One-step Closure, Weak One-step Closure and Meet Continuity

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Abstract

This paper studies the weak one-step closure and one-step closure properties concerning the structure of Scott closures. We deduce that every quasicontinuous domain has weak one-step closure and show that a quasicontinuous poset need not have weak one-step closure. We also constructed a non-continuous poset with one-step closure, which gives a negative answer to an open problem posed by Zou et al. Finally, we investigate the relationship between weak one-step closure property and one-step closure property and prove that a poset has one-step closure if and only if it is meet continuous and has weak one-step closure.

Keywords: Weak one-step closure, One-step closure, Quasicontinuous domain, Quasicontinuous poset, Continuous poset

1 Introduction

The Scott topology is an intrinsic topology on posets, which is the most important topology in domain theory. Scott proved that a domain endowed with the Scott topology is sober. It is well known that a poset is continuous if and only if its Scott closed set lattice is a completely distributive lattice. In [7], Zhao introduced the weak one-step closure property in order to obtain some characterizations of $Z$-continuous posets. In [6], Zou et al. proposed the one-step closure property and proved that every continuous poset has one-step closure. They asked whether all posets with one-step closure are continuous. Since every continuous poset is quasicontinuous, it is natural to wonder whether every quasicontinuous poset also has one-step closure.

In this paper we shall answer the above problems and investigate other aspects of weak one-step closure and one-step closure properties. We give the outline of this paper below.

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In Section 3, we prove that every quasicontinuous domain has weak one-step closure and show, by a counterexample, that a quasicontinuous poset may not have weak one-step closure. In Section 4, we give a negative answer to the problem posed by Zou et al. in [6]. In Section 5, we prove that a poset has one-step closure if and only if it is meet continuous and has weak one-step closure.

Some problems are posed for further investigation.

2 Preliminaries

We now recall some basic notions and results to be used later. We refer the readers to [3], [2] for more about these.

Let $P$ be a poset. For any subset $A$ of $P$, let $\uparrow A = \{y \in P : x \leq y \text{ for some } x \in A\}$ and $\downarrow A = \{y \in P : y \leq x \text{ for some } x \in A\}$. A nonempty subset $D$ of $P$ is directed, denoted by $D \subseteq^\uparrow P$, if every finite subset of $D$ has an upper bound in $D$. The supremum (infimum) of a subset $A$ of $P$, if exists, means the least upper (greatest lower) bound of $A$ in $P$ and will be denoted by $\sup A$ (inf $A$, resp.) A semilattice is a poset in which every nonempty finite subset has an inf; the dual notion is the sup semilattice. A Scott open subset of $P$ is an upper set $U$ of $P$ such that, for every directed subset $D$ of $P$ such that sup $D$ exists and is in $U$, there is a $d \in D$ such that $d \in U$. The complements of Scott open subsets are called Scott closed sets. The collection of all Scott open subsets of $P$ form a topology on $P$, which is called the Scott topology of $P$ and denoted by $\sigma(P)$. The collection of all Scott closed subsets of $P$ is denoted by $\Gamma(P)$. The space $(P, \sigma(P))$ is simply written as $\Sigma P$. For any $A \subseteq P$, we write $cl(A)$ as the Scott closure of $A$ (the closure of $A$ with respect to the Scott topology). We denote the set of all finite subsets of a poset $P$ by $Fin(P)$. The Smyth preorder on the set of all subsets of $P$ is given by $G \leq H$ if $\uparrow H \subseteq \uparrow G$. We say that $G$ is way below $H$ and write $G \ll H$ if for every directed subset $D \subseteq P$, sup $D \in \uparrow H$ implies $D \cap \uparrow G \neq \emptyset$. We write $G \ll x$ for $G \ll \{x\}$ and $\uparrow G = \{x \in L \mid G \ll x\}$. For $x, y \in P$, $x$ is way-below $y$, denoted by $x \ll y$, if for any directed subset $D$ of $P$ for which sup $D$ exists, $y \leq \sup D$ implies $D \cap \uparrow x \neq \emptyset$.

A poset $P$ is directed complete if sup $D$ exists for all $D \subseteq^\uparrow P$. A directed complete poset will be called a dcpo.

A subset $A$ of a topological space is saturated if $A$ is the intersection of all open sets containing $A$. For a topological space $X$, the set of all compact saturated subsets of $X$ is denoted by $Q(X)$. We write $\mathfrak{R} \subseteq_{fl} Q(X)$ represents that $\mathfrak{R}$ is filtered. We denote the set of all open sets of space $X$ by $\mathcal{O}(X)$. On $Q(X)$, we consider the upper Vietoris topology generated by the sets $\Box U = \{K \in Q(X) \mid K \subseteq U\}$, where $U \in \mathcal{O}(X)$.

Definition 2.1 ([6]) A poset $P$ is said to have one-step closure if $cl(A) = A'$ holds for any $A \subseteq P$, where $A' = \{x \in P \mid \exists D \subseteq^\uparrow \downarrow A, x = \sup D\}$.

Definition 2.2 ([2]) A poset $P$ is meet continuous if for any $x \in P$ and any directed set $D$ of $P$ with sup $D$ existing, $x \leq \sup D$ implies $x \in cl(\downarrow D \cap \downarrow x)$.

Remark 2.3 For a semilattice $L$, one can prove that it is meet continuous if and only if it satisfies $\inf\{x, \sup D\} = \sup_{d \in D} \inf\{x, d\}$ for any $x \in L$ and any directed set $D \subseteq L$ with sup $D$ existing.

Definition 2.4 ([2]) A poset $P$ is quasicontinuous, if for every $x \in P$,

1. $fin(x) = \{F \mid F \in Fin(P), F \ll x\}$ is a directed family;

2. $\uparrow x = \bigcap_{F \in fin(x)} \uparrow F$ for any $x \in P$.

A quasicontinuous dcpo is called a quasicontinuous domain.

For any quasicontinuous domain $P$, the family $\{\uparrow F : F \subseteq P \text{ is finite}\}$ is a base of the Scott topology on $P$ ([2]).

Definition 2.5 ([2]) A space $X$ is well-filtered if for each filter basis $\mathcal{C}$ of compact saturated sets of $X$ and each open set $U$ with $\bigcap \mathcal{C} \subseteq U$, there is a $K \in \mathcal{C}$ such that $K \subseteq U$. 
Definition 2.6 ([1]) The set $\mathbb{R}$ of all real numbers equipped with the topology having \{\([x, y) \mid x < y, x, y \in \mathbb{R}\)\} as a base is called the Sorgenfrey line, which is denoted by $\mathbb{R}_l$.

3 Weak one-step closure

By [6], every continuous poset has one-step closure. However, a quasicontinuous poset may not have one-step closure. In this section, we consider a weaker property, called weaker one-step closure. We prove that every quasicontinuous domain has the weak one-step closure, but a quasicontinuous poset need not have this property.

Definition 3.1 A poset $P$ is said to have the weak one-step closure if for any $A \subseteq P$, it holds that $\text{cl}(A) = A''$, where $A'' = \{x \in P \mid \exists D \subseteq \uparrow \downarrow A, x \leq \sup D\}$

Remark 3.2 In [7], Zhao introduced the Definition 3.1 for an arbitrary set system, and called it one-step closure. To be consistent with the paper [6], here we call this property weak one-step closure.

Theorem 3.3 Every quasicontinuous dcpo has weak one-step closure.

Proof. It suffices to show that $\text{cl}(A) \subseteq A''$ for any subset $A$ of $L$. To this end, let $x \in \text{cl}(A)$, $F \in \text{fin}(L)$ with $x \in \uparrow F$. Then $\uparrow F$ is Scott open as $L$ is quasicontinuous. Hence $\uparrow F \cap A \neq \emptyset$, which implies that $F \cap \downarrow A \neq \emptyset$. Thus $(F \cap \downarrow A)_{F \in \text{fin}(x)}$ is a filtered family (with respective to the Smyth preorder) of nonempty finite subsets of $L$. By Rudin’s Lemma ([7]), there exists a directed subset $D$ of $\bigcup_{F \in \text{fin}(x)} F \cap \downarrow A$ such that $D \cap (F \cap \downarrow A) \neq \emptyset$ for any $F \in \text{fin}(x)$. Also, since $L$ is a quasicontinuous domain, $\{\uparrow F \mid F \in \text{fin}(x)\}$ is a neighborhood basis of $x$. This indicates that $x \in \text{cl}(D) = \downarrow \sup D$. Note that $D \subseteq \downarrow A$. We conclude that $x \in A''$. Hence $\text{cl}(A) \subseteq A''$. \qed

The following example shows that the converse conclusion of Theorem 3.3 is not true.

Example 3.4 Let $L = (\mathbb{N} \times \mathbb{N}) \cup \{\top\}$. Define order $\leq$ on $L$ as follows:

(i) $(m, n) \leq (s, t)$ if and only if $m = s$ and $n \leq t$;
(ii) $x \leq \top$ for all $x \in L$.

It is well known that $L$ is a dcpo and not quasicontinuous. However, we can easily verify that $L$ has weak one-step closure.

Note that this dcpo $L$ does not have one-step closure.

The dcpo $L$ is illustrated in Figure 1.

![Fig.1. A non-quasicontinuous domain that has weak one-step closure.](image)

The following example shows that a quasicontinuous poset may not have weak one-step closure.

Example 3.5 Let $L = (\mathbb{N} \times (\mathbb{N} \cup \{\omega\})) \cup \mathbb{N}$. We define an order $\leq$ on $L$ as follows:

For any $x, y \in L$, $x \leq y$ if and only if one of the following holds:
(i) \( x = (m, n_1), y = (m, n_2), n_1 \leq n_2; \)
(ii) \( x = (m, n_1), y = (m, \omega); \)
(iii) \( x, y \in \mathbb{N} \) and \( x \leq y \) in \( \mathbb{N}; \)
(iv) \( x = (m, n), y \in \mathbb{N}, y \geq n, m \geq 2; \)
(v) \( x = (1, n_1), y = (m_2, \omega), m_2 \geq n_1; \)
(vi) \( x = (m_1, n), y = (m_2, n), m_1 \leq m_2, m_1 \geq 2. \)

\( L \) can be illustrated in Figure 2. Then \( L \) is a quasicontinuous poset, but \( L \) does not have weak one-step closure.

To see this, first note that \((1, \omega) \in cl(\mathbb{N}) = L \) and \((1, \omega) \notin \mathbb{N}^*\). Hence, \( L \) does not have weak one-step closure. It remains to show that \( L \) is quasicontinuous.

(i) For \((1, \omega), \) we have \( \{(1, n) \mid n \in \mathbb{N}\} \subseteq \uparrow (1, \omega) \) and \((1, \omega) = \sup_{n \in \mathbb{N}} (1, n).\)

(ii) For each \((1, n) \in L. \) Let \( F_{n,m} = \{((1, n), (2, m)) \mid m \in \mathbb{N}\}. \) Then \( \{F_{n,m} \mid m \in \mathbb{N}\} \subseteq fin(1, n) \) and is a filtered base with \( \uparrow (1, n) = \bigcap_{m \in \mathbb{N}} \uparrow F_{n,m}. \)

(iii) For each \((m, n) \in L \) with \( m \in \mathbb{N} \) and \( m \geq 2, \) we see easily that \((m, n) \ll (m, n). \) In addition, each \((m, \omega) \) with \( m \geq 2 \) is the supremum of the directed set \( \{(m, n) : n \in \mathbb{N}\} \) of compact elements.

All these together show that \( L \) is quasicontinuous.

![Fig.2. A quasicontinuous poset does not have weak one-step closure.](image)

4 One-step closure

In [6], Zou, Li and Ho showed that every continuous poset has one-step closure. They asked whether \( L \) is continuous if it has one-step closure. We now give a counterexample for their problem. We begin with a lemma which is crucial for further study.

**Lemma 4.1** If \( X \) is a well-filtered space and \( Q(X) \) endowed with the upper Vietoris topology is first-countable, then \((Q(X), \supseteq)\) has one-step closure.

**Proof.** Let \( A \subseteq Q(X) \) and \( K \in cl(A). \) The fact that \( Q(X) \) equipped with the upper Vietoris topology is first-countable implies that there exists a countable neighborhood basis \( B_K = \{\Box U_n \mid n \in \mathbb{N}\} \) of \( K \) and \( \Box U_{n+1} \subseteq \Box U_n \) for any \( n \in \mathbb{N}. \)

Claim 1: \( \Box U \subseteq \Box V \) implies \( U \subseteq V \) for any \( U, V \in O(X). \)
Let $x \in U$. Then $\uparrow x \in \square U \subseteq \square V$. In other words $\uparrow x \subseteq V$. So $U \subseteq V$ holds.

From Theorem 5.8 in [5], we know that the upper Vietoris topology coincides with the Scott topology on $(Q(X), \supseteq)$. It follows that $\square U_n \cap A \neq \emptyset$ for any $\square U_n \in \mathcal{B}_K$ due to the assumption that $K \in \text{cl}(A)$. Choose $K_n \in \square U_n \cap A$ for any $n \in \mathbb{N}$. We define $Q_n = K \cup \bigcup_{m \geq n} K_m$ for any $n \in \mathbb{N}$.

Claim 2: $Q_n \in Q(X)$ for each $n \in \mathbb{N}$.

As a union of saturated sets, $Q_n$ is a saturated. It suffices to verify that $Q_n$ is compact. Let $\{W_i : i \in I\}$ be a family of open sets of $X$ such that $Q_n \subseteq \bigcup_{i \in I} W_i$. Then $K \subseteq \bigcup_{i \in I} W_i$. As $K$ is compact, there exists $F_1 \in \text{Fin}(I)$ such that $K \subseteq \bigcup_{i \in F_1} W_i$. Then, there exists $\square U_{n_0} \in \mathcal{B}_K$ such that $K \in \square U_{n_0} \subseteq \bigcup_{i \in F_1} W_i$.

We consider the following two cases:

Case 1. $n_0 \leq n$: For any $m \geq n \geq n_0$, then $K_m \subseteq U_m \subseteq U_{n_0}$ by Claim 1. Hence, $Q_n \subseteq U_{n_0} \subseteq \bigcup_{i \in F_1} W_i$.

Case 2. $n_0 > n$: We can obtain that $\bigcup_{m \geq n_0} K_m \subseteq U_{n_0} \subseteq \bigcup_{i \in F_1} W_i$ by the similar proof to Case 1.

Note that $\bigcup_{i = n}^{n_0 - 1} K_i \in Q(X)$ and $\bigcup_{i = n}^{n_0 - 1} K_i \subseteq \bigcup_{i \in I} W_i$. This means that there exists $F_2 \in \text{Fin}(I)$ such that $\bigcup_{i = n}^{n_0 - 1} K_i \subseteq \bigcup_{i \in F_2} W_i$. Therefore, $Q_n \subseteq \bigcup_{i \in F_1 \cup F_2} W_i$.

Claim 3: $K = \sup_{n \in \mathbb{N}} Q_n = \bigcap_{n \in \mathbb{N}} Q_n$.

It is easy to see that $K \subseteq \bigcap_{n \in \mathbb{N}} Q_n$. For the converse, suppose $x \in \bigcap_{n \in \mathbb{N}} Q_n$. We claim that $x \in K$. Assume $x \notin K$. This manifests $x \cap K = \emptyset$. In other words, $K \subseteq X \setminus x$. It follows that there exists $n \in \mathbb{N}$ such that $K \in \square U_n \subseteq \square X \setminus x$. Through Claim 2, we can conclude that $x \in Q_n \subseteq U_n \subseteq X \setminus x$, which contradicts $x \notin x$.

Note that $K_n \in A$ for any $n \in \mathbb{N}$ and $Q_n \subseteq K_n$ (with respect to the reverse inclusion order). So $(Q_n)_{n \in \mathbb{N}}$ is a directed subset of $\downarrow A$ whose supremum equals $K$. Hence, $Q(X)$ has one-step closure.

The conclusion given in the next theorem answers the question from Zou et al.

Theorem 4.2 For the Sorgenfrey line $\mathbb{R}_l$, $Q(\mathbb{R}_l)$ has one-step closure and $Q(\mathbb{R}_l)$ is not continuous.

Proof. By Example 5.18 of [5], we know that the poset $Q(\mathbb{R}_l)$ is not continuous. The space $\mathbb{R}_l$ is Hausdorff, thus well-filtered (every Hausdorff space is sober and every sober space is well-filtered). Hence, by Lemma 4.1, $Q(\mathbb{R}_l)$ has one-step closure.

5 The relationship between weak one-step closure and one-step closure

In this section, we investigate the relationship between weak one-step closure and one-step closure.

The following lemma justify the term “weak one-step closure”.

Lemma 5.1 If a poset $P$ has one-step closure, then it has weak one-step closure.

Proof. It suffices to prove that $\text{cl}(A) = A''$ for any subset $A$ of $P$. From the definition of one-step closure, we have $\text{cl}(A) = A'$. One sees obviously that $A'' \subseteq \text{cl}(A)$. Let $x \in \text{cl}(A)$. Then $x \in A'$.

It follows that there exists $D \subseteq \uparrow A$ such that $x = \text{sup} D$, i.e., $x \in A''$. The converse of Lemma 5.1 is not true.

Example 5.2 Let $L = \mathbb{N} \cup \{\omega, a\}$, where $\mathbb{N}$ denotes all natural numbers. We define an order $\leq$ on $L$ by $x \leq y$ if and only if:

(i) $x, y \in \mathbb{N}$ and $x \leq y$ holds in $\mathbb{N}$, or

(ii) $x \in L$ and $y = \omega$.

Then $L$ can be easily illustrated in Figure 3. If $\mathbb{N}' = \mathbb{N} \cup \{\omega\} \subseteq \mathbb{N}'' = L$. Thus $L$ does not have one-step closure. But it is easy to check that $L$ has weak one-step closure.

It is then natural to wonder under what conditions, a poset has one-step closure if it has weak one-step closure. We shall prove that if a poset has weak one-step closure, then it has one-step closure if and only if it is meet continuous.
Lemma 5.3 Let $L$ be a poset. Then the following statements are equivalent:

1. $A'$ is a lower set for any $A \subseteq L$;
2. $D'$ is a lower set for any directed subset $D$ of $L$.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Assume $x \leq y \in A'$. Then there exists $D \subseteq A$ such that $y = \sup D$. This means that $x \leq y \in D'$. It follows that $x \in \downarrow D' = D'$. So we have that there exists a directed subset $E$ of $\downarrow D$ such that $x = \sup E$. Note that $E \subseteq \downarrow D \subseteq \downarrow A$. Therefore, $x \in A'$.

If $L$ has one-step closure, then for any subset $A \subseteq L$, $\text{cl}(A) = A'$, so it is a lower set.

In [6], Zou, Li and Ho proved that $L$ is meet continuous if $L$ has one-step closure. We now deduce this result using a weak assumption.

Lemma 5.4 Let $L$ be a poset. If $D' = \downarrow D'$ for any $D \subseteq L$, then $L$ is meet-continuous.

Proof. Let $x \in L$, $D \subseteq L$ with $\sup D$ existing. If $x \leq \sup D$, then $x \in \downarrow D' = D'$. This means that there exists a directed subset $E$ of $\downarrow D$ such that $x = \sup E$. Note that $E \subseteq \downarrow x \cap \downarrow D$. This implies that $x \in \text{cl}(\downarrow x \cap \downarrow D)$. Therefore, $L$ is meet-continuous.

Corollary 5.5 Every poset with one-step closure is meet continuous.

Corollary 5.6 Let $L$ be a meet continuous semilattice. Then $D'$ is a lower set for any directed subset $D$ of $L$. Moreover, if $L$ has weak one-step closure, then $L$ has one step closure.

Proof. From Lemma 5.8, it suffices to prove that $D'$ is a lower set for any directed subset $D$ of $L$. Suppose $x \leq y \in D'$. Then there exists a directed subset $E$ of $\downarrow D$ with $y = \sup E$. The fact that $L$ is a meet continuous semilattice implies that $x = \sup_{e \in E} \inf\{x, e\}$. It is noteworthy that $\{\inf\{x, e\} \mid e \in E\}$ is a directed subset of $\downarrow D$. This means that $x \in D'$. Therefore, $D'$ is a lower set.

Corollary 5.7 Let $L$ be a meet continuous sup-semilattice. Then $D'$ is a lower set for any directed subset $D$ of $L$.

Proof. Let $x \leq y \in D'$. This means that there exists a directed subset $E$ of $\downarrow D$ such that $y = \sup E$. Since $L$ is meet continuous, $x \in \text{cl}(\downarrow x \cap \downarrow E)$. It follows that $\downarrow x = \text{cl}(\downarrow x \cap \downarrow E)$, and hence $x = \sup(\downarrow x \cap \downarrow E)$. Let $G = \{\sup F : F \subseteq (\downarrow x \cap \downarrow E) \text{ and } F \text{ is finite}\}$. Then $G$ is a directed subset of $\downarrow x \cap \downarrow E$ with $\sup G = \sup(\downarrow x \cap \downarrow E) = x$. Clearly $G \subseteq \downarrow D$, thus $x \in D'$, showing that $D'$ is lower.

Lemma 5.8 Let $L$ be a poset with weak one-step closure. If $D'$ is a lower set for any directed subset $D$ of $L$, then $L$ has one-step closure.

Proof. This follows immediately from Definition 3.1, Definition 2.1 and Lemma 5.3.
Lemma 5.9 Let L be a meet continuous poset with weak one-step closure. Then L has one-step closure.

Proof. By Lemma 5.8, it suffices to show that $D'$ is a lower set for any directed $D \subseteq L$. Suppose $x \leq y \in D'$. Then, there exists a directed subset $E$ of $\downarrow D$ such that $y = \sup E$. Since $L$ is meet continuous, we have $x \in \text{cl}(\downarrow x \cap \downarrow E)$. Since $L$ has weak one-step closure, there is a directed $K \subseteq \downarrow x \cap \downarrow E$ such that $x \leq \sup K$. But, trivially $\sup K \leq x$, hence $x = \sup K$. In addition, $K \subseteq \downarrow E \subseteq \downarrow D$, so $x \in D'$. Therefore $D'$ is a lower set. □

From Lemma 5.1 and Lemma 5.9 we deduce the following result.

Theorem 5.10 A poset has one-step closure if and only if it is meet continuous and has weak one-step closure.

Since every poset having one-step closure is meet continuous, we have the following natural problem.

Problem 5.11 Is there a meet continuous poset that does not have one-step closure.

We have already given in Section 4 an example of a non-continuous poset that has one-step closure. We now confirm that an exact poset with one-step closure is continuous.

Definition 5.12 ([4]) Let $x, y$ be elements of a poset $P$. We say that $x$ is weakly way-below $y$, denoted by $x \ll_w y$, if for any directed subset $D$ of $P$ for which $\sup D$ exists, $y = \sup D$ implies $D \cap \uparrow x \neq \emptyset$. A poset $P$ is called exact if for any $x \in P$, $\downarrow_w x = \{ y \in P \mid y \ll_w x \}$ is directed and $\sup \downarrow_w x = x$.

Theorem 5.13 Let $L$ be a poset. Then the following statements are equivalent:

1. $L$ is continuous;
2. $L$ has one-step closure and is exact;
3. $A'$ is a lower set for any $A \subseteq L$ and $L$ is an exact poset.
4. $D'$ is a lower set for any directed subset $D$ of $L$ and $L$ is an exact poset.

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) are all obvious.

(4) ⇒ (1) From the definition of exact posets and continuous posets, it suffices to prove that $x \ll_w y$ implies that $x \ll y$ for any $x, y \in L$. Let $D$ be a directed subset of $L$ with $y \leq \sup D$, it follows that $\sup D \in D'$. We have that $y \in D'$ since $D'$ is a lower set. This means that there exists $E \subseteq \uparrow D$ such that $y = \sup E$. The assumption that $x \ll_w y$ reveals that $\uparrow x \cap E \neq \emptyset$. Hence, $\uparrow x \cap D \neq \emptyset$. □

The following is a problem concerning the connections among the concepts of meet continuity, exactness and continuity of posets.

Problem 5.14 Let $L$ be a meet continuous and exact poset. Must $L$ be continuous?

Although we cannot solve the above problems, we have the following corollary by Corollary 5.6 and Corollary 5.7.

Corollary 5.15 Let $L$ be a meet continuous semilattice or sup-semilattice. Then $L$ is continuous iff $L$ is an exact poset.

Proposition 5.16 Let $L, M$ be two posets. If $L$ is a Scott retract of $M$, which has one-step closure, then $L$ has one-step closure.

Proof. Since $L$ is a Scott retract of $M$, we have that there exists two Scott continuous maps $s : L \to M$ and $r : M \to L$ such that $\text{id}_L = r \circ s$. Now let $x \in L$, $A \subseteq L$ with $x \in \text{cl}(A)$. It follows that $s(x) \in \text{cl}(s(A))$ by the Scott continuity of $s$. We know that there exists $D \subseteq \uparrow s(A)$ such that $s(x) = \sup D$ since $M$ has one-step closure. This implies that $x = r \circ s(x) = r(\sup D) = \sup r(D)$ from the Scott continuity of $r$. Note that $r(D) \subseteq \downarrow r(s(A)) = \downarrow A$. This means that $L$ has one-step closure. □
Lemma 5.17 Let $L, M$ be two posets. If $L$ is a Scott retract of $M$, which has weak one-step closure, then $L$ has weak one-step closure.

Proof. The proof is similar to Proposition 5.16.

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