Jet Rates in Deep Inelastic Scattering at Small $x$ ${}^{*}$

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Abstract

The recent results of Forshaw and Sabio Vera on small-$x$ jet rates to order $\alpha_s^3$ are extended to all orders, for any number of jets. A simple generating function is obtained.

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1 Introduction

The summation of logarithms of $1/x$ in deep inelastic structure functions at small values of Bjorken $x$ leads to the Balitskii-Fadin-Kuraev-Lipatov (BFKL) equation \cite{1, 2}, which in the leading approximation sums terms of order $[\alpha_s \ln(1/x)]^n$. Recently the next-to-leading terms have also been computed \cite{3, 4}.

The usual derivation of the BFKL equation assumes the dominance of multi-Regge kinematics (i.e. strong ordering in the Sudakov variables). This is valid for the calculation of the totally inclusive structure functions, in which one sums over all hadronic final states. When studying the structure of the final states, however, one should take account of QCD coherence, which effectively imposes an angular ordering constraint on the emission of soft gluons \cite{5, 6, 7}. The resulting ‘CCFM’ formulation \cite{5} reduces to the BFKL equation for the inclusive structure functions, but leads in general to different exclusive multi-gluon distributions from those expected from multi-Regge kinematics \cite{8}.

In a recent paper, Forshaw and Sabio Vera \cite{9} showed that, in the leading log $x$ (LL$x$) approximation, to third order in $\alpha_s$, the rates for emission of fixed numbers of ‘resolved’ final-state gluons, together with any number of unresolvable ones, are the same in the multi-Regge (BFKL) and coherent (CCFM) approaches. Here ‘resolved’ means having a transverse momentum larger than some fixed value $\mu_R$, and the LL$x$ approximation means keeping only terms that have two large logarithms for each power of $\alpha_s$, at least one of which is $\ln(1/x)$. In this approximation, each resolved gluon can be equated to a single jet, since to resolve it into more than one jet would cost extra powers of $\alpha_s$ with no corresponding powers of $\ln(1/x)$.

The present paper extends the work of Forshaw and Sabio Vera to all orders, for any number of resolved gluons. The BFKL and CCFM formulations are shown to give the same jet rates in LL$x$ approximation to all orders. The factorization of collinear singularities is demonstrated, and a simple generating function for the jet multiplicity distribution is obtained.

2 Multi-Regge (BFKL) analysis

As in Ref. \cite{9}, we start from the unintegrated structure function of a single gluon, $f(x, k)$, which in the exclusive form of the BFKL approach satisfies the equation

$$f(x, k) = \delta(1-x)\delta^2(k) + \bar{\alpha}_s \int\frac{dq}{q^2} \int dz\Delta_R(z, k)\theta(q-\mu) f(x/z, q+k) .$$

Here $\bar{\alpha}_s = 3\alpha_s/\pi$, $k$ is the transverse momentum of the gluon probed in the deep inelastic scattering, $q$ is that of an emitted gluon, $\mu$ is a collinear cutoff (which cancels in the inclusive treatment) and $\Delta_R$ is the Regge form factor

$$\Delta_R(z, k) = \exp\left(-\bar{\alpha}_s \ln \frac{1}{z} \ln \frac{k^2}{\mu^2}\right) .$$
The contribution to the structure function at scale \( Q \) where
\[
\mu < q_i < Q
\]
and
\[
x_i = \prod_{j=1}^{i} z_j , \quad k_i = -\sum_{j=1}^{i} q_j .
\]

The contribution from emission of \( n \) gluons is obtained by iteration,
\[
f^{(n)}(x, k) = \prod_{i=1}^{n} \int \frac{d^2 q_i d z_i}{\pi q_i^2 z_i} \bar{\alpha}_s \Delta_R(z_i, k_i) \theta(q_i - \mu) \delta(x - x_n) \delta^2(k - k_n) ,
\]
where
\[
x_i = \prod_{j=1}^{i} z_j , \quad k_i = -\sum_{j=1}^{i} q_j .
\]

The contribution to the structure function at scale \( Q \) is then obtained by integrating over all \( \mu < q_i < Q \):
\[
F^{(n)}(x, Q, \mu) = \prod_{i=1}^{n} \int_{\mu^2}^{Q^2} \frac{dq_i^2}{q_i^2} \frac{d\phi_i}{2\pi} \frac{dz_i}{z_i} \bar{\alpha}_s \Delta_R(z_i, k_i) \delta(x - x_n) .
\]

### 2.1 One-jet rate

Now consider the effect of requiring one emitted gluon, say the \( j \)th, to have \( q_j > \mu_R \) while all the others have \( q_i < \mu_R \). This defines the contribution of one resolved gluon (plus \( n - 1 \) unresolved), \( F^{(n,1\text{jet})} \):
\[
F^{(n,1\text{jet})}(x, Q, \mu_R, \mu) = \sum_{j=1}^{n} \int_{\mu_R^2}^{Q^2} \frac{dq_j^2}{q_j^2} \frac{d\phi_j}{2\pi} \frac{dz_j}{z_j} \bar{\alpha}_s \Delta_R(z_j, k_j) \cdot \prod_{i \neq j}^{n} \int_{\mu^2}^{\mu_R^2} \frac{dq_i^2}{q_i^2} \frac{d\phi_i}{2\pi} \frac{dz_i}{z_i} \bar{\alpha}_s \Delta_R(z_i, k_i) \delta(x - x_n) .
\]

Notice that for \( i < j \) the contribution is identical to the \( (j - 1) \)-gluon contribution to the structure function evaluated at \( x = x_{j-1} \) and \( Q = \mu_R \). On the other hand for \( i \geq j \) we have \( k_i = q_j \) in leading logarithmic approximation, and so the \( q_i \) integrations become trivial:
\[
F^{(n,1\text{jet})}(x, Q, \mu_R, \mu) = \sum_{j=1}^{n} \int_0^{1} dx_{j-1} F^{(j-1)}(x_{j-1}, \mu_R, \mu) \int_{\mu_R^2}^{Q^2} \frac{dq_j^2}{q_j^2} \frac{dz_j}{z_j} \bar{\alpha}_s \Delta_R(z_j, q_j) \cdot \prod_{i = j+1}^{n} \int \frac{dz_i}{z_i} 2\bar{\alpha}_s S \Delta_R(z_i, q_j) \delta(x - x_n)
\]
where \( S = \ln(\mu_R/\mu) \).

To carry out the \( z_i \) integrations it is convenient to use a Mellin representation,
\[
F_{\omega} = \int_0^{1} dx x^\omega F(x) ,
\]
so that
\[
F_{\omega}^{(n,1\text{jet})}(Q, \mu_R, \mu) = \frac{1}{2S} \sum_{j=1}^{n} F_{\omega}^{(j-1)}(\mu_R, \mu) \int_{\mu_R^2}^{Q^2} \frac{dq_j^2}{q_j^2} \left[ 2\bar{\alpha}_s S \int \frac{dz}{z} z^\omega \Delta_R(z, q_j) \right]^{n-j+1}
\]
Summing over all \(j\) and \(n\) gives the total one-jet contribution,

\[
F^{(1\text{jet})}_\omega(Q, \mu_R, \mu) = F_\omega(\mu_R, \mu) \int_{\mu_R}^Q \frac{dq}{q} H_\omega(q, \mu_R)
\]

where

\[
H_\omega(q, \mu_R) = \frac{2\bar{\alpha}_s}{\omega + 2\bar{\alpha}_s \ln(q/\mu_R)},
\]

and hence

\[
F^{(1\text{jet})}_\omega(Q, \mu_R, \mu) = F_\omega(\mu_R, \mu) \ln\left(1 + \frac{2\bar{\alpha}_s}{\omega} T\right)
\]

where \(T = \ln(Q/\mu_R)\). The asymptotic behaviour of the structure function is

\[
F_\omega(\mu_R, \mu) \sim \left(\frac{\mu_R^2}{\mu^2}\right)^{\gamma(\alpha_s, \omega)} = \exp[2S\gamma(\alpha_s, \omega)]
\]

where \(\gamma(\alpha_s, \omega)\) is the Lipatov anomalous dimension:

\[
\gamma(\alpha_s, \omega) = \frac{\bar{\alpha}_s}{\omega} + 2\zeta(3) \left(\frac{\bar{\alpha}_s}{\omega}\right)^4 + \ldots
\]

Taking the leading term in \(\gamma\) gives the result of Ref. [9], extended to all orders:

\[
F^{(1\text{jet})}_\omega(Q, \mu_R, \mu) = \exp\left(\frac{2\bar{\alpha}_s}{\omega} S\right) \ln\left(1 + \frac{2\bar{\alpha}_s}{\omega} T\right)
\]

\[
= \frac{2\bar{\alpha}_s}{\omega} T + \left(\frac{2\bar{\alpha}_s}{\omega}\right)^2 \left[S - \frac{1}{2} T^2\right] + \left(\frac{2\bar{\alpha}_s}{\omega}\right)^3 \left[\frac{1}{3} T^3 - \frac{1}{2} T^2 S + \frac{1}{2} T S^2\right] + \left(\frac{2\bar{\alpha}_s}{\omega}\right)^4 \left[\frac{1}{6} T S^3 - \frac{1}{4} T S^2 + \frac{1}{3} T^3 S - \frac{1}{4} T^4\right] + \ldots
\]

Notice that the collinear-divergent part (the \(S\)-dependence) factorizes out, and the fraction of events with one jet is given by the cutoff-independent function

\[
F^{(1\text{jet})}_\omega(Q, \mu_R) = \frac{F^{(1\text{jet})}_\omega(Q, \mu_R, \mu)}{F_\omega(Q, \mu)} = \ln\left(1 + \frac{2\bar{\alpha}_s}{\omega} T\right) \exp[-2T\gamma(\alpha_s, \omega)].
\]

### 2.2 Multi-jet rates

Now suppose we resolve \(r\) gluons with transverse momenta \(q_j > \mu_R\). To leading logarithmic accuracy, the Regge form factors beyond the first of these have \(k_i\) fixed at the largest of the \(q_j\)'s resolved so far, and therefore Eq. (10) becomes

\[
F^{(r\text{jet})}_\omega(Q, \mu_R, \mu) = F_\omega(\mu_R, \mu) \prod_{j=1}^r \int_{\mu_R}^Q \frac{dq_j}{q_j} H_\omega(k_j, \mu_R)
\]

where \(k_j = \max_{i \leq j} \{q_i\}\). Introducing \(t = \ln(q/\mu_R)\), we have the general problem of evaluating

\[
G^{(r)}(T) \equiv \prod_{j=1}^r \int_0^T H_\omega(\max_{i \leq j} \{t_i\}) dt_j.
\]
Defining \(G^{(0)}(T) = 1\) and introducing the generating function
\[
G_\omega(u, T) = \sum_{r=0}^{\infty} u^r G^{(r)}_\omega(T),
\]
we have
\[
G_\omega(u, T) = \exp \left[ \int_0^T dt \left( uH_\omega(t) + u^2 t H^2_\omega(t) + u^3 t^2 H^3_\omega(t) + \cdots \right) \right] = \exp \left[ \int_0^T dt \frac{uH_\omega(t)}{1 - utH_\omega(t)} \right].
\]

From Eq. (11) we find in this case
\[
H_\omega(t) = \frac{2\bar{\alpha}_s}{\omega + 2\bar{\alpha}_s t}, \quad G_\omega(u, T) = \left[ 1 + (1 - u) \frac{2\bar{\alpha}_s}{\omega} T \right]^{\frac{u}{1-u}}.
\]

Thus the \(r\)-jet rate is given by
\[
R^{(r\text{ jet})}_\omega(Q, \mu_R) = \frac{F^{(r\text{ jet})}_\omega(Q, \mu_R, \mu)}{F_\omega(Q, \mu)} = \left. \frac{1}{r!} \frac{\partial^r}{\partial u^r} R_\omega(u, T) \right|_{u=0}, \]
where the jet-rate generating function \(R_\omega\) is given by
\[
R_\omega(u, T) = \exp \left( - \frac{2\bar{\alpha}_s}{\omega} T \right) \left[ 1 + (1 - u) \frac{2\bar{\alpha}_s}{\omega} T \right]^{\frac{u}{1-u}}.
\]

The remarkably simple expression (23) is the main result of the present paper. We shall see that the same result is obtained from the CCFM formulation of small-\(x\) dynamics. After convolution with the measured gluon structure function, it gives the predicted jet rates in the LL\(x\) region \(\ln(1/x) \gg T = \ln(Q/\mu_R) \gg 1\), to all orders in \(\alpha_s\).

Expanding to fourth order, we have explicitly
\[
\begin{align*}
R^{(0\text{ jet})}_\omega &\approx 1 - \frac{2\bar{\alpha}_s}{\omega} T + \frac{1}{2} \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^2 - \frac{1}{6} \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^3 + \frac{1}{24} \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^4 \\
R^{(1\text{ jet})}_\omega &\approx \frac{2\bar{\alpha}_s}{\omega} T - \frac{3}{2} \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^2 + \frac{4}{3} \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^3 - \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^4 \\
R^{(2\text{ jet})}_\omega &\approx \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^2 - \frac{13}{6} \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^3 + \frac{23}{8} \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^4 \\
R^{(3\text{ jet})}_\omega &\approx \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^3 - \frac{35}{12} \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^4 \\
R^{(4\text{ jet})}_\omega &\approx \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^4.
\end{align*}
\]

Note that these sum to unity as expected.
From the generating function (23) we can deduce other interesting quantities to all orders, for example the mean number of jets,

\[ \langle r \rangle = \left. \frac{\partial}{\partial u} R_\omega(u, T) \right|_{u=1} = \frac{2\bar{\alpha}_s}{\omega} T + \frac{1}{2} \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^2, \]

and the mean square fluctuation in this number,

\[ \langle r^2 \rangle - \langle r \rangle^2 = \frac{2\bar{\alpha}_s}{\omega} T + \frac{3}{2} \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^2 + \frac{2}{3} \left( \frac{2\bar{\alpha}_s}{\omega} T \right)^3. \]

In general, the \( p \)th central moment of the jet multiplicity distribution is a polynomial in \( \bar{\alpha}_s T/\omega \) of degree \( 2p - 1 \), indicating that the distribution becomes relatively narrow in the limit of very small \( x \) and large \( Q/\mu_R \).

### 3 Analysis including coherence (CCFM)

Taking account of QCD coherence gives the angular ordering constraint \( q_i > z_{i-1}q_{i-1} \) and in place of Eq. (5) one obtains [5]

\[ F^{(n)}(x, Q, \mu) = \prod_{i=1}^{n} \int_0^{Q^2} dq_i^2 \frac{d\phi_i}{2\pi} \int_0^{z_i} \frac{dz_i}{z_i} \bar{\alpha}_s \Delta(z_i, k_i, q_i) \theta(q_i - z_{i-1}q_{i-1}) \delta(x - x_n), \]  

(27)

where we set \( z_0q_0 = \mu \) and \( \Delta \) is the ‘non-Sudakov’ form factor

\[ \Delta(z, k, q) = \exp \left( -\bar{\alpha}_s \ln \frac{1}{z} \ln \frac{k^2}{zq^2} \right). \]

(28)

Corresponding to Eq. (7) we now have for one resolved emission

\[ F^{(n, 1\text{jet})}(x, Q, \mu_R, \mu) = F^{(j-1)}(x_{j-1}, \mu_R, \mu) \sum_{j=1}^{n} \int_{\mu_R}^{Q^2} dq_j^2 \int_0^{\mu_R/q_j} dz_j z_j \bar{\alpha}_s \Delta(z_j, q_j, q_j) \delta(x - x_n). \]  

(29)

Performing the Mellin transformation and summing over \( j \) and \( n \), we obtain in place of Eq. (10)

\[ F^{(1\text{jet})}(Q, \mu_R, \mu) = F(\mu_R, \mu) \int_{\mu_R}^{Q} dq H_\omega(q, q, \mu_R) \]  

(30)

where

\[ H_\omega(k, q, \mu_R) = 2\bar{\alpha}_s \int_0^{\mu_R/q} \frac{dz}{z} \Delta(z, k, q) K_\omega(z, k, q, \mu_R), \]  

(31)

\[ K_\omega(z, k, q, \mu_R) = 1 + 2\bar{\alpha}_s \int_{zq}^{\mu_R} \frac{dq'}{q'} \frac{dz'}{z'} \Delta(z', k, q', \mu_R). \]  

(32)
The CCFM treatment predicts the same behaviour as BFKL for the structure function, and so the factor $F_\omega(\mu_R, \mu_R)$ in Eq. (30) still has the form (13). The explicit solution of Eq. (32) is

$$K_\omega(z, k, q, \mu_R) = \frac{\omega + 2\bar{\alpha}_S \ln(k/zq)}{\omega + 2\bar{\alpha}_S \ln(k/\mu_R)}.$$  \hfill (33)

Introducing $s = \ln(k/\mu_R)$, $t = \ln(q/\mu_R)$, we can write the expression in Eq. (31) as

$$H_\omega(s, t) = \frac{2\bar{\alpha}_S}{\omega + 2\bar{\alpha}_S s} \exp \left( \bar{\alpha}_S (t^2 - 2st) + \omega t \right).$$  \hfill (34)

The exponential factor contributes only sub-leading corrections, i.e. terms with fewer factors of $\ln(1/x)$ than of $\alpha_s$, which we are neglecting. Substituting in Eq. (30), we therefore find that to the same precision the CCFM result for the one-jet contribution is equal to the BFKL prediction (10).

To find the multi-jet fractions we use Eq. (19) with

$$G^{(r)}(T) = \prod_{j=1}^{r} \int_0^T H_\omega \left( \max_{i \leq j} \{ t_i \}, t_j \right) dt_j,$$  \hfill (35)

$H_\omega$ now being a function of two variables, given by Eq. (34). However, $H_\omega$ depends only on its first argument, apart from the negligible exponential factor. Therefore the CCFM rates for more than one jet also have the BFKL values, given by Eq. (23). This completes the all-orders extension of the results of Ref. [9].

4 Conclusions

The above results show that the multi-jet rates in deep inelastic scattering at small $x$, as defined here, are sufficiently inclusive quantities to be insensitive to the differences between the BFKL and CCFM formulations of small-$x$ dynamics at the leading-logarithmic level. In both cases they are given by the simple generating function (23). Differences would be expected at the sub-leading level and in more differential quantities such as multi-jet rapidity correlations [10].

Bearing in mind the importance of sub-leading corrections to structure functions at small $x$ [3, 4], one would not expect these result to be of direct phenomenological relevance, although they may provide a useful cross-check on the results of numerical simulations of small-$x$ final states [11, 12, 13].

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