Group testing schemes from codes and designs
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Abstract

In group testing, simple binary-output tests are designed to identify a small number $t$ of defective items that are present in a large population of $N$ items. Each test takes as input a group of items and produces a binary output indicating whether the group is free of the defective items or contains one or more of them.

In this paper we study a relaxation of the combinatorial group testing problem. A matrix is called $(t, \epsilon)$-disjunct if it gives rise to a nonadaptive group testing scheme with the property of identifying a uniformly random $t$-set of defective subjects out of a population of size $N$ with false positive probability of an item at most $\epsilon$. We establish a new connection between $(t, \epsilon)$-disjunct matrices and error correcting codes based on the dual distance of the codes and derive estimates of the parameters of codes that give rise to such schemes. Our methods rely on the moments of the distance distribution of codes and inequalities for moments of sums of independent random variables. We also provide a new connection between group testing schemes and combinatorial designs.

Keywords: Group testing, disjunct matrices, error-correcting codes, constant weight codes, dual distance, combinatorial designs.

I. INTRODUCTION

Suppose that the elements of a finite population of size $N$ contain a small number of defective elements. The elements are tested in groups, and the collection of tests is said to form a group testing scheme if the outcomes of the tests enable one to identify any defective configuration size at most $t$. Let the number of tests in a group testing scheme be $M$. Then constructing a non-adaptive group testing scheme is equivalent to constructing a binary test matrix of dimensions $M \times N$ where the $(i, j)$-th entry is 1 if the $i$th test includes the $j$th element and is 0 otherwise. Each row of the matrix corresponds to a test, and the result of this test is positive if the indices of ones in the row have a nonempty intersection with the indices of the defective configuration. The smallest possible number of tests in terms of the total number of subjects $N$ and the maximum number of defective elements $t$ is known to satisfy $M = \Omega\left(\frac{t^2}{\log t} \log N\right)$; see [7], [9], [30].

A construction of group testing schemes using error-correcting codes and code concatenation appeared in the foundational paper by Kautz and Singleton [15]. They introduced a two-level construction in which a $q$-ary ($q > 2$) Reed-Solomon code is concatenated with a unit-weight binary code. The resulting vectors are used as columns of the testing matrix. Since every symbol of the Reed-Solomon codeword is replaced by a binary vector of Hamming weight one, the overall code is formed of codewords of a fixed Hamming weight. Overall, with $M = O(t^2 \log^2 N)$ tests this scheme identifies any defective

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configuration of size up to \( t \). More importantly, this construction offered a general method of obtaining test matrices from error-correcting codes. Many later constructions of group testing schemes also rely on Reed-Solomon codes and code concatenations; among them [8], [13], [24], [32]. An explicit constructions of non-adaptive group testing schemes with \( M = O(t^2 \log N) \) was presented in [27].

It has been suggested to construct schemes that permit a small probability of error (i.e., allowing false positives). Such schemes were considered under the name of almost disjunct matrices or weakly separated designs in [20], [21], [37] and independently in [18]. With this relaxation it is possible to reduce the number of tests to be proportional to \( t \log N \) [37]; however, this result is not constructive. In terms of constructive results, a scheme with \( t^{3/2} \sqrt{\log N} \) tests was presented in [22]. The recent work of Gilbert et al. [10] suggests a way to construct weakly separated designs with \( O(t \text{poly}(\log N)) \) tests by partitioning the subjects into blocks of equal size and using optimal non-adaptive tests independently for each block. Finally, an explicit (non-probabilistic) construction of nonadaptive schemes with the number of tests proportional to \( t \log^2 N/\log t \) was presented in [23]. All these works allow a small probability of existence of false positives.

The construction of Kautz and Singleton [15] and many others above are based on constant weight error-correcting codes. Estimates of the parameters of the group testing schemes from constant weight codes were obtained using the minimum distance of the code [15] and more recently using the average distance of the code [22], [23].

Our present work takes a different approach, relating construction of almost disjunct matrices and the dual distance of codes. Our main contribution consists of a refined analysis of constructions of group testing schemes that relates the number of tests \( M \) to the dual distance of the (constant weight) code and moments of the distance distribution.

The main ideas of our analysis involve techniques from coding theory, algebraic combinatorics, and probability, and are as follows. We form a group testing matrix whose columns are the codewords of a binary constant weight code \( C \). A simple observation given in Prop. 4 connects the probability \( P \) of violating the group testing recovery condition to the distribution of distances in the code \( C \) (the probability comes assuming the defective elements to be uniformly distributed). This enables us to transform the problem of estimating \( P \) to the question of estimating moments of the distance distribution of the code. In the next step, we consider examples of group testing schemes obtained from several known families of nonbinary codes via the Kautz-Singleton map. We obtain explicit expressions for the moments of order that does not exceed the dual distance of the nonbinary code (the strength of the combinatorial design formed by the code). Classical inequalities for the moments of sums of independent random variables give upper bounds on the moments, resulting in upper estimates of the probability \( P \).

Continuing the above line of thought, we turn to binary constant weight codes. We again connect the problem of estimating the probability \( P \) with a question about moments of the distance distribution of constant weight codes. Using simple facts from the theory of association schemes, we find an exact expression for the moments of order less than the dual distance of the code. Bounds for the probability \( P \) are again obtained using classical inequalities from probability theory. We note that, in the case of constant weight codes, the dual distance is related to the strength of the design formed by the codewords, and so the problem of constructing good almost disjunct matrices can be expressed in terms of combinatorial designs. This approach to the construction of group testing schemes does not involve the Kautz-Singleton construction.

Apart from [22], [23], the connection between error-correcting codes and weakly separated designs was known only for the very specific family of maximum distance separable codes [2], [18], for which much more than the dual distance of the binary code is known.
A. Contributions of this paper

In this work we estimate parameters of almost disjunct matrices using the distance distribution of codes.

1) Matrices from nonbinary codes: In this part, we show that group testing schemes from Reed-Solomon (RS) codes and codes on algebraic curves have the following parameters while allowing a small probability of false positives (that approaches zero with $N$ for RS codes and with the size of the code alphabet $q$ for the other cases):

- **RS codes**
  - $t = q^2$
  - $N = q^{2q^2}$
  - $M = O(\max\{t^2, t(\log N/\log t)^3\})$

- **Hermitian codes ($q$ is a prime power)**
  - $t = q^2$
  - $N = q^{2q^2}$
  - $M = q^5$

- **Suzuki codes ($q$ is a power of 2)**
  - $t = q^2$
  - $N = (2q)^{2q^3+q+1}$
  - $M = 8q^6$

Note that in the case of RS codes, we directly give an expression for $M$ in terms of $t$ and $N$ rather than individual expressions. This can be done because the relation between $N, t,$ and $q$ is given in the form of an inequality (see (13) and the discussion after it below). In the other two examples, the values of the parameters $N, t,$ and $M$ are uniquely determined by the value of $q$.

2) Almost disjunct matrices from constant weight codes: Our main contribution in this part can be summarized as follows. We show that a constant weight code of dual distance $d'$ gives rise to a group testing scheme that can, with $O(t(d')^2)$ tests, identify all items in a random defective configuration of size $t$ with probability of false positive for an element (outside of defective set) at most $\exp(-d')$ (see, Corollary 12).

To be able to construct almost disjunct matrices, we turn to known results about the existence of constant weight codes with large dual distance (i.e., of combinatorial designs; see Def. 3 and the discussion after it). This problem was addressed in a recent paper [17]. According to it, there exist codes of length $M$, size $N$, and dual distance $d'$ such that,

$$d' = \Omega\left(\frac{\log N}{\log(M/d')}\right).$$

(1)

Relying on this result, we obtain group testing schemes with $O(t\log^2 N/\log^2 t)$ tests that can identify all items in a random defective configuration of size $t$ with probability of false positive for an element at most $\exp(-\log N/\log t)$; see Corollary [12]. Comparing this with [23], while we improve on the number of tests by a factor of $\log t$, we obtain a slower decline of the probability of false positives, which was $O(1/poly(N))$ therein.

A caveat about the result of [17] given in (1) is that it is an existence claim rather than an explicit construction. However, we emphasize that the main new idea of our work is the connection between the dual distance of codes and group testing properties. Hence whenever constructions of combinatorial designs are available, we can use them as building blocks for constructing group testing scheme via the approach discussed in the paper. Indeed, recently we have found [36] that constant weight codes formed by the fixed-weight codewords of BCH code perform consistently better compared to the random test matrices in terms of false positives, which gives empirical evidence that combinatorial designs are good candidates for nonadaptive group testing (some more details about [36] are given in the concluding Section [IV]).

B. Plan of the paper

The paper is organized as follows. Some preliminaries about group testing and the mathematical tools that we use are provided in Section [II] Group testing schemes from nonbinary codes with large dual
distance are discussed in Section [III-A] where the main result is Theorem 5, and schemes from binary constant weight codes (designs) are discussed in Section [III-C]. Here the main results are Theorem 10 and Corollaries 12 and 13.

II. PRELIMINARIES

A. Group testing schemes

Define the support of a vector supp(x), x ∈ F_q^n as the set of coordinates where x has nonzero entries. The support of a set of vectors X = {x_i, i ≥ 1} is the union of supports ∪_{i≥1} supp(x_i).

Definition 1: An M × N binary matrix A is called t-disjunct if the support of any of its columns is not contained in the union of the supports of any other t columns.

It is easy to see that a t-disjunct matrix gives a group testing scheme that identifies any defective set up to size t. Conversely, any group testing scheme that identifies any defective set up to size t must be a (t − 1)-disjunct matrix [6]. To a great advantage, disjunct matrices support a simple identification algorithm that runs in time O(Nt). Indeed, any element that participates in a test with a negative outcome is not defective. After we perform all the tests and weed out all the non-defective elements, the disjunctness property of the matrix guarantees that all the remaining elements are defective.

A few words on notation. Let [N] := {1, 2, . . . , N} and let P_t(N) denote the set of t-subsets of [N]. The usual notation for probability Pr is used to refer a probability measure which will be understood from the context. Separate notation will be used for some frequently encountered probability spaces. In particular, we use Pr_t to denote the uniform probability distribution on P_t(N). If we need to choose a random t-subset I and a random index in [N] \ I, we use the notation Pr_t(I).

Definition 2: For any ε > 0, an M × N matrix A with columns a_1, a_2, . . . , a_N, is called (t, ε)-disjunct if

Pr_t({I ∈ P_t(N), j ∈ [N] \ I : supp(a_j) ⊆ ∪_{k∈I} supp(a_k)}) ≤ ε.

In other words, the union of supports of a randomly and uniformly chosen subset of t columns of a (t, ε)-disjunct matrix does not contain the support of any other random column with probability at least 1 − ε.

Note that, this definition crucially differs from that of (t, ε)-disjunct matrices appeared in [23]. In the definition of [23], the probability Pr_t({I ∈ P_t(N), ∀ j ∈ [N] \ I : supp(a_j) ⊆ ∪_{k∈I} supp(a_k)}) must be upper bounded by ε, instead of what we have.

The next fact follows from the definition of disjunct matrix and the decoding procedure [6, p. 134].

Proposition 1: A (t, ε)-disjunct matrix defines a group testing scheme that can identify all items in a random defective configuration of size t, and with probability at most ε identifies any randomly chosen item outside of the defective configuration as defective (false-positive).

Remark 1: This definition implies that the probability that there is a false positive is bounded above by ε(N − t), so as N increases, we need that ε goes to zero at least linearly with N.

For a fixed N, unless ε < 1/√N, the average number of false positives in the scheme given by a (t, ε)-disjunct matrix will be greater than the actual number of defectives. However even in that case, the tests will output a subset of [N] of a vanishingly small proportion that includes all of the t defective items.

B. Codes and the Kautz-Singleton construction

Let Q = {a_1, . . . , a_q} be a finite set (an alphabet). A code of length M is a subset of the set Q^M. The minimum Hamming distance between distinct codewords of C is called the distance of the code. We use
the notation \((M, N, d)\) to refer to a code \(C\) of length \(M\), cardinality \(N\) and distance \(d\). If \(\mathbb{F}\) is a finite field and \(C\) is a linear subspace of the vector space \(\mathbb{F}^M\), we call \(C\) a linear code. If all the codewords of the code \(C\) contain exactly \(w\) nonzero entries, we call it a constant weight code and use the notation \((M, N, d, w)\) (clearly, constant weight codes cannot be linear).

Kautz and Singleton [15] made two observations. First, they noted that constant weight binary codes give rise to disjunct matrices. More precisely, the following is true.

**Proposition 2:** An \((M, N, d, w)\) constant weight binary code \(C\) provides a \(t\)-disjunct matrix, where

\[
t = \left\lfloor \frac{w-1}{d/2} \right\rfloor.
\]

**Proof:** Write the codewords of \(C\) as the columns of an \(M \times N\) matrix. The intersection of supports of any two columns has size at most \(w - d/2\). Hence if \(w > t(w - d/2)\), the support of any column will not be contained in the union of supports of any \(t\) other columns.

This proposition implies that a group testing scheme can be obtained from constant weight codes with large distance. In [15], it is observed that such codes can be obtained from non-constant-weight \(q\)-ary codes in which every symbol is replaced by its binary indicator vector in the alphabet. More formally, let \(c = (c_1, \ldots, c_M) \in C\) and let \(c_i = a_{ij}\), where \(a_{ij} \in \mathbb{Q}, i = 1, \ldots, M\). The Kautz-Singleton map transforms \(c\) to a binary vector of length \(qM\) by mapping \(c_i, i = 1, \ldots, M\) to a binary vector of length \(q\) that contains 1 in position \(a_{ij}\) and zeros elsewhere. We note that the image of this map applied to any code is a binary constant weight code, and that applying this map to any two \(q\)-ary vectors \(c_1, c_2\) results in a pair of binary vectors with Hamming distance twice the distance \(d(c_1, c_2)\). Applying the Kautz-Singleton map to a \(q\)-ary \((n, N, d)\) code \(C\), we obtain a constant weight code with the parameters \((M = qn, N, 2d, w = n = M/q)\).

C. Distance distribution of codes

Let us recall some concepts related to the distance distribution of codes. More information about them can be found, for instance, in [4], [5] or [19]. Let \(C \subset \mathbb{F}^n_q\) be a code. The distance distribution of \(C\) is the set of numbers \((A_0, A_1, \ldots, A_n)\), where

\[
A_i = \frac{1}{|C|} |\{(x, y) \in C^2 : d(x, y) = i\}|, \quad i = 0, 1, \ldots, n
\]

is the average number of ordered pairs of codewords with Hamming distance \(i\). Clearly, the distance of \(C\) equals the smallest \(i \geq 1\) such that \(A_i > 0\).

Define the **dual distance** \(d'\) of \(C\) as follows:

\[
d'(C) = \min\{j \geq 1 : A'_j := \sum_{i=0}^{n} A_i K_j(i) > 0\},
\]

where \(K_j(i)\) is the value of the Krawtchouk polynomial of degree \(j\) [19 p.129] given by

\[
K_j(i) = \sum_{l=0}^{j} (-1)^l \binom{j}{l} \binom{n-i}{j-l} (q-1)^{i-l}.
\]

If \(C\) is a linear code, then \(d'\) equals the distance of the dual code \(C^\perp\). We also note that \(A'_0 = 1\).

Similar definitions can be given for binary constant weight codes. Let \(J^w_M\) be the set of all binary vectors (of length \(M\)) with \(w\) ones and let \(C \subset J^w_M\) be a code. We use the notation \(C(M, N, d, w)\) to
refer to a constant weight code of length $M$, cardinality $N$, distance $d$ and weight $w$. Define the distance distribution of the constant weight code $C$ as follows:

$$b_i = \frac{1}{|C|} |\{(x, y) \in C^2 : w - |\text{supp}(x) \cap \text{supp}(y)| = i\}|$$

for $i = 0, 1, \ldots, w$. The dual distance $d'$ of the constant weight code $C$ is defined as

$$d'(C) = \min \left\{ j \geq 1 : b'_j := \frac{1}{|C|} \sum_{i=0}^{w} b_i Q_j(i) > 0 \right\},$$

where $Q_j(i)$ is the value of the Hahn polynomial of degree $j$; see (29) and [19, p.545].

As is well known [4], a binary constant weight code with dual distance $d' = r + 1$ forms a combinatorial design of strength $r$.

**Definition 3:** An $r$-design (in more detail, an $r$-$(n, w, \lambda)$ design) is a collection $C$ of $w$-subsets of an $n$-set $V$, called blocks, such that every $r$ elements of $V$ are contained in the same number $\lambda$ of blocks. An $r$-design is also called a design of strength $r$.

Below we use designs of a given strength to construct almost disjunct matrices. The use of combinatorial designs for constructing disjunct matrices is not new, see, e.g., Sect. 7.4 of [6]. Special cases of designs have been used to construct nonadaptive group testing matrices for some particular parameters [34, Sec. 11.3], [3, Ch.56]. However the conclusion in [6, p. 146], is that disjunct matrices obtained from designs are of little interest because the number of tests $M$ in such constructions is too large compared to the number of subjects $N$. All of the cited works estimate the parameters of the design that give rise to a disjunct matrix. We take a different approach, using designs formed by codes to analyze the distance distribution of codes and estimate the probability that a chosen $t$-set of columns violates the disjunct condition. As a result, we are able to construct almost disjunct matrices with a small number of tests $M$ compared to $N$.

**D. Sums of independent random variables**

Here we list several classical results about sums of independent random variables and bounds on the moments of such sums. We begin with an inequality due to Hoeffding.

**Theorem 3:** [12, Thm. 4] Let $C$ be a finite set of numbers. Let $X_i, i = 1, \ldots, t$ be random samples without replacement from $C$ and let $Y_i, i = 1, \ldots, t$ be random samples with replacement from $C$. If the function $f(x)$ is continuous and convex, then

$$E f \left( \sum_{i=1}^{t} X_i \right) \leq E f \left( \sum_{i=1}^{t} Y_i \right).$$

We also use the following inequalities for moments of sums of independent random variables. Let $\{X_i, i = 1, \ldots, t\}$ be a sequence of independent random variables with zero means. Then the following **Marcinkiewicz–Zygmund inequality** holds true [28]:

$$E \left| \sum_{i=1}^{t} X_i \right|^\ell \leq C_\ell t^{\ell/2-1} \sum_{i=1}^{t} E|X_i|^\ell, \quad \ell \geq 2,$$

where $C_\ell$ depends only on $\ell$. In particular, one can take $C_\ell \leq (3\sqrt{2})^{\ell/2}$. Assume in addition that $E|X_i|^\ell < \infty$ for all $\ell > 2$. Then the following **Rosenthal inequality** holds true [29]:

$$\max \left\{ \sum_{i=1}^{t} E|X_i|^\ell, \left( \sum_{i=1}^{t} E|X_i|^2 \right)^{\ell/2} \right\}$$
where the factor $K_\ell$ does not depend on $t$. Moreover, one can take $K_\ell = (2\ell/\log \ell)^\ell$ [14].

Observe that inequalities (5), (6) belong to a large group of Khinchine-type inequalities and their extensions to martingales. It is possible to further optimize the constant in (6) as well as to establish other versions of (5) and (6), see e.g., [14], [25]. The choice of the inequality depends on the relation between the parameters of the group testing scheme, and below we do not attempt to optimize the constants for the large number of possible cases.

### III. Almost disjunct matrices from codes

The main contribution of this paper is a refined analysis of the distance distribution of codes that gives rise to almost disjunct matrices (see Def. 2). The dual distance of the code defined in sections III-A and III-C plays an important role in the analysis.

Our goal is to design an $M \times N$ matrix $A$ such that $t$ randomly chosen columns do not contain the support of any other of its columns. As above, we form the matrix using the codewords of an $(M, N, d, w)$ constant weight code $C$ as the columns. Let the codewords of $C$ be $x_1, x_2, \ldots, x_N$ and let $d_{ij} := d(x_i, x_j)$. Having in mind the design of almost disjunct matrices, we begin with the following extension of Prop. 2.

Denote by $P_A(t, N)$ the probability of violating the conditions of Def. 2

$$P_A(t, N) := P_{R_t}(\{I \in P_t(N), j \in [N] \setminus I : \text{supp}(x_j) \subseteq \bigcup_{k \in I} \text{supp}(x_k)\}).$$

It is clear, any matrix $A$ is $(t, P_A(t, N))$-disjunct.

**Proposition 4:** The following estimate holds true:

$$P_A(t, N) \leq P_{R_t}\left(\{I \in P_t(N), j \in [N] \setminus I : w \leq \sum_{k \in I} (w - d_{jk}/2)\}\right).$$

In the next sections we develop new ways of analyzing almost disjunct matrices from various families of codes, and also connect them with combinatorial designs. In particular, we examine two different ways of constructing almost disjunct matrices from codes using the above proposition.

#### A. Almost disjunct matrices from nonbinary codes

In this section we estimate the probability $P$ in (7) for nonbinary linear codes used in the Kautz-Singleton construction.

Consider a $q$-ary $(n, N, d)$ code. To construct a group testing (almost disjunct) matrix we map every symbol of the codeword to a binary vector of $(q - 1)$ 0s and one 1 in the location that corresponds to the value of the symbol. Applying this mapping, we obtain a set of binary vectors of length $M$ and constant weight $w = n = M/q$. The parameters of the resulting binary constant weight code $D$ are $(M = qn, N, 2d, w = n = M/q)$.

The main result proved in this part is given by the following statement.

**Theorem 5:** Let $C$ be a $q$-ary $(n, N)$ code with dual distance $d'$ and let $M$ be an $M \times N$ matrix constructed from it using the Kautz-Singleton mapping. If $t \leq q$, then $M$ is a $(t, \epsilon)$-disjunct matrix with

$$\epsilon \leq B(\ell, t) \left(\frac{e\ell(q - 1)}{2n(q - t)^2}\right)^{\ell/2} \sum_{i=0}^{\ell/2} \left(\frac{(q - 1)\ell}{2ne}\right)^i,$$

(8)
for any even $\ell < d'$, where $B(\ell, t) = \min \left\{ (18t\ell)^{\ell/2}, t^\ell \right\}$.

The examples given below show that it is possible to choose specific code families so that estimate (8) is nontrivial. To prove Theorem 5 we need several auxiliary statements.

Choose two codewords from the code $C$ randomly and uniformly with replacement, and denote by $Z$ the random variable whose value is the distance between these codewords. Clearly, the distribution of $Z$ is given by

$$\Pr(Z = i) = \frac{A_i}{N}, \quad i = 0, \ldots, n.$$  

Define $\theta := (q - 1)/q$.

We have the following proposition.

**Proposition 6:** Let $t \leq q$. Then

$$P_A(t, N) \leq B(\ell, t) \left( \frac{q}{n(q-t)} \right)^\ell \mathbb{E}(\theta n - Z)^\ell, \quad (9)$$

where $\ell \geq 2$ is an even integer.

**Proof:** Let $C$ be the $q$-ary code defined before the theorem, and let $D$ be the code obtained from $C$ by applying the Kautz-Singleton mapping. Given two codewords $x_j, x_k \in C$ let $\delta_{jk} = d(x_j, x_k)$ and let $d_{jk} = 2\delta_{jk}$ be the distance between their images in $D$. We will estimate from above the right-hand side in (7). With the current notation, the condition in (7) becomes $\sum_{k \in I} \delta_{jk} \leq n(t - 1)$. Using the assumption $t \leq q$ yields $1 - \frac{t}{q} \geq 0$, and we can relax the inequality in (7) to the following estimate:

$$P_A(t, N) \leq P_{K^t}(\left\{ I \in \mathcal{P}_t(N), j \in [N] \setminus I : \sum_{k \in I} (\theta n - \delta_{jk}) \geq n(1 - \frac{t}{q}) \right\})$$

$$\leq P_{K^t}(\left\{ I \in \mathcal{P}_t(N), j \in [N] \setminus I : \left( \sum_{k \in I} (\theta n - \delta_{jk}) \right)^\ell \geq n^\ell \left(1 - \frac{t}{q}\right)^\ell \right\})$$

$$\leq \left(n - \frac{nt}{q}\right)^{-\ell} \mathbb{E}\left( \sum_{k \in I} (\theta n - \delta_{jk}) \right)^\ell.$$  

Here the second line follows because $\ell$ is even and on the third line we use the Markov inequality. Observe that $\delta_{jk}$ are random variables that denote the distance between two codewords $x_j$ and $x_k$, $1 \leq k \leq t$ chosen randomly from without replacement. Let $\mu_{jk}$ denote the random variable corresponding to $\delta_{jk}$ when the codewords are chosen with replacement, and note that $\mu_{jk}$ and $\mu_{j',k'}$ are independent whenever $j \neq j'$ or $k \neq k'$. Using Theorem 6 we have

$$P_A(t, N) \leq \left(n - \frac{nt}{q}\right)^{-\ell} \mathbb{E}\left( \sum_{k \in I} (\theta n - \mu_{jk}) \right)^\ell. \quad (10)$$

Let us estimate the numerator in (10) using the Minkowski inequality (the triangle inequality for the $\ell$-norm), again keeping in mind that $\ell$ is even. We obtain

$$P_A(t, N) \leq \left( \frac{\sum_{k \in I} \left( \mathbb{E}(\theta n - \mu_{jk})^\ell \right)^{1/\ell}}{n^\ell (1 - \frac{t}{q})^\ell} \right) \frac{t^\ell \mathbb{E}(\theta n - Z)^\ell}{n^\ell (1 - \frac{t}{q})^\ell},$$

Now let us estimate the right-hand side of (10) relying on the Marcinkiewicz-Zygmund inequality. We obtain

$$P_A(t, N) \leq \frac{(3\sqrt{2})^{\ell/2} t^{\ell/2 - 1} \sum_{k \in I} \mathbb{E}(\theta n - \mu_{jk})^\ell}{n^\ell (1 - \frac{t}{q})^\ell} = \frac{(18t\ell)^{\ell/2} \mathbb{E}(\theta n - Z)^\ell}{n^\ell (1 - \frac{t}{q})^\ell}. \quad (11)$$
This completes the proof.  

Our next step will be to estimate the moments in (9).  

Lemma 7: Let \( C \) be a code over \( \mathbb{F}_q \) of length \( n \) and size \( N \) with dual distance \( d' \). For any \( \ell < d' \),  

\[
\frac{1}{N} \sum_{j=0}^{n} (j - \theta n)\ell A_j = \sum_{j=0}^{n} (j - \theta n)\ell \binom{n}{j} \theta^j (1 - \theta)^{n-j}.
\]  

(11)

Proof: Let \( C \subset \mathbb{F}_q^n \) be a \( q \)-ary code with distance distribution \( A_0, A_1, \ldots, A_n \) and let \( A'_i, i = 0, 1, \ldots, n \) be the dual distance distribution defined in (2). The following Pless power moment identities hold true \[26], [19, p.131]:

\[
\sum_{j=0}^{n} j^r A_j = \sum_{j=0}^{n} (-1)^j A'_j \left( \sum_{\nu=0}^{r} \nu! S(r, \nu) q^{k-\nu} (q-1)^{\nu-j} \binom{n-j}{n-\nu} \right)
\]  

(12)

where \( S(r, \nu) \) are the Stirling number of second kind,  

\[
S(r, \nu) = \frac{1}{\nu!} \sum_{i=0}^{\nu} \frac{(1)_{\nu-i} (i)_r}{i!} q_i
\]

when \( \nu \leq r \) and \( S(r, \nu) = 0 \) otherwise. Taking \( r < d' \), we find that on the right-hand side of (12) the sum on \( \nu \) is zero unless \( j < d' \), but then \( A'_j = 0 \) except for \( A'_0 = 1 \). Using this in (12), we obtain

\[
\sum_{j=0}^{n} j^r A_j = \sum_{\nu=0}^{r} \nu! S(r, \nu) q^{k-\nu} (q-1)^{\nu-j} \binom{n-j}{n-\nu}.
\]  

Substituting the definition of \( S(r, \nu) \) and simplifying, we find that

\[
\sum_{j=0}^{n} j^r A_j = \sum_{j=0}^{n} j^r \binom{n-j}{j} \theta^j (1 - \theta)^{n-j}.
\]

The lemma follows immediately.  

Another proof would be to use the general result for symmetric association schemes \[36\] established below together with the properties of the Hamming association scheme. We believe that readers familiar with association schemes will have no difficulty filling in the details.  

Let us bound above the right hand side of (11).  

Lemma 8: Let \( 1/2 < p < 1 \). Then,

\[
\mu_n(2r) := \sum_{j=0}^{n} \left( \frac{j - np}{\sqrt{p(1-p)}} \right)^{2r} \binom{n}{j} p^j (1-p)^{n-j} < (ner)^r \sum_{i=0}^{r} \left( \frac{pr}{(1-p)ne} \right)^i.
\]

Remark 2: When \( \frac{p}{1-p} \leq \frac{n}{r} \), the expression on the right hand side above can be further simplified. Namely, \( \frac{pr}{ne(1-p)} \leq e^{-1} \), and we have,

\[
\mu_n(2r) \leq \frac{1}{1 - e^{-1}} (ner)^r.
\]

Proof of Theorem 5 Using Lemma 7 and Lemma 8 with \( p = \theta \), we immediately obtain, for any even \( \ell < d' \),

\[
\mathbb{E}(Z - \theta n)^\ell \leq \left( \frac{ne\ell\theta}{2q} \right)^{\ell/2} \sum_{i=0}^{\ell/2} \left( \frac{(q-1)\ell/2}{ne} \right)^i.
\]

Substituting this estimate in (9) and rearranging, we obtain the bound (8).  

\[\square\]
B. Examples

Let us examine a few specific choices of the outer $q$-ary codes. In each case our goal is to choose the parameters so that the quantity in (8) is small, and to examine the parameters of the resulting almost disjunct matrices and group testing schemes.

1) Reed-Solomon codes: In this case, $n = q - 1$, $\ell \approx \log_q N$. If $t > 18\log N$, then the minimum in $B(\ell, t)$ is attained for the first term, otherwise for the second one. For instance, consider the first option. From (8) we obtain

$$P_A(t, N) < \left( \frac{9e\ell^2 t}{(q - t)^2} \right)^{\ell/2} \left( \frac{\ell}{2e} \right)^{\ell/2+1} \approx \frac{\ell}{2e} \left( \frac{2.13\ell^{3/2}\sqrt{t}}{q - t} \right)^\ell. \tag{13}$$

Thus the probability $P$ is small if we take $q > 2.13\ell^{3/2}\sqrt{t} + t$. Note that for Reed-Solomon codes we have $M = q(q - 1)$. Overall we obtain $M = O\left( \max\{t^2, t(\log_q N)^3\} \right)$.

2) Algebraic-geometric codes: Let us consider two examples that rely on codes on algebraic curves.

a) Hermitian codes: Let $q_0$ be a power of a prime and let $r$ be an integer such that

$$q_0^2 - q_0 - 2 \leq r \leq q^3.$$

There exists a family of linear $q$-ary codes, $q = q_0^2$, constructed on Hermitian curves [35, Sec. 8.3]. The parameters of the codes are as follows:

- length $n = q_0^3$
- cardinality $N = q_0^{2(r+1-q(q-1)/2)}$
- dual distance $d' \geq r + q_0 + 2 - q_0^2$.

In particular, choosing $r = q_0^2$, we obtain $d' \geq q_0 + 2$ and $N \approx q_0^{2q_0^2}$. This suffices to ensure that the quantity in (8) is small, i.e., the matrix formed by using Hermitian codes in the Kautz-Singleton construction is almost disjunct. Assuming that the number of defectives $t < q_0^2$, we obtain $M = q_0^5$ for the number of tests.

b) Suzuki codes: Similar results are obtained if we take codes constructed on Suzuki curves [11]. Namely, let $q = 2q_0^2$, $q_0 = 2^m$ and let $r$ be an integer such that $2q_0(q - 1) - 2 < r < q^2$. There exists a family of linear $q$-ary codes with the parameters

- length $n = q^2$
- cardinality $N = q^{r+1-q_0(q-1)}$
- dual distance $d' \geq r - 2(q_0(q - 1) - 1)$.

Choosing $r = 2q_0q$, we obtain $d' \geq 2q_0 + 2$, so we can take $\ell = (1/4)n^{1/4}$. Substituting this in (8), we see that $P \sim n^{-\ell/4}$ for any $t \leq q/2$, which allows us to choose the number of defectives $t = O(n^{1/2}) = O(q)$. With this choice of the parameters, testing matrices obtained from Suzuki codes using the Kautz-Singleton construction are guaranteed to have the almost disjunct property. With the above choice of $r$ we have $t = q/2$, $N = q^{2q_0^2+q_0+1}$ and $M = nq = q^3$.

C. Almost disjunct matrices from constant weight codes

In this part we study properties of matrices constructed from constant weight codes with a known value of the dual distance $d'$, i.e., combinatorial designs of strength $d' - 1$.

The scheme of the proof is similar to the previous section. Beginning with Prop. 4, we will estimate the probability of a false positive using moments of the distance distribution of constant weight codes. First
let us show that if \( r < d' \), then the \( r \)th central moment of distance distribution equals the \( r \)th moment of the distance distribution of the sphere of weight \( w \).

**Theorem 9:** Let \( C \) be a constant weight code of weight \( w \), length \( M \), distance distribution \( \{b_i, i = 0, \ldots, w\} \) and dual distance \( d' \). Let \( X \) be a hypergeometric random variable with pmf and moments

\[
f_X(i) = \frac{\binom{w}{i} \binom{M-w}{M-i}}{\binom{M}{w}}, \quad i = 0, 1, \ldots, w; \quad \mathbb{E}X = \frac{w^2}{M}, \quad \text{Var}(X) = \frac{w^2(M-w)^2}{M^2(M-1)}.
\]  

As long as \( r < d' \),

\[
\sum_{i=0}^{w} \left( \frac{w(M-w)}{M} - i \right)^r b_i = |C| \mathbb{E}(\langle X - \mathbb{E}X \rangle^r).
\]  

**Proof:** See the appendix.

**Remark 3:** Note that \( \overline{d} := w(M-w)/M \) is the average pairwise distance in the set of all binary vectors of weight \( w \), so equality (15) gives the moments of the distance distribution about the mean. In this sense it is analogous to the corresponding result for the Hamming space [11]. Irrespective of the value of \( r \), the left-hand side of (15) is always greater than or equal to the right-hand side (this can be seen from (36) and (37) in the Appendix). This result was first proved in [31] using analytic methods, and is known as the Sidelnikov inequality. A similar inequality for general symmetric association schemes is proved in [35, p.55] using a combinatorial approach which we adopt in our proof of Theorem 9.

Let \( Z := (1/2)d(x, y) \), where \( d \) is the Hamming distance, be the random variable defined by two random vectors chosen from a constant weight code (with replacement). We have \( \Pr(Z = i) = b_i/|C| \). Moreover, if \( d' \geq 3 \) then Theorem 9 implies that the central moments of \( Z \) up to order \( d' - 1 \) are the same that the central moments of \( X \), so

\[
\mathbb{E}Z = \overline{d}, \quad \text{Var}(Z) = \overline{d}^2/(M-1).
\]  

The main result of this section is given in the next theorem.

**Theorem 10:** Let \( C \) be an \( (M, N, d, w) \) constant weight code with dual distance \( d' \) and let \( w < M/2 \). Let \( t \) be the maximum number of defective items and suppose that \( t < M/w \). For any even \( \ell < d' \) the group testing scheme constructed from \( C \) is \( (t, \epsilon) \)-disjunct for

\[
\epsilon \leq B(\ell, t) \left( \frac{\ell(M-w)}{2(M-tw)^2} \right)^{\ell/2} \sum_{i=0}^{\ell/2} \left( \frac{2w^2}{\ell} \right) \frac{(M-w)^{\ell-i}}{2^i \ell^i},
\]  

where \( B(\ell, t) = \min\{18\ell^2/\ell^3, t\} \). In addition, if \( M \geq \max\{4w^2/\ell^2, w + 2w^2/\ell\} \) then the matrix is \( (t, \epsilon) \)-disjunct with

\[
\epsilon \leq t \left( \frac{2\ell^2(M-w)}{(\log \ell)w(M-tw)} \right)^{\ell/2}.
\]  

In the case of \( \ell = 2 \) we can use the following bound instead of the above,

\[
\epsilon < \frac{t}{M-1} \frac{(M-w)^2}{(M-tw)^2}.
\]  

**Remarks:** We comment on the conditions for the bounds (17)-(18) to be nontrivial in Corollary 11 below. As for (19), it gives a good bound for large \( M \) and small or slowly growing \( w \) and \( t \).

**Proof:** For all \( k \in I \) and an random index \( j \in [N] \setminus I \), let \( \xi_{jk} \) be the random value of \( d_{jk}/2 = d(x_j, x_k) \) when the vectors \( x_j, x_k, k \in I \) are chosen from \( C \) randomly and uniformly without replacement.
and let \(\eta_{jk}\) denote the same quantity when the vectors are chosen with replacement. The variables \(\eta_{jk}, \eta_{jk}'\) are independent whenever \(k \neq k'\), and each of them is stochastically equivalent to the random variable \(Z\) defined above. Proceeding similarly to the proof of Proposition 6, we obtain

\[
P_A(t, N) \leq P_{R_t} \left( \left\{ I \in \mathcal{P}_t(N), j \in [N] \ : w \leq \sum_{k \in I} (w - \xi_{jk}) \right\} \right)
\]

\[
= P_{R_t} \left( \left\{ I \in \mathcal{P}_t(N), j \in [N] \ : \sum_{k \in I} (\theta - \xi_{jk}) \geq w \left( 1 - \frac{tw}{M} \right) \right\} \right)
\]

\[
\leq \left( w - \frac{tw^2}{M} \right)^{-\ell} \mathbb{E} \left( \sum_{k \in I} (\theta - \xi_{jk}) \right)^\ell
\]

\[
\leq \left( w - \frac{tw^2}{M} \right)^{-\ell} \mathbb{E} \left( \sum_{k \in I} (\theta - \eta_{jk}) \right)^\ell,
\]

(20)

where in the last step we used (14) and where \(\ell, 2 \leq \ell < d'\) is an even number.

Now let us use the Minkowski inequality to obtain

\[
P_A(t, N) \leq \left( w - \frac{tw^2}{M} \right)^{-\ell} \left( \sum_{k \in I} (\mathbb{E}(\theta - \eta_{jk})^\ell)^{1/\ell} \right)^\ell
\]

\[
= \left( w - \frac{tw^2}{M} \right)^{-\ell} t^\ell \mathbb{E}(\theta - Z)^\ell
\]

\[
\leq \left( w - \frac{tw^2}{M} \right)^{-\ell} t^\ell \mathbb{E}((X - \mathbb{E}X)^\ell),
\]

(21)

where \(X\) is the hypergeometric random variable defined by (14).

As before, we can also estimate the right-hand side of (20) relying on the Marzinkiewicz-Zygmund inequality (5). We obtain

\[
P_A(t, N) \leq \left( w - \frac{tw^2}{M} \right)^{-\ell} \mathbb{E} \left( \sum_{k \in I} (\theta - \eta_{jk}) \right)^\ell
\]

\[
\leq \left( w - \frac{tw^2}{M} \right)^{-\ell} (3\sqrt{2})^\ell t^\ell/2^\ell/m/2-1 \sum_{k \in I} \mathbb{E}(\theta - \eta_{jk})^\ell
\]

\[
\leq \left( w - \frac{tw^2}{M} \right)^{-\ell} (18\ell t)^\ell/2^\ell \mathbb{E}(\theta - X)^\ell.
\]

Taking this estimate together with (21) we obtain the bound

\[
P_A(t, N) \leq \min \{ (18\ell t)^\ell/2^\ell, (1 - \frac{tw}{M})^\ell \} \mathbb{E}(\theta - X)^\ell
\]

(22)

It is evident that \(X = \sum_{i=1}^w X_i\), where \(X_1, \ldots, X_w\) denote values of random samples drawn without replacement from a population of \(M\) values with \(w\) ones and \(M - w\) zeros. Now using Theorem 5 we claim that

\[
\mathbb{E}((X - \mathbb{E}X)^\ell) \leq \mathbb{E}((Y - \mathbb{E}Y)^\ell),
\]

(23)

where \(Y = \sum_{i=1}^w Y_i\), and \(Y_1, \ldots, Y_w\) denote samples from the same population with replacement. This makes the \(Y_i\)'s into independent Bernoulli(\(\frac{w}{M}\)) random variables, and

\[
\mathbb{E}((Y - \mathbb{E}Y)^\ell) = \sum_{j=0}^w \left( j - \frac{w^2}{M} \right)^\ell \left( \frac{w}{M} \right)^j \left( 1 - \frac{w}{M} \right)^{w-j}
\]
\[
\sum_{j=0}^{w} \left( w \left( 1 - \frac{w}{M} \right) - j \right)^{\ell} \left( w \left( 1 - \frac{w}{M} \right) \right)^{w-j} \left( 1 - \frac{w}{M} \right)^{j} < \left( \frac{w^2 \ell}{2M} \left( 1 - \frac{w}{M} \right) \right)^{\ell/2} \sum_{i=0}^{\ell/2} \left( \frac{(M-w)\ell}{2eM^2} \right)^i,
\]

(24)

where in the last step we have used Lemma 8 with \( p = 1 - w/M \). Using (24) in (22) and rearranging, we obtain for the probability of the false positive the estimate (17).

Before proving (18), consider the case \( \ell = 2 \). Here the calculation is simpler because we can directly substitute the value of the variance of \( Z \).

Indeed, by independence

\[
E \left( \sum_{k \in I} (\vartheta - \eta_{jk}) \right)^2 = \sum_{k \in I} \text{Var}(\eta_{jk}) = \frac{t\vartheta^2}{M-1}.
\]

(25)

Using this in (20) and simplifying, we obtain (19).

Now let us prove (18), estimating the moment of the sum of random variables using Rosenthal's inequality (6). We have

\[
E \left( \sum_{k \in I} (\vartheta - \eta_{jk}) \right)^{\ell} \leq K_{\ell} \max \left\{ \sum_{k \in I} E(\vartheta - \eta_{jk})^{\ell}, \left( \sum_{k \in I} E(\vartheta - \eta_{jk})^{2} \right)^{\ell/2} \right\}
\]

(26)

To estimate the maximum, we will take an upper bound on the first term given by (24) and show that it is greater than the second term. Assume that \( M \geq w + 2ew^2/\ell \), then the largest term under the sum in (24) is the last one, and we obtain

\[
E((Y - EY)^{\ell}) \leq \frac{\ell}{2} \left( \frac{\ell(M-w)}{2M} \right)^{\ell},
\]

and therefore,

\[
\sum_{k \in I} E(\vartheta - \eta_{jk})^{\ell} \leq \frac{t\vartheta^2}{2} \left( \frac{\ell(M-w)}{2M} \right)^{\ell}.
\]

(27)

The value of the second term on the right-hand side of (26) is found directly from (25) and equals

\[
\left( \sum_{k \in I} \text{Var}(\eta_{jk}) \right)^{\ell/2} = \left( \frac{t\vartheta^2}{M-1} \right)^{\ell/2}.
\]

(28)

As is easily checked, the right-hand side of (27) is greater than (28) if \( M \geq 4w^2t/\ell^2 \). Therefore the right-hand side of (27) also provides an upper bound on the maximum in (6). Substituting it in (20), we obtain the claimed bound (18).

In the next corollary we state the conditions for the bounds of Theorem 10 to guarantee that we obtain almost disjunct matrices. We focus on the estimate (18), but a similar claim can be also made with respect to the bound (17).

**Corollary 11:** Suppose that \( w > 2\ell^2/\log \ell \) and \( M > \max \{ 4w^2t/\ell^2, w + 2ew^2/\ell, wt \log \ell \} \). Then the codewords of a binary constant weight code of length \( M \), cardinality \( N \), weight \( w \) and dual distance \( d' > \ell \) form an \((t, \epsilon)\)-disjunct matrix with \( \epsilon \) approaching zero exponentially with the increase of \( \ell \).

**Proof:** Under the stated assumptions the probability of a false positive can be bounded above by (18), and the term \((\cdot)^{\ell}\) in that expression approaches zero with increasing \( \ell \).

It is possible to consolidate the restrictions in \( M \) in this corollary by making further assumptions on the relation of \( t \) and \( w \), but we prefer to leave this statement in the most general form that arises from our
estimation method. This corollary also implies that the number of tests $M$ that suffices to for the matrix to be almost disjunct behaves as $M = O(\max(t\ell^2 / \log^2 \ell, \ell^3 / \log^2 \ell, t\ell^2)) = O(\ell^2 \max(\ell / \log^2 \ell, t))$. Hence, we have the following corollary.

**Corollary 12:** A constant weight code of length $M$ and size $N$ with dual distance greater than $\ell$ provides a $(t, \exp(-\ell))$ disjunct matrix with $t = \Omega(M/\ell^2)$.

The dual distance of a code (or strength of a design) of course depends on its length and size. Note in particular that if we assume $\ell = \Omega(\log N)$, then the probability that there exist false positives can be made to approach 0 (using an union bound on the $N-t$ elements) with total number of tests being $M = O(t \log^2 N)$.

It remains to use results about the existence of $\ell$-designs. Major progress in the existence problem has been achieved in recent years. For instance, due to the result of [17, Theorem 1.3] it is known that there exist $\ell$-designs with $
abla^N \leq c\ell \log(cM/\ell)$, for some constant $c > 0$, which supports the assumption of $\ell = \Omega(\log N)$ made above. We note that this result is existential and not an explicit construction. Substituting it into Corollary 12 we obtain the following result.

**Corollary 13:** There exist nonadaptive group testing schemes constructed from combinatorial designs with $O(t \log^2 N / \log^2 t)$ tests that can identify all items in a random defective configuration of size $t$ with probability of false positive for an element at most $\exp(-\log N / \log t)$.

Another breakthrough result is due to [16]. For a design with the parameters $\ell, M, w, \lambda$ to exist it is necessary that $(\ell - i)$ divides $(M - i)$ for all $i = 0, 1, \ldots, \ell$. It has been shown in [16] that these conditions are also sufficient, i.e., combinatorial designs exist whenever their parameters satisfy the natural divisibility constraints. This advance provides a large supply of objects for the construction of almost disjunct matrices following the analysis in this section.

**IV. OUTLOOK**

We introduced a general method of constructing explicit almost disjunct matrices from nonbinary and constant weight codes. The advantage of the introduced approach is related to its universal nature. While there are limitations on the applicability of the bounds obtained, in many cases they guarantee existence of almost disjunct matrices. Moreover, under certain assumptions stated in the paper, the constructed matrices rely on a number of tests of the order $O(t \cdot \text{polylog} N)$, matching the best known results. Experimentation shows that some of the simple constructions from constant weight codes behave very well in identifying random defectives. In particular, in [36] we have conducted experiments with testing matrices constructed from constant weight subcodes of binary BCH codes. Simulations showed that the matrices constructed from all the vectors of fixed weight of binary BCH codes of length $M = 63$ and 127 and weight $w = 3, 5, 7$ perform consistently better than random binary matrices in terms of the number of false positives. Another observation in [36] concerns the estimates of the probability of false positives $P_A(t, N)$ given by Theorem 10. Namely, while for the described range of the parameters, Eqns. 18 and 19 give nontrivial estimates of $P_A(t, N)$, in particular, better than those implied by [23], experimental results give much lower rates of false positives. This shows that the methods introduced here fail to account for the actual performance of some constant weight code matrices, leaving the derivation of better bounds as an open problem.

1 Note that due to the lower bound in [6] the order relations obtained here cannot be improved within the frame of the method considered.
APPENDIX: PROOF OF LEMMA

Proof: Let us define i.i.d. random variables $X_i$, $i = 1, \ldots, n$ in the following way:

$$X_i = \begin{cases} \sqrt{(1-p)/p} & \text{with probability } p \\ -\sqrt{p/(1-p)} & \text{with probability } 1-p. \end{cases}$$

Note that $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$. Let $X = \sum_{i=1}^{n} X_i$. Clearly, with probability $(n)_{r}(1-p)^{n-i}$,

$$X = i\sqrt{(1-p)/p} - (n-i)\sqrt{p/(1-p)} = \frac{i-np}{\sqrt{p(1-p)}}.$$

Hence,

$$\mu_n(2r) = \mathbb{E}X^{2r}.$$

Let us estimate the right-hand side. We have

$$\mathbb{E}X^{2r} = \sum_{i_1, \ldots, i_{2r}} \mathbb{E}X_{i_1} \cdots X_{i_{2r}}.$$

We use the fact that the variables $X_i$ are independent. If at least one of the $X_i$s appears only once then the corresponding monomial is zero. So it may be assumed that each index appears at least twice in the expectations that contribute to the sum. In particular, there are at most $r$ distinct $X_i$s that can appear. Suppose that $r-t$ such terms appear. Let $N_t$ be the number of ways one can assign integers $i_1, \ldots, i_{2r} \in \{1, \ldots, n\}$ such that each $i_j$ appears, at least twice and exactly $r-t$ integers appear. Using the fact that $X_i$s have unit variance and $|X_{i_j}| \leq \sqrt{p/(1-p)}$, we have,

$$\mathbb{E}X^{2r} \leq \sum_{t=0}^{r} N_t (\sqrt{p/(1-p)})^{2t} = \sum_{t=0}^{r} N_t \left( \frac{p}{1-p} \right)^t.$$

A crude bound on $N_t$ gives,

$$N_t \leq \left( \frac{n}{r-t} \right) (r-t)^{2r} \leq \left( \frac{ne}{r-t} \right)^{r-t} (r-t)^{2r}.$$ 

Hence,

$$\mathbb{E}X^{2r} \leq \sum_{t=0}^{r} (ne)^{r-t} (r-t)^{r+t} \left( \frac{p}{1-p} \right)^t \leq (ner)^r \sum_{t=0}^{r} \left( \frac{pr}{ne(1-p)} \right)^t.$$

APPENDIX: PROOF OF THEOREM

We prove Theorem 9 assuming we are given a $(n, N, d, w)$ constant weight code.

1. (Johnson association scheme) We will use some simple properties of the Johnson association scheme [1], [4]. Recall that $J_n^w$ denotes the set of all binary vectors of length $n$ and Hamming weight $w \leq n/2$. Let $C \subset J_n^w$ be a code whose distance distribution $(b_0 = 1, b_1, \ldots, b_w)$ is defined in (3) above. Define the Hahn polynomial of degree $k = 0, 1, \ldots, w$ by its values as follows:

$$Q_k(i) = \frac{\mu_k}{v_i} E_i(k),$$

(29)
where \( E_i(x), i = 0, 1, \ldots, w \) is the Eberlein polynomial, and

\[
v_i = \binom{w}{i} \binom{n-w}{n-i}, \quad \mu_i = \binom{n}{i} - \binom{n}{i-1}.
\]

(30)

are the valencies and the multiplicities of the scheme \( J_w^w \). The polynomials \( (Q_k, k = 0, 1, \ldots, w) \) form an orthogonal system on the set \( \{0, 1, \ldots, w\} \). The explicit expression for \( Q_k(i) \) is well known \[4, \text{p.}48], \[1, \text{pp.} 217-220\]. Below we need only the expression for the constant \( Q_0 \equiv 1 \) and the linear polynomial, given by

\[
Q_1(i) = (n-1)\left(1 - \frac{ni}{w(n-w)}\right).
\]

(31)

Define the dual distance distribution of \( C \) by

\[
b_j' = \frac{1}{|C|} \sum_{i=0}^{w} b_i Q_j(i), \quad j = 0, 1, \ldots, w
\]

and notice that \( b_0' = 1 \). By Delsarte’s inequalities \[4, \text{Thm.} 3.3\] the coefficients \( b_j' \) are nonnegative. The Hahn polynomials satisfy the relation

\[
Q_i(l)Q_j(l) = \sum_{k=0}^{m} q_{ij}^k Q_k(l)
\]

(32)

where the numbers \( q_{ij}^k \) are called the Krein parameters of the scheme. Importantly, we have \( q_{ij}^k \geq 0 \) \[4, \text{Lemma} 2.4\]. In fact, the matrices \( (E_i(k)) \) and \( (Q_k(i)) \), \( i, k = 0, 1, \ldots, w \) form the first and the second eigenvalue matrices of the scheme \( J_w^w \). This implies the following relations:

\[
\sum_{k=0}^{w} E_k(j)Q_i(k) = \binom{n}{w} \delta_{ij}
\]

(33)

\[
E_k(0) = v_k
\]

(34)

(for the proofs see Thm. 3.5 and Prop. 3.4 of \[1\]).

2. The calculation below is in part inspired by \[35\]. On account of (32) we can write for all \( i = 1, \ldots, w \)

\[
Q_1(i)^2 = \sum_{k=0}^{2} q_{11}^k Q_k(i)
\]

and generally

\[
Q_1(i)^r = \sum_{k=0}^{r} \beta_k(r) Q_k(i), \quad (r \geq 2)
\]

(35)

where \( \beta_k(r) \geq 0 \) are some nonnegative coefficients that can be easily computed by orthogonality. Now consider

\[
\sum_{i=0}^{w} Q_1(i)^r b_i = \sum_{i=0}^{w} \sum_{k=0}^{r} \beta_k(r) Q_k(i)b_i
\]

\[
= \sum_{k=0}^{r} \beta_k(r) \sum_{i=0}^{w} Q_k(i)b_i
\]
\[
|C| \sum_{k=0}^{r} \beta_k(r) b_k' = |C| \left( \beta_0(r) + \sum_{k=d'}^{r} \beta_k(r) b_k' \right). \tag{36}
\]

Let us compute \(\beta_0(r)\). We have

\[
\sum_{i=0}^{w} v_i Q_1(i)^r = \sum_{k=0}^{r} \beta_k(r) \sum_{i=0}^{w} v_i Q_k(i) = \sum_{k=0}^{r} \beta_k(r) \sum_{i=0}^{w} E_k(0)^r Q_k(i) = \beta_0(r) \left( \frac{n}{w} \right). \tag{30}
\]

Thus, using (30) we obtain

\[
\beta_0(r) = \frac{1}{\left( \frac{n}{w} \right)} \sum_{i=0}^{w} \binom{w}{i} \left( \frac{n-w}{i} \right) Q_1(i)^r.
\]

Let \(0 \leq r \leq d' - 1\), then from (36) and (31) we have

\[
\sum_{i=0}^{w} \left( 1 - \frac{n i}{w(n-w)} \right)^r b_i = \frac{|C|}{\binom{n}{w}} \sum_{i=0}^{w} \binom{w}{i} \left( \frac{n-w}{i} \right) \left( 1 - \frac{n i}{w(n-w)} \right)^r \tag{37}
\]

Equality (15) follows upon rewriting the right-hand side of (37) as follows:

\[
\sum_{i=0}^{w} \binom{w}{i} \frac{(n-w)}{i} \left( 1 - \frac{n i}{w(n-w)} \right)^r = \sum_{i=0}^{w} \binom{w}{i} \frac{(n-w)}{i} \left( 1 - \frac{n w - i}{w(n-w)} \right)^r
\]

\[
= \left( \frac{n}{w(n-w)} \right)^r \sum_{i=0}^{w} \binom{w}{i} \frac{(n-w)}{i} \left( 1 - \frac{w^2}{n} \right)^r.
\]

This completes the proof of Theorem 9.

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