MEASURES THAT DEFINE A COMPACT CAUCHY TRANSFORM

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ABSTRACT. The aim of this work is to provide a geometric characterization of the positive Radon measures $\mu$ with compact support on the plane such that the associated Cauchy transform defines a compact operator from $L^2(\mu)$ to $L^2(\mu)$. It turns out that a crucial role is played by the density of the measure and by its Menger curvature.

1. Introduction

In what follows we will identify the plane with the complex field $\mathbb{C}$. Let $\mu$ be a positive Radon measure on $\mathbb{C}$ with compact support and without atoms. For $\epsilon > 0$, $f \in L^1_{\text{loc}}(\mu)$ and $z \in \mathbb{C}$ we set

$$C_{\epsilon,\mu} f(z) := \int_{|z-w| > \epsilon} \frac{f(w)}{z-w} d\mu(w).$$

We define the Cauchy transform operator $C_{\mu}$ in a principal value sense, i.e., as the limit

$$C_{\mu} f(z) := \lim_{\epsilon \to 0} C_{\epsilon,\mu} f(z)$$

for every $z$ such that the above limit exists. We say that the Cauchy transform is bounded from $L^2(\mu)$ to $L^2(\mu)$ if the truncated operators $C_{\epsilon,\mu}$ are bounded uniformly in $\epsilon$. As a consequence of the work of Mattila and Verdera (see [9] or the book by Tolsa [14, Chapter 8]), the Cauchy transform is bounded from $L^2(\mu)$ to $L^2(\mu)$ if and only if the truncated operators $\{C_{\epsilon,\mu}\}_{\epsilon}$ converge as $\epsilon$ tends to 0 in the weak operator topology of the space of bounded linear operators from $L^2(\mu)$ to $L^2(\mu)$. Moreover, if we denote as $C^w_{\mu}$ the limit of the aforementioned net, for all $f \in L^2(\mu)$ and for almost all $z$, the principal value $C_{\mu} f(z)$ exists and it coincides with $C^w_{\mu} f(z)$. This is a peculiarity of the Cauchy transform and it does not hold for every singular integral operator. Now, it makes sense to introduce the following definition.

**Definition 1.** We say that the Cauchy transform is compact from $L^2(\mu)$ to $L^2(\mu)$ if it is bounded in $L^2(\mu)$ and $C^w_{\mu}$ is compact as an operator from $L^2(\mu)$ to $L^2(\mu)$.

As a consequence of the results we cited, one may replace $C^w_{\mu}$ in Definition 1 with the principal value $C_{\mu}$. A useful tool to study the Cauchy transform of a measure $\mu$ is the so-called Menger curvature $c(\mu)$, that was first related to the Cauchy transform in [10] and [11]. Denoting by $R(z, w, \zeta)$ the radius of the circumference passing through $z, w$ and $\zeta$, and defining

$$c^2_{\mu}(z) := \int \int \frac{1}{R(z, w, \zeta)^2} d\mu(w) d\mu(\zeta),$$

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the Menger curvature of $\mu$ is defined as

$$c^2(\mu) := \int c_\mu^2(z) d\mu(z).$$

Let $d, n \in \mathbb{N}$ with $n \leq d$. Given a cube $Q$ in $\mathbb{R}^d$, we denote by $l(Q)$ its side length and by

$$(1.1) \quad \Theta^\mu_n(Q) := \frac{\mu(Q)}{l(Q)^n}$$

its $n$-dimensional density. If $z \in \mathbb{R}^d$, we define the upper density of $\mu$ at $z$ as

$$(1.2) \quad \Theta^\mu_\ast(z) := \limsup_{l(Q) \to 0} \Theta^\mu_n(Q),$$

where $Q$ spans over the cubes centered at $z$. Replacing the superior limit with the inferior limit we get the definition of the lower density $\Theta^\mu_{\ast \mu}(z)$. If $\Theta^\mu_{\ast \mu}(z) = \Theta^\mu_n(z)$, we denote that common value as $\Theta^\mu_n(z)$ and call it ”density of $\mu$ at the point $z$”. In the case $d = 2$ and $n = 1$, for brevity we write $\Theta^\mu_n(z) := \Theta^1_n(\mu)$ and we omit the index $n$ from the notation for the upper and lower densities at any point.

The aim of the present work is to characterize the measures $\mu$ on the plane such that its associated Cauchy transform defines a compact operator from $L^2(\mu)$ into $L^2(\mu)$. Not much literature is available concerning compactness for Singular Integral Operators in the context of Euclidean spaces equipped with a measure different from the Lebesgue measure. We point out that a $T(1)$—like criterion for the compactness of Calderón-Zygmund operators in Euclidean spaces is available due to the work of Villarroya [16]. We denote by $K(L^2(\mu), L^2(\mu))$ the space of compact operators from $L^2(\mu)$ to $L^2(\mu)$. We will see that a crucial condition to get a compact Cauchy transform is to require that

$$\Theta^\mu_\ast(z) = 0$$

for every $z \in \mathbb{C}$. Our main result is the following.

**Theorem 1.** Let $\mu$ be a compactly supported positive Radon measure on $\mathbb{C}$ without atoms.

The following conditions are equivalent:

(a) $\mathcal{C}_\mu$ is compact from $L^2(\mu)$ to $L^2(\mu)$.

(b) the two following properties hold:

(1) $\Theta^\mu_\ast(z) = 0$ uniformly, which means that the limit in (1.2) is 0 uniformly in $z \in \mathbb{C}$.

(2) $c^2(\mu, Q)/\mu(Q) \to 0$ as $l(Q) \to 0$, where $\mu, Q$ stands for the restriction of $\mu$ to the cube $Q$.

(c) the truncated operators $\mathcal{C}_{\epsilon, \mu}$ converge as $\epsilon \to 0$ in the operator norm of the space of bounded linear operators from $L^2(\mu)$ to $L^2(\mu)$.

We remark that the proof of the theorem relies on the $T(1)$—theorem for the Cauchy transform (see [14]) and that one could replace the cubes with balls in condition (b), as well as in (1.1).

Theorem 1 can be generalized to higher dimensions taking into consideration the $n$-Riesz transform $\mathcal{R}^n_\mu$ on $\mathbb{R}^d$ for $n \leq d$ in place of the Cauchy transform. If $\mu$ is a compactly supported positive Radon measure on $\mathbb{R}^d$ without atoms, $\epsilon > 0$, $f \in L^1_{\text{loc}}(\mu)$ and $z \in \mathbb{R}^d$, the truncated Riesz transform is defined as

$$\mathcal{R}^n_{\epsilon, \mu}f(z) := \int_{|z-w| > \epsilon} \frac{x - y}{|x - y|^{n+1}} f(y) d\mu(y).$$
As in the case of the Cauchy transform, thanks again to the result in [9], the weak limit $\mathcal{R}^{n,w}_\mu$ of $\mathcal{R}^{n}_\mu$ as $\epsilon \to 0$ exists provided the $\mathcal{R}^{n}_\epsilon$ are uniformly bounded on $L^2(\mu)$, and we can understand the compactness of the Riesz transform as in Definition 1. The main difference with the Cauchy transform is that the only case in which boundedness is known to imply that the principal value exists is for $n = d - 1$. This is a consequence of [12].

In this more general context, Theorem 1 reads as follows.

**Theorem 2.** Let $\mu$ be a compactly supported positive Radon measure on $\mathbb{R}^d$ without atoms. The following conditions are equivalent:

(a) $\mathcal{R}^{n}_\mu$ is compact from $L^2(\mu)$ to $L^2(\mu)$.

(b) the two following properties hold:

1. $\Theta_{\mu}^{n-1,*}(z) = 0$ uniformly in $z \in \mathbb{R}^d$.
2. $||\mathcal{R}^{n}_\mu\chi_Q||_{L^2(\mu,Q)}/\mu(Q) \to 0$ as $l(Q) \to 0$.

(c) the truncated operators $\mathcal{R}^{n}_{\epsilon,\mu}$ converge as $\epsilon \to 0$ in the operator norm of the space of bounded linear operators from $L^2(\mu)$ to $L^2(\mu)$.

Theorem 2 can be proved with minor changes of the proof that we will discuss for the case of the Cauchy transform. Combining condition (b) in Theorem 2 with [9, Theorem 1.6], we can infer that if $\mathcal{R}^{n}_\mu$ is compact then the principal value $\mathcal{R}^{n}_\mu(x)$ exists for $\mu$–almost every $x$.

The work is structured as follows. In Section 2 we deal with two toy models: first we show a direct proof of the non-compactness of the Cauchy transform of the one dimensional Lebesgue measure on a segment. Then, we prove that the Cauchy transform of a disc endowed with the planar Lebesgue measure is compact. In Section 3 we prove Theorem 1. As an application of this result, Section 4 is devoted to the discussion of the case of the general planar Cantor sets. We conclude the exposition with a remark on the generalization of the main theorem to other Singular Integral Operators.

**Notation.** Throughout this work, we use the standard notations $A \lesssim B$ if there exits an absolute positive constant $C$ such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$.

Given an operator $T : X \to Y$, we use the notation $||T||_{X \to Y}$ for its operator norm.

2. THE CAUCHY TRANSFORM ON A SEGMENT AND ON THE DISC

It may be worth recalling the following property of compact operators: if $X$ and $Y$ are Banach spaces, $T : X \to Y$ is a compact operator and $\{u_k\}_k$ is a sequence in $X$ such that $u_k \to u$ for some $u \in X$ (weak convergence), then $Tu_k \to Tu$ (strongly) in $Y$. We will use this property both for the proof of the following proposition and for the proof of the main theorem. Let us start by considering the Cauchy transform on a segment. Given an interval $I$ on the real line, we denote by $\mathcal{H}^1$ the 1–dimensional Hausdorff measure and use the notation $L^2(I) := L^2(\mathcal{H}^1(I \times \{0\}))$. Without loss of generality, we analyze the case $I = [0, 1]$.

**Proposition 1.** Let $\mu := \mathcal{H}^1([0, 1] \times \{0\})$. The Cauchy transform $C_\mu$ is not a compact operator from $L^2(\mu)$ into $L^2(\mu)$.

**Proof.** Let $C_\mu$ be the Cauchy transform of the measure $\mu := \mathcal{H}^1([0, 1] \times \{0\})$, acting on functions belonging to $L^2([0, 1])$.

For $k \in \mathbb{N}$, let us define the function $f_k : \mathbb{R} \to \mathbb{R}$ as

$$f_k(x) := 2^{(k-1)/2}(\chi_{[1/2^k, 1/2^{k-2}]}(x) - \chi_{[1/2^k, 1/2^{k-2}]}(x)).$$
Notice that \( \|f_k\|_{L^2([0,1])} = \|f_k\|_{L^2(\mathbb{R})} = 1 \) and that \( \{f_k\} \) converges to 0 in the weak topology of \( L^2([0,1]) \). However, \( \{f_k\} \) does not converge in the strong topology of \( L^2([0,1]) \).

Let us denote by \( Hf_k \) the Hilbert transform of \( f_k \)
\[
Hf_k(x) := \text{p.v.} \int \frac{f_k(y)}{x-y} \, dy
\]
for \( x \in \mathbb{R} \). We claim that \( Hf_k \) does not converge to 0 in the strong topology of \( L^2([0,1]) \).

Hence \( C_\mu = H \) is not compact in \( L^2(\mu) \).

A well known fact regarding Hilbert transform (see e.g. [13]) is that
\[
\|Hf\|_{L^2(\mathbb{R})} = \pi \|f\|_{L^2(\mathbb{R})}
\]
for every \( f \in L^2(\mathbb{R}) \).

The following argument proves that \( \|C_\mu f_k\|_{L^2([0,1])} = \|Hf_k\|_{L^2([0,1])} \) tends to \( \pi \) for \( k \to \infty \).

It is clearly enough to show that
\[
(2.1) \quad \|Hf_k\|_{L^2((1, +\infty))}^2 \to 0 \quad \text{for} \quad k \to \infty
\]
and
\[
(2.2) \quad \|Hf_k\|_{L^2((-, 0])}^2 \to 0 \quad \text{for} \quad k \to \infty.
\]

To prove (2.1), first notice that for \( y \in \text{supp} \ f_k \) and \( x \geq 1 \), it holds that \( |x-y| \geq |x-3/4| \). Then
\[
\|Hf_k\|_{L^2((1, +\infty))}^2 = \int_1^{+\infty} \int_{1/2 - 2^{-k}}^{1/2 + 2^{-k}} \frac{|f_k(y)|}{|x-y|} \, dx \, dy
\]
\[
\leq \int_1^{+\infty} \frac{1}{|x-\frac{3}{4}|^2} \left( \int_{1/2 - 2^{-k}}^{1/2 + 2^{-k}} |f_k(y)|^2 \, dy \right) \, dx
\]
\[
\leq 2^{-k+1} \int_{1/2 - 2^{-k}}^{1/2 + 2^{-k}} \frac{1}{|x-\frac{3}{4}|^2} \, dy \lesssim 2^{-k},
\]
which gives (2.1). The proof of (2.2) is analogous. \( \square \)

Now we turn to analyze the Cauchy transform on the disc. Let \( D := D(0,1) = \{z \in \mathbb{C} : |z| < 1\} \) and let \( \epsilon > 0 \). Let \( \mu = dA \) be the 2-dimensional Lebesgue measure restricted to \( D \).

**Lemma 1.** The operator \( C_{\epsilon, \mu} : L^2(dA) \to L^2(dA) \) is compact for every \( \epsilon > 0 \).

**Proof.** Let \( z, w \in \mathbb{C} \) and let \( K_\epsilon(z, w) := \chi_{D(z, \epsilon)}(w)/(w-z) \). By the Hilbert-Schmidt’s Theorem (see [3, Theorem 6.12]), to prove the lemma it is enough to show that the integral
\[
\int_D |K_\epsilon(z, w)|^2 \, dA(z)
\]
converges. This occurs in our case because
\[
\int_D |K_\epsilon(z, w)|^2 \, dA(z) \leq \frac{A(D)}{\epsilon^2} = \frac{\pi}{\epsilon^2},
\]
and so the proof is complete. \( \square \)
For \( f \in L^2(dA) \) let us define
\[
C_\mu^\epsilon f(z) := C_\mu f(z) - C_{\epsilon, \mu} f(z).
\]
By Lemma 1, to prove that \( C_\mu \) belongs to \( K(L^2(dA), L^2(dA)) \) it suffices to prove that \( \|C_\mu^\epsilon\|_{L^2(dA) \to L^2(dA)} \to 0 \) as \( \epsilon \to 0 \). Indeed, this implies that \( \{C_{\epsilon, \mu}\}_{\epsilon > 0} \) converges in operator norm to the Cauchy transform, which proves that it is compact.

3. The proof of Theorem 1

3.1. Necessary conditions for the compactness. In order to prove the necessity of the conditions in Theorem 1, we argue by contradiction: assuming that there exists a sequence of cubes \( \{Q_j\}_j \) such that \( l(Q_j) \to 0 \) but \( \limsup_j \Theta_\mu^1(Q_j) > 0 \), we will prove that the Cauchy transform does not define a compact operator on \( L^2(\mu) \).

We recall that a necessary condition to have the \( L^2(\mu) \)-boundedness of \( C_\mu \) is that \( \mu \) has linear growth (see [4]). In particular we choose to denote by \( C_0 \) a positive constant such that
\[
\mu(Q) \leq C_0 l(Q)
\]
for every cube in \( \mathbb{R}^2 \).

Let us state a technical lemma that we are going to use to prove Theorem 1. The proof is a variant of Lemma 2.3 in [6].

Lemma 2. Suppose that there is a sequence of cubes \( \{Q_j\} \) such that \( l(Q_j) \to 0 \) and
\[
\limsup_j \Theta_\mu^1(Q_j) \equiv \Theta > 0.
\]
Let $Q$ be a cube in $\{Q_j\}$ such that $\Theta_\mu^1(Q) \geq \Theta/2$. There exist $C_1, C_1' \in \mathbb{N}$ both bigger than 1 such that we can find two cubes $Q'$ and $Q''$ with side length $l(Q)/C_1$ and with the following properties

1. $\text{dist}(Q', Q'') \approx l(Q')$.
2. $\min(\mu(Q'), \mu(Q'')) \geq l(Q)/C_1'$.

**Proof.** Let us argue by contradiction. Let us split $Q$ into a grid of $C_2^2$ equal cubes of side length $l(Q)/C_1$ whose sides are parallel to the sides of $Q$; we denote this collection of cubes as $D$. Let us assume that each couple of cubes $Q', Q'' \in D$ is such that either they touch (so that $\text{dist}(Q', Q'') = 0$) or $\min(\mu(Q'), \mu(Q'')) \leq l(Q)/C_1'$. By construction we have that

\[ \sum_{\tilde{Q} \in D} \mu(\tilde{Q}) = \mu(Q) = \Theta(Q)l(Q). \]  

Now let us consider the family

$G := \{ \tilde{Q} \in D : \mu(\tilde{Q}) \geq \frac{l(Q)}{C_1'} \}$. 

By hypothesis, all the cubes in $G$ must be contained in a single cube of side length $3l(Q)/C_1$ that we denote as $P$. The growth condition (3.1) gives

\[ \mu(P) \leq C_0 l(P) = 3C_0 l(Q)/C_1, \]

so that

\[ \sum_{\tilde{Q} \in G} \mu(\tilde{Q}) \leq \frac{3C_0}{C_1} l(Q). \]

For those cubes of $D$ not belonging to $G$ we can write

\[ \sum_{\tilde{Q} \in D \setminus G} \mu(\tilde{Q}) \leq \frac{C_2^2}{C_1} l(Q). \]

By hypothesis we have that $\Theta(Q) \geq \Theta/2$. Then, gathering (3.2), (3.3) and (3.4) we get the inequality

\[ \frac{C_2^2}{C_1} + \frac{3C_0}{C_1} \geq \frac{\Theta}{2}. \]

Choosing $C_1$ and $C_1'$ big enough, (3.5) gives a contradiction. \hfill $\Box$

**Remark 2.** Using the growth condition for the measure $\mu$, the condition (2) in the statement of Lemma 2 actually implies that $Q'$ and $Q''$ are such that

\[ \mu(Q') \approx \mu(Q'') \approx \mu(Q). \]

As a consequence of the proof of Lemma 2, it is not difficult to see that we can choose $Q'$ and $Q''$ arbitrarily small. This will lead to a contradiction.

Given a cube cube $Q$, we define the function $\varphi_Q := \chi_Q/\mu(Q)^{1/2}$. We have that $\|\varphi_Q\|_{L^2(\mu)} = 1$ for every cube and that

$\varphi_Q \to 0$
weakly in $L^2(\mu)$ for every sequence of cubes $\{Q_j\}_j$ such that $l(Q_j) \to 0$.

Now, taking $Q$, $Q'$ and $Q''$ as in Lemma 2, we can write

$$
|\langle C_\mu \varphi_{Q'}, \varphi_{Q''} \rangle| \leq \|C_\mu \varphi_{Q'}\|_{L^2(\mu)} \|\varphi_{Q''}\|_{L^2(\mu)} = \|C_\mu \varphi_{Q'}\|_{L^2(\mu)}.
$$

(3.7)

The proof of the necessity of the density condition of Theorem 1 follows from (3.7) if we can prove that $|\langle C_\mu \varphi_{Q'}, \varphi_{Q''} \rangle|$ is bounded from below away from 0; indeed, this would imply that $\|C_\mu \varphi_{Q'}\|_{L^2(\mu)}$ does not converge to 0, which contradicts the compactness of the Cauchy transform.

**Lemma 3.** Let $Q'$ and $Q''$ be as in Lemma 2. There exists a constant $c > 0$, independent on the side length of the cubes, such that

$$
\frac{\mu(Q)}{l(Q')} \leq |\langle C_\mu \varphi_{Q'}, \varphi_{Q''} \rangle|.
$$

**Proof.** Suppose without loss of generality that the centers of the cubes $Q'$ and $Q''$ are aligned with the real axis. By (3.6), we have that

$$
|\text{Re}(\langle C_\mu \varphi_{Q'}, \varphi_{Q''} \rangle) \approx \frac{1}{\mu(Q)} |\text{Re}(\langle C_\mu \chi_{Q'}, \chi_{Q''} \rangle)|.
$$

(3.8)

Suppose that $\text{Re}(z - w) > 0$ for every $z \in Q''$ and $w \in Q'$. Then

$$
|\text{Re}(\langle C_\mu \chi_{Q'}, \chi_{Q''} \rangle) = \text{Re} \int_{Q''} C_\mu \chi_{Q'}(z) d\mu(z) = \int_{Q', Q''} \frac{\text{Re}(z - w) d\mu(w) d\mu(z)}{|z - w|^2}.
$$

(3.9)

Lemma 2 ensures that, if $z \in Q''$ and $w \in Q'$, we have that $|z - w| \approx l(Q) \approx l(Q')$, so that, using (3.8),(3.9) we have

$$
|\text{Re}(\langle C_\mu \chi_{Q'}, \chi_{Q''} \rangle) \geq \frac{\mu(Q)^2}{l(Q')^2}.
$$

(3.10)

The Lemma follows from (3.10) and (3.8).

The following lemma gives a necessary condition for the Cauchy transform of a measure to be compact in terms of the curvature.

**Lemma 4.** Let $\mu$ be a compactly supported positive Radon measure on $\mathbb{C}$ without atoms. Suppose that $C_\mu$ defines a compact operator from $L^2(\mu)$ to $L^2(\mu)$. Then

$$
\frac{c^2(\mu,Q)}{\mu(Q)} \to 0
$$

as $l(Q) \to 0$.

**Proof.** Let $Q$ be an arbitrary cube in $\mathbb{R}^2$. From a formula due to Tolsa and Verdera (see [15], Theorem 2) applied to the measure $\mu_{\cdot}Q$, we have that

$$
\|C_\mu \chi_Q\|_{L^2(\mu_{\cdot}Q)} = \frac{\pi^2}{3} \int_Q \frac{\theta_{\mu}(z)^2}{1} d\mu(z) + \frac{1}{6} c^2(\mu_{\cdot}Q).
$$

(3.11)

Since we suppose $C$ to be compact, we proved that $\theta_{\mu}(z) = 0$ for every $z \in \mathbb{R}^2$, so that the integral in the right hand side of (3.11) vanishes. Consider a sequence of cubes $\{Q_j\}_j$ such that $l(Q_j) \to 0$ as $j \to \infty$. As before, if we define $\varphi_j := \chi_{Q_j}/\mu(Q_j)^{1/2}$, we have that

$$
\varphi_j \to 0
$$
weakly in $L^2(\mu)$. Then, since we suppose the Cauchy transform to be compact, we have that

$$||C_\mu \varphi_j||_{L^2(\mu)}^2 \to 0$$

for $j \to \infty$. The inequalities

$$||C_\mu \chi_{Q_j}||_{L^2(\mu)}^2 \leq ||C_\mu \chi_{Q_j}||_{L^2(\mu)}^2 \leq \mu(Q_j)||C_\mu \varphi_j||_{L^2(\mu)}^2,$$

and (3.11) conclude the proof of the Lemma.

3.2. Sufficient conditions for the compactness. The proof that we present now relies on the $T(1)$-theorem of David and Journe. More specifically, we prove that proper truncates of the Cauchy transform are compact operators and, then, we estimate the operator norm of the difference between $C_\mu$ and those truncates.

Let $\mu$ be a positive Radon measure with compact support in $\mathbb{C}$. Let $z \in \text{supp} \mu$ and let $Q_z$ be a square containing the support of $\mu$ and centered at $z$. Let $l(Q_z)$ denote its side length. For $j \in \mathbb{N}$ we denote as $Q_j(z)$ the square centered at $z$ and with side-length $2^{-j}l(Q_z)$. Moreover, we define

$$\Delta_j(z) := Q_j(z) \setminus Q_{j+1}(z).$$

Exploiting Hilbert-Schmidt’s Theorem, a proof analogous to the one of Lemma 1 shows that the truncated operator

$$T_j f(z) := \int_{\Delta_j(z)} \frac{f(w)}{z - w} d\mu(w)$$

is a compact operator from $L^2(\mu)$ to $L^2(\mu)$. Let us define

$$C^N_\mu f(w) := \sum_{j=0}^{N-1} T_j f(w)$$

and show that, under the hypothesis on the measure reported in the statement of Theorem 1, it converges in the $L^2(\mu) \to L^2(\mu)$ operator norm to the Cauchy transform. This will prove that $C_\mu \in K(L^2(\mu), L^2(\mu))$.

The $T(1)$–Theorem [see 14, Chapter 3] provides the estimate

$$||C_\mu - C^N_\mu||_{L^2(\mu) \to L^2(\mu)} \leq \sup_{z \in \text{supp} \mu} \sup_{\tilde{Q} \subseteq Q_N(z)} \Theta(\tilde{Q}) + \sup_{z \in \text{supp} \mu} \sup_{\tilde{Q} \subseteq Q_N(z)} \frac{||C \chi_{\tilde{Q}}||_{L^2(\mu, \tilde{Q})}}{\mu(\tilde{Q})^{1/2}} \equiv I_N + II_N. \tag{3.12}$$

First, $I_N \to 0$ as $N \to \infty$ by the hypothesis (2) of Theorem 1 on the density of $\mu$.

To show that $II_N \to 0$ as $N \to \infty$, it suffices to recall formula (3.11), which yields

$$||C_\mu \chi_{\tilde{Q}}||_{L^2(\mu, \tilde{Q})} \simeq c^2(\mu, \tilde{Q}).$$

The ratio $c^2(\mu, \tilde{Q})/\mu(\tilde{Q})$ has the correct behavior due to the condition (2) of Theorem 1. This concludes the proof of the equivalence of the conditions (a) and (b). In order to complete the proof of the theorem, it suffices to observe that the equivalence of (b) and (c) follows from (3.12).
4. An example: a generalized planar Cantor set

As an application of Theorem 1 we analyze the particular case of the planar Cantor sets (see e.g. [5, p. 87]). Let \( Q^0 := [0, 1]^2 \) be the unit square and let \( \lambda := \{ \lambda_n \}_{n=1}^{\infty} \) be a sequence of non-negative numbers such that \( 0 \leq \lambda_n \leq 1/2 \) for every \( n = 1, 2, \ldots \). The Cantor set is defined by means of an inductive construction:

- define 4 squares \( \{ Q^1_j \}_{j=1}^{4} \) of side length \( \lambda_1 \) such that each one of them contains a distinct vertex of \( Q_0 \) and call \( E_1 := \bigcup_{j=1}^{4} Q^1_j \).
- iterate the first step for each of the 4 cubes but using \( \lambda_2 \) as a scaling factor. As a result we get \( 2^4 = 16 \) squares of side length \( \lambda_2 = \lambda_1 \lambda_2 \). We denote those squares as \( \{ Q^2_j \}_{j=1}^{16} \).
- as a result of \( n \) analogous iterations, at the \( n \)-th step we get a collection of \( 4^n \) cubes \( \{ Q^n_j \}_{j=1}^{4^n} \) whose side length is \( \sigma_n := \prod_{j=1}^{n} \lambda_j \) and a set \( E_n := \bigcup_{j=1}^{4^n} Q^n_j \).

The planar Cantor set is defined as

\[
E = E(\lambda) := \bigcap_{n=1}^{\infty} E_n.
\]

We denote by \( p \) the canonical probability measure associated with \( E(\lambda) \). In particular, \( p \) is uniquely identified by imposing that \( p(Q) = 4^{-n} \) for every square that composes \( E_n \). We denote by \( C_p \) the Cauchy transform associated with the measure \( p \).

Let \( \theta_k := 2^{-k} \sigma_k^{-1} \). It is known (see e.g. [14], Lemma 4.29) that for the probability measure on the Cantor set, it holds that

\[
c_p^2(x) \approx \sum_{k=0}^{\infty} \theta_k^2
\]

for every \( x \in E(\lambda) \).

As a consequence of Theorem 1, \( C_p \) is compact from \( L^2(p) \) to \( L^2(p) \) if and only if \( \sum_{k=0}^{\infty} \theta_k^2 \) converges. This condition holds if and only if \( C_p \) is bounded from \( L^2(p) \) to \( L^2(p) \) (see [8]).

5. A counterexample to Theorem 1 for other kernels

A natural question is to ask if any analogue Theorem 1 holds also for other singular integral operators of the form

\[
Tf(z) = \int_{B} K(z, w) f(w) d\mu(w),
\]

where \( K \) is a kernel in a proper class and the singular integral operator has to be understood in the usual sense. For a kernel good enough so that the T(1)-theorem applies, similar considerations as the ones for the sufficiency in the proof of Theorem 1 apply. In particular, in order to have \( T \) is compact from \( L^2(\mu) \) to \( L^2(\mu) \) it suffices to require

1. \( \theta^*_n(z) = 0 \) for every \( z \in \mathbb{C} \).
2. \( \|T \chi_Q\|_{L^2(\mu, Q)}/\mu(Q) \rightarrow 0 \) as \( l(Q) \rightarrow 0 \).
3. \( \|T^* \chi_Q\|_{L^2(\mu, Q)}/\mu(Q) \rightarrow 0 \) as \( l(Q) \rightarrow 0 \).

However, these conditions turn out not to be necessary even in easy cases. An immediate example that shows that the density condition (1) is not necessary is the operator with kernel

\[
K(z, w) = \frac{\text{Im}(z - w)}{|z - w|^2}.
\]
and the measure \( \mu = \mathcal{H}^1(\{(0,1) \times \{0\}\}) \).

This operator (trivially) belongs to \( K(L^2(\mu), L^2(\mu)) \) even though \( \mu \) has positive linear density at each point of \((0,1) \times \{0\}\).

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