Probability Theory

Some remarks about the positivity of random variables on a Gaussian probability space

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Abstract

Let \((W, H, \mu)\) be an abstract Wiener space and let \(L \in L_{\log L}(\mu)\) is a positive random variable. Using the measure transportation of Monge–Kantorovitch, we prove that the operator corresponding to the kernel of the projection of \(L\) on the second Wiener chaos is lower bounded by a semi-positive Hilbert–Schmidt operator.

Résumé

Quelques remarques sur la positivité des variables aléatoires définies sur un espace gaussien. Soit \((W, H, \mu)\) un espace de Wiener abstrait et soit \(L \in L_{\log L}\) une variable aléatoire positive. A l’aide de la théorie de transport de mesure de Monge–Kantorovitch, nous montrons que le noyau de la projection de \(L\) dans le second chaos de Wiener est un opérateur de spectre inférieurement borné et que l’opérateur correspondant est inférieurement borné par un opérateur Hilbert–Schmidt semi-positif.

Version française abrégée

Soit \((W, H, \mu)\) un espace de Wiener abstrait : \(W\) est un Fréchet séparable localement convexe, \(\mu\) est une mesure gaussienne dont le support est \(W\) et \(H\) est l’espace de Cameron–Martin dont le produit scalaire et la norme sont notés respectivement \((\cdot, \cdot)_{H}\) et \(|\cdot|_{H}\). On notera \(\nabla\) la fermeture par rapport à \(\mu\) de la dérivée dans la direction de \(H\). En particulier, pour un espace hilbertien \(M\), \(D_{2,k}(M)\) est l’espace des classes d’équivalence de fonctions...
mesurables, à valeurs dans \( M \), dont les dérivées d’ordre \( k \in \mathbb{N} \) sont de carré intégrable par rapport à la norme du produit tensoriel Hilbert–Schmidt \( M \otimes H^{\otimes k} \), où \( H^{\otimes k} \) est l’espace des \( k \)-tensors Hilbert–Schmidt ; si \( M = \mathbb{R} \) alors nous noterons \( \mathbb{D}_{2,k} \) au lieu de \( \mathbb{D}_{2,k}(\mathbb{R}) \) (cf. [4,11,15]). On notera \( \delta \) l’adjoint de \( \nabla \) par rapport à \( \mu \), qui est une application continue de \( \mathbb{D}_{2,1}(M \otimes H^{\otimes k+1}) \) dans \( \mathbb{D}_{2,1}(M \otimes H^{\otimes k}) \). Noter que \( \delta \circ \nabla \) est l’opérateur d’Ornstein–Uhlenbeck, il sera noté \( L \). A l’aide de l’inégalité de Meyer, on peut définir les espaces de Sobolev d’ordre négatif \( (\mathbb{D}_{p,a}, \alpha \in \mathbb{R}, \ p > 1) \) et on note \( \mathbb{D}' = \bigcup_{p>1, \ a \in \mathbb{R}} \mathbb{D}_{p,a} \), qui est dual de l’espace \( \mathbb{D} = \bigcap_{p>1, \ a \in \mathbb{R}} \mathbb{D}_{p,a} \) (cf. [11,15]).

Quand \( W \) est l’espace de Wiener classique, i.e., \( W = C_0([0,1], \mathbb{R}) \), \( H = H_1([0,1], \mathbb{R}) \) (i.e., les primitives des éléments de \( L^2([0,1], dr) \)) il est bien connu que chaque élément \( L \) de \( L^2(\mu) \) admet une décomposition unique

\[
L = E[L] + \sum_{n=1}^{\infty} I_n(L_n),
\]

où \( L_n \in L^2([0,1]^n) \) et ce dernier représente les fonctions symétriques et de carré intégrables sur \([0,1]^n\). Soit \( H^{\otimes n} \) le produit tensoriel symétrique d’ordre \( n \) de \( H \), qui est isomorphe à \( L^2([0,1]^n) \). Si on note \( i_n, n \geq 1 \), cet isomorphisme, on peut montrer facilement que \( I_n(L_n) = \delta^n(i_n(L_n)) \), où \( \delta^n = (\nabla^n)^* \) par rapport à \( \mu \). Avec ces relations, on peut monter à partir de la formule de Taylor que

\[
L = E[L] + \sum_{n=1}^{\infty} \frac{1}{n!} \delta^n(E(\nabla^n L)),
\]

cf. [10,12,14,17] et aussi [15,16].

Soit \( v \) une autre probabilité, notons par \( \Sigma(\mu, v) \) l’ensemble des probabilités sur \( W \times W \) de marginales \( \mu \) et \( v \). On note \( J \) la fonctionnelle définie sur \( \Sigma(\mu, v) \) par

\[
J(\beta) = \int_{W \times W} |x - y|^2 d\beta(x, y).
\]

Dans le cas où \( W \) est de dimension finie, le problème de Monge–Kantorovitch consiste à trouver une mesure \( \gamma \in \Sigma(\mu, v) \) telle que la distance de Wasserstein

\[
d_H^2(\mu, v) = \inf \left\{ J(\beta) : \beta \in \Sigma(\mu, v) \right\}
\]

soit atteinte en \( \gamma \). Ce problème a été résolu dans [1] en dimension finie (cf. aussi [3] pour un survol rapide). Nous l’avons résolu dans [6,7] (cf. aussi [8]) quand la dimension de \( H \) est infinie. Exploquons plus précisément le cas particulier qui sera utilisé dans cette Note : si \( v \) est de la forme \( dv = L d\mu \), alors il existe une fonction \( \varphi \) appelée le potentiel de transport, appartenant à \( \mathbb{D}_{2,1} \), telle que \( T : W \to W \) définie par \( T = I_W + \nabla \varphi \) satisfasse \( T \mu = v \) et telle que \( \varphi = (I_W \times T) \mu \) soit l’unique mesure dans \( \Sigma(\mu, v) \) satisfaisant \( J(\gamma) = d_H^2(\mu, v) \). De plus \( \varphi \) est \( L \)-convexe : une variable aléatoire \( f : W \to \mathbb{R} \cup \{\infty\} \) est dite \( r \)-convexe, \( r \in \mathbb{R}, \) si \( \frac{r}{2} h^2 + f(w + h) \) est convexe sur \( H \) à valeurs dans \( L^0(\mu) \) [5] ; si \( r = 0 \), on l’appelle \( H \)-convexe. De même \( f \) est \( H \)-concave ou \( H \)-log-concave si, respectivement \(-f \) est \( H \)-convexe ou \(-\log f \) est \( H \)-convexe. Avec les hypothèses ci-dessus \( T \) admet un inverse p.s., noté \( S \), de la forme \( S = I_W + \eta \). De plus si \( \nabla \) est fermeable par rapport à \( v \) alors \( \eta : W \to H \) est de la forme

\[
\eta = \nabla \psi \quad \text{si} \quad \psi \in L^2(\mu) \quad \text{et} \quad \eta \quad \text{\( v \)-différentiable dans la direction de} \ H.
\]

Notons que nous avons déjà démontré dans [9] que \( \varphi \) est un élément de \( \mathbb{D}_{2,2} \) au lieu de \( \mathbb{D}_{2,1} \) si la densité \( L \in L\log L \) est \( H \)-concave. Cela rend possible le calcul du jacobien

\[
A = \det_2(I_H + V^2 \varphi) \exp \left\{ -\mathcal{L} \varphi - \frac{1}{2} |\nabla \varphi|^2_H \right\},
\]

où \( \det_2(I_H + V^2 \varphi) \) est le déterminant modifié de Carleman–Fredholm (cf. [2,16]).

1. Main results

Here is the first notable result of this Note:
Theorem 1.1. Assume that $L \in L^2(\mu)$ is a positive random variable and let $\varphi$ be the forward potential function associated to the Monge–Kantorovitch problem in $\Sigma(\mu, \nu)$, where $d\nu = \frac{1}{E[L]} L \, d\mu$. Then the following operator inequality holds true:

$$\frac{1}{2E[L]} \left\{ E[\nabla^2 L] - \frac{E[\nabla L] \otimes E[\nabla L]}{E[L]} \right\} \geq E[\nabla^2 \varphi].$$

(1)

Proof. Let us note first that, even if $\varphi$ is not in $D_{2,2}$, then the term $E[\nabla^2 \varphi]$ is a well-defined Hilbert–Schmidt operator since the constants are elements of the space of the test functions $D = \bigcap_{p,k} D_{p,k}$. Without loss of generality, we may assume that $E[L] = 1$. Let then $\nu$ be the measure $d\nu = L \, d\mu$. Since $E[L \log L] < \infty$, the Wasserstein distance $d_H(\mu, \nu) < \infty$, consequently, there exists a 1-convex map $\varphi \in D_{2,1}$ such that the transformation $T = I_W + \nabla \varphi$ solves the problem of Monge and the measure $(I \times T) \mu$ is the unique solution of Monge–Kantorovitch problem on $\Sigma(\mu, \nu)$.

Let $\rho(\delta h)$ denote the Wick exponential $\rho(\delta h) = \exp(\delta h - \frac{1}{2} |h|^2_H)$. For any $t \in \mathbb{R}$, we have

$$E[L \rho(\delta(\delta h))] = E[\rho(\delta(\delta h)) \circ T] = E\left[\exp\left(t\delta h - \frac{t^2}{2} |h|^2_H\right) \circ T\right].$$

A first order differentiation of this equality at $t = 0$ gives:

$$E[\nabla L, h]_H = E[(\nabla \varphi, h)_H],$$

for any $h \in H$, hence

$$E[\nabla L] = E[\nabla \varphi].$$

(2)

The second order differentiation at $t = 0$ and the integration by parts formula, which follows from the fact that $\delta = \nabla^*$, gives

$$E[(\nabla^2 L, h \otimes h)_2] = E\left[(\delta h + (\nabla \varphi, h)_H)^2 - |h|^2_H\right] = E[2\delta h(\nabla \varphi, h)_H + (\nabla \varphi, h)_H^2] = E[2(\nabla^2 \varphi, h \otimes h)_2 + (\nabla \varphi, h)_H^2],$$

for any $h \in H$, where $(\cdot, \cdot)_2$ denotes the Hilbert–Schmidt scalar product. Hence combining this with the relation (2) gives

$$E[\nabla^2 L] = E[\nabla \varphi \otimes \nabla \varphi] + 2E[\nabla^2 \varphi] \geq E[\nabla \varphi] \otimes E[\nabla \varphi] + 2E[\nabla^2 \varphi] = E[\nabla L] \otimes E[\nabla L] + 2E[\nabla^2 \varphi]$$

(3)

and the inequality (1) follows. □

Remark 1. Note that the inequality of Theorem 1.1 is different in spirit from the results of [13].

We can extend the inequality (1) as follows:
Corollary 1.2. Assume that \( m \in \mathcal{D}' \) is a positive distribution and denote again by \( m \) the Radon measure on \( W \) which corresponds to it (cf. [15]). Let \( m_2 \) be the projection of the distribution \( m \) to the second Wiener chaos, which is equal to \( \frac{1}{2} \delta^2 M_2 \), where \( M_2 \) is the element of \( H \otimes H \) defined by \( M_2(h \otimes k) = (m, \delta^2 (h \otimes k)) \). If the Wasserstein distance \( d_{H}(\mu, m) \) is finite, we have again
\[
\frac{1}{2m(W)} \left\{ M_2 - \frac{M_1 \otimes M_1}{m(W)} \right\} \geq E[\nabla^2 \psi],
\]
where \( M_1 \in H \) is defined by \( \langle M_1, h \rangle_H = (m, \delta h), h \in H \).

Proof. It suffices to apply Theorem 1.1 to the case \( P_t m \), where \( P_t \) is the Ornstein–Uhlenbeck semigroup. Then, from [8], the corresponding transport map \( \varphi_t \) converges to the transport map \( \varphi \) corresponding to the Monge–Kantorovitch problem for \( \Sigma(\mu, m) \) in \( \mathbb{D}_{2,1} \), as \( t \to 0 \). \( \square \)

We have also a weaker inequality whose difference with respect to (1) is that the Hilbert–Schmidt operator \( 2E[\nabla^2 \psi] \) is replaced by the identity operator of \( H' \):

Proposition 1.3. For any positive random variable \( L \in L^2(\mu) \), the following inequality is valid:
\[
I_H + \frac{1}{E[L]} E[\nabla^2 L] \geq \frac{1}{E[L]^2} E[\nabla L] \otimes E[\nabla L],
\]
where \( I_H \) denotes the identity operator of \( H \). In particular the projection of \( L \) in the second order Wiener chaos divided by the expectation of \( L \) is 1-convex.

Proof. Again, we may suppose that \( E[L] = 1 \). Let \( l(t) = E[L \rho(\delta(t h))], h \in H \), then we have
\[
l'(0)^2 \leq |h|^2_{H} + l''(0) \tag{5}
\]
To see this inequality it suffices to remark that \( \lambda^2 - 2\lambda \delta h + (\delta h)^2 \geq 0 \) for any \( \lambda \geq 0 \), hence taking the expectation with respect to \( L d\mu \) is again positive. Hence the discriminant of the second order polynomial in \( \lambda \) should be negative and this proves (5). To complete the proof of the proposition, it suffices to remark that \( l''(0) = E[\nabla_h L] \) and that \( l''(0) = \text{trace}(E[\nabla^2 L](h \otimes h)) \). The 1-convexity of \( \delta^2 \left( \frac{E[\nabla^2 L]}{2} \right) \) is immediate. \( \square \)

Proposition 1.3 extends also to the positive elements of \( \mathcal{D}' \) and it is to be noted that in this case we do not need the hypothesis about the finiteness of the Wasserstein distance:

Corollary 1.4. Assume that \( m \in \mathcal{D}' \) is a positive distribution and denote again by \( m \) the Radon measure on \( W \) which corresponds to it (cf. [15]). Using the notations of Corollary 1.2, we have again
\[
I_H + \frac{1}{m(W)} M_2 \geq \frac{1}{m(W)^2} M_1 \otimes M_1.
\]
In particular, the projection of \( m \) in the second Wiener chaos is 1-convex.

Proof. It suffices to regularize again \( m \) with \( P_t \) and then pass to the limit as \( t \to 0 \). \( \square \)

Remark 2.
(i) In the one dimensional case \( W = H = \mathbb{R} \), Theorem 1.1 says that, for an \( L \geq 0 \), with \( E[L] = 1 \) and with \( d_H^2(\mu, L \cdot \mu) < \infty \), we have the following inequality:
\[
\int_{\mathbb{R}} L(x)(x^2 - 1)\mu(dx) - \left( \int_{\mathbb{R}} xL(x)\mu(dx) \right)^2 \geq \int_{\mathbb{R}} \varphi(x)(x^2 - 1)\mu(dx),
\]
where \( \varphi \) is the forward transport function.
(ii) For the case $W = H = \mathbb{R}^n$, the equality (3) in the proof of Theorem 1.1 implies that

$$
\sum_{i=1}^{n} 1/2 E[\nabla^2 L](e_i \otimes e_i) = \frac{1}{2} E[\Delta L] = \frac{1}{2} E[|\nabla \phi|^2_H] + E[\Delta \phi] = \frac{1}{2} d_H^2(\mu, L \cdot \mu) + E[\Delta \phi],
$$

where $(e_i, 1 \leq i \leq n)$ is an orthonormal basis of $\mathbb{R}^n$. Note that, although $\Delta$ is a singular operator in the infinite dimensional case, the term

$$
E\left[\frac{1}{2} \Delta L - \Delta \phi\right] = \frac{1}{2} d_H^2(\mu, L \cdot \mu)
$$

is significant.

(iii) On the other hand, let us note that the inequality of Proposition 1.3 reduces to the Cauchy–Schwarz inequality.

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