MACNEILLE COMPLETION AND BUCHHOLZ’ OMEGA RULE FOR PARAMETER-FREE SECOND ORDER LOGICS

KAZUSHIGE TERUI

Research Institute for Mathematical Sciences, Kyoto University, Kitashirakawa-Oiwakecho, Sakyoku, Kyoto 606-8502, Japan.
e-mail address: terui@kurims.kyoto-u.ac.jp

ABSTRACT. Buchholz’ ω-rule is a way to give a syntactic, possibly ordinal-free proof of cut elimination for various subsystems of second order arithmetic. Our goal is to understand it from an algebraic point of view. Among many proofs of cut elimination for higher order logics, Maehara and Okada’s algebraic proofs are of particular interest, since the essence of their arguments can be algebraically described as the (Dedekind-)MacNeille completion together with Girard’s reducibility candidates. Interestingly, it turns out that the ω-rule, formulated as a rule of logical inference, finds its algebraic foundation in the MacNeille completion.

In this paper, we consider the parameter-free fragments \( \{ \text{LIP}_n \}_{n<\omega} \) of the second order intuitionistic logic, that correspond to the arithmetical theories \( \{ \text{ID}_n \}_{n<\omega} \) of iterated inductive definitions up to \( \omega \). In this setting, we observe a formal connection between the ω-rule and the MacNeille completion, that leads to a way of interpreting second order quantifiers in a first order way in Heyting-valued semantics, called the ω-interpretation. Based on this, we give an algebraic proof of cut elimination for \( \text{LIP}_n \) for every \( n < \omega \) that can be locally formalized in \( \text{ID}_n \). As a consequence, we obtain an equivalence between the cut elimination for \( \text{LIP}_n \) and the 1-consistency of \( \text{ID}_n \) that holds in a weak theory of arithmetic.

INTRODUCTION

This paper is concerned with cut elimination for subsystems of second order logics. It is of course very well known that the full second order classical/intuitionistic logics admit cut elimination. Then why are we interested in their subsystems? A primary reason is that proving cut elimination for a subsystem is often very hard if one is sensitive to the metatheory within which (s)he works. This is witnessed by the vast literature in the traditional proof theory. In fact, proof theorists are not just interested in proving cut elimination itself, but in identifying a characteristic principle \( P \) (e.g. ordinals, combinatorial principles and inductive definitions) for each system of logic, arithmetic and set theory, by proving cut

Key words and phrases: Algebraic cut elimination, Parameter-free second order logic, MacNeille completion, Omega-rule.

Supported by KAKENHI 25330013.
elimination within a weak metatheory (e.g. \( \text{PRA}, \text{I} \Sigma_1 \) or \( \text{RCA}_0 \)) extended by \( P \). Our long-span goal is to understand those hard proofs and results from an algebraic perspective.

**Various proofs of cut elimination.** One can distinguish several types of cut elimination proofs for higher order logics/arithmetic: (i) syntactic proofs by ordinal assignment (e.g. Gentzen’s consistency proof for \( \text{PA} \)), (ii) syntactic but ordinal-free proofs, (iii) semantic proofs based on Schütte’s semivaluation or its variants (e.g. \[31\]), (iv) algebraic proofs based on completions (the list is not intended to be exhaustive). Historically (i) and (iii) precede (ii) and (iv), but (i) is not easy to follow up due to the heavy use of ordinal notations, while (iii) is not completely satisfactory for computer scientists since it involves *reductio ad absurdum* and weak König’s lemma, that would destruct the proof structure: the output cut-free proof may have nothing to do with the input proof. Hence we address (ii) and (iv) in this paper.

For (ii), a very useful and versatile technique is Buchholz’ \( \Omega \)-rule. Although introduced in the context of ordinal analysis \[11\], the technique itself can be understood without recourse to any ordinals \[12, 3, 3\]. Still, the \( \Omega \)-rule is notoriously complicated and is hard to grasp its meaning at a glance. Even its semantic soundness is not clear at all. While Buchholz gives an account based on the BHK interpretation \[11\], we will try to give an algebraic account in this paper.

For (iv), there is a very conspicuous algebraic proof of cut elimination for higher order logics which may be primarily ascribed to Maehara \[24\] and Okada \[26, 28\]. In contrast to (iii), these algebraic proofs are fully constructive; no use of reductio ad absurdum or any nondeterministic principle. More importantly, it extends to proofs of normalization for proof nets and typed lambda calculi \[27\]. While their arguments can be described in various dialects (e.g. phase semantics of linear logic), apparently most neutral and most widely accepted would be to speak in terms of algebraic completions: the essence of their arguments can be described as the *(Dedekind-)*MacNeille completion together with Girard’s reducibility candidates, as we will explain in Appendix \[C\].

**Contents of this paper.** Having a syntactic technique on one hand and an algebraic methodology on the other, it is natural to ask the relationship between them. To make things concrete, we consider the *parameter-free* fragments \{\( \text{LIP}_n \)\}_{n<\omega} of the standard sequent calculus \( \text{LI}_2 \) for the second order intuitionistic logic. These fragments altogether constitite an intuitionistic counterpart of the classical sequent calculus studied in \[34\]. Although we primarily work on the intuitionistic basis, all results in this paper (except Proposition \[4.4\]) carry over to the classical logic too.

As we will see, cut elimination based on the \( \Omega \)-rule technique works for \( \text{LIP}_n \) for every \( n < \omega \). Moreover, it turns out to be intimately related to the MacNeille completion in that the \( \Omega \)-rule in our setting is not sound in Heyting-valued semantics in general, but is sound when the underlying algebra is the MacNeille completion of the Lindenbaum algebra. This observation leads to a curious way of interpreting second order formulas in a first order way, that we call the \( \Omega \)-interpretation. The basic idea already appears in Altenkirch and Coquand \[6\], but ours is better founded and accommodates the existential quantifier too.

The \( \Omega \)-interpretation in conjunction with the MacNeille completion gives rise to an algebraic proof of (partial) cut elimination for \( \text{LIP}_n \), that is comparable with Aehlig’s result \[1\] for the parameter-free, negative fragments of second order Heyting arithmetic. The \( \Omega \)-interpretation is essentially first order. In particular, it does not employ the reducibility
candidates. Hence it is “locally” formalizable in (the intuitionistic version of) $\text{ID}_n$. This leads to a correspondence between $\text{ID}$-theories in arithmetic and parameter-free logics, that we call the Takeuti correspondence: the cut elimination for $\text{LIP}_n$ is equivalent to the $1$-consistency of $\text{ID}_n$.

Outline. The rest of this paper is organized as follows. In Section 1 we recall some basics of the MacNeille completion. In Section 2 we review the theories of iterated inductive definitions up to $\omega$, introduce the parameter-free systems $\text{LIP}_n$ ($n < \omega$), and prove one direction of the Takeuti correspondence between $\text{LIP}_n$ and $\text{ID}_n$. In Section 3 we turn to the algebraic side and establish a connection between the $\Omega$-rule and the MacNeille completion, that leads to the concept of $\Omega$-interpretation. In Section 5 we sketch a (local) formalization of our algebraic argument, that establishes the Takeuti correspondence between $\text{LIP}_n$ and $\text{ID}_n$ for every $n < \omega$.

Remark 0.1. One often distinguishes cut elimination from cut admissibility (or cut eliminability). While the former gives a concrete procedure, the latter only ensures the existence of a cut-free derivation. Although our algebraic argument will only establish cut admissibility, we prefer to use the word cut elimination. A reason is that the statement of cut admissibility is $\Pi^0_2$, so a concrete procedure can be extracted from its proof (especially noting that our proof is fully constructive). Of course, this does not ensure that the procedure respects proof equivalence in any sense. Hence we do not make any formal claim on this point.

1. MacNeille completion

Let $A = \langle A, \land, \lor \rangle$ be a lattice. A completion of $A$ is an embedding $e : A \to B$ into a complete lattice $B = \langle B, \land, \lor \rangle$. We often assume that $e$ is an inclusion map so that $A$ is a subalgebra of $B$ (notation: $A \subseteq B$).

Here are two examples.

- Let $[0, 1]_Q := [0, 1] \cap \mathbb{Q}$ be the chain of rational numbers in the unit interval (seen as a lattice). Then it admits an obvious completion $[0, 1]_Q \subseteq [0, 1]$.

- Let $A$ be a Boolean algebra. Then it also admits a completion $e : A \to A^\sigma$, where $A^\sigma := \langle \wp(\text{uf}(A)), \cap, \cup, -, A, \emptyset \rangle$, the powerset algebra on the set of ultrafilters of $A$, and $e(a) := \{ u \in \text{uf}(A) : a \in u \}$.

A completion $A \subseteq B$ is $\lor$-dense if $x = \lor\{ a \in A : a \leq x \}$ holds for every $x \in B$. It is $\land$-dense if $x = \land\{ a \in A : x \leq a \}$. A $\lor$-dense and $\land$-dense completion is called a MacNeille completion (or a Dedekind-MacNeille completion). This means that any $B$-element can be approximated from above and below by $A$-elements. The following is a classical result [8, 30].

Theorem 1.1. Every lattice $A$ has a MacNeille completion unique up to isomorphism. Any MacNeille completion is regular, that is, preserves all joins and meets that already exist in $A$.

Coming back to the previous examples:
• $[0,1]_Q \subseteq [0,1]$ is MacNeille, since $x = \inf \{a \in Q : x \leq a\} = \sup \{a \in Q : a \leq x\}$ for any $x \in [0,1]$. It is regular since if $q = \lim_{n \to \infty} q_n$ holds in $Q$, then it holds in $\mathbb{R}$ too.

• $e : A \rightarrow A^\sigma$ is not regular when $A$ is an infinite Boolean algebra. In fact, the Stone space $uf(A)$ is compact, so collapses any infinite covering of a closed set into a finite one. It is actually a canonical extension, that has been extensively studied in ordered algebra and modal logic \cite{23, 20, 19}.

MacNeille completions behave better than canonical extensions in the preservation of existing limits, but the price to pay is the loss of generality. Let $DL (\mathcal{HA}, \mathcal{BA}, \text{resp.})$ be the variety of distributive lattices (Heyting algebras, Boolean algebras, resp.).

**Theorem 1.2.**

• $DL$ is not closed under MacNeille completions \cite{17}.

• $\mathcal{HA}$ and $\mathcal{BA}$ are closed under MacNeille completions.

• $\mathcal{BA}$ is the only nontrivial proper subvariety of $\mathcal{HA}$ closed under MacNeille completions \cite{9}.

As is well known, completion is a standard algebraic way to prove conservativity of extending first order logics to higher order ones. The above result indicates that MacNeille completions work for classical and intuitionistic logics, but not for proper intermediate logics. On the other hand, one finds many varieties closed under MacNeille completions when one moves to the realm of substructural logics \cite{15}. See \cite{35} for a comprehensive account on the MacNeille completions.

Now an easy but crucial observation follows.

**Proposition 1.3.** A completion $A \subseteq B$ is MacNeille iff the rules below are valid:

$$
\begin{align*}
\{a \leq y\}_{a \leq x} & \quad \text{where } x, y \text{ range over } B \text{ and } a \text{ over } A.
\end{align*}
$$

The left rule has infinitely many premises indexed by the set $\{a \in A : a \leq x\}$. It states that if $a \leq x$ implies $a \leq y$ for every $a \in A$, then we may conduce $x \leq y$. This is valid just in case $x = \bigvee \{a \in A : a \leq x\}$. Likewise, the right rule states that if $y \leq a$ implies $x \leq a$ for every $a \in A$, then $x \leq y$. This is valid just in case $y = \bigwedge \{a \in A : y \leq a\}$.

As we will see, the above looks very similar to the $\Omega$-rule. This provides a link between lattice theory and proof theory.

2. **Takeuti correspondence between logic and arithmetic**

There is a tight correspondence between systems of higher order logics and those of arithmetic. A well-known example in computer science is that the numerical functions representable in System F coincide with the provably total functions of the second order Peano arithmetic $\text{PA}_2$. In this paper, we rather focus on another type of correspondence, which we call the Takeuti correspondence, that concerns with cut elimination in logic and 1-consistency in arithmetic. One of our goals is to provide an easily accessible proof to the correspondence between the arithmetical theories of iterated inductive definitions (up to $\omega$) and the parameter-free fragments of $\text{LI}_2$.

We first give some background on arithmetic and second order logics (in Subsections 2.1 and 2.2), then introduce the parameter-free systems and examine their expressive power (in Subsections 2.3 and 2.5).
2.1. Arithmetic. Let $\Sigma_1^1$, $\text{PA}$ and $\text{PA2}$ be respectively the first order arithmetic with $\Sigma_1^0$ induction, that with full induction, and the second order arithmetic with full induction and comprehension (also called $\mathbb{Z}_2$). Given a theory $T$ of arithmetic, $T[X]$ denotes the extension of $T$ with a fresh set variable $X$ and atomic formulas of the form $X(t)$. An expression of the form $\lambda x.\varphi(x)$ with $\varphi$ a formula and $x$ a variable is called an abstract. Given an abstract $\tau = \lambda x.\varphi(x)$ and a term $t$, $\tau(t)$ stands for the formula $\varphi(t)$.

A great many subsystems of $\text{PA2}$ are considered in the literature. For instance, the system $\Pi^1_1$-$\text{CA}_0$ is obtained by restricting the induction and comprehension axiom schemata to $\Pi^1_1$ formulas. Even weaker are the theories of iterated inductive definitions $\text{ID}_n$ with $n < \omega$, that are obtained as follows.

$\text{ID}_0$ is just $\text{PA}$. To obtain $\text{ID}_{n+1}$, consider a formula $\varphi(X,x)$ in $\text{ID}_n[X]$ which contains no free variables other than $X$ and $x$, and no negative occurrences of $X$. It defines a monotone map $\varphi^N : \varphi(N) \rightarrow \varphi(N)$ sending a set $X \subseteq \mathbb{N}$ to $\{n \in \mathbb{N} : \mathbb{N} \models \varphi(X,n)\}$. By the Knaster-Tarski fixed point theorem, $\varphi^N$ has a least fixed point $I^N_{\varphi}$. Hence it is reasonable to add a unary predicate symbol $I_\varphi$ for each such $\varphi$ to the language of $\text{ID}_n$ and axioms

$$(lp_1) \quad \varphi(I_\varphi) \subseteq I_\varphi,$$

$$(lp_2) \quad \varphi(\tau) \subseteq \tau \rightarrow I_\varphi \subseteq \tau,$$

for every abstract $\tau = \lambda x.\xi(x)$ in the new language. Here $\varphi(I_\varphi)$ is a shorthand for the abstract $\lambda x.\varphi(I_\varphi, x)$ and $\tau_1 \subseteq \tau_2$ is for $\forall x.\tau_1(x) \rightarrow \tau_2(x)$. The induction schema is extended to the new language. This defines the system $\text{ID}_{n+1}$. Notice that $\text{ID}_{n+1}$ does not involve any set variable. It is purely a first order theory of arithmetic. Finally, let $\text{ID}_{<\omega}$ be the union of all $\text{ID}_n$ with $n < \omega$.

Clearly $\text{ID}_{<\omega}$ can be seen as a subsystem of $\Pi^1_1$-$\text{CA}_0$. In fact, any fixed point atom $I_\varphi(t)$ can be replaced by its second order definition

$$I^*_\varphi(t) := \forall X \forall x(\varphi(X,x) \rightarrow X(x)) \rightarrow X(t).$$

This makes the axioms of $\text{ID}_{<\omega}$ all provable in $\Pi^1_1$-$\text{CA}_0$.

The converse is not strictly true, but it is known that $\text{ID}_{<\omega}$ has the same proof-theoretic strength and the same arithmetical consequences with $\Pi^1_1$-$\text{CA}_0$ (see [13]).

Let us point out that typical use of an inductive definition is to define a provability predicate. Let $T$ be a sequent calculus system, and suppose that we are given a formula $\varphi(X,x)$ saying that there is a rule in $T$ with conclusion sequent $x$ (coded by a natural number) and premises $Y \subseteq X$. Then $I^N_{\varphi}$ gives the set of all provable sequents in $T$. Notice that the premise set $Y$ can be infinite. It is for this reason that $\text{ID}$-theories are suitable metatheories for infinitary proof systems. See [13] for more on inductive definitions.

Finally, let $\text{HA}_2$ be the second order Heyting arithmetic, and $\text{ID}_n^i$ the intuitionistic counterpart of $\text{ID}_n$ obtained by changing the underlying logic to the first order intuitionistic logic. Thus $\text{ID}_0^i = \text{HA}$ (the first order Heyting arithmetic). The following result is well known (see [13] for the second statement).

**Theorem 2.1.** $\text{PA2}$ and $\text{HA}_2$ prove exactly the same $\Sigma_2^1$ sentences. Hence the 1-consistency of $\text{PA2}$ is equivalent to that of $\text{HA}_2$ (provably in $\Sigma_1^2$). The same holds for $\text{ID}_n$ and $\text{ID}_n^i$ for every $n < \omega$.

Notice that the statement of 1-consistency (any provable $\Sigma_1^0$ sentence is true) and that of cut elimination are both $\Pi_2^1$. Hence it does not matter much for our purpose whether the logic is classical or intuitionistic.
2.2. Second order intuitionistic logic. In this subsection, we formally introduce sequent calculus $\text{LI}_2$ for the second order intuitionistic logic with full comprehension, that is an intuitionistic counterpart of Takeuti’s classical calculus $G^1\text{LC}$ for the second order classical logic [32].

Consider a language $L$ that consists of (first order) function symbols and predicate symbols. A typical example is the language $L_{\text{PA}}$ of Peano arithmetic, which contains a predicate symbol $=$, constant 0, successor $s$ and function symbols for all primitive recursive functions. Let

- $\text{VAR}$: a countable set of term variables $x, y, z, \ldots$
- $\text{VAR}$: a countable set of set variables $X, Y, Z, \ldots$
- $\text{Tm}(L)$: the set of first order terms $t, u, v, \ldots$ over $L$.

The set $\text{FM}(L)$ of second order formulas is defined by:

$$\varphi, \psi ::= p(\overline{t}) \mid X(t) \mid \bot \mid \varphi \ast \psi \mid Qx.\varphi \mid QX.\varphi,$$

where $\overline{t}$ is a list $t_1, \ldots, t_n$ of terms over $L$, $p$ is an $n$-ary predicate symbol in $L$, $\ast \in \{\land, \lor, \to\}$ and $Q \in \{\forall, \exists\}$. We define $\top ::= \bot \to \bot$. When the language $L$ is irrelevant, we write $\text{Tm} := \text{Tm}(L)$ and $\text{FM} := \text{FM}(L)$. Given $\varphi$, let $\text{FV}(\varphi)$ and $\text{Fv}(\varphi)$ be the set of free set variables and that of free term variables in $\varphi$, respectively.

We assume the standard variable convention that $\alpha$-equivalent formulas are syntactically identical, so that substitutions can be applied without variable clash. A term substitution is a function $\circ : \text{VAR} \to \text{Tm}$. Given $\varphi \in \text{FM}$, the substitution instance $\varphi^\circ$ is defined as usual. Likewise, a set substitution is a function $\bullet : \text{VAR} \to \text{ABS}$, where $\text{ABS} := \{\lambda x.\varphi : \varphi \in \text{FM}\}$. Instance $\varphi^\bullet$ is obtained by replacing each atomic formula $X(t)$ with $X^\bullet(t)$.

A sequent of $\text{LI}_2$ is of the form $\Gamma \Rightarrow \Pi$, where $\Gamma$ is a finite set of $\text{LI}_2$-formulas and and $\Pi$ is either the empty set or a singleton of an $\text{LI}_2$-formula. We write $\Gamma, \Delta$ to denote $\Gamma \cup \Delta$. The inference rules of $\text{LI}_2$ are given in Figure [1]. We write $\text{LI}_2 \vdash \Gamma \Rightarrow \Pi$ (resp. $\text{LI}_2 \vdash \text{cf} \Gamma \Rightarrow \Pi$) if the sequent $\Gamma \Rightarrow \Pi$ is provable (resp. cut-free provable) in $\text{LI}_2$.

In the sequel, we will build parameter-free logical systems upon the first order fragment of $\text{LI}_2$. Let $\text{Fm} \subseteq \text{FM}$ be the set of formulas without second order quantifiers. The ordinary sequent calculus $\text{LI}$ for the first order intuitionistic logic can be obtained from $\text{LI}_2$ by restricting the formulas to $\text{Fm}$ and by removing the rules for the second order quantifiers.

It is well-known that the cut elimination theorem for $G^1\text{LC}$ or $\text{LI}_2$ implies the consistency of $\text{PA}_2$ or $\text{HA}_2$ finitistically (or in $\text{IΣ}_1$, formally speaking). We also have the converse, when consistency is replaced with $1$-consistency meaning that all provable $\Sigma^0_1$ sentences are true (also called $\Sigma^0_1$-soundness).

**Theorem 2.2.** Let $\text{CE}(G^1\text{LC})$ be a $\Pi^0_2$ sentence stating that $G^1\text{LC}$ admits cut elimination, and $1\text{CON}(\text{PA}_2)$ a $\Pi^0_2$ sentence stating that $\text{PA}_2$ is 1-consistent. Then:

$$\text{IΣ}_1 \vdash \text{CE}(G^1\text{LC}) \leftrightarrow 1\text{CON}(\text{PA}_2).$$

Actually the above theorem holds even if cut elimination is replaced with partial cut elimination saying that any sequent $\Gamma \Rightarrow \Pi$ provable in $\text{LIP}_n$ has a cut-free derivation provided that $\Gamma \cup \Pi \subseteq \text{Fm}$.

An even stronger result holds too, as pointed out by [7] on the basis of Päppinghous’ theorem [29]: complete cut elimination is equivalent to partial cut elimination in the above sense. Let $\text{CE}_{\text{Fm}}(G^1\text{LC})$ be a $\Pi^0_2$ sentence that expresses the statement of partial cut elimination for $G^1\text{LC}$.
\[
\begin{array}{ll}
\Gamma, \varphi \Rightarrow \varphi & (\text{id}) \\
\top, \Gamma \Rightarrow \Pi & (\bot \text{ left}) \\
\varphi_1, \Gamma \Rightarrow \Pi & (\land \text{ left}) \\
\varphi_1 \land \varphi_2, \Gamma \Rightarrow \Pi & (\lor \text{ left}) \\
\varphi_1, \Gamma \Rightarrow \Pi & (\rightarrow \text{ left}) \\
\varphi(t), \Gamma \Rightarrow \Pi & (\forall x \text{ left}) \\
\varphi(x), \Gamma \Rightarrow \Pi & (\exists x \text{ left}) \\
\varphi(\tau), \Gamma \Rightarrow \Pi & (\forall X \text{ left}) \\
\varphi(Y), \Gamma \Rightarrow \Pi & (\exists X \text{ left}) \\
\Gamma \Rightarrow \varphi, \varphi, \Gamma \Rightarrow \Pi & (\text{cut}) \\
\Gamma \Rightarrow \top & (\bot \text{ right}) \\
\Gamma \Rightarrow \varphi_1 \land \varphi_2 & (\land \text{ right}) \\
\Gamma \Rightarrow \varphi_1 \rightarrow \varphi_2 & (\rightarrow \text{ right}) \\
\Gamma \Rightarrow \varphi(y) & (\forall x \text{ right}) \\
\Gamma \Rightarrow \exists x \varphi(x) & (\exists x \text{ right}) \\
\Gamma \Rightarrow \forall X \varphi(X) & (\forall X \text{ right}) \\
\Gamma \Rightarrow \exists X \varphi(X) & (\exists X \text{ right})
\end{array}
\]

Figure 1: Inference rules of LI2

**Theorem 2.3.** $\text{IS}_{1} \vdash \text{CE}(G^{1}\text{LC}) \Leftrightarrow \text{CE}_{\text{FM}}(G^{1}\text{LC})$.

**Remark 2.4.** The forward implication of Theorem 2.3 is due to Takeuti [33], while the backward one is a folklore (see [21] and [7]).

The same holds when PA2 is replaced by HA2 (because of Theorem 2.1), and/or $G^{1}\text{LC}$ is replaced by LI2 (because they admit essentially the same proof of cut elimination).

The paper [7] also mentions the following correspondence:

\[
\text{IS}_{1} \vdash \text{CE}(G^{1}\text{LC}(\Pi_{n}^{1})) \Leftrightarrow 1\text{CON}(\Pi_{n}^{1}\text{-CA}_{0})
\]

for every $n < \omega$, where $G^{1}\text{LC}(\Pi_{n}^{1})$ is the fragment of $G^{1}\text{LC}$ obtained by restricting the abstract $\tau$ in rules ($\forall X \text{ left}$) and ($\exists X \text{ right}$) to $\Pi_{n}^{1}$ abstracts.

This sort of correspondence between 1-consistency in arithmetic and cut elimination in logic may be called the Takeuti correspondence. A goal of this paper is to provide Takeuti correspondences between the theories HA, ID_{1}, ID_{2},... of inductive definitions and certain parameter-free fragments of LI2 that are to be introduced next.
Lemma 2.5. If $\phi$ has a free occurrence of $X$, then $\phi \not\in FMP$. By induction on the structure of $\phi$. 

Proof. We let $\phi$ not have a free occurrence of $X$. Consider the set 

$$ FMP = \{ \forall \star \mid LIP \phi \} $$

where $\star \in \{\wedge, \lor, \rightarrow\}$, $Q \in \{\forall, \exists\}$ and $\xi$ is any formula in $FMP_{n-1}$ such that $\text{FV}(\xi) \subseteq \{X\}$. We let 

$$ \text{ABS}_n := \{ \lambda x. \phi : \phi \in FMP_n \}.$$ 

An important property is the closure under substitution:

**Lemma 2.5.** If $\phi(X) \in FMP_n$ and $\tau \in \text{ABS}_n$, then $\phi(\tau) \in FMP_n$.

**Proof.** By induction on the structure of $\phi(X)$. If it is an atom $X(t)$, then $\phi(\tau) = \tau(t) \in FMP_n$. The induction steps for first order connectives are easy. If $\phi = QX.\xi$, then it does not have a free occurrence of $X$, so $\phi(\tau) = \phi \in FMP_n$. 

Sequent calculus $LIP_n$ for the parameter-free second order intuitionistic logic at level $n$ is obtained from $LIP_0$, $LIP_1$, $LIP_2, \ldots$ of $LIP_2$. They are parameter-free because any formula of the form $\forall X.\xi$ or $\exists X.\xi$ is second order closed, meaning that it does not contain any set parameter.

First, we write $FMP_{-1}$ for $Fm$ (the first order formulas) for convenience. For each $n \geq 0$, the set $FMP_n$ of parameter-free second order intuitionistic formulas at level $n$ is defined by:

$$ \phi, \psi ::= p(\bar{t}) \mid X(t) \mid \bot \mid \phi \star \psi \mid QX.\phi \mid QX.\xi,$$

where $\star \in \{\wedge, \lor, \rightarrow\}$, $Q \in \{\forall, \exists\}$ and $\xi$ is any formula in $FMP_{n-1}$ such that $\text{FV}(\xi) \subseteq \{X\}$. We let 

$$ \text{ABS}_n := \{ \lambda x. \phi : \phi \in FMP_n \}.$$ 

An important property is the closure under substitution:

**Lemma 2.5.** If $\phi(X) \in FMP_n$ and $\tau \in \text{ABS}_n$, then $\phi(\tau) \in FMP_n$.

**Proof.** By induction on the structure of $\phi(X)$. If it is an atom $X(t)$, then $\phi(\tau) = \tau(t) \in FMP_n$. The induction steps for first order connectives are easy. If $\phi = QX.\xi$, then it does not have a free occurrence of $X$, so $\phi(\tau) = \phi \in FMP_n$. 

Sequent calculus $LIP_n$ for the parameter-free second order intuitionistic logic at level $n$ is obtained from $LIP_0$, $LIP_1$, $LIP_2, \ldots$ of $LIP_2$. Most importantly, when one applies rules ($\forall X$ left) and ($\exists X$ right), both the main formula $QX.\phi$ and the minor formula $\phi(\tau)$ must belong to $FMP_n$.

Let $FMP_{<\omega}$ be the union of all $FMP_n$ and $LIP_{<\omega}$ the sequent calculus associated to it.

**Remark 2.6.** The idea of restricting to the parameter-free formulas dates back to [34], which introduces a similar condition called isolatedness. It also appears in more recent papers, such as [12, 6, 1].

A typical formula in $FMP_0$ is 

$$ N(t) := \forall X. \text{Sub}(X) \land \text{Suc}(X) \land X(0) \rightarrow X(t), $$

where $\text{Sub}(X) := \forall xy. x = y \land X(x) \rightarrow X(y)$ and $\text{Suc}(X) := \forall x. X(x) \rightarrow X(s(x))$. Given a formula $\phi$, let $\phi^N$ be the formula obtained by replacing all first order quantifiers $Qx$ with $Qx \in N$. That is, we replace $\forall x. \phi$ with $\forall x. N(x) \rightarrow \phi$, and $\exists x. \phi$ with $\exists x. N(x) \land \phi$. It is clear that if $\phi$ is a first order formula, then $\phi^N$ belongs to $FMP_0$.

On the other hand, the standard second order definitions of positive connectives $\{\exists, \lor\}$: 

$$ \exists X.\phi(X) := \forall Y. \forall X. (\phi(X) \rightarrow Y(\ast)) \rightarrow Y(\ast), $$

$$ \phi \lor \psi ::= \forall Y. (\phi \rightarrow Y(\ast)) \land (\psi \rightarrow Y(\ast)) \rightarrow Y(\ast), $$

with $Y \not\in \text{FV}(\phi, \psi)$ and $\ast$ a constant, are no longer available. They are not parameter-free (unless $\phi$ and $\psi$ are free of set variables in the latter formula). Hence restricting to the negative fragment $\{\forall, \land, \rightarrow\}$ causes a serious loss of expressivity in our parameter-free setting.
2.4. Expressivity of parameter-free logics. Let us now briefly examine the expressivity of \( \text{LIP}_0 \). In the following, we consider terms and formulas over the language \( L_{\text{PA}} \).

It is not hard to see that \( \text{LIP}_0 \) proves

\[
N(0), \quad \text{Suc}(N), \quad \text{Sub}(N), \quad \Gamma_{eq} \Rightarrow \text{Sub}(\tau),
\]

where \( \tau \) is any abstract of the form \( \lambda x.\varphi^N(x) \) with \( \varphi \in \text{Fm} \) and \( \Gamma_{eq} \) consists of some equality axioms for predicate and function symbols. Moreover, the principle of mathematical induction is also available in \( \text{LIP}_0 \).

Lemma 2.7. \( \text{LIP}_0 \) proves

\[
\Gamma_{eq} \Rightarrow [\forall x.(\varphi(x) \rightarrow \varphi(s(x))) \land \varphi(0) \rightarrow \forall y.\varphi(y)]^N
\]

for every formula \( \varphi \) in \( \text{Fm} \), where \( \Gamma_{eq} \) is a set of some equality axioms.

Proof. Let \( \tau := \lambda x.\varphi^N(x) \land N(x) \). We claim that \( \text{LIP}_0 \) proves

(1) \( \Gamma_{eq}, [\forall x.(\varphi(x) \rightarrow \varphi(s(x)))]^N, \varphi^N(0), \text{Sub}(\tau) \land \text{Suc}(\tau) \land \tau(0) \rightarrow \tau(y) \Rightarrow \varphi^N(y) \).

First, \( \Gamma_{eq} \Rightarrow \text{Sub}(\tau) \) follows from \( \text{Sub}(N) \) and \( \Gamma_{eq} \Rightarrow \text{Sub}(\lambda x.\varphi^N(x)) \), which are both provable. Moreover, we can easily prove

\[
\varphi^N(0) \Rightarrow \varphi^N(0) \land N(0),
\]

\[
[\forall x.\varphi(x) \rightarrow \varphi(s(x))]^N \Rightarrow \forall x.\varphi^N(x) \land N(x) \rightarrow \varphi^N(s(x)) \land N(s(x)),
\]

by using \( N(0) \) and \( \text{Suc}(N) = \forall x.\text{N}(x) \rightarrow \text{N}(s(x)) \). Hence we have (1). Now the desired formula is obtained by (1) and some elementary reasoning. \( \square \)

Thus \( \text{LIP}_0 \) can simulate reasoning in the first order Heyting arithmetic \( \text{HA} \) (see Appendix A for the detail).

Theorem 2.8. \( \text{I}\Sigma_1 \vdash \text{CE} (\text{LIP}_0) \rightarrow \text{1CON}(\text{HA}) \).

Proof. Suppose that \( \text{HA} \) proves a \( \Sigma^0_1 \) sentence \( \varphi \). We then have \( \text{LIP}_0 \vdash \Gamma \Rightarrow \varphi \), where \( \Gamma \) consists of some \( \Pi^0_1 \) axioms of \( \text{PA} \) (see Appendix A). Notice that the sequent only consists of first order formulas. Hence assuming the (partial) cut elimination for \( \text{LIP}_0 \), we obtain a cut-free derivation of it in \( \text{LI} \). By the standard soundness argument one can verify that \( \varphi \) is true. Moreover, all the reasoning can be done in a finitistic way, so is formalizable in \( \text{I}\Sigma_1 \). \( \square \)

2.5. Expressivity at higher levels. We next proceed to the expressivity of \( \text{LIP}_n \) with \( n > 0 \). Consider the second order definition of a least fixed point:

\[
I_{\varphi}(t) := \forall X.\text{Sub}(X) \land \forall x.(\varphi(X,x) \rightarrow X(x)) \rightarrow X(t).
\]

This is parameter-free and belongs to \( \text{FMP}_n \) provided that \( \varphi \in \text{FMP}_{n-1} \) and \( \text{FV}(\varphi) \subseteq \{X\} \). Moreover, it satisfies the axioms \( (\text{lfp}_1) \) and \( (\text{lfp}_2) \).

Lemma 2.9. Let \( \varphi(X,x) \) be a formula in \( \text{FMP}_{n-1} \) such that \( \text{FV}(\varphi) \subseteq \{X\} \) and \( X \) occurs only positively in it. Then \( \text{LIP}_n \) proves

\[
(\text{lfp}_1') \quad \forall x.\varphi(I_{\varphi}, x) \rightarrow I_{\varphi}(x),
(\text{lfp}_2') \quad \forall x.(\varphi(\tau, x) \rightarrow \tau(x)) \rightarrow \forall y(I_{\varphi}(y) \rightarrow \tau(y))
\]

for every \( \tau \in \text{ABS}_n \).
Proof. For \((lfp_1^I)\), notice that \(\text{LIP}_n\) proves

\[\text{Sub}(X), \forall x.\varphi(X, x) \rightarrow X(x), I_\varphi(y) \Rightarrow X(y).\]

Since \(X\) has only positive occurrences in \(\varphi(X, x)\),

\[\text{Sub}(X), \forall x.\varphi(X, x) \rightarrow X(x), \varphi(I_\varphi, y) \Rightarrow \varphi(X, y)\]

can be proved by induction on the structure of \(\varphi\). Hence

\[\text{Sub}(X), \forall x.\varphi(X, x) \rightarrow X(x), \varphi(I_\varphi, y) \Rightarrow X(y).\]

From this, we obtain \((lfp_1^I)\) by rules \((\forall x \text{ right}), (\rightarrow \text{ right})\) and \((\forall X \text{ right})\).

\((lfp_1^I)\) is obtained from

\[\text{LIP}_n \vdash \forall x.(\varphi(\tau, x) \rightarrow \tau(x)) \rightarrow \tau(y), \forall x.(\varphi(\tau, x) \rightarrow \tau(x)) \Rightarrow \tau(y)\]

by rule \((\forall X \text{ left})\) and some first order inferences. \(\square\)

Based on this, we translate each \(\text{ID}_n\)-formula \(\varphi\) into a formula \(\varphi^I \in \text{FMP}_n\) such that \(\text{FV}(\varphi^I) = \text{FV}(\varphi)\). It proceeds by induction on \(n\). For \(n = 0\), we let \(\varphi^I := \varphi^N\). For \(n > 0\), we replace each fixed point atom \(I_\xi(t)\) of \(\text{ID}_n\) with \(I_{\xi^I}(t)\), where \(\xi = \xi(X, x) \in \text{FMP}_{n-1}\) and \(\text{FV}(\xi) \subseteq \{X\}\). We also replace each first order quantifier \(Qx\) with \(Qx \in \mathbb{N}\).

Theorem 2.8 can be generalized to an arbitrary level.

**Theorem 2.10.** For every \(n < \omega\), \(\text{I}^\omega \vdash \text{CE}(\text{LIP}_n) \rightarrow 1\text{CON}(\text{ID}_n)\).

Proof. \(\text{LIP}_n\) proves \(\text{Sub}(I_\xi)\) for every \(\xi \in \text{FMP}_{n-1}\) with \(\text{FV}(\xi) \subseteq \{X\}\), so proves \(\Gamma_{eq} \Rightarrow \text{Sub}(\varphi^I)\) too for every \(\text{ID}_n\)-formula \(\varphi\). Hence Lemma 2.7 can be extended to all formulas of the form \(\varphi^I \in \text{FMP}_n\). Furthermore, \(\text{LIP}_n\) proves \((lfp_1^I)\) and \((lfp_1^I)\) by Lemma 2.9. Thus \(\text{LIP}_n\) can simulate reasoning in \(\text{ID}_n^I\). Therefore we can argue as in the proof of Theorem 2.8. \(\square\)

The converse implication can be obtained by proving cut elimination for \(\text{LIP}_n\) “locally” within \(\text{ID}_n^I\), that is, by proving

\[\text{LIP}_n \vdash \Gamma \Rightarrow \Pi \quad \text{implies} \quad \text{ID}_n^I \vdash \text{LI} \vdash^{cf} \Gamma \Rightarrow \Pi.\]

Thus the claim is that \(\text{ID}_n^I\) proves cut elimination for \(\text{LIP}_n\) “sequent-wise.” More precisely, one has to show that for each derivation \(\pi\) of \(\Gamma \Rightarrow \Pi\) in \(\text{LIP}_n\), there is a derivation \(\pi'\) in \(\text{ID}_n^I\) of a \(\Sigma_1^0\) sentence saying that \(\text{LI} \vdash^{cf} \Gamma \Rightarrow \Pi\). Moreover, \(\pi'\) should be obtained from \(\pi\) primitive recursively.

Thus assuming that \(\text{ID}_n^I\) is 1-consistent, we obtain a statement of cut elimination:

\[\text{LIP}_n \vdash \Gamma \Rightarrow \Pi \quad \text{implies} \quad \text{LI} \vdash^{cf} \Gamma \Rightarrow \Pi.\]

This motivates us to prove cut elimination for parameter-free logics “locally” within \(\text{ID}\)-theories.

As before, it is sufficient to prove partial cut elimination to establish the Takeuti correspondence. Moreover, Theorem 2.11 holds for \(\text{LIP}_n\) too, since the argument by Pippinghaus [29] can be restricted to a parameter-free fragment without any problem.

**Theorem 2.11.** For every \(n < \omega\), \(\text{I}^\omega \vdash \text{CE}(\text{LIP}_n) \leftrightarrow \text{CE}_{\text{Fm}}(\text{LIP}_n)\).

We are thus led to proving partial cut elimination, that is often simpler than proving complete cut elimination.
3. **Introduction to Ω-rule.** Cut elimination in a higher order setting is tricky, since a principal reduction step

\[
\frac{\Gamma \Rightarrow \phi(Y)}{\Gamma \Rightarrow \forall X.\phi(X)} \quad (\forall X \text{ right}) \quad \frac{\forall X.\phi(X) \Rightarrow \Pi}{\phi(\tau) \Rightarrow \Pi} \quad (\forall X \text{ left}) \quad \text{(cut)}
\]

may yield a bigger cut formula so that one cannot simply argue by induction on the complexity of the cut formula. The Ω-rule, introduced by [11], is an alternative of rule \( (\forall X \text{ left}) \) that allows us to circumvent this difficulty.

For illustration, let us first consider a naive implementation of the Ω-rule into our setting. We extend the first order calculus \( LI \) by enlarging the set of formulas to \( FMP_0 \) and by adding rules \( (\forall X \text{ right}) \) and \( \{ \Delta, \Gamma \Rightarrow \Pi \} \rightarrow \Delta \in |\forall X.\phi|_0 \)

where \( |\forall X.\phi|_0 \) consists of finite sets \( \Delta \subseteq \text{fin} \ Fm \) such that \( LI \vdash cf \Delta \Rightarrow \phi(Y) \) holds for some \( Y \notin \text{FV}(\Delta) \).

Rule \( (\Omega_0 \text{ left}) \) has infinitely many premises indexed by \( |\forall X.\phi|_0 \). In this respect it looks similar to the characteristic rules of the MacNeille completions (Proposition 1.3). In Section 4 we will provide a further link between them.

\( (\Omega_0 \text{ left}) \) is as strong as rule \( (\forall X \text{ left}) \) of \( LIP_0 \). To see this, consider a provable formula \( \forall X.\phi \Rightarrow \phi(\tau) \) in \( LIP_0 \). Let \( \Delta \in |\forall X.\phi|_0 \), that is, \( \Delta \Rightarrow \phi(Y) \) has a cut-free derivation \( \pi_{\Delta} \) in \( LI \) for some \( Y \notin \text{FV}(\Delta) \). Then there is a derivation \( \pi_{\Delta}^\tau \) of \( \Delta \Rightarrow \phi(\tau) \) in the extended system obtained by substituting \( \tau \) for \( Y \). Hence we have:

\[
\frac{\Delta \Rightarrow \phi(\tau)}{\forall X.\phi \Rightarrow \phi(\tau)} \quad (\Omega_0 \text{ left})
\]

Moreover, rule \( (\Omega_0 \text{ left}) \) suggests a natural step of cut elimination. Consider a cut:

\[
\frac{\Gamma \Rightarrow \phi(\tau)}{\Gamma \Rightarrow \forall X.\phi(X)} \quad (\forall X \text{ right}) \quad \frac{\{ \Delta \Rightarrow \Pi \} \Delta \in |\forall X.\phi|_0}{\forall X.\phi \Rightarrow \Pi} \quad (\text{cut})
\]

Arguing inductively, assume that \( \Gamma \subseteq \text{Fm} \) and \( \Gamma \Rightarrow \phi(Y) \) is cut-free provable in \( LI \). Then \( \Gamma \) belongs to \( |\forall X.\phi|_0 \), so the conclusion \( \Gamma \Rightarrow \Pi \) is just one of the infinitely many premises. Thus the above derivation reduces to:

\[
\frac{\pi_{\Gamma}}{\Gamma \Rightarrow \Pi}
\]

It looks fine so far. However, rule \( (\Omega_0 \text{ left}) \) cannot be combined with the standard rules for the first order quantifiers.

**Proposition 3.1.** System \( LI + (\forall X \text{ right}) + (\Omega_0 \text{ left}) \) is inconsistent.
Proof. Consider formula \( \varphi := X(c) \rightarrow X(x) \) with \( c \) a constant. We claim that \( \forall X.\varphi \Rightarrow \bot \) is provable. Let \( \Delta \in [\forall X.\varphi]_0 \), that is, \( \text{LI} \vdash [\forall X.\varphi]_0 \Rightarrow [Y(c) \rightarrow Y(x)] \) for some \( Y \notin \text{FV}(\Delta) \). Notice that the sequent is first order, \( \Delta \) and \( Y(c) \rightarrow Y(x) \) do not share any predicate symbol/variable, and \( Y(c) \rightarrow Y(x) \) is not provable. Hence Craig’s interpolation theorem yields \( \Delta \Rightarrow \bot \). From this, \( \forall X.\varphi \Rightarrow \bot \) follows by \( (\Omega_0 \text{ left}) \), and so \( \exists x.\forall X.\varphi \Rightarrow \bot \). On the other hand, \( \Rightarrow \exists x.\forall X.\varphi \) is also provable. Hence we obtain \( \bot \).

The primary reason for inconsistency is that \( (\Omega_0 \text{ left}) \) is not closed under term substitutions, while the standard treatment of first order quantifiers assumes that all rules are closed under term substitutions. Hence we have to weaken first order quantifier rules to obtain a consistent system. A reasonable way is to replace \( (\forall x \text{ right}) \) and \( (\exists x \text{ left}) \) with Schütte’s \( \omega \)-rules, which are infinitary (see Figure 2).

Remark 3.2. Buchholz’ later paper [12] includes a proof of (partial) cut elimination for a parameter-free subsystem \( \text{BI}_1^- \) of analysis that can be understood without recourse to ordinals. It is extended to complete cut elimination for the same system by [4], and to complete cut elimination for \( \Pi_1^- \text{-CA}_0 + \text{BI} \) (bar induction) by [3]. The \( \Omega \)-rule further finds applications in modal fixed point logics [22, 25]. It is used to show the strong normalization of analysis that can be understood without recourse to ordinals. As an advantage, one obtains a concrete operator for cut elimination and reduces the complexity of inductive definition: the original semiformal system can be defined by an inductive definition on a bounded formula, while ours requires a \( \Pi_1^0 \) formula. However, this point is irrelevant to the subsequent argument.

3.2. Syntactic cut elimination by \( \Omega \)-rules. We here give a syntactic proof of partial cut elimination for \( \text{LIP}_n \) for every \( n \geq 0 \). The crucial step is to define an infinitary sequent calculus \( \text{LI}\Omega_n \) for each \( n \) based on the \( \Omega \)-rules.

Let \( \text{LI}\Omega_{n-1} \) := \( \text{LI} \) for convenience. Provided that \( \text{LI}\Omega_{n-1} \) has been defined, the sequent calculus \( \text{LI}\Omega_n \) is defined as follows. Each sequent consists of formulas in \( \text{FMP}_n \), and the inference rules are \( (\text{id}) \), \( (\text{cut}) \), the rules for propositional connectives in Figure 1 and the rules for quantifiers given in Figure 2.

Some remarks are in order. First, notice that rules \( (\forall x \text{ right}) \) and \( (\exists x \text{ left}) \) are replaced with infinitary rules \( (\omega \text{ right}) \) and \( (\omega \text{ left}) \). Second, \( \text{LI}\Omega_n \) contains not just one, but all of \( (\Omega_0 \text{ left}), \ldots, (\Omega_n \text{ left}) \). Similarly for other \( \Omega \)-rules. The reason is that \( \text{LI}\Omega_n \) has to be an extension of \( \text{LI}\Omega_{n-1} \). Notice that each index set \( [\forall X.\varphi]_k \) consists of finite sets \( \Delta \subseteq \text{fin} \text{FMP}_{k-1} \), and \( [\exists X.\varphi]_k \) consists of sequents \( \Delta \Rightarrow \Lambda \) such that \( \Delta \cup \Lambda \subseteq \text{fin} \text{FMP}_{k-1} \). Finally, \( \text{LI}\Omega_n \) contains superfluous rules \( (\Omega_k \text{ left}) \) and \( (\Omega_k \text{ right}) \) for each \( k = 0, \ldots, n \), the former being derivable by combining \( (\forall X \text{ right}), (\Omega_k \text{ left}) \) and \( (\text{cut}) \). These are nevertheless included for a technical reason.

The partial cut elimination theorem will be established by a series of lemmas.

Lemma 3.3 (Substitution). Let \( n \geq 0 \). \( \text{LI}\Omega_n \vdash \Gamma \Rightarrow \Pi \) implies \( \text{LI}\Omega_n \vdash \Gamma^* \Rightarrow \Pi^* \) for every set substitution \( \bullet : \text{VAR} \rightarrow \text{ABS}_n \).

Proof. By induction on \( n \) and on the structure of the derivation. Let us treat only two cases.
Lemma 3.4 (Embedding). LIP \vdash \Gamma \Rightarrow \Pi \text{ implies } \text{LI}\Omega_n \vdash \Gamma^\circ \Rightarrow \Pi^\circ \text{ for every term substitution } \circ : \text{Var} \rightarrow \text{Tm}. 

We write $\Delta = \Delta(k)$ where

\[ \forall x. \varphi(x), \Gamma \Rightarrow \Pi \text{ (}\forall x \text{ left)} \]

and

\[ \exists x. \varphi(x), \Gamma \Rightarrow \Pi \text{ (}\exists x \text{ right)} \]

Update the given substitution $\bullet$ by letting $Y^\bullet := Z$ (fresh variable), so that $Z \notin \text{FV}(\Gamma^\bullet)$. By the induction hypothesis we have $\Gamma^\bullet \Rightarrow \forall x. \varphi(x)$, noting that $\text{FV}(\varphi(Y)) \subseteq \{Y\}$. We therefore obtain $\Gamma^\circ \Rightarrow \forall x. \varphi(x)$ as required.

(2) The derivation ends with

\[ \exists X. \varphi(X), \Gamma \Rightarrow \Pi \text{ (}\exists X \text{ left)} \]

where $k = 0, \ldots, n$ and

\[ \forall X. \varphi(X)|_k := \{\Delta : \text{LI}\Omega_k \vdash \Delta \Rightarrow \varphi(Y) \text{ for some } Y \notin \text{FV}(\Delta)\} \]

\[ \exists X. \varphi(X)|_k := \{(\Delta \Rightarrow \Lambda) : \text{LI}\Omega_k \vdash \Delta \Rightarrow \varphi(Y), \Lambda \Rightarrow \Pi \text{ for some } Y \notin \text{FV}(\Delta, \Lambda)\}. \]

Figure 2: Inference rules of LI\Omega_n for quantifiers
Proof. By structural induction on the derivation. We only consider two cases.

(1) The derivation ends with
\[ \frac{\Gamma \Rightarrow \varphi(y) \quad y \notin \text{Fv}(\Gamma)}{\Gamma \Rightarrow \forall x.\varphi(x)} \]
(\forall x \text{ right}).

By the induction hypothesis, we have \( \Gamma^o \Rightarrow \varphi^o(t) \) for every \( t \in Tm \). Hence \( \Gamma^o \Rightarrow (\forall x.\varphi(x))^o \) is obtained by rule (\( \omega \text{ right} \)).

(2) The derivation ends with (\( \forall \text{ left} \)). It suffices to show that \( \text{LI}_n \vdash \forall X.\varphi(X) \Rightarrow \varphi(\tau) \) for any \( \varphi \in \text{FMP}_n \) and \( \tau \in \text{ABS}_n \). We are going to use rule (\( \Omega \text{ left} \)). So let \( \Delta \in \forall X.\varphi|_n \), that is, \( \text{LI}_{n-1} \vdash \Delta \Rightarrow \varphi(Y) \) for some \( Y \notin \text{FV}(\Delta) \). Then \( \text{LI}_n \vdash \Delta \Rightarrow \varphi(Y) \) since \( \text{LI}_n \) is an extension of \( \text{LI}_{n-1} \), so \( \text{LI}_n \vdash \Delta \Rightarrow \varphi(\tau) \) by Lemma 3.3. Hence we obtain the desired sequent by (\( \Omega \text{ left} \)).

Lemma 3.5 (Cut elimination for \( \text{LI}_n \)). \( \text{LI}_n \vdash \Gamma \Rightarrow \Pi \) implies \( \text{LI}_n \vdash \Pi \).

This can be proved by a rather standard means, because any principal cut between \( \Omega \text{ left} \) and \( \forall \text{ right} \), that is the most crucial case, can be absorbed into rule (\( \tilde{\Omega} \text{ left} \)). A detailed proof will be given in Appendix B.

Lemma 3.6 (Collapsing). \( \text{LI}_n \vdash \Pi \Rightarrow \Pi \) implies \( \text{LI}_{n-1} \vdash \Pi \Rightarrow \Pi \), provided that \( \Gamma \cup \Pi \subseteq \text{FMP}_{n-1} \).

Proof. By structural induction on the cut-free derivation of \( \Gamma \Rightarrow \Pi \) in \( \text{LI}_n \).

(1) The derivation ends with
\[ \frac{\Gamma \Rightarrow \varphi(Y) \quad \Delta, \Gamma \Rightarrow \Pi \quad \Delta \in \forall X.\varphi|_n \quad Y \notin \text{FV}(\Gamma)}{\Gamma \Rightarrow \Pi} \]
(\( \tilde{\Omega}_n \text{ left} \)).

We have \( \text{LI}_{n-1} \vdash \Pi \Rightarrow \varphi(Y) \) by the induction hypothesis, noting that \( \Gamma \cup \{ \varphi(Y) \} \subseteq \text{FMP}_{n-1} \). Hence \( \Gamma \in \forall X.\varphi|_n \), so \( \Gamma \Rightarrow \Pi \) is among the premises. Therefore \( \text{LI}_{n-1} \vdash \Pi \Rightarrow \Pi \) by the induction hypothesis again.

(2) The derivation ends with (\( \tilde{\Omega}_k \text{ left} \)) with \( k < n \). It is straightforward from the induction hypotheses.

(3) \( n = 0 \) and the derivation ends with
\[ \frac{\{ \Gamma \Rightarrow \varphi(t) \}_{t \in Tm}}{\Gamma \Rightarrow \forall x.\varphi(x)} \]
(\( \omega \text{ right} \)).

We choose a variable \( y \) such that \( y \notin \text{Fv}(\Gamma) \). We have \( \text{LI} \vdash \Pi \Rightarrow \varphi(y) \) by the induction hypothesis, so the conclusion sequent is obtained by (\( \forall x \text{ right} \)).  

Other cases are treated similarly. □

Lemmas 3.4, 3.5 and 3.6 constitute a syntactic proof of partial cut elimination for \( \text{LIP}_n \). From a metatheoretical point of view, the most significant part is to define a provability predicate for \( \text{LI}_n \). For \( n = -1 \), a provability predicate for \( \text{LI}_n \) can be defined in \( \text{ID}_0 = \text{HA} \) as usual, since the proof system is finitary.
For \( n = 0 \), observe that one can define a formula \( \text{Step}(X, x) \) in \( \text{HA}[X] \) such that
\[
\text{Step}(X, \Gamma) \iff \Gamma \Rightarrow \Pi
\]
where \( \Gamma \Rightarrow \Pi \) is obtained from some \( Y \subseteq X \) by applying a rule of \( \text{LI} \Omega_0 \),
where \( \Gamma \) is a suitable coding function and \( X \) is supposed to be a set of (the codes of) sequents. Notice that the above formula relies on a provability predicate for \( \text{LI} \).
Now let \( \text{LI}_0 := I_{\text{Step}} \), that is available in \( \text{ID}_1^i \). We then have
\[
\text{LI}_0(\Gamma \Rightarrow \Pi) \iff \text{LI} \vdash \Gamma \Rightarrow \Pi.
\]

For \( n > 0 \), a provability predicate \( \text{LI}_n \) can be defined by relying on \( \text{LI}_{n-1}, \ldots, \text{LI}_1 \),
thus in \( \text{ID}_{n+1} \). Once suitable provability predicates have been defined, the rest of the proof can be smoothly formalized, since it mostly proceeds by structural induction on the derivation (see also Appendix B). Hence we obtain:

**Theorem 3.7** (Syntactic cut elimination for \( \text{LIP}_n \)). Let \( n \geq 0 \) and \( \Gamma \cup \Pi \subseteq \text{Fm} \). Then \( \text{LIP}_n \vdash \Gamma \Rightarrow \Pi \) implies \( \text{LI} \vdash \text{cf} \Gamma \Rightarrow \Pi \). Moreover, this fact can be proved in \( \text{ID}_{n+1}^i \).

Observe that it is impossible to prove it within \( \text{ID}_n^i \), because of Theorem 2.10 and the second incompleteness theorem.

### 4. \( \Omega \)-rule and MacNeille completion

In this section, we establish a formal connection between the \( \Omega \)-rule and the MacNeille completion. Let us start by introducing algebraic semantics for full second order calculus \( \text{LI} \).

Let \( L \) be a language. A (complete) Heyting-valued prestructure for \( L \) is \( \mathcal{M} = \langle A, \mathcal{M}, D, \mathcal{L} \rangle \)
where \( A = \langle A, \land, \lor, \rightarrow, \top, \bot \rangle \) is a complete Heyting algebra, \( M \) is a nonempty set (term domain), \( \emptyset \neq D \subseteq A^M \) (abstract domain) and \( \mathcal{L} \) consists of a function \( f^M : M^n \rightarrow M \)
for each \( n \)-ary function symbol \( f \in L \) and \( p^M : M^n \rightarrow A \) for each \( n \)-ary predicate symbol \( p \in L \).

Thus \( p^M \) is an \( A \)-valued subset of \( M^n \).

It is not our purpose to systematically develop a model theory for the intuitionistic logic. We will use prestructures only for proving conservative extension and cut elimination.

Hence we only consider term models below, in which \( M = \text{Fm} \) and \( f^M(\bar{t}) = f(\bar{t}) \). This assumption simplifies the interpretation of formulas a lot.

A **valuation** on \( \mathcal{M} \) is a function \( \mathcal{V} : \text{VAR} \rightarrow D \). The interpretation of formulas \( \mathcal{V} : \text{FM} \rightarrow A \) is inductively defined as follows:
\[
\begin{align*}
\mathcal{V}(p(\bar{t})) & := p^M(\bar{t}) \quad \mathcal{V}(X(t)) & := \mathcal{V}(X)(t) \\
\mathcal{V}(\bot) & := \bot \quad \mathcal{V}(\varphi \ast \psi) & := \mathcal{V}(\varphi) \ast \mathcal{V}(\psi) \\
\mathcal{V}(\forall x. \varphi(x)) & := \bigwedge_{t \in \text{Fm}} \mathcal{V}(\varphi(t)) \quad \mathcal{V}(\exists x. \varphi(x)) & := \bigvee_{t \in \text{Fm}} \mathcal{V}(\varphi(t)) \\
\mathcal{V}(\forall X. \varphi) & := \bigwedge_{F \in D} \mathcal{V}[F/X](\varphi) \quad \mathcal{V}(\exists X. \varphi) & := \bigvee_{F \in D} \mathcal{V}[F/X](\varphi)
\end{align*}
\]
where \( \ast \in \{ \land, \lor, \rightarrow \} \) and \( \mathcal{V}[F/X] \) is an update of \( \mathcal{V} \) that maps \( X \) to \( F \).

\( \mathcal{M} \) is called a Heyting-valued **structure** if \( \mathcal{V}(\tau) \in D \) holds for every valuation \( \mathcal{V} \) and every abstract \( \tau \in \text{ABS} \).

Clearly \( \mathcal{M} \) is a Heyting-valued structure if \( \mathcal{D} = A^{\text{Fm}} \). Such a structure is called **full**.

Given a sequent \( \Gamma \Rightarrow \Pi \), let
\[
\begin{align*}
\mathcal{V}(\Gamma) & := \bigwedge \{ \mathcal{V}(\varphi) : \varphi \in \Gamma \}, \\
\mathcal{V}(\Pi) & := \bigvee \{ \mathcal{V}(\psi) : \psi \in \Pi \}.
\end{align*}
\]

It is routine to verify:
Lemma 4.1 (Soundness). If $\text{LI}2 \vdash \Gamma \Rightarrow \Pi$, then $\Gamma \Rightarrow \Pi$ is valid, that is, $\mathcal{V}(\Gamma^\circ) \leq \mathcal{V}(\Pi^\circ)$ holds for every valuation $\mathcal{V}$ on every Heyting structure $\mathcal{M}$ and every term substitution $\circ$.

To illustrate the use of algebraic semantics, let us have a look at a proof of an elementary fact that $\text{LI}2$ is a conservative extension of $\text{LI}$.

Let $\mathbf{L}$ be the Lindenbaum algebra for $\text{LI}$, that is, $\mathbf{L} := \langle \text{Fm}/\sim, \land, \lor, \neg, \rightarrow, \top, \bot \rangle$ where $\varphi \sim \psi$ iff $\text{LI} \vdash \varphi \leftrightarrow \psi$. The equivalence class of $\varphi$ with respect to $\sim$ is denoted by $[\varphi]$. $\mathbf{L}$ is a Heyting algebra in which

\[(*) \quad [\forall x.\varphi(x)] = \bigwedge_{t \in \text{Tm}} [\varphi(t)], \quad [\exists x.\varphi(x)] = \bigvee_{t \in \text{Tm}} [\varphi(t)]\]

hold. Given a sequent $\Gamma \Rightarrow \Pi$, elements $[\Gamma]$ and $[\Pi]$ in $\mathbf{L}$ are naturally defined.

Let $\mathbf{G}$ be a regular completion of $\mathbf{L}$. Then $\mathcal{M}(\mathbf{G}) := \langle \mathbf{G}, \text{Tm}, \mathbf{G}^\text{Tm}, \mathcal{L} \rangle$ is a full Heyting structure, where $\mathcal{L}$ consists of a $\mathbf{G}$-valued predicate $p^{\mathcal{M}(\mathbf{G})}$ defined by $p^{\mathcal{M}(\mathbf{G})}(\vec{t}) := [p(\vec{t})]$ for each $p \in \mathbf{L}$ (in addition to the interpretations of function symbols). Define a valuation $\mathcal{I}$ by $\mathcal{I}(\vec{X})(t) := [X(t)]$. We then have $\mathcal{I}(\varphi) = [\varphi]$ for every $\varphi \in \text{Fm}$ by regularity (be careful here: $(*)$ may fail in $\mathbf{G}$ if it is not regular).

Now, suppose that $\text{LI}2$ proves $\Gamma \Rightarrow \Pi$ with $\Gamma \cup \Pi \subseteq \text{Fm}$. Then we have $\mathcal{I}(\Gamma) \leq \mathcal{I}(\Pi)$ by Lemma 4.1.1, so $[\Gamma] \leq [\Pi]$, that is, $\text{LI} \vdash \Gamma \Rightarrow \Pi$. This proves that $\text{LI}2$ is a conservative extension of $\text{LI}$.

Although this argument cannot be fully formalized in $\text{HA}2$ because of Gödel’s second incompleteness, it does admit a local formalization in $\text{PA}2$. In contrast, the above argument, when applied to $\text{LIP}_0$, cannot be locally formalized in the arithmetical counterpart $\text{HA}$. The reason is simply that $\text{HA}$ does not have second order quantifiers, which are needed to write down the definitions of $\mathcal{V}(\forall X.\varphi)$ and $\mathcal{V}(\exists X.\varphi)$. To circumvent this, we will make a crucial observation that $\mathcal{V}(\forall X.\varphi)$ and $\mathcal{V}(\exists X.\varphi)$ admit alternative first order definitions if the completion is MacNeille. It is here that one finds a connection between the MacNeille completion and the $\Omega$-rule.

Theorem 4.2. Let $\mathbf{L}$ be the Lindenbaum algebra for $\text{LI}$ and $\mathbf{L} \subseteq \mathbf{G}$ a regular completion. $\mathcal{M}(\mathbf{G})$ and $\mathcal{I}$ are defined as above. For every sentence $\forall X.\varphi$ in $\text{FMP}_0$, the following are equivalent.

1. $\mathcal{I}(\forall X.\varphi) = \bigvee \{ a \in \mathbf{L} : a \leq \mathcal{I}(\forall X.\varphi) \}$.
2. $\mathcal{I}(\forall X.\varphi) = \bigvee \{ [\Delta] \in \mathbf{L} : \Delta \in \mathcal{V}(\forall X.\varphi) \}$.
3. The inference below is sound for every $\varphi \in \mathbf{G}$:

\[\frac{\{ \mathcal{I}(\Delta) \leq y \} \Delta \in \mathcal{V}(\forall X.\varphi)}{\mathcal{I}(\forall X.\varphi) \leq y}\]

If $\mathbf{G}$ is the MacNeille completion of $\mathbf{L}$, all the above hold.

Proof. ((1) $\iff$ (2)) Let $a = [\Delta]$. It is sufficient to show

\[a \leq \mathcal{I}(\forall X.\varphi) \iff \Delta \in \mathcal{V}(\forall X.\varphi)\]

Suppose that $a \leq \mathcal{I}(\forall X.\varphi) = \bigwedge_{F \in \mathbf{G}^\text{Tm}} \mathcal{I}(F/X)(\varphi)$. We choose $Y \notin \text{FV}(\Delta)$ and define $F_Y \in \mathbf{G}^\text{Tm}$ by $F_Y(t) := [Y(t)]$ for every $t \in \text{Tm}$. We then have

\[[\Delta] \leq \mathcal{I}(\forall X.\varphi) \leq \mathcal{I}(F_Y/X)(\varphi(X)) = [\varphi(Y)]\]

that is, $\text{LI} \vdash \Delta \Rightarrow \varphi(Y)$. By the cut elimination for $\text{LI}$, we obtain $\text{LI} \vdash_{cf} \Delta \Rightarrow \varphi(Y)$. Hence $\Delta \in \mathcal{V}(\forall X.\varphi)$.
Conversely, suppose that $\mathit{LI} \vdash^f \Delta \Rightarrow \varphi(Y)$ with $Y \notin \text{FV}(\Delta)$. It implies $[\Delta] = \mathcal{I}(\Delta) = \mathcal{I}[F/Y](\Delta) \leq \mathcal{I}[F/Y](\varphi(Y))$ for every $F \in G^{\text{TM}}$ by Lemma 4.1. Hence $[\Delta] \leq \mathcal{I}(\forall X.\varphi(X))$.

((2) $\Rightarrow$ (3)) Assume the premises of (3). This means that we have $[\Delta] = \mathcal{I}(\Delta) \leq y$ for every $\Delta \in |\forall X.\varphi|_0$. Hence the conclusion $\mathcal{I}(\forall X.\varphi) \leq y$ follows by (2).

((3) $\Rightarrow$ (2)) Let $y := \bigvee \{[\Delta] \in \mathbf{L} : \Delta \in |\forall X.\varphi|_0 \}$. Then $\mathcal{I}(\Delta) = [\Delta] \leq y$ holds for every $\Delta \in |\forall X.\varphi|_0$, so $\mathcal{I}(\forall X.\varphi) \leq y$ by rule (3).

On the other hand, $\Delta \in |\forall X.\varphi|_0$ implies $[\Delta] \leq \mathcal{I}(\forall X.\varphi)$ as proved above. Hence we also have $y \leq \mathcal{I}(\forall X.\varphi)$.

Finally, suppose that $\mathbf{L} \subseteq \mathbf{G}$ is a MacNeille completion. Then (1) holds by $\mathbf{V}$-density. So (2) and (3) hold too.

The equivalence in Theorem 4.2 is quite suggestive, since (3) is an algebraic interpretation of rule $(\Omega_0 \text{ left})$, while (1) is a characteristic of the MacNeille completion. Equation (2) suggests a way of interpreting second order formulas without using second order quantifiers at the meta-level. All these are true if the completion is MacNeille.

Remark 4.3. Essentially the same as (2) has been already observed by Altenkirch and Coquand [6] in the context of lambda calculus (without making any explicit connection to the $\Omega$-rule and the MacNeille completion). Indeed, they consider a logic which roughly amounts to the negative fragment of our $\mathbf{LIP}_0$ and employ equation (2) to give a “finitary” proof of a (partial) normalization theorem for a parameter-free fragment of System F (see also [2, 5] for extensions). However, their argument is technically based on a downset completion, that is not MacNeille. As is well known, such a naive completion does not work well for the positive connectives $\{\exists, \lor\}$. In contrast, when $\mathbf{L} \subseteq \mathbf{G}$ is a MacNeille completion, we also have

$$\mathcal{I}(\exists X.\varphi) = \bigwedge \{[\Delta] \rightarrow [\Lambda] \in \mathbf{L} : (\Delta \Rightarrow \Lambda) \in |\exists X.\varphi|_0 \}.$$

We thus claim that the insight by Altenkirch and Coquand is further augmented and better understood if one employs the MacNeille completion instead of the downset completion (or the filter completion).

As a consequence of Theorem 4.2, it is possible to give an algebraic proof to the conservativity of $\mathbf{LIP}_0$ over $\mathbf{LI}$, that can be locally formalized in $\mathbf{HA}$.

The argument proceeds as follows. Let $\mathbf{L}$ be the Lindenbaum algebra for $\mathbf{LI}$ and $\mathbf{G}$ be the MacNeille completion of $\mathbf{L}$. Then $\mathcal{M}(\mathbf{G}) := (\mathbf{G}, \text{TM}, \mathbf{G}^{\text{TM}}, \mathcal{L})$ is a full Heyting structure. Define a valuation $\mathcal{I}$ by $\mathcal{I}(X)(t) := [X(t)]$ as before. To extend it inductively to the $\text{FMP}_0$ formulas, we use the clauses

$$\mathcal{I}(\forall X.\varphi) := \bigvee \{[\Delta] \in \mathbf{L} : \Delta \in |\forall X.\varphi|_0 \}$$

$$\mathcal{I}(\exists X.\varphi) := \bigwedge \{[\Delta] \rightarrow [\Lambda] \in \mathbf{L} : (\Delta \Rightarrow \Lambda) \in |\exists X.\varphi|_0 \}.$$

Soundness holds with respect to this interpretation by Theorem 4.2. Hence by the same argument as before, we may conclude that $\mathbf{LI2}$ is a conservative extension of $\mathbf{LI}$. We will not discuss formalization in $\mathbf{HA}$ here, as stronger results on cut elimination will be formalized in Section 5.

It is interesting to see that the second order $\forall$ is interpreted by the first order $\bigvee$, while the second order $\exists$ is by the first order $\bigwedge$. We call this style of interpretation the $\Omega$-interpretation, that is the algebraic side of the $\Omega$-rule, and that will play a key role in
the next section. We conclude our discussion by reporting a counterexample for the general soundness.

**Proposition 4.4.** There is a Heyting-valued structure in which $(\Omega_0 \text{ left})$ is not sound.

**Proof.** Let $A$ be the three-element chain $\{0 < 0.5 < 1\}$ seen as a Heyting algebra. Here the implication $\rightarrow$ is defined by:

\[
\begin{align*}
a \rightarrow b & := \top \text{ if } a \leq b, \\
& := b \text{ otherwise.}
\end{align*}
\]

Consider the language that only consists of a term constant $\ast$. Then a full Heyting-valued structure $A := \langle A, \text{Tm}, A^\text{Tm}, \mathcal{L} \rangle$ is naturally obtained. Let $\varphi := (X(*) \rightarrow \bot) \lor X(*)$. We then have $V(\forall X.\varphi) = 0.5$ for every valuation $V$. In fact, $V(\varphi) = 1$ if $V(X(\ast)) = 0$ or $1$, and $V(\varphi) = 0.5$ if $V(X(\ast)) = 0.5$.

Now consider the following instance:

\[
\Delta \Rightarrow \bot \quad \forall X.\varphi \Rightarrow \bot (\Omega_0 \text{ left})
\]

We claim that it is not sound provided that $V(X(t)) = 0$ for every $X \in \text{VAR}$ and $t \in \text{Tm}$. Suppose that $\Delta \in |\forall X.\varphi|_0$, i.e., $\text{LI} \vdash \Delta \Rightarrow \varphi(Y)$ with $Y \notin \text{FV}(\Delta)$. Then $V(\Delta) \leq V[F/X](\varphi)$ for every $F \in A^\text{Tm}$ by Lemma 4.1. Hence

\[
V(\Delta) \leq \bigwedge_{F \in A^\text{Tm}} V[F/X](\varphi) = V(\forall X.\varphi) = 0.5.
\]

But $\Delta$ is first order and does not involve any predicate symbol, so only takes value 0 or 1 by the assumption on $V$ (and the fact that $\{0 < 1\}$ is a Heyting subalgebra of $A$). Hence $V(\Delta) = 0$, that is, all the premises $\Delta \Rightarrow \bot$ are satisfied. However, $V(\forall X.\varphi) = 0.5 > 0$, that is, the conclusion $\forall X.\varphi \Rightarrow \bot$ is not satisfied.

This invokes a natural question. Is it possible to find a Boolean-valued counterexample? In other words, is the $\Omega$-rule classically sound? This question is left open.

5. **Algebraic cut elimination**

This section is devoted to an algebraic proof of cut elimination for parameter-free logics. After introducing a general concept of Heyting frame in Subsection 5.1, we consider a syntactic frame build upon cut-free provability in Subsection 5.2. A soundness argument then establishes the cut elimination theorem in Subsection 5.3. A small improvement is given in Subsection 5.4, that will be important when formalizing our proof in $\text{ID}$-theories in Section 6. An algebraic proof of cut elimination for $\text{LI2}$ due to [24, 28] is given in Appendix C for a comparison.

5.1. **Polarities and Heyting frames.** We begin with a very old concept due to Birkhoff [10], that provides a uniform framework for both MacNeille completion and cut elimination.

A *polarity* $W = \langle W, W', R \rangle$ (a.k.a. *formal context*) consists of two sets $W, W'$ and a binary relation $R \subseteq W \times W'$. Given $X \subseteq W$ and $Z \subseteq W'$, let

\[
X^p := \{ z \in W' : x \mathbin{R} z \text{ for every } x \in X \}, \quad Z^\prec := \{ x \in W : x \mathbin{R} z \text{ for every } z \in Z \}.
\]
Lemma 5.1. If \( X \lor \) we write \( \gamma \). In the sequel, we also make use of the property so induces a closure operator \( \gamma \). Nonempty \( X \in \mathbb{R} \times \) We only verify \((\varepsilon, \gamma)\) holds for any \( X, Y \subseteq W \). Note that \( X \subseteq W \) is closed iff there is \( Z \subseteq W' \) such that \( X = Z^{\lt} \). In the sequel, we also make use of the property \( \gamma(x) := \gamma(\{x\}), x^{\lhd} := \{x\}^{\lhd} \) and \( z^{\lhd} := \{z\}^{\lhd} \). Let \( G(W) := \{X \subseteq W : X = \gamma(X)\} \), \( X \land Y := X \cap Y, X \lor Y := \gamma(X \cup Y), \top := W \) and \( \bot := \gamma(\emptyset) \).

**Lemma 5.2.** If \( W \) is a polarity, then \( W^+ := \langle G(W), \land, \lor \rangle \) is a complete lattice.

This is just a well-known fact. See [16] for instance.

The lattice \( W^+ \) is not always distributive because of the use of \( \gamma \) in the definition of \( \lor \). To ensure distributivity, we have to impose a further structure on \( W \).

A Heyting frame is \( W = \langle W, W', R, \circ, \varepsilon, \bot \rangle \), where

- \( \langle W, W', R \rangle \) is a polarity,
- \( \langle W, \circ, \varepsilon \rangle \) is a monoid,
- function \( \langle W \times W' \rightarrow W' \) satisfies

\[
x \circ y R z \iff y R x \setminus z
\]

for every \( x, y \in W \) and \( z \in W' \),

- the following inferences are valid:

\[
\frac{x \circ y R z}{y \circ x R z} (e) \quad \frac{\varepsilon R z}{x R z} (w) \quad \frac{x \circ x R z}{x R z} (c)
\]

Clearly \( x R z \) is an analogue of a sequent and \((e), (w)\) and \((c)\) correspond to exchange, weakening and contraction rules. By removing some/all of them, one obtains a residuated frame that works for substructural logics as well [18, 15].

**Lemma 5.2.** If \( W \) is a Heyting frame, \( W^+ := \langle G(W), \land, \lor, \rightarrow, \top, \bot \rangle \) is a complete Heyting algebra, where \( X \rightarrow Y := \{y \in W : x \circ y \in Y \text{ for every } x \in X\} \).

**Proof.** First of all, observe that any \( X \in G(W) \) is closed under \((e), (w)\) and \((c)\), that is, the following inferences are all valid:

\[
\frac{x \circ y \in X}{y \circ x \in X} (e) \quad \frac{x \in X}{x \circ y \in X} (w) \quad \frac{x \circ x \in X}{x \in X} (c)
\]

We only verify \((w)\). Suppose that \( x \in X \) and \( z \in X^{\lhd} \). Then \( x R z \), i.e., \( x \circ \varepsilon R z \). So \( \varepsilon R x \setminus z \) and \( y R x \setminus z \) by \((w)\). Hence \( x \circ y R z \). Since this holds for every \( z \in X^{\lhd} \), we conclude \( x \circ y \in X^{\lor \lhd} = X \).
Next, we show that $X \to Y \in \mathcal{G}(W)$ whenever $Y \in \mathcal{G}(W)$. This can be shown by proving

$$X \to Y = (X \backslash Y^>)^\triangledown$$

where $X \backslash Y^> := \{ x \in W' : x \in X, z \in Y^> \}$.

For the forward direction, let $y \in X \to Y$, $x \in X$ and $z \in Y^>$. Then $x \circ y \in Y$, so $x \circ y \in Y$, hence $y \circ x \in X \backslash z$. Since this holds for every $x \circ z \in X \backslash Y^>$, we conclude $y \in (X \backslash Y^>)^\triangledown$.

For the backward direction, let $y \in (X \backslash Y^>)^\triangledown$, $x \in X$ and $z \in Y^>$. Then we have $y \circ x \in Z$, $y \in Z$, so $y \circ x \in Z$. Since this holds for every $z \in Y^>$, we have $x \circ y \in Y^>$. Since this holds for every $y \in X$, we conclude $y \in X \to Y$.

We now prove that

$$X \cap Y \subseteq Z \iff X \subseteq Y \to Z$$

holds for every $X,Y,Z \in \mathcal{G}(W)$. For the forward direction, let $x \in X$ and $y \in Y$. Then $x \circ y \in X \cap Y$ by (e) and (u), so $x \circ y \in Z$ by assumption. Since this holds for every $y \in Y$, we have $X \subseteq Y \to Z$.

For the backward direction, let $x \in X \cap Y$. Then $x \circ x \in Z$ by assumption, so $x \in Z$ by (c). This proves $X \cap Y \subseteq Z$.

Polarities and Heyting frames are handy devices to obtain MacNeille completions. Let $\mathbf{A} = \langle A, \wedge, \vee, \to, \top, \bot \rangle$ be a Heyting algebra. Then $W_\mathbf{A} := \langle A, \leq, \wedge, \top, \to \rangle$ is a Heyting frame. Notice that the third condition for a Heyting frame above amounts to $x \land y \leq z$ iff $y \leq x \to z$. For the next theorem, note that the closure operator $\gamma$ can be seen as a map $\gamma : \mathbf{A} \to W_\mathbf{A}^+$ sending $a \in A$ to $a^\triangledown$.

**Theorem 5.3.** If $\mathbf{A}$ is a Heyting algebra, then $\gamma : \mathbf{A} \to W_\mathbf{A}^+$ is a MacNeille completion.

**Proof.** It is easy to see that $\gamma(a) = a^\triangledown$ holds for every $a \in A$. Based on this, we can show that $\gamma(\bot) = \gamma(0)$ and $\gamma(a \star b) = \gamma(a) \star \gamma(b)$ for $\star \in \{ \land, \lor, \to \}$. Furthermore, $a \leq b$ holds if and only if $\gamma(a) \leq \gamma(b)$, meaning that $\gamma$ is an embedding.

Let us verify that the completion is MacNeille. Let $X \in \mathcal{G}(W_\mathbf{A})$. For $\lor$-density, we have

$$X = \gamma \left( \bigcup \{ \gamma(a) : a \in X \} \right) = \bigvee \{ \gamma(a) : \gamma(a) \leq X \}.$$  

For $\land$-density, notice that $X = \bigcap \{ a^\triangledown : a \in X^> \}$ and $\gamma(a) = a^\triangledown$. Hence

$$X = \bigland \{ \gamma(a) : X \leq \gamma(a) \}.$$  

\[ \square \]

### 5.2. A syntactic frame for cut elimination.

We now start an algebraic proof of (partial) cut elimination for $\text{LIP}_{n+1}$ (with $n \geq -1$). Although we have already given a proof of cut elimination in Subsection 3.2, the proof does not formalize in $\text{ID}_{n+1}$ but only in $\text{ID}_{n+2}$ (even locally). Our goal here is to give another proof that locally formalizes in $\text{ID}_{n+1}$.

What we actually do is to prove that

$$\text{LIP}_{n+1} \vdash \Gamma \Rightarrow \Pi \quad \text{implies} \quad \text{LI}_{n} \vdash_{cf} \Gamma \Rightarrow \Pi$$

provided that $\Gamma \cup \Pi \subseteq \text{FMP}_n$. In particular when $n = -1$, this means that $\text{LIP}_0 \vdash \Gamma \Rightarrow \Pi$ implies $\text{LI} \vdash_{cf} \Gamma \Rightarrow \Pi$ provided that $\Gamma \cup \Pi \subseteq \text{Fm}$. When $n \geq 0$, we may combine it with Lemma 3.6 to obtain partial cut elimination for $\text{LIP}_{n+1}$. Notice that any use of a
propositional predicate at level \( n + 1 \) is avoided here. It is for this reason that the argument locally formalizes in \( \text{ID}^k_{n+1} \).

To begin with, let \( \wp_{\text{fin}}(\text{Fm}) \) be the set of finite sets of first order formulas, so that \( \langle \wp_{\text{fin}}(\text{Fm}), \cup, 0 \rangle \) is a commutative idempotent monoid. Let \( \text{SEQ} \) be the set of sequents that consist of formulas in \( \text{FMP}_n \). There is a natural map \( \llangle : \wp_{\text{fin}}(\text{Fm}) \times \text{SEQ} \rightarrow \text{SEQ} \) defined by \( \Gamma \llangle (\Sigma \Rightarrow \Pi) := (\Gamma, \Sigma \Rightarrow \Pi) \). So

\[
\text{CF} := \langle \wp_{\text{fin}}(\text{Fm}), \text{SEQ}, \Rightarrow^{cf}, \cup, 0, \llangle \rangle
\]
is a Heyting frame, where the binary relation \( \Rightarrow^{cf} \) is defined by

\[
\Gamma \Rightarrow^{cf} (\Sigma \Rightarrow \Pi) \iff \text{LI}_{\Omega n} |\Rightarrow^{cf} \Gamma, \Sigma \Rightarrow \Pi.
\]

In fact, rules (e), (c) are automatically satisfied because the monoid is commutative and idempotent. Rule (w) is satisfied since the weakening rule is admissible in \( \text{LI}_{\Omega n} \). Finally, we have:

\[
\Delta \cup \Gamma \Rightarrow^{cf} (\Sigma \Rightarrow \Pi) \iff \Delta \Rightarrow^{cf} \Gamma \llangle (\Sigma \Rightarrow \Pi).
\]

In the following, we write \( \varphi \) for sequent \( (\emptyset \Rightarrow \varphi) \in \text{SEQ} \). Thus \( \Gamma \Rightarrow^{cf} \varphi \) simply means \( \text{LI}_{\Omega n} |\Rightarrow^{cf} \Gamma \Rightarrow \varphi \).

\( \text{CF} \) is a frame in which \( \Gamma \in \Pi^\varphi \) holds iff \( \Gamma \Rightarrow^{cf} \Pi \). In particular, \( \varphi \in \varphi^\varphi \) always holds, so

\[
(*) \quad \varphi \in \gamma(\varphi) \subseteq \varphi^\varphi.
\]

It should also be noted that each \( X \in \mathcal{G}(\text{CF}) \) is closed under weakening because of (w): if \( \Delta \in X \) and \( \Delta \subseteq \Sigma \), then \( \Sigma \in X \).

This yields a full Heyting-valued structure

\[
\mathcal{C}F := \langle \text{CF}^{+}, Tm, \mathcal{G}(\text{CF})^{Tm}, L \rangle,
\]

where \( L \) is defined by \( p^{CF}((i)) := \gamma(p(i)) \) for each predicate symbol \( p \in L \).

Let \( \mathcal{I} : \text{VAR} \rightarrow \mathcal{G}(\text{CF})^{Tm} \) be a valuation defined by \( \mathcal{I}(X(t)) := \gamma(X(t)) \). This can be extended to an interpretation \( \mathcal{I} : \text{FMP}_0 \rightarrow \mathcal{G}(\text{CF}) \) by induction, employing the \( \Omega \)-interpretation technique discussed in Section 4.

\[
\begin{align*}
\mathcal{I}(X(t)) & := \mathcal{I}(X)(t) = \gamma(X(t)), & \mathcal{I}(p(i)) & := p^{CF}(i) = \gamma(p(i)), \\
\mathcal{I}(\bot) & := \bot, & \mathcal{I}(\varphi \ast \psi) & := \mathcal{I}(\varphi) \ast \mathcal{I}(\psi), \\
\mathcal{I}(\forall x. \varphi(x)) & := \bigwedge_{t \in Tm} \mathcal{I}(\varphi(t)), & \mathcal{I}(\exists x. \varphi(x)) & := \bigvee_{t \in Tm} \mathcal{I}(\varphi(t)), \\
\mathcal{I}(\forall X. \varphi) & := \gamma([\forall X. \varphi]^{n+1}), & \mathcal{I}(\exists X. \varphi) & := [\exists X. \varphi]^{\varphi^\varphi}_{n+1},
\end{align*}
\]

where \( \ast \in \{ \land, \lor, \rightarrow \} \). Notice our specific choice of level \( n + 1 \) in the definitions of \( \mathcal{I}(\forall X. \varphi) \) and \( \mathcal{I}(\exists X. \varphi) \). It can be flexibly changed, however, because of the following property.

**Lemma 5.4.** For every \( 0 \leq k \leq n \),

\[
\gamma([\forall X. \varphi]^{n+1}) = \gamma([\forall X. \varphi]^{k}), \quad [\exists X. \varphi]^{\varphi^\varphi}_{n+1} = [\exists X. \varphi]^{\varphi^\varphi}_{k}
\]

hold if \( \forall X. \varphi, \exists X. \varphi \in \text{FMP}_k \).

**Proof.** It is clear that the inclusion \( \supseteq \) holds for the first equation. So let \( \Sigma \in [\forall X. \varphi]^{n+1} \) and \( (\Gamma \Rightarrow \Pi) \in [\forall X. \varphi]^{\varphi^\varphi}_{k} \). The former means that \( \Sigma \Rightarrow^{cf} \varphi(Y) \) with \( Y \not\in \text{FV}(\Delta) \), while the latter means that \( \Delta, \Gamma \Rightarrow^{cf} \Pi \) for any \( \Delta \in [\forall X. \varphi]^{k} \). Hence we obtain \( \Sigma, \Delta \Rightarrow^{cf} \Pi \) by rule (\( \Omega_k \) left). Therefore \( \Sigma \in [\forall X. \varphi]^{\varphi^\varphi}_{n+1} \).

For the second equation, the inclusion \( \subseteq \) holds because \( [\exists X. \varphi]^{k} \subseteq [\exists X. \varphi]^{n+1} \). So let \( \Sigma \in [\exists X. \varphi]^{\varphi^\varphi}_{k} \) and \( (\Gamma \Rightarrow \Pi) \in [\exists X. \varphi]^{n+1} \). The former means that \( \Sigma, \Delta \Rightarrow^{cf} \Lambda \) for any
(Δ ⇒ Λ) ∈ |∃X.φ|_{k+1}, while the latter means that φ(Y), Λ ⇒ Π with Y \notin FV(Γ, Π). Hence we obtain Σ, Γ ⇒ cf Π by rule (Ω_k right). Therefore Σ ∈ |∃X.φ|_{n+1}.

Now a crucial lemma follows (called the “main lemma” in [28]).

Lemma 5.5. For every formula ξ in FMP_n, ξ ∈ I(ξ) ⊆ ξ^c.

Proof. By induction on the structure of ξ.

1. ξ is an atomic formula. We have I(ξ) = γ(ξ), hence the claim holds by (∗) above.

2. ξ = ϕ ∧ ψ. We first show ϕ ∧ ψ ∈ I(ϕ ∧ ψ) = I(ϕ) ∩ I(ψ). Let (Γ ⇒ Π) ∈ I(ϕ)^Δ. Then the induction hypothesis ϕ ∈ I(ϕ) yields ϕ, Γ ⇒ cf Π, so ϕ ∧ ψ, Γ ⇒ cf Π by rule (∧ left). That is, ϕ ∧ ψ ∈ I(ϕ)^cf = I(ϕ). Likewise, ϕ ∧ ψ ∈ I(ψ). So ϕ ∧ ψ ∈ I(ϕ ∧ ψ).

   We next show I(ϕ ∧ ψ) ⊆ (ϕ ∧ ψ)^c. Let Γ ∈ I(ϕ) ∩ I(ψ). Then we have Γ ⇒ cf ϕ and Γ ⇒ cf ψ by the induction hypotheses. So Γ ⇒ cf ϕ ∧ ψ by rule (∧ right). That is, Γ ∈ (ϕ ∧ ψ)^c.

3. ξ = ϕ ∨ ψ. We first show ϕ ∨ ψ ∈ I(ϕ ∨ ψ) = γ(I(ϕ) ∪ I(ψ)). Let (Γ ⇒ Π) ∈ (I(ϕ) ∪ I(ψ))^Δ. Then the induction hypotheses ϕ ∈ I(ϕ) and ψ ∈ I(ψ) yield ϕ, Γ ⇒ cf Π and ψ, Γ ⇒ cf Π. Hence ϕ ∨ ψ, Γ ⇒ cf Π by rule (∨ left). That is, ϕ ∨ ψ, Γ ⇒ cf Π = I(ϕ ∨ ψ).

   We next show I(ϕ ∨ ψ) ⊆ (ϕ ∨ ψ)^c. Let Γ ∈ I(ϕ) ∪ I(ψ), say Γ ∈ I(ϕ). Then Γ ⇒ cf ϕ by the induction hypothesis. Hence Γ ⇒ cf ϕ ∨ ψ by rule (∨ right). That is, Γ ∈ (ϕ ∨ ψ)^c. This proves that I(ϕ ∨ ψ) ⊆ (ϕ ∨ ψ)^c.

4. ξ = ϕ → ψ. We first show ϕ → ψ ∈ I(ϕ → ψ). Let Σ ∈ I(ϕ) and (Γ ⇒ Π) ∈ I(ψ)^Δ. Then Σ ⇒ cf ϕ and ψ, Γ ⇒ cf Π by the induction hypotheses. Hence Σ, ϕ → ψ, Γ ⇒ cf Π by rule (→ left). Since this holds for any (Γ ⇒ Π) ∈ I(ψ)^Δ, we have Σ, ϕ → ψ ∈ I(ψ)^cf = I(ψ). Since this holds for any Σ ∈ I(ϕ), we conclude ϕ → ψ ∈ I(ϕ → ψ).

   We next show I(ϕ → ψ) ⊆ (ϕ → ψ)^c. Let Γ ∈ I(ϕ) → I(ψ). Since ϕ ∈ I(ϕ) and I(ψ) ⊆ ψ^c by the induction hypotheses, we have ϕ, Γ ⇒ cf ψ. Hence Γ ⇒ cf ϕ → ψ by rule (→ right). That is, Γ ∈ (ϕ → ψ)^c.

5. ξ = ∀x.ϕ(x). We first show \forall x.ϕ(x) ∈ I(∀x.ϕ(x)) = \bigcap_{t \in Tm} I(ϕ(t)). Let t ∈ Tm and (Γ ⇒ Π) ∈ I(ϕ(t))^Δ. Then the induction hypothesis ϕ(t) ∈ I(ϕ(t)) yields ϕ(t), Γ ⇒ cf Π, so ∀x.ϕ, Γ ⇒ cf Π by rule (\forall x left). That is, ∀x.ϕ ∈ I(ϕ(t))^cf = I(ϕ(t)). Since it holds for any t ∈ Tm, we conclude ∀x.ϕ ∈ \bigcap_{t \in Tm} I(ϕ(t)) = I(∀x.ϕ(x)).

   We next show I(∀x.ϕ(x)) ⊆ (∀x.ϕ(x))^c. Let Γ ∈ \bigcap_{t \in Tm} I(ϕ(t)). The proof splits into two cases.

   (i) If n = -1, we choose a variable y such that y \notin Fv(Γ). Then Γ ∈ I(ϕ(y)). We have Γ ⇒ cf ϕ(y) by the induction hypothesis, so Γ ⇒ cf ∀x.ϕ(x) by rule (∀x right). That is, Γ ∈ (∀x.ϕ(x))^c.

   (ii) If n ≥ 0, we have Γ ∈ I(ϕ(t)) ⊆ ϕ(t)^c for every t ∈ Tm. Hence Γ ⇒ cf ϕ(t), so Γ ⇒ cf ∀x.ϕ(x) by rule (ω right). That is, Γ ∈ (∀x.ϕ(x))^c.

6. ξ = ∃x.ϕ(x). We first show ∃x.ϕ(x) ∈ I(∃x.ϕ(x)). Let (Γ ⇒ Π) ∈ (\bigcup_{t \in Tm} I(ϕ(t)))^Δ. The proof splits into two cases.

   (i) If n = -1, we choose a variable y such that y \notin Fv(Γ). Then Γ ∈ I(ϕ(y)). We have Γ ⇒ cf ϕ(y) by the induction hypothesis, so Γ ⇒ cf ∀x.ϕ(x) by rule (∀x right). That is, Γ ∈ (∀x.ϕ(x))^c.

   (ii) If n ≥ 0, we have Γ ∈ I(ϕ(t)) ⊆ ϕ(t)^c for every t ∈ Tm. Hence Γ ⇒ cf ϕ(t), so Γ ⇒ cf ∀x.ϕ(x) by rule (ω right). That is, Γ ∈ (∀x.ϕ(x))^c.
• If $n = -1$, choose a variable $y$ such that $y \notin \text{Fv}(\Gamma, \Pi)$. Since $\varphi(y) \in \mathcal{I}(\varphi(y)) \subseteq \bigcup_{t \in \text{Tm}} \mathcal{I}(\varphi(t))$ by the induction hypothesis, we have $\varphi(y), \Gamma \Rightarrow^c \Pi$. Hence $\exists x. \varphi, \Gamma \Rightarrow^c \Pi$ by rule ($\exists x$ left). That is, $\exists x. \varphi \in (\bigcup_{t \in \text{Tm}} \mathcal{I}(\varphi(t)))^{\omega <} = \mathcal{I}(\exists x. \varphi(x))$.

• If $n \geq 0$, we have $\varphi(t) \in \mathcal{I}(\varphi(t)) \subseteq \bigcup_{t \in \text{Tm}} \mathcal{I}(\varphi(t))$ for every $t \in \text{Tm}$. Hence $\varphi(t), \Gamma \Rightarrow^c \Pi$, and so $\exists x. \varphi, \Gamma \Rightarrow^c \Pi$ by rule ($\exists x$ left). That is, $\exists x. \varphi \in \mathcal{I}(\exists x. \varphi(x))$.

We next show $\mathcal{I}(\exists x. \varphi(x)) \subseteq (\exists x. \varphi(x))^c$. Let $\Gamma \in \bigcup_{t \in \text{Tm}} \mathcal{I}(\varphi(t))$, say $\Gamma \in \mathcal{I}(\varphi(t))$. Then $\Gamma \Rightarrow^c \varphi(t)$ by the induction hypothesis. Hence $\Gamma \Rightarrow^c \exists x. \varphi$ by rule ($\exists x$ right). That is, $\Gamma \in (\exists x. \varphi)^c$. This proves that $\mathcal{I}(\exists x. \varphi(x)) \subseteq (\exists x. \varphi(x))^c$.

(7) $\xi = \bot$. Omitted.

(8) $\xi = \forall X. \varphi$. We have $\forall X. \varphi \in |\forall X. \varphi|_{n+1} \subseteq \mathcal{I}(\forall X. \varphi)$, since $\forall X. \varphi \Rightarrow \varphi(Y)$ is cut-free provable in $\text{LIP}_n$. We also have $|\forall X. \varphi|_{n+1} \subseteq (\forall X. \varphi)^c$ by rule ($\forall X$ right). Hence the claim follows.

(9) $\xi = \exists X. \varphi$. Similarly to (8). \hfill \square

5.3. Algebraic cut elimination by $\Omega$-interpretation. Given a sequent $\Gamma \Rightarrow \Pi$ in $\text{FMP}_{n+1}$, let

\[
\mathcal{I}(\Gamma) := \begin{cases} 
\emptyset & \text{if } \Gamma = \emptyset, \\
\bigcap_{\varphi \in \Gamma} \mathcal{I}(\varphi) & \text{otherwise},
\end{cases}
\]

\[
\mathcal{I}(\Pi) := \begin{cases} 
\gamma(\emptyset) & \text{if } \Pi = \emptyset, \\
\mathcal{I}(\varphi) & \text{if } \Pi = \{\varphi\}.
\end{cases}
\]

as in Section 4. We then have $\Gamma \in \mathcal{I}(\Gamma)$ and $\mathcal{I}(\Pi) \subseteq \Pi^c$ by Lemma 5.5.

Our next goal is to show that the interpretation $\mathcal{I}$ is sound for $\text{LIP}_{n+1}$. The proof consists of two steps.

Fix a set variable $X_0$ and $F \in \mathcal{G}(\text{CF})^{\text{Tm}}$. Define interpretation $\mathcal{I}_F : \text{FMP}_n \rightarrow \mathcal{G}(\text{CF})$ similarly to $\mathcal{I}$, except that $\mathcal{I}_F(X_0(0)) := F(0)$. Notice that we have $\mathcal{I}_F(\varphi) = \mathcal{I}(\varphi)$ if $X_0 \notin \text{Fv}(\varphi)$.

Lemma 5.6. If $\text{LIP}_n \vdash \Gamma \Rightarrow \Pi$, then $\mathcal{I}_F(\Gamma) \subseteq \mathcal{I}_F(\Pi)$.

Proof. When $n = -1$, this follows from the standard soundness theorem for the first order intuitionistic logic. So assume that $n \geq 0$. The proof proceeds by structural induction on the derivation. Let us only consider the cases for second order quantifiers.

(1) The derivation ends with

\[
\frac{\Gamma \Rightarrow \varphi(Y), Y \notin \text{Fv}(\Gamma)}{\Gamma \Rightarrow \forall X. \varphi(X)} (\forall X \text{ right}).
\]

Let $\Delta \in \mathcal{I}_F(\Gamma)$. We may assume that $Y \neq X_0$ and $Y \notin \text{Fv}(\Delta)$, since otherwise we can rename $Y$ to a new set variable. By the induction hypothesis and Lemma 5.5, we have $\Delta \in \mathcal{I}_F(\varphi(Y)) = \mathcal{I}(\varphi(Y)) \subseteq \varphi(Y)^c$, that is, $\Delta \Rightarrow^c \varphi(Y)$. Hence $\Delta \in |\forall X. \varphi|_{n+1} \subseteq \mathcal{I}_F(\forall X. \varphi)$.

(2) The derivation ends with

\[
\frac{\varphi(Y), \Gamma \Rightarrow \Pi \quad Y \notin \text{Fv}(\Gamma, \Pi)}{\exists X. \varphi(X), \Gamma \Rightarrow \Pi} (\exists X \text{ left}).
\]
We assume \( \Gamma = \emptyset \) for simplicity. Let \((\Delta \Rightarrow \Lambda) \in \mathcal{I}_F(\Pi)^\circ\). We may also assume that \(Y \neq X_0\) and \(Y \notin \text{FV}(\Delta, \Lambda)\). By the induction hypothesis and Lemma 5.5, \(\varphi(Y) \in \mathcal{I}(\varphi(Y)) \subseteq \mathcal{I}(\Pi)\), so \(\varphi(Y), \Delta \Rightarrow^\circ \Lambda\). That is, \(\mathcal{I}_F(\Pi)^\circ \subseteq \mathcal{I}(\exists X.\varphi(X))\). Hence we conclude that \(\mathcal{I}_F(\exists X.\varphi(X)) \subseteq \mathcal{I}_F(\Pi)\).

(3) The derivation ends with
\[
\{ \Delta, \Gamma \Rightarrow \Pi \}_{\Delta \in \exists X.\varphi|_k} (\Omega_k \text{ left}).
\]
We assume \( \Gamma = \emptyset \) for simplicity. We are going to use Lemma 5.4. So let \( \Delta = \Delta(X_0) \in \exists X.\varphi|_k \) and \((\Sigma \Rightarrow \Lambda) \in \mathcal{I}_F(\Pi)^\circ\). The former means that \( \text{LIP}_{\Omega k-1} \vdash \Delta \Rightarrow \varphi(Y) \) for some \( Y \notin \text{FV}(\Delta) \). We have \( \text{LIP}_{\Omega k-1} \vdash \Delta(Z) \Rightarrow \varphi(Y) \) with \( Z \) fresh by Lemma 3.3, so \( \Delta(Z) \in \exists X.\varphi|_k \). Hence \( \Delta(Z) \Rightarrow \Pi \) is among the premises. By the induction hypothesis, \( \mathcal{I}_F(\Delta(Z)) \subseteq \mathcal{I}_F(\Pi) \). Since \( \Delta(Z) \in \mathcal{I}(\Delta(Z)) = \mathcal{I}_F(\Delta(Z)) \) by Lemma 5.5, we have \( \Delta(Z), \Sigma \Rightarrow^\circ \Lambda \). From this, \( \Delta(X_0), \Sigma \Rightarrow^\circ \Lambda \) follows by Lemma 3.3. Since this holds for any \((\Sigma \Rightarrow \Lambda) \in \mathcal{I}_F(\Pi)^\circ\), we have \( \Delta \in \mathcal{I}_F(\Pi)^{\circ \circ} = \mathcal{I}_F(\Pi) \). Finally we conclude
\[
\mathcal{I}_F(\forall X.\varphi) = \mathcal{I}(\forall X.\varphi) = \gamma(\forall X.\varphi|_k) \subseteq \mathcal{I}_F(\Pi)
\]
by Lemma 5.4.

(4) The derivation ends with
\[
\{ \Gamma, \Delta \Rightarrow \Lambda \}_{\Delta \in \exists X.\varphi|_k} (\Omega_k \text{ right}).
\]
Again we are going to use Lemma 5.4. So let \((\Delta \Rightarrow \Lambda) \in \exists X.\varphi|_k \) and \( \Sigma \in \mathcal{I}(\Gamma) \). We may assume that \( X_0 \notin \text{FV}(\Delta, \Lambda) \), since otherwise it can be renamed by a fresh variable as in (3) above. By the induction hypothesis, \( \mathcal{I}_F(\Gamma, \Delta) \subseteq \mathcal{I}_F(\Delta) \). Since \( \Delta \in \mathcal{I}(\Delta) = \mathcal{I}_F(\Delta) \) and \( \mathcal{I}_F(\Lambda) \subseteq \Lambda^{\circ} \) by Lemma 5.5, we obtain \( \Sigma, \Delta \Rightarrow^\circ \Lambda \). This shows that
\[
\mathcal{I}_F(\Gamma) \subseteq \exists X.\varphi|^{\circ} = \mathcal{I}(\exists X.\varphi) = \mathcal{I}_F(\exists X.\varphi)
\]
by Lemma 5.4.

(5) The derivation ends with \((\Omega_k \text{ left})\) or \((\Omega_k \text{ right})\). We do not have to consider these cases, since these are derivable from other rules. \( \square \)

**Lemma 5.7.** If \( \text{LIP}_{n+1} \vdash \Gamma \Rightarrow \Pi \), then \( \mathcal{I}(\Gamma^o) \subseteq \mathcal{I}(\Pi^o) \) holds for every term substitution \( \circ \).

**Proof.** The proof is very similar to that of the previous lemma. The main difference is that we have to consider rules \((\forall X \text{ left})\) and \((\exists X \text{ right})\) instead of \((\Omega_k \text{ left})\) and \((\Omega_k \text{ right})\). So let us deal with only these two rules. For simplicity, we assume that \( \circ \) is an identity substitution.

(1) The derivation ends with rule \((\forall X \text{ left})\). It is sufficient to show that \( \mathcal{I}(\forall X.\varphi(X)) \subseteq \mathcal{I}(\varphi(\tau)) \) for any \( \forall X.\varphi(X) \in \text{FMP}_{n+1} \) and \( \tau \in \text{ABS}_{n+1} \). Let \( \Delta \in \forall X.\varphi|_{n+1} \), that is, \( \Delta \Rightarrow^\circ \varphi(X_0) \) with \( X_0 \) fresh. Define \( F \in \mathcal{G}(\text{CF})^\tau \) by \( F(t) := \mathcal{I}(\tau(t)) \). By Lemma 5.6, we have \( \mathcal{I}_F(\Delta) \subseteq \mathcal{I}_F(\varphi(X_0)) \). Notice that:
- \( \mathcal{I}_F(\Delta) = \mathcal{I}(\Delta) \) because \( X_0 \notin \text{FV}(\Delta) \).
- \( \Delta \in \mathcal{I}(\Delta) \) by Lemma 5.5, noting that \( \Delta \subseteq \text{FMP}_n \).
- \( \mathcal{I}_F(\varphi(X_0)) = \mathcal{I}(\varphi(\tau)) \) by induction on the structure of \( \varphi(X_0) \).
Hence we have $\Delta \in I(\varphi(\tau))$. This shows that $I(\forall X\varphi(X)) = \gamma(\forall X\varphi|_{n+1}) \subseteq I(\varphi(\tau))$.

(2) The derivation ends with rule ($\exists X$ right). It is sufficient to show that $I(\varphi(\tau)) \subseteq I(\exists X\varphi(X))$ for any $\exists X\varphi(X) \in \text{FMP}_{n+1}$ and $\tau \in \text{ABS}_{n+1}$. Let $(\Delta \Rightarrow \Lambda) \in [\exists X\varphi|_{n+1}]$, that is, $\varphi(X_0), \Delta \Rightarrow^c \Lambda$ with $X_0$ fresh. Define $F \in G(\text{CF})^{\text{TM}}$ by $F(t) := I(\tau(t))$. By Lemma 5.6 $I_F(\varphi(X_0), \Delta) \subseteq I_F(\Lambda)$. As before, we have:

- $I_F(\Delta) = I(\Delta)$ and $I_F(\Lambda) = I(\Lambda)$.
- $\Delta \in I(\Delta)$ and $I(\Lambda) \subseteq \Lambda^c$.
- $I_F(\varphi(Y)) = I(\varphi(\tau))$.

Together, this means that $\Sigma, \Delta \Rightarrow^c \Lambda$ holds for any $\Sigma \in I(\varphi(\tau))$ and any $(\Delta \Rightarrow \Lambda) \in [\exists X\varphi]_{n+1}$. Hence we conclude that $I(\varphi(\tau)) \subseteq I(\exists X\varphi(X))$. $\square$

By combining Lemmas 5.5 and 5.7 we obtain an algebraic proof of (partial) cut elimination.

**Theorem 5.8.** $\text{LIP}_{n+1} \vdash \Gamma \Rightarrow \Pi$ implies $\text{LI}_\Theta \vdash \varphi \Rightarrow \Pi$ provided that $\Gamma \cup \Pi \subseteq \text{FMP}_n$.

**Proof.** We have $I(\Gamma) \subseteq I(\Pi)$ by Lemma 5.7 while $\Gamma \in I(\Gamma)$ and $I(\Pi) \subseteq \Pi^c$ by Lemma 5.5. Hence $\Gamma \Rightarrow^c \Pi$, that is, $\text{LI}_\Theta \vdash \Gamma \Rightarrow \Pi$. $\square$

Together with Lemma 3.6 it leads to:

**Theorem 5.9** (Algebraic cut elimination for $\text{LIP}_{n+1}$). For every $n \geq -1$, $\text{LIP}_{n+1} \vdash \Gamma \Rightarrow \Pi$ implies $\text{LI} \vdash^c \varphi \Rightarrow \Pi$ provided that $\Gamma \cup \Pi \subseteq \text{Fm}$.

By combining this with Theorem 2.11 we may obtain complete cut elimination too.

**Corollary 5.10.** For every $n \geq -1$ and every sequent $\Gamma \Rightarrow \Pi$ of $\text{LIP}_{n+1}$, $\text{LIP}_{n+1} \vdash \Gamma \Rightarrow \Pi$ implies $\text{LIP}_{n+1} \vdash^c \varphi \Rightarrow \Pi$.

### 5.4. A small modification.

Theorem 5.8 gives a self-contained proof of the reduction from $\text{LIP}_{n+1}$ to cut-free $\text{LI}_\Theta$. On the other hand, it is also possible to prove the same thing relying on the fact that $\text{LI}_\Theta$ admits cut elimination (Lemma 3.5). This approach provides a simpler interpretation for the formulas in $\text{FMP}_n$ that will play an important technical role in the next section. It will also help clarify the essence of our algebraic argument so far.

**Lemma 5.11.** Assume that $\text{LI}_\Theta$ admits cut elimination. Then

$$I(\varphi) = \gamma(\varphi) = \varphi^c$$

holds for every $\varphi \in \text{FMP}_n$. As a consequence,

$$\gamma(\varphi \ast \psi) = \gamma(\varphi) \ast \gamma(\psi), \quad \gamma(\forall x\varphi(x)) = \bigcap_{t \in \text{TM}} \gamma(\varphi(t)), \quad \gamma(\exists x\varphi(x)) = \gamma\left( \bigcup_{t \in \text{TM}} \varphi(\varphi(t)) \right)$$

hold for every $\ast \in \{\land, \lor, \to\}$ and $\varphi, \psi \in \text{FMP}_n$.

**Proof.** By Lemma 5.5 we have $\gamma(\varphi) \subseteq I(\varphi) \subseteq \varphi^c$. Hence it suffices to show that $\varphi^c \subseteq \gamma(\varphi)$. Let $\Sigma \in \varphi^c$ and $(\Delta \Rightarrow \Lambda) \in \varphi^\Theta$. Then we have $\Sigma \Rightarrow^c \varphi$ and $\varphi, \Delta \Rightarrow^c \Lambda$, so $\Sigma, \Delta \Rightarrow^c \Lambda$ by cut elimination. Since this holds for any $(\Delta \Rightarrow \Lambda) \in \varphi^\Theta$, we conclude $\Sigma \in \gamma(\varphi)$. $\square$
Lemma 5.11 gives us a connection with the Lindenbaum algebra. Let us assume $n = -1$ for simplicity. Because of the lemma, we may restrict the underlying set $\mathcal{G}(\text{CF})$ of the Heyting algebra $\text{CF}^+$ to $\{\gamma(\varphi) : \varphi \in \text{Fm}\}$. This results in a Heyting subalgebra $\text{CF}^+_0$.

Notice that
$$\gamma(\varphi) \subseteq \gamma(\psi) \iff \textit{LI} \vdash \varphi \Rightarrow \psi.$$ 

Hence $\text{CF}^+_0$ is isomorphic to the Lindenbaum algebra for $\text{LI}$ (L in Section 4). Moreover it is not hard to see that $\text{CF}^+$ is the MacNeille completion of $\text{CF}^+_0$. To sum up:

**Proposition 5.12.** If $n = -1$, $\text{CF}^+$ is isomorphic to the MacNeille completion of the Lindenbaum algebra for $\text{LI}$.

Hence it seems reasonable to describe the essence of our argument as

MacNeille completion $+$ $\Omega$-interpretation.

This can be compared with the essence of the algebraic proof of cut elimination for $\text{LI}_2$ due to \cite{24,25}:

MacNeille completion $+$ reducibility candidates.

See Appendix C for the latter.

### 6. Formalizing cut elimination

In this section we outline how to formalize our proof of cut elimination for $\text{LIP}_{n+1}$ locally within $\text{ID}^i_{n+1}$. It consists of two steps (Subsections 6.1 and 6.2).

#### 6.1. Formalization in $\text{ID}^i_{n+1}$ (1)

Recall that the syntactic proof of cut elimination for $\text{LIP}_n$ (in Subsection 3.2) relies on a cut-free provability predicate $\text{LIO}^\text{cf}_k$ for $k = -1, \ldots , n$, which are definable in $\text{ID}^i_{n+1}$. Concretely, there is an $\text{ID}^i_{n+1}$-formula $\text{LIO}^\text{cf}_k(x)$ whose intended meaning is

$$\text{LIO}^\text{cf}_k(\langle \Gamma \Rightarrow \Pi \rangle) \iff \text{LI} \vdash \langle \Gamma \Rightarrow \Pi \rangle.$$ 

Likewise, we are given formulas $\text{LI} \Sigma_k^\Omega(x)$, $\text{LI} P_{n+1}(x)$ and $\text{LI} \gamma^\text{cf}_f(x)$ in $\text{ID}^i_{n+1}$ with analogous meanings. Notice that $\text{LIP}_{n+1}(x)$ and $\text{LI} \gamma^\text{cf}_f(x)$ are actually $\Sigma^0_1$ formulas of $\text{HA}$, since the proof systems are finitary. We assume that Lemma 3.5 for $\text{LI} \Sigma_k^\Omega_n$ has been already formalized in $\text{ID}^i_{n+1}$ (that is part of Theorem 3.7). Thus:

$$\text{ID}^i_{n+1} \vdash \forall x \in \text{SEQ}. \text{LI} \Sigma_k^\Omega_n(x) \Rightarrow \text{LI} \gamma^\text{cf}_f(x),$$

where $\text{SEQ}(x)$ is a unary predicate for the set of (codes of) $\text{LI} \Sigma_n$-sequents.

Let us now turn to the algebraic side. Recall that our syntactic frame $\text{CF}$ is defined in terms of cut-free provability in $\text{LIO}_n$. Thus the closure operator $\gamma$ can be formalized as follows. Given a set variable $X$, let

$$\gamma(x, X) := \forall y \in \text{SEQ}. \exists z \in X. \text{LI} \gamma^\text{cf}_n(\langle \Gamma z \Rightarrow \gamma y \rangle) \Rightarrow \text{LI} \gamma^\text{cf}_n(\langle \Gamma x \Rightarrow \gamma y \rangle),$$

where $\langle \Gamma z \Rightarrow \gamma y \rangle$ is a function symbol in variables $x, y$, whose intended meaning is a function that returns $\langle \Gamma \Sigma, \Gamma \Rightarrow \Pi \rangle$ given $\langle \Gamma \Sigma \rangle$ and $\langle \Gamma \Rightarrow \Pi \rangle$ as inputs. The intended meaning of $\gamma$ is that $\gamma(\langle \Delta, X \rangle)$ iff $\Delta \in \gamma(X)$.

Based on this, we can define an $\text{ID}^i_{n+1}$-formula $\text{Ip}_\varphi(x, y)$ for each $\varphi = \varphi(y) \in \text{FMP}_{n+1}$ such that

$$\text{Ip}_\varphi(\langle \Gamma \Delta, \Gamma \bar{t} \rangle) \iff \Delta \in \mathcal{I}(\varphi(\bar{t})).$$


This is possible because of the \(\Omega\)-interpretation technique, that allows us to interpret second order quantifiers by first order ones. It is not very hard to formalize Lemma 5.5: 
\[
\xi(i) \in \mathcal{I}(\xi(i)), \\
\Delta \in \mathcal{I}(\xi(i)) \quad \text{implies} \quad \Delta \Rightarrow \xi(i),
\]
for every \(\xi(x) \in \text{FMP}_n\), \(i \in \text{Tm}\) and \(\Delta \subseteq \text{fin} \text{FMP}_n\).

**Lemma 6.1.** For every formula \(\xi = \xi(y)\) in \(\text{FMP}_n\), \(\text{ID}^i_{n+1}\) proves
\[
\forall \vec{y} \in \text{Tm}. \text{Ip}_\xi(\neg\xi(\vec{y}) \land \neg\vec{y}), \\
\forall \vec{y} \in \text{Tm}. \forall z \in \text{Fset}. \text{Ip}_\xi(z, \vec{y}) \rightarrow \text{LI} \Omega^c_{n}(\neg\varphi \Rightarrow \xi(\vec{y}) \land \neg\vec{y}).
\]

Here \(\text{Tm}(x)\) and \(\text{Fset}(x)\) are unary predicates for the set of (codes of) terms and the set of (codes of) finite formula sets \(\Gamma, \Delta, \ldots\). In addition, \(\neg\xi(\vec{y}) \land \neg\vec{y}\) expresses a function in variables \(\vec{y}\) that returns \(\neg\xi(i)\) when given \(\vec{y}\) as inputs. \(\neg\varphi \Rightarrow \xi(\vec{y})\) should be understood accordingly.

Now the backbone of our argument is Lemma 5.7. So suppose that a derivation \(\pi_0\) of \(\Gamma_0 \Rightarrow \Pi_0\) in \(\text{LIP}_{n+1}\) is given, where \(\Gamma_0 \cup \Pi_0 \subseteq \text{FMP}_n\). Since there are only finitely many formulas occurring in \(\pi_0\), we obtain a formula \(\text{Ip}(x, y, \vec{z})\) such that
\[
\text{Ip}(\neg\Delta, \neg\Gamma, \neg\vec{t}) \iff \Delta \in \mathcal{I}(\Gamma(i))
\]
for any \(\Delta \subseteq \text{fin} \text{FMP}_n\) and \(\Gamma = \Gamma(\vec{y})\) that consists of formulas occurring in \(\pi_0\).

We would like to show that \(\text{ID}^i_{n+1}\) proves that \(\mathcal{I}(\Gamma_0) \subseteq \mathcal{I}(\Pi_0)\) (formally expressed by using predicate \(\text{Ip}\) above). If successful, we may further obtain
\[
\text{ID}^i_{n+1} \vdash \text{LI} \Omega^c_{n}(\neg\Gamma_0 \Rightarrow \Pi_0)
\]
with the help of Lemma 6.1 (see the proof of Theorem 5.8). Combined with a formalized proof of Lemma 3.6, we will be able to conclude
\[
\text{ID}^i_{n+1} \vdash \text{LI} \Omega^c_{n}(\neg\Gamma_0 \Rightarrow \Pi_0).
\]

The hardest part of the whole work is to properly formalize Lemma 5.6, which is a prerequisite for Lemma 5.7. We will argue that it is indeed possible in the next subsection.

### 6.2. Local formalization in \(\text{ID}^i_{n+1}\) (2)

Before addressing Lemma 5.6, a bit of preliminary is needed.

We fix a variable \(X_0\), a formula \(\forall X.\varphi_0(X)\) and abstract \(\tau_0\) that occur in the derivation \(\pi_0\) of \(\Gamma_0 \Rightarrow \Pi_0\) in \(\text{LIP}_{n+1}\). Lemma 5.6 is invoked by letting \(F := \mathcal{I}(\tau_0)\) and by considering a cut-free derivation of \(\Delta \Rightarrow \varphi_0(X_0)\) in \(\text{LIO}_n\) with \(X_0 \notin \text{FV}(\Delta)\) (see case (1) in the proof of Lemma 5.7). Actually, there is also a dual case corresponding to case (2), but let us forget about it for simplicity. So, there is an \(\text{ID}^i_{n+1}\)-formula \(F(x, y)\) whose intended meaning is that
\[
F(\neg\Delta, \neg\Gamma, \neg\vec{t}) \iff \Delta \in F(t) \iff \Delta \in \mathcal{I}(\tau(t)).
\]

We define the *subformula relation* to be the transitive reflexive closure of the following:
\[
\varphi, \psi \subseteq \varphi \times \psi, \quad \varphi(x) \subseteq Qx.\varphi(x),
\]
where \(\times \in \{\land, \lor, \rightarrow\}\) and \(Q \in \{\forall, \exists\}\). That is, \(QX.\varphi(X)\) does not have any proper subformula. Clearly the set \(\text{Sf}(\varphi_0)\) of subformulas of \(\varphi_0\) is finite. Hence as before, there is a formula \(\text{Ip}_F(x, y, \vec{z})\) such that
\[
\text{Ip}_F(\neg\Delta, \neg\Gamma, \neg\vec{t}) \iff \Delta \in \mathcal{I}_F(\Gamma(i))
\]
for any $\Delta \subseteq_{\text{fin}} \FMP_n$ and $\Gamma = \Gamma(\vec{y}) \subseteq_{\text{fin}} \Sf(\varphi_0)$.

Let $\SEQ(\varphi_0)$ be the (finite) set of sequents that consist of formulas in $\Sf(\varphi_0)$. Then Lemma 5.6 can be formalized as follows.

**Lemma 6.2.** $\ID^{i+1}_{n+1}$ proves the following statement (suitably formalized): for any sequent $\Gamma(\vec{y}) \Rightarrow \Pi(\vec{y})$ in $\SEQ(\varphi_0)$, for any terms $\vec{t}$ and for any $\Delta \subseteq_{\text{fin}} \FMP_n$ with $X_0 \not\in \FV(\Delta)$,

$$\LIO_\Delta \vdash^{cf} \Delta, \Gamma(\vec{t}) \Rightarrow \Pi(\vec{t}) \quad \text{implies} \quad \gamma(\Delta) \cap I_F(\Gamma(\vec{t})) \subseteq I_F(\Pi(\vec{t})).$$

Also, for any $\Gamma(\vec{y}) \subseteq_{\text{fin}} \Sf(\varphi_0)$, for any terms $\vec{t}$ and for any sequent $\Delta \Rightarrow \Pi$ with $\Delta \cup \Pi \subseteq \FMP_n$ and $X_0 \not\in \FV(\Delta, \Pi)$,

$$\LIO_\Delta \vdash^{cf} \Gamma(\vec{t}), \Delta \Rightarrow \Lambda \quad \text{implies} \quad I_F(\Gamma(\vec{t})) \cap \gamma(\Delta) \subseteq \Lambda^\triangleq.$$

**Proof.** By structural induction on the cut-free derivation (that is available in $\ID^{i+1}_{n+1}$). Since $\gamma(\Delta) = I_F(\Delta)$ and $\Lambda^\triangleq = I_F(\Lambda)$ by Lemma 5.11, the first statement amounts to:

$$\LIO_\Delta \vdash^{cf} \Delta, \Gamma(\vec{t}) \Rightarrow \Pi(\vec{t}) \quad \text{implies} \quad I_F(\Delta, \Gamma(\vec{t})) \subseteq I_F(\Pi(\vec{t})).$$

Hence the proof of Lemma 5.6 can be formalized almost straightforwardly. $\square$

We also have to ensure that the whole construction is primitive recursive.

**Lemma 6.3.** Given $\forall X. \varphi_0(X) \in \FMP_{n+1}$ and $\tau_0 \in \ABS_{n+1}$, there is a derivation $\pi$ of the following statement (formalized in $\ID^{i+1}_{n+1}$): for any $\Delta \subseteq_{\text{fin}} \FMP_n$ with $X_0 \not\in \FV(\Delta)$,

$$\LIO_\Delta \vdash^{cf} \Delta \Rightarrow \varphi_0(X_0) \quad \text{implies} \quad \Delta \in I_F(\varphi_0(X_0)).$$

Moreover, $\pi$ is computable from $\forall X. \varphi_0(X)$ and $\tau$ primitive recursively.

This is certainly true since all the reasoning is constructive and parametric in $\forall X. \varphi_0(X)$ and $\tau_0$.

Let us now come back to the derivation $\pi_0$ of $\Gamma_0 \Rightarrow \Pi_0$ in $\LIP_{n+1}$. In the proof of Lemma 5.7, Lemma 5.6 is invoked finitely many times depending on $\pi_0$. Moreover, we can verify that $\ID^{i+1}_{n+1}$ proves

$$I_F(\varphi_0(Y)) = I(\varphi_0(\tau_0)).$$

Hence Lemma 5.7 can be formalized as follows.

**Lemma 6.4.** Let $\pi_0$ be a derivation of $\Gamma_0 \Rightarrow \Pi_0$ in $\LIP_{n+1}$. There is a derivation $\pi_1$ of the statement $I(\Gamma_0) \subseteq I(\Pi_0)$ formalized in $\ID^{i+1}_{n+1}$. Moreover, $\pi_1$ is computable from $\pi_0$ primitive recursively.

This is again a matter of routine work.

As explained before, this lemma together with Lemma 6.1 and (a formalized version of) Lemma 3.6 gives rise to a proof of partial cut elimination for $\LIP_{n+1}$ locally formalized in $\ID^{i+1}_{n+1}$. Let us record this fact (with $m := n + 1$).

**Theorem 6.5.** $\IΣ_1$ proves the statement that for every sequent $\Gamma \Rightarrow \Pi$ of $\LI$,

$$\LIP_m \vdash \Gamma \Rightarrow \Pi \quad \text{implies} \quad \ID^i_m \vdash \LIF(\Gamma \Rightarrow \Pi^\prime).$$

Assuming the 1-consistency of $\ID^i_m$, we obtain

$$\LIP_m \vdash \Gamma \Rightarrow \Pi \quad \text{implies} \quad \LIF \vdash^{cf} \Gamma \Rightarrow \Pi,$$

that is nothing but a statement of partial cut elimination. Hence by combining it with Theorems 2.10 and 2.11, we finally obtain:
Theorem 6.6 (Takeuti correspondence between $\text{ID}_m^i$ and $\text{LIP}_m$). For every $m < \omega$,

$$\Sigma_1 \vdash \text{CE}(\text{LIP}_m) \iff 1\text{CON}(\text{ID}_m^i).$$

In the above theorem, $\text{ID}_m^i$ can be replaced by $\text{ID}_m$ by Theorem 2.1. Also, $\text{LIP}_m$ can be replaced by its classical counterpart since our proof of cut elimination works for classical systems with some minor changes.

Remark 6.7. It is not our original idea to combine a syntactic argument based on the $\Omega$-rule with a semantic argument to save one inductive definition. For instance, Aehlig [1] employs Tait’s computability predicate defined on a provability predicate based on the $\Omega$-rule. He works on the parameter-free, negative fragments of second order Heyting arithmetic without induction, and proves partial cut elimination in the corresponding $\text{ID}$-theories. His result is comparable with ours, but our approach based on the MacNeille completion works for logical systems with the full set of connectives (recall that second order definitions of positive connectives $\{\lor, \exists\}$ are not available in the parameter-free setting). Moreover, it works for classical logical systems too (because the variety of Boolean algebras is closed under MacNeille completions).

7. Conclusion

In this paper, we have brought the $\Omega$-rule technique originally developed in arithmetic into the logical setting, and studied it from an algebraic perspective. We have found an intimate connection with the MacNeille completion (Theorem 4.2), that is important in two ways. First, it provides an unexpected link between ordered algebra and proof theory. Second, it inspires an algebraic form of the $\Omega$-rule, called the $\Omega$-interpretation, that can be used to give an algebraic proof of cut elimination for $\text{LIP}_m$ (with $m < \omega$). As we have argued in Subsection 5.4, the essence of our approach could be summarized as

MacNeille completion + $\Omega$-interpretation.

This combination, together with some syntactic arguments, leads to a cut elimination proof which is locally formalizable in $\text{ID}_m^i$.

An outcome is the Takeuti correspondence between $\text{ID}$-theories and parameter-free logics (Theorem 6.6):

$$\Sigma_1 \vdash \text{CE}(\text{LIP}_m) \iff 1\text{CON}(\text{ID}_m^i).$$

This result should not be surprising for proof theorists at all, although we do not find any work formally proving this in the literature (either for the intuitionistic or classical logic).

Our emphasis rather lies in the methodological aspect. The algebraic approach works fine not just for full second order logics but also for their parameter-free fragments. Moreover, it works uniformly both for the intuitionistic and classical logics because of a purely algebraic reason: the variety of Heyting algebras and that of Boolean algebras are both closed under MacNeille completions (Theorem 1.2).

Our intuitionistic sequent calculus $\text{LIP}_{<\omega}$ roughly corresponds to the classical calculus studied in [33]. Hence what we have achieved in this paper is to algebraically reformulate Takeuti’s classical cut elimination theorem that accounts for the 1-consistency of $\Pi^1_1$-$\text{CA}_0$ [33]. Our hope is to expand this algebraic approach to more recent advanced results in proof theory, although we are not optimistic at all.
ACKNOWLEDGMENT

The author is grateful to Ryota Akiyoshi for useful comments.

REFERENCES

[1] K. Aehlig. Induction and inductive definitions in fragments of second order arithmetic. *Journal of Symbolic Logic*, 70:1087–1107, 2005.
[2] K. Aehlig. Parameter-free polymorphic types. *Annals of Pure and Applied Logic*, 156:3–12, 2008.
[3] R. Akiyoshi. An ordinal-free proof of the complete cut-elimination theorem for \( \Pi^1_1 \)-CA+BI with the \( \omega \)-rule. *IfCoLog Journal of Logics and their Applications*, 4(4):867–883, 2017.
[4] R. Akiyoshi and G. Mints. An extension of the Omega-rule. *Archive for Mathematical Logic*, 55(3):593–603, 2016.
[5] R. Akiyoshi and K. Terui. Strong normalization for the parameter-free polymorphic lambda calculus based on the Omega-rule. *Proceedings of FSCD 2016*, 5:1–15, 2016.
[6] T. Altenkirch and T. Coquand. A finitary subsystem of the polymorphic \( \lambda \)-calculus. *Proceedings of TLCA 2001*, 22–28, 2001.
[7] T. Arai. Cut-eliminability in second order logic calculi. *Annals of the Japan Association for Philosophy of Science*, 27:47–60, 2018.
[8] B. Banaschewski. Hüllensysteme und Erweiterungen von Quasi-Ordnungen. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 2: 35–46, 1956.
[9] J. Harding and G. Bezhanishvili. MacNeille completions of Heyting algebras. *The Houston Journal of Mathematics*, 30(4):937–952, 2004.
[10] G. Birkhoff. *Lattice Theory*. AMS, 1940.
[11] W. Buchholz. The \( \Omega_{\mu+1} \)-rule. In [13], 188–233, 1981.
[12] W. Buchholz. Explaining the Gentzen-Takeuti reduction steps. *Archive for Mathematical Logic*, 40:255–272, 2001.
[13] W. Buchholz, S. Feferman, W. Pohlers and W. Sieg. *Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies*, LNM 897, Springer, 1981.
[14] W. Buchholz and K. Schütte. *Proof Theory of Impredicative Subsystems of Analysis*, Bibliopolis, 1988.
[15] A. Ciabattoni, N. Galatos and K. Terui. Algebraic proof theory for substructural logics: cut-elimination and completions. *Annals of Pure and Applied Logic*, 163(3):266-290, 2012.
[16] B.A. Davey and H.A. Priestley. *Introduction to lattices and order, second edition*. Cambridge University Press, 2002.
[17] N. Funayama. On the completion by cuts of distributive lattices. *Proceedings of the Imperial Academy, Tokyo*, 20:1–2, 1944.
[18] N. Galatos and P. Jipsen. Residuated frames with applications to decidability. *Transactions of the AMS*, 365(3):1219–1249, 2013.
[19] M. Gehrke and J. Harding. Bounded lattice expansions. *Journal of Algebra*, 238(1):345–371, 2001.
[20] M. Gehrke and B. Jönsson. Bounded distributive lattice expansions. *Mathematica Scandinavica*, 94(1):13–45, 2004.
[21] J.-Y. Girard. *Proof theory and logical complexity, Vol. I*. Bibliopolis, 1987.
[22] G. Jäger and T. Studer. A Buchholz rule for modal fixed point logics. *Logica Universalis* 5(1):1–19, 2011.
[23] B. Jönsson and A. Tarski. Boolean algebras with operators I. *American Journal of Mathematics*, 73: 891–939, 1951.
[24] S. Maehara. Lattice-valued representation of the cut-elimination theorem. *Tsukuba Journal of Mathematics*, 15(9):509–521, 1991.
[25] G. Mints and T. Studer. Cut-elimination for the \( \mu \)-calculus with one variable. *Fixed Points in Computer Science*, 77: 47–54, 2012.
The purpose of this section is to prove Lemma \ref{lemma:A.1}, which is used in the proof of Theorem \ref{thm:2.8}.

First of all, recall that the axioms of $\text{PA}$ (and $\text{HA}$) consist of the equality axioms, $\forall xy, s(x) = s(y) \to x = y$, $\forall x, s(x) \neq 0$, the defining axioms for primitive recursive functions as well as the induction axioms. All are $\Pi^0_1$ sentences except the last ones.

**Lemma A.1.** Given a term $t$, let $N_t(x) := N(t(x))$. LIP$_0$ proves

$$\forall x \in N. N_t(x) \to N_t(s(x)) \land N_t(0) \to \forall y \in N. N_t(y).$$

This can be proved similarly to Lemma \ref{lemma:2.7}.

Given a list $\bar{x} = x_1, \ldots, x_n$ of variables, we denote the list $N(x_1), \ldots, N(x_n)$ by $N(\bar{x})$.

In the following lemmas, when we write $t(\bar{x})$ or $\varphi(\bar{x})$, we assume that all free variables in $t$ or $\varphi$ are included in $\bar{x}$.

**Lemma A.2.** For every term $t(\bar{x})$ over $L_{\text{PA}}$, LIP$_0$ proves

$$N(\bar{x}), \Gamma \Rightarrow N(t(\bar{x})), $$

where $\Gamma$ consists of some $\Pi^0_1$ axioms of $\text{PA}$.

**Proof.** We have already seen that LIP$_0$ proves $N(0)$ and $N(x) \Rightarrow N(s(x))$. Hence it is sufficient to show that LIP$_0 \vdash N(\bar{x}) \Rightarrow N(f(\bar{x}))$ for each symbol $f$ for a primitive recursive function. We only consider a simplified case, where $f$ is defined from constant $c$ and binary function $h$ by the sentence:

$$\text{Def}(f) := f(0) = c \land \forall x. f(s(x)) = h(x, f(x)).$$

We further assume that the claim has been proved for $c$ and $h$. That is, LIP$_0$ proves

$$\Gamma \Rightarrow N(c), \quad N(x), N(y), \Gamma \Rightarrow N(h(x, y)).$$
From the former, we obtain
\[ \Gamma, \text{Def}(f) \Rightarrow N(f(0)) \]
by using \textit{Sub}(N). From the latter, we obtain \( \Gamma, \text{Def}(f), N(x), N(f(x)) \Rightarrow N(f(s(x))) \), so
\[ \Gamma, \text{Def}(f) \Rightarrow \forall x \in N. N(f(x)) \Rightarrow N(f(s(x))). \]
\[ \Gamma, \text{Def}(f) \Rightarrow \forall y \in N.N(f(y)). \]
Therefore,
\[ \text{LIP}_0 \vdash N(y), \Gamma, \text{Def}(f) \Rightarrow N(f(y)) \]
as required.

Once Lemma A.2 has been proved, it is routine to prove the relativization lemma below.

**Lemma A.3** (Relativization). \( \text{LI} \vdash \Gamma \Rightarrow \Pi \) implies \( \text{LIP}_0 \vdash N(\bar{x}), \Gamma^\Pi \Rightarrow \Pi^N \), where \( Fv(\Gamma, \Pi) \subseteq \{ \bar{x} \} \).

Now let us put things together.

**Lemma A.4.** If \( \text{HA} \) proves a \( \Sigma^0_1 \) sentence \( \varphi \), then \( \text{LIP}_0 \) proves \( \Gamma \Rightarrow \varphi \) where \( \Gamma \) consists of some \( \Pi^0_1 \) axioms of \( \text{PA} \).

**Proof.** If \( \text{HA} \) proves \( \varphi \), \( \text{LI} \) proves \( \Gamma, \Delta \Rightarrow \varphi \) where \( \Gamma \) consists of \( \Pi^0_1 \) axioms and \( \Delta \) of induction axioms. By Lemmas 2.7 and A.3 we obtain \( \text{LIP}_0 \vdash \Gamma^\Pi \Rightarrow \varphi^N \). Since each \( \psi \in \Gamma \) is \( \Pi^0_1 \) and \( \varphi \) is \( \Sigma^0_1 \), we have \( \psi \Rightarrow \psi^N \) and \( \varphi^N \Rightarrow \varphi \), so that we finally obtain \( \text{LIP}_0 \vdash \Gamma \Rightarrow \varphi \). \( \square \)

### Appendix B. Proof of Lemma B.5

First, we define the \textit{rank} of each formula \( \varphi \in \text{FMP}_n \), denoted by \( \text{rank}(\varphi) \), as follows:

- \( \text{rank}(\bot) = \text{rank}(X(t)) = \text{rank}(p(\bar{t})) = \text{rank}(\forall X.x) = \text{rank}(\exists X.x) := 0 \),
- \( \text{rank}(\varphi \ast \psi) := \max\{\text{rank}(\varphi), \text{rank}(\psi)\} + 1 \ast \in \{\land, \lor, \rightarrow\} \),
- \( \text{rank}(\forall x.\varphi) = \text{rank}(\exists x.\varphi) := \text{rank}(\varphi) + 1 \).

Given an ordinal \( \alpha \leq \omega \), we write \( \vdash_\alpha \Gamma \Rightarrow \Pi \) if \( \Gamma \Rightarrow \Pi \) has a derivation in \( \text{LI}_\Omega_n \) in which all cut formulas are of rank strictly less than \( \alpha \). Thus \( \vdash_\omega \Gamma \Rightarrow \Pi \) means \( \text{LI}_\Omega_n \vdash \Gamma \Rightarrow \Pi \), and \( \vdash_0 \Gamma \Rightarrow \Pi \) means \( \text{LI}_\Omega_n \vdash_{cf} \Gamma \Rightarrow \Pi \).

**Lemma B.1.** Let \( m < \omega \). Suppose that \( \text{rank}(\varphi) \leq m \), \( \vdash_m \varphi, \Gamma \Rightarrow \Pi \) and \( \vdash_m \Gamma \Rightarrow \varphi \), where in the derivation of \( \Gamma \Rightarrow \varphi \) the main formula of the last inference step is the indicated \( \varphi \). Then \( \vdash_m \Gamma \Rightarrow \Pi \).

**Proof.** Let \( \pi_l \) be the derivation of \( \Gamma \Rightarrow \varphi \) and \( \pi_r \) that of \( \varphi, \Gamma \Rightarrow \Pi \). We argue by structural induction on \( \pi_r \). Let us only verify a few cases.

1. The main formula of the last inference of \( \pi_r \) is not \( \varphi \). In this case, the claim follows immediately from the induction hypothesis.

2. \( \pi_l \) and \( \pi_r \) respectively end with
   \[ \frac{\Gamma, \varphi_1 \Rightarrow \varphi_2}{\Gamma \Rightarrow \varphi_1 \Rightarrow \varphi_2} \quad (\rightarrow \text{right}), \quad \frac{\varphi_1 \rightarrow \varphi_2, \Gamma \Rightarrow \varphi_1 \varphi_2, \varphi_1 \rightarrow \varphi_2, \Gamma \Rightarrow \Pi}{\varphi_1 \rightarrow \varphi_2, \Gamma \Rightarrow \Pi} \quad (\rightarrow \text{left}), \]
where we assume that the upper sequents of rule \( \rightarrow \) left contain \( \varphi_1 \rightarrow \varphi_2 \) in the antecedent. The argument would be simpler if the formula is absent.

By the induction hypothesis, we have \( \vdash_m \Gamma \Rightarrow \varphi_1 \) and \( \vdash_m \varphi_2, \Gamma \Rightarrow \Pi \). Hence applying (cut) on \( \varphi_1 \) and \( \varphi_2 \), we obtain \( \vdash_m \Gamma \Rightarrow \Pi \), noting that \( \text{rank}(\varphi_i) \leq m \) for \( i = 1, 2 \).

(3) \( \pi_l \) and \( \pi_r \) respectively end with

\[
\frac{\{ \Gamma \Rightarrow \varphi(t) \}_{t \in T_m} \quad \varphi(t), \forall x.\varphi, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \forall x.\varphi \quad \forall x.\varphi, \Gamma \Rightarrow \Pi} \quad \text{(\( \forall x \) left)}.
\]

By the induction hypothesis, we have \( \vdash_m \varphi(t), \Gamma \Rightarrow \Pi \). Hence applying (cut) on \( \varphi(t) \), we obtain \( \vdash_m \Gamma \Rightarrow \Pi \), noting that \( \text{rank}(\varphi(t)) < m \).

(4) \( \pi_l \) and \( \pi_r \) respectively end with

\[
\frac{\Gamma \Rightarrow \varphi(Y) \quad \{ \Delta, \forall X.\xi, \Gamma \Rightarrow \Pi \}_{\Delta \in \forall X.\xi|k} \quad \forall X.\xi, \Gamma \Rightarrow \Pi \quad \text{(\( \forall X \) right)}}{\Gamma \Rightarrow \forall X.\varphi} \quad \text{(\( \forall X \) left)}.
\]

By the induction hypothesis, we have \( \vdash_m \Delta, \Gamma \Rightarrow \Pi \) for every \( \Delta \in \forall X.\xi|k \). Hence we may apply

\[
\frac{\Gamma \Rightarrow \varphi(Y) \quad \{ \Delta, \Gamma \Rightarrow \Pi \}_{\Delta \in \forall X.\xi|k} \quad \forall X.\xi, \Gamma \Rightarrow \Pi \quad \text{(\( \forall X \) left)}}{\Gamma \Rightarrow \Pi} \quad \text{(\( \forall X \) left)}.
\]

Lemma B.2. Let \( m < \omega \). Suppose that \( \text{rank}(\varphi) \leq m \), \( \vdash_m \varphi, \Gamma \Rightarrow \Pi \) and \( \vdash_m \Gamma \Rightarrow \varphi \). Then \( \vdash_m \Gamma \Rightarrow \Pi \).

Proof. By structural induction on the derivation of \( \Gamma \Rightarrow \varphi \). If the main formula of the last inference is \( \varphi \), it follows from Lemma [B.1]. Suppose otherwise. For instance, suppose that the derivation ends with

\[
\frac{\{ \Delta, \Gamma \Rightarrow \varphi \}_{\Delta \in \forall X.\xi|k} \quad \forall X.\xi, \Gamma \Rightarrow \varphi \quad \text{(\( \forall X \) left)}}{\Gamma \Rightarrow \Pi} \quad \text{(\( \forall X \) left)}.
\]

By the induction hypothesis, we have \( \vdash_m \Delta, \Gamma \Rightarrow \Pi \) for every \( \Delta \in \forall X.\xi|k \). Hence \( \vdash_m \forall X.\xi, \Gamma \Rightarrow \Pi \) by rule (\( \forall X \) left).

Lemma B.3. Let \( m < \omega \). If \( \vdash_{m+1} \Gamma \Rightarrow \Pi \), then \( \vdash_m \Gamma \Rightarrow \Pi \).

Proof. By structural induction on the derivation. Suppose that it ends with

\[
\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi} \quad \text{(cut)}.
\]

By the induction hypothesis, we have \( \vdash_m \Gamma \Rightarrow \varphi \) and \( \vdash_m \varphi, \Gamma \Rightarrow \Pi \). Moreover \( \text{rank}(\varphi) \leq m \). Hence \( \vdash_m \Gamma \Rightarrow \Pi \) by Lemma [B.2].

Lemma B.4. If \( \vdash_\omega \Gamma \Rightarrow \Pi \), then \( \vdash_0 \Gamma \Rightarrow \Pi \).

Proof. By structural induction on the derivation. Suppose that it ends with

\[
\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi} \quad \text{(cut)}.
\]

By the induction hypothesis, we have \( \vdash_0 \Gamma \Rightarrow \varphi \) and \( \vdash_0 \varphi, \Gamma \Rightarrow \Pi \). Let \( \text{rank}(\varphi) = m \), then we have \( \vdash_{m+1} \Gamma \Rightarrow \Pi \). Hence applying Lemma [B.3] \( m + 1 \) times, we obtain \( \vdash_0 \Gamma \Rightarrow \Pi \).
This completes the proof of Lemma 3.3.

APPENDIX C. ALGEBRAIC PROOF OF CUT ELIMINATION FOR LI2

We here outline an algebraic proof of cut elimination for the full second order calculus LI2 that we attribute to Maehara [24] and Okada [26, 28]. This will be useful for comparison with the parameter-free case LIP_{n+1}, that we have addressed in the main text.

Let \( \varphi_{\text{fin}}(FM) \) be the set of finite sets of second order formulas, so that \( \langle \varphi_{\text{fin}}(FM), \cup, \emptyset \rangle \) is a commutative idempotent monoid, and \( \text{SEQ} \) be the set of sequents of LI2. As before, we have \( \langle \varphi_{\text{fin}}(FM) \times \text{SEQ} \rightarrow \text{SEQ} \rangle \) defined by \( \Gamma \langle (\Sigma \Rightarrow \Pi) := (\Gamma, \Sigma \Rightarrow \Pi) \). So

\[
\text{CF} := \langle \varphi_{\text{fin}}(FM), \text{SEQ}, \Rightarrow^{cf}, \cup, \emptyset \rangle
\]

is a Heyting frame, where \( \Gamma \Rightarrow^{cf} (\Sigma \Rightarrow \Pi) \) iff LI2 \( \vdash^{cf} \Gamma, \Sigma \Rightarrow \Pi \).

Define a Heyting-valued prestructure \( \mathcal{CF} := (\text{CF}^+, \text{Tm}, \mathcal{D}, \mathcal{L}) \) by \( p^{\mathcal{CF}}(\bar{t}) := \gamma(p(\bar{t})) \) for each predicate symbol \( p \) and

\[
\mathcal{D} := \{ F \in \mathcal{G}(\text{CF})^{\text{Tm}} : F \text{ matches some } \tau \in \text{ABS} \},
\]

where \( F \text{ matches } \tau \text{ just in case } \tau(t) \subset F(t) \subseteq \tau(t)^\subset \text{ holds for every } t \in \text{Tm} \). This choice of \( \mathcal{D} \subseteq \mathcal{G}(\text{CF})^{\text{Tm}} \) is a logical analogue of Girard's reducibility candidates as noticed by Okada.

For instance, given a set variable \( X \), define \( F_X \in \mathcal{G}(\text{CF})^{\text{Tm}} \) by \( F_X(t) := \gamma(X(t)) \). Then \( X(t) \in F_X(t) \subseteq X(t)^\subset \). Hence \( F_X \) matches \( X = \lambda x. X(x) \), so belongs to \( \mathcal{D} \).

Given a set substitution \( \bullet : \text{VAR} \rightarrow \text{ABS} \) and a valuation \( \mathcal{V} : \text{VAR} \rightarrow \mathcal{D} \), we say that \( \mathcal{V} \text{ matches } \bullet \) if \( \mathcal{V}(X) \text{ matches } X^\bullet \in \text{ABS} \) for every \( X \in \text{VAR} \). That is, \( X^\bullet(t) \in \mathcal{V}(X(t)) \subseteq X^\bullet(t)^\subset \) holds for every \( X \in \text{VAR} \) and \( t \in \text{Tm} \).

Lemma C.1. Let \( \bullet \) be a set substitution and \( \mathcal{V} \) a valuation that matches \( \bullet \). Then for every \( \varphi \in \text{FM} \),

\[
\varphi^\bullet \in \mathcal{V}(\varphi) \subseteq \varphi^{\bullet^\subset}.
\]

Proof. By induction on the structure of \( \varphi \).

(1) \( \varphi \) is an atom \( X(t) \). By assumption we have \( X^\bullet(t) \in \mathcal{V}(X(t)) \subseteq X^\bullet(t)^\subset \).

(2) The outermost connective of \( \varphi \) is first order. Similar to the proof of Lemma 5.5

(3) \( \varphi = \forall X. \psi(X) \). We first prove \( \langle \forall X. \psi \rangle^\bullet \in \mathcal{V}(\forall X. \psi) = \bigcap_{F \in \mathcal{D}} \mathcal{V}[F/X](\psi) \). So let \( F \in \mathcal{D} \) that matches \( \tau \in \text{ABS} \). We update substitution \( \bullet \) by letting \( X^\bullet := \tau \) so that \( \bullet \) matches \( \mathcal{V}[F/X] \). By the induction hypothesis \( \psi^\bullet(\tau) \in \mathcal{V}[F/X](\psi) \). Hence for any \( (\Gamma \Rightarrow \Pi) \in \mathcal{V}[F/X](\psi) \), we have \( \psi^\bullet(\tau), \Gamma \Rightarrow^{cf} \Pi \). So \( \langle \forall X. \psi \rangle^\bullet, \Gamma \Rightarrow^{cf} \Pi \) by rule (\( \forall X \) left). Therefore \( \langle \forall X. \psi \rangle^\bullet \in \mathcal{V}(\forall X. \psi) \). This proves \( \langle \forall X. \psi \rangle^\bullet \in \mathcal{V}(\forall X. \psi) \).

We next prove \( \mathcal{V}(\forall X. \psi) \subseteq \langle \forall X. \psi \rangle^{\bullet^\subset} \). Let \( \Gamma \in \mathcal{V}(\forall X. \psi) \). Choose a variable \( Y \) such that \( Y \notin \text{FV}(\Gamma) \). By the induction hypothesis, we have \( \Gamma \in \mathcal{V}[F_Y/X](\psi) \subseteq \psi^\bullet(\psi)^\subset \), that is, \( \Gamma \Rightarrow^{cf} \psi^\bullet(\psi) \). Hence \( \Gamma \Rightarrow^{cf} \langle \forall X. \psi \rangle^\bullet \) by rule (\( \forall X \) right). This proves \( \Gamma \in \langle \forall X. \psi \rangle^{\bullet^\subset} \).

(3) \( \varphi = \exists X. \psi(X) \). We first prove \( \langle \exists X. \psi \rangle^\bullet \in \mathcal{V}(\exists X. \psi) = \bigcup_{F \in \mathcal{D}} \mathcal{V}[F/X](\psi) \). Let \( (\Gamma \Rightarrow \Pi) \in \bigcup_{F \in \mathcal{D}} \mathcal{V}[F/X](\psi) \) and choose a variable \( Y \) such that \( Y \notin \text{FV}(\Gamma, \Pi) \). By the induction hypothesis, we have \( \psi^\bullet(Y) \in \mathcal{V}[F_Y/X](\psi) \subseteq \bigcup_{F \in \mathcal{D}} \mathcal{V}[F/X](\psi) \), so \( \psi^\bullet(Y), \Gamma \Rightarrow^{cf} \Pi \). Hence \( \langle \exists X. \psi \rangle^\bullet, \Gamma \Rightarrow^{cf} \Pi \) by rule (\( \exists X \) left). Therefore \( \langle \exists X. \psi \rangle^\bullet \in \mathcal{V}(\exists X. \psi) \).
We next prove $V(\exists X.\psi) \subseteq (\exists X.\psi)^\bullet$. Let $\Gamma \in \bigcup_{F \in D} V[F/X](\psi)$, that is, $\Gamma \in V[F/X](\psi)$ for some $F \in D$ (that match $\tau$). By the induction hypothesis, $\Gamma \in V[F/X](\psi) \subseteq (\psi^*(\tau))^\triangledown$, so $\Gamma \Rightarrow^{\text{cf}} \psi^*(\tau)$. Hence $\Gamma \Rightarrow^{\text{cf}} (\exists X.\psi)^\bullet$ by rule $(\exists X$ right), so $\Gamma \in (\exists X.\psi)^{\bullet\triangledown}$. This proves $V(\exists X.\psi) \subseteq (\exists X.\psi)^{\bullet\triangledown}$.

As a consequence:

**Lemma C.2.** $CF$ is a Heyting structure.

**Proof.** Let $V$ be a valuation and $\tau$ an abstract. Our goal is to show that $V(\tau) \in D$, that is, there is some $\tau_0$ such that $\tau_0(t) \in V(\tau(t)) \subseteq \tau_0(t)^\triangledown$ holds for every $t \in Tm$. Here $V(\tau)$ is defined by $V(\tau)(t) := V(\tau(t))$.

Since $V$ is a valuation into $D$, every set variable $X$ is associated with an abstract $\tau_X$ so that $V(X)$ matches $\tau_X$. Define a set substitution $\bullet$ by $X^{\bullet} := \tau_X$. Then $V$ matches $\bullet$. Hence by Lemma C.1, we obtain $\tau^{\bullet}(t) \in V(\tau(t)) \subseteq \tau^{\bullet}(t)^\triangledown$. Thus $\tau^\bullet$ is the desired abstract.

Next, define a valuation $I$ by $I(X) := F_X$. We then have $X(t) \in I(X(t)) = \gamma(X(t)) \subseteq X(t)^\triangledown$ for every $X \in \text{VAR}$ and $t \in Tm$, so $I$ matches the identity substitution. Hence we have $\varphi \in I(\varphi) \subseteq \varphi^\triangledown$ for every formula $\varphi \in \text{FM}$.

**Theorem C.3** (Algebraic cut elimination for LI2). For every sequent $\Gamma \Rightarrow \Pi$, the following are equivalent.

1. $\Gamma \Rightarrow \Pi$ is provable in LI2.
2. $\Gamma \Rightarrow \Pi$ is valid in all Heyting-valued structures.
3. $\Gamma \Rightarrow \Pi$ is cut-free provable in LI2.

**Proof.** ((1) $\Rightarrow$ (2)) By Lemma 4.1.
((2) $\Rightarrow$ (3)) By $\Gamma \in I(\Gamma) \subseteq I(\Pi) \subseteq \Pi^\triangledown$.
((3) $\Rightarrow$ (1)) Trivial.