Variational theory for the resonant $T$-curvature equation

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Abstract

We study the resonant prescribed $T$-curvature problem on a compact 4-dimensional Riemannian manifold with boundary. We derive sharp energy and gradient estimates of the associated Euler-Lagrange functional to characterize the critical points at infinity of the associated variational problem under a non-degeneracy on a naturally associated Hamiltonian function. Using this, we derive a Morse type lemma around the critical points at infinity. Using the Morse lemma at infinity, we prove new existence results of Morse theoretical type. Combining the Morse lemma at infinity and the Liouville version of the Barycenter technique of Bahri-Coron\cite{13} developed in \cite{46}, we prove new existence results under a topological hypothesis on the boundary of the underlying manifold, and the entry and exit sets at infinity.

Key Words: $T$-curvature, $Q$-curvature, Morse Theory, Critical points at infinity, Barycenter Technique.

AMS subject classification: 53C21, 35C60, 58J60, 55N10.

1 Introduction and statement of the results

On a four-dimensional compact Riemannian manifolds with boundary $(M, g)$, there exists a fourth-order operator $P_g$ called Paneitz operator discovered by Paneitz\cite{49} and an associated curvature quantity $Q_g$ called $Q$-curvature introduced by Branson-Oersted\cite{14}. The Paneitz operator $P_g$ and the $Q$-curvature $Q_g$ are defined in terms of the Ricci tensor $Ric_g$ and the scalar curvature $R_g$ of $(M, g)$ by

$$P_g^4 = \Delta_g^2 - div_g \left(\frac{2}{3} R_g g - 2Ric_g \nabla_g\right); \quad Q_g = -\frac{1}{12}(\Delta_g R_g - R_g^2 + 3|Ric_g|^2),$$

where $div_g$ is the divergence and $\nabla_g$ is the covariant derivative of with respect to $g$.

On the other hand, Chang-Qing\cite{23} have discovered an operator $P_g^3$ which is associated to the boundary $\partial M$ of $M$ and a curvature quantity $T_g$ naturally associated to $P_g^3$. They are defined by the formulas

$$P_g^3 = \frac{1}{2} \frac{\partial \Delta_g}{\partial n_g} + \Delta_g \frac{\partial}{\partial n_g} - 2H_g \Delta_g + L_g(\nabla \nabla_g, \nabla_g) + \nabla_g H_g, \nabla_g + (F_g - \frac{R_g}{3}) \frac{\partial}{\partial n_g},$$

$$T_g = -\frac{1}{12} \frac{\partial R_g}{\partial n_g} + \frac{1}{2} R_g H_g - <G_g, L_g> + 3H_g^3 - \frac{1}{3} tr_g(L_g^3) - \Delta_g H_g,$$

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As was asked in [2], a natural question is whether every compact four-dimensional Riemannian manifold \( (M,g) \) carries a conformal metric \( g_u = e^{2u}g \) such that
\[
\int_M (Q_g + \frac{|W_g|^2}{8}) dV_g + \oint_{\partial M} (T_g + Z_g) dS_g = 4\pi^2 \chi(M)
\]
where \( W_g \) denote the Weyl tensor of \( (M,g) \) and \( Z_g \) is given by the following formula
\[
Z_g = R_g H_g - 3H_g Ric_{g,n} + \hat{g}^{ac} \hat{g}^{bd} R_{g,anbn}L_{g,cd} - \hat{g}^{ac} \hat{g}^{bd} R_{g,abcde}L_{g,cd} + 6H_g^3 - 3H_g^4 + \frac{1}{2} |L_g|^2 + tr_g(L_g^3)
\]
with \( tr_g \) denoting the trace with respect to the metric induced on \( \partial M \) by \( g \) (namely \( \hat{g} \)) and \( \chi(M) \) the Euler-Poincaré characteristic of \( M \). Concerning the quantity \( Z_g \), we have that it vanishes when the boundary is totally geodesic and \( \oint_{\partial M} Z_g dV_g \) is always conformally invariant, see [25]. Thus, setting
\[
\kappa_{(P^4,P^3)} := \kappa_{(P^4,P^3)}[g] := \int_M Q_g dV_g + \oint_{\partial M} T_g dS_g
\]
we have that thanks to [2], and to the fact that \( |W_g|^2 dV_g \) is pointwise conformally invariant, \( \kappa_{(P^4,P^3)} \) is a conformal invariant (which justifies the notation used above). We remark that \( 4\pi^2 \) is the total integral of the \( (Q,T) \)-curvature of the standard four-dimensional Euclidean unit ball \( \mathbb{B}^4 \).

As was asked in [2], a natural question is whether every compact four-dimensional Riemannian manifold with boundary \( \overline{(M,g)} \) carries a conformal metric \( g_u \) for which the corresponding \( Q \)-curvature \( Q_{g_u} \) is zero, the corresponding \( T \)-curvature \( T_{g_u} \) is prescribed function and such that \( (\overline{M},g_u) \) has minimal boundary. Thanks to [1], this problem is equivalent to finding a smooth solution to the following BVP:
\[
\begin{cases}
P^4_g u + 2Q_g = 0 & \text{in } M, \\P^3_g u + T_g = K e^{3u} & \text{on } \partial M, \
\frac{\partial u}{\partial n_g} + H_g u = 0 & \text{on } \partial M.
\end{cases}
\]
where \( K : \partial M \to \mathbb{R}_+ \) is a positive smooth function on \( \partial M \).

Since we are interested to find a metric in the conformal class of \( g \), then we can assume that \( H_g = 0 \), since this can be always obtained through a conformal transformation of the background metric. Thus, we are lead to solve the following BVP with Neumann homogeneous boundary condition:
\[
\begin{cases}
P^4_g u + 2Q_g = 0 & \text{in } M, \\P^3_g u + T_g = K e^{3u} & \text{on } \partial M, \\
\frac{\partial u}{\partial n_g} = 0 & \text{on } \partial M.
\end{cases}
\]
As a Liouville type problem, the assumption ker

$$\text{ker}$$

of equation (4) and of the associated Euler-Lagrange

$$F$$

we have integration by part implies

$$\mathbb{P}^4_g(u,v) = \int_M \left( \Delta_g u \Delta_g v + \frac{2}{3} R_g \nabla_g u \cdot \nabla_g v \right) dv_g - 2 \int_M R_g \left( \nabla_g u, \nabla_g v \right) dv_g$$

(5)

and is clearly a bilinear form on \( \mathcal{H}_g \). We set

$$\text{ker} \mathbb{P}^4_g := \{ u \in \mathcal{H}_g : \mathbb{P}^4_g(u,v) = 0, \forall v \in \mathcal{H}_g \}$$

On the other hand, standard regularity theory implies that smooth solutions to (4) can be found by looking at critical points of the geometric functional

$$\mathcal{E}_g(u) = \mathbb{P}^4_g(u,u) + 4 \int_M Q_g udV_g + 4 \oint_{\partial M} T_g u dS_g - \frac{4}{3} \kappa(p^4,p^3) \log \oint_{\partial M} K e^{2\alpha} dS_g, \quad u \in \mathcal{H}_g.$$
and define
\[ L_K(A) := -\sum_{i=1}^{k} (F_i^A)^{1/2} L_{\hat{g}}((F_i^A)^{1/2})(a_i), \]
where
\[ L_{\hat{g}} := -\Delta_{\hat{g}} + \frac{1}{8} R_{\hat{g}} \]
is the conformal Laplacian associated to \( \hat{g} \). We also set
\[ F_{\infty} := \{ A \in \text{Crit}(F_K) : L_K(A) < 0 \}, \]
\[ i_{\infty}(A) := 4k - 1 - \text{Morse}(A, F_K), \]
and define
\[ m^k_i := \text{card}\{ A \in \text{Crit}(F_K) : i_{\infty}(A) = i \}, \ i = 0, \ldots, 4k - 1, \]
where \( \text{Morse}(F_K, A) \) denotes the Morse index of \( F_K \) at \( A \). We point out that for \( k \geq 2, m^k_i = 0 \) for \( 0 \leq i \leq k - 2 \).

For \( k \geq 2 \), we use the notation \( B_{k-1}(\partial M) \) to denote the set of formal barycenters of order \( k - 1 \) of \( \partial M \), namely
\[ B_{k-1}(\partial M) := \{ \sum_{i=1}^{k-1} \alpha_i a_i : a_i \in \partial M, \ \alpha_i \geq 0, i = 1, \ldots, k - 1, \ \sum_{i=1}^{k-1} \alpha_i = 1 \}. \]
Furthermore, we define
\[ c_{p}^{k-1} = \text{dim} H_p(B_{k-1}(\partial M)), \ p = 1, \ldots, 4k - 5, \]
where \( H_p(B_{k-1}(\partial M)) \) denotes the \( p \)-th homology group of \( B_{k-1}(\partial M) \) with \( \mathbb{Z}_2 \) coefficients. Finally, we say
\[ (ND) \ \text{ holds if } \ F_K \text{ is a Morse function and for every } A \in \text{Crit}(F_K), \ L_K(A) \neq 0. \]

Now, we are ready to state our existence results of Morse theoretical type starting with the critical case, namely when \( k = 1 \).

**Theorem 1.1.** Let \( (\overline{M}, g) \) be a compact 4-dimensional Riemannian manifold with boundary \( \partial M \) and interior \( M \) such that \( H_g = 0, \ker P^4 g \cong \mathbb{R} \) and \( \kappa_{(p^4, p^3)} = 4\pi^2 \). Assuming that \( K \) is a smooth positive function on \( \partial M \) such that \( (ND) \) holds and the system
\[
\begin{align*}
    m^1_0 &= 1 + x_0, \\
    m^1_i &= x_i + x_{i-1}, & i = 1, \ldots, 3, \\
    0 &= x_3 \\
    x_i &\geq 0, & i = 0, \ldots, 3
\end{align*}
\]
has no solutions, then \( K \) is the \( T \)-curvature of a Riemannian metric on \( \overline{M} \) conformally related to \( g \) with zero \( Q \)-curvature in \( M \) and zero mean curvature on \( \partial M \).

The system \([10]\) not having a solution traduces the violation of a strong Morse type inequalities (SMTI) for the critical points at infinity of \( E_g \). Since (SMTI) imply Poincare-Hopf type formulas, then we have Theorem \([11]\) implies the following Poincare-Hopf index type result.
Corollary 1.2. Let $(\overline{M}, g)$ be a compact 4-dimensional Riemannian manifold with boundary $\partial M$ and interior $M$ such that $H_g = 0$, $\ker \mathbb{P}^{4,3}_g \simeq \mathbb{R}$ and $\kappa_{(p^4, p^3)} = 4\pi^2$. Assuming that $K$ is a smooth positive function on $\partial M$ such that (ND) holds and

$$\sum_{A \in \mathcal{F}_\infty} (-1)^{i_\infty(A)} \neq 1,$$

then $K$ is the $T$-curvature of a Riemannian metric on $\overline{M}$ conformally related to $g$ with zero $Q$-curvature in $M$ and zero mean curvature on $\partial M$.

The formula (17) says that the Euler number of the space of variations is different from the total contribution of the true critical points at infinity and is of global character. Localizing the arguments of Corollary 1.2 in the case of the presence of a jump in the Morse index of the critical points of the Hamiltonian function $F_K$, we have the following extension of Corollary 1.2.

Theorem 1.3. Let $(\overline{M}, g)$ be a compact 4-dimensional Riemannian manifold with boundary $\partial M$ and interior $M$ such that $H_g = 0$, $\ker \mathbb{P}^{4,3}_g \simeq \mathbb{R}$ and $\kappa_{(p^4, p^3)} = 4\pi^2$ and $K$ be a smooth positive function on $\partial M$ satisfying the non degeneracy condition (ND). Assuming that there exists a positive integer $1 \leq l \leq 3$ such that

$$\sum_{A \in \mathcal{F}_\infty, i_\infty(A) \leq l-1} (-1)^{i_\infty(A)} \neq 1$$

and

$$\forall A \in \mathcal{F}_\infty, \quad i_\infty(A) \neq l,$$

then $K$ is the $T$-curvature of a Riemannian metric on $\overline{M}$ conformally related to $g$ with zero $Q$-curvature in $M$ and zero mean curvature on $\partial M$.

In the supercritical case, i.e. $k \geq 2$, the Euler-Lagrange functional $E_g$ is not bounded from below, and taking into account the topological contribution of very large negative sublevels of $E_g$, we have the following analogue of Theorem 1.1.

Theorem 1.4. Let $(\overline{M}, g)$ be a compact 4-dimensional Riemannian manifold with boundary $\partial M$ and interior $M$ such that $H_g = 0$, $\ker \mathbb{P}^{4,3}_g \simeq \mathbb{R}$, and $\kappa_{(p^4, p^3)} = 4k\pi^2$ with $k \geq 2$. Assuming that $K$ is a smooth positive function on $\partial M$ such that (ND) holds and the following system

$$0 = x_0,$$

$$m_1^k = x_1,$$

$$m_i^k = c_{i-1}^k + x_i + k_{i-1}, \quad i = 2, \ldots, 4k - 4,$$

$$m_i^k = x_i + k_{i-1}, \quad i = 4k - 3, \ldots, 4k - 1,$$

$$0 = x_{4k-1},$$

$$x_i \geq 0, \quad i = 0, \ldots, 4k - 1,$$

has no solutions, then $K$ is the $T$-curvature of a Riemannian metric on $\overline{M}$ conformally related to $g$ with zero $Q$-curvature in $M$ and zero mean curvature on $\partial M$.

Remark 1.5. The presence of the number $c_{i-1}^k = \dim H_{i-1}(B_{i-1}(\partial M))$ in (18) account for the contribution of the topology of very negative sublevels of $E_g$. The relation between the topology of very negative sublevels of the Euler-Lagrange functional of Liouville type problems and the space of formal barycenters was first observed by Djadli-Malchiodi [29].

As in the critical case, we have that Theorem 1.4 implies the following Poincaré-Hopf index type criterion for existence.
Corollary 1.6. Let \((\bar{M}, g)\) be a compact 4-dimensional Riemannian manifold with boundary \(\partial M\) and interior \(M\) such that \(H_g = 0\), \(\ker \mathbb{P}^{4,3}_g \cong \mathbb{R}\), and \(\kappa(\mathbb{P}_4, \mathbb{P}_3) = 4\kappa\pi^2\) with \(k \geq 2\). Assuming that \(K\) is a smooth positive function on \(\partial M\) such that \((ND)\) holds and

\[
\sum_{A \in \mathcal{F}_\infty} (-1)^{i_\infty(A)} \neq \frac{1}{(k-1)!} \Pi_{i=1}^{k-1} (i - \chi(\partial M)),
\]

then \(K\) is the \(T\)-curvature of a Riemannian metric on \(\bar{M}\) conformally related to \(g\) with zero \(Q\)-curvature in \(M\) and zero mean curvature on \(\partial M\).

As in the critical case, we have that a localization of the arguments of Corollary 1.6 implies the following jumping index type result.

Theorem 1.7. Let \((\bar{M}, g)\) be a compact 4-dimensional Riemannian manifold with boundary \(\partial M\) and interior \(M\) such that \(H_g = 0\), \(\ker \mathbb{P}^{4,3}_g \cong \mathbb{R}\), \(\kappa(\mathbb{P}_4, \mathbb{P}_3) = 4\kappa\pi^2\) with \(k \geq 2\), and let \(K\) be a smooth positive function on \(\partial M\) satisfying the non degeneracy condition \((ND)\). Assuming that there exists a positive integer \(1 \leq l \leq 4k - 1\) and \(A^l \in \mathcal{F}_\infty\) with \(i_\infty(A^l) \leq l - 1\) such that

\[
\sum_{A \in \mathcal{F}_\infty, i_\infty(A) \leq l - 1} (-1)^{i_\infty(A)} \neq \frac{1}{(k-1)!} \Pi_{j=1}^{k-1} (j - \chi(\partial M))
\]

and

\[
\forall A \in \mathcal{F}_\infty, \quad i_\infty(A) \neq l,
\]

then \(K\) is the \(T\)-curvature of a Riemannian metric on \(\bar{M}\) conformally related to \(g\) with zero \(Q\)-curvature in \(M\) and zero mean curvature on \(\partial M\).

The Morse theoretical results stated above depend only the Morse Lemma at infinity around true critical points at infinity (see Lemma 3.21) which justify the condition \(\mathcal{L}_K < 0\) in the definition of \(\mathcal{F}_\infty\). However, our existence result of algebraic topological type are based on the Morse lemma at infinity around all critical points at infinity. Thus, to state our existence result of algebraic topological type, we need first to introduce the neighborhood of potential critical points at infinity of \(\mathcal{E}_g\). In order to do that, we first fix \(\nu\) to be a positive and small real number, \(\Lambda\) to be a large positive constant, and \(R\) to be a large positive constant too. Next, for \(\epsilon\) small and positive, and \(\Theta \geq 0\), we denote by \(V(k, \epsilon, \Theta)\) the \((k, \epsilon, \Theta)\)-neighborhood of potential critical points at infinity, namely

\[
V(k, \epsilon, \Theta) := \{ u \in \mathcal{H}_{\alpha, \beta} : \exists a_1, \ldots, a_k \in \partial M, a_1, \ldots, a_k > 0, \quad \lambda_1, \ldots, \lambda_k > 0, \beta_1, \ldots, \beta_k \in \mathbb{R},
\]

\[
|u - \overline{\nu}_{(Q,T)} - \sum_{i=1}^{k} \alpha_i \phi_{\alpha_i, \lambda_i} - \sum_{r=1}^{\tilde{k}} \beta_r (v_r - (\overline{v}_r)_{(Q,T)})|_{\mathbb{P}^{4,3}} < \epsilon, \quad \sum_{i=1}^{k} \alpha_i = k, \quad \alpha_i \geq 1 - \nu,
\]

\[
|\lambda_i| \leq \frac{1}{\epsilon}, i = 1, \ldots, k, \quad \frac{2}{\Lambda} \leq \frac{\lambda_i}{\lambda_j} \leq \frac{2}{\Lambda}, i, j = 1, \ldots, k, \beta_r \leq \Theta, \quad r = 1, \ldots, \tilde{k},
\]

and

\[
|\lambda_id_j(a_i, a_j) \geq 4\bar{C}R \text{ for } i \neq j, \]

where \(\bar{C}\) is as in (39), the \(\phi_{\alpha_i, \lambda_i}\)'s are as in (43), \(\tilde{k}\) is as in (29), the \(v_r\)'s are defined as in (30), the \((\overline{v}_r)_{(Q,T)}\)'s are as in (27), and \(|| \cdot ||_{\mathbb{P}^{4,3}}\) is defined as in (33).

As observed by Chen-Lin [26] for Liouville type problems, the minimization at infinity of Bahri-Coron [13] for Yamabe type problems has the following analogue for our problem. For \(\Theta \geq 0\), there exists \(\epsilon_0 = \epsilon_0(\Theta)\) small and positive such that \(\forall 0 < \epsilon \leq \epsilon_0\), we have

\[
V(k, \epsilon, \Theta) \in V(k, \epsilon, \Theta)\), the minimization problem \(\min_{B^k_{\epsilon, \Theta}} ||u - \overline{\nu}_{(Q,T)} - \sum_{i=1}^{k} \alpha_i \phi_{\alpha_i, \lambda_i} - \sum_{r=1}^{\tilde{k}} \beta_r (v_r - (\overline{v}_r)_{(Q,T)})|_{\mathbb{P}^{4,3}}\)

has a unique solution, up to permutations, where \(B^k_{\epsilon, \Theta}\) is defined as follows

\[
B^k_{\epsilon, \Theta} := \{(\alpha, A, \lambda, \beta) \in \mathbb{R}^k_+ \times (\partial M)^k \times \mathbb{R}^k_+ \times \mathbb{R}^k_+ : \sum_{i=1}^{k} \alpha_i = k, \alpha_i \geq 1 - \nu, \lambda_i \geq \frac{1}{\epsilon}, i = 1, \ldots, k,
\]

\[
|\beta_r | \leq \Theta, r = 1, \ldots, \tilde{k}, \lambda_id_j(a_i, a_j) \geq 4\bar{C}R, \quad i \neq j, i, j = 1, \ldots, k, \}
\]
The selection map \( s_k \) is defined by \( s_k : V(k, \epsilon, \Theta) \rightarrow (\partial M)^k/\sigma_k \) as follows

\[
(23) \quad s_k(u) := A, \quad u \in V(k, \epsilon, \Theta), \quad \text{and } A \text{ is given by } (30).
\]

We denote the critical points at infinity of \( \mathcal{E}_g \) by \( z^\infty \) and use the notation \( M_{\infty}(z^\infty) \) for their Morse indices at infinity, \( W_u(z^\infty) \) for their unstable manifolds and \( W_s(z^\infty) \) for their stable manifolds, where \( z \) is the corresponding critical point of \( \mathcal{F}_K \). Furthermore, we denote by \( x^\infty \) the "true" ones, namely \( \mathcal{L}_K(x) < 0 \) and the \( y^\infty \) the "false" ones, namely \( \mathcal{L}_K(y) > 0 \). Moreover, we define \( S \) to be the following invariant set

\[
S := \bigcup_{M_{\infty}(z^\infty), M_{\infty}(z^\infty) \geq 4k-4+k} W_u(z^\infty) \cap W_s(z^\infty).
\]

We also define \( S^\infty \) to be the part of \( S \) at infinity, namely

\[
S^\infty := \bigcup_{M_{\infty}(z^\infty), M_{\infty}(z^\infty) \geq 4k-4+k} W_u(z^\infty) \cap W_s(z^\infty),
\]

where \( W_u(z^\infty) \) denotes the exit set from \( S^\infty \) starting from a false critical point at infinity \( y^\infty \).

Similarly, we denote by \( S^\infty_+ \) the entry set to \( S^\infty \) after having exited \( S^\infty \) through a set contained in \( S^\infty \) and entering into \( S^\infty \) through a true critical point at infinity \( x^\infty \). Finally to state the result of algebraic topological type resulting from the Liouville version of the Barycenter technique of Bahri-Coron developed in [46], we need the existence of

\[
0 \neq O_{\delta M}^g \in H^3(\partial M).
\]

Indeed, we prove:

**Theorem 1.8.** Let \((\overline{M}, g)\) be a compact 4-dimensional Riemannian manifold with boundary \( \partial M \) and interior \( M \) such that \( H_g = 0 \), \( \ker \mathbb{P}^{4,3}_g \simeq \mathbb{R} \), and \( \kappa_{(P^4, P_3)} = 4k\pi^2 \) with \( k \geq 2 \). Assuming that \( K \) is a smooth positive function on \( \partial M \) such that \( (ND) \) holds and either there is no \( x^\infty \) with \( M_{\infty}(x^\infty) = 4k-4+k \) or \( s_k^*(O_{\delta M}^g) \neq 0 \) in \( H^3(S^\infty) \) and \( s_k^*(O_{\delta M}^g) = 0 \) in \( H^3(S^\infty_+ \cup S^\infty) \), then \( K \) is the T-curvature of a Riemannian metric conformally related to \( g \) with zero \( Q \)-curvature in \( M \) and zero mean curvature on \( \partial M \).

As in [46], Theorem 1.8 implies the following corollary.

**Corollary 1.9.** Let \((\overline{M}, g)\) be a compact 4-dimensional Riemannian manifold with boundary \( \partial M \) and interior \( M \) such that \( \ker \mathbb{P}^{4,3}_g \simeq \mathbb{R} \) and \( \kappa_{(P^4, P_3)} = 4k\pi^2 \) with \( k \geq 2 \). Assuming that \( K \) is a smooth positive function on \( \partial M \) such that \( (ND) \) holds and that every critical point \( x \) of \( \mathcal{F}_K \) of Morse index 0 or 1 satisfies \( \mathcal{L}_K(x) < 0 \), then \( K \) is the T-curvature of a Riemannian metric conformally related to \( g \) with zero \( Q \)-curvature in \( M \) and zero mean curvature \( \partial M \).

## 2 Notation and preliminaries

In the following, for a Riemannian metric \( \bar{g} \) on \( \partial M \) and \( p \in \partial M \), we will use the notation \( B^g_p(r) \) to denote the geodesic ball with respect to \( \bar{g} \) of radius \( r \) and center \( p \). We also denote by \( d_{\bar{g}}(x,y) \) the geodesic distance with respect to \( \bar{g} \) between two points \( x \) and \( y \) of \( \partial M \), \( \exp_{\bar{g}} \) the exponential map with respect to \( \bar{g} \) at \( x \in \partial M \). \( \text{inj}_{\bar{g}}(\partial M) \) stands for the injectivity radius of \( (\partial M, \bar{g}) \). \( dV_{\bar{g}} \) denotes the Riemannian measure associated to the metric \( \bar{g} \). Furthermore, we recall that \( \nabla_{\bar{g}}, \Delta_{\bar{g}}, R_{\bar{g}} \) will denote respectively the covariant derivative, the Laplace-Beltrami operator, the scalar curvature and Ricci curvature with respect to \( \bar{g} \). For simplicity, we will use the notation \( B_{\bar{g}}(r) \) to denote \( B^g_p(r) \), namely \( B_{\bar{g}}(r) = B^g_p(r) \). \( (\partial M)^2 \) stands for the cartesian product \( \partial M \times \partial M \), while \( \text{Diag}(\partial M) \) is the diagonal of \( (\partial M)^2 \).
Similarly, for a Riemannian metric $\hat{g}$ on $\overline{M}$, we will use the notation $B_p^{\hat{g}}(r)$ to denote the half geodesic ball with respect to $\hat{g}$ of radius $r$ and center $p \in \partial M$. We also denote by $d_{\hat{g}}(x,y)$ the geodesic distance with respect to $\hat{g}$ between two points $x$ and $y$ of $\overline{M}$, $\exp_{\hat{g}}^\hat{g}$ the exponential map with respect to $\hat{g}$ at $x \in \partial M$. $\text{inj}_\hat{g}(\overline{M})$ stands for the injectivity radius of $(\overline{M}, \hat{g})$, $dV_\hat{g}$ denotes the Riemannian measure associated to the metric $\hat{g}$, and $dS_{\hat{g}}$ the Riemannian measure associated to $\tilde{g} := \hat{g}|_{\partial M}$, namely $dS_{\hat{g}} = dV_{\tilde{g}}$. Furthermore, we recall that $\nabla_{\hat{g}}$, $\Delta_{\hat{g}}$, $R_{\hat{g}}$ will denote respectively the covariant derivative, the Laplace-Beltrami operator, the scalar curvature and Ricci curvature with respect to $\hat{g}$. For simplicity, we will use the notation $B_p^g(r)$ to denote $B_p^{\hat{g}}(r)$, namely $B_p^g(r) = B_p^{\hat{g}}(r)$, $p \in \partial M$.

For $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, $\theta \in [0,1]$, $L^p(\partial M)$, $W^{k,p}(\partial M)$, $C^k(\partial M)$, and $C^{k,\theta}(\partial M)$ stand respectively for the standard Lebesgue space, Sobolev space, $k$-continuously differentiable space and $k$-continuously differential space of Hölder exponent $\theta$, all with respect $g$. Similarly, $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, $\theta \in [0,1]$, $L^p(\partial M)$, $W^{k,p}(\partial M)$, $C^k(\partial M)$, and $C^{k,\theta}(\partial M)$ stand respectively for the standard Lebesgue space, Sobolev space, $k$-continuously differentiable space and $k$-continuously differential space of Hölder exponent $\theta$, all with respect $\tilde{g}$.

Given a function $u \in L^1(M) \cap L^1(\partial M)$, $\bar{u}_{\partial M}$ and $\overline{\Pi}_{(Q,T)}$ are defined by

$$\bar{u}_{\partial M} = \frac{\int_{\partial M} u(x) dS_g}{\text{Vol}_g(\partial M)},$$

with $\text{Vol}_g(\partial M) = \int_{\partial M} dS_g$ and

$$\overline{\Pi}_{(Q,T)} = \frac{1}{4k\pi^2} \left( \int_M Q_g u dV_g + \int_{\partial M} T_g u dS_g \right).$$

Given a generic Riemannian metric $\hat{g}$ on $\partial M$ and a function $F(x,y)$ defined on an open subset of $(\partial M)^2$ which is symmetric and with $F(\cdot, \cdot) \in C^2$ with respect to $\hat{g}$, we define $\frac{\partial F(a,a)}{\partial a} := \frac{\partial F(x,a)}{\partial x}|_{x=a} = \frac{\partial F(x,a)}{\partial y}|_{y=a} = 0$, and $\Delta_{\hat{g}} F(a_1,a_2) := \Delta_{\hat{g}} \hat{g} F(x,a_2)|_{x=a_1} = \Delta_{\hat{g}} \hat{g} F(a_2,y)|_{y=a_1}$.

For $\epsilon > 0$ and small, $\lambda \in \mathbb{R}$, $\lambda \geq \frac{1}{\epsilon}$, and $\alpha \in \partial M$, $O_{\lambda,\epsilon}(1)$ stands for quantities bounded uniformly in $\lambda$, and $\epsilon$, and $O_{\alpha,\epsilon}(1)$ stands for quantities bounded uniformly in $a$ and $\epsilon$. For $l \in \mathbb{N}^*$, $O_l(1)$ stands for quantities bounded uniformly in $l$ and $O_l(1)$ stands for quantities which tends to 0 as $l \rightarrow +\infty$. For $\epsilon$ positive and small, $a \in \partial M$ and $\lambda \in \mathbb{R}_+$ large, $\lambda \geq \frac{1}{\epsilon}$, $O_{\alpha,\epsilon}(1)$ stands for quantities bounded uniformly in $a$, $\lambda$, and $\epsilon$. For $\epsilon$ positive and small, $p \in \mathbb{N}^*$, $\lambda := (\lambda_1, \ldots, \lambda_p) \in (\mathbb{R}_+)^p$, $\lambda_i \geq \frac{1}{\epsilon}$ for $i = 1, \ldots, p$, and $A := (a_1, \ldots, a_p) \in M^p$ (where $(\mathbb{R}_+)^p$ and $(\partial M)^p$ denotes respectively the cartesian product of $p$ copies of $\mathbb{R}_+$ and $\partial M$), $O_{\alpha,\lambda}(1)$ stands for quantities bounded uniformly in $A$, $\lambda$, and $\epsilon$. Similarly for $\epsilon$ positive and small, $p \in \mathbb{N}^*$, $\alpha := (\alpha_1, \ldots, \alpha_p) \in (\mathbb{R})^p$, $\lambda_i \geq \frac{1}{\epsilon}$ for $i = 1, \ldots, p$, $A := (a_1, \ldots, a_p) \in (\partial M)^p$ (where $(\mathbb{R})^p$ denotes the cartesian product of $p$ copies of $\mathbb{R}$, $O_{\alpha,A,\epsilon}(1)$ will mean quantities bounded from above and below independent of $\alpha$, $A$, $\lambda$, and $\epsilon$. For $x \in \mathbb{R}$, we will use the notation $O(x)$ to mean $|x|O(1)$ where $O(1)$ will be specified in all the contexts where it is used. Large positive constants are usually denoted by $\bar{c}$ and the value of $C$ is allowed to vary from formula to formula and also within the same line. Similarly small positive constants are also denoted by $c$ and their value may varies from formula to formula and also within the same line.

We say $\mu \in \mathbb{R}$ is an eigenvalue of the $P^4_g$ to $P^3_g$ operator on $\mathcal{H}_{\overline{g}}^{\mu}$ if there exists $0 \neq v \in W^{2,2}(M)$ such that

$$\begin{cases}
P^4_g v = 0 & \text{in } M, \\
P^3_g v = \mu v & \text{on } \partial M, \\
\frac{\partial v}{\partial n_{\hat{g}}} = 0 & \text{on } \partial M.
\end{cases}$$

By abuse of notation, we call $v$ in (28) an eigenfunction associated to $\mu$. We call $\bar{k}$ the number of negative eigenvalues (counted with multiplicity) of the $P^4_g$ to $P^3_g$ operator on $\mathcal{H}_{\overline{g}}^{\mu}$. We point out that $\bar{k}$ can be zero, but it is always finite. If $\bar{k} \geq 1$, then we will denote by $E_- \subset \mathcal{H}_{\overline{g}}^{\mu}$ the direct sum of
the eigenspaces corresponding to the negative eigenvalues of the \(P_g^4\) to \(P_g^3\) operator on \(H_{\frac{1}{2}}\). The dimension of \(E_-\) is of course \(\bar{k}\), i.e

\[
\bar{k} = \dim E_.
\]

(29)

On the other hand, we have the existence of a basis of eigenfunctions \(v_1, \ldots, v_{\bar{k}}\) of \(E_-\) satisfying

\[
\begin{align*}
P_g^4 v_r &= 0 & \text{in } M, \\
P_g^3 v_r &= \mu_r v_r & \text{on } \partial M, \\
\frac{\partial v_r}{\partial n_g} &= 0 & \text{on } \partial M.
\end{align*}
\]

(30)

where \(\mu_r\)'s are the eigenvalues of the operator \(P_g^4\) to \(P_g^3\) on \(H_{\frac{1}{2}}\) counted with multiplicity. We define \(P_{g,+}^{4,3}\) by

\[
P_{g,+}^{4,3}(u,v) = P_g^4(u,v) - 2 \sum_{r=1}^{\bar{k}} \mu_r \left( \oint_{\partial M} uu_r dS_g \right) \left( \oint_{\partial M} vv_r dS_g \right).
\]

(32)

\(P_{g,+}^{4,3}\) is obtained by just reversing the sign of the negative eigenvalue of \(P_{g,+}^{4,3}\). We set also

\[
\|u\|_{P_{g,+}^{4,3}} := \sqrt{P_{g,+}^{4,3}(u,u)}, \quad (u,v)_{P_{g,+}^{4,3}} = P_{g,+}^{4,3}(u,v),
\]

where \(P_{g,+}^{4,3}\) is defined as in (32). We can choose \(v_1, \ldots, v_{\bar{k}}\) so that they constitute a \((\cdot,\cdot)_{P_{g,+}^{4,3}}\)-orthonormal basis for \(E_-\). We denote by \(\nabla^{P_{g,+}^{4,3}}\) the gradient with respect to \((\cdot,\cdot)_{P_{g,+}^{4,3}}\).

For \(t > 0\), we define the following perturbed functional

\[
(\mathcal{E}_g)_t(u) := P_{g,+}^{4,3}(u,u) + 4t \int_M Q_g u dV_g + 4t \oint_{\partial M} T_g udS_g - \frac{4}{3} \chi_0(p_t^+, p_t^\pm) \log \oint_M Ke^{3u}dS_g, \quad u \in H_{\frac{1}{2}}^g.
\]

\(\bar{B}_r^k\) will stand for the closed ball of center 0 and radius \(r\) in \(\mathbb{R}^k\). \(S^{k-1}\) will denote the boundary of \(\bar{B}_r^k\). Given a set \(X\), we define \(X \times \bar{B}_r^k\) to be the cartesian product \(X \times \bar{B}_r^k\) where the tilde means that \(X \times \partial B_r^k\) is identified with \(\partial B_r^k\).

In the sequel also, \((\mathcal{E}_g)^c\) with \(c \in \mathbb{R}\) will stand for \((\mathcal{E}_g)^c := \{u \in H_{\frac{1}{2}}^g : \mathcal{E}_g(u) \leq c\}\). For \(X\) a topological space, \(H_*(X)\) will denote the singular homology of \(X\), \(H^*(X)\) for the cohomology, and \(\chi(X)\) the Euler characteristic of \(X\), with all \(\mathbb{Z}_2\) coefficients.

As above, in the general case, namely \(\bar{k} \geq 0\), for \(\epsilon\) small and positive, \(\bar{\beta} := (\beta_1, \ldots, \beta_{\bar{k}}) \in \mathbb{R}^\bar{k}\) with \(\beta_i\) close to 0, \(i = 1, \ldots, \bar{k}\) (where \(\mathbb{R}^k\) is the empty set when \(\bar{k} = 0\)), \(\bar{\lambda} := (\lambda_1, \ldots, \lambda_{\bar{k}}) \in (\mathbb{R}^+)^\bar{k}\), \(\lambda_i \geq \frac{1}{4}\) for \(i = 1, \ldots, p\), \(\bar{\alpha} := (\alpha_1, \ldots, \alpha_{\bar{p}}) \in \mathbb{R}^\bar{p}\), \(\alpha_i\) close to 1 for \(i = 1, \ldots, p\), and \(A := (\alpha_1, \ldots, \alpha_{\bar{p}}) \in (\partial M)^\bar{p}\), \(p \in \mathbb{N}^*, w \in H_{\frac{1}{2}}^g\) with \(\|w\|_{P_{g,+}^{4,3}}\) small, \(O_{A,\bar{\alpha},\bar{\lambda},\bar{\beta},\epsilon}(1)\) will stand quantities bounded independent of \(\bar{\alpha}, \bar{\lambda}, \bar{\beta}\) and \(\epsilon\), and \(O_{A,\bar{\alpha},\bar{\lambda},\bar{\beta},w,\epsilon}(1)\) will stand quantities bounded independent of \(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}, w\) and \(\epsilon\).

For point \(b \in \mathbb{R}^3\) and \(\lambda\) a positive real number, we define \(\delta_{b,\lambda}\) by

\[
\delta_{b,\lambda}(y) := \log \left( \frac{2\lambda}{1 + \sqrt{\lambda^2 + |y-b|^2}} \right), \quad y \in \mathbb{R}^3.
\]

(35)

The functions \(\delta_{b,\lambda}\) verify the following equation

\[
(\Delta_{\mathbb{R}^3})^\frac{3}{2} \delta_{b,\lambda} = 2e^{3\delta_{b,\lambda}} \quad \text{in } \mathbb{R}^3.
\]

(36)

Using the existence of conformal Fermi coordinates, we have that, for \(a \in \partial M\) there exists a function \(u_a \in C^\infty(M)\) such that

\[
g_a = e^{2u_a} g \quad \text{verifies} \quad \det g_a(x) = 1 + O(d_{g_a}(x,a)^m) \quad \text{for} \quad x \in B_{a}^{g_a,+}(g_a).
\]

(37)
with $0 < \varrho_a < \frac{\min_a(M)}{10}$. Moreover, we can take the families of functions $u_a, g_a$ and $\varrho_a$ such that
\begin{equation}
\text{the maps } a \to u_a, g_a \text{ are } C^1 \text{ and } \varrho_a \geq \varrho_0 > 0,
\end{equation}
for some small positive $\varrho_0$ satisfying $\varrho_0 < \inf\{\frac{\min_a(M)}{10}, \frac{\min_o(DM)}{10}\}$, and
\begin{equation}
\|u_a\|_{C^0(M)} = O_a(1), \quad \frac{1}{C^2}g \leq g_a \leq C^{2}g,
\end{equation}
\begin{equation}
 u_a(x) = O_a(d^2 g_a(a, x)) = O_a(d^2 g(a, x)) \text{ for } x \in B^{\frac{3}{\varrho}}(\varrho_0) \cup B_n(\frac{\varrho_0}{C}), \text{ and}
\end{equation}
\begin{equation}
 u_a(a) = 0, \quad R_{g_a}(a) = 0, \quad \frac{\partial u_a}{\partial n}(a) = 0,
\end{equation}
for some large positive constant $C$ independent of $a$. For $a \in \partial M$, and $r > 0$, we set
\begin{equation}
 exp_a := exp_{\hat{\varrho}_a} \quad \text{and} \quad B_a^\varrho(r) := B^\varrho_{\hat{\varrho}}(r).
\end{equation}
Now, for $0 < \varrho < \min\{\frac{\min_a(\partial M)}{4}, \varrho_0\}$ where $\varrho_0$ is as in [38], we define a smooth cut-off function satisfying the following properties:
\begin{equation}
\left\{
\begin{aligned}
\chi_\varrho(t) &= t \quad \text{for } t \in [0, \varrho], \\
\chi_\varrho(t) &= 2\varrho \quad \text{for } t \geq 2\varrho, \\
\chi_\varrho(t) &= \varrho \quad \text{for } t \in [\varrho, 2\varrho].
\end{aligned}
\right.
\end{equation}
Using the cut-off function $\chi_\varrho$, we define for $a \in \partial M$ and $\lambda \in \mathbb{R}_+$ the function $\hat{\delta}_{a,\lambda}$ as follows
\begin{equation}
\hat{\delta}_{a,\lambda}(x) := \log \left( \frac{2\lambda}{1 + \lambda^2 \chi^2(\varrho_0 (x, a))} \right).
\end{equation}
For every $a \in \partial M$ and $\lambda \in \mathbb{R}_+$, we define $\varphi_{a,\lambda}$ to be the unique the solution of
\begin{equation}
\left\{
\begin{aligned}
P_g^a \varphi_{a,\lambda} + \frac{2}{\varrho} Q_g &= 0 \quad \text{in } M, \\
P_g^a \varphi_{a,\lambda} + \frac{1}{\varrho} T_g &= 4\pi^2 \frac{e^{\chi(\varrho_0 + \varrho_0)(x, a)}}{e^{\chi(\varrho_0 + \varrho_0)(x, a)}} dS_g \quad \text{in } \partial M, \\
\frac{\partial \varphi_{a,\lambda}}{\partial n}(a) &= 0, \\
(\varphi_{a,\lambda})(Q, T) &= 0.
\end{aligned}
\right.
\end{equation}
Next, let $S(a, x); (a, x) \in \partial M \times \overline{M}$ be defined by
\begin{equation}
\left\{
\begin{aligned}
P_g^a S(a, \cdot) + \frac{2}{\varrho} Q_g(\cdot) &= 0 \quad \text{in } M, \\
P_g^a S(a, \cdot) + \frac{1}{\varrho} T_g(\cdot) &= 4\pi^2 \delta_{a}(\cdot) \quad \text{on } \partial M, \\
\frac{\partial S(a, \cdot)}{\partial n}(a) &= 0 \quad \text{on } \partial M, \\
\int_M S(a, x)Q_g(x)dV_g(x) &= 0.
\end{aligned}
\right.
\end{equation}
Then
\begin{equation}
G(a, \cdot) = S(a, \cdot)|_{\partial M}.
\end{equation}
is a Green’s function of the $P_g^a + \frac{2}{\varrho} Q_g(\cdot)$ to $P_g^a + \frac{1}{\varrho} T_g(\cdot)$ operator on $\mathcal{H}_{\varrho_a}$. Thus, we have the integral representation: $\forall u \in \mathcal{H}_{\varrho_a}$ such that $P_g^a u + \frac{2}{\varrho} Q_g = 0$,
\begin{equation}
 u(x) - \overline{u}(Q, x) = \frac{1}{4\pi^2} \int_{\partial M} G(x, y)P_g^a u(y), \quad x \in \partial M.
\end{equation}
Moreover, $G$ decomposes as follows
\begin{equation}
 G(a, x) = \log \left( \frac{1}{\chi^2(\varrho_0 (a, x))} \right) + H(a, x),
\end{equation}
where $H$ is the regular part of $G$. Furthermore, we have

$$G \in C^\infty((\partial M)^2 - \text{Diag}(\partial M)), \quad \text{and } H \in C^{\alpha,\beta}((\partial M)^2) \quad \forall \beta \in (0,1).$$

B symmetry of $H$, we have

$$\frac{\partial F(a_1,\cdots,a_k)}{\partial a_i} = \frac{2}{3} \frac{\nabla_{\partial F} F^A(a_i)}{F^A(a_i)}, \quad i = 1,\cdots,k.$$  

Next, setting

$$l_K(A) := \sum_{i=1}^k \left( \frac{\Delta_{\partial F} F^A(a_i)}{F^A(a_i)} - \frac{3}{4} R_g(a_i)(F^A(a_i))^2 \right),$$

we have

$$l_K(A) = 6L_K(A), \quad \forall A \in \text{Crit}(F_K).$$

For $k \geq 2$, we denote by $B_k(\partial M)$ the set of formal barycenters of $\partial M$ of order $k$, namely

$$B_k(\partial M) := \{ \sum_{i=1}^k \alpha_i \delta_{a_i}, a_i \in \partial M, \alpha_i \geq 0, i = 1,\cdots,k, \sum_{i=1}^k \alpha_i = k \}.$$  

Finally, we set

$$A_{k,k} := \widetilde{B_k(\partial M)} \times \bar{B}_1,$$

and

$$A_{k-1,k} := \widetilde{B_{k-1}(\partial M)} \times \bar{B}_1.$$  

### 3 Blow-up analysis and critical points at infinity

This section deals with the blowup analysis of sequences of vanishing viscosity solutions of the type

$$\begin{cases}
P_g^4 u_t + 2tQ_g = 0 & \text{in } M, \\
P_g^3 u_t + tT_g = t^2 K e^{3u} & \text{on } \partial M, \\
\frac{\partial u_t}{\partial n_g} = 0 & \text{on } \partial M.
\end{cases}$$

with $t_l \to 1$ under the assumption $\ker P_{g,4}^4 \simeq \mathbb{R}$ and $\kappa_{(P^4,P^3)} = 4k\pi^2$ with $k \geq 1$ and their use to characterize the critical points at infinity of $E_g$.

#### 3.1 Blow-up analysis

The local behaviour of blowing up sequences of solutions of (55) is understood. In fact, in [44], we prove the following lemma.

**Lemma 3.1.** Assuming that $(u_l)$ is a blowing up sequence of solutions to (55), then up to a subsequence, there exists $k$ converging sequence of points $(x_l,i) \in \mathbb{N}, x_l,i \in \partial M$ with limits $x_i \in \partial M, i = 1,\cdots,k$, $k$ sequences $(\mu_{i,l})_{l \in \mathbb{N}} i = 1,\cdots,k$ of positive real numbers converging to 0 such that the following hold:

a) $$\frac{d_g(x_{l,i},x_{l,j})}{\mu_{i,l}} \to +\infty \quad i \neq j \quad i,j = 1,\cdots,k$$ and $$t_l K(x_{l,i}) \mu_{i,l}^3 e^{3u_l(x_{l,i})} e^{-3\log 2} = 2.$$  

b) $$v_{i,l}(x) = u_l(\exp_{x_{l,i}}^{\mu_{i,l}}(\mu_{i,l}x)) - u_l(x_{l,i}) + \log 2 \to V_0(x) \quad \text{in } C^4_{\text{loc}}(\mathbb{R}_+^4).$$  

There exists $C > 0$ such that
\[
\inf_{i=1,\ldots,k} d_g(x_{i,l},x)^3 e^{3u_l(x)} \leq C \quad \forall x \in \partial M, \quad \forall l \in \mathbb{N}.
\]

Moreover, the scaling parameters $\lambda_{i,l} := \mu_{i,l}^{-1}$ are comparable, namely there exists a large positive constant $\Lambda_0$ such that
\[
\Lambda_0^{-1} \lambda_{j,l} \leq \lambda_{i,l} \leq \Lambda_0 \lambda_{j,l}, \quad \forall i,j
\]
Furthermore, we have that the following estimate around the blow up points holds
\[
u_l(y) + \frac{1}{3} \log \frac{t_lK_l(x_{i,l})}{2} = \log \frac{2\lambda_{i,l}}{1 + \lambda_{i,l}^2 (d_g(x_{i,l},y))^2} + O(d_g(y, x_{i,l})), \quad \forall y \in B_{x_{i,l}}(\eta).
\]

To prove Proposition 3.2 as it is standard for Liouville type problems, one starts with the uniform isolation of blowing-up points. Indeed, we have

**Lemma 3.3.** Assuming that $(w)_{i \in \mathbb{N}}$ is a bubbling sequence of solutions to BVP (55), then keeping the notations in Lemma 3.1, we have that the points $x_{i,l}$ are uniformly isolated, namely there exists $0 < \eta_k < \frac{\eta_0}{10}$ (where $\eta_0$ is as in (38)) such that for $l$ large enough, there holds
\[
d_g(x_{i,l}, x_{j,l}) \geq 4C\eta_k, \quad \forall i \neq j = 1, \ldots, k.
\]

**Proof.** The proof use the integral method of Step 4 in [42] and hence we will be sketchy in many arguments. As in [42], we first fix $\frac{1}{3} < \nu < \frac{3}{4}$, and for $i = 1, \ldots, k$, we set
\[
\bar{u}_{i,l}(r) = Vol_\delta(\partial B_{x_i}(r))^{-1} \int_{\partial B_{x_i}(r)} u_l(x)d\sigma(x), \quad \forall 0 \leq r < inj_\delta(\partial M),
\]
and
\[
\psi_{i,l}(r) = r^{4\nu} \exp(4\bar{u}_{i,l}(r)), \quad \forall 0 \leq r < inj_\delta(\partial M).
\]
Furthermore, as in [42], we define $r_{i,l}$ as follows
\[
r_{i,l} := \sup\{R_{\nu}\mu_{i,l} \leq r \leq \frac{R_{i,l}}{2} \text{ such that } \psi_{i,l}(r) < 0 \text{ in } [R_{\nu}\mu_{i,l}, r]\};
\]
where \( R_{i,l} := \min_{j \neq i} d_g(x_{i,l}, x_{j,l}) \). Thus, by continuity and the definition of \( r_{i,l} \), we have that

\[
\psi'_{i,l}(r_{i,l}) = 0
\]

Now, as in \cite{42}, to prove (59), it suffices to show that \( r_{i,l} \) is bounded below by a positive constant in dependent of \( l \). Thus, we assume by contradiction that (up to a subsequence) \( r_{i,l} \to 0 \) as \( l \to +\infty \) and look for a contradiction. In order to do that, we use the integral representation formula (46) and argue as in Step 4 of \cite{42} to derive the following estimate

\[
\psi'_{i,l}(r_{i,l}) \leq (r_{i,l})^{3\nu-1} \exp(\tilde{u}_{i,l}(r_{i,l})) \left( 3\nu - 2C + o_l(1) + O_l(r_{i,l}) \right).
\]

with \( C > 1 \). So from \( \frac{1}{2} < \nu < \frac{2}{3} \), \( C > 1 \) and \( r_{i,l} \to 0 \) as \( l \to +\infty \), we deduce that for \( l \) large enough, there holds

\[
\psi'_{i,l}(r_{i,l}) < 0.
\]

Thus, (61) and (62) lead to a contradiction, thereby concluding the proof of (59). Hence, the proof of the Lemma is complete.

The next step to derive Proposition \ref{prop:3.2} is to establish its weak \( O(1) \) -version.

**Lemma 3.4.** Assuming that \((u_l)_{l \in \mathbb{N}} \) is a bubbling sequence of solutions to BVP (55), then keeping the notations in Lemma \ref{lemma:3.1} and Lemma \ref{lemma:3.3}, we have that for \( l \) large enough, there holds

\[
u(x) + \frac{1}{3} \log \frac{t_l K(x_i)}{2} = \log \frac{2\lambda_{i,l}}{1 + \nu^2_l(d_{g,s_l}(x, x_i))^2} + O(1), \quad \forall x \in B^x_{2r_l}(\eta_k),
\]

up to choosing \( \eta_k \) smaller than in Lemma \ref{lemma:3.3}.

**Remark 3.5.** We point out that the comparability of the scaling parameters \( \lambda_{i,l} \)'s follows directly from Lemma \ref{lemma:3.4}.

**Proof.** We are going to use the method of \cite{47}, hence we will be sketchy in many arguments. Like in \cite{47}, thanks to Lemma \ref{lemma:3.3}, we will focus only on one blow-up point and called it \( x \in \partial M \). Thus, we are in the situation where there exists a sequence \( x_l \in \partial M \) such that \( x_l \to x \) with \( x_l \) local maximum point for \( u_l \) on \( \partial M \) and \( u_l(x_l) \to +\infty \). Now, we recall \( g_x = e^{2u_x}g \) and choose \( \eta_l \) such that \( 20\eta_l < \min \{ \text{inj}_g(\partial M), q_0, \eta_k, d \} \) with \( 4d \leq r_{i,l} \) where \( r_{i,l} \) is as in the proof of Lemma \ref{lemma:3.3}. Next, we let \( \tilde{w}_x \) be the unique solution of the following boundary value problem

\[
\begin{cases}
P_{g_x}^4 \tilde{w}_x = P_{g_x}^4 u_x & \text{in } M, \\
P_{g_x}^3 \tilde{w}_x = P_{g_x}^3 u_x & \text{on } \partial M, \\
\tilde{w}_x |_{\partial M} = 0 & \text{on } \partial M, \\
\tilde{w}_x |_{\bar{B}_{2r_l}^{2r_l}(\eta_l)} = 0.
\end{cases}
\]

Using standard elliptic regularity theory and \cite{69}, we derive

\[
\tilde{w}(y) = O(d_g(y, x)) \quad \text{in } B_{2r_x}(2\eta_l).
\]

On the other hand, using the conformal covariance properties of the Paneitz operator and of the Chang-Qing one, see \cite{1}, we have that \( \hat{u}_l := u_l - \tilde{w}_x \) satisfies

\[
\begin{cases}
P_{g_x}^4 \hat{u}_l + 2 \hat{T}_l = 0 & \text{in } M, \\
P_{g_x}^3 \hat{u}_l + \hat{T}_l = t_l K e^{3\hat{u}_l} & \text{on } \partial M, \\
\hat{u}_l |_{\partial M} = 0 & \text{on } \partial M.
\end{cases}
\]

with

\[
\hat{Q}_l = t_l e^{-4\hat{w}} Q_g + \frac{1}{2} P_{g_x}^4 \hat{w} \quad \text{and} \quad \hat{T}_l = t_l e^{-3\hat{w}} T_g + P_{g_x}^3 \hat{w}.
\]
Next, as in [47], we are going to establish the classical sup+inf-estimate for $\hat{u}_l$, since thanks to [63], all terms coming from $\hat{u}_x$ can be absorbed on the right hand side of [63]. Now, we are going to rescale the functions $\hat{u}_l$ around the points $x$. In order to do that, we define $\varphi_l : B^2_0(\eta_l \mu_l^{-1}) \to B^2_x(\eta_l)$ by the formula $\varphi_l(z) := \mu_l z$ and $\mu_l$ is the corresponding scaling parameter given by Lemma 3.4. Furthermore, as in [47], we define the following rescaling of $\hat{u}_l$

$$v_l := \hat{u}_l \circ \varphi_l + \log \mu_l + \frac{1}{3} \log \frac{t_l K(x)}{2}.$$

Using the Green’s representation formula for and the method of the method of [47], we get

$$v_l(z) + 2 \log |z| = O(1), \text{ for } z \in B^3_0(\eta_l \mu_l^{-1}) - B^3_0(-\log \mu_l).$$

Now, we are going to show that the estimate (66) holds also in $\overline{B^3_0}(-\log \mu_l)$. To do so, we use Lemma 3.1 and the same arguments as in [47] to deduce

$$v_l(z) + 2 \log |z| = O(1), \text{ for } z \in \overline{B^3_0}(-\log \mu_l).$$

Now, combining (66) and (67), we obtain

$$v_l(z) + 2 \log |z| = O(1), \text{ for } z \in \overline{B^3_0}(-\log \mu_l).$$

Thus scaling back, namely using $y = \mu_l z$ and the definition of $v_l$, we obtain the desired $O(1)$-estimate. Hence the proof of the Lemma is complete.

**Proof of formula (58) of Proposition 3.2**

We are going to use the method of [47], hence we will be sketchy in many arguments. Now, let $V_0$ be the unique solution of the following conformally invariant integral equation

$$V_0(z) = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \log \frac{|y|}{|z-y|} e^{3V_0(y)} dy + \log 2, \quad V_0(0) = \log 2, \nabla V_0(0) = 0.$$

Next, we set $w_l(z) = v_l(z) - V_0(z)$ for $z \in B^3_0(\eta_l \mu_l^{-1})$, and use Lemma 3.4 to infer that

$$|w_l| \leq C \text{ in } B^3_0(\eta_l \mu_l^{-1}).$$

On the other hand, it is easy to see that to achieve our goal, it is sufficient to show

$$|w_l| \leq C \mu_l |z| \text{ in } B^3_0(\eta_l \mu_l^{-1}).$$

To show (70), we first set

$$\Lambda_l := \max_{z \in \Omega_l} \frac{|w_l(z)|}{\mu_l(1+|z|)}$$

with

$$\Omega_l = \overline{B^3_0}(\eta_l \mu_l^{-1}).$$

We remark that to show (70), it is equivalent to prove that $\Lambda_l$ is bounded. Now, let us suppose that $\Lambda_l \to +\infty$ as $l \to +\infty$, and look for a contradiction. To do so, we will use the method of [47]. For this, we first choose a sequence of points $z_l \in \Omega_l$ such that $\Lambda_l = \frac{|w_l(z_l)|}{\mu_l(1+|z_l|)}$. Next, up to a subsequence, we have that either $z_l \to z^*$ as $l \to +\infty$ (with $z^* \in \mathbb{R}^3$) or $|z_l| \to +\infty$ as $l \to +\infty$. Now, we make the following definition

$$\hat{w}_l(z) := \frac{w_l(z)}{\Lambda_l \mu_l(1+|z|)},$$

and have

$$|\hat{w}_l(z)| \leq \frac{1}{1+|z|} \left( \frac{1+|z|}{1+|z|} \right),$$

and

$$|\hat{w}_l(z_l)| = 1.$$
Now, we consider the case where the points \( z_l \) escape to infinity.

**Case 1**: \(|z_l| \to +\infty\)

In this case, using the integral representation \([46]\) with respect to \( g_x \) and the method of \([47]\), we obtain

\[
\bar{w}_l(z_l) = \frac{1}{2\pi^2} \int_{\Omega_l} \log \frac{|\xi|}{|z_l - \xi|} \left( \frac{O(1)(1 + |\xi|)^{-5}}{(1 + |\xi|)} + \frac{O(1)(1 + |\xi|)^{-5}}{\Lambda_l(1 + |\xi|)} \right) d\xi + o(1).
\]

Now, using the fact that \(|z_l| \to +\infty\) as \( l \to +\infty\), one can easily check that

\[
\bar{w}_l(z_l) = \frac{1}{2\pi^2} \int_{\Omega_l} \log \frac{|\xi|}{|z_l - \xi|} \left( \frac{O(1)(1 + |\xi|)^{-5}}{(1 + |\xi|)} + \frac{O(1)(1 + |\xi|)^{-5}}{\Lambda_l(1 + |\xi|)} \right) d\xi = o(1).
\]

Hence, we reach a contradiction to \((72)\).

Now, we are going to show that, when the points \( z_l \to z^* \) as \( l \to +\infty\), we reach a contradiction as well.

**Case 2**: \( z_l \to z^* \)

In this case, using the assumption \( z_l \to z^* \), the Green’s representation formula, and the method of \([47]\), we obtain that up to a subsequence

\[
\bar{w}_l \to w \text{ in } C^1_{loc}(\mathbb{R}^3) \text{ as } l \to +\infty,
\]

and

\[
\bar{w}_l(z) = \frac{1}{2\pi^2} \int_{\Omega_l} \log \frac{|\xi|}{|z - \xi|} \left( \frac{K \circ \varphi_l(\xi)}{K \circ \varphi_l(0)} e^{3\varphi_l(\xi)} \bar{w}(\xi) d\xi + \frac{1}{\Lambda_l \mu(1 + |\xi|)} \int_{\Omega_l} \log \frac{|\xi|}{|z - \xi|} O(\mu(1 + |\xi|)^{-5}) d\xi \right.
\]

\[
\left. + \frac{O(1)}{\Lambda_l(1 + |\xi|)} \right)
\]

where \( e^{3\varphi_l} := \int_0^1 e^{3(su_l + (1-s)v_0)} ds \). Thus, appealing to \((73)\) and \((74)\), we infer that \( w \) satisfies

\[
w(z) = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \log \frac{|\xi|}{|z - \xi|} e^{3\varphi_l(\xi)} w(\xi) d\xi
\]

Now, using \((71)\), we have that \( w \) satisfies the following asymptotics

\[
|w(z)| \leq C(1 + |z|).
\]

On the other hand, from the definition of \( v_l \), it is easy to see that

\[
w(0) = 0, \quad \text{and} \quad \nabla w(0) = 0.
\]

So, using \((75)-\(77)\), and observing that Lemma 3.7 in \([47]\) holds for dimension 3, we obtain

\[w = 0.\]

However, from \((72)\), we infer that \( w \) satisfies also

\[
|w(z^*)| = 1
\]

So we reach a contradiction in the second case also. Hence the proof of the lemma is complete.  

Because of the lack of understanding of the blowing PS-sequences for Louiville type problems, the role of the PS-sequences can be replaced by the vanishing viscosity solutions of the type of \([55]\) via the following Bahri-Lucia’s deformation lemma.

**Lemma 3.6.** Assuming that \( a, b \in \mathbb{R} \) such that \( a < b \) and there is no critical values of \( E_g \) in \([a, b]\), then there are two possibilities

1) Either

\[ (E_g)^a \text{ is a deformation retract of } (E_g)^b. \]

2) Or there exists a sequence \( t_l \to 1 \) as \( l \to +\infty \) and a sequence of critical point \( u_l \) of \( (E_g)_{t_l} \) verifying \( a \leq E_g(u_l) \leq b \) for all \( l \in \mathbb{N}^* \), where \( (E_g)_{t_l} \) is as in \([53]\) with \( t \) replaced by \( t_l \).
On the other hand, setting

\[ V_R(k, \epsilon, \eta) := \{ u \in H^1_\# : \exists a_1, \ldots, a_k \in \partial M, \ \lambda_1, \ldots, \lambda_k > 0, \ |u - \bar{U}_{Q,T} - \sum_{i=1}^{k} u_{a_i, \lambda_i}|_{p^{1,3}} = \] (79)

\[ O \left( \sqrt{\sum_{i=1}^{k} \frac{1}{\lambda_i}} \right) \]  \[ \lambda_i \geq \frac{1}{\epsilon}, \frac{2}{\Lambda} \leq \frac{\lambda_i}{\lambda_j} \leq \frac{\Lambda}{2}, \] and \[ d_g(a_i, a_j) \geq 4C\eta \] for \( i \neq j \},

where \( C \) is as in (39), \( L \) as in (20), \( O(1) := O_{A, \lambda, u, \epsilon}(1) \) meaning bounded uniformly in \( \bar{\lambda} := (\lambda_1, \ldots, \lambda_k), A := (a_1, \ldots, a_k), u, \epsilon \), we have as in (47) that Proposition 3.2 implies the following one.

**Lemma 3.7.** Let \( \epsilon \) and \( \eta \) be small positive real numbers with \( 0 < 2\eta < \varrho \) where \( \varrho \) is as in (11). Assuming that \( u_l \) is a sequence of blowing up critical point of \( (E_g)_l \) with \( (u_l)_{Q,T} = 0, l \in \mathbb{N} \) and \( t_l \rightarrow 1 \) as \( l \rightarrow +\infty \), then there exists \( l_{\epsilon, \eta} \) a large positive integer such that for every \( l \geq l_{\epsilon, \eta} \), we have \( u_l \in V_R(k, \epsilon, \eta) \), and for the definition of \( V_R(k, \epsilon, \eta) \), see (79).

Finally, as in (47), we have that Lemma 3.6 and Lemma 3.7 implies the following one.

**Lemma 3.8.** Assuming that \( \epsilon \) and \( \eta \) are small positive real numbers with \( 0 < 2\eta < \varrho \), then for \( a, b \in \mathbb{R} \) such that \( a < b \), we have that if there is no critical values of \( E_g \) in \( [a, b] \), then there are two possibilities

1) Either

\[ (E_g)^a \] is a deformation retract of \( (E_g)^b \).

2) Or there exists a sequence \( t_l \rightarrow 1 \) as \( l \rightarrow +\infty \) and a sequence of critical point \( u_l \) of \( (E_g)_l \) (for its definition see (31)) verifying \( a \leq E_g(u_l) \leq b \) for all \( l \in \mathbb{N}^* \) and \( l_{\epsilon, \eta} \) a large positive integer such that \( u_l \in V_R(k, \epsilon, \eta) \) for all \( l \geq l_{\epsilon, \eta} \), and for the definition of \( V_R(k, \epsilon, \eta) \), see (79).

### 3.2 Energy and gradient estimates at infinity

In this subsection, we present energy and gradient estimates needed to characterize the critical points at infinity of \( E_g \). We start with a parametrization of infinity. Indeed, as a Liouville type problem, we have that for \( \eta \) a small positive real number with \( 0 < 2\eta < \varrho \), there exists \( \epsilon_0 = \epsilon_0(\eta) > 0 \) such that \( \forall 0 < \epsilon \leq \epsilon_0 \), we have

\[ (80) \quad \forall u \in V_R(k, \epsilon, \eta), \text{ the minimization problem } \min_{B_{\epsilon, \eta}} \| u - \bar{U}_{Q,T} - \sum_{i=1}^{k} \alpha_i u_{a_i, \lambda_i} - \sum_{r=1}^{k} \beta_r (v_r - (v_r)_{Q,T}) \|_{p^{1,3}} \]

has a unique solution, up to permutations, where \( B_{\epsilon, \eta} \) is defined as follows

\[ (81) \quad B_{\epsilon, \eta} := \{ (\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in \mathbb{R}^k \times (\partial M)^k \times (0, +\infty)^k \times \mathbb{R}^k : |\alpha_i - 1| \leq \epsilon, \lambda_i \geq \frac{1}{\epsilon}, i = 1, \ldots, k, \]

\[ \geq \frac{1}{\epsilon}, i = 1, \ldots, k, \]

\[ d_g(a_i, a_j) \geq 4\epsilon\eta, i \neq j, \beta_r \leq R, r = 1, \ldots, k \} \]

Moreover, using the solution of (80), we have that every \( u \in V_R(k, \epsilon, \eta) \) can be written as

\[ (82) \quad u - \bar{U}_{Q,T} = \sum_{i=1}^{k} \alpha_i u_{a_i, \lambda_i} + \sum_{r=1}^{k} \beta_r (v_r - (v_r)_{Q,T}) + w, \]

where \( w \) verifies the following orthogonality conditions

\[ (83) \quad \bar{U}_{Q,T} = \langle \varphi_{a_1, \lambda_1}, w \rangle_{p^{1,3}} = \langle \varphi_{a_2, \lambda_2}, w \rangle_{p^{1,3}} = \langle \varphi_{a_k, \lambda_k}, w \rangle_{p^{1,3}} = \langle v_r, w \rangle_{p^{1,3}} = 0, i = 1, \ldots, k, \]

\[ r = 1, \ldots, k \]

and the estimate

\[ (84) \quad \| w \|_{p^{1,3}} = O \left( \sum_{i=1}^{k} \frac{1}{\lambda_i} \right), \]
where here $O(1) := O_{\tilde{a}, A, \bar{\lambda}, \bar{\beta}, \epsilon, \eta}(1)$. Furthermore, the concentration points $a_i$, the masses $\alpha_i$, the concentrating parameters $\lambda_i$ and the negativity parameter $\beta_r$ in (82) verify also
\[
d_\beta(a_i, a_j) \geq 4C\eta, \ i \neq j = 1, \cdots, k, \ \frac{1}{\Lambda} \leq \frac{\lambda_i}{\lambda_j} \leq \Lambda, \ i, j = 1, \cdots, k, \ \lambda_i \geq \frac{1}{\epsilon}, \ \text{and}
\]
(85)
\[
\sum_{r=1}^{k} |\beta_r| + \sum_{i=1}^{k} |\alpha_i - 1|\sqrt{\log \lambda_i} = O \left( \sum_{i=1}^{k} \frac{1}{\lambda_i} \right)
\]
with still $O(1)$ as in (83).

Because of the translation invariant property of $\mathcal{E}_g$ and the parametrization (82), to derive energy estimate in $V_R(k, \epsilon, \eta)$ we start with the following lemma.

Lemma 3.9. Assuming that $\eta$ is a small positive real number with $0 < 2\eta < \varrho$ where $\varrho$ is as in (11), and $0 < \epsilon \leq \epsilon_0$ where $\epsilon_0$ is as in (80), then for $a_i \in M$ concentration points, $\alpha_i$ masses, $\lambda_i$ concentration parameters ($i = 1, \cdots, k$), and $\beta_r$ negativity parameters ($r = 1, \cdots, k$) satisfying (85), we have
\[
\mathcal{E}_g \left( \sum_{i=1}^{k} \alpha_i \phi_{a_i, \lambda_i} + \sum_{r=1}^{k} \beta_r (v_r - (\bar{v}_r)_{Q,T}) \right) = C_0^k - 8\pi^2 F_K(a_1, \cdots, a_k) + 2 \sum_{r=1}^{k} \mu_r \beta_r^2
\]
\[
+ \sum_{i=1}^{k} (\alpha_i - 1)^2 \left[ 16\pi^2 \log \lambda_i + 8\pi^2 H(a_i, a_i) + C_1^k \right]
\]
\[
+ 8\pi^2 \sum_{i=1}^{k} (\alpha_i - 1) \left[ \sum_{r=1}^{k} 2\beta_r (v_r - (\bar{v}_r)_{Q,T})(a_i) + \sum_{j=1, j \neq i}^{k} (\alpha_j - 1) G(a_i, a_j) \right]
\]
\[
- \frac{c^1 8\pi^2}{9} \sum_{i=1}^{k} \frac{1}{\lambda_i} \left( \frac{\Delta_{a_i} F_i^A(a_i)}{F_i^A(a_i)} - \frac{3}{4} R_{\bar{y}}(a_i) \right)
\]
\[
+ \frac{c^1 8\pi^2}{9} \sum_{i=1}^{k} \frac{\tilde{\gamma}_i}{\lambda_i} \left( \frac{\Delta_{a_i} F_i^A(a_i)}{F_i^A(a_i)} - \frac{3}{4} R_{\bar{y}}(a_i) \right)
\]
\[
+ \frac{16\pi^2}{3} \sum_{i=1}^{k} \log(1 - \tilde{\gamma}_i) + O \left( \sum_{i=1}^{k} |\alpha_i - 1|^3 + \sum_{r=1}^{k} |\beta_r|^3 + \sum_{i=1}^{k} \frac{1}{\lambda_i^2} \right),
\]
where $O(1)$ means here $O_{\bar{a}, A, \bar{\lambda}, \bar{\beta}, \epsilon, \eta}(1)$ with $\bar{a} = (a_1, \cdots, a_k)$, $A := (a_1, \cdots, a_k)$, $\bar{\lambda} := (\lambda_1, \cdots, \lambda_k)$, $\bar{\beta} := (\beta_1, \cdots, \beta_k)$ and for $i = 1, \cdots, k$,
\[
\tilde{\gamma}_i := 1 - \frac{k \tilde{\gamma}_i}{\Gamma}, \quad \Gamma := \sum_{i=1}^{k} \tilde{\gamma}_i, \quad \tilde{\gamma}_i := \tilde{c}_i \lambda_i^{6 \alpha_i - 3} F_i^A(a_i) G_i(a_i),
\]
with
\[
\tilde{c}_i := \int_{\mathbb{R}^3} \frac{1}{(1 + |y|^2)^{3\alpha_i}} dy
\]
\[
G_i(a_i) := e^{3((\alpha_i - 1) H(a_i, a_i) + \sum_{j=1, j \neq i}^{k} (\alpha_j - 1) G(a_j, a_i))} e^{\frac{3}{4} \sum_{j=1, j \neq i}^{k} \Delta_{a_j} G(a_j, a_i)} e^{\frac{3}{4} \sum_{j=1}^{k} \Delta_{a_j} H(a_j, a_i)}
\]
\[
C_0^k \text{ is a real number depending only on } k, \ C_1^k \text{ is a real number and } c^1 \text{ is a positive real number and for the meaning of } O_{\bar{a}, A, \bar{\lambda}, \bar{\beta}, \epsilon, \eta}(1).
\]

Proof. The proof is the same as the one Lemma 4.1 in (11) replacing Lemma 10.1- Lemma 10.4 in (11) by Lemma 5.1- Lemma 5.4

Concerning the gradient estimates of $\mathcal{E}_g$ in $V_R(k, \epsilon, \eta)$, we have in the directions of the scaling parameters :
Lemma 3.10. Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \rho \) where \( \rho \) is as in (11), and \( \epsilon \leq \epsilon_0 \) where \( \epsilon_0 \) is as in (80), then for \( \alpha_i \in \partial M \) concentration points, \( \alpha_i \) masses, \( \lambda_i \) concentration parameters \((i = 1, \cdots, k)\) and \( \beta_r \) negativity parameters \((r = 1, \cdots, \bar{k})\) satisfying (85), we have that for every \( r = 1, \cdots, \bar{k} \), there holds

\[
\begin{align*}
\left\langle \nabla^{p+3}_g \mathcal{E}_g \left( \sum_{i=1}^{k} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{(v_r)_{Q,T}}) \right), \lambda_j \frac{\partial \varphi_{a_j, \lambda_j}}{\partial \lambda_j} \right\rangle_{p+3} &= 16\pi^2 \alpha_j \tau_j \\
- c^2 8\pi^2 &\left( \frac{\Delta_{g_{a_i}} F^A_i(a_i)}{F^A_i(a_i)} - \frac{3}{4} R_g(a_i) \right) - \frac{16\pi^2 \lambda_j}{\lambda_j} \tau_j \Delta_{g_{a_j}} H(a_j, a_j) \\
&- \frac{16\pi^2}{\lambda_j} \sum_{i=1,i\neq j}^{k} \tau_i \Delta_{g_{a_j}} G(a_j, a_i) + c^2 8\pi^2 \sum_{i=1}^{k} \beta_i |a_i|^3 \left( \frac{\Delta_{g_{a_i}} F^A_i(a_i)}{F^A_i(a_i)} - \frac{3}{4} R_g(a_i) \right) \\
+ O \left( \sum_{i=1}^{k} |\alpha_i - 1|^2 + \sum_{r=1}^{\bar{k}} |\beta_r|^3 + \sum_{i=1}^{k} \frac{1}{\lambda_j^3} \right),
\end{align*}
\]

where \( A := (a_1, \cdots, a_k) \), \( O(1) \) as in Lemma 3.5, \( c^2 \) is a positive real number, and for \( i = 1, \cdots, k \),

\[
\tau_i := 1 - \frac{k \tilde{\gamma}_i}{D}, \quad D := \int_{\partial M} K(x)e^{\alpha(\sum_{i=1}^{k} \alpha_i \varphi_{a_i, \lambda_i}(x) + \sum_{r=1}^{\bar{k}} \beta_r v_r(x))} dS_g(x),
\]

with \( \tilde{\gamma}_i \) as in Lemma 3.4.

Proof. The proof is the same as the one Lemma 5.1 in [1] replacing Lemma 10.1- Lemma 10.4 in [1] by Lemma 5.4 Lemma 5.4.

As in [1], Lemma 5.11 implies the following corollary.

Corollary 3.11. Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \rho \) where \( \rho \) is as in (11), and \( 0 < \epsilon \leq \epsilon_0 \) where \( \epsilon_0 \) is as in (80), then for \( \alpha_i \in \partial M \) concentration points, \( \alpha_i \) masses, \( \lambda_i \) concentration parameters \((i = 1, \cdots, k)\) and \( \beta_r \) negativity parameters \((r = 1, \cdots, \bar{k})\) satisfying (85), we have

\[
\begin{align*}
\left\langle \nabla^{p+3}_g \mathcal{E}_g \left( \sum_{i=1}^{k} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{(v_r)_{Q,T}}) \right), \sum_{i=1}^{k} \lambda_i \frac{\partial \varphi_{a_i, \lambda_i}}{\partial \lambda_i} \right\rangle_{p+3} &= \\
\sum_{i=1}^{k} c^3 8\pi^2 &\left( \frac{\Delta_{g_{a_i}} F^A_i(a_i)}{F^A_i(a_i)} - \frac{3}{4} R_g(a_i) \right) \\
&+ O \left( \sum_{i=1}^{k} |\alpha_i - 1|^2 + \sum_{r=1}^{\bar{k}} |\beta_r|^3 + \sum_{i=1}^{k} \tau_i^3 + \sum_{i=1}^{k} \frac{1}{\lambda_j^3} \right),
\end{align*}
\]

where \( A := (a_1, \cdots, a_k) \), \( O(1) \) as in Lemma 3.5, \( c^3 \) is a positive real number, and for \( i = 1, \cdots, k \), \( \tau_i \) is as in Lemma 3.4.

Proof. The proof uses the strategy of the proof of Corollary 5.2 in [1] replacing Lemma 5.1 in [1] by its counterpart Lemma 3.10.

For the gradient estimate in the directions of mass concentrations, we have:

Lemma 3.12. Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \rho \) where \( \rho \) is as in (11), and \( 0 < \epsilon \leq \epsilon_0 \) where \( \epsilon_0 \) is as in (80), then for \( \alpha_i \in \partial M \) concentration points, \( \alpha_i \) masses, \( \lambda_i \) concentration parameters \((i = 1, \cdots, k)\) and \( \beta_r \) negativity parameters \((r = 1, \cdots, \bar{k})\) satisfying (85), we
have that for every \( j = 1, \cdots, k \), there holds
\[
\left\langle \nabla^{p,3} \mathcal{E}_g \left( \sum_{i=1}^{k} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - \overline{v_r})_{(Q,T)} \right), \varphi_{a_j, \lambda_j} \right\rangle_{\text{r}4,3} = 4 \mu_i \beta_l + O \left( \sum_{i=1}^{k} |\alpha_i - 1| + \sum_{i=1}^{k} |\tau_i| \right) + O \left( \sum_{i=1}^{k} \frac{1}{\tau_i^2} \right),
\]
where \( O(1) \) as in Lemma 3.3 and \( C_2 \) is a real number.

**Proof.** It follows from the same arguments as in Lemma 5.3 in [1]. \( \blacksquare \)

Concerning the gradient estimate in the directions of the points of concentrations, we have:

**Lemma 3.13.** Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \varrho \) where \( \varrho \) is as in [11], and \( 0 < \varepsilon \leq \varepsilon_0 \) where \( \varepsilon_0 \) is as in (80), then for \( a_i \in \partial M \) concentration points, \( \alpha_i \) masses, \( \lambda_i \) concentration parameters \( (i = 1, \cdots, k) \), and \( \beta_r \) negativity parameters \( (r = 1, \cdots, \tilde{k}) \) satisfying (85), we have that for every \( j = 1, \cdots, k \), there holds
\[
\left\langle \nabla^{p,3} \mathcal{E}_g \left( \sum_{i=1}^{k} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - \overline{v_r})_{(Q,T)} \right), \frac{1}{\lambda_j} \partial \varphi_{a_i, \lambda_i} / \partial a_j \right\rangle_{\text{r}4,3} = \frac{c^2 32 \pi^2}{\lambda_j} \nabla \mathcal{F}^A_{\partial \mathcal{C}^A}(a_j) + O \left( \sum_{i=1}^{k} |\alpha_i - 1|^2 \right) + O \left( \sum_{i=1}^{k} \frac{1}{\tau_i^2} + \sum_{r=1}^{\tilde{k}} |\beta_r|^2 + \sum_{i=1}^{k} \tau_i^2 \right),
\]
where \( A := (a_1, \cdots, a_k) \), \( O(1) \) is as in Lemma 3.9, \( c^2 \) is as in Lemma 3.10 and for \( i = 1, \cdots, k \), \( \tau_i \) is as in Lemma 3.11.

**Proof.** The proof is the same as the one of Lemma 5.4 in [1]. \( \blacksquare \)

Concerning the gradient estimate in the directions of the negativity parameters, we have:

**Lemma 3.14.** Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \varrho \) where \( \varrho \) is as in [11], and \( 0 < \varepsilon \leq \varepsilon_0 \) where \( \varepsilon_0 \) is as in (80), then for \( a_i \in \partial M \) concentration points, \( \alpha_i \) masses, \( \lambda_i \) concentration parameters \( (i = 1, \cdots, k) \), and \( \beta_r \) negativity parameters \( (r = 1, \cdots, \tilde{k}) \) satisfying (85), we have that for every \( l = 1, \cdots, k \), there holds
\[
\left\langle \nabla^{p,3} \mathcal{E}_g \left( \sum_{i=1}^{k} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - \overline{v_r})_{(Q,T)} \right), v_l - \overline{v_l}_{(Q,T)} \right\rangle_{\text{r}4,3} = 4 \mu_l \beta_l + O \left( \sum_{i=1}^{k} |\alpha_i - 1| + \sum_{i=1}^{k} |\tau_i| \right) + O \left( \sum_{i=1}^{k} \frac{1}{\tau_i^2} \right),
\]
where \( O(1) \) is as in Lemma 3.3 and for \( i = 1, \cdots, k \), \( \tau_i \) is as in Lemma 3.11.

**Proof.** It follows from the same arguments as in the proof of Lemma 5.5 in [1]. \( \blacksquare \)
3.3 Finite-dimensional reduction

In this subsection, we complete the energy estimate of $E_{\eta}$ on $V_{R}(k, \epsilon, \eta)$ via Lyapunov finite dimensional type reduction and second variation arguments. First of all, we have:

**Proposition 3.15.** Assuming that $\eta$ is a small positive real number with $0 < 2\eta < \varrho$ where $\varrho$ is as in (93), and $0 < \epsilon \leq \epsilon_{0}$ where $\epsilon_{0}$ is as in (93) and $u = u_{Q,T} + \sum_{r=1}^{k} \alpha_{i} \varphi_{a_{i}, \lambda_{i}} + \sum_{r=1}^{k} \beta_{r} (v_{r} - (v_{r})_{(Q,T)}) + w \in V_{R}(k, \epsilon, \eta)$ with $w$, the concentration points $a_{i}$, the masses $\alpha_{i}$, the concentrating parameters $\lambda_{i}$ ($i = 1, \ldots, k$), and the negativity parameters $\beta_{r}$ ($r = 1, \ldots, k$) verifying (83)-(85), then we have

$$\mathcal{E}_{g}(u) = \mathcal{E}_{g} \left( \sum_{i=1}^{k} \alpha_{i} \varphi_{a_{i}, \lambda_{i}} + \sum_{r=1}^{k} \beta_{r} (v_{r} - (v_{r})_{(Q,T)}) \right) - f(w) + Q(w) + o(||w||_{p+1}^{2}),$$

where

$$f(w) := 16\pi^{2} \int_{\partial M} K e^{3} \sum_{i=1}^{k} \alpha_{i} \varphi_{a_{i}, \lambda_{i}} + 3 \sum_{r=1}^{k} \beta_{r} v_{r} |w| dS_{g},$$

and

$$Q(w) := ||w||_{p+3}^{2} - 24\pi^{2} \int_{\partial M} K e^{3} \sum_{i=1}^{k} \alpha_{i} \varphi_{a_{i}, \lambda_{i}} + 3 \sum_{r=1}^{k} \beta_{r} v_{r} w^{2} dS_{g}.$$
Furthermore, \( \bar{\theta} \in [0, 1] \), and

\[
\int_{\partial M} K e^{3 \sum_{i=1}^{k} \alpha_i \varphi_i + \lambda_i + 3 \sum_{r=1}^{k} \beta_r v_r} e^{3 \theta w} w^{g} = O \left( \|w\|^q_{g^{\partial M}} \left( \sum_{i=1}^{k} \lambda_i^{3+q} \right) \right),
\]

where \( \theta \in [0, 1] \), and

\[
\int_{\partial M} K e^{3 \sum_{i=1}^{k} \alpha_i \varphi_i + \lambda_i + 3 \sum_{r=1}^{k} \beta_r v_r} \left( e^{3w} - 1 - 3w - \frac{9}{2} w^2 \right) dV_g = o \left( \|w\|_{g^{\partial M}}^q \left( \sum_{i=1}^{k} \lambda_i^3 \right) \right).
\]

where here \( o(1) \) and \( O(1) \) are as in Proposition 3.15.

**Proof.** The proof is the same as the one of Lemma 6.2 in [1] replacing Lemma 10.1 by its counterpart Lemma 5.1.

Still as in [1], the second lemma that we need for the proof of Proposition 3.15 read as follows:

**Lemma 3.17.** Assuming the assumptions of Proposition 3.15 then there holds the following estimate

\[
\int_{\partial M} K e^{3 \sum_{i=1}^{k} \alpha_i \varphi_i + \lambda_i + 3 \sum_{r=1}^{k} \beta_r v_r} w dS_g = O \left( \|w\|_{g^{\partial M}} \left( \sum_{i=1}^{k} \frac{1}{\lambda_i} + \sum_{i=1}^{k} |\alpha_i - 1| \log \lambda_i + \sum_{r=1}^{k} |\beta_r| + \sum_{i=1}^{k} \frac{\log \lambda_i}{\lambda_i} \right) \right).
\]

**Proof.** If follows from the same arguments as in the proof of Lemma 6.3 in [1] replacing Lemma 10.1 by its counterpart Lemma 5.1.

Finally, as in [1], the third and last lemma that we need for the proof of Proposition 3.15 is the following one.

**Lemma 3.18.** Assuming the assumptions of Proposition 3.15 then for every \( i = 1, \ldots, k \), there holds

\[
\tau_i = O \left( \sum_{j=1}^{k} \frac{1}{\lambda_j} \right).
\]

**Proof.** The proof is the same as the one Lemma 6.4 in [1] replacing Lemma 5.1 by Lemma 5.10.

**Proof of Proposition 3.15.**

It follows from the same arguments as in the proof of Lemma 6.1 in [1] replacing Lemma 6.2-Lemma 6.4 in [1] by Lemma 5.16-Lemma 6.13 and Lemma 10.1 in [1] by Lemma 5.1. Furthermore, Lemma 10.6 and Lemma 10.7 in [1] are replaced by Lemma 5.9 and Lemma 5.10.

Now, as in [1], we have that Proposition 3.15 implies the following direct corollaries.

**Corollary 3.19.** Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \phi \) where \( \phi \) is as in (80), \( 0 < \epsilon \leq \epsilon_0 \) where \( \epsilon_0 \) is as in (80) and \( u := \sum_{i=1}^{k} \alpha_i \varphi_i + \lambda_i + \sum_{r=1}^{k} \beta_r v_r \) with the concentration points \( \alpha_i \), the masses \( \alpha_i \), the concentrating parameters \( \lambda_i \) (\( i = 1, \cdots, k \)) and the negativity parameters \( \beta_r \) (\( r = 1, \cdots, \bar{k} \)) satisfying (80), then there exists a unique \( \bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in E_{A, \bar{\lambda}} \) such that

\[
\mathcal{E}_{g} \left( u + \bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \right) = \min_{w \in E_{A, \lambda}, u + w \in V_{h}(\epsilon, \eta)} \mathcal{E}_{g}(u + w),
\]

where \( \bar{\alpha} := (\alpha_1, \cdots, \alpha_k), A := (a_1, \cdots, a_k), \bar{\lambda} := (\lambda_1, \cdots, \lambda_k) \) and \( \bar{\beta} := (\beta_1, \cdots, \beta_k) \).

Furthermore, \( (\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \rightarrow \bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in C^1 \) and satisfies the following estimate

\[
1 \leq \frac{1}{C} ||\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})||_{p^{\partial M}}^2 \leq \left| f \left( \bar{\alpha}, A, \bar{\lambda}, \bar{\beta} \right) \right| \leq C ||\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})||_{p^{\partial M}},
\]

for some large positive constant \( C \) independent of \( \bar{\alpha}, A, \bar{\lambda}, \bar{\beta} \), hence

\[
||\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})||_{p^{\partial M}} = O \left( \sum_{i=1}^{k} \frac{1}{\lambda_i} + \sum_{i=1}^{k} |\alpha_i - 1| \log \lambda_i + \sum_{r=1}^{k} |\beta_r| + \sum_{i=1}^{k} \frac{\log \lambda_i}{\lambda_i} \right).
\]
Corollary 3.20. Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \varrho \) where \( \varrho \) is as in (41), \( 0 < \epsilon \leq \epsilon_0 \) where \( \epsilon_0 \) is as in (50), and \( u_0 := \sum_{i=1}^{k} a_i^0 \varphi_{a_i^0,\lambda_i^0} + \sum_{r=1}^{\tilde{k}} \beta_r^0 (v_r - (\bar{v}_r)_{Q,T}) \) with the concentration points \( a_i^0 \), the masses \( \lambda_i^0 \), the concentrating parameters \( \lambda_i \) (i = 1, \ldots, k) and the negativity parameters \( \beta_r^0 \) \( (r = 1, \ldots, \tilde{k}) \) satisfying (55), then there exists an open neighborhood \( \mathcal{U} \) of \((\tilde{\alpha}^0, A^0, \tilde{\lambda}^0, \tilde{\beta}^0)\) with \( \tilde{\alpha}^0 := (\alpha_1^0, \ldots, \alpha_k^0), A^0 := (a_1^0, \ldots, a_k^0), \lambda^0 := (\lambda_1^0, \ldots, \lambda_k^0) \) and \( \beta^0 := (\beta_1^0, \ldots, \beta_{\tilde{k}}^0) \) such that for every \((\tilde{\alpha}, A, \tilde{\lambda}, \tilde{\beta}) \in \mathcal{U}\) with \( \tilde{\alpha} := (\alpha_1, \ldots, \alpha_k), A := (a_1, \ldots, a_k), \lambda := (\lambda_1, \ldots, \lambda_k) \), \( \beta := (\beta_1, \ldots, \beta_k) \), and the \( a_i \), the \( \alpha_i \), the \( \lambda_i \) \( (i = 1, \ldots, k) \) and the \( \beta_r \) \( (r = 1, \ldots, \tilde{k}) \) satisfying (55), and \( w \) satisfying (55) with \( \sum_{i=1}^{k} a_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - (\bar{v}_r)_{Q,T}) + w \in \bar{V}_R(k, \epsilon, \eta) \), we have the existence of a change of variable

\[
(102) \quad w \mapsto V
\]

from a neighborhood of \( \bar{w}(\tilde{\alpha}, A, \tilde{\lambda}, \tilde{\beta}) \) to a neighborhood of \( 0 \) such that

\[
E_g \left( \sum_{i=1}^{k} a_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - (\bar{v}_r)_{Q,T}) + w \right) =
\]

\[
E_g \left( \sum_{i=1}^{k} a_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - (\bar{v}_r)_{Q,T}) + \bar{w}(\tilde{\alpha}, A, \tilde{\lambda}, \tilde{\beta}) \right) + \frac{1}{2} \beta^2 E_g \left( \sum_{i=1}^{k} a_i^0 \varphi_{a_i^0,\lambda_i^0} + \sum_{r=1}^{\tilde{k}} \beta_r^0 (v_r - (\bar{v}_r)_{Q,T}) + \bar{w}(\tilde{\alpha}^0, A^0, \tilde{\lambda}^0, \tilde{\beta}^0) \right) (V, V),
\]

Thus, as in [1], with this new variable, it is easy to see that in \( \bar{V}_R(k, \epsilon, \eta) \) we have a splitting of the variables \((\tilde{\alpha}, A, \tilde{\lambda}, \tilde{\beta})\) and \( V \), namely that one can decrease the Euler-Lagrange functional \( E_g \) in the variable \( V \) without touching the variable \((\tilde{\alpha}, A, \tilde{\lambda}, \tilde{\beta})\) by considering just the flow

\[
(104) \quad \frac{dV}{dt} = -V.
\]

So, as in [1], and for the same reasons, to develop a Morse theory for \( E_g \) is equivalent to do one for the functional

\[
(105) \quad E_g(\tilde{\alpha}, A, \tilde{\lambda}, \tilde{\beta}) := E_g \left( \sum_{i=1}^{k} a_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - (\bar{v}_r)_{Q,T}) + \bar{w}(\tilde{\alpha}, A, \tilde{\lambda}, \tilde{\beta}) \right),
\]

where \( \tilde{\alpha} = (\alpha_1, \ldots, \alpha_k), A = (a_1, \ldots, a_k), \lambda = (\lambda_1, \ldots, \lambda_k) \) and \( \beta = (\beta_1, \ldots, \beta_{\tilde{k}}) \) with the concentration points \( a_i \), the masses \( \alpha_i \), the concentrating parameters \( \lambda_i \) \( (i = 1, \ldots, k) \) and the negativity parameters \( \beta_r \) \( (r = 1, \ldots, \tilde{k}) \) satisfying (55), and \( \bar{w}(\tilde{\alpha}, A, \tilde{\lambda}, \tilde{\beta}) \) is as in Corollary 3.19.

Finally, we have the following energy estimate of \( E_g \) on \( \bar{V}_R(k, \epsilon, \eta) \).

Lemma 3.21. Under the assumptions of Proposition 3.15 \( \forall u = \bar{v}(Q,T) + \sum_{i=1}^{k} a_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r -
In this subsection, we derive a Morse Lemma at infinity for \( \mathcal{E}_g \). As in [1], in order to do that, we first construct a pseudo-gradient for \( \bar{\mathcal{E}}_g(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \), where \( \bar{\mathcal{E}}_g(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \) is defined as in (105) exploiting the gradient estimates derived previously. Indeed, we have:

**Proposition 3.22.** Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \varrho \) where \( \varrho \) is as in [11], and \( 0 < \epsilon \leq \epsilon_0 \) where \( \epsilon_0 \) is as in [20], then there exists a pseudogradient \( \mathcal{W}_g \) of \( \bar{\mathcal{E}}_g(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \) such that

1) For every \( u := \sum_{i=1}^k \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^k \beta_r (v_r - (v_r)_{(Q,T)}) \in V_R(k, \epsilon, \eta) \) with the concentration points \( a_i \), the masses \( \alpha_i \), the concentrating parameters \( \lambda_i \) (\( i = 1, \cdots, k \)) and the negativity parameters \( \beta_r \) (\( r = 1, \cdots, k \)) satisfying (80), there holds

\[
\left\langle -\nabla_{p^{4,3}}^4 \mathcal{E}_g(u), \mathcal{W}_g \right\rangle_{p^{4,3}} \geq c \left( \sum_{i=1}^k \frac{1}{\lambda_i} + \sum_{i=1}^k \frac{|\nabla_g F_i^A(a_i)|}{\lambda_i} + \left| \sum_{i=1}^k \alpha_i - 1 \right| + \left| \sum_{i=1}^k \tau_i \right| + \left| \sum_{r=1}^k |\beta_r| \right| \right),
\]

and for every \( u := \sum_{i=1}^k \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^k \beta_r \lambda_i (v_r - (v_r)_{(Q,T)}) + \bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in V_R(k, \epsilon, \eta) \) with the concentration points \( a_i \), the masses \( \alpha_i \), the concentrating parameters \( \lambda_i \) (\( i = 1, \cdots, k \)) and the negativity parameters \( \beta_r \) (\( r = 1, \cdots, k \)) satisfying (80), and \( \bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \) is as in [94], there holds

\[
\left\langle -\nabla_{p^{4,3}}^4 \mathcal{E}_g(u + \bar{w}), \mathcal{W}_g + \frac{\partial \bar{w}}{\partial (\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})} \right\rangle_{p^{4,3}} \geq c \left( \sum_{i=1}^k \frac{1}{\lambda_i} + \sum_{i=1}^k \frac{|\nabla_g F_i^A(a_i)|}{\lambda_i} + \left| \sum_{i=1}^k \alpha_i - 1 \right| + \left| \sum_{i=1}^k |\tau_i| \right| + \left| \sum_{r=1}^k |\beta_r| \right| \right),
\]

where \( c \) is a small positive constant independent of \( A := (a_1, \cdots, a_k), \bar{\alpha} := (\alpha_1, \cdots, \alpha_k), \bar{\lambda} := (\lambda_1, \cdots, \lambda_k), \bar{\beta} := (\beta_1, \cdots, \beta_k) \) and \( \epsilon \).

2) \( \mathcal{W}_g \) is a \( \| \cdot \|_{p^{4,3}} \)-bounded vector field and is compactifying outside the region where \( A \) is very close to a critical point \( B \) of \( \mathcal{F}_K \) satisfying \( \mathcal{L}_K(B) < 0 \).

\[\text{Proof.}\] It follows directly from Lemma 3.19 formula (103) and Proposition 3.15. \( \blacksquare \)
Proof. It follows from the same arguments as in the proof of Proposition 8.1 in [1] replacing formulas (52)-(54), Lemma 5.1, Corollary 5.2 and Lemma 5.3-Lemma 5.5 in [1] with (49)-(51), Lemma 3.10, Corollary 3.11 and Lemma 3.12-Lemma 3.14. Furthermore, Lemma 4.1, Lemma 7.1, and Lemma 10.5 in [1] are replaced by Lemma 8.9, Lemma 8.15 and Lemma 8.21.

Now, as in [1], we have that Proposition 3.22 implies the following characterization of the critical points at infinity of $E_g$.

Corollary 3.23. 1) The critical points at infinity of $E_g$ correspond to the “configurations” $\alpha_i = 1$, $\lambda_i = +\infty$, $\tau_i = 0$, $i = 1, \cdots, k$, $\beta_r = 0$, $r = 1, \cdots, k$; $A$ is a critical point of $F_K$ and $V = 0$, and we denote them by $z^\infty$ with $z$ being the corresponding critical point of $F_K$.

2) The “true” critical points at infinity of $E_g$ are the $z^\infty$ satisfying $L_K(z) < 0$ and we denote them by $x^\infty$ with $x$ being the corresponding critical point of $F_K$.

3) The “false” critical points at infinity of $E_g$ are the $z^\infty$ satisfying $L_K(z) > 0$ and we denote them by $y^\infty$ with $y$ being the corresponding critical point of $F_K$.

4) The $E_g$-energy of a critical point at infinity $z^\infty$ denoted by $J_g(z^\infty)$ is given by

$$J_g(z^\infty) = C_0^k - 8\pi^2 F_K(z_1, \cdots, z_k)$$

where $z = (z_1, \cdots, z_k)$ and $C_0^k$ is as in Lemma 3.24.

Proof. Point 1)- Point 3) follow from [11], Lemma 8.8, the discussions right after [10.3], and Proposition 3.22 while Point 4) follows from Point 1) combined with [11] and Lemma 3.24.

Finally, we are going to conclude this subsection by establishing an analogue of the classical Morse lemma for both “true” and “false” critical points at infinity. In order to do that, we first remark that, as in [1], the arguments of Proposition 3.22 implies that $V_\infty := \{u \in V_{\hat{R}}(k, \epsilon, \eta) : I_K(A) < 0, \forall r \in \{1, \cdots, k\} \ |\beta_r| \leq 2\hat{C}_0 \left( \sum_{i=1}^k \frac{|\nabla_g F^\infty(a_i)}{\lambda_i} + \sum_{i=1}^k |\alpha_i - 1| + \sum_{i=1}^k |\tau_i| + \sum_{i=1}^k \frac{1}{\lambda_i^2} \right) \}$ and $V_\infty := \{u \in V_R(k, \epsilon, \eta) : I_K(A) > 0, \forall r \in \{1, \cdots, k\} \ |\beta_r| \leq 2\hat{C}_0 \left( \sum_{i=1}^k \frac{|\nabla_g F^\infty(a_i)}{\lambda_i} + \sum_{i=1}^k |\alpha_i - 1| + \sum_{i=1}^k |\tau_i| + \sum_{i=1}^k \frac{1}{\lambda_i^2} \right) \}$ (where $\hat{C}_0, \hat{C}_0$ and $C_0$ are large positive constants) are respectively a neighborhood of the “true” and “false” critical points at infinity of the variational problem. Hence, as in [1], [10.1], Corollary 3.20, Lemma 3.24 and classical Morse lemma imply the following Morse type lemma for a “true” critical point at infinity.

Lemma 3.24. (Morse lemma at infinity near a “true” one)

Assuming that $\eta$ is a small positive real number with $0 < 2\eta < \varrho$ where $\varrho$ is as in [11], $0 < \epsilon \leq \epsilon_0$ where $\epsilon_0$ is as in (54) and $u_0 := \sum_{i=1}^k \alpha_i^0 \varphi_{a_i} \lambda_i^0 + \sum_{r=1}^k \beta_r^0 (v_r - (v_r)_{Q,T}) + \hat{\nu}((\bar{a}^0, A^0, \lambda^0, \beta^0)) \in V_{\hat{R}}(k, \epsilon, \eta)$ (where $\bar{a}^0 := (a_1^0, \cdots, a_k^0)$, $A^0 := (a_1^0, \cdots, a_k^0)$, $\lambda := (\lambda_1^0, \cdots, \lambda_k^0)$ and $\beta^0 := (\beta_1^0, \cdots, \beta_k^0)$) with the concentration points $a_i^0$, the masses $\alpha_i^0$, the concentrating parameters $\lambda_i^0$ ($i = 1, \cdots, k$) and the negativity parameters $\beta_r^0$ ($r = 1, \cdots, k$) satisfying (54) and furthermore $A^0 \in \text{Crit}(F_K)$, then there exists an open neighborhood $U$ of $(\bar{a}^0, A^0, \lambda^0, \beta^0)$ such that for every $(\bar{a}, A, \lambda, \beta) \in U$ with $\bar{a} := (a_1, \cdots, a_k)$, $A := (a_1, \cdots, a_k)$, $\lambda := (\lambda_1, \cdots, \lambda_k)$, $\beta := (\beta_1, \cdots, \beta_k)$, and the $\alpha_i$, the $\lambda_i$ ($i = 1, \cdots, k$) and the $\beta_r$ ($r = 1, \cdots, k$) satisfying (54), and $w$ satisfying (58) with $u = (\bar{a}_{Q,T}) \sum_{i=1}^k \alpha_i \varphi_{a_i} \lambda_i + \sum_{r=1}^k \beta_r (v_r - (v_r)_{Q,T}) + w \in V_{\hat{R}}(k, \epsilon, \eta)$, we have the existence of a change of variable

$$\begin{align*}
\alpha_i &\rightarrow s_i, \ i = 1, \cdots, k, \\
A &\rightarrow \bar{A},
\bar{A}_+, \bar{A}_- \\
\lambda_i &\rightarrow \theta_1, \\
\tau_i &\rightarrow \tau_i, \ i = 2, \cdots, k, \\
\beta_r &\rightarrow \beta_r, \\
V &\rightarrow \bar{V},
\end{align*}$$

(109)
such that

\[ \mathcal{E}_g(u) = \mathcal{E}_g \left( \sum_{i=1}^{k} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r \left( v_r - \bar{(v_r)}_{(Q,T)} \right) + w \right) = -|\hat{A}_-|^2 + |\hat{A}_+|^2 + \sum_{i=1}^{k} s_i^2 - \sum_{r=1}^{\tilde{k}} \beta_r^2 + \sum_{i=2}^{\tilde{k}} \theta_i^2 + ||\bar{V}||^2 \]

where \( \hat{A} = (\hat{A}_-, \hat{A}_+) \) is the Morse variable of the map \( \mathcal{J}_g : (\partial M)^k \setminus F((\partial M)^k) \rightarrow \mathbb{R} \) which is defined by the right hand side of (110). Hence a “true” critical point at infinity \( x^\infty \) of \( \mathcal{E}_g \) has Morse index at infinity \( M_\infty(x^\infty) = i_\infty(x) + \tilde{k} \).

Similarly, and for the same reasons as above, we have the following analogue of the classical Morse lemma for a “false” critical point at infinity.

**Lemma 3.25. (Morse lemma at infinity near a “false” one)**

Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \varrho \) where \( \varrho \) is as in (11), \( 0 < \epsilon \leq \epsilon_0 \) where \( \epsilon_0 \) is as in (59) and \( v_0 := \sum_{i=1}^{k} \alpha_i^0 \varphi_{a_i, \lambda_i}^0 + \sum_{r=1}^{\tilde{k}} \beta_r^0 (v_r - \bar{(v_r)}_{(Q,T)}) + \bar{w}((\hat{\alpha}^0, \hat{A}^0, \hat{\lambda}^0, \hat{\beta}^0)) \in V_+(k, \epsilon, \eta) \) (where \( \hat{\alpha}^0 := (a_{1}^0, \ldots, a_{k}^0) \), \( \hat{A}^0 := (\lambda_{1}, \ldots, \lambda_{k}) \) and \( \hat{\beta}^0 := (\beta_{1}, \ldots, \beta_{k}) \)) with the concentration points \( a_0^1 \), the masses \( \alpha_i^1 \), the concentrating parameters \( \lambda_i^1 \) (\( i = 1, \ldots, k \)) and the negativity parameters \( \beta_r \) (\( r = 1, \ldots, \tilde{k} \)) satisfying (58) and furthermore \( A^0 \in \text{Crit}(F_\mathcal{K}) \), then there exists an open neighborhood \( U \) of \( (\hat{\alpha}^0, \hat{A}^0, \hat{\lambda}^0, \hat{\beta}^0) \) such that for every \( (\hat{\alpha}, \hat{A}, \hat{\lambda}, \hat{\beta}) \in U \) with \( \hat{\alpha} := (a_{1}, \ldots, a_{k}) \), \( \hat{A} := (\lambda_{1}, \ldots, \lambda_{k}) \) and \( \hat{\beta} := (\beta_{1}, \ldots, \beta_{k}) \), and the \( a_i \), the \( \lambda_i \) (\( i = 1, \ldots, k \)) and the \( \beta_r \) (\( r = 1, \ldots, \tilde{k} \)) satisfying (58), and \( w \) satisfying (59) with \( u = \bar{w}(Q,T) + \sum_{i=1}^{k} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r (v_r - \bar{(v_r)}_{Q^n}) + w \in V_+(k, \epsilon, \eta) \), we have the existence of a change of variable

\[
\alpha_i \rightarrow s_i, \quad i = 1, \ldots, k, \\
A \rightarrow \hat{A} = (\hat{A}_-, \hat{A}_+) \\
\lambda_1 \rightarrow \theta_1, \\
\tau_i \rightarrow \theta_i, \quad i = 2, \ldots, k, \\
\beta_r \rightarrow \beta_r, \\
V \rightarrow \bar{V},
\]

such that

\[ \mathcal{E}_g(u) = \mathcal{E}_g \left( \sum_{i=1}^{k} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\tilde{k}} \beta_r \left( v_r - \bar{(v_r)}_{(Q,T)} \right) + w \right) = -|\hat{A}_-|^2 + |\hat{A}_+|^2 + \sum_{i=1}^{k} s_i^2 - \sum_{r=1}^{\tilde{k}} \beta_r^2 + \sum_{i=2}^{\tilde{k}} \theta_i^2 + ||\bar{V}||^2, \]

where \( \hat{A} = (\hat{A}_-, \hat{A}_+) \) is the Morse variable of the map \( \mathcal{J}_g : (\partial M)^k \setminus F((\partial M)^k) \rightarrow \mathbb{R} \) which is defined by the right hand side of (118). Hence a “false” critical point at infinity \( y^\infty \) of \( \mathcal{E}_g \) has Morse index at infinity \( M_\infty(y^\infty) = i_\infty(y) + 1 + \tilde{k} \).

## 4 Proof of existence theorems

In this section, we show how the Morse lemma at infinity implies the main existence results via strong Morse type inequalities or Barycenter technique of Bahri-Coron.

### 4.1 Topology of very high and negative sublevels of \( \mathcal{E}_g \)

We study the topology of very high sublevels of \( \mathcal{E}_g \) and its very negative ones. We start with the very high sublevels of \( \mathcal{E}_g \) and first derive the following Lemma.

**Lemma 4.1.** Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \varrho \) where \( \varrho \) is as in (11), then there exists \( \mathcal{C}_0^k := \mathcal{C}_0^k(\eta) \) such that for every \( 0 < \epsilon \leq \epsilon_0 \) where \( \epsilon_0 \) is as in (59), there holds

\[ \forall \epsilon \in (0, \epsilon_0), \quad V(k, \epsilon, \eta) \subset (\mathcal{E}_g)^{\mathcal{C}_0} - (\mathcal{E}_g)^{-\mathcal{C}_0}. \]
Next, combining Proposition 3.7 and the latter lemma, we have the following corollary.

**Corollary 4.2.** There exists a large positive constant $\hat{C}_1^k$ such that

$$\text{Crit}(\mathcal{E}_g) \subset (\mathcal{E}_g)^{\hat{C}_1^k} - (\mathcal{E}_g)^{-\hat{C}_1^k}.$$ 

**Proof.** It follows, via a contradiction argument, from the fact that $\mathcal{E}_g$ is invariant by translation by constants, Proposition 3.7, and Lemma 4.1. \[\square\]

Now, we are ready to characterize the topology of very high sublevels of $\mathcal{E}_g$. Indeed, as in [1] and for the same reasons, we have that Lemma 3.8, Lemma 4.1, and Corollary 4.2 imply the following one which describes the topology of very high sublevels of the Euler-Lagrange functional $\mathcal{E}_g$.

**Lemma 4.3.** Assuming that $\eta$ is a small positive real number with $0 < 2\eta < \varrho$ where $\varrho$ is as in (11), then there exists a large positive constant $L^k := L^k(\eta)$ with $L^k > 2 \max(\hat{C}_0^k, \hat{C}_1^k)$ such that for every $L \geq L^k$, we have that $(\mathcal{E}_g)^{-L}$ has the same homotopy type as $B_k(\partial M)$ if $k \geq 2$ and $\bar{k} = 0$, as $A_{k-1}$ if $k \geq 2$ and $\bar{k} \geq 1$ and as $S^{k-1}$ if $k = 1$ and $\bar{k} \geq 1$, where $\hat{C}_0^k$ is as in Lemma 4.1 and $\hat{C}_1^k$ is as in Lemma 4.2.

Next, we turn to the study of the topology of very negative sublevels of $\mathcal{E}_g$ when $k \geq 2$ or $\bar{k} \geq 1$. Indeed, as in [1] and for the same reasons, we have that the well-known topology of very negative sublevels in the nonresont case (see [44]), Proposition 3.7, Lemma 4.1 and Corollary 4.2 imply the following lemma which gives the homotopy type of the very negative sublevels of the Euler-Lagrange functional $\mathcal{E}_g$.

**Lemma 4.4.** Assuming that $k \geq 2$ or $\bar{k} \geq 1$, and $\eta$ is a small positive real number with $0 < 2\eta < \varrho$ where $\varrho$ is as in (11), then there exists a large positive constant $L_{k,\bar{k}} := L_{k,\bar{k}}(\eta)$ with $L_{k,\bar{k}} > 2 \max(\hat{C}_0^k, \hat{C}_1^k)$ such that for every $L \geq L_{k,\bar{k}}$, we have that $(\mathcal{E}_g)^{-L}$ has the same homotopy type as $B_{k-1}(\partial M)$ if $k \geq 2$ and $\bar{k} = 0$, as $A_{k-1}$ if $k \geq 2$ and $\bar{k} \geq 1$ and as $S^{k-1}$ if $k = 1$ and $\bar{k} \geq 1$, where $\hat{C}_0^k$ is as in Lemma 4.1 and $\hat{C}_1^k$ is as in Lemma 4.2.

However, as in [10], to prove Theorem 1.8 we need a further information about the topology of very negative sublevels of $\mathcal{E}_g$. In order to derive that, we first make some definitions. For $p \in \mathbb{N}^*$ and $\lambda > 0$, we define

$$f_p(\lambda) : B_p(\partial M) \rightarrow \mathcal{H}_{\eta}$$

as follows

$$f_p(\lambda)(\sum_{i=1}^{p} \alpha_i \delta_{a_i}) := \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda}, \quad \sigma = \sum_{i=1}^{p} \alpha_i \delta_{a_i} \in B_p(\partial M),$$

with the $\varphi_{a_i, \lambda}$’s defined by (43). Furthermore, when $\bar{k} \geq 1$, for $\Theta > 0$, we define

$$\Psi_{p,\bar{k}}(\lambda, \Theta) : A_{p,\bar{k}} \rightarrow \mathcal{H}_{\eta}$$

as follows

$$\Psi_{p,\bar{k}}(\lambda, \Theta)(\sigma, s) := \begin{cases} \varphi_s + f_p(\lambda)(\sigma) & \text{for } |s| \leq \frac{1}{4}, \quad \sigma \in B_p(\partial M), \\ \varphi_s + f_p(2\lambda - 1 + 4(1-\lambda)|s|)(\sigma) & \text{for } \frac{1}{4} \leq |s| \leq \frac{1}{2}, \quad \sigma \in B_p(\partial M), \\ \varphi_s + 2(1 - f_p(1)(\sigma))|s| + 2f_p(1) - 1 & \text{for } |s| \geq \frac{1}{2}, \quad \sigma \in B_p(\partial M), \end{cases}$$

where $\varphi_s$ is defined by the following formula

$$\varphi_s = \Theta \sum_{r=1}^{\bar{k}} s_r (v_r - (\overline{v_r})_{(Q,T)}),$$

with $s = (s_1, \ldots, s_{\bar{k}})$. As in [10], concerning the $f_p(\lambda)$’s, we have the following estimates.
Lemma 4.5. Assuming that $p \in \mathbb{N}^*$, then we have

1) If $p < k$, then for every $L > 0$, there exists $\lambda^L_p > 0$ such that for all $\lambda \geq \lambda^L_p$, we have

$$f_p(\lambda)(B_p(\partial M)) \subset (\mathcal{E}_g)^{-L}.$$ 

2) If $p = k$, then there exist $\hat{C}_k > 0$ and $\lambda_k > 0$ such that for all $\lambda \geq \lambda_k$, we have

$$f_k(\lambda)(B_k(\partial M)) \subset (\mathcal{E}_g)^{\hat{C}_k}.$$ 

3) There exists $\hat{C}_k > 0$ such that up to taking $\epsilon_0$ smaller, where $\epsilon_0$ is given by (80), we have that for every $0 < \epsilon \leq \epsilon_0$, there holds

$$V(k, \epsilon) \subset (\mathcal{E}_g)^{\hat{C}_k}.$$ 

Proof. It follows from the same arguments as in the proof of Lemma 3.1 in [16] by using Lemma 5.1 and Lemma 5.6. Lemma 5.8.

Still, as in [16], we have the following estimates for the $\Psi_p(\lambda, \Theta)$’s when $k \geq 1$.

Lemma 4.6. Assuming that $p \in \mathbb{N}^*$, then we have

1) If $1 \leq p < k$, then for every $L > 0$, there exists $\lambda^L_{p,k} > 0$ and $\Theta^L_{p,k} > 0$ such that for all $\lambda \geq \lambda^L_{p,k}$, we have

$$\Psi_{p,k}(\lambda, \Theta^L_{p,k})(A_{p,k}) \subset (\mathcal{E}_g)^{-L}.$$ 

2) If $p = k$ and $\Theta > 0$, then there exists $C^\Theta_{k,k} > 0$, $\lambda^\Theta_{k,k} > 0$, such that for every $\lambda \geq \lambda^\Theta_{k,k}$, we have

$$\Psi_{k,k}(\lambda, \Theta)(A_{k,k}) \subset (\mathcal{E}_g)^{C^\Theta_{k,k}}.$$ 

3) If $\Theta > 0$ and there exists $C^\Theta_{k,k} > 0$ such that up to taking $\epsilon_0$ smaller, where $\epsilon_0$ is given by (80), we have that for every $0 < \epsilon \leq \epsilon_0$, there holds

$$V(k, \epsilon, \Theta) \subset (\mathcal{E}_g)^{C^\Theta_{k,k}}.$$ 

Proof. It follows from the same arguments as in the proof of Lemma 4.1 in [16] by using Lemma 4.5.

On the other hand, as in [16], Lemma 4.3 and Lemma 4.5 imply the following one:

Lemma 4.7. Assuming that $k \geq 2$, $\bar{k} = 0$, and $L \geq L_{k,0}$, then there exists $\lambda^L_{k-1}$ such that for all $\lambda \geq \lambda^L_{k-1}$, we have

$$f_{k-1}(\lambda) : B_{k-1}(\partial M) \rightarrow (\mathcal{E}_g)^{-L}$$

is well defined and induces an isomorphism in homology.

Furthermore, still as in [16], we have also that Lemma 4.4 and Lemma 4.5 imply the following one:

Lemma 4.8. Assuming that $k \geq 2$, $\bar{k} \geq 1$, $L \geq L_{k,\bar{k}}$, then there exists $\lambda^L_{k-1,\bar{k}} > 0$ and $\Theta^L_{k-1,\bar{k}} > 0$ such that for all $\lambda \geq \lambda^L_{k-1,\bar{k}}$, we have

$$\Psi_{k-1,\bar{k}}(\lambda, \Theta^L_{k-1,\bar{k}}) : A_{k-1,\bar{k}} \rightarrow (\mathcal{E}_g)^{-L}$$

is well defined and induces an isomorphism in homology.

4.2 Morse theoretical type results

Proof of Theorem 4.1 Theorem 4.7

The proof is the same as the one of Theorem 1.1-Theorem 1.6 in [11] by using Lemma 3.8 and Proposition 3.22 Corollary 3.23 Lemma 3.24-3.26 Lemma 4.1 Corollary 4.2 Lemma 4.3 and Lemma 4.4 combined with the works of Bahri-Rabinowitz [12], Karel-Karoui [37] and Malchiodi [39].
4.3 Algebraic topological type results

In order to carry the algebraic topological argument for existence, as in [46], we need the following lemma.

**Lemma 4.9.** Assuming that \( (ND) \) holds, \( s_k^*(O_{\partial M}) \neq 0 \) in \( H^3(S^\infty) \) and \( s_k^*(O_{\partial M}) = 0 \) in \( H^3(S^\infty_+ \cup S^\infty_-) \), then there exists \( 0 \neq \tilde{O}_{\partial M} \in H^3(S) \) such that

\[
i^*(\tilde{O}_{\partial M}) = s_k^*(O_{\partial M}),
\]

where \( i : S^\infty \to S \) is the canonical injection.

**Proof.** It follows from the same arguments as in the proof of Lemma 3.6 in [46] by using the analysis of Section 3.

**Proof of Theorem 1.8**

The proof is the same as the one in Theorem in [46] by using the algebraic topological tools [52, 54], characterization of the critical points at infinity of \( E_g \) established in Section 3 and Lemma 4.9.

5 Appendix

**Lemma 5.1.** Assuming that \( \epsilon \) is positive and small, \( a \in \partial M \) and \( \lambda \geq \frac{1}{\epsilon} \), then

1) \[
\varphi_{a, \lambda}(\cdot) = \delta_{a, \lambda}(\cdot) + \log \frac{\lambda}{2} + H(a, \cdot) + \frac{1}{2\lambda^2} \Delta g_a H(a, \cdot) + O \left( \frac{1}{\lambda^3} \right) \quad \text{on} \quad \partial M
\]

2) \[
\lambda \frac{\partial \varphi_{a, \lambda}(\cdot)}{\partial \lambda} = \frac{2}{1 + \lambda^2 \chi^2_g(d_{\partial a}(a, \cdot))} - \frac{1}{\lambda^2} \Delta g_a H(a, \cdot) + O \left( \frac{1}{\lambda^3} \right) \quad \text{on} \quad \partial M,
\]

3) \[
\frac{1}{\lambda} \frac{\partial \varphi_{a, \lambda}(\cdot)}{\partial a} = \chi_g(d_{\partial a}(a, \cdot)) \chi'_g((d_{\partial a}(a, \cdot)) \frac{2\lambda \epsilon p^{-1}(-1)}{1 + \lambda^2 \chi^2_g(d_{\partial a}(a, \cdot))} + \frac{1}{\lambda} \frac{\partial H(a, \cdot)}{\partial a} + O \left( \frac{1}{\lambda^3} \right) \quad \text{on} \quad \partial M,
\]

where \( O(1) \) means \( O_{a, \lambda, \epsilon}(1) \) and for it meaning see Section 3.

**Lemma 5.2.** Assuming that \( \epsilon \) is small and \( d \) positive, \( a \in M, \lambda \geq \frac{1}{\epsilon}, \) and \( 0 < 2\eta < \varrho \) with \( \varrho \) as in [11], then there holds

\[
\varphi_{a, \lambda}(\cdot) = G(a, \cdot) + \frac{1}{2\lambda^2} \Delta g_a G(a, \cdot) + O \left( \frac{1}{\lambda^3} \right) \quad \text{on} \quad \partial M \setminus B_a^\infty(\eta),
\]

\[
\lambda \frac{\partial \varphi_{a, \lambda}(\cdot)}{\partial \lambda} = \frac{1}{\lambda^2} \Delta g_a G(a, \cdot) + O \left( \frac{1}{\lambda^3} \right) \quad \text{on} \quad \partial M \setminus B_a^\infty(\eta),
\]

and

\[
\frac{1}{\lambda} \frac{\partial \varphi_{a, \lambda}(\cdot)}{\partial a} = \frac{1}{\lambda} \frac{\partial G(a, \cdot)}{\partial a} + O \left( \frac{1}{\lambda^3} \right) \quad \text{on} \quad \partial M \setminus B_a^\infty(\eta),
\]

where \( O(1) \) means \( O_{a, \lambda, \epsilon}(1) \) and for it meaning see Section 3.

**Lemma 5.3.** Assuming that \( \epsilon \) is small and positive, \( a \in \partial M \) and \( \lambda \geq \frac{1}{\epsilon} \), then there holds

\[
P^4_g \left( \varphi_{a, \lambda}, \varphi_{a, \lambda} \right) = 16\pi^2 \log \lambda - 8\pi^2 C_0 + 8\pi^2 \frac{\partial H(a, a)}{\partial a} + O \left( \frac{1}{\lambda^3} \right),
\]

\[
P^4_g \left( \varphi_{a, \lambda} \lambda \frac{\partial \varphi_{a, \lambda}}{\partial \lambda} \right) = 8\pi^2 - \frac{8\pi^2}{\lambda^2} \Delta g_a H(a, a) + O \left( \frac{1}{\lambda^3} \right),
\]

\[
P^4_g \left( \varphi_{a, \lambda}, \frac{1}{\lambda} \frac{\varphi_{a, \lambda}}{\partial a} \right) = \frac{8\pi^2}{\lambda^2} \frac{\partial H(a, a)}{\partial a} + O \left( \frac{1}{\lambda^3} \right),
\]

where \( C_0 \) is a positive constant depending only on \( n \), \( O(1) \) means \( O_{a, \lambda, \epsilon}(1) \) and for its meaning see Section 3.
Lemma 5.4. Assuming that $\epsilon$ is small and positive $a_i, a_j \in \partial M$, $d_g(a_i, a_j) \geq 4C\eta$, $0 < 2\eta < \rho$, \( \frac{1}{4} \leq \frac{\lambda_i}{\lambda_j} \leq \Lambda \), and $\lambda_i, \lambda_j \geq \frac{1}{4}, C$ as in (39), and $\rho$ as in (41), then there hold

\[
P^4,3_g (\varphi_{a_i, \lambda_i}, \varphi_{a_j, \lambda_j}) = 8\pi^2 G(a_j, a_i) + \frac{4\pi^2}{\lambda_i} \Delta \varphi_{a_i, \lambda_i} G(a_j, a_i) + \frac{4\pi^2}{\lambda_j} \Delta \varphi_{a_j, \lambda_j} G(a_j, a_i) + O\left(\frac{1}{\lambda_i^2} + \frac{1}{\lambda_j^2}\right),
\]

where $O(1)$ means bounded by positive constants form below and above independent of $\epsilon, a$, and $\lambda$.

Lemma 5.5. Assuming that $\epsilon > 0$ is very small, we have that for $a \in \partial M, \lambda \geq \frac{1}{\epsilon}$, there holds

\[
\|\frac{\partial \varphi_{a, \lambda}}{\partial \lambda}\|_{p=4} = \tilde{O}(1),
\]

\[
\|\frac{1}{\lambda} \frac{\partial \varphi_{a, \lambda}}{\partial a}\|_{p=4} = \tilde{O}(1),
\]

and

\[
\|\frac{1}{\sqrt{\log \lambda}} \varphi_{a, \lambda}\|_{p=4} = \tilde{O}(1),
\]

where $\tilde{O}(1)$ means bounded by positive constants form below and above independent of $\epsilon, a$, and $\lambda$.

Lemma 5.6. 1) If $\epsilon$ is small and positive, $a \in \partial M, p \in \mathbb{N}^*$, and $\lambda \geq \frac{1}{\epsilon}$, then there holds

\[
C^{-1}\lambda^{0p-3} \leq \int_{\partial M} e^{p\varphi_{a, \lambda}} dS_g \leq C\lambda^{6p-3},
\]

where $C$ is independent of $a, \lambda$, and $\epsilon$.

2) If $\epsilon$ is positive and small, $a_i, a_j \in \partial M, \lambda \geq \frac{1}{\epsilon}$ and $\lambda d_g(a_i, a_j) \geq 4CR$, then we have

\[
P^4,3_g (\varphi_{a_i, \lambda}, \varphi_{a_j, \lambda}) \leq 8\pi^2 G(a_i, a_j) + O(1),
\]

where $O(1)$ means here $O_{A, \lambda, \epsilon}(1)$ with $A = (a_i, a_j)$ and for the meaning of $O_{A, \lambda, \epsilon}(1)$, see Section 4.

3) If $\epsilon$ is positive and small, $a_i, a_j \in \partial M, \lambda_i, \lambda_j \geq \frac{1}{\epsilon}, \frac{1}{4} \leq \frac{\lambda_i}{\lambda_j} \leq \Lambda$ and $\lambda_i d_g(a_i, a_j) \geq 4CR$, then we have

\[
P^4,3_g (\varphi_{a_i, \lambda}, \varphi_{a_j, \lambda}) \leq 8\pi^2 G(a_i, a_j) + O(1),
\]

where $O(1)$ means here $O_{A, \lambda, \epsilon}(1)$ with $A = (a_i, a_j)$ and $\lambda = (\lambda_i, \lambda_j)$ and for the meaning of $O_{A, \lambda, \epsilon}(1)$, see Section 4.

Lemma 5.7. Let $p \in \mathbb{N}^*$, $R$ be a large positive constant, $\epsilon$ be a small positive number, $\alpha_i \geq 0, i = 1, \cdots, p, \sum_{i=1}^p \alpha_i = k, \lambda \geq \frac{1}{\epsilon}$ and $u = \sum_{i=1}^p \alpha_i \varphi_{a_i, \lambda}$. Assuming that there exist two positive integer $i, j \in \{1, \cdots, p\}$ with $i \neq j$ such that $\lambda d_g(a_i, a_j) \leq \frac{R}{4C}$, where $C$ is as in (39), then we have

\[
E_g(u) \leq E_g(v) + O(\log R),
\]

with

\[
v := \sum_{k \leq p, k \neq i, j} \alpha_k \varphi_{a_k, \lambda} + (\alpha_i + \alpha_j) \varphi_{a_i, \lambda},
\]

where $O(1)$ stand for $O_{\bar{A}, A, \lambda, \epsilon}(1)$, with $\bar{A} = (\alpha_1, \cdots, \alpha_p)$ and $A = (a_1, \cdots, a_p)$, and for the meaning of $O_{\bar{A}, A, \lambda, \epsilon}(1)$, we refer the reader to Section 2.
Lemma 5.8. 1) If $\epsilon$ is positive and small, $a_i, a_j \in \partial M$, $\lambda \geq \frac{1}{A}$ and $\lambda d_2(a_i, a_j) \geq 4CR$, then

$$\varphi_{a_i, \lambda} \cdot a_j \cdot G(a_j, \cdot) + O(1) \text{ in } B_{a_i}^G \left( \frac{R}{\lambda} \right),$$

where here $O(1)$ means here $O_{A, \lambda, \epsilon}(1)$, with $A = (a_i, a_j)$, and for the meaning of $O_{A, \lambda, \epsilon}(1)$, see Section 3.

2) If $\epsilon$ is positive and small, $a_i, a_j \in \partial M$, $\lambda_i, \lambda_j \geq \frac{1}{A}$, $\frac{1}{A} \leq \lambda_i \leq \Lambda$, and $\lambda_i d_2(a_i, a_j) \geq 4CR$, then

$$\varphi_{a_i, \lambda, \cdot}(\cdot) = G(a_j, \cdot) + O(1) \text{ in } B_{a_i}^G \left( \frac{R}{\lambda_i} \right),$$

where here $O(1)$ means here $O_{A, \lambda, \epsilon}(1)$, with $A = (a_i, a_j)$, $\tilde{\lambda} = (\lambda_i, \lambda_j)$ and for the meaning of $O_{A, \lambda, \epsilon}(1)$, see Section 3.

Lemma 5.9. There exists $\Gamma_0$ and $\bar{\Lambda}_0$ two large positive constant such that for every $a \in \partial M$, $\lambda \geq \bar{\Lambda}_0$, and $w \in F_{a, \lambda} := \{ w \in H_{\text{g, } \bar{\omega}, \partial M} : \langle \varphi_{a, \lambda}, w \rangle_{p_{4, 3}} = \langle v_r, w \rangle_{p_{4, 3}} = 0, r = 1 \cdots, \bar{k}\}$, we have

$$\int_{\partial M} e^{3\lambda_{a, \lambda}} w^2 dV_{g_a} \leq \Gamma_0\|w\|^2_{p_{4, 3}}. \quad (124)$$

Lemma 5.10. Assuming that $\eta$ is a small positive real number with $0 < 2\eta < \varphi$ where $\varphi$ is as in (11), then there exists a small positive constant $c_0 := c_0(\eta)$ and $\Lambda_0 := \Lambda_0(\eta)$ such that for every $a_i \in \partial M$ concentration points with $d_2(a_i, a_j) \geq 4CR$ where $C$ is as in (39) for every $\lambda_i > 0$ concentration parameters satisfying $\lambda_i \geq \Lambda_0$, with $i = 1, \cdots, k$, and for every $w \in E_{a, \lambda}^* = \bigcap_{j=1}^k E_{a, \lambda_j}^*$, with $A := (a_1, \cdots, a_k)$, $\tilde{\lambda} := (\lambda_1, \cdots, \lambda_k)$ and $E_{a, \lambda}^* \subset w(Q, T) = \{ v_r \in H_{\text{g, } \bar{\omega}, \partial M} : \langle \varphi_{a, \tilde{\lambda}}, v \rangle_{p_{4, 3}} = \langle \partial a_{\lambda_i, \lambda_j}, v \rangle_{p_{4, 3}} = \langle \partial a_{\lambda_i, \lambda_j}, v \rangle_{p_{4, 3}} = \langle \varphi_{a, \lambda}, w \rangle_{p_{4, 3}} = 0, r = 1 \cdots, \bar{k}\}$, there holds

$$\|w\|^2_{p_{4, 3}} - \sum_{i=1}^k \int_{\partial M} e^{3\lambda_{a, \lambda}} w^2 dS_{g_{a_i}} \geq c_0\|w\|^2_{p_{4, 3}}. \quad (125)$$

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