Black Holes and Causal Structure in Anti-de Sitter Isometric Spacetimes

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Abstract

The observation that the 2+1 dimensional BTZ black hole can be obtained as a quotient space of anti-de Sitter space leads one to ask what causal behaviour other such quotient spaces can display. In this paper we answer this question in 2+1 and 3+1 dimensions when the identification group has one generator. Among other things we find that there does not exist any 3+1 generalization of the rotating BTZ hole. However, the non-rotating generalization exists and exhibits some unexpected properties. For example, it turns out to be non-static and to possess a non-trivial apparent horizon.

1 Introduction

In 1992 Bañados et al \cite{1} reported a new 2+1 dimensional solution to Einstein's equations in vacuum with a negative cosmological constant. As all such solutions this one was locally isometric to anti-de Sitter space (adS-space)—the negatively curved analogue of Minkowski space—and could even be constructed from it simply by means of an identification of points \cite{2}. The surprising fact with this solution, so easily obtained, was that it described a black hole—a very "minimal" one, containing just the necessary ingredients in order to fulfil the defining properties. Hence this solution gave rise to a rather large amount of papers (see Carlip \cite{3} and Mann \cite{4} and references therein).

However, not much has been said about its 3+1 dimensional analogues. In ref. \cite{5} some examples of such generalizations were given, and ref. \cite{6} discusses adS black holes in even higher dimensions, especially the 4+1 case. Here we intend to give a more systematic treatment of the subject, following the method of appendix A in ref. \cite{2} to classify the different spacetimes that can be obtained by making identifications in 3+1 dimensional adS-space using a one dimensional subgroup of its symmetry group SO(2,3). We will see that the different cases will exhibit a great variety of causal structures, most of which, unfortunately, involve naked regions with closed timelike curves. In particular, we will show that there does not exist any 3+1 dimensional generalization of the rotating BTZ black hole. It is only the non-rotating special case that has a 3+1 dimensional counterpart.

The route in this paper will be as follows. First we will explain how to classify all spacetimes obtainable as a quotient of adS-space by one of its one generator subgroups. This amounts to classifying the elements in SO(2,3), or equivalently, the Killing fields of

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3+1 adS-space. Given a particular Killing field we will see how to deduce the causal structure of the corresponding (identified) spacetime. Then we will perform the classification of SO(2,3), and compare the result with the classification of SO(2,2), performed in ref. [2]. A particular choice of coordinates will enable us to visualize our conclusions concerning the different causal structures. At the end we will take a closer look at the only 3+1 dimensional black hole that can be obtained by the methods used in this paper—the 3+1 dimensional analogue of the non-rotating BTZ black hole. That solution actually turns out to be non-static, and to contain an apparent horizon which does not coincide with the event horizon.

2 Causal structure and symmetries

As mentioned above, the BTZ black hole can be obtained by an identification of points in adS-space. More specifically, all points that are mapped into each other under a particular discrete symmetry Γ of adS-space. In other words, the BTZ-solution could be considered as the quotient space \([\text{adS}] / \mathcal{G}_\Gamma\), where \(\mathcal{G}_\Gamma\) is the group generated by Γ: \(\mathcal{G}_\Gamma = \{\Gamma^n; n \in \mathbb{Z}\}\). Furthermore, this Γ is obtainable by exponentiating a local symmetry of SO(2,2)—the symmetry group of 2+1 adS-space—that is, Γ = \(e^{\alpha \xi}\) where \(\xi\) is a Killing field and \(\alpha\) is a finite parameter fixing the “size” of Γ.

So by simply taking the quotient space of adS-space with a particular Γ we can obtain such interesting objects as black holes. The question then arises, what other causal structures could be obtained by choosing Γ differently? The goal of this paper is to answer this question in 2+1 and in 3+1 dimensions.

There is one point about the notation that should be made clear from the beginning: If \(\mathcal{G}_\Gamma\) does not act properly discontinuously, which for example may be the case if the generating Killing field contains rotation, then, strictly speaking, we cannot take the quotient \([\text{adS}] / \mathcal{G}_\Gamma\) without first removing points from adS-space, and maybe extend it to some covering space. In what follows we will ignore such technicalities and just write \([\text{adS}] / \mathcal{G}_\Gamma\) even if \(\mathcal{G}_\Gamma\) fails to act properly discontinuously. In each case the meaning should be clear anyway.

We must also clarify what we mean by “different Γ:s”. Of course, we are not interested in comparing symmetries that differ only by a global isometry, because they have to yield the same spacetime when used for identification. Thus, if two generators \(\Gamma_1\) and \(\Gamma_2\) satisfy

\[\Gamma_2 = T^{-1}\Gamma_1 T \quad T \in \text{SO}(2,d-1)\] (1)

where \(d\) is the spacetime dimension (3 or 4 here), they will be considered to correspond to the same symmetry. Our task is then to classify SO(2,\(d-1\)), or equivalently the Killing fields for adS-space, up to this equivalence relation. For \(d = 3\) this was done in ref. [2] and in the next section we will compare that result with the corresponding one for \(d = 4\).

For now, consider this to be done. The next question then is how we, given a Killing field \(\xi\), can deduce the causal structure of \([\text{adS}] / \mathcal{G}_\Gamma\), where \(\mathcal{G}_\Gamma\) is generated by \(\Gamma = e^{\alpha \xi}\) for some fixed \(\alpha\). First, obviously enough, if \(\xi\) is timelike in some region in adS-space, then, when we identify points connected by \(e^{\alpha \xi}\), closed timelike curves will result in this region, consisting simply of the (closed) Killing field lines there. In general, there could be closed timelike curves extending outside this region too, but at least a small segment of them

\[^1\text{True adS-space has a cyclic time and hence contains closed timelike curves in itself. In this paper we will actually be working with its universal covering, see equations (5) and the following comments.}\]
would have to be inside this region. Therefore, an observer not entering the region where \( \xi \) is timelike, that is, not passing the hypersurface \( \xi^\mu \xi_\mu = 0 \), would not be able to travel on a closed timelike curve. Thus, to clarify the causal structure of \( [adS]/G_\Gamma \), we would like to know how different observers, in the region where \( \xi \) is spacelike, are related to the surface where \( \xi \) is null. For this reason, from now on we adopt the following definitions for the quotient space \( [adS]/\{e^{n\alpha \xi}; n \in \mathbb{Z}\} \) for some Killing field \( \xi \) and a fixed \( \alpha \):

First, the hypersurface \( \xi^\mu \xi_\mu = 0 \) will be called the *singularity*. This definition is consistent with the language used in ref. [1] and [2]. In what follows we will be interested in the causal structure with respect to this singularity. This of course actually means, the causal structure with respect to the region where timelike curves may close, something that the reader who feels uncomfortable with this terminology may wish to keep in mind.

The horizons are then defined as usual. The *event horizon* is the boundary of the past of the future null infinity, \( J^{-}(\mathcal{I}^+) \), where the word “infinity” actually refers to the part of infinity where \( \xi \) is spacelike, that is, the part relevant to an observer not allowed to pass the singularity. When an observer passes this horizon he loses his chances to reach infinity. If there are several disconnected infinities there is one event horizon for each of them.

The *inner horizon* is the boundary to the future of the singularity, \( J^{+}(\{\xi^\mu \xi_\mu = 0\}) \). An observer passing this horizon will be able to see the singularity, without falling into it.

Note that to apply these definitions, that is, to find the causal structure in \( [adS]/G_\Gamma \), the only thing we need is the behaviour of the generating Killing field \( \xi \) in adS-space. The parameter \( \alpha \) will not affect the causality and we do not have to obtain an explicit metric for the quotient space, or know precisely which points that are identified with each other. We do not even have to know very much about the Killing field itself, only the properties of the hypersurface \( \xi^\mu \xi_\mu = 0 \), especially near infinity.

Let us see what we can say about this singularity hypersurface in general. Consider the scalar field defined by \( f(x) = \xi^\mu \xi_\mu \), where \( \xi \) is some Killing field. The normal to hypersurfaces where \( f(x) \) is constant is

\[
\nabla_\rho f(x) = \nabla_\rho \xi^\mu \xi_\mu = 2 \xi^\mu \nabla_\rho \xi_\mu
\]

which implies that

\[
\xi^\rho \nabla_\rho f(x) = 2 \xi^\rho \xi^\mu \nabla_\rho \xi_\mu = 0
\]

since \( \xi \) is a Killing field: \( \nabla_{(\rho \xi_\mu)} = 0 \). Thus \( \xi \) is orthogonal to the normal of the hypersurfaces with \( f(x) \) constant, and hence lies in such surfaces. In particular, the Killing field is tangent to the singularity \( f(x) = 0 \). More intuitively, this has to be the case, since otherwise the singularity would not be mapped into itself under the isometry corresponding to \( \xi \).

From this we see that except where \( \xi^\mu = 0 \), the singularity surface has to be either timelike or lightlike, since it has a tangent that is null. If it is timelike it necessarily is naked with respect to some region, that is, visible for some observers in the spacetime.

For the 2+1 dimensional case we can draw even more restrictive conclusions about the singularity. Later, in order to visualize our results, we will use a conformal picture of adS-space, that is, coordinates where points infinitely far away from the origin are mapped into a finite distance, and thus, where infinity itself is represented as a hypersurface. The Killing field is well-defined in this hypersurface by a limit procedure, and if it is non-zero there it has to be tangent to infinity since otherwise adS-space would not be mapped into itself by the corresponding symmetry. Now, if the singularity and the infinity meet each
other, they do so in a one dimensional line, and since the Killing field is tangent to both of them, this line has to be a Killing field line. Further, since the field is null at the singularity, this line must be lightlike. (This, in turn, is very restrictive for the shape of a possible event horizon.)

In 3+1 dimensions we can not draw such a conclusion, because there the singularity meets infinity in a two dimensional surface, which then must have a null direction. This is not very restrictive because this surface could still be either timelike or lightlike.

3 Classification of adS-isometries

Up to now, what we have said is quite general. Not only did we avoid specifying a particular symmetry or Killing field to be used for the identification—we did not even use any properties of the playground itself: Anti-de Sitter space. Actually, we have not even defined it yet. The previous section holds for every quotient space of the form \([\mathcal{M}, g_{\mu\nu}] / \{\Gamma^n; n \in \mathbb{Z}\}\) where \(\Gamma\) is generated by a continuous symmetry \(\xi\) on the spacetime \([\mathcal{M}, g_{\mu\nu}]\).

Now we will specialize to adS-space and discuss its isometries. This will enable us, in the next section, to draw conclusions about the different causal structures in all quotient spaces of the form \([\text{adS}]/e^{\alpha \xi}\). We begin by defining 3+1 adS-space as the hyperboloid

\[
X^2 + Y^2 + Z^2 - U^2 - V^2 = -1
\]  

embedded in the flat 5 dimensional space with metric

\[
ds^2 = dX^2 + dY^2 + dZ^2 - dU^2 - dV^2
\]

The 2+1 dimensional case is defined by the same equations but with \(Z = 0\).

A natural set of “base Killing vector fields” are given by

\[
J_{ab} = x_b \frac{\partial}{\partial x^a} - x_a \frac{\partial}{\partial x^b}
\]

where \(x^a = (U, V, X, Y, Z)\)

in terms of which a general Killing field \(\xi\) may be written as

\[
\xi = \frac{1}{2} \omega^{ab} J_{ab} = \omega^{ab} x_b \partial_a
\]

where \(\omega^{ab} = -\omega^{ba}\). Hence, a general Killing field may be characterized with the matrix \(\omega^a_b\). In fact, since

\[
\xi x^c = \omega^a_b x^b \partial_a x^c = \omega^c_b x^b
\]

it is simply the generator of the infinitesimal symmetry transformation \(\delta x^a = \epsilon \xi x^a\):

\[
x'^a = x^a + \delta x^a = x^a + \epsilon \omega^a_b x^b = (\delta^a_b + \epsilon \omega^a_b) x^b
\]

To classify the isometries of adS-space then amounts to classifying the matrices \(\omega^a_b\) up to the equivalence relation \([\mathbb{I}]\), that is,

\[
\omega'^a_b = (T^{-1})^a_c \omega^c_d T^d_b
\]
where \( T^a_b \in \text{SO}(2, d - 1) \). However, we could equally well say that we want to classify all antisymmetric matrices \( \omega_{ab} \) up to similarity (that is, with an arbitrary \( T \)), since for such matrices (2) is fulfilled if and only if \( T^a_b \in \text{SO}(2, d - 1) \).

Now, any diagonalizable matrix may be uniquely characterized, up to similarity, by its eigenvalues. Hence, if every \( \omega^a_b \) were diagonalizable, which they of course are not, they could be classified according to their eigenvalues. In our case we have to use a slightly more sophisticated method than that. Recall that any matrix \( M \) may be written as \( PNP^{-1} \) where \( N \) is not necessarily diagonal, but at least on Jordan’s normal form, that is (for 5 dimensions)

\[
N = \begin{pmatrix}
\lambda_1 & a_1 & 0 & 0 & 0 \\
0 & \lambda_2 & a_2 & 0 & 0 \\
0 & 0 & \lambda_3 & a_3 & 0 \\
0 & 0 & 0 & \lambda_4 & a_4 \\
0 & 0 & 0 & 0 & \lambda_5
\end{pmatrix}
\]

where each \( a_i \) is 0 or 1, and where, if \( a_k = 1 \) for some \( k \), then \( \lambda_k = \lambda_{k+1} \).

For definiteness, suppose that \( a_1 = 1, a_2 = a_3 = a_4 = 0 \), and hence that \( \lambda_1 = \lambda_2 = \lambda \). Then there exist only one eigenvector corresponding to this eigenvalue, let us call it \( \mathbf{x} \): \( M\mathbf{x} = \lambda \mathbf{x} \). But it is easy to see that there exist another vector \( \mathbf{y} \) (simply the second column in the transformation matrix \( P \)) such that \( M\mathbf{y} = \lambda \mathbf{y} + \mathbf{x} \). Hence, operating with \( M \) on any vector in the subspace spanned by \( \mathbf{x} \) and \( \mathbf{y} \), yields a vector in the same subspace. Therefore one says that \( \mathbf{x} \) and \( \mathbf{y} \) span a 2 dimensional invariant subspace corresponding to eigenvalue \( \lambda \). Similarly, if \( a_1 = a_2 = 1, a_3 = a_4 = 0 \) and hence \( \lambda_1 = \lambda_2 = \lambda_3 \), \( M \) would have a 3 dimensional invariant subspace.

From this we conclude that \( \omega^a_b \) can be uniquely specified, up to similarity, by its eigenvalues and the dimensions (and signatures) of its invariant subspaces corresponding to these, because this information is sufficient to fix the matrix \( N \).

To actually perform this classification is tedious. However, the observation that if \( \lambda \) is an eigenvalue to \( \omega^a_b \), then \(-\lambda, \lambda^* \) and \(-\lambda^* \) are also eigenvalues, greatly facilitates the calculation. For this, and for an explicit demonstration of the algebraic manipulations, see the appendix of ref. [2] where this classification is performed for \( \text{SO}(2, 2) \). We quote their result in table 1, and the corresponding result for \( \text{SO}(2, 3) \) in table 2. For the rest of this section we will compare and explain some features of these tables.

The second and third columns give the characteristics of each type: The eigenvalues (in terms of the real parameters \( a \) and \( b \)) and the dimension of the invariant subspaces, respectively. The latter is directly reflected in the numbering of the types. The last column provides an explicit Killing field, given values of the \( a \)s and \( b \)s. Note that, because of these parameters, each type contains a 0-, 1- or 2-parameter family of non-equivalent fields.

Comparing the tables, we see that there are two more types in the \( \text{SO}(2,3) \) case: Types \( I_d \) and \( V \). It is clear that they cannot exist for \( \text{SO}(2,2) \) because they demand 3 spatial dimensions—all of the three spatial indices 2, 3 and 4 occur in their Killing fields. Further, for types \( III_a \) and \( III_b \) there is a 1-parameter family of Killing fields in table 2. When the parameter is zero they both coincide with the corresponding (0-parameter) fields in table 1.

\(^2\)Strictly speaking, if \( \omega_{ab} \) and \( \omega^a_b \) are anti-symmetric (3) is fulfilled if and only if \( T^a_b \in \text{O}(2, d - 1) \), that is, the determinant of \( T \) could be \(-1 \). In practice this means that the classification below does not distinguish between two Killing fields that are the mirror image of each other, and they will belong to the same type in table 1 or 2 below.

\(^3\)In ref. [2] types \( III_a \) and \( III_b \) are referred to as \( III^+ \) and \( III^- \) respectively, because for \( \text{SO}(2,2) \) they
As a curiosity, note that neither $SO(2,2)$ nor $SO(2,3)$ can have a 4 dimensional invariant subspace. When one performs the classification that turns out to be inconsistent, in both cases, with the signature of the metric.

4 Horizons in the quotient space

In this section we will examine the causal structure for all quotient spaces $[adS]/G$, systematically by choosing the generator $\xi$ of $\Gamma$ from all different types found in the last section. To do this we will use the tools developed in section 2, were we defined the singularity and the horizons in terms of the hypersurface $\xi^\mu \xi_\mu = 0$.

First, however, we will introduce a set of coordinates that will enable us to visualize our conclusions. Since we wish to be able to understand horizons and such, the best would be to find some Penrose-like coordinates, that is, coordinates in which infinity is mapped into a finite distance and where all lightlike curves have slope 1. Unfortunately, in general there does not exist such coordinates in higher dimensions than 1+1, and since we want coordinates for 3 and even 4 spacetime dimensions we have to relax one of these requirements.

In the spherical coordinates given below—$t$, $\rho$, $\phi$ and (in 4 dimensions) $\theta$—spacelike infinity is mapped into the cylindrical surface $\rho = 1$. On the other hand the slope of the null curves differ with the radius $\rho$, but for a given point each null curve through that point has the same slope irrespectively of its direction.

\[
\begin{align*}
X & = \frac{2\rho}{1 - \rho^2} \sin \theta \cos \phi \\
Y & = \frac{2\rho}{1 - \rho^2} \sin \theta \sin \phi \\
Z & = \frac{2\rho}{1 - \rho^2} \cos \theta \\
U & = \frac{1 + \rho^2}{1 - \rho^2} \cos t \\
V & = \frac{1 + \rho^2}{1 - \rho^2} \sin t
\end{align*}
\]

This parametrizes the defining hypersurface for 3+1 adS-space. The radial coordinate $\rho$ takes values between 0 and 1 while $\phi$ and $\theta$ are the usual azimuthal and polar angles, respectively. Since we are going to discuss causal behaviour we will think of the time as “unwound”, that is $-\infty < t < \infty$. Otherwise we would have closed timelike curves already from the start. (Formally this means that we are actually working with the universal covering of adS-space.) The metric becomes

\[
\begin{align*}
ds^2 = - \left( \frac{1 + \rho^2}{1 - \rho^2} \right)^2 dt^2 + \frac{4}{(1 - \rho^2)^2} (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2)
\end{align*}
\]

First, let us concentrate on the 2+1 dimensional case for which we simply put $\theta = \pi$ in equations $\mathbf{[3]}$ and $\mathbf{[5]}$. Then these coordinates describe adS-space as the interior of only differ by the signature of the invariant subspace.

\footnote{We will choose coordinates which meet the requirement of mapping all points in spatial infinity into a finite distance. For a different choice, which fails to satisfy this, but where the other requirement is met instead—that the slope of null curves is 1—see ref. $\mathbf{[4]}$.}
| Type | Eigenvalues | Inv.subspace | Killingfield |
|------|-------------|--------------|--------------|
| Ia   | $\lambda, -\lambda, \lambda^*, -\lambda^*$ | $\lambda = a + ib, a \neq 0, b \neq 0$ | $b(J_{01} + J_{23}) - a(J_{03} + J_{12})$ |
| Ib   | $a_1, -a_1, a_2, -a_2$ | | $a_1J_{12} + a_2J_{03}$ |
| Ic   | $ib_1, -ib_1, ib_2, -ib_2$ | | $b_1J_{01} + b_2J_{23}$ |
| IIa  | $a, -a$ | 2 dim. to $a$ 2 dim. to $-a$ | $a(J_{03} + J_{12}) + J_{01} - J_{02} - J_{13} + J_{23}$ |
| IIb  | $ib, -ib$ | 2 dim. to $ib$ 2 dim. to $-ib$ | $(b - 1)J_{01} + (b + 1)J_{23} + J_{02} - J_{13}$ |
| IIIa | 0 | 3 dim. to 0 with sign. $(+ -)$ | $J_{23} - J_{03}$ |
| IIIb | 0 | 3 dim. to 0 with sign. $(+ -)$ | $J_{02} - J_{01}$ |

Table 1: Classification of SO(2,2)

| Type | Eigenvalues | Inv.subspace | Killingfield |
|------|-------------|--------------|--------------|
| Ia   | $\lambda, -\lambda, \lambda^*, -\lambda^*, 0$ | $\lambda = a + ib, a \neq 0, b \neq 0$ | $b(J_{01} + J_{23}) - a(J_{03} + J_{12})$ |
| Ib   | $a_1, -a_1, a_2, -a_2, 0$ | | $a_1J_{12} + a_2J_{03}$ |
| Ic   | $ib_1, -ib_1, ib_2, -ib_2, 0$ | | $b_1J_{01} + b_2J_{23}$ |
| Id   | $a, -a, ib, -ib, 0$ | | $aJ_{03} + bJ_{24}$ |
| IIa  | $a, -a, 0$ | 2 dim. to $a$ 2 dim. to $-a$ | $a(J_{03} + J_{12}) + J_{01} - J_{02} - J_{13} + J_{23}$ |
| IIb  | $ib, -ib, 0$ | 2 dim. to $ib$ 2 dim. to $-ib$ | $(b - 1)J_{01} + (b + 1)J_{23} + J_{02} - J_{13}$ |
| IIIa | $a, -a, 0$ | 3 dim. to 0 with sign. $(+ -)$ | $-aJ_{14} + J_{23} - J_{03}$ |
| IIIb | $ib, -ib, 0$ | 3 dim. to 0 with sign. $(+ -)$ | $-bJ_{34} + J_{02} - J_{01}$ |
| V    | 0 | 5 dim. to 0 | $-J_{01} - J_{03} - J_{12} - J_{14} + J_{23} + J_{34}$ |

Table 2: Classification of SO(2,3)
an infinitely long cylinder whose surface \( \rho = 1 \) represents spatial infinity. Every constant time slice of this cylinder yields the Poincaré disk model of a negatively curved surface of infinite area, which among other things means that the geodesics in such slices are arcs of circles meeting the boundary \( \rho = 1 \) at right angles.

Because of our “unwinding” of the time parameter all timelike geodesics exhibit a periodicity of \( 2\pi \), generically spiraling through the cylinder. Each lightlike geodesic start and end in spatial infinity during a finite (coordinate) time interval. In particular, it takes a time \( \pi \) for a lightray to travel from one side of the cylinder, through the origin \( \rho = 0 \), to the opposite side. This behaviour is expected since, as is well known, adS-space fails to be globally hyperbolic; information keeps flowing in from infinity and there do not exist any Cauchy surfaces. Actually, it is this property of adS-space that renders possible the appearance of interesting causal structures in its quotient spaces.

In order to understand the 3+1 dimensional case we just have to think about each constant time slice of the cylinder as, not a Poincaré disk, but a Poincaré ball. The geodesics in this ball still consist of all arcs of circles meeting its boundary at right angles.

Now, at last, we are ready to investigate the causal structure of \([\text{adS}]_{/G_\Gamma}\) where the generator to \( \Gamma \) belongs to one of the possible Killing field types found in the last section. In order to make the procedure clear we will describe the first case in some detail, but then merely state the results together with some comments.

**Type I\( a \)**
Using equation (4) to write the Killing field from table 1 or 2 in terms of the embedding coordinates, we have

\[
\xi^\mu = (-aY - bV) \frac{\partial}{\partial U} + (-aX + bU) \frac{\partial}{\partial V} + (-aV + bY) \frac{\partial}{\partial X} + (-aU - bX) \frac{\partial}{\partial Y}
\]

for both 4 and 5 dimensions. Using equation (2) this gives the norm

\[
\xi^\mu \xi_\mu = (a^2 - b^2)(Z^2 + 1) - 4ab(YV - UX)
\]

from which we see that the Killing field is everywhere timelike when \( a = 0, b \neq 0 \), and

![Figure 1: The type \( I_a \) singularity surface (for \( a = b \)) is shown during a time interval \( \Delta t = \pi \) using the cylinder coordinates \( \rho, \phi \) and \( t \) introduced in the text. The boundary of the cylinder represents spatial infinity. The type \( I_a \) Killing field is timelike on one side of this coreset screw like surface, and spacelike on the other.](image-url)
everywhere spacelike when \( a \neq 0, b = 0 \). However, when both \( a \) and \( b \) are non-zero we get both a timelike and a spacelike region and thus a singularity surface \( \xi^\mu \xi_\mu = 0 \) in between, according to our terminology from section 2.

In figure 1 we have drawn this singularity in the special case \( a = b \) for 2+1 adS-space using the coordinates introduced above. It is a corkscrew shaped surface meeting the infinity in two lines spiraling around the cylinder; these are lightlike geodesics as they have to be according to the analysis in section 2. When \( a \neq b \) these lines remain the same, but the singularity between them then bulge in one or the other direction, making either the timelike or the spacelike region larger.

Since, in the part where \( \xi \) is spacelike, infinity is connected and remains for all \( t:\text{s} \), it is always possible for an observer to escape to it. Thus there does not exist an event horizon in this case. Since the singularity is timelike, it is visible everywhere and therefore there is no inner horizon either.

These results extend to the 3+1 dimensional case. Then the singularity in each Poincaré ball (that is, at each time \( t \)) consists of a surface again dividing it into one timelike and one spacelike part. This singularity only rotates in time around the axis \( \theta = 0 \), and hence, for the same reasons as before, there can be no horizons in this case either.

![Figure 2: The static BTZ black hole: Type I\(_b\) with \( a_1 = 0 \) (or \( a_2 = 0 \)). Regions of spacelike Killing field last only for finite coordinate time intervals \( \pi \). Such a region is bounded by spatial infinity and the singularity surface shown in the left adS-cylinder. The corresponding event horizon is depicted to the right.](image-url)
**Type Ib**

Let us consider the 2+1 dimensional case first. This choice for $\xi$ then yields the BTZ black hole. When either $a_1$ or $a_2$ equals zero we get the static non-rotating case, that is, only then the quotient space admits a timelike hypersurface orthogonal Killing field. The singularity surface—described by $U = Y$ if we put $a_1 = 0$—is depicted to the left in figure 2. We see that at the time symmetric moment $t = 0$ it consists of two opposite points on the border of the cylinder, that is, on the infinity. From these points it grows up forward and backward in time, at each instant consisting of two opposite geodesics. These grow larger and larger until they come together at $t = \pm \frac{\pi}{2}$. Therefore the relevant region $\xi^\mu \xi^\nu > 0$ only exists for a finite coordinate time interval $\Delta t = \pi$, and not all of it is visible from infinity. Thus, there exists an event horizon which turns out to be described by $V = X$. It is the same type of surface as the singularity, only rotated $\pi / 2$ around the symmetry axis of the cylinder and translated $\pi / 2$ in coordinate time, see the right cylinder in figure 2. However, it is not possible for an observer to see the singularity before he hits it, and hence there is no inner horizon.

When both $a_1$ and $a_2$ are non-zero one obtains the more general rotating BTZ black hole. As is seen from figure 3 the singularity surfaces do not meet each other at $t = \pm \pi / 2$.

![Figure 3: When both $a_1$ and $a_2$ are non-zero type Ib yields the rotating BTZ black hole. The singularity surface opens up at $t = \pm \frac{\pi}{2}$, and the spacelike Killing field region is not interrupted any longer at these coordinate times. As a result, there are not only event horizons but also inner horizons.](image)

![Figure 4: The rotating BTZ black hole as a time series of Poincaré disks. The shaded regions indicates where the Killing field is timelike and may be thought of as excluded from the spacetime. The event horizon is dashed, while the inner horizon is dotted.](image)
any longer; an observer sitting in the center of the cylinder never hits the singularity. Therefore the quotient space will contain not only event horizons, one for each connected part of infinity, but also inner horizons, one for each singularity surface. Since each connected part of infinity is the same as in the non-rotating case, each event horizon again will be the surface depicted to the right in figure 2. The inner horizons happens to look the same as the singularity surface in the non-rotating case, to the left in figure 2. For clarity, both the singularity and the two horizons are drawn in figure 4 in a time series beginning with the time symmetrical Poincaré disk at \( t = 0 \).

Let us now see how these results generalize to 3+1 dimensions. The singularity surface in the non-rotating case, together with the resulting event horizon, is depicted in the time series in figure 5. This series could be thought of as obtained simply by rotating the Poincaré disk at each instant for the 2+1 non-rotating case (figure 2) in such a way that the opposite singularity geodesics, or arcs of circles, instead becomes segments of spheres; the horizon becomes the cylindrical membrane in the figure, connecting these spheres. Note, however, that in the quotient space this cylindrical horizon actually is a torus, on account of the identification. Superficially all this seems very similar to the 2+1 case: The singularity surfaces again meet at \( t = \pm \pi/2 \), and the event horizon grows up from \( t = 0 \). But a closer analysis reveals some important differences. Most notably this black hole turns out to be non-static and to possess a non-trivial apparent horizon; these properties will be discussed further in the next section.

What is the 3+1 counterpart to the rotating BTZ black hole? As in the 2+1 case the singularity starts out at \( t = 0 \) from two oppositely lying points at infinity. The two surfaces growing up from these points move inwards as time passes until they reach their closest position at \( t = \pi/2 \), after which they recede from each other again. At this moment, depicted in figure 6, they touch in two points, lying opposite to each other at infinity. In 2+1 dimensions this “touching” is enough to make the infinity disconnected, giving rise to one event horizon for each part of it. Here, on account of the third space dimension, the infinity remains connected, and therefore there will be no event horizons. Thus, it is not possible to generalize the rotating BTZ-black hole to 3+1 dimensions, because the analogue does not contain any event horizons.

However, there is an inner horizon for each part of the singularity, which looks the same as the singularity surface in figure 5.

![Figure 5](image)

Figure 5: The 3+1 dimensional version of the non-rotating BTZ hole shown as a time series of Poincaré balls. The opposite spherical segments are the singularity, and the cylindrical membrane connecting these is the event horizon.
Figure 6: The Poincaré ball at $t = \pm \frac{\pi}{2}$ for the 3+1 version of the rotating BTZ hole. At this moment the singularity surfaces have reached their closest position, but there is still infinity left in between. This is in contrast to the situation in the 2+1 dimensional counterpart, the last Poincaré disk in the series of figure 4.

**Type Ic**
The singularity surface only exists when both $b_1$ and $b_2$ are non-zero and $|b_2| > |b_1|$. It turns out to be independent of coordinate time $t$, and therefore looks the same in every constant time Poincaré disk or ball. In 2+1 dimensions it simply is a circle centered around $\rho = 0$ in the Poincaré disk whose radius depends on $b_1$ and $b_2$. In 3+1 dimensions it is a cylinder centered around the axis $\theta = 0$ (or $\pi$) in the Poincaré ball. There do not exist any horizons in either case.

**Type Id**
This case only exists in 3+1 dimensions. For $b = 0$ it coincides with the non-rotating BTZ black hole (type $I_b$ with $a_1 = 0$). As in that case the singularity surfaces grows up from two oppositely lying points in the infinity at $t = 0$. However, for non-zero $b$, these surfaces do not come together in $t = \pi/2$; they merely touch at $\rho = 0$. In particular, the infinity is connected and exist for all times $t$, and hence there will be no event horizons. But there will be inner horizons, again of the same shape as the singularity in figure 5, one for each of the disconnected singularity surfaces.

**Type IIa**
When $a = 0$ this Killing field is null everywhere, and we can not talk about a singularity surface at all. For $a \neq 0$ it gives rise to what Bañados *et al.*\(^1\,\,\,2\) call “the extremal $J = M$ black hole”, because in their setting it appears as a natural limit of the general type $I_b$ black hole. However, actually there are no event horizons in this case since the infinity is connected and lasts forever in coordinate time. On the other hand, the very complicated looking singularity surface consists of disconnected parts, and so there will be inner horizons.

This terminology of calling something a black hole that actually is not may seem strange, but we remind the reader that this is the situation also for the asymptotically anti-de Sitter extremal Kerr black hole. This is not a black hole either, in the sense of the definition, due to the null infinity of adS-space being a “vertical line” (referring to its appearance in a Penrose diagram). Namely, in the extremal Kerr solution there is nothing that interrupts this vertical line infinity, and so there can be no event horizons. Likewise, one could argue that the usual extremal Kerr black hole, that is, the asymptotically flat one, is a real black hole only because of the structure of Minkowskian infinity, consisting of a past and a future null infinity. This makes the infinity for the extremal Kerr solution being a “zig-zag line” consisting of alternate future and past infinities, hence giving rise to event horizons.

Note, therefore, that this type provides an example of a case where the infinity structure
of adS-space makes it harder to form a black hole. This is contrary to the impression given by the fact that the BTZ-hole exists only in adS-background—its counterpart in Minkowski background, the Misner space, contains no horizons.

**Type IIb**
There are no horizons in this case either. In 2+1 dimensions, at each moment, the singularity is a loop in the Poincaré disk touching infinity at one point. This loop whirls around as time passes without changing its shape. When the third spatial dimension is included the loop becomes a rather complicated surface in the Poincaré ball. However, its time dependence still only consists of a “rigid rotation” around the axis $\theta = 0$ (or $\pi$) and thus, since the singularity has to be a timelike surface by the general arguments in section 2, it can not give rise to any horizons.

**Type IIIa**
For $a = 0$ this type exists for both 2+1 and 3+1 dimensions. In the former case it gives rise to the extremal $M = 0$ black hole in the terminology introduced by Banados et al [2]; the singularity is the null surface shown in figure 7. Actually, $\xi^\mu \xi_\mu > 0$ on both sides of this surface, but it has fix points (that is, it equals zero) along the dashed lines in the figure.

![Figure 7](image)

Figure 7: The 2+1 dimensional type IIIa singularity. The Killing field actually is spacelike on both sides of this surface, but it has fix points (that is, it equals zero) along the dashed lines in the figure.
of this surface, and therefore there is no region in the quotient space containing closed
timelike curves. However, it still makes sense to call it a singularity, because it consists of
closed null lines, and furthermore, $\xi$ has a line of fixpoints at $Y = 0$, $X = -U$ (the dashed
line in the figure) making the quotient space singular there (non-Hausdorff, to be precise).

It is always possible for an observer to escape to infinity, so there is no event horizon.
It is not possible for an observer to see the future part of the singularity before he hits it,
while the past part is visible from everywhere. Thus there is no inner horizon either.

The extremal $M = 0$ BTZ black hole easily generalizes to 3+1 dimensions. Just take
every constant time Poincaré disk in figure 7 and rotate it to obtain the Poincaré ball,
in such a way that the singularity becomes a segment of a sphere instead of an arc of a
circle. This sphere segment then bounces to-and-fro in the Poincaré ball in time. For the
same reasons as before there are no horizons.

The case $a \neq 0$ only exists in the 3+1 case. The singularity then consists of two
ellipsoidal segments bouncing around in the Poincaré ball. Again, there is no event horizon.
But since these segments, with a period of $\Delta t = \pi$, shrinks into two points at infinity, the
singularity consists of several disconnected parts. To each of them there is a corresponding
inner horizon, again with the shape of the singularity surface in figure 5.

Type IIIb
In the 2+1 case, that is, when $b = 0$, the Killing field is everywhere timelike and there is
closed timelike lines through every point in the quotient space. When $b \neq 0$ we get the
singularity surface illustrated in figure 8, which, despite its fancy appearance, does not
give rise to any horizons.

Type V
This 3+1 dimensional type does not give rise to any horizons either. The singularity could
be described as a sheet with a kink rocking up and down in the Poincaré ball.

5 A closer look at the 3+1 black hole

In the previous section we found and catalogized all possible causal structures in spacetimes
of the form $\text{[adS]/G}_\Gamma$. In the 3+1 dimensional case, although we did find some strange
looking singularities which in many cases gave rise to inner horizons, there was only one
case containing an event horizon: Type I b when either $a_1$ or $a_2$ vanishes. Actually, the
most surprising conclusion we could draw was a negative one: There does not exist any
3+1 version of the rotating BTZ black hole, because the third space dimension provides
extra infinity to which an observer always might escape. Thus, it is only the non-rotating
case that admits a 3+1 analogue. However, this analogue has some interesting properties
which it does not share with the 2+1 case. In this section we will investigate these new
features.

First, we may ask what are the symmetries of our black hole. When making the
identification $\Gamma = e^{\alpha \xi}$ in adS-space we lose all Killing fields that are not single valued
in the quotient space, that is, those fields that are not mapped into themselves under $\Gamma$.
This means that the remaining Killing fields will be exactly those which commute with $\xi$
[3]. Therefore $\xi$ itself obviously has to be a symmetry of the quotient space, and, since
both SO(2,2) and SO(2,3) have rank 2, at least one more Killing field should survive the
identification.

For our non-rotating black holes $\xi$ equals $J_{03}$, putting $a_1 = 0$ for definiteness. Consid-
Figure 8: The type $III_b$ ($b \neq 0$) singularity. This series show the singularity during an interval $\Delta t = \pi$. Having reached the position in the last picture the surface oscillates back again following this time series backwards.

ering the 2+1 case first, the only field commuting with $J_{03}$ apart from itself is $J_{12}$, as is easily shown from equation (4). This field is timelike for $X > V$, which is—recalling from section 4 that $X = V$ is the event horizon—precisely the exterior region. Furthermore, $J_{12}$ is hypersurface orthogonal, and thus the 2+1 case must be static, describing an eternal black hole. In particular, the area of the event horizon does not change with time, and can be shown to be $(\frac{\alpha^2}{\pi^2})^2$ (for $\Gamma = e^{\alpha_{03}}$).

The situation for the 3+1 black hole is different. It follows from equation (4) that the Killing fields commuting with $J_{03}$ apart from itself are $J_{12}$, $J_{14}$ and $J_{24}$, so these then are our symmetries. Of these $J_{12}$, $J_{14}$ and $J_{24}$ are generators of the 2+1 dimensional Lorentz group SO(1,2), something that is most easily seen from their explicit form (5). Hence, since the fourth symmetry $J_{03}$ commutes with all of them, the full symmetry group for our 3+1 black hole must be $SO(1,2) \times U(1)$.

The event horizon—depicted in figure 5 as the cylindrical membrane connecting the growing singularity surface segments—is given by $V^2 = X^2 + Z^2$. In contrast with the 2+1 case, there is no Killing field that is timelike everywhere in the exterior region

$$V^2 < X^2 + Z^2$$

Indeed, the only candidates would be of the form $J_{12} \cos \chi + J_{14} \sin \chi$, since $J_{03}$, the identification field itself, is by construction spacelike in the interesting region, whereas $J_{24}$ is a pure rotation, hence also spacelike. But this combination is timelike only where

$$|V| < |X \cos \chi + Z \sin \chi|$$
For a given $\chi$, this inequality does not cover the whole exterior region; only for $Z = X \tan \chi$ it is fulfilled all the way in to the event horizon. Hence, although the union of all regions for different values of $\chi$ covers precisely the exterior region, there do not exist any Killing field that by itself is timelike everywhere outside the event horizon, and we may conclude that the black hole is non-static. Equivalently, we could say that the horizon is not a Killing horizon, that is, its generators do not belong to one single Killing field.

Intuitively this may be seen already from figure 5. The circumference of the cylindrical horizon obviously increases in time, while its length must be the same as the length of the event horizon in the 2+1 case: $(\frac{a}{2\pi})^2$ and constant in time. Thus its area is increasing in time, and the spacetime has to be non-static. Moreover, the fact that the horizon is growing only in one of its directions shows that it must be shearing.

As noted already in the previous section the horizon actually is a torus on account of the identification, and the spacetime could be said to describe a growing toroidal black hole. The reader may feel suspicious against this non-trivial topology of the horizon. But actually, there is nothing else to expect, since the whole spacetime has topology $T^2 \times R$ and infinity itself is, at each instant, a torus.

The observation that the event horizon is growing lead us to one further question: Where is the apparent horizon? In the 2+1 dimensional BTZ-black hole this horizon coincides with the event horizon (as discussed in ref. [3] and [8]). This actually has to be the case for all static black holes. However, by a theorem of Hawking and Ellis [1] (carefully discussed and proved in ref. [10])—which states that the apparent horizon must be marginally trapped—it cannot coincide with a growing event horizon. Thus, our 3+1 black hole, despite its trivial nature of being locally anti-de Sitter, must contain a non-trivial apparent horizon. After defining some relevant concepts we will devote the rest of this section to finding its location.

First, a trapped surface is a spacelike surface such that, for each of its points, the two families of lightrays orthogonal to it converge in their future direction. Actually, there is nothing unusual with such a surface; a particularly clear example is provided by the intersection of two backward lightcones in Minkowski space: The two families of lightrays orthogonal to this intersection are nothing else than the lightcone generators themselves, which, of course, converge to the top of the cones.

The interesting situation occurs when the trapped surface is closed, that is, compact and without edge; such a surface is called a closed trapped surface. By the singularity theorems formulated and proved in the sixties by Hawking and Penrose (see e.g. ref. [11]) such surfaces signal the existence of a future singularity. Actually, in the proof of these theorems one assumes the existence, not of closed trapped surfaces, but of the slightly different concept of closed outer trapped surfaces. For these only the outgoing family of lightrays converge. Also, they have to be the boundary to a volume, something that actually is not required for a closed trapped surface. Finally, a closed marginally outer trapped surface is a closed outer trapped surface except that the outgoing family of lightrays has zero convergence.

Now, consider a foliation $\Sigma$ of the spacetime. For each slice $\Sigma_i$ in this foliation, the apparent horizon is defined as the boundary to the region in $\Sigma_i$ containing closed outer trapped surfaces. As already mentioned, the apparent horizon can be shown to be marginally outer trapped. Another well-known result is that it has to reside inside the black hole, that is, behind the event horizon.

Note that this definition, which is the conventional one, is strongly foliation dependent; the apparent horizons with respect to two different foliations do in general not give the
same 3-surface in spacetime. One could of course define an “invariant” apparent horizon as the boundary to the spacetime region containing all closed outer trapped surfaces that exist in the spacetime. However, we do not know of any theorems concerning such a horizon, and it would probably be a very difficult task to actually locate it in a given situation. Thus we will stick to the conventional definition.

The first thing to do then is to fix a suitable foliation. Our choice will be governed by the requirement that it should preserve all symmetries of the spacetime. This will actually make our choice unique, and in order to find it, note that three of the symmetries—\( J_{12}, J_{14}, \text{and} \ J_{24} \)—act as \( \text{SO}(1,2) \) on the \( \{V,X,Z\} \)-subspace of the embedding space \( \mathbb{R}^3 \), hence leaving the hyperboloids \( X^2 + Z^2 - V^2 = c \) invariant. The fourth symmetry \( J_{03} \) only affects the \( \{U,Y\} \)-plane, and therefore these hyperboloids are preserved by the whole symmetry group. For \( c < 0 \) they are spacelike and—recalling inequality (8) for the exterior region—cover precisely the interior of the black hole. This is enough, since we only expect to find closed trapped surfaces inside the black hole anyway.

In order to make the foliation explicit, let us parametrize the interior region with the so called Schwarzschild coordinates \( r, t, \varphi, \psi \) (due to the Schwarzschild like properties of the resulting metric (11)):

\[
\begin{align*}
X &= -\sqrt{1 - r^2} \sinh t \cos \varphi \\
Y &= r \sinh \psi \\
Z &= -\sqrt{1 - r^2} \sinh t \sin \varphi \\
U &= r \cosh \psi \\
V &= -\sqrt{1 - r^2} \cosh t
\end{align*}
\] (10)

Here \( r \in (0,1) \) acts as a timelike coordinate, whereas \( t \geq 0 \) acts as a radial one (!). Coordinates \( \varphi \) and \( \psi \) are angular variables, corresponding to the closed symmetries \( J_{24} \) and \( J_{03} \), respectively. As a result of the identification with \( \Gamma = e^{\alpha J_{03}} \), \( \psi \) takes values only between 0 and \( 2\pi \alpha \), these being identified. (These coordinates may be extended to the exterior region \( (r > 1) \) by changing sign inside and outside the square root in \( X, Z \) and \( V \), and making the exchange \( \sinh t \leftrightarrow \cosh t \), see e.g. ref. [2].) The metric becomes

\[
d s^2 = -\frac{1}{1 - r^2} d r^2 + \left(1 - r^2\right) d t^2 + r^2 d \varphi^2 + \left(1 - r^2\right) \sinh^2 t \, d \varphi^2
\] (11)

As the ordinary Schwarzschild metric this one is singular at \( r = 0 \) and \( r = 1 \). These values correspond to the singularity and the horizon, respectively. The symmetry preserving foliation found above is given simply by slices of constant \( r \).

Now then, how do we find the apparent horizon with respect to this constant \( r \) foliation? The first thing to note is that, since our slicing respects the spacetime symmetries—a property that makes it unique—the apparent horizon clearly has to do that too. But this means that it actually has to coincide with one of these constant \( r \) slices. The question is only for which value of \( r \) this happens.

In order to solve this problem we will use the method described for example in ref. [12]. First, introduce the lightlike coordinates \( u \) and \( v \) expressed in the Schwarzschild parameters \( r \) and \( t \) as

\[
\begin{align*}
u &= \arctan \left( e^{-t} \sqrt{\frac{1 - r}{1 + r}} \right) \\
v &= \arctan \left( e^t \sqrt{\frac{1 - r}{1 + r}} \right)
\end{align*}
\] (12)
Figure 9: A qualitative picture of the interior region clarifying the relation between the Schwarzschild coordinates $r$ and $t$, and the light-like ones $u$ and $v$ used in the text to find the apparent horizon.

Since $0 < r < 1$ and $t \geq 0$ we have that $0 < u < \frac{\pi}{4}$ and $0 < v < \frac{\pi}{2}$. Directly from equations (12) we also have that $v > u$. In figure 9 the meaning of $u$ and $v$, and their relation to $r$ and $t$, is clarified. The metric (11) goes into

$$
\begin{align*}
\frac{1}{\cos^2(u-v)}( -4dudv + \cos^2(u+v)d\psi^2 + \sin^2(u-v)d\varphi^2 )
\end{align*}
$$

(13)

Next, consider the lightlike surfaces of constant $u$. Their lightlike normal field is given by

$$
l^\alpha = -\nabla^\alpha u
$$

(14)

The minus sign makes the field future directed. This means that $l^\alpha$ could be viewed as the outward future directed lightlike normals of 2-surfaces of constant $r$ and $t$. These surfaces, being closed in $\psi$ and $\varphi$, are all toroidal, and, because of their constant radius $t$, symmetric with respect to these variables. We will find all such symmetric trapped surfaces first, and then argue that taking into account non-symmetric ones as well would not affect the resulting apparent horizon.

The divergence $\vartheta$ of the outward directed lightrays orthogonal to these constant $\{r, t\}$ surfaces are then given by

$$
\vartheta = \nabla_\alpha l^\alpha = -\nabla_\alpha \nabla^\alpha u
$$

(15)

But, to be honest, this statement does not follow only from the observation made above that $l^\alpha$ is the normal field to such surfaces. Namely, the divergence of lightrays starting out from a 2-surface should only depend on its extrinsic curvature, or in our case, how the normal field changes in the $\psi$ and $\varphi$ directions along the surface. But why does not the divergence (15) acquire contributions from the $u$ and $v$ directions as well? To understand this, note that the metric $\tilde{g}^{\alpha\beta}$ for an $\{u,v\}$-plane may be written

$$
\tilde{g}^{\alpha\beta} = f(u,v)(l^\alpha m^\beta + m^\alpha l^\beta)
$$

with $l^\alpha$ as in (14) and $m^\alpha = -\nabla^\alpha v$. Let $D_\alpha$ denote the covariant derivative with respect to this metric. Then

$$
D_\alpha l^\alpha = \tilde{g}^{\alpha\beta} D_\alpha l_\beta = f(u,v)(l^\alpha m^\beta + m^\alpha l^\beta) D_\alpha l_\beta
$$
But since \( l^\alpha \) is a gradient field \( D_\alpha l_\beta = D_\beta l_\alpha \), and we get
\[
D_\alpha l^\alpha = 2f(u,v)l^\alpha m^\beta D_\beta l_\alpha = 0
\]
where the last equality follows since \( l^\alpha \) is null. Thus the divergence of \( l^\alpha \) gets no contribution from the \( u \) and \( v \) directions.

Having clarified the meaning of equation (15) we now express it in terms of the Schwarzschild parameters \( r \) and \( t \) (a calculation which is facilitated by changing \( u \) and \( v \) for \( \tau = v + u \) and \( \zeta = v - u \)):
\[
(1 - r^2)(\sinh 2t - 2r \cosh 2t) = 2r^3 - \vartheta r \sqrt{1 - r^2} \sinh t(1 + (1 - r^2) \sinh^2 t) \tag{16}
\]
The divergence \( \vartheta \) of a given closed 2-surface of constant radius \( t \) in the slice \( r \) has to satisfy this relation. For this surface to be trapped \( \vartheta \) must be less than zero; \( \vartheta = 0 \) gives a marginally trapped surface. Note that \( \vartheta \leq 0 \) clearly implies that the right-hand side of equation (16) is positive. However, the left-hand side is negative for \( \frac{1}{2} < r < 1 \), since \( \cosh x > \sinh x \) for all \( x \). Thus there can be no trapped surfaces of this type for \( r > \frac{1}{2} \).

On the other hand, putting \( \vartheta = 0 \), we can solve equation (16) explicitly for the radius \( t \) as a function of \( r \):
\[
\cosh 2t = \frac{4r^4 - 3r^2 + 1}{(1 - 4r^2)(1 - r^2)} \tag{17}
\]
This has real solutions for \( 0 < r < \frac{1}{2} \), and as \( r \) approaches the upper limit \( \frac{1}{2} \), \( t \) becomes infinite. Thus, for each value of \( r \) between 0 and \( \frac{1}{2} \), there exists a marginally trapped surface of radius \( t \) given by this equation.

The two last paragraphs show that closed trapped surfaces of constant radius exist for all \( r < \frac{1}{2} \), but not for any larger \( r \). Now we will argue that this is true, not only for these \( \psi \) and \( \varphi \) symmetric surfaces, but for all closed trapped surfaces with respect to the \( r \)-foliation. Namely, suppose that there exists a non-symmetric closed surface \( \mathcal{T} \) for \( r > \frac{1}{2} \) whose divergence \( \vartheta_{\mathcal{T}} \leq 0 \). The radius \( t \) varies along \( \mathcal{T} \) as a function of \( \psi \) and \( \varphi \), \( t = t(\psi, \varphi) \). This function reaches its maximum value \( t_{\text{max}} \) for some angles \( \psi_0 \) and \( \varphi_0 \), \( t_{\text{max}} = t(\psi_0, \varphi_0) \). Now, consider the symmetrical closed surface whose constant radius equals \( t_{\text{max}} \). Clearly, this surface encloses \( \mathcal{T} \), touching it in \( (\psi_0, \varphi_0) \). Since it by construction lies outside \( \mathcal{T} \), its outward orthogonal lightrays in \( (\psi_0, \varphi_0) \) must converge at least as fast as the ones corresponding to \( \mathcal{T} \). Hence, its divergence \( \vartheta \leq \vartheta_{\mathcal{T}} \leq 0 \), and we would have found it in the analysis above concerning such symmetrical surfaces. But we did not, and we may conclude that there do not exist any closed trapped surfaces, symmetrical or non-symmetrical, for \( r > \frac{1}{2} \).

We have then shown that the apparent horizon with respect to the \( r \)-foliation lies at \( r = \frac{1}{2} \). In the Penrose-diagram in figure 10 we have drawn it together with the singularity and the event horizon. The \( \psi \) and \( \varphi \) directions are suppressed in this diagram, which means that each point actually represents a torus.

### 6 Some concluding remarks

In this paper we have classified and discussed the possible causal structures that can be obtained by identifying points in \( \text{adS} \)-space connected by one of its symmetries. The method we have used rests on the fact that the resulting spacetime may be written as a quotient space \( \text{adS}/G \), and therefore clearly should not depend on which points are considered...
as identified with which—only the symmetry $\Gamma$ matters while the choice of fundamental region is irrelevant. In particular, as we saw in section 2, in order to understand the causal behaviour it is sufficient to find the hypersurface in $\text{adS}$-space where the Killing field corresponding to $\Gamma$ is null.

Using these ideas we have generalized the different 2+1 dimensional BTZ black holes to 3+1 dimensions. While the causal properties of the two types of extremal cases—$M = 0$ and $M = J$ in the notation of Banados et al [2]—remains essentially unchanged as one extra dimension is added, the non-extremal ones change dramatically: The rotating black hole simply disappears, its event horizon being destroyed by the extra dimension, while the non-rotating one becomes non-static!

The 3+1 version of the non-rotating BTZ hole actually describes a growing black hole with a toroidal event horizon. Furthermore. as shown in the previous section, it possesses a non-trivial apparent horizon—non-trivial in the sense that it does not coincide with the event horizon, as is the case for example in the ordinary BTZ hole. Thus, it has almost all characteristics of a realistic black hole formed from gravitational collapse. Perhaps its most unphysical property is the torus topology of infinity. On the other hand, the other properties are largely due to precisely this toroidal nature. For example, the closed trapped surfaces found above could not be both closed and trapped was it not for their torus topology. This is because in $\text{adS}$-space all trapped surfaces are non-closed (either $\mathbb{R}^2$ or $\mathbb{R}^1 \times S^1$) and by means of identifications one could close these into toruses, but never—as would be more realistic for a trapped surface—into spheres.

Due to the, at the same time, simple and rich nature of this black hole—simple because of its metric being that of $\text{adS}$-space, and rich in the sense of describing a growing black hole with an apparent horizon—it may turn out to be valuable as a toy model in different areas. For example, it could play an important role in research concerning quantum field theory and black hole physics.

But we also think that our methods are of some interest in their own right. Usually one deduces the causal structure of a spacetime by studying its metric. Then one has to worry about a number of coordinate related problems, such as extendability of the spacetime or
whether a singularity is real or only signals a breakdown of coordinates. However, as we have shown here, for spacetimes of the form \([\mathcal{M}, g^{\mu\nu}] / \mathcal{G}_\Gamma\) (where the properties of \([\mathcal{M}, g^{\mu\nu}]\) are assumed to be known) it is not necessary to go through this type of considerations—one does not even have to write down an explicit metric.

However, our discussion has only concerned taking the quotient with groups \(\mathcal{G}_\Gamma\) generated by a single generator. What if \(\mathcal{G}_\Gamma\) is generated by two or more generators? Are the same methods still applicable? First, if \(\mathcal{G}_\Gamma\) is Abelian, generated by \(\Gamma_1, \Gamma_2\), etc., then \([\mathcal{M}, g^{\mu\nu}] / \mathcal{G}_\Gamma\) could be equivalently constructed as the quotient by one generator at a time, that is, as \((([\mathcal{M}, g^{\mu\nu}] / \mathcal{G}_{\Gamma_1}) / \mathcal{G}_{\Gamma_2})\) etc., because, since the generators commute, \(\Gamma_2\) will be a symmetry also of \((([\mathcal{M}, g^{\mu\nu}] / \mathcal{G}_{\Gamma_1})\). Our method could then be applied to one generator at a time, and thus extended to the case with two or more commuting identifications.

On the other hand, for non-Abelian groups \(\mathcal{G}_\Gamma\) this “quotient decomposition” does not work because if \(\Gamma_1\) and \(\Gamma_2\) are non-commuting, then \(\Gamma_2\) is not a symmetry of \([\mathcal{M}, g^{\mu\nu}] / \mathcal{G}_{\Gamma_1}\) (unless in exceptional cases, in which the argument of the previous paragraph goes through as before). Still, of course, \([\mathcal{M}, g^{\mu\nu}] / \mathcal{G}_\Gamma\) is well-defined, but the problem is that in the generic case it would not be clear how to define the singularity. The view adopted in this paper, that the singularity is the hypersurface beyond which the identification Killing field \(\xi\) is timelike does not make sense any more when there are more than one such field.

The reader may wonder why not the boundary to the region where at least one of the generators in \(\mathcal{G}_\Gamma\) are timelike would do as a definition for the singularity. Even if the consequences of such a definition surely could be analysed, remember that part of the motivation for defining the singularity as the boundary to the region with timelike (identification) Killing flow, was that every closed timelike curve in the quotient space actually must close behind that surface, and that the exclusion of this “timelike” region therefore guarantees the absence of closed timelike curves. But when several identifications are performed, then, even if each of the corresponding Killing fields are purely spacelike, the resulting spacetime may contain closed timelike curves anyway. (A striking example of this is the “Gott time machine”, see ref. [13].) For this reason, the motivation for our definition of the singularity partly disappears for two or more non-commuting identifications.

Another possible definition for such situations—more in accord with the one adopted in this paper—is the boundary to the smallest region (in some sense) that one would have to remove from spacetime in order to get rid of all closed timelike curves. But it is difficult to see how the concept of “smallest region” could be made precise, and even if it could, it is not clear whether such a definition would give a unique singularity.

We must conclude that it seems hard to construct any sensible definition when the generators are non-commuting, except in special cases. One such special case is when the identification Killing fields have fixpoints, which then give rise to singularities in the quotient space. Specific examples of this are provided by the multi black holes discussed by Brill [14] and Steif [15], or the related black wormholes studied in ref. [8]. Both of these constructions make use of two or more of the static BTZ-type Killing fields: Type \(I_b\) when \(a_1\) or \(a_2\) equals zero. However, because of what we have said here, we do not think it is possible to generalize these spacetimes to rotating multi black holes or wormholes with rotation, simply because the involved Killing fields would then not have any fixpoints (except at infinity) and it would not any longer be clear as to what the singularity is.
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