Expected Coverage of Bayesian Confidence Intervals for the Mean of a Poisson Statistic in Measurements with Background

Ilya Narsky *
Physics Department, Southern Methodist University, Dallas, TX, 75275-0175, USA
(January 8, 2022)

Abstract

Expected coverage and expected length of 90% upper and lower limit and 68.27% central intervals are plotted as functions of the true signal for various values of expected background. Results for several objective priors are shown, and formulas for calculation of confidence intervals are obtained. For comparison, expected coverage and length of frequentist intervals constructed with the unified approach of Feldman and Cousins and a simple classical method are also shown. It is assumed that the expected background is accurately known prior to performing an experiment. The plots of expected coverage and length are provided for values of signal and background typical for particle experiments with small numbers of events.

*Tel.: 1 607 254 2778; Fax: 1 607 255 8062; E-mail: narsky@mail.lns.cornell.edu; Mail: Wilson Laboratory, SMU Group, Ithaca, NY, 14853, USA.
I. INTRODUCTION

An intense discussion of various methods for setting confidence limits has led to a number of recent publications in the particle physics community. Many such papers discuss methods for confidence interval construction in experiments with small numbers of signal events. The outcome of such an experiment is modeled by a Poisson statistic

\[ f(n|s) = e^{-(s+b)}(s+b)^n/n! , \]  

where \( f(n|s) \) is the conditional probability of observing \( n \) events given the signal \( s \). It is assumed in this note that the expected background \( b \) is accurately known before performing an experiment and can be treated as constant.

Expected coverage and interval length are important properties of a method. These properties were studied in Refs. \[1–3\], but there the discussion was mostly focused on Bayesian methods with a uniform prior. In this paper I compare expected coverage and interval length for several objective Bayesian methods and the unified approach \[4\] of Feldman and Cousins.

The expected coverage for a specific method is defined as

\[ C(s) = \sum_{n=0}^{\infty} I(n, s)f(n|s) ; \]  

\[ I(n, s) = \begin{cases} 
1 & \text{if } s_1(n) \leq s \leq s_2(n) ; \\
0 & \text{otherwise}.
\end{cases} \]

where \([s_1(n), s_2(n)]\) is the confidence interval constructed for the number of events \( n \) with this method. The expected length is given by

\[ L(s) = \sum_{n=0}^{\infty} (s_2(n) - s_1(n)) f(n|s) . \]  

For lower limit intervals \([s_1, +\infty)\) the expected length is infinite, and instead of the length \((3)\) an expected distance from zero to the lower limit is plotted:

\[ L_1(s) = \sum_{n=0}^{\infty} s_1(n)f(n|s) . \]  

Coverage is mostly a frequentist concept. In a Bayesian analysis one is not concerned about coverage as much as about adequately representing one’s objective or subjective prior belief. This, of course, should not prevent us from investigating properties of expected coverage for various Bayesian methods. Another important question is how one should define the desired coverage. Due to the discreteness of the Poisson pdf \((1)\), none of the methods provides exact coverage for all values of \( s \). In a frequentist approach one typically requires at least minimal coverage for every value of the true signal \( s \), and thus frequentist intervals overcover on the average. This overcoverage is not an intrinsic feature of the frequentist interval construction but merely a consequence of the empirical conservatism that many physicists find desirable. Statistics literature has examples of consistent frequentist methods \[3,6\] that lack this minimal coverage property. One can argue that a method that gives the requested mean coverage should be preferred over the one that always overcovers.
No matter what the solution to this problem is, further discussion of this issue is beyond the scope of this paper.

While confidence intervals and the expected coverage are invariant under metric transformation, the expected length and mean coverage are not. A transformation of metric, of course, involves a corresponding Jacobian transformation of the prior density in a Bayesian approach. For example, if the confidence interval for $s$ is $[s_1(n), s_2(n)]$, then a confidence interval for the quantity $1/s$ is obtained by a simple inversion: $[1/s_2(n), 1/s_1(n)]$, and the expected coverage has the same functional dependence upon signal $s$. But the expected interval length and mean coverage apparently change due to the choice of a new metric.

II. BAYESIAN INTERVALS

In a Bayesian approach one has to assume a prior probability density function $\pi(s)$ for unknown signal and then obtain a posterior pdf

$$\pi(s|n) = \frac{f(n|s)\pi(s)}{\int_{0}^{\infty} f(n|s)\pi(s)ds}.$$  

A confidence interval $[s_1, s_2]$ is then found by solving equations

$$\left\{ \begin{array}{l} \alpha_1 = \int_0^{s_1} \pi(s|n)ds ; \\
\alpha_2 = \int_{s_2}^{\infty} \pi(s|n)ds ; \end{array} \right.$$  

for $s_1$ and $s_2$. Here, $\alpha_1$ and $\alpha_2$ are the probabilities of the left and right tails of the posterior pdf, respectively, and

$$CL = 1 - \alpha_1 - \alpha_2$$  

is the confidence level of the constructed interval $[s_1, s_2]$. For example, to compute a 90% upper limit, one should put $\alpha_1 = 0$ and $\alpha_2 = 0.1$; for a 90% lower limit these values are $\alpha_1 = 0.1$ and $\alpha_2 = 0$; and to obtain a 68.27% central interval, one should use $\alpha_1 = \alpha_2 = 0.15865$.

In particle physics it has been customary to choose a “non-informative” or “objective” prior $\pi(s)$, i.e., a prior that claims absence of any knowledge about the true signal $s$. The three most popular candidates for an objective prior in a measurement without background are a flat pdf $\left[1\right] \left[1\right]$, Jeffreys’ prior $1/s$ $\left[1\right] \left[2\right] \left[1\right] \left[2\right]$ and the $1/\sqrt{s}$ prior $\left[3\right]$. The last two were derived from first principles, and the motivation for a flat prior is merely that it is “obvious”. If non-zero background is expected, one can apply the same first principles and obtain $1/(s+b)$ and $1/\sqrt{s+b}$ for the two derived prior pdf’s, respectively. One can also argue $\left[4\right]$ that the $1/\sqrt{s}$ prior should be used even if non-zero background is expected because our prior knowledge of background should not affect our prior ignorance about signal. The latter argument can be applied to the $1/s$ prior as well, but this prior gives a divergent posterior pdf for non-zero background $b > 0$ and is therefore unacceptable. The discussion below is confined to these four prior pdf’s: flat, $1/(s+b)$, $1/\sqrt{s}$ and $1/\sqrt{s+b}$.

For a prior distribution

$$\pi(s) = 1/s^m ; \quad 0 \leq m < 1 ;$$  

for
the posterior pdf is given by
\[
\pi(s|n) = \frac{e^{-s} s^{-m}(s+b)^n/n!}{\sum_{k=0}^{n} b^k \Gamma(n-k-m+1)/(k!(n-k)!)} ,
\] (9)
and the confidence interval can be computed using
\[
\begin{align*}
\alpha_1 &= 1 - \frac{\sum_{k=0}^{n} b^k \Gamma(n-k-m+1,s_1)}{\sum_{k=0}^{n} b^k \Gamma(n-k-m+1)} ; \\
\alpha_2 &= \frac{\sum_{k=0}^{n} b^k \Gamma(n-k-m+1,s_2)}{\sum_{k=0}^{n} b^k \Gamma(n-k-m+1)} ;
\end{align*}
\] (10)
where
\[
\Gamma(p, \mu) = \int_{\mu}^{\infty} x^{p-1} e^{-x} dx ; \quad p > 0 ; \quad \mu \geq 0 ;
\] (11)
is an incomplete gamma-function, and \(\Gamma(p) = \Gamma(p, 0)\) is the standard gamma-function.

For a prior distribution
\[
\pi(s) = 1/(s+b)^m ; \quad 0 \leq m \leq 1 ;
\] (12)
a similar derivation leads to the posterior pdf
\[
\pi(s|n) = \frac{e^{-(s+b)}(s+b)^{n-m}}{\Gamma(n-m+1,b)} .
\] (13)
The confidence interval is then given by
\[
\begin{align*}
\alpha_1 &= 1 - \frac{\Gamma(n-m+1,s_1+b)}{\Gamma(n-m+1,b)} ; \\
\alpha_2 &= \frac{\Gamma(n-m+1,s_2+b)}{\Gamma(n-m+1,b)} .
\end{align*}
\] (14)
At \(m = 0\) (flat prior) or \(b = 0\) (no background) the posterior distributions (9) and (13) are, of course, identical. The posterior pdf (13) is divergent at \(m = 1\) (Jeffreys' prior) and \(n = 0\) (no events observed). I assume that this is equivalent to producing an interval of zero length and set \(I(0, s)\) in Eqn. (2) to zero.

### III. FREQUENTIST INTERVALS

For comparison, expected coverage and length of confidence intervals constructed with the unified approach [4] of Feldman and Cousins and a simple classical method are also plotted. The simple classical approach, to which I will refer below as “standard”, assumes a confidence interval of the form \([0, s_2]\) for upper limit calculation and that of the form \([s_1, +\infty)\) for a lower limit. Rules for construction of frequentist confidence belts are outlined in a famous paper [15] by Neyman. The confidence belt \([n_1(s), n_2(s)]\) for the standard classical method is given by
\[
\begin{align*}
n_1 &= \text{smallest integer for which } \sum_{k=0}^{n_1} f(k|s) > \alpha_2 ; \\
n_2 &= \text{largest integer for which } \sum_{k=n_2}^{\infty} f(k|s) > \alpha_1 .
\end{align*}
\] (15)
Here, the same convention as in the previous section is used for \(\alpha_1\) and \(\alpha_2\), e.g., to compute a 90% upper limit, one should put \(\alpha_1 = 0\) and \(\alpha_2 = 0.1\) etc. This belt is then used to construct a confidence interval \([s_1(n), s_2(n)]\) for a specific number of events \(n\) in accordance with Ref. [13].
IV. DISCUSSION

Expected coverage and interval length for the four objective priors and two frequentist methods are plotted versus true signal in the range $0 \leq s \leq 10$ with a step 0.001 for 4 different values of expected background $b = 0, 1, 3,$ and 6. The plots are shown in Figs. 1-12. The expected length was computed by summing terms in Eqn. (3) from 0 to 100 which was enough to achieve a good accuracy for the chosen values of $s$ and $b$.

Both frequentist methods produce intervals of at least minimal coverage because this conservative requirement was used in construction of their confidence belts. Without background the Bayesian method with a flat prior and the standard classical procedure give identical upper limit values, and this is also true for the Bayesian with Jeffreys’ prior and the standard classical methods in case of lower limits. At any background the Bayesian method with a flat prior provides at least minimal coverage for upper limit intervals and undercovers for lower limits, and this is reversed for Jeffreys’ prior. None of the Bayesian procedures provides minimal coverage for all values of the true signal and for all types of confidence intervals. In terms of coverage, the $\frac{1}{\sqrt{s + b}}$ prior is the most versatile choice among the Bayesian methods. It provides a reasonable mean coverage for all types of confidence intervals while the $\frac{1}{\sqrt{s}}$ prior tends to undercover for upper limits and overcover for lower limits. For central intervals the expected coverage of all Bayesian methods oscillates about the required confidence level. The Bayesian method with a flat prior always gives longer central and upper limit intervals than that with the $\frac{1}{\sqrt{s + b}}$ prior, and intervals produced with this prior are in turn longer than those obtained with Jeffreys’ $1/(s + b)$ prior. The $\frac{1}{\sqrt{s}}$ prior produces short intervals at small signal values but, as the signal increases, these intervals become longer than those produced by the $\frac{1}{\sqrt{s + b}}$ and $1/(s + b)$ priors. Expected lengths for all Bayesian methods approach each other asymptotically as the signal increases. The unified approach produces short 90% intervals at large signal values, but one should keep in mind that here the expected length of a two-sided unified interval is plotted as opposed to lengths of one-sided intervals for all other methods.

V. ACKNOWLEDGEMENTS

Thanks to Robert Cousins for useful discussion and reading the draft of this paper. Thanks to Jim Berger for useful comments.
REFERENCES

[1] V. Innocente and L. Lista, *Nucl. Instr. and Meth.* A 340 (1994) 396.
[2] O. Helene, *Phys. Rev.* D 60 (1999) 037901.
[3] G. Zech, *Confronting classical and Bayesian confidence limits to examples*, hep-ex/0004011.
[4] G.J. Feldman and R.D. Cousins, *Phys. Rev.* D 57 (1998) 3873.
[5] E.B. Wilson, *J. of the Amer. Stat. Assoc.* 22 (1927) 209.
[6] A. Agresti and B.A. Coull, *The Amer. Statistician* 52 (1998) 119.
[7] T. Bayes, *Facsimiles of Two Papers by Bayes* (Hafner Publ. Co., 1963).
[8] P.S. Laplace, *Philosophical Essay on Probabilities* (Dover Publications, 1951).
[9] O. Helene, *Nucl. Instr. and Meth.* 212 (1983) 319.
[10] H. Jeffreys, *Theory of Probability* (3rd ed., Oxford at Clarendon, 1961).
[11] H. Jeffreys, *Proc. of the Royal Soc. of London* A 196 (1946) 453.
[12] E.T. Jaynes, *IEEE Trans. on Systems Sci. and Cyber.* SSC-4, 3 (1968) 227.
[13] G.E.P. Box and G.C. Tiao, *Bayesian Inference in Statistical Analysis* (Wiley Classics, 1992).
[14] J. Linnemann, Talk at the Workshop on Confidence Limits, Fermilab, March 2000; [http://conferences.fnal.gov/cl2k/](http://conferences.fnal.gov/cl2k/)
[15] J. Neyman, *Philos. Trans. of the Royal Soc. of London* A 236 (1937) 333.
FIG. 1. Expected coverage of 90% upper limit (solid) and lower limit (dashed) intervals for the expected background $b = 0$. For the unified approach the expected coverage of 90% intervals is plotted.
FIG. 2. Expected coverage of 68.27% central intervals for the expected background $b = 0$. 
FIG. 3. Expected coverage of 90% upper limit (solid) and lower limit (dashed) intervals for the expected background $b = 1$. For the unified approach the expected coverage of 90% intervals is plotted.
FIG. 4. Expected coverage of 68.27% central intervals for the expected background $b = 1$. 
FIG. 5. Expected coverage of 90% upper limit (solid) and lower limit (dashed) intervals for the expected background $b = 3$. For the unified approach the expected coverage of 90% intervals is plotted.
FIG. 6. Expected coverage of 68.27% central intervals for the expected background $b = 3$. 
FIG. 7. Expected coverage of 90% upper limit (solid) and lower limit (dashed) intervals for the expected background $b = 6$. For the unified approach the expected coverage of 90% intervals is plotted.
FIG. 8. Expected coverage of 68.27% central intervals for the expected background $b = 6$. 
FIG. 9. Expected length of 90% upper limit (upper bunch of curves) and lower limit (lower bunch of curves) intervals on the left and expected length of 68.27% central intervals on the right. The expected background is $b = 0$. 
FIG. 10. Expected length of 90% upper limit (upper bunch of curves) and lower limit (lower bunch of curves) intervals on the left and expected length of 68.27% central intervals on the right. The expected background is $b = 1$. 
FIG. 11. Expected length of 90% upper limit (upper bunch of curves) and lower limit (lower bunch of curves) intervals on the left and expected length of 68.27% central intervals on the right. The expected background is $b = 3$. 
FIG. 12. Expected length of 90% upper limit (upper bunch of curves) and lower limit (lower bunch of curves) intervals on the left and expected length of 68.27% central intervals on the right. The expected background is $b = 6$. 