Second Quantization of the Stueckelberg Relativistic Quantum Theory and Associated Gauge Fields

L.P. Horwitz

School of Physics
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University, Ramat Aviv, Israel

Abstract. The gauge compensation fields induced by the differential operators of the Stueckelberg-Schrödinger equation are discussed, as well as the relation between these fields and the standard Maxwell fields. An action is constructed and the second quantization of the fields carried out using a constraint procedure. Some remarks are made on the properties of the second quantized matter fields.

I. Introduction.

There has been considerable progress in recent years in the study of the relativistic quantum theory proposed (for a single particle) by Stueckelberg\(^1\) and generalized by Horwitz and Piron\(^2\) to many body systems. Collins and Fanchi\(^3\) arrived at this structure by considering the relativistic current conservation law, and Fanchi\(^4\) has made many studies of the properties of the theory. In particular, he has shown that there is no Klein paradox.\(^5\) It has been shown\(^2\) that the Newton-Wigner operator,\(^6\) originally constructed for on-shell wave equations, emerges from the Stueckelberg theory, which is intrinsically off-shell, as the operator

\[
\mathbf{x}_{op} = \mathbf{x} - \frac{1}{2}\{t, \mathbf{p}\},
\]

in a direct integral representation over masses, at each mass value. The Landau-Peierls\(^7\) uncertainty relation, \(\Delta p \Delta t \geq \hbar/c\) has also been shown to arise as an exact mathematical bound\(^8\) for the dispersion of the operator

\[
t_{op} = t - \frac{1}{2}\{x, E\},
\]

which has an evident semiclassical interpretation.

The two body quantum relativistic bound state problem for spinless particles with invariant action-at-a-distance potentials has been solved\(^9\); the differential equation for the radial part of the wave function (for which the variable is the spacelike invariant separation \(\rho = \sqrt{(x_1 - x_2)^2 - (t_1 - t_2)^2}\)) is identical to the non-relativistic radial Schrödinger equation with potential of the same functional form. The eigenvalues of the reduced motion Hamiltonian therefore coincide with the Schrödinger energy spectrum, but correspond to

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mass shifts induced by the interaction. The observed center of mass energy of the system is given by (see ref. 9 for further discussion)

\[ E_n = \sqrt{M^2 + 2Mk_n}, \]  

where \( k_n \) is the (Schrödinger) spectrum of the reduced motion, and \( M \) is a dimensional scale parameter corresponding to the sum of the Galilean limiting masses of the two particles of the system. For small excitations, \( E_n \cong M + k_n \), up to terms of higher order in \( 1/c \) (relativistic corrections).

I wish to review here the general gauge structure of the theory, the relation of the gauge fields generated by the Stueckelberg-Schrödinger equation to the Maxwell fields, and their second quantization based on a constraint formalism\(^{10,11}\).

II. The Gauge Fields.

I begin with the statement of the Stueckelberg-Schrödinger equation for the wave function representing a free one-particle state,

\[ i\frac{\partial}{\partial \tau} \psi_\tau(x) = \frac{p_\mu p^\mu}{2M} \psi_\tau(x), \]  

where \( x \equiv x^\mu = (x, t) \), and \( \tau \) is the invariant parameter of evolution. The operator \( p^\mu \) is realized by \(-i\partial/\partial x_\mu\) in this representation. We recognize that, in addition to the coordinate \( x \), \( t \) is also an observable in the quantum theory, for which there is a self-adjoint time operator. This structure is necessary for the manifest covariance of the quantum theory for which, under the action of the Lorentz group, the space and time variables transform (linearly) among each other.* The standard relativistic quantum field theories admit the Lorentz transformation in a consistent way by interpreting both \( x \) and \( t \) as parameters. As we shall see, the second quantization of the Stueckelberg theory has the same property, but includes the evolution parameter \( \tau \) for the description of dynamical processes (it does not necessarily correspond to a label for spacelike surfaces, as in the formulation of Schwinger and Tomonaga\(^{12}\), or to the proper time of any system). The coordinate representations generated by these fields are over space-time, as for the one-body quantum theory of Stueckelberg.\(^1\)

If we admit a local gauge transformation

\[ \psi'_\tau(x) = e^{iA(x, \tau)} \psi_\tau(x), \]  

* Note that the parameters of the Lorentz group are not associated with the position or velocity of a particle. The association is often made classically when one describes the particle in terms of a motion induced on a particle at rest by transforming to a moving frame. The acceleration of a particle due to forces cannot be accounted for in this way, since this would involve transformation to a non-inertial frame, going beyond the applicability of special relativity. Accelerated motions of the particle are accounted for in the Stueckelberg theory as a result of covariant dynamical equations.
the equation (2.1) must be altered in order to admit gauge covariance by the addition of gauge compensation fields. The modified equation reads

\[(i \frac{\partial}{\partial \tau} + e_0 a_5) \psi_\tau(x) = \frac{(p^\mu - e_0 a^\mu)(p_\mu - e_0 a_\mu)}{2M} \psi_\tau(x), \] (2.3)

which is clearly gauge covariant if

\[a_\mu' = a_\mu + \frac{1}{e_0} \partial_\mu \Lambda, \]

\[a_5' = a_5 + \frac{1}{e_0} \partial_5 \Lambda, \] (2.4)

which we can write in terms of a five-component field \((\alpha = 0, 1, 2, 3, 5; \partial_5 \equiv \partial_\tau)\)

\[a_\alpha' = a_\alpha + \frac{1}{e_0} \partial_\alpha \Lambda. \] (2.5)

The field strengths

\[f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha \] (2.6)

are gauge invariant, and field equations of second order can be generated if the term \(f_{\alpha\beta} f^{\alpha\beta}\) is present in the Lagrangian. In order to determine the coefficients in the model Lagrangian that we shall write down, we shall need some information about the conserved current associated with Eq. (2.3).

Differentiating the probability density \(\rho_\tau(x) = |\psi_\tau(x)|^2\) with respect to \(\tau\), and using Eq. (2.3), one finds that

\[\partial_\mu j_\mu^\tau = \frac{1}{2M i} \left\{ \psi_\tau^\ast \left[ (\partial^\mu - ie_0 a^\mu) \psi_\tau \right] - \left[ (\partial^\mu - ie_0 a^\mu) \psi_\tau^\ast \psi_\tau \right] \right\}. \] (2.8)

Since \(\rho_{\pm \infty}(x) = 0\) \(^{1,14,15}\) (pointwise), it follows from (2.7) that the integrated current

\[J^\mu = \int_{-\infty}^{\infty} d\tau \ j_\mu^\tau(x) \] (2.9)

satisfies

\[\partial_\mu J^\mu(x) = 0, \] (2.10)

and may therefore be identified with the Maxwell current. Writing a Lagrangian density which generates the Stueckelberg-Schrödinger equation as a field equation, along with the second order equations for the gauge compensation fields, one finds that these field equations have the form

\[\partial_\beta f^{\alpha\beta} \propto j^\alpha, \]

so that

\[j^\alpha \propto \partial_\tau f^{\mu\tau} + \partial_\nu f^{\mu\nu}. \]
Integrating this relation over $\tau$, with vanishing boundary condition on the field strengths at $\tau \to \pm \infty$ for any finite spacetime $x$, we see that the $\tau$-integral of the fields satisfy Maxwell equations, with the Maxwell current as the source. We therefore identify the Maxwell fields with the zero modes of what we shall call the pre-Maxwell fields, i.e.,

$$A^\mu(x) = \int_{-\infty}^{\infty} d\tau \, a^\mu(x, \tau).$$  \hfill (2.11)

Defining the Fourier representation

$$a^\mu(x, \tau) = \int dse^{i\sigma \tau} \hat{a}^\mu(x, s),$$  \hfill (2.12)

we see that the Maxwell limit corresponds to a field $\hat{a}^\mu(x, s)$ with support in a small interval $\Delta s$ around zero, so that

$$a^\mu(x, \tau) \sim \Delta s \hat{a}^\mu(x, 0).$$  \hfill (2.13)

In the Maxwell theory, the field $A^\mu(x)$ must have dimension ($\ell$ is length) $\ell^{-1}$, and therefore the dimension of $a^\mu$ must be $\ell^{-2}$, and that of $e_0$, $\ell$. Then the dimension of $f^{\alpha\beta}$ is $\ell^{-3}$, so that the second order form occurring in the Lagrangian has dimension $\ell^{-6}$. The action integral provides a factor $d\tau d^4 x$, of dimension $\ell^5$, and therefore there must be a coefficient of dimension $\ell$, which we shall call $\lambda$, for the quadratic term in field strengths. It then follows that in the Maxwell limit,

$$e_0 a^\mu(x, \tau) \sim e_0 \Delta s A^\mu(x),$$

so that this width sets the scale for the Maxwell limit of the theory, i.e., $\Delta s e_0$ corresponds to the Maxwell electric charge. We shall see below that it must coincide with $\frac{1}{\lambda}$, the dimensional parameter introduced in the Lagrangian*.

The action for the quantized fields is then (see ref. 11 for further discussion, and a review of the standard Maxwell case by this method) ($d^5 x \equiv d\tau d^4 x$)

$$S = \int d^5 x \left\{ -\frac{\lambda}{4} f^{\alpha\beta} f_{\alpha\beta} + \frac{i}{2} \left\{ \psi^\dagger \frac{\partial \psi}{\partial \tau} - \frac{\partial \psi^\dagger}{\partial \tau} \psi \right\} ight.$$  

$$- \frac{1}{2M} \psi^\dagger (\partial^\mu - ie_0 a^m u)(\partial_\mu - ie_0 a_\mu) \psi$$  

$$+ e_0 \psi^\dagger a_5 \psi - G \partial_\alpha a^\alpha + \frac{1}{2\lambda} G^2 \right\},$$  \hfill (2.14)

where we have omitted explicit $x, \tau$ dependences; the operator-valued function $G(x)$ is a ghost field, providing a canonical conjugate to $a_5$ (as for $A_0$ in the usual Maxwell action).

The classical field equations corresponding to this action, in the absence of the ghost field, take the form

$$\lambda \partial_\alpha f^{\alpha\beta} = e_0 j^\beta,$$  \hfill (2.15)

* In a more complete theory, this parameter should therefore emerge from a dynamical condition.
so that we see that $e_0/\lambda$ is to be identified with the Maxwell electric charge, and hence in the Maxwell limit, the inverse correlation length $\Delta s \sim 1/\lambda$. It would be of interest to study the Ward identities of this theory, and establish the relation of this structure to charge renormalization.

We further note from (2.15) that, using a generalization of the Lorentz gauge, $\partial_\alpha a^\alpha = 0$, the equation for the source-free case becomes a d’Alembert equation with an additional second derivative with respect to $\tau$. The differential operator has the form

$$\partial_\tau \partial^\tau - \partial^2_t + \Delta = 16 \sigma \partial_\tau^2 - \partial^2_t + \Delta,$$

where $\sigma = \pm$ is the signature of the $\tau$ variable in the wave equation ($\Delta$ is the Laplacian). Just as for the emergence of electromagnetism formally from the non-relativistic Schrödinger theory, the evolution parameter enters the manifold of the resulting wave equation. Positive signature, corresponding to $O(4,1)$ invariance of the homogeneous equations, corresponds to a field with real mass $s$ under Fourier transform, and with negative signature, to a tachyonic wave equation, with $O(3,2)$ invariance. The Green’s functions for these wave equations have been worked out. The integral over $\tau$ for the tachyonic part of the Green’s function vanishes, so there is no violation of causality in the transmission of information defined through Maxwell fields. The theory is, however, capable of establishing dynamical correlations which are spacelike.

The canonical conjugate momenta are defined as

$$\pi^\mu = \frac{\delta L}{\delta(\partial_\tau a_\mu)} = -\lambda f^{5\mu},$$

$$\pi^5 = \frac{\delta L}{\delta(\partial_\tau a_5)} = -\sigma G(x),$$

$$\pi_\psi = \frac{\delta L}{\delta(\partial_\tau \psi)} = i\psi^\dagger.$$

The Gauss law obtained from the classical equations (2.15) is

$$\lambda \partial_\mu f^{5\mu} = e_0 \rho,$$

so that we should impose on physical states that

$$\langle \partial_\mu \pi^\mu + j^5 \rangle = 0,$$

i.e., that the Gauss law holds. This requirement can be satisfied by imposing $G^{(+)}|\nu\rangle = 0$ (note that $G(x)$ satisfies $(\sigma \partial^2_\tau - \partial^2_t + \Delta)G = 0$, so that it is a free field, and the $\pm$ frequency parts can be isolated). The stability of this condition requires that

$$\dot{G}^{(+)}|\nu\rangle = 0.$$

We shall see that this condition is satisfied if the Gauss law is true, so that the theory is consistent.
To generate the time evolution, we carry out the Legendre transform of the Lagrangian to obtain the Hamiltonian of the system, with the result that

\[ K = K_\gamma + K_m + K_{\gamma m}, \quad (2.19) \]

where

\[
K_\gamma = \int d^4x \left\{ -\frac{1}{2\lambda} \pi^\mu \pi_\mu - \frac{\lambda}{4} f^{\mu\nu} f_{\mu\nu} \\
+ \pi^\mu (\partial_\mu a^5) - \pi^5 (\partial_\mu a^\mu) - \frac{\sigma}{2\lambda} (\pi^5)^2 \right\},
\]

\[
K_m = \frac{\sigma}{2M} \int d^4x \psi^\dagger \partial_\mu \partial^\mu \psi,
\]

and

\[
K_{\gamma m} = \sigma \int d^4x \left\{ -e_0 \psi^\dagger a_5 \psi - \frac{ie_0}{2M} \psi^\dagger [2a^\mu a_\mu + \partial_\mu a^\mu] \psi \\
- \frac{e_0^2}{2M} \psi^\dagger a^\mu a_\mu \right\}.
\]

The canonical commutation relations are

\[
[\pi^\alpha(x), a_\beta(y)] = -i \delta^\alpha_\beta \delta(x - y)
\]

\[
[i \psi^\dagger(x), \psi(y)] = -i \delta(x - y). \quad (2.20)
\]

It then follows that

\[
\hat{G} = i[K, G] = -\sigma (\rho + \partial_\mu \pi^\mu), \quad (2.21)
\]

and hence the stability of the condition (2.18) implies the Gauss law. The condition that the commutator of \( \hat{G} \) with the Hamiltonian vanish in physical states, i.e., the stability of Gauss law, is satisfied as well; one uses, after taking the commutator of \( \partial_\mu \pi^\mu \) with \( K \), the current conservation law (2.7). The operator

\[
e^{i\chi} = \exp i \int d^4x \Lambda(x) (\partial_\mu \pi^\mu(x) + \rho(x))
\]

commutes with the Hamiltonian in physical states, and can be used to generate a gauge transformation which eliminates the part of the \( a^\mu \) field which is parallel to \( \partial^\mu \Lambda \) in the field functional \( \Psi(a_\perp^\mu, a_\parallel^\mu, a^5, \psi) \), i.e., the part \( a_\parallel^\mu \). Consider the following three cases.\(^{11}\)

**Case 1:** \( k^\mu \) timelike.

In this case, the component \( a^0 \) can be eliminated in the frame \( k^\mu = (k^0, 0, 0, 0) \), leaving the “Coulomb” potential \( a^5 \) and three polarizations \( a_i \). The polarization space is then a positive norm representation of O(3).

**Case 2:** \( k^\mu \) spacelike.

In the frame \( k^\mu = (0, 0, 0, k^3) \), one can eliminate \( a^3 \), and leaving the components \( a^0, a^1, a^2 \). These directions span the indefinite space representing O(2,1). The Casimir operator \( N = M_{12}^2 - M_{01}^2 - M_{02}^2 \) is invariant (under the dynamical action of the Hamiltonian, which is O(3,1) invariant), and the sign of its expectation value should therefore be preserved.
under evolution. The states of polarization with negative norm removed is therefore a stable invariant subspace. The zero-norm components can be removed as in Case 3.

Case 3: $k^\mu$ lightlike.

In this case, one can eliminate $a_0$ and $a_\parallel$ together, leaving only two transverse polarizations, as in the Maxwell theory. The zero norm states are eliminated by means of the Gupta-Bleuler condition, as is well known in the usual electromagnetic theory.

We see that the off-shell photons, which are massive (or tachyonic), have three polarization degrees of freedom. It is therefore important to prove that for the equilibrium black-body radiation field, which shows a specific heat characteristic of just two polarization states, that the off-shell photons do not contribute. Although it could be expected that off-shell photons are important at the walls, where emission and absorption take place, the volume contribution to the specific heat would not show this effect, but careful estimates must be carried out. There are many other places where observable phenomena might exist, such as deep inelastic scattering experiments, and these will be discussed elsewhere.

Carrying out the transformation (2.22) on the Hamiltonian, only polarization degrees of freedom remain\(^{11}\), as for the usual Maxwell case. There is, as in the usual theory, where it emerges as an instantaneous Coulomb interaction, an additional residual term of the form

$$\langle K_c \rangle = \left\langle -\frac{1}{2\lambda} \int d^4x d^4y \, \partial_\mu \pi_\mu(x)G(x-y)\partial_\nu \pi_\nu(y) \right\rangle$$

$$= \frac{e^2}{2\lambda} \int d^4x d^4y \, \langle \rho_\tau(x)G(x-y)\rho_\tau(y) \rangle + \text{const}, \quad (2.23)$$

where we have used the Gauss law, and $G(x-y)$ is a Green’s function for the d’Alembert operator.

We have discussed above an interpretation linking the parameter $\lambda$ with a correlation length of the fields in the Maxwell limit. Taking this correlation into account, it was argued in ref. 11 that, in the classical limit, when the relative motion of the particles is not too large, (2.23) becomes equivalent to the Fokker action\(^{17}\).

3. Conclusions and Remarks.

I have reviewed the second quantization of the gauge fields generated by the Stueckelberg-Schrödinger equation, and shown how the resulting Hamiltonian can be represented in terms of the polarization (physical) fields up to an additional term which is approximately related to the Fokker action. Just as in the usual case, where the additional term represents the instantaneous Coulomb field, our construction\(^{11}\) provides an extra term that corresponds to a “pre-theory”, i.e., a theory that could be effective before quantum effects become important.

The quantum matter fields satisfy the (equal $\tau$) canonical commutation relations

$$[\psi_\tau(x), \psi_{\tau}^\dagger(y)] = \delta^4(x-y) \quad (3.1)$$

Evidently, these fields generate a Fock space of a different type than that generated by on-shell fields associated with a Klein-Gordon equation. They create and annihilate particles of arbitrary mass; the quantum states are constructed over wave functions which restrict
these masses according to the dynamical equations of the system. In the limit in which we may think of restricting these masses to definite values, it is of interest to see how this Fock space deforms to the usual one. To see this, let us take the Fourier transform, and express (3.1) in the form

\[ [\psi_\tau(p), \psi_\tau^\dagger(p')] = \delta^4(p-p') \]  

(3.2)

Integrating both sides over \( E \), the right side becomes \( \delta^3(p-p') \); to carry out the integral over the left side, we use the fact that the momentum must remain fixed in the differentiation, and therefore only the mass can be varied (in the formula \( E = \sqrt{p^2 + m^2} \)), and hence

\[ dE = \frac{1}{2E} dm^2. \]  

(3.3)

If we call

\[ \hat{\psi}(p) = \sqrt{dm^2} \psi(p) \bigg|_{E=\sqrt{p^2+m^2}} \]  

(3.4)

we see that the resulting commutation relations are

\[ [\hat{\psi}_\tau(p), \hat{\psi}_\tau^\dagger(p')] = 2E \delta^3(p-p'). \]  

(3.6)

In a similar way, the space-time commutation relations (3.1) vanish for \( x^0 \neq y^0 \). Integrating over an infinitesimal interval \( dx^0 \), the right hand side becomes \( \delta^3(x-y) \), and we may absorb factors \( \sqrt{dx^0} \) into each space-time field at equal time. We therefore recover the usual equal time commutation relations. A more complete discussion of the equal time limit of the on mass shell theory will be given elsewhere.

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