Double groupoids, matched pairs and then matched triples

Ronald Brown
School of Computer Science
Bangor University
Gwynedd LL57 1UT, U.K.

January 19, 2013

1 Introduction

In this note we show that the known relation between double groupoids and matched pairs of groups may be extended, or seems to extend, to the triple case. The references give some other occurrences of double groupoids.

I hope that someone can pursue these ideas further.

2 Double groupoids

A \textit{double groupoid} is simply a groupoid object internal to the category of groupoids. Thus it consists of a set \( G \) with two groupoid structures, say \( \circ_1, \circ_2 \) which satisfy the \textit{interchange law}, namely the condition that there is only one way of evaluating the composition

\[
\begin{bmatrix}
   x & y \\
   z & w \\
\end{bmatrix} \xrightarrow{1} \quad \xrightarrow{2}
\]

This gives the formula on a line

\[(x \circ_1 z) \circ_2 (y \circ_1 w) = (x \circ_2 y) \circ_1 (z \circ_2 w), \tag{2}\]

for all \( x, y, z, w \in G \) such that all the compositions on both sides are defined.

3 Matched pairs of groups

The aim of this section is to give an exposition of matched pairs of groups in a style in keeping with the geometry of double groupoids, and where the notation is suggestive for higher dimensions. For information on matched pairs, see \cite{12}. I think this relationship is well known but currently do not have a reference.
Let the group $G$ be given as the product $MN$ of subgroups $M, N$ such that $M \cap N = 1$. Then each element $g$ of $G$ can be uniquely written as $mn, m \in M, n \in N$. By taking inverses we see that $g$ can also be written uniquely as $n'm', n' \in N, m' \in M$. We therefore write for $m \in M, n \in N$

$$mn = \overline{mnm}.$$ 

We write this pictorially as

\[ \begin{array}{c}
\vdots \\
m \\
\downarrow \\
\vdots \\
m^n \\
\downarrow \\
n \\
\downarrow \\
\vdots \\
\end{array} \quad \begin{array}{c}
\vdots \\
\vdots \\
\downarrow \\
\vdots \\
\overline{mnm} \\
\downarrow \\
\overline{n} \\
\downarrow \\
\vdots \\
\end{array} \]

The advantage of this approach is that this easily gives rules for products and these operations, as follows.

Consider the diagram:

\[ \begin{array}{c}
\vdots \\
l \\
\downarrow \\
\vdots \\
l^{m_n} \\
\downarrow \\
l^{m} \\
\downarrow \\
\vdots \\
\end{array} \quad \begin{array}{c}
\vdots \\
\vdots \\
\downarrow \\
\vdots \\
l^{m_n} \\
\downarrow \\
\overline{m_n} \\
\downarrow \\
\overline{n} \\
\downarrow \\
\vdots \\
\end{array} \quad \begin{array}{c}
\vdots \\
\vdots \\
\downarrow \\
\vdots \\
\overline{m_n} \\
\downarrow \\
\overline{n} \\
\downarrow \\
\vdots \\
\end{array} \quad \begin{array}{c}
\vdots \\
\vdots \\
\downarrow \\
\vdots \\
\overline{m_n} \\
\downarrow \\
\overline{n} \\
\downarrow \\
\vdots \\
\end{array} \]

We obtain immediately that:

$$m^{np} = (m^n)^p, \quad m(np) = m^n(m^n p).$$

Putting $p = n^{-1}$ in this equation gives

$$(mn)^{-1} = m^n (n^{-1}).$$

Consider the diagram:

\[ \begin{array}{c}
\vdots \\
l \\
\downarrow \\
\vdots \\
l^{m} \\
\downarrow \\
l^{m_n} \\
\downarrow \\
\vdots \\
\end{array} \quad \begin{array}{c}
\vdots \\
\vdots \\
\downarrow \\
\vdots \\
\overline{m} \\
\downarrow \\
\overline{n} \\
\downarrow \\
\vdots \\
\end{array} \]

This gives the rules:

$$lmn = l(mn), \quad (lm)^n = (l^m n)m^n.$$ 

In this case we deduce

$$(m^n)^{-1} = (m^{-1})^m.$$ 

We also note that

$$n^{-1}m^{-1} = (m_n)^{-1}(m_n)^{-1}.$$ 

If we replace $n^{-1}$ with $n$, and $m^{-1}$ with $m$, and write $\overline{m} = m^{-1}, \overline{n} = n^{-1}$, then we deduce that

$$nm = (\overline{m} \overline{n})^{-1}(\overline{m} \overline{n})^{-1} = (m^{\overline{n}})(m^{\overline{n}}).$$ (3)
We can express this in the language of double groupoids by saying that the set $M \times N$ may be given the two action groupoid structures $M \rtimes N$ and $M \ltimes N$. That is, in accordance with the pictures above, we have

$$(m, n) \circ_2 (m^n, p) = (m, np) \in M \rtimes N,$$

$$(l, m^n) \circ_1 (m, n) = (lm, n) \in M \ltimes N.$$ 

This double groupoid has the special property that given any two adjacent edges then there is a unique square filling them. Such double groupoids are well known to give rise to a groupoid, by filling in according to the following diagram:

leading to the definition in our case that

$$(m, n)(l, p) = (ml^n, n^p).$$

4 Triple groupoids

Here is an initial experiment on the ideas. We have three groups $M, N, P$ for which

- $M$ operates on the left of $N$ and of $P$
- $N$ operates on the right of $M$ and the left of $P$
- $P$ operates on the right of $M$ and of $N$.

The cubical model is then of the following form:

where

$$A = m^np = (m^p)^n,$$

$$B = m^np(n^p) = (m^n)^{m^p},$$

$$C = mn\text{p} = m^p(m^p).$$
At this stage one can begin to see the notation becoming impossible to handle, for example to describe the reconstitution of a group from the above data. We therefore adopt a \TeX style approach to superscripts and to left and right actions and write

\[ ^\wedge mn = m^n, \quad m^\wedge n = m^n. \]

That is a $^\wedge$ before a term raises it to a superscript. The above picture then becomes as follows:

\[
\begin{aligned}
A &= m^\wedge(np) = (m^\wedge((^\wedge n)p))^\wedge(n^\wedge p), \\
B &= ^\wedge(m^\wedge(^\wedge np))(n^\wedge p) = (^\wedge mn)^\wedge(^\wedge(m^\wedge n)p), \\
C &= ^\wedge(mn)p = (^\wedge mn)(^\wedge(m^\wedge n)p).
\end{aligned}
\]

This allows us to describe a possible formula for the composition in the group associated with a matched triple. First, the previous formula (3) for $nm$ may now be written as

\[ nm = (m^\wedge(^\wedge \bar{m} \bar{n}))(^\wedge(^\wedge \bar{m} \bar{n})n). \tag{4} \]

Even this is too cumbersome, and we therefore define two pairings by

\[
\begin{aligned}
m \not\rightarrow n &= m^\wedge(^\wedge \bar{m} \bar{n}), \\
m \not\leftarrow n &= (^\wedge(^\wedge \bar{m} \bar{n}))n.
\end{aligned}
\]

Now adjoining two cubes suggests the formula for a group composition:

\[ (m, n, p)(\mu, \nu, \pi) = (m((\mu \not\rightarrow p) \not\rightarrow n), ((\mu \not\rightarrow p) \not\leftarrow n)(\nu \not\rightarrow (\mu \not\leftarrow p)), (\nu \not\rightarrow (\mu \not\leftarrow p))\pi). \]

There remains to check that this satisfies the group axioms. This should be done as far as possible not by manipulating these formulae on a line but by examining the required cubical diagrams.

This situation should occur for a matched triple of subgroups $M, N, P$ of $G$ in which each subpair is a matched pair.
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