\textbf{\textit{\aleph_0}-Extended Supergravity and Chern-Simons Theories}\textsuperscript{1}

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\textbf{Abstract}

We give generalizations of extended Poincaré supergravity with \textit{arbitrarily many} supersymmetries in the absence of central charges in three-dimensions by gauging its intrinsic global \textit{SO}(N) symmetry. We call these \textit{\aleph_0} (Aleph-Null) supergravity theories. We further couple a non-Abelian supersymmetric Chern-Simons theory and an Abelian topological BF theory to \textit{\aleph_0} supergravity. Our result overcomes the previous difficulty for supersymmetrization of Chern-Simons theories beyond \textit{N} = 4. This feature is peculiar to the Chern-Simons and BF theories including supergravity in three-dimensions. We also show that dimensional reduction schemes for four-dimensional theories such as \textit{N} = 1 self-dual supersymmetric Yang-Mills theory or \textit{N} = 1 supergravity theory that can generate \textit{\aleph_0} globally and locally supersymmetric theories in three-dimensions. As an interesting application, we present \textit{\aleph_0} supergravity Liouville theory in two-dimensions after appropriate dimensional reduction from three-dimensions.

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1. Introduction

Recently there have been new developments in globally supersymmetric theories in three dimensions (3D) or lower dimensions [1] based on $\mathcal{G}\mathcal{R}(d,N)$ algebras leading to the use of quantities called $L$ and $R$ matrices satisfying a certain anti-commutator algebra which generalizes the usual Clifford algebra. Representations of these algebras enable us to construct a theory of on-shell 3D representations as well as off-shell 1D representations. In particular scalar multiplets with arbitrarily large numbers of supersymmetries have been constructed [2]. We can call these systems $\aleph_0$ (alephnull) supersymmetry, since in the limit $N \to \infty$ they can accommodate infinitely many supersymmetries. An interesting question then is whether there is a similar technique applicable to Chern-Simons (CS) theories. If the answer is affirmative, then the subsequent question is whether those globally supersymmetric theories can be coupled to supergravity with $\aleph_0$ supersymmetry.

As a matter of fact, there has been indication that supergravity theories in 3D can be interpreted as CS theories, in particular, with infinitely many extended supersymmetries [3]. In a paper in a similar direction P. Howe et al. [4] it is found that there exist infinitely many local supersymmetries for on-shell Poincaré supergravity, with two sets of vector fields, one set gauging the group $SO(p) \otimes SO(q)$ with $N = p + q$ and another gauging the central charges [4]. This system is analogous to the conformal supergravity with arbitrary number of extended supersymmetries [5][6].

Independent of these developments within 3D, there has been another important observation [7] about “strong-weak coupling duality” between 3D superstring and 4D superstring theories in order to understand the vanishing of the cosmological constant in 4D. According to this scenario, the reason we have exactly zero cosmological constant in 4D even after supersymmetry breaking is due to the duality between these 4D theories and 3D superstring theories in which supersymmetry keeps the zero cosmological constant, while the usual mass degeneracy between bosons and fermions is lifted.

Considering these recent developments, it is crucial to find supersymmetric non-Abelian CS and topological BF theories [8] that can couple to the $\aleph_0$ Poincaré supergravity. We will try to combine the two different recent theories, i.e., one with arbitrary number of global supersymmetries in terms of $L$ and $R$ matrices [1], and supergravity theories based on CS formulations [3].

In our previous paper [6] we found an apparent barrier that prevented going beyond $N = 4$ supersymmetric non-Abelian CS theories. In the present paper, we will present two ways to bypass this difficulty, maintaining the on-shell closure of the gauge algebra by virtue of vanishing field strengths of the gravitini. We will see the minimal field content needed for
on-shell closure of Poincaré supergravity with no central charges [4]. After understanding this extended supergravity, we also perform its coupling to supersymmetric CS theory, a topological BF theory [8], and also to a tensor multiplet with arbitrary number of supersymmetries. As by-products and interesting applications, we also present $\aleph_0$ supergravity Liouville theory in 2D, as well as $\aleph_0$ BF theory in 3D.

2. $\aleph_0$ Supergravity in 3D

We start with reviewing the on-shell $\aleph_0$ extended supergravity in 3D [3]. This supergravity multiplet consists of the dreibein and the gravitini $(e^m_\mu, \psi^A_\mu)$, where the indices $\mu, \nu, \ldots = 0, \ldots, 3$ are the curved coordinates, while $m, n, \ldots = (0), (1), (2)$ are local Lorentz coordinates. Relevantly the signature of our space-time is $(+, -, -)$, while our $\gamma$-matrices satisfy $\gamma^{mn} = i\epsilon^{mn}$ with $\epsilon^{(0)(1)(2)} = +1$, and $\gamma^m\gamma^n + \gamma^n\gamma^m = 2\eta^{mn}$. We use the indices $A, B, \ldots = 1, \ldots, N$ for the $N$-extended supersymmetries. Since our formulation is on-shell, we do not have any additional gauge fields such as $A_{AB}$ or $C_{AB}$ presented in the off-shell formulation in ref. [4].

The supersymmetry transformation rules and the invariant lagrangian [3] are similar to the most standard form of $N = 1$ supergravity in 4D [9] or 3D [4][10]:

$$\delta_Q e^m_\mu = -i(\tau^A \gamma^m \psi^A_\mu) ,$$

$$\delta_Q \psi^A_\mu = \partial_\mu e^A + \frac{1}{4} \tilde{\omega}^{mn}(e, \psi)\gamma_{mn}^A e^A \equiv D_\mu(\tilde{\omega})e^A ,$$

(2.1)

where $\omega^{rs}_\mu$ has the $\psi$-torsion like 4D [11]:

$$\tilde{\omega}^{rs}_\mu(e, \psi) = \frac{1}{2} (C^{rs}_\mu - C^{sr}_\mu + C_{\mu}^{sr}) , \quad C_{\mu}^{mn} \equiv \partial_\mu e^m_\nu - \partial_\nu e^m_\mu + i (\overline{\psi}^A_\mu \gamma^m \psi^A_\nu) .$$

(2.2)

with the invariant lagrangian:

$$\mathcal{L}_{\aleph_0 SG} = -\frac{1}{4} e R(\tilde{\omega}) - \frac{1}{4} e^{\mu\nu\rho} (\overline{\psi}^A_\mu \gamma^\rho \psi^A_\nu) ,$$

(2.3)

where $\mathcal{R}^{A}_{\mu\nu}$ is the gravitino field strength:

$$\mathcal{R}^{A}_{\mu\nu} \equiv D_\mu(\tilde{\omega})\psi^A_\nu - D_\nu(\tilde{\omega})\psi^A_\mu$$

(2.4)

Since this system has been presented in the past [3] we do not repeat the details. Such infinitely many supersymmetries are possible due to the peculiar feature of 3D namely both the dreibein and gravitini have no physical degrees of freedom. The closure of two supersymmetries on the dreibein is the usual one, while that on the gravitino yields a peculiar
“extra” transformation on $\psi^A$, i.e., $[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)] = \delta_P(i\tau^A\gamma^m\epsilon^2A) + \delta_E$ with

$$
\delta_E\psi^A_\mu = +ie^{-1}\epsilon^\rho_\mu A_{\mu AB}\gamma_\nu\widetilde{R}_{\rho B} + \frac{1}{12} A_{\mu AB}\widetilde{R}_{\mu B}
- \frac{3}{8} e^{-1}\epsilon^\rho_\mu S_1^\nu A_{\mu AB}\widetilde{R}_{\rho B} + \frac{1}{8} e^{-1}\epsilon^\rho_\mu S_1^\nu\widetilde{R}_{\rho A}
+ S_{2\mu AB}(i\gamma_\nu\widetilde{R}^\nu B) + S_{2\nu AB}(i\gamma_\mu\widetilde{R}_\nu^B - i\gamma_\nu\widetilde{R}_\mu^B)
$$

(2.5)

where

$$
\widetilde{R}_{\mu A} \equiv e^{-1}\epsilon^\rho_\mu R_{\nu\rho} A^\nu,
A_{1 AB} \equiv \frac{3}{8}(\epsilon_1[A\epsilon_2 B]) = -A_{1 AB},
S_{0 AB} \equiv -i\left(\epsilon_1^A \gamma^B\epsilon_2\right) = +S_{0 B A}.
$$

(2.6)

The lagrangian (2.2) is invariant under these extra symmetries (2.5), as is easily confirmed. This is natural because these extra symmetries can be regarded as the on-shell vanishing terms in the commutator algebra [11], since the gravitino field equations are simply $\mathcal{R}_{\mu\nu} = 0$ anyway.

Compared also with the algebra presented in [4], our system lacks vector fields due to the absence of central charges gauged by $C_{\mu AB}$ in the former. Since the two sets of vector fields, i.e., one for $SO(p) \otimes SO(q)$ and another for the central charges [4], appear in pair in the lagrangian, it is natural that our system does not have any of these vector fields. In any case, our system has the minimal field content for the Poincaré algebra with no central charges.

We now present the following generalizations of the “minimal” $\mathcal{R}_0$ supergravity above, which have not been given in the past to our knowledge. First we can gauge the global $SO(N)$ symmetry by the gauge field $B_{\mu AB}$ and an additional vector field $C_{\mu AB}$. The resulting multiplet $(\epsilon^{m \mu}, \psi^A, B_{\mu AB}, C_{\mu AB})$ can be obtained from the supergravity multiplet in [4] by identifying their central charges $Z^{ij}$ identified with the $SO(N)$ generators $T^{ij}$. We first include the $SO(N)$ minimal coupling in all the derivatives such as

$$
2D_{[\mu \psi^A_\nu]} = 2\left(D_{[\mu (\bar{\psi}^A_\nu)} + \bar{g}B_{[\mu AB}\psi^A_\nu]\right) \equiv \bar{R}_{\mu A}^\nu,
$$

(2.7)

and the similarly for $D(\bar{\psi}^A_\mu)\epsilon^A$ in (2.1) by $D_\mu \epsilon^A$. We also add the transformations

$$
\delta_Q B_{\mu AB} = \frac{1}{2}(\tau^{A[I] A^\nu_\mu R_{\mu \nu B]} + \frac{1}{2} e^{-1}\epsilon^\rho_\mu R_{\rho \sigma A} B)]
$$

$$
\delta_Q C_{\mu AB} = -(\tau^{A[I] A^\nu_\mu R_{\mu \nu B]}),
$$

(2.8)

together with the new additional terms in the gravitino transformation:

$$
\delta_Q \psi^A_\mu = \partial^A_\mu + \frac{1}{4} \bar{g}_\mu^m = m_n \epsilon^A + \bar{g} A_{\mu AB} \epsilon^B - \bar{g} e^{-1}\epsilon^\rho_\mu \bar{H}_{\rho \sigma AB}^B + i\bar{g} \gamma_\nu \epsilon^B \bar{H}_{\mu \nu AB},
$$

(2.9)
where

\[ H_{\mu \nu}^{AB} \equiv (\partial_\mu C_\nu^{AB} + 2B_\mu^{[A}C_\nu^{C]}B) - (\mu \leftrightarrow \nu) \]  

(2.10)

looks like a field strength and is actually covariant under the \( SO(N) \), but different from the proper \( SO(N) \) gauge field strength:

\[ G_{\mu \nu}^{AB} = (\partial_\mu B_\nu^{AB} + B_\mu^{AC} B_\nu^{CB}) - (\mu \leftrightarrow \nu) \]  

(2.11)

As usual, the hatted quantities are supercovariant, \( e.g. \)

\[ \hat{H}_{\mu \nu}^{AB} \equiv H_{\mu \nu}^{AB} + \frac{1}{2} (\psi_\mu^{[A} \psi_\nu^{B]}) \]  

(2.12)

Finally our lagrangian has an explicit \( \tilde{g} \)-term like a \( BF \) lagrangian [8][4]:

\[ \mathcal{L}_{8\text{SG}, \tilde{g}} = -\frac{1}{4} eR(\tilde{\omega}) - \frac{1}{4} \epsilon^{\mu \nu \rho} (\bar{\psi}_\mu^{A} \tilde{R}_\rho^{A}) + \tilde{g} \epsilon^{\mu \nu \rho} C_\mu^{AB} G_\rho^{AB} \]  

(2.13)

and relevantly \( G_\rho^{AB} \) is the “field strength” of \( C_\mu^{AB} \).

The on-shell closure of this multiplet is easy to confirm:

\[ [\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_P(\bar{\epsilon}_1^A \gamma^m \epsilon_2^A) + \delta_G(\bar{\epsilon}_1^A \bar{\epsilon}_2^B) \]  

(2.14)

where \( \delta_Q \) is the \( O(N) \) gauge transformation acting as

\[ \delta_G A_\mu^{AB} = D_\mu A^{AB} \]  

\[ \delta_G C_\mu^{AB} = D_\mu A^{AB} \]  

(2.15)

Note that \( D_\mu \) is \( SO(N) \) covariant with the minimal coupling by \( A_\mu^{AB} \). Even though both of these fields transform in the same way under \( SO(N) \), there will be no problem in 3D for the same reason given in the context of \( N = 4 \) CS theory in [6]. To put it differently, we can identify the central charges with the \( SO(N) \) generators, when there is only one \( SO(N) \) symmetry. Needless to say, we can always go back to the \emph{minimal} supergravity field content by turning off the \( SO(N) \) coupling: \( \tilde{g} \to 0 \).

There is another generalization which has not been given in literature. We can include additional vector and a spinor fields \( A_\mu^I \) and \( \lambda \) with a supersymmetric CS form. Now the new field content is \( (\epsilon_\mu^m, \psi_\mu^A, A_\mu^{AB}, B_\mu^{AB}, C_\mu^{AB}, \lambda^A) \), where \( B_\mu^{AB} \) is the gauge field for gauging \( SO(N) \). We use the indices \( A, B, \ldots = 1, \ldots, N \) for the vectorial representation of \( SO(N) \).

Our lagrangian

\[ \mathcal{L}_{8\text{SG}, g, \tilde{g}} = -\frac{1}{4} eR(\tilde{\omega}) + \frac{1}{4} \epsilon^{\mu \nu \rho} (\bar{\psi}_\mu^{A} R_{\nu \rho}^{A}) + \tilde{g} \epsilon^{\mu \nu \rho} C_\mu^{AB} G_\rho^{AB} \]

\[ + \frac{1}{2} \tilde{g} \tilde{h} (\bar{\lambda}^A \lambda^B) + \frac{1}{2} \tilde{g} \tilde{h} \epsilon^{\mu \nu \rho} \left( F_\mu^{AB} A_\rho^{AB} - \frac{2}{3} A_\mu^{AB} A_\nu^{BC} A_\rho^{CA} \right) \]  

(2.16)

\footnote{Any “extra” transformation involved is skipped here.}
is invariant under the supertranslation rule for this multiplet

\[ \delta_Q e^m = -i \left( \tilde{\epsilon}^A \gamma^m \psi^A \right) \]

\[ \delta_Q \psi^A = D_\mu (\tilde{\omega}) e^A + \tilde{g} B_{\mu}^{AB} e^B + \tilde{g} \epsilon_{\mu}^{\rho \sigma} e^B \hat{H}_{\rho \sigma}^{AB} + i \tilde{g} \gamma^\mu e^B \tilde{H}_{\mu}^{AB} \]

\[ \delta_Q B_{\mu}^{AB} = \frac{i}{2} \left( \tilde{\epsilon}^A \gamma^\mu R_{\mu \nu} [B] + \frac{1}{2} \epsilon_{\mu}^{\rho \sigma} \left( \tilde{\epsilon}^A R_{\rho \sigma}^{B} \right) + i \tilde{h} \left( \tilde{\epsilon}^A \gamma_\mu \lambda^B \right) \right) \]

\[ \delta_Q C_{\mu}^{AB} = + \frac{1}{2} \left( \tilde{\epsilon}^A \psi_{\mu}^B \right) + i \tilde{h} \left( \tilde{\epsilon}^A \gamma_\mu \lambda^B \right) \]

\[ \delta_Q A_{\mu}^{AB} = i \left( \tilde{\epsilon}^A \gamma_\mu \lambda^B \right) \]

\[ \delta_Q \lambda^A = - \gamma^{\mu \nu} e^B \left( F_{\mu \nu}^{AB} + G_{\mu \nu}^{AB} + H_{\mu \nu}^{AB} \right) + \frac{i}{2} \tilde{g} \left( \tilde{\epsilon}^B \gamma_\mu \psi_{\nu}^B \right) \lambda^A + \frac{i}{2} \left( \tilde{\epsilon}^A R_{\mu \nu}^{B} \right) \left( \gamma_\mu \psi_{\nu}^B \right) . \]

The \( \tilde{g} \) and \( \tilde{h} \) are coupling constants, and in particular the former is the \( SO(N) \) coupling. Therefore if we switch off \( \tilde{g} \to 0 \), then the system is reduced to the \textit{minimal} \( N_0 \) supergravity. If we keep non-zero \( \tilde{g} \), while taking the limit \( \tilde{h} \to 0 \), the two fields \( A_{\mu}^{AB} \) and \( \lambda^A \) will be removed. Our system is thus a combination of the usual supersymmetric CS action made of the \( A \) and \( \lambda \)-fields and the \( SO(N) \) gauged \( N_0 \) supergravity.

The relevant (super)field strengths are defined by

\[ R_{\mu \nu}^{AB} \equiv \left( \partial_{\mu} \psi_{\nu}^A + \frac{1}{4} \tilde{\omega}_{\mu}^{mn} \gamma_{mn} \psi_{\nu}^A + \tilde{g} B_{\mu}^{AB} \psi_{\nu}^B \right) - (\mu \leftrightarrow \nu) \]

\[ (D_\mu \psi_{\nu}^A + \tilde{g} B_{\mu}^{AB} \psi_{\nu}^B) - (\mu \leftrightarrow \nu) \equiv D_{\mu} \psi_{\nu}^A - D_{\nu} \psi_{\mu}^A \],

\[ F_{\mu \nu}^{AB} \equiv \left( \partial_{\mu} A_{\nu}^{AB} + A_{\mu}^{AC} A_{\nu}^{CB} \right) - (\mu \leftrightarrow \nu) \]

\[ G_{\mu \nu}^{AB} \equiv \left( \partial_{\mu} B_{\nu}^{AB} + \tilde{g} B_{\mu}^{AC} B_{\nu}^{CB} \right) - (\mu \leftrightarrow \nu) \]

\[ H_{\mu \nu}^{AB} \equiv \left( \partial_{\mu} C_{\nu}^{AB} + \tilde{g} B_{\mu}^{AC} C_{\nu}^{CB} + \tilde{g} B_{\mu}^{BC} C_{\nu}^{AC} \right) - (\mu \leftrightarrow \nu) \]

\[ \tilde{F}_{\mu \nu}^{AB} \equiv F_{\mu \nu}^{AB} - 2i \left( \tilde{\psi}_{[\mu}^{[A} [\gamma_{\nu]}^{\rho]} R_{\rho]^{B} \right) \]

\[ \tilde{G}_{\mu \nu}^{AB} \equiv G_{\mu \nu}^{AB} - i \left( \tilde{\psi}_{[\mu}^{[A} [\gamma_{\nu]}^{\rho]} R_{\rho]^{B} \right) + e_{\mu}^{\rho \sigma} \left( \tilde{\psi}_{\nu}^{[A} [R_{\rho \sigma}^{B]} \right) \]

\[ - 2i \tilde{h} \left( \tilde{\psi}_{[\mu}^{[A} [\gamma_{\nu]}^{\rho]} \lambda^{B} \right) \]

\[ \tilde{H}_{\mu \nu}^{AB} \equiv H_{\mu \nu}^{AB} - \frac{1}{2} \left( \tilde{\psi}_{[\mu}^{[A} [\gamma_{\nu]}^{\rho]} \lambda^{B} \right) - 2i \tilde{h} \left( \tilde{\psi}_{[\mu}^{[A} [\gamma_{\nu]}^{\rho]} \lambda^{B} \right) . \]

3. \( N_0 \) SCS Theory Coupled to \( N_0 \) SG

Once the \( N_0 \) supergravity is realized, the next interesting question is its couplings to any “matter” multiplet. The easiest case is the CS theory, which has the simplest lagrangians in general. To this end, we have to establish a vector multiplet with arbitrary number of supersymmetries. This can be easily done, once we notice the duality transformation
connecting a scalar multiplet to a possible vector multiplet, because in 3D a vector is dual to a scalar. As a matter of fact, using the result in [1], we can establish our $\mathbb{N}_0$ non-Abelian vector multiplet $(A_{\mu}^I, \lambda_i^I)$ coupled to $\mathbb{N}_0$ supergravity:

$$
\delta_Q A_{\mu}^I = + \frac{1}{2\sqrt{2}} \sum_j (L^A)_{ij} (\tau^A \gamma_{\mu} \lambda_j^I) ,
$$

$$
\delta_Q \lambda_i^I = - \frac{1}{2\sqrt{2}} \sum_j (R^A)_{ij} (\gamma^\mu \epsilon^A \hat{F}_{\mu\nu} j^I) ,
$$

(3.1)

where $i, j, \ldots$ are for the adjoint representation for the non-Abelian gauge group, while $i, j, \ldots = 1, 2, \ldots, d$ are for the representation of the $d \times d$ matrices $L$ and $R$, which satisfy the relationships [1] below. These are the defining conditions for these matrices,

$$
\sum_k [(L^A)_{ik} (R^B)_{kj} + (L^B)_{ik} (R^A)_{kj}] = -2\delta^{AB} \delta_{ij} ,
$$

(3.2)

The contraction with respect the $i, j, \ldots$ indices always need the explicit summation symbols such as $\sum_i$ for the reason to be seen later. As has been pointed out in ref. [1], we can always construct these $L$ and $R$ matrices for arbitrary $N$, by choosing a sufficiently large $d$-dimensional representation. In particular, when $N = 8$ or 6 (mod. 8), these matrices coincide with the Clifford algebra construction given in ref. [6].

The field strength of the vector field is defined by

$$
F_{\mu\nu}^I \equiv \partial_{\mu} A_{\nu}^I - \partial_{\nu} A_{\mu}^I + f^{IJK} A_{\mu}^J A_{\nu}^K ,
$$

(3.3)

where $f^{IJK}$ is the structure constant of the non-Abelian gauge group. Due to the third term here with the index $i$ repeated two times, we need always the explicit summation symbol for these indices to avoid confusion. In other words, the $i$-index appearing in (3.3) should be regarded as not obeying the Einstein-summation convention. As usual, the hatted field strength $\hat{F}_{\mu\nu}^I$ denotes its supercovariantization:

$$
\hat{F}_{\mu\nu}^I \equiv F_{\mu\nu}^I + \left[ \frac{1}{2\sqrt{2}} \sum_j (L^A)_{ij} (\gamma^A \lambda^I) - \langle \mu \nu \rangle \right] .
$$

(3.4)

Even though we have $d$ multiple gauge fields for a single gauge group, this will not pose any problem. As a matter of fact, we have already encountered an exactly the same structure for the case of $N = 4$ CS theory in ref. [6].

The gravitino-dependent term in (3.1) is the effect of local supersymmetry, which does not pose any problem about the closure of the gauge algebra, as will be seen shortly. The
gravitino-independent terms can be easily obtained, based on the knowledge about the case of scalar multiplet with the global $\aleph_0$ supersymmetries.

The invariant lagrangian for our CS theory with $\aleph_0$ supersymmetries is

$$L_{\aleph_0}^{\text{CS}} = \frac{1}{2} m \epsilon^{\mu\rho\sigma} \sum_i \left( F_{\mu\nu i}^I A_{\rho i}^I - \frac{1}{3} f^{IJK} A_{\mu i}^I A_{\nu i}^J A_{\rho i}^K \right) + me \sum_i (\bar{\lambda}_i^I \lambda_i^I) .$$

As usual in any CS theory, the coefficient $m$ should be quantized for a gauge group with non-trivial $\pi_3$-homotopy, e.g.,

$$m = \frac{n}{8\pi} \quad (n = \pm 1, \pm 2, \cdots) .$$

A key equation useful for the invariance check of (3.5) is the arbitrary variation of the field strength:

$$\delta F_{\mu\nu i}^I = D_\mu (\delta A_{\nu i}^I) - D_\nu (\delta A_{\mu i}^I) ,$$

where the covariant derivative $D_\mu$ for an arbitrary vector $V_{\nu i}^I$ with the index $i$ is defined by

$$D_\mu V_{\nu i}^I \equiv \partial_\mu V_{\nu i}^I + f^{IJK} A_{\mu i}^J V_{\nu i}^K - \{^\rho_\mu_{\nu i} \} V_{\rho i}^I .$$

Here the absence of the symbol $\sum_i$ for the second term implies the index $i$ is not summed. Relevantly the gauge covariance of the field strength under our gauge transformation

$$\delta G A_{\mu i}^I = D_\mu \Lambda_i^I ,$$

has the desirable form:

$$\delta G F_{\mu\nu i}^I = -f^{IJK} \Lambda_i^J F_{\mu\nu i}^K .$$

Again there is no summation over $i$ in the r.h.s.

The closure of the gauge algebra (3.1) at the local level is also essentially the same as the global case, because the field equation $\lambda_i^I = 0$ delete all the on shell effect with the gravitino field. The on-shell closure yields $[ \delta Q(\epsilon_1), \delta Q(\epsilon_2) ] = \delta_P (\tau^A_1 \gamma^m \epsilon^A_1) + \delta_E$, where $\delta_E$ now implies the extra symmetry on the vector fields

$$\delta_E A_{\mu i}^I = e^{-1} \epsilon_\rho^{\rho\sigma} \sum_j a_{ij} F_{\rho\sigma j}^I , \quad (a_{ij} = -a_{ji}) ,$$

with the antisymmetric parameters $a_{ij}$, leaving the CS lagrangian (3.3) invariant desirably.\footnote{We can of course simply discard these extra symmetry terms, regarding them as the on-shell vanishing terms.}

We can further generalize our system to a product of different gauge groups: $G_1 \otimes G_2 \otimes \cdots \otimes G_d$, where $G_i$ $(i = 1, \ldots, d)$ are different gauge groups where $d$ is exactly the same as the dimensions of the $L$ and $R$ matrices. Accordingly (3.2) and (3.5) can be generalized to
\[ F_{\mu\nu}^{I_i} \equiv \partial_{\mu}A_{\nu}^{I_i} - \partial_{\nu}A_{\mu}^{I_i} + f^{I_iJ_iK_i}A_{\mu}^{J_i}A_{\nu}^{K_i}, \quad (3.12) \]

\[ \mathcal{L}_{\mathbb{N}_0\text{CS}} = \sum_i \left[ \frac{1}{2} m_i \epsilon_{\mu\nu\rho} \left( F_{\mu\nu}^{I_i}A_{\rho}^{I_i} - \frac{1}{3} f^{I_iJ_iK_i}A_{\mu}^{J_i}A_{\nu}^{K_i} A_{\rho}^{I_i} \right) + m_i \epsilon \left( \bar{\lambda}_i^{I_i} \lambda_i^{I_i} \right) \right], \quad (3.13) \]

where for a fixed index \( i \), the \( I_i, J_i, \ldots \) indices serve as dummy indices, and the quantization of the coefficients \( m_i \) can depend on each gauge group \( G_i \). The \( f^{I_iJ_iK_i} \) is the structure constant for \( G_i \). Note the peculiar role played by the \( i \)-index, which does not merely represent a product of groups of \( G_i \), due to the multiple \( \aleph_0 \) supersymmetries (3.1). In other words, the superficially simple-looking lagrangian (3.5) actually embraces infinitely many supersymmetries as hidden symmetries!

We stress here again the non-trivial feature of the non-Abelian SCS theory, namely even though the field strength term in the lagrangian (3.5) vanishes, the action still has topological meaning due to the non-Abelian term, in particular when the gauge group has non-trivial \( \pi_3 \)-homotopy groups. On top of that, we have established a system with \( \aleph_0 \) local supersymmetries.

4. Dimensional Reduction to \( \aleph_0 \) Theories

It is worthwhile to mention the important relationship of the \( \aleph_0 \) SCS theory with the 4D self-dual supersymmetric Yang-Mills (SDSYM) theory, which is the consistent background for the \( N = 2 \) open superstring [12]. The importance of the SDSYM theory is due to the general conjecture that all the supersymmetric integrable systems in \( D \leq 3 \) are generated by the SDSYM theory [13], which is the “supersymmetrization” of the non-supersymmetric conjecture by M.F. Atiyah [14]. As a matter of fact, a recent study [1] shows that a set of conjectural \( \aleph_0 \) supersymmetric integrable equations can be embedded into the \( \aleph_0 \) supersymmetric YM theory in 3D. Even though ref. [1] suggested that the 4D SDSYM theory does not seem to generate arbitrary number of \( \aleph_0 \) supersymmetries in 3D, we are going to show that there is a dimensional reduction scheme, such that the 4D SDSYM theory with finite \( N \) indeed generates infinitely many supersymmetries.

Our scheme of dimensional reduction is much similar to the method used in ref. [15], namely we can think of a torus compactification of \( N = 4 \) SDSYM [13] in 4D on \( \mathbb{R}^3 \otimes S^1 \). Here instead of directly using the \( N = 4 \) SDSYM in 4D, we use a \( N = 1 \) SDSYM in 4D [13] obtained from the former by some truncation of fields, and its action is

\[ I_{\text{SDSYM}}^{N=1} = \int d^4x \int d^2\hat{\theta} \hat{\Lambda}^{\hat{m}\hat{I}}\hat{W}_{\hat{m}\hat{I}}, \quad (4.1) \]

\[ = \int d^4x \left[ -\frac{1}{2} \hat{G}^{\hat{m}\hat{I}} \left( \hat{F}_{\hat{m}\hat{I}} - \frac{1}{2} \hat{\epsilon}_{\hat{m}\hat{n}\hat{\bar{r}}\hat{\bar{s}}} \hat{F}_{\hat{r}\hat{s}\hat{I}} \right) + i\hat{\rho}^{\hat{\alpha}\hat{I}}(\hat{\bar{\Gamma}}^{\hat{m}})_{\hat{\alpha}\hat{\beta}} \hat{D}_{\hat{m}}\hat{\bar{\lambda}}_{\hat{I}}^{\hat{\alpha}} + \hat{\varphi}^{\hat{I}}\hat{D}^{\hat{I}} + \hat{\psi}^{\hat{m}\hat{I}}\hat{\lambda}_{\hat{m}\hat{I}} \right]. \]
As usual [6], all the \textit{hatted} quantities and indices refer to 4D.

We now apply exactly the same dimensional reduction scheme as eqs. (3.3) through (3.13) in ref. [6], \textit{except} that now we introduce multiple gauge groups $G_{\text{total}} = G \otimes G \otimes \cdots \otimes G = G^d$ with the superfields $\hat{A}_i(z)^I_i (i = 1, 2, \ldots, d)$, where $I_i$ is for the adjoint representation for the $i$-th gauge group in $G^d$. However, we can equivalently use these $i$-indices as $\hat{A}_i(z)^I$, distinguishing the superfields. By this prescription for the torus compactification on $\mathbb{R}^3 \otimes S^1$, we get the action

$$I_{\text{N}=1}^{\text{SDSYM} \rightarrow \text{N}=1}^{\text{DR}} = \frac{1}{2} m \int d^3 x \int d^2 \theta \left[ A^i(z) W_i(z) - \frac{i}{6} (\gamma^m)^{\beta\gamma} A_{\beta i}^I(z) A_{\gamma i}^J(z) A_{mi}^K(z) \right]$$

$$= \int d^3 x \sum_i \left[ \frac{1}{2} m e^{\mu\rho\gamma} \left( F^{\mu\nu}_i A_{\rho i}^I - \frac{1}{3} f^{IJK} A_{\mu i}^I A_{\nu i}^J A_{\rho i}^K \right) + m \hat{\lambda}_i^I \lambda_i^I \right],$$

which is nothing but our $\mathbb{N}_0$ SCS (3.5)!

When there were no $i$-summation, this action would be just an $N = 1$ SCS theory in 3D. However, due to this $i$-summation, the system has more \textit{hidden} supersymmetries than expected, promoted to $\mathbb{N}_0$ supersymmetries under (3.1). The important point here is that even though we have originally $N = 1$ supersymmetry from the dimensional reduction, we have ended up with hidden promoted supersymmetries of arbitrary number. It is due to the \textit{on-shell} supersymmetry in the system that such promotions are possible.

Before concluding, we mention a similar dimensional reduction/truncation for the $\mathbb{N}_0$ supergravity. For the $\mathbb{N}_0$ supergravity we use the usual $N = 1$ supergravity in 4D [11] instead of SDSYM as the original theory, and we perform the dimensional reduction/truncation on $\mathbb{R}^3 \otimes S^1$. The $N = 1$ supergravity in 4D has the lagrangian [11]

$$\mathcal{L}_{SG}^{4D, N=1} = -\frac{1}{4} R(\hat{\omega}) + \frac{i}{2} \bar{\psi}_\mu \hat{\gamma}^\mu \hat{\rho} \hat{D}_\rho (\hat{\omega}) \hat{\psi}_\rho .$$

We use the first-order formalism [11], regarding $\hat{\omega}_\mu^{\hat{\rho}\hat{s}}$ as an independent variable, in order to simplify our dimensional reduction/truncation which is similar to (4.2):

$$\hat{\psi}_\mu(x, y) = \sqrt{2} \sum_{A=1}^\infty \psi^A(x) \cos(2\pi A y) , \quad \hat{\psi}_3(x, y) = 0 ,$$

$$\hat{e}_\mu^m = e_\mu^m(x) , \quad \hat{e}_3^3 = 1 , \quad \hat{e}_3^m = 0 , \quad \hat{e}_m^3 = 0 ,$$

$$\hat{\omega}_{\mu}^{mn} = \omega_{\mu}^{mn}(x) . \quad \hat{\omega}_3^{mn} = 0 , \quad \hat{\omega}_m^{(3)n} = 0 .$$

Here $\check{x}^\mu = x^\mu (\mu = 0, 1, 2)$ and $\check{x}^3 = y (0 \leq y < 1)$ coordinates represent the 3D part and the “extra” coordinate in 4D for the reduction/truncation. Notice that we setup the $y$-dependence only for the gravitino field.
Performing now our dimensional reduction/truncation, we get the $\mathcal{N}_0$ supergravity in the first-order formalism in 3D:

$$I_{SG}^{4D, N=1} = \int d^3x \int_0^1 dy \mathcal{L}_{SG}^{4D, N=1}$$

$$= \int d^3x \mathcal{L}_{\mathcal{N}_0SG}^{3D}$$

$$= \int d^3x \left[ -\frac{1}{4} R(\omega) + \sum_A \frac{i}{2} \bar{\psi}_\mu^A \gamma^{\mu\nu\rho} D_\nu(\omega) \psi_\rho^A \right].$$

Note that all the terms under the $y$-integration are always bilinear, since we are using the first-order formalism, and we have used the relations for $A, B \in \{1, 2, \ldots\}$ like

$$2 \int_0^1 dy \cos(2\pi Ay) \cos(2\pi By) = \delta_{AB}, \quad \int_0^1 dy \cos(2\pi Ay) \sin(2\pi By) = 0. \ (4.6)$$

We can also truncate any of $\psi_\mu^A$, so that the summation in (4.5) is a finite one from $A = 1$ to a finite but arbitrary $N$.

An interesting point here is that even though the original gravitino had finite degrees of freedom in 4D, it yields an infinite number of gravitini with infinitely many supersymmetries in 3D! In other words, the $\mathcal{N}_0$ supersymmetries emerge as hidden symmetries out of our dimensional reduction/truncation from 4D. This is possible thanks to the peculiar property of 3D where a supergravity multiplet has no physical degree of freedoms.\footnote{If we try a similar procedure in a higher-dimensional supergravities, such as from 11D to 10D, we lose consistent supersymmetries in 10D after the dimensional reduction. This is because on-shell $\mathcal{N}_0$ supergravity is not possible in 10D. To put it differently, the dimensional reduction scheme (4.4) does not maintain supersymmetries in general higher-dimensions.}

5. $\mathcal{N}_0$ BF Theories Coupled to SG

It is now straightforward to consider another important theory in 3D, namely BF theory. For this purpose we need two independent Abelian vector multiplets $(A_{\mu i}, \lambda_i)$ and $(B_{\mu i}, \chi_i)$, with the supertranslation rules

$$\delta_Q A_{\mu i} = + \frac{1}{2\sqrt{2}} \sum_j (L^A)_{ij} (\tau^A \gamma_\mu \lambda_j),$$

$$\delta_Q \lambda_i = - \frac{1}{2\sqrt{2}} \sum_j (R^A)_{ij} (\gamma^{\mu\nu} e^A) \tilde{F}_{\mu\nu j} + \frac{i}{2} \lambda_i (\tau^A \gamma^\mu \psi^A),$$

$$\delta_Q B_{\mu i} = + \frac{1}{2\sqrt{2}} \sum_j (L^A)_{ij} (\tau^A \gamma_\mu \chi_j),$$

$$\delta_Q \chi_i = - \frac{1}{2\sqrt{2}} \sum_j (R^A)_{ij} (\gamma^{\mu\nu} e^A) \tilde{G}_{\mu\nu j} + \frac{i}{2} \chi_i (\tau^A \gamma^\mu \psi^A). \ (5.1)$$
where we are considering only Abelian vector multiplet. The field strengths are defined by

\[ F_{\mu \nu i} \equiv \partial_\mu A_{\nu i} - \partial_\nu A_{\mu i} , \quad G_{\mu \nu i} \equiv \partial_\mu B_{\nu i} - \partial_\nu B_{\mu i} , \] (5.2)

and their hats denote the supercovariantizations as (3.4). The structure of this multiplet is similar to (3.1) except for the terms with gravitini, which depend on the structure of the lagrangian, like the auxiliary-field terms vanishing by the field equations.

The \( \mathbb{N}_0 \) BF lagrangian is given by

\[ L_{\mathbb{N}_0 BF} = \frac{1}{2} \epsilon^{\rho \mu \nu} \sum_i B_{\rho i} F_{\mu \nu i} + e \sum_i \left( \tilde{\lambda}_i \chi_i \right) . \] (5.3)

The invariance check of (5.3) is easy, because the only effect by local supersymmetry is the \( \psi \chi \lambda \)-terms arising from the second term, which cancel by themselves by the help of the gravitino-dependent terms in (5.1).

We give also an alternative \( \mathbb{N}_0 \) BF theory based on the on-shell 3D, \( \mathbb{N}_0 \) -supersymmetric vector multiplet with field content \( (A_\mu, B_i^j, \lambda_\alpha, \hat{\lambda}_{\alpha k}^k) \). The variations of these fields are given by [1]

\[
\begin{align*}
\delta_Q A_\mu &= -i \epsilon^\alpha \gamma_\mu \lambda_\alpha^1, \\
\delta_Q B_i^j &= \epsilon^\alpha \left( f_{1i}^j \lambda_\alpha + (L_1)_i^j \hat{\lambda}_{\alpha k}^k \right), \\
\delta_Q \lambda_\alpha &= \epsilon^\beta \left( f_{1i}^j \lambda_\alpha + d_{i j} (f_{1i}^j) \lambda_\alpha \right), \\
\delta_Q \hat{\lambda}_{\alpha k}^k &= \epsilon^\beta \left( f_{1i}^j \lambda_\alpha + d_{i j} (f_{1i}^j) \lambda_\alpha \right).
\end{align*}
\] (5.4)

The existence of this on-shell supersymmetric representation suggests that there is another 3D, \( \mathbb{N}_0 \) -supersymmetric vector multiplet that is dual to the one above in such a way that a 3D, \( \mathbb{N}_0 \) -supersymmetric BF action exists. This purported theory in the special case of \( N = 4 \) has already been constructed [16]. In the following, we generalize this result to all values of \( N \).

The first step in our generalization is to note that the fields of our expected on-shell dual 3D, \( \mathbb{N}_0 \) -supersymmetric vector multiplet can be written in the form \( (B_\mu, \beta_\alpha^1, \beta_{\alpha k}^k, d_i^j) \). We want this vector multiplet to be dual to the one above in the sense that its components can appear in an action that contains the usual BF coupling between \( A_\mu \) and \( B_\mu \). For this purpose we write,

\[ L'_{\mathbb{N}_0 BF} = \frac{1}{2} \epsilon^{\mu \nu \rho} B_\mu F_{\nu \rho} - \beta^1 \lambda_\alpha - d^{-1} \beta_{\alpha k}^k \hat{\lambda}_{\alpha k}^k + d^{-1} d_i^j B_i^j , \] (5.5)

and in such a way that the action is a supersymmetric invariant. The requirement that this action is left invariant under a supersymmetric variation can be used to determine the
appropriate variations for \((B_\mu, \beta_\alpha, \tilde{\beta}_\alpha, d_\mu)\)

\[
\begin{align*}
\delta_Q B_\mu &= -i\epsilon^{\alpha_1} (\gamma_\mu)_{\alpha\beta} \beta_1^\beta \\
\delta_Q \beta_\alpha &= \epsilon_\alpha \left[ i \frac{1}{2} (\gamma^\mu)_{\alpha\beta} \delta_{\mu\nu} F_{\nu\nu}(B) - d^{-1} (f_{1j})_i j d_i^j \right] \\
\delta_Q \tilde{\beta}_\alpha &= \epsilon_\alpha \left[ d_k^j (L_1)_j^k + d^{-1} (L_1)_k^i (f_{1j})_i j d_i^j \right] \\
\delta_Q d_\mu^j &= i\epsilon^{\alpha_1} (\gamma^\mu)_{\alpha\gamma} \partial_\mu \left[ \beta^\gamma (f_{1j})_i j + \tilde{\beta}^\gamma (R_1)_j^i \right].
\end{align*}
\] (5.6)

In closing this section, we note that the existence of the this 3D, \(\aleph_0\)-supersymmetric BF action together with the existence of 3D, \(\aleph_0\)-supersymmetric scalar multiplets should naturally lead to 3D, \(\aleph_0\)-supersymmetric anyonic models. A further challenge will be to investigate the further existence of 3D, \(\aleph_0\)-supersymmetric CS actions that possess anyonic extensions. Finally we note that given the action in (5.5), we expect a further duality transformation exists that permits the last term to be replaced by BF-type terms. (See the two different \(N = 2\) theories of [6].)

6. \(\aleph_0\) Supergravity Liouville Theory in 2D

We have so far discussed \(\aleph_0\) supergravity theories only in 3D. As an interesting application of such theories, we perform the dimensional reductions of them to get 2D \(\aleph_0\) theories. A typical example we give here is \(\aleph_0\) supergravity Liouville theory. In this paper, we skip all the details of the dimensional reduction but only the final results which will be of more interest for other applications.

Our metric in 2D is \((\eta_{mn}) = (\eta_{00}, \eta_{11}) = (1, -1)\), and accordingly we have \(\gamma^{mn} = +\eta^{mn} + e^{mn}\gamma_5\). Our multiplets are the \(\aleph_0\) supergravity \((\epsilon^m_\mu, \psi^A_\mu)\) and a dilaton multiplet \((\varphi, \chi^A)\). Here the indices \(A, B, \ldots = 1, ..., N \rightarrow \infty\) are for the \(N \rightarrow \infty\)-extended supersymmetries. The invariant lagrangian \(\mathcal{L}_0 + \mathcal{L}_g\) for our \(\aleph_0\) -extended supergravity coupled to \(\aleph_0\) Liouville theory is given by

\[
\begin{align*}
e^{-1} \mathcal{L}_0 &= + \varphi R - 2e^{-1} \epsilon_{\mu\nu} \overline{\chi} \gamma_5 R_{\mu\nu} + \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{i}{2} (\overline{\chi} \gamma^\mu D_\mu \chi) \\
&\quad - \frac{1}{2} (\overline{\psi}_{\mu} \gamma^{\mu\nu} \gamma^\nu \chi) (\partial_\nu \varphi + \tilde{D}_\nu \varphi), (6.1)\\
e^{-1} \mathcal{L}_g &= - 8g^2 e^\varphi + 4ige^{\varphi/2} (\overline{\psi}_{\mu} \gamma^\mu \chi) - 8ge^{-1} e^{\varphi/2} \epsilon_{\mu\nu} (\overline{\psi}_{\mu} \gamma_5 \psi_{\nu}) + ge^{\varphi/2} (\overline{\chi} \chi).
\end{align*}
\]
where \((\chi \gamma_5 R) \equiv (\chi^A \gamma_5 R_{\mu
u})\), etc. Eq. (6.1) is invariant under the supersymmetry

\[
\delta_Q e^m_{\mu} = -2i(\chi^A \gamma^m \psi^A_{\mu}) \equiv -2i(\chi^m \psi_{\mu}) ,
\]

\[
\delta_Q \psi^A_{\mu} = \partial_{\mu} \epsilon^A + \frac{1}{2} \omega_{\mu} \gamma_5 \epsilon^A - \frac{i}{32} \gamma_{\mu} \gamma_{\nu} \epsilon^B (\chi^A \gamma^B_{\nu} \chi^B) 
+ \frac{1}{4} \gamma_{\nu} \chi^A (\chi^B_{\nu} \psi^B_{\mu}) + \frac{1}{4} \gamma^\nu (\chi^B_{\nu} \psi^B_{\mu})
\]

\[
\delta_Q \varphi = (\chi^A \chi^A) \equiv (\chi^A)
\]

\[
\delta_Q \chi^A = -i\gamma^\mu \epsilon^A \hat{D}_{\mu} \varphi + 4g e^{\varphi/2} \epsilon^A - \frac{1}{8} \gamma^\mu \epsilon^B (\chi^A \gamma^B) ,
\]

\[
\delta_Q \omega_{\mu} \equiv +ie^{-1}e^{\rho\sigma}(\bar{\epsilon}^A \gamma_{\mu} R_{\rho\sigma})
\]

Here we have defined

\[
R \equiv +2e^{-1}\epsilon^{\mu\nu} \partial_{\mu} \omega_{\nu} , \quad R_{\mu\nu}^A \equiv D_{\mu} \psi^A_{\nu} - D_{\nu} \psi^A_{\mu} ,
\]

\[
D_{\mu} \epsilon^A \equiv \partial_{\mu} \epsilon^A + \frac{1}{2} \omega_{\mu} \gamma_5 \epsilon^A , \quad \omega_{\mu} \equiv -e^{-1}e^{\rho\sigma} (e_{\mu}^m \partial_{\rho} e_{\sigma m} + \bar{\epsilon} \gamma_{\mu} \psi_{\rho\sigma})
\]

The constant \(g\) controls the potential term with the exponential function of the dilaton \(\varphi\) as usual in a Liouville theory in 2D. Note that in the case of simple supersymmetry \((N = 1)\), all the fermionic bilinear terms in (6.2) disappear.

This invariant lagrangian and the supersymmetry transformation rules are fixed by the usual method, namely cancelling the derivative on the parameter in the supersymmetric transformation of fermionic field equations by adding fermionic quartic terms to the lagrangian and the fermionic bilinear terms to the transformation rules for fermions. Notice that there is no explicit quartic terms in the lagrangian or bilinear fermions in the supersymmetry of fermions when \(g = 0\).

7. Concluding Remarks

In our paper we have presented an amusing result that on-shell Poincaré supergravity in 3D can be extended up to \(N = \infty\) with the minimal field content only with the dreibein and the gravitini which we call minimal \(N_0\) supergravity. This system can be further coupled to CS as well as BF theories. This result is also consistent with the recent on-shell results in [4], when central charges are present.

We have overcome the previous difficulty with supersymmetrizing CS theories beyond \(N = 4\) [6], by introducing the \(L\) and \(R\)-matrices. In the on-shell formulation, all the vanishing terms by field equations can be also understood as the extra symmetries [6]. This prescription seems possible only in 3D due to the special property of the dreibein and gravitini.
which are essentially non-physical, making the on-shell closure of the gauge algebra simple. The usual upper limit for the number of supersymmetries does not apply to 3D because of these non-physical dreibein and gravitini. If we try to couple supergravity to “physical” scalar multiplet with the usual kinetic terms, we encounter an obstruction against consistent Noether couplings. This fact also shows the importance of the non-physical property of all the fields as the key feature of the system. In the context of conformal supergravity [6], a similar phenomenon in 3D has been already encountered, in which arbitrary number of supersymmetries up to infinity are allowed. To our knowledge, however, our system is the first example with $\aleph_0$ Poincaré supersymmetries, including also the “matter” multiplets.

We can re-interpret our results for $\aleph_0$ SCS theories as follows. Reviewing (3.5), we can regard it just as an $N = 1$ SCS theory for a product of the same Yang-Mills gauge group: $G^n \equiv G \otimes G \otimes \cdots \otimes G$, and we are re-labeling $A_{\mu I}^I$ as $A_{\mu i}^I$ etc. as given in [6]. However, such a system has hidden enlarged supersymmetries promoted to $N = n$, where $n$ coincides that in $G^n$, and the enlarged supersymmetry is characterized by the $L$ and $R$ matrices in (3.2). As a matter of fact, this new observation has overcome the previous difficulty to go beyond $N = 4$ in ref. [6].

Our result in three-dimensional systems is natural also from the viewpoint of strong-weak coupling duality suggested by Witten [7]. The appearance of arbitrarily many supersymmetric gravitino fields in three-dimensions may be understood as a reminiscent of “dimensional reduction” of some four-dimensional theory, in which there is a finite number of gravitini. Upon a compactification of such a theory on $\mathbb{R}^3 \times S^1$ with an $S^1$ of an infinitely large radius will yield a set of infinitely many gravitini. Thus from the 4D viewpoint this limit is understood as the weak coupling, while from the 3D viewpoint this limit can be shown to be equivalent to taking the string coupling $\lambda \to \infty$ limit [7]. In an ordinary “reduction” into other dimensions, this usually fails due to the inconsistency with the couplings of gravitini. The special feature of the three-dimension is that the gravitini are no longer physical fields, but rather non-propagating fields, that enable us to construct couplings to CS and BF theories which also have only non-physical fields.

This feature can be more elucidated, when we compare it with the conformal supergravity in 2D [17]. In the latter, we do not have any field equations such as the vanishing gravitino field strengths, so that even on-shell closure of gauge algebra was not manifest, and therefore we needed more field to realize its closure off-shell. It is very peculiar to the 3D theories, where lagrangians produce vanishing field strengths, which make the closure of the gauge

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7This is not “reduction” in its strict sense, because the radius of the $S^1$ will be infinity instead of zero.
algebra manifest.

The importance of 3D systems of this type is being increasingly recognized. Most recently, it has been suggested [18] that the appearance of the non-perturbative potential of 4D heterotic string theory is governed by 3D physics. An intriguing question to ask is whether there exists an $\aleph_0$ string-like theory that incorporates all of such theories.

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**Added Note in Proof**

After the completion of this paper, the authors became aware of a work (Devchand and Ogievetsky, “Interacting Fields of Arbitrary Spin and $N > 4$ Supersymmetric Self-Dual Yang-Mills Equations” ICTP preprint IC/96/88, hep-th/9606027) which reports to prove the existence of $\aleph_0$ supersymmetric self-dual Yang-Mills theory in 4D. This provides independent and additional support for the new class of integrable systems proposed in reference [1].
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