A FAMILY OF FUNCTIONS ASSOCIATED WITH THREE TERM RELATIONS AND EISENSTEIN SERIES

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Abstract. In this paper, for $a \in \mathbb{C}$, we investigate functions $g_a$ and $\psi_a$ associated with three term relations. $g_a$ is defined by means of function $\psi_a$. By using these functions, we obtain some functional equations related to the Eisenstein series and the Riemann zeta function. Also we find a generalized difference formula of function $g_a$.

1. Introduction

Recently, many authors has studied on period functions and three term relations. The period functions are real analytic functions $\psi(x)$ which satisfy three term relations, for $t \in \mathbb{R}$,

$$\psi(x) = \psi(x + 1) + \frac{1}{(x + 1)^2} \psi\left(\frac{x}{x + 1}\right),$$

where $s = \frac{1}{2} + it$ (cf. [4], [9]).

Let $\mathbb{H} = \{z \in \mathbb{C}: \text{Im}(z) > 0\}$. The period function is also associated with a periodic and holomorphic function $f$ defined by

$$f(z) = \psi(z) + \frac{1}{z^{2s}} \psi\left(-\frac{1}{z}\right),$$

where $z \in \mathbb{H}$.

In [4], Bettin and Conrey studied on the case of real analytic Eisenstein series. For these, the periodic function $f$ turns out to be essentially

$$\sum_{n=1}^{\infty} \sigma_{2s-1}(n)e^{2\pi inz},$$

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1671
where, for \( n \in \mathbb{N} \) and \( a \in \mathbb{C} \),

\[
\sigma_a(n) = \sum_{d \mid n} d^a.
\]

A Maass wave form on the full modular group \( \Gamma = \text{PSL}(2, \mathbb{Z}) \) is a smooth \( \Gamma \)-invariant function \( u \) from the upper half plane \( \mathbb{H} \) to \( \mathbb{C} \) which is small as \( y \to \infty \) and satisfies \( \Delta u = \lambda u \) for some \( \lambda \in \mathbb{C} \), where

\[
\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
\]

is the hyperbolic Laplacian (cf. [8], [9]).

Maass forms have many applications in a number of areas of mathematics such as number theory, dynamical systems and quantum chaos (cf. [9]).

In [7], Lewis showed that there exists a one-to-one correspondence between the space of even Maass wave forms with eigenvalue \( \lambda = s(1-s) \) and the space of holomorphic functions on \( \mathbb{C}' = \mathbb{C} \setminus (-\infty, 0] \) satisfying three term relation together with a suitable growth condition.

In [9], Lewis and Zagier investigated the properties of general solutions of the three term relations, which is called periodlike functions. Also they were interested both in describing the totality of periodlike functions and in determining sufficient conditions for such a function to be the period function of a Maass form.

Let \( u(z) \) be a Maass wave form with spectral parameter \( s \). Then Lewis and Zagier defined the associated period function \( \psi \) in the upper and lower half-planes by the formula

\[
\psi(z) \doteq \pm \sum_{n=1}^{\infty} n^{(s-1)/2} A_{\pm n} \left( e^{\pm 2\pi i nz} \pm \frac{z^{2s} e^{\pm 2\pi i n/z}}{z} \right)
\]

(Here, the symbol \( \doteq \) denotes equality up to a factor depending only on \( s \)). On the other hand, the original definition of the period function as given (in the even case) in [7] was represented by an integral transform; namely

\[
\psi_1(z) \doteq \int_0^{\infty} z^{s} (z^2 + t^2)^{-s-1} u(it) dt , \quad (\text{Re}(z) > 0)
\]

where we have written “\( \psi_1 \)” instead of “\( \psi \)” to avoid ambiguity.

In [5], Bruggeman gave a cohomological interpretation of theory of period functions and therefore the theory in the Maass context was developed.

In this paper, we focus on the results of Bettin and Conrey in [4]. By using these results, for \( a \in \mathbb{C} \), we obtain some functional equations related to function \( g_a \) where \( g_a \) is associated with period function \( \psi_a \). In final section, we give generalized difference formulas related to \( g_a \).

In [4], for \( a \in \mathbb{C} \), Bettin and Conrey gave a relation between extended Eisenstein series \( E_{a+1} \) and period function \( \psi_a \). Therefore they obtained the
function $g_a$ associated with $\psi_a$. Also, for $\text{Re}(\tau) > 0$ and $|z| < \tau$, they gave the Taylor series of $g_a(z)$ around $\tau$.

For $a > 2$, the Eisenstein series are defined by (cf. [2], [10], [11])

$$G_a(z) = \sum_{m,n=-\infty \atop (m,n)\neq 0}^{\infty} \frac{1}{(mz+n)^a}$$

and also the Eisenstein series has a Fourier expansion, for $k \geq 2$ and $a = 2k$, given by (cf. [2])

$$G_{2k}(z) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)e^{2\pi inz}.$$ 

In this expansion, for $\sigma > 1$ and $s = \sigma + it$, we define Riemann zeta function (cf. [3], [12])

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$ 

For $s = 1$, the Riemann zeta function is the harmonic series which diverges to $\infty$ and satisfies the following properties

$$\zeta(s) = 2^s\pi^{s-1} \sin \left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

and

$$\zeta(-n) = -\frac{B_{n+1}}{n+1},$$

where Bernoulli numbers and gamma function are denoted by $B_n$ and $\Gamma(s)$, respectively (cf. [1], [12]).

The Laurent series of $\zeta(s)$ in a neighborhood of its pole $s = 1$ has the form:

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} \gamma_n(s-1)^n,$$

where $\gamma$ is the Euler-Mascheroni constant and $\gamma_n$ is also expressed in (cf. [6]):

$$\gamma_n = \lim_{m \to \infty} \left\{ \sum_{k=1}^{m} \frac{(\log k)^n}{k} - \frac{(\log m)^{n+1}}{n+1} \right\}.$$ 

In special case of $n = 0$, we have $\gamma_0 = \gamma$.

2. Some functional equations related to function $g_a$

Let $\text{Im} z > 0$. We consider the function $E_{a+1}$ defined by

$$E_{a+1}(z) = 1 + \frac{2}{\zeta(-a)} S_a(z),$$
where
\[ S_a(z) = \sum_{n=1}^{\infty} \sigma_a(n)e^{2\pi inz}. \]

For \( 2 \leq k \in \mathbb{N} \) and \( a = 2k - 1 \), we arrive at the following well known property:
\[ E_{2k}(z) - \frac{1}{z^{2k}} E_{2k}\left(-\frac{1}{z}\right) = 0. \]

If we extend \( 2k \) to any complex number \( a + 1 \), the equation (2.1) is no longer true. However, Bettin and Conrey investigate the properties of the function
\[ \psi_a(z) = E_{a+1}(z) - \frac{1}{z^{a+1}} E_{a+1}\left(-\frac{1}{z}\right). \]

In this section, we obtain some functional equations related to \( g_a \) by using the following theorem:

**Theorem 2.1** (cf. [4]). Let \( \text{Im} z > 0 \) and \( a \in \mathbb{C} \). Then \( \psi_a \) satisfies the three term relation
\[ \psi_a(z + 1) - \psi_a(z) + \frac{1}{(z + 1)^{1+a}} \psi_a\left(\frac{z}{z + 1}\right) = 0 \]
and extends to an analytic function on \( \mathbb{C}' = \mathbb{C} \setminus (-\infty, 0] \) via the representation
\[ \psi_a(z) = \frac{i}{\pi z} \left( \frac{\zeta(1-a)}{\zeta(-a)} - \frac{1}{z^{1+a}} \cot \frac{\pi a}{2} \right) + \frac{g_a(z)}{\zeta(-a)}, \]
where
\[ g_a(z) = -2 \sum_{1 \leq n \leq M} (-1)^n B_{2n} \frac{B_{2n}}{(2n)!} \zeta(1-2n-a)(2\pi z)^{2n-1} \]
\[ + \frac{1}{\pi i} \int_{(-\frac{1}{2}) - 2M} \zeta(s)\zeta(s-a)\Gamma(s) \frac{\cos \frac{\pi s}{2}}{\sin \frac{\pi s}{2}} (2\pi z)^{-s} ds \]
and \( M \) is any integer greater or equal to \( -\frac{1}{2} \min(0, \text{Re} a) \).

For \( a \to 0^+ \), we get
\[ \psi_0(z) = -2 \frac{(-\gamma + \log 2\pi z)}{\pi iz} - 2ig_0(z), \]
where
\[ g_0(z) = \frac{1}{\pi i} \int_{(-1/2)} \zeta(s)\zeta(1-s) \frac{\cos \frac{\pi s}{2}}{\sin \frac{\pi s}{2}} z^{-s} ds. \]

In Theorem 2.1, Bettin and Conrey also showed that there exists a function \( \psi_a \neq 0 \) which satisfies the three term relation. Three term relation is a special case of the following functional equation:
\[ \phi_a(z + 1) - \phi_a(z) + \frac{1}{(z + 1)^{1+a}} \phi_a\left(\frac{z}{z + 1}\right) = f_a(z). \]
In the following theorem, we show the existence of functions \( f \neq 0 \) and \( \phi \neq 0 \) such that the equation (2.5) holds.

Theorem 2.2. Let \( \text{Im} z > 0 \) and \( a \in \mathbb{C} \). Then, we have

\[
\frac{1}{(z + 1)^{1+a}} g_a \left( \frac{z}{z + 1} \right) + g_a(z + 1) - g_a(z) = \frac{\zeta(1-a)}{\pi z (z + 1)} + \frac{\zeta(-a) \cot \frac{\pi a}{2}}{(z + 1)^{1+a}} - \frac{\zeta(1-a)}{\pi z (z + 1)^a}.
\]

Proof. From equation (2.3), we know that

\[
\psi_a(z) = \frac{i}{\pi z} \frac{\zeta(1-a)}{\zeta(-a)} - \frac{1}{z^{1+a}} \cot \frac{\pi a}{2} + \frac{i g_a(z)}{\zeta(-a)}.
\]

Then

\[
\frac{1}{(z + 1)^{1+a}} \psi_a \left( \frac{z}{z + 1} \right) = \frac{i}{\pi z (z + 1)^a} \frac{\zeta(1-a)}{\zeta(-a)} - \frac{1}{z^{1+a}} \cot \frac{\pi a}{2} + \frac{i}{\zeta(-a) (z + 1)^{1+a}} g_a \left( \frac{z}{z + 1} \right)
\]

and

\[
\psi_a(z + 1) = \frac{i}{\pi (z + 1)} \frac{\zeta(1-a)}{\zeta(-a)} - \frac{1}{(z + 1)^{1+a}} \cot \frac{\pi a}{2} + \frac{i g_a(z + 1)}{\zeta(-a)}
\]

where

\[
(2.6) \quad \psi_a(z + 1) - \psi_a(z) + \frac{1}{(z + 1)^{1+a}} \psi_a \left( \frac{z}{z + 1} \right) = 0.
\]

By using the above equations, we arrange (2.6) as a functional equation represented by \( g_a \).

Remark 2.3. In Theorem 2.2, consider \( z \to z - 1 \). Then,

\[
\lim_{a \to 0} \frac{1}{\zeta(-a)} \left\{ \frac{1}{z^{a+1}} g_a \left( \frac{z-1}{z} \right) + g_a(z) - g_a(z - 1) \right\}
\]

\[
= \lim_{a \to 0} \frac{\zeta(1-a)}{\zeta(-a)} \left\{ \frac{1}{\pi(z-1)} - \frac{1}{\pi z} \right\} + \lim_{a \to 0} \frac{\cot \frac{\pi a}{2}}{z^{1+a}} - \lim_{a \to 0} \frac{\zeta(1-a)}{\zeta(-a) \pi(z-1)z^a}
\]

\[
= - \lim_{a \to 0} \left\{ \frac{\zeta(1-a)}{\zeta(-a)} \frac{1}{\pi z} - \frac{1}{z^{1+a}} \cot \frac{\pi a}{2} \right\} + \frac{1}{\pi (z - 1)} \lim_{a \to 0} \frac{\zeta(1-a)(z^a - 1)}{\zeta(-a) z^a}.
\]

By using the properties of (1.1) and (1.2) related to zeta function, we get

\[
(2.7) \quad \lim_{a \to 0} \left\{ \frac{\zeta(1-a)}{\zeta(-a)} \frac{1}{\pi z} - \frac{1}{z^{1+a}} \cot \frac{\pi a}{2} \right\} = 2 \left( -\gamma + \log 2\pi z \right)
\]

and

\[
(2.8) \quad \lim_{a \to 0} \zeta(1-a)(z^a - 1) = - \log z.
\]
From (2.7) and (2.8), we arrive at

\[
\frac{1}{z}g_0 \left( \frac{z-1}{z} \right) + g_0(z) - g_0(z-1) = -\gamma + \log 2\pi z - \frac{1}{\pi(z-1)} \log z.
\]

**Lemma 2.4.** Let \( \text{Im} z > 0 \) and \( a \in \mathbb{C} \). Then, we have

\[
\psi_a(z) + \frac{1}{z^{a+1}} \psi_a \left( -\frac{1}{z} \right) = (1 + (-1)^a) E_{a+1}(z).
\]

**Proof.** If we consider \( z \to -1/z \) in equation (2.2), we get

\[
(2.9) \quad \frac{1}{z^{a+1}} \psi_a \left( -\frac{1}{z} \right) = \frac{1}{z^{a+1}} E_{a+1} \left( -\frac{1}{z} \right) + (-1)^a E_{a+1}(z).
\]

By combining the equations (2.2) and (2.9), we arrive at the desired result. \( \square \)

**Theorem 2.5.** Let \( \text{Im} z > 0 \) and \( a \in \mathbb{C} \). Then, we have

\[
g_a(z) + \frac{1}{z^{a+1}} g_a \left( -\frac{1}{z} \right) = -i \zeta(-a)(1 + (-1)^a) E_{a+1}(z) - \frac{\zeta(1-a)}{\pi} \left( \frac{1}{z} - \frac{1}{z^a} \right) - \zeta(-a) \left( -1 \right)^a \frac{1}{z^{1+a}} \cot \frac{\pi a}{2}.
\]

**Proof.** If we consider \( z \to -1/z \) in Theorem 2.1, we get

\[
\psi_a \left( -\frac{1}{z} \right) = -\frac{i z \zeta(1-a)}{\pi \zeta(-a)} + i(-1)^a z^{1+a} \cot \frac{\pi a}{2} + \frac{i}{\zeta(-a)} \left( -1 \right)^a \frac{1}{z^{1+a}} g_a \left( -\frac{1}{z} \right)
\]

or

\[
(2.10) \quad \frac{1}{z^{1+a}} \psi_a \left( -\frac{1}{z} \right) = -\frac{i}{\pi z^a} \frac{z \zeta(1-a)}{\zeta(-a)} + i(-1)^a \cot \frac{\pi a}{2} \frac{1}{z^{1+a}} \psi_a \left( -\frac{1}{z} \right) + \frac{i}{\zeta(-a)} \left( g_a(z) + \frac{1}{z^{a+1}} g_a \left( -\frac{1}{z} \right) \right).
\]

By combining equations (2.3) and (2.10), we have

\[
(1 + (-1)^a) E_{a+1}(z) = \frac{i z \zeta(1-a)}{\pi \zeta(-a)} \left( \frac{1}{z} - \frac{1}{z^a} \right) + i \left( -1 \right)^a \zeta(-a) \left( -1 \right)^a \frac{1}{z^{1+a}} \cot \frac{\pi a}{2} + \frac{i}{\zeta(-a)} \left( g_a(z) + \frac{1}{z^{a+1}} g_a \left( -\frac{1}{z} \right) \right).
\]

After some elementary calculation, we arrive at the desired result. \( \square \)

In Theorem 2.5, we also show that there exist the functions \( \phi^* \neq 0 \) and \( f^* \neq 0 \) such that they satisfy the following functional equation:

\[
\phi^*_a(z) - \frac{1}{z^{a+1}} \phi^*_a \left( -\frac{1}{z} \right) = f^*_a(z).
\]
Remark 2.6. By using Theorem 2.5, we get
\[
\lim_{a \to 0} \frac{1}{\zeta(-a)} \left\{ g_a(z) + \frac{1}{z^{a+1}} g_a \left( \frac{1}{z} \right) \right\} = -i \lim_{a \to 0} \left( 1 + \frac{(-1)^a}{\pi z} \right) E_a(z) - \lim_{a \to 0} \left\{ \zeta(-a) \frac{1}{\pi z} - (-1)^a \cot \frac{\pi a}{2} \right\} \]
or
\[
-2 \left\{ g_0(z) + \frac{1}{z} g_0 \left( \frac{1}{z} \right) \right\} = -2i E_1(z) - 2 \left( -\gamma + \log 2 \pi z \right) + \lim_{a \to 0} \left\{ \zeta(-a) \frac{1}{\pi z^a} - (-1)^a \cot \frac{\pi a}{2} \right\}.
\]
By using the properties of (1.1) and (1.2) related to zeta function, we get
\[
\lim_{a \to 0} \left\{ -\frac{2}{\pi} \zeta(-a) - (-1)^a \cot \frac{\pi a}{2} \right\} = -\frac{2\gamma}{\pi} + \frac{2}{\pi} \lim_{a \to 0} \left( 1 - (-1)^a \right)
\]
\[
= -\frac{2\gamma}{\pi} - 2i.
\]
Therefore, we arrive at the following result:
\[
g_0(z) + \frac{1}{z} g_0 \left( \frac{1}{z} \right) = \frac{\gamma}{\pi} + i(1 + E_1(z)) + \frac{-\gamma + \log 2 \pi z}{\pi z}.
\]

In the following theorem, we consider
\[
f_a(z) = \frac{1}{(z+1)^{1+a}} g_a \left( \frac{z}{z+1} \right) + g_a(z) + g_a(z).
\]

**Theorem 2.7.** The function \( f_a \) has Taylor series as follows:
\[
f_a(z) = \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{\zeta(-a) (n+a) \cot \frac{\pi a}{2} - \frac{\zeta(1-a)}{\pi} (1 - \frac{n+a}{n-a-1})}{2} \right\} z^n,
\]
where \( |z| < 1 \).

**Proof.** We know that
\[
(2.11) \quad \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{z+1},
\]
where \( |z| < 1 \).

If we take the \( a \)-th derivative of the above series, we have
\[
(2.12) \quad \sum_{n=0}^{\infty} (-1)^n \binom{n+a}{a} z^n = \frac{1}{(z+1)^{a+1}},
\]
where \( |z| < 1 \).
Then, by using the series (2.11) and (2.12), we get
\[
f_a(z) = \frac{\zeta(1-a)}{\pi z} \sum_{n=0}^{\infty} (-1)^n n^a \cot \frac{\pi n}{2} \sum_{n=0}^{\infty} (-1)^n \left( \frac{n+a}{a-1} \right) n^a z^n
\]
\[
\quad - \frac{\zeta(1-a)}{\pi z} \sum_{n=0}^{\infty} (-1)^n \left( 1 - \left( \frac{n+a}{a-1} \right) \right) n^a \cot \pi a \frac{n}{2} \sum_{n=0}^{\infty} (-1)^n \left( \frac{n+a}{a-1} \right) n^a z^n
\]
\[
\quad + \frac{\zeta(-a)\cot \pi a}{2} \sum_{n=0}^{\infty} (-1)^n \left( \frac{n+a}{a} \right) n^a \sum_{n=0}^{\infty} (-1)^n \left( \frac{n+a}{a-1} \right) n^a z^n.
\]
Therefore, we obtain the Taylor series of function \( f_a \) around \( z_0 = 0 \). \( \square \)

3. Generalized difference formulas related to function \( g_a \)

In this section, by using the properties of the function \( \psi_a \), we obtain some generalized difference formulas related to the function \( g_a \).

**Lemma 3.1.** Let \( N \in \mathbb{N} \) and \( a \in \mathbb{C} \). Then we have
\[
\psi_a(z) - \psi_a(z + N + 1) = \sum_{k=1}^{N+1} \frac{1}{(z+k)^{1+a}} \psi_a \left( \frac{z+k-1}{z+k} \right).
\]

**Proof.** We use the iteration \( z \rightarrow z + 1 \) as follows:
\[
\psi_a(z) - \psi_a(z + 1) = \frac{1}{(z+1)^{1+a}} \psi_a \left( \frac{z}{z+1} \right),
\]
\[
\psi_a(z + 1) - \psi_a(z + 2) = \frac{1}{(z+2)^{1+a}} \psi_a \left( \frac{z+1}{z+2} \right),
\]
\[
\psi_a(z + 2) - \psi_a(z + 3) = \frac{1}{(z+3)^{1+a}} \psi_a \left( \frac{z+2}{z+3} \right),
\]
\[
\vdots
\]
\[
\psi_a(z + N) - \psi_a(z + N + 1) = \frac{1}{(z+N+1)^{1+a}} \psi_a \left( \frac{z+N}{z+N+1} \right).
\]
By combining the above equations, we arrive at the desired result. \( \square \)
Theorem 3.2. Let $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Then we have

$$
\sum_{k=1}^{N+1} \frac{1}{(z+k)^{1+a}} g_a \left( \frac{z+k-1}{z+k} \right) = \frac{(N+1)\zeta(1-a)}{\pi z(z+N+1)} + g_a(z) - g_a(z+N+1)
$$

or

$$
\sum_{k=1}^{N+1} \frac{\zeta(-a) \cot \frac{\pi a}{2}}{(z+k)^{1+a}} \left( \frac{1}{(z+k-1)^{1+a}} - \frac{1}{z^{1+a}} \right)
$$

Proof. By using the equations (2.3) and (3.1), we have

$$
\sum_{k=1}^{N+1} \frac{1}{(z+k)^{1+a}} \left\{ \frac{i(z+k)\zeta(1-a)}{\pi(z+k-1)\zeta(-a)} - \frac{i(z+k)^{1+a}}{(z+k-1)^{1+a}} \cot \frac{\pi a}{2} \right. \\
+ \left. \frac{i(N+1)\zeta(1-a)}{\pi z(z+N+1)\zeta(-a)} + i \left( \frac{1}{(z+N+1)^{1+a}} - \frac{1}{z^{1+a}} \right) \right\}
$$

or

$$
\sum_{k=1}^{N+1} \frac{1}{(z+k)^{1+a}} g_a \left( \frac{z+k-1}{z+k} \right)
$$

because of

$$
\frac{1}{(z+N+1)^{1+a}} - \frac{1}{z^{1+a}} + \sum_{k=1}^{N+1} \frac{1}{(z+k-1)^{1+a}} = \sum_{k=1}^{N+1} \frac{1}{(z+k)^{1+a}}.
$$

Then, we arrive at the desired result. \qed

In Theorem 3.2, taking the limit $a \to 0$, we have

$$
\sum_{k=1}^{N+1} \frac{1}{(z+k)^{1+a}} g_a \left( \frac{z+k-1}{z+k} \right) = g_0(z) - g_0(z+N+1) + \lim_{a \to 0} \frac{(N+1)\zeta(1-a)}{\pi z(z+N+1)}
$$
\[
+ \lim_{a \to 0} \sum_{k=1}^{N+1} \left\{ \frac{\zeta(-a) \cot \frac{\pi a}{z+k} - \frac{\zeta(1-a)}{\pi(z+k-1)}}{z+k} \right\}
\]

\[
geq g_0(z) - g_0(z + N + 1) - \lim_{a \to 0} \sum_{k=1}^{N+1} \frac{1}{z+k} \left\{ \frac{\cot \frac{\pi a}{2}}{2} + \frac{\zeta(1-a)}{\pi} \right\}
\]

because of

\[
\frac{1}{z} - \frac{1}{z+N+1} - \sum_{k=1}^{N+1} \frac{1}{z+k-1} = - \sum_{k=1}^{N+1} \frac{1}{z+k}
\]

and \(\zeta(0) = -1/2\).

We know that

\[
(3.2) \quad \lim_{a \to 0} \left\{ \frac{\cot \frac{\pi a}{2} + \frac{\zeta(1-a)}{\pi}}{2} \right\} = \gamma/\pi.
\]

Therefore, we obtain the following corollary:

**Corollary 3.3.** Let \(N \in \mathbb{N}\). Then, we have

\[
(3.3) \quad g_0(z) - g_0(z + N + 1) = \sum_{k=1}^{N+1} \frac{1}{z+k} \left\{ \frac{\gamma}{\pi} + g_0 \left( \frac{z + k - 1}{z+k} \right) \right\}.
\]

By using the equation (2.4), for \(k \in \{1, 2, \ldots, N+1\}\), we have

\[
\frac{1}{z+k} \psi_0 \left( \frac{z+k-1}{z+k} \right)
\]

\[
= \frac{1}{z+k} \left\{ -\gamma + \log \left( \frac{2\pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \right) \right\} - \frac{2i}{\gamma} g_0 \left( \frac{z+k-1}{z+k} \right).
\]

By summing from 1 to \(N + 1\), we deduce from the above that

\[
\sum_{k=1}^{N+1} \frac{1}{z+k} \psi_0 \left( \frac{z+k-1}{z+k} \right)
\]

\[
= -2 \sum_{k=1}^{N+1} \left\{ -\gamma + \log \left( \frac{2\pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \right) \right\} - \frac{2i}{\gamma} \sum_{k=1}^{N+1} \frac{1}{z+k} g_0 \left( \frac{z+k-1}{z+k} \right).
\]

By using the equations (3.1) and (3.3), we get

\[
\psi_0(z) - \psi_0(z + N + 1)
\]

\[
= -2i \left\{ g_0(z) - g_0(z+N+1) \right\} + \frac{2i}{\pi} \sum_{k=1}^{N+1} \log \left( \frac{2\pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \pi \left( \frac{z+k-1}{z+k} \right)^2 \right) \right\}
\]

\[
- \frac{2i\gamma}{\pi} \sum_{k=1}^{N+1} \left( \frac{1}{z+k-1} - \frac{1}{z+k} \right).
\]
Corollary 3.4. Let \( N \in \mathbb{N} \). Then, we have
\[
\sum_{k=1}^{N+1} \frac{1}{z + k - 1} \log \left( 2\pi \left( \frac{z + k - 1}{z + k} \right) \right) = \frac{1}{2} \left( \psi_0(z) - \psi_0(z + N + 1) \right).
\]

Proof. From equation (2.6), we know
\[
\frac{1}{z} \left( \frac{1}{z + N + 1} \right) + \pi \left( g_0(z) - g_0(z + N + 1) \right) = \frac{\pi i}{2} \left( \psi_0(z) - \psi_0(z + N + 1) \right).
\]

Lemma 3.5. Let \( a, \lambda \in \mathbb{C} \) and \( \lambda \neq 0 \). Then, we have
\[
(3.4) \quad \frac{\psi_a \left( \frac{z}{z + \lambda} \right) - \psi_a \left( \frac{z + 1}{z + \lambda} \right)}{\psi_a \left( \frac{z}{z + \lambda} \right)} = \lambda^{a+1} \left( \frac{\psi_a (z + \lambda - 1) - \psi_a (z + \lambda)}{\psi_a \left( \frac{z + \lambda - 1}{z + \lambda} \right)} \right).
\]

Proof. From equation (2.6), we know
\[
\frac{\psi_a (z) - \psi_a (z + 1)}{\psi_a \left( \frac{z}{z + 1} \right)} = \frac{1}{(z + 1)^{a+1}}.
\]
For \( z \to z/\lambda \) (\( 0 \neq \lambda \in \mathbb{C} \)), we get
\[
\frac{\psi_a \left( \frac{z}{z + \lambda} \right) - \psi_a \left( \frac{z + 1}{z + \lambda} \right)}{\psi_a \left( \frac{z}{z + \lambda} \right)} = \frac{\lambda^{a+1}}{(z + \lambda)^{a+1}}
\]
\[
= \lambda^{a+1} \left( \frac{\psi_a (z + \lambda - 1) - \psi_a (z + \lambda)}{\psi_a \left( \frac{z + \lambda - 1}{z + \lambda} \right)} \right).
\]

Then, we have the desired result. \( \square \)

Theorem 3.6. Let \( a, \lambda \in \mathbb{C} \) and \( N \in \mathbb{N} \). Then, we have
\[
\sum_{k=1}^{N+1} \left\{ \left( \frac{1}{(z + \lambda)^{a+1}} \right) \times \left( \frac{1}{(z + k + \lambda)^{a+1}} \right) \right\} \frac{\pi \lambda^a \zeta (1 - a)}{\pi z (z + N \lambda)} + \left( \frac{1}{(z + N \lambda)^{a+1}} - \frac{1}{z^{a+1}} \right) \zeta (-a) \cot \frac{\pi a}{2}
\]
\[
+ \frac{1}{\lambda^{a+1}} \left( g_a \left( \frac{z}{\lambda} \right) - g_a \left( \frac{z + N \lambda}{\lambda} \right) \right).
\]

Proof. From equation (3.4), we use the iteration \( z \to z + \lambda \) as follows:
\[
\psi_a \left( \frac{z}{z + \lambda} \right) - \psi_a \left( \frac{z + 1}{z + \lambda} \right) = \lambda^{a+1} \left( \frac{\psi_a \left( \frac{z}{z + \lambda} \right) - \psi_a \left( \frac{z + 1}{z + \lambda} \right)}{\psi_a \left( \frac{z}{z + \lambda} \right)} \right) \{ \psi_a (z + \lambda - 1) - \psi_a (z + \lambda) \},
\]
\[
\psi_a \left( \frac{z + \lambda}{\lambda} \right) - \psi_a \left( \frac{z + 2\lambda}{\lambda} \right) = \lambda^{a+1} \frac{\psi_a \left( \frac{z + \lambda}{\lambda} \right)}{\psi_a \left( \frac{z + 2\lambda}{\lambda} \right)} \{ \psi_a (z + 2\lambda - 1) - \psi_a (z + 2\lambda) \},
\]
\[
\psi_a \left( \frac{z + 2\lambda}{\lambda} \right) - \psi_a \left( \frac{z + 3\lambda}{\lambda} \right) = \lambda^{a+1} \frac{\psi_a \left( \frac{z + 2\lambda}{\lambda} \right)}{\psi_a \left( \frac{z + 3\lambda}{\lambda} \right)} \{ \psi_a (z + 3\lambda - 1) - \psi_a (z + 3\lambda) \},
\]
\[
\vdots
\]
\[
\psi_a \left( \frac{z + (N-1)\lambda}{\lambda} \right) - \psi_a \left( \frac{z + N\lambda}{\lambda} \right) = \lambda^{a+1} \frac{\psi_a \left( \frac{z + (N-1)\lambda}{\lambda} \right)}{\psi_a \left( \frac{z + N\lambda}{\lambda} \right)} \{ \psi_a (z + N\lambda - 1) - \psi_a (z + N\lambda) \}.
\]

By combining the above equations, we get
\[(3.5)\]
\[
\frac{\psi_a (\pi) - \psi_a \left( \frac{z + N\lambda}{\lambda} \right)}{\lambda^{a+1}} = \sum_{k=1}^{N} \frac{\psi_a \left( \frac{z + k\lambda - 1}{z + k\lambda} \right)}{\psi_a \left( \frac{z + k\lambda}{z + k\lambda} \right)} \{ \psi_a (z + k\lambda - 1) - \psi_a (z + k\lambda) \}.
\]

By using the equation (2.3), we arrange the equation (3.5). Then, we arrive at the desired result. \(\square\)

In Theorem 3.6, taking the limit \(a \to 0\), we have
\[
\sum_{k=1}^{N} \left\{ \frac{\left( \frac{z + k\lambda - 1}{z + k\lambda} \right)}{(z + k\lambda)(z + k\lambda)} \times \frac{(\zeta(1-a) + \frac{\pi}{2} \cot \frac{\pi a}{2} + \pi(z + k\lambda - 1)g_0(z + k\lambda))}{(\zeta(1-a) + \frac{\pi}{2} \cot \frac{\pi a}{2} + \pi(z + k\lambda - 1)g_0(z + k\lambda))} \right\} = \frac{N\lambda}{z(z + N\lambda)} \lim_{a \to 0} \left( \frac{\zeta(1-a) + \frac{\pi}{2} \cot \frac{\pi a}{2}}{\zeta(a)} \right) + \frac{1}{\lambda} \left( g_0 \left( \frac{z}{\lambda} \right) - g_0 \left( \frac{z + N\lambda}{\lambda} \right) \right).
\]

By using (3.2), we arrive at the following corollary:

**Corollary 3.7.** Let \(\lambda \in \mathbb{C}\) and \(N \in \mathbb{N}\). Then, we have
\[
\sum_{k=1}^{N} \left\{ \frac{\left( \frac{z + k\lambda - 1}{z + k\lambda} \right)}{(z + k\lambda)(z + k\lambda)} \times \frac{\gamma(z + k\lambda) + \pi(z + (k-1)\lambda)g_0(z + (k-1)\lambda)}{\gamma(z + k\lambda) + \pi(z + k\lambda - 1)g_0(z + k\lambda - 1)} \right\} = \frac{N\lambda}{\pi z(z + N\lambda)} + \frac{1}{\lambda} \left( g_0 \left( \frac{z}{\lambda} \right) - g_0 \left( \frac{z + N\lambda}{\lambda} \right) \right).
\]

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