Geometric Unification of Classical Physics (I):
Equations of Motion, both Standard and Extended
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To the memory of the late Dr. Marsha Torr, for the invaluable support that she gave to this author at crucial moments in the development of a program to which this paper belongs.

Abstract. We show the most unexpected result that the form of the Lorentz force is inescapable in Finsler geometry, as it is always present in the autoparallels. It follows that Finslerian refibrations – if your chosen bundle is not Finslerian ab initio – reveal a special relation between the Lorentz force and the autoparallels, which is hidden in standard differential geometry (The \( q/m \) factor should not be seen as a problem, but as an effect of the asymmetry inherent in the “particle in a field picture”, certainly to be solved in due time).

It emerges that Finslerian torsions, \( \Omega^\mu \equiv d\omega^\mu - \omega^\nu \wedge \omega^\mu_\nu = R^\mu_\nu\lambda \omega^\nu \wedge \omega^\lambda + S^\mu_\nu l_\lambda \), \((\lambda = (0, l_i) = 0, 1, 2, 3)\), yield three sectors. An electrodynamics sector, defined by \( \Omega^i = 0 \) and \( S^0_\nu l_i = 0 \). A “dark matter sector”, for lack of a better name, defined by \( \Omega^i = 0 \), \( R^\mu_\nu\lambda = 0 \), \( S^i_\nu l_i \neq 0 \). It affects the equation of the autoparallels by endowing them with additional terms, not only for the “force part of the equation, but also for its momentum part.” (If this is on the right track, there is no hope for velocity curves in galaxies to be explained by tinkering with just the force terms; it may be a matter of the format itself of the equation of motion). A third sector, the one of the \( \Omega^i \), we call dark energy sector, since this part of the torsion does not contribute to the equations of motion but still contributes to the energy equations (next paper).

A physical looking differential geometry canonically emerges in Finsler bundles on metrics, not necessarily Finslerian, of Lorentz signature. The next steps will be endowing it with teleparallelism and replacing exterior differentiation with Kähler’s exterior-interior differentiation.
1 Back to 1905

In 1905, Einstein did what was possible at the time to deal with the issue of finding the dynamics underlying electromagnetic theory, namely relativistic mechanics. Present mathematics allows us to do much better, by starting with a geometric setting that did not exist at the time. A first step in the argument is the adoption for physics of Finsler bundles on metrics of Lorentz signature, further specification of the metric being unnecessary in early stages of the argument. We then see the emerging from it not only of electrodynamics but also of a more comprehensive physical theory that in turn pays back and suggests an enriched differential geometry that looks increasingly like the Kähler version of quantum physics.

In these bundles one can not only accommodate the electromagnetic sector but — allow me to borrow names — “dark matter” and “dark energy” sectors. If this sounds preposterous, just provide your own interpretation to the way in which different “covariant sectors” of Finslerian torsions relate or fail to relate to autoparallels, viewed as equations of motion of particles. An even more important aspect of this new restart for the physics of after 1905 is that the unification of gravity with the so expanded theory is not to be sought but actually unavoidable. In this paper, the equation of the motion will make this clear.

Finsler bundles can be seen as refibrations of standard bundles, after first considering the latter ones simply as topological manifolds. If, in addition to refibrating the spacetime’s standard frame bundle, one lifts a standard connection with torsion to the corresponding Finsler bundle, the form of the Lorentz force emerges in the equations of the autoparallels for the components of type $R$ of $\Omega^0$. This feature of the refibration is not possible in pre-Finslerian differential geometry, as $\Omega^0$ does not remain unchanged under Lorentz transformations, which then ceases to be the group in the fibers. This does not necessarily clash with the Lorentz group. It simply happens that the boosts without rotation act on “reduced (specifically four-dimensional) vectors” tangent to the 7-D base space of the Finslerian bundle of spacetime.

Putting things differentially, the group in the fibers of this bundle is the rotation group in three dimensions. $\Omega^0$ thus behaves as a scalar field in 7-D under tangent Lorentz transformations, the boosts simply move the scalar-valued differential form $\Omega^0$ (In this case, we reserve the term vector-valued for $\Omega^0$ from one point to another in that 7-D manifold. When the $\Omega^i$ are all zero, rotations thus amount to a change of coordinates in the expression for
\(\Omega^0\), only component of the torsion in this case. Because of this, the impact of the metric on the dynamics recedes to the background for this special torsion in this (Finslerian) setting, though it certainly does not vanish; coordinate changes per se do not mean much, provided that one keeps track of who they relate to measurements.

The physical results that we are about to obtain in this and the next two papers advocate the use of Finsler bundles, or, in due time, something that will supersede them. But there is also a mathematical argument for the study of geometry in these bundles. It has not been realized – mostly because one has not cared – that Finsler bundles are canonical for the Lorentz signature, like the Riemannian bundles are canonical for the positive-definite signature. This is so because of the following. Spacetime tangent vectors are lifted to the Finsler bundle as equivalence classes of 7-D tangent vectors that are put in a one-to-one correspondence with the 4-vectors. This takes place by first choosing a special dimension to define those equivalence classes. But there are not special directions when the metric is positive definite. Temporal dimensions are special. Not counting null-dimensions, there is one and only one special direction in any set of four independent spacetime directions.

What we have just said should be enough to start understanding the following brief description of the first three papers in this series. The present one (I) will deal with autoparallels as equations of the motion. It will be followed by a second one (II), dealing with the subjects of first pair of Maxwell’s equations, emergence of totally geometric Einstein’s equations and gravitational energy-momentum tensor. In a third paper (III), we deal with the Kählerian generalization (the term Kählerian being used here in the sense of work by E. Kähler in 1960-1962, not in 1933) of Cartan’s differential geometry, as well as the development of the electromagnetic energy-momentum tensor and a deeper understanding of the second pair of Maxwell’s equations. In this process, III brings classical mechanics closer to quantum mechanics, by virtue of the adoption of the differentiation with which Kähler developed an improved version of Dirac’s relativistic quantum mechanics.

In II, we assume teleparallelism (TP), as Einstein did [1], though in Finsler geometry in our case. Hence, in the course of the three papers just mentioned, we get from 1905 to what Einstein tried to achieve in the late 1920’s but failed. In the process, we encounter another Einstein theme. In 1933, he proposed the thesis of, to use his own words, logical homogeneity of theoretical physics and differential geometry [2]. Of course, the term differential geometry has to be understood more comprehensively that has been the case traditionally. At some point, it will not seem appropriate to
refer to the issues addressed as pertaining to differential geometry, whether extended or not. As a counterpart, we shall already find in the extended differential geometry and its application to the analysis of the second pair of Maxwell’s equations that quantum mechanical issues such as the acquiring mass by massless fields (Higgs mechanics) is already a classical issue. This is a reflection of the fact that we are dealing with a classical field theory where we are not dealing with particles in a field, except opportunistically in this paper (since autoparallels are assumed to play the role of trajectories of particles). We are rather dealing with fields, special configurations of which may become particles. But this mechanism belongs to the Kähler calculus \[3\] as projections upon ideals of wave functions that are not spinors \[4\]. So, in classical physics, we get only scratch the surface of the Higgs mechanics. Quantum physics will come much later in this series, far beyond this first miniseries of three papers.

2 Introduction

This is the first in a long series of papers that develop Einstein’s thesis of logical homogeneity, which is to be interpreted as meaning that the field equations have to be equations of structure and Bianchi identities of some geometry. The mathematics of his time were not sufficiently developed for the task, but they are now. We live at a moment of history where the mathematics can guide us beyond our wildest expectations, once we have assumed that the connection of spacetime is Finslerian, rather than pseudo-Riemannian or pseudo-Riemannian with torsion. Please view this paper as a down payment on the credibility of that statement. Another two papers, written in parallel with this one, provide support to our claim and are almost ready. If this miniseries of three papers provides the right guidance, theoretical physics can be put in a much sounder mathematical and philosophical basis than is presently the case.

A long series of papers will nevertheless be needed to get the program to the point where the formalisms of classical and quantum physics are connected at the level of fundamental equations. We will use the first three papers to show major results in gravitational physics and the classical version of electrodynamics. (Impatient readers can google “Alterman 2016”, “Alterman 2017” and “Alterman 2018” to catch a glimpse of the present authors efforts in the quantum physics area). By the third paper, we shall have started to use in classical physics the calculus that Kähler used for quantum mechanics \[3\]. But we shall see that it is not just a matter of language, since
the scope of the geometry used — connections on Finsler bundles — allows one to see electrodynamics as a theory of the electromagnetic field created by charges, but where field, charge and current are manifestations of the torsion of space-time.

The exploits of the paradigm and the extraordinary precision of electrodynamics would seem to constitute a strong argument against such type of an effort (Let us simply retort that the Newtonian solar system also was very precise). But there are very important basic problems in the paradigm, which are swept under the rug. For instance, the canonical electromagnetic energy-momentum tensor is wrong (More on this in the next paper). Still worse, a variety of reasons are given to justify the absence of a gravitational energy-momentum tensor in Einstein’s theory of general relativity. In this case, the problem lies in that a wrong decision was made by the physics community in the years 1917-1920, namely the adoption of the Levi-Civita connection. We discuss this at length in the second paper in the series, where we specialize specifically to (Finslerian) teleparallelism (TP).

The Finslerian setting allows one to integrate the two indices of the electromagnetic field into the three indices of the torsion. There is then the extraordinary revelation, which we reported in the abstract, on the relation of the autoparallels (i.e. lines of constant direction) to the Lorentz force and to the splitting of the torsion into three sectors of interest. The use of Finsler bundles is not just a matter of trusting the potential for physics of the results that we shall obtain in this and the next few papers. It simply happens that the Lorentz signature — not the Euclidean one — is the canonical signature of Finsler bundles (see Section 4 of [5]). The form of the Lorentz force is then the canonical one for Finsler geometry.

In each of the first three papers in this series, we shall demonstrate the suitability for geometrization and unification purposes of overlooked areas of advanced mathematics. In this paper, the area is Finslerian bundles. In the second paper, it will be teleparallelism (TP), which nobody has tried in that bundle. In a third paper — guided by electrodynamics — a first inroad will be made into replacing the exterior calculus with the Kähler calculus in differential geometry. Hence, in the course of the first two papers, electrodynamics emerges in mathematics. It then pays back by making evident a natural generalization of differential geometry through the calculus that Kähler used for his version of quantum mechanics.

The contents of the paper is organized as follows. In section 3, we provide a birds eye view of differential geometry on Finsler bundles [5]. In section 4, we use the relatively simple case of general relativity to show how autoparallels are computed in that geometry. It will help to understand in
Section 5 the treatment of the equations of the motion unified for classical electrodynamics and gravitation. For that we only use torsions whose $S$ terms vanish. In section 6, the rich structure of Finslerian geometry allows one to go beyond Maxwell-Lorentz electrodynamics through the $S$ sector of the torsion that accompanies the subsector for electrodynamics, i.e. the components associated with what we have called dark matter. In the concluding remarks of section 7, we call attention to the fact that there still are three quarters of the components of the torsion that do not enter the autoparallels, but that nevertheless enter the energy equations (See the second paper). So, this emerging physical theory has its own dark energy section. Do not blame the author for these coincidences. He is just the messenger.

3 A bird’s eye view of Finslerian differential geometry

In 1922, Cartan stated: “The metric does not contain all the geometric structure of the space ... one may define the space by its structure equations” [6]. These equations constitute a differential system for the connection. We may thus reformulate Cartan and state that the geometric reality of a space resides in its connection, term here used to refer to $(\omega^\mu, \omega_\lambda^\nu)$ and not just to the $\omega_\nu^\lambda$. Hence the view of Finsler geometry as the geometry of a metric more general than Riemannian is adequate for global issues, but is totally insufficient for general purposes (Practitioners of global differential geometry may nevertheless use the canonical connection of the metric for their purposes).

Differential forms constitute the appropriate tool for differential geometry. To our knowledge, only American practitioners of Finsler geometry use differential forms, but they are only interested in global problems. That leaves only the Cartan-Clifton formulation of Finsler geometry for our purposes. That can be found in [5], where we explain it, and why the true author of the paper is the differential topologist Y. H. Clifton, and not its nominal ones.

The basis for the Clifton approach to Finsler geometry is the seminal Cartan paper of 1934 on the subject [7]. With great effort (as he confessed to the present author), Clifton was able to interpret it in terms of frame bundles. To be specific, let us consider spacetime. Its standard pseudo-orthonormal frame bundle is a 10-D manifold fibred over a 4-D base, i.e. over spacetime. The corresponding Finsler bundle is the same topological space fibred over a 7-D manifold. The latter is the so called sphere bundle
or bundle of directions. These names are not very fortunate because they evoke a Euclidean rather than a Lorentzian connotation. Suffice to say that a natural set of coordinates are the \((t, x^i, u^i)\), where the \(u^i\)'s are the so-called velocity coordinates, or parameters of the Lorentz boosts without rotation. On curves that are natural liftings of spacetime curves, they satisfy the conditions \(dx^i - u^i dt = 0\), and thus \(u^i = dx^i/dt\). Hence the term velocity coordinates follows.

Let us understand what the equations of structure

\[
\begin{align*}
  d \omega^\mu - \omega^\nu \wedge \omega^\mu_{\nu} &= R^\mu_{\pi \rho} \omega^\pi \wedge \omega^\rho, \\
  d \omega^\nu - \omega^\lambda \wedge \omega^\nu_{\lambda} &= R^\nu_{\mu \pi} \omega^\pi \wedge \omega^\mu,
\end{align*}
\]

mean. From a frame bundle perspective, the left hand sides, when expressed in terms of a basis of differential 2-forms, do not have terms \(\omega^\mu \wedge \omega^\nu \wedge \omega^\lambda \wedge \omega^\rho\). In the presence of a metric, equations (1)-(2) have to be complemented with \(\omega^\mu_{\nu} + \omega^\nu_{\mu} = 0\), equivalently, therefore, \(\omega^0_0 = \omega^0\) and \(\omega^i_j = -\omega^j_i\).

The Finslerian equations of structure read

\[
\begin{align*}
  d \omega^\mu - \omega^\nu \wedge \omega^\mu_{\nu} &= R^\mu_{\pi \rho} \omega^\pi \wedge \omega^\rho + S^\mu_{\pi \rho} \omega^\pi \wedge \omega^\rho, \\
  d \omega^\nu - \omega^\lambda \wedge \omega^\nu_{\lambda} &= R^\nu_{\mu \pi} \omega^\pi \wedge \omega^\mu + S^\nu_{\mu \pi} \omega^\pi \wedge \omega^\mu + T^\nu_{\mu \pi \rho} \omega^\pi \wedge \omega^\rho.
\end{align*}
\]

(The summation over the \(T\) terms is for a complete set of independent \(\omega^0_0 \wedge \omega^0\) forms). Terms containing at least one factor \(\omega^m\) are absent on the right hand side. So, although the terms on the left hand sides are not horizontal, their combinations on the left hand sides of (3) and (4) are.

In the case of the properly Euclidean signature, the role of the index zero is taken by any arbitrary index. In the Lorentzian case, the natural choice for the “special role coordinate” is time. This is what we mean when we state that the Lorentzian signature is the canonical signature of Finslerian geometry. It provides the special coordinate.

Some remarks of interest (if not already obvious by now) are:

(a) The \(\omega^0\)'s are independent of the \(\omega^\mu\)'s on sections of the Finsler bundle. In other words, they are horizontal.

(b) The \(e_\mu\)'s dual to the \(\omega^\mu\)'s on sections of the same bundle, as the \(\omega^\mu\)'s themselves, do not coincide with the \(e_\mu\)'s and \(\omega^\mu\)'s on sections of the standard bundle. In particular, the four velocity, \(u^\mu e_\mu\), is simply \(e_0\) on sections of the Finsler bundle.

(c) On sections of the Finsler bundle, the \(\omega^\mu\)'s can be written as

\[
\omega^0 = l dt + \Lambda^r \sigma^r, \quad \omega^i = A^i_r \sigma^r, \quad r = 7
\]
where $\sigma^r = dx^r - u^r dt$ and where $l, \Lambda_r$ and $A_i^r$ depend on the $x$’s and $u$’s. The second equations (5) are special in that they are linear combinations of just the three differential forms $\sigma^i$. The equations $\sigma^i = 0$ are the so-called natural lifting conditions. Without further ado, let us remark that, on curves, all differentials are linear functions of just one, say $dt$. In this case, $\sigma^i = 0$ yields $u^i = dx^i/dt$.

(d) A consequence of (b): although the $R_{\nu\lambda}^\mu \omega^\nu \wedge \omega^\lambda$ retain this form when lifted to sections of the Finsler bundle, this is a different set of $R_{\nu\lambda}^\mu$’s since the $\omega^\mu$’s on sections are changed by the lifting; the $\omega^i$ do not take in the usual bundle the form $A_i^r \sigma^r$. These remarks are of great importance for understanding that the form of the Lorentz force is always present on sections of the Finsler bundle, even for liftings of arbitrary connections on the usual bundle.

(e) On natural liftings, $\omega^0 = l dt$, $\omega^i = 0$. Hence, on such curves, the metric can be replaced with

$$ds = l dt \quad \text{or} \quad \omega^0, \text{mod} \sigma^i, \quad (6)$$

4 The Finslerian form of the equations of motion of general relativity

In this section, we acquaint readers with computations of the equations of motion in Finsler geometry, where the autoparallels or lines of constant direction are given by

$$\omega_0^m = 0, \quad \omega^l = 0 = \sigma^l. \quad (7)$$

When the torsion is zero, we shall write $\omega_0^m$ as $\alpha_0^m$, and shall thus have

$$0 = d\omega^0 - \omega^i \wedge \alpha_0^i. \quad (8)$$

We need to compute $\alpha_0^0$. Using equations (5) in (8), we obtain

$$0 = (d\Lambda_i - l_i dt + \alpha_0^0 A_i^r) \wedge \sigma^i + (\Lambda_m - l_m) dt \wedge du^m, \quad (9)$$

where subscripts “,” and “,” mean partial derivatives with respect to $x$ and $u$ respectively. We have used that $l_m dx^m \wedge dt = l_m \sigma^m \wedge dt$. Equation (9) can further be written as

$$\alpha_i \wedge \sigma^i + (\Lambda_m - l_m) dt \wedge du^m = 0, \quad (10)$$
where $\alpha_i$ is defined as

$$\alpha_i \equiv d\Lambda_i - l_i dt + \alpha^0_r A^r_i.$$  \hspace{1cm} (11)

Since there are no $dx^l$ factors in $dt \wedge du^m$, the two terms in (10) must be zero. From the second term, we get

$$\Lambda_m = l.m,$$  \hspace{1cm} (12)

The annulment of the first term implies,

$$\alpha_i = C_{im} \sigma^m, \quad \text{with} \quad C_{im} = C_{mi}. \hspace{1cm} (13)$$

We now assume the equation satisfied by all natural lifting conditions, namely $\sigma^m = 0$. Thus, $\alpha_i = 0$ and, therefore,

$$0 = d\Lambda_i - l_i dt + \alpha^0_r A^r_i = dl_i - l_i dt + \alpha^0_r A^r_i,$$  \hspace{1cm} (14)

where we have used (12). Since the autoparallels satisfy $\alpha^r_0$ when the torsion is zero, we finally have

$$l_i dt = dl_i,$$  \hspace{1cm} (15)

or, equivalently

$$\frac{\partial l}{\partial x^i} = \frac{d}{dt} \frac{\partial l}{\partial u^i},$$  \hspace{1cm} (16)

which are the Euler-Lagrange equations of motion in general relativity [Recall (6)]. We have followed this route to get to Eq. (16) in order to facilitate the understanding of the contents of the next section.

In order to connect with more familiar concepts, let us be aware of the fact that Eqs. (5) give the $\omega^\mu$’s for affine connections in Finsler geometry. We have not yet required that $(\omega^0)^2 - \sum (\omega^i)^2$ be an invariant. Much less has one required that this invariant be $dt^2 - \sum (dx^i)^2$. In the last case, the coefficient $l$ is readily computed. We then have

$$\omega^0 = \gamma(dt - u_i dx^i) = \gamma dt - \gamma u_i (\sigma^i + u^i dt),$$  \hspace{1cm} (17)

where $\gamma = (1 - u^2)^{-1}$, and, therefore,

$$l = \gamma - \gamma u_i u^i = \gamma (1 - u^2) = \sqrt{1 - u^2} = \gamma^{-1}. \hspace{1cm} (18)$$

In flat spacetime, we get the following standard result for the spatial components of the kinetic 4-momentum:

$$\frac{\partial l}{\partial u^i} = \frac{-u_i}{\sqrt{1 - u^2}} = -\gamma u_i = p_i. \hspace{1cm} (19)$$

The presence of the Lagrangian in equations (5) has thus been illustrated.
5 Equation of motion geometrically unified for electrodynamics and gravitation

We shall now consider the effect of torsion on the equations of the autoparallels in Finsler bundles. The temporal component of the torsion is to be read from (3). The sets of $\omega^\mu$'s constitute bases of differential 1-forms dual to bases of tangent vectors.

In order to remove clutter in the process of identifying the presence of the Lorentz force, we assume $S_{\mu i}^0$ equal to zero. We also redefine the components of $\Omega_i^0$ as per the equation

$$\Omega_0^0 = R_{0\mu}^0 (\omega^\mu \wedge \omega^\nu) = R_{0i}^0 dt \wedge dx^i + \frac{1}{2} R_{lm}^0 dx^l \wedge dx^m. \tag{20}$$

The parenthesis around $\omega^\mu \wedge \omega^\nu$ means that we are summing over a basis of independent differential 2-forms $\omega^\mu \wedge \omega^\nu$. On the right hand side, on the other hand, we are summing over all $l$ and $m$. We shall later use $\omega_l^0 = \omega_l^0$ and $\omega_l^i = -\omega_i^l$, since we have simply changed coordinates and have not done anything to change the basis of tangent vectors.

At this point, we have, instead of equation (8),

$$0 = d\omega^0 - \omega^i \wedge \omega_i^0 - R_{0i}^0 dt \wedge dx^i - \frac{1}{2} R_{lm}^0 dx^l \wedge dx^m. \tag{21}$$

The development of the first two terms in (21) gives rise to the right hand side of (9), with $\alpha_i^0$ replaced with $\omega_i^0$. We now proceed to develop the $dx$'s in terms of the $\sigma$'s, recalling that $dx^i = \sigma^i + u^i dt$.

We write the third term on the right hand side of (21) as $-R_{0i}^0 dt \wedge \sigma^i$. The fourth term evolves as follows:

$$-\frac{1}{2} R_{lm}^0 (\sigma^l + u^l dt) \wedge (\sigma^m + u^m dt)$$

$$= -\frac{1}{2} [R_{lm}^0 \sigma^l \wedge \sigma^m - R_{lm}^0 \sigma^l \wedge u^m dt - R_{lm}^0 u^l dt \wedge \sigma^m]. \tag{22}$$

On the right of (22), let us collect all the terms where $\sigma^1$. We get $(R_{12}^0 - R_{31}^0 \sigma^3) \wedge \sigma^1$. Doing the same with $\sigma^2$ and $\sigma^3$, we shall have counted each term twice. Hence the sum of all the $\sigma^l \wedge \sigma^m$ terms yields

$$\frac{1}{2} (R_{0ij}^0 \sigma^j - R_{0kij}^0 \sigma^k) \wedge \sigma^i, \tag{23}$$

with sum over cyclic permutations. In the remainder of the right hand side of (22), the factor $\sigma^1$ will emerge in

$$(R_{12}^0 u^2 - R_{31}^0 u^3) dt \wedge \sigma^1.$$
Hence, when the contributions of $\sigma^2$ and $\sigma^3$ are also taken into account, we get
\begin{equation}
(R'_{ij}u^i - R'_{ki}u^k)dt \land \sigma^i.
\end{equation}
(24)

By bringing all these results to (21), we obtain
\begin{align}
0 &= [d\Lambda_i - l_i dt + \omega^0_r A^r_i - R'_{0i}dt + (R'_{ij}u^j - R'_{ki}u^k)dt
\nonumber
+ \frac{1}{2}(R'_0 \sigma^j - R'_k \sigma^k)dt] \land \sigma^i + (\Lambda_m - l_m)dt \land du^m.
\end{align}
(25)

Now as before this implies that $\Lambda_m = l_m$ and that the square bracket must be a linear combination of the $\sigma^m$'s, i.e. $c_{mi} \sigma^m$ with $c_{mi} = c_{im}$. Since the autoparallels satisfy the conditions $\sigma^j = 0$ and $\omega^r_0 = 0$, with $\omega^r_0 = \omega^r_0$, we conclude that they satisfy the equation
\begin{equation}
0 = dl_i - [l_i + R'_{0i} + (R'_{0k}u^k - R'_{ij}u^j)]dt.
\end{equation}
(26)

This is the equation of motion unified for the gravitational and electromagnetic forces if we identity $R'_{0i}$ with $(q/m)E_i$, and $R'_{ij}$ and $R'_{ki}$ with $(q/m)B_k$ and $(q/m)B_j$, respectively.

Some readers may be thinking that the factor $q/m$, which differs from particle to particle, will invalidate the argument that the $R$ part of the torsion is to be identified with the electromagnetic field. The right interpretation is that “far from the core of a particle” the electromagnetic field must indeed be identified with the torsion, but, at the position of the particle, the torsion is not dictated by the outside field but will be the torsion that defines the specific particle. It is a challenge for this emerging theory the obtaining of the factor $q/m$ from the torsion field. But the right way would be to do so after contact has been made with the quantum mechanics consistent with this geometric picture for classical physics. We thus read from this result that
\begin{equation}
R^0_{\mu\nu} = F_{\mu\nu} = \begin{bmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & -B_z & B_y \\
-E_y & B_z & 0 & -B_x \\
-E_z & -B_y & B_x & 0
\end{bmatrix}
\end{equation}
(27)

without a $q/m$ factor.

We no longer need to use primed quantities. It is understood that the basis of differential forms for the $R$'s and the $F$'s will be the same, which in our computation became the $(dt, dx^i)$, as per equation (20). The relations (27) are basis independent, except that $\Omega^0$ pertains to canonical bases, i.e. orthonormal when both a metric and a connection define a structure. The fact
that only the energy equations will involve exchanges between vector-valued quantities of different valuedness makes electrodynamics largely impervious to what specific Lorentzian connection underlies our developments. This has as concomitant, only seen retrospectively, that our return to 1905 may focus on the issue of the mathematical structure of electrodynamics, disregarding the issue of the speed of light and related considerations on relations of simultaneity and synchronization procedures.

At this point in the argument, we are interested in the relation between geometric and physical formulas. The present problem consists in identifying geometric equations with key equations of electrodynamics, making abstraction for the moment of the presence of mass, charge and current. As we obtain further corroboration of (27), we become increasingly independent of formulas in the paradigm and let the mathematics speak. This course of action will show us that terms like interactions, potentials, sectors, cosmological constant, etc. are not the most suitable terms to represent what is going on. But we shall temporarily use those terms for enhancing the parallelism between the paradigm and the physical theory that emerges here.

Decades ago, Ringermacher [8] and the present author [9] guessed a torsion which did not make sense in Riemannian with torsion geometry, but which, we both believed, was a crude version of something very profound and thus had to be viewed as indicating the direction of future research. Somebody well versed on the Finsler bundle would have realized that our result was rigorously meaningful in this bundle. That somebody was the differential topologist Yeaton H. Clifton [See our report in [5]]. It was not immediately realized by any of us that, as we have just shown, the form of the Lorentz force is canonical in the Finsler bundle of spacetime, which brings about the consideration that follows. In the usual bundle, one has to find a specific torsion that could be identified —even if not quite correctly— with the electromagnetic field. But all “Riemannian connections with torsion” can be lifted to the Finsler bundle, where all of them yield the form of the Lorentz force in the autoparallels. So, the lifting achieves something very special, namely that all of them had in common, though hidden, the Lorentz force (The difference between liftings of difference connections lies in details of the terms within the force, not in its form; recall that the lifting of a standard connection to the Finsler bundle changes the coefficients $R_{\nu\lambda}^\mu$)

Let us say it again. Because the electromagnetic field is confined to the temporal component of the tangent index of the torsion, Maxwell’s equations do not depend on the connection of spacetime. Thus scalar-valued differential forms suffice for this purpose. This confirms a remark by Cartan in 1924 [2] (as if his remarks needed confirmation!) that the most fundamen-
tal form of Maxwell’s equations is the one in terms of integrals, not in point form. This is consistent with the formulation of electrodynamics in terms of differential forms, when these are viewed as functions of hypersurfaces rather than as antisymmetric multilinear functions of vectors. Cartan went on to say that Maxwell’s equations do not contain all of classical electrodynamics. They have to be complemented with relations involving energy and momentum, and this is where the non-scalar-valuedness explicitly enters electrodynamics (We used the term explicitly because, as we said, the vector valuedness is implicit in $\Omega^0$, which was central to this paper, even though it looked as if only scalar-valued differential forms were needed for our dealings.)

The non-vanishing of $S^0_{\mu i}$ and $\Omega^i$ will certainly reveal a deeper reality (See next section). This by itself indicates that, as we are about to do and shall continue to do in the next two papers, one should further explore the consequences of Finslerian geometry with torsion, and continue the study of the deep synergy existing not just between classical electrodynamics and Finsler geometry, but also of the latter with less known sectors of the paradigm. In the next section, we shall deal with one of them.

### 6 Beyond Electrodynamics

In this section, we start to obtain geometric terms beyond those of classical field theory. They concern non-electromagnetic components of the torsion, and one of them actually involves the acceleration, like the time derivative of the momentum does. In order to deal with this situation, the term sector helps. So, we shall speak of dark matter, dark energy, Higgs and cosmological sectors. The justification for the use of the term Higgs sector has to do with the fact that it deals with the emergence of mass from fields. It is of the essence already in classical physics, so that one does not have a classical theory of particles (classical mechanics) on the one hand, and a classical theory of fields on the other. The use of the term cosmological sector is due to the fact that the cosmological term seem to be just the tip of a cosmological iceberg. In this section, we have said all what we have to say about dark matter at this point. We reserve for the next paper incipient considerations about the other three sectors.

It is not strange that the $\Omega^i$ do not contribute to the equation of the autoparallels, since $\Omega = \Omega^\mu e_{\mu}$ and only $e_0$ contributes to the 4-velocity in the Finsler bundle, $u = e_0$. But, what about the $S^0_{\mu i}$ terms? Their contribution is not immediately identifiable with anything in the paradigm,
except possibly for what goes by the name of dark matter. Hence our choice of the name dark matter for what we are about to say in this section. Thus let us suppose that we again proceed as in sections 3 and 4, but with the full $\Omega^0$.

In order not to get lost in manipulating a large number of terms, we shall keep track of the combinations of terms that become zero by virtue of the autoparallelism conditions $\sigma^i = 0$ and $\omega^i_0 = 0$. We can express the $\omega^i_0$ as

$$\omega^i_0 = a^i dt + b^i_l dx^l + c^i_m du^m,$$

(28)

Before using this in $-S^0_{\mu i} \omega^\mu \wedge \omega^i_0$, we write this term as

$$... = -S^0_{0i} \omega^i_0 \wedge \omega^0_l - S^0_{0i} \Lambda^i \sigma^r \wedge \omega^i_0 - S^0_{li} A^m \sigma^m \wedge \omega^i_0.$$

(29)

The last two terms on the right hand side of (29) will contribute to the new $\alpha_i$ by virtue of the $\sigma$ factors in them. When we then make $\omega^i_0$ equal to zero, they vanish and thus fail to contribute to the equation of the autoparallels. Hence, we have to take care of just the first term, where we use (28):

$$S^0_{0i} l dt \wedge (a^i dt + b^i_l dx^l + c^i_m du^m) = -S^0_{0i} l dt \wedge \sigma^i - S^0_{0m} l c^i_m dt \wedge du^i).$$

(30)

which fits the pattern

$$... (\ldots)_l \wedge \sigma^l + (\ldots)_i \ dt \wedge du^i = 0.$$  

(31)

By virtue of the contribution by the last term in (30), we shall have

$$\Lambda_i = l_i + S^0_{0m} l c^i_m$$

(32)

instead of $\Lambda_i = l_i$ The $dl_{i\dot{i}}$ in equation (26) will be replaced with

$$dl_{i\dot{i}} + d(S^0_{0m} l c^i_m).$$

(33)

Hence, the acceleration will emerge not only from $dl_{i\dot{i}}$, but also from the dependence of $S^0_{0m} l c^i_m$ on the $u^l$.

The Finsler bundle may not be the ideal way of representing the potential physical implications that we have uncovered through its use. In a much later paper, we shall advocate a canonical Kaluza-Klein space without compactification, where a “unit vector” for the fifth dimension embodies the velocity coordinates. This is not an ad hoc structure. Although he did not stop at discussing it, Cartan showed that differential geometry is ONLY a theory of moving frames, not of particles and frames [10], pp. 7-8. That
Kaluza-Klein structure, which has in five dimensions a null metric equivalent to the standard metric, brings in the particles at par with the frames.

The additional terms that we have just found may be appropriate for situations like, for example, a current in a metal, which is a quantum mechanical system. More importantly, the additional acceleration term is of a nature completely different from the standard one. If this term is in the right track, there is no hope for velocity curves in galaxies to be explained by tinkering with the force terms, like is the case with the postulation of dark matter. It may not be just a “matter of matter”, whether dark or not, but also of the format itself of the equation of motion. We shall nevertheless use the term dark matter sector or simply dark sector to refer to anything that transcends the format and not just the contents of the force part of the equation of motion.

The Finsler bundle may not be the ideal way of representing the potential physical implications that we have uncovered through its use. In a much later paper, we shall advocate a canonical 5-D Kaluza-Klein space without compactification of the fifth dimension, where a “unit vector” embodies the velocity coordinates. This may seem ad hoc, but it is not. Although he did not stop at discussing it, Cartan showed in 1922 that differential geometry is ONLY a theory of moving frames, not of particles and frames [10], pp. 7-8. The said Kaluza-Klein structure, which has in five dimensions a null metric equivalent to the standard metric brings in the particles at par with the frames. It has 4-D subspaces respectively suited for classical and quantum physics.

The additional terms that we have found in the development of the equations of the autoparallels may be appropriate for situations like, for example, a current in a metal, which is a quantum mechanical system. More importantly, if the additional acceleration term represents nature, there is no hope for velocity curves in galaxies to be explained by tinkering with the force terms. Its explanation may not be just a “matter of matter”, whether dark or not, but also of the format itself of the equation of motion. We shall nevertheless use the term dark matter sector or simply dark sector to refer to anything that transcends the form and not just the force terms in the equation of motion.

7 Closing Remarks

The results obtained commit us to at least inquire whether electrodynamics can be geometrized through the use of Finslerian differential geometry. It
emerges that the electromagnetic field is not the only classical field other than the gravitational one. It is to be replaced with the much richer torsion field. It should follow equations more comprehensive than Maxwell’s since, if one of the $\Omega^i$ is involved, so are the other two.

A challenge for this emerging theory is the derivation of the factor $q/m$. For that, one has to connect with quantum physics. In paper III, the mathematics of quantum mechanics, though not quantum mechanics itself, will suggest itself from classical physics. But this enrichment of the mathematics used is not enough per se enough to extract from field theory the $q/m$ factor, since classical physics does not tell us what makes a field be structured as a particle. It will take us several papers to get to where we can start to address this issue. In the meantime, readers may google Alterman 2016 and Alterman 2017 for websites of two events where different papers and other documents speak of the Kähler calculus and its implications for quantum mechanics. See also, still in raw form, a first approach to particle physics with this calculus in papers by this author in the arXiv. All of it is directly related to spacetime. No auxiliary bundles are or will be involved.

Under physical assumptions of a very general nature and the development of a mathematics which (a) starts with a vision by Leibniz, (b) was first developed by Grassmann, (c) was “corrected” and further developed by Clifford and specially E. Cartan, and (d) under Kähler’s unification of the developments in (c), an extraordinary physics-unification-looking theory emerges. It will take us three additional papers to explain (IV to VI) that, underlying the unity of the physics, there is a unity of what we might call physical mathematics. It is then much easier to undertake a less mystical version of undertake quantum mechanics, formulated by Kähler. If this is not a potential new paradigm, nature, as represented by the present paradigm, will be jealous.

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