Numerical solution of singularly perturbed delay differential equations using gaussian quadrature method

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Abstract. In this paper, a quadrature technique is employed for the solution of singularly perturbed delay differential equation. A first-order neutral type delay differential equation is achieved, which is asymptotically equivalent to the given singularly perturbed delay differential equation. Then Gaussian quadrature two-point formula is implemented on the first order equation to get a tridiagonal. Thomas algorithm is used to solve the resulting tri-diagonal system. The proposed method is implemented on model example, for different value of delay parameter and perturbation parameter. Maximum absolute errors are tabulated with a comparison to authorize the method. Theoretical convergence of the method is discussed. The layer behaviour is discussed using the graphical histrionics.

1. Introduction
An ordinary differential equation in which the highest derivative is multiplied by a small parameter and containing delay term is called singularly perturbed delay differential equation. In such problems, usually there are thin transition layers, where the solution varies rapidly or jumps abruptly, while away from the layers, the solution behaves regularly and varies slowly.

In the modelling of a system in control theory, the presence of small time parasitic parameters like resistances, moments of inertia, capacitances and inductances increase the order and stiffness of these systems. The suppression of these small constants result in the reduction of the order of the system. Such systems are termed as singular perturbation systems and when these systems take into account the past history as well as the present state, of the physical system, then they are called singularly perturbed delay differential equations. The elaborate study of differential-difference equations, with the presence of deviating terms, which gives large amplitudes and exhibit oscillations, were carried out by Lange and Miura [1-3]. Mohapatra and Natesan [4] advised a numerical method comprising of upwind finite difference operator on an adaptive grid and an arc-length monitor function for the solution of a class of singularly perturbed differential-difference equations with small delay and shift terms. A comprehensive numerical work had been initiated by Kadalbajoo and et al. [5–10]. Rao and Chakaravarthy [11] planned a tri-diagonal finite difference method for differential-difference equations of convection-diffusion type. Geng et al.[12] elucidate an improved kernel method for accurate approximation of solutions to singularly perturbed differential-difference equations, with small delay parameter.

2. Description of the Method
Consider the singular perturbed delay differential equation is form

\[ \varepsilon x''(t) + a(t)x'(t-d) + b(t)x(t) = f(t) \]  \hspace{1cm} (1)

The boundary conditions \[ x(t) = \phi(t), \quad -\delta \leq t \leq 0 \] and \[ x(1) = \beta \] \hspace{1cm} (2)
where $0 < \varepsilon < 1$ is small positive perturbation parameter, $0 < \delta < 1$ is small delay parameter, $a(t)$, $b(t)$, $f(t)$ and $\phi(t)$ are sufficiently smooth functions and $\beta$ is a constant. Furthermore, $\delta$ is delay parameter such that for all $x \in [0,1]$. When $\delta$ is zero Eq. (1) reduces to a singular perturbation problem which with small $\varepsilon$, exhibits layers and turning points depending upon the coefficient of convection term. The layer behaviour of the problem under consideration is maintained only for $\delta \neq 0$ but sufficiently small i.e. $\delta$ is of $O(\varepsilon)$. When the delay parameter exceeds the perturbation parameter i.e. $\delta$ is of $O(\varepsilon)$, the layer behaviour of the solution is no longer maintained, rather the solution exhibits an oscillatory behaviour or diminished behaviour.

3. Numerical scheme

3.1. Left End Boundary Layer Problem

This assumption implies that the boundary layer will be at the left end, that is $t_{i+1} \leq t \leq t_i$. Using Taylor’s expansions of the retarded term. We have

$$x'(t-\varepsilon) = x'(t) - \varepsilon x''(t)$$

implies

$$\varepsilon x''(t) = x'(t) - x'(t-\varepsilon)$$

Substituting Equation (3) put in equation (1), it is replaced by an asymptotically first order delay neutral type differential equation, then we get

$$x'(t) = x'(t-\varepsilon) - a(t)x'(t-\delta) - b(t)x(t) + f(t)$$

The domain $[p, q]$ is partitioned into $N$ equal subintervals of mesh size $h = \frac{q-p}{N}$ so that $t_i = p + ih$, $i = 0, 1, \ldots, N$ are the mesh points. Integrating Eq. (4) with respect to $t$ from $t_i$ to $t_{i+1}$, we get

$$\int_{t_i}^{t_{i+1}} x'(t) dt = \int_{t_i}^{t_{i+1}} ((x'(t-\varepsilon) - a(t)x'(t-\delta) - b(t)x(t) + f(t)) dt$$

$$x_{i+1} - x_i = \int_{t_i}^{t_{i+1}} x'(t-\varepsilon) dt - \int_{t_i}^{t_{i+1}} a(t)x'(t-\delta) dt - \int_{t_i}^{t_{i+1}} b(t)x(t) dt + \int_{t_i}^{t_{i+1}} f(t) dt$$

Using Gaussian two-point quadrature formula, we have

$$\int_{t_i}^{t_{i+1}} F(t) dt = F\left(\frac{1}{\sqrt{3}}\right) + F\left(-\frac{1}{\sqrt{3}}\right)$$

For any continuous and differentiable function $F(x)$ in an arbitrary interval $[t_i, t_{i+1}]$, the Gaussian two-point quadrature formula becomes

$$\int_{t_i}^{t_{i+1}} F(t) dt = \frac{h}{2} \left(F(t_i + k) + F(t_{i+1} - k)\right)$$
where \( k = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{3}} \right) \). Using Eq. (6) in Eq. (5), we have

\[
\begin{align*}
x_i - x_{i+1} &= x(t_{i+1}) - x(t_i) - a(t_i) x(t_i) - a(t_{i+1}) x(t_{i+1}) \\
&\quad + \frac{h}{2} \left[ a'(t_{i+1} - k) x(t_{i+1} - k) + a'(t_i - k) x(t_i + k - \delta) \right] \\
&\quad - \frac{h}{2} \left[ b(t_{i+1} - k) x(t_{i+1} - k) + b(t_i + k) x(t_i + k) \right] \\
&\quad + \frac{h}{2} \left[ f(t_{i+1} - k) + f(t_i + k) \right]
\end{align*}
\]

(7)

Using the linear interpolation for \( x(t_{i+1} - \epsilon) \), \( x(t_i - \epsilon) \), \( x(t_i + k) \), \( x(t_{i+1} - k) \), and \( x(t_{i+1} - k - \delta) \) the Eq. (7) reduces to

\[
\begin{align*}
&\left[ \frac{\epsilon}{h} - a(t_i) \left( \frac{\delta}{h} \right) - a'(t_i + k) \left( \frac{k - \delta}{2} \right) - b(t_i + k) \left( \frac{k}{2} \right) \right] x_{i+1} \\
&\quad + \left[ -\frac{2\epsilon}{h} + a(t_{i+1}) \left( \frac{\delta}{h} \right) - a(t_i) \left( \frac{h - \delta}{h} \right) - a'(t_{i+1} - k) \left( \frac{k + \delta}{2} \right) - a'(t_i + k) \left( \frac{h + \delta}{2} \right) \right] x_i \\
&\quad + \left[ \frac{\epsilon}{h} + a(t_{i+1}) \left( \frac{h - \delta}{h} \right) - a'(t_{i+1} - k) \left( \frac{h - k - \delta}{2} \right) + b(t_{i+1} - k) \left( \frac{h - k}{2} \right) \right] x_{i-1} \\
&\quad = \frac{h}{2} \left[ f(t_{i+1} - k) + f(t_i + k) \right]
\end{align*}
\]

(8)

Hence Eq. (8) can be rewritten in a three term recurrence relation as follows:

\[
W_i y_{i+1} + U_i y_i + V_i y_{i-1} = H_i
\]

(9)

Where

\[
\begin{align*}
W_i &= \left[ \frac{\epsilon}{h} - a(t_i) \left( \frac{\delta}{h} \right) + a'(t_i + k) \left( \frac{k - \delta}{2} \right) - b(t_i + k) \left( \frac{k}{2} \right) \right] \\
U_i &= \left[ -\frac{2\epsilon}{h} + a(t_{i+1}) \left( \frac{\delta}{h} \right) - a(t_i) \left( \frac{h - \delta}{h} \right) - a'(t_{i+1} - k) \left( \frac{k + \delta}{2} \right) - a'(t_i + k) \left( \frac{h + \delta}{2} \right) \right] \\
V_i &= \left[ \frac{\epsilon}{h} + a(t_{i+1}) \left( \frac{h - \delta}{h} \right) - a'(t_{i+1} - k) \left( \frac{h - k - \delta}{2} \right) + b(t_{i+1} - k) \left( \frac{h - k}{2} \right) \right] \\
H_i &= \frac{h}{2} \left[ f(t_{i+1} - k) + f(t_i + k) \right]
\end{align*}
\]

The tridiagonal system Eq. (9) is solved efficiently by using Thomas Algorithm.
3.2. Right End Boundary Layer Problem

Using Taylor’s series expression of \( x'(t + \varepsilon) \) gives \( x'(t + \varepsilon) = x'(t) + \varepsilon x'' \)
implies \( \varepsilon x'(t) \approx x'(t + \varepsilon) - x'(t) \) (10)

Consequently, Eq. (1) is replaced by an asymptotically equivalent first order delay neutral type
differential equation:

\[ x'(t) = x'(t + \varepsilon) + a(t)x'(t - \delta) + b(t)x(t) - f(t) \] (11)

Integrating equation (11) on \( x_{i-1} \) to \( x_i \) on both side we get

\[ \int_{x_{i-1}}^{x_i} x'(t) \, dt = \int_{x_{i-1}}^{x_i} \left( x'(t + \varepsilon) + a(t)x'(t - \delta) + b(t)x(t) - f(t) \right) \, dt \] (12)

Using Gaussian quadrature two-point formula for any continuous and differentiable function \( F(x) \) in an
arbitrary interval \([t_{i-1}, t_i]\), we get

\[ \int_{t_{i-1}}^{t_i} F(t) \, dt = \frac{h}{2} \left( F(t_{i-1} - k) + F(t_i + k) \right) \] (13)

Using Eq. (13), from Eq. (12) we get

\[
x(t_{i-1}) - x(t_{i-1} + \varepsilon) = x(t_i + \varepsilon) - x(t_{i+1} + \varepsilon) + a(t_{i-1})x(t_i - \delta) - a(t_{i+1})x(t_{i+1} - \delta) + \frac{h}{2} \left[ a'(t_{i-1} - k)x(t_{i-1} - k) - a'(t_{i+1} + k)x(t_{i+1} + k) \right] + \frac{h}{2} \left[ b(t_{i-1} - k)x(t_{i-1} - k) + b(t_{i+1} + k)x(t_{i+1} + k) \right] - \frac{h}{2} \left[ f(t_{i-1} - k) + f(t_{i+1} + k) \right]
\] (14)

Using linear interpolation for the terms \( x(t_{i-1} + \varepsilon), x(t_i + \varepsilon), x(t_{i+1} - k), x(t_i + k), x(t_{i-1} - \varepsilon), x(t_{i+1} + \varepsilon), x(t_i - \delta), x(t_{i+1} - \delta), x(t_{i-1} + k - \delta) \) and \( x(t_i - k - \delta) \), we get

\[
\begin{align*}
\left\{ \frac{\varepsilon}{h} - a(t_{i-1}) \left( 1 + \frac{\delta}{h} \right) + a'(t_{i-1} + k) \left( \frac{h - k - \delta}{2} \right) + b(t_{i-1} + k) \left( \frac{h - k}{2} \right) \right\} x_{i-1} \\
+ \left\{ -\frac{2\varepsilon}{h} + a(t_i) \left( 1 + \frac{\delta}{h} \right) + a(t_{i+1}) \delta \frac{h}{h} - a'(t_i - k) \left( \frac{h + k + \delta}{2} \right) + a'(t_{i+1} + k) \left( \frac{k - \delta}{2} \right) + b(t_i - k) \left( \frac{h - k}{2} \right) + b(t_{i+1} + k) \left( \frac{k}{2} \right) \right\} x_i \\
+ \left\{ \frac{\varepsilon}{h} - a(t_{i+1}) \left( \frac{\delta}{h} \right) + a'(t_i - k) \left( \frac{\delta + k}{2} \right) + b(t_i - k) \left( \frac{k}{2} \right) \right\} x_{i+1} \\
= \frac{h}{2} \left[ f(t_i - k) + f(t_{i+1} + k) \right]
\end{align*}
\] (15)

From Eq. (15), we can re write the tridiagonal system

\[ W_i y_{i-1} + U_j y_j + V_i y_{i+1} = H_i \quad \text{for} \quad 1 \leq i \leq N - 1 \] (16)

where
The system of Eq. (16) is solved by using the Thomas algorithm [7].

4. Convergence analysis:

The convergence analysis of the method described in Section 2 is considered in this section. Inserting the boundary conditions we write the system of equations in the matrix form as:

\[
(D + P)x + Q + T(h) = 0
\]  

(17)

Where 

\[
D = \begin{bmatrix}
\frac{\varepsilon}{h} & \frac{-2\varepsilon}{h} & \frac{\varepsilon}{h} \\
\frac{\varepsilon}{h} & \frac{-2\varepsilon}{h} & \frac{\varepsilon}{h} \\
0 & \frac{\varepsilon}{h} & \frac{-2\varepsilon}{h} \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

And 

\[
P = \begin{bmatrix}
v_1 & w_1 & 0 & 0 & \cdots & 0 \\
z_2 & v_2 & w_2 & 0 & \cdots & 0 \\
0 & z_3 & v_3 & w_3 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & 0 & z_{N-1} & v_{N-1}
\end{bmatrix}
\]
where

\[ Z_i = -a(t_{i\alpha}) \frac{\delta}{h} + \frac{k}{2} (a'(t_{i\alpha}+k) - b(t_{i\alpha}+k)) \]

\[ V_i = \left\{ \begin{align*}
& -a(t_{i\alpha}) \frac{\delta}{h} - a \left(1 - \frac{\delta}{h}\right) - a'(t_{i\alpha} - k) \left(\frac{k + \delta}{2}\right) - a'(t_{i\alpha} + k) \left(\frac{h-k-\delta}{2}\right) + b(t_{i\alpha} + k) \left(\frac{h+k}{2}\right) \\
& + b(t_{i\alpha} + k) \left(\frac{h+k}{2}\right)
\end{align*} \right\} \]

\[ W_i = \left\{ a(t_{i\alpha}) \left(1 - \frac{\delta}{h}\right) - a'(t_{i\alpha} - k) \left(\frac{h-k+\delta}{2}\right) + b(t_{i\alpha} - k) \left(\frac{h-k}{2}\right) \right\} \]

\[ Q = \left\{ q_1 \left(\frac{\varepsilon}{h} + \delta_i\right) + \gamma_0 q_0, q_2, q_{N-2}, q_{N-1} + \left(\frac{\varepsilon}{h} + W_{N-1}\right) \gamma_1 \right\} \]

where \( q_i = \frac{h}{2} \{ f(t_{i\alpha} - k) + f(t_{i\alpha} + k) \} \) where \( i = 1, 2, ..., N-1 \)

The local truncation error \( T(h) \) associated with the proposed scheme is

\[ T(h) = -\frac{h}{2} \left( a'(t_{i\alpha} - k) + a'(t_{i\alpha} + k) \right) \gamma_i + \frac{\delta h}{2} \left( a'(t_{i\alpha} - k) + a'(t_{i\alpha} + k) \right) \gamma'_i + O(h^2) \]

For \( k_1 = \left\{ \frac{\varepsilon}{h} - a(t_{i\alpha}) \left(\frac{\delta}{h}\right) + a'(t_{i\alpha} + k) \left(\frac{k-\delta}{2}\right) - b(t_{i\alpha} + k) \left(\frac{k}{2}\right) \right\} \)

\[ k_2 = \left\{ \frac{-2\varepsilon}{h} + \frac{a(t_{i\alpha}) \left(\frac{\delta}{h}\right)}{a(t_{i\alpha})} - \frac{a(t_{i\alpha}) \left(1 - \frac{\delta}{h}\right) - a'(t_{i\alpha} - k) \left(\frac{k+\delta}{2}\right) - a'(t_{i\alpha} + k) \left(\frac{h-k-\delta}{2}\right)}{2} \right\} \]

\[ k_3 = \left\{ \frac{\varepsilon}{h} + a(t_{i\alpha}) \left(1 - \frac{\delta}{h}\right) - a'(t_{i\alpha} - k) \left(\frac{h-k+\delta}{2}\right) + b(t_{i\alpha} - k) \left(\frac{h-k}{2}\right) \right\} \]

and \( X = [x_1, x_2, x_3, ..., x_{N-1}]^T \), \( T(h) = [t_1, t_2, t_3, ..., t_{N-1}]^T \), \( O = [0, 0, 0, ..., 0]^T \) are associated vector of equation (1)

Let \( X = [x_1, x_2, x_3, ..., x_{N-1}]^T \equiv X \) which satisfies the equation

\[ (D + P)x + Q = 0 \]  \hspace{1cm} (18)

Let \( e_i = x_i - X_i \), \( i=1(1)N-1 \) be the discretization error so that \( E = [e_1, e_2, e_3, ..., e_{N-1}]^T = x - X \)

Subtract Eq. (17) – Eq. (18), we get the error equation

\[ (D + P)E = T(h) \]  \hspace{1cm} (19)

Let \( |P(t)| \leq c_1 \) and \( |Q(t)| \leq c_2 \) where \( c_1, c_2 \) is a positive constant if \( p_{i,j} \) be the \( (i, j) \)th element of \( P \),

then \( |P_{i,j}| = |w_i| \leq \left| a(t_{i\alpha,i}) \left(1 - \frac{\delta}{h}\right) - a'(t_{i\alpha,i} - k) \left(\frac{h-k+\delta}{2}\right) + b(t_{i\alpha,i} - k) \left(\frac{h-k}{2}\right) \right| C_1 \)

where \( i = 1, 2, 3, ..., N-2 \).
\[ |P_{i+1}| = |Z_{i}| \leq \left( -a(t_{i}) \left( \frac{\delta}{h} \right) + \left( \frac{k - \delta}{2} \right) a'(t_{i} + k) - b(t_{i} + k) \left( \frac{k}{2} \right) \right) C_{2} \text{ where } i = 2, 3, 4, \ldots, N - 1. \]

Thus for sufficiently small \( h \), we have
\[ |P_{i+1}| < \epsilon \text{ for } i = 1, 2, 3, \ldots, N - 2 \tag{20} \]
and
\[ |P_{i+1}| < \epsilon, \text{ for } i = 2, 3, \ldots, N - 1 \tag{21} \]
Hence \( (D + P) \) is irreducible.

Let \( S_{i} \) be the sum of the elements of the \( i^{th} \) row of the matrix \( (D + P) \) then we have
\[
\bar{S}_{i} = \sum_{j=1}^{N-1} M_{ij} = \left\{ \begin{array}{l}
-\frac{\epsilon}{h} + a(t_{i}) - a(t_{i+1}) + \frac{\delta}{h} \left( a'(t_{i} + k) - a'(t_{i+1} - k) \right) + a'(t_{i} + k) + b(t_{i} + k) \left( \frac{k}{2} \right) \\
+ \frac{h}{2} \left( b(t_{i} + k) + b(t_{i} + k) - \delta \right)
\end{array} \right. \]
for \( i = 1; \)
\[
\bar{S}_{i} = \sum_{j=1}^{N-1} M_{ij} = a(t_{i}) - a(t_{i+1}) + \frac{h}{2} \left( b(t_{i} + k) + b(t_{i} + k) - \delta \right) \]
for \( i = N - 1 \)
\[
\bar{S}_{i-1} = \sum_{j=1}^{N-1} M_{i-1,j} = a(t_{i-1}) - a(t_{i}) + \frac{h}{2} \left( b(t_{i} + k) + b(t_{i} + k) - \delta \right) \]
for \( i = 2, 3, 4, \ldots, N - 2 \)

Let \( C_{i} = \min \{ p(t) \} \) and \( C_{i}^{*} = \max \{ p(t) \} \). Since \( 0 < i < 1 \) and \( \epsilon \propto o(h) \) its verify that for sufficiently small \( h \), \( (D + P) \) is monotonic.
Hence \( (D + P)^{-1} \) exists and \( (D + P)^{-1} \geq 0 \), thus from Eq. (19) we get
\[ \|E\| \leq \| (D + P)^{-1} \| \|T\| \tag{22} \]
For sufficiently small \( h \) we have
\[ \bar{S}_{i} > (a(t_{i}) - a(t_{i+1})) \text{ for } i = 1 \Rightarrow \bar{S}_{i} > \text{ constants} \]
\[ \bar{S}_{i} > (a(t_{i}) - a(t_{i+1})) \text{ for } i = 1, 2, 3, \ldots, N - 1 \Rightarrow \bar{S}_{i} > \text{ constants} \]
\[ \bar{S}_{i} > B_{i} \text{ for } i = 2, 3, 4, \ldots, N - 2 \]

Let \( (D + P)^{-1} \) be the \( (i, k)^{th} \) elements of \( (D + P)^{-1} \) and we define \( \| (D + P)^{-1} \| = \max \sum_{k=1}^{N-1} (D + P)_{i,k}^{-1} \) and
\[ \|T(h)\| = \max \|T(h)\| \tag{23} \]
Since \( (D + P)^{-1}_{i,k} \geq 0 \) and \( \sum_{k=1}^{N-1} (D + P)^{-1}_{i,k} \bar{S}_{i} = 1 \) for \( i = 1, 2, 3, \ldots, N - 1 \)
\[ (D + P)^{-1}_{i,k} \leq \frac{1}{S_{i}} \leq \frac{1}{C} \tag{24} \]
\[ (D + P)^{-1}_{i,N-1} \leq \frac{1}{S_{N-1}} \leq \frac{1}{C} \tag{25} \]
Furthermore, \( \sum_{i=1}^{N-2} (D + P)^{-1}_{(i,k)} \leq \frac{1}{\min \lambda_j} \leq \frac{1}{C} \) for \( i = 1, 2, 3, \ldots, N-2 \) \hfill (26)

With the help of Eqs. (23)-(26), from Eq. (22), we get \( \|E\| \leq O(h) \).

Hence, the method is first order convergent. In similar steps, we can analyze the convergence in other cases.

5. Numerical Experiments

The proposed method is validated on two numerical examples with left layer and three examples with right layer, for various values of perturbation parameter and delay parameter. Maximum absolute errors for the examples are computed, tabulated and compared with the results of Kadalbajoo and Sharma [4], Patidar and Sharma [6]. For the examples with variable coefficients, maximum absolute errors are calculated using the double mesh principle [7], \( E^N = \max_{0 \leq i \leq N} |y_i - y_{2i}^N| \).

**Example 1.** Consider an example of singularly perturbed delay differential equation with left layer
\[
\varepsilon x''(t) + x'(t - \delta) - x(t) = 0; \quad t \in [0, 1]
\]
with \( x(0) = 1 \) and \( x(1) = 1 \).

The exact solution is
\[
x(t) = \frac{(1 - e^{m_1})e^{m'} + (e^{m} - 1)e^{m'}}{(e^m - e^{m'})}
\]
where \( m_1 = \left(-1 - \sqrt{1 + 4(\varepsilon - \delta)} \right)/2(\varepsilon - \delta) \) and \( m_2 = \left(-1 + \sqrt{1 + 4(\varepsilon - \delta)} \right)/2(\varepsilon - \delta) \).

The maximum absolute error as shown in the Table 1 and graphically in Figure 1.

**Example 2.** we consider an example of variable coefficient singularly perturbed delay differential equation with left layer \( \varepsilon x'(t) + \varepsilon x'(t - \delta) - tx(t) = 0 \) with \( x(0) = 1, x(1) = 1 \).

The maximum absolute error as shown in the Table 2 and graphically in Figure 2.

**Example 3.** Consider a singularly perturbed delay differential equation with right layer:
\[
\varepsilon x''(t) - x'(t - \delta) - x(t) = 0; \quad t \in [0, 1] \quad \text{with} \quad x(0) = 1 \quad \text{and} \quad x(1) = -1.
\]

The exact solution is:
\[
x(t) = \frac{(1 + e^{m_2})e^{m'} - (e^{m} + 1)e^{m'}}{(e^{m} - e^{m'})}
\]
where \( m_1 = \left(1 - \sqrt{1 + 4(\varepsilon + \delta)} \right)/2(\varepsilon + \delta) \) and \( m_2 = \left(1 + \sqrt{1 + 4(\varepsilon + \delta)} \right)/2(\varepsilon + \delta) \).

Maximum absolute error as shown in the Table 3 and graphically shows in Figure 3.

**Example 4.** we consider an example of variable coefficient singularly perturbed delay differential equation with right layer: \( \varepsilon x''(t) - (1 + t)x'(t - \delta) - e^{-\varepsilon} x(t) = 0, \quad \text{with} \quad x(0) = 1, x(1) = 1 \).
The maximum absolute errors in the solution are shown in Table 4 with $\delta = 0.5e$ using double mesh principle for different values of $\varepsilon$ and $N$. Figure 4 shows the graphical solution of the problem.

Table 1. The maximum absolute errors of Example 1.

| $\varepsilon / N$ | 16   | 32   | 64   | 128  | 256  | 512  |
|-------------------|------|------|------|------|------|------|
| **Proposed method** |      |      |      |      |      |      |
| $10^{-9}$         | 0.40E-02 | 0.20E-02 | 0.10E-02 | 0.51E-03 | 0.26E-03 | 0.13E-03 |
| $10^{-10}$        | 0.40E-02 | 0.20E-02 | 0.10E-02 | 0.51E-03 | 0.26E-03 | 0.13E-03 |
| $10^{-11}$        | 0.40E-02 | 0.20E-02 | 0.10E-02 | 0.51E-03 | 0.26E-03 | 0.13E-03 |
| $10^{-12}$        | 0.39E-02 | 0.20E-02 | 0.98E-03 | 0.47E-03 | 0.21E-03 | 0.81E-04 |
| $10^{-13}$        | 0.37E-02 | 0.17E-02 | 0.74E-03 | 0.29E-03 | 0.29E-03 | 0.18E-03 |

Results in [10]

| $\varepsilon / N$ | 16   | 32   | 64   | 128  | 256  | 512  |
|-------------------|------|------|------|------|------|------|
| $10^{-9}$         | 0.11E-01 | 0.57E-02 | 0.29E-02 | 0.14E-02 | 0.72E-03 | 0.36E-03 |
| $10^{-10}$        | 0.11E-01 | 0.57E-02 | 0.29E-02 | 0.14E-02 | 0.72E-03 | 0.36E-03 |
| $10^{-11}$        | 0.11E-01 | 0.57E-02 | 0.29E-02 | 0.14E-02 | 0.72E-03 | 0.36E-03 |
| $10^{-12}$        | 0.11E-01 | 0.57E-02 | 0.29E-02 | 0.14E-02 | 0.72E-03 | 0.36E-03 |
| $10^{-13}$        | 0.11E-01 | 0.57E-02 | 0.29E-02 | 0.14E-02 | 0.72E-03 | 0.36E-03 |

Table 2. The maximum absolute errors of Example 2

| $\varepsilon / N$ | 16   | 32   | 64   | 128  | 256  | 512  |
|-------------------|------|------|------|------|------|------|
| **Proposed method** |      |      |      |      |      |      |
| $10^{-9}$         | 0.66E-02 | 0.42E-02 | 0.23E-02 | 0.12E-02 | 0.64E-03 | 0.32E-03 |
| $10^{-10}$        | 0.66E-02 | 0.42E-02 | 0.23E-02 | 0.12E-02 | 0.64E-03 | 0.32E-03 |
| $10^{-11}$        | 0.66E-02 | 0.42E-02 | 0.23E-02 | 0.12E-02 | 0.64E-03 | 0.32E-03 |
| $10^{-12}$        | 0.66E-02 | 0.42E-02 | 0.23E-02 | 0.12E-02 | 0.64E-03 | 0.32E-03 |
| $10^{-13}$        | 0.66E-02 | 0.42E-02 | 0.23E-02 | 0.12E-02 | 0.64E-03 | 0.32E-03 |

Results in [10]

| $\varepsilon / N$ | 16   | 32   | 64   | 128  | 256  | 512  |
|-------------------|------|------|------|------|------|------|
| $10^{-9}$         | 0.18E-01 | 0.93E-02 | 0.47E-02 | 0.24E-02 | 0.12E-02 | 0.59E-03 |
| $10^{-10}$        | 0.18E-01 | 0.93E-02 | 0.47E-02 | 0.24E-02 | 0.12E-02 | 0.59E-03 |
| $10^{-11}$        | 0.18E-01 | 0.93E-02 | 0.47E-02 | 0.24E-02 | 0.12E-02 | 0.59E-03 |
| $10^{-12}$        | 0.18E-01 | 0.93E-02 | 0.47E-02 | 0.24E-02 | 0.12E-02 | 0.59E-03 |
| $10^{-13}$        | 0.18E-01 | 0.93E-02 | 0.47E-02 | 0.24E-02 | 0.12E-02 | 0.59E-03 |
Table 3. The maximum errors of Example 3 with $\varepsilon = 0.1$ for different values of $\delta$ and $h$

| $\delta$ \(h\rightarrow\) | \(10^2\)   | \(10^3\)   | \(10^4\)   | \(10^5\)   |
|-----------------------------|------------|------------|------------|------------|
| Present method              |            |            |            |            |
| 0.000                       | 0.023594   | 0.0024574 | 0.00024676| 0.000024713|
| 0.007                       | 0.022105   | 0.0022960 | 0.00023050| 0.000023085|
| 0.015                       | 0.020589   | 0.0021355 | 0.00021434| 0.000021467|
| 0.025                       | 0.018988   | 0.0019637 | 0.00019704| 0.000019735|
| Results in [8]              |            |            |            |            |
| 0.000                       | 0.17855305 | 0.02387951 | 0.00247645 | 0.00024857 |
| 0.007                       | 0.11763184 | 0.01395171 | 0.00142255 | 0.00014254 |
| 0.015                       | 0.08351560 | 0.00944039 | 0.00095984 | 0.0009615  |
| 0.025                       | 0.06147531 | 0.00678868 | 0.00068675 | 0.0006876  |

Table 4. The maximum absolute error of example 4

| $\varepsilon$ \(N\)   | 16   | 32   | 64   | 128  | 256  | 512  |
|--------------------------|------|------|------|------|------|------|
| Proposed method          |      |      |      |      |      |      |
| \(10^{-9}\)              | 0.23E-02 | 0.12E-02 | 0.63E-03 | 0.32E-03 | 0.16E-03 | 0.81E-04 |
| \(10^{-10}\)             | 0.23E-02 | 0.12E-02 | 0.63E-03 | 0.32E-03 | 0.16E-03 | 0.81E-04 |
| \(10^{-11}\)             | 0.23E-02 | 0.12E-02 | 0.63E-03 | 0.32E-03 | 0.16E-03 | 0.81E-04 |
| \(10^{-12}\)             | 0.23E-02 | 0.12E-02 | 0.63E-03 | 0.32E-03 | 0.16E-03 | 0.81E-04 |
| \(10^{-13}\)             | 0.23E-02 | 0.12E-02 | 0.63E-03 | 0.32E-03 | 0.16E-03 | 0.81E-04 |
| Results in [10]          |      |      |      |      |      |      |
| \(10^{-9}\)              | 0.20E-01 | 0.10E-01 | 0.54E-02 | 0.28E-02 | 0.14E-02 | 0.70E-03 |
| \(10^{-10}\)             | 0.20E-01 | 0.10E-01 | 0.54E-2 | 0.28E-02 | 0.14E-02 | 0.70E-03 |
| \(10^{-11}\)             | 0.20E-01 | 0.10E-01 | 0.54E-02 | 0.28E-02 | 0.14E-02 | 0.70E-03 |
| \(10^{-12}\)             | 0.20E-01 | 0.10E-01 | 0.54E-02 | 0.28E-02 | 0.14E-02 | 0.70E-03 |
| \(10^{-13}\)             | 0.20E-01 | 0.10E-01 | 0.54E-02 | 0.28E-02 | 0.14E-02 | 0.70E-03 |

6. Discussions and conclusion
In this paper, we have discussed a numerical integration method for singularly perturbed delay differential equations, whose solutions exhibit layer behaviour on left-end and right end of the interval.
Gaussian quadrature two-point formula and linear interpolation has been used to obtain a tri-diagonal system. The tri-diagonal system has been solved using Thomas algorithm. To confirm the pertinence of the method, worthy examples from literature have been solved for diverse values of $\varepsilon$ and $\delta$, where the choice of the delay parameter $\delta$ is not unique. To validate the proposed method, we compared the maximum absolute errors in the solution of the examples with Kadalbajoo and Sharma [8], Patidar and Sharma et. al. [10].

The effect of negative shift on the boundary layer solutions has been investigated and presented in graphs. It is observed that, as the value of the negative shift, $\delta$ increases, the thickness of the layer decreases in the left end layer and increases in right end boundary layer. Theoretical convergence of the proposed method has also been secure. It can be observed that the accuracy predicted can always be achieved with very little computational acitivity with linear convergence.

**Figure 1** of Example 1 $\varepsilon = 0.1$ and $h=2^7$

**Figure 2** of Example 2 $\varepsilon = 0.1$ and $h=2^7$

**Figure 3** of Example 3 $\varepsilon = 0.1$ and $h=2^7$

**Figure 4** of Example 4 $\varepsilon = 0.1$ and $h=2^7$
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