The limiting behavior of the Liu-Yau quasi-local energy

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Abstract

The small- and large-sphere limits of the quasi-local energy recently proposed by Liu and Yau are carefully examined. It is shown that in the small-sphere limit, the non-vacuum limit of the Liu-Yau quasi-local energy approaches the expected value \( \frac{4\pi}{3} r^3 \mathcal{I}(\mathbf{e}_0, \mathbf{e}_0) \). Here, \( \mathcal{I} \) is the energy-stress tensor of matter, \( \mathbf{e}_0 \in T_pM \) is unit time-like and future-directed at the point \( p \) located at the center of the small sphere of radius \( r \) in the limit \( r \to 0 \). In vacuum, however, the limiting value of the Liu-Yau quasi-local energy contains the desired limit \( \frac{\sqrt{15}}{\sqrt{3}} \mathcal{B}(\mathbf{e}_0, \mathbf{e}_0, \mathbf{e}_0, \mathbf{e}_0) \), where \( \mathcal{B} \) is the Bel-Robinson tensor, as well as an extra term. In the large-sphere limit at null infinity, for isolated gravitational sources, the Liu-Yau quasi-local energy is shown to recover the Bondi mass and Bondi news flux, in space-times that are asymptotically empty and flat at null infinity. The physical validity of the Liu-Yau model in view of these results is discussed.
I. INTRODUCTION

Among a handful of unsettled puzzles at the foundation of Einstein’s general theory of relativity, the very basic notion of energy-momentum seems to be of everlasting interest. Despite the triumph of the proof of the positivity of the total gravitational energy at both spatial and null infinity [31], there is a lack of a well-defined notion of the local gravitational energy-momentum density. In fact, the equivalence principle, or, the existence of the normal co-ordinate system, prohibits any non-trivial point-wise localizable density. Consequently, only quasi-local quantities are meaningful. Although various proposals have been put forward (see, for example, [32] for a fairly complete and up-to-date review.), it should be noted that the study of quasi-local quantities is still rather premature in the sense that no truly axiomatic framework has been distilled from physics. A generally accepted strategy of studying quasi-local quantities is to devise such quantities and to check that they recover, in certain limiting situations, known properties. To be more specific, take for example the quasi-local energy-momentum. It has been proposed [8], [32] that it satisfy the following empirical criteria:

(C1) Causality:
Quasi-local energy-momentum is future-directed and non-space-like, provided that matter, if any, satisfies the dominant energy condition in the region enclosed by $S$.

(C2) Positivity:
Quasi-local energy is positive and is monotone in a suitable sense but vanishes in flat space-times.

(C3) Limiting behaviors:
(a) Standard sphere limit: Quasi-local energy recovers the standard value of mass enclosed in $S$ for $S \approx S^2$ in a spherically symmetric space-time. In particular, for a sphere centered at the origin of Schwarzschild space-time, it coincides with the Schwarzschild mass parameter.

(b) Marginally trapped surface limit: Quasi-local mass agrees with the irreducible mass $\sqrt{\frac{\text{Area}(S)}{16\pi}}$.
(c) Small sphere limits:

(i) in non-vacuum: on a small sphere of radius \( r \) centered at any arbitrary point \( p \) in the space-time \( M \), the quasi-local energy-momentum recovers the energy-momentum of the matter observed by an equivalence class of instantaneous observers (the meaning of which is made precise in Sec. III C), characterized by a unit time-like and future-directed \( e_0 \in T_p M \), namely, 
\[
\frac{4\pi}{3} r^3 \mathcal{T}(\bullet, e_0),
\]
where \( \mathcal{T} \) is the energy-stress tensor.

(ii) in vacuum: the quasi-local energy-momentum yields the analogue of the gravitational energy-momentum observed by the class of observers given in (i) in terms of the Bel-Robinson tensor \( \mathcal{B} \), namely, 
\[
\frac{r^5}{90} \mathcal{B}(\bullet, e_0, e_0, e_0).
\]

(d) Large sphere limits:

(i) at spatial infinity: quasi-local energy-momentum approaches the Arnowitt-Deser-Misner (ADM) energy-momentum in an asymptotically flat space-like hypersurface.

(ii) at null infinity: quasi-local energy-momentum reproduces the standard Bondi mass \( E_{BS} \) and news flux \( \frac{\partial}{\partial t} E_{BS} \).

In fact, in the large sphere limits, quasi-local quantities are no longer truly quasi-local as \( S \) contains an infinite measure. A perhaps more proper term here would be quasi-global.

A similar scrutiny of the quasi-local angular momentum is technically more involved partially because the very definition of the quasi-local angular momentum in several contexts is yet to be unanimously agreed upon. Tentative investigations have been carried out in the past decades with few definitive outcomes (see [32] for an overview). The present work is thus focused on studying the quasi-local energy, only.

Like any other construction of quasi-local energy, Liu-Yau’s is subject to reasonable reality checks in order to be a physically sound candidate. Among (C1)-(C3) listed above, (C1), (C2), (C3)-(a), (b) and (d)-(i) have been discussed [21], [24], [22]. The examination of other limiting behaviors of Liu-Yau quasi-local energy—the main body of the present work—is presented in the following sections.

This article is structured as follows. In Sec. II, a specific model of the quasi-local energy proposed by Liu and Yau [21] is reviewed. The small- and large-sphere limits of the Liu-Yau quasi-local energy are closely examined, in Sec. III and Sec. IV respectively. Also considered, along the same lines, is the possibility of generalizing the notion of quasi-local energy for non-isolated gravitational sources. A summary is given in Sec. V.
II.  DEFINITION OF THE LIU-YAU QUASI-LOCAL ENERGY

The Liu-Yau quasi-local energy originated as a continuation of Yau’s mathematical work on the positivity of black hole mass [36], [8]. The definition of the Liu-Yau quasi-local energy is reviewed here for completeness.

A.  Physical part

Consider a closed orientable space-like 2-surface $S$ embedded in $M$. $\forall p \in S$, $\exists$ null frame $\{X_i\}^4_{i=1} \in CT_pM$ with its dual $\{\theta^i\}^4_{i=1} \in CT^*_pM$ adapted to $S$, such that $CT_pM = CT_pS \oplus CT_pS^\perp$, where $CT_pS = \text{span}_C\{X_3, X_4\}$ and $CT_pS^\perp = \text{span}_C\{X_1, X_2\}$. (Shorthand notation $AE = E \otimes_R A$, where $A$ is a field and $E$ is a bundle, is used throughout. For example, for $A = C$, $CE$ is the complexified bundle of $E$. When $A = R$, however, it is obvious that $RE = E$.) The mean curvature vector of $S$ at $p$ in $M$ is then $H = -2\mu X_1 - 2\rho X_2$, where $\mu = \theta^1(DX_3X_4)$ and $\rho = \theta^2(DX_4X_3)$. It is worth noticing that the norm of $H$ is, however, independent of the choice of moving frames. Indeed, $\|H\| = \sqrt{8\rho\mu}$, where $\rho\mu > 0$ for space-like $H$. The physical part of the quasi-local energy is then defined as

$$E_{\text{phys}}(S) \equiv \frac{1}{8\pi} \int_S \|H\|\Omega,$$

where $\Omega$ is the volume form of $S$.

B.  Reference part

Suppose that $S$, equipped with a Riemannian metric $g^S$ and Levi-Civita connection $D^S$, has positive sectional curvature. Then by Weyl’s embedding theorem [34], there exists a unique isometric embedding $\iota_1 : (S, g^S, D^S) \hookrightarrow (M^\circ, g^\circ, D^\circ)$, up to isometries of $\mathbb{R}^3$, such that the second fundamental form $II^\circ$ is solely determined by $g^S$ and is positive definite on $S$. The composition of $\iota_1$ with a successive embedding $\iota_2 : \mathbb{R}^3 \hookrightarrow \mathbb{R}^3_1$ gives the reference embedding $\iota \equiv \iota_2 \circ \iota_1 : S \hookrightarrow \mathbb{R}^3_1$ of $S$ into Minkowski space-time $M^\circ$.

The same construction as in Sec. II A gives the mean curvature vector, $H^\circ$, of $S$ at $p$ in $\mathbb{R}^3_1$ whose norm is $\|H^\circ\| = \sqrt{8\rho^\circ\mu^\circ}$, where $\rho^\circ\mu^\circ > 0$. Hence the reference part of the
quasi-local energy is naturally defined as

\[ E_{\text{ref}}(S) \equiv \frac{1}{8\pi} \int_S \|H^o\|\Omega. \]

A caveat is emphasized in [24] and [22] to avoid any misleading interpretations. Unless \( S \) lies in a space-like hypersurface \( \Sigma \subset M \), it would be highly unnatural to require that \( S \) be isometrically embedded into a space-like hypersurface in \( M^o \). That is, the absence of this additional hypothesis may result in positive quasi-local energy even in \( M^o \).

One of the merits of the embedding scheme described above is that \( \iota_1 \) is unique up to isometries of \( R^3 \) and that \( H^o \) is therefore well-defined. However, it is in general non-trivial to obtain a complete solution to the full set of integrability conditions for the sequence of embeddings of \( S \) in \( M^o \). An alternative approach is to consider the co-dimension 2 embedding \( \iota^o : S \hookrightarrow M^o \) at a possible expense of uniqueness (up to isometries of \( R^3_1 \)) unless extra restrictions are imposed. The existence of \( \iota^o \) is, nonetheless, guaranteed for any conformally flat \( S \), as shown in [24]. In Sec. [III D], embeddings of this kind are realized as the null-cone reference. In Sec. [IV C], the asymptotic version of such embeddings is studied in the large-sphere limit at null infinity.

C. The definition of the Liu-Yau quasi-local energy

Definition 2.1
The \textit{Liu-Yau quasi-local energy} associated with the 2-surface \( S \) is

\[ E(S) \equiv E_{\text{ref}}(S) - E_{\text{phys}}(S). \]

III. THE SMALL-SPHERE LIMIT

This section is devoted to gauging Liu-Yau quasi-local energy \( E(S) \) against criterion \textbf{C3-(c)} when \( S \) is a small sphere, as defined below. It is shown that \( E(S) \) satisfies \textbf{C3-(c)-(i)} for non-vacuum, but deviates, in vacuum, from the expected value in \textbf{C3-(c)-(ii)} by an extra term of which the physical nature is yet to be explored.
A. Construction of the small sphere

The small sphere around an arbitrary point in the space-time is a space-like level set of the null cone emanating from that point. \( \forall p \in M, \exists \) a normal neighborhood \( U \) of \( p \) in \( M \) which uniquely determines a star-shaped neighborhood \( \tilde{U} \) of 0 in \( T_p M \), such that the exponential map \( \exp_p \) is a diffeomorphism of \( \tilde{U} \) onto \( U \) whose inverse is denoted by \( \exp_p^{-1} \). Without loss of generality, it is assumed that \( \tilde{U} \) is small enough such that the null cut locus \( \tilde{C}_N^+(p) \not\subset \tilde{U} \) and hence that \( C_N^+(p) \not\subset U \). For a given orthonormal basis \( \{e_i\}_{i=0}^{n-1} \) for \( T_p M \) (\( n = 4 \) when \( M \) represents a space-time), with \( (e_i, e_j) = \delta_{ij}e_j, (i, j = 0, \ldots, n-1) \) and its dual basis \( \{e^i\}_{i=0}^{n-1} \) for \( T_p^* M \), the normal (Cartesian) co-ordinate system of the connection \( D \) on \( U \) is defined by \( \xi \equiv (x^0, \ldots, x^{n-1}) \in \mathfrak{X}(U, \mathbb{R}^n) \): \( x^i = z^i \circ \exp_p^{-1}, i = 0, \ldots, n-1 \). Correspondingly, such a normal co-ordinate system \( (x^0, \ldots, x^{n-1}) \) determined by \( \{e_i\}_{i=0}^{n-1} \) assigns to each point \( q \in U \) co-ordinates with respect to basis \( \{e_i\}_{i=0}^{n-1} \) of the pull-back \( \exp_p^{-1}(q) \in \tilde{U} \subset T_p M \) via \( \exp_p^{-1}(q) = \sum_{i=0}^{n-1} x^i(q)e_i, \forall q \in U \). Thus, the normal (Cartesian) co-ordinate system \( (x^0, \ldots, x^{n-1}) \) on \( U \) induces a Cartesian co-ordinate basis \( \left\{ \frac{\partial}{\partial x^i} \right\}_{i=0}^{n-1} \in \mathfrak{X}(U) \).

Define the Lorentz radius function \( \delta \in \mathfrak{X}(U, \mathbb{R}) \) on \( M \) at \( p \) as \( \delta(q) \equiv |\exp_p^{-1}(q)|, \forall q \in U \) and consider the geodesic ball \( \delta^{-1}(c) = \left\{ q \in U : \sum_{i=0}^{n-1} \epsilon_i(x^i(q))^2 = c^2 \right\} \subset U \) in normal co-ordinates for sufficiently small \( |c| \geq 0 \) and a hypersurface \( (x^0)^{-1}(t) \) for a given \( t \in \mathbb{R} \). Then \( S(c, t) \equiv \delta^{-1}(c) \cap (x^0)^{-1}(t) = r^{-1}(\sqrt{c^2 - x^0(t)^2}) \) is a closed sub-manifold of \( M \), where \( r \in \mathfrak{X}(U, [0, +\infty)) \) by \( r \equiv \sqrt{c^2 - x^0(t)^2} \) is the radial co-ordinate of the spherical normal co-ordinates. In particular, for \( c = 0 \), \( \delta^{-1}(0) = \mathbb{J}^+(p, U) \) and thus \( \forall q \in U, \exists \text{ null } X_1 = \exp_p^{-1}(q) \in \tilde{U} \), such that the ray \( \rho : [0, a_0) \rightarrow \tilde{U} \) by \( \rho(x^0) \equiv x^0 X_1, x^0 \in [0, a_0) \), where \( [0, a_0) \) is the maximal domain of \( \rho \), uniquely determines a radial null geodesic from \( p \) to \( q \), \( \gamma^{X_1} : [0, a_0) \rightarrow U \) by \( \gamma^{X_1} = \exp_p \circ \rho \). The superscript \( X_1 \) stresses that \( \gamma \) is the local flow of \( X_1 \), which is understood hereafter and the superscript will most often be suppressed when confusion is unlikely. It is noted in passing that as \( c = 0 \), \( x^0 = r \), hence \( \gamma \) is well affinely parameterized by \( r \in [0, a_0) \). The local null-cone \( \Lambda(p) \equiv \delta^{-1}(0) - p \) is then foliated by \( \mathfrak{F}_\Lambda = \left\{ \gamma|_{[0, a_0)} : \lim_{r \rightarrow 0} \dot{\gamma}(r) = \exp_p^{-1}(q), q \in \Lambda(p) \right\} \).

**Definition 3.1**

For every given \( t \), \( S(c, t) = S(0, r) \cong S^2 \), as a regularly embedded space-like sub-manifold of \( M \), is the desired small sphere. For economy of notation, \( S(0, r) \) will almost always be
abbreviated as \( S(r) \).

**B. Moving frames on the small sphere**

The regular embedding of \( S(r) \) into \( M \) suggests an envisaged choice of adapted orthonormal null frames on \( U \cap \Lambda(p) \) as in Sec. II. For every given \( r = x^0 \in (0, a_0) \), \( \forall q \in S(r) \hookrightarrow M \), \( \exists \) null frame \( \{X_i(q)\}_{i=1}^n \in CW_q \), where \( W_q = T_qM \cap T_qM^\perp \), with its dual \( \{\theta^i(q)\}_{i=1}^n \in CW_q^* \) adapted to \( S \), such that \( CT_qM = CT_qS(r) \oplus CT_qS(r)^\perp \), where \( CT_qS(r) = \text{span}_C\{X_3(q), X_4(q)\} \) and \( CT_qS(r)^\perp = \text{span}_C\{X_1(q), X_2(q)\} \). The principal bundle of null frames \( F(U \cap \Lambda(p)) \) thus possesses a reduced structure group \( G = \mathbb{C}^\times \cong \text{GL}(1, \mathbb{C}) \subset \text{SL}(2, \mathbb{C}) \), where the right group action \( F(U \cap \Lambda(p)) \times G \longrightarrow F(U \cap \Lambda(p)) \) is given by \( u \cdot z = (X_1', \ldots, X_4') = (X_1, \ldots, X_4)A(z) \), where \( u = (X_1, \ldots, X_4) \in CW_q \), \( \forall q \in U \cap \Lambda(p) \), \( z \in G \), and \( A(z) = \text{diag}(|z|^2, i\frac{1}{|z|^2}, \frac{1}{|z|^2}, \overline{z}) \in \text{GL}(1, \mathbb{C}) \cong G \). Such construction of adapted null frames is known, in physics literature, as the Geroch-Held-Penrose (GHP) formalism [14], [11].

A few geometrical properties follow almost transparently from the preceding construction. For pedagogical purposes, however, it is considered helpful to first recall some facts about degenerate sub-manifolds of semi-Riemannian manifolds [20], which will be preliminary for Sec. IV as well.

Denoted by \( \mathbb{A}K \) the degenerate bundle over a degenerate sub-manifold \((H, g_H)\) of a semi-Riemannian manifold \((M, g)\) equipped with a Levi-Civita connection \( D \). It is known that \( \mathbb{A}K = \mathbb{A}TH \cap \mathbb{A}TH^\perp \) and \( \mathbb{A}K^\perp = \mathbb{A}TH + \mathbb{A}TH^\perp \). In particular, when \((H, g_H)\) is a null sub-manifold of a Lorentzian manifold \((M, g)\), \( \mathbb{A}K \) is the unique null \( \mathbb{A} \)-line sub-bundle of \( \mathbb{A}TH \). Furthermore, if \( H \) is a null hypersurface, then \( \mathbb{A}K = \mathbb{A}TH^\perp \).

**Definition 3.2**

\((H, g_H)\) is irrotational if \( DU \in \text{End}_{\mathbb{A}}(\mathbb{A}TH) \), \( \forall U \in \Gamma(\mathbb{A}K) \), where the bundle homomorphism \( DU \in \text{Hom}_{\mathbb{A}}(\mathbb{A}TH, \mathbb{A}TM|_H) \), for a given \( U \in \Gamma(\mathbb{A}K) \), is defined as \( DU(X) \equiv D_XU \), \( \forall X \in \mathbb{A}TH \).

The following properties are immediate consequences of the above series of definitions.

**Proposition 3.3**

1. If \((H, g_H)\) is irrotational, then \( DU \in \text{End}_{\mathbb{A}}(\mathbb{A}TH) \) is self-adjoint, \( \forall U \in \Gamma(\mathbb{A}K) \), i.e., \( g_H(DU(X), Y) = g_H(X, DU(Y)) \), \( \forall X, Y \in \mathbb{A}TH \).
(2) If \((H, g_H)\) is integrable and \(A \cdot K = A \cdot T H^\perp\) (or, equivalently, \(A \cdot K^\perp = A \cdot T H\)), then \((H, g_H)\) is irrotational.

Proof:

(1) A straightforward calculation.
(2) follows from (1). ■

**Corollary 3.4**

Every degenerate hypersurface \((H, g_H)\) is irrotational. ■

**Definition 3.5**

\(U \in \Gamma(A \cdot K)\) is pregeodesic if \(\exists f \in \mathcal{F}(H, R), \) such that \(D_U U = f U. \) \((H, g_H)\) is geodesic if every \(U \in \Gamma(A \cdot K)\) is pregeodesic.

**Corollary 3.6**

If \((H, g_H)\) is irrotational, then it is geodesic.

Proof: compatibility and torsion-free properties of \(D\). ■

In the present context, \(\Lambda(p)\) is a degenerate (in fact, null) hypersurface of \(M\). Thus, with \(A = C\), the following lemma, which will be used frequently throughout later calculations, is readily seen to hold.

**Lemma 3.7**

(1) \(\varepsilon \equiv \frac{1}{2} \left( \theta^1(D_X X_1) - i \theta^3(D_X X_3) \right) = 0 \) and \(\kappa \equiv \theta^2(D_X X_3) = 0\).
(2) \(\rho \equiv -\theta^2(D_X X_3)\) and \(\mu \equiv \theta^1(D_X X_4)\) are real.
(3) \(\tau - \overline{\tau} - \beta = 0\) and \(\pi - \alpha - \overline{\beta} = 0\), where \(\tau \equiv -\theta^2(D_X X_3)\), \(\pi \equiv -\theta^2(D_X X_4)\) and \(\alpha + \beta \equiv -\theta^2(D_X X_2)\)

Proof:

(1) Since \(X_1 = \dot{\gamma}\) for the radial null geodesic \(\gamma, D_X X_1 = 0\). Hence the orthonormality of the null frame implies, for the Levi-Civita connection \(D\), that \(\theta^2(D_X X_2) = 0, \theta^3(D_X X_3) = 0\), and thus the claim.
(2) Since \(\Lambda(p)\) is a null hypersurface of \(M\), it follows immediately from Proposition 3.3 that \(\theta^3(D_X X_3) = \theta^4(D_X X_4)\).
(3) Similar to (2). ■

To better demonstrate the geometry of the local null-cone and the small sphere, it is instructive to introduce angular co-ordinate functions \((\theta, \varphi) \in \mathcal{F}(U \cap (\Lambda(p) - \{\gamma_0, \gamma_\pi\}), (0, \pi) \times [0, 2\pi))\), where \(\gamma_0\) and \(\gamma_\pi \in \mathcal{F}_A\) are two disconnected leaves (aka generators) of \(\mathcal{F}_A\). Together with \(x^0\) and \(r\), the local null-cone \((\Lambda(p) - \{\gamma_0, \gamma_\pi\})\) comes equipped with spherical normal
co-ordinates $\varsigma \equiv (x^0, r, \vartheta, \varphi)$. Gauss’ lemma then guarantees that $\{\frac{\partial}{\partial x^0}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \vartheta}, \frac{\partial}{\partial \varphi}\} \in \mathcal{X}(U \cap (\Lambda(p) - \{\gamma_0, \gamma_\pi\}))$ constitutes a set of mutually orthogonal spherical co-ordinate basis. In particular, for every fixed $r \in (0, a_0)$, the small sphere $S(r)$ can be parameterized by $(\vartheta, \varphi)$ or, as $S^2 \cong \mathbb{C}P^1$, by the stereographic co-ordinates $(\zeta, \bar{\zeta})$, with the understanding that the poles are not covered.

In the spherical normal co-ordinates $\varsigma$, the null orthonormal frame at every point has co-ordinate representation: $X_i(q) = \sum_{a=1}^{n} X_i^a(q) \frac{\partial}{\partial x^a} |_{q}$, $i = 1, \ldots, n$, $\forall q \in U \cap (\Lambda(p) - \{\gamma_0, \gamma_\pi\})$. One noticeable simplification in the co-ordinate representation of $X_i(q)$ is furnished by the foliation $\mathcal{F}_\Lambda$. Recall from Sec. III A that $\forall \gamma \in \mathcal{F}_\Lambda$, $\dot{\gamma} = X_1$, and that the affine parameter of $\gamma$ can be arranged to be one of the spherical normal co-ordinates, namely $\varsigma^r = r$. Therefore, along the local flow, $\gamma^X_1$, of $X_1$ in $U$, it is clear by definition that $1 = D_{X_1} r = X_1 r$. Then, $X_1^a = \delta^a_r$ in the co-ordinate representation of $X_1$, and thus $X_1 = 1 \cdot \frac{\partial}{\partial r}$.

C. Remarks on the vertex of the null cone

Notice, by definition of normal co-ordinates, that $x^0(p) = r(p) = 0$, whereas the angular part of $\varsigma$ becomes degenerate at $p$. Consequently, the moving frames cannot be extended even continuously to $p$. It is then necessary to examine the directional dependence of the limiting process as $r \to 0$.

**Definition 3.8**

The blow-up $\hat{\Lambda}(p)$ of $\Lambda(p)$ at $p$ is defined as $\hat{\Lambda}(p) \equiv \Lambda(p) \cup_\pi S^2$ with the contraction $\pi : S^2 \to p$. Hence $\hat{\Lambda}(p)$ and $\delta^{-1}(0)$ are homotopy equivalent. Correspondingly, the blow-up $\hat{U}_0$ of $\hat{U}$ at $0$ is simply the diffeomorphic pull-back of $\hat{\Lambda}(p)$ by $\exp_p^{-1}$, i.e., $\hat{U}_0 = (\hat{U} - 0) \cup_\pi \hat{S}^2$, where $\pi : \hat{S}^2 \to 0$ is the corresponding contraction in $T_p M$.

Now, for a given orthonormal basis $\{e_i\}_{i=0}^{n-1}$ of $T_p M$, it is natural to construct the spherical radial basis vector $e_r(\vec{\vartheta}, \vec{\varphi}) = \sin \vec{\vartheta} \cos \vec{\varphi} e_1 + \sin \vec{\vartheta} \sin \vec{\varphi} e_2 + \cos \vec{\vartheta} e_3$, where $(\vec{\vartheta}, \vec{\varphi}) \in (0, \pi) \times [0, 2\pi)$ represents the corresponding angular co-ordinates of $\hat{S}^2$ obtained via the dual basis $\{x^i\}_{i=0}^{n-1}$ of $T_p^* M$. Then the direction-dependent limit of the null basis $\{X_i\}_{i=1}^{n} \in \Gamma(\mathcal{C}K) at \hat{S}^2$ coincides with $X_1(p, \vec{\vartheta}, \vec{\varphi}) = e_0 + e_r$, $X_2(p, \vec{\vartheta}, \vec{\varphi}) = \frac{1}{\sqrt{2}}(e_0 - e_r)$, $X_3(p) = \frac{1}{\sqrt{2}}(e_1 + ie_2)$, and $X_4 = \vec{\nabla}_3$, $\forall (\vec{\vartheta}, \vec{\varphi}) \in \hat{S}^2$.

Physically, the blow-up of $p$ defines an equivalence class of spatially isotropic instantaneous observers at $p$. It suffices to consider only linear isometries of $T_p M$, which constitute
a subgroup, $\mathcal{L} = O(3, 1, \mathbb{R})$, of the full isometry group $I(T_p M) = \mathcal{L} \rtimes \mathbb{R}^n$.

**Definition 3.9**

An *instantaneous observer* at $p$ is an ordered pair $(p, e_0)$, where $e_0 \in T_p M$ is unit time-like and future-directed. $(p, e_0)$ is said to be *spatially isotropic* if the stabilizer of $e_0$ in $\mathcal{L}$ is $\mathcal{L}_{e_0} = O(3, \mathbb{R})$.

**Definition 3.10**

An *equivalence class*, denoted as $\mathcal{O}_p(e_0)$, of spatially isotropic instantaneous observers at $p$ is defined as the orbit of $\mathcal{L}_{e_0}$ in $T_p M$.

It will be seen in Sec. III E that it is $\mathcal{O}_p(e_0)$ who measures the quasi-local energy in the small sphere limit at $p$.

While $\tilde{S}^2$ is a topological 2-sphere, whether or not it can be realized in the small-sphere limit as a metric 2-sphere depends upon whether $p$ is a curvature singularity, i.e., whether $p \in M$. Analysis in the cases where $p$ is of “elementary singularity” is miserably complicated as no explicit Riemannian metric on the limiting sphere $\tilde{S}^2$ is available. Efforts have been channeled towards appealing to series expansions for small asphericity in $\tilde{S}^2$, but have not proven to be as promising as expected. On the other hand, when the curvature at $p$ is finite, it is always possible to choose $\tilde{S}^2$ as a metric 2-sphere as the limiting small sphere. Therefore, only the latter situation is considered in the present work.

**D. Reference embedding**

The reference part of the quasi-local energy associated with the 2-surface $S(r)$ is solely determined by the intrinsic properties of $S(r)$ although it appears to be defined in an extrinsic manner as in Sec. III B. The apparent inconsistency is reconciled by virtue of the integrability conditions for $S(r)$, particularly, as a sub-manifold of $M$. It turns out that for the current problem at hand, the Gauss equation alone is sufficient.

As described in Sec. III B, consider the embedding $\iota^o : (S(r), g^S, D^S) \hookrightarrow (M^o, g^o, D^o)$ of $S(r)$ of dimension $m = 2$ into Minkowski $(M^o, g^o)$ with flat connection $D^o$. Now, $\forall p \in S(r)$, with the usual identification of $\mathbb{A}T_p S(r)$ and $\mathbb{A}T_p S(r)^\perp$ with sub-spaces of $\mathbb{A}T_{\iota^o p} M$, $\mathbb{A}T_{\iota^o p} M = \mathbb{A}T_p S(r) \oplus \mathbb{A}T_p S(r)^\perp$. Here, it is assumed that $T_p S(r)^\perp = \text{span}_{\mathbb{R}}\{T, N\}$, with orthonormal frame $\{T, N\}$ and associated co-frame $\{\theta^T, \theta^N\}$, where $T$ is chosen to be time-like and $N$ space-like. As a side remark, the standard space-time orthonormal frame field

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is used in $TM$ ($A = \mathbb{R}$), whereas the null frame field is used in $CTM$ ($A = \mathbb{C}$). The same remark applies dually for the co-frames. The transformation between the frame $\{T, N\}$ and the null frame $\{X_1, X_2\}$ as in Sec. III B is, by convention, fixed by $(T, N) = (X_1, X_2)\mathfrak{U}$, where $\mathfrak{U} = \begin{pmatrix} 1/2 & 1/2 \\ 1 & -1 \end{pmatrix} \in SO(1, 1, \mathbb{C})$. Together with any given orthonormal frame $\{E_j\}_{j=1}^m \in AT_pS(r)$ and its co-frame $\{\omega^j\}_{j=1}^m$, established are an adapted orthonormal frame $\{E_j\}_{j=1}^n \in AT_{e_p}M$ and its co-frame $\{\omega^j\}_{j=1}^n \in AT^*_{e_p}M$ satisfying

$$\overline{E}_j = \begin{cases} T : & j = 1 \\ N : & j = 2 \\ E_j : & j = 3, 4 \end{cases}$$

and

$$\overline{\omega}^j = \begin{cases} \theta^T : & j = 1 \\ \theta^N : & j = 2 \\ \omega^j : & j = 3, 4 \end{cases}$$

Alternatively, if null frames are used,

$$\overline{E}_j = \begin{cases} X_j : & j = 1, 2 \\ X_j : & j = 3, 4 \end{cases}$$

and

$$\overline{\omega}^j = \begin{cases} \theta^j : & j = 1, 2 \\ \theta^j : & j = 3, 4 \end{cases}$$

Connections $D^S$ and $D^o$ are related by the shape tensor $s^\perp$. By definition, $\forall x_p, y_p \in T_pS(r), D^o x_p y = D^S_{x_p} y + s^\perp(x_p, y_p)$, where $s^\perp(x_p, y_p) = II^T(x_p, y_p)T + II^N(x_p, y_p)N$, with $II^T(x_p, y_p) = \theta^T(D^o x_p y)$ and $II^N(x_p, y_p) = \theta^N(D^o x_p y)$, where $y$ is the extension of $y_p$ in the neighborhood of $p$.

The fully contracted Gauss equation for the embedding $(S(r), g^S, D^S) \hookrightarrow (M^o, g^o, D^o)$ can be written as

$$0 = S_{S(r)} + \tau_2 \left[ \sum_{j=3}^4 \sum_{l=3}^4 II^N(E_l, E_j)II^N(\omega^l, \omega^j) - (II^N)^2 \right]$$

$$+ \tau_1 \left[ \sum_{j=3}^4 \sum_{l=3}^4 II^T(E_l, E_j)II^T(\omega^l, \omega^j) - (II^T)^2 \right],$$

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where \( \tau_1 = g(T, T) \), \( \tau_2 = g(N, N) \), and \( H^r = \sum_{i=3}^4 H^r(\omega^i, E_i) \), \((r = N, T)\), or, for brevity,
\[
\bar{\tau}_2 \left[ (II^N)^2 - II^N \cdot II^N \right] + \bar{\tau}_1 \left[ (II^T)^2 - II^T \cdot III^T \right] - S_{S(r)} = 0. \tag{1}
\]
Further simplification of the integrability conditions comes from the use of the so-called null-cone reference (aka light-cone reference). The same procedure as in Sec. III A applies for the construction of \( \Lambda^o(p) \) in the reference space-time \( M^o \) only now, it is a bona fide null cone as \( M^o \) is Minkowski. One of the immediate benefits is that the leaves of \( \mathcal{F}_{A^o} \) are shear-free, i.e., \(-\sigma^o = \theta^2(D^o_{X_3}X_3) = II^N(X_3, X_3) + II^T(X_3, X_3) = 0 = -\overline{\sigma}^o = \theta^2(D^o_{X_4}X_4) = II^N(X_4, X_4) + II^T(X_4, X_4) \). Then, Eq. (1) becomes
\[
\bar{\tau}_2 \left[ (II^N)^2 - 2II^N(X_3, X_4)II^N(X_4, X_3) \right] + \bar{\tau}_1 \left[ (II^T)^2 - 2II^T(X_3, X_4)II^T(X_4, X_3) \right] - S_{S(r)} = 0. \tag{2}
\]
Lemma 3.7-(2), when applied to \( \Lambda^o(p) \), yields \( \overline{\rho}^o = \rho^o \) and \( \overline{\mu}^o = \mu^o \). Hence, \( II^N(X_3, X_4) = \theta^o(D^o_{X_3}X_4) = -\frac{1}{2}\overline{\rho}^o + \mu^o = -\frac{1}{2}\rho^o + \overline{\rho}^o = II^N(X_4, X_3) \). Similarly, \( II^T(X_3, X_4) = II^T(X_4, X_3) \). In fact, finer results of \( \rho^o \) and \( \mu^o \) can be obtained by basis transformation \( \Omega \): \( \rho^o = -\theta^2(D^o_{X_4}X_3) = -II^T(X_4, X_3) + II^N(X_4, X_3) = \frac{1}{2}(-II^T + II^N) \) and, by the same token, \( \mu^o = \frac{1}{4}(II^T + II^N) \). Hence, \( \|H^o\| = \sqrt{8\rho^o\mu^o} = \sqrt{(II^N)^2 - (II^T)^2} \). On the other hand, \( 2II^N(X_3, X_4)II^N(X_4, X_3) = \frac{1}{2}(II^N)^2 \) and \( 2II^T(X_3, X_4)II^T(X_4, X_3) = \frac{1}{2}(II^T)^2 \). Therefore, Eq. (2) can be written as
\[
(II^N)^2 - (II^T)^2 - 2S_{S(r)} = 0,
\]
which is the ultimate integrability condition needed here. It then follows that \( \|H^o\| = \sqrt{2S_{S(r)}} \), intrinsically determined by the (positive) sectional curvature of \( S(r) \) as desired.

**E. Limiting process**

As declared in Sec. III C, it is always assumed that curvature at \( p \) is finite. Thus, the blow-up of \( p \), \( S^2 \), is a metric 2-sphere. Since the connection at \( p \) is flat, the leading term in every quantity is always its value in Minkowski space-time. The limiting process is then a straightforward but tedious radial expansion along the leaves of \( \mathcal{F}_A \) of the quasi-local energy associated with the small sphere \( S(r) \) having a standard \( S^2 \) as the limiting sphere at \( p \). Such expansions have been carried out, up to various accuracy order, within the Newman-Penrose
(NP) formalism \cite{25} and are well documented in the existing literature. In what follows, expansions of related quantities are quoted without proof as details can be found in, for example, Refs. \cite{18}, \cite{9}, \cite{6}.

To differentiate among the behaviors of the quasi-local energy in non-vacuum and vacuum cases, the small-sphere limit is taken separately.

1. Non-vacuum

It turns out that the expansions of the relevant variables are needed only along the leaves of $\mathcal{F}_\Lambda$ in the small-sphere limit. In the spherical normal co-ordinates $\varsigma$ on the local null cone of $p$, the following most pertinent quantities are expanded, in the Newman-Penrose formalism, in power series of the radial co-ordinate $r$.

$$
S_{S(r)} = 2r^{-2} + S_{S(r)}^{(0)} + O(r)
$$

$$
\rho = -r^{-1} + \frac{1}{3} r \phi_{00}^{0} + O(r^2)
$$

$$
\mu = -\frac{1}{2} r^{-1} +
\frac{1}{2} r \left[ \psi_{2}^{0} + \psi_{2}^{0} + 2 \Lambda^{0} + \frac{2}{3} \phi_{11}^{0} - \frac{1}{3} \phi_{00}^{0} \right] + O(r^2)
$$

$$
\Omega = \Omega_{0} r^{2} \left[ 1 - \frac{1}{3} r^{2} \phi_{00}^{0} + O(r^{3}) \right],
$$

where $\Omega_{0}$ is the volume form of a metric 2-sphere $S_{0}$. Consequently, the quasi-local energy in non-vacuum can be written as

$$
E = \frac{1}{8\pi} \int_{S_{r}} \left( \sqrt{2S_{S(r)}} - \sqrt{8\rho\mu} \right) \Omega
$$

$$
= \frac{1}{2} \int_{S_{0}} \left[ \frac{1}{3} \left( \phi_{00}^{0} + \phi_{10}^{0} + \phi_{01}^{0} - 2 \phi_{00}^{0} \psi_{0}^{0} + 2 \psi_{0}^{0} \phi_{00}^{0} \right) +
\left( \psi_{0}^{2} + \psi_{0}^{2} + 2 \Lambda^{0} + \frac{2}{3} \phi_{11}^{0} - \frac{1}{3} \phi_{00}^{0} \right) + \frac{1}{3} \phi_{00}^{0} \right] \frac{\Omega_{0}}{4\pi} + O(r^{4}).
$$

Most of the terms in the integral vanish for one of two reasons shown below. The validity of the following lemma is well known, although it is sketched here with a slightly more formal proof.

\footnote{It is worth pointing out that Eq.(B18) in \cite{6} is a misprint, although the following Eqs.(B19) and (B20), are nevertheless, correct.}
Lemma 3.11
(a) \( \int_{S_0} (\partial_0 f) \Omega_0 = 0, \forall f \in \mathcal{C}^k(-1) \) and \( \int_{S_0} (\bar{\partial}_0 g) \Omega_0 = 0, \forall g \in \mathcal{C}^k(+1) \), where \( \mathcal{C}^k(s) \) is the sheaf of germs of spin weight \( s \) \( \mathbb{C} \)-valued functions.
(b) \( \int_{S_0} \text{Re}(\psi_2^0) \Omega_0 = 0. \)

Proof:

(1) An observation made in [12] indicates that, on \( S_0 \cong \mathbb{C}P^1 \), the elliptic operator \( \bar{\partial} \) is merely \( \partial \) in disguise. Hence, \( \bar{\partial} : \mathcal{C}^k(-1) \rightarrow \mathcal{C}^k(0) \) is essentially the same as \( \partial : \mathcal{E}^{0,1} \rightarrow \mathcal{E}^{1,1} \), where \( \mathcal{E}^{p,q} \) is the sheaf of germs of \( \mathbb{C} \)-valued forms of type \( (p,q) \). A similar argument holds for the conjugate operators. Then it is more transparent that the integral vanishes essentially by virtue of Stokes’ theorem.

(2) By definition, \( \text{Re}(\psi_2) = -\frac{1}{2} \theta^4(W_{X_1}X_2X_1), \) where \( W \) is the Weyl tensor. Then, using the basis transformation \( \Phi \) at \( p \), i.e., on \( \widetilde{S}^2 \), \( g(W_{X_1}X_2X_1, X_2) = \frac{1}{4} \mathcal{W}((e_0 + e_r) \wedge (e_0 - e_r), (e_0 + e_r) \wedge (e_0 - e_r)) = \mathcal{W}(e_0 \wedge e_r, e_0 \wedge e_r) = a^2 \mathcal{W}(e_0 \wedge e_1, e_0 \wedge e_1) + b^2 \mathcal{W}(e_0 \wedge e_2, e_0 \wedge e_2) + c^2 \mathcal{W}(e_0 \wedge e_3, e_0 \wedge e_3) + 2ab \mathcal{W}(e_0 \wedge e_1, e_0 \wedge e_2) + 2ac \mathcal{W}(e_0 \wedge e_1, e_0 \wedge e_3) + 2bc \mathcal{W}(e_0 \wedge e_2, e_0 \wedge e_3). \) Here, with a slight abuse of notation, \( W \) is also represented by \( \mathcal{W} \in \text{SymBil}(\wedge^2 T_p M \times \wedge^2 T_p M, \mathbb{R}) \), such that \( \text{Ric}(\mathcal{W}) = 0 \) [13]. In terms of the spherical harmonics \( Y_i^m(\bar{\partial}, \bar{\varphi}), a = \sqrt{\frac{2\pi}{3}}(Y_1^1 - Y_1^{-1}), b = i \sqrt{\frac{2\pi}{3}}(Y_1^1 + Y_1^{-1}), \) and \( c = \sqrt{\frac{2\pi}{3}} Y_1^0, \) where \( (\bar{\partial}, \bar{\varphi}) \in \widetilde{S}^2 \) and \( \{e_i\}_{i=0}^{3} \in T_p M \) are as given in Sec. III.C. Now it follows from the orthogonality of spherical harmonics and the trace-free character of the Weyl tensor that the integral vanishes identically. ■

Now, it is straightforward to establish the following:

\[
E(S) = \frac{r^3}{2} \int_{S_0} \left[ \frac{1}{3} \phi_{00}^0 + 2 \Lambda^0 + \frac{2}{3} \phi_{111}^0 \right] \Omega_0 + O(r^4)
\]

\[
= \frac{r^3}{2} \left[ \frac{1}{3} \text{Ric}(e_0, e_0) + \frac{1}{6} S \right] + O(r^4)
\]

\[
= \frac{4\pi}{3} r^3 \mathfrak{I}(e_0, e_0) + O(r^4),
\]

where \( \mathfrak{I} \) is the energy-stress tensor of matter. As anticipated, in non-vacuum, the leading contribution of \( E(S) \) in the small sphere limit at \( p \) comes from the energy of matter, observed by \( g_p(e_0) \) (c.f. Definition 3.10). This is physically reasonable because any form of gravitational energy-momentum that is quadratic in curvature enters at higher orders in \( r \).

2. Vacuum

In the vacuum case, higher order expansions are inevitably necessary. It is, nevertheless, a straightforward calculation to obtain the following expansions:
\[
S_{S(r)} = 2r^{-2} + S_{S(r)}^{(0)} + r S_{S(r)}^{(1)} + r^2 S_{S(r)}^{(2)} + O(r^3)
\]
\[
\rho = -r^{-1} + \frac{1}{45}r^3 \psi_0^0 \psi_0 + O(r^4)
\]
\[
\mu = -\frac{1}{2} r^{-1} + \frac{1}{2} r (\psi_0^2 + \bar{\psi}_2) + \frac{1}{3} r^2 (\psi_2^1 + \bar{\psi}_2^1) + r^3 \left( \frac{1}{360} \psi_0^0 \psi_0^0 - \frac{1}{40} \bar{\partial}_0 (\bar{\psi}_0 \psi_1^0 + 4 \bar{\psi}_1^1) - \frac{1}{40} \bar{\partial}_0 (\bar{\psi}_0^0 \psi_0^0 + 4 \bar{\psi}_0^1) - \frac{1}{4} S_{S(r)}^{(2)} \right) + O(r^4),
\]
\[
\Omega = \Omega_0 r^2 \left[ 1 - \frac{1}{90} r^4 \psi_0^0 \psi_0 + O(r^5) \right],
\]

where
\[
S_{S(r)}^{(0)} = -\frac{4}{3} (\bar{\partial}_0 \psi_1^0 + \bar{\partial}_0 \psi_1^1),
\]
\[
S_{S(r)}^{(1)} = -\frac{5}{6} (\bar{\partial}_0 \psi_1^0 + \bar{\partial}_0 \psi_1^1),
\]
\[
S_{S(r)}^{(2)} = \frac{1}{45} \psi_0^0 \psi_0^0 - \frac{3}{5} \bar{\partial}_0 \psi_1^0 - \frac{3}{5} \bar{\partial}_0 \psi_1^1 - \frac{17}{90} \bar{\partial}_0 (\bar{\psi}_0^0 \psi_0^1) - \frac{17}{90} \bar{\partial}_0 (\bar{\psi}_0^0 \psi_1^1).
\]

In light of Lemma 3.11, the quasi-local energy in vacuum becomes
\[
E(S) = \frac{1}{8\pi} \int_{S(r)} (\sqrt{2S_{S(r)}} - \sqrt{8\rho\mu}) \Omega
\]
\[
= \frac{r^5}{72} \int_{s_0} \psi_0^0 \psi_0^0 \Omega_0^2 \frac{\Omega_0}{4\pi} - \frac{3}{4} \frac{r^5}{2} \int_{s_0} (\text{Re} \psi_2^0)^2 \frac{\Omega_0}{4\pi} + O(r^6)
\]
\[
= \frac{r^5}{90} \mathcal{B}(e_0, e_0, e_0, e_0) - \frac{3}{4} \frac{r^5}{2} \int_{s_0} (\text{Re} \psi_2^0)^2 \frac{\Omega_0}{4\pi} + O(r^6),
\]
\[
= \frac{r^5}{90} \mathcal{B}(e_0, e_0, e_0, e_0) - \frac{r^5}{40} \left( \mathcal{E}^2(e_1, e_1) + \mathcal{E}^2(e_1, e_2) + \mathcal{E}^2(e_1, e_3) + \mathcal{E}^2(e_2, e_3)
\]
\[
- \mathcal{E}(e_2, e_2) \mathcal{E}(e_3, e_3) \right) + O(r^6),
\]

where \(\mathcal{B}\) is the Bel-Robinson tensor and \(\mathcal{E}(e_i, e_j) = -g(W_{e_ie_je_0}, e_i), i, j = 1, 2, 3\), is the symmetric (aka electric) part of the Weyl tensor. In the leading order, \(O(r^5)\), the first term is the gravitational energy measured by \(\mathcal{O}_p(e_0)\), whereas the second term comes from the power series expansion in \(r\) of \(\sqrt{2S_{S(r)}}\) in the integrand of \(E(S)\). The physical interpretation of this additional term remains unclear.
IV. THE LARGE-SPHERE LIMIT AT NULL INFINITY

For the sake of simplicity and also of highlighting the physics, the discussion is restricted to perfectly isolated sources in an empty (i.e., Ricci-flat) space-time. An attempt at generalization to non-isolated gravitational sources is briefly mentioned in Sec. IV F.

A. Construction of the large sphere at null infinity

Several definitions and elementary properties of null infinity are collected here for the purpose of unifying terminology and notation.

Definition 4.1

A \( C^r \) \( (r \geq 0) \) asymptote of a differentiable manifold \( M \) is an ordered triple \((\tilde{M}, i, f_\Omega)\), where \( \tilde{M} \) is a manifold with boundary \( \partial \tilde{M} \), \( i : M \hookrightarrow \tilde{M} \) is embedding by inclusion, and \( f_\Omega : \partial \tilde{M} \times [0, +\infty) \longrightarrow \tilde{M} \), such that \( f_\Omega(x, 0) = x, \forall x \in \partial \tilde{M} \), is a collar \[17\].

The existence of collars in the differentiable category is easily shown \[17\]. That boundaries of \( C^0 \) manifolds have collars is, however, far from obvious but is proved to be true \[7\]. A collar \( f_\Omega \) can essentially be characterized, with recourse to the partition of unity, if necessary, by \( \Omega \in \mathfrak{F}(U, [0, +\infty)) \), having 0 as its regular value so that \( \Omega^{-1}(0) = \partial \tilde{M} \), where \( U \subset \tilde{M} \) is a neighborhood of \( \partial \tilde{M} \). Hence, a collar is denoted hereafter simply by \( \Omega \).

A richer structure of the asymptote becomes available when a differentiable manifold \( M \) possesses a semi-Riemannian metric \( g \) and a Levi-Civita connection \( D \). In particular, in the Lorentzian category, it is a space-time, denoted by an ordered triple \((M, g, D)\). The collar \( \Omega \) of the asymptote of a space-time then becomes crucially intertwined with the geometric structure.

Definition 4.2

A \( C^r \) \( (r \geq 0) \) asymptote of a space-time \((M, g, D)\) is a \( C^r \) asymptote of \( M \) with a Lorentzian metric \( \tilde{g} \) and a Levi-Civita connection \( \tilde{D} \) associated to \( \tilde{M} \).

There exist in the literature various ways of defining the asymptotic structure of null infinity, most of which are essentially equivalent. It is attempted here yet another formulation that might be more appropriate in logic. Elaborate discussions of the topological properties of simple space-times can be found in, for example, \[27\], whereas more geometric properties
are recorded in, for example, [20]. For brevity, only future null infinity is considered for the past null infinity can be treated dually.

**Definition 4.3**

The *null infinity* \( \mathcal{I} \) of the space-time \((M, g, D)\) is a (not necessarily connected) null submanifold of \( \partial \tilde{M} \) that is orientable and time-orientable.

In fact, the time orientation of \( M \), and thus of \( \tilde{M} \), induces a compatible time orientation of \( \mathcal{I} \) so that \( \mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^- \), where \( \mathcal{I}^\pm = \mathcal{I} \cap I^\pm(\tilde{M}, \tilde{M}) = \mathcal{I} - I^\mp(\tilde{M}, \tilde{M}) \) are, respectively, future \((\mathcal{I}^+)\) and past \((\mathcal{I}^-)\) null infinity. Note that each of the \( \mathcal{I}^\pm = \mathcal{I} - \mathcal{I}^\mp \) is relatively clopen in \( \mathcal{I} \) and thus is a connected component of \( \mathcal{I} \).

In particular, \( \mathcal{I} \) has a collar, which, with a slight abuse of notation, is also denoted by \( \Omega \), as null infinity is the only piece of \( \partial \tilde{M} \) that is of interest here. Moreover, the conformal properties of null structure in the Lorentzian category can be exploited to provide remarkable convenience in the analysis of the asymptotic structure of a space-time.

**Definition 4.4**

A space-time \((M, g, D)\) is *asymptotically empty and flat at null infinity* if

1. It is conformally diffeomorphic to its asymptote at null infinity with \( i^*\tilde{g} = \Omega^2 g \) and \( i^*\tilde{d}\Omega \neq 0 \), where \( \tilde{d} \) is the exterior derivative in \( \tilde{M} \);
2. \( \exists \mathcal{I}_0 = \{ q \in \mathcal{I} : q \text{ is strongly causal} \} \subset \mathcal{I} \), which is thus open in \( \tilde{M} \) and is relatively open in \( \mathcal{I} \);
3. \( \Omega^{-2}\text{Ric} \) admits a \( C^r \) extension to \( \Omega^{-1}(0) \);
4. \( \mathcal{I}_0 \approx S^2 \times I \), where \( I \subset \mathbb{R} \) is a connected component of \( \mathbb{R} \).

\((M, g, D)\) is said to be *asymptotically Minkowskian at null infinity* when \( I = \mathbb{R} \).

Definition 4.4 implies that \( \mathcal{I}_0 \) can be causally separated by a space-like surface \( S \approx S^2 \subset \mathcal{I}_0 \). Hence there exists a tubular neighborhood \( \mathcal{N}_2 \) of \( S \) in \( \mathcal{I}_0 \), i.e., there exists a line bundle \((E, S, \pi_S)\), where \( E = TS^\perp \), and a \( C^r \) diffeomorphism \( \psi : E \to \mathcal{N}_2 \), with \( \psi(0_x) = x, \forall x \in S \), such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\psi} & \mathcal{N}_2 \\
\pi_S \downarrow & & \downarrow \text{ret} \\
S & \xrightarrow{\text{ret}} & \mathcal{N}_2
\end{array}
\]

commutes [17], where \( \text{ret} : \mathcal{N}_2 \rightarrow S \) is a retraction. Then, \( \mathcal{N}_2 \) is foliated by curves that intersect \( S \) transversely. Furthermore, since \( \mathcal{I}_0 \) is a null hypersurface in \( \tilde{M} \), it is irrotational and thus geodesic by Corollary 3.4 and 3.6. Hence, \( \forall \) future-directed null curve
\( \tilde{\lambda} : \mathcal{E}_2 \rightarrow \mathcal{I}_0 \), which can be parameterized to be a null geodesic, \( \exists a \in \mathcal{E}_2 \), such that \( \tilde{\lambda} \cap_q S \), where \( q = \tilde{\lambda}(a) \). On the other hand, as \( \tilde{\lambda}(a) \) is null, \( \tilde{\lambda}(a) \notin T_qS \), which is space-like. Thus, either \( q \notin S \) or \( q \in S \) and \( \tilde{\lambda}_a(T_a\mathcal{E}_2) + T_qS = T_q\mathcal{I}_0 \). For notational purposes, denote \( X_2 = \tilde{\lambda} \), then \( X_2 \in \Gamma(\mathcal{C}L_2) \), where \( L_2 = T\mathcal{I}_0 \cap T\mathcal{I}_0^\perp \) is the only null line bundle over \( \mathcal{I}_0 \). In fact, with a little hindsight, the normal bundle \( E \) over \( S \) might as well be chosen to be \( L_2 \).

Now, \( \mathcal{N}_2 \) can be co-ordinatized by \( \varrho_2 = (u, \vartheta, \varphi) \in \mathfrak{F}(\mathcal{N}_2, \mathcal{E}_2 \times (0, \pi) \times [0, 2\pi]) \), with the usual understanding that the poles of \( S^2 \) are not covered. Clearly \( S^2 \) is a non-empty locally acausal compact connected topological 2-sub-manifold of \( \mathcal{I}_0 \), hence is sometimes called a “cut” of \( \mathcal{I}_0 \). The set of all such cuts is denoted by \( \mathcal{E}_{\mathcal{I}_0} \).

To define the large sphere, first recall that a subset \( F \subseteq M \) is a causal (resp. chronological) future set in a space-time \( M \) if \( J^+(F, M) \subseteq F \) (resp. \( I^+(F, M) \subseteq F \)). For a given compact subset \( K \subseteq M \), \( J^+(K, \tilde{M}) \) is a causal future set of \( \tilde{M} \). Consider \( \tilde{A}(K) = J^+(K, \tilde{M}) \). It is shown in [27] that \( \tilde{A}(K) \neq \emptyset \) and that \( \tilde{A}(K) \) is a compact achronal embedded \( C^0 \) sub-manifold of \( \tilde{M} \) with boundary \( \partial \tilde{A}(K) = \tilde{A}(K) \cap \mathcal{I}_0^+ \neq \emptyset \). With the differentiable structure prescribed above, \( \tilde{A}(K) \) is a closed null sub-manifold in \( \tilde{M} \) of co-dimension 1 with boundary \( \partial \tilde{A}(K) \). Therefore, \( \tilde{A}(K) \) has a tubular neighborhood \( \mathcal{N}_1 \) in \( \tilde{M} \). Similar to the treatment of \( \mathcal{N}_2 \), \( \tilde{A}(K) \) is geodesic and \( \mathcal{N}_1 \) is foliated by null geodesics \( \tilde{\gamma} : \mathcal{E}_1 \rightarrow \tilde{A}(K) \) such that \( X_1 = \tilde{\gamma} \in \Gamma(\mathcal{C}L_1) \), where \( L_1 = T\tilde{A}(K) \cap T\tilde{A}(K)^\perp \) is the only null line bundle over \( \tilde{A}(K) \).

**Definition 4.5**

The large sphere near future null infinity is a compact 2-surface \( K \subseteq M \), such that \( \tilde{A} \) is a neat sub-manifold [17] and that \( \tilde{A}(K) \cap \mathcal{I}_0^+ \in \mathcal{E}_{\mathcal{I}_0} \).

According to Definition 4.5, \( \forall x \in \partial \tilde{A} \), \( T_x\tilde{A}(K) \not\subseteq T_x\mathcal{I}_0^+ \), i.e., \( \tilde{A}(K) \) is nowhere tangent to \( \mathcal{I}_0^+ \), or, notationally, \( \tilde{A}(K) \cap \mathcal{I}_0^+ \). On the other hand, \( \mathcal{I}_0^+ \) has a collar, \( \Omega \) say, which restricts to a collar, \( \Omega_{\tilde{A}(K)} \), on \( \partial \tilde{A}(K) \) in \( \tilde{A}(K) \). Hence, \( \exists \) a null foliation of \( \mathcal{N}_1 \) each of whose generators, \( \tilde{\gamma} \), is affinely parameterized by \( \Omega_{\tilde{A}(K)} \) so that \( D_{X_1}\Omega_{\tilde{A}(K)} = 1 \), where \( X_1 = \tilde{\gamma} \), and that \( \mathcal{L}_{X_1} \vartheta = \mathcal{L}_{X_1} \varphi = 0 \), where \( \mathcal{L}_{X_1} \) is the Lie derivative along the flow of \( X_1 \). It is then natural to adopt \( \tilde{\varrho} = (u, \vartheta, \varphi, \tilde{\varphi}) \in \mathfrak{F}(\mathcal{I}_0^+, \mathcal{E}_2 \times [0, +\infty) \times (0, \pi) \times [0, 2\pi]) \) as a co-ordinate chart on the neighborhood of \( \mathcal{I}_0^+ \) in the asymptote. By the definition of a neat sub-manifold, \( \tilde{A}(K) \) is covered by the chart \( (\tilde{\varrho}, \tilde{U}) \) of \( \tilde{M} \) such that \( \tilde{A} \cap \tilde{U} = \tilde{\varrho}^{-1}(u = u_0) \), where \( u_0 \in \mathcal{E}_2 \). Consequently, \( \tilde{A} \) admits an adapted co-ordinate system \( \tilde{\varrho}_1 = (u_0, \Omega_{\tilde{A}(K)}, \vartheta, \varphi) \in \mathfrak{F}(\tilde{A}, [0, +\infty) \times (0, \pi) \times [0, 2\pi]) \).

It is equally convenient to carry out the analysis in the asymptote using the local chart
\(\tilde{g}\), as done in, for example, [23]. However, for the purposes of studying “physical fields” [15], it is more often useful to work in the original space-time \((M, g, D)\). The latter approach is taken in what follows.

Most asymptotic behaviors of the original space-time near null infinity come almost for free because \((M, g, D)\) is conformally diffeomorphic to its asymptote. For example, \(\mathcal{N}_1 \cap M\) in \(M\) is foliated instead by null geodesics (possibly after reparameterization of null geodesics) \(\gamma : (b, +\infty) \rightarrow M\) such that \(\tilde{\gamma} = i \circ \gamma\) [2]. Let \(r\) be the affine parameter of \(\gamma\), then \(D_{X_1}r = 1\). The local co-ordinate chart in \(M\) on the neighborhood of null infinity is simply \(\varrho = (u, r, \vartheta, \varphi)\), which is known, in physics literature, as Bondi-type co-ordinates, although in the original Bondi co-ordinates [4], \(r\) is chosen to be a luminosity distance parameter as opposed to an affine parameter as used here.

B. Moving frames on the large sphere

The seemingly pedantic construction in Sec. IV A exhibits its advantages now when it comes to setting up adapted moving frames on \(\tilde{A}\); almost the same moving frames as used in Sec. III B can be applied in parallel for the large sphere \(K\). The only difference lies in the obvious fact that rather than the blow-up sphere \(S^2\), the limit sphere now is \(K_0 = \tilde{A} \cap \mathcal{I}^+_0 \approx S^2\). Correspondingly, all quantities in the Newman-Penrose formalism are expanded in powers of \(\frac{1}{r}\).

C. Reference embedding

Given a generic space-time \((M, g, D)\), it is generally unlikely that a large sphere \(K\) in the asymptotic null region of \(M\) could be embedded into a genuine null cone in Minkowski reference space-time that is emanated from one single point. Thus the embedding scheme in Sec. III D becomes inappropriate in the large-sphere limit. However, it is possible to isometrically embed \(\tilde{A}\) into the asymptotic null region of the Minkowski space-time \((M^\circ, g^\circ, D^\circ)\) such that the sectional curvature of \(K\), when calculated via the Gauss equation, is preserved regardless of which ambient manifold into which \(K\) is embedded. Next, recall from Sec. IV A that the leaves of the foliation of \(\mathcal{N}_1\) form a null congruence of \(X_1\). It is assumed [35] that the shear of such null congruence is the same at \(K_0\) in both \(M\) and \(M^\circ\). The same construction
as in Sec. [IV A] produces a local co-ordinate chart $\mathcal{o} = (u, r^o, \vartheta, \varphi)$ of the Bondi-type in $M^o$ on the neighborhood of $\mathcal{I}_0^+$ that differs from $\mathcal{g}$ only by a possible reparameterization of the null congruence, registered by $r^o$ in $\mathcal{o}$.

Asymptotic expansions of the Newman-Penrose variables in both $M$ and $M^o$ can be performed at one stroke. In $M^o$, $\psi_i^o = 0$, $i = 0, \ldots, 4$ for $M^o$ is flat. Some of the most pertinent expansions are listed below:

\[ \rho = -\frac{1}{r} - \frac{\sigma^0\bar{\sigma}^0}{r^3} - \left(\frac{\sigma^0\bar{\sigma}^0}{r^3}\right)^2 - \frac{1}{6}(\sigma^0\bar{\psi}_0 + c.c.) + O(r^{-6}) \]
\[ \mu = -\frac{1}{2r} - \frac{\psi^0_2 + \sigma^0\bar{\sigma}^0 + \bar{\sigma}^0\sigma^0}{r^2} + O(r^{-3}) \]
\[ \Omega = \Omega_0 r^2 \left(1 - \frac{\sigma^0\bar{\sigma}^0}{r^2}\right) + O(r^{-2}) \]
\[ \rho^o = -\frac{1}{r^o} - \frac{\sigma^0\bar{\sigma}^0}{r^{o3}} + O(r^{o-5}) \]
\[ \mu^o = -\frac{1}{2r^o} - \frac{\bar{\sigma}^0\sigma^0}{r^{o2}} + O(r^{o-3}) \]

where quantities with superscripts or subscripts 0 represent their corresponding asymptotic values at $K_0$, which are not to be confused with those with superscripts $o$ in the reference space-time $(M^o, g^o, D^o)$.

Expansions of the full set of Newman-Penrose variables in both $M$ and $M^o$, when applied to the Gauss equation, establish an equality of the sectional curvature of $K$, calculated via two different embeddings, from which the relation between $r$ and $r^o$ can be read off: $r^o = r + (\partial_0^o\sigma^o + \bar{\sigma}^0_0\sigma^o - \bar{\sigma}^0_0\sigma^0 - \bar{\sigma}^0_0\sigma^0) + O(r^{-1})$. Now the assumption $\sigma^0|_{K_0} = \sigma^o|_{K^o}$ leads to a much simpler relation $r^o = r + O(r^{-1})$.

**D. Bondi-mass loss**

After the preparatory work from previous sections, the calculation of the quasi-local energy in the large sphere limit now becomes completely transparent:

\[ E(K) = \frac{1}{8\pi} \int_{K} [\sqrt{8\rho^o\mu^o} - \sqrt{8\rho\mu}] \Omega \]
\[ = -\frac{1}{4\pi} \int_{K_{0}} (\psi^0_2 + \sigma^0\bar{\sigma}^0) \Omega_0 + O(r^{-1}) \]
\[ = E_{BS} + O(r^{-1}), \]
where \( E_{BS} = -\frac{1}{4\pi} \int_{K_0} (\psi^0_2 + \sigma^0 \dot{\sigma}^0)\Omega_0 \) is the Bondi mass loss.

### E. Energy flux

The energy flux through \( K \) is defined as the rate of change in the quasi-local energy \( E(K) \) in the time-like direction characterized by \( T \). Here, \( T(q) \) is related to \( \{X_i(q)\}_{i=1}^4 \in \mathbb{C}T_0 K^\perp \), \( \forall q \in K \) by the same basis transformation \( \mathfrak{U} \) as in Sec. III D and agrees with the generator of the time translation subgroup of the Bondi-Metzner-Sachs (BMS) group at \( \mathcal{I}^+ \). Hence,

\[
\frac{\partial E^{\text{phys}}(K)}{\partial t} = \frac{1}{8\pi} \int_K \mathcal{L}_T(\sqrt{8\rho\mu})
\]

\[
= \frac{1}{8\pi} \int_K \left[ \sqrt{8\rho\mu} (\mathcal{L}_T\Omega) + (T \sqrt{8\rho\mu})\Omega \right]
\]

\[
= \frac{1}{8\pi} \int_K \left[ \sqrt{8\rho\mu}(2\mu - \rho) + \left( (X_2 + \frac{1}{2}X_1) \sqrt{8\rho\mu} \right) \right] \Omega
\]

\[
= \frac{1}{8\pi} \int_{K_0} 2 \frac{\partial}{\partial u} (\psi^0_2 + \sigma^0 \dot{\sigma}^0 + \bar{\sigma}^0 \dot{\sigma}^0)\Omega_0 + O(r^{-1}),
\]

in which \( \iota : K \hookrightarrow M \) is the embedding map that induces \( \iota^*(\mathcal{L}_X \theta^3 \wedge \theta^4) = -2\rho \theta^3 \wedge \theta^4 \) and \( \iota^*(\mathcal{L}_X \theta^3 \wedge \theta^4) = 2\mu \theta^3 \wedge \theta^4 \) by straightforward calculations.

Very similarly,

\[
\frac{\partial E^{\text{ref}}(K)}{\partial t} = \frac{1}{8\pi} \int_K \mathcal{L}_T(\sqrt{8\rho\mu})
\]

\[
= \frac{1}{8\pi} \int_{K_0} 2 \frac{\partial}{\partial u} \bar{\sigma}^0 \dot{\sigma}^0 \Omega_0 + O(r^{-1}),
\]

where, again, \( r^o = r + O(r^{-1}) \) is used in the co-ordinate representation of \( X_1 \) and \( X_2 \).

Recall that the embedding scheme is tacitly chosen so that \( \sigma^0 |_{K_0} = \sigma^0 |_{K_0} \) and that \( \psi^0_3 \in \mathcal{B}(-1) \). Then, with the help of the Newman-Penrose equations \( \dot{\psi}^0_2 + \frac{1}{\sqrt{2}} \bar{\sigma}^0 \dot{\psi}^0_3 - \sigma^0 \dot{\psi}^0_4 = 0 \) and \( \dot{\psi}^0_4 + \bar{\sigma}^0 = 0 \), the flux of quasi-local energy in the large-sphere limit at \( \mathcal{I}^+ \) is

\[
\frac{\partial E(K)}{\partial t} = \frac{\partial E^{\text{ref}}(K)}{\partial t} - \frac{\partial E^{\text{phys}}(K)}{\partial t}
\]

\[
= -\frac{1}{4\pi} \int_{K_0} \dot{\sigma}^0 \bar{\sigma}^0 \Omega_0 + O(r^{-1}),
\]

in which the leading term is precisely the flux of Bondi news.
F. Remarks on generalization to non-isolated systems

Applying the notion of quasi-local quantities, in general, to non-isolated gravitational systems may incur failure to satisfy, for example, criterion C3-d. The obstruction to such generalizations largely lies in the difficulty of having a well-behaved or, well-described large-sphere limit. It suffices to analyze the problem at null infinity; the situation at the spatial infinity, if treated with care, is quite similar. Intuitively, it is conceivable that the non-isolated source has to eventually run off the boundaryless manifold and wreck the topological structure of the asymptote described in Sec. IV A. However, the following example illustrates a more subtle cause.

Definition 4.6

Given two quartic polynomials \( G(x) = 1 - x^2 - 2mAx^3 - e^2A^2x^4 \) and \( F(y) = -G(-y) \), where \( m \geq 0, e \in \mathbb{R}, \) and \( A > 0 \), assume that all of the roots of \( G(x) \) are distinct among which at least two are real (i.e., \( e \neq 0 \) or \( mA < \frac{1}{\sqrt{27}} \)), denoted by \( x_2 < x_1 \), such that \( G(x) \geq 0, \forall x \in [x_2, x_1] \). The (charged) C-metrics, denoted by \( C_1 = C_1(m, A, e) \), are a 3-parameter family of space-times of Petrov type-D, satisfying

1. topologically, \( C_1 \cong P_1 \times Q_1 \), where \( P_1 \cong \mathbb{R} \times \mathbb{R}^+ \) and \( Q_1 \cong S^2 \);

2. metrically, \((C_1, g_1, D_1)\) is conformally diffeomorphic to its asymptote. In a local chart \( \nu_{A_1} = (t, y, x, z) \), set \( \kappa \) to be a real constant, \( \tilde{C}_1 \cong \tilde{P}_1 \times \tilde{Q}_1 \), where \((\tilde{P}_1, \tilde{f}_1)\) is given by \( \tilde{P}_1 \cong \mathbb{R} \times [-x, +\infty) \) and \( \tilde{f}_1 = -F(y)dt \otimes dt + \frac{dy \otimes dy}{F(y)} \), \((\tilde{Q}_1, \tilde{h}_1)\) is given by \( \tilde{Q}_1 \cong (x_2, x_1) \times [0, 2\pi\kappa) \) and \( \tilde{h}_1 = \frac{dx \otimes dx}{G(x)} + G(x)dz \otimes dz \). Hence, \( \tilde{g}_1 = \tilde{f}_1 + \tilde{h}_1 \). The collar is defined by \( \Omega^{-1} = A(x+y) \in \mathbb{R}^+ \).

The following lemma exposes one of the most peculiar features of the C-metrics, namely the so-called “conical singularity” or “nodal singularity” \([19], [1]\) at one of the boundaries of the annulus \( \tilde{Q}_1 \) (in the proof, \( x_1 \)) that cannot be compactified in the \( C^0 \) category for \( r > 0 \).

Lemma 4.7

\( \tilde{Q}_1 \cong S^2 \) only in the \( C^0 \) category but \( \tilde{Q}_1 \cong D^2 \) in the \( C^r \) \((r > 0)\) category,

Proof:

Consider another local chart \( \nu_{B_1} = (t, y, \vartheta, \varphi) \) which is \( C^r \) compatible with \( \nu_{A_1} \), where \( r \geq 0, \vartheta(x) = \int_{x_2}^x \frac{dx'}{\sqrt{G(x')}} \) and \( \varphi = \kappa^{-1}z \). Put \( \rho(\vartheta) = \sqrt{G(x(\vartheta))} \) and \( \vartheta_0 = \vartheta(x_1) \). Then the graph of \((\vartheta, \rho(\vartheta))\) is \((0, \vartheta_0) \times (0, \rho_M)\), for some \( \rho_M > 0 \). Clearly, \( \rho(0) = \rho(\vartheta_0) = 0 \). Now \( \tilde{Q}_1 \cong (0, \vartheta_0) \times [0, 2\pi) \).

1. \( r = 0 \), simply extend continuously the domain of \( \vartheta \) to compactify the annulus \( \tilde{Q}_1 \).
(2) $r > 0$, the standard Bertrand-Puiseux test \[30\], when applied on $\tilde{Q}_1$, shows that \[19\] the $C^r$-differentiable structure is preserved at the boundary of the annulus $\tilde{Q}_1$ if and only if $\kappa^{-1} = \left. \frac{d\rho}{d\theta} \right|_{\theta = 0}$. However, $\kappa^{-1} = \left. \frac{d\rho}{d\theta} \right|_{\theta = 0}$ unless $m = 0$ or $|e| = m > \frac{1}{4A}$. Thus, without loss of generality, set $\kappa^{-1} = \left. \frac{d\rho}{d\theta} \right|_{\theta = 0}$. Then, except for two 2-parameter families of electrovac solutions, $m = 0$ or $|e| = m > \frac{1}{4A}$, only one of the boundaries of the annulus can be compactified such that $\tilde{Q}_1 \cong [x_2, x_1] \times [0, 2\pi) \cong D^2$. ■

Lemma 4.7 clearly shows that a generic C-metric is not asymptotically empty and flat at null infinity in the sense of Definition 4.4. The remedy comes out of a key observation \[1\] that $C_1$ is not maximal. Hence, a maximal extension of $C_1$ leads to $C = C_1 \cup C_2$, where $C_2$ is an identical replicate of $C_1$. In the asymptote of $C_2$, $\tilde{C}_2 \approx \tilde{P}_2 \times \tilde{Q}_2$, where Lemma 4.7 applies except that in the local chart $v_{B_2}$ of $\tilde{Q}_2$ the opposite boundary of the annulus $\tilde{Q}_2$ is compactified, i.e., $\tilde{Q}_2 \cong (x_2, x_1] \times [0, 2\pi) \cong D^2$ in the $C^r$ category for $r > 0$. Therefore, as shown in \[1\], in $C$, or its asymptote $\tilde{C}$, $v_B = \{v_{B_1}, v_{B_2}\}$ constitutes an atlas for $\tilde{Q} = \tilde{Q}_1 \cup \tilde{Q}_2 \cong S^2$ and that $\mathcal{I} \cong S^2 \times I$, where $I \cong \mathbb{R}$ except for two null generators corresponding to precisely the boundaries of the annulus $\tilde{Q}$. But these two generators can be arranged such that they both are incomplete in the future or in the past.

The above analysis demonstrates that for a non-isolated source modeled by the C-metrics, the topological and geometrical properties of $\mathcal{I}$ as in Definition 4.4 can certainly be retained almost as for the isolated sources. The real difficulty, nonetheless, arises at a technical level when practical calculations are carried out. Evaluations of the Bondi mass or energy flux inevitably involve integrations on $\tilde{Q} \in \mathcal{C}_\mathcal{I}$, which has to be covered by two charts. This task usually turns out to be analytically intractable \[10\].

**V. SUMMARY AND DISCUSSION**

The investigation of the limiting behavior of the Liu-Yau quasi-local energy is carried out. Such an analysis could be utilized to provide an appropriate certification for the Liu-Yau’s proposal as a physically sound candidate for the quasi-local energy. Preliminary results that are considered new include:

- In the small-sphere limit, the leading term in the quasi-local energy measured by the equivalence class of spatially-isotropic instantaneous observers $O_p(e_0)$ at an arbitrary
point \( p \) in non-vacuum is found to be \( \frac{4\pi}{3}r^3\mathcal{T}(e_0, e_0) \), where \( \mathcal{T} \) is the energy-stress tensor of matter and \( r \) is the radius of the small sphere in the limit \( r \to 0 \).

- In vacuum, however, the gravitational quasi-local energy measured by \( \mathcal{E}_p \) gains an extra term in the leading order, in addition to the currently known limit \( \frac{r^5}{90}\mathcal{B}(e_0, e_0, e_0, e_0) \), where \( \mathcal{B} \) is the Bel-Robinson tensor.

The occurrence of the extra term is considered as an example of how the quasi-local energy depends rather crucially upon the choice of the reference embedding. Since the co-dimension 2 embedding of the 2-surface \( S \) into the reference space-time \( M^0 \) is in general non-unique, it is plausible that an embedding scheme other than the null-cone reference may result in a different limiting behavior. Moreover, the currently known limit \( \frac{r^5}{90}\mathcal{B}(e_0, e_0, e_0, e_0) \) in vacuum is actually model dependent and usually variable to reference embedding (for example, [6]). Therefore, it is contemplated that the mismatch in the small-sphere limit in vacuum does not seem to serve as a strong piece of evidentiary support to rule out Liu-Yau’s model.

- In the large-sphere limit at null infinity of an asymptotically empty and flat space-time, the Liu-Yau quasi-local energy is found to coincide, in radiating scenarios, with the Bondi mass loss and the news flux. A tentative generalization of the quasi-local energy to non-isolated sources encounters technical difficulties at null infinity in the example of the C-metric.

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