The vertex-rainbow index of a graph

Yaping Mao†

Department of Mathematics, Qinghai Normal University, Xining, Qinghai 810008, China

Abstract

The $k$-rainbow index $r_x^k(G)$ of a connected graph $G$ was introduced by Chartrand, Okamoto and Zhang in 2010. As a natural counterpart of the $k$-rainbow index, we introduced the concept of $k$-vertex-rainbow index $r_{vx}^k(G)$ in this paper. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an $S$-Steiner tree or a Steiner tree connecting $S$ (or simply, an $S$-tree) is a such subgraph $T = (V', E')$ of $G$ that is a tree with $S \subseteq V'$. For $S \subseteq V(G)$ and $|S| \geq 2$, an $S$-Steiner tree $T$ is said to be a vertex-rainbow $S$-tree if the vertices of $V(T) \setminus S$ have distinct colors. For a fixed integer $k$ with $2 \leq k \leq n$, the vertex-coloring $c$ of $G$ is called a $k$-vertex-rainbow coloring if for every $k$-subset $S$ of $V(G)$ there exists a vertex-rainbow $S$-tree. In this case, $G$ is called vertex-rainbow $k$-tree-connected. The minimum number of colors that are needed in a $k$-vertex-rainbow coloring of $G$ is called the $k$-vertex-rainbow index of $G$, denoted by $r_{vx}^k(G)$. When $k = 2$, $r_{vx}^2(G)$ is nothing new but the vertex-rainbow connection number $r(vc)(G)$ of $G$. In this paper, sharp upper and lower bounds of $sr_{vx}^k(G)$ are given for a connected graph $G$ of order $n$, that is, $0 \leq sr_{vx}^k(G) \leq n - 2$. We obtain the Nordhaus-Guddum results for 3-vertex-rainbow index, and show that $r_{vx}^3(G) + r_{vx}^3(G) = 4$ for $n = 4$ and $2 \leq r_{vx}^3(G) + r_{vx}^3(G) \leq n - 1$ for $n \geq 5$. Let $t(n, k, \ell)$ denote the minimal size of a connected graph $G$ of order $n$ with $r_{vx}^k(G) \leq \ell$, where $2 \leq \ell \leq n - 2$ and $2 \leq k \leq n$. The upper and lower bounds for $t(n, k, \ell)$ are also obtained.

Keywords: vertex-coloring; connectivity; vertex-rainbow $S$-tree; vertex-rainbow index; Nordhaus-Guddum type.

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†E-mail: maoyaping@ymail.com
1 Introduction

The rainbow connections of a graph which are applied to measure the safety of a network are introduced by Chartrand, Johns, McKeon and Zhang [6]. Readers can see [6, 7, 9] for details. Consider an edge-coloring (not necessarily proper) of a graph $G = (V, E)$. We say that a path of $G$ is rainbow, if no two edges on the path have the same color. An edge-colored graph $G$ is rainbow connected if every two vertices are connected by a rainbow path. The minimum number of colors required to rainbow color a graph $G$ is called the rainbow connection number, denoted by $rc(G)$. In [15], Krivelevich and Yuster proposed a similar concept, the concept of vertex-rainbow connection. A vertex-colored graph $G$ is vertex-rainbow connected if every two vertices are connected by a path whose internal vertices have distinct colors, and such a path is called a vertex-rainbow path. The vertex-rainbow connection number of a connected graph $G$, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make $G$ vertex-rainbow connected.

For more results on the rainbow connection and vertex-rainbow connection, we refer to the survey paper [21] of Li, Shi and Sun and a new book [22] of Li and Sun. All graphs considered in this paper are finite, undirected and simple. We follow the notation and terminology of Bondy and Murty [2], unless otherwise stated.

For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an $S$-Steiner tree or a Steiner tree connecting $S$ (or simply, an $S$-tree) is a such subgraph $T = (V', E')$ of $G$ that is a tree with $S \subseteq V'$. A tree $T$ in $G$ is a rainbow tree if no two edges of $T$ are colored the same. For $S \subseteq V(G)$, a rainbow $S$-Steiner tree (or simply, rainbow $S$-tree) is a rainbow tree connecting $S$. For a fixed integer $k$ with $2 \leq k \leq n$, the edge-coloring $c$ of $G$ is called a $k$-rainbow coloring if for every $k$-subset $S$ of $V(G)$ there exists a rainbow $S$-tree. In this case, $G$ is called rainbow $k$-tree-connected. The minimum number of colors that are needed in a $k$-rainbow coloring of $G$ is called the $k$-rainbow index of $G$, denoted by $rx_k(G)$. When $k = 2$, $rx_2(G)$ is the rainbow connection number $rc(G)$ of $G$. For more details on $k$-rainbow index, we refer to [3, 8, 12, 18, 19].

Chartrand, Okamoto and Zhang [9] obtained the following result.

**Theorem 1** [8] For every integer $n \geq 6$, $rx_3(K_n) = 3$.

As a natural counterpart of the $k$-rainbow index, we introduce the concept of $k$-vertex-rainbow index $rvx_k(G)$ in this paper. For $S \subseteq V(G)$ and $|S| \geq 2$, an $S$-Steiner tree $T$ is said to be a vertex-rainbow $S$-tree or vertex-rainbow tree connecting $S$ if the vertices of $V(T) \setminus S$ have distinct colors. For a fixed integer $k$ with $2 \leq k \leq n$, the vertex-coloring $c$ of $G$ is called a $k$-vertex-rainbow coloring if for every $k$-subset $S$ of $V(G)$ there exists a vertex-rainbow $S$-tree. In this case, $G$ is called vertex-rainbow $k$-tree-connected. The minimum number of colors that are needed in a $k$-vertex-rainbow coloring of $G$ is called the $k$-vertex-rainbow index of $G$, denoted by $rvx_k(G)$. When $k = 2$, $rvx_2(G)$ is nothing new but the vertex-rainbow connection number $rvc(G)$ of $G$. It follows, for every nontrivial
connected graph $G$ of order $n$, that

$$rvx_2(G) \leq rvx_3(G) \leq \cdots \leq rvx_n(G).$$

Let $G$ be the graph of Figure 1 (a). We give a vertex-coloring $c$ of the graph $G$ shown in Figure 1 (b). If $S = \{v_1, v_2, v_3\}$ (see Figure 1 (c)), then the tree $T$ induced by the edges in $\{v_1u_1, v_2u_1, u_1u_4, u_4v_3\}$ is a vertex-rainbow $S$-tree. If $S = \{u_1, u_2, v_3\}$, then the tree $T$ induced by the edges in $\{u_1u_2, u_2u_4, u_4v_3\}$ is a vertex-rainbow $S$-tree. One can easily check that there is a vertex-rainbow $S$-tree for any $S \subseteq V(G)$ and $|S| = 3$. Therefore, the vertex-coloring $c$ of $G$ is a 3-vertex-rainbow coloring. Thus $G$ is vertex-rainbow 3-tree-connected.

In some cases $rvx_k(G)$ may be much smaller than $rx_k(G)$. For example, $rvx_k(K_{1,n-1}) = 1$ while $rx_k(K_{1,n-1}) = n - 1$ where $2 \leq k \leq n$. On the other hand, in some other cases, $rx_k(G)$ may be much smaller than $rvx_k(G)$. For $k = 3$, we take $n$ vertex-disjoint cliques of order 4 and, by designating a vertex from each of them, add a complete graph on the designated vertices. This graph $G$ has $n$ cut-vertices and hence $rvx_3(G) \geq n$. In fact, $rvx_3(G) = n$ by coloring only the cut-vertices with distinct colors. On the other hand, from Theorem [1] it is not difficult to see that $rx_3(G) \leq 9$. Just color the edges of the $K_n$ with, say, color 1, 2, 3 and color the edges of each clique with the colors 4, 5, $\cdots$, 9.

Steiner tree is used in computer communication networks (see [14]) and optical wireless communication networks (see [13]). As a natural combinatorial concept, the rainbow index and the vertex-rainbow index can also find applications in networking. Suppose we want to route messages in a cellular network in such a way that each link on the route between more than two vertices is assigned with a distinct channel. The minimum number of channels that we have to use is exactly the rainbow index and vertex-rainbow index of the underlying graph.

The Steiner distance of a graph, introduced by Chartrand, Oellermann, Tian and Zou [8] in 1989, is a natural generalization of the concept of classical graph distance. Let $G$ be a connected graph of order at least 2 and let $S$ be a nonempty set of vertices of $G$. Then the Steiner distance $d(S)$ among the vertices of $S$ (or simply the distance of $S$) is the minimum size among all connected subgraphs whose vertex sets contain $S$. Let $n$ and $k$ be two integers with $2 \leq k \leq n$. The Steiner $k$-eccentricity $e_k(v)$ of a vertex $v$ of $G$ is
defined by $e_k(v) = \max \{d(S) : S \subseteq V(G), |S| = k, \text{ and } v \in S\}$. The Steiner $k$-diameter of $G$ is $sdiam_k(G) = \max \{e_k(v) : v \in V(G)\}$. Clearly, $sdiam_k(G) \geq k - 1$.

Then, it is easy to see the following results.

**Proposition 1** Let $G$ be a nontrivial connected graph of order $n$. Then $rvx_k(G) = 0$ if and only if $sdiam_k(G) = k - 1$.

**Proposition 2** Let $G$ be a nontrivial connected graph of order $n$ ($n \geq 5$), and let $k$ be an integer with $2 \leq k \leq n$. Then

$$0 \leq rvx_k(G) \leq n - 2.$$ 

**Proof.** We only need to show $rvx_k(G) \leq n - 2$. Since $G$ is connected, there exists a spanning tree of $G$, say $T$. We give the internal vertices of the tree $T$ different colors. Since $T$ has at most two leaves, we must use at most $n - 2$ colors to color all the internal vertices of the tree $T$. Color the leaves of the tree $T$ with the used colors arbitrarily. Note that such a vertex-coloring makes $T$ vertex-rainbow $k$-tree-connected. Then $rvx_k(T) \leq n - 2$ and hence $rvx_k(G) \leq rvx_k(T) \leq n - 2$, as desired. \hfill \qed

**Observation 1** Let $K_{s,t}$, $K_{n_1,n_2,\ldots,n_k}$, $W_n$ and $P_n$ denote the complete bipartite graph, complete multipartite graph, wheel and path, respectively. Then

1. For integers $s$ and $t$ with $s \geq 2, t \geq 1$, $rvc(K_{s,t}) = 1$.
2. For $k \geq 3$, $rvx_k(K_{n_1,n_2,\ldots,n_k}) = 1$.
3. For $n \geq 4$, $rvx_k(W_n) = 1$.
4. For $n \geq 3$, $rvx_k(P_n) = n - 2$.

Let $\mathcal{G}(n)$ denote the class of simple graphs of order $n$ and $\mathcal{G}(n,m)$ the subclass of $\mathcal{G}(n)$ having graphs with $n$ vertices and $m$ edges. Give a graph parameter $f(G)$ and a positive integer $n$, the Nordhaus-Gaddum ($\text{NG}$) Problem is to determine sharp bounds for: (1) $f(G) + f(\overline{G})$ and (2) $f(G) \cdot f(\overline{G})$, as $G$ ranges over the class $\mathcal{G}(n)$, and characterize the extremal graphs. The Nordhaus-Gaddum type relations have received wide attention; see a recent survey paper [1] by Aouchiche and Hansen.

Chen, Li and Lian [10] gave sharp lower and upper bounds of $rx_k(G) + rx_k(\overline{G})$ for $k = 2$. In [11], Chen, Li and Liu obtained sharp lower and upper bounds of $rvx_k(G) + rvx_k(\overline{G})$ for $k = 2$. In Section 2, we investigate the case $k = 3$ and give lower and upper bounds of $rvx_3(G) + rvx_3(\overline{G})$.

**Theorem 2** Let $G$ and $\overline{G}$ be a nontrivial connected graph of order $n$. If $n = 4$, then $rvx_3(G) + rvx_3(\overline{G}) = 4$. If $n \geq 5$, then we have

$$2 \leq rvx_3(G) + rvx_3(\overline{G}) \leq n - 1.$$ 

Moreover, the bounds are sharp.
Let \( s(n, k, \ell) \) denote the minimal size of a connected graph \( G \) of order \( n \) with \( rx_k(G) \leq \ell \), where \( 2 \leq \ell \leq n-1 \) and \( 2 \leq k \leq n \). Schiermeyer [24] focused on the case \( k = 2 \) and gave exact values and upper bounds for \( s(n, 2, \ell) \). Later, Li, Li, Sun and Zhao [17] improved Schiermeyer’s lower bound of \( s(n, 2, 2) \) and got a lower bound of \( s(n, 2, \ell) \) for \( 3 \leq \ell \leq \lceil \frac{n}{2} \rceil \).

In Section 3, we study the vertex case. Let \( t(n, k, \ell) \) denote the minimal size of a connected graph \( G \) of order \( n \) with \( rvx_k(G) \leq \ell \), where \( 2 \leq \ell \leq n-2 \) and \( 2 \leq k \leq n \). We obtain the following result in Section 3.

**Theorem 3** Let \( k, n, \ell \) be three integers with \( 2 \leq \ell \leq n - 3 \) and \( 2 \leq k \leq n \). If \( k \) and \( \ell \) have the different parity, then

\[
 n - 1 \leq t(n, k, \ell) \leq n - 1 + \frac{n - \ell - 1}{2}.
\]

If \( k \) and \( \ell \) have the same parity, then

\[
 n - 1 \leq t(n, k, \ell) \leq n - 1 + \frac{n - \ell}{2}.
\]

2 Nordhaus-Guddum results

To begin with, we have the following result.

**Proposition 3** Let \( G \) be a connected graph of order \( n \). Then the following are equivalent.

1. \( rvx_3(G) = 0 \);
2. \( sdiam_3(G) = 2 \);
3. \( n - 2 \leq \delta(G) \leq n - 1 \).

**Proof.** For Proposition 3, \( rvx_3(G) = 0 \) if and only if \( sdiam_3(G) = 2 \). So we only need to show the equivalence of (1) and (3). Suppose \( n - 2 \leq \delta(G) \leq n - 1 \). Clearly, \( G \) is a graph obtained from the complete graph of order \( n \) by deleting some independent edges. For any \( S = \{u, v, w\} \subseteq V(G) \), at least two elements in \( \{uv, vw, uw\} \) belong to \( E(G) \). Without loss of generality, let \( uv, vw \in E(G) \). Then the tree \( T \) induced by the edges in \( \{uv, vw\} \) is an \( S \)-Steiner tree and hence \( d_G(S) \leq 2 \). From the arbitrariness of \( S \), we have \( sdiam_3(G) \leq 2 \) and hence \( sdiam_3(G) = 2 \). Therefore, \( rvx_3(G) = 0 \).

Conversely, we assume \( rvx_3(G) = 0 \). If \( \delta(G) \leq n - 3 \), then there exists a vertex \( u \in V(G) \) such that \( d_G(u) \leq n - 3 \). Furthermore, there are two vertices, say \( v, w \), such that \( uv, uw \notin E(G) \). Choose \( S = \{u, v, w\} \). Clearly, any rainbow \( S \)-tree must occupy at least a vertex in \( V(G) \setminus S \), which implies that \( rvx_3(G) \geq 1 \), a contradiction. So \( n - 2 \leq \delta(G) \leq n - 1 \).

After the above preparation, we can derive a lower bound of \( rvx_3(G) + rvx_3(G) \).
Lemma 1 Let \( G \) and \( \overline{G} \) be a nontrivial connected graph of order \( n \). For \( n \geq 5 \), we have \( \rvx_3(G) + \rvx_3(\overline{G}) \geq 2 \). Moreover, the bound is sharp.

Proof. From Proposition 2, we have \( \rvx_3(G) \geq 0 \) and \( \rvx_3(\overline{G}) \geq 0 \). If \( \rvx_3(G) = 0 \), then we have \( n - 2 \leq \delta(G) \leq n - 1 \) by Proposition 3 and hence \( \overline{G} \) is disconnected, a contradiction. Similarly, we can get another contradiction for \( \rvx_3(\overline{G}) = 0 \). Therefore, \( \rvx_3(G) \geq 1 \) and \( \rvx_3(\overline{G}) \geq 1 \). So \( \rvx_3(G) + \rvx_3(\overline{G}) \geq 2 \).

To show the sharpness of the above lower bound, we consider the following example.

Example 1: Let \( H \) be a graph of order \( n-4 \), and let \( P = a, b, c, d \) be a path. Let \( G \) be the graph obtained from \( H \) and the path by adding edges between the vertex \( a \) and all vertices of \( H \) and adding edges between the vertex \( d \) and all vertices of \( H \); see Figure 2 (a). We now show that \( \rvx_3(G) = \rvx_3(\overline{G}) = 1 \). Choose \( S = \{a, b, d\} \). Then any \( S \)-Steiner tree must occupy at least one vertex in \( V(G) \setminus S \). Note that the vertices of \( V(G) \setminus S \) in the tree must receive different colors. Therefore, \( \rvx_3(G) \geq 1 \). We give each vertex in \( G \) with one color and need to show that \( \rvx_3(G) \leq 1 \). It suffices to prove that there exists a vertex-rainbow \( S \)-tree for any \( S \subseteq V(G) \) with \( |S| = 3 \). Suppose \( |S \cap V(H)| = 3 \). Without loss of generality, let \( S = \{x, y, z\} \). Then the tree \( T \) induced by the edges in \( \{xa, ya, za\} \) is a vertex-rainbow \( S \)-tree. Suppose \( |S \cap V(H)| = 2 \). Without loss of generality, let \( x, y \in S \cap V(H) \). If \( a \in S \), then the tree \( T \) induced by the edges in \( \{xa, ya\} \) is a vertex-rainbow \( S \)-tree. If \( b \in S \), then the tree \( T \) induced by the edges in \( \{xa, ya, ab\} \) is a vertex-rainbow \( S \)-tree. Suppose \( |S \cap V(H)| = 1 \). Without loss of generality, let \( x \in S \cap V(H) \). If \( a, b \in S \), then the tree \( T \) induced by the edges in \( \{xa, ab\} \) is a vertex-rainbow \( S \)-tree. If \( b, c \in S \), then the tree \( T \) induced by the edges in \( \{xd, cd, bc\} \) is a vertex-rainbow \( S \)-tree. If \( a, c \in S \), then the tree \( T \) induced by the edges in \( \{xa, ab, bc\} \) is a vertex-rainbow \( S \)-tree. Suppose \( |S \cap V(G')| = 0 \). If \( a, b, c \in S \), then the tree \( T \) induced by the edges in \( \{ab, bc\} \) is a vertex-rainbow \( S \)-tree. If \( a, b, d \in S \), then the tree \( T \) induced by the edges in \( \{ab, cd\} \) is a vertex-rainbow \( S \)-tree. From the arbitrariness of \( S \), we conclude that \( \rvx_3(G) \leq 1 \). Similarly, one can also check that \( \rvx_3(\overline{G}) = 1 \). So \( \rvx_3(G) + \rvx_3(\overline{G}) = 2 \).

We are now in a position to give an upper bound of \( \rvx_3(G) + \rvx_3(\overline{G}) \). For \( n = 4 \), we have \( G = \overline{G} = P_4 \) since we only consider connected graphs. Observe that \( \rvx_3(G) = \).
Observation 2 Let $G, \overline{G}$ be connected graphs of order $n$ ($n = 4$). Then $rvx_3(G) + rvx_3(\overline{G}) = n$.

For $n \geq 5$, we have the following upper bound of $rvx_3(G) + rvx_3(\overline{G})$.

Lemma 2 Let $G, \overline{G}$ be connected graphs of order $n$ ($n = 5$). Then $rvx_3(G) + rvx_3(\overline{G}) \leq n - 1$.

Proof. If $G$ is a path of order 5, then $rvx_3(G) = 3$ by Observation 1. Observe that $sdiam_3(\overline{G}) = 3$. Then $rvx_3(\overline{G}) \leq 1$ and hence $rvx_3(G) + rvx_3(\overline{G}) \leq 4$, as desired.

![Figure 3: Graphs for Lemma 2](image)

If $G$ is a tree but not a path, then we have $G = H_1$ since $\overline{G}$ is connected (see Figure 3 (a)). Clearly, $rvx_3(G) \leq 2$. Furthermore, $\overline{G}$ consists of a $K_2$ and a $K_3$ and two edges between them (see Figure 3 (a)). So we assign color 1 to the vertices of $K_2$ and color 2 to the vertices of $K_3$, and this vertex-coloring makes the graph $G$ vertex-rainbow 3-tree-connected, that is, $rvx_3(\overline{G}) \leq 2$. Therefore, $rvx_3(G) + rvx_3(\overline{G}) \leq 4$, as desired.

Suppose that both $G$ and $\overline{G}$ are not trees. Then $e(G) \geq 5$ and $e(\overline{G}) \geq 5$. Since $e(G) + e(\overline{G}) = e(K_5) = 10$, it follows that $e(G) = e(\overline{G}) = 5$. If $G$ contains a cycle of length 5, then $G = \overline{G} = C_5$ and hence $rvx_3(G) = rvx_3(\overline{G}) = 2$. If $G$ contains a cycle of length 4, then $G = H_2$ (see Figure 3 (b)). Clearly, $rvx_3(G) = rvx_3(\overline{G}) = 2$. If $G$ contains a cycle of length 3, then $G = \overline{G} = H_3$ (see Figure 3 (c)). One can check that $rvx_3(G) = rvx_3(\overline{G}) = 2$. Therefore, $rvx_3(G) + rvx_3(\overline{G}) = 4$, as desired.

Lemma 3 Let $G$ be a nontrivial connected graph of order $n$, and $rvx_3(G) = \ell$. Let $G'$ be a graph obtained from $G$ by adding a new vertex $v$ to $G$ and making $v$ be adjacent to $q$ vertices of $G$. If $q \geq n - \ell$, then $rvx_3(G') \leq \ell$. 


Proof. Let \( c : V(G) \to \{1, 2, \cdots, \ell\} \) be a vertex-coloring of \( G \) such that \( G \) is vertex-rainbow 3-tree-connected. Let \( X = \{x_1, x_2, \cdots, x_q\} \) be the vertex set such that \( vx_i \in E(G') \). Set \( V(G) \setminus X = \{y_1, y_2, \cdots, y_{n-q}\} \). We can assume that there exist two vertices \( y_j, y_k \) such that there is no vertex-rainbow tree connecting \( \{v, y_j, y_k\} \); otherwise, the result holds obviously.

We define a minimal \( S \)-Steiner tree \( T \) as a tree connecting \( S \) whose subtree obtained by deleting any edge of \( T \) does not connect \( S \). Because \( G \) is vertex-rainbow 3-tree-connected, there is a minimal vertex-rainbow tree \( T_i \) connecting \( \{x_i, y_j, y_k\} \) for each \( x_i \ (i \in \{1, 2, \cdots, q\}) \). Then the tree \( T_i \) has four types; see Figure 4. For the type shown in (c), the Steiner tree \( T_i \) connecting \( \{x_i, y_j, y_k\} \) is a path induced by the edges in \( E(P_1) \cup E(P_2) \) and hence the internal vertices of the path \( T_i \) must receive different colors. Therefore, the tree induced by the edges in \( E(P_1) \cup E(P_2) \cup \{vx_i\} \) is a vertex-rainbow tree connecting \( \{v, y_j, y_k\} \), a contradiction. So we only need to consider the other three cases shown in Figure 4 (a), (b), (d). Obviously, \( T_i \cap T_j \) may not be empty. Then we have the following claim.

Claim 1: No other vertex in \( \{x_1, x_2, \cdots, x_q\} \) different from \( x_i \) belong to \( T_i \) for each \( 1 \leq i \leq q \).

Proof of Claim 1: Assume, to the contrary, that there exists a vertex \( x'_i \in \{x_1, x_2, \cdots, x_q\} \) such that \( x'_i \neq x_i \) and \( x'_i \in V(T_i) \). For the type shown in Figure 4 (a), the Steiner tree \( T_i \) connecting \( \{x_i, y_j, y_k\} \) is a path induced by the edges in \( E(P_1) \cup E(P_2) \) and hence the internal vertices of the path \( T_i \) receive different colors. If \( x'_i \in V(P_1) \), then the tree induced by the edges in \( E(P_1') \cup E(P_2) \cup \{vx_i\} \) is a vertex-rainbow tree connecting \( \{v, y_j, y_k\} \) where \( P_1' \) is the path between the vertex \( x'_i \) and the vertex \( y_j \) in \( P_1 \), a contradiction. If \( x'_i \in V(P_2) \), then the tree induced by the edges in \( E(P_2) \cup \{vx_i\} \) is a vertex-rainbow tree connecting \( \{v, y_j, y_k\} \), a contradiction. The same is true for the type shown in Figure 4 (b). For the type shown in Figure 4 (c), the Steiner tree \( T_i \) connecting \( \{x_i, y_j, y_k\} \) is a tree induced by the edges in \( E(P_1) \cup E(P_2) \cup E(P_3) \) and hence the internal vertices of the tree \( T_i \) receive different colors. Without loss of generality, let \( x'_i \in V(P_1) \). Then the tree induced by the edges in \( E(P_1') \cup E(P_2) \cup E(P_3) \) is a vertex-rainbow tree connecting \( \{v, y_j, y_k\} \) where \( P_1' \) is the path between the vertex \( x'_i \) and the vertex \( v \) in \( P_1 \), a contradiction. \( \blacksquare \)
From Claim 1, since there is no vertex-rainbow tree connecting \( \{ v, y_{j1}, y_{j2} \} \), it follows that there exists a vertex \( y_k \) such that \( c(x_l) = c(y_k) \) for each tree \( T_i \), which implies that the colors that are assigned to \( X \) are among the colors that are assigned to \( V(G) \setminus X \). So \( rvx_3(G) = \ell \leq n - q \). Combining this with the hypothesis \( q \geq n - \ell \), we have \( rvx_3(G) = n - q \), that is, all vertices in \( V(G) \setminus X \) have distinct colors. Now we construct a new graph \( G' \), which is induced by the edges in \( E(T_1) \cup E(T_2) \cup \cdots \cup E(T_q) \).

**Claim 2:** For every \( y_i \) not in \( G' \), there exists a vertex \( y_s \in G' \) such that \( y_i y_s \in E(G) \).

**Proof of Claim 2:** Assume, to the contrary, that \( N(y_i) \subseteq \{ x_1, x_2, \ldots, x_q \} \). Since \( G \) is vertex-rainbow 3-tree-connected, there is a vertex-rainbow tree \( T \) connecting \( \{ y_i, y_{j1}, y_{j2} \} \). Let \( x_r \) be the vertex in the tree \( T \) such that \( x_r \in N_G(y_i) \). Then tree induced by the edges in \( (E(T) \setminus \{ y_i x_r \}) \cup \{ vx_r \} \) is a vertex-rainbow tree connecting \( \{ v, y_{j1}, y_{j2} \} \), a contradiction.

From Claim 2, \( G[y_1, y_2, \ldots, y_{n-q}] \) is connected. Clearly, \( G[y_1, y_2, \ldots, y_{n-q}] \) has a spanning tree \( T \). Because the tree \( T \) has at least two pendant vertices, there must exist a pendant vertex whose color is different from \( x_1 \), and we assign the color to \( x_1 \). One can easily check that \( G \) is still vertex-rainbow 3-tree-connected, and there is a vertex-rainbow tree connecting \( \{ v, y_{j1}, y_{j2} \} \). If there still exist two vertices \( y_{j3}, y_{j4} \) such that there is no vertex-rainbow tree connecting \( \{ v, y_{j3}, y_{j4} \} \), then we do the same operation until there is a vertex-rainbow tree connecting \( \{ v, y_{j3}, y_{j4} \} \) for each pair \( y_{j3}, y_{j4} \in \{ 1, 2, \ldots, n - q \} \). Thus \( G' \) is vertex-rainbow 3-tree-connected. So \( rvx(G') \leq \ell \).

**Proof of Theorem 2:** We prove this theorem by induction on \( n \). By Lemma 2, the result is evident for \( n = 5 \). We assume that \( rvx_3(G) + rvx_3(\overline{G}) \leq n - 1 \) holds for complementary graphs on \( n \) vertices. Observe that the union of a connected graph \( G \) and its complement \( \overline{G} \) is a complete graph of order \( n \), that is, \( G \cup \overline{G} = K_n \). We add a new vertex \( v \) to \( G \) and add \( q \) edges between \( v \) and \( V(G) \). Denoted by \( G' \) the resulting graph. Clearly, \( \overline{G'} \) is a graph of order \( n + 1 \) obtained from \( \overline{G} \) by adding a new vertex \( v \) to \( \overline{G} \) and adding \( n - q \) edges between \( v \) and \( V(G) \).

**Claim 3:** \( rvx_3(G') \leq rvx_3(G) + 1 \) and \( rvx_3(\overline{G'}) \leq rvx_3(\overline{G}) + 1 \).

**Proof of Claim 3:** Let \( c \) be a \( rvx_3(G) \)-vertex-coloring of \( G \) such that \( G \) is vertex-rainbow 3-tree-connected. Pick up a vertex \( u \in N_G(v) \) and give it a new color. It suffices to show that for any \( S \subseteq V(G') \) with \( |S| = 3 \), there exists a vertex-rainbow \( S \)-tree. If \( S \subseteq V(G) \), then there exists a vertex-rainbow \( S \)-tree since \( G \) is vertex-rainbow 3-tree-connected. Suppose \( S \not\subseteq V(G) \). Then \( v \in S \). Without loss of generality, let \( S = \{ v, x, y \} \). Since \( G \) is vertex-rainbow 3-tree-connected, there exists a vertex-rainbow tree \( T' \) connecting \( \{ u, x, y \} \). Then the tree \( T \) induced by the edges in \( E(T') \cup \{ u, v \} \) is a vertex-rainbow \( S \)-tree. Therefore, \( rvx_3(G') \leq rvx_3(G) + 1 \). Similarly, \( rvx_3(\overline{G'}) \leq rvx_3(\overline{G}) + 1 \).
From Claim 3, we have $rvx_3(G') + rvx_3(G) \leq rvx_3(G) + 1 + rvx_3(G) + 1 \leq n + 1$. Clearly, $rvx_3(G') + rvx_3(G) \leq n$ except possibly when $rvx_3(G') = rvx_3(G) + 1$ and $rvx_3(G) = rvx_3(G) + 1$. In this case, by Lemma [3], we have $q \leq n - rvx_3(G) - 1$ and $n - q \leq n - rvx_3(G) - 1$. Thus, $rvx_3(G) + rvx_3(G) \leq (n - 1 - q) + (q - 1) = n - 2$ and hence $rvx_3(G') + rvx_3(G) \leq n$, as desired. This completes the induction. 

To show the sharpness of the above bound, we consider the following example.

**Example 2:** Let $G$ be a path of order $n$. Then $rvx_3(G) = n - 2$. Observe that $sdiam_3(G) = 3$. Then $rvx_3(G) = 1$, and so we have $rvx_3(G) + rvx_3(G) = (n - 2) + 1 = n - 1$.

### 3 The minimal size of graphs with given vertex-rainbow index

Recall that $t(n, k, \ell)$ is the minimal size of a connected graph $G$ of order $n$ with $rvx_k(G) \leq \ell$, where $2 \leq \ell \leq n - 2$ and $2 \leq k \leq n$. Let $G$ be a path of order $n$. Then $rvx_k(G) \leq n - 2$ and hence $t(n, k, n - 2) \leq n - 1$. Since we only consider connected graphs, it follows that $t(n, k, n - 2) \geq n - 1$. Therefore, the following result is immediate.

**Observation 3** Let $k$ be an integer with $2 \leq k \leq n$. Then

$$t(n, k, n - 2) = n - 1.$$ 

A rose graph $R_p$ with $p$ petals (or $p$-rose graph) is a graph obtained by taking $p$ cycles with just a vertex in common. The common vertex is called the center of $R_p$. If the length of each cycle is exactly $q$, then this rose graph with $p$ petals is called a $(p, q)$-rose graph, denoted by $R_{p,q}$. Then we have the following result.

**Proof of Theorem 3:** Suppose that $k$ and $\ell$ has the different parity. Then $n - \ell - 1$ is even. Let $G$ be a graph obtained from a $(\frac{n-\ell-1}{2}, 3)$-rose graph $R_{\frac{n-\ell-1}{2}, 3}$ and a path $P_{\ell+1}$ by identifying the center of the rose graph and one endpoint of the path. Let $w_0$ be the center of $R_{\frac{n-\ell-1}{2}, 3}$, and let $C_i = w_0v_iu_iw_0$ ($1 \leq i \leq \frac{n-\ell-1}{2}$) be the cycle of $R_{\frac{n-\ell-1}{2}, 3}$. Let $P_{\ell+1} = w_0w_1 \cdots w_\ell$ be the path of order $\ell + 1$. To show the $rvx_k(G) \leq \ell$, we define a vertex-coloring $c : V(G) \to \{0, 1, 2, \cdots, \ell - 1\}$ of $G$ by

$$c(v) = \begin{cases} 
  i, & \text{if } v = w_i \ (0 \leq i \leq \ell - 1); \\
  1, & \text{if } v = u_i \text{ or } v = v_i \ (1 \leq i \leq \frac{n-\ell-1}{2}); \\
  1, & \text{if } v = w_\ell.
\end{cases}$$

One can easily see that there exists a vertex-rainbow $S$-tree for any $S \subseteq V(G)$ and $|S| = 3$. Therefore, $rvx_k(G) \leq \ell$ and $t(n, k, \ell) \leq n - 1 + \frac{n-\ell-1}{2}$.

Suppose that $k$ and $\ell$ has the same parity. Then $n - \ell$ is even. Let $G$ be a graph obtained from a $(\frac{n-\ell}{2}, 3)$-rose graph $R_{\frac{n-\ell}{2}, 3}$ and a path $P_\ell$ by identifying the center of the
rose graph and one endpoint of the path. Let \( w_0 \) be the center of \( R_{\frac{n-\ell}{2}, 3} \), and let \( C_i = w_0v_iu_iw_0 \) (1 \( \leq i \leq \frac{n-\ell}{2} \)) be the cycle of \( R_{\frac{n-\ell}{2}, 3} \). Let \( P_\ell = w_0w_1 \cdots w_{\ell-1} \) be the path of order \( \ell \). To show the \( rvx_k(G) \leq \ell \), we define a vertex-coloring \( c : V(G) \rightarrow \{0, 1, 2, \cdots, \ell - 1\} \) of \( G \) by
\[
c(v) = \begin{cases} 
  i , & \text{if } v = w_i \ (0 \leq i \leq \ell - 1); \\
  1 , & \text{if } v = u_i \text{ or } v = v_i \ (1 \leq i \leq \frac{n-\ell}{2})
\end{cases}
\]
One can easily see that there exists a vertex-rainbow \( S \)-tree for any \( S \subseteq V(G) \) and \( |S| = 3 \). Therefore, \( rvx_k(G) \leq \ell \) and \( t(n, k, \ell) \leq n - 1 + \frac{n-\ell}{2} \).

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