Resonances near the real axis for manifolds with hyperbolic trapped sets

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RESONANCES NEAR THE REAL AXIS FOR MANIFOLDS WITH
HYPERBOLIC TRAPPED SETS

By EMMANUEL SCHENCK

Abstract. For manifolds Euclidian at infinity, we show that for some compact perturbations of the
Laplacian with hyperbolic classical trapped set, there are strips below the real axis where the resonance
counting function grows sub-linearly. We also provide an inverse result, showing that the knowledge
of the scattering poles can give some information about the Hausdorff dimension of the trapped set
when the classical flow satisfies the Axiom-A condition.

1. Introduction. Let $M$ be a $C^\infty$ manifold of dimension $n \geq 2$ which
agrees with $\mathbb{R}^n$ outside a compact set:

$$M = M_0 \sqcup M_1, \quad M_0 \cong \mathbb{R}^n \setminus B(0, R_0), \quad R_0 > 1.$$ 

Here, $M_1$ is a smooth compact manifold with connected boundary $\partial M_1 \cong S^{n-1}$,
and we can then see $M$ as a compact perturbation of $\mathbb{R}^n$. We assume that $M$ is
equipped with a positive density $dx$ which coincides with the Lebesgue measure
on $M_0$. We will be interested in the scattering theory on $M$ for a positive elliptic
self-adjoint pseudodifferential operator $P$ of order 2. The operator $P$ is supposed
to agree with the Euclidian Laplacian on $M_0$:

$$P|_{M_0} = -\Delta|_{\mathbb{R}^n \setminus B(0, R_0)}.$$ 

The total symbol $\sigma_P(x, \xi)$ of $P$ is assumed to be classical, in the Kohn-Nirenberg
class $S^2(M)$, and with vanishing sub-principal symbol:

$$\sigma_{\text{sub}}(P) = 0.$$ 

If $p \in C^\infty(T^*M)$ denotes the principal symbol of $P$, the classical Hamiltonian flow
$e^{tH_p} : T^* M \to T^* M$ is given by

$$\left( \begin{array}{c} \dot{x} \\ \dot{\xi} \end{array} \right) = H_p(x, \xi) = \left( \begin{array}{c} \partial_\xi p(x, \xi) \\ -\partial_x p(x, \xi) \end{array} \right), \quad (x, \xi) \in T^* M.$$ 

It is symplectic and preserves the energy layers $p^{-1}(E) \subset T^* M, \ E > 0$. By defi-
nition, the trapped set $\Gamma_E$ at energy $E > 0$ is the set of points that do not escape to
infinity in the future, nor in the past:

\[ \Gamma_E = \{ \rho \in T^*M : p(\rho) = E \text{ and } e^{tH_P(\rho)} \to \infty, \ t \to \pm \infty \} \subset T^*M. \]

Our main assumption in this article is that \( \Gamma_E \neq \emptyset \) for \( |E - 1| \) small enough, and if \( \Gamma \overset{\text{def}}{=} \Gamma_1 \), then

\[ e^{tH_P} : \Gamma \longrightarrow \Gamma \text{ is hyperbolic}. \]

The standard situation to be kept in mind is the metric scattering, where \( M \) is endowed with a Riemannian metric \( g \) and

\[ P = -\Delta_g, \quad g|_{M_0} = g_{\text{Eucl}}, \quad g|_{M_1 \setminus \partial M_1} \text{ is negatively curved}. \]

The main result of this article deals with the resonance counting function in strips of finite size \( s > 0 \) below the real axis. Under the precise assumptions in Theorem 1, we show that this resonance counting function satisfies:

\[ \exists s_0 > 0, \ \forall s \geq s_0, \quad \# \{ \lambda \text{ resonance of } P : |\text{Im} \lambda| \leq s, \ |\text{Re} \lambda| \leq r \} \geq C r^{1 - \frac{s}{2}} + C', \quad \alpha > 0. \]

Resonance counting functions in strips have been intensively studied since the conjecture of Lax and Phillips [LP89], and we will briefly review below the literature on this subject.

The resolvent \((P - \lambda^2)^{-1}\) continues meromorphically as an operator \( L^2_{\text{comp}} \to L^2_{\text{loc}} \) from \( \text{Im } z > 0, \ z^2 \notin \text{Spec}_{pp}(P) \) where it is analytic to \( \mathbb{C} \) when \( n \) is odd, and the logarithmic cover \( \Lambda_{\mathbb{C}} \) of \( \mathbb{C} \) when \( n \) is even [SZ91]. The poles of the meromorphic continuation of the resolvent are called the resonances, or scattering poles of \( P \), and they are the objects that replace the usual spectrum of \( P \) when \( M \) is compact.

We will denote by \( \mathcal{R}_M(P) \) the resonances of \( P \) on the manifold \( M \). Remark also that there is \( k \in \mathbb{N} \) such that \( 1_{\mathbb{R}^n \setminus B(0,R_0)}(P+i)^{-k} \) is of trace class. In this way, the black box scattering formalism of Sjöstrand and Zworski applies, in particular there are Poisson formula for resonances in this context, both for odd and even dimensions, see [SZ94, Zwo98] and Section 2 below.

Resonances close to the real axis are of particular interest, for instance in studying the local energy decay for the solutions of the wave equation \((\partial_t^2 + P)u = 0\). The trapping properties of the classical flow \( e^{tH_P} \) have a direct consequence on the repartition of these scattering poles in the lower half plane: in the non-trapping case \( \Gamma = \emptyset \), it has been shown that there are pole-free regions of logarithmic sizes below the real axis [MS78, MS82]. On the other hand, when Lax and Phillips first published their monograph [LP89], they conjectured that if \( \Gamma \neq \emptyset \), there should exist a sequence of resonances \((\lambda_i)_{i \in \mathbb{N}}\) converging to the real axis, namely

\[ \lim_{i \to +\infty} \text{Im } \lambda_i = 0. \]
Ikawa [Ika82] showed that this conjecture was generally incorrect, as he established a strip with no resonances below the real axis in the case of Euclidian scattering in $\mathbb{R}^3$ by two compact, disjoint convex obstacles. Ikawa formulated then what is known to be the modified Lax-Phillips conjecture: for a trapping scattering problem where by convention the resonances are located in the lower half plane, there is $\alpha < 0$ such that the strip

$$S_{\alpha} = \{ z \in \mathbb{C} : \alpha \leq \text{Im} z \leq 0 \}$$

contains infinitely many resonances. A good amount of results have been obtained so far concerning this conjecture, which turned out to be true in various settings [Gér88, Far95, SV96, Sto09, Pet02], see also the survey article of Sjöstrand [Sjö97].

However, the precise distribution of the scattering poles near the real axis is still badly understood. Unlike the counting function inside disks of radius $r \to \infty$, which in our settings reads [SZ91]:

$$C^{-1}r^n \leq \# \{ \lambda \in \mathcal{R}_M(P), |\lambda| \leq r \} \leq Cr^n, \quad C > 0,$$

there is no such asymptotics for the counting function in a fixed strip below the real axis. Sjöstrand [Sjö90] first proved an upper bound of fractal type for potential scattering in a semiclassical framework. Building on this work, Guillopé, Lin and Zworski [GLZ04] have proven geometric upper bounds for $P = -\Delta_g$ acting on $L^2(G\setminus \mathbb{H}^{n+1})$ where $G$ is a convex, co-compact Schottky group:

$$\forall \alpha > 0, \exists C > 0, \quad \# \{ \lambda \in \mathcal{R}_G(-\Delta_g) : |\text{Im} \lambda| \leq \alpha, |\text{Re} \lambda| \leq r \} \leq C r^{1+\delta}$$

where $\delta = \dim \Lambda(G)$ is the dimension of the limit set of $G$. This upper bound has led to conjecture a lower bound of the same order, conjecture which is known as the fractal Weyl law. Unfortunately, few lower bounds for the counting function in a strip are known. For $X$ a surface of constant negative curvature, Guillopé and Zworski [GZ99] have shown that for any (small) $\varepsilon > 0$ and (large) $A > 0$, there is a constant $C_\varepsilon > 0$ and a sequence $r_i \to \infty$ such that

$$\# \{ \lambda \in \mathcal{R}_X(-\Delta) : |\text{Im} \lambda| \leq C_\varepsilon, |\text{Re} \lambda| \leq r_i \} \geq Ar_i^{1-\varepsilon}.$$  

(1.1)

Even if this lower bound can be generalized in higher dimensions for the same type of manifolds, it is not sensitive to the size of the trapped set, and is not even optimal in the elementary case of the hyperbolic cylinder (with a single trapped orbit), where a lower bound is computable and linear as the resonances are distributed on a lattice.

For scattering by disjoint convex obstacles in dimension $n = 3$, lower bounds on the number of scattering poles in strips below the real axis have been obtained by Farhy [Far95] and later by Petkov [Pet02] who proved bounds similar to (1.1) and (1.1), building on earlier works of Ikawa [Ika85].
All of the above results about estimating the counting function in strips rely on a trace formula, which connects resonances to periodic orbits of the flow $e^{tH_p} : \Gamma \to \Gamma$. Under a reasonable assumption meaning roughly that the length spectrum of $e^{tH_p}$ is not too much clustered (see the definition immediately below), our first result can be stated as follows:

**Theorem 1.** Let $M$ be a $C^\infty$ manifold of dimension $n \geq 2$ such that

$$M = M_0 \cup M_1, \quad M_0 \simeq \mathbb{R}^n \setminus B(0,R_0), \quad R_0 > 1, \quad M_1 \text{ compact}.$$ 

Let $P$ be a second order elliptic, positive, selfadjoint pseudodifferential operator as above, and assume that:

(i) if $M$ has even dimension, 0 is not a resonance of $P$,

(ii) the Hamiltonian flow $e^{tH_p} : \mathcal{E}^*M \to \mathcal{E}^*M$ has a non-empty trapped set on which it is hyperbolic, and its length spectrum is minimally separated.

Then there are positive constants $C, \Theta, \epsilon_0$ such that for any $\epsilon \leq \epsilon_0$, there is $r_0(\epsilon) > 0$ with

$$\# \{ \lambda \in \mathcal{R}_M(P) : |\text{Im} \, \lambda| \leq \frac{2n}{\epsilon} \text{ and } |\text{Re} \, \lambda| \leq r \} \geq Cr_{1-\epsilon\Theta}, \quad \forall r \geq r_0(\epsilon).$$

More can be said about the constant $\Theta$, for this we refer to Section 4. These lower bounds are however still far from the already known fractal upper bounds, but they are the first explicit lower bounds for such systems. Let us discuss now our assumption about the length spectrum of $e^{tH_p}$, which we denote by

$$\mathcal{L} \overset{\text{def}}{=} \{ \ell > 0, \exists \rho \in \mathcal{E}^*M \text{ with } e^{\ell H_p}(\rho) = \rho \}.$$ 

The set of periodic orbits will be denoted by

$$\mathcal{P} \overset{\text{def}}{=} \{ \rho \in \mathcal{E}^*M : \exists t \in \mathbb{R} \text{ with } e^{tH_p}(\rho) = \rho \}.$$ 

**Definition 2.** We will say that $\mathcal{L}$ is minimally separated, if for some $s_+ > 0$ the following property holds true:

For any $C_0 > 0$, there is $T_0 > 0$ such that if $T \geq T_0$, any interval $[T - 1/2, T]$ contains at least one sequence of consecutive periods $\ell_1 < \cdots < \ell_k$ with $k \geq 3$ and

$$\ell_2 - \ell_1 \geq e^{-s_+T}, \quad \ell_k - \ell_{k-1} \geq e^{-s_+T}, \quad \ell_{k-1} - \ell_2 \leq e^{-C_0T}.$$ 

In other words, given $C_0 > 0$, for $T$ large enough we can always find in $[T - 1/2, T]$ two gaps of size $e^{-s_+T}$ in the length spectrum with between them either a single period, or a group of periods that spread over a distance at most $e^{-C_0T}$. The value $1/2$ is irrelevant in this definition and has been chosen for later convenience. What is important is that it is independent of $T$.

This assumption is reasonable for several reasons. As an important example, much more is true in dimension $\geq 3$ for the length spectrum of a compact (even
finite volume) hyperbolic manifold: every pair of distinct primitive periods is exponentially bounded from below [DJ16]. In particular, this would apply to our settings if $M_1$ is built from such hyperbolic manifolds. For compact surfaces in constant negative curvature, Dolgopyat and Jakobson note that this strong property remains true for a dense set in the corresponding Teichmüller space.

For general hyperbolic flows, like Anosov flows on basic sets, the number of periodic orbits with length $\leq T$ grows exponentially fast as $T \to \infty$. For instance, if $\Gamma$ is a basic set and $e^{tH_P} : \Gamma \to \Gamma$ is weak mixing, then [MS04]:

$$\lim_{T \to \infty} \frac{1}{T} \log \# \{ \gamma \in \mathcal{P} : \ell(\gamma) \in [T - 1/2, T] \} = h_{\text{top}} > 0,$$

where $h_{\text{top}} > 0$ denotes the topological entropy. Hence, if $s_+ > h_{\text{top}}$, by a box principle the number of gaps of size $e^{-s_+T}$ in $\mathcal{L} \cap [T - 1/2, T]$ grows exponentially as $T \to \infty$. This means that many occurrences of two consecutive periods separated by at least $e^{-s_+T}$ appear, however it could happen that (possibly many) periods cluster together with arbitrary small gaps between them. Note that $\mathcal{L}$ being minimally separated does not exclude this possibility either.

It is plausible that this condition can be satisfied in variable negative curvature: Anosov flows are structurally stable, which means that small $C^r$ perturbations ($r \geq 1$) of the flow are still Anosov, while the lengths of the periodic orbits would not be preserved in general.

Our second result deals with a topological pressure related to the infinitesimal unstable Jacobian $J^u$ on the trapped set, defined by

$$J^u(\rho) \overset{\text{def}}{=} \frac{d}{dt} \bigg|_{t=0} \log \det d\Phi_t^{\rho} |_{E^u_{\rho}},$$

where $E^u_{\rho}$ is the unstable subbundle at $\rho \in \Gamma$, see Section 3. We will consider the pressure of $-\frac{1}{2}J^u$ with respect to the flow $e^{tH_P} : \Gamma \to \Gamma$, which we will denote by

$$\Pr^u_{\Gamma} \overset{\text{def}}{=} \Pr_{\Gamma} \left( -\frac{1}{2}J^u \right).$$

A precise definition of this pressure is given in (3.4) below, see also [NZ09, Section 3.3] for a general discussion in our context. This pressure is always a real number, the sign of which may vary according to the geometry of the trapped set: it is heuristically considered that $\Gamma$ is large, or “thick”, if $\Pr^u_{\Gamma} \geq 0$, and small, or “filamentary”, if $\Pr^u_{\Gamma} < 0$. This consideration comes from the fact that the map $t \mapsto \Pr_{\Gamma} (-tJ^u)$ is decreasing for $t \geq 0$, and that in the case $\dim M = 2$, the unique root $t_u > 0$ of Bowen’s equation is related to $d_H(\Gamma)$, the Hausdorff dimension of $\Gamma$:

$$\Pr_{\Gamma} (-t_uJ^u) = 0 \iff d_H(\Gamma) = 2t_u + 1, \quad \dim M = 2.$$
In dimension $n \geq 3$, there is no simple relation between $d_H(\Gamma)$ and $t_u$, unless the flow $e^{tH_p}$ is conformal in the stable and unstable directions [PS01]. Nonnenmacher and Zworski [NZ09] have established the important role played by this pressure concerning the presence of a spectral gap near the real axis for hyperbolic scattering systems similar to those we consider in this article, but in a semiclassical framework. Translated to our settings, they showed that if $\text{Pr}^u_{\Gamma} < 0$, then there is a spectral gap near the real axis, namely for any $\varepsilon > 0$ and $\alpha_\varepsilon = \text{Pr}^u_{\Gamma} + \varepsilon$,

$$\#\{\lambda \in \mathcal{R}_M(P) : \lambda \in S_{\alpha_\varepsilon}\} < \infty.$$  

To state our second result, let us denote by $\text{Tr}u(t)$ the distributional trace of the wave group $u(t) = \cos t\sqrt{F}$, see Section 2 below. As a result, the pressure $\text{Pr}^u_{\Gamma}$ can be computed from the knowledge of this trace when the periodic orbits are dense in $\Gamma$ (here no assumption on the length spectra is needed):

**Theorem 3.** Let $M, P$ be as above, and assume that the flow $e^{tH_p} : \Gamma \to \Gamma$ is Axiom-A. We have:

$$\text{Pr}_\Gamma \left( - \frac{1}{2} J^u \right) = \lim_{T \to +\infty} \frac{1}{T} \log \left( \limsup_{\Xi \to +\infty} \sup_{\lambda \in [\exp T, e^{\exp T}]} \langle \text{Tr}u, f_{\lambda,T} \rangle \right).$$

The function $f_{\lambda,T} \in C^\infty_0(\mathbb{R}_+)$ is given by $f_{\lambda,T}(t) = \cos(\lambda t)\phi\left(\frac{t - T + 1/2}{t/2}\right)$, where $\phi \in C^\infty_0(\mathbb{R})$ is positive and equal to 1 near 0.

In other words, the pressure $\text{Pr}^u_{\Gamma}$ is a scattering invariant in our geometric settings: for instance if $n$ is odd, the trace part in the right-hand side can be computed directly from the resonances $\mathcal{R}_M(P)$ using the Poisson formula. The situation is however more subtle if $n$ is even, as the Poisson formula is not exact, see (4.9).

This theorem has also an interesting inverse scattering result for corollary once we have in mind the relationship between $\text{Pr}^u_{\Gamma}$ and the Hausdorff dimension of $\Gamma$. For this, define first the stable infinitesimal Jacobian by

$$J^s(\rho) \overset{\text{def}}{=} \left. \frac{d}{dt} \right|_{t=0} \log \det d\Phi_t^s|_{E^s_\rho},$$

and denote by $t^s \geq 0$ the root of Bowen’s equation $\text{Pr}_\Gamma(t^s J^s) = 0$. We recall that $t_u + t_s + 1 = d_H(\Gamma)$ if $e^{tH_p} : \Gamma \to \Gamma$ is conformal in the stable and unstable directions [PS01]. Theorem 3 implies:

**Corollary 4.** Let $M$ and $P$ be as above, with $e^{tH_p} : \Gamma \to \Gamma$ Axiom-A. Let $\text{Tr}u(t)$ be the distributional wave trace of $P$, $d_H(\Gamma)$ the Hausdorff dimension of $\Gamma$, and $f_{\lambda,T}$ be as in Theorem 3. Assume either that:

(i) $M$ has dimension 2, or
(ii) \( M \) has dimension \( n \geq 3 \), the flow \( e^{tH_P} \) is conformal in the stable and unstable directions, and \( t_u = t_s \).

Then,

\[
\lim_{T \to +\infty} \frac{1}{T} \log \left( \limsup_{\Xi \to \infty} \sup_{\lambda \in [e^{\exp T}, e^{\exp \Xi}]} \langle \text{Tr} u, f_{\lambda, T} \rangle \right) < 0 \iff d_H(\Gamma) < 2.
\]

In particular if \( \dim M \) is odd or \( P \) has no resonances at 0, the simple knowledge of the scattering poles gives an information about the Hausdorff dimension of the trapped set in this geometrical framework. Indeed, as noted above the distribution \( \text{Tr} u(t) \) is either completely determined by the resonances in odd dimensions, or in even dimensions with no resonances at 0, the terms in \( \text{Tr} u(t) \) which are not determined by the resonances produce an irrelevant \( O(1) \) term in the formula \( \langle \text{Tr} u, f_{\lambda, T} \rangle \) in the limit \( T \to +\infty \), see (2.3) below.

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2. The trace formula for resonances. As stated in the introduction, we assume that the symbol \( \sigma_P(x, \xi) \) of \( P \) belongs in the class \( S^2(M) \), where

\[
S^m(M) \equiv \{ a \in C^\infty(T^*M) : |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|} \}.
\]

The hypothesis that \( P \) is classical means that in each coordinate chart, \( \sigma_P \) admits an asymptotic expansion

\[
\sigma_P(x, \xi) \sim \sum_{j=0}^{\infty} p_{2-j}(x, \xi)
\]

with \( p_{2-j} \in C^\infty(T^*M) \) homogeneous of degree \( 2-j \). If \( P \) satisfies \( \sigma_{\text{sub}}(P) = 0 \), it implies in particular that if \( Q = \sqrt{P} \) is the square root of the unique self-adjoint extension of \( P \) in \( L^2(M) \) given by the spectral theorem, then \( \sigma_{\text{sub}}(Q) = 0 \) as well.

The operator \( Q = \sqrt{P} \) is then a pseudodifferential operator of order 1, with principal symbol \( q = \sqrt{p} \). The Hamiltonian vector field satisfies \( H_q = \frac{1}{2} H_P \) on \( T^*M \). We will denote by

\[
\Phi^t \overset{\text{def}}{=} e^{tH_q} : T^*M \longrightarrow T^*M
\]

the Hamiltonian flow of \( H_q \), and \( \Phi^t : \Gamma \to \Gamma \) is still hyperbolic. We now recall the various Poisson formula available in our context. Let \( T > R_0 > 0 \) and \( \chi = \chi_T \in C^\infty_0(M) \) with \( \chi \geq 0, \chi = 1 \) in \( M \setminus (\mathbb{R}^n \setminus B(0, R_0 + T)) \), \( \chi = 0 \) on \( \mathbb{R}^n \setminus B(0, R_0 + 2T) \subset M_0 \). In particular, \( \chi = 1 \) near \( M_1 \), and \( P \) coincides with \( -\Delta \) on the support.
of $1 - \chi$. We will be interested in the wave group

$$u(t) \overset{\text{def}}{=} \cos t\sqrt{P},$$

and the truncated free wave group

$$(1 - \chi)u_0(t)(1 - \chi), \quad u_0(t) = \cos t\sqrt{-\Delta}.$$ 

Denote by $C^\infty_T(\mathbb{R}_+)$ the set of functions in $C^\infty_0(\mathbb{R}_+)$ supported on the interval $[0, T]$. Because of the properties of $\chi$ and the finite speed of propagation of the waves in $\mathbb{R}^n$, the operator

$$(2.1) \quad \int (u(t) - (1 - \chi)u_0(1 - \chi))\rho(t)dt$$

is a smoothing operator with compactly supported kernel and defines a tempered distribution $\text{Tr} u(t)$ on $C^\infty_T(\mathbb{R}_+)$ by the formula

$$\text{Tr} u: \begin{cases} C^\infty_T(\mathbb{R}_+) \to \mathbb{C} \\ \rho \mapsto \langle \text{Tr} u, \rho \rangle \overset{\text{def}}{=} 2\text{Tr} \int (u(t) - (1 - \chi)u_0(1 - \chi))\rho(t)dt. \end{cases}$$

The idea that it is possible to relate the above trace to the resonances of $P$ originated in the work of Bardos, Guillot and Ralston [BGR82], followed by Melrose [Mel82], see also Sjöstrand and Zworski [SZ94] for the general presentation in the black box formalism. The Poisson formula established by these authors make this relation explicit, and the formula required by the framework of this article is due to Sjöstrand and Zworski in odd dimensions [SZ94], and Zworski [Zwo98] in even dimensions.

If $n$ is odd, the formula reads

$$(2.2) \quad \text{Tr} u(t) = \sum_{\lambda \in \mathbb{R}_M(P)} m_\lambda e^{-|\theta|\lambda}, \quad t \neq 0,$$

where $m_\lambda$ is the multiplicity of $\lambda$ as a pole of the resolvent of $P$.

When $n$ is even, the formula is no longer as simple, and we follow [Zwo98] for the presentation. Let $\rho \in [0, \pi/2]$ and denote by

$$\Lambda_\rho = \{r e^{i\theta}: r > 0, |\theta + 2k\pi| < \rho, k \in \mathbb{Z}\}$$

$$\cup \{r e^{i\theta}: r > 0, |\theta - \pi + 2k\pi| < \rho, k \in \mathbb{Z}\}$$

a conic open neighborhood of the real axis on the logarithmic plane with cut $i\mathbb{R}_+$. Finally let $\sigma(\lambda)$ be the scattering phase of $P$ and $\psi \in C^\infty_0([0, 1])$ be equal to 1.
near 0. Then

$$
\text{Tr} u(t) = \sum_{\lambda \in \Lambda_p} m(\lambda) e^{i|t|\lambda} + \sum_{\lambda^2 \in \text{Spec}_{pp}(P) \cap \mathbb{R}_-} m(\lambda) e^{i|t|\lambda} + m(0) + 2 \int_0^\infty \psi(\lambda) \frac{d\sigma}{d\lambda}(\lambda) \cos t\lambda d\lambda + v_{\rho,\psi}(t), \ n \text{ even},
$$

where $v_{\rho,\psi} \in C^\infty(\mathbb{R} \setminus \{0\})$ satisfies $\partial^k v_{\rho,\psi} = O(t^{-N})$ for all $k, N$ and $t \to \infty$. We refer to [Zwo98] and the references given there for the definition and properties of the scattering phase $\sigma(\lambda)$. In particular, if 0 is not a resonance of $P$, then $\sigma'(\lambda)$ is smooth near 0.

On the other hand, the Duistermaat-Guillemin trace formula [DG75] gives informations on the left-hand side of (2.2) and (2.3) and show precisely that the distribution $\text{Tr} u(t)$ has singularities located at times that correspond to length of periodic orbits of the flow $\Phi^t$ in $\Gamma$. The next paragraph makes this more precise.

### 2.1. Semiclassical formulation and localization of the trace.

In this article we will consider test functions $\rho$ in (2.1) which are supported inside a bounded interval near a time $T > 0$, where ultimately $T \to +\infty$. The finite speed of propagation implies that if $\Pi \in C^\infty_0(M)$ is equal to 1 in a neighborhood of

$$
B_T \overset{\text{def}}{=} M_1 \cup (B(0, R_0 + 2T) \setminus B(0, R_0)),
$$

then the operator (2.1) has a smooth Schwartz kernel included in $B_T \times B_T$, so we might simply look at $\Pi(u - (1 - \chi)u_0 (1 - \chi))\Pi$ to compute the trace:

$$
\langle \text{Tr} u, \rho \rangle = 2 \int \int \Pi(x) (u(t, x, x) - (1 - \chi(x))u_0(t, x, x)(1 - \chi(x)))\Pi(x) \rho(t) dt dx.
$$

Two specific test functions $\rho$ will be used: the first one for Theorem 1 and the second one for Theorem 3. We will choose them of the form

$$
\rho(t) = \cos(\lambda t) \phi^{(i)}(t), \quad i \in \{1, 2\}
$$

where $\lambda > 0$ will be large and $\phi^{(i)}$ is a real, compactly supported function. To define $\phi^{(i)}$, let first $\phi \in C^\infty_0([-1, 1])$ be a positive function, with $0 \leq \phi \leq 1 = \phi(0)$. We will assume that $\phi = 1$ on $[-3/4, 3/4]$ and define

$$
\phi^{(i)}(t) = \phi \left( \frac{t - b_i}{a_i} \right), \quad a_i, b_i > 0, \ i = 1, 2.
$$

For the first test function $\phi^{(1)}$, we will choose a parameter $\beta > 0$ such that for some $\epsilon > 0$ small but fixed, we have

$$
T = \epsilon \log \beta \leq \epsilon \log \lambda \quad \text{and} \quad \beta \leq \lambda \leq \beta + 1.
$$
Hence the two parameters $\lambda, \beta$ are not independent: in practice, for the proof of Theorem 1 we will choose an arbitrary (large) value of $\beta > 0$, and then adjust $\lambda$ accordingly, as it will be explained in Section 4. The value of $a_1$ and $b_1$ will be chosen such that

\begin{equation}
  b_1 \in [T - 1, T], \quad a_1 = e^{-J_+T} = \beta^{-\epsilon J_+}, \quad T - 1 \leq b_1 \pm a_1 \leq T
\end{equation}

where $J_+ > 0$ is a fixed constant that depends only on $M$ and $p$ that will be defined later in (4.2). The precise value of $b_1$ needed for Theorem 1 will also be defined in Section 4.

For the second test function $\varphi^{(2)}$, we will only assume that $T \leq \epsilon \log \lambda$ and set

\begin{equation}
  b_2 = T - \frac{1}{2}, \quad a_2 = \frac{1}{2}.
\end{equation}

For convenience, we will rather consider the unitary groups

$$
U(t) = e^{-itQ}, \quad U_0(t) = e^{-it\sqrt{-\Delta}}
$$

and work with the (half) wave equation written on the form $(\frac{1}{i} \frac{\partial}{\partial t} + Q)f = 0$. The next proposition establishes a correspondence for the wave groups $U$, $u$ and $u_0$.

**Proposition 5.** Let us write $U(t, x, y)$ for the Schwartz kernel of $U(t)$. Then,

\begin{equation}
  2 \text{Tr} \int \left( u(t) - (1 - \chi)u_0(1 - \chi) \right) \cos(\lambda t) \varphi^{(i)}(t) dt
  = \text{Re} \left( \int \Pi(x)U(t, x, x)\Pi(x) e^{i\lambda t} \varphi^{(i)}(t) dx dt \right) + O(\lambda^{-\infty}), \quad \lambda \to +\infty.
\end{equation}

The proof is postponed in Section 7.4. Roughly speaking, in the limit $\lambda \to \infty$, it is sufficient to consider the term involving $u$ in the trace, since when subtracting the cut-off free wave group, the part involving $u_0$ does not contribute significantly to the trace as $(1 - \chi)u_0(t)(1 - \chi)$ has a singular support in time reduced to 0. For convenience, in the next section we will write

$$
\langle \text{Tr} U, \rho \rangle \overset{\text{def}}{=} \int \Pi(x)U(t, x, x)\Pi(x)\rho(t) dx dt
$$

and the above proposition simply states that

$$
\langle \text{Tr} u(t), \cos(\lambda t) \varphi^{(i)}(t) \rangle = \text{Re} \langle \text{Tr} U, e^{i\lambda t} \varphi^{(i)}(t) \rangle + O(\lambda^{-\infty}).
$$

The study of the above traces in the limit $\lambda \to \infty$ is more easily treated with semiclassical microlocal analysis, so we present now a semiclassical formulation of the problem—see [JPT07] for an similar reformulation, and [Zwo10] for a complete introduction to semiclassical analysis. We note that if $h \overset{\text{def}}{=} \lambda^{-1}$, then $hQ = \sqrt{h^2P}$ and $hQ_0 = \sqrt{-h^2\Delta}$ are classical $h$-semiclassical operators of order
1. Our symbol classes for the semiclassical calculus can now depend on \( h \), and we will use the following notation:

\[
S_{\delta}^{m,k}(M) \overset{\text{def}}{=} \left\{ a \in C^\infty (T^* M \times [0, 1]) : |\partial_x^\alpha \partial_{\xi}^\beta a(x, \xi, h)| \leq C_{\alpha \beta} h^{k-\delta(|\alpha|+|\beta|)} \langle \xi \rangle^{m-|\beta|} \right\}.
\]

The assumption that the subprincipal symbol of \( P \) vanishes implies that

\[
hQ = \text{Op}_h(\sqrt{p}) + h^2 \text{Op}_h(q_2), \quad q_2 \in S_{0}^{-1,0}(M),
\]

where \( \text{Op}_h(\cdot) \) denotes the Weyl quantization. In the following we will write \( q = \sqrt{p} \) for the principal symbol of \( Q \). As a result, the wave groups

\[
u(t) = u_h(t) = \cos \left( h^{-1} t \sqrt{h^2 p} \right), \quad u_{0, h}(t) = \cos \left( h^{-1} t \sqrt{h^2 \Delta} \right)
\]
as well as \( U(t) \) and \( U_0(t) \) can be seen as semiclassical Fourier integral operators (albeit with trivial dependance on \( h \)), and we are lead to conduct the semiclassical analysis of the oscillatory integral

\[
\iint \Pi(x) \left( u_h(t, x, x) - (1-\chi(x)) u_{0, h}(t, x, x) (1-\chi(x)) \right) \\
\times \Pi(x) \cos \left( h^{-1} t \phi^{(i)}(t) \right) dt, \quad h \to 0.
\]

We conclude this section with the microlocalisation of the trace near \( \mathcal{E}^* M \).

From the fact that the principal symbol of \( Q \) and \( Q_0 \) is homogeneous of degree 1, it is easy to see [DG75, Section 1] that the operators

\[
\int \Pi e^{-\frac{i}{h}(tQ-1)} \Pi \phi^{(i)}(t) dt \quad \text{and} \quad \int \Pi (1-\chi) e^{-\frac{i}{h}(tQ_0-1)} (1-\chi) \Pi \phi^{(i)}(t) dt
\]

are microlocalized in phase space around \( \mathcal{E}^* M \). As a result, if \( f \in C^\infty_0(\mathbb{R}) \) is supported in an interval of the form \([1-\delta/2, 1+\delta/2]\) for some \( \delta > 0 \), and is identically equal to 1 near 1, then we have

\[
\text{Tr} \left( \int \Pi f(hQ)(U(t) - (1-\chi)U_0(t)(1-\chi)) \Pi e^{\frac{i}{h} t \phi^{(i)}(t)} dt \right) \\
= \text{Tr} \left( \int \Pi (U(t) - (1-\chi)U_0(t)(1-\chi)) \Pi e^{\frac{i}{h} t \phi^{(i)}(t)} dt \right) + \mathcal{O}(h^{\infty}).
\]

Here the remainder is actually \( \mathcal{O}_{N,\delta}(h^{N(1-\epsilon J_+)} \right) \) for all \( N \in \mathbb{N} \), due to the fact that

\[
\left\| \frac{d^N}{dt^N} \phi^{(1)}(t) \right\|_{L^\infty} = \mathcal{O}(\lambda^{N\epsilon J_+}), \quad \left\| \frac{d^N}{dt^N} \phi^{(2)}(t) \right\|_{L^\infty} = \mathcal{O}(1).
\]
2.2. Long-time trace formula. To prove Theorems 1 and 3, our main tool is a long-time trace formula, applied to a well-chosen test function. This formula is simply a generalization of the Duistermaat-Guillemin trace formula for test functions with support that can vary with $\lambda \to +\infty$, in a non-compact setting.

**Proposition 6.** Denote by $\mathcal{P}$ the set of periodic bicharacteristics of the flow $\Phi^t : \Gamma \to \Gamma$, and for $\gamma \in \mathcal{P}$, call $\ell^\#(\gamma)$ its primitive length and $P_{\gamma}$ its Poincaré map. Let $\phi^{(i)}, i \in \{1, 2\}$ be as in (2.4) with $a_i, b_i$ satisfying (2.6), (2.7). There is $\tilde{\epsilon} > 0$ and $C = C_{M,p,\phi} > 0$ such that for any $\epsilon \leq \tilde{\epsilon}$, we have the following expansion when $\lambda \to +\infty$:

\[
\langle \text{Tr} U, e^{it\lambda} \phi^{(i)} \rangle = \sum_{\gamma \in \mathcal{P}} e^{i\lambda \ell(\gamma)} \frac{\ell^\#(\gamma)}{\sqrt{|1 - P_{\gamma}|}} \phi^{(i)}(\ell(\gamma)) + O_{M,p}(\lambda^{-\mu}), \quad \mu = 1 - C\epsilon > 0.
\]

The particularity of this trace formula, albeit fairly classical, is twofolds: the number of periodic orbits that are considered can be large, for instance it can grow exponentially when $i = 2$, and their length diverge with $\lambda \to +\infty$. Also, the remainder in $\lambda$ is precisely controlled with respect to $\epsilon$. In [JPT07], a similar formula is derived for the Laplacian on a compact surface with negative curvature.

3. Hyperbolic and Axiom-A Hamiltonian flows. Hyperbolic dynamical systems have been extensively studied in the last decades, see [KH95] for a comprehensive introduction and many additional properties of such systems. Our main assumption in this article is that

\[ \Phi^t : E^*M \longrightarrow E^*M \]

is hyperbolic when restricted to the trapped set $\Gamma$. The set $\Gamma$ is a compact space, invariant under the flow. By definition of hyperbolicity, for any $\rho \in \Gamma$, the tangent space $T_{\rho}E^*M$ splits into flow, stable and unstable subspaces

\[ T_{\rho}E^*M = \mathbb{R}H_q \oplus E^s_{\rho} \oplus E^u_{\rho}. \]

The spaces $E^s_{\rho}$ and $E^u_{\rho}$ are $n - 1$ dimensional, and are preserved under the flow map:

\[ \forall t \in \mathbb{R}, \quad d\Phi^t_{\rho}(E^s_{\rho}) = E^s_{\Phi^t(\rho)}, \quad d\Phi^t_{\rho}(E^u_{\rho}) = E^u_{\Phi^t(\rho)}. \]

Moreover, there exist $\mu, C > 0$ such that on $E^*M$,

(i)

\[ \|d\Phi^t_{\rho}(v)\| \leq Ce^{-\mu t} \|v\|, \quad \text{for all } v \in E^s_{\rho}, \ t \geq 0. \]
(ii)

\( \| d\Phi^{-t}_\rho (v) \| \leq C e^{-\mu t} \| v \|, \) for all \( v \in E^u_{\rho}, t \geq 0. \)

There is a metric near \( \Gamma \), called the adapted metric, such that one can take \( C = 1 \) in the above equations. This metric can be extended to the total energy layer \( E^* M \) in a way that it coincides with the standard Euclidean metric outside \( T^* M_1 \).

The adapted metric on \( E^* M \) induces a volume form \( \Omega_\rho \) on any \( n-1 \) dimensional subspace of \( T(E^*_\rho M)_\rho \). Using \( \Omega_\rho \), we can define the unstable Jacobian at \( \rho \) for time \( t \). We set

\[ \det \left. d\Phi^t \right|_{E^u_\rho \to E^u_{\Phi^t(\rho)}} = \frac{\Omega_{\Phi^t(\rho)}(d\Phi^t v_1 \land \cdots \land d\Phi^t v_{n-1})}{\Omega_{\rho}(v_1 \land \cdots \land v_{n-1})}, \]

where \( (v_1, \ldots, v_{n-1}) \) can be any basis of \( E^u_{\rho} \). The infinitesimal unstable Jacobian is defined by

\[ J^u(\rho) = \frac{d}{dt} \bigg|_{t=0} \log \det \left. d\Phi^t \right|_{E^u_\rho}, \]

where \( d\Phi^t : E^u_\rho \to E^u_{\Phi^t(\rho)} \).

The unstable Jacobian at \( \rho \in T^* M \) will be denoted by

\[ e^{\lambda^+_t}(\rho) = \det \left. d\Phi^t \right|_{E^u_\rho} = e^{\int_0^t J^u(\Phi^{s}(\rho)) ds} \xrightarrow{t \to +\infty} +\infty. \]

The flow \( \Phi^t : E^* M \to E^* M \) is said to be Axiom-A if periodic orbits are dense in \( \Gamma \). We recall that hyperbolic sets are structurally stable, namely there is \( \delta > 0 \) such that \( \forall E \in [1 - \delta, 1 + \delta], \Gamma_E \) is a hyperbolic set for \( \Phi^t_{|p^{-1}(E)} : \Gamma_E \to \Gamma_E \). Hence in the thickened unit energy layer

\[ E^* M^\delta \overset{\text{def}}{=} \{ \rho \in T^* M : |q(\rho) - 1| \leq \delta \}, \]

the dynamics is uniformly hyperbolic [KH95], provided \( \delta \) is sufficiently small.

Let \( d \) be the distance function associated with the adapted metric. This will lead to consider the open balls around \( \rho_0 \in E^* M \) defined by \( B_{\rho_0}(\varepsilon) \overset{\text{def}}{=} \{ \rho \in T^* M : d(\rho, \rho_0) \leq \varepsilon \} \), where the distance is measured with the adapted metric and \( \varepsilon > 0 \) is small enough so that \( B_{\rho_0}(\varepsilon) \subset E^* M^\delta \). We end this paragraph by noting that there are some constants \( \varepsilon_0, C_+, K_+ > 0 \) depending only on \( p \) and \( M \) such that for any \( \rho_0 \in E^* M \) and \( \varepsilon < \varepsilon_0 \) we have

\[ \text{diam} \left. \Phi^t \right|_{B_{\rho_0}(\varepsilon)} \leq C_+ e^{K_+ t} \varepsilon \]

where the distance is again measured with the adapted metric.
3.1. The topological pressure for Axiom A flows. Let $f \in C^0(T^*M)$. For every $\varepsilon > 0$ and $T > 0$, a set $E \subset \Gamma \subset T^*M$ is $(\varepsilon, T)$ separated if for every $x, y \in E$ with $x \neq y$, $d(\Phi^t(x), \Phi^t(y)) > \varepsilon$ for some $t \in [0, T]$. Since $\Gamma$ is compact, the cardinal of such a set is always bounded, but it may grow exponentially with $T$. Set

$$Z_T(\Phi, f, \varepsilon) = \sup \left( \sum_{x \in E} \int_0^T f(\Phi^s(x)) \, ds \right)$$

where the supremum is taken over all $(\varepsilon, T)$ separated subsets of $\Gamma$. The topological pressure of the flow $\Phi^t : \Gamma \to \Gamma$, with respect to the function $f$ is defined by

$$(3.4) \quad \text{Pr}_\Gamma(f) = \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log Z_T(\Phi, f, \varepsilon).$$

If the flow $\Phi^t$ is Axiom A on $\Gamma$, it is possible to express the topological pressure by using the periodic orbits (see [KH95, Chapters 18 and 20]). More precisely,

$$\text{Pr}_\Gamma(f) = \lim_{T \to \infty} \frac{1}{T} \log \sum_{\gamma \in \mathcal{P} : \ell(\gamma) \leq T} e^{\int_0^T f} , \quad \Phi^t : \Gamma \to \Gamma \text{ is Axiom A}.$$

If $\text{Pr}_\Gamma(f) > 0$, then one has the even more precise asymptotics [Bow72]:

$$\sum_{\gamma \in \mathcal{P} : \ell(\gamma) \leq T} e^{\int_0^T f} = e^{T \text{Pr}_\Gamma(f)} \left( 1 + o(1) \right) , \quad T \to +\infty.$$

4. Counting scattering poles. This entire section is devoted to the proof of Theorem 1, assuming Proposition 6. The trace formula (2.9) can be used to derive a lower bound on the number of scattering poles near the real axis. This approach is the main method available to obtain lower bounds in resonance counting problems. However, the dependence on $\beta$ and $\lambda$ in the test function here is not standard as we will potentially deal with many periodic orbits of diverging length when these parameters tend to $+\infty$. Through this section, we then assume that the length spectrum of $\Phi^t$ is minimally separated.

4.1. Lower bound for the trace formula up to Ehrenfest time. Let $\text{h}_{\text{top}} \geq 0$ denote the topological entropy of the flow, and define

$$(4.1) \quad \Theta^u_+ = \limsup_{t \to \infty} \frac{1}{2t} \log \sup_{\rho \in \Gamma \subset E^*M} \det \left| 1 - d\Phi^t|_{E^u_\rho} \right| > 0.$$ 

Let $s_+$ be as in Definition 2 and choose a positive constant $J_+$ such that

$$(4.2) \quad J_+ > \max \left( \Theta^u_+, \text{h}_{\text{top}}, s_+ \right) > 0.$$
Now fix $\epsilon > 0$ small enough so that Proposition 6 holds with a function $\phi^{(1)}$ as in (2.4), with $a_1 = \beta^{-\epsilon J_+}$ and $b_1 \in ]T - 1, T[$ to be defined. Take then

$$C_0 \overset{\text{def}}{=} 2 \epsilon^{-1}, \quad \text{hence} \quad T = \epsilon \log \beta \rightarrow e^{-C_0 T} = \beta^{-2}. $$

Since for Theorem 1 we assumed that $\mathcal{L}$ is minimally separated, for $\beta$ large enough we can always find an interval $I \subset [T - 1, T]$ such that

$$I = I_{-1} \sqcup I_0 \sqcup I_1$$

with $I_{-1}, I_1$ open of size $e^{-J_+ T}$, $I_{\pm} \cap \mathcal{L} = \emptyset$, $I_0 = [\ell_0, \ell_1]$ with $\ell_0, \ell_1 \in \mathcal{L}$ and either

$$\ell_0 = \ell_1 \quad \text{or} \quad \ell_1 - \ell_0 \leq \beta^{-2}.$$

Let now set $b_1 = \ell_0$. It is then clear that the test function $\phi^{(1)}$ defined as in (2.4) is supported on $I$, and it satisfies $\phi^{(1)}(\ell_0) = 1$. The next proposition gives an estimate on the right-hand side of (2.9), and to keep in mind that $\phi^{(1)}$ depends on the parameter $\beta$ we write $\phi_\beta = \phi^{(1)}$ in the following.

**Proposition 7.** Let $\phi_\beta = \phi^{(1)}$ be as in (2.4) with $b_1, a_1$ as above. Then there are $c_0, \beta_0 > 0$ such that for any $\beta \geq \beta_0$ and $\epsilon$ sufficiently small, there is a choice of the parameter $\lambda$ such that (2.5) holds true and furthermore,

$$\langle \text{Tr} u(t), \cos(\lambda t) \phi_\beta(t) \rangle \geq c_0 \lambda^{-\epsilon \Theta^u_+}$$

where $\Theta^u_+ > 0$ is given in (4.1). The constant $c_0$ depends on $M$ and $p$.

**Proof.** From Proposition 5 it is enough to estimate $\text{Re} \langle \text{Tr} U, e^{i \lambda t} \phi_\beta \rangle$. Applying the trace formula to the test function $e^{i \lambda t} \phi_\beta(t)$ and taking the real part of both sides yields to

$$\text{Re} \langle \text{Tr} U, e^{i \lambda t} \phi_\beta \rangle = \sum_{\gamma \in \mathcal{P}} \cos \left( \lambda \ell(\gamma) \right) \frac{\ell^2(\gamma)}{\sqrt{1 - \mathcal{P}_\gamma}} \phi_\beta(\ell(\gamma)) + \mathcal{O}_{M, p, \phi}(\lambda^{-\mu}).$$

The difficulty to get lower bounds from this formula usually come from the highly oscillatory terms $\cos(\lambda \ell(\gamma))$. However from our choice of $a_1$ and $b_1$, the number of periods in the support of $\phi_\beta$ is either one, or the periods spread out over a size $\leq \beta^{-2}$ near $\ell_0$. It follows that once $\beta$ is chosen, the value of $\ell_0$ is determined and we can easily adjust $\lambda = \lambda(\beta)$ such that for $\beta$ sufficiently large, then

$$\cos \left( \lambda \ell_0 \right) = 1, \quad \lambda, \beta \text{ satisfy (2.5)}. $$
Note that if \( \{ \ell \in \mathcal{L} \cap \text{supp} \phi_{\beta} \} = \ell_0 \leq \cdots \leq \ell_k \) and \( k > 0 \), then for \( 0 < i \leq k \) we have

\[
|\cos(\lambda \ell_i) - \cos(\lambda \ell_0)| \leq \lambda \beta^{-2} \left( \frac{1}{2} \right), \quad \beta \text{ large enough}.
\]

From this we conclude that \( \forall \ell \in \mathcal{L} \cap \text{supp} \phi_{\beta} \) we have

\[
\cos(\lambda \ell) \geq \frac{1}{2}.
\]

From now on, and until the end of Section 4, we work with this precise value of \( \lambda = \lambda(\beta) \).

We then deduce from (4.3) that

\[
\text{Re}(\text{Tr} U, e^{i \lambda t} \phi_{\beta}) \geq \frac{1}{2} \sum_{\ell(\gamma) \in \text{supp} \phi_{\beta}} \frac{\ell^\sharp(\gamma)}{\sqrt{|1 - P_{\gamma}|}} \phi_{\beta}(\ell(\gamma)) + \mathcal{O}_{M,p,\phi}(\lambda^{-\mu}).
\]

Let us analyze further the right-hand side when \( \beta, \lambda \to \infty \). By the definition of \( \Theta_+^u \) in (4.2), we have that

\[
\sqrt{|1 - P_{\gamma}|} = \left| \det(1 - d\Phi^{\ell(\gamma)}|_{E^u}) \det(1 - d\Phi^{\ell(\gamma)}|_{E^u}) \right|^{\frac{1}{2}} < \frac{1}{c_0} e^{T \Theta_+^u}
\]

for some \( c_0 > 0 \). Since \( \phi_{\beta} > 0 \), \( \phi_{\beta} = 1 \) near \( \ell_0 \) and recalling that \( T \leq \epsilon \log \lambda \), we can keep only the orbits such that \( \ell(\gamma) = \ell_0 \) to get

\[
\sum_{\ell(\gamma) \in \text{supp} \phi_{\beta}} \frac{\ell^\sharp(\gamma)}{\sqrt{|1 - P_{\gamma}|}} \phi_{\beta}(\ell(\gamma)) \geq c_0 \sharp \{ \gamma : \ell(\gamma) = \ell_0 \} \lambda^{-\epsilon \Theta_+^u}.
\]

Now, from proposition 6 we see that the term \( \mathcal{O}_{M,p,\phi}(\lambda^{-\mu}) \) is \( o(\lambda^{-\epsilon \Theta_+^u}) \) if \( \epsilon \) is sufficiently small and can then be absorbed in the constant \( c_0 \), which concludes the proof. \( \square \)

### 4.2. Proof of Theorem 1: \( n \) odd.

In this section we still write \( \phi_{\beta} = \phi^{(1)} \).

If \( n \) is odd, then applying the distributional trace \( \text{Tr} u \) to \( \cos(\lambda t) \phi_{\beta}(t) \), we get from (2.2):

\[
\langle \text{Tr} u, \cos(\lambda t) \phi_{\beta}(t) \rangle = \left\langle \sum_{\lambda_j \in \mathfrak{M}(P)} e^{-\lambda_j t}, \cos(\lambda t) \phi_{\beta}(t) \right\rangle = \sum_{\lambda_j \in \mathfrak{M}(P)} \sum_{\nu = \pm 1} a_1 \hat{\phi}(a_1(\lambda_j + i \nu)) e^{-ib_1(\lambda_j + i \nu)} = \sum_{\lambda_j \in \mathfrak{M}(P)} \sum_{\nu = \pm 1} \hat{\phi}_{\beta}(\lambda_j + i \nu),
\]
where
\[ \hat{\phi}_\beta(\tau) \overset{\text{def}}{=} a_1 \hat{\phi}(a_1 \tau) e^{-ib_1 \tau}. \]

We are now in position to prove our main result. The Paley-Wiener estimate for \( \hat{\phi}_\beta \) reads:
\[ |\hat{\phi}_\beta(\tau)| \leq C_K a_1 e^{(b_1 - a_1) \text{Im} \tau} \left(1 + |a_1 \tau|\right)^{-K}, \quad \text{for any } K > 0, \text{ Im} \tau \leq 0. \]

Recall that \( a_1 = \beta^{-J_+} \) and \( b \geq \epsilon \log \beta - 1 \). This implies that for any \( K > 0 \) there is \( C_K > 0 \) such that for \( \beta \) large enough,
\[ |\hat{\phi}_\beta(\tau)| \leq C_K a_1 e^{\epsilon \log \beta \text{Im} \tau} \frac{1}{(1 + a_1 |\tau|)^K}. \]

Let \( s > 0 \), and define
\[ B_s \overset{\text{def}}{=} \{ z \in \mathbb{C} : -s \leq \text{Im} z \leq 0 \}. \]

We rewrite the sum over \( j \) as
\[
\sum_{\lambda_j \in \mathcal{R}_M(P)} \hat{\phi}_\beta(\lambda_j - \lambda) + \hat{\phi}_\beta(\lambda_j + \lambda) = \sum_{\lambda_j \in B_s} \sum_{\epsilon = \pm 1} \hat{\phi}_\beta(\lambda_j + \epsilon \lambda) \\
+ \sum_{\lambda_j \not\in B_s} \sum_{\epsilon = \pm 1} \hat{\phi}_\beta(\lambda_j + \epsilon \lambda).
\]

For the deep resonances outside the strip \( B_s \), we use the Paley-Wiener estimate to write, for \( \beta > 0 \) large enough,
\[
|\hat{\phi}_\beta(\lambda_j \pm \lambda)| \leq C_{K, \epsilon, s} \beta^{-J_+ \epsilon} e^{-\epsilon \frac{1}{2} |\text{Im}(\lambda_j)| \log \beta} \left(1 + \beta^{-\epsilon J_+} |\lambda_j \pm \lambda|\right)^{-K} \\
\leq C_{K, \epsilon, s} \beta^{-J_+ \epsilon} e^{-\frac{1}{2} |\text{Im}(\lambda_j)| \log \beta} \left(1 + \beta^{-J_+ \epsilon} ||\lambda_j| - \lambda||\right)^{-K}.
\]

Denote now the counting function in the lower half-plane by
\[ N(\lambda) \overset{\text{def}}{=} \sharp\{ \lambda_j \in \mathcal{R}_M(P) : |\lambda_j| \leq \lambda \} \]
and introduce the counting measure
\[ dN(t) = \sum_{\lambda_j \in \mathcal{R}_M(P)} \delta(t - |\lambda_j|) \, dt. \]
The sum over the resonances outside $B_s$ can be rewritten using this counting measure:
\[
\left| \sum_{\lambda_j \not\in B_s} \hat{\phi}_\beta(\lambda_j + \lambda) \right| \leq \sum_{\lambda_j \not\in B_s} C_{K,\epsilon,s}\beta^{-J_+}\epsilon e^{-\frac{\epsilon}{2}|\text{Im}(\lambda_j)|}\log(1 + (|\lambda_j| - \lambda)\beta^{-J_+}\epsilon)^{-K} \\
\leq C_{K,\epsilon,s} \int_s^{+\infty} \frac{\beta^{-s}\epsilon^{-J_+}\epsilon}{(1 + |t - \lambda|\beta^{-J_+}\epsilon)K}dN(t) \\
\leq C_{K,\epsilon,s} \int_s^{+\infty} \frac{t^n\beta^{-(J_+ + \frac{2}{3})}\epsilon}{(1 + |t - \lambda|\beta^{-J_+}\epsilon)K}dt \\
\leq C_{K,\epsilon,s}\lambda^n\beta^{-(J_+ + \frac{2}{3})}\epsilon \leq C'_{K,\epsilon,s}\lambda^{-s}(\frac{\epsilon}{2} + J_+)^{\epsilon} = o(\lambda^{-\epsilon\Theta_+^u}).
\]

In the last line, we have used a standard estimate $dN([0,t]) \leq Ct^n$ valid in our geometric settings and the fact that $|\lambda - \beta| \leq 1$. Finally, it is easily checked that

\[
(4.5) \\
\epsilon > 0 has to be chosen small. If \( s \) is chosen according to \((4.5)\), we have obtained
\[
\sum_{\lambda_j \in B_s} \sum_{i=\pm 1} \hat{\phi}_\beta(\lambda_j + i\lambda) = \sum_{\lambda_j \in B_s} \sum_{i=\pm 1} \hat{\phi}_\beta(\lambda_j + i\lambda) + o(\lambda^{-\epsilon\Theta_+^u})
\]
and it remains to estimate the sum over resonances in $B_s$. For this we adapt the technique of [SZ93]. Note first that if $\tau = r + i\sigma$ with $-s \leq \sigma \leq 0$ and $|r| \to \infty$, then
\[
\hat{\phi}_\beta(r + i\sigma) = \hat{\phi}_\beta(r) + O_{K,s}\left( (r\beta^{\epsilon J_+})^{-K} \right).
\]

Then, if $x \in \mathbb{R}$ and still $-s \leq \sigma \leq 0$, we have
\[
|\hat{\phi}_\beta(x + i\sigma)| \leq a_1 e^{\sigma b_1} |\hat{\phi}(a_1(x + i\sigma))| \\
\leq a_1 |\hat{\phi}(a_1 x)| + \left| (|\hat{\phi}(a_1(x + i\sigma)) - \hat{\phi}(a_1 x)|) \right|.
\]

A straightforward computation using the Paley-Wiener estimate and \((4.5)\) shows that the second term of the right-hand side of the above equation is $O_{\epsilon,M,p}(a_1(a_1x)^{-K})$ for any $K \in \mathbb{N}$. So we have
\[
|\hat{\phi}_\beta(x + i\sigma)| \leq a_1 |\hat{\phi}(a_1 x)| + O_{\epsilon,M,p}(a_1(a_1x)^{-K}), \forall K \in \mathbb{N}.
\]
In particular, if we write
\[ \psi(x) \overset{\text{def}}{=} \sup_{-s \leq \sigma \leq 0} \left| a_1 \hat{\phi}(a_1(x + i \sigma)) \right| \]
it is clear that \( \psi(x) \geq a_1 |\hat{\phi}(a_1 x)| \), so we have obtained
\[ \psi(x) = a_1 |\hat{\phi}(a_1 x)| + O_{\epsilon, M, p}(a_1 (a_1 x)^{-K}), \quad \forall K \in \mathbb{N}. \]  

Let us define now a measure on the real line that counts the resonances in the strip \( B_s \) by
\[ d\mu_s(x) = \sum_{\lambda_j \in \mathcal{R}_M(P) \cap B_s} \delta(x - r_j), \quad r_j = \text{Re} \lambda_j. \]  

We can then write
\[ \left| \sum_{\lambda_j \in \mathcal{R}_M(P) \cap B_s} \sum_{\iota = \pm 1} \hat{\phi}_\beta(\lambda_j + \iota \lambda) \right| \leq \sum_{\lambda_j \in \mathcal{R}_M(P) \cap B_s} \sum_{\iota = \pm 1} \psi(r_j + \iota \lambda) \]
\[ = \int_{\mathbb{R}} \sum_{\iota = \pm 1} \psi(x + \iota \lambda) d\mu_s(x) \]
\[ = \psi \ast \mu_s(-\lambda) + \psi \ast \mu_s(\lambda). \]

We first examine the term \( \psi \ast \mu_s(\lambda) \). For \( \nu > 0 \) arbitrary, decompose this term as
\[ \psi \ast \mu_s(\lambda) = \int_{\lambda + \lambda^{\nu + \epsilon} J_+}^{\lambda - \lambda^{\nu + \epsilon} J_+} \psi(x - \lambda) d\mu_s(x) + \int_{-\infty}^{\lambda - \lambda^{\nu + \epsilon} J_+} \psi(x - \lambda) d\mu_s(x) \]
\[ + \int_{\lambda + \lambda^{\nu + \epsilon} J_+}^{+\infty} \psi(x - \lambda) d\mu_s(x). \]

The Paley-Wiener estimate for \( \hat{\phi}_\beta \) implies that
\[ \int_{-\infty}^{-\lambda^{\nu + \epsilon} J_+} a_1 |\hat{\phi}(a_1 x)| dx = O_K(\lambda^{-K \nu}), \quad \int_{\lambda^{\nu + \epsilon} J_+}^{+\infty} a_1 |\hat{\phi}(a_1 x)| dx = O_K(\lambda^{-K \nu}). \]

Since \( \nu > 0 \), notice that these terms are \( O(\lambda^{-\infty}) \). These equations, together with (4.6) and the bound \( d\mu_s([-t, t]) = O(t^n) \), imply now that
\[ \int_{-\infty}^{\lambda - \lambda^{\nu + \epsilon} J_+} \psi(x - \lambda) d\mu_s(x) = O_K(\lambda^{n - K \nu}), \]
\[ \int_{\lambda + \lambda^{\nu + \epsilon} J_+}^{+\infty} \psi(x - \lambda) d\mu_s(x) = O_K(\lambda^{n - K \nu}). \]

We could write exactly the same type of estimates for \( \psi \ast \mu_s(-\lambda) \) but now for intervals of the form \([-\infty, -\lambda - \lambda^{\nu + \epsilon} J_+] \) and \([-\lambda + \lambda^{\nu + \epsilon} J_+, +\infty[\), and altogether
this yields to
\[ c_0 \lambda^{-\epsilon \Theta^u_+} \leq \int_{\lambda-\lambda^{\nu+\epsilon J_+}}^{\lambda+\lambda^{\nu+\epsilon J_+}} \psi(x-\lambda) \, d\mu_s(x) + \int_{-\lambda-\lambda^{\nu+\epsilon J_+}}^{-\lambda+\lambda^{\nu+\epsilon J_+}} \psi(x+\lambda) \, d\mu_s(x) \\
+ \mathcal{O}_K (\lambda^{n-K \nu}) + o (\lambda^{-\epsilon \Theta^u_+}). \]

Using (4.6), if \( \beta \) (and then \( \lambda \)) is large enough this can be rewritten as
\[ \lambda^{\epsilon (J_+ - \Theta^u_+)} \leq C_{M,p} \mu_s \left( \left[ \lambda - \lambda^{\nu+\epsilon J_+}, \lambda + \lambda^{\nu+\epsilon J_+} \right] \cup \left[ -\lambda - \lambda^{\nu+\epsilon J_+}, -\lambda + \lambda^{\nu+\epsilon J_+} \right] \right), \]
\[ C_{M,p} > 0. \]

At this point, we can not use directly a Tauberian argument to conclude, since the above counting estimate is a priori valid only for the precise value of \( \lambda \) such that (4.4) holds true once \( \beta \) is chosen. However, it is not hard to see that for an arbitrary value of \( \beta \) sufficiently large, we have
\[ \left[ \lambda - \lambda^{\nu+\epsilon J_+}, \lambda + \lambda^{\nu+\epsilon J_+} \right] \subset \left[ \beta - \beta^{2\nu+\epsilon J_+}, \beta + \beta^{2\nu+\epsilon J_+} \right] \]
and we finally get that
\[ \lambda^{\epsilon (J_+ - \Theta^u_+)} \leq C_{M,p} \mu_s \left( \left[ \beta - \beta^{2\nu+\epsilon J_+}, \beta + \beta^{2\nu+\epsilon J_+} \right] \right). \]

We are now in position to recall a general Tauberian Lemma due to Sjöstrand and Zworski:

**Lemma 8.** [SZ93, Lemma p. 849] Let \( \mu_s \) be a discrete counting measure as in (4.7). Assume that for some \( \delta \in (0, 1) \), \( c, r_0 > 0 \) and \( \kappa + \delta \geq 0 \) we have for \( r \geq r_0 \),
\[ \mu \left( \left[ r, r + r^\delta \right] \right) \geq cr^{\kappa+\delta}. \]

Then there is \( c_1, c_2 > 0 \) such that
\[ \mu \left( [0, r] \right) \geq c_1 r^{1+\kappa} - c_2. \]

This lemma can be used to conclude the proof of Theorem 1 in the following way. First if \( I \) is a positive interval, we denote by \( -I \) the symmetric interval with respect to 0, and we define \( \tilde{\mu}_s (I) = \mu_s (I) + \mu_s (-I) \) which is now a measure on the real axis. The main counting estimate (4.8) can now be rewritten
\[ C_{M,p} \tilde{\mu}_s \left( \left[ \beta - \beta^{\nu+\epsilon J_+}, \beta + \beta^{\nu+\epsilon J_+} \right] \right) \geq \beta^{\epsilon (J_+ - \Theta^u_+)} . \]
For any \( \varepsilon_0 > 0 \), if \( r = \beta - \beta^{\nu + \varepsilon J^+} \), \( \delta = \nu + \varepsilon J^+ + \varepsilon_0 \) and \( \beta \) is sufficiently large, then 
\[ [\beta - \beta^{\nu + \varepsilon J^+}, \beta + \beta^{\nu + \varepsilon J^+}] \subset [r, r + r^\delta] \] so we can write
\[
\tilde{\mu}_s([r, r + r^\delta]) \geq \frac{1}{C_{M,p}} r^{\epsilon(J^+ - \Theta^{\nu}_+)}.
\]
A straightforward application the above lemma with \( \kappa = -\nu - \varepsilon \Theta^{\nu}_+ - \varepsilon_0 \) shows that there is \( r_0, C > 0 \) such that if \( r \geq r_0 \) then
\[
\mu_s([-r, r]) = \tilde{\mu}_s([0, r]) \geq C r^{1 - \nu - \Theta^{\nu}_+ - \varepsilon_0} \geq C r^{1 - 3\Theta^{\nu}_+}
\]
since \( \nu, \varepsilon_0 \) are arbitrary and can then be chosen to be equal to \( \varepsilon \Theta^{\nu}_+ \). If we set \( s = \frac{2n}{\epsilon} \), \( \Theta = 3\Theta^{\nu}_+ \) and recall (4.2), (4.5), Theorem 1 is then proved.

**4.3. Proof of Theorem 1, \( n \) even.** Again in this section, \( \phi_{\beta} = \phi^{(1)} \). If \( n \) is even, using (2.3), the trace applied to the test function \( \cos(\lambda t) \phi_{\beta}(t) \) gives
\[
\langle \text{Tr} u, \cos(\lambda t) \phi_{\beta}(t) \rangle = \sum_{\lambda_j \in \Lambda_{\rho}} \sum_{\iota = \pm 1} \hat{\phi}_{\beta}(\lambda_j + \iota \lambda) + \sum_{\lambda^2 \in \text{Spec}_{pp}(P) \cap \mathbb{R}_{-}, \text{Im} \lambda < 0} \hat{\phi}_{\beta}(\lambda_j + \iota \lambda) + O(\lambda^{-\epsilon J^+})
\]
(4.9)
since the three last terms in (2.3) are controlled by the size of the support of \( \phi_{\beta} \) if \( P \) has no resonance at 0, see [Zwo98, Section 3].

The principle is exactly the same as for \( n \) odd. We split the sum over the resonances \( \lambda \in \Lambda_{\rho} \) and \( \lambda^2 \in \text{Spec}_{pp}(P) \cap \mathbb{R}_{-}, \text{Im} \lambda < 0 \) in two parts. The first one deals with the resonances located outside a strip of width \( s \) so that (4.5) holds true with the same arguments. The second one deals with the resonances inside this strip, where we get now an equality of the form
\[
0 < \lambda^{-\Theta^{\nu}_+} \leq C_{M,p}(\mu_s([\lambda - \lambda^{\nu + \varepsilon J^+}, \lambda + \lambda^{\nu + \varepsilon J^+}] \cup [-\lambda - \lambda^{\nu + \varepsilon J^+}, -\lambda + \lambda^{\nu + \varepsilon J^+}])
\]
+ \( O(\lambda^{-\infty}) + O(\lambda^{-\epsilon J^+}) \)
where the last remainder comes from the terms in the second line in (2.3). Since we have chosen \( J^+ > \Theta^{\nu}_+ \), the last two terms can be absorbed into the left-hand side, and we conclude the proof as for the case \( n \) odd.

**5. The Topological Pressure as a scattering invariant.** In this section we give the proof of Theorem 3. Let us fix some \( T > 0 \), and for a given value of \( \Xi > 1 \) such that
\[
e^T + \varepsilon_{hyp} T \log 2 \leq e^{\Xi T},
\]
consider a range of values for $\lambda$ given by

$$T \leq \log \log \lambda \leq \Xi T.$$  

(5.1)

For convenience, we write

$$I_{\Xi}(T) = \left[ e^{\exp T}, e^{\exp \Xi T} \right].$$

If $T$ is large enough, for any choice of $\lambda$ in this interval we have $T \leq \epsilon \log \lambda$ and we can apply Proposition 6, with $\phi^{(2)}(t)$ as it has been defined in (2.4) and (2.7). Applying the long-time trace formula and taking the real part, we obtain:

$$\Re \left( \text{Tr} U, e^{i \lambda t} \phi^{(2)}(t) \right) = \sum_{\gamma \in \mathcal{P}} \cos \left( \lambda \ell(\gamma) \right) \frac{\ell^{\sharp}(\gamma)}{\sqrt{|1 - P_\gamma|}} \phi^{(2)}(\ell(\gamma)) + O_{M,p}(\lambda^{-\mu}).$$

An upper bound for the left-hand side is easily derived: the cosine and $\phi^{(2)}(\ell(\gamma))$ terms are bounded by 1, which gives an upper bound of the right-hand side independent of $\lambda$ and $\Xi$, so we finally get

$$\limsup_{\Xi \to \infty} \sup_{\lambda \in [e^{\exp T}, e^{\exp \Xi T}]} \Re \left( \text{Tr} U, e^{i \lambda t} \phi^{(2)} \right) \leq \sum_{\gamma \in \mathcal{P}, \ell(\gamma) \in [T - 1, T]} \frac{\ell^{\sharp}(\gamma)}{\sqrt{|1 - P_\gamma|}} + C_{M,p} e^{-\mu e^{\Xi T}}.$$  

(5.2)

To obtain a lower bound, we must again control the oscillating terms. Note that with the above choice of $a_2$ and $b_2$, $\phi^{(2)}$ has support in $[T - 1, T]$, and is equal to 1 in $[T - 5/4, T - 1/4]$. Define

$$ \mathcal{J}_T = \{ \gamma \in \mathcal{P} : \ell(\gamma) \in [T - 1, T] \}, \quad \mathcal{J}'_T = \{ \gamma \in \mathcal{P} : \ell(\gamma) \in [T - 5/4, T - 1/4] \}$$

and denote by $\nu(T)$ the number of distinct lengths of the orbits in $\mathcal{J}_T$. Observe now that

$$\nu(T) \leq \# \mathcal{J}_T \leq e^{h_{\text{top}} T} \leq e^{\Xi T}.$$ 

A Dirichlet box principle can now be used [JPT07]: if $\{r_1, \ldots, r_{\nu(T)}\}$ are $\nu(T)$ distinct positive real numbers, for any constant $m > 0$ there is $\lambda_0 \in [m, 2^{\nu(T)} m]$ such that the following equation holds true:

$$1 \leq j \leq \nu(T), \quad \left| \lambda_0 r_j \mod 2\pi \right| \leq \frac{1}{2}.$$ 

This immediately yields to $\cos(\lambda_0 r_j) \geq \frac{1}{2}$ for any $r_j$. Now take $m = e^{\alpha T}$ with $\alpha > 0$. In particular, with $m = e^{\alpha T}$ this implies that $\lambda_0$ varies in the interval

$$[e^{\alpha T}, e^{\alpha T} + \log 2 e^{h_{\text{top}} T}].$$
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which is included in (5.1). For this special value of \( \lambda = \lambda_0 \), we have

\[
\frac{1}{2} \sum_{\gamma \in J_T'} \frac{\ell^u(\gamma)}{\sqrt{|1 - P_\gamma|}} - C_{M,p} e^{-\mu t} \leq \text{Re}\langle \text{Tr} U, e^{i \lambda_0 t} \phi^{(2)} \rangle
\]

\[
\leq \limsup_{\Xi \to \infty} \sup_{\lambda \in I_{\Xi}(T)} \text{Re}\langle \text{Tr} U, e^{i \lambda t} \phi^{(2)} \rangle.
\]

Combining this equation with (5.2), we obtain

\[
\frac{1}{2} \sum_{\gamma \in J_T'} \frac{\ell^u(\gamma)}{\sqrt{|1 - P_\gamma|}} - C_{M,p} e^{-\mu t} \leq \limsup_{\Xi \to \infty} \sup_{\lambda \in I_{\Xi}(T)} \text{Re}\langle \text{Tr} U, e^{i \lambda t} \phi^{(2)} \rangle
\]

\[
\leq \sum_{\gamma \in J_T'} \frac{\ell^u(\gamma)}{\sqrt{|1 - P_\gamma|}} + C_{M,p} e^{-\mu t}.
\]

The sums over periodic orbits in the above equation can be related to the topological pressure \( \text{Pr}_u^\Gamma \). First, recall that by definition, we have

\[
\det d\Phi^t_\rho|_{E^\rho_u} = e^{\int_0^t J^u(\Phi^s(\rho)) \, ds}.
\]

On the other hand, the hyperbolicity of the flow implies that there are constants \( K_i, C_i > 0, 1 \leq i \leq 4 \), independent of the point \( \rho \), with

\[
C_1 e^{K_1 t} \leq \det d\Phi^t_\rho|_{E^\rho_u} \leq C_2 e^{K_2 t}, \quad C_3 e^{-K_3 t} \leq \det d\Phi^t_\rho|_{E^\rho_s} \leq C_4 e^{-K_4 t}.
\]

This means that if \( \gamma \) is a periodic orbit of length \( t \) (large), then \( \det(1 - P_\gamma) \) is dominated by the unstable Jacobian, namely

\[
\det (1 - P_\gamma) = e^{\int_0^t J^u(\Phi^s(\rho)) \, ds} \left(1 + \mathcal{O}(e^{-\tilde{K} t})\right)
\]

for some \( \tilde{K} > 0 \) independent of \( \gamma \). Now, for some \( \delta > 0 \) consider the function \( f \in C^0(\Gamma) \) given by

\[
f = -\frac{1}{2} J^u + (1 + \delta) |\text{Pr}_\Gamma^u|
\]

where we have written \( \text{Pr}_\Gamma^u \overset{\text{def}}{=} \text{Pr}_\Gamma(\frac{1}{2} J^u) \). By construction we have \( \text{Pr}_\Gamma(f) > 0 \), so from the results recalled in section 3.1, we deduce the existence of \( C > 0 \) such that for \( T \) large enough,

\[
\frac{1}{C} e^{T \text{Pr}_\Gamma(f)} \leq \sum_{\gamma \in J_T} e^{f_{\gamma} f} \leq C e^{T \text{Pr}_\Gamma(f)}.
\]
and this can be rewritten as
\[
\frac{1}{C} e^{T|Pr^{n}_{T} + (1+\delta)|Pr^{n}_{T}|} \leq \sum_{\gamma \in \mathcal{P}: \ell(\gamma) \in J_{T}} e^{\int_{\gamma} f} \leq C e^{T|Pr^{n}_{T} + (1+\delta)|Pr^{n}_{T}|}.
\]

Thus, there is \( C > 0 \) such that for \( T \) large enough,
\[
\frac{1}{C} e^{TPr^{n}_{T}} \leq \sum_{\gamma \in \mathcal{P}: \ell(\gamma) \in J_{T}} e^{\int_{\gamma} \frac{1}{2} J^{u}_{\gamma} e^{(\ell(\gamma) - T)(1+\delta)|Pr^{n}_{T}|}} \leq C e^{TPr^{n}_{T}}
\]
and we obtain for some other \( C = C_{M,p,\delta} > 0 \) that
\[
\frac{1}{C} e^{TPr^{n}_{T}} \leq \sum_{\gamma \in \mathcal{P}: \ell(\gamma) \in J_{T}} e^{\int_{\gamma} \frac{1}{2} J^{u}_{\gamma}} \leq C e^{TPr^{n}_{T}},
\]
again for \( T \) sufficiently large. Finally in view of (5.3) and the fact that \( \inf_{\gamma \in \mathcal{P}} \ell(\gamma) \leq \ell^{a}(\gamma) \leq T \), we conclude that
\[
\frac{1}{C} \left( \inf_{\gamma \in \mathcal{P}} \ell(\gamma) \right) e^{TPr^{n}_{T}} \leq \sum_{\gamma \in \mathcal{P}: \ell(\gamma) \in J_{T}} \frac{\ell^{a}(\gamma)}{\sqrt{T - T_{\gamma}}} \leq CT e^{TPr^{n}_{T}}.
\]
It is easy to see that the same inequality (again with a different constant \( C \)) holds for periodic orbits in \( J_{T} \). Combining these inequalities, and noting that \( e^{-\mu e^{T}} = o(e^{TPr^{n}_{T}}) \) if \( T \) is sufficiently large, we end up with
\[
\frac{1}{C} \left( \inf_{\gamma \in \mathcal{P}} \ell(\gamma) \right) e^{TPr^{n}_{T}} \leq \limsup_{\Xi \to \infty} \sup_{\lambda \in I_{\Xi}(T)} \Re \langle \text{Tr} U, e^{i\lambda T} \phi^{(2)} \rangle \leq C T e^{TPr^{n}_{T}}.
\]
We only need to take the logarithm of the above equation and divide by \( T \) to recover (1.2) in the limit \( T \to \infty \), and Theorem 3 is proved since we can replace \( \Re \langle \text{Tr} U, e^{i\lambda T} \phi^{(2)} \rangle \) by \( \langle \text{Tr} u, \cos(\lambda t) \phi^{(2)} \rangle \) thanks to Proposition 5.

6. Hyperbolic trapped set and long-time generating functions.

6.1. Separation of periodic orbits. In this section, we consider periodic orbits of the flow in \( E^{*}M \) with length in the interval \( \ell(\gamma) \in [T - 1, T] \), where \( T \leq \epsilon \log h^{-1} \). For \( \gamma \) a periodic orbit and \( \epsilon > 0 \), denote by
\[
\Theta(\gamma, \epsilon) \text{ def } = \{ \rho \in E^{*}M, \, d(\rho, \gamma) < \epsilon \}
\]
an open tubular \( \epsilon \)-neighborhood \( \gamma \) in the adapted metric for \( E^{*}M \). We first state a fact which is essentially a direct consequence of the well-known Anosov shadowing Lemma [KH95].
LÉMA 9. There are constants \( \delta_0, C > 0 \) depending only on \( M \) and \( p \) such that if \( \gamma, \gamma' \) are periodic orbits of the flow \( \Phi^t \) in \( \mathcal{E}^*M \) with length in the interval \([\ell, \ell + \delta_0]\), then
\[
\Theta(\gamma, C e^{-K+\ell}) \cap \Theta(\gamma', C e^{-K+\ell}) = \emptyset,
\]
unless \( \gamma' = -\gamma \), in which case the two sets are identical.

Proof. From Darboux theorem, there is a symplectic chart \((y, \eta) = (y^1, \ldots, y^n; \eta^1, \ldots, \eta^n)\) near every \( \rho \in \mathcal{E}^*M \) such that \( \rho \equiv (0,0), \eta_1 = q - 1 \) and \( \frac{\partial}{\partial y_1}(0,0) = H_q(\rho) \). The unit energy layer is obtained by imposing \( \eta_1 = 0 \), so for \( \varepsilon, \bar{\varepsilon} > 0 \) two small enough constants, every \( \rho \in \Gamma \subset \mathcal{E}^*M \) has a neighborhood in \( \mathcal{E}^*M \) diffeomorphic to the flow-box
\[
\mathcal{F}(\bar{\varepsilon}, \varepsilon) = [-\varepsilon, \bar{\varepsilon}] \times U_\rho(\varepsilon) \subset \mathbb{R}^{2n-1},
\]
where \([ -\varepsilon, \bar{\varepsilon}] \) denotes the local coordinate along the flow direction, and \( U_\rho(\varepsilon) \subset \mathbb{R}^{2n-2} \) is a cross-section of the flow made of an open ball of radius \( \varepsilon > 0 \). Let now \( \gamma \in \mathcal{E}^*M \) be a periodic orbit of length \( \ell \), and we assume that for some \( \tau \in ]0, \bar{\varepsilon}[, \) the tubular neighborhood \( \Theta(\gamma, \varepsilon) \) contains another orbit \( \gamma' \neq \gamma \) of length \( \ell + \tau \). We will show that if \( \varepsilon, \bar{\varepsilon} \) are sufficiently small, we will get a contradiction. We can choose two points \( \rho, \rho' \in \Theta(\gamma, \varepsilon) \) such that \( \rho \in \gamma, \rho' \in \gamma' \) and in the flow box \( \mathcal{F}(\bar{\varepsilon}, \varepsilon) \) centered around \( \rho = (0,0) \), the point \( \rho' \) has coordinates \( (0, x') \), and then \( \Phi^t(\rho') = (-\tau, x') \). On the other hand, we know that
\[
d(\Phi^t(\rho), \Phi^t(\rho')) \leq C_+ e^{tK} d(\rho, \rho') \leq C_+ e^{tK} \varepsilon.
\]
Hence if \( \varepsilon \leq C_+ e^{-tK} \varepsilon, \) then \( d(\Phi^t(\rho), \Phi^t(\rho')) \leq \bar{\varepsilon} \) for \( 0 \leq t \leq \ell \). From these data, we will construct an infinite, discrete \( \varepsilon - \text{pseudo orbit} \) of the flow near \( \gamma \).

For this, let us divide the interval \([0, \ell + \tau]\) into subintervals \( \left[ i_0, i_1 \right] \cup \cdots \cup \left[ i_{N-1}, i_N \right] \) such that \( i_0 = 0, i_N = \ell + \tau, i_{k+1} - i_k = \tau \) if \( k \leq N - 1 \) and \( i_N - i_{N-1} \leq \tau \). We define a sequence of points \( (z_k)_{k \in \mathbb{N}} \) in \( \mathcal{E}^*M \), and a sequence of associated times \( (\delta_k)_{k \in \mathbb{N}} \) by
\[
z_k = \Phi^{(k \mod N)\tau}(\rho'), \quad \delta_k = \tau, \quad k \in \mathbb{Z}.
\]
By construction, \( (z_k, \tau)_{k \in \mathbb{Z}} \) is an \( \varepsilon - \text{pseudo orbit} \) of the flow, as \( d(z_{k+1}, \Phi^\tau(z_k)) = 0 \) if \( k \neq N - 1 \mod N \), and if \( k = N - 1 \mod N \), we see that
\[
d(\Phi^\tau(z_k), z_{k+1}) = d(\Phi^\tau(\rho'), \rho') \leq \varepsilon, \quad r = \tau - |i_N - i_{N-1}|.
\]
On the other hand, we have clearly \( d(z_k, \gamma') = 0 \leq \bar{\varepsilon} \) for all \( k \in \mathbb{Z} \). Recalling (6.1) and the choice of \( \varepsilon \), we also see that \( d(z_k, \gamma) \leq \bar{\varepsilon} \) for any \( k \). Finally, \( (z_k, \tau)_{k \in \mathbb{Z}} \) is an infinite \( \varepsilon - \text{pseudo-orbit} \) which is \( \varepsilon - \text{shadowed by} \) \( \gamma \) and \( \gamma' \) which are supposedly distinct. But the Anosov shadowing Lemma [KH95, Chapter 18] ensures the existence of \( \delta_0 > 0 \) such that if \( \varepsilon \leq \delta_0 \), then any \( \varepsilon - \text{pseudo orbit} \) is \( \varepsilon - \text{shadowed} \)
by a unique genuine orbit. Hence if we chose ε ≤ δ₀ we must have ρ = ρ' and the Lemma is proved up to shrink δ₀ further, so that no orbits multiple of each other can have lengths in an interval of size δ₀. □

6.2. Hamilton-Jacobi around Ehrenfest time. Consider the flow Φᵗ : T*M → T*M for 0 ≤ t ≤ T where again T ≤ ε log h⁻¹. In this section, examine how the (local) Hamilton-Jacobi theory of generating functions for the has an invertible differential at the point (6.2)

Let examine how the (local) Hamilton-Jacobi theory of generating functions for the flow Φᵗ apply when T → +∞.

Since we will work locally, we first need to control the size of open sets evolved by the flow until such large times. As above, let ρ₀ ∈ Γ. From the growth of balls under the dynamics (3.3), we notice that if we consider ε' > 0, ε(t) = e⁻ᵀ for L > 0 fixed but such that Bₚ₀(ε(t)) ⊂ E*ᵦ for all t ≥ 0, then

\[ \text{diam } Φᵗ(Bₚ₀(ε(t))) \leq ε' \]

if Cₑ⁽⁺(K⁺ₐ⁻L)₀ ≤ ε', which can be satisfied for sufficiently large times only if L > K⁺. Hence if L is sufficiently large, then for any t ∈ [T - 1, T], Φᵗ(Bₚ₀(e⁻ᵀ)) ⊂ T*M can be parametrized by a single local coordinates patch in ℝ²ⁿ.

We now recall a particular choice of coordinates near ρ₀ which is well adapted to the dynamics of the flow Φᵗ for times t ≤ ε log h⁻¹, essentially built on classical results of symplectic geometry, see for instance [GS94, Chapters 5 and 9]. For U ⊂ T*M we denote by

\[ \text{Graph } Φᵗ|_U \overset{\text{def}}{=} \{(ρ, σ), ρ ∈ U, σ = Φᵗ(ρ)\} ⊂ T*M × T*M. \]

Let ε₀, t₀ ∈ (0, 1), possibly depending on h, and 0 ≤ t₀ < T be such that t₀ + ε₀ ≤ T, and consider for ρ₀ = (y₀, η₀) ∈ E*ᵦ the open set

\[ U(ρ₀, t₀, ε₀) = Bₚ₀(ε₀) × \bigcup_{t₀ - tₙ < s < t₀ + tₙ} Φˢ(Bₚ₀(ε₀)) × ]t₀ - tₙ, t₀ + tₙ[ \]

\[ = \bigcup_{t₀ - tₙ < s < t₀ + tₙ} \text{Graph } Φˢ|_{Bₚ₀(ε₀)} × ]t₀ - tₙ, t₀ + tₙ[ \subset T*M × T*M × ℝ. \]

Without loss of generality, for ε₀, tₙ > 0 sufficiently small, we can perform a symplectic change of variables in Bₚ₀(ε₀) such that the projection map

\[ (6.2) \quad \pi_G : \text{Graph } Φᵗ|_{Bₚ₀(ε₀)} \ni (y, η, x, ξ, t) \mapsto (x, η, t), \quad Φᵗ(y, η) = (x, ξ) \]

has an invertible differential at the point (y₀, η₀, x₀, ξ₀, t₀), and this is equivalent to say that the upper-left block of dΦᵗ₀(y₀, η₀) is invertible. By the local inverse theorem, π_G is then invertible near (y₀, η₀, x₀, ξ₀, t₀) but we want to estimate quantitatively the size of such a neighborhood. More precisely, we have:

**Proposition 10.** There is L₀ > 0 such that for any ρ₀ ∈ E*ᵦ, t₀ > 0 with t₀ + hₑL₀ ≤ ε log h⁻¹, there are local coordinates on U₀ = Bₚ₀(hₑL₀) ⊂ E*ᵦ and
near $\Phi^t_0(U_0)$ with the property that the projection map

$$\text{Graph } \Phi^t|_{U_0} \ni (y, \eta, x, \xi) \mapsto (x, \eta), \quad \Phi^t(y, \eta) = (x, \xi)$$

is a diffeomorphism for $|t - t_0| \leq \h^{-L_0}$.

**Proof.** Set $\Phi^t_0(x_0, \xi_0) = (x_0, \xi_0)$. In a local chart in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, let $U \subset \mathbb{R}^{2n+1}$ be an open ball of radius $r$ around $(y_0, \eta_0, t_0)$ and $V \subset \mathbb{R}^{2n+1}$ an open neighborhood of $(x_0, \eta_0, t_0)$. Here if $z, z' \in U$, there are $c, c' > 0$ depending only on $M$ and $p$ such that if $\|\cdot\|$ denotes the usual Euclidian norm on $\mathbb{R}^{2n+1}$ and $d$ the adapted metric on $T^*M$, then

$$(6.3) \quad c\|z - z'\| \leq d(\rho, \rho') + |t - t'| \leq c'\|z - z'\|.$$  

What we want is an estimate on $r$ for which $F \overset{\text{def}}{=} \pi_G \circ \Phi^t : U \to V$ is one-to-one on its image, knowing that $d_{\rho_0, t_0}F \in GL(2n + 1)$. We will follow the lines of the usual proof of the local inverse theorem. Let us write $z = (y, \eta, t)$ and $\xi = (x, \eta, t)$. Without loss of generality, assume that $z_0 = (y_0, \eta_0, t_0) = 0$, $\xi_0 = (x_0, \eta_0, t_0) = 0$. In local coordinates, it makes sense to write, for $\xi$ fixed near $F(z_0)$:

$$G(z) = (d_{z_0} F)^{-1} (F(z) - \xi) - z, \quad z \in U.$$  

Note that $d_{z_0} G = 0$, so by continuity there is $r' > 0$ such that $\|z\| \leq r' \Rightarrow \|d_z G\| \leq 1/4$. To estimate $r'$, we first recall some standard flow estimates. Since $d_t \Phi^t = H_p \Phi^t$, estimates using Gronwall inequality [Zwo10, Lemma 11.11] show that for $t \geq 0$, the derivatives of $\Phi^t$ with respect to the phase space and time variables grow at most exponentially in time, namely if $(x, \xi) \in T^*M$ satisfies $p(x, \xi) \leq R$ for some $R > 0$, then there is constant $C_{M, p, R}$ such that for any multi-index $\alpha \in \mathbb{N}^{2n+1}$,

$$(6.4) \quad \sup_{(x, \xi) \in T^*M : p(x, \xi) \leq R} |\partial^\alpha \Phi^t(x, \xi)| \leq C_{\alpha} e^{C_{M, p, R} |\alpha||t|}, \quad C_{\alpha} > 0, \ t \in \mathbb{R}.$$  

This implies using a Taylor expansion of $dF$ around $z_0$ together with (6.4) that for some $C > 0$, we have $\|d_z F - d_{z_0} F\| \leq C e^{CT}$, $\|(d_{z_0} F)^{-1}\| \leq C e^{CT}$ and finally, up to increase $C$ again,

$$\|z\| \leq r' \overset{\text{def}}{=} C^{-1} e^{-CT} \Longrightarrow \|d_z G\| \leq \frac{1}{4}.$$  

Now $\|G(z)\| \leq \|d_{z_0} F^{-1} \xi\| + \frac{1}{4}\|z\| < 1/2$ if, say, $\|d_{z_0} F^{-1} \xi\| \leq 1/4$. This is for instance satisfied as soon as $\xi \in B(0, \frac{1}{4C} e^{-CT})$. Hence $G$ is contracting on a ball of radius $r'$ and the unique fixed point satisfies $F(z) = \xi$. We then obtain an invertible mapping from a ball $\{|z| < r\}$ on its image as soon as $r < r'$ and $\| \xi\| = \|F(z)\| \leq r'/4$ which is achieved for $r = L^{-1} e^{-LT}$ and some $L > C$. Hence for

$$U = B(z_0, r) \subset \mathbb{R}^{2n+1}, \quad F : U \to F(U)$$
is one-to-one and it is actually a diffeomorphism. Coming back to $T \leq \epsilon \log h^{-1}$, this finally implies in view of (6.3) the existence of $L_0 > 0$ depending only on $p$ and $M$ such that if $h$ is small enough, the projection $\pi_G$ defined in (6.2) is invertible at $(\rho, \Phi^t(\rho), t) \equiv (y, \eta, x, \xi, t)$ if

$$
\rho \in T^* M, \quad d(\rho_0, \rho) \leq e^{-L_0 T} \leq \epsilon_h = e^{-L_0 \epsilon \log h^{-1}} = h^{L_0},
$$

$$
|t - t_0| \leq t_h = h^{L_0}.
$$

The open set $B_{\rho_0}(\epsilon_h)$ is then of size $O(h^{L_0 \epsilon})$ in phase space. It follows that if $\epsilon > 0$ is small enough, we will be able to (semiclassically) microlocalize operators in such neighborhoods and use the semiclassical calculus.

A classical consequence of the preceding proposition is the existence of a local generating function $\varphi_0(t, x, \eta)$ for the flow $\Phi^t$ defined near $\rho_0 \in \Gamma$ up to times $\epsilon \log h^{-1}$. For $(y, \eta) \in B_{\rho_0}(h^{L_0})$, and $t = t_0$, Proposition 10 implies that there is a first generating function $\tilde{\varphi}_0(x, \eta)$ such that

$$
\Phi^{t_0} : (\partial_\eta \tilde{\varphi}_0(x, \eta), \eta) \mapsto (x, \partial_x \tilde{\varphi}_0(x, \eta))
$$

and using $(x, \eta, t)$ as coordinates on $\mathcal{U}(\rho_0, t_0, \epsilon)$, we can define

$$
\varphi_0(t, x, \eta) = \tilde{\varphi}_0(x, \eta) - \int_{t_0}^t q(x, \xi(s, x, \eta)) \, ds
$$

where as above, $\Phi^t(y, \eta) = (x, \xi)$. We readily check that $\varphi_0(t, x, \eta)$ is now a generating function for the flow for times $|t - t_0| \leq h^{L_0}$,

$$
(6.5)
y = \partial_\eta \varphi_0(t, x, \eta), \quad \xi = \partial_x \varphi_0(t, x, \eta), \quad \Phi^t : (\partial_\eta \varphi_0(t, x, \eta), \eta) \mapsto (x, \partial_x \varphi_0(t, x, \eta))
$$

and furthermore $\varphi_0$ satisfies the Hamilton-Jacobi equation

$$
(6.6)
\partial_t \varphi_0(t, x, \eta) + q(x, \partial_x \varphi_0(t, x, \eta)) = 0.
$$

The above construction can be done in a particular choice of coordinates in $\mathcal{U}(\rho_0, t_0, h^{L_0})$ which is very useful when considering the hyperbolic dynamics at stake near the trapped set. In [NZ09, Lemmas 4.3 and 4.4], Nonnenmacher and Zworski show the existence of a coordinate chart $(y, \eta) = (y^1, \ldots, y^n, \eta^1, \ldots, \eta^n)$ near $\rho_0 \equiv (y_0, \eta_0) \in \Gamma$ and $(x, \xi) = (x^1, \ldots, x^n, \xi^1, \ldots, \xi^n)$ near $(x_0, \xi_0) \equiv \Phi^{t_0}(\rho_0)$ for which the projection $\pi_G$ has a bijective differential near $(\rho_0, \Phi^{t_0}(\rho_0), t_0)$ and which is furthermore well adapted to the dynamics, in the sense that

$$
\frac{\partial}{\partial y^1}(\rho_0) = H_q(\rho_0), \quad \eta^1 = q - 1, \quad \frac{\partial}{\partial \xi^1}(\Phi^{t_0}(\rho_0)) = H_q(\Phi^{t_0}(\rho_0)),
$$
and moreover the unstable and stable spaces at \((y_0, \eta_0)\) are given by

\[
E_{\rho_0}^u = \text{Span} \left( \frac{\partial}{\partial y^2}(y_0, \eta_0), \ldots, \frac{\partial}{\partial y^n}(y_0, \eta_0) \right),
\]

\[
E_{\rho_0}^s = \text{Span} \left( \frac{\partial}{\partial \eta^2}(y_0, \eta_0), \ldots, \frac{\partial}{\partial \eta^n}(y_0, \eta_0) \right),
\]

and similar equations for the unstable and stable subspaces at \(\Phi_t^0(\rho_0)\). In these charts, we have time and energy coordinates for the index 1, and “stable” and “unstable” coordinates for the indices \(\geq 2\). We usually write \(u = (y_2, \ldots, y^n)\) and \(s_0 = (\eta_2, \ldots, \eta^n)\) for the stable and unstable coordinates. Now consider in these coordinates the following family of horizontal Lagrangian leaves

\[
\Lambda_\eta = \{ (y, \eta), y \in \pi B_{\rho_0}(h^\epsilon L_0), \ (y, \eta) \in B_{\rho_0}(h^\epsilon L_0) \}.
\]

Note that these Lagrangian manifolds are isoenergetic since \(\Lambda_\eta \subset p^{-1}(1 + \eta^1) \subset T^*M\). They also have nice properties when evolved by the flow \(\Phi_t\), and we extract from [NZ09] the ones which will be used in the present article.

**Proposition 11.** (adapted from [NZ09, Section 5]) Let \(\rho_0 \in \Gamma\), and \(\Lambda_\eta\) the above family of Lagrangian manifolds in the coordinate system adapted to the dynamics. If \(0 \leq t \leq T\), then the map

\[
\pi \circ \Phi^t|_{\Lambda_\eta} : \begin{cases} 
\pi \Lambda_\eta \rightarrow \pi \Phi^t(\Lambda_\eta) \\
y \mapsto y(t)
\end{cases}
\]

is well defined and invertible. Moreover, the differential matrix \(\frac{\partial y}{\partial y(t)}\) is uniformly bounded in time:

\[
\exists C_{M,p} > 0, \forall t \in [0, T], \quad \left\| \frac{dy}{dy(t)} \right\| \leq C_{M,p}
\]

and satisfies the following estimate on its domain of definition:

\[
C e^{-\lambda_t^+(\rho_0)} \leq \left| \det \frac{dy}{dy(t)} \right| \leq \frac{1}{C} e^{-\lambda_t^+(\rho_0)}, \quad 0 \leq t \leq T
\]

where \(C = C(M,p) > 0\) and the unstable Jacobian \(\lambda_t^+\) has been defined in Section 3.

This proposition is similar to what is known as the inclination lemma [KH95]: in the chosen coordinate system, these horizontal Lagrangian are transverse to the stable foliation, which explains their stretching along the unstable manifold when evolved by the flow. This property will be of crucial importance when determining estimates in \(C^k\) norm for the symbol of an oscillatory integral representation of \(U(t)\) up to time \(T\).
Our dynamical setup near the trapped set is then as follows. Let \( \rho_0 \in \Gamma \), and consider an orbit \( \gamma \) issuing from \( \rho_0 \) and of length \( \ell \), with \( \ell \leq \epsilon \log h^{-1} \). Let \( V_0 = B_{\rho_0}(h^{L_0}) \subset T^*M \) be an open neighborhood of \( \rho_0 \) of size \( h^{L_0} \) for some \( L_0 > 0 \) large enough, depending on \( M \) and \( p \) but independent of \( \epsilon \), so that Proposition 10 applies. In particular, we can find a sequence of times \( (t_k)^{0 \leq k \leq K} \) with \( t_0 = 0, t_K = \ell, \ K = O(h^{-\epsilon L_0} \log h^{-1}) \) and a chain of neighborhoods

\[
V_k = \bigcup_{t \in [t_k - h^{L_0}, t_k + h^{L_0}]} \Phi^t(V_0)
\]

with a coordinate set \( J_k : V_0 \times V_k \rightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n} \) such that there is a local generating function \( \varphi^k(t, x, \eta) \) for the flow for times near \( t_k \), more precisely:

\[
\Phi^t : \begin{cases} V_0 \mapsto \Phi^t(V_0), & t \in [t_k - h^{L_0}, t_k + h^{L_0}] \\
(\partial_\eta \varphi^k(t, x, \eta), \eta) \mapsto (x, \partial_x \varphi^k(t, x, \eta))
\end{cases}
\]

and \( \varphi^k \) also satisfies the Hamilton-Jacobi equation (6.6).

### 7. Proof of Proposition 6

To prove Proposition 6, we will adopt the same method as the original one developed by Duistermaat and Guillemin [DG75], namely we will represent the Schwartz kernel of \( U(t) \) locally by oscillatory integrals, and perform a stationary phase expansion in the integral on the right-hand side of (2.8). There is however an important difference with the situation considered in [DG75], since our test function has support localized around Ehrenfest times \( \leq \epsilon \log \lambda \). In this section, without further notifications, \( C \) will denote a positive constant depending only on \( M \) and \( p \).

#### 7.1. Microlocalization and local integral representations

Let \( \rho_0 \in \Gamma \), and \( \gamma \) the orbit issuing from \( \rho_0 \) for positive times. Let \( \chi \in C_0^\infty(T^*M) \) being equal to one near \( \rho_0 \). We assume that \( \chi \) is supported in a neighborhood of \( \rho_0 \) of size \( O(h^{L_0}) \) where \( L_0 > 0 \) is as in the preceding section, namely it is large enough so that Propositions 10 and 11 hold true. We again stress that \( L_0 \) is independent of \( \epsilon \), so if \( \epsilon < \delta L_0^{-1} \), we can ensure that \( \chi \in S^{0,0}_\delta(T^*M) \) for some \( \delta < 1/2 \). Defining \( V_0 = \text{supp} \chi \), we can find local generating function for the flow restricted to \( V_0 \), up to times \( T \leq \epsilon \log h^{-1} \). Let us call \( A_0^w = \text{Op}_h(\chi) \). The goal of this section is to build a local integral representation of \( U(t)A_0^w \) for times \( t \in [0, T] \).

If \( u \in L^2(M) \), we can use a partition of unity and assume without loss of generality that \( u \) is compactly supported, so that in a local chart,

\[
A_0^w u(y) = \frac{1}{(2\pi h)^n} \int \exp(h^{y-z, \eta}) \chi \left( \frac{y+z}{2}, \eta \right) u(z) d\eta dz = \int \exp(h^{y, \eta}) u_\eta(y) d\eta
\]
where
\[ u_{\eta}(y) \overset{\text{def}}{=} \frac{1}{(2\pi h)^n} \int e^{-\frac{i}{h} (z,\eta)} \chi \left( \frac{y + z}{2} \right) u(z) \, dz. \]

In this way, we have decomposed \( A^w_0 u \) into a sum of “momentum” Lagrangian states \( e^{i \frac{(y,\eta)}{h}} u_{\eta}(y) \) depending on a parameter \( \eta \). Note that because \( \chi \in C^\infty_0(T^* M) \), the parameter \( \eta \) belongs to a compact set so this integral is always well defined. These Lagrangian states are purely horizontal, their Lagrangian manifold being given by

\[ \Lambda^0_\eta = \{ (y, \eta) \in T^* M, y \in \text{supp} \, u_{\eta} \}. \]

Hence \( V_0 = \text{supp} \, \chi \) is foliated by such horizontal Lagrangian manifolds, and this foliation is used on the microlocal level in the above decomposition of \( A^w_0 u(y) \). The important fact here is that in the coordinates adapted to the dynamics around \( \rho_0 \), the manifolds \( \Lambda^0_\eta \) are transverse to the stable foliation and the inclination property applies when evolving these manifolds under the Hamiltonian flow.

For each time-window \( t \in [\tau_k, \tau_{k+1}] \) we will find an oscillatory integral representation of the operator \( U(t)A^w_0 \), which will be denoted by \( U_k(t)A^w_0 \), such that

\[ \text{Tr} \left( \Pi \left( U(t) - U_k(t) \right) A^w_0 \right) = O(h^n). \]

The next proposition makes this much more precise, and is the key technical tool needed to prove our main theorem.

**PROPOSITION 12.** Let \( \gamma \) be an orbit of length \( \ell \leq T \) starting at \( \rho_0 \in E^* M \). Let \( \chi \in C^\infty_0(T^* M) \), \( V_0 = \text{supp} \, \chi \) and \( A^w_0 \) be as above. Let \( (V_k)_{0 \leq k \leq K}, (\varphi^k)_{0 \leq k \leq K} \) be as in (6.8) and (6.9). For any \( N \in \mathbb{N} \) large enough, \( 0 \leq k \leq K \) and \( t \in [t_k - h^{\epsilon L_0}, t_k + h^{\epsilon L_0}] \), there is a sequence of local Fourier integral operators \( U_k(t) \) and \( \mathcal{R}^N_k(t) \) such that

\[ U_k(t)A^w_0 = U(t)A^w_0 + \mathcal{R}^N_k(t)A^w_0 \quad \text{and} \quad \text{Tr} \left( (\Pi \mathcal{R}^N_k(t)A^w_0) \right) = O(h^{-N/3}). \]

Furthermore, the operators \( U_k(t) \) are of the form

\[ U_k(t)A^w_0 u(x) = \int e^{\frac{i}{h} \varphi^k(t,x,\eta) - (y,\eta)} a^k_h(t,x,\eta) (A^w_0 u)(y) \, dy \, d\eta, \]

\[ t \in [t_k - h^{\epsilon L_0}, t_k + h^{\epsilon L_0}], \]

where the symbol \( a^k_h \) is compactly supported and admits the asymptotic expansion:

\[ a^k_h(t,x,\eta) = \sum_{j=0}^{N-1} h^j a^k_j(t,x,\eta), \quad N \geq 1. \]
The principal symbol $a^k_0$ is given by

$$a^k_0 = i^{\sigma_k} \left| \det \partial^2_{x\eta} \varphi^k \right|^{\frac{1}{2}}$$

where $\sigma_k$ is an integer of Maslov type (see (7.10) below).

7.1.1. Sketch of proof of Proposition 12. The proof of the preceding proposition relies on a standard WKB method adjusted for our purpose. Let $\gamma(t) \subset E^* M$ with $0 \leq t \leq \epsilon \log h^{-1}$. From Section 6.2, we have at our disposal a sequence of open patches in $T^* M$ that cover $\gamma(t)$, centered at $t_0, \ldots, t_K$ and for which there is a generating function of the flow in the sense of (6.5). If $A^w_0$ is an operator that microlocalizes in the first patch around $\rho_0 = \gamma(0)$, then $U(t_i)A^w_0$ is microlocalized in the patch centered around $\Phi^t_i(\rho_0)$. In each patch around $\gamma(t_i)$, the WKB procedure can produce a local integral representation for $U(t_i)$ thanks to the generating function and the usual Duistermaat-Hörmander transport equations for the symbols. By identifying two such representations at the intersection of two consecutive patches modulo a remainder term, we can obtain at the end a local integral representation valid up to time $\epsilon \log h^{-1}$. For our purpose to take a trace, the principal symbol must be computed exactly for all times (namely, including the Maslov factors), and the remainders must also be small in the trace class norm. The main difficulty in this construction is due to the fact that $t$ can be of order $\epsilon \log h^{-1}$, which means that all the symbols and remainder terms must be controlled as $t \sim \epsilon \log h^{-1}$. In particular, the hyperbolicity of the trapped set requires the use of symbol classes $S^{0,0}_{\epsilon C}$ for a given constant $C > 0$ that depends only on $M$ and the symbol $p$, as it has already been noted above. But provided that if $\epsilon$ is small enough, we are always in tractable symbol classes $S^{0,0}_{1/2}$ and microlocal calculus is then always available.

7.1.2. Changing symbols and phase functions along a bicharacteristic. The content of the proposition is standard if $k = 0$ and $t = O(1)$, see [Zwo10, Chapter 10]. Writing $u_0 = A^w_0 u$ we have

$$U(t)u_0 = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{\hbar}(\varphi^0(t,x,\eta) - \langle y,\eta \rangle)} a^0(t,x,\eta,\hbar)u_0(y)dyd\eta$$

where

$$a^0 \sim \sum_{j=0}^{\infty} \hbar^j a^0_j, \quad a^0(t,x,\eta) = \left| \det \partial^2_{x\eta} \varphi^0(t,x,\eta) \right|^{\frac{1}{2}}, \quad \sigma_0 = 0,$$

(7.3)
and the symbols $a_j^0$ are smooth and compactly supported in all variables (in particular, $x \in \pi \Phi^t(V_0)$, $\pi : T^* M \to M$). We take
\[
U_0(t)u_0 = \frac{1}{(2\pi h)^n} \int e^{i(tx,\eta)} \sum_{j=0}^{N-1} h^j a_j^0(t,x,\eta)u_0(y)dy d\eta
\]
for arbitrary $N \in \mathbb{N}$. As a result, there is a (semiclassical) local Fourier integral operator $Z_0^N(t)$ such that
\[
(\hbar D_t + hQ)A_0^w = Z_0^N(t)A_0^w, \quad t \in [-h^{\epsilon L_0}, h^{\epsilon L_0}].
\]
The operator $Z_0^N(t)$ arises from the usual transport equations, it depends on the symbols $a_0^0(t), \ldots, a_{N-1}^0(t)$ and their derivatives and since its symbol is uniformly of order $O(h^N)$ on its support, we get that $\text{Tr}(\Pi Z_0^N(t)A_0^w) = O(h^{N-n}) = O(h^{N/2})$ if $N$ is sufficiently large. Duhamel formula gives
\[
\mathcal{R}_0^N(t) = \int_0^t U(t-s)Z_0^N(s)ds
\]
and (7.1) follows from the trace estimate of $\Pi Z_0^N(t)A_0^w$.

We will now construct $U_k$ and $\mathcal{R}_k^N$ by induction, assuming that $U_{k-1}, \mathcal{R}_{k-1}^N$ have been constructed for time $t \in [t_{k-1} - h^{\epsilon L_0}, t_{k-1} + h^{\epsilon L_0}]$. For this we will regularly change the phase function along the chain of neighborhoods $V_k$ defined above, so that locally in $t, x, \eta$, we still have a good oscillatory integral representation. For all $k \leq K$, the times $t_k$ can be chosen so that $V_{k-1}$ and $V_k$ always intersect, for instance around a point $\rho_{k-1}$ such that
\[
\rho_{k-1} = \Phi^{\tau_{k-1}}(\rho_0), \quad \tau_{k-1} \in [t_{k-1} - h^{\epsilon L_0}, t_{k-1} + h^{\epsilon L_0}].
\]
Note that $\Phi^{\tau_{k-1}}(V_0) \subset V_{k-1} \cap V_k$.

We start by defining the operator $U_k(\tau_{k-1})$, which will be our initial data from which $U_k(\tau_{k-1} + s)$ for $s \geq 0$ will be constructed. For $N > 1$, our induction hypothesis allows to write
\[
(U_{k-1}(t)u_0)(x) = \frac{1}{(2\pi h)^n} \int e^{i(x,\eta)} a_{k-1}^{k-1}(t,x,\eta)u_0(y)dy d\eta
\]
for $t \in [t_{k-1} - h^{\epsilon L_0}, t_{k-1} + h^{\epsilon L_0}]$, where
\[
a_{k-1}^{k-1} = \sum_{j=0}^{N-1} h^j a_j^{k-1}
\]
is compactly supported in $(x,\eta)$. We want first to change the phase function in the operator $U_{k-1}(\tau_{k-1})$, and use $\varphi^k$ instead of $\varphi^{k-1}$: this is possible since at time
\(\tau_{k-1}\), these phase functions are both generating functions for the flow with the same initial conditions.

To change the phase function from \(\phi^{k-1}\) to \(\phi^k\), we apply the original method developed by Hörmander (reduction of the number of fibre variables, fiber-preserving mappings and adjunction of quadratic forms). This process is fairly long, so we refer the reader to the thorough exposition in the original article [Hör71], in particular Sections 3.1 and 3.2—see also [Dui96]. This is also precisely the method used in [DG75, p. 68]. As a result, there is a sequence of symbols \((\tilde{a}_j^k)_{0 \leq j < N}\) which are determined by equations of the form

\[
\tilde{a}_j^k(x,\eta) = \sum_{\nu=0}^j Z_{j,\nu}^{k} a_{j-\nu}^{k-1}(\tau_{k-1},x,\eta)
\]

where the differential operators \(Z_{j,\nu}^{k}\) are of order \(2\nu\) for \(0 \leq \nu \leq j < N\) with coefficients independent of \(h\) and uniformly bounded with respect to \(k\), such that we can define a local Fourier integral \(\tilde{U}_k\) by the formula

\[
\tilde{U}_k(x,y) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\phi^k(\tau_{k-1},x,\eta)-(y,\eta))} \sum_{j=0}^{N-1} h^j \tilde{a}_j^k(x,\eta) d\eta,
\]

which defines locally the same distribution as \(U_{k-1}(\tau_{k-1})\) up to a remainder of arbitrary order. In particular, there is another Fourier integral operator \(S_{k-1}^N\) with a compactly supported symbol \(s_{k-1}^N(x,\eta,h)\) that satisfies

\[
(\tilde{U}_k - U_{k-1}(\tau_{k-1})) A_0^w = S_{k-1}^N A_0^w
\]

and its symbol \(s_{k-1}^N\) obey estimates of the form:

\[
\|s_{k-1}^N\|_{C^0} \leq h^N \sum_{j=0}^{N-1} C_j \|a_{j-1}^{k-1}\|_{C^{2(N-j)+m}}, \quad 0 \leq m \leq n+1, \quad C_j > 0.
\]

The celebrated transition equation for the principal symbols given by

\[
\tilde{a}_0^k = e^{\frac{i\pi}{h}(\text{sgn} \partial^2_\eta \phi^{k-1} - \text{sgn} \partial^2_\eta \phi^k)} a_0^{k-1}(\tau_{k-1}) \sqrt{D_{k,k-1}},
\]

where \(\sqrt{D_{k,k-1}}\) is a positive factor which has an interpretation as a quotient of half densities on the Lagrangian manifold \(\Lambda \subset T^*(M \times M)\) formed by the local graph of \(\Phi^{\tau_k}\), where \(T^*(M \times M)\) is endowed with the usual twisted symplectic form [Hör71, Section 3.2]. In particular, since \(a_0^0 > 0\) we have

\[
\tilde{a}_0^k = |a_0^k| \prod_{i=1}^k e^{\frac{i\pi}{h}(\text{sgn} \partial^2_\eta \phi^{i-1} - \text{sgn} \partial^2_\eta \phi^i)}.
\]
We now simply set
\[ U_k(\tau_{k-1}) \overset{\text{def}}{=} \vec{U}_k, \quad a^j_k(\tau_{k-1}, x, y) \overset{\text{def}}{=} \vec{a}^j_k(x, \eta), \quad 0 \leq j \leq N - 1. \]

We have then a transition at time \( \tau_{k-1} \) given by
\[
U_k(\tau_{k-1}) A^w_0 = U_{k-1}(\tau_{k-1}) A^w_0 + S^N_{k-1} A^w_0
\]
where \( U_k(\tau_{k-1}) \) is exactly of the form (7.2).

### 7.1.3. Computation of the symbol from the transport equations.
It remains to define \( U_k(\tau_{k-1} + t) \) for \( \tau_{k-1} + t \in [t_k - h^{\ell_{L_0}}, t_k + h^{\ell_{L_0}}] \). In practice, we will use the symbol \( \vec{a}^k = a^k(\tau_{k-1}) \) as an initial data, and show that there is a local Fourier integral operator \( Z^N_k(t) \) such that we have the following evolution problem for the unknown symbol \( a^k(t) \):
\[
\begin{align*}
(hD_t + hQ) U_k(t) u_0 &= Z^N_k(t) u_0 \\
U_k(\tau_{k-1}) u_0 &= U_{k-1}(\tau_{k-1}) u_0 + S^N_{k-1} u_0.
\end{align*}
\]

The method to solve these equations is standard [Dui96, Zwo10], but note however that in our case \( t \) can ultimately depends on \( h \), so we have to control the whole process in time as well. To estimate \((hD_t + hQ) U_k\), we write
\[ \psi^k(t, x, y, \eta) = \varphi^k(t, x, \eta) - \langle y, \eta \rangle \]
and compute:
\[
hD_t e^{\frac{\imath}{h} \psi^k(t, x, y, \eta)} a^k(t, x, \eta) = \partial_t \varphi^k e^{\frac{\imath}{h} \psi^k(t, x, y, \eta)} a^k(t, x, \eta) \\
\quad + e^{\frac{\imath}{h} \psi^k(t, x, y, \eta)} \frac{h}{\imath} \partial_t a^k(t, x, \eta).
\]

From now on we fix \( \eta \) and \( y \) as parameters and consider \( hQ e^{\frac{\imath}{h} \psi^k} a^k \). Let us write
\[ hQ = q^w + h^2 q_2^w \]
where the exponent \( w \) is a shorthand to denote the Weyl quantization, \( q = \sqrt{p} \in S^{1,0}_0 \) is the principal symbol of \( hQ \), and \( q_2 \in S^{-1,0}_0 \). Since \( q \) is homogeneous of order 1 in \( \xi \), a straightforward computation using the Weyl quantization of linear symbols gives
\[
q^w e^{\frac{\imath}{h} \psi^k} a^k = e^{\frac{\imath}{h} \psi^k} \left( q(x, \partial_x \varphi^k) + \frac{h}{\imath} \left( X + \frac{1}{2} \text{div} X \right) \right) a^k
\]
where

\[ X = \sum_{j=1}^{n} \partial_{\xi_j} q(x, \partial_x \varphi^k) \partial_{x_j}. \]

Taking into account the eikonal equation (6.6), we have altogether

\[ (hD_t + hQ) e^{\frac{i}{\hbar} \psi} a^k = e^{\frac{i}{\hbar} \psi} \left( hD_t a^k + \frac{\hbar}{2} \left( X + \frac{1}{2} \text{div} X \right) a^k + \hbar^2 q_2^w a_k \right). \]

The last term is of lower order term compared to \( h(D_t a^k - i(X + \frac{1}{2} \text{div} X) a^k) \), so we can impose the Duistermaat-Hörmander transport equations that must be satisfied by the amplitude \( a^k \):

\[
\begin{cases}
(\partial_t + X + \frac{1}{2} \text{div} X)a_0^k = 0 \\
(\partial_t + X + \frac{1}{2} \text{div} X)a_j^k = \frac{\hbar}{4}q_2^w a_{j-1}^k
\end{cases}
\]

with initial conditions

\[ a_j^k(\tau_{k-1}) = \sum_{\nu=0}^{j} Z_{j,\nu} a_{j-\nu}^k(\tau_{k-1}). \]

If the above transport equations are satisfied, we end up with

\[
(hD_t + hQ) \left( \frac{1}{(2\pi \hbar)^n} \int e^{\frac{i}{\hbar} \psi^k(t,x,y,\eta)} \sum_{j=0}^{N-1} \hbar^j a_j^k(t,x,\eta) d\eta \right)
= \frac{\hbar^N}{(2\pi \hbar)^n} \int e^{\frac{i}{\hbar} \psi^k(t,x,y,\eta)} q_2^w a_{N-1}^k \eta \overset{\text{def}}{=} Z_N^k(t).
\]

(7.9)

It then remains to solve the transport equations. For this, consider the two Lagrangian manifolds

\[ \Lambda_0^\eta = \{ (y,\eta) : y \in \pi V_0 \}, \quad \Lambda_\eta(t) = \{ (x,\partial_x \varphi^k(t,x,\eta)) : x \in \pi V_k \} \]

where by construction

\[ \Lambda_0^\eta(t) = \Phi^t(\Lambda_0^\eta). \]

Define also the flow

\[
\Phi^{s,s'} \colon \begin{cases} 
\pi \Lambda_\eta(s) \rightarrow \pi \Lambda_\eta(s + s') \\
x \mapsto \pi \Phi_{s'}(x, \partial_x \varphi^k(s, x, \eta))
\end{cases}
\]
The first transport equation expresses the classical fact that seen as a half-density on \( \Lambda_\eta(t) \) parametrized by \( x \), the amplitude \( a_0^k|dx|^{\frac{1}{2}} \) is invariant by this flow since

\[
\left( \partial_t + X + \frac{1}{2} \text{div} \, X \right) a_0^k|dx|^{\frac{1}{2}} = \left( \partial_t + L_X \right) (a_0^k|dx|^{\frac{1}{2}})
\]

where \( L_X \) denotes the Lie derivative, so

\[
\left( \phi_{\tau_k-1,t}^k \right)^* a_0^k = a_0^k(\tau_{k-1},x)|dx|^{\frac{1}{2}}
\]

\[
\iff a_0^k(\tau_{k-1} + t, x) = a_0^k(\tau_{k-1}, (\phi_{\tau_k-1,t}^k)^{-1}(x))|\text{det} \, dx \phi_{\tau_k-1,t}^k|^{-\frac{1}{2}}.
\]

In particular, since \( (\phi_{\tau_k-1,t}^k)^{-1} \) maps \( x \) to \( x_{k-1} \), we have

\[
\prod_{j=0}^{k-2} (\phi_{\tau_j,\tau_{j+1}}^k)^{-1}(\phi_{\tau_k-1,t}^k)^{-1} : x \mapsto \partial_\eta \varphi^k(t, x, \eta)
\]

and an immediate induction using the previous equation and (7.7) shows that

\[
a_0^k = i^{\sigma_k} \prod_{i=0}^{k} |\text{det} \, dx \phi_{\tau_{i-1},\tau_i}^i|^{-\frac{1}{2}} = i^{\sigma_k} |\text{det} \, \partial_{\eta} \varphi^k|^{\frac{1}{2}}
\]

where

\[
i^{\sigma_k} = \prod_{i=1}^{k} e^{\frac{i}{2} (\text{sgn} \, \partial_\eta \varphi^{i-1} - \text{sgn} \, \partial_\eta \varphi^i)}.
\]

By convenience, we will write

\[
a_0^k(\tau_{k-1} + t) = T_{\tau_k-1}^t a_0^k(\tau_{k-1})
\]

and call \( T \) the transport operator. The higher order terms are easily obtained by

\[
a_j^k(\tau_{k-1} + t) = T_{\tau_k-1}^t a_j^k(\tau_{k-1}) + \int_0^t T_{\tau_k-1}^{t-s} a_j^k(\tau_{k-1}) \, ds
\]

and we can compute the time-dependant symbol as long as we stay in the framework described at the end of Section 6.2, namely for \( t \) such that \( \tau_{k-1} + t \in [t_k - h^L_0, t_k + h^L_0] \).

To evaluate the difference \( U_k(\tau_{k-1} + t) - U(\tau_{k-1} + t) \), we proceed by induction again: assuming that

\[
U_{k-1}(t) A_0^w = U(t) A_0^w + \mathcal{R}_k^{N,k-1}(t) A_0^w
\]
holds true for \( t \in [t_{k-1} - h^{\epsilon L_0}, t_{k-1} + h^{\epsilon L_0}] \), we will construct \( \mathcal{R}_k^N(t) \) for \( t \in [t_k - h^{\epsilon L_0}, t_k + h^{\epsilon L_0}] \). If \( t = \tau_{k-1} \), it is natural to define in view of (7.8):

\[
\mathcal{R}_k^N (\tau_{k-1}) \overset{\text{def}}{=} \mathcal{S}_k^{N-1} + \mathcal{R}_k^{N} (\tau_{k-1}).
\]

Hence we can rewrite the equation satisfied by \( U_k(t) \):

\[
\begin{aligned}
\left\{ \begin{array}{l}
(hD_t + hQ) U_k(t) A_0^w = Z_k^N(t) A_0^w \\
U_k(\tau_{k-1}) A_0^w = U(\tau_{k-1}) A_0^w + \mathcal{R}_k^N (\tau_{k-1}) A_0^w.
\end{array} \right.
\end{aligned}
\]

This allow to compare \( U_k(t) A_0^w \) with \( U(t) A_0^w \) for \( t = \tau_{k-1} + t' \in [t_k - h^{\epsilon L_0}, t_k + h^{\epsilon L_0}] \) thanks to the Duhamel formula:

\[
\begin{aligned}
U_k(\tau_{k-1} + t') A_0^w &= U(t') U_k(\tau_{k-1}) A_0^w + \int_0^{t'} U(t' - s) Z_k^N(s) A_0^w ds \\
&= U(\tau_{k-1} + t') A_0^w + U(t') \mathcal{R}_k^N (\tau_{k-1}) A_0^w \\
&+ \int_0^{t'} U(t'-s) Z_k^N(s) A_0^w ds.
\end{aligned}
\]

From this we define

\[
\mathcal{R}_k^N (\tau_{k-1} + t') = U(t') \mathcal{R}_k^N (\tau_{k-1}) + \int_0^{t'} U(t'-s) Z_k^N(s) ds.
\]

**7.1.4. Estimation of the remainder terms in the trace class norm.** To complete the proof of the proposition, we now have to estimate the trace of \( \Pi \mathcal{R}_k^N(t) A_0^w \) for all \( k \leq K \) and \( t \in [t_k - h^{\epsilon L_0}, t_k + h^{\epsilon L_0}] \). Equations (7.12) and (7.13) show that for this purpose, it is enough to estimate the traces of \( \Pi \mathcal{S}_k^{N-1} A_0^w \) and \( \Pi Z_k^N(t) A_0^w \) for all \( k \leq K \) and \( t \in [t_k - h^{\epsilon L_0}, t_k + h^{\epsilon L_0}] \). These operators have compactly supported kernels, and to estimate their traces in view of (7.5), (7.6) and (7.9) it is then sufficient to have \( C^0 \) estimates on their Schwartz kernels, hence on the symbols of \( \mathcal{S}_k^{N-1} \) and \( Z_k^N \). The definition of these operators clearly indicates that it is necessary to estimate the \( C^\ell \) norms of the symbols \( a_j^k(t) \), which is the purpose of the next lemma.

**Lemma 13.** With the above notations, let \( a_j^k(t, x, \eta) \) be the \( j \)-th term of the symbol of \( U_k(t) \), and denote by \( \rho = (y, \eta) \) the unique point in \( V_0 \subset T^* M \) such that \( \pi \Phi^t(y, \eta) = x \). For the principal symbol, the following estimate hold true:

\[
\| a_0^k(t) \|_{C^0} \leq C \sup_{\rho \in V_0} e^{-\lambda^t_{\rho}}(\rho), \quad \| a_0^k(t) \|_{C^\ell} \leq C\ell(k+1)^\ell, \quad t \in [t_k - h^{\epsilon L_0}, t_k + h^{\epsilon L_0}].
\]
For the higher order symbols, we have

$$\|a_j^k(t)\|_{C^\ell} \leq C_{j,\ell}(k+1)^{\ell+3j}, \quad t \in [t_k - h^L_0, t_k + h^L_0].$$

The constants $C, C_\ell, C_{j,\ell}$ depend only on $M$ and $p$.

**Proof.** We consider first the principal symbol. We will use the following notations: for $0 \leq i \leq k$, we write $\delta t_i \overset{\text{def}}{=} \tau_i - \tau_{i-1}$ with the convention $\tau_{-1} = 0$, and set

$$(x_i, \xi_i) = \Phi^{\delta t_i}(x_{i-1}, \xi_{i-1}) \in V_i, \quad 0 \leq i \leq k$$

again with the convention $(x_{-1}, \xi_{-1}) = (y, \eta)$. For sake of notational simplicity we will consider only the case $t = \tau_k$ (hence $x = x_k$), the modification of the following proof for $t = \tau_{k-1} + t' \neq \tau_k$ being immediate. We recall that $|\det d_x \phi_i^{j-1, \delta t_i}|^{-\frac{1}{2}} = |\det \frac{\partial x_{i-1}}{\partial x_i}|^{-\frac{1}{2}}$ and hence we can rewrite the absolute value of the principal symbol as

$$\prod_{i=0}^k |\det d_x \phi_i^{j-1, \delta t_i}|^{-\frac{1}{2}} = \prod_{i=0}^k |\det \frac{\partial x_{i-1}}{\partial x_i}|^{-\frac{1}{2}} = \left| \det \frac{\partial y}{\partial x_k} \right|^{-\frac{1}{2}},$$

and the $C^0$ estimate is a direct consequence of Proposition 11.

Before proceeding to the estimates of the derivatives, we start with a remark concerning the map $\phi_i^{j-1, \delta t_i} : x_{i-1} \to x_i$. For a given value of the parameter $\eta$, this map is invertible and since $(x_i, \xi_i) = \Phi^{\tau_i}(y, \eta)$, it also induces an invertible map $g_i^{j-1, \delta t_i} : \xi_{i-1} \to \xi_i$, which differential $dg_i^{j-1, \delta t_i}$ is uniformly contracting in the adapted coordinates [NZ09, p. 190]. Note however that in our case, the upper bound on the differential of this map depends on $h$, and is actually of the form $1 - O(h^L_0 < 1$ since out time-step is $\delta t_i = O(h^L_0)$. As a result, the Jacobian matrices $\frac{\partial \xi_{i,j}}{\partial \xi_i}$ are uniformly bounded for $i, j \geq 0$: these estimates are similar to the estimate (6.7) and they are obtained exactly in the same way, using the fact that $\xi$ are the “stable” coordinates as in the standard proof of the stable/unstable manifold theorem for hyperbolic flows [KH95, Chapters 6 and 17]. Let us sketch here the main argument: on a Poincaré section $\Sigma$ in a small neighborhood of $\Phi^{\delta t}(x, \xi)$, the Poincaré map of the flow for time $\delta t$ has a differential in the adapted coordinates of the form

$$d\kappa(\rho_0)|_\Sigma = \begin{pmatrix} A & 0 \\ 0 & tA^{-1} \end{pmatrix} + O(\varepsilon), \quad \varepsilon \ll 1.$$ 

In these coordinates, we then have

$$\delta x = (A + O(\varepsilon)) \delta x \quad \text{and} \quad \delta \xi = (tA^{-1} + O(\varepsilon)) \delta \eta.$$
Due to hyperbolicity, there is $\nu > 1$ uniform near $\Gamma$ such that for $\varepsilon$ small enough, $\|A^{-1} + \mathcal{O}(\varepsilon)\|, \|tA^{-1} + \mathcal{O}(\varepsilon)\| \leq \nu$: namely, these matrices are respectively uniformly expanding and uniformly contracting. Iterating this property from $\delta \xi_i$ to $\delta \xi_{i+j}, i + j = O(\varepsilon \log h^{-1})$, it is possible to control the remainder terms (and here we refer the reader to [NZ09, Proposition 5.1]) and get for instance that

$$
\delta \xi_{i+j} = \prod_{k=1}^{j} \left( tA_k^{-1} + \mathcal{O}_k(\varepsilon) \right) \delta \xi_i = \mathcal{O}(1) \delta \xi_i.
$$

This implies that the Jacobian matrices $\frac{\partial \xi_{i+j}}{\partial \xi_j}$ are uniformly bounded for $i, j \geq 0$.

For higher derivatives, let us call for $0 \leq i \leq k$

$$
D_i = D_i(V_o) = \sup_{x \in \Phi^{i}(V_o)} |\det \partial x (\phi_{\tau_{i-1}, \delta t_i})|^{\frac{1}{2}},
$$

$$
f_i = |\det \partial x \phi_{\tau_{i-1}, \delta t_i}|^{\frac{1}{2}}, \quad D_i = \prod_{0 \leq j, i} D_j.
$$

Note that $D_i \leq 1$ for all $i$ since the map $\phi_{\tau_{i-1}, \delta t_i} : x_{i-1} \to x_i$ is uniformly expanding. We start with one derivative, and for this we consider the two cases

$$
\left\{ \begin{array}{l}
\frac{\partial a_{0}^{k}(x_k, \eta)}{\partial x_k} = \frac{\partial f_k}{\partial x_k} \prod_{i \neq k} f_i + \frac{\partial f_{k-1}}{\partial x_{k-1}} \frac{\partial x_{k-1}}{\partial x_k} \prod_{i \neq k-1} f_i + \cdots + \frac{\partial f_0}{\partial x_0} \frac{\partial x_0}{\partial x_k} \prod_{i > 0} f_i,

\frac{\partial a_{0}^{k}(x_k, \eta)}{\partial \eta} = \frac{\partial f_k}{\partial \xi_k} \prod_{i \neq k} f_i + \frac{\partial f_{k-1}}{\partial \xi_{k-1}} \frac{\partial \xi_{k-1}}{\partial \eta} \prod_{i \neq k-1} f_i + \cdots + \frac{\partial f_0}{\partial \xi_0} \prod_{i > 0} f_i.
\end{array} \right.
$$

Using the uniform bounds for the Jacobian matrices $\frac{\partial x_{k-1}}{\partial x_k}, \frac{\partial \xi_{k-1}}{\partial \eta}$ and the fact that $\frac{\partial f_{k-1}}{\partial x_{k-1}}, \frac{\partial f_{k-1}}{\partial \xi_{k-1}}$ are uniformly bounded by constants that depends only on $M$ and $p$, we end up with

$$
\|a_{0}^{k}\|_{C_{x, \eta}^{1}} \leq C_{0,1}(k + 1) \quad \text{and} \quad \|a_{0}^{k}\|_{C_{0}^{1}} \leq C_{0,1}(k + 1).
$$

The same procedure exactly can be applied by induction to show the bounds for higher derivatives using the chain rule (see [NZ09, pp. 179–182]): if $\alpha = (\alpha_{x, 1}, \ldots, \alpha_{x, n}; \alpha_{\eta, 1}, \ldots, \alpha_{\eta, n})$ is a multi index of length $|\alpha|$, we obtain that

$$
\|a_{0}^{k}\|_{C_{x, \eta}^{\alpha}} \leq C_{0,\alpha}(k + 1)^{|\alpha|},
$$

and this concludes the proof for the $C^\ell$ estimate of the principal symbol.

For higher order symbols $a_{j}^{k}, j \geq 1$, we proceed by induction, on both $j$ and $k$. The principle is here slightly different compared to [NZ09] since our transitions at times $\tau_{k}$ are made differently. Assume the estimates have been established for
\( a_j^k(\tau_k) \) and all \( 0 \leq j \leq N - 1 \), and consider \( a_j^{k+1}(\tau_{k+1}) \). Let \( \ell \in \mathbb{N}^{2n} \) be a multi-index and consider the \( C^\ell = C^\ell_{x,\eta} \) norm of \( a_j^{k+1}(\tau_{k+1}) \). From the transport equation (7.11), the transitions at times \( \tau_k \) and the induction hypothesis we have

\[
\|a_j^{k+1}(\tau_{k+1})\|_{C^\ell} \\
\leq C_{M,p,\ell} D_{k+1} \left( \|a_j^{k+1}(\tau_k)\|_{C^\ell} + \|a_j^{k+1}(\tau_k)\|_{C^\ell+2} \right) \\
\leq C_{M,p,\ell} D_{k+1} \left( \sum_{\nu=0}^j \|Z_j^{k+1} a_j^k \tau_k\|_{C^\ell} + \sum_{\nu=0}^{j-1} \|Z_j^{k+1} a_j^{k-\nu} \tau_k\|_{C^\ell+2} \right) \\
\leq C_{M,p,\ell} D_{k+1} \left( \sum_{\nu=0}^j C_{\ell,\nu} \|a_j^k \tau_k\|_{C^\ell+2\nu} + \sum_{\nu=0}^{j-1} C_{\ell,\nu} \|a_j^{k-\nu} \tau_k\|_{C^\ell+2+2\nu} \right) \\
\leq C_{M,p,\ell,j} D_{k+1} \left( \frac{D_k ((k+1)^{\ell+3j} + (k+1)^{\ell+3j-1})} \right) \\
\leq C_{M,p,\ell,j} D_{k+1} (k+2)^{\ell+3j} \leq C_{M,p,\ell,j} (k+2)^{\ell+3j}
\]

and this concludes the proof of the lemma. \( \square \)

We now come back to the estimate of the trace of \( \Pi S_{k-1}^N A_0^w \). Let \( \Pi S_{k-1}^N A_0^w(\cdot,\cdot) \) denotes its Schwartz kernel. Since the support of the diagonal embedding \( x \mapsto \Pi S_{k-1}^N A_0^w(x,x) \) has a volume of order \( \mathcal{O}(\log h^{-1}) \), it follows from the properties of \( \Pi \) and \( A_0^w \) that there exists \( C(h) = \mathcal{O}(\log h^{-1}) \) independent of \( k \) such that

\[
|\text{Tr} \Pi S_{k-1}^N A_0^w| \leq C(h) \|\Pi S_{k-1}^N A_0^w(\cdot,\cdot)\|_{C^0}.
\]

Now since the symbol of \( S_{k-1}^N \) satisfies (7.6), we can estimate the symbols \( a_j^{k-1} \) thanks to the preceding lemma, and we see that since \( t \leq \epsilon \log h^{-1} \) and \( k \leq K = \mathcal{O}(h^{-\epsilon} \log h^{-1}) \), the Schwartz kernel of \( \Pi S_{k-1}^N A_0^w \) satisfies

\[
\|\Pi S_{k-1}^N A_0^w(\cdot,\cdot)\|_{C^0} \leq C_{M,p,N,\epsilon} \frac{h^{N(1-C\epsilon)}}{(2\pi h)^n} = C_{M,p,N,\epsilon} \mathcal{O}(h^{N})
\]

if \( \epsilon \) and \( N^{-1} \) are small enough. In this case, this implies that \( |\text{Tr} \Pi S_{k-1}^N A_0^w| = \mathcal{O}(h^{N}) \) for large enough \( N \).

The estimate for \( \text{Tr} \Pi Z_k^N(t) A_0^w \) is obtained in the very same way. The operator \( Z_k^N(t) \) is a Fourier integral operator with a symbol given by \( q_2^w a_{N-1}^k(t) \) and we immediately get that

\[
\|\Pi Z_k^N(t) A_0^w(\cdot,\cdot)\|_{C^0} \leq C_{M,p,N,\epsilon} h^{N-n-C\epsilon} \|a_{N-1}^k(t)\|_{C^\alpha}
\]
where \( \alpha = \alpha_{M,p} > 0 \). Using Lemma 13, we conclude that for \( \epsilon, N^{-1} \) small enough, we have

\[
\| \Pi Z^N_k(t)A_0^w(\cdot, \cdot) \|_{C^0} = O_{M,p,N,\epsilon}(h^{N/2})
\]

uniformly for \( t \in [t_k - h^{L_0}, t_k + h^{L_0}] \), and this again implies that

\[
\left| \text{Tr} \Pi Z^N_k(t)A_0^w \right| = O_{M,p,N,\epsilon}(h^{N/3})
\]

for \( \epsilon, N^{-1} \) small enough and \( t \in [t_k - h^{L_0}, t_k + h^{L_0}] \).

For the final estimate involving \( R^N_k(t) \), we use a recursion based on (7.12) and (7.13). Indeed, if we define \( \delta_k \overset{\text{def}}{=} \tau_k - \tau_{k-1} \), then we have

\[
R^N_k(\tau_{k-1} + t') = U(t')S^N_{k-1} + U(t')(U(\delta_{k-1})R^N_{k-1}(\tau_{k-2}) + \int_0^{\delta_{k-1}} U(\delta_{k-1} - s)Z^N_{k-1}(s)ds) + \int_0^{t'} U(t' - s)Z^N_k(s)ds.
\]

We can iterate this formula further down to \( k = 0 \), to obtain that

\[
| \text{Tr} \Pi R^N_k(\tau_{k-1} + t')A_0^w | \leq C_{M,p,N,\epsilon} \sum_{\nu=0}^k | \text{Tr} \Pi S^N_k A_0^w | + \sup_{t \in [0, \delta_\nu]} | \text{Tr} \Pi Z^N_k(t)A_0^w | \leq C_{M,p,N,\epsilon}h^{-L_0\epsilon}(\log h^{-1})O(h^{N/2}) = O(h^{N/3})
\]

if \( \epsilon, N^{-1} \) are small enough, thanks to the estimates we have established just above. This concludes the proof of Proposition (12).

**7.2. Microlocal partition and computation of the wave trace.** In this section we complete the proof of Proposition 6 for the test function \( \phi^{(1)} \). In order to use the local representations of \( U(t) \) with \( h \)-oscillatory integral developed in the preceding section, we will first define a specific cover of the phase space in a neighborhood of \( E^*M \).

**7.2.1. Microlocalization around periodic orbits, and completion of the cover.** Recall that \( \phi^{(1)} \) has support of the form \( [t_0 - \frac{h^{\epsilon J_+}}{2}, t_0 + \frac{h^{\epsilon J_+}}{2}] \subset [T-1, T] \).

For the proof of Proposition 6, it is not necessary to control the elements of the length spectrum in \( \text{supp} \phi^{(1)} \), so we simply assume here that (2.5) holds true with no further precisions. Since \( h^{\epsilon J_+} < \delta_0 \) if \( h \) is small enough, we are sure from Lemma 9 (up to enlarge \( L_0 \) a bit) that if \( \gamma \in E^*M \) satisfies \( \ell(\gamma) \in \text{supp} \phi^{(1)} \), then the...
tubular neighborhood $\Theta(\gamma, h^{\ell L_0})$ in $\mathcal{E}^* M$ does not contain another periodic orbit with length in $\text{supp} \phi^{(1)}$.

For such a $\gamma$ with $\ell(\gamma) \in \text{supp} \phi^{(1)}$, take a sequence of points $(\rho_i)_{0 \leq i \leq N_\gamma}$ of $\gamma$, such that the open balls $B_{\rho_i}(h^{\ell L_0})$ form a chain of neighborhoods in $T^* M$ covering $\gamma$. In particular, $N_\gamma = O(h^{-\varepsilon L_0} \log h^{-1})$. We will denote by $B_\gamma = \bigcup_{i=1}^{N_\gamma} B_{\rho_i}(h^{\ell L_0})$ the open cover of $\gamma$ obtained with these balls.

Consider now the set

$$
\mathcal{E}^* M_{\text{res}}^{\delta/2} \defeq (\mathcal{E}^* M^{\delta/2} \cap T^*(\text{supp II})) \setminus \bigcup_{\gamma: \ell(\gamma) \in \text{supp} \phi^{(1)}} B_{\rho_i}(h^{\ell L_0})
$$

and choose an open cover of $\mathcal{E}^* M_{\text{res}}^{\delta/2}$ by open balls of size $h^{\ell L_0}$. We denote by $(W_j)_{j \in \mathcal{J}}$ the open sets of this cover, and we can arrange that $\bigcup_j W_j \subset \mathcal{E}^* M^{\delta}$, so $(W_j)_{j \in \mathcal{J}}$ and $(B_{\gamma})_{\ell(\gamma) \in \text{supp} \phi^{(1)}}$ form a cover of $\mathcal{E}^* M^{\delta/2} \cap T^*(\text{supp II})$ which stays included in $\mathcal{E}^* M^{\delta}$. Without loss of generality, we can also require that if $\rho \in W_j$ for some $j \in J$, then then $d(\rho, \gamma) > \frac{1}{2} h^{\ell L_0}$: in this way, the intersection of each $W_j$ with periodic orbits in $\mathcal{E}^* M$ of length in $\text{supp} \phi^{(1)}$ is always empty.

We then choose a partition of unity adapted to the full cover of $\mathcal{E}^* M^{\delta/2} \cap T^* \text{supp II}$ we have obtained with the sets $(B_{\gamma})_{\ell(\gamma) \in \text{supp} \phi^{(1)}}$ and $(W_j)_{j \in \mathcal{J}}$. In particular, for each orbit $\gamma$ and $i \leq N_\gamma$, we can choose the points $\rho_i$ and the functions $\chi_{\gamma i} \in C^0_0(B_{\rho_i}(h^{\ell L_0}))$ such that near $\gamma$, the functions $\chi_{\gamma i}$ form (local) partition of unity with

$$
(7.14) \quad \sum_{i=1}^{N_\gamma} \chi_{\gamma i}(\rho) = 1, \quad \rho \in \Theta\left(\gamma, \frac{1}{2} h^{\ell L_0}\right).
$$

It is clear that if we choose $\varepsilon > 0$ small enough such that $L_0 \varepsilon < \frac{1}{2}$, the $\chi_{\gamma i}$ can be constructed by rescaling $h$-independent functions by $h^{\ell L_0}$, so we can have $\chi_{\gamma i} \in S^{0,0}_{\ell L_0}(T^* M)$. Remark that with the notations of Section 7.1, we can take $V_0 \equiv B_{\rho_i}(h^{\ell L_0})$ for each $i$, and find local generating functions for the flow around $\gamma$, starting in $V_0$. Finally, let

$$
\chi_\gamma \defeq \sum_{i=0}^{N_\gamma} \chi_{\gamma i}.
$$

We denote by $(\bar{\chi}_j)_{j \in \mathcal{J}}$ the functions in $C^\infty_0(T^* M)$ supported in the $(W_j)_{j \in \mathcal{J}}$ which complete the partition of unity. Again, $\bar{\chi}_j \in S^{0,0}_{\ell L_0}(T^* M)$. By quantization, all these functions produce a quantum partition of unity microlocalized inside.
$E^* M^\delta \cap T^* \text{supp } \Pi$, and we have:

$$\text{Tr} \int \Pi f(hQ)U(t) e^{\frac{i}{\hbar} \phi^{(1)}(t)} dt \Pi$$

$$= \text{Tr} \int \Pi U(t) \left( \sum_{\gamma: \ell(\gamma) = \ell_0} \text{Op}_h(\chi_\gamma) \right) e^{\frac{i}{\hbar} \phi^{(1)}(t)} dt$$

$$+ \text{Tr} \int \Pi U(t) \left( \sum_{j \in J} \text{Op}_h(\tilde{\chi}_j) \right) e^{\frac{i}{\hbar} \phi^{(1)}(t)} dt + O(h^\omega),$$

and the same equality holds true with $(1 - \chi)U_0(t)(1 - \chi)$ replacing $U(t)$.

### 7.2.2. Stationary phase near periodic orbits.

In this section also, $C$ will here denote a positive constant depending only on $M, p$ if written without further notifications. Still for $\gamma \in \mathcal{P}$ such that $\ell(\gamma) \in \text{supp } \phi^{(1)}$, let us define

$$\text{Tr}(\gamma, i) \overset{\text{def}}{=} \text{Tr} \int \Pi U(t) \text{Op}_h(\chi_{\gamma, i}) \Pi e^{\frac{i}{\hbar} \phi^{(1)}(t)} dt,$$

and

$$\text{Tr}_\chi \overset{\text{def}}{=} \sum_{i=1}^{N_\gamma} \text{Tr}(\gamma, i).$$

The preliminary work in the above sections allows to represent $\text{Tr}(\gamma, i)$ by oscillatory integrals. Locally in $t, x, y, \eta$, we have from Proposition 12:

$$\Pi U(t) \text{Op}_h(\chi_{\gamma, i}) \Pi = \frac{1}{(2\pi \hbar)^2n} \int e^{\frac{i}{\hbar}(\varphi(t, x, \eta) - \langle y, \eta \rangle)} e^{\frac{i}{\hbar}(y - z, \xi)} a_h(t, x, \eta) \left( \frac{y + z}{2}, \xi \right) dy \eta d\xi$$

since working with local charts, we have $\Pi \equiv 1$ on $\text{supp } \chi_{\gamma, i}$. In the sequel to simplify the notations, we will omit the terms $\Pi$ in the computations.

**Lemma 14.** We have

$$U(t) \text{Op}_h(\chi_\gamma)$$

$$= \frac{1}{(2\pi \hbar)^n} \int e^{\frac{i}{\hbar}(\varphi(t, x, \eta) - \langle x, \eta \rangle)} a_h(t, x, \eta) \left( \chi_{\gamma, i}(x, \eta) + h^{\omega_1 + r_{\gamma, i}}(x, \eta) \right) d\eta$$

where $r_{\gamma, i} \in S_{\epsilon L_0}^0(T^* M)$ and $\omega_1 = 1 - C\epsilon > 0$ if $\epsilon$ is small enough.

**Proof.** This is a straightforward application of the stationary phase expansion in the $y, \xi$ variables in (7.15), noting that $\chi_{\gamma, i} \in S_{\epsilon L_0}^0(T^* M)$. 

\qed
The content of the next lemma is classical and consists in the stationary phase expansion of the trace localized around the periodic orbits. We refer to [CdV73a, CdV73b, DG75, SZ02] for additional details and references. We recall that in our case, we must check carefully the expansion in $h = \lambda^{-1}$ as we work until times which are of Ehrenfest type.

**Lemma 15.** Let $\gamma \in P$ with $\ell(\gamma) \in \text{supp} \phi^{(1)}$ as above. Then

$$\text{Tr}_{\chi_\gamma} = e^{i \frac{\ell(\gamma)}{\pi}} \frac{\ell^2(\gamma)}{\sqrt{1-P}} \phi^{(1)}(\ell(\gamma)) + O(h^{\alpha_2})$$

where $\ell^2(\gamma)$ is the primitive length of $\gamma$ and $\alpha_2 = 1 - C\epsilon > 0$ if $\epsilon$ is small enough.

**Proof.** For a given term $\text{Tr}(\gamma, i)$, Proposition 12, Lemma 14, and the hypothesis on $\chi_{\gamma^i}$ and $\text{supp} \phi^{(1)}$ allow to find a single phase function $\varphi$ and a symbol $a_{h,N} = \sum_{j=0}^{N-1} h^j a_j$ such that we can compute the trace via the formula

$$\text{Tr}(\gamma, i) = \frac{1}{(2\pi h)^n} \int e^{i \langle \varphi(t, x, \eta) - \langle x, \eta \rangle + t \rangle} a_{h,N}(t, x, \eta) \chi_{\gamma^i}(x, \eta) \phi^{(1)}(t) d\eta dx dt$$
$$+ \frac{h^{\alpha_1}}{(2\pi h)^n} \int e^{i \langle \varphi(t, x, \eta) - \langle x, \eta \rangle + t \rangle} a_{h,N}(t, x, \eta) r_{\gamma^i}(x, \eta) \phi^{(1)}(t) d\eta dx dt$$
$$+ O(h^{N-3}).$$

Here we have used the fact that $\Pi = 1$ on $\text{supp} x a_{h}^N$. We would like to evaluate this expression via the stationary phase in $x, \eta, t$. The critical points satisfy

$$\partial_{\eta} \varphi(t, x, \eta) = x,$$
$$\partial_{x} \varphi(t, x, \eta) = \eta,$$
$$1 + \partial_{t} \varphi(t, x, \eta) = 0.$$

(7.16)

In view of Proposition 10, this means that at the critical points $(x_c, \eta_c, t_c)$ we have $\Phi^{t_c}(x_c, \eta_c) = (x_c, \eta_c)$ and $q(x_c, \eta_c) = 1$, which is precisely the equation defining a periodic orbit on the unit energy level of length $t_c \in \text{supp} \phi^{(1)}$. Hence these critical points form a closed, non-degenerate submanifold of $T^* M$ of dimension 1. Following [CdV73b, DG75] we will use the clean version of the stationary phase theorem by using local coordinates transverse to $\gamma$. Denote by $\rho_i$ the central point of $B_{\rho_i}(h^{t_c} L_0) = \text{supp} \chi_{\gamma^i}$, and remark first that because of $\chi_{\gamma^i}$, the piece of orbit that form the critical points of the above integral lies in a small neighborhood of $\rho_i$, say

$$W_{\chi_\gamma} = \{ \Phi^s(\rho_0), |s| \leq O(h^{t_c} L_0) \}.$$
Choose local coordinates \((x_1, \ldots, x_n; \xi_1, \ldots, \xi_n)\) near \(\rho_i\) such that \(H_p = \partial/\partial x_i\) at the periodic point, and denote by \(W_{\chi\gamma}^\perp\) the set described by the transversal coordinates \((x_2, \ldots, \xi_n)\). We can write

\[
(7.17)
\]

\[
\text{Tr}(\gamma, i) = \frac{1}{(2\pi h)^n} \int \int_{W_{\chi\gamma}^\perp \times W_{\chi\gamma}^\perp} e^{i\psi(t, x, y, \eta)} a_{h, N}(t, x, \eta) \chi_{\gamma i}(x, \eta) \phi^{(1)} \, dx \, dt \, d\eta
\]

\[
+ \frac{h^{\alpha_1}}{(2\pi h)^n} \int \int_{W_{\chi\gamma}^\perp \times W_{\chi\gamma}^\perp} e^{i\psi(t, x, y, \eta)} a_{h, N}(t, x, \eta) r_{\gamma}(x, \eta) \phi^{(1)} \, dt \, d\eta \, dx
\]

\[
+ O(h^{N})
\]

where \(\psi(t, x, y, \eta) \triangleq \varphi(t, x, \eta) - \langle x, \eta \rangle + t\). We apply the (usual) stationary phase in the variables \((t, \eta_1, x_i, \eta_i)\) for \(i > 1\). The Hessian of the phase \(\psi(t, x, y, \eta)\) is given by

\[
\text{Hess} \psi(t, x, x, \eta) = \left( \begin{array}{cccc}
\partial^2_x \varphi & \partial^2_{xt} \varphi & \partial^2_{\eta t} \varphi \\
\partial^2_{tx} \varphi & \partial^2_x \varphi & \partial^2_{\eta x} \varphi - \text{Id} \\
\partial^2_{t\eta} \varphi & \partial^2_{\eta x} \varphi - \text{Id} & \partial^2_{\eta \eta} \varphi
\end{array} \right).
\]

Using (6.5) and the Hamilton-Jacobi equation (6.6), we find that \(\text{Hess} \psi\) is block-diagonal in the decomposition \(\text{Span}(\partial_t, \partial_{\xi_1}) \oplus \text{Span}_{i>1}(\partial_{x_i}, \partial_{\xi_i})\), and

\[
(7.18)
\]

\[
\text{Hess} \psi|_{W_{\chi\gamma}^\perp} = \left( \begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & \partial^2_x \varphi & \partial^2_{\eta x} \varphi - \text{Id} \\
0 & 0 & \partial^2_{x\eta} \varphi - \text{Id} & \partial^2_{\eta \eta} \varphi
\end{array} \right).
\]

A standard computation [SZ02] shows that

\[
d\Phi^t|_{W_{\chi\gamma}^\perp} = \left( \begin{array}{cc}
\left(\partial^2_{\eta x} \varphi \right)^{-1} & -\left(\partial^2_{\eta x} \varphi \right)^{-1} \partial^2_{\eta \eta} \varphi \\
\partial^2_{x\eta} \varphi \left(\partial^2_{\eta x} \varphi \right)^{-1} & \partial^2_{x\eta} \varphi - \partial^2_{xx} \left(\partial^2_{\eta x} \varphi \right)^{-1} \partial^2_{\eta \eta} \varphi
\end{array} \right)_{\partial_x, \partial_\eta \in \text{TW}_{\chi\gamma}^\perp}
\]

where by convenience, we have written \(\partial^2_{x\eta} \varphi \left(\partial^2_{\eta x} \varphi \right)^{-1}\) when \(\partial_x, \partial_\eta \in \text{TW}_{\chi\gamma}^\perp\). Again, matrices equalities show that

\[
\det(P_\gamma - \text{Id}) \equiv \det(d\Phi^t|_{W_{\chi\gamma}^\perp} - \text{Id})
\]

\[
(7.19)
\]

\[
= \left(\det\left(\partial^2_{\eta x} \varphi \right)^{-1}\right) \det \left( \begin{array}{cc}
\partial^2_x \varphi & \partial^2_{xx} \varphi - \text{Id} \\
\partial^2_{x\eta} \varphi - \text{Id} & \partial^2_{\eta \eta} \varphi
\end{array} \right)
\]

where \(P_\gamma\) is the Poincaré map of the orbit \(\gamma\). Note also that in the chosen coordinate system, \(\partial^2_{x\eta} \varphi = \partial^2_{\eta x} \varphi \) since \(\partial^2_{xx} \varphi = 1\) and \(\partial^2_{x\eta} \varphi = 0\) if \(i > 1\).
Without loss of generality we can assume that $\tau_{N,\gamma} < t_{N,\gamma} \overset{\text{def}}{=} \ell(\gamma)$, $t_{N,\gamma}$ being the last time in the subdivision chosen to apply Proposition 12. Recall that

$$\sigma_{N,\gamma} = \sum_{i=1}^{N,\gamma} \frac{1}{2} \left( \text{sgn} \partial^2_{x\theta} \varphi_i^{i-1} - \text{sgn} \partial^2_{x\theta} \varphi_i^i \right).$$

From Proposition 12, the principal symbol has the form

$$a_0 = i^{\sigma_{N,\gamma}} \left| \det \partial^2_{x\theta} \varphi^\perp \right|^{\frac{1}{2}} = i^{\sigma_{N,\gamma}} \left| \det \partial^2_{x\theta} \varphi^\perp \right|^{\frac{1}{2}}$$

as noted above. Hence from (7.19) we have

$$\left| \det \partial^2_{x\theta} \varphi^\perp \right|^{\frac{1}{2}} \det (P_\gamma - \text{Id}) \left|^{\frac{1}{2}} \right. = \left| \det \text{Hess} \psi \right|_{W_{\gamma,\gamma}^\perp} \left|^{\frac{1}{2}} \right.$$  

Applying the stationary phase principle in the integral (7.17), we will then get an expansion where the leading term coming from $a_0$ gives precisely $\left| \det (P_\gamma - \text{Id}) \right|^{\frac{1}{2}}$ modulo a phase factor. From the symbol estimates in Lemma 13 and the property

$$\frac{d^j}{dt^j} \phi(1) = O(\beta^{j\epsilon J_+}) = O(h^{-j\epsilon J_+}),$$

the remainder in the stationary phase expansion at first order together with the higher order symbols $a_j$ with $2 \leq j \leq N - 1$ at the critical points have a contribution of order $O(h^\alpha)$ with $\alpha = 1 - C_{M,p,\phi} \epsilon < 0$ if $\epsilon$ is small enough, and finally

$$\text{Tr}(\gamma, i) = \int_{W_{\gamma,\gamma}^\perp} \frac{i^{\sigma_{N,\gamma}} \text{sgn Hess } \psi \left|_{W_{\gamma,\gamma}^\perp} \right|^{\frac{1}{2}} \phi(1)(\ell'(\gamma)) dx_1 + O(h^{1-C_\epsilon}) + O(h^N).$$

To conclude the proof of the long time trace formula, we need now to take care of the phase factor in the above expression.

**Lemma 16.** The integer $\mu(\gamma) \overset{\text{def}}{=} \sigma_{N,\gamma} + \frac{1}{2} \text{sgn Hess } \psi \left|_{W_{\gamma,\gamma}^\perp} \right|$ is the Maslov index of the orbit $\gamma$, and the hyperbolicity of the flow implies that $\mu(\gamma) = 0$.

**Proof:** Let us first recall a couple of facts about the Hörmander and Hörmander-Kashiwara indices [LV89, Dui76]. If $L_0, L_1, L_2$ is a triple of Lagrangian planes in a symplectic vector space $(S,\omega)$, consider the quadratic form on $L_0 \oplus L_1 \oplus L_2$ given by

$$Q(v_0, v_1, v_2) = \omega(v_0, v_1) + \omega(v_1, v_2) + \omega(v_2, v_0).$$
The Hörmander-Kashiwara index of this triple is defined by

$$\text{sgn} \left( L_0, L_1, L_2 \right) \overset{\text{def}}{=} \text{sgn} Q.$$ 

In the particular case where $L_1$ is transversal to $L_2$, the projection $\pi_{12}$ on $L_2$ along $L_1$ is well defined and we are reduced to compute the signature of a quadratic form on $L_0$ only [LV89]:

(7.20) $$\text{sgn}(L_0, L_1, L_2) = -\text{sgn} \omega(\pi_{12} \cdot \cdot)|_{L_0 \times L_0}.$$ 

In this transversal situation, one also has the property that sgn is antisymmetric in its variables. Finally, if $M_1, M_2; L_1, L_2$ are Lagrangian planes, the Hörmander index is defined by

$$s(M_1, M_2; L_1, L_2) = \frac{1}{2} (\text{sgn}(M_1, M_2, L_1) - \text{sgn}(M_1, M_2, L_2)).$$

Now let $V = \{(0, \delta \xi)\}$ be the tangent space to the fiber in $T^* M$ and $H_i = \{(\delta x, 0)\}$ the horizontal space in the $i$-th local representation of $U(t)$. These Lagrangian planes are transversal, and from

$$\left( \partial_\eta \varphi^i(t, x, \eta) \right) \overset{\Phi^i}{\longrightarrow} (x, \partial_x \varphi^i(t, x, \eta)), \quad y = \partial_\eta \varphi^i(t, x, \eta), \quad \xi = \partial_x \varphi^i(t, x, \eta),$$

we have $\partial^2_\eta \varphi_i : V \to H_i$ and its graph is precisely $(d\Phi^i)^{-1}(V)$. Then (7.20) implies that

$$\text{sgn} \partial^2_\eta \varphi_i = \text{sgn} \left( H_i, V, (d\Phi^i)^{-1}(V) \right)$$

and this yields (see also [DG75]):

$$\frac{1}{2} \left( \text{sgn} \partial^2_\eta \varphi^{i-1} - \text{sgn} \partial^2_\eta \varphi^i \right) = s(H_{i-1}, H_i; (d\Phi^{i-1})^{-1}(V), V).$$

Consider the product symplectic manifold $(X, \sigma) \overset{\text{def}}{=} (T^* M \times T^* M, \omega_1 - \omega_2)$ where $\omega_i$ are the canonical symplectic forms on the factors, and set

(7.21) $$H \times V \overset{\text{def}}{=} (H \oplus \{0\}) \times (\{0\} \oplus V) \subset T^*_{(x, \xi; y, \eta)}(T^* M \times T^* M).$$

Since $V$ and $H$ are transversal, it follows that $V$ and $(d\Phi^i)^{-1}(V)$ are transversal too, so [Dui76, Corollary 3.3] says that if $\Delta$ denotes the diagonal in $T^* X$, then

$$s(H_{i-1}, H_i; (d\Phi^{i-1})^{-1}(V), V) = -s(H_{i-1} \times V, H_i \times V; \Delta, \text{Graph}(d\Phi^{i-1}))$$

$$= -s(\text{Graph}(d\Phi^{i-1}), \Delta; H_{i-1} \times V, H_i \times V)$$

$$= \frac{1}{2} \text{sgn}(\text{Graph}(d\Phi^{i-1}), \Delta, H_i \times V)$$

$$- \frac{1}{2} \text{sgn}(\text{Graph}(d\Phi^{i-1}), \Delta, H_{i-1} \times V).$$
where we used that $\Delta$ is always transversal to $H \times V$ and the antisymmetry of $\text{sgn}$ in that case.

In view of our choice (7.21), consider now the general graph $(\Phi^t(\rho), \rho) \subset X$ near a base point $\rho = (y, \eta)$, $t \leq t_{N_\gamma}$. We can here apply (7.20) to compute $\text{sgn}(\text{Graph}(d\Phi^t), \Delta, H \times V)$, using for $\pi_{12}$ the projection on $\Delta$ along $H \times V$. As above, we can take $(x, \eta)$ as coordinates on the graph of $\Phi^t$, and then:

$$\text{Graph } d\Phi^t = (\delta x, (\partial_x^2 \varphi) \delta x + (\partial^2_{x\eta} \varphi) \delta \eta; \delta x + (\partial^2_{\eta \varphi} \delta \eta, \delta \eta)).$$

A straightforward computation using (7.20) with $L_0 = \text{Graph}(d\Phi^t)$, $L_1 = \Delta$ and $L_2 = H \times V$, the canonical form $\omega_1 - \omega_2$ on $T^*X$ and the decomposition

$$T^*M = \text{Span}(H_p) \oplus \text{Span}(\partial_t, \partial_{\xi_1}) \oplus \text{Span}_{i>1}(\partial_{x_i}, \partial_{\xi_i})$$

yields directly to

$$\text{sgn}(\text{Graph}(d\Phi^t), \Delta, H \times V) = -\text{sgn} \left( \begin{array}{cc} \partial_x^2 \varphi & \partial^2_{\eta \varphi} \delta \eta - \text{Id} \\ \partial^2_{x\eta} \varphi - \text{Id} & \partial^2_{\eta \varphi} \end{array} \right).$$

If we take $t = t_{N_\gamma}$ in the above equation, we obtain from (7.18) the identity

$$\text{sgn}(\text{Graph}(d\Phi^{t_{N_\gamma}}), \Delta, H_{N_\gamma} \times V) = -\text{sgn} \text{Hess } \psi|_{W_{\Delta_{t_{N_\gamma}}}}.$$

We can sum up the preceding discussion to get:

$$\mu(\gamma) = -\frac{1}{2} \text{sgn}(\text{Graph}(d\Phi^{t_{N_\gamma}}), \Delta, H_{N_\gamma} \times V)$$

$$+ \frac{1}{2} \sum_{i=1}^{N_\gamma} \text{sgn}(\text{Graph}(d\Phi^{t_i-1}), \Delta, H_i \times V)$$

$$- \text{sgn}(\text{Graph}(d\Phi^{t_{i-1}}), \Delta, H_{i-1} \times V)$$

$$= -\text{sgn}(\text{Graph}(d\Phi^{t_0}), \Delta, H_0 \times V)$$

$$- \sum_{i=1}^{N_\gamma} \text{sgn}(\text{Graph}(d\Phi^{t_i}), \Delta, H_i \times V)$$

$$- \text{sgn}(\text{Graph}(d\Phi^{t_{i-1}}), \Delta, H_i \times V)$$

$$\sum_{i=0}^{N_\gamma} \text{s}(\Delta, H_i \times V; \text{Graph}(d\Phi^{t_i}), \text{Graph}(d\Phi^{t_{i-1}}))$$

(7.22)

where we have set $t_{-1} = 0$ and used the fact that

$$\text{sgn}(d\Phi^0, \Delta, H_0 \times V) = \text{sgn}(\Delta, \Delta, H_0 \times V) = 0.$$
The integer in the last line of (7.22) is called the *Maslov index* of the symplectic curve \( t \mapsto \text{Graph} \Phi^t \) for \( 0 \leq t \leq t_{N_\gamma} \), and this number is actually independent of the subdivision \( t_0, \ldots, t_{N_\gamma} \) [Dui76, LV89].

Consider then in \( X \) the reference Lagrangian manifold \( \Lambda_0 = M \times F \) where \( M \) is the base manifold in the first factor, and \( F \) the fibre in the second factor. In local coordinates, \( \Lambda_0 = \{(x, \{\xi\}, \{y, \eta\})\} \). The tangent space of \( \Lambda_0 \) at \((x, \xi; y, \eta)\) is a Lagrangian plane in \( T^*_{(x, \xi; y, \eta)}X \) which is precisely \( H \times V \) in the above notations. Now the mapping

\[
(7.23) \quad t \mapsto \text{sgn} \left( \text{Graph} \left( d\Phi^t \right), \Delta, H \times V \right)
\]

is *not* continuous: it jumps by \( \pm 2 \) if \( \text{Graph}(d\Phi^t) \) crosses the singular cycle \( \Lambda^1(X, \Lambda_0) \) attached to \( \Lambda_0 \), in other words, if there is \( t \) such that the projection \( \pi_G : \text{Graph} d\Phi^t \to H \times V \) is singular [Dui96]. But this never happens for hyperbolic flows, as the vertical fibre bundle \( F \) is transverse to the weak unstable foliation. Indeed, in the coordinate system adapted to the dynamics introduced above, the weak unstable manifold \( \mathbb{R}H_{\rho}(\rho) \oplus E^u_{\rho} \) is canonically identified to \( H_{t=0} = T^*_\rho M \).

Now \( \Lambda(0) = (\{y\}, \eta) \) is a Lagrangian manifold in \( T^* \mathcal{M} \) which has tangent space precisely equal to \( F \), which is transverse to \( H_{t=0} \) and Proposition 11 then says that \( \Lambda(t) = \Phi^t(\Lambda(0)) \) is a Lagrangian manifold in \( T^* \mathcal{M} \) that projects diffeomorphically (locally) onto the weak unstable manifold \( \mathbb{R}H_{\rho}(\Phi^t(\rho)) \oplus E^u_{\Phi^t(\rho)} \equiv H_t \) at all times \( t > 0 \). Equivalently, this means that \( \text{Graph} \Phi^t \) always projects locally diffeomorphically to \( M \times F \) for all times. Hence the mapping (7.23) is constant:

\[
\forall i \geq 0, \quad \text{sgn} \left( \text{Graph} \left( d\Phi^{t_{i+1}} \right), \Delta, H_i \times V \right) = \text{sgn} \left( \text{Graph} \left( d\Phi^{t_i} \right), \Delta, H_i \times V \right),
\]

and we immediately get that \( \mu(\gamma) = 0 \).

Finally, by the principle of stationary phase at the first order, we get

\[
\text{Tr}(\gamma, i) = \int_{W_{\chi_{\gamma i}}} \frac{1}{|\det(1 - P_\gamma)|^{1/2}} (1 + \mathcal{O}(h^{1-C_\epsilon})) \phi^{(1)}(\ell(\gamma)) dx_1 + \mathcal{O}(h^{N_\gamma})
\]

\[
= \frac{1}{|1 - P_\gamma|^{1/2}} e^{i\ell(\gamma)} \phi^{(1)}(\ell(\gamma)) \int_{\gamma} \chi_{\gamma i} (1 + \mathcal{O}(h^{1-C_\epsilon})) ds + \mathcal{O}(h^{N_\gamma}).
\]

Summing up over \( i \), we are left in view of (7.14) with

\[
\text{Tr} \Pi U(t) \text{Op}_h (\chi_{\gamma}) \phi^{(1)}(t) \Pi = \frac{\ell^2(\gamma)}{|1 - P_\gamma|^{1/2}} e^{i\ell(\gamma)} \phi^{(1)}(\ell(\gamma)) + \mathcal{O}(h^{1-C_\epsilon}) + \mathcal{O}(h^{N_\gamma}),
\]

and this concludes the proof of the Lemma for large enough \( N \).
Finally, observing that the number of periodic orbit that have length in $\text{supp } \phi^{(1)}$ cannot exceed $e^{h \text{top} \ell_0} = O(h^{-C\epsilon})$, we have

$$\text{Tr} \int \Pi f(hQ)U(t)\Pi e^{\hat{\pi}t} \phi^{(1)}(t) dt = \sum_{\gamma: \ell(\gamma) \in \text{supp } \phi^{(1)}} e^{\frac{i\ell(\gamma)}{h}} \frac{\ell^2(\gamma)}{\sqrt{|1-P\gamma|}} \phi^{(1)}(\ell(\gamma)) + O(h^{1-C\epsilon})$$

$$+ \text{Tr} \int \Pi U(t) \left( \sum_{j \in J} \text{Op}_h(\bar{\chi}_j) \right) e^{\hat{\pi}t} \phi^{(1)}(t) dt$$

and it remains to check that the last term does not contribute to the wave trace in the limit $h \to 0$, as it is expected.

### 7.2.3. Remaining contributions to the wave trace.

To complete the proof of Proposition 6, we indicate briefly how to deal with the terms microlocalized outside the periodic orbits with length in $\text{supp } \phi^{(1)}$. The operator $\Pi U(t) \sum_{j \in J} \text{Op}_h(\bar{\chi}_j)$ is a sum of semiclassical Fourier integral operators, and because of the crucial fact that the time involved is always $\leq \epsilon \log h^{-1}$ for sufficiently small $\epsilon$, they can again be represented by local oscillatory integrals with a compactly supported Schwartz kernel of the form

$$\int e^{iS_j(t,x,y,\eta)} b_j(t,x,y,\eta,h) d\eta.$$

Here $S_j$ is a generating function of the canonical relation

$$C = \{(t,e),(x,\xi),(y,\eta)) : (x,\xi),(y,\eta) \in T^*M \setminus 0,$$

$$(t,e) \in T^*\mathbb{R} \setminus 0, \ e + q(x,\xi) = 0, \ (x,\xi) = \Phi_t(y,\eta) \}$$

and $b_j \sim \sum h^k b_{j,k}$. Such an integral representation can for instance be obtained by composing $O(t)$ times a Fourier integral operator quantizing the time 1 symplectic transformation $\Phi^1 : T^*M \to T^*M$ after microlocalizing with $\text{Op}_h(\bar{\chi}_j)$. From the transport equations (7.11) one can show by applying crudely the chain rule inductively (exactly as in Lemma 13) that there is a positive constant $C = C_{M,p}$ depending only on $M$ and $p$ (via $\Phi^1$) such that

$$(7.24) \quad \|b_{j,k}\|_{C^\ell} \leq C_{j,k,\ell}(t + 1)^{\ell + 3k} C^{(\ell + 1)}.$$

Actually this type of estimate is true even without any hyperbolicity assumption, which essentially allows to replace the term $C^{(\ell + 1)}$ by $e^{-tC_{M,p}}$ for some $C_{M,p} > 0$ using Proposition 11. It follows that we can perform integrations by parts in the
integral
\[ \int e^{\frac{i}{\hbar}(S_j(t,x,x,\eta)+t)} b_j(t,x,x,\eta,h)\phi^{(1)}(t) d\eta dx dt \]
since the critical equations
\[ \partial_\eta S_j(t,x,x,\eta) = x, \quad \partial_x S_j(t,x,x,\eta) + \partial_y S_j(t,x,x,\eta) = 0, \quad 1 + \partial_t S_j(t,x,x,\eta) = 0 \]
cannot be satisfied, due to the fact that \( \text{supp} \tilde{\chi}_j \) do not contain any point belonging to a periodic orbit with unit energy and length in \( \text{supp} \phi^{(1)} \). As a result, we finally have
\[ \text{Tr} \int \Pi f(hQ) U(t) \left( \sum_{j \in J} \text{Op}_h(\tilde{\chi}_j) \right) \phi^{(1)}(t) dt = \mathcal{O}(h^{\infty}) \]
or more precisely, \( \mathcal{O}_N(h^{N(1-C\epsilon)}) \) for some \( C > 0 \) and any \( N \in \mathbb{N} \) in view of (7.24).
Indeed the number of terms in the sum is bounded above by \( \mathcal{O}(h^{-C\epsilon}) \), due to the fact that the support of \( \Pi \) has size \( \log h^{-1} \) and that the \( \tilde{\chi}_j \) are localized in balls of size \( O(h^{\epsilon L_0}) \).

Finally, the preceding construction applies identically when replacing \( U(t) \) by the truncated free wave group \( (1-\chi)U_0(1-\chi) \), and for the same reasons as above we have:
\[ \text{Tr} \int \Pi f(hQ)(1-\chi) U_0(t)(1-\chi) \Pi e^{\frac{i}{\hbar}} \phi^{(1)}(t) dt = \mathcal{O}(h^{\infty}). \]
To summarize, we have shown in this section that
\[ \text{Tr} \int \Pi f(hQ)(U(t) - (1-\chi)U_0(t)(1-\chi)) \Pi e^{\frac{i}{\hbar}} \phi^{(1)}(t) dt \]
\[ = \sum_{\gamma: \ell(\gamma) = \ell_0} \frac{e^{\frac{i}{\hbar} \ell(\gamma)} \ell^4(\gamma)}{\sqrt{|1 - P_\gamma|}} \phi^{(1)}(\ell(\gamma)) + \mathcal{O}(h^{1-C\epsilon}) + \mathcal{O}(h^{\infty}), \]
and this concludes the proof of Proposition 6 for \( \phi^{(1)} \).

7.3. The case of \( \phi^{(2)} \). Even if the principle of the proof of Proposition 6 is the same with \( \phi^{(2)} \), some significant changes must be made in order to take into account the fact that now \( \text{supp} \phi^{(2)} = [T-1, T] \) contains exponentially many periodic orbits as \( T \to +\infty \).
Let \( \delta_0 \) be as in Lemma 9. To divide the time interval \( [T-1, T] \) into subintervals of size \( \delta_0 \), let \( K_0 = \left\lfloor \delta_0^{-1} \right\rfloor + 1 \) and
\[ (J_k)_{0 \leq k \leq K_0 - 1} = [T - 1 + k\delta_0, T - 1 + (k + 1)\delta_0], \quad J_{K_0} = [T - 1 + \left\lfloor \delta_0^{-1} \right\rfloor \delta_0, T]. \]
Now we can find some functions $f_k \in C^\infty_0(\mathbb{R})$ such that $f_k$ is supported near $J_k$ and furthermore they realize a partition of unity near $\text{supp} \phi^{(2)}$, namely

$$\forall t \in [T - 1, T], \quad \sum_{k=0}^{K_0} f_k(t) \phi^{(2)}(t) = \phi^{(2)}(t).$$

(7.25)

Up to shrink $\delta_0$ and enlarge $K_+$ further, we can assume without loss of generality if $T$ is large enough that if $\gamma, \gamma' \in \mathcal{P}$ are such that $\ell(\gamma), \ell(\gamma') \in \text{supp} f_k$, then the neighborhoods $\Theta(\gamma, e^{-K_+})$ and $\Theta(\gamma', e^{-K_+})$ are disjoint. Note also that since $T \leq \log \log h^{-1}$, these neighborhoods are now of size $O(1/\log h)$.

We can then proceed to the proof of the long time trace formula as in the preceding section, working first with

$$\phi_k \overset{\text{def}}{=} f_k \phi^{(2)}$$

and then summing up the contributions according to (7.25), the point being that by construction, we can isolate microlocally periodic orbits whose length is in $\text{supp} \phi_k$. For each $\gamma$ such that $\ell(\gamma) \in \text{supp} \phi_k$, we can define again the cutoff functions $\chi_{\gamma, i} \in C^\infty_0(B_{\rho_i}(e^{-TK_+}))$ forming a partition of unity around $\gamma$ in phase space, and write:

$$\Pi f(hQ)U(t)\phi^{(2)}(t) \Pi = \sum_k \Pi U(t) \left( \sum_{\gamma: \ell(\gamma) \in \text{supp} \phi_k} \text{Op}_h(\chi_{\gamma}) + \sum_{j \in J_k} \text{Op}_h(\tilde{\chi}_j) \right) \phi_k(t) \Pi$$

where again the $\tilde{\chi}_j$ form a partition of unity associated to a cover of

$$\mathcal{E}^* M^{\delta/2}_{\text{res}} \overset{\text{def}}{=} \left( \mathcal{E}^* M^{\delta/2} \cap T^*(\text{supp} \Pi) \right) \setminus \bigcup_{\gamma: \ell(\gamma) \in \text{supp} \phi_k} \bigcup_{i=1}^{N_{\gamma}} B_{\rho_i}(e^{-TK_+})$$

which stays away in $T^*M$ from the orbits with length in $\text{supp} \phi_k$. We can define as well

$$\text{Tr}(\gamma, k, i) \overset{\text{def}}{=} \text{Tr} \int \Pi f(hQ)U(t) \text{Op}_h(\chi_{\gamma, i}) \Pi e^{i\pi t} \phi_k(t) dt,$$

Since the orbits $\gamma$ entering in $\text{Tr}(\gamma, k, i)$ are microlocally isolated from each other, we can perform the stationary phase as in the preceding section to get

$$\text{Tr}(\gamma, k, i) = \frac{1}{|1 - P_{\gamma}|^{1/2}} e^{i\ell(\gamma)} \phi_k(\ell(\gamma)) \int_{\gamma} \chi_{\gamma, i}(1 + O(h^{1-C_\epsilon})) ds$$
and adding the contributions microlocalized outside the periodic orbits, this yields
again to
\[
\text{Tr} \int \Pi f(hQ)U(t)\Pi e^{\frac{i}{h}\lambda t} \phi_k(t)dt = \sum_{\gamma: \ell(\gamma) \in \text{supp} \phi_k} e^{\ell(\gamma) \gamma} \frac{\ell(\gamma)}{\sqrt{1 - P}} \phi_k(\ell(\gamma)) \\
+ O(h^{1-C\epsilon}) + O(h^\infty).
\]
Summing up the contributions in k conclude the proof in view of (7.25).

7.4. Proof of Proposition 5. The arguments of the above sections shows that we have
\[
\text{Tr} \int \Pi f(hQ)U(t)\Pi e^{-\lambda t} \phi^{(i)}(t)dxdt = O(\lambda^{-\infty})
\]
because the critical equations in the stationary phase expansion cannot be satisfied in this case since \( t, \lambda > 0 \). So
\[
\text{Re} \left( \text{Tr} \int \Pi f(hQ)U(t)\Pi e^{i\lambda t} \phi^{(i)}(t)dt \right) = \text{Re} \left( \text{Tr} \int \Pi f(hQ)U(t)\Pi 2\cos(\lambda t)\phi^{(i)}(t)dt \right) + O(\lambda^{-\infty}) \\
= 2\text{Tr} \int \Pi f(hQ)u(t)\Pi \cos(\lambda t)\phi^{(i)}(t)dt + O(\lambda^{-\infty}).
\]
From the above sections we also know that
\[
\text{Tr} \int_R \Pi (1 - \chi) f(hQ)U_0(t)(1 - \chi)\Pi e^{\pm i\lambda t} \phi^{(i)}(t)dt = O(\lambda^{-\infty})
\]
which allows to conclude the proof of the proposition.

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