Real-number Computability from the Perspective of Computer Assisted Proofs in Analysis

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Abstract

We present an interval approach to real-number computations. In some aspects it is similar to existing ones. However, those aspects in which our attitude differs give it several advantages. First, we do not need any oracles and impractical or non-realistic structures; we carry out calculations in a way it is done in real-life practice (e.g. in computer assisted proofs in analysis). Second, the interval point of view allows us to consider various kinds of global information. Apparently, the latter has not been treated in the literature.

Keywords: computable analysis, interval arithmetic, computer assisted proofs

1. Introduction and motivation

There are various approaches to the definition of computability and complexity of problems in continuous mathematics, e.g. numerically solving algebraic equations, ODEs or PDEs. However the standard numerical computations are not rigorous, as a finite computer is unable to represent the continuum. Depending on how they idealize the non-rigorous computations performed on the existing computers, various authors (see for example [1, 5, 6, 10, 31]) propose different approaches to the machine model used and to the notion of computability of objects.

We are not fully satisfied with these attempts. Our point of departure are the existing computer assisted proofs in analysis (we will use the acronym CAPA), like Lanford’s proof of the Feigenbaum conjectures for the period doubling cascade [12] or other CAPAs in dynamics (see for example [2, 4, 16–19, 29, 30, 32, 33]) and our personal experience with the CAPD library [7] for rigorous integration of ODEs.

1.1. Intervals as a natural phenomenon

The majority of problems in analysis cannot be solved analytically, hence approximate numerical methods are developed. These methods usually give flawed solutions and sometimes give rise to misleading mathematical conclusions (see for example [9]).

Example 1. Consider an ODE and the problem of finding $x(t, x_0)$, a trajectory starting from the given point $x_0 \in \mathbb{R}$ at time $t_0 = 0$. Apart from very special cases this problem cannot be solved analytically and we are compelled to use numerical methods. Even if we were able to represent $x_0$ and $t$ with an infinite precision as the actual real numbers, an inaccuracy would appear as a consequence of a method. The result we obtain is a set, for example, a product of intervals, which contains the
error bound. This set becomes the initial condition for the next time step. Therefore we need an arithmetic on sets (or intervals) in real number computations with rigorous control of errors.

Using intervals to describe real numbers is essential for obtaining rigorous computations, where we can be sure that actual results belong to the computed intervals. To achieve this the following strategy is used: the computed right-end of an interval should be a representable number not less than the actual right end, and analogously, the computed left end should be a representable number not greater than the actual left end. This is automatically realized by the interval arithmetic [20, 24], see also Section Appendix A.

The main body of the paper is organized in the following way. In Section 2 we present some foundations for representable numbers. Section 3 explains the CAPA point of view on rigorous computations. In Section 4 we briefly outline Ko’s approach to the real numbers computability [10, 11] and indicate some of its problems from the point of view of CAPA. In Section 5 we discuss the notion of the interval computable function. Finally, in Section 7 we briefly present and comment on the existing models of real computations from the CAPA perspective.

For the sake of clarity all technicalities are covered in the appendices. In Appendix Appendix A we recall the basic notions of interval arithmetic. The model of an interval machine is presented in the Appendix Appendix B.

1.2. Notations

For the ease of reference, we put major notations here; some of them may only become clear when they appear in their proper context in further sections.

Let \( \Sigma \) be any finite and non-empty set. Define

\[
\Sigma^0 = \{ \varepsilon \} \quad \text{(the language consisting only of the empty string)},
\]

\[
\Sigma^1 = \Sigma,
\]

\[
\Sigma^{n+1} = \{ wv : w \in \Sigma^i \text{ and } v \in \Sigma \} \quad \text{for each } i > 0.
\]

We use \( \Sigma^* \) to denote the usual Kleene closure, i.e.

\[
\Sigma^* = \bigcup_{i \in \mathbb{N}} \Sigma^i = \{ \varepsilon \} \cup \Sigma \cup \Sigma^2 \cup \Sigma^3 \cup \ldots.
\]

We denote by \( \mathbb{Z} \) the set of integer numbers, by \( \mathbb{Q} \) the set of rational numbers and by \( \mathbb{D} \) the set of dyadic numbers, i.e. numbers in the form \( p/2^q \) where \( p, q \in \mathbb{Z} \). Note that dyadic numbers correspond to floating point numbers: a floating point notation \( m \cdot 2^e \) is equivalent to a dyadic number \( m/2^{-e} \).

By \( \hat{R} \) we denote the set of representable numbers and by \( \hat{\hat{R}} \) the set of all representations. We do not specify any particular set; we just assume some properties as pointed out in Section 2. By \( X \) we denote the set of all intervals with endpoints belonging to \( \hat{\hat{R}} \) and we call them representable intervals.

By \( s_c \) we denote a step function with a discontinuity in point \( c \), i.e. any function of the form

\[
s_c(x) = \begin{cases} 
  a_1, & x > c \\
  a_2, & x \leq c,
\end{cases}
\]

where \( a_1, a_2 \in \mathbb{R} \). Note that it is not the values \( a_1, a_2 \) but just the position of discontinuity that is important in our considerations.
Chapter 2. Real numbers and their representations

It is a common approach to computability on real numbers to distinguish between computable and non-computable numbers. This very problem is not what we are interested in: we do exact calculations on simple representations and infer properties of the entire sets.

Note that there is a significant difference in what we want to attain: whether we want to be able to compute (any digit of) a number, or to represent it (write down the entire number in some way). Thus computable numbers are not the same as representable numbers. However, there is no general definition of representable real numbers. The program (machine) computing a number can be one of its possible representations. A common representation is floating-point numbers with a finite mantissa of fixed length, which gives just a finite set of numbers (a subset of dyadic numbers).

Computers are discrete machines unable to represent the continuum of real numbers. Therefore they operate on a subset of \( \mathbb{R} \) called the set of representable numbers, which can be written as finite strings over a finite alphabet. This assumption leads to a fundamental statement about set of representable numbers.

Remark 1. The set of representable numbers is countable (finite or infinite).

Definition 1. Let \( \Sigma \) be a finite alphabet. We say that a function \( r : \mathbb{R} \to \mathcal{P}(\Sigma^*) \) is a representation if

\[
\forall x, y \in \mathbb{R} : x \neq y \implies r(x) \cap r(y) = \emptyset.
\]

A representation of a real number \( x \) with respect to \( r \) is any element of the set \( r(x) \). A number \( x \) is representable if \( r(x) \neq \emptyset \).

Example 2. In dyadic notation \( 1/2 \) can have different representations as

\[
\frac{1}{2}, \frac{2}{2^2}, \frac{4}{2^3}, \ldots
\]

Thus the function \( r \) is

\[
r(x) = \begin{cases} 
\left\{ \frac{p2^k}{2^{2^k}} : k \in \mathbb{N} \right\}, & \text{if } x = \frac{p}{2^q} \text{ with } p \text{ odd} \\
\emptyset, & \text{otherwise.}
\end{cases}
\]

By \( R_r \) we denote the set of representable numbers with respect to the representation \( r \); then

\[
\hat{R}_r = \bigcup_{x \in \mathbb{R}} r(x)
\]

is a set of all representations with respect to \( r \).

It does not make sense to consider the computability of \( r \), but we still want to be able to decide whether a given string over \( \Sigma \) represents a number or not.

We do not require that \( R_r \) is closed under arithmetic operations used since commonly used representations do not satisfy this requirement. For example, the dyadic number representation is not closed under division, whilst floating point representation is closed neither under multiplication, nor division. Therefore we state a weaker condition.

Definition 2. Let \( r \) be a representation. Whenever we use an operation on representable numbers, say \( f : R_{r_m} \to R_r \), it has to be implemented faithfully:

\[
\forall \overline{x} \in R_{r_m} : r(f(\overline{x})) \neq \emptyset \implies \forall \overline{w} \in r(\overline{x}) : \hat{f}(\overline{w}) \in r(f(\overline{x})),
\]

where \( \hat{f} : \hat{R}_{r_m} \to \hat{R}_r \) is the implementation of \( f \).
Remark 2. From our point of view, \( r \) is an interesting representation if it satisfies the following conditions:

1. \( \forall w \in \Sigma^* \) it is decidable whether \( w \) is a representation of some number \( x \), i.e. if there exists \( x \) such that \( w \in r(x) \);
2. \( \forall z \in \mathbb{Z} : r(z) \neq \emptyset \);
3. operations we want to use can be implemented faithfully (see Definition 2).

Since all results we present are independent of the specific set of representable numbers, further throughout the paper we do not specify a representation \( r \) and we just call the set of representable numbers \( R \) and the set of representations \( \hat{R} \).

Remark 3. Moreover, since in practice we usually identify (real) numbers with their representations, whenever possible, we use this convention in the present paper: we say representable real number meaning a representation of that number.

To illustrate the ideas we often use the example of rational numbers or their finite subset. These two cases are important because the former is infinite and the latter corresponds to the floating point arithmetic realized in the present day computers.

3. CAPAs in practice and a model of computation

In his book \cite{Weihrauch2000} Weihrauch says: Let us consider computable analysis as a mathematical theory of those real functions (...) which can be computed by physical machines like digital computers. Since we do not know the precise meaning of ‘computable by physical machines’, every mathematical investigation in computable analysis must be based on a model of computation. Such a model of computation is not ‘true’ or ‘false’ but can merely be more or less realistic, powerful, expressive, elucidating or useful in practice according to the specific situation.

However, we are not seeking the ultimate boundary of physical computation and we are not satisfied with unrealistic models or models that do not provide rigorous calculations.

From the point of view of CAPAs, it does not matter if a real number is computable or not. Each CAPA operates on representable numbers (usually some subset of rationals) and on intervals with representable ends, and it collects properties of all (from some range of interest) real numbers.

There is a lot of mathematically interesting questions, which can be reduced to a finite number of strict inequalities between computable functions on some simple compacts in \( \mathbb{R}^n \). Obviously in each CAPA of some mathematical interest there is a substantial theoretical (i.e. non-algorithmic) work required to reduce it to the inequalities to be checked rigorously.

To check a finite number of strict inequalities the finite precision computations are often sufficient, hence one can obtain true mathematical statements even for objects, which might not be computable.

Below we give some examples of questions amenable to CAPA.

Example 3. Given a computable continuous function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and a convex set \( I \), we are interested if there is a fixed-point of \( f \) in \( I \), i.e. \( \exists x_0 \in I : f(x_0) = x_0 \).

By the Brouwer fixed-point theorem, if \( f(I) \subset I \), then there exists \( x_0 \in I \). However we need stable (i.e. resistant to small perturbations) assumptions thus we require that \( I \) is a “simple” set (e.g. a ball or a cube) and \( f(I) \subset \text{int} I \). It is obvious that the last condition can be formulated in terms of strong inequalities.

Notice that the computations are carried out on representable numbers, whilst the fixed-point need not be a computable number \cite{Weihrauch2000}.
Example 4 (see [13, 14]). Rigorous integration of ODEs. Assume we have initial value problem in $\mathbb{R}^n$:

$$\begin{align*}
\dot{x} &= f(t, x) \\
x(0) &= x_0 \in \mathbb{R}^n,
\end{align*}$$

where $f \in C^\infty$. We start with $Z = [x_0, x_0]$. Each time step of the rigorous Lohner integration algorithm has two stages. In the first stage we choose a set $W$ (using a heuristic procedure), such that:

- $Z \subset W$
- a solution starting from a point in $Z$ exists in $W$ for $t \in [0, h]$, i.e. we check the condition $\varphi([0, h], Z) \subset W$, which can be reduced to a finite number of inequalities.

In the second stage the solution after the time $h$ is obtained by evaluation in interval arithmetic of the Taylor formula on $Z$; remainder term needs to be evaluated on $W$. The sum of these two intervals is the new value of $Z$.

The use of interval arithmetic (see Appendix A) is crucial to CAPAs, which are not based on the symbolic computations, as it takes care of the round-off errors and it allows evaluations of functions on arguments belonging to a set. This also provides a different kind of information in the analysis of the complexity of problems; in [10] or [27, 28] only values of a function (or its derivatives) at a point are allowed. However, it is also possible to have a CAPA without using the interval arithmetic, i.e. performing all computations in a finite precision arithmetic and then running a separate program (or doing pen and paper calculations) to estimate the influence of the round-off errors. This is the case of the work [16, 17]. This approach does not differ conceptually from the one based on the systematic use of the interval arithmetic.

In each approach to computability and complexity of problems in analysis there is always an attempt to define the notion of a computable mathematical object, for example number, function, set etc. These definitions are different in various approaches. In the present work we adopt the following principle:

**The object is computable in the CAPA world, if we can verify various mathematical statements about it using the computer rigorously.**

It often happens that some properties of mathematical objects are amenable to a CAPA and others are not. Therefore it makes sense to discuss the computability up to a certain degree and the difference between computable and non-computable is blurred.

4. Ko’s approach

The literature contains various approaches to computability on real numbers. We cover the most important of them in Appendix A. However, since Ko’s ideas are apparently a good starting point for the complexity theory for the rigorous computations of ODE solutions, we present them briefly here; cf. [8, 10, 11, 21].

When discussing the computable real functions, Ko first introduces the idea of computable numbers (Definition 3), then the set of functions binary converging to $x$ (Definition 4) and finally the computable functions (Definition 5).

**Definition 3.** We say a real number $x$ is computable if there is a computable function $\varphi : \mathbb{N} \to \mathbb{D}$ such that for all $n \in \mathbb{N}$, $|\varphi(n) - x| \leq 2^{-n}$. 

5
\textbf{Definition 4.} For each real number \( x \), a function \( \varphi : \mathbb{N} \rightarrow \mathbb{D} \) is said to binary converge to \( x \) if it satisfies the condition that for all \( n \in \mathbb{N} \), \( \varphi(n) \in \{m/2^n, \ m \in \mathbb{Z}\} \) and \( |\varphi(n) - x| \leq 2^{-n} \). Let \( \text{CF}_x \) (Cauchy function) denote the set of all functions binary converging to \( x \).

Intuitively, Turing machine \( M \) with one oracle computes a real function \( f \) in the following way:

(a) The input \( x \) to \( f \), represented by some \( \varphi \in \text{CF}_x \), is given to \( M \) as an oracle.

(b) The output precision \( 2^{-n} \) is given in the form of integer \( n \) (or, in unary notation, a string \( 0^n \)) as the input to \( M \).

(c) The computation of \( M \) usually takes two steps, though sometimes these two steps may be repeated an indefinite number of times:

(i) \( M \) computes, from the output precision \( 2^{-n} \), the required input precision \( 2^{-m} \),

(ii) \( M \) queries the oracle to get \( \varphi(m) \), such that \( |\varphi(m) - x| \leq 2^{-m} \), and computes from \( \varphi(m) \) an output \( d \in \mathbb{D} \) with \( |d - f(x)| \leq 2^{-n} \).

The precise definition is as follows.

\textbf{Definition 5. [10, Def. 2.11] A real function }\( f : \mathbb{R} \rightarrow \mathbb{R} \) \textit{is computable if there is a function-oracle TM (Turing machine) }\( M \) \textit{such that for each }\( x \in \mathbb{R} \) \textit{and each }\( \varphi \in \text{CF}_x \), \textit{the function }\( \psi \) \textit{computed by }\( M \) \textit{with oracle }\( \varphi \) (i.e. \( \psi(n) = M^\varphi(n) \)) \textit{is in }\( \text{CF}_{f(x)} \). \textit{We say that }\( f \) \textit{is computable on interval }\([a, b]\) \textit{if the above condition holds for all }\( a \in [a, b] \).

\[ \psi_x(n) = \begin{cases} 1, & \text{if the } n\text{-th Turing machine stops on } n \\ 0, & \text{otherwise} \end{cases} \]

and the binary Cauchy representation is \( b_x(n) = \sum_{k=1}^{n} \psi(k) \cdot 2^{-k} \). It is clear that there is no Turing machine computing \( b_x \).

From our point of view the use of uncomputable oracles to describe computable functions is a superfluous construct. In Section 4 we give a definition of s-IAC functions, which is constructive, and turns out to be equivalent to Ko’s definition.

Next we argue that if \( f \) is computable, then the Turing Machine \( M \) from Definition 5 performs the estimation of \( f \) on some interval.
The oracle-Turing machine for the computable function (in the sense of Definition 5) for given \( n \) and binary representation of the number \( x \) up to \( m(n) \) binary places, returns \( f(x) \) up to an error \( 2^{-n} \). Thus, in terms of intervals, the machine performs two operations: seeks for the right value of \( m(n) \) and then computes

\[
\trunc_n(x) + 2^{-m(n)}[-1, 1] \subset f(x) + 2^{-n} \cdot [-1, 1].
\]

The question is how this inclusion is achieved? In Section 6 we discuss an algorithm for the interval machine (equipped with an interval arithmetic), which has a built-in mechanism to return a rigorous bound for \( f(I) \), where \( I \) is an interval (or product of intervals) and \( f \) is an elementary function.

5. Computable functions

The goal of this section is to discuss the notion of a real function computable on the interval machine. The interval machine is a simple RAM-like model, equivalent to the Turing machine. It has one fundamental advantage though: it is equipped with an arithmetic which operates on intervals. This is absolutely crucial for CAPAs. For the detailed definition of the interval machine see Appendix B.

Let us recall that by \( \mathbb{R} \) we denote the set of representable reals on which the interval arithmetic is built. Denote by \( \mathcal{RI} \) the class of representable interval functions, i.e. the class of functions computed by interval machines (see Definition 13). For the following definitions we need the convention mentioned in Remark 3 let \( [\alpha, \beta] \in \mathcal{X} \) and \( f : \mathbb{R} \to \mathbb{R} \). By \( f([\alpha, \beta]) \) we mean a value of \( f((\alpha, \beta)) \), where \( [\alpha, \beta] \subset \mathbb{R} \) and \( \alpha \in r(\alpha) \) and \( \beta \in r(\beta) \).

We start with two definitions of a computable function. We distinguish between weak and strong computability; we abbreviate these terms as \( w \)-IAC and \( s \)-IAC. We also distinguish between computability and \( \varepsilon \)-computability. The former is stronger since we demand that some properties hold regardless of accuracy \( \varepsilon \); we denote it with an \( \varepsilon \) prefix: \( \varepsilon w \)-IAC and \( \varepsilon s \)-IAC.

5.1. Weak interval arithmetic computable (\( w \)-IAC) functions

**Definition 6.** Let us fix \( \varepsilon > 0 \). We say that a function \( f : \mathbb{R} \supset \text{dom}(f) \to \mathbb{R} \) is \( \varepsilon w \)-IAC if there exists \( F \in \mathcal{RI} \) such that

\[
\forall x \in \text{dom}(f) \quad \exists I \in \mathcal{X}, I \subset \text{dom}(f) : \ x \in I, \ f(I) \subset F(I), \ \text{diam}(F(I)) \leq \varepsilon. \tag{1}
\]

We say that \( f \) is \( w \)-IAC if there exists \( F \in \mathcal{RI} \) such that

\[
\forall \varepsilon > 0 \quad \forall x \in \text{dom}(f) \quad \exists I \in \mathcal{X}, I \subset \text{dom}(f) : \ x \in I, \ f(I) \subset F(\varepsilon, I), \ \text{diam}(F(\varepsilon, I)) \leq \varepsilon. \tag{2}
\]

The above definition says that a function \( f \) is \((\varepsilon)w\)-IAC, if there exists an algorithm \( F \) (i.e. a program of an interval machine) that computes \( f \) with desired/requested/proper precision. In the definition, we require that for any (even non-computable) \( x \in \text{dom}(f) \) there exists an interval \( I \), for which the algorithm \( F \) returns a value suitably close to \( f(x) \). If we omit pathological functions as for example

\[
f(x) = \begin{cases} \omega, & x \in R, \\ 1, & x \in \mathbb{R} \setminus R, \end{cases}
\]

where \( \omega \) denotes an undefined value, the process of choosing \( I \) is algorithmic, as is shown by the next theorem.

**Theorem 1.** Let \( f : \mathbb{R} \supset \text{dom}(f) \to \mathbb{R} \) be such that its domain is a sum (possibly infinite) of representable intervals.
Let us fix $\varepsilon > 0$. The function $f$ is $\varepsilon$w-IAC iff there exists $F \in \mathcal{RI}$ such that there is an algorithm to establish a sequence (potentially infinite) of representable intervals $I_1, I_2, \ldots$ such that

$$\text{dom}(f) = \bigcup_{i \in \mathbb{N}} I_i \quad \text{and} \quad \forall i : [f(I_i) \subset F(I_i) \land \text{diam}(F(I_i)) \leq \varepsilon]. \quad (3)$$

The function $f$ is w-IAC iff there exists $F \in \mathcal{RI}$ such that for any given $\varepsilon > 0$ there is an algorithm to establish a sequence (potentially infinite) of representable intervals $I_1, I_2, \ldots$ such that

$$\text{dom}(f) = \bigcup_{i \in \mathbb{N}} I_i \quad \text{and} \quad \forall i : [f(I_i) \subset F(\varepsilon, I_i) \land \text{diam}(F(\varepsilon, I_i)) \leq \varepsilon]. \quad (4)$$

**Proof** The idea of the proof is exactly the same in both cases (computability and $\varepsilon$-computability), thus we write $F([\varepsilon, I])$ to emphasize that the first parameter of $F$ is optional.

$(\Rightarrow)$ We assume (in Remark 1) that there are countably many representable numbers, hence there are countably many representable intervals (i.e. intervals with representable endpoints). To obtain countable covering of $\text{dom}(f)$ we inspect every representable interval (using for example primitive recursive Cantor pairing function) checking if it is contained in $\text{dom}(f)$ and $\text{diam}(F([\varepsilon, I])) < \varepsilon$. By the assumption that $f$ is $(\varepsilon)$w-IAC, we know that for any $x \in \text{dom}(f)$ the proper representable interval exists, thus we will find it in a finite number of steps.

Intervals that satisfy the requirements form the desired covering of $\text{dom}(f)$.

$(\Leftarrow)$ We assume that $f$ satisfies (3) or (4). Let us take any $x \in \text{dom}(f)$. Since $\text{dom}(f)$ is a sum of representable intervals $I_1, I_2, \ldots$, the number $x$ belongs to some $I_i$ and by (3) (or (4)) we know that the required $I \in \mathcal{X}$ exists.

**Remark 4.** Definition 4 and equivalent conditions at Theorem 4 carry over to multidimensional setting by demanding that $I_i$ become elementary multidimensional interval sets, for example products of representable intervals or some other simple sets.

**Theorem 2.** Assume $f : \mathbb{R} \supset \text{dom}(f) \to \mathbb{R}$. If $f$ is w-IAC then it is continuous at every non-representable point.

**Proof** As $f$ is w-IAC, by Definition 6 there exists the corresponding $F \in \mathcal{RI}$. Let us fix $\varepsilon_0 > 0$ and $x_0 \in \text{dom}(f)$; now there exists $I_0 \in \mathcal{X}$ such that $x_0 \in I_0$, $I_0 \subset \text{dom}(f)$, $f(I_0) \subset F(\varepsilon_0, I_0)$ and $\text{diam}(F(\varepsilon_0, I_0)) < \varepsilon_0$.

If $x_0 \in \mathbb{R} \setminus R$, then $x_0 \in \text{int}(I_0)$ (recall that $\mathcal{X}$ is the set of intervals with representable endpoints), hence $\exists \delta > 0 : B(x_0, \delta) \subset I_0$, where $B(x_0, \delta)$ is an open ball with a center at $x_0$ and radius $\delta$. Now, $\forall x : |x - x_0| < \delta \Rightarrow x \in I_0$, thus

$$|f(x) - f(x_0)| \leq \text{diam}(f(I_0)) \leq \text{diam}(F(\varepsilon, I)) \leq \varepsilon.$$

Hence, $f$ is continuous at $x_0$. 

Below we consider a family of step functions $s_c$ and we prove that if $c$ is representable, then $s_c$, although discontinuous, is w-IAC.
Example 6. Consider a step function \( s_c(x) = \begin{cases} 0, & \text{if } x \leq c \\ 1, & \text{if } x > c \end{cases} \). If \( c \) is a representable number then \([c, c] \in X\). Thus for \( s_c \) there exists an interval extension \( F \in RI \), such that \( \operatorname{diam}(F) < \varepsilon \):

\[
F(I) = \begin{cases} [1, 1], & \text{if } I \subseteq [c, c] \\ [2, 2], & \text{if } I > [c, c] \end{cases}.
\]

For \( I \in X \) with \( I \subseteq [c, c] \) or \( I > [c, c] \), we have \( \operatorname{diam}(F(I)) = 0 \).

Assume that \( c \) is not a representable number. Then any interval \( I \) such that \( c \in I \) has a form \([\alpha, \beta]\), where \( \alpha < c < \beta \). Thus for any interval extension \( F \) of the function \( s_c \) and for any \( I \) such that \( c \in I \), \( s_c(I) = \{1, 2\} \subset F(I) \). Hence \( \operatorname{diam}(F(I)) \geq 1 > \varepsilon \).

Therefore, for \( \varepsilon < 1 \), \( s_c \) is \( \varepsilon \)-w-IAC if and only if \( c \) is representable.

The following example is a \( w \)-IAC function which is discontinuous for all \( x \in \mathbb{D} \setminus \{0\} \).

Example 7. Let us consider a function \( f : (0, 1) \to \mathbb{R} \):

\[
f(x) = \begin{cases} \frac{1}{x^2}, & x = \frac{m}{n} \text{ is an irreducible fraction} \\ 0, & \text{otherwise}. \end{cases}
\]

Assume that we know an algorithm (which is obvious to write) computing \( \max_{x \in D \cap I}(f(x)) \). The function \( f \) is \( w \)-IAC since its interval extension is accomplished by \( F \in RI \):

\[
F(I) = \begin{cases} [0, \max_{x \in D \cap I}(f(x))], & \text{if } \left(\text{left}(I) \neq \text{right}(I)\right) \text{ then } \left(\text{left}(I), \text{right}(I) \in \mathbb{D}\right) \text{ then } f(\text{right}(I)) \text{ else } [0, 0]).
\end{cases}
\]

The following remark shows that the set of \( w \)-IAC functions in the sense of Definition 6 is not closed under composition.

Remark 5. Composition of two \( w \)-IAC functions need not be a \( w \)-IAC function. For example, let \( g(x) = \sin(x) \). Using the series expansion we can prove that \( \sin(x) \) is \( w \)-IAC. Observe that function \( s_0 \circ g \) is discontinuous at \( \pi \), hence by Theorem 2 it cannot be \( w \)-IAC.

The above example shows that the class of \( w \)-IAC functions in the sense of Definition 6 does not coincide with the ones computable in the sense of Definition 5, which must be continuous.

5.2 Strong interval arithmetic computable (s-IAC) functions

The goal of this subsection is to define the class of s-IAC functions which will turn out to be the same as the computable functions in the sense of Definition 5.

Definition 7. Let us fix \( \varepsilon > 0 \). We say that a function \( f : \mathbb{R} \supset \text{dom}(f) \to \mathbb{R} \) is \( \varepsilon \)-s-IAC if there exists \( F \in RI \) such that for any compact \( K \in X \), \( K \subset \text{dom}(f) \) there exists an algorithm to establish a finite sequence \( I_1, \ldots, I_n \in X \) such that

\[
K \subset \bigcup_{i=1}^{n} I_i \subset \text{dom}(f) \quad \text{and} \quad \forall i \in \{1, \ldots, n\} : \left| f(I_i) \subset F(I_i) \land \operatorname{diam}(F(I_i)) \leq \varepsilon \right|.
\]  

We say that \( f \) is \( s \)-IAC if there exists \( F \in RI \) such that for any given \( \varepsilon > 0 \) and for any compact set \( K \subseteq X \), \( K \subset \text{dom}(f) \) there exists an algorithm to establish a finite sequence \( I_1, \ldots, I_n \in X \) such that

\[
K \subset \bigcup_{i=1}^{n} I_i \subset \text{dom}(f) \quad \text{and} \quad \forall i \in \{1, \ldots, n\} : \left| f(I_i) \subset F(\varepsilon, I_i) \land \operatorname{diam}(F(\varepsilon, I_i)) \leq \varepsilon \right|.
\]
Note that a remark analogous to Remark 4 is valid in this case.

If \( R = \mathbb{Q} \) (i.e. arithmetic with infinite precision), then it is easy to see that basic arithmetic operations (+, \( \cdot \) and \( 1/x \)), function \( \sqrt{\cdot} \) and many elementary analytic functions like \( \exp \) or trigonometric functions are s-IAC. However in the case of \( R \) being floating point numbers of finite size (this is the hardware standard in the present day computers), these operations are only \( \varepsilon \)-s-IAC (on compact domains, with \( \varepsilon \to 0 \) when the number of bits in the representation is increasing to infinity).

The following lemma states that in conditions (5) and (6) we can assume all \( I_j \)'s to have positive diameter, i.e. we can remove all degenerate intervals.

**Lemma 1.** Assume that \( K \in \mathcal{X} \) and \( K \subset \bigcup_{i=1}^{n} I_i \), where \( I_i \in \mathcal{X} \). Then \( K \subset \bigcup_{i \in Z} I_i \), where \( i \in Z \) iff \( \text{diam}(I_i) > 0 \).

**Proof** Let \( K = [a, b] \). Assume that we have an interval \( I_j \) with zero diameter, i.e. \( I_j = [x, x] \). We would like to remove it from the covering of \( K \).

We have two cases:

1. There exists \( I_i \) such that \( x \in \text{Int} I_i \), then we can discard \( I_j \) from the covering.
2. For all intervals \( I_i \), \( x \notin \text{Int} I_i \).

   We argue that there exist \( c_1 < x < c_2 \), such that the intervals \([c_1, x] \) and \([x, c_2] \) belong to the covering (if \( a = x \) or \( x = b \) then we just need one of these intervals).

   Existence of the interval on the left: let us consider all intervals \( I_i = [l_i, r_i] \) such that \( l_i < x \). Obviously, \( r_i \leq x \), otherwise \( x \notin \text{Int} I_i \). The largest of such \( r_i \), say \( r_{i_0} \), must be equal to \( x \), otherwise the open interval \((r_{i_0}, x)\) will be not covered by \( \bigcup I_i \).

   The interval on the right is obtained analogously.

**Theorem 3.** Assume that \( f : [a, b] \to \mathbb{R} \) is s-IAC. Then \( f \) is continuous.

**Proof** Let us fix \( \varepsilon \) and let \( [a, b] = \bigcup_{i=1}^{n} I_i \), where \( \{I_i\} \) are as in condition (6). For every \( x \in [a, b] \) we set

\[
U_x = \bigcup\{I_i : i = 1, \ldots, n, \ x \in I_i\}.
\]

It is easy to see that \( x \in \text{Int} U_x \) (see proof of Lemma 1) and for all \( y \in U_x \), \( |f(y) - f(x)| \leq \varepsilon \). Indeed if \( y \in U_x \) then there is \( I_i \), such that \( y, x \in I_i \), hence

\[
|f(x) - f(y)| \leq \text{diam}(f(I)) \leq \text{diam}(F(I)) \leq \varepsilon.
\]

**Lemma 2.** Assume that \( K = [a, b] \) and \( K \subset \bigcup_{i=1}^{n} I_i \), where \( I_i = [l_i, r_i] \) with \( l_i < r_i \). Let \( \delta = \min_{i=1, \ldots, n} \text{diam}(I_i) \).

Assume \( K = [a, b] \subset \bigcup_{i=1}^{n} I_i \), where \( I_i = [l_i, r_i] \) with \( l_i < r_i \). Let \( \delta = \min_{i=1, \ldots, n} \text{diam}(I_i) \). If \( x, y \in K \) and \( |x - y| \leq \delta \), then there exist \( k, j \) such that \( x \in I_j \), \( y \in I_k \) and \( I_j \cap I_k \neq \emptyset \).

**Proof** If there exists \( i \), such that \( x, y \in I_i \), then take \( j = k = i \).

Assume now that such \( i \) does not exists. Without loss of generality, assume that \( x < y \). Let \( j \) be such that \( r_j = \max\{r_i : i = 1, \ldots, n, \ x \in I_i\} \). Obviously \( r_j < y \), otherwise \( x, y \in I_j \). Analogously, let \( k \) be such that \( l_k = \min\{l_i : i = 1, \ldots, n, \ y \in I_i\} \). Obviously \( l_k > x \), otherwise \( x, y \in I_k \).

We show that \( r_j \geq l_k \) meaning that \( I_j \cap I_k \neq \emptyset \). Assume the contrary. Let \( z = (r_j + l_k)/2 \). Consider \( I_z = [l_z, r_z] \), such that \( z \in I_z \). From the assumption, \( x \notin I_z \) and \( y \notin I_z \), hence \( I_z \subset (x, y) \) and \( \text{diam}(I_z) < \delta \), which contradicts the definition of \( \delta \). Therefore \( I_j \cap I_k \neq \emptyset \).
The next theorem shows that the information about the modulus of continuity can be obtained for $\varepsilon$-s-IAC functions.

**Theorem 4.** Let us fix any $\varepsilon > 0$. Assume that $f : [a, b] \to \mathbb{R}$ is $\varepsilon$-s-IAC, i.e. there exists a finite covering of $[a, b]$ required by Definition 7 of a nondegenerate interval (see Lemma 1) $[a, b] = \bigcup_{i=1}^{n} I_i$. Let $\delta = \min_{i=1,\ldots,n} \text{diam} (I_i)$. Then for all $x, y \in [a, b]$ such that $|x - y| \leq \delta$, $|f(x) - f(y)| \leq 2\varepsilon$.

**Proof** Let us take any $x, y \in [a, b]$, $|x - y| \leq \delta$. By Lemma 2 there exist $I_i, I_j$ such that $x \in I_i$, $y \in I_j$ and $I_i \cap I_j \neq \emptyset$. Let us take any $z \in I_i \cap I_j$. Then $|f(x) - f(z)| \leq \varepsilon$ and $|f(y) - f(z)| \leq \varepsilon$, therefore $|f(x) - f(y)| \leq 2\varepsilon$.

The above theorems are not surprising. In Ko’s approach computable functions are continuous with computable modulus of continuity on compact domains.

Below we show that Definition 7 is equivalent to the one given by Ko (see Section 4). However, there is a significant difference between these approaches. We do not need any oracles to ask for a desired number of digits of some non-computable reals. We follow computations on finite numbers being ends of an interval, but conclusions concern all real numbers belonging to the interval.

**Theorem 5.** Assume we have an interval arithmetic with infinite precision. Function $f : K \to \mathbb{R}$, where $K \in X$, is s-IAC iff it is computable in the sense of Ko, i.e. Definition 5.

**Proof**

$\Rightarrow$ Assume $f$ is s-IAC. Let $F \in \mathcal{R}L$ and $T$ be a Turing machine (an algorithm) computing the partition in Definition 7. We define the function-oracle Turing machine $M$ as follows,

- on input we have $n \in \mathbb{N}$ and an oracle $\varphi \in CF_x$
- we call $T$ with parameter $\varepsilon = 2^{-(n+2)}$ to obtain a sequence $\{I_i\}_{i=1}^{L}$, such that $K \subset \bigcup_{i=1}^{L} I_i$, $f(I_i) \subset F(\varepsilon, I_i) \land \text{diam} (F(\varepsilon, I_i)) < \varepsilon$
- let $\delta = \min_{i=1,\ldots,L} \text{diam} (I_i)$ (by Lemma 1 we can assume that we do not have intervals with zero length)
- let $m$ be the smallest such that $2^{-m} \leq \delta$,
- return any dyadic number $d$ from $F(\varepsilon, I_j)$, where $I_j$ is any interval containing $\varphi(m)$.

To complete the proof we need to show that $|f(x) - d| < 2^{-n}$. Since $|\varphi(m) - x| \leq 2^{-m} \leq \delta$, then by Lemma 2 it follows that there exists $k_1, k_2$ such that $\varphi(m) \in I_{k_1}$, $x \in I_{k_2}$ and $I_{k_1} \cap I_{k_2} \neq \emptyset$. Let us take any $y \in I_{k_1} \cap I_{k_2}$. We have

\[ |f(x) - d| \leq |f(x) - f(y)| + |f(y) - f(\varphi(m))| + |f(\varphi(m)) - d| \leq \text{diam} (F(\varepsilon, I_{k_2})) + \text{diam} (F(\varepsilon, I_{k_1})) + \text{diam} (F(\varepsilon, I_j)) \leq 3 \cdot 2^{-(n+2)} < 2^{-n}.

$\Leftarrow$

Instead of Ko’s definition, we use an equivalent one given by Pour-El, Caldwell and Shepherdson [22, 23]:

**Definition 8.** The real function $f$ is said to be $P$-computable on the bounded set $I$ iff there exists a recursively enumerable sequence $(P_n(x))$ of polynomials with rational coefficients such that for all $x$ in $I$ for which $f(x)$ is defined, and all $n$

\[ |f(x) - P_n(x)| < 2^{-n}.\]
A sequence \((P_n(x))\) of polynomials in \(x\) is *recursively enumerable* if there exist recursive functions \(q, r, s, d\) such that
\[
P_n(x) = \sum_{j=0}^{d(n)} (-1)^{(s(n))_j} \frac{(q(n))_j}{1 + (r(n))_j} x^j,
\]
where \((a)_j\) is the exponent of the \(j^{th}\) prime in the prime factorization of \(a\).

It means that there exists a Turing machine, by Remark 8 equivalent to an interval machine \(PI_f\), that for given \(n\) returns a polynomial \(P_n\). The algorithm for \(F \in RI\) looks as follows:

**Input:** \(E = [\varepsilon, \varepsilon], K \in X\)

1. Calculate \(n\) such that \(2^{-n} \leq \varepsilon/4\).
2. Run \(PI_f\) on \(n\) obtaining a polynomial \(P_n\).

**Output:** \(P_n(K) + [-2^{-n}, 2^{-n}]\).

Since for any \(x \in K:\ |f(x) - P_n(x)| < 2^{-n}\) and \(F(\varepsilon, K) = P_n(K) + [-2^{-n}, 2^{-n}]\), then \(f(K) \subset F(\varepsilon, K)\).

The question is if for any \(\text{dom}(f) \supset K \in X\) there exists an algorithm to establish a partition of \(K = \bigcup_{i=1}^n I_i\) such that \(\text{diam}(F(\varepsilon, I_i)) < \varepsilon\). A solution is for example BSA (the algorithm discussed in detail in Section 6), which in this situation looks as follows:

\[
\text{BSA}_{P_n}(\varepsilon, I) : \begin{cases} 
\text{if diam } P_n(I) < \varepsilon/2 \text{ then return } I \\
\text{else } \\
\text{bisect } I = I_L \cup I_R; \\
\text{BSA}_{P_n}(\varepsilon, I_L); \\
\text{BSA}_{P_n}(\varepsilon, I_R);
\end{cases}
\]

Since basic arithmetic operations (of which polynomials are constructed) are \(s\)-IAC with infinite precision arithmetic, the algorithm stops after a finite number of steps.

### 5.3. Summary

To summarize the properties of different definitions of interval arithmetic computability, let us remark:

1. \(\varepsilon\) interval arithmetic computability (i.e. \(\varepsilon_w\)-IAC and \(\varepsilon_s\)-IAC functions) results from the use of finite precision arithmetic; the non-\(\varepsilon\) variant is only possible when infinite precision arithmetic is used (with the only exception: a constant function);

2. interval arithmetic strong computability (i.e. \(\varepsilon_s\)-IAC and \(s\)-IAC functions) implies the continuity of functions; non-strong variant allows discontinuous functions to be handled.

### 6. Binary subdivision algorithm

The goal of this section is to emphasize two issues which are neglected in other approaches. When using the interval arithmetic

1. information about the value of the function in chosen points is not the only accessible information; we can ask for an estimate of \(f(I)\), i.e. a set \(J\), such that \(f(I) \subset J\);
2. there is no need for any kind of global information, like the knowledge of the Lipschitz constant or the modulus of continuity.

To illustrate the above points we describe a binary subdivision algorithm (BSA), which can be easily implemented on an interval machine. As an example of use of BSA, we chose the problem of evaluation of the range of a function with desired accuracy. However, BSA might be used in many applications, for example to compute \( \int_I f(x)dx \), \( \sup_{x \in I} f(x) \) or \( \inf_{x \in I} f(x) \) with desired precision.

Before presenting the BSA algorithm let us stress that the interval arithmetic, by its design, for any rational function \( f : \mathbb{R}^n \to \mathbb{R} \) and any interval \( I \subset \mathcal{X} \) computes an interval \( J \), such that \( f(I) \subset J \). This is achieved by replacing all real variables in \( f \) by the interval ones and executing the interval arithmetic evaluation of the expression defining \( f \). The same method can be applied to estimate the range of an analytic function with an explicit series expansion and the remainder term.

This straightforward approach suffers from a serious drawback, the so-called dependency problem: it might significantly overestimate the result. The example below illustrates this phenomenon.

**Example 8.** We want to estimate the range of \( f(x) = e^{-x} \) for \( I = [0, h] \), where \( h > 0 \) is a representable number. We consider a series expansion for \( f(x) \), i.e.

\[
e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots
\]

Let us denote by \( \exp \) the interval arithmetic realization of a ‘naive’ summation of the above series term by term.

It is easy to see that

\[
\sup \exp([-0, h]) \geq 1 + \frac{h^2}{2!} + \frac{h^4}{4!} + \ldots = \cosh h = \frac{e^h + e^{-h}}{2},
\]

\[
\inf \exp([-0, h]) \leq 1 - h - \frac{h^3}{3!} - \ldots = 1 - \sinh h = 1 - \frac{e^h - e^{-h}}{2}.
\]

Observe that we have inequalities because of possible truncation errors.

Therefore we obtain

\[
\left[ 1 - \frac{e^h - e^{-h}}{2}, \frac{e^h + e^{-h}}{2} \right] \subset \exp([-0, h]).
\]

(7)

We have for \( h \to 0 \)

\[
\text{diam } \exp([-0, h]) \geq e^h - 1 = h + \frac{h^2}{2!} + O(h^3), \quad \text{diam } \left( e^{-[0,h]} \right) = 1 - e^{-h} = h - \frac{h^2}{2!} + O(h^3),
\]

(8)

hence

\[
\text{diam } \exp([-0, h]) - \text{diam } \left( e^{-[0,h]} \right) \geq h^2 + O(h^3).
\]

(9)

Thus to estimate effectively the range of \( e^{-x} \) on \( I = [0, h] \), with \( h \) large, it is not enough to take \( \exp(-I) \), since interval arithmetic gives us a highly overestimated result: \( \text{diam } \exp([-0, h]) \supset e^h - 1 \) and it tends to infinity with \( h \to \infty \). Fortunately, the overestimation of the diameter of the result is \( O(h^2) \) with \( h \to 0 \), if we disregard the truncation errors. This observation is the basic idea behind the design of the binary subdivision algorithm in Section 6.1.

6.1. BSA for the range evaluation on the interval machine

The algorithm works for any \((\varepsilon)s\)-IAC function \( f : \mathbb{R} \to \mathbb{R} \), where \( \varepsilon \) is optional. For \( f \) there exists \( F \in \mathfrak{RI} \) such that \( f(J) \subset F([\varepsilon], J) \).

The input of the algorithm consists of:
• the interval for which we estimate the range of f;

• an accuracy bound \( \varepsilon > 0 \), which for an interval machine has to be an interval, i.e. \( E = [\varepsilon, \varepsilon] \).

The function F is not an input parameter; it is a sub-procedure for an interval machine implementing the BSA. F may not be defined for some intervals, for example when dividing by an interval containing zero or when diameter of an interval exceeds the local convergence radius for an analytic function. However, we assume that the Turing Machine computing \( F([\varepsilon], I) \) always stops and decides whether the computation is successful and \( F([\varepsilon], I) \) is defined.

Here is the algorithm:

**Input:** \( I = [a, b] \), \( E = [\varepsilon, \varepsilon] \)

1. \( S := \emptyset \) (an empty interval).
2. Define the initial partition of \([a, b]\) by setting \([a_0, a_1] = [a, b]\) and mark this interval as **bad**.
3. Assume that \([a, b]\) has already been partitioned into \( m \) intervals, i.e.,
   \[
   [a, b] := [a_0, a_1] \cup [a_1, a_2] \cup \ldots \cup [a_{m-1}, a_m]
   \]
   and there exists \( J \subset \{1, \ldots, m\} \) such that \([a_{i-1}, a_i]\) is **bad** for \( i \in J \), while other intervals are **proper**. Let \( K = [a_{k-1}, a_k] \) be the longest **bad** interval.

If \( K \) cannot be bisected (this may happen when \( \text{diam}(K) = 0 \) or the midpoint of \( K \) is not a representable number), then stop and return **failure**.

Otherwise:
   (a) bisect \( K \) into \( K_L \) and \( K_R \),
   (b) remove \( K \) from the partition and add both \( K_L \) and \( K_R \),
   (c) for \( Z = K_L \) and \( Z = K_R \) do the following:
      – compute \( F([\varepsilon], Z) \),
      – if \( F([\varepsilon], Z) \) is defined and \( \text{diam}(F([\varepsilon], Z)) \leq \varepsilon \), then \( Z \) is **proper** and set \( S = \text{hull}(S \cup F([\varepsilon], Z)) \), else \( Z \) is **bad**.

Repeat step 3 as long as the set \( J \) of **bad** intervals is not empty.

**Output:** \( S \).

Note that the partition produced during the execution of the BSA establishes strong \( \varepsilon \)-computability of \( f \) on \( I \), thus we can state an obvious theorem:

**Theorem 6.** BSA stops and \( f(I) \subset S \subset \overline{B}(f(I), \varepsilon) \).

Obviously, if we know more about a function \( f \) then the range can be computed faster using other approaches. For example for an analytic function computable in polynomial time (with respect to \( \log_2(1/\varepsilon) \)) the maximum value on the compact interval can be computed in polynomial time by searching for all roots of \( f'(x) = 0 \) (see [10, Sec. 6.2]).

**Remark 6.** BSA with infinite precision interval arithmetic allows to compute any rational function on any compact interval with arbitrary precision. Therefore with such interval arithmetic these functions are s-IAC in the sense of Definition[7]. Analogous results can be easily proved for other elementary functions \( \exp, \sin, \cos \ etc. \ and \ their \ compositions. \)**
7. Existing models of computation

In Section 4 we have already discussed Ko’s approach in order to prepare for the CAPA inspired approach given in previous sections. Here we will discuss other models prevailing in the literature, from the point of view of CAPA.

There are approaches that consider functions on all real numbers: real-RAM and bit models; on the other hand, the are settings (belonging to a field of constructive mathematics) which work on (specifically defined) computable numbers only: e.g. Banach/Mazur (cf. [15]) or Pour-El/Richards (cf. [23]). We are interested in computations that allow the use of all real numbers and not just a computable subset thus we skip the latter constructive approach. We follow [6] and [31] in a short presentation of different models.

7.1. Real-number models

In real-number models the reals are considered real, i.e. actual objects. The model of computation is a generalized Random Access Machine with two kinds of registers: for natural and for real numbers. The two approaches we briefly touch upon, BSS and IBC, are examples of the real-number models.

7.1.1. Blum-Shub-Smale (BSS) model

In [5] the authors point to the incompatibility between the discrete world of computer computations and the continuous nature of calculus and numerical analysis. Since the discrete computer cannot cope with the continuous mathematics, they decide to change the computation model. This idealization makes perfect sense in many contexts, as it allows to put the classical numerical analysis on a firm basis. In the BSS approach arithmetic operations are performed on real numbers that are stored with infinite precision. It is obvious that no physical model implements this approach, however this is nearly true when manipulating polynomials or in a lot of tasks of numerical linear algebra.

In the BSS model the authors define computable functions to be the ones produced by programs using the arithmetic operations and the inequalities for branching. Therefore a domain of computable function $f$ in the BSS model is a possibly countable collection of semi-algebraic sets $\{A_i\}$ and $f$ is rational on $A_i$.

There are functions obviously non-computable on present computers, but computable in the BSS model. For example the function:

$$f(x) = \begin{cases} 
1, & x \in \mathbb{Q} \\
0, & x \not\in \mathbb{Q} \text{ and } x^2 \in \mathbb{Q} \\
\omega, & \text{otherwise}
\end{cases}$$

where $\omega$ means “non-defined”. This means that program does not stop for such input.

On the other hand, there are functions computable with some error $\varepsilon$, classified as non-computable in the BSS model; for example the square root or the exponential function. This is not a fundamental issue; it is just a matter of definition; see [6] where a suitable extension of definition of computability of the real function inside the BSS model is given.

Observe that, even in the context of the BSS model, for example when dealing with solutions of ODEs, the intervals might be needed to carry out the estimates. In that case according to the spirit of the BSS model the set of representable numbers to be used in the construction of the interval arithmetic should be $\mathbb{R}$ (see Section [Appendix A] for the brief discussion of the interval arithmetic).

7.1.2. The Information Based Complexity

The focus of Information Based Complexity (IBC) [26, 28] is on the complexity of the problems in continuous mathematics. There is no explicit need for the definition of computable functions in the
IBC. The lower bound for the complexity of some problems, for example the computation of a definite integral, is obtained by considering the following (optimization) problem: given the values of \( f \) or its derivatives at \( n \) points (evaluation nodes) find the radius of the ball containing all possible values of the integral and minimize this radius over all possible locations of the evaluation nodes. This gives a minimal error \( \varepsilon(n) \), which any program must make if it asks just for \( n \) values of function or its derivatives at some points. The \( n(\varepsilon) \), which is inverse to \( \varepsilon(n) \), give then the lower bound for the complexity of computing the definite integral with the precision \( \varepsilon \). If then one can find an algorithm \( A \) using some oracles returning the values of the function or its derivatives, for which the cost matches the lower bound \( n(\varepsilon) \), then the claim can be made that the obtained function \( n(\varepsilon) \) is the complexity of the problem and \( A \) is an optimal algorithm (under the given type of admissible information). The cost computation in the IBC depends on the computation model (...) the complexity of a problem can be totally different in real number and bit models; cf. [26].

From the perspective of the interval machine the IBC approach makes typically unjustified assumption about the kind of information a computer program may ask for. As it was stressed in Section [5] for the computable functions on the interval machine we can ask not only for the values of the functions or derivatives at some points, but also for the estimates of these values over some interval sets. But for this to make sense, one needs to define a notion of the computable function and computation model, which the IBC appears to deliberately avoid while striving for the largest possible generality and to be largely independent of the particular computation model assumed.

7.2. Bit-models

7.2.1. Type-2 Theory of Effectivity (TTE)

Type-2 Theory of Effectivity (TTE) described by K. Weihrauch in [31] provides a general framework for investigating problems in computable analysis.

The basic TTE computability concepts are implemented via naming systems. Although the TTE is more realistic than the BSS model, even this attempt diverges from (clashes/conflicts with) practice:

1. Our main objection is the infeasibility of TTE representations of mathematical functions (both input and output) as infinite strings. Programs in real life do not generate output bit by bit; in order to get a new decimal digit on output, we rather need to run the program again with parameters forcing the better precision.

2. In numerical practice functions are represented as programs or as a composition of basic operations (e.g. piecewise rational functions), thus representing them in a sophisticated theoretical way might be expensive. This cost can be neglected in the context of computability, but not complexity.

3. Real computers operate on finite representations thus simple discontinuous functions as

\[
s_1(x) = \begin{cases} 
1, & x \geq 1 \\
0, & \text{otherwise},
\end{cases}
\]

are obviously computable from the CAPA viewpoint (see also our definition of computable functions in Section [5]), while the TTE classifies them as uncomputable.

7.2.2. Ko’s approach

K.I. Ko ([10, 11]) applies ideas from the recursion theory and the classical complexity theory real functions. His approach is equivalent to Weihrauch’s (see for example [31], Theorem 9.4.3). We have already presented this approach in Section [4].
8. Conclusions

In the paper we present the idea of interval computability which has some advantages over other models of real computation:

1. it operates on finitely represented numbers being the ends of intervals;
2. it allows to obtain interval information, i.e. values of functions not only at a point, but also on the whole interval;
3. it allows rigorous interval calculations (which are crucial for CAPAs) giving the certainty that all real numbers (whether computable or not) in the interval possess a property in question;
4. it has an implementation in the form of various packages for rigorous numerics (e.g. CAPD, see [7]).

Computable functions are defined as functions computed (with arbitrary accuracy) by an interval machine which is Turing equivalent. The set of these functions clearly depends on the representation used.

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Appendix A. Basic terms and concepts of interval arithmetic

Let us recall that by \( X \) we denote the set of all intervals with endpoints belonging to \( \hat{\mathbb{R}} \). Some of the notations and definitions related to intervals are taken from the book [24]. In particular, we use capital letters to denote intervals and their endpoints are marked with a line below or above the letter: \( X = [\underline{X}, \overline{X}] \). Two intervals \( X \) and \( Y \) are said to be equal, if they are the same sets i.e. \( X = Y \iff (\underline{X} = \underline{Y} \wedge \overline{X} = \overline{Y}) \).

We say that \( X \) is degenerate if \( \underline{X} = \overline{X} \). Such an interval contains a single representable number \( x \).

By convention, we identify a degenerate interval \([\underline{x}, \overline{x}]\) with the representable number \( x \). In this sense, we may write such equations as \( 0 = [0, 0] \).

**Definition 9.** The class of interval predicates \( \mathcal{C} \) consists of:

- interval equality and inequalities:
  \[
  X = Y \iff \underline{X} = \underline{Y} \wedge \overline{X} = \overline{Y}
  \]
  \[
  X < Y \iff \underline{X} < \underline{Y}
  \]

- interval set inclusion:
  \[
  X \subset Y \iff (\underline{Y} \leq \underline{X} \wedge \overline{X} \leq \overline{Y}).
  \quad (A.1)
  \]

Observe that when applied to interval arguments it might happen that neither \( x \leq y \) nor \( y > x \) holds. We also have that if \( X < Y \), then for all \( x \in X \) and for all \( y \in Y \), \( x < y \). However from \( X = Y \) we cannot infer that \( x = y \) for all \( x \in X \) and \( y \in Y \). This has consequences for programming using interval arithmetic.

**Definition 10.** An interval hull of \( S \subset \mathbb{R} \) is the smallest closed interval containing \( S \), i.e. \[
\text{hull}(S) = [\inf_{x \in S} x, \sup_{x \in S} x].
\]

A function \( F : \mathbb{R} \supset \text{dom}(t) \to \mathbb{R} \) is called an interval function, where \( \mathbb{R} \) denotes the set of all intervals with real endpoints. For any function \( f : \mathbb{R} \to \mathbb{R} \) there exists an interval extension function \( F \) such that \[
F(X) = \text{hull}\{ y \mid \exists x \in X : f(x) = y \},
\]
where \( X \subset \mathbb{R} \), \( X \) interval. Note that the opposite is not true; e.g. there is no \( f : \mathbb{R} \to \mathbb{R} \) such that the constant interval function \( F(X) = [-1, 1] \) is its extension.

**Definition 11.** We define interval extensions of arithmetic operations (for division we assume that \( 0 \notin Y \)):

\[
\begin{align*}
X + Y &= [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}] \\
X - Y &= [\underline{X} - \underline{Y}, \overline{X} - \overline{Y}] \\
X \cdot Y &= [\min\{X \cdot Y, X \cdot \overline{Y}, \overline{X} \cdot Y, \overline{X} \cdot \overline{Y}\}, \max\{X \cdot Y, X \cdot \overline{Y}, \overline{X} \cdot Y, \overline{X} \cdot \overline{Y}\}] \\
X/Y &= X \cdot [1/\overline{Y}, 1/\underline{Y}] \\
&= [\min\{X/Y, X/\overline{Y}, \overline{X}/Y, \overline{X}/\overline{Y}\}, \max\{X/Y, X/\overline{Y}, \overline{X}/Y, \overline{X}/\overline{Y}\}],
\end{align*}
\]

Note that interval operations defined above do not take into account that, even if the endpoints are representable numbers, the results of the arithmetic operation might not be representable (this is the case of the floating point arithmetic present on today’s machines). As mentioned in Section 1.1, we
require that actual results belong to the computed intervals; to achieve this we require that computer implementation \(\hat{\circ}\) of a theoretical interval operation \(\circ\) satisfies

\[
\forall X, Y \in \mathcal{X} \quad X \circ Y \subseteq X \hat{\circ} Y \subseteq X, \quad \text{where} \quad \circ \in \{+, -, \cdot, /\}.
\]  

(A.2)

and for division we assume that \(0 \not\in Y\).

**Definition 12.** We define the unary basic operations:

1. the left endpoint of an interval \(X\) is given by \(\text{left}(X) = [X, X] \equiv X\).

2. the right endpoint of \(X \in \mathcal{X}\) is given by \(\text{right}(X) = X\).

3. the absolute value of \(X \in \mathcal{X}\), denoted \(|X|\), is the maximum of the absolute values of its endpoints:

\[
|X| = \max\{|X|, |\overline{X}|\}.
\]

Note that \(|x| \leq |X|\) for every \(x \in X\).

Note that for \(X \in \mathcal{X}\) basic unary operations are functions \(\mathcal{X} \to \mathcal{X}\).

**Appendix A.1. Reasonable arithmetic**

If we do not state any requirement for the quality of interval arithmetic, it can give arbitrarily large error in the implementation of real functions. Thus we discard unreasonable implementations and we want to be as realistic as possible. We require the computer implementation to be reasonable, i.e. \(\text{diam}(X \hat{\circ} Y)\) to be as small as can possibly be obtained. In floating point arithmetic this is achieved by an application of the directed rounding when performing the non-rigorous arithmetic operations, i.e. when computing the left end of an interval the numbers are rounded down, whilst for the right end they are rounded up.

**Definition 13.** The computer implementation is reasonable, if for finite \(R\), the result of an operation \(X \circ Y\) is an interval with left endpoint being the greatest representable number less than or equal to \(\overline{X} \circ Y\) and right endpoint the least representable number greater than or equal to \(X \circ \overline{Y}\). If \(R = \mathbb{Q}\), then \(X \circ Y = X \hat{\circ} Y\).

**Definition 14.** A set of reasonable computer implementations of interval extensions of \(\{+, -, \cdot, /\}\) is called an interval arithmetic. If \(R = \mathbb{Q}\), we say that we have the interval arithmetic with infinite precision.

**Appendix B. Model of an interval machine**

When a CAPA is performed, we deal with the really existing computers and no idealizations are made, beside believing that the processor and compilers work as promised in the documentation. For this reason it might be worth to formalize the model of a machine, which will correspond to the reality of CAPAs. We call it an interval machine. It is equipped with arithmetic which operates on intervals as described in Section **Appendix A**.

An interval machine \(M\) over representations \(\hat{R} \subseteq \Sigma^*\), where \(\Sigma\) is a fixed, finite alphabet, is a program that can be presented as a flow diagram (a finite connected graph) with nodes representing instructions. Instructions are described in Section **Appendix B.1** and the execution of the program in Section **Appendix B.2**.
Appendix B.1. A program of an interval machine

A program contains nodes of the following form:

1. Exactly one input node start with input variables $X_1, \ldots, X_n$ of type $\mathcal{X}$ (i.e. variables store representable intervals) and exactly one output node stop, where we specify intervals returned as shown in Figure B.1. An input node has one outgoing edge, and no incoming edges; an output node has no outgoing edges, but at least one incoming edge.

2. Assignment nodes as shown in Figure B.2.
   The right hand side can be a constant $[c_1, c_2]$ or one of the following basic operations (denoted by $F(X)$ in Figure B.2):
   • any basic interval arithmetic operations (see Definition 11),
   • any of the unary basic operation (see Definition 12),
   • a variable,
   • a value popped from a stack (see description of stack operations at 4).

   In an instruction $Z \leftarrow [c_1, c_2]$ we assume that both $c_1, c_2$ are representable numbers, where $c_1 \leq c_2$. On the left hand side, $Z$ can be a new variable or a previously existing one.

   Figure B.2: Assignment nodes.

3. Branching nodes as shown in Figure B.3. The predicate $C$ is one of the interval predicates (see Definition 9). If $C(Y)$ is satisfied, then we use a branch yes; if it is not satisfied, we use a branch no; if it is not defined, the result of the operation is not defined. Note that $Y$ can be a vector of variables.

   Figure B.3: A branching node

4. Stack operations as shown in Figure B.4. There are three kinds of stack operations:
   • empty($S$) – checking if $S$ is an empty stack; if $S$ does not exist, then the result of the operation is undefined,
   • push($S, X$) – push a value $X$ onto a stack $S$; if stack $S$ does not exist, a new one is created,
   • pop($S$) – returns a value form the top of a stack $S$; it removes the returned value from the top. This has to be a right hand side of an assignment operation. If stack $S$ is empty or it does not exist, then the result of the operation is undefined.
A notation $Z \leftarrow F(X)$ is also an abbreviation of the whole block of instructions computing the function $F$, assuming that $F$ can indeed be computed with elementary instructions of the machine.

For the sake of clarity, we adopt a convention that a stack and a variable cannot share a name.

In any situation not described above, the operation of the machine is undefined.

Internal memory is allocated for variables, while the external memory is allocated for the implementation of stacks. There can be any finite number of variables and stacks used in the machine.

**Appendix B.2. Computations on an interval machine**

We assume that on input data $X_1, \ldots, X_n \in X$ machine $M$ executes a program with sequence of instructions (nodes) $\{P_0, P_1, \ldots, P_m, P_{m+1}\}$.

1. **Initial state:** The program starts at the $P_0 = \text{START}$ with input data $X_1, \ldots, X_n$.

2. **Intermediate state:** The program executes consequently instructions from $\{P_1, \ldots, P_m\}$. If current instruction is $P_i$, then it modifies variables and stacks according to that instruction and proceeds to the next instruction $P_{i+1}$.

3. **Final state:** The program ends with $P_{m+1} = \text{STOP}$ and the output $Y_1, \ldots, Y_k \in X$.

**Definition 15.** We say that an interval machine $M$ computes a function $F : X^n \to X^k$ if for any $(X_1, \ldots, X_n) \in \text{dom}(f)$ starting at the initial state with an input data $X_1, \ldots, X_n$, it arrives at the final state and returns $(Y_1, \ldots, Y_k) \in X^k$, such that $(Y_1, \ldots, Y_k) = F(X_1, \ldots, X_n)$.

**Example 9.** Let us consider a function

$$f(x, y) = \begin{cases} 0 & \text{if } (x + y \leq 0) \\ 1 & \text{else} \end{cases}$$

An interval extension of $f$ is a function

$$F(I_x, I_y) \equiv \begin{cases} [0, 0] & \text{if } (I_x + I_y \leq [0, 0]) \\ [1, 1] & \text{else} \\ [0, 1] & \text{else} \end{cases}.$$ 

A program of an interval machine computing $F$ is presented on Figure B.5.

**Appendix B.3. Statements about the power of the interval machine**

Notice that the model of interval machine is ideologically very similar to a Random Access Machine in its traditional form, which is Turing equivalent. Hence the next two remarks (we omit the proofs as the results are fairly intuitive). The conclusion from these remarks is that an interval machine does not introduce any new operations. However, it has the advantage of specific notation tailored to the interval arithmetic.

**Remark 7.** If $F$ is a function computed by an interval machine $M_F$, then there exists a deterministic Turing machine $T_F$ computing $F$.

**Remark 8.** For any RAM machine computing a function, there exists an interval machine computing the same function, where the integer values $n$ are represented as degenerate intervals $[n, n]$. 

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The power of the interval machine from the point of view of CAPA is summarized by the following statements:

1. Let $f: \mathbb{R}^n \supset \text{dom}(f) \rightarrow \mathbb{R}^m$ be a rational function (i.e. the function that can be represented as a straight-line program evaluating $f$ using interval arithmetic) and let $K \subset \text{dom}(f)$ be a compact set that can be covered by cubes of arbitrarily small diameter with representable endpoints. Then $\sup_{x \in K} f(x)$ can be estimated from above with arbitrary accuracy (for the one-dimensional case see Remark 6).

2. Every mathematical theorem which can be reduced to a finite number of strict inequalities between rational functions on some simple (“computable”) compact in $\mathbb{R}^n$ can be proved as CAPA on the interval machine.

The converse of the second statement is obviously not true; see for example the symbolic computations leading to identities about trigonometric functions.