ON SMALL GAPS AMONG PRIMES

FRED B. HOLT AND HELGI RUDD

Abstract. A few years ago we identified a recursion that works directly with the gaps among the generators in each stage of Eratosthenes sieve. This recursion provides explicit enumerations of sequences of gaps among the generators, which are known as constellations.

As the recursion proceeds, adjacent gaps within longer constellations are added together to produce shorter constellations of the same sum. These additions or closures correspond to removing composite numbers that are divisible by the prime for that stage of Eratosthenes sieve. Although we don’t know where in the cycle of gaps a closure will occur, we can enumerate exactly how many copies of various constellations will survive each stage.

In this paper, we study these systems of constellations of a fixed sum. Viewing them as discrete dynamic systems, we are able to characterize the populations of constellations for sums including the first few primorial numbers: 2, 6, 30.

Since the eigenvectors of the discrete dynamic system are independent of the prime – that is, independent of the stage of the sieve – we can characterize the asymptotic behavior exactly. In this way we can give exact ratios of the occurrences of the gap 2 to the occurrences of other small gaps for all stages of Eratosthenes sieve.

1. Introduction

We work with the prime numbers in ascending order, denoting the $k^{th}$ prime by $p_k$. Accompanying the sequence of primes is the sequence of gaps between consecutive primes. We denote the gap between $p_k$ and $p_{k+1}$ by $g_k = p_{k+1} - p_k$. These sequences begin

\[
p_1 = 2, \quad p_2 = 3, \quad p_3 = 5, \quad p_4 = 7, \quad p_5 = 11, \quad p_6 = 13, \quad \ldots
\]
\[
g_1 = 1, \quad g_2 = 2, \quad g_3 = 2, \quad g_4 = 4, \quad g_5 = 2, \quad g_6 = 4, \quad \ldots
\]

A number $d$ is the difference between prime numbers if there are two prime numbers, $p$ and $q$, such that $q - p = d$. There are already many interesting results and open questions about differences between prime numbers;

Date: 8 Jan 2014– version 1.1.
1991 Mathematics Subject Classification. 11N05, 11A41, 11A07.
Key words and phrases. primes, twin primes, gaps, prime constellations, Eratosthenes sieve, primorial numbers.
a seminal and inspirational work about differences between primes is Hardy and Littlewood’s 1923 paper [2].

A number $g$ is a gap between prime numbers if it is the difference between consecutive primes; that is, $p = p_i$ and $q = p_{i+1}$ and $q - p = g$. Differences of length 2 or 4 are also gaps; so open questions like the Twin Prime Conjecture, that there are an infinite number of gaps $g_k = 2$, can be formulated as questions about differences as well.

A constellation among primes [6] is a sequence of consecutive gaps between prime numbers. Let $s = c_1c_2 \cdots c_k$ be a sequence of $k$ numbers. Then $s$ is a constellation among primes if there exists a sequence of $k + 1$ consecutive prime numbers $p_ip_{i+1} \cdots p_{i+k}$ such that for each $j = 1, \ldots, k$, we have the gap $p_{i+j} - p_{i+j-1} = c_j$. Equivalently, $s$ is a constellation if for some $i$ and all $j = 1, \ldots, k$, $c_j = g_{i+j}$.

We will write the constellations without marking a separation between single-digit gaps. For example, a constellation of 24 denotes a gap of $g_k = 2$ followed immediately by a gap $g_{k+1} = 4$. For the small primes we will consider explicitly, most of these gaps are single digits, and the separators introduce a lot of visual clutter. We use commas only to separate double-digit gaps in the cycle. For example, a constellation of 2, 10, 2 denotes a gap of 2 followed by a gap of 10, followed by another gap of 2.

In [3] we introduced a recursion that works directly on the gaps among the generators in each stage of Eratosthenes sieve. These are the generators of $Z \mod p^#$ in which $p^#$ is the product of the prime numbers from 2 through $p$, known as the primorial of $p$. For a constellation $s$, this recursion enables us to enumerate exactly how many copies of $s$ occur in the $k^{th}$ stage of the sieve. We denote this number of copies of $s$ as $N_s(p_k)$.

For example, after the primes 2, 3, and 5 and their multiples have been removed, we have the cycle of gaps $G(5^#) = 64242462$. This cycle of 8 gaps sums to 30. In this cycle, for the constellation $s = 2$, we have $N_2(5) = 3$. For the constellation $s = 242$, we have $N_{242}(5) = 1$. The cycle of gaps $G(p^#)$ has $\phi(p^#)$ gaps that sum to $p^#$.

In [4] we assumed that copies of a constellation were approximately uniformly distributed within the cycle of gaps $G(p^#)$, from which we could then estimate the numbers of these constellations that survive to occur as constellations among prime numbers. For a few select constellations we compared our estimates to actual counts up through $10^{12}$. For these constellations, our estimates in [4] appear to have the correct asymptotic behavior, but our estimates also seem to have a systematic error correlated with the number of gaps in the constellation.

In this paper, we identify a discrete dynamic system that provides exact counts of a gap and its driving terms, which are constellations that under
successive closures produce the gap at later stages of the sieve. These raw counts grow superexponentially, and so to better understand their behavior we take the ratio of a raw count to the number of gaps \( g = 2 \) at each stage of the sieve.

For a gap \( g \) that has driving terms of lengths \( 2 \leq j \leq J \), we form a vector of initial values \( \bar{w}_{p_0} \), whose \( j^{th} \) entry is the ratio of the number of driving terms for \( g \) of length \( j \) in \( G(p_0\#) \) to the number of gaps \( 2 \) in this cycle of gaps. Recasting the discrete dynamic system to work with these ratios, we have

\[
\bar{w}_{p_k} = M_J|_{p_k} \cdot \bar{w}_{p_{k-1}}
\]

The matrix \( M_J \) does not depend on the gap \( g \). It does depend on the prime \( p_k \), and we use the exponential notation \( M_J^k \) to indicate the product of the \( M \)'s over the indicated range of primes.

Although the matrix \( M_J \) depends on the prime \( p_k \), its eigenvectors do not. We are therefore able to give a simple exact expression of the dynamic system that reveals its asymptotic behavior. We show that as \( p_k \longrightarrow \infty \), the following ratios describe the relative frequency of occurrence of gaps in Eratosthenes sieve:

| ratio \( N_g/N_2 \) | gaps \( g \) with this ratio |
|---------------------|-----------------------------|
| 1 : 2               | 4, 8, 16, 32 |
| 2 : 6               | 12, 18, 24 |
| 2.6 : 30            |

The ratios discussed in this paper give the exact values of the relative frequencies of various gaps and constellations as compared to the number of gaps \( 2 \) at each stage of Eratosthenes sieve. As the sieving process continues, if the closures are at all fair, then these ratios should also be good approximations to the relative occurrence of these gaps and constellations as gaps among primes.

2. Recursion on Cycle of Gaps

In the cycle of gaps, the first gap corresponds to the next prime. In \( G(5\#) \) the first gap \( g_1 = 6 \), which is the gap between 1 and the next prime, 7. The next several gaps are actually gaps between prime numbers. In the cycle of gaps \( G(p_k\#) \), the gaps between \( p_{k+1} \) and \( p_{k+1}^2 \) are in fact gaps between prime numbers.

There is a simple recursion which generates \( G(p_{k+1}\#) \) from \( G(p_k\#) \). This recursion and many of its properties are developed in [3]. We include only the concepts and results we need for developing the material in this paper.
The recursion on the cycle of gaps consists of three steps.

R1. The next prime \( p_{k+1} = g_1 + 1 \), one more than the first gap;
R2. Concatenate \( p_{k+1} \) copies of \( G(p_#) \);
R3. Add adjacent gaps as indicated by the elementwise product \( p_{k+1} \ast G(p_#) \): let \( i_1 = 1 \) and add together \( g_{i_1} + g_{i_1+1} \); then for \( n = 1, \ldots, \phi(N) \), add \( g_j + g_{j+1} \) and let \( i_{n+1} = j \) if the running sum of the concatenated gaps from \( g_{i_n} \) to \( g_j \) is \( p_{k+1} \ast g_n \).

**Example:** \( G(7#) \). To illustrate this recursion, we construct \( G(7#) \) from \( G(5#) = 64242462 \).

R1. Identify the next prime, \( p_{k+1} = g_1 + 1 = 7 \).
R2. Concatenate seven copies of \( G(5#) \):

\begin{align*}
64242462 & \quad 64242462 \quad 64242462 \quad 64242462 \quad 64242462 \quad 64242462 \quad 64242462
\end{align*}

R3. Add together the gaps after the leading 6 and thereafter after differences of \( 7 \ast G(5#) = 42, 28, 14, 28, 14, 28, 42 \):

\[
G(7#) = 6 + 42, 28, 14, 28, 14, 28, 42 + 62
\]

The final difference of 14 wraps around the end of the cycle, from the addition preceding the final 6 to the addition after the first 6.

We summarize a few properties of the cycle of gaps \( G(p#) \), as established in \[3\]. The cycle of gaps ends in a 2, and except for this final 2, the cycle of gaps is symmetric. In constructing \( G(p_{k+1}#) \), each possible addition of adjacent gaps in \( G(p_k#) \) occurs exactly once.

2.1. **Numbers of constellations.** The power of the recursion on the cycle of gaps is seen in the following theorem, which enables us to calculate the number of occurrences of a constellation \( s \) through successive stages of Eratosthenes sieve.

**Theorem 2.1.** (from \[3\]) Let \( s \) be a constellation of \( j \) gaps in \( G(p#) \), such that \( j < p_{k+1} - 1 \) and \( \sigma(s) < 2p_{k+1} \). Let \( S \) be the set of all constellations \( \bar{s} \) which would produce \( s \) upon one addition of adjacent gaps in \( \bar{s} \). Then the number \( N_s(p) \) of occurrences of \( s \) in \( G(p#) \) satisfies the recurrence

\[
N_s(p_{k+1}) = (p_{k+1} - (j + 1)) \cdot N_s(p_k) + \sum_{\bar{s} \in S} N_{\bar{s}}(p_k)
\]
Figure 1. This figure illustrates the initial conditions and driving terms for calculating the numbers of copies of the gaps 6, 8, 10, 12 in $G(p^\#)$. The entries in this chart indicate the constellation $s$, its length $j$; the prime for which the constellation occurs in $G(p^\#)$ and which satisfies the conditions of Theorem 2.1, and the number $N = N_s(p)$ of occurrences of the constellation in $G(p^\#)$. From these figures we can derive the recursive count $N_s(q)$ for primes $q > p$. For the gap 30, the system of driving terms goes out to length $j = 8$.

3. The Dynamic System

Figure [1] illustrates the initial conditions for the gaps 2, 4, 6, 8, 10, and 12, and their driving terms. Note that the initial conditions are not predicated
Figure 2. This figure illustrates the dynamic system of Theorem 2.1 through stages of the recursion for $G(p^\#)$, using just the counts of gaps and their driving terms. The action of the system at each stage of the recursion is independent of the specific gap and its driving terms. Below the diagram for the system, we record the initial conditions for a set of gaps at $p_0 = 13$. From this information we can derive the recursive count $N_s(q)$ for primes $q > p_0$. Since the raw counts are superexponential, we take the ratio of the count for each constellation to the simplest counts $N_2(p) = N_4(p)$.

For larger gaps, these systems of driving terms become more unwieldy. For a gap $g$, we don’t need to identify all of the individual constellations of length $j$ that sum to $g$. All we need is a count of these constellations. So our diagram in Figure 1 becomes simpler, as shown in Figure 2.

Recall that $g = 2$ has no driving terms, so

$$N_2(p_k) = (p_k - 2) \cdot N_2(p_{k-1}).$$

Let $n_{s,j}(p)$ be the number of all constellations of length $j$ that either are copies of $s$ itself (if $j$ equals the length of $s$) or are driving terms for
s, in \( G(p\#) \). As the recursion continues, these numbers \( n_{s,j} \) grow superexponentially by factors of \((p - j - 1)\). To make the numbers and analysis manageable over many stages of the recursion, we normalize by the number of 2’s, \( N_2(p) = N_4(p) \). We define

\[
w_{s,j}(p) = \frac{n_{s,j}(p)}{N_2(p)}.
\]

Anticipating our work with \( g = 30 \) below, let us use \( p_0 = 13 \) for our initial conditions. The prime \( p = 13 \) is the first prime for which the conditions of Theorem 2.1 are satisfied for \( g = 30 \). In \( G(13\#) \) we have the following initial values.

| gap  | \( n_{g,j}(13) \): driving terms of length \( j \) in \( G(13\#) \) |
|-----|-----------------------------------------------------|
|     | \( g \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2, 4 | 1485 |  |  |  |  |  |  |  |  |  |
| 6 | 1690 | 1280 |  |  |  |  |  |  |  |  |
| 8 | 394 | 902 | 189 |  |  |  |  |  |  |  |
| 10 | 438 | 1164 | 378 |  |  |  |  |  |  |  |
| 12 | 188 | 1276 | 1314 | 192 |  |  |  |  |  |  |
| 14 | 58 | 536 | 900 | 288 |  |  |  |  |  |  |
| 16 | 12 | 252 | 750 | 436 | 35 |  |  |  |  |  |
| 18 | 8 | 256 | 1224 | 1272 | 210 |  |  |  |  |  |
| 20 | 0 | 24 | 348 | 960 | 600 | 48 |  |  |  |  |
| 22 | 2 | 48 | 312 | 784 | 504 |  |  |  |  |  |
| 24 | 0 | 20 | 258 | 928 | 1260 | 504 |  |  |  |  |
| 26 | 0 | 2 | 40 | 322 | 724 | 448 | 84 |  |  |  |
| 28 | 0 | 0 | 36 | 344 | 794 | 528 | 80 |  |  |  |
| 30 | 0 | 0 | 10 | 194 | 1066 | 1784 | 816 | 90 |  |  |
| 32 | 0 | 0 | 0 | 12 | 200 | 558 | 523 | 172 | 20 |  |

For \( g = 6 \) there are driving terms of length \( j = 2 \), so we have a 2-dimensional system.

\[
\begin{bmatrix}
w_{6,1} \\
w_{6,2}
\end{bmatrix}
\|_{p_k} =
\begin{bmatrix}
\frac{p_k-2}{p_k-2} & 1 \\
0 & \frac{p_k-3}{p_k-2}
\end{bmatrix}
\cdot
\begin{bmatrix}
w_{6,1} \\
w_{6,2}
\end{bmatrix}
\|_{p_{k-1}}
\]

\[
= \begin{bmatrix}
1 & b_1 \\
0 & a_2
\end{bmatrix}
\cdot
\begin{bmatrix}
w_{6,1} \\
w_{6,2}
\end{bmatrix}
\|_{p_{k-1}}
\]

We have the system matrix

\[
M_2 = \begin{bmatrix}
1 & b_1 \\
0 & a_2
\end{bmatrix}
\]

with \( b_1 = b_1(p) = \frac{1}{p-2} \) and \( a_2 = a_2(p) = \frac{p-3}{p-2} \). We will often suppress the explicit dependence of \( a_i \) and \( b_i(p) \) on the prime \( p \), but a consequence is that multiplication among these parameters does not commute.

Formulated in this way, we can use common methods of analysis for dynamic systems, except that the values of the matrix entries depend on the
progression of primes. Again we caution that we have qualified the expo-
nential notation, to mean the product of a parameter over the appropriate
sequence of prime numbers. Let

\[
\begin{bmatrix}
w_{6,1} \\
w_{6,2}
\end{bmatrix}
_{p_k}
= M_2|_{p_k}
\begin{bmatrix}
w_{6,1} \\
w_{6,2}
\end{bmatrix}
_{p_{k-1}}
= M_2^k
\begin{bmatrix}
w_{6,1} \\
w_{6,2}
\end{bmatrix}
_{p_0}
\]

To understand the relative occurrence of 6’s to 2’s in the large, we examine
the matrices \(M_2^k\).

\[
M_2^k = \begin{bmatrix}
1 & \beta^{(k)}_{12} \\
0 & a_2^k
\end{bmatrix}
\]

with initial values \(\beta_{12} = b_1(17) = \frac{1}{15}\), \(a_2 = \frac{14}{15}\), and powers

\[
\beta^{(k)}_{12} = 1 \cdot \beta^{(k-1)}_{12} + \frac{1}{p_k - 2} \cdot a_2^{k-1}
\]

and \(a_2^k = \frac{p_k - 3}{p_k - 2} a_2^{k-1} = \prod_{q=p_1}^{p_k} \frac{q - 3}{q - 2}\)

The limit of the ratios \(w_{6,j}\) is determined by the limit of products of the
system matrix

\[
M_2^\infty = \begin{bmatrix}
1 & \beta^{(\infty)}_{12} \\
0 & a_2^\infty
\end{bmatrix}
= \begin{bmatrix}
1 & \lim_{k \to \infty} \beta^{(k)}_{12} \\
0 & \lim_{k \to \infty} \prod_{q=17}^{p_k} \frac{q - 3}{q - 2}
\end{bmatrix}.
\]

For \(g = 8\) and \(g = 10\), there are driving terms up to length 3, so we have
a 3-dimensional system. The system matrix is

\[
M_3 = \begin{bmatrix}
1 & b_1 & 0 \\
0 & a_2 & b_2 \\
0 & 0 & a_3
\end{bmatrix}
\]

with \(b_1\) and \(a_2\) as before in \(M_2\), \(b_2 = b_2(p) = \frac{2}{p - 2}\) and \(a_3 = a_3(p) = \frac{p - 4}{p - 2}\).

Powers of \(M_3\) will be upper triangular

\[
M_3^k = \begin{bmatrix}
1 & \beta^{(k)}_{12} & \beta^{(k)}_{13} \\
0 & a_2^k & \beta^{(k)}_{23} \\
0 & 0 & a_3^k
\end{bmatrix}
= M_3|_{p_k} \cdot M_3^{k-1},
\]
with the following recursive definitions:

\begin{align*}
(1) \quad a_2^k &= \prod_{q=17}^{p_k} \frac{q-3}{q-2} \\
(2) \quad a_3^k &= \prod_{q=17}^{p_k} \frac{q-4}{q-2} \\
\beta_{12}^{(k)} &= 1 \cdot \beta_{12}^{(k-1)} + \frac{1}{p_k - 2} \cdot a_2^{k-1} \\
\beta_{23}^{(k)} &= \frac{p_k - 3}{p_k - 2} \cdot \beta_{23}^{(k-1)} + \frac{2}{p_k - 2} \cdot a_3^{k-1} \\
\beta_{13}^{(k)} &= 1 \cdot \beta_{13}^{(k-1)} + \frac{1}{p_k - 2} \cdot \beta_{23}^{(k-1)}
\end{align*}

Since we will later be comparing these values to $w_{30, j}$, we calculate initial conditions using $p_0 = 13$. We can then use calculations of the system parameters in $M_3^k$ to obtain the ratios $w_{8, j}$ and $w_{10, j}$ for large primes. With $p_0 = 13$, we have calculated the system parameters through $p_k = \hat{p} = 999,999,999,989$. See Figure 3. For this value of $p_k$, we calculate the following values.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{System_parameters_to_prime_q=999,999,999,989, with p0=13.png}
\caption{This figure illustrates the values of the system parameters for $M_3^k$ as the value of $p_k$ runs from 17 to 999,999,999,989. With the parameters $\beta_{12}^{(k)}$ and $\beta_{13}^{(k)}$, we can calculate the ratios $w_{p, j}$ for the gaps 6, 8, 10 up through $G(999,999,999,989\#)$.}
\end{figure}
For \( p_0 = 13 \)

| \( g \) | \( w_{g,1}(13) \) | \( w_{g,2}(13) \) | \( w_{g,3}(13) \) |
|-------|----------------|----------------|----------------|
| 6     | 1.13804714     | 0.86195286     | 0              |
| 8     | 0.26531987     | 0.60740741     | 0.12727273     |
| 10    | 0.29494949     | 0.78383838     | 0.25454545     |

For \( p_k = \hat{p} = 999,999,999,989 \)

\[
\begin{align*}
\alpha_{1} &= 1 \\
\beta_{12} &= 0.89793248 \\
\beta_{13} &= 0.80606493
\end{align*}
\]

\[
w_{6,1}(\hat{p}) = 1.91202 \quad w_{8,1}(\hat{p}) = 0.91332 \quad w_{10,1}(\hat{p}) = 1.20396
\]

This data tells us that in \( G(999,999,999,989\#) \), which covers the interval \( \hat{p} = 999,999,999,989 \) to \( \hat{p}\# \approx 10^{434294060804} \), the ratio of gaps \( g = 6 \) to gaps \( g = 2 \) is \( w_{6,1}(\hat{p}) = 1.91202 \). The number of gaps \( g = 10 \) has surpassed the gaps \( g = 2 \) with a ratio of \( w_{10,1}(\hat{p}) = 1.20396 \), but the gaps \( g = 8 \) still lag the number of gaps \( g = 2 \) with a ratio \( w_{8,1}(\hat{p}) = 0.91332 \).

4. General system

The general form of this dynamic system, for gaps or constellations with driving terms of length \( j \leq J \) is

\[
\begin{bmatrix}
w_{g,1} \\
\vdots \\
w_{g,J}
\end{bmatrix}_{p_k} =
\begin{bmatrix}
1 & b_1 & 0 & \cdots & 0 \\
0 & a_2 & b_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & a_{J-1} & b_{J-1} & 0 \\
0 & \cdots & 0 & a_J
\end{bmatrix}_{p_k}
\begin{bmatrix}
w_{g,1} \\
\vdots \\
w_{g,J}
\end{bmatrix}_{p_{k-1}}
\]

\[
\begin{bmatrix}
w_{g,1} \\
\vdots \\
w_{g,J}
\end{bmatrix}_{p_k} = M_J^{k} \begin{bmatrix}
w_{g,1} \\
\vdots \\
w_{g,J}
\end{bmatrix}_{p_0}
\]

Each \( w_{g,j}(p_k) \) is the ratio of the number of driving terms of length \( j \) for the gap \( g \), to the number of gaps 2 in the cycle of gaps \( G(p_k\#) \). In particular, \( w_{g,1}(p_k) \) is the ratio of the number of gaps \( g \) to gaps 2 at this stage of the recursion.

\( M_J \) is a banded matrix that depends on the iteration \( p_k \) but not on the gap \( g \).

\[
\begin{align*}
b_j &= \frac{j}{p-2} \\
a_j &= \frac{p-j-1}{p-2}
\end{align*}
\]
While $M_J$ is banded, $M_J^k$ becomes upper triangular.

$$M_J^k = \begin{bmatrix}
1 & \beta_{12}^{(k)} & \beta_{13}^{(k)} & \cdots & \beta_{1J}^{(k)} \\
0 & a_2^k & \beta_{23}^{(k)} & \cdots & \beta_{2J}^{(k)} \\
& \ddots & \ddots & \ddots & \vdots \\
0 & & a_{J-1}^k & \beta_{J-1,J}^{(k)} & \\
0 & & & 0 & a_J^k
\end{bmatrix}$$

with

$$\beta_{ij}^{(k)} = \begin{cases}
  a_i \cdot \beta_{ij}^{(k-1)} + b_i \cdot a_{j-1}^{k-1} & \text{if } j = i + 1 \\
  a_i \cdot \beta_{ij}^{(k-1)} + b_i \cdot \beta_{i+1,j}^{(k-1)} & \text{if } j > i + 1
\end{cases}$$

Note that the multiplication on the right-hand side does not commute, since the value of each factor depends on the respective value of the prime $p$ as indicated by its position in the product.

$M_J^k$ applies to all constellations whose driving terms have length $j \leq J$; and we continue to use the exponential notation to denote the product over the sequence of primes from $p_1$ to $p_k$: e.g.

$$M_J^k = M_J|_{p_k} \cdot M_J|_{p_{k-1}} \cdots M_J|_{p_1}.$$ 

With $M_J^k$ we can calculate the ratios $w_{g,J}(p_k)$ for the complete system of driving terms, relative to the population of the gap 2, for the cycle of gaps $\mathcal{G}(p_k^#)$ (here, $p_k$ is the $k$th prime after $p_0$). With $J = 3$ we calculated above the ratios for $g = 6, 8, 10$. For $g = 12$ we need $J = 4$, and for $g = 30$, we need $J = 8$.

Fortunately, we can completely describe the eigenstructure for $M_J|_{p}$, and even better – the eigenvectors for $M_J$ do not depend on the prime $p$. This means that we can use the eigenstructure to describe the behavior of this iterative system as $k \to \infty$.

### 4.1. Eigenstructure of $M_J$

We list the eigenvalues, the left eigenvectors and the right eigenvectors for $M_J$, writing these in the product form

$$M_J = R \cdot \Lambda \cdot L$$

with $LR = I$. 
With \(a_j\) and \(b_j\) as defined in Equation 3, for \(J = 4\) we have

\[
M_4 = \begin{bmatrix}
1 & b_1 & 0 & 0 \\
0 & a_2 & b_2 & 0 \\
0 & 0 & a_3 & b_3 \\
0 & 0 & 0 & a_4
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & -1 & 1 & -1 \\
0 & 1 & -2 & 3 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & a_2 & 0 & 0 \\
0 & 0 & a_3 & 0 \\
0 & 0 & 0 & a_4
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Note that while the eigenvalues of \(M_4\) depend on the prime \(p\) (through the \(a_j\)), the eigenvectors do not. Thus the matrix \(M_k^4\) can be written

\[
M_k^4 = RA^k L
\]

\[
= \begin{bmatrix}
1 & -1 & 1 & -1 \\
0 & 1 & -2 & 3 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & a^k_2 & 0 & 0 \\
0 & 0 & a^k_3 & 0 \\
0 & 0 & 0 & a^k_4
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

With \(J = 4\) we can calculate the ratio of the gap \(g = 12\) to the gap \(g = 2\) in the cycle of gaps. For initial conditions at \(p_0 = 13\), we have

\[
N_2(13) = 1485 \quad N_{12}(13) = 188 \quad w_{12,1}(13) = 188/1485
\]

\[
n_{12,2}(13) = 1276 \quad w_{12,2}(13) = 1276/1485
\]

\[
n_{12,3}(13) = 1314 \quad w_{12,3}(13) = 1314/1485
\]

\[
n_{12,4}(13) = 192 \quad w_{12,4}(13) = 192/1485
\]

To determine the ratios after \(k\) iterations of the recursion, we apply \(M_k^4\).

\[
M_k^4 \cdot \bar{w}|_{13} = \begin{bmatrix}
1 & -1 & 1 & -1 \\
0 & 1 & -2 & 3 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & a^k_2 & 0 & 0 \\
0 & 0 & a^k_3 & 0 \\
0 & 0 & 0 & a^k_4
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & -1 & 1 & -1 \\
0 & 1 & -2 & 3 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
2 \quad 4480/1485 a^k_2 \\
0 \quad 1890/1485 a^k_3 \\
0 \quad 192/1485 a^k_4
\end{bmatrix}
\]

Focusing just on the ratio \(w_{12,1}\) of the occurrences of gap \(g = 12\) to \(g = 2\), we see that

\[
w_{12,1}(p_k) = 2 - \frac{4480}{1485} a^k_2 + \frac{1890}{1485} a^k_3 - \frac{192}{1485} a^k_4
\]

which converges to \(w_{12,1}(p_\infty) = 2\) as rapidly as \(a^k_2 \rightarrow 0\). In Figure 3 we observe that \(a^k_2\) still has a value around 0.1 for \(p_k \sim 10^{12}\).
For the general system $M_J$, the upper triangular entries of $R$ and $L$ are binomial coefficients, with those in $R$ of alternating sign; and the eigenvalues are the $a_j$.

\[
R_{ij} = \begin{cases} 
(-1)^{i+j} \binom{j-1}{i-1} & \text{if } i \leq j \\
0 & \text{if } i > j
\end{cases}
\]

\[
\Lambda = \text{diag}(1, a_2, \ldots, a_J)
\]

\[
L_{ij} = \begin{cases} 
\binom{j-1}{i-1} & \text{if } i \leq j \\
0 & \text{if } i > j
\end{cases}
\]

For any vector $\bar{w}$, multiplication by the left eigenvectors (the rows of $L$) yields the coefficients for expressing this vector of initial conditions over the basis given by the right eigenvectors (the columns of $R$):

\[
\bar{w} = (L_1.\bar{w})R_1 + \cdots + (L_J.\bar{w})R_J
\]

Lemma 4.1. For any gap $g$ with initial ratios $\bar{w}_0$, the ratio of occurrences of this gap $g$ to occurrences of the gap $2$ in $G(p^\#)$ as $p \to \infty$ converges to the sum of the initial ratios across the gap and all its driving terms:

\[
w_{g,1}(\infty) = L_1.\bar{w}_0 = \sum_j w_{g,j}|_{p_0}.
\]

Proof. Let $g$ have driving terms up to length $J$. Then the ratios $\bar{w}|_p$ are given by the iterative linear system

\[
\bar{w}|_p = M_J^p.\bar{w}_0.
\]

From the eigenstructure of $M_J$, we have

\[
\bar{w}_0 = (L_1.\bar{w}_0)R_1 + (L_2.\bar{w}_0)R_2 + \cdots + (L_J.\bar{w}_0)R_J,
\]

and so

\[
M_J^k.\bar{w}_0 = (L_1.\bar{w}_0)R_1 + a_2^k(L_2.\bar{w}_0)R_2 + \cdots + a_J^k(L_J.\bar{w}_0)R_J.
\]

We note that $L_1 = [1 \cdots 1]$, $\lambda_1 = 1$, and $R_1 = e_1$; that the other eigenvalues $a_j^k \to 0$ with $a_j^k > a_{j+1}^k$. Thus as $k \to \infty$ the terms on the righthand side decay to 0 except for the first term, establishing the result.

With Lemma 4.1 and the initial values in $G(13^\#)$ tabulated above, we can calculate the asymptotic ratios of the occurrences of the gaps $g = 6, 8, \ldots, 32$ to the gap $g = 2$, and we provide the intermediate values at $\hat{p} = 999,999,999,989$ to give a sense of the rate of convergence.
Values of $a^k_j$ at $\hat{p} = 999,999,999,989$

| $j$ | $a^k_2$ | $a^k_3$ | $a^k_4$ | $a^k_5$ | $a^k_6$ | $a^k_7$ | $a^k_8$ | $a^k_9$ |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|
| 0   | 0.102067517997789430000 | 0.0101999689756664110000 | 0.0009959226991829460000 | 0.00094770935314020220000 | 0.0000876214163461868090000 | 0.000078408120499455720000 | 0.0000067575616112121770000 | 0.0000000557283548473588330000 |

From these values, we see the decay of the $a^k_j$ toward 0, but $a^k_2$ and $a^k_3$ are still making significant contributions when $p_k \approx 10^{12}$.

5. Observations and Conclusions

We recall that these ratios apply to the gaps in the cycle of gaps $G(p^\#)$. These ratios are representative of the gaps that will survive to become gaps between prime numbers \([3, 4]\), but they are not direct calculations of the gaps among primes.

To calculate the ratio $w_{g,1}(p_k)$, which gives the relative number of occurrences of the gap $g$ to the gap 2 at the stage of Eratosthenes sieve for $p_k$, we only need the parameters $\beta_{1j}^{(k)}$ from the top row of $M^k_J$, and the initial values $w_{g,j}(p_0)$.

$$w_{g,1}(p_k) = w_{g,1}(p_0) + \sum_{j=2}^{J} \beta_{1j}^{(k)} \cdot w_{g,j}(p_0).$$

Given the simple eigenstructure of $M^k_J$, we can compute the $\beta_{1j}^{(k)}$ from $M^k = R L^k$.

Brent \([1]\) computed the Hardy and Littlewood estimates \([2]\) for the occurrences of gaps among primes for gaps $g = 2, 4, \ldots, 80$, in the range $10^6$ to $10^9$. In the table below, we compare the actual ratios of the occurrences of the gaps from 4...32 to the occurrences of the gap 2; to the ratios in the predictions as computed by Brent; to the ratios of occurrences in the cycle of gaps $G(45053^\#)$ – we chose this prime as a representative whose square is approximately $2 \times 10^9$; to the ratios in the cycle of gaps for $\hat{p} = 999,999,999,989$; and to the asymptotic value.
Counts and ests over $[10^6, 10^9]$

| gap | actual count | actual ratio-to-2 | Brent-IL ratios | $w_{g,1}(45053)$ | $w_{g,1}(\hat{p})$ | $w_{g,1}(\infty)$ |
|-----|--------------|------------------|----------------|-----------------|-----------------|-----------------|
| 2   | 3416337      | 1.0000058        | 1.000000       | 1.000000        | 1.000000        | 1               |
| 4   | 3416536      | 1.778584         | 1.778548       | 1.773251       | 1.912023        | 2               |
| 6   | 6076242      | 0.787258         | 0.786805       | 0.781874       | 0.913321        | 1               |
| 8   | 2689540      | 0.1017958        | 0.101769       | 1.010457       | 1.203964        | 1.3333          |
| 10  | 3477688      | 1.305770         | 1.305407       | 1.305407       | 1.704932        | 2               |
| 12  | 4460952      | 0.539530         | 0.539307       | 0.530094       | 0.795251        | 1               |
| 14  | 1843216      | 0.979448         | 0.979564       | 0.959984       | 1.536000        | 2               |
| 16  | 3346123      | 0.533215         | 0.533624       | 0.519616       | 0.952118        | 1.3333          |
| 18  | 1821641      | 0.458827         | 0.458646       | 0.447082       | 0.801923        | 1.1111          |
| 20  | 1567507      | 0.692201         | 0.691456       | 0.670242       | 1.352488        | 2               |
| 22  | 2364792      | 0.327371         | 0.327304       | 0.315738       | 0.701375        | 1.0909          |
| 24  | 1118410      | 0.356525         | 0.356576       | 0.343838       | 0.769263        | 1.2             |
| 26  | 1218009      | 0.200023         | 0.199842       | 0.190052       | 0.555727        | 1               |

The values $w_{g,1}$ are the actual ratios between the numbers of these gaps at the corresponding stage of Eratosthenes sieve. So these ratios, when computed exactly, represent the exact proportions of the relative occurrences among these gaps.

If there are significant deviations from these ratios among gaps in the cycle compared to the ratios of those that survive to be gaps among primes over this range, what can we understand about the mechanism that would selectively close gaps of certain values?

This column $w_{g,1}(\hat{p})$ provides the ratios in $G(999,999,999,989\#)$, which covers the interval $\hat{p} = 999,999,999,989$ to $\hat{p}\# = 10^{434294060804}$. As the recursion continues, many closures will occur within this range. The final column $w_{g,1}(\infty)$ provides the asymptotic ratios of the occurrences of the indicated gap to the occurrences of the gap 2.

To understand the convergence to $w_{g,1}(\infty)$, from the eigenstructure of $M_k$ we can approximate $w_{g,1}(p_k)$ by truncating:

$$w_{g,1}(p_k) \approx 1 \sum_{j=1}^{J} w_{g,j}(p_0) - a_2^J \sum_{j=2}^{J} (j-1)w_{g,j}(p_0) + a_3^J \sum_{j=3}^{J} \binom{j-1}{2} w_{g,j}(p_0) \ldots$$

Note that for $G(999,999,999,989\#)$ the value of $a_2^J \approx 0.1$, so the convergence to 0 is very gradual.

This work supports the conjecture that 30 eventually is a more common gap among primes than 6. In the table above, we see that asymptotically
there are in the cycles of gaps for Eratosthenes sieve twice as many 6’s as 2’s and 2\(\frac{3}{2}\) times as many 30’s as 2’s. However, even at the prime \(\hat{p} \approx 10^{12}\), these ratios are \(w_{6,1}(\hat{p}) = 1.91202\) and \(w_{30,1}(\hat{p}) = 1.580455\). Truncating \(w_{g,1}(p_k)\) as suggested and using the initial conditions for \(g = 6\) and \(g = 30\) in \(G(13\#)\), we see that 30’s will outnumber 6’s in Eratosthenes sieve when \(a_k^2 < 0.07\).

The asymptotic ratios appear to follow the formula:

\[
    w_{g,1}(\infty) = \prod_{q\mid g, q > 2} \frac{q - 1}{q - 2}.
\]

It would be interesting to see whether this formula holds up for larger gaps, since it provides supporting evidence for Conjecture B in [2]; these ratios among gaps hold asymptotically in Eratosthenes sieve. From Lemma 4.1, this means that for a given set of prime factors (no matter what the powers on these factors), any gap with this same set of prime factors has the same total number of driving terms in any stage of Eratosthenes sieve that satisfies the conditions of Theorem 2.1.

One more observation about primorial gaps and their driving terms. Since the length of \(G(5\#)\) is 8 with sum 30, in all subsequent cycles of gaps the sum of every constellation of length 8 will be at least 30. Since \(n_{30,8}(13\#) = 90\), there are 90 complete copies of \(G(5\#)\) in \(G(13\#)\). Complete copies are only preserved for \(G(5\#), G(3\#),\) and \(G(2\#)\). These are preserved since the elementwise products in step R3 of the recursion are large enough to pass completely over one of the copies concatenated in step R2. Starting with \(G(7\#)\), the primorial 7\# is larger than any of the elementwise products, and no complete copies of these longer cycles are preserved in their entirety.

References

1. R.P. Brent, The distribution of small gaps between successive prime numbers, Math. Comp. 28 (1974), 315–324.
2. G.H. Hardy and J.E. Littlewood, Some problems in ‘partitio numerorum’ iii: On the expression of a number as a sum of primes, G.H. Hardy Collected Papers, vol. 1, Clarendon Press, 1966, pp. 561–630.
3. F.B. Holt, Expected gaps between prime numbers, arXiv 0706.0888v1, 6 June 2007.
4. F.B. Holt and H. Rudd, Estimating constellations among primes - I. uniformity, arXiv 1312.2165, 8 Dec 2013.
5. H.L. Montgomery and R.C. Vaughan, On the distribution of reduced residues, Annals of Math., 2nd series 123 (1986), no. 2, 311–333.
6. H. Riesel, Prime numbers and computer methods for factorization, 2 ed., Birkhauser, 1994.