ψ - VECTORS FOR THREE DIMENSIONAL MODELS.

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Abstract
In this paper we apply the method of ψ - vectors to the three dimensional statistical models. This method gives the correspondence between the Bazhanov – Baxter model and its vertex version. Considering ψ - vectors for the Planar model we obtain its self – duality.

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1. Introduction

Recently the vertex version of the Bazhanov – Baxter model (BBM) [1] has been obtained in Ref. [2]. There we showed the thermodynamic equivalence between the vertex model and Interaction Round a Cube (IRC for the shortness) type BBM. As the vertex model, it contains the Hietarinta’s and Korepanov’s models [3, 4].

There are at least two types of vertex – IRC correspondence. The first one is the Wu - Kadanoff duality, which is valid when it is possible to regard the vertex variables of a weight function as combinations of the IRC spins (see Ref. [3] for details). The second one is a three dimensional modification of the $\psi$ vectors method [5, 6, 10].

In section 2 we fix the notations for the Tetrahedron equations of both vertex and IRC types and give some evident notations for the $\psi$ - vectors. Necessary notations and definitions are given in section 3. Next, in section 4 we give the explicit expressions for the $\psi$ - vectors in BBM and show the explicit equivalence between vertex and IRC types. In section 5 we consider the Planar model [7] and find the $\psi$ - vectors for it. For this model the vertex and the IRF type Boltzmann weights are the same.

Note that this paper does not contain explicit description of our calculations, and for the details of the calculations see the collection of $\omega$-hypergeometrical formulas in the Appendix of Ref. [2].

2. Tetrahedron equation and definition of $\psi$ - vectors.

We consider two forms of the Tetrahedron equation (TE): vertex and IRC. The vertex form is the following:

$$\sum R^{k_1,k_2,k_3}_{i_1,i_2,i_3} R^{j_1,k_4,k_5}_{k_1,i_4,i_5} R^{m_{j_2} j_4 k_6}_{k_2 k_4 k_5} R^{m_{j_3} j_5 j_6}_{k_3 k_5 k_6} =$$

$$= \sum R^{m_{k_3},k_5,k_6}_{i_3,i_5,i_6} R^{m_{k_2} k_4 j_6}_{i_2 k_4 k_5} R^{m_{j_1} j_3}_{i_1 k_5 k_6} R^{j_1 j_2 j_3}_{k_1 k_2 k_3}. \quad (2.1)$$
The IRC type TE is
\[ W'(a_1|d, c_{13}, c_{14}|b_2, a_3|c_{34})W(d|a_2, a_3, c_{34}, c_{24}, c_{23}|b_1) \] (2.2)

This equation differs from the standard equation (2.2) in Ref. [8] by a simple reordering of the spins.

With each \( R \) and \( W \) we associate an oriented trihedron, described by three ordered dihedral angles \( \theta_1, \theta_2, \theta_3 \), or, equivalently, by the corresponding planar angles \( a_1, a_2, a_3 \). These trihedrons will be regarded as the spectral arguments of the corresponding \( R \) and \( W \). With TE we associate an (oriented) tetrahedron. Complete solutions of TE are parameterized by six angles of the tetrahedron (five of them are independent):

\[
\begin{align*}
(R, W) &= (R, W)(\theta_1, \theta_2, \theta_3), \\
(R', W') &= (R, W)(\theta_1, \theta_4, \theta_5), \\
(R'', W'') &= (R, W)(\pi - \theta_2, \theta_4, \theta_6) \\
(R''', W''') &= (R, W)(\theta_3, \pi - \theta_5, \theta_6).
\end{align*}
\] (2.3)

The ordering of the dihedral angles is natural with respect to numbering of the spaces and differs from that in the standard equation (2.2) in Ref. [8].

For each vertex in (2.3) let \( a_i^\# \) be the corresponding planar angles:

\[
\begin{align*}
(\theta_1, \theta_2, \theta_3) &\mapsto (a_1, a_2, a_3), \\
(\theta_1, \theta_4, \theta_5) &\mapsto (a_1', a_2', a_3'), \\
(\pi - \theta_2, \theta_4, \theta_6) &\mapsto (a_1'', a_2'', a_3''), \\
(\theta_3, \pi - \theta_5, \theta_6) &\mapsto (a_1'', a_2'', a_3'''),
\end{align*}
\] (2.4)

and this correspondence will be implied below.

Vectors \( \psi \) and \( \bar{\psi} \) are defined as solutions of the following equations:

\[
\sum_{k_1,k_2,k_3} R_{i_1,i_2,i_3}^{k_1,k_2,k_3} \psi_1(k_1|c, h, c, d)\psi_2(k_2|d, b, h, f)\psi_3(k_3|h, g, c, b) = \\
= \sum_{a} \psi_1(i_1|a, b, g, f)\psi_2(i_2|e, g, c, a)\psi_3(i_3|d, a, e, f)W(a|e, f, g|b, c, d|h),
\] (2.5)
\[
\sum_h W(a|e, f, g|b, c, d|h)\psi_1(j_1|e, h, c, d|f)\psi_2(j_2|d, b, h, f)\psi_3(j_3|h, g, c, b) = \\
= \sum_{k_1, k_2, k_3} \psi_1(k_1|a, b, g, f)\psi_2(k_2|e, g, c, a)\psi_3(k_3|d, a, e, f)R^{j_1, j_2, j_3}_{k_1, k_2, k_3},
\]

(2.6)

where \(R\) and \(W\) must obey TEs. The geometry demands the tetrahedron – like parameterization for \(\psi\) and \(\overline{\psi}\):

\[
(R, W) = (R, W)(\theta_1, \theta_2, \theta_3),
\]

\[
(\psi_1, \overline{\psi}_1) = (\psi, \overline{\psi})(\theta_1, \theta_4, \theta_5),
\]

\[
(\psi_2, \overline{\psi}_2) = (\psi, \overline{\psi})(\pi - \theta_2, \theta_4, \theta_6)
\]

\[
(\psi_3, \overline{\psi}_3) = (\psi, \overline{\psi})(\theta_3, \pi - \theta_5, \theta_6).
\]

(2.7)

TEs and the equations for the psi vectors have two popular limits: the static limit, when in each \((R, W)(\theta_1, \theta_2, \theta_3)\) \(\theta_1 + \theta_2 + \theta_3 = \pi\), and the planar limit, when in each \((R, W)(a_1, a_2, a_3)\) \(a_2 = a_1 + a_3\).

### 3. Notations and definitions

We tried to make the list of definitions and notations shortest. Here we give only definition of \(w\) function. Its properties the Reader can find in Ref. [2]. Let

\[
\omega^{1/2} = \exp(\pi i / N).
\]

(3.1)

Taking \(p\) to be a point on a Fermat curve \(\Upsilon\), so that there defined three complex numbers \(x(p), y(p), z(p)\), constrained by the Fermat equation

\[
x(p)^N + y(p)^N = z(p)^N,
\]

(3.2)

and \(a\) to be an element of \(Z_N\), define

\[
\frac{w(p|a)}{w(p|0)} = \prod_{s=1}^{a} \frac{y(p)}{z(p) - x(p)\omega^s}.
\]

(3.3)
The absolute value of \( w(p|0) \) we define through
\[
\prod_{a=0}^{N-1} w(p|a) = 1. \tag{3.4}
\]

Branches of \( y(p) \) and \( w(p|0) \) are arbitrary in general, but it is convenient to choose them appropriately. Below all points \( p \)-s will be defined so that (when it is possible, we will omit the argument \( p \) for the shortness)
\[
-2\pi/N < \text{Arg}(x/z) < 0 \quad \text{and} \quad -\pi/N < \text{Arg}(y/z) < \pi/N. \tag{3.5}
\]

This subregion in \( \Upsilon \) we call \( \Upsilon_0 \). For \( p \in \Upsilon_0 \) we define \( w(p|0) \) as follows
\[
w(p|0) = \left( \frac{y}{z} \right)^{\frac{N-1}{2}} \frac{1}{d(\omega x/z)} = \left( \frac{x}{y} \right)^{\frac{N-1}{2}} \Phi_0^{-1} d(z/x), \tag{3.6}
\]
where \( \Phi_0 = \exp\left(i\pi(N-1)(N-2)/6N\right) \).
\[
\tag{3.7}
\]

It is implied in (3.8) that
\[
-\pi < \text{Im} \log(z) \leq \pi. \tag{3.9}
\]

Defined \( w \) functions have the following property:
\[
w(p|a)w(Op|-a)\bar{\Phi}(a) = 1, \quad a \in \mathbb{Z}_N, \tag{3.10}
\]
where automorphism \( O : \Upsilon_0 \to \Upsilon_0 \) is defined as
\[
x(Op) = z(p), \quad y(Op) = \omega^{1/2}y(p), \quad z(Op) = \omega x(p), \tag{3.11}
\]
and
\[
\bar{\Phi}(a) = \omega^{a(a-N)/2} \exp\left(i\pi(N^2-1)/6N\right), \quad \prod_{a \in \mathbb{Z}_N} \bar{\Phi}(a) = 1. \tag{3.12}
\]

There are a lot of several identities for the \( w \) functions, closely connected with the basic \( q \) hypergeometric series. Most useful of them are the so called Star – Square relation and \((\tau \rho)^2\) transformation. These identities are cumbersome enough, so we do not write them here, and the Reader can find them in Ref. [2].
4. $\psi$ - vectors for BBM

Recall the definition of the vertex and IRC weights for BBM. The forms of the weights are taken from Refs. [1, 2, 9]. First, for given trihedron define four points $p_i = p_i(a_1, a_2, a_3)$:

$$x_{p_1} = \omega^{-1/2} \exp(i \frac{a_3}{N}) \sqrt{\frac{\sin \beta_1}{\sin \beta_2}}; y_{p_1} = \exp(i \frac{\beta_1}{N}) \sqrt{\frac{\sin a_3}{\sin \beta_2}};$$

$$x_{p_2} = \omega^{-1/2} \exp(i \frac{a_3}{N}) \sqrt{\frac{\sin \beta_2}{\sin \beta_1}}; y_{p_2} = \exp(i \frac{\beta_2}{N}) \sqrt{\frac{\sin a_3}{\sin \beta_1}};$$

$$x_{p_3} = \omega \exp(i \frac{a_3}{N}) \sqrt{\frac{\sin \beta_3}{\sin \beta_0}}; y_{p_3} = \exp(-i \frac{\beta_3}{N}) \sqrt{\frac{\sin a_3}{\sin \beta_0}};$$

$$x_{p_4} = \omega \exp(i \frac{a_3}{N}) \sqrt{\frac{\sin \beta_0}{\sin \beta_3}}; y_{p_4} = \exp(-i \frac{\beta_0}{N}) \sqrt{\frac{\sin a_3}{\sin \beta_3}};$$

$$z_{p_i} = 1, i = 1, 2, 3, 4; (4.1)$$

where the linear excesses are

$$\beta_0 = \pi - \frac{a_1 + a_2 + a_3}{2}, \beta_i = \frac{a_j + a_k - a_i}{2}. \quad (4.2)$$

Let $\rho_k$ be the normalization factors:

$$\rho_k = \frac{1}{N} \left( \frac{\sin a_k}{2 \cos \beta_0 / 2 \ldots \cos \beta_3 / 2} \right)^{N+1}. \quad (4.3)$$

The vertex weight is given by

$$R^{j_1,j_2,j_3}_{i_1,i_2,i_3} = \delta_{j_2+j_3,i_2+i_3} \omega^{j_3(j_1-i_1)} \rho_3 w(p_1|i_1 - i_2) w(p_2|j_1 - j_2) w(p_3|i_1 - j_2) w(p_4|j_1 - i_2). \quad (4.4)$$

Introducing the normalization factor $\rho_3$, we make $R$ symmetrical with respect to cube symmetry group (see Ref. [2]) and restore Bazhanov – Baxter’s normalization of the model (see Ref. [1]).

Define other four points $q_i(a_1, a_2, a_3) = Op_i(a_1, a_3, a_2)$ The IRC weight is given by

$$W(a|e, f, g|b, c, d|h|\theta_1, \theta_2, \theta_3) =$$

$$= \rho_2 \sum_{\sigma} \frac{w(q_1|f - a + \sigma)w(q_3|h - c + \sigma)}{w(q_1|d - e + \sigma)w(q_2|b - g + \sigma)} \omega^\sigma(c + g - a - c). \quad (4.5)$$
This weight differs from the Bazhanov – Baxter weight, the correspondence is

$$W_B(a|e, f, g|b, c, d|h|\theta_1^B, \theta_2^B, \theta_3^B) = W(a|f, g, e|c, d, b|h|\theta_1, \theta_2, \theta_3)$$  \hspace{1cm} (4.6)

where

$$\theta_1 = \theta_2^B, \quad \theta_2 = \theta_3^B, \quad \theta_3 = \theta_1^B.$$  \hspace{1cm} (4.7)

The formulae for $\psi$ and $\overline{\psi}$ are:

$$\psi(\sigma|e, h, c, d) = \frac{w(s|\sigma + e - c)}{w(t|\sigma + d - h)} \omega^{(h-c)},$$

$$\overline{\psi}(\sigma|a, b, g, f) = \frac{w(s'|\sigma + f - b)}{w(t'|\sigma + a - g)} \omega^{(g-b)},$$  \hspace{1cm} (4.8)

where if these $(\psi, \overline{\psi}) = (\psi, \overline{\psi})(\theta_1, \theta_2, \theta_3)$ then

$$s = q_4(a_2, \pi - a_3, \pi - a_1), \quad t = q_1(a_2, \pi - a_3, \pi - a_1),$$

$$s' = q_3(a_2, \pi - a_3, \pi - a_1), \quad t' = q_2(a_2, \pi - a_3, \pi - a_1).$$  \hspace{1cm} (4.9)

Note that a natural IRF – type $L$ – operator

$$L(a|e, f, g|b, c, d|h)_{\theta_1, \theta_2, \theta_3} = \rho_1 \sum_{\sigma} \psi(\sigma|e, h, c, d)\overline{\psi}(\sigma|a, b, g, f)$$  \hspace{1cm} (4.10)

is equivalent to the $(\tau \rho)^2$ transformed weight $W$ (4.5). The analogous vertex type $L$ – operator

$$L_{i, c, e, c, e - d}^{j, h - d, e - h} = \rho_1 \psi(i|e, h, c, d)\overline{\psi}(j|e, h, c, d)$$  \hspace{1cm} (4.11)

is also equivalent to the appropriately transformed vertex weight $R$ (4.4).

The proof of equations (2.5,2.6) are simple. For the left hand side of (2.5) one has to make $(\tau \rho)^2$ over $k_1$ and after this the Star – Square summation formula over $k_2$ (see Ref. [2] for the meaning of these charms). In the right hand side one has to sum over the spin $a$. The final expressions are to coincide.
5. \( \psi \)-vectors for the Planar Model.

The Planar model is considered in Ref. [7]. Recall the definition of it. Define another four points for the trihedron \((a_1, a_2, a_3)\):

\[
r_i = \left( \exp\left(-i \frac{\beta_i}{N}\right), \omega^{1/4} \sqrt{2 \sin \beta_i}, \exp\left(i \frac{\beta_i}{N}\right) \right).
\]  

(5.1)

Consider the planar limit of the TE, i.e. the case when \(\beta_2 = 0\) for each weight. The vertex weight is

\[
\mathcal{R}_{j_1, j_2, j_3}^{i_1, i_2, i_3} = \delta_{j_2, i_1 + i_3} \delta_{i_2, j_1 + j_3} \omega^{j_1(i_3 - j_3)} \frac{w(r_1|i_3 - j_3)w(r_3|i_1 - j_1)}{w(Or_0|j_2 - i_2)}.
\]  

(5.2)

IRC form of this weight is

\[
W(a|e, f, g|b, c, d|h) = \omega^{(h - e)(a - d - g + h)} \frac{w(r_1|a - d - g + h)w(r_3|b - a - h + e)}{w(Or_0|b - d - g + e)}. 
\]  

(5.3)

The Planar model is self-dual in the sense that

\[
\mathcal{R}_{b - a, g - e, a - d}^{h - e, b - d, g - h} = W(a|e, f, g|b, c, d|h).
\]  

(5.4)

\( \psi \)-vectors for the weights (5.2) and (5.3) are

\[
\psi(\sigma|a, b, c, d) = w(v|\sigma + a - b)\omega^{\sigma(d - b)} \phi(a, b, c, d),
\]

\[
\overline{\psi}(\sigma|a, b, c, d) = \frac{\omega^{\sigma(c - a)}}{w(u|\sigma + a - b)} \overline{\phi}(a, b, c, d),
\]  

(5.5)

where for each \( \phi \) and \( \overline{\phi} \) there are two independent choices:

\[
\phi(a, b, c, d) = \omega^{(a - b)(a - c)} \text{ or } \omega^{(a - b)(d - b)},
\]

\[
\overline{\phi}(a, b, c, d) = \omega^{(a - b)(b - d)} \text{ or } \omega^{(a - b)(c - a)}.
\]  

(5.6)

It is useful to choose the second expressions for the phases \( \phi \) and \( \overline{\phi} \). The equations for the \( \psi \)-vectors are equivalent to a pair of \((\tau\rho)^2\) transformations.
According to the scheme (2.7) the arguments of $\psi$ vectors are (omitting the middle argument $a_2 = a_1 + a_3$):

$$v(a_1, a_3) = \left( \exp\left(-\frac{\text{i} a_1}{N}\right) \sqrt{\frac{\sin a_3}{\sin a_2}}, \exp\left(\text{i} \frac{a_3}{N}\right) \sqrt{\frac{\sin a_1}{\sin a_2}}, 1 \right),$$

$$u(a_1, a_3) = \left( \omega^{-1} \exp\left(\text{i} \frac{a_1}{N}\right) \sqrt{\frac{\sin a_3}{\sin a_2}}, \exp\left(-\text{i} \frac{a_3}{N}\right) \sqrt{\frac{\sin a_1}{\sin a_2}}, 1 \right).$$

(5.7)

The $\psi$ - vectors are in the same time $L$ operators for the $R$ matrix (5.2):

$$L_{i_1,i_2,i_3}^{j_1,j_2,j_3} = \delta_{j_1,i_3-i_2} \delta_{j_1,i_3-j_2} w(v|i_1-j_1) \omega^{j_2(i_1-j_1)}$$

(5.8)

and analogically for $\overline{\psi}$. With the help of psi vectors one can restore the $W$ weight (5.3).

$$n_1 \sum_{\sigma} \psi_v(\sigma|e,h,c,d) \overline{\psi_u}(\sigma|a,b,g,f) = W(a|e,f,g|b,c,d|h) \omega^{-(a-b)(d-h)-(a-g)(h-e)}.$$

(5.9)

The phase factor in the right hand side is the gauge factor for TE and the normalization factor

$$n_k^{-1} = \sqrt{N} \exp\left(i \pi \frac{N^2 - 1}{12N}\right) \left(2 \frac{\sin a_i \sin a_j}{\sin a_k}\right)^{\frac{N+1}{2N}}$$

(5.10)

6. Discussion

In two dimensional statistical systems the application of the $\psi$ vectors method gives the excellent results. First, the IRF type models, corresponding to the $R$ matrices of the simple Lie algebras, admit the elliptic deformation even if the $R$ matrices are trigonometric [6]. Second, the $\psi$ vectors for the cyclic representations of $U_q(A_n)$ allow one to construct new models [10]. Contrary to this situation in $D = 2$, we still have not succeeded in any application of the three dimensional $\psi$ vectors.

In three dimensions we have three types of Boltzmann weights. Two of them, vertex and IRC, were considered above. The third type Boltzmann weights with twelve spin variables are connected with the scattering straight strings [11]. Taking into account this third type of the spin structure, one
can modify the given scheme of the \( \psi \) vectors in different ways. Perhaps, in the case when \( \psi \) vectors for some modified scheme exist, they would give us a method of constructing new models.

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