A FEYNMAN-KAC FORMULA FOR STOCHASTIC DIRICHLET PROBLEMS

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Abstract. A representation formula for solutions of stochastic partial differential equations with Dirichlet boundary conditions is proved. The scope of our setting is wide enough to cover the general situation when the backward characteristics that appear in the usual formulation are not even defined in the Itô sense.

1. Introduction

The goal of the article is to present a Feynman-Kac formula for the solutions of stochastic partial differential equations (SPDEs). For deterministic PDEs such a probabilistic interpretation of the solution proved to be a remarkably useful tool to prove results that are either not available or are rather more difficult to obtain by purely analytic methods. It is hence not an unreasonable hope that a representation formula can also help in the stochastic case to obtain further information about the solutions. To indicate why obtaining Feynman-Kac formulae for SPDEs is not straightforward, let us recall a simple deterministic case. Take the 1-dimensional stochastic differential equations, parametrized by \( t \) and \( x \),

\[
dX^{t,x}_s = \sigma_s(X^{t,x}_s) \, dB_s \quad \text{for } s \in [0,t], \quad X^{t,x}_t = x,
\]

where \( B \) is a standard Wiener process and \( dB_s \) is its backward Itô differential. The solution \( X \) - or rather its continuous modification in \( s,t,x \) - is often referred to as the backward characteristic. Under some mild conditions on \( \sigma \) and \( \psi \), \( u_t(x) := \mathbb{E}\psi(X^{t,x}_0) \)

satisfies the Cauchy problem

\[
\partial_t u_t(x) = \frac{1}{2} \sigma^2_t(x) \Delta u_t(x), \quad u_0(x) = \psi(x).
\]

Now if we start from an initial value problem for SPDEs, in general - and in particular for the important example of the Zakai equation - the coefficients will be random and adapted to a forward filtration. Since in the noise evolves in reversed time, it becomes an equation in which the direction of the randomness in the coefficients and that of the noise do not match: the interpretation of a solution of such an equation and the subsequent analysis needed to prove the validity the formula is problematic.

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When the equation is given on the whole space, this difficulty can be overcome by an elegant argument through fully degenerate SPDEs, see [Kry92], and one obtains a representation in which the role of the backward flows are taken over by spatial inverses of forward flows. While this gives some idea how a representation should look like when the equation is considered with some boundary conditions, the argument itself breaks down: the Dirichlet problem for degenerate equations is ill-posed. Here we take a more pragmatic approach and ‘build up’ the representation formula from situations where the coefficients are deterministic and one can make sense of the backward characteristics. We note that the case of deterministic coefficients in a simplified setting were considered previously in [FS90], and indeed the first step in our proof is quite similar to that in [FS90], whose method in turn is based on [KR86].

As an application of the formula, we get an estimate the ‘localization’ error one makes when imposing artificial boundary conditions to problems that are originally given on the whole space. The reason why this is of interest is that often the particular model that one wants to study, and gets the equation from, has no natural boundary conditions but is expected to vanish at infinity. One then may think then that setting the value to be zero on the boundary of a large enough domain is a good approximation of the original problem, and this is what we confirm and make precise below.

The article is structured as follows. We continue with introducing some notations, after which in Section 2 the necessary objects for the Feynman-Kac formula are introduced and in Theorem 2.2 the representation formula is stated. In Section 3 we collect some auxiliary results, and in Section 4 we give the proof of Theorem 2.2. Section 5 contains the above mentioned application for the localization error.

**Notations.** Given a $d$-dimensional stochastic differential equation (SDE),

\[ dX^i_t = \alpha^i_t(X_t) \, dt + \sum_k \beta^i_k(X_t) \, dw^k_t, \quad i = 1, 2, \ldots, d \]

(2)

driven by a (possibly infinite) sequence of Wiener processes, the corresponding stochastic flow on an interval $[0, T]$ is a continuous random field $(X_{s,t}(x))_{0 \leq s \leq t \leq T, x \in \mathbb{R}^d}$ such that for all $s$ and $x$, the process $(X_{s,t}(x))_{s \leq t \leq T}$ is a solution of the equation (2) with initial condition $X_{s,s}(x) = x$, and that furthermore for all $0 \leq s \leq t \leq v \leq T$ and $x \in \mathbb{R}^d$,

\[ X_{t,v}(X_{s,t}(x)) = X_{s,v}(x). \]

When emphasizing the direction of the equation, one may also refer to it as the forward flow, distinguishing it from backward flows, which are the analogous objects for equations involving backward Itô differentials. The existence of stochastic flows is known in quite large generality, see [Kun97], [Kun84]. Moreover, also under quite general assumptions, the mappings $X_{s,t}$ are diffeomorphisms from $\mathbb{R}^d$ onto itself, and hence one can also talk about the inverse flow $(X_{s,t}^{-1}(x))_{0 \leq s \leq t \leq T, x \in \mathbb{R}^d}$.

The derivative of a function $f$ on $\mathbb{R}^d$ with respect to $x^i$ is denoted by $D_i$. We denote by $C^0$ the space of continuous functions, and by $C^\alpha$ the space of Hölder
continuous functions with exponent $\alpha \in (0, 1)$. For $\alpha \in [1, \infty)$, the space $C^\alpha$ consists of functions $v$ such that $D_l v \in C^{\alpha - |\alpha|}$ for all multiindex $l$ with length at most $|\alpha|$. For $p \geq 2$, $L_p$ denotes the usual Lebesgue space of generalized functions integrable to the $p$-th power, and $W^m_p$ the Sobolev space of generalized functions from $L_p$ whose distributional partial derivatives up to order $m$ are also generalized functions from $L_p$. When talking about an infinite sequence of functions $g = (g^K)_{K \in \mathbb{N}}$ belonging to a function space $C^\alpha$ or $W^m_p$, we always understand $g \in C^\alpha(l_2)$ or $g \in W^m_p(l_2)$, respectively. For a probability space with a product measure $P \otimes \hat{P}$, the notation $\mathbb{E}\hat{P}X = EX$ for integrable random variables $X$. Unless it is indicated otherwise, the summation convention with respect to repeated indices is used throughout the paper.

2. FORMULATION AND MAIN RESULT

Let $D \subset \mathbb{R}^d$ be a bounded $C^2$-domain, $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(\mathcal{F}_t)_{t \geq 0}$ be a filtration, and $(w^K_t)_{t \geq 0, K = 1, 2, ...}$ be a sequence of independent $(\mathcal{F}_t)$-Wiener martingales. The filtration is assumed to satisfy the “usual conditions”, i.e., $\mathcal{F}_0$ contains every event of probability zero, and $\mathcal{F}_t = \cap_{s \leq t} \mathcal{F}_s$. The predictable $\sigma$-algebra on $[0, \infty) \times \Omega$ is denoted by $\mathcal{P}$.

We consider the following initial-boundary value problem

\[
\begin{cases}
    du = [Lu + f] \, dt + [M^K u + g^K] \, dw^K_t & \text{on } [0, T] \times D, \\
    u = 0, & \text{on } (0, T) \times \partial D, \\
    u_0 = \psi,
\end{cases}
\]

where the differential operators $L$ and $M$ are given by

\[
L \varphi = \frac{1}{2}(\sigma \sigma^* + \rho \rho^*)^{ij} D_i D_j \varphi + b^i D_i \varphi + c \varphi,
\]
\[
M^K \varphi = \sigma^{ik} D_i \varphi + \mu^K \varphi,
\]

with coefficients $\rho, b, c, \sigma, \mu$, and initial and free data $\psi, f, g$, defined for $(t, \omega, x)$ from $[0, \infty) \times \Omega \times \mathbb{R}^d$, such that they vanish for $x \notin D^1 := \{x : d(x, D) \leq 1\}$. They are subject to the following assumptions, for some $\alpha > 0$.

Assumption 2.1. There exists a $\lambda > 0$ such that for all $t, \omega$, and $x \in D^{1/2}$,

\[
(\rho \rho^*)_t(x) \geq \lambda I
\]

in the sense of positive semidefinite matrices, where $I$ is the identity matrix, and $\rho^*$ is the transpose of $\rho$.

Assumption 2.2. The coefficients $\rho, \sigma, b, c, \mu$ are predictable functions with values in $C^{2+\alpha}(\mathbb{R}^{d \times d}), C^{2+\alpha}(l_2^d), C^{1+\alpha}(\mathbb{R}^d), C^\alpha(\mathbb{R})$, and $C^{1+\alpha}(l_2)$, respectively; bounded uniformly in $t$ and $\omega$ by a constant $K$. 

Assumption 2.3. The initial value, $ψ$ is an $F_0$-measurable random variable with values in $C^α$. The free data, $f$ and $g$, are predictable processes with values in $C^α$ and $C^{1+α}(L_2)$, respectively, such that

$$\mathbb{E}\left(\left|ψ\right|^2_{L^2} + \int_0^T \left|f_t\right|^2_{L^2} + \left|g_t\right|^2_{L^2} \, dt \right) \leq K.$$

The above assumptions are more than sufficient to get from the general solution theory of SPDEs on domains in [Kim04] that the problem (3) admits a unique solution $u$ belongs to $L_2(\Omega, C([0,T], L_2(D))) \cap L_2([0,T] \times \Omega, \mathcal{F}, H^1_0(D))$, the equality

$$(u_t, ϕ) = (ψ, ϕ) + \int_0^t (a^{ij} D_j u_s - D_i ϕ) + ((b^i + D_j a^i j) D_i u + cu_s + f, ϕ) \, ds$$

$$+ \int_0^t (σ^{ik} D_i u_s + µ^k u_s, ϕ) \, dw^k_s$$

holds for all $ϕ$ smooth and compactly supported function on $D$ almost surely for all $t \in [0,T]$, and $(u_t(x))_{t \in [0,T], x \in D}$ is a continuous random field. Here $(\cdot, \cdot)$ denotes the $L_2(\mathbb{R}^d)$ inner product.

To introduce the representation of the solution $u$, let $(\hat{w}^t_i)_{t \in [0,T]}$ be the $d$-dimensional Wiener process on the standard Wiener space $(\hat{Ω}, \hat{\mathcal{F}}, \hat{P})$, where $\hat{Ω} = C([0,T], \mathbb{R}^d)$, $\hat{\mathcal{F}} = \mathcal{B}(\hat{Ω})$, and $\hat{P}$ is the Wiener measure. The associated forward problem to the problem (3) are given by the SDE,

$$dY_t = β_t(Y_t) \, dt - σ^k_t(Y_t) \, dw^k_t - ρ^k_t(Y_t) \, dw^k_t, \quad t \in [0,T],$$

$$Y \mid Y_0 = y \in \mathbb{R}^d,$$

on the completion of the probability space $(\Omega \times \hat{Ω}, \mathcal{F} \otimes \hat{\mathcal{F}}, P \otimes \hat{P})$ where for $t \in [0,T], y \in \mathbb{R}^d$,

$$β_t(y) = -b_t(y) + σ^k_t(y)D_iσ^k_t(y) + ρ^i_t(y)D_iρ^k_t(y) + σ^k_t(y)µ^k_t(y),$$

and $σ^k, \rho^k$ stand for the column vectors $(σ^{1k}, \ldots, σ^{dk}), (ρ^{i1}, \ldots, ρ^{id})$, respectively. We shall also use the notation $P = P \otimes \hat{P}$. Taking the stochastic flow $(Y_{s,t}(y))_{0 \leq s \leq t \leq T, y \in \mathbb{R}^d}$ defined by (4), one can define the random times, for $t \in [0,T], x \in \mathbb{R}^d$

$$γ_{t,x} = \sup\{s \in [0,t] : (s, Y^{-1}_{s,t}(x)) \notin (0,T) \times D\},$$

that is, the exit time of the inverse characteristic starting from $t, x$. Note however, that $γ$ is not a stopping time in general with respect to either of the forward or backward filtrations.

Finally, introduce the processes $η$ and $U$ by

$$dη_t(y) = \tilde{c}_t(Y_{0,t}(y))η_t(y) \, dt + µ^k_t(Y_{0,t}(y)) η_t(y) \, dw^k_t, \quad η_0(y) = 1,$$

$$dU_t(y) = \{\tilde{c}_t(Y_{0,t}(y))U_t(y) + \tilde{f}_t(Y_{0,t}(y))\} \, dt$$

$$+ \{µ^k_t(Y_{0,t}(y))U_t(y) + g^k_t(Y_{0,t}(y))\} \, dw^k_t, \quad U_0(y) = 0$$

where

$$\tilde{c}_t(x) := c_t(x) - σ_t^{ki} D_i µ_t^k(x), \quad \tilde{f}_t(x) = f_t(x) - σ_t^{ki} D_i g_t^k.$$
It is straightforward to check by Kolmogorov’s criterion that \( (\eta_t(y))_{t \in [0,T], y \in \mathbb{R}^d} \) and \( (U_t(y))_{t \in [0,T], y \in \mathbb{R}^d} \) have continuous versions, for which we use the same notation. The ‘right-hand-side’ of the Feynman-Kac formula will then read as

\[
v_t(x) := \mathbb{E}^P \left( (\psi \eta_t)(Y_{0,t}^{-1}(x))1_{\gamma_{t,x}=0} + (U_t - U_{\gamma_{t,x}} \frac{n}{\eta_{\gamma_{t,x}}})(Y_{0,t}^{-1}(x)) \right).
\]  

(8)

Remark 2.1. Note that integrating out the \( \hat{\omega} \) variable gives (a version of) the conditional expectation given \( \mathcal{F} \). Using then the explicit expressions for \( \eta \) and \( U \), the formula can be written in the more familiar form

\[
\mathbb{E} \left[ \psi(y)e^{\varphi_t(y)}1_{\tau=0} + \left( \int_\tau^t \hat{f}_s(Y_{0,s}(y))e^{-\varphi_s(y)} \, ds \right) + \int_\tau^t g^k_{s}(Y_{0,s}(y))e^{-\varphi_s(y)} \, dw_s^k \right] \left. \right|_{\tau=\gamma_{t,x}} \left. \right|_{y=Y_{0,t}^{-1}(x)} \left( \mathcal{F} \right)
\]

where \( \varphi_t(y) = \int_0^t (\hat{\epsilon}_s - (1/2)|\mu_s|^2)(Y_{0,s}(y)) \, ds + \int_0^t \mu_s(Y_{0,s}(y)) \, dw_s^k \).

Fubini’s theorem tells us that (8) is meaningful for \( dt \otimes dx \otimes dP \)-almost every \( t,x,\omega \), in particular, there is an event of full probability on which \( v_t(x) \) is well defined for almost all \( t,x \). To talk about \( v \) as a random field however, we need a slightly better property, given by the following proposition.

**Proposition 2.1.** Under Assumptions 2.2-2.3, there exists an event of full probability on which the right-hand-side of (8) exists for all \( t,x \), and it is jointly measurable in \( \omega,t,x \).

The proof of this is given in Section 3. We are now in a position to state the main result.

**Theorem 2.2.** Under Assumptions 2.2-2.3, \( u_t(x) = v_t(x) \) for all \( t, dx \otimes dP \)-almost everywhere.

3. Preliminaries

**Proof of Proposition 2.2.** Consider the random fields

\[
U_{t}^{(n,m)}(x) = (\psi \eta_t)(Y_{0,t}^{-1}(x))1_{\gamma_{t,x}} + (U_t - U_{\gamma_{t,x}} \frac{n}{\eta_{\gamma_{t,x}}})(Y_{0,t}^{-1}(x)),
\]

\[
U_t^{(m)}(x) = (\psi \eta_t)(Y_{0,t}^{-1}(x))1_{\gamma_{t,x}} + (U_t - U_{\gamma_{t,x}} \frac{n}{\eta_{\gamma_{t,x}}})(Y_{0,t}^{-1}(x)),
\]

\[
U_t(x) = (\psi \eta_t)(Y_{0,t}^{-1}(x))1_{\gamma_{t,x}=0} + (U_t - U_{\gamma_{t,x}} \frac{n}{\eta_{\gamma_{t,x}}})(Y_{0,t}^{-1}(x)),
\]

for \( n,m \in \mathbb{N} \), where

\[
1_{\gamma_{t,x}}^{(m)}(x) := 1_{x < 0} + (1 - mx)1_{x \in [0,1/m]},
\]

\[
\gamma_{t,x}^{(n)} := n \int_0^{1/n} \gamma_{t,x}(\delta) \, d\delta,
\]

with the notation \( D_\delta = \{ x \in D : d(x, \partial D) > \delta \} \) and

\[
\gamma_{t,x}(\delta) = \sup\{ s \in [0,t] : (s,Y_{s,t}^{-1}(x)) \notin (0,T] \times D_\delta \}.
\]
Lemma 3.2. One has, First notice that it suffices to prove the statement when one modifies the where converges to \(0\) Let \(\gamma\) converges to \(s\), is dominated by \(\omega\) which is integrable in \(\hat{\omega}\), By Fubini's theorem there is an event \(\tilde{\Omega}\) of full probability on which for almost for almost all \(t, x, U_t^{(n, m)}(x)\) is measurable as a function of \(\hat{\omega}\). Since \(U_t^{(n, m)}(x)\) is continuous, this actually holds for all \(t, x\). Since \(\gamma_t(x)\) is right-continuous in \(x\), the functions \(\gamma_t(x)\) converge to \(\gamma_t(x)\), and so \(U_t^{(n, m)}(x)\) converge to \(U_t^{(m)}(x)\) for all \(t, x\). In particular, for \(\omega \in \tilde{\Omega}\), \(U_t^{(m)}(x)(\omega, \hat{\omega})\) is a measurable function of \(\hat{\omega}\) for all \(t, x\). Taking then the \(m \to \infty\) limit, this holds for \(U\) as well. Therefore \(U_t(x)\) is a measurable function that is dominated by

\[
\sup_{(t,y)\in[0,T] \times D^1} |\psi|_I(y) + 2 \sup_{(s,t,y)\in[0,T] \times D^1} |U_t^{n/m}|(y),
\]

which is integrable in \(\tilde{\omega}\) for almost all \(\omega\), and therefore so is \(U_t(x)\).

The following limit theorem is known, see e.g \[Kun84\], \[LM15\].

Lemma 3.1. Let \(\rho^n\), \(a^n\), \(\mu^n\), and \(b^n\) be coefficients satisfying Assumption 2.2 for \(n = 0, 1, \ldots\) such that

\[
|\rho^n - \rho^0|_{C^{2+\alpha}(D^1)} + |\sigma^n - \sigma^0|_{C^{2+\alpha}(D^1)} + |\mu^n - \mu^0|_{C^{1+\alpha}(D^1)} + |b^n - b^0|_{C^{1+\alpha}(D^1)}
\]

converges to 0 in measure with respect to \(dt \otimes dP\) as \(n \to \infty\). Then

\[
\lim_{n \to \infty} \mathbb{E} \sup_{t \in [0,T]} |Y_{0,t}^n - Y_{0,t}^0|^2_{C^{1+\alpha}(D^1)} = 0,
\]

\[
\lim_{n \to \infty} \mathbb{E} \sup_{t \in [0,T]} |Y_{0,t}^{n-1} - Y_{0,t}^{0-1}|^2_{C^{1+\alpha}(D^1)} = 0.
\]

Define the set of trajectories that 'touch' the boundary at some point as

\[
T_D = \cup_{t > 0} T_D(t),
\]

where

\[
T_D(t) = \{ f \in C([0, t], \mathbb{R}^d) : \exists s \in [0, t], \epsilon > 0 such that f_s \in \partial D; \forall r \in [(s - \epsilon) \vee 0, (s + \epsilon) \wedge t] f_r \in \bar{D}. \}
\]

Lemma 3.2. One has, \(dx \otimes dP \otimes d\hat{P}\)-almost surely

\[
(Y_{0,s}(x))_{s \in [0,T]} \notin T_D.
\]

Proof. First notice that it suffices to prove the statement when one modifies the definition of \(T_D\) to, say, \(T_D'\), by changing \(s \in [0, t]\) to \(s \in [0, t]\) in the definition of \(T_D(t)\). Indeed, the trajectory \((Y_{0,s}(x))_{s \in [0,T]}\) may only belong to \(T_D \setminus T_D'\) if \(Y_{0,T}(x)\) is \(\partial D\), in other words, for

\[
(x, \omega, \hat{\omega}) \in \{(x, \omega, \hat{\omega}) : x \in Y_{0,T}^{-1}(\partial D)(\omega, \hat{\omega})\},
\]
and the latter set is of measure 0. The function
\[ d(x) = d(x, \partial D) \text{ if } x \in \bar{D}, \quad d(x) = -d(x, \partial D) \text{ if } x \notin D \]
is \( C^2 \) in a neighbourhood of \( \partial D \), see e.g. \cite{GT83}. It is also easy to see that \( |\nabla d_D| \) is separated away from zero in a neighbourhood of \( \partial D \). One can then find a globally \( C^2 \) function \( \tilde{d} \) which agrees with \( d \) on a neighbourhood of \( \partial D \) and is separated away from zero outside that neighbourhood. Defining \( Z_s := \tilde{d}(Y_{0,s}(x)) \), we have
\[ (Y_{0,s}(x))_{s \in [0,T]} \notin \mathcal{T}_D' \iff (Z_s)_{s \in [0,T]} \notin \mathcal{T}_{\mathbb{R}^+}. \]
The process \( Z \) has Itô differential
\[ dZ_t = b_t \, dt + \sigma_t \, d\tilde{w}_t \]
with the Wiener process \( \tilde{w} = (w, \tilde{w}) \) and with some bounded predictable functions \( b \) and \( \sigma \). Moreover, \( d\langle Z \rangle_t \geq \lambda |Z| \, dt \) for some positive constants \( \lambda \) and \( \delta \). Define the stopping times \( \tau_0 = 0 \) and for \( i \geq 0 \)
\[ \tau_{2i+1} = \inf\{ s \geq \tau_{2i} : |Z_s| \geq \delta \} \land T, \quad \tau_{2i+2} = \inf\{ s \geq \tau_{2i+2} : |Z_s| \leq \delta/2 \} \land T. \]
Note that the hitting times of 0 of \( Z \) can only occur on the intervals \( [\tau_{2i}, \tau_{2i+1}] \).
Let us define \( \tilde{b}_t^i = \tilde{b}_{(t+\tau_{2i}, \tau_{2i+1})}, \tilde{\sigma}_t^i = \tilde{\sigma}_{(t+\tau_{2i}, \tau_{2i+1})} \), and \( \tilde{w}_t^i = w_{t+\tau_{2i}} \). Then for each \( i \geq 0 \),
\[ Z_t^i := \int_0^t \tilde{b}_s^i \, ds + \int_0^t \tilde{\sigma}_s^i \, d\tilde{w}_s \]
is a semimartingale with respect to the filtration \( (\mathcal{F}_{\tau_{2i+1}^a})_{s \geq 0} \), satisfying \( d\langle Z^i \rangle_t \geq \lambda \, dt \). Moreover, if \( (Z_s^i)_{s \in [0,T]} \in \mathcal{T}_{\mathbb{R}^+}^a \), then \( (Z_s^i)_{s \geq 0} \in \mathcal{T}_{\mathbb{R}^+}^a \) for some \( i \geq 0 \) and for one of \( a = \delta/2, -\delta/2, \) or \( -Z_0 \). Fixing \( i \) and \( a \) like so, to show that \( (Z_s^i)_{s \geq 0} \in \mathcal{T}_{\mathbb{R}^+}^a \) has probability zero, we may change to an equivalent measure and hence by a Girsanov transform we may assume that \( b = 0 \). Moreover, the probability also doesn’t change if we perform a time change whose derivative is separated from 0 and \( \infty \), and so it actually suffices to see that \( \tilde{P}((B_s)_{s \geq 0} \in \mathcal{T}_{\mathbb{R}^+}^a) = 0 \) for a standard 1-dimensional Brownian motion. This is however known, and follows from
\[ \tilde{P}((B_s)_{s \geq 0} \in \mathcal{T}_{\mathbb{R}^+}^a) \leq \sum_{r,q \in \mathbb{Q}^+} \tilde{P}(\min_{s \in [r,q]} B_s = a), \]
and recalling that since the random variable \( \min_{s \in [r,q]} B_s \) is absolutely continuous (in fact, with explicitly known density), and hence each term in the above sum is 0.
\[ \square \]

**Proposition 3.3.** For all \( t \in [0,T] \), \( dx \otimes \tilde{P} \otimes \tilde{P} \)-almost everywhere,
\[ (Y_{s,t}^{-1}(x))_{s \in [0,t]} \notin \mathcal{T}_D. \]  \hspace{1cm} (9)

**Proof.** By the previous lemma we can write
\[ 0 = \int_{D^1} \tilde{P}((Y_{0,s}(y))_{s \in [0,t]} \in \mathcal{T}_D) \, dy \]
\[ E \int_{D} 1_{(Y_{0,s}(y)) \in [0,t] \in T} dy = E \int_{D} 1_{(Y_{0,s}^{-1}(Y_{0,t}(x))) \in [0,t] \in T} |\det \nabla Y_{0,t}(x)| dx. \]

After interchanging the integral and expectation we conclude that for almost all \( x \),
\[ E[1_{(Y_{0,s}(Y_{0,t}^{-1}(x))) \in [0,t] \in T} |\det \nabla Y_{0,t}^{-1}(x)|] = 0, \]
and since, \( \inf_{x \in D} |\det \nabla Y_{0,t}^{-1}(x)| \) is almost surely nonzero, the indicator is almost surely 0, which proves the claim. □

Proposition 3.4. Let \( \{f_{i}\}_{i \in I} \) be a uniformly integrable family of real-valued functions on a product of two measure spaces \((A, \mu)\) and \((B, \nu)\). Then \( \{f_{i}(a, \cdot)\}_{i \in I} \) is uniformly integrable for almost all \( a \in A \).

Proof. This is an easy consequence of de la Vallée Poussin’s theorem: we have a function \( G \) such that
\[ \lim_{t \to \infty} \frac{G(t)}{t} = \infty \]
and
\[ \sup_{i \in I} \int G(f_{i}(a, b)) \, d\mu(a) \, d\nu(b) < \infty. \]

Then by Fubini’s theorem, for almost all \( a \in A \),
\[ \sup_{i \in I} \int G(f_{i}(a, b)) \, d\nu(b) < \infty, \]
which, by the converse direction of de la Vallée Poussin’s theorem, proves the claim. □

4. Proof of Theorem 2.2

Step 1. First consider the case when, further to the assumptions of the theorem, all coefficients and data are deterministic and do not depend on time. This was considered in [FS90] with further assuming \( f = g = 0 \) and \( \psi|_{\partial D} = 0 \). The proof consists of two main steps: (a) establish a representation formula in terms of the appropriate backward flow (b) rewrite the formula in terms of the inverse flow, using the relationship between backward and inverse flows from [Kun84]. Part (a) follows very similarly to [KRS86] and [FS90], and is based on the Feynman-Kac formula for the deterministic PDEs
\[
\begin{cases}
  d\tilde{u} = [L\tilde{u} + f + q^{k}(M^{k}\tilde{u} + g^{k})] \, dt & \text{on } [0,T] \times D, \\
  \tilde{u} = 0, & \text{on } (0,T] \times \partial D, \\
  \tilde{u}_{0} = \psi, & \text{on } (0,T].
\end{cases}
\]
for arbitrary \( q \in C^{\infty}([0,T], l_{2}) \). We therefore not give the details, but we note that for this representation it is not required that \( \psi \) has limit at the boundary, and hence neither is this assumption needed for Theorem 2.2. One obtains the formula through the backward characteristics
\[
\begin{equation}
  dX_{t} = (b_{t}(X_{t}) - \sigma_{t}^{k} \mu_{t}^{k}(X_{t})) \, dt + \sigma_{t}^{k}(X_{t}) \, d\tilde{w}_{t}^{k} + \rho_{t}^{r}(X_{t}) \, d\tilde{w}_{t}^{r},
\end{equation}
\]
where \( \hat{d} \) denotes the backward Itô differential, defined as in \cite{Kun84}. Considering the corresponding backward flow \( (X_{t,s}(x))_{0 \leq s \leq t \leq T, x \in \mathbb{R}^d} \), the formula then reads as
\[
\begin{align*}
u_t(x) &= \mathbb{E}[\hat{P}^{\hat{\psi}}(X_{t,0}(x))e^{\hat{\xi}_0(x)}1_{\tau_{t,x} = 0} + \int_t^{\tau_{t,x}} f_s(X_{t,s}(y))e^{\hat{\xi}_s(x)} ds \\
&+ \int_t^{\tau_{t,x}} g^k_s(X_{t,s}(x))e^{\hat{\xi}_s(x)} \hat{dw}_s^k],
\end{align*}
\]
for all \( t, dx \otimes dP \text{-a.e.} \), where
\[
\tau_{t,x} = \sup\{s \in [0,t] : (s, X_{t,s}^{-1}(x)) \notin (0,T] \times D\}
\]
and \( \hat{\xi}_s(x) = \int_s^t (c_s - (1/2)|\mu_s|^2)(X_{t,s}(x)) ds + \int_t^s \mu^k(X_{t,s}(x)) \hat{dw}_s^k \).

For part (b) we give the full details here, partially because the transformation of the terms coming from the forcing is not trivial, and partially in order to correct a slight miscalculation in \cite{FS90} which in fact affects the formula therein itself (cf. the definition \( \hat{\eta} \) of \( \eta \) and (2.7) in \cite{FS90} of the corresponding term \( \mu \)). To this end, it is useful to introduce
\[
\begin{align*}
d\hat{\eta}^B_t &= c_t(X_t)\eta^B_t dt + \mu^k(X_t)\eta^B_t \hat{dw}_t^k \\
d\hat{\mu}^B_t &= f_t(X_t)\eta^B_t dt + g^k_t(X_t)\eta^B_t \hat{dw}_t^k
\end{align*}
\]
and view \( (Z^1, \ldots, Z^{d+2}) := (X^1, X^2, \ldots, X^d, \eta^B, U^B) \) as a stochastic flow on \( \mathbb{R}^{d+2} \).

We can write, with the notation \( \bar{x} = (x,1,0) \),
\[
u_t(x) = \mathbb{E}[\hat{P}^{\hat{\psi}}((Z_{t,0}^1, \ldots, Z_{t,0}^d)(\bar{x})))Z_{t,0}^{d+1}(\bar{x})1_{\tau_{t,x} = 0} + Z_{t,\tau_{t,x},x}(\bar{x})]. \tag{11}
\]

We now invoke Theorem II.6.1. from \cite{Kun84}. It states that \( Z \) can be obtained as the inverse of the forward flow \( V = (V^1, \ldots, V^{d+2}) \), the coefficient of whose equation can be obtained from those of \( Z \). Substituting in the formula, we get that
\[
V^j(x^1, \ldots, x^{d+2}) = Y^j(x^1, \ldots, x^d)
\]
for \( j = 1, \ldots, d \) (in particular, \( \tau_{t,x} = \gamma_{t,x} \)), and the equations for the last two coordinates read as
\[
\begin{align*}
dV^{d+1}_t &= (-c_t + |\mu_t|^2 + \sigma^k t D_t \mu^k_t)(V^{d-1}_t - V^{d+1}_t) dt - \mu^k_t(V^{d-1}_t - V^{d+1}_t) dw^k_t, \\
dV^{d+2}_t &= (-d_t + \sigma^k D_t \mu^k_t)(V^{d-1}_t - V^{d+1}_t) dt - g^k_t(V^{d-1}_t - V^{d+1}_t) dw^k_t
\end{align*}
\]
where \( V^{d-} \) denotes the first \( d \) coordinates of \( V \). Let us also introduce the processes
\[
\begin{align*}
d\tilde{\eta}_t(x) &= (-c_t + |\mu_t|^2 + \sigma^k t D_t \mu^k_t)(Y_{0,t}(x))\tilde{\eta}_t(x) dt - \mu^k_t(Y_{0,t}(x))\tilde{\eta}_t(x) dw^k_t, \\
\tilde{\eta}_0(x) &= 1 \\
d\tilde{U}_t(x) &= (-d_t + \sigma^k D_t \mu^k_t)(Y_{0,t}(x))\tilde{\eta}_t(x) dt - g^k_t(Y_{0,t}(x))\tilde{\eta}_t(x) dw^k_t, \\
\tilde{U}_0(x) &= 0. \tag{12}
\end{align*}
\]
These processes look very similar to \( V^{d+1}, V^{d+2} \), and indeed the relations between the two notions can be expressed as
\[
V_{s,t}^{d+1}(y^1, \ldots, y^{d+2}) = y^{d+1} \tilde{\eta}_s^0(Y_{0,s}^{-1}(y^1, \ldots, y^d)),
\]
\[ V_{s,t}^{d+2}(y^1, \ldots, y^{d+2}) = y^{d+2} + \frac{y^{d+1}}{\eta_t(Y_{0,t}^{s}(y^1, \ldots, y^d))}(\tilde{U}_t - \tilde{U}_s)(Y_{0,s}^{-1}(y^1, \ldots, y^d)). \]

Also note that simple applications of Itô’s formula yield that \( \tilde{\eta}_t(x) = 1/\eta_t(x) \) and \( \tilde{\Omega}_t(x) = -U_t(x)/\eta_t(x) \). Hence we can also write
\[
\begin{align*}
V_{s,t}^{d+1}(y^1, \ldots, y^{d+2}) &= y^{d+1} + \frac{y^d}{\eta_t(Y_{0,t}^{s}(y^1, \ldots, y^d))}, \\
V_{s,t}^{d+2}(y^1, \ldots, y^{d+2}) &= y^{d+2} - y^d + \frac{y^{d-1}}{\eta_t(Y_{0,t}^{s}(y^1, \ldots, y^d))}(\tilde{U}_t - \tilde{U}_s)(Y_{0,s}^{-1}(y^1, \ldots, y^d)).
\end{align*}
\]

Now when we write down the inverse of \( V \) at the point \( \tilde{x} \), we can express the last two coordinates of the inverse in terms of \( \eta \) and \( U \):
\[
((V_{s,t}^{-1})^1, \ldots, (V_{s,t}^{-1})^d)(\tilde{x}) = ((V_{s,t}^{-1})^{-1})^1, \ldots, (V_{s,t}^{-1})^d)(x^1, \ldots, x^d)
\]
\[
(V_{s,t}^{-1})^{d+1}(\tilde{x}) = \frac{\eta_t}{\eta_t}(Y_{0,t}^{-1}(x^1, \ldots, x^d))
\]
\[
(V_{s,t}^{-1})^{d+2}(\tilde{x}) = (V_{s,t}^{-1})^{d+1}(\tilde{x})(\frac{\eta_t}{\eta_t}U_t - U_s)(Y_{0,t}^{-1}(x^1, \ldots, x^d))
\]
\[
= (U_t - \frac{\eta_t}{\eta_t}U_s)(Y_{0,t}^{-1}(x^1, \ldots, x^d)).
\]

Hence substituting \( V^{-1} \) in place of \( Z \) in (11), we recognize the right-hand-side as \( v_t(x) \), and thus get the claim, for deterministic data and coefficients.

**Step 2.** One can then easily extend the formula to the case when all the coefficients and data are of the form
\[
a = \sum_{i=1}^{n} a_i 1_{A_i}
\]
for some \( n \geq 1 \), deterministic smooth functions \( a_i \) of the spatial variable \( x \), and \( F_0 \)-measurable events \( A_i \). The set of functions of this form will be denoted by \( \mathcal{H}(F_0) \).

The next case to consider is when \( \psi \in \mathcal{H}(F_0) \) and all other data and coefficient are of the form
\[
a = \sum_{i=1}^{n} \tilde{a}_i 1_{[t_{i-1}, t_i)}
\]
for some \( n \geq 1 \), functions \( \tilde{a}_i \in \mathcal{H}(F_{t_{i-1}}) \), and times \( 0 = t_0 < t_1 < \cdots < t_n = T \). The set of functions of this form will be denoted by \( \mathcal{H} \). We demonstrate the argument for \( n = 2 \), the generalization of which is straightforward. For \( t \leq t_1 \) we are in the previous situation, so we need only consider a fixed \( t \in (t_1, T] \). The probability measure \( \hat{P} \) on \( \hat{\Omega} \) induces probability measures \( \hat{P}^{(1)} \) and \( \hat{P}^{(2)} \) on \( \hat{\Omega}^{(1)} = C([0, t_1], \mathbb{R}^d) \) and \( \hat{\Omega}^{(2)} = C([t_1, T], \mathbb{R}^d) \) by the mappings
\[
(\hat{w}_t)_{t \in [0,T]} \mapsto (\hat{w}_t^{(1)} := \hat{w}_t)_{t \in [0,t_1]}, \quad (\hat{w}_t)_{t \in [0,T]} \mapsto (\hat{w}_t^{(2)} := \hat{w}_t - w_{t_1})_{t \in [t_1,T]},
\]
under which \( \hat{w}_t^{(1)} \) and \( \hat{w}_t^{(2)} \) are Wiener processes. We shall also use the notations \( \gamma^{(i)}_{t,x}, \eta^{(i)}_t(x), \) and \( U_t^{(i)}(x) \) for \( i = 1, 2 \), that are defined similarly to \( \gamma_t(x), \eta_t(x), \) and \( U_t(x) \), but with ‘initial time’ \( t_{i-1} \) instead of \( 0 \), ‘terminal time’ \( t_i \) instead of \( T \).

By applying the formula in the already established cases, on one hand we get that \( u_{t_1} = v_{t_1} \) holds \( dx \otimes P \)-almost everywhere, in other words, for an event \( \Omega \) of
full probability and \( \omega \in \tilde{\Omega} \), \( u_{t_1} \) and \( v_{t_1} \) differ on a set \( R(\omega) \subset \mathbb{R}^d \) of measure 0. On the other hand we can write
\[
u_t(x) = \mathbb{E}^{\hat{P}(2)} \left((u_{t_1} \eta_t(2))(Y_{t_1,t}(x))1_{\gamma_{t,x} = t_1} + (U_t(2) - U_t(2) \cdot \eta_{t,2}(2))(Y_{t_1,t}(x))\right)
\]
dx \otimes P\text{-}almost everywhere. Clearly, \((u_{t_1} \eta_t(2))(Y_{t_1,t}(x)) \) and \((u_{t_1} \eta_t(2))(Y_{t_1,t}(x))\) only differ by a finite random field \( e(x) \) which may be nonzero only on
\[\{\omega, \hat{\omega}, x : \omega \notin \hat{\Omega} \text{ or } x \in Y_{t_1,t}(R(\omega))\}.
\]
Since \( \sup_x |\nabla Y_{t_1,t}(x)| < \infty \) almost surely, this set has measure 0, and therefore \( \mathbb{E}^{\hat{P}(2)} e(x) = 0, \) dx \otimes P\text{-}almost everywhere. Thus, we have
\[
u_t(x) = \mathbb{E}^{\hat{P}(2)} \left((u_{t_1} \eta_t(2))(Y_{t_1,t}(x))1_{\gamma_{t,x} = t_1} + (U_t(2) - U_t(2) \cdot \eta_{t,2}(2))(Y_{t_1,t}(x))\right)
\]
dx \otimes P\text{-}almost everywhere.

The concatenation mapping (that is, “gluing” \( \hat{\omega}^{(1)} \) and \( \hat{\omega}^{(2)} \) together) from \( \hat{\Omega}^{(1)} \times \hat{\Omega}^{(2)} \) to \( \hat{\Omega} \) maps the measure \( \hat{P}^{(1)} \times \hat{P}^{(2)} \) to \( \hat{P} \). Under this mapping
(i) The flow \( Y \) on \([0,t_0]\) driven by \( \hat{\omega}^{(1)} \) and the one on \([t_0,T]\) driven by \( \hat{\omega}^{(2)} \) also glue together to form a flow on \([0,T]\), driven by \( \hat{\omega} \),
(ii) On \( \{\gamma_{t,x} > t_1\} \), one has \( \gamma_{t,x} = \gamma_{t,x}^{(2)} \), while on \( \{\gamma_{t,x} = t_1\} \), one has \( \gamma_{t,x} = \gamma_{t_1,y}|_{y=Y_{t_1,t}(x)} \),
(iii) For \( t \geq t_1 \), one has \( \eta_t(y) = \eta_t^{(1)}(y) \eta_t^{(2)}(Y_{0,t_1}(y)) \),
(iv) For \( t \geq t_1 \), one has \( U_t(y) = U_t^{(2)}(Y_{0,t_1}(y)) + U_t^{(1)}(y) \frac{\eta_t(y)}{\eta_t^{(1)}(y)} \).

From these properties, along with of the flow identity \( Y_{r,t}(Y_{s,t}(x)) = Y_{r,t}(x) \), the following identities follow easily:
\[
U_t^{(2)}(Y_{t_1,t}(x)) = U_t(Y_{0,t}(x)) - U_t^{(1)}(Y_{0,t}(x)) \frac{\eta_t(Y_{0,t}(x))}{\eta_t^{(1)}(Y_{0,t}(x))},
\]
\[
U_t(Y_{0,t}(x)) = U_t^{(2)}(Y_{t_1,t}(x)) + U_t^{(1)}(Y_{0,t}(x)) \eta_t^{(2)}(Y_{t_1,t}(x)),
\]
\[
\eta_t(Y_{0,t}(x)) = \eta_t^{(1)}(Y_{0,t}(x)) \eta_t^{(2)}(Y_{t_1,t}(x)).
\]

Therefore, substituting in (14) the definition of \( v_{t_1} \), we can write
\[
u_t(x) = \mathbb{E}^{\hat{P}} \left(1_{\{\gamma_{t,x} > t_1\}} \left[(U_t^{(2)} - U_t^{(2)} \cdot \eta_{t,2}^{(2)})(Y_{t_1,t}(x))\right] + 1_{\{\gamma_{t,x} = t_1\}} \left[\left((\psi \eta_t^{(1)})(Y_{0,t_1}(\cdot))1_{\gamma_{t,x} = 0}\right) \eta_t^{(2)}(\cdot)\right](Y_{t_1,t}(x))\right)
\]
we can write

\[ + \{ (U_{t_1}^{(1)} - U_{t_1}^{(2)}) \frac{n_t^{(1)}}{\eta_{t_1}^{(1)}} (Y_{0,t_1}^{-1}(\cdot)) \eta_{t_1}^{(2)}(\cdot) \} (Y_{t_1}^{-1}(x)) \]

\[ + (U_t^{(2)} - U_t^{(2)}) \frac{n_t^{(2)}}{\eta_{t_1}^{(2)}} (Y_{t_1}^{-1}(x)) \]}

\[ = \mathbb{E}^{P} \left( \mathbf{1}_{\{ \gamma_{1,t,x}^{(2)} > t_1 \}} \left[ (U_t - U_{\gamma_{t,x}} \frac{n_t}{\eta_{\gamma_{t,x}}}y_{0,t}) (Y_{0,t}^{-1}(x)) \right] \right. \]

\[ + \mathbf{1}_{\{ \gamma_{1,t,x}^{(2)} = t_1 \}} \left[ (\psi \eta_t)(Y_{0,t}^{-1}(x)) \mathbf{1}_{\gamma_{t,x} = 0} \right] \]

\[ + U_t^{(1)} (Y_{0,t}^{-1}(x)) \eta_t^{(2)} (Y_{t_1}^{-1}(x)) - (U_t^{(1)} \frac{n_t}{\eta_{t_1}^{(1)}}) (Y_{0,t}^{-1}(x)) \]

\[ + U_t^{(2)} (Y_{t_1,t}^{-1}(x)) \]}

as claimed. The proof of the formula for data and coefficients from the class \( \mathcal{H} \) is hence finished.

**Step 3.** For the general case, take coefficients and data \( \rho^n, \sigma^n, \mu^n, b^n, \psi^n, f^n, \) and \( g^n \) of class \( \mathcal{H} \) such that they satisfy Assumptions 222234 \( |c - c^n|_{C^a} \to 0 \) in measure with respect to \( dt \otimes dP \),

\[ |\psi - \psi^n|_{C^a} + \int_0^T |f_t - f_t^n|^2_{C^a} + |g_t - g_t^n|^2_{C^{1+a}} dt \to 0 \]

in probability, and the remaining coefficients converge as in the condition of Lemma 3.1. The existence of such approximation is well-known and follows from standard arguments. From the previous parts we can write

\[ u_t^n(x) = \mathbb{E}^{P} \left( (\psi^n \eta^n_t) (Y_{0,t}^{n,-1}(x)) \mathbf{1}_{\gamma_{t,x}^{(n) = 0}} + (U_t^n - U_t^{n} \frac{n_t^n}{\eta_{t,x}^{n}} \eta_{t,x}^{n}) (Y_{0,t}^{n,-1}(x)) \right). \]}

\[ dx \otimes dP \)-almost everywhere for every \( n \), where \( \gamma^n, \eta^n, \) and \( U^n \) are defined analogously to 5, 6, and 7. The left-hand-side of (20) converges to \( u_t(x) \) almost surely for each \( (t,x) \in [0,T] \times D \), by the theory of SPDEs of domains, see Kry94, Kim04.
For the convergence of the right-hand-side, first note that by Proposition 3.3 we may replace it by
\[
\mathbb{E}^P 1_{(Y_{s,t}^{-1}(x))_{s\in[0,t]}\notin\mathcal{T}_D} \left(\psi^m \eta_{t,x}^n (Y_{s,t}^{-1}(x)) 1_{\gamma_{t,x}^n=0} + (U_t^n - \eta_{t,x}^n \eta_{t,x}^n) (Y_{s,t}^{-1}(x))\right).
\]
By Vitali’s convergence theorem it suffices to prove that for all \(t, dx \otimes dP\)-almost everywhere, the quantity under the sign \(\mathbb{E}^P\)
(i) converges \(\hat{P}\)-a.s.
(ii) is uniformly integrable in \(\hat{\omega}\).
Moreover, recalling also Proposition 3.3 instead of (ii) it actually suffices to prove that the family
\[
\sup_{(t,y)\in[0,T]\times D} |\psi^m \eta_{t,x}^n|(y) + \sup_{(s,t,y)\in[0,T]^2\times D} |U_{s,t}^n \eta_{s,t}^n|\end{equation}
\]is uniformly integrable in \((\omega, \hat{\omega})\). Since we have uniform (in \(n\)) bounds on the coefficients of the SDEs (6), (7), the uniform integrability follows from standard moment bounds, see e.g. [Kry80]
Concerning (i), from Lemma 3.1 we have that that the inverse flow trajectories \((Y_{s,t}^{-1}(x))_{s\in[0,t]}\) converge to \((Y_{s,t}^{-1}(x))_{s\in[0,t]}\) in the supremum norm. If furthermore \((Y_{s,t}^{-1}(x))_{s\in[0,t]} \notin \mathcal{T}_D\), then \(\gamma_{t,x}^n\) also converges to \(\gamma_{t,x}\) and \(1_{\gamma_{t,x}^n=0}\) to \(1_{\gamma_{t,x}=0}\). For the convergence of the other terms it suffices to see that \(\eta^n\), \(1/\eta^n\), and \(U^n\) converge along a subsequence uniformly in space and time, and hence when substituting in the space-time parameters convergent quantities, in our case \(Y_{0,t}^{-1}(x)\) and \(\gamma_{t,x}^n\), the resulting quantity also converges. The proof for the uniform convergence is virtually identical for \(\eta^n\), \(1/\eta^n\), and \(U^n\), so we only detail the first. Let \(p > 1/\alpha\), \(\Lambda_n\) denote a \(1/n^p\)-net of \([0,T] \times D\) and \(\Pi_n\) a function \([0,T] \times D \to \Lambda_n\) such that \(|\Pi_n(t,x) - (t,x)| \leq 1/n^p\) for all \((t,x)\in[0,T]\times D\). Since \(\eta_{t,x}^n(x)\) converges in \(L_1(\Omega)\) to \(\eta_{t,x}(x)\) for all \((t,x)\), we can find a subsequence \((k(n))_{n\in\mathbb{N}}\) such that
\[
\sum_{(t,x)\in\Lambda_n} \mathbb{E}|\eta_{k(n)}^n(x) - \eta_{t,x}(x)| \leq n^{-3}.
\]
Therefore, by Markov’s inequality
\[
\check{P}(A^n_m) := \check{P}(\max_{(t,x)\in\Lambda_n} |\eta_{k(n)}^n(x) - \eta_{t,x}(x)| \geq mn^{-1}) \leq n^{-2} m^{-1}.
\]
Also, we have \(\mathbb{E}(|\eta_{k(n)}^n|_{C^{\alpha/2}} + |\eta|_{C^{\alpha/2}})^2 \leq N\) for all \(n\), with some constant \(N = N(d,\alpha,K,T,D)\). Applying Markov’s inequality again, we have, for any \(\delta > 0\),
\[
\check{P}(B^n_m) := \check{P}(|\eta_{k(n)}^n|_{C^{\alpha/2}} + |\eta|_{C^{\alpha/2}} \geq mn^{1/2+\delta}) \leq N n^{-1-2\delta} m^{-2}.
\]
For each \(m\), we can therefore write on \(C^m := \cap_{n\in\mathbb{N}} (A^n_m)^c \cap (B^n_m)^c\), for all \(n \in \mathbb{N}\) and for any \((t,x)\),
\[
|\eta_{k(n)}^n(x) - \eta_{t,x}(x)| \leq |\eta_{k(n)}^n(\Pi_n(t,x)) - \eta(\Pi_n(t,x))| + (n^{-p})^{\alpha/2}(|\eta_{k(n)}^n|_{C^{\alpha/2}} + |\eta|_{C^{\alpha/2}})
\leq mn^{-1} + mn^{-p\alpha/2+1/2+\delta}.
\]
If $\delta < (p\alpha - 1)/2$, which we can achieve, then the right-hand side goes to 0, uniformly in $t$ and $x$. It remains to notice that $\bar{P}(C^m) \geq 1 - N'm^{-1}$ with some constant $N' = N'(N, \delta)$ and therefore the uniform convergence holds on the set $\cup_{m \in \mathbb{N}} C^m$ of full probability. In other words, the set of $\omega$-s where the uniform convergence holds $\bar{P}$-almost surely, has probability 1, which finishes the proof.

\[ \square \]

**Remark 4.1.** As it is seen from the proof, one could also write the formula in terms of $\tilde{\eta}$ and $\tilde{U}$, as defined in (12)-(13). In fact, from the inversion of the flows this would be somewhat more natural, but the formula as written is more consistent with the existing literature, e.g. [FS90], [Kry92], [LM15].

5. **Localization errors for artificial boundary conditions**

Let us turn to an application of the formula. In this section we consider equations on the whole space

\[
\begin{align*}
\frac{du}{dt} &= [Lu + f] dt + [M^k u + g^k] dw^k_t \quad \text{on } [0, T] \times \mathbb{R}^d, \quad u_0 = \psi.
\end{align*}
\]  

(21)

We are interested how close to $u$ is the solution of the truncated problem

\[
\begin{align*}
\begin{cases}
\frac{du^R}{dt} &= [Lu^R + f] dt + [M^k u^R + g^k] dw^k_t \\
u^R &= 0, \\
u^R_0 &= \psi.
\end{cases} \quad \text{on } [0, T] \times \partial B_R,
\end{align*}
\]  

(22)

The differential operators $L$ and $M$ have the same form as in (3), and while our assumptions are similar to Assumptions 2.1-2.3, due to some differences and for the convenience of the reader we state them separately.

**Assumption 5.1.** There exists a $\lambda > 0$ such that for all $t, \omega, x$,

\[
(p\rho^*)_t(x) \geq \lambda I
\]

in the sense of positive semidefinite matrices, where $I$ is the identity matrix.

**Assumption 5.2.** The coefficients $\rho, \sigma, b, c, \mu$ are predictable functions with values in $C^2(\mathbb{R}^{d \times d}), C^2([l_2]^d), C^1(\mathbb{R}^d)$, $C^1(\mathbb{R})$, and $C^2([l_2])$, respectively, bounded uniformly in $t$ and $\omega$ by a constant $K$.

**Assumption 5.3.** For some $p > d$, the initial value, $\psi$ is an $\mathcal{F}_0$-measurable random variable with values in $W^1_p$. The processes $f$ and $g$ are predictable with values in $W^1_p$ and $W^2_p$, respectively, such that

\[
\mathcal{K}_{1,p}(\psi, f, g) := |\psi|_{W^1_p} + \|f\|_{L^p([0,T], W^1_p)} + \|g\|_{L^p([0,T], W^2_p)} < \infty
\]

almost surely.

Introduce the shorthand $B_R = [0, T] \times B_R$. The result on localization of linear equations reads as follows.

**Theorem 5.1.** Let Assumptions 5.1-5.3 hold. Then for any $R > 1, q > 1, \varepsilon \in (0, 1)$, and $\nu \in (0, 1)$, one has

\[
\mathbb{E}\|u - u^R\|_{L^\infty(B_{R-\varepsilon R^\nu})} \leq N e^{-\delta R^2\varepsilon} \mathbb{E}^{1/q} \mathcal{K}_{1,p}(\psi, f, g),
\]  

(23)
where the constants $N, \delta > 0$ depend on $p, q, \varepsilon, \nu, \lambda, d, K, T$.

**Remark 5.1.** It should be noted that in the generality considered here, for the localized equation (22) there are not known approximating schemes with optimal rate, and hence it is likely preferable to use the localization of [GG16]. Therein, even though all data have compact support, the localized equation still can be considered on the whole space, and be approximated as such (see e.g. the full discretization scheme in [GG16]). One advantage of the method presented here is that coercivity is preserved, in fact, the equation itself does not change at all. Therefore, if a specific equation has efficient schemes on domains (which usually do strongly rely on coercivity), then this type of localization can be favourable.

**Remark 5.2.** We also note that while the extension of the above error estimate to nonlinear equation is not an easy task in this generality, Theorem 5.1 still can be a useful tool in nonlinear situations. For example, take some sufficiently nice functions $\bar{f}$ and $\bar{g}$ mapping from $\mathbb{R}$ to $\mathbb{R}$, let $u$ be the solution of (21) with $f$ and $g$ replaced by the semilinear terms $\bar{f}(u)$ and $\bar{g}(u)$, and similarly change the equation (22) for $u^R$. If one then defines $\tilde{u}^R$ as the solution of (22) with $f$ and $g$ replaced by $\bar{f}(u)$ and $\bar{g}(u)$, respectively, then Theorem 5.1 gives a bound for $u - \tilde{u}^R$. It then remains to estimate $\tilde{u}^R - u^R$, which is perhaps a challenging task in general, but under some additional assumptions on the operators $L$ and $M$ - which, as mentioned above, are necessary anyway to be able to approximate the localized problem - it may not be insurmountable. This direction is left for future work.

Before turning to the proof, let us recall some estimates from a Sobolev space theory of degenerate equations in [GGK14]: under Assumptions 5.2-5.3, one has for all $q \in (0, \infty)$,

$$
E\|u\|^q_{L_\infty([0,T],W^1_p)} \leq N \mathcal{E}\{(\psi,f,g)\},
$$

(24)

where $N$ depends only on $p, q, \lambda, d, K$. We also invoke a probability estimate for the flows from [GG16]. While in fact in [GG16], this is only proved for $\varepsilon = 1$, this slight generalization is straightforward.

**Lemma 5.2.** Let $Y$ be as in (4) and define the event

$$
H_R := \left[ \sup_{(t,x) \in B_{R-(\nu/2)R^\varepsilon}} |Y_{0,t}^{-1}(x)| > R - (\nu/2)R^\varepsilon \right] \cup \left[ \sup_{(t,x) \in B_{R-(\nu/2)R^\varepsilon}} |Y_{0,t}(x)| > R \right].
$$

Then

$$
P(H_R) \leq Ne^{-\delta R^\varepsilon},
$$

where $N$ and $\delta > 0$ depend only on $\lambda, d, K, T, \nu$, and $\varepsilon$.

**Proof of Theorem 5.1.** We may and will assume that the coefficients are smooth enough so that Assumption 2.2 is satisfied. Indeed, if the estimate is obtained for such smoothed coefficients, the passage to the limit is justified by [Kry99] (for $u$) and by [Kim04] (for $u^R$). The constant $N$ may change from line to line, but always has the dependence specified in the theorem.
Let \( Y \) be as above, \( \eta \) and \( U \) as in (3)-(7), and \( \gamma^R \) as in (3), with \( D = B_R \). By Theorem 2.2 and recalling the representation on the whole space by [Kry92] we have that
\[
\begin{align*}
U_t(x) &= \mathbb{E}^P \left( (\psi \eta_t)(Y_{0,t}^{-1}(x)) + U_t(Y_{0,t}^{-1}(x)) \right), \\
U_t^R(x) &= \mathbb{E}^P \left( (\psi \eta_t)(Y_{0,t}^{-1}(x)) \mathbf{1}_{\gamma_{s,t} = 0} + (U_t - U_{\gamma_{s,t}} \eta_{\gamma_{s,t}})(Y_{0,t}^{-1}(x)) \right).
\end{align*}
\]

Take a parameter \( \bar{p} \geq 1 \), with which we will eventually tend to infinity. Denote the quantities under the \( \mathbb{E}^P \) sign by \( \mathcal{U}_t(x) \) and \( \mathcal{U}_t^R(x) \), respectively, the norm in \( L_\infty([0,T], L_p(\partial_B(B_{R_{\bar{p}R}))) \) by \( \| \cdot \|_{(R)} \), and note that on the complement of \( H_R, \mathcal{U}_t(x) = \mathcal{U}_t^R(x) \) for all \((t,x) \in B_{R_{\bar{p}R}}\). By Minkowski and Hölder inequalities and Lemma 5.2,
\[
\mathbb{E} \| u - u^R \|_{(R)} = \mathbb{E} \| \mathbb{E}^P(\mathcal{U} - \mathcal{U}^R) \|_{(R)} \\
\leq \mathbb{E} \| \mathbb{E}^P \| \mathcal{U} - \mathcal{U}^R \|_{(R)} \\
= \mathbb{E} \| \mathcal{U} - \mathcal{U}^R \|_{(R)} \\
\leq (P(H_R))^{1/q} \mathbb{E}^{1/q} \| \mathcal{U} - \mathcal{U}^R \|_{(R)}^{q} \\
\leq N e^{-\delta R^2 \varepsilon} (\mathbb{E}^{1/q} \| \mathcal{U} \|_{(R)}^{q} + \mathbb{E}^{1/q} \| \mathcal{U}^R \|_{(R)}^{q}), \tag{25}
\]
where \( q \in (1, \infty) \) and \( q' = q/(q - 1) \). At this stage we can make use of the fact, see again [Kry92], that \( \mathcal{U} \) is in fact a solution of the fully degenerate SPDE
\[
\frac{d\mathcal{U}}{dt} = [\mathcal{U} + f] dt + [M^k \mathcal{U} + g^k] d\mathcal{W}_t^k + \rho^i \partial_i \mathcal{U} d\mathcal{W}_t^i \quad \text{on } [0,T] \times \mathbb{R}^d, \quad \mathcal{U}(0) = \psi. \tag{26}
\]
By elementary inequalities, Sobolev’s embedding, and (24), we have
\[
\mathbb{E} \| \mathcal{U} \|_{(R)}^{q} \leq N(2R)^{d/q} \mathbb{E} \| \mathcal{U} \|_{L_\infty(B_R)}^{q} \\
\leq N(2R)^{d/q} \mathbb{E} \| \mathcal{U} \|_{L_\infty([0,T] \times \mathbb{R}^d)}^{q} \\
\leq N(2R)^{d/q} \mathbb{E} \| \mathcal{U} \|_{L_\infty([0,T], W^1_p)}^{q} \\
\leq N(2R)^{d/q} \mathbb{E} \mathcal{K}_1(\psi, f, g)^q =: \mathcal{E}^q. \tag{27}
\]
As for \( \mathcal{U}^R \), let us write
\[
\mathbb{E} \| \mathcal{U}^R \|_{(R)}^{q} \leq 3^{q-1} (\mathbb{E} \| \mathcal{V}^1 \|_{(R)}^{q} + \mathbb{E} \| \mathcal{V}^2 \|_{(R)}^{q} + \mathbb{E} \| \mathcal{V}^3 \|_{(R)}^{q}),
\]
where
\[
\begin{align*}
\mathcal{V}^1_t(x) &= (\psi \eta_t)(Y_{0,t}^{-1}(x)), \\
\mathcal{V}^2_t(x) &= U_t(Y_{0,t}^{-1}(x)), \\
\mathcal{V}^3_t(x) &= U_{\gamma_{s,t}}(Y_{0,t}^{-1}(x)),
\end{align*}
\]
Applying again the representations on the whole space, we have that \( \mathcal{V}^1, \mathcal{V}^2, \) and \( \mathcal{V}^3 \) are solutions of equations of type (26), with the data \((\psi, f, g)\) replaced by \((\psi, 0, 0), (0, f, g), \) and \((1, 0, 0), \) respectively. Hence (24) yields estimates of type (27) for \( \mathcal{V}^1 \) and \( \mathcal{V}^2 \). One can also verify by direct calculation (see e.g. [GG14]) that the field \((t,x) \mapsto (\mathcal{V}^1_t(x))/(1 + |x|^2)\) is also a solution of an equation of type (26), with data
Fatou’s lemma. This yields (23), keeping in mind that $q < p$ and since the right-hand side doesn’t depend on $\overline{M}$. Gerencsér [GG16], Flandoli [FS90], and M. Gerencsér [GG14].

Together with (27) and (25) this yields, for $\overline{M}$, get

\begin{equation}
\mathbb{E}[|V^5(x)|] = |\eta_{\overline{M},x}^{-1}(Y_{\overline{M},x}(x))| \leq \sup_{(s,y) \in \mathcal{B}_R} |\eta_{s}^{-1}(Y_{0,s}(y))|.
\end{equation}

By Itô’s formula, $\eta^{-1}$ is the solution of an SDE of the same type as $\eta$, in fact its differential was already given in the proof of Theorem 2.2, see [12]. Hence, the field $(\eta_{s}^{-1}(Y_{0,s}(y)))_{s \in [0,T], y \in \mathbb{R}^d}$ is again a solution of an equation of type (26), with the data $(\psi, f, g)$ replaced by $(1, 0, 0)$, and the zero order coefficients $(c, \mu)$ replaced by $(-c + |\mu|^2, -\mu)$. Hence we can estimate its supremum norm as in (25), and we can write

\begin{equation}
\mathbb{E}[|U^R|_R\leq \mathcal{E} + N R^2 \mathbb{E}^{1/q_3} \|V^3\|_R^q.
\end{equation}

Finally, $V^3$ can be treated similarly:

\begin{equation}
|V^3(x)| = |U_{\gamma,t,x}(Y_{\gamma,t,x}^{-1}(x))| \leq \sup_{(s,y) \in \mathcal{B}_R} |U_s(Y_{0,s}(y))|,
\end{equation}

One can recognize the right-hand-side as $\|V^2\|_{\mathcal{B}_R}$, which is estimated as in (27). We can therefore conclude

\begin{equation}
\mathbb{E}^{1/q_3}|U^R|_R \leq \mathcal{E} + N (2R)^{4+d/\beta}\mathbb{E}^{1/q_3} \|V^2\|_R \leq N (2R)^{4+d/\beta} \mathbb{E}^{1/q_3} K_{m,p}^3(\psi, f, g).
\end{equation}

Together with (24) and (25) this yields, for $\overline{M}$, get

\begin{equation}
\mathbb{E}\left[ u - u^R \right]_{L_\infty([0,T], \mathcal{L}_p(\mathcal{B}_{R-\delta'}R^2))} \leq N e^{-\delta' R^2} R^{4+d/\beta} \mathbb{E}^{1/q_3} K_{m,p}^3(\psi, f, g),
\end{equation}

and since the right-hand side doesn’t depend on $\overline{M}$, we can take the limit $\overline{M} \to \infty$ by Fatou’s lemma. This yields [23], keeping in mind that $q \in (1, \infty)$ was arbitrary and that $R^{4+d} \leq N e^{\delta' R^2}$ for any $\delta' > 0$. 

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