A MATHEMATICAL ANALYSIS OF DIRAC EQUATION PHYSICS

H.O.Cordes©2014

To the Memory of Lars Hörmander

Abstract. This paper analyzes time-propagation of Dirac observables — using Heisenberg representation — in the light of various pseudodifferential operator algebras. We have discussed such matters earlier (cf. [Co3], [Co15, Co16]), observing the elegant relation to classical physics coming into play, also giving insight into a (sort of) magnetic moment, representing the spin.

Presently we analyze this more carefully — looking at the Physical aspects. Our theory gives (i) a mechanical angular momentum (the spin) and (ii) another real 3-vector travelling with the particle with magnetic properties (its motion guided by the magnetic field around it, but not in the proper relativistic way). This questions the interpretation of the magnetic moment of the particle being generated by rotation of the charge, as suggested by macroscopic arguments.

All the above was proven under assumptions on potentials, making them vanish at infinity. But we now also look at a Dirac particle under the influence of a plane polarized X-ray-wave, trying to analyze the Compton effect. What we can derive there might also be surprising: Looking at the total energy \(E\) and the orbital momentum coordinate \(P_1\) in the direction of the radiation, we find that these two observables are coupled. Their time propagation shows a number of discrete possibilities: Either there is no change in time, of both \(E\) and \(P_1\) or there is a change by \(n\hbar\nu\) of \(E\) and \(n\hbar\nu/c\) of \(P_1\) with an integer \(n = 1, 2, \ldots\) — with same \(n\) for \(E\) and \(P_1\). This is valid for large frequencies — i.e., large values of the momentum coordinates.

We need not point out the possible interpretation: There may be a collision of the electron-positron-particle with one — or two — or \(n\) — Photons of total energy \(\hbar\nu\) each, effecting a sudden change of energy and momentum. Observe, this does not require any use of QFT.

Keywords: Precisely predictable Observables; Dirac photons without quantizing the EM-field; Magnetic spin.

AMS Subject Classification: 81CXX, 35L45, 35S99, 47G05, 78A15.

1. Introduction

In this paper we try to apply rigorous mathematics to analyze two different physical problems, attached to Dirac’s first order symmetric hyperbolic \(4 \times 4\)-system of partial differential equations, using calculus of pseudodifferential operators, resp. Fourier integral operators. In sections 3 through 6 we have a class of electro-magnetic potentials vanishing at \(|x| = \infty\), including the Coulomb potential with its singularity smoothened out. In sections 7 to 12 we deal with a Dirac particle under the (time-dependent) potential of an electro-magnetic wave, such as occurring at the Compton effect.
In the first case we mainly focus on the spin of the particle: We can establish a mechanical spin, as a 3-vector, travelling with the particle, behaving just like a mechanical angular momentum should, in this relativistic environment.

But, on the other hand, there is another 3-vector \( \vec{\kappa} \), also travelling with the particle, with its motion along the particles orbit entirely determined by the two components \( B \) and \( \dot{x} \times E \), combined in a way not expected for the magnetic field, the moving particle see’s. Actually if either \( E = 0 \) or \( B = 0 \), then the movement of \( \vec{\kappa} \) fits that of a magnetic moment. But then there is a difference in strength of these two magnetic moments by a factor \((1 + \sqrt{1 - \dot{x}^2})\). That factor will be \( \approx 2 \), for relativistically small \( \dot{x} \).

While we think that, perhaps a better mathematical construction might correct this, so that the vector \( \vec{\kappa} \) might be regarded as the magnetic moment generated by the spinning charge of the particle, we are left open, with this problem.

In the second case — an electron under an X-ray-wave — we also end up with a contradiction to general expectation: a possible mathematical rediscovery of simple (or multiple) collision between the Dirac particle and photons of energy \( h \nu \) and momentum \( h \nu / c \), from Dirac’s and Maxwell’s equations only. Looking at old standard text, such as Sommerfeld [So1], ch.1, sec.7, this was believed to be impossible to explain from Dirac- or electro-magnetic wave theory. But we believe now, it probably can be explained — and without using second quantization, i.e., without quantizing the electro-magnetic field.

The organization of the paper seems clear, after these remarks. In sec.2 we give some basics of Dirac’s equation; in sec.3 we try to give hints about 3 different algebras of pseudodifferential operators, with the main effort on explaining various asymptotically convergent Leibniz formulas: the asymptotic convergence to be regarded none other than that of the well known Hankel-asymptotic expansions for Bessel-functions at infinity: totally divergent, but still extremely useful.

Unfortunately, as a retired mathematician, working alone, we feel quite helpless in examining the huge physical literature on the subject. We are very grateful to have available the large reference section in the book of B. Thaller [Th1] of 1992 on Dirac’s equation, but apologize in advance to anyone who might have worked in similar directions without our knowledge.

2. Elementary Facts on Dirac Operators

We depart from the non-relativistic Dirac equation \( \dot{\psi} + iH\psi = 0 \) with \( \dot{\psi} = \partial\psi/\partial t \), and the ‘Dirac operator’

\[
(2.1) \quad H = \sum_{j=1}^{3} \alpha_j(D_j - A_j(t,x)) + \beta + V(t,x) \quad : D_j = \frac{i}{t} \partial/\partial x_j ,
\]

with a set \( \alpha_j, \beta \) of self-adjoint \( 4 \times 4 \)-(Dirac)-matrices satisfying

\[
(2.2) \quad \alpha_j \alpha_l + \alpha_l \alpha_j = 2 \delta_{j,l} , \quad \beta^2 = 1 , \quad \alpha_j \beta + \beta \alpha_j = 0 , \quad j,l = 1,2,3,
\]

and with real-valued potentials \( V(t,x), A_j(t,x) \), \( j = 1,2,3 \).

The first order differential operator \( H \) in the 3 variables \( x_1, x_2, x_3 \) has ‘symbol’

\[
(2.3) \quad h(t,x,\xi) = \sum_{j=1}^{3} \alpha_j(\xi_j - A_j(t,x)) + \beta + V(t,x) \quad : D_j = \frac{i}{t} \partial/\partial x_j ,
\]

\(^1\)Reading in R.Becker [Be1], p.85, we note that there also seems to be a factor 2-discrepancy in the theoretical interpretation of the Einstein-de Haas experiment, and electron spin.
so that we may write \( H = h(t, x, D) \). For the mathematics of the differential equation \( \dot{\psi} + iH\psi = 0 \) the spectral behaviour of the \( 4 \times 4 \)-matrix-valued function \( h(t, x, \xi) \) is important. Clearly \( h(t, x, \xi) \), as a self-adjoint \( 4 \times 4 \)-matrix, has real eigenvalues. We get \( (h - \mathbf{V})^2 = 1 + |\xi - \mathbf{A}|^2 = |\xi - \mathbf{A}|^2 \), a scalar multiple of 1, as a consequence of relations (2.2). Accordingly, \( h \) can only have the eigenvalues \( \lambda_{\pm} = \mathbf{V} \pm (\xi - \mathbf{A}) \), and the orthogonal projections on corresponding eigenspaces are given by

\[
(2.4) \quad p_{\pm}(t, x, \xi) = \frac{1}{2} (1 \pm \frac{1}{\langle \xi - \mathbf{A}(t, x) \rangle} h(t, x, \xi)).
\]

A calculation shows that both eigenspaces are two-dimensional, for every \( t, x, \xi \).

There even will be a need for a unitary \( 4 \times 4 \)-matrix \( \Upsilon \) diagonalizing the self-adjoint \( h(t, x, \xi) \), then also supplying a natural orthonormal set of eigenvectors. For this we introduce the \( 4 \times 4 \)-matrix

\[
(2.5) \quad \Upsilon(t, x, \xi) = \frac{1}{\sqrt{(1 + \nu_0)(1 + \nu_0)}} (1 + \nu_0 - \beta\alpha
\nu_0) , \quad v(x, \xi) = \frac{\xi - \mathbf{A}(t, x)}{\langle \xi - \mathbf{A}(t, x) \rangle} , \quad \nu_0(x, \xi) = \frac{1}{\langle \xi - \mathbf{A}(t, x) \rangle}.
\]

Using (2.2) again, a calculation shows that we have

\[
(2.6) \quad \Upsilon^* \Upsilon = 1 , \quad \Upsilon^* h \Upsilon = \mathbf{V}(t, x) + \langle \xi - \mathbf{A}(t, x) \rangle \beta \quad \text{for all } t, x, \xi.
\]

Accordingly, the matrix \( \Upsilon \) will diagonalize \( h(t, x, \xi) \) for every \( t, x, \xi \), if we select a set of Dirac matrices such that \( \beta \) equals the diagonal matrix with entries \( 1, 1, -1, -1 \).

Actually, we are going to use two kinds of Dirac matrices \( \alpha_j, \beta \). Introducing the \( 2 \times 2 \)-Pauli matrices

\[
(2.7) \quad \sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

we may define

\[
(2.8) \quad \alpha = \begin{pmatrix} 0 & i \sigma \\ -i \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

writing the \( 4 \times 4 \)-matrices as \( 2 \times 2 \)-matrices of \( 2 \times 2 \)-blocks. This indeed checks with the conditions (2.2), while, indeed, \( \beta \) is the diagonal matrix with entries as desired above.

Another set of Dirac matrices will be used in sections 7-11. There we set

\[
(2.9) \quad \alpha_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha_2 = i \begin{pmatrix} 0 & \sigma \sigma_3 \\ -\sigma \sigma_3 & 0 \end{pmatrix}, \quad \alpha_3 = i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

again checking with (2.2). The set (2.9) will not have \( \beta \) diagonal but, instead, have \( \alpha_1 \) with that property, this being helpful when we use the positive \( x_1 \)-direction as the direction of an incoming X-ray. The set (2.9) may be related to (2.8) by conjugating each matrix (2.8) with a certain constant real orthogonal \( 4 \times 4 \)-matrix.

The lemma, below, is valid for any choice of Dirac matrices \( \alpha, \beta \) satisfying (2.2), and the corresponding projections \( p_{\pm}(\xi) \) of (2.4), setting \( \mathbf{V} = \mathbf{A}_j = 0 \), \( j = 1, 2, 3 \). Its proof is a calculation.

**Lemma 2.1.** We have

\[
(2.10) \quad p_{\pm}(\xi) \alpha_j p_{\pm}(\xi) = \pm s_j(\xi) p_{\pm}(\xi) , \quad j = 1, 2, 3 , \quad p_{\pm}(\xi) \beta p_{\pm}(\xi) = \pm s_0(\xi) p_{\pm}(\xi) ,
\]

where we have set \( s_j(\xi) = \xi_j / |\xi| \), \( s_0(\xi) = 1 / |\xi| \), \( j = 1, 2, 3 \).
It is known that the Dirac equation \( \dot{\psi} + H \psi = 0 \) has a solution \( \psi(t, x) \) satisfying \( \psi(0, x) = \psi_0(x) \) where \( \psi_0(x) \) may be any complex 4-vector-valued function satisfying \( \int |\psi_0(x)|^2 dx < \infty \). In fact, we get
\[
(2.11) \quad \int |\psi(t, x)|^2 dx = \int |\psi_0(x)|^2 dx, \quad \text{for all } t.
\]
Defining a linear operator \( U(t) \) in the Hilbert space \( \mathcal{H} \) of squared integrable 4-vector-functions by setting \( U(t)\psi_0(x) = \psi(t, x) \) one finds that \( U(t) \) is unitary. We call \( U(t) \) the propagator of Dirac’s equation.

Coming to Quantum Mechanics, we first notice that one may introduce physical units for length, time, energy and electrical charge making \( \hbar = c = m_e = |e| = 1 \), denoting charge and mass of the electron by \( e \) and \( m_e \). That will give the Dirac operator the form (2.1).

A ‘state’ (of the electron-positron system) then is described by a unit-vector in \( \mathcal{H} \) — a 4-vector-function \( \psi_0(x) \) with \( \| \psi \|^2 = \int |\psi_0(x)|^2 dx = 1 \). The observable quantities — called ‘observables’ — are given by (unbounded) self-adjoint operators (acting on a subspace of \( \mathcal{H} \)). The theory predicts the statistical expectation value
\[
(2.12) \quad \bar{A}_{\psi_0} = \langle \psi_0, A\psi_0 \rangle
\]
for the observable \( A \) in the state \( \psi_0 \), where \( \langle ., . \rangle \) denotes the inner product in the Hilbert space \( \mathcal{H} \).

One may predict such expectation-value of the observable \( A \) for a future time, starting with the state \( \psi_0 \) at time \( t = 0 \), by using the state \( \psi_t(x) = \psi(t, x) \), with above solution \( \psi(t, x) \) of the Dirac equation, starting with \( \psi_0 \) at \( t = 0 \). Or else, we get
\[
(2.13) \quad \bar{A}_{\psi_t} = \langle \psi_t, A\psi_t \rangle = \langle U(t)\psi_0, AU(t)\psi_0 \rangle = \langle \psi_0, A_t\psi_0 \rangle = \bar{A}_t\psi_0,
\]
with above ‘propagator’ of Dirac’s equation, setting \( A_t = U^*(t)AU(t) \).

So, for future predictions of \( A \) in the state \( \psi_0 \) at \( t = 0 \), we either must obtain the solution \( \psi_t = \psi(t, x) \) of Dirac’s equations, or else, the observable \( A_t = U^*(t)AU(t) \). Traditionally, getting \( \psi_t \) is called the ‘Schrödinger representation’, and, getting \( A_t \) the ‘Heisenberg representation’.

While a general unbounded self-adjoint operator of \( \mathcal{H} \) will qualify as observable, we should emphasize the two observables \( x \) and \( D \) (with components \( x_j, D_j, \ j = 1, 2, 3, \) known as location and momentum. In classical theory knowledge of location and momentum will completely determine the state of the point-system we consider here. In Quantum Mechanics, we find that the — so-called — dynamical observables all are built from combinations of \( D \) and (functions of) \( x \) : they are differential operators.

The Fourier transform \( F \), defined as
\[
(2.14) \quad F\psi(\xi) = \psi^\wedge(\xi) = (2\pi)^{-3/2} \int dx e^{-ix\xi} \psi(x),
\]
will define a unitary operator of \( \mathcal{H} \) with the property that
\[
(2.15) \quad FDF^* = \text{multiplication by } x, \quad FXF^* = -D.
\]
We observe that our quantum theory might just as well be performed by using the Fourier transformed states $\psi^\wedge$ and observables $A^\wedge = \mathcal{F}A\mathcal{F}^*$ instead of $\psi$ and $A$. We then might speak of the momentum representation, since then the momentum observables $D$ will be ‘diagonal’ (i.e., will be multiplication operators).

For a differential operator observable $A$ the operator $A_t = U^*(t)AU(t)$ in general will not be a differential operator. But we find it a rewarding problem to look at observables with the property that $A_t$ is a pseudodifferential operator.

3. Some Global Pseudodifferential Operator Algebras on $\mathbb{R}^3$

We will discuss here the calculus of $\psi$do-s of 3 special algebras of pseudodifferential operators (abbrev. $\psi$do-s).

Note, the location observables (of multiplication by) $x_j$ and momentum observables $D_l$ generate an algebra of differential operators (containing all linear combinations of finite products of these operators). Clearly $D_j$ and $x_j$ do not commute — we get $[D_j, x_j] = \frac{1}{i}$. These differential operators may be written in the form

$$L = \sum a_\theta(x)D^\theta, \text{ or also as }, L = \sum D^\theta a_\theta(x),$$

using multi-index notation, where $a_\theta(x)$ and $\tilde{a}_\theta(x)$ usually are different functions.

Calculations among differential operators then are governed by the so-called Leibniz formulas.

Generally we decide to use the first form of (3.1) when writing a differential operator, keeping multiplications to the left of differentiations. For a polynomial $a(x, \xi) = \sum a_\theta(x)\xi^\theta$ in $\xi$ we write

$$a(x, D) = \sum a_\theta(x)D^\theta, \text{ or also as }, L = \sum D^\theta a_\theta(x),$$

and calling $a(x, \xi)$ the symbol of the differential operator $a(x, D)$.

Lemma 3.1. (Leibniz formulas) Let $A = a(x, D)$, $B = b(x, D)$ then $AB = C = c(x, D)$, $A^* = \tilde{a}(x, D)$ with symbols given by the formulas

$$c(x, \xi) = \sum_{j=0}^{\infty} \sum_{|\theta|=j} (-i)^{|\theta|} \theta! \partial_\theta^\xi a(x, \xi) \partial_\theta^\xi b(x, \xi), \text{ or } \tilde{a}(x, \xi) = \sum_{j=0}^{\infty} \sum_{|\theta|=j} (-i)^{|\theta|} \theta! \partial_\theta^\xi \partial_\theta^\xi a^*(x, \xi).$$

The sums in (3.3) are finite, since the derivatives $\partial_\theta^\xi$ of a polynomial in $\xi$ vanish as soon as $|\theta|$ is larger than its order. The formulas are easily verified for $a(x, \xi), b(x, \xi)$ polynomials of order 0 or 1. Then an induction proof can be given.

With the Leibniz formulas we then can control sums, products and adjoints of differential operators.

It was the merit of Hörmander [Hoe2] to design a technique for extending this calculus of differential operators to a larger class of symbols, no longer being polynomials in $\xi$, then getting a class of pseudodifferential operators, and providing a meaning to the Leibniz formulas. We are using this technique here, in a slightly different form, for construction of some (global) algebras of $\psi$do-s.

First of all we use the Fourier transform (2.13) and (2.14) to write the action of (3.2) as

$$a(x, D)u(x) = \frac{1}{(2\pi)^{3}} \int d\xi \int dy e^{i\xi(x-y)} a(x, \xi)u(y).$$
We again use \( \psi_q \cap \omega > \) for all multi-indices \( x, \xi \) defined and smooth for all 6 variables independent — in the variables \( x \) (with order \( m_2 \)) and \( \xi \) (with order \( m_1 \)). There are two orders then combined into a (double-)order \( m = (m_1, m_2) \).

On the other hand, the (larger) class \( \psi_q \) will contain all \( a(x, \xi) \) such that all \( x \)-derivatives \( \partial_x^\alpha a(x, \xi) \) are of polynomial growth — order \( m \) — in the variables \( \xi \) with constants \( c_0 \) of def. 3.2 independent of \( x, \theta, \iota \).

Finally, the class \( \psi_p \) consists of all \( a(x, \xi) \) in \( \psi q \) which are independent of \( x_2, x_3 \) and periodic (with period \( 2\pi/\omega \) in \( x_1 \), with a given fixed (circular) frequency \( \omega = 2\pi \nu \).

To be precise, let us restate this as follows.

**Definition 3.3.** (i) The class \( \psi_c \) of symbols (we call ‘strictly classical’) consists of all functions \( a(x, \xi) \) defined and smooth for all 6 variables \( x, \xi \) and such that

\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq c_{0,1}(1 + |\xi|)^{m_1-|\beta|}(1 + |x|)^{m_2-|\alpha|}
\]

for all multi-indices \( \theta, \iota \), and all \( x, \xi \in \mathbb{R}^3 \) with constants \( c_{0,1} \) independent of \( x, \xi \).

The class of all such functions \( a(x, \xi) \), for a given order \( m = (m_1, m_2) \) will be denoted by \( \psi_c^m \). We also define \( \psi_c = \psi\infty = \cup_m \psi^m, \; \psi_{-\infty} = \cap_m \psi^m \).

(ii) The class \( \psi_q \) consists of all smooth functions \( a(x, \xi) \), defined for \( (x, \xi) \in \mathbb{R}^6 \) such that

\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq c_{0,1}(1 + |\xi|)^{m-|\beta|} \quad \text{for some } m \in \mathbb{R} \text{ and all } \iota, \theta, \; x, \xi.
\]

We again use \( \psi_q^m \) for the class of symbols of order \( m \), and define \( \psi_q = \psi\infty = \cup_m \psi^m, \; \psi_{-\infty} = \cap_m \psi^m \).

(iii) The class \( \psi_p \) consists of all \( a(x_1, \xi) \in \psi_q \), independent of \( x_2, x_3 \) and \( 2\pi/\omega \)-periodic in \( x_1 \), where \( \omega > 0 \) is some given fixed positive number; we again set \( \psi_p = \psi\infty = \cup_m \psi^m, \; \psi_{-\infty} = \cap_m \psi^m \).
We refer to [Co5], ch.1 for a proof of the fact that the integrals at right of (3.4)-(3.5) exist, in the order stated, whenever \( u \in S \) and \( a(x, \xi) \in \psi q \), defining a continuous operator \( A = a(x, D) \) on the space \( S \) — and then also on the space \( S' \) of temperate distributions. The classes of such operators then will be called \( Op\psi c \), \( Op\psi p \), \( Op\psi q \), etc.

We again must refer to ch.1 of [Co5] to see that there are Leibniz formulas with integral remainder valid, in the sense that, for product and adjoint among operators \( a(x, D), b(x, D) \in Op\psi q \), the differences \( c(x, \xi) - \sum_{0}^{N} \cdots , \bar{a} - \sum_{0}^{N} \cdots \) in (3.3) may be expressed as certain integrals, involving very singular integrals (called ‘finite parts’) of derivatives of the symbols involved. Using these we then get the following result.

**Theorem 3.4.** \( Op\psi c = \cup Op\psi c_{m} \) and \( Op\psi q = \cup Op\psi q_{m} \) are adjoint invariant graded algebras. The Leibniz formulas (3.3) for product and adjoint hold in the sense of asymptotic convergence (mod \( Op\psi c_{-\infty} \)) and (mod \( Op\psi q_{-\infty} \)), resp., of the infinite series \( \sum_{j=0}^{\infty} \) occurring. The classes \( Op\psi c_{-\infty} \) and \( Op\psi q_{-\infty} \) are two-sided \(*\)-ideals of \( Op\psi c \) and \( Op\psi q \), respectively.

In thm. 3.4 we used the following concepts.

**Definition 3.5.** (i) A sequence \( \{a_{n}(x, \xi) \in \psi c\} \) is said to converge asymptotically (mod \( \psi c_{-\infty} \)) to \( a(x, \xi) \) if the order \( m = (m_{1}, m_{2}) \) of the difference \( a(x, \xi) - a_{n}(x, \xi) \) tends to \((-\infty, -\infty)\) as \( n \to \infty \). Then also we shall say that \( A_{n} = a_{n}(x, D) \) tends to \( A = a(x, D) \) asymptotically (mod \( Op\psi c \)).

(ii) A sequence \( \{a_{n}(x, \xi) \in \psi q\} \) is said to converge asymptotically (mod \( \psi q_{-\infty} \)) to \( a(x, \xi) \) if the order of the difference \( a(x, \xi) - a_{n}(x, \xi) \) tends to \(-\infty\) as \( n \to \infty \). Then also we shall say that \( A_{n} = a_{n}(x, D) \) tends to \( A = a(x, D) \) asymptotically (mod \( Op\psi q \)).

The essence of the proof of thm.3.4 then will be that the ‘integral remainders’ representing the differences \( c(x, \xi) - \sum_{0}^{N} \cdots , \bar{a} - \sum_{0}^{N} \cdots \) in (3.3), must be shown to be symbols of orders tending to \(-\infty\), as \( N \to \infty \).

We also need

**Proposition 3.6.** Let \( r = c \) or \( r = q \). For any sequence of symbols \( \{a_{j}(x, \xi) : j = 0, 1, 2, \cdots \} \) with \( a_{j} \in \psi r_{m}^{l} \), where \( m^{l} \to -\infty \) resp. \( m^{l} \to -\infty \), \( l = 1, 2 \), there exists a symbol \( a(x, \xi) \in \psi r_{m}^{0} \) such that

\[
(3.8) \quad a(x, \xi) = \sum_{j=0}^{\infty} a_{j}(x, \xi)( \text{mod} \psi r_{-\infty})
\]

A proof (a la Hoermander) may be found in [Co5], ch.1, lemma 6.4, p.75. (Or else, cf.[Co16] footnote 18 on p.18) (for \( r = c \) only, but it may be adapted for \( r = q \)).

**Proposition 3.7.** (i) The class \( Op\psi c_{-\infty} \) consists of all integral operators \( Ku(x) = \int_{R^{3}} k(x, y)u(y)dy \) with kernel \( k(x, y) \) in \( S(R^{3}) \).

(ii) The class \( Op\psi q_{-\infty} \) consists of all \( \psi do-s \) \( C = c(x, D) \) with symbol \( c \) having all \( x \)-derivatives belonging to \( S \) in the \( \xi \)-variable, uniformly for all \( x \in R^{3} \).

\^{3}The assumptions made in ch.1 of [Co5] match ours here for \( Op\psi c \), but not for \( Op\psi q \). However, we checked in detail, that the arguments used there may be literally extended to the case of \( Op\psi q \), as shall be lined out explicitly in [Co17].
For the proof of (i) cf. [Co5], ch.3, prop.3.4 on p.111. (ii) is just a reformulation of the definition of $\psi_{q-\infty}$.

Finally, among results about $\psi do$-s, we also need to look at a representation of $\psi do$-s involving both representations (3.1) — i.e., allowing multiplications left and right from differentiations. This means generalizing (3.4) by writing

\[
a(M_t, M_r, D)u(x) = \frac{1}{(2\pi)^3} \int d\xi \int dy e^{\xi(x-y)} a(x, y, \xi) u(y) ,
\]

where the symbol $a(x, y, \xi)$ now depends on 9 variables $x, y, \xi \in \mathbb{R}^3$, and satisfies the estimates

\[
|\partial_x \partial_y \partial_{\xi} a(x, y, \xi)| \leq c_{0,\lambda,\theta}(1 + |\xi|)^{m-|\theta|} \quad \text{for some } m \in \mathbb{R} , \quad \text{all } x, y, \xi , \quad \text{all } \lambda, \theta .
\]

The class of all smooth $a(x, y, \xi)$ defined over $\mathbb{R}^3$ satisfying (3.10) will be denoted by $\psi qlr_m$, with $\psi qlr = \cup_m \psi qlr_m$. The notation $a(M_t, M_r, D)$ seeks to remind of the fact that we have $a(M_t, M_r, D) = p(x) r(D) q(x)$ for $a(x, y, \xi) = p(x) q(y) r(\xi)$.

Such operators $a(M_t, M_r, D)$ belong to $\text{Op}_c \psi q_m$, if the symbol $a(x, y, \xi)$ satisfies (3.10), and there exists a Leibniz formula (asymptotic (mod $\text{Op}_c \psi_{q-\infty}$)) defining a symbol $b \in \psi q_m$ such that $a(M_t, M_r, D) = b(x, D)$. Again, this is a matter of slightly adapting things around f'la. (5.5) on p.70 of [Co5].

We shall have to deal intensively with operators of this kind in sections 11 and following. It then even will be necessary to discuss some facts regarding Fourier integral operators with symbol and phase functions in $\psi qlr$. For more detail we refer to sect.12.

4. **Time-Independent Potentials vanishing at $\infty$**

We return to the Dirac equation and will assume here that the potentials $\mathbf{A}_j, \mathbf{V}$ of $H$ in (2.1) do not depend on $t$, and will have the limit zero, as $|x| \to \infty$. Moreover, we shall assume that $\mathbf{V}(x)$ and $\mathbf{A}_j(x), j = 1, 2, 3$ are of polynomial growth, order $-1$. We then get $H \in \text{Op}_c(1,0)$, and

\[
h(x, \xi) = \sum_{j=1}^{3} \alpha_j (\xi_j - \mathbf{A}_j(x)) + \beta + \mathbf{V}(x) .
\]

The propagator $U(t)$ then may be written as $U(t) = e^{-iHt}$; it commutes with $H$ for every $t$. However, it does not belong to $\text{Op}_c$. In [Co3],[Co16] (and in numerous other articles) we then asked the question for observables $A$ with the property that the Heisenberg transform $A_t = e^{iHt} A e^{-iHt}$ belongs to $\text{Op}_c$, for all $t$. In essence this implies that $A = a(x, D)$ has its symbol $a(x, \xi)$ commuting with the symbol $h(x, \xi)$ of $H$, for very large $|x| + \xi$. Recall, the matrix $h(x, \xi)$ has the two eigenvalues $\lambda_{\pm}(x, \xi) = \mathbf{V}(x) \pm (\xi - \mathbf{A}(x))$, of multiplicity 2 each, and their spectral projections $p_{\pm}(x, \xi)$ of (2.4) separate the states belonging to electron and positron, respectively. The fact that $a(x, \xi)$ must commute with $h(x, \xi)$ implies that $a(x, \xi)$ takes the spaces of electron and positron states into themselves — in some weakened sense. Clearly, this should be a desirable property, in view of the various contradictions or paradoxes in older literature, stemming from violation of this property.

In earlier publications we were using the name *precisely predictable* for observables $A$ with $A_t \in \text{Op}_c$, and we proposed that the rule (2.12) of predicting the statistical expectation-value should be applicable only to precisely predictable observables. While total energy and (often also) total angular momentum trivially are precisely predictable, other observables — like $x_j$ and $D_j$ do not have this property, but they
are approximately predictable — with a preset error — in the sense that there are precisely predictable observables in their close neighbourhood.

Here we will attempt to describe the essentials of the theory, omitting a discussion of a large amount of technical proofs, already discussed in close detail in [Co16].

Suppose \( A_t = e^{iHt}Ae^{-iHt} \) belongs to \( Op\psi c_m \), for some fixed \( m = (m_1, m_2) \), and all \( t \). So, we have \( A_t = e^{iHt}Ae^{-iHt} = a_t(x, D) \). Assume also that the time-derivative \( \dot{a}_t(x, \xi) \) exists and belongs to \( \psi c_{m-e^2} \) where \( e^2 = (0, 1) \). Differentiating for \( t \) we get

\[
\dot{A}_t = iHe^{iHt}Ae^{-iHt} - ie^{iHt}Ae^{-iHt}H = i[H, A_t].
\]

Since \( H \) and \( A_t \) are \( \psi do-s \), by assumption, we may use the Leibniz formula of lemma 3.1 to obtain a symbol for the commutator \([H, A_t] = HA_t - A_tH\). We get

\[
[1, h, a_t] = [h, a_t] - i\{h, a_t\} - \frac{1}{2!}\{h, a_t\}_2 + \frac{i}{3!}\{h, a_t\}_3 + \cdots,
\]

where we use the (generalized) Poisson brackets

\[
\{h, a_t\} = [h, a_1] = h[\xi a_t|x - a_t|\xi x] , \quad \{h, a_t\}_2 = h[\xi\xi a_t|x\xi - a_t|\xi\xi x] , \quad \text{etc.}
\]

In (4.3) the terms at right have orders \( m + e^1, m + e^1 - e, m + e^1 - 2e, \cdots \), with \( e^1 = (1, 0) \), so, the asymptotic sum mod \( \psi c_{-\infty} \) exists, by prop.3.6. With (4.3) we may express (4.2) symbol-wise in the form

\[
\dot{a}_t = i\{h, a_t\} + \{h, a_t\} - \frac{i}{2!}\{h, a_t\}_2 - \frac{1}{3!}\{h, a_t\}_3 + \cdots.
\]

Proposition 4.1. If we have \( A_t = e^{iHt}Ae^{-iHt} = a_t(x, D) \), where \( a_t(x, \xi) \in \psi c_m \), \( \dot{a}_t(x, \xi) \in \psi c_{m-e^2} \), then the commutator \([h(x, \xi), a_t(x, \xi)]\) — naturally being of order \( m + e^1 \), since \( h \in \psi c_e \) — must have the (lower) order \( m - e^2 \).

Indeed, all terms in (4.5), except the term involving \([h, a_t]\), have order \( m - e^2 \) (or lower), hence \([h, a_t]\) also must be of order \( m - e^2 \).

So, indeed, we get \([h(x, \xi)/\xi), (a_t(x, \xi)/\langle x \rangle^{m_1}[\xi]^{m_2}) = O((\langle x \rangle)^{-1}) \), i.e., this commutator vanishes as \(|x| + |\xi| \to \infty\).

Vice versa, (4.5) suggests, that we might attempt construction of a precisely predictable \( A = a(x, D) \in Op\psi c \) by starting with a (self-adjoint) \( q(x, \xi) \in \psi c_m \) with the property that \([h(x, \xi), q(x, \xi)] = 0 \) for all \( x, \xi \), and then trying to find a \( z(x, \xi) \in \psi c_{m-e} \) such that \( a = q + z \) satisfies (4.5). Noting that the terms at right of (4.5) are of order \((m + e^1), (m + e^1) - e, (m + e^1) - 2e, (m + e^1) - 3e \cdots \) with \( e = (1, 1) \), we might neglect all terms at right of (4.5) but the first two, then getting an equation valid modulo \( \psi c_{m-e^2-e} \) only:

\[
\dot{a}_t = i[h, a_t] + \{h, a_t\} \mod \psi c_{m-e^2-e}.
\]

Let us assume that we also have \( a_t(x, \xi) = q_t(x, \xi) + z_t(x, \xi) \) with \([h(x, \xi), q_t(x, \xi)] = 0 \) \( \forall x, \xi \), where \( q_t(x, \xi) \in \psi c_m \), \( z_t \in \psi c_{m-e} \), \( q_t \in \psi c_{m-e^2} \), \( z_t \in \psi c_{m-e^2-e} \). Then we may omit further terms, vanishing or being of order \( m - e^2 - e \):

\[
\dot{q}_t = i[h, z_t] + \{h, q_t\} \mod \psi c_{m-e^2-e}.
\]
We start an iteration by assuming \((4.6)\) as a sharp equation — not only modulo \(\psi c_{m_0}\). Assuming \(q_t\) known we obtain an equation for \(z_t\):

\[
(4.7) \quad [h, z_t] = i((\{h, q_t\} - \dot{q}_t)).
\]

Attempting to solve this matrix-commutator equation for \(z\) we observe the following:

**Proposition 4.2.** Equation \((4.7)\) has no solution, unless the right hand side \(Z_t = i((\{h, q_t\} - \dot{q}_t))\) satisfies

\[
(4.8) \quad p_+((\{h, q_t\} - \dot{q}_t)p_+ = 0, \quad p_-((\{h, q_t\} - \dot{q}_t)p_- = 0, \quad \text{for all } x, \xi.
\]

If \((4.8)\) holds, then an infinity of solutions is given by

\[
(4.9) \quad z_t = \frac{1}{\lambda_+ - \lambda_-}(p_+Z_t p_+ - p_-Z_t p_-) + c_t = \frac{1}{2}\langle \xi - A(x) \rangle (p_+Z_t p_+ - p_-Z_t p_-) + c_t,
\]

with the eigenvalues \(\lambda_\pm\) of \(h(x, \xi)\), where \(c_t(x, \xi)\) may be any symbol commuting with \(h(x, \xi)\) — i.e., we must have \(c_t = p_+c_t p_+ + p_-c_t p_-\).

The proposition is easily verified, using facts on spectral projections: \(p_+ + p_- = 1\), \(p_+^2 = p_+\), \(p_-^2 = p_-\), \(p_+p_- = p_-p_+ = 0\), \(h = \lambda_+p_+ + \lambda_-p_-\).

The interesting fact now is that — while we know \(q_t\) only for \(t = 0\) (where we should have \(q_0 = q\)), the solvability conditions \((4.8)\) will resolve into a set of partial differential equations determining \(q_t\) for all \(t\), from its initial-value \(q_0\), so that we then indeed may use \((4.9)\) to obtain the desired \(z_t\) (including \(z = z_0\)). Moreover, this set of differential equations relates to the classical equations determining the propagation of the particle, as we shall see.

Of course, this will only supply a solution to equation \((4.7)\), not the real thing \((4.5)\). However, then, we shall set up an iteration, getting us a solution of \((4.5)\) modulo \(\psi c_{-\infty}\), using prop.3.6. In combination with prop.3.7 this indeed will be enough to construct a precisely predictable observable \(a(x, D) = q(x, D) + z(x, D)\) in \(Op\psi c_m\), with lower order \(z\), starting from an arbitrarily given symbol \(q \in \psi c_m\), commuting with \(h\).

There is a mountain of technicalities in our way, all discussed in detail in \([Co16]\). Here we shall focus on the above first step, solving eq. \((4.7)\).

Let us try to evaluate the conditions \((4.8)\). The assumption \([h, q] = 0\) implies that \(q = q^+ + q^-\), where \(q^+ = p_+q_+\), \(q^- = p_-q_-\). We first work with a simplifying assumption that \(q^+\) and \(q^-\) are scalar multiples of \(p_+\) and \(p_-\), resp., a condition trivially satisfied by symbols being scalar multiples of the \(4 \times 4\) unit matrix. In that case we shall be successful if we assume the same for \(q_t^+ = p_+q_t^+\), \(q_t^- = p_-q_t^-\). So, we first look at the special case where

\[
(4.10) \quad q_t = q_t^+p_+ + q_t^-p_- \text{ with scalar (complex-valued) symbols } q^+, q^-.
\]

**Proposition 4.3.** With above assumptions on \(q_t\) we get

\[
(4.11) \quad p_+\{h, q_t\}p_+ = \{\lambda_+, q_t^+\}p_+ = \{\lambda_-, q_t^-\}p_-.
\]

The proof is a calculation (cf. \([Co16]\), p.93). Applying this to \((4.8)\), using \((4.10)\), these equations assume the form

\[
(4.12) \quad \hat{q}_t^+ = \{\lambda_+, q_t^+\}, \quad \hat{q}_t^- = \{\lambda_-, q_t^-\},
\]

with the eigenvalues \(\lambda_\pm(x, \xi) = V(x) \pm (\xi - A(x))\) of \(h(x, \xi)\), noted in sec.1.
Two things are interesting here: First of all, the two equations (4.8) have split into separate equations for \( q_+^t \) and \( q_-^t \) — the first involves only \( q_+^t \), the second only \( q_-^t \). Secondly, both these equations now are first order partial differential equations for a scalar dependent variable:

\[
q_+^t = \lambda_{+\vert_\xi} q_{+\vert_\xi}^t - \lambda_{+\vert_\omega} q_{+\vert_\xi}^t, \quad q_-^t = \lambda_{-\vert_\xi} q_{-\vert_\xi}^t - \lambda_{-\vert_\omega} q_{-\vert_\xi}^t.
\]

Solving the initial-value problem for equations (4.13) is a simple matter, just involving ordinary differential equations: For the first equation (4.13) look at the first order system of 6 ODE-s

\[
(4.14+) \quad \dot{x} = \lambda_{+\vert_\xi}, \quad \dot{\xi} = -\lambda_{+\vert_\omega}, \quad \lambda_+ = \mathbf{V}(x) + (\xi - \mathbf{A}(x)),
\]

in the 6 unknown functions \( x(t), \xi(t) \), of the single variable \( t \). Given any initial real 6-vector \((x^0, \xi^0)\) there is a unique curve \( x(t), \xi(t) \) in \( \mathbb{R}^6 \) solving (4.14+), passing through \((x^0, \xi^0)\) at \( t = 0 \). In fact, the entire ‘phase space’ \( \mathbb{R}^6 \) is filled with such ‘orbits’ with no two of them intersecting.

We then may look at the first (4.13) along such a curve \( x(t), \xi(t) \). Substituting (4.14+) we get

\[
(4.15) \quad \partial_\xi q_{+\vert_\tau}^t (x(t), \xi(t)) = \partial_\xi q_{+\vert_\tau}^t (x(t), \xi(t)) \dot{x}(t) + \partial_\xi q_{+\vert_\tau}^t (x(t), \xi(t)) \dot{\xi}(t),
\]

amounting to \( \frac{d}{dt} q_{+\vert_\tau}^t (x(t), \xi(t)) = 0 \). Or else, \( q_{+\vert_\tau}^t (x(t), \xi(t)) \) must be a constant — independent of \( t \) — along any such curve.

Here we consider the flow defined by the system (4.14+): For any fixed \( t \) introduce the diffeomorphism \( \nu_t^+ : \mathbb{R}^6 \rightarrow \mathbb{R}^6 \) defined by letting \((x, \xi)\) move along the solution curve of (4.14+) through it for a distance \( t \) (positive or negative, according to the sign of \( t \)). Then consider the expression \( q_{+\vert_\tau}^t (\nu_{t\rightarrow} (x, \xi)) = q_{+\vert_\tau}^t (x_{t\rightarrow}(x, \xi), \xi_{t\rightarrow}(x, \xi)) \), as a function of \( \tau \).

This function is constant — independent of \( \tau \), as a consequence of the above. Thus, setting \( \tau = t \) and \( \tau = 0 \), and using that \( v_0(x, \xi) = (x, \xi) \), we get

\[
(4.16) \quad q_t(x, \xi) = q_t(v_0(x, \xi)) = q_0(v_t(x, \xi)) = q(v_t(x, \xi)).
\]

So — since \( q_0 = q \) is given, we indeed have obtained a well defined \( q_t(x, \xi) = q(v_t(x, \xi)) \) as the only possible candidate for solving (4.7).

**Observation 4.4.** *It should be noted here that the differential equations are the classical equations of motion for a (spinless) electron moving in the electromagnetic field defined by \( \mathbf{V} \) and \( \mathbf{A} \).*

A similar discussion — of course — will hold for the second condition (4.8), resulting in another (Hamiltonian) system

\[
(4.14-\) \quad \dot{x} = \lambda_{-\vert_\xi}, \quad \dot{\xi} = -\lambda_{-\vert_\omega}, \quad \lambda_- = \mathbf{V}(x) - (\xi - \mathbf{A}(x)),
\]

a corresponding flow \( \nu_t^- (x, \xi) \) and a \( q_{-\vert_\tau}^t (x, \xi) = q(\nu_{t\rightarrow}^- (x, \xi)) \). In each case we also get a \( z_{+\vert_\tau}^t \), \( z_{-\vert_\tau}^t \) from (4.9), and a \( q_t + z_t \) solving (4.6), thus completing the first step of our iteration. The flow \( \nu_t^- \) will describe the classical motion of a spinless positron.

A discussion of the elements of the proof of thm.4.5, below, is given in [Co16], chapters 4 and 5. We also might refer to [Co3] and [Co5] where the same facts are discussed.
The more general case, where \( q^+ \), \( q^- \) are not necessarily multiples of the identity, is more complicated — and, perhaps, more interesting, since magnetic spin-problems will appear. It will be discussed in the next following section.

**Theorem 4.5.** Assume a given symbol \( q \in \psi c_m \) with \([h(x,\xi), q(x,\xi)] = 0 \forall (x,\xi)\), and such that we have

\[
p^+ q p^+ = q^+(x,\xi) p^+, \quad p^- q p^- = q^-(x,\xi) p^-.
\]

Then there exists a symbol \( a_t(x,\xi) = q_t(x,\xi) + z_t(x,\xi) \in \psi c_m \), for all \( t \), satisfying

\[
a_t(x, D) = e^{iHt} a_0(x, D) e^{-iHt},
\]

and such that \( z_t \in \psi c_{m-}\) while

\[
q_t(x,\xi) = q^+(\nu^+ p(x,\xi)) p_t(x,\xi) + q^-(\nu^- p(x,\xi)) p_t(x,\xi),
\]

with the two flows \( \nu^+ \), \( \nu^- \) generated by the classical motions of the spinless electron and positron, respectively.

The symbol \( z_t(x,\xi) \) may be chosen such that \( a_t(x,\xi) \) is self-adjoint for all \( x,\xi \). Then the operator \( A = a(x, D) = q(x, D) + z_0(x, D) \) is a precisely predictable observable.

---

4Explicitly, the system (4.14+) for \( \lambda_+ = (\xi - A) + V \) looks like this:

\[
\dot{x} = \frac{1}{\sqrt{1 - \lambda_+^2}} (\xi - A(x)) \quad \dot{\lambda} = \frac{1}{\xi - A(x)} \sum_j (\xi_j - A_j(x)) A_{ji}(x) - V_{ji}(x).
\]

The first equation may be solved for \( \xi - A \): We get

\[
\xi - A = \frac{\dot{x}}{\sqrt{1 - \lambda_+^2}} \quad \xi - A = \frac{1}{\sqrt{1 - \lambda_+^2}}
\]

Equating the derivative \( \dot{\xi} \) of (4.18) with the second (4.17) gives

\[
\frac{\dot{x}}{\sqrt{1 - \lambda_+^2}} + \partial_t A(x(t)) = -V_{ix} + \sum_j \dot{x} j A_{ji}(x).
\]

In (4.19) we get \( \partial_t A(x(t)) = \sum_i \dot{x} i A_{ix}(x(t)) \). Now we use the relation

\[
\dot{x} \times \text{curl } A = \sum_i (\dot{x} i A_{ix} - \dot{x} i A_{ix})
\]

As a consequence (4.19) assumes the form

\[
\frac{\dot{x}}{\sqrt{1 - \lambda_+^2}} = -V_{ix}(x(t)) + \dot{x} \times \text{curl } A(x(t)).
\]

But electric and magnetic field \( E \) and \( H \) as functions of \( A \) and \( V \) are given by the formulas

\[
E = -A_{ix} - \text{grad } V, \quad B = \text{curl } A,
\]

and the relativistic mass (of the particle with rest mass \( 1 \)) will be \( \frac{1}{\sqrt{1 - \lambda_+^2}} \) in the physical units we employ here. Accordingly (4.20) reads

\[
\frac{\dot{x}}{\sqrt{1 - \lambda_+^2}} = E + \dot{x} \times B.
\]

Clearly, this exactly describes the acceleration of the electron under the force of the (time-dependent) electromagnetic field acting on it.

5This condition simply means that \( q(x,\xi) \) is a scalar multiple of the identity in each of the two eigenspaces \( S_\pm = S_\pm(x,\xi) \) of the symbol \( b(x,\xi) \).
A General Commuting \(q(x, \xi)\) and a Magnetic 3-Vector \(\kappa(x, \xi)\)

In this section we shall discuss the more general case where \(q\) is not scalar in the eigenspaces of \(H\). We still look for a solution of the commutator equation (4.7), i.e.,

\[
[h, z_t] = i\{h, q_t\} - q_t^* = 0
\]

keeping in mind prop. 4.2 with solvability condition (4.8). But we must replace prop.4.3:

**Proposition 5.1.** We get

\[
\begin{align*}
(5.1+) & \quad p_+\{h, q_t\}p_+ = p_+\{\lambda_+, q_t^+\}p_+ + 2\{\zeta\}p_+\{p_+, q_t^+\}p_+ , \\
(5.1-) & \quad p_-\{h, q_t\}p_- = p_-\{\lambda_-, q_t^-\}p_- - 2\{\zeta\}p_-\{p_-, q_t^-\}p_- ,
\end{align*}
\]

with \(\zeta = \xi - A(x)\), \(q_t^+ = p_+q_TP^+\), \(q_t^- = p_-q_TP^-\).

**Proof** Clearly we have

\[
(5.2) \quad p_+\{h, q_t\}p_+ = p_+\{\lambda_+, q_t\}p_+ + p_+\{\lambda_-, q_t\}p_+ = Z_1 + Z_2 ,
\]

where \(Z_1 = p_+\{\lambda_+, q_t\}p_+ + \lambda_+p_+\{p_+, q_t\}p_+\), while

\[
Z_2 = \lambda_-\{p_+q_Tp_+ - \lambda_+p_+q_Tp_+ + \lambda_-p_-q_Tp_+ - \lambda_-p_-q_Tp_-\}p_+ + \lambda_-p_-\{p_+, q_t\}p_+ ,
\]

since \(p_+p_- = p_-p_+ = 0\). Also, \(p_+ + p_- = 1\) implies \(p_+|\xi| = -p_+|\xi|\) and \(p_-|\xi| = -p_-|\xi|\), hence, \(\{p_+, q_t\} = -\{p_+, q_t\}\), so that, \(Z_2 = -\lambda_-p_+\{p_+, q_t\}p_+\). Together we get

\[
(5.3) \quad p_+\{h, q_t\}p_+ = p_+\{\lambda_+, q_t\}p_+ + (\lambda_+ - \lambda_-)p_+\{p_+, q_t\}p_+ .
\]

Simplifying (5.3) we first recall that \(\lambda_+ - \lambda_- = 2\{\zeta\} = 2\{\xi - A(x)\}\). Furthermore we get \(q_t = q_t^+ + q_t^-\), where \(p_+\{\lambda_+, q_t^-\}p_+ = 0\), since \(\lambda_+\) is a scalar, so that \(\{\lambda_+, q_t^-\} = \lambda_-q_t^-\), \(\lambda_-q_t^-\), and \(p_+q_t^-p_+ = p_+q_t^-p_+ = 0\), implied by \(p_+q_t^- = 0\Rightarrow p_+q_t^- = -p_+q_t^-\), etc. So, in the first term at right of (5.3) we may replace \(q_t\) by \(q_t^+\).

The same follows for the second term, so that (5.1+) follows: Indeed, we get

\[
(5.4+) \quad p_+\{p_+, q_t^-\}p_+ = p_+p_+|\xi|q_t^-p_+ = -p_+q_Tp_+ = -p_+q_Tp_+ + p_+p_+|\xi|q_t^-p_+ = 0 ,
\]

where we again used that \(p_+q_T = q_Tp_+ = 0\) implies \(p_+q_Tp_+ = -p_+q_Tp_+ = q_Tp_+\). A similar argument yields (5.1)-, q.e.d.

After prop.5.1 it is clear that we again have split the two solvability conditions (4.8) into separate systems for \(q_{\pm}^\pm\): The first cdn. involves only \(q_{\pm}^+\), the second only \(q_{\pm}^-\). Using (5.1) and differentiating along the solution curves of the Hamiltonian system (4.14+) for \(\lambda_+\) we may rewrite the first (4.8) as

\[
(5.4+) \quad p_+q_{\pm}^+p_+ - 2\{\zeta\}p_+\{p_+, q_{\pm}^+\}p_+ = 0 ,
\]

where "\(\partial\)" denotes the directional derivative \(\partial_t - \lambda_{\pm}|\xi|\partial_x + \lambda_{\pm}|\xi|\partial_\xi\), used for \(\lambda_+\). Similarly

\[
(5.4-) \quad p_-q_{\pm}^-p_- + 2\{\zeta\}p_-\{p_-, q_{\pm}^-\}p_- = 0 ,
\]

with "\(\partial\)" for \(\lambda_-\).

In the case of a \(q(x, \xi)\) scalar in the two eigenspaces \(S_{\pm}(x, \xi)\), as discussed in thm.4.5, we only needed the two eigenvalues \(\lambda_{\pm}(x, \xi)\) to set up our first approximation. In the present more general case we will obtain explicit \(2 \times 2\) matrices of \(q_{\pm}(x, \xi)\) with respect to a natural orthonormal base of \(S_{\pm}(x, \xi)\) of the
symbol \( h(x, \xi) \). Getting restricted to only use the Dirac matrices \( \alpha_j, \beta \) of (2.8), so that \( \beta \) is the diagonal matrix defined there, we recall the orthogonal matrix \( \Upsilon \) of (2.5), known to satisfy (2.6), i.e.,

\[
(5.5) \quad h(x, \xi) \Upsilon(x, \xi) = \Upsilon(x, \xi)(V(x) + (\xi - A(x))\beta), \quad \Upsilon = \frac{1}{\sqrt{2(1 + v_0)}} \begin{pmatrix} 1 + v_0 & -i\sigma v \\ -i\sigma v & 1 + v_0 \end{pmatrix},
\]

with \( v_0 = \langle \xi \rangle^{-1}, v = \xi/\langle \xi \rangle \), \( \xi = \xi - A(x) \). We may rewrite this as

\[
(5.6) \quad h(x, \xi) \Upsilon_\pm(x, \xi) = \lambda_\pm(x, \xi) \Upsilon_\pm(x, \xi), \quad \text{with } \Upsilon_+ = \begin{pmatrix} 1 + v_0 \\ -i\sigma v \end{pmatrix}, \quad \Upsilon_- = \begin{pmatrix} -i\sigma v \\ 1 + v_0 \end{pmatrix}.
\]

The columns of the \( 4 \times 2 \)-matrices \( \Upsilon_\pm(x, \xi) \) are eigenvectors to \( \lambda_\pm(x, \xi) \), of length \( \sqrt{2(1 + v_0)} \), and mutually orthogonal.

We then have \( q^\pm_\xi(x, \xi) \) represented by the \( 2 \times 2 \)-matrices

\[
(5.7) \quad \kappa^\pm_j = \left((\kappa^\pm_{jl})_{j,l=1,2}\right) = \frac{1}{2(1 + v_0)} \Upsilon^\pm q^\pm_\xi \Upsilon^\pm.
\]

Writing \( \Upsilon_\pm = (\varphi_1^\pm, \varphi_2^\pm) \) column-wise, we may introduce the \( 4 \times 4 \)-matrices

\[
(5.8) \quad p^\pm_{jl} = \frac{1}{2(1 + v_0)} \langle \varphi_j^\pm | \varphi_l^\pm \rangle, \quad \varphi_j^\pm \langle \varphi_l^\pm, \varphi_l^\pm \rangle = \varphi_l^\pm \varphi_l^{\pm*},
\]

and then get

\[
(5.9) \quad q^\pm_\xi = \sum_{j,l=1}^2 \kappa^\pm_{jl} p^\pm_{jl}, \quad p^\pm = p^\pm_{11} + p^\pm_{22}.
\]

We now use (5.9) to translate (5.4+) into a \( 2 \times 2 \)-matrix form. Using that \( p^2 = p \) for "\( \pm \)" implies \( p_\pm p^\prime_\pm p_\pm = 0 \) for any directional derivative "\( \alpha_\mu \)", (5.4+) and (5.9) yield

\[
(5.10) \quad \sum_j p_{jl} \kappa^\prime_{jl} + \sum_j \kappa_{jl}(pp^\prime_{jl}p) - 2(\xi) \sum_j \kappa_{jl} p\{p, p_{jl}\}p = 0,
\]

where we restricted to "\( + \)" and dropped the "\( + \)" in notation. Evidently, the first term of (5.10) has the matrix \( (\kappa^\prime_{jl}) \). The matrices of the other two terms may be written as \( W^\prime_{\xi} \kappa_\xi \) with a certain linear map \( W^\prime_{\xi} \) taking \( 2 \times 2 \)-matrices to \( 2 \times 2 \)-matrices. Thus (5.10) may be written as

\[
(5.11) \quad (\kappa^\prime_j)^\prime + W^\prime_{\xi} \kappa^\prime_\xi = 0
\]

with "\( \alpha_\mu \)" of (5.4+). Using the hamiltonian system (4.14+) this again will turn into a system of 4 ODE-s along the classical electron-particle flow for the 4 scalar functions \( \kappa^\prime_{t,jl}(x(t), \xi(t)) \).

**Proposition 5.2.** Relation (5.11) may be rewritten as

\[
(5.12) \quad (\kappa^\prime_j)^\prime + [\Theta^+, \kappa^\prime_\xi] = 0,
\]

with the directional derivative "\( \alpha_\mu \)" of (5.4+) and the \( 2 \times 2 \)-matrix commutator \([, ,]\), where the \( 2 \times 2 \)-matrix \( \Theta^+ \) is defined as

\[
(5.13) \quad \Theta^+ = \frac{1}{2(1 + v_0)}(\Upsilon^+_+ \Upsilon^+_+ - 2(\xi) \Upsilon^+_+ p^\pm_{\xi(P^\pm_{\xi})\Upsilon^+_+} Y).
\]

**Proof:** Indeed, (dropping "\( + \)"), and with "\( \alpha_\mu \)" any directional derivative) we have \( pp^\prime_{jl} = p_{jl} \), hence \( pp^\prime_{jl} + p^\prime_{jl} p_{jl} \Rightarrow (1 - p)p^\prime_{jl} = p^\prime_{jl} + p_{jl}, \) also, \( pp^\prime = p^\prime(1 - p) \) as already used. Thus \( pp_{\xi(P^\pm_{\xi})p} = pp_{\xi}(1 - p)p_{\xi_{\xi}p} = p_{\xi}(pp_{\xi_{\xi}p})p_{\xi} \), and, similarly, \( pp_{\xi(P^\pm_{\xi})p} = p_{\xi}(pp_{\xi_{\xi}p})p_{\xi} \). This will give

\[
(5.14) \quad \sum_j \kappa_{jl} p\{p, p_{jl}\}p = [pp_{\xi(P^\pm_{\xi})p}, \sum_j \kappa_{jl} p_{jl}] = [pp_{\xi(P^\pm_{\xi})p}, q^\pm_\xi],
\]
showing that the last term in (5.10) has the desired commutator form giving the second term at right of (5.13)

For the second term of (5.10) note that \( p'_{q} = q'_{p} \langle \chi_{1} + \varphi_{j} \rangle \langle \chi'_{1} \rangle \), where we wrote \( \chi_{1} = \frac{1}{2(1+υ)} \varphi_{j} \), for a moment. The \( \varphi_{j} \) and \( \chi_{1} \) satisfy \( \langle \chi_{1}, \varphi_{j} \rangle = δ_{ij} \), implying that \( \langle \chi'_{1}, \varphi_{j} \rangle = -\langle \chi_{1}, \varphi'_{j} \rangle \). The coefficients of the \( 2 \times 2 \)-matrix of \( pp'_{p} \) then will be \( \langle \chi_{p}, (p'_{q} + \varphi_{j}) \rangle \langle \chi'_{1} \rangle \varphi_{q} \rangle = \langle \chi_{p}, \varphi'_{j} \rangle δ_{ij} - δ_{pq} \langle \chi_{1}, \varphi'_{j} \rangle \). Accordingly, the second term of (5.10) will give

\[
\sum \kappa_{j} \chi_{p} (p'_{q} + \varphi_{j}) \langle \chi'_{1} \rangle \varphi_{q} \rangle = \sum \kappa_{pq} \chi_{1} \varphi_{q} \rangle ,
\]

giving the first term at right of (5.13). Q.E.D.

Of course there is an analogous consideration for “-” which will be left to the reader.

Here let us pass from the \( 2 \times 2 \)-matrix representation of \( q_{t}^{\pm} \) to the so-called Garding-Wightman representation of \( 2 \times 2 \)-matrices:

**Lemma 5.3.** Every complex \( 2 \times 2 \)-matrix \( a = (a_{jk}) \) may be uniquely written in the form

\[
a = k_{0} + \bar{k} \sigma , \quad \text{where} \quad k_{0} = \frac{1}{2} \text{trace} (a) \ , \quad k_{j} = \frac{1}{2} \text{trace} (\sigma_{j} a) \ , \quad j = 1, 2, 3 ,
\]

with the Pauli-matrices \( \sigma_{j} \) of (2.7), where \( k_{0}, \bar{k} \) are real if and only if \( a \) is self-adjoint.

The proof of lemma 5.3 is trivial.

If we substitute \( k_{+} = k_{10} + \sigma \bar{k} \), \( \Theta^{+} = -\frac{i}{2} (F_{0} + \sigma \bar{F}) \) into (5.12) we get

\[
k'_{t} = 0 \ , \quad (\bar{k}_{t})' + \bar{F} \times \bar{k}_{t} = 0 .
\]

Here we used the well known formula

\[
(\sigma \eta)(\sigma \eta) = \xi \eta + i \sigma \cdot (\xi \times \eta) \ , \quad \xi, \eta \in \mathbb{R}^3 .
\]

The first equation (5.16) states what we already know from sec.3: If \( q_{t} \) is a scalar in \( S_{p} \) the we have \( k_{+} \) a multiple of the identity, so that \( k_{+} = k_{10} \) while \( \bar{k}_{t} = 0 \). So, \( k_{t} \) is constant on the flow \( \nu_{t}^{+} \). Assuming that the corresponding also holds for \( q_{t}^{-} \) we then again get the statement of thm.4.5.

For the second equation (5.16), we again involve the system (4.14+) of ODE-s and its flow \( \nu_{t}^{+} \). We get

\[
d \frac{d}{dt} \bar{k}_{t} (\nu_{t}(x, \xi)) = \bar{F} (\nu_{t}(x, \xi)) \times \bar{k}_{t} (\nu_{t}(x, \xi)) = 0 ,
\]

along any solution curve \( x(t), \xi(t) \) of the system (4.14+). With the flow \( \nu_{t} \) we get

\[
d \frac{d}{d\tau} \bar{k}_{t-\tau} (\nu_{t}(x, \xi)) = \bar{F} (\nu_{t}(x, \xi)) \times \bar{k}_{t-\tau} (\nu_{t}(x, \xi)) = 0 ,
\]

a system of 3 ODE-s in 3 unknown functions of the variable \( \tau \). We know the solution \( \bar{k}_{t-\tau} (\nu_{t}(x, \xi)) \) at \( \tau = t \) where it becomes \( \bar{k}_{0} (\nu_{t}(x, \xi)) \) with the matrix \( k_{0} = q_{0}^{+} = q^{+} \). Thus \( \bar{k}_{t-\tau} (\nu_{t}(x, \xi)) \) is completely determined for all \( \tau \), and especially for \( \tau = 0 \), where we get \( \bar{k}_{t} (x, \xi) \). The components of \( \bar{k} \) remain symbols in \( \psi_{cm} \), as a consequence of our discussion in [Co16], ch.5. Corresponding statements hold for “-”, and the existence result of thm.4.5 will be following again.
Observation 5.4. The second (5.15) appears interesting from a different viewpoint: Clearly the expression $\frac{d}{dt} \mathbf{\hat{n}}_t - \mathbf{\hat{n}}_{t-\tau}(v_t(x, \xi))|_{\tau=0}$ may be interpreted as the rate of change (in time) of the real 3-vector $\mathbf{\hat{n}}_t(x, \xi)$ progressing on its orbit through $(x, \xi)$, while subtracting the orbital rate of change. According to (5.15), this vector equals a vector product $-\mathbf{\hat{F}}(x, \xi) \times \mathbf{\hat{n}}_t(x, \xi)$ with a certain 3-vector $-\mathbf{\hat{F}}(x, \xi)$.

As will be shown in sec.6, below, the vector $\mathbf{\hat{F}}(x, \xi)$ will be a linear combination of magnetic vectors — the magnetic induction $\mathbf{B}(x, \xi)$ and a vector of the form $\dot{x} \times \mathbf{E}$, at $(x, \xi)$, where we used (4.18) to replace $\zeta = \xi - \mathbf{A}(x)$ by $\dot{x}$ — the velocity of the particle.

So, we might have reason to regard the vector $\mathbf{\hat{n}}_t^j(x, \xi)$ as a magnetic moment vector, traveling with the particle — since it reacts to the fields at the location $(x, \xi)$ of the particle. But, as we shall find, the magnetic field, this vector ‘sees’, is not the relativistic field of the moving particle at the point $(x, \xi)$. So, while we are tempted to interpret $\mathbf{\hat{n}}_t$ as a magnetic spin-vector, traveling with the particle, there will be some paradoxes appearing, possibly to be eliminated by a better setup?  

6. Extension of Theorem 4.5

It now will be a matter of a (lengthy) calculation to verify that the vector $\mathbf{\hat{F}}$ plays the role of a magnetic field vector.

Proposition 6.1. The 3-vector $\mathbf{\hat{F}}$ is explicitly given as

$$\mathbf{\hat{F}} = \frac{1}{(\zeta)(1+\langle\zeta\rangle)}(-\zeta \times \mathbf{E} + \frac{1}{\langle\zeta\rangle}(\langle\zeta\rangle \mathbf{B} - (\langle\zeta\rangle \mathbf{B})) - \frac{1}{\langle\zeta\rangle^2}(\mathbf{B} + \frac{1}{1+\langle\zeta\rangle}(\langle\zeta\rangle \mathbf{B}))$$

with $\zeta = \xi - \mathbf{A}(x)$ and the field vectors

$$\mathbf{E} = -\text{grad} \mathbf{V}, \mathbf{B} = \text{curl} \mathbf{A}.$$

Proof. To simplify calculations, we note that the matrices $\Theta$ occur only in the commutator of equation (5.13). When we evaluate them we may omit any additive term giving a scalar multiple of the $2 \times 2$-identity matrix, because its contribution to the commutator will vanish. We shall write ‘$a = b(mod \ 1)$’ if $b - a$ is a scalar multiple of the $2 \times 2$-identity matrix. In other words, the term $\mathbf{F}_0$ of the decomposition of $\Theta$ is irrelevant, hence shall be ignored.

Again we shall focus on “+”, and shall omit +sub-(super-)scripts in notation wit some exceptions.

Let us write $\Omega_+ = \frac{1}{2(1+v_0)} \mathbf{Y}_+$, then we get

$$\mathbf{Y}_+ = (1+v_0)(-\sigma\gamma), \ \Omega_+ = \frac{1}{2}(-\sigma\gamma)\gamma, \ \text{with} \ \gamma = \frac{v}{1+v_0} = \frac{\zeta}{1+\langle\zeta\rangle},$$

where we recall that $v_0 = 1/\langle\zeta\rangle$, $v = (v_1, v_2, v_3)$, $v_j = \zeta_j/\langle\zeta\rangle$, $\zeta_j = \xi_j - \mathbf{A}_j(x)$, $j = 1, 2, 3$.

First we look at (the $2 \times 2$-matrix)

$$\Theta^\gamma = \frac{1}{2(1+v_0)} \mathbf{Y}_+^\gamma \mathbf{Y}_+^\gamma = \Omega_+^\gamma \mathbf{Y}_+^\gamma = -\Omega_+^\gamma \mathbf{Y}_+^\gamma,$$

recalling that we have $\Omega_+^\gamma \mathbf{Y}_+ = 1$, hence $\Omega_+^\gamma \mathbf{Y}_+^\gamma = -\Omega_+^\gamma \mathbf{Y}_+^\gamma$. From (6.3) we get $\Omega_+^\gamma = \frac{1}{2}(-\sigma\gamma)$, and,

$$\Theta^\gamma = -\frac{1}{2}(1+v_0)(\sigma\gamma)(\sigma\gamma) = -\frac{1}{2}(1+v_0)i\gamma(\gamma \times \gamma) \ (mod \ 1).$$

using (5.17) again.

\[\text{Note, we have} \ \zeta = \xi - \mathbf{A} = \frac{\dot{x}}{\sqrt{1-x^2}}, \ \langle\zeta\rangle = (\xi - \mathbf{A}) = \frac{\dot{x}}{\sqrt{1-x^2}} \ \text{by (4.19), if we relate (x, \xi) to (x, \dot{x}) using the classical equations of motion of (4.14+).}\]
Note, $\gamma = \zeta/(1 + \zeta)$ is a scalar multiple of $\zeta = \xi - \mathbf{A}$, hence $\gamma \times \gamma' = \frac{1}{2(1+\zeta)}\zeta \times \zeta'$, since $\zeta \times \zeta = 0$. Thus we get - all (mod 1) -

$$\Theta^\gamma = \frac{i}{2} \langle \zeta - \mathbf{A} \rangle (\zeta - \mathbf{A})' \sigma \cdot (\zeta - \mathbf{A}) \times (\zeta - \mathbf{A})' \ .$$

Next we calculate

$$\langle \zeta - \mathbf{A} \rangle' = \{ \partial_t - \sum_j \lambda_{\xi, j} \partial_x j + \sum j \lambda_{x, j} \partial_{\xi, j} \} (\zeta - \mathbf{A}) \ ,$$

where $\lambda_{\xi, j} = (\xi_j - \mathbf{A}_j)/(\zeta - \mathbf{A})$ and $\lambda_{x, j} = \mathbf{V}_{,x_j} - \sum_j \mathbf{A}_{,x_j} (\xi_j - \mathbf{A}_j)/(\zeta - \mathbf{A}) \ .$ The result is this:

$$\langle \xi_k - \mathbf{A}_k \rangle' = -\mathbf{V}_{,x_k} + \sum_j \frac{\xi_j + \mathbf{A}_j}{\zeta - \mathbf{A}} (\mathbf{A}_{,x_j} - \mathbf{A}_{,x_k}) \ , \ k = 1, 2, 3.$$

The last term equals $-\frac{1}{\zeta - \mathbf{A}} (\zeta \times (\zeta - \mathbf{A}))_k$. Thus we have

$$\zeta' = (\zeta - \mathbf{A})' \mathbf{V}_{,x} - \frac{1}{\zeta - \mathbf{A}} \zeta \times \mathbf{V} = -\mathbf{E} - \frac{1}{\zeta} \mathbf{B} \times \zeta \ ,$$

and we get

$$\zeta \times \zeta' = -\zeta \times \mathbf{E} - \frac{1}{\zeta} \zeta \times (\mathbf{B} \times \zeta) = -\zeta \times \mathbf{E} - \frac{1}{\zeta} (|\zeta|^2 \mathbf{B} - (\zeta, \mathbf{B}) \zeta) \ .$$

All together we get

$$\Theta^\gamma = \frac{i}{2} \langle \zeta - \mathbf{A} \rangle (\zeta - \mathbf{A})' \sigma \cdot (\zeta \times \mathbf{E} + \frac{1}{\zeta} (|\zeta|^2 \mathbf{B} - (\zeta, \mathbf{B}) \zeta)) \ .$$

Next we set out to calculate the other part $-2\langle \zeta \rangle \Theta^1$ of the matrix $\Theta$ of (5.13). Here it might be some help to go back and write

$$2\langle \zeta \rangle pp_{|x}p = pp_{|x}h_{,x} \ ,$$

noting that $2\langle \zeta \rangle pp_{|x}p = \lambda_{p} + pp_{|x}p = \lambda_{p} - pp_{|x}p = 0$ while $pp_{|x}p = 0$.

We get

$$2\langle \zeta \rangle \Theta^1 = \frac{1}{2} (1 + v_0) (1, i\sigma_\gamma) pp_{|x}h_{0}(\zeta)_{,x}(-\frac{1}{i\sigma_\gamma}) \ (mod \ 1)$$

with $h_{0}(\zeta) = \alpha_{\zeta} + \beta_{\zeta}$, since the term $\mathbf{V}_{,x} (1, i\sigma_\gamma)(-\frac{1}{i\sigma_\gamma}) = \mathbf{V}_{,x} (1 + |\gamma|^2)$ is scalar, using (5.17).

Now we get $p = \frac{1}{2} (1 + \frac{h_{0}(\zeta)}{v_0}) = \frac{1}{2} (1 + v_0 h_{0}(\zeta))$, hence $p_{|x} h_{0}(\zeta) = \frac{1}{2} (v_0 v_{,x} v_{|x} h_{0}(\zeta) + \frac{1}{2} v_0 \alpha_{x})$ where the first term at right will generate a scalar multiple of 1, hence may be ignored. Also, $h_{0}(\zeta)_{,x} = (\sum \alpha_{x}(\xi_j - \mathbf{A}_j) + \beta_{x})(\xi_j - \mathbf{A}_j) = \alpha_{x} \mathbf{A}_{,x} \ .$ Substituting into (6.13) we get

$$2\langle \zeta \rangle \Theta^1 = -\frac{v_0}{4} (1 + v_0)(1, i\sigma_\gamma)(\sum j \mathbf{A}_{,x_j} \alpha_{j x})(-\frac{1}{i\sigma_\gamma}) \ .$$

But we have

$$\sum j \mathbf{A}_{,x_j} \alpha_{j x} = \text{div} \ A - i\rho \cdot \text{curl} \ A \ ,$$

with $\rho = (\sigma^0_0)$, where again the first term may be ignored, when we substitute this into (6.14). We get

$$2\langle \zeta \rangle \Theta^1 = -\frac{v_0}{4} (1 + v_0)(1, i\sigma_\gamma) \rho \cdot \mathbf{B}(-\frac{1}{i\sigma_\gamma}) \ .$$
A matrix calculation then gives

\[
(1, i\sigma\gamma)^T \begin{pmatrix} \sigma B & 0 \\ 0 & \sigma B \end{pmatrix} \begin{pmatrix} 1 \\ -i \sigma\gamma \end{pmatrix} = (\sigma B) + (\sigma\gamma) = \sigma((1 - |\gamma|^2)B + 2\gamma B\gamma). 
\]

We have \(1 - |\gamma|^2 = \frac{2}{1 + \langle \zeta \rangle}\), so (5.17) equals

\[
\frac{2}{1 + \langle \zeta \rangle} \left( B + \frac{1}{1 + \langle \zeta \rangle} (B\zeta)\zeta \right). 
\]

All together we then get

\[
2\langle \zeta \rangle \Theta^1 = -\frac{i}{2} \frac{1}{\langle \zeta \rangle(1 + \langle \zeta \rangle)} \sigma(\zeta \times \varepsilon + \frac{1}{\langle \zeta \rangle}(|\zeta|^2 B - (\zeta, B)\zeta)). 
\]

Collecting things, up to here: We have

\[
\Theta = \Theta^1 - 2\langle \zeta \rangle \Theta^1
\]

with \(\Theta^1\) of (5.19) and

\[
\Theta^1 = \frac{i}{2} \frac{1}{\langle \zeta \rangle(1 + \langle \zeta \rangle)} \sigma(\zeta \times \varepsilon + \frac{1}{\langle \zeta \rangle}(|\zeta|^2 B - (\zeta, B)\zeta)). 
\]

We then may write

\[
\Theta = -\frac{i}{2} \sigma \mathcal{F}
\]

to get (6.1), proving prop.6.1, q.e.d.

**Theorem 6.2.** We consider (time-independent) local potentials \(V(x), A(x)\) satisfying (3.6) with \(m = (0, -1)\) as described early in sec.4. Assume we have a symbol \(q(x, \xi) \in \psi_{c_m}\) such that \(q(x, \xi)\) commutes with \(h(x, \xi)\) for all \(x, \xi\). Let \(\kappa^+ (x, \xi)\) and \(\kappa^- (x, \xi)\) be the matrices representing \(q(x, \xi)\) in its two eigenspaces \(S_{\pm}(x, \xi)\), with respect to the orthonormal bases given by the columns of the \(4 \times 4\)-matrix \(\Upsilon(x, \xi)\) of (2.5) with Dirac matrices \(\alpha, \beta\) of (2.8), and let \(\kappa^+_0\) be the matrices representing \(q(x, \xi)\) in its two eigenspaces \(S_{\pm}(x, \xi)\), with respect to orthonormal bases linked to the diagonalization (2.6) of \(h(x, \xi)\), as follows:

(i) We have

\[
\text{trace } q^+_t(x, \xi) = \text{trace } q^+(\nu^+_t(x, \xi)) \text{, } \text{trace } q^-_t(x, \xi) = \text{trace } q^-(\nu^-_t(x, \xi)),
\]

where \(\nu^+_t : \mathbb{R}^6 \to \mathbb{R}^6\) is the flow, letting each point \((x, \xi)\) wander along the solution \((x(t), \xi(t))\) of (4.14±) for a time-length \(t\) counted positive or negative. Here we should remind of the fact that the system (4.14±) may be rewritten as a set of second order equations in \(x\) only of the form

\[
(\frac{\dot{x}}{\sqrt{1 - \dot{x}^2}})^* = \varepsilon + \dot{x} \times B,
\]

for \(\lambda = \lambda_+\), and,

\[
(\frac{\dot{x}}{\sqrt{1 - \dot{x}^2}})^* = -\varepsilon - \dot{x} \times B,
\]

for \(\lambda = \lambda_-\), with electrical field strength \(\varepsilon\) and magnetic induction \(B\) induced by \(V\) and \(A\).
(ii) The two real 3-vectors $\vec{\kappa}^\pm_{\tau \rightarrow \tau}(\nu_T(x, \xi))$ will satisfy the equations
\begin{align}
(6.24+) & \quad \frac{1}{\sqrt{1 - 2^2}} \frac{d}{d\tau} \vec{\kappa}^+_{\tau \rightarrow \tau}(\nu_T)\big|_{\tau=0} = \vec{\kappa}^+_t \times \mathcal{B}^\sim, \quad \text{where} \quad \mathcal{B}^\sim = \mathcal{B} + \frac{1}{1 + \sqrt{1 - 2^2}} \hat{x} \times \mathcal{E}, \\
(6.24-) & \quad \frac{1}{\sqrt{1 - 2^2}} \frac{d}{d\tau} \vec{\kappa}^-_{\tau \rightarrow \tau}(\nu_T)\big|_{\tau=0} = -\vec{\kappa}^-_t \times \mathcal{B}^\sim, \quad \text{where} \quad \mathcal{B}^\sim = \mathcal{B} + \frac{1}{1 + \sqrt{1 - 2^2}} \hat{x} \times \mathcal{E},
\end{align}
with initial-values $\vec{\kappa}^+_0 = \vec{\kappa}^+_0$, $\vec{\kappa}^-_0 = \vec{\kappa}^-_0$.

(iii) Formulas (6.22) and (6.24) are valid only asymptotically, modulo $\psi_{c_m-e}$, assuming that the initial symbol $q(x, \xi)$ belongs to $\psi_{c_m}$. That is, they may be trusted if either $|x|$ is large or if $\hat{x} \approx 1 = \text{velocity of light} − \text{or both}.$

However, an infinite sequence of improvements can be constructed, by solving (iteratively) a system of differential equations similar to (5.16), leading to exact symbols $a = q + z$, $a_t = q_t + z_t$ with (6.22), (6.24) being true asymptotically, modulo $\psi_{c-\infty}$.

7. The $\vec{\kappa}$-vectors of Total Angular Momentum

Most of the dynamical observables, generally considered, are scalar in $\mathbb{C}^4$, so also scalar in the two eigenspaces $S_\pm$, implying that the two vectors $\vec{\kappa}_i^\pm$ will vanish identically, for all $x, \xi$. An exception is the total angular momentum defined as $J = S + L$, where $L = x \times D$ is the orbital angular momentum while $S = \frac{1}{2}(\sigma_0^\alpha)$ usually is interpreted as the (mechanical) spin of the particle. It is known that the self-adjoint operator $J$ commutes with $H$, assuming that $\mathbf{A} = 0$, $\mathbf{V} = \mathbf{V}(|x|)$, so that $e^{iHt}J e^{-iHt} = J$. So, $J$ is precisely predictable, if $\mathbf{V}(|x|)$ satisfies our assumptions. On the other hand, the spin $S$, as defined above, certainly is not precisely predictable. Neither is $L$, although thm.4.5 allows construction of a lower order correction $L_{\text{corr}}$, such that $L + L_{\text{corr}}$ is precisely predictable. Note, we have $L \in \text{Op}\psi_{c(1,1)}$, hence $L_{\text{corr}} \in \text{Op}\psi_{c(0,0)}$. We may write $J = (J + L_{\text{corr}}) + (S - L_{\text{corr}})$ and then reinterpret the (precisely predictable) observable $S_{\text{corr}} = S - L_{\text{corr}} \in \text{Op}\psi_{c(0,0)}$ as the spin. Checking this symbol-wise one finds that (modulo lower order) we get
\begin{equation}
\text{symb}(S_{\text{corr}}) = p_+(x, \xi) \ S \ p_-(x, \xi) + p_-(x, \xi) \ S \ p_-(x, \xi),
\end{equation}
where the right hand side makes sense also for general potentials, and then commutes with $h(x, \xi)$ also for general potentials, not necessarily $(0, \mathbf{V}(|x|))$. We then proposed to generally redefine the spin observable, using the right hand side of (7.1).

Here we are interested only in the two vectors $\vec{\kappa}^\pm$ for the (corrected) spin and the total angular momentum. Note, the orbital angular momentum $L$ is scalar in $S_\pm$, hence will not contribute to the $\vec{\kappa}^\pm$. So, both $J$ and $S_{\text{corr}}$ have the same $\vec{\kappa}^\pm$-vectors. In fact, it suffices to just calculate the $2 \times 2$-matrices $\kappa^\pm$ of the (uncorrected) spin-observable $S = \frac{1}{2}(\sigma_0^\alpha)$, and then calculate its corresponding vectors $\vec{\kappa}^\pm$.

**Proposition 7.1.** Looking at the $2 \times 2$-matrices $\kappa^\pm(x, \xi)$ of the matrices $p_+(x, \xi)S_j p_+(x, \xi)$ and $p_-(x, \xi)S_j p_-(x, \xi)$ for a spin component $S_j$ with respect to the orthonormal bases of $S_\pm$ used in sec.5 and sec.6, we get
\begin{equation}
\kappa^\pm = \kappa^\mp = \frac{1}{2} \sqrt{1 - \hat{x}^2} \{ \sigma_j + \frac{1}{\sqrt{1 - \hat{x}^2(1 + \sqrt{1 - \hat{x}^2})}} \hat{x}_j(\hat{x}\sigma) \},
\end{equation}
where we have replaced the $\xi$-variable by $\hat{x}$ with the relation $\hat{x} = \lambda_\xi \Leftrightarrow \xi = \mathbf{A}(x) + \frac{\hat{x}}{\sqrt{1 - \hat{x}^2}}$, as this was done in the two earlier sections.
Using (7.2) we then at once obtain the components of the vectors $\vec{r}^{\pm}$ by using (5.15):

\[
(7.3) \quad \vec{r}^{\pm} = \frac{1}{2} \sqrt{1 - \dot{x}^2} \{ \delta_j l + \frac{1}{\sqrt{1 - \dot{x}^2}(1 + \sqrt{1 - \dot{x}^2})} \dot{x}_j \dot{x}_l \}
\]

To express this alternately:

\[
(7.4) \quad \vec{r}^{\pm} = \frac{1}{2} \sqrt{1 - \dot{x}^2} \{ \delta_j l - \frac{1}{\sqrt{1 - \dot{x}^2}(1 + \sqrt{1 - \dot{x}^2})} \dot{x}_j \dot{x}_l \} + \frac{1}{2} \frac{\dot{x}_j \dot{x}_l}{\dot{x}^2}, \quad \text{as } \dot{x} \neq 0.
\]

**Observation 7.2.** At speed $\dot{x} = 0$ the three vectors $\vec{r}^j$ are just the three unit vectors $\vec{r}^{\pm} = \frac{1}{2} \dot{x}^j$ (with $\dot{x}^j = 1$ in $j$-th row, and zero elsewhere) — except for a factor $\frac{1}{2}$. At arbitrarily speeds $\dot{x}$ there will be a relativistic shortening in the perpendicular directions, and no shortening in the parallel direction — with respect to $\dot{x}$.

**Proof of prop.7.1.**

We discuss the "+" case only, with "+" going similarly. Using (6.3) we get $\kappa^+ = \Upsilon^+_\gamma S\Omega_+\gamma$ with $\gamma$ as stated there. That is, we get

\[
(7.5) \quad \kappa^+ = \kappa^- = \frac{1}{4\xi} (1 + \xi) \{ \sigma_j + (\gamma \sigma) \sigma_j (\gamma \sigma) \}, \quad \gamma = \frac{\xi}{1 + \xi}, \quad \xi = \xi - A(x).
\]

A calculation gives

\[
(7.6) \quad (\gamma \sigma) \sigma_j (\gamma \sigma) = 2 \gamma_j (\gamma \sigma) - \gamma^2 \sigma_j,
\]

\[
(7.7) \quad \sigma_j + (\gamma \sigma) \sigma_j (\gamma \sigma) = (1 - \gamma^2) \sigma_j + 2 \gamma_j (\gamma \sigma) = \frac{2}{1 + \xi} \{ \sigma_j + \frac{1}{1 + \xi} \xi_j (\xi \sigma) \}, \quad \text{so},
\]

\[
(7.8) \quad \kappa^+ = \kappa^- = \frac{1}{4\xi} \{ \sigma_j + \frac{1}{1 + \xi} \xi_j (\xi \sigma) \},
\]

Transforming onto the variable $\dot{x}$ again we get the desired equation (7.2). Q.E.D.

8. **An Electron under Electro-Magnetic Radiation**

We next consider a time-dependent Dirac operator of the form

\[
(8.1) \quad H = \alpha_1 D_1 + \alpha_2 (D_2 - \varepsilon_0 \sin \omega (x_1 - t)) + \alpha_3 D_3 + \beta,
\]

where we use the Dirac matrices $\alpha, \beta$ of (2.9). Symbolwise we may write $H = h(t, x, D)$ with $h(t, x, \xi) = h_0(\xi) - \varepsilon_0 \alpha_2 \sin \omega (x_1 - t)$, $h_0(\xi) = \alpha \xi + \beta$.

Clearly we then have the potentials $V = A_1 = A_2 = 0$, $A_2 = \varepsilon_0 \sin \omega (x_1 - t)$. The corresponding electro-magnetic field then is defined as

\[
(8.2) \quad E = -\dot{A} - \text{grad } V = \varepsilon_0 \omega \cos \omega (x_1 - t)(0, 1, 0)^T, \quad B = \text{curl } A = \varepsilon_0 \omega \cos \omega (x_1 - t)(0, 0, 1)^T,
\]

corresponding to a plane polarized wave of (circular) frequency $\omega$ propagating in the positive $x_1$-direction, with $E$ and $B$ oscillating in the $(x_1, x_2)$- and $(x_1, x_3)$-plane, respectively.

This Dirac operator $H$ does not belong to Opψc. But it will belong to the class $\psi p_1$ of def.3.3 (iii). Since $H = H(t)$ now depends on $t$, the propagator $U(t)$ no longer is an exponential function. However, due to the special form of time-dependency, we find that $U(t)$ is a product of two exponentials:
Proposition 8.1. The propagator $U(t)$ such that $U(t)\psi_0 = \psi(t, x)$ solves $\dot{\psi} + iH(t)\psi = 0$ (with $H(t)$ of (8.1)), and $\psi(0, x) = \psi_0(x)$, has the form

$$U(t) = T_{-t}e^{-iKt}, \quad \text{with } K = H(0) - D_1,$$

and the translation $T_t\psi(x) = \psi(x_1 + t, x_2, x_3)$.

Moreover, the propagator $U(\tau, t)$ solving the problem with initial-values at $t = \tau$ may be written as

$$U(\tau, t) = T_{-t}e^{-iK(t-\tau)}T_\tau.$$

Proof. We get $(T_{-t}H(0)T_t\psi(x)) = (H_0\psi)(x) - \alpha_2(T_{-t}A_2(x)T_t\psi)(x) = H(t)$, since $H_0$ is translation invariant. Thus we may write $\dot{\psi} + iH(t)\psi = 0$ as

$$T_t\dot{\psi} + iH(0)T_t\psi = 0.$$

Here we set $\chi(t, x) = T_t\psi(t, x) = \psi(t, x + te^1)$, and use that

$$\chi(t, x) = \partial_t(\psi(t, x_1 + t, x_2, x_3)) = T_t\dot{\psi}(t, x) + \partial_x\chi(t, x).$$

Equation (8.4) then may be written as

$$\dot{\chi} + i(H(0) - D_1)\chi = 0.$$

In other words, the substitution $\chi(t, x) = T_t\psi = \psi(t, x + te^1)$ converts the Dirac equation into equ. (8.6), where now the operator $H(t)$ of (8.1) is replaced by the (time-independent) operator

$$K = H(0) - D_1 = H_0 - \alpha_2A_2(x_1) - D_1.$$

It is evident then that (8.6) will be solved by

$$\chi(t, x) = e^{-itK}\chi(0, x).$$

Or else, we may write this as

$$\psi = T_{-t}e^{-iKt}\psi_0,$$

proving (8.2), while (8.3) then follows trivially. Q.E.D.

Note, for this Dirac operator, the total energy $H(t)$ is not constant — it fluctuates periodically, with period $2\pi/\omega$. For $t = 0$ the spectral decomposition of $K$, not of $H(0)$ will provide the split between electron and positron. The spectral theory of $K$ can be worked out explicitly. We shall find that $K$ has continuous spectrum along all of $\mathbb{R}$. But there is a strong singularity at $t = 0$. We shall set

$$H = H_e \oplus H_p,$$

with the spectral spaces $H_e$, $H_p$ of $K$ belonging to the intervals $(0, \infty)$ and $(-\infty, 0)$ respectively. Then $H_e$ and $H_p$ are defined as the spaces of electron states and positron states, resp., at $t = 0$.

It may be seen that these spaces converge towards the well known electron and positron spaces for $H_0 = \alpha D + \beta$ as the amplitude $\varepsilon_0$ tends to 0, so that $H(t) \to H_0$.

As time $t$ progresses, the spaces $H_e$, $H_p$ will change; at time $t$ we will set

$$H_e(t) = T_{-t}H_e, \quad H_p(t) = T_{-t}H_p.$$
Indeed, a state \( \psi_0 \in \mathcal{H}_e \) will propagate to \( \psi(t, x) = U(t)\psi_0 = T_{-t}e^{-iKt}\psi_0 \), where \( e^{-iKt}\psi_0 \in \mathcal{H}_e \), since \( e^{-iKt} \) leaves all spectral spaces of \( K \) invariant. So, it follows that \( \psi(t, .) \in \mathcal{H}_e(t) \) — indeed, an electron state remains an electron state. Similar with positron states.

Regarding prediction of expectation values, things remain as discussed earlier: For a state \( \psi_0 \in \mathcal{H} \) and an observable \( A \) we get the expectation value \( \langle \psi_0, A\psi_0 \rangle \). For a future time then, if \( \psi_t = U(t)\psi_0 \) or also \( A_t = U^*(t)AU(t) \) the predicted expectation value then will be \( \langle \psi_t, A\psi_t \rangle = \langle \psi_0, A_t\psi_0 \rangle \), marking Schrödinger or Heisenberg representation.

**Lemma 8.2.** We have

\[(8.12) \quad U^{-1}(t)H(t)U(t) = H(0) + e^{iKt}D_1e^{-iKt} - D_1 = K + U^{-1}(t)D_1U(t).\]

That is, the changes of expectation values of total energy and of momentum component \( D_1 \) at time \( t \) are related: Defining \( A_t = U^*(t)AU(t) \) for an arbitrary observable \( A \), we get

\[(8.13) \quad (H(t))_t - H(0) = (D_1)_t - D_1.\]

**Proof.** We get

\[(8.14) \quad e^{iKt}T_{-t}e^{-iKt} = e^{iKt}H(0)e^{-iKt} = e^{iKt}Ke^{-iKt} + e^{iKt}D_1e^{-iKt} = K + (D_1)_t.\]

Q.E.D.

We shall need details of the spectral theory of the operator \( K \) but will discuss this in a later section. Right now let us focus on an attempt to repeat the procedures of earlier sections, regarding potentials vanishing at \( |x| = \infty \), for the present Dirac operator \( H(t) \) of (2.1). As already observed, we no longer have \( H(t) \in \text{Op}\psi_c \), but rather have \( H(t) \in \text{Op}\psi_{p_1} \subset \text{Op}\psi_q \), with the larger symbol classes of sec.3.

With some exceptions we then shall focus entirely on time-propagation of symbols of the form \( q(\xi) \) — independent of \( x \), with \( q \in \psi_{c(m,0)} \), and with \( q(\xi) \) commuting with \( h_0(\xi) = a_\xi + \beta \), for all \( \xi \). Of special interest will be the case of \( q(\xi) = \xi_\ell \) (and also \( q(\xi) = \xi_j \), \( j = 2,3 \)), — that is, of the momentum observables.

For such a symbol \( q(x) \) the operator \( q(D) \) is translation invariant: Especially we get \( T_{-t}q(D)T_t = q(D) \), implying that

\[(8.15) \quad (q(D))_t = U^*(t)q(D)U(t) = e^{iKt}q(D)e^{-iKt}.\]

Therefore our attempt to repeat earlier arguments for the case of a \( q \in \psi_c \) will focus on the assignment \( a(x, \xi) \to a_\ell(x, D) = e^{iKt}a(x, D)e^{-iKt} \) equivalent to the ODE-initial-value problem

\[(8.16) \quad a_\ell(x, D) = i[K, a_\ell(x, D)] \quad \text{as} \quad -\infty < t < \infty \quad \text{a}_0(\xi, x) \quad \text{given}.\]

The theorem, below, will address the initial-value problem (8.16) modulo \( \psi_{c(m,0)} \). We shall require another lengthy argument involving calculus of Fourier integral operators (to be discussed in sec’s 11 f.) to also cover the corresponding Heisenberg transform \( U^*(t)AU(t) \). However, the results of sec.10, below, addressing only the case of a simple photon-collision, will not be affected by these more complicated things.

**Theorem 8.3.** Given any self-adjoint (4 \times 4\text{-}matrix-valued) symbol \( q(\xi) \in \psi_{c(m,0)} \), independent of the location variable \( x \), depending on the momentum variable \( \xi \) only, and such that the commutator \( [h_0(\xi), q(\xi)] = h_0(\xi)q(\xi) - q(\xi)h_0(\xi) \) vanishes, for all \( \xi \).
I) There exists a (lower order) ‘correction symbol’ \( z(t_1, \xi) \in \psi_{m-1} \) with 
\[
[h_0(\xi), z(t_1, \xi)]_+ = h_0(\xi)z(t_1, \xi) + z(t_1, \xi)h_0(\xi) = 0 \quad \text{for all } t_1, \xi, \quad \text{such that the initial-value problem } (8.16) 
\]
with \( a_0(x, \xi) = q(\xi) + z(t_1, \xi) \) admits a solution \( a_0(x, \xi) \) modulo \( \psi_{-\infty} \) of the form
\[
(8.17) 
a_t(x, D) = q_t(x, D) + z_t(x, D) \quad (\text{mod } Op\psi_{-\infty}) ,
\]
where \( q_t(x_1, \xi) \in \psi_m , \quad [h_0(\xi), q_t(x_1, \xi)] = 0 , \forall t_1, \xi , \quad z_t(x_1, \xi) \in \psi_{m-1} , \quad [h_0(\xi), z_t(x_1, \xi)]_+ = 0 , \quad \forall t_1, \xi , \quad \dot{q}_t(x_1, \xi), \dot{z}_t(x_1, \xi) \in \psi_{m-1}, \quad \text{and, } q_0(x_1, \xi) = q(\xi) , \quad z_0(x_1, \xi) = z(x_1, \xi). 

II) The symbols \( q_t(x_1, \xi), z_t(x_1, \xi) \) have \( x_1 \)-Fourier-series-expansions
\[
(8.18) 
q_t(x_1, \xi) = \sum q_{t,n}(\xi)e^{in\omega x_1} , \quad z_t(x_1, \xi) = \sum z_{t,n}(\xi)e^{in\omega x_1} , \quad q_{t,n}, z_{t,n} \in \psi_m ,
\]
where the sums over \( n \) are finite if looked at modulo \( \psi_{m-j} \), for every \( j = 1, 2, \ldots \). That is, for every \( j = 1, 2, \ldots \) only a finite number of the coefficients \( q_{t,n}, z_{t,n} \) are not in \( \psi_{m-j} \).

Accordingly, the corresponding \( \psi \)-do-s are of the form
\[
(8.19) 
q_t(x_1, D) = \sum e^{in\omega x_1}q_{t,n}(D) , \quad z_t(x_1, D) = \sum e^{in\omega x_1}z_{t,n}(D) .
\]

III) In momentum space — looking at the Fourier transformed operators \( q_t(x_1, D)^\wedge = Fq_t(x_1, D)F^\ast \), \( z_t(x_1, D)^\wedge = Fz_t(x_1, D)F^\ast \) — \( f \)'s (8.19) assume the form
\[
(8.20) 
q_t(x_1, D)^\wedge = \sum T_{-n\omega}q_{t,n}(\xi) , \quad z_t(x_1, D)^\wedge = \sum T_{-n\omega}z_{t,n}(\xi) ,
\]
with the translation operator \( T_{\kappa}u(x) = u(x_1 + \kappa, x_2, x_3) \).

IV) In general the “corrected operator” \( A(t) = q(D) + z(x_1 - t, D) \) of (8.17) may not be self-adjoint, so, it may not count as an observable. However, we may take the self-adjoint operator
\[
(8.21) 
\hat{A}(t) = \frac{1}{2}\{A(t) + A^\ast(t)\} ,
\]
noting that
\[
(8.22) 
q_t(x_1, D)^\ast = \sum e^{in\omega x_1}q_{t,-n}^\ast(D + n\omega e^1) , \quad z_t(x_1, D)^\ast = \sum e^{in\omega x_1}z_{t,-n}^\ast(D + n\omega e^1) ,
\]
so that
\[
(8.23) 
\hat{A}_t = U^\ast(t)\hat{A}(t)U(t) = \hat{q}_t(x_1, D) + \hat{z}_t(x_1, D)
\]
with
\[
(8.24) 
\hat{q}_t(x_1, \xi) = \sum \hat{q}_{t,n}(\xi)e^{in\omega x_1} , \quad \hat{z}_t(x_1, \xi) = \sum \hat{z}_{t,n}(\xi)e^{in\omega x_1} , \quad \text{where}
\]
\[
\hat{q}_{t,n}(\xi) = \frac{1}{2}\{q_{t,n}(\xi) + q_{t,-n}^\ast(\xi + n\omega e^1)\} , \quad \hat{z}_{t,n}(\xi) = \frac{1}{2}\{z_{t,n}(\xi) + z_{t,-n}^\ast(\xi + n\omega e^1)\} .
\]
In particular note that
\[
(8.25) 
A(t) = q(D) + \hat{z}_0(x_1 - t, D) ,
\]
with \( \hat{z}_0 \) of (8.24) for \( t = 0 \), now is self-adjoint, hence counts as an observable.

On the other hand, it is important to emphasize that we no longer have \( [h_0(\xi), \hat{q}_t(x_1, \xi)] = [h_0(\xi), \hat{z}_t(x_1, \xi)]_+ = 0 \), although both still are symbols of one order lower than required.

V) Going into momentum space again, we find that
\[
(8.26) 
\hat{q}(x_1, D)^\wedge = \frac{1}{2}\sum T_{-n\omega}\{q_{t,n}(D) + q_{t,-n}^\ast(D + n\omega e^1)\} , \quad \hat{z}_t(x_1, D)^\wedge = \frac{1}{2}\sum T_{-n\omega}\{z_{t,n}(D) + z_{t,-n}^\ast(D + n\omega e^1)\} ,
\]
In contrast to our procedure of previous sections — where we were simplifying previously published things, we shall attempt to discuss a full proof of thm.8.3 in sections below.

9. The Photon Hypothesis

Note, in thm.8.3 we were including the Fourier transformed operators, defined as \( A^\wedge = FAF^{-1} \) for an important reason: This will transform us to the momentum representation, where the momentum observables \( D_j \) appear as multiplication operators \( \psi^\wedge \rightarrow \xi_j \psi^\wedge (\xi) \). Formally, a \( \psi do a(x, D) \) will have \( (a(x, D))^\wedge = a(-M_e, D) \), with notation as in (3.9). Especially, we get

\[
\left( e^{-in\omega_1} a(D) \right)^\wedge = T_{n\omega} a(\xi) .
\]

This latter formula we find interesting: Looking at (8.20) it appears that, for a \( q(D) \) as in thm.8.3 the Heisenberg transformed \( (a_t(x, D))^\wedge \) splits up into a (discrete) sum of terms consisting of products \( T_{n\omega} f(\xi) \). So, these terms have their momentum variable translated by an integer multiple of \( n\omega \) in the \( x_1 \)-direction — the direction of our radiation. Recalling our constants \( \hbar = c = m_e = |e| = 1 \), we get dimensions right when we claim this \( n\omega \) as an integer multiple of \( \hbar \omega/c = \hbar \nu/c \). With that, there arises the suspicion that this points to a collision of the electron (positron) with a discrete number of particles, all having momentum \( \hbar \nu/c \) — so, with Photons?

We will work on such assumption, when we now sketch a proof of thm.8.3, focusing on the special case of \( q(\xi) = \xi_j \), \( j = 1, 2, 3 \). At the same time this will prepare us for the proof of the general case.

Recalling the operator \( K = H(0) - D_1 \) of (8.7), we consider the expression \( A_t = e^{iKt} A e^{-iKt} \) and assume that \( A_t = a_t(x, D) \) is a \( \psi do \), for all \( t \), and then write

\[
\dot{a}_t(x, D) = i[K, a_t(x, D)] ,
\]

then seeking to write this symbolwise, assuming that we work with symbols \( a(x, \xi) \in \psi p \), as defined in def.2.3(iii), independent of \( x_2, x_3 \).

**Proposition 9.1.** For a \( \psi do C = c(x_1, D) \in \psi p_m \) we have

\[
\text{symbol}([K, C]) = \left[ h_0(\xi), c(x_1, \xi) \right] - \epsilon_0 \sin \omega_1 [\alpha_2, c(x_1, \xi)] - i(\alpha_1 - 1)c_{1|2} - i\frac{\epsilon_0}{2} \alpha_2 X c(x_1, \xi) , \quad \text{where}
\]

\[
h_0(\xi) = \alpha_\xi + \beta \quad X c(x_1, \xi) = \{ [c(x_1, \xi + \omega^1) - c(x_1, \xi)]e^{i\omega_1} + [c(x_1, \xi) - c(x_1, \xi - \omega^1)]e^{-i\omega_1} \} .
\]

**Proof.** For \( H_0 = h_0(D) \) we get

\[
\text{symbol} ([H_0, C]) = [h_0(\xi), c(x_1, \xi)] - i \sum \alpha_j c_{|j} (x_1, \xi) , \quad \text{symbol} ([D_1, C]) = -i\epsilon_0 c_{|2} (x_1, \xi) .
\]

by using the Leibniz formula (3.3) (with the infinite series there breaking off). For the term \( \epsilon_0 \alpha_2 \sin \omega_1 \), we proceed directly. For \( [\sin \omega_1 x, c(x_1, D)] \) get

\[
c(x_1, D)(u(x) \sin \omega_1) = (2\pi)^{-3/2} \int d\xi e^{ix\xi} (u\sin(\omega_1)\wedge (\xi)c(x_1, \xi) ,
\]

where \( (\sin \omega_1)^\wedge (\xi) = (2\pi)^{-3/2} \int du u(x) e^{-ix(\xi + \omega^1)} = u^\wedge (\xi + \omega^1) \), hence

\[
(u\sin \omega_1)^\wedge = \frac{1}{2}(u^\wedge (\xi + \omega^1) - u^\wedge (\xi - \omega^1)) , \quad \text{so that}
\]

\[
c(x_1, D)(u(x) \sin \omega_1) = \frac{1}{2i}(2\pi)^{-3/2} \int d\xi e^{ix\xi} \{ e^{i\omega_1}; c(x_1, \xi + \omega^1) - e^{-i\omega_1}; c(x_1, \xi - \omega^1) \} u^\wedge (\xi) .
\]
Accordingly \([\sin \omega x_1, c(x_1, D)]\) has the symbol

\[ i \frac{1}{2} (e^{i \omega x_1}(c(x_1, \xi + \omega e^1) - c(x_1, \xi)) + e^{-i \omega x_1}(c(x_1, \xi) - c(x_1, \xi - \omega e^1))) \].

So, we get (9.3), q.e.d.

With prop.9.1 and (9.2) we then conclude that the symbol \(a_t\) of \(A_t = e^{i K t} a(x_1, D)e^{-i K t}\) must satisfy the equation

\[ \dot{a}_t(x_1, \xi) = i[h_0(\xi), a_t(x_1, \xi)] + (\alpha_1 - 1) j_{zt}(x_1, \xi) + (Za_t)(x_1, \xi), \]

with \((Ze)(x_1, \xi) = -i \varepsilon_0 \sin \omega x_1[\alpha_2, c(x_1, \xi)] + \frac{\varepsilon_0}{2} \alpha_2 (X e)(x_1, \xi),\)

assuming that \(A_t\) and \(\dot{A}_t\) belong to \(O \psi p\).

We note that (9.8) is a differential equation in the variables \(t, x_1\), but also is governed by the commutator \([h_0, a_t]\) representing a term of order \(m + 1\), assuming \(a_t \in \psi p_m\). Decomposing again

\[ a_t = a_t^+ + a_t^- + a_t^\pm + a_t^\top, \]

where \(a_t^+ = p_+ a_t p_+, a_t^- = p_- a_t p_-\), \(a_t^\pm = p_+ a_t p_-\), \(a_t^\top = p_- a_t p_+\),

we get

\[ ([h_0, a_t])^\pm = 2 \langle \xi \rangle a_t^\pm, ([h_0, a_t])^\top = -2 \langle \xi \rangle a_t^\top, ([h_0, a_t])^+ = ([h_0, a_t])^- = 0. \]

With \(q_t = a_t^+ + a_t^-\), \(z_t = a_t^\pm + a_t^\top\) we get \(a_t = q_t + z_t\) where \([h_0, q_t] = 0\), \([h_0, z_t]_+ = 0\).

Since all terms in (9.8) but the commutator-term are of order \(m\) or less we conclude that

\[ z_t = \frac{1}{2 \langle \xi \rangle} \left\{ ([h_0, a_t])^\pm - ([h_0, a_t])^\top \right\} \in \psi p_{m-1}. \]

So, we have proven this:

**Proposition 9.2.** If an operator \(A = a(x_1, D) \in \psi p_m\) has the above property that \(A_t = e^{i K t} A e^{-i K t} = a_t(x_1, D) \mod \psi q_{-\infty}\), where \(a_t\) and \(\dot{a}_t\) belong to \(\psi p_m\) \(\mod \psi q_{-\infty}\) then (9.9), (9.10), (9.11) lead to a decomposition \(a_t(x_1, \xi) = q_t(x_1, \xi) + z_t(x_1, \xi)\) where \(q_t \in \psi p_m\), \(z_t \in \psi p_{m-1}\) all \(\mod \psi q_{-\infty}\) while \([h_0, q_t] = 0\), \([h_0, z_t]_+ = 0\).

In particular this decomposition applies to the case \(t = 0\), so that also \(\mod \psi q_{-\infty}\) \(a(x_1, \xi) = q(x_1, \xi) + z(x_1, \xi)\) where \(q \in \psi p_m\), \(z \in \psi p_{m-1}\) while \([h_0, q] = 0\), \([h_0, z]_+ = 0\).

Vice versa, focusing on construction of \(\psi do-s\) of the form \(a(D)\) with \(e^{i K t} a(D)e^{-i K t} \in \psi p\), it is clear then that we might start with \([h_0, a] = 0\), and then have to add a "lower order correction" \(z(x_1, \xi) \in \psi p_{m-1}\) (and with \([h_0, z]_+ = 0\) to make above equ. (9.8) possible.

For this task we will use an iteration, starting with a given initial self-adjoint \(q(\xi)\) commuting with \(h_0(\xi)\), the construction seeking for a \(z_t\) of lower order and a commuting \(q_t\) with \(q_0 = q\) such that \(a_t = q_t + z_t\) will solve (9.8) with higher and higher accuracy, as \(|\xi| \to \infty\).

Remembering that (9.8) is an equation for a \(4 \times 4\) matrix-function \(a_t\) we distinguish three steps, to be iterated infinitely:
Step I We omit some lower order terms in (9.8), then trying to solve that as a sharp equation.

Step II: We multiply the (simplified) (9.8) left and right by \( p_+ \) (and left and right by \( p_- \)) obtaining two differential equations to be solved. That will get us an approximate \( q_t \).

Step III: We multiply (9.8) left and right by \( p_+ \) and \( p_- \), respectively (or by \( p_- \) and \( p_+ \), resp.). That will give us equations to obtain an approximate \( z_t \).

These steps, applied alternately, in iteration, will result in an infinite sequence of improvements satisfying eq. (9.8) modulo \( \psi_{p_{m-j}} \) only, for \( j = 1, 2, \ldots \). Then an asymptotic limit (mod \( \psi_{-\infty} \) (in the sense of prop. 3.7) must be taken to obtain an \( \alpha_t^\infty = \tau_t^\infty + z_t^\infty \) solving (5.6) modulo \( \psi_{-\infty} \).

With such \( \alpha_t^\infty(x_1, \xi) \in \psi_{p_m} \) we then define the operator \( A^\infty_0 = \alpha_t^\infty(x_1, D) \), and then define

\[
B_t = e^{-iKt} A^\infty_t e^{iKt} - A^\infty_0.
\]

Clearly we get \( B_0 = 0 \), while

\[
\dot{B}_t = e^{-iKt} C_t e^{iKt}, \quad C_t = \dot{A}^\infty_t - i[K, A^\infty_t].
\]

Here the expression \( C_t \) belongs to \( Op\psi_{-\infty} \), since its symbol satisfies (9.8) modulo \( \psi_{-\infty} \). It follows that

\[
e^{-iKt} A^\infty_t e^{iKt} - A^\infty_0 = B_t = \int_0^t d\tau e^{-iK\tau} C_\tau e^{iK\tau},
\]

hence

\[
e^{iKt} A^\infty_0 e^{-iKt} - A^\infty_t = \int_0^t e^{i(t-\tau)K} C_\tau e^{-i(t-\tau)K}.
\]

Here we are facing a slight difficulty:

**Observation 9.3.** Note, the above \( C_t \) is the error occurring in our procedure of solving the ODE-initial-value problem (8.16). That error belongs to \( Op\psi_{-\infty} \) — its differentiation order is \( -\infty \). Since it is a \( \psi_{do} \), its momentum representation [i.e., its Fourier transform] only provides a negligible contribution if applied to functions with support for very large \( \xi \).

On the other hand, the error \( \alpha_t^\infty - e^{iKt} A^\infty_0 e^{-iKt} = \Gamma_t \) is given by

\[
\Gamma_t = \int_0^t e^{i\tau K} C_{t-\tau} e^{-i\tau K}.
\]

We shall show in sec. 13, below, that this kind operator belongs to \( Op\psi_{-\infty} \) if we assume that \( P_+ C_\tau P_- = P_- C_\tau P_+ = 0 \) for all \( \tau \in [0, t] \), where \( P_+ \), \( P_- \) denote the orthogonal projections onto the spaces \( \mathcal{H}_e \) and \( \mathcal{H}_p \) of electron (positron) states, resp.

The projections \( P_+ \), \( P_- \), as spectral projections of \( K \), commute with \( K \) and with \( e^{iKt} \). Thus, if we introduce a ‘commuting part’ \( \kappa(c)(R) = P_+ R P_+ + P_- R P_- \), for general operators \( R \), then we get

\[
e^{iKt} \kappa(c)(A^\infty_0) e^{-iKt} = \kappa(c)(A^\infty_t) + \Gamma^\infty_t,
\]

where then \( \Gamma^\infty_t = \int_0^t e^{i\tau K} \kappa(c)(C_{t-\tau}) e^{-i\tau K} \in Op\psi_{-\infty} \) also is a \( \psi_{do} \), so that the right hand side of (9.17) indeed is a \( \psi_{do} \) in \( Op\psi_{-\infty} \).
We shall see later that $P_+, P_-$ are $\psi do$s in $Op\hat{\psi}q_0$, and that the passage $R \rightarrow \kappa_c(R)$ to the commuting part may be carried into the infinite series of thm 8.3 with little or no change. In particular, the discussion in thm. 10.4, involving only the first and second terms of these infinite series — i.e., only a single collision between a Dirac particle and a photon — will not be affected at all.

Actually, the projections $p_+(D), p_-(D)$ used in our iteration are close to $P_+, P_-$, resp., as shall be seen, so that the commuting terms at each step of the iteration are almost commuting with respect to $P_+, P_-$. It is easy then to return to our propagator $U(t) = T_{-1}e^{-iKt}$ of the Dirac operator (8.1): Just rewrite (9.17) as

$$U^*(t)(T_{-1}\kappa_c(A^\infty_0)T_1)U(t) = \kappa_c(A^\infty_t) + \Gamma^\infty_t. \tag{9.18}$$

Setting $\dot{A}_t = \kappa_c(A^\infty_0) = \dot{a}_t(x_1, D) \in Op\hat{\psi}q_m$ we shall get $T_{-1}\kappa_c(A^\infty_0)T_1 = \dot{a}_0(x_1 - t, D) \in Op\hat{\psi}q_m$.

**Proposition 9.4.** We have

$$U^*(t)\dot{a}_0(x_1 - t, D)U(t) = \dot{a}_t(x_1, D) + \Gamma^\infty_t \text{ with } \Gamma^\infty_t \in Op\hat{\psi}q_{-\infty}. \tag{9.19}$$

Here the problem remains to relate $\dot{a}_0(x_1, t)$ to the given symbol $q(\xi)$ of thm.8.3. We shall discuss that in more detail in sec 13, after we control the operators $P_+, P_-$.  

### 10. The Momentum Observables $D_1, D_2, D_3$

Focusing on the 3 momentum coordinates as observables, we start with the initial self-adjoint symbol $q(\xi) = \xi_j \in \psi\hat{c}(m, 0)$ with $m = 1$, for fixed $j = 1, 2, 3$. where $j = 1$ will give the momentum coordinate in the direction of our radiation. In particular we recall (8.13), i.e.,

$$(H(t))_t - H(0) = (D_1)_t - D_1, \tag{10.1}$$

indicating a relation between the development of the observables $H(t)$ and $D_1$, looking at their Heisenberg transforms.

We then want to apply thm.8.3 to the special cases of $q(\xi) = \xi_j , j = 1, 2, 3$, and also discuss the details of the iteration, completing the proof of thm.8.3.

So, in (9.8), we set $a_t = q_t + z_t$, where $q_t \in \psi p_1$, $z_t \in \psi p_0$ and $[h_0(\xi), q_t(x_1, \xi)] = 0$, $[h_0(\xi), z_2(x_1, \xi)] = 0$, for all $x_1, \xi$. In that substitution we tend to ignore all terms of order $m - 1 (= 0$ for $q = \xi_j$). In addition, $z_t$ also will be regarded as of order $m - 1$, and will be ignored, a fact to be confirmed later on, after solving for $q_t, z_t$ modulo $\psi p_0$ — assuming that initially, at $t = 0$, we have $q_0(x_1, \xi) = \xi_j , j = 1, 2, 3$.

**Proposition 10.1.** The operation $c(x, \xi) \rightarrow (Xc)(x, \xi)$ (with $X$ of (9.3)) lowers the differentiation order $m$ of $c \in \psi p_m$ by one unit — to $\psi p_{m-1}$.

Also, if a symbol $M(x, \xi)$ commutes with $h_0(\xi) = \alpha \xi + \beta$ then we get

$$p_+ [\alpha, M]p_+(x, \xi) = p_- [\alpha, M]p_-(x, \xi) = 0. \tag{9.2}$$

Indeed, looking at (9.3) we observe that $c(x, \xi + \omega e^1) - c(x, \xi) = \int_{-\infty}^{0} d\nu \xi c(x, \xi + \nu e^1)$ has differentiation order $m - 1$ if $c(x, \xi)$ has order $m$. Similar with the second term in (9.3), so that $(Xc)$ has order $m - 1$. For the second statement we observe that $p_+ [\alpha, M]p_+ = [p_+ \alpha, p_+ M]$, since $[h_0, M] = 0$ implies $[p_+, M] = 0$. But we know that $p_+ \alpha_2 p_+ = -p_- \alpha_2 p_- = s_2(\xi) = \xi_2/\{\xi\}$ is a scalar (cf. lemma 2.1). So $p_+ [\alpha, M]p_+ = [p_+ \alpha, p_+ M] = [s_2, M] = 0$. Similar for $p_- confirming the statement.
We get

\[ \dot{q}_t = i[h_0, z_t] + (\alpha_1 - 1)q_t|x_t, + Z(q_t) \quad (\mod \psi p_0) . \]

Here we apply the multiplication \( p_+ \{XX\} p_+ \) of \('step II', noting that \( p_+ [h_0, z_t] p_+ = 0 \), and that \( p_+ Z(q_t) p_+ \in \psi p_0 \), due to prop.10.1, so that (10.2) simplifies to

\[ \dot{q}_t^+ = (s_1 - 1)q_t^+\] (mod \( \psi p_0 \)).

The sharp D.E. (10.2') with initial-value \( q_0^+ (x_1, \xi) = \xi_j p_+ (\xi) \) has the unique solution \( q_t^+ (x_1, \xi) = \xi_j p_+ (\xi) \).

Similarly we get \( q_t^- (x_1, \xi) = \xi_j p_- (\xi) \).

So, we will get just

\[ q_t (x_1, \xi) = q_t^+(\xi) + q_t^-(\xi) = \xi_j (p_+(\xi) + p_-(\xi)) = \xi_j , \quad j = 1, 2, 3 . \]

Next we apply step III - multiplying \( p_+ \{XX\} p_- \) with \( a_t = q(\xi) + z_t(x_1, \xi) \) in (9.8), using that \( q_t \) is independent of \( x \) and \( t \), and that

\[ p_+ [h_0, c] p_- = 2(\xi)c^\pm , \quad p_- [h_0, c] p_+ = -2(\xi)c^\mp , \]

we get

\[ \dot{z}_t^\pm = 2i(\xi)z_t^\pm + ((\alpha_1 - 1)z_t|x_t, z_t^\pm - i\varepsilon_0 \sin \omega x_1 \{[\alpha_2^\pm, q] + 2s_2(\xi)z_t^\pm \} + \frac{\varepsilon_0}{2}(\alpha_2 X a_1)^\pm . \]

Assuming that \( \dot{z}_t \) also is of order \( m - 1 \) and omitting all terms of order \( m - 1 \) this reads

\[ 2i(\xi)z_t^\mp = i\varepsilon_0 \sin \omega x_1 [\alpha_2^\mp, q] \quad (\text{modulo } \psi p_{m-1}) . \]

Since division by \( \langle \xi \rangle \) lowers the order by 1 we thus get (also, repeating the procedure with \( p_- \{XX\} p_+ \))

\[ \dot{z}_t^\pm = \frac{\varepsilon_0}{2(\xi)}[\alpha_2^\pm(\xi), q(\xi)] \sin \omega x_1 \in \psi p_{m-1} , \quad z_t^\mp = -\frac{\varepsilon_0}{2(\xi)}[\alpha_2^\mp(\xi), q(\xi)] \sin \omega x_1 \in \psi p_{m-1} . \]

Both, \( z_t^\pm \) and \( z_t^\mp \) are approximations modulo \( \psi p_{m-2}, m=1 \), to be improved in the next iteration.

**Remark 10.2.** Note that our \( z_t^\pm \), \( z_t^\mp \) of (10.6) also are independent of \( t \), just as the \( q_t = q \), so that \( \dot{z}_t = 0 \) while also \( z_t^\pm|x_t, z_t^\mp|x_t \in \psi p_{m-1} \), so that (10.5') indeed is satisfied modulo \( \psi p_{m-1} \).

In our special case where \( q(\xi) \) is scalar – so that it commutes with the matrices \( \alpha^\pm(\xi) \) – we even get \( z_t^\pm = z_t^\mp = 0 \).

With \( z_t^\pm = z^\pm \), \( z_t^\mp = z^\mp \) of (10.6) (independent of \( t \)) we then get

\[ z_t = z^\pm + z^\mp + z_t^+ + z_t^- , \]

where \( z_t^+ \), \( z_t^- \in \psi p_{m-1} \) still remain undetermined – they will be fixed in the next iteration.

For the next iteration we return to steps I and II: With above \( q_t = q \) and \( z_t \) of (10.7) we set

\[ a_t = (q + z_t) + v_t , \quad \text{where } v_t \in \psi c_{m-2} , \]

recalling that \( z_t \) still has the free symbols \( z_t^+ \) and \( z_t^- \) belonging to \( \psi p_{m-1} \), so that we may assume \( v_t^+ = v_t^- = 0 \). Substituting into (9.8) and multiplying \( p_+ \{XX\} p_+ \) we get

\[ \dot{z}_t^+ = (s_1(\xi) - 1)z_t^+ + a_1^+(\xi)z_t^+ + a_1^-(\xi)v_t^+ + (Z(q + z_t + v_t))^+(x, \xi) , \]
where we used that \( \dot{q}^+ = q_1^+ = e_1^+ = 0 \). We want to look at (10.9) modulo \( \psi_m \), hence will drop all terms of order \( m - 2 \):

\[
(10.10) \quad \dot{z}_1^+ = (s_1(\xi) - 1)z_1^+ + \alpha_1^+(\xi)z_1^+ + (Z(q + z_1^+ + z_1^-))^+(x_1, \xi),
\]

keeping in mind that \( z_i \) is independent of \( x_2, x_3 \), also that - for \( c_i = z_i^+, z_i^- \) we have \( Z(c_i) \) of order \( m - 2 \), by prop.10.1.

Relation (10.10) again will be regarded as a differential equation for \( z_i^+ \). We may write it as

\[
(10.11) \quad \partial_t z_i^+(x_1 - t(s_1(\xi) - 1), \xi) = F_i(x_1 - t(s_1(\xi) - 1), \xi),
\]

with \( F_i(x_1, \xi) = \alpha_1^+(\xi)z_1^+ + (Z(q + z_1^+ + z_1^-))^+(x_1, \xi) \).

This (with initial value \( z_0^+ (x_1, \xi) \)) is solved by integration; we get

\[
(10.12) \quad z_i(x_1 - t(s_1(\xi) - 1), \xi) = \int_0^t d\tau F_i(x_1 - \tau(s_1(\xi) - 1), \xi).
\]

Substituting \( x_1 - t(s_1(\xi) - 1) \) by \( x_1 \) will give us

\[
(10.13) \quad z_i(x, \xi) = z_0^+(x_1, \xi) + \int_0^t d\tau F_i(x_1 + (t - \tau)(s_1(\xi) - 1), \xi).
\]

We assume \( z_0^+ = 0 \) as to leave the original commutative part \( q = q_0 \) untouched. Then we get

\[
(10.14) \quad z_i^+(x_1, \xi) = \int_0^t d\tau F_{1-\tau}(x_1 + \tau(s_1(\xi) - 1), \xi).
\]

We still simplify our \( F_i \) of (10.11), omitting more terms of order \( m - 2 \): Write

\[
(10.15) \quad p_+ Z(q + z_1^+ + z_1^-)p_+ = \frac{e_0}{2} p_+ (\alpha_2 X(q + z_1^+ + z_1^-))p_+ - i e_0 \sin \omega x_1 + [o_2, q + z_1^+ + z_1^-]p_+.
\]

Applying prop.10.1 we may omit \( z_1^+ \) and \( z_1^- \) in the first term, at right and \( q \) in the second term, so that

\[
(10.16) \quad F_i(x_1, \xi) = \alpha_1^+(\xi)z_1^+ + \frac{e_0}{2} p_+ (\alpha_2 X(q))p_+ - i e_0 \sin \omega x_1 + [o_2, z_1^+ + z_1^-]p_+.
\]

The last term still simplifies: \( p_+[o_2, z_1^+ + z_1^-]p_+ = \alpha_2^+ z_1^- - z_1^- \alpha_2^- \), so, we get

\[
(10.16') \quad F_i(x_1, \xi) = \alpha_1^+(\xi)z_1^+ + \frac{e_0}{2} p_+ (\alpha_2 X(q))p_+ - i e_0 \sin \omega x_1 (\alpha_2^+ z_1^- - z_1^- \alpha_2^-).
\]

Due to (10.6) this \( F_i \) is independent of \( t \). It belongs to \( \psi_{m-1} \), and it is a finite sum \( \sum_{j=0}^{2} f_j(\xi)e^{j\omega x_1} \) with certain \( f_j(\xi) \in \psi_{m-1,0} \). We may write the integrand of (10.14) as \( \sum_{j=0}^{2} e^{j\omega x_1} f_j^+(\xi) \). So, (6.16) then assumes the form

\[
(10.17+) \quad z_i^+(x_1, \xi) = \sum_{j=0}^{2} e^{j\omega x_1} f_j^+(\xi) \int_0^t d\tau e^{ij\omega x_1(s_1(\xi) - 1)} , f_j^+ \in \psi_{m-1}.
\]

The integrals \( \int_0^t d\tau e^{ij\omega x_1(s_1(\xi) - 1)} \) in (10.17+) belong to \( \psi_{0} \) - they may be evaluated explicitly, of course.

So, \( z_i^+ \) of (10.17+) indeed belongs to \( \psi_{m-1} \).

A similar procedure, using the multiplication \( p_-(XX)p_- \) will lead to construction of a \( z_i^- \) of the form

\[
(10.17-') \quad z_i^- = \sum_{j=0}^{2} e^{j\omega x_1} f_j^- (\xi) \int_0^t d\tau e^{-ij\omega x_1(s_1(\xi) + 1)} , f_j^- \in \psi_{m-1}.
\]

Four our iteration it is important to note that, while the \( x_1 \)-Fourier series expansion of \( z_1^+ \), \( z_1^- \) extended only from \(-1\) to \(+1\), it now will go from \(-2\) to \(+2\). One will see that all future such correction symbols
have finite sums, but with range increasing while the order decreases to \(-\infty\). As a consequence, even the asymptotic infinite sum to be defined eventually will have only a finite number of terms not of order \(\mu\), for any \(\mu \in \mathbb{R}\).

We now have \(q_t = q\) and \(z_t = z^+ + z^- + z_t^+ + z_t^-\) completely determined, up to an error in \(\psi p_{m-1}\) and \(\psi p_{m-2}\), respectively. Applying step III again then will result in corrections (mod \(\psi p_{m-2}\)) called \(v_t^+\) and \(v_t^-\) for \(z^\pm\) and \(z^\mp\); we use the multiplication \(p_+ \{XX\} p_-\), omitting terms of order \(m-1\), getting \(v_t^\pm\) as a quotient \((\psi p_{m-1})/\langle \xi \rangle\), where we must use that \(v_t \in \psi c_{m-1}\), and confirm this later on the calculated \(v_t\), recalling that division by \(\langle \xi \rangle\) preserves \(\psi p\) and lowers the \(\psi p\)-order by 1. Similar for \(v_t^\mp\) using \(p_- \{XX\} p_+\).

After obtaining the corrections \(v_t^+\) and \(v_t^-\) we still may introduce correction symbols \(v_t^+, v_t^- \in \psi p_{m-2}\) (so far held zero) together with new corrections \(w_t^+, w_t^- \in \psi p_{m-3}\) and start over with step I and step II on \(a_t = q + z_t + v_t + w_t\).

We have discussed the above for general \(q(\xi)\) to fill in the iteration, used for the proof of thm.8.3. It should be clear now, how this will go, and we regard that proof complete.

However, we must remind of the fact that this \(a_t(x, \xi)\) of (8.17) only solves the initial value problem (8.16) modulo \(\psi q_{-\infty}\); it will not yet lead to the Heisenberg transform of \(a_0(x, D)\) as a \(\psi do\ \psi a_t(x, D)\) modulo \(\psi q_{-\infty}\). We have indicated the steps necessary in sec.9 (cf. Obs.9.3). Still, we will continue to also apply thm.8.3 to \(q(\xi) = \xi_j\), noting that an argument of sec.13, below will get us to the same expansion (mod \(\psi q_{-1}\)) for our Heisenberg transform.

For the special \(q(\xi) = \xi_j\), we have in the present section, we get \(z_t^+ = z_t^- = 0\). For \(\xi_2, \xi_3\) we just get

\[
(10.18) \quad a_t(x_1, \xi) = \xi_j \pmod{\psi p_{-1}}, \quad \text{for all } t, \quad \text{as } j = 2, 3.
\]

So, the observables \(D_2, D_3\) will not change in time, modulo \(\psi q_{-1}\).

For \(q = \xi_1\) (10.14) assumes the form

\[
(10.19+) \quad z_t^+(x_1, \xi) = \varepsilon_0 \omega s_2(\xi)p_+(\xi) \int_0^t d\tau \cos \omega(x_1 + \tau s_1(\xi) - 1))
\]

\[
= \frac{\varepsilon_0}{2} \omega s_2(\xi)p_+(\xi)\{\gamma_t(\xi)e^{i\omega x_1} + \bar{\gamma}_t(\xi)e^{-i\omega x_1}\}
\]

with

\[
(10.20) \quad \gamma_t(\xi) = \int_0^t d\tau e^{i\omega \tau(s_1(\xi) - 1)}.
\]

Similarly,

\[
(10.19-) \quad z_t^-(x_1, \xi) = -\varepsilon_0 \omega s_2(\xi)p_-(\xi) \int_0^t d\tau \cos \omega(x_1 - \tau s_1(\xi) + 1))
\]

\[
= -\frac{\varepsilon_0}{2} \omega s_2(\xi)p_-(\xi)\{\gamma_t(-\xi)e^{i\omega x_1} + \bar{\gamma}_t(-\xi)e^{-i\omega x_1}\},
\]

with \(\gamma_t(\xi)\) of (10.20).

In this way we have calculated our symbol \(a_t = q_t + z_t^+ + z_t^-\) modulo \(\psi q_{-1}\), for the observable \(D_1\). Of course there will be terms modulo \(\psi p_{-2} \cdots\) with stronger and stronger decay as \(|\xi| \to \infty\), but the above lists all terms of order 0 for the operator \(D_1\). The \(a_t\) thus obtained will not give a self-adjoint \((D_1)_t\), but we have pointed out how to remedy this.

We summarize
Theorem 10.3. Regarding the symbols \( a_t = q_t + z_t \) and \( \tilde{a}_t = \tilde{q}_t + \tilde{z}_t \) for the 3 observables \( D_1, D_2, D_3 \) modulo \( \psi_{q-1} \), we get

\[
a_t(x_1, \xi) = \xi_j \pmod{\psi_{q-1}}, \quad \text{for all } t, \quad \text{as } j = 2, 3,
\]

that is, for \( j = 2, 3 \), we have

\[
a_t(x_1, \xi) = \xi_j, \quad z_t = z_t^+ = z_t^- = 0.
\]

For \( j = 1 \) we get (as formulas modulo \( \psi_{q-1} \))

\[
a_t(x_1, \xi) = \xi_1 + \frac{\xi_0}{2} \omega s_2(\xi) \{ (\gamma_1(\xi)e^{i\omega x_1} + \gamma_1(\xi)e^{-i\omega x_1})p_+(\xi) - (\gamma_1(-\xi)e^{i\omega x_1} + \gamma_1(-\xi)e^{-i\omega x_1})p_-(\xi) \},
\]

In particular, calculating mod \( \psi_{q-1} \), the correction term for self-adjointness of \( a_t(x, D) \) also vanishes, so that \( a_t(x_1, D) \) already is self-adjoint modulo \( \psi_{q-1} \).

We come to the following:

Theorem 10.4. Set \( \theta(\xi) = \frac{1}{2}(1 - s_1(\xi)) \), evaluate (above) \( \gamma_1(\xi) = te^{-i\omega\theta(\xi)t} \varphi(\omega\theta(\xi)t) \), with \( \varphi(\kappa) = \frac{\sin\kappa}{\kappa} \).

Then we have

\[
(H(t))_t - H(0) = (D_1)_t - D_1 = 0,
\]

\[
\varepsilon_{\omega} \omega t \cos(\omega(x_1 - t\theta(D))s_2(D)\varphi(\omega\theta(D)t)p_+(D)
- \varepsilon_{\omega} \omega t \cos(\omega(x_1 - t\theta(-D))s_2(D)\varphi(\omega\theta(-D))p_-(D))
\]

a relation valid modulo \( Op \psi_{q-1} \) (also, with \( D_1 \) in \( (D_1)_t \) entered only mod \( Op \psi_{q-1} \)) — cf.thm.13.3.

The proof is a calculation, mainly focusing on self-adjointness (mod \( Op \psi_{q-1} \)) of the corresponding operator terms.

Remark 10.5. Recall again: A special argument, as sketched at end of sec.9, accessible only through the spectral theory of the operator \( K \), is needed to derive thm’s 10.3 and 10.4, after clearing thm.8.3. This is to be discussed in sec.13, below.

Observation 10.6. It is clear that the first term at right of (10.23) addresses the electron part of the state, while the second term addresses positrons. The symbol of the electron part may be rewritten as

\[
\varepsilon_{\omega} \omega t \cos(\omega(x_1 - t\theta(\xi))s_2(\xi)\varphi(\omega\theta(\xi)t)p_+(\xi) = \frac{\xi_0}{2\theta(\xi)} s_2(\xi) \{ \sin(\omega(x_1 - \theta(\xi)t) - \sin \omega x_1) \}.
\]

Note the right hand side is a difference of a time-independent term and a term propagating like a wave with speed \( 2\theta(\xi) \). For large \( |\xi| \) — as dominant here — we have \( s_1(\xi) \approx \xi_1/|\xi| = \cos \lambda \), with the angle \( \lambda \) between the vector \( \xi \) and the radiation direction \( \xi_1 \). It follows that \( 2\theta(\xi) \approx (1 - \cos \lambda) = 2\sin(\lambda/2). \)

In other words, this propagation speed will display the same dependence on the direction as Compton’s wave-length dependence (cf. Sommerfeld [So1], p.50).

Clearly this term, marking a single collision with a photon, is of one order lower than the original observable. The further terms (we shall not calculate), will be of lower and lower order, hence of lesser and lesser probability since we deal with large \( |\xi| \).

Notice also: the term (10.25) vanishes for \( t = 0 \), marking the fact, that we do not need a correction \( z(x_1, t) \) for our present \( q(\xi) = \xi_j \), when working only mod \( \psi_{q-1} \).
11. Spectral Theory of the Operator $K = H(0) - D_1$

So far, regarding the proof of thm.9.4, we have solved the differential equation $\dot{u} = i \lambda u$, symbol $([K, A_1])$ modulo $\psi_{q-\infty}$. But, in order to get back to our desired $A_1 = e^{iKt}Ae^{-iKt} = a_t(x, D)$ (mod $\psi_{q-\infty}$), we now will have to involve Fourier integral operators. Actually, we shall get a representation of $e^{-iKt}$ as a sum of two Fourier integral operators, if we just invoke the spectral theorem for the self-adjoint operator $K$. In fact, this even brings about the additional advantage that the two FIO-s obtained are mutually orthogonal in our Hilbert space: their products vanish.

Considering the spectral theory of the operator $K$, we may separate off the variables $x_2, x_3$, since the coefficients of $K$ are only dependent on $x_1$. In other words, we may take the Fourier transform with respect to $\tilde{x} = (x_2, x_3)$. This leads us to a new operator

$$K = (\alpha_1 - 1)D_1 + (\xi_2 - A_2(x_1))\alpha_2 + \xi_3\alpha_3 + \beta.$$  

Recall, we are using the matrices $\alpha, \beta$ of (2.9). Thus we may write (11.1) block-matrix-wise as

$$K = \begin{pmatrix} 2i\partial & ip \\ -iq & 0 \end{pmatrix}, \quad p = \sigma_3(\xi_2 - A_2(x_1)) + \sigma_2\xi_3 - i, \quad q = \sigma_3(\xi_2 - A_2(x_1)) + \sigma_2\xi_3 + i,$$

with $\partial = \partial_x$, this being the $\tilde{x}$-Fourier-transformed operator $K$ of (8.7).

Writing $\partial_x f = f'$, and $\psi = (\psi,c)$, the equation $K\psi = \lambda\psi$ dissolves into this:

$$-2u' - i\lambda u = pv, \quad qu = i\lambda v.$$

As earlier, let $P(\tau) = \sigma_3(\xi_2 - A_2(\tau)) + \sigma_2\xi_3$. We observe that

$$pq = 1 + (\xi_2 - A_2(x_1))^2 + \xi_3^2 = 1 + P(x_1)^2 = \langle P(x_1) \rangle^2,$$

is a scalar. So, in particular,

$$p^{-1} = \frac{1}{1 + P^2(x_1)}q; \quad q^{-1} = \frac{1}{1 + P^2(x_1)}p.$$

The two equations (11.3) combine into one (scalar) first order differential equation

$$u' = -\frac{i}{2}(\lambda - \frac{1}{\lambda})\langle P \rangle^2 u.$$

for the variable $u$ only. Equation (11.6) is solved by

$$u(x_1, \tilde{\xi}) = e^{-i\frac{1}{\lambda} x_1 + \frac{1}{\lambda} \int_0^1 \langle P \rangle^2(r) dr} c, \quad c \in \mathbb{C}^2.$$

Once we have $u$ explicitly we may use the second (11.3) to also get $v$. All together we get

$$\psi(x_1, \tilde{\xi}, \lambda) = \langle \psi \rangle(x_1, \tilde{\xi}) = (\psi_{\frac{1}{\lambda} c}) e^{-i\frac{1}{\lambda} x_1 + \frac{1}{\lambda} \int_0^1 \langle P \rangle^2(r) dr} c = c(\lambda, \tilde{\xi}) \in \mathbb{C}^2,$$

where $c$ is independent of $x_1$.

Looking at (11.8) we observe that $\psi$, as a function of $x_1$, never will be $L^2(\mathbb{R})$, except for vanishing $c$. Thus there will not be any point-eigenvalues of the operator of $x_1$. On the other hand, there should be continuous spectrum on all of $\mathbb{R}$ since (for $c$ constant in $\lambda$) an integral $\int d\lambda \psi$, will be $L^2(\mathbb{R})$ defining a wave-packet.

One might see that there is some ‘separation at $\lambda = 0$’ in this continuous spectrum, insofar as the function $\psi(x_1, \tilde{\xi}, \lambda)$ becomes very discontinuous there. Indeed, the point $\lambda = 0$ here separates the line $-\infty < \lambda < \infty$ into the half-lines $(-\infty, 0)$ and $(0, \infty)$. The corresponding partition of unity

$$1 = P_{(-\infty, 0)} + P_{(0, \infty)}$$
with spectral projections $P_\Delta$ of $K$ will generate the split into electron states and positron states: We may write (with $\mathcal{H} = L^2(\mathbb{R})$

$$\mathcal{H}_\sigma = \{u \in \mathcal{H} : P_+ u = u\} \cup \mathcal{H}_\rho = \{u \in \mathcal{H} : P_- u = u\},$$

where

$$P_+ = \hat{F}^{-1}P_{(0,\infty)}\hat{F}, \quad P_- = \hat{F}^{-1}P_{(-\infty,0)}\hat{F},$$

with the $\hat{F}$-Fourier transform $\hat{F}$.

We now want to get the explicit spectral projections of $K$ of (11.2). A practical way to achieve this is a technique of complex analysis developed by Titchmarsh [Ti1].

Recalling the resolvent representation of spectral projections:

For a self-adjoint $N \times N$-matrix $X$, we may obtain the spectral projection $P_\Delta$ for any closed interval $\Delta$ of the real axis by the formula

$$P_\Delta = \frac{1}{2\pi i} \int_{\Gamma} (X - \lambda)^{-1} d\lambda,$$

where $\Gamma$ denotes any simple closed (positively oriented) curve in the complex plane encircling all eigenvalues on $\Delta$ but none of the others. Indeed, this is true, because, if $\varphi_1, \ldots, \varphi_N$ denotes an orthonormal base of eigenvectors to eigenvalue $\lambda_1, \ldots, \lambda_N$ then we may write

$$\mathcal{K} = \sum_{j=1}^{N} \frac{1}{\lambda_j - \lambda} \varphi_j \langle \varphi_j \rangle .$$

Then the residue theorem will imply (11.12).

Then we assume $\varphi_1, \ldots, \varphi_N$ are orthonormal and

$$\mathcal{P}_\Delta = -\frac{1}{\pi} \lim_{\varepsilon \to 0, \varepsilon > 0} \Im \{ \int_{\Delta} d\lambda (X - (\lambda - i\varepsilon))^{-1} \},$$

setting $\Im A = \frac{1}{2}(A - A^*)$ for any matrix $A$.

Formula (11.14) also holds for unbounded self-adjoint linear operators like our $K$ above – for a more detailed discussion note the book [Ti1] of Titchmarsh.

To implement (11.14) for $K$ of (11.2) we set up the resolvent ODE $K\psi - \lambda \psi = \chi$, $\psi = (u)$, $\chi = (\hat{F})$, so that $\chi = (K - \lambda)^{-1} \psi$. That is, we must solve the system

$$2iu' + ipv - \lambda u = f, \quad -iqu - \lambda v = g,$$

simplifying to

$$2iu' - (\lambda - \frac{1}{\chi}(P)^2)u = f + \frac{i}{\chi}g, \quad v = -\frac{1}{\chi}(g + iqu).$$

We must pick the unique solution in $L^2(\mathbb{R})$, assuming that $if - \frac{1}{\chi}pg \in L^2(\mathbb{R})$: Here we assume $\lambda = \mu - i\varepsilon$, $\varepsilon > 0$; then the homogeneous equation $-2u' - i(\lambda - \frac{1}{\chi}(P)^2)u = 0$ is solved by

$$u = ce^{-\frac{1}{\chi}(\lambda x_1 - \frac{\varepsilon}{\lambda} \rho(x_1))} = ce^{-\frac{1}{\chi}x_1(\lambda - \frac{1}{\lambda})} = ce^{-\frac{1}{\chi}x_1(1-i/|\lambda|^2)} e^{-\frac{1}{\chi}x_1(1+i/|\lambda|^2)}.$$
with $\rho(x_1) = \int_0^{x_1} (P(\tau))^2 d\tau$, and $\iota(\tau) = \rho(\tau)/\tau$. Here $u$ of (11.17) and its inverse vanish exponentially as $x_1 \to \infty$, and as $x_1 \to -\infty$, respectively. Hence the solution of (11.16) in $L^2$ will be

$$u = -\frac{1}{2} e^{-\frac{1}{\lambda}(\lambda^{\frac{1}{2}}(x_1)-\frac{1}{\lambda}(\rho(x_1)-\rho(\tau)))}$$

We also need (11.18) for the adjoint $(K - (\mu - i\varepsilon))^{-1} = (K - (\mu + i\varepsilon))^{-1}$, So, we also must set $\lambda = \mu + i\varepsilon$, $\varepsilon > 0$. Then the $L^2$-solution of the ODE will change to this:

$$u = \frac{1}{2} e^{-\frac{1}{\lambda}(\lambda^{\frac{1}{2}}(x_1)-\frac{1}{\lambda}(\rho(x_1)-\rho(\tau)))}$$

We now must take the difference of the two operators in (11.18) and (11.19), setting $\lambda = \mu - i\varepsilon$ in (11.18) and $\lambda = \mu + i\varepsilon$ in (11.19), with same $\mu, \varepsilon$, $\varepsilon > 0$ small; then that difference should be integrated $d\mu$ over an interval $\Delta = [\mu_1, \mu_2] \subset \mathbb{R}$, not containing 0. Then we should let $\varepsilon > 0$, $\varepsilon \to 0$, to, finally, get a constant multiple of the spectral projection $P_\Delta$ for $K$.

We shall set $\lambda = \mu - i\varepsilon$ in (11.18) and work with $\lambda = \mu + i\varepsilon$ in (11.19). Then we introduce the 'Greens-function-type expressions'

$$H^1(\mu, x_1, \tau) = e^{-\frac{1}{\lambda}(\lambda^{\frac{1}{2}}(x_1)-\frac{1}{\lambda}(\rho(x_1)-\rho(\tau)))}$$ as $\tau < x_1$,

$$H^1(\mu, x_1, \tau) = e^{-\frac{1}{\lambda}(\lambda^{\frac{1}{2}}(x_1)-\frac{1}{\lambda}(\rho(x_1)-\rho(\tau)))}$$ as $\tau > x_1$,

$$H^2(\mu, x_1, \tau) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}(\lambda^{\frac{1}{2}}(x_1)-\frac{1}{\lambda}(\rho(x_1)-\rho(\tau)))}$$ as $\tau < x_1$,

$$H^2(\mu, x_1, \tau) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}(\lambda^{\frac{1}{2}}(x_1)-\frac{1}{\lambda}(\rho(x_1)-\rho(\tau)))}$$ as $\tau > x_1$.

and

$$H^3(\mu, x_1, \tau) = \frac{1}{\lambda^2} e^{-\frac{1}{\lambda}(\lambda^{\frac{1}{2}}(x_1)-\frac{1}{\lambda}(\rho(x_1)-\rho(\tau)))}$$ as $\tau < x_1$,

$$H^3(\mu, x_1, \tau) = \frac{1}{\lambda^2} e^{-\frac{1}{\lambda}(\lambda^{\frac{1}{2}}(x_1)-\frac{1}{\lambda}(\rho(x_1)-\rho(\tau)))}$$ as $\tau > x_1$.

With these three functions, let

$$\omega = \left(\frac{\chi}{\chi}\right) = -\frac{1}{2\pi} i \{(K - \lambda)^{-1} - (K - \lambda)^{-1}\} \chi, \chi = \left(\frac{\chi}{\chi}\right).$$

We then get

$$w = \frac{1}{4\pi} \int_{-\infty}^{+\infty} H^1(\mu, x_1, \tau) f(\tau) d\tau + \frac{i}{4\pi} \int_{-\infty}^{+\infty} H^2(\mu, x_1, \tau) p(\tau) g(\tau) d\tau$$

$$z = \frac{\varepsilon}{\pi |\lambda|^2} g(x_1) - \frac{i}{4\pi} q(x_1) \int_{-\infty}^{+\infty} H^2(\mu, x_1, \tau) f(\tau) d\tau + \frac{1}{4\pi} q(x_1) \int_{-\infty}^{+\infty} H^3(\mu, x_1, \tau) p(\tau) g(\tau) d\tau.$$
Writing \( P_{\Delta} = ((P_{\Delta}^{ij}))_{j,\ell=1,2} \) as a \( 2 \times 2 \)-block matrix, acting on \( \xi = (\xi^1) \), we get
\[
P_{\Delta}^{11} f = \frac{1}{4\pi} \int_{\Delta} d\tau e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} f(\tau) ,
\]
(11.26) \[
P_{\Delta}^{12} g = \frac{i}{4\pi} \int_{\Delta} \frac{d\lambda}{\lambda} \int d\tau e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} p(\tau) g(\tau) ,
\]
\[
P_{\Delta}^{21} f = -\frac{i}{4\pi} q(x_1) \int_{\Delta} \frac{d\lambda}{\lambda} \int d\tau e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} f(\tau) ,
\]
\[
P_{\Delta}^{22} g = \frac{1}{4\pi} q(x_1) \int_{\Delta} \frac{d\lambda}{\lambda} \int d\tau e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} p(\tau) g(\tau) .
\]

Being in control of the spectral projections of the operator \( K \), we may apply the spectral theorem, for a representation \( G(\lambda) = \int G(\lambda) dP_{\lambda} \), where \( G(\lambda) \) denotes any function of the real variable \( \lambda \). Accounting for the singularity at \( \lambda = 0 \) we write
\[
G(K) \psi = \int_{-\infty}^{\infty} G(\lambda) dP_{\lambda} \psi = \int_0^0 G(\lambda) dP_{\lambda} \psi + \int_{0}^{+\infty} G(\lambda) dP_{\lambda} \psi = (G(K))_{-} \psi + (G(K))_{+} \psi ,
\]
(11.27) Clearly then we may use (11.26) to express the differential \( dP_{\lambda} \) by \( d\lambda \). For \((G(K))_{+} = G = ((G_{jl}))_{j,\ell=1,2}\) and \( \psi = (\xi^1) \) we then get
\[
(G(K))^{11}_+ f = \frac{1}{4\pi} \int_0^0 \frac{d\lambda}{\lambda} G(\lambda) \int_{-\infty}^{\infty} d\tau e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} f(\tau) ,
\]
(11.28) \[
(G(K))^{12}_+ g = \frac{i}{4\pi} \int_0^0 \frac{d\lambda}{\lambda} G(\lambda) \int_{-\infty}^{\infty} d\tau e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} p(\tau) g(\tau) ,
\]
\[
(G(K))^{21}_+ f = -\frac{i}{4\pi} q(x_1) \int_0^0 \frac{d\lambda}{\lambda} G(\lambda) \int_{-\infty}^{\infty} d\tau e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} f(\tau) ,
\]
\[
(G(K))^{22}_+ g = \frac{1}{4\pi} q(x_1) \int_0^0 \frac{d\lambda}{\lambda} G(\lambda) \int_{-\infty}^{\infty} d\tau e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} p(\tau) g(\tau) ,
\]
and corresponding formulas for \( G_{-}(K) \), where \( \int_0^0 d\tau \) has been replaced by \( \int_{-\infty}^{0} d\tau \).

In (11.28) we interchange integrals and write \( G_{jl} = (G(K))^{jl}_+ \), \( G_{jl} = (G(K))^{jl}_- \):
\[
G_{11} f(x_1) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\tau f(\tau) \int_0^0 d\lambda e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} G(\lambda) ,
\]
(11.29) \[
G_{12} g(x_1) = \frac{i}{4\pi} \int_{-\infty}^{\infty} d\tau g(\tau) p(\tau) \int_0^0 \frac{d\lambda}{\lambda} e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} G(\lambda) ,
\]
\[
G_{21} f(x_1) = -\frac{i}{4\pi} q(x_1) \int_{-\infty}^{\infty} d\tau f(\tau) \int_0^0 \frac{d\lambda}{\lambda} e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} G(\lambda) ,
\]
\[
G_{22} g(x_1) = \frac{1}{4\pi} q(x_1) \int_{-\infty}^{\infty} d\tau g(\tau) p(\tau) \int_0^0 \frac{d\lambda}{\lambda} e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} G(\lambda) .
\]
Note, for the term \((G(K))_{-}\) of (11.27) we get the same kind of formulas — the difference being that the inner integral now extends from \( -\infty \) to \( 0 \), instead from \( 0 \) to \( \infty \):
\[
G_{11} f(x_1) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\tau f(\tau) \int_{-\infty}^0 d\lambda e^{-\frac{1}{2}(\xi^1_1-\tau)(\lambda-\xi^2_{12})} G(\lambda) ,
\]

MATHEMATICAL ANALYSIS OF DIRAC EQUATION PHYSICS
(11.29)  \[ G_{12}g(x_1) = \frac{i}{4\pi} \int_{-\infty}^{\infty} d\tau g(\tau)p(\tau) \int_{-\infty}^{0} \frac{d\lambda}{\lambda} e^{-\frac{i}{\lambda}(x_1-\tau)(\lambda-i\frac{\lambda}{2})} G(\lambda) , \]

\[ G_{21}f(x_1) = -\frac{i}{4\pi} q(x_1) \int_{-\infty}^{\infty} d\tau f(\tau) \int_{-\infty}^{0} \frac{d\lambda}{\lambda} e^{-\frac{i}{\lambda}(x_1-\tau)(\lambda-i\frac{\lambda}{2})} G(\lambda) , \]

\[ G_{22}g(x_1) = \frac{1}{4\pi} q(x_1) \int_{-\infty}^{\infty} d\tau g(\tau)p(\tau) \int_{-\infty}^{0} \frac{d\lambda}{\lambda^2} e^{-\frac{i}{\lambda}(x_1-\tau)(\lambda-i\frac{\lambda}{2})} G(\lambda) . \]

Here we would like to transform the inner integrals. Substitute

\[ \lambda - i^2 \frac{1}{\lambda} = 2\mu , \quad \lambda = \mu \pm i\sqrt{\mu^2 + \mu^2} , \quad d\lambda = \pm \frac{\lambda d\mu}{\sqrt{\lambda^2 + \mu^2}} , \]

to be used with both (11.29) and (11.29-). With \( \lambda = \mu + i\sqrt{\mu^2 + \mu^2} \) we get an invertible map \( \mu \leftrightarrow \lambda \) with \( \lambda > 0 \) and

\[ \lambda = 0 \leftrightarrow \mu = -\infty , \quad \lambda = \infty \leftrightarrow \mu = \infty , \]

useful for (11.29), while \( \lambda = \mu - i\sqrt{\mu^2 + \mu^2} \) implies \( \lambda < 0 \) and gives an invertible map with \( \mu \leftrightarrow \lambda \) and

\[ \lambda = 0 \leftrightarrow \mu = \infty , \quad \lambda = -\infty \leftrightarrow \mu = -\infty . \]

So, (11.31) is useful for a transformation of (11.29) while (11.31-) will work for (11.29-).

For the 4 inner integrals \( I_{ij} \) we get

\[ I_{11} = \int_{-\infty}^{\infty} (1 + \frac{\mu}{\sqrt{\lambda^2 + \mu^2}}) d\mu e^{-i\mu(x_1-\tau)} G(\mu + \sqrt{\lambda^2 + \mu^2}) , \]

\[ I_{12} = I_{21} = \int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{\lambda^2 + \mu^2}} e^{-i\mu(x_1-\tau)} G(\mu + \sqrt{\lambda^2 + \mu^2}) , \]

\[ I_{22} = \int_{-\infty}^{\infty} \frac{\sqrt{\lambda^2 + \mu^2}}{\lambda^2 \sqrt{\lambda^2 + \mu^2}} d\mu e^{-i\mu(x_1-\tau)} G(\mu + \sqrt{\lambda^2 + \mu^2}) , \]

and

\[ I_{11} = \int_{-\infty}^{\infty} (1 - \frac{\mu}{\sqrt{\lambda^2 + \mu^2}}) d\mu e^{-i\mu(x_1-\tau)} G(\mu - \sqrt{\lambda^2 + \mu^2}) , \]

\[ I_{12} = I_{21} = -\int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{\lambda^2 + \mu^2}} e^{-i\mu(x_1-\tau)} G(\mu - \sqrt{\lambda^2 + \mu^2}) , \]

\[ I_{22} = -\int_{-\infty}^{\infty} \frac{\sqrt{\lambda^2 + \mu^2} + \mu}{\lambda^2 \sqrt{\lambda^2 + \mu^2}} d\mu e^{-i\mu(x_1-\tau)} G(\mu - \sqrt{\lambda^2 + \mu^2}) . \]

We substitute (11.35±) into (11.32±) and interchange integrals again, renaming integration variables \( (\tau, \mu) \to (y_1, -\xi_1) \):

\[ G_{11}f(x_1) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{+\infty} dy_1 e^{i\xi_1(y_1-y_1)} G(-\xi_1 + \sqrt{y_1 + \xi_1^2})(1 - \frac{\xi_1}{\sqrt{y_1 + \xi_1^2}}) f(y_1) , \]

\[ G_{12}g(x_1) = \frac{i}{4\pi} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{+\infty} dy_1 e^{i\xi_1(y_1-y_1)} G(-\xi_1 + \sqrt{y_1 + \xi_1^2}) \frac{1}{\sqrt{y_1 + \xi_1^2}} p(y_1)g(y_1) , \]

\[ G_{21}f(x_1) = -\frac{i}{4\pi} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{+\infty} dy_1 e^{i\xi_1(y_1-y_1)} G(-\xi_1 + \sqrt{y_1 + \xi_1^2}) \frac{1}{\sqrt{y_1 + \xi_1^2}} q(x_1) f(y_1) , \]

\[ G_{22}g(x_1) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{+\infty} dy_1 e^{i\xi_1(y_1-y_1)} G(-\xi_1 + \sqrt{y_1 + \xi_1^2}) \frac{1}{\sqrt{y_1 + \xi_1^2}} q(y_1) g(y_1) . \]
\[
G_{22}g(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} dy_1 e^{i\xi_1(x_1-y_1)} G(-\xi_1 + \sqrt{\eta^2 + a^2})(1 + \frac{\xi_1}{\sqrt{\eta^2 + a^2}})g(x_1)p(y_1)g(y_1),
\]
and
\[
G_{11}f(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} dy_1 e^{i\xi_1(x_1-y_1)} G(-\xi_1 - \sqrt{\eta^2 + a^2})f(y_1),
\]

(11.33) \quad \left\{ \begin{array}{l}
G_{12}g(x) = -\frac{i}{4\pi} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} dy_1 e^{i\xi_1(x_1-y_1)} G(-\xi_1 - \sqrt{\eta^2 + a^2})(1 + \frac{\xi_1}{\sqrt{\eta^2 + a^2}})g(x_1)p(y_1)g(y_1), \\
G_{21}f(x) = \frac{i}{4\pi} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} dy_1 e^{i\xi_1(x_1-y_1)} G(-\xi_1 - \sqrt{\eta^2 + a^2})(1 + \frac{\xi_1}{\sqrt{\eta^2 + a^2}})f(y_1), \\
G_{22}g(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_1 e^{i\xi_1(x_1-y_1)} G(-\xi_1 - \sqrt{\eta^2 + a^2})g(x_1)p(y_1)g(y_1).
\end{array} \right.
\]

In (11.33) we recall that
\[
c^2(x_1, y_1) = (\rho(x_1) - \rho(y_1))/(x_1 - y_1) = (\xi_2 - c(x_1, y_1)^2 + \xi_3^2 + a(x_1, y_1)^2),
\]
with
\[
c(x_1, y_1) = \frac{1}{x_1 - y_1} \int_{x_1}^{y_1} A_2(t) dt, \quad d(x_1, y_1) = \frac{1}{x_1 - y_1} \int_{x_1}^{y_1} A_2'(t) dt, \quad a^2 = 1 + d - c^2,
\]
by a calculation.

Notice that (11.33) already gives (the \(\tilde{F}\)-transform) of \(G(K)\) as a sum of 2 one-dimensional ‘formal’ \(\psi\)-do-s. To get back to the operator \(K\) of (8.7) we must replace \(\psi(x_1) = (\delta(x_1))\) in above formulas by
\[
\frac{1}{2\pi} \int e^{-i\xi_1 y} \psi(x_1, y) dy,
\]
and then apply the inverse \(\tilde{F}\)-transform to the \(G\)\(\hat{\psi}\).

**Theorem 11.1.** For the operator \(K\) of (8.7) and a function \(G(\lambda) : (R \rightarrow C) \) we have \(G(K) = (G(K))_+ + (G(K))_-\) in the sense of (11.27) where \((G(K))_+ = ((G_{ji}))_{j,i=1,2} \) and \((G(K))_- = ((G_{ji}))_{j,i=1,2} \), with
\[
G_{11}f(x) = \frac{1}{16\pi^3} \int d\xi \int dy e^{ix(y-x)} G(-\xi_1 + \sqrt{\eta^2 + a^2})(1 - \frac{\xi_1}{\sqrt{\eta^2 + a^2}})f(y),
\]
(11.37)
\[
G_{12}g(x) = \frac{i}{16\pi^3} \int d\xi \int dy e^{ix(y-x)} G(-\xi_1 + \sqrt{\eta^2 + a^2})\frac{1}{\sqrt{\eta^2 + a^2}}p(y_1)g(y),
\]
\[
G_{21}f(x) = -\frac{i}{16\pi^3} \int d\xi \int dy e^{ix(y-x)} G(-\xi_1 + \sqrt{\eta^2 + a^2})\frac{1}{\sqrt{\eta^2 + a^2}}f(y),
\]
\[
G_{22}g(x) = \frac{1}{16\pi^3} \int d\xi \int dy e^{ix(y-x)} G(-\xi_1 + \sqrt{\eta^2 + a^2})\sqrt{\eta^2 + a^2}\xi_1\frac{1}{\sqrt{\eta^2 + a^2}}g(x_1)g(y_1),
\]
and
\[
G_{11}f(x) = \frac{1}{16\pi^3} \int d\xi \int dy e^{ix(y-x)} G(-\xi_1 - \sqrt{\eta^2 + a^2})(1 + \frac{\xi_1}{\sqrt{\eta^2 + a^2}})f(y),
\]
(11.37)
\[
G_{12}g(x) = -\frac{i}{16\pi^3} \int d\xi \int dy e^{ix(y-x)} G(-\xi_1 - \sqrt{\eta^2 + a^2})\frac{1}{\sqrt{\eta^2 + a^2}}p(y_1)g(y),
\]
\[
G_{21}f(x) = \frac{i}{16\pi^3} \int d\xi \int dy e^{ix(y-x)} G(-\xi_1 - \sqrt{\eta^2 + a^2})\frac{1}{\sqrt{\eta^2 + a^2}}f(y),
\]

\[
G_{22}g(x) = \frac{1}{16\pi^3} \int d\xi \int dy e^{ix(y-x)} G(-\xi_1 - \sqrt{\eta^2 + a^2})\sqrt{\eta^2 + a^2}\xi_1\frac{1}{\sqrt{\eta^2 + a^2}}g(x_1)g(y_1).
\]
\[ G_{22} g(x) = \frac{1}{16\pi} \int d\xi \int d\eta e^{i(x-y)} G(-\xi_1 - \sqrt{\eta^2 + a^2}) \frac{\sqrt{\eta^2 + a^2} - \xi_1}{(\eta^2 + a^2)\sqrt{\eta^2 + a^2}} q(x_1) p(y_1) g(y) . \]

again using the vectors \( \eta = (\xi_1, \xi_2 - c(x_1, y_1), \xi_3) \), \( \tilde{\eta} = (\eta_2, \eta_3) \)

Clearly the operators \( G(K)_{\pm} \) are formal \( \psi do-s \), with their symbol containing the factors \( G(-\xi_1 \pm \sqrt{\eta^2 + a^2}) \). But it will depend on the choice of the function \( G(\lambda) \) whether these will be operators belonging to one of our classes \( Op\psi q_m \). We shall find that true if we choose \( G(\lambda) \equiv 1 \) but false for \( G(\lambda) = e^{-i\lambda} \). In the latter case the operators assume a form we shall call \( \text{FIo-} \psi do-s \).

12. A Class of Global Fourier Integral Operators

The functions \( G(\lambda) \), most important for us here, are \( G(\lambda) \equiv 1 \), \( G(\lambda) = \lambda \), \( G(\lambda) = e^{-i\lambda} \). For \( G(\lambda) \equiv 1 \) the operators \( (G(K))_{\pm} \) will give the two projections \( P_+, P_- \) separating the spaces of electron and positron states, at \( t = 0 \).

Clearly they appear as formal (left-right-multiplying) \( \psi do-s \)

\[ P_{\pm} = p_{\pm}(M_1, M_r, D) = ((p_{\pm}^\dagger(M_1, M_r, D)) \]

with the 2 \times 2-block-symbols

\[
(12.1) \quad p_{11}^+(x, y, \xi) = \frac{1}{2}(1 - \frac{\xi_1}{\sqrt{\eta^2 + a^2}}) , \quad p_{12}^+(x, y, \xi) = \frac{i}{2} \frac{1}{\sqrt{\eta^2 + a^2}} p(y_1) ,
\]

\[
(12.1-1) \quad p_{11}^-(x, y, \xi) = \frac{i}{2} \frac{1}{\sqrt{\eta^2 + a^2}} q(x_1) , \quad p_{12}^-(x, y, \xi) = \frac{1}{2} (\frac{\xi_1}{\sqrt{\eta^2 + a^2}} + i) q(x_1) p(y_1) ,
\]

\[
= \frac{1}{2}(1 + \frac{\xi_1}{\sqrt{\eta^2 + a^2}}) ) , \quad p_{12}^+(x, y, \xi) = \frac{i}{2} \frac{1}{\sqrt{\eta^2 + a^2}} p(y_1) ,
\]

\[
p_{21}^+(x, y, \xi) = \frac{i}{2} \frac{1}{\sqrt{\eta^2 + a^2}} q(x_1) , \quad p_{22}^+(x, y, \xi) = \frac{1}{2} (\frac{\xi_1}{\sqrt{\eta^2 + a^2}} - i) q(x_1) p(y_1) .
\]

We should remind of the fact that we have \( c = c(x_1, y_1) \), \( d = d(x_1, y_1) \) given by (11.35) , or, explicitly, for \( A_2(\tau) = \varepsilon_0 \sin \omega \tau \), by

\[
(12.2) \quad c(x_1, y_1) = \varepsilon_0 \sin \omega \frac{x_1 + y_1}{2} , \quad d(x_1, y_1) = \frac{\varepsilon_0^2}{2} - \frac{\varepsilon_0^2}{2} \cos \omega (x_1 + y_1) \phi(\omega (x_1 - y_1)) ,
\]

with \( \phi(\kappa) = \sin \kappa / \kappa \), and that \( a^2 = 1 + d - c^2 \).

With that it is easily confirmed that we have all the symbols (12.1),(12.1-) belonging to \( \psi q l r_0 \), as defined by the estimates (3.10), so that we verified that \( P_+, P_- \in \text{Op}\psi q_0 \). In addition, with (12.1),(12.1-), we have obtained explicit \( \psi q l r_0 \)-symbols of the operators \( P_{\pm} \) — valid for the Dirac matrices (2.9) only.

Of course, we have \( P_+ + P_- = 1 \). Looking at above symbols (12.1),(12.1-) we note that, indeed, \( p_{11}^+ + p_{11}^- = 1 \), \( p_{12}^+ + p_{12}^- = p_{21}^+ + p_{21}^- = 0 \) for all \( x, y, \xi \), but

\[
(12.3) \quad p_{22}^+ + p_{22}^- = \frac{q(x_1) p(y_1)}{\eta^2 + a^2} .
\]

The latter symbol is not \( \equiv 1 \), although it turns out to be \( \equiv 1 \) for \( A_2 \equiv 0 \), and, also, for general \( A_2(\tau) \), and \( x_1 = y_1 \). This points to the fact that the assignment \( a(x, y, \xi) \to a(M_1, M_r, D) \) is not bi-unique: an
operator $\Lambda = \{M_1, M_r, D\}$ may be represented by many different symbols $b(x, y, \xi)$, exactly one of them independent of $y$, then giving $a(M_1, M_r, D) = b(x, D) \in Op\psi q$.

There will be an asymptotic Leibniz formula, to get this $b(x, \xi)$ from $a(x, y, \xi)$:

$$b(x, \xi) = \sum_{j=0}^{\infty} \frac{1}{j!} \{(-i\partial_y \partial_\xi)^j a(x, y, \xi)\}_{z=y} \mod{\psi q}$$.

Applying this to the symbol (12.3) it is found that the term of order 0 at right of (12.4) is $\equiv 1$, thus, at least, confirming that $P_+ + P_- = 1 \mod{Op\psi q_{-1}}$.

Applying now thm.11.1 to the function $G(\lambda) = e^{-i\lambda t}$ we again obtain $e^{-iKt}$ as a sum of two ‘formal’ $\psi do$-s, given by (11.37) and (11.37-). Their $\psi dolr$-symbols are given by

$$e^{-it(\xi_1 + \sqrt{\eta^2 + a^2})} p_+(x_1, y_1, \xi) \quad \text{and} \quad e^{-it(\xi_1 - \sqrt{\eta^2 + a^2})} p_-(x_1, y_1, \xi)$$,

with $p_\pm$ of (12.1).

Evidently the symbols (12.5) do not belong to $\psi qlr$ — any derivative landing on the exponential factor producing no decay in the required sense. However, referring to [Co5], p.53, we observe that these $\psi do$-symbols (12.5) still belong to the space $ST$ defined there. As a consequence, the ‘finite-part-singular integrals’ defined there still exist; we have the Beals formulas as well as the Leibniz formulas with integral reminder of ch.1, sec’s 4 and 5 valid, although no asymptotically convergent Leibniz formulas can be derived, for a $\psi do$-calculus.

Actually, a different interpretation then is customary: Following Hörmander [Hoe4] such operators are written in the form

$$Au(x) = \frac{1}{(2\pi)^r} \int d\xi \int dy e^{i\varphi(x, y, \xi)} a(x, y, \xi) u(\xi)$$,

with a symbol $a(x, y, \xi) \in \psi q$, as before, and with a (real-valued) ‘phase function’ $\varphi(x, y, \xi)$. In our present case we will have

$$\varphi(x, y, \xi) = \xi(x-y) + \varphi_0(x, y, \xi) \quad \text{with} \quad \varphi_0 = t(\xi_1 \pm \sqrt{\eta^2 + a^2})$$;

note, we have $\varphi(x, y, \xi)$, $\varphi_0(x, y, \xi) \in \psi qlr$.

Hörmander introduced the name ‘Fourier integral operator’ (abbrev. FIO) for operators of this form. One may find an extensive theory of ‘local’ FIO-s — applicable only to functions defined in a bounded subdomain of $\mathbb{R}^3$, or also on a compact manifold [cf. also Egorov [Eg1], Maslov[Ms1], Buslaev[But1] for development of general ideas]. When applied to a function $u(x)$, local FIO-s will move singularities of $u(x)$. A given local FIO can be given by many different symbols and phase functions. Composition of two local FIO-s will give a local FIO again, with construction of new phase function and symbol involving an interesting but complicated theory, not concerning us here.

The kind of ‘$global$’ FIO-s over $\mathbb{R}^3$, we have here, has been studied by Sandro Coriasco [Cr1] [Cr2], although only for phase functions and symbols in $\psi c$ — not in $\psi q$, as we require. In [Cr1] we find results for composition of our kind of FIO-s, but only for special phase functions: They cover the case of $AB$, $BA$ where $B$ is a $\psi do$ [it has $\varphi_0 \equiv 0$, and also the case $A^*BA$, again with $B$ a $\psi do$: Then $A^*BA$ also is a $\psi do$. Essential ingredient of the discussion is the fact that $e^{i\varphi(x, y, \xi)}$ is a symbol in $\psi clr_0$ whenever $\psi(x, y, \xi) \in \psi c_0$, while this is not true for $\psi$ of order $> 0$.

Although Coriasco discusses only the case where symbols and phase functions belong to $\psi clr$, we find that his results extend to symbols and phase functions in $\psi qlr$ if the asymptotic convergence modulo
Proposition 13.2. \( q \) should have to carry the operation \( R \) (or, had pointed out at the end of sec.9 that we should replace \( U \) between thm.8.3 and its application to obtain the operator (13.2).

Observation 13.1. Looking at the quantum mechanical application: We are mainly interested in predicting an observable \( R \) in a pure electron (or pure positron) state; that is in a state \( \psi \) (or, \( P \cdot \psi = \psi \)). If \( P_+ \psi = \psi \) then the expectation value for an observable \( R = q(D) \) in \( Op\psi_q \), at time \( t \), may be written as

\[(12.8) \quad e^{iKt} P_+ C P_+ e^{-iKt} \in Op\psi_q , \quad e^{iKt} P_- C P_- e^{-iKt} \in Op\psi_q , \]

and, likewise,

\[(12.9) \quad U^*(t) P_+ C t P_+ U(t) \in Op\psi_q , \quad U^*(t) P_- C t P_- U(t) \in Op\psi_q , \quad C_t = T_{-t} C T_t , \]

for the propagator \( U(t) \) of our Dirac equation, with \( H(t) \) of (8.1), and the projections \( P_{\pm} = T_{-t} P_{\pm} T_t \) onto the electron (positron) spaces \( H_e(t) \), \( H_p(t) \) at time \( t \) of (8.11).

13. Returning to the Heisenberg Transform

Finally, after gaining control on the FIO-analysis of the operators \( e^{-iKt} \) we now may address the gap between thm.8.3 and its application to obtain the operator \( U^*(t) a_0^\infty(x, D) U(t) \) as a \( \psi do Op\psi q \). We had pointed out at the end of sec.9 that we should replace \( A\infty_t = a_0^\infty(x, D) \) by the operator \( \kappa_c(A_t^\infty) = P_+ A_t^\infty P_+ + P_- A_t^\infty P_- \), then landing at (9.17), with its remainder \( \Gamma_t^\infty \in Op\psi_{q-\infty} \). Then, however, we should have to carry the operation \( R \to \kappa_c(R) \) into the asymptotic expansions (mod \( \psi_{q-\infty} \) of thm.8.3. In particular we already stated that the initial expansions of thm’s 10.3 and 10.4 will not change by passing from \( A_t^\infty \) to \( \kappa_c(A_t^\infty) \).

Observation 13.1. Looking at the quantum mechanical application: We are mainly interested in predicting an observable \( R \) in a pure electron (or pure positron) state; that is in a state \( \psi \) satisfying \( P_+ \psi = \psi \) (or, \( P_- \psi = \psi \)). If \( P_+ \psi = \psi \) then the expectation value for an observable \( R = q(D) \) in \( Op\psi_q \), at time \( t \), may be written as

\[(13.1) \quad \langle U(t) \psi, RU(t) \psi \rangle = \langle \psi, e^{iKt} T_{-t} q(D) T_{-t} e^{-iKt} \psi \rangle = \langle \psi, e^{iKt} P_+ R P_+ e^{-iKt} \psi \rangle = \langle \psi, \kappa_c(R) e^{-iKt} \psi \rangle , \]

using that \( q(D) \) is translation invariant, i.e., \( T_{-t} q(D) T_{-t} = q(D) \), and that \( P_- \psi = P_- P \psi = 0 \), giving \( P_+ R P_+ \psi = (P_+ R P_+ + P_- R P_-) \psi = \kappa_c(R) \psi \).

So, the operator \( e^{iKt} \kappa_c(R) e^{-iKt} \) really is governing prediction of \( R = q(D) \) in the sense of the Heisenberg transform, for all times. And, according to thm. 12.1, this operator belongs to \( Op\psi q \), at all \( t \).

Proposition 13.2. With the symbols \( p_{\pm}(\xi) = \frac{1}{2} (1 \pm \frac{1}{\sqrt{\eta^2 + a^2}} h_0(\xi)) \), \( h_0 = a \xi + \beta \), we have

\[(13.2) \quad P_+ - p_+(D) \in Op\psi_{q-1} , \quad P_- - p_-(D) \in Op\psi_{q-1} , \]

Proof. Clearly we obtain a block-matrix representations of the symbols \( p_{\pm}(\xi) \) by setting \( A_2 = 0 \) in (12.1) and (12.1-), where then \( \eta^2 + a^2 = \langle \xi \rangle^2 \). Also, modulo \( \psi_{q-1} \), we may replace the terms \( p_{22} \) and \( p_{22} \) by \( \frac{1}{2} (1 \pm \frac{1}{\sqrt{\eta^2 + a^2}}) \), as already noted in (12.3). Looking at (12.2) we observe that the functions \( a, c, d \) are all bounded with all their \( x_1, y_1 \)-derivatives.

Taking the differences (13.2) we then note that

\[(13.3) \quad \frac{1}{\langle \xi \rangle} - \frac{1}{\sqrt{\eta^2 + a^2}} = \frac{d^2 - 2c \xi}{\langle \xi \rangle \sqrt{\eta^2 + a^2} \left\{ \langle \xi \rangle + \sqrt{\eta^2 + a^2} \right\}} \in \psi_{q-2} . \]
This, and similar observations will indeed show the statement, q.e.d.

Now let us come back to formulas (10.21),(10.22),(10.23): According to our arguments, so far, this was just a rewriting of (8.17), with its following Fourier series expansion, for the special case of \( q(D) = D_j \), \( j = 1, 2, 3 \), listing the terms of order 0 and 1 explicitly, while ignoring all terms of order less than 0. But, recall, this only solves the initial-value problem (8.16) modulo \( O\phi q_{-\infty} \); it does not make \( e^{-iKt}a_{0}(x,D)e^{-iKt} \) a \( \psi do \) in \( O\phi q \).

On the other hand, looking at (9.17) — now established, since we proved thm. 12.1, it is clear that we get

\[
e^{iKt}\kappa_{c}(A_{0}^{\infty})e^{-iKt} - \kappa_{c}(A_{1}^{\infty}) \in O\phi q_{-\infty}.
\]

In order to get our formula on Heisenberg’s transform, modulo \( O\phi q_{-1} \) it then will be a matter of showing that the passing from \( A_{1}^{\infty} \) to \( \kappa_{c}(A_{1}^{\infty}) \) will only produce errors in \( O\phi q_{-1} \).

Note, f’la (10.23) may be written as

\[(13.5) \quad a_{t}(x_{1}, \xi) = \epsilon_1 + f_{+}(x_{1}, \xi)p_{+}(\xi) + f_{-}(x_{1}, \xi)p_{-}(\xi) \mod \psi_{-1}
\]

with scalar symbols \( f_{\pm}(x_{1}, \xi) \in \psi_{0} \).

Using (13.2) we may write (13.5) as

\[(13.6) \quad a_{t}(x_{1}, D) = D_{1} + f_{+}(x_{1}, D)P_{+} + f_{-}(x_{1}, D)P_{-} \mod \psi_{-1}
\]

Here we get

\[(13.7) \quad \kappa_{c}(D_{1}) = D_{1} + [P_{+}, D_{1}]P_{+} + [P_{-}, D_{1}]P_{-} = D_{1} + [(P_{+} - p_{+}(D)), D_{1}] + [(P_{-} - p_{-}(D)), D_{1}]
\]

since \( p_{\pm}(D) \) commute with \( D_{1} \). Clearly the last two terms in (13.7) belong to \( O\phi q_{-1} \), since the differences \( P_{\pm} - p_{\pm}(D) \) are \( O\phi q_{-1} \) while \( D_{1} \in \psi q_{1} \) and because the commutator with the scalar \( D_{1} \) still has order of the sum of orders decreased by 1. Thus we get \( \kappa_{c}(D_{1}) - D_{1} \in O\phi q_{-1} \).

Next,

\[(13.8) \quad \kappa_{c}(f_{+}(x_{1}, D)P_{+}) = f_{+}(x_{1}, D)P_{+} + [P_{+}, f_{+}(x_{1}, D)]P_{+}
\]

and similarly for \( f_{-}(x_{1}, D)P_{-} \), where again the commutators of \( P_{\pm} \) with the scalar operators \( f_{\pm}(x_{1}, D) \) are of order -1.

As a consequence we get

\[(13.9) \quad \kappa_{c}(a_{t}(x, D)) = a_{t}(x, D) \mod \psi_{-1}
\]

With the above we repeat the result of thm 10.4:

**Theorem 13.3.** Set \( \theta(\xi) = \frac{1}{2}(1 - s_{1}(\xi)) \), evaluate (above) \( \gamma_{t}(\xi) = te^{-i\omega\theta(\xi)}\epsilon(\omega\theta(\xi)t) \), with \( \epsilon(\kappa) = \frac{1}{\kappa} \). For any observable \( R \) write \( R_{t} = U^{*}(t)RU(t) \), with the propagator \( U(t) \) of the Dirac equation \( \psi + iH(t)\psi = 0 \), with the Dirac operator \( H(t) \) of (8.1), marking a Dirac particle under the influence of a plane polarized electro-magnetic wave in the \( x_{1} \)-direction.

Then we have \( (H(t))_{t} - H(0) = (D_{1})_{t} - D_{1} \) where \( (D_{1} + r_{t}(x, D))_{t} \), with a suitable \( \psi do \) \( r_{t}(x, D) \in O\phi q_{-1} \) is a \( \psi do \) in \( O\phi q_{1} \) satisfying

\[
(D_{1} + r_{t}(x, D))_{t} - D_{1} = \epsilon_{0}\omega\epsilon(\omega(x_{1} - t\theta(D)))s_{1}(D)\epsilon(\omega\theta(D)t)p_{+}(D)
\]

\[
-\epsilon_{0}\omega\epsilon(\omega(x_{1} - t\theta(-D)))s_{2}(D)\epsilon(\omega\theta(-D)t)p_{-}(D)
\]

(13.10)
a relation valid modulo $O(p_q - 1)$.

We might point again to observation 10.6, above: For our conjecture that the two terms at right of (13.10) mark the possibility of a collision between the Dirac particle and a ‘photon’, we can offer only two reasons: (i) the fact that — in the momentum representation — these terms mark a shift of energy by $\pm \hbar \nu$ and of momentum by $\pm \hbar \nu/c$, while multiple collisions will shift by discrete integer multiples of that; (ii) that a directional shift of propagation speed will enter, similar in nature as that observed by Compton for the shift of wavelength.

Perhaps others might see more details, in these matters.

References

[Be1] R. Becker, Theorie der Electrizitaet; Bd.2, B.G. Teubner Verlag, Leibzig 1949.
[BLT] N. N. Bogoliubov, A. A. Logunov and I. T. Todorov, Introduction to Axiomatic Quantum Field Theory, Benjamin, Reading, Massachusetts, 1975.
[Bu1] V.S. Buslaev, The generating integral and the canonical Maslov operator in the WKB-method; Funct. anal. i ego pril., 3:3 (1969), 17-31. English translation: Funct. Anal. Appl., 3 (1969), 181-193.
[CZ] A.P. Calderon and A. Zygmund, Singular integral operators and differential equations; Amer. J. Math. 79 (1957) 901-921.
[Co1] H.O. Cordes, On pseudodifferential operators and smoothness of special Lie group representations; Manuscripta Math. 28 (1979) 51-69.
[Co2] H.O. Cordes, Elliptic pseudo-differential operators - an abstract theory; Springer Lecture Notes Math. Vol. 756, Springer Berlin Heidelberg New York 1979
[Co3] H.O. Cordes, Remarks about observables for the quantum mechanical harmonic oscillator; Operator Theory, Adv., Appl., 191 305-321 2009.
[Co4] H.O. Cordes, Spectral theory of linear differential operators and comparison algebras; London Math. Soc. Lecture Notes No.76 (1987); Cambridge Univ. Press; Cambridge.
\[\text{Siemron, Programm Luisenschule, Berlin (1890) [Jahrbuch ueber die Fortschritte der Math. (1890) 840-842.]}\]

\[\text{So1} \quad \text{A.Sommerfeld, Atombau und Spektrallinien, vol.1. 5th ed. Braunschweig, Viehweg and Sons, 1931.}\]

\[\text{So2} \quad \text{A.Sommerfeld, Atombau und Spektrallinien, Vol.2. Braunschweig Vieweg and Sons, 1931.}\]

\[\text{Sn1} \quad \text{Sonine, Math. Ann. 16 (1880) 38f.}\]

\[\text{St1} \quad \text{Struve, Mem. de l’Acad.Imp.des Sci. de St Peterburg (7) 30 (1882) no. 8; Ann. der Physik, (8) 17 (1882) 1008-1016.}\]

\[\text{Ta1} \quad \text{M. Taylor, Pseudodifferential operators; Princeton Univ. Press., Princeton, NJ 1981.}\]

\[\text{Ta2} \quad \text{M.Taylor, Partial differential equations; Vol.I,II,III; Springer New York Berlin Heidelberg 1991.}\]

\[\text{Th1} \quad \text{B.Thaller, The Dirac equation; Springer 1992 Berlin Heidelberg New York.}\]

\[\text{Ti1} \quad \text{E.C.Titchmarsh, Eigenfunction expansions associated with second order differential equations Part 1, 2-nd ed.; Clarendon Press, Oxford 1962.}\]

\[\text{Ti2} \quad \text{E.C.Titchmarsh, Eigenfunction expansions associated with second order differential equations Part 2 [PDE]; Oxford Univ. Press 1958.}\]

\[\text{Tr1} \quad \text{F. Treves, Introduction to pseudodifferential and Fourier integral operators, Vol’s I,II; Plenum Press, New York London 1980.}\]

\[\text{Un1} \quad \text{A. Unterberger, A calculus of observables on a Dirac particle, Annales Inst. Henri Poincaré (Phys. Théor.), 69 (1998) 189-239.}\]

\[\text{Un2} \quad \text{A. Unterberger, Quantization, symmetries and relativity; Contemporary Math. 214, AMS (1998), 169-187.}\]

\[\text{Wa1} \quad \text{G.N.Watson, A Treatise on the Theory of Bessel Functions; Cambridge Univ. Press, 1922.}\]

\[\text{We1} \quad \text{A.Weinstein, A symbol class for some Schrödinger equations on \(\mathbb{R}^n\); Amer. J. Math. (1985) 1-21.}\]

\[\text{Wk1} \quad \text{J.Walker, The Analytical Theory of Light, Cambridge 1904 392-395.}\]

\[\text{W} \quad \text{E. Wichmann. Quantenphysik. Braunschweig: Viehweg und Sohn, 1985.}\]

\[\text{YM} \quad \text{C.N.Yang and R.L.Mills, Conservation of isotopic spin and isotopic gauge invariance; Phys.Rev. 96 (1954) 191-195.}\]

Emeritus Professor
Department of Mathematics
University of California
Berkeley, CA 94720, U.S.A.
E-mail: cordes@math.berkeley.edu

(Received: 4 May, 2014)