Relativistic particle, fluid and plasma mechanics coupled to gravity

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Abstract

In this introductory review article, we explore the special relativistic equations of particle motions and the consequent derivation of Einstein’s famous formula $E = mc^2$. Next, we study the special relativistic electromagnetic field equations and generalizations of Lorentz equations of motion for charged particles. We then introduce the special relativistic gravitational field as a symmetric second order tensor field. Particle motions in the presence of static gravity are explored which could be used to study planetary dynamics, revealing perihelion shifts. Next, we investigate the system of consisting of pressureless plasmas and neutral perfect fluids coupled to the gravitational field. In that arena, we derive the relativistic Euler equation. Finally, we investigate the relativistic dynamics of a perfect fluid plasma and extensions to viscous flow and derive the relativistic Navier-Stokes equation.

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1 Introduction

A century has elapsed since the momentous discovery of the special theory of relativity by Einstein [1]. This theory has inspired subsequent pursuit of the general theory of relativity as a novel theory of gravitation [2]. It has also led to the understanding of the relativistic wave equation of an electron [3]. In the last five decades, relativity has also inspired the gauge field theories of subatomic particle interactions which have spectacular experimental confirmations [4]. However, applications of relativity theory into applied mathematical problems are almost non-existent. We venture to write this review article mainly to attract the academic attentions of applied mathematicians to this fascinating branch of modern theoretical science.

There exist well established special relativistic particle mechanics, relativistic fluid mechanics and electrodynamics. In the special theory of relativity, spacetime is assumed to be a flat differentiable manifold. Since Einstein’s gravitational theory involves a curved pseudo-Riemannian manifold, special relativistic dynamics of various macroscopic systems usually must omit gravitation. This is generally valid as the gravitational field is extremely weak in comparison to the other force fields involved. However, a linearized version of Einstein’s theory of gravitation can be incorporated within the framework of special relativity. Our present review article aims at such a treatment of various dynamical systems. It is hoped that such a review will serve as a useful introduction to the field for practitioners of non relativistic fluid mechanics as well as those wishing to study weak-field gravitating particles, fluids and plasmas such as are found in various astrophysical systems. As mentioned above, it is written with applied mathematicians as the primary intended audience. However, we hope this review will be useful to the wider audience as well. No previous knowledge of special relativity or Minkowski tensors is assumed.

The electrodynamics and mechanics of relativistic continuous media, coupled to gravitation, has many interesting applications in astrophysics (for examples, see [5]) and other areas of general relativity and high-energy physics [6]. As well, the interested reader is referred to the books [7] and references therein.

In section-II, notations for vectors and tensors in three and four dimensions are laid out in a leisurely fashion. In the third section, particle mechanics (mainly special relativistic) is discussed in a nut-shell. We do derive Einstein’s famous equation $E = mc^2$ in this section.

In section-IV, we discuss Maxwell’s equations of electromagnetic fields. These equations are known to be already relativistic! However, Lorentz’s equation of motion requires some minor modification for the relativistic conversion.

In section-V, a special relativistic version of the gravitational field equations is investigated. It involves a symmetric second order tensor field in spacetime to represent the gravitational force. We then couple the gravitational field with: (i) an incoherent dust, (ii) an electrically charged dust (pressureless plasma), and (iii) a perfect fluid. In the following section we specialize furthermore to static gravitational fields. Planetary motions are investigated in the static field and an approximation to the famous perihelion shift is derived. Moreover, we also explore a perfect fluid in the presence of static external gravitation, deriving the relativistic Euler equation in the
process.

In the last section, we generalize preceding investigations to more complicated materials (plasmas with pressure and viscosity) and derive the relativistic Navier-Stokes equation. Furthermore, generalizations to curvilinear coordinates and orthonormal (or physical) components are also touched upon.

In deformable media, a stress tensor, \( \sigma_{ij}(x^1, x^2, x^3) \equiv \sigma_{ji}(x^1, x^2, x^3) \) indicate equilibrium conditions by the satisfaction of the equations \( \sum_{j=1}^{3} \frac{\partial \sigma_{ij}(x^1, x^2, x^3)}{\partial x^j} = 0 \). In the special relativistic generalization, an energy-momentum-stress tensor \( T_{\mu\nu}(x^1, x^2, x^3, x^4) \equiv T_{\nu\mu}(x^1, x^2, x^3, x^4) \) is introduced. Moreover, dynamical equilibrium of deformable bodies or fluids are characterized by conditions \( \sum_{\nu=1}^{4} \frac{\partial T_{\mu\nu}(x^1, x^2, x^3, x^4)}{\partial x^\nu} = 0 \). These equations are of utmost importance throughout the paper in deriving relativistic continuity equations and relativistic equations for streamlines.

### 2 Notations and conventions

The three-dimensional physical space is assumed to be Euclidean. A typical point in this space is denoted by \( x := (x^1, x^2, x^3) \in \mathbb{R}^3 \). We use mostly a Cartesian coordinate system.

The spacetime continuum, \( M_4 \), is assumed to be a flat differentiable manifold admitting Minkowskian coordinate systems \[8\][10]. (These are generalizations of Cartesian coordinates.) Relative to a Minkowski coordinate system, an idealized point event in \( M_4 \) can be mapped uniquely into the point \( x := (x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \). Here, \( (x^1, x^2, x^3) \) indicates the spatial coordinates in the Euclidean \( \mathbb{R}^3 \), whereas \( x^4 \) is the speed of light times the time coordinate, i.e. \( x^4 = ct \).

Roman indices are used for three-dimensional spatial components of vectors and tensors (the components in the Euclidean \( \mathbb{R}^3 \)). Greek indices are used for the four dimensional spacetime components of vectors and tensors. Einstein’s summation convention is followed for both Roman and Greek indices.

We shall furnish some simple examples. three-dimensional vectors are denoted by a **bold** letter. For example, the vector \( \mathbf{v} \) has components \( v^i; i \in \{1, 2, 3\} \). The Kronecker delta: \( \delta^i_j, \delta_{ij}, \delta^{ij} \) are all entries of the \( 3 \times 3 \) unit matrix \([I]_{3 \times 3}\). Therefore,

\[
\delta^i_j \equiv \delta_{ij} \equiv \delta^{ij} := \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}
\] (1)

The inner product of two (three-dimensional) vectors \( \mathbf{v}, \mathbf{w} \) and the length \( |\mathbf{v}| \) are given by

\[
\mathbf{v} \cdot \mathbf{w} = \delta_{ij}v^iw^j := \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij}v^iw^j \quad \text{and} \quad |\mathbf{v}| := +\sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\delta_{ij}v^iv^j} = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2}. \] (2a, 2b)
The totally anti-symmetric numerical (oriented) tensor $\epsilon_{ijk}$ is defined by the components:

$$
\epsilon_{ijk} := \begin{cases} 
+1 & \text{for } (ijk) \text{ an even permutation of (123)}, \\
-1 & \text{for } (ijk) \text{ an odd permutation of (123)}, \\
0 & \text{otherwise}.
\end{cases}
$$

We can express some familiar vector calculus notions with the help of $\epsilon_{ijk}$. For example,

$$
\epsilon_{ijk} v^j w^k = (v \times w)^i, 
$$

$$
\epsilon_{ijk} \frac{\partial B^j(x)}{\partial x^k} = - (\nabla \times B)^i.
$$

In the four-dimensional Minkowski spacetime, the metric tensor components are furnished by

$$
\begin{align*}
\delta_{\mu\nu} &\equiv d_{\nu\mu} := \begin{cases} 
\delta_{ij} & \text{for } \mu = i \text{ and } \nu = j, \\
-1 & \text{for } \mu = \nu = 4, \\
0 & \text{otherwise}.
\end{cases} \\
[d_{\mu\nu}] &= [d_{\nu\mu}]^{-1}, \\
d^{\mu\nu}d_{\nu\lambda} &= \delta_\lambda^\mu = d_{\lambda\nu}d^{\nu\mu}.
\end{align*}
$$

For the four-dimensional vector components, the lowering of indices is accomplished by:

$$
u_\alpha := d_{\alpha\beta} u^\beta,
$$

so that it follows:

$$
\begin{align*}
u_i &= d_{i\nu} u^\nu = \delta_{ij} u^j, \\
u_4 &= d_{4\nu} u^\nu = -u^4, \\
u^\alpha &= d^{\alpha\beta} u_\beta.
\end{align*}
$$

The four-dimensional inner product between two vectors is provided by:

$$
\begin{align*}
d_{\mu\nu} a^\mu b^\nu &= \delta_{ij} a^i b^j - a^4 b^4, \\
d_{\mu\nu} a^\mu a^\nu &= (a^1)^2 + (a^2)^2 + (a^3)^2 - (a^4)^2.
\end{align*}
$$

Vectors are characterized as timelike, spacelike and null by:

$$
\begin{align*}
d_{\alpha\beta} v^\alpha v^\beta < 0 & \text{ for a timelike vector, } \\
d_{\alpha\beta} v^\alpha v^\beta > 0 & \text{ for a spacelike vector, } \\
d_{\alpha\beta} v^\alpha v^\beta = 0 & \text{ for a null vector.}
\end{align*}
$$
3 Particle mechanics (Newtonian and Relativistic)

We shall start with a very brief review of Newtonian mechanics. For the sake of simplicity, we restrict ourselves to the case of a single point particle with mass \( m > 0 \).

Let the parameterized motion curve be given by

\[ x^i = X^i(t) \]  

(11)

where \( x^i \) are Cartesian coordinates of the Euclidean space \( E_3 \) and \( t \) is the time variable. Let the three components of the force vector be given by \( f^i(t, x, v) \), which are functions of seven real variables. The components \( v^i \) represent the velocity variables.

Newton’s equations of motion for a single particle are provided by the well known equations:

\[
m \frac{d^2 X^i(t)}{dt^2} = f^i(t, x, v)|_{x^i = X^i(t), v^i = dX^i/dt}
\]  

(12)

These equations imply that

\[
\frac{d}{dt} \left[ \frac{1}{2} m \delta_{ij} \frac{dX^i(t)}{dt} \frac{dX^j}{dt} \right] = \delta_{ij} \left[ v^i f^j(t, x, v) \right]|_{\cdots}
\]  

(13)

The above equations are physically interpreted as “the rate of increase of kinetic energy of the particle is equal to the rate of work performed by the external force”.

We note that Newton’s equations of motion \([13]\) remain unchanged in form (or covariant) under the coordinate transformations:

\[
\hat{x}^a = c^a + r^a_b x^b, \quad [R]_{3\times3} := [r^a_b], \quad [R]^T [R] = [I]_{3\times3}.
\]  

(14a) (14b) (14c)

The equation \([14c]\) defines an orthogonal matrix \([R]_{3\times3}\). The set of transformations \([14a]\) constitute the six parameter group \( \mathcal{IO}(3; \mathbb{R}) \), the isometry group of Euclidean three-space \( E_3 \).

Consider another transformation, namely a special Galilean transformation:

\[
\hat{x}^1 = x^1 - v^1 t, \\
\hat{x}^2 = x^2, \quad \hat{x}^3 = x^3, \\
\hat{t} = t.
\]  

(15)

The new (hatted) frame is moving with constant velocity \( v^1 \) along the \( x^1 \)-axis relative to the old frame. The Newtonian motion laws \([13]\) remain unchanged in form by the Galilean transformations \([15]\).

In the nineteenth century, Michelson and Morley performed some sophisticated experiments regarding light propagation in vacuum \([9]\). The startling outcome of their results was that the speed of light does not change due to any (constant)
motion of either the source or observer. Newton’s ideas of absolute space and absolute time (inherent in [12] and [15]) are incompatible with Michelson and Morley’s experimental findings [9]. In 1905 Einstein solved this puzzle by the revolutionary ideas that space and time are relative in regards to any motion [1]. However, a combined spacetime continuum, $M_4$, is still absolute. The appropriate generalization of the three-dimensional Cartesian coordinates are the four-dimensional Minkowskian coordinates [8] [10]. Moreover, the correct generalization of the transformations in (14a) are furnished by (figure 1):

\[
\hat{x}^\alpha = c^\alpha + l^\alpha_\beta x^\beta, \quad \text{(16a)}
\]
\[
[L]_{4\times 4} := [l^\alpha_\beta], \quad \text{(16b)}
\]
\[
[D]_{4\times 4} \equiv [d_{\mu\nu}], \quad \text{(16c)}
\]
\[
[L]^T [D] [L] = [D], \quad \text{(16d)}
\]
\[
[d^{\mu\nu}] := [D]^{-1} \equiv [D] = [d_{\mu\nu}]. \quad \text{(16e)}
\]

Figure 1: The inhomogeneous Lorentz transformation in spacetime.

A typical example of the above transformation (with $x^4 = ct$) is provided by:

\[
\begin{align*}
\hat{x}^1 &= x^1 - \left(\frac{v^1}{c}\right) x^4 = x^1 - v^1 t + O \left(\frac{(v^1)^2}{c^2}\right), \\
\hat{x}^2 &= x^2, \\
\hat{x}^3 &= x^3, \\
\hat{x}^4 &= x^4 - \left(\frac{v^4}{c}\right) x^1 = ct + O \left(\frac{v^1}{c^2}\right), \\
\hat{t} &= t + O \left(\frac{1}{c^2}\right).
\end{align*}
\]
We make the following comments on the above transformations:

(i) The equation (17) is called the Lorentz (or “boost”) transformation [8] [10].

(ii) It is the correct generalization of the Galilean transformation (15) for a moving frame.

(iii) The above transformation implies contraction of length measurements in the moving frame.

(iv) It can bring about the “slowing” of time measurements (time dilation) for a moving observer.

(v) The speed, \( |v| \) of the moving observer must be strictly less than \( c \), the speed of light.

The set of transformations in (16a) constitutes a continuous group known as the inhomogeneous Lorentz group or the Poincaré group. It is a ten parameter group denoted by \( IO(3,1;\mathbb{R}) \).

It follows from (16d) that \( \det [L] = \pm 1 \). Therefore, the inverse matrix exists and is denoted by

\[
[a^\alpha_\beta]_{4\times4} \equiv [A] := [L]^{-1},
\]

\[
l^\mu_\alpha a^\nu_\beta = a^\mu_\alpha l^\nu_\beta = \delta^\mu_\beta. \tag{18}
\]

Now we shall define Minkowskian tensor fields in the flat spacetime manifold \( M_4 \). These are defined by the transformation properties [8], [10], [11]:

\[
\hat{T}^{\alpha_1...\alpha_r}_{\beta_1...\beta_s}(\hat{x}) = l^{\alpha_1}_{\gamma_1}...l^{\alpha_r}_{\gamma_r} a^{\mu_1}_{\beta_1}...a^{\mu_s}_{\beta_s} T^{\gamma_1...\gamma_r}_{\mu_1...\mu_s}(x). \tag{19}
\]

Here, the coefficients \( l^{\alpha}_\gamma \), \( a^\mu_\beta \) are defined by (16a) and (18). The tensor fields in (19) are of order \( r+s \), where \( r \) is the contravariant order and \( s \) is the covariant order. Note that the right hand side of (19) condenses a sum of \( 4^{r+s} \) terms! Moreover, these tensor fields are assumed to be twice continuously differentiable. The restriction of the tensor fields on a parameterized curve in \( M_4 \) satisfies the transformation rules:

\[
\hat{T}^{\alpha_1...\alpha_r}_{\beta_1...\beta_s}(\hat{x})|_{\hat{x}=\hat{x}(\tau)} = l^{\alpha_1}_{\gamma_1}...l^{\alpha_r}_{\gamma_r} a^{\mu_1}_{\beta_1}...a^{\mu_s}_{\beta_s} T^{\gamma_1...\gamma_r}_{\mu_1...\mu_s}(x)|_{x=\chi(\tau)}. \tag{20}
\]

As simple examples, we consider the numerical second order tensors \( d_{\mu\nu} \), \( d^{\alpha\beta} \) in (16e) and (16f) respectively. By the rules (19) we deduce that

\[
\hat{d}_{\alpha\beta} = a^\mu_\alpha a_\beta^\nu d_{\mu\nu} = d_{\alpha\beta},
\]

\[
\hat{d}^{\alpha\beta} = l^\mu_\alpha l^\nu_\beta d^{\mu\nu} = d^{\alpha\beta}. \tag{21}
\]

These special tensor components retain their numerical values under the rules (20).

Now, let us consider a 0 + 0 order or scalar field \( W(x) \) which is twice differentiable. Further, let it satisfy the wave equation:

\[
\Box W(x) := d^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} = \left[ \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2} - \frac{1}{c^2} \frac{\partial^2}{(\partial t)^2} \right] W(\ldots) = 0. \tag{22}
\]
We can prove from (20) (which in this case reads $\hat{W}(\hat{x}) = W(x)$), (21), (16a) and the chain rule of differentiation that (22) implies:

$$\hat{\square} \hat{W}(\hat{x}) = \delta_{\alpha\beta} \partial^2 \hat{W}(\hat{x}) \partial^2 \hat{x} \partial^2 \hat{t} \partial^2 \hat{t} = \left[ \frac{1}{(\partial \hat{x})^2} \frac{\partial^2}{(\partial \hat{x})^2} + \frac{\partial^2}{(\partial \hat{x})^2} - \frac{1}{c^2} \frac{\partial^2}{(\partial \hat{t})^2} \right] \hat{W}(\hat{x}) = 0. \quad (23)$$

Since the speeds of wave propagation in (22) and (23) are both $c$, the speed of light, we conclude that the speed of such a wave remains invariant under any motion of an observer characterized by (17).

Now we discuss another important feature of the tensor field in (19) (or (20)). The four-dimensional tensor field equation

$$T^{\gamma_1...\gamma_r}_{\mu_1...\mu_s}(x) = 0 \quad (24)$$

hold if and only if the transformed components satisfy

$$\hat{T}^{\alpha_1...\alpha_r}_{\beta_1...\beta_s}(\hat{x}) = 0. \quad (25)$$

This statement physically signifies that a natural law expressible by the vanishing of a tensor field remains unaltered in any rotated, reflected or moving frame. The main mathematical postulate of the special theory of relativity is that the natural laws must be expressed as tensor field equations in spacetime.

Now we shall study the relativistic particle mechanics. For that purpose we have to introduce the exact definition of a parameterized curve in $M_4$ and its physical interpretation. It is easier to consider the corresponding parameterized curve in the coordinate space $\mathbb{R}^4$. See figure 2.

![Figure 2: A parametrized curve in spacetime.](image)

Let a differentiable parameterized curve into the Minkowskian coordinate space $\mathbb{R}^4$ be characterized be characterized by

$$x^\mu = \chi^\mu(u), \quad u_1 \leq u \leq u_2. \quad (26)$$
Here, the functions $\mathbf{X}^\#_\mu$ are assumed to be continuously twice differentiable. Suppose that the curve represents physically the history of an idealized “point” clock moving in spacetime. The proper (or actual) time flow of the clock along the motion curve is given by \[8\], \[10\]

$$s = S^\#(u) := \frac{1}{c} \int_{u_1}^{u} \sqrt{-d_{\alpha\beta} \frac{dX^\#_\alpha(w)}{dw} \frac{dX^\#_\beta(w)}{dw}} dw. \quad (27)$$

Here, we have tacitly assumed that

$$0 < \sqrt{-d_{\alpha\beta} \frac{dX^\#_\alpha(u)}{du} \frac{dX^\#_\beta(u)}{du}} \equiv c \frac{dS^\#(u)}{du} \quad (28)$$

for all $u \in [u_1, u_2]$. The suppositions above are based on physical principles. Consider for example a free particle in the spacetime. The history curve, or world line, of such a particle would be a straight line with some slope. An observer “traveling” with this particle, feeling no acceleration can justifiably claim that he or she is not moving but it is the surroundings which are moving. Therefore, to the observer, he or she is not moving in space but only in time. Generalizing this, a straight time-like world line therefore represents the appropriate time axis for an observer on this trajectory.

Physically, \[28\] implies that the curve is timelike and the actual speed along the curve, as measured by an observer whose time axis is given by the $x^4$ axis in figure 2, is always less than the speed of light.

The integral \[27\], which defines the proper time, $s$, is invariant (or scalar) with respect to transformations \[16a\]. Moreover, the integral \[27\] is invariant under any smooth reparameterization of the curve characterized by

$$x^\mu = \mathbf{X}^\#_\mu(u) = \mathbf{X}^\mu(y),$$
$$y = Y(u), \quad \frac{dY(u)}{du} \neq 0,$$
$$Y(u_1) := y_1 \leq y \leq y_2 := Y(u_2). \quad (29)$$

Chosing the parameter $y = s$, the proper time, we obtain from \[27\], \[28\] and \[29\] that

$$x^\mu = \mathbf{X}^\#_\mu(u) = \mathbf{X}^\mu(s), \quad s_1 = 0 \leq s \leq s_2, \quad (30a)$$
$$s = S(s) = \frac{1}{c} \int_{0}^{s} \sqrt{-d_{\alpha\beta} \frac{dX^\alpha(w)}{dw} \frac{dX^\beta(w)}{dw}} dw, \quad (30b)$$
$$1 = \frac{dS(s)}{ds} = \frac{1}{c} \sqrt{-d_{\alpha\beta} \frac{dX^\alpha(s)}{ds} \frac{dX^\beta(s)}{ds}}, \quad (30c)$$
$$d_{\alpha\beta} \frac{dX^\alpha(s)}{ds} \frac{dX^\beta(s)}{ds} \equiv -c^2 < 0. \quad (30d)$$

Here, $\frac{dX^\alpha(s)}{ds}$ are the four components of the relativistic velocity along the motion curve.
In case the motion curve \( x^\mu = \chi^\mu(s) \) is continuously twice differentiable, which we shall always assume, the differentiation of (30d) yields
\[
d_{\alpha\beta} \frac{d\chi^\alpha(s)}{ds} \frac{d^2\chi^\beta(s)}{ds^2} \equiv 0. \tag{31}\]

Therefore, in the proper time parametrization, the four-acceleration is always (Minkowskian) orthogonal to the four-velocity!

In case we reparameterize the curve by choosing \( y = t \), the usual or coordinate time, we derive from (27), (28) and (29) that
\[
x^\mu = \chi^\mu(u) =: \mathbf{X}^\mu(t), \quad x^i = \mathbf{X}^i(t) := ct, \tag{32a}\]
\[
s = \mathcal{S}(t) = \frac{1}{c} \int_{t_1}^t \sqrt{-d_{\alpha\beta} \frac{d\mathbf{X}^\alpha(w)}{dw} \frac{d\mathbf{X}^\beta(w)}{dw}} \, dw, \tag{32b}\]
\[
\frac{d\mathcal{S}(t)}{dt} = \frac{1}{c} \sqrt{-d_{\alpha\beta} \frac{d\mathbf{X}^\alpha(t)}{dt} \frac{d\mathbf{X}^\beta(t)}{dt}} > 0, \tag{32c}\]
\[
\left[ \frac{d\mathcal{S}(t)}{dt} \right]^2 = 1 - \frac{1}{c^2} \left[ \delta_{ij} \frac{d\mathbf{X}^i(t)}{dt} \frac{d\mathbf{X}^j(t)}{dt} \right]. \tag{32d}\]

Recall that the Newtonian 3-velocity variables in (12) are
\[
v^i = V^i(t) := \frac{d\mathbf{X}^i(t)}{dt}, \quad |v|^2 = |\mathbf{V}(t)|^2 = \delta_{ij} V^i(t) V^j(t) \geq 0. \tag{33}\]

Thus, we deduce from (32d) and (33) that
\[
\left[ \frac{d\mathcal{S}(t)}{dt} \right]^2 = 1 - \frac{|\mathbf{V}(t)|^2}{c^2}, \quad 0 < \frac{d\mathcal{S}(t)}{dt} = + \sqrt{1 - \frac{|\mathbf{V}(t)|^2}{c^2}} \leq 1. \tag{34}\]

Now we investigate the relationships among Newtonian 3-velocity components with the corresponding relativistic 4-velocity components. Using the chain rule of differentiation, we obtain from (30a), (30b), (30c), (32a), (32b), (32c) and (31) that
\[
\frac{d\mathbf{X}^i(t)}{dt} = \frac{d\mathcal{S}(t)}{dt} \frac{dX^i(s)}{ds}, \tag{35a}\]
\[
\frac{dX^i(s)}{ds} = \frac{V^i(t)}{\sqrt{1 - \frac{|\mathbf{V}(t)|^2}{c^2}}}, \tag{35b}\]
\[
\frac{dX^4(s)}{ds} = \frac{1}{\left[ \frac{d\mathcal{S}(t)}{dt} \right] \frac{d\mathbf{X}^4(t)}{dt} = \frac{c}{\sqrt{1 - \frac{|\mathbf{V}(t)|^2}{c^2}}} > c. \tag{35c}\]
Now we shall generalize the Newtonian equations of motion (12) into the relativistic arena. We postulate, as a generalization of (12), the special relativistic equations of motion as [8], [10]:

\[
m \frac{d^2 \mathcal{X}^\alpha(s)}{ds^2} = \mathcal{F}^\alpha(x; u)|_{x^\alpha = \mathcal{X}^\alpha(s), \ u^\alpha = \frac{d\mathcal{X}^\alpha(s)}{ds}}.
\]  

(36)

Here, \((x) := (x^1, x^2, x^3, x^4)\) and \((u) := (u^1, u^2, u^3, u^4)\). Each of the four components \(\mathcal{F}^\alpha\) is a function of eight real variables and represents physically the relativistic force. (We have tacitly assumed that the mass \(m > 0\) remains unchanged along the motion curve in spacetime.) It follows from (31), (35b, 35c) and (36) that

\[
d_{\alpha\beta} \mathcal{F}^\alpha(...)|_{.} \frac{d\mathcal{X}^\beta(s)}{ds} \equiv 0,
\]

(37a)

\[
\mathcal{F}^4(...)|_{.} \equiv \delta_{ij} F^i(...) \frac{d\mathcal{X}^j(s)}{ds} \left[ \frac{d\mathcal{X}^4(s)}{ds} \right]^{-1} \equiv \delta_{ij} \mathcal{F}^i(...),
\]

(37b)

Now we shall compare the relativistic equations (36) with the Newtonian equations (12) and (13). Let us digress slightly. Suppose that we have a differentiable function \(f\) defined along the motion curve. By the equations (30a), (32a) and (34) and the chain rule, we get

\[
\overline{f}(t) := f[S(t)] = f(s),
\]

(38a)

\[
\frac{df(s)}{ds} = \frac{1}{\sqrt{1 - \frac{|V(t)|^2}{c^2}}} \frac{df(t)}{dt}.
\]

(38b)

Substituting (38b) into the first three equations in (36) we deduce that

\[
\frac{d}{dt} \left[ \frac{mV^i(t)}{\sqrt{1 - \frac{|V(t)|^2}{c^2}}} \right] = \sqrt{1 - \frac{|V(t)|^2}{c^2}} \mathcal{F}^i(...)
\]

(39a)

\[
m \frac{d^2 \mathcal{X}^i(t)}{dt^2} = \mathcal{F}^i(...) + \mathcal{O} \left( \frac{1}{c^2} \right).
\]

(39b)

Comparing (39a) with (12) we conclude that the Newtonian momentum components, \(mV^i(t)\) have to be modified into \(mV^i(t)/\sqrt{1 - \frac{|V(t)|^2}{c^2}}\) in relativity. Moreover, the Newtonian force components relate to relativistic force components by the equations:

\[
f^i(t, x, v) \bigg|_{x^i = \mathcal{X}(t), v^i = \frac{d\mathcal{X}(t)}{dt}} = \sqrt{1 - \frac{|V(t)|^2}{c^2}} \mathcal{F}^i(x, u) \bigg|_{x^4 = ct, u^4 = v^4 / \sqrt{1 - \frac{|V(t)|^2}{c^2}}}.
\]

(40)
The fourth equation in (36) yields, with (35c), (37b), (38b) and (40)

\[ m \frac{d}{dt} \left[ \frac{c}{\sqrt{1 - \frac{|V(t)|^2}{c^2}}} \right] = \sqrt{1 - \frac{|V(t)|^2}{c^2}} f^4(\ldots), \quad (41a) \]

or, \[ \frac{d}{dt} \left[ \frac{mc^2}{\sqrt{1 - \frac{|V(t)|^2}{c^2}}} \right] = \delta_{ij} [v^i f^j(\ldots)] \ldots. \quad (41b) \]

Expanding (41b) for \(|V(t)|/c < 1\), we obtain that

\[ \frac{d}{dt} \left[ mc^2 + \frac{1}{2} m |V(t)|^2 + O \left( \frac{1}{c^2} \right) \right] = \delta_{ij} [v^i f^j(\ldots)] \ldots. \quad (42) \]

Comparing the above equation with the corresponding Newtonian equation (13), we conclude that the instantaneous energy \( E(|V(t)|) \) of the particle must be furnished by

\[ E(|V(t)|) = \frac{mc^2}{\sqrt{1 - \frac{|V(t)|^2}{c^2}}} = mc^2 + \frac{m}{2} |V(t)|^2 + O \left( \frac{1}{c^2} \right). \quad (43) \]

In the limit \(|V| \to 0\), we derive that

\[ E(0) = mc^2. \quad (44) \]

The above equation, which is the most famous formula of modern science, reveals the enormous rest energy of a massive particle. (The rest energy associated with a single 1 kilogram object is approximately \(9 \times 10^{16}\) Joules, enough to meet New York City’s energy requirements for more than seven months!)

### 4 Electromagnetic fields

Maxwell’s equations of electromagnetic fields are, in Gaussian units, the following:

\[ c \epsilon_{ijk} \frac{\partial B^j(x, t)}{\partial x^k} = -4\pi j_i(x, t) - \frac{\partial E_i(x, t)}{\partial t}, \quad (45a) \]

\[ \frac{\partial E_i(x, t)}{\partial x^i} = 4\pi \sigma(x, t), \quad (45b) \]

\[ \frac{\partial B_i(x, t)}{\partial x^i} = 0, \quad (45c) \]

\[ c \epsilon_{ijk} \frac{\partial E^j(x, t)}{\partial x^k} = \frac{\partial B^i(x, t)}{\partial t}, \quad (45d) \]

Here, \( E^i(\ldots) \) and \( B^i(\ldots) \) stand for the electric and magnetic field components respectively. Also, The charge density and the current density components are denoted by \( \sigma(\ldots) \) and \( j_i(\ldots) \) respectively.
Another popular system of units, which we shall *not* employ here, is the Systeme Internationale (SI). In these units, Maxwell’s equations read:

\[
\begin{align*}
\epsilon_{ijk} \frac{\partial B^j(x, t)}{\partial x^k} &= -\mu_0 j_i(x, t) - \frac{1}{c^2} \frac{\partial E_i(x, t)}{\partial t}, \\
\frac{\partial E_i(x, t)}{\partial x^i} &= \frac{\sigma(x, t)}{\epsilon_0}, \\
\frac{\partial B^i(x, t)}{\partial x^i} &= 0, \\
\epsilon_{ijk} \frac{\partial E_j(x, t)}{\partial x^k} &= \frac{\partial B^i(x, t)}{\partial t}.
\end{align*}
\]

(46a)

The constants \(\epsilon_0\) and \(\mu_0\) respectively represent the permittivity and permeability of free space with the relation \(c = 1/\sqrt{\epsilon_0\mu_0}\).

The equations (45a - 45d) imply that the charge current conservation equation:

\[
\frac{\partial j^i(x, t)}{\partial x^i} + \frac{\partial \sigma(x, t)}{\partial t} = 0 \quad (47)
\]

must be satisfied.

The energy density of the electromagnetic field is characterized by

\[
u(x, t) := \frac{1}{8\pi} \delta_{ij} \left[ E_i(\ldots) E^j(\ldots) + B^i(\ldots) B^j(\ldots) \right].
\]

(48)

The momentum density of the electromagnetic field is provided by

\[
S_i(x, t) := \frac{1}{4\pi} \epsilon_{ijk} E^j(x, t) B^k(x, t)
\]

(49)

The corresponding vector field \(S(x, t)\) is also known as the Poynting vector.

Maxwell’s *electromagnetic stress tensor* is furnished by

\[
M_{ij}(x, t) = \frac{1}{4\pi} \left\{ \left[ E_i(x, t) E_j(x, t) - \frac{1}{2} \delta_{ij} \delta^{kl} E_k(x, t) E_l(x, t) \right] \\
+ \left[ B_i(x, t) B_j(x, t) + \frac{1}{2} \delta_{ij} \delta^{kl} B_k(x, t) B_l(x, t) \right] \right\}
\]

(50)

The components of the *Lorentz force* on a charged particle (of net charge \(e\)) is given by

\[
f_i(t, x, v) := e \left[ E_i(x, t) + \epsilon_{ijk} \frac{v^j}{c} B^k(x, t) \right].
\]

(51)

Note that this equation yields

\[
v^i f_i(t, x, v) = e E_i(x, t) v^i,
\]

(52)

indicating the well known result that the magnetic field makes no contribution to the rate of work.
Now, we shall obtain the special relativistic versions of the various electromagnetic equations. Following Minkowski [13], we define the four-dimensional electromagnetic field tensor components as:

\[
[F_{\mu\nu}(x)] := \begin{bmatrix}
0 & B_3(x) & -B_2(x) & E_1(x) \\
-B_3(x) & 0 & B_1(x) & E_2(x) \\
B_2(x) & -B_1(x) & 0 & E_3(x) \\
-E_1(x) & -E_2(x) & -E_3(x) & 0
\end{bmatrix} \equiv [-F_{\nu\mu}(x)].
\]

(53)

It should be mentioned that Minkowski first unified electric and magnetic fields by the definition (53) of the four-dimensional anti-symmetric tensor $F_{\mu\nu}(x)$. The above is one of several possible definitions for the tensor $F_{\mu\nu}$. We define the relativistic charge-current components by:

\[
J^i(x) := j^i(x, t),
\]

\[
J^4(x) := c \sigma(x, t).
\]

(54)

The Maxwell equations (45a-45d), along with definitions (53) and (54), neatly boil down to [12]

\[
\frac{\partial F^{\mu\nu}(x)}{\partial x^\nu} = \frac{4\pi}{c} J^\mu(x),
\]

(55a)

\[
\frac{\partial F^{\mu\nu}(x)}{\partial x^\lambda} + \frac{\partial F^{\nu\lambda}(x)}{\partial x^\mu} + \frac{\partial F^{\lambda\mu}(x)}{\partial x^\nu} = 0
\]

(55b)

\[
\frac{\partial J^\mu(x)}{\partial x^\nu} = 0.
\]

(55c)

Outside of charged matter, Maxwell’s equations are summarized by:

\[
\frac{\partial F^{\mu\nu}(x)}{\partial x^\nu} = 0,
\]

\[
\frac{\partial F^{\mu\nu}(x)}{\partial x^\lambda} + \frac{\partial F^{\nu\lambda}(x)}{\partial x^\mu} + \frac{\partial F^{\lambda\mu}(x)}{\partial x^\nu} = 0,
\]

(56)

which together imply

\[
\Box F_{\mu\nu}(x) = \left[ \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2} - \frac{1}{c^2} \frac{\partial^2}{(\partial t)^2} \right] F_{\mu\nu}(x) = 0.
\]

(57)

Since the above equations are tensor field equations in the four dimensional spacetime, Maxwell’s equations outside matter were already relativistic even before Einstein’s discovery of the special theory of relativity! Moreover, the equation (57) implies that electromagnetic waves propagate with the speed of light. Since the wave operator $\Box$ is a relativistic invariant (see equation (23)), we can conclude that the speed of electromagnetic wave propagation remains unchanged under the boost transformation (16a) characterizing a moving observer. Thus, Michelson and Morley’s experimental puzzle is logically explained.
Now we shall unify energy density, momentum density and Maxwell’s stress tensor by defining a relativistic electromagnetic energy-momentum-stress tensor:

\[ M^{\alpha \beta}(x) := \frac{1}{4\pi} \left[ F^{\lambda \alpha}(x)F^{\beta \lambda}(x) - \frac{1}{4}d^{\alpha \beta}F_{\mu \nu}(x)F^{\mu \nu}(x) \right] \equiv M^{\beta \alpha}, \quad (58a) \]

\[ \left[ M^{\alpha \beta}(x) \right] = \left[ -M^{ij}(x, t) \quad cS_i(x, t) \quad cS_j(x, t) \quad u(x, t) \right]. \quad (58b) \]

Working out the covariant divergence, \( \frac{\partial M^{\beta \alpha}}{\partial x^\beta} \), using \( (55a) \), \( (55b) \) and \( (58a) \) yields

\[ \frac{\partial M^{\beta \alpha}}{\partial x^\beta} = \frac{1}{4\pi} \left[ F^{\lambda \alpha}(x) \frac{\partial F^{\beta \lambda}(x)}{\partial x^\beta} + F^{\beta \lambda}(x) \frac{\partial F^{\alpha \lambda}(x)}{\partial x^\beta} - \frac{1}{2}F^{\mu \nu}(x)\frac{\partial F^{\mu \nu}(x)}{\partial x^\alpha} \right] \]

\[ = \frac{1}{c} F^{\lambda \alpha}J_\lambda + \frac{1}{8\pi} F^{\beta \lambda} \left[ \frac{\partial F^{\alpha \lambda}(x)}{\partial x^\beta} + \frac{\partial F^{\alpha \beta}(x)}{\partial x^\lambda} + \frac{\partial F^{\beta \lambda}(x)}{\partial x^\alpha} \right] \quad (59) \]

Now, the relativistic Lorentz equation for a charged particle with mass \( m \) and charge \( e \) are taken to be (compare with \( (36) \))

\[ m \frac{d^2x^\alpha(s)}{ds^2} = F^\alpha(x(s)) \frac{dx^\lambda(s)}{ds}, \quad (60a) \]

\[ \frac{d^2x^\alpha(s)}{ds^2} = e \left[ F^\alpha(x(s)) \frac{dx^\lambda(s)}{ds} - cF_{\alpha \lambda}(x(s)) \right] \equiv 0, \quad (60b) \]

\[ \sqrt{1 - \frac{\lvert V(t) \rvert^2}{c^2}}F_i(x(t)) = e \left[ F_{ij}(x, t)\frac{dx^j(t)}{dt} + cF_{14}(x(t)) \right] \]

\[ = e \left[ E_i(x, t) + \epsilon_{ijk}v^j cB^k(x, t) \right]. \quad (60c) \]

The right hand side of \( (60c) \) yields the correct Lorentz force given in equation \( (51) \).

Now we shall briefly introduce the electromagnetic four-potential. According to the converse Poincaré lemma \[14\], the equations \( (55b) \) imply that there exists relativistic field components \( A^\mu(x) \) of class \( C^3 \) such that

\[ F_{\mu \nu}(x) = \frac{\partial A_\nu(x)}{\partial x^\mu} - \frac{\partial A_\mu(x)}{\partial x^\nu}, \quad (61) \]

so that

\[ E_i(x, t) = \frac{\partial A_i}{\partial x^t} - \frac{1}{c} \frac{\partial A_i}{\partial t}, \quad (62a) \]

\[ B^i(x, t) = \frac{1}{2} \epsilon^{ijk} \left[ \frac{\partial A_j}{\partial x^k} - \frac{\partial A_k}{\partial x^j} \right]. \quad (62b) \]

We note that the four-potential components \( A_\mu(x) \) are not unique. We can make a (local) gauge transformation:

\[ A'_\mu(x) = A_\mu(x) - \frac{\partial \Lambda(x)}{\partial x^\mu}, \quad (63) \]

\[ F'_{\mu \nu}(x) \equiv F_{\mu \nu}(x). \]
Here, $\Lambda(x)$ is an arbitrary function of class $C^4$. We can perform a gauge transformation such that the function $\Lambda(x)$ satisfies the partial differential equation

$$\Box \Lambda(x) = \frac{\partial A^\mu}{\partial x^\mu}. \quad (64)$$

Thus, from equation (63) and (64), we get

$$\frac{\partial A'^\mu}{\partial x^\mu} = \frac{\partial A^\mu}{\partial x^\mu} - \Box \Lambda = 0. \quad (65)$$

The above condition on $A'^\mu$ is called the Lorentz gauge condition. In this gauge, Maxwell’s equations (55a - 55c) reduce, by (63) to

$$d^\mu \lambda \frac{\partial}{\partial x^\lambda} \left[ \frac{\partial A'^\nu}{\partial x^\nu} \right] - \Box A'^\mu = - \Box A'^\mu = 4\pi J^\mu(x) \quad (66a)$$

$$\frac{\partial J^\mu(x)}{\partial x^\mu} = 0. \quad (66b)$$

The relativistic inertial energy-momentum-stress tensor for the charged matter is given by:

$$[I^\mu_\nu(x)] := \begin{bmatrix} \rho(x)U^i(x)U^j(x) & \rho(x)U^i(x)U^4(x) \\ \rho(x)U^i(x)U^4(x) & \rho(x)(U^4(x))^2 \end{bmatrix} \equiv [I^\nu_\mu(x)], \quad (67a)$$

$$d^\mu_\nu U^\mu(x)U^\nu(x) \equiv -c^2. \quad (67b)$$

Here, $\rho(x)$ represents the invariant (or proper) mass density and $U^\mu(x)$ are components of the four-velocity field. Thus, $\rho(x)U^i(x)$ are related to the three-momentum density components and $\rho(x)[U^4(x)]^2$ is the energy density.

Observing the similarity between $M^{\alpha\beta}(x)$ in (58b) and $I^{\alpha\beta}(x)$ in (67a), we define the total energy-momentum stress field for a charged material as

$$T^{\alpha\beta}(x) := I^{\alpha\beta}(x) + M^{\alpha\beta}(x) \equiv I^{\beta\alpha}(x) + M^{\beta\alpha}(x) \equiv T^{\beta\alpha}(x), \quad (68)$$

Using (58a), (59), (67a-67b) and (68) we derive that:

$$\frac{\partial T^{\alpha\beta}(x)}{\partial x^\beta} = \rho u^\beta \frac{\partial u^\alpha}{\partial x^\beta} + u^\alpha \frac{\partial (\rho u^\beta)}{\partial x^\beta} + \frac{1}{c} F^\lambda_\alpha J^\lambda. \quad (69)$$

We postulate that a physical conservation law to hold:

$$\frac{\partial T^{\alpha\beta}(x)}{\partial x^\beta} = 0. \quad (70)$$

Moreover, we consider the case of a charged dust, i.e.

$$J^\lambda(x) = \sigma_0(x)U^\lambda(x). \quad (71)$$
Here, $\sigma_0(x)$ is the proper charge density (the charge density measured in the co-moving frame of the charge). The equation (69), with (67b), yields

$$\frac{\partial}{\partial x^\beta} \left[ \rho(x) U^\beta(x) \right] = 0, \quad (72a)$$

$$\rho U^\beta(x) \frac{\partial U^\alpha(x)}{\partial x^\beta} = \frac{\sigma_0(x)}{c} U^\lambda(x) F^\alpha_\lambda. \quad (72b)$$

The equations (72a) stands for the continuity of the material flow, whereas the equation (72b) represents the Lorentz equation (60a) for the charged dust.

Following the discussion on particle mechanics, we define the three-dimensional velocity field components by:

$$v^i = V^i(x, t) := c \frac{U^i(x)}{U^4(x)}, \quad U^4(x) = \frac{c}{\sqrt{1 - \frac{|V(x)|^2}{c^2}}},$$

$$v^i = V^i(x, t) = \sqrt{1 - \frac{|V(x, t)|^2}{c^2}} U^i(x), \quad \text{and} \quad J^4(x) = \sigma_0(x) U^4(x) = c \sigma(x). \quad (73)$$

The spatial components (72b) provide:

$$\rho \left\{ \frac{\partial}{\partial t} \left[ \frac{v^i}{\sqrt{1 - \frac{|V(x)|^2}{c^2}}} \right] + v^k \frac{\partial}{\partial x^k} \left[ \frac{v^i}{\sqrt{1 - \frac{|V(x)|^2}{c^2}}} \right] \right\} = \sigma_0 \left[ E^i + \epsilon^{ijk} v_j c B_k \right]. \quad (74)$$

Using notations of three-dimensional vector calculus, the equation (74) can be cast into the more familiar form:

$$\rho \left\{ \frac{\partial}{\partial t} \left[ \frac{V(x, t)}{\sqrt{1 - \frac{|V(x, t)|^2}{c^2}}} \right] + v \cdot \nabla \left[ \frac{V(x, t)}{\sqrt{1 - \frac{|V(x, t)|^2}{c^2}}} \right] \right\} = \sigma_0 \left[ E + \frac{v}{c} \times B \right]. \quad (75)$$

5 Special relativistic gravitational fields

5.1 Introduction

The static Newtonian gravitational potential, $W(x)$, satisfies:

$$\nabla^2 W(x) := \delta^{ij} \frac{\partial^2 W(x)}{\partial x^i \partial x^j} = \begin{cases} 4\pi G \rho(x) & \text{inside a material body,} \\ 0 & \text{in vacuum;} \end{cases} \quad (76a)$$

$$W(x) = -G \int_{\text{body}} \frac{\rho(x')}{|x - x'|} d^3 x', \quad (76b)$$

$$\rho(x') \geq 0, \quad W(x) \leq 0. \quad (76c)$$

Here, $G$ is the Newtonian constant of gravitation.
The well-known equations of motion of a particle of mass \( m > 0 \) in the external gravitational field \( W(x) \) are furnished by:

\[
\begin{align*}
  m \frac{d\mathbf{V}(t)}{dt} &= - m \nabla W(x)|_{\mathbf{x}(t)}, \quad (77a) \\
  \frac{dV^i(t)}{dt} &= - \frac{\partial W(x)}{\partial x^i} |_{\mathbf{x}(t)}. \quad (77b)
\end{align*}
\]

The natural relativistic generalization of the Newtonian potential is a four-dimensional scalar field. Such a field has been considered but the scalar field theory yields incorrect planetary orbits \([15]\). Next in order of complication, we should consider a four-dimensional vector field as in equation \((66a)\). However, material sources of such a field repel each other (like similar electric charges). Thus, vector fields are ruled out as viable candidates for a relativistic gravitational theory. Next in order of complication is the four-dimensional second order tensor field. This type of field is a potential candidate as the relativistic generalization of the mass density (within a factor of \( c^2 \)) is the energy density. As discussed in the previous section, the energy density is a component of a symmetric rank two tensor, the energy-momentum-stress tensor. Therefore, it is natural to consider a potential which is a symmetric second rank tensor field with the energy-momentum-stress tensor as its source. The first-order approximation of Einstein’s general relativity theory of gravity yields a second rank theory similar to that presented below. The field equations, along with supplementary conditions are summarized as the following:

\[
\begin{align*}
  \Box \phi_{\mu\nu}(x) &= - 2\kappa T_{\mu\nu}, \quad \kappa := \frac{8\pi G}{c^4}, \quad (78a) \\
  \frac{\partial T^\mu\nu(x)}{\partial x^\nu} &= 0, \quad (78b) \\
  \frac{\partial \phi^{\mu\nu}(x)}{\partial x^\nu} &= 0, \quad (78c) \\
  \phi_{\mu\nu}(x, x^4) &= \frac{2G}{c^4} \int_{\text{body}} T_{\mu\nu}(x', x^4 - |x - x'|) \frac{d^3x'}{|x - x'|}, \quad (78d)
\end{align*}
\]

Physically, the components of the tensor, \( T_{\mu\nu} \), represent the total density of energy, \( c \)-times momentum and stress of the source material. (A special example was furnished in the equation \((68)\).) The component \( T_{44}(x) \), \( T_{i4}(x) \) and \( T_{ij}(x) \) represent energy density, momentum density and stress (or rate of stress) density respectively. The equation \((78a-78c)\) are analogous to the electromagnetic equations \((66a - 66b)\) and \((65)\) respectively.

Now we shall express the equation of motion of a test particle of mass \( m > 0 \) in a gravitational field. The equations, though complicated, are aesthetically pleasing. Firstly, it is convenient to introduce a related second order symmetric tensor field by the definition:

\[
g_{\mu\nu}(x) := d_{\mu\nu} + \phi_{\mu\nu}(x) - \frac{1}{2} d_{\mu\nu} \phi^{\alpha\beta} \phi_{\alpha\beta}(x) \equiv g_{\nu\mu}(x), \quad (79)
\]
For weak gravitational fields, $|\phi_{\mu \nu}(x)| << 1$ and the determinant of $g_{\mu \nu}(x)$ is close to $-1$. Thus, there exists a unique inverse matrix defined by

$$[g^{\mu \nu}(x)] := [g_{\mu \nu}(x)]^{-1},$$

(80)

Treating formally $g_{\mu \nu}(x)$ as a “metric tensor”, we define the associated Christoffel symbols as

$$\{ \frac{\alpha}{\beta \gamma} \} := \frac{1}{2}g^{\alpha \lambda}(x) \left[ \frac{\partial g_{\gamma \lambda}(x)}{\partial x^\beta} + \frac{\partial g_{\lambda \beta}(x)}{\partial x^\gamma} - \frac{\partial g_{\beta \gamma}(x)}{\partial x^\lambda} \right] = \{ \frac{\alpha}{\gamma \beta} \}.$$  

(81)

In our linear theory, the above are components of a third order four-dimensional tensor with forty independent components!

We postulate that the special relativistic equations of motion, subject only to gravity, must follow geodesic paths of the “metric” $g_{\mu \nu}(x)$:

$$\frac{d^2X^\alpha(s)}{ds^2} + \left\{ \frac{\alpha}{\beta \gamma} \right\}_{|X(s)} \frac{dX^\beta(s)}{ds} \frac{dX^\gamma(s)}{ds} = 0.$$ 

(82)

Note that these equations of motion are independent of mass, a property in common with the Newtonian theory. These semi-linear equations distantly resemble the Lorentz equations (60a) and generalize Newton’s equations (77b) considerably. Geodesic equations (82) admit the exact first integral

$$g_{\mu \nu}(x)|_{X(s)} \frac{dX^\mu(s)}{ds} \frac{dX^\nu(s)}{ds} \equiv \text{constant}.$$ 

(83)

Since, in the limit $\phi_{\mu \nu}(x) \rightarrow 0$, the equations (83) reduce to (30d) and (67b), we must choose

$$g_{\mu \nu}(x)|_{X(s)} \frac{dX^\mu(s)}{ds} \frac{dX^\nu(s)}{ds} = -c^2.$$ 

(84)

The generalization of (82) in the presence of gravitational as well as non-gravitational force components $F^\alpha(x, u)$ is given by

$$m \frac{d^2X^\alpha(s)}{ds^2} = -m \left\{ \frac{\alpha}{\beta \gamma} \right\}_{|X(s)} \frac{dX^\beta(s)}{ds} \frac{d^2X^\gamma(s)}{ds} + F^\alpha(x, u)|_{X(s), dX(s)/ds},$$

(85a)

$$g_{\alpha \beta}(x)|_{X(s)} \frac{dX^\alpha(s)}{ds} F^\beta(x, u)|_{\_} = 0.$$ 

(85b)

(Compare the above equations with (36) and (37a)).

With the help of the “metric” tensor $g_{\mu \nu}(x)$, we can define the familiar co-
variant derivative \[11\] and the following consequences:

\[
\nabla_\alpha A_\beta (x) := \frac{\partial A_\beta (x)}{\partial x^\alpha} + \left\{ \begin{array}{c}
\beta \\
\alpha \gamma
\end{array} \right\} A^\gamma (x),
\]

(86a)

\[
\nabla_\alpha A_\beta (x) := \frac{\partial A_\beta (x)}{\partial x^\alpha} - \left\{ \begin{array}{c}
\gamma \\
\alpha \beta
\end{array} \right\} A_\gamma (x),
\]

(86b)

\[
\nabla_\alpha T_{\beta\gamma} (x) := \frac{\partial T_{\beta\gamma} (x)}{\partial x^\alpha} - \left\{ \begin{array}{c}
\lambda \\
\alpha \beta
\end{array} \right\} T_{\lambda\gamma} (x) - \left\{ \begin{array}{c}
\lambda \\
\alpha \gamma
\end{array} \right\} T_{\alpha\lambda} (x),
\]

(86c)

\[
\nabla_\alpha g_{\beta\gamma} (x) \equiv 0, \quad \nabla_\alpha g^{\beta\gamma} (x) \equiv 0, \quad \nabla_\alpha \delta_\beta^\gamma \equiv 0,
\]

(86d)

\[
\nabla_\alpha d_{\beta\gamma} \not\equiv 0, \quad \nabla_\alpha d^\beta_\gamma \not\equiv 0.
\]

(86e)

### 5.2 Gravitational field of an incoherent dust

The gravitational equations (78a - 78c) are all linear. Splitting the solution into its vacuum and inhomogeneous parts we express:

\[
\phi_{\mu\nu} (x) = \phi_{(I)\mu\nu} (x) + \phi_{(0)\mu\nu} (x),
\]

(87a)

\[
\Box \phi_{(I)\mu\nu} (x) = -2\kappa T_{\mu\nu} (x), \quad \frac{\partial T_{\mu\nu} (x)}{\partial x^\nu} = 0, \quad \frac{\partial \phi_{(I)\mu\nu} (x)}{\partial x^\nu} = 0,
\]

(87b)

\[
\Box \phi_{(0)\mu\nu} (x) = 0, \quad \frac{\partial \phi_{(0)\mu\nu} (x)}{\partial x^\nu} = 0,
\]

(87c)

\[
\phi_{\mu\nu} (x) = \phi_{(0)\mu\nu} (x) + \frac{2G}{c^4} \int_{\text{test body}} T_{\mu\nu} \left( x', x^4 - |x - x'| \right) \frac{1}{|x - x'|} d^3x'.
\]

(87d)

The above is the most general solution of the partial differential equations (78a - 78c). Here \( \phi_{(I)\mu\nu} (x) \) represents the particular solution due to the test body material with energy momentum stress tensor \( T_{\mu\nu} (x) \) and \( \phi_{(0)\mu\nu} (x) \) represents the vacuum solution created by any external sources (see figure (3)).

Ignoring the internal field of the test body, we modify (79) and (84) to

\[
g_{(0)\mu\nu} := d_{\mu\nu} + \phi_{(0)\mu\nu} (x) - \frac{1}{2} d_{\mu\gamma} d^{\alpha\beta} \phi_{(0)\alpha\beta} (x),
\]

(88a)

\[
g_{(0)\mu\nu} (x) U^\mu (x) U^\nu (x) \equiv -c^2,
\]

(88b)

\[
\tau_\mu := g_{(0)\mu\nu} u^\nu, \quad \tau_\mu u_\mu \equiv -c^2.
\]

(88c)

For an incoherent dust coupled to gravity, the energy-momentum-stress tensor is taken to be

\[
T^{\mu\nu} (x) := \rho (x) U^\mu (x) U^\nu (x) + \Theta^{\mu\nu} (x, \rho, u^\alpha, \phi_{(0)\gamma\sigma}),
\]

(89)

Here \( \rho (x) \) is the proper mass density, \( U^\mu (x) \) is the four velocity-field of the test particle and \( \Theta^{\mu\nu} (x, \rho, u^\alpha, \phi_{(0)\gamma\sigma}) \) indicates the interaction of the test body with the external gravitational field (from this point onward we will write this term as \( \Theta^{\mu\nu} (x) \)).
The conservation equation (78b) along with (89) yields:

\[
0 = \bar{\pi}_\mu \frac{\partial T^{\mu\nu}}{\partial x^\nu} = \bar{\pi}_\mu \left[ u^\mu \frac{\partial (\rho u^\nu)}{\partial x^\nu} + \rho u^\nu \frac{\partial u^\mu}{\partial x^\nu} + \frac{\partial \Theta^{\mu\nu}}{\partial x^\nu} \right]
\]
\[
= -c^2 \frac{\partial (\rho u^\nu)}{\partial x^\nu} + \rho u^\nu \bar{\pi}_\mu \left[ \frac{\partial u^\mu}{\partial x^\nu} + \left\{ \frac{\mu}{\nu} \lambda \right\}_{(0)} u^\lambda \right] + \bar{\pi}_\mu \left[ \frac{\partial \Theta^{\mu\nu}}{\partial x^\nu} - \rho \left\{ \frac{\lambda}{\nu} \lambda \right\}_{(0)} u^\nu u^\lambda \right].
\]

Note that the middle terms in the above equation vanishes via

\[
\bar{\pi}_\mu \left[ \frac{\partial u^\mu}{\partial x^\nu} + \left\{ \frac{\mu}{\nu} \lambda \right\}_{(0)} u^\lambda \right] = \bar{\pi}_\mu \nabla_{(0)\nu} u^\mu = \frac{1}{2} \nabla_{(0)\nu} \left[ g_{(0)\mu\nu} u^\mu u^\nu \right] = -\frac{1}{2} \nabla_{(0)\nu} (c^2) \equiv 0.
\]

At this stage there are several possible choices to enforce the conservation law (90). We stipulate the condition:

\[
\frac{\partial \Theta^{\mu\nu}}{\partial x^\nu} = \rho \left\{ \frac{\lambda}{\alpha} \beta \right\}_{(0)} u^\nu u^\lambda,
\]

as this choice yields the correct physical laws as dictated by the continuity equation (16):

\[
\frac{\partial}{\partial x^\nu} \left[ \rho(x) U^\nu(x) \right] = 0.
\]

The right hand side of (92) therefore represents the gravitational force components due to external sources.
With the help of equations (73), we can express (93) in a slightly more familiar form:
\[ \partial \frac{\partial}{\partial t} \left[ \frac{\rho}{\sqrt{1 - \frac{|v|^2}{c^2}}} \right] + \nabla \cdot \left[ \frac{\rho v}{\sqrt{1 - \frac{|v|^2}{c^2}}} \right] = 0. \] (94)

Substituting (92) and (93) into the conservation equation (78b) with (86a) and (89) we obtain
\[ u^\nu \left[ \partial_{x^\nu} u^\mu + \left\{ \begin{array}{c} \mu \\ \nu \\ \lambda \end{array} \right\} (0) u^\lambda \right] = u^\nu \nabla_{(0)\nu} u^\mu = 0. \] (95)

For the streamlines, we need to solve the system of first-order ordinary differential equations:
\[ \frac{dX^\mu(s)}{ds} = U^\alpha(x| \chi(s)). \] (96)
(We assume the validity of the Lifshitz conditions 17 on the right-hand-side of (95).) Using the above equation in (95) we derive that
\[ \frac{d^2X^\mu(s)}{ds^2} + \left\{ \begin{array}{c} \mu \\ \nu \end{array} \right\} (0) \frac{dX^\nu(s)}{ds} \frac{dX^\lambda(s)}{ds} = 0. \] (97)
Therefore, streamlines follow “geodesics” of the external metric \( g_{\mu\nu}(x) \). The Christoffel symbols mediate the external gravitational forces on a test particle.

5.3 Gravitational field of a charged dust

A fluid with vanishing pressure is known as a “dust”. In this case, the combined electromagnetic and gravitational field equations are investigated 18. Using (55a - 55c), (71), (58a), (67a) and (87a 87c), we obtain the pertinent equations to be:
\[ \frac{\partial F^{\mu\nu}}{\partial x^\nu} = 4\frac{\pi}{c} \sigma(0) u^\mu, \] (98a)
\[ \frac{\partial F^{\mu\nu}}{\partial x^\nu} + \frac{\partial F^{\nu\lambda}}{\partial x^\mu} + \frac{\partial F^{\lambda\mu}}{\partial x^\nu} = 0, \] (98b)
\[ \frac{\partial (\sigma(0) u^\nu)}{\partial x^\mu} = 0, \] (98c)
\[ \phi_{\alpha\beta}(x) = \phi_{(I)\alpha\beta}(x) + \phi_{(0)\alpha\beta}(x), \] (98d)
\[ \Box \phi_{(0)\alpha\beta} = \frac{\partial \phi_{(0)\alpha\beta}}{\partial x^\beta} = 0, \] (98e)
\[ \Box \phi_{(I)\alpha\beta} = -2\kappa T_{\alpha\beta} := -2\kappa [\rho u_\alpha u_\beta + \Theta_{\alpha\beta} + M_{\alpha\beta}], \] (98f)
\[ \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0, \] (98g)
\[ \frac{\partial \phi_{(I)\alpha\beta}}{\partial x^\beta} = 0. \] (98h)
Using (98f), (71) and (69), the conservation equation (98g) yields:

\[ 0 = \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = u^\alpha \frac{\partial (\rho u^\beta)}{\partial x^\beta} + \rho u^\beta \frac{\partial u^\alpha}{\partial x^\beta} + \frac{\partial \Theta^{\alpha\beta}}{\partial x^\beta} + \frac{\sigma(0)}{c} u^\beta F_\beta^\alpha. \] \hspace{1cm} (99)

Imposing the condition (92) on \( \Theta^{\alpha\beta}(x) \), the above equation (99) leads to

\[ u^\alpha \frac{\partial (\rho u^\beta)}{\partial x^\beta} + \rho u^\beta \nabla^{(0)} u^\alpha + \frac{\sigma(0)}{c} u^\beta F_\beta^\alpha = 0. \] \hspace{1cm} (100)

Now, the equations (86d) and (88b) provide:

\[ 2\pi_\alpha \nabla^{(0)} u^\alpha = g^{(0)\alpha\gamma} \left[ u^\gamma \nabla^{(0)} u^\alpha + u^\alpha \nabla^{(0)} u^\gamma \right] = \nabla^{(0)} \left[ g^{(0)\alpha\gamma} u^\alpha u^\gamma \right] = \nabla^{(0)} \left[ -c^2 \right] = 0. \] \hspace{1cm} (101)

\( (\nabla^{(0)} \) is the covariant derivative defined with respect to the metric \( g^{(0)\mu\nu} \), defined in (88a).) Therefore, (101), (90) and (100) yield

\[ \frac{\partial (\rho u^\beta)}{\partial x^\beta} = \frac{\sigma(0)}{c^2} \pi_\alpha u^\beta F_\beta^\alpha. \] \hspace{1cm} (102)

Substituting (102) into (100), we finally obtain

\[ \rho u^\beta \nabla^{(0)} u^\alpha + \frac{\sigma(0)}{c} u^\beta F_\beta^\gamma \left[ \delta^\gamma_\alpha + \frac{1}{c^2} \pi_\alpha u^\gamma \right] = 0. \] \hspace{1cm} (103)

Hence, the stream lines of a charged dust satisfying (96) must pursue trajectories governed by the equations of motion:

\[ \left\{ \rho(x) \left[ \frac{d^2 X^\alpha(s)}{ds^2} + \left\{ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right\} \frac{dX^\beta(s)}{ds} \frac{dX^\gamma(s)}{ds} \right] + \frac{\sigma(0)}{c} F_\beta^\gamma(x) \left[ \delta^\gamma_\alpha + \frac{1}{c^2} g(0)_{\gamma\mu} \frac{dX^\alpha(s)}{ds} \frac{dX^\mu(s)}{ds} \right] \frac{dX^\beta(s)}{ds} \right\} \bigg|_{x^\alpha = X^\alpha(s)} = 0. \] \hspace{1cm} (104)

These are the modified relativistic Lorentz equations of motion in the presence of external gravitational fields. (Compare with the equations (72b.) and (74).)
5.4 Gravitational field of a perfect fluid

Following the prescription in the previous sections, a special relativistic perfect fluid in a gravitational field is governed by

\[ \phi_{\alpha\beta}(x) = \phi^{(1)\alpha\beta}(x) + \phi^{(0)\alpha\beta}(x), \]

\[ \Box \phi^{(0)\alpha\beta}(x) = 0 = \frac{\partial \phi^{(0)\alpha\beta}}{\partial x^\beta}, \]

\[ \Box \phi^{(1)\alpha\beta}(x) = -2\kappa T^{\alpha\beta}(x) := -2\kappa \left[ (\rho(x) + \frac{p(x)}{c^2}) u^\alpha u^\beta + p(x) g^{\alpha\beta}(0) + \Theta^{\alpha\beta}(x) \right], \]

\[ \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0, \]

\[ \frac{\partial \Theta^{\alpha\beta}}{\partial x^\beta} = 0. \]

Here, \( p(x) \) is the pressure.

The conservation equation (105d) implies from (105c) that

\[ 0 = \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = u^\alpha \frac{\partial}{\partial x^\beta} \left[ (\rho + \frac{p}{c^2}) u^\beta \right] + \left( \rho + \frac{p}{c^2} \right) u^\beta \nabla^\alpha_{\beta} u^\alpha + \frac{\partial}{\partial x^\beta} \left[ p g^{\alpha\beta}(0) \right]
\]

\[ + \left[ \frac{\partial \Theta^{\alpha\beta}}{\partial x^\beta} - \left( \rho + \frac{p}{c^2} \right) \left\{ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right\}_{(0)} u^\beta u^\gamma \right]. \]

Following the previous section, reasonable physics demands that we stipulate

\[ \frac{\partial \Theta^{\alpha\beta}}{\partial x^\beta} = \left( \rho + \frac{p}{c^2} \right) \left\{ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right\}_{(0)} u^\beta u^\gamma. \]

Note that the right-hand-side denotes the external gravitational forces on the effective mass density \((\rho + \frac{p}{c^2})\). Now, the equation (106) yields

\[ u^\alpha \frac{\partial}{\partial x^\beta} \left[ (\rho + \frac{p}{c^2}) u^\beta \right] + \left( \rho + \frac{p}{c^2} \right) u^\beta \nabla^\alpha_{\beta} u^\alpha = -\frac{\partial}{\partial x^\beta} \left[ p g^{\alpha\beta}(0) \right]. \]

Contracting this equation with \( \tau_\alpha \) and using (88b) and (101), we deduce the continuity equation,

\[ \frac{\partial}{\partial x^\beta} \left[ (\rho + \frac{p}{c^2}) u^\beta \right] = \frac{1}{c^2} \tau_\alpha \frac{\partial}{\partial x^\beta} \left( p g^{\alpha\beta}(0) \right). \]

Using this equation in (108), we derive the relativistic Euler equation

\[ (\rho + \frac{p}{c^2}) u^\beta \nabla^\alpha_{\beta} u^\alpha = -\left[ \delta^\alpha_\gamma + \frac{\tau_\alpha u^\alpha}{c^2} \right] \frac{\partial}{\partial x^\beta} \left( p g^{\gamma\beta}(0) \right). \]
On a typical streamline (given by (96)) of the perfect fluid, the following equations of motion hold:

\[
\rho + \frac{p}{c^2} \left[ \frac{d^2 \chi^\alpha(s)}{ds^2} + \left\{ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right\}_{(0)} \frac{d \chi^\beta(s)}{ds} \frac{d \chi^\gamma(s)}{ds} \right]_{|x^\alpha = \chi^\alpha(s)} = - \left[ \delta_\alpha^\gamma + \frac{1}{c^2} g^{(0)\gamma\mu} \frac{d \chi^\alpha(s)}{ds} \frac{d \chi^\mu(s)}{ds} \right] \left\{ \frac{\partial}{\partial x^\beta} \left[ p g^{\gamma\beta}_{(0)} \right] \right\}_{|x^\alpha = \chi^\alpha(s)}. \tag{111}
\]

6 Static external gravitational fields

The special relativistic gravitational fields may easily be compared with Newtonian theory in this case. The relevant relativistic equations read:

\[
\begin{align*}
\phi_{(0)ij}(x) &\equiv 0, \quad \phi_{(0)i4}(x) \equiv 0, \tag{112a} \\
\phi_{(0)44}(x) &\neq 0, \quad \frac{\partial \phi_{(0)44}(x)}{\partial x^4} \equiv 0, \tag{112b} \\
d^{\alpha\beta} \phi_{(0)\alpha\beta}(x) &= - \phi_{(0)44}(x), \tag{112c} \\
g_{(0)ij}(x) &= \left[ 1 + \frac{1}{2} \phi_{(0)44}(x) \right] \delta_{ij}, \tag{112d} \\
g_{(0)i4}(x) &\equiv 0, \quad g_{(0)44}(x) = -1 + \frac{1}{2} \phi_{(0)44}(x). \tag{112e}
\end{align*}
\]

(We have made use of equations (88a) to derive the above.)

The equations (87c) reduce Laplace’s equation:

\[
\nabla^2 \phi_{(0)44}(x) = 0. \tag{113}
\]

Comparing (113) with (76a), (76b) and (87d) we identify \( \phi_{(0)44}(x) \) with the Newtonian gravitational potential via

\[
\phi_{(0)44}(x) = - \frac{4}{c^2} W(x) = \frac{4}{c^2} |W(x)| \geq 0. \tag{114}
\]

We also note that for a four-dimensional vector field \( T^\alpha(x) \), the equations (112d), (112e) and (114) yield

\[
- g_{(0)\alpha\beta}(x) T^\alpha(x) T^\beta(x) = \left[ 1 - \frac{1}{2} \phi_{(0)44}(x) \right] \left[ T^4(x) \right]^2 - \left[ 1 + \frac{1}{2} \phi_{(0)44}(x) \right] \delta_{ij} T^i(x) T^j(x)
\]

\[
= \left[ 1 - \frac{2|W(x)|}{c^2} \right] \left[ T^4(x) \right]^2 - \left[ 1 + \frac{2|W(x)|}{c^2} \right] \delta_{ij} T^i(x) T^j(x). \tag{115}
\]

We have implicitly assumed that \( 2 \frac{|W(x)|}{c^2} < 1 \). The physical components of the vector field \( T^\alpha(x) \) are

\[
\bar{T}^i(x) := \sqrt{1 + \frac{2|W(x)|}{c^2}} T^i(x),
\]

\[
\bar{T}^4(x) := \sqrt{1 - \frac{2|W(x)|}{c^2}} T^4(x). \tag{116}
\]
Notice that these components are just the components of $\mathcal{T}_\alpha(x)$ projected into the corresponding orthonormal coordinates (or frame) of the metric $g_{0\mu\nu}$.

6.1 Test particle motions in external static gravitational fields

In a previous section it was noted that any stream line of incoherent dust follows the “geodesic” equation (37). Along any of these time-like geodesics, the equation (84) implies that

$$g_{0\mu\nu}(x) \frac{d\mathcal{X}^\mu(s)}{ds} \frac{d\mathcal{X}^\nu(s)}{ds} \equiv -c^2. \quad (117)$$

The above condition modifies the equation (30d). For the sake of consistency, we must alter the definition (37) for the proper time along a time-like curve by:

$$s = S^\#(u) := \frac{1}{c} \int_{u_1}^{u} \sqrt{-g_{0\mu\nu}(x)|.\cdot g_{0\mu\nu}(x)|} \frac{d\mathcal{X}^\#(u)}{dw} \frac{d\mathcal{X}^\#(w)}{dw} dw, \quad (118a)$$

$$s = S(t) := \frac{1}{c} \int_{t_1}^{t} \sqrt{-g_{0\mu\nu}(x)|.\cdot g_{0\mu\nu}(x)|} \frac{d\mathcal{X}^\#(w)}{dw} \frac{d\mathcal{X}^\#(w)}{dw} dw, \quad (118b)$$

$$s = S(s) := \frac{1}{c} \int_{0}^{s} \sqrt{-g_{0\mu\nu}(x)|.\cdot g_{0\mu\nu}(x)|} \frac{d\mathcal{X}^\#(w)}{dw} \frac{d\mathcal{X}^\#(w)}{dw} dw. \quad (118c)$$

The equation (30d) is modified by (115) and (117) into

$$\left[1 + 2\left|W(x)\right| c^2 \right] \delta_{ij} \frac{d\mathcal{X}^i(s)}{ds} \frac{d\mathcal{X}^j(s)}{ds} - \left[1 - 2\left|W(x)\right| c^2 \right] \left[\frac{d\mathcal{X}^4(s)}{ds}\right]^2 \equiv -c^2 < 0, \quad (119)$$

while the equation (34) is changed by (119) into

$$\frac{ds}{dt} = \frac{dS(t)}{dt} = \sqrt{1 - 2\left|W(x)\right| c^2} - \left[1 + 2\left|W(x)\right| c^2 \right] \left|\mathbf{V}(t)\right|^2 c^2 \leq 1. \quad (120)$$

The above equation reveals the time dilation along a moving dust particle in an external gravitational field.

Similarly, the equations (35b) and (35c) change over into:

$$U^i(s) = \frac{d\mathcal{X}^i(s)}{ds} = \frac{V^i(t)}{\sqrt{1 - 2\left|W(x)\right| c^2} - \left[1 + 2\left|W(x)\right| c^2 \right] \left|\mathbf{V}(t)\right|^2 c^2}}, \quad (121a)$$

$$U^4(s) = \frac{d\mathcal{X}^4(s)}{ds} = \frac{c}{\sqrt{1 - 2\left|W(x)\right| c^2} - \left[1 + 2\left|W(x)\right| c^2 \right] \left|\mathbf{V}(t)\right|^2 c^2}},. \quad (121b)$$

Here, $V^i(t) = \frac{d\mathcal{X}^i(t)}{dt}$ are the components of the “coordinate” velocity which in general differ from the “measurable” velocity components, $\overline{V}^i(t) := \sqrt{1 + 2\left|W(x)\right| c^2} \frac{d\mathcal{X}^i(t)}{dt} |.\cdot$. 27
Now, the geodesic equations (97) are derivable from a variational principle [11]. A free particle’s motion is governed by a purely kinetic Lagrangian, \( \frac{1}{2} m \delta_{ij} V^i V^j \) in non-relativistic mechanics. The relativistic analogue is \( \frac{1}{2} m g_{\alpha\beta} u^\alpha u^\beta \). In the presence of gravity, the gravitational coupling arises from lowering the index with the metric \( g_{\alpha\beta} \) and therefore, using (112d), (112e) and (114) the Lagrangian (per unit mass) becomes

\[
L(x, u) := \frac{1}{2} \left[ 1 + 2 \frac{|W(x)|}{c^2} \right] \delta_{ij} u^i u^j - \frac{1}{2} \left[ 1 - 2 \frac{|W(x)|}{c^2} \right] (u^4)^2 = \frac{1}{2} \delta_{ij} u^i u^j - W(x) \left( \frac{u^4}{c} \right)^2 - \frac{(u^4)^2}{2} - \frac{W(x)}{c^2} \delta_{ij} u^i u^j. \tag{122}
\]

The relativistic Euler-Lagrange equations are given by

\[
\frac{\partial L(x, u)}{\partial x^\alpha} \bigg|_{x^\alpha = \chi^\alpha(s), u^\alpha = \frac{d\chi^\alpha}{ds}} - \frac{d}{ds} \left[ \frac{\partial L(x, u)}{\partial u^\alpha} \bigg|_{.} \right] = 0. \tag{123}
\]

The Newtonian Lagrangian for the corresponding Newtonian theory is

\[
L_N(x, v) := \frac{1}{2} \delta_{ij} v^i v^j - W(x), \tag{124}
\]

giving rise to the equations of motion (77b).

For small velocities, by the equations (121a), (122) matches (124) except for the term \(-\frac{(u^4)^2}{2}\). This term represents the large rest energy contribution which is manifest in the relativistic physics.

It will be instructive to investigate the fourth equation of (123). Since \( x^4 \) is a cyclic variable in the Lagrangian (122), the corresponding equation of motion admits the first integral:

\[
\frac{\partial L(x, u)}{\partial u^4} \bigg|_{.} = - \left[ 1 + \frac{2W(x)}{c^2} \right] U^4(s) = \text{const.}, \tag{125a}
\]

or,

\[
\frac{1}{c^2} \left[ 1 + \frac{2W(x)}{c^2} \right] \sqrt{1 + \frac{2W(x)}{c^2} - \frac{1}{c^2} \left[ 1 - \frac{2W(x)}{c^2} \right] \delta_{ij} V^i V^j} = \frac{\mathcal{E}}{c} = \text{const.} \tag{125b}
\]

For small velocities and weak gravitational fields, equation (125b) yields:

\[
\mathcal{E} = c^2 + \left[ \frac{1}{2} |V|^2 + W(x) \right] \bigg|_{.} + \mathcal{O} \left( \frac{1}{c^2} \right). \tag{126}
\]

It is not difficult to interpret the above equation. The constant \( \mathcal{E} \) represents the conserved total energy per unit mass. The first term on the RHS is the large rest energy of the unit mass. The second and third terms are the usual kinetic and potential energies respectively.
6.1.1 An example from planetary motion

Here we investigate planetary motions due to the spherically symmetric gravitational field of the sun. The usual potential function is furnished by:

\[ W(x) = \frac{-GM}{\sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2}}. \]  

(127)

with \( M > 0 \) being the solar mass. Employing spherical polar coordinates in space, the equations (122) and (127) yield the Lagrangian

\[ L(x, u) = \frac{1}{2} \left( 1 + \frac{2GM}{c^2 r} \right) \left[ (u^r)^2 + r^2 (u^\theta)^2 + r^2 \sin^2 \theta (u^\varphi)^2 \right] - \frac{1}{2} \left( 1 - \frac{2GM}{c^2 r} \right) (u^4)^2. \]  

(128)

The Euler-Lagrange equations (122), from (128) admit uniplanar motions characterized by:

\[ \theta = \Theta(s) = \frac{\pi}{2}. \]  

(129)

In this case, the conservation of energy equation (125a) reads:

\[ \left[ 1 - \frac{2GM}{c^2 r} \right] U^4(s) = \frac{E}{c}. \]  

(131)

The presence of another cyclic coordinate, \( \varphi \), leads to the conservation of angular momentum:

\[ \frac{\partial L(0)(..)}{\partial u_\varphi} \bigg|_{..} = \frac{r^2}{2} \left( 1 + \frac{2GM}{c^2 r} \right) u^\varphi \bigg|_{..} = h = \text{const.} \]  

(132)

Substituting (129), (131) and (132) into (119), we obtain

\[ \left[ \left( 1 + \frac{2GM}{c^2 r} \right) (u^r)^2 + \frac{h^2}{r^2 (1 + \frac{2GM}{c^2 r})} - \frac{E^2}{c^2 (1 - \frac{2GM}{c^2 r})} \right] \bigg|_{..} = -c^2. \]  

(133)

We reparameterize the curve by the following:

\[ r = \mathcal{R}(s) = \mathcal{R}(\varphi), \]  

(134)

\[ \frac{d\mathcal{R}(s)}{ds} = u^\varphi \bigg|_{..} = \frac{\mathcal{R}(\varphi)}{r^2 \left( 1 + \frac{2GM}{c^2 r} \right)} \frac{d\mathcal{R}(\varphi)}{d\varphi}. \]

It is useful at this point to make the following coordinate transformation:

\[ y = \frac{1}{r}, \quad y = Y(\phi), \]  

(135)

\[ y' := \frac{dY(\phi)}{d\varphi}. \]
With (134) and (135), the equation (133) reduces to
\[
\left[ 1 - \frac{2GMyc}{c^2} \right] \left[ (y')^2 + y^2 \right] + \frac{c^2}{h^2} \left( 1 - \frac{2GMyc}{c^2} \right) = \frac{E^2}{h^2c^2}. \tag{136}
\]

The above first-order equation can be solved by quadrature. However, to extract physically important effects, we differentiate the equation (136) to get
\[
y'' + y = \left( 1 + \frac{4GMyc}{c^2} \right) \left[ \frac{GM}{h^2} + \frac{2GMyc}{c^2} \left( (y')^2 + y^2 \right) \right] + O\left( \frac{1}{c^4} \right) \\
= \frac{GM}{h^2} + 4 \left( \frac{GM}{hc} \right)^2 y + \frac{2GMyc}{c^2} \left( (y')^2 + y^2 \right) + O\left( \frac{1}{c^4} \right). \tag{137}
\]

The equation (137) may be solved by the perturbative expansion:
\[
y = y_0 + \frac{y_1}{c^2} + \frac{y_2}{c^4} + ... \tag{138}
\]
Using this expansion in (137), we obtain
\[
y = y_0 + \frac{y_1}{c^2} + ... = \frac{GM}{h^2} \left[ 1 + e \cos(\varphi - \overline{\omega}) \right] + \frac{4(GM)^3}{c^2h^4} e \varphi \sin(\varphi - \overline{\omega}) + ... . \tag{139}
\]
Here, the constants of integration $e$ and $\overline{\omega}$ represent the eccentricity and perihelion angle of the orbit respectively. Combining the first two terms in (139), we conclude that
\[
y_0 + \frac{y_1}{c^2} = \frac{GM}{h^2} \left[ 1 + e \cos(\varphi - \overline{\omega} - \delta \overline{\omega}) \right] + (\text{higher order}) , \tag{140}
\]
where
\[
\delta \overline{\omega} := \arctan \left[ 4\varphi \left( \frac{GM}{h} \right)^2 \right] = 4 \left( \frac{GM}{h} \right)^2 \varphi + (\text{higher order}).
\]
Thus, the elliptic orbit precesses and the perihelion angle changes (see figure 4). This gravitational perihelion shift per revolution is given by
\[
\Delta \overline{\omega} = 8\pi \left( \frac{GM}{h} \right)^2. \tag{141}
\]
For the planet Mercury, the above amount yields a little over 57$''$ per century! However, the full non-linear theory of general relativity predicts the observed amount of almost exactly 43$''$ per century \footnote{The actual observed perihelion precession of Mercury is approximately 5600$''$ per century. When effects such the attraction due to the Newtonian gravitational field of the other planets are taken into account, along with the fact that the Earth is not an inertial frame of reference, a residual 43$''$ per century persists. The origin of this residual precession was a mystery until general relativity was formulated in 1915 \cite{2}.}.\footnote{1}
6.2 Perfect fluid in static gravity

In this case, the equations (112d), (112e) and (115) yield, for the metric tensor,

\[ g_{ij}(0) = \left[ 1 - \frac{2W(x)}{c^2} \right] \delta_{ij}, \quad g_{(0)44}(x) \equiv 0, \quad g_{(0)44}(x) = - \left[ 1 + \frac{2W(x)}{c^2} \right]^{-1}. \]  

The corresponding non-zero Christoffel symbols are provided by:

\[ \left\{ \begin{array}{l} \gamma^{ik} \\ j \end{array} \right\}_{(0)} = \frac{1}{c^2} \left[ 1 - \frac{2W(x)}{c^2} \right]^{-1} \delta^{il} \delta_{jk} \left( \frac{\partial W(x)}{\partial x^l} - \frac{\partial W(x)}{\partial x^j} - \frac{\partial W(x)}{\partial x^k} \right), \]  

\[ \left\{ \begin{array}{l} 4 \\ i \end{array} \right\}_{(0)} = \frac{1}{c^2} \left( \frac{\partial W(x)}{\partial x^i} \right), \quad \left\{ \begin{array}{l} 4 \\ 4 \\ i \end{array} \right\}_{(0)} = \frac{1}{c^2} \left[ 1 - \frac{2W(x)}{c^2} \right]^{-1} \delta_{ij} \frac{\partial W(x)}{\partial x^j}. \]  

The continuity equation (109) for a perfect fluid yields

\[ \frac{\partial}{\partial x^b} \left[ \left( \rho + \frac{p}{c^2} \right) u^b \right] + \frac{\partial}{\partial x^4} \left[ \left( \rho + \frac{p}{c^2} \right) u^4 \right] = \frac{1}{c^2} g_{(0)\alpha\gamma} u^\gamma \left[ \frac{\partial}{\partial x^b} \left( pg_{(0)\alpha} \right) + \frac{\partial}{\partial x^4} \left( pg_{(0)4} \right) \right]. \]  

Using (143), the equation (144) leads to

\[ \frac{\partial}{\partial x^b} \left[ \left( \rho + \frac{p}{c^2} \right) u^b \right] + \frac{\partial}{\partial x^4} \left[ \left( \rho + \frac{p}{c^2} \right) u^4 \right] = \frac{1}{c^2} \left[ 1 - \frac{2W(x)}{c^2} \right] u^b \frac{\partial}{\partial x^b} \left[ p \left( 1 - \frac{2W(x)}{c^2} \right)^{-1} \right]. \]
Substituting (121a) and (121b) into (145), we derive that

\[
\frac{\partial}{\partial x^b} \left[ \left( \rho + \frac{p}{c^2} \right) \frac{v^b}{\sqrt{1 + \frac{2W(x)}{c^2} - \left( 1 - \frac{2W(x)}{c^2} \right) \frac{|v|^2}{c^2}}} \right] + \frac{\partial}{\partial t} \left[ \frac{\left( \rho + \frac{p}{c^2} \right)}{\sqrt{1 + \frac{2W(x)}{c^2} - \left( 1 - \frac{2W(x)}{c^2} \right) \frac{|v|^2}{c^2}}} \frac{v^b}{c^2} \right] = 1 \frac{c^2}{c^2} \left[ 1 - \frac{2W(x)}{c^2} \right] \frac{v^b}{c^2} \partial_x^b \left( pg_{b\gamma}(0) \right) + \frac{\partial}{\partial x^4} \left( pg_{4\gamma}(0) \right) \right]_{..}.
\]

Expanding the above in powers of \( c^{-2} \), we can express

\[
\frac{\partial}{\partial x^b} \left[ \left( \rho + \frac{p}{c^2} \right) \left( 1 - \frac{W(x)}{c^2} + \frac{|v|^2}{2c^2} \right)^2 v^b \right] + \frac{\partial}{\partial t} \left[ \left( \rho + \frac{p}{c^2} \right) \left( 1 - \frac{W(x)}{c^2} + \frac{|v|^2}{2c^2} \right) \right] = 1 \frac{c^2}{c^2} \left[ 1 - \frac{2W(x)}{c^2} \right] \frac{v^b}{c^2} \partial_x^b + O \left( \frac{1}{c^4} \right).
\]

In more familiar notation the above equation reads

\[
\nabla \cdot [\rho \mathbf{v}] + \frac{\partial \rho}{\partial t} = \frac{1}{c^2} \left\{ -p \nabla \cdot \mathbf{v} + \nabla \cdot \left[ \rho \left( W(x) - \frac{|v|^2}{2} \right) \mathbf{v} \right] - \frac{\partial p}{\partial t} + \frac{\partial}{\partial t} \left[ \rho \left( W(x) - \frac{|v|^2}{2} \right) \right] \right\} + O \left( \frac{1}{c^4} \right).
\]

The relativistic correction terms are all collected on the right hand side of (148).

Now we shall investigate the equations of stream lines of a perfect fluid in the external gravitational fluid. Equations (111) provide

\[
\left( \rho + \frac{p}{c^2} \right) \left[ \frac{d^2X^i(s)}{ds^2} + \left\{ \begin{array}{c} i \\
\end{array} \right\} \frac{dX^j(s)}{ds} \frac{dX^k(s)}{ds} \right] + \left\{ \begin{array}{c} i \\
\end{array} \right\} \frac{dX^i(s)}{ds} \left( \rho \right) (0)^{2} = - \left[ \delta^i_k + \frac{1}{c^2} \left( g(0)_{\gamma j} \frac{dX^j(s)}{ds} + g(0)_{\gamma k} \right) \frac{dX^i(s)}{ds} \right] \left( \frac{\partial}{\partial x^b} \left( pg_{b\gamma}(0) \right) + \frac{\partial}{\partial x^4} \left( pg_{4\gamma}(0) \right) \right) \right]_{..}.
\]

(149)
We parameterize the curve by the rule:

\[
\frac{d}{ds} = \left[ \frac{dS(t)}{dt} \right]^{-1} \frac{d}{dt} = \frac{1}{\sqrt{1 + \frac{2W(x)}{c^2} - \left(1 - \frac{2W(x)}{c^2}\right) \frac{|v|^2}{c^2}}} \frac{d}{dt}.
\]

Then, from (150) and (151) the following is obtained:

\[
\left( \rho + \frac{p}{c^2} \right) \left\{ \frac{dV^i(t)}{dt} - \frac{V^i(t)}{2} \frac{d}{dt} \left[ \ln \left(1 + \frac{2W(x)}{c^2} - \left(1 - \frac{2W(x)}{c^2}\right) \frac{|v|^2}{c^2} \right) \right] \right\} + \frac{1}{c^2} \left(1 - \frac{2W(x)}{c^2}\right)^{-1} \delta^i_{jk} \frac{\partial V^j}{\partial x^k} - \delta^i_k \frac{\partial V^j}{\partial x^j} - \delta^i_j \frac{\partial V^k}{\partial x^k} V^j V^k + \left(1 - \frac{2W(x)}{c^2}\right)^{-1} \delta^i_{ij} \frac{\partial V^i}{\partial x^j} \right\} |.. \\
= - \left\{ \left[1 + \frac{2W(x)}{c^2} - \left(1 - \frac{2W(x)}{c^2}\right) \frac{|v|^2}{c^2} \right] \delta^i_k + \frac{1}{c^2} \left(1 - \frac{2W(x)}{c^2}\right) \delta_{kj} V^i V^j \right\} \times \left\{ \delta^{ik} \frac{\partial}{\partial x^a} \left[ p \left(1 - \frac{2W(x)}{c^2}\right)^{-1} \right] \right\} - \frac{1}{c^2} \left(1 + \frac{2W(x)}{c^2}\right) V^i \left\{ \frac{\partial}{\partial t} \left[ p \left(1 + \frac{2W(x)}{c^2}\right)^{-1} \right] \right\} |.. \right. \\
\]

Using (142) and (143) along with (149), we deduce that

\[
\left( \rho + \frac{p}{c^2} \right) \left[ \frac{d^2 \lambda^i(s)}{ds^2} + \frac{1}{c^2} \left(1 - \frac{2W(x)}{c^2}\right)^{-1} \left( \delta_{jk} \delta^i_l \frac{\partial W}{\partial x^l} - \delta^i_k \frac{\partial W}{\partial x^j} - \delta^i_j \frac{\partial W}{\partial x^k} \right) \frac{d \lambda^j(s) d \lambda^k(s)}{ds} \right] + \frac{1}{c^2} \left(1 - \frac{2W(x)}{c^2}\right)^{-1} \delta^i_{ij} \frac{\partial \lambda^i}{\partial x^j} \left( \frac{d \lambda^i}{ds} \right)^2 \right\} |.. \\
= - \left\{ \left[1 + \frac{2W(x)}{c^2} - \left(1 - \frac{2W(x)}{c^2}\right) \frac{|v|^2}{c^2} \right] \delta^i_k + \frac{1}{c^2} \left(1 - \frac{2W(x)}{c^2}\right) \delta_{kj} V^i V^j \right\} \times \left\{ \delta^{ik} \frac{\partial}{\partial x^a} \left[ p \left(1 - \frac{2W(x)}{c^2}\right)^{-1} \right] \right\} - \frac{1}{c^2} \left(1 + \frac{2W(x)}{c^2}\right) V^i \left\{ \frac{\partial}{\partial t} \left[ p \left(1 + \frac{2W(x)}{c^2}\right)^{-1} \right] \right\} |.. \right. \\
\]

Expanding in inverse powers of \(c^2\), we re-write the above equation of the stream-lines (152) as (19):

\[
\left\{ \rho(x) \left[ \frac{dV(t)}{dt} + \nabla W(x) \right] + \nabla p(x) \right\} |.. \\
= - \frac{1}{c^2} \left\{ \rho \left[ \left( V(t) \cdot \frac{dV(t)}{dt} - 3V(t) \cdot \nabla W \right) V(t) + (2W + |V(t)|^2) \nabla \right] W + p \left( \frac{dV(t)}{dt} + 3 \nabla W \right) + (4W - |V(t)|^2) \nabla p + \left( V(t) \cdot \nabla p + \frac{\partial p}{\partial t} \right) V(t) \right\} |.. + O \left( \frac{1}{c^4} \right). \\
\]

(153)
7 Generalizations to complicated materials and curvilinear coordinates

7.1 Perfect fluid plasma

In this section we maintain field equations (105a - 105e) for the fields $\phi(I)_{\alpha\beta}(x)$ and $\phi(0)_{\alpha\beta}(x)$. We also retain equations (88a) and (88c) for the exterior metric $g(0)_{\mu\nu}(x)$ and the normalized vector $u^\mu$. Moreover, we stipulate the same equation as (107) for the interacting energy momentum stress tensor $\Theta^{\alpha\beta}(x)$.

A charged fluid or plasma satisfies the electromagnetic field equations (55a-55c). The energy momentum stress tensor for this system is given by

$$T^{\alpha\beta}_{(nvp)}(x) := \left[ \rho(x) + \frac{p(x)}{c^2} \right] u^\alpha(x) u^\beta(x) + \left[ \rho(x) + \frac{p(x)}{c^2} \right] g^{\alpha\beta}(0)(x) + \Theta^{\alpha\beta}(x) + M^{\alpha\beta}(x), \tag{154}$$

where the subscript $(nvp)$ denotes “non-viscous plasma”. The quantity $M^{\alpha\beta}(x)$ is furnished by (58a). Therefore, equation (105d) implies that

$$0 = \frac{\partial T^{\alpha\beta}_{(nvp)}(x)}{\partial x^\beta} = u^\alpha \frac{\partial}{\partial x^\beta} \left[ \left( \rho + \frac{p}{c^2} \right) u^\beta \right] + \frac{\partial}{\partial x^\beta} \left( \rho + \frac{p}{c^2} \right) \nabla^\beta \left( \frac{u^\alpha}{c^2} \right) + \frac{\partial}{\partial x^\beta} \left( p g^{\alpha\beta}(0)(x) \right) + \frac{J^\beta}{c} F^\beta \tag{155}.$$

Here, we have used equation (107) for $\frac{\partial \Theta^{\alpha\beta}(x)}{\partial x^\beta}$.

Multiplying (155) by $\bar{u}_\alpha$ (and summing), we derive the plasma continuity equation:

$$\frac{\partial}{\partial x^\beta} \left[ \left( \rho + \frac{p}{c^2} \right) u^\beta \right] = \frac{\bar{u}_\alpha}{c^2} \left[ \frac{\partial}{\partial x^\beta} \left( p g^{\alpha\beta}(0)(x) \right) + \frac{J^\beta}{c} F^\beta \right]. \tag{156}$$

Substituting this last equation into (155), we deduce the generalized Euler equation

$$\left( \rho + \frac{p}{c^2} \right) u^\beta \nabla^\beta u^\alpha + \left( \delta^{\alpha}_{\gamma} + \frac{\bar{u}_\gamma u^\alpha}{c^2} \right) \left[ \frac{\partial}{\partial x^\beta} \left( p g^{\beta\gamma}(0)(x) \right) + \frac{J^\gamma}{c} F^\gamma \right] = 0. \tag{157}$$

From (157), the equations for a stream line emerges as

$$\left\{ \left[ \rho + \frac{p}{c^2} \right] \frac{d^2 x^\alpha(s)}{ds^2} + \left\{ \begin{array}{c} \alpha \\ \beta \gamma \end{array} \left( \begin{array}{c} d x^\beta(s) \\ d x^\gamma(s) \end{array} \right) \right\} \left( \begin{array}{c} \frac{d x^\beta(s)}{ds} \\ \frac{d x^\gamma(s)}{ds} \end{array} \right) \right\} \bigg|_{x^\alpha = \lambda^\alpha(s)} = - \left\{ \left[ \delta^{\alpha}_{\gamma} + \frac{g(0)_{\mu\gamma}}{c^2} \frac{d x^\mu(s) d x^\alpha(s)}{ds} \right] \left[ \frac{\partial}{\partial x^\beta} \left( p(x) g^{\beta\gamma}(0)(x) \right) + \frac{J^\gamma(x)}{c} F^\gamma(x) \right] \right\} \bigg|_{..}. \tag{158}$$
One may add viscosity to the above system. For such a fluid, the energy momentum stress tensor is

\[
T^{\alpha\beta}(x) := T^{\alpha\beta}_{\text{(nvp)}}(x) - \eta(x) \left\{ \left[ g^{\alpha\sigma}(x) + \frac{u^\alpha(x) u^\sigma(x)}{c^2} \right] \frac{\partial u^\beta(x)}{\partial x^\sigma} + \left[ g^{\beta\sigma}(x) + \frac{u^\beta(x) u^\sigma(x)}{c^2} \right] \frac{\partial u^\alpha(x)}{\partial x^\sigma} \right\} + \left[ \frac{2}{3} \eta(x) - \zeta(x) \right] \left[ g^{\alpha\beta}(x) + \frac{u^\alpha(x) u^\beta(x)}{c^2} \right] \frac{\partial u^\gamma(x)}{\partial x^\sigma}.
\] (159)

Here, \( \eta(x) \) and \( \zeta(x) \) represent the shear viscosity and bulk viscosity respectively. Using (105d), we obtain, from (159) the following equation

\[
0 = \frac{\partial T^{\alpha\beta}(x)}{\partial x^\beta} = \frac{\partial T^{\alpha\beta}_{\text{(nvp)}}(x)}{\partial x^\beta} \left\{ \eta \left[ \left( g^{\alpha\sigma}(x) + \frac{u^\alpha(x) u^\sigma(x)}{c^2} \right) \frac{\partial u^\beta(x)}{\partial x^\sigma} + \left( g^{\beta\sigma}(x) + \frac{u^\beta(x) u^\sigma(x)}{c^2} \right) \frac{\partial u^\alpha(x)}{\partial x^\sigma} \right] \right\} + \left( \zeta - \frac{2}{3} \eta \right) \left[ g^{\alpha\beta}(x) + \frac{u^\alpha(x) u^\beta(x)}{c^2} \right] \frac{\partial u^\gamma(x)}{\partial x^\sigma} \right\}. \]
(160)

Here we have again stipulated the equation (107).

Multiplying (160) by \( u^\alpha(x) \), we derive the continuity equation:

\[
\frac{\partial}{\partial x^\beta} \left[ \left( \rho + \frac{p}{c^2} \right) u^\beta \right] = \frac{\partial}{\partial x^\beta} \left[ \left( \rho + \frac{p}{c^2} \right) u^\beta \right] \left\{ \frac{\partial}{\partial x^\beta} \left( pg^{\alpha\beta}(0) \right) + J^{\beta}_{\gamma} F_{\beta}^{\alpha} \right\} - \frac{\partial}{\partial x^\beta} \left( \eta \left[ \left( g^{\alpha\sigma}(x) + \frac{u^\alpha(x) u^\sigma(x)}{c^2} \right) \frac{\partial u^\beta(x)}{\partial x^\sigma} + \left( g^{\beta\sigma}(x) + \frac{u^\beta(x) u^\sigma(x)}{c^2} \right) \frac{\partial u^\alpha(x)}{\partial x^\sigma} \right] \right\} + \left( \zeta - \frac{2}{3} \eta \right) \left[ g^{\alpha\beta}(x) + \frac{u^\alpha(x) u^\beta(x)}{c^2} \right] \frac{\partial u^\gamma(x)}{\partial x^\sigma} \right\}.
\] (161)

Substituting (161) into (160), we deduce the generalized Navier-Stokes equation:

\[
\left( \rho + \frac{p}{c^2} \right) u^\beta \nabla^\beta_{\alpha}(x) = \left( \delta^\beta_{\gamma} + \frac{\eta}{c^2} u^\alpha \right) \left\{ - \frac{\partial}{\partial x^\beta} \left( pg^{\gamma\beta}(0) \right) - J^{\beta}_{\gamma} F_{\beta}^{\alpha} \right\} + \frac{\partial}{\partial x^\beta} \left( \eta \left[ \left( g^{\gamma\sigma}(x) + \frac{u^\gamma(x) u^\sigma(x)}{c^2} \right) \frac{\partial u^\beta(x)}{\partial x^\sigma} + \left( g^{\beta\sigma}(x) + \frac{u^\beta(x) u^\sigma(x)}{c^2} \right) \frac{\partial u^\gamma(x)}{\partial x^\sigma} \right] \right\} + \left( \zeta - \frac{2}{3} \eta \right) \left[ g^{\gamma\beta}(x) + \frac{u^\gamma u^\beta(x)}{c^2} \right] \frac{\partial u^\gamma(x)}{\partial x^\sigma} \right\}.
\] (162)

### 7.2 Curvilinear coordinates and orthonormal frames

Now we shall introduce curvilinear spacetime coordinates by transformation equations:

\[
\hat{x}^\alpha = \hat{X}^\alpha(x),
\]

\[
\frac{\partial (\hat{x}^1, \hat{x}^2, \hat{x}^3, \hat{x}^4)}{\partial (x^1, x^2, x^3, x^4)} \neq 0,
\]

\[
x^\alpha = X^\alpha(\hat{x}).
\] (163)
Here, we have assumed that the functions $\hat{X}^{\alpha}$ are of class $C^2$. The transformation rules for tensor fields are furnished by

$$\hat{T}^{\alpha\beta}_{\mu\nu..}(\hat{x}) = \frac{\partial \hat{X}^{\alpha}(\hat{x})}{\partial x^\gamma} \frac{\partial \hat{X}^{\beta}(\hat{x})}{\partial x^\delta} \cdots \frac{\partial X^\rho(\hat{x})}{\partial \hat{x}^\mu} \frac{\partial X^\sigma(\hat{x})}{\partial \hat{x}^\nu} T^{\gamma\delta..}_{\rho\sigma..}(x). \quad (164)$$

The coordinate transformation (163) generate a new metric tensor field as follows:

$$\hat{g}_{\alpha\beta}(\hat{x}) := g_{\mu\nu} \frac{\partial X^\mu(\hat{x})}{\partial \hat{x}^\alpha} \frac{\partial X^\nu(\hat{x})}{\partial \hat{x}^\beta}, \quad \left[ \hat{g}^{\alpha\beta}(\hat{x}) \right] := \left[ g_{\alpha\beta}(\hat{x}) \right]^{-1}. \quad (165)$$

The Christoffel symbols are given by (compare with (81))

$$\Gamma^{\gamma\delta..}_{\rho\sigma..}(\hat{x}) := \frac{1}{2} \hat{g}^{\alpha\sigma}(\hat{x}) \left[ \frac{\partial \hat{g}_{\gamma\sigma}(\hat{x})}{\partial \hat{x}^\beta} + \frac{\partial \hat{g}_{\sigma\beta}(\hat{x})}{\partial \hat{x}^\gamma} - \frac{\partial \hat{g}_{\beta\gamma}(\hat{x})}{\partial \hat{x}^\sigma} \right]. \quad (166)$$

The covariant derivatives are defined by (compare with (86a - 86d))

$$\hat{\nabla}_\alpha \hat{T}^{\mu\nu..}_{\rho\sigma..}(\hat{x}) := \frac{\partial \hat{T}^{\mu\nu..}_{\rho\sigma..}(\hat{x})}{\partial \hat{x}^\alpha} + \left\{ \begin{array}{c} \mu \\ \alpha \beta \end{array} \right\} \hat{T}^{\mu\beta..}_{\rho\sigma..}(\hat{x}) + \left\{ \begin{array}{c} \nu \\ \alpha \beta \end{array} \right\} \hat{T}^{\mu\alpha..}_{\rho\sigma..}(\hat{x}) + \cdots - \left\{ \begin{array}{c} \beta \\ \alpha \rho \end{array} \right\} \hat{T}^{\mu\nu..}_{\beta\sigma..}(\hat{x}) - \left\{ \begin{array}{c} \beta \\ \alpha \sigma \end{array} \right\} \hat{T}^{\mu\nu..}_{\rho\beta..}(\hat{x}) - \cdots. \quad (167)$$

If we replace the various tensor fields $T^{\gamma\delta..}_{\rho\sigma..}(x)$ by $\hat{T}^{\alpha\beta}_{\mu\nu..}(\hat{x})$ (in (164)), and $\hat{\nabla}_\alpha \hat{T}^{\mu\nu..}_{\rho\sigma..}(\hat{x})$ (in (167)), then all the constitutive equations are expressed in curvilinear spacetime coordinates $\hat{x}^\alpha$.

However, for the sake of applications, we restrict ourselves to spatial curvilinear coordinates only. In that case the following restriction is placed on equations (163)

$$\hat{x}^i = \hat{X}^i(x), \quad \hat{x}^4 = \hat{X}^4(x, x^4) := x^4, \quad x^i = X^i(x). \quad (168)$$

The tensor transformation rules (163) then simplify in the obvious way:

$$\hat{T}^{ij..}_{kl4..}(\hat{x}, \hat{x}^4) = \frac{\partial \hat{X}^i(\hat{x})}{\partial x^\alpha} \frac{\partial \hat{X}^j(\hat{x})}{\partial x^\beta} \cdots \frac{\partial X^m(\hat{x})}{\partial \hat{x}^k} \frac{\partial X^n(\hat{x})}{\partial \hat{x}^l} \cdots T^{ab..}_{mn4..}(x, x^4), \quad \hat{T}^4_4(\hat{x}, \hat{x}^4) = T^4_4(x, x^4), \quad (169)$$

and the metric tensor field in (165) reduces to

$$\hat{g}_{ij}(\hat{x}) = \delta_{kl} \frac{\partial X^k(\hat{x})}{\partial \hat{x}^i} \frac{\partial X^l(\hat{x})}{\partial \hat{x}^j}, \quad \hat{g}_{44}(\hat{x}, x^4) \equiv 0, \quad \hat{g}_{44}(\hat{x}, \hat{x}^4) \equiv -1. \quad (170)$$
The Christoffel symbols in (166) boil down to
\[
\begin{cases}
\frac{i}{j k} = \frac{1}{2} \delta^{i a} (\hat{x}) \left[ \frac{\partial \hat{g}_{k a} (\hat{x})}{\partial x^j} + \frac{\partial \hat{g}_{a j} (\hat{x})}{\partial x^k} - \frac{\partial \hat{g}_{j k} (\hat{x})}{\partial x^a} \right], \\
\frac{4}{j k} = \left\{ \begin{array}{ll}
4 & j 4 \\
4 & 4 4
\end{array} \right\} \equiv 0.
\end{cases}
\]
(171)

The laws of covariant differentiation (167) imply that
\[
\begin{align*}
\hat{\nabla}_j \hat{T}^{a_4} b_4 (\hat{x}, \hat{x}^4) &= \frac{\partial \hat{T}^{a_4} b_4 (\hat{x}, \hat{x}^4)}{\partial \hat{x}_j} + \left\{ \begin{array}{ll}
\frac{a}{j k} & \hat{T}^{k_4} b_4 (\hat{x}, \hat{x}^4) - \left\{ \begin{array}{ll}
k & j b \\
4 & 4 4
\end{array} \right\} \hat{T}^{a_4} k_4 (\hat{x}, \hat{x}^4), \\
\frac{4}{j k} &= \frac{\partial \hat{T}^{a_4} b_4 (\hat{x}, \hat{x}^4)}{\partial x^j} = \frac{\partial \hat{T}^{a_4} b_4 (\hat{x}, \hat{x}^4)}{\partial x^k}.
\end{align*}
\]
(172)

If we now replace various tensor fields \( T^{\mu \nu}_{\alpha \beta..} \) occurring in the constitutive equations (153 - 154) by \( \hat{T}^{ab..} \), \( \hat{T}^{a_4..} \), \( \hat{T}^{a_4 4..} \), and \( \hat{\nabla}_j \hat{T}^{ab..} (\hat{x}, \hat{x}^4) \), \( \frac{\partial}{\partial x^\gamma} \hat{T}^{ab..} (\hat{x}, \hat{x}^4) \) etc., then we have converted all relevant equations into spatial curvilinear coordinates.

In physical applications, usually orthonormal or physical components of a tensor are necessary [23]. For that purpose we introduce three orthonormal vectors, \( \hat{x}_A (\hat{x}) \) \( A \in \{1, 2, 3\} \), in space. These vectors satisfy the orthonormality conditions:
\[
\hat{g}_{ij} (\hat{x}) \lambda^i_A (\hat{x}) \lambda^j_B (\hat{x}) = \delta_{AB}.
\]
(173)

We define the inverse entries by
\[
[\mu^A_i] := [\lambda^i_A]^{-1}, \\
\lambda^i_A \mu^A_i = \delta^i_j,
\]
(174)

(Here, the summation convention is also followed for capital roman indices.)

By (173) and (174) we obtain
\[
\hat{g}_{ij} (\hat{x}) = \delta_{AB} \mu^A_i (\hat{x}) \mu^B_j (\hat{x}).
\]
(175)

The tensor transformation rules (164) lead to
\[
\hat{T}^{AB..}_{CD..} (\hat{x}, \hat{x}^4) = \mu^A_a (\hat{x}) \mu^B_b (\hat{x}) \lambda^i_A (\hat{x}) \lambda^j_B (\hat{x}) \lambda^k_i (\hat{x}) \lambda^l_j (\hat{x}) \lambda^m_k (\hat{x}) \lambda^n_l (\hat{x}) \hat{T}^{ab..}_{cd..} (\hat{x}, \hat{x}^4).
\]
(176)

Instead of Christoffel symbols, we require Ricci rotation coefficients [23] for the connexion. These are defined by:
\[
\gamma_{ABC} (\hat{x}) := \hat{g}_{jl} (\hat{x}) \left( \hat{\nabla}_k \lambda^l_A (\hat{x}) \lambda^i_B (\hat{x}) \lambda^j_C (\hat{x}) \right) \equiv -\gamma_{BAC} (\hat{x}),
\]
\[
\gamma^A_{BC} (\hat{x}) := \delta^A_j \gamma_{JBC} (\hat{x}) = \gamma_{ABC} (\hat{x}).
\]
(177)
The appropriate covariant derivatives can be characterized by:

\[ \hat{\nabla}_J \hat{T}^{A4}_{B4}(\hat{x}, \hat{x}^4) := \lambda^J_4(x) \frac{\partial}{\partial \hat{x}^i} \hat{T}^{A4}_{B4}(\hat{x}, \hat{x}^4) - \gamma^A_{D4J}(\hat{x}) \hat{T}^{D4}_{B4} + \gamma^D_{BJ}(\hat{x}) \hat{T}^{A4}_{D4}. \]  

(178)

If we now replace the tensor fields \( T^{\mu\nu\ldots}_{\alpha\beta}(x) \), \( \frac{\partial}{\partial x^\gamma} T^{\mu\nu\ldots}_{\alpha\beta}(x) \) appearing in the constitutive equations (154 - 162) by \( \hat{T}^{AB\ldots}_{CD\ldots}(\hat{x}, \hat{x}^4) \), \( \hat{T}^{A4\ldots}_{B4\ldots}(\hat{x}, \hat{x}^4) \), etc., then we have transformed all the required equations into the orthonormal or physical frame. Physical measurements correspond to quantities expressed in this frame.

Before concluding, we shall now explore a special example which is most useful for continuum mechanics. This example involves orthogonal coordinate systems in Euclidean three-space. The equations (170) reduce to

\[ [\hat{g}_{ij}(\hat{x})] = \begin{bmatrix} [h_1(\hat{x})]^2 & 0 & 0 \\ 0 & [h_2(\hat{x})]^2 & 0 \\ 0 & 0 & [h_3(\hat{x})]^2 \end{bmatrix}, \]

(179)

\( h_i(x) > 0, \quad \sqrt{\det [\hat{g}_{ij}]} = h_1(\hat{x})h_2(\hat{x})h_3(\hat{x}) > 0. \)

The non-zero Christoffel symbols, from (171) and (179) are summarized by:

\[ \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \frac{\partial}{\partial \hat{x}^i} \ln h_1, \quad \text{etc.} \]

\[ \begin{bmatrix} 1 \\ 12 \end{bmatrix} = \frac{\partial}{\partial \hat{x}^2} \ln h_1, \quad \text{etc.} \]

(180)

\[ \begin{bmatrix} 1 \\ 22 \end{bmatrix} = -h_2 [h_1]^{-2} \frac{\partial}{\partial \hat{x}^i} \ln h_2, \quad \text{etc.} \]

The orthonormal (or physical) vector components from (173) and (179) are given by

\[ \lambda^i_A(\hat{x}) = [h_i(\hat{x})]^{-1} \delta^i_A, \quad \mu^4_A(\hat{x}) = h_i(\hat{x})\delta^4_A, \]

(181)

and the non-zero Ricci rotation coefficients from (177), (180) and (181) are furnished by:

\[ \gamma_{ABC} := \delta_{AE} \gamma^E_{BC}, \quad \gamma_{(A)(B)(C)} := \gamma_{ABC} \equiv \gamma_{BAC}, \]

\[ \gamma(2)(1)(2) = -\frac{\partial}{\partial \hat{x}^1} \ln h_2, \quad \gamma(3)(1)(3) = -\frac{\partial}{\partial \hat{x}^3} \ln h_3, \]

(182)

\[ \gamma(1)(2)(1) = -\frac{\partial}{\partial \hat{x}^2} \ln h_1, \quad \gamma(3)(2)(3) = -\frac{\partial}{\partial \hat{x}^3} \ln h_3, \]

\[ \gamma(1)(3)(1) = -\frac{\partial}{\partial \hat{x}^3} \ln h_1, \quad \gamma(2)(3)(2) = -\frac{\partial}{\partial \hat{x}^2} \ln h_2. \]
Also, the gradient of a scalar field is given by
\[ \hat{\nabla}_i \hat{\phi}(\hat{x}) = \frac{\partial}{\partial \hat{x}^i} \hat{\phi}(\hat{x}), \quad \hat{\nabla}_A \hat{\phi}(\hat{x}) = \lambda^i_A \frac{\partial}{\partial \hat{x}^i} \hat{\phi}(\hat{x}) = (h_A)^{-1} \frac{\partial}{\partial \hat{x}^A} \hat{\phi}(\hat{x}), \]
\[ \hat{\nabla}_{(1)} \hat{\phi}(\hat{x}) = (h_1)^{-1} \frac{\partial}{\partial \hat{x}^1} \hat{\phi}(\hat{x}) \neq \hat{\nabla}_{1} \hat{\phi}(\hat{x}). \]  
(183)

In (183) there is no summation over \( A \).

The divergence of a vector field is furnished by
\[ \hat{\nabla}^i \hat{T}^i(\hat{x}) \equiv \hat{\nabla}^A \hat{T}^A(\hat{x}) = (h_1 h_2 h_3)^{-1} \left\{ \frac{\partial}{\partial \hat{x}^i} \left[ (h_1 h_2 h_3) \hat{T}^i(\hat{x}) \right] \right\}. \]  
(184)

The curl is given by:
\[ \left[ \hat{\nabla} \times \hat{A} \right]^i := \frac{1}{2} \epsilon^{ijk} \left[ \hat{\nabla}_j \hat{A}_k - \hat{\nabla}_k \hat{A}_j \right] = \frac{1}{2(h_1 h_2 h_3)} \epsilon^{ijk} \left[ \frac{\partial \hat{A}_k(\hat{x})}{\partial \hat{x}^j} - \frac{\partial \hat{A}_j(\hat{x})}{\partial \hat{x}^k} \right], \]
\[ \left[ \hat{\nabla} \times \hat{A} \right]^B := \frac{1}{2} \epsilon^{BCD} \left[ \lambda^i_C \frac{\partial}{\partial \hat{x}^i} \left( \hat{A}_D(\hat{x}) \right) - \lambda^i_D \frac{\partial}{\partial \hat{x}^i} \left( \hat{A}_C(\hat{x}) \right) \right] \]
\[ = \frac{1}{2} \epsilon^{BCD} \left[ (h_C)^{-1} \frac{\partial}{\partial \hat{x}^C} \left( \hat{A}_D(\hat{x}) \right) - (h_D)^{-1} \frac{\partial}{\partial \hat{x}^D} \left( \hat{A}_C(\hat{x}) \right) \right], \]  
(185)

Finally, \( \left[ \hat{\nabla} \times \hat{A} \right]^1 \neq \left[ \hat{\nabla} \times \hat{A} \right]^{(1)}. \)

The Laplacian operator is furnished by
\[ \nabla^2 \hat{W}(\hat{x}) := \hat{g}^{ij} \hat{\nabla}_i \hat{\nabla}_j \hat{W}(\hat{x}) \equiv \delta^{AB} \hat{\nabla}_A \hat{\nabla}_B \hat{W}(\hat{x}) \]  
(186)
\[ = \frac{1}{(h_1 h_2 h_3)} \left[ \frac{\partial}{\partial \hat{x}^1} \left( \frac{h_1 h_2}{h_1} \frac{\partial \hat{W}(\hat{x})}{\partial \hat{x}^1} \right) + \frac{\partial}{\partial \hat{x}^2} \left( \frac{h_3 h_1}{h_2} \frac{\partial \hat{W}(\hat{x})}{\partial \hat{x}^2} \right) + \frac{\partial}{\partial \hat{x}^3} \left( \frac{h_1 h_2}{h_3} \frac{\partial \hat{W}(\hat{x})}{\partial \hat{x}^3} \right) \right]. \]
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