THETA RANK, LEVELNESS, AND MATROID MINORS

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Abstract. The Theta rank of a finite point configuration \( V \) is the maximal degree necessary for a sum-of-squares representation of a non-negative linear function on \( V \). This is an important invariant for polynomial optimization that is in general hard to determine. We study the Theta rank and levelness, a related discrete-geometric invariant, for matroid base configurations. It is shown that the class of matroids with bounded Theta rank or levelness is closed under taking minors. This allows for a characterization of matroids with bounded Theta rank or levelness in terms of forbidden minors.

We give the complete (finite) list of excluded minors for Theta-1 matroids which generalizes the well-known series-parallel graphs. Moreover, the class of Theta-1 matroids can be characterized in terms of the degree of generation of the vanishing ideal and in terms of the psd rank for the associated matroid base polytope.

We further give an excluded minor characterization for \( k \)-level graphs and we investigate the excluded minors for graphs of Theta rank 2.

1. Introduction

Let \( V \) be a configuration of finitely many points in \( \mathbb{R}^n \). A linear function \( \ell(x) = \delta - \langle c, x \rangle \) which is non-negative on \( V \) is called \( k \)-sos with respect to \( V \) if there exist polynomials \( h_1, \ldots, h_s \in \mathbb{R}[x_1, \ldots, x_n] \) such that \( \deg h_i \leq k \) and

\[
\ell(v) = h_1^2(v) + h_2^2(v) + \cdots + h_s^2(v)
\]

for all \( v \in V \). The **Theta rank** \( \text{Th}(V) \) of \( V \) is the smallest \( k \geq 0 \) such that every non-negative linear function is \( k \)-sos with respect to \( V \). The Theta rank was introduced in \([\text{GPT10}]\) as a measure for the ‘complexity’ of linear optimization over \( V \) using tools from polynomial optimization. If \( V \) is given as the solutions to a system of polynomial equations, then the size of a semidefinite program for the (exact) optimization of a linear function over \( V \) is of order \( O(n^{\text{Th}(V)}) \). For many practical applications, for example in combinatorial optimization, an algebraic description of \( V \) is readily available and the semidefinite programming approach is the method of choice. Clearly, situations with high Theta rank render the approach impractical.

We are interested in

\[
\mathcal{V}_k^\text{Th} := \{ V \text{ point configuration} : \text{Th}(V) \leq k \}.
\]

As \( V \) is finite and \( \ell(x) \) non-negative on \( V \), we may interpolate \( \sqrt{\ell(x)} \) over \( V \) by a single polynomial which shows that \( \text{Th}(V) \leq |V| - 1 \). This, however, is a rather crude estimate as the 0/1-cube \( V = \{0, 1\}^n \) has Theta rank 1.

Let \( \ell(x) \) be a non-negative linear function. The subconfiguration \( V' = \{ v \in V : \ell(v) = 0 \} \) is called a face of \( V \) with supporting hyperplane \( H = \{ x \in \mathbb{R}^n : \ell(x) = 0 \} \). If \( V' \neq V \) is inclusion-maximal, then \( V' \) is called a **facet** and \( H \) (and equivalently \( \ell(x) \)) **facet-defining**. If \( V \) is a full-dimensional point configuration then \( H \) and \( \ell(x) \), up to positive scaling, are unique. It follows from basic convexity that \( \text{Th}(V) \) is the smallest \( k \) such that all facet-defining \( \ell(x) \) are
We recall the notion of matroids and the associated geometric objects in Section 2. The following are equivalent:

(i) $V_M$ has Theta rank $1$ or, equivalently, is $2$-level;
(ii) $M$ has no minor isomorphic to $M(K_4)$, $W^3$, $Q_6$, or $P_6$;
(iii) $M$ can be constructed from uniform matroids by taking direct sums or 2-sums;
(iv) The vanishing ideal $I(V_M)$ is generated in degrees $\leq 2$;
(v) The base polytope $P_M$ has minimal psd rank.

Part (ii) yields a complete and, in particular, finite list of excluded minors whereas (iii) give a synthetic description of this class of matroids. The parts (iv) and (v) are proved in Section 4. The former states that 2-level matroids are precisely those matroids $M$ for which the base configuration $V_M$ is cut out by quadrics (Theorem 4.4). This contrasts the situation for general point configurations (Example 8). The psd rank of a polytope $P$ is the smallest ‘size’ of a spectrahedron that linearly projects to $P$. The psd rank was studied in [GPT13, GRT13] and it was shown that the psd rank $\text{Psd}(P)$ is at least $\dim P + 1$. Part (v) shows that the 2-level matroids are exactly those matroids for which the psd rank of the base polytope $P_M = \text{conv}(V_M)$ is minimal. Again, this is in strong contrast to the psd rank of general polytopes.

In Section 5 we restrict to the class of graphs $G$. We give a complete list of excluded minors for $k$-level graphs (Theorem 5.6). The classes of 3-level and 4-level graphs appear in works of Halin (see [Die90, Ch. 6]) and Oxley [Oxl89]. In particular, the wheel with 5 spokes $W_5$ is shown to have Theta rank 3. Combined with results of Oxley [Oxl89], this yields a finite list of candidates for a complete characterization of Theta-2 graphs. We currently do not know how to extend the excluded minor characterization of $k$-levelness to general matroids.

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2. Point configurations and matroids

In this section we study properties of Theta rank and levelness related to the geometry of the point configuration. In particular, we investigate the behavior of these invariants under
taking sub-configurations. We recall basic notions from matroid theory and associated point configurations and polytopes.

2.1. Theta rank, levelness, and face-hereditary properties. The definitions of levelness and Theta rank make only reference to the affine hull of the configuration $V$ and thus neither depend on the embedding nor on a choice of coordinates. To have it on record we note the following basic property.

**Proposition 2.1.** The levelness and the Theta rank of a point configuration are invariant under affine transformations.

That this does not hold for (admissible) projective transformations is clear for the levelness and for the Theta rank follows from Theorem 1.1.

**Proposition 2.2.** Let $V_1 \subset \mathbb{R}^{d_1}$ and $V_2 \subset \mathbb{R}^{d_2}$ be point configurations. Then the Theta rank satisfies $\text{Th}(V_1 \times V_2) = \max(\text{Th}(V_1), \text{Th}(V_2))$. The same is true for $\text{Lev}(V_1 \times V_2)$.

**Proof.** A linear function $\ell(x, y)$ is facet defining for $V_1 \times V_2$ if and only if $\ell(x, 0)$ is facet defining for $V_1$ or $\ell(0, y)$ is facet defining for $V_2$. Thus any representation (1.1) lifts to $\mathbb{R}[x, y]$. \Halmos

The Theta rank as well as the levelness of a point configuration are not monotone with respect to taking subconfigurations as can be seen by removing a single point from $\{0, 1\}^d$. However, it turns out that monotonicity holds for subconfigurations induced by supporting hyperplanes.

Let us call a collection $P$ of point configurations face-hereditary if it is closed under taking faces. That is, $V \cap H \in P$ for any $V \in P$ and supporting hyperplane $H$ for $V$.

**Lemma 2.3.** The classes $\mathcal{V}_k^{\text{Th}}$ and $\mathcal{V}_k^{\text{Lev}}$ are face-hereditary.

**Proof.** Let $V \subset \mathbb{R}^d$ be a full-dimensional point configuration and $H = \{p \in \mathbb{R}^d : g(p) = 0\}$ a supporting hyperplane such that the affine hull of $V' := V \cap H$ has codimension 1. Let $\ell(x)$ be facet defining for $V'$. Observe that $\ell(x)$ and $\ell_\delta(x) := \ell(x) + \delta g(x)$ give the same linear function on $V'$ for all $\delta$. For

$$\delta = \max\{\frac{-\ell(v)}{g(v)} : v \in V \setminus V'\}$$

$\ell_\delta(x)$ is non-negative for $V$. Hence any representation (1.1) of $\ell_\delta$ over $V'$ yields a representation for $\ell$ over $V'$. Moreover, the levelness of $\ell_\delta$ gives an upper bound on the levelness of $\ell$. \Halmos

It is interesting to note that these properties are not hereditary with respect to arbitrary hyperplanes. Indeed, consider the point configuration

$$V = \{(0, 1)^n \times \{-1, 0, 1\}\} \setminus \{0\}$$

It can be easily seen that $\text{Th}(V) = \text{Lev}(V) = -1 = 2$. The hyperplane $H = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ is not supporting and $V' = V \cap H = \{0, 1\}^n \setminus \{0\}$. The linear function $\ell(x) = x_1 + \cdots + x_n - 1$ is facet-defining for $V'$ with $n$ levels. As for the Theta rank, any representation (1.1) yields a polynomial $f(x) = \ell(x) - \sum h_i^2(x)$ of degree $2k$ that vanishes on $V'$ and $f(0) = -1 - \sum h_i^2(0) < 0$. For $n > 4$, the following proposition assures that $\text{Th}(V') \geq 3$.

**Proposition 2.4.** Let $V' = \{0, 1\}^n \setminus \{0\}$ and $f(x)$ a polynomial vanishing on $V'$ and $f(0) \neq 0$. Then $\deg f \geq n$.

**Proof.** For a monomial $x^\alpha$, let $\alpha = \{i : \alpha_i > 0\}$ be its support. For any point $v \in V'$ it follows that $v^\alpha = v^\tau = \prod_{i \in \tau} v_i$, since all the points have $0/1$ coordinates. Hence, we can assume that $f$ is of the form $f(x) = \sum_{\tau \subseteq [n]} c_\tau x^\tau$ for some $c_\tau \in \mathbb{R}$, $\tau \subseteq [n]$. Moreover $c_{\emptyset} = f(0) \neq 0$ and without loss of generality we can assume $c_{\emptyset} = 1$. Any point $v \in V'$ is of the form $v = 1_\sigma$ for some $\emptyset \neq \sigma \subseteq [n]$ and we calculate

$$0 = f(v) = \sum_{\emptyset \neq \tau \subseteq \sigma} c_\tau.$$

It follows that $c_\tau$ satisfies the defining conditions of the Möbius function of the Boolean lattice and hence equals $c_\tau = (-1)^{|\tau|}$ for all $\tau \subseteq [n]$. In particular $c_{[n]} \neq 0$ which finishes the proof. \Halmos
2.2. Matroids and basis configurations. We now introduce the combinatorial point configurations that are our main object of study. Matroids and their combinatorial theory are a vast subject and we refer the reader to the book by Oxley [Oxl11] for further information.

Definition 2.5. A matroid of rank $k$ is a pair $M = (E, \mathcal{B})$ consisting of a finite ground set $E$ and a collection of bases $\emptyset \neq \mathcal{B} \subseteq \binom{E}{k}$ satisfying the basis exchange axiom: for $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$ there is $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}$.

A set $I \subseteq E$ is independent if $I \subseteq B$ for some $B \in \mathcal{B}$. The rank of $X$, denoted by $\text{rank}_M(X)$, is the cardinality of the largest independent subset contained in $X$. The circuits of $M$ are the inclusion-minimal dependent subsets. An element $e$ is called a loop if $\{e\}$ is a circuit. We say that $e, f \in E$ are parallel if $\{e, f\}$ is a circuit. A matroid is simple if it does not contain loops or parallel elements. A flat of a matroid is a set $F \subseteq E$ such that $\text{rank}(F) < \text{rank}(F \cup e)$ for all $e \in E \setminus F$.

A particular class of matroids that we will consider are the graphic matroids. To a graph $G = (V, E)$ we associate the matroid $M(G) = (E, \mathcal{B})$. The bases are exactly the spanning forests of $G$. The running example for this section is the following.

Example 1. Let $G$ be the graph

The graphic matroid $M = M(G)$ has ground set $E = \{1, 2, 3, 4\}$, rank$(M) = 2$, and bases

$\mathcal{B}(G) = \{12, 13, 14, 23, 24\}$.

The dual matroid $M^*$ of the matroid $M = (E, \mathcal{B})$ is the matroid defined by the pair $(E, \mathcal{B}^*)$ where $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$. A coloop of $M$ is an element which is a loop of $M^*$. Equivalently it is an element which appears in every basis of $M$.

If $e \in E$ is not a coloop, we define the deletion as $M \setminus e := (E \setminus e, \{B \in \mathcal{B} : e \notin B\})$. If $e$ is a coloop, then the bases of $M \setminus e$ are $\{B \setminus e : B \in \mathcal{B}\}$. Dually, if $e \in E$ is not a loop, we define the contraction as $M/e := (E \setminus e, \{B \setminus e : e \in B \in \mathcal{B}\})$. These operations can be extended to subsets $X \subseteq E$ and we write $M \setminus X$ and $M/X$, respectively. We also define the restriction of $M$ to a subset $X \subseteq E$ as $M|_X := M \setminus (E \setminus X)$. Note that $(M \setminus X)^* = M^*/X$. A minor of $M$ is a matroid obtained from $M$ by a sequence of deletion and contraction operations. The subclass of graphic matroids is closed under taking minors but not under taking duals.

To each matroid we associate a point configuration representing the set of bases. For a fixed ground set $E$ let us write $1_X \in \{0,1\}^E$ for the characteristic vector of $X \subseteq E$.

Definition 2.6. Let $M = (E, \mathcal{B})$ be a matroid. The base configuration of $M$ is the point configuration

$V_M := \{1_B : B \in \mathcal{B}\} \subset \mathbb{R}^E$.

The base polytope of $M$ is $P_M := \text{conv}(V_M)$.

The dual $M^*$ is obtained by taking the complements of bases. The corresponding base configuration is thus

$V_{M^*} = 1 - V_M$. \hspace{1cm} (2.1)

In particular, $V_M$ and $V_{M^*}$ are related by an affine transformation.

Observe that $V_M$ is not a full-dimensional point configuration. Indeed, $V_M$ is contained in the hyperplane $\sum_{e \in E} x_e = \text{rank}(E)$. In order to determine the dimension of $V_M$ we need to consider the relations among elements of $E$: $e_1, e_2 \in E$ are related if there exists a circuit of $M$ containing both. This is an equivalence relation and the equivalence classes are called the
connected components of $M$. Let us write $c(M)$ for the number of connected components.
The matroid $M$ is connected if $c(M) = 1$.
Let $M_1$ and $M_2$ be matroids with disjoint ground sets $E_1$ and $E_2$. The collection
$$\mathcal{B} := \{ B_1 \cup B_2 : B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2) \}.$$ is the set of bases of a matroid on $E_1 \cup E_2$, called the direct sum of $M_1$ and $M_2$ and denoted
by $M_1 \oplus M_2$. The corresponding base configuration is exactly the Cartesian product
$$V_{M_1 \oplus M_2} = V_{M_1} \times V_{M_2}.$$ (2.2)
If $E_1, \ldots, E_r \subseteq E$ are the connected components of $M$, then
$$M = \bigoplus_{i} M|_{E_i}.$$ Thus, showing that
$$\dim V_M = |E| - 1$$ if $M$ is connected proves the following.

**Proposition 2.7.** The smallest affine subspace containing $V_M$ is of dimension $|E| - c(M)$.

For a subset $X \subseteq E$ let us write $\ell_X(x) = \sum_{e \in X} x_e$. For $A \subseteq E$ we then have $\ell_X(1_A) = |A \cap X|$. Hence
$$\text{rank}_M(X) = \max_{v \in P_M} \ell_X(v).$$ For $X \subseteq E$ we define the supporting hyperplane
$$H_M(X) := \{ x \in \mathbb{R}^E : \ell_X(x) = \text{rank}_M(X) \}.$$ The corresponding faces of $V_M$ (or equivalently of $P_M$) are easy to describe.

**Proposition 2.8 (Edm70).** For a matroid $M = (E, \mathcal{B})$ and a subset $X \subseteq E$, we have
$$V_M \cap H_M(X) = V_{M|_{X \oplus M/X}} = V_{M|X} \times V_{M/X}.$$ Let us illustrate this on our running example.

**Example 2** (continued). The graph given in Example 1 yields a connected matroid on 4 elements
and hence a 3-dimensional base configuration. The corresponding base polytopes is this:

![Base Polytopes](image)

The 5 bases correspond to the vertices of $P_M$. We considered the subset $\{3, 4\}$ whose associated
face $M(G)|_{\{3,4\}} \times M(G)/\{3, 4\}$ is the quadrilateral facet of the polytope, and the subset $\{1, 2\}$
whose associated face $M(G)|_{\{1,2\}} \times M(G)/\{1, 2\}$ is the vertex $(1, 1, 0, 0)$.

We define the following families of matroids:
$$\mathcal{M}_k^{\text{lev}} := \{ M \text{ matroid} : \text{Lev}(V_M) \leq k \},$$
$$\mathcal{M}_k^{\text{th}} := \{ M \text{ matroid} : \text{Th}(V_M) \leq k \}.$$ We will say that a matroid $M$ is of Theta rank or level $k$ if the corresponding base configuration
$V_M$ is. Now combining Proposition 2.8 with Lemma 2.3 proves the main theorem of this section.

**Theorem 2.9.** The classes $\mathcal{M}_k^{\text{th}}$ and $\mathcal{M}_k^{\text{lev}}$ are closed under taking minors.

**Proof.** Using Proposition 2.8 repeatedly on one-element sets shows that for every minor $N$ of
$M$ there is a supporting hyperplane such that $V_M \cap H$ is affinely isomorphic to $V_N$. Lemma 2.3
assures us that $\text{Th}(V_M) \leq \text{Th}(V_M)$. \qed

Let us analogously define the classes $\mathcal{G}_k^{\text{th}}$ and $\mathcal{G}_k^{\text{lev}}$ of graphic matroids of Theta rank and levelness
bounded by $k$. These are also closed under taking minors and the Robertson–Seymour’s theorem
([RS04]) asserts that there is a finite list of excluded minors characterizing each class.
In the remainder of the section we will recall the facet-defining hyperplanes of $V_M$ which will also show that all faces of $V_M$ correspond to direct sums of minors. The facial structure of $V_M$ has been of interest originally in combinatorial optimization [Edm70] (see also [Sch03, Ch. 40]) and later in geometric combinatorics and tropical geometry [AK06, FS05, Kim10].

**Theorem 2.10.** Let $M = (E, \mathcal{B})$ be a connected matroid. For every facet $U \subset V_M$ there is a unique $\emptyset \neq S \subset E$ such that $U = V_M \cap H_M(S)$. Conversely, a subset $\emptyset \neq S \subset E$ gives rise to a facet if and only if

(i) $S$ is a flat such that $M|_S$ as well as $M/S$ are connected;
(ii) $S = E \setminus e$ for some $e \in E$ such that $M|_S$ as well as $M/S$ are connected.

In [FS05] the subsets $S$ in (i) were called facets and we stick to this name. In our study of the Theta rank and the levelness of base configurations, the following asserts that we will only need to consider facets. For brevity, a $k$-level facet refers to a facet whose corresponding facet is $k$-level.

**Proposition 2.11.** Let $M$ be a connected matroid and $S = E \setminus e$. Then $\ell_S(x)$ is 2-level for $V_M$ and hence 1-sos.

**Proof.** Let $r$ be the rank of $M$. Restricted to the affine hull of $V_M$, we have that $\ell_S(x)$ and $r - x_e$ induce the same linear function. As $V_M$ is a 0/1-configuration, it follows that $\ell_S(x)$ is 2-level.

**Example 3.** The facets of the running example are four triangles and one square. The four triangles correspond to the two sets $\{1, 2, 4\}, \{1, 2, 3\}$ of cardinality $|E| - 1$ and the two facets $\{2\}, \{1\}$, while the square corresponds to the facet $\{3, 4\}$. We have already described in the previous example the square facet. In the picture we highlight two triangular facets, the first one (green) corresponding to the facet $\{1\}$, the second one (red) to the set $\{1, 2, 4\}$.

A seemingly trivial but useful class of matroids is given by the uniform matroids $U_{n,k}$ for $0 \leq k \leq n$ given on ground set $E = \{1, \ldots, n\}$ and bases $\mathcal{B}(U_{n,k}) = \{B \subseteq E : |B| = k\}$.

**Proposition 2.12.** Uniform matroids are 2-level and hence have Theta rank 1.

**Proof.** The base polytope of $U_{n,k}$ is also known as the $(n, k)$-hypersimplex and is given by

\[ P_{U_{n,k}} = \text{conv}\{1_B : B \subseteq E, |E| = k\} = \{x \in \mathbb{R}^E : 0 \leq x_e \leq 1, \sum_e x_e = k\}. \]

The facet-defining linear functions are among the functions $\{\pm \ell_{(e)}(x) = \pm x_e : e \in E\}$ which can take only two different values on 0/1-points.

### 3. 2-Level Matroids

In this section we investigate the excluded minors for the class of 2-level matroids and, by Theorem 1.1, equivalently the matroids of Theta rank 1. In this case we can give the complete and in particular finite list of forbidden minors. We start by showing that we can exclude matroids with few elements and of small rank.

**Proposition 3.1.** Let $M = (E, \mathcal{B})$ be a matroid. If $\text{rank}(M) \leq 2$ or $|E| \leq 5$, then $M$ is 2-level.
The case rank($M$) = 1 is trivial since there is no proper flacet. On the other hand, if rank($M$) = 2 the proper flacet are necessarily flacets of rank 1. The linear function $\ell_F$ for any such flacet $F$ only takes values in $\{0, 1\}$ and thus is 2-level. By (2.1) and Proposition 2.1, $M$ and $M^*$ have the same Theta rank and levelness. If $|E| \leq 5$, then either $M$ or $M^*$ is of rank $\leq 2$. □

A first example of a matroid of levelness $\geq 3$ is given by the graphic matroid associated to the complete graph $K_4$.

**Proposition 3.2.** The graphic matroid $M(K_4)$ is 3-level.

Proof. Let $F = \{1, 2, 3\}$ corresponding to the labelled example shown below. Both the contraction of $F$ and the restriction to $F$ are connected (or biconnected on the level of graphs) and thus $F$ is a flacet with $\ell_F(x) = x_1 + x_2 + x_3 = 2$. The spanning trees $B_1 = \{1, 5, 6\}$ and $B_2 = \{4, 5, 6\}$ satisfy $|F \cap B_2| < |F \cap B_1| < \text{rank}(F)$ which shows that $M(K_4)$ is at least 3-level. To see that $M(K_4)$ is at most 3-level we notice that every proper flacet $F$ has rank $\leq \text{rank}(M(K_4)) - 1 = 2$ and hence $\ell_F(x)$ can take at most three different values. □

Before analyzing other matroids we quickly recall a geometric representation of certain matroids of rank 3: The idea is to draw a diagram in the plane whose points correspond to the elements of the ground set. Any subset of 3 elements constitute a basis unless they are contained in a depicted line.

**Example 4.** Let us consider the graph $K_4$ and its geometric representation as a matroid:

![Diagram of K_4](https://via.placeholder.com/150)

Thus the geometric representation consists only of the four lines associated to the 3-circuits of $K_4$.

Starting from the geometric representation of $M(K_4)$ we define three new matroids by removing one, two or three lines of the representation and we call them respectively $W^3$, $Q_6$ and $P_6$. None of these matroids is graphic, but we can easily draw their geometric representations:

![Geometric representations of W^3, Q_6, and P_6](https://via.placeholder.com/150)

**Proposition 3.3.** The matroids $W^3$, $Q_6$, and $P_6$ are 3-level.

Proof. Let $M$ be any of the three given matroids and consider $F = \{1, 2, 3\}$. It is easy to check that $M|_F \cong U_{3,1}$ and $M/F \cong U_{3,2}$ which marks $F$ as a flacet. The vertices of the matroid polytopes associated to the bases $\{4, 5, 6\}$, $\{1, 4, 6\}$, $\{1, 2, 6\}$ lie on distinct hyperplanes parallel to $H_M(F) = \{\ell_F(x) = \text{rank}_M(F)\}$. Therefore the matroids are at least 3-level. Since rank($M$) = 3, we can use the same argument as in the proof of Proposition 3.2. □

The list of excluded minors for $\mathcal{M}_2^{\text{lev}}$ so far includes $M(K_4)$, $W_3^3$, $Q_6$, and $P_6$. To show that this list is complete, we will approach the problem from the constructive side and consider how to synthesize 2-level matroids. We already saw that $\mathcal{M}_2^{\text{lev}}$ is closed under taking direct sums. We will now consider three more operations that retain levelness. Let $M_1 = (E_1, B_1)$ and
$M_2 = (E_2, B_2)$ be matroids such that \{p\} = $E_1 \cap E_2$. We call $p$ a base point. If $p$ is not a coloop of both, then we define define the series connection $S(M_1, M_2)$ with respect to $p$ as the matroid on ground set $E_1 \cup E_2$ and with bases $B = \{B_1 \cup B_2 : B_1 \in B_1, B_2 \in B_2, B_1 \cap B_2 = \emptyset\}$.

We also define the parallel connection with respect to $p$ as the matroid $S(M_1^*, M_2^*)^*$ provided $p$ is not a loop of both. Notice that $S(M_1, M_2)$ contains both $M_1$ and $M_2$ as a minor.

The operations of series and parallel connection, introduced by Brylawski [Bry71], are inspired by the well-known series and parallel operations on graphs. The following example illustrates the construction in the graphic case.

Example 5. Let us consider again the two graphic matroids $U_{3,2}$ and $M(K_4)$. Their series connection is the following graph:

\[
\begin{aligned}
S(\cdot, E) &= \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \\
M_1 &= \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \\
M_2 &= \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}
\end{aligned}
\]

An extensive treatment of these two operations is given in [Oxl11, Sect. 7.1]. We focus here on the geometric properties from which many combinatorial consequences can be deduced. For the following result, we write $E_1 \cup E_2 = (E_1 \cup E_2 \cup \{p_1, p_2\}) \setminus \{p\}$ for the disjoint union of $E_1$ and $E_2$.

Lemma 3.4. Let $M_1 = (E_1, B_1)$ and $M_2 = (E_2, B_2)$ be matroids with \{p\} = $E_1 \cap E_2$ not a coloop of both. Then the base polytope $P_S$ of the series connection $S = S(M_1, M_2)$ is linearly isomorphic to

$$
(P_{M_1} \times P_{M_2}) \cap \{x \in \mathbb{R}^{E_1 \cup E_2} : x_{p_1} + x_{p_2} \leq 1\}.
$$

Proof. It is clear that the base configuration $V_S$ is isomorphic to $V' = (V_{M_1} \times V_{M_2}) \cap \{x \in \mathbb{R}^{E_1 \cup E_2} : x_{p_1} + x_{p_2} \leq 1\}$ under the linear map $\pi : \mathbb{R}^{E_1 \cup E_2} \to \mathbb{R}^{E_1 \cup E_2}$ given by $\pi(1_{p_1}) = \pi(1_{p_2}) = 1_p$ and $\pi(1_e) = 1_e$ otherwise. Indeed, let $r_i = \text{rank}(M_i)$, then a linear inverse is given by $s : \mathbb{R}^{E_1 \cup E_2} \to \mathbb{R}^{E_1 \cup E_2}$ with $s(x)_{p_1} = r_1 - \ell_{E_i}(x)$ for $i = 1, 2$ and the identity otherwise.

It is therefore sufficient to show that the vertices of

$$
P' = (P_{M_1} \times P_{M_2}) \cap \{x \in \mathbb{R}^{E_1 \cup E_2} : x_{p_1} + x_{p_2} \leq 1\}.
$$

are exactly the points in $V'$. Clearly $V'$ is a subset of the vertices and any additional vertex of $P'$ would be the intersection of the relative interior of an edge of $P_{M_1} \times P_{M_2}$ with the hyperplane $H = \{x : x_{p_1} + x_{p_2} = 1\}$. However, every edge of $P_{M_1} \times P_{M_2}$ is parallel to some $1_e - 1_f$ for $e, f \in E_1$ or $e, f \in E_2$. Thus every edge of $P_{M_1} \times P_{M_2}$ can meet $H$ only in one of its endpoints which proves the claim.

It is interesting to note that the operation that related $P_{M_1}$ and $P_{M_2}$ to $P_{S(M_1, M_2)}$ is exactly a subdirect product in the sense of McMullen [McM76]. From the description of $P_{S(M_1, M_2)}$ we instantly get information about the Theta rank and levelness of the series and parallel connection.

Corollary 3.5. Let $S = S(M_1, M_2)$ be the series connection of matroids $M_1$ and $M_2$. Then

$\text{Th}(S) = \max(\text{Th}(M_1), \text{Th}(M_2))$.

The same holds true for the parallel connection as well as the levelness.

Proof. Lemma 3.4 shows that the facet-defining linear functions of $P_S$ are among those of $P_{M_1} \times P_{M_2}$ and $\ell(x) = x_{p_1} + x_{p_2}$. However, by the characterization of the bases of $S$, $\ell(x)$ can take only values in \{0, 1\}. Hence, $\text{Lev}(V_S) = \text{Lev}(V_{P_{M_1} \times P_{M_2}})$ and Proposition 2.2 finishes the proof.
Corollary 3.6. The classes $\mathcal{M}_k^{\text{Th}}$ and $\mathcal{M}_k^{\text{Lev}}$ are closed under taking series and parallel connections.

The most important operation that we will need is derived from the series connection. Let $M_1 = (E_1, B_1)$ and $M_2 = (E_2, B_2)$ be matroids with $E_1 \cap E_2 = \{p\}$. If $p$ is not a coloop for both, then we define the 2-sum

$$M_1 \oplus_2 M_2 := S(M_1, M_2)/p.$$ 

This is the matroid on the ground set $E = (E_1 \cup E_2) \setminus p$ and with bases

$$B := \{B_1 \cup B_2 \setminus p : B_1 \in B_M, B_2 \in B_N, p \in B_1 \triangle B_2\}$$

where $B_1 \triangle B_2$ is the symmetric difference.

The 2-sum is an associative operation for matroids which defines, by analogy to the direct sum, the 3-connectedness: a connected matroid $M$ is 3-connected if and only if it cannot be written as a 2-sum of two matroids each with fewer elements than $M$.

Example 6. Let us consider the 2-sum of a matroid $U_3, 2 \oplus_2 M(K_4)$: both matroids are graphic, therefore we can illustrate the operation for the corresponding graphs.

To perform the 2-sum we select an element for each matroid, while in the picture it looks like we also need to orient the chosen element. This is the case only because we are drawing an embedding of a graphic matroids; in fact the structure given by the vertices is forgotten when we look at the matroid. Whitney’s 2-Isomorphism Theorem [Oxl11, Thm. 5.3.1] clarifies that the matroid structure does not depend on the orientation we decide for the chosen elements.

We will need the following two properties of 2-sums.

Lemma 3.7 ([CO03, Lem. 2.3]). Let $M$ be a 3-connected matroid having no minor isomorphic to any of $M(K_4)$, $W^3$, $Q_6$, $P_6$. Then $M$ is uniform.

Lemma 3.8 ([Oxl11, Thm. 8.3.1]). Every matroid that is not 3-connected can be constructed from 3-connected proper minors of itself by a sequence of direct sums and 2-sums.

We can finally give a complete characterization of the class $\mathcal{M}_2^{\text{Lev}} = \mathcal{M}_1^{\text{Th}}$.

Theorem 3.9. For a matroid $M$ the following are equivalent.

(i) $M$ has Theta rank 1.
(ii) $M$ is 2-level.
(iii) $M$ has no minor isomorphic to $M(K_4)$, $W^3$, $Q_6$, or $P_6$.
(iv) $M$ can be constructed from uniform matroids by taking direct or 2-sums.

Proof. (i) $\Rightarrow$ (ii) is just Theorem 1.1. (ii) $\Rightarrow$ (iii) follows from Theorem 2.9 and Proposition 3.3. Let $M$ be a matroid satisfying (iii). If $M$ is 3-connected, then $M$ is uniform by Lemma 3.7. If $M$ is not 3-connected, then Lemma 3.8 shows that it satisfies (iv). Finally, uniform matroids have Theta rank 1 by Proposition 2.12. Theta rank $\leq k$ is retained by series connection (Corollary 3.5) and, by definition, also by the 2-sum.

Example 7. If we look at the family of 2-level graphic matroids, the only excluded minor is the graph $K_4$. The class of graphs which do not contain $K_4$ as a minor is the well-known class of series-parallel graphs $\mathcal{G}_{\text{SP}}$. The theorem implies $\mathcal{G}_2^{\text{Lev}} = \mathcal{G}_{\text{SP}}$.

The theorem can also be used to learn about the facial structure of the matroid base polytopes. A polytope $P$ is called $k$-simplicial if every face of dimension $k - 1$ is a simplex.
Corollary 3.10. Let $M$ be a matroid. If the base polytope $P_M$ is 6-simplicial, then $M$ has Theta rank 1.

Proof. Assume that $M$ is not a 2-level matroid. Then, by Theorem 3.9, $M$ has a minor $N \in \{M(K_4), W^3, Q_6, P_6\}$ and, in particular, $P_N$ as a face of $P_M$. Now $N$ is connected on 6 elements and thus $\dim P_N = 5$ but $P_N$ is not a simplex. \qed

4. Generation and psd rank

In this section we study two further face-hereditary properties of point configurations that are intimately related to Theta-1 configurations.

4.1. Degree of generation. For a point configuration $V \subset \mathbb{R}^d$, the vanishing ideal of $V$ is

$$I(V) := \{ f(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_d] : f(v) = 0 \text{ for all } v \in V \}.$$

We say that $V$ is of degree $\leq k$ if the ideal $I(V)$ has some set of generators of degree $\leq k$. We write $\text{Gen}(V) = k$ for the maximal degree in any minimal generating set for $I(V)$. We define

$$\mathcal{V}^\text{Gen}_k := \{ V \text{ point configuration} : \text{Gen}(V) \leq k \}$$

It is clear that $\text{Gen}(V)$ is an affine invariant and, since all point configurations are finite, we get

Proposition 4.1. The class $\mathcal{V}^\text{Gen}_k$ is face-hereditary.

Proof. Let $H = \{ p : \ell(p) = 0 \}$ be a supporting hyperplane for $V$. The vanishing ideal of $V' = V \cap H$ is the ideal generated by $I(V)$ and $\ell(\mathbf{x})$. Since $\ell(\mathbf{x})$ is linear, this then shows that $\text{Gen}(V') \leq \text{Gen}(V)$. \qed

The relation to point configurations of Theta rank 1 is given by the following proposition which is implicit in [GPT10].

Proposition 4.2. If $V \subset \mathbb{R}^d$ be a point configuration of Theta rank 1, then $\text{Gen}(V) \leq 2$.

Proof. By Theorem 1.1 we infer that the points $V$ are in convex position and the polytope $P = \text{conv}(V)$ is 2-level. We may assume that the configuration is spanning and hence up to affine equivalence, the polytope is given by

$$P = \{ p \in \mathbb{R}^d : -1 \leq \ell_i(p) \leq 1 \text{ for all } i = 1, \ldots, n \}$$

for unique linear functions $\ell_i(\mathbf{x}) = \sum c_i x_i$. We claim that $I(V)$ is generated by the quadrics $\ell_i(\mathbf{x})^2 - 1$ for $i = 1, \ldots, n$. First note that these polynomials span a real radical ideal. Indeed, by choosing appropriate coordinates it can be shown that the set $U = \{ p \in \mathbb{R}^d : \ell_i(p)^2 = 1, i = 1, \ldots, n \}$ is a subset of $[-1, +1]^d$ and in particular smooth and real. Since every $v \in V$ satisfies $\ell_i(v) = \pm 1$, we have $U \subseteq V$. Conversely, any $u \in U$ is a vertex of $P$ and thus contained in $V$. \qed

Example 8. To see that generation in degrees $\leq 2$ is necessary for Theta rank 1 but not sufficient, consider the planar point configuration $V = \{(1,0), (0,1), (2,0), (0,2)\}$. The configuration is clearly not 2-level and hence not Theta 1, however the vanishing ideal $I(V)$ is generated by $x_1x_2$ and $(x_1 + x_2 - 1)(x_1 + x_2 - 2)$ which implies $\text{Gen}(V) \leq 2$. 

![Example Diagram](https://example.com/example.png)
The vanishing ideals of base configurations are easy to write down explicitly.

**Proposition 4.3.** Let \( M = (E, B) \) be a matroid of rank \( r \). The vanishing ideal for \( V_M \) is generated by
\[
x_e^2 - x_e \text{ for all } e \in E, \quad \ell_E(x) - r, \quad x^C \text{ for all circuits } C \subset E.
\]

**Proof.** Any solution to the first two sets of equations is of the form \( 1_B \) for some \( B \subset E \) with \( |B| = r \). For the last set of equations, we note that \( (1_B)^C = 0 \) for all circuits \( C \) if and only if \( B \) does not contain a circuit. This is equivalent to \( B \in \mathcal{B} \). □

Let us write \( \mathcal{M}_k^{\text{Gen}} \) for the class of matroids \( M \) with \( \text{Gen}(V_M) \leq k \). The previous proposition is a little deceiving in the sense that it suggests a direct connection between the size of circuits and the degree of generation. This is not quite true. Indeed, let \( G = K_4 \setminus e \) be the complete graph on 4 vertices minus an edge. Then \( M(G) \) has a circuit of cardinality 4 but \( M(G) \in \mathcal{M}_2^{\text{ev}} \subseteq \mathcal{M}_2^{\text{Gen}} \) by Theorem 3.9 and Proposition 4.2. The main result of this section is that for base configurations the condition of Proposition 4.2 is also sufficient.

**Theorem 4.4.** Let \( M \) be a matroid. Then \( V_M \) is Theta 1 if and only if \( \text{Gen}(V_M) \leq 2 \).

**Proof.** In light of Proposition 4.2, we already know that \( \mathcal{M}_1^{\text{Th}} \subseteq \mathcal{M}_2^{\text{Gen}} \). Now, if \( M \in \mathcal{M}_2^{\text{Gen}} \setminus \mathcal{M}_1^{\text{Th}} \), then \( M \) has a minor isomorphic to \( M(K_4) \), \( P_6 \), \( Q_6 \), or \( W_3 \). Since \( \mathcal{M}_2^{\text{Gen}} \) is closed under taking minors, the following proposition yields a contradiction. □

**Proposition 4.5.** \( M(K_4) \), \( W_3 \), \( Q_6 \), and \( P_6 \) are not in \( \mathcal{M}_2^{\text{Gen}} \).

**Proof.** For a point configuration \( V \subset \mathbb{R}^n \), let \( I \subset \mathbb{R}[x_1, \ldots, x_n] \) be its vanishing ideal. If \( I \) is generated in degrees \( \leq k \), then so is any Gröbner basis of \( I \) with respect to a degree-compatible term order. The claim can now be verified by, for example, using the software Macaulay2 [GS]. □

**Psd rank and minimality.** Let \( S^m \subset \mathbb{R}^{m \times m} \) be the vector space of symmetric \( m \times m \) matrices. The **psd cone** is the closed convex cone \( S^m_+ = \{ A \in S^m : A \text{ positive semidefinite} \} \).

**Definition 4.6.** A polytope \( P \subset \mathbb{R}^d \) has a **psd-lift** of size \( m \) if there is a linear subspace \( L \subset S^m \) and a linear projection \( \pi : S^m \to \mathbb{R}^d \) such that \( P = \pi(S^m_+ \cap L) \). The **psd rank** \( \text{Psd}(P) \) is the size of a smallest psd-lift.

Psd-lifts together with lifts for more general cones were introduced by Gouveia, Parrilo, and Thomas [GPT13] as natural generalization of polyhedral lifts or extended formulations. Let us define \( \mathcal{V}_k^{\text{Psd}} \) as the class of point configurations \( V \) in convex position such that \( \text{conv}(V) \) has a psd-lift of size \( \leq k \). In [GRT13] it was shown that for a \( d \)-dimensional polytope \( P \) the psd rank is always \( \geq d + 1 \). A polytope \( P \) is called **psd-minimal** if \( \text{Psd}(P) = \dim P + 1 \). We write \( \mathcal{V}_k^{\text{Psd}} \) for the class of psd-minimal (convex position) point configurations.

**Proposition 4.7.** The classes \( \mathcal{V}_k^{\text{Psd}} \) and \( \mathcal{V}_k^{\text{Psd}} \) are face-hereditary.

**Proof.** Let \( V \in \mathcal{V}_k^{\text{Psd}} \) and let \( (L, \pi) \) be a psd-lift of \( P = \text{conv}(V) \). For a supporting hyperplane \( H \) we observe that \( (L \cap \pi^{-1}(H), \pi) \) is a psd-lift of \( P \cap H \) of size \( m \).

Let \( P \) be psd-minimal and let \( F = P \cap H \) a face of dimension \( \dim F = \dim P - 1 \). If \( F \) is not psd-minimal, then by [GRT13, Prop. 3.8], \( \text{Psd}(F) \geq \text{Psd}(F) + 1 > \dim F + 2 > \dim P + 1 \). □

A characterization of psd-minimal polytopes in small dimensions was obtained in [GRT13] and, in particular, the following relation was shown.

**Proposition 4.8.** Let \( V \) be a point configuration in convex position. If \( \text{Th}(V) = 1 \), then \( P = \text{conv}(V) \) is psd-minimal.
In [GRT13] an example of a psd-minimal polytope that is not 2-level is given, showing that the condition above is sufficient but not necessary. The main result of this section is that the situation is much better for base configurations.

**Theorem 4.9.** Let $M$ be a matroid. If the base polytope $P_M = \text{conv}(V_M)$ is psd-minimal, then $\text{Th}(M) = 1$.

In light of Proposition 4.8 it remains to show that there is no psd-minimal matroid $M$ with $\text{Th}(M) > 1$. Since $\mathcal{V}_{\text{psd}}^{\text{min}}$ is face-hereditary, it is sufficient to show that the excluded minors $M(K_4)$, $W^3$, $Q_6$, and $P_6$ are not psd-minimal.

In order to do so, we need to recall the connection to slack matrices and Hadamard square roots developed in [GRT13]. For a more coherent picture of the relations in particular to cone factorizations we refer to the papers [GPT13, GRT13]. Let $P$ be be a polytope with vertices $v_1, \ldots, v_t$ and facet-defining linear functions $\ell_j(x) = \beta_j - \langle a_j, x \rangle$ for $j = 1, \ldots, f$. The slack matrix of $P$ is the non-negative matrix $S_P \in \mathbb{R}^{t \times f}$ with $(S_P)_{ij} = \beta_j - \langle a_j, v_i \rangle$ for $i = 1, \ldots, t$ and $j = 1, \ldots, f$. A Hadamard square root of $S_P$ is a matrix $H \in \mathbb{R}^{t \times f}$ such that $(S_P)_{ij} = H_{ij}^2$ for all $i, j$. Moreover, we define $\text{rank}_{\sqrt{\cdot}} S_P$ as the smallest rank among all Hadamard square roots. The following is the main connection between Hadamard square roots and the psd-rank.

**Theorem 4.10 ([GRT13, Thm. 3.5]).** A polytope $P$ is psd-minimal if and only if $\text{rank}_{\sqrt{\cdot}} (S_P) = \dim P + 1$.

Thus, we will complete the proof of Theorem 4.9 by showing that the slack matrices for the forbidden minors have Hadamard square roots of rank $\geq 7$. We start with a technical result.

**Proposition 4.11.** The matrix

$$A_0 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

has $\text{rank}_{\sqrt{\cdot}} A_0 = 4$.

**Proof.** Every Hadamard square root of $A_0$ is of the form

$$H = \begin{pmatrix} 0 & y_1 & y_2 & y_3 \\ y_4 & 0 & y_5 & y_6 \\ y_7 & y_8 & 0 & y_9 \\ y_{10} & y_{11} & y_{12} & 0 \end{pmatrix}$$

with $y_i^2 = 1$, $i = 1, \ldots, 12$. Claiming that $\text{rank}_{\sqrt{\cdot}} A_0 = 4$ is equivalent to the claim that every Hadamard square root $H$ is non-singular. Using the computer algebra software Macaulay2 [GS] it can be checked that the ideal

$$I = \langle y_1^2 - 1, \ldots, y_{12}^2 - 1, \det H \rangle \subseteq \mathbb{C}[y_1, \ldots, y_{12}]$$

contains 1 which excludes the existence of a rank-deficient Hadamard square root.

**Proposition 4.12.** Let $P = P_M$ the base polytope for $M \in \{M(K_4), W^3, Q_6, P_6\}$. Then $\text{rank}_{\sqrt{\cdot}} (S_P) \geq 7$.

**Proof.** We explicitly give the argument for $M = M(K_4)$ and $P = P_M$. The other cases are analogous. It will be sufficient to find a $7 \times 7$-submatrix $A$ of $S_P$ with $\text{rank}_{\sqrt{\cdot}} (N) \geq 7$. Consider...
the following collection of bases and facets of $M$:

\[
B_1 = \{1, 2, 4\} \quad F_1 = \{1\} \\
B_2 = \{1, 2, 5\} \quad F_2 = \{2\} \\
B_3 = \{1, 2, 6\} \quad F_3 = \{3\} \\
B_4 = \{1, 3, 6\} \quad F_4 = \{4\} \\
B_5 = \{1, 4, 6\} \quad F_5 = \{5\} \\
B_6 = \{1, 5, 6\} \quad F_6 = \{6\} \\
B_7 = \{2, 4, 6\} \quad F_7 = \{3, 4, 5\}
\]

and the induced submatrix of $S_P$

\[
A = \begin{pmatrix}
\{1, 2, 4\} & \{2\} & \{3\} & \{4\} & \{5\} & \{3, 4, 5\} \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Then $\text{rank}_{\sqrt{\cdot}}(A) = 7$ if and only if

\[
\begin{vmatrix}
0 & 0 & \pm 1 & 0 & \pm 1 & \pm 1 \\
0 & 0 & \pm 1 & \pm 1 & 0 & \pm 1 \\
0 & \pm 1 & 0 & \pm 1 & 0 & 0 \\
0 & \pm 1 & \pm 1 & 0 & \pm 1 & 0 \\
0 & \pm 1 & \pm 1 & \pm 1 & 0 & 0 \\
\end{vmatrix} = \pm
\begin{vmatrix}
0 & \pm 1 & 0 & \pm 1 & \pm 1 \\
0 & \pm 1 & \pm 1 & 0 & \pm 1 \\
0 & \pm 1 & \pm 1 & \pm 1 & 0 \\
\end{vmatrix} \neq 0
\]

The last determinant is of the form $a + \sqrt{2}b$ for some integers $a, b$. To check that this determinant is nonzero, we can check that $b$ is nonzero. By Laplace expansion, this is the case if

\[
\begin{vmatrix}
0 & \pm 1 & \pm 1 \\
0 & \pm 1 & \pm 1 \\
\pm 1 & 0 & \pm 1 \\
\pm 1 & \pm 1 & \pm 1 \\
\pm 1 & \pm 1 & \pm 1 \\
\end{vmatrix} = \pm
\begin{vmatrix}
0 & \pm 1 & \pm 1 \\
\pm 1 & 0 & \pm 1 \\
\pm 1 & \pm 1 & \pm 1 \\
\end{vmatrix} \neq 0
\]

The latter is exactly the claim that the matrix $A_0$ of Proposition 4.11 has rank$_{\sqrt{\cdot}}(A_0) = 4$. □

5. Higher level graphs

In this section we study the class $\mathcal{G}_k^{\text{lev}}$ of $k$-level graphs for arbitrary $k$. The Robertson-Seymour theorem assures that the list of forbidden minors characterizing $\mathcal{G}_k^{\text{lev}}$ is finite and we give an explicit description in the next subsection. In Section 5.2, we focus on the class of 3-level graphs which is characterized by exactly one forbidden minor, the wheel $W_4$ with 4 spokes. The class of $W_4$-minor-free graphs was studied by Halin and we recover its building blocks from levelness considerations. In Section 5.3 we focus on the class of graphs with Theta rank 2. Forbidden minors for this class can be obtained from the structure of 4-level graphs.

5.1. Excluded minors for $k$-level graphs. A consequence of Theorem 3.9 is that a graph $G$ is 2-level if and only if $G$ does not have $K_4$ as a minor. In order to give a characterization of $k$-level graphs in terms of forbidden minors, we first need to view $K_4$ from a different angle.

**Definition 5.1.** The **cone** over a graph $G = (V, E)$ with apex $w \notin V$ is the graph $\text{cone}(G) = (V \cup \{w\}, E \cup \{wv : v \in V\})$. 
Let us denote by $C_n$ the $n$-cycle. Thus, we can view $K_4$ as the cone over $C_3$. As in the previous section, we only need to consider graphic matroids $M(G)$ which are connected. In terms of graph theory these correspond exactly to biconnected graphs. For a facet $F$ let us denote by $V_F \subseteq V$ the vertices covered by $F$.

**Proposition 5.2.** Let $G = (V, E)$ be a biconnected graph and $F \subseteq E$ a facet with $|E \setminus F| \geq 2$. Then $G|_{E \setminus F}$ is a vertex-induced subgraph.

**Proof.** By contradiction, suppose that $e \in E \setminus F$ is an edge with both endpoints in $V_F$. Since $F$ is a facet, $G/F$ is a biconnected graph with loop $e$. This contradicts $|E \setminus F| \geq 2$. □

The definition of facets requires the graph $G/F$ to be biconnected. This, in turn, implies that $G|_{E \setminus F}$ is connected. Let us write $C(F) := \{uv \in E : u \in V_F, v \notin V_F\}$ for the induced cut. Moreover, let us write $\overline{F} := E \setminus (F \cup C(F))$. The next result allows us to find $k$-level minors in a $k$-level graph.

**Lemma 5.3.** Let $G$ be a biconnected graph and $F$ a $k$-level facet. Then $F$ is a $k$-level facet of the graph $G/\overline{F}$.

**Proof.** Let $H = G/\overline{F}$. It follows from the definition of facets, that $G|_{\overline{F}}$ is connected and thus $H/F = G/(F \cup \overline{F})$ is biconnected. Moreover $H|_F = G|_F$ is biconnected and therefore $F$ is a facet of $H$.

For the levelness of $F$, observe that it cannot be bigger than $k$. Let $T_1 \subseteq E$ be a spanning tree such that the restriction to the connected graph $G|_{E \setminus F}$ is also a spanning tree. In particular, $|T_1 \cap F|$ is minimal among all spanning trees. It now suffices to show that there is a sequence of spanning trees $T_1, T_2, \ldots, T_k \subseteq E$ with $|T_i \cap F| = |T_1 \cap F| + i - 1$ for all $i = 1, \ldots, k$ and such that $T_i \cap \overline{F} = T_j \cap \overline{F}$ for all $i, j$. The contractions $T_i/\overline{F}$ then show that $F$ is at least $k$-level for $H$.

If $T_i \cap F$ is not a spanning tree for $G|_F$, then pick $e \in F \setminus T_i$ such that $e$ connects two connected components of $(V_F, T_i \cap F)$. Since $T_i$ is a spanning tree, there is a cycle in $T_i \cup e$ that uses at least one cut edge $f \in C(F) \cap T_i$. Hence $T_{i+1} = (T_i \setminus e) \cup f$ is the new spanning tree with the desired properties.

The contraction of $\overline{F}$ in $G$ gives a graph with vertices $V_F \cup \{w\}$, where $w$ results from the contraction of $\overline{F}$.

**Proposition 5.4.** Let $G = (V, E)$ be a simple, biconnected graph and let $w$ be a vertex such that the set of edges $F$ of $G - w$ is a facet. Then $F$ is $k$-level if and only if $\deg(w) \leq k$.

**Proof.** Let $E_w$ be the edges incident to $w$. For a spanning tree $T \subseteq E$, we have $\ell_F(1_T) = |F \cap T| = |T| - |E_w \cap T|$. Hence, $F$ is $k$-level if and only if there are at most $k$ spanning trees $T_1, \ldots, T_k$ such that every $T_i$ uses a different number of edges from $E_w$. Since every spanning tree of $G/F$ lifts to a spanning tree of $G$, it suffices to prove the result for $G/F$ which has two vertices and $m \leq k$ parallel edges. For such a graph, the claim is obvious. □

It follows from Proposition 5.4 that the cone over a biconnected graph on $k$ vertices has a $k$-level facet. The next result gives a strong converse to this observation. A graph $G$ is called **minimally biconnected** if $G \setminus e$ is not biconnected for all $e \in E$. For more background on this class of graphs we refer to [Plu68] and [Dir67].

**Proposition 5.5.** Let $G$ be a simple, biconnected graph with a vertex $w$ such that the set of edges $F$ not incident to $w$ is a facet. If $\text{Lev}(F) = k$, then $G$ has a minor cone($H$) where $H$ is a minimally biconnected graph on $k$ vertices.

**Proof.** Let $m = |V_F|$. By Proposition 5.4, $\deg(w) = k$ and thus $m \geq k$. By removing edges if necessary, we can assume that $F$ is minimally biconnected. By a result of Tutte (see [Oxl11, Thm. 4.3.1]) the contraction of any edge of $F$ leaves a biconnected graph. Contract an edge
such that at most one endpoint is connected to \( w \). The new edge set \( F' \) is still a \( k \)-level facet. By iterating these deletion-contraction steps, we obtain a cone over \( F' \) with apex \( w \).

**Theorem 5.6.** A graph \( G \) is \( k \)-level if and only if \( G \) has no minor \( \text{cone}(H) \) where \( H \) is a minimally biconnected graph on \( k + 1 \) vertices.

**Proof.** Let \( G = (V,E) \) be a graph and \( F \subseteq E \) a facet with levelness \( \text{Lev}(F) = m > k \). By Lemma 5.3, we may assume that \( F \) is the set of edges not incident to some \( w \in V \). By Proposition 5.5, we may also assume that \( G|_F \) is minimally biconnected on \( m \) vertices. Now, \( G|_F \) contains a minor \( H \) that is minimally biconnected on \( k + 1 \) vertices and hence \( G \) contains \( \text{cone}(H) \) as a minor. \( \square \)

5.2. **The class of 3-level graphs.** According to Theorem 5.6, the excluded minors for \( G_3^{\text{lev}} \) are cones over minimally biconnected graphs on 4 vertices. The only minimally biconnected graph on 4 vertices is the 4-cycle and hence the excluded minor is \( W_4 \). The only minimally biconnected graph on 4 vertices with exactly \( k \) spokes is \( C_4 \). In general, let us write \( \text{cone}(G) \). Proposition 5.5, we may also assume that \( G|_F \) is minimally biconnected on \( m \) vertices. Now, \( G|_F \) contains a minor \( H \) that is minimally biconnected on \( k + 1 \) vertices and hence \( G \) contains \( \text{cone}(H) \) as a minor. \( \square \)

We start with the observation that by Lemma 3.8 and Corollary 3.5, we may restrict to 3-connected, simple graphs. Recall that a graph \( G \) is 3-connected if every vertex is incident to exactly \( k \) edges.

**Proposition 5.7.** A 3-level, 3-connected simple graph is 3-regular.

**Proof.** A graph \( G \) with a vertex of degree at most 2 cannot be 3-connected. If there is a vertex \( w \) of degree at least 4, then \( G-w \) is biconnected. It follows that the set of edges \( F \) not incident to \( w \) form a facet and Proposition 5.4 yields the claim. \( \square \)

The following well-known result (see [Oxl11, Thm 8.8.4]) puts strong restrictions on minimally 3-connected matroids. A \( n \)-whirl is the matroid of the \( n \)-wheel \( W_n = \text{cone}(C_n) \) with the additional basis being the rim of the wheel \( B = E(C_n) \).

**Theorem 5.8 (Tutte’s wheels and whirl theorem).** Let \( M = (E,B) \) be a 3-connected matroid. Then the following are equivalent:

(i) For all \( e \in E \) neither \( M \setminus e \) nor \( M/e \) is 3-connected;
(ii) \( M \) is a \( n \)-whirl or \( n \)-wheel, for some \( n \).

We will come back to whirls in the next section. For now, we note that the only minimally 3-connected graphs are the wheels. Moreover note that every 3-regular simple graph must have an even number of vertices \( (3|V(G)| = 2|E(G)|) \).

**Lemma 5.9.** Let \( G \) be a 3-connected 3-regular simple graph with at least 6 vertices. Then \( G \) is at least 4-level.

**Proof.** By assumption \( G \) cannot be a wheel. By Theorem 5.8, there must be an edge \( e \) such that \( G \setminus e \) or \( G/e \) is 3-connected. Now, \( G \setminus e \) has a degree-2 vertex for all \( e \in E \) and hence is not 3-connected. On the other hand, \( G/e \) has either a degree-2 vertex (not counting multiple edges) or a vertex of degree 4 which contradicts Proposition 5.4. \( \square \)

**Corollary 5.10.** \( K_4 \) is the only 3-level, 3-connected simple graph.

The following gives a complete characterization of level 3 graphs.

**Theorem 5.11.** For a graph \( G \) the following are equivalent.

(i) \( G \) has no minor isomorphic to \( W_4 \);
(ii) \( G \) is 3-level;
(iii) \( G \) can be constructed from cycles, \( C_3^* \), and \( K_4 \) by taking direct or 2-sums.
Proof. (i) ⇔ (ii) is Theorem 5.6 together with the fact that $C_4$ is the unique minimally biconnected graph on 4 vertices. (ii) ⇒ (iii) follows from Corollary 5.10. (iii) ⇒ (ii) follows from Corollary 3.5 and (2.2).

By inspecting the building blocks for 2-level (Example 7) and 3-level graphs, it is tempting to think that the building blocks of $k$-level graphs are given by the building blocks and the forbidden minors of $G_{k-1}^{\text{lev}}$. This turns out to be false even for $G_{k}^{\text{lev}}$. Indeed Lev($K_5$) = 4 and we cannot obtain it as a sequence of direct sums and 2-sums of cycles, $C_4, K_4 = W_3$, and $W_4$.

5.3. 4-level and Theta-2 graphs. A further hope one could nourish is that 3-level graphs coincide with the graphs of Theta rank 2. This would be the case if and only if $\text{Th}(W_4) = 3$. The only facet $F$ of $W_n$ with $\text{Lev}(F) > 3$ is determined by the rim of the wheel $F = E(C_n)$. To find a sum-of-squares representation of $\ell_F(x)$ for the basis configuration $V_{M(W_n)}$ of $W_n$, we may project onto the coordinates of $F$ which coincides with the configuration of forests of $C_n$. Now, every subset of $E(C_n)$ is independent except for the complete cycle $I = E(C_n)$. Hence the configuration of forests is given by $\{0,1\}^n \setminus 1$ and the linear function in question is $\ell(x) = n - 1 - \sum_i x_i$. For $n = 4$, a sum-of-squares representation (1.1) of degree $\leq 2$ can be easily found computationally.

Towards a list of excluded minors for $G_2^\text{Th}$, we focus on the class of 4-level graphs. Using Theorem 5.6 we easily find the two excluded minors for $G_4^{\text{lev}}$:

The first graph is the 5-wheel $W_5$, the second graph is the cone over $K_{2,3}$ and is called $A_3 \setminus x$ in [Oxl89]. The next result states that this is the right class to study.

Proposition 5.12. The wheel $W_5$ has Theta rank 3.

Proof. Let $F = E(C_5)$ be the edges of the rim of the wheel which is a flat of rank 4. This is the unique facet of levelness 5 and it is sufficient to show that $4 - \ell_F(x)$ is not 2-sos with respect to the spanning trees $V = V_{M(W_5)}$ of $W_5$. Arguing by contradiction, let us suppose that there are polynomials $h_1(x), \ldots, h_m(x)$ of degree $\leq 2$ such that

$$f(x) := 4 - \ell_F(x) - h_1(x)^2 - \cdots - h_1(x)^2$$

is identically zero on $V$.

Consider the point $p = 1_F$. This is not a basis of $M(W_5)$ and a polynomial separating $p$ from $V$ is given by $f$. That is, by construction $f$ is a polynomial that vanishes on $V$ and $f(p) \leq -1 \neq 0$. Now we may compute a degree-compatible Gröbner basis of the vanishing ideal $I = I(V)$ using Macaulay2 [GS]. Evaluating the elements of the Gröbner basis at $p$ shows that the only polynomials not vanishing on $p$ are of degree 5. As $\deg(f) \leq 4$ by construction, this yields a contradiction.

The proof suggests an interesting connection to Tutte’s wheels and whirls theorem (Theorem 5.8): For $n = 4$ it states that the vanishing ideal of the $n$-wheel $I(W_n)$ is generated by $I(W^n)$ and a unique polynomial of degree $n$. This should be viewed in relation to Proposition 2.4: Projecting $V_{W^n}$ and $V_{W_n}$ onto the coordinates of $F = E(C_n)$ yields $\{0,1\}^n$ and $\{0,1\}^n \setminus 1$, respectively.

Oxley [Oxl89] determined that the class of 3-connected graphs not having $W_5$ as a minor consists of 17 individual graphs and 4 infinite families. The graph $A_3 \setminus x$ is clearly among these graphs and is a minor of the 4 infinite families as well as three further ones. This proves the following result.
Theorem 5.13. Every 4-level graph is obtained by direct and 2-sums of cycles, \( C_3^* \), and the following 14 graphs:

\[
K_4 \quad W_4 \quad K_{3,3} \quad K_5 \setminus e \quad (K_5 \setminus e)^* \\
K_5 \quad H_6 \quad A_3 \setminus \{x, y\} \quad Q_3 \quad J_1 \\
J_2 \quad K_{2,2,2} \quad H_7 \quad J_3
\]

As \( A_3 \setminus x \) is Theta-2, a complete list of excluded minor has to be extracted from the 17 graphs plus 4 families in [Oxl89]. As a last remark, we note that the Theta-1 graphs are given by series-parallel graphs. The property of begin Theta-2 however is independent of planarity.

Proposition 5.14. The graphs \( K_5 \) and \( K_{3,3} \) have Theta rank 2.

Proof. For both cases we use the idea that for a given facet \( F \subseteq E \), we may project the basis configuration \( V \) onto the coordinates given by \( F \) and find a 2-sos representation of the linear function rank\((F) - \sum_i x_i \).

For the graph \( K_{3,3} \), the only facets of levelness > 3 are given by 4-cycles. Projecting onto these coordinates yields \( \{0,1\}^4 \setminus 1 \) which is a point configuration of Theta rank 2.

For the complete graph \( K_5 \), we note that the only facets \( F \) of levelness > 3 are given by the edges of an embedded \( K_4 \). For such a facet, we might equivalently consider \( \ell_{E \setminus F}(x) - 1 \geq 0 \).

Projecting onto \( E \setminus F \) again yields \( \{0,1\}^4 \setminus 0 \). \( \Box \)

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