THIRD HANKEL DETERMINANT FOR A CLASS OF STARLIKE FUNCTIONS ASSOCIATED WITH EXPONENTIAL FUNCTION

KUNAL JOSHI AND S. SIVAPRASAD KUMAR

ABSTRACT. Recently, the subclass of starlike functions associated with exponential function $e^z$, given by $S^*_e = \{ f(z) \in S : zf'(z)/f(z) \prec e^z, (z \in D) \}$ was introduced and studied by Mendiratta et al. In this article, we obtain sharp bound for third Hankel determinant for the class $S^*_e$, by improving the already known results, in this direction. In our methodology, we use the newly obtained expression for the fourth coefficient of Carathéodory functions. Univalent functions; Starlike functions; Hankel determinant; Sharp bound; Exponential function.

1. Introduction

In simple words univalent means one to one or injective mapping. Whereas, an univalent function in context of complex analysis refers to a regular (i.e, holomorphic) or a meromorphic function with atmost a simple pole, which is a one-one mapping. We can consider them on various domains of definition, like on a Reimann surface, but we confine our attention to the class $S$. $S$ is a class of holomorphic and normalised univalent functions defined on a unit disk $\mathbb{D} := \{ z : |z| < 1 \}$ which consists of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

Let $\mathcal{P}$ denote the class of analytic functions $p$, with positive real part when defined on the domain $\mathbb{D}$, normalised by

$$p(z) = 1 + c_1 z^2 + c_2 z^2 + c_3 z^2 \ldots$$

This is generally known as the Carathéodory class.

We can observe that if $p(z) \in \mathcal{P}$, then a Schwarz function $\omega(z)$ exists such that $\omega(0) = 0$, $|\omega(z)| \leq 1$ and we have the equality

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)}, (z \in \mathbb{D}).$$

We say that a function $f(z)$ is subordinate to a function $g(z)$ if there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| \leq 1$ such that

$$f(z) = g(w(z)), (z \in \mathbb{D}).$$

where $f(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$ and we write

$$f(z) \prec g(z), (z \in \mathbb{D}).$$
Among various subclasses of $S$, the most important are $S^*$, class of starlike functions and $C$, class of convex functions. Ma and Minda [15] considered the most general form of starlike and convex functions

$$S^*(\varphi(z)) = \{ f(z) \in S : \frac{zf'(z)}{f(z)} \prec \varphi(z) \}$$

(1.1)

and

$$C(\varphi(z)) = \{ f(z) \in S : 1 + \frac{zf'''(z)}{f''(z)} \prec \varphi(z) \},$$

(1.2)

where $\varphi(z)$ is analytic in $\mathbb{D}$, belongs to Carathéodory class $P$, $\varphi(\mathbb{D})$ is starlike with respect to $\varphi(0) = 1$ and is geometrically symmetric with respect to the real axis with $\varphi'(0) > 0$.

If we take $\varphi(z) = (1 + z)/(1 - z)$ then (1.1) and (1.2) reduce to the class Starlike and Convex functions respectively.

We define $q-th$ Hankel determinant for an analytic function, where $q, n \in \mathbb{N}$ as follows:

$$H_q(n) = \begin{bmatrix} a_n & \cdots & a_{n+q-1} \\ \vdots & \ddots & \vdots \\ a_{n+q-1} & \cdots & a_{n+2q-2} \end{bmatrix},$$

If we take $n = 1$ and $q = 3$, then $H_3(n)$ becomes the third Hankel determinant, given by

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

(1.3)

For functions in $S$ with bounded boundary, Noor [17] studied the rate of growth of $H_q(n)$ as $n \to \infty$.

It is known that the estimation of $H_3(1)$ or the third Hankel determinant is the hardest and most tedious work. Initially Babalola [2] published the first article on $H_3(1)$ in which he estimated the same for the subclasses of $S$, which are $S^*$, $C$ and $R$. Later following the trend, authors began to publish similar articles for various subclasses of holomorphic and univalent functions, refer [1, 4, 5, 10, 20], this in turn opened a new scope for research in this field. Interesting to note here is how various authors discovered some interesting classes for the said property of Hankel determinant.

Currently study of bi-univalent function has picked up pace, in this field Srivastava et al. [26] explored various results related to Hankel determinant for the class of bi-univalent functions defined by q-derivative and discovered some interesting properties. Furthermore, continuing his work with other co-authors in [24] he investigated the class of close to convex functions to calculate the Hankel determinant associated with the lemniscate of Bernoulli. Continuing the trend, he in [25] explored and did some research on Toeplitz forms and Hankel determinant for q-starlike functions associated with bounded domains. Some interesting domains on which the Hankel determinant for starlike class was estimated by including k-Fibonacci numbers, Shafiq et al. [21] estimated the bounds of the third Hankel determinant for the same. To read more on this work, one can refer [22, 6, 7, 3, 11].
According to the trend on how researchers were computing the bounds, we need to first obtain the sharp bounds of the initial coefficients \(a_2, a_3, a_4 \text{ and } a_5\), Feketo Szegő functional, second Hankel determinant \(H_2(2)\) and the functional \(|a_4 - a_2a_3|\) which lets say is \(L\). Then at last we can use triangular inequality in the equation (1.3) to finally get the upper bound of the third hankel determinant as follows

\[
|H_3(1)| \leq |a_3||H_2(2)| + |a_4||L| + |a_5||H_2(1)|,
\]

The important point is that this computation does not yield the sharp bound of \(|H_3(1)|\). However, if we wish to compute the sharp bounds of \(|H_3(1)|\) and \(|H_2(3)|\) the calculations would become very tedious. But for the class of starlike functions, Raza and Malik \[19\] found out the sharp bounds of \(|H_3(1)|\) and \(|H_2(2)|\) and also the upper bounds of \(|H_3(1)|\). That leaves the estimate of the sharp bound of \(|H_3(1)|\) for the class of starlike functions an open problem to solve. Recently the formula for the coefficient \(c_4\) was obtained in \[12\], which has opened up a new path to solve this problem of third Hankel determinant and the interesting part is that this path yields the sharp bounds for most of the cases although the general case is yet to be solved. In this attempt, Kwon and Sim \[13\] tried to improve the estimate of the third Hankel determinant for the class of starlike functions. Furthermore, the sharp bound of \(|H_3(1)|\) for the class

\[
T(\alpha) = \{ f \in A : \text{Re}(f(z)/z) > \alpha, \alpha \in [0,1) \}
\]

was estimated by Kowalczyk et al. \[8\]. Further, with other authors, Kowalczyk et al. \[9\] estimated the sharp bound for \(|H_3(1)|\) for the class of convex functions. Later Zaprawa \[27\] also estimated the bounds (sharp) of \(|H_2(3)|\) for the family of typically real functions.

In this research paper we will calculate \(|H_3(1)|\) for the subclass of starlike functions associated with the exponential function, i.e, \(\varphi(z) = e^z\). The class is given as follows,

\[
S^*_e = \{ f(z) \in S : \frac{zf''(z)}{f'(z)} < e^z, (z \in D) \}
\]

This class was introduced by Mendiratta et al. \[16\]. Initially, Zhang et al. \[29\] estimated the bound of \(H_3(1)\) as 0.565, then in an attempt to improve the result Shi et al. \[23\] and Zaprawa \[28\] estimated the bounds (sharp) of \(H_3(1)\) as 0.5004 and 0.385 respectively. Although they were improving the result but still couldn’t achieve the sharp bound.

In this paper we will establish the sharp result for the same.

2. Sharp estimation of \(|H_3(1)|\) for \(S^*_e\)

We first need the formulae for \(c_2, c_3\) \[14\] and \(c_4\) \[12\] which is given by the following lemma:

**Lemma 2.1.** Let \(p \in \mathcal{P}\) and of the form \(1+\sum_{n=1}^{\infty} c_n z^n\). Then

\[
2c_2 = c_1^2 + \delta(4 - c_1^2),
\] (2.1)
4c_3 = c_1^2 + 2c_1(4 - c_1^2)\delta - c_1(4 - c_1^2)\delta^2 + 2(4 - c_1^2)(1 - |\delta|^2)\alpha \quad (2.2)

and

8c_4 = c_1^4 + (4 - c_1^2)\delta(c_1^2(\delta^2-3\delta+3)+4\delta)-4(4-c_1^2)(1-|\delta|^2)(c_1(\delta-1)\alpha+\delta^2\alpha^2-(1-|\alpha|^2)\beta), \quad (2.3)

where |\alpha| \leq 1, |\beta| \leq 1 and |\delta| \leq 1.

Theorem 2.1. If \( f \in S^*_\nu \). Then we have

\[ |H_3(1)| \leq \frac{1}{9}. \quad (2.4) \]

The bound is sharp.

Proof.

Let \( f \in S^*_\nu \), given by \( f(z) = z + \sum_{n=2}^{\infty} b_n z^n \), then from [18], we have

\[ b_2 = \frac{c_1}{2}, \quad b_3 = \frac{1}{16}(c_1^2 + 4c_2), \quad b_4 = \frac{1}{48} \left( -\frac{c_1^3}{6} + 2c_1c_2 + 8c_3 \right), \quad (2.5) \]

\[ b_5 = \frac{c_1^4 - 12c_1^2c_2 + 24c_1c_3 + 44c_4}{1152}, \quad (2.6) \]

From equation (1.3), we get

\[ H_3(1) = 2b_2b_3b_4 - b_3^3 - b_4^2 + b_5b_5 - b_2b_5. \quad (2.7) \]

We know that under rotation, \( \mathcal{P} \) is invariant, therefore the value of \( c_1 \) lies in the interval \([0, 2]\). Now substituting the values of \( b_i \)'s in (2.7), we have

\begin{align*}
H_3(1) &= \frac{1}{331776} \left( -211c_1^6 + 420c_1^4c_2 + 2544c_1^3c_3 + 10944c_1c_2c_3 - 144c_1^2(13c_2^2 + 54c_4) \\
&\quad - 576(9c_2^2 + 16c_3 - 18c_2c_4) \right)
\end{align*}

Let \( c = c_1 \), using the equalities (2.1), (2.2), (2.3) and upon simplification, we finally get

\[ H_{3}(1) = \frac{1}{331776} \left( \phi_1(c, \delta) + \phi_2(c, \delta)\alpha + \phi_3(c, \delta)\alpha^2 + \zeta(c, \delta, \alpha)\beta \right) \]

Where \( \alpha, \beta, \delta \in \overline{D} \),

\[ \phi_1(c, \delta) = -13c^6 + 78c^4t\delta - 1296c^2t^2\delta^2 + 120c^4t^2\delta^2 - 36c^2t^2\delta^2 - 324c^4t^2\delta^3 - 360c^2t^2\delta^3 + 72c^2t^2\delta^4 \]

\[ \phi_2(c, \delta) = tr(408c^3 + 1296c^3\delta + 720ct\delta - 288ct\delta^2) \]

\[ \phi_3(c, \delta) = tr(-2304t - 288t|\delta|^2 + 1296c^2t) \]

and

\[ \zeta(c, \delta, \alpha) = trs(-1296c^2 + 2592t\delta), \]

where \( t = 4 - c^2, r = 1 - |\delta|^2, s = 1 - |\alpha|^2. \)
Now, by taking $x = |\delta|$, $y = |\alpha|$ and using the fact $|\beta| \leq 1$, we get
\[ |H_3(1)| \leq \frac{1}{331776} \left( |\phi_1(c, \delta)| + |\phi_2(c, \delta)|y + |\phi_3(c, \delta)|y^2 + |\zeta(c, \delta, \alpha)| \right) \leq M(c, x, y) \] (2.8)

Next, we find the solution of inequalities (2.9) and (2.10). Let
\[ \mu(x) = 8(8 - x)/(25 - 2x) \]
then we can see that $\mu(x)$ is a decreasing function as $\mu'(x) < 0$ for all $x \in (0, 1)$. Therefore from (2.10), $c^2 = \mu(x)$ achieves its maximum value at 1, i.e., $1.56 > 1$ for all $x \in (0, 1)$. But from (2.9) we can deduce that $54c^2(1 - x) \leq 17c^3$, which does not gives us $c > 1$ for all $x \in (0, 1)$. Therefore no solution, hence no critical point.

(2) Next we consider interior of all the six faces of the cuboid $S$. On the face

\[ \lambda_1(c, y) = M(c, 0, y) = \frac{13c^6 + 408c^3(4 - c^2)y + 2304(4 - c^2)^2y^2 + 1296c^2(4 - c^2)(1 - y^2)}{331776} \] (2.11)

On solving $\frac{\partial \lambda_1(c, y)}{\partial c} = 0$ and $\frac{\partial \lambda_1(c, y)}{\partial y} = 0$, we get no solution for $c \in (0, 2)$ and $y \in (0, 1)$, therefore no critical points.
Working on the same line we get max $|M(c, x, y)|$ as 0.0398426 at $x = 1$, $y = 0$ but no solution at, $y = 1$, $c = 0$ and $c = 2$.

(3) Next we consider the edges of the cuboid $S$. At $x = 0$ and $y = 1$, $M(c, x, y)$ reduces to

$$\lambda_2(c) = M(c, 0, 1) = \frac{13c^6 + 408c^3(4 - c^2) + 2304(4 - c^2)^2}{331776}$$  \hspace{1cm} (2.12)$$

On solving $\frac{\partial \lambda_2(c)}{\partial c} = 0$ we get no solution for $c$ in $(0, 2)$, hence no critical point in this case.

Working on the same line we get max $|M(c, x, y)|$ as 0.0159535 at $x = 0$, $y = 0$; 0.0398426 at $x = 1, y = 0$ and $x = 1, y = 1$; 0.0481125 at $c = 0, y = 0$ and no critical point at $x = 0, y = 1$; $x = 0, c = 0; c = 0, y = 1$.

(4) At last we consider the vertices. We get $|M(c, x, y)|$ as 0 at $x = 0, y = 0, c = 0; x = 1, y = 0, c = 0; x = 1, y = 1, c = 0; 1/9 at x = 0, y = 1, c = 0$ $13/5164 \approx 0.00223598211$ at other points.

From above three cases, we get $|M(c, x, y)| \leq 1/9$, therefore from (2.8)

$$|H_3(1)| \leq \frac{1}{9}.$$  

We also observe for the function

$$f(z) = z(\exp \int_0^z (\phi(z^3) - 1)/(z) \, dz) = z + \frac{z^4}{3} + \frac{5z^7}{36} + \frac{17z^{10}}{324} + \cdots.$$ 

we get $|H_3(1)| = 1/9$, which gives us our extremal function for the sharpness.

3. Conclusion

In the present present, we not only have improved earlier known results on the bound of the third Hankel determinant for the well known class $S^*_e$, but also estimated the sharp bound for the same.

Acknowledgement

Authors thank the Geometric Function Theory Group and research scholar Mr. Kamaljeet Gangania of the Department of Applied Mathematics, DTU for completion of this work.

Conflict of interest

The authors declare that they have no conflict of interest.
REFERENCES

[1] Altinkaya, S and Yalçın, SIBEL, Third Hankel determinant for Bazilevic functions, Advances in Math, 5, 91–96, 2016

[2] Babalola, Kunle Oladeji, On $H_3(1)$ Hankel determinant for some classes of univalent functions, arXiv preprint [arXiv:0910.3779], 2009

[3] Banga, Shagun and Kumar, S Sivaprasad, The sharp bounds of the second and third Hankel determinants for the class, Mathematica Slovaca, 70, 4, 849–862, 2020

[4] Bansal, Deepak and Maharana, Sudhananda and Prajapati, Jugal Kishore, Third order Hankel determinant for certain univalent functions, Journal of the Korean Mathematical Society, 52, 6, 1139–1148, 2015

[5] Cho, Nak Eun and Kowalczyk, B and Kwon, OS and Lecko, A and Sim, YJ, Some coefficient inequalities related to the Hankel determinant for strongly starlike functions of order alpha, J. Math. Inequal, 11, 2, 429–439, 2017

[6] Khan, Muhammad Ghaffar and Ahmad, Bakhtiar and Murugusundaramoorthy, Gangadharan and Chinram, Ronnason and Mashwani, Wali Khan, Applications of modified sigmoid functions to a class of starlike functions, Journal of Function Spaces, 2020, 2020

[7] Khan, Muhammad Ghaffar and Ahmad, Bakhtiar and Sokol, Janusz and Muhammad, Zubair and Mashwani, Wali Khan and Chinram, Ronnason and Petchkaew, Pattarawan, Coefficient problems in a class of functions with bounded turning associated with Sine function, European Journal of Pure and Applied Mathematics, 14, 1, 53–64, 2021

[8] Kowalczyk, Bogumila and Lecko, Adam and Lecko, Milledenia and Sim, Young Jae, The sharp bound of the third Hankel determinant for some classes of analytic functions, Bulletin of the Korean Mathematical Society, 55, 6, 1859–1868, 2018

[9] Kowalczyk, Bogumila and Lecko, Adam and Srivastava, HM, A note on the Fekete-Szegő problem for close-to-convex functions with respect to convex functions, Publications de l’Institut Mathematique, 101, 115, 143–149, 2017

[10] Krishna, D Vamshee and Venkateswarlu, B and RamReddy, T, Third Hankel determinant for bounded turning functions of order alpha, Journal of the Nigerian Mathematical Society, 34, 2, 121–127, 2015

[11] Kumar, S Sivaprasad and Gangania, Kamaljeet, A cardioid domain and starlike functions, Analysis and Mathematical Physics, 11, 2, 2021

[12] Kwon, Oh Sang and Lecko, Adam and Sim, Young Jae, On the fourth coefficient of functions in the Carathéodory class, Computational Methods and Function Theory, 18, 2, 307–314, 2018

[13] Kwon, Oh Sang and Sim, Young Jae, The sharp bound of the Hankel determinant of the third kind for starlike functions with real coefficients, Mathematics, 7, 8, 721, 2019

[14] Libera, Richard J and Złotkiewicz, Eligiusz J, Early coefficients of the inverse of a regular convex function, Proceedings of the American Mathematical Society, 85, 2, 225–230, 1982

[15] Ma, Wancang, A unified treatment of some special classes of univalent functions, Proceedings of the Conference on Complex Analysis, 1992, 1992, International Press Inc.

[16] Mendiratta, Rajni and Nagpal, Sumit and Ravichandran, V, On a subclass of strongly starlike functions associated with exponential function, Bulletin of the Malaysian Mathematical Sciences Society, 38, 1, 365–386, 2015

[17] Noor, K Inayat, Hankel determinant problem for the class of functions with bounded boundary rotation, Revue Roumaine de Mathématiques Pures et Appliquées, 28, 8, 731–739, 1983
[18] Ravichandran, V and Verma, Shelly, Bound for the fifth coefficient of certain starlike functions, Comptes Rendus Mathematique, 353, 6, 505–510, 2015

[19] Raza, Mohsan and Malik, Sarfraz Nawaz, Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli, Journal of Inequalities and Applications, 2013, 1, 1–8, 2013

[20] Shanmugam, G and Stephen, B Adolf and Babalola, Kunle Oladeji, Third Hankel determinant for alpha-starlike functions, Gulf Journal of Mathematics, 2, 2, 2014

[21] Shafiq, Muhammad and Srivastava, Hari M and Khan, Nazar and Ahmad, Qazi Zahoor and Darus, Maslina and Kian, Samiha, An upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with k-Fibonacci numbers, Symmetry, 12, 6, 1043, 2020

[22] Shi, Lei and Ghaffar Khan, Muhammad and Ahmad, Bakhtiar, Some geometric properties of a family of analytic functions involving a generalized q-operator, Symmetry, 12, 2, 291, 2020

[23] Shi, Lei and Srivastava, Hari Mohan and Arif, Muhammad and Hussain, Shehzad and Khan, Hassan, An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function, Symmetry, 11, 5, 598, 2019

[24] Srivastava, Hari M and Ahmad, Qazi Zahoor and Darus, Maslina and Khan, Nazar and Khan, Bilal and Zaman, Naveed and Shah, Hasrat Hussain, Upper bound of the third Hankel determinant for a subclass of close-to-convex functions associated with the lemniscate of Bernoulli, Mathematics, 7, 9, 848, 2019

[25] Srivastava, Hari M and Ahmad, Qazi Zahoor and Khan, Nasir and Khan, Nazar and Khan, Bilal, Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain, Mathematics, 7, 2, 181, 2019

[26] Srivastava, Hari M and Altınkaya, Şahsene and Yalcın, Sibel, Hankel determinant for a subclass of bi-univalent functions defined by using a symmetric q-derivative operator, Filomat, 32, 2, 503–516, 2018

[27] Zaprawa, Paweł, Second Hankel determinants for the class of typically real functions, Abstract and Applied Analysis, 2016, 2016, Hindawi

[28] Zaprawa, Paweł, Hankel determinants for univalent functions related to the exponential function, Zaprawa, Paweł, Symmetry, 11, 10, 1211, 2019

[29] Zhang, Hai-Yan and Tang, Huo and Niu, Xiao-Meng, Third-order Hankel determinant for certain class of analytic functions related with exponential function, Symmetry, 10, 10, 501, year2018

DEPARTMENT OF APPLIED MATHEMATICS, DELHI TECHNOLOGICAL UNIVERSITY, DELHI–110042, INDIA

Email address: 2k18mc059@dtu.ac.in

DEPARTMENT OF APPLIED MATHEMATICS, DELHI TECHNOLOGICAL UNIVERSITY, DELHI–110042, INDIA

Email address: spkumar@dce.ac.in