Transport coefficients from the boson Uehling-Uhlenbeck equation

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Kinetic equations provide a means to derive microscopic expressions for the transport coefficients appearing in the equations of fluid hydrodynamics. The transport coefficients for dilute gases at high temperature can be computed using the Boltzmann equation. However, when the temperature is lowered enough that quantum degeneracy begins to affect the behavior of the gas, one must use the Uehling-Uhlenbeck (U-U) equation [1,2], which is a semiclassical extension of the Boltzmann equation that accounts for the quantum statistics of the particles.

There are two traditional approaches to computing transport coefficients; the method outlined by Chapman and Enskog [3–6], and a method due to Resibois [7], which directly uses the microscopic hydrodynamic modes of the system. In subsequent sections, we use the Resibois method to compute the transport coefficients of dilute degenerate boson gases, and we give explicit numerical values for their transport coefficients just above the Bose-Einstein condensation temperature. Although we focus on boson gases, it is worth noting that there has been interest recently in the viscosity of some exotic systems whose relaxation is also governed by the U-U equation. These include the aftermath of high-energy collisions, and recent predictions of a universal limit on the ratio of viscosity and entropy density [3,8–11].

The Resibois method relates the transport coefficients to the spectral decomposition of the collision operator, and the discrete eigenvalues (relaxation rates) of the collision operator are central. For gases that interact via hard-core potentials or contact potentials, the eigenvalues of the Boltzmann and the Uehling-Uhlenbeck collision operators have a known upper cutoff [12] and the eigenvalues rapidly converge towards this value. When an upper cutoff exists, the contribution to the transport coefficients from higher eigenmodes can not be neglected. For gases that interact via a softer potential, such as \( V(r) \sim \frac{1}{r^\alpha} \) for \( n > 4 \), no cutoff exists in the discrete spectrum of the collision operator and simple approximations can be used in obtaining transport coefficients. In subsequent sections, we obtain exact values for the transport coefficients of classical hard sphere gases and ultracold boson gases, and we show how the Eucken number [6,13] of classical gases changes as the interaction varies from that of a Maxwell gas \((n = 4)\) to a hard sphere interaction \((n \to \infty)\), which is equivalent to a contact interaction at low temperature.

Typically, in the past, the transport coefficients of dilute classical hard sphere gases have been obtained by truncating a Sonine polynomial expansion at low order [4,6,14]. As we will show, this truncation neglects the effect of the upper cutoff on collision operator eigenvalues and gives an Eucken number that differs little from that of a classical Maxwell gas.

In Sec. II we introduce the U-U equation and in Sec. III, we linearize it and introduce an abstract velocity-space basis that will be used for calculation. In Sec. IV, we derive microscopic expressions for the hydrodynamic frequencies by relating them to the collision operator, and use them to obtain explicit expressions for the bulk viscosity, thermal conductivity, and shear viscosity of the gas. We also discuss the proper treatment of the eigenvalue “cutoff” that is present for hard sphere and contact interactions. In Sec. V, we present the results of our numerical calculation of the transport coefficients and justify our treatment of the eigenvalue cutoff. We conclude by summarizing our results in Sec. VI.

I. INTRODUCTION

II. UEHLING-UHLENBECK EQUATION

We consider a dilute gas of noncondensed bosons that interact via a contact potential \( U(r) = U_0 \delta^3(r) \), where \( U_0 = 4\pi\hbar^2a/m \), \( \hbar \) is Planck’s constant, \( a \) is the \( s \)-wave scattering length, and \( m \) is the mass of a particle. The derivation of a kinetic equation for this system is discussed in Ref. [15]. Above the critical temperature for Bose-Einstein condensation, the dynamics of the gas is governed by the Uehling-Uhlenbeck (U-U) kinetic equation, which can be written

\[
\frac{\partial f_1}{\partial t} + \frac{p_1}{m} \cdot \nabla f_1 = -\mathcal{C}[f_1],
\]

where \( p_1 \) and \( r \) are momentum and position of the bosons, respectively, and \( f_1 = f(r,p_1,t) \) is the phase space number density of bosons in the phase space volume \( p_1 \to p_1 + dp_1 \) and \( r \to r + dr \) at time \( t \). It is normalized so that \( N = \int \frac{dp_1}{2\pi \hbar} \int f(r,p_1,t) \), where \( N \) is the average number of particles in the gas.
The quantity $C[f_1]$ is the collision integral and is defined
\[ C[f_1] = \frac{d^2}{m^2 \pi^2 \hbar^2} \int dp_x dp_y dp_z \delta^3(p_1 + p_2 - p_3 - p_4) \times \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) f_1 f_2 (1 + f_3)(1 + f_4) \times \epsilon_1 = (p_1^2)/(2m). \]
where $\epsilon_1$ is $p_1^2/(2m)$.

The collision integral explicitly conserves particle number, momentum, and energy. This can be seen from the five integrals $\int dp_x dp_y dp_z f_1$, $\int dp_x dp_y dp_z f_2$, $\int dp_x dp_y dp_z f_3$, $\int dp_x dp_y dp_z f_4$, and $\int dp_x dp_y dp_z f_1 f_2 f_3 f_4$. Integrating $f_1$ over momentum gives the equilibrium particle density $n_0$ of the gas
\[ n_0 = \int \frac{dp_1}{(2\pi \hbar)^3} f_1^0(p_1) = \frac{1}{\lambda_T^3} \text{Li}_{3/2}(z), \]
where $\lambda_T = \sqrt{\frac{2\pi \hbar^2}{mk_BT}}$ is the thermal wavelength, $z = e^{\mu/(k_BT)}$ is the fugacity and $\text{Li}_{3/2}(z)$ is a polylogarithm. Polylogarithms appear repeatedly for degenerate gases and are defined by
\[ \text{Li}_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{z^t}{e^t - z}. \]
For simplicity we will use the notation $\sigma_n = \text{Li}_{n+1}(z)$.

It is useful to introduce a dimensionless momentum $c_1 = p_1/(mv_T)$ where $v_T = \sqrt{2k_BT/m}$. Then the Uehling-Uhlenbeck equation can be written
\[ \frac{\partial f_1}{\partial t} + v_T c_1 \cdot \nabla f_1 = -\gamma C[f_1], \]
where the dimensionless collision integral $C[f_1]$ is given by
\[ C[f_1] = \frac{1}{\pi \sqrt{2\pi} \hbar^2} \int dc_x dc_y dc_z \delta^3(c_1 + c_2 - c_3 - c_4) \times \delta(c_1^2 + c_2^2 - c_3^2 - c_4^2) f_1 f_2 (1 + f_3)(1 + f_4) - (1 + f_1)(1 + f_2)f_3 f_4, \]
and $f_1 = f(r, c_1, t)$. The overall rate constant is
\[ \gamma = \frac{8ma^2 z(kaT)^3}{\pi \hbar^3}, \]
which now depends on the fugacity and equilibrium temperature.

III. LINEARIZED U-U EQUATION

In computing transport coefficients, it is sufficient to consider the relaxation of the gas when it is close to equilibrium. In that case, the distribution function $f(r, c_1, t)$ will be a slowly varying function of $r$, and will be close to its equilibrium value. We can then linearize the U-U equation by writing $f(r, c_1, t) = f_1^0 + f_1^0(1 + f_1^0)\Phi(r, c_1, t)$,
where $f_1^0 = z/(\epsilon_1^2 - z)$ and $\Phi(r, c_1, t)$ contains information about the small deviations from equilibrium and satisfies $|\Phi| \ll 1$. We substitute this expression for $f(r, c_1, t)$ into (7) and neglect terms of quadratic and higher order in $\Phi$ and obtain
\[ \frac{\partial \Phi_1}{\partial t} + v_T c_1 \cdot \nabla \Phi_1 = -\gamma C[\Phi_1], \]
where $\Phi_1 = \Phi(r, c_1, t)$ and
\[ C[\Phi_1] = \frac{1}{z \pi \sqrt{2\pi} \hbar^2} \int dc_x dc_y dc_z \delta^3(c_1 + c_2 - c_3 - c_4) \times \delta(c_1^2 + c_2^2 - c_3^2 - c_4^2) f_1^0 f_2 (1 + f_3^0)(1 + f_4^0) - \Omega(f_1^0) + f_3 f_4, \]
is the linearized collision operator.

Since Eq. (9) is linear, we may take the Fourier transform of both the space and time dependence of (9) to obtain
\[ -i\omega \phi_1 + i v_T (k \cdot c_1) \phi_1 = -\gamma C[\phi_1], \]
where $\phi_1 = \phi(k, c_1, \omega) = \int dr / \int d\Phi [\Phi(r, c_1, t)] e^{-i(k \cdot r + i\omega t)}$. Equation (11) governs the relaxation of deviations from equilibrium with wave vector $k$ and frequency $\omega$.

A. Momentum basis

Since $\Omega$ is linear, deviations $\phi(k, c_1, \omega)$ will not be coupled to those of different wave vectors or frequencies and we may suppress the dependence of $\phi(k, c_1, \omega)$ on $k$ and $\omega$. We can greatly simplify the notation in the following calculations by introducing an abstract vector $|\Phi\rangle$ and making the interpretation $\phi(c_1) = |\Phi\rangle c_1 \phi(\Phi)$, where $|\Phi\rangle$ represents an abstract momentum basis. We define the inner product between two abstract vectors as
\[ \langle \chi | \phi \rangle \equiv \int dc_1 w(c_1) \chi^\ast(c_1) \phi(c_1), \]
where the weighting factor $w(c_1)$ is defined
\[ w(c_1) = \frac{1}{\pi \sqrt{2\pi} \hbar^2} f_1^0(c_1)[1 + f_1^0(c_1)]. \]
If $C[\phi]$ is interpreted as $\langle \Phi | \hat{C} | \phi \rangle$, this definition of the weighting function makes the collision operator symmetric in the sense that $\langle \chi | \hat{C} | \phi \rangle = \langle \Phi | \hat{C} | \chi \rangle$. In the momentum basis, the collision operator $\hat{C}$ can then be expressed as
\[ \hat{C} = \int dc_1 \int dc_3 w(c_1) w(c_3) |\Phi\rangle c_1 \chi 1 c_3 \langle c_3 | C[|c_1\rangle \langle c_3 |] |c_3\rangle |c_1\rangle, \]
where $C[|c_1\rangle \langle c_3 |] = \langle |c_1\rangle | \hat{C} | c_3 \rangle$ is the “kernel function” of $\hat{C}$.

It will be convenient to introduce an orthonormal “angle” basis, $|c, l, m\rangle$, which is defined by
\[ \langle c_1 | c, l, m \rangle = \frac{1}{c_1 \sqrt{w(c_1)}} \delta(c_1 - c) Y_{l}^{m}(\theta_1, \phi_1). \]
The collision operator can be expressed in the angle basis as
\[ \hat{C} = 2\pi \int dc_1 \int dc_2 \sum_{l,m} |c_1, l, m\rangle \chi 1 c_2 \sum_{l,m} |c_1, l, m\rangle c_2 \langle c_1 | c_2 \rangle e^{i c_1 c_2} w(c_1) w(c_2) C[|c_1\rangle \langle c_2 |] |c_2\rangle |c_1\rangle, \]

\[ \langle c_1 | c, l, m \rangle = \frac{1}{c_1 \sqrt{w(c_1)}} \delta(c_1 - c) Y_{l}^{m}(\theta_1, \phi_1). \]
The collision operator can be expressed in the angle basis as
\[ \hat{C} = 2\pi \int dc_1 \int dc_2 \sum_{l,m} |c_1, l, m\rangle \chi 1 c_2 \sum_{l,m} |c_1, l, m\rangle c_2 \langle c_1 | c_2 \rangle e^{i c_1 c_2} w(c_1) w(c_2) C[|c_1\rangle \langle c_2 |] |c_2\rangle |c_1\rangle, \]

\[ \langle c_1 | c, l, m \rangle = \frac{1}{c_1 \sqrt{w(c_1)}} \delta(c_1 - c) Y_{l}^{m}(\theta_1, \phi_1). \]
where

\[ C^l(c_1, c_2) = \int_{-1}^{1} d(\hat{c}_1 \cdot \hat{c}_2) C(c_1, c_2) P_l(\hat{c}_1 \cdot \hat{c}_2) \]  

(17)

and \( P_l \) is a Legendre polynomial. The function \( C^l(c_1, c_2) \) is the angular kernel function. In the angle basis, \( \hat{C} \) is diagonal in \( l \) and \( m \), a fact that follows from the rotational symmetry of \( C(c_1, c_2) \) [16].

We may now express the linearized U-U equation as an operator equation. Without any loss of generality, we can assume that the wave vector \( \mathbf{k} = k\hat{c}_z \), where \( \hat{c}_z \) is a unit vector along the \( z \) direction. The U-U equation then takes the form

\[ -i\omega |\phi\rangle + iv_T k \hat{c}_z |\phi\rangle = -\gamma \hat{C} |\phi\rangle, \]  

(18)

where \( \hat{c}_z \) is now an operator defined by \( \hat{c}_z |\mathbf{c}\rangle = c_z |\mathbf{c}\rangle \). This form of U-U equation allows us to determine the microscopic frequencies as a perturbation expansion in powers of \( k \).

B. Spectrum of the collision operator

The spectrum of the collision operator plays a fundamental role in both the derivation of the hydrodynamic equations and the calculation of transport coefficients. By inspection, we can see that the collision operator has five zero eigenvalues, which correspond to the five conserved quantities (particle number, momentum, and energy) in a two-body collision. The nonzero eigenvalues are all positive and, for hard sphere or contact interactions, approach a finite limiting value \( \lambda_M = \frac{\sigma}{c} \) [12,16].

The representation of \( \hat{C} \) in the angle basis shows that the eigenfunctions of the collision operator are states of definite \( l \) and \( m \). This allows us to use the indices \( n, l, m \) to label the eigenfunctions \( |\phi_{n,l,m}\rangle \) of \( \hat{C} \) and write

\[ \langle \phi_{n',l',m'} | \phi_{n,l,m} \rangle = \delta_{n,n'} \delta_{l,l'} \delta_{m,m'}, \]  

(19)

where \( \phi_{n,l,m}(c, \theta, \varphi) = |\mathbf{c}| \phi_{n,l,m}(c) \), where \( |\mathbf{c}| = |c, \theta, \varphi| \) in spherical coordinates, then the radial part of the eigenfunction is given by

\[ \phi_{n,l}(c) = c \sqrt{w(c)} \int d\Omega Y_{l}^{*m}(\theta, \varphi) \phi_{n,l,m}(c, \theta, \varphi). \]  

(20)

For example, the particle number conservation eigenfunction is \( \phi_{0,0,0}(c) = 1 \) and the radial part is \( \phi_{0,0}(c) = c \sqrt{4\pi w(c)} \). The radial part of the eigenfunctions satisfy the orthogonality condition

\[ \int_{0}^{\infty} dc \phi_{n,l}^{*}(c) \phi_{n',l}(c) = \delta_{n',n}. \]  

(21)

IV. RELATION OF TRANSPORT COEFFICIENTS TO EIGENVALUES

In order to relate the transport coefficients to the eigenvalues of the collision operator, we will equate frequencies obtained from the linearized hydrodynamic equations to frequencies derived directly from the U-U equation.

A. Hydrodynamic frequencies

A derivation of the linearized hydrodynamic equations and the hydrodynamic normal mode frequencies for a dilute fluid at temperatures above the critical temperature for Bose-Einstein condensation can be found in Ref. [14]. For such a fluid there are five microscopically conserved quantities and therefore five hydrodynamic normal modes. The normal mode frequencies are as follows.

\[ \omega_1 = \omega_2 = -\frac{ik^2 \eta}{mn_0}, \]  

(22a)

\[ \omega_3 = -\frac{ik^2 \kappa}{n_0 c_p}, \]  

(22b)

\[ \omega_4 = -kc_s - \frac{ik^2}{2mn_0} \left( \zeta + \frac{4}{3} \eta + \frac{4m \kappa \sigma_2}{15k_B \sigma_3} \right), \]  

(22c)

\[ \omega_5 = kc_s - \frac{ik^2}{2mn_0} \left( \zeta + \frac{4}{3} \eta + \frac{4m \kappa \sigma_2}{15k_B \sigma_3} \right), \]  

(22d)

where \( \eta \) is the shear viscosity, \( \kappa \) is the thermal conductivity, and \( \zeta \) is the bulk viscosity. The speed of sound is given by

\[ c_s = \sqrt{\frac{5k_B T \sigma_3}{3m\sigma_2}}, \]  

(23)

and the specific heats at constant pressure and constant density are given by

\[ c_p = \frac{5k_B \sigma_0 \sigma_4}{2\sigma_2^2} \left( \frac{5\sigma_0 \sigma_4 - 3\sigma_2^2}{2\sigma_0 \sigma_2} \right) \]  

(24)

and

\[ c_n = \frac{3k_B}{2} \left( \frac{5\sigma_0 \sigma_4 - 3\sigma_2^2}{2\sigma_0 \sigma_2} \right), \]  

(25)

respectively.

B. Microscopic frequencies

To obtain microscopic expressions for the hydrodynamic frequencies, we apply standard Rayleigh-Schroedinger degenerate perturbation theory to the Eq. (18), using the wave vector \( k \) as the small parameter and \( v_T k \hat{c}_z \) as the perturbation. As a first step, we expand \( \omega \) and \( |\phi\rangle \) in powers of \( k \) so that

\[ \omega = \omega^{(0)} + k \omega^{(1)} + k^2 \omega^{(2)} + \cdots \]  

and \( |\phi\rangle = |\phi^{(0)}\rangle + k |\phi^{(1)}\rangle + k^2 |\phi^{(2)}\rangle + \cdots \) and we then substitute these equations into Eq. (18) and require that the coefficients of each power of \( k \) vanish separately.

In the limit when \( k = 0 \), Eq. (18) can be written

\[ -i\omega^{(0)} |\phi^{(0)}\rangle = -\gamma \hat{C} |\phi^{(0)}\rangle. \]  

(26)

From this it is clear that the unperturbed eigenvectors \( |\phi^{(0)}\rangle \) are just the eigenvectors of the collision operator \( \hat{C} \). Let us denote the eigenvalues of the collision operator as \( \lambda_{\beta} \) so that

\[ \hat{C} |\phi^{(0)}\rangle = \lambda_{\beta} |\phi^{(0)}\rangle. \]  

(27)

We will denote the five degenerate “zero” eigenvalues of \( \hat{C} \) by \( \beta = 1, \ldots, 5 \). The remaining positive nondegenerate eigenvalues of \( \hat{C} \) will be denoted by \( \beta = 6, \ldots, \infty \). The five
eigenvectors of $\hat{C}$ with $\beta = 1, \ldots, 5$ represent microscopically conserved quantities. When properly normalized [using Eq. (12)] they are given by

$$
\langle \epsilon | \phi^{(0)}_1 \rangle = 1, \tag{28a}
$$
$$
\langle \epsilon | \phi^{(0)}_2 \rangle = \frac{2\sigma_0}{\sqrt{3(5\sigma_4 - 3\sigma_2)}} (c^2 - 3\sigma_2), \tag{28b}
$$
$$
\langle \epsilon | \phi^{(0)}_3 \rangle = \frac{2\sigma_0}{\sigma_2} c_s, \tag{28c}
$$
$$
\langle \epsilon | \phi^{(0)}_4 \rangle = \frac{2\sigma_0}{\sigma_2} c_y, \tag{28d}
$$
$$
\langle \epsilon | \phi^{(0)}_5 \rangle = \frac{2\sigma_0}{\sigma_2} c_z. \tag{28e}
$$

Since $\lambda_\beta = 0 (\beta = 1, \ldots, 5)$, we must apply degenerate perturbation theory [14,17] to determine the linear combinations of $|\phi^{(0)}_\beta\rangle (\beta = 1, \ldots, 5)$ appropriate for a convergent perturbation expansion. They are

$$
|\psi^{(0)}_1\rangle = |\phi^{(0)}_0\rangle, \tag{29a}
$$
$$
|\psi^{(0)}_2\rangle = |\phi^{(0)}_4\rangle, \tag{29b}
$$
$$
|\psi^{(0)}_3\rangle = -\sqrt{1 - \alpha^2} |\phi^{(0)}_1\rangle + \alpha |\phi^{(0)}_2\rangle, \tag{29c}
$$
$$
|\psi^{(0)}_4\rangle = \frac{1}{\sqrt{2}} \left( -\alpha |\phi^{(0)}_1\rangle - \sqrt{1 - \alpha^2} |\phi^{(0)}_2\rangle + |\phi^{(0)}_3\rangle \right), \tag{29d}
$$
$$
|\psi^{(0)}_5\rangle = \frac{1}{\sqrt{2}} \left( \alpha |\phi^{(0)}_1\rangle + \sqrt{1 - \alpha^2} |\phi^{(0)}_2\rangle + |\phi^{(0)}_3\rangle \right). \tag{29e}
$$

where $\alpha = \sqrt{3\sigma_2/(5\sigma_0\sigma_4)}$. For $\beta > 6$, $|\psi^{(0)}_\beta\rangle = |\phi^{(0)}_\beta\rangle$.

The perturbation expansion for the eigenfrequencies of the linearized U-E equation can now be written

$$
\omega_\beta = -i\gamma \lambda_\beta + v_T k |\psi^{(0)}_\beta\rangle \langle \psi^{(0)}_\beta| + \frac{i v_T^2 k^2}{\gamma} \sum_{\beta', \lambda_{\beta'} \neq \lambda_\beta} \frac{|\psi^{(0)}_\beta\rangle \langle \psi^{(0)}_{\beta'}| |\psi^{(0)}_{\beta'}\rangle \langle \psi^{(0)}_\beta|}{\lambda_{\beta'} - \lambda_\beta} + \cdots. \tag{30}
$$

We will only need to consider the hydrodynamic modes $\beta = 1, \ldots, 5$. The restriction in Eq. (30) that $\lambda_{\beta'} \neq \lambda_\beta$ indicates that $\beta' \geq 6$. With these considerations, we may rewrite Eq. (30) as

$$
\omega_\beta = v_T k |\psi^{(0)}_\beta\rangle \langle \psi^{(0)}_\beta| + \frac{i v_T^2 k^2}{\gamma} \sum_{\beta', \mu_{\beta'} = 0}^{\beta_{\text{max}}} \frac{|\psi^{(0)}_\beta\rangle \langle \psi^{(0)}_{\beta'}| |\psi^{(0)}_{\beta'}\rangle \langle \psi^{(0)}_\beta|}{\mu_{\beta'} - \mu_{\beta}} + \cdots. \tag{31}
$$

Since the eigenvalues $\lambda_{\beta'}$ are all positive, the term of order $k^2$ lies on the negative imaginary axis and the hydrodynamic modes to decay towards global equilibrium.

C. First-order corrections

The first-order corrections to the hydrodynamic frequencies are given by

$$
\omega^{(1)}_\beta = v_T k |\psi^{(0)}_\beta\rangle \langle \psi^{(0)}_\beta|, \tag{32}
$$

It is straightforward to compute these quantities in the angle basis and we find

$$
\omega^{(1)}_1 = \omega^{(1)}_2 = \omega^{(1)}_3 = 0 \quad \text{and} \quad -\omega^{(1)}_4 = \omega^{(1)}_5 = v_T \sqrt{5\sigma_4/\sigma_2}. \tag{33}
$$

The frequencies $\omega^{(1)}_4$ and $\omega^{(1)}_5$ correspond to sound waves traveling at the speed $c_s$, which matches the result (23) obtained from hydrodynamics.

D. Second-order corrections

The second-order corrections to the hydrodynamic frequencies $\omega_\beta (\beta = 1, \ldots, 5)$ are given by

$$
\omega^{(2)}_\beta = -i v_T^2 \sum_{\gamma, \beta', \mu_{\beta'}} |\psi^{(0)}_\beta\rangle \langle \psi^{(0)}_{\beta'}| |\psi^{(0)}_{\beta'}\rangle \langle \psi^{(0)}_\beta|, \tag{34}
$$

This expression is only of practical value when a small number of terms are needed for the sum to converge. In the case of the classical Maxwell gas, this is guaranteed because only a limited number of inner products $|\psi^{(0)}_\beta\rangle \langle \psi^{(0)}_{\beta'}|$ are nonzero. For other classical soft potentials, the convergence comes from the steadily increasing $\lambda_{\beta'}$.

When the eigenvalue spectrum has a cutoff, as it does for hard spheres and contact interactions [16], there are an infinite number of terms with $\lambda_{\beta'} \approx \lambda_M$, and when they are summed they give a finite contribution to the transport coefficient. We can rearrange the summation in Eq. (34) to simplify the computation of the large number of terms that must be included. We rewrite Eq. (34) in the form

$$
\omega^{(2)}_\beta = -i v_T^2 \sum_{\gamma} \frac{|\psi^{(0)}_\beta\rangle \langle \psi^{(0)}_{\beta'}| |\psi^{(0)}_{\beta'}\rangle \langle \psi^{(0)}_\beta|}{\lambda_M - \mu_{\beta'}} + \frac{i v_T^2 k^2}{\gamma} \sum_{\beta', \mu_{\beta'} = 0}^{\beta_{\text{max}}} \frac{|\psi^{(0)}_\beta\rangle \langle \psi^{(0)}_{\beta'}| |\psi^{(0)}_{\beta'}\rangle \langle \psi^{(0)}_\beta|}{\mu_{\beta'} - \mu_{\beta}}, \tag{35}
$$

where $\mu_{\beta'} = (1/\lambda_{\beta'} - 1/\lambda_M)^{-1}$ and $\beta_{\text{max}}$ is chosen so that $\lambda_{\beta_{\text{max}}} \approx \lambda_M$. In contrast to Eq. (34), only the first few terms contribute to the sum in Eq. (35), because $\mu_{\beta'}$ rapidly becomes large. As we will show in Sec. V, Eq. (35) must be used to obtain correct results for hard spheres and contact interactions. In Appendix A, we show in more detail how Eq. (35) is obtained from Eq. (34).

E. Microscopic expressions for the transport coefficients

We now equate the microscopic frequencies to the hydrodynamics frequencies. As we show in Appendix B, the transport coefficients can be written as

$$
\eta \xi = \frac{\sigma_4}{2\sigma_3} + \frac{8\pi\sigma_0}{15\xi} \sum_{n=0}^{n_{\text{max}}} |Q_{n,2}|^2, \tag{36}
$$
The lower limits of the \( n \) summations come from the condition that \( \lambda_{n,l} > 0 \). The upper limits of the \( n \) summations, \( n_{\text{max}} \), are the analogy to \( \tilde{\rho}_{\text{max}} \) for a specific value of \( l \).

The expression for the bulk viscosity \( \zeta \) implies that it is zero, because \( S_n \) is identically zero for \( n \geq 2 \). This is easily seen if we note that \( c^3 \sqrt{w(c)} \) is a linear combination of \( \psi_{0,0}^{(0)}(c) \) and \( \psi_{1,0}^{(0)}(c) \) [see Eq. (20)], and thus the function \( c^3 \sqrt{w(c)} \) is orthogonal to \( \psi_{n,0}^{(0)}(c) \) for \( n \geq 2 \). Only the two values \( S_0 \) and \( S_1 \) are nonzero, but these are not included in the sum because \( \lambda_{0,0} \) and \( \lambda_{1,0} \) are zero. The fact that a monatomic gas has zero bulk viscosity is well known [18,19]. Here we have shown that this result holds true in a quantum treatment of the noncondensed Bose gas.

V. NUMERICAL COMPUTATION OF TRANSPORT COEFFICIENTS

Numerical computation of transport coefficients requires a numerical computation of the eigenvalues and eigenvectors of the collision operator. We form a matrix representation of the collision operator and diagonalize it [16,20]. Our results are obtained by performing the calculation with several different matrix sizes, and extrapolating these results to an infinite matrix size.

In Fig. 1 we show our numerical results for the transport coefficients of a degenerate noncondensed boson gas. The transport coefficients are plotted versus the reduced temperature \( T/T_C \), where \( T_C \) is the critical temperature for Bose-Einstein condensation. Note that the transport coefficients have an overall dependence on \( \sqrt{T} \), which is divided out in the dimensionless transport coefficients. We also plot the Eucken number, which is defined as \( mk / (\eta c_n) \), where \( c_n \) is given in Eq. (25).

Filled circles (black) depict the “exact” U-U equation result of Eqs. (36) and (37) (with contact potential). The straight solid (blue) line depicts the U-U value as \( T/T_C \rightarrow \infty \) (the Boltzmann limit). The straight dashed (red) line depicts the approximate classical result obtained by using only the lowest order sonine polynomial [6,14]. The open circles (gray) show the results obtained by using Eq. (34) instead of Eq. (35) and ignoring the cutoff behavior.

The transport coefficients for the degenerate boson gas are indistinguishable from the classical Boltzmann values for temperatures above 20\( T_C \), but become sensibly different for \( T < 10T_C \). As one approaches \( T_C \), the shear viscosity decreases to a value that is approximately 82% of its value at high temperature. The thermal conductivity shows a minimum around \( T = 1.5T_C \) at approximately 95% of its high-temperature values and then increases as the temperature approaches \( T_C \), perhaps anticipating the appearance of second sound [14,21] below \( T_C \).

The Eucken number is useful for comparing our results to other systems and experiments, because it is independent of the overall interaction strength. To test the validity of our result, we compare Eucken numbers in the classical limit. We have calculated the Eucken number for soft potentials (\( V(r) \sim r^{-n} \)), as a function of \( n \), in the classical limit (\( T/T_C \rightarrow \infty \)). This is accomplished by using a generalization of the polynomial method for the Maxwell gas [20]. The results of this calculation are shown in Fig. 2. With a potential that goes as \( r^{-4} \), the Eucken number is 5/2. As the power of \( 1/r \) is increased, the Eucken number slightly increases at first and then drops toward the hard sphere result (thick black line), which takes account of the cutoff. At very high powers of \( 1/r \), increasingly large matrix sizes are needed to see convergence toward the hard
We have calculated the transport coefficients of a Bose gas interacting with a contact potential as they vary with temperature. Overall, we find that the transport coefficients of the Bose gas decrease faster than a classical gas with decreasing temperature, owing to the increased scattering produced by the Bose enhancement factors in the collision integral. We also observe a sharp increase in the thermal conductivity as the temperature nears $T_C$, which may be a precursor of the onset of second sound in the condensate. Using the U-U equation instead of the Boltzmann equation does not change the fact the the bulk viscosity for a monatomic gas is zero.

Finally, we have shown that when an eigenvalue cutoff exists, as it does for contact interactions in the U-U collision operator, it is important to treat the cutoff correctly when calculating transport properties.

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APPENDIX A: EXPRESSION FOR FREQUENCY CORRECTIONS IN THE PRESENCE OF AN EIGENVALUE CUTOFF

In this appendix, we demonstrate how Eq. (35) is derived from Eq. (34). We begin with the crucial assumption that beyond a certain order ($\beta'$) the numerically calculated eigenvalues/eigenvectors become erroneous due to truncation errors in the matrix representation of the collision operator. The highest reliable eigenvalue/eigenvector pair will be labeled with $\beta_{\text{max}}$. The rapid convergence of eigenvalues to their limiting value makes it acceptable to replace the remaining eigenvalues with the limiting value $\lambda_M$.

We seek a form of Eq. (34) that does not explicitly depend on the eigenvalues or eigenvectors with $\beta > \beta_{\text{max}}$. To do this, we first split the sum in Eq. (34) into two parts,

$$\omega_{\beta}^{(2)} = -\frac{i\nu^2 k^2}{\gamma} \sum_{\beta' = 6}^{\beta_{\text{max}}} \langle \psi_{\beta'} (0) | \hat{c}_i (0) | \psi_{\beta'} (0) \rangle \langle \psi_{\beta'} (0) | \hat{c}_i (0) | \psi_{\beta'} (0) \rangle + \frac{1}{\lambda M} \sum_{\beta' > \beta_{\text{max}}} \infty \langle \psi_{\beta'} (0) | \hat{c}_i (0) | \psi_{\beta'} (0) \rangle \langle \psi_{\beta'} (0) | \hat{c}_i (0) | \psi_{\beta'} (0) \rangle , (A1)$$

where the first term accounts for the eigenmodes which are given accurately from the numerical calculation, and the second term contains all higher-order eigenmodes, which are not accurately determined by the numerical calculation.

To simplify the second term on the right-hand side of Eq. (A1), we rewrite it as

$$\sum_{\beta' > \beta_{\text{max}}} \infty \langle \psi_{\beta'} (0) | \hat{c}_i (0) | \psi_{\beta'} (0) \rangle \langle \psi_{\beta'} (0) | \hat{c}_i (0) | \psi_{\beta'} (0) \rangle = \sum_{\beta' = 1}^{\infty} \langle \psi_{\beta'} (0) | \hat{c}_i (0) | \psi_{\beta'} (0) \rangle \langle \psi_{\beta'} (0) | \hat{c}_i (0) | \psi_{\beta'} (0) \rangle$$

VI. CONCLUSION

We have derived an expression for the transport coefficients of a monatomic dilute Bose gas that obeys the Uehling-Uhlenbeck kinetic equation. Our expression relates the transport coefficients to a spectral decomposition of the linearized collision operator. This illuminates the relation between the collision operator, the microscopic hydrodynamic modes, and the transport coefficients.
The first term is identified as \((\psi_\beta^0 | i | \psi_\beta^0)\), due to the fact that the eigenvectors \((| \psi_\beta^0 \rangle\) with \(\mu_\beta = 1, \ldots, \infty\) form a complete set. The second term is identified as \((\psi_\beta^0 | i | \psi_\beta^0)^2\), due to the fact that the eigenvectors \(| \psi_\beta^0 \rangle\) satisfy \((\psi_\beta^0 | i | \psi_\beta^0) = \delta_{\beta,\beta'}\langle \psi_\beta^0 | i | \psi_\beta^0 \rangle\) for \(\mu_\beta = 1, \ldots, \infty\). The third term can be combined with the first term of Eq. (A1).

Using these simplifications, we obtain

\[
\omega_\beta^{(2)} = -\frac{i v^2 k^2}{\gamma} \left[ \frac{\Delta_\beta}{\lambda_M} + \Omega_\beta \right]
\]

where \(\mu_\beta = \left( \frac{\lambda_M}{\gamma} \right)^{-1} \).

This approximate expression for \(\omega_\beta^{(2)}\) does not depend on any of the unreliable eigenmodes or eigenvalues. It is approximate because in deriving it, we have replaced a portion of the eigenvalues with \(\lambda_M\). The appropriateness of this replacement may be judged by examining the values \(\mu_\beta\), which quickly become large as the eigenvalues approach \(\lambda_M\). A negative value of \(\mu_\beta\) indicates that \(\lambda_M\) is erroneous and we use this as the condition for determining \(\beta_{\text{max}}\).

**APPENDIX B: CALCULATION OF FREQUENCY CORRECTIONS**

In this appendix, we give some details of the calculations needed to obtain Eqs. (36), (37) and (38) from the general expression Eq. (35).

The second-order corrections to the hydrodynamic frequencies can be written as

\[
\omega_\beta^{(2)} = -\frac{i v^2 k^2}{\gamma} \left[ \frac{\Delta_\beta}{\lambda_M} + \Omega_\beta \right],
\]

where we have defined

\[
\Delta_\beta = \langle \psi_\beta^0 | i | \psi_\beta^0 \rangle - \langle \psi_\beta^0 | i | \psi_\beta^0 \rangle^2,
\]

and

\[
\Omega_\beta = \sum_{\beta' = 6}^{\beta_{\text{max}}} \frac{\langle \psi_\beta^0 | i | \psi_\beta^0 \rangle \langle \psi_\beta^0 | i | \psi_\beta^0 \rangle}{\mu_\beta}.
\]

Or, more explicitly,

\[
\Omega_\beta = \sum_{\mu_\beta} \sum_{\lambda_M} \sum_{\mu_\beta} \frac{\langle \psi_\beta^0 | i | \psi_\beta^0 \rangle \langle \psi_\beta^0 | i | \psi_\beta^0 \rangle}{\mu_\beta}. \tag{B4}
\]

Simplification of Eqs. (B2) and (B4) can be done relatively quickly by using the angle basis defined in Sec. III A. First we must express the zeroth-order eigenstates in terms of the \(|\psi_\lambda^0\rangle\) basis. They are

\[
\langle \psi_{\mu \lambda}^0 | \psi_{\mu \lambda}^0 \rangle
\]

\[
\langle \psi_{\mu \lambda}^0 | i | \psi_{\mu \lambda}^0 \rangle = \sqrt{\frac{4 \pi \sigma_0 w(c)}{\lambda^2}} \delta_{\mu, \lambda} \delta_{\mu, \lambda} - \delta_{m, -\lambda}, \tag{B5a}
\]

\[
\langle \psi_{\mu \lambda}^0 | i | \psi_{\mu \lambda}^0 \rangle = \sqrt{\frac{4 \pi \sigma_0 w(c)}{\lambda^2}} \delta_{\mu, \lambda} \delta_{\mu, \lambda} + \delta_{m, -\lambda}, \tag{B5b}
\]

\[
\langle \psi_{\mu \lambda}^0 | i | \psi_{\mu \lambda}^0 \rangle = \sum_{\lambda M} \left( \epsilon^2_{\lambda M} \right) \delta_{\mu, \lambda} \delta_{\mu, \lambda} - c \epsilon_{\mu, \lambda} \delta_{m, -\lambda}, \tag{B5c}
\]

\[
\langle \psi_{\mu \lambda}^0 | i | \psi_{\mu \lambda}^0 \rangle = \sum_{\lambda M} \left( \epsilon^2_{\lambda M} \right) \delta_{\mu, \lambda} \delta_{\mu, \lambda} + c \epsilon_{\mu, \lambda} \delta_{m, -\lambda}, \tag{B5d}
\]

\[
\langle \psi_{\mu \lambda}^0 | i | \psi_{\mu \lambda}^0 \rangle = \sum_{\lambda M} \left( \epsilon^2_{\lambda M} \right) \delta_{\mu, \lambda} \delta_{\mu, \lambda} + c \epsilon_{\mu, \lambda} \delta_{m, -\lambda}.
\]

Expressing the operator \(\hat{c}_\lambda\) in the \(|\psi_{\mu \lambda}^0\rangle\) basis provides an economical way to evaluate \(\Delta_\beta\) and \(\Omega_\beta\). We can do this by inserting identity operators to obtain

\[
\hat{c}_\lambda = \int_0^\infty \frac{dc}{l_0} \sum_{l, m} \int_0^\infty \frac{dc'}{l', m'} \sum_{l', m'} \int dc_1 |l, m\rangle \langle c, l, m| \langle c_1 |l, m_1\rangle |c, l, m_1\rangle |c', l', m'\rangle \langle c', l', m' |, \tag{B6}
\]

We then use the definitions (15) to get

\[
\hat{c}_\lambda = \int_0^\infty \frac{dc}{l_0} \sum_{l, m} \sum_{l', m'} |c, l, m\rangle \langle c_1 |l, m_1\rangle \langle c_1 |l, m_1\rangle |c, l, m_1\rangle |c', l', m'\rangle \langle c', l', m' |. \tag{B7}
\]

Performing the angular integration in the bracketed term involves Wigner 3-j symbols, but can be simplified to

\[
\hat{c}_\lambda = \int_0^\infty \frac{dc}{l_0} \sum_{l=0}^l \sum_{m=-l}^{l} |c, l, m\rangle \langle c_1 |l, m_1\rangle \langle c_1 |l, m_1\rangle |c, l, m_1\rangle \langle c', l', m' |. \tag{B8}
\]

where \(J_{l+1, m}(c, l+1, -m) + J_{l, m}(c, l-1, -m)\).

Calculating \(\Delta_\beta\) using the expressions (B5) and (B8), one obtains

\[
\Delta_1 = \Delta_2 = \frac{\sigma_3}{2 \sigma_2}, \tag{B9}
\]

\[
\Delta_3 = \frac{7 \sigma_3 \sigma_2 - 5 \sigma_4^2}{2 \sigma_2}, \tag{B10}
\]

\[
\Delta_4 = \frac{7 \sigma_3 \sigma_2 - 4 \sigma_4^2}{12 \sigma_2 \sigma_4}. \tag{B11}
\]

To demonstrate the method of calculation for \(\Omega_\beta\), we outline the calculation of \(\Omega_1\) below. Starting with Eq. (B4) with \(\beta = 1\),
we begin by inserting the expression (B8) for both occurrences of $\hat{c}_1$ and using the relations in Eqs. (B5) and (19) to obtain

$$\Omega_1 = \frac{4\pi \sigma_0}{3\sigma_2} \sum_{n,\lambda_n > 0} \sum_{l,m} \frac{1}{\mu_{n,1}} \sum_{l_1, m_1, l_2, m_2}$$

$$\times \int_0^\infty dc_1 c_1^2 \sqrt{w(c_1)\psi_{n,l}(c_1)} \int_0^\infty dc_2 c_2^3 \sqrt{w(c_2)\psi_{n,l}^*(c_2)}$$

$$\times \delta_{l,1} \left( \delta_{m,1} - \delta_{m_1,1} \right) \delta_{m,-m_1} \delta_{l,1} \left( \delta_{m_2,1} - \delta_{m_2,-1} \right) \delta_{m,m_2}$$

$$\times \left( \langle J_{l_1+1,m_1} \delta_{l_1,l_1+1} + J_{l_1,m_1} \delta_{l_1,l_1-1} \rangle \langle J_{l_2+1,m_2} \delta_{l_2,l_2+1} \rangle + J_{l_2,m_2} \delta_{l_2,l_1+1} \right).$$

(B12)

Performing the summations and some minor simplifications, we get

$$\Omega_1 = \frac{8\pi \sigma_0}{15\sigma_2} \sum_{n=0}^{n_{\text{max}}} \left| Q_n \right|^2,$$

(B13)

where $Q_n$ is given in Eq. (39). In a similar calculation, we find that $\Omega_2 = \Omega_1$ and that

$$\Omega_3 = \frac{16\pi \sigma_2^2 \sigma_0}{15\sigma_4(5\sigma_4 \sigma_0 - 3\sigma_2^2)} \sum_{n=1}^{n_{\text{max}}} \left| R_n \right|^2.$$

(B14)

$$\Omega_4 = \frac{8\pi \sigma_0}{45\sigma_2} \sum_{n=0}^{n_{\text{max}}} \left| R_n \right|^2$$

$$+ \frac{16\pi \sigma_0}{45\sigma_2} \sum_{n=0}^{n_{\text{max}}} \left| Q_n \right|^2 + \frac{4\pi \sigma_0}{9\sigma_2} \sum_{n=0}^{n_{\text{max}}} \left| S_n \right|^2,$$

(B15)

where $R_n$ and $S_n$ are those given in Eqs. (40) and (41).

In carrying out the calculation of $\Omega_3$ for this five cases ($\beta = 1, \ldots, 5$), we find that there is only one remaining summation. The range of this summation must be restricted so that $0 < \lambda_n < \lambda_M$. We define $n_{\text{max}}$ in analogy with $\beta_{\text{max}}$ as the largest value of $n$ for which the numerically computed $\lambda_n$ is less than $\lambda_M$ for a given value of $l$.

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