On the automorphism group of tube type real symmetric domains

Fernando De Oliveira

Institut Élie Cartan de Nancy (IECN), Nancy-Université, CNRS, INRIA, Boulevard des Aiguillettes, B.P. 239, 54506 Vandoeuvre-lès-Nancy, France

Abstract
The aim of this note is to explain a generalization to the real case of a well known result on the automorphism group of an unbounded tube type symmetric domain in a complex vector space of finite dimension.

Keywords: Jordan algebra, Jordan triple system, bounded symmetric domain, partial Cayley transform, symmetric cone, tube type domain

1. Introduction

Let \( \mathcal{D} = G/K \) be a (complex) bounded symmetric domain of tube type in a finite dimension complex vector space \( V \). The tangent space at the origin is a positive hermitian Jordan triple system \( (V, \{\}) \) and \( V \) is also endowed to a structure of semisimple complex Jordan algebra. Moreover, there exists an euclidian real form \( V^+ \) of the Jordan algebra \( V \) and a Cayley transform \( \gamma \) such that \( \gamma(\mathcal{D}) = T_{\Omega} \), \( T_{\Omega} \) being the tube domain \( T_{\Omega} = \Omega \oplus iV^+ \) where \( \Omega \) means the symmetric cone of invertible squares of \( V^+ \) (see [4]). It is well known (see [1, Theorem X.5.6]) that the automorphism group \( \gamma \circ G \circ \gamma^{-1} \) of \( T_{\Omega} \) is generated by \( H(\Omega), N^+ \) and the inversion map of the Jordan algebra \( V \), where

\[
N^+ = \{ x \mapsto x + iv | v \in V^+ \}, \quad H(\Omega) = \{ g \in \text{GL}(V^+) | g\Omega = \Omega \}.
\]

This result can be generalized to any real bounded symmetric domain of tube type. This work is essentially based on [1] and [4].

Email address: deolivei@iecn.u-nancy.fr ()

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2. Algebraic framework

Let \( V = V^+ \oplus V^- \) be the Cartan decomposition of a real semisimple Jordan algebra \( V \) of finite dimension with Cartan involution \( \star \). That is

\[
V^\pm = \{ x \in V | x^\star = \pm x \}
\]

and the trace form \( \alpha : (x, y) \mapsto \text{Tr}(L(xy)) \) is positive definite on \( V^+ \) and negative definite on \( V^- \), where the \( L(x) \) are the multiplication operators on \( V \). This implies that \( V^+ \) is a semisimple euclidian Jordan algebra. For \( x, y \in V \), define

\[
x \triangleleft y := L(xy^\star) + [L(x), L(y^\star)].
\]

Then a simple calculation shows that \( V \) equipped with the triple product

\[
\{x, y, z\} := x \triangleleft y(z)
\]

is a positive real Jordan triple system, that is \( (V, \{\}) \) is a real Jordan triple system and the trace form \( \beta : (x, y) \mapsto \text{Tr}(x \triangleleft y) \) is positive definite on \( V \). Observe that \( \alpha(x, y^\star) = \beta(x, y) \) so if \( f^\star \) means the adjoint operator of \( f \in \text{End}(V) \) with respect to \( \alpha \), then \( f^\star \) means the adjoint of \( f \) with respect to \( \beta \), where \( f^\star \) is define as the conjugation of \( f \) by the Cartan involution \( \star \).

Distinguish the two structures of \( V \) by noting \( V' \) the Jordan triple structure of \( V \).

Let \( P, Q \) be the quadratic representations of \( V, V' \) respectively, define as

\[
P : x \mapsto (P(x) = 2L(x)^2 - L(x^2)),
\]

\[
Q : x \mapsto (Q(x) : y \mapsto P(x)y^\star).
\]

Let now \( \text{Str}(V), \text{Str}(V') \) be the structure groups and \( \text{Aut}(V), \text{Aut}(V') \) the automorphism groups of \( V \) and \( V' \) respectively, that is to say

\[
\text{Str}(V) := \{ g \in \text{GL}(V) | P(gx) = gP(x)^\star g \},
\]

\[
\text{Aut}(V) := \{ g \in \text{GL}(V) | g(xy) = (gx)(gy) \},
\]

\[
\text{Str}(V') := \{ g \in \text{GL}(V) | Q(gx) = gQ(x)^\star g^\star \},
\]

\[
\text{Aut}(V') := \{ g \in \text{GL}(V) | g\{x, y, z\} = \{gx, gy, gz\} \}.
\]
We see easily that if $g$ is an element of $\text{Str}(V)$ then $\dot{g} = g^{-1}P(ge)$. Also, $\text{Aut}(V)$ is the isotropy group $\text{Str}(V)^e$ of $e$ in $\text{Str}(V)$, and we have

$$\text{Str}(V) = \{ g \in \text{GL}(V) | j \circ g \circ j \in \text{GL}(V) \}$$

(see e.g. [1, Propositions VIII.2.4 and VIII.2.5]). Moreover, if $O(V, \beta)$ means the orthogonal group of $V$ with respect to $\beta$, then

$$\text{Aut}(V') = \text{Str}(V') \cap O(V, \beta).$$

Furthermore, a simple computation shows that $\text{Str}(V') = \text{Str}(V)$.

3. Tube type domains and automorphism groups

A result of O. Loos says that there exists a real bounded symmetric domain of tube type $\mathcal{D}$ such that the tangent space at the origin of $\mathcal{D}$ is $V'$ and, the domain $\mathcal{D}$ is equivalent to a unbounded real symmetric domain, the tube $T_\Omega$ define as

$$T_\Omega := \Omega \oplus V^-,$$

$\Omega$ being the symmetric cone of invertible squares of $V^+$ (see [4, Theorem 10.8]). The equivalence is given by the partial Cayley transform $\gamma_e$, define on

$$\{ x \in V | e - x \text{ is invertible} \}$$

by

$$\gamma_e(x) = (e + x)(e - x)^{-1} = -e + 2(e - x)^{-1},$$

where $e$ means the identity element of the Jordan algebra $V$. The geodesic symmetry around $0 \in \mathcal{D}$ is $-\text{Id}_V$ and then, the geodesic symmetry around $e = \gamma_e(0) \in T_\Omega$ is the inversion map $j : x \mapsto x^{-1}$ of the algebra $V$, which means that

$$j \circ \gamma_e = \gamma_e \circ (-\text{Id}_V).$$

The bounded domain $\mathcal{D}$ is homogeneous under the action of a connected semisimple Lie group $G$ without center : $\mathcal{D} = G/K$, $K$ being the isotropy group of $0$ in $G$ (see [4, 11.14]). One has that $K$ is the identity component of $\text{Aut}(V')$ (see [4, Lemme 2.11]). Now, we can define the automorphism group $L$ of the tube domain $T_\Omega$ as

$$L := \gamma_e \circ G \circ \gamma_e^{-1}.$$
The base point of $T_\Omega$ is $e$ so $T_\Omega = L/U$ with $U = \gamma_e \circ K \circ \gamma_e^{-1}$, so $U$ is the isotropy group of $e$ in $L$.

For $v \in V^-$, let $t_v : x \mapsto x + v$ be the translation by $v$ and define $N^+ := \{t_v | v \in V^-\}$. The translation group $N^+$ appears as an abelian subgroup of $L$ isomorphic to $V^-$. Finally, let $G(\Omega)$ be the identity component of the group

$$\{g \in \text{GL}(V) | g\Omega = \Omega, g^* = g\} = \{g \in \text{GL}(V) | g\Omega = \Omega, g^* = g\}.$$

Then $G(\Omega)$ acts naturally on $T_\Omega$ as $a + v_- \mapsto ga + gv_-$ and thus we identify $G(\Omega)$ with a subgroup of $L$.

We shall prove the following theorem.

**Theorem 1.** The group $L$ is generated by $G(\Omega)$, $N^+$ and $j$.

4. Affine transformations of $T_\Omega$

Recall that an element $x$ of the Jordan algebra $V$ is invertible if and only if the operator $P(x)$ is and therefore $x^{-1} = P(x)^{-1}x$.

The group $G(\Omega)$ acts transitively on the cone $\Omega$. Indeed, let $x \in V^+$ be an invertible element. Since $P(x)^* = P(x^*) = P(x)$, the restriction to $V^+$ of $P(x)$ is in $\text{GL}(V^+)$. We have also $P(x)e = x^2 \in \Omega$ and so $P(x)\Omega$ is the connected component of $x^2$ in the group $(V^+)^\times$ of invertible elements of $V^+$, none other than $\Omega$. Hence

$$\Omega = \{x^2 | x \in (V^+)^\times\} = \{P(x)e | x \in (V^+)^\times\} \subset G(\Omega)e \subset \Omega.$$

For very good and complete explanations on euclidian Jordan algebras and symmetric cones, see [1].

**Lemma 1.** We have the decomposition

$$L = N^+ G(\Omega) U.$$

**Proof.** Let $g \in L$ and $x = g \cdot e := x + y \in \Omega \oplus V^-$. Then there exists $h \in G(\Omega)$ such that $x = he$ and thus we have

$$x = t_y \circ h(e).$$
The transformation $g' := h^{-1} \circ t_y^{-1} \circ g$ of $T_\Omega$ satisfies $g'(e) = e$, that is to say $g' \in U$ and

$$g = t_y \circ h \circ g'$$

is the desired decomposition. \(\square\)

The next step is the characterization of the group of affine transformations of $T_\Omega$, which is the key of the proof of the theorem 1.

According to [4, Proposition 2.2], the group $K$ is the normalizer of $-\text{Id}_V$ in $G$, therefore $U$ is the normalizer of $j$ in $L$:

$$U = \{ g \in L | j \circ g \circ j = g \}.$$  

By consequence,

$$U \cap \text{GL}(V) \subset \text{Str}(V)^e = \text{Aut}(V)$$

so if $g \in U \cap \text{GL}(V)$ then $t'g = g^{-1}$. Let $g = \gamma_e \circ k \circ \gamma_e^{-1} \in U$ with $k \in K$. We have

$$g^* = (\gamma_e \circ k \circ \gamma_e^{-1})* = \gamma_e^* \circ k^* \circ \gamma_e^*^{-1}$$

$$= \gamma_e \circ t'k^{-1} \circ \gamma_e^{-1}$$

$$= \gamma_e \circ k \circ P(k^{-1}e) \circ \gamma_e^{-1}.$$  

By linearity of $g$, we get

$$0 = g(0) = \gamma_e k \gamma_e^{-1}(0)$$

thus

$$\gamma_e^{-1}(0) = k \gamma_e^{-1}(0).$$

Since $\gamma_e^{-1}(0) = e$, we obtain $ke = e$ and $g^* = g$. Besides, a linear element $g$ of $U$ satisfies $t'g^*^{-1} = g$ and $ge = e$, which implies

$$U \cap \text{GL}(V) \subset \text{Aut}(V')^e,$$

$\text{Aut}(V')^e$ being the isotropy group of $e$ in $\text{Aut}(V')$. Observe that

$$\text{Aut}(V')^e = \text{Aut}(V) \cap O(V, \beta) = \{ g \in \text{Aut}(V) | g^* = g \}.$$  

Conversely, we have $\text{Aut}(V')^e \subset \text{Aut}(V)$ thus any element $g$ of $\text{Aut}(V')^e$ verifies $\gamma_e \circ g \circ \gamma_e^{-1}$, which means

$$\text{Aut}(V')^e = \gamma_e \circ \text{Aut}(V')^e \circ \gamma_e^{-1} \subset U.$$
After all, we get

\[ U \cap \text{GL}(V) = \text{Aut}(V')e. \]

Moreover, one has \( \text{Aut}(V')e = G(\Omega)e \), \( G(\Omega)e \) being the isotropy group of \( e \) in \( G(\Omega) \). Indeed, \( G(\Omega)e \subset U \cap \text{GL}(V) = \text{Aut}(V')e \) and conversely, for \( g \in \text{Aut}(V') \subset \text{Aut}(V) \) and \( x \in \Omega \) (\( x = y^2 \) for some \( y \in (V^+)^* \)), we have

\[ gx = g(y^2) = (gy)^2 \in \Omega \]

thus \( \text{Aut}(V')e \subset G(\Omega)e \). We just proved the following lemma:

**Lemma 2.** The following equalities hold:
\[
U \cap \text{GL}(V) = \text{Aut}(V')e = G(\Omega)e.
\]

We can now give the main result of this section:

**Proposition 1.** Let \( M \in \text{GL}(V) \) and \( v \in V \). Then the affine transformation \( x \mapsto Mx + v \) belongs to \( L \) if and only if \( M \in G(\Omega) \) and \( v \in V^- \). In particular, the connected component of the identity of the affine transformations group of \( T_\Omega \) is the semi-direct product \( G(\Omega) \ltimes N^+ \).

**Proof.** The part ‘if’ of the equivalence is obvious. For the converse, suppose that \( x \mapsto Mx + v \) belongs to \( L \) and first show that \( M \in G(\Omega) \). According to 1, exists \( v \in V^- \), \( h \in G(\Omega) \) and \( g \in U \) such that \( M = t_v \circ h \circ g = h \circ t_{h^{-1}v} \circ g \). By linearity of \( h^{-1} \circ M \), the map \( g \) is an affine transformation which can be written \( x \mapsto g'x + u \) with \( g' \in \text{GL}(V) \) and \( u \in V^- \). First, we note that the equality \( ge = e \) implies \( g'e = e - u \in T_\Omega \). Now, we know that we have \( g = ggj \), thus for every \( \varepsilon > 0 \) one has

\[ \varepsilon g'e + u = g(\varepsilon e) = ggj(\varepsilon e) = \varepsilon(g'e + \varepsilon u)^{-1}. \]

By continuity of these maps, we get \( u = 0 \). Hence \( g \in \text{GL}(V) \) and 2 gives \( M \in G(\Omega) \). It remains to prove that \( v \) is in \( V^- \). The inverse map \( x \mapsto M^{-1}x - M^{-1}v \) of \( x \mapsto Mx + v \) is also in \( L \) and for \( \varepsilon > 0 \),

\[ M(\varepsilon e) + v \in T_\Omega. \]
We obtain \( v \in \overline{T_0} = \overline{\Omega} \oplus V^- \) through continuity of \( M \). Let \( p_{V^+} \) be the projection operator on \( V^+ \). Then \( p_{V^+}(v) \in \overline{\Omega} \) and also \( p_{V^+}(-M^{-1}v) \in \overline{\Omega} \). But \( M = M^* \) thus \( p_{V^+}(-M^{-1}v) = -M^{-1}p_{V^+}(v) \). Moreover, since \( M \) stabilizes \( \overline{\Omega} \), we get \( -p_{V^+}(v) \in \overline{\Omega} \) and hence \( p_{V^+}(v) = 0 \), that is \( v \in V^- \). \( \square \)

5. Proof of the theorem

For the sequel, we need some additional elements. Let

\[
\Sigma := \{ x \in V | x \text{ is invertible and } x^{-1} = x^* \};
\]

\( \Sigma \) is not empty because it contains the orbit \( Ke \). Indeed, for \( k \in K \), \( ke \) is invertible and we have

\[
(k e)^{-1} = k^{-1} e = k^* e = (k e)^*.
\]

The set \( \Sigma \) corresponds in \( V' \) to the set of maximal tripotents (see [4, 11.10]). For \( x \in \Sigma \) we define \( S_x = K x ; S_x \) is the connected component of \( x \) in \( \Sigma \).

**Lemma 3.** We have

\[
\gamma_e^{-1}(V^-) = \{ x \in \Sigma | e - x \text{ is invertible} \} \subset S_{-e}.
\]

In particular, \( \gamma_e^{-1}(V^-) \) is a nonempty open connected set, dense in \( S_{-e} \).

**Proof.** Let \( x \in \gamma_e^{-1}(V^-) \), \( x = e - 2(v + e)^{-1} = (v - e)(v + e)^{-1} \) for some \( v \in V^- \). Then \( x \) and \( e - x \) are invertible elements and

\[
x^{-1} = (v + e)(v - e)^{-1} = -(v + e)(e - v)^{-1}
\]

\[
= (v - e)^*[(v + e)^{-1}]^* = x^*.
\]

Conversely, let \( x \in \Sigma \) be such that \( e - x \) is invertible. Then

\[
\gamma_e(x)^* = \gamma_e(x^*) = -e + 2(e - x^*)^{-1}
\]

\[
= -e + 2(e - x^{-1})^{-1} = -e - 2x(e - x)^{-1}
\]

\[
= -\gamma_e(x)
\]
and hence \( \gamma_e(x) \in V^- \). We get
\[
\gamma_e^{-1}(V^-) = \{ x \in \Sigma | \det(e - x) \neq 0 \}
\]
where \( \det \) means the determinant function of \( V \). The map \( \det \) is polynomial so is continuous and thus, \( \gamma_e^{-1}(V^-) \) is an open set in \( \Sigma \). The open set \( \gamma_e^{-1}(V^-) \) is also connected because \( V^- \) is connected and the map \( \gamma_e^{-1} \) is continuous on \( V^- \). Finally, \( \gamma_e^{-1}(V^-) \) contains \(-e\) thus it is included in \( S_{-e} \).

We define now a binary relation \( \triangledown \) on \( V \) called transversality relation, as
\[
x \triangledown y \iff \det(x - y) \neq 0.
\]

For \( x \in \Sigma \), we define also \( x_\triangledown := \{ y \in \Sigma | x \triangledown y \} \). From 3, \( e_\triangledown = \gamma_e^{-1}(V^-) \subset S_{-e} \).

If \( x = ke \in S_e = Ke \) then for all \( y \in \Sigma \),
\[
\det(x - y) \neq 0 \iff \det(e - k^{-1}y) \neq 0 \iff k^{-1}y \in \gamma_e^{-1}(V^-)
\]
and hence
\[
x_\triangledown = k\gamma_e^{-1}(V^-) \subset S_{-x} = S_{-e}
\]
is an open connected set, dense in \( S_{-e} \). Consequently, for all \( x, y \) in \( S_e \) we have \( x_\triangledown \cap y_\triangledown \neq \emptyset \).

**Lemma 4.** For all \( x, y \) in \( S_e \), exists \( k \in K \) satisfying \(-kx \triangledown e \) and \(-ky \triangledown e \).

**Proof.** We deduce from the above that for all \( x, y \in S_e \), exists \( z \in S_{-e} \) such that \( x \triangledown z \) and \( y \triangledown z \). We can also choose \( k \in K \) such that \( kz = -e \). The elements \(-kx \) and \(-ky \) are thus transverse to \( e \). \( \square \)

**Proposition 2.** Let \( g \in U \) and \( h := \gamma_e^{-1}g\gamma_e \in K \). If \( h^{-1}e \triangledown e \) then \( g \in N^+G(\Omega)N^+ \).

**Proof.** From 3, the condition \( \det(e - h^{-1}e) \neq 0 \) ensures the existence of an element \( v \) of \( V^- \) verifying \( h^{-1}e = \gamma_e^{-1}(v) \). Let \( x \in T_\Omega \). Then
\[
g(x) = \gamma_e(h\gamma_e^{-1}(x)) = -e + 2(e - h\gamma_e^{-1}(x))^{-1} = -e + 2(h(\gamma_e^{-1}(v) - \gamma_e^{-1}(x)))^{-1}.
\]
But \( h \) belongs to \( K \) so
\[
(h(\gamma_e^{-1}(v) - \gamma_e^{-1}(x)))^{-1} = \, \! \! 'h^{-1}(\gamma_e^{-1}(v) - \gamma_e^{-1}(x))^{-1}
= h^*(\gamma_e^{-1}(v) - \gamma_e^{-1}(x))^{-1}.
\]
Thus
\[
g(x) = -e + 2h^*(\gamma_e^{-1}(v) - \gamma_e^{-1}(x))^{-1}
= -e - h^*((e + v)^{-1} - (e + x)^{-1})^{-1}.
\]

Using the following Hua identity (see e.g. [1, Exercice 5, Chapitre II])
\[
a^{-1} - b^{-1} = (a + P(a)(b - a)^{-1}
\]
with \( a = e + v \) and \( b = e + x \), we get
\[
g(x) = -e - h^*(e + v + P(e + v)(x - v)^{-1})
= -e - h^*(e + v) - h^*P(e + v)(x - v)^{-1}
= -h^*P(e + v)(j \circ t_v(x)) - e - h^*(e + v).
\]

The proposition 1 gives \( g \circ t_v \circ j \in N^+G(\Omega) \) and hence \( g \in N^+G(\Omega)jN^+ \). \( \square \)

We now have all the elements to prove the theorem 1, we recall :

**Theorem.** The group \( L \) is generated by \( G(\Omega), N^+ \) and \( j \).

**Proof.** Using 2, it suffices to prove that every element \( g \in U \) such that
\[\det(e - h^{-1}e) = 0\]
is generated by \( N^+, G(\Omega) \) and \( j \), where \( h = \gamma_e^{-1}g\gamma_e \in K \).

Let \( g \) be such an element. From 4, we can find \( k \in K \) satisfying
\[\det(e + k^{-1}h^{-1}e) \neq 0 \quad \text{and} \quad \det(e + k^{-1}e) \neq 0.\]

By 2 we get already \( \tilde{g} := \gamma_e \circ (-k) \circ \gamma_e^{-1} = \gamma_e \circ k \circ \gamma_e^{-1} \circ j \) is generated by \( N^+, G(\Omega) \) and \( j \). Exists also \( v \in V^\circ \) verifying \((-hk)^{-1}e = \gamma_e^{-1}(v)\). Let now \( x \in T_{\Omega} \). One has
\[
g(x) = \gamma_e(h\gamma_e^{-1}(x)) = -e + 2(e - h\gamma_e^{-1}(x))^{-1}
= -e + 2(-hk\gamma_e^{-1}(v) - h\gamma_e^{-1}(x))^{-1}
= -e + 2(-hk(\gamma_e^{-1}(v) + k^{-1}\gamma_e^{-1}(x)))^{-1}
= -e - 2h^*k^*(\gamma_e^{-1}(v) + k^{-1}\gamma_e^{-1}(x))^{-1}
= -e - 2h^*k^*(\gamma_e^{-1}(v) - \gamma_e^{-1}\tilde{g}^{-1}(x))^{-1}.\]
Repeating the calculus of the proof of 2, we obtain that $g \circ \tilde{g}$ belongs to $N^+G(\Omega)N^+$ and hence $g$ is generated by $N^+$, $G(\Omega)$ and $\tilde{g}$. 

6. Relation between \textit{Str}(V) and $G(\Omega)$

If $V^− = \{0\}$, that is if $V = V^+$ is an euclidian Jordan algebra, then $T_\Omega = \Omega$. Also the condition $g^* = g$ for $g \in \text{GL}(V)$ is empty. This implies that $G(\Omega) = \{g \in \text{GL}(V^+)\mid g\Omega = \Omega\}$ and

$$\{g \in \text{Str}(V)\mid g^* = g\} = \text{Str}(V^+) = \pm G(\Omega)$$

by [1, Proposition VIII.2.8].

Let now $D$ be a complex bounded symmetric domain of tube type as in the section 1. It is the particular case where $V^− = iV^+$, that is to say when the domain $D$ is considered as real. The condition $g^* = g$ for $g \in \text{GL}(V)$ is equivalent to the existence of a unique $\tilde{g} \in \text{GL}(V^+)$ satisfying $g(u + iv) = \tilde{g}u + i\tilde{g}v$ for all $u, v \in V^+$. Then

$$\{g \in \text{Str}(V)\mid g^* = g\} = \{g \in \text{GL}(V)\mid g^* = g \text{ and } \tilde{g} \in \text{Str}(V^+)\} \cong \text{Str}(V^+)$$

and

$$G(\Omega) = \{g \in \text{GL}(V)\mid g^* = g \text{ and } \tilde{g} \in H(\Omega)\} \cong H(\Omega).$$

Using the identity $\text{Str}(V^+) = \pm H(\Omega)$, we have hence again

$$\{g \in \text{Str}(V)\mid g^* = g\} = \pm G(\Omega).$$

In fact, the last equality is always true.

**Lemma 5.** An element $x$ of $V^+$ belongs to $\Omega \cup (−\Omega)$ if and only if $P(x)$ is positive definite.

**Proof.** Let $x \in \Omega \cup (−\Omega)$ ; $x = \pm y^2$ for some $y \in (V^+)^*$ and thus

$$P(x) = P(\pm P(y)e) = P(y)^2$$

is positive definite. Conversely, let $x \in V^+$ be such that $P(x)$ is positive definite. First, $P(x)^* = P(x)$ so the restriction to $V^+$ of $P(x)$ (ever noted
$P(x)$ belongs to $\text{GL}(V^+)$. Moreover, exists a Jordan frame $\{c_j\}$ of $V^+$ and real numbers $\{\lambda_j\}$ such that $x = \sum_j \lambda_j c_j$. If $V^+ = \sum_{i,j} V^+_{ij}$ means the Peirce decomposition of $V^+$ with respect to $\{c_j\}$, then $P(x)$ is $\lambda_i \lambda_j$ on $V^+_{ij}$. The positivity condition prove that $\lambda_i \lambda_j > 0$ for all $i, j$. Consequently, $\lambda_i > 0$ or $\lambda_i < 0$ for all $i$ and hence $x \in \Omega \cup (-\Omega)$. 

**Proposition 3.** The following equality holds:

$$\{g \in \text{Str}(V) | g^* = g\} = \pm G(\Omega).$$

**Proof.** Let $g \in \text{Str}(V)$ be such that $g^* = g$ and let $x \in \Omega$. Then $gx \in V^+$ and

$$P(gx) = gP(x)^t g$$

is positive definite. From 5 we get $gx \in \Omega \cup (-\Omega)$. Since $\Omega$ is connected, we find $g\Omega \subset \Omega$ or $g\Omega \subset -\Omega$. It is the same for $g^{-1}$ so $g\Omega = \Omega$ or $g\Omega = -\Omega$. Therefore,

$$\{g \in \text{Str}(V) | g^* = g\} \subset \pm G(\Omega).$$

Let now $g \in G(\Omega)$. We can find $y \in (V^+)^\times$ such that $ge = P(y)e$ and thus $P(y)^{-1} g \in G(\Omega)^e$. So we write $g = P(y)k$ with $y \in (V^+)^\times$ and $k \in G(\Omega)^e$. As $P(y) \in \text{Str}(V)$ verifying $P(y)^* = P(y)$ and since $G(\Omega)^e = \text{Aut}(V) \cap O(V, \beta) \subset \{g \in \text{Str}(V) | g^* = g\}$, we get $G(\Omega) \subset \{g \in \text{Str}(V) | g^* = g\}$. Observing that $-\text{Id}_V \in \{g \in \text{Str}(V) | g^* = g\}$, we obtain the result. 

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