Theory of electric-field-induced spin accumulation and spin current in the two-dimensional Rashba model

V.V. Bryksin

Physical Technical Institute, Politekhnicheskaya 26, 194021 St. Petersburg, Russia

P. Kleinert

Paul-Drude-Intitut für Festkörperelektronik,

Hausvogteiplatz 5-7, 10117 Berlin, Germany

(Dated: May 1, 2018)

Abstract

Based on the spin-density-matrix approach, both the electric-field-induced spin accumulation and the spin current are systematically studied for the two-dimensional Rashba model. Eigenmodes of spin excitations give rise to resonances in the frequency domain. Utilizing a general and physically well-founded definition of the spin current, we obtain results that differ remarkably from previous findings. It is shown that there is a close relationship between the spin accumulation and the spin current, which is due to the prescription of a quasi-chemical potential and which does not result from a conservation law. Physical ambiguities are removed that plagued former approaches with respect to a spin-Hall current that is independent of the electric field. For the clean Rashba model, the intrinsic spin-Hall conductivity exhibits a logarithmic divergence in the low-frequency regime.

PACS numbers: 72.25.-b 73.23.-b 73.50.Bk
I. INTRODUCTION

It has been anticipated that the spin degree of freedom of charge carriers potentially provides additional functionality to electronic devices. Recent progress aims at spintronic applications that rely on the capability to manipulate electron spin polarizations in nonmagnetic semiconductors. Of particular interest are efficient injection mechanisms of spins in semiconductors at room temperature. In this respect, the proposal by Murakami et al. [1] and Sinova et al. [2] of generating dissipationless transverse spin currents by a driving electric field has attracted considerable attention. The observation of this spin-Hall effect has been reported in recent experiments on GaAs and related materials [3, 4]. Many theoretical studies of this effect focused on the Rashba spin-orbit interaction since this type of spin-charge coupling can be easily controlled by an electrical gate. The original conclusion concerning the existence of an universal intrinsic spin-Hall conductivity [2] in clean systems was reexamined in more detail by treating effects of the elastic impurity scattering. Based on numerical [17, 18] and analytical results derived from the Keldysh [12, 19] and Kubo [14, 20, 21] formalism, it has been concluded that vertex corrections lead to a vanishing zero-frequency spin-Hall current in the thermodynamic limit of the linear Rashba model. After long debates many researchers finally arrived at the same conclusion so that there seems to be agreement now that the intrinsic zero-frequency spin current is finite and universal for a free two-dimensional electron gas but vanishes in impure systems for an arbitrary ratio of spin splitting and the impurity scattering rate.

Despite this consensus there are still challenging problems to be addressed referring to a proper definition of the spin current. This issue has recently been treated by Zhang et al. [22] and Sugimoto et al. [23]. The authors pointed out that the "conventional" spin current, which is defined as the product of spin and velocity operators, loses its physical foundation when the spin-orbit coupling is present. The main deficiency of this definition is related to the absence of a conservation law of spins. Therefore, a physically motivated definition was suggested [22, 23] that relates the spin current to the time derivative of the spin displacement. It was argued that under quite general conditions this effective spin current satisfies the continuity equation so that it is measurable as a spin accumulation. While for the charge transport both definitions completely agree to each other, there is a remarkable discrepancy between them with respect to the spin current due to the torque dipole contribution [22].
Results for the spin Hall conductivity derived from this physically motivated definition of the spin current remarkably differ from previous findings based on the “conventional” definition. Unfortunately, the authors did not apply direct perturbational techniques to calculate this spin-Hall current. Rather, they worked out a special calculational schema for the determination of the torque dipole density, which they used to complement the “conventional” spin-Hall current to a conserved quantity. In view of the long and successful history to derive the current from the time derivative of the dipole moment [24], we prefer the application of the standard procedure. What is needed in this approach is nothing but the density matrix, which is determined from quantum-kinetic equations. Starting from the physical definition of the spin-Hall current, we calculate the frequency dependent spin polarization and spin-Hall conductivity by analytically solving the kinetic equations for the spin-density matrix. The obtained results do not agree with previous findings deduced from the ”conventional” definition of the spin-Hall current. Furthermore, physical inconsistencies with respect to a spin-Hall current component that is independent of the electric field as well as the relationship between the spin accumulation and the spin current are puzzled out. For the spin-Hall conductivity of a clean two-dimensional electron gas with Rashba type spin-orbit coupling, our exact calculation does not reproduce any universal value but yields a logarithmic divergency in the low frequency limit. In addition, the frequency dependent spin-Hall conductivity, which is obtained by a controlled perturbational approach, differs even qualitatively from previous results [12].

II. THE KINETIC EQUATIONS

We consider a two-dimensional electron gas in the presence of Rashba spin-orbit interaction with amplitude $\alpha$. The system with an applied in-plane electric field $\vec{E}$ (which is oriented along the $x$ axis) is described by the Hamiltonian

$$
H_0 = \sum_{k,\lambda} a_{k\lambda}^\dagger [\varepsilon_k - \varepsilon_F] a_{k\lambda} - \sum_{k,\lambda,\lambda'} (\hbar \vec{\omega}_k \cdot \vec{\sigma}_{\lambda\lambda'}) a_{k\lambda}^\dagger a_{k\lambda'}
$$

$$
- e\vec{E} \sum_{k,\lambda} \nabla_\kappa a_{k-1/2\lambda}^\dagger a_{k+1/2\lambda} \bigg|_{\kappa=0},
$$

where we introduced the abbreviations

$$
\varepsilon_k = \frac{\hbar^2 k^2}{2m}, \quad \vec{\omega}_k = \frac{\hbar}{m} (K \times k), \quad K = \frac{ma}{\hbar^2} \vec{e}_z.
$$

(2)
\(m, \varepsilon_F, \) and \(\vec{\sigma}\) denote the effective mass, the Fermi energy, and the vector of Pauli matrices, respectively. \(a^\dagger_{k\lambda}\) and \(a_{k\lambda}\) are creation and annihilation operators with quasimomentum \(k = (k_x, k_y, 0)\) and spin \(\lambda\). We are going to calculate the time-dependent density matrix

\[ f^\lambda_{\lambda'}(k, k' \mid t) = \langle a^\dagger_{k\lambda} a_{k'\lambda'} \rangle_t, \tag{3} \]

which is more conveniently expressed by its physical representation

\[ f(k, \kappa \mid t) = \sum_\lambda f^\lambda_{\lambda}(k, \kappa \mid t), \quad \vec{f}(k, \kappa \mid t) = \sum_{\lambda,\lambda'} f^\lambda_{\lambda'}(k, \kappa \mid t) \vec{\sigma}_{\lambda\lambda'}, \tag{4} \]

in the \(k, \kappa\) space, where \(k \rightarrow k + \kappa/2\) and \(k' \rightarrow k - \kappa/2\). \(\kappa\) refers to a possible inhomogeneity of the charge and/or spin distribution. Using the Liouville equation, the quantum-kinetic equations for the components of the density matrix are straightforwardly derived. From the result

\[
\begin{align*}
\frac{\partial f}{\partial t} - i\hbar \left(\frac{\kappa \cdot k}{m} f - \frac{\hbar}{m} K(f \times \kappa) + \frac{e}{\hbar} \vec{E} \nabla_k f\right) &= \sum_{\lambda,\lambda_1,\lambda_2} \sum_{k'} \left\{ f^{\lambda_1}_{\lambda_2}(k', \kappa \mid t) W^{\lambda_1\lambda_2}_{\lambda_2\lambda}(k', k, \kappa) - f^{\lambda_1}_{\lambda_2}(k, \kappa \mid t) W^{\lambda_1\lambda_2}_{\lambda_2\lambda}(k, k', \kappa) \right\} I, \\
\frac{\partial \vec{f}}{\partial t} - i\hbar \left(\frac{\kappa \cdot k}{m} \vec{f} + 2(\vec{\omega} \times \vec{f}) + \frac{\hbar}{m} (K \times \kappa) f + \frac{e}{\hbar} \vec{E} \nabla_k \vec{f} \right) &= \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \sum_{k'} \left\{ f^{\lambda_1}_{\lambda_2}(k', \kappa \mid t) W^{\lambda_1\lambda_3}_{\lambda_2\lambda_4}(k', k, \kappa) - f^{\lambda_1}_{\lambda_2}(k, \kappa \mid t) W^{\lambda_1\lambda_3}_{\lambda_2\lambda_4}(k, k', \kappa) \right\} \vec{\sigma}_{\lambda_3\lambda_4} \equiv \vec{I},
\end{align*}
\]

it is concluded that not only scattering but also any inhomogeneity \((\kappa \neq 0)\) couples the charge \((f)\) and spin \((\vec{f})\) degrees of freedom to each other. Consequently, any accumulation of charges induces a spin response and vice versa. The left-hand side of these equations was derived and discussed in Ref. [25]. The scattering probabilities \(W^{\lambda_1\lambda_2}_{\lambda_2\lambda_4}\) on the right hand sides of Eqs. (5) and (6) comprise both elastic and inelastic scattering-in and scattering-out contributions, which satisfy a sum rule. We shall restrict the consideration to elastic scattering described by the Hamiltonian

\[ H_{int} = u \sum_{k,k'} \sum_{\lambda} a^\dagger_{k\lambda} a_{k'\lambda}, \tag{7} \]

with \(u\) denoting the magnitude of the short-range impurity potential. For time-dependent phenomena, we prefer the treatment of the Laplace transformed kinetic Eqs. (5) and (6).
Within the Born approximation, we obtain for the scattering probabilities the exact result

\[
W_{\lambda_2,\lambda_3}^{\lambda_1}(k', k, \kappa | s) = \frac{u^2}{\hbar^2} \int_0^\infty dt \exp \left[ -st + \frac{i}{\hbar} (\varepsilon_{k'} - \varepsilon_k - \varepsilon_{k+k/2}) t \right]
\]

\[
\times \left[ \cos \left( \omega_{k' - k/2} t \right) \delta_{\lambda_1\lambda_3} - i \frac{\sigma_{\lambda_1\lambda_3} \cdot \vec{\omega}_{k' - k/2}}{\omega_{k' - k/2}} \sin \left( \omega_{k' - k/2} t \right) \right]
\]

\[
\times \left[ \cos \left( \omega_{k+k/2} t \right) \delta_{\lambda_1\lambda_2} + i \frac{\sigma_{\lambda_1\lambda_2} \cdot \vec{\omega}_{k+k/2}}{\omega_{k+k/2}} \sin \left( \omega_{k+k/2} t \right) \right] + k \leftrightarrow k',
\]

with \( s \) denoting the variable of the Laplace transformation. As usual, it is assumed that corrections due to the \( \kappa \) expansion of the scattering probabilities are small compared to corresponding contributions on the left-hand side of the kinetic equations. Furthermore, for weak spin-orbit coupling, we may restrict to the lowest-order contributions in \( \omega_k t \). Adopting these approximations, the collision integrals are expressed by

\[
I = \frac{1}{\tau} (\vec{f} - \vec{f}) - \frac{\hbar \omega_k}{\tau} \frac{\partial^2}{\partial \varepsilon_k^2} \hbar \omega_k \vec{f} + \frac{1}{\tau} \frac{\partial}{\partial \varepsilon_k} \hbar \omega_k \cdot \vec{f}
\]

\[
- \frac{\hbar \omega_k}{\tau} \frac{\partial}{\partial \varepsilon_k} \vec{f} + 4 \hbar u^2 \sum_{k'} \vec{f}(k') \frac{\vec{\omega}_{k'} \times \vec{\omega}_k}{(\varepsilon_k - \varepsilon_{k'})^3},
\]

\[
\bar{I} = \frac{1}{\tau} (\vec{f} - \vec{f}) + \frac{\hbar \omega_k}{\tau} \frac{\partial^2}{\partial \varepsilon_k^2} \hbar \omega_k \vec{f} + \frac{1}{\tau} \frac{\partial}{\partial \varepsilon_k} \hbar \omega_k \vec{f} - \frac{\hbar \omega_k}{\tau} \frac{\partial}{\partial \varepsilon_k} \vec{f}
\]

\[
+ \frac{\hbar}{\tau} \left[ \vec{\omega}_k \times \left( \frac{\partial^2}{\partial \varepsilon_k^2} \hbar \omega_k \vec{f} \right) \right] - 2 u^2 \sum_{k'} \frac{[(\vec{\omega}_k + \vec{\omega}_{k'}) \times \vec{f}(k')]}{(\varepsilon_k - \varepsilon_{k'})^2} + 4 \hbar u^2 \sum_{k'} \vec{f}(k') \frac{\vec{\omega}_{k'} \times \vec{\omega}_k}{(\varepsilon_{k'} - \varepsilon_k)^3},
\]

where the scattering time \( \tau \) is calculated from

\[
\frac{1}{\tau} = \frac{2\pi u^2}{\hbar} \sum_{k'} \delta(\varepsilon_{k'} - \varepsilon_k).
\]

\( \vec{f}(k) \) means an average over the angle of the vector \( k \). In the Eqs. (9) and (10) corrections appear, which result from virtual transitions that are not linked to the scattering time \( \tau \). The treatment of these contributions as well as higher-order corrections in the spin-orbit coupling \( \alpha \) goes beyond the scope of this paper. Restricting to the lowest-order scattering contributions, assuming an initial thermodynamic equilibrium state, and focusing on weak spin-orbit coupling so that \( \hbar^2 K/(m\varepsilon_k) \ll 1 \), the kinetic equations are expressed by

\[
sf - \frac{i\hbar}{m} (\kappa \cdot k) \vec{f} - \frac{i\hbar}{m} K (\vec{f} \times \kappa) + \frac{e\vec{E} \cdot \nabla_k}{\hbar} \vec{f} = \frac{1}{\tau} (\vec{f} - \vec{f}) + n(\varepsilon_k),
\]

\( (9) \) and \( (10) \) corrections appear, which result from virtual transitions that are not linked to the scattering time \( \tau \). The treatment of these contributions as well as higher-order corrections in the spin-orbit coupling \( \alpha \) goes beyond the scope of this paper. Restricting to the lowest-order scattering contributions, assuming an initial thermodynamic equilibrium state, and focusing on weak spin-orbit coupling so that \( \hbar^2 K/(m\varepsilon_k) \ll 1 \), the kinetic equations are expressed by

\[
sf - \frac{i\hbar}{m} (\kappa \cdot k) \vec{f} - \frac{i\hbar}{m} K (\vec{f} \times \kappa) + \frac{e\vec{E} \cdot \nabla_k}{\hbar} \vec{f} = \frac{1}{\tau} (\vec{f} - \vec{f}) + n(\varepsilon_k),
\]

\( (9) \) and \( (10) \) corrections appear, which result from virtual transitions that are not linked to the scattering time \( \tau \). The treatment of these contributions as well as higher-order corrections in the spin-orbit coupling \( \alpha \) goes beyond the scope of this paper. Restricting to the lowest-order scattering contributions, assuming an initial thermodynamic equilibrium state, and focusing on weak spin-orbit coupling so that \( \hbar^2 K/(m\varepsilon_k) \ll 1 \), the kinetic equations are expressed by

\[
sf - \frac{i\hbar}{m} (\kappa \cdot k) \vec{f} - \frac{i\hbar}{m} K (\vec{f} \times \kappa) + \frac{e\vec{E} \cdot \nabla_k}{\hbar} \vec{f} = \frac{1}{\tau} (\vec{f} - \vec{f}) + n(\varepsilon_k),
\]
\[ s \tilde{f} + 2(\tilde{\omega}_k \times \tilde{f}) - \frac{i\hbar}{m} (\kappa \cdot \vec{k}) \tilde{f} + \frac{i\hbar}{m} (\vec{K} \times \kappa) f + \frac{e\vec{E}}{\hbar} \nabla_k \tilde{f} \]

\[ = \frac{1}{\tau} (\tilde{f} - \tilde{f}) + \frac{1}{\tau} \frac{\partial}{\partial \varepsilon_k} \tilde{f} \hbar \tilde{\omega}_k - \frac{\hbar \tilde{\omega}_k}{\tau} \frac{\partial}{\partial \varepsilon_k} \tilde{f} - \hbar \tilde{\omega}_k \frac{\partial n(\varepsilon_k)}{\partial \varepsilon_k}, \tag{13} \]

where \( n(\varepsilon_k) \) denotes the initial equilibrium charge density.

### III. GENERAL EXPRESSIONS FOR THE CHARGE AND SPIN CURRENTS

The Laplace-transformed density matrix \( \hat{f}(\vec{k}, \kappa | s) = \{ f(\vec{k}, \kappa | s), \tilde{f}(\vec{k}, \kappa | s) \} \) contains all the information needed to determine all kinetic observables. In particular, the quantity

\[ f(s) = \sum_k f(\vec{k}, \kappa | s) |_{\kappa=0} \tag{14} \]

represents nothing but the conserved charge density. Exactly in the same way, the total magnetic moment is calculated from

\[ \tilde{f}(s) = \sum_k \tilde{f}(\vec{k}, \kappa | s) |_{\kappa=0}. \tag{15} \]

When an electric field is applied to the system, these quantities become time dependent and allow the treatment of relaxation processes.

Other quantities of interest are the current of charge carriers and spins, which are obtained from

\[ \tilde{j}(s) = -ise \sum_k \nabla_\kappa \tilde{f}(\vec{k}, \kappa | s) |_{\kappa=0}, \tag{16} \]

and

\[ \tilde{j}^s(s) = -is \frac{1}{2} \sum_k \nabla_\kappa \otimes \tilde{f}(\vec{k}, \kappa | s) |_{\kappa=0}, \tag{17} \]

respectively. In Eq. (17), \( \otimes \) denotes the dyadic product. These definitions are fundamental and sufficiently general. In the time domain, the spatial versions of these equations describe the temporal evolution of the carrier or spin displacement, i.e., the center-of-mass velocity of the wave packet. The expression for the charge-carrier current in Eq. (16) is also applicable for systems without any spatial dispersion. If the interaction Hamiltonian that describes elastic or inelastic scattering commutes with the dipole operator of the carriers than the definition in Eq. (16) becomes completely equivalent to

\[ j_i(t) = \frac{e}{\hbar} \sum_k \frac{\partial \varepsilon_k}{\partial k_i} f(\vec{k} | t) + \frac{e}{\hbar} \sum_{k,j} \frac{\partial^2 \varepsilon_k}{\partial k_i \partial k_j} \left( \vec{K} \times \tilde{f}(\vec{k} | t) \right)_j, \tag{18} \]
in which the $\kappa$ dependence does no longer occur. We want to stress that the situation for the spin transport is completely different and more subtle. For a system without any spatial dispersion, it is in general not possible to express the spin current as defined in Eq. (17) by the density matrix $\hat{f}(k, \kappa | s) \rvert_{\kappa=0}$ alone. What is really needed is the $\kappa$ derivative of this function at $\kappa = 0$. Notwithstanding this fact, most researchers defined the spin current $\hat{j}^s(s)$ as the symmetrized product of the spin and velocity operators $\{\hat{\sigma}, \hat{v}_k\}_+/4$, where $\hat{v} = \nabla_k H_0/\hbar$ and $H_0$ is obtained from Eq. (1) for $\vec{E} = 0$. For the Rashba model, this definition of the spin current takes the for

$$\hat{j}^s(s) = \frac{1}{2\hbar} \sum_k \nabla_k \varepsilon_k \otimes \hat{f}(k, \kappa | s) \rvert_{\kappa=0}. \quad (19)$$

As shown below, the definition in Eq. (19) leads to a stationary spin-Hall current that does not depend on the electric field. In addition, recent studies \[22, 23\] clearly demonstrated that results derived from the Eqs. (17) and (19) considerably differ from each other and that the measurable quantity is related to Eq. (17). Furthermore, we note that the diffusion tensor is likewise obtained from derivatives of the density matrix $\hat{f}(k, \kappa | s)$ with respect to $\kappa_i$ at $\kappa = 0$. Although the extension of our approach to the treatment of the diffusion coefficient is straightforward, we want to confine ourselves to the analysis of spin currents.

The definitions of measurable quantities in Eqs. (14) to (18) already set up our calculational schema as an iteration with respect to $\kappa$. As we restrict ourselves to elastic scattering, the spectral functions can be calculated for a given energy over which one finally integrates. This procedure is applicable only in the linear response regime, where carrier heating and energy relaxation due to nonlinear field effects do not play an essential role. Therefore, we treat only first-order corrections in the electric field. The analytic solution of the kinetic Eqs. (12) and (13), which is straightforward but cumbersome is presented in the Appendix.

IV. SPIN ACCUMULATION

To begin with the solution of the kinetic equation, which is derived in the Appendix, is used for the calculation of the spin accumulation. From Eq. (A8), we obtain for the vector of the field-induced components of the spin-density matrix

$$\vec{f}(s) = -\frac{e \hbar}{ms} (K \times \vec{E}) \sum_k n'(\varepsilon_k) \frac{2\tau \omega_k^2}{\sigma^2 s\tau + 2\omega_k^2(2s\tau + 1)}, \quad (20)$$

7
with $\sigma = s + 1/\tau$. According to Eq. (20), the in-plane spin accumulation is calculated by a $k$ integral over poles. Under the condition $\omega_k \tau > 1$, when one expects sharp resonances, the positions of these poles are given by

$$ (s\tau)_1 = -\frac{1}{2}, \quad (s\tau)_{2,3} = -\frac{3}{4} \pm 2i\omega_k \tau. $$

The resonance is most pronounced at zero temperature ($T = 0$) and when $\omega_k \tau \gg 1$. Depending on the Rashba coupling constant $\alpha$ and on the carrier density, the resonance at $2\omega_k$ ($F_k$ denotes the Fermi wave vector) is located in the THz regime. Switching from the Laplace to the frequency domain ($s \rightarrow -i\omega$) and considering zero temperature, we obtain from Eq. (20)

$$ f^y(\omega) = \frac{eE \tau K}{\pi \hbar \omega \tau} \frac{2i\omega_k^2}{\left[ 4\omega_k^2 - (\omega + i/\tau)^2 \right] + 2i\omega_k^2}, $$

with $\omega_k = \hbar Kk_F/m$. It is a striking coincidence that the denominator in this expression for the spin accumulation completely agrees with the denominator in the spin-Hall conductivity calculated by Mishchenko et al. [12]. The associated spin excitation gives rise to resonances in the spin accumulation. An example is shown in Fig. 1 for some spin-orbit coupling parameters $\omega_k \tau$. According to the $k$-integral in Eq. (20), the sharp peak is increasingly washed out with increasing temperature. The zero-frequency limit of the spin accumulation $f^y(\omega = 0)$ agrees with the result published by Edelstein [26].

Applying an inverse Laplace transformation to Eq. (20), the time evolution of the spin accumulation after the electric field is switched on can be determined. Numerical results are
shown in Fig. 2 (solid lines) and compared with the following analytical solution (dashed line)

\[ f_y(t) = eE\tau \frac{K}{\pi \hbar} \left[ 1 - \exp \left( -\frac{t}{2\tau} \right) - \frac{\sin(2\omega_{kF} t)}{2\omega_{kF}\tau} \exp \left( -\frac{3t}{4\tau} \right) \right], \]

valid under the condition \( \omega_{kF}\tau \gg 1 \). As expected, for systems with a weak Rashba spin-orbit coupling, the steady state spin accumulation is only reached after a sufficiently long time. With decreasing \( \omega_{kF}\tau \), weak oscillations are strongly suppressed.

V. CHARGE CURRENT

According to the consideration in Section III, the longitudinal current is calculated from the time derivative of the dipole operator [cf. Eq. (16)]. Taking into account Eq. (A12) in the Appendix, we immediately obtain for the longitudinal charge current

\[ j_x(s) = -4e^2E \frac{\epsilon n'}{\sigma sm} \sum_k \frac{\omega_{kF}^2 n'}{\sigma^2 s \tau + 2\omega_{kF}^2 (2s \tau + 1)}, \]

which can be used to calculate the frequency response at zero temperature (\( n' \) denotes the derivative \( d\epsilon_k/d\epsilon_k \)). The result in the frequency domain

\[ j_x(\omega) = \frac{2\epsilon_F \tau}{\pi \hbar^2} \frac{e^2 E}{1 - i\omega\tau} + e^2E\tau \frac{K^2}{\pi m \omega \tau} \frac{2i\omega_{kF}^2}{4\omega_{kF}^2 - (\omega + i/\tau)^2 + 2i\omega_{kF}^2}, \]

is composed of two contributions. The first one is expressed by the well-known Drude conductivity. The second one is due to the spin-orbit interaction and exhibits the same resonant denominator as the spin accumulation in Eq. (22). At \( \omega = 0 \), Eq. (25) reproduces the result published in Ref. [8]. The measurement of the resonant longitudinal charge current contribution allows the determination of the Rashba coupling constant \( \alpha \). The alternative formulation in Eq. (18) exactly reproduces Eq. (24).

VI. SPIN-HALL EFFECT

To introduce the spin-Hall effect, let us first shortly recapitulate the main findings derived in the literature based on the "conventional" definition of the spin current in Eq. (19).
The treatment of the spin current within this framework reveals an anomalousness, which in our opinion was not duly noticed by many researchers. The approach predicts a non-vanishing stationary spin-Hall current that is independent of the electric field. Indeed, inserting Eq. (A5) into the expression in Eq. (19) for the spin current, we obtain

\begin{equation}
    j_y^x(s) = \frac{\hbar K}{2m_s} \sum_k \varepsilon_k n'(\varepsilon_k),
\end{equation}

which leads to the constant \(x\) component of the spin-Hall current \(j_y^x(\omega) = -K\varepsilon_F/(2\pi\hbar)\) in the frequency domain at zero temperature. The absence of any resonance indicates that there is no relationship between this fictitious spin-Hall current and the spin accumulation in Eq. (22).

The field-induced spin-Hall current is calculated from its definition in Eq. (19) and by taking into account Eq. (A8). For zero temperature, we obtain for the frequency-dependent spin-Hall current

\begin{equation}
    j_y^z(\omega) = -\frac{eE}{2\pi\hbar} \frac{\omega\tau\omega_{k_F}^2}{4\omega_{k_F}^2 - (\omega + i/\tau)^2 + 2i\omega_{k_F}^2},
\end{equation}

which was recently derived by applying the Keldysh approach [12]. Here, the same resonant denominator appears as in the longitudinal charge current [Eq. (25)] and the spin accumulation [eq. (22)]. For a free electron gas (\(\tau \to \infty\)), Eq. (27) simplifies to

\begin{equation}
    j_y^z(\omega) = -\frac{eE}{2\pi\hbar} \frac{\omega_{k_F}^2}{4\omega_{k_F}^2 - \omega^2},
\end{equation}

which was previously obtained by Erlingsson et al. [13]. Finally the steady state spin-Hall current (\(\omega \to 0\)) of the clean Rashba model is given by the universal value [1, 2]

\begin{equation}
    j_y^z(\omega = 0) = -\frac{eE}{8\pi\hbar}.
\end{equation}

Although many authors confirmed these results, there remain some reservations. First of all, the definition of the "conventional" spin-Hall current led to considerable confusion and to serious doubts on its experimental relevance [15, 22, 23]. The main difficulty results from the fact that the spin is not a conserved quantity. As it has been claimed recently, a proper definition of the spin current requires a careful analysis of the torque density. It is assumed that this quantity may complement the above fictitious current to a conserved spin current. The prerequisite for such a construction is given, when the averaged spin-torque density
vanishes in the bulk \cite{22}. It has been argued that this condition is fortunately satisfied for many spin models treated in the literature. This conserved spin current has the advantage that it can be measured via the spin accumulation to which it is related by the continuity equation. It has been pointed out that this quantity is straightforwardly calculated from the time derivative of the spin displacement (Eq. (5) in Ref. \cite{22}). A firm foundation of this approach provides the definition of the spin current in Eq. (17) and the kinetic equation treated in Section II and solved in the Appendix.

Based on the physically motivated definition of the spin current in Eq. (17) and using the analytical solutions of the kinetic equations presented in the Appendix, we shall restart the study of the spin-Hall current of the Rashba model. First, it is noted that we also get a spin-Hall current contribution that is independent of the electric field. From Eq. (17) and \cite{A11}, we obtain a result

$$j_y^{\sigma}(s) = -\frac{\hbar K}{m} \sum_k n(\varepsilon_k) \frac{\omega_k^2 \tau}{\sigma^2 s\tau + 2 \omega_k^2 (2s\tau + 1)} \tag{30}$$

that differs from Eq. (26) in many respects. First of all, this current contribution is closely related to the spin accumulation in Eq. (20). A first evidence for this conclusion is the appearance of the same resonant denominator. However, this relationship goes even deeper and has a firm physical foundation, which becomes obvious by comparing the analytical solution for $\vec{f}_{0E}$ [Eq. (A7)] with its counterpart for $\vec{f}_{\kappa 0}$ [Eq. (A10)]. One solution is obtained from the other one by the replacement $\kappa \rightarrow -ieE\vec{e}_x \partial_{\varepsilon}$, where the derivative with respect to the energy applies to the charge density $n$. Transforming back this replacement to the spatial dependence, we obtain $\nabla \rightarrow \nabla + e\vec{E} \partial_{\varepsilon}$ \cite{12}, which gives the general recipe of a quasi-chemical potential to translate spatial inhomogeneities to internal field fluctuations and vice versa. It is this connection and not the conservation of spins that establishes the close relationship between the field-induced spin accumulation and a field-independent spin current, which contrary to Eq. (26) vanishes in the stationary regime. The spin-Hall current in Eq. (30) is interpreted as the response to the initial time evolution of the spin accumulation after the electric field is switched on. When the spin accumulation reaches its stationary value (cf. Fig. 2), the related spin-Hall current component disappears.

Starting from the physically motivated definition of the spin current in Eq. (17) and using the analytic solution in Eq. (A13) derived in the Appendix, we obtain the following general
result for the Laplace transformed spin-Hall current

\[ j^z_y(s) = \frac{eE\tau}{h} \left( \frac{hK}{m} \right)^2 \sum_k n(\varepsilon_k) \frac{4\omega_k^2(1 + 2s\tau) + 2\sigma^2\omega_k^2(1 + 3s\tau) - \sigma^4s\tau}{[\sigma^2s\tau + 2\omega_k^2(2s\tau + 1)]^2 \left[ \sigma^2s\tau + 4\omega_k^2(s\tau + 1) \right]} \]  

(31)

which differs from all previous results in many respects. First, the denominator is composed of three factors, the zeros of which characterize spin eigenmodes. The in-plane spin precision is characterized by poles resulting from \( 1/\left( \sigma^2s\tau + 4\omega_k^2(s\tau + 1) \right)^2 \). Resonances of this kind appear both in the spin accumulation and in the charge current. In addition, there is the factor \( \sigma^2s\tau + 4\omega_k^2(s\tau + 1) \) in the denominator of Eq. (31) that is associated with out-of-plane spin eigenmodes. The most striking discrepancy results from the \( k \) sum over the entire spin-orbit coupled Fermi sea, whereas only contributions from the Fermi surface are needed in charge transport problems. All states contribute to the time-dependent spin-Hall current. This astonishing result refers both to the spin current in [Eq. (30)] and in [Eq. (31)].

To clarify the origin of this peculiarity, we note that the spin current is not due to displacements of carriers but induced by the change of the magnetic moment. This situation is quite similar to the diamagnetism in normal metals, which is also determined by all states in the entire Brillouin zone \[28\]. Finally, we point out that the \( k \) integral in Eq. (31) leads to a logarithmic singularity, the due treatment of which requires a careful consideration of the kinetic equations in the limit \( \omega_k \to 0 \).

Applying an integration by parts, we obtain an equivalent expression for the spin-Hall current, in which the logarithmic contributions are singled out and in which at zero temperature the \( k \) integral is artificially fixed at the Fermi surface:

\[
\begin{align*}
j^z_y(s) &= -\frac{eE\tau}{2h} \sum_k n'(\varepsilon_k) \left\{ \frac{2\omega_k^2(2s\tau + 1)}{[\sigma^2s\tau + 2\omega_k^2(2s\tau + 1)]} \\
&\quad + \frac{(1 + s\tau)(8s\tau + 3)}{2(2s\tau + 1)} \ln \left[ 1 + \frac{2\omega_k^2(2s\tau + 1)}{\sigma^2s\tau} \right] - \frac{8(s\tau)^2 + 15s\tau + 6}{4(s\tau + 1)} \ln \left[ 1 + \frac{4\omega_k^2(s\tau + 1)}{\sigma^2s\tau} \right] \right\} 
\end{align*}
\]

(32)

From this equation, it is concluded that in the steady state \( (s \to 0) \) the spin-Hall current vanishes for the impure Rashba model of two-dimensional electrons. This finding agrees
completely with former conclusions derived from the "conventional" definition of the spin-Hall current [12].

Applying an inverse Laplace transformation, the time evolution of \( j^z_y \) can be studied. Fig. 3 shows an example for the time-dependence of the spin-Hall current, which is induced by switching on the electric field at \( t = 0 \). Depending on the coupling parameter \( \omega_{k_F} \tau \), strong oscillations of the spin-Hall current initially develop, which are completely damped out after a couple of scattering times.

If time-dependent electric fields are applied, the spin-Hall conductivity \( \sigma_{sH} \) becomes nonzero. The frequency-dependent spin-Hall conductivity is obtained from Eq. (32) by an analytic continuation \( (s \rightarrow -i\omega) \). Numerical results for the real and imaginary part of \( \sigma_{sH} \) are shown in Fig. 4 and 5 by thick solid lines. We focus on the zero-temperature case and compare with previous results obtained from the "conventional" definition of the spin-Hall current [12] [dashed lines as calculated from Eq. (27)]. Both approaches predict a sharp resonance in the ac spin-Hall conductivity, when the condition \( \omega_{k_F} \tau \gg 1 \) is satisfied. The enhancement of the spin-Hall current appears at \( \omega = 2\omega_{k_F} \). As seen from Fig. 4, remnants of this feature survive even in the case \( \omega_{k_F} \tau \lesssim 1 \). In the limit \( \omega_{k_F} \tau \ll 1 \), both approaches agree and result in

\[
j^z_y(\omega) = -\frac{e E}{2\pi \hbar} \left( \frac{\omega_{k_F} \tau}{1 - i\omega \tau} \right)^2,
\]

(33)
which is plotted by the dash-dotted line in Fig. 4. From Eq. (33), the non-analytic behavior of the spin-Hall current becomes obvious because the approximation fails in predicting zero spin-Hall current in the limit \((\omega \to 0)\), which requires the general result in Eq. (31).

Comparing the solid and dashed lines in Figs. 4 and 5, it is tempting to conclude that there is no qualitative difference between the results of both approaches. That this conclusion is only partly true shows a consideration of clean samples \((\tau \to \infty)\). In this case, our approach becomes completely exact and we obtain

\[
j^z_y(\omega) = \frac{eE}{8\pi\hbar} \left\{ \frac{1}{2} \ln \left( 1 - \frac{4\omega^2_{k_F}}{\omega^2} \right) + \frac{\omega^4}{(4\omega^2_{k_F} - \omega^2)^2} + \frac{\omega^2}{2(4\omega^2_{k_F} - \omega^2)} - \frac{1}{2} \right\},
\]

which gives a logarithmic divergency at vanishing frequency

\[
j^z_y(\omega \to 0) = -\frac{eE}{8\pi\hbar} \ln \frac{\omega}{2\omega_{k_F}}.
\]

Most previous approaches predict in this case a finite universal spin-Hall conductivity \(\sigma_{sH} = -e/(8\pi\hbar)\) \[1, 2\]. In contrast, based on a physically motivated definition of the spin-Hall current and on an exact procedure, we obtain neither zero nor an universal value but a logarithmic divergency for the spin-Hall conductivity at zero frequency.

Unfortunately, recent calculations \[22\] of the physical spin-Hall conductivity in the clean limit of the non-interacting Rashba model do not agree with our exact result in Eq. (35). The authors started from the same physically motivated definition of the spin-Hall current (Eq. (5) in Ref. \[22\]) and calculated \(j^z_y\) via the sourceless continuity equation. However, strictly speaking, the spin current, which is directly calculated from the \(\kappa\) derivative of the averaged distribution function \(\overline{f}(k, \kappa \mid t)\), is not related to source but to vortex fields. To illustrate the situation, let us treat the set of kinetic equations for \(\overline{E} = 0\) and to first-order in \(\kappa\). These equations contain two curl contributions namely \(\sim [K \times \kappa] \overline{f}(k, \kappa \mid s)\) and \(\sim [\kappa \times \overline{f}(k, \kappa \mid s)]\), which do not enter any continuity equation. The first vector describes the coupling between charge and spin degrees of freedom and gives rise to the field-independent spin-Hall current. If an electric field is switched on, the replacement \(\kappa \to \kappa - ie\overline{E}\partial_\kappa\) generates a contribution \(\sim -ie[K \times \overline{E}]\partial_\kappa \overline{f}(k, \kappa \mid s)\), which is responsible for the appearance of the field-induced spin accumulation. For the mean values \(\overline{f}(\varepsilon \mid s) = \overline{f}(k, \kappa \mid s) \mid_{\kappa=0}\) and \(\hat{j}(\varepsilon \mid s) = is\partial_\kappa \otimes \overline{f}(k, \kappa \mid s) \mid_{\kappa=0} /2\), which are used to express the spin accumulation and the field-independent spin-Hall current, respectively, we obtain a relationship of the form

\[
\frac{s}{2} f^i(\varepsilon \mid s)n = eE_j^i(\varepsilon \mid s)n',
\]
which is confirmed by our approach [compare Eq. (20) and Eq. (30)]. This equation is not valid for the "conventional" field-independent spin-Hall current in Eq. (26). The interrelation in Eq. (36) is based on the assumption that the field effects can be accounted for by a quasi-chemical potential in the electronic part of the density matrix. For $\alpha = 0$, it is known that this supposition leads to the Einstein relation between the mobility and the diffusion coefficient. Recently, this hypothesis has also been accepted for systems with spin-orbit interaction [12, 29, 30], although its application becomes more subtle due to the spin splitting of the energy bands. An analogous use of quasi-chemical potentials for the treatment of field effects on the spin density requires further justification.

The appearance of the logarithmic dependence of the ac spin-Hall conductivity in the disordered two-dimensional Rashba model remind us very much of the logarithmic quantum corrections in the theory of weak localization [31]. Based on scaling arguments, it was shown that the ac conductivity of a disordered two-dimensional electron gas exhibits a logarithmic divergency in the zero-frequency limit [$\sigma \sim \ln(\omega \tau)$]. Our result for the spin-Hall conductivity in Eq. (35) is comparable to this dependence observed in the completely other field of weak localization.

VII. SUMMARY

Based on the kinetic equations for the spin-density matrix of the two-dimensional Rashba model, we treated the electric-field-induced spin accumulation and spin transport in the linear response regime. At zero temperature, the frequency dependence of both the spin accumulation and the longitudinal charge-carrier transport exhibit a sharp resonance at $\omega = 2\omega_{k_F}$ ($\omega_{k_F} = \alpha k_F / \hbar$, with $\alpha$ being the spin-orbit coupling constant), which is due to eigenmodes of spin excitations. The measurement of this resonance should be possible. It allows the determination of the Rashba coupling constant $\alpha$. Similar resonances are expected to appear also in the $k$-cubed Rashba model for 2D holes and the Luttinger model for 3D holes.

For the charge-carrier transport, there are two completely equivalent procedures to calculate the conductance. The same results are obtained by starting either from the symmetrized product of the density and velocity operators or the time derivative of the dipole moment. Unfortunately, this equivalence no longer holds for the spin transport. The spin is not a
conserved quantity. Therefore, it is a serious problem to chose a proper definition for the 
spin current that does not lose its physical foundation. Most researchers preferred a definition 
of the spin-Hall conductivity, which revealed rather unconventional properties so that 
serious doubts arose on its experimental relevance. Recently, a physically motivated definition 
of the spin current has been suggested [22] that resolved a number of difficulties of 
former approaches. It has been argued that the proper effective spin current is inevitably 
defined as the time derivative of the spin displacement. Applying this definition, results for 
the spin-Hall conductivity are obtained that are drastically different from previous findings. 
First, the approach predicts a field-independent spin-Hall current that reflects the initial 
variation of the spin accumulation after the electric field is switched on. Contrary to previ-
ous results, this specific spin current contribution disappears in the steady state. Its physical 
origin is due to the initial time evolution of the spin polarization. Furthermore, for a clean 
two-dimensional Rashba model, we obtain a spin-Hall conductivity that exhibits a logarith-
mic dependence at low-frequencies and not an universal constant value. This observation, 
which is an exact result, reminds us on the well-known frequency-dependent conductivity of 
a disordered two-dimensional system in the theory of weak localization.

Acknowledgments

Partial financial support by the Deutsche Forschungsgemeinschaft and the Russian Founda-
tion of Basic Research under the grant number 05-02-04004 is gratefully acknowledged.

APPENDIX A: SOLUTION OF THE KINETIC EQUATIONS

The kinetic Eqs. (12) and (13) are solved by an perturbational approach with respect to 
\( \mathbf{E} \) and \( \mathbf{k} \). This calculation exploits the formal exact solutions of these equations given by

\[
\bar{f} = \frac{\sigma r - 2 \bar{\omega}_k \times r + 4 \bar{\omega}_k (\bar{\omega}_k \cdot r) / \sigma}{\sigma^2 + 4 \omega_k^2},
\]

(A1)

with \( \sigma = s + 1/\tau \), \( r = R + \bar{f}/\tau \) and

\[
R = \frac{i \hbar}{m} (\kappa \cdot k) \bar{f} - \frac{i \hbar}{m} (K \times \kappa) \bar{f} - \frac{e \mathbf{E}}{\hbar} \nabla_k \bar{f}
\]

\[
+ \frac{1}{\tau} \frac{\partial}{\delta \varepsilon_k} \frac{\hbar \omega_k}{\tau} \frac{\partial}{\delta \varepsilon_k} \bar{f}.
\]

(A2)
For the angle-averaged spin-density matrix, we obtain

\[ \overline{\mathbf{f}} = \sigma_\tau \frac{\sigma \mathbf{R} - 2 \overline{\omega}_k \times \mathbf{R} + 4 \overline{\omega}_k (\overline{\omega}_k \cdot \mathbf{R})/\sigma}{\sigma^2 \sigma + 2 \omega_k^2 (2 \sigma \tau + 1)}, \]  

(A3)

for their \( x, y \) components, while for the \( z \) component it follows

\[ \overline{\mathbf{f}}^z = \sigma_\tau \frac{\sigma \mathbf{R} - 2 \overline{\omega}_k \times \mathbf{R}}{\sigma^2 \sigma + 4 \omega_k^2}. \]  

(A4)

The lowest-order solutions in \( E = 0 \) and \( \kappa = 0 \) (the corresponding elements of the density matrix are denoted by \( f_{00} \) and \( \overline{f}_{00} \)) are easily obtained

\[ f_{00} = \frac{n(\varepsilon_k)}{s}, \quad \overline{f}_{00} = -\hbar \overline{\omega}_k \frac{n'}{s}, \quad \overline{f}_{00} = 0, \]  

(A5)

where \( n' \) denotes the derivative with respect to \( \varepsilon_k \). Both quantities \( f_{00} \) and \( \overline{f}_{00} \) do not depend on time and are therefore conserved under the condition of thermodynamic equilibrium. For the derivation of this result it was necessary to consider the spin-orbit coupling in the collision integral [the second and third term on the right-hand side of Eq. (13)]. Next, let us calculate the lowest-order correction due to the electric field \((E \neq 0, \kappa = 0)\). Taking into account Eqs. (12) and (13) together with Eq. (A3), we obtain

\[ f_{0E} = -\frac{eE h_k x}{m} \frac{n'}{s}, \quad \overline{f}_{0E} = 0, \]  

(A6)

\[ \overline{f}_{0E} = \overline{\omega}_k (E k_x) \frac{h_k^2 n''}{m s} + n' \tau \frac{\sigma \overline{\omega}_E - 2 \overline{\omega}_k \times \overline{\omega}_E + 4 (\overline{\omega}_k \cdot \overline{\omega}_E) \overline{\omega}_k / \sigma}{[\sigma^2 \sigma + 2 \omega_k (2 \sigma \tau + 1)]}, \]  

(A7)

\[ \overline{f}_{0E} = \frac{\overline{\omega}_E}{s} \left\{ (\varepsilon_k n')' - \frac{2 \sigma \tau \omega_k^2 n'}{\sigma^2 \sigma + 2 \omega_k (2 \sigma \tau + 1)} \right\}, \]  

(A8)

where we used the abbreviation \( \overline{\omega}_E = e \hbar (K \times \overline{E})/m \). Next, we calculate \( f_{\kappa 0} \) and \( \overline{\mathbf{f}}_{\kappa 0} \) by collecting the first-order contributions in \( \kappa \) and by setting \( \overline{\mathbf{E}} = 0 \). Neglecting corrections, which are of the order \( \hbar^2 K/m \varepsilon_k \), we obtain

\[ f_{\kappa 0} = \frac{i \hbar}{m s} (\kappa \cdot k) n, \]  

(A9)

\[ \overline{f}_{\kappa 0} = -i \overline{\omega}_k (k \cdot \kappa) \frac{h_k n''}{m s} - i n \tau \frac{\sigma \overline{\omega}_k \cdot \overline{\omega}_k - 2 \overline{\omega}_k \times \overline{\omega}_k + 4 (\overline{\omega}_k \cdot \overline{\omega}_k) \overline{\omega}_k / \sigma}{[\sigma^2 \sigma + 2 \omega_k (2 \sigma \tau + 1)]} \]  

(A10)

\[ \overline{f}_{\kappa 0} = -\frac{i \overline{\omega}_k}{s} \left\{ (\varepsilon_k n')' - \frac{2 \sigma \tau \omega_k^2 n}{\sigma^2 \sigma + 2 \omega_k^2 (2 \sigma \tau + 1)} \right\}. \]  

(A11)
There is an interesting symmetry between the vectors $\vec{f}_\kappa \vec{0}$ and $\vec{f}_E \vec{0}$. From Eq. (A10), the electric-field-induced contribution in Eq. (A7) is obtained by the replacement $\kappa \to -ieE\vec{e}_x \partial_k$, where the derivative refers specifically to the charge density $n$. Finally, we need the first-order corrections in $\kappa$ and $E$ of the angle-averaged component of the density matrix, which are expressed by

$$\overline{f}_{\kappa E} = -\frac{i\kappa_x}{s} \left\{ \frac{\hbar K}{m} \overline{f}_{\kappa E} + \frac{eE}{m\sigma s} [2(\varepsilon n)' - n] \right\}, \quad (A12)$$

$$\overline{f}_{z \kappa E} = 2i\kappa_y \frac{eE\tau}{\hbar} \left( \frac{\hbar K}{m} \right)^2 \tau n(\varepsilon_k) \frac{4\omega_k^4(1 + 2s\tau) + 2\sigma^2\omega_k^2(1 + 3s\tau) - \sigma^4s\tau}{[\sigma^2s\tau + 2\omega_k(2s\tau + 1)]^2 [\sigma^2s\tau + 4\omega_k^2(s\tau + 1)]}. \quad (A13)$$

[1] S. Murakami, N. Nagaosa, and S. C. Zhang, 301, 1348 (2003).

[2] J. Sinova, D. Culcer, Q. Niu, N. A. Sinitsyn, T. Jungwirth, and A. H. MacDonald, Phys. Rev. Lett. 92, 126603 (2004).

[3] Y. K. Kato, R. C. Myers, A. C. Gossard, and D. D. Awschalom, Science 306, 1910 (2004).

[4] J. Wunderlich, B. Kaestner, J. Sinova, and T. Jungwirth, Phys. Rev. Lett. 94, 047204 (2005).

[5] M. I. Dyakonov and V. I. Perel, JETP Lett. 13, 467 (1971).

[6] L. S. Levitov, Y. V. Nazarov, and G. M. Eliashberg, Zh. Eksp. Teor. Fiz. 88, 229 (1985).

[7] E. I. Rashba, Phys. Rev. B 70, 201309 (2004).

[8] J. Inoue, G. E. W. Bauer, and L. W. Molenkamp, Phys. Rev. B 67, 033104 (2003).

[9] J. Inoue, G. E. W. Bauer, and L. W. Molenkamp, Phys. Rev. B 70, 041303 (2004).

[10] D. Culcer, J. Sinova, N. A. Sinitsyn, T. Jungwirth, A. H. MacDonald, and Q. Niu, Phys. Rev. Lett. 93, 046602 (2004).

[11] J. Schliemann and D. Loss, Phys. Rev. B 69, 165315 (2004).

[12] E. G. Mishchenko, A. V. Shytov, and B. I. Halperin, Phys. Rev. Lett. 93, 226602 (2004).

[13] S. I. Erlingsson, J. Schliemann, and D. Loss, Phys. Rev. B 71, 035319 (2005).

[14] O. V. Dimitrova, Phys. Rev. B 71, 245327 (2005).

[15] S. Zhang and Z. Yang, Phys. Rev. Lett. 94, 066602 (2005).

[16] A. Khaetskii, cond-mat/0408136 (2004).

[17] D. N. Sheng, L. Sheng, Z. Y. Weng, and F. D. M. Haldane, Phys. Rev. B 72, 153307 (2005).
[18] K. Nomura, J. Sinova, N. A. Sinitsyn, and A. H. MacDonald, Phys. Rev. B 72, 165316 (2005).
[19] S. Y. Liu, X. L. Lei, and N. J. M. Horing, cond-mat/0506189 (2005).
[20] O. Chalaev and D. Loss, Phys. Rev. B 71, 245318 (2005).
[21] R. Raimondi and P. Schwab, Phys. Rev. B 71, 033311 (2005).
[22] P. Zhang, J. Shi, D. Xiao, and Q. Niu, cond-mat/0503505 (2005).
[23] N. Sugimoto, S. Onoda, S. Murakami, and N. Nagaosa, cond-mat/0503475 (2005).
[24] H. Böttger and V. Bryksin, *Hopping Conduction in Solids* (Akademie Verlag, Berlin, 1985).
[25] E. Mishchenko and B. I. Halperin, Phys. Rev. B 68, 045317 (2003).
[26] V. M. Edelstein, Solid State Commun. 73, 233 (1990).
[27] C. Zhang and Z. Ma, Phys. Rev. B 71, 121307 (2005).
[28] A. Abrikosov, *Introduction to the theory of normal metals* (Solid State Physics, Suppl. 12, p. 182, Academic Press, New York and London, 1972).
[29] A. A. Burkov, A. S. Nunez, and A. H. MacDonald, Phys. Rev. B 70, 155308 (2004).
[30] O. Bleibaum, Phys. Rev. B 69, 205202 (2004).
[31] P. Lee and T. Ramakrishnan, Rev. Mod. Phys. 57, 287 (1985).