined as usual. \( \vec{L} \cdot \vec{M} \) then denotes the standard inner product on \( \mathbb{R}^3 \) which on the Lie algebra corresponds to \(-\frac{1}{2}\) trace.

As an orientation we remind on the asymptotic behavior of the dyon solution found by Julia and Zee [4] (see also [5] for a comprehensive account).

\[
A^a_k(r \to \infty) \propto \varepsilon_{aki} \frac{n^i}{r} + \frac{\alpha}{r^2} + O(1/r^2) \quad (2a)
\]
\[ A_0^a(r \to \infty) \propto n^a + \beta/r + O(1/r) \tag{2b} \]
\[ \phi^a(r \to \infty) \propto n^a + \gamma/r + O(1/r) \tag{2c} \]

where \( \alpha, \beta, \) and \( \gamma \) are certain constants and \( O(1/r^n) \) denotes terms with falloff faster than \( 1/r^n \). In the Hamiltonian formulation the (redundant) phase space \( \Gamma \) is labeled by \( \vec{A}_i \), the Higgs field \( \vec{\phi} \) and their momenta \( \vec{\pi}_i \) respectively. Asymptotically the Higgs field is required to approach the so-called Higgs vacuum which is defined by \( \vec{\phi} \) assuming values in the vacuum manifold \( S_H = \{ \vec{\phi} / \| \vec{\phi} \| = a \} \) (the two-sphere of radius \( a \) in the Lie algebra, where \( a \) sets the symmetry braking scale) and be covariantly constant: \( D_\mu \vec{\phi} = 0 \). The boundary condition for the Higgs field is thus given by a map \( \phi_\infty : S_\infty \to S_H \) whose degree (also called winding number) \( m \) is identified with the monopole number. For example, the “radial” map \((2c)\) has degree \( m = 1 \). But since \( \vec{\phi} \) approaches a radially independent value \( \vec{\phi}_\infty \) (i.e. depending only on \( n^i = x^i/r \)), its partial derivatives \( \partial_k \vec{\phi} \) must approach zero. Since the covariant derivatives must also approach zero, \( \vec{A}_k \) must also approach zero. This is exemplified by \((2)\), where the only asymptotically non vanishing gauge potential is \( \vec{A}_0 \), which in the canonical theory becomes the generator for gauge transformations.

A Lie algebra valued map \( \vec{\Lambda} \) defines an infinitesimal gauge transformation according to

\[ \delta \vec{A}_k = D_k \vec{\Lambda} \tag{3a} \]
\[ \delta \vec{\phi} = -\vec{\Lambda} \times \vec{\phi} \tag{3b} \]

In order to preserve the asymptotic behavior of \( \vec{\phi} \) the asymptotically non vanishing part must be proportional to \( \vec{\phi} \):

\[ \vec{\Lambda}(r \to \infty) = \eta \vec{\phi} + \frac{\lambda(\omega)}{r} + O(1/r) \tag{3c} \]

Inserting this into \((3a)\) shows that \( \delta A_k \) falls off faster than \( 1/r \) if \( \partial_k \eta \) falls off faster than \( 1/r \). The phase space function that generates the transformations \((3)\) is given by

\[ I^\Lambda = \int_{\mathbb{R}^3} d^3x \left\{ \vec{\pi}^k \cdot D_k \vec{\Lambda} + (\vec{\pi} \times \vec{\phi}) \cdot \vec{\Lambda} \right\} \tag{4} \]

In contrast, the Gauss law reads

\[ \vec{G} = -D_k \vec{\pi}^k + \vec{\pi} \times \vec{\phi} = 0 \tag{5} \]
so that
\[ \int_{\mathbb{R}^3} d^3 x \, \vec{\Lambda} \cdot \vec{G} = I^\Lambda + \int_{S_\infty} d\omega (\vec{\pi^k} n_k) \cdot \vec{\Lambda} \] (6)

This tells us that the Gauss constraint generates only those gauge transformations for which the surface term in (6) vanishes. Since we wish to allow a $1/r^2$ falloff for the field strength, this means that $\Lambda$ must approach zero at infinity. The redundancy group $\mathcal{R}$, which was defined to be the group generated by the Gauss constraint, is thus seen to be given by the identity component of all asymptotically trivial gauge transformations. Their group will thus be called $\mathcal{G}_F$ ($F$ to remind on the falloff condition) and its identity component $\mathcal{G}^0_F$. On the other hand, gauge transformations that asymptotically approach rotations about the Higgs field are still allowed. One says that asymptotically the gauge group is broken from $SO(3)$ to $U(1)$, where $U(1)$ labels the rotation angle about the Higgs field. This group of residual gauge transformations contains as a subgroup those that asymptotically assume a constant value in $U(1)$. These we take as our invariance group $\mathcal{I}$, which we now call $\mathcal{G}_\infty$, where the subscript $\infty$ reminds us on the explicit dependence on the asymptotic Higgs field. Modulo the asymptotically trivial gauge transformations, these invariances consist of what one might call the *global* $U(1)$, since $\mathcal{G}_\infty/\mathcal{G}_F \cong U(1)$ (throughout we use the symbol $\cong$ to denote structural isomorphisms). Formally we can characterize $\mathcal{G}_\infty$ by saying that the maps in $\mathcal{G}_\infty$ extend to the one point compactification $(\mathbb{R}^3, \infty) \cong S^3$, with $\infty$ being the point ‘infinity’. This restriction does in fact not imply a loss of generality concerning those aspects we are interested in here, as we will briefly point out at the end of this section. In the sequel we maintain the symbol $\mathcal{S}$ for the symmetry group in each case.

After these general remarks, we now wish to determine the symmetry group $\mathcal{S}$. As just pointed out, the quotient $\mathcal{G}_\infty/\mathcal{G}_F$ is given by the group $U(1)$ whose points label the rotation value (about the Higgs field) at infinity. The group $\mathcal{G}_\infty$ can in fact be regarded as the principal fiber bundle with fiber $\mathcal{G}_F$ and base $U(1)$:

\[ \mathcal{G}_F \xrightarrow{i} \mathcal{G}_\infty \xrightarrow{p} U(1) \] (7)

where the projection map $p$ just evaluates the functions in $\mathcal{G}_\infty$ at $\infty$. Associated to it is the bundle obtained by taking as structure group the group of connected components. It is obtained from (7) by taking the quotient of the total space with
respect to $\mathcal{G}^0_F$. We identify $\mathcal{G}_F$ with the space of based maps $(\mathbb{R}^3, \infty) \rightarrow (SO(3), e)$ where $e$ denotes the identity in $SO(3)$ and where by $(\mathbb{R}^3, \infty)$ we mean the already mentioned $S^3$ compactification of $\mathbb{R}^3$ with basepoint $\infty$. The different connected components are thus labeled by the homotopy group $\pi_3(SO(3)) \cong \mathbb{Z}$, which is generated by the standard covering map $S^3 \rightarrow \mathbb{R}P^3$ of degree two.* We thus obtain the associated bundle

$$Z \cong \mathcal{G}_F/\mathcal{G}^0_F \xrightarrow{i} \mathcal{G}_\infty/\mathcal{G}^0_F \cong S$$

$$\xrightarrow{p} \mathcal{G}_\infty/\mathcal{G}_F \cong U(1)$$

We wish to know how these two abelian groups, $Z$ and $U(1)$, are combined topologically. This can be deduced by looking at the nontrivial piece of the exact sequence for the bundle (8):

$$1 \rightarrow \pi_1(S) \xrightarrow{p} Z \xrightarrow{\partial} Z \xrightarrow{i} S/S^0 \rightarrow 1$$

$$\xrightarrow{\downarrow iso} \pi_1(U(1)) \xrightarrow{\downarrow iso} \mathcal{G}_F/\mathcal{G}^0_F$$

Here the homomorphism $\partial_*$ can be described as follows: Take a loop $\gamma_t$, $t \in [0, 1]$, in $U(1)$ based at the identity $e$ whose homotopy class $[\gamma_t]$ generates $\pi_1(U(1)) \cong \mathbb{Z}$. Let $\epsilon \in p^{-1}(e)$ be the identity of $S$. Now lift $\gamma_t$ to a curve $\bar{\gamma}_t$ in $S$, so that $\bar{\gamma}_0 = \epsilon$. The end point $\bar{\gamma}_1$ lies in the fiber $p^{-1}(e)$ which we identify with $Z$. We thus write $\bar{\gamma}_1 = k \in Z$. Since the fibers are discrete, $k$ only depends on the homotopy class $[\gamma_t]$ and we have $\partial_*(\gamma_t) := k$. It is easy to see that this map is in fact a homomorphisms from $Z$ to $Z$ which is hence given by $\partial_*(n) = kn$. We do not yet know what the integer $k$ is. Here we have:

**Lemma.** The homomorphism $\partial_*$ is given by $n \mapsto mn$, where $m$ is the monopole number.

**Proof.** We must prove that the generator of $Z \cong \pi_1(\mathcal{G}_\infty/\mathcal{G}_F)$ is mapped to $m$ times the generator of $Z \cong \mathcal{G}_F/\mathcal{G}^0_F \cong \pi_3(SO(3))$. To do this, we represent the monopole

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* If one defines winding number as the degree, this is at variance with some statements in the literature. Let us therefore recall that the definition of degree is given by the sum over the signs of the Jacobian determinant for all preimage points of some regular value. There is simply no map of odd degree from $S^3$ to $\mathbb{R}P^3$. 

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configuration in the following form, into which each configuration of monopole number $m \neq 0$ may be smoothly deformed: Take $m$ disjoint open 3-balls in $\mathbb{R}^3$, $B_i$, $i = 1, \ldots, m$, and inside each of them a concentric and slightly smaller closed 3-ball $B'_i$. The zeros of the Higgs field occur precisely at the $m$ centers of these balls. In the regions between the balls, $S_i := B_i - B'_i$, the Higgs field is radially pointing with respect to the local centers. It points either in an outward direction, in which case the monopole number is positive, or inward if the monopole number is negative. We now take a real valued $\mathcal{C}^\infty$ function, $\rho$, that assumes the constant value one outside all balls $B_i$ and zero inside the balls $B'_i$ and only depends on the radial coordinate in each $S_i$ where it is strictly monotonic. We then define a one parameter family of maps $\bar{\gamma}_t : \mathbb{R}^3 \rightarrow SO(3)$, $t \in [0, 1]$, by

$$\bar{\gamma}_t(x) := \exp \left\{ 2\pi t \rho(x) \frac{\phi^a(x)}{\|\phi(x)\|} T_a \right\}$$

where $T_a$ denote the generators of $SO(3)$. This one parameter family of maps just define the lifted curve $\bar{\gamma}_1$ mentioned earlier. $\bar{\gamma}_1$ maps the exterior region $\mathbb{R}^3 - \bigcup_{i=1}^m B_i$ and the interior region $\bigcup_{i=1}^m B'_i$ onto the identity. If we collapse each boundary sphere $\partial B_i$ and $\partial B'_i$ of $S_i$ to a point, the resulting 3-sphere is wrapped twice onto $SO(3)$, since in $S_i$ opposite directions with angles adding to $2\pi$ are mapped to the same point in $SO(3)$. The map $\bar{\gamma}_1$ has thus degree $2m$ so that it represents the element $m \in \mathcal{G}_F / \mathcal{G}_F^0$, denoted by $[\bar{\gamma}_1]$. On the other hand, the maps $\bar{\gamma}_t$ define a loop $\gamma_t$ in $\mathcal{G}_\infty / \mathcal{G}_F$ whose homotopy class, $[\gamma_t]$, generates $\pi_1(\mathcal{G}_\infty / \mathcal{G}_F)$. From the discussion of the map $\partial_*$ we know that $\partial_* [\gamma_t] = [\bar{\gamma}_1]$. This proves the Lemma for $m \neq 0$. For $m = 0$ we take only two concentric balls $B$ and $B'$ and the field configuration so that outside $B'$ the Higgs field assumes a constant value in $S_H$. As before we take a function $\rho$, build the 1-parameter family of maps (10) and have $\partial_* [\gamma_t] = [\bar{\gamma}_1]$. But now the map $\bar{\gamma}_1$ has zero degree $\bullet$

The exactness of (9) allows us to immediately infer from this Lemma the triviality of $\pi_1(S)$ and $S/S^0 \cong \mathbb{Z}_{|m|}$, for $m \neq 0$. For $m = 0$ it implies $\pi_1(S) \cong \mathbb{Z}$ and $S/S^0 \cong \mathbb{Z}$. For $m \neq 0$ this means that the holonomy group of the bundle (8) is the subgroup of integers in $\mathcal{G}_F / \mathcal{G}_F^0$ divisible by $m$. The symmetry group is thus given by the quotient

$$S \cong \mathbb{Z} \times \mathbb{R}^\infty$$

where the group $\mathbb{Z}$ in the denominator is generated by $(m, 2\pi)$. But this quotient is isomorphic to $\mathbb{Z}_{|m|} \times \mathbb{R}$, as the following isomorphism explicitly shows (equivalence
classes referring to the $Z$-quotient are denoted by square brackets)

\[ \theta : Z|_{m|} \times I R \rightarrow \frac{Z \times I R}{Z} \]

\[ \theta(n, r) = [(n, \frac{2\pi n}{m} + r)] \]  

(12)

**Theorem.** The symmetry group $S$ of a dyon configuration with monopole number $m \neq 0$ is isomorphic to $Z|_{m|} \times I R$. The interpretation of these factors may be taken from (8) and (11).

Due to the existence of discrete symmetries, the global $U(1)$ turns out to be neither identical to, nor a subgroup of the symmetry group. Rather, the subgroup $mZ \subset Z$ (denoting the integers divisible by $m$) extended $U(1)$ to become the universal covering group $I R$, which is non compact. This might be taken as topological origin for the possibility of fractional charge in the quantum theory of the model discussed here [2]. We emphasize how crucially this depends on a careful separation of redundancy transformations (defined to be those generated by the Gauss constraints) from among all allowed Gauge transformations. For example, at the end of Ref. [2] it was remarked that – in our notation – generators of $G_F/G_0^F$ do not really deserve to be called “topologically non-trivial” since they are still in the identity component of $G_\infty$. But in quantum theory states are only required to be annihilated by the redundancy group, so that the relevant topology is that of $G_F$ and not $G_\infty$. If we treated the transformations in $G_F/G_0^F$ as redundancies, we only would allow representations of $S$ that restricted to the trivial representation on $G_F/G_0^F$ and there would be no fractional charges. If at all, restrictions on the representations of the general symmetry group must be explained on physical grounds. This could, for example, come about if one tries to implement another group action on the state space as a symmetry. It might then happen that the sectors for the irreducible representations of $G_F/G_0^F$ (labelled by $theta \in S^1$) do not reduce the action of an additional symmetry group, except for specific values of $theta$. Well known is that $CP$ exchanges the sectors $\theta$ and $-\theta$ so that an implementation of $CP$ symmetry selects the $\theta$ values $0$ or $\pi$ [6].

Finally, we comment on our restriction of $G_\infty$ to include only asymptotically constant rotations. One could indeed envisage more general choices, in which $G_\infty/G_F$ could in fact become an infinite dimensional group. For example, one could require the maps in $G_\infty$ to extend to a $2$-sphere compactification. In this case $I R^3$ becomes the interior of a closed $3$-ball whose $2$-sphere boundary, $S_\infty$,  

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now corresponds to infinity. \( \mathcal{G}_\infty/\mathcal{G}_F \) can then be identified with a mapping space \( M : S_\infty \to U(1) \) which forms a group under pointwise multiplication. Our old choice would correspond to the \( U(1) \) subgroup of constant maps. However, given any fixed point \( x \in S_\infty \), we have a mapping \( P : M \to U(1) \) defined by evaluation at the point \( x : \sigma \mapsto \sigma(x) \). Now, this map in fact induces isomorphisms on the homotopy groups. To see this, we also introduce \( M_x = \{ \sigma \in M / \sigma(x) = e \} \) where \( e \) denotes the unit element in \( U(1) \). For the space of base point preserving maps \( M_x \) it is indeed very easy to see that \( \pi_n(M_x) \cong \pi_{n+2}(U(1)) \cong 1 \). On the other hand, we have the fibration

\[
\begin{array}{ccc}
M_x & \overset{i}{\rightarrow} & M \\
\downarrow \quad & & \downarrow P \\
U(1) & \overset{}{\rightarrow} & \end{array}
\]

whose associated exact sequence tells us that the maps \( P_*^{(n)} : \pi_n(M) \to \pi_n(U(1)) \) are isomorphisms for all \( n \geq 0 \). This implies that the discussion following (8) indeed captures all the nontrivial topological implications presented by the discrete group group \( \mathcal{G}_F/\mathcal{G}_F^0 \). It extends the \( U(1) \) subgroup of asymptotically constant rotations about the Higgs field in the way indicated by formula (11).
General Relativity

In this last section we briefly indicate the applicability of the foregoing to general relativity. Here we cannot attempt to account for all technicalities so that some arguments are necessarily somewhat sketchy. Like gauge theories, general relativity also deals with long ranged field configurations which are usually taken to be asymptotically flat. The canonical variables are given by a Riemannian metric, $g_{ik}$, and its conjugate momentum $\pi^{ik}$. Both these tensor fields are defined on a 3-manifold $\Sigma$, the carrier space for the Cauchy data. We assume $\Sigma$ to be without boundary and to possess only one asymptotic region, that is, it contains a compact set outside which it is homeomorphic to the complement of a closed ball in $\mathbb{R}^3$. After space-time is developed from the initial data, the momenta $\pi^{ik}$ can be expressed as a linear function of the extrinsic curvature $K_{ik}$ of $\Sigma$ in space-time. The condition of asymptotic flatness says that for large distances there exists a coordinate chart in which the canonical variables have the following falloff behavior:

$$g_{ik}(x) = \delta_{ik} + \frac{a(n^i)}{r} + O(1/r)$$

$$\pi^{ik}(x) = \frac{b(n^i)}{r^2} + O(1/r^2)$$  \hspace{1cm} (14)

where $n^i = x^i/r$. In addition, the functions $a$ and $b$ must be even respectively odd under parity: $a(-n^i) = a(n^i)$ and $b(-n^i) = -b(n^i)$ (see [7] or [8]). Diffeomorphisms on $\Sigma$ must respect the asymptotic behavior (14). The phase space function that generates infinitesimal such transformations is given by ($\nabla$ denotes the Levi-Civita covariant derivative with respect to the metric $g_{ik}$)

$$I^{\xi} = 2 \int_\Sigma d^3x \nabla_i \xi_k \pi^{ik}$$  \hspace{1cm} (15)

In comparison, the diffeomorphism constraint reads

$$D^k = -2 \nabla_i \pi^{ik} = 0$$  \hspace{1cm} (16)

so that

$$\int_\Sigma d^3x \xi_k D^k = I^{\xi} - 2 \int_{S_\infty} n_i \pi^{ik} \xi_k$$  \hspace{1cm} (17)

Formulae (15)-(17) are just the analogs of (4)-(6) respectively. The diffeomorphism constraints thus only generates the identity component of the asymptotically trivial
diffeomorphisms, for which the surface term in (17) vanishes. In contrast, the identity component of the diffeomorphisms compatible with the asymptotic conditions (14) are generated by vector fields of the asymptotic form

$$\xi_i = \varepsilon_{ijk} \rho^j x^k + \tau_i + O(1)$$

(18)

where $\tau^i$ and $\rho^i$ are constant in the asymptotic chart and represent rigid translations and rotations respectively with respect to the asymptotically euclidean structure. As before, the existence of discrete symmetries may cause the symmetry group to be different from the euclidean group $E_3$. To see this, we remark that it is sufficient to consider only the rotational part and discard the translations within the allowed diffeomorphisms since the mechanism described is entirely topological in nature and thus insensitive to contractible parts of the group. This is just as in the gauge theoretic case. In both cases it is sufficient to retain only the maximal compact subgroup ($SO(3)$ here, $U(1)$ there). This allows us to make use of the formal convenience that all these diffeomorphisms extend to the 1-point compactification $\bar{\Sigma} = \Sigma \cup \infty$. The allowed diffeomorphisms then fix $\infty$ and reduce to $SO(3)$ rotations (with respect to the preferred frame defined by the asymptotic chart) on the tangent space at this point. We call this group, which is now our invariance group, $\mathcal{D}_\infty$. The redundancy group, which is generated by the diffeomorphism constraint, is given by the identity component of those diffeomorphisms that not only fix $\infty$, but also induce the identity map on the tangent space. We call it $\mathcal{D}^0_F$, $F$ for frame-fixing and $0$ to denote the identity component. It is clear that $\mathcal{D}_\infty/\mathcal{D}^0_F \cong SO(3)$ which gives us the analog of (7):

$$\mathcal{D}_F \xrightarrow{\sim} \mathcal{D}_\infty \xrightarrow{p} SO(3)$$

(19)

Here the projection map $p$ is just the evaluation of the tangent map at $\infty$. On the other hand, the symmetry group $\mathcal{S}$ is defined by $\mathcal{S} = \mathcal{D}_\infty/\mathcal{D}^0_F$. Again we wish to know how the discrete normal subgroup $\mathcal{D}_F/\mathcal{D}^0_F$ combines topologically with $SO(3)$ to form $\mathcal{S}$. In full analogy to (8) we have

$$\mathcal{D}_F/\mathcal{D}^0_F \rightarrow \mathcal{D}_\infty/\mathcal{D}^0_F \cong \mathcal{S} \xrightarrow{p} SO(3)$$

(20)
In distinction to (8), we did not indicate what the group $D_F/D_F^0$ is, since this depends on the topology of the underlying 3-manifold $\overline{\Sigma}$. We can nevertheless continue the analogy, and write down the final piece of the exact sequence associated with (20)

$$\begin{array}{ccccccccc}
1 & \rightarrow & \pi_1(S) & \xrightarrow{p_*} & Z_2 & \xrightarrow{\partial} & D_F/D_F^0 & \xrightarrow{i} & S/S^0 & \rightarrow & 1 \\
& & & \downdownarrows_{iso} & \pi_1(SO(3)) & & & & & \end{array}$$

With the obvious adaptations we can almost literally transfer the discussion following Eqn. (9) for the map $\partial_*$. A more geometric description that is analogous to the one surrounding formula (10) goes as follows: Pick a loop $\gamma_t$, $t \in [0,1]$, in $SO(3)$ whose homotopy class $[\gamma_t]$ generates $\pi_1(SO(3)) \cong Z_2$. Take a closed embedded ball $B \subset \overline{\Sigma}$ centered at $\infty$ with standard spherical polar coordinates $(r, \theta, \varphi)$ with $0 \leq r \leq 2$, i.e., $r = 2$ corresponds to the boundary $\partial B$. Let the ball $r \leq 1$ be called $B'$. Let further $\rho$ be a smooth monotonic function $\mathbb{R} \to \mathbb{R}$ so that $\rho(x) = 0$ for $x \leq 0$ and $\rho(x) = 1$ for $x \geq 1$. Define a curve $\bar{\gamma}_t$ in $D_\infty$ in the following way: outside $B$ $\bar{\gamma}_t$ is the identity and inside $B$ it is defined by $\bar{\gamma}_t(r, \theta, \varphi) := (r, \theta, \varphi + 2\pi t \rho(2 - r))$. This defines a lift of $\gamma_t$ into $D_\infty$. The end point, $\bar{\gamma}_1$, corresponds to the identity map inside $B'$ and thus defines an element in $D_F$. It is called a rotation parallel to the spheres $\partial B$ and $\partial B'$, or, since $\infty \in B'$, simply a ‘rotation at infinity’. We denote by $[\bar{\gamma}_1] \in D_F/D_F^0$ its mapping class within in $D_F$. The map $\partial_*$ is now defined by $\partial_*(\gamma_t) = [\bar{\gamma}_1]$. This map is well defined and in fact a homomorphism.

Comparing (9) and (21) we see that now the third entry in the sequence is $Z_2$ rather than $Z$ which leaves us with only two possibilities:

Case I. $\pi_1(S) \cong Z_2$ and Image($\partial_*$) = $\{1\}$ (the trivial group). In this case the bundle (20) is trivial. Geometrically this means that the rotation at infinity is an element of $D_F^0$, i.e., in the identity component. The symmetry group is then just given by

$$S \cong SO(3) \times \{D_F/D_F^0\}$$

Case II. $\pi_1(S) \cong \{1\}$ and Image($\partial_*$) $\cong Z_2$. In this case the bundle is non-trivial with $Z_2$ holonomy. Geometrically this means that the rotation at infinity is not in the identity component $D_F^0$. We have

$$S \cong \frac{SU(2) \times \{D_F/D_F^0\}}{Z_2}$$
where the $Z_2$ in the denominator is ‘diagonal’, that is, it is generated by $(-1, -1)$ where the left $-1$ generates the center in $SU(2)$ and the right $-1$ some (in fact central) $Z_2$ subgroup of $D_F/D_F^0$ which is generated by a rotation at infinity.

Now, whether case I or II is realized depends purely on the topology of the manifold $\bar{\Sigma}$. Both possibilities occur, and it is established for all known 3-manifolds under which case they fall. The working result is in fact easy to communicate: Let $\bar{\Sigma}$ be any of the presently known compact orientable 3-manifolds. It falls under case I, if and only if it is the connected sum of handles ($S^1 \times S^2$) and lens spaces ($L(p, q)$) (the 3-sphere is included here as the space $L(1, 1)$). For more details we refer to [9] and references therein. The interesting thing about case II is that the symmetry group simply does not have an $SO(3)$ rotational subgroup. Manifolds in this class may therefore be termed spinorial. It has been argued that these manifolds could give rise to odd half-integer angular momentum states in quantum gravity which would thus be stabilized against decay in pure gravity [10] (see also [11] for a more recent survey). A general investigation on the structure of the group $D_F/D_F^0$ for arbitrary 3-manifolds $\bar{\Sigma}$ will appear elsewhere [12].

Summary

We started with the observation that in gauge or diffeomorphism invariant theories, not all gauge transformations correspond to redundancies in the presence of long ranged configurations. Rather, it is the Gauss constraint that declares some of the formally present degrees of freedom to be physically non existent. But it only generates the identity component of asymptotically trivial transformations, leaving out the long ranging ones with preserve the asymptotic structure imposed by boundary conditions as well as those not in the identity component of the asymptotically trivial ones. These should be considered as proper physical symmetries which act on physically existing degrees of freedom. For example, asymptotic $U(1)$ gauge rotations of a dyon with non vanishing electric charge, or asymptotic spatial rotations of a black hole with non vanishing angular momentum cost physical action. In order to establish the structure of these symmetry groups one needs to take into account the asymptotically trivial transformations not connected to the identity. These generally do not form a factor of the full symmetry group, but are rather positioned in a topologically non trivial way so as to reduce the number of connected components. In our examples this was described by formulae (11) and (23). As a result, the group $U(1)$ was turned into its non compact universal cover
\(\mathbb{R}\). In gravity, the change from the \(SO(3)\) of spatial rotations to its universal cover \(SU(2)\) was only a possibility (although quite generic), depending on the topology of the underlying 3-manifold \(\tilde{\Sigma}\).

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