On Evolution Equations for Marginal Correlation Operators

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Abstract. This paper is devoted to the problem of the description of nonequilibrium correlations in quantum many-particle systems. The nonlinear quantum BBGKY hierarchy for marginal correlation operators is rigorously derived from the von Neumann hierarchy for correlation operators that give an alternative approach to the description of states in comparison with the density operators. A nonperturbative solution of the Cauchy problem of the nonlinear quantum BBGKY hierarchy for marginal correlation operators is constructed.

Key words: nonlinear quantum BBGKY hierarchy; von Neumann hierarchy; correlation operator; density matrix; quantum many-particle system.

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1 Introduction

The importance of the mathematical description of correlations in numerous problems of the modern statistical mechanics is well-known. Among them in particular, we refer to such fundamental problems as the problem of quantum measurements and of a description of collective behavior of interacting particles by quantum kinetic equations [1–10]. Owing to the intrinsic complexity and richness of these problems, primarily it is necessary to develop an adequate mathematical theory of underlying evolution equations.

The goal of this paper is to derive rigorously the evolution equations for marginal correlation operators that give an equivalent approach to the description of the evolution of states in comparison with marginal density operators governed by the quantum BBGKY hierarchy and to construct a solution of the corresponding Cauchy problem.

We briefly outline the results and structure of the paper. In introductory section 2 we set forth an approach to the description of the evolution of states of quantum many-particle systems within the framework of correlation operators governed by the von Neumann hierarchy [11,12]. In section 3 we introduce the notion of marginal correlation operators. To justify this notion, we discuss in detail the motivation of the description of states within the framework of marginal correlation operators or in other words, the origin of the microscopic description of correlations in quantum many-particle systems. Then we rigorously derive the nonlinear quantum BBGKY hierarchy for marginal correlation operators from the von Neumann hierarchy for correlation...
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operators. The nonlinear quantum BBGKY hierarchy gives an alternative method of the description of the evolution of states of infinitely many particles in comparison with the quantum BBGKY hierarchy for the marginal density operators \([13, 14]\). In section 4 we construct the nonperturbative solution of the Cauchy problem of the nonlinear quantum BBGKY hierarchy.

In section 5 we conclude with some observations and perspectives for future research in the light of the results we present here.

2 Preliminary facts: the von Neumann hierarchy

We consider a quantum system of a non-fixed, i.e. arbitrary but finite, number of identical (spinless) particles with unit mass \(m\). We adopt the usual convention that \(H\) is a self-adjoint operator with the domain \(D\) of \(H\) of the \(n\)-particle Hilbert space \(H_n\), where \(H_n\) is a tensor product of \(n\) Hilbert spaces \(H\) and we adopt the usual convention that \(H_0 = \mathbb{C}\). The Hamiltonian \(H_n\) of the \(n\)-particle system is a self-adjoint operator with the domain \(\mathcal{D}(H_n) \subset H_n\)

\[
H_n = \sum_{i=1}^{n} K(i) + \sum_{i_1 < i_2 = 1}^{n} \Phi(i_1, i_2),
\]

where \(K(i)\) is the operator of a kinetic energy of the \(i\) particle and \(\Phi(i_1, i_2)\) is the operator of a two-body interaction potential. In particular on functions \(\psi_n\) that belong to the subspace \(L^2_0(\mathbb{R}^m) \subset \mathcal{D}(H_n) \subset L^2(\mathbb{R}^m)\) of infinitely differentiable symmetric functions with compact supports the operator \(K(i)\) acts according to the formula: \(K(i)\psi_n = -\frac{\hbar^2}{2m} \Delta \psi_n\), where \(2\pi\hbar\) is a Planck constant, and for the operator \(\Phi\) we have: \(\Phi(i_1, i_2)\psi_n = \Phi(q_{i_1}, q_{i_2})\psi_n\), respectively.

We assume that the function \(\Phi(q_{i_1}, q_{i_2})\) is symmetric with respect to permutations of arguments and it is translation-invariant bounded function.

States of a system of the Maxwell-Boltzmann particles belong to the space \(\mathfrak{L}^1(\mathcal{F}_H) = \bigoplus_{n=0}^{\infty} \mathfrak{L}^1(\mathcal{H}_n)\) of sequences \(f = (f_0, f_1, \ldots, f_n, \ldots)\) of trace-class operators \(f_n \equiv f_n(1, \ldots, n) \in \mathfrak{L}^1(\mathcal{H}_n)\) and \(f_0 \in \mathbb{C}\), that satisfy the symmetry condition: \(f_n(1, \ldots, n) = f_n(i_1, \ldots, i_n)\) for arbitrary \((i_1, \ldots, i_n) \in (1, \ldots, n)\), equipped with the norm

\[
\|f\|_{\mathfrak{L}^1(\mathcal{F}_H)} = \sum_{n=0}^{\infty} \|f_n\|_{\mathfrak{L}^1(\mathcal{H}_n)} = \sum_{n=0}^{\infty} \text{Tr}_{1, \ldots, n} |f_n(1, \ldots, n)|,
\]

where \(\text{Tr}_{1, \ldots, n}\) are partial traces over 1, \ldots, \(n\) particles \([14]\). We denote by \(\mathfrak{L}^0_0(\mathcal{F}_H)\) the everywhere densely set in \(\mathfrak{L}^1(\mathcal{F}_H)\) of finite sequences of degenerate operators with infinitely differentiable kernels with compact supports.

We describe states of a system by means of sequences \(g(t) = (g_0, g_1(t, 1), \ldots, g_n(t, 1, \ldots, n), \ldots) \in \mathfrak{L}^1(\mathcal{F}_H)\) of the correlation operators \(g_n(t), n \geq 1\). The evolution of all possible states is
d\frac{ds}{dt}g_s(t, Y) = \mathcal{N}(Y \mid g(t)),
\quad \text{where the following notations are used:}
\mathcal{N}(Y \mid g(t)) = -\mathcal{N}_s(Y)g_s(t, Y) + \sum_{P: Y = X_1 \cup X_2} \sum_{i_1 \in X_1} \sum_{i_2 \in X_2} (-\mathcal{N}_\text{int}(i_1, i_2))g_{|X_1|(t, X_1)}g_{|X_2|}(t, X_2),
\quad \sum_{P: Y = X_1 \cup X_2} \text{is the sum over all possible partitions } P \text{ of the set } Y \equiv (1, \ldots, s) \text{ into two nonempty mutually disjoint subsets } X_1 \subset Y \text{ and } X_2 \subset Y,
\quad \text{the operator } (-\mathcal{N}_s) \text{ is defined on } \mathcal{L}^1_0(\mathcal{H}_s) \text{ by the formula}
(-\mathcal{N}_s(Y))f_s \equiv -\frac{i}{\hbar}(H_sf_s - f_sH_s),
\quad \text{is the generator of the von Neumann equation } [15] \text{ and the operator } (-\mathcal{N}_\text{int}) \text{ is defined by}
(-\mathcal{N}_\text{int}(i_1, i_2))f_s \equiv -\frac{i}{\hbar}(\Phi(i_1, i_2)f_s - f_s\Phi(i_1, i_2)).

Hereafter we use the following notations: \{\{X_1\}, \ldots, \{X_{|P|}\}\} \text{ is a set, elements of which are } |P| \text{ mutually disjoint subsets } X_i \subset Y \equiv (1, \ldots, s) \text{ of the partition } P : Y = \bigcup_{i=1}^{|P|} X_i, \text{ i.e. } |\{\{X_1\}, \ldots, \{X_{|P|}\}\}| = |P|. \text{ In view of these notations we state that } (\{Y\}) \text{ is the set consisting of one element } Y = (1, \ldots, s) \text{ of the partition } P (|P| = 1) \text{ and } |\{(Y)\}| = 1. \text{ We introduce the declusterization mapping } \theta : \{(X_1), \ldots, \{X_{|P|}\}\} \to Y, \text{ by the following formula: } 
\theta(\{X_1\}, \ldots, \{X_{|P|}\}) = Y. \text{ For example, let } X \equiv (1, \ldots, s + n), \text{ then for the set } (\{Y\}, X \setminus Y) \text{ it holds: } \theta(\{Y\}, X \setminus Y) = X.

On the space \mathcal{L}^1(\mathcal{H}_n) \text{ we also introduce the mapping: } \mathbb{R} \ni t \mapsto \mathcal{G}_n(-t)f_n, \text{ which is generated by the solution of the von Neumann equation of } n \text{ particles } [15, 16]
\mathcal{G}_n(-t)f_n \equiv e^{-\frac{i}{\hbar}tH_n}f_ne^{\frac{i}{\hbar}tH_n}.\text{(6)}

This mapping is an isometric strongly continuous group that preserves positivity and self-adjointness of operators [15]. On \mathcal{L}^1_0(\mathcal{H}_n) \subset \mathcal{D}(-\mathcal{N}_n) \text{ the infinitesimal generator of group } (6) \text{ is determined by operator } (4).

A solution of the Cauchy problem (1)-(2) is given by the following expansion [11-12, 17]
g_s(t, Y) = \mathcal{G}(t; Y|g(0)) \equiv \sum_{P: Y = \bigcup X_i} \mathfrak{A}_{|P|}(t, \{X_1\}, \ldots, \{X_{|P|}\}) \prod_{X_i \subset P} g_{|X_i|}(0, X_i), \quad s \geq 1,
\quad \text{where } \sum_{P: Y = \bigcup X_i} \text{ is the sum over all possible partitions } P \text{ of the set } Y \equiv (1, \ldots, s) \text{ into } |P| \text{ nonempty mutually disjoint subsets } X_i \subset Y, \text{ the evolution operator } \mathfrak{A}_{|P|}(t) \text{ is the } |P|\text{th-order
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The evolution of states of infinite-particle quantum systems is traditionally described by the marginal (or s-particle) density operators governed by the quantum BBGKY hierarchy [13,14]. In this section we introduce the marginal correlation operators that give an equivalent approach to the description of the evolution of such states and describe the nonequilibrium correlations in quantum systems. We also rigorously derive the nonlinear quantum BBGKY hierarchy for marginal correlation operators from the von Neumann hierarchy [11] for correlation operators.
3.1 Marginal correlation operators and marginal density operators

In the capacity of an example of a mean-value functional of observables [12] we consider the definition of the mean-value functional of the additive-type observable $A^{(1)} = (0, a_1(1), \ldots, \sum_{n=1}^\infty a_1(i_1), \ldots)$

$$\langle A^{(1)}(t) \rangle = \frac{1}{n!} \sum_{n=0}^\infty \langle a_1(1)g_{1+n}(t, 1, \ldots, 1+n) \rangle,$$

(12)

where the operators $g_{1+n}(t), n \geq 0$, are determined by expansions (7), and the functional of the dispersion for this type of observables

$$\langle (A^{(1)} - \langle A^{(1)} \rangle)^2 \rangle(t) =$$

$$= \sum_{n=0}^\infty \frac{1}{n!} \langle a_1(1) - \langle A^{(1)} \rangle^2(t) \rangle g_{1+n}(t, 1, \ldots, 1+n) +$$

$$+ \sum_{n=0}^\infty \frac{1}{n!} \langle a_1(1) a_1(2) g_{2+n}(t, 1, \ldots, 2+n) \rangle.$$

(13)

For $A^{(1)} \in \mathcal{L}(\mathcal{F}_H)$ and $g \in \mathcal{L}^1(\mathcal{F}_H)$ functionals (12), (13) exists.

Following to formula (13), we introduce the marginal correlation operators by the series

$$G_s(t, 1, \ldots, s) = \sum_{n=0}^\infty \frac{1}{n!} \langle a_1(1) a_1(2) \rangle g_{s+n}(t, 1, \ldots, s+n) = \sum_{n=0}^\infty \frac{1}{n!} \langle a_1(1) a_1(2) g_{2+n}(t, 1, \ldots, 2+n) \rangle.$$

(14)

where the sequence $g_{s+n}(t, 1, \ldots, s+n), n \geq 0$, is a solution of the Cauchy problem of the von Neumann hierarchy (11). According to estimate (9), series (14) exists and the estimate holds: $\|G_s(t)\|_{\mathcal{L}(\mathcal{F}_H)} \leq s!(2e)^s c^s \sum_{n=0}^\infty (2e)^n c^n$. Thus, macroscopic characteristics of fluctuations of observables are determined by marginal correlation operators (14) on the microscopic level

$$\langle (A^{(1)} - \langle A^{(1)} \rangle)^2 \rangle(t) = \langle a_1(1) a_1(2) \rangle + \langle a_1(1) a_1(2) g_{1+n}(t, 1, \ldots, 1+n) \rangle +$$

$$+ \langle a_1(1) a_1(2) g_{2+n}(t, 1, \ldots, 2+n) \rangle.$$

(15)

Traditionally marginal correlation operators are introduced by means of the cluster expansions of the marginal density operators $F_s(t), s \geq 1$, governed by the quantum BBGKY hierarchy (13)

$$F_s(t, Y) = \sum_{P: Y = \bigcup_i X_i} \prod_{X_i \subset P} \langle X_i \rangle(t, X_i), \quad s \geq 1,$$

where $\sum_{P: Y = \bigcup_i X_i}$ is the sum over all possible partitions $P$ of the set $Y \equiv (1, \ldots, s)$ into $|P|$ nonempty mutually disjoint subsets $X_i \subset Y$. Hereupon solutions of cluster expansions (15)

$$G_s(t, Y) = \sum_{P: Y = \bigcup_i X_i} (-1)^{|P|-1} (|P|-1)! \prod_{X_i \subset P} \langle X_i \rangle(t, X_i), \quad s \geq 1,$$

(16)

are interpreted as the operators that describe correlations of many-particle systems. Thus, marginal correlation operators (16) are cumulants (semi-invariants) of the marginal density operators.
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As follows from formula (12) and its generalization [12] the marginal density operators \( F_s(t) \) are defined in terms of the correlation operators of clusters of particles \( g^{(s)}(t) = (g_{1+0}(t, \{ Y \}), \ldots, g_{1+n}(t, \{ Y \}, s + 1, \ldots, s + n), \ldots) \) by the expansion

\[
F_s(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} g_{1+n}(t, \{ Y \}, s + 1, \ldots, s + n), \quad s \geq 1, \quad (17)
\]

where the sequence \( g_{1+n}(t, \{ Y \}, s + 1, \ldots, s + n), n \geq 0, \) is a solution of the Cauchy problem of the von Neumann hierarchy for correlation operators of particle clusters [12], namely

\[
g_{1+n}(t, \{ Y \}, X \setminus Y) = G(t; \{ Y \}, X \setminus Y | g(0)) = \sum_{\mathcal{P}: (|Y|, X \setminus Y) = \bigcup_{i} X_i} \mathcal{A}_{|\mathcal{P}|}(t, \{ \theta(X_1) \}, \ldots, \{ \theta(X_{|\mathcal{P}|}) \}) \prod_{X_i \subset \mathcal{P}} g_{|X_i|}(0, X_i), \quad s \geq 1, n \geq 0, \quad (18)
\]

where \( \mathcal{A}_{|\mathcal{P}|}(t) \) is the \(|\mathcal{P}|\)th-order cumulant defined by formula (8). According to estimate (9), series (17) exists and the estimate holds: \( \|F_s(t)\|_{\mathcal{L}^1(\mathcal{H}_s)} \leq e^3 c \sum_{n=0}^{\infty} e^{3n} c^n \). We note that every term of marginal correlation operator expansion (14) is determined by the \((s + n)\)-particle correlation operator (7) as contrasted to marginal density operator expansion (17) which is defined by the \((1 + n)\)-particle correlation operator (18).

The correlation operators of particle clusters \( g^{(s)}(t) = (g_{1+0}(t, \{ Y \}), \ldots, g_{1+n}(t, \{ Y \}, X \setminus Y), \ldots) \in \mathcal{L}^1(\oplus_{n=0}^{\infty} \mathcal{H}_{s+n}) \) can be expressed in terms of correlation operators of particles (7)

\[
g_{1+n}(t, \{ Y \}, X \setminus Y) = \sum_{\mathcal{P}: (|Y|, X \setminus Y) = \bigcup_{i} X_i} (-1)^{|\mathcal{P}|-1}(|\mathcal{P}| - 1)! \prod_{X_i \subset \mathcal{P}} \sum_{\theta(X_i) = \bigcup_j Z_{ji}} \prod_{Z_{ij} \subset \mathcal{P}^\prime} g_{|Z_{ji}|}(t, Z_{ji}). \quad (19)
\]

In particular case \( n = 0 \), i.e. the correlation operator of a cluster of \(|Y|\) particles, these relations take the form

\[
g_{1+0}(t, \{ Y \}) = \sum_{\mathcal{P}: Y = \bigcup_{i} X_i} \prod_{X_i \subset \mathcal{P}} g_{|X_i|}(t, X_i).
\]

By the way we observe on that cluster expansions (15) follow from definitions (14) and (17) in consequence of relations (19) between correlation operators of particle clusters and correlation operators of particles.

The marginal \((s\text{-particle})\) density operators (17) are determined by the Cauchy problem of the quantum BBGKY hierarchy [13]

\[
\frac{d}{dt} F_s(t, Y) = -\mathcal{N}_s(Y) F_s(t, Y) + \sum_{i \in Y} \text{Tr}_{s+1}(-\mathcal{N}_\text{int}(i, s + 1)) F_{s+1}(t), \quad (20)
\]

\[
F_s(t) \big|_{t=0} = F_s(0), \quad s \geq 1. \quad (21)
\]

We remind that usually the marginal density operators \( F_s(t), s \geq 1 \), are defined by the well-known formula of the nonequilibrium grand canonical ensemble [18, 19] in terms of the density
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operators \( D = (I, D_1(t), \ldots, D_n(t), \ldots) \) governed by the von Neumann equations (the quantum Liouville equation)

\[
F_s(t, Y) = (I, D(t))^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} D_{s+n}(t, X),
\]

where \( (I, D(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1,\ldots,n} D_n(t) \) is a normalizing factor, \( I \) is the identity operator and \( Y \equiv (1, \ldots, s), X \equiv (1, \ldots, s + n) \). Thus, along with the definition within the framework of the non-equilibrium grand canonical ensemble the marginal density operators can be defined within the framework of dynamics of correlations that allows to give the rigorous meaning of the states for more general classes of operators than the trace-class operators.

If \( F(0) \in \mathcal{L}_1^1(\mathcal{F}_\mathcal{H}) = \bigoplus_{n=0}^{\infty} \alpha_n \mathcal{L}_1^1(\mathcal{H}_n) \) and \( \alpha > e \), then for \( t \in \mathbb{R} \) a unique solution of the Cauchy problem \((20) - (21)\) of the quantum BBGKY hierarchy exists and is given by the expansion \((11), (20)\)

\[
F_s(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \mathfrak{A}_{1+n}(t, \{Y\}, X\setminus Y) F_{s+n}(0, X), \quad s \geq 1,
\]

(22)

where the \((1 + n)\)th-order cumulant \( \mathfrak{A}_{1+n}(t) \) of groups of operators \((0)\) is defined by

\[
\mathfrak{A}_{1+n}(t, \{Y\}, X\setminus Y) = \sum_{P:((Y), X\setminus Y)=\bigcup_i X_i} (-1)^{|P|-1}(|P| - 1)! \prod_{X_i \in P} G_{\theta(X_i)}(-t, \theta(X_i)),
\]

(23)

\( \sum_P \) is the sum over all possible partitions \( P \) of the set \((\{Y\}, X\setminus Y)\) into \(|P|\) nonempty mutually disjoint subsets \( X_i \subset (\{Y\}, X\setminus Y) \).

Formally, the evolution equations for marginal correlation operators are derived from the quantum BBGKY hierarchy for marginal density operators \((20)\) on basis of expression \((16)\). Then the evolution of all possible states of quantum many-particle systems obeying the Maxwell-Boltzmann statistics with the Hamiltonian \((11)\) can be described within the framework of marginal correlation operators governed by the nonlinear quantum BBGKY hierarchy

\[
\frac{d}{dt} G_s(t, Y) = \mathcal{N}(Y | G(t)) + \text{Tr}_{s+1} \sum_{i \in Y} (-\mathcal{N}_{\text{int}}(i, s + 1)) (G_{s+1}(t, Y, s + 1) +
\]

(24)

\[
+ \sum_{P : (Y, s + 1) = X_1 \cup X_2, i \in X_1; s + 1 \in X_2} G_{|X_1|}(t, X_1) G_{|X_2|}(t, X_2),
\]

\[
G_s(t, Y) |_{t=0} = G_s(0, Y), \quad s \geq 1.
\]

(25)

In equation \((24)\) the operator \( \mathcal{N}(Y | G(t)) \) is generator of the von Neumann hierarchy \((11)\) defined by formula \((3)\), i.e.

\[
\mathcal{N}(Y | G(t)) \doteq (-\mathcal{N}_{\text{c}}(Y)) G_s(t, Y) +
\]

(26)

\[
+ \sum_{P : Y = X_1 \cup X_2} \sum_{i_1 \in X_1} \sum_{i_2 \in X_2} (-\mathcal{N}_{\text{int}}(i_1, i_2)) G_{|X_1|}(t, X_1) G_{|X_2|}(t, X_2),
\]
where the operators \((-N_s)\) and \((-N_{\text{int}})\) are defined by formulas \((4)\) and \((5)\) respectively, \(\sum_{P: Y = X_1 \cup X_2} \) is the sum over all possible partitions \(P\) of the set \(Y \equiv (1, \ldots, s)\) into two nonempty mutually disjoint subsets \(X_1 \subset Y\) and \(X_2 \subset Y\), and \(\sum_{P: (Y, s+1) = X_1 \cup X_2}\) is the sum over all possible partitions of the set \((Y, s+1)\) into two mutually disjoint subsets \(X_1\) and \(X_2\) such that \(i\)th particle belongs to the subset \(X_1\) and \((s+1)\)th particle belongs to \(X_2\). As far as we know hierarchy \((24)\) was introduced by M.M. Bogolyubov \([13]\) and in the papers of J. Yvon \([21]\) and M.S. Green \([22]\) for systems of classical particles.

Another method of the justification of evolution equations for marginal correlation operators consists in their derivation from the von Neumann hierarchy for correlation operators \((1)\) on the basis of definition \((14)\).

### 3.2 The derivation of the nonlinear quantum BBGKY hierarchy

In this section we establish that marginal correlation operators \((14)\) are governed by the nonlinear quantum BBGKY hierarchy \((24)\). With this aim we differentiate by time variable the marginal correlation operators defined by series \((14)\) in the sense of the pointwise convergence on the space \(\mathfrak{B}^1(\mathcal{H}_n)\)

\[
\frac{d}{dt} G_s(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \left( (-N_{s+n}(X)) g_{s+n}(t, X) + \sum_{P: X = X_1 \cup X_2} \sum_{i_1 \in X_1} \sum_{i_2 \in X_2} (-N_{\text{int}}(i_1, i_2)) g_{|X_1|}(t, X_1) g_{|X_2|}(t, X_2) \right),
\]

where we use the notations: \(X \equiv (1, \ldots, s+n)\), \(Y \equiv (1, \ldots, s)\), \(\sum_{P: Y = X_1 \cup X_2}\) is the sum over all possible partitions \(P\) of the set \(Y \equiv (1, \ldots, s)\) into two nonempty mutually disjoint subsets \(X_1 \subset Y\) and \(X_2 \subset Y\), and operators \((-N_{s+n})\) and \((-N_{\text{int}})\) are defined by formulas \((4)\) and \((5)\) respectively. Taking into account the equality

\[-N_{s+n}(X) = -N_s(Y) - N_n(Y \setminus Y) + \sum_{i_1 \in Y} \sum_{i_2 \in X \setminus Y} (-N_{\text{int}}(i_1, i_2)),\]

and the identity

\[\text{Tr}_{s+1, \ldots, s+n} (-N_n(Y \setminus Y)) g_{s+n}(t) = 0,\]

and according to the symmetry property of the correlation operators \(g_{s+n}(t, X)\), for the first term in the right-hand side of identity \((26)\) in terms of definition \((14)\) we obtain

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} (-N_{s+n}(X)) g_{s+n}(t, 1, \ldots, s+n) =
\]

\[= -N_s(Y) G_s(t, Y) + \sum_{n=1}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \sum_{i_1 \in Y} \sum_{i_2 \in X \setminus Y} (-N_{\text{int}}(i_1, i_2)) g_{s+n}(t, X) =
\]

\[= -N_s(Y) G_s(t, Y) + \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n+1} \sum_{i_1 \in Y} (-N_{\text{int}}(i_1, s+1)) g_{s+n+1}(t, X, s+n+1) =
\]
Taking into account that the equality is true
\[ g \sum \left(-N_{\text{int}}(i_1, i_2)\right) G_{s+1}(t, 1, \ldots, s + 1). \]

Let us consider successively the following four parts of the second term of the right-hand side of identity (26)

\[
\sum_{P: X = X_1 \cup X_2} \sum_{i_1 \in X_1} \sum_{i_2 \in X_2} (-N_{\text{int}}(i_1, i_2)) = \sum_{P: X = X_1 \cup X_2} \left( \sum_{i_1 \in Y \cap X_1} \sum_{i_2 \in Y \cap X_2} + \sum_{i_1 \in X \cap Y \cap X_1} \sum_{i_2 \in X \cap Y \cap X_2} \right) (-N_{\text{int}}(i_1, i_2)).
\]

Taking into account that the equality is true
\[
\sum_{P: X = X_1 \cup X_2, Y \cap X_1 \neq \emptyset, Y \cap X_2 \neq \emptyset} g_{(X_1)}(t, X_1) g_{(X_2)}(t, X_2) =
\sum_{P: Y = Y_1 \cup Y_2, Y_1 \neq \emptyset, Y_2 \neq \emptyset} g_{Y_1+|Z|}(t, Y_1, Z) g_{Y_2+|(X \setminus Y)|}(t, Y_2, (X \setminus Y) \setminus Z),
\]

and the validity of the following equality (according to the symmetry property of operators \(g_n(t)\))
\[
\text{Tr}_{s+1, \ldots, s+n} \sum_{Z \subset X \setminus Y} g_{Y_1+|Z|}(t, Y_1, Z) g_{Y_2+|(X \setminus Y)|}(t, Y_2, (X \setminus Y) \setminus Z) =
\text{Tr}_{s+1, \ldots, s+n} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} g_{Y_1+|Z|}(t, Y_1, Z) g_{Y_2+|(X \setminus Y)|}(t, Y_2, (X \setminus Y) \setminus Z),
\]

for the first part of equality (28) of the second term of identity (26) it holds
\[
\text{Tr}_{s+1, \ldots, s+n} \sum_{P: X = X_1 \cup X_2} \sum_{i_1 \in Y \cap X_1} \sum_{i_2 \in Y \cap X_2} (-N_{\text{int}}(i_1, i_2)) g_{(X_1)}(t, X_1) g_{(X_2)}(t, X_2) =
\text{Tr}_{s+1, \ldots, s+n} \sum_{P: Y = Y_1 \cup Y_2} \sum_{i_1 \in Y_1 \cup Y_2} \sum_{i_2 \in Y_1 \cup Y_2} (-N_{\text{int}}(i_1, i_2)) \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \times
g_{Y_1+|Z|}(t, Y_1, Z) g_{Y_2+|(X \setminus Y)|}(t, Y_2, (X \setminus Y) \setminus Z).
\]

Then in terms of definition (14) the last expression takes the form
\[
\sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \sum_{P: Y = Y_1 \cup Y_2} \sum_{i_1 \in Y_1 \cup Y_2} \sum_{i_2 \in Y_1 \cup Y_2} (-N_{\text{int}}(i_1, i_2)) \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \times
g_{Y_1+|Z|}(t, Y_1, Z) g_{Y_2+|(X \setminus Y)|}(t, Y_2, (X \setminus Y) \setminus Z).
\]
Hence for the first part of equality (28) of the second term of the right-hand side of identity (26) we have
\[
\sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,...,s+n} \sum_{P:X=X_1 \cup X_2} \sum_{i_1 \in X \cap X_1} \sum_{i_2 \in X \cap X_2} (-\mathcal{N}_{\text{int}}(i_1, i_2)) g_{|X_1|}(t, X_1) g_{|X_2|}(t, X_2) = \tag{29}
\]
\[
= \sum_{P:Y=Y_1 \cup Y_2} \sum_{i_1 \in Y_1} \sum_{i_2 \in Y_2} (-\mathcal{N}_{\text{int}}(i_1, i_2)) G_{|Y_1|}(t, Y_1) G_{|Y_2|}(t, Y_2).
\]

Similarly the second and third parts of equality (28) of the second term of the right-hand side of identity (26) are expressed in terms of definition (14) in the form
\[
\sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,...,s+n} \sum_{P:X=X_1 \cup X_2} \left( \sum_{i_1 \in X \cap X_1} \sum_{i_2 \in X \cap X_2} + \right) (-\mathcal{N}_{\text{int}}(i_1, i_2)) g_{|X_1|}(t, X_1) g_{|X_2|}(t, X_2) = \tag{30}
\]
\[
= \sum_{i \in Y} \text{Tr}_{s+1} (-\mathcal{N}_{\text{int}}(i, s + 1)) \sum_{P: (Y, s + 1) = X_1 \cup X_2, i \in X_1; s + 1 \in X_2} G_{|X_1|}(t, X_1) G_{|X_2|}(t, X_2),
\]
where \( \sum_{P: (Y, s + 1) = X_1 \cup X_2, i \in X_1; s + 1 \in X_2} \) is the sum over all possible partitions of the set \((Y, s + 1)\) into two mutually disjoint subsets \(X_1\) and \(X_2\) such that \(i\)th particle belongs to the subset \(X_1\) and \((s + 1)\)th particle belongs to \(X_2\).

Then taking into account the following identity for the fourth part (28) of the second term of (26)
\[
\sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,...,s+n} \sum_{P:X=X_1 \cup X_2} \sum_{i_1 \in X \cap X_1} \sum_{i_2 \in X \cap X_2} (-\mathcal{N}_{\text{int}}(i_1, i_2)) g_{|X_1|}(t, X_1) g_{|X_2|}(t, X_2) = 0,
\]
and identities (29), (30), for the second term of the right-hand side of (26) it holds
\[
\sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,...,s+n} \sum_{P:X=X_1 \cup X_2} \sum_{i_1 \in X \cap X_1} \sum_{i_2 \in X \cap X_2} (-\mathcal{N}_{\text{int}}(i_1, i_2)) g_{|X_1|}(t, X_1) g_{|X_2|}(t, X_2) = \tag{31}
\]
\[
= \sum_{P:Y=Y_1 \cup Y_2} \sum_{i_1 \in Y_1} \sum_{i_2 \in Y_2} (-\mathcal{N}_{\text{int}}(i_1, i_2)) G_{|Y_1|}(t, Y_1) G_{|Y_2|}(t, Y_2) +
\]
\[
+ \sum_{i \in Y} \text{Tr}_{s+1} (-\mathcal{N}_{\text{int}}(i, s + 1)) \sum_{P: (Y, s + 1) = X_1 \cup X_2, i \in X_1; s + 1 \in X_2} G_{|X_1|}(t, X_1) G_{|X_2|}(t, X_2).
\]

In consequence of identities (27) and (31) we finally derive
\[
\frac{d}{dt} G_s(t, Y) = -\mathcal{N}_s(Y) G_s(t, Y) + \sum_{P:Y=Y_1 \cup Y_2} \sum_{i_1 \in Y_1} \sum_{i_2 \in Y_2} (-\mathcal{N}_{\text{int}}(i_1, i_2)) G_{|Y_1|}(t, Y_1) G_{|Y_2|}(t, Y_2) +
\]
\[
+ \sum_{i \in Y} \text{Tr}_{s+1} (-\mathcal{N}_{\text{int}}(i, s + 1)) \sum_{P: (Y, s + 1) = X_1 \cup X_2, i \in X_1; s + 1 \in X_2} G_{|X_1|}(t, X_1) G_{|X_2|}(t, X_2).
\]
where the operator \( p \) formula (Evolution of marginal correlation operators 12 group \( \Phi \) obeying quantum statistics with the Hamiltonian correlation operators defined by expansion (14) we treat as the hierarchy of evolution equations, which governs the marginal correlation operators of quantum many-particle systems.

We also formulate the nonlinear quantum BBGKY hierarchy in case of many-particle systems obeying quantum statistics with the Hamiltonian

\[
H_n = \sum_{i=1}^{n} K(i) + \sum_{k=1}^{n} \sum_{i_1 < \ldots < i_k} \Phi^{(k)}(q_{i_1}, \ldots, q_{i_k}),
\]

where \( \Phi^{(k)} \) is a \( k \)-body interaction potential. We introduce the symmetrization operator \( S^+_n \) and the anti-symmetrization operator \( S^-_n \) on \( \mathcal{H}^{\otimes n} \) which are defined on the space \( \mathcal{H}_n \) by the formula

\[
S^\pm_n = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} (\pm 1)^{|\pi|} p_{\pi},
\]

where the operator \( p_{\pi} \) is a transposition operator of the permutation \( \pi \) from the permutation group \( \mathcal{S}_n \) of the set \( \{1, \ldots, n\} \) and \( |\pi| \) denotes the number of transpositions in the permutation \( \pi \). The operators \( S^\pm_n \) are orthogonal projectors, i.e. \( (S^\pm_n)^2 = S^\pm_n \), ranges of which are correspondingly the symmetric tensor product \( \mathcal{H}^+_n \) and the antisymmetric tensor product \( \mathcal{H}^-_n \) of \( n \) Hilbert spaces \( \mathcal{H} \). We denote by \( \mathcal{F}^+_n = \bigoplus_{n=0}^{\infty} \mathcal{H}^+_n \) and \( \mathcal{F}^-_n = \bigoplus_{n=0}^{\infty} \mathcal{H}^-_n \) the Bose and Fermi Fock spaces over the Hilbert space \( \mathcal{H} \) respectively.

The evolution of all possible states of many-particle systems of bosons or fermions is described within the framework of marginal correlation operators \( G(t) = (G_0, G_1(t), \ldots, G_n(t), \ldots) \in \mathfrak{L}^1(\mathcal{F}^\pm_\mathcal{H}) \) governed by the following nonlinear quantum BBGKY hierarchy

\[
\frac{d}{dt} G_s(t, Y) = -\mathcal{N}_s(Y) G_s(t, Y) + \sum_{\substack{P: \text{ } Y = \bigcup_{|P| > 1} X_i, \
Z_1 \subseteq X_1, Z_2 \subseteq X_2, \
\ldots \ldots \ldots \ldots}} \big( -\mathcal{N}_s^{(|Z_1|)}(Z_1, \ldots, Z_{|P|}) \big) S^\pm_{X_1 \in P} \prod_{X_i \in P} G_{|X_i|}(t, X_i) + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{1 \leq j_1 < \ldots < j_k} \text{Tr}_{s+1, \ldots, s+1+n-k} \left( -\mathcal{N}_s^{(n+1)}(j_1, \ldots, j_k, s + 1, \ldots, s + 1 + n-k) \right) \times S^\pm_{s+1+n-k} \prod_{X_i \in P} G_{|X_i|}(t, X_i),
\]

where notations accepted above are used and the operator \( -\mathcal{N}_s^{(k)} \) acts on \( \mathfrak{L}_0^1(\mathcal{H}_s) \subset \mathfrak{L}_1^1(\mathcal{H}_s) \) according to the formula

\[
(-\mathcal{N}_s^{(k)}(i_1, \ldots, i_k)) f_s = -\frac{i}{\hbar} (\Phi^{(k)}(i_1, \ldots, i_k) f_s - f_s \Phi^{(k)}(i_1, \ldots, i_k)).
\]
We emphasize that the evolution of marginal correlation operators of both finitely and infinitely many quantum particles is described by initial-value problem of the nonlinear quantum BBGKY hierarchy (24) (or (32)). For finitely many particles the nonlinear quantum BBGKY hierarchy is equivalent to the von Neumann hierarchy (1).

4 A nonperturbative solution of the nonlinear quantum BBGKY hierarchy

In this section we construct a solution of the initial-value problem of the nonlinear quantum BBGKY hierarchy (24). A nonperturbative solution is represented in form of an expansion over particle clusters which evolution is governed by the corresponding-order cumulant (semi-invariant) of nonlinear groups of operators (7) generated by the von Neumann hierarchy (1). A solution representation in the form of the perturbation (iteration) series of hierarchy (24) is derived as a result of applying of analogs of the Duhamel equation to cumulants of groups of operators (7) of constructed solution.

4.1 A solution in case of chaos initial data

To construct a nonperturbative solution of the Cauchy problem (24)-(25) of the nonlinear quantum BBGKY hierarchy we first consider its structure for physically motivated example of initial data, namely, initial data satisfying a chaos property

\[ G_s(t,Y)|_{t=0} = G_1(0,1)\delta_{s,1}, \quad s \geq 1, \]  

where \( \delta_{s,1} \) is a Kronecker symbol. Chaos property (34) means the absence of state correlations in a system at the initial time.

According to definition (14) and solution expansion (11), in the case under consideration the following relation between the marginal correlation operators and correlation operators is true

\[ G_s(t,Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \mathfrak{A}_{s+n}(t,1,\ldots,s+n) \prod_{i=1}^{s+n} g_1(0,i), \]  

where \( \mathfrak{A}_{s+n}(t) \) is \((s+n)th\)-order cumulant (8). In consequence of relation (35) we finally derive

\[ G_s(t,Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \mathfrak{A}_{s+n}(t,1,\ldots,s+n) \prod_{i=1}^{s+n} G_1(0,i), \quad s \geq 1. \]  

If \( \|G_1(0)\|_{\mathcal{L}^1(\mathcal{H})} \leq (2e)^{-1} \), series (37) converges, since for cumulants (8) the estimate holds (11)

\[ \|\mathfrak{A}_n(t)f\|_{\mathcal{L}^1(\mathcal{H}_n)} \leq n! e^n \|f\|_{\mathcal{L}^1(\mathcal{H}_n)}. \]
From the structure of series (37) it is clear that in case of absence of correlations at initial instant in a system the correlations generated by the dynamics of quantum many-particle systems are completely governed by cumulants (8) of groups of operators (6).

Thus, the cumulant structure of solution (7) of the von Neumann hierarchy (11) induces the cumulant structure of solution expansion (37) of the initial-value problem of the quantum nonlinear BBGKY hierarchy for marginal correlation operators.

The evolution equations which satisfy expression (37) are derived similarly to the derivation of hierarchy (24) given in section 3.2 on the base of definition (36).

We note, that in case of initial data (10) solution (11) of the Cauchy problem (1)-(2) of the von Neumann hierarchy may be rewritten in another representation. For

$$g_1(t, 1) = \mathfrak{A}_1(t, 1)g_1(0, 1).$$

Then, within the context of the definition of the first-order cumulant, $\mathfrak{A}_1(-t)$, and the dual group of operators $\mathfrak{A}_1(t)$, we express the correlation operators $g_s(t)$, $s \geq 2$, in terms of the one-particle correlation operator $g_1(t)$ using formula (11). Hence for $s \geq 2$ formula (11) is represented in the form of the functional with respect to one-particle correlation operators

$$g_s(t, Y | g_1(t)) = \hat{\mathfrak{A}}_s(t, Y) \prod_{i=1}^{s} g_1(t, i), \quad s \geq 2,$$

where $\hat{\mathfrak{A}}_s(t, Y)$ is $sth$-order cumulant (8) of the scattering operators

$$\hat{G}_s(t, Y) = G_s(-t, Y) \prod_{i=1}^{s} G_1(t, i), \quad s \geq 1. \quad (38)$$

The generator of the scattering operator $\hat{G}_t(Y)$ is determined by the operator

$$\frac{d}{dt}\hat{G}_s(t, Y)|_{t=0} = \sum_{k=2}^{s} \sum_{i_1, \ldots, i_k=1}^{s} (-N_{\text{int}}^{(k)}(i_1, \ldots, i_k)),$$

where the operator $(-N_{\text{int}}^{(k)})$ acts on $\mathfrak{L}_0^1(\mathcal{H}_s) \subset \mathfrak{L}_0^1(\mathcal{H}_s)$ according to formula (33).

Similar representation of a solution holds for marginal correlation operators (37) which forms inherently the basis of the kinetic description of the evolution. In this case the marginal correlation functionals $G_s(t, Y | G_1(t))$, $s \geq 2$, are represented by the expansions

$$G_s(t, Y | G_1(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \mathfrak{V}_{1+n}(t, \theta(\{Y\}), s + 1, \ldots, s + n) \prod_{i=1}^{s+n} G_1(t, i), \quad (39)$$

where the operator $G_1(t, i)$ is given by (36) for $s = 1$, and we use the notion of the declusterization mapping defined in section 2 [20]. In expansion (39) the $(1+n)th$-order evolution operator $\mathfrak{V}_{1+n}(t)$ is defined by the formula [9]
Evolution of marginal correlation operators

\[ \mathfrak{V}_{1+n}(t, \theta(\{Y\}, X \setminus Y)) = n! \sum_{k=0}^{n} (-1)^k \sum_{n_1=1}^{n} \cdots \sum_{n_{k-1}=1}^{n-n_1-\ldots-n_{k-1}} \frac{1}{(n-n_1-\ldots-n_k)!} \times \]
\[ \times \hat{\mathfrak{A}}_{s+n-n_1-\ldots-n_k}(t, \{Y\}, s+1, \ldots, s+n-n_1-\ldots-n_k) \times \]
\[ \times \prod_{j=1}^{k} \sum_{D_j: \exists \bigcup_{j \in D_j} x_{i_j}, \bigcup_{|D_j| \leq s+n-n_1-\ldots-n_j} 1} \frac{1}{|D_j|!} \sum_{i \neq \exists |D_j|=1} \prod_{i \in D_j \subset X_{i_j}} \frac{1}{|X_{i_j}|!} \hat{\mathfrak{A}}_{1+|X_{i_j}|}(t, i_j, X_{i_j}), \]

where \( \sum_{D_j: \exists \bigcup_{j \in D_j} x_{i_j}} \) is the sum over all possible dissections \( D_j \) of the linearly ordered set \( Z_j \equiv (s+n-n_1-\ldots-n_j+1, \ldots, s+n-n_1-\ldots-n_{j-1}) \) on no more than \( s+n-n_1-\ldots-n_j \) linearly ordered subsets, and the operator \( \hat{\mathfrak{A}}_{1+n}(t) \) is the \((1+n)\)-order cumulant \( \mathfrak{S} \) of the scattering operators \( \mathfrak{S} \). For example, the lower orders evolution operators \( \mathfrak{V}_{1+n}(t, \theta(\{Y\}), s+1, \ldots, s+n) \), \( n \geq 0 \), have the form

\[ \mathfrak{V}_1(t, \theta(\{Y\})) = \hat{\mathfrak{A}}_s(t, \theta(\{Y\})), \]
\[ \mathfrak{V}_2(t, \theta(\{Y\}), s+1) = \hat{\mathfrak{A}}_{s+1}(t, \theta(\{Y\}), s+1) - \hat{\mathfrak{A}}_s(t, \theta(\{Y\})) \sum_{i=1}^{s} \hat{\mathfrak{A}}_2(t, i, s+1), \]

and in case of \( s = 2 \), it holds

\[ \mathfrak{V}_1(t, \theta(\{1, 2\})) = \hat{\mathfrak{G}}_2(t, 1, 2) - I. \]

We point out also that in case of chaos initial data solution expansion \( \mathfrak{S} \) of the quantum BBGKY hierarchy \( \mathfrak{S} \) for marginal density operators differs from solution expansion \( \mathfrak{S} \) of the nonlinear quantum BBGKY hierarchy \( \mathfrak{S} \) for marginal correlation operators only by the order of the cumulants of the groups of operators of the von Neumann equations \( \mathfrak{S} \).\( \mathfrak{S} \)

\[ F_s(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \mathfrak{A}_{1+n}(t, \{Y\}, X \setminus Y) \prod_{i=1}^{s+n} F_i(0, i), \quad s \geq 1, \quad (40) \]

where \( \mathfrak{A}_{1+n}(t) \) is the \((1+n)th\)-order cumulant \( \mathfrak{S} \). Series \( \mathfrak{S} \) converges under the condition: \( \|F_1(0)\|_{\mathcal{L}^1(\mathcal{H})} \leq e^{-1} \).

### 4.2 The structure of a nonperturbative solution expansion

The direct method of the construction of a solution of the nonlinear quantum BBGKY hierarchy \( \mathfrak{S} \) in the form of nonperturbative expansion consists in its derivation on the basis of expansions \( \mathfrak{S} \) from nonperturbative solution \( \mathfrak{S} \) of initial-value problem of the quantum BBGKY hierarchy \( \mathfrak{S} \). Following stated above approach, we derive a formula for a solution of the quantum nonlinear BBGKY hierarchy for marginal correlation operators in case of general initial data on the basis of definition \( \mathfrak{S} \) and nonperturbative solution \( \mathfrak{S} \) of initial-value problem of the von Neumann hierarchy \( \mathfrak{S} \). With this aim on \( f_n \in \mathfrak{S}(\mathcal{H}) \) we introduce an analogue of the annihilation operator

\[ (af)_s(1, \ldots, s) = \text{Tr}_{s+1}f_{s+1}(1, \ldots, s, s+1), \quad s \geq 1, \quad (41) \]
and, therefore we have

\[(e^{\pm a}f)_s(1, \ldots, s) = \sum_{n=0}^{\infty} \frac{(\pm 1)^n}{n!} \text{Tr}_{s+1, \ldots, s+n} f_{s+n}(1, \ldots, s + n).\]

According to definition \((14)\) of the marginal correlation operators, i.e.

\[G(t) = e^a g(t),\]

where the sequence \(g(t)\) is a solution of the von Neumann hierarchy for correlation operators defined by group \((7)\), i.e. \(g(t) = G(t \mid g(0))\), and to the equality: \(g(0) = e^{-a}G(0)\), we finally derive

\[G(t) = e^a G(t \mid e^{-a}G(0)).\] (42)

To set down formula \((42)\) in componentwise form we observe, that the following equality holds

\[
\prod_{X_i \in \mathcal{P}} (e^{-a}G(0)\rvert_{X_i})(X_i) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \text{Tr}_{s+n+1, \ldots, s+n+k} \sum_{k_1=0}^{k} \frac{k!}{k_1!(k - k_1)!} \ldots
\]

\[
\ldots \sum_{k_{\left|\mathcal{P}\right| - 2} = 0}^{k_{\left|\mathcal{P}\right| - 2}} \frac{k_{\left|\mathcal{P}\right| - 2}!}{k_{\left|\mathcal{P}\right| - 1}!(k_{\left|\mathcal{P}\right| - 2} - k_{\left|\mathcal{P}\right| - 1})!} G_{|X_1| + k - k_1}(0, X_1, s + n + 1, \ldots, s + n + k - k_1) \ldots
\]

\[
\ldots G_{|X_{\left|\mathcal{P}\right| - 1}| + k_{\left|\mathcal{P}\right| - 1} - 1}(0, X_{\left|\mathcal{P}\right|}, s + n + k - k_{\left|\mathcal{P}\right| - 1} + 1, \ldots, s + n + k).
\] (43)

Then according to formulas \((12)\) and \((7)\), for \(s \geq 1\) we have

\[G_s(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \sum_{P : (1, \ldots, s+n) = \bigcup_i X_i} \mathcal{A}_{|P|}(t, \{X_1\}, \ldots, \{X_{|P|}\}) \prod_{X_i \in \mathcal{P}} (e^{-a}G(0)\rvert_{X_i})(X_i),\]

where \(\mathcal{A}_{|P|}(t)\) is \(|P|th\)-order cumulant \((8)\), and as a result of the validity of equality \((13)\) for sequence \((12)\) we obtain

\[G_s(t, 1, \ldots, s) =
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \sum_{P : (1, \ldots, s+n-k) = \bigcup_i X_i} \mathcal{A}_{|P|}(t, \{X_1\}, \ldots, \{X_{|P|}\}) \times
\]

\[
\times \sum_{k_1=0}^{k} \frac{k_1!}{k_1!(k - k_1)!} \ldots \sum_{k_{\left|\mathcal{P}\right| - 2} = 0}^{k_{\left|\mathcal{P}\right| - 2}} \frac{k_{\left|\mathcal{P}\right| - 2}!}{k_{\left|\mathcal{P}\right| - 1}!(k_{\left|\mathcal{P}\right| - 2} - k_{\left|\mathcal{P}\right| - 1})!} G_{|X_1| + k - k_1}(0, X_1,
\]

\[
s + n - k + 1, \ldots, s + n - k_1) \ldots G_{|X_{\left|\mathcal{P}\right| - 1}| + k_{\left|\mathcal{P}\right| - 1} - 1}(0, X_{\left|\mathcal{P}\right|}, s + n - k_{\left|\mathcal{P}\right| - 1} + 1, \ldots, s + n).
\]

Consequently the solution expansion of the nonlinear quantum BBGKY hierarchy has the following structure

\[G_s(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1, \ldots, s+n} U_{1+n}(t; \{Y\}, s + 1, \ldots, s + n \mid G(0)), \quad s \geq 1,\] (44)
where we introduce the notion of the \((1+n)\text{-th}\) order reduced cumulant \(U_{1+n}(t)\) of nonlinear groups of operators \((7)\):

\[
U_{1+n}(t; \{Y\}, s + 1, \ldots, s + n \mid G(0)) \equiv \\
\sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \sum_{P: (\theta(\{1,\ldots,s\}, s+1,\ldots,s+n-k)=\cup X_i} \mathcal{A}_{|P|}(t, \{X_1\}, \ldots, \{X_{|P|}\}) \times \\
\times \sum_{k_1=0}^{k} \frac{k!}{k_1!(k-k_1)!} \cdots \sum_{k_{|P|-1}=0}^{k_{|P|-2}} \frac{k_{|P|-2}!}{k_{|P|-1}!(k_{|P|-2} - k_{|P|-1})!} G_{X_1|X_1+k-k_1}(0, X_1, \\
s + n - k + 1, \ldots, s + n - k_1) \cdots G_{X_{|P|}+k_{|P|-1}}(0, X_{|P|}, s + n - k_{|P|-1} + 1, \ldots, s + n).
\]

We give simplest examples of reduced nonlinear cumulants \((45)\):

\[
U_1(t; \{Y\} \mid G(0)) = \mathcal{G}(t; Y \mid G(0)) = \\
\sum_{P: \{1,\ldots,s\}=\cup X_i} \mathcal{A}_{|P|}(t, \{X_1\}, \ldots, \{X_{|P|}\}) \prod_{X_i \in P} G_{|X_i|}(0, X_i),
\]

\[
U_2(t; \{Y\}, s + 1 \mid G(0)) = \\
\sum_{P: \{Y,s+1\}=\cup X_i} \mathcal{A}_{|P|}(t, \{X_1\}, \ldots, \{X_{|P|}\}) \prod_{X_i \in P} G_{|X_i|}(0, X_i) - \\
- \sum_{P: \{1,\ldots,s\}=\cup X_i} \mathcal{A}_{|P|}(t, \{X_1\}, \ldots, \{X_{|P|}\}) \sum_{|X_j|} \prod_{j=1}^{X_i \in P} G_{|X_i|}(0, X_i) G_{|X_j|+1}(0, X_j, s + 1).
\]

We remark that in case of solution expansion \((22)\) of the quantum BBGKY hierarchy, an analog of reduced cumulant \((45)\) is the reduced cumulant of groups of operators \((7)\) defined by formula \((14)\):

\[
U_{1+n}(t; \{Y\}, s + 1, \ldots, s + n) \equiv \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \mathcal{G}_{s+n-k}(-t).
\]

### 4.3 Reduced cumulants of nonlinear groups of operators

We indicate some properties of reduced nonlinear cumulants \((45)\) of groups of operators \((7)\). According to formula \((14)\) and properties of cumulants \((8)\), namely \(\mathcal{A}_{n}(0) = I\delta_{n,1}\), the following equality holds

\[
U_{1+n}(0; \{Y\}, s + 1, \ldots, s + n \mid G(0)) = \\
\sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \mathcal{A}_{1}(0, \{1, \ldots, s + n - k\}) G_{s+n}(0, 1, \ldots, s + n) = \\
= G_{s+n}(0, 1, \ldots, s + n) \delta_{n,0},
\]

and hence the marginal correlation operators determined by series \((14)\) satisfy initial data \((25)\).
In case of \( n = 0 \) for \( f \in \mathfrak{A}_0^1(\mathcal{F}_t) \) in the sense of the norm convergence of the space \( \mathfrak{A}_0^1(\mathcal{H}_s) \) the infinitesimal generator of first-order reduced cumulant (15) coincides with generator (3) of the von Neumann hierarchy (1):

\[
\lim_{t \to 0} \frac{1}{t} \left( \mathbf{U}(t; \{Y\} \mid f) - f_s(Y) \right) = \mathcal{N}(Y \mid f), \quad s \geq 1,
\]

where the operator \( \mathcal{N}(Y \mid f) \) is defined by formula (3). In case of \( n = 1 \) for second-order reduced cumulant (15) in the same sense we obtain the following equality

\[
\text{Tr}_{s+1} \lim_{t \to 0} \frac{1}{t} \left( \mathbf{U}(t; \{Y\}, s + 1 \mid f) = \sum_{i \in Y} \text{Tr}_{s+1}(-\mathcal{N}_\text{int}(i, s + 1))(f_{s+1}(t, Y, s + 1) + \right.

\sum_{P: (Y, s + 1) = X_1 \cup X_2, i \in X_1; s + 1 \in X_2} f_{[X_1]}(t, X_1)f_{[X_2]}(t, X_2),
\]

where notations are used as above for hierarchy (24), and for \( n \geq 2 \) as a consequence of the fact that we consider a system of particles interacting by a two-body potential, it holds

\[
\text{Tr}_{s+1, \ldots, s+n} \lim_{t \to 0} \frac{1}{t} \left( \mathbf{U}(t; \{Y\}, s + 1, \ldots, s + n \mid f) = 0. \right.
\]

In case of initial data satisfying a chaos property, i.e. \( G^{(1)}(0) \equiv (0, G_1(0,1), 0, \ldots) \), for the \((1+n)\)th-order reduced cumulant we have

\[
\mathbf{U}_{1+n}(t; \{Y\}, s + 1, \ldots, s + n \mid G^{(1)}(0)) = \mathfrak{A}_{s+n}(t, 1, \ldots, s + n) \prod_{i=1}^{s+n} G_1(0, i),
\]

i.e. the only summand that gives contribution to the result is the one with \( k = 0 \) and \( |P| = s+n \), since otherwise there is at least one operator \( G_\alpha(0) \) with \( s \geq 2 \) in the last product. We note that in section 4.1 the same result was obtained using the properties of solution of the von Neumann hierarchy (1).

For the \((1+n)\)th-order reduced cumulant (45) the following inequality holds

\[
\|\mathbf{U}_{1+n}(t; \{Y\}, s + 1, \ldots, s + n \mid f)\|_{\mathfrak{A}^1(\mathcal{H}_{s+n})} \leq 2n!s!(2\pi e^2)^{s+n}c^{s+n}, \quad (46)
\]

where \( c \equiv \max_{P: (1, \ldots, s+n-k)} \max_{k, k_1, \ldots, k_{|P|-1} \in \mathfrak{M}(\mathcal{H}_{s+n-k+1}, \ldots, \mathfrak{M}(\mathcal{H}_{s+n-k+1}, |X|, s+n-k, |P|) \times \mathfrak{M}(s+n-k, \ldots, s+n), \ldots, \mathfrak{M}(s+n-k, \ldots, s+n)} \).

To prove this inequality we first remark that for cumulant (8) the following estimate holds

\[
\|\mathfrak{A}_{|P|}(t, \{X_1\}, \ldots, \{X_{|P|}\})f_{n}\|_{\mathfrak{A}^1(\mathcal{H}_{s+n})} \leq |P|! |P|! \|f_{n}\|_{\mathfrak{A}^1(\mathcal{H}_{s+n})}. \quad (47)
\]

Indeed, we have

\[
\|\mathfrak{A}_{|P|}(t, \{X_1\}, \ldots, \{X_{|P|}\})f_{n}\|_{\mathfrak{A}^1(\mathcal{H}_{s+n})} \leq \sum_{P': (\{X_1\}, \ldots, \{X_{|P|}\}) = \bigcup_k Z_k} (|P'|-1)! \prod_{Z_k \subseteq P'} G_{\theta(Z_k)}(-t, \theta(Z_k))f_{n}\|_{\mathfrak{A}^1(\mathcal{H}_{s+n})} =
\]

\[
= \|f_{n}\|_{\mathfrak{A}^1(\mathcal{H}_{s+n})} \sum_{l=1}^{|P|} s(|P|, l)(l-1)!,
\]

where
where \( s([P], l) \) are the Stirling numbers of second kind and we use the isometric property of the groups \( G_n(t), n \geq 1 \). Estimate (47) holds as a consequence of the inequality

\[
\sum_{i=1}^{[P]} s([P], l)(l-1)! \leq [P]! e^{[P]}.
\]

Then owing to estimate (47), for the \((1 + n)th\)-order reduced cumulant (45) we have

\[
\| U_{1+n}(t; \{ Y \}, s + 1, \ldots, s + n \mid f) \|_{\mathfrak{L}^1(\mathcal{H}_{s+n})} \leq
\]

\[
\leq \sum_{k=0}^{[P]-2} \frac{n!}{k!(n-k)!} \sum_{P: (1, \ldots, s+n-k) = \bigcup_i X_i} [P]! e^{[P]} \sum_{k_1=0}^{k} \frac{k!}{k_1!(k-k_1)!} \cdots
\]

\[
\sum_{k_{[P]-1}=0}^{[P]-1} \frac{k_{[P]-2}!}{(k_{[P]-2} - k_{[P]-1})!} \| f|_{X_1|+k-1} \|_{\mathfrak{L}^1(\mathcal{H}_{(s+n-k)})} \cdots \| f|_{X_{[P]}|+k_{[P]-1}} \|_{\mathfrak{L}^1(\mathcal{H}_{(s+n-k)})} \leq
\]

\[
\leq \sum_{k=0}^{n} \frac{n!}{(n-k)!} \sum_{P: (1, \ldots, s+n-k) = \bigcup_i X_i} [P]! e^{2[P]-1} e^{[P]}.
\]

As result of using of the definition of the Stirling numbers of second kind \( s(s+n-k, l) \) and the inequalities

\[
\sum_{k=0}^{n} \frac{n!}{(n-k)!} \sum_{P: (1, \ldots, s+n-k) = \bigcup_i X_i} [P]! e^{2[P]-1} = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \sum_{l=1}^{s+n-k} s(s+n-k, l)! e^{2l-1} \leq
\]

\[
\leq \sum_{k=0}^{n} \frac{n!(s+n-k)!}{(n-k)!} e^{3(s+n-k)} \leq 2n! s!(2e^3)^{s+n},
\]

we obtain estimate (46).

Thus, according to estimate (46), for initial data from the space \( \mathfrak{L}^1(\mathcal{H}_n) \) series (41) converges under the condition: \( \varepsilon \equiv \max_{n \geq 1} \| G_n(0) \|_{\mathfrak{L}^1(\mathcal{H}_n)} < (2e^3)^{-1} \), and the following inequality holds

\[
\| G_s(t) \|_{\mathfrak{L}^1(\mathcal{H}_s)} \leq 2s! (2e^3)^s \sum_{n=0}^{\infty} (2e^3)^n c^n. \tag{48}
\]

A solution of the Cauchy problem of the nonlinear quantum BBGKY hierarchy for marginal correlation operators (24) is determined by the following one-parametric mapping

\[
\mathbb{R} \ni t \to \mathcal{U}(t \mid f) = e^{\delta \mathcal{G}(t \mid e^{-\alpha} f)}, \tag{49}
\]

which is defined on the space \( \mathfrak{L}^1(\mathcal{F}_H) \) owing to estimate (48), and has the group property

\[
\mathcal{U}(t_1 \mid \mathcal{U}(t_2 \mid f)) = \mathcal{U}(t_2 \mid \mathcal{U}(t_1 \mid f)) = \mathcal{U}(t_1 + t_2 \mid f).
\]

Indeed, according to definition (41) and taking into consideration the group property of the mapping \( \mathcal{G}(t \mid \cdot) \), we obtain

\[
\mathcal{U}(t_1 + t_2 \mid f) = e^{\delta \mathcal{G}(t_1 + t_2 \mid e^{-\alpha} f)} = e^{\delta \mathcal{G}(t_1 \mid \mathcal{G}(t_2 \mid e^{-\alpha} f)} =
\]

\[
= e^{\delta \mathcal{G}(t_1 \mid e^{-\alpha} \mathcal{G}(t_2 \mid e^{-\alpha} f)} = e^{\delta \mathcal{G}(t_1 \mid e^{-\alpha} \mathcal{U}(t_2 \mid f))} = \mathcal{U}(t_1 \mid \mathcal{U}(t_2 \mid f)).
\]
To construct the generator of the strong continuous group $\mathcal{U}(t; Y \mid \cdot)$ we differentiate it in the sense of the norm convergence on the space $\mathcal{L}^1(\mathcal{H}_s)$

$$
\frac{d}{dt} \mathcal{U}(t; Y \mid f)|_{t=0} = \frac{d}{dt} (e^a \mathcal{G}(t \mid e^{-a} f))_{s}(Y)|_{t=0} =
$$

$$
= \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \mathcal{N}(X \mid \mathcal{G}(t \mid e^{-a} f))|_{t=0} = (e^a \mathcal{N}(\cdot \mid e^{-a} f))_{s}(Y),
$$

where $\mathcal{N}(\cdot \mid f)$ is a generator of the von Neumann hierarchy (11) defined by formula (3) on the subspaces $\mathcal{L}^1_0(\mathcal{H}_s) \subset \mathcal{L}^1(\mathcal{H}_s)$, $s \geq 1$, or in the componentwise form

$$
(e^a \mathcal{N}(\cdot \mid e^{-a} f))_{s}(Y) = \mathcal{N}(Y \mid f) + \text{Tr}_{s+1} \sum_{i \in Y} \left(-\mathcal{N}_{\text{int}}(i, s + 1)\right)(f_{s+1}(Y, s + 1) + \quad (50)
$$

+ \sum_{P: (Y, s + 1) = X_1 \cup X_2, \quad i \in X_1; s + 1 \in X_2} \left(\mathcal{N}_{\text{int}}(i_1, i_2)\right)(e^{-a} f)_{|X_1}(X_1)(e^{-a} f)_{|X_2}(X_2)),
$$

where we use notations as above for formula (24), and transformations similar to equalities (27) and (31) have been applied.

Indeed, to set down a generator of mapping (49) in componentwise form we observe that according to definitions (41) and (3), the following equality holds

$$
(e^a \mathcal{N}(\cdot \mid e^{-a} f))_{s}(Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \left(-\mathcal{N}_{s+n}(X)(e^{-a} f)_{s+n}(X) +
$$

+ \sum_{P: X = X_1 \cup X_2, \quad i_1 \in X_1; i_2 \in X_2} \left(\mathcal{N}_{\text{int}}(i_1, i_2)\right)(e^{-a} f)_{|X_1}(X_1)(e^{-a} f)_{|X_2}(X_2)).
$$

Then in view of formulas (41) and (43) we have

$$
(e^{-a} f)_{|X_1}(X_1)(e^{-a} f)_{|X_2}(X_2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \text{Tr}_{s+n+1,\ldots,s+n+k} \sum_{k_1=0}^{k} \frac{k!}{k_1!(k-k_1)!} f_{|X_1+k-k_1}(X_1,
$$

$s + n + 1, \ldots, s + n + k - k_1) f_{|X_2+k_1}(X_2, s + n + k - k_1 + 1, \ldots, s + n + k),
$$

and as a result we obtain

$$
(e^a \mathcal{N}(\cdot \mid e^{-a} f))_{s}(1, \ldots, s) =
$$

$$
= \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \left(-\mathcal{N}_{s+n-k}(1, \ldots, s + n - k) f_{s+n}(X) +
$$

+ \sum_{P: (1,\ldots,s+n-k) = X_1 \cup X_2, \quad i_1 \in X_1; i_2 \in X_2} \left(\mathcal{N}_{\text{int}}(i_1, i_2)\right) \sum_{k_1=0}^{k} \frac{k!}{k_1!(k-k_1)!} \times
$$

$$
\times f_{|X_1+k-k_1}(X_1, s + n - k + 1, \ldots, s + n - k_1) f_{|X_2+k_1}(X_2, s + n - k_1 + 1, \ldots, s + n)).
$$

Therefore the first term of this series is generator (3) of the von Neumann hierarchy

$$
I_1 \equiv (-\mathcal{N}_{s}) f_{s}(Y) + \sum_{P: Y = X_1 \cup X_2, \quad i_1 \in X_1; i_2 \in X_2} \sum_{i_1 \in X_1} \sum_{i_2 \in X_2} \left(\mathcal{N}_{\text{int}}(i_1, i_2)\right) f_{|X_1}(X_1)f_{|X_2}(X_2),
$$
and, as stated above, it coincides with the generator of the first order cumulant (45). For the second term of series (50) we have

\[ I_2 \equiv \text{Tr}_{s+1} \left( (-\mathcal{N}_{s+1}(Y, s + 1)) f_{s+1} - (-\mathcal{N}_s(Y)) f_{s+1} + \right. \\
+ \sum_{P: (Y, s+1) = X_1 \cup X_2; i_1 \in X_1; i_2 \in X_2} \sum (-\mathcal{N}_{\text{int}}(i_1, i_2)) f_{|X_1|}(X_1) f_{|X_2|}(X_2) - \\
- \sum_{P: Y = X_1 \cup X_2; i_1 \in X_1; i_2 \in X_2} \sum (-\mathcal{N}_{\text{int}}(i_1, i_2)) (f_{|X_1|+1}(X_1, s + 1) f_{|X_2|}(X_2) + \\
+ f_{|X_1|}(X_1) f_{|X_2|+1}(X_2, s + 1)) \right). 

Taking into account equality (28) in case of the set \((Y, s + 1)\), it holds

\[
\sum_{P: (Y, s+1) = X_1 \cup X_2; i_1 \in X_1; i_2 \in X_2} (-\mathcal{N}_{\text{int}}(i_1, i_2)) f_{|X_1|}(X_1) f_{|X_2|}(X_2) = \\
= \sum_{P: (Y, s+1) = X_1 \cup X_2; i_1 \in X_1 \cap Y; i_2 \in X_2 \cap Y} (-\mathcal{N}_{\text{int}}(i_1, i_2)) f_{|X_1|}(X_1) f_{|X_2|}(X_2) + \\
+ \sum_{P: (Y, s+1) = X_1 \cup X_2; i_1 \in X_1 \cap Y} (-\mathcal{N}_{\text{int}}(i_1, s + 1)) f_{|X_1|}(X_1) f_{|X_2|}(X_2) = \\
= \sum_{P: Y = Y_1 \cup Y_2; i_1 \in Y_1; i_2 \in Y_2} (-\mathcal{N}_{\text{int}}(i_1, i_2)) (f_{|Y_1|+1}(Y_1, s + 1) f_{|Y_2|}(Y_2) + \\
+ f_{|Y_1|}(Y_1) f_{|Y_2|+1}(Y_2, s + 1)) + \sum_{i \in Y} (-\mathcal{N}_{\text{int}}(i, s + 1)) \sum_{P: (Y, s+1) = X_1 \cup X_2; i \in X_1; s + 1 \in X_2} f_{|X_1|}(X_1) f_{|X_2|}(X_2),
\]

where \(\sum_{P: (Y, s+1) = X_1 \cup X_2; s + 1 \in X_2}\) is the sum over all possible partitions of the set \((Y, s + 1)\) into two mutually disjoint subsets \(X_1\) and \(X_2\) such that \((s + 1)th\) particle index belongs to set \(X_2\). As a result we obtain

\[ I_2 = \text{Tr}_{s+1} \left( \sum_{i \in Y} (-\mathcal{N}_{\text{int}}(i, s + 1)) f_{s+1} + \\
+ \sum_{i \in Y} (-\mathcal{N}_{\text{int}}(i, s + 1)) \sum_{P: (Y, s+1) = X_1 \cup X_2; i \in X_1; s + 1 \in X_2} f_{|X_1|}(X_1) f_{|X_2|}(X_2) \right), \]

i.e. this term coincides with the generator of second-order cumulant (45).

In case of a two-body interaction potential other terms of series (50) are identically equal to zero. This statement is a consequence of the structure of expansion (51) and of the fact that its third term equals zero. Indeed as a result of regrouping terms in the expression of the third
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Thus, we conclude the validity of formula \((50)\) in case of a two-body interaction potential which describes the structure of the infinitesimal generator of mapping \((49)\) in the general case.

4.4 An existence theorem

For an abstract initial-value problem of hierarchy \((24)\) in the space \(L^1(\mathcal{F}_H)\) the following theorem is true.

**Theorem 1.** If \(\max_{n \geq 1} \| G_n(0) \|_{L^1(\mathcal{H}_n)} < (2e^3)^{-1}\), then in case of bounded interaction potentials for \(t \in \mathbb{R}\) a solution of the Cauchy problem of the nonlinear quantum BBGKY hierarchy \((24)\)-(\(28)\) is determined by expansion \((44)\). If \(G_n(0) \in L^1_0(\mathcal{H}_n) \subset L^1(\mathcal{H}_n)\), it is a strong (classical) solution and for arbitrary initial data \(G_n(0) \in L^1(\mathcal{H}_n)\) it is a weak (generalized) solution.

**Proof.** It will be recalled that according to estimate \((18)\), series \((41)\) converges under the condition: \(\max_{n \geq 1} \| G_n(0) \|_{L^1(\mathcal{H}_n)} < (2e^3)^{-1}\). To prove that a strong solution of the nonlinear BBGKY hierarchy \((24)\) is given by expansion \((44)\) we first differentiate it over time variable in the sense of a pointwise convergence on the space \(L^1(\mathcal{H}_n)\), i.e. for every function from the domain \(\psi_s \in \mathcal{D}(\mathcal{H}_s) \subset \mathcal{H}_s\). Taking into account the group property of mapping \((49)\) generated by expansion \((44)\) and properties of reduced nonlinear cumulants \((45)\) of groups of operators \((7)\), for \(G_n(0) \in L^1_0(\mathcal{H}_n) \subset L^1(\mathcal{H}_n), n \geq 1\), we obtain

\[
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left( \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{Tr}_{s+1,\ldots,s+n}(U_{1+n}(\Delta t; \{Y\}, X \setminus Y | G(t)) - G_s(t, Y)) \psi_s \right) = \mathcal{N}(Y | G(t)) \psi_s + \sum_{i \in Y} \left( -\mathcal{N}_{\text{int}}(i, s + 1) \right) (G_{s+1}(t, Y, s + 1) + \sum_{i \in Y} G_{[X_1]}(t, X_1) G_{[X_2]}(t, X_2) \psi_s,
\]

where expansion \((44)\) is denoted by the symbol \(G_s(t, Y)\). Since \(G_n(0) \in L^1_0(\mathcal{H}_n) \subset L^1(\mathcal{H}_n), n \geq 1\), then using equality \((51)\), in the sense of the norm convergence in \(L^1(\mathcal{H}_n)\) we finally establish
the validity of the equality

$$\lim_{\Delta t \to 0} \text{Tr}_{1,\ldots,s} \left[ \frac{1}{\Delta t} \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} U_{1+n}(t + \Delta t; \{Y\}, X \setminus Y \mid G(0)) - \right.$$  

$$- \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} U_{1+n}(t; \{Y\}, X \setminus Y \mid G(0)) -$$  

$$- \left( \mathcal{N}(Y \mid G(t)) + \text{Tr}_{s+1} \sum_{i \in Y} (-\mathcal{N}_{\text{int}}(i, s + 1))(G_{s+1}(t, Y, s + 1) +$$  

$$+ \sum_{P : (Y, s + 1) = X_1 \cup X_2, \ i \in X_1; s + 1 \in X_2} G_{|X_1|}(t, X_1)G_{|X_2|}(t, X_2)) \right) \bigg| = 0,$$

which means that a strong solution of the nonlinear BBGKY hierarchy (24) is given by expansion (44) in case of initial data from the subspaces $\mathcal{L}_0^1(\mathcal{H}_n) \subset \mathcal{L}_1^1(\mathcal{H}_n)$, $n \geq 1$.

Let us give a sketch of the prove that in case of arbitrary initial data $G_n(0) \in \mathcal{L}_1^1(\mathcal{H}_n)$, $n \geq 1$, expansion (44) is a weak solution of the initial-value problem (24)-(25). To this end we introduce the functional

$$(f, G(t)) \doteq \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1,\ldots,s} f_s(Y) G_s(t, Y),$$  

where $f = (0, f_1, \ldots, f_n, \ldots) \in \mathcal{L}_0(\mathcal{F}_n)$ is the finite sequence of degenerate bounded operators with infinitely times differentiable kernels with compact supports. For $G_n(0) \in \mathcal{L}_1^1(\mathcal{H}_n)$ and $f_n \in \mathcal{L}_0(\mathcal{H}_n)$ functional (52) exists.

We transform functional (52) to the following form

$$(f, G(t)) = (f, e^a G(t \mid e^{-a} G(0))) = (e^{a^+} f, G(t \mid e^{-a} G(0))),$$  

where the operator $a$ is defined by (41) and on $f_s \in \mathcal{L}_0(\mathcal{H}_s)$ the operator $a^+$ is defined by the formula (an analog of the creation operator)

$$(a^+ f)_s(Y) \doteq \sum_{j=1}^{s} f_{s-1}(Y \setminus (j)).$$

To differentiate obtained functional (53) with respect to the time variable we use the corresponding result (12) of the differentiation of group (7) of the von Neumann hierarchy (1). As a result we derive that

$$\frac{d}{dt}(f, G(t)) = \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1,\ldots,s} \left( \mathcal{N}_s(Y)(e^{a^+} f)_s(Y) G(t, Y \mid e^{-a} G(0)) +

+ \sum_{P : Y = X_1 \cup X_2, i_1 \in X_1; i_2 \in X_2} \mathcal{N}_{\text{int}}(i_1, i_2)(e^{a^+} f)_s(Y) G(t, X_1 \mid e^{-a} G(0)) G(t, X_2 \mid e^{-a} G(0)) \right).$$
Taking into account the structure of expansion \((44)\) of the nonlinear quantum BBGKY hierarchy solution, for \(f_s \in \mathcal{L}_0(\mathcal{H}_s), \ s \geq 1\), the following equality holds

\[
\frac{d}{dt}(f_s, G(t)) = \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1,\ldots,s} \left( (\mathcal{N}_s(Y)f_s(Y) + \sum_{i,j=1}^{s} \mathcal{N}_{\text{int}}(i,j)f_{s-1}(Y \setminus (j)))G_s(t, Y) + \right.
\]

\[
+ \sum_{P : Y = X_1 \cup X_2} \sum_{i \in X_1} \sum_{j \in X_1} \sum_{i \neq j} \mathcal{N}_{\text{int}}(i,j)f_s(Y)G_{|X_1|}(t, X_1)G_{|X_2|}(t, X_2) + \n\]

\[
+ \sum_{i,j=1}^{s} \mathcal{N}_{\text{int}}(i,j)f_{s-1}(Y \setminus (j)) \sum_{P : Y = X_1 \cup X_2} G_{|X_1|}(t, X_1)G_{|X_2|}(t, X_2) \right).
\]

This equation means that in case of arbitrary initial data \(G_n(0) \in \mathcal{L}_1(\mathcal{H}_n), \ n \geq 1\), a weak solution of the initial-value problem \((24)-(25)\) is given by expansion \((44)\).

\[\square\]

### 4.5 Remark: the nonlinear Vlasov hierarchy

We give comments on the mean field asymptotic behavior \([23]\) of constructed solution \((44)\).

Let us suppose the existence of the mean field limit of initial state in the following sense

\[
\lim_{\epsilon \to 0} \| \epsilon^s G_n(0) - g_n(0) \|_{\mathcal{L}_1(\mathcal{H}_n)} = 0, \quad n \geq 1.
\]

Then there exists the mean field limit \(g_s(t, 1, \ldots, s), \ s \geq 1\), of marginal correlation operators \((44)\)

\[
\lim_{\epsilon \to 0} \| \epsilon^s G_s(t) - g_s(t) \|_{\mathcal{L}_1(\mathcal{H}_s)} = 0, \quad s \geq 1,
\]

which is governed by the nonlinear Vlasov quantum hierarchy

\[
\frac{d}{dt} g_s(t, Y) = \sum_{i \in Y} (-\mathcal{N}(i)) g_s(t, Y) + \text{Tr}_{s+1} \sum_{i \in Y} (-\mathcal{N}_{\text{int}}(i, s+1)) \times \n\]

\[
\times (g_{s+1}(t, Y, s+1) + \sum_{P : (Y, s+1) = X_1 \cup X_2} g_{|X_1|}(t, X_1)g_{|X_2|}(t, X_2)), \quad s \geq 1,
\]

where notations similar to hierarhy \((24)\) are used.

If initial data satisfies chaos property, then we establish

\[
\lim_{\epsilon \to 0} \| \epsilon^s G_s(t) \|_{\mathcal{L}_1(\mathcal{H}_s)} = 0, \quad s \geq 2,
\]

since solution expansions \((37)\) for marginal correlation operators are defined by the \((s + n)\)th-order cumulants as contrasted to solution expansions \((22)\) for marginal density operators defined by the \((1 + n)\)th-order cumulants and in the consequence of the following formula on an asymptotic perturbation of cumulants of groups of operators \([24]\)

\[
\lim_{\epsilon \to 0} \| \frac{1}{\epsilon^s} \mathfrak{A}_{s+n}(t, 1, \ldots, s + n) f_{s+n} \|_{\mathcal{L}_1(\mathcal{H}_{s+n})} = 0.
\]
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In case of \(s = 1\) provided that (54) we have

\[
\lim_{\epsilon \to 0} \| \epsilon G_1(t) - g_1(t) \|_{L^1(H)} = 0,
\]

where for finite time interval the limit one-particle marginal correlation operator \(g_1(t, 1)\) is given by the norm convergent on the space \(L^1(H)\) series

\[
g_1(t, 1) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_n} \text{Tr}_{2,\ldots,n+1} G_1(-t + t_1, 1)(-\mathcal{N}_{\text{int}}(1, 2)) \prod_{j_1=1}^{2} G_1(-t_1 + t_2, j_1) \ldots \\
\ldots \prod_{i_n=1}^{n} G_1(-t_n + t_n, i_n) \sum_{k_n=1}^{n} (-\mathcal{N}_{\text{int}}(k_n, n + 1)) \prod_{j_n=1}^{n+1} G_1(-t_n, j_n) \prod_{i=1}^{n+1} g_1(0, i),
\]

which obviously coincides with iteration series of the Vlasov quantum kinetic equation [10]. For bounded interaction potential (1) series (57) is norm convergent on the space \(L^1(H)\) under the condition:

\[
t < t_0 = (2 \| \Phi \|_{L^2(H_2)} \| g_1(0) \|_{L^1(H)})^{-1}.
\]

In view of the validity of limit (56) from the Vlasov nonlinear quantum hierarchy (55) we also conclude that limit one-particle marginal correlation operator (57) is governed by the Cauchy problem of the Vlasov quantum kinetic equation

\[
\frac{d}{dt} g_1(t, 1) = -\mathcal{N}(1)g_1(t, 1) + \text{Tr}_2(-\mathcal{N}_{\text{int}}(1, 2))g_1(t, 1)g_1(t, 2),
\]

and consequently for pure states we derive the Hartree equation.

Thus, the nonlinear Vlasov quantum hierarchy (55) describes the evolution of initial correlations.

5 Conclusion

In the paper the origin of the microscopic description of non-equilibrium correlations of quantum many-particle systems obeying the Maxwell-Boltzmann statistics has been considered. The nonlinear quantum BBGKY hierarchy (24) for marginal correlation operators was introduced. It gives an alternative approach to the description of the state evolution of quantum infinite-particle systems in comparison with quantum BBGKY hierarchy for marginal density operators [13, 14]. The evolution of both finitely and infinitely many quantum particles is described by initial-value problem of the nonlinear quantum BBGKY hierarchy (24) and in case of finitely many particles the nonlinear quantum BBGKY hierarchy is equivalent to the von Neumann hierarchy [1].

A nonperturbative solution of the nonlinear quantum BBGKY hierarchy is constructed in the form of expansion (44) over particle clusters which evolution is governed by corresponding-order cumulant (45) of the nonlinear groups of operators generated by solution (7) of the von Neumann hierarchy [1]. We established that in case of absence of correlations at initial time the correlations generated by the dynamics of quantum many-particle systems (37) are completely determined by cumulants (8) of groups of operators (6).
Thus, the cumulant structure of solution (7) of the von Neumann hierarchy (1) induces the cumulant structure of solution expansion (44) of initial-value problem of the nonlinear quantum BBGKY hierarchy (24).

We emphasize that intensional Banach spaces for the description of states of infinite-particle systems, which are suitable for the description of the kinetic evolution or equilibrium states, are different from the exploit spaces [14], [19]. Therefore marginal correlation operators from the space of trace-class operators describe finitely many quantum particles. In order to describe the evolution of infinitely many particles we have to construct solutions for initial data from more general Banach spaces than the space of sequences of trace-class operators. For example, it can be the space of sequences of bounded translation invariant operators which contains the marginal density operators of equilibrium states [25]. In that case every term of the solution expansion of the nonlinear quantum BBGKY hierarchy (11) contains the divergent traces, which can be renormalized due to the cumulant structure of solution expansion (45).

The mean field asymptotic behavior of constructed solution (44) is governed by the nonlinear Vlasov quantum hierarchy (55). In such approximation this hierarchy describes the evolution of initial correlations and in case of its absence the nonlinear Vlasov hierarchy (55) is equivalent to the Vlasov quantum kinetic equation (58).

Following to the paper [12] the obtained results can be also generalized on many-particle systems obeying the Fermi-Dirac and Bose-Einstein statistics (32).

References

[1] A. Arnold, *Mathematical properties of quantum evolution equations*. Lecture Notes in Math. **1946**, (2008), 45-110.

[2] C. Bardos, B. Ducomet, F. Golse, A.D. Gottlieb and N.J. Mauser, *The TDHF approximation for Hamiltonians with m-particle interaction potentials*. Commun. Math. Sci. **5**, (2007), 1-9.

[3] L. Erdős, B. Schlein and H.-T. Yau, *Derivation of the cubic nonlinear Schrödinger equation from quantum dynamics of many-body systems*. Invent. Math. **167**, (3), (2007), 515-614.

[4] L. Erdős, B. Schlein and H.-T. Yau, *Derivation of the Gross-Pitaevskii Equation for the Dynamics of Bose-Einstein Condensate*. Ann. of Math., **172**, (2010), 291-370.

[5] J. Fröhlich, S. Graffi and S. Schwarz, *Mean-field and classical limit of many-body Schrödinger dynamics for bosons*. Commun. Math. Phys. **271**, (2007), 681-697.

[6] A. Michelangeli, *Role of scaling limits in the rigorous analysis of Bose-Einstein condensation*. J. Math. Phys. **48**, (2007), 102102.

[7] F. Pezzotti and M. Pulvirenti, *Mean-field limit and semiclassical expansion of quantum particle system*. Ann. Henri Poincaré. **10**, (2009), 145-187.

[8] L. Saint-Raymond, *Kinetic models for superfluids: a review of mathematical results*. C. R. Physique. **5**, (2004), 6575.
[9] V.I. Gerasimenko and Zh.A. Tsvir, *A description of the evolution of quantum states by means of the kinetic equation*. J. Phys. A: Math. Theor. **43**, (48), (2010), 485203.

[10] V.I. Gerasimenko, *Heisenberg picture of quantum kinetic evolution in mean-field limit*, Kinet. Relat. Models, **4**, (1), (2011), 385-399.

[11] V.I. Gerasimenko and V.O. Shtyk, *Evolution of correlations of quantum many-particle systems*. J. Stat. Mech. Theory Exp. **3**, (2008), P03007, 24p.

[12] V.I. Gerasimenko and D.O. Polishchuk, *Dynamics of correlations of Bose and Fermi particles*. Math. Meth. Appl. Sci. **34**, (1), (2011), 76-93.

[13] M.M. Bogolyubov, *Lectures on Quantum Statistics. Problems of Statistical Mechanics of Quantum Systems*. Kyiv, 1949 (in Ukrainian).

[14] D.Ya. Petrina, *Mathematical Foundations of Quantum Statistical Mechanics. Continuous Systems*. Kluwer, 1995.

[15] R. Dautray and J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*. 5, Springer-Verlag, 1992.

[16] O. Bratelli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*. 2, Springer, 1997.

[17] V.I. Gerasimenko, *Groups of operators for evolution equations of quantum many-particle systems*. Oper. Theory Adv. Appl. **191**, (2009), 341-355.

[18] V.I. Gerasimenko and D.Ya. Petrina, *A mathematical description of the evolution of the states of infinite systems of classical statistical mechanics*. Russ. Math. Surv., **38**, (5), (1983), 3-58.

[19] C. Cercignani, V.I. Gerasimenko and D.Ya. Petrina, *Many-Particle Dynamics and Kinetic Equations*. Kluwer, 1997.

[20] D.O. Polishchuk, *BBGKY hierarchy and dynamics of correlations*. Ukrainian J. Phys. **55**, (5), (2010), 593-598.

[21] J. Yvon, *La theorie statistique des fluides et l’équation d’état*. Actualites Scientifiques et Industrielles, **49**, (203). Paris: Hermann, 1935.

[22] M.S. Green, *Boltzmann equation from the statistical mechanical point of view*. J. Chem. Phys. **25**, (5), (1956), 836-855.

[23] H. Spohn, *Kinetic equations from Hamiltonian dynamics: Markovian limits*. Rev. Modern Phys. **53**, (1980), 569-615.

[24] T. Kato, *Perturbation Theory for Linear Operators*. Springer-Verlag, 1995.

[25] J. Ginibre, *Some applications of functional integrations in statistical mechanics*. (in *Statistical Mechanics and Quantum Field Theory*. Gordon and Breach, 1971), 329-427.