We elaborate the statistical field theory of Turbulence suggested in the previous paper [1]. We clarify and simplify the basic Energy pumping equation of that theory and study mathematical properties of singular field configuration (instanton) which determine the tails of PDF for the velocity circulation around large loop $C$ in isotropic turbulence at highest Reynolds numbers. Explicit analytic solution is found for the Clebsch instanton in an Euler equation for a planar loop circulation problem. This solution for vorticity is has a term proportional to a delta function in normal direction to the minimal surface bounded by the loop. The smoothing of $\delta$ functions in the vorticity in the full Navier-Stokes equations is investigated and exponential profile of smoothed singularity is found. The PDF for circulation is now an infinite sum of decreasing exponential terms $\exp\left(-n|w|\right) \sqrt{\frac{n}{|w|}}$, with $w = \frac{\Gamma}{\Gamma_0[C]}$, and $\Gamma_0[C] \sim \sqrt{A_C}$ with minimal area $A_C$. The leading term fits with adjusted $R^2 = 0.9999$ the PDF tail found in DNS over more than six orders of magnitude. The area dependence of the ratio of the circulation moments $M_8/M_6$ fits with adjusted $R^2 = 0.9996$ the DNS in inertial range of square loop sizes from 100 to 500 Kolmogorov scales. Thus, our theory explains DNS with high degree or confidence. For a flat loop we derive two-dimensional integral equation for the dependence of the scale $\Gamma_0[C]$ of circulation as a function of the shape of the loop (aspect ratio for rectangular loop).

**Keywords** Turbulence · Instanton · Exact Solution · Euler · Clebsch

### 1 Introduction. Statistical Approach to The Turbulence

The statistical approach to the turbulence suggested in [1] is based on a concept of an effective distribution for vorticity in a cell in a certain domain in space, with all remaining vorticity cells acting as a thermostat. This is similar to a step leading to a Gibbs distribution from a microcanonical distribution.

In our case the energy $H = \frac{1}{2} \int d^3r \nu_\alpha^2$ is not an integral of motion, there is instead a steady energy flow from the boundary of the system, which is dissipated at smallest scales, where vorticity dominates in the Navier-Stokes equations and Euler dynamics break.

We do not assume any specific models of this energy flux except that it originates at the boundary by some random forces. The energy balance equation which follows from Navier-Stokes equation was reduced in [1] to an unfamiliar form

$$\int_V d^3r \nu_\alpha^2(r) - f_\alpha \int_V d^3r e_{\alpha\beta\gamma} r_\beta \omega_\gamma(r) = 0$$  \hspace{1cm} \text{(1)}$$

where the external random force $f$ is related to pressure distribution over an infinite bounding sphere (with $\bar{n}$ being its normal vector)

$$f_\alpha = - \langle n_\alpha n_\beta \partial_\beta p \rangle_{n \in S_2}$$  \hspace{1cm} \text{(2)}$$
The green flat blob in that picture represents our instanton, – singular vorticity sheet which we are going to study in this paper.

It is assumed that the net vorticity is zero

\[ \int_V d^3r \omega(r) = 0 \]  \hspace{1cm} (3)

in which case the integral \( \int_V d^3r e_{\alpha \beta \gamma} r_\beta \omega_\gamma(r) \) is translation invariant.

In virtue of the Central Limit Theorem we expect this force to be a Gaussian random vector with isotropic distribution

\[ P(\vec{f}) \propto \exp \left( -\frac{\vec{f}^2}{2\sigma} \right) \]  \hspace{1cm} (4)

Our formula for the energy flow is additive with respect to the vorticity cells and is dominated by the regions of high vorticity.

If we single out the contribution from a particular vorticity cell where we are studying vorticity distribution, averaged over all configurations of remaining vorticity cells (thermostat), we get an effective Hamiltonian for a subsystem in a volume \( V_s \) under study

\[ H^{eff} = \int_{V_s} d^3r \omega_\alpha^2(r) - f_\alpha \int_{V_s} d^3r e_{\alpha \beta \gamma} r_\beta \omega_\gamma(r) \]  \hspace{1cm} (5)

which enters our distribution as \( \exp (-\lambda H^{eff}) \) with some Lagrange multiplier \( \lambda \) analogous to \( \beta = 1/T \) in the Gibbs distribution. This \( \lambda \) arises as a derivative of an entropy of the thermostat configurations as a function of the energy flow. In our case these are configurations of the vorticity in its steady state (Generalized Beltrami Flow, or GBF).

The transformation from the microcanonical distribution with (1) imposed as a delta function in the phase space to the canonical distribution (Gibbs) is discussed in Appendix A for reader’s convenience, though it must have been published many times in appropriate textbooks.

We should also keep in mind that in the Biot-Savart integral for the velocity field there is an additional contribution \( \vec{v}^T(r_0) \) from the thermostat as well as from the cell we are studying

\[ \vec{v}(r_0) = \frac{1}{4\pi} \int_{V_s} d^3r \vec{\omega}(r) \times \nabla \frac{1}{|r-r_0|} + \vec{v}^T(r_0) \]  \hspace{1cm} (6)

This effective Gibbs distribution is acting on top of the steady flow conditions GBF. In space of all vorticity configurations there is certain invariant measure in Clebsch coordinates, selecting every such flow with equal weight and eliminating the unstable flows (with bad Lyapunov indexes for vorticity field).

This measure was constructed and discussed in detail in [1]. We do not need all this heavy machinery here, as we are going to study just the leading WKB approximation for the vorticity PDF in presence of large circulation over large flat loop in coordinate space.
There is also some new understanding of the thermostat which we are bringing in now, namely the boundary condition at infinity, at this remote sphere where random forces are creating energy flow.

We argue that the thermostat velocity should decay as follows (with \( \hat{r} \) being unit vector directed at \( \vec{r} \))

\[
v_{T\alpha}(\vec{r}) \to \gamma f_\beta \partial_\alpha \frac{1}{|\vec{r}|} = \gamma \frac{3 r_\alpha \hat{r}_\beta f_\beta - f_\alpha}{|\vec{r}|^3}
\]

with constant \( \gamma \) having dimension \( r^3 t \). One may easily check that vorticity is zero for this velocity field up to higher order corrections at large \( |\vec{r}| \) and that this field is incompressible.

The energy flow \( \mathcal{E}^T \) into the thermostat at the boundary (infinite sphere) with external pressure

\[
p^{\text{ext}} = |\vec{r}| g(\hat{r});
\]

\[
g(\hat{r}) = r_\alpha \partial_\alpha p^{\text{ext}};
\]

\[
\mathcal{E}^T = \int_{S_2} d\sigma_\alpha v_{T\alpha}^{\text{ext}} p^{\text{ext}} = 8 \pi \gamma f^2
\]

\[
\langle \mathcal{E}^T \rangle = 24 \pi \gamma \sigma
\]

which determines this new parameter \( \gamma \)

\[
\gamma = \frac{\langle \mathcal{E}^T \rangle}{24 \pi \sigma}
\]

So, the random forces working on a remote sphere induced mass flow through the surface, netting to zero as it should in steady state.

Note that we did not assume anything about velocity or vorticity in the thermostat at finite distances, just its asymptotic behavior on the infinite boundary. We demanded that velocity has a pole at infinity compatible with incompressibility and matching the energy flow \( \mathcal{E}^T \) produced by random forces.

We are using here specific coordinate frame centered at the origin, and we use the sphere as a boundary. With some obvious generalizations the results should come out the same for arbitrary shape off the boundary surface as it was discussed in [1].

In summary, we assume two components of vorticity field, both decreasing as \( 1/|\vec{r}|^3 \): the localized vorticity cell (singular vorticity sheet in a limit of large circulation flow), and the background thermostat vorticity, spread over space.

The thermostat velocity is involved in receiving and passing the power generated by the outside random forces on a remote sphere. The localized cell is receiving the energy flow and dissipating it on a vorticity singularities (which singularities, as we shall see later, are smeared by viscosity to become Zeldovich pancakes of the viscous width).

The inertial range in our theory is a physical space rather than symbolic range of wavelengths: this is a space between the bounding sphere, where the energy flow originates, and the vorticity peaks where it is dissipated.

Weak background vorticity is spread over this space, and dissipation there is proportional to vanishing viscosity, whereas at peaks this small viscosity is compensated by large density of vorticity.

We describe this as a stationary distribution with Gaussian force as a vector parameter, but this can be regarded as a random process, describing time evolution of velocity field at large distances, where nonlinear effects disapper

\[
\partial_\alpha v_{T\alpha}^\prime(\vec{r}, t) = -\partial_\alpha p(\vec{r}, t);
\]

\[
p(\vec{r}, t) = -\gamma f_\beta(t) \partial_\beta \frac{1}{|\vec{r}|}.
\]

This internal pressure \( p(\vec{r}, t) \) describes time evolution of our random forces \( f_\beta(t) \) regarded as a time series. At large time these forces obey Gauss distribution. So, instead of averaging over time we average over this distribution as usual in stochastic differential equations.

This is different from a Kolmogorov scenario, but maybe it is time to move on after 80 years of praying to great Andrey Kolmogorov. In a feat of intuition he discovered the heart of the turbulence phenomena, but as it turns out, turbulence has some other body parts as well, and some of these parts are easier to study than the others.

The picture described here seems adequate to the high Reynolds DNS [2] as far as the circulation distribution is concerned. The critical phenomena taking place in turbulent flow at smaller spatial scales in absence of large circulation, are so far beyond the reach in our approach.
At the qualitative level we may view these multi-scale fluctuations as coming from singular vorticity structures (surfaces) of various spatial scales, uniformly distributed over space. This is similar to duality in four dimensional field theory. There, too, fluctuating fields in a strong coupling phase are equivalent to weakly fluctuating strings which are two-dimensional surfaces in space-time.

So, we are not trying to deny the complex multi-fractal distributions of local vorticity and velocity differences. Obviously, these phenomena are real – but we simply found the conditions when these fluctuations are decoupling from the main singular flow. We found the way to bypass this complexity and get some exact relations for other observable quantities by using dual language of singular vorticity sheets.

By the way, nobody has proven that the multi-fractal scaling phenomena are even universal – the physics of the ensemble of the vorticity structures of varying sizes could depend of the specifics of the energy pumping on a bounding sphere, simply because the farther away the more influence comes from the velocity correlation growing as $r^{2/3}$. In our case there are special reasons which we discuss, for the circulation PDF to come out universal up to scale factors.

The interaction between vorticity sheet and thermostat is described by a master equation which we derive and solve below.

Once again, our picture is anisotropic and our coordinate frame is fixed, simply because we are studying conditional probability for large velocity circulation around some large loop in coordinate space. Local velocity fluctuations play little role in this situation, it is all dominated by some steady singular flow, parametrized by global random force, implicitly describing stochastic process.

## 2 Clebsch instanton

We found in [1] multi-valued fields with nontrivial topology which are relevant to large circulation asymptotic behavior. Vorticity is parametrized by famous Clebsch coordinates

$$\omega_\alpha = e_{\alpha\beta\gamma} \partial_\beta \phi_1 \partial_\gamma \phi_2$$  \hspace{1cm} (15)

and velocity is related to vorticity by a Biot-Savart integral

$$v_\alpha(r) = -e_{\alpha\beta\gamma} \partial_\beta \int d^3r' \frac{\omega_\gamma(r')} {4\pi |r - r'|}$$  \hspace{1cm} (16)

### 2.1 Gauge Invariance and Clebsch Confinement

There are some gauge transformations (canonical transformation in terms of Hamiltonian system, or area preserving diffeomorphisms geometrically) which leave vorticity invariant

$$\phi_\alpha(r) \Rightarrow G_\alpha(\phi(r))$$

$$\det \frac{\partial G_\alpha}{\partial \phi_b} = \frac{\partial (G_1, G_2)}{\partial (\phi_1, \phi_2)} = 1.$$  \hspace{1cm} (18)

Infinitesimal version of these transformation is

$$\delta \phi_\alpha = \epsilon e_{ab} \frac{\partial h}{\partial \phi_b}$$  \hspace{1cm} (19)

with arbitrary function $h(\phi_1, \phi_2)$.

The conventional time evolution for Clebsch fields in Euler Hamiltonian dynamics is just a passive convection

$$\partial_t \phi_\alpha = -v_\alpha \partial_\alpha \phi_a$$  \hspace{1cm} (20)

We, however, generalize this evolution by adding time dependent gauge transformation which produce equivalent Clebsch fields

$$\partial_t \phi_\alpha = -v_\alpha \partial_\alpha \phi_a + e_{ab} \frac{\partial h}{\partial \phi_b}$$  \hspace{1cm} (21)

Independently of the gauge function $h(\phi)$ the vorticity satisfies the same equations

$$\partial_t \omega_\alpha = \omega_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta \omega_\alpha$$  \hspace{1cm} (22)

This is a direct consequence of gauge invariance of the Clebsch parametrization of vorticity.
The Turbulence phenomenon in fluid dynamics in Clebsch variables resembles the color confinement in QCD. We have no Yang-Mills gauge field here, but instead we have nonlinear Clebsch field participating in gauge transformations. These transformations are global as opposed to local gauge transformations in QCD, but the common part is that this symmetry stays unbroken and leads to confinement of Clebsch field.

The description of Clebsch field as nonlinear waves \[3\] which was appropriate at large viscosity, or weak turbulence, quickly gets hopelessly complex when one tries to go beyond the K41 law into fully developed turbulence. The basic assumption \[3\] of the Gaussian distribution of Clebsch field breaks down at small viscosity.

The small viscosity in Navier-Stokes equations is a nonperturbative limit, like the infra-red phenomena in QCD, when the waves combine into non-local and nonlinear structures best described as solitons or instantons.

Nobody managed to explain color confinement in gauge theories as a result of strong interaction of gluon waves. On the contrary, the topologically nontrivial field configurations such as monopoles in 3D gauge theory and instantons in 4D led to the understanding of the color confinement.

This is what we are doing here as well, except our singular solutions are not point like singularities but rather singular vorticity sheets.

Vorticity sheets (so called Zeldovich pancakes \[4\]), were extensively discussed in the literature in the context of the cosmic turbulence. Superficially they look similar to my instantons but at closer look there are some important distinctions. For one thing they are unrelated to the minimal surfaces, and for another one, they seem to have no topological numbers.

The general physics of the “frozen” vorticity in incompressible flow, collapsing in the normal direction and expanding along the surface, is essentially the same. What is different here is an explicit singular solution with its tangent and normal components at the minimal surface, the Clebsch field topology and its consequences for the circulation PDF.

The relevance of classical solutions in nonlinear stochastic equations to the intermittency phenomena (tails of the PDF for observables) was noticed back in the 90-ties \[5\] when it was used \[6\] to explain intermittency in Burgers equation. However, nobody succeeded in finding the instanton solution in 3D fluid dynamics until now.

2.2 Discontinuity at the Minimal Surface

Let us now describe the proposed stationary solutions of Euler equations in Clebsch variables.

Our Clebsch field \(\phi_2\) has \(2\pi n\) discontinuity across the minimal surface \(S_C\) bounded by \(C\). As it was argued in \[1\] the minimal surface is compatible with Clebsch parametrization of conserved vorticity directed at its normal in linear vicinity of the surface.

We parametrize the minimal surface \[\#\] as a mapping to \(R_3\) from the unit disk in polar coordinates \(\rho, \alpha\)

\[S_C : \vec{r} = \vec{X}(\xi), \ \xi = (\rho, \alpha)\]  

In the linear vicinity of the surface

\[\delta S_C : \vec{r} = \vec{X}(\xi) + z\vec{n}(\xi)\]  

the Clebsch field \(\phi_2\) is discontinuous

\[\phi_2 (\vec{r} \in \delta S_C) = m\alpha + 2\pi n\theta(z) + O(z^2); \ m, n \in Z\]  

while the other component is continuous

\[\phi_1 (\vec{r} \in \delta S_C) = \Phi(\xi) + O(z^2)\]  

The vorticity has the delta-function singularity at the surface:

\[\Omega(\xi) = m\frac{\partial\Phi(\xi)}{\partial\rho}\sqrt{|\text{det} g|}\]  

\[\vec{n} = \frac{\partial \vec{X}}{\partial \rho} \times \frac{\partial \vec{X}}{\partial \alpha}\sqrt{|\text{det} g|};\]  

\[\text{there is a mathematical theory, initiated by Weierstrass, relating these surfaces on three dimensions to a pair of analytic functions. We reproduce it in \[1\] in modern field theory jargon.}\]
If you study the vorticity conservation
\[ \partial_\alpha \omega_\alpha (r \in \delta S_C) = 0 \] (31)
you will arrive at the self-consistency equation [1]
\[ \partial_\alpha n_\alpha = 0 \] (32)
corresponding to the mean curvature being zero at the minimal surface.

This delta term in vorticity is orthogonal to the normal vector to the surface and thus does not contribute to the flux through the minimal surface, so this flux is still determined by the second (regular) term and circulation is related to this
\[ \Phi(\xi) \]
\[ \Gamma_C = \oint_C v_\alpha dr_\alpha = \int_S d\phi_1 \wedge d\phi_2 = m \int_0^{2\pi} (\Phi(1, \alpha) - \Phi(0, \alpha)) d\alpha \] (33)
The Stokes theorem ensures that the flux through any other surface bounded by the loop \( C \) would be the same, but in that case the singular tangent component of vorticity would also contribute. The simplest computation corresponds to choosing the flux through the minimal surface.

The instanton velocity field reduces to the surface integral
\[ v_\beta^{\text{inst}}(r) = 2\pi n \left( \delta_{\beta \gamma} \partial_\alpha - \delta_{\alpha \beta} \partial_\gamma \right) \int_{D_C} d\sigma_\gamma(\xi) \partial_\alpha \Phi(\xi) \frac{1}{4\pi |\vec{X}(\xi) - \vec{r}|} \] (34)

We are assuming that the Clebsch field falls off outside the surface so that vorticity is present only in an infinitesimal layer surrounding this surface. In this case only the delta function term contributes to the Biot-Savart integral though only a regular term contributes to the circulation.

Let us now consider the steady flow Clebsch equations derived in [1], which we call the master equation:
\[ v_\alpha \partial_\alpha \phi_a = e_{ab} \frac{\partial h(\phi)}{\partial \phi_b} \] (35)

Here the gauge function \( h(\phi) \) is arbitrary, and must be determined from consistency of the equation.

The master equation is much simpler than the vorticity equations for GBF.

The leading term in these equations near the minimal surface is the normal flow restriction
\[ v_\alpha(r)n_\alpha(r) = 0; r \in S_C \] (36)
which annihilates the \( \delta(z) \) term on the left side of \( \text{[35]} \). The next order terms will already involve the gauge function \( h(\phi) \).

The simplest case of our instanton is that of a flat loop in 3D space, which we assume to be in \( x, y \) plane. The minimal surface is a part \( D_C \) of \( x, y \) plane bounded by this flat loop.

The generic formula \( \text{[34]} \) simplifies here (here \( i, j = 1, 2 \)):
\[ v_1^{\text{inst}}(r_0) = 0, \] (37)
\[ v_3^{\text{inst}}(r_0) = \frac{n}{2} \int_{D(C)} d^2r \sqrt{g} g^{ij} \partial_i \Phi(r) \partial_j \frac{1}{|r - r_0|} \] (38)

The vanishing tangent velocity means that the regular part of equation \( \text{[35]} \) is satisfied identically with \( h = 0 \).

As for the singular part, proportional to \( \delta(z) \) it requires \( v_3(r) = 0 \).

In fact, there is always extra contribution \( v_3^T(r_0) \) to the normal velocity from the 3D Biot-Savart integral of over vorticity in the thermostat cells (see \( \text{[1]} \)). So, correct equation reads
\[ v_3(r_0) = v_3^T(r_0) + \frac{n}{2} \int_{D(C)} d^2r \sqrt{g} g^{ij} \partial_i \Phi(r) \partial_j \frac{1}{|r - r_0|} = 0 \] (39)

### 3 Smeared Vorticity and Dissipation in Navier-Stokes Equations

The square of delta function entering the dissipation from the instanton has to be smeared at viscous scales. This smearing comes from the viscosity terms in the Navier-Stokes equation:
\[ \partial_t \omega_\alpha = \nu \partial^2 \omega_\alpha + \omega_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta \omega_\alpha \] (40)
In the steady flow the right side must vanish. The $\delta'$-terms coming from the $v_\beta \partial_\beta \omega_\alpha$ term cancel by themselves in virtue of vanishing normal velocity at the surface.

The singular $\delta(z)$ terms must balance the first term $\nu \partial^2 \omega_\alpha$ at $z \to 0$. For that purpose the $z$ dependence must match. The only function smearing $\delta(z)$ and proportional to itself after two derivatives is

$$\delta(h, z) = \frac{1}{2h} \exp \left( -\frac{|z|}{h} \right) \quad (41)$$

The perturbation term in the steady Navier-Stokes equations coming from viscosity is

$$2\pi n e_{\alpha \beta \gamma} \partial_\beta \Phi(\xi) n_\gamma(\xi) \partial^2_\beta \delta(h, z) = 2\pi n \frac{\nu}{h^2} e_{\alpha \beta \gamma} \partial_\beta \Phi(\xi) n_\gamma(\xi) \delta(h, z) \quad (42)$$

With this term present, we introduce viscosity correction to vorticity

$$\delta \omega_\alpha = \frac{\nu}{h^2} \delta(h, z) w_\alpha(\xi) \quad (43)$$

and we find in the first order by matching the delta-terms:

$$w_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta w_\alpha + 2\pi n e_{\alpha \beta \gamma} \partial_\beta \Phi(\xi) n_\gamma(\xi) = 0 \quad (44)$$

There is a solution with $w_\alpha$ belonging to the tangent plane

$$w_k \partial_k v_i - v_k \partial_k w_i + 2\pi n e_{ik} \partial_k \Phi = 0 \quad (45)$$

Now we can take a limit at $h = \nu / \Lambda \to 0$

$$\nu \delta(h, z)^2 = \frac{\Lambda}{4h} \exp \left( -\frac{2|z|}{h} \right) \to \frac{\Lambda}{4} \delta(z) \quad (46)$$

From above viscosity correction we see that $\Lambda$ must go to zero with viscosity faster then $\sqrt{\nu}$

$$\Lambda \ll \sqrt{\nu} \quad (47)$$

This brings us to the effective Hamiltonian, which we now can compute as an anomaly at small viscosity

$$H_{\text{eff}} = \int_{S_{\text{min}}} d\sigma(\xi) \left( \frac{\Lambda}{4} \bar{F}^2(\xi) - \bar{F} \left( (\bar{r} \bar{n}) \bar{F} - (\bar{r} \bar{F}) \bar{n} \right) \right) \quad (48)$$

$$\bar{F} = 2\pi n \bar{\nabla} \Phi \times \bar{n} \quad (49)$$

As we see in the end of this paper, this vanishing $\Lambda$ is necessary for the existence of critical phenomena in PDF distribution.

4 Instanton On Flat Surface

4.1 Infinite Plane

In an infinite $R_2$ plane $z = 0$ the equation is a convolution in Fourier space, which leads to an obvious solution

$$v^T_z(r_0, 0) = \int_{R_2} d^2 k e^{i \bar{k} r_0} \tilde{v}^T_z(k_x, k_y) \quad (50)$$

$$\tilde{v}^T_z(k_x, k_y) = \int_{-\infty}^{\infty} dk_z \tilde{v}^T_z(k_x, k_y, k_z) \quad (51)$$

$$\Phi(\bar{r}) = -\frac{1}{\pi n} \int_{R_2} d^2 k e^{i \bar{k} \bar{r}} \tilde{v}^T_z(k_x, k_y) \sqrt{k_x^2 + k_y^2} \quad (52)$$

$$= -\frac{1}{2\pi^2 n} \int_{R_2} d^2 r \frac{v^T_z(r')}{|\bar{r} - \bar{r}'|} \quad (53)$$

This solution will apply to an arbitrary planar loop in a limit when it blows up to infinity, as long as the point $\bar{r}'$ is far away from its boundary, at fixed distance from the center of the loop.
4.2 Semiplane

Let us now deal with the upper semi-plane. We are left with the integral equation

\[-\frac{n}{2} \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \partial_{x} \Phi(x, y) \partial_{y} \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = v_z^T(x_0, y_0)\]  

(54)

Let us look for a solution as a combined Laplace-Fourier transform

\[\Phi(x, y) = \int_{0}^{\infty} d\eta e^{-\eta y} \int_{-\infty}^{\infty} dke^{i k x} \tilde{\Phi}(k, \eta)\]  

(55)

and use the Fourier representation

\[\frac{1}{|\vec{r} - \vec{r}_0|} = \int d^2 q e^{i \vec{q} \cdot (\vec{r}_0 - \vec{r})} \]  

(56)

\[v_z^T(x_0, y_0) = \int d^2 q v_z^T(q_x, q_y) e^{i \vec{q} \cdot \vec{r}_0} \]  

(57)

Integration over \(x\) produces \(2\pi \delta(k - q_x)\) which leaves us with the following integral equation

\[-\frac{n}{2} \int_{0}^{\infty} d\eta q_x^2 + \eta q_y \tilde{\Phi}(q_x, \eta) = \sqrt{q_x^2 + q_y^2} v_z^T(q_x, q_y)\]  

(58)

Let us introduce an analytic function (with \(q_x\) as a parameter)

\[F(q_x, z) = -\frac{n}{2} \int_{0}^{\infty} d\eta \frac{q_x^2 - \eta z}{\eta - z} \tilde{\Phi}(q_x, \eta)\]  

(59)

and define discontinuity on real axis as

\[\Im_z f(z) \equiv \frac{f(z + i 0) - f(z - i 0)}{2i}; \quad z \in R_1\]  

(60)

Then we find that discontinuity of \(F(q_x, z)\) is related to \(\tilde{\Phi}(q_x, \eta)\)

\[\Im_z F(q_x, z) = \frac{\pi n}{2} \tilde{\Phi}(q_x, z) (q_x^2 - z^2) ; \quad z \in R_1\]  

(61)

Our integral equation reduces to the condition

\[F(q_x, -i q_y) = \sqrt{q_x^2 + q_y^2} v_z^T(q_x, q_y)\]  

(62)

Assuming there is analytic continuation of \(v_z^T(q_x, q_y)\) to the upper semi-plane we find analytic solution

\[F(k, z) = \sqrt{k^2 - z^2} v_z^T(k, i z)\]  

(63)

\[\tilde{\Phi}(k, \eta) = \frac{2 \Im \pi n F(k, \eta)}{\pi n (k^2 - \eta^2)}\]  

(64)

Note that there is a difference between discontinuity and imaginary part of the function at the upper side of the cut in presence of some complex external parameters. We are literally finding the difference between the values of the function at two sides of the cut and dividing it by \(2i\). In the same way \(\Re \eta\) reduces to the average of two values on each side of the cut.

It is instructive to see how this solution gets back to the one for an infinite plane far away from the boundary \(y = 0\). The \(\eta\) integral in (55) with \(\Phi\) given by (63) is, in fact a contour integral surrounding all singularities of

\[F(k, z) \]  

(65)

except for the pole at \(z = |k|\).
We are interested in the limit of large area, i.e. at $R \to \infty$. Formally, at $R \to \infty$ the solution for an infinite plane minus the pole term with double Fourier representation for both $\Phi(y)$ with the solution for an infinite plane minus the pole term which at large $R$ vanishes.

This equation (with $J$ for $\Phi$) is valid for arbitrary radius $R$ (in fact, as $k^2$), so that this correction goes to zero away from the boundary and we recover an infinite plane solution.

To study analytic properties of velocity field $\tilde{v}^T_z$ in Fourier space, let us express it in terms of vorticity (we now restore the position $z_0 \neq 0$ of our $x - y$ plane)

$$\tilde{v}^T_z(q_x, q_y) = \int_{-\infty}^{\infty} dq_x \frac{\epsilon}{\sqrt{q_x^2 + q_y^2 + q_z^2}} e^{i q_z z_0}$$

Using this representation we find that

$$\lim_{\epsilon \to 0^+} \int_{q_x>0} \frac{\epsilon}{\sqrt{q_x^2 + q_y^2 + q_z^2}} e^{i q_z z_0} \to \pi \left( i k \tilde{\omega}_y(k, i |k|, 0) + |k| \tilde{\omega}_x(k, i |k|, 0) \right)$$

Note that this result does not depend of the arbitrary position $z_0$ of our plane in $R_3$.

### 4.3 Disk

In case of a disk with radius $R$ we have the integral equation

$$-\frac{n}{2} \int_{x^2 + y^2 < R^2} dx dy \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = \tilde{v}_z^T(x, y)$$

With double Fourier representation for both $\Phi(x, y)$, $v_z^T$ it reduces to

$$\int d^2 k \Phi(k) \frac{|k|}{|q|} J_1 \left( |k| - |q| R \right) = \frac{2\pi |q|}{n} \tilde{v}_z^T(q)$$

This equation (with $J_1(x)$ being the Bessel function) is valid for arbitrary radius $R$ of the circle.

We are interested in the limit of large area, i.e. at $R \to \infty$. Formally, at $R = \infty$ the convolution would lead to an the same solution we have found for an infinite plane.

This solution is valid inside the domain, at any fixed distance from the origin. However, at the vicinity of the bounding loop (circle in our case), the solution is different. Namely, near the border we may treat it as a semi-plane in the limit of large $R$, which means small curvature. So, there the previous solution applies.

As a consequence, the bulk effects, when the integral over area are involved, like in effective Hamiltonian, the Fourier solution \[52\] can be used, but for the circulation we should use the semi-plane solution.

Let us see how this transition happens. Shifting the origin to a point $0, -R$ on a circle, we have the inequality

$$x^2 + (-R + y)^2 < R^2$$

which at large $R$ compared to $x, y$ is equivalent to a parabola

$$y > \frac{x^2}{2R}$$

Using as before the Laplace-Fourier transform \[55\] we have this time

$$\Delta(\sigma, k) = \frac{1}{\sqrt{2\pi \sigma}} \exp \left( -\frac{k^2}{2\sigma} \right)$$

At large positive $y$ the dominant $k \sim 1/y \to 0$ in the last term and as we shall see below $\Omega(k)$ goes to zero at least as $|k|$ (in fact, as $k^2$), so that this correction goes to zero away from the boundary and we recover an infinite plane solution.
This smeared delta function in the limit $R \to \infty$ becomes the ordinary delta function

$$\lim_{\sigma \to 0; R \sigma > 0} \Delta(\sigma, k) = \delta(k)$$  \hspace{1cm} (77)

after which we recover the semi-plane integral equation we solved before.

In coordinate space the above infinite plane solution \cite{52} applies in a limit of $R \to \infty$ at fixed coordinates $r = (x, y)$ as measured from the center of the disk.

The semi-plane solution in this circle centered frame reads at $|x| \ll R, R + y \ll R$

$$\Phi(x, y) \to \int_{-\infty}^{\infty} dk e^{\imath k x} \int_0^{\infty} d\eta e^{-\eta(R+y)} 2\sqrt{R \eta} \frac{\sqrt{k^2 - \eta^2} \tilde{v}^T(k, \imath \eta)}{\pi n (k^2 - \eta^2)}$$ \hspace{1cm} (78)

In particular, it applies precisely at the circle $y = - R$. Let us study the difference between the solution at the circle and that at its center: this difference will define the circulation below.

As we discussed before, the difference is the contribution from the missing pole term at $\eta = |k|$ when this integral is rewritten as a contour integral and the contour rotated to imaginary axis. The integral reduces to

$$\Phi(0, -R) = -\frac{1}{\pi n} \int d^2 k \frac{\tilde{v}^T_z(k_x, k_y)}{\sqrt{k_x^2 + k_y^2}} - \frac{\pi}{n} \int_{-\infty}^{\infty} \frac{dk}{|k|} (\imath k \tilde{\omega}_y(k, \imath |k|, 0) + |k| \tilde{\omega}_x(k, \imath |k|, 0))$$ \hspace{1cm} (79)

At the center we have an old solution, for an infinite plane

$$\Phi(0, 0) = -\frac{1}{\pi n} \int d^2 k \frac{\tilde{v}^T_z(k_x, k_y)}{\sqrt{k_x^2 + k_y^2}}$$ \hspace{1cm} (80)

This results in a difference

$$\Phi(0, -R) - \Phi(0, 0) = -\frac{\pi}{n} \int_{-\infty}^{\infty} \frac{dk}{|k|} (\imath k \tilde{\omega}_y(k, \imath |k|, 0) + |k| \tilde{\omega}_x(k, \imath |k|, 0))$$ \hspace{1cm} (81)

This computation was restricted to specific point $(0, -R)$ on a circle. We argue that result is independent of the point on a circle. In virtue of axial rotational symmetry of our problem this move from one point on a circle to another one would be equivalent to a rotation around center of the circle by a certain angle $\theta$. But the velocity $v^T_z(q_x, q_y, q_z)$ in \cite{68} is manifestly rotation invariant.

Therefore, the circulation reduced to $\theta$ integration of a constant

$$\Gamma = m \oint d\theta \left( \Phi(0, -R) - \Phi(0, 0) \right) = \frac{4\pi m}{n} \int_{0}^{\infty} dk \left( \Im \tilde{\omega}_y(k, \imath |k|, 0) - \Re \tilde{\omega}_x(k, \imath |k|, 0) \right)$$ \hspace{1cm} (82)

Finally, in coordinate space, integrating over $k$

$$\tilde{\omega}_i(k, \imath |k|, 0) = \frac{1}{(2\pi)^3} \int d^3 r e^{\imath k x - |k| y} \omega_i(r)$$ \hspace{1cm} (83)

$$\int_0^{\infty} dk e^{\imath k x - |k| y} = \frac{y + \imath x}{y^2 + x^2}$$ \hspace{1cm} (84)

we find

$$\Gamma = \frac{m}{2\pi n} \int d^3 r \frac{x \omega_y(\hat{r}) - y \omega_x(\hat{r})}{x^2 + y^2}$$ \hspace{1cm} (85)

There must be a better way to derive such a simple answer, without solving singular integral equations.

This integral can be simplified further. It involves vorticity integrated along the normal direction

$$\xi_i(x, y) = \int_{-\infty}^{\infty} dz \omega_i(x, y, z)$$ \hspace{1cm} (86)

In polar coordinates only the angular component $\xi_\theta$ of this vector contributes here:

$$\Gamma = \frac{m}{2\pi n} \int d\theta \int_0^{\infty} d\rho \xi_\theta(\rho, \theta)$$ \hspace{1cm} (87)
4.4 Minimization Problem

There is a way to reduce our master equation to a minimization of a quadratic form. Let us make the integral transformation

\[ \Phi(\vec{r}) = \gamma f_z n|z| \int_{D_C} d^2 r' \frac{H(\vec{r}')}{2\pi|\vec{r} - \vec{r}'|} \]

and we are arrive at the equation (with constant parameter \( z \) being the distance from the plane to the origin)

\[
\frac{1}{4\pi^2} \int_{D_C} d^2 r' \partial_\alpha \frac{1}{|\vec{r}' - \vec{r}|} \int_{D_C} d^2 r'' H(\vec{r}'') \partial_\alpha' \frac{1}{|\vec{r}'' - \vec{r}'|} = R(\vec{r}, z)
\]

\[ R(\vec{r}, z) = \frac{|z| (2z^2 - (\vec{r} - \vec{r}_0)^2)}{2\pi((\vec{r} - \vec{r}_0)^2 + z^2)^{3/2}} \to \delta^2(\vec{r} - \vec{r}_0); \ z \to 0 \]

The last equation followed from the asymptotic form (7) of the thermostat velocity field at the distances much larger than the region of vorticity support in the thermostat. This function at \( z \ll R \) can be replaced by a 2d delta function.

We have now a symmetric kernel here

\[ K(\vec{x}, \vec{y}) = \frac{1}{4\pi^2} \int_{D_C} d^2 r \partial_\alpha \frac{1}{|\vec{x} - \vec{r}|} \partial_\alpha' \frac{1}{|\vec{y} - \vec{r}|} \]

and we observe that this problem is equivalent to minimization of quadratic form

\[ Q[H] = -H(0) + \frac{1}{2} \int_{D_C} d^2 x \int_{D_C} d^2 y H(\vec{x}) K(\vec{x}, \vec{y}) H(\vec{y}) \]

For numerical solution it pays to introduce the vector function linearly related to \( H \)

\[ F_\alpha[H, \vec{r}] = \frac{1}{2\pi} \int_{D_C} d^2 r' H(\vec{r}') \partial_\alpha' \frac{1}{|\vec{r}' - \vec{r}|} \]

and rewrite the quadratic form as

\[ Q[H] = -H(0) + \frac{1}{2} \int_{D_C} d^2 r F^2_\alpha[H, \vec{r}] \]

This \( F_\alpha[H, \vec{r}] \) is proportional to \( \partial_\alpha \Phi(\vec{r}) \). Thus, the quadratic part of our target functional is just a kinetic energy of a free scalar field, but it is the linear term which forces us to use \( H(\vec{r}) \) as an unknown.

In order for \( \Phi(\vec{r}) \) and its gradients to remain finite at the boundary \( C \) the new field \( H \) should satisfy Dirichlet boundary condition

\[ H(C) = 0 \]

There is also the vanishing net vorticity restriction. It can be written as 2-vector constraint

\[ \int_{D_C} d^2 r F_\alpha[H, \vec{r}] = 0 \]

Coulomb poles disappeared from this problem, being replaced by weaker, logarithmic singularities (see the next section).

The parameter \( z \) remains as a free parameter. In DNS we average over positions of the square over all physical space (hypercube). In virtue of the translation invariance on a hypercube with periodic boundary conditions, such averaging does not affect results.

Here, we, also can average solution for the circulation over some range of \( z \) where vorticity is present. This will only affect the normalization of \( \Gamma \) without changing the shape of the distribution nor the dependence of the Clebsch fields on the position at the minimal surface.

\[ ^2 \text{As for the position } \vec{r}_0 \text{ of the center of our instanton in the } x - y \text{ plane, it remains another free parameter. We have to investigate this issue later and now we just assume } |\vec{r}_0| \ll R \text{ and simply use } \delta^2(\vec{r}) \text{ as a right side of the master equation.} \]
4.5 Numerical Method

Now, we assume that the function $H(\vec{x})$ is a smooth function on a surface. Then the following numerical approach would work.

Let us cover the domain $D_C$ by a square grid step 1 and assume that there are large number of these squares inside the loop. Let us approximate the loop by the loop drawn on this grid, passing through its cites.

Eventually we shall tend the area of $D_C$ to infinity, in which case this quantization will become irrelevant.

Now let us approximate $H(\vec{r})$ by its value at the center $\vec{c} \square$ inside each square $\square$

$$H(\vec{r} \in \square) \approx h_{\square} = H(\vec{c}_{\square})$$  \hspace{1cm} (97)

The resulting integral over the square is calculable:

$$F_\alpha([\square], \vec{r}) = \frac{1}{2\pi} \int_{\square} d^2r' \partial_\alpha' \frac{1}{|\vec{r'} - \vec{r}|} = \sum_{i=0}^{3} (-1)^i A_\alpha (\vec{V}_i - \vec{r})$$  \hspace{1cm} (98)

Here $\vec{V}_i, i = 0, 1, 2, 3$ are the vertices of $\square$, counted anticlockwise starting with the left lowest corner $\vec{V}_0$ and

$$A_\alpha(\vec{r}) = \frac{1}{2\pi} \arctanh \hat{r}_\alpha; \hat{r} = \frac{\vec{r}}{|\vec{r}|};$$  \hspace{1cm} (99)

Thus we get an approximation

$$F_\alpha[H, \vec{r}] \approx \sum_{\square \in D_C} h_{\square} F_\alpha([\square], \vec{r})$$  \hspace{1cm} (100)

After that the target functional $Q[H]$ becomes an ordinary quadratic form of a vector $h_{\square}, \square \in D_C$, including $H(0) = h_{\square_0}$ corresponding to a central square.

The integral $\int_{D_C} d^2r$ in (106) converges (there is logarithmic singularity in $A_\alpha(\vec{r})$ at $r_\alpha \to \pm |\vec{r}|$, but it is integrable).

We have to compute symmetric matrix

$$\langle \square_1 | M | \square_2 \rangle = \int_{D_C} d^2r F_\alpha([\square_1], \vec{r}) F_\alpha([\square_2], \vec{r})$$  \hspace{1cm} (101)

and the linear term

$$- \int d^2r R(\vec{r}, z) H(\vec{r}) \approx -h_{\square_0}$$  \hspace{1cm} (102)

where $\square_0$ is the square surrounding the origin (the middle of the domain).

These integrals for the matrix elements as well as the linear term are calculable with 5 significant digits using adaptive cubature library [7], based on recursive subdivision of the multidimensional cube [8]. We wrote parallel code which works fast enough for millions of squares on a supercomputer.

For numerical stabilization we replaced the singular logarithm function in (99) by cutoff function at $\epsilon = 10^{-6}$

$$A_\alpha(\vec{r}) \approx \ln \left(1 + \hat{r}_\alpha, \epsilon\right) - \ln \left(1 - \hat{r}_\alpha, \epsilon\right)$$  \hspace{1cm} (103)

$$\ln(x, \epsilon) = \ln \left(\max(|x|, \epsilon)\right)$$  \hspace{1cm} (104)

$$B(y, x, \epsilon) = \arcsinh \left(\frac{y}{\max(|x|, \epsilon)}\right)$$  \hspace{1cm} (105)

We also added to our target the stabilizer:

$$Q[\vec{h}] = -h_{\square_0} + \frac{1}{2} \sum_{\square_1, \square_2} h_{\square_1} \langle \square_1 | M | \square_2 \rangle h_{\square_2} + \frac{1}{2} \lambda(M) \sum_{<\square_1, \square_2>} (h_{\square_1} - h_{\square_2})^2$$  \hspace{1cm} (106)

Here $<\square_1, \square_2>$ denote squares sharing a side and

$$\lambda(M) = \max |\delta M|$$  \hspace{1cm} (107)
is maximal absolute error in computation of numerical integrals for matrix elements of $M$, in our case $\lambda \sim 10^{-6}$.

There are also two constraints:

$$C_1 : \sum_{\square} h_\square \int_{D_C} d^2 r F_\alpha(\square, r) = 0;$$  \hspace{1cm} (108)

$$C_2 : h_\square = 0; \forall \square \in C;$$  \hspace{1cm} (109)

Once the matrix $M$ is computed, the solution for the grid weights $h_\square$ is given by the minimum of quadratic form $Q$ with conditions $C_1, C_2$

$$h_\square = \arg \min_{C_1, C_2} [Q]$$  \hspace{1cm} (111)

As for the symmetric positive definite matrix inversion, there are fast parallel libraries available in C++, so this does not look like a serious problem even for the grids with million squares.

We are planning to perform this computation for a rectangle on a supercomputer and compare to available DNS data.

The universal functional $G[C]$ in terms of these coefficient $h_\square$ reads

$$G[C] = \frac{\sum_{\square \in D_C} h_\square \int_{D_C} d^2 r e_{\alpha \beta} r_\alpha F_\beta(\square, \vec{r})}{\sqrt{\int_{D_C} d^2 r}}$$  \hspace{1cm} (112)

As expected, this functional is scale invariant.

Also note that in the limit of large size of the domain, when the number $N$ of grid squares goes to infinity, the coefficients $h_\square$ decrease as $1/N$.

In this limit, our sum over squares becomes the Riemann sum for an integral (93).

The reason for exactly computing the integrals over elemental squares with constant $H(\vec{r})$ inside each square was the Coulomb singularity. Resulting function $A_\alpha(\vec{r})$ has only a logarithmic singularity, rather than the pole in Coulomb potential.

By exactly computing the integrals we accelerated the convergence to a local limit $N \to \infty$. With Riemann sums for Coulomb kernel the errors would be $O\left(1/\sqrt{N}\right)$, but with replacing $H(\vec{r})$ by its values at the center the relative errors are related to second derivatives which is $O\left(1/N\right)$. So, with accessible $N \sim 10^6$ at modern supercomputers we expect to get 5 significant digits, which is beyond the statistical and systematic errors of the DNS at achievable lattices $24K^3$.

The hardest part of this computation is numerical integration needed for the kernel $\langle \square_1 | M | \square_2 \rangle$ for all the squares $\square_1, \square_2$. It has $O\left(N^3\right)$ complexity where $N$ is the number of squares inside $D_C$. Still, with $N \sim 100$ this (parallel) computation using adaptive cubature library [7] takes less than a minute on my server with 24 cores.

5 Circulation PDF

In this section we are going to repeat the line of argument in the previous work, but with much more clarity and without extra assumptions we made there.

The vorticity and velocity fields as determined from the homogeneous GBF equations have arbitrary overall scale $Z$. We can normalize the vorticity by minimizing over the zero mode factor $\omega \Rightarrow Z\omega$ the effective Hamiltonian

$$H_{\epsilon ff}^{eff} = \min_Z \left( Z^2 H_{\epsilon ff}^{eff} - Z H_{1}^{eff} \right)$$  \hspace{1cm} (113)

$$H_{2}^{eff} = \int_{V_s} d^3 r \omega_\alpha^2 (r)$$  \hspace{1cm} (114)

$$H_{1}^{eff} = \int_{V_s} d^3 r \vec{r} \times \omega$$  \hspace{1cm} (115)

$$Z = \frac{H_{1}^{eff}}{2H_{2}^{eff}}$$  \hspace{1cm} (116)

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After this global renormalization, our solution for $\Phi$, which was linearly related to the normal global velocity at the surface will also get the same factor.

The circulation was computed above for a flat loop. Let us use other exact results to compute effective Hamiltonian. The quadratic part is given by integral over momenta

$$H_{\text{eff}}^2 = \pi^2 \Lambda \int d^2k |\tilde{v}_Z^T (k)|^2$$

(118)

This integral converges at $k \to 0$ where $\tilde{v}_Z^T (k)$ has a finite limit, so the size $R$ can be set to infinity. It is then determined by the large $k$ (i.e. microscopic) features of the thermostat, which are not calculable here, but just lead to an unknown constant.

We need the ratio $Z$ of linear part to the quadratic part

$$Z = \Sigma[C] \frac{f_2^2}{\sigma} \mu(\tilde{f})$$

(119)

where

$$\mu(\tilde{f}) = \frac{2 \langle \mathcal{E}_T^T \rangle}{\Lambda \int d^2k |\tilde{v}_Z^T (k)|^2}$$

(120)

At small external force we can leave in this susceptibility $\mu(\tilde{f})$ only a leading term $\mu = \mu(0)$.

Such an expansion would be justified if, just like in a critical phenomena in statistical physics, the susceptibility $\mu$ would grow to infinity to compensate small value of external force.

This is what happens in a ferromagnet near the Curie point, when infinitesimal external magnetic field is enhanced by large susceptibility, resulting in a spontaneous magnetization.

In our theory this happens because $\Lambda \ll \sqrt{\nu}$ becomes small. This enhances the leading term $\langle P_b \rangle \propto \sigma^2$ in $\mu(\tilde{f})$ so that the higher terms of expansion would be negligible.

Let us proceed with our computation. We have our result (85) for the circulation. All we need is to multiply it now by a zero mode normalization factor $Z$. We find

$$\Gamma = \frac{m}{\pi} \frac{f_2^2}{2\pi} \frac{\Sigma[C]}{\sigma \mu}$$

(121)

Note that our formula for circulation represents a homogeneous functional of the thermostat vorticity distribution. When one multiplies this vorticity by a scale factor, $\mu$ scales inversely with this factor, compensating the scale of remaining integral. This happened because we integrated out the zero mode in our global vorticity GBF.

However, the external random force, which determines the energy pumping, did not factor out by this rescaling. This happened because the circulation and the effective energy flow are determined by different parts of the vorticity distribution. The energy flow determines the asymptotic behavior at large coordinates, near the boundary sphere, whereas the circulation as well as dissipation is determined by a microscopic scales of the vorticity of the thermostat.

The critical phenomenon, which is transformation of the Gaussian distribution to an exponential one, happens because of our singular instanton solution, which produces another factor of Gaussian force multiplying this one through the interaction of an instanton with the thermostat background.

This second factor comes about because of the infrared divergency of the integral $\int d^2r \epsilon_{ij} \partial_i \partial_j \Phi$, relating this integral to the power produced by random forces at the bounding sphere.

Resulting square of Gaussian variable in above formula for the circulation transforms the Gaussian distribution to the exponential one.

Also, we observe that the sign of $\Gamma$ is proportional to the sign of the ratio of winding numbers $\frac{m}{n}$.

Clearly, in addition to solution with winding numbers $m, n$ there are always mirror solutions with $m, -n$ and $-m, n$. 

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The effective Hamiltonian at this solution in is exactly the same as for the positive $m$, so the contributions from these flows must be added.\footnote{However, at the next level of the WKB approximation there will be some pre-exponential factors related to the determinant of the linear operator of small fluctuations around the instanton background. These weights are unknown so far, but we expect the mirror symmetry with regard to the change of orientation of the loop $m \Rightarrow -m$.}

This provides the negative branch of circulation PDF. The sign of $\Sigma[C]$ is not well defined, but sum over positive and negative winging numbers covers this up, so we shall in the following imply an absolute value

$$\Sigma[C] \rightarrow |\Sigma[C]|$$

(123)

Summing up contribution from both signs we obtain an explicit formula for a Wilson loop

$$\exp\left(i \gamma \frac{\Gamma C}{\bar{\mu} \Sigma[C]}\right) = \frac{1}{2} \left(W\left(\frac{m}{n} \gamma\right) + W\left(-\frac{m}{n} \gamma\right)\right)$$

(124)

$$W(\gamma) = \frac{1}{\sqrt{1 - i \gamma}}$$

(125)

This corresponds to exact asymptotic law

$$P(\Gamma) \propto \sqrt{\left|\frac{n}{m} \Gamma \bar{\mu} \Sigma[C]\right|} \exp\left(-\left|\frac{n \Gamma}{m \bar{\mu} \Sigma[C]}\right|\right)$$

(126)

The functional $\Sigma[C]$ is completely universal and calculable in terms of the eigensystem of Laplace equation inside the loop $C$. All non-universal parameters of the thermostat and random forces are hidden in the scale factor $\bar{\mu}$.

It is assumed that both the circulation and the area are large compared to the viscous scales, and by definition of the WKB approximation we were considering the tails of distribution, at $|\Gamma| \gg \Sigma[C]$.

In that region the (even) moments $M_p = \langle \Gamma^p \rangle$ grow as $\Gamma^{(p + \frac{1}{2})}$.

Another interesting prediction we have here is a nontrivial dependence of the circulation scale $\Sigma[C]$ from the shape of the loop $C$, given by the function $G[C]$.

This function can be computed numerically using the Laplace eigensystem for the loop under study. In particular, for the rectangle the eigensystem is elementary, so there is no problem to compute this function.

We postpone numerical study till the next publication with Kartik Iyer, where we are going to systematically compare this theory with recent DNS.

Note that small value of the variance $\sigma$ of random force is compensated by large susceptibility $\bar{\mu} \sim \frac{1}{\lambda} \gg \frac{1}{\sqrt{\nu}}$ to produce finite slope in exponential law. This matching requires $\langle P_b \rangle \sim \sigma \sim \Lambda \ll \sqrt{\nu}$.

This is a justification for our expansion in powers of the random force. The leading constant term was enhanced by large susceptibility $\bar{\mu}$, after which the higher terms will be relatively small at $\bar{f} \sim \sqrt{\sigma}$. The next term in $\mu(\bar{f})$ would be $f^2 \sim \sigma$ as compared to the leading constant one.

6 Topology of Instanton and Circulation PDF

The quantization of the circulation in a classical problem deserves further attention.

One may wonder what are the physical values of the winding numbers $m, n$. Maybe only the lowest levels are stable, and higher ones must be discarded?

If you consider effective Hamiltonian contribution from this instanton \footnote{However, at the next level of the WKB approximation there will be some pre-exponential factors related to the determinant of the linear operator of small fluctuations around the instanton background. These weights are unknown so far, but we expect the mirror symmetry with regard to the change of orientation of the loop $m \Rightarrow -m$.} you observe that it does not depend of winding numbers as the solution for $\Phi$ does not depend of $m$ and is inversely proportional to $n$.

Therefore, only the circulation depends of winding numbers through their ratio $p = \frac{n}{m}$. In general case we have to sum over all $m, n$ with yet unknown weights

$$\exp\left(i \gamma \frac{\Gamma C}{\sqrt{A_C}}\right) \propto \sum_{n,m \in \mathbb{Z}, n,m \neq 0} W\left(\frac{m}{n} \gamma\right) A_{n,m}$$

(127)
These weights $\mathcal{A}_{n,m}$ would come from the functional determinant arising from integration over fluctuations around the instanton in our effective field theory in [1]. This is a hard mathematical task, though in principle doable, just as it was for instantons in gauge field theories.

The PDF tail from each term would be

$$
\frac{1}{\sqrt{|\Gamma| \mu \Sigma(C)}} \exp \left( - \frac{n}{m} \left| \frac{|\Gamma|}{\mu \Sigma(C)} \right| \right) \mathcal{A}_{n,m} \sqrt{\frac{n}{m}}
$$

(128)

If we sum over all rational numbers $q = \frac{n}{m}$, the exponential decay would become power-like contrary to numerical experiments [2] which strongly support a single exponential.

So, there is still something we do not understand about our measure on GBF: there are some topological super-selection rules on top of the steadiness of the flow and minimization of effective Hamiltonian.

The topological invariant which depends of these winding numbers was suggested in [1]. Consider the circulation $\Gamma_{\delta C(\alpha)}$, around the infinitesimal loop $\delta C(\alpha)$ which encircles our loop at some point with angular variable $\alpha$ (Fig [1]). It is straightforward to compute

$$
\Gamma_{\delta C(\alpha)} = \oint_{\delta C(\alpha)} \phi_1 \, d\phi_2 = 2\pi n \phi_1
$$

(129)

Clearly, this circulation stays finite in a limit of shrinking loop $\delta C$ because of singular vorticity at the loop $C$.

Now, integrating this over $d\phi_2 = m \, d\alpha$ we get our original circulation

$$
\oint \Gamma_{\delta C(\alpha)} \, d\phi_2(\alpha) = 2\pi n \oint \phi_1 \, d\phi_2 = 2\pi n \Gamma_C
$$

(130)

Geometrically, this is a volume of the solid torus in Clebsch space mapped from the tube made by sweeping the infinitesimal disk around our loop (see Fig [2]). This volume stays finite in the limit of shrinking tube and equals $2\pi n$ times the velocity circulation $\Gamma_C$ in original space $R_3$.

This circulation by itself is an oriented area inside the loop in Clebsch space, which area is $m$ times the geometric area, as the area is covered $m$ times by the instanton field.

Let us look at the topology of the mapping from the physical space to the Clebsch space. We cut out of $R_3$ the infinitesimal solid torus around our loop – this remaining space topologically also represents a solid torus. We cut this solid torus along the minimal surface $S_C$ bounded by $C$. So to say, this surface now is a portal from one universe to a parallel one.

The Clebsch field $\phi_\alpha$ varies inside this solid torus and changes by $2\pi n \delta \alpha_2$ when crossing the surface $S_C$.

In other words, $\phi_2$ covers $n$ times the circle $S_1$ when it goes around this solid torus.

This is the first cycle. The second one would correspond to the loop around the origin in polar coordinates we used. Topologically this means the following.

Let us cut another infinitesimal tube out of our solid torus, the one passing along the normal direction through the origin $\rho = 0$ of polar coordinates at the minimal surface. In case of a planar loop this is the $z$ axis.

In general case such polar coordinates and such origin always exist on a minimal surface described by Enneper-Weierstrass parametrization

$$
\begin{align*}
\vec{X}(\rho, \theta) &= \vec{F}(\rho e^{i \theta}) \\
\vec{F}'(z) &= \left\{ \frac{1}{2} (1 - g^2) f, \frac{1}{2} (1 + g^2) f, g f \right\}
\end{align*}
$$

(131)

(132)

with $g(z), f(z)$ being analytic functions inside the unit circle $|z| < 1$. (see Fig [3] with $f = 1, g = z$).

With the tube cut out of the solid torus it is now a space between two tori. Rather than a bagel, it is now a circular water hose (not a very useful thing but imaginable). Its boundary is $T_2 \times T_2$. It is not contractible into a circle, unlike the solid torus, but it is contractible to two circles, just like a simple torus.

Its homology is therefore $H_2(T) = \mathbb{Z} \times \mathbb{Z}$, which means that there could be two integer winding numbers corresponding to these two cycles.

However, at the second look, this topology is unacceptable, as it introduces a fictitious singularity at $\rho = 0$, which is a singularity of the spatial coordinates, but need not be a singularity of the field.
With the first tube, surrounding the loop $C$ this was a real singularity of the flow, but this singularity at the axis of cylindrical coordinates does not really exist.

The Clebsch field near the polar coordinate origin $x = y = 0$ at the surface would have to behave as $(x + iy)^m$ in order to have this winding number without introducing a singularity at the origin. There is no physical reason nor any free parameters to tune for vanishing of $m - 1$ derivatives at the origin.

Without cutting out the second tube, our solid torus has homology of a circle $H_2(S_1) = \mathbb{Z}$ which means that there is only one winding number $n$, the first one.

There is another way to arrive at the same conclusion. Topology of the Clebsch field was analysed in previous work [10] and it was concluded that there is a helicity

$$H = \int d^3r v_\alpha \omega_\alpha$$  \hspace{1cm} (133)

which is characterized by an integer. In our previous work [11] we computed helicity for our instanton in some general way and we found that it was proportional to the winding number $n$.

$$H = 2\pi n \oint_{\tilde{C}} \tilde{\phi}_3 d\phi_1$$  \hspace{1cm} (134)

Here $\tilde{\phi}_3$ is a third Clebsch field parametrizing velocity

$$v_\alpha = -\phi_2 \partial_\alpha \phi_1 + \partial_\alpha \tilde{\phi}_3$$  \hspace{1cm} (135)

This supports our argument that $n$ has some topological meaning but $m$ does not.

We therefore restrict ourselves with solutions with

$$m = \pm 1$$  \hspace{1cm} (136)

which have quantized helicity but no fictitious axial singularities.

7 Discussion. Do we have a theory yet?

We identified the instanton mechanism of enhancement of infinitesimal random force in Euler equation and demonstrated how this enhancement takes place at small viscosity.

The required random force needed to create the energy flow and asymptotic exponential distribution of circulation, has the variance $\sigma \sim \sqrt{\nu}$. This small force is enhanced by large susceptibility $\mu \sim 1/\sqrt{\nu}$. This large susceptibility can be traced back to the singular behavior of the vorticity field at the minimal surface in the Euler limit of Navier-Stokes equations.

We presented an explicit solution for the shape of circulation PDF generated by instanton. We claim it is realized in high Reynolds flows for the large loops and large circulations, not as a model, but rather as an exact asymptotic law. We confirmed the dependence $|\Gamma| \propto \sqrt{\mathcal{A}}$ predicted earlier [11] based on the Loop equations. The raw data from [2] were compared with this prediction. We took the ratio of the moments $M_p = \langle \Gamma^p \rangle$ at largest available $p$ and defined the circulation scale as $\mathcal{S} = \sqrt{\frac{M_8}{M_6}}$.

We fitted using Mathematica® $S(r)$ as a function of the size $r = \frac{\pi}{\eta}$ of the square loop measured in the Kolmogorov scale $\eta$. The quality of a linear fit was very high with adjusted $R^2 = 0.9996$. The so called ANOVA table summarizes this fit as follows

| DF | SS     | MS     | F-Statistic | P-Value |
|----|--------|--------|-------------|---------|
| R  | 2.30567| 2.30567| 22984.4     | 4.008150477989548*^-15 |
| Error | 0.000802519 | 0.000100315 | Null | Null |
| Total | 9 | 2.30648 | Null | Null |

(137)

The linear fit is shown at Fig.4, 5. The errors are most likely artifacts of harmonic random forcing at a 8K cubic lattice.

This is not to say that some other nonlinear formulas cannot fit this data equally well or maybe even better, for example fitting $\log S$ by $\log R$ would produce very good linear fit with the slope 1.1 instead of our 1.1. Data fitting cannot derive the physical laws – it can only verify them against some null hypothesis. This is especially true in presence of few percent of systematic errors related to finite size effects and harmonic quasi random forcing. We believe that distinguishing between 1.1 and 1 is an over-fit in such case.
Contrary to some of my early conjectures, there is no universality in the area law, though there is a universal shape of decay of PDF, and the singular vorticity at the minimal surface is responsible for that decay.

There is a universal scaling factor $G|C|$, multiplying the square root of the area inside the loop, which determines the exponential slope as a functional of the shape of the loop.

We wrote down linear integral equation which determines this factor for any shape of the loop using the Laplace eigensystem inside this loop. For the observed rectangular shape the eigensystem is elementary and integrals calculable, so we can compute this function with high accuracy when the DNS data will be available to match with.

The PDF is given by sum over positive integer winding numbers $n$

$$P(\Gamma) = \int_{-\infty}^{\infty} d\gamma \frac{e^{-i\gamma\Gamma}}{2\pi} \exp \left( i\gamma \oint_C dr \alpha \right) \langle \exp \left( i\gamma \oint_C dr \alpha \right) \rangle$$

$$\propto \frac{1}{\sqrt{\Gamma|\mu|C}} \sum_{n=1}^{\infty} \exp \left( -n \frac{|\Gamma|}{\mu|C|} \right) A_{n,1}\sqrt{n}$$

Negative winding numbers are responsible for another branch of the PDF, so that resulting PDF is an even function of circulation at large $|\Gamma|$. Pre-exponential factors $A_{n,1}$ come from the next order in WKB approximation, – the functional integral over the linear perturbations around our instanton.

The instanton expansion like this one was computed exactly in certain CFT in two and four dimensions. In general, it is a difficult task in gauge theories as the multi-instanton solution is quite complex. Our Abelian theory with multi-instanton simply corresponding to higher winding number in the same solution, is supposed to be much simpler than that of gauge theories.

Obviously, at large circulation only the $n = 1$ term remains, matching numerical experiments. We found that our formula fits the latest data by Kartik Iyer within error bars of DNS with adjusted $R^2 = 0.9999$, (see Fig.6, 7, 8).

The ANOVA table is as good as it gets

|       | DF  | SS  | MS  | F-Statistic | P-Value |
|-------|-----|-----|-----|-------------|---------|
| $x$   | 1   | 474.656 | 474.656 | 34991.5  | 3.3022757147320016*^-50 |
| Error | 32  | 0.434077 | 0.0135649 | Null      | Null     |
| Total | 33  | 475.091 | Null | Null | Null |

With circulation here being the sum of normal components of large number of local vorticities over the minimal surface, it is nontrivial for this circulation to have an exponential distribution, regardless of the local vorticity PDF as long as it has finite variance.

The Central Limit theorem tells us that unless these local vorticities are all strongly correlated, resulting flux (i.e. circulation) will have a Gaussian distribution. The deviation from this Gaussian distribution is the essence of critical phenomena in statistical mechanics, and we definitely have it here, in Turbulence.

The spectacular violation of this Gaussian distribution in the DNS with seven decades of exponential tails, strongly suggest that there are large spatial structures with correlated vorticity, relevant for these tails.

In this paper, developing the previous one, we identified these spatial structures as coherent vorticity spread thin over minimal surface. We computed resulting PDF matching this DNS including pre-exponential $1/\sqrt{|\Gamma|}$ factor.

The sum over integers emerges here by the same mechanism as in Planck’s distribution in quantum physics. There we had to sum over all occupation numbers in Bose statistics. Here we sum over all winding numbers of the Clebsch field across the minimal surface in physical space.

In Bose statistics the discreteness of quantum numbers is related to the compactness of the domain for the corresponding degree of freedom.

In our case this also follows from compactness of the domain for the Clebsch fields, varying on a torus. The velocity circulation in physical space becomes the area inside oriented loop in Clebsch space, related to one of the cycles of that torus.

The physical reason why the multi-valued Clebsch fields are acceptable in a real world with single-valued velocity field is the unbroken gauge invariance, or Clebsch confinement. Clebsch fields are unobservable, just like quarks or gluons.

\[5\] Again, some nonlinear power fit with log/log slope different from 1 could also fit these data, but as we mentioned above, with systematic errors present we cannot reliably distinguish linear law from power close to 1.
So, do we have a theory of turbulence? Not yet IMHO, but we may be getting there. There are still some issues to be clarified and some computations to be made and some limits to be proven to exist.

Once again I am appealing to young string theorists: come and help me! You are missing all the fun. This is no less beautiful than conformal field theories or matrix models. You would understand it and you can develop it into a Theory of Turbulence.

8 Acknowledgements

I am grateful to Nikita Nekrasov for helping me understand the topology of Clebsch field as well as the statistical equilibrium between vorticity cells.

Useful discussions with Eugene Kuznetzov and Victor Yakhot helped me understand better the duality of wave-instanton pictures in Clebsch field theory as well as the properties of Zeldovich pancakes.

I also benefited from discussions with Kartik Iyer and Katepalli Sreenivasan regarding numerical simulations. This theory perfectly matches their numerical experiments.

Sasha Polyakov read the draft of this paper and we had a productive discussion, helping me understand the meaning of this solution.

This work is supported by a Simons Foundation award ID 686282 at NYU.

A Saddle Point Integral for Energy Surface Constraint

Let us consider the micro-canonical distribution for some large system consisting of the subsystem $H_1 = H[p_1, q_1]$ with phase space volume $d\Gamma_1 = d\Gamma[p_1, q_1]$ and a thermostat $H_2 = H[p_2, q_2]$ with phase space volume $d\Gamma_2 = d\Gamma[p_2, q_2]$:

$$Z = \int_C \frac{d\lambda}{2\pi i} \exp(\lambda E) \int d\Gamma_1 \exp(-\lambda H_1) \int d\Gamma_2 \exp(-\lambda H_2)$$  \hspace{1cm} (141)

The integration contour $C$ here goes along the imaginary axis, providing thus the Fourier representation of the delta function which constraints the distribution to the energy surface $H_1 + H_2 = E$.

In case of the ordinary statistical mechanics $H[p, q]$ is the Hamiltonian and $d\Gamma[p, q] = \prod dp dq$ is the linear phase space volume. Resulting distribution would be an ordinary Gibbs distribution $\exp(-\beta H_1)$.

However, the mathematical mechanism behind this transformation from the delta function to the exponent is fairly general. It applies to arbitrary (maybe nonlinear) measure $d\Gamma$ and arbitrary (maybe non-positive) Hamiltonian as long as it is bounded from below in the infinite phase space.

In case of Turbulence we are applying this transformation to the system where $[p, q]$ stand for Clebsch variables parametrizing vorticity $\vec{\omega} = \nabla p \times \nabla q$ and $H[p, q]$ contains quadratic as well as linear terms in vorticity

$$H[p, q] = h_2 - h_1;$$  \hspace{1cm} (142)

$$h_1 = \bar{f} \int d^3r \bar{r} \times \vec{\omega};$$  \hspace{1cm} (143)

$$h_2 = \int d^3r \nu \bar{\omega}^2$$  \hspace{1cm} (144)

Important property of this $H[p, q]$ is that it is bounded from below: at large absolute value of scaling factor $\rho$ in vorticity $\bar{\omega} = \rho \omega^0$ the first term dominates and this term is positive definite.

The measure $d\Gamma[p, q]$ in case of proposed Field Theory of Turbulence $[1]$ is restricted to so called Generalized Beltrami Flow. Explicit form of this measure is not relevant, but it is important that it is invariant with respect to scale transformations $\bar{\omega} \rightarrow \rho \bar{\omega}$.

This scaling factor $\rho$ represents the zero mode of this measure, and there is an implicit integration over $\rho$ in (141) in both phase space integrals there. These integrals over zero modes will converge only if $\Re \lambda > 0$, for which purpose the integration contour $C$ must be shifted to the right semi-plane by an infinitesimal amount.

Finally, the energy $E_0 = 0$ in case of turbulence, as this is in fact, the net energy flow including both pumping and dissipation so it adds up to zero.
After all these comments we can proceed with computation, and it goes the same way in both cases: Gibbs and Turbulence.

Namely, we are looking for a saddle point in the one-dimensional integral over $\lambda$.

$$Z = \int d\Gamma_1 \Omega[p_1, q_1];$$  \hspace{1cm} (145)

$$\Omega[p_1, q_1] = \int_C \frac{d\lambda}{2\pi i} \exp (\lambda E - \lambda H_1 + S(\lambda))$$ \hspace{1cm} (146)

$$\exp (S(\lambda)) = \int d\Gamma_2 \exp (-\lambda H_2)$$ \hspace{1cm} (147)

The saddle point equation

$$\Omega[p_1, q_1] \propto \frac{1}{\sqrt{S''(\lambda)}} \exp (\lambda E - \lambda H_1 + S(\lambda));$$ \hspace{1cm} (148)

$$S'(\lambda) + E - H_1 = 0$$ \hspace{1cm} (149)

Now, assuming that $H_1 \ll H_2$ (there is an infinite thermostat $H_2$ and finite subsystem $H_1$ under study) we can approximate $\lambda$ as solution of universal equation (independent of $p_1, q_1$)

$$S'(\lambda_0) + E = 0;$$ \hspace{1cm} (150)

After that, up to universal factors

$$\Omega[p_1, q_1] \propto \exp (-\lambda_0 H_1)$$ \hspace{1cm} (151)

Now, this $\lambda_0$ in case of thermodynamics is given by inverse temperature $\lambda_0 = \beta$. In our case it is some parameter characterizing the thermostat $H_2$. By varying the energy pumping to the thermostat we can vary this parameter in the same way as we vary the temperature in the thermodynamics.

As it is evident from this computation, this saddle point, if it exists, can only be at real positive $\lambda_0$, as the integral $\int d\Gamma_2 \exp (-\lambda H_2)$ converges only in the right semi-plane (the zero mode blows in the left semi-plane).

Now, the saddle point equation can be also rewritten as

$$E(\lambda_0) = \langle H_2 \rangle = \frac{\int d\Gamma_2 H_2 \exp (-\lambda_0 H_2)}{\int d\Gamma_2 \exp (-\lambda_0 H_2)}$$ \hspace{1cm} (152)

Note that $E(\lambda_0)$ is a monotonously decreasing function as

$$E'(\lambda_0) = -\left(\langle H_2 \rangle - \langle H_2 \rangle^2\right) < 0;$$ \hspace{1cm} (153)

This also means that the entropy $S(\lambda)$ is convex function

$$S''(\lambda) = -E'(\lambda) > 0$$ \hspace{1cm} (154)

As a consequence, the factor of $i$ in $\frac{1}{\sqrt{-S''(\lambda)}} = \frac{i}{\sqrt{S''(\lambda)}}$ which arises from the Gaussian integration around the saddle point, cancels the factor of $i$ in denominator of original integral.

To be more precise, when we move the integration contour $C$ to the saddle point, we have to direct it along the steepest descent path. In our case this path goes in imaginary direction, as the second derivative $S''(\lambda)$ is positive. Thus, we have

$$\Omega[p_1, q_1] = \int_{-\infty}^{\infty} \frac{dz}{2\pi} \exp ((\lambda_0 + i z)(E - H_1) + S(\lambda_0 + i z))$$ \hspace{1cm} (155)

$$\rightarrow Z_0 \exp (-\lambda_0 H_1);$$ \hspace{1cm} (156)

$$Z_0 = \sqrt{\frac{2\pi}{S''(\lambda_0)}} \exp (\lambda_0 E + S(\lambda_0))$$ \hspace{1cm} (157)

Therefore the Gibbs weight $\Omega[p_1, q_1]$ is real and positive as it should be.
Let us now study the important issue of existence and uniqueness of this saddle point.

In case of the thermodynamics $H_2$ is positive definite, and by varying $\lambda_0$ along positive axis we go from the region of high energies (small $\lambda$) to the region of low energies (high $\lambda$). So, the expectation value monotonously varies from zero to infinity and at some point it crosses the level $E$ (only once).

In the case of Turbulence $E = 0$. Let us study the $\lambda_0$ dependence of $E(\lambda_0)$.

At small positive $\lambda_0$ expectation value is positive and grows as $\frac{1}{\lambda_0}$, thanks to the zero mode $\rho \sim \frac{1}{\sqrt{\lambda_0}}$ which leads to domination of $h_2 \sim \frac{1}{\lambda_0}$:

$$E(\lambda_0) \to \langle h_2 \rangle > 0;$$

$$\lambda_0 \to 0^+;$$

(158)

(159)

At large $\lambda_0$ this expectation value is dominated by its own saddle point, minimizing $H_2$:

$$E(\lambda_0) \to \left( \min_\rho (\rho^2 h_2 - \rho h_1) \right) =$$

$$- \left( \frac{h_1^2}{4h_2} \right) < 0;$$

$$\lambda_0 \to +\infty$$

(160)

(161)

(162)

This limit is negative, therefore at some positive $\lambda_0$ there is a solution of equation $E(\lambda_0) = 0$. As the function $E(\lambda_0)$ is monotonously decreasing, this solution is unique.

This concludes our proof.

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Figure 1: The infinitesimal loop $\delta C$ (red) encircling original loop $C$ (blue).
Figure 2: The solid torus mapped into Clebsch space
Figure 3: The Enneper’s Minimal surface
Figure 4: Linear fit of the circulation scale $S = \sqrt{\frac{\nu s}{M_8}}$ (with $M_p = \langle \Gamma^p \rangle$) as a function of the $R = a/\eta$ for inertial range $100 \leq R \leq 500$. Here $a$ is the side of the square loop $C$ and $\eta$ is a Kolmogorov scale. The linear fit $S = -0.073404 + 0.00357739 R$ is almost perfect: adjusted $R^2 = 0.999909$.
Figure 5: The relative residuals $\frac{\delta S}{S}$ of the linear fit are shown as a function of the side of the square. The smooth harmonic wave suggests that these errors are affected by harmonic wave forcing on a $16K$ lattice rather than genuine oscillations in infinite isotropic system.
Figure 6: $\log P(x)$ (red dots) together with fitted line $\log P \approx -0.000526724x - 4.3711 - 0.5 \log(x) \pm 0.116469$, $1300 < x < 28000$. Here $x = \frac{|\Gamma|}{\nu}$. Last two points have low statistics in DNS and were discarded from fit. Remaining data match the theoretical formula within statistical errors of DNS. Adjusted $R^2 = 0.999929$. 
Figure 7: Subtracting the slope. $0.000526724x + \log P(x)$ (red dots) together with fitted line $-4.3711 - 0.5 \log(x)$, $1300 < x < 28000$. Here $x = \frac{|\Gamma|}{\nu}$. We see that the pre-exponential factor $1/\sqrt{|\Gamma|}$ fits the data, though with less accuracy after subtracting the leading term.
Figure 8: Relative residuals of the log fit of PDF. The harmonic wave behavior suggests that these are artefacts of harmonic random forcing on a $16K^3$ cubic lattice rather than genuine oscillations in infinite isotropic system. Such residuals do not imply contradictions with the theory.