CHARACTERIZATION OF PARTIAL HAMILTONIAN OPERATORS AND RELATED FIRST INTEGRALS

Rehana Naz\textsuperscript{a} and Fazal M. Mahomed\textsuperscript{b}

\textsuperscript{a}Centre for Mathematics and Statistical Sciences
Lahore School of Economics
Lahore 53200, Pakistan

\textsuperscript{b}DST-NRF Centre of Excellence in Mathematical and Statistical Sciences
School of Computer Science and Applied Mathematics
University of the Witwatersrand
Johannesburg, Wits 2050, South Africa

Abstract. We focus on partial Hamiltonian systems for the characterization of their operators and related first integrals. Firstly, it is shown that if an operator is a partial Hamiltonian operator which yields a first integral, then so does its evolutionary representative. Secondly, extra operator conditions are provided for a partial Hamiltonian operator in evolutionary form to yield a first integral. Thirdly, characterization of partial Hamiltonian operators and related first integral conditions are provided for the partial Hamiltonian system. Applications to mechanics are presented to illustrate the theory.

1. Introduction. The classical correspondence between symmetries and first integrals for the canonical Hamiltonian equations has been vigorously studied by several authors including Marsden and Weinstein [9], Kozlov [4], Olver [14], Dorodnitsyn and Kozlov [2] and Mahomed and Roberts [7]). The link between Hamiltonian symmetries and their first integrals was first promulgated by the Italian mathematician Levi-Civita [6] (see the insightful translation by Saccomandi and Vitolo [15]). A Hamiltonian symmetry in evolutionary form (see, e.g. [14]) determines a first integral of the canonical Hamilton equations up to a time-dependent function. Dorodnitsyn and Kozlov [2] alluded to some apparent disadvantages of this approach. They [2] looked at the Hamiltonian symmetries which are not restricted to phase space and also transform the time variable $t$. A direct method to construct first integrals for suitable gauge terms that require integration was proposed by [2]. The two approaches [6, 14, 2] utilized for the construction of a first integral once the symmetry is known fails to unveil the first integral uniquely. In the first approach as in [14], the first integrals are obtained up to a time dependent function whereas in the second as in [2], the first integrals are established up to a suitable gauge term. Recently, Mahomed and Roberts [7] showed how one can determine the first integral uniquely by providing an extra integrability condition on the first integral for the first method and giving the integrability conditions on the gauge term for the second method. It is shown, as a consequence (see [7]) that both the methods are in fact equivalent.

\textit{2010 Mathematics Subject Classification.} 37K05, 70S05, 70S10.

\textit{Key words and phrases.} Partial Hamiltonian system, partial Hamiltonian operators, first integral.

* Corresponding author: Fazal M. Mahomed.
The partial Hamiltonian approach occurs naturally in economic growth theory. Naz et al [10, 11] developed a partial Hamiltonian approach in order to construct first integrals of partial Hamilton systems arising in economic growth theory. However, this was established up to some gauge terms. Also Naz [12] has presented some applications of partial Hamiltonian approach in mechanics and other areas of applied mathematics. A partial Lagrangian method was initiated in Kara et al [3]. Following up on this the partial or discount free Lagrangian approach [13] has been developed to derive the first integrals and closed-form solutions in economic growth theory by using a Lagrangian formulation that naturally arises in this context.

In this work, we focus on the characterization of partial Hamiltonian operators and their related first integrals in a unique manner. This adds to the literature in multiple ways. Firstly, it is shown that if an operator is a partial Hamiltonian operator that yields a first integral, then so does its evolutionary representative. Secondly, extra operator conditions are provided for a partial Hamiltonian operator in evolutionary form to yield a first integral. Thirdly, characterization of partial Hamiltonian operators and related first integral conditions are provided for the partial Hamiltonian systems.

The layout of the paper is as follows. Preliminaries on known aspects of the partial Hamiltonian method are presented in Section 2. In Section 3, the formulas are provided in order to find the partial Hamiltonian operators in evolutionary form and the first integrals of a partial Hamiltonian system in order to result in a first integral. Moreover, extra conditions that give rise to a first integral are provided for the partial Hamiltonian operator in evolutionary form. Characterization of Hamiltonian operators and related first integral conditions for partial Hamiltonian systems are given in Section 4. In Section 5, applications taken from mechanics are presented to show the effectiveness of the approach proposed here. Finally, conclusions are presented in Section 6.

2. Preliminaries. Again as in many papers on the Hamiltonian formalism we let $t$ be the independent variable (which is usually the time) and $(q, p) = (q^1, ..., q^n, p_1, ..., p_n)$ the phase space coordinates. In applications to equations of mechanics, $q^1, ..., q^n$ are taken as the position coordinates and $p_1, ..., p_n$ are called the conjugate momenta. The following notions and results are adapted from [14, 2, 7, 10, 11].

**Definition 1.** The Euler operator $\delta/\delta q^i$ and the variational operator $\delta/\delta p_i$ are

\[
\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - D_t \frac{\partial}{\partial \dot{q}^i}, \quad i = 1, \cdots, n, \tag{1}
\]

and

\[
\frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i} - D_t \frac{\partial}{\partial \dot{p}_i}, \quad i = 1, \cdots, n, \tag{2}
\]

where

\[
D_t = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} + \cdots, \tag{3}
\]

is the total derivative operator with respect to $t$. As usual the summation convention is implemented for repeated indices here and in the sequel.

The variables $t, q, p$ are considered independent and only connected by the differential relations

\[
\dot{p}_i = D_t(p_i), \quad \dot{q}^i = D_t(q^i), \quad i = 1, 2, \cdots, n. \tag{4}
\]
A partial Hamiltonian system satisfies (see [10])

\[
\dot{q}^i = \frac{\partial H}{\partial p_i}, \\
\dot{p}_i = -\frac{\partial H}{\partial q_i} + \Gamma_i, \quad i = 1, \ldots, n,
\]  

(5)

where we view \( \Gamma_i \) as a nonzero function of \( t, p_i, q_i \) in general and \( H \) is a partial Hamiltonian function (see [10, 11]). In the event \( \Gamma_i = 0 \) for all \( i \), then we have the usual canonical Hamiltonian system.

The partial Hamiltonian system naturally arises in economics (see, e.g. [1]).

The operator \( X \) which is a generator of Hamiltonian symmetry (see e.g. [14]) is

\[
X = \xi(t, q, p) \frac{\partial}{\partial t} + \eta^i(t, q, p) \frac{\partial}{\partial q^i} + \zeta_i(t, q, p) \frac{\partial}{\partial p_i}.
\]  

(6)

This operator can also be written in characteristic form as

\[
X = \xi D_t + \tilde{\eta}^i \frac{\partial}{\partial q^i} + \tilde{\zeta}_i \frac{\partial}{\partial p_i},
\]  

(7)

where \( \tilde{\eta}^i \) and \( \tilde{\zeta}_i \) are the characteristic functions given by

\[
\tilde{\eta}^i = \eta^i - \xi \dot{q}^i, \quad \tilde{\zeta}_i = \zeta_i - \xi \dot{p}_i, \quad i = 1, \ldots, n.
\]  

(8)

The operator (7) has evolutionary representative [14]

\[
\bar{X} = \tilde{\eta}^i \frac{\partial}{\partial q^i} + \tilde{\zeta}_i \frac{\partial}{\partial p_i},
\]  

(9)

or is called the canonical form of the operator \( X \). The operators (7) and (9) are equivalent since \( X - \bar{X} = \xi D_t \).

The first prolongation of the operator \( X \) is given by

\[
X^{[1]} = \xi D_t + \tilde{\eta}^i \frac{\partial}{\partial q^i} + \tilde{\zeta}_i \frac{\partial}{\partial p_i} + D_t(\tilde{\eta}^i) \frac{\partial}{\partial q^i} + D_t(\tilde{\zeta}_i) \frac{\partial}{\partial p_i}.
\]  

(10)

**Definition 2.** A first integral \( I \) of the partial Hamiltonian system (5) is obtained from the relation

\[
D_t I = 0,
\]  

(11)

where \( I = I(t, q, p) \), which is satisfied on all solutions to (5).

We also have the case that

\[
D_t I = Q^i(\dot{p}_i + H_{q^i} - \Gamma_i) + R_i(q^i - H_{p^i})
\]  

(12)

which is called the characteristic form of the conservation law (11) with the functions \( Q^i \) and \( R_i, i = 1, \ldots, n \), the associated characteristic functions. When the condition (11) is satisfied on the solutions to (5), \( I \) is referred to as a first integral of the system (5).

Hamiltonian symmetries in evolutionary or canonical form have been considered before (see, e.g. [14]). Furthermore, Hamiltonian symmetries have also been studied in [2] and the Hamiltonian version of Noether’s theorem is provided therein. Naz et al [10] developed a partial Hamiltonian approach to derive first integrals of partial Hamiltonian systems arising in economic growth theory. Naz [12] has amply illustrated that the formulas for the partial Hamiltonian approach can well be invoked
to deduce first integrals of many partial Hamiltonian systems other than those in economics.

**Definition 3** (see [10]). An operator $X$ of the form as given in (6) is said to be a partial Hamiltonian operator corresponding to a partial Hamiltonian $H(t, q, p)$ if there exists a function $B(t, q, p)$ such that

\[
\zeta_i \frac{\partial H}{\partial p_i} + p_i D_t(\eta^i) - X(H) - HD_t(\xi) = D_t(B) + (\eta^i - \xi \frac{\partial H}{\partial p_i})(-\Gamma_i) \tag{13}
\]

is satisfied on all solutions to system (5).

The following theorem is essential for the construction of first integrals for our system (5).

**Theorem 1** (see [10]). The first integral corresponding to the system (5) associated with a partial Hamiltonian operator $X$ is determined from

\[
I = p_i \eta^i - \xi H - B, \tag{14}
\]

for some $B(t, p, q)$ which is a gauge-like function.

If $\Gamma_i = 0$ and $B = B(t, p, q)$, then this formula (14) is valid for an invariant Hamiltonian action up to divergence [2] as well.

3. **Partial Hamiltonian operators in evolutionary form.** In this section, we present the partial Hamiltonian operators (see [10]) in evolutionary form. The partial Hamiltonian identity is provided and is utilized to derive the operator determining equations in evolutionary form.

The operator $X$ as in (6) is a generator of symmetry of the partial Hamiltonian system (5) if

\[
D_t(\eta^i) - \dot{q}^i D_t(\xi) - X(\frac{\partial H}{\partial p_i}) = 0, \tag{15}
\]

\[
D_t(\zeta_i) - \dot{p}_i D_t(\xi) + X(\frac{\partial H}{\partial q^i} - \Gamma_i) = 0, \quad i = 1, \ldots, n \tag{16}
\]

are satisfied on the system (5).

The proof of this is rather simple. The action of the generator $X$ on the first equation of system (5) easily gives

\[
X(\dot{q}^i - \frac{\partial H}{\partial p_i}) = 0, \tag{17}
\]

which straightforwardly yields (15). The action of the generator $X$ on the second equation of system (5) results in

\[
X(\dot{p}_i + \frac{\partial H}{\partial q^i} - \Gamma_i) = 0, \tag{18}
\]

and this directly provides (16).

**Lemma 1.** If operator $X$ defined as in (6), is a partial Hamiltonian operator that yields a first integral, then so does the evolutionary representative $\bar{X} = X - \xi D_t$.

**Proof.** If $X$ is a partial Hamiltonian operator yielding a first integral, then it satisfies the condition (13). Introducing $X = \bar{X} + \xi D_t$, $\zeta_i = \bar{\zeta}_i + \xi \bar{p}_i$ and $\eta^i = \bar{\eta}^i + \xi \bar{q}^i$ in equation (13), we have

\[
\bar{\zeta}_i \bar{q}^i + p_i D_t(\bar{\eta}^i) - \bar{X} H - D_t \bar{B} + \bar{\eta}^i \bar{\Gamma}_i = 0 \tag{19}
\]
with
\[ B = \xi H + B - \xi p_i q^i. \]

\[ B = \xi H + B - \xi p_i q^i. \]

**Lemma 2.** The identity
\[ \tilde{\zeta} q^i + p_i D_i (\tilde{\eta}^i) - \tilde{X}(H) - D_i(B) + \tilde{\eta}^i \Gamma_i \]
\[ = -\tilde{\eta}^i (\dot{p_i} + \frac{\partial H}{\partial q^i} - \Gamma_i) + \tilde{\zeta}_i (q^i - \frac{\partial H}{\partial p_i}) + D_t (p_i \tilde{\eta}^i - \tilde{B}) \]
holds for any smooth functions \( H(t, q, p) \) and suitable functions \( \tilde{B} \) and \( \Gamma_i \).

**Proof.** This identity easily follows from direct computations. \( \square \)

**Theorem 2.** If \( \tilde{X} \) is a partial Hamiltonian operator in evolutionary form, then \( \tilde{X} \) satisfies the following conditions
\[ D_i (\tilde{\zeta}_i) + \tilde{X} \left( \frac{\partial H}{\partial q^i} \right) = \tilde{\eta}^i \Gamma_{iq}, \tag{21} \]
\[ D_t (\tilde{\eta}^i) - \tilde{X} \left( \frac{\partial H}{\partial p_j} \right) = -\tilde{\eta}^i \Gamma_{ip_j} \tag{22}, \]
where \( i, j = 1, \ldots, n \), on system (5).

**Proof.** The variational derivatives \( \frac{\delta}{\delta p_j} \) and \( \frac{\delta}{\delta q^i} \) of identity (20) results in
\[ \frac{\delta}{\delta p_j} \left( \tilde{\zeta}_i q^i + p_i D_i (\tilde{\eta}^i) - \tilde{X}(H) - D_i(B) + \tilde{\eta}^i \Gamma_i \right) \]
\[ = D_t (\tilde{\eta}_j) - \tilde{X} \left( \frac{\partial H}{\partial p_j} \right) + \tilde{\eta}^i \Gamma_{ip_j} \]
\[ - \frac{\partial \tilde{\eta}^i}{\partial p_j} (\dot{p_i} + \frac{\partial H}{\partial q^i} - \Gamma_i) + \frac{\partial \tilde{\zeta}_i}{\partial p_j} (q^i - \frac{\partial H}{\partial p_i}) \]
\[ = D_t (\tilde{\zeta}_j) - \tilde{X} \left( \frac{\partial H}{\partial q^j} \right) + \tilde{\eta}^i \Gamma_{iq_j} \]
\[ - \frac{\partial \tilde{\eta}^i}{\partial q^j} (\dot{p_i} + \frac{\partial H}{\partial q^i} - \Gamma_i) + \frac{\partial \tilde{\zeta}_i}{\partial q^j} (q^i - \frac{\partial H}{\partial p_i}) \]
Equations (23) and (24) yield
\[ \frac{\delta}{\delta p_j} (\tilde{\zeta} q^i + p_i D_i (\tilde{\eta}^i) - \tilde{X}(H) - D_i(B) + \tilde{\eta}^i \Gamma_i) \big|_{q^i} = \frac{\partial H}{\partial q^j} \cdot \dot{p_i} + \frac{\partial H}{\partial p_j} - \Gamma_i = 0, \tag{25} \]
\[ \frac{\delta}{\delta q^j} (\tilde{\zeta} q^i + p_i D_i (\tilde{\eta}^i) - \tilde{X}(H) - D_i(B) + \tilde{\eta}^i \Gamma_i) \big|_{q^i} = \frac{\partial H}{\partial q^j} \cdot \dot{p_i} + \frac{\partial H}{\partial p_j} - \Gamma_i = 0, \tag{26} \]
provided operator conditions (21) and (22) hold. This completes the proof. \( \square \)

4. **Characterization of operators and related first integral conditions.** In this section, we provide a characterization of the Hamiltonian operator which directly corresponds to a first integral and as a consequence obtain the extra conditions on the first integral.

**Theorem 3.** Necessary and sufficient conditions that the operator \( X \) of the form (7), yields a first integral of system (5) is that the characteristics \( \tilde{\eta}^i \) and \( \tilde{\zeta}_i \) of \( X \)
are also the characteristics of the conservation law of system (5) and additionally satisfies the relations
\[
\frac{\partial \bar{\eta}^i}{\partial q^j} + \frac{\partial \bar{\zeta}_j}{\partial p_i} = 0, \quad (27)
\]
\[
\frac{\partial \bar{\zeta}_i}{\partial q^j} - \frac{\partial \bar{\eta}^j}{\partial p_i} = 0, \quad i, j = 1, \ldots, n, \quad (29)
\]
as well as the operator conditions (21) and (22).

Proof. Any conservation law of the partial Hamiltonian system (5) can be written in the equivalent characteristic form as
\[
D_t I = Q^i(t, q, p)(\dot{p}_i + H_{q^i} - \Gamma_i) + R_i(t, q, p)(\dot{q}^i - H_{p_i}) \quad (30)
\]
for suitable multipliers $Q^i$ and $R_i$ yet to be determined. First, we act with the Euler operator $\delta/\delta q^i$, $j = 1, \ldots, n$ on equation (30) and this yields
\[
\frac{\partial Q^i}{\partial q^j}(\dot{p}_i + H_{q^i} - \Gamma_i) + Q^i(H_{q^i} - H_{q^j}) + \frac{\partial R_i}{\partial q^j}(\dot{q}^i - H_{p_i}) - R_i H_{p_i q^j} + \frac{\partial Q^i}{\partial p_i} H_{q^i} = \frac{\partial R_j}{\partial p_i} - \frac{\partial R_i}{\partial q^j} H_{q^i} - \Gamma_i \frac{\partial \bar{\eta}^j}{\partial q^i} - \bar{\eta}^i \Gamma_{iq^j}, \quad j = 1, \ldots, n. \quad (31)
\]
where $\delta^i_j$ is the Kronecker delta. Separating the terms contains $\dot{q}^i$ and $\dot{p}_i$ from equation (31) gives
\[
\frac{\partial Q^i}{\partial q^j} - \frac{\partial R_j}{\partial p_i} = 0, \quad (32)
\]
\[
\frac{\partial R_i}{\partial q^j} - \frac{\partial R_j}{\partial p_i} = 0, \quad i, j = 1, \ldots, n \quad (33)
\]
and the remaining terms of equation (31) then results in
\[
Q^i H_{q^i q^j} - R_i H_{p_i q^j} - \frac{\partial R_j}{\partial t} - \frac{\partial R_j}{\partial q^i} H_{p_i} + \frac{\partial Q^i}{\partial q^j} H_{q^i} - \Gamma_i \frac{\partial Q^i}{\partial q^j} - Q^i \Gamma_{iq^j} = 0, \quad j = 1, \ldots, n. \quad (34)
\]
Equation (34) with the aid of (32) and (33) reduces to
\[
Q^i H_{q^i q^j} - R_i H_{p_i q^j} - \frac{\partial R_j}{\partial t} - \frac{\partial R_j}{\partial q^i} H_{p_i} + \frac{\partial R_j}{\partial p_i} H_{q^i} - \Gamma_i \frac{\partial Q^i}{\partial q^j} - Q^i \Gamma_{iq^j} = 0, \quad j = 1, \ldots, n. \quad (35)
\]
This is precisely one half of the operator conditions, viz.
\[
D_t(\bar{\zeta}_j) + \bar{X}\left(\frac{\partial H}{\partial q^j}\right) = \bar{\eta}^j \Gamma_{iq^j}, \quad j = 1, \ldots, n, \quad (36)
\]
which hold on the solutions of (5), if and only if following characteristics are chosen
\[
Q^i = \bar{\eta}^i, \quad R_i = -\bar{\zeta}_i, i = 1, \ldots, n \quad (37)
\]
Equations (32) and (33), with the aid of (37), provide the required conditions (27) and (28).

The equation (30) takes the following form

$$D_t I = \bar{\eta}^i (\dot{q}_i + H_{q^i} - \Gamma_i) - \bar{\zeta}_i (q^i - H_{p^i}).$$

To complete the proof of the second part, the action of the variational derivative operator with respect to $p$, $\delta/\delta p_j$, $j = 1, \ldots, n$ on (38) results in

$$\frac{\partial \bar{\eta}^i}{\partial p_j} (\dot{p}_i + H_{q^i} - \Gamma_i) + \bar{\eta}^j (H_{q^j} - \Gamma_{ip})$$

$$- D_t (\delta_i^j \bar{\eta}^i) - \frac{\partial \bar{\zeta}_i}{\partial p_j} (q^i - H_{p^i}) + \bar{\zeta}_i H_{p^i p^j} = 0, j = 1, \ldots, n$$

After expansion and setting the coefficients of $\dot{p}_i$ and $\dot{q}^i$ to zero yield

$$\frac{\partial \bar{\eta}^i}{\partial p_j} - \frac{\partial \bar{\eta}^j}{\partial p_i} = 0,$$  \hspace{1cm} (40)

$$\frac{\partial \bar{\eta}^j}{\partial q^i} + \frac{\partial \bar{\zeta}_i}{\partial p_j} = 0, \hspace{0.5cm} i, j = 1, \ldots, n$$  \hspace{1cm} (41)

The rest of the terms in (39) are

$$\bar{\eta}^i H_{q^j p^j} + \bar{\zeta}_i H_{p^i p^j} - \frac{\partial \bar{\eta}^j}{\partial t} + H_{p^i} \frac{\partial \bar{\zeta}_i}{\partial p_j} + (H_{q^i} - \Gamma_i) \frac{\partial \bar{\eta}^j}{\partial p_j} - \bar{\eta}^j \Gamma_{ip} = 0, j = 1, \ldots, n$$  \hspace{1cm} (42)

Equation (42), with the aid of (40) and (41) yield the second half of the operator conditions of (5)

$$D_t (\bar{\eta}^i) - \bar{X} \left(\frac{\partial H}{\partial p_j}\right) = -\bar{\eta}^j \Gamma_{ip}, \hspace{0.5cm} i = 1, \ldots, n$$  \hspace{1cm} (43)

which are satisfied on the solutions to (5). Note that equation (41) is the same as what we deduced in the first part of the proof after interchanging $i$ and $j$ due to symmetry in the indices. This completes the proof.

**Remark 1.** It is clear from Theorem 3 that for $n = 1$ the conditions (27), (28), (29), (21) and (22) reduce to just one relation, viz.

$$\frac{\partial \bar{\eta}}{\partial q} + \frac{\partial \bar{\zeta}_i}{\partial p} = 0,$$  \hspace{1cm} (44)

and

$$D_t (\bar{\eta}) + \bar{X} \left(\frac{\partial H}{\partial q}\right) = \bar{\eta} \Gamma_q,$$  \hspace{1cm} (45)

$$D_t (\bar{\zeta}) - \bar{X} \left(\frac{\partial H}{\partial p}\right) = -\bar{\eta} \Gamma_p.$$

**Theorem 4.** For each partial Hamiltonian operator $X = \bar{X} + \xi D_t$ which satisfies the conditions (27), (28) and (29), there corresponds a first integral $I$ which is determined uniquely (up to an ignorable constant) from

$$I_{q^i} = -\bar{\zeta}_i, \hspace{0.5cm} I_{p^i} = \bar{\eta}^i, \hspace{0.5cm} i = 1, \ldots, n$$

$$I_t = \bar{\eta}^i (H_{q^i} - \Gamma_{i}) + \bar{\zeta}_i H_{p^i}. $$  \hspace{1cm} (47)
Proof. If the partial Hamiltonian generator $X$ satisfies (27), (28), (29) and operator conditions (21), (22), then

$$D_t I = \bar{\eta}^i (\dot{p}_i + H_{q^i - \Gamma_i}) - \bar{\zeta}_i (\dot{q}_i - H_{p_i}),$$

holds. The expansion of the left hand side of (48) and equating it to the right hand side of (48) straightforwardly gives the results (47). This completes the proof.

5. Applications. We take examples from the literature to illustrate our results.

1. It is always opportune to start with the simple harmonic oscillator equation which has partial Hamiltonian

$$H = \frac{p^2}{2}$$

and partial Hamiltonian system

$$\dot{q} = p, \quad \dot{p} = -q.$$  \hspace{1cm} (49)

Here $\Gamma = -q$ and Theorem 2 results in

$$D_t \bar{\eta} + \bar{\zeta} = 0, \quad D_t \bar{\eta} - \bar{\zeta} = 0,$$

where $\bar{\eta} = \eta - \xi p$ and $\bar{\zeta} = \zeta + \xi q$. If one confines $\xi$ and $\eta$ to be independent of $p$, then by solving (50) one ends up with eight operators which are of point type in $(t,q)$ space. Only the following five

$$X_1 = \partial_t, \quad X_2 = \sin 2t \partial_t + q \cos 2t \partial_q - (p \cos 2t + 2q \sin 2t) \partial_p, \quad X_3 = \cos 2t \partial_t - q \sin 2t \partial_q + (p \sin 2t - 2q \cos 2t) \partial_p, \quad X_4 = \sin t \partial_q + \cos t \partial_p, \quad X_5 = \cos t \partial_q - \sin t \partial_p$$

(51)

satisfy Theorem 3 which gives only the first condition as $n = 1$ here which is

$$\frac{\partial \bar{\eta}}{\partial q} + \frac{\partial \bar{\zeta}}{\partial p} = 0.$$  \hspace{1cm} (52)

Just to mention that although $X = q \partial_q + p \partial_p$ is an operator that identically satisfies (50), it does not satisfy condition (52) and thus does not provide a first integral. Important to note that the operators (51) are also point symmetries of our system (49) due to $\Gamma$ being independent of $p$. These operators arise in reference [7] as well. However, we stress that here we have taken a partial Hamiltonian system with partial Hamiltonian $H = \frac{p^2}{2}$. It turns out that for the $n$-dimensional harmonic oscillator system one has the similar results. The partial Hamiltonian in this case is

$$H = \frac{1}{2} \sum_{j=1}^{n} p_j^2$$

(53)

and now the partial Hamiltonian system is

$$\dot{q}^i = p_i, \quad \dot{p}_i + q^i = 0, i = 1, \ldots, n.$$  \hspace{1cm} (54)

with $\Gamma_i = -q^i$. The partial Hamiltonian operators which satisfy Theorem 3 are again $(n^2 + 3n + 6)/2$ operators which are as stated in [7] (see Example 3.). Let us now demonstrate how one can compute a first integral via Theorem 4 by knowledge of an operator which satisfies Theorem 4. This is manifestly straightforward. We consider the operator $X_2$ in (51). For this operator $\bar{\eta} = q \cos 2t - p \sin 2t$ and $\bar{\zeta} = -p \cos 2t - q \sin 2t$. Therefore the relations of Theorem 4 become

$$I_q = p \cos 2t + q \sin 2t, \quad I_p = q \cos 2t - p \sin 2t, \quad I_t = q(q \cos 2t - p \sin 2t) - p(p \cos 2t + q \sin 2t).$$  \hspace{1cm} (55)
The solution of this system is easy enough and yields
\[ I = pq \cos 2t + \frac{1}{2} q^2 \sin 2t - \frac{1}{2} p^2 \sin 2t \] (56)
which is the desired first integral. This is obtained up to an ignorable constant. Likewise, one can deduce all remaining integrals using Theorem 4 by simple integration.

2. A modified Emden equation which was investigated in Leach [5] and Kara et al [3] using Lagrangian formulations is the following nonlinear second-order ODE
\[ \ddot{q} + \frac{2}{t} \dot{q} = 3q^5. \] (57)
This can be written as a partial Hamiltonian system as
\[ \dot{q} = p, \]
\[ \dot{p} = -\frac{2}{t} p + 3q^5, \] (58)
with \( \Gamma = -\frac{2}{t} p + 3q^5 \) and partial Hamiltonian \( H = \frac{1}{2} p^2 \). By utilizing the condition of Theorems 2 and 3, a partial Hamiltonian operator of (58) works out to be
\[ X = 2t^3 \partial_t - t^2 q \partial_q. \] (59)
Invoking Theorem 4 we have
\[ I_q = -t^2 p + 6t^3 q^5, \]
\[ I_p = -t^2 q - 2t^3 p, \]
\[ I_t = (-t^2 q - 2t^3 p)(\frac{2}{t} p - 3q^5) + (t^2 p - 6t^3 q^5)p. \] (60)
The solution of (60) is the known [5] first integral
\[ I = -t^2 pq + t^3 q^6 - t^3 p^2. \] (61)

3. Mahomed and Moitsheki [8] in their study utilized the plane polar form of the generalized Ermakov system, viz.
\[ \ddot{r} - r \dot{\theta}^2 = \frac{F(\theta)}{r^3}, \]
\[ r \ddot{\theta} + 2r \dot{\theta} = \frac{G(\theta)}{r^3}, \] (62)
where \( F \) and \( G \) are arbitrary functions of \( \theta \). A partial Hamiltonian for the system (62) was invoked in [8] and is of the form
\[ H = \frac{1}{2} p_1^2 + \frac{1}{2} r^2 p_2^2 + \frac{1}{2} F(\theta) \]
provided \( G + F'/2 \neq 0 \), where the partial Hamiltonian system is
\[ \dot{r} = p_1, \]
\[ \dot{\theta} = p_2/r^2, \]
\[ \dot{p}_1 = p_2^2/r^3 + F(\theta)/r^3, \]
\[ \dot{p}_2 = G(\theta)/r^2. \] (64)
Here we have
\[ \Gamma_1 = 0, \Gamma_2 = \frac{1}{2} F'/r^2 + G/r^2 \] (65)
Note that if $G + F'/2 = 0$, then system (64) becomes a canonical Hamiltonian system. Both these cases were looked at in detail in [8] using the approach of [10]. They worked out all the possibilities for which the system (64) admits first integrals. In order to illustrate the beauty of Theorem 4, we use one of their cases when $H$ is partial Hamiltonian, viz.

$$F(\theta) = G(\theta) \frac{c_1 \sin \theta + c_2 \cos \theta}{c_1 \cos \theta - c_3 \sin \theta},$$

where $c_1$ and $c_3$ are nonzero constants. If $F$ and $G$ are related by (66), the authors [8] obtained the partial Hamiltonian operators

$$X_1 = (b_1 \cos \theta - b_2 \sin \theta) \frac{\partial}{\partial \theta} - \frac{1}{r} (b_1 \sin \theta + b_2 \cos \theta) \frac{\partial}{\partial \theta},$$

$$X_2 = p_2 \frac{\partial}{\partial \theta},$$

with $b_1$ and $b_2$ related by

$$b_1 = c_1 + c_2 t, \quad b_2 = c_3 + c_2 c_4 t / c_1.$$  

The application of Theorem 2, for $X_1$, indeed results in

$$D_1 \eta^1 - \zeta_1 = 0,$$

$$D_1 \eta^1 + \frac{2}{r} p_2 \eta^1 = \frac{1}{r} \zeta_2$$

which in turn gives rise to $\zeta_1$ and $\zeta_2$ (Theorem 3 holds as well) and hence Theorem 4 yields the relations

$$I_r = -c_2 \cos \theta + c_2 c_3 \sin \theta / c_1 + \frac{p_2}{r^2} (b_1 \sin \theta + b_2 \cos \theta),$$

$$I_\theta = -p_1 (b_1 \sin \theta + b_2 \cos \theta) - \frac{p_2}{r} (b_1 \cos \theta - b_2 \sin \theta) + r(c_2 \sin \theta + c_2 c_3 \cos \theta / c_1),$$

$$I_{p_1} = b_1 \cos \theta - b_2 \sin \theta,$$

$$I_{p_2} = -\frac{1}{r} (b_1 \sin \theta + b_2 \cos \theta),$$

$$I_t = p_1 (c_2 \cos \theta - c_2 c_3 \sin \theta / c_1) - \frac{p_2}{r} (c_2 \sin \theta + c_2 c_3 \cos \theta / c_1).$$  

The integration of system (70) is straightforward and gives precisely

$$I_1 = (p_1 b_1 - rc_2) (c_1 \cos \theta - c_3 \sin \theta) - p_2 b_1 (c_1 \sin \theta + c_3 \cos \theta) / r$$

which confirms the results of [8]. For $X_2$ one gets the angular momentum type integral quite quickly. That is

$$I_2 = \frac{1}{2} p_2^2 - \int G(\theta) d\theta.$$  

4. Now we briefly consider the inverse problem. That is given a first integral of a partial Hamiltonian system, how does one connect an operator or operators that give rise to the said integral. We take the angular momentum type first integral which occurs in general for the generalized Ermakov system (64) in polar form. This is again given by (72). We utilize Theorem 4. After differentiating (71) with respect to each of $r, \theta, p_1, p_2$, we immediately arrive at

$$\check{\zeta}_1 = 0, \quad \check{\zeta}_2 = G(\theta), \quad \check{\eta}^1 = 0, \quad \check{\eta}^2 = p_2.$$  

Thus we have a liberty of choice in $\xi$ since each of the quantities in (73) are expressed in terms of $\xi$ in view of the relations (8). However, there is no value of $\xi$ such that the operator $X$ associated with (72) has $\eta^1$ or $\eta^2$ independent of $p_1$ or $p_2$. One
can observe this from $X_2$ already in Example 3. Hence the operator $X$ which is a point operator in $(t, r, \theta, p_1, p_2)$ space here will not be a point type operator in a Lagrangian context but rather a generalized operator with derivative dependency. We moreover notice that an integral can have more than one operator associated with it. For the angular momentum type integral above there are many operators which will gives rise to it by Theorem 4.

These examples amply suffice in illustrating the main results as encapsulated in Theorems 2, 3 and 4.

6. Conclusions. We have focused on the partial Hamiltonian systems in terms of characterization of their operators and related first integrals. Firstly, it is shown that if an operator is a partial Hamiltonian operator which yields a first integral, then so does its evolutionary representative. Secondly, extra operator conditions are provided for a partial Hamiltonian operator in evolutionary form to yield a first integral. Thirdly, characterization of partial Hamiltonian operators and related first integral conditions are provided for the partial Hamiltonian system.

We have then applied this theory to familiar examples of mechanics starting with a linear paradigm. Then the modified Emden equation and generalized Ermakov system in polar form.

The approach we have developed here also shows that if one has any first integral that arises from a partial Hamiltonian system, then there is at least one operator which corresponds to this first integral which is the inverse problem for partial Hamiltonian systems. We illustrated the result with the angular momentum type integral of the generalized Ermakov system. Here we had many operators associated with this integral. This result provides a correspondence between first integrals and partial Hamiltonian operators for systems that have partial Hamiltonian formulation.

Acknowledgments. FM is thankful to the N.R.F. of South Africa for research grants and enabling environment for research. Also he wishes to register his deep gratitude to Dr. Shahid Amjad Chaudhry of LSE, Pakistan for great hospitality during time there when much of this work was commenced and fruitfully completed.

REFERENCES

[1] A. C. Chiang, Elements of Dynamic Optimization, McGraw Hill, New York, 1992.
[2] V. Dorodnitsyn and R. Kozlov, Invariance and first integrals of continuous and discrete Hamiltonian equations, J. Eng. Math., 66 (2010), 253–270.
[3] A. H. Kara, F. M. Mahomed, I. Naeem and C. Wafo Soh, Partial Noether operators and first integrals via partial Lagrangians, Math. Methods in the Applied Sciences, 30 (2007), 2079–2089.
[4] V. V. Kozlov, Integrability and nonintegrability in Hamiltonian mechanics, Russ. Math. Surveys, 38 (1983), 1–76.
[5] P. G. L. Leach, First integrals for the modified Emden equation $\ddot{q} + \alpha(t)\dot{q} + q^n = 0$, J. Math. Phys. 26 (1985), 2510–2514.
[6] T. Levi-Civita, Interpretazione gruppale degli integrali di un sistema canonico, Rend. Acc. Lincei, ser. III, 8 (1899), 235–238.
[7] F. M. Mahomed and J. A. G. Roberts, Characterization of Hamiltonian symmetries and their first integrals, International Journal of Non-Linear Mechanics, 74 (2015), 84–91.
[8] K. S. Mahomed and R. J. Moitsheki, First integrals of generalized Ermakov systems via the Hamiltonian formulation, International Journal of Modern Physics B, 30 (2016), 1640019, 12 pp.
[9] J. E. Marsden and A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys., 5 (1974), 121–130.
[10] R. Naz, F. M. Mahomed and A. Chaudhry, A partial Hamiltonian approach for current value Hamiltonian systems, *Commu. Nonlinear. Sci. Numer. Simulat.*, **19** (2014), 3600–3610.

[11] R. Naz, A. Chaudhry and F. M. Mahomed, Closed-form solutions for the Lucas-Uzawa model of economic growth via the partial Hamiltonian approach, *Commu. Nonlinear. Sci. Numer. Simulat.*, **30** (2016), 299–306.

[12] R. Naz, The applications of the partial Hamiltonian approach to mechanics and other areas, *International Journal of Non-Linear Mechanics*, **86** (2016), 1–6.

[13] R. Naz, F. M. Mahomed and A. Chaudhry, A partial Lagrangian method for dynamical systems, *Nonlinear Dynamics*, **84** (2016), 1783–1794.

[14] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1993.

[15] G. Saccomandi and R. Vitolo, A Translation of the T. Levi-Civita paper: Interpretazione Gruppale degli Integrali di un Sistema Canonico, *Regul. Chaotic Dyn.*, **17** (2012), 105–112, *arXiv:1201.2388v1*.

Received November 2016; revised April 2017.

E-mail address: drrehana@lahoreschool.edu.pk
E-mail address: Fazal.Mahomed@wits.ac.za