How does the geodesic rule really work for global symmetry breaking first order phase transitions?

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The chain of events usually understood to lead to the formation of topological defects during phase transitions is known as the Kibble mechanism. A central component of the mechanism is the so-called “geodesic rule”. Although in the Abelian Higgs model the validity of the geodesic rule has been questioned recently, it is known to be valid on energetic grounds for a global U(1) symmetry breaking transition. However, even for these globally symmetric models no dynamical analysis of the rule has been carried to this date, and some points as to how events proceed still remain obscure.

This paper tries to clarify the dynamics of the geodesic rule in the context of a global U(1) model. With an appropriate ansatz for the field modulus we find a family of analytical expressions, phase walls, that accounts for both geodesic and nongeodesic configurations. We then show how the latter ones are unstable and decay into the former by nucleating pairs of defects. Finally, we try to give a physical perspective of how the geodesic rule might really work in these transitions.

I. INTRODUCTION

According to the current models in particle physics, symmetry breaking phase transitions are expected to have occurred in the early stages of the evolution of the universe. There are two mechanisms by which these transitions may have taken place: either by the formation of bubbles of the new phase within the old one (i.e., first order phase transition), or by spinodal decomposition (i.e., second order transition). Although we still do not know which one did actually take place in each particular case, for the electroweak phase transition for instance common opinion inclines more towards the first of these two possibilities. As the first order transition scenario would have it, bubbles of the new phase nucleated within the old one (the nucleation process being described by instanton methods as far as the WKB approximation remains valid [3]), and subsequently expanded and collided with each other until they occupied all of the available volume at the time at which the transition was completed. In the process of bubble collision though, the possibility arises that regions of the old phase become trapped within the new one, giving birth to topologically stable localized energy concentrations known as topological defects (for recent reviews see refs. 3), much in the same way in which these structures are known to appear in condensed matter phase transitions.

From a theoretical point of view, topological defects will appear whenever a symmetry group G is spontaneously broken to a smaller group H such that the resulting vacuum manifold \( M = G/H \) has a non-trivial topology: cosmic strings for instance (vortices in two space dimensions) will form whenever the first homotopy group of \( M \) is non-trivial, i.e., \( \pi_1(M) \neq 1 \). To see how this could happen in detail, let us consider the Lagrangian

\[
\mathcal{L} = \partial_\mu \Phi \partial^\mu \Phi - V(\Phi)
\]

for a complex scalar field \( \Phi \). Let us assume that \( V \) is of the type \( V = \frac{1}{4}(\Phi^2 - \eta^2)^2 \), and that its parameters are functions of the temperature such that at high temperatures \( \Phi = 0 \) is the only minimum of \( V \), while at zero temperature all the \( |\Phi| = \eta \) states correspond to different degenerate minima. Then the structure of the vacuum manifold will be that of \( S^1 \). \( \pi_1(S^1) \neq 1 \) however, and thus we can form non-contractible loops in the vacuum manifold. That is, the model admits cosmic string solutions.

The way in which these configurations would actually arise during a phase transition is via the Kibble mechanism [3]. In the context of a first order transition the basic ideas behind it are that bubbles are nucleated with random phases of the field, and that when two regions in which the phase takes different values encounter each other the phase should interpolate between this two regions following a geodesic path in the vacuum manifold—the so-called geodesic rule. A possible scenario for vortex formation in two space dimensions would then look like this: three bubbles with respective phases of 0, \( 2\pi/3 \), and \( 4\pi/3 \) collide simultaneously. Then, if we walk from the first bubble to the second one the geodesic rule tells us that the phase has to grow from 0 to \( 2\pi/3 \) along our path (as opposed to decreasing from 0 to \(-4\pi/3\)). In the same manner, walking from the second bubble to the third one we will see the phase grow from \( 2\pi/3 \) to \( 4\pi/3 \), and if we finally walk from the third bubble back to the first one again we will see a change from \( 4\pi/3 \) to \( 2\pi \). The phase will have thus wound up by \( 2\pi \) in our path, having traversed the whole of the vacuum manifold.

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once along the way. Continuity of the field everywhere in the region inside our path demands then that the field be zero at some point in this region, namely the vortex core. In the limit in which the bubbles extend to infinity, outwards from the center of collision, removal of this vortex would cost us an infinite amount of energy, since it would involve unwinding the field configuration over an infinite volume. The vortex is thus said to be topologically stable. In three space dimensions, the resulting object would obviously be a string, rather than a vortex. Clearly though there are other ways in which strings could be formed. Collisions of more than three bubbles could also lead to string formation, or, for instance, two of the bubbles could collide first, the third one hitting them only at some later time while the phase is still equilibrating within the other two. This event in particular will be far more likely than a simultaneous three way (or higher order) collision, and it is probably the dominating process by which strings are formed (especially if nucleation probabilities are as low as required for WKB methods to be valid). When two bubbles collide, the phase will try to reach a homogeneous distribution within the single true vacuum cavity formed after the collision. However, in the absence of coupling between the scalar and the plasma that would also be present in a cosmological setting, this process is never completed—essentially because the velocity at which the phase propagates inside the bubbles is the same as that with which the bubble walls expand. Thus, this does not substantially modify the picture of defect formation described above.

The geodesic rule is therefore central in that not only gives us a heuristic view of how bubbles interact after they collide, but it also yields an estimate of the density of defects produced in a phase transition, i.e., about one per every three bubbles (times the probability that the three bubbles have the correct phases of course). If one is concerned about the density of defects that are present after a cosmological phase transition, the question of the validity of the rule is therefore of obvious importance. Despite its importance however, the rule still presents serious problems in its understanding. For globally symmetric models the role of the geodesic rule was studied by Srivastava some time ago. In he established its validity within these type of models mainly on energetic grounds, the geodesic path being the least energetic of all the possible paths that the phase can follow. Recently however a number of authors have argued for and against the validity of the rule or observed what seems to be a breakdown of the rule in numerical simulations, within the context of the Abelian Higgs model. In this case it is rather less clear whether one can reason on energetic grounds since the phase difference between two points is not a gauge invariant quantity by itself. In principle therefore one could always choose a gauge in which the phase is zero everywhere, and it would not be immediately obvious how the geodesic rule would work in this situation. The goal of this paper is to take a step back and try to give a dynamical account of how the geodesic rule may actually work for a globally U(1) symmetric model. In order to do this we will use a type of analytical solutions, hereafter referred to as phase walls, by which the phase interpolates between its values within two bubbles. We will also resort to numerical simulations to study the stability of different phase wall configurations, and finally we will discuss the physical settings in which non-geodesic phase walls can appear. The paper is organized as follows: in section II we give a brief account of the model and the results previously found in relevant for this paper. We then extend our treatment in section III to include the case in which the scalar field is not dissipatively coupled and we have 2+1 dimensions. In section IV we study the stability of non-geodesic phase walls. In V we study the influence of our work in three bubble collisions and topological defect nucleation, and in VI we present a general view of how the geodesic rule might work in globally symmetric models. Finally, the conclusions and the implications that our analysis may have for the Abelian Higgs mode are presented in section VII.

II. 1 DIMENSIONAL PHASE WALLS IN A DAMPING ENVIRONMENT

A. expression for the interpolating phase

Consider the Lagrangian for a complex field Φ. We will use the same form of potential that was used in, that is,

\[ V = \lambda \left( \frac{1}{2} \Phi^2 (\Phi - \eta)^2 - \frac{\epsilon}{3} |\Phi|^3 \right). \]  (2)

This is just a quartic potential with a minimum at |Φ| = 0 (the false vacuum), and a set of minima connected by a U(1) transformation (true vacuum) at |Φ| = \( \rho_\nu \equiv \frac{\eta}{2} \sqrt{3 + \epsilon + \sqrt{(3 + \epsilon)^2 - 8}} \), towards which the false vacuum will decay via bubble nucleation. It is the dimensionless parameter \( \epsilon \) that is responsible for breaking the degeneracy between the true and the false vacua.

The equations of motion for this system are then

\[ \partial_\mu \partial^\mu \Phi = -\partial V/\partial \Phi. \]  (3)

For the potential (2), approximate solutions of (3) exist for small values of \( \epsilon \), the so-called thin wall regime, and are of the form

\[ |\Phi| = \frac{\rho_\nu}{2} \left[ 1 - \tanh \left( \frac{\sqrt{\lambda \eta}}{2} (\chi - R_0) \right) \right], \]  (4)

where \( R_0 \) is the bubble radius at nucleation time and \( \chi^2 = -t^2 - x^2 \). The bubble then grows with increasingly fast speed and its walls quickly reach velocities of order 1. In however the authors were concerned about a...
model with overdamped motion of the walls due to the interaction with a surrounding plasma. In order to model this effect, they inserted a frictional term for the modulus of the field in the equation of motion, namely

\[ \partial_t \partial^\mu \Phi + \gamma |\Phi| e^{i\theta} = -\partial V/\partial \Phi, \]

(5)

where \( |\Phi| = \partial \Phi/\partial t \), \( \theta \) is the phase of the field, and \( \gamma \) stands for the friction coefficient—which will serve as parameter under which we will hide our lack of knowledge about the detailed interaction between the wall and the plasma. It can be shown (see \[8\]) that this equation does indeed possess a solution that shows the desired type of overdamped motion for the bubble walls. In the thin wall limit, this solution can be written as

\[ \rho = \frac{\rho_v}{2} \left[ 1 - \tanh \left( \frac{\sqrt{\lambda} \eta (x - \psi_{\text{ter}} - R_0)}{2 - \sqrt{1 - \psi_{\text{ter}}^2}} \right) \right], \]

(6)

which is simply a spherically symmetric Lorentz-contracted moving domain wall with a velocity \( \psi_{\text{ter}} \) of the form \( \psi_{\text{ter}} \sim \varepsilon \delta_m/\gamma \rho_v^2 \), where \( \delta_m = 1/\sqrt{\lambda} \) is the bubble wall thickness.

In two bubble collisions, the phase will at first—well before the bubble walls actually collide, at least in a zero temperature situation—interpolate between the values that it takes in each bubble by means of a phase wall situated at the midpoint in between the bubbles. This can be seen by taking the value of the modulus of the field \( \rho \) far away from the bubbles as that given by the ansatz

\[ \Phi(\text{bubble}1 + \text{bubble}2) \equiv \rho e^{i\theta} \approx \Phi(\text{bubble}1) + \Phi(\text{bubble}2) \equiv \rho_1 e^{i\theta_1} + \rho_2 e^{i\theta_2}. \]

(7)

Note that we will assume that \( \Phi(\text{bubble}1) \) yields only a correct approximation for \( \rho \), and not for \( \theta \). For \( \rho_1, \rho_2 \) we will take an exponentially decaying ansatz in a 1 dimensional approximation

\[ \rho_1 \approx \rho_v e^{(x + vt - x_0)/\delta_m}, \]

\[ \rho_2 \approx \rho_v e^{-(x - vt + x_0)/\delta_m}, \]

(8)

where \( v \) is the terminal velocity of the walls, \( \psi_{\text{ter}} \) above, and the bubble centers are situated at \( \pm x_0 \), with \( R \leq x_0, R \gg |x_0 - R| \gg \delta_m \) (\( R \) being the bubbles radii). That is, we are basically asking that the distance from the bubble walls to the origin \( |x_0 - R| \) be several times \( \delta_m \) (so that we fields decay exponentially there), and to have thin wall bubbles with \( R \) large enough so that the 1-d approximation is valid. Then \( \Phi(\text{bubble}1) \) yields for the combined modulus

\[ \rho^2 \approx \rho_v^2 e^{2(vt - x_0)/\delta_m} \{ 2 \cosh(2x/\delta_m) + 2 \cos(\theta_1 - \theta_2) \}. \]

(9)

We can now use this expression in the equation of motion for the phase \( \theta \), to get

\[ \dot{\theta} - \theta'' + \frac{2v}{\delta_m} \dot{\theta} - \frac{2\delta_m^{-1} \sinh(2x/\delta_m)}{\cosh(2x/\delta_m) + \cos(\theta_1 - \theta_2)} \theta' = 0, \]

(10)

where \( \theta' \equiv \partial \theta/\partial x, \dot{\theta} \equiv \partial \theta/\partial t \). Note that for an initial phase difference \( \theta_1 - \theta_2 = \pi \) the modulus of \( \Phi(\text{bubble}1) \) is zero at the midpoint between the bubbles. In this case then the denominator in the last term of (10) goes to zero due to the fact that the phase has the shape of a step function as we go over the origin, switching discontinuously from \( \theta_1 \) to \( \theta_2 \). On the other extreme, if the initial phase difference is zero there is of course no dynamics to it. We will then focus in an intermediate situation, and find solutions to (10) for a phase difference of \( \pi/2 \). Using the ansatz \( \theta = T(t)X(x) \), we can separate variables to obtain:

\[ \ddot{T} + \frac{2v}{\delta_m} \dot{T} + k^2 T = 0, \]

\[ X'' + \frac{2}{\delta_m} \tanh(\frac{2x}{\delta_m}) X' + k^2 X = 0, \]

(11a)

(11b)

\( k \) being the separation constant. The equation for \( T \) is obviously that of a damped oscillator. For \( k = 0 \), a \( T = \text{const.} \) solution exists consistent with our boundary condition that \( \theta \) goes to \( \theta_{1,2} \) as \( x \to \mp \infty \) at all times, while, for \( k \neq 0 \) all solutions will present unwanted temporal dependence temporal dependence of \( \theta \) even as \( x \to \mp \infty \). (Note that in case the lifetime of any transient configuration that might appear is only of the order \( \delta_m/v \), since the equation for \( T \) is that of a damped oscillator). We take \( k = 0 \) then, whence an immediate integration of (11a) leads to the equation for \( X' \) (or equivalently \( \theta' \) since \( T = \text{const.} \))

\[ \theta' = \frac{K}{\cosh(2x/\delta_m)}, \]

(12)

with \( K \) being an integration constant with dimensions (length)\(^{-1}\). This equation admits a family of general solutions given by the static sine-Gordon kink form

\[ \theta = K \delta_m \arctan(\exp(2x/\delta_m)) + D = C \arctan(\exp(2x/\delta_m)) + D, \]

(13)

with \( K \delta_m = C \), as can be easily checked by direct differentiation. (Note that this is not the original form in which these solutions appeared in \[8\], the expression found there is equivalent to this standard sine-Gordon form, though). The solution to the equation of motion for the phase \( \Phi(X) \) interpolating from say \( \theta_1 = 0 \) at \( x \to -\infty \) to \( \theta_2 = \pi/2 \) at \( x \to +\infty \) will then simply be

\[ \theta(x) = \arctan(\exp(\frac{2x}{\delta_m})), \]

(14)

which clearly shows the structure of a phase wall placed at the origin and of width closely related to the bubble wall’s width \( \delta_m \)—although this relation will change as the
phase difference between the bubbles in (10) change. It is also clear from this expression that this wall interpolates from $\theta_1$ to $\theta_2$ following the shortest (geodesic) path in the vacuum manifold, since $\theta$ monotonically increases from 0 to $\pi/2$.

Since $\theta$ admits sine-Gordon solutions though, it must be possible to write its equation of motion (13) in a sine-Gordon manner. Let us use the form of $\theta(x)$ given by (13) and assume without loss of generality that the phase at $x \to -\infty$ is zero, so that $D = 0$. It is then not difficult to see that

$$\sqrt{\frac{C^2}{2\delta_m^2} \left[ 1 - \cos \left( \frac{\theta(x)}{C} \right) \right]} = \frac{C}{\delta_m \cosh (2x/\delta_m)}.$$

(15)

Therefore, with $V(\theta) = \frac{C^2}{4\delta_m^2} \left[ 1 - \cos \left( 4\theta/C \right) \right]$, the expression

$$\frac{d\theta}{dx} = \sqrt{2V(\theta)}.$$

(16)

is a sine-Gordon equation which has (13) as its general solution. $V$ has its minima located at $\theta = mC\pi/2$, $m$ being an integer. Note though that (14) is a static sine-Gordon equation, it thus only contains the kink and antikink sectors (apart from the vacuum of course) and its solutions only connect adjacent vacua.

Going back to our particular problem, where $\Delta \theta = \theta_2 - \theta_1 = \pi/2$, we can now use the fact that the physical phase $\theta$ is defined only up to modulo $2\pi$ to choose $\theta_2 = \pi/2 + 2\pi n$, with $n = -2, -1, 0, 1, 2, ...$ being an integer. It immediately follows that (14) is not the only solution consistent with our physical boundary conditions. In fact there is an infinite family of solutions to (12) having $\Delta \theta = \pi/2 + 2\pi n$ which will satisfy our requirements, of which (14) is only the $n = 0$ case. An interesting case corresponds to $n = -1$. This phase wall forces $\theta$ to monotonically decrease from $\theta_1 = 0$ at $x \to -\infty$ to $\theta_2 = -3\pi/2$ at $x \to +\infty$, and therefore follows the shortest non-geodesic path in the vacuum manifold between $\theta_1$ and $\theta_2$. Its form is

$$\theta(x) = -3 \arctan(\exp(\frac{2x}{\delta_m})).$$

(17)

Lower negative values of $n$ will correspond to phase walls that completely wind around the vacuum manifold $-|n|+1$ times in the negative $\theta$ direction before going from 0 to $-3\pi/2$, whereas positive values of $n$ will lead to phase walls carrying a (positive) winding number $n$. (Hereafter in the paper, when using the word “winding” to denote the phase change across a phase wall it should always be understood that there is no phase circulation around any closed path in physical space, as opposed to the times in which “winding” is used to refer to the circulation of the phase around a vortex or antivortex). The expression giving $\theta(x)$ for a general value of $n$ will then simply be

$$\theta(x) = [1 + 4n] \arctan(\exp(\frac{2x}{\delta_m})).$$

(18)

### III. PHASE WALLS IN 2 SPATIAL DIMENSIONS WITHOUT DISSIPATION

We will now extend the analysis carried out previously to the case in which there is no dissipative coupling of the scalar field to the rest of the matter present in the model. Also, and since we will be needing it in order to be accurate in the numerical simulations that we will perform, we will introduce 2 spatial dimensions in the problem.

The problem of two bubbles approaching each other and colliding in 2+1 dimensions is invariant under the SO(1,1) Lorentz group. If we take the bubbles to nucleate initially at $(0,\pm x_0,0)$ with coordinates $(t,x,y)$ and signature $(-,+,+)$, then the symmetry of the problem dictates that we must have $\rho = \rho(x,\tau)$ and $\theta = \theta(x,\tau)$, where $\tau^2 = y^2 - t^2$. Because of the particular expression that we will use as the approximate value of the modulus of the field though, a different set of coordinates turn out to be more convenient. We define new coordinates $u, v, \alpha$

$$x = x_0 \cosh u \cos v,$$

(19a)

$$y = x_0 \sinh u \sin v \cos \alpha,$$

(19b)

$$t = x_0 \sinh u \sin v \sinh \alpha.$$  

(19c)

Note that

$$\tau = x_0 \sinh u \sin v,$$

(20)

thus to preserve Lorentz invariance in these coordinates is simply to say that $\rho$ and $\theta$ can not depend on $\alpha$. The range of variation of $u, v, \alpha$ will be

$$u \geq 0; \quad -\pi/2 \leq v \leq \pi/2; \quad \alpha \geq 0,$$

(21)

and in terms of these coordinates the equation of motion for $\theta$ will read (dropping already any dependence on $\alpha$)

$$\partial_{uu} \theta + \partial_u \theta + \cosh u \cdot \partial_v \theta + \cot v \cdot \partial_v \theta = \frac{2}{\rho} \left( \partial_u \rho \partial_u \theta + \partial_v \rho \partial_v \theta \right) = 0.$$

(22)

Now we have to specify an ansatz for $\rho$. As in the case before, we will take the modulus of the field between the bubbles to be given by the modulus of the sum of the two individual bubble fields. Thus,

$$|\Phi|^2 = \rho^2 = |\rho_1 e^{i\theta_1} + \rho_2 e^{i\theta_2}|^2 = \rho_1^2 + \rho_2^2 + 2\rho_1 \rho_2 \cos(\Delta \theta).$$

(23)

Let us now define the invariant distance of any point to the bubble nucleation events, $\chi_{1,2}$

$$\chi_{1,2} = \sqrt{(x \pm x_0)^2 + y^2 - t^2},$$

(24)

where the plus sign holds for $\chi_1$ and the minus for $\chi_2$. Looking back at (14) it is obvious that outside the bubbles and away of the their walls we should have
\[ \rho_{1,2} \simeq \rho_1 e^{-(\chi_{1,2} - R_0)/\delta_m}. \]  
(25)

Taking this into (23), setting already \( \Delta \theta = \pi/2 \), and writing the result in terms of \( u \) and \( v \) leads to an expression for \( \rho \)

\[ \rho^2 = 2\rho_0^2 e^{2R_0/\delta_m} e^{-2x_0 \delta_m \cosh u \cos v}, \]  
(26)

as can be checked by trivial, if lengthy, manipulations. Then plugging this form for \( \rho \) into (22) results in

\[ \partial_{uv} \theta + \partial_{vu} \theta + \left[ \cosh u - \frac{2x_0}{\delta_m} \sinh u \right] \partial_u \theta + \left[ \cot v - \frac{2x_0}{\delta_m} \sin v \tanh \left( \frac{2x_0}{\delta_m} \cos v \right) \right] \partial_v \theta = 0 \]  
(27)

In spite of the appearance of this equation, it is actually not too difficult to find the type of solutions that we are looking for, at least within some approximation. Note that we expect the phase walls to have their central axis located at \( x = 0 \), the line equidistant to the bubble centers, which in \( u, v \) coordinates corresponds to \( v = \pi/2 \). If we expand the second bracket in powers of \( v \) around \( v_0 = \pi/2 \) it is easy to see that neglecting the \( \cot v \) term when compared to the other term in the bracket amounts to neglecting terms of order 1 or smaller when compared to terms of order \( (2x_0/\delta_m)^2 \) or higher in the coefficients of the series. Since within the approximation that we are working with \( x_0/\delta_m \gg 1 \), we conclude that to an accuracy \( 1/(2x_0/\delta_m)^2 \) we can safely neglect the first term in that bracket. Assuming as ansatz that the solutions that we are looking for depend only in \( v \) leads then to

\[ \partial_{uv} \theta - \frac{2x_0}{\delta_m} \sin v \tanh \left( \frac{2x_0}{\delta_m} \cos v \right) \partial_v \theta = 0 \]  
(28)

which indeed has phase wall solutions of the form

\[ \theta = |4n + 1| \arctan(\exp(\frac{2x_0}{\delta_m} \cos v)), \]  
(29)

or, in terms of \( \chi_{1,2} \)

\[ \theta = |4n + 1| \arctan(\exp(\frac{\chi_1 - \chi_2}{\delta_m})). \]  
(30)

Expression (30) is clearly again a sine-Gordon kink, differing from (18) only in its argument. Note however that the fact that (30) depends on \( \chi_1 - \chi_2 \) implies that it does not represent a planar wall. This is depicted in figure 4, where we show constant phase lines for the geodesic phase wall interpolating from 0 to \( \pi/2 \). Thus, lines of constant phase will have a curvature which will depend on the distance to the bubble centers (in the figure the bubble centers are situated at \( x_0 = \pm 10 \)). If we expand \( \chi_{1,2} \) in terms of \( \chi/x_0 \) though, where \( \chi = (t, x, y) \), (30) reduces to the 1-dimensional expression (18) above for as long as \( (x/x_0)^2 \sim (y/x_0)^2 \sim (t/x_0)^2 \ll 1 \). As in the case studied in the previous section, \( n = 0 \) will correspond to the geodesic phase wall solution, while higher (or lower) values of \( n \) lead to phase walls that wind around the vacuum manifold of the model \( n \) (or \( |n| - 1 \)) times. As before, \( n = -1 \) corresponds to the shortest non-geodesic path. We have thus shown that the existence of phase wall solutions is a general feature of the model.

IV. PHASE WALL STABILITY ANALYSIS

Since an analytical approach proved to be too involved, we resorted to numerical simulations in order to investigate the problem of the stability of non-geodesic phase walls— the geodesic one being obviously stable. We used a straightforward leapfrog method with second order accuracy to evolve in time different initial configurations corresponding to phase walls of various winding numbers. In order to better understand the process of phase wall decay we first studied the evolution of the 1-dimensional solutions, then carried on the study to two dimensional configurations.

We begin in figure 3 by plotting, using polar coordinates in field space, the fate of the shortest non-geodesic wall through a series of snapshots. In (a) we can see the initial configuration of the phase at \( t = 0 \), going from \( \theta = 0 \) to \( \theta = -3\pi/2 \). Subsequently we see how in the very core of the wall, the field starts decreasing its modulus, eventually crossing the origin (\( \rho = 0 \)), and finally settling down in the geodesic configuration, in which \( \theta \) goes from \( \theta = 0 \) to \( \theta = \pi/2 \), by \( t = 2 \) in units of the bubble wall thickness \( \delta_m \). In figure 3 we have depicted the unwinding of a \( n = -2 \) wall. In (a) we can see how the initial configuration has, as expected, an extra \( 2\pi \) winding over the \( n = -1 \) wall previously seen. Note however in this case the decay proceeds by unwinding first this extra \( 2\pi \) loop, after which we are left with a configuration equivalent to that of a \( n = 1 \) wall which in turn decays as described above. The unwinding of the two \( 2\pi \) is shown in figure 3. In it we can see how the loop is progressively displaced and shrunk, until the field crosses the origin at the midpoint between the bubble centers. After that, the field reconfigures itself. The gradient energy acts as a tension on the phase loop, which eventually evolves into a cusp. This cusp then divides into two sharp kinks that move away from each other. As we can see then the main features that seem to emerge from these results are that a phase wall of winding \( n \) will cross the zero field value \( |n| \) times in its decay, decreasing \( n \) in steps of one, and that the time scale of the decay is of the order of \( \delta_m \), becoming progressively shorter as the winding increases.

Using these results it is not too difficult to convince oneself of the fact that in two spatial dimensions we should in general expect the decay process to produce pairs of vortices and antivortices. Let us think for instance of the symmetry axis of a \( n = -1 \) phase wall, that is, the line of points that are equidistant to the bubb-
ble centers. (For a 1-dimensional wall, the symmetry axis is of course reduced to a single point: the central point equidistant to the bubble centers). In two space dimensions we may think of the $n = -1$ wall decay as being the result of all the points in this axis crossing the zero field value —each following a similar process as the 1-dimensional case— and reeqnarraying to follow the geodesic path. The points along the wall’s symmetry axis are not all equivalent to one another though, since the distance to the bubble centers changes as we move along the axis. Therefore the field does not cross the zero value everywhere along the axis simultaneously, but rather some points will go through this value before others. In particular, given the symmetry of the problem it seems clear that the central point of the axis, with the smallest distance to the bubble centers, will cross the zero first. Once the field has switched to the geodesic path along the line that joins the bubble centers, the two points immediately above and below the central point will cross the zero themselves, then the ones respectively above and below these two, and so on. Therefore, as the phase shifts towards the geodesic path we always have a pair of points on the wall axis at which the field takes the false vacuum value, both of them quickly moving up and down respectively. Since the direction of the field in the region that has already switched to the geodesic path is opposite to its direction immediately above and below it, we end up having a net circulation of the phase around these two points. We have thus produced a vortex-antivortex pair along the phase wall axis, the two defects moving away from each other.

In figure 5(a) we can see how this process takes place. In it we can see the contour lines of the modulus of the field, while the phase is represented by the angle that the arrows form with the $x$ axis. The length of the arrows is proportional to the modulus of the field at the point at which it is drawn, except in the central region with $\rho < 0.001$ where they have been magnified to a common length so that we can appreciate the decay process. Throughout the simulations the true vacuum value of the field is $\rho_{tv} \sim 1$. In figure 5(b) we have the initial $n = -1$ configuration, if we follow the change of the phase horizontally along the central $y = 0$ line for instance we can see how the phase steadily decreases from $0$ to $-3\pi/2$. In figure 5(c) we see how the field starts decreasing its modulus, especially in the central region between the two bubbles. Eventually, as described above, the central point will cross the $\rho = 0$ value first. After this has occurred the points above and below it will follow suit, then the ones respectively above and below these two, and so on. Due to the opposing phase directions above and below the $\rho = 0$ points though a vortex-antivortex pair will form, as can be seen in 5(c). As successive points along the axis cross the zero, the defects move further and further away from each other, as shown in 5(d). Note how on 5(d) the phase still has some reeqnarrayment left to do in the region that has crossed the zero field value. It is below zero right before the wall axis, and over $\pi/2$ right after it as we move from left to right. This situation is analogous to what we see in figure 1(b) and (c) in the 1-dimensional case. Eventually, the geodesic configuration is attained.

In figure 5 we can see the same process taking place for the $n = -2$ phase wall. As was the case with the 1-dimensional solution, this configuration first decays by unwinding its extra $2\pi$ to become a $n = -1$ wall. In order to do this however it must cross the $\rho = 0$ value as we saw in 1 dimensions. The $2\pi$ unwinding will thus make the field cross its zero value, first at the origin, then generate two zeroes of the field that move away from each other along the wall axis following a process analogous to the one just seen. As before thus, a pair of defects is created in this decay. Although not shown in the figures, the $n = -1$ wall will then in turn decay to the geodesic configuration generating a second pair of defects. Therefore, the final unwinding of the $n = -2$ wall produces two pairs of defects. The rule that immediately seems to suggest itself is that a phase wall of winding $n$ will produce precisely $|n|$ pairs of defects in its decay. The simple reason behind it would be that for such a wall the field needs to cross $|n|$ times the $\rho = 0$ value in order to complete its decay, and that each time that the field crosses the zero value at the central point a vortex-antivortex pair is produced. One should be careful though. Note that the decay of the $n = -2$ wall to the $n = -1$ configuration sends and vortex into the lower half axis of the wall, whereas the decay of the $n = -1$ wall sends an antivortex into the same region. Since global vortices and antivortices attract each other one could worry that these pairs could tend to annihilate, especially in the case of high $n$ walls where the defects will be produced within a short distance of each other. It would seem though that such an annihilation can not actually happen, since each defect carries away as phase circulation around it one of the winding factors present in the wall. (i.e., think for instance of the $n = -2$ case. The successive vortex and antivortex produced in the decay can not annihilate each other because, if they did, the phase would have to go from the region in which the wall has not decayed yet into the region in which it has, having no zeros around which it can wind in between). It is unclear at the moment whether this same argument holds for higher $n$ walls however. More simulations should perhaps be carried to test this case.

V. NON GEODESIC PHASE WALLS AND THREE BUBBLE COLLISIONS: TOPOLOGICAL DEFECT NUCLEATION

If non geodesic phase walls can indeed be formed in two bubble collisions and they subsequently decay by nucleating vortices, then the next obvious question is: what will be the impact of these processes on the probability of topological defect becoming trapped in a region of true vacuum when three bubbles collide?. The standard
picture of defect nucleation described in the introduction roughly yields a prediction of 1 defect per every three bubbles (times a proportionality constant to account for the probability of the three bubbles having the appropriate phase). Let us think of three colliding bubbles with phases such that they would nucleate a defect following the geodesic rule: is the defect still formed when the phase chooses a non-geodesic path between two of the bubbles? Alternatively, if the initial phases would not yield a vortex following the geodesic rule, will a vortex be nucleated if a non-geodesic wall is formed between any two bubbles? Clearly, these questions can potentially alter the standard prediction for vortex nucleation probability. We have therefore simulated these two scenarios to find out their answers.

In figure 6(a) we can see the results of the first case. In it we have three bubbles with phases $\pi/6$, $5\pi/6$ and $3\pi/2$ that in principle would nucleate a vortex following the geodesic rule. The two upper bubbles however have collided slightly before, hitting the third one a little later, and the phase has wound up between them following a $n = -1$ phase wall. As we can see in figure 6(a) though, the field does not reach its zero value anywhere in the region between the three bubble centers, as it would be the case if the phase were to follow the geodesic path everywhere. This is the key of the question here: if the geodesic rule were to be holding, the vortex predicted by it would appear in the region between the three bubbles as soon as they make contact with each other, in this case however the vortex is not there to begin with. As we have seen though, the phase wall has to decay producing a vortex-antivortex pair. In the following figures we can then see how the vortex produced in the decay ends up resting between the three bubbles, while the antivortex escapes “upwards” and away from the bubbles, into a region that gets exponentially closer and closer to the false vacuum. Thus, in terms of stable vortices—that is, vortices surrounded by regions of true vacuum—the net result is exactly the same as the one predicted by the geodesic rule. There is however, one extra antivortex that has been expelled into the near-false vacuum region.

The second case is depicted in figure 6. Clearly the phases inside the bubbles would not produce a vortex should the geodesic rule hold. As before, a $n = -1$ phase wall has formed between the upper two bubbles, and in this case the field already has a zero in the region between the bubble centers as we can see in figure 6(a). When the phase wall unwinds, the resulting vortex annihilates with the existing antivortex, therefore yielding again the same result as the geodesic rule would predict: no stable defects are formed. Again, the other antivortex resulting from the phase wall decay escapes away from the bubbles.

Although we have not carried out simulations with phase walls of higher winding, it is easy to convince oneself that the results will not change for higher $|n|$ configurations. For instance, to reproduce with a higher winding an initial situation analogue to the one in figure 6(a) we would need a phase wall with $n = -3$. However, as this wall decays first to $n = -2$ and then to $n = -1$ it nucleates into the region between the bubbles first a vortex, then an antivortex. Obviously this pair annihilates as it gets trapped between the bubbles, thus leaving us with the same initial state as the one we have studied. The only difference would lie in the number of defects that are expelled away from the bubbles.

VI. A MODEL FOR HOW THE GEODESIC RULE WORKS IN PHASE TRANSITIONS WITH GLOBAL U(1) SYMMETRY

Since the non-geodesic phase walls and their decay described in the sections above were found to form in the region between the bubbles where the modulus has an exponentially low value, the preceding considerations would seem to be only an academic exercise. In this section we will however attempt to give a physical picture of how the geodesic rule could work in models with global symmetry.

We will start by assuming that the physical set up in which the process takes place is that of a first order phase transition at finite temperature. The oscillations in the value of the field around its false vacuum value due to the finite temperature effects will yield a background thermal field. These field fluctuations will each have a phase, and we will denote the mean value of their modulus over space and time as $<\rho_{th}>$. They will also have an average size (radius) and lifetime, $<\xi>$ and $<\tau>$. We will also assume that $<\rho_{th}>$ is still well below the true vacuum value of the field $\rho_{tv}$, so that the thermal fluctuations do not strongly affect the field inside each bubble and the values of the phase of each of them, $\theta_1$ and $\theta_2$, remain well defined and largely unaffected by the fluctuations. It is against this thermal background that the bubble fields, $\phi_1$ and $\phi_2$, will propagate. The $\rho_{1,2}$ will obviously have the shape of the bubble profile, but any trace of them will be lost soon after their value becomes lower than $<\rho_{th}>$ since beyond the bubble walls they decay exponentially with a typical decay distance $\delta_m$. Therefore we will assume that beyond a few $\delta_m$ from the point at which $\rho_{1,2} < <\rho_{th}>$ the thermal background field will effectively erase any presence of the bubble fields. Figure 7 exemplifies this picture. The two bubbles will first collide along the line that joins their centers. Roughly, this will happen at about the time when their combined modulus at the point equidistant to the centers $\rho_d$—given by the ansatz (1) evaluated at $x = \theta$—reaches a value equal to the thermal fluctuation average $\rho_d < <\rho_{th}>$, and the whole collision process will take place on a region that we will characterize by a length $d$. Although $d$ will most likely be of the order of a few $\delta_m$, we will leave it largely unspecified for the time being. As we will see this will not have a big impact in our conclusions as far as this simple argument is concerned. When the collision takes place the phase has to decide which path will it follow in the vacuum manifold in order to interpolate from $\theta_1$
to \( \theta_2 \). Obviously the value of the phase of the thermal fluctuations in the collision region will have an important influence on what configuration is finally chosen, since by continuity the phase will be forced to interpolate between \( \theta_1 \) and \( \theta_2 \) going through the value that it takes at the collision region. After that, \( \rho_0 \) will rise above \( \rho_{th} \) exponentially fast and as far the thermal effects are concerned the configuration chosen by the phase at that point will remain frozen thereafter. We will work in three spatial dimensions, and as elsewhere in the paper we will take \( \theta_1 = 0 \) and \( \theta_2 = \pi/2 \). This case is of particular interest because since \( \Delta \theta \) will always lie randomly between 0 and \( \pi \) (modulo \( 2 \pi \)), the average phase difference between any two given bubbles will precisely be \( \Delta \theta = \pi/2 \). We will examine two limiting cases.

As our first case let us examine a slow collision scenario in which the time scale of the collision \( t_{col} \) (i.e., the time needed by the bubbles to expand their radii by an amount \( d \)) is much larger than the average lifetime of the fluctuations \( \tau \). If \( t_{col} \ll \tau \), fluctuations appearing between the bubbles when they are at a distance \( d \) or less away from each other will be energetically favored to have a phase that falls within the geodesic path from \( \theta_1 \) to \( \theta_2 \). Under these conditions, we will suppose that the probability \( p_n \) of choosing any given configuration with winding \( n \) is weighted by a Boltzmann type of distribution, \( p_n = e^{-\beta E_n} \) where as usual \( \beta = 1/T \) (in natural units, \( k = 1 \)), and \( E_n \) is the energy corresponding to the given configuration. Thus, the relative suppression factor for non-geodesic walls of winding \( n \) with respect to the geodesic configuration will be given by \( e^{-\beta \Delta E} \), where \( \Delta E = E_n - E_0 \). Since both solutions have the same background modulus for the field though, the gradient of the modulus does not contribute to this energy difference, and neither does the potential energy since the potential depends only on the modulus. If we then assume that the collision region is small enough so that the three dimensional energy density can be approximated by its one dimensional limit, the total energy difference will simply be given by the one dimensional integral times the cross sectional area of the collision region, \( d^2 \). As was seen at the end of section II this approximation will be valid for as long as \((d/x_0)^2 \ll 1\), with \( x_0 \) being the distance to the bubble centers, which will be justified in our case. Thus, as an order of magnitude estimate we will have

\[
\Delta E \sim d^2 \int_{-\infty}^{+\infty} \rho_n^2 (\partial \theta_n^2 - \partial \theta_0^2) \, dx \sim < \rho_{th} >^2 d^2 \int_{-\infty}^{+\infty} (\partial \theta_n^2 - \partial \theta_0^2) \, dx, \tag{31}
\]

since the phase gradients are exponentially close to zero outside of the phase wall, where at the moment in which the collision takes place \( \rho_0 \sim < \rho_{th} > \). We can now use the well known results for the sine-Gordon kinks to get

\[
\Delta E \sim < \rho_{th} >^2 d^2 \int [(4n + 1)^2 - 1]. \tag{32}
\]

This is the key expression that will tell us how suppressed the non-geodesic configurations will be. We can see how phase walls with high winding get quickly suppressed, as one would have expected. As expected too, formation of non-geodesic walls is also strongly suppressed for sufficiently high values of \( < \rho_{th} > \). This agrees with the standard argument which states that at least for transitions with global symmetry the geodesic rule must hold since it yields the least energetic configurations. Note however how, for sufficiently low values of \( < \rho_{th} > \) this argument doesn’t hold any more and production of non-geodesic walls could take place, in particular those with low winding. The underlying physical reason is simply that the energy density of the phase wall is given by \( \rho^2 \partial \theta^2 \), therefore, if the wall is placed in a region were \( \rho^2 \) is small the energy difference between the different configurations will also be small. Once the phase has chosen an interpolating path, the modulus of the bubble field will quickly rise against the thermal background, and the influence of the thermal fluctuations will become negligible as stated above. The non-geodesic walls will subsequently decay, the pairs of defects generated by their decay being expelled into the thermal background. It would seem though that in this case we can expect to form non-geodesic phase walls only for small values of \( < \rho_{th} > \).

As our second limit let us examine the fast collision scenario and suppose that \( t_{col} \ll \tau \). In this case the last fluctuation on the scale of the collision region appeared well before the bubbles were close to it. Let us then make the naive assumption that, largely unaffected by the bubbles, fluctuations in the collision region pick up phase values \( \theta_p \) at random from 0 to \( 2\pi \) at the time at which they nucleate, with all values having the same probability. Since by definition the geodesic path from \( \theta_1 \) to \( \theta_2 \) is the shortest one on the vacuum manifold, the field fluctuations in the collision region are more likely to have a phase that falls out of the geodesic path than inside of it. If, whatever value chosen for \( \theta_p \), the paths travelled by the phase to interpolate from \( \theta_1 \) to \( \theta_p \) and then from \( \theta_p \) to \( \theta_2 \) are geodesic, then it is easy to see that any value of \( \theta_p \) between \( \pi \) and \( 3\pi/2 \) will lead to the formation of a \( n = -1 \) phase wall. That is, the probability \( p(n) \) of forming a non-geodesic phase wall in this case will be simply

\[
p(n) = \frac{\pi/2}{2\pi} = \frac{1}{4}. \tag{33}
\]

Note that this is an estimate from below, since phase walls with winding other than \(-1\) are not being consid-
ered at all in this simple picture. (They would correspond to non-geodesic interpolations between the bubble phases and $\theta_p$.) Note also how in this limit the fact that for any two bubbles $\overline{\Delta \theta} = \pi/2$, as pointed above, also implies that the average probability of forming a non-geodesic wall between any two bubbles will be 0.25, since the fact that $0 \leq \Delta \theta \leq \pi$ with all values having equal probability automatically implies that $0 \leq p(ng) < 0.5$ (we exclude the case $\Delta \theta = \pi$ since in that case any of the two paths is geodesic), and that $p(ng) = 0.25$. As before however, any non-geodesic phase wall will quickly decay into the geodesic configuration, expelling vortices and antivortices into the background.

The simple argument we have just presented would seem to indicate then that in the case of slow collisions the geodesic rule is established mostly by the thermal suppression of non-geodesic configurations, while for fast collisions the rule holds because non-geodesic walls are unstable and decay into the geodesic one. Since in both limits the geodesic rule holds, one would also expect this to be the case in the (more physical) intermediate regime in which $t_{col} \lesssim \tau$.

### VII. CONCLUSIONS

Although the geodesic rule is known to hold for first order phase transitions that break a global U(1) symmetry, a dynamical account as to how the field settles into the geodesic configuration was so far missing in the literature. In this paper we have attempted to give precisely such an account, in the hope that some of the findings and the methods here developed could shed some light on the more difficult questions concerning the validity of the rule in the Abelian-Higgs model. Using a family of analytical solutions that represent the phase as it interpolates between its values inside two different bubbles, i.e., phase walls, we have found both the solution corresponding to the geodesic path as well as infinitely many more which wind around the vacuum manifold $n$ times before reaching the desired value. After finding the expression for the phase walls in 2+1 dimensions, we turned our attention to the question of the stability of the non-geodesic configurations. Our analysis shows that, as one would expect, non-geodesic phase walls are unstable, decaying into the geodesic configuration in times of the order of $\delta_{\mu}$.

The decay of the non-geodesic states takes place via the nucleation of vortex-antivortex pairs that quickly move away from each other, and the simple rule that seems to be the result from our study is that a phase wall of winding $n$ will produce precisely $|n|$ pairs of defects in its decay. Although this production of defect pairs does not alter the geodesic rule standard prediction of stable defects, nucleation in three bubbles collisions, it is perhaps worth pointing out that for first order phase transitions with supercooling (where $< \rho_{th} >$ will be low and $< \tau >$ large) a significant number of bubble collisions may potentially result in the production of non-geodesic phase walls and their subsequent decay. Thus, it would seem that for these transitions one could think of the thermal background as of a region in which a gas of defects and antidefects exist. Since the average modulus $< \rho_{th} >$ of this thermal field will be low in supercooled transitions, the origin of these defects could in general be due to both thermal fluctuations of the field and to their production in phase walls decay. For as long as the second mechanism is not dominant, one could think of this gas as being roughly in equilibrium, since, in a large volume thermal fluctuations would be creating and annihilating defects and anti-defects with roughly the same probability. If production via phase wall decay becomes dominant though, the density of defects in the thermal background will tend to grow as the transition proceeds. A number of questions are raised by this scenario. For instance, what will be the final fate of all these defects as the transition reaches its end? Will they all annihilate one another or will a significant fraction survive the transition? Although in a large volume the average number of vortices will be equal to that of antivortices, in a single smaller volume vortices and antivortices need not exactly cancel each other out. What will happen if a three bubble collision traps a net winding of the phase in the basin between the bubbles? Also, could the presence of this defect gas in the thermal background have an impact on models in which baryogenesis is induced by topological defects?

As far as the connection with the Abelian-Higgs model is concerned, some of the results that we have found could be of interest. The motivation for our study lies partly in the hope that one could export to the gauge case the same type of techniques that we have developed to study the global model. This could be done by defining some gauge invariant form of the phase difference (i.e., a la Kibble-Vilenkin for instance [8]), where $\Delta \theta = \int \frac{dx^k D_k \theta}{\sqrt{-g}}$, with $D_k \theta = \partial_k \theta + e A_k$, and $e, A_k$ being the gauge coupling constant and gauge field respectively) and studying its equations of motion. It is also worth pointing out the close resemblance that some of our results bear to those of [4]. In this work, the authors found out that both in the case of the Abelian-Higgs model and the sin $\theta_W = 0$ limit of the Weinberg-Salam model there may be enough energy released from the collision of the bubble walls as to drive the modulus of the Higgs away from its vacuum expectation value and make it oscillate around zero. Extra regions of symmetry restoration are thus produced, and the field oscillations as they decay lead to the formation

\[ i.e., \text{defects surrounded by regions of true vacuum} \]
of multiple windings. In their simulations, they found that the validity of the geodesic rule seemed to depend not only on the gauge coupling, but also in the initial bubble separation and the phase difference between the bubbles. Although in our model the energy needed to take the phase wall away from the geodesic configuration is provided by the background thermal fluctuations, rather than wall collision, note that our two dimensional phase walls also show a dependence both on the distance between the bubble centers and on the initial phase difference. (Although only for $\Delta \theta = \pi/2$ do we have an analytical solution, solutions for other values of $\Delta \theta$ are in principle obtainable numerically). Furthermore, high winding phase walls decay via field fluctuations around the false vacuum value, nucleating multiple defects as they decay.

VIII. ACKNOWLEDGMENTS

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FIG. 1. Lines of constant $\theta$ for a geodesic phase wall going from $\theta = 0$ to $\theta = \pi/2$. The lines are labelled in units of $\Delta \theta$. Thus, a label of 0.5 implies a value of the phase of $\pi/4$, and so on. The bubble centers are placed at $x_0 = \pm 10$, and we have taken $\delta_m = 1$. A lower value of $|x_0|$ would make the contour lines show more curvature for the same axis scale, whereas for larger values the wall would look flatter on this scale.

FIG. 2. The decay of a $n = -1$ phase wall is shown in polar coordinates in field space. Distance to the origin gives the modulus of the field, while the angle corresponds to the phase difference. In (a) the initial configuration at $t = 0$ is shown. (b) corresponds to $t = 0.4$, (c) to $t = 0.6$, and finally (d), in which we can see that the geodesic wall is the final state, corresponds to $t = 2$. Time is measured in units of the bubble wall thickness $\delta_m$.

FIG. 3. We see here the decay of the $n = -2$ phase wall into a configuration equivalent to a $n = -1$ wall. As before, the initial $t = 0$ configuration is shown in (a). Subsequent images correspond to: (b) $t = 0.1$, (c) $t = 0.15$, and (d) $t = 0.2$.

FIG. 4. The shrinkage of the extra $2\pi$ loop is shown here in greater detail. Images correspond to: (a) $t = 0$, (b) $t = 0.11$, (c) $t = 0.13$ by which time the field has already crossed the origin and we no longer have a $2\pi$ phase change around the loop, (d) $t = 0.14$ where we see how the loop has been stretched into a cusp, (e) $t = 0.16$ where the cusp has divided into two sharp kinks that move away from each other, and finally (f) $t = 0.18$.

FIG. 5. A $n = -1$ two dimensional phase wall decays into the geodesic configuration. Solid lines indicate contour lines of the field modulus, while the phase is given by the angle between the arrows and the $x$ axis. The length of the arrows is proportional to the value of the modulus at the point they are drawn, except in the central region occupied by the phase wall, where they have been magnified to a common length. The initial configuration at $t = 0$ is shown in (a). In (b) $t = 0.3$, and we see the modulus of the field quickly decreasing in the central region. In (c) $t = 0.304$, the field has crossed the origin in field space at the central point, and its modulus is now growing there. As described in the text this produces a vortex-antivortex pair, we can see the vortex somewhat above $y = 1$, and the antivortex at $y = -1$. In (d) $t = 0.308$, the defects keep moving away from each other as the field increases its modulus in the central region.

FIG. 6. A 2-dimensional $n = -2$ phase wall decays into the $n = -1$ configuration. The initial configuration is shown in (a) at $t = 0$. To see the extra $2\pi$ factor in the phase follow its change along a horizontal line, the arrows turning always in a clockwise direction. In (b) $t = 0.108$, the field is decreasing its value in the central region, in (c) $t = 0.109$ and the vortex (around $y = -1$) and antivortex ($y = 1$) have already appeared. Note how between the defects the phase is now increasing across the wall from almost $-\pi$ to almost $-\pi/2$, subsequently decreasing again until it reaches $-3\pi/2$. In (d), $t = 0.111$, the defects keep moving away from each other, and the field increases its modulus in the central region.
FIG. 7. A $n = -1$ phase wall between two bubbles with phases $5\pi/6$ (left) and $\pi/6$ (right) decays in the presence of a third bubble with phase $3\pi/2$. In (a) $t = 0$, (b) $t = 0.34$, (c) $t = 0.384$, and (d) $t = 0.8$. The decay sends a vortex into the region between the three bubbles that gets trapped there. The (upper) antivortex is expelled away from the bubbles.

FIG. 8. A $n = -1$ wall decays in the presence of a third bubble. The three bubbles would not form a defect according to the geodesic rule. The net result of the process agrees with that prediction. In (a) $t = 0$, we can already see how, as a consequence of the presence of the $n = -1$ wall we already have an antivortex in the region between the bubble centers. The wall subsequently decays, in (b) $t = 0.34$, and (c) $t = 0.372$, sending a vortex into the interior region. This vortex and the antivortex annihilate each other, leaving in (d) $t = 0.8$ the final configuration predicted by the geodesic rule.

FIG. 9. Two bubbles about to collide expand against a fluctuating thermal field background. Since the bubble fields decay exponentially fast beyond the walls, any trace of them is quickly lost once they get below the thermal field level. The thermal fluctuations are presumed not to be strong enough to significantly alter the bubble fields inside the walls however.
