OF BAK’S UNITARY GROUP OVER GRADED RINGS

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Abstract: For an associative ring $R$ with identity, we study the absence of $k$-torsion in $\NK_1 GQ(R)$; Bass nil-groups for the general quadratic or Bak’s unitary groups. By using a graded version of Quillen–Suslin theory we deduce an analog for the graded rings.

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1. Introduction

Let $R$ be an associative ring with identity element 1. When $R$ is commutative, we define $\SK_1(R)$ as the kernel of the determinant map from the Whitehead group $K_1(R)$ to the group of units of $R$. The Bass nil-group $\NK_1(R) = \ker(K_1(R[X]) \to K_1(R)); X = 0$. i.e., the subgroup consisting of elements $[\alpha(X)] \in K_1(R[X])$ such that $[\alpha(0)] = [1]$. Hence $K_1(R[X]) \cong \NK_1(R) \oplus K_1(R)$. The aim of this paper is to study some properties of Bass nil-groups $\NK_1$ for the general quadratic groups or Bak’s unitary groups.

It is well-known that for many rings, e.g. if $R$ is regular Noetherian, Dedekind domain, or any ring with finite global dimension, the group $\NK_1(R)$ is trivial. On the other hand, if it is non-trivial, then it is not finitely generated as a group. e.g. if $G$ is a non-trivial finite group, the group ring $\mathbb{Z}G$ is not regular. In many such cases, it is difficult to compute $\NK_1(\mathbb{Z}[G])$. In [19], D.R. Harmon proved the triviality of this group when $G$ is finite group of square free order. C. Weibel, in [20], has shown the non-triviality of this group for $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/4,$ and $D_4$. Some more results are known for finite abelian groups from the work of R.D. Martin; cf. [16]. It is also known (cf. [12]) that for a general finite group $G$, $\NK_1(R[G])$ is a torsion group for the group ring $R[G]$. In fact, for trivial $\NK_1(R)$, every element of finite order of $\NK_1(R[G])$ is some power of the cardinality of $G$. For $R = \mathbb{Z}$, this is a result of Weibel. In particular, if $G$ is a finite $p$-group ($p$ a prime), then every element of $\NK_1(\mathbb{Z}[G])$ has $p$-primary order. In [17], J. Stienstra showed that $\NK_1(R)$ is a $W(R)$-module, where $W(R)$ is the ring of big Witt vectors (cf. [11] and [19]). Consequently, in [18], §3, C. Weibel observed that if $k$ is a unit in $R$, then $\SK_1(R[X])$ has no $k$-torsion, when $R$ is a commutative local ring. Note that if $R$ is a commutative local ring then $\SK_1(R[X])$ coincides with $\NK_1(R)$; indeed, if $R$ is a local ring then $\SL_n(R) = \mathbb{E}_n(R)$ for all $n > 0$. Therefore, we may replace $\alpha(X)$ by $\alpha(X)\alpha(0)^{-1}$ and assume that $[\alpha(0)] = [1]$. In [7], the first author extended Weibel’s result for arbitrary associative rings. In this paper we prove the analog result for $\lambda$-unitary Bass nil-groups, viz. $\NK_1 GQ^\lambda(R, \Lambda)$, where $(R, \Lambda)$ is the form ring as introduced by A. Bak in [1]. The main ingredient for our proof is an analog of Higman linearisation (for a subclass of Bak’s unitary group) due to V. Kopeiko; cf. [15]. For the general linear groups, Higman linearisation (cf. [3]) allows us to show that $\NK_1(R)$ has a unipotent representative. The same result is not true in general for the unitary nil-groups.
Kopeiko’s results in [15] explain a complete description of the elements of $N_{K^1}(R, \Lambda)$ that have (unitary) unipotent representatives. Followings are the main results in this article.

**Theorem 1.1.** Let $\alpha(X) = [\begin{pmatrix} A(X) & B(X) \\ C(X) & D(X) \end{pmatrix}] \in N_{K^1}(R, \Lambda)$ with $A(X) \in GL_r(R[X])$ for some $r \in \mathbb{N}$. Then $[\alpha(X)]$ has no $k$-torsion if $kR = R$.

And, an analog for the graded rings:

**Theorem 1.2.** Let $R = R_0 \oplus R_1 \oplus \ldots$ be a graded ring. Let $k$ be a unit in $R_0$. Let $N = N_0 + N_1 + \cdots + N_r \in M_r(R)$ be a nilpotent matrix, and $I$ denote the identity matrix. If $[(I + N)]^k = [I]$ in $K^1(R, \Lambda)$, then $[I + N] = [I + N_0]$.

In the proof of [12] we have used a graded version of Quillen–Suslin’s local-global principle for Bak’s unitary group over graded rings. This unify and generalize the results proved in [5], [7], [9], and [10].

**Theorem 1.3.** (Graded local-global principle) Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be a graded ring with identity 1. Let $\alpha \in GQ(2n, R, \Lambda)$ be such that $\alpha \equiv I_{2n} \pmod{R_+}$. If $\alpha_m \in EQ(2n, R_m, \Lambda_m)$, for every maximal ideal $m \in \text{Max}(C(R_0))$, then $\alpha \in EQ(2n, R, \Lambda)$.

### 2. Preliminaries

Let $R$ be an associative ring with identity element 1. Let $M(n, R)$ denote the additive group of $n \times n$ matrices, and $GL(n, R)$ denote the multiplicative group of $n \times n$ invertible matrices. Let $e_{ij}$ be the matrix with 1 in the $ij$-th position and 0’s elsewhere. The elementary subgroup of $GL(n, R)$ plays a key role in classical algebraic $K$-theory. We recall,

**Definition 2.1.** Elementary Group $E(n, R)$: The subgroup of all matrices in $GL(n, R)$ generated by $\{E_{ij}(\lambda) : \lambda \in R, i \neq j\}$, where $E_{ij}(\lambda) = I_n + \lambda e_{ij}$, and $e_{ij}$ is the matrix with 1 in the $ij$-position and 0’s elsewhere.

**Definition 2.2.** For $\alpha \in M(r, R)$ and $\beta \in M(s, R)$, the matrix $\alpha \perp \beta$ denotes its embedding in $M(r + s, R)$ (here $r$ and $s$ are even integers in the non-linear cases), given by

$$\alpha \perp \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$ 

There is an infinite counterpart: Identifying each matrix $\alpha \in GL(n, R)$ with the large matrix $(\alpha \perp \{1\})$ gives an embedding of $GL(n, R)$ into $GL(n + 1, R)$. Let $GL(R) = \bigcup_{n=1}^{\infty} GL(n, R)$, and $E(R) = \bigcup_{n=1}^{\infty} E(n, R)$ be the corresponding infinite linear groups.

As a consequence of classical Whitehead Lemma (cf.[3]) due to A. Suslin, one gets

$$[GL(R), GL(R)] = E(R).$$

**Definition 2.3.** The quotient group

$$K_1(R) = \frac{GL(R)}{[GL(R), GL(R)]} = \frac{GL(R)}{E(R)}$$

is called the Whitehead group of the ring $R$. For $\alpha \in GL(n, R)$, let $[\alpha]$ denote its equivalence class in $K_1(R)$.

In the similar manner we define $K_1$ group for the other types of classical groups; viz., the symplectic Whitehead group $K_1Sp(R)$ and the orthogonal Whitehead group $K_1O(R)$.

This paper explores a uniform framework for classical type groups over graded structures. Let us begin by recalling the concept of form rings and form parameter as introduced by A. Bak in [1]. This allows us to give a uniform definition for classical type groups.
Definition 2.4. (Form rings): Let $R$ be an associative ring with identity, and with an involution $- : R \to R$, $a \mapsto \overline{a}$. Let $\lambda \in C(R)$ be the center of $R$, with the property $\lambda \overline{\lambda} = 1$. We define two additive subgroups of $R$, viz.

$$\Lambda_{\text{max}} = \{ a \in R \mid a = -\lambda \overline{a} \} \quad \text{and} \quad \Lambda_{\text{min}} = \{ a - \lambda \overline{a} \mid a \in R \}.$$ 

One checks that for any $x \in R$, $\Lambda_{\text{max}}$ and $\Lambda_{\text{min}}$ are closed under the conjugation operation $a \mapsto xax$.

A $\lambda$-form parameter on $R$ is an additive subgroup $\Lambda$ of $R$ such that $\Lambda_{\text{min}} \subseteq \Lambda \subseteq \Lambda_{\text{max}}$, and $\overline{x} \Lambda x \subseteq \Lambda$ for all $x \in R$, i.e., a subgroup between two additive groups which is also closed under the conjugation operation. A pair $(R, \Lambda)$ is called a form ring.

To define Bak’s unitary group or the general quadratic group, we fix a central element $\lambda \in R$ with $\lambda \overline{\lambda} = 1$, and then consider the form $\psi_n = (0 \ I_n \ I_n \ 0)$. For more details, see [7], and [8].

Bak’s Unitary or General Quadratic Groups $GQ$:

$$GQ(2n, R, \Lambda) = \{ \sigma \in \text{GL}(2n, R, \Lambda) | \overline{\psi_n} \sigma = \psi_n \}.$$ 

Elementary Quadratic Matrices: Let $\rho$ be the permutation, defined by $\rho(i) = n + i$ for $i = 1, \ldots, n$. For $a \in R$, and $1 \leq i, j \leq n$, we define

$$q_{\varepsilon_{ij}}(a) = I_{2n} + ae_{ij} - \lambda \overline{\rho(i) \rho(j)} \quad \text{for } i \neq j,$$

$$q_{r_{ij}}(a) = \begin{cases} I_{2n} + ae_{ij} - \lambda \overline{\rho(i) \rho(j)} & \text{for } i \neq j \\ I_{2n} & \text{for } i = j \end{cases},$$

$$q_{l_{ij}}(a) = \begin{cases} I_{2n} + ae_{ij} - \lambda \overline{\rho(i) \rho(j)i} & \text{for } i \neq j \\ I_{2n} & \text{for } i = j \end{cases}.$$ 

(Note that for the second and third type of elementary matrices, if $i = j$, then we get $a = -\lambda \overline{a}$, and hence it forces that $a \in \Lambda_{\text{max}}(R)$. One checks that these above matrices belong to $GQ(2n, R, \Lambda)$; cf. [1].)

$n$-th Elementary Quadratic Group $EQ(2n, R, \Lambda)$:

The subgroup generated by $q_{\varepsilon_{ij}}(a), q_{r_{ij}}(a)$ and $q_{l_{ij}}(a)$, for $a \in R$ and $1 \leq i, j \leq n$. For uniformity we denote the elementary generators of $EQ(2n, R, \Lambda)$ by $\eta_{ij}(\ast)$.

Stabilization map: There are standard embeddings:

$$GQ(2n, R, \Lambda) \rightarrow GQ(2n + 2, R, \Lambda)$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Hence we have

$$GQ(R, \Lambda) = \lim_{\rightarrow} GQ(2n, R, \Lambda).$$

It is clear that the stabilization map takes generators of $EQ(2n, R, \Lambda)$ to the generators of $EQ(2(n + 1), R, \Lambda)$. Hence we have

$$EQ(R, \Lambda) = \lim_{\rightarrow} EQ(2n, R, \Lambda).$$
There are standard formulas for the commutators between quadratic elementary matrices. For details, we refer [1] (Lemma 3.16). In later sections there are repeated use of those relations. The analogue of the Whitehead Lemma for the general quadratic groups (cf. [1]) due to Bak allows us to write:

\[ [GQ(R, \Lambda), GQ(R, \Lambda)] = [EQ(R, \Lambda), EQ(R, \Lambda)] = EQ(R, \Lambda). \]

Hence we define the **Whitehead group** of the general quadratic group

\[ K_1 GQ = \frac{GQ(R, \Lambda)}{EQ(R, \Lambda)}. \]

And, the Whitehead group at the level \( m \)

\[ K_{1,m} GQ = \frac{GQ_m(R, \Lambda)}{EQ_m(R, \Lambda)} \]

where \( m = 2n \) in the non-linear cases.

Let \((R, \Lambda)\) be a form ring. We extend the involution of \( R \) to the ring \( R[X] \) of polynomials by setting \( \overline{X} = X \). As a result we obtain a form ring \((R[X], \Lambda[X])\).

**Definition 2.5.** The kernel of the group homomorphism

\[ K_1 GQ(R[X], \Lambda[X]) \to K_1 GQ(R, \Lambda) \]

induced from the form ring homomorphism \((R[X], \Lambda[X]) \to (R, \Lambda) : X \mapsto 0\) is denoted by \( NK_1 GQ(R, \Lambda) \). We often say it as Bass nilpotent unitary \( K_1 \)-group of \( R \), or just unitary nil-group.

From the definition it follows that

\[ K_1 GQ(R[X], \Lambda[X]) = K_1 GQ(R, \Lambda) \oplus NK_1 GQ(R, \Lambda). \]

In this context, we will use following two types of localizations, mainly over graded ring \( R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots \).

1. Principal localization: for a non-nilpotent, non-zero divisor \( s \) in \( R_0 \) with \( \overline{s} = s \), we consider the multiplicative subgroup \( S = \{1, s, s^2, \ldots\} \), and denote localized form ring by \((R_s, \Lambda_s)\).
2. Maximal localization: for a maximal ideal \( m \in \text{Max}(R_0) \), we take the multiplicative subgroup \( S = R_0 - m \), and denote the localized form ring by \((R_m, \Lambda_m)\).

**Blanket assumption:** We always assume that \( 2n \geq 6 \).

Next, we recall the well-known **“Swan–Weibel’s homotopy trick”**, which is the main ingredient to handle the graded case. Let \( R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots \) be a graded ring. An element \( a \in R \) will be denoted by \( a = a_0 + a_1 + a_2 + \cdots \), where \( a_i \in R_i \) for each \( i \), and all but finitely many \( a_i \) are zero. Let \( R_+ = R_1 \oplus R_2 \oplus \cdots \). Graded structure of \( R \) induces a graded structure on \( M_n(R) \) (ring of \( n \times n \) matrices).

**Definition 2.6.** Let \( a \in R_0 \) be a fixed element. We fix an element \( b = b_0 + b_1 + \cdots \) in \( R \) and define a ring homomorphism \( \epsilon : R \to R[X] \) given by

\[ \epsilon(b) = \epsilon(b_0 + b_1 + \cdots) = b_0 + b_1 X + b_2 X^2 + \cdots + b_i X^i + \cdots. \]

Then we evaluate the polynomial \( \epsilon(b)(X) \) at \( X = a \) and denote the image by \( b^+(a) \) i.e., \( b^+(a) = \epsilon(b)(a) \). Note that \( (b^+(x))^+(y) = b^+(xy) \). Observe, \( b_0 = b^+(0) \). We shall use this fact frequently.

The above ring homomorphism \( \epsilon \) induces a group homomorphism at the \( GL(2n, R) \) level for every \( n \geq 1 \), i.e., for \( \alpha \in GL(2n, R) \) we get a map

\[ \epsilon : GL(2n, R, \Lambda) \to GL(2n, R[X], \Lambda[X]) \]

defined by

\[ \alpha = \alpha_0 \oplus \alpha_1 \oplus \alpha_2 \oplus \cdots \mapsto \alpha_0 \oplus \alpha_1 X \oplus \alpha_2 X^2 \cdots, \]

where \( \alpha_i \in M(2n, R_i) \). As above for \( \alpha \in R_0 \), we define \( \alpha^+(a) \) as

\[ \alpha^+(a) = \epsilon(\alpha)(a). \]
Notation 2.7. By $\text{GQ}(2n, R[X], \Lambda[X], (X))$ we shall mean the group of all quadratic matrices over $R[X]$, which are $I_{2n}$ modulo $(X)$. Also if $R$ is a graded ring, then by $\text{GQ}(2n, R, \Lambda, (R_+))$ we shall mean the group of all quadratic matrices over $R$ which are $I_{2n}$ modulo $R_+$.

The following lemma highlights very crucial fact which we use (repeatedly) in the proof of “Dilation Lemma”.

Lemma 2.8. Let $R$ be a Noetherian ring and $s \in R$. Then there exists a natural number $k$ such that the homomorphism $\text{GQ}(2n, R, \Lambda, s^k R) \rightarrow \text{GQ}(2n, R_+, \Lambda_k)$ (induced by localization homomorphism $R \rightarrow R_+$) is injective. Moreover, it follows that the induced map $\text{EQ}(2n, R, \Lambda, s^k R) \rightarrow \text{EQ}(2n, R_+, \Lambda_k)$ is injective.

For the proof of the above lemma we refer [14], Lemma 5.1. Recall that any module finite ring $R$ is direct limit of its finitely generated subrings. Also, $G(R, \Lambda) = \lim_{\rightarrow} G(R_+, \Lambda_k)$, where the limit is taken over all finitely generated subring of $R$. Thus, one may assume that $C(R)$ is Noetherian. Hence one may consider module finite (form) rings $(R, \Lambda)$ with identity.

Now we recall few technical definitions and useful lemmas.

Definition 2.9. A row $(a_1, a_2, \ldots, a_n) \in R^n$ is said to be unimodular if there exists $(b_1, b_2, \ldots, b_n) \in R^n$ such that $\sum_{i=1}^{n} a_i b_i = 1$. The set of all unimodular rows of length $n$ is denoted by $\text{Um}_n(R)$.

For any column vector $v \in (R^{2n})^t$ we define the row vector $\tilde{v} = \overline{v} \psi_n$.

Definition 2.10. We define the map $M : (R^{2n})^t \times (R^{2n})^t \rightarrow M(2n, R)$ and the inner product $\langle \cdot, \cdot \rangle$ as follows:

$$M(v, w) = v.\tilde{w} - \overline{\Lambda} w.\tilde{v}$$

$$\langle v, w \rangle = \tilde{v}.w$$

Note that the elementary generators of the groups $\text{EQ}(2n, R, \Lambda)$ are of the form $I_{2n} + M(\ast_1, \ast_2)$ for suitably chosen standard basis vectors.

Lemma 2.11. (cf. [1]) The group $E(2n, R, \Lambda)$ is perfect for $n \geq 3$, i.e.,

$$[\text{EQ}(2n, R, \Lambda), \text{EQ}(2n, R, \Lambda)] = \text{EQ}(2n, R, \Lambda).$$

Lemma 2.12. For all elementary generators of $\text{GQ}(2n, R, \Lambda)$ we have the following splitting property: for all $x, y \in R$,

$$\eta_{ij}(x + y) = \eta_{ij}(x)\eta_{ij}(y).$$

Proof: See pg. 43-44, Lemma 3.16, [1].

Lemma 2.13. Let $G$ be a group, and $a_i, b_i \in G$, for $i = 1, 2, \ldots, n$. Then for $r_i = \Pi_{j=1}^{n} a_{ij}$, we have $\Pi_{i=1}^{n} r_i b_i^{-1} \Pi_{i=1}^{n} a_{i} = 1$.

Lemma 2.14. The group $\text{GQ}(2n, R, \Lambda, R_+) \cap \text{EQ}(2n, R, \Lambda)$ generated by the elements of the type $\varepsilon \eta_{ij}(*)^{e_{ij}}$, where $\varepsilon \in \text{EQ}(2n, R, \Lambda)$ and $* \in R_+$.

Proof: Let $\alpha \in \text{GQ}(2n, R, \Lambda, R_+) \cap \text{EQ}(2n, R, \Lambda)$. Then we can write

$$\alpha = \Pi_{k=1}^{n} \eta_{jk}^{*(a_k)}(a_k)$$

for some element $a_k \in R$, $k = 1, \ldots, r$. We can write $a_k$ as $a_k = (a_0)_k + (a_+)_k$ for some $(a_0)_k \in R_0$ and $(a_+)_k \in R_+$. Using Lemma 2.12, we can write $\alpha$ as

$$\alpha = \Pi_{k=1}^{n} \eta_{jk}^{*(a_0)_k}(a_0)_k (\eta_{jk}^{*(a_+)_k}(a_+)_k).$$

Let $e_t = \Pi_{k=1}^{t} \eta_{jk}^{*(a_0)_k}$ for $1 \leq t \leq r$. By the Lemma 2.13 we have

$$\alpha = \Pi_{k=1}^{n} e_t \eta_{jk}^{*(a_+)_k} (e^{-1}_k \Pi_{k=1}^{n} \eta_{jk}^{*(a_0)_k}).$$

Let us write $A = \Pi_{k=1}^{n} e_t \eta_{jk}^{*(a_+)_k} (e^{-1}_k)$ and $B = \Pi_{k=1}^{n} \eta_{jk}^{*(a_0)_k}$. Hence $\alpha = AB$. Let ‘over-line’ denotes the quotient ring modulo $R_+$. Now going modulo $R_+$, we have...
\[ \bar{\alpha} = \overline{AB} = \overline{AB} = \overline{I_{2n}B} = \overline{I_{2n}}, \] the last equality holds as \( \alpha \in GQ(2n, R, \Lambda, R_{+}) \). Hence, 
\( B = \overline{I_{2n}} \). Since the entries of \( B \) are in \( R_{0} \), it follows that \( B = I_{2n} \). Therefore it follows that
\[ \alpha = \Pi_{k=1}^{n} (\alpha_{k}) \varepsilon_{k}^{-1}. \]
\[ \square \]

3. Quillen–Suslin Theory for Bak’s Group over Graded Rings

3.1. Local–Global Principle.

Lemma 3.1. Let \((R, \Lambda)\) be a form ring and \(v \in \text{EQ}(2n, R, \Lambda)e_{1}\). Let \(w \in R^{2n} \) be a column vector such that \( \langle v, w \rangle = 0 \). Then \( I_{2n} + M(v, w) \in \text{EQ}(2n, R, \Lambda) \).

Proof: Let \( v = \varepsilon_{1} \). Then we have \( I_{2n} + M(v, w) = \varepsilon(I_{2n} + M(e_{1}, w_{1}))\varepsilon^{-1} \), where \( w_{1} = \varepsilon^{-1}w \). Since \( \langle e_{1}, w_{1} \rangle = \langle v, w \rangle = 0 \), we have \( w_{1}^{T} = (w_{11}, \ldots, w_{1n-1}, 0, \ldots, w_{12n}) \). Therefore, since \( \lambda \varepsilon = 1 \), we have
\[ I_{2n} + M(v, w) = \prod_{1 \leq i \leq n \leq n-1} \varepsilon q_{1n}(-\lambda_{i}q_{1n+i}q_{j}(-\lambda_{i}q_{j})q_{1n}^{-1}(\ast)\varepsilon^{-1} \]

Lemma 3.2. Let \( R \) be a graded ring. Let \( \alpha \in \text{EQ}(2n, R, \Lambda) \). Then for every \( a \in R_{0} \) one gets \( \alpha^{+}(a) \in \text{EQ}(2n, R, \Lambda) \).

Proof: Let \( \alpha = \Pi_{k=1}^{n} (1_{2n} + aM(e_{ik}, e_{jk})) \), where \( a \in R \) and \( i \geq 1 \). Hence for \( b \in R_{0} \), we have \( \alpha^{+}(b) = \Pi_{k=1}^{n} (1_{2n} + a^{+}(b)M(e_{ik}, e_{jk})) \). Now taking \( v = e_{i} \) and \( v = a^{+}(b)e_{j} \) we have \( \langle v, w \rangle = 0 \) and \( I_{2n} + M(v, w) = I_{2n} + a^{+}(b)M(e_{i}, e_{j}) \) which belongs to \( \text{EQ}(2n, R, \Lambda) \) by Corollary 3.1. Hence we have \( \alpha^{+}(b) \in \text{EQ}(2n, R, \Lambda) \) for \( b \in R_{0} \).

Lemma 3.3. (Graded Dilation Lemma) Let \( \alpha \in GQ(2n, R, \Lambda) \) with \( \alpha^{+}(0) = 1_{2n} \) and \( \alpha_{s} \in \text{EQ}(2n, R_{s}, \Lambda_{s}) \) for some non-zero-divisor \( s \in R_{0} \). Then there exists \( \beta \in \text{EQ}(2n, R, \Lambda) \) such that
\[ \beta_{s}^{+}(b) = \alpha_{s}^{+}(b) \]
for some \( b = s^{l} \) and \( l \gg 0 \).

Proof: Let \( \alpha_{s} \in \text{EQ}(2n, R_{s}, \Lambda_{s}) \) with \( \alpha_{s}^{+}(0) = 1_{2n} \). Then one gets
\[ \alpha_{s}^{+}(b + d)\alpha_{s}^{+}(d)^{-1} \in \text{EQ}(2n, R, \Lambda) \]
for some \( s, d \in R_{0} \) and \( b = s^{l}, l \gg 0 \).

Proof: Since \( \alpha_{s} \in \text{EQ}(2n, R_{s}, \Lambda_{s}) \), we have \( \alpha_{s}^{+}(X) \in \text{EQ}(2n, R_{s}[X], \Lambda_{s}[X]) \). Let
\[ \beta^{+}(X) = \alpha^{+}(X + d)\alpha^{+}(d)^{-1}, \]
where \( d \in R_{0} \). Then we have
\[ \beta_{s}^{+}(X) \in \text{EQ}(2n, R_{s}[X], \Lambda_{s}[X]) \]
and \( \beta^{+}(0) = 1_{2n} \). Hence by Lemma 3.3, we have, there exists \( b = s^{l}, l \gg 0 \), such that \( \beta^{+}(bX) \in \text{EQ}(2n, R[X], \Lambda[X]) \). Putting \( X = 1 \), we get our desired result.

Proof of Theorem 1.3 – Graded Local–Global Principle:
Since \( \alpha_{m} \in \text{EQ}(2n, R_{m}, \Lambda_{m}) \) for all \( m \in \text{Max}(C(R_{0})) \), for each \( m \) there exists \( s \in C(R_{0}) \setminus m \) such that \( \alpha_{s} \in \text{EQ}(2n, R_{s}, \Lambda_{s}) \). Using Noetherian property we can consider a finite cover of \( C(R_{0}) \), say \( s_{1} + \cdots + s_{r} = 1 \). From Lemma 3.3 we have \( \alpha_{s_{1}}^{+}(b_{1}) \in \text{EQ}(2n, R, \Lambda) \) for some \( b_{1} = s_{1}^{l} \) with \( b_{1} + \cdots + b_{r} = 1 \). Now consider \( \alpha_{s_{1}} \cdots s_{r} \), which is the image of \( \alpha \) in \( R_{s_{1}}\cdots s_{r} \).
By Lemma 2.8 \( \alpha \mapsto \alpha_{s_1 s_2 \ldots s_r} \) is injective. Hence we can perform our calculation in \( R_{s_1 s_2 \ldots s_r} \) and then pull it back to \( R \).

\[
\alpha_{s_1 s_2 \ldots s_r} = \alpha_{s_1 s_2 \ldots s_r} (b_1 + b_2 + \cdots + b_r) = ((\alpha_{s_1})_{s_2} s_3 \ldots)^+ (b_2 + \cdots + b_r)^{-1} \cdots ((\alpha_{s_r})_{s_1} s_{r-1} \ldots) (b_{r+1} + \cdots + b_r)^{-1} + (0)^{-1}
\]

Observe that \( ((\alpha_{s_1})_{s_2} \ldots s_r)^+ (b_1 + \cdots + b_r) ((\alpha_{s_1})_{s_2} \ldots s_r)^+ (b_{r+1} + \cdots + b_r)^{-1} \in EQ(2n, R, \Lambda) \) due to Lemma 3.3 (here \( s_i \) means we omit \( s_i \) in the product \( s_1 \ldots s_i \ldots s_r \), and hence \( \alpha_{s_1 s_2 \ldots s_r} \in EQ(2n, R_{s_1 \ldots s_r}, \Lambda_{s_1 \ldots s_r}) \)). This proves \( \alpha \in EQ(2n, R, \Lambda) \).

3.2. Normality and Local–Global. Next we are going to show that if \( K \) is a commutative ring with identity and \( R \) is an associative \( K \)-algebra such that \( R \) is finite as a left \( K \)-module, then the normality criterion of elementary subgroup is equivalent to the Local-Global principle for quadratic group. (One can also consider \( R \) as a right \( K \)-algebra.)

Lemma 3.5. (Bass; cf [4]) Let \( A \) be an associative \( B \)-algebra such that \( A \) is finite as a left \( B \)-module and \( B \) be a commutative local ring with identity. Then \( A \) is semilocal.

Theorem 3.6. (cf [7]) Let \( A \) be a semilocal ring (not necessarily commutative) with involution. Let \( v \in \text{Um}_{2n}(A) \). Then \( v \in e_i EQ(2n, A) \). In other words the group \( EQ(2n, A) \) acts transitively on \( \text{Um}_{2n}(A) \).

Before proving the next theorem we need to recall a theorem from [7]:

Theorem 3.7. (Local–Global Principle) Let \( A \) be an associative \( B \)-algebra such that \( A \) is finite as a left \( B \)-module and \( B \) be a commutative ring with identity. If \( \alpha(X) \in GQ(2n, A[X], \Lambda[X]) \), \( \alpha(0) = I_{2n} \) and \( \alpha_m(X) \in EQ(2n, A_m[X], \Lambda_m[X]) \) for every maximal ideal \( m \in \text{Max}(B) \), then \( \alpha \in EQ(2n, A[X], \Lambda[X]) \).

Theorem 3.8. Let \( K \) be a commutative ring with unity and \( R = \bigoplus_{i=0}^{\infty} R_i \) be a graded \( K \)-algebra such that \( R_0 \) is finite as a left \( K \)-module. Then for \( n \geq 3 \) the following are equivalent:

1. \( EQ(2n, R, \Lambda) \) is a normal subgroup of \( GQ(2n, R, \Lambda) \).
2. If \( \alpha \in GQ(2n, R, \Lambda) \) with \( \alpha^+(0) = I_{2n} \) and \( \alpha_m \in EQ(2n, R_m, \Lambda_m) \) for every maximal ideal \( m \in \text{Max}(K) \), then \( \alpha \in EQ(2n, R, \Lambda) \).

Proof: (1) \( \Rightarrow \) (2) We have proved the Lemma 3.3 for any form ring with identity and shown that the local-global principle is a consequence of Lemma 3.1. So, the result is true in particular if we have \( EQ(2n, R, \Lambda) \) is a normal subgroup of \( GQ(2n, R, \Lambda) \).

(2) \( \Rightarrow \) (1) Since polynomial rings are special case of graded rings, the result follows by using the Theorem 3.7. Let \( \alpha \in EQ(2n, R, \Lambda) \) and \( \beta \in GQ(2n, R, \Lambda) \). Then we have \( \alpha \) can be written as product of the matrices of the form \( (I_{2n} + \beta M(s_1, s_2)\beta^{-1}) \), with \( s_1, s_2 = 0 \) where \( s_1 \) and \( s_2 \) are suitably chosen basis vectors. Let \( \gamma = \beta s_1 \). Then we can write \( \beta s_2 \) as a product of the matrices of the form \( I_{2n} + M(v, w) \) for some \( v \in R_{2n}^* \). We must show that each \( I_{2n} + M(v, w) \in EQ(2n, R, \Lambda) \).

Consider \( \gamma = I_{2n} + M(v, w) \). Then \( \gamma^+(0) = I_{2n} \). By Lemma 3.3 we have the ring \( S^{-1} R \) is semilocal where \( S = \Lambda \setminus m \) and \( m \in \text{Max}(K) \). Since \( v \in \text{Um}_{2n}(R) \), then by Theorem 3.6 we have \( v \in EQ(2n, S^{-1} R, S^{-1} \Lambda) \). Therefore by applying Lemma 3.3 to the ring \( (S^{-1} R, S^{-1} \Lambda) \), we have \( \gamma_m \in EQ(2n, R_m, \Lambda_m) \) for every maximal ideal \( m \in \text{Max}(K) \). Hence by hypothesis we have \( \gamma \in EQ(2n, R, \Lambda) \). This completes the proof.

Remark 3.9. We conclude that the local-global principle for the elementary subgroups and their normality properties are equivalent.
4. Bass Nil Group \( \text{NK}_4 \text{GQ}(R) \)

In this section recall some basic definitions and properties of the representatives of \( \text{NK}_4 \text{GQ}(R) \). We represent any element of \( M_{2n}(R) \) as \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), where \( a, b, c, d \in M_n(R) \).

For \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) we call \( (a, b) \) the upper half of \( \alpha \). Let \( (R, \lambda, \Lambda) \) be a form ring. By setting \( \overline{\Lambda} = \{ \overline{a} : a \in \Lambda \} \) we get another form ring \( (R, \overline{\Lambda}, \overline{\Lambda}) \). We can extend the involution of \( R \) to \( M_n(R) \) by setting \( (a_{ij})^* = (\overline{a}_{ji}) \).

**Definition 4.1.** Let \( \alpha = (a_{ij}) \in M_n(R) \) is said to be \( \Lambda \)-Hermitian if \( \alpha = -\lambda^* \alpha^* \) and all the diagonal entries of \( \alpha \) are contained in \( \Lambda \). A matrix \( \beta \in M_n(R) \) is said to be \( \overline{\Lambda} \)-Hermitian if \( \beta = -\overline{\lambda} \beta^* \) and all the diagonal entries of \( \beta \) are contained in \( \overline{\Lambda} \).

**Remark 4.2.** A matrix \( \alpha \in M_n(R) \) is \( \Lambda \)-Hermitian if and only if \( \alpha^* \) is \( \overline{\Lambda} \)-Hermitian. The set of all \( \Lambda \)-Hermitian matrices forms a group under matrix multiplication.

**Lemma 4.3.** [15 Example 2] Let \( \beta \in GL_n(R) \) be a \( \Lambda \)-Hermitian matrix. Then the matrix \( \alpha^* \beta \alpha \) is \( \Lambda \)-Hermitian for every \( \alpha \in GL_n(R) \).

**Definition 4.4.** Let \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be a matrix. Then \( \alpha \) is said to be a \( \Lambda \)-quadratic matrix if one of the following equivalent conditions holds:

1. \( \alpha \in \text{GQ}(2n, R, \Lambda) \) and the diagonal entries of the matrices \( a^*c, b^*d \) are in \( \Lambda \),
2. \( a^*d + \lambda cb^*d = I_n \) and the matrices \( a^*c, b^*d \) are \( \Lambda \)-Hermitian,
3. \( \alpha \in \text{GQ}(2n, R, \Lambda) \) and the diagonal entries of the matrices \( ab^*, cd^* \) are in \( \Lambda \),
4. \( \lambda d^* + \lambda bc^* = I_n \) and the matrices \( ab^*, cd^* \) are \( \Lambda \)-Hermitian.

**Remark 4.5.** The set of all \( \Lambda \)-quadratic matrices of order \( 2n \) forms a group called \( \Lambda \)-quadratic group. We denote this group by \( \text{GQ}^\Lambda(2n, R, \Lambda) \). If \( 2 \in R^* \), then we have \( \Lambda_{\min} = \Lambda_{\max} \). In this case notions of quadratic groups and notions of \( \Lambda \)-quadratic groups coincides.

Also this happens when \( \Lambda = \Lambda_{\max} \). Hence quadratic groups are special cases of \( \Lambda \)-quadratic groups. Other classical groups appear as \( \Lambda \)-quadratic groups in the following way. Let \( R \) be a commutative ring with trivial involution. Then

\[
\text{GQ}^\Lambda(2n, R, \Lambda) = \begin{cases}
\text{Sp}_{2n}(R), & \text{if } \lambda = -1 \text{ and } \Lambda = \Lambda_{\max} = R \\
\text{O}_{2n}(R), & \text{if } \lambda = 1 \text{ and } \Lambda = \Lambda_{\min} = 0
\end{cases}
\]

And for general linear group \( GL_n(R) \), we have, \( GL_n(R) = \text{GQ}^\Lambda(2n, H(R), \Lambda = \Lambda_{\max}) \), where \( H(R) \) denotes the ring \( R \oplus R^op \) with \( R^op \) is the opposite ring of \( R \) and the involution on \( H(R) \) is defined by \( (x, y) = (y, x) \). Thus the study of \( \Lambda \)-quadratic matrices unifies the study of quadratic matrices.

We recall following results from [15].

**Lemma 4.6.** Let \( \alpha = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_{2n}(R) \). Then \( \alpha \in \text{GQ}^\Lambda(2n, R, \Lambda) \) if and only if \( a \in GL_n(R) \) and \( d = (a^*)^{-1} \).

**Proof:** Let \( \alpha \in \text{GQ}^\Lambda(2n, R, \Lambda) \). In view of (2) of Definition 4.4 we have, \( a^*d = I_n \). Hence \( a \) is invertible and \( d = (a^*)^{-1} \). Converse holds by (2) of Definition 4.4. \( \square \)

**Definition 4.7.** Let \( \alpha \in GL_n(R) \) be a matrix. A matrix of the form \( \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^*)^{-1} \end{pmatrix} \) is denoted by \( \mathbb{H}(\alpha) \) and is said to be hyperbolic.

**Remark 4.8.** In a similar way we can show that matrices of the form \( T_{12}(\beta) := \begin{pmatrix} I_n & \beta \\ 0 & I_n \end{pmatrix} \) is \( \Lambda \)-quadratic matrix if and only if \( \beta \) is \( \overline{\Lambda} \)-Hermitian. And the matrix of the form \( T_{21}(\gamma) := \begin{pmatrix} I_n & 0 \\ \gamma & I_n \end{pmatrix} \) is \( \Lambda \)-quadratic matrix if and only if \( \gamma \) is \( \Lambda \)-Hermitian.
Likewise in the quadratic case we can define the notion of \( \Lambda \)-elementary quadratic groups in the following way:

**Definition 4.9.** The \( \Lambda \)-elementary quadratic group is denoted by \( \text{EQ}_\Lambda^\lambda(2n, R, \Lambda) \) and defined by the group generated by \( \alpha \in \text{GL}_n(R) \), \( T_{12}(\beta) \) and \( \beta \) is \( \bar{\Lambda} \)-Hermitian and \( T_{21}(\gamma) \) is \( \gamma \bar{\Lambda} \)-Hermitian.

**Lemma 4.10.** Let \( A = \begin{pmatrix} \alpha & \beta \\
0 & \delta \end{pmatrix} \in \text{M}_{2n}(R) \). Then \( A \in \text{GQ}_\Lambda^\lambda(2n, R, \Lambda) \) if and only if \( \alpha \in \text{GL}_n(R), \delta = (\alpha^*)^{-1} \) and \( \alpha^{-1} \bar{\beta} \) is \( \bar{\Lambda} \)-Hermitian. In this case \( A \equiv \mathbb{H}(\alpha) \) (mod \( \text{EQ}_\Lambda^\lambda(2n, R, \Lambda) \)).

**Proof:** Let \( A \in \text{GQ}_\Lambda^\lambda(2n, R, \Lambda) \). Then by (4) of Definition 4.4 we have \( \alpha \sigma^* = I_n \) and \( \alpha \bar{\beta}^* \) is \( \Lambda \)-Hermitian. Hence \( \alpha \) is invertible and \( \delta = (\alpha^*)^{-1} \). For \( \alpha^{-1} \bar{\beta} \), we get

\[
(\alpha^{-1} \bar{\beta})^* = \bar{\beta}'(\alpha^{-1})^* = \alpha^{-1}(\alpha \sigma^*)(\alpha^{-1})^*,
\]

which is \( \Lambda \)-Hermitian by Lemma 4.3. Hence \( \alpha^{-1} \bar{\beta} \) is \( \bar{\Lambda} \)-Hermitian. Conversely, the condition on \( A \) will fulfill the condition (4) of Definition 4.4. Hence \( A \) is \( \Lambda \)-quadratic. Since \( \alpha^{-1} \bar{\beta} \) is \( \bar{\Lambda} \)-Hermitian,

\[
T_{12}(\alpha^{-1} \bar{\beta}) \in \text{EQ}_\Lambda^\lambda(2n, R, \Lambda)
\]

and \( AT_{12}(\alpha^{-1} \bar{\beta}) = \mathbb{H}(\alpha) \). Thus \( A \equiv \mathbb{H}(\alpha) \) (mod \( \text{EQ}_\Lambda^\lambda(2n, R, \Lambda) \)).

A similar proof will prove the following:

**Lemma 4.11.** Let \( B = \begin{pmatrix} \alpha & 0 \\
\gamma & \delta \end{pmatrix} \in \text{M}_{2n}(R) \). Then \( B \in \text{GQ}_\Lambda^\lambda(2n, R, \Lambda) \) if and only if \( \alpha \in \text{GL}_n(R), \delta = (\alpha^*)^{-1} \) and \( \gamma \) is \( \Lambda \)-Hermitian. In this case

\[
B \equiv \mathbb{H}(\alpha) \pmod{\text{EQ}_\Lambda^\lambda(2n, R, \Lambda)}.
\]

**Lemma 4.12.** Let \( \alpha = \begin{pmatrix} a & b \\
c & d \end{pmatrix} \in \text{GQ}_\Lambda^\lambda(2n, R, \Lambda) \). Then

\[
\alpha \equiv \mathbb{H}(\alpha) \pmod{\text{EQ}_\Lambda^\lambda(4n, R, \Lambda)}
\]

if \( a \in \text{GL}_n(R) \). Moreover, if \( a \in \text{E}_n(R), \) then \( \alpha \equiv \mathbb{H}(\alpha) \pmod{\text{EQ}_\Lambda^\lambda(2n, R, \Lambda)} \).

**Proof:** By same argument as given in Lemma 4.10 we have \( \alpha^{-1} b \) is \( \Lambda \)-Hermitian. Hence

\[
T_{12}(\alpha^{-1} b) \in \text{EQ}_\Lambda^\lambda(2n, R, \Lambda),
\]

and consequently \( \alpha T_{12}(\alpha^{-1} b) = \begin{pmatrix} a & 0 \\
c & d' \end{pmatrix} \in \text{GQ}_\Lambda^\lambda(2n, R, \Lambda) \)

for some \( d' \in \text{GL}_n(R) \). Hence by Lemma 4.11 we get

\[
\alpha T_{12}(\alpha^{-1} b) \equiv \mathbb{H}(\alpha) \pmod{\text{EQ}_\Lambda^\lambda(2n, R, \Lambda)}.
\]

Hence \( \alpha \equiv \mathbb{H}(\alpha) \pmod{\text{EQ}_\Lambda^\lambda(2n, R, \Lambda)} \).

**Definition 4.13.** Let \( \alpha = \begin{pmatrix} a_1 & b_1 \\
c_1 & d_1 \end{pmatrix} \in \text{M}_{2r}(R), \beta = \begin{pmatrix} a_2 & b_2 \\
c_2 & d_2 \end{pmatrix} \in \text{M}_{2s}(R) \). As before, we define \( \alpha \perp \beta \), and consider an embedding

\[
\text{GQ}_\Lambda^\lambda(2n, R, \Lambda) \to \text{GQ}_\Lambda^\lambda(2n + 2, R, \Lambda), \quad \alpha \mapsto \alpha \perp I_2.
\]

We denote \( \text{GQ}_\Lambda^\lambda(R, \Lambda) = \bigcup_{n=1}^{\infty} \text{GQ}_\Lambda^\lambda(2n, R, \Lambda) \) and \( \text{EQ}_\Lambda^\lambda(R, \Lambda) = \bigcup_{n=1}^{\infty} \text{EQ}_\Lambda^\lambda(2n, R, \Lambda) \).

In view of quadratic analog of Whitehead Lemma, we have the group \( \text{EQ}_\Lambda^\lambda(R, \Lambda) \) coincides with the commutator of \( \text{GQ}_\Lambda^\lambda(R, \Lambda) \). Therefore the group

\[
K_1 \text{GQ}_\Lambda^\lambda(R, \Lambda) := \frac{\text{GQ}_\Lambda^\lambda(R, \Lambda)}{\text{EQ}_\Lambda^\lambda(R, \Lambda)}
\]

is well-defined. The class of a matrix \( \alpha \in \text{GQ}_\Lambda^\lambda(R, \Lambda) \) in the group \( K_1 \text{GQ}_\Lambda^\lambda(R, \Lambda) \) is denoted by \([\alpha]\). In this way we obtain a \( K_1 \)-functor \( K_1 \text{GQ}_\Lambda^\lambda \) acting form the category of form rings to the category of abelian groups.
Remark 4.14. Likewise in the quadratic case, the kernel of the group homomorphism
\[ K_1 GQ^\lambda(R[X], \Lambda[X]) \to K_1 GQ^\lambda(R, \Lambda) \]
induced from the form ring homomorphism \((R[X], \Lambda[X]) \to (R, \Lambda); X \mapsto 0\) is denoted by \(NK_1 GQ^\lambda(R, \Lambda)\). Since the \(\Lambda\)-quadratic groups are subclass of the quadratic groups, the Local-global principle holds for \(\Lambda\)-quadratic groups. We use this throughout for the next section.

5. Absence of torsion in \(NK_1 GQ^\lambda(R, \Lambda)\)

In this section we give the proof of Theorem 4.1 and Theorem 5.2. In [14], the proof of the theorem for the linear case is based on two key results, viz. the Higman linearisation, and a lemma on polynomial identity in the truncated polynomial rings. Here we recall the lemma with its proof to highlight its connection with the big Witt vectors. Recently, in [15], V. Kopeiko deduced an analog of Higman linearisation process for a subclass of the general quadratic groups.

Definition 5.1. For an associative ring \(R\) with unity we consider the truncated polynomial ring
\[ R_t = \frac{R[X]}{(X + 1)^t}. \]

Lemma 5.2. (cf. [8], Lemma 4.1) Let \(P(X) \in R[X]\) be any polynomial. Then the following identity holds in the ring \(R_t\):
\[ (1 + X^r P(X)) = (1 + X^r P(0))(1 + X^{r+1}Q(X)), \]
where \(r > 0\) and \(Q(X) \in R[X]\), with \(\deg(Q(X)) < t - r\).

Proof: Let us write \(P(X) = a_0 + a_1 X + \cdots + a_t X^t\). Then we can write \(P(X) = P(0) + X P'(X)\) for some \(P'(X) \in R[X]\). Now, in \(R_t\)
\[ (1 + X^r P(0))(1 + X^r P(0))^{-1} = (1 + X^r P(0) + X^{r+1} P'(X))(1 + X^r P(0))^{-1} \]
\[ = 1 + X^{r+1} P'(X)(1 - X^r P(0) + X^{2r}(P(0))^2 - \cdots) \]
\[ = 1 + X^{r+1} Q(X) \]
where \(Q(X) \in R[X]\) with \(\deg(Q(X)) < t - r\). Hence the lemma follows.

Remark. Iterating the above process we can write for any polynomial \(P(X) \in R[X]\),
\[ (1 + X P(X)) = \prod_{i=1}^t (1 + a_i X^i) \]
in \(R_t\), for some \(a_i \in R\). By ascending induction it will follow that the \(a_i\)'s are uniquely determined. In fact, if \(R\) is commutative then \(a_i\)'s are the \(i\)-th component of the ghost vector corresponding to the big Witt vector of \((1 + X P(X)) \in W(R) = (1 + X R[[X]])^\times\). For details see [11], §I.

Lemma 5.3. Let \(R\) be a ring with \(1/k \in R\) and \(P(X) \in R[X]\). Assume \(P(0)\) lies in the center of \(R\). Then
\[ (1 + X^r P(X))^{k^r} = 1 \Rightarrow (1 + X^r P(X)) = (1 + X^{r+1} Q(X)) \]
in the ring \(R_t\) for some \(r > 0\) and \(Q(X) \in R[X]\) with \(\deg(Q(X)) < t - r\).

Following result is due to V. Kopeiko, cf. [15].

Proposition 5.4. (Higman linearisation) Let \((R, \Lambda)\) be a form ring. Then, every element of the group \(NK_1 GQ^\lambda(R, \Lambda)\) has a representative of the form
\[ [a; b, c]^n = \left( \frac{bX}{-cX^n} \right) \in GQ^\lambda(2r, R[X], \Lambda[X]) \]
for some positive integers \(r\) and \(n\), where \(a, b, c \in M_r(R)\) satisfy the following conditions:
1. the matrices \(b\) and \(ab\) are Hermitian and also \(ab = ba^*\),
Let \( a \alpha \) and hence we have \([7]\), we have \([I r \lambda] \). By Corollary 5.5, we have \([\mathbb{H}(I_r - aX)] \) if \((I_r - aX) \in \text{GL}_r(R)\).

Hence we have \([\alpha = [\mathbb{H}(I_r - aX)]]) in \( \text{NK}_1 \mathcal{G}^\lambda(R, \Lambda) \).

Proportion 5.4. Therefore since the first corner matrix \( A(X) \in \text{GL}_s(R[X]) \), then we have \((I_r - aX) \in \text{GL}_s(R[X])\). By Corollary 5.5 we have \([\alpha = [\mathbb{H}(I_r - aX)]]). Now let \([\alpha] \) be a \( k \)-torsion. Then we have \([\mathbb{H}(I_r - aX)] \) is a \( k \)-torsion. Since \((I_r - aX) \) is invertible, it follows that \( a \) is nilpotent. Let \( a^{k+1} = 0 \). Since \([I_r - aX]^k = [I] in K_1 \mathcal{G}^\lambda(R[X], \Lambda[X])\), then by arguing as given in \([7]\), we have \([I_r - aX = [I] in K_1 \mathcal{G}^\lambda(R[X], \Lambda[X])]\). This completes the proof.

Proof of Theorem 1.2 – (Graded Version):
Consider the ring homomorphism \( f : R \rightarrow R[X] \) defined by
\[
f(a_0 + a_1 + \ldots) = a_0 + a_1 X + \ldots.
\]

Then
\[
([I + N]^k = [I] \Rightarrow f([I + N]^k) = f([I + N])^k = [I]
\Rightarrow ([I + N_0 + N_1 X + \ldots + N_r X^r] )^k = [I].
\]

Let \( m \) be a maximal ideal in \( R_0 \). By Theorem 1.1 we have
\[
([I + N_0 + N_1 X + \ldots + N_r X^r] ) = [I]
\]
in \( \text{NK}_1 \mathcal{G}^\lambda((R)_m, \Lambda_m) \). Hence by using the local-global principle we conclude
\[
[I + N] = [I + N_0]
\]
in \( \text{NK}_1 \mathcal{G}^\lambda(R, \Lambda) \), as required.

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