Josephson and proximity effects on the surface of a topological insulator

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We investigate Josephson and proximity effects on the surface of a topological insulator on which superconductors and a ferromagnet are deposited. The superconducting regions are described by the conventional BCS Hamiltonian, rather than the superconducting Dirac Hamiltonian. Junction interfaces are assumed to be dirty. We obtain analytical expressions of the Josephson current and the proximity-induced anomalous Green’s function on the topological insulator. The dependence of the Josephson effect on the junction length, the temperature, the chemical potential and the magnetization is discussed. It is also shown that the proximity-induced pairing on the surface of a topological insulator includes even and odd frequency triplet pairings as well as a conventional s-wave one.

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I. INTRODUCTION

Topological insulator offers a new state of matter topologically different from the conventional band insulator. Edge channels or surface states of the topological insulator are topologically protected and described by Dirac fermions at low energies. The nature of the surface Dirac fermion of the topological insulator manifests itself in interesting phenomena such as the quantized magneto-electric effect, giant spin rotation, magnetic properties of the surface states, magnetization dynamics, magneto-transport phenomena, and superconducting proximity effect.

There have been a great and increasing interest on topological insulators attached to superconductors. In particular, Majorana fermions emerging in these systems have been intensively investigated. When superconductor/ferromagnet junctions are deposited on the topological insulators, surface Dirac fermions acquire a domain wall structure of the mass. At the domain wall, Majorana fermions emerge as a zero energy bound state. Majorana fermions have received much interest from the viewpoint of fundamental physics and also fault-tolerant quantum computing due to their exotic properties. It has been also shown that Majorana bound states crucially influence the Josephson effect. The current-phase relation shows $4\pi$ periodicity, i.e., $\sin(\phi/2)$ with the phase difference across the junction $\phi$. In previous works, it is assumed that the Dirac fermions become superconducting due to the proximity effect, and the junctions between superconducting and normal Dirac fermions are considered. In this paper, we take a different modeling of the same system. We consider the coupling between conventional superconductors and a topological insulator, rather than that between superconducting and normal (or magnetic) Dirac fermions. Namely, tunneling between the Shr"odinger electrons and the Dirac fermions is explicitly taken into account. Here, the superconductors are topologically trivial and hence, in this setup, there appear no Majorana fermions.

In this paper, we study Josephson and proximity effects on the surface of a topological insulator on which superconductors and a ferromagnet are deposited. The superconducting regions are described by the conventional BCS Hamiltonian, rather than superconducting Dirac electrons. We consider disordered junction interfaces in contrast to the previous works. We obtain analytical expressions of the Josephson current and the proximity-induced anomalous Green’s function on the topological insulator. The dependence of the Josephson effect on the junction length, the temperature, the chemical potential and the magnetization is discussed. It is also shown that the proximity-induced pairing on the surface of a topological insulator includes even and odd frequency triplet pairings as well as a conventional s-wave one.

Previous works on the Josephson effect on the surface of a topological insulator are mostly based on the approach which takes into account only the contribution from the Andreev bound states. This holds for short junctions with $d \ll \xi$, where $d$ and $\xi$ are the junction length and the superconducting coherence length, respectively. In this paper, we adopt the functional integral method which is applicable to any length of the junction, thus allowing to study the asymptotic behavior of the Josephson current for $d \to \infty$.

II. FORMULATION

We consider superconductor/topological insulator/superconductor junctions where a ferromagnet is also attached to the topological insulator, as shown in Fig. 1 (a). Junctions without the ferromagnetic region (Fig. 1 (b)) can be considered just by setting the exchange field to zero in the ferromagnetic region. The total Hamiltonian of the system reads

$$H = H_L + H_R + H_M + H_T$$

where...
FIG. 1: (Color online) Schematic of the model.

\[ H_{L(R)} = \sum_{k \in L(R)} \phi_{kL(R)}^\dagger \xi_{kL(R)} \sigma_0 \otimes \tau_3 \phi_{kL(R)} + \sum_{k \in L(R)} \phi_{kL(R)}^\dagger \left[ \Delta e^{-i \varphi_{L(R)} \tau_3} \sigma_0 \otimes \tau_1 \right] \phi_{kL(R)}, \]

\[ H_M = \sum_k \phi_k^\dagger \left[ \hbar v_F \left( k_y \sigma^y - k_x \sigma^x \right) \otimes \tau_3 + m \cdot \sigma \otimes \tau_0 - \mu \sigma_0 \otimes \tau_3 \right] \phi_k, \]

\[ H_T = \sum_{k,k'} \phi_{kL}^\dagger \left[ t e^{i (k-k') \cdot r_L} \sigma_0 \otimes \tau_3 \right] \phi_k + \sum_{k,k'} \phi_{kR}^\dagger \left[ t e^{i (k-k') \cdot r_R} \sigma_0 \otimes \tau_3 \right] \phi_k + h.c., \]

with \( \xi_{kL(R)} = \frac{\hbar^2 k^2_{L(R)}}{2m} - \mu_{L(R)} \) and \( \phi_{kL(R)}^\dagger = (c_{kL(R)}^\dagger, c_{-kL(R)}^\dagger, ic_{kL(R)}^\dagger, -ic_{-kL(R)}^\dagger) \). Here, \( \Delta \) and \( \varphi_{L(R)} \) are the magnitude of the gap function and the phase of the left (right) superconductor, respectively. Also, \( m \) is the exchange field, and \( \sigma \) and \( \tau \) are Pauli matrices in spin and Nambu spaces, respectively. \( H_{L(R)} \) represents the Hamiltonian on the left (right) superconductor, while \( H_M \) is the Dirac Hamiltonian with the exchange field. Note that the superconductors are described by the Schrödinger electrons and topologically trivial. Hence, in this setup, no Majorana fermions emerge. \( H_T \) is the tunneling Hamiltonian between the superconductors and the surface of the topological insulator which is treated as a perturbation. \( r_{L(R)} \) is the position of the interface between the left (right) superconductor and the topological insulator. We consider the incoherent tunneling model where the spin is conserved but the momentum is not conserved upon tunneling at the interface. This modeling is applicable to junctions with imperfect insulating barriers. In real space representation, the tunneling matrix element reads \( t \delta(r - r_{L(R)}) \). The average of the position vectors is assumed to give \( \langle r_R - r_L \rangle = d \frac{\hbar}{m} \). The calculated results are averaged over the positions of \( r_L \) and \( r_R \) at the interfaces.

The partition function is then given by

\[ Z = \int D\tilde{\psi} D\psi \exp \left[ -\sum_k \bar{\psi} \left(-G_0^{-1} + \hat{T} \right) \psi \right] \]

where \( \bar{\psi} = (\tilde{\phi}_{kL}, \tilde{\bar{\phi}}_{kL}, \tilde{\phi}_{kR}, \tilde{\bar{\phi}}_{kR}) \). \( G_0 \) is the bulk Green’s function while \( \hat{T} \) is a tunneling matrix. See the Appendix for their explicit forms. The free energy of the system can be calculated as \( F = -T \ln Z \) where \( T \) is the temperature of the system. The leading contribution to the Josephson current is given by the fourth order with respect to the tunneling Hamiltonian (see the Appendix for the details of the calculation). The Josephson current is then calculated as...
\[ I = -\frac{2e}{\hbar} \frac{\partial F}{\partial \varphi} = -\frac{4e}{h} T^4 \sin(\varphi + 2m_y d/\hbar \nu F) \sum_{\omega_n} \frac{(\nu V \Delta)^2}{\omega_n^2 + \Delta^2} \left[ |\hbar \nu F k_F|^2 |K_1(k_F d)|^2 - (\omega_n^2 + \mu^2 - m_z^2) |K_0(k_F d)|^2 \right] \] (6)

where \( \nu, V, \omega_n, \text{ and } K_\nu(z) (\nu = 0, 1) \) are, respectively, the density of states at the Fermi level, the area of the surface of the topological insulator sandwiched between the superconductors, the fermionic Matsubara frequency, and the modified Bessel function. Also, \( k_F \) is defined by \( \hbar \nu F k_F = \sqrt{(\omega_n^2 - \mu)^2 + m_z^2} \) and the branch is taken so that \( \text{Re} k_F > 0 \). Here, \( \varphi = \varphi_R - \varphi_L \) is the phase difference across the junction. It is seen that the Josephson effect is independent of \( m_x \), and \( m_y \) shifts the phase difference.

The critical current \( I_C \) can be written as

\[ \frac{-eI_CR}{T_C} = \frac{T}{T_C} \left( \frac{d}{\hbar \nu F} \right)^2 \sum_{\omega_n} \frac{\Delta^2}{\omega_n^2 + \Delta^2} \left[ |\hbar \nu F k_F|^2 |K_1(k_F d)|^2 - (\omega_n^2 + \mu^2 - m_z^2) |K_0(k_F d)|^2 \right] \] (7)

where \( T_C \) is the superconducting transition temperature and

\[ R^{-1} = \frac{4e^2}{\hbar} \left( \frac{\nu V^2}{d} \right)^2. \] (8)

III. RESULTS

A. Josephson effect

In what follows, we will study the critical Josephson current using Eq. (7). We consider a temperature dependence of the gap of the BCS type modeled by23

\[ \Delta(T) = \Delta(0) \tanh \left( 1.74 \sqrt{T_C/T - 1} \right). \] (9)

1. The effect of the exchange field

Here, let us study the effect of the exchange field. As seen from Eq. (7), the Josephson effect is independent of \( m_x \), and \( m_y \) shifts the phase difference. Since the inplane exchange field corresponds to the shift of the momentum23 the effect of the inplane exchange field can be reduced to the phase factor (which can be seen by proper transformations in Eq. (A5)) and hence we find the phase shift proportional to \( m_y \).

To see the effect of the \( z \)-component of the exchange field, \( m_z \), we plot the dependence of the critical Josephson current on \( m_z \) in Fig. 2 for \( T/T_C = 0.1, d/\xi = 1 \) and \( \mu/T_C = 100 \) where \( \xi = \hbar \nu F/T_C \) is the superconducting coherence length. With increasing \( m_z \), \( I_C \) increases and for \( m_z > \mu \), the current is strongly suppressed. This

FIG. 2: Critical Josephson current as a function of \( m_z \) for \( T/T_C = 0.1, d/\xi = 1 \) and \( \mu/T_C = 100 \).

FIG. 3: Critical Josephson current as a function of temperature of the system with \( m_z = 0 \) and \( \mu/T_C = 100 \) for (a) \( d/\xi = 0.1 \) and (b) \( d/\xi = 5 \).
been clarified, here let us consider the junction with characteristics. Since the effects of the exchange field have been considered, here let us consider the junction with $m = 0$. This corresponds to the junction illustrated in Fig. 4(b).

Figure 4 depicts $T$ dependence of the critical Josephson current with $\mu/T_C = 100$ for (a) $d/\xi = 0.1$ and (b) $d/\xi = 5$. For short normal segment $d/\xi = 0.1$, the behavior is similar to that of the conventional Josephson junctions through an insulating barrier, i.e., $\tanh(\Delta/2T)^{15}$. For large $d$, the critical current shows an exponential decay. This can be also obtained as follows. Using the asymptotic form of the modified Bessel function for $|z| \gg 1$:

$$K_{\nu}(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left[ 1 + \frac{(4\nu^2 - 1)}{8z} + \ldots \right], \quad (10)$$

for $m_z = 0, Td/hv_F \gg 1$ and $\mu \gg T$, we have

$$\frac{-eI_C R}{T} \sim \frac{1}{2\mu T_C} \frac{(\pi T \Delta)^2}{(\pi T)^2 + \Delta^2} e^{-2\pi d/hv_F}. \quad (11)$$

Notice that the $n = 0$ component in the Matsubara frequencies has a dominant contribution to the Josephson current. This form shows a typical exponential decay of the critical Josephson current for $Td/hv_F \gg 1$: the asymptotic behavior of the Josephson current (the exponential decay) has the same form as that governed by the Schrödinger electrons $^{20,44}$.

In Fig. 4, we show $d$ dependence of the critical Josephson current for $\mu/T_C = 100$ and several temperatures. For large $d$ or at high temperature, we see an exponential decay of the critical Josephson current. This is also consistent with the above analytical expression.

Figure 5 shows $\mu$ dependence of the critical Josephson current for $T/T_C = 0.1$ and several $d$. It is found that with the increase of $\mu$, the critical current monotonically decreases. This is because the proximity effect is suppressed by increasing the chemical potential $\mu$ as will be shown in Eq. (17). For large $d$, the Josephson current is inversely proportional to the chemical potential as seen from Eq. (14). Experimentally, the chemical potential can be tuned by chemical doping $^{46}$ or gating $^{47}$.

Recently, Josephson supercurrent through a topological insulator surface state has been observed $^{45}$. The dependence on $T$ and $d$ shown in Figs. 5 and 4 is qualitatively consistent with the experimental data. A quantitative difference would come from the fact that the bulk states of the topological insulator also contribute to the Josephson current because the chemical potential of the sample used in Ref. 45 probably crosses the bulk bands.

**B. Proximity effect**

In this subsection, we will investigate the proximity effect in a topological insulator/s-wave superconductor junction. Proximity effect in this junction has been investigated in Refs. 31, 53. The tunneling between the superconductor and a bulk topological insulator has been considered, and the validity of the Fu-Kane model has been discussed, based on mostly numerical approaches $^{21,23}$.

2. **Superconductor/topological insulator/superconductor junction**

Now, we will investigate the Josephson junction characteristics. Since the effects of the exchange field have been clarified, here let us consider the junction with...
functions can be calculated as

\[ \sigma \]

generality, we can set \( \varphi_L = 0 \) and \( r_L = 0 \). The Green functions can be calculated as

\[ G = \int D\bar{\psi}D\psi\bar{\psi} \exp \left[ -\sum_{\{k\}} \bar{\psi} (-G_0^{-1} + \hat{T}) \psi \right] \] (12) where

\[ G = \begin{pmatrix} G_L & 0 \\ 0 & G_M \end{pmatrix}, \quad G_0 = \begin{pmatrix} G_L & 0 \\ 0 & G_M \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} 0 & T_{12} \\ T_{21} & 0 \end{pmatrix}. \] (14)

The leading contribution is given by the second order with respect to the tunneling matrix. The anomalous Green’s function on the surface of the topological insulator \( f_M^s \) in the second order in \( t \) can be represented as

\[ f_M^s = -t^2 g_M(k, \omega_n) g_M(k, \omega_n) \sum_{k_x} f_L(k_L, \omega_n) \] (15)

\[ = \frac{\pi \nu \Delta t^2}{\sqrt{\omega_n^2 + \Delta^2}} \times \left( \omega_n^2 + \mu^2 + (\hbar v_F k_L \cdot \sigma + 2i\omega_n \mathbf{m} \cdot \sigma + 2\hbar v_F (k_L \times \mathbf{m}) \cdot \sigma \right) \] (16)

with \( k_L = (k_y, -k_x, 0) \). Here, spin-singlet pairing is proportional to the unit matrix in spin space while spin-triplet pairing is proportional to the Pauli matrix \( \sigma \). Therefore, it is seen that both singlet and triplet pairings are induced on the surface of the topological insulator. The generation of the triplet pairing reflects the symmetry breaking in spin space.\(^{48}\) In particular, for \( \mathbf{m} = 0 \), we have

\[ f_M^s = \frac{\pi \nu \Delta t^2}{\sqrt{\omega_n^2 + \Delta^2}} \times \left( \omega_n^2 + \mu^2 + (\hbar v_F k_L \cdot \sigma \right) \] (17)

We see that in the limit of \( \mu \to \infty \), we have \( f_M^s \to 0 \). This explains the suppression of the Josephson current with \( \mu \) in Fig. 5.\(^{5} \) It is also found that even in the absence of the exchange field, triplet pairing is induced on the surface if \( \mu \neq 0 \), which is consistent with Refs. 31,32 (see also Ref. 49). In previous works, it is assumed that by attaching an \( s \)-wave superconductor to a topological insulator, the same \( s \)-wave superconductivity is induced on the surface.\(^{31,32}\) Here, we find that not only \( s \)-wave singlet superconductivity but, in general, triplet \( p \)-wave superconductivity is also induced on the surface of the topological insulator.\(^{31,32}\) Also, we have clean surface states on the topological insulator. If the surface is in the diffusive regime, it is expected that odd frequency triplet \( s \)-wave superconductivity is induced on the topological insulator.\(^{50}\)

Let us focus on the case with \( \mu = 0 \) but finite exchange field. The anomalous Green’s function then becomes

\[ f_M^s = F(k_x, k_y, \omega_n) \left[ -\omega_n^2 - (\hbar v_F k_L)^2 + m^2 + 2i\omega_n \mathbf{m} \cdot \sigma + 2i\hbar v_F (k_L \times \mathbf{m}) \cdot \sigma \right], \] (18)

\[ F(k_x, k_y, \omega_n) = \frac{\pi \nu \Delta t^2}{\sqrt{\omega_n^2 + \Delta^2}} \times \frac{1}{\omega_n^2 + (\hbar v_F)^2 \{ (k_y + m_x)^2 + (k_x - m_y)^2 \} + m_z^2} \] (19)

Note that \( F(k_x, k_y, \omega_n) \) is an even function of \( k(= (k_x, k_y, 0)) \) and \( \omega_n \). We find that the component propor-
tional to $-\omega_n^2 - (\hbar v_F k)^2 + m^2$ represents a singlet $s$-wave superconductivity while that proportional to $2i\omega_n \mathbf{m} \cdot \mathbf{\sigma}$ is triplet and odd in $\omega_n$, namely odd frequency triplet $s$-wave pairing.\(^{34,35}\) The component proportional to $2i(k_1 \times \mathbf{m}) \cdot \mathbf{\sigma}$ corresponds to a triplet $p$-wave superconductivity.

IV. SUMMARY

In this paper, we have investigated Josephson and proximity effects on the surface of a topological insulator on which superconductors and a ferromagnet are deposited. We have described the superconducting regions by the conventional BCS Hamiltonian, rather than the superconducting Dirac Hamiltonian. We have presented analytical expressions of the Josephson current and the proximity-induced anomalous Green’s function on the topological insulator. The dependence of the Josephson effect on the junction length, the temperature, the chemical potential and the magnetization has been discussed. It has been also shown that the proximity-induced pairing on the surface of a topological insulator includes even and odd frequency triplet pairings as well as a conventional $s$-wave one.

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Appendix A: Calculation of the free energy

Here, we present the details of the calculation of the free energy of the junctions. The unperturbed Green’s function $G_0$ is represented by a $12 \times 12$ matrix as

$$G_0^{-1} = \begin{pmatrix} G_L^{-1} & 0 & 0 \\ 0 & G_M^{-1} & 0 \\ 0 & 0 & G_R^{-1} \end{pmatrix}. \quad (A1)$$

$$G_L(R) = \begin{pmatrix} i\omega_n + \xi_{k_L(R)} \tau_0 & \tau_1 & \Delta e^{-i\phi L(R) \tau_3} \tau_0 \\ \omega_n^2 + \xi_{k_L(R)}^2 + \Delta^2 & 0 \\ \omega_n - \mu + \hbar v_F (k_z - m_z) & 0 \end{pmatrix}, \quad (A2)$$

$$G_M = \begin{pmatrix} 0 & 0 \\ \omega_n - \mu - \hbar v_F (k_z - m_z) & 0 \end{pmatrix}. \quad (A3)$$

By performing the functional integral, we have the free energy of the junctions of the form

$$F = -T \ln Z = -T \text{Tr} \ln \left[ -G_0^{-1} + \hat{T} \right] \quad (A4)$$

where $\hat{T}$ is the tunneling matrix given by

$$\hat{T} = \begin{pmatrix} 0 & T_{12} & 0 \\ T_{21} & 0 & T_{23} \\ 0 & T_{32} & 0 \end{pmatrix}. \quad (A5)$$

with $T_{12} = t e^{i(k_2 - k_1) \cdot \mathbf{r}_L} \tau_0 \otimes \tau_3 = T_{12}^*$ and $T_{23} = t e^{i(k_3 - k_2) \cdot \mathbf{r}_R} \tau_0 \otimes \tau_3 = T_{23}^*$. The leading contribution is given by the fourth order with respect to the tunneling element, which is calculated as

$$F \approx -\frac{T}{4} \text{Tr} \left( \hat{G}_0 \hat{T} \right)^4. \quad (A6)$$

$$= -T \sum_{k_L, k, k_L, k', \omega_n} G_L(k_L, \omega_n) T_{12} G_M(k, \omega_n) T_{23} G_R(k_L, \omega_n) T_{32} G_M(k', \omega_n) T_{21} \quad (A7)$$

$$= 2T t^4 \text{Tr} \text{Re} \left[ \sum_{k_L, k, k_R, k', \omega_n} e^{i(k-k') \cdot (\mathbf{r}_R - \mathbf{r}_L)} f_L(k_L, \omega_n) g_M(k, \omega_n) f_R(k, \omega_n) g_M(k', \omega_n) \right]. \quad (A8)$$

$$= -2T t^4 \cos(\varphi + 2m_0 \hbar v_F) \sum_{\omega_n} \frac{(\nu V \Delta)^2}{\omega_n^2 + \Delta^2} \left[ \hbar v_F k_F |K_1(k_F d)|^2 - (\omega_n^2 + \mu^2 - m_0^2) |K_0(k_F d)|^2 \right]. \quad (A9)$$

Here, we have used the following relations

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \cos \varphi} d \varphi, \quad K_0(ak) = \int_0^\infty \frac{x J_0(ax)}{x^2 + k^2} dx, \quad K_1(x) = -\frac{d}{dx} K_0(x). \quad (A10)$$
for $a > 0$ and $\Re k > 0$, where $J_0(x)$ is the Bessel function.