LINEAR STRUCTURE ON CALABI-YAU MODULI SPACES

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ABSTRACT

We show that the formal moduli space of a Calabi-Yau manifold $X^n$ carries a linear structure, as predicted by mirror symmetry. This linear structure is canonically associated to a splitting of the Hodge filtration on $H^n(X)$.

We begin by establishing terminology. In this paper a Calabi-Yau $n$-manifold means a compact complex manifold $X$ such that

(i) $X$ has trivial canonical bundle $K_X = \Omega^n_X = \mathcal{O}_X$ ;

(ii) $X$ carries no holomorphic vector fields, i.e. $H^0(\Theta_X) = 0$ ;

(iii) $X$ is Kählerian, i.e. admits some Kähler metric.

(Note that (ii) is equivalent to the universal cover of $X$ having no flat factors – cf. [P]).

Recently such manifolds, long of interest to mathematicians, have received a great deal of attention on the part of physicists, because of their role in conformal field theories: see [Y] for a collection of papers in this vein. One consequence of said attention has been the emergence of a number of what might be called ‘physical facts’ about Calabi-Yau manifolds: these are mathematical assertions which physicists regard as established facts while at least some mathematicians would regard them as likely, and often extremely intriguing, but not rigorously proven in the usual mathematical sense.

*Supported in part by NSF under DMS-9202050
One particularly interesting such physical fact is the assertion, common in the physics literature (cf. [FL] and references therein), that the moduli space of (complex structures on) Calabi-Yau manifolds, at least in dimension $n = 3$, should admit a canonical linear structure or flat coordinates: indeed this assertion would seem to be a consequence of the conjectured Mirror Symmetry, because the moduli of complex structures on $X$ would correspond to the moduli of complexified Kähler structures on a mirror $Y$ of $X$ if $Y$ exists, and this latter moduli is just an open subset in a complex vector space.

On the other hand, consideration of the case $n = 2$, where $X$ is a $K3$ surface, and the usual description of the moduli of $X$ in terms of periods, makes it seem rather unlikely that such a linear structure can exist depending holomorphically only on the ‘bare’ manifold $X$.

The purpose of this paper is to prove that a canonical linear structure on the formal moduli on $X$ can indeed be constructed, in agreement, apparently, with the predictions of Mirror Symmetry. To give a precise statement we need some more terminology.

Let $X$ be a Calabi-Yau $n$-manifold. We identify $H^n(X, \mathbb{C}) = H^n_{DR}(X)$ as the hypercohomology of the holomorphic DeRham complex

$$
\Omega_X : \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X.
$$

This gives rise to the Hodge filtration $F^*H^n_{DR}(X)$, which corresponds to the stupid filtration $F^*(\Omega_X)$. By a splitting on $X$ we shall mean a collection of complex vector subspaces \(\{H^{p,q} \subset H^n_{DR}(X) : p + q = n\}\) which split the Hodge filtration. We do not impose the condition that $H^{p,q} = \overline{H^{q,p}}$. Thus the set of all splitting on $X$ forms a Zariski open subset of a suitable product of complex Grassmannians. It contains a canonical element given by $\{H^{p,q} = F^p \cap \overline{F^q}\}$, which however does not vary holomorphically with $X$. [In this connection, the following problem arises naturally and is apparently unsolved.

Problem O. Is there a natural map $K_C(X) \to \{splittings\}$ on the complexified Kähler cone, which itself varies holomorphically with $X$?

Mirror Symmetry would suggest that the answer should be affirmative: Because $\oplus H^{p,q}(X) = \oplus H^{p,q}(Y)$ for a mirror $Y$ and the latter depends holomorphically on the complex moduli of $Y$, i.e. the Kähler moduli of $X$.

Now let $M$ be the component of $[X]$ in the moduli space. By a theorem of Viehweg, $[V]$, $M$ is a quasi-projective variety. On the other hand let $M$ be the canonical universal
formal deformation of $X$, as constructed in [R2] (so the formal neighborhood of $[X]$ in $\mathcal{M}$ is just $\mathcal{M}/\text{Aut}(X)$). By the theorem of Bogomolov-Tian-Todorov, $\mathcal{M}$ is smooth (this will be reproved below). Put

$$ T = T_{[X]}\mathcal{M} = H^1(X, \Theta_X), \Theta_X = \text{holomorphic tangent sheaf.} $$

**Theorem 1.** Given a split Calabi-Yau manifold $(X, \{H^{p,q}\})$, we have a canonical isomorphism

$$ \mathcal{M} \simeq \text{Spf}(\mathbb{C}[[T^*]]) $$

**Remarks.** 1. If Problem O above were solved affirmatively, the proof below should yield similarly a linear structure on the formal moduli $\tilde{\mathcal{M}}$ of the pair $(X$, complexified complex structure $J)$.

2. The linear structure we obtain is a formal one and we have not addressed questions of convergence. Even if this were resolved and a local linear structure on $\tilde{\mathcal{M}}$, say, obtained, this would seem to depend on the particular point $(X, J)$.

We begin the proof of Theorem 1 by recalling and amplifying some notions and constructions from [R2] concerning higher-order deformations and, in particular, relating them to DeRham cohomology and the Gauss-Manin connection.

For a topological space $X$, we denote by $X\langle m \rangle$ its $m$-fold very symmetric product, defined as the space of all nonempty subsets of $X$ of cardinality $\leq m$, with the topology induced from the Cartesian product $X^m$ or the symmetric product $X_m$, i.e. such that we have a diagram of topological quotients.

$$ \pi_m \downarrow \quad \xymatrix{ X^m \ar[r]_{\eta_m} \ar[d] & X_m \ar[r] & X \langle m \rangle. } $$

Given a sheaf $\mathcal{L}$ of complex Lie algebras on $X$, we construct as in [R1, R2] the associated Jacobi complex $J_m(\mathcal{L})$ on $X\langle m \rangle$. This construction generalizes naturally to $\mathcal{L}$-modules, as well as to complexes of such. Explicitly, let $(\mathcal{E}', d')$ be a complex of $\mathcal{L}$-modules. We define the Jacobi bicomplex $J^\cdot_\cdot = J_m(\mathcal{L}, \mathcal{E}')$ on $X\langle m \rangle \times X$ as follows. First, note the natural embedding

$$ X = \Delta(X) \hookrightarrow X\langle 1 \rangle \times X \subseteq X\langle m \rangle \times X. $$

Set

$$ J^a_{-b} = J_{-b}^\cdot(\mathcal{L}) \otimes \mathcal{E}_a^b, \quad b > 0. $$
where

\[ J_m^{-b}(\mathcal{L}) = (q_b)_* \left((\mathcal{L} \boxtimes \cdots \boxtimes \mathcal{L})^-\right), \]

\[ J^{a,-b} = \mathcal{E}^a|_X, \quad b = 0. \]

Making this into a bicomplex, the horizontal differentials are given by

\[ d^{a,-b} = (-1)^b \text{id} \otimes d^a \]

while the vertical ones are defined by the usual formula from Lie algebra homology

\[ \partial^{a,-b}(t_1 \times \cdots \times t_b \times e) = \frac{(-1)^a}{b!} \text{Res} \left( \sum_{\sigma \in S_b} \text{sgn}(\sigma) \left( t_{\sigma(1)} \times \cdots \times [t_{\sigma(b-1)} t_{\sigma(b)}] \times e - t_{\sigma(1)} \times \cdots \times t_{\sigma(b-1)} \times (t_{\sigma(b)} e) \right) \right) \]

Note that if \( \mathcal{E}^- = \mathbb{C} \) with trivial \( \mathcal{L} \)-action, then

\[ J_m(\mathcal{L}, \mathcal{E}^-) = J_m(\mathcal{L}) \boxtimes \mathbb{C} \oplus \mathbb{C}_X. \]

(Incidentally, in this paper we will identify a bicomplex with the associated simple complex and in particular permit a map of bicomplexes to only preserve total degree.)

The case we will be interested in here is where \( X \) is a compact complex manifold, \( \mathcal{L} \) is (essentially) the Lie algebra \( \Theta_X \) of holomorphic vector fields, and \( \mathcal{E}^- = \Omega^-_X \) is the holomorphic DeRham complex, on which \( \Theta_X \) acts by Lie derivative. The resulting bicomplex \( J_m(\Theta_X, \Omega^-_X) \), which might be called the Jacobi-DeRham bicomplex of \( X \), looks like

\[
\begin{array}{cccccccc}
\mathcal{O}_X & \rightarrow & \Omega^1_X & \rightarrow & \cdots & \cdots & \Omega^n_X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Theta_X \boxtimes \mathcal{O}_X & \rightarrow & & \rightarrow & & \rightarrow & \Theta_X \boxtimes \Omega^n_X \\
\end{array}
\]

\( (\Theta_X \boxtimes \cdots \boxtimes \Theta_X)^- \boxtimes \mathcal{O}_X \rightarrow \cdots \cdots \rightarrow (\Theta_X \boxtimes \cdots \boxtimes \Theta_X)^- \boxtimes \Omega^n_X \)

By the Poincaré lemma, we have a quasi-isomorphism

\[ J_m(\Theta_X, \Omega^-_X) \sim J_m(\Theta_X) \boxtimes \mathbb{C} \oplus \mathbb{C}_X \quad (1) \]

In essence, this quasi-isomorphism is nothing but the Gauss-Manin connection, as we proceed to explain.
Define the $m$-th prolongation of the DeRham cohomology group $H^r(X) = \mathbb{H}^r(\Omega^X)$ by the formula (in which $\mathbb{H}^\cdot$ denotes Kunneth components):

$$D^m H^r(X) = \mathbb{H}^{0,r}(X\langle m + 1 \rangle, J^m_m(\Theta_X, \Omega_X)).$$

This terminology is justified by the following result, which can be proven by the method of [R2] (proof omitted).

**Theorem 2.** Let $X_m/R_m$ be the canonical $m$-th order deformation of $X$ as in [R2]. Then we have a canonical isomorphism

$$D^m H^r(X) \simeq Diff^m_{R_m}(H^r_{DR}(X_m/R_m)^*, \mathbb{C}).$$

Note that these groups form a tower

$$H^r(X) = D^0 H^r(X) \subseteq D^1 H^r(X) \subseteq \cdots \subseteq D^m H^r(X)$$

and we have exact sequences

$$0 \to D^{m-1} H^r(X) \to D^m H^r(X) \to S_m T \otimes H^r(X)$$

By (1), we have

$$D^m H^r(X) \simeq T^{(m)} R_m \otimes H^r(X) \Theta H^r(X) = Diff^m(R_m, \mathbb{C}) \otimes H^r(X),$$

where $T^{(m)} R_m = Diff^m_{+}(R_m, \mathbb{C}) = \mathbb{H}^0(X\langle m \rangle, J^m_m(\Theta_X))$, reflecting the Gauss-Manin connection on $H^r_{DR}(X_m/R_m)$. This isomorphism can also be seen, even on the homotopy level, directly from the bicomplex $J^m_m(\Theta_X, \Omega_X)$, using the Cartan formula the Lie derivative of differential forms:

$$L_t = i_t \circ d + d \circ i_t$$

where $t \in \Theta_X$ and $i_t$ denotes interior multiplication by $t$. Indeed we may define a homotopy splitting $\gamma$ of the natural projection

$$J^m_m(\Theta_X, \Omega_X) \to J^m_m(\Theta_X, \Omega_X)/G^0 J^m_m(\Theta_X, \Omega_X),$$

where $G^\cdot$ denotes the vertically stupid filtration on a bicomplex and the RHS is viewed as all the terms of vertical degree $< 0$ in $J^m_m(\Theta_X, \Omega_X)$, by the formula

$$\gamma_a^{a+b} = \text{id} \quad b \geq 2$$

$$\gamma_a^{a-1} = \text{id} \otimes i$$
where \( j : \Theta_X \boxtimes \Omega^n_X \to \Omega^{n-1}_X \) is restriction followed by interior multiplication.

The cohomology map associated to \( \gamma \) yields the Gauss-Manin splitting of the inclusion

\[
H^r(X) \subseteq D^m H^r(X).
\]

A similar construction yields a homotopy equivalence

\[
J_m(\Theta_X, \Omega_X) / G^0 J_m(\Theta_X, \Omega_X) \sim J_m(\Theta_X) \boxtimes \Omega_X
\]

so that \( D^m H^r(X) \simeq H^r(X) \oplus (T^{(m)} R_m) \otimes H^r(X) \).

Now let \( X \) be a Calabi-Yau manifold, and fix some holomorphic volume form \( \Phi \). Let \( \widehat{\Theta}_X \subset \Theta_X \) denote the subsheaf of divergence-free vector fields, i.e. those annihilating \( \Phi \) via Lie derivative. As \( \Phi \) is unique up to a constant \( \widehat{\Theta} \) is independent of the choice of \( \Phi \). As in [R2], there is a canonical formal moduli \( \widehat{M} \) for the pair \((X, \Phi)\) and we have

\[
T^{(m)} \widehat{M} = \ker^0 J_m(\widehat{\Theta}).
\]

Next we replace the DeRham complex \( \Omega^*_X \) by its quasi-isomorphic subcomplex \( \Omega^*_{X,0} \) defined by

\[
\Omega^i_{X,0} = \Omega^0_X, \quad i \leq n - 2
\]

\[
= \widehat{\Omega}^{n-1}_X, \quad i = n - 1 \quad \text{(i.e. the closed \((n-1)\)-forms)}
\]

\[
= 0, \quad i = n.
\]

This forms a complex of \( \widehat{\Theta} \)-modules and as above one may form the Jacobi bicomplex \( J_m(\widehat{\Theta}_X, \Omega^*_{X,0}) \) and establish a canonical isomorphism

\[
\mathbb{H}^r(X \langle m + 1 \rangle, J^*_m(\widehat{\Theta}_X, \Omega^*_{X,0}) = \widehat{D}^m H^r(X) = \text{Diff}^m_{(X, \Omega^*_{X,0})} (H^r_{DR}(X, \Omega^*_{X,0}), \mathbb{C})
\]

where \( \widehat{X}_m = X_m x_{R_m} \) is the canonical \( m \)-th order deformation of \((X, \Phi)\). Note that by Hodge theory (resp. smoothness of \( \widehat{M} \)), both hypercohomology spectral sequences for the bicomplex \( J^*_m(\widehat{\Theta}_X, \Omega^*_{X,0}) \) degenerate at \( E_1 \).

Now note that interior multiplication by \( \Phi \) yields an isomorphism

\[
\widehat{\Theta} \to \widehat{\Omega}^{n-1}_X.
\]
and by the Cartan formula we have
\[ i_{[t_1, t_2]} \Phi = d(i_{t_1} \wedge t_2 \Phi) \quad t_1, t_2 \in \hat{\Theta}. \]

This implies that the entire complex \( J_{m+1}(\hat{\Theta})[-n+1, -1] \), viewed vertically and pulled back to \( X(m) \times X \), is isomorphic to a direct summand, viz. the ‘fully alternating’ part of the subcomplex
\[ F^{n-1} J_m(\hat{\Theta}_X, \Omega^0_{X,0}), \quad F^* = \text{horizontally stupid filtration}. \]

Now to complete the proof of Theorem 1 it will suffice, by the results of [R2], to construct canonical—in terms of the given data \((X, \{H^{p,q}\})\)—isomorphisms
\[ T^{(m)} M \cong \bigoplus_{i=1}^m S'_i T. \]

As
\[ \hat{T} = H^1(\hat{\Theta}_X) = \Phi^{-1} F^{n-1} H^n(X) = \Phi^{-1} (H^{n,0} \oplus H^{n-1,1}) = \Phi^{-1} H^{n,0} \oplus T, \]

it will suffice to construct suitable isomorphisms
\[ (**)_m \quad \hat{T}^{(m)} := T^{(m)} \hat{M} \cong \bigoplus_{i=1}^m S'_i \hat{T}. \]

We will construct these by induction on \( m \), together with isomorphisms
\[ (**)_m \quad \hat{D}^{m-2} H^n(X) \cong \bigoplus_{i=0}^{m-2} \bigoplus_{p=0}^m S'_i \hat{T} \otimes H^{p,q}. \]

For \( m = 1 \) there is nothing to prove. For \( m = 2 \), \((**)_0\) is already given so it suffices to construct \((*)_2\). To this end, define a map of bicomplexes
\[ \varphi_2 : J_2(\hat{\Theta})[-n+1, -1] \to J'_0(\hat{\Theta}, \Omega^0_{X,0}) = \Omega^0_{X,0} \]

by
\[ \varphi_2^{-1} : J_2^{-1}(\hat{\Theta}) \to \Omega^{n-1}_{X,0} = \hat{\Omega}^{n-1}_X \]
\[ \varphi_2^{-1}(t) = i_t(\Phi), \]
\[ \varphi_2^{-2} : J_2^{-2}(\hat{\Theta}) \to \Omega^{n-2}_{X,0} = \Omega^{n-2}_X \]
\[ \varphi_2^{-2}(t_1 \wedge t_2) = i_{t_1} \wedge i_{t_2}(\Phi). \]
On cohomology, this yields a diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \hat{T} & \rightarrow & \hat{T}^{(2)} & \rightarrow & S_2\hat{T} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & F^{n-1}H^n(X) & \rightarrow & H^n(X) & \rightarrow & H^n(X)/F^{n-1}H^n(X) & \rightarrow & 0.
\end{array}
\]

As we are given a splitting of the bottom row and the left vertical arrow is an isomorphism, we get a splitting of the top row. Note that for \(n = 3\) the right vertical arrow is the so-called Yukawa coupling.

Now inductively, assuming \((\ast)_m\) and \((\ast\ast)_{m-2}\) are done, we firstly obtain \((\ast\ast)_{m-1}\) (and even \((\ast\ast)_m\)) from the Gauss-Manin isomorphism

\[
\hat{D}^iH^n(X) \simeq H^n(X) \otimes \hat{T}^i \otimes H^n(X).
\]

To obtain \((\ast)_{m+1}\), define a morphism of bicomplexes

\[
\varphi_{m+1} : J_{m+1}(\hat{\Theta})[-n+1, -1] \rightarrow J_{m-1}(\hat{\Theta}, \Omega_{X,0})
\]

by

\[
\varphi_{m+1}^j : J_{m+1}^j(\hat{\Theta}) \rightarrow J_{m-1}^{n-1-j+1}(\hat{\Theta}, \Omega_{X,0}) \quad i \leq m
\]

\[
\varphi_{m+1}^{-j} (t_1 \times \cdots \times t_j) = \sum_{k=1}^{j} (-1)^k t_1 \times \cdots \times \hat{t}_k \times \cdots \times t_j \times i_{t_j}(\Phi),
\]

\[
\varphi_{m+1}^{-m-1} : J_{m+1}^{-m-1}(\hat{\Theta}) \rightarrow J_{m-1}^{n-2-m+1}(\hat{\Theta}, \Omega_{X,0})
\]

\[
\varphi_{m+1}^{-m-1} (t_1 \times \cdots \times t_{m+1}) = \sum_{i,j} (-1)^i j t_1 \times \cdots \times \hat{t}_j \times \cdots \times \hat{t}_j \times \cdots \times t_{m+1} \times i_{t_i \wedge t_j}(\Phi).
\]

Taking cohomology, we get a diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \hat{T}^{(m)} & \rightarrow & \hat{T}^{m+1} & \rightarrow & S_{m+1}\hat{T} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \hat{D}^{m-1}F^{n-1}H^n(X) & \rightarrow & \hat{D}^{m-1}H^n(X) & \rightarrow & \hat{D}^{m-1}(H^n(X)/F^{n-1}H^n(X)) & \rightarrow & 0.
\end{array}
\]

By induction, we have a splitting of the bottom row and, what’s more, of the left vertical arrow. This yields a splitting of the top row, completing the induction step.

**Remarks.** 1. A similar construction can be used to give a simple a priori proof of the smoothness of \(\hat{M}\) or \(M\), i.e., the degeneration at \(E_1\) of the hypercohomology spectral
sequence of $J_m(\Theta_X)$: the point is that the coboundary maps factor through the exterior derivative, which induces the zero map on cohomology. (That the degeneration is equivalent to unobstructedness of the corresponding moduli problem is a general fact, implicitly proven in [R2].)

2. For $X$ symplectic, i.e. an $Sp(n)$-manifold, it is interesting to compare the above construction with that of ‘formal twistors’ in [R1]: the latter lifts an isotropic vector $\alpha \in H^1(X, \Theta)$ to a formal deformation, and, unlike the former, is manifestly (canonical and) holomorphic in $(X, \alpha)$. However, at least in the K3 case, the two do, in fact agree, which suggests that they should agree in general (in the (symplectic, isotropic) case). We hope to return to this point elsewhere.

Note: After this work was done, we were informed by Professor M. Green that he and Professor P. Griffiths had also obtained results on this problem. We are grateful to Professor Green for his subsequent incisive comments on the manuscript.
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