STABLY BALANCED AND RELATIVELY RIGID CURVES ON PROJECTIVE HYPERSURFACES

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ABSTRACT. A curve $C$ on a variety $X$ is stably balanced if the slopes of the Harder-Narasimhan filtration of its normal bundle $N$ are contained in an interval of length $1$. We show a general curve of any genus and large enough degree in $\mathbb{P}^n$ is stably balanced. For each $d \geq n$ we construct some regular families of pairs $(C, X)$ of the expected dimension with $X$ a hypersurface of degree $d$ in $\mathbb{P}^n$ and $C$ a stably balanced curve on $X$, such that whenever $\chi(N) \leq 0$, $C$ is rigid as subvariety of $X$ and the family has codimension $h^1(N)$ in the space of hypersurfaces.

CONFLICT OF INTEREST STATEMENT

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INTRODUCTION

The normal bundle of curves in projective space, and in hypersurfaces in projective space, has been studied by a number of authors in recent years, following the work of Sacchiero [13], who proved that the normal bundle $N_{C/\mathbb{P}^n}$ of a general nondegenerate rational curve of degree $e$ in projective space is balanced, i.e. a direct sum of line bundles degrees $a, a + 1$ where $a = e + [(2e - 2)/(n - 1)]$. The balance property is essentially equivalent to the interpolation property, i.e. that $C$ can be deformed to go through the expected number of general points, and in that guise can be extended in various ways to curves of higher genus. It implies, e.g. that the embedding is regular, i.e. $H^1(N_{C/\mathbb{P}^n}) =$
Several authors have established balancedness properties for curves both rational and irrational, both in $\mathbb{P}^n$ and in projective hypersurfaces, see e.g. [5], [10], [11]. This work has largely been restricted to Fano hypersurfaces $X$, i.e. those of degree $d \leq n$ in $\mathbb{P}^n$, due to the fact that for higher $d$ usually $N = N_{C/X}$ has $\chi(N) < 0$ which forces $C$ to be obstructed in $X$; in fact one would then expect that $C$ is rigid in $X$, i.e. $H^0(N) = 0$. Some results on rigid curves in non-Fano, e.g. Calabi-Yau hypersurfaces were obtained in [12].

Another important property of a bundle such as $N$ is (semi-) stability, which makes sense whether or not $N$ has sections. For elliptic normal curves, of degree $n+1$ on $\mathbb{P}^n$, semi-stability of the normal bundle was established by Ein-Lazarsfeld [6]. For curves in $\mathbb{P}^3$, stability of the normal bundle was studied by numerous authors (see references in [4]). For curves of genus 2 or more, stability of the normal bundle was finally proven in most cases by Coskun-Larson-Vogt [4] [3]. The analogous statement for $\mathbb{P}^n, n \geq 4$ is not known.

Here we aim to study normal bundles from the point of view of stability and, motivated by the case of rational curves, we consider a property weaker than stability that we call stable balance. This stipulates that the slopes of graded the Harder-Narasimhan or stable filtration of $N$ should all lie in some interval of length 1. Stable balance is obviously equivalent to balance for rational curves, and is meaningful for any bundle regardless of positivity or sections. The relation of stable balance and even stability to balance in the sense used in [1] and [11] is not clear.

The main results of this paper may be summarized as follows.

- For any genus $g \geq 0$ and $n \geq 4$, a general curve of genus $g$ and sufficiently high degree in $\mathbb{P}^n$ has stably balanced normal bundle (Theorem 6).
- For each $d \geq n \geq 4$ and each $(e, g)$ in a suitable range depending on $n, d$, there exists a regular family of pairs $(C, X)$ where $X$ is a hypersurface of degree $d$ in $\mathbb{P}^n$ and $C$ is a curve of degree $e$ and genus $g$ on $X$ with stably balanced normal bundle $N$, such that either
  - $C$ is regular on $X$ and $X$ is general (Case $\chi(N) \geq 0$; Theorems 8 and 15) or
  - $C$ is rigid on $X$ and the family of hypersurfaces $X$ is of codimension $-\chi(N)$ in the space of all hypersurfaces of degree $d$ (Case $\chi(N) \leq 0$; Theorem 15).

Here

\[ \chi(N) = e(n+1-d) + (n-4)(1-g). \]

- A general elliptic curve of degree $2n-2$ or $\geq 3n-3$ in $\mathbb{P}^n$ has semistable normal bundle (Theorem 18).
- If $n = 2m + 1, m \geq 2$, a general elliptic curve of degree $\ell m$ with $\ell = 4$ or $\ell \geq 6$ in $\mathbb{P}^n$ has stable normal bundle (Theorem 20).

In more detail, the contents of the paper as as follows. After recalling some facts on fans, their hypersurfaces and degenerations in §1 some properties of, and operations on
stably balanced bundles are developed in §2. In particular, we study elementary modifications and specialization/generalization. Then in §3 we prove in Theorem 6 that the normal bundle of a general curve of any genus and sufficiently large degree in $\mathbb{P}^n, n \geq 4$, previously shown to be balanced [11], is stably balanced. The proof is based on a suitable 'elliptic-toothed comb' degeneration, together with the Ein-Lazarsfeld result applied on the comb’s teeth. Then in §4 we consider curves on general hypersurfaces of degree $n$ in $\mathbb{P}^n$ and prove a similar result, Theorem 8.

Next we turn to non-Fano hypersurfaces $X \subset \mathbb{P}^n$ of degree $d > n$. Here the situation changes because the normal bundle $N$ is generally not regular nor can $X$ be general (if it carries a general curve). Therefore we study deformations of the pair $(C, X)$ in $\mathbb{P}^n$ which can nevertheless be regular, and we implicitly view them as fibred over deformations of $C$ rather than of $X$. We define a suitable normal sheaf $N_{C/X/\mathbb{P}^n}$ controlling embedded deformations of the pair, such that the vanishing of $H^1(N_{C/X/\mathbb{P}^n})$ is equivalent to the pair $(C, X)$ having unobstructed deformations in $\mathbb{P}^n$ of the expected dimension $h^0(N_{C/X/\mathbb{P}^n}) = \chi(N_{C/X/\mathbb{P}^n})$ (the cohomology of degree $> 1$ usually vanishes automatically). The curve $C \subset X$ is said to be rigid-regular if $H^1(N_{C/X/\mathbb{P}^n}) = 0$ and $H^0(N_{C/X}) = 0$. This implies that the space of deformations of the pair $(C, X)$ in $\mathbb{P}^n$, the so-called flag Hilbert scheme, is a smooth subvariety of codimension $h^1(N_{C/X})$ in the deformation space of $X$.

These notions will be discussed in §5 whose main result, Theorem 15, is that for any $d > n \geq 4$ there is a range of $(e, g)$ for which the general (nonspecial) curve of degree $e$ and genus $g$ admits a stably balanced and rigid-regular embedding in a hypersurface of degree $d$ in $\mathbb{P}^n$. The proof uses degeneration and relies in an essential way on maximal rank theorems of Hartshorne-Hirschowitz [7] for general skew lines and Ballico-Ellia [2] for general nonspecial curves.

An appendix gives a short proof of an analogue of the Ein-Lazarsfeld result, showing that the normal bundle of general elliptic curves of certain degrees is semistable. For elliptic curves of certain degrees in odd-dimensional projective spaces we show the normal bundle is actually stable.

Note that the degree, hence also the genus, of any rigid-regular curve of a hypersurface of degree $d$ is a priori bounded by a function on $d$.

It seems likely that the results can be extended to hypersurfaces of degree $d < n$, but this is unproven. Also, it remains an open question in genus $g \geq 2$ whether the normal bundles to the curves we construct may actually be stable or semistable rather than just stably balanced. However Example 2 suggests that this may be false for stability ($n \geq 4$) or semi-stability ($n \geq 5$).

**Conventions.** We work over an arbitrary algebraically closed field. As general references for deformation theory, see the books by Sernesi [14] or Hartshorne [8].
The slope of a bundle $E$ on a curve, i.e. $\deg(E)/\text{rk}(E)$ is denoted by $\mu(E)$.

1. **Fan and Quasi-cone Degenerations**

See [10] for details. We recall that a *fan* (also called a 2-fan) is a reducible normal-crossing variety of the form

$$P_0 = P_1 \cup E \cup P_2$$

where

$$P_1 = B_p \mathbb{P}^n, P_2 = \mathbb{P}^n$$

and $E \subset P_1$ is the exceptional divisor and $E \subset P_2$ is a hyperplane. The family

$$\mathcal{P} = B_{(p,0)}(\mathbb{P}^n \times \mathbb{A}^1)$$

is called a standard fan degeneration and realizes $P_0$ as the special fibre in a family with general fibre $\mathbb{P}^n$.

A *hypersurface of type $(d_1, d_2)$* in $P_0$ has the form

$$X_0 = X_1 \cup Z \cup X_2$$

where

$$X_1 \in |d_1H_1 - d_2E|_{P_1}, X_2 \in |d_2H_2|_{P_2}, X_1 \cap Z = X_2 \cap Z$$

($H_1, H_2$ are the respective hyperplanes). If $d_2 = d_1 - 1$, $X_0$ is said to be of *quasi-cone type*. Given a family $\mathcal{X} \subset \mathbb{P}^n \times \mathbb{A}^1$ of hypersurfaces of degree $d_1$ whose special fibre has multiplicity $d_2$ at $p$, its birational transform $\mathcal{X} \subset \mathcal{P}$ is a family of hypersurfaces in $\mathbb{P}^n$ specializing to one of type $(d_1, d_2)$.

2. **Stably Balanced Bundles**

Given a vector bundle $E$ on a curve $C$, we denote by

$$(E_\bullet) = (0 \neq E_1 \subset E_2...)$$

its Harder-Narasimman (HN) filtration, characterized by the property that

$$\mu_{\text{max}}(E) = \mu_1(E) := \mu(E_1)$$

is the largest slope among slopes of subbundles $E_1 \subset E$, and $E_1$ is maximal of slope $\mu_1(E)$ (i.e. $E_1$ is the maximal-rank, maximal-slope subbundle and as such is unique and semi-stable); and likewise for $\mu_{i+1}(E) := \mu(E_{i+1}/E_i), \forall i \geq 1$. Thus we have a strictly decreasing slope sequence

$$\mu_{\text{max}}(E) > ... > \mu_{\text{min}}(E)$$

(unless $E$ is semi-stable, in which case $\mu_{\text{max}}(E) = \mu_{\text{min}}(E)$). The interval $[\mu_{\text{min}}(E), \mu_{\text{max}}(E)]$ is called the slope interval of $E$ and its width $\mu_{\text{max}}(E) - \mu_{\text{min}}(E)$ is called the slope width.
E is said to be stably balanced if its slope width is \( \leq 1 \). Note that the slope \( \mu(E) \) is contained in the slope interval, hence if \( E \) is stably balanced then
\[
\mu_{\text{max}}(E) - 1 \leq \mu(E) \leq \mu_{\text{min}}(E) + 1.
\]
If \( C = \mathbb{P}^1 \), stable balance is equivalent to balance. It is noted in [11], Lemma 8 that if the graded of the HN filtration of \( E \) are direct sums of line bundles with degrees in some interval \([a, a + 1], a \geq 2g - 1\) (so that \( E \) is stably balanced), then \( E \) is balanced. Otherwise, the relation between stable balance and balance in the sense used, e.g. in [11] is not understood for \( g > 0 \).

Our first result is about the behavior of stable balance under general modifications. This was essentially done in [10] for \( C = \mathbb{P}^1 \) and the discussion there largely extends to the general case.

By a semi-general (resp. general) down modification of \( E \) we mean an exact sequence
\[ 0 \to E' \to E \to \tau \to 0 \]
such that \( \tau \) is of the form \( \bigoplus \mathbb{k}(p_i)^{s_i} \) for some collection (resp. a general collection) of distinct points \( p_1, ..., p_k \in C \), and such that the map
\[ E|_{p_i} \to \mathbb{k}(p_i)^{s_i} \]
is a general quotient for \( i = 1, ..., k \).

**Lemma 1.** A semi-general down modification of a stably balanced bundle is stably balanced.

**Proof.** We may assume the modification \( E \to \tau \) is at a single point \( p \). Set \( t = \ell(\tau), r = \text{rk}(E) \). Thus \( t \leq r \). By generality we may assume that for every subbundle \( E_i \subset E \) in the HN filtration of \( E \), the map \( E_i|_p \to \tau \) has maximal rank. Let \( E_i \subset E_j \) be members of the HN filtration of \( E \), of respective ranks \( r_i < r_j \) and \( E'_i \subset E'_j \) their (saturated) intersections with \( E' \) (we only need the case \( j = i + 1 \)). Note that \( E_i, E_j \) are the respective saturations of \( E'_i, E'_j \) in \( E \), hence every pair of saturated subbundles of \( E' \) is obtained this way. There are the following cases.

**Case 1:** \( r_j \leq t \) and \( E_i|_p \to \tau \) injective.

In this case \( E'_i = E_i(-p), E'_j = E_j(-p) \) and
\[ \mu(E'_j/E'_i) = \mu(E_j/E_i) - 1. \]

**Case 2:** \( r_i \leq t < r_j, E_i|_p \to \tau \) injective and \( E_j|_p \to \tau \) surjective. This case can happen for at most one pair \((i, i + 1)\).

In this case \( E'_i = E_i(-p) \) and \( E'_j \subset E_j \) has colength \( t \). We compute that
\[ \mu((E'_j/E'_i)) = \mu(E_j/E_i) - \frac{r_i}{r_j - r_i}. \]
Case 3: $t < r_i$ and $E_i|_p \to \tau$ surjective.

In this case $E'_i \subset E_i, E'_j \subset E_j$ both have colength $t$ and we have

$$\mu(E'_i/E'_j) = \mu(E_i/E_i).$$

The 'case number' increases (nonstrictly) as we go up the filtration.

Note that $\mu_{\text{max}}(E') \leq \mu_{\text{max}}(E)$. Now assume to begin with that $E$ is semistable. Then for any quotient $E'/G$, we have

$$\mu(E) \leq \mu(E/G) \leq \mu(E'/G) + 1.$$ 

Thus $\mu_{\text{min}}(E') \geq \mu(E) - 1$, so $E'$ is stably balanced.

In the general case consider the HN filtration

$$0 = E_0 \subset E_1 \subset ... \subset E_a = E.$$ 

and the corresponding filtration $E'_\bullet$ on $E'$. There is a unique $k$ such that the map $E_i|_p \to \tau$ is injective for $i \leq k$ and surjective, non-injective for $i > k$. The we have

$$E'_i = E_i(-p), \mu(E'_i/E'_{i-1}) = \mu(E_i/E_{i-1}) - 1, i < k,$$

$$\mu(E_k/E_{k-1}) - 1 \leq \mu_{\text{min}}(E'_k/E'_{k-1}) \leq \mu_{\text{max}}(E'_k/E'_{k-1}) \leq \mu(E_k/E_{k-1})$$

$$\mu(E'_i/E'_{i-1}) = \mu(E_i/E_{i-1}), i > k.$$ 

Now $E'$ admits a filtration, not necessarily the HN filtration, with semistable gradeds consisting of

$$E_i(-p)/E_{i-1}(-p), i = 1, ..., k - 1; E_i/E_{i-1}, i = k + 1, ..., a;$$

plus the HN gradeds of $E'_k/E'_{k-1}$. By the semistable case just discussed, the slopes of these all lie in the interval $[\mu(E_k/E_{k-1}) - 1, \mu(E_k/E_{k-1})]$, hence $E'$ is stably balanced. □

Thus, semi-general elementary modifications stay within the class of stably balanced bundles. As the following results shows, this is false for the class of (semi)stable bundles.

Example 2. In any genus, if $E = 2O$ and $\tau = k(p)$, then $E$ is semistable but $E' = O \oplus O(-p)$ is stably balanced but not semistable. So in that sense the result is sharp.

It appears that a semi-general modification of a rank-2 stable bundle is stable, which may in part explain the result of [4] on the stability of normal bundles of curves in $\mathbb{P}^3$. However, in higher rank it is possible for $E$ to be stable while $E'$ is not, at least in genus $g \geq 2$. We start with an example of a non-stable but maybe semi-stable modification of a stable rank-3 bundle. Consider a general extension

$$0 \to O \xrightarrow{\epsilon} F \to G \to 0$$

where $G$ is a line bundle of degree 1 and the extension corresponds to a general element $\epsilon \in H^1(G^{-1}) = H^0(G \otimes K)^*$. I claim that $F$ is stable. In fact, consider any line bundle $L$
of degree $\geq \mu(F) = 1/2$ i.e. $\deg(L) \geq 1$. If $\deg(L) > 1$ or $\deg(L) = 1$ and $L \neq G$ then clearly $H^0(F \otimes L^{-1}) = 0$. If $L = G$, the same is true by nontriviality of the extension. Thus $F$ is stable.

Now consider an extension

$$(1)\quad 0 \to \mathcal{O} \to E \to F \to 0$$

corresponding to an element $\eta \in H^1(\tilde{F}) = H^0(F \otimes K)^*$ which comes from $H^1(\tilde{G})$, i.e.

fits in an exact diagram

$$(2)\quad \begin{array}{cccc}
0 & \to & \mathcal{O} & \to & E & \to & F & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O} & \to & E_G & \to & G & \to & 0.
\end{array}$$

Let $L$ be any line bundle with $\deg(L) \geq \mu(E) = 1/3$ (i.e. $\deg(L) > 0$). As we have seen $H^0(F \otimes L^{-1}) = 0$, hence $H^0(E \otimes L^{-1}) = 0$. So $E$ has no rank-1 destabilizing subsheaf.

Now we have an exact sequence

$$0 \to F \to \Lambda^2 E \to G \to 0$$

which is none other than the dual of (1), twisted by $G$, and we know $H^0(F \otimes L^{-1}) = 0$. Therefore an argument similar to the above shows that we have $H^0(\Lambda^2 E \otimes L^{-1}) = 0$, for any $L$ of degree $> 0$, so $E$ has no rank-2 destabilizing subsheaf. Thus $E$ is stable (even cohomologically stable in the sense of [6]).

Now a general corank-1 modification $E'$ of $E$ at $p$ fits on an exact sequence

$$0 \to \mathcal{O}(-p) \to E' \to F \to 0$$

which corresponds to an element of $H^1(\tilde{F} \otimes \mathcal{O}(-p))$, As with the original element $\eta$ this comes from an element of $H^1(\tilde{G} \otimes \mathcal{O}(-p))$ corresponding to a modification $E'_G$ of $E_G$. Therefore the section $s$ of $F$ lifts to $E'$. Then $E'$ has slope 0 but $H^0(E') \neq 0$ so $E'$ is not stable (but maybe semi-stable).

For an example of a non-semistable modification consider instead a similar extension

$$(3)\quad 0 \to 2\mathcal{O} \to E_2 \to F \to 0$$

then a similar argument as above shows that for any line bundle $L$ with $\deg(L) \geq \mu(E_2) = 1/4$, we have $H^0(E_2 \otimes L^{-1}) = H^0(\Lambda^3 E_2 \otimes L^{-1}) = 0$. Moreover we have an exact sequence

$$0 \to \mathcal{O} \to \Lambda^2 E_2 \to E_1 \to 0$$

where $E_1$ is analogous to $E$ above. Then a similar argument again shows $H^0(\Lambda^2 E_2 \otimes L^{-1}) = 0$. Therefore $E_2$ cannot admit a destabilizing subsheaf of rank 1, 2 or 3, hence it is stable.
Now a general corank-2 modification at $p$, $E'_2$, fits in an exact sequence
\[ 0 \to 2\mathcal{O}(-p) \to E'_2 \to F \to 0 \]
and as above we have $H^0(E'_2) \neq 0$ while $\mu(E'_2) = -1/4$, so $E'_2$ is not semistable. \hfill \Box

Next we come to an important technique for constructing stably balanced bundles via degeneration. We begin with the following situation:

\[ \pi : C \to B \]
is a family of curves over a smooth curve with all fibres $C_t$ smooth except

\[ C_0 = C_1 \cup_p C_2 \]
with $C_1, C_2$ smooth and transverse; $\mathcal{E}$ is a vector bundle on $C$, and we set $E_i = \mathcal{E}|_{C_i}$, $i = 1, 2$, $E_t = E_{C_t}$.

**Lemma 3.** Notations as above, assume $E_1$ is semistable and $E_2$ is stably balanced. Then a general $E_t$ is stably balanced.

**Proof.** Let $[\alpha, \beta]$ be the slope interval of $E_2$ and let $F_t$ be a subbundle of the general fibre $E_t$ ($F_t$ may possibly be defined only after a base-change, though actually we may choose for $F_t$ the maximal-rank, maximal-slope subbundle and then no base-change is required). We will show that

\[ \mu(F_t) \leq \mu(E_1) + \beta. \tag{4} \]

Then applying this result to the dual will show that

\[ \mu(F_t) \geq \mu(E_1) + \alpha. \]

Thus, the slope interval of $E_t$ is contained in $[\mu(E_1) + \alpha, \mu(E_1) + \beta]$, proving our result.

Now to prove (4), after a suitable base-change $B' \to B$ and replacing $C$ by $C' = C \times_B B'$, we may assume given a torsion-free subsheaf $\mathcal{F} \subset \mathcal{E}$ extending $F_t$. Then after further base-change and blowing up we may assume $\det(F_t) = \wedge^s F_t$ extends to an invertible sheaf $\mathcal{L}$ on $C'$ and that the map $\mathcal{L} \to \wedge^s \mathcal{E}$ vanishes only on an effective divisor $Z$ supported on the special fibre so the induced map $\mathcal{L}(Z) \to \wedge^s \mathcal{E}$ is nowhere vanishing. Then we have

\[ s\mu(F_t) = \deg(\mathcal{L}(-Z)|_{C_0}) \leq \deg(\mathcal{L}|_{C_1}) + \deg(\mathcal{L}|_{C_2}) \leq s(\mu(E_1) + \beta). \]

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The same result with the same proof also holds for a comb-shaped curve:
Lemma 4. Let \( C_0 \) be a connected nodal curve of the form \( D \cup C_1 \cup ... \cup C_m \) with \( D \cap C_i \) a single point, \( i = 1, ..., m \), and no other singularities. Let \( C \to B \) be a family as above with special fibre \( C_0 \) and general fibre \( C_t \), and let \( E \) a vector bundle on \( C \) such that the restriction of \( E \) on all but one of the components \( D, C_1, ..., C_m \) is semistable, and the restriction is stably balanced on the remaining component. Then the restriction \( E|_{C_t} \) is stably balanced.

Remark 5. Note that a stably balanced bundle \( E \) of slope \( \mu(E) \leq -1 \) must have \( \mu_{\text{max}}(E) < 0 \), hence \( H^0(E) = 0 \).

A lci subvariety \( C \subset X \) is said to be stably balanced if the normal bundle \( N = N_{C/X} \) is, and rigid if \( H^0(N) = 0 \). When \( C \) is a curve of genus \( g \) and \( X \) is \( n \)-dimensional, we have

\[
\mu(N) = (-C.K_X + 2g - 2)/(n-1).
\]

Consequently, if \( C \) is stably balanced and \( 2g - 2 \leq C.K_X - n + 1 \) then \( C \) is rigid.

3. Stably Balanced Curves in Projective Space

Theorem 6. For given \( n \geq 3, g \geq 0 \), a general curve of genus \( g \) and degree \( e \) in \( \mathbb{P}^n \) is stably balanced provided either

(i) \( e = \epsilon + (n+1)g \) where either

\[
(ia)1 + (g - 1/2)(n - 2) \leq \epsilon < g(n - 2) + n/2,
\]

or \( n \) is even and

\[
(ib) \epsilon = 1 + (g + 1/2)(n - 2);
\]

or

(ii) \( e = \epsilon + (n + 1)g + (n - 1)(\gamma + 1) + 2 \) where \( \gamma \geq 0 \) and either

\[
(iiia) 1 + (g + 1/2)(n - 2) \leq \epsilon < (g + 1)(n - 2) + n/2
\]

or \( n \) is even and

\[
(iiib) \epsilon = (g + 1)(n - 2) + n/2
\]

or

(iii) \( g \geq 1 \) and

\[
e \geq (2n - 1)g + 2n - 2.
\]

Proof. Since better results are known for \( g = 0 \) we assume \( g > 0 \). The proof is based on Lemma 4 and the stability theorem of Ein-Lazarsfeld [6]. We consider a fan

\[
P_1 = B_p \mathbb{P}^n, P_2 = \mathbb{P}^n, P_0 = P_1 \cup E P_2
\]

and use the usual fan degeneration of \( \mathbb{P}^n \) to \( P_0 \). We begin with Case (i). We construct a balanced rational curve \( C_1 \subset P_1 \) as follows. Start with a balanced rational curve \( C_1' \subset \mathbb{P}^n \) of degree \( \epsilon \) and lift it to a curve \( C_1^* \subset P_1 \) of \( \mathbb{P}^n \)-degree \( e_1 = \epsilon + g \), i.e, \( C_1^* \) meets...
transversely in \( g \) points and maps to a curve of degree \( \epsilon + g \) in \( \mathbb{P}^n \). Using the fact that \( P_1 = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}(1) \oplus \mathcal{O}) \), it is easy to check that the relative tangent bundle \( T_{P_1/\mathbb{P}^{n-1}} \) restricts on \( C_1^\ast \) to \( \mathcal{O}(\epsilon + 2g) \). Then we have the standard normal sequence

\[
0 \to T_{P_1/\mathbb{P}^{n-1}}|_{C_1^\ast} \to N_{C_1^\ast/P_1} \to N_{C_1^\ast/\mathbb{P}^{n-1}} \to 0
\]

where the right member has slope \( (n\epsilon - 2)/(n - 2) \) and our hypotheses imply that the latter is at most \( \epsilon + 2g - 1 \), so by Lemma 18 of [11], \( C_1^\ast \subset P_1 \) is balanced.

By construction, \( C_1^\ast \) meets \( E \) in \( g \) distinct points \( p_1, \ldots, p_g \). Now let \( C_{2,1}, \ldots, C_{2,g} \subset \mathbb{P}^n \) be general elliptic normal curves of degree \( n + 1 \) through \( p_1, \ldots, p_g \) respectively. By [6], the normal bundle \( N_{C_{2,i}/P_2} \) is semi-stable. Now set

\[
C_2 = \bigcup C_{2,i}
\]

and attach rulings \( R_{i,j} \subset P_1 \) through the remaining points of \( C_2 \cap E \), and finally set

\[
C_1 = C_1^\ast \cup \bigcup R_{i,j}.
\]

Because \( H^1(N_{C_1/P_1}(-E)) = H^1(N_{C_2/P_2}(-E)) = 0 \), it is easy to see as in [10] that the resulting curve is smoothable to a curve of genus \( g \) and degree \( e = e_1 + ng = \epsilon + (n + 2)g \) in \( \mathbb{P}^n \) and by Lemma 4 the smoothed curve is stably balanced.

Now in Case (ii) the proof is similar except that in addition to the \( g \) elliptics \( C_{2,i} \) we also attach a rational curves \( C_{2,0} \subset P_2 \) of degree \( n + \gamma(n - 1) \), with normal bundle that is perfectly balanced, hence semistable, that meets \( C_1^\ast \) in 1 point, then attach rulings \( R_k \) at the other points of \( C_{2,0} \).

For Case (iii) we take \( C_1^\ast \) perfectly balanced in \( P_1 \), hence with semistable normal bundle, and \( C_{2,0} \) merely balanced of genus 0 and degree \( e_{2,0} \geq n \).

\[ \square \]

**Corollary 7.** For all \( n \geq 4, g \geq 0 \), a general curve of genus \( g \) and degree \( \geq (2n - 1)(g + 1) \) in \( \mathbb{P}^n \) is stably balanced and balanced.

**Proof.** Stable balance follows from Case (iii) of Theorem 6 above, while balance is shown in [11], §3, Corollary 28. \[ \square \]

4. **Anticanonical Hypersurfaces**

Next we construct some good, in particular stably balanced, curves on hypersurfaces \( X \) of degree \( n \) in \( \mathbb{P}^n \). This result will be used later as the starting point of an induction on the degree of \( X \).

**Theorem 8.** (i) For all \( g \geq 0, n \geq 4, e \geq n^2 + gn(n - 2) \), there exists a balanced and stably balanced curve \( C \) of degree \( e \) and genus \( g \) on a general hypersurface of degree \( n \) in \( \mathbb{P}^n \).
(ii) If moreover
\[
\left(\frac{2n-2}{n-1}\right) > e + n
\]
then \( C \) is a general curve of its degree and genus and satisfies \( H^1(\mathcal{I}_C/\mathbb{P}^n(n)) = 0 \).

Remark 9. The vanishing of \( H^1(\mathcal{I}_C/\mathbb{P}^n(n)) \) clearly imposes an upper bound on the degree \( e \). The range of \( e \) permitted in Theorem 8 (ii) is an integer interval which is nonempty provided \( g < \left(\frac{2n-2}{n-1}\right)/(n-1)(2n-1) \).

Proof. The curve \( C \) is constructed exactly as the curve in Theorem 21, §4 in [11], where it is shown to be balanced, so for Part (i) it's just a matter of showing that \( C \) is stably balanced. This follows directly from Corollary 7, combined with the following Lemma:

**Lemma 10.** Suppose there exists a stably balanced curve of degree \( e_* \) and genus \( g \) in \( \mathbb{P}^{n-1} \), \( n \geq 4 \). Then for all \( e < [(n-1)(e_*-1), (n-1)e_*] \) there exists a stably balanced curve of genus \( g \) and degree \( e \) on a general hypersurface of degree \( n \) in \( \mathbb{P}^n \).

Proof of Lemma. The proof closely mimics that of Theorem 21, §4, in [11]. We start with a stably balanced curve \( C_* \subset \mathbb{P}^{n-1} \) of degree \( e_* \) and genus \( g \) (which may if necessary be assume balanced as well). We then construct a suitable fan of quasi-cone type \((n, n-1)\) of the form
\[
X_0 = X_1 \cup_Z X_2
\]
where \( X_1 = B_YP^{n-1}, Y = F_{n-1} \cap F_0 \) and \( Z \) is the birational transform of \( F_{n-1} \). We choose the complete intersection \( Y \) so as to meet \( C_* \) in \( a \) points \( p_1, ..., p_a \), where
\[
e = e_* (n-1) - a, 0 \leq a \leq n-1.
\]

Then let \( C_1 \subset X_1 \) be the birational transform of \( C_* \). As noted in loc. cit. the tangent spaces to \( Y \) in \( p_1, ..., p_a \) may be assume general, so \( N_{C_1/X_1} \) is a semi-general down modification of \( N_{C_*/\mathbb{P}^{n-1}} \), hence stably balanced by Lemma [11]. Then we complete the proof as in loc. cit. by attaching lines with trivial normal bundle
\[
L_i \subset X_2, i = 1, ..., e_* (n-1) - a.
\]

QED Lemma.

Note that because \( \chi(N_{C_*/\mathbb{P}^{n-1}}(- (n-1))) > 0 \), \( C_* \) being balanced implies
\[
H^1(N_{C_*/\mathbb{P}^{n-1}}(- (n-1))) = 0.
\]
This implies that \( C_* \cap F_{n-1} \) consists of general points on \( F_{n-1} \). Now \( C_1 \cap E \) may be identified with a subset of \( C_* \cap F_{n-1} \), therefore it too consists of general points. Consequently, the lines \( L_i \) may be assumed general.

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Now for Part (ii), we use the Maximal Rank theorems of Ballico-Ellia \cite{2} and Hartshorne-Hirschowitz \cite{7}, applied respectively to \( C^* \) and the line collection \( C_2 \). Our numerical assumption implies that
\[
\binom{2n-2}{n-1} > (n-1)e_1 + 1 - g
\]
and therefore \( H^1(\mathcal{I}_{C^*}(n-1)L)) = 0 \) by maximal rank \cite{2}, where \( L \) is the hyperplane from \( \mathbb{P}^{n-1} \). Now we have \( L = H - E \) where \( H \) is the pullback of a hyperplane via \( P_1 \to \mathbb{P}^n \). Let \( M = \mathcal{O}(n, n-1) \) be the line bundle on \( \mathbb{P}_{0} \) of the form
\[
M = \mathcal{O}_{P_1}(nH - (n-1)E) \cup \mathcal{O}_{P_2}(n-1),
\]
which is a specialization of \( \mathcal{O}_{\mathbb{P}^n}(n) \). Then we have an exact sequence
\[
0 \to \mathcal{I}_{C_1}((n-1)L) \to \mathcal{I}_{C_0}(M) \to \mathcal{I}_{C_2}((n-1) \to 0)
\]
and our degree hypothesis plus maximal rank \cite{7} yields \( H^1(\mathcal{I}_{C_2}(n-1)) = 0 \), hence
\[
H^1(\mathcal{I}_{C_0}(M)) = 0.
\]
As \( C_0 \subset X_0 \) generalizes to \( C' \subset X' \), the latter generalizes to \( H^1(\mathcal{I}_{C}(n)) \), so the latter vanishes as well. Now use the exact sequence \cite{6} together with the identifications \cite{7} for \( C' \subset X' \subset \mathbb{P}^n \). Then the vanishing if \( H^1(\mathcal{I}_{C'}/\mathbb{P}^n(n)) \) yields surjectivity of \( H^0(N_{C'/X'}/\mathbb{P}^n) \to H^0(N_{C'/\mathbb{P}^n} \) so \( (C', X') \) general yields \( C' \) general.

\( \square \)

Part (ii) will be used in the discussion of higher-degree hypersurfaces in the next section.

5. Higher-degree hypersurfaces

Here we consider curves \( C \) on hypersurfaces \( X \) of degree \( d \geq n + 1 \) in \( \mathbb{P}^n \). This case is of a different character because generally \( \chi(N_{C/X}) < 0 \) so \( H^1(N_{C/X}) \) cannot vanish, nor can \( X \) be general. Consequently at least the smoothing issue has to be handled differently. We shall do so by studying deformations and specializations of the triple \( (C, X, \mathbb{P}^n) \).

5.1. Deformations of pairs. Compare \cite{14}. Let \( C \subset X \) be an inclusion of smooth projective varieties. There are corresponding deformation functors and spaces
\[
\text{Def}(C/X) \to \text{Def}(C, X) \to \text{Def}(X)
\]
parametrizing deformations of, respectively, \( C \) fixing \( X \), the pair \( (C, X) \), the variety \( X \). These correspond to an exact sequence of sheaves on \( X \)
\[
0 \to T_X(- \log C) \to T_X \to N_{C/X} \to 0.
\]
In fact, $T_X$ is a sheaf of Lie algebras and $T_X(-\log C)$ is a subalgebra sheaf, which endows $N = N_{C/X}$ with the structure of Lie atom (see [9]), which yields the corresponding deformation theory, i.e. deformations of $C$ in a fixed $X$. Thus the complex $N[-1]$ is endowed with a bracket in the derived category and deformations (resp. obstructions to deformations) of $C$ fixing $X$ are in $H^0(N)$ (resp. $H^1(N)$); for the pair $(C, X)$ the corresponding groups are $H^1(T_X(-\log C)), H^2(T_X(-\log C))$.

5.2. Embedded deformations of pairs. Let $X \subset P$ be a locally complete intersection with $P$ smooth, and let $C \subset X$ be a locally complete intersection in $X$. Then the normal bundles $N_{C/P}$ and $N_{X/P}$ admit $N_{C/P}|_C$ as a common quotient. Define a coherent sheaf $N_{C/X/P}$ on $X$ by the exact sequence

$$0 \to N_{C/X/P} \to N_{C/P} \oplus N_{X/P} \to N_{X/P}|_C \to 0.$$  

(5)

Then $N_{C/X/P}$ controls deformations of the pair $(C, X)$ as subvarieties of $P$, and we have an exact sequence

$$0 \to N_{C/X} \to N_{C/X/P} \to N_{X/P} \to 0.$$  

We will say that the pair $(C \subset X)$ of subvarieties is rigid-regular in $P$ provided the following vanishings hold.

$$H^0(N_{C/X}) = 0, H^1(N_{C/P/X}) = 0, H^1(N_{X/P}) = 0.$$  

The following follows from general deformation/obstruction theory (see e.g. [14]):

**Lemma 11.** Assume $(C, X)$ is rigid-regular in $P$. Then near $(C, X)$, the Hilbert scheme of pairs in $P$ embeds as a smooth subvariety of codimension $-\chi(N_{C/X}) = h^1(N_{C/X})$ in the Hilbert scheme of $X$ in $P$.

There is another exact sequence

$$0 \to N^C_{X/P} \to N_{C/X/P} \to N_{C/P} \to 0$$  

(6)

where $N^C_{X/P} \subset N_{X/P}$ is the subsheaf corresponding to deformations of $X$ fixing $C$ (or equivalently, deformations of the birational transform of $X$ in the blowup of $P$ in $C$).

Now suppose $X$ is a hypersurface of degree $d$ with equation $f$ in $P = \mathbb{P}^n$. We can identify

$$N_{X/P} = \mathcal{O}_X(d), N^C_{X/P} = \mathcal{I}_C/X(d) = \mathcal{I}_C/P(d)/f\mathcal{O}_P.$$  

(7)

Now assume $C, X, P$ are smooth, and let $T_{P/X/C} \subset T_P$ denote the subsheaf consisting of local vector fields tangent to both $X$ and $C$. Then we have an exact sequence

$$0 \to T_{P/X/C} \to T_P \to N_{C/X/P} \to 0$$  

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and also

\[ 0 \rightarrow T_P(-X) \rightarrow T_{P/X/C} \rightarrow T_X(-\log C) \rightarrow 0 \]

Then from the cohomology sequence we conclude

**Lemma 12.** Assume the vanishings

\[ H^1(N_{C/X/P}) = 0, H^2(T_P) = 0, H^3(T_P(-X)) = 0. \]

Then \( C \) is rigid-regular in \( X \), i.e. the pair \((C, X)\) has unobstructed deformations of dimension \( h^1(T_X(-\log C)) \).

**Example 13.** Let \( P = \mathbb{P}^n, n \geq 5 \) and \( X \subset P \) a smooth hypersurface of degree \( d \). Then if \( C \subset X \) is a curve of degree \( e \) and genus \( g \) with \( H^1(N_{C/X/P}) = 0 \), the pair \((C, X)\) has unobstructed deformations of dimension \( h^1(T_X/C) \). If \((C, X)\) is relatively regular in \( P \), the family of hypersurfaces carrying deformations of \( C \) is smooth of codimension \( e(d - n - 1) + (n - 4)(g - 1) \) near \( X \).

### 5.3. Singular case.

Consider a lci diagram of connected normal-crossing varieties

\[
C_0 = C_1 \cup C_2 \subset X_0 = X_1 \cup_Z X_2 \subset P_0 = P_1 \cup_Q P_2 \subset \mathcal{P}
\]

where each \( C_i, X_i, P_i \) and \( \mathcal{P} \) are smooth and there is a morphism \( \pi : \mathcal{P} \rightarrow B \) to a smooth curve such that \( P_0 = \pi^{-1}(0) \). Then a general fibre \( P_t = \pi^{-1}(t) \) is smooth. We have an exact sequence

\[ 0 \rightarrow N_{C_0/X_0/P_0} \rightarrow N_{C_0/X_0/P} \rightarrow \mathcal{O}_{X_0} \rightarrow 0. \]

We have an isomorphism \( H^0(\mathcal{O}_{X_0}) \simeq H^0(\mathcal{O}_Z) \) and \( \mathcal{O}_Z \) can be identified with the \( \mathcal{T}_{X_0}^1 \) which restricts on \( C_0 \) to \( \mathcal{T}_{C_0}^1 \), it follows that a deformation of \((C_0, X_0)\) in \( \mathcal{P} \) is a smoothing of \( C_0 \) and \( X_0 \) iff its image in \( \mathcal{O}_{X_0} \) is nonzero. Then from the long cohomology sequence we conclude

**Lemma 14.** Notations as above, assume

\[ H^1(N_{C_0/X_0/P_0}) = 0 = H^1(\mathcal{O}_{X_0}). \]

Then \((C_0, X_0, P_0)\) is smootheable to \((C_1, X_1, P_1)\) in a general fibre \( P_t \) of \( \pi \), i.e. there is a smooth variety \( T \) of dimension \( h^0(N_{C_0/X_0/P}) \), a smooth morphism \( T \rightarrow B \) with a point \( 0' \) over \( 0 \), and a triple \((C, \mathcal{X}, \mathcal{P} \times_B T)\) flat over \( T \), with special fibre \((C_0, X_0, P_0)\) and general fibre \((C_i, X_i, P_i)\) with \( C_i, X_i, P_i \) smooth and moreover \( H^1(N_{C_1/X_1/P_1}) = 0 \).

As for the vanishing in question, note that

\[ N_{C_0/X_0/P_0}|_{X_i} = N_{C_i/X_i/P_i}, i = 1, 2 \]

hence we have an exact sequence

\[ 0 \rightarrow N_{C_1/X_1/P_1}(-Z) \rightarrow N_{C_0/X_0/P_0} \rightarrow N_{C_2/X_2/P_2} \rightarrow 0. \]
5.4. Results. The main result of this section is the following

**Theorem 15.** Assume

\[
\begin{align*}
    d &\geq n + 1 \geq 5, \\
g &\geq 1, \\
e &> 2g + 2n, \\
\left(\frac{d + n - 2}{n}\right) &> ed/2.
\end{align*}
\] (8)

Then there exists a rigid-regular stably balanced pair \((C, X)\) in \(\mathbb{P}^n\) where \(C\) is a general smooth curve of genus \(g\) and degree \(e\) and \(X\) is a hypersurface of degree \(d\).

**Remark 16.** Assumptions as in Theorem 15, the family of pairs \((C, X)\) where \(C\) has maximal rank is clearly irreducible of the expected dimension, hence so is the corresponding family of hypersurfaces \(X\). As subfamily of the family of all hypersurfaces, the latter family has codimension

\[
h^1(N_{C/X}) = -\chi(N) = (d - n - 1)e + (n - 4)(g - 1).
\]

**Example 17.** There are stably balanced, rigid-regular curves of every degree \(5 < e < 72\) and genus \(0 < g < e/5 - 5\) on smooth quintic fourfolds.

**Proof of Theorem.** We use induction on \(d \geq n + 1\) and a suitable quasi-cone degeneration in a fan degeneration of \(\mathbb{P}^n\) as in §1. We construct a suitable curve on a quasi-cone degeneration as in §5.3.

\[
C_0 = C_1 \cup C_2 \subset X_0 = X_1 \cup Z X_2 \subset P_0 = P_1 \cup Q P_2.
\]

Thus \(P_1 = B_p \mathbb{P}^n\) with exceptional divisor \(Q, P_2 = \mathbb{P}^n\) with \(Q \subset P_2\) a hyperplane, \(X_1 \subset P_1\) is the blowup of a hypersurface of degree \(d\) with multiplicity \(d - 1\) at \(p\), and \(X_2 \subset P_2\) is a hypersurface of degree \(d - 1\). Via projection from \(p\) we may realize \(X_1\) as \(B_Y \mathbb{P}^{n-1}\) where \(Y\) is a \((d - 1, d)\) complete intersection.

We now assume \(d \geq n + 1\) and work by induction on \(d\). Assume to begin with that \(e\) is even. We first construct \(C_2 \subset X_2 \subset P_2\). Begin with a general curve \(C_2 \in \mathbb{P}^n = P_2\) of genus \(g \geq 0\) and degree \(e_2 = e/2 > g + n\) satisfying the inequalities (8). Note that these imply

\[
\left(\frac{d + n - 1}{n}\right) > e_2(n + 1) + 1 - g.
\]
By the Maximal Rank Theorem of Ballico-Ellia [2], the restriction map

\[ H^0(\mathcal{O}_P^n(d-2)) \to H^0(\mathcal{O}_{C_2}(d-2)h) \]

is surjective and we may choose a general hypersurface \( X_2 \subset \mathbb{P}^n \) of degree \( d - 1 \) containing \( C_2 \), which will moreover be smooth. Moreover by induction or, in the initial case \( d = n + 1 \), by Theorem \([8]\) the normal bundle \( N_{C_2/X_2} \) is stably balanced.

Furthermore, by the theorem of Atanasov-Larsen-Yang \([1]\), we have \( H^1(N_{C_2/P_2}(-h)) = 0 \). From this, together with the basic exact sequence (5) it follows that

\[ H^1(N_{C_2/X_2/P_2}(-h)) = 0 \]

and also that by varying \( C_2, X_2 \) while fixing \( Z = X_2 \cap Q \), the points \( W \setminus C_2 \cap Z = \{p_1, \ldots, p_{e_2}\} \) will be \( e_2 \) general points on \( Z \).

Now to construct \( C_1 \subset X_1 \subset P_1 \), identify \( Z \) with \( F_{d-1} \subset \mathbb{P}^{n-1} \) and let \( \ell_i \subset \mathbb{P}^{n-1} \) be a general line through \( p_i \). By the Maximal Rank Theorem of Hartshorne and Hirschowitz [7], the restriction map

\[ H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d-1)) \to \bigoplus H^0(\mathcal{O}_{c_i}(d-1)) \]

is surjective. Let \( F_d \subset \mathbb{P}^{n-1} \) be a general hypersurface containing \( \ell_i \cap F_{d-1} \setminus p_i \) but not \( p_i \) for \( i = 1, \ldots, e_2 \). Now I claim that for any \( q \in \ell_i \setminus p_i, i = 1, \ldots, e_2 \), \( F_d \) has general tangent hyperplane at \( q \). Indeed this is clear because given \( q \) we may take an \( F_d \) of the form \( F_{d-1}' \cup F_1 \) with \( F_1 \) general trough \( q \) and \( F_{d-1}' \) through all the other points. Therefore a general \( F_d \) has general tangent hyperplane at the given \( q \), hence also for all \( q \) simultaneously.

Finally let \( Y = F_{d-1} \cap F_d \) and \( X_1 = B_Y \mathbb{P}^{n-1} \subset P_1 \) as above. Note that the lines \( \ell_i \) lift to conics \( D_i \) on \( X_1 \) and we set

\[ C_1 = D_1 \cup \ldots \cup D_{e_2} \]

Note that by the claim above, the normal bundle \( N_{D_1/X_1} \) is a general modification of \( N_{L_i/\mathbb{P}^{n-1}} \), hence it is balanced. In particular, \( H^0(N_{C_1/X_1}) = 0 \).

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Now on \( P_1 \) we have, where \( H \) is a hyperplane from \( \mathbb{P}^n \),

\[ X_1 \in |dH - (d-1)Q| = |dL + Q| \]

hence

\[ N_{X_1/P_1} = \mathcal{O}(dL + Z) = \mathcal{O}(2d - 1)L - E_Y \]

From this it follows easily that \( H^1(N_{X_1/P_1}) = 0 \). Also clearly \( H^1(N_{C_1/P_1}) = 0 \). Moreover by the aforementioned Maximal Rank Theorem of Hartshorne and Hirschowitz [7], the restriction map

\[ H^0(P_1, dL + Q) \to H^0(C_1, (dL + Q)|_{C_1}) \]
is surjective. Hence we conclude that

\[ H^1(N_{C_1/X_1/P_1}) = 0. \]

Therefore by the obvious exact sequence

\[ 0 \to N_{C_2/X_2/P_2}(-1) \to N_{C_0/X_0/P_0} \to N_{C_1/X_1/P_1} \to 0 \]

(or just as well, the analogous one with \(C_1, C_2\) etc. interchanged), we conclude

\[ H^1(N_{C_0/X_0/P_0}) = 0. \]

Finally we claim that \(C_0\) is rigid on \(X_0\), hence likewise for the smoothing. To see this note that

\[ N_{C_0/X_0}|_{C_i} = N_{C_i/X_i}, i = 1, 2 \]

and as we have seen \(N_{C_1/X_1}|_{D_i}\) is a general modification of \(N_{D_i/P^{n-1}}\) and in particular it is strictly negative and its upper subspace at \(p_i\) is general. As for \(N_{C_2/X_2}\), in the initial case \(d = n + 2\), we know it is balanced and we have \(\chi(C_2/X_2) = n - 4 < e_2\) so \(h^0(N_{C_2/X_2}) < e_2\). In case \(d > n + 2\) we have inductively that \(h^0(N_{C_2/X_2}) = 0\). Thus in all cases it follows that

\[ H^0(N_{C_0/X_0}) = 0. \]

Moreover an argument as in the proof of [10], Theorem 6 or [12], Theorem 1 (essentially, blowing down the tails \(D_i\)) shows that for a smoothing \(C' \subset X'\) of \(C_0 \subset X_0\), the normal bundle \(N_{C'/X'}\) is essentially a deformation of a semi-general down modification of \(N_{C_2/X_2}\) corresponding to the upper subspaces of \(T_{p_i}D_i\), and consequently by Lemma \(\Pi\), \(C' \subset X'\) is stably balanced. This completes the proof of Theorem \(\text{[15]}\) in case \(e\) is even.

The proof for \(e = 2e_2 - 1\) odd is similar except we make \(D_{e_2}\) a 'line' (i.e. a ruling of \(P_1/P^{n-1}\)) instead of conic by making \(F_d\) go through the point \(p_{e_2}\) as well. \(\Box\)

6. APPENDIX: NORMALLY SEMISTABLE ELLIPTICS

An important role in our results is played by the result of Ein-Lazarsfeld [6] on semistability of the normal bundle of an elliptic normal curve, of degree \(n + 1\) in \(P^n\). Here we will give a short self-contained proof of an analogous result, showing semistability of the normal bundle of a general elliptic curve of degree \(2n - 2\) or \(\geq 3n - 3\) in \(P^n\). This result, which is apparently new for \(n \geq 4\), can be used in place of that of [6] to give a slightly weaker version of Theorem [6] requiring higher curve degrees.

**Theorem 18.** A general elliptic curve of degree \(e = 2n - 2\) or \(e \geq 3n - 3\) in \(P^n, n \geq 3\), has semistable normal bundle.
Proof. We will do the case \( n = 2m \) even, \( n \geq 4 \) as the case \( n \) odd is similar and simpler (see comments at the end of the proof). Assume first that \( e = 2n - 2 \). Consider a fang degeneration

\[ P_0 = P_1 \cup_E P_2 \]

where

\[ P_1 = B_{\mathbb{P}^m} \mathbb{P}^n \supset E_1 = \mathbb{P}^m \times \mathbb{P}^{m-1} \]

\[ P_2 = B_{\mathbb{P}^{m-1}} \mathbb{P}^n \supset E_2 = \mathbb{P}^m \times \mathbb{P}^{m-1} \]

in which \( E_1 \subset P_1, E_2 \subset P_2 \) are exceptional divisors and \( P_0 \) is constructed via an isomorphism \( E_1 \simeq E \simeq E_2 \). There is a standard smoothing of \( P_0 \) to \( \mathbb{P}^n \). Consider curves

\[ C_1 \subset P_1, C_2 \subset P_2 \]

with each being a birational transform of a rational normal curve, such that

\[ C_1 \cap E = C_2 \cap E = \{ p, q \}. \]

Then \( C_0 = C_1 \cup C_2 \) is a nodal, lci curve in \( P_0 \) and smooths out to an elliptic curve \( C_0 \) of degree \( 2n - 2 \) in \( \mathbb{P}^n \) whose normal bundle is a deformation of \( N_{C_0/P_0} = N_{C_1/P_1} \cup N_{C_2/P_2} \). Now using Lemma 31 of [11], we may assume each \( C_i \) is balanced in \( P_i, i = 1, 2 \), which means that

\[ N_{C_i/P_i} = \mathcal{O}(2m + 2) \oplus (2m - 2) \mathcal{O}(2m + 1), \]

\[ N_{C_2/P_2} = (2m - 2) \mathcal{O}(2m + 1) \oplus \mathcal{O}(2m). \]

Consequently,

\[ \wedge^i N_{C_i/P_i} = \binom{2m - 2}{i - 1} \mathcal{O}(i(2m + 1) + 1) \oplus \binom{2m - 2}{i} \mathcal{O}(i(2m + 1)), \]

\[ \wedge^i N_{C_2/P_2} = \binom{2m - 2}{i} \mathcal{O}(i(2m + 1)) \oplus \binom{2m - 2}{i - 1} \mathcal{O}(i(2m + 1) - 1). \]

Note that the slope of \( N_{C_0/P_0} \), hence that of \( N_{C_0/\mathbb{P}^n} \) equals \( 2n + 2 = 4m + 2 \). Now we have natural identifications

\[ N_{C_i/P_i}|_p = T_p E_i, i = 1, 2 \]

and likewise for \( q \). Now choosing the identification \( T_p E_1 \simeq T_p E_2 \) sufficiently general, we may assume the upper subspaces of \( \wedge^i N_{C_1/P_1}|_p \) and \( \wedge^i N_{C_2/P_2}|_p \) are in general position, and this implies that \( \wedge^i N_{C_0/P_0} \) has no line subbundle of bidegree \( (i(2m + 1) + 1, i(2m + 1)) \). This implies that the total degree of any line subbundle of \( \wedge^i N_{C_0/P_0} \) is at most \( i(4m + 2) \) which is equal to the slope of \( \wedge^i N_{C_0/\mathbb{P}^n} \). Now a line subbundle \( M \) of \( \wedge^i N_{C_0/\mathbb{P}^n} \)
specializes either to a line subbundle, or to a non-inverible subsheaf $M_0$ of $\wedge^i N_{C_0/P_0}$ which has rank 1 on each component, and in the latter case we have
\[ \deg(M) < \deg(M_0|_{C_1}/(\text{torsion})) + \deg(M_0|_{C_2}/(\text{torsion})). \]
Therefore in either case we have
\[ \deg(M) \leq i(2m + 2). \]
Since this is true for all $i$, it follows that $N_{C_*/P^n}$ is (cohomologically) semistable (assuming $n$ is even and $e = 2n - 2$).

For the case $n$ even and $e \geq 2n - 3$ we take $C_1$ rational normal as above and $C_2$ rational of degree $e_2 \geq 2n - 1$, and again use the appropriate cases of Lemma 31 of [11]. Then the proof proceeds similarly.

Finally for $n = 2m + 1$ odd we just take $P_1 \simeq P_2 \simeq B_{p=\mathbb{P}^n}$ and proceed similarly. \qed

**Remark 19.** As we show in the next section, a deeper analysis of the gluing of the normal bundles of $C_1$ and $C_2$ at $p$ and $q$ shows in some cases that the normal bundle of $C_0$, hence of $C_*$, is in fact stable rather than just semi-stable.

### 7. Stability

Before we prove Theorem 20, we need to introduce some notation. Let $P_0 = P_1 \cup E_1 P_1$ where $E_1$ is the space of hyperplanes through $P_1$. Thus $P_0$ is defined by an isomorphism $E_1 \simeq E_2$ interchanging the $P_i$ (‘horizontal’) and $\hat{P}_i$ (‘vertical’) factors.

We assume first that $e = 2n - 2 = 4m$. Consider a curve $C_0 = C_1 \cup_{p,q} C_2 \subset P_0$
where \( C_i \subset P_i \) is the birational transform of a rational normal curve (degree \( n \)). Then \( C_0 \subset P_0 \) smooths to an elliptic curve \( C' \) of degree \( 2n - 2 \) in \( \mathbb{P}^n \), and we claim that normal bundle \( N_{C'}/\mathbb{P}^n \) is (cohomologically) stable. By [11], Lemma 31, we have

\[
N_{C_i}/P_i \simeq 2mO(2m + 2).
\]

Following the proof of Theorem 18 above, our first goal is to prove that \( N_{C_0}/P_0 \) has no ‘semi-destabilizing’ subbundles of type \( (2m + 2, 2m + 2) \).

We have isomorphisms

\[
H^0(N_{C_i}/P_i(-2m - 2)) \xrightarrow{e^i_p} T_pE_i, H^0(N_{C_i}/P_i(-2m - 2)) \xrightarrow{e^i_q} T_qE_i.
\]

It is easy to check that \( \eta^i : e^i_q(e^i_p)^{-1} \) sends the vertical subspace \( T_p\mathbb{P}^m_i = T_p(E_i/\mathbb{P}^m) \subset T_pE_i \) to a ‘quasi-horizontal’ subspace of \( T_qE_i \), i.e. one having trivial intersection with the vertical subspace (though not necessarily equal to \( T_p\mathbb{P}^m_i \)).

Now given points \( p, q \), the map \( \eta \) depends on the choice of rational normal curve transform \( C_i \) through \( p, q \) and we have

\[
N_{C_i}/P_i(-p - q) \simeq 2mO(2m).
\]

In particular the restriction map

\[
H^0(N_{C_i}/P_i(-p - q)) \rightarrow N_{C_i}/P_i(-p - q)|_{\{p, q\}}
\]

is surjective, hence

\[
H^0(N_{C_i}/P_i(-p - q)(-p)) \rightarrow N_{C_i}/P_i(-p - q)|_{\{q\}}
\]

is surjective. This means that we can move the \( C_i \) through \( p, q \), and with a given vertical tangent direction \( k.v \in T_pE_i \) and any given quasi-horizontal tangent direction at \( q \), i.e. the induced map

\[
\omega^{i}_{p,q} = (\text{projection}) \circ \eta^i : T_p\mathbb{P}^m_i \rightarrow T_q\mathbb{P}^m_i
\]

is general up to scalar multiple. Likewise with \( p, q \) interchanged.

Now if \( N_{C_0}/P_0 \) has a line subbundle of type \( (2m + 2, 2m + 2) \), it would mean that the map \( \omega^1_{p,q} \circ \omega^2_{q,p} \) has a fixed vector up to scalars which by general choice of \( C_1, C_2 \) we may assume is not the case. Thus \( N_{C_0}/P_0 \) has no semi-destabilizing invertible subsheaves. Then by an argument as in the proof of Theorem 18, \( N_{C_0}/P_0 \) has no destabilizing non-invertible rank-1 subsheaves either.

Next, again as in the proof of Theorem 18 above, we may apply a similar argument to show there are no rank-1 de-semistabilizing subsheaves of \( \wedge^i N_{C_0}/P_0 \) for all \( i \). Therefore we may conclude, similarly as in the above proof, that \( N_{C_0}/P_0 \), and its smoothings, are stable. This completes the proof in case \( e = 4m \).
In case $e = \ell m$, $\ell \geq 6$ we use the same $C_2$ but for $C_1$ we take the transform of a rational curve of degree $(\ell - 2)m + 1$, which is again perfect. Using [11], Lemma 31 again, we conclude as above. \hfill \Box

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