PRYM-TYURIN VARIETIES VIA HECKE ALGEBRAS

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ABSTRACT. Let $G$ denote a finite group and $\pi : Z \to Y$ a Galois covering of smooth projective curves with Galois group $G$. For every subgroup $H$ of $G$ there is a canonical action of the corresponding Hecke algebra $\mathbb{Q}[H \backslash G/H]$ on the Jacobian of the curve $X = Z/H$. To each rational irreducible representation $W$ of $G$ we associate an idempotent in the Hecke algebra, which induces a correspondence of the curve $X$ and thus an abelian subvariety $P$ of the Jacobian $JX$. We give sufficient conditions on $W$, $H$, and the action of $G$ on $Z$ for $P$ to be a Prym-Tyurin variety. We obtain many new families of Prym-Tyurin varieties of arbitrary exponent in this way.

1. Introduction

A Prym-Tyurin variety of exponent $q$ is by definition a principally polarized abelian variety $(P, \Xi)$, for which there exists a smooth projective curve $C$ and an embedding $P \hookrightarrow JC$ into the Jacobian of $C$ such that the restriction of the canonical polarization $\Theta$ of $JC$ is the $q$-fold of $\Xi$:

$$\Theta|_P = q\Xi.$$ 

One point of interest in these varieties is that the structure of a Prym-Tyurin variety allows to study the geometric properties of the underlying abelian varieties via curve theory.

According to the Theorem of Matsusaka-Ran [BL, 11.8.1] Prym-Tyurin varieties of exponent 1 are exactly the canonically polarized products of Jacobians. Welters showed in [W] that Prym-Tyurin varieties of exponent 2 are exactly the classical Prym varieties or some of their specializations. The Abel-Prym-Tyurin map of $(P, \Xi)$ is the following composition of maps

$$\alpha_P : C \xrightarrow{\alpha_C} JC = \widehat{JC} \to \widehat{P} = P,$$

where $\alpha_C$ denotes the usual Abel map and $\widehat{A}$ the dual of an abelian variety $A$. The property of being a Prym-Tyurin variety can then be expressed by certain properties of the curve $\alpha_P(C)$ in $P$. One can deduce a criterion (see [BL Welters’ criterion 12.2.2]) for $(P, \Xi)$ to be a Prym-Tyurin variety in term of a curve in $P$. Using this it is easy to see, using complete intersection curves on the Kummer variety of $P$, that every principally polarized abelian variety of dimension $g$ is a Prym-Tyurin variety of exponent $2^{g-1}(g-1)!$.

The problem is to construct Prym-Tyurin varieties of small exponent $\geq 3$. There are in fact not many examples. A series of examples was given by Kanev in [K2] using Weyl groups of type $A_n, D_n, E_6$ and $E_7$. Generalizing the construction of [LRR], Salomon gave

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in examples of Prym-Tyurin varieties of arbitrary exponent using some special graph constructions. There are a few other examples which shall not be mentioned here.

The construction of these examples are based on the following two facts: (i) The use of correspondences: The set of abelian subvarieties of a polarized abelian variety is in canonical bijection to the set of symmetric (with respect to the Rosati involution) idempotents of its endomorphism algebra. On the other hand, it follows almost from the definitions that the obvious map

$$\text{Div}_\mathbb{Q}(C \times C) \longrightarrow \text{End}_\mathbb{Q}(JC)$$

from the algebra of rational correspondences of $C$ to the endomorphism algebra of its Jacobian is surjective. Hence one can describe an abelian subvariety of $JC$ by correspondences. (ii) Kanev’s Criterion: In [K1] Kanev showed that an abelian subvariety of $JC$ is a Prym-Tyurin variety of exponent $q$ if it is given by an integral effective fixed-point free correspondence on $C$ whose associated endomorphism of $JC$ satisfies the equation

$$(1.1) \quad x^2 + (q - 2)x - (q - 1) = 0.$$ 

This is a direct generalization of Wirtinger-Mumford’s construction of classical Prym varieties and, in fact, most Prym-Tyurin varieties are given by constructing integral symmetric fixed point free correspondences satisfying (1.1). There is an analogous result for correspondences with some fixed points, due to Ortega [O], which also can be applied.

Our construction of such correspondences is, roughly speaking, as follows: Let $G$ denote a finite group and $\pi : Z \rightarrow Y$ be a Galois covering of smooth projective curves with Galois group $G$. The group action induces a homomorphism of the group algebra $\mathbb{Q}[G]$ into the endomorphism algebra $\text{End}_\mathbb{Q}(JZ)$. We identify the elements of $\mathbb{Q}[G]$ with their images in $\text{End}_\mathbb{Q}(JZ)$. Hence any $\alpha \in \mathbb{Q}[G]$ defines an abelian subvariety $\text{Im}(\alpha)$ of $JZ$. For the details we refer to Section 3.1. On the other hand, every element $\alpha = \sum_{g \in G} \alpha_g g \in \mathbb{Q}[G]$ defines a rational correspondence on $Z$, namely

$$D_\alpha = \sum_{g \in G} \alpha_g \Gamma_g \in \text{Div}_\mathbb{Q}(Z \times Z).$$

where $\Gamma_g$ denotes the graph of the automorphism $g$ of $Z$. The problem is to find integral symmetric fixed-point free correspondences satisfying (1.1) in this way. For this we proceed as follows:

For simplicity we first describe a special case. Let $V$ be an irreducible complex representation of $G$ and $W$ the associated rational irreducible representation. For any subgroup $H$ of $G$, the covering $\pi$ factorizes via a covering $\varphi_H : X \rightarrow Y$ where $X$ denotes the quotient $Z/H$. The Hecke algebra $\mathbb{Q}[H \backslash G/H]$ is a subalgebra of $\mathbb{Q}[G]$ and the above homomorphism induces a homomorphism

$$\mathbb{Q}[H \backslash G/H] \longrightarrow \text{End}_\mathbb{Q}(JX).$$

which factorizes via $\text{Div}_\mathbb{Q}(X \times X)$. Then we associate to the pair $(H, W)$ an integral symmetric correspondence $\mathcal{D}_{H, W}$ on $X$ in a canonical way. It is defined via the projectors of $\mathbb{Q}[G]$ given in [CaRo]. Our essential assumption is

$$(1.2) \quad H \text{ is a subgroup satisfying } \dim V^H = 1 \text{ and maximal with this property,}$$

where $V^H$ denotes the fixed subspace of $V$ under the action of $H$.\["""
In fact, under this assumption we can show (Proposition 3.9) that a modified version $K_X$ of $D_{H,W}$ is an effective integral symmetric correspondence on $X$, which in the special case $Y = \mathbb{P}^1$ satisfies equation (1.1). Moreover we can compute its fixed points in terms of the ramification of $\pi : Z \to \mathbb{P}^1$.

In order to formulate the result, we use the following notation: We denote by $K_V$ the field generated by the values of the character of $V$. Moreover, if $C$ denotes a conjugacy class of cyclic subgroups of the group $G$, a branch point $y \in Y$ of the covering $\pi$ is called of type $C$, if the stabilizer of any point $z$ in the fibre $\pi^{-1}(y)$ is a subgroup in the class $C$. We define the geometric signature of the covering $\pi : Z \to Y$ to be the tuple $[g; (C_1, m_1), \ldots, (C_t, m_t)]$, where $g$ is the genus of the quotient curve $Y$, the covering $\pi$ has a total of $\sum_{j=1}^{t} m_j$ branch points, and exactly $m_j$ of them are of type $C_j$ for $j = 1, \ldots, t$ (see [R]). Using this we prove the following

**Theorem.** Let $W$ denote a nontrivial rational irreducible representation of $G$, with associated complex irreducible representation $V$, and $H$ a subgroup of $G$ satisfying (1.2). Suppose that the action of $G$ has geometric signature $[0; (C_1, m_1), \ldots, (C_t, m_t)]$ satisfying

\[
\sum_{j=1}^{t} m_j \left(q [K_V : \mathbb{Q}] (\dim V - \dim V^{G_j}) - ([G : H] - |H \backslash G/G_j|)\right) = 0,
\]

for an integer $q$ given in terms of the algebraic data, where $G_j$ is of class $C_j$. Then the correspondence $K_X$ defines a Prym-Tyurin variety of exponent $q$.

Using the theorem one obtains Kanev’s examples and several more, some of which are included in Sections 5.1 and 5.2. However we could not find examples that exhibited the full force of the theorem; we were interested in finding Prym-Tyurin varieties of small exponent bigger than 2, using complex representations whose field of definition properly contains $\mathbb{Q}$, as these would provide entirely new examples, different from the existing ones. In fact, we wrote a computer program which for many groups $G$ and all of their subgroups and irreducible complex representations such that $\mathbb{Q} \subseteq K_V$ verifies the hypothesis of the theorem. In this way we found many examples of exponents 1 and 2, which however are not interesting from our point of view.

The new idea was to start, instead of one irreducible representation, with several representations satisfying some additional conditions (see Hypothesis 3.7) and generalize the above result to Theorem 4.9. This gave in fact many examples, most of them new, and which also include the above mentioned examples of Salomon. For details see Section 5.3 and the forthcoming paper [CLRR], in which more examples are given and these Prym-Tyurin varieties will be investigated in detail.

The contents of the paper are as follows. Section 2 contains some algebraic preliminaries. The essential result is Proposition 2.3 which describes the coefficients of a certain idempotent of the group algebra in term of the character of $V$. In Section 3 we define the correspondence $K_X$ and derive its properties. Section 4 contains the proof of our main result Theorem 4.9. We also compare our construction with the construction of [K2]. Finally in Section 5 we give some examples.
We suppose throughout that the curves are defined over any algebraically closed field of characteristic 0. Moreover the curves always will be smooth and projective.

2. Algebraic preliminaries

2.1. The group algebra. Let $G$ be a finite group. In order to fix the notation, we start by recalling some basic properties of representations of $G$ (see [CR]). For any field $K$ of characteristic 0 we denote by $K[G]$ the group algebra of $G$ over $K$. It is a semisimple algebra, whose simple components correspond one-to-one to the irreducible $K$-representations of $G$. We may identify the elements of $K[G]$ with the $K$-valued functions on $G$. From this point of view, the multiplication in $K[G]$ is convolution

$$(f_1f_2)(g) = \sum_{g_1g_2 = g} f_1(g_1)f_2(g_2)$$

for $f_1, f_2 \in K[G]$ and $g \in G$. In this paper the field $K$ will be either $\mathbb{C}$ or $\mathbb{Q}$.

For any complex irreducible representation $V$ of $G$, we denote by $\chi_V$ its character, by $L_V$ its field of definition and by $K_V$ the subfield $K_V = \mathbb{Q}(\chi_V(g) \mid g \in G)$. $L_V$ and $K_V$ are finite abelian extensions of $\mathbb{Q}$. We denote by

$$m_V = [L_V : K_V]$$

the Schur index of $V$. For any automorphism $\varphi$ of $L_V/\mathbb{Q}$ we denote by $V^{\varphi}$ the representation conjugate to $V$ by $\varphi$.

If $\mathcal{W}$ is a rational irreducible representation of $G$, then there exists a complex irreducible representation $V$ of $G$, uniquely determined up to conjugacy in $\text{Gal}(L_V/\mathbb{Q})$, such that

$$\mathcal{W} \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\varphi \in \text{Gal}(L_V/\mathbb{Q})} V^{\varphi} \cong m_V \bigoplus_{\tau \in \text{Gal}(K_V/\mathbb{Q})} V^\tau$$

We call $V$ the complex irreducible representation associated to $\mathcal{W}$.

The central idempotent $e_V$ of $\mathbb{C}[G]$ that generates the simple subalgebra of $\mathbb{C}[G]$ corresponding to $V$ and the central idempotent $e_{\mathcal{W}}$ of $\mathbb{Q}[G]$ that generates the simple subalgebra of $\mathbb{Q}[G]$ corresponding to $\mathcal{W}$ are given by

$$(2.1) \quad e_V = \frac{\dim V}{|G|} \sum_{g \in G} \chi_V(g^{-1})g \quad \text{and} \quad e_{\mathcal{W}} = \frac{\dim V}{|G|} \sum_{g \in G} \text{tr}_{K_V/\mathbb{Q}}(\chi_V(g^{-1}))g.$$

Lemma 2.1. If $e$ is an idempotent of $K[G]$, the map

$$\text{End}_{K[G]}(K[G]e) \to eK[G]e, \quad \varphi \mapsto \varphi(e)$$

is an anti-isomorphism of $K$-algebras.

Proof. Note first that $\varphi(e) = \varphi(e \cdot e) = e \cdot \varphi(e) \in eK[G]e$. For the proof that the map is “anti”, suppose that $\varphi_i(e) = f_i e$ for $i = 1, 2$. Then

$$\varphi_1\varphi_2(e) = \varphi_1(\varphi_2(e)) = \varphi_1(f_2e) = f_2\varphi_1(e) = f_2e f_1 e = \varphi_2(e)\varphi_1(e).$$

Finally, for any $efe \in eK[G]e$, the map $\varphi \in \text{End}_{K[G]}(K[G]e)$, defined by $\varphi(f'e) = f'efe$ for all $f' \in K[G]$, maps to $efe$. Hence the map is bijective. \qed
2.2. The Hecke algebra of a subgroup. Let $H$ be a subgroup of $G$. The element

$$p_H = \frac{1}{|H|} \sum_{h \in H} h$$

is the central idempotent of $K[H]$ corresponding to the trivial representation of $H$. Moreover, the left ideal $K[G]p_H$ defines the $K$-representation of $G$ induced by the trivial representation of $H$ for any field $K$ as above. In the sequel we denote this representation by $\rho_H$.

The $K$-algebra $p_H K[G]p_H$, considered as a subalgebra of $K[G]$, consists of the $K$-valued functions on $G$ which are constant on each double coset $HgH$ of $H$ in $G$. It is called the Hecke algebra over $K$ of $H$ in $G$, and it is usually denoted by $K[H \backslash G / H]$.

The idempotent $e_V$ of (2.1) is central in $\mathbb{C}[G]$. This implies that the element

$$f_{H,V} = p_H e_V = e_V p_H$$

is an idempotent of $\mathbb{C}[G]$ (or zero) which satisfies

- $hf_{H,V} = f_{H,V} = f_{H,V}h$ for all $h \in H$,
- the left ideal $\mathbb{C}[G]f_{H,V}$ defines the representation $V$ with multiplicity $\dim V^H$.

The last property follows from the fact that $\rho_H \simeq \mathbb{C}[G]p_H$ and $V$ occurs in $\rho_H$ with multiplicity $\dim V^H$.

Similarly, since the idempotent $e_W$ of (2.1) is central in $\mathbb{Q}[G]$, the element

$$f_{H,W} := p_H e_W = e_W p_H$$

is an idempotent of $\mathbb{Q}[G]$ (or zero) which satisfies

- $hf_{H,W} = f_{H,W} = f_{H,W}h$ for all $h \in H$,
- the left ideal $\mathbb{Q}[G]f_{H,W}$ defines the representation $W$ with multiplicity $\frac{\dim V^H}{m_V}$.

The last property and the fact that $\frac{\dim V^H}{m_V}$ is an integer follow from the equation $\dim V^H = \langle \rho_H, V \rangle$ and the fact that $\rho_H$ is a rational representation. For a complete proof see [CaRo, Theorem 4.4].

According to [H, Theorem 5.1.7] there exists a set of representatives

$$\{g_{ij} \in G \mid i = 1, \ldots, d \text{ and } j = 1, \ldots, n_i\}$$

for both the left cosets and right cosets of $H$ in $G$, such that

$$G = \bigsqcup_{i=1}^d H g_{i1} H \quad \text{and} \quad H g_{i1} H = \bigsqcup_{j=1}^{n_i} g_{ij} H = \bigsqcup_{j=1}^{n_i} H g_{ij}$$

are the decompositions of $G$ into double cosets, and of the double cosets into right and left cosets of $H$ in $G$. Moreover, we assume $g_{11} = 1_G$.

Clearly $\chi_V(g^{-1}) = \chi_V(g)$ is an algebraic integer for all $g \in G$ ([CR Corollary 30.11]). This implies that $\text{tr}_{K_V / Q}(\chi_V(g)) = \text{tr}_{K_V / Q}(\chi_V(g^{-1}))$ is an integer for every $g \in G$. We conclude

$$\sum_{h \in H} \text{tr}_{K_V / Q}(\chi_V(h g_{ij}^{-1})) = \sum_{h \in H} \text{tr}_{K_V / Q}(\chi_V(g_{ij} h)),$$

and use this to prove the following lemma.
Lemma 2.3. For any $1 \leq i \leq d$ and $1 \leq j \leq n_i$ we have

\begin{equation}
(2.6) \quad a_i := \sum_{h \in H} \text{tr}_{K_V/Q}(\chi_V(hg_{ij}^{-1})) = \sum_{h \in H} \text{tr}_{K_V/Q}(\chi_V(hg_{i1}^{-1})) \in \mathbb{Z}.
\end{equation}

Proof. Since $g_{ij}$ and $g_{i1}$ are in the same double coset of $H$ in $G$, there exist $k, k' \in H$ such that $g_{ij} = kg_{i1}k'$. Hence for all $h \in H$ we have $g_{ij}h = kg_{i1}(k'hk)k^{-1}$, which implies

\[ \chi_V(g_{ij}h) = \chi_V(g_{i1}(k'hk)) \]

for all $h \in H$. Now the map $g_{ij}H \rightarrow g_{i1}H$, $g_{ij}h \mapsto g_{i1}k'hk$ is a bijection, which gives

\begin{equation}
(2.7) \quad \sum_{h \in H} \chi_V(g_{ij}h) = \sum_{h \in H} \chi_V(g_{i1}h).
\end{equation}

Now applying $\text{tr}_{K_V/Q}$ and (2.5) twice gives the assertion. \qed

Now consider the following element of $\left[\frac{1}{|H|}\mathbb{Z}[G]\right]$,

\[ f_{H,W} := \sum_{i=1}^{d} a_i \sum_{j=1}^{n_i} g_{ij}p_H. \]

Lemma 2.3. (a): $f_{H,W} = \frac{\dim V}{|G|} F_{H,W}$;
(b): $F_{H,W} = p_H \sum_{i=1}^{d} a_i \sum_{j=1}^{n_i} g_{ij}$;
(c): $f_{H,W}$ and $F_{H,W}$ are elements of the Hecke algebra $\mathbb{Q}[H\backslash G/H]$.

Proof. (a): Using Lemma 2.2 we have,

\[ f_{H,W} = e_W p_H \]

\[ = \frac{\dim V}{|G||H|} \sum_{i=1}^{d} \sum_{j=1}^{n_i} \left( \sum_{k \in H} \text{tr}_{K_V/Q}(\chi_V((g_{ij}k)^{-1})) \right) g_{ij}k \left( \sum_{h \in H} h \right) \]

\[ = \frac{\dim V}{|G||H|} \sum_{i=1}^{d} \sum_{j=1}^{n_i} \left( \sum_{k \in H} \text{tr}_{K_V/Q}(\chi_V(kg_{ij}^{-1})) \right) g_{ij} \left( \sum_{h \in H} h \right) \]

\[ = \frac{\dim V}{|G||H|} \sum_{i=1}^{d} \sum_{j=1}^{n_i} \left( \sum_{k \in H} \text{tr}_{K_V/Q}(\chi_V(kg_{i1}^{-1})) \right) g_{ij} \left( \sum_{h \in H} h \right) = \frac{\dim V}{|G|} F_{H,W}. \]

For the proof of (b) we start with $f_{H,W} = p_H e_W$ and proceed as above. Finally, according to Lemma 2.1 and (2.5) the coefficients of $f_{H,W}$ are the same on double cosets, which gives (c). \qed

2.3. The integers $a_i$. The following lemma is a generalization of [McD] 91.60 p.391], where it is proved under the additional assumption that $(G, H)$ is a Gelfand pair. Recall that $(G, H)$ is called a Gelfand pair, if $\dim V^H = 0$ or 1 for every complex irreducible representation $V$ of $G$. This is equivalent to the fact that the Hecke algebra $\mathbb{C}[H\backslash G/H]$ is commutative.

Let $V$ be a complex irreducible representation of $G$ and let $(\cdot, \cdot)$ denote any $G$-invariant hermitian scalar product on $V$ (unique up to a positive constant).
Lemma 2.1 gives

\[(v, gv) = \frac{(v, v)}{|H|} \sum_{h \in H} \chi_V(h g^{-1})\]

for all \(g \in G\).

**Proof.** It follows from Schur’s character relations (see e.g. [BL, Proposition 13.6.4]) that

\[l_v = \frac{\dim V}{|G|(v, v)} \sum_{g \in G} (v, gv)g\]

is a primitive idempotent in \(\mathbb{C}[G]\) such that \(\mathbb{C}[G]l_v\) affords the representation \(V\).

We claim that \(l_v \in \mathbb{C}[H/G/H]\). In fact, \(hl_v = \frac{\dim V}{|G|(v, v)} \sum_{g \in G} (hv, hvg)h g = l_v\), and similarly \(h_l h = l_v\) for all \(h \in H\). Therefore \(l_v\) is constant on double cosets of \(H\) in \(G\), which gives the assertion.

Now \(l_v \in \mathbb{C}[G]e_V\) implies \(l_v = p_H l_v p_H \in p_H \mathbb{C}[G]e_V p_H = f_{H,V} \mathbb{C}[G]f_{H,V}\). On the other hand, Lemma 2.1 gives

\[f_{H,V} \mathbb{C}[G]f_{H,V} \simeq \text{End}_{\mathbb{C}[G]}(\mathbb{C}[G]f_{H,V}),\]

and, as we saw after (2.3), the left ideal \(\mathbb{C}[G]f_{H,V}\) provides the representation \(V\) with multiplicity \(\dim V^H = 1\). Hence Schur’s lemma implies that \(f_{H,V} \mathbb{C}[G]f_{H,V}\) is a one-dimensional complex vector space. Therefore \(l_v\) and \(f_{H,V}\) differ at most by a constant. So in order to complete the proof, it suffices to compare the coefficients of \(1 \in G\) in \(l_v\) and \(f_{H,V}\).

Now

\[f_{H,V} = \frac{\dim V}{|G||H|} \sum_{g \in G} \sum_{h \in H} \chi_V(g^{-1})gh = \frac{\dim V}{|G||H|} \sum_{g \in G} \left( \sum_{h \in H} \chi_V(hg^{-1}) \right) g\]

Hence the coefficient of \(1 \in G\) in \(f_{H,V}\) is

\[\frac{\dim V}{|G|} \left( \frac{1}{|H|} \sum_{h \in H} \chi_V(h) \right) = \frac{\dim V}{|G|} \langle V|H, 1_H \rangle_H = \langle V, \rho_H \rangle_G = \frac{\dim V}{|G|} \dim V^H = \frac{\dim V}{|G|}\]

which coincides with the coefficient of \(1 \in G\) of \(l_v\). Hence \(f_{H,V} = l_v\). Comparing the coefficients in this equation completes the proof of the proposition. \(\square\)

As a consequence we obtain the following inequality for the integers \(a_i\) of (2.6).

**Proposition 2.5.** Let the representatives \(g_{ij}\) of the cosets of \(H\) in \(G\) be as above and assume \(\dim V^H = 1\). Then for every \(1 \leq i \leq d\) the integer \(a_i\) satisfies

\[a_i \leq a_1 = [K_V : \mathbb{Q}]|H|\]

with equality if and only if \(g_{i1}\) fixes \(V^H\).

**Proof.** First consider \(a_1\). Since we assumed \(g_{11} = 1_G\), we have

\[a_1 = \sum_{h \in H} tr_{K_V/\mathbb{Q}}(\chi_V(h)) = \sum_{h \in H} \sum_{\psi \in \text{Gal}(K_V/\mathbb{Q})} \varphi(\chi_V(h)) = \sum_{\varphi \in \text{Gal}(K_V/\mathbb{Q})} \varphi(\sum_{h \in H} \chi_V(h))\]

But \(\sum_{h \in H} \chi_V(h) = |H| \dim V^H = |H|\) and hence \(a_1 = [K_V : \mathbb{Q}]|H|\).
For the proof of the inequality note that the Schur index \( m_V \) is a divisor of \( \dim V^H = 1 \), which implies \( L_V = K_V \). Hence the representation \( V \) is defined over \( K_V \), i.e. there is a \( G \)-representation \( V_{K_V} \) over \( K_V \) such that \( V = V_{K_V} \otimes \mathbb{C} \).

Consider the \( K_V \)-vector space

\[
\mathcal{V} = \bigoplus_{\varphi \in \text{Gal}(K_V/\mathbb{Q})} V_{K_V}^\varphi,
\]

where \( V_{K_V}^\varphi \) is the representation conjugate to \( V_{K_V} \) by \( \varphi \). We denote the element of \( V_{K_V}^\varphi \) corresponding to \( v \in V_{K_V} \) by \( v^\varphi \) and observe that

\[
(\alpha v)^\varphi = \varphi(\alpha)v^\varphi
\]

for all \( \alpha \in K_V \). The group \( \text{Gal}(K_V/\mathbb{Q}) \) acts on \( \mathcal{V} \) by permuting the coordinates, and, if \( \text{Gal}(K_V/\mathbb{Q}) = \{ \varphi_1 = 1, \varphi_2, \ldots, \varphi_{[K_V:\mathbb{Q}]} \} \), then its fixed set is the subset of \( \mathcal{V} \) given by

\[
\mathcal{W} = \{(v, v^{\varphi_2}, \ldots, v^{\varphi_{[K_V:\mathbb{Q}]}}) \mid v \in V_{K_V} \}.
\]

This is a rational subvector space of \( \mathcal{V} \) of dimension \( \dim V \cdot [K_V : \mathbb{Q}] \), which affords the rational irreducible representation defined by \( V \). In order to see this, choose a basis \( \{v_1, \ldots, v_s\} \) of \( V_{K_V} \) and a basis \( \{\eta_1, \ldots, \eta_s\} \) for \( K_V/\mathbb{Q} \). Then \( \mathcal{W} \) is the rational vector space with basis

\[
w_{i,k} = (\eta_k v_i, \ldots, (\eta_k v_i)^{\varphi_s}) = \varphi_s(\eta_k) v_i^{\varphi_s}.
\]

Furthermore, it is clear that \( \mathcal{V} \simeq \mathcal{W} \otimes_{\mathbb{Q}} K_V \).

Certainly, for any nonzero vector \( v \in V_K \) we may choose a \( G \)-invariant hermitian scalar product \( (\cdot, \cdot) \) on \( V_{K_V} \) such that \((v, v)\) is a (positive) rational number. Then clearly \( S : \mathcal{W} \times \mathcal{W} \to \mathbb{Q} \), defined by

\[
S(w_1, w_2) = \text{tr}_{K_V/\mathbb{Q}}(v_1, v_2)
\]

for any \( w_i = (v_i, v_i^{\varphi_2}, \ldots, v_i^{\varphi_{[K_V:\mathbb{Q}]}}), \) is a \( G \)-invariant scalar product on the \( \mathbb{Q} \)-vector space \( \mathcal{W} \). In particular, choosing \((\cdot, \cdot)\) for a fixed nonzero \( v \in V_K^H \), and letting \( w = (v, v^{\varphi_2}, \ldots, \varphi_{[K_V:\mathbb{Q}]}) \in \mathcal{W} \), we have by Proposition 2.4

\[
S(w, g_i w) = \text{tr}_{K_V/\mathbb{Q}}(v, g_i v) = \text{tr}_{K_V/\mathbb{Q}}(v, v) \sum_{h \in H} \chi_V(h g_i^{-1}) \frac{(v, v)}{|H|} a_i
\]

for \( i = 1, \ldots, d \). Since \( S \) is positive definite, symmetric and \( G \)-invariant, we conclude

\[
0 \leq S(g_i w - w, g_i w - w) = 2(S(w, w) - S(w, g_i w)) = \frac{2(v, v)}{|H|} (a_i - a_i)
\]

with equality if and only if \( g_i w = w \), which completes the proof of the proposition. \( \square \)

3. The correspondences

3.1. The set up. Let \( Z \) be a smooth projective curve such that the group \( G \) acts on \( Z \). Let \( JZ \) denote the Jacobian of \( Z \) and \( \text{End}_\mathbb{Q}(JZ) = \text{End}(JZ) \otimes_\mathbb{Z} \mathbb{Q} \) its endomorphism algebra. The group action induces an algebra homomorphism

\[
\mathbb{Q}[G] \to \text{End}_\mathbb{Q}(JZ).
\]
Since this homomorphism is canonical, we will denote the elements of $\mathbb{Q}[G]$ and their images by the same letter. In particular we consider elements of $\mathbb{Z}[G]$ as endomorphisms of $JZ$. For any $\alpha \in \mathbb{Q}[G]$ we define

$$\text{Im}(\alpha) := \text{Im}(m\alpha) \subset JZ,$$

where $m$ is a positive integer such that $m\alpha \in \text{End}(JZ)$. It is an abelian subvariety of $JZ$ which certainly does not depend on the chosen integer $m$.

Now consider a subgroup $H$ of $G$. If we denote the quotients of $Z$ by $H$ and $G$ by $X = Z/H$ and $Y = Z/G$ respectively, we have the following diagram

$$(3.1)$$

Since $\mathbb{Q}[H \backslash G/H] = p_H \mathbb{Q}[G]p_H$, the homomorphism $\mathbb{Q}[G] \rightarrow \text{End}_{\mathbb{Q}}(JZ)$ restricts to an algebra homomorphism

$$\mathbb{Q}[H \backslash G/H] \rightarrow \text{End}_{\mathbb{Q}}(p_H(JZ)).$$

Now the pull-back map $\pi_H^* : JX \rightarrow p_H(JZ)$ and the restriction of the norm map $\text{Nm}_{\pi_H} : p_H(JZ) \rightarrow JX$ are isogenies satisfying $\text{Nm}_{\pi_H} \circ \pi_H^* = |H|1_{JX}$. This implies that the composition

$$(3.2)$$

$$\varepsilon : \mathbb{Q}[H \backslash G/H] \rightarrow \text{End}_{\mathbb{Q}}(JX), \quad \varphi \mapsto \frac{1}{|H|} \text{Nm}_{\pi_H} \circ \varphi \circ \pi_H^*$$

is a homomorphism of $\mathbb{Q}$-algebras.

3.2. The Hecke ring $\frac{1}{|H|}\mathbb{Z}[H \backslash G/H]$. Let $H_1 = H, H_2, \ldots, H_d$ denote the double cosets of $H$ in $G$. Consider for $i = 1, \ldots, d$ the elements of $\mathbb{Q}[H \backslash G/H]$ defined by

$$F_i := \frac{1}{|H|} \sum_{g \in H_i} g = \sum_{j=1}^{n_i} g_{ij} p_H = p_H \sum_{j=1}^{n_i} g_{ij}$$

with $p_H$ as in (2.2) and representatives $g_{ij}$ as chosen in subsection 2.2. We define

$$\frac{1}{|H|}\mathbb{Z}[H \backslash G/H]$$

as the free $\mathbb{Z}$-module with basis $F_1, \ldots, F_d$. So any element $F \in \frac{1}{|H|}\mathbb{Z}[H \backslash G/H]$ can be written as

$$F = \sum_{i=1}^{d} \alpha_i F_i = \sum_{i=1}^{d} \sum_{j=1}^{d_i} \alpha_i g_{ij} p_H = p_H \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{d_i} g_{ij}$$

with uniquely determined integer constants $\alpha_i$. The following lemma justifies the notation.

**Lemma 3.1.** $\frac{1}{|H|}\mathbb{Z}[H \backslash G/H]$ is a $\mathbb{Z}$-algebra of rank $d$ with unit element $F_1 = p_H$. 
Proof. It remains to show that $F_i F_j \in \frac{1}{|H|} \mathbb{Z}[H \backslash G/H]$ for all $i$ and $j$. Now it is easy to see that

$$F_i F_j = \sum_{k=1}^{d} c_{ijk} F_k$$

with $c_{ijk} = \frac{|H_i \cap g_k H_j^{-1}|}{|H|}$

where $g_k$ is any element of $H_k$. Since both $H_i$ and $g_k H_j^{-1}$ are unions of left cosets of $H$ in $G$, the constants $c_{ijk}$ are integers which implies the assertion. \hfill \blacksquare

We call $\frac{1}{|H|} \mathbb{Z}[H \backslash G/H]$ the Hecke ring of the subgroup $H$ of $G$.

3.3. The homomorphism $\frac{1}{|H|} \mathbb{Z}[H \backslash G/H] \to \text{Div}(X \times X)$. It is well known that the canonical homomorphism $\mathbb{Q}[G] \to \text{End}_{\mathbb{Q}}(J^2 Z)$ of Section 3.1 factorizes via the algebra of $\mathbb{Q}$-correspondences $\text{Div}_\mathbb{Q}(Z \times Z)$. In this subsection we show that the homomorphism $\epsilon$ of (3.2) factorizes via the algebra $\text{Div}_\mathbb{Q}(X \times X)$. To see this, note first that every element $F = \sum_{i=1}^{d} \alpha_i F_i \in \mathbb{Q}[H \backslash G/H]$ induces a $\mathbb{Q}$-correspondence on $Z$ defined by

$$ \mathcal{D}_F = \frac{1}{|H|} \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} \sum_{h \in H} \Gamma_{g_j h} = \frac{1}{|H|} \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} \Gamma_{g_j} \sum_{h \in H} \Gamma_h \in \text{Div}_\mathbb{Q}(Z \times Z)$$

where $\Gamma_g \subset Z \times Z$ denotes the graph of $g$ for any $g \in G$, and we use the multiplication of the ring $\text{Div}_\mathbb{Q}(Z \times Z)$. The corresponding map $Z \to \text{Div}_\mathbb{Q} Z$ is given by

$$ \mathcal{D}_F(z) = \frac{1}{|H|} \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} \sum_{h \in H} g_{ij} h(z) = \frac{1}{|H|} \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} \sum_{h \in H} h g_{ij}(z), $$

since the $g_{ij}$ are representatives for both the left and right cosets of $H$ in $G$. In particular we have for every $h \in H$ and $z \in Z$,

$$ (3.3) \quad \mathcal{D}_F(h(z)) = \mathcal{D}_F(z). $$

As every $\mathbb{Q}$-correspondence on $Z$, the correspondence $\mathcal{D}_F$ pushes down to a correspondence on $X$, namely

$$ \overline{\mathcal{D}}_F := \frac{1}{|H|} (\pi_H \times \pi_H)_* \mathcal{D}_F \in \text{Div}_\mathbb{Q}(X \times X). $$

The following proposition shows that $\overline{\mathcal{D}}_F$ is actually an integral correspondence for any $F \in \frac{1}{|H|} \mathbb{Z}[H \backslash G/H]$.

**Proposition 3.2.** The map $F \mapsto \overline{\mathcal{D}}_F$ is a homomorphism of rings

$$ \frac{1}{|H|} \mathbb{Z}[H \backslash G/H] \to \text{Div}(X \times X). $$

If $F = \sum_{i=1}^{d} \alpha_i F_i$, we have for any $x \in X$

$$ (3.4) \quad \overline{\mathcal{D}}_F(x) = \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} \pi_H(g_{ij}(z)) $$

where $z \in Z$ is a preimage of $x$. 
Proof. First note that the right hand side of (3.4) is independent of the choice of the preimage $z$ of $x$. If $z'$ is another preimage, there is an $h \in H$ such that $z' = h(z)$. The choice of the $g_{ij}$ as a simultaneous set of representatives for the left and right cosets of $H$ in $G$ implies for any $i$, $1 \leq i \leq d$,

$$
\sum_{j=1}^{n_i} \pi_H(g_{ij}(z')) = \sum_{j=1}^{n_i} \pi_H(hg_{ij}(z)) = \sum_{j=1}^{n_i} \pi_H(h(g_{ij}(z))) = \sum_{j=1}^{n_i} \pi_H(g_{ij}(z)).
$$

Multiplying by $\alpha_i$ and summing over all $i$ gives the assertion.

Hence the right hand side of (3.4) defines a set-theoretical correspondence $\overline{D}_F$. In order to show that $\overline{D}_F$ is an algebraic correspondence on $X$ and equals $\frac{1}{|H|}(\pi_H \times \pi_H)_* D_F$, it suffices to show that $((\pi_H \times \pi_H)_* D_F)(x) = |H|D_F(x)$ for all $x \in X$.

Using the special properties of the representatives $g_{ij}$ and choosing a fixed preimage $z$ of $x$, we have

$$(\pi_H \times \pi_H)_* D_F(x) = \sum_{h \in H} \pi_H(D_F(h(z))) = |H| \cdot \pi_H(D_F(z))$$

$$= \sum_{h \in H} \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} \pi_H(hg_{ij}(z))$$

$$= |H| \cdot \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} \pi_H(g_{ij}(z)) = |H| \cdot D_F(x).$$

It remains to see that the map $F \mapsto \overline{D}_F$ is a ring-homomorphism. But it is easily seen using (3.3) several times that $\overline{D}_{F_F'}(x) = \overline{D}_F \overline{D}_{F'}(x)$ for any $F, F' \in \frac{1}{|H|} \mathbb{Z}[H\backslash G/H]$ and any $x \in X$.

Since $F_1, \ldots, F_d$ are also a $\mathbb{Q}$-basis of the Hecke algebra $\mathbb{Q}[H\backslash G/H]$, the homomorphism of Proposition 3.2 extends to a homomorphism of $\mathbb{Q}$-algebras $\mathbb{Q}[H\backslash G/H] \rightarrow \text{Div}_\mathbb{Q}(X \times X)$. Together with the definition of the homomorphism $\epsilon : \mathbb{Q}[H\backslash G/H] \rightarrow \text{End}_\mathbb{Q}(JX)$, we obtain

**Corollary 3.3.** The homomorphism of Proposition 3.2 extends to a homomorphism of $\mathbb{Q}$-algebras $\mathbb{Q}[H\backslash G/H] \rightarrow \text{Div}_\mathbb{Q}(X \times X)$, which factorizes the homomorphism $\epsilon : \mathbb{Q}[H\backslash G/H] \rightarrow \text{End}_\mathbb{Q}(JX)$.

Consider now a rational irreducible representation $W$ of $G$ with associated complex irreducible representation $V$. From subsection 2.2 we conclude that

$$F_{H,W} = \sum_{i=1}^{d} a_i F_i \in \frac{1}{|H|} \mathbb{Z}[H\backslash G/H]$$

with integers $a_i$ given by (2.6). Hence we may apply Proposition 3.2 to $F_{H,W}$ to conclude the first part of the following proposition.

**Proposition 3.4.** $\overline{D}_{H,W} := \overline{D}_{F_{H,W}}$ is an integral symmetric correspondence on $X$, of degree 0 if $W$ is non-trivial and of degree $|G|$ if $W$ is trivial.
Proof. According to Proposition 3.2, $\overline{D}_{H,W}$ is an integral correspondence on $X$. Applying (2.5), (2.6) and Lemma 2.2 we have for the degree,

$$\deg \overline{D}_{H,W} = \sum_{i=1}^{d} a_{i} n_{i} = \sum_{i=1}^{d} n_{i} \sum_{h \in H} \text{tr}_{K_{V}/Q}(\chi_{V}(hg_{i}^{-1}))$$

$$= \sum_{g \in G} \text{tr}_{K_{V}/Q}(\chi_{V}(g))$$

$$= \begin{cases} 0 & \text{if } \mathcal{W} \text{ is non-trivial;} \\ |G| & \text{if } \mathcal{W} \text{ is trivial.} \end{cases}$$

where the zero in the last equation is just the fact that the representation $\mathcal{W}$ and the trivial representation of $G$ have zero scalar product.

It remains to show that the correspondence $\overline{D}_{H,W}$ is symmetric. For any $x \in X$ choose a preimage $z \in Z$. Then $\pi_{H}(g_{ij}(z))$ appears with multiplicity $a_{i}$ in $\overline{D}_{H,W}(x)$. It suffices to show that $x$ appears with the same multiplicity in $\overline{D}_{H,W}(\pi_{H}(g_{ij}(z)))$.

By definition we have

$$\overline{D}_{H,W}(\pi_{H}(g_{ij}(z))) = \sum_{k=1}^{d} a_{k} \sum_{l=1}^{n_{k}} \pi_{H}(g_{kl}(g_{ij}(z))).$$

We are interested in its coefficient of $x = \pi_{H}(z)$. But $\pi_{H}(g_{kl}(g_{ij}(z))) = x$ if and only if $g_{kl}g_{ij} \in H$, which is the case if and only if

$$g_{kl} \in Hg_{ij}^{-1}$$

Hence $Hg_{kl} = Hg_{ij}^{-1}H$ and it suffices to show that for any $g \in G$, $F_{H,W}$ has the same coefficient on the double cosets $HgH$ and $Hg^{-1}H$. But this follows from equation (2.5) and (2.6) or, to be more precise, from the fact that $\text{tr}_{K_{V}/Q}(\chi_{V}(g)) = \text{tr}_{K_{V}/Q}(\chi_{V}(g^{-1}))$.

This completes the proof of the proposition. \qed

**Proposition 3.5.** For any two different rational irreducible representations $\mathcal{W}$ and $\tilde{\mathcal{W}}$ of $G$ we have

(a): \hspace{0.5cm} $\overline{D}^{*}_{H,W} = \frac{|G|}{\dim V} \overline{D}_{H,W}$.

(b): \hspace{0.5cm} $\overline{D}_{H,W} \overline{D}_{H,\tilde{W}} = 0$.

**Proof.** (a) follows from Lemma 2.3 and Proposition 3.2 together with the fact that $f_{H,W} = \frac{|G|}{\dim V} F_{H,W}$ is an idempotent. Similarly, (b) is a consequence of the same results together with the fact that $f_{H,W}$ and $f_{H,\tilde{W}}$ are orthogonal idempotents. \qed

### 3.4. The trace correspondence

We denote by $T_{X/Y} := \varphi_{*}^{*} \varphi_{H}$ the trace correspondence of the covering $\varphi_{H} : X \rightarrow Y$. So for any $x \in X$,

$$T_{X/Y}(x) = \sum_{i=1}^{d} \sum_{j=1}^{n_{i}} \pi_{H}(g_{ij}(z)).$$
where \( z \in Z \) is a preimage of \( x \). On the other hand, if \( V_0 = W_0 \) denotes the trivial representation of \( G \), according to (3.4) and the proof of Proposition 3.4 we have

\[
\mathcal{D}_{H,V_0}(x) = \sum_{i=1}^{d} |H| \sum_{j=1}^{n_i} \pi_H(g_{ij}(z))
\]

which implies

\[
T_{X/Y} = \frac{1}{|H|} \mathcal{D}_{H,V_0}.
\]

Hence as a special case of Proposition 3.5 we obtain

**Corollary 3.6.** (a) \( T_{X/Y}^2 = [G : H]T_{X/Y} \);

(b) For any nontrivial rational irreducible representation \( W \) of \( G \),

\[
\mathcal{D}_{H,W}T_{X/Y} = T_{X/Y} \mathcal{D}_{H,W} = 0.
\]

3.5. The Kanev correspondence. Let \( W_1, \ldots, W_r \) denote nontrivial pairwise non-isomorphic rational irreducible representations of the group \( G \) with associated complex irreducible representations \( V_1, \ldots, V_r \). We make the following hypothesis for the \( W_i \) and a subgroup \( H \) of \( G \).

**Hypothesis 3.7.** For all \( k, l = 1, \ldots, r \) we assume

a) \( \dim V_k = \dim V_l =: n \) fixed,

b) \( K_{V_k} = K_{V_l} =: L \) fixed,

c) \( \dim V_k^H = 1 \),

d) \( H \) is maximal with property c), that is, for every subgroup \( N \) of \( G \) with \( H \subsetneq N \) there is an index \( k \) such that \( \dim V_k^N = 0 \).

Recall the correspondence \( \mathcal{D}_{H,W_k} \) given by

\[
\mathcal{D}_{H,W_k}(x) = \sum_{i=1}^{d} a_{ki} \sum_{j=1}^{n_i} \pi_H(g_{ij}(z))
\]

for all \( x \in X \) and \( z \in Z \) with \( \pi_H(z) = x \), and where the \( a_{ki} \) are the integers

\[
a_{ki} = \sum_{h \in H} \text{tr}_{L/\mathbb{Q}}(\chi_{V_k}(hg_{i1}^{-1})).
\]

Now consider the correspondence

\[
\mathcal{D} := \sum_{k=1}^{r} \mathcal{D}_{H,W_k}.
\]

Denoting \( b_i := \sum_{k=1}^{r} a_{ki} \) for \( i = 1, \ldots, d \) we have

\[
\mathcal{D}(x) = \sum_{i=1}^{d} b_i \sum_{j=1}^{n_i} \pi_H(g_{ij}(z)).
\]

Then set

\[
b := \gcd\{b_1 - b_i \mid 2 \leq i \leq d\}.
\]
Denoting by $\Delta_X \in \text{Div}(X \times X)$ the identity correspondence, we define the Kanev correspondence on $X$ associated to $W_1, \ldots, W_r$ (which we omit in the notation) by

$$K_X := \Delta_X - \frac{1}{b}D + \left(\frac{b_1}{b} - 1\right)T_{X/Y}.$$  

**Lemma 3.8.** Suppose $W_1, \ldots, W_r$ are nontrivial pairwise non-isomorphic rational irreducible representations of $G$ satisfying the Hypothesis 3.7 and recall $L = K_{V_i}$ for all $i$. Then $K_X$ is an integral effective symmetric correspondence on $X$ of degree

$$\text{deg } K_X = 1 + \frac{r|G|}{b}[L : \mathbb{Q}] - [G : H].$$

**Proof.** Since for all $k$, $V_k$ is not the trivial representation, but satisfies $\text{dim } V_k^H = 1$, it follows from Proposition 2.5 that $a_{ki} \leq a_{k1}$ for all $i, k$. Therefore $b_i \leq b_1$ for all $i$. Moreover, the maximality condition d) of $H$ together with Proposition 2.5 imply that actually $b_i < b_1$ for all $i \geq 2$. Therefore $b$ is a positive integer.

Now choose for every $x \in X$ a preimage $z \in Z$. Then

$$K_X(x) = x - \frac{1}{b} \sum_{i=1}^{d} b_i \sum_{j=1}^{n_i} \pi_H(g_{ij}(z)) + \left(\frac{b_1}{b} - 1\right) \sum_{i=1}^{d} \sum_{j=1}^{n_i} \pi_H(g_{ij}(z))$$

$$= x + \sum_{i=1}^{d} \left(\frac{b_1 - b_i}{b} - 1\right) \sum_{j=1}^{n_i} \pi_H(g_{ij}(z)) = \sum_{i=2}^{d} \left(\frac{b_1 - b_i}{b} - 1\right) \sum_{j=1}^{n_i} \pi_H(g_{ij}(z)).$$

Hence $K_X$ is integral and effective. The symmetry of $K_X$ is a consequence of Proposition 3.4. Finally, using that $\text{deg } D = 0$ according to Proposition 3.4 as well as Proposition 2.5 we have

$$\text{deg } K_X = 1 + \left(\frac{r|G|}{b}\right)\text{deg } T_{X/Y}$$

$$= 1 + \left(\frac{r|G|}{b}\right)[L : \mathbb{Q}] - [G : H] = 1 + \frac{r|G|}{b}[L : \mathbb{Q}] - [G : H].$$

**Proposition 3.9.** Suppose $W_1, \ldots, W_r$ are nontrivial pairwise non-isomorphic rational irreducible representations of $G$ satisfying the Hypothesis 3.7. Then the correspondence $K_X$ satisfies the cubic equation

$$K_X - \Delta_X)(K_X + (q - 1)\Delta_X)(K_X - \text{deg } K_X \cdot \Delta_X) = 0$$

where $\Delta_X$ is the diagonal in $X \times X$ and $q$ the positive integer

$$q = \frac{|G|}{b \cdot n}.$$  

**Proof.** Note first that Corollary 3.6 and (3.8) imply

$$K_X T_{X/Y} = \left(1 + \left(\frac{b_1}{b} - 1\right)[G : H]\right)T_{X/Y} = \text{deg } K_X \cdot T_{X/Y}.$$
Now using the definition of $\mathcal{K}_X$, Proposition 3.5, Corollary 3.6 and equation 3.6 we get
\[
(\Delta_X - \mathcal{K}_X)^2 = \left(\frac{1}{b} \mathcal{D} - \left(\frac{b_1}{b} - 1\right) \mathcal{T}_{X/Y}\right)^2 \\
= \frac{1}{b} \mathcal{D}^2 + \left(\frac{b_1}{b} - 1\right) \mathcal{T}_{X/Y}^2 \\
= \frac{|G|}{b^2 \cdot n} \mathcal{D}^2 + \left(\frac{b_1}{b} - 1\right)^2 [G : H] \mathcal{T}_{X/Y}
\]

Defining $q$ by the right hand side of (3.10), this gives
\[
(3.12) \quad (\mathcal{K}_X - \Delta_X)^2 + q(\mathcal{K}_X - \Delta_X) + c \cdot \mathcal{T}_{X/Y} = 0
\]
where $c$ denotes the rational number
\[
c = (1 - \frac{b_1}{b}) \left(\frac{b_1}{b} - 1\right)[G : H] + \frac{|G|}{b \cdot n}.
\]

Multiplying (3.12) by $\mathcal{K}_X - \deg \mathcal{K}_X \cdot \Delta_X$ and applying (3.11), we get (3.9).

It remains to show that $q$ is an integer. For this consider a general point $y \in Y$. The action of $\mathcal{K}_X$, respectively of $\mathcal{T}_{X/Y}$, on the fibre $\varphi^{-1}_H(y)$ is described by a square integral matrix of size $[G : H]$ denoted by $M_{\mathcal{K}_X}$, respectively by $M_{\mathcal{T}_{X/Y}}$. If we denote $N = M_{\mathcal{K}_X} - E$, where $E$ denotes the identity matrix of size $[G : H]$, equation (3.12) implies
\[
(3.13) \quad N(N + qE) = -cM_{\mathcal{T}_{X/Y}}.
\]

We will complete the proof by showing that the rational number $-q$ is an eigenvalue of the integral matrix $N$. If $c = 0$, then equation (3.13) implies that $N$ satisfies the equation $x(x + q) = 0$. Since $N$ is not the zero matrix, then either $x + q$ or $x(x + q)$ is the minimal polynomial for $N$ and we are done. If $c \neq 0$, then, according to (3.9) and (3.13), the minimal polynomial of $N$ is $x(x + q)(x - (\deg \mathcal{K}_X - 1))$ and we conclude in the same way that $q$ is an integer. \(\square\)

Remark 3.10. Equation (3.13) implies that $cM_{\mathcal{T}_{X/Y}}$ is an integral matrix. Since all entries of the matrix $M_{\mathcal{T}_{X/Y}}$ are equal to 1, we conclude that $c$ is also an integer.

Corollary 3.11. With the notation of Proposition 3.9 the following conditions are equivalent:
(a): $q = 1$,
(b): $\rho_H \simeq \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_r$,
(c): $\mathcal{K}_X = 0$.

Proof. Note first that
\[
\chi_{\rho_H}(1_G) = [G : H] \geq 1 + rn[L : \mathbb{Q}].
\]

According to Proposition 3.9 $q = 1$ if and only if $|G| = bn$. Since $\mathcal{K}_X$ is effective, this is equivalent to
\[
0 \leq \deg \mathcal{K}_X = 1 + rn[L : \mathbb{Q}] - [G : H] \leq 0.
\]

This implies all assertions. \(\square\)
4. Prym varieties

4.1. The Prym variety $P_{\mathcal{D}}$. Let the notation be as in the last section. So $H$ is a subgroup of $G$. Moreover denote by $\mathcal{W} := (\mathcal{W}_1, \ldots, \mathcal{W}_r)$ an $r$-tuple of nontrivial pairwise non-isomorphic rational irreducible representations of the group $G$ with associated complex irreducible representations $V_1, \ldots, V_r$ satisfying Hypothesis [3.7]. Let $\delta_{\mathcal{D}}$ denote the endomorphism of the Jacobian $JX$ associated to the correspondence $\mathcal{D}$. Then we get the following statement, as a direct consequence of Proposition 3.5 and the definitions.

Proposition 4.1.

$$\delta_{\mathcal{D}}^2 = \frac{|G|}{n} \delta_{\mathcal{D}}.$$ 

We denote by

$$P_{\mathcal{D}} := \text{Im}(\delta_{\mathcal{D}})$$

the image of the endomorphism $\delta_{\mathcal{D}}$ in the Jacobian $JX$ and call it the (generalized) Prym variety associated to the correspondence $\mathcal{D}$. Our aim is to investigate the restriction of the canonical polarization of $JX$ to $P_{\mathcal{D}}$ in some cases.

Let us denote by $\tau_{X/Y} \in \text{End}(JX)$ the endomorphism associated to the trace correspondence $\mathcal{T}_{X/Y}$. The Prym variety of the covering $\varphi_H : X \to Y$ is defined by

$$\text{Prym}(X/Y) := \ker(\tau_{X/Y})^0 \subset JX.$$ 

According to Corollary [3.6] we have $\tau_{X/Y} \delta_{\mathcal{D}} = 0$. This implies

Proposition 4.2. The Prym variety $P_{\mathcal{D}}$ is a subvariety of the Prym variety of the covering $\varphi_H$:

$$P_{\mathcal{D}} \subset \text{Prym}(X/Y).$$

Let $\kappa_X$ denote the endomorphism of $JX$ associated to the correspondence $\mathcal{K}_X$. According to Corollary [3.6] we have $\kappa_X \tau_{X/Y} = \tau_{X/Y} \kappa_X = (1 + (\frac{b}{G} - 1)[G : H]) \tau_{X/Y}$. Hence $\kappa_X$ restricts to an endomorphism $\tilde{\kappa}_X$ of $\text{Prym}(X/Y)$.

Proposition 4.3. (a) $P_{\mathcal{D}} = \text{Im}(1_{\text{Prym}(X/Y)} - \tilde{\kappa}_X)$;

(b) $\tilde{\kappa}_X$ satisfies the quadratic equation

$$\tilde{\kappa}_X^2 + (q - 2)\tilde{\kappa}_X - (q - 1)1_{\text{Prym}(X/Y)} = 0$$

with $q = \frac{|G|}{b \cdot n}$.

Proof. According to (3.7) we have $\delta_{\mathcal{D}}|_{\text{Prym}(X/Y)} = b(1_{\text{Prym}(X/Y)} - \tilde{\kappa}_X)$, which gives (a). Finally, (b) is a consequence of (3.12).

□

Proposition 4.4.

$$\dim P_{\mathcal{D}} = \frac{1}{q} \left( g(X) + [G : H] - 1 - \frac{r|G|}{b}[L : \mathbb{Q}] + \frac{1}{2}(\kappa_X \cdot \Delta_X) \right).$$

Proof. According to Proposition 4.1 the element $\frac{n}{|G|} \delta_{\mathcal{D}}$ is the symmetric idempotent corresponding to the abelian subvariety $P_{\mathcal{D}}$ of $JX$. Hence [BL] Corollary 5.3.10] gives

$$\dim P_{\mathcal{D}} = \text{tr}_a \left( \frac{n}{|G|} \delta_{\mathcal{D}} \right) = \frac{b \cdot n}{|G|} \text{tr}_a(1_{JX} - \kappa_X) = \frac{1}{q} \left( g(X) - \frac{1}{2} \text{tr}_r(\kappa_X) \right).$$
On the other hand, according to \cite[Proposition 11.5.2]{BL} we have for the rational trace of $\kappa_X$,

$$\text{tr}(\kappa_X) = 2 \deg K_X - (K_X \cdot \Delta_X).$$

So (3.8) gives the assertion. \hfill \Box

4.2. **Variation of $H$.** Up to now we considered a fixed subgroup $H$ of $G$. For any $r$-tuple of nontrivial pairwise non-isomorphic rational irreducible representations $W := (W_1, \ldots, W_r)$ of the group $G$ with associated complex irreducible representations $V_1, \ldots, V_r$ satisfying Hypothesis 3.7, we associated a correspondence $\mathcal{D}$ of the curve $X$ and an abelian subvariety of $J_X$, the Prym variety $P_{\mathcal{D}}$.

Suppose now, we are given two subgroups $H_1$ and $H_2$ of $G$ such that $W$ satisfies Hypothesis 3.7 for both of them. For $i = 1, 2$ denote by $X_i$ the curve $Z/H_i$, and by $\mathcal{D}_i$ the associated correspondence. The following proposition shows how the corresponding Prym varieties $P_{\mathcal{D}_i}$ are related.

**Proposition 4.5.** (a): With the above notations the abelian varieties $P_{\mathcal{D}_1}$ and $P_{\mathcal{D}_2}$ are isogenous.

(b): If in addition $H_1$ and $H_2$ are conjugate in $G$, the canonical isomorphism $X_1 \to X_2$ induces an isomorphism of polarized abelian varieties $P_{\mathcal{D}_1} \to P_{\mathcal{D}_2}$.

**Proof.** We only give a sketch, since we do not need the result in the sequel. First note that Proposition 3.5 (b) reduces the proof to the case of one representation, i.e. to $W = W_1$. For (a) let $f$ denote a primitive idempotent of $\mathbb{Q}[G]_{e W}$ considered as an element of $\text{End}_{\mathbb{Q}}(J_Z)$. Then it is easy to see that $P_{\mathcal{D}_i}$ is isogenous to $\text{Im}(f) \subset J_Z$. The proof of (b) is a straightforward computation. \hfill \Box

**Remark 4.6.** An example of Proposition 4.5 with $H_1$ and $H_2$ not conjugate but still each one satisfying Hypothesis 3.7 is provided by the Alternating group of degree 4, with $V$ the standard representation and the subgroups $H_1$ of order 2 and $H_2$ of order 3.

4.3. **The case $r = 1$.** Let $W$ denote a rational irreducible representation of $G$. Recall from \cite{CaRo} that for any rational irreducible representation $W$ and any subgroup $H$ of $G$ there is a uniquely determined abelian subvariety of the Jacobian $J_X$, called the isotypical component associated to $W$ (see also \cite{LRe}). It is isogenous to $B_{W}^{\dim V^H}$, where $B_W$ is the image of a primitive idempotent of the group algebra $\mathbb{Q}[G]$ in $J_X$ corresponding to a minimal left ideal of the simple subalgebra of $\mathbb{Q}[G]$ defined by $W$. The abelian subvariety $B_W$ is only determined up to isogeny in general.

**Proposition 4.7.** Let $W$ denote the rational irreducible representation of $G$ associated to $V$ with $\dim V^H = 1$. Then $P_{\mathcal{D}_W} = B_{W}^{\dim V^H}$ is the isotypical component of $J_X$ associated to $W$.

**Proof.** This is a consequence of \cite[Proposition 5.2]{CaRo}: On the one hand $P_{\mathcal{D}_W}$ is the isotypical component associated to $W$ in $J_X$ and on the other hand it is shown that $P_{\mathcal{D}_W}$ is isogenous to $B_{W}^{\dim V^H}$. The assumption $\dim V^H = 1$ implies $m_V = 1$ and thus the assertion. \hfill \Box
4.4. **Fixed points of** $K_X$. As before, $W_1, \ldots, W_r$ denote nontrivial pairwise non-isomorphic rational irreducible representations of the group $G$ with associated complex irreducible representations $V_1, \ldots, V_r$ satisfying Hypothesis 3.7. The number of fixed points of the correspondence $K_X$ is by definition the intersection number $(K_X \cdot \Delta_X)$. It depends on the type of the Galois covering $\pi: Z \to Y$. In order to express it in terms of the type of $\pi$, we use the notion of geometric signature as defined in the introduction.

**Proposition 4.8.** Suppose the action of the group $G$ on the curve $Z$ has geometric signature $[0; (C_1, m_1), \ldots, (C_t, m_t)]$. Then the number of fixed points of the correspondence $K_X$ of the curve $JX$ is given by

$$(K_X \cdot \Delta_X) = \sum_{j=1}^{t} m_j \left( q[L : Q] \sum_{i=1}^{r} (\dim V_i - \dim V_i^{G_j}) - ([G : H] - |H\backslash G/G_j|) \right).$$

where $G_j$ denotes any subgroup in the class $C_j$.

**Proof.** According to [R, Corollary 3.4] we have for the genus of $X$,

$$g(X) = 1 - [G : H] + \frac{1}{2} \sum_{j=1}^{t} m_j ([G : H] - |H\backslash G/G_j|).$$

Since the representations $W_i$ are pairwise non-isomorphic, we certainly have

$$\dim P_D = \sum_{i=1}^{r} \dim P_{D_{W_i}}.$$

where $P_{D_{W_i}} = \text{Im}(\delta_{W_i})$ and $\delta_{W_i}$ is the element of $\text{End}_Q(JX)$ associated to the correspondence $D_{W_i}$. Now [R, Theorem 5.12] gives, using Proposition 4.7 and the fact that the Schur index of $V_i$ is 1,

$$\dim P_{D_{W_i}} = [L : Q] \left( \frac{1}{2} \sum_{j=1}^{t} m_j (\dim V_i - \dim V_i^{G_j}) - n \right).$$

Hence

$$\dim P_D = [L : Q] \sum_{i=1}^{r} \left( \frac{1}{2} \sum_{j=1}^{t} m_j (\dim V_i - \dim V_i^{G_j}) - n \right).$$

Inserting these equations into (4.2) we get

$$(K_X \cdot \Delta_X) = 2 \left( \frac{|G|}{b \cdot n} \dim P_D - g(X) - [G : H] + 1 + \frac{r |G|}{b } [L : Q] \right)$$

$$= \sum_{j=1}^{t} m_j \left( q[L : Q] \sum_{i=1}^{r} (\dim V_i - \dim V_i^{G_j}) - ([G : H] - |H\backslash G/G_j|) \right).$$

□
4.5. **Prym-Tyurin varieties.** Recall that an abelian subvariety $P$ of $JX$ is called a **Prym-Tyurin variety of exponent** $q$ in $JX$, if the restriction of the canonical principal polarization of $JX$ to $P$ is the $q$-fold of a principal polarization of $P$. The main result of the paper is the following theorem.

**Theorem 4.9.** Let $W_1, \ldots, W_r$ denote nontrivial pairwise non-isomorphic rational irreducible representations of the group $G$ with associated complex irreducible representations $V_1, \ldots, V_r$ satisfying Hypothesis 3.7. Suppose that the action of the finite group $G$ has geometric signature $[0; (C_1, m_1), \ldots, (C_t, m_t)]$ satisfying

$$
\sum_{j=1}^t m_j \left( q[L : \mathbb{Q}] \sum_{i=1}^r (\dim V_i - \dim V_i^{G_j}) - ([G : H] - |H \backslash G/G_j|) \right) = 0.
$$

where $G_j$ is of class $C_j$ and $q = \frac{|G|}{b^n}$. Then $P_{\mathcal{D}}$ is a Prym-Tyurin variety of exponent $q$ in $JX$.

Note that the theorem gives a method to construct Prym-Tyurin varieties.

**Proof.** Since $g(Y) = 0$, we have $P(X/Y) = JX$. The correspondence $\mathcal{K}_X$ on the curve $X$ is integral, symmetric and fixed-point free according to Proposition 4.3. Moreover the associated endomorphism $\kappa_X$ of $JX$ satisfies equation (4.1). Hence the assertion is a consequence of Kanev’s criterion (see e.g. [BL, Theorem 12.9.1]).

□

4.6. **Relation to Kanev’s construction.** In this subsection we compare our construction of the Prym variety $P_{\mathcal{D}}$ to Kanev’s original construction. Let us first recall the original construction.

Let $\mathcal{W}$ be an absolutely rational irreducible representation of the group $G$. Suppose $G$ acts on a lattice $\Lambda$ of maximal rank in $\mathcal{W}$. Moreover let $w \in \mathcal{W}$ be a weight; that is a nonzero vector satisfying $gw - w \in \Lambda$ for all $g \in G$. Then there exists a uniquely determined negative definite $G$-invariant symmetric bilinear form $(\cdot, \cdot)$ on $\mathcal{W}$ such that (i) $(w|\Lambda) \subset \mathbb{Z}$ and (ii) any negative definite $G$-invariant form with (i) is an integer multiple of $(\cdot, \cdot)$.

Now let $\pi : Z \rightarrow \mathbb{P}^1$ be a simply ramified Galois covering with Galois group $G$. Consider the subgroup $H := \text{Stab}_G(w)$, define $X := Z/H$, and let $\varphi_H : X \rightarrow \mathbb{P}^1$ denote the canonical map. Then for any pair $(\Lambda, w)$ as above, Kanev defines a symmetric effective correspondence on the curve $X$, which induces an abelian subvariety $P_K = P_K(\Lambda, w)$ of $JX$. We call it the **Prym variety** in $JX$ associated to the pair $(\Lambda, w)$. It is shown in [LR] that the abelian subvariety $P_K$ of $JX$ is given by the image of the element $s_w \in \text{End}_\mathbb{Q}(JX)$ associated to the correspondence $S_w(x)$ on $X$, defined for every $x \in X$ with preimage $z \in Z$ by (see [LR] Proposition 2.2 and its proof)

$$
S_w(x) = |H|^2 \sum_{i=1}^d \sum_{j=1}^{n_i} (w|g_{ij}w) \pi_H(g_{ij}(z)).
$$

Here $\{g_{ij} \mid i = 1, \ldots, d; j = 1, \ldots, n_i\}$ is a set of representatives of the left and right cosets of $H$ in $G$.

Now let the notation be as in Section 4. Consider the special case $r = 1$, $V = \mathcal{W} \otimes \mathbb{C}$ with an absolutely rational irreducible representation $\mathcal{W}$ and finally $H$ a subgroup of $G$. 

satisfying \( \dim V^H = 1 \). Clearly \( \dim W^H = 1 \). For any nonzero vector \( w \in W^H \) the subgroup of \( W \) generated by the elements \( gw - w \) for all \( g \in G \) is a \( G \)-invariant lattice \( \Lambda \) of maximal rank in \( W \) for which \( w \) is a weight. Hence the Prym variety associated to the pair \( (\Lambda, w) \) is well defined. With these assumptions we have

**Proposition 4.10.** The Prym variety associated to the pair \( (\Lambda, w) \) coincides with the Prym variety \( P_D \) as defined in Section 4.1:

\[
P_K(\Lambda, w) = P_D.
\]

**Proof.** Note first that \( P_D \) does not depend on the choice of the bilinear form \((\cdot, \cdot)\) used in Section 2. The statement of Proposition 2.5 is independent of the choice of the form \((\cdot, \cdot)\). Hence we may choose the bilinear form in such a way that

\[
(w, w) = \frac{|H|}{b},
\]

with \( b = \gcd\{a_i - a_1 | 2 \leq i \leq d\} \) as in (3.6).

Then we have, according to Proposition 2.4 and equations (2.7) and (2.6),

\[
(w, g_{ij}w - w) = \frac{a_i - a_1}{b}.
\]

This implies that Kanev’s distinguished form \((\cdot | \cdot)\) is just the negative of our form \((\cdot, \cdot)\).

Now the Prym variety \( P_D \) is defined as the image of the element \( \delta_D \in \text{End}_Q(JX) \) associated to the correspondence \( D \), which according to equation (3.5) is given by

\[
D_{H,W}(x) = \sum_{i=1}^d a_i \sum_{j=1}^{n_i} \pi_H g_{ij}(z) = \sum_{i=1}^d b(w, g_{ij}w) \sum_{j=1}^{n_i} \pi_H g_{ij}(z) \quad \text{(by (4.5))}
\]

Comparison with (4.4) implies that \( s_w \) and \( \delta_D \) are rational multiples of each other, and thus have the same image in \( JX \). \( \square \)

**Remark 4.11.** (a): Proposition 4.10 means that Kanev’s and our constructions have nonzero intersection. Kanev’s construction is more general in the sense that he does not assume that \( \dim V^{\text{Stab}(w)} = 1 \). Our construction is more general in the sense that we do not have to assume \( W \) absolutely irreducible. Moreover, \( W \) can be a tuple of representations, which actually gives interesting examples, as we will see in Section 5.3.

(b): Our construction of abelian subvarieties of \( JX \) can be carried out without the hypotheses “\( \dim V^H = 1 \) and \( H \) maximal with this property” and “\( Z/G = \mathbb{P}^1 \)” (and this is valid in fact for any rational irreducible representation \( W \)) by just considering a primitive
idempotent $f$ in $Q[G]e_W$ that is invariant under multiplication by $p_H$ on both sides; it can be proven that such idempotents exist if $\langle p_H, V \rangle \neq 0$. Then $P = \text{Im}(f)$ is an abelian subvariety of $JX$. Hence our construction also has points in common with the one by Merindol [M], again generalizing to non absolutely irreducible representations.

The purpose of our additional assumptions is to allow the study of the polarization of $P$.

5. Examples

5.1. Kanev’s Examples. In [K2] Kanev constructed families of Prym-Tyurin varieties for the Weyl groups $G = W(R)$ for the root systems $R$ of type $A_n$, $D_n$, $E_6$ and $E_7$ using the construction outlined at the beginning of Section 4.6. In these cases $W$ is the root representation of the group $G$, $\Lambda$ is the root lattice and $w$ a minuscule fundamental weight. It is easy to check that in all these cases the subgroup $H = \text{Stab}_G(w)$ satisfies $\dim W^H = 1$ and is maximal with this property. So, according to Proposition 4.9 the corresponding Prym-Tyurin varieties can be also constructed by our method.

5.2. Additional Prym-Tyurin varieties for Weyl groups. In order to construct his Prym-Tyurin varieties, Kanev [K2] starts with a covering $\varphi_H : X \to \mathbb{P}^1$ with simplest ramification. In our terminology this means that the geometric signature of the covering $\pi : Z \to Y$ only contains the conjugacy class of reflections in the Weyl group. Using equation (4.3) one can work out exactly the types of conjugacy classes to which Theorem 4.9 applies. We call these classes admissible conjugacy classes for the triple $(G, H, V)$, where $G = W(R)$, $V$ is the root representation and $H$ a subgroup of $G$ satisfying Hypothesis 3.7. Using coverings with these more general geometric signatures we get other families of Prym-Tyurin varieties. In fact they are of different dimensions. However certainly these varieties can also be constructed generalizing Kanev’s approach.

One idea to construct new Prym-Tyurin varieties is as follows: For each root system $R$ of type $A_n$, $D_n$, $E_6$ and $E_7$, denote by $G$ its Weyl group. In each of these cases $G$ has a unique subgroup of index 2, which will be denoted by $\tilde{G}$; in fact, if we choose the basis $\{\alpha_i\}$ for the corresponding root system as in Bourbaki [Bo], then $G$ is generated by the reflections $r_{\alpha_i}$ and $\tilde{G}$ is generated by the elements of order three in $G$ that are products of two such reflections $s_{ij} = r_{\alpha_i}r_{\alpha_j}$ for $\alpha_i$ and $\alpha_j$ connected by an edge in the corresponding Dynkin diagram. Let us denote by $C_2$ the class of a subgroup $G_2$ generated by one of the $r_{\alpha_i}$, and by $C_3$ the class of a subgroup $G_3$ generated by one of the $s_{ij}$.

**Proposition 5.1.** Assume $Z$ is a curve with $\tilde{G}$-action and geometric signature $[0; (C_3, m)]$. Let $V$ denote the restriction to $\tilde{G}$ of the root representation of $G$.

Then the triples $(\tilde{G}, H, V)$, with $H$ as in the following table, satisfy Hypothesis 3.7, and the corresponding abelian subvarieties of $J(X)$, $X = Z/H$, are Prym-Tyurin varieties of exponent $q$ as in the table.
\[ \tilde{G} \quad H \quad q \]

| $\widetilde{A}_n$ = $\text{Alt}(n+1)$ | $S_k \times S_{n+1-k}$ $\cap$ $\text{Alt}(n+1)$, $1 \leq k \leq n$ | \((n-1)!\) \((k-1)!((n-k))!\) |
| $\widetilde{D}_n$ | $((\mathbb{Z}/2\mathbb{Z})^{n-2} \times S_{n-1}) \cap \widetilde{D}_n$ | 2 |
| $\widetilde{D}_n$ | $S_n \cap \widetilde{D}_n$ | $2^{n-3}$ |
| $\widetilde{E}_6$ | $\widetilde{D}_5$ | 6 |
| $\widetilde{E}_7$ | $\widetilde{E}_6$ | 12 |

**Proof.** The proof is a straightforward application of Theorem 4.9. We use for it the computer program mentioned in the introduction. We omit the details. \(\square\)

Note that we have obtained new families of Prym Tyurin varieties, with the same exponent as in the examples of Section 5.1, and that the representation $V$ is in each case absolutely irreducible. By considering other representations of the same groups, we can find examples with other exponents, and also cases where the restriction is not absolutely irreducible. Some examples are given below.

The group $\widetilde{W}D_5$ has a unique representation of degree six; it is absolutely irreducible, and there is no Prym-Tyurin variety associated to this representation that may be constructed using our methods. However, when restricted to $\widetilde{W}D_5$ it becomes a rational representation $\mathcal{W}$ with associated complex representation $V$ of degree three and $[K_V : \mathbb{Q}] = 2$. Furthermore, there exists a (unique conjugacy class of) subgroup $H$ of order 80 of $\widetilde{W}D_5$ such that $(\widetilde{W}D_5, H, \mathcal{W})$ satisfies Hypothesis 3.7. Again we omit the details of the proof of the following proposition.

**Proposition 5.2.** Consider the triple $(\widetilde{W}D_5, H, \mathcal{W})$, then the conjugacy classes admissible for it are the two conjugacy classes $C_6, C'_6$ of cyclic subgroups of order 6 and the unique conjugacy class $C_3$ of order 3, each of them generates the group $\widetilde{W}D_5$. Moreover, if $m_3, m_6, m'_6$ are the number of branch points of type $C_3, C_6, C'_6$ respectively, then the associated Prym-Tyurin variety has exponent 2 and dimension $2(m_3 + m_6 + m'_6) - 6$.

Similarly, for the irreducible rational representation $\mathcal{W}_4$ of dimension 4 of $\widetilde{W}D_5$, there exists a subgroup $H_2$ of order 96 such that $(\widetilde{W}D_5, H_2, \mathcal{W}_4)$ satisfies Hypothesis 3.7. The same conjugacy classes as above are admissible for this triple, and the associated Prym-Tyurin variety has exponent 3 and dimension $m_3 + m_6 + m'_6 - 4$. exponent 3 and dimension $m_3 + m_6 + m'_6 - 4$.

**5.3. Other examples.** As mentioned in the introduction, we obtain many new families of Prym-Tyurin varieties, if we consider the case $r > 1$ with the notation of Hypothesis 3.7. We only mention two series. The first one shows how to construct Salomon’s examples via Theorem 4.9.
Consider the triple \((G = S_n \times S_n, H, W = (V \boxtimes V_0, V_0 \boxtimes V))\), where \(H\) is a subgroup of index \(n^2\), \(V\) is the root representation of \(W A_{n-1} = S_n\), and \(V_0\) is the trivial representation of \(S_n\). This triple satisfies the Hypothesis 3.7.

By choosing the geometric signature of type \([0; (C_1, m_1), (C_2, m_2)]\) where \(C_1\) is the class of \(G_1 = \langle(1 2), 1_{s_n}\rangle\) and \(C_2\) the class of \(G_2 = \langle(1_{s_n}, (1 2))\rangle\), we reobtain Salomon’s examples of Prym-Tyurin varieties of exponent \(n\). If we choose a different signature, and there are many possibilities (for instance another admissible group is \(G_3 = \langle((1 2), (1 2))\rangle\)), we obtain new families of Prym-Tyurin varieties of exponent \(n\) for each natural \(n\). Examples may also be constructed with non absolutely irreducible representations.

For the second series of examples let \(D_p\) denote the dihedral group of order \(2p\) with an odd prime \(p\). It has \(\frac{p-1}{2}\) complex irreducible representations of degree two, each with \(L_V = K_V = \mathbb{Q}[w_p + w_p^{-1}]\), where \(w_p\) denotes a \(p\)-th root of unity. They are all associated to the same rational irreducible representation. Call any one of them \(V\).

Now consider the group
\[
G = D_p \times D_p \times \mathbb{Z}/2\mathbb{Z} = \langle x_1, y_1 : x_1^p, y_1^2, (x_1 y_1)^2 \rangle \times \langle x_2, y_2 : x_2^p, y_2^2, (x_2 y_2)^2 \rangle \times \langle z : z^2 \rangle,
\]
the subgroup
\[
H = \langle y_1, y_2, z \rangle,
\]
and the pair of rational irreducible representations \(W_j, j = 1, 2, \) of \(G\) associated to \(V \boxtimes V_0 \boxtimes V_1\) and \(V_0 \boxtimes V \boxtimes V_1\) respectively, where \(V\) is as above, and \(V_0\) and \(V_1\) denote the trivial representations for \(D_p\) and \(\mathbb{Z}/2\mathbb{Z}\) respectively. Then one checks that the triple \((G, H, W = (W_1, W_2))\) satisfies the conditions of Hypothesis 3.7. Furthermore, the classes of the subgroups \(G_1 = \langle y_1 \rangle, G_2 = \langle y_2 \rangle, G_3 = \langle y_1 z \rangle, G_4 = \langle y_2 z \rangle, G_5 = \langle x_1 \rangle, G_6 = \langle x_2 \rangle, G_7 = \langle x_1 z \rangle, G_8 = \langle x_2 z \rangle,\) and \(G_9 = \langle z \rangle\) are admissible for the triple \((G, H, W)\). Then one can apply Theorem 4.9 to deduce the following proposition, where for the details of the proof we refer to [CLRR].

**Proposition 5.3.** Let the notation be as above and let \(Z\) be a curve with \(G\)-action and geometric signature \([0; (G_1, m_1), \ldots, (G_9, m_9)]\) with \(m_i\) even for \(1 \leq i \leq 8\). Then, if \(\ell_1 := m_1 + m_3 + 2(m_5 + m_7)\) and \(\ell_2 := m_2 + m_4 + 2(m_6 + m_8)\) are both \(\geq 6\), the associated abelian subvarieties of \(JX\), with \(X = Z/H\) as usual, are Prym-Tyurin varieties of exponent \(p\) and dimension \(\frac{p-1}{4}(\ell_1 + \ell_2 - 8)\).

**References**

[BL] Ch. Birkenhake, H. Lange: *Complex Abelian Varieties*. 2nd ed, Grundl. der Math. Wiss., 302, Springer (2004).

[Bo] N. Bourbaki: *Groupes et algèbres de Lie*. Chapitres 4, 5 et 6, Hermann (1968).

[CaRo] A. Carocca, R. E. Rodríguez: *Jacobians with group actions and rational idempotents*. J. Algebra 306 (2006), 322-343.

[CLRR] A. Carocca, H. Lange, A. Rojas, R. E. Rodríguez: *Prym-Tyurin varieties using self-products of groups*. Submitted (2008).

[CR] C.W. Curtis, I. Reiner: *Representation Theory of Finite Groups and Associative Algebras*. John Wiley (1988).

[H] M. Hall: *Combinatorial Theory* (Second Edition). John Wiley (1986).

[K1] V. Kanev: *Principal polarizations of Prym-Tyurin varieties*. Compos. Math. 64 (1987), 243-270.
V. Kanev: *Spectral curves and Prym-Tyurin varieties I*. Proc. of the Egloffstein conf. 1993, de Gruyter, 151-198 (1995).

H. Lange, S. Recillas: *Abelian varieties with group action*. J. reine angew. Math. 575 (2004), 135-155.

H. Lange, S. Recillas, A. Rojas: *A family of Prym-Tyurin varieties of exponent 3*. Journ. of Alg. 289 (2005), 594-613.

H. Lange, A. Rojas: *A Galois-theoretic approach to Kanev’s correspondence*. Manuscripta Math. 125 (2008), no. 2, 225-240.

MacDonald: *Symmetric functions and Hall polynomials*. 2nd ed, Oxford Science Publ., Clarendon Press (1998).

J.-Y. Mérimdeol: *Variétés de Prym d’un revêtement galoisien*. J. reine Ang. Math. 461 (1995), 49-61.

A. Ortega: *Prym-Tyurin Varieties coming from correspondences with fixed points*. J. Algebra 311 (2007), 268-356.

A. M. Rojas: *Group actions on Jacobian varieties*. Rev. Mat. Iber. 23 (2007), no. 2, 397-420.

R. Salomon: *Prym varieties associated to graphs*. J. Algebra 313 (2007), 828-845.

G.E. Welters: *Curves of twice the minimal class on principally polarized abelian varieties*. Indag. Math. 49 (1987), 87-109.

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