Asymptotic results on the moments of the ratio of the random sum of squares to the square of the random sum

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Abstract: Let \( \{X_1, X_2, \ldots \} \) be a sequence of positive independent and identically distributed random variables of Pareto-type with index \( \alpha > 0 \) and let \( \{N(t); t \geq 0\} \) be a mixed Poisson process independent of the \( X_i \)'s. For \( t \geq 0 \), define:

\[
T_{N(t)} := \frac{X_1^2 + X_2^2 + \cdots + X_{N(t)}^2}{(X_1 + X_2 + \cdots + X_{N(t)})^2}
\]

if \( N(t) \geq 1 \) and \( T_{N(t)} := 0 \) otherwise.

We derive the limiting behavior of the \( k \)th moment of \( T_{N(t)} \), \( k \in \mathbb{N} \), by using the theory of functions of regular variation and an integral representation for \( \mathbb{E} \{ T_{N(t)}^k \} \). We also point out the connection between \( T_{N(t)} \) and the sample coefficient of variation which is a popular risk measure in practical applications.

Keywords: Function of regular variation; Laplace transform; Mixed Poisson process; Pareto-type distribution; Risk measure.

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1 Introduction

Let \( \{X_1, X_2, \ldots \} \) be a sequence of independent and identically distributed (i.i.d.) positive random variables with distribution function \( F \) and let \( \{N(t); t \geq 0\} \) be a counting process independent of the \( X_i \)'s. For \( t \geq 0 \), define:

\[
T_{N(t)} := \frac{X_1^2 + X_2^2 + \cdots + X_{N(t)}^2}{(X_1 + X_2 + \cdots + X_{N(t)})^2}
\]

if \( N(t) \geq 1 \) and \( T_{N(t)} := 0 \) otherwise.

Denote by \( T_n \) the random variable \( T_{N(t)} \) when the counting process \( \{N(t); t \geq 0\} \) is non-random. An asymptotic analysis of \( T_n \) is provided by Albrecher and Teugels \[1\], assuming the distribution function \( F \) of \( X_1 \) to be of Pareto-type with positive index \( \alpha \). In particular, they determine limits in distribution for the properly normalized quantity \( T_n \) and derive the limiting behavior of arbitrary moments of \( T_n \), generalizing earlier results pertaining to \( \mathbb{E} T_n \) by Fuchs et al. \[2\].

In this paper, we focus on moment convergence in deriving the asymptotic behavior of the \( k \)th moment of \( T_{N(t)} \), \( k \in \mathbb{N} \), under the mixed Poisson assumption for the counting process \( \{N(t); t \geq 0\} \). The distribution function \( F \) of \( X_1 \) is still assumed to be of Pareto-type with positive index \( \alpha \). The appropriate

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definitions are recalled in Section 2 along with some properties which prove to be useful later on. The results of the paper are obtained by using the theory of functions of regular variation (e.g. Bingham et al. [3]) and an integral representation for $E \{ T_N^k \}$ in terms of the probability generating function of $N(t)$ and the Laplace transform of $X_1$, following in that the basis for the analysis in Albrecher and Teugels [4].

When $X_1$ has a Pareto-type distribution function $F$ with positive index $\alpha$, its moment of order $\beta > 0$ is:

$$
\mu_\beta := \mathbb{E} \left\{ X_1^\beta \right\} = \beta \int_0^\infty x^{\beta-1} (1 - F(x)) \, dx \leq \infty
$$

which is finite if $\beta < \alpha$ but infinite whenever $\beta > \alpha$.

As pointed out by Albrecher and Teugels [4], both the numerator and the denominator defining $T_N(t)$ exhibit an erratic behavior if $\mu_1 = \infty$, whereas this is the case only for the numerator if $\mu_1 < \infty$ and $\mu_2 = \infty$. When $X_1$ has a Pareto-type distribution function $F$ with positive index $\alpha$, then $\mu_1 < \infty$ if $\alpha > 1$ while $\mu_2 < \infty$ as soon as $\alpha > 2$. Since the asymptotic behavior of $T_N(t)$ is influenced by the finiteness of $\mu_1$ and/or $\mu_2$, different kinds of results show up depending on the range of $\alpha$ as presented in Section 3. In Section 4, we give some concluding remarks. Incidentally, we point out the link existing between $T_N(t)$ and the sample coefficient of variation of a random sample $X_1, \ldots, X_N(t)$ from a positive random variable $X$ of sample size $N(t)$ from an integer valued distribution. In a forthcoming paper, we will take advantage from this link to derive asymptotic properties of the sample coefficient of variation.

2 Preliminaries

Recall that a counting process $\{N(t); t \geq 0\}$ is called a mixed Poisson process if $\{N(t); t \geq 0\} = \{N(\Lambda t); t \geq 0\}$, where the mixing random variable $\Lambda$ is positive and $\{N(t); t \geq 0\}$ is a homogeneous Poisson process with intensity 1 independent of $\Lambda$. For each fixed $t \geq 0$, the random variable $N(t)$ has a mixed Poisson distribution, with mixing distribution the distribution function $H$ of $\Lambda$, given by:

$$
p_n(t) := \mathbb{P}[N(t) = n] = \mathbb{E} \left\{ (\Lambda t)^n e^{-\Lambda t} \right\} = \int_0^\infty \frac{\Lambda^n}{n!} e^{-\Lambda t} dH(\Lambda), \quad n \in \mathbb{N}.
$$

If the distribution function $H$ is degenerate at a single point $\lambda \in (0, \infty)$ then the counting process is a homogeneous Poisson process with intensity $\lambda$. The latter plays a crucial role in practical applications. In particular, it is the most popular among all claim number processes in the actuarial literature. Also, the mixed Poisson process, introduced to actuaries by Dubourdieu [4], has always been very popular among (re)insurance modelers. It has found many applications in (re)insurance mathematics because of its flexibility, its success in actuarial data fitting and its property of being more dispersed than the Poisson process. For a general overview on mixed Poisson processes, we refer to the monograph by Grandell [6].

Another way of highlighting the role of the random variable $\Lambda$ can be expressed by the observation that:

$$
\frac{N(t)}{t} \overset{a.s.}{\longrightarrow} \Lambda \quad \text{as} \quad t \to \infty
$$

where $\overset{a.s.}{\longrightarrow}$ stands for almost sure convergence.

For a fixed time $t \geq 0$, the probability generating function of $N(t)$ is denoted by $Q_t(.)$ and satisfies:

$$
Q_t(z) := \mathbb{E} \left\{ z^{N(t)} \right\} = \sum_{n=0}^{\infty} p_n(t) z^n = \mathbb{E} \left\{ e^{-t(1-z)\Lambda} \right\}, \quad |z| \leq 1.
$$

The $r$th derivative of $Q_t(z)$ with respect to $z$ is denoted by $Q_t^{(r)}(z)$ and is defined for $|z| < 1$. It can be expressed in terms of expectations as:

$$
Q_t^{(r)}(z) = r! \mathbb{E} \left\{ \binom{N(t)}{r} z^{N(t)-r} \right\} = t^r \mathbb{E} \left\{ e^{-t(1-z)\Lambda} \Lambda^r \right\}.
$$
We define the auxiliary quantities \( q_r(w) := \mathbb{E} \{ e^{-w \Lambda^r} \} \), \( r \in \mathbb{N}, w \geq 0 \), where \( q_r(0) = \mathbb{E} \{ \Lambda^r \} \leq \infty \). Notice that for all \( 0 \leq w < t \) and \( r \in \mathbb{N} \), the following identity holds:

\[
\frac{1}{t^r} Q_t^{(r)} \left( 1 - \frac{w}{t} \right) = q_r(w)
\]

where the right-hand side does no longer depend on \( t \).

Before giving an easy but useful result on the moment condition for \( \Lambda \), note that for any \( \beta > 0 \):

\[
\int_0^\infty w^{\beta - 1} q_r(w) \, dw = \Gamma(\beta) \mathbb{E} \{ \Lambda^{-\beta} \}
\]

where \( \Gamma(\cdot) \) denotes the gamma function.

**Lemma 1** Let \( \Lambda \) be a positive random variable with distribution function \( H \). Then for all \( 0 < r \leq s \), \( \mathbb{E} \{ \Lambda^s \} < \infty \Rightarrow \mathbb{E} \{ \Lambda^r \} < \infty \) and \( \mathbb{E} \{ \Lambda^{-s} \} < \infty \Rightarrow \mathbb{E} \{ \Lambda^{-r} \} < \infty \).

**Proof:**

Let \( 0 < r \leq s \). Assume that \( \mathbb{E} \{ \Lambda^s \} < \infty \). Then:

\[
\mathbb{E} \{ \Lambda^r \} = \int_0^1 \lambda^r \, dH(\lambda) + \int_1^\infty \lambda^r \, dH(\lambda) \leq \mathbb{P}[\Lambda \leq 1] + \mathbb{E} \{ \Lambda^s \} < \infty.
\]

Now, assume that \( \mathbb{E} \{ \Lambda^{-s} \} < \infty \). Then:

\[
\mathbb{E} \{ \Lambda^{-r} \} = \int_0^1 \lambda^{-r} \, dH(\lambda) + \int_1^\infty \lambda^{-r} \, dH(\lambda) \leq \mathbb{E} \{ \Lambda^{-s} \} + \mathbb{P}[\Lambda > 1] < \infty.
\]

Recall that the process \( \{ X_i; i \geq 1 \} \) consists of i.i.d. positive random variables with distribution function \( F \). As specified above, our asymptotic results are derived under the condition that \( F \) is of Pareto-type with positive index \( \alpha \), or equivalently that \( F \) has a regularly varying tail at \( \infty \) with negative index \( -\alpha \). This means that the tail of \( F \) satisfies:

\[
1 - F(x) \sim x^{-\alpha} \ell(x) \quad \text{as} \quad x \to \infty
\]

where \( \alpha > 0 \) and \( \ell \) is slowly varying at \( \infty \).

Recall that a measurable and ultimately positive function \( f \) on \( \mathbb{R}_+ \) is regularly varying at \( \infty \) with index \( \gamma \in \mathbb{R} \) (written \( f \in RV^{-\gamma}_0 \)) if for all \( x > 0 \), \( \lim_{t \to \infty} f(tx)/f(t) = x^\gamma \). When \( \gamma = 0 \), \( f \) is said to be slowly varying at \( \infty \). Similarly, a function \( g \) on \( \mathbb{R}_+ \) is regularly varying at \( 0 \) with index \( \gamma \in \mathbb{R} \) (written \( g \in RV^{\gamma}_0 \)) if for all \( x > 0 \), \( \lim_{t \to 0} g(tx)/g(t) = x^\gamma \). When \( \gamma = 0 \), \( g \) is said to be slowly varying at \( 0 \). For a textbook treatment on the theory of functions of regular variation, we refer to Bingham et al. [3].

It is well-known that condition [2] appears as the essential condition in the extremal domain of attraction problem of extreme value theory. For a recent treatment, see Beirlant et al. [2]. When \( \alpha \in (0, 2) \), the condition [2] is also necessary and sufficient for \( F \) to belong to the additive domain of attraction of a non-normal stable law with exponent \( \alpha \) (e.g. Theorem 8.3.1 of Bingham et al. [3]).

The common Laplace transform of the \( X_i \)'s is defined and denoted by:

\[
\varphi(s) := \mathbb{E} \{ e^{-sX_1} \} = \int_0^\infty e^{-sx} \, dF(x), \quad s \geq 0.
\]

We denote by \( \varphi^{(n)}(s) \) the \( n \)th derivative of \( \varphi(s) \) with respect to \( s \). By Lemma 3.1 of Albrecher and Teugels [11] and Bingham-Doney’s lemma (e.g. Theorem 8.1.6 of Bingham et al. [3]), the asymptotic behavior of \( \varphi^{(n)} \) at the origin when \( F \) satisfies [2] is the following.
Lemma 2 If the distribution function $F$ of $X_1$ satisfies $1 - F(x) \sim x^{-\alpha} \ell(x)$ as $x \to \infty$ for some $\ell \in RV_0^\infty$ and $\alpha > 0$, then:

$$(-1)^n \varphi^{(n)}(s) \sim \begin{cases} \alpha \Gamma(n - \alpha) s^{\alpha-n} \ell \left( \frac{1}{s} \right) & \text{if } n > \alpha \\ \alpha \hat{\ell} \left( \frac{1}{s} \right) & \text{if } n = \alpha \text{ and } \mu_n = \infty \\ \mu_n & \text{if } n < \alpha \text{ or if } n = \alpha \text{ and } \mu_n < \infty \end{cases}$$

where $\hat{\ell}(x) = \int_0^x \ell(u) \frac{du}{u} \in RV_0^\infty$.

Now, we give our results.

3 Results

We start by deriving an integral representation for the $k$th moment of $T_{N(t)}$ under the mixed Poisson assumption for the counting process $\{N(t); t \geq 0\}$. Note that we do not make any assumption on the distribution function $F$ of $X_1$.

Lemma 3 Let $t \geq 0$ and $k \in \mathbb{N}\{0\}$ be fixed. Assume that $\{N(t); t \geq 0\}$ is a mixed Poisson process. The $k$th moment of $T_{N(t)}$ is then given by:

$$E \left\{ T_{N(t)}^k \right\} = \sum_{n=0}^{\infty} p_n(t) E \left\{ T_{N(t)}^k \mid N(t) = n \right\} = \sum_{n=0}^{\infty} p_n(t) E \left\{ T_n^k \right\}$$

(3)

with:

$$B_t(k_1, \ldots, k_r) := \int_0^\infty s^{2k-1} \prod_{i=1}^{r} \varphi^{(2k_i)}(s) Q_t^{(r)}(\varphi(s)) \, ds. \quad (4)$$

Proof: Using Lemma 2.1 of Albrecher and Teugels, we easily derive:

$$E \left\{ T_{N(t)}^k \right\} = \sum_{n=0}^{\infty} p_n(t) E \left\{ T_{N(t)}^k \mid N(t) = n \right\} = \sum_{n=0}^{\infty} p_n(t) E \left\{ T_n^k \right\}$$

$$= \sum_{n=0}^{\infty} p_n(t) \sum_{r=1}^{k} \frac{k!}{n!} \prod_{i=1}^{r} \left( \frac{n}{r} \right) \int_0^\infty s^{2k-1} \prod_{i=1}^{r} \varphi^{(2k_i)}(s) \varphi^{n-r}(s) \, ds$$

$$= \sum_{r=1}^{k} \frac{k!}{\prod_{i=1}^{r} k_i! (2k_i - 1)!} \int_0^\infty s^{2k-1} \prod_{i=1}^{r} \varphi^{(2k_i)}(s) \sum_{n=0}^{\infty} p_n(t) \left( \frac{n}{r} \right) \varphi^{n-r}(s) \, ds$$

$$= \sum_{r=1}^{k} \frac{k!}{\prod_{i=1}^{r} k_i! (2k_i - 1)!} \int_0^\infty s^{2k-1} \prod_{i=1}^{r} \varphi^{(2k_i)}(s) Q_t^{(r)}(\varphi(s)) \, ds. \quad \blacksquare$$

We rewrite $B_t(k_1, \ldots, k_r)$ in a more convenient form which proves useful later on. Defining $\psi(s) := \varphi^{-1}(1 - s)$ and substituting $s = \psi \left( \frac{u}{n} \right)$ in (4) lead to:

$$B_t(k_1, \ldots, k_r) = \int_0^\infty s^{2k-1} \prod_{i=1}^{r} \varphi^{(2k_i)}(s) Q_t^{(r)}(\varphi(s)) \, ds$$
\[ \int_0^t \psi^{2k-1} \left( \frac{w}{t} \right) \prod_{i=1}^r \varphi^{(2k_i)} \left( \psi \left( \frac{w}{t} \right) \right) Q_i(r) \left( 1 - \frac{w}{t} \right) \frac{d}{dw} \psi \left( \frac{w}{t} \right) dw \]

\[ = -t^{-r-1} \varphi^{(2k_i)} \left( \frac{w}{t} \right) \prod_{i=1}^r \varphi^{(2k_i)} \left( \psi \left( \frac{1}{t} \right) \right) \int_0^t \left( \psi \left( \frac{w}{t} \right) \right) \frac{d}{dw} \psi \left( \frac{w}{t} \right) \prod_{i=1}^r \varphi^{(2k_i)} \left( \psi \left( \frac{1}{t} \right) \right) \varphi^{(1)} \left( \psi \left( \frac{1}{t} \right) \right) \frac{Q_i(r)}{t^r} \psi \left( \frac{1}{t} \right) \prod_{i=1}^r \varphi^{(2k_i)} \left( \psi \left( \frac{1}{t} \right) \right) \varphi^{(1)} \left( \psi \left( \frac{1}{t} \right) \right) q_r(w) \mathbb{1}_{[0,1]}(w) dw \]

\[ = f_t(k_1, \ldots, k_r) \int_0^t g_t(w; k_1, \ldots, k_r) dw \]

with:

\[ f_t(k_1, \ldots, k_r) := -t^{-r-1} \varphi^{(2k_i)} \left( \frac{w}{t} \right) \prod_{i=1}^r \varphi^{(2k_i)} \left( \psi \left( \frac{1}{t} \right) \right) \]

and:

\[ g_t(w; k_1, \ldots, k_r) := \left( \frac{\psi \left( \frac{w}{t} \right)}{\psi \left( \frac{1}{t} \right)} \right)^{2k-1} \prod_{i=1}^r \varphi^{(2k_i)} \left( \psi \left( \frac{1}{t} \right) \right) q_r(w) \mathbb{1}_{[0,1]}(w). \]

From now on, the distribution function \( F \) of \( X_1 \) is assumed to satisfy (2) for some \( \alpha > 0 \). Here is the first of our main results pertaining to moment convergence for \( T_{N(t)} \). It concerns the case \( \alpha \in (0,1) \).

**Theorem 1** Assume that \( X_1 \) is of Pareto-type with index \( \alpha \in (0,1) \). Let \( \{N(t); t \geq 0\} \) be a mixed Poisson process with mixing random variable \( \Lambda \). If \( \mathbb{E} \{ \Lambda^\epsilon \} < \infty \) and \( \mathbb{E} \{ \Lambda^{-\epsilon} \} < \infty \) for some \( \epsilon > 0 \), then for any fixed \( k \in \mathbb{N} \setminus \{0\} \):

\[ \lim_{t \to \infty} \mathbb{E} \{ T_N^k(t) \} = \frac{k!}{(2k-1)!} \sum_{r=1}^k \alpha^{r-1} \Psi^r(1-\alpha) G(r,k) \]

where \( G(r,k) \) is the coefficient of \( x^k \) in the polynomial \( \left( \sum_{i=1}^{k-r+1} \frac{\Gamma(2k-\alpha)}{\Gamma(1-\alpha)} x^r \right)^r \).

**Proof:**

Let \( k \in \mathbb{N} \setminus \{0\} \) and \( \alpha \in (0,1) \) be fixed. Since \( 1 - F(x) \sim x^{-\alpha} \ell(x) \) as \( x \to \infty \), it follows from Corollary 8.1.7 in [3] that \( 1 - \varphi(s) \sim \Gamma(1-\alpha) s^\alpha \ell(\frac{1}{s}) \). Hence, we easily deduce \( s = 1 - \varphi(s) \sim \Gamma(1-\alpha) \psi^\alpha(s) \ell(1/\psi(s)) \) which leads to:

\[ \lim_{s \to 0} s^{-1} \psi^\alpha(s) \ell \left( \frac{1}{\psi(s)} \right) = \frac{1}{\Gamma(1-\alpha)}. \]

Relation (5) learns us that \( \psi \) is regularly varying at 0 with index \( 1/\alpha \). For \( n > \alpha \), we consequently get \( \varphi^{(n)} \circ \psi \in \mathcal{R}^{0}_{1-\frac{n}{\alpha}} \) since \( \varphi^{(n)} \in \mathcal{R}^{0}_{1-\frac{n}{\alpha}} \) by Lemma 1. \( \psi \in \mathcal{R}^{0}_{1/\alpha} \) and \( \lim_{s \to 0} \psi(s) = 0 \).

Using Potter’s theorem (e.g. Theorem 1.5.6 of Bingham et al. [3]), we therefore obtain the following upper bound for the integrand in \( B_r(k_1, \ldots, k_r) \). Set \( \delta_r := \frac{2k_r}{1-\alpha} \) with \( \zeta = \epsilon \) if \( \epsilon \in (0,1) \) or \( \zeta \in (0,1) \) otherwise. For this chosen \( \delta_r > 0 \), there exists \( C_r = C_r(\delta_r) > 1 \) such that for all \( t > 0 \):

\[ g_t(w; k_1, \ldots, k_r) \leq C_r w^{-2k-1} \max \left\{ w^{\delta_r}, w^{\delta_r - \delta_r} \right\}^r \prod_{i=1}^r \varphi^{(2k_i)} \left( \psi \left( \frac{1}{t} \right) \right) q_r(w) \]

\[ = C_r w^{-r+1} \max \left\{ w^{\zeta}, w^{\zeta - \zeta} \right\} q_r(w) =: h(w). \]

Now, \( \int_0^\infty h(w) dw < \infty \) if and only if \( \int_0^1 w^{r-1-\zeta} q_r(w) dw < \infty \) and \( \int_1^\infty w^{r-1+\zeta} q_r(w) dw < \infty \).

Since \( \zeta \in (0,1) \) and \( \zeta \leq \epsilon \), we use (11) together with Lemma 11 to get:

\[ \int_0^1 w^{r-1-\zeta} q_r(w) dw \leq \int_0^\infty w^{r-1-\zeta} q_r(w) dw = \Gamma(r-\zeta) \mathbb{E} \{ \Lambda^\zeta \} < \infty \]
and:
\[
\int_1^\infty w^{r-1+\zeta} q_r(w) \, dw \leq \int_0^\infty w^{r-1+\zeta} q_r(w) \, dw = \Gamma(r+\zeta) \mathbb{E}\{\Lambda^{-\zeta}\} < \infty.
\]
Hence, the function \( h \) is integrable.

Finally, \( \lim_{t \to \infty} g_t(w; k_1, \ldots, k_r) = w^{2k-1} w^{-r/k} w^{r-1} q_r(w) = w^{r-1} q_r(w) \). Thus, applying Lebesgue's theorem on dominated convergence and using (1), we deduce:
\[
\lim_{t \to \infty} \int_0^\infty g_t(w; k_1, \ldots, k_r) \, dw = \int_0^\infty w^{r-1} q_r(w) \, dw = (r-1)!. 
\]
Using Lemma 2 and relation (5), we get:
\[
\tilde{\alpha} \in \mathbb{C} 
\]
so that:
\[
\lim_{t \to \infty} f_t(k_1, \ldots, k_r) = \frac{\alpha^{r-1} \prod_{i=1}^r \Gamma(2k_i - \alpha)}{\Gamma^r(1-\alpha)}. 
\]

Therefore, we obtain:
\[
\lim_{t \to \infty} B_t(k_1, \ldots, k_r) = \frac{(r-1)! \alpha^{r-1} \prod_{i=1}^r \Gamma(2k_i - \alpha)}{\Gamma^r(1-\alpha)}. 
\]
Summing up over all \( r \in \{1, \ldots, k\} \) in (6), we arrive at:
\[
\lim_{t \to \infty} \mathbb{E}\{T_{N(t)}^k\} = \frac{k!}{(2k-1)!} \sum_{r=1}^k \frac{\alpha^{r-1}}{r \Gamma^r(1-\alpha)} \sum_{k_1, \ldots, k_r \geq 1, k_1 + \cdots + k_r = k} \prod_{i=1}^r \frac{\Gamma(2k_i - \alpha)}{k_i!}. 
\]

Finally, Albrecher and Teugel [1] have observed that:
\[
G(r, k) := \sum_{k_1, \ldots, k_r \geq 1 \atop k_1 + \cdots + k_r = k} \prod_{i=1}^r \frac{\Gamma(2k_i - \alpha)}{k_i!}
\]
can be read off as the coefficient of \( x^k \) in the \( r \)-fold product \( \left( \sum_{i=1}^{k-r+1} \frac{\Gamma(2i-\alpha)}{i!} x^i \right)^r. \]

Our next result deals with the case \( \alpha = 1 \) and \( \mu_1 = \infty \).

**Theorem 2** Assume that \( X_1 \) is of Pareto-type with index \( \alpha = 1 \) and that \( \mu_1 = \infty \). Let \( \{N(t); t \geq 0\} \) be a mixed Poisson process with mixing random variable \( \Lambda \). If \( \mathbb{E}\{\Lambda^r\} < \infty \) and \( \mathbb{E}\{\Lambda^{-r}\} < \infty \) for some \( \epsilon > 0 \), then for any fixed \( k \in \mathbb{N} \setminus \{0\} \):
\[
\mathbb{E}\{T_{N(t)}^k\} \sim \frac{1}{2k-1} \frac{\ell(a_1)}{\ell(a_1)} \text{ as } \ t \to \infty
\]
where \( \ell(x) = \int_0^\infty x^{-\alpha} q_r(w) \, dw \in \mathbb{R}_0^\infty \) and \( (a_t)_{t > 0} \) is a sequence defined by \( \lim_{t \to \infty} t a_t^{-1} \ell(a_t) = 1. \)

**Proof**: Let \( k \in \mathbb{N} \setminus \{0\} \) be fixed. Since \( \mu_1 = \infty \) and \( 1 - F(x) \sim x^{-1} \ell(x) \) for some \( \ell \in \mathbb{R}_0^\infty \), it follows by
Lemma 2 that $\varphi^{(1)}(s) \sim -\ell \left( \frac{1}{s} \right)$ and then that $1 - \varphi(s) \sim s \ell \left( \frac{1}{s} \right)$ with $\ell(x) = \int_0^x \frac{t}{\psi(t)} \, dt \in RV_0^\infty$. Since $s = 1 - \varphi(\psi(s)) \sim \psi(s) \ell \left( 1/\psi(s) \right)$, we obtain:

$$
\lim_{s \to 0} s^{-1} \psi(s) \ell \left( \frac{1}{\psi(s)} \right) = 1.
$$

(6)

Relation 4 learns us that $\psi \in RV_1^0$, leading to $\lim_{s \to 0} \psi(s) = 0$. For $n \geq 2$, we consequently get $\varphi^{(n)} \circ \psi \in RV_{1-n}^0$ since $\varphi^{(n)} \in RV_{1-n}^0$ by Lemma 2 Moreover, $\varphi^{(1)} \circ \psi \in RV_0^0$ since $\varphi^{(1)} \in RV_0^0$.

Using Potter's theorem, we therefore obtain the following upper bound for the integrand in $B_t(k_1, \ldots, k_r)$. Set $\delta_r := \frac{\zeta}{2k+r}$ with $\zeta = \epsilon$ if $\epsilon \in (0, 1)$ or $\zeta \in (0, 1)$ otherwise. For this chosen $\delta_r > 0$, there exists $C_r = C_r(\delta_r) > 1$ such that for all $t > 0$:

$$
g_t(w; k_1, \ldots, k_r) \leq C_r w^{2k-1} \left( \max \left\{ w^{\delta_i}, w^{-\delta_i} \right\} \right)^{2k-1} w^{r-2k} \left( \max \left\{ w^{\delta_i}, w^{-\delta_i} \right\} \right)^{r+1} q_r(w) = C_r w^{r-1} \max \left\{ w^{\delta_i}, w^{-\delta_i} \right\} q_r(w) =: h(w).
$$

Now, $\int_0^\infty h(w) \, dw < \infty$ if and only if $\int_1^\infty w^{r-1-\zeta} q_r(w) \, dw < \infty$ and $\int_1^\infty w^{r-1+\zeta} q_r(w) \, dw < \infty$.

Since $\zeta \in (0, 1)$ and $\zeta \leq \epsilon$, we use 10 together with Lemma 11 to get:

$$
\int_0^1 w^{r-1-\zeta} q_r(w) \, dw \leq \int_0^\infty w^{r-1-\zeta} q_r(w) \, dw = \Gamma(r - \zeta) \mathbb{E} \left\{ \Lambda^\zeta \right\} < \infty
$$

and:

$$
\int_1^\infty w^{r-1+\zeta} q_r(w) \, dw \leq \int_0^\infty w^{r-1+\zeta} q_r(w) \, dw = \Gamma(r + \zeta) \mathbb{E} \left\{ \Lambda^{-\zeta} \right\} < \infty.
$$

Hence, the function $h$ is integrable.

Finally, $\lim_{t \to \infty} g_t(w; k_1, \ldots, k_r) = w^{2k-1} w^{r-2k} q_r(w) = w^{r-1} q_r(w)$. Thus, applying Lebesgue's theorem on dominated convergence and using 11, we deduce:

$$
\lim_{t \to \infty} \int_0^{\infty} g_t(w; k_1, \ldots, k_r) \, dw = \int_0^{\infty} w^{r-1} q_r(w) \, dw = (r - 1)!.
$$

Using Lemma 2 relation 3 and $\varphi^{(1)}(\psi(s)) \sim \ell \left( 1/\psi(s) \right)$, we get:

$$
B_t(k_1, \ldots, k_r) \sim \int_0^t \left( \frac{\ell(1/\psi(s))}{\ell(1/\psi(s))} \right)^r.
$$

Therefore, we obtain:

$$
B_t(k_1, \ldots, k_r) \sim \lim_{t \to \infty} \left( \frac{\ell(1/\psi(s))}{\ell(1/\psi(s))} \right)^r.
$$

Since $\lim_{t \to \infty} \ell(1/\psi(s)) \ell(1/\psi(s)) = 0$, only the summand with $r = 1$ contributes to the dominating asymptotic term of 3. Hence, we get:

$$
\mathbb{E} \left\{ T_{N(t)}^k \right\} \sim \frac{1}{2k-1} \frac{\ell(1/\psi(s))}{\ell(1/\psi(s))} \text{ as } t \to \infty.
$$

One easily notes that relation 16 is equivalent to $\lim_{t \to \infty} t \psi \left( \frac{1}{t} \right) \ell \left( 1/\psi \left( \frac{1}{t} \right) \right) = 1$. Defining a sequence $(a_t) \to 0$ by $\lim_{t \to \infty} t a_t^{-1} \ell(a_t) = 1$ implies that $\psi \left( \frac{1}{t} \right) \sim \frac{1}{a_t}$. By virtue of the uniform convergence theorem
for slowly varying functions (e.g. Theorem 1.2.1 of Bingham et al. [3]), we thus get \( \ell \left( 1/\psi \left( \frac{t}{s} \right) \right) \sim \ell(a_t) \) and \( \hat{\ell} \left( 1/\psi \left( \frac{1}{t} \right) \right) \sim \hat{\ell}(a_t) \). Consequently, we finally arrive at:

\[
\mathbb{E} \left\{ T_{N(t)}^k \right\} \sim \frac{t}{2k - 1} \frac{\ell(a_t)}{\ell(a_t)} \text{ as } t \to \infty.
\]

In the following result, the case \( \alpha \in (1, 2) \) (including \( \alpha = 1 \) if \( \mu_1 < \infty \)) is of interest.

**Theorem 3** Assume that \( X_1 \) is of Pareto-type with index \( \alpha \in (1, 2) \) (including \( \alpha = 1 \) if \( \mu_1 < \infty \)). Let \( \{N(t); t \geq 0\} \) be a mixed Poisson process with mixing random variable \( \Lambda \). For any fixed \( k \in \mathbb{N} \setminus \{0\} \), if \( \mathbb{E} \{\Lambda^\prime\} < \infty \) and \( \mathbb{E} \{\Lambda^{-k(\alpha - 1) - \epsilon}\} < \infty \) for some \( \epsilon > 0 \), then:

\[
\mathbb{E} \left\{ T_{N(t)}^k \right\} \sim \frac{\alpha}{\mu_1^k} B(2k - \alpha, \alpha) \mathbb{E} \{\Lambda^{1-\alpha}\} t^{1-\alpha} \ell(t) \text{ as } t \to \infty
\]

where \( B(., .) \) denotes the beta function.

**Proof:**

Let \( k \in \mathbb{N} \setminus \{0\} \) and \( \alpha \in [1, 2] \) be fixed. Since \( \mu_1 < \infty \), it follows that \( \varphi^{(1)}(0) = -\mu_1 \) and \( 1 - \varphi(s) \sim \mu_1 s \).

Hence, \( s = 1 - \varphi(s) \sim s_{s=0} \mu_1^k \psi(s) \) and we deduce that \( \psi(s) \sim s_{s=0} \mu_1^{-1} \in \text{RV}_1^0 \). Obviously, we have \( \psi(\cdot) \sim s_{s=0} \mu_1^{-1} \in \text{RV}_1^0 \). Consequently, we get \( \varphi(\cdot) \circ \psi \in \text{RV}_0^0 \) and \( \varphi^{(n)} \circ \psi \in \text{RV}_0^0 \) for \( n > \alpha \), since \( \varphi^{(n)} \in \text{RV}_\alpha^0 \) by Lemma 2. Hence, \( \psi \in \text{RV}_1^0 \) and \( \lim_{s \to 0} \psi(s) = 0 \).

Using Potter’s theorem, we therefore obtain the following upper bound for the integrand in \( B_t(k_1, \ldots, k_r) \).

Set \( \delta_r := \frac{\zeta}{r} \) with \( \zeta = \epsilon \) if \( \epsilon \in (0, 1) \) or \( \zeta \in (0, 1) \) otherwise. For this chosen \( \delta_r > 0 \), there exists \( C_r = C_r(\delta_r) > 1 \) such that for all \( t > 0 \):

\[
g_t(w; k_1, \ldots, k_r) \leq C_r w^{2k - 1} \left( \max \left\{ w^{\delta_r}, w^{-\delta_r} \right\} \right)^{2k} w^{\alpha - 2k} \left( \max \left\{ w^{\delta_r}, w^{-\delta_r} \right\} \right)^{r + 1} q_r(w) =: h(w).
\]

Now, \( \int_0^\infty h(w) \, dw < \infty \) if and only if \( \int_0^1 w^{\alpha - 1 - \zeta} q_r(w) \, dw < \infty \) and \( \int_1^\infty w^{\alpha - 1 + \zeta} q_r(w) \, dw < \infty \).

Since \( \zeta \in (0, 1) \) and \( \zeta \leq \epsilon \), we use 1 together with Lemma 1 to get:

\[
\int_0^1 w^{\alpha - 1 - \zeta} q_r(w) \, dw \leq \int_1^{\infty} w^{\alpha - 1 - \zeta} q_r(w) \, dw \leq \Gamma(r - \zeta) \mathbb{E} \{\Lambda^{\zeta}\} < \infty
\]

and since \( -k(\alpha - 1) - \epsilon \leq -r(\alpha - 1) - \zeta < 0 \):

\[
\int_1^\infty w^{\alpha - 1 + \zeta} q_r(w) \, dw \leq \int_0^{\infty} w^{\alpha - 1 + \zeta} q_r(w) \, dw = \Gamma(r + \zeta) \mathbb{E} \{\Lambda^{1-r(\alpha - 1) - \zeta}\} < \infty.
\]

Hence, the function \( h \) is integrable.

Finally, \( \lim_{t \to \infty} g_t(w; k_1, \ldots, k_r) = w^{2k - 1} w^{\alpha - 2k} q_r(w) = w^{\alpha - 1} q_r(w) \). Thus, applying Lebesgue’s theorem on dominated convergence and using 1, we deduce:

\[
\lim_{t \to \infty} \int_0^\infty g_t(w; k_1, \ldots, k_r) \, dw = \int_0^\infty w^{\alpha - 1} q_r(w) \, dw = \Gamma(r) \mathbb{E} \{\Lambda^{r(1-\alpha)}\}.
\]
Since $\ell \left( \frac{x}{s} \right) \in RV_0^0$, the uniform convergence theorem for slowly varying functions states that $\ell \left( \frac{x}{s} \right) \sim \ell \left( \frac{1}{x} \right)$ uniformly on each compact $x$-set in $(0, \infty)$. Since $\lim_{s \to 0} \frac{s}{\psi(s)} = \mu_1 \in (0, \infty)$, we consequently get $\ell \left( \frac{1}{\psi(s)} \right) = \ell \left( \frac{1}{\psi(s)} \frac{s}{\psi(s)} \right) \sim \ell \left( \frac{1}{s} \right)$. This together with Lemma 2 and $\varphi^{(1)}(\psi(s)) \sim -\mu_1$ yields:

$$f_t(k_1, \ldots, k_r) \sim_{t \to \infty} \frac{\alpha^r}{\mu_1^r} \prod_{i=1}^{r} \Gamma(2k_i - \alpha) t^{r-1} \psi^{r-1} \left( \frac{1}{t} \right) \ell^r \left( \frac{1}{\psi\left( \frac{1}{t} \right) t} \right)$$

Therefore, we obtain:

$$B_t(k_1, \ldots, k_r) \sim_{t \to \infty} \left( \frac{\alpha}{\mu_1^r} \right)^r \Gamma(r\alpha) E \left\{ \Lambda^{r(1-\alpha)} \right\} \prod_{i=1}^{r} \Gamma(2k_i - \alpha) t^{r(1-\alpha)} \ell^r(t).$$

When $\alpha = 1$, we have $\ell(t) = o(1)$ since $\mu_1 < \infty$. Hence, the first order asymptotic behavior of $\psi_{\ell}^t$ is solely determined by the term with $r = 1$ and we obtain:

$$E \left\{ T^k_{N(t)} \right\} \sim \frac{\alpha \Gamma(2k - \alpha)}{\mu_1^2} E \left\{ \Lambda^{1-\alpha} \right\} t^k \ell(t) \quad \text{as} \quad t \to \infty.$$

Finally, note that $E \left\{ \Lambda^{1-\alpha} \right\} < \infty$ since $-k(\alpha - 1) - \epsilon < 1 - \alpha < 0$ if $\alpha \neq 1$ and $1 - \alpha = 0$ if $\alpha = 1$. ■

We pass to the case $\alpha > 2$.

**Theorem 4** Assume that $X_1$ is of Pareto-type with index $\alpha > 2$. Let $\{N(t); t \geq 0\}$ be a mixed Poisson process with mixing random variable $\Lambda$. Let $k \in \mathbb{N}\setminus\{0\}$ be fixed. If $k = 1$, assume further that $E \left\{ \Lambda^{-1-\epsilon} \right\} < \infty$ for some $\epsilon > 0$. If $k \neq 1$, assume further that $E \left\{ \Lambda^{k-2+\epsilon} \right\} < \infty$ and $E \left\{ \Lambda^{1-2k-\epsilon} \right\} < \infty$ for some $\epsilon > 0$. Then for $k < \alpha - 1$:

$$E \left\{ T^k_{N(t)} \right\} \sim \left( \frac{\mu_2}{\mu_1^2} \right)^k E \left\{ \Lambda^{-k} \right\} t^{-k} \quad \text{as} \quad t \to \infty \quad (7)$$

and for $k > \alpha - 1$:

$$E \left\{ T^k_{N(t)} \right\} \sim \frac{\alpha}{\mu_1^2} B(2k - \alpha, \alpha) E \left\{ \Lambda^{1-\alpha} \right\} t^{1-\alpha} \ell(t) \quad \text{as} \quad t \to \infty. \quad (8)$$

If $k = \alpha - 1$, then:

(i) (4) holds if $\ell(x) = o(1)$ (and in particular if $\mu_{k+1} < \infty$);

(ii) $E \left\{ T^k_{N(t)} \right\} \sim \left( \frac{\mu_2}{\mu_1^2} \right)^k + C \frac{B(k-1, 0)}{\mu_1^2} \frac{k}{\mu_1} E \left\{ \Lambda^{-k} \right\} t^{-k} \quad \text{as} \quad t \to \infty$ holds if $\lim_{x \to \infty} \ell(x) = C$ for a positive constant $C$;

(iii) (5) holds otherwise.

**Proof**:

Let $k \in \mathbb{N}\setminus\{0\}$ and $\alpha > 2$ be fixed. Since $\mu_1 < \infty$, it follows that $\varphi^{(1)}(0) = -\mu_1$ and $1 - \varphi(s) \sim \mu_1 s$. Hence, $s = 1 - \varphi(\psi(s)) \sim \mu_1 \psi(s)$ and we deduce that $\psi(s) \sim \frac{s}{\mu_1 s} \in RV_0^0$, leading to $\lim_{s \to 0} \psi(s) = 0$. Since $\varphi^{(1)}(s) \sim -\mu_1 \in RV_0^0$, we easily get $\varphi^{(1)} \circ \psi \in RV_0^0$. Moreover, by Lemma 2 $\varphi^{(n)} \in RV_{\alpha-n}$ for $n > \alpha$ and $\varphi^{(n)} \in RV_0^0$ for $n \leq \alpha$. Consequently, we get $\varphi^{(n)} \circ \psi \in RV_{\alpha-n}$ for $n > \alpha$ and $\varphi^{(n)} \circ \psi \in RV_0^0$ for $n \leq \alpha$. Consequently, we get $\varphi^{(n)} \circ \psi \in RV_{\alpha-n}$ for $n > \alpha$ and $\varphi^{(n)} \circ \psi \in RV_0^0$ for $n \leq \alpha$.
for $n \leq \alpha$.

For simplicity, we first assume that $\alpha \notin \mathbb{N}$. Using Potter’s theorem, we obtain the following upper bound for the integrand in $B_t(k_1, \ldots, k_r)$. Set $\delta_0 := \frac{\epsilon}{2k+r}$ with $\epsilon \in (0, 1)$ or $\zeta \in (0, 1)$ otherwise. For this chosen $\delta_0 > 0$, there exists $C_r = C_r(\delta_0) > 1$ such that for all $t > 0$:

$$g_t(w; k_1, \ldots, k_r) \leq C_r w^{2k-1} (\max\{w^{\delta_0}, w^{-\delta_0}\})^{2k-1} w^{r_1 \alpha - 2u_1} (\max\{w^{\delta_0}, w^{-\delta_0}\})^{r_1 + 1} q_r(w)$$

where $r_1$ denotes the number of integers among $\{k_1, \ldots, k_r\}$ that are greater than $\alpha/2$ and $u_1$ is the sum of these.

Now, $\int_0^\infty h(w) \, dw < \infty$ if and only if only $\int_0^1 w^{2k-1+r_1 \alpha - 2u_1 - \zeta} q_r(w) \, dw < \infty$ and $\int_1^\infty w^{2k-1+r_1 \alpha - 2u_1 + \zeta} q_r(w) \, dw < \infty$.

We have $2 - 2k \leq r_1 \alpha - 2u_1 \leq 0$. Indeed, if $r_1 = 0$ then obviously $r_1 \alpha - 2u_1 = 0$. Now, if $r_1 \neq 0$ (i.e. $r_1 \geq 1$) then $r_1 \alpha - 2u_1 < 0$ on the one hand, and $r_1 \alpha - 2u_1 \geq \alpha - 2k \geq \alpha - 2k$ on the other hand. Consequently, $1 \leq 2k + r_1 \alpha - 2u_1 \leq 2k - 1$.

Since $\zeta \in (0, 1)$ and $\zeta \leq \epsilon$, we get $0 < r - 2 + \zeta \leq k - 2 + \epsilon$ if $r \geq 2$ and $1 - 2k - \epsilon \leq -1 + \zeta < 0$ if $r = 1$. Moreover, $1 - 2k - \epsilon \leq r - 2k - \zeta < 0$. Therefore, using (1) together with Lemma (1) leads to:

$$\int_0^1 w^{2k-1+r_1 \alpha - 2u_1 - \zeta} q_r(w) \, dw \leq \int_0^1 w^{1-\zeta} q_r(w) \, dw \leq \Gamma(2 - \zeta) \mathbb{E} \left\{ \Lambda^{r-2+\zeta} \right\} < \infty$$

and:

$$\int_1^\infty w^{2k-1+r_1 \alpha - 2u_1 + \zeta} q_r(w) \, dw \leq \int_1^\infty w^{2k-1+\zeta} q_r(w) \, dw \leq \Gamma(2k + \zeta) \mathbb{E} \left\{ \Lambda^{r-2k-\zeta} \right\} < \infty.$$

When $k = 1$, obviously $r = 1$ and we get $-1 - \epsilon \leq -1 - \zeta < -1 + \zeta < 0$. The condition $\mathbb{E} \left\{ \Lambda^{-1+\epsilon} \right\} < \infty$ is thus sufficient for $\mathbb{E} \left\{ \Lambda^{-1+\zeta} \right\} < \infty$ and $\mathbb{E} \left\{ \Lambda^{-1-\zeta} \right\} < \infty$.

Hence, the function $h$ is integrable.

Finally, $\lim_{t \to \infty} g_t(w; k_1, \ldots, k_r) = w^{2k-1+r_1 \alpha - 2u_1} q_r(w) = w^{2(k-u_1)+r_1 \alpha - 1} q_r(w)$. Thus, applying Lebesgue’s theorem on dominated convergence and using (1), we deduce:

$$\lim_{t \to \infty} \int_0^\infty g_t(w; k_1, \ldots, k_r) \, dw = \int_0^\infty w^{2(k-u_1)+r_1 \alpha - 1} q_r(w) \, dw = \Gamma(2(k-u_1) + r_1 \alpha) \mathbb{E} \left\{ \Lambda^{r-2(k-u_1)-r_1 \alpha} \right\}.$$
It remains to determine the dominating asymptotic term among all possible $B_i(k_1, \ldots, k_r)$. For $r_1 > 0$, the largest exponent is achieved with $r_1 = 1$, $u_1 = k$ and thus $r = 1$, so that the asymptotic order is $t^{1-\alpha} \ell(t)$. Note that $r_1 > 0$ is possible for $2k > \alpha$ only. For $r_1 = 0$, obviously $r = k$ (which implies $k_1 = \cdots = k_r = 1$) dominates, leading to the asymptotic order $t^{-k}$. Hence, the asymptotically dominating power among all $B_i(k_1, \ldots, k_r)$ is given by $\max\{1-\alpha, -k\}$. From this, we see that when $k < \alpha - 1$, $r = k$ dominates and we obtain from (4):

$$
\mathbb{E} \left\{ T_N^k(t) \right\} \sim \frac{k! \mu_2^k \Gamma(2k) \mathbb{E} \left\{ \Lambda^{-k} \right\}}{(2k-1)! \mu_1^{2k}} t^{-k} = \left( \frac{\mu_2}{\mu_1^2} \right)^k \mathbb{E} \left\{ \Lambda^{-k} \right\} \frac{t^{-k}}{k} \quad \text{as} \quad t \to \infty.
$$

Alternatively, when $k > \alpha - 1$, the term with $r = 1$ dominates and we find:

$$
\mathbb{E} \left\{ T_N^k(t) \right\} \sim \frac{k! \alpha \Gamma(2k-\alpha) \Gamma(\alpha) \mathbb{E} \left\{ \Lambda^{1-\alpha} \right\}}{k! (2k-1)! \mu_1^\alpha} t^{1-\alpha} \ell(t) = \frac{\alpha}{\mu_1^\alpha} B(2k-\alpha, \alpha) \mathbb{E} \left\{ \Lambda^{1-\alpha} \right\} \frac{t^{1-\alpha}}{k} \ell(t) \quad \text{as} \quad t \to \infty
$$

which is the same expression as the one obtained in Theorem 3 for $\alpha \in [1, 2)$ and $\mu_1 < \infty$.

The above conclusions also hold for $\alpha \in \mathbb{N}$ as long as $k \neq \alpha - 1$. Nevertheless, just note that instead of (6) we have the following by virtue of Lemma 4:

$$
\ell(t) \sim \frac{\alpha^{r_1+r_2} K_3}{\mu_1^{2(k-1)+\alpha}} t^{r_2-2(k-1)-r_1} \ell^{r_1}(t) \ell^{r_2}(t)
$$

where $\ell(x) = \int_0^x \frac{\ell(u)}{u} \, du \in RV_0^\infty$, $K_3 := \prod_{j \in \{i : 2k_i \leq \alpha, \mu_2k_i < \infty\}} \mu_2 k_i$ and $r_2 := \text{card}\{i : 2k_i = \alpha, \mu_2k_i = \infty\}$, with $\text{card}\{i : 2k_i \leq \alpha, \mu_2k_i < \infty\} = r - r_1 - r_2$.

When $k = \alpha - 1$, the slowly varying function $\ell$ determines which of the two terms $t^{1-\alpha} \ell(t)$ (corresponding to $r = 1$) and $t^{-k}$ (corresponding to $r = k$) dominates the asymptotic behavior. If $\ell(x) = o(1)$, which is in particular fulfilled if $\mu_{k+1} < \infty$, the second term dominates. If $\lim_{x \to \infty} \ell(x) = C$ for a positive constant $C$, then both terms matter. Otherwise, the first term dominates.

To end the proof, it remains to check that $\mathbb{E} \{ \Lambda^{-k} \} < \infty$ for $k \leq \alpha - 1$ and that $\mathbb{E} \{ \Lambda^{1-\alpha} \} < \infty$ for $k > \alpha - 1$. When $k > \alpha - 1$, we have $1 - 2k - \epsilon < -k < 1 - \alpha < 0$. Otherwise, we have $1 - 2k - \epsilon < -k < 0$. Thus, we conclude by using Lemma 4.

Finally, we consider the case $\alpha = 2$.

**Corollary 1** Assume that $X_1$ is of Pareto-type with index $\alpha = 2$. Let $\{N(t) ; t \geq 0\}$ be a mixed Poisson process with mixing random variable $\Lambda$.

(i) If $\mathbb{E} \{ \Lambda^{-1-\epsilon} \} < \infty$ for some $\epsilon > 0$, then:

$$
\mathbb{E} \left\{ T_N(t) \right\} \sim \begin{cases}
\frac{\mu_2 \mathbb{E} \{ \Lambda^{-1} \}}{\mu_1^2 \ell(t)} & \text{if } \mu_2 < \infty \\
\frac{2 \mathbb{E} \{ \Lambda^{-1} \}}{\mu_1^2} \ell(t) & \text{if } \mu_2 = \infty
\end{cases}
\quad \text{as} \quad t \to \infty
$$

where $\ell(x) = \int_0^x \frac{\ell(u)}{u} \, du \in RV_0^\infty$.

(ii) For any fixed $k \geq 2$, if $\mathbb{E} \{ \Lambda^{-2k+\epsilon} \} < \infty$ and $\mathbb{E} \{ \Lambda^{1-2k-\epsilon} \} < \infty$ for some $\epsilon > 0$, then:

$$
\mathbb{E} \left\{ T_N^k(t) \right\} \sim \frac{\mathbb{E} \{ \Lambda^{-1} \}}{\mu_1^2 (k-1) (2k-1)} \frac{\ell(t)}{t} \quad \text{as} \quad t \to \infty.
$$
Using (1), we deduce:

\[ B_t(1) = \int_0^\infty s \varphi^{(2)}(s) Q_t^{(1)}(\varphi(s)) \, ds \]

\[ = \frac{-\psi \left( \frac{1}{t} \right)}{\varphi^{(1)} \left( \psi \left( \frac{1}{t} \right) \right) \varphi^{(2)} \left( \psi \left( \frac{1}{t} \right) \right)} \int_0^\infty \frac{\psi \left( \frac{w}{t} \right) \varphi^{(2)} \left( \psi \left( \frac{w}{t} \right) \right)}{\varphi^{(1)} \left( \psi \left( \frac{w}{t} \right) \right)} q_1(w) \, I_{[0,t]}(w) \, dw. \]

Since \( \mu_1 < \infty \), it follows that \( \varphi^{(1)}(0) = -\mu_1 \) and \( 1 - \varphi(s) \sim \mu_1 s \). Hence, \( s = 1 - \varphi(s) \sim -\mu_1 \psi(s) \) and we deduce that \( \psi(s) \sim \frac{1}{\mu_1} \in \mathbb{RV}_1^0 \), leading to \( \lim_{s \to 0} \psi(s) = 0 \). Since \( \varphi^{(1)}(s) = -\mu_1 \in \mathbb{RV}_0^0 \), we easily get \( \varphi^{(1)} \circ \psi \in \mathbb{RV}_0^0 \). Moreover, \( \varphi^{(2)}(s) \sim 2 \bar{\ell} \left( \frac{1}{s} \right) \in \mathbb{RV}_0^0 \) by Lemma 2 and as a consequence \( \varphi^{(2)} \circ \psi \in \mathbb{RV}_0^0 \), where \( \bar{\ell}(x) = \int_0^x \frac{d(w)}{w} \, dw \in \mathbb{RV}_0^\infty \).

Using Potter’s theorem, we therefore obtain the following upper bound for the integrand in \( B_t(1) \). Set \( \delta := \frac{\epsilon}{2} \) with \( \zeta = \epsilon \) if \( \epsilon \in (0,1) \) or \( \zeta \in (0,1) \) otherwise. For this chosen \( \delta > 0 \), there exists \( C = C(\delta) > 1 \) such that for all \( t > 0 \):

\[ g_t(w; 1) \leq C \, w \left( \max \left\{ w^\delta, w^{-\delta} \right\} \right)^3 q_1(w) = C \, w \max \left\{ w^\zeta, w^{-\zeta} \right\} q_1(w) =: h(w). \]

Now, \( \int_0^\infty h(w) \, dw < \infty \) if and only if \( \int_0^1 w^{1-\zeta} q_1(w) \, dw < \infty \) and \( \int_1^\infty w^{1+\zeta} q_1(w) \, dw < \infty \).

Since \( \zeta \in (0,1) \) and \( \zeta < \epsilon \), we get \( -1 - \epsilon \leq -1 - \zeta < 1 + \zeta < 0 \). Therefore, using Lemma 1 together with Lemma 1 leads to:

\[ \int_0^1 w^{1-\zeta} q_1(w) \, dw \leq \int_0^\infty w^{1-\zeta} q_1(w) \, dw = \Gamma(2 - \zeta) \, E \{ \Lambda^{-1-\zeta} \} < \infty \]

and:

\[ \int_1^\infty w^{1+\zeta} q_1(w) \, dw \leq \int_0^\infty w^{1+\zeta} q_1(w) \, dw = \Gamma(2 + \zeta) \, E \{ \Lambda^{-1-\zeta} \} < \infty. \]

Hence, the function \( h \) is integrable.

Finally, \( \lim_{t \to \infty} g_t(w; 1) = w \, q_1(w) \). Thus, applying Lebesgue’s theorem on dominated convergence and using (1), we deduce:

\[ \lim_{t \to \infty} \int_0^\infty g_t(w; 1) \, dw = \int_0^\infty w \, q_1(w) \, dw = E \{ \Lambda^{-1} \}. \]

By virtue of the uniform convergence theorem for slowly varying functions, we get \( \bar{\ell}(1/\psi(s)) \sim \bar{\ell} \left( \frac{1}{s} \right) \)

This together with \( \varphi^{(1)}(\psi(s)) \sim -\mu_1 \) and \( \varphi^{(2)}(\psi(s)) \sim 2 \bar{\ell}(1/\psi(s)) \) yields:

\[ f_t(1) \sim \frac{2 \bar{\ell}(t)}{\mu_1^2 t}. \]

Consequently, we obtain:

\[ E \{ T_N(t) \} = B_t(1) \sim \frac{2 E \{ \Lambda^{-1} \} \bar{\ell}(t)}{\mu_1^2 t} \text{ as } t \to \infty. \]

Finally, note that \( E \{ \Lambda^{-1} \} < \infty \) since \( -1 - \epsilon < -1 \).

We end by a remark.
Remark As in Albrecher and Teugels [1], the integral representation approach that we use in this paper does not permit to get a general asymptotic result for $E\left\{ T_{N(t)}^k \right\}$ when $F$ in the additive domain of attraction of a normal law, i.e. when $F$ has a slowly varying truncated variance. Note that if the distribution function $F$ of $X_1$ is as in Corollary [4] then $F$ is in the additive domain of attraction of a normal law.

4 Conclusion

In this paper, we have derived the limiting behavior of arbitrary moments of $T_{N(t)}$ assuming the distribution function $F$ of $X_1$ to be of Pareto-type with index $\alpha > 0$ and the counting process $\{N(t); t \geq 0\}$ to be mixed Poisson. Different results have shown up depending on the range of $\alpha$.

In the special setting where the distribution function $H$ of the mixing random variable $\Lambda$ is degenerate at the point 1, our results are similar to those derived by Albrecher and Teugels [1] who assume the counting process to be non-random. This is basically explained by the fact that the important quantities $q_r(w)$ are the same in these two cases.

The coefficient of variation of a positive random variable $X$ is defined and denoted by:

$$CoVar(X) := \sqrt{\frac{\mathbb{V}X}{\mathbb{E}X}}$$

where $\mathbb{V}X$ denotes the variance of $X$. This risk measure is frequently used in practice and is very popular among actuaries.

From a random sample $X_1, \ldots, X_{N(t)}$ from $X$ of random size $N(t)$ from an integer valued distribution, the coefficient of variation $CoVar(X)$ is naturally estimated by the sample coefficient of variation of $X$ defined and denoted by:

$$\hat{CoVar}(X) := \frac{S}{\bar{X}}$$

where $\bar{X} := \frac{X_1 + \cdots + X_{N(t)}}{N(t)}$ is the sample mean and $S^2 := \frac{1}{N(t)} \sum_{i=1}^{N(t)} (X_i - \bar{X})^2$ is the sample variance.

The properties of the sample coefficient of variation $\hat{CoVar}(X)$ are usually studied in assuming the finiteness of sufficiently many moments of $X$. The existence of moments of $X$ is not always guaranteed in practical applications. Hence, it is useful to investigate asymptotic properties of $\hat{CoVar}(X)$ also in these cases and it turns out that this can be done by using results on $T_{N(t)}$.

Indeed, it appears that the quantity $T_{N(t)}$ is a basic ingredient in the study of the sample coefficient of variation since the following holds:

$$\hat{CoVar}(X) = \sqrt{N(t) T_{N(t)} - 1}.$$  \hspace{1cm} (10)

In a forthcoming paper, we will touch on the question of convergence in distribution for the appropriately normalized quantity $T_{N(t)}$ when $X$ is of Pareto-type with positive index $\alpha$ and the counting process $\{N(t); t \geq 0\}$ is mixed Poisson. Thanks to relation $[10]$, asymptotic properties of the sample coefficient of variation will be derived, even when the first moment and/or the second moment of $X$ do not exist.

Incidentally, it will also be seen how the methodology can be adapted to derive asymptotic properties of another risk measure, the sample dispersion. Recall that the value of the dispersion allows to compare the volatility with respect to the Poisson case.

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