The niche graphs of doubly partial orders

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Abstract

The competition graph of a doubly partial order is known to be an interval graph. The competition-common enemy graph of a doubly partial order is also known to be an interval graph unless it contains a cycle of length 4 as an induced subgraph. In this paper, we show that the niche graph of a doubly partial order is not necessarily an interval graph. In fact, we prove that, for each $n \geq 4$, there exists a doubly partial order whose niche graph contains an induced subgraph isomorphic to a cycle of length $n$. We also show that if the niche graph of a doubly partial order is triangle-free, then it is an interval graph.

Keywords: niche graph; doubly partial order; interval graph

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1 Introduction

Throughout this paper, all graphs and all digraphs are simple.

Given a digraph $D$, if $(u, v)$ is an arc of $D$, we call $v$ a prey of $u$ and $u$ a predator of $v$. The competition graph $C(D)$ of a digraph $D$ is the graph which has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exists a common prey of $u$ and $v$ in $D$. The notion of competition graph is due to Cohen [3] and has arisen from ecology. Competition graphs also have applications in coding, radio transmission, and modelling of complex economic systems. (See [13] and [15] for a summary of these applications.) Since Cohen introduced the notion of competition graph, various variations have been defined and studied by many authors (see the survey articles by Kim [9] and Lundgren [11]). One of its variants, the competition-common enemy graph (or CCE graph) of a digraph $D$ introduced by Scott [16] is the graph which has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exist both a common prey and a common predator of $u$ and $v$ in $D$. Another variant, the niche graph of a digraph $D$ introduced by Cable et al. [1] is the graph which has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exists a common prey or a common predator of $u$ and $v$ in $D$.

A graph $G$ is an interval graph if we can assign to each vertex $v$ of $G$ a real interval $J(v) \subset \mathbb{R}$ such that whenever $v \neq w$,

$$vw \in E \text{ if and only if } J(v) \cap J(w) \neq \emptyset.$$ 

The following theorem is a well-known characterization for interval graphs.

Theorem 1 ([7]). A graph is an interval graph if and only if it is a chordal graph and it has no asteroidal triple.

Cohen [3,4] observed empirically that most competition graphs of acyclic digraphs representing food webs are interval graphs. Cohen’s observation and the continued preponderance of examples that are interval graphs led to a large literature devoted to attempts to explain the observation and to study the properties of competition graphs. Roberts [14] showed that every graph can be made into the competition graph of an acyclic digraph by adding isolated vertices. (Add a vertex $i_{\alpha}$ corresponding to each edge $\alpha = \{a, b\}$ of $G$, and draw arcs from $a$ and $b$ to $i_{\alpha}$.) He then asked for a characterization of acyclic digraphs whose competition graphs are interval graphs. The study of acyclic digraphs whose competition graphs are interval graphs led to several new problems and applications (see [5, 6, 10, 12]).
We introduce some notations for simplicity. A cycle of length $n$ is denoted by $C_n$. For two vertices $x$ and $y$ in a graph $G$, we write $x \sim y$ in $G$ when $x$ and $y$ are adjacent in $G$. For each point $x$ in $\mathbb{R}^2$, we denote its first coordinate by $x_1$ and the second coordinate by $x_2$.

We define a partial order $\prec$ on $\mathbb{R}^2$ by

$$x \prec y \text{ if and only if } x_1 < y_1 \text{ and } x_2 < y_2.$$  

For $x, y, z \in \mathbb{R}^2$, $x, y \prec z$ (resp. $x, y \succ z$) means $x \prec z$ and $y \prec z$ (resp. $x \succ z$ and $y \succ z$). For vertices $x$ and $y$ in $\mathbb{R}^2$, we write

$$x \searrow y \quad \text{if } x_1 \leq y_1 \text{ and } y_2 \leq x_2$$  

$$x \searrow y \quad \text{if } x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$  

A digraph $D$ is called a **doubly partial order** if there exists a finite subset $V$ of $\mathbb{R}^2$ such that

$$V(D) = V \text{ and } A(D) = \{(v, x) \mid v, x \in V, x \prec v\}.$$  

We may embed each of the competition graph, the CCE graph, and the niche graph of a doubly partial order $D$ in $\mathbb{R}^2$ by locating each vertex at the same position as in $D$. We will always assume that $D$, its competition graph, CCE graph, and niche graph are embedded in $\mathbb{R}^2$ in natural way.

For two vertices $x$ and $y$ of a doubly partial order $D$, if there is a vertex of $D$ in the region

$$\{z \in \mathbb{R}^2 \mid z \prec (\min\{x_1, y_1\}, \min\{x_2, y_2\})\}$$  

$$\cup \quad \{z \in \mathbb{R}^2 \mid z \succ (\max\{x_1, y_1\}, \max\{x_2, y_2\})\}$$  

(see Figure 1), then, by definition, $x$ and $y$ are adjacent in the niche graph of $D$.  

![Figure 1: The region related to the adjacency of $x$ and $y$](image-url)
The competition graph of a doubly partial order is an interval graph, and the CCE graph of a doubly partial order is also an interval graph if it is $C_4$-free:

**Theorem 2** ([2]). The competition graph of a doubly partial order is an interval graph.

**Theorem 3** ([8]). The CCE graph of a doubly partial order is an interval graph unless it contains $C_4$ as an induced subgraph.

It is natural to ask if another important variant of the competition graph, the niche graph, of a doubly partial order is an interval graph. In this paper, we show that for each $n \geq 4$, there is a doubly partial order whose niche graph contains an induced subgraph isomorphic to $C_n$, which implies that the niche graph of a doubly partial order is not necessarily an interval graph. Then we show that if the niche graph of a doubly partial order is triangle-free, then it is an interval graph.

### 2 Main results

We will show that the niche graph of a doubly partial order is not necessarily an interval graph. We first prove the following lemma.

For $c \in \mathbb{R}$, let $L_c := \{v \in \mathbb{R}^2 \mid v_1 + v_2 = c\}$ and $Z^2 := \{v \in \mathbb{R}^2 \mid v_1, v_2 \in \mathbb{Z}\}$. Given a vertex $v$ in a graph $G$, we denote by $\Gamma_G(v)$ the neighborhood of $v$ in $G$.

**Lemma 4.** Let $V$ be a finite subset of $\mathbb{R}^2$ satisfying

$$V \cap Z^2 \subseteq L_c \cup L_c + 2 \quad \text{and} \quad V \setminus Z^2 \subseteq \bigcup_{c \leq c' < c + 2} L_{c'}$$

for some $c \in \mathbb{R}$. Suppose that $u_1 + 1 \neq v_1$ or $u_2 - 1 \neq v_2$ for two vertices $u, v$ of $V \cap Z^2$ with $u_1 \leq v_1$. Then $u \not\sim v$ in the niche graph of the doubly partial order $D$ associated with $V$.

**Proof.** We prove by contradiction. Suppose that there exist two vertices $u, v \in V \cap Z^2$ with $u_1 \leq v_1$ such that $u_1 + 1 \neq v_1$ or $u_2 - 1 \neq v_2$ but $u \sim v$ in the niche graph of $D$. Since $u \sim v$, there exists a vertex $a \in V$ such that either $a < u, v$ or $u, v < a$. Since $a \in V$,

$$c \leq a_1 + a_2 \leq c + 2. \quad (2.1)$$

Suppose that $\{u, v\} \not\subseteq L_c$ and $\{u, v\} \not\subseteq L_{c+2}$. Then either $u \in L_{c+2}$ and $v \in L_c$, or $u \in L_c$ and $v \in L_{c+2}$. This implies that

$$\min\{u_1 + u_2, v_1 + v_2\} = c \quad \text{and} \quad \max\{u_1 + u_2, v_1 + v_2\} = c + 2.$$
If \( a \prec u, v \), then \( a_1 + a_2 < \min\{u_1 + u_2, v_1 + v_2\} = c \), which contradicts (2.1). If \( u, v \prec a \), then \( a_1 + a_2 > \max\{u_1 + u_2, v_1 + v_2\} = c + 2 \), which contradicts (2.1) again. Therefore either \( \{u, v\} \subset L_c \) or \( \{u, v\} \subset L_{c+2} \).

Now suppose that \( \{u, v\} \subset L_c \). If \( a \prec u, v \), then \( a_1 + a_2 < u_1 + u_2 = c \), which is a contradiction to (2.1). Therefore it must hold that \( u, v \prec a \). Then it is easy to check that
\[
a_1 + a_2 > v_1 + u_2. \tag{2.2}
\]
Since \( u \neq v \) and \( c = u_1 + u_2 = v_1 + v_2, u_1 \neq v_1 \). By the assumption that \( u_1 \leq v_1 \), it is true that \( u_1 < v_1 \). Since \( c = u_1 + u_2 = v_1 + v_2, u_2 > v_2 \). In addition, from the assumption that \( u_1 + 1 \neq v_1 \) or \( u_2 - 1 \neq v_2 \), we have \( v_1 - u_1 \geq 2 \) or \( u_2 - v_2 \geq 2 \). If \( v_1 - u_1 \geq 2 \), then, by (2.2), \( a_1 + a_2 > v_1 + u_2 \geq u_1 + u_2 + 2 = c + 2 \), which contradicts (2.1). If \( u_2 - v_2 \geq 2 \), then, by (2.2), \( a_1 + a_2 > v_1 + u_2 \geq v_1 + v_2 + 2 = c + 2 \), which is a contradiction. Therefore it must hold that \( \{u, v\} \subset L_{c+2} \).

If \( u, v \prec a \), then \( c + 2 = u_1 + u_2 < a_1 + a_2 \), which is a contradiction to (2.1). Therefore it must hold that \( a \prec u, v \). Then
\[
a_1 + a_2 < u_1 + v_2. \tag{2.3}
\]
Since \( u \neq v \), \( u_1 \leq v_1 \), and \( c + 2 = u_1 + u_2 = v_1 + v_2 \), it is true that \( u_1 < v_1 \) and \( v_2 > u_2 \). Since \( u_1 + 1 \neq v_1 \) or \( u_2 - 1 \neq v_2 \), we have \( v_1 - u_1 \geq 2 \) or \( u_2 - v_2 \geq 2 \). If \( v_1 - u_1 \geq 2 \), then, by (2.3), \( a_1 + a_2 < u_1 + v_2 \leq v_1 + v_2 - 2 = c \), which is a contradiction. If \( u_2 - v_2 \geq 2 \), then, by (2.3), \( a_1 + a_2 < u_1 + v_2 \leq u_1 + u_2 - 2 = c \), which is a contradiction.

Hence \( u \) and \( v \) are not adjacent in the niche graph of \( D \). \( \square \)

**Theorem 5.** For any integer \( n \geq 4 \), there is a doubly partial order whose niche graph contains \( C_n \) as an induced subgraph.

**Proof.** We construct a doubly partial order \( D_n \) for each integer \( n \geq 4 \). For any \((i, j) \in \mathbb{R}^2\), let \( X_{(i,j)} := \{(i-1, j-1), (i,j), (i+1, j+1)\} \). For an integer \( k \) with \( k \geq 2 \), we define a finite subset \( W_k \) of \( \mathbb{R}^2 \) as follows:
\[
W_k \cap \mathbb{Z}^2 := \{(i, k - 1 - i), (i + 1, k - i) \mid i = 0, 1, \ldots, k - 2\}
\]
\[
W_k \setminus \mathbb{Z}^2 := \{(i - \frac{1}{3}, k - i - \frac{1}{3}), (i + \frac{1}{3}, k - i + \frac{1}{3}) \mid i = 1, 2, \ldots, k - 2\} \quad (k \geq 3)
\]
and \( W_2 \setminus \mathbb{Z}^2 = \emptyset \). Let \( A_k \) be the sequence of vertices of \((W_k \cap \mathbb{Z}^2) \cup \{(0, k)\}\) listed as follows:
\[
(k - 2, 1), (k - 3, 2), \ldots, (i, k - 1 - i), \ldots, (2, k - 3), (1, k - 2), (0, k - 1), (k - 2, 1), (k - 1, k), (2, k - 1), \ldots, (i + 1, k - i), \ldots, (k - 2, 3), (k - 1, 2),
\]
\((*)\)
Let $G_k$ be the niche graph of a doubly partial order associated with $X_{(0,k)} \cup W_k$. First, we will show that the sequence $A_k$ is a path of length $2k - 2$ as an induced subgraph in $G_k$. In $G_k$, we can easily check the following:

(i) For $i = 0, 1, \ldots, k - 3$, the vertex $(i + 1 + \frac{1}{2}, k - 1 - i + \frac{1}{2})$ of $W_k \setminus Z^2$ is a common predator of the $(k - 1 - i)$th vertex $(i, k - 1 - i)$ and the $(k - i)$th vertex $(i + 1, k - 2 - i)$;

(ii) For $i = 0, 1, \ldots, k - 3$, the vertex $(i + 1 - \frac{1}{2}, k - 1 - i - \frac{1}{2})$ of $W_k \setminus Z^2$ is a common prey of the $(k + i + 1)$st vertex $(i + 1, k - i)$ and the $(k + i + 2)$nd vertex $(i + 2, k - 1 - i)$;

(iii) The vertex $(1, k + 1)$ is a common predator of the $k$th vertex $(0, k)$ and the $(k - 1)$st vertex $(0, k - 1)$;

(iv) The vertex $(-1, k - 1)$ is a common prey of the $k$th vertex $(0, k)$ and the $(k + 1)$st vertex $(1, k)$.

By (i) through (iv), the $i$th vertex and the $j$th vertex of the sequence $A_k$ are adjacent in $G_k$ if $|i - j| = 1$, and so $A_k$ forms a path of length $2k - 2$ in $G_k$.

In addition, the sequence $A_k$ is a path of length $2k - 2$ as an induced subgraph in $G_k$. To see why, we will show that the $i$th vertex and the $j$th vertex of $A_k$ are not adjacent in $G_k$ if $|i - j| \geq 2$. Take the $i$th vertex and the $j$th vertex of $A_k$ with $|i - j| \geq 2$ and denote them by $x$ and $y$. Suppose that $k = i$ or $j$. Then the $k$th vertex of $A_k$ is $(0, k)$ and it is easy to check that

$$
\Gamma_{G_k}((0, k)) = \{(1, k), (0, k - 1), (-1, k - 1), (1, k + 1)\}.
$$

Since $(1, k)$ and $(0, k - 1)$ are the $(k + 1)$st vertex and $(k - 1)$st vertex of $A_k$, respectively, and $(-1, k - 1)$ and $(1, k + 1)$ are not vertices of $A_k$, we conclude that $x \not\sim y$ in this case.

Suppose that $i \neq k$ and $j \neq k$. Without loss of generality, we may assume that $x_1 \leq y_1$. Note that $W_k$ satisfies that

$$
W_k \cap Z^2 \subseteq L_{k-1} \cup L_{k+1} \quad \text{and} \quad W_k \setminus Z^2 \subseteq \bigcup_{k-1 < c' < k+1} L_{c'}.
$$

Since $|i - j| \geq 2$, $x_1 + 1 \neq y_1$ or $x_2 - 1 \neq y_2$ by the definition of $A_k$. Then, by Lemma 4, $x \not\sim y$ in the niche graph of the doubly partial order associated with $W_k$. Therefore $x \not\sim y$ in the subgraph of $G_k$ induced by $W_k$. It remains to show that $x$ and $y$ have neither a common prey nor a common predator in $X_{(0,k)} = \{(-1, k - 1), (0, k), (1, k + 1)\}$. The set of predators
or prey of \((-1, k - 1)\) in \(A_k\) is \(\{(0, k), (1, k)\}\). These two vertices are \(k\)th and \((k + 1)\)st vertices of \(A_k\) and so \((-1, k - 1)\) cannot be a common prey or a common predator of \(x\) and \(y\). The set of predators or prey of \((0, k)\) in \(A_k\) is \(\{(-1, k - 1), (1, k + 1)\}\) and so \((0, k)\) cannot be a common prey or a common predator of \(x\) and \(y\). The set of predators or prey of \((1, k + 1)\) in \(A_k\) is \(\{(0, k), (0, k - 1)\}\). These two vertices are \(k\)th and \((k - 1)\)st vertices of \(A_k\) and so \((-1, k - 1)\) cannot be a common prey or a common predator of \(x\) and \(y\). Hence we conclude that the \(i\)th vertex and the \(j\)th vertex of \(A_k\) are not adjacent in \(G_k\) if \(|i - j| \geq 2\).

Now we are ready to give a construction of a doubly partial order \(D_n\) for each integer \(n \geq 4\). Suppose that \(n = 2k\) for some integer \(k \geq 2\). Let

\[
V_n : = X_{(0,k)} \cup X_{(k-1,1)} \cup W_k
\]

and \(D_n\) be the doubly partial order associated with \(V_n\). We will show that the vertices of \((W_k \cap \mathbb{Z}^2) \cup \{(0,k), (k-1,1)\}\) form \(C_n\) without chord in the niche graph of \(D_n\). See Figure 2 for an illustration. Let \(N_n\) be the niche graph of \(D_n\).

Note that \(X_{(k-1,1)} = \{(k-2,0), (k-1,1), (k,2)\}\). Consider the sequence \(A_k\) defined in (1). It is not difficult to check that none of vertices in \(X_{(k-1,1)}\) can be a common prey or a common predator of two vertices of \(A_k\). Thus by the previous argument, \(A_k\) forms a path as an induced subgraph of \(N_n\). On the other hand, in the niche graph \(N_n\) of \(D_n\), it can easily be checked that

\[
\Gamma_{N_n}((k-2,0)) = \{(k-1,1), (k,2), (k-1,2)\};
\]

\[
\Gamma_{N_n}((k,2)) = \{(k-1,1), (k-2,0), (k-2,1)\};
\]

\[
\Gamma_{N_n}((k-1,1)) = \{(k-2,0), (k,2), (k-2,1), (k-1,2)\}.
\]

Thus, the vertices of \(A_k\) together with \((k-1,1)\) form a cycle of length \(2k = n\) as an induced subgraph.

Now we assume that \(n\) is an odd integer with \(n \geq 5\). Then \(n = 2k + 1\) for some integer \(k \geq 2\). Let

\[
V_n : = X_{(0,k)} \cup X_{(k+1,1)} \cup W_k
\]

and \(D_n\) be the doubly partial order associated with \(V_n\). See Figure 3 for an illustration. Note that \(X_{(k+1,1)} = \{(k,0), (k+1,1), (k+2,2)\}\).

Consider the sequence \(A_k\) defined in (1). Then it is not hard to check that none of vertices in \(X_{(k+1,1)}\) is a common prey or a common predator of two vertices of \(A_k\). Thus, by the previous argument, \(A_k\) is a path as an induced subgraph of \(N_n\).
Figure 2: A doubly partial order $D_8$ and the niche graph of $D_8$. Note that the thick edges form a cycle of length 8 as an induced subgraph of the graph.
Figure 3: A doubly partial order $D_9$ and the niche graph of $D_9$. The thick edges form a cycle of length 9 as an induced subgraph of the graph.
It can easily be checked that
\[
\begin{align*}
\Gamma_{N_n}((k,0)) &= \{(k+1,1), (k-2,1)\}; \\
\Gamma_{N_n}((k+1,1)) &= \{(k,0), (k+2,2), (k-2,1)\}; \\
\Gamma_{N_n}((k+2,2)) &= \{(k+1,1), (k-1,2)\}.
\end{align*}
\]

Thus the first vertex \((k-2,1)\) of \(A_k\) is the only vertex in \(A_k\) adjacent to \((k+1,1)\). In addition, the \((2k-1)\text{st}\) vertex \((k-1,2)\) of \(A_k\) are the only vertex in \(A_k\) adjacent to \((k+2,2)\). Since \((k+1,1)\) and \((k+2,2)\) are adjacent, the vertices of sequence \(A_k\) together with \((k+2,2)\) and \((k+1,1)\) form a cycle of length \(2k+1 = n\) as an induced subgraph. Hence \(N_n\) contains \(C_n\) as an induced subgraph. \(\square\)

Theorems \([\text{II}]\) and \([\text{I}]\) tell us that the niche graph of a doubly partial order is not necessarily an interval graph. However if the niche graph of a doubly partial order is triangle-free, then it is an interval graph. To show that, we start with the following lemma:

**Lemma 6.** Let \(D\) be a doubly partial order. Suppose that the niche graph \(G\) of \(D\) is triangle-free. Then if \(x \sim y, y \sim z\) in \(G\), and \(x_1 \leq z_1\), then \(x \searrow y \searrow z\).

**Proof.** Since \(x \sim y\) and \(y \sim z\) in \(G\), there are vertices \(a\) and \(b\) such that either \(a \prec x, y\) or \(x, y \prec a\) and either \(b \prec y, z\) or \(y, z \prec b\). Suppose that \(a \prec x, y\) and \(y, z \prec b\). Then \(a \prec y \prec b\) and so \(a \prec b\). Therefore \(a\) is a common prey of \(x, y\), and \(b\), and so \(x, y\) and \(b\) form a triangle in \(G\), which is a contradiction. Similarly, if \(x, y \prec a\) and \(b \prec y, z\), then we reach a contradiction. Hence either \((1) a \prec x, y\) and \(b \prec y, z\), or \((2) x, y \prec a\) and \(y, z \prec b\). In each case, we show that \(x_1 \leq y_1 \leq z_1\). To show by contradiction, we consider two subcases \((A) x_1 > y_1\) and \((B) y_1 > z_1\) in each case.

**Case 1.** \(a \prec x, y\) and \(b \prec y, z\).

**Subcase A.** \(y_1 < x_1\).

If \(z_2 \leq x_2\), then \(b_1 < y_1 < x_1\) and \(b_2 < z_2 \leq x_2\) which imply that \(b \prec x\). Then \(b \prec x, y, z\) and so \(x, y\), and \(z\) form a triangle in \(G\), which is a contradiction. If \(z_2 > x_2\), then \(a_1 < y_1 \leq x_1 \leq z_1\) and \(a_2 < x_2 < z_2\) which imply that \(a \prec z\). Then \(a \prec x, y, z\) and so \(x, y\), and \(z\) form a triangle in \(G\), which is a contradiction.

**Subcase B.** \(z_1 < y_1\).

If \(x_2 < y_2\), then \(x \prec y\) and so \(x, a, b \prec y\). Now suppose that \(y_2 \leq x_2\) and \(y_2 \leq z_2\). If \(x_1 \leq z_1\), then \(a_1 < x_1 \leq z_1\) and \(a_2 < y_2 \leq z_2\) which imply that \(a \prec z\). Then \(a \prec x, y, z\) and so \(x, y\), and \(z\) form a triangle in \(G\), which
is a contradiction. If \( z_2 < y_2 \), then \( z < y \) and so \( z, a, b < y \). Now suppose that \( y_2 \leq x_2 \) and \( y_2 \leq z_2 \). If \( z_1 < x_1 \), then \( b_1 < z_1 < x_1 \) and \( b_2 < y_2 \), which imply that \( b \prec x \). Then \( b \prec x, y, z \) and so \( x, y, z \) form a triangle in \( G \), which is a contradiction.

Case 2. \( x, y \prec a \) and \( y, z \prec b \).

Subcase A. \( y_1 < x_1 \).

If \( y_2 < x_2 \), then \( y \prec x \) and so \( y \prec x, a, b \). Then \( x, a, b \) form a triangle, which is a contradiction. If \( y_2 < z_2 \), then \( y \prec z \) and so \( y \prec z, a, b \). Then \( z, a, b \) form a triangle, which is a contradiction. Now suppose that \( x_2 \leq y_2 \) and \( z_2 \leq y_2 \). If \( x_1 \leq z_1 \), then \( x_1 \leq z_1 < b_1 \) and \( x_2 \leq y_2 < b_2 \), which imply that \( x \prec b \). Then \( x, y, z \prec b \) and so \( x, y, z \) form a triangle in \( G \), which is a contradiction. If \( z_1 < x_1 \), then \( z_1 < x_1 < a_1 \) and \( z_2 \leq y_2 < a_2 \), which imply that \( z \prec a \). Then \( x, y, z \prec a \) and so \( x, y, z \) form a triangle in \( G \), which is a contradiction.

Subcase B. \( z_1 < y_1 \).

If \( x_2 < z_2 \), then \( x_1 \leq y_1 < b_1 \) and \( x_2 < z_2 \leq b_2 \) which imply that \( x \prec b \). Then \( x, y, z \prec b \) and so \( x, y, z \) form a triangle in \( G \), which is a contradiction. If \( x_2 \geq z_2 \), then \( z_1 \leq y_1 < a_1 \) and \( z_2 \leq x_2 < a_2 \) which imply that \( z \prec a \). Then \( x, y, z \prec a \) and so \( x, y, z \) form a triangle in \( G \), which is a contradiction.

Thus we can conclude that \( x_1 \leq y_1 \leq z_1 \) in each case. In addition, it cannot happen \( x_1 = y_1 = z_1 \). To see why, let \( c \) be an element of \( \{a, b\} \) with smallest second component and \( d \) be the element of \( \{a, b\} \setminus \{c\} \). Suppose that \( a \prec x, y \) and \( b \prec y, z \). Since \( x_1 = y_1 = z_1 \), we have \( c \prec x, y, z \) and so \( x, y, z \) form a triangle. Similarly, if \( x, y \prec a \) and \( y, z \prec b \), then \( x, y, z \prec d \) and so \( x, y, z \) create a triangle. Therefore it holds that (1) \( x_1 = y_1 < z_1 \), (2) \( x_1 = y_1 = z_1 \), or (3) \( x_1 < y_1 < z_1 \). In the following, we show that \( x_2 \geq y_2 \geq z_2 \) in these three cases.

Case 1. \( x_1 = y_1 < z_1 \)

Suppose that \( x_2 < y_2 \). If \( x, y \prec a \) and \( y, z \prec b \), then \( x, y, z \prec b \). If \( a \prec x, y \) and \( b \prec y, z \), and \( z_2 \geq x_2 \), then \( b \prec x, y, z \). If \( a \prec x, y \) and \( b \prec y, z \), and \( z_2 \geq x_2 \), then \( a \prec x, y, z \). Therefore we reach a contradiction, and so it must hold that \( x_2 \geq y_2 \). Suppose that \( y_2 < z_2 \). If \( a \prec x, y \) and \( b \prec y, z \), then, since \( b_1 < y_1 = x_1 \) and \( b_2 < y_2 \leq x_2 \), we have \( b \prec x, y, z \). If \( x, y \prec a \) and \( y, z \prec b \), then, since \( y \prec a, b \) and \( y \prec z \), we have \( y \prec a, b, z \). Therefore we reach a contradiction, and so it must hold that \( y_2 \geq z_2 \). Thus \( x_2 \geq y_2 \geq z_2 \).

Case 2. \( x_1 < y_1 = z_1 \)

Suppose that \( y_2 < z_2 \). If \( a \prec x, y \) and \( b \prec y, z \), then \( a \prec x, y, z \). If \( x, y \prec a \) and \( y, z \prec b \), then \( x, y, z \prec b \). If \( x, y \prec a \) and \( y, z \prec b \)
and \( z_2 < x_2 \), then \( x, y, z \prec a \). Therefore we reach a contradiction, and so it must hold that \( y_2 \geq z_2 \). Suppose that \( x_2 < y_2 \). If \( x, y \prec a \) and \( y, z \prec b \), then, since \( z_1 = y_1 < a_1 \) and \( z_2 \leq y_2 < a_2 \), we have \( x, y, z \prec a \). If \( a \prec x, y \) and \( b \prec y, z \), then, since \( a, b \prec y \) and \( x \prec y \), we have \( x, a, b \prec y \). Therefore we reach a contradiction, and so it must hold that \( x_2 \geq y_2 \). Thus \( x_2 \geq y_2 \geq z_2 \).

Case 3. \( x_1 < y_1 < z_1 \).

Suppose that \( x_2 < y_2 \). Then \( x \prec y \). If \( a \prec x, y \) and \( b \prec y, z \), then, since \( z_1 = y_1 < a_1 \) and \( z_2 \leq y_2 < a_2 \), we have \( x, y, z \prec a \). Therefore we reach a contradiction, and so it must hold that \( x_2 \geq y_2 \). Thus \( x_2 \geq y_2 \geq z_2 \).

Theorem 7. Let \( D \) be a doubly partial order. Suppose that the niche graph of \( D \) is a triangle-free graph. Then each component of the niche graph of \( D \) is a path.

Proof. Let \( G \) be the niche graph of a doubly partial order \( D \). First, we will show that \( G \) is a forest. Suppose that there is a cycle \( C \) of length \( n \). We may assume that \( x \) is a vertex such that its first component \( x_1 \) is the minimum among those of vertices of \( C \). Since \( G \) is triangle-free, \( n \geq 4 \) and so there exist 4 distinct vertices \( x, y, z, w \) such that \( x \sim y \), \( y \sim z \), \( w \sim x \). Let \( u \) be the vertex of \( C \) such that \( u \sim w \) and \( u \neq x \). By the choice of \( x \), \( x_1 \leq u_1 \) and \( x_1 \leq z_1 \). Then, since \( xwu \) and \( xyz \) are paths in \( G \), \( x \sim w \) and \( x \sim y \) by Lemma 6. If \( y_1 \geq w_1 \), then, by Lemma 6, \( w \sim x \), which implies that \( x = w \). If \( y_1 < w_1 \), then \( y \sim x \), which implies that \( y = x \). Thus we reach a contradiction in either case. Hence \( G \) is a forest.

In the following, we will show that \( \text{deg}_G(v) \leq 2 \) for any vertex \( v \). Suppose that there is a vertex \( u \) such that \( \text{deg}_G(u) \geq 3 \). Let \( x, y \) and \( z \) be three distinct neighbors of \( u \). Without loss of generality, we may assume that \( x_1 \leq y_1 \leq z_1 \). Since \( xuy \) and \( yuz \) are paths in \( G \), \( x \sim u \) and \( y \sim z \) by Lemma 6. Then \( u \sim y \) and \( y \sim u \) and so \( y = u \), which is a contradiction. Hence each component of the niche graph of \( D \) is a path.

By Theorem 1 and Theorem 7, the following theorem holds.

Theorem 8. The niche graph of a doubly partial order is an interval graph unless it contains a triangle.
3 Concluding remarks

We have shown that the niche graph of a doubly partial order is not necessarily an interval graph by constructing a doubly partial order whose niche graph contains a cycle an induced subgraph for each integer $n \geq 4$. Then we tried to find a doubly partial order such that its niche graph does not contain a cycle of length at least 4 as an induced subgraph and it is not an interval graph, but in vain. Accordingly, we would like to ask whether or not such a doubly partial order exists.

Eventually, it remains open to characterize doubly partial orders whose niche graphs are interval graphs.

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