Embedding Conditional Event Algebras Into Temporal Calculus of Conditionals

Jerzy Tyszkiewicz\textsuperscript{1,2}
Achim Hoffmann\textsuperscript{2}
Arthur Ramer\textsuperscript{2}

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\textsuperscript{1} Institute of Informatics, Warsaw University, Banacha 2, 02-097 Warszawa, Poland.
E-mail jty@mimuw.edu.pl.
Supported by the Polish Research Council KBN grant 8 T11C 027 16.

\textsuperscript{2} School CSE, UNSW, 2052 Sydney, Australia.
E-mail \{jty|achim|ramer\}@cse.unsw.edu.au.
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Abstract

In this paper we prove that all the existing conditional event algebras (abbreviated cea in this paper) embed into the three-valued extension (TL|TL) of temporal logic of discrete past time, which the authors of this paper have proposed in [TRH01] as a general model of conditional events.

First of all, we discuss the descriptive incompleteness of the cea’s. In this direction, we show that some important notions, like independence of conditional events, cannot be properly addressed in the cea framework, while they can be precisely formulated and analyzed in the (TL|TL) setting.

We also demonstrate that the embeddings allow one to use the native (TL|TL) algorithms for computing probabilities of complex conditional expressions of the embedded cea’s, and that these algorithms can outperform those previously known.
1 Preliminaries and statement of the problem

1.1 The problem of conditional objects

Probabilistic reasoning [Pea88] is the basis of Bayesian methods of expert system inferences, of knowledge discovery in databases, and in several other domains of computer, information, and decision sciences. The model of conditioning and conditional objects we discuss serves equally to reason about probabilities over a finite domain $X$, or probabilistic propositional logic with a finite set of atomic formulae.

Computing of conditional probabilities of the form $\Pr(X \mid Y_1, \ldots, Y_n)$ and, by extension of conditional beliefs, is well understood. Attempts of defining first the conditional objects of the basic form $X \mid Y$, and then defining $\Pr(X \mid Y)$ as $\Pr((X \mid Y))$ were proposed, without much success, by some of the founders of probability [Boo57, dF72]. They were taken up systematically only about 1980 [Ada86, Lew76, ES94, GGNR91, GN95]. The development was slow, both because of logical difficulties — interpretation of conditionals, and even more because the computational model is difficult to construct. (While $a \mid b$ appears to stand for a sentence 'if $b$ then $a$' and the probability is $\Pr(a \mid b) = \Pr(a \land b) / \Pr(b)$, there is no obvious calculation for $\Pr(a(b\mid c))$, nor intuitive meaning for $(a \mid (b\mid c))$, $(a \mid b) \land (c\mid d)$, and the like.)

The idea of defining conditional objects was entertained by some founders of modern probability [Boo57, dF72], but generally abandoned since introduction of the measure-theoretic model. It was revived mostly by philosophers in 1970’s [Ada86, vF77] with a view towards artificial intelligence reasoning. Formal computational models came in the late 1980’s [Cal87, GNW91] with only one, based on formal fractions and three-valued indicator functions, used for few actual calculations of conditionals and their probabilities. That model may give results whose values are open to questions [Cal94].

The authors of this paper have developed in the companion paper [TRH01] a temporal calculus (TL/TL) of conditionals, based on the early ideas of de Finetti [dF72].

In the present paper we show that all the major previously existing systems of conditionals, the so called conditional event algebras (see [GMN97] or Sect. 1.2 in the current paper), embed isomorphically into (TL/TL). Looking at them as fragments of (TL/TL), we demonstrate their insufficient expressive power and other defects in their construction.

They attempt to provide certain kind of a model of the logic of conditional expressions, built up from simple conditionals of the form $(a \mid b)$ with the connectives: conjunction, disjunction and complementation.

However, the semantical objects assigned to the expressions are not required to be of probabilistic nature, so they fail to provide methods to verify the chosen structure experimentally.

The structure of conditionals is not determined functorially by the space of
nonconditional events. Moreover, very restricted setting of cea’s does not allow one to address many important questions, like stochastic independence of complex conditionals. In the (TL|TL) setting independence can be precisely defined and analyzed, unlike in the cea formalism, (cf. Theorem 29 below).

Finally, we use the algorithms for calculating probabilities in (TL|TL), which stem from the highly developed algorithms for calculating limiting probabilities in Markov chains, and apply them to the embedded cea’s. It appears that these algorithms clearly outperform the previously known ones for the important product space cea [Goo94]. Consequently, we believe that our (TL|TL) can be used as a single alternative to each of the major cea’s considered in the literature so far, superior to each of them, in the sense of expressive power, clarity of logical and semantical structure, and, last but not least, availability of efficient algorithms.

2 The tools

2.1 (TL|TL), Moore machines and Markov chains

We describe briefly the construction of temporal conditionals, presented in detail in [TRH01].

Let $\mathcal{E} = \{a, b, c, d, \ldots\}$ be a finite set of basic events, and let $\Sigma$ be the Boolean algebra generated by $\mathcal{E}$, and $\Omega$ the set of atoms of $\Sigma$. Consequently, $\Sigma$ is isomorphic to the powerset of $\Omega$, and $\Omega$ is isomorphic to the powerset of $\mathcal{E}$. Any element of $\Sigma$ will be considered as an event, and, in particular, $\mathcal{E} \subseteq \Sigma$.

The union, intersection and complementation in $\Sigma$ are denoted by $a \cup b$, $a \cap b$ and $a^\complement$, respectively. The least and greatest elements of $\Sigma$ are denoted $\emptyset$ and $\Omega$, respectively. However, sometimes we use a more compact notation, replacing $\cap$ by a juxtaposition. When we turn to logic, it is customary to use yet another notation: $a \lor b$, $a \land b$ and $\neg a$, respectively. In this situation $\Omega$ appears as true and $\emptyset$ as false, but 1 and 0, respectively, are incidentally used, as well.

$3 = \{0, 1, \bot\}$ is the set of truth values, interpreted as true, false and undefined, respectively. The subset of $3$ consisting of 0 and 1 will be denoted $2$.

Let us first define temporal logic of linear discrete past time, called TL. The formulas are built up from the set $\mathcal{E}$ (the same set of basic events as before), interpreted as propositional variables here, and are closed under the following formula formation rules:

1. Every $a \in \mathcal{E}$ is a formula of temporal logic.
2. If $\varphi, \psi \in \text{TL}$, then their boolean combinations $\varphi \lor \psi$, $\neg \varphi$ are in TL.

   The other Boolean connectives: $\land, \rightarrow, \leftrightarrow, \ldots$ can be defined in terms of $\neg$ and $\lor$, as usual.

3. If $\varphi, \psi \in \text{TL}$, then their past tense temporal combinations $\bullet \varphi$ and $\varphi \text{Since } \psi$ are in TL, where $\bullet \varphi$ is spelled “previously $\varphi$.”

A model of temporal logic is a sequence $M = s_0, s_1, \ldots, s_n$ of states, each state being a function from $E$ (the same set of basic events as before) to the boolean values $\{0, 1\}$. Note that a state can be therefore understood as an atomic event from $\Omega$, and $M$ can be thought of as a word from $\Omega^\dagger$. The states of $M$ are ordered by $\leq$, and $s + 1$ denotes the successor state of $s$. We adopt the convention that, unless explicitly indicated otherwise, a model is always of length $n + 1$, and thus $n$ is always the last state of a model.

For every state $s$ of $M$ we define inductively what it means that a formula $\varphi \in \text{TL}$ is satisfied in the state $s$ of $M$, symbolically $M, s \models \varphi$.

1. $M, s \models a$ iff $s(a) = 1$

2. $M, s \models \neg \varphi : \iff M, s \not\models \varphi$,
   
   $M, s \models \varphi \lor \psi : \iff M, s \models \varphi$ or $M, s \models \psi$.

3. $M, s \models \bullet \varphi : \iff s > 0$ and $M, s - 1 \models \varphi$;

   $M, s \models \varphi \text{Since } \psi : \iff (\exists t \leq s)(M, t \models \psi$ and $(\forall t < w \leq s)M, w \models \varphi$).

The syntactic abbreviations $\blacksquare \varphi$ and $\Diamond \varphi$ are of common use in TL. They are defined by $\Diamond \varphi \equiv \text{true Since } \varphi$ and $\blacksquare \varphi \equiv \neg \Diamond \neg \varphi$. The first of them is spelled “once $\varphi$” and the latter “always in the past $\varphi$”.

Their semantics is then equivalent to

$M, s \models \blacksquare \varphi \iff (\forall t \leq s)M, t \models \varphi$;

$M, s \models \Diamond \varphi \iff (\exists t \leq s)M, t \models \varphi$.

**Theorem 1 (see [Eme90]).** The set of valid TL formulas is complete in PSPACE. The set of valid TL formulas with $\blacksquare$ and $\Diamond$ as the only temporal connectives is complete in coNP.

$(\text{TL} | \text{TL})$ is the logic of formulas of the form $(\varphi | \psi)$, where $\varphi, \psi \in \text{TL}$. $(\text{TL} | \text{TL})$ is a 3-valued extension of TL, and $(\varphi | \psi)$ is

1. true in $M, n$ iff $M, n \models \varphi \land \psi$. 

0). false in $\mathcal{M}, n \models \neg \varphi \land \psi$.

$\bot$). undefined in $\mathcal{M}, n \models \neg \psi$.

A 3-valued Moore machine $\mathfrak{A}$ is a five-tuple $\mathfrak{A} = (Q, \Omega, \delta, h, q_0)$, where

where $Q$ is its set of states, $\Omega$ (the same set of atomic events as before) is the input alphabet, $q_0 \in Q$ is the initial state and $\delta : Q \times \Omega \to Q$ is the transition function, and $h$ is the output function $Q \to 3$.

Formally, to describe the computation of $\mathfrak{A}$ we extend $\delta$ to a function $\hat{\delta} : Q \times \Omega^+ \to Q$ in the following way:

$$\hat{\delta}(q, w) = \begin{cases} 
\delta(q, w) & \text{if } |w| = 1 \\
\delta(\hat{\delta}(q, v), \omega) & \text{if } w = v\omega.
\end{cases}$$

$\mathfrak{A}$ computes a function $f_{\mathfrak{A}} : \Omega^+ \to 3^+$ defined by

$$f_{\mathfrak{A}}(\omega_1 \omega_2 \ldots \omega_n) = h(\hat{\delta}(q_0, \omega_1))h(\hat{\delta}(q_0, \omega_1 \omega_2)) \ldots h(\hat{\delta}(q_0, \omega_1 \omega_2 \ldots \omega_n))$$

(note that $|f_{\mathfrak{A}}(\omega_1 \omega_2 \ldots \omega_n)| = n$, as desired)

We picture $\mathfrak{A}$ as a labeled directed graph, whose vertices are elements of $Q$, labeled by their values under $h$, the function $\delta$ is represented by directed edges labeled by elements of $\Omega$: the edge labeled by $\omega \in \Omega$ from $q \in Q$ leads to $\delta(q, \omega)$. The initial state is typically indicated by an unlabeled edge “from nowhere” to this state.

As the letters of the input word $w \in \Omega^+$ come in one after another, we walk in the graph, always choosing the edge labeled by the letter we receive. At each step it reports to the outside world the value $h(q)$ of the state $q$ in which it is at the moment.

Drawing Moore machines, we almost always make certain graphical simplification: we merge all the transitions joining the same pair of states into a single transition, labeled by the union (evaluated in $\Sigma$) of all the labels. Sometimes we go even farther and drop the label altogether from one transition, which means that all the remaining input letters follow this transition. It is known for deterministic finite automata \[HU79\], and extends easily to Moore machines with the same proof, that for any such device there is a unique (up to isomorphism) minimal (with respect to the number of states) device of the same kind, which accepts the same language (computes the same function, respectively). Moreover, this minimal device can be obtained from any such device as a quotient automaton/machine, i.e., by dividing the state space by some equivalence relation. For details, including a very efficient algorithm to perform minimization, see \[HU79\].
**Definition 2.** A Moore machine $A$ is called *counter-free* if there is no word $w \in \Omega^+$ and no states $q_1, q_2, \ldots, q_s$, $s > 1$, such that $\hat{\delta}(q_1, w) = q_2, \ldots, \hat{\delta}(q_{s-1}, w) = q_s, \hat{\delta}(q_s, w) = q_1$.

Sometimes we use an extension of Moore machines—*Moore machines with $\epsilon$-moves.* In a deterministic finite automaton an $\epsilon$-move is a transition between two states done without intervention of any letter from the input. By necessity, to maintain the deterministic character of the automaton, an $\epsilon$-move must not be combined with any other transitions starting from the same state. For Moore machines, we adopt the convention that after performing an $\epsilon$-move, no symbol is appended to the output.

E.g., for the Moore machine with $\epsilon$-moves $A$ below

![](image)

we have $f_A(\omega\omega\omega) = 0\perp0$ and $f_A(\omega\omega\omega) = 0\perp1$.

It is known that any function computable by a Moore machine with $\epsilon$-moves can be computed by a Moore machine without $\epsilon$-moves. Thus using $\epsilon$-moves we do not achieve greater generality. However, some transformations of the machines can be very conveniently represented by introducing $\epsilon$-moves.

For us, Markov chains are a synonym of *Markov chains with stationary transitions and finite state space.*

Formally, given a finite set $I$ of *states* and a fixed function $p : I \times I \rightarrow [0, 1]$ satisfying $(\forall i \in I) \sum_{j \in I} p(i, j) = 1$, the *Markov chain* with state space $I$ and transitions $p$ is a sequence $X = X_0, X_1, \ldots$ of random variables $X_n : W \rightarrow I$, such that

$$
\Pr(X_{n+1} = j | X_n = i) = p(i, j).
$$

The standard result of probability theory is that there exists a probability triple $(W, \mathcal{M}, \Pr)$ and a sequence $X$ such that (1) is satisfied. $W$ is indeed the space of infinite sequences of ordered pairs of elements from $I$, and $\Pr$ is a certain product measure on this set.

One can arrange the values $p(i, j)$ in a matrix $\Pi = (p(i, j); i, j \in I)$. Of course, $p(i, j) \geq 0$ and $\sum_{j \in I} p(i, j) = 1$ for every $i$. Every real square matrix $\Pi$ satisfying these conditions is called *stochastic.*

The initial distribution of $X$ is that of $X_0$, which can be conveniently represented by a vector $\Xi_0 = (p(i); i \in I)$. Its choice is independent from the function $p(i, j)$. 
It follows by a simple calculation that
\[
\Pr(X_{n+1} = j | X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1) = \Pr(X_{n+1} = j | X_n = i),
\]
which is called the Markov property.

For our purposes, it is convenient to imagine the Markov chain \(X\) in another, equivalent form: Let \(K_I\) be the complete directed graph on the vertex set \(I\). First we randomly choose the starting vertex in \(I\), according to the initial distribution. Next, we start walking in \(K_I\); at each step, if we are in the vertex \(i\), we choose the edge \((i, j)\) to follow with probability \(p(i, j)\). If we define \(X_n = (\text{the vertex in which we are after } n\text{ steps})\), then \(X_n\) is indeed the same \(X_n\) as in (1).

So we will be able to draw Markov chains. Doing so, we will often omit edges \((i, j)\) with \(p(i, j) = 0\).

### 2.2 Conditional objects and conditional events

Let for \((\varphi|\psi) \in (\text{TL}|\text{TL})\) the function \(c = c_{(\varphi|\psi)} : \Omega^+ \to 3\) be defined by

\[
c(w) = \begin{cases} 
1 & \text{if } (\varphi|\psi) \text{ is true in } (w, n) \\
0 & \text{if } (\varphi|\psi) \text{ is false in } (w, n) \\
\bot & \text{if } (\varphi|\psi) \text{ is undefined in } (w, n)
\end{cases}
\]

\(\mathbb{C}\) is the set of all functions \(c : \Omega^+ \to 3\) definable in \((\text{TL}|\text{TL})\), and \(\mathbb{C}_+\) is the set of all functions \(c_+ : \Omega^+ \to 3^+\) computable by counter-free \(3\)-valued Moore machines.

\(\mathbb{C}\) and \(\mathbb{C}_+\) are isomorphic under the mapping \(\mathbb{C}_+ \ni c_+ \mapsto c \in \mathbb{C}\) defined by

\[
c(w) = \text{last-letter-of}(c_+(w)).
\]

The sets \(\mathbb{C}\) and \(\mathbb{C}_+\) are regarded as two representations of conditional objects. We have yet another representation, denoted \(\mathbb{C}_\infty\): it consists of functions \(\Omega^\infty \to 3^\infty\), and \(c_\infty\), the third representation of the same conditional, is an infinite sequence of values of \(c\) on all finite nonempty prefixes of \(w\).

We will be using the name conditional events to refer to conditionals considered with a probability space in the background.

**Definition 3** (Conditional event \([\text{PRH01]}\)). Let \(c \in \mathbb{C}\) be a conditional object over \(\Omega\) and let \(\mathcal{A}\) be any counter-free Moore machine with output function \(h\), computing \(c_+\). Suppose \(\Omega\) is endowed with a probability space
structure \( (\Omega, \mathcal{M}, \Pr) \). The conditional object \( c \) becomes then a sequence \( [c] = [c]_1, [c]_2, \ldots \) of random variables \( [c]_n : \Omega^\infty \to 3 \), defined by the formula

\[
[c]_n(w) = c(\text{prefix-of-length-}n\text{-of}(w)),
\]

where \( \Omega^\infty \) is considered with the product probability structure.

We call \( [c] \) the conditional event associated with \( c \).

Moreover, in presence of probability space structure \( A \) becomes a Markov chain \( \mathcal{X}(A) \) (by replacing labels of the transitions by their probabilities under \( \Pr \) in the diagram of \( A \)), and then \( [c] = h(\mathcal{X}) \), where \( h \) is the output function of \( A \).

In particular, \( \Pr([c]_n = 1) \) is the probability that at time \( n \) the conditional object \( c \) is true, \( \Pr([c]_n = 0) \) is the probability that at time \( n \) the conditional object \( c \) is false, and \( \Pr([c] = \bot) \) is the probability that at time \( n \) the conditional object \( c \) is undefined.

**Definition 4 (asymptotic probability [TRH01])**. We define the asymptotic probability at time \( n \) of a conditional event \( c \in C \) by the formula

\[
\Pr_n(c) = \frac{\Pr([c]_n = 1)}{\Pr([c]_n = 0 \text{ or } 1)}.
\]

If the denominator is 0, \( \Pr_n(c) \) is undefined.

The asymptotic probability of \( c \) is

\[
\Pr(c) = \lim_{n \to \infty} \Pr_n(c),
\]

provided that \( \Pr_n(c) \) is defined for all sufficiently large \( n \) and the limit exists. If \( \varphi \in TL \) then we write \( \Pr(\varphi) \) for \( \Pr((\varphi|true)) \).

**Theorem 5 (Bayes’ Formula [TRH01])**. Let \((\varphi|\psi)\) be a conditional object over \( \Omega \) endowed with a probability space structure \((\Omega, \mathcal{M}, \Pr)\), and let \( A \) be any counter-free Moore machine with output function \( h \), computing \((\varphi|\psi)\).

1. For every state \( i \) of \( \mathcal{X}(A) \) the probability \( \lim_{n \to \infty} \Pr(X_n = i) \) exists.

2. For every \( * \in 3 \) the probability \( \lim_{n \to \infty} \Pr([((\varphi|\psi)]_n = *) \) exists.

3. The Bayes’ Formula

\[
\Pr((\varphi|\psi)) = \frac{\Pr(\varphi \land \psi)}{\Pr(\psi)}
\]

holds whenever the right-hand-side above is well-defined, i.e., \( \Pr(\psi) > 0 \).
3 Connectives of conditionals

If we wish to extend the classical two-valued conjunction to conditionals, we are faced with the problem of synchronization. Indeed, what is easy in 2-valued world becomes messy in 3-valued world. The problem is that the conditionals need not become defined synchronously. For the classical conjunction this problem does not exist, because both arguments are always defined. Now we have to resolve the question how to define the conjunction, when some of the arguments are undefined.

3.1 Present tense connectives

Let us recall that present tense connectives are those, whose definition in (TL|TL) does not use temporal connectives, and therefore depends on the present, only. Equivalently, an n-ary present tense connective is completely characterized by a function $3^n \rightarrow 3$.

Here are several possible choices for the conjunction, which is always defined as a pointwise application of the following 3 valued functions. Above we display the notation for the corresponding kind of conjunction.

| $x \land_{SAC} y$ | $x \land_{GNW} y$ | $x \land_{Sch} y$ |
|-------------------|-------------------|-------------------|
| $x \\ y \ \ 0 \ \ 1 \ \ \bot$ | $x \\ y \ \ 0 \ \ 1 \ \ \bot$ | $x \\ y \ \ 0 \ \ 1 \ \ \bot$ |
| 0 0 0 0 | 0 0 0 0 | 0 0 0 0 |
| 1 0 1 1 | 1 0 1 1 | 1 0 1 1 |
| $\bot$ 0 1 $\bot$ | $\bot$ 0 $\bot$ | $\bot$ 0 $\bot$ |

$\neg x$

| $\neg x$ |
|----------|
| 0 1 |
| 1 0 |
| $\bot$ |

| $x \lor_{SAC} y$ | $x \lor_{GNW} y$ | $x \lor_{Sch} y$ |
|-------------------|-------------------|-------------------|
| $x \\ y \ \ 0 \ \ 1 \ \ \bot$ | $x \\ y \ \ 0 \ \ 1 \ \ \bot$ | $x \\ y \ \ 0 \ \ 1 \ \ \bot$ |
| 0 0 1 0 | 0 0 1 0 | 0 0 1 0 |
| 1 1 1 1 | 1 1 1 1 | 1 1 1 1 |
| $\bot$ 0 1 $\bot$ | $\bot$ $\bot$ 1 $\bot$ | $\bot$ $\bot$ 1 $\bot$ |

They can be equivalently described by syntactical manipulations in (TL|TL). The reduction rules are as follows:
\[(a|b) \land_{SAC} (c|d) = (abcd \lor abd \lor cd \lor b \lor d)\]
\[(a|b) \land_{GNW} (c|d) = (abcd|a^c d \lor c^d \lor abcd)\]
\[(a|b) \land_{Sch} (c|d) = (abcd|bd)\]
\[\sim_0 (a|b) = (a^c|b)\]
\[(a|b) \lor_{SAC} (c|d) = (ab \lor cd|b \lor d)\]
\[(a|b) \lor_{GNW} (c|d) = (ab \lor cd|ab \lor cd \lor bd)\]
\[(a|b) \lor_{Sch} (c|d) = (ab \lor cd|bd).\]

The first is based on the principle “if any of the arguments becomes defined, act!”. A good example would be a quotation from [Cal97]:

“One of the most dramatic examples of the unrecognized use of compound conditioning was the first military strategy of our nation. As the Colonialists waited for the British to attack, the signal was ‘One if by land and two if by sea’. This is the conjunction of two conditionals with uncertainty!”

Of course, if the above was understood as a conjunction of two conditionals, the situation was crying for the use of \(\land_{SAC}\), whose definition has been proposed independently by Schay, Adams and Calabrese (the author of the quotation).

The conjunction \(\land_{GNW}\) represents a moderate approach, which in case of an apparent evidence for 0 reports 0, but otherwise it prefers to report unknown in a case of any doubt. Note that this conjunction is essentially the same as lazy evaluation, known from programming languages.

Finally, the conjunction \(\land_{Sch}\) is least defined, and acts (classically) only if both arguments become defined. It corresponds to the strict evaluation.

We have given an example for the use of \(\land_{SAC}\). The uses of \(\land_{GNW}\) and \(\land_{Sch}\) can be found in any computer program executed in parallel, which uses either lazy or strict evaluation of its logical conditions. And indeed both of them happily coexist in many programming languages, in that one of them is the standard choice, the programmer can however explicitly override the default and choose the other evaluation strategy.

This seems to suggest that neither of the three choices discussed in this paragraph is the conjunction of conditionals. There are indeed many possible choices, and all of them have their own merits. And indeed already the original system of Schay consisted of five operations: \(\sim_0, \land_{SAC}, \lor_{SAC}, \land_{Sch}\) and \(\lor_{Sch}\). Moreover, he was aware that these operations still do not make the algebra functionally complete (even in the narrowed sense, restricted to

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1NB, if the British had decided to attack from both directions, but not simultaneously, we would have probably discovered temporal conditionals much earlier.
defining only operations which are undefined for all undefined arguments).
And in order to remedy this he suggested to use one of several additional
operators, one of them being $\land_{\text{GNW}}$. So for him all those operations could
oxist in one system.

**Present tense re-conditioning** Calabrese [Ca90] and Goodman, Nguyen
and Walker [GN95] proposed their own extensions of the conditioning oper-
ator to 3, hence making it available for re-conditioning in SAC and GNW,
respectively. The definitions are

| $x \mid \text{SAC} \ y$ | $x \mid \text{GNW} \ y$ |
|----------------------|----------------------|
| $x \setminus y$      | $x \setminus y$      |
| 0                    | 0                    |
| 1                    | 0                    |
| ⊥                    | ⊥                    |
| 0                    | ⊥                    |
| 1                    | ⊥                    |
| ⊥                    | ⊥                    |

3.2 Past tense connectives

Now we consider connectives, whose definitions refer to the strict past of
their arguments. We continue to consider conjunction, which we use as a
kind of model example.

**Examples of past tense connectives** The following are connectives very
close to the conjunction and disjunction of the product space $\text{cea}$ introduced
in [Goo94], defined by the rule: the conjunction is defined and true iff
both of its arguments have been defined, and moreover the historically first
values of its two arguments have been both 1. Otherwise it is defined and
false. Disjunction is defined similarly. They use the “Russian roulette”
approach to repeating experiments.

In the language of (TL|TL) $(a \mid b) \land_{\text{PS}} (c \mid d)$ can be expressed by

$$(\Diamond (a \land b \land \neg \Diamond b)) \land (\Diamond (c \land d \land \neg \Diamond d)) \mid \text{true},$$

and the definition of and $(a \mid b) \lor_{\text{PS}} (c \mid d)$ is similar. They seem complicated,
but can be simplified, what we do below, and the Moore machine represen-
tations are again much simpler and easier to analyze.

To simplify the (TL|TL) representation above, let us define $\text{first}(a \mid b)$ to be

$$\text{first}(a \mid b) := (\Diamond (a \land b \land \neg \Diamond b)) \mid \text{true}.$$

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See the discussion on embedding $\text{cea}$’s in our model below.
It is convenient to denote by $\text{first}_2(a|b)$ the first argument of $\text{first}(a|b)$. Then the conjunction can be easily and effectively described as $(\text{first}_2(a|b) \land \text{first}_2(c|d)|\text{true})$ (where $\land$ is the classical conjunction of temporal logic). The minimal Moore machine of $\text{first}(a|b)$ is depicted in Fig. 2 below.

4 Embedding of existing cea’s and their incompleteness

In this section we want to discuss the problem of embedding existing cea’s into our model, and on that basis, the problem of defining natural connectives among conditionals in general.

4.1 Syntax

We assume the following syntax of the flat conditional expressions. The set of all such expressions will be denoted $L$. The set of these expressions is the smallest set, containing all simple conditionals of the form $(x|y)$, where $x, y \in \Omega$, and closed under two-ary (infix) operations $\land, \lor$ and one unary prefix operation $\sim$.

If we require the closure under one additional binary operation $(\cdot|\cdot)$ (which shouldn’t be mixed up with the parenthesis-bar-parenthesis construction appearing in simple conditionals), we obtain the set of full conditional expressions, denoted $L^\dagger$.

4.2 Conditional event algebras

According to [GMN97], a conditional event algebra (cea in short) over a probability space $(\Omega, \mathcal{M}, \text{Pr})$ is a space (but not necessarily a probability space) $(\Omega_0, \mathcal{M}_0, \text{Pr}_0)$, extending $(\Omega, \mathcal{M}, \text{Pr})$, together with a function $(\cdot|\cdot) : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}_0$ such that

- $\mathcal{M}_0$ is an algebra of the signature of boolean algebras.
- $(a|b) = (a \land b|b)$ for all $a, b \in \mathcal{M}$.
- The subalgebra of $(\Omega_0, \mathcal{M}_0, \text{Pr}_0)$ consisting of the elements $(a|\Omega)$ is isomorphic to $(\Omega, \mathcal{M}, \text{Pr})$ under the bijection $a \mapsto (a|\Omega)$.
- $\text{Pr}_0((a|b)) = \text{Pr}(a \land b)/\text{Pr}(b)$ for $a, b \in \mathcal{M}$, $\text{Pr}(b) > 0$.
- Certain equalities hold among the $\text{Pr}_0$-probabilities.
- The $\text{Pr}_0$-probabilities for $\land, \lor$ and $\ell$ of elements of the form $(a_i|b_i)$ for $a_i, b_i \in \mathcal{M}$ are effectively computable from the set of $\text{Pr}$-probabilities of all the boolean combinations of the elements $a_i, b_i$. 
Figure 1: Moore machine of $\langle\langle a|b \wedge c|d\rangle\rangle_{PS}$

Figure 2: Moore machine of $\langle\langle a|b\rangle\rangle_{PS} = \text{first}(a|b)$. 
How we understand cea’s. It is readily seen, that any particular cea over any \((\Omega, M, \Pr)\) can be equivalently considered as a mapping assigning elements of \(M_o\) and probabilities to flat conditional expressions. In this sense, cea is a kind of a model of a logic, whose syntax are the flat conditional expressions. This is the way we understand cea’s and this is the level on which we will criticize them.

Probabilistic models and cea’s. First of all, we do not think that the algebraic structure of a cea is particularly important. The language of flat conditional expressions does not have equality, so what is really crucial are the probabilities. The algebraic structure can indeed be an obstacle while assigning probabilities (this is perhaps why the authors of the earlier papers devoted so much attention to it), but otherwise we are not so much interested in it.

Next, what a cea assigns to a conditional expression is definitely too little. Apart from an element of \(M_0\) and probability, an experiment should be determined to verify experimentally the value of the probability. This means, that the objects assigned to conditional expressions should be events in a probabilistic space, i.e., the triple \((\Omega_o, M_o, \Pr_o)\) should be indeed a probability space. Moreover, the experiments for simple conditionals \((a|b)\) should correspond to the natural experiments one performs to learn conditional probabilities. In such experiments one can typically measure another probabilistic parameters, like, e.g., the probability that \((a|b)\) is defined. We think such additional parameters should be assigned to compound conditional events, too. Of course, in the existing algebras we have hints concerning it, hidden in their universes and other details of the constructions, provided by the inventors, but the very definition of a cea does not require the additional parameters to be even defined, let alone to satisfy any reasonable properties. Strictly speaking, the signature of cea’s is too small. In its present shape it permits existence of other, isomorphic algebras, where all the additional information is lost.

Last, but not least, in the applications of classical probability theory one often encounters problems in modeling, typically of the following form: one has an event, whose meaning is completely clear (it is known, when it happens and when it doesn’t), but there is a problem of specifying the probability space structure, and sometimes different choices lead to different values for the probability of the same event. In such circumstances one can only experimentally decide which of the models is the correct one. A standard example is the difference between the so called statistics of: Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac, considered in quantum physics, and the unsuccessful search for any elementary particle, which would satisfy the first statistics, seemingly the most natural one among them (therefore we have bosons and fermions, but we do not have maxwellons in physics) [Fel68].
The tremendous success of probability theory in applications seems to suggest this is the right way of creating a mathematical model of a real life situation. However, the \textit{cea} offers us another challenge. Even after coming up with the proper model of probability space of unconditional events, one still has a lot of work to choose the right model for conditional events. In plain words, it means that the structure modeling conditional events over a given probabilistic space of unconditional events, should be functorial: given the former space, the space of conditional events ought to be uniquely determined. Here the \textit{cea’s} again fall short of satisfying this requirement, because there are many known \textit{cea’s}, and each of them has its own definition of \((a|b) \land (c|d)\), with its own probability, and all of them derive their definitions from certain first principles. The answer is almost obvious — the signature is too small, and should consist of many different conjunctions, disjunctions and negations, and perhaps lots of other connectives, which do not have any natural counterparts in the nonconditional world.

\textbf{(TL|TL) as a solution.} We believe that our system of temporal conditional events addresses all of the problems we have indicated above. Our conditional events belong to a normal probability space. They are indeed stochastic processes — projections of Markov chains, and all their probabilistic properties (many more than just the bare probability) can be verified experimentally.

To the contrary, all the \textit{cea’s} we call present tense in this paper have a natural representation as algebras of 3-valued indicator functions \cite[Chapter 3]{GNW91}. Recalling that every event from \(\Sigma\) can be equivalently characterized by its 2-valued \textit{indicator random variable}, which is nothing but the characteristic function, we should naturally expect that the lifting of \(\Pr\) to the space of 3-valued indicator functions, should consist of \textit{3-valued random variables}, while the definition of a \textit{cea} requires just elements of a strange algebra \(\mathfrak{M}_o\) and numbers. In case of our temporal conditionals, we naturally expect conditionals to be functions \(\Omega^\infty \to 3^\infty\), and hence the lifting of \(\Pr\) to consist of random functions, which are nothing but \textit{stochastic processes}, the choice we actually have made. This is the natural pattern one should follow, and this is where the scalability of the model is hidden. E.g., nothing could prevent us from considering continuous time stochastic processes as models of conditional events, if need be.

The space of conditional events is uniquely determined by the probabilistic space of nonconditional events, and has many more natural connectives than just three. In fact, we are not very original here: already Schay \cite{Sch68} proposed a system with five connectives, and considered adding even more of them.

Finally, to make our argument complete, we consider the major \textit{cea’s} below, and show that they embed in our models, in many different ways. For
the product space *cea*, we employ the embeddings to demonstrate that this *cea* has some further defects. The existence of multiple embeddings shows that the structure of a *cea*’s isn’t functorial just because there are many known *cea*’s, but because the structure of a *cea* does not prescribe the way the conditionals relate to nonconditional events. Additionally, the product space *cea* is unable to characterize independence of conditional events by means of equalities of probabilities.

4.3 Present tense *cea*’s

In our framework, there is a distinctive class of *cea*’s, which we call *present tense* *cea*’s. The definitions of their connectives refer to the present of the process, only, hence the name. Consequently, such a connective applied to simple conditionals yields simple conditionals, again. Among such systems are SAC, proposed independently by Schay [Sch68], Adams [Ada86] and Calabrese [Cal87] (who extended it by an operator for re-conditioning), and GNW proposed by Goodman [Goo87] and Goodman, Nguyen and Walker [GNW91] (later Goodman and Nguyen [GN95] proposed a re-conditioning operator for GNW). They assume that all boolean combinations of simple conditional expressions yield simple conditional expressions again. The definitions of their connectives are given in Section 3.1. They do form *cea*’s. Equivalently, [GNW91, Chapter 3], these algebras can be characterized as algebras of 3-valued indicator functions, and their connectives are then characterized by mappings from certain Cartesian power of 3 into 3. We take the second point of view, and define their semantics as follows. Our definition assigns to every conditional expression $e$ a 3-valued indicator function $\langle \langle e \rangle \rangle_{SAC}$ and $\langle \langle e \rangle \rangle_{GNW}$, respectively. For us, indicator functions are nothing else than present tense conditionals. (Originally these algebras do not involve time.) So, we give the definition by describing translations $\langle \langle \cdot \rangle \rangle_{SAC}, \langle \langle \cdot \rangle \rangle_{GNW} : \mathcal{L} \rightarrow \mathcal{C}$. We use in the translations present tense connectives of conditionals defined in (6).

The definition of SAC:

\[
\begin{align*}
\langle \langle (a|b) \rangle \rangle_{SAC} &= (a|b), \\
\langle \langle e \land e' \rangle \rangle_{SAC} &= \langle \langle e \rangle \rangle_{SAC} \land \langle \langle e' \rangle \rangle_{SAC}, \\
\langle \langle e \lor e' \rangle \rangle_{SAC} &= \langle \langle e \rangle \rangle_{SAC} \lor \langle \langle e' \rangle \rangle_{SAC}, \\
\langle \langle \neg e \rangle \rangle_{SAC} &= \neg 0 \langle \langle e \rangle \rangle_{SAC}, \\
\langle \langle (e|e') \rangle \rangle_{SAC} &= \langle \langle e \rangle \rangle_{SAC}| \langle \langle e' \rangle \rangle_{SAC}.
\end{align*}
\] (9)
The definition of GNW:

\[
\langle\langle (a|b) \rangle\rangle_{GNW} = (a|b), \\
\langle\langle e \land e' \rangle\rangle_{GNW} = \langle\langle e \rangle\rangle_{GNW} \land \langle\langle e' \rangle\rangle_{GNW}, \\
\langle\langle e \lor e' \rangle\rangle_{GNW} = \langle\langle e \rangle\rangle_{GNW} \lor \langle\langle e' \rangle\rangle_{GNW}, \\
\langle\langle \sim e \rangle\rangle_{GNW} = \sim \langle\langle e \rangle\rangle_{GNW}, \\
\langle\langle (e|e') \rangle\rangle_{GNW} = (\langle\langle e \rangle\rangle_{GNW}|\langle\langle e' \rangle\rangle_{GNW}).
\]

(10)

Given a probability space \((\Omega, \mathcal{F}(\Omega), \Pr)\), the probability of a conditional expression \(e\) is \(\Pr_{\text{sac}}(e) = \Pr(\langle\langle e \rangle\rangle_{\text{sac}} = 1)/\Pr(\langle\langle e \rangle\rangle_{\text{sac}} \neq \bot)\), and similarly \(\Pr_{\text{GNW}}(e) = \Pr(\langle\langle e \rangle\rangle_{\text{GNW}} = 1)/\Pr(\langle\langle e \rangle\rangle_{\text{GNW}} \neq \bot)\), provided that the denominators are nonzero.

All theses systems are readily seen to embed in our system of conditionals. In fact, if one represents them in the form of reduction rules as in (6), they do even embed syntactically in the \((\text{TL}|\text{TL})\) logic. They are present tense because they do not contain temporal connectives.

4.4 Product space \(cea\)

However, there is another \(cea\), called the product space \(cea\), which is not present tense. In order to analyze it and show that it can be interpreted in our model, we have to give the definition.

The semantics is as follows:

Beginning with \((\Omega, \mathcal{M}, \Pr)\), we form its countable power \(\Omega^\infty\) endowed with the product measure. The cylinder \(\underbrace{b \times \cdots \times b}_{j} \times a \times \hat{\Omega} \times \Omega \times \cdots \subseteq \Omega^\infty\) for \(a, b \in \Sigma\) is denoted \(\overbrace{b^j \times a \times \hat{\Omega}}\).

Define the semantics function \(\langle\langle \cdot \rangle\rangle_{PS}: PS \to \mathcal{P}(\Omega^\infty)\) by

\[
\langle\langle (a|b) \rangle\rangle_{PS} = \bigcup_{i=0}^{\infty} (\Omega \setminus b)^i \times (b \cap a) \times \hat{\Omega}, \\
\langle\langle e \land e' \rangle\rangle_{PS} = \langle\langle e \rangle\rangle_{PS} \land \langle\langle e' \rangle\rangle_{PS}, \\
\langle\langle e \lor e' \rangle\rangle_{PS} = \langle\langle e \rangle\rangle_{PS} \lor \langle\langle e' \rangle\rangle_{PS}, \\
\langle\langle \sim e \rangle\rangle_{PS} = \Omega^\infty \setminus \langle\langle e \rangle\rangle_{PS}.
\]

\(\mathcal{M}_o\) of the product space \(cea\) is then the subalgebra of the (boolean) algebra \(\langle\langle \mathcal{P}(\Omega^\infty), \cup, \cap, (\Omega^\infty \setminus \cdot) \rangle\\rangle\), generated by all elements \(b^j \times a \times \hat{\Omega}\) where \(a, b \in \Sigma\), and \(\Pr_o\) is the product measure.

There are indeed two versions of PS: one defined in the paper [Goo94], where equality of two conditionals is understood as true equality of sets, and another, defined in [GMN97], where the equality of conditionals is understood as equality almost everywhere, i.e., two conditional events of PS are equal
iff their symmetric difference has probability 0. The latter is therefore not logical, since it depends on the particular probability space structure.

The probabilities assigned to the elements of \( PS \) are those according to the infinite product of \( \Pr \).

### 4.5 First embedding

In order to construct the first embedding of \( PS \) into \( C \) by defining two operations \( \sigma(\cdot) : \mathcal{L} \to TL \) and \( \tau(\cdot) : \mathcal{L} \to (TL|TL) \) as follows:

\[
\begin{align*}
\sigma((a|b)) &= \text{first}_2(a|b) \\
\sigma(e \land e') &= \sigma(e) \land \sigma(e') \\
\sigma(e \lor e') &= \sigma(e) \lor \sigma(e') \\
\sigma(\neg e) &= \neg \sigma(e) \\
\tau(e) &= (\sigma(e)|\text{true})
\end{align*}
\]

\( \tau(e) \) (or, more formally, the conditional from \( C \) represented by the former) is the desired embedding.

**Lemma 6.** For every expression \( e \in \mathcal{L} \)

1. For every word \( w \in \Omega^\infty \) holds \( (\tau(e))_\infty(w) \in 2^\infty \);
2. Suppose \( (a_1|b_1), \ldots, (a_n|b_n) \) are all simple conditionals occurring in \( e \). Suppose \( w \in \Omega^\infty \) is so that \( b_{i_1}, \ldots, b_{i_k} \) are all events among \( b_1, \ldots, b_n \) which happen in the sequence \( w \), and all of them happen not later than at time \( m \). Then of the word \( (\tau(e))_\infty(w) \) is constant beginning since time \( m \).
3. \( \langle \langle e \rangle \rangle_{PS} = \{ w \in \Omega^\infty \mid (\tau(e))_\infty(w) \) is eventually constant \( 1 \} \);
4. \( \Pr_o(e) = \Pr(\tau(e)) \);

**Proof.** The proof of 1., 2. and 3. goes by simultaneous induction w.r.t. \( e \). For \( e = (a|b) \) they follows from a simple analysis of the definition of \( \langle \langle e \rangle \rangle_{PS} \) and \( \text{first}_2(a|b) \).

Now consider \( e = \neg e' \) and assume by induction that 1., 2. and 3. hold for \( e' \). Since \( \sigma(\neg e) = \neg \sigma(e') \), we have 1. and 2. immediately.

Moreover, \( (\neg \sigma(e')|\text{true})_\infty(w) \) is eventually constant \( 1 \) iff \( (\sigma(e')|\text{true})_\infty(w) \) is eventually constant \( 0 \), which by 1. and 2. for \( e' \) is equivalent to the fact that \( (\sigma(e')|\text{true})_\infty(w) \) is not eventually constant \( 1 \). This concludes the induction step of 3.

Induction steps for the other connectives are equally simple.

We turn now to 4. Suppose \( (a_1|b_1), \ldots, (a_n|b_n) \) are all simple conditionals occurring in a conditional expression \( \varphi \). W.l.o.g. assume \( \Pr(b_i) > 0 \) for
\( i = 1, \ldots, k \) and \( \Pr(b_i) = 0 \) for \( i = k + 1, \ldots, n \). (We permit \( k = 0 \) and \( k = n \), in which cases either all \( b_i \) are impossible, or all of them have positive probability.)

Represent \( \Omega^\infty \) as a disjoint union of sets

\[
A_n := \{ w \in \Omega^\infty / \text{ \( b_1, \ldots, b_k \) happen in } w \text{ and the first time they all have already happened is } n \} \\
\text{ and the set } A_\infty := \{ w \in \Omega^\infty / \text{ not all of } b_1, \ldots, b_k \text{ happen in } w \}. \text{ All these sets are clearly measurable.}
\]

It is not hard to verify, either, that

\[
\lim_{n \to \infty} \sum_{m \in \mathbb{N}} \Pr_n(A_m) = 1.
\]

These two equalities imply 4. immediately, since for every \( n \)

\[
\sum_{m \in \mathbb{N}, A_m \subseteq \langle \langle e \rangle \rangle_{PS}} \Pr_n(A_m) \leq \Pr_n(\tau(e)) \leq 1 - \sum_{m \in \mathbb{N}, A_m \cap \langle \langle e \rangle \rangle_{PS} = \emptyset} \Pr_n(A_m).
\]

\[\Box\]

The following theorem follows now instantly.

**Theorem 7.** \( \tau(\cdot) \) is an embedding of the PS cea into \( \mathcal{C} \), in the sense that for any underlying probability space, and any conditional expressions \( e, e' \), \( \tau(e) = \tau(e') \iff \langle \langle e \rangle \rangle_{PS} = \langle \langle e' \rangle \rangle_{PS} \), and \( \Pr(\tau(e)) = \Pr_\circ(e) \). \[\Box\]

### 4.6 Reverse embeddings

**Definition 8.** The reverse of a word \( w = \omega_1 \ldots \omega_n \in \Omega^+ \), denoted \( w^R \), is \( \omega_n \omega_{n-1} \ldots \omega_1 \).

Now consider a conditional \( c \in \mathcal{C} \). Then \( c^R \in \mathcal{C} \) is a conditional defined by

\[
c^R(w) := c(w^R).
\]

The class of languages definable in TL is reverse-closed [Eme90], i.e., if \( L = \{ w / w, |w| \models \varphi \} \) for some \( \varphi \in \mathsf{TL} \), then \( L^R = \{ w^R / w \in L \} = \{ w / w, |w| \models \psi \} \) for some \( \varphi \in \mathsf{TL} \). It follows that \( c^R \) is indeed a conditional in our sense.
Theorem 9. For every $\ast \in \mathfrak{I}$

$$\Pr([c]_n = \ast) = \Pr([c^R]_n = \ast).$$

Consequently, $\Pr_o(c) = \Pr(c^R)$.

Proof. $\Pr$ is understood here as a product measure, which is insensitive to the order of its coordinates. It follows that any cea, which can be at all isomorphically embedded in our stochastic process model, has at least two embeddings, which are reverses of each other. The only exception is when the embedding is invariant under reverse, i.e., when each conditional $c$ in the image of the embedding satisfies $c(w) = c(w^R)$ for all $w \in \Omega^+$. However, it seems unlikely that any reasonable embedding has this property. In particular, the natural embeddings of the cea’s we consider here are not of this kind.

As a matter of example, we consider here PS. For the PS conjunction, its informal description of its reverse representation in $\mathcal{C}$ is that it is always defined and true iff the most recent defined values of its both arguments were 1.

In (TL|TL), we have that $\tau^R((a|b) \land (c|d))$ (the reverse of the embedding $\tau(\cdot)$ defined in (11)) is defined by $((b^R\text{Since}(a \land b)) \land (d^R\text{Since}(c \land d))|\text{true})$, whose Moore machine is depicted below.

The precise definition of the reverse embedding of PS is as follows: first, we take the original conditional expression $e = e((a_1|b_1), \ldots, (a_m|b_m))$ and replace every $(a_i|b_i)$ occurring in it by $b_i^R\text{Since}(a_i \land b_i)$, obtaining $e' \in \text{TL}$, and then define $\tau^R(e) := (e'|\text{true})$. Formally:

$$\begin{align*}
\sigma^R((a|b)) &= b^R\text{Since}(a \land b) \\
\sigma^R(e \land e') &= \sigma^R(e) \land \sigma^R(e') \\
\sigma^R(e \lor e') &= \sigma^R(e) \lor \sigma^R(e') \\
\sigma^R(\neg e) &= \neg\sigma^R(e) \\
\tau^R(e) &= (\sigma^R(e)|\text{true}).
\end{align*}
$$

(12)

For this particular embedding, we have the following consequence of Theorem 9 (and the simple fact that reversing is an automorphism of the whole (TL|TL))

Theorem 10. $\tau^R(\cdot)$ is an embedding of the PS cea into $\mathcal{C}$, in the sense that for any underlying probability space, and any conditional expressions $e, e'$, $\tau^R(e) = \tau^R(e')$ iff $\langle \langle e \rangle \rangle_{PS} = \langle \langle e' \rangle \rangle_{PS}$, and $\Pr(\tau^R(e)) = \Pr_{o}(e)$.
Figure 3: Moore machine of $\tau^R((a|b) \land (c|d))$. 
4.7 Sparse reverse embedding

We give a new, radically different interpretation of PS in $\mathcal{C}$. The main difference is that it is not an embedding. We are going to present a way to interpret PS expressions in $\mathcal{C}$ so that, for any probability space $(\Omega, \mathcal{F}(\Omega), \Pr)$, the $\Pr_\omega$-probability of a PS-expression is equal to the asymptotic probability of its interpretation. However, expressions which yield equal element of the PS $cea$, may well give distinct conditional events in $\mathcal{C}$, although, as said before, these expressions will have equal asymptotic probabilities.

First of all, we redefine the meaning of simple conditionals $(a|b)$. According to the new embedding, they are represented by $(\text{TL} \mid \text{TL})$ formula

$$(a \land \neg (\# \neg b \lor \neg \neg \bullet \text{true}) \mid b \lor \neg \neg \bullet b).$$

Denote this formula by $(a|Sb)$.

It is essentially the simple conditional $(a|b) \in (\text{TL} \mid \text{TL})$, except that it is defined and false until $b$ becomes true for the very first time, and since then behaves exactly like $(a|b)$ does in $(\text{TL} \mid \text{TL})$. The manipulation is necessary to accommodate the PS principle, that degenerate simple conditionals, like $(a|0)$, do have probability, and that it is 0.

We now define the sparse reverse interpretation of PS in $\mathcal{C}$ as follows: First, we take the original conditional expression $e = e((a_1|b_1), \ldots, (a_m|b_m))$ and set $\tau^{RS}(e) = (\sigma^R(e) \mid \bigvee_{i=1}^m (b_i \lor \neg \neg \bullet b_i))$, where $\sigma^R(\cdot)$ has been defined in (12).

The conjunction and disjunction are defined precisely when at least one of the arguments is defined, so they resemble the connectives of SAC in this respect, but instead of assigning the other arguments default values when they are undefined, like SAC does, their most recent defined values are always used, instead. Here is an example Moore machine, in which we use $\epsilon$-moves. The graphical representation in Fig. 4 and Fig. 5 shows that the new operation is closely related to the reversed product space conjunction, as can be expected from the shape of the $(\text{TL} \mid \text{TL})$ representation.

**Lemma 11.** If $\Pr(b_i) = 0$ for at least one $1 \leq i \leq m$, then $\Pr(\tau^{RS}(e)) = \Pr(\tau(e))$.

**Proof.** In this case, assuming $\Pr(b_i) = 0$, we have that $\bigvee_{i=1}^m (b_i \lor \neg \neg \bullet b_i)$ is true with probability 1, hence $\llbracket \tau^{RS}(e) \rrbracket = \llbracket \tau^R(e) \rrbracket$ with probability 1. Now the thesis follows immediately from Theorem [12] \(\square\)

**Lemma 12.** If $\Pr(b_i) > 0$ for all $1 \leq i \leq m$, then $\Pr(\tau^{RS}(e)) = \Pr_\omega(\tau(e))$.

**Proof.** Let $e = e((a_1|b_1), \ldots, (a_n|b_n))$. We prove $\Pr(\tau^R(e)) = \Pr(\tau^{RS}(e))$, which is, by Theorem [12], equivalent to what we have to show.

Denote $t = \text{the first moment} m$ when all the $b_i$’s have already been defined.
Figure 4: Moore machine of $\tau_{RS}^R((a|b) \land (c|d))$, part 1. This part of the machine is the transient part of the Markov chain, when $\Pr(b), \Pr(d) > 0$ (Lemma 12) and the whole reachable part when at least one of these probabilities is 0 (Lemma 13). Subscripts of the state labels indicate the most recent value of $(a|b)$ and $(c|d)$, respectively.
Figure 5: Moore machine of $\tau^{RS}((a|b) \land (c|d))$, part 2. This part of the machine is the (only) ergodic class of the Markov chain, when $\Pr(b), \Pr(d) > 0$ (Lemma 12), and is unreachable when at least one of these probabilities is 0 (Lemma 11).

Subscripts of the state labels indicate the most recent value of $(a|b)$ and $(c|d)$, respectively.
Figure 6: Moore machine of \( \tau_{RS}(\{a|b\} \land \{c|d\}) \), part 3. This part of the machine is the table of the transitions from the “transient” part (Fig. 4) to the “ergodic” part (Fig. 5).

Let us note that, if \( t < n \), then the event \( [\tau_{RS}(e)]_n = \perp \) is independent of the whole history of \( [\tau_{RS}(e)] \) up to time \( n-1 \). This is so because the decision whether \( [\tau_{RS}(e)]_n \) is defined or not depends solely on the present time values of \( b_i \)'s, and their present time values become independent of the (strict) past, when the condition \( \bigvee_{i=1}^n \neg \blacksquare b_i \) becomes for the first time false, because it remains then false forever, and the “given” part of \( \tau_{RS}(e) \) does not contain any other time modalities.

Moreover, whenever \( [\tau_{RS}(e)](\omega_1 \ldots \omega_n) \neq \perp \), then in fact \( \tau_{RS}(e)(\omega_1 \ldots \omega_n) = \tau^R(e)(\omega_1 \ldots \omega_n) \), which is clear from the syntactic representation of both conditional objects.

Denote for convenience
\[
g = \Pr([\tau_{RS}(e)]_n = \perp|t < n) = 1 - \Pr(\bigvee_{i=1}^n b_i) < 1,\]
as well as and \( c = \tau^R(e) \) and \( c^S = \tau_{RS}(e) \).

Fix \( \varepsilon > 0 \). Let \( M \) be large enough to have \( \Pr(t \geq M) < \varepsilon \). Let \( N \) be a large integer, and let \( n \) satisfy \( n - N > M \).

We have then by the independence
\[ \Pr([c]_n = 1) \geq \sum_{i=0}^{N} \Pr([c^S]_{n-i} = 1) \Pr([c^S]_{n-i+1} = \bot) \cdots \Pr([c^S]_n = \bot) - \varepsilon, \]

\[ \Pr([c]_n = 0) \geq \sum_{i=0}^{N} \Pr([c^S]_{n-i} = 0) \Pr([c^S]_{n-i+1} = \bot) \cdots \Pr([c^S]_n = \bot) - \varepsilon, \]

where the \( \varepsilon \) error terms are caused by the event that \( t \geq M \).

Because each of the \( \Pr(\ldots) \) expressions above tends to a limit as \( n \) approaches infinity, we get

\[
\Pr(c) \geq \sum_{i=0}^{N} \lim_{n \to \infty} \Pr([c^S]_{n-i} = 1) \lim_{n \to \infty} \Pr([c^S]_{n-i+1} = \bot) \cdots \Pr([c^S]_n = \bot) - \varepsilon,
\]

\[
= \sum_{i=0}^{N} \lim_{n \to \infty} \Pr([c^S]_n = 1) \lim_{n \to \infty} \Pr([c^S]_n = \bot) \cdots \Pr([c^S]_n = \bot) - \varepsilon
\]

\[
= \sum_{i=0}^{N} \lim_{n \to \infty} \Pr([c^S]_n = 1)(\lim_{n \to \infty} \Pr([c^S]_n = \bot))^i - \varepsilon
\]

\[
= \sum_{i=0}^{N} \lim_{n \to \infty} \Pr([c^S]_n = 1)q^i - \varepsilon
\]

\[
= \frac{1 - q^N}{1 - q} \lim_{n \to \infty} \Pr([c^S]_n = 1) - \varepsilon,
\]

and similarly

\[
1 - \Pr(c) \geq \frac{1 - q^N}{1 - q} \lim_{n \to \infty} \Pr([c^S]_n = 0) - \varepsilon.
\]

In the limit \( N \to \infty \) both inequalities become

\[
\Pr(c) \geq \frac{1}{1 - q} \lim_{n \to \infty} \Pr([c^S]_n = 1) - \varepsilon
\]

\[
1 - \Pr(c) \geq \frac{1}{1 - q} \lim_{n \to \infty} \Pr([c^S]_n = 0) - \varepsilon,
\]

hence
$\Pr(c) \geq \frac{1}{1 - q} \lim_{n \to \infty} \Pr([c^S]_n = 1) - \varepsilon$

$\Pr(c) \leq 1 - \frac{1}{1 - q} \lim_{n \to \infty} \Pr([c^S]_n = 0) + \varepsilon$

$= \frac{1}{1 - q} (1 - q - \lim_{n \to \infty} \Pr([c^S]_n = 0)) + \varepsilon$

$= \frac{1}{1 - q} \lim_{n \to \infty} \Pr([c^S]_n = 1) + \varepsilon$,

because $1 = \Pr([c^S]_n = 1) + \Pr([c^S]_n = 0) + \Pr([c^S]_n = \bot)$, in which the last term is constant equal $q$.

We took an arbitrary $\varepsilon > 0$, therefore indeed

$$\Pr(c) = \frac{1}{1 - q} \lim_{n \to \infty} \Pr([c^S]_n = 1),$$

and likewise

$$1 - \Pr(c) = \frac{1}{1 - q} \lim_{n \to \infty} \Pr([c^S]_n = 0).$$

From the last two equalities the equality $\Pr(c) = \Pr(c^S)$ follows immediately. $\square$

Summing up,

**Theorem 13.** For every conditional expression $e$ and every probability assignment to the elements in $\Omega$, $\Pr_o(e) = \Pr(\tau RS(e))$. $\square$

The very important consequence of the theorem is the following:

**Corollary 14.** The formalism of PS cea, seen as a logic of conditionals, is unable to determine certain probabilistic characteristics, other than the asymptotic probability, associated with stochastic processes.

**Proof.** PS possesses two interpretations in $\mathcal{C}$, one of which consists entirely of always defined temporal conditionals (the $\tau(\cdot)$ embedding), while the second contains conditionals which are defined with asymptotic probability strictly less than 1 (the $\tau RS(\cdot)$ interpretation). $\square$

Another similar example of deficiencies of the PS cea can be found below, Theorem 28.

Let us note that the $\tau RS(\cdot)$ interpretation does not preserve the algebraic structure of the PS cea, in general. Indeed, already $\langle\langle(0|a)\rangle\rangle_{PS} = \langle\langle(0|b)\rangle\rangle_{PS}$, while $\tau RS((0|a)) = (a|sb)$ and $\tau RS((0|b)) = (0|sb)$ represent different conditional objects, for $a \neq b, a, b \in E$. 

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Theorem 28. Let us note that the $\tau RS(\cdot)$ interpretation does not preserve the algebraic structure of the PS cea, in general. Indeed, already $\langle\langle(0|a)\rangle\rangle_{PS} = \langle\langle(0|b)\rangle\rangle_{PS}$, while $\tau RS((0|a)) = (a|sb)$ and $\tau RS((0|b)) = (0|sb)$ represent different conditional objects, for $a \neq b, a, b \in E$. 

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5 Advantages of (TL|TL) conditionals

5.1 Complexity and proof systems

The paper \cite{Goo94} asks for the proof systems for various three-valued logics, appearing in the context of the theory of conditionals.

In order to discuss this issue, we use the machinery of complexity theory. All the necessary definitions can be found in \cite{HU79}.

As we prove below, for all of the major cea’s, the sets of weak tautologies are \(\text{coNP}\) complete.

In the light of the above results, there is little hope for a practically useful proof system for the most prominent systems among GNW and SAC. Indeed, unless \(\mathbb{NP} = \text{coNP}\), a very unlikely complexity-theoretic collapse, for every sound and complete proof system for the two above logics, there must be weak tautologies of length \(n\) such that their shortest proofs are of superpolynomial length w.r.t. \(n\), for infinitely many \(n\).

Temporal logic is known to be \(\mathbb{PSPACE}\)-complete, Theorem \ref{temporal-complexity}. It is therefore obvious that the set of weak tautologies of (TL|TL), consisting of all expressions \((\varphi|\psi)\) such that for every \(w \in 3^+\), \((\varphi|\psi)(w) \in \{1, \bot\}\) (equivalently: that \(\psi \rightarrow \varphi\) is a tautology of TL), is \(\mathbb{PSPACE}\)-complete, too.

Although it is commonly believed that \(\text{coNP} \subset \mathbb{PSPACE}\), from practical standpoint both admit exponential time algorithms, and no better ones are known. Consequently, the practical algorithmic difference between cea’s and (TL|TL) is not so crucial. The advantage of considering cea’s as subsystems of (TL|TL) stems from the fact that a lot is known about proof systems for temporal logic — unlike for cea’s.

We are not interested in the complexity of cea’s as logics involving terms \((a|b)\) as atoms, but rather as 3-valued logics. To explain the difference, let us note that in the calculus of any of the cea’s one can easily restrict atoms to be two-valued (e.g., by using only atoms of the form \((a|1)\)), and thus all the complexity questions trivialize, the sets of tautologies of all the logics are \(\text{coNP}\)-complete. Here, we assume that the atoms can always assume all three logical values, and are effectively variables. Hence we indeed view our (weak) tautologies as a kind of (weak) meta-tautologies, i.e., formulas which evaluate to either 1 or \(\bot\), no matter what the arguments are.

Formally, for this section we modify \(L\) and \(L^\dagger\) replacing simple conditionals \((a|b)\) by variables \(p_1, p_2, \ldots\) in conditional expressions.

Likewise, given a valuation \(v : \{p_1, p_2, \ldots\} \rightarrow 3\), we let \(\langle \cdot \rangle_v^\text{SAC}\) and \(\langle \cdot \rangle_v^\text{GNW}\) assign values in 3 to expressions in \(L^\dagger\). These values are determined by the equations in \ref{valuation-eq} and \ref{valuation-eq}.

An expression \(e\) is a weak tautology of SAC cea (GNW cea, respectively) iff \(\langle e \rangle_v^\text{SAC} \neq 0\) (\(\langle e \rangle_v^\text{GNW} \neq 0\), respectively) for every \(v\).

\(e\) is a strong tautology of SAC (GNW, respectively), iff the above values are 1 for every \(v\).
We consider the complexity problem of determining if an expression $e$ is a weak tautology according to each of the considered cea’s, considering also some syntactical restrictions put on the syntactical shape of $e$.

We do not consider strong tautologies, which is explained by the following.

**Proposition 15.** In GNW and SAC there are no strong tautologies.

**Proof.** All connectives have value $\perp$ if all their arguments are $\perp$, for both SAC and GNW. \qed

Occasionally, we want to consider $L\mid$ as the syntax of classical logic. In this case, given a valuation $v : \{p_1, p_2, \ldots\} \to 2$, the value $\langle\langle e\rangle\rangle_v^{\text{CL}} \in 2$ is computed according to the classical rules, where the conditioning $|$ is understood as reverse implication: $(a|b)$ is $a \leftarrow b$.

**Pure conditional parts.** We consider here pure conditional fragments of SAC and GNW, i.e., expressions in which the only connective used is $\mid$.

It shows that unlimited use of re-conditioning leads to coNP-completeness of the weak tautology problem.

**Theorem 16.** It is a coNP-complete problem to determine if an expression $e \in L\mid$ involving only re-conditioning is a weak tautology of SAC.

It is an coNP-complete problem to determine if an expression $e \in L\mid$ involving only re-conditioning is a weak tautology of GNW.

**Proof.** It is obvious that the sets of weak tautologies in both cases are in coNP. So it remains to prove their hardness in this complexity class.

It is easily seen that the (re-)conditioning operators of both GNW and SAC satisfy the following property: the equivalence relation $\approx$ on $3$ identifying $1$ with $\perp$ is a congruence of $\mathfrak{A} = (3, |_{\text{SAC}})$ and $\mathfrak{B} = (3, |_{\text{GNW}})$, and the quotient algebras both $\mathfrak{A} = (3, |_{\text{SAC}})/\approx$ and $\mathfrak{B} = (3, |_{\text{GNW}})/\approx$ are isomorphic to the 2-element algebra with the reversed classical implication $\langle2, \leftarrow\rangle$. The natural epimorphism $\eta_\mathfrak{A} : \mathfrak{A} \to \langle2, \leftarrow\rangle$ sends $1$ and $\perp$ to $1$, and $0$ to $0$, and the definition of $\eta_\mathfrak{B}$ is identical.

Therefore a pure conditional expression is a weak tautology of either of the considered cea’s iff it is a classical tautology, after its (re-)conditioning operator is replaced by the reversed classical implication. The classical formula resulting from this replacement is denoted $\tilde{e}$.

We have to prove that $e$ is not a weak tautology iff $\tilde{e}$ is not a tautology. Let $v$ be any valuation of the variables of $e$ in $3$. Now we use the natural epimorphism $\eta_\mathfrak{A}$ and get

$$\langle\langle \tilde{e}\rangle\rangle_v^{\text{CL}} = \eta_\mathfrak{A}(\langle\langle e\rangle\rangle_v^{\text{SAC}}).$$

So if one of the values above can be 0, the other can be, as well, which establishes the desired equivalence.
Since it is known that the tautologies of the classical propositional logic of pure implication are coNP complete \cite{Heu95}, the claim follows. \hfill \qed

As a by-product we have

**Corollary 17.** The sets of pure conditional weak tautologies of SAC and GNW are identical. \hfill \qed

**Flat parts.** Here we consider SAC and GNW without re-conditioning. We can define the following NP-complete problem 3CNF-SAT.

Given: an expression $e \in \mathbb{L}$ of the following syntactical form:

$$e = (\ell_{11} \lor \ell_{12} \lor \ell_{13}) \land (\ell_{21} \lor \ell_{22} \lor \ell_{23}) \land \cdots \land (\ell_{s1} \lor \ell_{s2} \lor \ell_{s3}),$$  \hspace{1cm} (13)

where each of the $\ell_{ij}$ is either $p_j$ or $\sim p_j$.

The NP-complete problem is: given $e$ of the above shape, determine if $e$ is satisfiable, i.e. if there exists $v$ such that $\langle e \rangle^v_{CL} = 1$.

It follows that it is coNP-complete to determine, given $e$ as above, if $e$ is not satisfiable, i.e., whether $\langle \sim e \rangle^v_{CL} = 1$ for every $v$.

In order to prove coNP-completeness of the sets of weak tautologies of either of the cea’s, we have to construct a polynomial time computable transformation $e \mapsto \bar{e}$ translating $e$ of the form (13) into $\bar{e}$ of the form conforming to the restriction set in the respective theorem, and such that $\sim e$ is not satisfiable in the classical sense iff $\bar{e}$ is a weak tautology of the respective logic.

**Theorem 18.** It is an coNP-complete problem to determine if an expression $e \in \mathbb{L}$ is a weak tautology of SAC.

It is an coNP-complete problem to determine if an expression $e \in \mathbb{L}$ is a weak tautology of GNW.

**Proof.** It is easily seen that the connectives $\land_{GNW}$ and $\lor_{GNW}$ satisfy again the property that the equivalence relation $\approx$ on $\mathbb{3}$ identifying $1$ with $\bot$ is a congruence of the algebra with the above functions, and the quotient algebra $\langle \mathbb{3}, \land_{GNW}, \lor_{GNW} \rangle / \approx$ is isomorphic to the classical $\langle \mathbb{2}, \land, \lor \rangle$. This fails about the negation, however.

As the negation is applied to atoms only in 3CNF-SAT, we do not have to use the negation of GNW directly. Instead, we introduce new variables to denote the negations, and force them to behave correctly outside of the translation of $e$.

Formally, let the mapping $e \mapsto e'$ from the classical propositional logic into GNW be defined by replacing unnegated atoms $p$ in $e$ by $\bar{p}$ and negated atoms $\sim p$ by $\bar{p}$. Concerning binary connectives, we leave $\land$ and $\lor$ untouched. Then let $\bar{e}$ be defined as $\sim (e' \land \bigwedge_{p}(\bar{p} \lor \bar{p}) \land (\sim \bar{p} \lor \sim \bar{p}))$, where $p$ in the big conjunction ranges over all propositional variables of $e$. 

Certainly the mapping \( e \mapsto \bar{e} \) is computable in polynomial time. In order to show the co\( \text{NP} \) completeness of the set of tautologies of GNW, it suffices to show two implications:

- if \( e \) is satisfiable classically, then \( \bar{e} \) is not a weak tautology of GNW.
- if \( \bar{e} \) is not a weak tautology of GNW, then \( e \) is satisfiable classically.

For the first item, assume that \( e \) is satisfiable, i.e., there is an assignment \( v \) of 0’s and 1’s to the propositional variables of \( e \) which makes \( e \) into 1. We construct a 3-valued assignment \( w \) which makes \( \bar{e} \) into 0. If \( v(p) = 1 \), we let \( w(\hat{p}) = 1 \) and \( w(\bar{p}) = 0 \). If \( v(p) = 0 \), we let \( w(\hat{p}) = 0 \) and \( w(\bar{p}) = 1 \). In \( e' \) each variable has under \( w \) exactly the value of the corresponding literal in \( e \) has under \( v \). So \( e' \) evaluates to 1, because connectives in GNW behave classically for classical arguments. In addition, each of the formulas \( (\hat{p} \lor \bar{p}) \land (\sim \hat{p} \lor \sim \bar{p}) \) evaluates to 1, so altogether \( \bar{e} \) evaluates to the \( \sim_0 \)-negation of the value to which \( e' \) does evaluate, which is 0, as desired.

For the second item, assume there is an assignment \( w \) of 0’s, 1’s and \( \perp \)’s to the propositional variables of \( \bar{e} \) which makes it 0. It follows that each of the terms \( (\hat{p} \lor \bar{p}) \land (\sim \hat{p} \lor \sim \bar{p}) \) must evaluate to 1 under \( w \). Therefore of each pair \( \hat{p}, \bar{p} \), one variable must be assigned 1 and the other 0 by \( w \), which can be checked by simple inspection of all possibilities. Moreover, \( e' \) must evaluate to 1 under \( w \), which is indeed 2-valued, by the previous observation. The connectives of GNW act classically for classical arguments, therefore \( e \) is indeed classically satisfiable, by the valuation \( v : p \mapsto w(\hat{p}) \).

This finishes the proof.

**Theorem 19.** The weak flat-conditional SAC is co\( \text{NP} \)-complete.

**Proof.** We are going to use the same proof idea as before. However, we have a small problem. The conjunction of SAC does not permit us to deduce, that if a conjunction of two formulas evaluates to 1, so does each of the components.

So instead of the original conjunction, we have to use some custom connective defined from the conjunction, disjunction and negation, which will act as a “good” conjunction, for which the inference does hold. It turn out, that the conjunction of GNW is not definable in SAC, but there is another connective we can use instead, and which is definable (we discuss the definability of connectives in SAC and GNW in another paper [CWTHR01]). Its definition is as follows:

\[
x \sqcap y \equiv [x \lor (y \land (x \lor \sim y))] \land [y \lor (x \land (y \lor \sim x))].
\]

It is not difficult (but tedious) to check, that \( \langle x \sqcap y \rangle_{\text{SAC}} \) has truth table
which is exactly what we need for our purposes. The only subtle point is that our $x \sqcup y$ is substantially longer than $|x| + |y|$. Indeed it is about 4 times longer. We do replace $\wedge$ by $\sqcap$ in very long conjunctions. However, if we represent this long conjunction as a balanced binary (parse) tree, i.e., insert brackets to obtain the structure

$$(((\ldots \sqcap \ldots) \sqcap (\ldots \sqcap \ldots)) \sqcap ((\ldots \sqcap \ldots) \sqcap (\ldots \sqcap \ldots)))$$

the depth of nesting of conjunctions is at most $\log_2 N$ of the number $N$ of clauses in the conjunction, and the total increase of length caused by the replacement is $4^{\text{depth of nesting}} = 4^{\log_2 N} = N^2$. Altogether, the resulting formula, using $\sqcap$ in place of $\wedge$, is still of polynomial size, and can be easily constructed in polynomial time, as needed.

5.2 Independence of conditional events

There has been a considerable amount of interest in the independence issue for conditional events, reflected in the $cea$ literature [Goo94, Cal97, Pea88]. The problem is that typically even for $a$ and $b$ mutually independent of $c$ and $d$ one does not have $\Pr((a|b) \land (c|d)) = \Pr((a|b)) \Pr((c|d))$. The only exception is PS, where this equality holds. The other variant of independence: $\Pr((a|b)|(c|d)) = \Pr((a|b))$ is undefined in some formalisms, due to the lack of re-conditioning operator, and fails in others. However, note that in the $cea$ framework one cannot obtain any proper characterization of independence, because there is no underlying probabilistic semantics, in which one could say which pairs of conditionals are independent and which aren’t, and then attempt to characterize this by equalities among probabilities. One feels that $(a|b)$ and $(c|d)$ should be independent for mutually independent arguments, but this is not more than a feeling, and there is no idea there what might make two conditionals independent when their arguments are not mutually independent, or when they are composite.

We can address this problem in our semantical setting. First of all, for $a$ and $b$ mutually independent of $c$ and $d$, the stochastic processes $[(a|b)]$ and $[(c|d)]$ are obviously independent. And of course, the independence of the stochastic processes is what the independence of conditionals should be. This remains true, no matter which $cea$ we consider. It is, however, a different story if this independence can be formally characterized in terms of equalities between
probabilities of conditionals in the *cea* under consideration. It appears that in the pure *cea* formalism this cannot be achieved, because in Theorem 29 below we show that independence is undefinable in the PS *cea*.

To be precise, the full independence of stochastic processes $X, Y$ means that the full histories of both processes are independent, which is different from the much less restrictive requirement that just the present time values should be independent. The first version is formalized by the requirement that $X_{+,t}$ and $Y_{+,t}$ are independent at any time $t > 0$, i.e., for any $w_1 \ldots w_t, v_1 \ldots v_t \in 3^t$ holds

$$
\Pr(X_1 = w_1, \ldots, X_t = w_t) = \Pr(X_1 = w_1, \ldots, X_t = w_t) \Pr(Y_1 = v_1, \ldots, Y_t = v_t).
$$

The weaker, present tense independence requires only that $X_t$ and $Y_t$ are independent at any time $t > 0$, i.e., that for any $w, v \in 3$ holds $\Pr(X_t = w, Y_t = v) = \Pr(X_t = w) \Pr(Y_t = v)$. To see the difference it is worth noting that for any present tense $(TL|TL)$ formula $((\varphi|\psi))$ the processes $[\varphi|\psi]$ are present tense independent, although of course they are easily seen to be dependent, unless the former is constant.

But let us note the following simple fact.

**Lemma 20.** If $c_1$ and $c_2$ are two present tense conditionals, they are independent iff they are present tense independent.

We know now what independence should mean. It is another story how to characterize it in terms of the asymptotic probability of conditionals.

First we prove the characterization for present tense independence at fixed time.

Let $\uparrow (a|b) := (b|true)$.

**Lemma 21.** Let $n$ be a fixed time instant. The following are equivalent:

- Random variables $[(a|b)]_n$ and $[(c|d)]_n$ are independent.
- The following four equalities hold:

$$
\Pr_n((a|b) \land_{Sch} (c|d)) = \Pr_n((a|b)) \Pr_n((c|d)) \quad (14)
$$
$$
\Pr_n((a|b) \land_{Sch} (\uparrow (c|d))) = \Pr_n((a|b)) \Pr_n((\uparrow (c|d))) \quad (15)
$$
$$
\Pr_n((\uparrow (a|b) \land_{Sch} (c|d))) = \Pr_n((\uparrow (a|b))) \Pr_n((c|d)) \quad (16)
$$
$$
\Pr_n((\uparrow (a|b) \land (\uparrow (c|d))) = \Pr_n((\uparrow (a|b))) \Pr_n((\uparrow (c|d))), \quad (17)
$$

where we assume an equation to hold in case when both sides are undefined.
Proof. \(\Downarrow\) Independence of random variables \([[(a|b)]_n\) and \([[(c|d)]_n\) implies, in particular, that

\[
\Pr([[a|b]]_n = 0, 1, [[c|d]]_n = 0, 1) = \Pr([[a|b]]_n = 0, 1) \Pr([[c|d]]_n = 0, 1),
\]

which is exactly equivalent to (17). The other consequences of independence are equalities

\[
\begin{align*}
\Pr([[a|b]]_n = 1, [[c|d]]_n = 1) &= \Pr([[a|b]]_n = 1) \Pr([[c|d]]_n = 1) \quad \text{(14')} \\
\Pr([[a|b]]_n = 1, [[c|d]]_n = 0, 1) &= \Pr([[a|b]]_n = 1) \Pr([[c|d]]_n = 0, 1) \quad \text{(15')} \\
\Pr([[a|b]]_n = 0, 1, [[c|d]]_n = 1) &= \Pr([[a|b]]_n = 0, 1) \Pr([[c|d]]_n = 1) \quad \text{(16')} 
\end{align*}
\]

which, divided by (17'), yield (14), (15) and (16), respectively. Note that if both sides of (17') are 0, then all the resulting equalities involve an undefined term on both sides, and hence hold, according to our convention.

\(\uparrow\) Let (17') (i.e., (17)) hold. If its both sides are 0, the random variables \([[(a|b)]_n\) and \([[(c|d)]_n\) are independent, because one of them is constant. So let us assume (17') holds and its both sides are nonzero. In particular, each of the (14'), (15') and (16') is defined on both sides, because the denominators are everywhere nonzero. Multiplying these equalities by (17'), we get (14'), (15') and (16'), respectively. It is now a matter of routine to prove that the independence of \([[(a|b)]_n\) and \([[(c|d)]_n\) follows from (14'), (15') and (16') and (17').

The lemma allows us to characterize independence for present tense conditionals.

**Theorem 22.** For present tense \(a|b\) and \(c|d\) the following are equivalent:

- Stochastic processes \([[(a|b)]\) and \([[(c|d)]\) are independent.

- The equalities (14)–(17) hold with \(\Pr_n\) replaced by \(\Pr\) in each term, where we again assume an equation to hold in case when both sides are undefined.

Proof. For present tense \(a|b\) the probability \(\Pr_n((a|b))\) is independent of \(n\), and is (of course) equal to \(\Pr((a|b))\). Now Lemmas 20 and 21 give us the desired equivalence. \(\square\)
The full characterization of independence for general temporal conditionals is not known at the moment. Most likely, if it at all exists, it must be nonuniform, in the sense that the number of equalities between probabilities depends in principle on the actual \((\varphi|\psi)\) and \((\zeta|\xi)\).

However, there is a quite general sufficient condition for independence, which can be (nonuniformly) characterized by equalities of asymptotic probability. Call two conditionals \(a, b\) strongly independent iff there exist stochastically independent Markov chains \(X\) and \(Y\) and projections \(h, g\) such that \([a] = h(X)\) and \([b] = g(Y)\).

**Theorem 23.** Strong independence of conditional events from \((\mathcal{TL}|\mathcal{TL})\) can be equivalently characterized by equations of asymptotic probability.

We begin with

**Lemma 24.** Let Markov chains \(X, Y\) have \(n\) and \(m\) states, respectively. If \(X\) and \(Y\) are independent until time \(mn + 1\), they are fully independent, i.e., if

\[
\operatorname{Pr}(X_1 = w_1, \ldots, X_t = w_t, Y_1 = v_1, \ldots, Y_t = v_t) = 
\operatorname{Pr}(X_1 = w_1, \ldots, X_t = w_t) \operatorname{Pr}(Y_1 = v_1, \ldots, Y_t = v_t) \quad (18)
\]

holds for all \(t \leq mn + 1\) and all sequences \(w_1, \ldots, w_t, v_1, \ldots, v_t\) of states of \(X\) and \(Y\), respectively, then \(X\) and \(Y\) are independent and \((18)\) holds indeed for all \(t\).

**Proof.** First of all, observe that \((X, Y) = (X_1, Y_1), (X_2, Y_2), \ldots\) is a Markov chain, as well.

Suppose that \((18)\) fails and that the least \(t\) for which it fails is \(t > mn + 1\) (because for \(t \leq mn + 1\) \((18)\) holds by assumption).

The in-equality

\[
\operatorname{Pr}(X_1 = w_1, \ldots, X_t = w_t, Y_1 = v_1, \ldots, Y_t = v_t) \neq
\operatorname{Pr}(X_1 = w_1, \ldots, X_t = w_t) \operatorname{Pr}(Y_1 = v_1, \ldots, Y_t = v_t) \quad (19)
\]

is by Markov property \((2)\) for \(X, Y\) and \((X, Y)\) equivalent to

\[
\operatorname{Pr}(X_1 = w_1, \ldots, X_{t-1} = w_{t-1}, Y_1 = v_1, \ldots, Y_{t-1} = v_{t-1}) \operatorname{Pr}(X_t = w_t | X_{t-1} = w_{t-1}) \neq 
\operatorname{Pr}(X_1 = w_1, \ldots, X_{t-1} = w_{t-1}) \operatorname{Pr}(X_t = w_t | X_{t-1} = w_{t-1}) \times 
\operatorname{Pr}(Y_1 = v_1, \ldots, Y_{t-1} = v_{t-1}) \operatorname{Pr}(Y_t = v_t | Y_{t-1} = v_{t-1}),
\]
which in turn is equivalent to

\[
\Pr \left( \begin{array}{c} X_t = w_t \\ Y_t = v_t \end{array} \right | \begin{array}{c} X_{t-1} = w_{t-1} \\ Y_{t-1} = v_{t-1} \end{array} \) \neq \Pr(X_t = w_t | X_{t-1} = w_{t-1}) \Pr(Y_t = v_t | Y_{t-1} = v_{t-1}),
\]

(20)

because \( t \) is the least one for which inequality holds, and so the non-conditional probabilities in the previous inequality cancel out.

Moreover, the canceling terms must be nonzero for the inequality to hold, which means \((w_{t-1}, v_{t-1})\) is reachable with positive probability from the initial state in \((X, Y)\). But therefore it must be reachable with positive probability in at most \( mn \) steps, because there are exactly so many states in \((X, Y)\). So let \((x_i, y_i), i = 1, \ldots, s \leq mn \) be a sequence of states of \((X, Y)\) leading to \((x_s, y_s) = (w_{t-1}, v_{t-1})\) with positive probability. By assumption

\[
\Pr \begin{cases} X_1 = x_1, & \ldots, \ X_s = x_s \\ Y_1 = y_1, & \ldots, \ Y_s = v_s \end{cases}
= \Pr(X_1 = x_1, \ldots, X_s = x_s) \Pr(Y_1 = y_1, \ldots, Y_s = v_s),
\]

(21)

because \( s \leq mn \). If we now multiply the above by (20), we get, by a calculation reverse to what we have done above, an instance of (19) with \( t \leq mn + 1 \), a contradiction.

\textbf{Lemma 25.} For given Markov chains \( X \) and \( Y \) and for a fixed time \( t \), fixed sequences \( w_1, \ldots, w_t \) and \( v_1, \ldots, v_t \) of states of \( X \) and \( Y \), respectively, the formula (18) can be equivalently characterized by equalities among asymptotic probabilities of certain conditionals, derived from \( X \) and \( Y \).

\textbf{Proof.} Let \( A \) and \( B \) be the deterministic finite automata, underlying \( X \) and \( Y \). For a state \( w \) of \( A \) let \( A_w \) be the Moore machine resulting from \( A \) by labeling the state \( w \) with 1 and all the remaining states with 0. Since all \( A_w \)'s are 2-valued, there exist TL formulas \( \alpha_w \), which are true precisely when the last symbol of the output of \( A_w \) is 1. Similarly we define \( B_v \) and \( \beta_v \).

Now (18) is equivalent to

\[
\Pr(\bullet^t \text{true} \land \neg \bullet^{t+1} \text{true} \land \bigwedge_{i=1}^t (\bullet^t \neg (\alpha_{w_i} \land \beta_{v_i}))) =
\]

\[
\Pr(\bullet^t \text{true} \land \neg \bullet^{t+1} \text{true} \land \bigwedge_{i=1}^t (\bullet^{t-i} (\alpha_{w_i}))) \times
\]

\[
\Pr(\bullet^t \text{true} \land \neg \bullet^{t+1} \text{true} \land \bigwedge_{i=1}^t (\bullet^{t-i} (\beta_{v_i}))),
\]
Each of the TL formulas asserts that it has been once that there was something $t - 1$ steps ago, but there was nothing $t$ steps ago (so we have been at time $t$ precisely), and we were in the prescribed states of the Markov chain in question at times: $t$, one step before that, $\ldots$, $t - 1$ steps before that.

What remains to be seen is that we can indeed choose some canonical Markov chains $\mathcal{X}$ and $\mathcal{Y}$ to represent $a$ and $b$, which are independent whenever $a$ and $b$ are strongly independent.

Let us recall, that any conditional event in our model is a projection of a Markov chain, derived from a Moore machine for the underlying conditional object. Since for every Moore machine there exists the minimal Moore machine computing the same function, in presence of probabilities, we thus always have the minimal Markov chain underlying any given conditional event.

**Lemma 26.** Let $a$ and $b$ be strongly independent. Then the minimal Markov chains for $a$ and $b$ are independent.

**Proof.** Let $\mathcal{X}$ and $\mathcal{Y}$ be two independent Markov chains, underlying $a$ and $b$. Applying the quotient construction to $\mathcal{X}$ and $\mathcal{Y}$ we pass to the minimal Markov chains underlying $a$ and $b$. The quotient construction is deterministic, and therefore it does not break independence (exactly like strong independence implies independence). It follows that the minimal Markov chains are independent, too.

**Proof of Theorem 23.** The conditional events $a$ and $b$ are strongly independent iff the minimal Markov chains underlying them are independent, by Lemma 26. The latter can be expressed equivalently by $(mn + 1)^{mn}$ conditions of the form (18) for minimal chains of $m$ and $n$ states, respectively, by Lemma 24. Each of these conditions in turn can be expressed equivalently by a single equality of asymptotic probabilities of certain conditional objects. This means that the strong independence of $a$ and $b$ can be equivalently characterized by a set of equalities among asymptotic probabilities of conditionals, which can be syntactically determined from $a$ and $b$ and do not depend on the probability space structure.

Of course, for conditionals which are themselves Markov chains for any probability assignment, strong independence is the same as independence. Therefore we have

**Corollary 27.** For conditionals which are themselves Markov chains for any probability assignment, independence can be characterized by equalities of asymptotic probabilities.
The conditionals to which this applies can be recognized by the property that their minimal Moore machine has at most one state labeled by each element of $3$ (and thus at most three states altogether). Present tense conditionals are of this kind, and thus we have an alternative proof of Theorem 22, which much less elegant set of equalities, however. But present tense conditionals do not exhaust all conditionals, which are Markov chains. An example is the conditional $(a|\Box((\Diamond a \rightarrow a^\Box) \land (\Diamond a^\Box \rightarrow a) \land (\neg \Diamond true \rightarrow a)))$, analyzed in [TRH01]. Its minimal Moore machine is depicted below.

![Moore machine](image)

**Figure 7:** Moore machine of $(a|\Box((\Diamond a \rightarrow a^\Box) \land (\Diamond a^\Box \rightarrow a) \land (\neg \Diamond true \rightarrow a)))$.

Therefore Theorem 23 is indeed stronger than Theorem 22.

Finally, we consider the question of PS $cea$, for which one might want a characterization of independence in terms of asymptotic probability. Here we give a negative answer.

**Theorem 28.** There exist two conditional expressions $e_1$ and $e_2$ and a probability space such that the embeddings $\tau(e_1)$ and $\tau(e_2)$ are independent, while their sparse reverse counterparts $\tau^{RS}(e)$ and $\tau^{RS}(e_2)$ are not independent.

**Proof.** Take $e_1 = e_2 = (0|a)$ and any probability space with $0 < \Pr(a) < 1$. Then $[\tau(e_1)]$ and $[\tau(e_2)]$ are constant processes, equal to 0, so they are (trivially) independent. However, already

\[
\Pr([\tau^{RS}(e_1)](w) = 0\perp) = \Pr([\tau^{RS}(e_1)](w) = 0\perp) > (\Pr([\tau^{RS}(e_1)](w) = 0\perp))^2 = \Pr([\tau^{RS}(e_1)](w) = 0\perp) \cdot \Pr([\tau^{RS}(e_2)](w) = 0\perp),
\]

where the inequality holds because $\Pr([\tau^{RS}(e_1)](w) = 0\perp) = \Pr(a)(1 - \Pr(a)) \neq 0, 1$. \qed
Corollary 29. There is no characterization of independence in PS $cea$ in terms of equalities of asymptotic probability.

Proof. Because both $\tau(\cdot)$ and $\tau^RS(\cdot)$ preserve all asymptotic probabilities of conditionals, both of them satisfy precisely the same equalities of asymptotic probabilities. So if there were a characterization of independence in terms of equalities of such probabilities, the interpretations of the two conditionals $(a|0)$ and $(a|0)$ above would have to be either independent in both cases, or dependent in both cases, while they are not, a contradiction. \qed

The consequence is that in the $cea$ formalism is not expressive enough to define independence of conditionals by means of equalities of asymptotic probabilities. Note however, that such a representation is certainly possible by means of equalities of probabilities and equalities of the algebraic structure. Indeed, PS $cea$ is boolean algebra with respect to its connectives $\land, \lor, \sim$ (as it is easily visible from its syntactic representation within (TL|TL)), and the equalities it satisfies enforce, that $Pr_o$ is an ordinary probability measure. Therefore independence is equivalent to the standard equality $Pr_o(\langle\langle e_1 \land e_2 \rangle\rangle_{PS}) = Pr_o(\langle\langle e_1 \rangle\rangle_{PS}) Pr_o(\langle\langle e_2 \rangle\rangle_{PS})$. What we have constructed are two non-boolean subsystems of (TL|TL), in which all the (asymptotic) probability assignments agree with those of $Pr_o$, and yet no set of equalities of probabilities can characterize the true probabilistic independence in both of them simultaneously.

5.3 Algorithms

Polynomial algorithm for PS $cea$. Let us see that our approach provides a nontrivial improvements to the algorithmic status of existing $cea$'s. We will demonstrate this by calculating the probabilities of conditional expressions, according to PS $cea$, in time polynomial in their size and exponential in the number of variables. (Note that the number of arguments for computation of the probability of an $n$-ary conditional is $2^n$, so the above indeed means computation polynomial in the size of the input.) In [Goo94] it is stated that the computation of the PS-probability of a conjunction of $n$ conditionals $(a_i|b_i)$, according to the method used in that paper, requires adding $\sum_{m=1}^{n} m! \cdot S_0(m, n) \cdot (2^{m+1} - 2)$ terms, each being a nonconditional probability of a conjunction of certain events $a_i$ and $b_i$. The number of summands, where $S_0(m, n)$ are Stirling’s number of the second kind, is of order $2^n \log n$. It is substantially more than about $c2^n$ one obtains for the present tense $cea$’s SAC and GNW, and has been stressed in [GMN97, p. 499] and in [Fox], since it strongly affects the usefulness of PS as a tool for applications. Using our approach we have instantly an algorithm to calculate the same probability in $2^{O(n)}$ steps. As a matter of fact, this applies to any conditional expression with $n$ arguments $(a_i|b_i)$, $i = 1 \ldots n$, as long as its length does not exceed $2^{O(n)}$. All the complexity bounds given here assume unit
cost of basic arithmetical operations: addition, multiplication, subtraction and division.

**Theorem 30.** There is an algorithm, computing the PS probability of an $n$-ary conditional expression of length $m$ in time polynomial in $\max(m, 2^n)$.

**Proof.** The first step of the algorithm on input expression $e$ is to construct the minimal Moore machine, computing $\tau(e)$.

**Lemma 31.** The minimal Moore machine of $\tau(e)$ for $n$-ary conditional expression $e$ has at most $3^n$ states.

**Proof.** The minimal Moore of first($a|b$) has 3 states (see (8) and Figure 2). By the definition of the $\tau(\cdot)$ embedding (Section 4.3), a Moore machine $A = (Q, \Omega, \delta, h, q_0)$ of $\tau(e)$ can be constructed as follows:

1. The set $Q$ of states of $A$ is the product $Q_1 \times \cdots \times Q_n$ of state sets of Moore machines $A_i = (Q_i, \Omega_i, \delta_i, h_i, q_{0i})$ of all expressions first($a_i|b_i$) occurring in $\tau(e)$.
2. The transition function of $A$ is defined coordinate-wise, i.e.,
   \[ \delta(\langle q_1, \ldots, q_n \rangle, \omega) = \langle \delta_1(q_1, \omega), \ldots, \delta_n(q_n, \omega) \rangle, \]

   the initial state is $q_0 = \langle q_{01}, \ldots, q_{0n} \rangle$, and, crucially,

   \[ h(\langle q_1, \ldots, q_n \rangle) = \hat{c}(h_1(q_1), \ldots, h_n(q_n)), \]

   where $\hat{c}(h_1(q_1), \ldots, h_n(q_n))$ is the classical logic evaluation of the expression $c$ on arguments $h_1(q_1), \ldots, h_n(q_n) \in \mathbb{2}$.

This product construction is well-known for automata theory, and it is immediate that it does the work. \[\square\]

So it is quite easy to construct, given $e \in \mathcal{L}$, the Moore machine of $\tau(e)$. Now we have to turn this Moore machine into a Markov chain. Assuming that all the probabilities of atomic events from $\Omega$ are given, we simply replace multiple transitions between the same states represented by the sum of their probabilities—a single number.

Furthermore, the Markov chain we obtain is absorbing, i.e., it has one-element ergodic classes. It can be proven by a straightforward induction on $n$ — the number of three element Moore machines we product. It follows [KS76, Chapter III] that we can use the following method to compute the limiting probability that the chain finally arrives at a state labeled by 1.

Clearly, in this situation we can collapse all absorbing states labeled 1 into a single such state.

Denote by $P$ the matrix $(p(i, j))$ of transition probabilities, by $Q$ the sub-matrix of rows and columns corresponding to transient states, and by $R$
the submatrix of rows corresponding to transient states and columns corresponding to absorbing states. Let $Id$ be a diagonal matrix with 1’s on the diagonal and 0’s elsewhere. Let $B = (Id - Q)^{-1}R$. Then the probability we are looking for is the entry in $B$ in the row corresponding to the initial state in in the column corresponding to the (only) absorbing state labeled by 1 in the Markov chain. Since all the calculations on matrices necessary to compute $B$ are doable in time polynomial in the size of the matrices, the total computation time is $(2^n)^O(1) = 2^{O(n)}$, as desired.

6 Summary

We have discussed the temporal calculus of conditional objects and conditional events (TL|TL) as a formalism alternative to conditional event algebras. We have shown that all the major conditional event algebras, including those of Schay-Adams-Calabrese, Goodman-Nguyen-Walker and the product space $cea$, embed isomorphically in (TL|TL). Moreover, (TL|TL) is superior to those formalisms in several ways:

- It provides natural, probabilistic semantics of conditionals, allowing one to construct experiments to evaluate all their interesting probabilistic parameters, unlike $cea$’s, which generally are not probability spaces, and which do not require certain probabilistic parameters to be defined at all.

- The construction of (TL|TL) is functorial, in the sense, that the underlying probabilistic space of nonconditional events determines the space of temporal conditional events uniquely, while $cea$’s generally are not unique.

- The formalism of (TL|TL) allows one to define and analyze independence of conditional events, which is difficult or impossible in $cea$’s.

- (TL|TL) offers better algorithms for calculation of probabilities, than those known previously for $cea$’s.

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