Mean Field Limit of Interacting Filaments for 3D Euler Equations

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Abstract
The 3D Euler equations, precisely local smooth solutions of class $H^s$ with $s > 5/2$ are obtained as a mean field limit of finite families of interacting curves, the so called vortex filaments, described by means of the concept of 1-currents. This work is a continuation of a previous paper, where a preliminary result in this direction was obtained, with the true Euler equations replaced by a vector valued non linear PDE with a mollified Biot–Savart relation.

Keywords 3D Euler equations · Vortex filaments · Currents · Mean field theory

Mathematics Subject Classification Primary 35Q31, 70F45 · Secondary 37C10, 76B47, 49Q15

1 Introduction

In our previous paper [6], we investigated a mean-field type convergence of a family of vortex lines (called vortex filaments) to a smoothed version of the 3D Euler equations, where the interaction between the vortex lines was suitably mollified, too. This present paper is a continuation of a previous paper with the purpose of proving convergence of the mollified vortex line dynamics to the true 3D Euler equations. The result is local in time, since we deal with relatively smooth solutions of 3D Euler equations in their vorticity formulation, which exist and are unique in Sobolev spaces $H^s$ for $s > \frac{3}{2}$. The passage from the mollified to the true Euler equations was missing in our previous work since we were unable to overcome...
a difficulty concerning the local time interval where the convergence should take place. It is quite clear that mollified and true Euler equations have unique $H^4$ local solutions on a common local time interval, as proved below in Theorem 10. However, it is not clear a priori that the mollified vortex dynamics has a local solution on an interval which is independent of the mollification. Under special assumptions on the mollification kernel, however, one can prove globality of the vortex dynamics: this is the key property used below to close the approximation result. Let us also mention the much more difficult open problem of dealing with the true vortex dynamics instead of the mollified one; however, this problem is unsolved even for a single vortex filament, so the question of convergence of a family of interacting filaments to Euler equations is premature.

The idea of a mean field of vortex filaments has been already investigated with some success in [18,19]. However, in those works the limit equation is not the Euler equations, due to some idealization in the vortex model. The filaments considered in these papers were assumed to be parallel. The present work is the first one that relates vortex filament models to the true Euler equations.

Our motivation comes from studying the Euler equations in $\mathbb{R}^3$

$$\begin{cases}
\partial_t v + (v \cdot \nabla)v + \nabla p = 0 \\
\nabla \cdot v = 0 \\
v(0, x) = v_0(x)
\end{cases}$$

where $v = v(t, x)$ denotes the velocity vector field and $p = p(t, x)$ the scalar pressure, for $x \in \mathbb{R}^3$, $t > 0$. If we denote by $\xi$ the vorticity field, $\xi = \nabla \times v$, then $\xi$ satisfies

$$\begin{cases}
\partial_t \xi + (v \cdot \nabla)\xi = (\xi \cdot \nabla)v \\
\nabla \cdot \xi = 0, \\
\xi(0, x) = \xi_0(x).
\end{cases}$$

The vorticity and velocity fields $\xi, v$ are related by the Biot–Savart formula

$$v(x) = (K * \xi)(x),$$

where $K$ is the singular matrix (at 0) given by

$$K(x) = \frac{-\Gamma}{4\pi |x|^3} \begin{pmatrix}
0 & x_3 & -x_2 \\
-x_3 & 0 & x_1 \\
x_2 & -x_1 & 0
\end{pmatrix}$$

and $\Gamma$ is the circulation associated to the velocity field $v$. The kernel $K$ can also be rewritten as follows: for $x, h \in \mathbb{R}^3$

$$K(x)h = \frac{\Gamma}{4\pi |x|^3} x \times h.$$

By using the Biot–Savart formula, the vorticity equation (2) becomes

$$\begin{cases}
\partial_t \xi + [(K * \xi) \cdot \nabla] \xi = (\xi \cdot \nabla)(K * \xi) \\
\nabla \cdot \xi = 0, \\
\xi(0, x) = \xi_0(x).
\end{cases}$$

As already mentioned, our aim is to prove a mean field result for Eq. (4), in a sense analogous to [10,21], when interacting point particles are replaced by interacting curves; see
the discussion about motivations in [6]. We consider curves which vary in time and which are parametrized by \( \sigma \in [0, 1] \). Given \( t \geq 0 \), we consider the family of curves
\[
\{ \gamma_{i,t}^{j \cdot N, \delta}(\sigma) ; \sigma \in [0, 1], \ i = 1, ..., N \}
\]
in \( \mathbb{R}^d \). For shortness we shall often write \( \{ \gamma_{i,t}^{j \cdot N, \delta} \} \).

These curves vary in time and interact through the equations
\[
\frac{\partial}{\partial t} \gamma_{i,t}^{j \cdot N, \delta}(\sigma) = \sum_{j=1}^{N} \alpha_{j}^{N} \int_{0}^{1} K^{\delta}(\gamma_{i,t}^{j \cdot N, \delta}(\sigma) - \gamma_{j,t}^{i \cdot N, \delta}(\sigma')) \frac{\partial}{\partial \sigma'} \gamma_{j,t}^{i \cdot N, \delta}(\sigma') d\sigma',
\]
subject to suitable initial condition. Here \( \alpha_{j}^{N} \) behaves as \( \frac{1}{N} \) when the number of curves \( N \) grows, and \( K^{\delta} : \mathbb{R}^3 \to \mathbb{R}^{3 \times 3} \) is a smooth matrix-valued function (precisely, we need \( K^{\delta} \) of class \( UC_{b}^{3}(\mathbb{R}^3, \mathbb{R}^{3 \times 3}) \), see [6]), which is defined as a mollification of the Biot–Savart kernel \( K \). We denote by \( \delta > 0 \) the mollification parameter as defined in Sect. 3.1. Notice that the curves \( \gamma_{i,t}^{j \cdot N, \delta} \) will also depend on \( \delta \). Indeed, without the mollification we are unable to prove the well-posedness of System (5). The final limit to Euler equations is not uniform in \( N \) and \( \delta \): we have to choose \( \delta \) depending on \( N \) (see (28)).

Let us briefly comment on previous investigations about vortex filaments dynamics. In our previous paper [6], as a byproduct of our mean field result, we were able to prove the well posedness of System (5) which is a system of \( N \) interacting filaments. We proved that there exist unique global solutions in the space of smooth curves. Previous results on the dynamic of one filament has been studied with some success in [1,4,5] where local in time solutions were shown to exist in some Sobolev spaces. Global solutions for the dynamic of one filament can be found in [2,8]. In all these papers, the study was focused on one filament and sometimes with some stochastic features. Numerical results can be found in [9,16] with some focus on Lagrangian numerical methods. A statistical approach was also developed by using some Gibbs measures. This approach has been used with some success in [19] for \( N \) filaments that are nearly parallel. For stochastic Brownian filaments, a similar approach was used in [3]. The Gibbs ensemble for these stochastic Brownian filaments were first described by [11,12].

As explained in [6], we associate to such a family of curves a distributional vector field (called a 1-current) defined as
\[
\xi_{i,t}^{N, \delta} = \sum_{j=1}^{N} \alpha_{j}^{N} \int_{0}^{1} \delta_{\gamma_{j,t}^{i \cdot N, \delta}(\sigma)} \frac{\partial}{\partial \sigma} \gamma_{j,t}^{i \cdot N, \delta}(\sigma) d\sigma,
\]
which plays the same role as the empirical measure in classical mean-field theory. This current also depends on \( \delta \).

In [6] it was proved that \( \xi_{i,t}^{N, \delta} \) converges, in appropriate topologies, to \( \xi_{i,t}^{\delta} \), the unique solution of a vector valued nonlinear PDE, of the form
\[
\frac{\partial \xi_{i,t}^{\delta}}{\partial t} + \left[ (K^{\delta} \ast \xi_{i,t}^{\delta}) \cdot \nabla \right] \xi_{i,t}^{\delta} = \xi_{i,t}^{\delta} \cdot \nabla (K^{\delta} \ast \xi_{i,t}^{\delta}).
\]
We will use the result on the regularized equation to establish a mean field result for the 3D Euler equations in their vorticity formulation (4).

The main result of this paper is Theorem 13, which states that, given \( \xi \) solution to (4), there exists a sequence of empirical measures \( \xi_{i,t}^{N, \delta} \) of the form (6) which converges toward \( \xi \). We base the proof of this theorem on the same ideas as the classical paper by Marchioro

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and Pulvirenti [21]. The convergence will be split in two parts. First, we proved in [6] for a fixed \( \delta \) the convergence
\[
\xi^{N,\delta} \rightarrow \xi^\delta, \quad \text{as } N \rightarrow +\infty
\]  
(8)

where \( \xi^{N,\delta} \) is defined in (6) and \( \xi^\delta \) is a solution to (7).

In what follows, we expand on that result, by showing that there exists a common time-interval, independent of \( \delta \), where there exist solutions of (4) and (7) and, where the convergence (8) takes place. The fundamental result to find this common interval is that (5) has a global in time solution for every \( \delta \). The second crucial ingredient for the proof of Theorem 13 is the convergence
\[
\xi^\delta \rightarrow \xi, \quad \text{as } \delta \rightarrow 0.
\]  
(9)

Hence, the convergence \( \xi^{N,\delta} \rightarrow \xi \) can be split in (8) and (9). However, the convergence in (8) is not uniform in \( \delta \). We keep this into account in the proof of Theorem 13 and we show that one can choose the regularization \( \delta \) depending on \( N \) such that the final convergence holds true. The rate of convergence of vortex filaments to solutions of Euler equations found in our main theorem has only a theoretical purpose; it depends on several approximations, sometimes in exponential way (like in the flow estimates). It depends also on the rate of convergence of initial conditions. In Sect. 4.5 we prove that given a smooth compact support initial condition for the 3D Euler equations, we may approximate it by a sequence of vortex filament currents; and the error of this approximation can be chosen arbitrarily small, to get an appropriate speed of convergence of solutions. However, the link between the two rates is far from being optimal. In view of numerical simulations, not addressed here, this link is a major and important problem that deserves investigation.

As far as the content of the paper, in Sect. 2 we briefly introduce the mathematical objects that we need in the remainder of the paper. In Sect. 3, the main results of our previous paper [6] are recalled. Moreover, a collection of refined estimates are given in Lemma 6. These estimates are necessary for the following section on mean field results for the true Euler equations (2). We begin Sect. 4 by recalling some known results on local classical solutions for the three dimensional Euler equations. We then proceed in proving the stability of the Euler equations under the regularization of the Biot–Savart Kernel \( K \), namely (9). We state and prove our main result in Theorem 13. We prove the mean field results both for the regularized and for the true 3D Euler equations. Finally, we address the approximation of the initial vorticity field of the true Euler equation by a sequence of vortex filaments currents to ensure that the set of approximations is not void.

2 Notations and Definitions

2.1 Spaces of Functions

We will often refer to \( \xi \) as a 1-current. Currents of dimension 1 (called 1-currents here) are linear continuous mappings on the space \( C_c^\infty(\mathbb{R}^d, \mathbb{R}^d) \) of smooth compact support vector fields of \( \mathbb{R}^d \), see for instance [13,15]. For a more in depth discussion about currents we refer to [6], especially Section 2 and the Appendix.

Given \( k, d, m \in \mathbb{N} \), we denote by \( C^k_b(\mathbb{R}^d, \mathbb{R}^m) \) the space of all functions \( f : \mathbb{R}^d \rightarrow \mathbb{R}^m \) that are of class \( C^k \), bounded together with their derivatives of order up to \( k \). By \( UC^3_b(\mathbb{R}^d, \mathbb{R}^m) \),...
we denote the subset of $C^3_b(\mathbb{R}^d, \mathbb{R}^m)$ of those functions $f$ such that $f$, $Df$ and $D^2f$ are also uniformly continuous.

We denote by $S(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions, defined as

$$S(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |x^\alpha f^{(\beta)}(x)| < +\infty \quad \forall \alpha, \beta \in \mathbb{Z}^d \right\}$$

### 2.2 Spaces of Measures

On the space $C_b(\mathbb{R}^3; \mathbb{R}^3) = C^0_b(\mathbb{R}^3; \mathbb{R}^3)$ of continuous and bounded vector fields on $\mathbb{R}^3$, denote the uniform topology by $\|\cdot\|_\infty$. Throughout the paper we shall always deal with the following Banach space of 1-currents:

$$\mathcal{M} := C^b_b(\mathbb{R}^3; \mathbb{R}^3)'$$

The topology induced by the duality will be denoted by $|\cdot|_\mathcal{M}$:

$$|\xi|_\mathcal{M} := \sup_{\|\theta\|_\infty \leq 1} |\xi(\theta)| .$$

As already seen in [6], to deal with approximation by filaments it is essential to consider the weak topology too

$$\|\xi\| = \sup\{\xi(\theta) \mid \|\theta\|_\infty + \text{Lip}(\theta) \leq 1\}$$

where Lip$(\theta)$ is the Lipschitz constant of $\theta \in C_b(\mathbb{R}^d, \mathbb{R}^d)$. We set

$$d\left(\xi, \tilde{\xi}\right) = \|\xi - \tilde{\xi}\|$$

for all $\xi, \tilde{\xi} \in \mathcal{M}$. The number $\|\xi\|$ is well defined and

$$\|\xi\| \leq |\xi|_\mathcal{M} ,$$

$d\left(\xi, \tilde{\xi}\right)$ satisfies the conditions of a distance. Convergence in the metric space $(\mathcal{M}, d)$ corresponds to weak convergence in $\mathcal{M}$ as dual to $C^b(\mathbb{R}^3, \mathbb{R}^3)$. We shall denote by $\mathcal{M}_w$ the space $\mathcal{M}$ endowed by the metric $d$. The unit ball in $(\mathcal{M}, \cdot|\mathcal{M})$ is complete with respect to $d$ (see [6] in the Appendix).

### 2.3 Push-Forward

Throughout the paper we will deal with the notion of pull-back and push-forward with respect to a differentiable function. Let $\theta \in C^b(\mathbb{R}^3, \mathbb{R}^3)$ be a vector field (test function) and $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be a map. When defined, the pull-back of $\theta$ is

$$(\varphi^*\theta)(x) = D\varphi(x)^T \vartheta(\varphi(x)) .$$

If $\varphi$ is of class $C^1(\mathbb{R}^3, \mathbb{R}^3)$, then $\varphi^*$ is a well defined bounded linear map from $C^b(\mathbb{R}^3, \mathbb{R}^3)$ to itself.

Given a 1-current $\xi \in \mathcal{M}$ and a smooth map $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, recall that the push-forward $\varphi_*\xi$ is defined as the current

$$(\varphi_*\xi)(\theta) := \xi(\varphi^*\theta) , \quad \theta \in C^b(\mathbb{R}^3, \mathbb{R}^3) .$$
3 Mollified Lagrangian Dynamics

In the previous paper [6], we analyzed the mean field of interacting filaments and the well posedness for the PDE (7) with initial condition $\xi_0 \in \mathcal{M}$. The relevant results of that paper are summarized in this section. Moreover, some more accurate estimates on the kernel and the flow are computed in order to get the mean field limit for the 3D Euler vorticity equation (2).

In order to prove that the nonlinear vector-valued PDE (7), with initial condition $\xi_0 \in \mathcal{M}$, has unique solutions in the space of currents, we adopt a Lagrangian point of view, i.e.: we examine the ordinary differential equation

$$\frac{d}{dt} X_t = (K^\delta \ast \xi_t)(X_t),$$  \hspace{1cm} (10)

consider the flow of diffeomorphisms $\phi^t, K^\delta \ast \xi$ generated by it and take the push forward of $\xi_0$ under this flow:

$$\xi_t = \phi^t_{\ast} K^\delta \ast \xi_0, \quad t \in [0, T].$$  \hspace{1cm} (11)

The pair of equations (10)–(11) defines a closed system for $(\xi_t)_{t \in [0, T]}$. We begin by giving the exact definition of the mollification $K^\delta$.

3.1 The Mollified Kernel

Let us now define how we obtain $K^\delta$ by smoothing the singular kernel $K$, defined in Eq. (3), by using a mollifier $\rho^\delta$. We denote by $K^\delta = \rho^\delta \ast K$ where $\rho^\delta = \delta^{-3} \rho(\frac{x}{\delta})$ where $\rho \in S(\mathbb{R}^3)$. We assume that $\hat{\rho}$, the Fourier transform of $\rho$, has compact support and that $\rho \geq 0$ and $\int_{\mathbb{R}^3} \rho(x) dx = 1$. As a consequence, the mollified kernel $K^\delta$ belongs to $C^\infty(\mathbb{R}^3, \mathbb{R}^3 \times \mathbb{R}^3)$ and its Fourier transform has compact support. For the remainder of the paper this definition of $K^\delta$ will be adopted. Let us mention, however, that the results of this section (see [6]) occur under a less restrictive assumption, more precisely under the assumption that $K^\delta \in UC^3_b(\mathbb{R}^3, \mathbb{R}^{3 \times 3})$.

If $\xi \in \mathcal{M}$ then $K^\delta \ast \xi$ is the vector field in $\mathbb{R}^3$ with $i$-component given by

$$\left(K^\delta \ast \xi\right)_i(x) = \left(K^\delta_{ij}(z)\right)(x \cdot \cdot) = \xi \left(K^\delta_{ij}(x \cdot \cdot)\right)$$

where $K^\delta_{ij}(z)$ is the vector $\left(K^\delta_{ij}(z)\right)_{j=1,2,3}$. We have

$$\left|\left(K^\delta \ast \xi\right)(x)\right| \leq \left\|\xi\right\|_{\mathcal{M}} \left\|K^\delta\right\|_{\infty}.\hspace{1cm}$$

Moreover, we also have

$$\left|\left(K^\delta \ast \xi\right)(x)\right| \leq \left\|\xi\right\| \left(\left\|K^\delta\right\|_{\infty} + \left\|DK^\delta\right\|_{\infty}\right).$$

3.2 Main Results for the Mollified Problem

In [6] the mean field limit to vortex filaments system (5)–(6) was analyzed. In this section we recall the main results from that paper.

**Theorem 1** (i) (Maximal solutions) For every $\xi_0 \in \mathcal{M}$, there is a unique maximal solution $\xi$ of the flow equations (10)–(11) in $C([0, T_{\xi_0}); \mathcal{M})$. 

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(ii) (Continuous dependence) If \( \xi^N_0 \to \xi_0 \) in \( \mathcal{M}_w \) and \([0, T_0]\) is a common time interval of existence and uniqueness for the flow equations (10)–(11) with initial conditions \( \xi^N_0 \) and \( \xi_0 \), then for the corresponding solutions \( \xi^N \) and \( \xi \) we have \( \xi^N \to \xi \) in \( C([0, T_0]; \mathcal{M}_w) \).

There is an equivalence between the Lagrangian formulation and the Eulerian formulation that is:

**Lemma 2** A function \( \xi \in C([0, T]; \mathcal{M}) \) is a current-valued solution for the PDE (7) if and only if it is given by

\[
\xi_t = \varphi^{t, K^\delta \ast \xi}_\mu \xi_0.
\]

The solution \( \xi \) has to be understood in the following sense:

**Definition 3** We say that \( \xi \in C([0, T]; \mathcal{M}) \) is a current-valued solution for the PDE (7) if for every \( \theta \in C^1_b(\mathbb{R}^3; \mathbb{R}^3) \) and every \( t \in [0, T] \), it satisfies

\[
\xi_t(\theta) - \int_0^t \xi_s \left( D\theta \cdot (K^\delta \ast \xi_s) \right) ds = \xi_0(\theta) + \int_0^t \xi_s \left( (D K^\delta \ast \xi_s)^T \cdot \theta \right) ds.
\]

Notice that the previous one is the weak formulation of (7), since \( K^\delta \) is divergence free.

**Theorem 4** (Global solutions) Let \( \xi_0 \in \mathcal{M} \) and assume that \( \xi \) is the maximal solution for the flow equations (10)–(11) as given by Theorem 1. If \( \xi_0 \) has a compact support, then for every \( t \in [0, T] \)

\[
\|K^\delta \ast \xi_t\|_{L^2} = \|K^\delta \ast \xi_0\|_{L^2} \tag{12}
\]

and

\[
\|K^\delta \ast \xi_t\|_{L^2} \leq C_1 \|\xi_0\|_{\mathcal{M}} e^{T\|K^\delta \ast \xi_0\|_{L^2}} \tag{13}
\]

As a consequence, we deduce that the solutions of (11) are global in time.

**Remark 5** For the general formulation of Theorems 1 and 4 the reader can refer to [6]. We would like to stress that it is necessary in Theorem 4 that the Fourier transform of \( K^\delta \) has compact support and \( \text{div}(K^\delta) = 0 \).

### 3.3 Properties of the Flow of Diffeomorphism

Let us denote by \( C^m_\delta := \|D^m K^\delta\|_\infty + \|D^{m+1} K^\delta\|_\infty \approx \frac{1}{\delta^{1+m}} \). By using the properties of the Kernel \( K^\delta \) and the estimates (12) and (13), it is not difficult to deduce that the following properties hold for every \( \xi_t, \tilde{\xi}_t \in \mathcal{M} \)

\[
\|K^\delta \ast \xi_t\|_{L^2} \leq C_0 \|\xi_0\|_{\mathcal{M}} \tag{14}
\]

\[
\|D K^\delta \ast \xi_t\|_{L^2} \leq C_1 \|\xi_0\|_{\mathcal{M}} \tag{15}
\]

\[
\|D^2 K^\delta \ast \xi_t\|_{L^2} \leq C_2 \|\xi_0\|_{\mathcal{M}} \tag{16}
\]

\[
\|K^\delta \ast \xi_t - K^\delta \ast \tilde{\xi}_t\|_{L^2} \leq C_3 \|\xi_t - \tilde{\xi}_t\| \tag{17}
\]

\[
\|D K^\delta \ast \xi_t - D K^\delta \ast \tilde{\xi}_t\|_{L^2} \leq C_4 \|\xi_t - \tilde{\xi}_t\| \tag{18}
\]

The flow \( \varphi^{t, K^\delta \ast \xi}_\mu \) associated to the ODE (10) satisfies the following properties.
Lemma 6 If $\xi \in C ([0, T]; \mathcal{M}_w)$, then the flow $\varphi^{t, K^\delta \ast \tilde{\xi}} : \mathbb{R}^3 \to \mathbb{R}^3$ is twice differentiable and satisfies, for all $t \in [0, T]$,

$$
\left\| D\varphi^{t, K^\delta \ast \tilde{\xi}} \right\|_{\infty} \leq e^{C^1_\delta T \| \xi_0 \|}
$$

(19)

$$
\left\| \varphi^{t, K^\delta \ast \tilde{\xi}} - \varphi^{t, K^\delta \ast \tilde{\xi}} \right\|_{\infty} \leq C^1_\delta \| \xi_0 \| e^{2C^1_\delta T \| \xi_0 \|} \int_0^T \| \xi_s - \tilde{\xi}_s \| ds
$$

(20)

and for every $x \in \mathbb{R}^3$

$$
\left\| D\varphi^{t, K^\delta \ast \tilde{\xi}} - D\varphi^{t, K^\delta \ast \tilde{\xi}} \right\|_{\infty} \leq C^2_\delta \| \xi_0 \| e^{2C^1_\delta T \| \xi_0 \|} \int_0^T \| \xi_s - \tilde{\xi} \| ds.
$$

(21)

Moreover, for every $x, y \in \mathbb{R}^3$,

$$
| D\varphi^{t, K^\delta \ast \tilde{\xi}} (x) - D\varphi^{t, K^\delta \ast \tilde{\xi}} (y) | \leq C^2_\delta T \| \xi_0 \| e^{2C^1_\delta (2T) \| \xi_0 \|} |x - y|.
$$

(22)

Proof See [6] for a proof in a more general setting but for the sake of completeness, we will perform the estimates in details.

Since the kernel $K^\delta$ is regular, then the flow map $\varphi^{t, K^\delta \ast \tilde{\xi}}$ is differentiable and we have that

$$
\frac{d}{dt} D\varphi^{t, K^\delta \ast \tilde{\xi}} (x) = D(K^\delta \ast \xi) \big( \varphi^{t, K^\delta \ast \tilde{\xi}} (x) \big) D\varphi^{t, K^\delta \ast \tilde{\xi}} (x).
$$

Hence,

$$
\left\| D\varphi^{t, K^\delta \ast \tilde{\xi}} (x) \right\| \leq e^{\int_0^t \| D(K^\delta \ast \xi) \| ds}
$$

Now, using the assumption (15) we get (19).

For the estimate (20), notice that

$$
\frac{d}{dt} \left( \varphi^{t, K^\delta \ast \tilde{\xi}} (x) - \varphi^{t, K^\delta \ast \tilde{\xi}} (x) \right) = K^\delta \ast \xi \left( \varphi^{t, K^\delta \ast \tilde{\xi}} (x) \right) - K^\delta \ast \xi \left( \varphi^{t, K^\delta \ast \tilde{\xi}} (x) \right)
$$

$$
= K^\delta \ast \xi \left( \varphi^{t, K^\delta \ast \tilde{\xi}} (x) \right) - K^\delta \ast \xi \left( \varphi^{t, K^\delta \ast \tilde{\xi}} (x) \right)
$$

$$
+ K^\delta \ast \xi \left( \varphi^{t, K^\delta \ast \tilde{\xi}} (x) \right) - K^\delta \ast \xi \left( \varphi^{t, K^\delta \ast \tilde{\xi}} (x) \right).
$$

Hence

$$
\left| \varphi^{t, K^\delta \ast \tilde{\xi}} (x) - \varphi^{t, K^\delta \ast \tilde{\xi}} (x) \right| \leq \int_0^t \left\| D(K^\delta \ast \xi) \| ds \left| \varphi^{t, K^\delta \ast \tilde{\xi}} (x) - \varphi^{t, K^\delta \ast \tilde{\xi}} (x) \right| ds
$$

$$
+ \int_0^t \| K^\delta \ast \xi_s - K^\delta \ast \xi_s \| \| D(K^\delta \ast \xi) \| ds.
$$

Thus, using Gronwall’s Lemma we get that

$$
\left| \varphi^{t, K^\delta \ast \tilde{\xi}} (x) - \varphi^{t, K^\delta \ast \tilde{\xi}} (x) \right| \leq \int_0^t \| K^\delta \ast \xi_s - K^\delta \ast \xi_s \| e^{\int_0^t \| D(K^\delta \ast \xi) \| ds} ds.
$$

Now, using again assumptions (15) and (17), we deduce (20).
In order to prove (21). Let us notice that
\[
\frac{d}{dt} \left( D\varphi^t, K^\delta \ast \xi (x) - D\varphi^t, K^\delta \ast \tilde{\xi} (x) \right) = D K^\delta \ast \xi_t (\varphi^t, K^\delta \ast \xi (x)) D\varphi^t, K^\delta \ast \xi (x) - D K^\delta \ast \xi_t (\varphi^t, K^\delta \ast \tilde{\xi} (x)) D\varphi^t, K^\delta \ast \tilde{\xi} (x)
\]
\[
= D K^\delta \ast \xi_t (\varphi^t, K^\delta \ast \xi (x)) D\varphi^t, K^\delta \ast \xi (x) - D K^\delta \ast \xi_t (\varphi^t, K^\delta \ast \xi (x)) D\varphi^t, K^\delta \ast \xi (x) + D K^\delta \ast \xi_t (\varphi^t, K^\delta \ast \tilde{\xi} (x)) D\varphi^t, K^\delta \ast \tilde{\xi} (x) - D K^\delta \ast \xi_t (\varphi^t, K^\delta \ast \tilde{\xi} (x)) D\varphi^t, K^\delta \ast \tilde{\xi} (x).
\]

Now, following very similar arguments to the ones used above, and by using the estimates (14), (18), (19) and (20), we get (21).

It is left to prove (22).
\[
\left| D\varphi^t, K^\delta \ast \xi (x) - D\varphi^t, K^\delta \ast \xi (y) \right| \leq \int_0^t \left| D K^\delta \ast \xi_s (\varphi^t, K^\delta \ast \xi (x)) D\varphi^t, K^\delta \ast \xi (x) - D K^\delta \ast \xi_s (\varphi^t, K^\delta \ast \xi (y)) D\varphi^t, K^\delta \ast \xi (y) \right| ds
\]
\[
\leq \int_0^t \left( D K^\delta \ast \xi_s (\varphi^s, K^\delta \ast \xi (x)) D\varphi^s, K^\delta \ast \xi (x) - D K^\delta \ast \xi_s (\varphi^s, K^\delta \ast \xi (y)) D\varphi^s, K^\delta \ast \xi (y) \right) ds
\]
\[
+ \sup_{s \in [0, t]} \left| D K^\delta \ast \xi_s (\varphi^s, K^\delta \ast \xi (x)) - D K^\delta \ast \xi_s (\varphi^s, K^\delta \ast \xi (y)) \right| ds
\]
\[
+ t \sup_{s \in [0, t]} \left( \left\| D\varphi^s, K^\delta \ast \xi \right\|_\infty^2 \left\| D^2 K^\delta \ast \xi \right\|_\infty \right) \left| x - y \right|.
\]

We now apply Gronwall’s Lemma and we get
\[
\left| D\varphi^t, K^\delta \ast \xi (x) - D\varphi^t, K^\delta \ast \xi (y) \right| \leq T \sup_{s \in [0, T]} \left( \left\| D\varphi^s, K^\delta \ast \xi \right\|_\infty^2 \left\| D^2 K^\delta \ast \xi \right\|_\infty \right) e^{T \sup_{s \in [0, T]} \left\| D K^\delta \ast \xi \right\|_\infty} \left| x - y \right|.
\]

Now, using (15), (16) and (19) we get (22).

\[\square\]

4 Mean Field Result for the 3D Euler Equations

Now, we are able to prove a mean field result for the 3D Euler equations. We have to do it in two steps. First we mollify the 3D Euler equations through the mollification of the Kernel \( K \). We approximate the vorticity \( \xi \) by the mollified one \( \xi^\delta \) and we use the mean field result for the vorticity \( \xi^\delta \) proved in [6] and we combine the two pieces. In what follows, \( H^s (\mathbb{R}^3 : \mathbb{R}^3) \) is the Sobolev space of vector valued functions, we denote by \( \| \cdot \|_{H^s} \) the associated norm. In particular for \( s = 0 \), \( H^0 (\mathbb{R}^3, \mathbb{R}^3) \) reduces to the Lebesgue space \( L^2 (\mathbb{R}^3, \mathbb{R}^3) \).

4.1 Local Posedness for the Euler Equations

It is known that (1) has local (in time) classical solutions:
Theorem 7 Let $v_0 \in H^m(\mathbb{R}^3, \mathbb{R}^3)$ with $m > 5/2$ such that $\text{div} \ v_0 = 0$. Then, there exists $T_0 > 0$ and there exists a unique $v \in C([0, T]; H^m(\mathbb{R}^3, \mathbb{R}^3))$ solution of (1).

The proof of this theorem can be performed through a fixed point argument and can be found for example in [7] and also in [20]. We will show here only how to get the estimate of the vorticity field in $H^{m-1}(\mathbb{R}^3, \mathbb{R}^3)$. We define the linear operator $\Lambda^s = \left(\frac{-\Delta}{\Lambda^s}\right)^{1/2}$ and its powers $\Lambda^s$. Hence $\Lambda^2 = -\Delta$. Note, in particular that $\Lambda^s$ maps $H^r(\mathbb{R}^3, \mathbb{R}^3)$ into $H^{r-s}(\mathbb{R}^3, \mathbb{R}^3)$.

For any $u, v, w \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ with $\nabla \cdot u = 0$ we have
\[
\langle u \cdot \nabla v, w \rangle = -\langle u \cdot \nabla w, v \rangle, \quad \langle u \cdot \nabla v, v \rangle = 0.
\] (23)

Define the commutator $[\Lambda^s, f]g = \Lambda^s(fg) - f \Lambda^s g$.

From [14] we have the following two lemmas.

Lemma 8 (Commutator lemma) Let $s > 0$, $1 < p < \infty$ and $p_2, p_3 \in (1, \infty)$ be such that
\[
\frac{1}{p} \geq \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{p} \geq \frac{1}{p_3} + \frac{1}{p_4}.
\]
Then
\[
\|[\Lambda^s, f]g\|_{L^p} \leq C \left(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}\right).
\]

Lemma 9 Let $s > \frac{3}{2}$ then $H^s(\mathbb{R}^3, \mathbb{R}^3)$ is a Banach algebra, that is there is a $c > 0$ such that for all $f, g \in H^s(\mathbb{R}^3)$
\[
\|\Lambda^s(fg)\|_{H^0} \leq C \|f\|_{H^s} \|g\|_{H^s}.
\]

In particular for $u, v \in \mathbb{R}^3$,
\[
[\Lambda^s, u] \cdot \nabla v = \Lambda^s((u \cdot \nabla)v) - (u \cdot \nabla)\Lambda^s v.
\]

Hence,
\[
\langle \Lambda^s((u \cdot \nabla)v), \Lambda^s v \rangle = \langle [\Lambda^s, u] \cdot \nabla v, \Lambda^s v \rangle + \langle (u \cdot \nabla)\Lambda^s v, \Lambda^s v \rangle = 0 \text{ by (23)}
\] (24)

Now the estimate on the vorticity field in $H^{m-1}(\mathbb{R}^3, \mathbb{R}^3)$ can be proved as in the proof of Theorem 10 below.

4.2 The Mollified 3D Euler Equations

We are interested in the 3D Euler equations in their vorticity formulation. Consider the vector field $\xi^\delta$ solution of
\[
\begin{aligned}
\partial_t \xi^\delta + (v^\delta \cdot \nabla)\xi^\delta &= (\xi^\delta \cdot \nabla)v^\delta \\
\nabla \cdot \xi^\delta &= 0, \\
\xi^\delta(0, x) &= \xi^\delta_0(x)
\end{aligned}
\] (25)
where
\[
v^\delta(t, x) = (K^\delta * \xi^\delta)(t, x),
\] (26)
Theorem 10 Let $\xi_0^\delta \in H^s(\mathbb{R}^3, \mathbb{R}^3)$ with $s > 3/2$. Then, there exists $T_0 > 0$, independent of $\delta$ and there exists a unique $\xi^\delta \in C([0, T]; H^s(\mathbb{R}^3, \mathbb{R}^3))$ solution of (25).

Proof Here we only prove that there exist $T_0, C > 0$, independent of $\delta$, such that $\sup_{0 \leq t \leq T_0} \|\xi^\delta(t)\|_{H^s} \leq C$.

Let us apply the operator $\Lambda^s$ to (25) and then multiply by $\Lambda^s \xi^\delta(t)$. By using (24) we obtain that

$$
\frac{1}{2} \frac{d}{dt} \|\Lambda^s \xi^\delta(t)\|_{H^0}^2 \leq \|\Lambda^s \xi^\delta(t)\|_{H^0} \|\Lambda^s \xi^\delta(t)\|_{H^0} + \|\Lambda^s \xi^\delta(t)\|_{H^0} \|\Lambda^s \xi^\delta(t)\|_{H^0}
$$

By using Lemma 8 with $p = p_2 = 2$, $p_1 = \infty$ and $p_3 = 6$ and $p_4 = 3$ and also Lemma 9, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|\Lambda^s \xi^\delta(t)\|_{H^0}^2 \leq \left( \|\nabla v^\delta \|_{L^\infty} \|\Lambda^s \xi^\delta(t)\|_{H^0} + \|\Lambda^s v^\delta \|_{L^6} \|\Lambda^s \xi^\delta(t)\|_{L^3} \right) \|\Lambda^s \xi^\delta(t)\|_{H^0}
$$

Since $s > \frac{3}{2}$, $H^s(\mathbb{R}^3, \mathbb{R}^3) \subset L^\infty(\mathbb{R}^3, \mathbb{R}^3)$, $H^1(\mathbb{R}^3, \mathbb{R}^3) \subset L^6(\mathbb{R}^3, \mathbb{R}^3)$ and $H^{1/2}(\mathbb{R}^3, \mathbb{R}^3) \subset L^3(\mathbb{R}^3, \mathbb{R}^3)$, then up to a constant that we omit, we have $\|\Lambda \xi^\delta(t)\|_{L^3} \leq \|\Lambda \xi^\delta(t)\|_{H^{1/2}} \leq \|\Lambda^s \xi^\delta(t)\|_0$, and $\|\nabla v^\delta \|_{L^\infty} \leq \|\Lambda^s v^\delta \|_{H^0}$; moreover, $\|\Lambda^s v^\delta \|_{L^6} \leq \|\Lambda^s v^\delta \|_{H^0}$.

Hence the previous sum of terms is bounded above (up to a constant) by

$$
\frac{1}{2} \frac{d}{dt} \|\Lambda^s \xi^\delta(t)\|_{H^0}^2 \leq C \|\Lambda^s v^\delta \|_{H^0} \|\Lambda^s \xi^\delta(t)\|_{H^0}^2
$$

Now, by using the identity (26), the definition of $K^\delta$, the properties of the convolution and the definition of $\rho$, we have

$$
\|\Lambda^s v^\delta \|_{H^0} = \|\Lambda^s (K^\delta * \xi^\delta)(t)\|_{H^0} = \|\Lambda^s (\rho^\delta * \rho^\delta)(t)\|_{H^0} \leq \|\rho^\delta * \Lambda^s v^\delta(t)\|_{H^0} \leq \|\Lambda^s (K * \xi^\delta)(t)\|_{H^0} \leq C \|\Lambda^s \xi^\delta(t)\|_{H^0}
$$

(27)

Let us mention that in the above inequality, we take into account that the linear map

$$
\xi \mapsto K * \xi
$$

is continuous from $H^s(\mathbb{R}^3, \mathbb{R}^3)$ into $H^{s+1}(\mathbb{R}^3, \mathbb{R}^3)$ and that the linear map

$$
g \mapsto \rho^\delta * g
$$

is equibounded in $\delta$ from $H^{s+1}(\mathbb{R}^3, \mathbb{R}^3)$ into $H^{s+1}(\mathbb{R}^3, \mathbb{R}^3)$.

By plugging inequality (27) in the previous estimate we get that

$$
\frac{d}{dt} \|\Lambda^s \xi^\delta(t)\|_{H^0}^2 \leq C \|\Lambda^s \xi^\delta(t)\|_{H^0}^3
$$

which implies that

$$
\frac{d}{dt} \|\Lambda^s \xi^\delta(t)\|_{H^0} \leq C \|\Lambda^s \xi^\delta(t)\|_{H^0}^2.
$$
then, after integrating
\[
\frac{1}{\| \Lambda^s \xi^\delta (0) \|_{H^0}} - \frac{1}{\| \Lambda^s \xi^\delta (t) \|_{H^0}} \leq Ct
\]

Hence, if we assume that \( T_0 < \frac{1}{C \| \xi^\delta (0) \|_{H^s}} \), then we have the estimate
\[
\sup_{0 \leq t \leq T_0} \| \xi^\delta (t) \|_{H^s}^2 \leq C.
\]

\[\square\]

4.3 Stability for the Regularized Solution

Now we are able to state the convergence of \( \xi^\delta \) to \( \xi \). We have from the previous section that
\[
\sup_{t \in [0, T_0]} \| \xi^\delta (t) \|_{H^s} \leq C
\]
for every \( \delta \in [0, 1) \).

**Theorem 11** If \( \xi^\delta (0) \rightarrow \xi (0) \) in \( L^2 (\mathbb{R}^3, \mathbb{R}^3) \) then \( \xi^\delta \rightarrow \xi \) in \( C \left( [0, T_0]; L^2 (\mathbb{R}^3, \mathbb{R}^3) \right) \), as \( \delta \rightarrow 0 \).

*In addition, if \( \| \xi^\delta (0) - \xi (0) \| \rightarrow 0 \), then \( \sup_{t \in [0, T]} \| \xi^\delta - \xi \| \rightarrow 0 \).*

**Proof** We have that
\[
\partial_t \left( \xi^\delta - \xi \right) + (v^\delta \cdot \nabla) \xi^\delta - (\xi \cdot \nabla) v = (\xi^\delta \cdot \nabla) v^\delta - (\xi \cdot \nabla) v.
\]
Hence
\[
\partial_t \left( \xi^\delta - \xi \right) + (v^\delta \cdot \nabla) \left( \xi^\delta - \xi \right) + (v^\delta - v) \cdot \nabla \xi = (\xi^\delta \cdot \nabla) (v^\delta - v) + ((\xi^\delta - \xi) \cdot \nabla) v.
\]
This implies, by using (23), that
\[
\frac{1}{2} \frac{d}{dt} \| \xi^\delta - \xi \|^2_{L^2(\mathbb{R}^3)} = - \int (v^\delta - v) \cdot \nabla \xi \cdot (\xi^\delta - \xi) \, dx + \int (\xi^\delta \cdot \nabla) (v^\delta - v) \cdot (\xi^\delta - \xi) \, dx + \int ((\xi^\delta - \xi) \cdot \nabla) v \cdot (\xi^\delta - \xi) \, dx \leq \| D\xi \|_{L^3} \| v^\delta - v \|_{L^6} \| \xi^\delta - \xi \|_{L^2} + \| \xi^\delta \|_\infty \| D (v^\delta - v) \|_{L^2} \| \xi^\delta - \xi \|_{L^2} + \| Dv \|_\infty \| \xi^\delta - \xi \|^2_{L^2}.
\]

Since \( H^s (\mathbb{R}^3, \mathbb{R}^3) \subset W^{1, 3} (\mathbb{R}^3, \mathbb{R}^3) \)
\[
\| D\xi \|_{L^3} \leq C \| \xi \|_{H^s} \leq C.
\]
Moreover, since $H^1(\mathbb{R}^3, \mathbb{R}^3) \subset L^6(\mathbb{R}^3, \mathbb{R}^3)$
\[
\|v^\delta - v\|_{L^6} \leq \|K^\delta * (\xi^\delta - \xi)\|_{H^1} + \|K^\delta * \xi - K * \xi\|_{H^1} \\
= \|\rho^\delta * K * (\xi^\delta - \xi)\|_{H^1} + \|\rho^\delta * v - v\|_{H^1} \\
\leq C \|K * (\xi^\delta - \xi)\|_{H^1} + \|\rho^\delta * v - v\|_{H^1} \\
\leq C \|\xi^\delta - \xi\|_{L^2} + \|\rho^\delta * v - v\|_{H^1} 
\]

In the same way we can estimate the term with the first order derivative,
\[
\|D (v^\delta - v)\|_{L^2} \leq \|DK^\delta * (\xi^\delta - \xi)\|_{L^2} + \|DK^\delta * \xi - DK * \xi\|_{L^2} \\
= \|\rho^\delta * DK * (\xi^\delta - \xi)\|_{L^2} + \|\rho^\delta * Dv - Dv\|_{L^2} \\
\leq C \|DK * (\xi^\delta - \xi)\|_{L^2} + \|\rho^\delta * Dv - Dv\|_{L^2} \\
\leq C \|\xi^\delta - \xi\|_{L^2} + \|\rho^\delta * Dv - Dv\|_{L^2} .
\]

Now, by using $\|\xi^\delta\|_{L^\infty} \leq C \|\xi\|_{H^1} \leq C$, and $\|Dv\|_{L^\infty} \leq C$, we get that
\[
\frac{1}{2} \frac{d}{dt} \|\xi^\delta - \xi\|_{L^2}^2 \leq C \left(\|\xi^\delta - \xi\|_{L^2} + \|\rho^\delta * v - v\|_{H^1}\right) \|\xi^\delta - \xi\|_{L^2} \\
+ C \left(\|\xi^\delta - \xi\|_{L^2} + \|\rho^\delta * Dv - Dv\|_{L^2}\right) \|\xi^\delta - \xi\|_{L^2} \\
+ C \|\xi^\delta - \xi\|_{L^2} \\
\leq C \|\xi^\delta - \xi\|_{L^2}^2 + C \|\rho^\delta * v - v\|_{H^1}^2 + C \|\rho^\delta * Dv - Dv\|_{L^2}^2 .
\]

By using Gronwall’s lemma and the fact that $\|\rho^\delta * v - v\|_{H^1}^2 \to 0$, $\|\rho^\delta * Dv - Dv\|_{L^2}^2 \to 0$ uniformly in $t$ completes the first part of the proof.

Finally we prove the second part of the theorem. Take a vector field $\theta$ such that $\|\theta\|_{\infty} + \text{Lip}(\theta) \leq 1$. From the identity
\[
\langle \xi^\delta (t) - \xi (t) , \theta \rangle = \langle \xi^\delta (0) - \xi (0) , \theta \rangle \\
+ \int_0^t \langle \xi^\delta (s) - \xi (s) , v^\delta (s) \cdot \nabla \theta \rangle \, ds \\
+ \int_0^t \langle \xi (s) , (v^\delta (s) - v (s)) \cdot \nabla \theta \rangle \, ds \\
- \int_0^t \langle v^\delta (s) - v (s) , \xi^\delta (s) \cdot \nabla \theta \rangle \, ds \\
- \int_0^t \langle v (s) , (\xi^\delta (s) - \xi (s)) \cdot \nabla \theta \rangle \, ds
\]
we deduce
\[
\|\xi^\delta (t) - \xi (t) , \theta \| \leq \|\xi^\delta (0) - \xi (0)\| \\
+ \int_0^t \left(\|v (s)\|_{L^2} + \|v^\delta (s)\|_{L^2}\right) \|\xi^\delta (s) - \xi (s)\|_{L^2} \, ds \\
+ \int_0^t \left(\|\xi (s)\|_{L^2} + \|\xi^\delta (s)\|_{L^2}\right) \|v^\delta (s) - v (s)\|_{L^2} \, ds.
\]
This implies \( \sup_{t \in [0, T]} \| \xi^\delta_t - \xi^\delta \| \to 0 \), using the convergence in \( L^2 \) proved in the first part of the theorem.

\[ \square \]

**Lemma 12** Given \( \delta, R > 0 \) there exists a constant \( C_{\delta, R} > 0 \) with the following property. If \( \xi_0, \tilde{\xi}_0 \) satisfy \( |\xi_0|_{\mathcal{A}} \leq R, |\tilde{\xi}_0|_{\mathcal{A}} \leq R \), and \( \xi^\delta_t, \tilde{\xi}^\delta_t \) are the corresponding solutions of equation (4) on \([0, T]\), then, globally in time we have

\[
\sup_{t \in [0, T]} \| \xi^\delta_t - \tilde{\xi}^\delta_t \| \leq C_{\delta, R} \| \xi_0 - \tilde{\xi}_0 \|.
\]

**Proof** Let \( t \in [0, T] \). We have

\[
\left| \xi^\delta_t - \xi^\delta \right| = \left| \varphi^t_{\delta} \xi_0 (\theta) - \varphi^t_{\delta} \tilde{\xi}_0 (\theta) \right|
\leq \varphi^t_{\delta} \left( - \xi_0 (\theta) + \left( \xi^\delta_t - \xi^\delta \right) - \left( \tilde{\xi}^\delta_t - \tilde{\xi}^\delta \right) \right) + \left| \varphi^t_{\delta} \xi^\delta_t (\theta) - \varphi^t_{\delta} \tilde{\xi}^\delta_t (\theta) \right|
\leq \left( \xi_0 - \tilde{\xi}_0 \right) \left( \varphi^t_{\delta} \xi^\delta_t (\theta) - \varphi^t_{\delta} \tilde{\xi}^\delta_t (\theta) \right) + \left\| \varphi^t_{\delta} \xi_0 (\theta) - \varphi^t_{\delta} \tilde{\xi}_0 (\theta) \right\|.
\]

Now, from the definition of push-forward and (19) we infer that

\[
\| \varphi^t_{\delta} \xi^\delta_t (\theta) \| = \| D\varphi^t_{\delta} \xi^\delta_t (\theta) (\varphi^t_{\delta} \xi^\delta_t (\theta)) \| \leq \| D\varphi^t_{\delta} \xi^\delta_t (\theta) \| \| \theta \| \leq e^{C_{\delta, T}} \| \xi_0 \| \| \theta \|.
\]

On the other hand,

\[
\left| D\varphi^t_{\delta} \xi^\delta_t (x) T (\varphi^t_{\delta} \xi^\delta_t (x)) - D\varphi^t_{\delta} \xi^\delta_t (y) T (\varphi^t_{\delta} \xi^\delta_t (y)) \right|
\leq \left| D\varphi^t_{\delta} \xi^\delta_t (x) T (\varphi^t_{\delta} \xi^\delta_t (y)) - D\varphi^t_{\delta} \xi^\delta_t (y) T (\varphi^t_{\delta} \xi^\delta_t (x)) \right|
+ \left| D\varphi^t_{\delta} \xi^\delta_t (y) T (\varphi^t_{\delta} \xi^\delta_t (x)) - D\varphi^t_{\delta} \xi^\delta_t (x) T (\varphi^t_{\delta} \xi^\delta_t (y)) \right|
\leq \| \theta \| \left( \| D\varphi^t_{\delta} \xi^\delta_t (x) - D\varphi^t_{\delta} \xi^\delta_t (y) \| + \| \theta \| \| D\varphi^t_{\delta} \xi^\delta_t (x) \| \| x - y \| \right).
\]

Hence, by using (19) and (22), we infer that

\[
\| \varphi^t_{\delta} \xi^\delta_t (\theta) \| \leq TC_{\delta}^2 \| \xi_0 \| e^{2C_{\delta} T} \| \xi_0 \| \| \theta \| + e^{C_{\delta} T} \| \tilde{\xi}_0 \| \| \theta \|.
\]

Moreover,

\[
\| \varphi^t_{\delta} \xi^\delta_t - \varphi^t_{\delta} \tilde{\xi}^\delta_t \| \leq \| D\varphi^t_{\delta} \xi^\delta_t (\theta) \| \| \theta \| \leq e^{C_{\delta} T} \| \tilde{\xi}_0 \| \| \theta \|.
\]

By using (19), (21) and (20) we get that

\[
\| \varphi^t_{\delta} \xi^\delta_t - \varphi^t_{\delta} \tilde{\xi}^\delta_t \| \leq \| \xi_0 \| e^{C_{\delta} T} \| \xi_0 \| \| \theta \| + e^{C_{\delta} T} \| \tilde{\xi}_0 \| \| \theta \|.
\]

\[ \wedge \]
By collecting all these estimates, we get that
\[
\|\xi_{t}^{\delta} - \tilde{\xi}_{t}^{\delta}\| \leq \|\xi_{0} - \tilde{\xi}_{0}\| (T C_{0}^{2} \|\xi_{0}\| + 2) e^{2C_{1}^{T} \|\xi_{0}\|} + \left(\left|\xi_{0}\right|_{M} \|\xi_{0}\| \int_{0}^{T} \|\xi_{s}^{\delta} - \tilde{\xi}_{s}^{\delta}\| ds\right) \\
\times \left( C_{0}^{1} \sqrt{e^{2T} C_{0}^{1} \|\xi_{0}\|} + C_{0}^{2} e^{C_{1}^{T} (\|\xi_{0}\| + \|\tilde{\xi}_{0}\|)} \left(1 + C_{K}^{0} C_{0}^{1} T \|\xi_{0}\| e^{C_{1}^{T} \|\xi_{0}\|}\right)\right)
\]

By using \(\xi_{0}\) and \(\tilde{\xi}_{0}\) we get that
\[
\|\xi_{t}^{\delta} - \tilde{\xi}_{t}^{\delta}\| \leq \|\xi_{0} - \tilde{\xi}_{0}\| (T C_{0}^{2} R + 2) e^{2C_{1}^{T} R} + \left( R^{2} \int_{0}^{T} \|\xi_{s}^{\delta} - \tilde{\xi}_{s}^{\delta}\| ds\right) \\
\times \left( C_{0}^{1} C_{0}^{1} + C_{0}^{2} \left(1 + C_{K}^{0} C_{0}^{1} T R e^{C_{1}^{T} R}\right)\right) e^{2T C_{1}^{R}}
\]
Set
\[
C^{*} := R^{2} \left( C_{0}^{1} C_{0}^{1} + C_{0}^{2} \left(1 + C_{K}^{0} C_{0}^{1} T R e^{C_{1}^{T} R}\right)\right) e^{2T C_{1}^{R}}
\]
and
\[
C_{*} := (T C_{0}^{2} R + 2) e^{2C_{1}^{T} R}
\]
Hence, by using Gronwall lemma we deduce that
\[
\sup_{t \in [0,T]} \|\xi_{t}^{\delta} - \tilde{\xi}_{t}^{\delta}\| \leq C_{*} \|\xi_{0} - \tilde{\xi}_{0}\| e^{T C^{*}}
\]
and this completes the proof by setting \(C_{\delta,R} := C_{*} e^{T C^{*}}\). \(\square\)

**4.4 Mean Field Result**

**Theorem 13** Let \(\xi_{0}^{N}\) be a sequence of currents and \(\xi_{0} \in H^{s}\), \(s > 3/2\), such that
\[
\lim_{N \to \infty} \|\xi_{0}^{N} - \xi_{0}\| = 0.
\]
Let \(\xi_{t}\) be the solution in \(H^{s}\) of equation (2) on the interval \([0, T_{0}]\) with initial condition \(\xi_{0}\). For every \(\delta > 0\) and positive integer \(N\), let \(\xi_{t}^{N,\delta}\) satisfy (6) and (5) on the interval \([0, T]\) with initial condition \(\xi_{0}^{N}\). Let \(R > 0\) be such that \(\|\xi_{0}\|_{M} \leq R\), \(\|\xi_{0}^{N}\|_{M} \leq R\), and let \(C_{\delta,R} > 0\) be the corresponding constant of Lemma 12. Let \(\delta_{N} \to 0\) be a sequence. If
\[
\lim_{N \to \infty} C_{\delta_{N},R} \|\xi_{0}^{N} - \xi_{0}\| = 0
\]
then
\[
\lim_{N \to \infty} \sup_{t \in [0,T]} \|\xi_{t}^{N,\delta_{N}} - \xi_{t}\| = 0.
\]

**Proof** Denote by \(\xi_{t}^{\delta}\) the solution on \([0, T]\) of Eq. (7) with initial condition \(\xi_{0}\). We have
\[
\|\xi_{t}^{N,\delta_{N}} - \xi_{t}\| \leq \|\xi_{t}^{N,\delta_{N}} - \xi_{t}^{\delta_{N}}\| + \|\xi_{t}^{\delta_{N}} - \xi_{t}\| \\
\leq C_{\delta_{N},R} \|\xi_{0}^{N} - \xi_{0}\| + \|\xi_{t}^{\delta_{N}} - \xi_{t}\|.
\]
Recall that \(\lim_{N \to \infty} \sup_{t \in [0,T]} \|\xi_{t}^{\delta_{N}} - \xi_{t}\| = 0\), from Theorem 11.
Then \(\lim_{N \to \infty} \sup_{t \in [0,T]} \|\xi_{t}^{N,\delta_{N}} - \xi_{t}\| = 0\). \(\square\)
4.5 Vector Fields as a Continuum of Curves

In this section we show a method to construct an approximation $\xi^N_0$ of a vector field $\xi_0$ which satisfies the assumptions of Theorem 13.

We will only consider the case of compact support, solenoidal (i.e. divergence free) vector-fields. However, using the same argument as in [17], Appendix A, one can show that any vector field can be approximated by compact-support, solenoidal vector fields. This shows that the assumptions of the lemma below are not restrictive and the lemma can be generalized to any solenoidal vector-field.

**Lemma 14** Let $\xi$ be a solenoidal vector field in $\mathbb{R}^3$, i.e. such that $\nabla \cdot \xi = 0$. Further assume that $\xi$ has compact support $K \subset \mathbb{R}^3$ and it is Lipschitz continuous. There exist curves $\gamma^{i,N} : [0, 1] \to \mathbb{R}^3$ and weights $\alpha^{i,N} \in \mathbb{R}$ such that,

$$\lim_{N \to \infty} \left\| \sum_{i=1}^N \alpha^{i,N} \theta(\gamma^{i,N}(\sigma)) \cdot \dot{\gamma}^{i,N}(\sigma) d\sigma - \int_K \theta(x) \cdot \xi(x) dx \right\| \to 0.$$

**Proof** Consider the ODE on $[0, 1]$,

$$\begin{cases}
\dot{\gamma}(\sigma) = \xi(\gamma(\sigma)) \\
\gamma(0) = x
\end{cases}$$

This ODE admits a flow of diffeomorphism which we will call $\Gamma_1(\sigma, x)$. Using standard arguments based on the uniqueness of solutions, one can show that support of $K$ is invariant through the flow, namely

$$\Gamma(\sigma, K) = K, \quad \forall \sigma \in [0, 1]. \quad (29)$$

In a similar way, since $\xi$ is solenoidal, we have that

$$|\det D_x \Gamma(\sigma, x)| = 1. \quad (30)$$

It follows from properties (29) and (30) that

$$\int_K \theta(x) \cdot \xi(x) dx = \int_K \theta(\Gamma(\sigma, x)) \cdot \xi(\Gamma(\sigma, x)) dx.$$

We call $\lambda$ the Lebesgue measure on $\mathbb{R}^3$. The measure $\frac{\lambda(K)}{\lambda(K)}$ is a probability measure on $K$. Hence, we can find points $x^{i,N} \in K$ such that $\frac{1}{N} \sum_{i=1}^N \delta_{x^{i,N}} \Rightarrow \frac{\lambda}{\lambda(K)}$. The weak convergence on a compact set is equivalent to the convergence in 1-Wasserstein metric, $W_1$, which is defined as

$$W_1(\mu, \nu) := \sup_{\|\psi\|_{\infty} + \text{Lip}(\psi) \leq 1} \left| \int \psi d\mu - \int \psi d\nu \right|$$

Where $\mu$ and $\nu$ are two probability measures on $K$ and $\psi \in C_b(K)$ with $\|\psi\|_{\infty} + \text{Lip}(\psi) \leq 1$, where Lip($\psi$) is the Lipschitz constant of $\psi$.

If $\theta$ is a Lipschitz-continuous vector field, with $\|\theta\| + \text{Lip}(\theta) \leq 1$, then $\theta(\Gamma(\sigma, x)) \cdot \xi(\Gamma(\sigma, x))$ is a Lipschitz-continuous and bounded function. For every $\sigma \in [0, 1]$, have that

$$\sup_{\|\theta\| + \text{Lip}(\theta) \leq 1} \left| \frac{\lambda(K)}{N} \sum_{i=1}^N \theta(\Gamma(\sigma, x^{i,N})) \cdot \xi(\Gamma(\sigma, x^{i,N})) - \frac{\lambda(K)}{N} \int_K \theta(\Gamma(\sigma, x)) \cdot \xi(\Gamma(\sigma, x)) dx \right| \frac{dx}{\lambda(K)} \leq C(\xi, \Gamma) W_1 \left( \frac{1}{N} \sum \delta_{x^{i,N}}, \frac{\lambda}{\lambda(K)} \right) \to 0, \text{ as } N \to \infty \quad (31)$$
The constant $C$ depends only on the supremum and the Lipschitz constants of $\xi$ and $\Gamma$, which are both bounded uniformly on $\sigma$.

We choose $\alpha^{i,N} := \frac{\lambda(K)}{N}$ as weights and $\gamma^{i,N} := \Gamma(\sigma, x^{i,N})$ as curves and we obtain the approximating sequence,

$$\sum_{i=1}^{N} \frac{\lambda(K)}{N} \int_{0}^{1} \theta(\Gamma(\sigma, x^{i,N})) \cdot \xi(\Gamma(\sigma, x^{i,N})) \, d\sigma = \sum_{i=1}^{N} \alpha^{i,N} \int_{0}^{1} \theta(\gamma^{i,N}(\sigma)) \cdot \dot{\gamma}^{i,N}(\sigma) \, d\sigma =: \xi^{N}(\theta)$$

The convergence $\|\xi^{N} - \xi\| \to 0$ is a direct consequence of (31).  

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