Reverse estimation theory, Complementarity between SLD and RLD, and monotone distances

Keiji Matsumoto

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Abstract

Many problems in quantum information theory can be viewed as interconversion between resources. In this talk, we apply this viewpoint to state estimation theory, motivated by the following observations.

First, a monotone metric takes value between SLD and RLD Fisher metric. This is quite analogous to the fact that entanglement measures are sandwiched by distillable entanglement and entanglement cost. Second, SLD add RLD are mutually complement via purification of density matrices, but its operational meaning was not clear.

To find a link between these observations, we define reverse estimation problem, or simulation of quantum state family by probability distribution family, proving that RLD Fisher metric is a solution to local reverse estimation problem of quantum state family with 1-dim parameter. This result gives new proofs of some known facts and proves one new fact about monotone distances.

We also investigate information geometry of RLD, and reverse estimation theory of a multi-dimensional parameter family.

1 Introduction

Many problems in quantum/classical information theory can be viewed as interconversion between a given resource and a 'standard' resource, and such viewpoint had turned out to be very fruitful. This manuscript will exploit this scenario in asymptotic theory of quantum estimation theory (with some comments on classical estimation theory).

Resource conversion scenario was first explored in axiomatic theory of entanglement measures. Entanglement is a kind of quantum non-locality, which cannot be explained by classical mechanical theory. Its effect is typically visible in so called maximally entangled states, which we regards as standard resources. The optimal asymptotic conversion ratio from maximally entangled state to a given state is called entanglement cost, while the optimal ratio for inverse conversion is called distillable entanglement. It is shown that all quantities which satisfies a set of reasonable axioms takes value between these two quantities.

It had been known that a monotone metric in quantum state space takes value between RLD and SLD Fisher metric. It had been also pointed out that these metrics are mutually 'complement', in the following sense: A mixed quantum state can be regarded as a reduced state of pure state in larger system. In this view, SLD Fisher metric of a quantum state space equals RLD Fisher metric in the space of quantum states in ancilla system.

In the manuscript, we link these two facts via resource conversion scenario, giving new proof of the former, and monotonicity of RLD and SLD Fisher metrics. We also prove similar statement for quantum version of relative entropy, which, to author's knowledge, is new. In the discussion, estimation corresponds to distillation of standard resource.

In above discussion, we need 'reverse estimation', which we formulate as reverse estimation of quantum state families: Given a family of quantum state, its reverse estimation is a CQ map and a family of probability distributions such that the output of the CQ map equals the quantum state family (In fact, we mainly consider local version of this, to make a complement to local estimation theory, which is equivalent to asymptotic estimation theory).

Next, we study local reverse estimation itself more in detail. Especially, we point out that, in general, local 'reverse estimation' is mathematically equivalent to local estimation with fixed set of observables. Straightforward calculation shows that optimal reverse estimation corresponds to P-representation. For the Q-representation corresponds to optimal estimation, gap between reverse estimation and estimation origins from uncertainty contained in coherent states.
2 SLD and RLD, Fisher information

In the manuscript, we restrict ourselves to finite (namely, d×d) dimensional Hilbert space ℋ, unless otherwise mentioned. The totality of density matrices is denoted by S(ℋ), and the totality rank r elements is denoted by S_r(ℋ). In the manuscript, r = d, unless otherwise mentioned. Unless otherwise mentioned, a parameterized family of quantum states, often denoted by ℳ = {ρ_θ; θ ∈ Θ ⊂ ℜ^m}, is assumed to be differentiable up to an arbitrary order.

Define a symmetric logarithmic derivative L^S_θ, and a right logarithmic derivative L^R_θ, as a solution to the matrix equation,

$$\partial_i \rho_θ = \frac{1}{2}(L^S_{θ,i} \rho_θ + \rho_θ L^S_{θ,i}) = L^R_{θ,i} \rho_θ,$$

where ∂_i := (∂/∂θ_i). If ρ_θ is strictly positive, L^S_θ exists and L^R_θ are uniquely defined in this way. If ρ_θ has zero eigenvalues, L^S_θ can still be defined, but not uniquely. L^R_θ exists (if and exists, unique) if and only if ∂_i ρ_θ has non-zero eigenvalues only in the support of ρ_θ.

Observe they are quantum equivalences of a classical logarithmic derivative, ∂_i log p_θ(x).

An SLD Fisher information matrix J^S_θ and RLD Fisher information matrix J^R_θ are defined as

$$J^S_{θ,i,j} = \text{Tr} ρ_θ L^S_{θ,i} L^S_{θ,j},$$
$$J^R_{θ,i,j} = \text{Tr} ρ_θ L^R_{θ,i} L^R_{θ,j}.$$  

They are quantum analogs of a classical Fisher information matrix,

$$J_{θ,i,j} := \sum_x p_θ(x) ∂_i log p_θ(x) ∂_j log p_θ(x).$$

For they are positive definite, they can be regarded as metric tensors introduced to the tangent space T_θ(ℳ), and the corresponding metrics are called SLD Fisher metric and RLD Fisher metric, respectively.

3 Duality between SLD and RLD, Reverse SLD

Denote by ℳ the totality of matrices with finite numbers of rows and columns. In the manuscript, an element of ℳ is considered as an ordered set of orthonormal state vectors which forms a convex decomposition of a mixed state, with their magnitudes corresponding to weights. Equivalently, an element of ℳ can be interpreted as a representation of a bipartite pure state, whose reduced density matrix to one of the parties equals a given density matrix. The totality of d×d elements of ℳ is denoted by ℳ_d.

We consider a map π form ℳ to S(ℋ),

$$π : ℳ → ℳ_d.$$  

An interpretation of this map is as follows. Let

$$ℳ = \{\sqrt{p_1} | φ_1⟩, \cdots, \sqrt{p_d} | φ_d⟩\},$$

then,

$$π(ℳ) = \sum_{i=1}^d p_i | φ_i⟩⟨φ_i|.$$  

Another interpretation would be given by taking correspondence,

$$| Φ_W⟩ = \sum_{i,j} w_{i,j} | e_i⟩ | f_j⟩,$$

where | e_i⟩ is an orthonormal basis in ℋ, and | f_j⟩ is an orthonormal basis in a Hilbert space ℋ’ for purification. Then,

$$π(ℳ) = \text{Tr}_{ℋ'} | Φ_W⟩⟨Φ_W|.$$  

Its differential map is denoted by π*: T(ℳ) → T(S(ℋ)), where T_W(ℳ) means a tangent space to ℳ at W, and T(ℳ) is a tangent bundle, or the union of T_W(ℳ), with ℳ’s running all over ℳ. An element of T(ℳ) is naturally represented by an element of ℳ by considering a parameterized family of an elements ℳ and differentiating with respect to a parameter. Denote such representation of an element X of T_W(ℳ) by M_X, or more explicitly,

$$M \frac{∂}{∂ζ} W \bigg|_W := 2 \frac{∂W_ζ}{∂ζ} W \bigg|_W.$$  

In that representation,

$$π_*(X) = \frac{1}{2} \{ W \left(M_X^T + (M_X)^T\right) W \},$$

which is easily understood recalling Leibnitz’s rule of differentiation of a product of two matrices.

Observe that these maps are not unique. First, the map π satisfies

$$π(WU) = π(W)$$  

where U is a matrix with UU^† = I (need not to be a unitary). Sometimes, this transform is refered to as a gauge transform. Correspondingly, the kernel of π*, denoted by K_W(ℳ), is

$$K_W(ℳ) = \{ X ; M_X = W A^K, \exists A^K = -A^{K†} \}.$$  

Denote an element of T_p(S(ℋ)) by X, and denote an SLD and RLD corresponding to X by L^S_p,X and L^R_p,X, respectively. We define two inverse maps of
π∗, which are denoted by \( h^S_W \) and \( h^R_W \) (subscript \( W \) is often dropped) as,
\[
Mh^S_W(X) = L^S_{\rho,X}W, \\
Mh^R_W(X) = L^R_{\rho,X}W,
\]
with \( \rho = \pi(W) \). It is easy to verify \( \pi∗ \circ h^S_W = \pi∗ \circ h^R_W = \text{id} \). Consider subspaces \( \mathcal{LS}_W(W) \) and \( \mathcal{LR}_W(W) \) of \( T_W(W) \) which are defined by,
\[
\mathcal{LS}_W(W) = \left\{ \tilde{X} : \tilde{X} = h^S_W(X), \exists X \in T_{\pi(\rho)}(S(H)) \right\}, \\
\mathcal{LR}_W(W) = \left\{ \tilde{X} : \tilde{X} = h^R_W(X), \exists X \in T_{\pi(\rho)}(S(H)) \right\}.
\]
It is easy to see
\[
L^R_{\rho,X}W = WA^R, \exists A^R = A^{Ri}.
\]
\( A^R \) is said to be the reverse SLD at \( W \).

Define a map \( \tilde{\pi} \) from \( W \) to \( S(H^\dagger) \) such that,
\[
\tilde{\pi}(W) = W^\dagger W - \pi(W) = \text{Tr}_H[\Phi_W](\Phi_W).
\]
Correspondingly, we can define \( \tilde{h}^S_W(X), \tilde{h}^R_W(X) \), \( \mathcal{LS}_W(W), \) and \( \mathcal{LR}_W(W) \), for which
\[
\mathcal{LS}_W(W) \supset \mathcal{LR}_W(W), \\
\mathcal{LR}_W(W) \supset \mathcal{LS}_W(W)
\]
holds. Especially, if \( d′ = r \), the LHS and the RHS coincide with each other. This means that RLD of the system corresponds to SLD of the ancilla system. Also, we have,
\[
\text{Tr}_pL^{Ri}L^R \leq \text{Tr}_p(\tilde{\pi}(W))A^R A^{Ri}.
\]
Especially, if \( d′ = r \), the equality holds. These relations are called duality between SLD and RLD.

### 4 Reverse estimation of quantum state family and RLD

The heart of quantum statistics is optimization of a measurement, i.e., choice of a measurement which converts a family of quantum states to the most informative classical probability distribution family. Let us denote by \( p^M_0 \) the probability distribution of measurement results of applied to \( \rho_0 \), and denote by \( J^M_0 \) the classical Fisher information matrix of the probability distribution family \( \{p^M_0 \} \). Then, it is known that, for a 1-dim quantum state family \( \mathcal{M} \),
\[
\max_{\mathcal{M}: \text{meas}} J^M_0 = J^S_\theta,
\]
or, \( J^S_\theta \) is the maximal amount of classical Fisher information extracted from the 1-dim quantum state family \( \{\rho_0 \} \) at \( \theta \). In other words, we consider a QC map which maximizes the output Fisher information.

Now, we consider the reverse of above, i.e., estimation of the 1-dim quantum state family \( \{\rho_0 \} \) at \( \theta_0 \) up to the first order, i.e., a pair \( (\Phi, \{\rho_0 \}) \) of the probability distribution family \( \{\rho_0 \} \) such that with a QC channel \( \Phi \), such that,
\[
\Phi(\rho_0) = \rho_0, \quad (1)
\]
\[
\left. \frac{d\Phi(\rho_0)}{d\theta} \right|_{\theta = \theta_0} = \left. \frac{d\rho_0}{d\theta} \right|_{\theta = \theta_0}.
\]
Our task is to optimize a pair \( (\Phi, \{\rho_0 \}) \), called local reverse estimation at \( \theta_0 \), to minimize Fisher information \( J_\theta \) of the input \( \{\rho_0 \} \).

A local reverse estimation of \( \{\rho_0 \} \) at \( \theta_0 \) is constructed as follows. Define a system of state vectors \( |\phi_1 \rangle, \cdots, |\phi_{d′} \rangle \), and a probability distribution \( \{p(i) \} \) by the equations,
\[
\rho_{\theta_0} = \sum_{x=1}^{d′} p(x) |\phi_x \rangle \langle \phi_x |.
\]
This corresponds to a QC map \( \Phi \) which outputs \( |\phi_x \rangle \) according to the input probability probability distribution \( p(x) \). Define real numbers \( \lambda_1, \cdots, \lambda_{d′} \) by
\[
\left. \frac{d\rho_0}{d\theta} \right|_{\theta = \theta_0} = \sum_{x=1}^{d′} \lambda_x p(x) |\phi_x \rangle \langle \phi_x |.
\]
and define \( \{p_0 \} \) by \( p_0(x) := p(x) + \lambda_x p(x)(\theta - \theta_0) \). Then, the pair \( (\Phi, \{p_0 \}) \) is a local reverse estimation, and any local reverse estimation is given in this way, essentially (i.e., modulo the difference of \( o(\theta - \theta_0) \)).

Define also
\[
W = [\sqrt{p(1)} |\phi_1 \rangle, \cdots, \sqrt{p(1)} |\phi_{d′} \rangle], \\
A = \text{diag}(\lambda_1, \cdots, \lambda_{d′}).
\]
Then, we have
\[
\left. \frac{d\rho_0}{d\theta} \right|_{\theta = \theta_0} = WA W^\dagger, \\
L^R_{\theta_0,1} W P = WA,
\]
with \( P \) being the projector onto the support of \( \tilde{\pi}(W) \).

The logarithmic derivative of \( \{p_0 \} \) at \( \theta = \theta_0 \) is
\[
\left. \frac{d\log p_0(x)}{d\theta} \right|_{\theta = \theta_0} = \lambda_x,
\]
and its Fisher information is,
\[
J_\theta = \sum_{x=1}^{d′} (\lambda_x)^2 p(x) = \text{Tr}_W WA W^\dagger \\
\geq \text{Tr}_W P A W^\dagger = J^R_{\theta_0}.
\]
The equality holds if \( P \) equals the identity, or \( d′ = r \). Hence, to simulate \( \{\rho_0 \} \) at the neighbor of \( \theta_0 \) up to the first order, we need classical Fisher information by the amount of \( J^R_{\theta_0} \).
Theorem 1

$$\max J_{\theta_0} = J^R_{\theta_0},$$

where maximization is taken over all the local reverse estimations of \( \{ \rho_0 \} \) at \( \theta_0 \).

5 Monotone metric revisited

It is known that SLD Fisher metric and RLD Fisher metric are monotone by application of CPT map, and any monotone metric takes value between SLD and RLD Fisher metric. In this section, we demonstrate operational meaning of SLD and RLD implies these properties in trivial manner.

First, monotonicity of SLD is trivial because the optimization of measurement applied to the family \( \{ \Lambda \rho_0 \} \) is equivalent to the optimization of measurement to \( \{ \rho_0 \} \) over all the restricted class of measurement of the form \( M \circ \Lambda \).

The monotonicity of RLD Fisher metric is proven in the similar manner. Given a local reverse estimation \( (\Phi, \{ \rho_0 \}) \) of \( \{ \rho_0 \} \) at \( \theta_0 \), \( (\Lambda \circ \Phi, \{ \rho_0 \}) \) is a local reverse estimation of the family \( \{ \Lambda \rho_0 \} \) at \( \theta_0 \). We may be able to improve this reverse estimation to reduce the amount of classical Fisher information of the probability distribution family. Thus the monotonicity of RLD Fisher metric is proved.

Also, we can prove that SLD Fisher metric is no larger than RLD by considering composition of the optimal local reverse estimation followed by the optimal measurement. This operation, being a CPT map, cannot increase classical Fisher information. For the initial classical Fisher information equals RLD Fisher information and the final one equals SLD Fisher information, we have the inequality.

Assume that a metric is not increasing by a QC channel, and coincides with classical Fisher information restricted to classical probability distributions. Then, this metric should be no smaller than SLD Fisher metric. Let us consider a 1-dim family \( \{ \rho_0 \} \). If one apply an optimal QC map, classical Fisher information \( J_\theta \) of the output probability distribution family \( \{ \rho_0 \} \) equals \( J^S_{\rho_0} \). Due to the latter assumption, \( g_{\rho_0} = J^S_{\rho_0} \). Therefore, the monotonicity by a QC channel \( g_{\rho_0} \geq g_{\rho_0} = J^S_{\rho_0} \).

Similarly, assume that a metric is not increasing by a CQ channel and coincides with classical Fisher information restricted to classical probability distributions. Then, the metric should be no larger than RLD Fisher metric. Consider an optimal local reverse estimation of the 1-dim family \( \{ \rho_0 \} \) at \( \theta \). Then, classical Fisher information \( J_\theta \) of the input probability distribution family \( \{ \rho_0 \} \) equals \( J^R_{\rho_0} \). Due to the latter assumption, \( g_{\rho_0} = J^R_{\rho_0} \). Therefore, the monotonicity by a CQ channel \( g_{\rho_0} \leq g_{\rho_0} = J^R_{\rho_0} \).

Altogether, if a metric is monotone non-increasing by application of QC and CQ maps, the metric takes value between SLD and RLD Fisher metric.

Theorem 2 Assume that a metric \( g \) coincide with classical Fisher information in the space of classical probability distributions. In addition, if \( g \) is monotone decreasing by a QC map, then \( g \) is larger than SLD Fisher metric. If \( g \) is monotone decreasing by a CQ map, then \( g \) is smaller than RLD Fisher metric.

6 Global reverse estimation

Let us define a global reverse estimation of a quantum state family \( \{ \rho_0 \} \) for a likelihood family \( \{ \rho_0 \} \) such that with a QC channel \( \Phi \) such that,

$$\Phi(p_0) = \rho_0, \forall \theta \in \Theta.$$  

This is equivalent to

$$\rho_\theta = \sum_i p_\theta(i) |\phi_i\rangle \langle \phi_i|,$$

$$= W_0 M_\theta W_0^\dagger,$$

where \( M_\theta = \text{diag}(p_\theta(x), \cdots, p_\theta(x)) \). Let

$$A^R_{\theta,i} = \text{diag}(\partial_i \log p_\theta(x), \cdots, \partial_r \log p_\theta(x)),$$

$$W_\theta = W_0 \sqrt{M_\theta}.$$

For 1-dim restriction of achieves RLD Fisher information, we have to have,

$$L^R_{\theta,i} W_\theta = W_\theta A^R_{\theta,i}.$$

Theorem 3 If \( \rho_0 \) is a full-rank matrix for all \( \theta \in \Theta \), the following three are equivalent.

(i) The state family \( \{ \rho_0 \} \) has a global reverse estimation such that its 1-dim restriction achieves RLD Fisher information at all \( \theta \in \Theta \).

(ii) \( \rho_\theta = W_0 M_\theta W_0^\dagger \), where \( M_\theta \) is a \( r \times r \) Hermitian matrix, and \( |M_{\theta_1}, M_{\theta_2}| = 0 \) for all \( \theta_1, \theta_2 \).

(iii) \( [L^R_{\theta,i}, L^R_{\theta',j}] = 0 \) for all \( i, j, \theta, \theta' \).

7 Two point reverse estimation

Now, we turn to reverse estimation of two quantum states, \( \rho, \sigma \), which is a pair \( (\Phi, \{ p_\lambda; \lambda = \rho, \sigma \}) \) of a CQ map and a probability distribution family such that \( \Phi(p_\rho) = \rho \) and \( \Phi(p_\sigma) = \sigma \). The problem discussed here is the minimization of the divergence between the probability distributions between \( p_\rho \) and \( p_\sigma \).
It is known that the divergence equals a integral of metric along a curve, \( \{ p_i^{(m)} = tp_\rho + (1-t)p_\sigma \} \),

\[
D(p_\rho||p_\sigma) = \int_0^1 \int_0^t J_i^{(m)} ds dt,
\]

where \( J_i^{(m)} \) is a Fisher information of the family \( \{ p_i^{(m)} \} \). This quantity is upper-bounded by

\[
D_R(\rho||\sigma) := \int_0^1 \int_0^t J_i^{R} ds dt,
\]

where \( J_i^{R} \) the RLD Fisher information of the family of quantum states \( \{ \Phi(p_i^{(m)}) \} \). Observe that, for any reverse estimation, we have

\[
\Phi(p_i^{(m)}) = \rho_i^{(m)} := t\rho + (1-t)\sigma.
\]

Hence, \( D(p_\rho||p_\sigma) \) is maximized if \( J_i^{(m)} \) is maximized at each \( t \). If \( J_i^{R} = J_i^{(m)} \) for all \( t \), \( 0 \leq t \leq 1 \), i.e., the reverse estimation is an optimal local reverse estimation at all \( t \), i.e., a minimal reverse estimation, the reverse estimation should be optimal. In the proof, a key point was that image of m-affine curve is also m-affine.

The integration (3) is computed by Hayashi:

\[
D_R(\rho||\sigma) = \text{Tr} \rho \log \rho^{2} - \rho^{2}.
\]

8 Monotone Divergence

Let \( D^Q(\rho||\sigma) \) be a quantity which coincides with classical divergence in the space of probability distributions, non-increasing by application of a CPT map, and is additive,

\[
D^Q(\rho_1 \otimes \rho_2||\sigma_1 \otimes \sigma_2) = D^Q(\rho_1||\sigma_1) + D^Q(\rho_2||\sigma_2).
\]

Then, in the almost the same way as monotone metric, we can conclude such quantity is upper-bounded by \( D^R(\rho||\sigma) \), and lower-bounded by

\[
D(\rho||\sigma) := -\text{Tr} \rho (\log \rho - \log \sigma).
\]

Assume that \( D^Q(\rho||\sigma) \) is monotone by a QC map, coincide with the classical divergence for the probability distributions, and is additive. It is known that there is a QC map such that the output probability distributions \( p_{\rho_\otimes}^{M_\alpha} \) and \( p_{\sigma_\otimes}^{M_\alpha} \) satisfies, \( D(\rho||\sigma) = -\frac{1}{n} D(\rho_{\rho_\otimes}^{M_\alpha}||\sigma_{\sigma_\otimes}^{M_\alpha}) + o(1) \). This implies

\[
D(\rho||\sigma) = \frac{1}{n} D^Q(\rho_{\rho_\otimes}^{M_\alpha}||\sigma_{\sigma_\otimes}^{M_\alpha}) + o(1)
\]

Here, tending \( n \to \infty \), we have \( D(\rho||\sigma) \leq D^Q(\rho||\sigma) \) (This part is done by Hayashi).

On the other hand, assume that \( D^Q(\rho||\sigma) \) is non-increasing by a CQ map, coincide with the classical divergence for the probability distributions. Then, letting \( \{ \Phi, \{ p_\lambda \} \} \) be an optimal reverse estimation,

\[
D_R(\rho||\sigma) = D(p_\rho||p_\sigma) = D^Q(p_\rho||p_\sigma)
\]

\[ \geq D^Q(\rho||\sigma). \]

Theorem 4 Assume that \( D^Q(\rho||\sigma) \) coincides with classical divergence for the probability distributions. In addition, if \( D^Q(\rho||\sigma) \) is additive and non-increasing by a QC map,

\[
D(\rho||\sigma) \leq D^Q(\rho||\sigma).
\]

On the other hand, if \( D^Q(\rho||\sigma) \) is non-increasing by a CQ map,

\[
D(\rho||\sigma) \leq D^R(\rho||\sigma).
\]

Can additivity assumption decrease the upper bound to the monotone divergences ? This cannot be true, for \( D^R(\rho||\sigma) \) is additive. On the other hand, if we remove the additivity assumption, the lower-bound can be increased.

9 Local reverse estimation of a multi-dimensional family

A local reverse estimation can be recast as follows. Under the constraint of

\[
A_i = U \tilde{A}_i U^†, \quad [ \tilde{A}_i^R, \tilde{A}_j^R ] = 0 \quad (i, j = 1, \cdots, m),
\]

we minimize

\[
\sum_{i,j} G_{ij} \text{Tr} \sqrt{U} \tilde{A}_i \tilde{A}_j U^† \sqrt{\rho}
\]

\[ = \sum_{i,j} G_{ij} \text{Tr} U^† \rho U \tilde{A}_i \tilde{A}_j U^†, \]

(5)

where \( U \) is an isometry from \( \mathcal{H}' \) to \( \mathcal{H} \), with \( \dim \mathcal{H} \geq \dim \mathcal{H}' \).

Here, note the analogy of this with the local state estimation, which gives same result as the first order asymptotic theory. Assume we measure set of observables \( X^i \) to estimate \( \theta^i \), i.e., \( X^i = \sum_k \tilde{\theta}^k \hat{M}_k \), where \( \{ \tilde{\theta}^k \} \) is an estimate of \( \theta^i \), and \( \{ M_k \} \) is a POVM for a measurement used for the estimation. Then, due to Naimark extension, we can find a set of observables \( \tilde{X}^i \) \( (i = 1, 2, \cdots, m) \) with

\[
X^i = U \tilde{X}^i U^†, \quad [ \tilde{X}^i, \tilde{X}^j ] = 0 \quad (i, j = 1, \cdots, m),
\]
and \[ \text{Tr} U^\dagger \rho U \bar{X}_i \bar{X}_j = \sum \frac{\partial_i \bar{X}_j}{\partial_i} \text{Tr} \rho M_\kappa. \]

Hence, establishing correspondence between \( X_i \) and \( A^R_\kappa \), our target function \( G \) corresponds to the weighted sum of the 'mean squared error' with the fixed set of observables. In other words, the problem is reduced to optimization of measurement in quantum estimation with the constraint \( X_i = \sum \bar{X}_j \bar{X}_j \).

In particular, consider asymptotic exact reverse estimation with corrective operation, i.e., the minimization of

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i,j} G_{ij} \text{Tr} U^\dagger \rho \otimes n U \bar{A}_i^R \bar{A}_j^R, \]

with the constraint

\[ \Phi (\rho^n_0) = \rho^n_0, \quad \Phi (\partial_i \rho^n_0) = \partial_i \rho^n_0. \]

Define

\[ X_i^n = \frac{1}{n} (\rho^n_0)^{-\frac{1}{2}} \partial_i (\rho^n_0)^{-\frac{1}{2}}, \]

\[ = \frac{1}{n} \sum_{k=1}^n I \otimes \cdots \otimes A^R_i \otimes I \cdots \otimes I. \]

Then, our target function is

\[ \min \liminf_{n \to \infty} \frac{1}{n} \sum_{i,j} G_{ij} \text{Tr} U^\dagger \rho \otimes n U \bar{X}_i^n \bar{X}_j^n, \]

where \( U \) runs over all isometry such that

\[ X_i^n = U \bar{X}_i^n U^\dagger, \quad [\bar{X}_i^n, \bar{X}_j^n] = 0, \quad (i, j = 1, \ldots, m). \]

This corresponds to the asymptotic lower bound to the weighted sum of mean square error of corrective measurements. Hence, the minimum is given using so-called Holevo bound. For we have

\[ \text{Tr} \rho A_i^R A_j^R = J_{i,j}^R, \]

due to Holevo bound, we have

\[ \min \liminf_{n \to \infty} \frac{1}{n} \sum_{i,j} G_{ij} \text{Tr} U^\dagger \rho \otimes n U \bar{X}_i^n \bar{X}_j^n \]

\[ = \text{Sp} \mathbb{R} J^R_\theta + \text{Spabs} G \mathbb{J}^R_\theta, \]

\[ = \min \{ \text{Sp} G J ; J \geq J^R_\theta \}. \quad (6) \]

Note

\[ \Im J^R_\theta = -\frac{1}{2} \text{Tr} \rho \left[ L_i^R, L_j^R \right], \]

and this quantity is a measure of non-commutativity of RLD's. If this quantity is larger, we need more classical Fisher information than the real part of RLD.

On the other hand, if the given state family is \( D \)-invariant in Holevo’s sense, the bound corresponding to the estimation is given,

\[ \max \{ \text{Sp} G J ; J \leq J^R_\theta \} = \text{Sp} \mathbb{R} J^R_\theta - \text{Spabs} G \mathbb{J}^R_\theta, \]

and the bound is achievable. This is smaller than the reverse estimation bound by \( \text{Spabs} G \mathbb{J}^R_\theta \).

**Example 5 (Gaussian state family)** A Gaussian state family is defined by

\[ \rho = \int \frac{dp dq}{2 \pi \sigma^2} e^{-\frac{(q-\theta)^2 + (p-\bar{\theta})^2}{2 \sigma^2}} | p, q \rangle \langle p, q | \]

This definition itself gives a global reverse estimation such that the coherent state \( | p, q \rangle \) is according to the Gaussian distribution with the variance \( \sigma^2 \) and the mean \( \theta = (\theta_1, \theta_2) \). Its input Fisher information is \( J = \sigma^{-2} I \), and

\[ \text{Sp} J = 2 \sigma^{-2}. \]

This in fact is optimal:

\[ \text{Sp} \mathbb{R} J^R_\theta + \text{Spabs} \mathbb{J}^R_\theta = 2 \sigma^{-2}, \]

where

\[ J^R = \frac{1}{(\sigma^2 + \hbar)^2} \left[ \sigma^2 + \frac{\hbar^2}{2}, -i \hbar/2, \sigma^2 + \frac{\hbar}{2} \right]. \]

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