Complex critical exponents for percolation transitions in Josephson-junction arrays, antiferromagnets, and interacting bosons

Rafael M. Fernandes and Jörg Schmalian
Ames Laboratory and Department of Physics and Astronomy, Iowa State University, Ames, IA 50011
(Dated: January 20, 2011)

We show that the critical behavior of quantum systems undergoing a percolation transition is dramatically affected by their topological Berry phase $2\pi \rho$. For irrational $\rho$, we demonstrate that the low-energy excitations of diluted Josephson-junctions arrays, quantum antiferromagnets, and interacting bosons are spinless fermions with fractal spectrum. As a result, critical properties not captured by the usual Ginzburg-Landau-Wilson description of phase transitions emerge, such as complex critical exponents, log-periodic oscillations and dynamically broken scale-invariance.

A fundamental aspect of the Ginzburg-Landau-Wilson (GLW) description of phase transitions is scale invariance, which relies on the absence of characteristic length and energy scales at criticality, leading to the concept of universality [1, 2]. For instance, near a quantum critical point (QCP), if a physical observable $O(T)$ transforms for an arbitrary scale transformation $b > 0$ according to $O(T) = b^{-x}O(b^{t}T)$, then we obtain a power-law temperature dependence $O(T) \propto T^{x/z}$, with universal critical exponent $x/z$. However, if scaling is valid only for powers of a discrete value $b_{0}$, it follows that $O(T) = T^{x/z}Q(\ln T)$, with $Q(t)$ a periodic function of period $z \ln b_{0}$. Fourier expansion of $Q(t)$ yields:

$$O(T) = \sum_{n=-\infty}^{\infty} \alpha_{n} T^{x/z + i\pi n / (z \ln b_{0})}$$

with constant coefficients $\alpha_{n}$. Thus, the system is characterized by a family of non-universal complex exponents. An invariant scale $b_{0}$, leading to this discrete scale invariance, is found in several critical systems that either are out of equilibrium or have an underlying built-in hierarchical structure (for a review, see [3]).

In this Letter, we show that complex critical exponents and log-periodic behavior appear in a variety of disordered systems close to a percolation QCP, such as Josephson-junction (JJ) arrays, quantum antiferromagnets (QAF) and interacting bosons. Rather than being related to non-equilibrium properties or to the fractality of the percolating cluster, in these systems the invariant scale $b_{0}$ emerges naturally in their low-energy excitation spectrum for certain values of their Berry phase $2\pi \rho$.

By calculating their specific heat and compressibility at the percolation threshold, we show that, for rational $\rho$, large clusters have the lowest excitation energies, giving rise to usual power-law behavior below a crossover temperature $T^{\star}$, which varies in a pronounced non-monotonic way with respect to $\rho$ (see Fig. 1). For irrational $\rho$, the low-energy properties are governed instead by degenerate clusters of intermediate sizes, leading to the breakdown of continuous scale invariance ($T^{\star} \to 0$). Remarkably, the sizes and energies of these resonating clusters depend solely on the continued-fraction expansion of $\rho$. As a result, when $\rho$ is a quadratic irrational, the periodicity of its continued-fraction expansion gives rise to an invariant scale $b_{0}$ and, consequently, to complex critical exponents.

To introduce a model for all the systems discussed above [2], consider an array of grains characterized by an XY order parameter $\Psi_{j} = |\Psi_{0}| \exp(i \theta_{j})$, with phase $\theta_{j}$ and fixed amplitude $|\Psi_{0}|$. The array is diluted on a regular lattice of dimension $d > 1$, characterized by a quenched random-site variable $\epsilon_{j}$ that takes the values 0 and 1 with probabilities $P$ and $(1 - P)$, respectively. We consider the Hamiltonian [2, 4, 5]:

$$H = U \sum_{i} \epsilon_{i} (n_{i} - \rho)^{2} - \sum_{ij} \epsilon_{i} \epsilon_{j} J_{ij} \cos(\theta_{i} - \theta_{j}),$$

where $n_{i} = -i \partial_{\theta_{i}}$. $\rho$ can be externally controlled and causes the phase to precess in time according to $\partial \theta_{j} / \partial t = 2U \rho$ (see Fig. 1). In JJ-arrays [4, 5], $\Psi_{j}$ denotes the superconducting order parameter, $J_{ij}$ is the Josephson coupling, $U$ is the charging energy and $\rho$, related to the AC-Josephson effect, can be changed by an external gate voltage. $\Psi_{j}$ can also represent planar quantum rotors, associated with the low-energy modes of QAF [2], where $\rho$ is proportional to a perpendicular external magnetic field. For systems of interacting bosons [6, 7], which can
be realized in optical experiments with cold atoms [8], 2U\rho corresponds to the chemical potential \mu.

The effects of percolative dilution on all these systems have been the subject of various experimental and numerical investigations [8,12]. Here, we focus on the critical properties at the percolation threshold \ Pc, where the density of clusters with \ s connected occupied sites varies as \ N (s) \propto s^{-\tau}, with \ 2 < \tau \equiv d / D_f + 1 \leq 2.5 \text{ and } D_f \text{ the fractal dimension of the percolating cluster [13]. At low temperatures } T \ll |J_{ij}| \text{ and deep in the ordered state of the clean system } (U < |J_{ij}|), the relative phase between grains inside each cluster is fixed, implying that the entire cluster is characterized by a global phase [14–16]. At \ Pc, contributions to the total specific heat of a single cluster arise from the coherent precession of its macroscopic quantities can be calculated by averaging over all clusters, i.e. \mathcal{O} (\rho, T) = \sum_s N (s) \mathcal{O}_{s} (\rho, T).

Let us first revisit the results for \ \rho = 0, where universal power-law behavior was previously found [14]. The low-temperature specific heat of a cluster is given by \ C_s (\rho = 0) \propto \exp (- U / s T), i.e. the typical excitation energy of a cluster decreases monotonically with its size, \ \varepsilon_s = U / s \ T. Then, the low-energy behavior is dominated by large clusters and we can replace the sum over \ s by an integral, obtaining, for \ T \ll U,

\[ C (\rho = 0, T) \propto T^{d/z}, \quad (4) \]

where the dynamic scaling exponent \ z = D_f is introduced. This result was also found in detailed computer simulations [17,18]. Consider now a finite Berry phase \ \rho \neq 0. Solving the problem of a single quantum rotor and using the scaling \ T \to sT, \ \rho \to s \rho \ derived from the action [3], we find the spectrum of a single cluster \ E_m = U (m - s \rho)^2 / s, with \ m \in \mathbb{Z}. Therefore, the lowest excitation energy is (see also [2,7]):

\[ \varepsilon_s = \frac{U}{s} (1 - 2 |\rho_s|), \quad (5) \]

where \ \rho_s = s \rho - \lfloor s \rho + \frac{1}{2} \rfloor \text{ and } \lfloor x \rfloor \text{ is the integer part of } x, i.e. \lfloor x + \frac{1}{2} \rfloor \text{ is the integer closest to } x, such that \ |\rho_s| \leq 1/2. Note that \ \rho_s \ depends on the droplet size in a highly non-monotonic way, reflecting the periodicity of the Berry phase (see Fig. 2). Now, not only large clusters yield small excitation energies, but also intermediate-size clusters with \ |\rho_s| \leq 1/2.

These low-energy excitations can be described by spinless fermions, yielding the well-known fermionic expressions \ C_1 (\omega, T) = (\omega / T)^2 n_F (\omega) n_F (- \omega) \text{ and } \kappa_{s} (\omega, T) = (1 / T) n_F (\omega) n_F (- \omega) \text{ for the single site specific heat and compressibility, with } n_F (\omega) \text{ the Fermi function. In Fig. 3 we demonstrate this numerically by comparing } C (T) \text{ obtained from the exact energy spectrum and from the fermionic expression only. We can also show this result analytically: Consider, for definiteness, } 0 \leq \rho \leq 1/2. The spectrum of a single rotor, including degeneracies, can be generated from a model of effective interacting fermions and bosons, with occupation numbers \ n_f = f^\dagger f \text{ and } n_b = b^\dagger b, respectively:

\[ H_0 = U (n_b + 1 - 2 \rho) n_f + \rho)^2. \quad (6) \]

For \ \rho = 0, one recovers the } N = 2 \text{ super-symmetric description of a rotor [19]. Expanding for } \rho \approx 1/2, we obtain instead } H_0 \simeq \varepsilon_1 n_f + U n_b + U n_b^2 + 2 \varepsilon_1 n_b n_f. \text{ While the excitation energy of single bosons is } U, \text{ the fermionic
excitation energy is $\varepsilon_1 = U (1 - 2\rho) \ll U$. Thus, at sufficiently low temperatures, bosons are diluted and the interaction terms can be neglected, implying that free spinless fermions are the dominant excitations of the system. The onset of power-law behavior is marked by $T^* \sim (d\omega)^{-1}$, and this regime prevails down to the lowest energies.

Let us now consider $\rho = p/q$ to be rational, i.e. $p$ and $q$ are integers with no common divisors. Due to the periodicity of the Berry phase term, the number of distinct values of $\rho$, is of the order of $q/2$, defining well separated branches in $\varepsilon_\tau$, all decaying as $s^{-1}$, with lowest branch $\varepsilon_\tau \approx U (qs)^{-1}$ (see inset of Fig. 2). For $T \ll U/q^2$, the problem is virtually the same as for $\rho = 0$, leading to a heat capacity dominated by very large clusters as given in Eq. (1) with the same exponent $z_\tau$, as shown in Fig. 3. For the compressibility, we obtain $\kappa (T) \propto T^{d/z_\tau - 1}$, with a Wilson ratio $\kappa T / C \approx 0.3$ for $d = 2$. Note, from Fig. 3, that the crossover temperature $T^*$ changes as $q^{-2}$ and is insensitive to $p$. Thus, systems with similar values of $\rho$ can have very different $T^*$ (see Fig. 1).

The natural question is: what happens in the regime $U/q^2 \ll T \ll U$ for very large $q$? For irrational $\rho$, $T^* \to 0$ and this regime prevails down to the lowest energies. Indeed, when $\rho$ is irrational, the sequence $\rho_n$ is uniformly distributed between $-1/2$ and $1/2$ [20], i.e. there are finite-size droplets with arbitrarily low excitation energy $\varepsilon_\tau$. It is convenient to introduce the averaged integrated density of fermionic states, $D (\omega) = \sum s s^{-1} \theta (\omega - \varepsilon_s)$; from the periodicity of $\varepsilon_s$ with respect to $\rho$ (i.e. summing over winding numbers), we obtain, for $\omega \ll U$:

$$D (\omega) = \zeta (\tau - 1) \frac{\omega}{U} + \sum s, \lambda \frac{f_{\text{saw}} (s\varepsilon_s)}{s^\tau},$$

(7)

where $x_\pm = \omega/U \pm 2\rho$, $\zeta (x)$ is the zeta function and $f_{\text{saw}} (x)$ is the sawtooth function, which has period $2$ and unit jumps at every odd integer: $f_{\text{saw}} (x) = (-x + 2n)/2$ for $2n - 1 < x < 2n + 1$. These jumps give rise to discontinuities in $D (\omega)$ at frequencies:

$$\omega_j = 2U \left| \rho - \frac{p_j}{q_j} \right|,$$

(8)

where $p_j$ is an odd integer and $q_j$ is even. At $\omega \ll U$, the fractions $p_j/q_j$ that satisfy Eq. (5) are the ones that best approximate $\rho$, i.e. the Diophantine approximants, which are given by the convergents of the continued fraction expansion of $\rho$ [21]. Physically, the jumps are a consequence of the existence of a set of energetically degenerate (i.e. “resonating”) clusters with sizes that are odd multiples of $q_j/2$, $s = (n + 1/2)q_j$ (see Fig. 2). Summing over all these clusters in Eq. (7), we find that each jump in $D (\omega)$ is given by $\Delta_j = q_j^{-1} \zeta (\tau) (2^\tau - 1)$.

Back to Eq. (7) we find that the regular part of the sawtooth function cancels out the linear in $\omega$ term. Thus, the frequency dependence of $D (\omega)$ is governed by the successive jumps $\Delta_j$ at $\omega_j$ of Eq. (8) and, consequently, by the sequence of convergents of the continued fraction expansion of $\rho$ with even denominator $q_j$. Although the determination of this sequence for an arbitrary irrational $\rho$ is an outstanding problem in number theory, it is simplified in the case of quadratic irrationals, which have periodic continued fraction expansions [21]. Then, one finds that the sequence of even $q_j$ is also periodic.

In fact, for quadratic irrationals with a single period $a \in \mathbb{Z}$, which are solutions of the algebraic equation $y^2 - ay - 1 = 0$, we find that if $q_j$ is even, so is $q_{j+N}$, with $N = 2 + \text{mod} (a, 2)$. Consequently, the distance between consecutive jumps is a constant in log-scale, $\ln (\omega_j/\omega_{j+1}) = 2\ln y_1$, as well as the ratio between their amplitudes, $\ln (\Delta_j/\Delta_{j+1}) = \tau \ln y_1$, where $y_1$ is the positive solution of the algebraic equation. Using these properties, we can show that $D (\omega)$ is a fractal function with fractal dimension $d_\omega = \tau/2$, characterized by a power-law decay in $\omega$ and periodic oscillations in $\ln \omega$. Since $C (T) = \int \omega \nu (\omega) C_1 (\omega, T)$, where $\nu = dD/d\omega$ is the density of states, we obtain:

$$C (T) = T^{d/z_\tau} A (\ln T),$$

(9)

where $z_\tau = 2D f_j/(D f_1 + d)$, i.e. $z_\tau > z_\tau$, and $A (t)$ is a periodic function of period $z_\tau \ln b_0 = 2\ln y_1$. For the compressibility, we find $\kappa (T) = T^{(d/z_\tau) - 1} B (\ln T)$, where $B (t)$ has the same period as $A (t)$. Thus, the system has complex critical exponents as in Eq. (1). In Fig.

![Figure 3: Specific heat $C$ as function of temperature $T$ for the rational $\rho = 55/89$. The dashed blue line is the exact result and the solid red line is the fermionic approximation. The onset of power-law behavior is marked by $T^*(\text{dotted line})$. The inset shows the log-log variation of $T^*$ with respect to the denominator $q$ for a series of rationals with comparable values, $\rho = \{\frac{5}{7}, \frac{8}{11}, \frac{21}{34}, \frac{52}{93}, \frac{144}{257}, \frac{897}{1597}\}$.

![Graph](image-url)
critical exponents

The breakdown of continuous scale invariance for irrational \( \rho \) can be attributed to the resonating clusters with arbitrarily low excitation energies, as they cause jumps in the entire spectrum, prohibiting to replace \( \sum_n \to \int ds \) in Eq. 7. For rational \( \rho \), such a replacement is allowed, leading to full scaling \( s \to s/(\hbar D) \) and to Eq. 4. Yet, when \( \rho \) is a quadratic irrational, the dynamically broken scale invariance is partially restored as discrete scale invariance. In this case, the periodic structure of the continued-fraction expansion of \( \rho \) gives rise to log-periodic relations between sizes \( s = (n + 1/2)q_j \) and energies \( \omega_j = 2U(\rho - p_j/q_j) \) of different sets of resonating clusters, establishing an invariant scale \( b_0 \) and complex critical exponents \( (d/z) + i\pi/(N\ln y_+) \) for \( C(T) \).

Going back to the contribution of the spin-waves, their density of states is \( \nu_{sw} \propto k^{d_s-1} \) at \( P_c \), where the fracton dimension \( d_s \) characterizes the spectrum of the eigenvalues \( k^2 \) of the Laplacian on the percolating cluster \( \Omega \). From the spin-waves dispersion \( \Omega_{s,k} = \varepsilon_k^2 + c^2 k^2 \) and the scaling properties of the fermionic density of states, we find \( \Omega_{sw} \propto OT^\phi \) for both \( O = C, \kappa \), with \( \phi = d_s - 1/2 \) for rational (irrational) \( \rho \). As \( d_s > 1 \) \([13]\), it follows that \( \phi > 0 \). Since the internal modes are sub-leading compared to the coherent modes, the spectrum of a single cluster depends solely on its size and not on its shape, in accordance to Eq. 3.

In summary, we solved the XY quantum-rotor problem at low \( T \) and close to the percolation threshold, which describes diluted systems as diverse as JJ-arrays with d.c.-bias voltage, canted QAF in a perpendicular magnetic field, and interacting bosons coupled to a particle reservoir. Their topological Berry phase \( 2\pi \rho \) dramatically alters the percolation QCP, since the low-\( T \) behavior is governed by emergent spinless fermions with fractal spectrum, giving rise to generally irregular \( \log T \)-oscillations of thermodynamic variables. While for irrational \( \rho \) they persist to \( T \to 0 \), for rational \( \rho = p/q \) they occur in the temperature range \( q^{-2} \lesssim T/U \lesssim 1 \), which can be broad for \( q \gg 1 \). Remarkably, for a quadratic irrational \( \rho \), they become regular, leading to complex critical exponents.

Our results demonstrate that the quantum criticality in disordered systems governed by a topological Berry phase is beyond the GLW paradigm of critical systems.

We thank M. Axenovich for pointing out Ref. 20 to us. This research was supported by the U.S. DOE, Office of BES, Materials Sciences and Engineering Division.

References:

[1] S.-K. Ma, Modern Theory of Critical Phenomena, Benjamin, Reading (1976).
[2] S. Sachdev, Quantum Phase Transitions, Cambridge Univ. Press (1999).
[3] D. Sornette, Phys. Rep. 297, 239 (1998).
[4] S. Doniach, Phys. Rev. B 24, 5063 (1981).
[5] M. P. A. Fisher et al., Phys. Rev. B 40, 546 (1989).
[6] P. B. Weichman and R. Mukhopadhyay, Phys. Rev. B 77, 214516 (2008).
[7] E. Altman, Y. Kafri, A. Polkovnikov, and G. Refael, Phys. Rev. Lett. 100, 170402 (2008).
[8] S. Ospelkaus et al., Phys. Rev. Lett. 96, 180403 (2006).
[9] J. L. Yum et al., Phys. Rev. Lett. 97, 215701 (2006).
[10] J. P. Lv et al., Phys. Rev. B 79, 104512 (2009).
[11] O. P. Vajk et al., Science 295, 1691 (2002).
[12] L. Wang and A. W. Sandvik, Phys. Rev. Lett. 97, 117204 (2006); Phys. Rev. B 81, 054417 (2010).
[13] D. Stauffer and A. Aharony, Introduction to Percolation Theory, Taylor & Francis, London, (1994).
[14] T. Vojta and J. Schmalian, Phys. Rev. Lett. 95, 237206 (2005).
[15] J. A. Hoyos et al., Phys. Rev. Lett. 99, 230601 (2007); T. Vojta et al., Phys. Rev. B 79, 024401 (2009).
[16] N. Bray-Ali, J. E. Moore, T. Senthil, and A. Vishwanath, Phys. Rev. B 73, 064417 (2006).
[17] A.W. Sandvik, Phys. Rev. Lett. 89, 177201 (2002).
[18] R. Yu, T. Roscilde, and S. Haus, Phys. Rev. Lett. 94, 197204 (2005).
[19] F. Correa and M. S. Plyushchay, Ann. Phys. 322, 2493.

Figure 4: (a) Frequency dependent integrated density of states \( D(\omega) \), as well as the temperature dependent (b) specific heat \( C(T) \) and (c) compressibility \( \kappa(T) \) for the system with \( \rho = \sqrt{2} \). Dashed lines show the underlying power-law behavior superimposed to the log-periodic oscillations. Inset is a zoom of \( D(\omega) \).
(2007).
[20] H. Weyl, Mathematische Annalen 77, 313 (1916).
[21] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Clarendon Press, Oxford (1960).