COMPOSITE FINITE-TIME CONVERGENT GUIDANCE LAW FOR MANEUVERING TARGETS WITH SECOND-ORDER AUTOPILOT LAG

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ABSTRACT

This paper aims to develop a new finite-time convergent guidance law for intercepting maneuvering targets accounting for second-order autopilot lag. The guidance law is applied to guarantee that the line of sight (LOS) angular rate converges to zero in finite time and results in a direct interception. The effect of autopilot dynamics can be compensated based on the finite-time backstepping control method. The time derivative of the virtual input is avoided, taking advantage of integral-type Lyapunov functions. A finite-time disturbance observer (FTDOB) is used to estimate the lumped uncertainties and high-order derivatives to improve the robustness and accuracy of the guidance system. Finite-time stability for the closed-loop guidance system is analyzed using the Lyapunov function. Simulation results and comparisons are presented to illustrate the effectiveness of the guidance strategy.

Key Words: Finite-time backstepping control, finite-time disturbance observer, second-order autopilot lag, guidance law, maneuvering targets.

I. INTRODUCTION

Guidance law for tactical missiles plays an important role in the performance of intercepting maneuvering targets. The well-known proportional navigation (PN) guidance law has been proven to be effective in terms of simplicity, performance, and ease of implementation [1–3]. Although the classical PN guidance law offers satisfactory performance for non-maneuvering or weakly maneuvering targets, it may not hold true in engaging highly maneuvering and agile targets, where the performance of the PN guidance law drastically degrades [33]. To this end, advanced variants of the PN guidance law, such as the augmented PN guidance law [1] are superior to the PN law. Guidance laws based on advanced nonlinear control theories and robust control theories, such as H-infinity guidance law [4], Lyapunov-based nonlinear guidance law [5], L2 gain guidance law [6], differential game guidance laws [7,8], and sliding mode control-based guidance laws [9–12], can also be found.

These laws generally require explicit target acceleration information. Nonetheless, knowledge of the target motion is not usually available from sensors on board. Thus, implementation of such guidance laws is restrictive. In this regard, target acceleration has been frequently regarded as external disturbance and compensated as feedforward term. Disturbance observer (DOB) is a well-known approach used to estimate disturbance of the system and improve the performance of the controller [13–15]. In [15], a guidance law is designed based on the integral sliding mode control method, and a nonlinear DOB is used to eliminate the influence of target maneuvers. A formulation of terminal guidance law for a missile intercepting maneuvering target is investigated in [16]; in this study, a nonlinear DOB is employed to improve the system’s performance and avoid the chattering phenomenon.

In many realistic scenarios such as defense against incoming high-speed tactical ballistic missiles (TBMs), the interception time of the end-game scenario is only several seconds. Therefore, the LOS angular rate is required to converge rapidly to zero to maintain the collision triangle. Finite-time convergent (FTC) control law is an effective feedback control methodology that exhibits a high robustness and convergence rate against uncertainty. The authors in [17] propose a novel finite-time convergent guidance law to prove that the LOS angular rate converges to zero or the presence of a small range around zero before interception. A nonsingular terminal sliding mode-based guidance law is developed in [18] to guarantee finite-time stability for maneuvering target interception. However, the undesired chattering phenomenon is the main limitation for implementing these guidance laws.

An autopilot lag exists between the command and acceleration missile achieved; this lag exerts an undesirable effect on the performance of the guidance system. The
integrated design of the missile autopilot and guidance system may improve guidance precision. In [19,20], guidance laws considering first-order autopilot dynamics are designed based on the backstepping method. However, missile autopilot is always considered a complex system, and the properties of autopilot can be well approximated by the second-order lag rather than the first-order lag [21–26]. The authors in [25] present a guidance law with second-order autopilot dynamics compensation, but only asymptotical stability is proven. Accounting for second-order autopilot lag, a finite-time convergent guidance law based on the standard backstepping method is proposed. However, missile autopilot can be well approximated by the second-order differentiations of intermediate controls are required analytic derivative of virtual control law in the derivation process.

Motivated by the above discussion, this work investigates a new composite finite-time convergent guidance law based on the standard backstepping control technique is developed in [24]. The required analytic differentiations of intermediate controls are obtained using a tracking differentiator.

This paper is organized as follows. Section II presents the preliminaries and problem formation. Section III focuses on the detailed design procedure of the proposed guidance law. Section IV presents the results of the closed-loop system stability. Section V states the simulation results and compares the proposed law with existing formulations. Finally, Section VI discusses the conclusions.

II. PRELIMINARIES AND PROBLEM FORMATION

2.1 Some lemmas

The following lemmas should be recalled because they play an important role in any subsequent study of finite-time convergent guidance law.

Lemma 1. [28]. Assume a positive definite, continuously differentiable function \( V(x) \) defined on \( U \subset \mathbb{R}^n \) and real numbers \( c > 0 \) and \( 0 < \alpha < 1 \), such that \( \dot{V}(x) + c(V(x))^\alpha \) is a negative semi-definite function on \( U \subset \mathbb{R}^n \); an area \( U_0 \subset \mathbb{R}^n \) that exists, and any \( V(x) \) starts from \( U_0 \subset \mathbb{R}^n \) can reach \( V(x) \equiv 0 \) in finite time

\[
T_{\text{reach}} \leq \frac{(V(x_0))^{1-\alpha}}{c(1-\alpha)}
\]

where \( V(x_0) \) is the initial value of \( V(x) \).

Lemma 2. [29]. If \( p_1 > 0 \) and \( 0 < p_2 \leq 1 \), then \( \forall x \in \mathbb{R}, \forall y \in \mathbb{R} \)

\[
|x|^{p_1}y^{p_2} \leq 2^{1-p_1}|x|^{p_1} - |y|^{p_2}
\]

Lemma 3. [29]. Let \( c \) and \( d \) be positive constants. Given any positive number \( \gamma > 0 \), the following inequality holds

\[
|x|^{c+y} \leq \frac{c}{c+d}y|x|^{c+d} + \frac{d}{c+d}y^{-c/d}|y|^{c+d}
\]

Lemma 4. [29]. For any real numbers \( x_i, i = 1, \ldots, n \), and \( 0 < b \leq 1 \), the following inequality holds

\[
(|x_1| + \cdots + |x_n|)^b \leq |x_1|^b + \cdots + |x_n|^b
\]

2.2 Nonlinear engagement of pursuit-evasion motion

The planar engagement scenario, which consists of a missile pursuing a maneuvering target, is shown in Fig. 1, where the symbols \( M \) and \( T \) denote a missile and a target, respectively; the flight path angles of the missile and target are represented by \( \gamma_M \) and \( \gamma_T \) respectively; \( r \) is the relative range between the target and missile; \( q \) is the LOS angle; the velocities of the missile and target are represented by \( V_M \) and \( V_T \) respectively, and the accelerations of the missile and target are represented by \( a_M \) and \( a_T \) respectively.

In practice, the end-game scenario has no thrust and aerodynamics serve as control input. We assume that the command acceleration of the missile is perpendicular to its velocity vector. The missile and the target are assumed to be point masses to simplify the dynamics of engagement. Furthermore, the magnitudes of velocity of the missile and the target are assumed to be constant.

Under the above-stated assumptions, the corresponding equations of motion that depict the engagement scenario are formulated as in [27].

\[
\dot{r} = V_T \cos(\gamma_T - q) - V_M \cos(\gamma_M - q)
\]

\[
\dot{q} = \frac{1}{r} [V_T \sin(\gamma_T - q) - V_M \sin(\gamma_M - q)]
\]

\[
\dot{\gamma}_M = \frac{a_M}{V_M}
\]
\[ \dot{\gamma}_T = \frac{a_T}{V_T} \tag{4} \]

Let \( V_r = \dot{r}, V_q = \dot{q} \), then, differentiating Eqs. 1–2 and rearranging the terms yield [30].

\[ \dot{V}_r = \frac{V_r^2}{r} + a_{Tr} - a_{Mr} \tag{5} \]

\[ \dot{V}_q = -\frac{V_r V_q}{r} + a_{Tq} - a_{Mq} \tag{6} \]

where, \( a_{Tr} = a_T \sin (q - \gamma_T), a_{Mr} = a_M \sin (q - \gamma_M) \), \( a_{Tq} = a_T \cos (q - \gamma_T) \), and \( a_{Mq} = a_M \cos (q - \gamma_M) \). \( a_{Mr} \) and \( a_{Tr} \) represent the projections of the missile and the target acceleration along the LOS, respectively. \( a_{Mq} \) and \( a_{Tq} \) are the projections of the missile and the target acceleration orthogonal to LOS, respectively. The target acceleration \( a_T \) is usually difficult to measure directly [30]. Therefore \( a_{Tr} \) and \( a_{Tq} \) are considered unknown disturbances.

Fig. 1. 2D target-missile engagement geometry.

Normally, the acceleration along missile velocity of most tactical missiles in the terminal phase cannot be controlled, and a well-known guidance strategy is to nullify the LOS angular rate \( \dot{q} \) [17]. Thus, the purpose of the guidance system is adjusting lateral acceleration \( a_M \) to keep \( \dot{q} \) in a small range around zero and end up with a successful interception. Moreover, to remove the combined effect of the nonlinear term and the unknown target maneuver, a new quantity is defined as

\[ h = -\frac{V_r V_q}{r} + a_{Tq} - a_M \cos(q - \gamma_M) + a_M \tag{7} \]

then 6 can be rewritten as

\[ \dot{V}_q = h - a_M \tag{8} \]

**Assumption 1.** Consider a positive constant \( \sigma \) and a positive integer \( m \), the lumped uncertainty \( h \) is piecewise continuous and satisfies

\[ \left| \frac{d^m h}{dt^m} \right| \leq \sigma \tag{9} \]

**Remark 1.** During the guided flight process, both the missile and target have physical limitations, namely, coefficients that compose lumped uncertainty are time-varying bounded. Thus, the assumption considered above is not restrictive.

In practice, the missile autopilot can be well approximated by the second-order system as [25].

\[ \ddot{a}_M = -2\xi_a \omega_a \dot{a}_M - \omega_a^2 a_M + \omega_a^2 a_{Mc} \tag{10} \]

where \( a_M \) denotes the achieved missile acceleration, \( a_{Mc} \) is the commanded acceleration of missile, and \( \xi_a \) and \( \omega_a \) denote the damping ration and natural frequency, respectively, of the missile autopilot. However, the performance of the missile is adversely affected by the autopilot lag between the commanded acceleration and the real acceleration of the missile. Compensation of dynamics is an efficacious way to remove the influence of system lag.

The objective of this paper is to design a guidance law in the presence of an unknown target maneuver and autopilot dynamics. Based on derived guidance law, the LOS angular rate of the close-loop system is regulated to zero in finite time.
III. GUIDANCE LAW DESIGN WITH FINITE-TIME CONVERGENCE

In this section, finite-time convergent guidance law is presented using the finite-time backstepping control technique and FTDOB. The systematic procedure of guidance law design and the proof of close-loop finite-stability are given by the following.

3.1 Control algorithm

We define state variables \( x_1 = V_q, x_2 = -a_M, \) and \( x_3 = -\dot{a}_M. \) An integrated guidance and control system can be written as

\[
\begin{align*}
\dot{x}_1 &= h + x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -2\zeta_a \omega_a x_3 - \omega_a^2 x_2 - \omega_a^3 a_M.
\end{align*}
\]

(11)

An FTDOB [31] is employed here to estimate disturbance and compensate control input to suppress the lumped uncertainty \( h. \) A third-order observer can then be constructed as

\[
\begin{align*}
\dot{z}_0 &= v_0 + x_2, v_0 = -\lambda_3 L^{1/4} |z_0 - x_1|^{1/4} \text{sgn}(z_0 - x_1) + z_1 \\
\dot{z}_1 &= v_1, v_1 = -\lambda_2 L^{1/3} |z_1 - v_0|^{2/3} \text{sgn}(z_1 - v_0) + z_2 \\
\dot{z}_2 &= v_2, v_2 = -\lambda_1 L^{1/2} |z_2 - v_1|^{1/2} \text{sgn}(z_2 - v_1) + z_3 \\
\dot{z}_3 &= -\lambda_0 L \text{sgn}(z_3 - v_2)
\end{align*}
\]

(12)

where \( \lambda_i > 0 (i = 0, 1, 2, 3) \) and \( L > 0 \) are the observer gains to be designed, with appropriate values of observer gains, \( z_0, z_1, z_2, z_3 \) approaches to \( V_q, h, \dot{h}, \ddot{h} \), respectively. According to [31], the following lemma is obtained.

Lemma 5. Under Assumption 1, the estimation error dynamics of FTDOB 12 is governed by

\[
\begin{align*}
\dot{\sigma}_0 &= -\lambda_3 L^{1/4} |\sigma_0|^{3/4} \text{sgn}(\sigma_0) + \sigma_1 \\
\dot{\sigma}_1 &= -\lambda_2 L^{1/3} |\sigma_1 - \sigma_0|^{2/3} \text{sgn}(\sigma_1 - \sigma_0) + \sigma_2 \\
\dot{\sigma}_2 &= -\lambda_1 L^{1/2} |\sigma_2 - \sigma_1|^{1/2} \text{sgn}(\sigma_2 - \sigma_1) + \sigma_3 \\
\dot{\sigma}_3 &= -\lambda_0 L \text{sgn}(\sigma_3 - \sigma_n - 1) + [-L, L]
\end{align*}
\]

(13)

where the inclusion \( \bar{h} \in [-L, L] \) is used in the last line, and \( \sigma_0 = z_0 - x_1, \sigma_1 = z_1 - h, \sigma_2 = z_2 - \dot{h}, \) and \( \sigma_3 = z_3 - \ddot{h} \) are the estimation errors. A time constant exists \( t_0 > 0 \) such that \( \sigma_i = 0 (i = 0, 1, 2, 3) \) for all \( t > t_0 \) because the observer is finite-time stable.

Remark 2. The principle selection of observer gains \( \lambda_i > 0 (i = 0, 1, 2, 3) \) and \( L > 0 \) can be followed from [31]. The convergent rate will be very fast when the observer gain \( L \) is large enough, but a very large value of \( L \) will lead to unexpected overshoot and an excessive acceleration command. Hence, \( L \) should be designed under certain circumstances.

Remark 3. The defect of lumped uncertainty concludes from error dynamics 13 that \( \sigma_0 = \sigma_1 = \sigma_2 = \sigma_3 = 0 \) always holds when the initial states are set as \( z_0(0) = V_q, z_1(0) = 0, z_2(0) = 0, \) and \( z_3(0) = 0. \) Hence, the FTDOB has no influence on the closed-loop system and the performance of the nominal system is retained. In addition, the uncertainty can be estimated regardless of control input, then the FTDOB and guidance law can be designed independently.

The integrated system can be expressed as follows

\[
\begin{align*}
\dot{x}_1 &= z_1 - (z_1 - h) + x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -2\zeta_a \omega_a x_3 - \omega_a^2 x_2 - \omega_a^3 a_M.
\end{align*}
\]

(14)

\[z_1\] will converge to uncertainty \( h \) when \( t > t_s. \) According to Lemma 5, the system reduces to

\[
\begin{align*}
\dot{x}_1 &= x_2 + z_1 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -2\zeta_a \omega_a x_3 - \omega_a^2 x_2 - \omega_a^3 a_M.
\end{align*}
\]

(15)

The nonlinear system 15 is in a parametric strict feedback form. The backstepping method is an effective way to cancel the influence of uncertainty for the aforementioned system. Similar to the traditional backstepping method, a finite-time backstepping control method proposed in [29] is used to deduce guidance law step-by-step as follows.

Step 1. Let \( x_1 = x_1, x_2 = x_2 + z_1, x_3 = x_3 + \dot{z}_1, \) and system 15 can be rewritten as

\[
\begin{align*}
\dot{\bar{x}}_1 &= \bar{x}_2 \\
\dot{\bar{x}}_2 &= \bar{x}_3 \\
\dot{\bar{x}}_3 &= -2\zeta_a \omega_a (\bar{x}_3 - \bar{z}_1) - \omega_a^2 (\bar{x}_2 - z_1) - \omega_a^3 a_M + \bar{z}_1
\end{align*}
\]

(16)

Let \( p = p/q \in (0, 1/3) \) and \( q_k = 1 - (k - 1)p, \) where \( p \) is an even integer and \( q \) is an odd integer. We have \( 1 = q_1 > q_2 > \cdots > q_{n + 1} > 0, q_{i + 1} = q_2 + q_i - 1, \) and \( q_i - q_{i + 1} = 1. \) A \( C^1 \), positive definite and proper Lyapunov function is chosen as follows [32]

\[
V_k(x_1, \ldots, x_k) = \sum_{l=1}^{k} W_l,
\]

(17)

\[W_l = \frac{1}{p} \left( x_l^{q_k} - (\bar{x}_l)^{q_k} \right)^{2 - q_k} ds\]

where \( \bar{x}_l \) represents the virtual control law, defined by
\[ x^*_1 = 0 \quad \xi_1 = x_1^{1/q_1} = x_1 \]
\[ \vdots \]
\[ x^*_k = -\kappa_k \xi^{q_k}_{k-1} \quad \xi_k = x_k^{1/q_k} - (\xi^*_k)^{1/q_k} \]  
with constants \( \kappa_k > 0 \).

Taking the derivative of \( V_1(x_1) \) along system 16 produces

\[ \dot{V}_1(x_1) = (x_1)^{2-q_1} (x_1^2 + x_2 - x_3^2) = \xi_1 (x_2 - x_3^2) - \kappa_1 (\xi^*_1)^{2-r} \]

(19)

Step 2. In the following proposition, we can obtain a useful property proven in [32].

**Proposition 1.** \( W_k(x_1, \ldots, x_k) \) is \( C^1 \), \( \partial W_k / \partial x_k = \xi_k^{q_k-1} \) and for \( l = 1, \ldots, k-1 \)

\[ \frac{\partial W_k}{\partial x_l} = -(2 - q_k) \frac{\partial (x_{l+1})^{1/q_{l+1}}}{\partial x_l} (s^{1/q_l} - (x_{l+1})^{1/q_{l+1}})^{2-q_k} ds \]

According to Proposition 1, computing the first-time derivative of \( V_2 \) gives

\[ \dot{V}_2 (x_1, x_2) = \dot{V}_1 \frac{\partial W_1}{\partial x_1} + \frac{\partial W_2}{\partial x_1} \dot{x}_1 + \frac{\partial W_2}{\partial x_2} \dot{x}_2 \]

\[ = -\kappa_1 \xi_1^{2-r} + \xi_1 (x_2 - x_3^2) + \frac{\partial W_2}{\partial x_1} \dot{x}_1 + \frac{\partial W_2}{\partial x_2} \dot{x}_2 \]

(20)

From Lemma 2 and Lemma 3, the equation is

\[ \xi_1 (x_2 - x_3^2) \leq 2 |\xi_1| |\xi_2|^{2-r} \leq 2 \left( \frac{2}{2-r} \right) |\xi_1|^{2-r} \]

\[ + \frac{2q_2}{2-r} |\xi_2|^{2-r} \leq 2c_21 |\xi_1|^{2-r} + c_22 |\xi_2|^{2-r} \]

(21)

where \( c_{21} > 2/(2-r) \) and \( c_{22} > 2q_2/(2-r) \) are positive constants.

We introduce the following proposition whose proof is given in the appendix to facilitate the construction of the finite-time controller and estimate the right-hand side of 20.

**Proposition 2.** \( p_{21} \) and \( p_{22} \) are positive constants such that

\[ \frac{\partial W_2}{\partial x_1} \dot{x}_1 \leq p_{21} |\xi_1|^{2-r} + p_{22} |\xi_2|^{2-r} \]

(22)

Substituting 21 and 22 into 20 gives

\[ \dot{V}_2 (x_1, x_2) \leq (-\kappa_1 + c_{21} + p_{21}) |\xi_1|^{2-r} + (c_{22} + p_{22}) |\xi_2|^{2-r} - \kappa_2 |\xi_2|^{2-r} + \xi_2^{2-q_1} (\bar{x}_3 - \bar{x}_3) \]

(23)

Step 3. According to Proposition 1, evaluating the first-time derivative of \( V_3 \) gives

\[ \dot{V}_3 (x_1, x_2, x_3) = \dot{V}_2 + \frac{\partial W_3}{\partial x_1} \dot{x}_1 + \frac{\partial W_3}{\partial x_2} \dot{x}_2 + \frac{\partial W_3}{\partial x_3} \dot{x}_3 \]

\[ + \frac{\partial W_3}{\partial x_1} \xi_1 \leq (-\kappa_1 + c_{21} + p_{21}) |\xi_1|^{2-r} + (c_{22} + p_{22}) |\xi_2|^{2-r} - \kappa_2 |\xi_2|^{2-r} + \xi_2^{2-q_1} (\bar{x}_3 - \bar{x}_3) \]

(24)

Similar to the derivative process in Step 2 we have

\[ \xi_2^{2-q_1} (\bar{x}_3 - \bar{x}_3) \leq c_{31} |\xi_2|^{2-r} + c_{32} |\xi_3|^{2-r} \]

(25)

where \( c_{31} > 2(2 - q_2)/(2 - r) \) and \( c_{32} > 2q_3/(2 - r) \) are positive constants.

The proposition whose proof is given in the appendix follows.

**Proposition 3.** \( p_{31}, p_{32}, \) and \( p_{33} \), are positive constants such that

\[ \frac{\partial W_3}{\partial x_1} \dot{x}_1 + \frac{\partial W_3}{\partial x_2} \dot{x}_2 + \frac{\partial W_3}{\partial x_3} \dot{x}_3 \leq p_{31} |\xi_1|^{2-r} + p_{32} |\xi_2|^{2-r} + p_{33} |\xi_3|^{2-r} \]

(26)

Substituting 25 and 26 into 24 gives

\[ \dot{V}_3 (x_1, x_2, x_3) \leq (-\kappa_1 + c_{21} + p_{21} + p_{31}) |\xi_1|^{2-r} \]

\[ + (-\kappa_2 + c_{22} + c_{31} + p_{22} + p_{32}) |\xi_2|^{2-r} \]

\[ + (c_{32} + p_{33}) |\xi_3|^{2-r} \]

\[ + \xi_2^{2-q_1} (2c_{31} \omega_{a_1} (\bar{x}_3 - \bar{z}_1) - \omega_{a_3} (\bar{x}_2 - z_1) - \omega_{a_2} (\bar{x}_2 - \bar{z}_3)) \]

(27)

The proposed acceleration command is obtained as

\[ a_{Mc} = -\frac{1}{\omega_{a_1}^2} (-\kappa_3 q_{31} + 2\xi_1 \omega_{a_1} (\bar{x}_3 - \bar{z}_1) + \omega_{a_3} (\bar{x}_2 - z_1) - \omega_{a_2} (\bar{x}_2 - \bar{z}_3)) \]

(28)

Finally, according to Lemma 5, \( z_1, z_2, \) and \( z_3 \) will converge to \( h, \bar{h}, \) and \( \bar{h} \) respectively, in finite time. Definitions of \( x_2, x_3, \) and the guidance law 28 can be replaced with

\[ a_{Mc} = -\frac{1}{\omega_{a_1}^2} (-\kappa_3 q_{31} + 2\xi_1 \omega_{a_1} (\bar{x}_3 - \bar{z}_1) + \omega_{a_3}^2 (\bar{x}_2 - z_1) - \omega_{a_2} (\bar{x}_2 - \bar{z}_3)) \]

(29)

By using integral-type Lyapunov functions, the analytic differentiation of virtual control law can be completely avoided in a backstepping-like procedure to induce proposed guidance law.

### 3.2 Closed-loop stability analysis

**Theorem 1.** Consider integrated system 15, the proposed guidance law 29 with FDOB 12 will drive the LOS angular rate to zero in finite time.
Proof. The nonlinear observer-controller structure is not usually suitable for the well-known separation principle. Therefore, the proof of Theorem 1 is divided into two main steps. First, the boundness of the system states is proven during the convergent phase of FTDOB based on the FTB function technique. Next, the closed-loop finite-time stability of the overall system will be presented with the Lyapunov function approach.

Step 1. The candidate FTB function for system 14 is defined as

\[
V = \frac{1}{2} \sum_{i=1}^{3} x_i^2 + \frac{1}{2} \sum_{i=0}^{3} z_i^2
\]  

(30)

The derivative of \( V \) along system trajectory 14 yields

\[
\dot{V} = x_1 x_i x_2 + x_3 x_3 + z_0 z_0 + z_1 z_1 + z_2 z_2
\]

(31)

With \( \forall x \in R, 0 < r < 1 \), the inequality |\( x_i^r | < 1 + |x| \) holds. Lemma 4 can deduce that

\[
|v_0| = \left| -\lambda_3 L^{1/4} |z_0 - x_1|^{3/4} \text{sgn}(z_0 - x_1) + z_0 \text{sgn}(z_0 - x_1) + |z_0| \leq \lambda_x L^{1/4} |z_0|^{3/4}
\]

(32)

Note that |\( x_i | \leq \sqrt{V}, i = 1, 2, 3, |z_i| \leq \sqrt{V}, \) and \( i = 0, 1, 2, 3 \). In addition, during the terminal phase, the closing velocity along and perpendicular to the LOS, as well as the missile-target relative distance, are bounded in any time interval [0, \( t \); hence, \( V_n, V_q \), and \( r \) all have limited values during any time interval [0, \( t \)]. The existence of positive constants \( \delta_n, L_v \) satisfies the following

\[
|v_0| \leq \delta_n \sqrt{V} + L_v
\]

(33)

Likewise, we conclude that

\[
|v_1| = \left| -\lambda_2 L^{1/3} |z_1 - v_1|^{2/3} \text{sgn}(z_1 - v_1)
\]

\[
+ z_2 |x_2| L^{1/3} |z_1 - v_1|^{2/3} + |z_2| L^{1/3} (|z_1| + 2)
\]

\[
+ |z_2| + \lambda_2 L^{1/3} |v_1| L^{1/3} (|z_1| + 2) + |z_2|
\]

\[
+ \lambda_2 L^{1/3} (\delta_n \sqrt{V} + L_v) \leq \delta_v \sqrt{V} + L_v
\]

(34)

\[
|v_2| = \left| -\lambda_1 L^{1/2} |z_2 - v_2|^{1/2} \text{sgn}(z_2 - v_2) + z_3 |x_2| L^{1/2} |z_2 - v_2|^{1/2} + |z_3| \leq \lambda_1 L^{1/2} (|z_2| + 2)
\]

\[
+ |z_3| + \lambda_1 L^{1/2} |v_1| L^{1/2} (|z_2| + 2) + |z_3|
\]

\[
+ \lambda_1 L^{1/2} (\delta_v \sqrt{V} + L_v) \leq \delta_v \sqrt{V} + L_v
\]

(35)

where \( \delta_n, \delta_v, L_n, L_v, \) are positive constants.

According to Lemma 1, the estimation error \( \sigma_1 \) is globally bounded, and substituting 33–36 into 37 leads to

\[
\dot{V} \leq \delta_1 V + \delta_2 \sqrt{V} + L_1
\]

(38)

where \( \delta_1, \delta_2, L_1 \) are three positive constants.

We must consider the following two cases to facilitate proving the theorem.

Case 1. \( V \geq 1 \). We have \( \sqrt{V} \leq V \), consequently, 38 becomes

\[
\dot{V} \leq (\delta_1 + \delta_2) V + L_1
\]

(39)

Case 2. \( 0 < V < 1 \). In such case, \( \sqrt{V} < 1 + V \), then, 38 becomes

\[
\dot{V} \leq (\delta_1 + \delta_2) V + \delta_2 + L_1
\]

(40)

Combining the preceding two cases implies that

\[
\dot{V} \leq \dot{\delta}_n V + L_v
\]

(41)

where \( \delta_n, L_v \) are two positive constants. The solution of inequality 41 in the time interval [0, \( t \)] is obtained as

\[
V \leq \left[ V(0) + \frac{L_v}{\delta_n} \right] e^{\delta_n t} - \frac{L_v}{\delta_n}
\]

(42)

where \( V(0) \) is the initial value of \( V \). 42 shows that the states of the system will be bounded in a finite time during the convergent phase of FTDOB.
Step 2. System 14 reduces to system 15 and guidance law 28 to 29 when $t \geq t_f$ because the error dynamics of FTDOB are finite-time stable and the close-loop guidance system will not escape to infinity. Substituting 29 into 27 yields
\begin{equation}
\dot{V}_3 \leq \left( -\kappa_1 + c_{21} + \bar{p}_{21} + \bar{p}_{31} \right) |\xi_1|^{2-\tau} + \left( -\kappa_2 + c_{22} + c_{31} + \bar{p}_{22} + \bar{p}_{32} \right) |\xi_2|^{2-\tau} + \left( c_{32} + \bar{p}_{33} \right) |\xi_3|^{2-\tau} - \kappa_3 |\xi_3|^{2-\tau} \tag{43}
\end{equation}

Let $\varepsilon_1 = \kappa_1 - (c_{21} + \bar{p}_{21} + \bar{p}_{31}) \geq 0$, $\varepsilon_2 = \kappa_2 - (c_{22} + c_{31} + \bar{p}_{22} + \bar{p}_{32}) \geq 0$, $\varepsilon_3 = \kappa_3 - (c_{32} + \bar{p}_{33}) \geq 0$ yields
\begin{equation}
\dot{V}_3 \leq -\varepsilon \left( |\xi_1|^{2-\tau} + |\xi_2|^{2-\tau} + |\xi_3|^{2-\tau} \right) \tag{44}
\end{equation}

with $\varepsilon = \min \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. According to Lemma 2 and a similar process in [32], we obtain for $i = 1, 2, 3$
\begin{equation}
\bar{V}_i \leq 2^{1-q_i} |\xi_i|^{2} + 2^{1-q_i} |\xi_2|^{2} + 2^{1-q_i} |\xi_3|^{2} \tag{45}
\end{equation}

This implies that
\begin{equation}
\dot{V}_3 \leq 2^{1-q_1} |\xi_1|^{2} + 2^{1-q_2} |\xi_2|^{2} + 2^{1-q_3} |\xi_3|^{2} \leq \bar{\varepsilon} \left( |\xi_1|^{2} + |\xi_2|^{2} + |\xi_3|^{2} \right) \tag{46}
\end{equation}

where $\bar{\varepsilon} = \max (2^{1-q_1}, 2^{1-q_2}, 2^{1-q_3})$ is a constant. Together with Lemma 4 yields
\begin{equation}
|\xi_1|^{2-\tau} + |\xi_2|^{2-\tau} + |\xi_3|^{2-\tau} = (|\xi_1|^{2})^{(2-\tau)/2} + (|\xi_2|^{2})^{(2-\tau)/2} + (|\xi_3|^{2})^{(2-\tau)/2} \geq \bar{\varepsilon}^{-\tau} V_3^{(2-\tau)/2} \tag{47}
\end{equation}

Hence
\begin{equation}
\dot{V}_3 + \bar{\varepsilon}^{-\tau} V_3^{(2-\tau)/2} \leq (1 - \varepsilon) \left( |\xi_1|^{2-\tau} + |\xi_2|^{2-\tau} + |\xi_3|^{2-\tau} \right) \tag{48}
\end{equation}

According to Lemma 1, if $\varepsilon \geq 1$, the closed-loop guidance system is finite-time stable. Therefore, the LOS angular rate will converge to zero in finite time and the control objective is achieved by invoking the definition of $\xi_i$.

IV. NUMERICAL SIMULATIONS

4.1 Simulations with varying missile velocity

In this section, numerical simulations are performed to demonstrate the efficacy of the proposed guidance law for various kinds of target maneuvers. Suppose the missile is equipped with an active radar seeker, providing LOS angle, LOS angular rate, range, and range rate information.

Although the engagement model is constructed with consideration that both the missile and the target velocities are constant, guidance law 29 also performs well with varying speeds. For simplicity, only varying missile velocity is considered here. Under this condition, 6 can be rewritten as
\begin{equation}
\dot{V}_q = -\frac{V_q}{r} + a_T - a_M + \dot{V}_M \sin (q - \gamma_M) \tag{49}
\end{equation}

Define $h = -\frac{V_q}{r} + a_T - a_M + \dot{V}_M \sin (q - \gamma_M)$ as the new lumped uncertainty; substituting into 49 yields
\begin{equation}
\dot{V}_q = h - a_M \tag{50}
\end{equation}

which has the same structure as shown in 8. Therefore, guidance law 29 still works well if lumped uncertainty $h$ is estimated accurately.

A more realistic intercept model [33] is presented by considering the effects of thrust and aerodynamics. Missile acceleration is normal in missile velocity, and the dynamics of missile velocity can be given in the body frame as
\begin{equation}
\dot{V}_M = \frac{T - D}{M} \tag{51}
\end{equation}

where $T$ is the thrust, $D$ is the drag force, and $M$ is its mass. The thrust profile is given by
\begin{equation}
T = \begin{cases} 
T_0, & t \leq t_b \\
0, & t > t_b 
\end{cases} \tag{52}
\end{equation}

The mass of the missile changes when the propulsion system is on, and it can be approximated as
\begin{equation}
M = m_i - \frac{m_p}{t_b} \tag{53}
\end{equation}

where $m_i$ and $m_p$ is the initial mass of missile and mass of propellant, respectively. The term $m_p/t_b$ represents the fuel mass flow rate, where $t_b$ is the burn time.

The profile of drag force is governed by
\begin{equation}
D = \frac{1}{2} \rho V_M^2 C_D A \tag{54}
\end{equation}

where $\rho$ is the atmosphere air density, $A$ denotes the reference area, and $C_D$ is the drag force coefficient. $C_D$ is given by a parabolic model as $C_D = C_{D0} + k C_L^2$; $C_{D0}$ and $C_L$ denote the zero lift drag coefficient and lift force coefficient, respectively, and $k$ is the parameter of the induced drag. As the lift force acting on the missile is given as $L = 0.5 \rho V_M^2 C_L$ and lateral acceleration is $a_e = L/M$, the resulting lift coefficient is expressed in $C_L = 2a_M M / (\rho V_M^2 A)$.

The initial engagement condition and related data required in closed system 11 are selected as: 1) initial missile-target relative distance: 14,142 m; 2) initial LOS
angle: \(\pi/4\) rad; 3) initial flight path angle of missile: \(\pi/4\) rad; 4) initial flight path angle of target: \(\pi\) rad; 5) initial missile velocity: 800 m/s; 6) initial target velocity: 500 m/s; 7) autopilot parameters: \(\xi_a = 0.8\), \(\omega_a = 10\text{rad/s}\); 8) maximum acceleration of missile achieved: 200 m/s\(^2\); 9) parameters of thruster: \(m_i = 165\text{kg}\), \(m_p = 15\text{kg}\), \(t_p = 5\text{s}\), and \(T_0 = 17640\text{N}\); and 10) parameters of aerodynamic force: \(C_{D_0} = 0.74\), \(k = 0.03\), \(A = 0.0324\text{m}^2\), and \(\rho = 0.909\text{kg/m}^3\). Three different target maneuver profiles are given below for simulations.

Case 1: Constant target maneuvers \(a_T = +80\text{m/s}^2\).
Case 2: Sudden evasive target maneuvers \(a_T = 0\text{m/s}^2\) for \(t < 2\text{s}\), \(a_T = -100\text{m/s}^2\) for \(2 < t < 4\text{s}\), and \(a_T = 100\text{m/s}^2\) for \(t > 4\text{s}\).
Case 3: Periodic target maneuvers \(a_T = -100\sin(2t)\text{m/s}^2\).

The design parameters for implementing proposed guidance law 29 are set as \(k_1 = 5\), \(k_2 = 20\), \(k_3 = 40\), and \(\tau = 2/39\). The selection of observer gains of FTDOB is given as: \(\lambda_0 = 1.1\), \(\lambda_1 = 1.5\), \(\lambda_2 = 5\), and \(\lambda_3 = 10\).

To make a better showcase, the adaptive sliding mode guidance law (ASMG) [11] and finite time convergent guidance law (FTCG) [17] are also considered in the simulation for comparison.

The ASMG law is defined as

\[
a_{Mc} = -N_a V_r \dot{q} + C \ddot{q}/(|\dot{q}| + \delta) \tag{55}
\]

where the effective navigation ratio \(N_a\) is usually 3-5, and \(C > 0\) is a design parameter. The parameters are selected as \(N_a = 4\) and \(C = 100\). The FTCG law is defined as

\[
a_{Mc} = -N_f V_r \dot{q} + \beta_1 \text{sgn}(\dot{q}) + \beta_2 |\dot{q}|^{\eta} \text{sgn}(\dot{q}) \tag{56}
\]

where \(N_f > 2\), \(\beta_1\), \(\beta_2 > 0\), and \(0 < \eta < 1\) are design parameters and selected as \(N_f = 4\), \(\beta_1 = 100\), \(\beta_2 = 30\), and \(\eta = 0.5\). However, the undesired violent chattering phenomenon will result from the property of discontinuous terms \(\beta_1 \text{sgn}(\dot{q})\). A saturation function \(\text{sat}(x)\) is used to replace the sign function to avoid this phenomenon, where

\[
\text{sat}(x) = \begin{cases} 
\text{sgn}(x), & |x| > \eta \\
\frac{1}{\eta} x, & |x| \leq \eta 
\end{cases} \tag{57}
\]
where $\eta$ is a small positive constant and is set as 0.005 in simulations.

The simulations are carried out for three different target acceleration profiles and the results of comparison are presented in the following figures: Fig. 2 is for Case 1; Fig. 3 is for Case 2; and Fig. 4 is for Case 3. The figures contain profiles of interception trajectories, LOS angular rates, acceleration of missile achieved, and uncertainty estimations. Compared with other guidance laws, the proposed guidance law offers better performance in all cases. The LOS angular rate is regulated to zero rapidly in finite time under the proposed guidance law, whereas ASMG only keeps the LOS rate boundness and becomes divergence as close to target. As a result of an initial error in estimation, the required acceleration of the missile for the proposed law is initially higher. However, the acceleration profile is more practically acceptable than other laws during most of the flight time.

Additionally, FTCG law can drive the LOS angular rate around zero in finite time, whereas the LOS rate cannot converge to a small region. The reason is that the chattering problem is addressed when FTCG law adopts the boundary layer technique. Thus, only motion around the sliding surface can be maintained. The chattering behavior of the LOS angular rate under the ASMG and FTCG laws are more apparent than the proposed law in Case 3, which is not desirable for the seeker system, because of the existence of the missile autopilot lag. In contrast, with effective autopilot lag compensation, the proposed guidance law displays better transient performance and attains a steady state earlier. The miss distances and interception time obtained by these guidance laws under the target acceleration cases are as given in Table I. From the table we can see that the interception time spent by the ASMG, FTCG, and proposed laws is similar for all target situations. However, the miss distances achieved by the proposed law are much less than other laws, because the proposed law can maintain the LOS angular rate at zero during most of the flight time.

### 4.2 Simulations with measurement noise

Sensor information is always affected by the noise in realistic scenarios, which leads to degradation of the
performance of the closed-loop guidance system [34], especially for the FTDOB. Simulations are carried out to test the performance of the proposed guidance law by considering corrupted measurements of the LOS angular rate and range. Measurement noises are assumed to be normally distributed with a zero mean, and corresponding standard deviations are given in Table II [35].

The simulation results in a noisy environment for Case 3 are presented in Fig. 5. From Fig. 5(d), it can be observed that the lumped uncertainty can be well estimated through FTDOB and estimation error remains bounded, even if measurement noises exist. FTDOB performance can improve slightly when the gain of FTDOB $L$ is set at twice the original one. Moreover, the LOS angular rate also regulates to a small region around zero and the

| Guidance laws | Miss distance (m) | $t_f$(s) | Miss distance (m) | $t_f$(s) | Miss distance (m) | $t_f$(s) |
|---------------|------------------|---------|------------------|---------|------------------|---------|
| ASMG          | 0.0781           | 11.028  | 0.0519           | 11.824  | 0.8095           | 13.337  |
| FTCG          | 0.0019           | 11.008  | 0.0018           | 11.852  | 0.0438           | 13.390  |
| Proposed      | $9 \times 10^{-6}$ | 10.947  | $7 \times 10^{-6}$ | 11.861  | $4 \times 10^{-4}$ | 13.407  |

| Parameter | Standard deviation |
|-----------|---------------------|
| $\dot{q}$ (rad/s) | $5 \times 10^{-5}$ |
| $r$ (m)    | 30.48 |

Table I. Miss distances and interception time.

Table II. Measurement noise.
The guidance law can guarantee that the LOS angular rate converges to zero in finite time. In the design, the guidance law is derived from the finite-time backstepping method and lumped uncertainty can be estimated incorporating with FTDOB. The time derivatives of virtual controls are avoided by applying integral-type Lyapunov functions. Simulations and comparisons of various engagement scenarios demonstrate the excellent interception provided by the proposed guidance law with autopilot dynamics.

V. CONCLUSION

In this work, a nonlinear finite time guidance law considering second-order autopilot lag is presented for maneuvering targets. Based on the finite-time backstepping control technique and the FTDOB method, the guidance law can guarantee that the LOS angular rate converges to zero in finite time. In the design, the guidance law is derived from the finite-time backstepping method and lumped uncertainty can be estimated incorporating with FTDOB. The time derivatives of virtual controls are avoided by applying integral-type Lyapunov functions. Simulations and comparisons of various engagement scenarios demonstrate the excellent interception provided by the proposed guidance law with autopilot dynamics.

REFERENCES

1. Zarchan, P., Tactical and Strategic Missile Guidance, 6th edn., Reston, Virginia: American Institute of Aeronautics & Astronautics Inc (2012).
2. Ghawghawe, S. N., and D. Ghose, “Pure proportional navigation against time-varying target maneuvers,” IEEE Trans. Aerosp. Electron. Syst., Vol. 32, No. 4, pp. 1336–1347 (1996).
3. Yang, C. D., and C. C. Yang, “A unified approach to proportional navigation,” *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 33, No. 2, pp. 557–567 (1997).

4. Yang, C. D., and H. Y. Chen, “Nonlinear H∞ robust guidance law for homing missiles,” *J. Guid. Control Dyn.*, Vol. 21, No. 6, pp. 882–890 (1998).

5. Lechevin, N., and C. A. Rabbath, “Lyapunov-based nonlinear missile guidance,” *J. Guid. Control Dyn.*, Vol. 27, No. 6, pp. 1096–1101 (2012).

6. Zhou, D., C. Mu, and T. Shen, “Robust guidance law with L2 gain performance,” *Trans. Jpn. Soc. Aeronaut. Space Sci.*, Vol. 44, No. 144, pp. 82–88 (2005).

7. Zhang, P., Y. Fang, F. Zhang, et al., “An adaptive weighted differential game guidance law,” *Chin. J. Aeronaut.*, Vol. 25, No. 5, pp. 739–746 (2012).

8. Shima, T., and O. M. Golan, “Linear quadratic differential games guidance law for dual controlled missiles,” *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 43, No. 3, pp. 834–842 (2007).

9. Brierley, S. D., and R. Longchamp, “Application of sliding-mode control to air-air interception problem,” *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 26, No. 2, pp. 306–325 (1990).

10. Babu, K. R., I. G. Sarma, and K. N. Swamy, “Switched bias proportional navigation for homing guidance against highly maneuvering targets,” *J. Guid. Control Dyn.*, Vol. 17, No. 6, pp. 1357–1363 (1994).

11. Zhou, D., C. Mu, and W. Xu, “Adaptive sliding-mode guidance of a homing missile,” *J. Guid. Control Dyn.*, Vol. 22, No. 4, pp. 589–594 (1999).

12. Moon, J., K. Kim, and Y. Kim, “Design of missile guidance law via variable structure control,” *J. Guid. Control Dyn.*, Vol. 24, No. 4, pp. 659–664 (2001).

13. Yang, J., S. Li, J. Su, et al., “Continuous nonsingular terminal sliding mode control for systems with mismatched disturbances,” *Automatica*, Vol. 49, No. 7, pp. 2287–2291 (2013).

14. Yang, J., S. Li, C. Sun, et al., “Nonlinear-disturbance-observer-based robust flight control for airbreathing hypersonic vehicles,” *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 49, No. 2, pp. 1263–1275 (2013).

15. Zhang, Z., C. Man, S. Li, et al., “Finite-time guidance laws for three-dimensional missile-target interception,” *Proc. Inst. Mech. Eng. Part G J. Aerospace Eng.*, Vol. 230, No. 2, pp. 392–403 (2015).

16. Zhang, Z., S. Li, and S. Luo, “Terminal guidance laws of missile based on ISMC and NDOB with impact angle constraint,” *Aerosp. Sci. Technol.*, Vol. 31, No. 1, pp. 30–41 (2013).

17. Zhou, D., S. Sun, K. L. Teo, et al., “Guidance laws with finite time convergence,” *J. Guid. Control Dyn.*, Vol. 32, No. 6, pp. 1436–1449 (2009).

18. Kumar, S. R., S. Rao, and D. Ghose, “Nonsingular terminal sliding mode guidance with impact angle constraints,” *J. Guid. Control Dyn.*, Vol. 37, No. 4, pp. 1114–1130 (2014).

19. Golestani, M., I. Mohammadzaman, and A. R. Vali, “Finite-time convergent guidance law based on integral backstepping control,” *Aerosp. Sci. Technol.*, Vol. 39, pp. 370–376 (2014).

20. Sun, S., D. Zhou, and W. Hou, “A guidance law with finite time convergence accounting for autopilot lag,” *Aerosp. Sci. Technol.*, Vol. 25, No. 1, pp. 132–137 (2013).

21. Rusnak, I., and L. Meir, “Modern guidance law for high-order autopilot,” *J. Guid. Control Dyn.*, Vol. 14, No. 5, pp. 1056–1058 (2015).

22. He, S., W. Wang, and D. Lin, “Adaptive backstepping impact angle guidance law accounting for autopilot lag,” *J. Aerosp. Eng.*, Vol. 30, No. 3, pp. 04016094 (2016).

23. Wu, L., W. Wang and S. Xiong, “Guidance law accounting for second-order dynamics of missile autopilot and impact angle constraints,” *IEEE Control and Decision Conference*, Changsha, China, pp. 2459–2464 (2014).

24. He, S., D. Lin, and J. Wang, “Robust terminal angle constraint guidance law with autopilot lag for intercepting maneuvering targets,” *Nonlinear Dyn.*, Vol. 81, No. 1–2, pp. 881–892 (2015).

25. Zhou, D., P. Qu, and S. A. Sun, “Guidance law with terminal impact angle constraint accounting for missile autopilot,” *J. Dyn. Syst. Meas. Control*, Vol. 135, No. 5, pp. 051009 (2013).

26. He, S., W. Wang, and J. Wang, “Adaptive backstepping impact angle control with autopilot dynamics and acceleration saturation consideration,” *Int. J. Robust Nonlinear Control*, Vol. 41, No. 7, pp. 1591–1601 (2017).

27. He, S., T. Song, and D. Lin, “Impact angle constrained integrated guidance and control for maneuvering target interception,” *J. Guid. Control Dyn.*, Vol. 40, No. 10, pp. 1–9 (2017).

28. Bhat, S. P., “Finite-time stability of continuous autonomous systems,” *SIAM J. Control Optim.*, Vol. 38, No. 3, pp. 751–766 (2000).

29. Qian, C., and W. Lin, “A continuous feedback approach to global strong stabilization of nonlinear systems,” *IEEE Trans. Autom. Control*, Vol. 46, No. 7, pp. 1061–1079 (2001).

30. Shtessel, Y. B., I. A. Shkolnikov, and A. Levant, “Guidance and control of missile interceptor using second-order sliding modes,” *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 45, No. 1, pp. 110–124 (2009).
31. Levant, A., “Higher-order sliding modes, differentiation and output-feedback control,” *Int. J. Control*, Vol. 76, No. 9–10, pp. 924–941 (2003).
32. Huang, X., W. Lin, and B. Yang, “Global finite-time stabilization of a class of uncertain nonlinear systems,” *Automatica*, Vol. 41, No. 5, pp. 881–888 (2005).
33. Phadke, S. B., and S. E. Talole, “Sliding mode and inertial delay control based missile guidance,” *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 48, No. 4, pp. 3331–3346 (2012).
34. He, S., W. Wang, and J. Wang, “Discrete-time super-twisting guidance law with actuator faults consideration,” *Asian J. Control*, Vol. 19, No. 5, pp. 1854–1861 (2017).
35. Vaddi, S., P. Menon and E. Ohlmeyer, “Target state estimation for integrated guidance-control of missiles,” *AIAA Guidance, Navigation and Control Conference and Exhibit*. Hilton Head, South Carolina, pp. 6838 (2007).

**VI. APPENDIX**

### 6.1 Proof for Key Propositions

#### 6.1.1 Proof of Proposition 2

The following equation can be concluded from Proposition 1

\[
\left| \frac{\partial W_2}{\partial x_1} \right| \leq (2 - q_2) |x_2 - x_1^*| |\xi_2|^{1-q_2} \left| \frac{\partial (x_1^{1/q_2})}{\partial x_1} \right| x_1
\]

Let \( \gamma_k \) be the following inequality

\[
|\gamma_k - \gamma_k^*| \leq 2\xi_k^{q_k}
\]

Then, substituting 59 into 58 gives

\[
\left| \frac{\partial W_2}{\partial x_1} \right| \leq a_2 |\xi_2^*| |\frac{\partial (x_1^{1/q_2})}{\partial x_1} | x_1
\]

where \( a_2 = 2(2 - q_2) \) is a positive constant.

From the definition of \( \gamma_k = -\kappa_1\xi_1^* \), \( \xi_1 = x_1^{1/q_1} = x_1 \), we have

\[
\left| \frac{\partial (x_1^{1/q_1})}{\partial x_1} \right| = \frac{1}{q_1} |x_2|
\]

and inequality \( |\gamma_k| \leq 2(\xi_2^{q_2} + \kappa_1 |\xi_1^{q_1}|^{q_2}) \) according to Lemma 4. Combining 61 and Lemma 3, we have

\[
\left| \frac{\partial W_2}{\partial x_1} \right| \leq a_2 |\xi_2^*| |\frac{1}{q_1} |x_2|
\]

where the last inequality is obtained using Lemma 3 and 4, \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \) are positive constants.
Considering 67, 68, and Lemma 3, we have the following

\[
\frac{\partial W_3}{\partial x_1} + \frac{\partial W_3}{\partial x_2} \leq a_3\gamma_1|\xi_3| + a_3\gamma_2|\xi_2| + a_3\gamma_3|\xi_1| + a_3\gamma_4|\xi_2| + a_3\gamma_5|\xi_1| + a_3\gamma_6|\xi_2|
\]

where \( p_{31} > 0, p_{32} > 0, \) and \( p_{33} > 0 \) are positive constants. This completes the proof.

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