RIGIDITY OF ACTIONS ON METRIC SPACES CLOSE TO THREE DIMENSIONAL MANIFOLDS

NOÉ BÁRCENAS AND MANUEL SEDANO-MENDOZA

Abstract. In this paper we propose a metric variation on the $C^0$-version of the Zimmer program for three manifolds. After a reexamination of the isometry groups of geometric three-manifolds, we consider homomorphisms defined on higher rank lattices to them and establish a dichotomy between finite image or infinite volume of the quotient. Along the way, we enumerate classification results for actions of finite groups on three manifolds where available, and we give an answer to a metric variation on topological versions of the Zimmer program for aspherical three-manifolds, as asked by Weinberger and Ye, which are based on the dichotomy established in this work and known topological rigidity phenomena for three manifolds. Using results by John Pardon and Galaz-García-Guijarro, the dichotomy for homomorphisms of higher rank lattices to isometry groups of three manifolds implies that a $C^0$-isometric version of the Zimmer program is also true for singular geodesic spaces closely related to three dimensional manifolds, namely three dimensional geometric orbifolds and Alexandrov spaces. A topological version of the Zimmer Program is seen to hold in dimension 3 for Alexandrov spaces using Pardon’s ideas.

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Zimmers program. The question on the nature of group homomorphisms $\rho : \Gamma \to \text{Diff}(M)$, between a finitely generated group and the group of diffeomorphisms of a compact, $n$-dimensional, smooth manifold is interesting in many contexts. Particularly, in a series of conjectures known as the Zimmer program [Zim87], [Fis11], [Fis20], concerning the question on whether the group homomorphism cannot have large image if the dimension of the manifold is small, relative to the rank of the group. More precisely, the Zimmer
program deals with groups \( \Gamma \) which are lattices in a semisimple algebraic Lie group of rank at least 2, which we will refer to in this introduction as **higher rank lattices** see [3]. As an example of this, in the recent result [BFH20], it is found that a homomorphism \( \rho : SL_{k+1}(\mathbb{Z}) \to \text{Diff}(M) \) must have finite image when \( k > n \) and \( k \geq 2 \), in this case the parameter \( k \) is the rank of the Lie group \( SL_{k+1}(\mathbb{R}) \). This result is greatly generalized for other higher rank semisimple lattices on [BFH21]. Another instance of the Zimmer program is the complete characterization of the action in critical dimensions, c.f. Conjectures 1.2 and 1.3 in [Fis11]. For example, when the dimension hypothesis is modified in the previous context to \( n = k + 1 \), and the hypothesis that the action preserves a finite volume and an affine connection is added, then [Fis11, Theorem 6.9] tells us that the action is conjugated to the standard linear action of \( SL_n(\mathbb{Z}) \) on \( \mathbb{R}^n/\mathbb{Z}^n \).

The \( C^0 \)-version of the Zimmer Program, as suggested in [Wei11], and [Ye20], [Ye19], asks roughly for changing the category of manifolds and morphisms in the Zimmer Program, from the smooth setting into a topological setting, that is, by considering a group homomorphism from a finitely generated group, and specifically a higher rank lattice, onto the group of homeomorphisms within a prescribed category (topological, smooth, piecewise linear, etc.). The following Conjecture is an example of a problem stated in this setting, found in [Ye20]:

**Problem 2.1.** Any group action of \( SL_{k+1}(\mathbb{Z}) \), with \( k \geq 2 \), on a closed, aspherical \( n \)-manifold by homeomorphisms factors trough a finite group if \( n < k \).

This variation of Zimmer Program is, as expected, much harder than the original one, and it has shown only small advances, such as the solution in the one-dimensional case by Hurtado and Deroin [DH20], and in the context of infinite homological actions on three-manifolds [FS00].

**Variations of the problem.** It is natural to explore analogous rigidity results outside of the differentiable category and into the category of metric spaces endowed with extra structure. As an example of this, in [Hae20] it is proved that any action of a higher rank lattice in a Gromov \( \delta \)-hyperbolic metric space must be elementary. Among many things, such result implies that any homomorphism \( \rho : \Gamma \to \text{Mod}(S) \) from a higher rank lattice onto the mapping class group of a compact surface is finite, a result first proved in [FM98].

The notion closer to manifolds for which we explore these rigidity results is that of Alexandrov spaces, which are metric spaces with a synthetic notion of curvature bounded from below. Alexandrov spaces include compact Riemannian manifolds and non-trivial modifications of them, such as orbifold quotients and Gromov-Hausdorff limits (see Section 3.2). In this paper we propose the following variation of Problem 2.1
**Problem 2.2.** Let $X$ be an $n$-dimensional, compact, Alexandrov space. Does a group homomorphism
\[
\rho : SL_{k+1}(\mathbb{Z}) \rightarrow \text{Homeo}(X),
\]
factor through a finite group if $k > n$? Can we obtain a classification of such actions in the case $k = n$ or $k = n+1$ if we ask the action to be isometric?

Of course, Problem 2.2 can be stated for general higher rank lattices with comparison bounds on the dimension and rank as in [Fis11, Conjecture 4.12].

Alexandrov spaces are rigid (in a sense analogous to Gromov’s rigid geometric structures [Fis11]) as their isometry groups are Lie groups with bounded dimension in terms of the dimension of the space [GGG13]. Moreover, 3-dimensional Alexandrov spaces which are sufficiently collapsed (c.f. section Alex), are in fact orbifolds over one of the eight Thurston geometries [GGGNnZ20]. These reduction phenomenae together with classification results for isometric actions on Thurston geometries, lead us to the following much more tractable problem:

**Problem 2.3.** Let $X$ be a 3-dimensional, compact Alexandrov space. Characterize any homomorphism
\[
\rho : \Gamma \rightarrow \text{Iso}(X),
\]
where $\Gamma$ is a higher rank lattice.

It is worth mentioning other rigidity results obtained for 3-dimensional Alexandrov spaces such as the proof of the Borel conjecture for sufficiently collapsed Alexandrov spaces [BNnZ21]. Finally, John Pardon’s proof of the Hilbert-Smith conjecture for three manifolds [Par13], can be extended to the singular case in the setting of Alexandrov spaces as it can be reduced to a local behaviour, leading to the following result:

**Theorem 2.4.** If $G$ is a locally compact, topological group, acting faithfully on a three dimensional Alexandrov space by homeomorphisms, then $G$ is a Lie group.

This result lead to the natural generalization of the Hilbert-Smith conjecture, which simply would ask if Theorem 2.4 is valid for $n$-dimensional Alexandrov spaces). A first approach to this generalization is to extend the result of [RS97], proving Hilbert-Smith conjecture for Lipschitz actions, where the difficulty lies on the extension of Yang’s Theorem (on the increase of dimension in the quotient for $p$-adic actions [Yan60] [BRW61]) to Alexandrov spaces.

**Main results and related discussions.** The main result of this paper, concerning Problem 2.3 is

**Theorem 2.5.** Let $\tilde{X}$ be a simply connected, homogeneous 3-dimensional manifold and let $H$ be a discrete group of isometries of $\tilde{X}$, such that $X = \tilde{X}/H$ has finite volume, then $X$ admits an infinite isometric action of a
higher rank lattice $\Gamma \subset G$ if and only if the group $\text{Iso}(X)$ contains the group $\text{SO}(3)$. Moreover, the semisimple Lie group $G$ is isotypic of type $\text{SO}(3)$, the lattice is uniform and $X$ is an orbifold over either $S^3$ or $\mathbb{R} \times S^2$.

Recall that an isotypic group of type $\text{SO}(3)$ is an algebraic group which is, up to finite covers and connected components, a product of copies of $\text{SO}(3)$ and $\text{SO}(2,1)$ (see Section 3). In [BNnZ21], a three dimensional Alexandrov space $X$ of curvature $\geq -1$ is said to be sufficiently collapsed, if there exist $D > 0$ and $\epsilon > 0$ such that the diameter of $X$ is less or equal to $D$, and the volume is strictly less than $\epsilon$. We include as a corollary of the results here a classification of the discrete groups acting by isometries on a three dimensional Alexandrov space with a sufficiently collapsed quotient.

Corollary 2.6. Assume that a discrete group $\Gamma$ acts by isometries on the three dimensional Alexandrov space $X$ such that the quotient $X/\Gamma$ is sufficiently collapsed with parameters $d, \epsilon$. Then, Theorem 2.8, together with the geometrization of 3-dimensional Alexandrov spaces provide a classification of the possible such $\Gamma$ within the lattices in the isometry groups.

As an immediate consequence of this theorem, we get the following corollary in the spirit of the Zimmer’s problem

Corollary 2.7. Let $\Gamma$ be a higher rank lattice acting by isometries on a finite volume, three dimensional orbifold $X$ (modelled over a homogeneous 3-manifold $X$), then the action factors through a finite group if either:

- $X$ is aspherical or,
- $\Gamma$ is non-uniform.

As an example of this, we have $\Gamma = \text{SL}_r(\mathbb{Z})$ with $r \geq 3$.

The proof of Theorem 2.5 relies on close, case by case examination of Thurston’s 3-dimensional geometries, their finite volume quotients and their corresponding isometry groups. The computations of such groups can be summarized in the following

Theorem 2.8. Let $\widetilde{X}$ be a simply connected, homogeneous 3-dimensional manifold and let $G$ be a discrete group of isometries of $\widetilde{X}$, such that $\widetilde{X}/G$ has finite volume. Then the isometry group $\text{Iso}(\widetilde{X}/G)$ has finitely many connected components, such that its connected component of the identity is isomorphic to

- a closed subgroup of $\mathbb{S}^1$, if $\widetilde{X}$ is either $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\text{SL}}_2(\mathbb{R})$ or $\text{Nil}$;
- a closed subgroup of $\text{SO}(3) \times \mathbb{S}^1$, if $\widetilde{X} = S^2 \times \mathbb{R}$;
- a closed subgroup of $\mathbb{R}^3/\mathbb{Z}^3$, if $\widetilde{X} = \mathbb{R}^3$;
- a closed subgroup of $\text{SO}(4)$, if $\widetilde{X} = S^3$.

Moreover, $\text{Iso}(\widetilde{X}/G)$ is finite if $\widetilde{X}$ is either $\mathbb{H}^3$ or $\text{Sol}$. 
Strategy of the proof and structure of the paper. Main Theorem 2.5 is proved in Section 3 using the classification of isometry groups of orbifolds given by Theorem 2.8, together with rigidity results of semisimple Lie groups. To prove Theorem 2.8, one first needs to understand finite volume quotients of Thurston’s three-dimensional geometries. Among such geometries, the most homogeneous ones are $S^3$, $\mathbb{H}^3$ and $\mathbb{R}^3$; and the remaining five present a more flexible description as fiber bundles

$$F \rightarrow X \rightarrow B,$$

where $B$ is a two-dimensional homogeneous geometry for $X$ either $\mathbb{H}^2 \times \mathbb{R}$, $\tilde{SL}_2(\mathbb{R})$ or $\text{Nil}$ and $B = \mathbb{R}$ for $X$ either $\text{Sol}$ or $S^2 \times \mathbb{R}$. In this context, a discrete group acting on the homogeneous space $X$, acts on the base space of the corresponding fiber bundle as well. The induced action on the base space of the fiber bundle presents a dual behavior given by the following Theorem, whose proof can be derived from the discussions on [Thu97].

**Theorem 2.9.** Let $G$ be a discrete group of isometries of any of the 3-dimensional geometric manifolds $\mathbb{H}^2 \times \mathbb{R}$, $\tilde{SL}_2(\mathbb{R})$ or $\text{Nil}$; then either

- $G$ projects to a discrete group of isometries of the base $B$ of the fiber bundle, or
- the orbifold $\tilde{X}/G$ has infinite volume.

Moreover, in the cases $\text{Sol}$ and $S^2 \times \mathbb{R}$, the projection to the base space is always discrete.

For the sake of completeness we present here a proof of Theorem 2.9. The proofs of Theorem 2.8 and Theorem 2.9 are carried out in a case by case setting on each Thurston geometry.

The structure of the paper is as follows: In Section 3 we present background material on three-dimensional Alexandrov spaces, the Hilbert Smith conjecture and semisimple Lie groups and their lattices. Sections 5 through 13 cover the proof of Theorem 2.9 and Theorem 2.8 on each individual three-dimensional geometry.

Theorem 2.9 can be used to obtain explicit characterizations of discrete groups acting on the corresponding three-dimensional geometry (in particular within the $\text{Nil}$ and $\text{Sol}$ cases), which lead to the proof of Theorem 2.8 in each case.

### 2.1. Concluding remarks and open questions

In this work we proposed a metric variation on the Zimmer program. The variation consisted in

- Strengthening the category of automorphisms of the action from $C^0$ to isometries.
- Relaxing the topological type of the spaces considered from smooth manifolds to Alexandrov spaces, which include three dimensional geometric orbifolds.
While Alexandrov spaces have an open dense subset which is a topological manifold, results related to Zimmer’s conjecture do not apply directly because the topological manifold is open, and the only rigidity results for actions on open manifolds in the spirit of the Zimmer program which are known to the authors are restricted to the one dimensional case [DH20]. In another instance of a complication, the manifold is not Riemannian, as Otsu-Shioya’s example shows [OS94].

There exist several instances of families of homeomorphisms of metric geodesic spaces for which rigidity results of actions of discrete groups can be proved. Among them we can consider, for a strengthening with respect to homeomorphisms and a weakening with respect to isometries:

- Quasiconformal homeomorphisms, as in the alternative proof of Mostow Rigidity Theorem by [Bou09].
- Bilipschitz homeomorphisms, as in the proof of the Hilbert-Smith Conjecture mentioned before [RS97].
- Quasimöbius homeomorphisms as in [BK02], where rigidity results for them have as a consequence the rigidity of actions of quasi-convex cocompact actions on CAT(−1)-spaces.

Moreover, the specific analytic and geometric characteristics of the class of homeomorphisms are exploited in the process of proving an action rigidity result in an analogous way to how we used the geometric structure of Alexandrov three spaces in this work, inspired by the proofs of Zimmer program results in the diffeomorphism case.

Let us introduce the notation

\[ \text{AUT}^\alpha(X) \]

for homeomorphisms of a metric space with a metric property \( \alpha \) and let us refer to a metric condition \( \alpha \) as a decoration in analogy with surgery theory, having at least the following examples in mind:

(i) The smooth case \( \alpha = \text{Diff} \), for diffeomorphisms of the smooth structure associated to a Riemannian metric on a smooth manifold with fixed metric.
(ii) The topological case \( \alpha = \text{Top} \), referring to homeomorphisms of a topological manifold.
(iii) The isometric case \( \alpha = \text{Iso} \), meaning isometries of the geodesic metric space associated to a geometric three manifold, orbifold or Alexandrov space.
(iv) The quasiconformal case \( \alpha = \text{QC} \), the referring to quasiconformal homemorphisms, \( \alpha = \text{QM} \), associated to quasimöbius homeomorphisms and \( \alpha = \text{BI} \) for bilipschitz homemorphisms of a geodesic length metric space as discussed in the paragraph above.

We can consider the problem of describing the behaviour of a group homomorphism
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\[ \Lambda \rightarrow \text{AUT}^\alpha(X) \]

to a homeomorphism group with decorations as described. Notice that 
\( \alpha = \text{Diff} \) is the Zimmer program as described for instance in \([\text{Fis11}], [\text{Fis20}], [\text{Zim84}]\). On the other hand \( \alpha = \text{Top} \) is the \( C^0 \)-Zimmer program as described in \([\text{Wei11}], [\text{Ye20}], [\text{Ye19}]\). Finally \( \alpha = \text{Iso} \) was the point of view adopted in this note, and \( \alpha = QC, QM, BI \) are as before.

We would like to finish the present note with the following two questions:

- To what extent a condition of prescribed curvature in metric spaces, such as the Alexandrov condition, or a choice of \( QC, QM \) or \( BI \)-structures can be seen as a rigid structure, in the sense of Gromov \([\text{Fis11}]\)?
- For which decorations of homeomorphisms \( \alpha \) is it possible to prove that homeomorphisms of a higher rank lattice \( \Lambda \) with respect to the dimension of an Alexandrov space \( X \)

\[ \Lambda \rightarrow \text{AUT}^\alpha(X) \]

or in general a metric measure space \( X \) with finite Hausdorff dimension less than the rank of \( \Lambda \) either factorize through a finite quotient of \( \Lambda \) or produce a quotient of infinite volume?

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3. PRELIMINARIES

3.1. Three Dimensional Manifolds. Recall that a closed three dimensional manifold \( M \) is prime if it cannot be written as a connected sum with summands not homeomorphic to the three dimensional sphere.

By the prime decomposition Theorem, 3.15 page 31 in \([\text{Hem04}]\), any closed three dimensional manifold can be written as a connected sum of prime factors.

 Recall that a model geometry is a simply connected smooth three manifold \( X \) together with a transitive action of a Lie group \( G \) on \( X \) with compact stabilizers such that \( G \) is maximal among groups acting smoothly and transitively on \( X \) with compact stabilizers.
A geometric structure on a three dimensional manifold $M$ is a diffeomorphism from $M$ to $X/\Gamma$ for some model geometry $X$, where $\Gamma$ is a discrete subgroup of $G$ acting freely on $X$;

It is a consequence of Thurston geometrization theorem due to Perelman, that any prime closed three dimensional manifold can be cut along three dimensional tori such that the interiors of the resulting manifold carries a geometric structure of finite volume.

The eight three dimensional geometries which admit at least one compact three manifold are:

(i) Spherical, where the three dimensional sphere is a representative.
(ii) Euclidean, where the flat three dimensional torus is an example,
(iii) Hyperbolic, where the three dimensional hyperbolic space is an example.
(iv) $\tilde{SL}_2(\mathbb{R})$, where an example is given by the unit tangent bundle in a Riemannian metric of the tangent bundle over a genus two surface.
(v) $Nil$, where an example is given by the mapping torus of a Dehn twist on the two dimensional torus.
(vi) $Sol$, where an example is given by a manifold which fibers over the line with fiber the plane.
(vii) $S^2 \times \mathbb{R}$.
(viii) $S^1 \times \mathbb{H}^2$.

A closed 3-manifold has a geometric structure of at most one of the 8 types above.

**Remark 3.1.** (Non-uniqueness of Geometric Structures on three manifolds)

- Finite volume non-compact manifolds may have more than one type of geometric structure. An example is the complement of the trefoil knot, which has hyperbolic structure and $\tilde{SL}_2$ structure.
- If the three manifold has infinite volume, it might carry many geometric structures, for example $\mathbb{R}^3$ is diffeomorphic to all aspherical model geometries.
- There exists an infinite number of geometric structures with no compact models; for example, the geometry of many non-unimodular 3-dimensional Lie groups, see remark 9.1 below.

3.2. Three dimensional Alexandrov Spaces. We will need some preliminaries on three dimensional Alexandrov spaces. For a general reference see [BBI01].

For the purposes of this work, Alexandrov spaces are well behaved metric spaces which have three main properties that we want to highlight here:

(i) They have an open dense subset which is topological manifold.
(ii) Their isometry groups are Lie groups.
(iii) The class of Alexandrov spaces is stable under Gromov-Hausdorff convergence.
(iv) They include orbifolds over Riemannian manifolds.

In slightly more detail, Alexandrov spaces are a synthetic generalization of complete Riemannian manifolds with a lower bound on sectional curvature. The generalization uses comparison triangles with respect to the model spaces $S^2_k$, which are simply connected, two dimensional complete Riemannian manifolds of constant curvature $k$. More precisely, for $k > 0$, $S^2_k$ is the sphere of radius $1/\sqrt{k}$, for $k < 0$, $S^2_k$ is the hyperbolic plane $\mathbb{H}^2(1/\sqrt{-k})$ of constant curvature $k$, and for $k = 0$, $S^2_k$ is the euclidean space $\mathbb{R}^2$.

Given a geodesic triangle in a geodesic length space $(X, d)$, with vertices $p, q, r \in X$, a comparison triangle in $S^2_k$ is a geodesic triangle $\overline{pq\bar{r}}$ having the same side lengths. The geodesic length space $(X, d)$ is said to satisfy the Topogonov property for $k \in \mathbb{R}$, if for each triple $p, q, r \in X$ of vertices of a geodesic triangle, and each point $s$ on the geodesic from $q$ to $r$, the inequality $d(p, s) \geq d(\overline{pq\bar{s}})$ holds, where $\overline{s}$ is the point on the geodesic side $\overline{q\bar{r}}$ of the comparison triangle with $d(\overline{pq\bar{s}}) = d(p, s)$.

**Definition 3.2.** A $n$-dimensional $k$-Alexandrov space is a complete, locally compact, length space of finite Hausdorff dimension $n$, such that the Topogonov Property is satisfied locally for $k$.

Topogonov’s globalization theorem tells us that the local and global Topogonov property are equivalent in $k$-Alexandrov spaces. By Gromov’s precompactness theorem, Alexandrov $n$-dimensional spaces arise as Gromov-Hausdorff limits of compact riemannian manifolds of dimension $n$ for which the sectional curvature is bounded below by $k$, and the diameter is bounded above by some fixed positive number $D$.

The class of $k$-Alexandrov spaces includes riemannian manifolds of sectional curvature bounded below by $k$, and several constructions including more general geodesic length spaces such as euclidean cones, suspensions, joins, quotients by isometric actions of compact Lie groups, and gluings along a submetry, see [GG16] section 2.2. From now on, we will omit the $k$ from the notation.

There exists a notion of angle between geodesics of an Alexandrov space, and a space of tangent directions at a given point $p$, denoted by $\Sigma_p$, can be defined as the completion of the metric space of equivalence classes of geodesics making a zero angle.

The space of tangent directions at a point $p$ in an Alexandrov space $X$, denoted by $\Sigma_p$, has the structure of a 1-Alexandrov space of Hausdorff dimension $\text{dim}(X) - 1$. There is a set $R_X \subset X$, called the set of metrically regular points, where a point $p$ belongs to $R_X$ if its direction space $\Sigma_p$ is isometric to the radius one sphere. The complement is called the set of metrically singular points and denoted by $S_X = X \setminus R_X$. There are examples of Alexandrov spaces whose space of metrically singular points is dense, as seen in an example constructed in [OS94] as a limit of Alexandrov spaces.
using baricentric subdivisions of a tetrahedron. However, for every Alexandrov space $X$, there is a dense subset of topologically regular points, whose space of directions are homeomorphic to a sphere (the set of topologically singular points is the complement of the set of topologically regular points). By Perelman’s conical neighborhood theorem, every point $p$ in an Alexandrov space has a neighborhood pointed homeomorphic to the euclidean cone over $\Sigma_p$, so that a locally compact, finite dimensional Alexandrov space has a dense subset which is a topological manifold.

In the specific case of dimension three, there are only two possibilities for the homeomorphic type of the space of directions, which is the two sphere $S^2$, for the topologically regular points and the real projective space $\mathbb{RP}^2$ for the topologically singular points. Let us summarize the basic structure of three dimensional Alexandrov spaces due to Galaz-García and Guijarro, compare Theorem 1.1 in page 5561 of [GGG15], and Theorem 3.1 and 3.2 in page 1196 of [GGG13]. See also [HS17].

**Theorem 3.3.** Let $X$ be a three dimensional Alexandrov space.

- The set of metrically regular points is a Riemannian three manifold.
- The set of topologically singular points is a discrete subset of $X$.
- If $X$ is closed, and positively curved Alexandrov space, that contains a topologically singular point, then $X$ is homeomorphic to the suspension of $\mathbb{RP}^2$.

A closed Alexandrov space is geometric if it can be written as a quotient of one of the eight geometries of Thurston under a cocompact lattice. The following theorem was proved as Theorem 1.6 in [GGG15] in page 5563. See also [HS17].

**Theorem 3.4.** A three dimensional Alexandrov space admits a geometric decomposition into geometric pieces, along spheres, projective planes, tori and Klein bottles.

We now direct our attention to the isometry group of three dimensional Alexandrov spaces.

**Theorem 3.5.** Let $X$ be an $n$-dimensional Alexandrov space of Hausdorff dimension $n$.

- The Isometry group of $X$ is a Lie Group which is compact if $X$ is compact as well.
- The dimension of the group of Isometries of $X$ is at most

$$\frac{n(n+1)}{2},$$

and the bound is attained if and only if $X$ is a Riemannian manifold.

**Proof.** The first part is proved as the main Theorem, 1.1 in [FY94]. The second part follows from the Van Dantzig-Van der Waerden Theorem [DVdW28].
• This is proved as Theorem 3.1 in page 570.

\[ \square \]

**Remark 3.6.** It is proved in [BZ07] that the same lower bound for the dimension of the isometry group holds in general for Riemannian orbifolds.

3.3. **Hilbert-Smith Conjecture.** The following conjecture was formulated as an extension of Hilbert’s 5th Problem:

**Conjecture 3.7 (Hilbert-Smith conjecture).** If \( G \) is a locally compact, topological group, acting faithfully on a topological manifold, then \( G \) is a Lie group.

See [Tao14] for a modern account.

As a consequence of structural theorems of locally compact groups, such as the Gleason-Yamabe theorem and its predecessor by Von Neumann [vN33], a counter-example to the Hilbert-Smith conjecture must contain a copy of a \( p \)-adic group \( \hat{\mathbb{Z}}_p \), for some \( p \), see [Lee97], thus giving the equivalent conjecture

**Conjecture 3.8 (Hilbert-Smith conjecture \( p \)-adic version).** For every prime \( p \), there are no faithful actions of the \( p \)-adic group \( \hat{\mathbb{Z}}_p \) on a topological manifold.

Conjecture 3.8 has been proven in different contexts. For example, if there is a notion of dimension which must be preserved, such as bi-Lipschitz actions of \( \hat{\mathbb{Z}}_p \) on Riemannian manifolds, where three notions of dimension coincide: Hausdorff dimension, cohomological dimension with integer coefficients and topological dimension. In such setting, the bi-Lipschitz condition tells us that the Hausdorff dimension on the quotient cannot decrease, but on the other hand a theorem by Yang [Yan60], tells us that the cohomological dimension of the quotient increases by two, leading to the following result:

**Theorem 3.9 (Repovš-Ščepin [RS97]).** There are no faithful actions by bi-Lipschitz maps of the \( p \)-adic group \( \hat{\mathbb{Z}}_p \) on a Riemannian manifold.

The stronger setting of topological actions is much harder and has been proven only for small dimensions

**Theorem 3.10 ([Par19], [Par13]).** For every prime \( p \), there are no faithful actions by homeomorphisms of the \( p \)-adic group \( \hat{\mathbb{Z}}_p \) on a topological manifold of dimension \( n \leq 3 \).

**Remark 3.11.** The \( p \)-adic group can be described as

\[
\hat{\mathbb{Z}}_p = \left\{ \sum_{n=0}^{\infty} a_n p^n : a_n \in \{0, 1, \cdots, p-1\} \right\}
\]

so that \( p^k \hat{\mathbb{Z}}_p \subset \hat{\mathbb{Z}}_p \) is an open, normal subgroup, with \( \hat{\mathbb{Z}}_p/p^k \hat{\mathbb{Z}}_p \cong \mathbb{Z}_p^{\times} \), giving the inverse limit description \( \hat{\mathbb{Z}}_p = \lim_{\leftarrow} \mathbb{Z}_p^{\times} \), moreover, the group \( \hat{\mathbb{Z}}_p \) is
homeomorphic to the Cantor space \( \{0, \cdots, p - 1\}^\mathbb{N} \). Observe that there is a topological 2-manifold with the cantor space \( 2^\mathbb{N} \) as its ends space, which is \( \Sigma = S^2 \setminus C \), where \( C \subset S^2 \) is a closed subset homeomorphic to \( 2^\mathbb{N} \). Thus, there is a faithful action of \( \mathbb{Z}_2^\infty \) on \( \text{End}(\Sigma) \cong 2^\mathbb{N} \) and every homeomorphism of \( \text{End}(\Sigma) \) extends to a homeomorphism of the surface \( \Sigma \), however, by Theorem 3.10 such extensions cannot be promoted to an action of \( \mathbb{Z}_2^\infty \) on the Freudenthal compactification.

Hence, the weaker version of the \( p \)-adic Hilbert-Smith conjecture for Alexandrov spaces holds, and we can consider the following conjecture:

**Conjecture 3.12.** If \( G \) is a locally compact, topological group, acting faithfully on a finite dimensional Alexandrov space by homeomorphisms, then \( G \) is a Lie group.

A consequence of Theorem 3.10 gives us a three dimensional case of this result

3.4. Lie Groups and Lattices. A real Lie group is a Hausdorff topological group which is a \( C^\infty \) smooth manifold for which the multiplication and inversion are smooth maps.

Recall that by Haar’s Theorem there exists up to a positive multiplicative constant, a unique countably additive, nontrivial measure \( \mu \) on the Borel subsets of \( G \) which is right translation invariant, has finite values on compact subsets, and is inner and outer regular.

**Definition 3.13.** A lattice in a Lie Group \( G \) is a discrete group \( \Lambda \) for which the quotient space \( G/\Lambda \) has finite measure.

The right translate \( \mu(g^{-1}) \) by an element in the group \( g^{-1} \) of a right invariant Haar measure \( \mu \) is a right invariant Haar measure, and hence there exists a real function \( \Delta \) satisfying

\[
\mu(g^{-1}S) = \Delta(g)\mu(S).
\]

**Definition 3.14.** A group is said to be unimodular if the function \( \Delta \) is the constant function 1.

**Example 3.15.** The following families of examples of Lie groups will be the main focus of the article.

- By the second Myers-Steenrod theorem [MS39], the isometry group of a smooth manifold is a Lie group.
- By the Montgomery-Zippin theorem [MZ74], if a topological group acts by isometries transitively on a finite dimensional, locally compact, connected and locally connected metric space, then it is a Lie group.
- By results of Bochner [Boc46], the isometry groups of a smooth manifold of constant negative Ricci curvature is finite.
On the other hand, a locally compact subgroup of $\mathcal{C}^2$ diffeomorphisms of a $\mathcal{C}^2$ manifold for which the trivial subgroup is the only subgroup with fixed points with nonempty interior must be a Lie group $[BM46]$.

In the subsequent sections of this article, we will examine the isometry groups of three manifolds and their lattices, as well as the isometry groups of orbifolds or Alexandrov spaces.

3.5. Discrete groups of isometries. If $\tilde{X}$ is a complete, simply connected, Riemannian manifold and $\Gamma \subset Iso(\tilde{X})$ a discrete subgroup of isometries, then $X/\Gamma$ has the structure of a complete Riemannian orbifold. The covering map $\rho : \tilde{X} \to X$ satisfies the property that $\rho(x) = \rho(y)$ if and only if $\gamma x = y$ for some $\gamma \in \Gamma$. An isometry $\phi : X \to X$ lifts to $\tilde{\phi} : \tilde{X} \to \tilde{X}$ such that $\rho \circ \tilde{\phi} = \phi \circ \rho$ and for every $\gamma \in \Gamma$ and $x \in \tilde{X}$ we have

$$\rho \circ \tilde{\phi}(\gamma x) = \phi \circ \rho(\gamma x) = \phi \circ \rho(x) = \rho \circ \tilde{\phi}(x),$$

thus there exist $\gamma' \in \Gamma$ such that

$$\tilde{\phi}(\gamma x) = \gamma' \tilde{\phi}(x),$$

that is $\tilde{\phi} \circ \gamma \circ \tilde{\phi}^{-1} = \gamma'$ and $\tilde{\phi} \in N = N_{Iso(\tilde{X})}(\Gamma)$. This tells us that we have the isomorphism

$$Iso(X) \cong N_{Iso(\tilde{X})}(\Gamma)/\Gamma.$$

**Proposition 3.16.** If $G$ is a Lie group and $\Gamma \subset G$ is a discrete subgroup with associated normalizer and centralizer subgroups

$$N = N_G(\Gamma), \quad Z = Z_G(\Gamma),$$

then the connected components of $N$ and $Z$ coincide. Moreover, if $Z_0$ denotes such connected component, the projection $\pi : N \to N/\Gamma$ is a covering Lie group homomorphism such that $\pi(Z_0) \subset N/\Gamma$ is the connected component of the identity.

**Proof.** If $g_t \in N$ is a 1-parameter subgroup and $\gamma \in \Gamma$, then $g_t \gamma g_{-t} = \gamma$ is a 1-parameter group in $\Gamma$, but as $\Gamma$ is discrete, $\gamma_t = \gamma$ and this tells us that $g_t \in Z$, so that $Z_0 = N_0$. Now, $N$ is a Lie group having $\Gamma$ as a normal, discrete subgroup so that the projection map

$$\pi : N \to N/\Gamma$$

is a homomorphism of Lie groups and a covering map. In particular, $\pi(N_0)$ is a connected, open Lie subgroup of the same dimension of $N/\Gamma$ and thus it is the connected component of the identity.

3.6. Lattices on semisimple Lie groups of higher rank. Recall that an algebraic $\mathbb{R}$-group is a subgroup $G_\mathbb{C} \subset GL_m(\mathbb{C})$ obtained as solutions of polynomial equations with coefficients over $\mathbb{R}$ and $G_\mathbb{R} = G_\mathbb{C} \cap GL_m(\mathbb{R})$ is a real Lie group. In this context we say that $G_\mathbb{R}$ is a real form of $G_\mathbb{C}$ or that $G_\mathbb{C}$ is a complexification of $G_\mathbb{R}$. The local structure of a Lie group is captured by its Lie algebra, so that two groups are locally isomorphic if and
only if they have isomorphic Lie algebras, and thus, they can be obtained one from the other by taking connected components and topological covers.

The class of semisimple Lie groups can be defined as the class of Lie groups which are constructed up to covers and connected components from algebraic \( \mathbb{R} \)-groups which split as products \( G_1 \times \cdots \times G_k \), where each factor \( G_j \) is simple. This definition is equivalent to other definitions of semisimple Lie groups available in the literature, see [Zim84].

Remark 3.17. Not every semisimple Lie group is an algebraic group as the group \( \text{SL}_2(\mathbb{R}) \) has a universal cover, denoted by \( \tilde{\text{SL}}_2(\mathbb{R}) \), which is homeomorphic to \( \mathbb{R}^3 \) and it cannot be embedded in any linear group \( \text{GL}_m(\mathbb{C}) \) as a Lie subgroup. In the same way, not every semisimple Lie group splits as a product of simple Lie groups, as the example \( \text{SO}(4) \) shows, but its universal cover is isomorphic to the product \( \text{SU}(2) \times \text{SU}(2) \). In general, given a connected semisimple Lie group \( G \), with center \( Z(G) \), then the quotient \( G/Z(G) \) is a connected, linear algebraic group which splits as a product of simple groups and it is locally isomorphic to \( G \). Thus it is common for some results to ask for the group to be centerless.

In the context of algebraic groups defined over a field \( k \), the concept of \( k \)-rank is the maximal abelian subgroup which can be diagonalized over \( k \). Thus, for a complex algebraic group, the \( \mathbb{C} \)-rank is the dimension of a maximal subgroup isomorphic to a complex torus \((\mathbb{C}^*)^l\) and we are particularly interested in the real rank of a real form. We can observe that the real rank of a product \( G_1 \times \cdots \times G_k \) is the sum of the real rank of its factors \( G_j \) and we can give some explicit examples.

Example 3.18. The following is a complete list, up to local isomorphism, of complex, simple Lie groups and some examples of their real forms:

(i) The group \( \text{SL}_n(\mathbb{C}) \), has \( \mathbb{C} \)-rank \( n - 1 \) and has the groups \( \text{SU}(p,q) \) and \( \text{SL}_n(\mathbb{R}) \) as real forms, with real rank equal to \( \min\{p,q\} \) and \( n - 1 \) respectively.

(ii) The group \( \text{SO}(n,\mathbb{C}) \) has \( \mathbb{C} \)-rank \( \left\lfloor \frac{n}{2} \right\rfloor \) and has the groups \( \text{SO}(p,q) \) as real forms, having real rank equal to \( \min\{p,q\} \).

(iii) The group \( \text{Sp}(2n,\mathbb{C}) \) has \( \mathbb{C} \)-rank \( n \) and has the groups \( \text{Sp}(p,q) \) and \( \text{Sp}(2n,\mathbb{R}) \) as real forms, with real rank equal to \( n \) and \( \min\{p,q\} \) respectively.

(iv) The exceptional complex groups \( G_2(\mathbb{C}), F_4(\mathbb{C}), E_6(\mathbb{C}), E_7(\mathbb{C}), E_8(\mathbb{C}) \) have \( \mathbb{C} \)-rank determined by the corresponding subindex.

Remark 3.19. Between the possible real forms of a complex semisimple Lie group, there is one and only one compact real form up to conjugacy and such compact form has a compact universal cover, so the compactness property survives in the process of passing to a cover. We can thus, speak of the compact factors of a real semisimple Lie group. Moreover, the rank of a compact Lie group, defined as the dimension of a maximal torus \((S^1)^l\) contained in the group, equals the rank of its complexification and has real rank equal to \( 0 \). Finally, given a compact, connected, Lie group \( C \), there is
a finite cover of $C$ that splits as $G \times T$, where $G$ is an algebraic semisimple Lie group, and $T$ is a torus.

**Definition 3.20 (Higher Rank Lattice).** A semisimple Lie group is said to have higher rank if its real rank is greater than or equal to 2. Moreover, if a semisimple Lie group has a complexification whose simple factors are all locally isomorphic, the group is called isotypic.

Isotypic Lie groups are important because we can construct irreducible lattices in them, which don’t split as a product of lattices in the simple factors.

**Example 3.21.** If $\sigma : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ is the non-trivial Galois automorphism and $Q(x, y, z, t) = x^2 + y^2 - \sqrt{2}z^2 - \sqrt{2}t^2$, $\sigma(Q) = x^2 + y^2 + \sqrt{2}z^2 + \sqrt{2}t^2$. The groups $G = SO(Q, \mathbb{R}) \cong SO(2, 2)$ and $K = SO(\sigma(Q), \mathbb{R}) \cong SO(4)$ are semisimple Lie groups, with $K$ compact and $G$ of real rank equal to 2. If we consider the integral points in $G$, that is, the group $\Gamma = SO(Q, \mathbb{Z}(\sqrt{2})) \subset G$, then the group, 

$$\hat{\Gamma} = \{(g, \sigma(g)) \in G \times K : g \in \Gamma\}$$

is discrete. In fact there is an $\mathbb{R}$-group $\mathbb{H}_C \subset GL_m(\mathbb{C})$ such that $\mathbb{H}_\mathbb{R} = G \times K$ and $\mathbb{H}_\mathbb{Z} = \hat{\Gamma}$, in particular, it is a lattice which is co-compact. As the projection $G \times K \to G$ has compact Kernel and maps $\hat{\Gamma}$ onto $\Gamma$, thus $\Gamma \subset G$ is discrete and thus, a co-compact lattice in $G$.

**Remark 3.22.** The previous example captures the general behaviour of irreducible lattices in isotypic semisimple Lie groups. In fact, isotypic, semisimple Lie groups are the only cases of semisimple Lie groups admitting irreducible lattices and such lattices are constructed with the method of the previous example, but with higher degree extension fields $k/\mathbb{Q}$. See [Mor15], Section 5.6 for the details of the previous example and the construction in general.

### 4. Detailed Strategy for the Proof of Main Theorem

We are interested in the particular case where given a three-manifold $X$, the universal cover $\tilde{X}$ is a homogeneous space, i.e. its group of isometries acts transitively on $\tilde{X}$, and $X$ has finite volume. A general setting where this is achieved is when we consider $G = \tilde{X}$ a simply-connected Lie group with a right-invariant (or left-invariant) riemannian metric and $X = G/\Gamma$, with $\Gamma \subset G$ a lattice subgroup (i.e. $\Gamma$ is a discrete subgroup such that $G/\Gamma$ has finite, left $G$-invariant volume). As there is an embedding $G \subset Iso(G)$, we have that $N_G(\Gamma) \subset N_{Iso(G)}(\Gamma)$, but it could happen that $N_G(\Gamma)/\Gamma$ is strictly smaller than $Iso(G/\Gamma)$. On the other hand, we can extend $\Gamma$ to a discrete subgroup of $Iso(G)$ which is not completely contained in $G$, so that the isometry group $Iso(G/\Gamma)$ is decreased.

In the following sections, we will examine this phenomenon in the Thurston Geometries, and determine the possible isometry groups of the corresponding finite-volume orbifolds.
The main result \(2.8\) is proved by a case by case schema organized around the eight geometries, which we present in a resumed form below.

Within the most homogeneous three of them (spherical, hyperbolic and euclidean), the Theorem is a consequence of classical results, which we gather from references and include the classification of spherical manifolds due to Seifert-Threlfall, Borel’s density Theorem for hyperbolic geometry, and Bieberbach’s theorems for the euclidean case, respectively.

In the cases of Nil and Sol, we elaborate arguments in this text which include the restrictive conditions for the existence of lattices in solvable groups, as well as an algebraically rigid classification of discrete subgroups of isometries of nilmanifolds. This is the most original contribution among the proofs presented in this note.

For the geometry \(\tilde{SL}_2(\mathbb{R})\), which is given as a non-trivial central extension of \(PSL_2(\mathbb{R})\) by \(\mathbb{Z}\), the arguments include an analysis of the behaviour of the fixed point set of discrete subgroups of isometries of the visual compactification of the hyperbolic plane.

The products \(S^2 \times \mathbb{R}\) and \(H^2 \times \mathbb{R}\) exhibit differences in the main argument. For the latter the projection to the hyperbolic factor is analyzed, and the result is reduced to the observation that the isometry groups of a two dimensional hyperbolic orbifold are finite, which are combined together with the fact that discrete subgroups of isometries on \(H^2 \times \mathbb{R}\) project to isometry groups of \(H^2\) producing finite volume orbifolds.

For the former, \(S^2 \times \mathbb{R}\), the key remark is that a discrete cocompact isometry group can be realized as subgroup of \(SO(3) \times S^1\). This result is followed from Tollefson’s classification results \([Tol74]\) of the groups acting on three manifolds with that geometry.

### 4.1. Overview of Classification Results

We notice that the classification of isometry groups presented here has as consequence classification results for finite group actions on three manifolds. There are three general kinds of behaviour:

- Within the geometries whose isometry groups are extensions involving a discrete and finite volume subgroup of \(SL_2(\mathbb{R})\), any finite group can act by isometries. This concerns the geometries \(H^3\), \(\tilde{SL}_2\) and \(H^2 \times \mathbb{R}\).
- For the case of spherical factors and the euclidean case, the classical results by Tollefson, Seifert-Threlfall and Bieberbach theorems exhaust the class of finite groups acting by isometries. This is recorded in \([McC02]\), \([Tol74]\), \([Now34]\). There exists a classification of free (topological) finite group actions in \([LSY93]\), \([HJL02]\).
- For Sol, any cyclic group can be realized as a consequence of the discussion of example \(9.8\). For the case Nil, \(6.9\) gives a classification depending on the results developed in this note.
Notice that there exists a classification of topological free actions of finite groups on Nil and Sol manifolds based on the $p$-rank and P.A. Smith theory \cite{Lee97,LSY93}.

4.2. **Proof Schema for Theorem 2.8.** In this short subsection we summarize the detailed chain of implications leading to the proof of Theorem 2.8.

**Euclidean Geometry**

According to the Bieberbach Theorems, there exists a discrete free abelian subgroup of translations $T$, which has rank less or equal to three. The assertion of theorem 2.8 for the isometry group of $\mathbb{R}^3/\Gamma$ will be verified by examining the rank of the translation subgroup $T$, and discarding rank two and one by producing a quotient of infinite volume.

**Nil Geometry.**

For the Nil geometry, Theorem 2.8 is a consequence of 6.10, characterizing lattices of infinite volume, and Proposition 6.9 giving an exact sequence between isometry groups.

**Spherical Geometry.**

The theorem is a direct consequence of the classification of finite group actions on three-manifolds, as well as the determination of the components of the isometry groups 7.3. Notice that there are neither non-discrete subgroups of isometries nor groups whose quotient shows infinite volume within the spherical geometry.

**$S^2 \times \mathbb{R}$ Geometry.**

The theorem is consequence of the splitting of the isometry groups, as well as the characterization of discrete subgroups of isometries in 8.3, finally concluding in 8.4.

**Sol Geometry.**

The theorem is stated as Corollary 9.7, which depends on the determination of centralizers in 9.6 and the determination of finite volume in 9.4.

**Hyperbolic Geometry.**

The theorem is direct consequence of Lemma 10.1, which is in turn consequence of Borel Density, or the preceding argumentation there. See also \cite{Boc46}.

**$\mathbb{H}^2 \times \mathbb{R}$ Geometry.** The theorem is stated in 13.6 and it is consequence of Theorem 12.1 stating the splitting of isometry groups of the factors.

**$\widetilde{SL_2(\mathbb{R})}$ Geometry.**
The Theorem is also stated in [13.6] and it is consequence of Lemma [10.1] and Proposition [13.5].

5. Euclidean Geometry

Recall that a three manifold is Euclidean if it is locally isometric to the Euclidean three dimensional space \( \mathbb{R}^3 \). The isometry group of the three dimensional space is the semidirect product \( E(3) = \mathbb{R}^3 \rtimes O(3) \).

Let \( \Gamma \) be a discrete subgroup of isometries \( E(3) \). It is a consequence of the Bieberbach Theorems, as interpreted by Nowacki [Now34], that there exists a free abelian group \( T \) of rank \( \leq 3 \) and having finite index in \( \Gamma \).

End of proof of Theorem 2.8 for euclidean geometry. We will now verify the assertion of theorem 2.8 for the isometry group of \( \mathbb{R}^3/\Gamma \) by examining the rank of the translation subgroup \( T \).

- If the rank of \( T \) is one, then \( \Gamma \) is a finite extension of \( \mathbb{Z} \), and \( \mathbb{R}^3/T \) is either the interior of a solid torus or the topological interior of a solid Klein Bottle, depending on the orientability, where the generator of \( T \) acts as a screwdriver isometry (combination of a rotation around an axis and a translation along a parallel direction). It follows that \( \mathbb{R}^3/T \) has infinite volume.
- If the rank of \( T \) is two, then \( \mathbb{R}^3/T \) is the total space of a line bundle over either the torus or the Klein bottle, and \( \mathbb{R}^3/T \) has infinite volume.
- If the rank of \( T \) is three, then the isometry group of \( E(3)/\Gamma \) is a finite extension of a rank three torus by a finite subgroup.

\[ \square \]

5.1. Classification. The classification of (topological) finite group actions on the torus by isometries has been concluded by work of Lee, Shin and Yokura [LSY93] and Ha, Jo, Kim and Lee [HJJKL02].

It follows from the Bieberbach theorems that any topological action on the torus is topologically conjugated to an isometry; moreover, by the fact that the three dimensional torus is sufficiently large in the sense of Heil and Waldhausen, [Wal68], any homotopy equivalence is homotopic to a homeomorphism, and any two homotopic homeomorphisms are isotopic.

Connected components

The isometry groups of co-compact euclidean orbifolds have been determined by Ratcliffe and Tschantz [RT15], in Theorem 1 and Corollaries 1 and 2 in pages 46 and 47, which we state now for later reference.

Theorem 5.1. The isometry group of a cocompact euclidean orbifold \( \mathbb{R}^3/\Gamma \) is a compact Lie group whose identity component is a Torus of dimension
equal the first Betti number of the group $\Gamma$, which corresponds to the rank of the abelian group $\Gamma/[\Gamma, \Gamma]$.

5.2. Examples. To understand why a compact quotient $\mathbb{R}^3/\Gamma$ could have as isometry group a torus of smaller dimension than 3, we can take a look at two examples in dimension two:

Example 5.2. The group $\mathbb{Z}^2$ is a discrete subgroup of $\text{Iso}(\mathbb{R}^2)$, such that the quotient $\mathbb{R}^2/\mathbb{Z}^2$, has the torus $N_{\mathbb{R}^2}(\mathbb{Z}^2)/\mathbb{Z}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ acting naturally by isometries, however the full isometry group $\text{Iso}(\mathbb{R}^2/\mathbb{Z}^2) \cong (\mathbb{R}^2/\mathbb{Z}^2) \rtimes D_4$ is bigger.

Example 5.3. We may extend the previous example to the group $\Lambda = \mathbb{Z}^2 \rtimes D_4$, which is a discrete subgroup of $\text{Iso}(\mathbb{R}^2)$, such that it is not completely contained in $\mathbb{R}^2$ and produces a compact quotient $\mathbb{R}^2/\Lambda$, homeomorphic to the 2-sphere $S^2$. To compute the isometry group, we observe the contentions

$$N_{\mathbb{R}^2}(\Lambda) = \{(n/2, n/2 + m) : n, m \in \mathbb{Z}\} \subset N_{\mathbb{R}^2}(\mathbb{Z}^2) = \mathbb{R}^2,$$

and $N_{\text{Iso}(\mathbb{R}^2)}(\Lambda) = \text{Aut}(\mathbb{Z}^2) \ltimes N_{\mathbb{R}^2}(\Lambda) \cong D_4 \ltimes N_{\mathbb{R}^2}(\Lambda)$, which gives us

$$N_{\text{Iso}(\mathbb{R}^2)}(\Lambda)/\Lambda \cong (D_4 \ltimes N_{\mathbb{R}^2}(\Lambda))/(D_4 \times \mathbb{Z}^2) \cong \mathbb{Z}_2.$$

This gives us a finite isometry group $\text{Iso}(\mathbb{R}^2/\Lambda) \cong \mathbb{Z}_2$. Observe that if $\sigma \in D_4$ and $v \in \mathbb{Z}^2$, then the commutator of these elements is $[\sigma, v] = \sigma(v) - v \in \mathbb{Z}^2$ and we can see that the commutator group $[\Gamma, \Gamma]$ contains a lattice subgroup of $\mathbb{R}^2$ which implies that $\Gamma/[\Gamma, \Gamma]$ is finite, verifying Theorem 5.1.

6. Nil geometry

6.1. Riemannian geometry of the Heisenberg group. If $\mathbb{F}$ is a commutative ring, denote by $H_3(\mathbb{F})$ the group of $3 \times 3$ upper triangular matrices over $\mathbb{F}$ with 1 in the diagonal, that is

$$H_3(\mathbb{F}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{F} \right\}.$$

The group $H_3(\mathbb{R})$ is a Lie group called the three dimensional Heisenberg group that fits into the exact sequence

$$1 \to \mathbb{R} \to H_3(\mathbb{R}) \to \mathbb{R}^2 \to 1,$$

where $\mathbb{R} \subset H_3(\mathbb{R})$ is its center. The three matrices

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

determine a canonical basis of the tangent space at the identity $T_1(H_3(\mathbb{R}))$, so that its translations by left-multiplications gives us a basis of left invariant
vector fields denoted by \( X_j \) with \( X_j(I) = e_j \). For a fixed element

\[
g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{H}_\mathbb{R},
\]

the vector fields at \( T_g(\mathbb{H}_\mathbb{R}) \) have expressions

\[
X_1(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2(g) = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3(g) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

If we consider the global coordinates

\[
\mathbb{R}^3 \to \mathbb{H}_\mathbb{R}, \quad (x, y, z) \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},
\]

then a vector \( v \in T_g(\mathbb{H}_\mathbb{R}) \) decomposes as

\[
v = v_1 e_1 + v_2 e_2 + v_3 e_3 = v_1 X_1(g) + v_2 X_2(g) + (v_3 - xv_2) X_3(g),
\]

so that the left-invariant metric in \( \mathbb{H}_\mathbb{R} \) having \( X_j(g) \) as an orthonormal basis is given in this coordinates as

\[
ds^2 = dx^2 + dy^2 + (dz - xdy)^2.
\]

Being left-invariant, this metric has \( \mathbb{H}_\mathbb{R} \) as a subgroup of isometries given by left multiplication

\[
L_g : \mathbb{H}_\mathbb{R} \to \mathbb{H}_\mathbb{R}, \quad L_g(h) = gh,
\]

for every \( g \in \mathbb{H}_\mathbb{R} \). Notice that there are other isometries that don’t come from left multiplication of \( \mathbb{H}_\mathbb{R} \). Such isometries form a group isometric to the orthogonal group \( O(2) \) generated by the reflection \( R(x, y, z) = (x, -y, -z) \) and the twisted rotations

\[
m : S^1 \times \mathbb{H}_\mathbb{R} \to \mathbb{H}_\mathbb{R}, \quad m_\theta(x, y, z) = (\rho_\theta(x, y), z + \eta_\theta(x, y)),
\]

where \( \rho_\theta \) is a rotation in the \( (x, y) \)-plane with angle \( \theta \), \( \eta_\theta \) is a polynomial function in \( x \) and \( y \) and trigonometric in \( \theta \).

The full isometry group \( Iso(\mathbb{H}_\mathbb{R}) \) can be described as a semi-direct product \( \mathbb{H}_\mathbb{R} \rtimes O(2) \), because

\[
m_\theta \circ L_g \circ m_\theta^{-1} = L_{m_\theta(g)}, \quad R \circ L_g \circ R = L_{R(g)},
\]

meaning that the exact exact sequence

\[
1 \to \mathbb{R} \to Iso(\mathbb{H}_\mathbb{R}) \to Iso(\mathbb{R}^2) \to 1,
\]

induced by the action on the quotient by the center \( \mathbb{H}_\mathbb{R}/\mathbb{R} \cong \mathbb{R}^2 \) splits off, see \cite{Sco83} for more details.

6.2. Examples. In this section we describe a series of ilustrative examples that capture the behaviour of every discrete subgroup of \( Iso(\mathbb{H}_\mathbb{R}) \).

Example 6.1. The group \( H_\mathbb{Z} \subset \mathbb{H}_\mathbb{R} \) is a discrete subgroup so that the exact sequence determining \( \mathbb{H}_\mathbb{R} \) induces the fiber-bundle structure

\[
\mathbb{R}/\mathbb{Z} \to \mathbb{H}_\mathbb{R}/H_\mathbb{Z} \to \mathbb{R}^2/\mathbb{Z}^2
\]
and thus $H_Z$ is a lattice subgroup of $H_R$ such that $H_R/H_Z$ is a compact Riemannian manifold. As the conjugation can be computed as
\[ g = (x, y, z), \quad g (n, m, p) g^{-1} = (n, m, p + xm - yn), \]
the normalizer in $H_R$ is $N_{H_R}(H_Z) = \{(n, m, p) : n, m \in \mathbb{Z}, p \in \mathbb{R}\}$. This gives us the isometries in the quotient
\[ S^1 \cong N_{H_R}(H_Z)/H_Z \hookrightarrow \text{Iso}(H_R/H_Z). \]

We consider now the normalizer of the Heisenberg group in $\text{Iso}(H_R)$. This can be determined as
\[ N_{\text{Iso}(H_R)}(H_Z) = \{(n, m, p) : n, m \in \mathbb{Z}, p \in \mathbb{R}\} \rtimes D_4, \]
where the Dihedral group $D_4$ is generated by the isometries
\[ m_{\pi/2}(n, m, p) = (-m, n, p - nm), \quad R(n, m, p) = (n, -m, -p), \]
so that what we get is $\text{Iso}(H_R/H_Z) \cong S^1 \rtimes D_4$.

We can modify this example by adding the dihedral group to the lattice, so that we have the fiber bundle structure
\[ \mathbb{R}/\mathbb{Z} \to H_R/(H_Z \times D_4) \to \mathbb{R}^2/(\mathbb{Z}^2 \times D_4) \cong S^2. \]
The discussion on Example 5.3 explains the last quotient in the sequence). Notice that we decreased the normalizer
\[ N_{\text{Iso}(H_R)}(H_Z \times D_4) = \{(n, m, l) : n, m, 2l \in \mathbb{Z}\} \rtimes D_4, \]
so that $\text{Iso}(H_R/(H_Z \times D_4)) \cong \mathbb{Z}_2$.

**Example 6.2.** Fix a positive integer $p \in \mathbb{N}$ and consider the lattice
\[ G_p = \left\{ \left( n, m, \frac{l}{p} \right) : n, m, l \in \mathbb{Z} \right\} \subset H_R, \]
which has as normalizer group in $H_R$ the group
\[ N_{H_R}(G_p) = \left\{ \left( \frac{n}{p}, \frac{m}{p}, r \right) : n, m \in \mathbb{Z}, r \in \mathbb{R} \right\}, \]
and normalizer group in $\text{Iso}(H_R)$, the group $N_{H_R}(G_p) \rtimes D_4$, with Dihedral group $D_4 = \langle m_{\pi/2}, R \rangle$ as before. The isometry group is characterized by the exact sequence
\[ 1 \to S^1 \to \text{Iso}(H_R/G_p) \to D_4 \rtimes (\mathbb{Z}_p \times \mathbb{Z}_p) \to 1 \]
and we recover the previous example by taking $p = 1$.

**Example 6.3.** Fix a positive integer $p \in \mathbb{N}$ and consider the lattice
\[ L_p = \left\{ \left( \frac{n}{2} + m, \frac{\sqrt{3}n}{2}, \frac{\sqrt{3}l}{2p} \right) : n, m, l \in \mathbb{Z} \right\} \subset H_R, \]
so that it has normalizer group in $H_R$
\[ N_{H_R}(L_p) = \left\{ \left( \frac{n}{2p} + \frac{m}{p}, \frac{\sqrt{3}n}{2p}, r \right) : n, m \in \mathbb{Z}, r \in \mathbb{R} \right\}. \]
As the group $L_p$ projects to a hexagonal lattice in $\mathbb{R}^2$, we should expect to have a Dihedral group $D_6$ normalizing $L_p$, however, the rotation $m_{\pi/3} : H_\mathbb{R} \to H_\mathbb{R}$ given by

$$m_{\pi/3}(x, y, z) = \left(\frac{1}{2}(x - \sqrt{3}y), \frac{1}{2}(y + \sqrt{3}x), z + \frac{\sqrt{3}}{8}(y^2 - x^2 - 2\sqrt{3}xy)\right),$$

doesn’t preserve $L_p$. To fix this, we must add a translation mixed with the rotation. Put $g = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right) \in H_\mathbb{R}$, then $\varphi = m_{\pi/3} \circ L_g \in N_{\text{Iso}(H_\mathbb{R})}(L_p)$, which can be verified using the relation $\varphi \circ L_h \circ \varphi^{-1} = L_{m_{\pi/3}(ghg^{-1})}$. We can describe the normalizer group of $L_p$ in $\text{Iso}(H_\mathbb{R})$ in terms of generators as

$$N_{\text{Iso}(H_\mathbb{R})}(L_p) = \langle L_g, \varphi, R : g \in N_{H_\mathbb{R}}(L_p) \rangle,$$

where $R(x, y, z) = (x, -y, -z)$, and so, we have the isometry group

$$1 \to S^1 \to \text{Iso}(H_\mathbb{R}/L_p) \to (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes D_6 \to 1,$$

where the dihedral group $D_6$ is generated by $\langle \varphi, R \rangle$.

**Example 6.4.** All the previous examples can be generalized as follows: Fix $p \in \mathbb{N}$ and $u, v \in \mathbb{R}^2$ linearly independent, so that $\Gamma = \{nu + mv : n, m \in \mathbb{Z}\} \subset \mathbb{R}^2$ is a lattice. If $(u, 0) \times (v, 0) = (0, 0, \lambda) \in \mathbb{R}^3$, then the group

$$M_p = \left\{\left(\frac{nu + mv, \lambda l}{p}\right) : n, m, l \in \mathbb{Z}\right\} \subset H_\mathbb{R}$$

is a lattice having normalizer group in $H_\mathbb{R}$

$$N_{H_\mathbb{R}}(M_p) = \left\{\left(\frac{n}{p}u + \frac{m}{p}v, r\right) : n, m \in \mathbb{Z}, \ r \in \mathbb{R}\right\}.$$ 

The lattice $\Gamma$ has an automorphism group $\text{Aut}(\Gamma) = \{0, \mathbb{Z}_2, D_4, D_6\}$, which is, if non-trivial, generated by a rotation with angle $\theta$ and a reflection. The whole normalizer group is given in terms of generators as

$$N_{\text{Iso}(H_\mathbb{R})}(M_p) = \langle L_g, \varphi, R : g \in N_{H_\mathbb{R}}(M_p) \rangle,$$

where $\varphi = m_g \circ L_w$. Here, $w = (w_0, 0) \in H_\mathbb{R}$ must be chosen so that if

$$w \times (u, 0) = (0, r_1), \quad w \times (v, 0) = (0, r_2),$$

then $m_g(u, r_1), m_g(v, r_2) \in M_p$. Thus, we have an isometry group of the quotient given by the exact sequence

$$1 \to S^1 \to \text{Iso}(H_\mathbb{R}/M_p) \to (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \text{Aut}(\Gamma) \to 1.$$

**Remark 6.5.** The previous examples give us the general strategy to compute the isometry group of a quotient $H_\mathbb{R}/G$, for $G \subset \text{Iso}(H_\mathbb{R})$ a discrete group of isometries. This strategy is as follows: $G$ projects to a discrete subgroup $\Gamma \subset \text{Iso}(\mathbb{R}^2)$ which has a finite index subgroup $\Gamma_0 \subset \mathbb{R}^2$, corresponding to a finite index subgroup $G_0 = G \cap H_\mathbb{R} \subset G$ and a lattice in $H_\mathbb{R}$. The normalizer of $G_0$ projects again a lattice in $\mathbb{R}^2$ and thus $\text{Iso}(H_\mathbb{R}/G_0)$ is an extension of a finite group $\mathbb{Z}_p \times \mathbb{Z}_p$ by $S^1$. The isometry group $\text{Iso}(H_\mathbb{R}/G)$ is just the previous group with an extra finite group of isometries, coming from the automorphisms of the lattice $\text{Aut}(\Gamma)$. This strategy fails if the projection to $\mathbb{R}^2$ is non-discrete, a possibility shown in the following two examples,
however, in the case where the quotient $H_R/G$ has finite volume, we will see that this pathological behaviour doesn’t occur.

Here we add two examples of discrete groups whose projected action onto $\mathbb{R}^2$ is non-discrete, these examples capture the general behaviour of discrete groups having this property as we will see in the next section.

**Example 6.6.** Consider $\varphi : \mathbb{N} \to S^1$, a homomorphism with dense image and $g = (0,0,1) \in H_R$ a generator of the center, so that the group

$$\{(g^n, \varphi(n)) : n \in \mathbb{N}\} \subset H_R \rtimes S^1 \cong \text{Iso}(H_R)$$

is a discrete subgroup of isometries of $H_R$ with dense projection onto $S^1 \cong SO(2)$ and in particular, with a non-discrete action on $\mathbb{R}^2$. In this example, the projected group leaves fixed the point $p = \frac{1}{1-\lambda} \in \mathbb{C} \cong \mathbb{R}^2$, where $\lambda = \varphi(1)$ and in particular, it is a group of rotations around such point.

**Example 6.7.** Given a scaling $0 < \varepsilon < 1$, consider the group generated by $(1,0,0), (\varepsilon,0,1) \in H_R \subset \text{Iso}(H_R)$ and $-1 \in S^1 \subset \text{Iso}(H_R)$. This is a discrete subgroup of $\text{Iso}(H_R)$, which projects to a non-discrete subgroup of $\text{Iso}(\mathbb{R}^2)$ leaving fixed the line $\{(x,0) : x \in \mathbb{R}\} \subset \mathbb{R}^2$.

**Remark 6.8.** The most symmetric lattices in $\mathbb{R}^2$ are the square and hexagonal lattices, having linear symmetry groups $D_4$ and $D_6$. Theorem 6.12 tells us that the generalizations of these lattices to $H_R$, described in Example 6.2 and Example 6.3 are the most symmetric finite volume quotients $H_R/G$, with isometry groups

$$1 \to S^1 \to \text{Iso}(H_R/G) \to (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes D \to 1,$$

with $D$ equal to $D_4$ and $D_6$ respectively, and $n \in \mathbb{N}$.

### 6.3. Classification of discrete subgroups of isometries

In this section $H_R$ denotes the Heisenberg Lie group considered as a Riemannian manifold with respect to the left-invariant metric constructed in the previous section. Here, we describe the conditions on which a discrete group on $\text{Iso}(H_R)$ induces a discrete action on the Euclidean plane $\mathbb{R}^2$.

**Proposition 6.9.** If $G$ is a discrete subgroup of isometries of $H_R$, then the exact sequence

$$1 \to \mathbb{R} \to \text{Iso}(H_R) \to \text{Iso}(\mathbb{R}^2) \to 1$$

induces an exact sequence

$$1 \to K \to G \to \Gamma \to 1,$$

where $\Gamma \subset \text{Iso}(\mathbb{R}^2)$ is either discrete or it is an abelian group leaving fixed either a point or a line. Moreover,

- (i) if $\Gamma \subset \text{Iso}(\mathbb{R}^2)$ has a finite index lattice, then $K \subset \mathbb{R}$ is a non-trivial discrete subgroup and
- (ii) if $\Gamma$ is non-discrete and leaves fixed a line, then there is a finite index subgroup of $G$ which is contained in $H_R$. 

Proof. Observe first that \( K = G \cap \mathbb{R} \) is a discrete subgroup of isometries of \( \mathbb{R} \) and so, if non-trivial, there is an isomorphism \( \mathbb{R}/K \cong S^1 \). The exact sequence 
\[ 1 \to S^1 \to H_{\mathbb{R}}/K \to \mathbb{R}^2 \to 1 \]
gives us
\[ 1 \to S^1 \to Iso(H_{\mathbb{R}})/K \to Iso(\mathbb{R}^2) \to 1, \]
which has compact kernel and thus, any discrete group in \( Iso(H_{\mathbb{R}})/K \) projects to a discrete group in \( Iso(\mathbb{R}^2) \). This argument tells us that if \( K \) is non-trivial then \( \Gamma \) is discrete in \( Iso(\mathbb{R}^2) \), because it is the projection of \( G/K \) with compact kernel, and \( G/K \) is always discrete in \( Iso(H_{\mathbb{R}})/K \).

Suppose from now on that \( K \) is trivial. If we identify \( \mathbb{R}^2 \cong \mathbb{C} \) as a Euclidean space, then we can realize the group of orientation preserving isometries of the plane \( \mathbb{R}^2 \) as the matrix group
\[ Iso^+(\mathbb{R}^2) \cong \left\{ \begin{pmatrix} \lambda & z \\ 0 & 1 \end{pmatrix} : \lambda, z \in \mathbb{C}, \ |\lambda| = 1 \right\} \]
with action
\[ \begin{pmatrix} \lambda & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda w + z \\ 1 \end{pmatrix}. \]

Observe that the restriction \( Iso^+(\mathbb{R}^2) \subset Iso(\mathbb{R}^2) \) reduces the discussion to a subgroup of index 2, which doesn’t alter the property of discreteness. We recall two important properties on commutators. First, commutators of two isometries give elements of pure translation part
\[ \left[ \begin{pmatrix} \lambda & z \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \beta & w \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & (z - w) + (\lambda w - \beta z) \\ 0 & 1 \end{pmatrix}, \]
which tells us that \([G,G]\) projects to a subgroup of \( Iso(\mathbb{R}^2) \) with only translation part, and so \([G,G] \subset H_{\mathbb{R}} \). Second, the commutator in \( H_{\mathbb{R}} \) satisfies the relation
\[ \left[ \begin{pmatrix} 1 & x & r \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & u & s \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 & xv - uy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]
which has the geometric interpretation: if two elements of \( H_{\mathbb{R}} \) project to the vectors \((x,y)\) and \((u,v)\), then its commutator is an element of the center \( \mathbb{R} = Z(H_{\mathbb{R}}) \) whose magnitude is the area of the projected vectors. As we are under the supposition that \( G \cap \mathbb{R} \) is trivial, the two previous relations on commutators tells us that \([G,G]\) is a commutative group and the corresponding projected group satisfies
\[ [\Gamma,\Gamma] \subset \left\{ \begin{pmatrix} 1 & rz_0 \\ 0 & 1 \end{pmatrix} : r \in \mathbb{R} \right\} \]
for some \( z_0 \in \mathbb{C} \). Suppose first that \( \Gamma \) is non-commutative. The commutation relation
\[ \left[ \begin{pmatrix} \lambda & z \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & (\lambda - 1)z_0 \\ 0 & 1 \end{pmatrix}, \]
and the hypothesis that all the translation elements of \([\Gamma,\Gamma]\) are linearly dependent give us the condition \( r = (1 - \lambda) \) for some \( r \in \mathbb{R} \) and as \(|\lambda| = 1\),
the only options are $\lambda = \pm 1$. As $\Gamma$ is non-commutative, there is at least one element that is not a translation, that is
\[
\begin{pmatrix}
-1 & z \\
0 & 1
\end{pmatrix} \in \Gamma,
\]
and without loss of generality, we can change $\Gamma$ by $h\Gamma h^{-1}$ (where $h$ is the translation by $1/2z$) so that in fact
\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix} \in \Gamma,
\]
this conjugation leaves $[\Gamma, \Gamma]$ invariant. Observe also that
\[
\left[ \begin{pmatrix}
\beta & w \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix} \right] = \begin{pmatrix}
1 & 2w \\
0 & 1
\end{pmatrix} \subset [\Gamma, \Gamma],
\]
implies by the same argument that $\beta = \pm 1$ and $w = sz_0$ for some $s \in \mathbb{R}$, and thus $\Gamma$ preserves the line generated by $z_0$. If on the other hand $\Gamma$ is commutative and contains an element of the form
\[
\begin{pmatrix}
\lambda & z \\
0 & 1
\end{pmatrix}, \quad \lambda \neq 1,
\]
this element has as a unique fixed point $\frac{x_0}{\lambda - 1}$. As $\Gamma$ is a commutative group, every element of $\Gamma$ must fix $\frac{x_0}{\lambda - 1}$, and thus, it consists of rotations around this point. If no such element exists, $\Gamma$ consists of elements with purely translation part, which tells us that $G \subset H_\mathbb{R}$. We observe that in this last case, two elements $a, b \in G$ which project to two linearly independent vectors in $\Gamma$ must satisfy that $e \neq [a, b] \in G \cap K$, which can’t happen by hypothesis, so $\Gamma$ is a subgroup of the group $\{r\omega : r \in \mathbb{R}\}$ for some $\omega \in \mathbb{C}$ and thus $\Gamma$ preserves the line generated by $\omega$. □

**Lemma 6.10.** Let $G$ be a discrete subgroup of isometries of $H_\mathbb{R}$ together with the projection to the isometry group of $\mathbb{R}^2$
\[
G \rightarrow \Gamma \subset Iso(\mathbb{R}^2).
\]
If $\Gamma$ preserves either a line or a point in $\mathbb{R}^2$, then the orbifold $H_\mathbb{R}/G$ has infinite volume.

**Proof.** Suppose first that $\Gamma$ preserves the line $\mathbb{R}v$, then as a consequence of either Bieberbach’s Theorem if $\Gamma$ is discrete, or as a consequence of the proof of Proposition 6.9 if $\Gamma$ is non-discrete, $G$ has a finite index subgroup that is contained in $H_\mathbb{R}$. Passing to a finite index subgroup doesn’t change the property of having finite co-volume so without loss of generality we may suppose that $G \subset H_\mathbb{R}$. There is a fundamental domain that has non-empty interior, given for example by the Dirichlet’s fundamental domain $\{q \in H_\mathbb{R} : d(q_0, q) < d(q_0, \gamma(q)), \gamma \in G \setminus \{e\}\}$, with respect to the Riemannian distance $d$, see [Rat19]. In particular there is a subset of the form
\[
D = \{(tv + sv^\perp, \lambda r_0) \subset \mathbb{C} \times \mathbb{R} : (s, t, \lambda) \in (-\varepsilon, \varepsilon)^3 + (s_0, t_0, \lambda_0)\} \subset H_\mathbb{R}
\]
such that no two elements of $D$ can be identified with an element of $G$. As $\Gamma$ preserves the line $\mathbb{R}v$, then we can see that no two elements of $D$ can be
identified with an element of $G$, where

$$\tilde{D} = \{(tv + sv^\perp, \lambda r_0) \subset \mathbb{C} \times \mathbb{R} : (s, t, \lambda) \in \mathbb{R} \times (-\varepsilon, \varepsilon)^2 + (0, t_0, \lambda_0)\}$$

but $\tilde{D} = \bigcup_j D_j$, where

$$D_j = \{(tv + sv^\perp, \lambda r_0) \subset \mathbb{C} \times \mathbb{R} : (s, t, \lambda) \in (-\varepsilon, \varepsilon)^3 + (s_j, t_0, \lambda_0)\} \subset H_{\mathbb{R}}$$

and every $D_j$ can be obtained by translating $D$ with an element of $H_{\mathbb{R}}$, thus

$$\text{Vol}(H_{\mathbb{R}}/G) \geq \text{Vol}(\tilde{D}) = \sum_j \text{Vol}(D_j) = \sum_j \text{Vol}(D) = \infty.$$ 

The second possibility is when $\Gamma$ is a commutative group preserving a point, that is, $\Gamma$ is conjugated to a subgroup of $SO(2)$. Again there is a fundamental domain of $G$ with non-empty interior and in particular, there is a subset

$$\Omega = \{(re^{i\theta}, sr_0) \subset \mathbb{C} \times \mathbb{R} : (r, \theta, s) \in (-\varepsilon, \varepsilon)^3 + (a, b, c)\} \subset H_{\mathbb{R}},$$

such that no two elements of $\Omega$ can be identified with an element of $G$. As $\Gamma$ acts only as rotations in the $\mathbb{C}$ plane, we can enlarge as before $\Omega$ to the subset

$$\tilde{\Omega} = \{(re^{i\theta}, sr_0) \subset \mathbb{C} \times \mathbb{R} : (r, \theta, s) \in \mathbb{R}_{>0} \times (-\varepsilon, \varepsilon)^2 + (0, b, c)\},$$

so that no two elements of $\tilde{\Omega}$ can be identified with an element of $G$. As before, we have a countable union of disjoints sets contained in $\tilde{\Omega}$ that are translated copies of $\Omega$, that is

$$\bigcup_i \Omega + (\omega_j, 0) \subset \tilde{\Omega}$$

and $\text{Vol}(H_{\mathbb{R}}/G) \geq \text{Vol}(\tilde{\Omega}) \geq \sum_j \text{Vol}(\Omega + (\omega_j, 0)) = \sum_j \text{Vol}(\Omega) = \infty$. □

**Lemma 6.11.** If $u, v \in \mathbb{R}^2$ are two linearly independent vectors, with $(u, 0) \times (v, 0) = (0, 0, \lambda) \in \mathbb{R}^3$ and $n \in \mathbb{N}$, $r, s \in \mathbb{R}$, then the group

$$G = \left\langle (u, r), (v, s), \left(0, 0, \frac{\lambda}{n}\right) \right\rangle \subset H_{\mathbb{R}}$$

is a lattice in $H_{\mathbb{R}}$. Conversely, every lattice in $H_{\mathbb{R}}$ can be obtained like this.

**Proof.** Observe that the center of $G$ is the subgroup $K = \left\{ \frac{\lambda p}{n} : p \in \mathbb{Z} \right\}$ and if $(x, y, z), (x, y, z') \in G$, then

$$(x, y, z)^{-1} \cdot (x, y, z') = (0, 0, z' - z) \in K,$$

so that for $k, l \in \mathbb{N}$ fixed, and

$$(u, r)^k \cdot (v, s)^l = (ku + lv, rk, sl)$$

the level set

$$\{(w, z) \in G : w = ku + lv\} = \left\{ (ku + lv, r_{n,m} + \frac{\lambda p}{n}) : p \in \mathbb{Z} \right\}$$

is discrete and thus, $G$ is a discrete subgroup of $H_{\mathbb{R}}$. If $\Gamma = \{ku + lv : k, l \in \mathbb{Z}\}$ denotes the projection of $G$ onto $\mathbb{R}^2$, then there is an exact sequence

$$1 \to K \to G \to \Gamma \to 1.$$
which induces the fiber bundle structure

\[ S^1 \cong \mathbb{R} / K \rightarrow H_{\mathbb{R}} / G \rightarrow \mathbb{R}^2 / \Gamma \cong S^1 \times S^1, \]

which tells us that \( H_{\mathbb{R}} / G \) is compact and thus, \( G \) is a lattice in \( H_{\mathbb{R}} \). Suppose now that \( L \subset H_{\mathbb{R}} \) is a lattice, then by Lemma 6.10 \( L \) projects to a lattice subgroup of \( \mathbb{R}^2 \), generated by two linearly independent vectors \( u', v' \in \mathbb{R}^2 \) such that \((u',0) \times (v',0) = (0,0,\lambda')\), with \( 0 \neq \lambda' \in \mathbb{R} \) and observe that if \( g = (u', r'), h = (v', s') \in L \), then their commutator is \([g,h] = (0,0,\lambda')\). As the intersection \( K' = G \cap Z(H_{\mathbb{R}}) \) is discrete and contains the non-trivial element \((0,0,\lambda') \in K'\), then there is an integer \( n' \in \mathbb{N} \) such that \( K' = \left\{ \frac{\lambda' n}{m} : p \in \mathbb{Z} \right\} \) and thus, the lattice \( L \) is generated by the set \( \left\{ (u',r'),(v',s'),(0,0,\frac{\lambda'}{n'}) \right\} \).

**Connected components**

**Theorem 6.12.** If \( G \subset Iso(H_{\mathbb{R}}) \) is a discrete subgroup such that \( H_{\mathbb{R}} / G \) has finite volume, then there is an exact sequence

\[ 1 \rightarrow C \rightarrow Iso(H_{\mathbb{R}} / G) \rightarrow F \rightarrow 1, \]

where \( F \) is a finite group, and \( C \subset S^1 \) is a closed subgroup. In particular, either \( Iso(H_{\mathbb{R}} / G) \) is finite, or it is a finite extension of \( S^1 \).

**Proof.** By proposition 6.9 and Lemma 6.10 the projection of \( G \) to \( Iso(\mathbb{R}^2) \) has a lattice \( \Gamma \subset \mathbb{R}^2 \) as a finite index subgroup. This is equivalent to the fact that \( L = G \cap H_{\mathbb{R}} \) is a lattice in \( H_{\mathbb{R}}' \) and a finite index subgroup in \( G \). By Lemma 6.11 there are \( u,v \in \mathbb{R}^2 \), \( \lambda,r,s \in \mathbb{R} \), with \( \lambda \neq 0 \), and \( n \in \mathbb{N} \), such that \( \Gamma = \{ ku + lv : k,l \in \mathbb{Z} \} \) and

\[ L = \left\langle (u,r),(v,s),\left(0,0,\frac{\lambda}{n}\right) \right\rangle \subset H_{\mathbb{R}}. \]

As seen in Example 6.11, the group \( N_{H_{\mathbb{R}}}(L) = \left\{ \left(\frac{nu + mw}{p}, v, r \right) : n,m \in \mathbb{Z}, r \in \mathbb{R} \right\} \) is the normalizer of \( L \) in \( H_{\mathbb{R}} \). Denote by \( Aut(\Gamma) \subset O(2) \) the subgroup that preserves the lattice \( \Gamma \) and observe that an element \( \varphi = \sigma \circ L_g \in Iso(H_{\mathbb{R}}) \) satisfies that \( \varphi \circ L_h \circ \varphi^{-1} = L_{\sigma(ghg^{-1})} \). As \( \sigma(ghg^{-1}) \) and \( \sigma(h) \) have the same projection onto \( \Gamma \), then if \( \varphi \) normalizes \( L \), \( \sigma \in Aut(\Gamma) \) and we have that

\[ 1 \rightarrow K \rightarrow G \rightarrow F' \times \Gamma \rightarrow 1, \]

for some subgroup \( F' \subset Aut(\Gamma) \) and \( K = L \cap \mathbb{R} \). As \( H_{\mathbb{R}} \) is normal in \( Iso(H_{\mathbb{R}}) \), we see that \( N_{Iso(H_{\mathbb{R}})}(G) \subset N_{H_{\mathbb{R}}}(G) \), and thus, by applying a trick as in Example 5.3, we may describe the greater normalizer as

\[ 1 \rightarrow H \rightarrow N_{Iso(H_{\mathbb{R}})}(G) \rightarrow F'' \times \Lambda \rightarrow 1, \]

with \( F'' \subset Aut(\Gamma) \) a finite group and \( \Lambda \subset \mathbb{R}^2 \) a lattice containing \( \Gamma \). Thus, the isometry group is calculated as

\[ 1 \rightarrow C = H / K \rightarrow N_{Iso(H_{\mathbb{R}})}(G) / G \rightarrow F = (F'' \times \Lambda) / (F' \times \Gamma) \rightarrow 1, \]

so that \( C \) is either finite, cyclic or \( S^1 \) and \( F \) is finite. \( \square \)
7. Spherical Geometry

This section is largely expository due to the fact that the verification of 2.8 in the spherical case consists of the comparison of the statement with the (fundamentally algebraic) classification of groups acting by isometries on three dimensional spherical manifolds and orbifolds. This concerns specifically the quotient orbifold of an action of a discrete group on a spherical three-manifold, that is, a quotient of the form

\[ M = S^3 / \Gamma, \]

for \( \Gamma \) a finite subgroup of \( O(4) \). The crucial point is that the classification of orbifolds up to orientation preserving isometry is equivalent to the classification of subgroups of \( O(4) \).

The following is a consequence of the classification of isometry groups of spherical 3-manifolds in [McC02], tables 2 and 3 in pages 173 and 176, relying on work of Mccullough and collaborators and ultimately going back to Seifert, Threlfall, Hopf and Hattori. See [HKMR12], chapter 1 for an account of these facts.

7.1. Classification.

Lemma 7.1. Up to finite subgroups, the isometry groups of spherical three manifolds are:

- \( SO(3) \).
- \( O(2) \).
- \( O(4) \).
- \( SO(4) \).
- \( SO(3) \).
- \( O(2) \times O(2) \).
- \( S^1 \times \mathbb{Z}/2 \cdot S^1 \).

In particular, these subgroups can be realized as closed subgroups of \( O(4) \).

For a complete list of isometry groups of spherical orbifolds, see Chapter 3 of [MS19].

Connected components

An important result by Hatcher [Hat83], originally conjectured by Smale states that the inclusion of the isometry group of \( S^3 \) into the group of diffeomorphisms is a homotopy equivalence.

The following result with contributions of many persons including (at least) Asano, Boileau, Bonahon, Birman, Cappell, Ivanov, Rubinstein, and Shaneson, is a consequence of research in mapping class groups and three-dimensional spherical manifolds. It is discussed with comments about attribution in [McC02], Theorem 1.1 in page 3.
Theorem 7.2. Let $M$ be a spherical manifold, then the inclusion of the group of isometries of $M$ into the group of diffeomorphisms induces a bijection on path components.

As of 2022, the following result in page 2 of [BK19] is a consequence of the study via Ricci flow methods of the homotopy type of the spaces of positive scalar curvature and the subspace of metrics which are locally isometric to either the round sphere $S^3$ or the round cylinder $S^2 \times \mathbb{R}$.

Theorem 7.3. Let $(M, g)$ be a riemannian manifold which is an isometric quotient of the three dimensional round sphere. Then, the inclusion of the isometry group into the diffeomorphism group is a homotopy equivalence.

The following theorem was proved in [MS19], using previous analysis of the authors of Seifert fibrations for spherical orbifolds. It is a consequence of tables 2 in page 1302, table 3 in page 1304 and table 4 in page 1308.

Theorem 7.4. Let $X$ be a spherical three-manifold, and let $\Gamma$ be a discrete group of $X$.

- The isometry groups of the orbifold $X/\Gamma$ are either closed subgroups of $SO_4$ or $PSO_4$, if the action is orientation preserving.
- The identity component of the isometry groups are $S^1$, $S^1 \times S^1$ or trivial for the orientation preserving case.

End of the proof of Theorem 2.8 for the spherical geometry. The result thus follows from Lemma 7.1.

\[ \square \]

8. $S^2 \times \mathbb{R}$ geometry.

A three dimensional manifold is said to have $S^2 \times \mathbb{R}$-geometry if its universal covering is homeomorphic to $S^2 \times \mathbb{R}$.

The determination of the discrete isometry groups of spaces with $S^2 \times \mathbb{R}$ geometry is a consequence of well-known facts, which we will gather here.

We recall the following result, proved in [KN96], Chapter VI, Theorem 3.5.

Theorem 8.1. Given a product of riemannian manifolds $M \times N$ with $M$ of constant sectional curvature and $N$ flat, the isometry group of $M \times N$ decomposes as a direct product $\text{Iso}(M) \times \text{Iso}(N)$. It follows that for a discrete subgroup $\Gamma \leq \text{Iso}(S^2 \times \mathbb{R}) \cong O(3) \times (\mathbb{R} \times \mathbb{Z}_2)$, the projection onto the second factor $\pi_\mathbb{R}(\Gamma) \leq \mathbb{R} \times \mathbb{Z}/2$ is a discrete subgroup.

We will use this splitting and the classification of finite groups acting on $S^2 \times \mathbb{R}$, which was proved in by Tollefson in page 61 of [Tol74], as follows:

Theorem 8.2. There exist only four three-manifolds covered by $S^2 \times \mathbb{R}$, namely: $S^2 \times S^1$, the non orientable $S^2$-bundle over $S^1$, $\mathbb{R}P^1 \times S^1$, and $\mathbb{R}P^2 \# \mathbb{R}P^2$. 

Moreover, the finite groups which act freely on $S^2 \times S^1$ are classified in [Tol74], Corollary 2. They are:

- $\mathbb{Z}/p$, producing quotient spaces homeomorphic to $S^2 \times S^1$ in the odd case, and $\mathbb{R}P^2$ in the even case as quotient space.
- $\mathbb{Z}/p \times \mathbb{Z}/2$, for $p$ even, producing a quotient space homeomorphic to $\mathbb{R}P^2$, and
- $D_n$, the dihedral group of order $2n$, producing $\mathbb{R}P^3 \# \mathbb{R}P^3$ as quotient space.

8.1. Classification of discrete groups of isometries. The previous example gives us the general behaviour for discrete groups of isometries on $S^2 \times \mathbb{R}$ as seen by the following Lemma

**Lemma 8.3.** If $\Gamma \subset Iso(S^2 \times \mathbb{R})$ is a discrete subgroup, then there is a finite group $F \subset O(3)$ and $\lambda \in \mathbb{R}$ such that the exact sequence

$$1 \to O(3) \to Iso(S^2 \times \mathbb{R}) \to \mathbb{R} \rtimes \mathbb{Z}_2$$

induces an exact sequence $1 \to F \to \Gamma \to L$, where $L$ is either $\lambda \mathbb{Z}$ or $\lambda \mathbb{Z} \rtimes \mathbb{Z}_2$.

**Proof.** As the group $O(3)$ is compact, the projection of the discrete group $\Gamma$ onto $Iso(\mathbb{R})$ is discrete, so it is of the form $\lambda \mathbb{Z}$ or $\lambda \mathbb{Z} \rtimes \mathbb{Z}_2$, for some $\lambda \in \mathbb{R}$. As $O(3)$ can be seen as a closed subgroup of $Iso(S^2 \times \mathbb{R})$, then the intersection of $\Gamma$ with $O(3)$ is a finite group, which we denote by $F$. The result thus follows from the product structure of $Iso(S^2 \times \mathbb{R})$. In fact, $\Gamma$ is generated by $F$, $\mathbb{Z}_2 \subset Iso(\mathbb{R})$ and the twisted translation subgroup $\{ (\sigma^n, n\lambda) \in O(3) \times \mathbb{R} : n \in \mathbb{N} \}$, for some $\sigma \in O(3)$. □

### Connected Components

**Theorem 8.4.** If $\Gamma \subset Iso(S^2 \times \mathbb{R})$ is a discrete subgroup, such that $(S^2 \times \mathbb{R})/\Gamma$ is compact, then $Iso((S^2 \times \mathbb{R})/\Gamma)$ is up to finite index, a closed subgroup of $SO(3) \times S^1$. In particular, the connected component of the identity of the isometry group of the quotient can only be one of the three possibilities:

$$SO(3) \times S^1, \quad S^1 \times S^1, \quad \text{or} \quad S^1.$$

**Proof.** By Lemma 8.3, the discrete group $\Gamma$ is generated by a finite group of $O(3)$ and a twisted translation as in Example 8.5. The isometry group is compact, so it has a finite number of connected components and by Proposition 8.10, the connected component of the identity can be computed using the centralizer, which always contains the $\mathbb{R}$-factor, so the result follows by examining the possible connected, closed subgroups of $SO(3)$. □

8.2. Example of a non discrete subgroup of isometries. We may observe that the projection onto the $S^2$ factor of a discrete group of isometries need not be discrete as the following example shows:

**Example 8.5.** If $\sigma \in SO(3)$ is a rotation with irrational angle along a fixed axis, so that the orbit $\{ \sigma^n(p) : n \in \mathbb{N} \}$ is dense in a circle, orthogonal to
the rotation axis, for almost every \( p \in S^2 \), then the group given by twisted translations

\[
\{(\sigma^n, n) \in O(3) \times \mathbb{R} : n \in \mathbb{N}\}
\]

is a discrete subgroup of \( Iso(S^2 \times \mathbb{R}) \) with non-discrete projection on \( Iso(S^2) \).

9. Sol Geometry

9.1. Riemannian Geometry of Three Dimensional Sol-manifolds.
Sol-geometry is given by the solvable Lie group of upper-triangular matrices

\[
S = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R} \right\},
\]

which decomposes as a semidirect product \( S = [S, S] \rtimes (S/[S, S]) \cong \mathbb{R}^2 \rtimes \mathbb{R} \). In global coordinates, the vector fields

\[
X_1(x, y, t) = (e^t, 0, 0), \quad X_2(x, y, t) = (0, e^{-t}, 0), \quad X_3(x, y, t) = (0, 0, 1),
\]

define a basis of left-invariant vector fields. We choose the left invariant Riemannian metric in \( S \) having this basis as orthonormal basis, so that in our global coordinates, the metric has the expresion

\[
ds^2 = e^{-2t}dx^2 + e^{2t}dy^2 + dt^2.
\]

The isometry group of this metric is generated by left translations

\[
L_g : S \to S, \quad L_g(h) = gh,
\]

and the group of reflections \((x, y, z) \to (\pm x, \pm y, \pm z)\), isomorphic to the Dihedral group \( D_4 \). In particular, \( Iso(S) \) has eight connected components, with the connected component of the identity isomorphic to \( S \) \([\text{Sco83}]\).

9.2. Existence of lattices. Consider the following

**Remark 9.1.** A Lie group admits a lattice subgroup if and only if it is unimodular \([\text{Rag07}]\), and so, not every solvable group admit lattice subgroups.

**Example 9.2.** A solvable group which is closely related to \( S \) considered here, is the group of orientation preserving affine transformations on \( \mathbb{R} \), given by

\[
\text{Aff}^+(\mathbb{R}) \cong \left\{ \begin{pmatrix} e^t & x \\ 0 & 1 \end{pmatrix} : x, t \in \mathbb{R} \right\}.
\]

We could try for example, to exponentiate the set

\[
\Lambda = \exp \left\{ \begin{pmatrix} n & m \\ 0 & 0 \end{pmatrix} : n, m \in \mathbb{N} \right\} = \left\{ \begin{pmatrix} e^n & (m/n)(e^n - 1) \\ 0 & 1 \end{pmatrix} : n, m \in \mathbb{N} \right\},
\]

however, such discrete set is not a subgroup and the group which generates is not discrete. The problem is that the group \( \text{Aff}^+(\mathbb{R}) \) is not unimodular, and in fact its modular function has the expression

\[
\Delta : \text{Aff}^+(\mathbb{R}) \to \mathbb{R}_+, \quad \Delta \left( \begin{pmatrix} e^t & x \\ 0 & 1 \end{pmatrix} \right) = e^t,
\]

which is non-trivial.
The solvable group $S$ is unimodular, so that it admits a lattice subgroup and an explicit way to construct a lattice is as follows: Consider a matrix $A \in SL_2(\mathbb{Z})$, such that $\text{tr}(A) > 2$ and the group 

$$
\Gamma_A = \left\{ \begin{pmatrix} A^n & Z \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}, \ Z \in M_{2\times1}(\mathbb{Z}) \right\} \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z}.
$$

Lemma 9.3. For every $A \in SL_2(\mathbb{Z})$, with $\text{tr}(A) > 2$, $\Gamma_A$ is conjugated in $SL_3(\mathbb{R})$ to a lattice subgroup in $S$, moreover, every lattice subgroup of $S$ is conjugated to one of such groups.

Proof. Suppose first that $A \in SL_2(\mathbb{Z})$, with $\text{tr}(A) > 2$, then there is a matrix $B \in SL_3(\mathbb{R})$ such that 

$$
BAB^{-1} = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix},
$$

for some $\lambda \neq 0$. We may define $A^t = B^{-1} \begin{pmatrix} e^{t\lambda} & 0 \\ 0 & e^{-t\lambda} \end{pmatrix} B$, so that $\Gamma_A$ is a discrete subgroup of the group 

$$
S_A = \left\{ \begin{pmatrix} A^t & Z \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}, \ Z \in M_{2\times1}(\mathbb{R}) \right\} \cong \mathbb{R} \rtimes \mathbb{R}^2,
$$

such that 

$$
1 \to \mathbb{R}^2/\mathbb{Z}^2 \to S_A/\Gamma_A \to \mathbb{R}/\mathbb{Z} \to 1,
$$

thus, $\Gamma_A$ is a lattice in $S_A$. Observe that we have an isomorphisms of Lie groups via the conjugation 

$$
S_A \to S, \quad X \mapsto \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} X \begin{pmatrix} B^{-1} & 0 \\ 0 & 1 \end{pmatrix},
$$

and thus, a lattice in $S$.

Before proceeding with the proof of this Lemma, we need to prove the following discrete projection Lemma:

Lemma 9.4. If $\Gamma \subset S$ is a discrete subgroup, then its projection $\Gamma \subset S/[S,S] \cong \mathbb{R}$ is also discrete.

Proof. An element $\gamma = (x,y,t)$, with $t \neq 0$, acts discretely by translations on the line $\left\{ \left( \frac{x}{1-e^t}, \frac{y}{1-e^t}, s \right) : s \in \mathbb{R} \right\} \subset S$, as $\gamma^n \left( \frac{x}{1-e^t}, \frac{y}{1-e^t}, s \right) = \left( \frac{x}{1-e^t}, \frac{y}{1-e^t}, s + nt \right)$. Thus, if $\Gamma$ is commutative, either $\Gamma \subset \mathbb{R}^2$ and its projection is trivial, or $\Gamma$ preserves a unique line on which it acts as translations and the action on this line is precisely the action on $\mathbb{R}$ of its projection, which must be discrete. If $\Gamma$ is non-commutative, then at least it has two elements $u = (a,b,0)$ and $\gamma = (x,y,t)$ with $t \neq 0$. Observe that if $b = 0$, then $\gamma u \gamma^{-1} = (e^t a - a, 0, 0)$ and by iterating conjugation we get a non-discrete subgroup of $\mathbb{R}^2 \cap \Gamma$ which is impossible and the same goes for the case $a = 0$. Thus $a,b \neq 0$ and $\Gamma \cap \mathbb{R}^2$ contains two linearly independent vectors, say $u$ and $v = \gamma u \gamma^{-1} = (e^t a, e^{-t} b, 0)$, which implies that $\Gamma \cap \mathbb{R}^2$ is cocompact in $\mathbb{R}^2$. $\Gamma/(\Gamma \cap \mathbb{R}^2)$ is discrete in $S/(\Gamma \cap \mathbb{R}^2)$ and the projection
\[ S/(\Gamma \cap \mathbb{R}^2) \to S/\mathbb{R}^2 \] has compact Kernel \( \mathbb{R}^2/(\Gamma \cap \mathbb{R}^2) \cong S^1 \times S^1 \), thus the corresponding projection of \( \Gamma/ (\Gamma \cap \mathbb{R}^2) \) into \( S/[S,S] \) is discrete. \( \square \)

Suppose now that \( \Gamma \subset S \) is a lattice subgroup, then by Lemma 9.4, the \( \Gamma \) projects to a non-trivial discrete group \( \mathbb{R} \), generated by an element \( e^{n\beta} \), with \( \beta \neq 0 \). The intersection \( \Gamma \cap \mathbb{R}^2 \) is a lattice, so that there are \( u, v \subset \mathbb{R}^2 \) linearly independent, such that \( \Gamma \cap \mathbb{R}^2 = \{ nu + mv : n, m \in \mathbb{Z} \} \). Take \( C \in GL_2(\mathbb{R}) \) the matrix sending \( \Gamma \cap \mathbb{R}^2 \) onto the canonical lattice \( \mathbb{Z}^2 \) and define the matrix \( A' = B \begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix} B^{-1} \). An element \( g = \begin{pmatrix} B^{-1}A'B & W \\ 0 & 1 \end{pmatrix} \in \Gamma \) must preserve \( \Gamma \cap \mathbb{R}^2 \), so that the element \( h = \begin{pmatrix} A' & BW \\ 0 & 1 \end{pmatrix} \in \Gamma \) must preserve \( \mathbb{Z}^2 \). Observe that the action of \( h \) in an element \( v = (n, m) \in \mathbb{Z}^2 \) is \( A'v + BW \in \mathbb{Z}^2 \), this implies that \( BW \in \mathbb{Z}^2 \) and \( A' \in SL_2(\mathbb{Z}) \). In particular, the group \( \Gamma \) is isomorphic to the group \( \Gamma_{A'} \) and the isomorphism is obtained by conjugation. \( \square \)

**Remark 9.5.** The existence of lattices in the Lie group \( S \) is related to the existence of a \( \mathbb{Q} \)-structure on \( S \). More precisely, if \( A \in SL_2(\mathbb{Z}) \), with \( \text{tr}(A) > 2 \), and \( c = \sqrt{\text{tr}(A)^2 - 4} \), then \( A \) is diagonalizable over the field \( \mathbb{Q}(c) \), that is, there is a matrix \( B \in SL_2(\mathbb{Q}(c)) \) such that \( BAB^{-1} \) is diagonal. If \( Q_{ij}(X) = X_{ij} \) is the linear map that gives the \((i,j)\)-entry, then the group \( G(k) = \left\{ \begin{pmatrix} X & Z \\ 0 & 1 \end{pmatrix} : Z \in M_{2 \times 1}(k), \ Q_{ij}(BXB^{-1}) = 0, \ i \neq j, i, j \in \{1, 2\} \right\} \), is algebraic subgroup of \( SL_3(\mathbb{R}) \), defined by polynomial equations with coefficients over \( \mathbb{Q}(c) \), such that \( G(\mathbb{R}) \cong S \) and \( G(\mathbb{Z}) = \Gamma_A \). Moreover, the Galois automorphism \( \sigma : \mathbb{Q}(c) \to \mathbb{Q}(c) \), defined by \( \sigma(c) = -c \), has a natural extension to automorphisms of matrices and polynomials, so that we have the embedding \( SL_3(\mathbb{Q}(c)) \to SL_3(\mathbb{R}) \times SL_3(\mathbb{R}) \), \( Y \mapsto (Y, \sigma(Y)) \), and a polynomial condition \( Q(Y) = 0 \) on \( Y \in SL_3(\mathbb{Q}(c)) \) is equivalent to the pair of polynomial conditions \( Q(Y) + \sigma(Q)(Y') = 0 \) and \( Q(Y')\sigma(Q)(Y') = 0 \) on \( (Y, Y') \in SL_3(\mathbb{R}) \times SL_3(\mathbb{R}) \), but the latter are polynomials with coefficients over \( \mathbb{Q} \) (this trick is called “restriction of scalars” \( \text{[Mor15]} \)).

**Lemma 9.6.** \( S \) has trivial center and the centralizer of a lattice group \( \Gamma \subset S \) is also trivial.

**Proof.** Take \( \gamma = (x, y, t) \) in the centralizer of \( \Gamma \) in \( S \), then as in the previous proposition \( \Gamma \cap [S, S] \) has a rank two subgroup, thus it contains at least a vector \( u = (a, b, 0) \) such that \( a, b \neq 0 \) and we have \( \gamma u \gamma^{-1} = (e^t a, e^{-t} b, 0) = (a, b, 0) \), which implies that \( t = 0 \). As \( \Gamma \) projects to a lattice group in \( S/[S, S] \cong \mathbb{R} \), then there is a \( \beta \in \Gamma \) such that \( \beta = (c, d, s) \) with \( s \neq 0 \) and thus \( \beta \gamma \beta^{-1} = (e^s x, e^{-s} y, 0) = \gamma = (x, y, 0) \).
which implies that $x = y = 0$ and $\gamma$ is the identity. A completely analogous computation shows that $S$ has trivial center. □

**Corollary 9.7.** If $\Gamma$ is a discrete group of isometries of $S$ such that $S/\Gamma$ has finite volume, then $S/\Gamma$ is compact and has finite isometry group.

**Proof.** As the connected component of the isometry group of $S$ is $S$ itself acting by left multiplications, $\Gamma$ is modulo a finite index subgroup a lattice in $S$ and it lies in an exact sequence

$$1 \to \Gamma_0 \to \Gamma \to \Gamma_1 \to 1$$

where $\Gamma_0 = \Gamma \cap [S, S]$ and $\Gamma/\Gamma_0 \cong \Gamma_1 \subset \mathbb{R}$. By Proposition 9.4 $\Gamma_1$ is a discrete subgroup, so this exact sequence induces a fiber bundle

$$\mathbb{R}^2/\Gamma_0 \to S/\Gamma \to \mathbb{R}/\Gamma_0,$$

so that $S/\Gamma$ has finite volume if and only if $\mathbb{R}^2/\Gamma_0$ and $\mathbb{R}/\Gamma_1$ are torus of the corresponding dimension and $S/\Gamma$ is compact. The isometry group of $S/\Gamma$ is a compact Lie group with connected component of the identity determined by the centralizer of $\Gamma$ in $S$ (Proposition 3.16) which is the trivial group by Lemma 9.6, thus the isometry group is a compact, zero-dimensional Lie group, i.e. finite. □

9.3. **Examples.**

**Example 9.8.** For $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $n \in \mathbb{N}$, consider the lattice $\Gamma_{A^n} = \mathbb{Z}^2 \rtimes_{A^n} \mathbb{Z}$. A matrix $Y = \begin{pmatrix} M & W \\ 0 & 1 \end{pmatrix} \in GL_3(\mathbb{R})$ normalizes $\Gamma_{A^n}$ if and only if $M = A^k$ for some $k \in \mathbb{Z}$ and $(I - A^n)W \in \mathbb{Z}^2$, so that if $\Lambda_n = (I - A^n)^{-1}\mathbb{Z}^2$, then the normalizer is $N_{Iso(S)}(\Gamma_{A^n}) = \mathbb{Z} \rtimes_A \Lambda_n$ and the isometry group is computed as

$$Iso(S/\Gamma_{A^n}) = (\Lambda_n/\mathbb{Z}^2) \rtimes_A \mathbb{Z}_n.$$

Three illustrative cases are

(i) $\Lambda_1 = \mathbb{Z}^2$, so that $Iso(S/\Gamma_A)$ is trivial;
(ii) $\det(I - A^2) = -5$, so that $\mathbb{Z}^2 \leq \Lambda_2 \leq \frac{1}{5}\mathbb{Z}^2$ and each contention is of index 5, in particular we have that $Iso(S/\Gamma_{A^2}) = \mathbb{Z}_5 \times \mathbb{Z}_2$;
(iii) $\Lambda_5 = \frac{1}{5}\mathbb{Z}^2$, so that $Iso(S/\Gamma_{A^5}) = (\mathbb{Z}_{11} \times \mathbb{Z}_{11}) \rtimes_A \mathbb{Z}_5$.

9.4. **Classification of free actions.**

**Remark 9.9.** The previous family of examples exhibits isometric actions of each finite cyclic group on a three dimensional solvmanifold. Moreover, we should notice that such actions are necessarily not free, since there exists a very rigid classification of free actions of finite groups on three dimensional manifolds with Nil and Sol structure, based on $p$-rank estimates and P.A. Smith Theory, [JL10], [KOS17].
10. Hyperbolic geometry

10.1. Normalizers of Fuchsian groups. Denote by $\mathbb{H}^n$ the n-dimensional hyperbolic space and recall that the isometry group $Iso(\mathbb{H}^n)$ is a non-compact semisimple Lie group that can be identified with the group $PO(n, 1)$. We begin by recalling the following properties of normalizers of discrete subgroups of isometries.

**Lemma 10.1.** If $\Gamma \subset Iso(\mathbb{H}^n)$ is a discrete subgroup such that $\mathbb{H}^n/\Gamma$ has finite volume, then the normalizer group

$$\Lambda = \{g \in Iso(\mathbb{H}^n) : ghg^{-1} = h, \forall h \in \Gamma \} \subset Iso(\mathbb{H}^n)$$

is discrete and $Iso(\mathbb{H}^n/\Gamma)$ is a finite group.

**Proof.** Passing to a finite cover doesn’t alter the outcome, so we may suppose that $\Gamma, \Lambda \subset O(n, 1)$. By Proposition 3.16, the connected component of $\Lambda$ lies inside the centralizer of $\Gamma$ in $O(n, 1)$. Let $g \in O(n, 1)$ centralizing $\Gamma$, then the polynomial

$$P_t : M_{n+1}(\mathbb{R}) \to M_{n+1}(\mathbb{R}), \quad P_t(X) = gXg^{-1} - X$$

vanishes at $\Gamma$ but by Borel’s density Theorem (see [Fur76]), $\Gamma$ is Zariski dense in $O(n, 1)$ and thus $P_t(O(n, 1)) = 0$ which tells us that $g$ lies in the center of $O(n, 1)$, which is finite. This tells us that $\Lambda$ is a discrete group that contains the lattice $\Gamma$, so $\Lambda$ is also a lattice in $O(n, 1)$. If $F_\Lambda, F_\Gamma \subset \mathbb{H}^n$ are fundamental domains of the groups $\Lambda$ and $\Gamma$ correspondingly, so we have that

$$|Iso(\mathbb{H}^n/\Gamma)| = |\Lambda/\Gamma| = Vol(F_\Gamma)/Vol(F_\Lambda) < \infty.$$  

□

**Remark 10.2.** The previous result is stated for hyperbolic manifolds in Corollary 3, Section 12.7 of [Rat19] and for hyperbolic orbifolds in [Rat99], where the hypotheses are that the discrete group is non elementary, geometrically finite and without fixed $m$-planes, for $m < n - 1$. In Lemma 10.1, we presented an argument using Zariski-density of the lattice group in $Iso(\mathbb{H}^n)$, which implies for example the non-existence of fixed $m$-planes. As seen in [Gre74], every finite group can be realized as the isometry group of a compact hyperbolic surface as in Lemma 10.1.

10.2. Rank of isometries and Lie groups acting on hyperbolic surfaces.

**Lemma 10.3.** If $\Sigma$ is a compact, orientable surface of genus $g \geq 2$, then there are no faithful actions of the compact group $S^1$ on $\Sigma$.

**Proof.** Suppose there is a faithful action $S^1 \times \Sigma \to \Sigma$, then perhaps after an averaging process, we may suppose that the action is isometric with respect to a Riemannian metric $h$. The existence of isothermal coordinates [UY17] tells us that there exists a complex structure in $\Sigma$ such that in holomorphic coordinates $z = x + iy$, the vector fields $\partial_x$ and $\partial_y$ are $h$-orthogonal. As the
\( S^1 \)-action is \( h \)-isometric, it preserves angles and orientation in the isothermal coordinates and thus it is an action by holomorphic transformations. By the uniformization Theorem, the universal cover of \( \Sigma \) is the hyperbolic semiplane \( \mathbb{H}^2 \subset \mathbb{C} \) and the holomorphic automorphisms of \( \Sigma \) lift to holomorphic automorphisms of \( \mathbb{H}^2 \) which also are isometric automorphisms with respect to the hyperbolic metric. As a consequence of this, we have that the \( S^1 \)-action preserves a hyperbolic metric in \( \Sigma \) which has finite volume, because \( \Sigma \) is compact, but this contradicts Lemma 10.1. □

**Corollary 10.4.** If \( \Sigma \) is a compact, orientable surface of genus \( g \geq 2 \) and \( h \) is a Riemannian metric in \( \Sigma \), then the isometry group \( \text{Iso}(\Sigma, h) \) is finite.

**Proof.** As \( \Sigma \) is compact, the isometry group \( G = \text{Iso}(\Sigma, h) \) is a compact Lie group. If \( g \) denotes the Lie algebra of \( G \), then for every \( X \in g \), the one parameter group \( \{\exp(tX)\} \) is a commutative group whose closure is a compact, commutative Lie group with connected component of the identity isomorphic to a product \( S^1 \times \cdots \times S^1 \). As a consequence of this and the fact that \( G \) has only has finitely many connected components, if \( G \) is infinite, then it has a closed subgroup isomorphic to \( S^1 \), but this is impossible as is shown in Lemma 10.3. □

10.3. **Non-Classification of finite hyperbolic groups of isometries.**

**Remark 10.5.** It is proved in [Koj88] that every finite group can be realized as the isometry group of a closed hyperbolic manifold of dimension three.

11. **Finer classification of 2-dimensional hyperbolic isometries.**

Recall that in dimension two, the group \( SL_2(\mathbb{R}) \) acts on \( \mathbb{H}^2 \) by isometries in the form of Möbius transformations, so that we have a realization of the orientation preserving isometries as \( \text{Iso}(\mathbb{H}^2) \cong PSL_2(\mathbb{R}) \).

11.1. **Classification of elements in \( SL_2 \) according to their fixed point sets on the visual compactification of \( \mathbb{H}^2 \).** We recall the classification of elements in \( SL_2(\mathbb{R}) \) An element \( A \in SL_2(\mathbb{R}) \) has as a characteristic polynomial \( p_A(x) = x^2 - tr(A)x + 1 \), and discriminant \( tr(A)^2 - 4 \). Thus, there are three dynamically different possibilities for the isometry of \( \mathbb{H}^2 \) generated by \( A \), characterized by the sign of \( tr(A) - 2 \):

- \( tr(A) - 2 > 0 \), where the matrix is conjugated to a diagonal matrix over \( \mathbb{R} \), and thus, the conjugated isometry is contained in the one parameter group of isometries generated by

\[
\left\{ \exp \left( \begin{array}{cc} t & 0 \\ 0 & -t \end{array} \right) = \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) : t \in \mathbb{R} \right\}.
\]

One isometry of this type is called hyperbolic, and the one-parameter group generated by this matrix is characterized by the property of having two fixed points in the boundary \( S^1 = \partial \mathbb{H}^2 \) and preserves a foliation determined by the two points and guided by the geodesic
that joins the two points (in the case of diagonal matrices, this is \(\{0, \infty\}\)).

- \(\text{tr}(A) - 2 = 0\), where the matrix is conjugated over \(\mathbb{R}\) to an upper triangular matrix, and thus, the conjugated isometry is contained in the one parameter group of isometries generated by

\[
\left\{ \exp \left( \begin{array}{cc} 0 & t \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right) : t \in \mathbb{R} \right\}.
\]

One isometry of this type is called parabolic, and the one-parameter group generated by this matrix is characterized by the property of having one fixed point in the boundary \(\partial \mathbb{H}^2\) and preserving the foliation of horocycles tangent to the fixed point (in the upper triangular case, the horocycles that are tangent to \(\infty\) are just horizontal lines).

- \(\text{tr}(A) - 2 < 0\), where the matrix is conjugated over \(\mathbb{R}\) to a rotation matrix, so that the conjugated isometry is contained in the one parameter group of isometries generated by

\[
\left\{ \exp \left( \begin{array}{cc} 0 & -t \\ t & 0 \end{array} \right) = \left( \begin{array}{cc} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{array} \right) : t \in \mathbb{R} \right\}.
\]

One isometry of this type is called elliptic, and the one-parameter group generated by this matrix is characterized by the property of having one fixed point in the interior of \(\mathbb{H}^2\) and preserving a foliation of circles.

### Lemma 11.1

If \(\alpha, \beta \in PSL_2(\mathbb{R})\) are two non-trivial elements, then

(i) \(\alpha\) and \(\beta\) commute if and only if \(\text{Fix}(\alpha) = \text{Fix}(\beta)\),
(ii) \(C(\alpha) = \{\beta \in PSL_2(\mathbb{R}) : \alpha\beta = \beta\alpha\} = \{\exp(tX) : t \in \mathbb{R}\}\), for some \(X \in \mathfrak{s}l_2(\mathbb{R})\). In particular \(C(\alpha)\) is isomorphic to either \(\mathbb{R}\) or \(S^1\).

**Proof.** Suppose \(\alpha\beta = \beta\alpha\), then \(\beta(\text{Fix}(\alpha)) = \text{Fix}(\alpha)\) and \(\alpha(\text{Fix}(\beta)) = \text{Fix}(\beta)\). If \(\alpha\) is parabolic or elliptic, then it has only one fixed point and thus \(\text{Fix}(\alpha) = \text{Fix}(\beta)\) and the same applies for \(\beta\) either parabolic or elliptic.

In the case where both \(\alpha\) and \(\beta\) are hyperbolic, we observe that \(\beta\) cannot interchange two distinct elements of the boundary \(S^1\), thus the property \(\beta(\text{Fix}(\alpha)) = \text{Fix}(\alpha)\) implies \(\text{Fix}(\alpha) = \text{Fix}(\beta)\). On the other hand, if \(\alpha\) and \(\beta\) have the same set of fixed points, then they are elements of the same one-parameter group, this is obvious when the fixed points are in standard configuration, that is \(\{0, \infty\}\), \(\{\infty\}\) or \(\{i\}\) according if the element is hyperbolic, parabolic or elliptic; and in general it can be seen via a conjugation of matrices by sending the fixed points to the standard configuration. In particular \(\alpha\beta = \beta\alpha\), because a one-parameter group is commutative and the result follows. \(\square\)

### 11.2. Discrete subgroups of Isometries of \(SL_2(\mathbb{R})\).

**Corollary 11.2.** If \(\Gamma \subset PSL_2(\mathbb{R})\) is a subgroup such that it has the identity element as an accumulation point (equivalently \(\Gamma\) is not a discrete subgroup) and \(\Lambda \subset \Gamma\) is a non-trivial, normal and discrete subgroup, then there exists \(\Gamma_1 \subset \Gamma\) commutative subgroup of finite index.
Proof. \( \Lambda \) is cyclic. As \( \Lambda \) is normal, for every \( \gamma \in \Gamma \), the conjugation induces an automorphism

\[
\Lambda \rightarrow \Lambda, \quad g \mapsto \gamma g \gamma^{-1},
\]

and as \( \Lambda \) is discrete and \( \Gamma \) has the identity element as an accumulation point, for every \( F \subset \Lambda \) finite set, there exist \( \gamma \in \Gamma \) close enough to the identity such that \( \gamma \neq e \) and \( \gamma g = g \gamma \), for every \( g \in F \). By the Lemma 11.1, the group generated by \( F \) is a discrete subgroup of the one-parameter group \( C(\gamma) \) and thus it is a cyclic group. For \( F_1 \subset F_2 \subset \Lambda \) any two distinct finite subsets, there are elements \( g_j \in \Lambda \) such that \( \langle g_j \rangle = \langle F_j \rangle \) and \( \langle F_1 \rangle \subset \langle F_2 \rangle \) which implies that \( g_1 = g_2^k \) for some \( k \) and in particular \( 0 < |g_2| < |g_1| \). Now \( \Lambda \) must be cyclic because otherwise we would have a sequence \( \{g_j\} \subset \Lambda \) obtained as the generators of subgroups generated by an increasing tower of finite subsets of \( \Lambda \) that converge to the identity.

Existence of \( \Gamma_1 \). Take \( \alpha \subset \Lambda \) a generator of the group and as \( \gamma \alpha \gamma^{-1} \) is again a generator of \( \Lambda \), for every \( \gamma \in \Gamma \), then the subgroup

\[
\Gamma_1 = \{ \gamma \in \Gamma : \gamma \alpha \gamma^{-1} = \alpha \}
\]

is a finite index subgroup of \( \Gamma \) (\( [\Gamma : \Gamma_1] \leq 2 \) if \( \Lambda \cong \mathbb{Z} \), and \( [\Gamma : \Gamma_1] \leq |\Lambda| \) if \( \Lambda \cong \mathbb{Z}/m\mathbb{Z} \)). Finally, by the Lemma 11.1 \( \Gamma_1 \) is commutative and the result follows.

11.3. Non-classification of finite groups of isometries.

Remark 11.3. It is proved in [Gre74] that every finite group can be realized as the isometry group of a compact hyperbolic surface.

12. \( \mathbb{H}^2 \times \mathbb{R} \)

Recall [KN96], Chapter VI, Theorem 3.5 that given a product of riemannian manifolds \( M \times N \) with \( M \) of constant sectional curvature and \( N \) flat, the isometry group of \( M \times N \) decomposes as a direct product, \( \text{Iso}(M) \times \text{Iso}(N) \). The following result gives us the isometry groups of finite volume quotients of \( \mathbb{H}^2 \times \mathbb{R} \) (see Theorem 13.6 for another proof):

12.1. Isometry groups of finite volume.

**Theorem 12.1.** If \( G \subset \text{Iso}(\mathbb{H}^2 \times \mathbb{R}) \) is a discrete subgroup such that \( (\mathbb{H}^2 \times \mathbb{R})/G \) has finite volume, then the group \( \text{Iso}(\mathbb{H}^2 \times \mathbb{R})/G \) is a finite extension of \( S^1 \)

**Proof.** Consider the exact sequence

\[
1 \rightarrow K \rightarrow G \rightarrow \Gamma \rightarrow 1,
\]

where \( K = G \cap \text{Iso}(\mathbb{R}) \) is a discrete subgroup of \( G \) and \( \Gamma \cong G/K \) is a subgroup of isometries of \( \mathbb{H}^2 \). If \( \Gamma \) is discrete as a subgroup of \( \text{Iso}(\mathbb{H}^2) \), then \( \mathbb{H}^2/\Gamma \) is an hyperbolic orbifold such that

\[
\mathbb{R}/K \rightarrow (\mathbb{H}^2 \times \mathbb{R})/G \rightarrow \mathbb{H}^2/\Gamma
\]
is a locally trivial fiber bundle and as \((\mathbb{H}^2 \times \mathbb{R})/G\) has finite volume, then 
\(\mathbb{R}/K \cong S^1\) and \(\Gamma\) is a Lattice subgroup of \(\text{Iso}(\mathbb{H}^2)\). In this case, we have an 
exact sequence of isometry groups

\[1 \to \text{Iso}(S^1) \to \text{Iso}(\mathbb{H}^2 \times \mathbb{R}/G) \to \text{Iso}(\mathbb{H}^2/\Gamma) \to 1,\]

where \(\text{Iso}(\mathbb{H}^2/\Gamma)\) is a finite group by Lemma 10.1 and thus \(\text{Iso}(\mathbb{H}^2 \times \mathbb{R}/G)\) is a finite extension of \(S^1\).

If \(\Gamma\) is not discrete as a subgroup of \(\text{Iso}(\mathbb{H}^2)\), we can see that the quotient
\((\mathbb{H}^2 \times \mathbb{R})/G\) cannot have finite volume. To see this, first observe that we 
have another exact sequence

\[1 \to \Lambda \to G \to \mathbb{L} \to 1,\]

where \(\Lambda = G \cap \text{Iso}(\mathbb{H}^2) \subset \Gamma\) is a discrete, normal subgroup and \(G/\Lambda \cong L \subset \text{Iso}(\mathbb{R})\). If \(\Lambda = 0\), then \(G \cong \mathbb{L}\) is commutative and thus \(\Gamma\) is commu-
tative. If instead \(\Lambda\) is non-trivial, then Corollary 11.2 tells us again that \(\Gamma\) 
is commutative (perhaps after passing to a finite index subgroup). In any 
case, \(G\) leaves a closed surface \(\zeta \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}\) fixed, where \(\zeta\) is a geodesic,
an horocycle or a circle (corresponding to the type of the iso-
metries of \(\Gamma\)). If \(\Gamma\) consists of parabolic or hyperbolic elements, then \(\Gamma\) acts discretely by 
Euclidean automorphisms in \(\zeta \times \mathbb{R} \cong \mathbb{R}^2\) so that by Bieberbach Theorem
\cite{Rat19}, \(\Gamma\) contains a finite index subgroup isomorphic to a sub-
group of \(\mathbb{Z}^2\) and in particular the fundamental domain of the \(G\)-action in \(\mathbb{H}^2 \times \mathbb{R}\) contains
a subset isometric to 
\[\{x + iy : a < x < b\} \times \{c, d\} \subset \mathbb{H}^2 \times \mathbb{R},\]

this implies that \((\mathbb{H}^2 \times \mathbb{R})/G\) doesn’t have finite volume. If \(\Gamma\) consists of 
elliptic elements, then \(G\) acts discretely by Euclidean automorphisms in 
\(\zeta \times \mathbb{R} \cong S^1 \times \mathbb{R}\), and thus as in the previous case, the \(G\)-action has a 
fundamental domain containing an open subset isomorphic to

\[\{(se^{i\theta}, r) \in \mathbb{D} \times \mathbb{R} : a < \theta < b, c < r < d\},\]

where \(\mathbb{D} \cong \mathbb{H}^2\) is the Poincaré disc model of the hyperbolic plane, and again
\((\mathbb{H}^2 \times \mathbb{R})/G\) doesn’t have finite volume. \(\square\)

13. \(\overline{SL}_2\) Geometry

13.1. Riemannian Geometry of \(PSL_2(\mathbb{R})\). Riemannian structure of Rec-
call that given a Riemannian manifolds \((M, g)\), there is a natural construc-
tion of a Riemannian metric tensor on the tangent bundle \(TM\) constructed
as follows: if \(\langle p, x \rangle \in TM\), and \((c(t), v(t)) \in TM\) is a smooth curve such that 
\(c(0) = p\) and \(v(0) = x\), then

\[\|(c'(0), v'(0))\|^2_{(p,x)} = \|d\pi_{(p,x)}((c'(0), v'(0)))\|^2_p + \left\|\frac{D}{dt}_{|t=0} v(t)\right\|^2_p,\]

where \(\pi : TM \to M\) is the projection, \(\frac{D}{dt} v(t)\) is the covariant derivative
along the curve \(c(t)\) and \(g(u, u)_p = \|u\|^2_p\). If \(X = c'(0)\) and \(Z = v'(0)\), in
local coordinates we have the formula
\[ \|(X, Z)\|^2_{(p,x)} = \|X\|^2_p + \|Z + X^j v^i \Gamma^k_{ij} \partial_k\|^2_p. \]

The vector \((X, Z)\) is called horizontal if \(c(t)\) is constant, and thus \(X = 0\), it is called vertical if it is orthogonal to every horizontal vector in which case \(Z = -X^j v^i \Gamma^k_{ij} \partial_k\). So, we have a decomposition in horizontal and vertical components as
\[ (X, Z) = (0, Z + X^j v^i \Gamma^k_{ij} \partial_k) + (X, -X^j v^i \Gamma^k_{ij} \partial_k). \]

If we take the global coordinates \((x, y)\) and so the projection \(\pi_{\text{SL}}\), the isometric action by Möbius transformations of \(\text{PSL}\) will define a left invariant metric in \(\text{SL}\). This action is transitive in the unitary tangent bundle \(T\mathbb{H}^2\), and the projection \(\pi: T\mathbb{H}^2 \to \mathbb{H}^2\) is just given by the projection in the first factor and we have global coordinates in each tangent plane \(\partial_1 = 1\) and \(\partial_2 = i\). If as before, \((X, Z)\) is a tangent vector to \(T\mathbb{H}^2\) at the point \((p, v) = (i, 1)\), then the orthogonal decomposition in horizontal and vertical components is given by
\[ (X, Z) = (0, Z - X^2 + X^1 i) + (X, X^2 - X^1 i). \]

The isometric action by Möbius transformations of \(\text{SL}(\mathbb{R})\) in \(\mathbb{H}^2\), induces the action in the tangent bundle
\[ \text{SL}(\mathbb{R}) \times T\mathbb{H}^2 \to T\mathbb{H}^2, \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \cdot (z, w) = \left(\frac{az + b}{cz + d}, \frac{w}{(cz + d)^2}\right). \]

This action is transitive in the unitary tangent bundle \(T^1\mathbb{H}^2 = \{(z, w) \in \mathbb{H}^2 : \|w\|_z = 1\}\), so the orbit of the point \((i, 1) \in T^1\mathbb{H}^2\) induces the diffeomorphism \(\phi: \text{PSL}(\mathbb{R}) \to T^1\mathbb{H}^2\) given explicitly by the formula
\[ \phi\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = \left(\frac{ai + b}{ci + d}, \frac{1}{(ci + d)^2}\right). \]

As this action is also isometric with respect to the previously defined metric, it will define a left invariant metric in \(\text{PSL}(\mathbb{R})\) that corresponds to an inner product in its tangent vector to the identity, naturally identified with the Lie algebra
\[ \mathfrak{sl}_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) : \text{tr}(A) = 0\}. \]

More precisely, if we consider the derivative \(d\phi\), we get the identification
\[ \Psi: \mathfrak{sl}_2(\mathbb{R}) \to T_{(1,1)}(T\mathbb{H}^2), \quad \Psi(X) = \frac{d}{dt}_{|t=0} \phi(\exp(tX)). \]

A basis of \(\mathfrak{sl}_2(\mathbb{R})\) is given by
\[ X_1 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right), \quad X_2 = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right), \quad X_3 = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right). \]
If $g_{i,j} = \exp(tX_j)$, then $\phi(g_{i,1}) = (e^{2ti}, e^{2ti})$, $\phi(g_{i,2}) = (i, e^{2ti})$ and

$$
\phi(g_{i,3}) = \left( \frac{\text{ch}(t)i + \text{sh}(t)}{\text{ch}(t) + i\text{sh}(t)}, \frac{1}{(\text{ch}(t) + i\text{sh}(t))^2} \right),
$$

where \( \text{ch}(t) \) and \( \text{sh}(t) \) denote the hyperbolic cosine and the hyperbolic sine correspondingly. If $\tilde{X}_j = \Psi(X_j)$, we have

$$
\tilde{X}_1 = (2i, 2), \quad \tilde{X}_2 = (0, 2i), \quad \tilde{X}_3 = (2, -2i),
$$

where we immediately see that $\tilde{X}_2$ is vertical and a direct computation tells us that $\tilde{X}_1$ and $\tilde{X}_3$ are horizontal and orthogonal. Thus $\{\frac{1}{2}X_1, \frac{1}{2}X_2, \frac{1}{2}X_3\}$ is an orthonormal basis in the corresponding inner product in $\mathfrak{sl}_2(\mathbb{R})$.

As the $PSL_2(\mathbb{R})$-action is given by holomorphic maps, it commutes with the action of $S^1$ given by rotations in each tangent plane

$$
S^1 \times T^1\mathbb{H}^2 \to T^1\mathbb{H}^2, \quad \eta \cdot (z, w) = (z, \eta w),
$$

as well as with the map $(z, w) \mapsto (\overline{z}, \overline{w})$. It is immediate that the previous maps act by isometries and in fact generate the whole isometry group. Thus, the isometry group $Iso(PSL_2(\mathbb{R}))$ is isomorphic to $PSL_2(\mathbb{R}) \times (S^1 \rtimes \mathbb{Z}_2)$, see [Sco83].

13.2. Groups of isometries of finite volume.

**Theorem 13.1.** If $\Gamma \subset Iso(PSL_2(\mathbb{R}))$ is a discrete group such that $PSL_2(\mathbb{R})/\Gamma$ has finite volume, then

$$
Iso(PSL_2(\mathbb{R})/\Gamma) \cong S^1 \rtimes F,
$$

where $F$ is a finite group.

**Proof.** Consider the projection into the simple factor

$$
P : Iso(PSL_2(\mathbb{R})) \to PSL_2(\mathbb{R}),
$$

as the Kernel of $P$ is compact and $\Gamma$ is a discrete subgroup, then $\Gamma_0 = P(\Gamma)$ is a discrete subgroup of $PSL_2(\mathbb{R})$ and $\Gamma_0 \cong \Gamma/F_0$, with $F_0 = \text{Ker}(P) \cap \Gamma$ a finite subgroup of $S^1 \rtimes \mathbb{Z}_2$. Observe that $\pi : PSL_2(\mathbb{R}) \to \mathbb{H}^2$ is a fiber bundle with fiber $S^1$ such that $\pi(\gamma x) = P(\gamma)\pi(x)$, so we have an induced projection

$$
\pi : PSL_2(\mathbb{R})/\Gamma \to \mathbb{H}^2/\Gamma_0,
$$

which implies that $\mathbb{H}^2/\Gamma_0$ has finite hyperbolic area. As we also have the identification $\pi : PSL_2(\mathbb{R})/\Gamma_0 \to \mathbb{H}^2/\Gamma_0$, we have that $\Gamma_0$ is a Lattice in $PSL_2(\mathbb{R})$. By Lemma [H1], we have that $\Gamma_0$ has finite index in $\Lambda = N_{PSL_2(\mathbb{R})}(\Gamma_0)$. Observe that if $\Gamma_1 = \Lambda/\Gamma_0$, then we have that $\Gamma \subset \Lambda \times S^1 \rtimes \mathbb{Z}_2$ and a bijection of sets

$$
(\Lambda \times S^1 \rtimes \mathbb{Z}_2)/\Gamma \cong (\Lambda \times S^1 \rtimes \mathbb{Z}_2)/\Gamma_0 \cong \Gamma/F_0,
$$

where $F$ is either $F_1$, or $F_1 \times \mathbb{Z}_2$, depending on whether $\Gamma$ contains the map $(z, w) \mapsto (\overline{z}, \overline{w})$ or not. Thus, we have that

$$
S^1 \subset N_{Iso(PSL_2(\mathbb{R}))}((\Lambda \times S^1 \rtimes \mathbb{Z}_2)/\Gamma \cong F \rtimes S^1,
$$

and the result follows. \(\square\)
13.3. Non-classification of finite group actions.

**Remark 13.2.** As seen in Remark 10.2, we can obtain every finite group as an isometry group of an hyperbolic surface, so that, the finite factor of the isometry group in Theorem 13.1, can be any finite group.

13.4. Isometries of the universal cover \( \widetilde{SL}_2(\mathbb{R}) \). The Lie group \( PSL_2(\mathbb{R}) \) is topologically the product \( S^1 \times \mathbb{R}^2 \), so that there is a simply connected Lie group denoted by \( \widetilde{SL}_2(\mathbb{R}) \) which is the topological universal cover of \( PSL_2(\mathbb{R}) \) and algebraically it is a non-split central extension by a cyclic group \( \mathbb{Z} \), more precisely, there is an exact sequence

\[
1 \to \mathbb{Z} \to \widetilde{SL}_2(\mathbb{R}) \to PSL_2(\mathbb{R}) \to 1,
\]

where \( \mathbb{Z} \subset \widetilde{SL}_2(\mathbb{R}) \) lies in the center. We can pull-back the metric tensor of \( PSL_2(\mathbb{R}) \), constructed in the previous section, to \( \widetilde{SL}_2(\mathbb{R}) \) to obtain the model of the homogeneous 3-dimensional geometry denoted by \( SL_2 \).

**Remark 13.3.** The isometry group of \( \widetilde{SL}_2(\mathbb{R}) \) can be characterized in three different ways. First, we have the homomorphism

\[
\widetilde{SL}_2(\mathbb{R}) \times \mathbb{R} \to \text{Iso}(\widetilde{SL}_2(\mathbb{R}))
\]

given by left and right multiplications, here \( \mathbb{R} \cong \widetilde{SO}(2) \) is the universal cover of the rotation group \( SO(2) \subset SL_2(\mathbb{R}) \), with Kernel \( \mathbb{Z} = \mathbb{R} \cap \widetilde{SL}_2(\mathbb{R}) \) being precisely the center of \( \widetilde{SL}_2(\mathbb{R}) \). The group \( \text{Iso}(\widetilde{SL}_2(\mathbb{R})) \) has two connected components and

\[
\text{Iso}(\widetilde{SL}_2(\mathbb{R}))(\mathbb{R}) \cong (\widetilde{SL}_2(\mathbb{R}) \times \mathbb{R}) / \mathbb{Z}
\]

is the component of the identity. In fact, we have an epimorphism

\[
\text{Iso}(\widetilde{SL}_2(\mathbb{R})) \to \text{Iso}(PSL_2(\mathbb{R})) \cong PSL_2(\mathbb{R}) \times (S^1 \times \mathbb{Z}_2),
\]

with kernel isomorphic to \( \mathbb{Z} \), however, the group \( \text{Iso}(\widetilde{SL}_2(\mathbb{R})) \) is no longer a product group. The left projection of the previous product gives us the second description in terms of a short exact sequence

\[
1 \to \mathbb{R} \to \text{Iso}(\widetilde{SL}_2(\mathbb{R}))(\mathbb{R}) \to PSL_2(\mathbb{R}) \to 1,
\]

and if we consider the groups \( \widetilde{SL}_2(\mathbb{R}) \) and \( \mathbb{R} \) as closed subgroups of \( \text{Iso}(\widetilde{SL}_2(\mathbb{R})) \), then we have the third description

\[
\text{Iso}(\widetilde{SL}_2(\mathbb{R}))(\mathbb{R}) = L(\widetilde{SL}_2(\mathbb{R})) R(\mathbb{R}),
\]

where \( L(\cdot) \) and \( R(\cdot) \) represent left and right multiplications in the group \( \widetilde{SL}_2(\mathbb{R}) \).

A discrete subgroup \( \Gamma \subset \text{Iso}(PSL_2(\mathbb{R})) \) can be lifted to a discrete subgroup \( \widetilde{\Gamma} \subset \text{Iso}(\widetilde{SL}_2(\mathbb{R})) \), so that \( \widetilde{SL}_2(\mathbb{R})/\widetilde{\Gamma} \cong PSL_2(\mathbb{R})/\Gamma \) and thus, we can compute \( \text{Iso}(\widetilde{SL}_2(\mathbb{R}))/\widetilde{\Gamma} \) with Theorem 13.1, however, not every discrete group of \( \text{Iso}(\widetilde{SL}_2(\mathbb{R})) \) can be obtained this way. In the next section we discuss the proof in the general setting for discrete groups of isometries in \( \widetilde{SL}_2(\mathbb{R}) \).
The following Lemma is well known and holds for every Lie group, but we include a proof of the case we need for the sake of completeness.

**Lemma 13.4.** If $G$ is a Lie group locally isomorphic to $\mathbb{R} \times SL_2(\mathbb{R})$, for example $G$ can be the isometry group of $SL_2(\mathbb{R})$ or $\mathbb{H}^2 \times \mathbb{R}$, then there exists a neighborhood of the identity $U \subset G$ such that $[U, U] \subset U$.

**Proof.** Observe first that this is a local property, so we only need to prove this for linear groups. As the $\mathbb{R}$ factor lies in the center, we have that

$$[g_0h_0] = [g, h], \quad \forall g_0, h_0 \in \mathbb{R}$$

and thus we only need to prove this for $SL_2(\mathbb{R})$. The commutator

$$\left[\begin{pmatrix} a & x \\ y & b \end{pmatrix}, \begin{pmatrix} c & z \\ w & d \end{pmatrix}\right] = \begin{pmatrix} t_1 & t_3 \\ t_4 & t_2 \end{pmatrix},$$

is defined by the relations

- $t_1 = 1 + xy + zw + xyzw + wxac + w^2x^2 - adxw - a^2zw + bxwd - yd^2x - zyd$,
- $t_2 = 1 + xy + zw + xyzw - xwbc - c^2xy + ayzc - zb^2w - zbcy + zybd + z^2y^2$,
- $t_3 = xac(d - c) - cx^2w + acz(a - b) - xwbd + zydx + z^2ya$,
- $t_4 = w^2xb + wxcy - bdw(b - a) - awyz + bdy(c - d) - dy^2z$.

So that if $0 \leq |x|, |y|, |z|, |w| < \varepsilon$ and $1 - \varepsilon < a, b, c, d < 1 + \varepsilon$, then there is a constant $C > 0$ independent of $\varepsilon$ such that $|t_3|, |t_4| < C\varepsilon^2$ and $|t_1 - 1|, |t_2 - 1| < C\varepsilon^2$. Thus, by choosing $\varepsilon > 0$ such that $C\varepsilon^2 < \varepsilon$, the neighborhood

$$U_\varepsilon = \left\{ \begin{pmatrix} a & x \\ y & b \end{pmatrix} : |x|, |y| < \varepsilon, |a - 1|, |b - 1| < \varepsilon \right\}$$

is stable under taking commutators. \qed

### 13.5. Isometry groups of finite volume.

**Proposition 13.5.** Let $H$ be a Lie group which is a central extension of $PSL_2(\mathbb{R})$ of the form

$$1 \rightarrow \mathbb{R} \rightarrow H \rightarrow PSL_2(\mathbb{R}) \rightarrow 1.$$

If $G \subset H$ is a discrete subgroup with induced exact sequence

$$1 \rightarrow K \rightarrow G \rightarrow \Gamma \rightarrow 1,$$

with $K \subset \mathbb{R}$, then either $\Gamma \subset PSL_2(\mathbb{R})$ is discrete or is an abelian subgroup leaving fixed a point, a geodesic or a horocycle in $\mathbb{H}^2$.

**Proof.** Denote by $p : H \rightarrow PSL_2(\mathbb{R})$ the projection and consider $U \subset H$ a neighborhood of the identity such that $[U, U] \subset U$ and $U \cap G = \{e\}$. We have that the group $L = \langle p(U) \cap \Gamma \rangle$ is a commutative subgroup of $PSL_2(\mathbb{R})$, to see why this is true take two elements $\alpha, \beta \in G$ such that $p(\alpha), p(\beta) \in p(U)$, then we may write those elements as $\alpha = \alpha_0\alpha_1$ and $\beta = \beta_0\beta_1$, where

$$\alpha_1, \beta_1 \in \mathbb{R}, \quad \alpha_0, \beta_0 \in U.$$
As \( \mathbb{R} \) lies in the center of \( H \) we have that \( [\alpha_0, \beta_0] = [\alpha, \beta] \in G \cap U \) and thus \( \alpha \) and \( \beta \) commute. Now, for every \( \alpha \in G \), choose a neighborhood of the identity \( U_\alpha \subset G \) such that \( [\alpha, U_\alpha] \subset U \), so that the elements of \( \Gamma \cap p(U_\alpha) \) commute with \( p(\alpha) \) (same argument as with the commutativity of \( L \)). Suppose that \( \Gamma \) is non-discrete, then \( L \) is a non-trivial commutative subgroup and for every \( \gamma = p(\alpha) \in \Gamma \), we have that \( \Gamma \cap p(U_\alpha) \) is a non-trivial subset that generates the group \( L \) and commutes with \( \gamma \). So, \( \Gamma \) commutes with \( L \) and thus, there exists an element \( X \in \mathfrak{sl}_2(\mathbb{R}) \) such that \( \Gamma \subset \exp(tX) : t \in \mathbb{R} \) and \( \Gamma \) leaves fixed a point, a geodesic or a horocycle, depending on the type of \( X \).

\[ \Gamma \subset \mathbb{L} = \{ \exp(tX) : t \in \mathbb{R} \} \]

and \( \Gamma \) leaves a fixed point, a geodesic or a horocycle, depending on the type of \( X \).

**Theorem 13.6.** If \( G \) is a discrete subgroup of isometries of \( X \) either \( \tilde{SL}_2(\mathbb{R}) \) or \( \mathbb{H}^2 \times \mathbb{R} \) such that \( X/G \) has finite volume, then the isometry group \( \text{Iso}(X/G) \) is a finite extension of \( S^1 \).

**Proof.** The exact sequence

\[ 1 \to \mathbb{R} \to \text{Iso}(X) \to PSL_2(\mathbb{R}) \to 1 \]

induces the sequence

\[ 1 \to K \to G \to \Gamma \to 1, \]

if \( \Gamma \) is non-discrete, then it preserves a geodesic, a point or a horocycle by Proposition 13.5 and we can see that \( X/G \) doesn’t have finite volume (as we did in Theorem 12.1). So, \( \Gamma \) is a discrete subgroup of isometries of the hyperbolic plane and we have a fiber bundle structure

\[ \mathbb{R} \to X \to \mathbb{H}^2 \]

so that the volume form decomposes as

\[ \int_{\mathbb{R}} \int_{\mathbb{H}^2} f \, d\mu dt = \int_X f \, dvol_X \]

where \( d\mu \) is the hyperbolic area form. If \( D \subset \mathbb{H}^2 \) is a fundamental domain of \( \Gamma \), then \( \pi^{-1}(D) = \hat{D} \) is such that \( g\hat{D} \cap \hat{D} \neq \emptyset \) only for \( g \in K = G \cap \mathbb{R} \). Thus for \( \Omega \subset \mathbb{R} \) fundamental domain of \( K \) in \( \mathbb{R} \) we have that \( \Omega \times D \) is a fundamental domain for \( G \) which implies that

\[ \text{Vol}(X/G) \geq \int_{\Omega} \int_D \xi = |\Omega| \times \mu(D), \]

and we have that \( \mu(A) < \infty \) and \( |\Omega| < \infty \) which implies that \( K = \mathbb{Z} \). Take \( \tilde{N} = N_{\text{Iso}(X)}(G) \) and \( N = N_{PSL_2(\mathbb{R})}(\Gamma) \), so that we have the exact sequence

\[ 1 \to N_0 \to \tilde{N} \to N \to 1 \]

(because \( gGg^{-1} = G \) projects to \( \overline{\Gamma} \overline{g}^{-1} = \Gamma \) and \( N_0 = \mathbb{R} \) because \( \mathbb{R} \) normalizes \( \overline{SL}_2(\mathbb{R}) \)), this sequence induces the exact sequence

\[ 1 \to N_0/\mathbb{Z} \to \tilde{N}/G \to N/\Gamma \to 1 \]

(to see that this sequence is exact observe that \( \pi(gG) = \overline{\Gamma} \) is well defined and surjective, the condition \( \pi(gG) = \Gamma \) holds if and only if \( \overline{\Gamma} \in \Gamma \) and thus \( g = [A, r] \) with \( A \in \Gamma \), this is because there is an element \( h = [A, s] \in G \),
thus \( gG = gh^{-1}G \) and \( gh^{-1} \in i(N_0/Z) \). This implies that the kernel of \( \pi \) is \( i(N_0/Z) \). Finally \( i(rZ) = G \) if and only if \( r \in G \), but \( G \cap \mathbb{R} = \mathbb{Z} \), so that \( r\mathbb{Z} = \mathbb{Z} \) and thus \( i \) is injective). This exact sequence can be written as
\[
1 \to S^1 \to \text{Iso}(X/G) \to \text{Iso}(\mathbb{H}^2/\Gamma) \to 1
\]
which implies the result because of the Lemma 10.1.

14. Corollaries of the Main Theorem

14.1. Actions of \( SL_k(Z) \) on aspherical three dimensional manifolds by isometries. We have the following affirmative solution to Problem 2.1.

**Theorem 14.1.** Any group action by isometries of \( SL_k+1(Z) \), with \( k \geq 3 \), on a closed, aspherical 3-manifold factors through a finite group.

14.2. Discrete groups acting with a sufficiently collapsed Alexandrov space as quotient.

**Theorem 14.2.** Assume that a discrete group \( \Gamma \) acts by isometries on the three dimensional Alexandrov space \( X \) such that the quotient \( X/\Gamma \) is sufficiently collapsed with parameters \( d, \) and \( \epsilon \). Then, Theorem 2.8 together with the geometrization of 3-dimensional Alexandrov spaces provide a classification of the possible such \( \Gamma \) within the lattices in the isometry groups.

14.3. Hilbert Smith Conjecture for three dimensional Alexandrov spaces. Let us recall Theorem 3.10.

**Theorem 14.3 (Par19, Par13).** For every prime \( p \), there are no faithful actions by homeomorphisms of the \( p \)-adic group \( \hat{\mathbb{Z}}_p \) on a topological manifold of dimension \( n \leq 3 \).

As seen in section 4.2, an Alexandrov space \( X \) has a closed subset \( S_X \), corresponding to topologically singular points and such that the set of regular points \( R_X = X \setminus S_X \) is an open-dense subset, having the structure of a topological manifold. An action by homeomorphisms on \( X \) must preserve the decomposition \( X = S_X \cup R_X \) and a continuous action of \( \hat{\mathbb{Z}}_p \) which is trivial on the regular points, is trivial on the whole space \( X \).

Hence, the weaker version of the \( p \)-adic Hilbert-Smith conjecture for Alexandrov spaces holds.

A consequence of Theorem 3.10 gives us

**Theorem 14.4.** If \( G \) is a locally compact, topological group, acting faithfully on a three dimensional Alexandrov space by homeomorphisms, then \( G \) is a Lie group.

**Remark 14.5.** As observed in previous section, there is a subset of metrically regular points which admits a compatible Riemannian metric, constructed in [OS94]. Thus, we have as a consequence of Theorem 3.9, that the \( p \)-adic group \( \hat{\mathbb{Z}}_p \) cannot act faithfully by bi-Lipschitz homeomorphisms.
However, we should be careful, as the set of metrically singular points can be dense, as seen in an example constructed in [OS94] as a limit of Alexandrov spaces, using baricentric subdivisions of a tetrahedron.

14.4. Non Existence of actions of Higher Rank Lattices by Isometries on three dimensional Geometric Orbifolds.

**Theorem 14.6.** Let $\Gamma$ be a higher rank lattice acting by isometries on a finite volume, three dimensional orbifold $X$ (modelled over a homogeneous 3-manifold $X$), then the action factors through a finite group if either:

- $X$ is aspherical or,
- $\Gamma$ is non-uniform.

As an example of this, we have $\Gamma = \text{SL}_r(\mathbb{Z})$ with $r \geq 3$.

14.5. Characterization of Higher Rank Lattices actions by isometries on three dimensional spherical Orbifolds.

**Remark 14.7.** As a consequence of the previous discussion, for every semisimple Lie group $G$ such that $G \times \text{SO}(4)$ is isotypic\(^1\), there is an irreducible lattice $\Gamma \subset G$ and an homomorphism $\Gamma \to \text{SO}(4)$ with dense image. In particular, such lattice acts by isometries on the round sphere $S^3$ with dense orbits. This tells us that there is no restriction on the dimension of the class of higher rank lattices which can act on the round sphere, but the type of such lattice is restricted. The same applies to the 3-orbifolds of the type $S^2 \times S^1$,

\[^{1}\text{For example, any product } G = G_1 \times \cdots \times G_k, \text{ where each } G_j \text{ is one of } \text{SO}(3,1), \text{SO}(2,2) \text{ or } \text{SO}(4,\mathbb{C}).\]"
**Corollary 14.10.** Let $X$ be a geometric 3-orbifold of finite volume, and $\Gamma$ a non-cocompact higher rank lattice in a semisimple Lie group $G$, then any action of $\Gamma$ in $X$ factors through a finite group.

As a particular example of the previous, any action of $SL_n(\mathbb{Z})$ in a geometric 3-orbifold of finite volume, factors through a finite group.

**Corollary 14.11.** Let $X$ be a geometric 3-orbifold of finite volume, then $X$ admits an isometric action of a higher rank lattice $\Gamma \subset G$ if and only if the group $\text{Iso}(X)$ contains the group $SO(3)$. Moreover, the semisimple Lie group $G$ is isotypic of type $SO(3)$ and the lattice is uniform.

Observe that the group $SO(4)$ factors locally as the product $SO(3) \times SO(3)$ and in fact, there is a copy of $SO(3)$ inside $SO(4)$, so that the previous Corollary includes at the same time examples like $X = S^3/\Lambda$ and $X = (S^2 \times \mathbb{R})/\Lambda$.

**References**

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[BFH16] Aaron Brown, David Fisher, and Sebastian Hurtado. Zimmer’s conjecture: subexponential growth, measure rigidity, and strong property (t). *Preprint*, 2016.

[BFH20] Aaron Brown, David Fisher, and Sebastian Hurtado. Zimmer’s conjecture for actions of $SL(m, \mathbb{Z})$. *Invent. Math.*, 221(3):1001–1060, 2020.

[BFH21] Aaron Brown, David Fisher, and Sebastian Hurtado. Zimmer’s conjecture for non-uniform lattices and escape of mass, 2021.

[BK02] Mario Bonk and Bruce Kleiner. Rigidity for quasi-Möbius group actions. *J. Differential Geom.*, 61(1):81–106, 2002.

[BK19] Richard Bamler and Bruce Kleiner. Ricci flow and contractibility of spaces of metrics. arXiv:1909.08710, 2019.

[BM46] Salomon Bochner and Deane Montgomery. Locally compact groups of differentiable transformations. *Ann. of Math. (2)*, 47:639–653, 1946.

[BNNZ21] Noé Bárcenas and Jesús Núñez Zimbrón. On topological rigidity of Alexandrov 3-spaces. *Rev. Mat. Iberoam.*, 37(5):1629–1639, 2021.

[Boc46] S. Bochner. Vector fields and Ricci curvature. *Bull. Amer. Math. Soc.*, 52:776–797, 1946.

[Bou09] Marc Bourdon. Quasi-conformal geometry and Mostow rigidity. In *Géométries à courbure négative ou nulle, groupes discrets et rigidités*, volume 18 of *Sémin. Congr.*, pages 201–212. Soc. Math. France, Paris, 2009.

[BRW61] G. E. Bredon, Frank Raymond, and R. F. Williams. $p$-adic groups of transformations. *Trans. Amer. Math. Soc.*, 99:488–498, 1961.

[BZ07] A. V. Bagaev and N. I. Zhukova. The isometry groups of Riemannian orbifolds. *Sibirsk. Mat. Zh.*, 48(4):723–741, 2007.

[DH20] Bertrand Deroin and Sebastian Hurtado. Non left-orderability of lattices in higher rank semi-simple lie groups, 2020. arXiv:2008.10687.

[DVdW28] D. van Dantzig and B. L. Van der Waerden. Über metrisch homogene Räume. *Abh. Math. Semin. Univ. Hamb.*, 6:367–376, 1928.

[Fis11] David Fisher. Groups acting on manifolds: around the Zimmer program. In *Geometry, rigidity, and group actions*, Chicago Lectures in Math., pages 72–157. Univ. Chicago Press, Chicago, IL, 2011.

[Fis20] David Fisher. Recent developments in the Zimmer program. *Notices Amer. Math. Soc.*, 67(4):492–499, 2020.
[FM98] Benson Farb and Howard Masur. Superrigidity and mapping class groups. *Topology*, 37(6):1169–1176, 1998.

[FS00] Benson Farb and Peter Shalen. Lattice actions, 3-manifolds and homology. *Topology*, 39(3):573–587, 2000.

[Fur76] Harry Furstenberg. A note on Borel’s density theorem. *Proc. Amer. Math. Soc.*, 55(1):209–212, 1976.

[FY94] Kenji Fukaya and Takao Yamaguchi. Isometry groups of singular spaces. *Math. Z.*, 216(1):31–44, 1994.

[GG16] Fernando Galaz-García. A glance at three-dimensional Alexandrov spaces. *Front. Math. China*, 11(5):1189–1206, 2016.

[GGG13] Fernando Galaz-Garcia and Luis Guijarro. Isometry groups of Alexandrov spaces. *Bull. Lond. Math. Soc.*, 45(3):567–579, 2013.

[GGG15] Fernando Galaz-Garcia and Luis Guijarro. On three-dimensional Alexandrov spaces. *Int. Math. Res. Not. IMRN*, (14):5560–5576, 2015.

[GGGNnZ20] Fernando Galaz-García, Luis Guijarro, and Jesús Núñez Zimbrón. Sufficiently collapsed irreducible Alexandrov 3-spaces are geometric. *Indiana Univ. Math. J.*, 69(3):977–1005, 2020.

[Gre74] Leon Greenberg. Maximal groups and signatures. In *Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, Ann. of Math. Studies, No. 79, pages 207–226. Princeton Univ. Press, Princeton, N.J., 1974.

[Hae20] Thomas Haettel. Hyperbolic rigidity of higher rank lattices. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(2):439–468, 2020. With an appendix by Vincent Guirardel and Camille Horbez.

[Hat83] Allen E. Hatcher. A proof of the Smale conjecture, Diff(S^3) ≃ O(4). *Ann. of Math. (2)*, 117(3):553–607, 1983.

[Hem04] John Hempel. *3-manifolds*. AMS Chelsea Publishing, Providence, RI, 2004. Reprint of the 1976 original.

[HJKL02] Ku Yong Ha, Jang Hyun Jo, Seung Won Kim, and Jong Bum Lee. Classification of free actions of finite groups on the 3-torus. *Topology Appl.*, 121(3):469–507, 2002.

[HKMR12] Sungbok Hong, John Kalliongis, Darryl McCullough, and J. Hyam Rubinstein. *Diffeomorphisms of elliptic 3-manifolds*, volume 2055 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[HS17] John Harvey and Catherine Searle. Orientation and symmetries of Alexandrov spaces with applications in positive curvature. *J. Geom. Anal.*, 27(2):1636–1666, 2017.

[JL10] Jang Hyun Jo and Jong Bum Lee. Group extensions and free actions by finite groups on solvmanifolds. *Math. Nachr.*, 283(7):1054–1059, 2010.

[KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication.

[Koj88] Sadayoshi Kojima. Isometry transformations of hyperbolic 3-manifolds. *Topology Appl.*, 29(3):297–307, 1988.

[KOS17] Daehwan Koo, Myungsung Oh, and Joonkook Shin. Classification of free actions of finite groups on 3-dimensional nilmanifolds. *Ann. of Math. (2)*, 185(3):1411–1440, 2017.

[Lee97] Joo Sung Lee. Totally disconnected groups, p-adic groups and the Hilbert-Smith conjecture. *Commun. Korean Math. Soc.*, 12(3):691–699, 1997.

[LSY93] Kyung Bai Lee, Joon Kook Shin, and Shoji Yokura. Free actions of finite abelian groups on the 3-torus. *Topology Appl.*, 53(2):153–175, 1993.

[McC02] Darryl McCullough. Isometries of elliptic 3-manifolds. *J. London Math. Soc. (2)*, 65(1):167–182, 2002.

[Mor15] Dave Witte Morris. *Introduction to arithmetic groups*. Deductive Press, [place of publication not identified], 2015.

[MS39] S. B. Myers and N. E. Steenrod. The group of isometries of a Riemannian manifold. *Ann. of Math. (2)*, 40(2):400–416, 1939.
[MS19] Mattia Mecchia and Andrea Seppi. Isometry groups and mapping class groups of spherical 3-orbifolds. *Math. Z.*, 292(3-4):1291–1314, 2019.

[MZ74] Deane Montgomery and Leo Zippin. *Topological transformation groups*. Robert E. Krieger Publishing Co., Huntington, N.Y., 1974. Reprint of the 1955 original.

[Now34] Werner Nowacki. Die euklidischen, dreidimensionalen, geschlossenen und offenen Raumformen. *Comment. Math. Helv.*, 7(1):81–93, 1934.

[OS94] Yukio Otsu and Takashi Shioya. The Riemannian structure of Alexandrov spaces. *J. Differential Geom.*, 39(3):629–658, 1994.

[Par13] John Pardon. The Hilbert-Smith conjecture for three-manifolds. *J. Amer. Math. Soc.*, 26(3):879–899, 2013.

[Par19] John Pardon. Totally disconnected groups (not) acting on two-manifolds. In *Breadth in contemporary topology*, volume 102 of *Proc. Sympos. Pure Math.*, pages 187–193. Amer. Math. Soc., Providence, RI, 2019.

[Rag07] M. S. Raghunathan. Discrete subgroups of Lie groups. *Math. Student*, (Special Centenary Volume):59–70 (2008), 2007.

[Rat99] John G. Ratcliffe. On the isometry groups of hyperbolic orbifolds. *Geom. Dedicata*, 78(1):63–67, 1999.

[Rat19] John G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, Cham, 2019. Third edition [of 1299730].

[RS97] Dusan Repovs and Evgenij Scepin. A proof of the hilbert-smith conjecture for actions by lipschitz maps. *Math. Ann.*, 308(2):361–364, 1997.

[RT15] John G. Ratcliffe and Steven T. Tschantz. On the isometry group of a compact flat orbifold. *Geom. Dedicata*, 177:43–60, 2015.

[Sco83] Peter Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.

[Tao14] Terence Tao. *Hilbert's fifth problem and related topics*, volume 153 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2014.

[Thu97] William P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[Tol74] Jeffrey L. Tollefson. The compact 3-manifolds covered by $S^2 \times R^1$. *Proc. Amer. Math. Soc.*, 45:461–462, 1974.

[Uy17] Masaaki Umehara and Kotaro Yamada. Differential geometry of curves and surfaces. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. Translated from the second (2015) Japanese edition by Wayne Rossman.

[vN33] J. von Neumann. Die Einführung analytischer Parameter in topologischen Gruppen. *Ann. of Math. (2)*, 34(1):170–190, 1933.

[Wal68] Friedhelm Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Ann. of Math. (2)*, 87:56–88, 1968.

[Wei11] Shmuel Weinberger. Some remarks inspired by the $c^0$ zimmer program. In *Geometry, rigidity, and group actions*, Chicago Lectures in Math., pages 262–282. Univ. Chicago Press, Chicago, IL, 2011.

[Yan60] Chung-Tao Yang. $p$-adic transformation groups. *Michigan Math. J.*, 7:201–218, 1960.

[Ye19] Shengkui Ye. Symmetries of flat manifolds, jordan property and the general zimmer program. *J. Lond. Math. Soc. (2)*, 100(3):1065–1080, 2019.

[Ye20] Shengkui Ye. A survey of topological’s zimmer programm. arXiv: arXiv:2002.01206, 2020.

[Zim84] Robert J. Zimmer. *Ergodic theory and semisimple groups*, volume 81 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[Zim87] Robert J. Zimmer. Actions of semisimple groups and discrete subgroups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 1247–1258. Amer. Math. Soc., Providence, RI, 1987.
Email address: barcenas@matmor.unam.mx, masedano@matmor.unam.mx

Centro de Ciencias Matemáticas. UNAM, Ap.Postal 61-3 Xangari. Morelia, Michoacán MEXICO 58089