Operator estimates for non-periodically perforated domains with Dirichlet and nonlinear Robin conditions: vanishing limit

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Abstract
We consider a general second order linear elliptic equation in a finely perforated domain. The shapes of cavities and their distribution in the domain are arbitrary and non-periodic; they are supposed to satisfy minimal natural geometric conditions. On the boundaries of the cavities we impose either the Dirichlet or a nonlinear Robin condition; the choice of the type of the boundary condition for each cavity is arbitrary. Then we suppose that for some cavities the nonlinear Robin condition is sign-definite in certain sense. Provided such cavities and ones with the Dirichlet condition are distributed rather densely in the domain and the characteristic sizes of the cavities and the minimal distances between the cavities satisfy certain simple condition, we show that a solution to our problem tends to zero as the perforation becomes finer. Our main result are order sharp estimates for the $L_2$- and $W^{1,2}_2$-norms of the solution uniform in the $L_2$-norm of the right hand side in the equation.

Keywords
Perforated domain · Non-periodic perforation · Operator estimates · Vanishing limit · Order sharp estimates

Mathematics Subject Classification
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1 Introduction

Elliptic boundary value problems in finely perforated domains are one of the classical objects in the modern homogenization theory. A typical formulation of such problems consists in an elliptic equation in a domain perforated by small closely spaced cavities with some boundary conditions on their boundaries. The main aim of the study is to characterize the behavior of the solutions to the considered problems as the perforation becomes finer. There are hundreds of works devoted to studying such problems and not trying to mention all of them, we just cite several books, where problems in perforated domains were considered [7, 10, 14–16], see also the references therein. Typical results obtained for the problems in perforated domains usually state a convergence of the solutions to those of some homogenized problems. The convergence is usually proved in $L_2$ or in $W^{1,2}$, in a weak or strong sense, for given right hand sides in the equations and boundary conditions.

During the last 20 years, a new direction is being developed in the homogenization theory devoted to so-called operator estimates. The matter is that the difference between the solutions to the perturbed and homogenized problems is estimated uniformly with respect to the $L_2$-norm of the right hand side in the equation. From the point of view of the spectral theory, in the case of linear equations such estimates correspond to the norm resolvent convergence and describe the convergence rates, while the classical results usually establishes the strong or weak resolvent convergence.

Operator estimates for problems in periodically perforated domains were established in [5, 11, 12, 17, 18, 20]. In first two papers on the boundaries of the cavities the Neumann condition was imposed. In [20], the perforation was considered as an example of a general approach on homogenization of abstract measures, and it also led to the Neumann conditions on the boundaries of the cavities. In all three papers [17, 18, 20], the sizes of the cavities were proportional to the distances between them. In [12], the boundaries of the cavities were subject to the Dirichlet conditions, their linear sizes were assumed to satisfy certain upper bound expressed in terms of the size of the periodicity cell. All cavities had the same shape but it was allowed to rotate them arbitrarily and to shift a little their positions. In paper [11] the perforation was pure periodic and the cavities were small balls with the Robin condition on the their boundaries. A similar setting was studied in [5], where on the boundaries of the cavities also the Dirichlet and Neumann conditions were imposed. In all these papers, the homogenized problems were found and various operator estimates were established. The sharpness of these estimates was not discussed.

The case of a non-periodic perforation was considered in [1]. Here the cavities were non-periodically distributed small balls with the Dirichlet or Neumann conditions on their boundaries and certain conditions were imposed on the sizes of these balls ensuring that under the homogenization the balls disappear. In addition, there was considered a case, when the sizes of the balls with the Dirichlet condition were not too small so that the solution vanished in the limit in a part of the domain covered by such balls. The main results in [1] were also operator estimates for the considered problems.

There are also a few papers on operator estimates in domains finely perforated along a given manifold [2–4]. Here the perforation was of a general non-periodic
structure, both the shapes of cavities and their distribution were arbitrary. Various cases of possible homogenized problems were considered and in all of them operator estimates were established. In some cases these estimates were shown to be order sharp.

In the present paper we consider a boundary value problem for a general linear second order elliptic equation in a perforated domain; the equation is not necessary formally symmetric and involves complex-valued coefficients. The perforation is of a general structure, namely, both the shapes and distribution of the holes are arbitrary and only minimal natural geometric conditions are imposed on the perforation. The minimal distance between the cavities is characterized by a small positive parameter $\varepsilon$, while a characteristic linear size of each cavity is $\varepsilon \eta$, where $\eta = \eta(\varepsilon)$ is some bounded function.

On the boundaries of the cavities we impose either the Dirichlet or a nonlinear Robin condition. The choice of type of the boundary condition for each cavity is arbitrary and in the general situation we have the mixing of boundary conditions, that is, the boundaries of some cavities are subject to the Dirichlet condition, while on the boundaries of other cavities the nonlinear Robin condition is imposed. We assume that on the boundaries of some cavities, the nonlinear Robin condition is sign-definite in certain sense and its strength is controlled by some function $\mu(\varepsilon) \geq 1$. We also suppose that such cavities together with the Dirichlet cavities are distributed rather densely in the domain. These two assumptions and also certain condition for the relation between $\varepsilon$, $\eta$ and $\mu$ then ensure that a solution to the considered vanishes in the limit $\varepsilon \to +0$, that is, as the perforation becomes finer. Namely, we establish operator estimates both in $L_2$- and $W^{1,2}_2$-norms uniformly in $L_2$-norm of the right hand side in the equation writing explicitly the convergence rate in terms of $\varepsilon$, $\eta$ and $\mu$. The convergence rate for the $L_2$-norm of the solution is twice better than that for the $W^{1,2}_2$-norm. We succeed to show that the found convergence rates are order sharp; this is adduced by appropriate examples. One more important feature of our results is that they are true under minimal assumptions for the perforation, which are expressed in geometric terms. They allow us to establish key local estimates, which can be interpreted as the spectral solidifying condition, which was one of the main assumptions in [1].

2 Problem and main results

Let $x = (x_1, \ldots, x_n)$ be Cartesian coordinates in $\mathbb{R}^n$ and $\Omega$ be a domain in $\mathbb{R}^n$, the boundary of which has the smoothness $C^2$. The domain $\Omega$ can be both bounded or unbounded; a particular case $\Omega = \mathbb{R}^n$ is also possible. In the paper, we study an elliptic boundary value problem in a domain obtained by perforating $\Omega$. This perforation and the problem are introduced as follows.

By $\varepsilon$ we denote a small positive parameter. We choose a family of points $M^\varepsilon_k \in \Omega$, $k \in \mathbb{M}^\varepsilon$, and a family of domains $\omega_k, \varepsilon \subset \mathbb{R}^d$, $k \in \mathbb{M}^\varepsilon$, where $\mathbb{M}^\varepsilon$ is some set of indices, which is at most countable. The domains $\omega_k, \varepsilon$ are bounded and have $C^1$-boundaries. We then denote:
\[ \omega_k^\varepsilon := \{ x : (x - M_k^\varepsilon)\varepsilon^{-1} \eta^{-1}(\varepsilon) \in \omega_{k,\varepsilon} \}, \quad k \in \mathbb{M}^\varepsilon, \quad \theta^\varepsilon := \bigcup_{k \in \mathbb{M}^\varepsilon} \omega_k^\varepsilon, \quad (2.1) \]

where \( \eta = \eta(\varepsilon) \) is some function such that \( 0 < \eta(\varepsilon) \leq 1 \). The aforementioned perforated domain is introduced as \( \Omega^\varepsilon := \Omega \setminus \theta^\varepsilon \). We shall formulate rigorous conditions for the geometry of perforation later, now we just say that the cavities \( \omega_k^\varepsilon \) are disjoint and at the same time the distances between the points \( M_k^\varepsilon \) are small.

The coefficients of the elliptic equation we shall study are complex-valued functions \( A_{ij}^\varepsilon = A_{ij}^\varepsilon(x), A_j^\varepsilon = A_j^\varepsilon(x), A_0^\varepsilon = A_0^\varepsilon(x) \) defined in the perforated domain \( \Omega^\varepsilon \), which are supposed to satisfy the conditions

\[
A_{ij}^\varepsilon, A_j^\varepsilon, A_0^\varepsilon \in L_\infty(\Omega^\varepsilon), \quad A_{ij}^\varepsilon = \overline{A_{ji}^\varepsilon}, \\
\sum^n_{i,j=1} A_{ij}^\varepsilon(x) \xi_i \xi_j \geq c_0 \sum^n_{j=1} |\xi_j|^2, \quad x \in \Omega^\varepsilon, \quad \xi_i \in \mathbb{C}, \quad (2.2)
\]

where \( c_0 > 0 \) is some fixed constant independent of \( \varepsilon, \xi \) and \( x \). The functions \( A_{ij}^\varepsilon, A_j^\varepsilon, A_0^\varepsilon \) are bounded uniformly in \( \varepsilon \) in the norm of \( L_\infty(\Omega^\varepsilon) \).

We introduce an arbitrary partition of the set \( \theta^\varepsilon \):

\[
\theta^\varepsilon_D := \bigcup_{k \in \mathbb{M}^\varepsilon_D} \omega_k^\varepsilon, \quad \theta^\varepsilon_R := \bigcup_{k \in \mathbb{M}^\varepsilon_R} \omega_k^\varepsilon, \quad \mathbb{M}^\varepsilon_D \cup \mathbb{M}^\varepsilon_R = \mathbb{M}^\varepsilon, \quad \mathbb{M}^\varepsilon_D \cap \mathbb{M}^\varepsilon_R = \emptyset. \quad (2.3)
\]

By \( a^\varepsilon = a^\varepsilon(x, u) \) we denote a measurable complex-valued function defined for \( x \in \partial \theta^\varepsilon_R \) and \( u \in \mathbb{C} \) obeying the Lipschitz conditions

\[
|a^\varepsilon(x, u_1) - a^\varepsilon(x, u_2)| \leq a^\varepsilon_0|u_1 - u_2|, \quad u_1, u_2 \in \mathbb{C}, \quad (2.4)
\]

where \( a^\varepsilon_0 \) is some constant independent of \( x, u_1, u_2 \) but depending on \( \varepsilon \). We also assume that the function \( a^\varepsilon(x, u) \) satisfies the estimate

\[
\text{Re} a^\varepsilon(x, u) \bar{u} \geq -c_1 \varepsilon \eta^{-n+1}(\varepsilon)|u|^2, \quad x \in \partial \theta^\varepsilon_R. \quad (2.5)
\]

where \( c_1 \) is some fixed constant independent of \( \varepsilon \) and \( \eta \); this constant can be positive or negative or zero.

In this paper we study the following boundary value problem:

\[
(L - \lambda) u_\varepsilon = f \quad \text{in} \quad \Omega^\varepsilon, \quad u_\varepsilon = 0 \quad \text{on} \quad \partial \Omega \cup \partial \theta^\varepsilon_D, \\
\frac{\partial u_\varepsilon}{\partial \nu} + a^\varepsilon(x, u_\varepsilon) = 0 \quad \text{on} \quad \partial \theta^\varepsilon_R. \quad (2.6)
\]

Here \( L \) is a differential expression:

\[
L := -\sum^n_{i,j=1} \frac{\partial}{\partial x_i} A_{ij}^\varepsilon \frac{\partial}{\partial x_j} + \sum^n_{j=1} A_j \frac{\partial}{\partial x_j} + A_0, \quad (2.7)
\]
\( f \in L^2(\Omega^\varepsilon) \) is an arbitrary function, \( \lambda \in \mathbb{C} \) is a fixed constant and the co-normal derivative is defined as
\[
\frac{\partial}{\partial \nu} = \sum_{i,j=1}^n A_{ij}^\varepsilon v_i \frac{\partial}{\partial x_j},
\]
where \( v_i \) are the components of the unit normal to \( \partial \theta^\varepsilon_D \) directed inside \( \theta^\varepsilon_D \).

The main aim of our study is to describe the behavior of a solution to problem (2.6) as \( \varepsilon \to +0 \). A solution is understood in the generalized sense. Given a domain \( Q \subseteq \mathbb{R}^n \) and a manifold \( S \subseteq Q \) of codimension one, let \( \dot{W}^1_2(Q, S) \) be the subspace of the Sobolev space \( W^1_2(Q) \) formed by the functions with the zero trace on the manifold \( S \). A generalized solution to problem (2.6) is a function \( u \in \dot{W}^1_2(\Omega^\varepsilon, \partial \Omega \cup \partial \theta^\varepsilon_D) \) satisfying the integral identity
\[
\mathbb{h}_d(u_\varepsilon, v) - \lambda(u_\varepsilon, v)_{L^2(\Omega^\varepsilon)} = (f, v)_{L^2(\Omega^\varepsilon)} \tag{2.8}
\]
for each \( v \in \dot{W}^1_2(\Omega^\varepsilon, \partial \Omega \cup \partial \theta^\varepsilon_D) \), where
\[
\mathbb{h}_d(u, v) := \mathbb{h}_A(u, v) + (a^\varepsilon(\cdot, u), v)_{L^2(\partial \theta^\varepsilon_D)},
\]
\[
\mathbb{h}_A(u, v) := \sum_{i,j=1}^n \left( A_{ij}^\varepsilon \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i} \right)_{L^2(\Omega^\varepsilon)} + \sum_{j=1}^n \left( A_j^\varepsilon \frac{\partial u}{\partial x_j}, v \right)_{L^2(\Omega^\varepsilon)} + (A_0^\varepsilon u, v)_{L^2(\Omega^\varepsilon)}.
\]

Now we are going to formulate exact conditions on the geometry of perforation. By \( B_r(M) \) we denote a ball in \( \mathbb{R}^n \) of a radius \( r \) centered at a point \( M \). In the vicinity of the boundaries \( \partial \omega_k,\varepsilon \) we introduce a local variable, which is the distance \( \tau \) measured along the normal to \( \partial \omega_k,\varepsilon \) directed outside \( \omega_k,\varepsilon \). Our first assumption concerns the general structure of the perforation.

A1 The points \( M^\varepsilon_k \) and the domains \( \omega_k,\varepsilon \) obey the conditions
\[
B_{R_1}(y_k,\varepsilon) \subseteq \omega_k,\varepsilon \subseteq B_{R_2}(0), \quad B_{\varepsilon R_3}(M^\varepsilon_k) \cap B_{\varepsilon R_3}(M^\varepsilon_j) = \emptyset,
\]
\[
dist(M^\varepsilon_k, \partial \Omega) \geq R_3, \quad k \neq j, \quad k, j \in \mathbb{M}^\varepsilon, \tag{2.9}
\]
where \( y_k,\varepsilon \) are some points and \( R_1 < R_2 < R_3 \) are some fixed constants independent of \( \varepsilon, \eta, k \) and \( j \). The sets \( B_{R_2}(0) \setminus \omega_k,\varepsilon \) are connected. For each \( k \in \mathbb{M}^\varepsilon_R \) there exist local variables \( s \) on \( \partial \omega_k,\varepsilon \) such that the variables \((\tau, s)\) are well-defined at least on \( \{ x \in \mathbb{R}^n \setminus \omega_k,\varepsilon : \text{dist}(x, \partial \omega_k,\varepsilon) \leq \tau_0 \} \subseteq B_{R_2}(0) \), where \( \tau_0 \) is a fixed constant independent of \( k \in \mathbb{M}^\varepsilon_R \) and \( \varepsilon \) and the Jacobians corresponding to passing from variables \( x \) to \((\tau, s)\) are separated from zero and bounded from above uniformly in \( \varepsilon, k \in \mathbb{M}^\varepsilon \) and \( x \) as \( 0 \leq \tau \leq \tau_0 \). The first derivatives of \( x \) in \((\tau, s)\) and of \((\tau, s)\) in \( x \) are continuous and bounded uniformly in \( \varepsilon, k \in \mathbb{M}^\varepsilon \) and \( x \) as \( 0 \leq \tau \leq \tau_0 \).

It is known that the behavior of a solution to problem (2.6), (2.7) is very sensible to the geometry of perforation and the structure of the coefficients in the equation and
boundary conditions. In this paper we consider just one of several possible typical cases, namely, the case when \( u_\varepsilon \) vanishes as \( \varepsilon \to +0 \). Such situation is ensured by the following additional conditions.

**A6** The set \( M_R^\varepsilon \) contains a subset \( M_{R,0}^\varepsilon \) such that

\[
\text{Re} \, a^\varepsilon(x, u) \bar{u} \geq \mu(\varepsilon) a^\varepsilon_k(x) |u|^2, \quad x \in \partial \omega_k^\varepsilon, \quad k \in M_{R,0}^\varepsilon, \tag{2.10}
\]

where \( \mu = \mu(\varepsilon) \geq 1 \) and \( a^\varepsilon_k(x) \) are some measurable functions obeying the conditions

\[
\alpha^\varepsilon_k \in L_\infty(\partial \omega_k^\varepsilon), \quad \alpha^\varepsilon_k \geq 0 \quad \text{a.e. on } \partial \omega_k^\varepsilon, \quad \| \alpha^\varepsilon_k \|^2_{L_2(\partial \omega_k^\varepsilon)} \leq c_2 (\varepsilon \eta)^{n-1}, \quad \| \alpha^\varepsilon_k \|_{L_1(\partial \omega_k^\varepsilon)} \geq c_3 (\varepsilon \eta)^{n-1}, \tag{2.11}
\]

where \( c_2, c_3 \) are some fixed positive constants independent of \( k \in M_{R,0}^\varepsilon, \varepsilon \) and \( \eta \).

**A5** There exists a fixed constant \( R_4 > 0 \) independent of \( \varepsilon \) and \( k \) such that

\[
\Omega \subseteq \bigcup_{k \in M_{R,0}^\varepsilon \cup M_D^\varepsilon} B_{R_4}(M_k^\varepsilon). \tag{2.12}
\]

We denote:

\[
\varkappa(\varepsilon) := | \ln \eta(\varepsilon) | + 1 \quad \text{as} \quad n = 2, \quad \varkappa(\varepsilon) := 1 \quad \text{as} \quad n \geq 3.
\]

Now we are in position to formulate our main result.

**Theorem 2.1** Let Conditions A1, A5, A6 be satisfied and

\[
\varepsilon \eta^{-n+1}(\varepsilon) \mu^{-1}(\varepsilon) + \varepsilon^2 \eta^{-n+2}(\varepsilon) \varkappa(\varepsilon) \to +0, \quad \varepsilon \to +0. \tag{2.13}
\]

Then there exists a fixed \( \lambda_0 \in \mathbb{R} \) independent of \( \varepsilon \) such that as \( \text{Re} \, \lambda \leq \lambda_0 \), problem (2.6), (2.7) is solvable for each \( f \in L_2(\Omega^\varepsilon) \) and each of its solutions satisfies the estimates:

\[
\| u_\varepsilon \|_{W_1^2(\Omega^\varepsilon)} \leq C \left( \varepsilon^2 \eta^{-n+2}(\varepsilon) (\mu^{-1}(\varepsilon) + \varepsilon \eta^{-n+1}(\varepsilon) \varkappa(\varepsilon)) \right) \| f \|_{L_2(\Omega^\varepsilon)}, \tag{2.14}
\]

\[
\| u_\varepsilon \|_{L_2(\Omega^\varepsilon)} \leq C \left( \varepsilon^2 \eta^{-n+2}(\varepsilon) (\mu^{-1}(\varepsilon) + \varepsilon \eta^{-n+1}(\varepsilon) \varkappa(\varepsilon)) \right) \| f \|_{L_2(\Omega^\varepsilon)}, \tag{2.15}
\]

where \( C \) is some constant independent of \( \varepsilon \) and \( f \). These estimates are order sharp.

**Theorem 2.2** Let Assumptions A1, A5 be satisfied, the set \( M_{R,0}^\varepsilon \) be empty, and

\[
\varepsilon^2 \eta^{-n+2}(\varepsilon) \varkappa(\varepsilon) \to +0, \quad \varepsilon \to +0. \tag{2.16}
\]

Then there exists a fixed \( \lambda_0 \in \mathbb{R} \) independent of \( \varepsilon \) such that as \( \text{Re} \, \lambda \leq \lambda_0 \), problem (2.6), (2.7) is solvable for each \( f \in L_2(\Omega^\varepsilon) \) and each of its solutions satisfies the estimates:
Operator estimates for non-periodically...  

**Fig. 1** Domain $\omega_{k,\varepsilon}$ and corresponding balls $B_{R_1}(y_{k,\varepsilon})$ and $B_{R_2}(0)$

\[
\|u_\varepsilon\|_{W^1_2(\Omega^\varepsilon)} \leq C \varepsilon^{-(n+1)(\varepsilon)} \varepsilon^{(n+2)(\varepsilon)} \|f\|_{L^2(\Omega^\varepsilon)},
\]

\[
\|u_\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C \varepsilon^{2(n+2)(\varepsilon)} \|f\|_{L^2(\Omega^\varepsilon)},
\]

where $C$ is some constant independent of $\varepsilon$ and $f$. These estimates are order sharp.

Let us discuss briefly the problem and main result. The first main feature is that the perforation we consider is of a general non-periodic structure. We only make Assumption A1, which is quite natural. The first two-sided inclusion in (2.9) means that the domains $\omega_{k,\varepsilon}$ are roughly of the same sizes: each of these domain can be put inside a fixed ball $B_{R_2}(0)$ and inside each of these domain a fixed ball of the radius $R_1$ can be inscribed, see Fig. 1. We stress that this does not impose any restrictions on the shapes of the domains $\omega_{k,\varepsilon}$ and they indeed can be very arbitrary. In view of the inequality $R_2 < R_3$, the second condition in (2.9) means that the cavities $\omega^\varepsilon_{k}$, defined in (2.1), do not intersect and there is a minimal distance $2R_3$ between the points $M^\varepsilon_{k}$, which can be regarded as a distance between the cavities. The third inequality in (2.9) ensures that the cavities do not intersect the boundary of the domain $\partial \Omega$, which is also a natural assumption, see Fig. 2. We note that these conditions do not imply that the distances between the points $M^\varepsilon_{k}$ are indeed of order $\sim \varepsilon$ since only the lower bounds for the distances are postulated. Some of the distances can be much larger than $\varepsilon$. It should also be said that the distribution of the points $M^\varepsilon_{k}$, and as a consequence, of the cavities $\omega^\varepsilon_{k}$, is very weakly restricted by (2.9) and it can be very arbitrary.

The connectedness of sets $B_{R_2}(0) \setminus \omega_{k,\varepsilon}$ is also a natural assumption. Indeed, if this is not the case for some $k$, then the part $B_{\varepsilon R_2(M^\varepsilon_{k})} \setminus \omega^\varepsilon_{k}$ of the domain $\Omega^\varepsilon$ contains a small piece not connected with the rest of $\Omega^\varepsilon$. Then the boundary value problem on this small piece becomes independent on the problem on the rest of $\Omega^\varepsilon$ and its solution do not influence the behavior of the solution on the rest of $\Omega^\varepsilon$. The
assumption on the local variables in the vicinity of the boundaries \( \partial \omega_k, \varepsilon \) for \( k \in \mathbb{M}^\varepsilon_{R,0} \) is more gentle and requires a certain uniform regularity of the shapes of boundaries \( \partial \omega_k, \varepsilon \) with respect to \( k \) and \( \varepsilon \). First, since the assumed smoothness of \( \partial \omega_k, \varepsilon \) is \( C^1 \), there is a tangential hyperplane to \( \partial \omega_k, \varepsilon \) at each point and the normal vector to this hyperplane is continuous. We additionally assume that the introduced local variables possess uniformly bounded continuous derivatives with respect to the initial Cartesian coordinates and the corresponding Jacobians are well-defined. Second, the uniform boundedness of these Jacobians is an additional regularity property. Roughly speaking, this regularity prohibits the situation when on some sequence of values \( k \) and \( \varepsilon \), the boundaries \( \partial \omega_k, \varepsilon \) highly oscillates or approximates some non-smooth surfaces. We also conjecture that if this is the case, this could influence and change the behaviour of solutions to problem (2.6).

Problem (2.6) involves general linear differential expression \( L \) defined in (2.7), which is not formally symmetric due to the presence of the first derivatives and due to the complex coefficients. The coefficients in this expression can also depend on the parameter \( \varepsilon \) and the only restrictions on the dependence are ones in (2.2). These are very weak conditions and this is why the dependence of the coefficients on \( \varepsilon \) can be very arbitrary including, for instance, a particular case of fast and non-periodically oscillating functions.

The Dirichlet boundary condition on \( \partial \Omega \) is imposed just for the definiteness and can be replaced by any other classical boundary condition. The main feature are the conditions on the boundaries of the cavities: these are the Dirichlet and the nonlinear Robin conditions. The choice of type of the boundary conditions is in fact arbitrary and is described by the partition of cavities in (2.3). The nonlinearity in the Robin condition should satisfy Lipschitz condition (2.4), which means that it has at most a linear growth. One more condition is inequality (2.5), which restricts the growth of the nonlinearity with respect to the small parameter \( \varepsilon \).

Extra conditions are Assumptions A6, A5. The former states that at least for some cavities with the nonlinear Robin condition the nonlinearity is positive definite in the sense of inequality (2.10), where the functions \( \alpha_k, \varepsilon \) should obey (2.11). The estimates for the \( L_1 \)- and \( L_2 \)-norms of these functions can be equivalently rewritten as

\[
\|\alpha_{k, \varepsilon}\|_{L_2(\partial \omega_k, \varepsilon)}^2 \leq c_2, \quad \|\alpha_{k, \varepsilon}\|_{L_1(\partial \omega_k, \varepsilon)} \geq c_3, \quad \alpha_{k, \varepsilon}(x) := \alpha^\varepsilon_k(M_k^\varepsilon + \varepsilon \eta x).
\]

Here the spaces \( L_1 \) and \( L_2 \) are considered on non-small surfaces \( \partial \omega_k, \varepsilon \) and these inequalities mean that the functions \( \alpha_{k, \varepsilon} \) should be bounded in \( L_2(\omega_k, \varepsilon) \)-norms uniformly in \( k \) and \( \varepsilon \) and should have an \( L_1(\omega_k, \varepsilon) \)-norm uniformly separated from zero. It is clear that these conditions are very weak and are obeyed by a very wide class of functions \( \alpha_{k, \varepsilon} \). The parameter \( \mu = \mu(\varepsilon) \) in (2.10) characterizes the dependence of the nonlinearity \( \alpha^\varepsilon \) on \( \partial \omega_k^\varepsilon \) with respect to \( \varepsilon \). This parameter behaves arbitrarily as \( \varepsilon \rightarrow +0 \), including a possible arbitrary growth.

Assumption A5 has a very simple geometric interpretation: the cavities with the Dirichlet condition and the nonlinear Robin condition obeying Assumption A6 should be distributed quite dense to ensure covering (2.12) of the domain \( \Omega \), see Fig. 3. This assumption is used at a final step in the proofs of Theorems 2.1, 2.2. Namely, we first make a series of certain local estimates for the solution \( u_\varepsilon \) in balls \( B_{\varepsilon R_4}(M_k^\varepsilon) \)
and then we glue these estimates into global inequalities (2.14), (2.15), (2.17), (2.18). Assumption A6 allow us to cover in this way the entire domain \( \Omega^\varepsilon \); while doing this, by Assumption A1 we also confirm that each point \( x \) of \( \Omega^\varepsilon \) is covered by finitely many balls \( B_{\varepsilon R_4}(M^\varepsilon_k) \) and the number of such balls for each point \( x \) is bounded uniformly in \( \varepsilon \) and \( x \).

Our main result states that provided convergence (2.13) holds, problem (2.6) is solvable and each solution tends to zero as \( \varepsilon \to +0 \). The convergence is established both in \( W^{1,2}_2 \)- and \( L_2 \)-norms and it is uniform in the \( L_2 \)-norm of the right hand side in the equation. A gentle point is that we do not state a unique solvability and the situation of multiple solutions is not excluded. In the proof of Theorem 2.1, the solvability is ensured by Assumption 2.1 and inequalities (2.4), (2.5); other assumptions and conditions are not used. Assumptions A6, A5 and convergence (2.13) guarantee inequalities (2.14), (2.15). The first of them is for the \( W^{1,2}_2(\Omega^\varepsilon) \)-norm of the solution and it describes the rate, at which the solution vanishes as \( \varepsilon \to +0 \) uniformly in the \( L_2(\Omega^\varepsilon) \)-norm of the right hand side. The second inequality is of the same nature but now the \( L_2(\Omega^\varepsilon) \)-norm of the solution is estimated. The second norm of the solution is weaker than the norm in (2.14) and the convergence rate is twice better.

If there is no cavities with the nonlinear Robin conditions obeying Assumption A6 but the cavities with the Dirichlet condition still satisfy Assumption A5, this case can be also treated and the result is formulated in Theorem 2.2. It is similar to that in Theorem 2.1, but now convergence (2.13) is replaced by (2.16) and this is reflected in the convergence rates in (2.17), (2.18).

An important feature of our main results is that the estimates in Theorems 2.1, 2.2 are order sharp. This is justified by appropriate examples provided in the proofs of these theorems.
3 Proofs

In this section we prove Theorems 2.1, 2.2.

3.1 Auxiliary lemmata

Here we prove a series of auxiliary lemmata, which play a crucial role in the proofs of Theorems 2.1, 2.2.

Lemma 3.1 Under Assumption A1 for each $k \in \mathbb{M}^e_D$ and each function $u \in \tilde{W}^1(B_{\varepsilon R_3}(M^e_k) \setminus \omega^e_k, \partial \omega^e_k)$ the estimate

$$\|u\|_{L^2(B_{\varepsilon R_3}(M^e_k) \setminus \omega^e_k, \cdot)} \leq C \varepsilon^{-\eta - 2} \kappa \|\nabla u\|_{L^2(B_{\varepsilon R_3}(M^e_k) \setminus \omega^e_k, \cdot)}$$

holds, where $C$ is a constant independent of $k, \varepsilon, \eta$ and $u$.

Proof Given a function $u \in \tilde{W}^1(B_{\varepsilon R_3}(M^e_k) \setminus \omega^e_k, \cdot)$, we continue it by zero inside $\omega^e_k$. Then the continuation is an element of $\tilde{W}^1(B_{\varepsilon R_3}(M^e_k) \setminus \omega^e_k, \cdot)$, its $L^2(B_{\varepsilon R_3}(M^e_k) \setminus \omega^e_k, \cdot)$-norm and $W^1_2(B_{\varepsilon R_3}(M^e_k) \setminus \omega^e_k, \cdot)$-norm coincide with $L^2(B_{\varepsilon R_3}(M^e_k) \setminus \omega^e_k, \cdot)$-norm and $W^1_2(B_{\varepsilon R_3}(M^e_k) \setminus \omega^e_k, \cdot)$-norm of the original function $u$, and in view of the first condition in (A1), the continuation vanishes at least on the ball $B_{\varepsilon R_1}(M^e_k + \varepsilon \eta y, \cdot)$. We keep the same notation $u$ for this continuation. By $r$ we denote the radius for spherical coordinates centered at the point $M^e_k + \varepsilon \eta y, \cdot$. Then for all $x \in B_{\varepsilon R_3}(M^e_k) \setminus B_{\varepsilon R_1}(M^e_k + \varepsilon \eta y, \cdot)$ by the Cauchy-Schwarz inequality we have:

$$|u(x)|^2 = \left| \int_{|x - M^e_k - \varepsilon \eta y, \cdot|}^{\varepsilon R_1} \frac{\partial u}{\partial r} dr \right|^2 \leq \int_{\varepsilon R_1}^{\varepsilon R_1} \frac{|x - M^e_k - \varepsilon \eta y, \cdot| dr}{r^{n-1}} \int_{\varepsilon R_1}^{\varepsilon R_1} |\nabla u|^2 r^{n-1} dr$$

$$\leq C(\varepsilon \eta)^{-\eta - 2} \kappa \int_{\varepsilon R_1}^{\varepsilon R_1} |\nabla u|^2 r^{n-1} dr,$$

where $C$ is some constant independent of $\varepsilon, \eta, k$ and $u$. Integrating this estimate over $B_{\varepsilon R_3}(M^e_k) \setminus B_{\varepsilon R_1}(M^e_k + \varepsilon \eta y, \cdot)$, we arrive at the statement of the lemma. The proof is complete. \qed

The next lemma was stated in [2, Lm. 3.1] and its proof was essentially based on the Cheeger’s estimate for the lowest positive eigenvalue of a Laplacian in a given domain provided in the original work [6]. However, as we found very recently, the Cheeger’s estimate has to be modified appropriately once we deal with the Neumann condition on some parts of the boundary, see, for instance, [9, Introduction], [13, Ch. 9, Cor. 9.7]. Despite the found gap in the proof of Lemma 3.1 in [2], the lemma is still true and here we give a corrected proof which is not based on the Cheeger’s estimate.
Lemma 3.2 Under Assumption A1, for each $k \in \mathbb{M}^\varepsilon$ and each function $u \in \tilde{W}^1_2(B_{R_3}(0) \setminus \omega_{k,\varepsilon}, \partial B_{R_3}(0))$ the estimate holds:

$$\|u\|_{L^2_2(B_{R_3}(0) \setminus \omega_{k,\varepsilon})} \leq C \|\nabla u\|_{L^2_2(B_{R_3}(0) \setminus \omega_{k,\varepsilon})},$$

where $C$ is a constant independent of $\varepsilon$, $k$ and $u$.

Proof We choose an arbitrary function $u \in \tilde{W}^1(B_{R_3}(0) \setminus \omega_{k,\varepsilon}, \partial B_{R_3}(0))$. Since the lowest eigenvalue of the Laplacian in $B_{R_3}(0) \setminus B_{R_2}(0)$ subject to the Dirichlet condition on $\partial B_{R_3}(0)$ and to the Neumann condition on $\partial B_{R_2}(0)$ is strictly positive, by the minimax principle we immediately obtain:

$$\|u\|_{L^2_2(B_{R_3}(0) \setminus B_{R_2}(0))} \leq C \|\nabla u\|_{L^2_2(B_{R_3}(0) \setminus B_{R_2}(0))};$$

(3.1) throughout the proof by $C$ we denote various inessential constants independent of $\varepsilon$, $k$, $u$ and the spatial variables.

We fix a positive number $\tau_1 \leq \frac{\tau_0}{10\sqrt{n}}$ and consider a lattice $\tau_1 \mathbb{Z}^n$ in $\mathbb{R}^n$. By $\Gamma_{k,\varepsilon}$ we denote a subset of this lattice defined as

$$\Gamma_{k,\varepsilon} := \{z \in \tau_1 \mathbb{Z}^n : z + 2\tau_1(0, 1)^n \subset B_{R_3}(0) \setminus \omega_{k,\varepsilon}\}.$$

The number of the points in the set $\Gamma_{k,\varepsilon}$ satisfies an obvious estimate

$$\#\Gamma_{k,\varepsilon} \leq \#\tau_1 \mathbb{Z}^n \cap B_{R_3}(0),$$

(3.2)
which is uniform in $k$ and $\varepsilon$. It also follows from the definition of $\Gamma_{k,\varepsilon}$ that choosing $\tau_1$ sufficiently small but fixed, we have the covering

$$B_{\frac{R_2+R_3}{2}}(0) \setminus \tilde{\omega}_{k,\varepsilon} \subseteq \bigcup_{z \in \Gamma_{k,\varepsilon}} (z + 2\tau_1(0, 1)^n),$$

$$\tilde{\omega}_{k,\varepsilon} := \left\{ x : \text{dist}(x, \omega_{k,\varepsilon}) \leq \frac{\tau_0}{2} \right\}, \quad \omega_{k,\varepsilon} \subseteq \tilde{\omega}_{k,\varepsilon}, \quad (3.3)$$

for all $\varepsilon$ and $k$. Since the domain $B_{R_3}(0) \setminus \omega_{k,\varepsilon}$ is connected by Assumption A1, the above covering yields that for each $z \in \Gamma_{k,\varepsilon} \cap B_{\frac{2R_2+R_3}{3}}(0)$ such that $\text{dist}(\partial\omega_{k,\varepsilon}, z + 2\tau_1(0, 1)^n) \geq \frac{\tau_0}{2}$ there exist points $y_j = y_j(z) \in \Gamma_{k,\varepsilon}$, $j = 0, \ldots, N(z)$, $y_N(z) = z$ such that

$$y_0(z) + 2\tau_1(0, 1)^n \subset B_{\frac{R_2+R_3}{2}}(0) \setminus B_{R_2}(0),$$

$$y_j(z) + 2\tau_1(0, 1)^n \subset B_{\frac{R_2+R_3}{2}}(0) \setminus \omega_{k,\varepsilon}, \quad y_j \neq y_p, \quad j \neq p, \quad (3.4)$$

and for each $j$ the coordinates of the points $y_j$ and $y_{j+1}$ differ at most by $\tau_1$. Due to estimate (3.2), the number $N(z)$ is bounded uniformly in $\varepsilon$, $k$ and $z$. For each two neighbouring points $y_j$ and $y_{j+1}$ the cubes $y_j + 2\tau_1(0, 1)^n$ and $y_{j+1} + 2\tau_1(0, 1)^n$ have a non-empty intersection, which contains at least an appropriate shift of the cube $\tau_1(0, 1)^n$, see Fig. 4.

For each $z \in \Gamma_{k,\varepsilon} \cap B_{\frac{2R_2+R_3}{3}}(0)$ such that $\text{dist}(\partial\omega_{k,\varepsilon}, z + 2\tau_1(0, 1)^n) \geq \frac{\tau_0}{2}$ we choose a corresponding point $y_0(z)$ obeying (3.4) and by (3.1) we have the estimate

$$\|u\|_{L_2(y+\tau_1(0,1)^n)} \leq C \|\nabla u\|_{L_2(B_{R_3}(0)\setminus\omega_{k,\varepsilon})} \quad (3.5)$$

with $y = y_0$. Then we choose corresponding points $y_j$ and suppose that for some $j$ estimate (3.5) holds with $y = y_j$. Then the cubes $y_{j+1} + 2\tau_1(0, 1)^n$ intersects with $y_j + 2\tau_1(0, 1)^n$ at least by a cube with side $\tau_1$, see Fig. 5; we denote this cube by $Q_j$. At least one of the vertices of $Q_j$ coincide with one of the vertices of $y_{j+1} + 2\tau_1(0, 1)^n$; we denote this vertex by $q_j$. For an arbitrary point $x \in y_{j+1} + 2\tau_1(0, 1)^n$ separated from $q_j$ by a distance at least $\frac{\tau_1}{2}$, we consider a segment $I(x)$ connecting the point $q_j$ and $x$ with a variable $t \in [0, |x-q_j|]$ on this segment. Then for $x \in y_{j+1} + 2\tau_1(0, 1)^n$ such that $|x - q_j| > \frac{\tau_1}{2}$ we obviously have

$$u(x) = \int_0^{\frac{|x-q_j|}{\tau_1}} \frac{\partial}{\partial t} u \chi_1(t) \, dt,$$

where the integration is taken along the segment $I(x)$ and $\chi_1 = \chi_1(t)$ is an infinitely differentiable function equalling to one as $t > \frac{\tau_1}{2}$ and vanishing as $t < \frac{\tau_1}{3}$. Then by
the Cauchy-Schwarz inequality we immediately get:

\[ |u(x)|^2 \leq C \int_0^{\frac{x-q_j}{2}} |\nabla u|^2 t^{n-1} \, dt + C \int_0^{\frac{2}{x}} |u|^2 t^{n-1} \, dt, \]

where the integration is again made along the segment \( I(x) \). Integrating the obtained inequality over \((y_j + 1 + 2\tau_1(0, 1))^n \setminus Q_j\), passing to the spherical coordinates and using (3.5), we find:

\[
\|u\|_{L_2((y_j + 1 + 2\tau_1(0, 1))^n \setminus Q_j)}^2 \leq C \|\nabla u\|_{W_2^1((y_j + 1 + 2\tau_1(0, 1))^n \setminus Q_j)}^2 + C \|u\|_{L_2(Q_j)}^2 \leq C \|\nabla u\|_{L_2(BR_3(0) \setminus \omega_{k,\varepsilon})}^2,
\]

and this implies (3.5) for \( y = y_j + 1 \). Hence, using the above described procedure of extending estimate (3.5) along the points \( y_j \) and summing up all obtained estimates over \( y \in \Gamma_{k,\varepsilon} \cap BR_3(0) \) such that

\[
\text{dist} \left( \partial \omega_{k,\varepsilon}, y + 2\tau_1(0, 1)^n \right) \geq \frac{\tau_0}{2},
\]

by (3.1) we finally get:

\[
\|u\|_{L_2(BR_3(0) \setminus \omega_{k,\varepsilon})} \leq C \|\nabla u\|_{L_2(BR_3(0) \setminus \omega_{k,\varepsilon})}, \quad \omega_{k,\varepsilon} := \left\{ x : \text{dist}(x, \omega_k) \leq \frac{7\tau_0}{10} \right\}.
\]

(3.6)

Let \( \chi_2 = \chi_2(\tau) \) be an infinitely differentiable cut-off function equalling to one as \( \tau < 8\tau_0/10 \) and vanishing as \( \tau > 9\tau_0/10 \). Then for \( x \) such that \( \tau < 7\tau_0/10 \) due to the regularity of the boundaries \( \partial \omega_{k,\varepsilon} \) postulated in Assumption A1 we have

\[
|u(x)|^2 = \left| \int_0^\tau \frac{\partial}{\partial \tau} u \chi_2 \, d\tau \right|^2 \leq C \int_0^\tau (|\nabla u|^2 + |u|^2) \, d\tau.
\]

(3.7)

Integrating this estimate over \( \omega_{k,\varepsilon} \setminus \omega_{k,\varepsilon} \) and using then (3.6), we obtain:

\[
\|u\|_{L_2(\omega_{k,\varepsilon} \setminus \omega_{k,\varepsilon})}^2 \leq C \|\nabla u\|_{L_2(\omega_{k,\varepsilon} \setminus \omega_{k,\varepsilon})}^2 + C \|u\|_{L_2(BR_3(0) \setminus \omega_{k,\varepsilon})}^2 \leq C \|\nabla u\|_{L_2(BR_3(0) \setminus \omega_{k,\varepsilon})}^2.
\]

This estimate and (3.6) imply the estimate in the statement of the lemma. The proof is complete. \( \square \)

**Lemma 3.3** Under Assumption A1 the problem
possesses a unique generalized solution for all $\varepsilon$ and $k \in \mathbb{M}^\varepsilon$. This solution belongs to $W^1_2(B_{R_3}(0) \setminus \omega_{k,\varepsilon})$ and satisfies the estimate

$$
\|X_{k,\varepsilon}\|_{W^1_2(B_{R_3}(0) \setminus \omega_{k,\varepsilon})} \leq C, \quad \|\frac{\partial X_{k,\varepsilon}}{\partial |x|}\|_{L^2(\partial B_{R_5}(0))} \leq C, \quad R_5 := \frac{R_2 + R_3}{2},
$$

(3.9)

with a constant $C$ independent of $x$, $k$ and $\varepsilon$.

**Proof** Due to the presence of the Dirichlet condition on $\partial B_{R_3}(0)$ in problem (3.8), it is uniquely solvable and the generalized solution belongs to $W^1_2(B_{R}(0) \setminus \omega_{k,\varepsilon})$. The corresponding integral identity
\[ \| \nabla X_{k,\varepsilon} \|^2_{L^2(B_{R_3}(0) \setminus \omega_{k,\varepsilon})} = \int_{B_{R_3}(0) \setminus \omega_{k,\varepsilon}} X_{k,\varepsilon} \, dx \]

and Lemma 3.2 by the Cauchy-Schwarz inequality give the estimates

\[ \| \nabla X_{k,\varepsilon} \|^2_{L^2(B_{R_3}(0) \setminus \omega_{k,\varepsilon})} \leq C \| X_{k,\varepsilon} \|_{L^2(B_{R_3}(0) \setminus \omega_{k,\varepsilon})} \leq C \| \nabla X_{k,\varepsilon} \|_{L^2(B_{R_3}(0) \setminus \omega_{k,\varepsilon})} \]

with constants \( C \) independent of \( \varepsilon \) and \( k \). These estimates imply the first inequality in (3.9). Standard smoothness improving estimates then yield

\[ \| X_{k,\varepsilon} \|_{W^2_2(B_{R_3+\delta}(0) \setminus B_{R_3-\delta}(0))} \leq C \| X_{k,\varepsilon} \|_{W^1_2(B_{R_3+2\delta}(0) \setminus B_{R_3-2\delta}(0))} \leq C \]

with \( \delta := \frac{1}{5}(R_3 - R_2) \) and a fixed constant \( C \) depending only on \( \delta \). This leads us to the second inequality in (3.9). The proof is complete. \( \square \)

**Lemma 3.4** Under Assumption A1, for each \( k \in \mathbb{M}_D^\varepsilon \) and each function \( u \in W^1_2(B_{R_3}(0) \setminus \omega_{k,\varepsilon}) \) obeying the condition

\[ \int_{B_{R_3}(0) \setminus \omega_{k,\varepsilon}} u(x) \, dx = 0 \quad (3.10) \]

the estimate holds:

\[ \| u \|_{L^2(B_{R_3}(0) \setminus \omega_{k,\varepsilon})} \leq C \| \nabla u \|_{L^2(B_{R_3}(0) \setminus \omega_{k,\varepsilon})}, \quad (3.11) \]

where \( C \) is a constant independent of \( \varepsilon, k \) and \( u \).

**Proof** Since the space \( C^\infty(B_{R_3}(0) \setminus \omega_{k,\varepsilon}) \) is dense in \( W^1_2(B_{R_3}(0) \setminus \omega_{k,\varepsilon}) \), it is sufficient to prove (3.11) only for smooth functions \( u \). Given such smooth function \( u \), we denote

\[ \langle u \rangle := \frac{1}{\text{mes}_n B_{R_3}(0) \setminus B_{R_5}(0)} \int_{B_{R_3}(0) \setminus B_{R_5}(0)} u(x) \, dx, \quad R_5 := \frac{R_2 + R_3}{2}, \quad u_\perp := u - \langle u \rangle, \]
where \( \text{mes}_n(\cdot) \) stands for the \( n \)-dimensional Lebesgue measure of a set. It is clear that

\[
\|u\|_{L^2(B_{R_5}(0) \setminus B_{R_5}(0))}^2 = |\langle u \rangle|^2 \text{mes}_n B_{R_5}(0) \setminus B_{R_5}(0) \\
+ \|u_\perp\|_{L^2(B_{R_5}(0) \setminus B_{R_5}(0))}^2 \int_{B_{R_5}(0) \setminus B_{R_5}(0)} u_\perp \, dx = 0.
\]

(3.12)

Since the second eigenvalue of the Neumann Laplacian in \( B_{R_5}(0) \setminus B_{R_5}(0) \) is strictly positive, by the minimax principle the function \( u_\perp \) satisfies the estimate

\[
\|u_\perp\|_{L^2(B_{R_5}(0) \setminus B_{R_5}(0))} \leq C \|\nabla u\|_{L^2(B_{R_5}(0) \setminus B_{R_5}(0))} ;
\]

(3.13)

throughout the proof by \( C \) we denote various inessential constants independent of \( u, k, \varepsilon \) and spatial variables. This estimate then obviously implies that

\[
\|u_\perp\|_{L^2(\partial B_{R_5}(0))} \leq C \|\nabla u\|_{L^2(B_{R_5}(0) \setminus B_{R_5}(0))}.
\]

(3.14)

We also observe that by (3.10)

\[
\langle u \rangle = -\frac{1}{\text{mes}_n B_{R_5}(0) \setminus B_{R_5}(0)} \int_{B_{R_5}(0) \setminus \omega_{k,\varepsilon}} u \, dx.
\]

(3.15)

We take the solution \( X_{k,\varepsilon} \) to problem (3.8) and integrate by parts:

\[
\text{mes}_n B_{R_5}(0) \setminus \omega_{k,\varepsilon} = - \int_{B_{R_5}(0) \setminus \omega_{k,\varepsilon}} \Delta X_{k,\varepsilon} \, dx = - \int_{\partial B_{R_5}(0)} \frac{\partial X_{k,\varepsilon}}{\partial |x|} \, ds.
\]

Using this identity, we make a similar integration by parts:

\[
\int_{B_{R_5}(0) \setminus \omega_{k,\varepsilon}} u \, dx = - \int_{B_{R_5}(0) \setminus \omega_{k,\varepsilon}} u \Delta X_{k,\varepsilon} \, dx
\]

\[
= - \int_{\partial B_{R_5}(0)} u \frac{\partial X_{k,\varepsilon}}{\partial |x|} \, ds + \int_{B_{R_5}(0) \setminus \omega_{k,\varepsilon}} \nabla u \cdot \nabla X_{k,\varepsilon} \, dx
\]

\[
= \langle u \rangle \text{mes}_n B_{R_5}(0) \setminus \omega_{k,\varepsilon} - \int_{\partial B_{R_5}(0)} u_\perp \frac{\partial X_{k,\varepsilon}}{\partial |x|} \, ds \\
+ \int_{B_{R_5}(0) \setminus \omega_{k,\varepsilon}} \nabla u \cdot \nabla X_{k,\varepsilon} \, dx.
\]
Comparing this identity with (3.15), we obtain:

\[
\langle u \rangle = \frac{1}{\text{mes}_n B_R(0) \setminus \omega_{k,e}} \left( \int_{\partial B_R(0)} u \frac{\partial X_{k,e}}{\partial |x|} \, ds - \int_{B_R(0) \setminus \omega_{k,e}} \nabla u \cdot \nabla X_{k,e} \, dx \right)
\]

and by estimates (3.9), (3.13), (3.14) we see that

\[
|\langle u \rangle| \leq C \|\nabla u\|_{L^2(B_R(0) \setminus \omega_{k,e})}.
\]

Therefore, in view of the first identity in (3.12) and estimates (3.13), (3.14),

\[
\|u\|_{L^2(B_R(0) \setminus B_R(0))} \leq C \|\nabla u\|_{L^2(B_R(0) \setminus B_R(0))}.
\]

(3.16)

Now we reproduce the proof of Lemma 3.2 and again introduce the lattice \( \Gamma_{k,e} \) and corresponding covering (3.3). Estimate (3.3) ensures (3.5) for cubes located in \( B_R(0) \setminus B_R(0) \) provided \( \tau_1 \) is chosen sufficiently small and fixed. Then as in the proof of Lemma 3.2 we extend estimate (3.5) to other cubes and for a tubular neighbourhood of \( \partial \omega_{k,e} \) getting finally

\[
\|u\|_{L^2(B_R(0) \setminus \omega_{k,e})} \leq C \|\nabla u\|_{L^2(B_R(0) \setminus B_R(0))}.
\]

This inequality and (3.16) yield (3.11). The proof is complete. \( \square \)

Lemma 3.5 Under Assumption A1 for all \( k \in \mathbb{M}^e \) and all \( u \in W^1_2(B_{\varepsilon \eta R_3(M_k^e) \setminus \omega_k^e}) \) obeying the identity

\[
\int_{B_{\varepsilon \eta R_3(M_k^e) \setminus \omega_k^e}} u(x) \, dx = 0
\]

(3.17)

the estimates

\[
\|u\|_{L^2(B_{\varepsilon \eta R_3(M_k^e) \setminus \omega_k^e})}^2 \leq C \varepsilon^2 \eta^2 \|\nabla u\|_{L^2(B_{\varepsilon \eta R_3(M_k^e) \setminus \omega_k^e})}^2,
\]

(3.18)

\[
\|u\|_{L^2(\partial \omega_k^e)}^2 \leq C \varepsilon \eta \|\nabla u\|_{L^2(B_{\varepsilon \eta R_3(M_k^e) \setminus \omega_k^e})}^2,
\]

(3.19)

hold, where \( C \) is a constant independent of the parameters \( k, \varepsilon, \eta \) and the function \( u \).

Proof Inequality (3.18) is easily obtained by passing to the function \( \tilde{u}(\xi) := u(M_k^e + \varepsilon \eta \xi) \) and applying then estimate (3.11) to this function as depending on the variable \( \xi \in B_R(0) \setminus \omega_{k,e} \). Then we apply inequality (3.7) to the function \( \tilde{u} \) and integrate it over \( \partial \omega_{k,e} \) using the regularity properties of the boundary \( \partial \omega_{k,e} \). This gives:

\[
\|\tilde{u}\|_{L^2(\partial \omega_{k,e})} \leq C \|\tilde{u}\|_{W^1_2(B_R(0) \setminus \omega_{k,e})} \leq C \|\nabla \tilde{u}\|_{L^2(B_R(0) \setminus \omega_{k,e})},
\]

(3.20)
where $C$ is some constant independent of $\varepsilon$, $k$, $\bar{u}$. Returning back to the function $u$, we arrive at (3.19). The proof is complete.

**Lemma 3.6** Under Assumption A1 for all $k \in \mathbb{M}_R$ and all $u \in W^1_2(B_{\varepsilon R_3}(M^\varepsilon_k) \setminus \omega^\varepsilon_k)$ the estimate

$$\| u \|_{L^2(B(\partial \omega_k^\varepsilon))} \leq C \left( \varepsilon \eta \| \nabla u \|_{L^2(B_{\varepsilon R_3}(M^\varepsilon_k) \setminus \omega^\varepsilon_k)} + \varepsilon^{-1} \eta^{-1} \| u \|_{L^2(B_{\varepsilon R_3}(M^\varepsilon_k) \setminus B_{\varepsilon R_2}(M^\varepsilon_k))} \right)$$

holds, where $C$ is a constant independent of the parameters $k$, $\varepsilon$, $\eta$ and the function $u$.

**Proof** In the case $n \geq 3$, this lemma was proved in [2], see Lemma 3.3 in this work. Since here we also deal with the case $n = 2$, we reproduce the proof of this lemma taking into consideration the mentioned case. We fix $k \in \mathbb{M}_R$ and all $u \in W^1_2(B_{\varepsilon R_3}(M^\varepsilon_k) \setminus \omega^\varepsilon_k)$ and let $\chi_3 = \chi_3(t)$ be an infinitely differentiable cut-off function equalling to one as $t \leq R_2$ and vanishing as $t \geq R_5$. Then the function

$$\tilde{u}(\xi) := \chi_3(|x - M^\varepsilon_k| \varepsilon^{-1} \eta^{-1})u(M^\varepsilon_k + \varepsilon \eta \xi)$$

is an element of $\hat{W}^1_2(B_{R_3}(0) \setminus \omega_{k, \varepsilon}, \partial B_{R_3}(0))$. Applying estimate (3.20) and Lemma 3.2, we obtain:

$$\| \tilde{u} \|_{L^2(B_{R_3}(0) \setminus \omega_{k, \varepsilon})} \leq C \| \nabla \tilde{u} \|_{L^2(B_{R_3}(0) \setminus \omega_{k, \varepsilon})};$$

throughout the proof by $C$ we denote various inessential constants independent of $\varepsilon$, $\eta$, $k$, $u$ and $\tilde{u}$. Returning back to the function $u$, we obtain:

$$\| u \|_{L^2(B_{R_3}(0) \setminus \omega_{k, \varepsilon})} \leq C \left( \varepsilon \eta \| \nabla u \|_{L^2(B_{\varepsilon R_3}(0) \setminus \omega^\varepsilon_k)} + \varepsilon^{-1} \eta^{-1} \| u \|_{L^2(B_{\varepsilon R_3}(0) \setminus B_{\varepsilon R_2}(0))} \right).$$

(3.21)

By $\chi_4 = \chi_4(t)$ we denote one more infinitely differentiable cut-off function equalling to one as $t \leq R_5$ and vanishing as $t \geq R_3$. For $x \in B_{\varepsilon \eta R_3}(M^\varepsilon_k)$ we have:

$$u(x) = \frac{\varepsilon^{-M^\varepsilon_k}}{\partial r} u(x) \chi_4(r \varepsilon^{-1}) dr,$$

where $r$ is the radius in the spherical coordinates centered as $M^\varepsilon_k$. By the Cauchy-Schwarz inequality for $x \in B_{\varepsilon \eta R_3}(M^\varepsilon_k)$ we then obtain:

$$\| u(x) \|^2 \leq C \int_{\varepsilon R_3}^{\varepsilon R_3} \frac{dt}{t^{n-1}} \int_{\varepsilon \eta R_5}^{\varepsilon \eta R_5} \| \nabla u \|^2 t^{n-1} dt + C \varepsilon^{-2} \int_{\varepsilon R_5}^{\varepsilon R_5} \frac{dt}{t^{n-1}} \int_{\varepsilon R_5}^{\varepsilon R_5} \| u \|^2 t^{n-1} dt$$

$$\leq C \varepsilon^{-n+2} \eta^{-n+2} \int_{\varepsilon \eta R_5}^{\varepsilon R_3} \| \nabla u \|^2 t^{n-1} dt + C \varepsilon^{-n} \int_{\varepsilon R_5}^{\varepsilon R_5} \| u \|^2 t^{n-1} dt.$$

(3.22)
We integrate the obtained inequality over $B_{\varepsilon \eta R_5} \setminus B_{\varepsilon \eta R_2} (M_k^\varepsilon)$ and we get:

$$\|u\|^2_{L^2(B_{\varepsilon \eta R_5} \setminus B_{\varepsilon \eta R_2} (M_k^\varepsilon))} \leq C \varepsilon^2 \eta^2 \varepsilon \|\nabla u\|^2_{L^2(B_{\varepsilon \eta R_3} (M_k^\varepsilon) \setminus \omega_{\varepsilon k})} + C \eta^n \|u\|^2_{L^2(B_{\varepsilon \eta R_3} (M_k^\varepsilon) \setminus \omega_{\varepsilon k})}.$$  

This estimate and (3.21) give the desired estimate in the statement of the lemma. 

\[ \square \]

### 3.2 Proof of Theorem 2.1

We begin the proof with checking the solvability of problem (2.6) for appropriate values of $\lambda$.

**Lemma 3.7** Under Assumption A1 there exists $\lambda_0 \in \mathbb{R}$ independent of $\varepsilon$ such that for $\text{Re} \, \lambda < \lambda_0$ problem (2.6) is solvable in $\mathring{W}^1_2(\Omega^\varepsilon, \partial \Omega \cup \partial \theta_{\varepsilon D})$ for each $f \in L^2(\Omega^\varepsilon)$.

**Proof** The proof is based on applying standard technique similar to the Browder-Minti theory. Namely, according to the general results in [19, Ch. VI, Sect. 18.4], [8, Ch. 1, Sect. 1.20], the solvability (not necessarily unique!) of problem (2.6) is ensured by the following conditions:

1. For all $u, v, w \in \mathring{W}^1_2(\Omega^\varepsilon, \partial \Omega \cup \partial \theta_{\varepsilon D})$ the scalar function $t \mapsto h_A(u + tv, w)$ is continuous;
2. The convergence holds:

$$\frac{\text{Re} \left( h_A(u, u) - \lambda \|u\|^2_{L^2(\Omega^\varepsilon)} \right)}{\|u\|^2_{W^1_2(\Omega^\varepsilon)}} \to +\infty \quad \text{as} \quad \|u\|^2_{W^1_2(\Omega^\varepsilon)} \to +\infty.$$

Let us check these conditions. It follows from the definition of the form $h_A(u, v)$ that for arbitrary $t_1 < t_2$ we have

$$h_A(u + t_2 v, w) - h_A(u + t_1 v, w) = (t_2 - t_1) h_A(v, w) + \left( a^\varepsilon(\cdot, u + t_2 v) - a^\varepsilon(\cdot, u + t_1 v), w \right)_{L^2(\partial \theta_{\varepsilon D})}. \quad (3.23)$$

Condition (2.4) then yields

$$\left| a^\varepsilon(\cdot, u + t_2 v) - a^\varepsilon(\cdot, u + t_1 v) \right| \leq a^\varepsilon_0 |v|(t_2 - t_1).$$

This estimate and (3.23) imply condition 1.

In order to check condition 2, we first observe that due to our assumptions on the coefficients $A_{ij}^\varepsilon$, $A_j^\varepsilon$, $A_0^\varepsilon$ the estimate

$$\text{Re} \, h_A(u, u) \geq \frac{3c_0}{4} \|\nabla u\|^2_{L^2(\Omega^\varepsilon)} - C \|u\|^2_{L^2(\Omega^\varepsilon)} \quad (3.24)$$
holds, where \( C \) is some absolute constant independent of \( \varepsilon \) and \( u \in W^1_2(\Omega^\varepsilon) \). It also follows from (2.5) and Lemma 3.6 that

\[
\text{Re} \left( a^\varepsilon \left( \cdot, u \right), u \right)_{L^2(\partial \Omega^\varepsilon)} \geq - c_1 \varepsilon \eta^{-n+1} \sum_{k \in \mathcal{M}^x} \| u \|_{L^2(\partial \omega^k)}^2
\]

\[
\geq - C \left( \varepsilon^2 \eta^{-n+2} \varkappa \| \nabla u \|_{L^2(B_{\varepsilon R_3}(M_k^x) \setminus \omega^k)}^2 + \| u \|_{L^2(B_{\varepsilon R_3}(M_k^x) \setminus B_{\varepsilon R_3}(M_k^{x_0}))}^2 \right),
\]

where \( C \) is some fixed constant independent of \( \varepsilon, \eta \) and \( u \in W^1_2(\Omega^\varepsilon) \). According to condition (2.13) we have \( \varepsilon^2 \eta^{-n+2} \varkappa \to 0 \) as \( \varepsilon \to +0 \). Hence, the above estimate and (3.24) yield:

\[
\text{Re} h_A(u, u) \geq \frac{c_0}{2} \| \nabla u \|_{L^2(\Omega^\varepsilon)}^2 - C \| u \|_{L^2(\Omega^\varepsilon)}^2,
\]

where \( C \) is some fixed constant independent of \( \varepsilon, \eta \) and \( u \in W^1_2(\Omega^\varepsilon) \). The obtained inequality implies condition 2 once we choose \( \text{Re} \lambda \leq \lambda_0 < C - 1 \). The proof is complete.

We proceed to proving estimates (2.14), (2.15). We write integral identity (2.8) using \( u_\varepsilon \) as a test function and take then the real part of the obtained relation. The result can be written as

\[
\text{Re} h_A(u_\varepsilon, u_\varepsilon) + \sum_{k \in \mathcal{M}^x \setminus \mathcal{M}^x_{R,0}} \left( a^\varepsilon \left( \cdot, u_\varepsilon \right), u_\varepsilon \right)_{L^2(\partial \omega^k)} + \sum_{k \in \mathcal{M}^x_{R,0}} \left( a^\varepsilon \left( \cdot, u_\varepsilon \right), u_\varepsilon \right)_{L^2(\partial \omega^k)} - \text{Re} \lambda \| u_\varepsilon \|_{L^2(\Omega^\varepsilon)}^2 = \text{Re}(f, u_\varepsilon)_{L^2(\Omega^\varepsilon)},
\]

(3.25)

Then it follows from inequality (3.24), (2.5), (2.10) and Lemma 3.6 with \( \delta = \frac{c_0}{4} \) that for the mentioned choice of the constant \( \text{Re} \lambda < \lambda_0 \) the inequality holds:

\[
\frac{c_0}{4} \| \nabla u_\varepsilon \|_{L^2(\Omega^\varepsilon)}^2 + \| u_\varepsilon \|_{L^2(\Omega^\varepsilon)}^2 + \mu(\varepsilon) \sum_{k \in \mathcal{M}^x_{R,0}} \left( \alpha_k u_\varepsilon, u_\varepsilon \right)_{L^2(\partial \omega^k)} \leq \| f \|_{L^2(\Omega^\varepsilon)} \| u_\varepsilon \|_{L^2(\Omega^\varepsilon)}.
\]

(3.26)

Our next step is to estimate the norm \( \| u_\varepsilon \|_{L^2(\Omega^\varepsilon)} \). The main idea is to cover the domain \( \Omega^\varepsilon \) by balls \( B_{\varepsilon R_3}(M_k^x) \), which is possible owing to Assumption A5, and then to make appropriate local estimates on each such ball intersected with \( \Omega^\varepsilon \).

For \( k \in \mathcal{M}^x_R \) we denote

\[
\langle u_\varepsilon \rangle := \frac{1}{\text{mes}_n B_{\varepsilon \eta R_3}(M_k^x) \setminus \omega^k} \int_{B_{\varepsilon R_3}(M_k^x) \setminus \omega^k} u_\varepsilon \, dx, \quad u^+_\varepsilon := u_\varepsilon - \langle u_\varepsilon \rangle.
\]

(3.27)
It is clear that the function \( u_{\varepsilon, k} \) satisfies condition (3.17) and hence, by Lemmas 3.5, 3.6,

\[
\| u_{\varepsilon} \|^2_{L^2(B_{\varepsilon\eta R_3}(M_k^\varepsilon) \setminus \omega_k, \varepsilon)} \leq C \varepsilon^2 \eta^2 \| \nabla u_{\varepsilon} \|^2_{L^2(B_{\varepsilon\eta R_3}(M_k^\varepsilon) \setminus \omega_k, \varepsilon)},
\]

(3.28)

\[
\| u_{\varepsilon} \|^2_{L^2(\partial \omega_k, \varepsilon)} \leq C \varepsilon \eta \| \nabla u_{\varepsilon} \|^2_{L^2(B_{\varepsilon\eta R_3}(M_k^\varepsilon) \setminus \omega_k, \varepsilon)},
\]

(3.29)

hereinafter in the proof by \( C \) we denote various inessential constants independent of \( \varepsilon, \eta, k, u_{\varepsilon} \) and spatial variables.

Employing assumptions (2.11) and Cauchy-Schwarz inequality, we obtain:

\[
\int_{\partial \omega_k} \alpha_k^\varepsilon |u_{\varepsilon}|^2 ds = |\langle u_{\varepsilon} \rangle|^2 \int_{\partial \omega_k} \alpha_k^\varepsilon ds + 2 \text{Re} \langle u_{\varepsilon} \rangle \int_{\partial \omega_k} \alpha_k^\varepsilon u_{\varepsilon}^\perp ds + \int_{\partial \omega_k} |\alpha_k^\varepsilon|^2 u_{\varepsilon}^\perp ds
\]

\[
\geq c_3 (\varepsilon \eta)^{n-1} |\langle u_{\varepsilon} \rangle|^2 - 2 |\langle u_{\varepsilon} \rangle| \int_{\partial \omega_k} |\alpha_k^\varepsilon| u_{\varepsilon}^\perp | ds
\]

\[
\geq \frac{c_3}{2} (\varepsilon \eta)^{n-1} |\langle u_{\varepsilon} \rangle|^2 - \frac{2}{c_3} (\varepsilon \eta)^{-n+1} \left( \int_{\partial \omega_k} |\alpha_k^\varepsilon| u_{\varepsilon}^\perp | ds \right)^2
\]

\[
\geq \frac{c_3}{2} (\varepsilon \eta)^{n-1} |\langle u_{\varepsilon} \rangle|^2 - \frac{2c_2}{c_3} \| u_{\varepsilon} \|^2_{L^2(\partial \omega_k, \varepsilon)},
\]

and therefore,

\[
\varepsilon^n \eta^2 |\langle u_{\varepsilon} \rangle|^2 \leq C \varepsilon \eta \left( \int_{\partial \omega_k} |\alpha_k^\varepsilon| u_{\varepsilon}^\perp | ds + \| u_{\varepsilon} \|^2_{L^2(\partial \omega_k, \varepsilon)} \right).
\]

This estimate and (3.29), (3.28) yield:

\[
\| u_{\varepsilon} \|^2_{L^2(B_{\varepsilon\eta R_3}(M_k^\varepsilon) \setminus \omega_k, \varepsilon)} \leq 2 |\langle u_{\varepsilon} \rangle|^2 \text{mes}_\eta (B_{\varepsilon\eta R_3}(M_k^\varepsilon) \setminus \omega_k, \varepsilon) + 2 \| u_{\varepsilon} \|^2_{L^2(B_{\varepsilon\eta R_3}(M_k^\varepsilon) \setminus \omega_k, \varepsilon)}
\]

\[
\leq C \left( \varepsilon \eta \int_{\partial \omega_k} |\alpha_k^\varepsilon| u_{\varepsilon}^\perp | ds + \varepsilon^2 \eta^2 \| \nabla u \|^2_{L^2(B_{\varepsilon\eta R_3}(M_k^\varepsilon) \setminus \omega_k, \varepsilon)} \right).
\]

(3.30)

Let \( \chi_5 = \chi_5(t) \) be an infinitely differentiable cut-off function equalling to one as \( t > R_3 \) and vanishing as \( t < R_2 \). For \( x \in B_{\varepsilon R_3}(M_k^\varepsilon) \setminus B_{\varepsilon\eta R_3}(M_k^\varepsilon) \) we have an obvious formula:

\[
u(x) = \int_{\varepsilon R_2} \frac{\partial}{\partial r} u \chi_5(r \varepsilon^{-1} \eta^{-1}) dr,
\]
where \( r \) is the radius in the spherical coordinates centered at \( M^\varepsilon_k \). Proceeding then as in (3.22), we easily obtain:

\[
|u(x)|^2 \leq C \varepsilon^{-n+2} \eta^{-n+2} \int_{\varepsilon \eta R_2} \int_{\varepsilon \eta R_3} |\nabla u|^2 r^{n-1} dr + \varepsilon^{-n} \eta^{n-1} \int_{\varepsilon \eta R_2} |u|^2 r^{n-1} dr.
\]

Integrating this estimate over \( B_{\varepsilon R_3}(M^\varepsilon_k) \setminus B_{\varepsilon \eta R_3}(M^\varepsilon_k) \) and using (3.30), we get

\[
\|u^\varepsilon\|^2_{L^2(B_{\varepsilon R_3}(M^\varepsilon_k) \setminus \omega_k, \varepsilon)} \leq C \left( \varepsilon^2 \eta^{-n+2} \kappa \|\nabla u\|^2_{L^2(B_{\varepsilon R_3}(M^\varepsilon_k) \setminus \omega_k, \varepsilon)} + \eta^{-n} \|\nabla u\|^2_{L^2(B_{\varepsilon \eta R_3}(M^\varepsilon_k) \setminus B_{\varepsilon \eta R_2}(M^\varepsilon_k))} \right)
\leq C \left( \varepsilon \eta^{-n+1} \int \alpha_{\varepsilon R}^k |u^\varepsilon|^2 ds + \varepsilon^2 \eta^{-n+2} \kappa \|\nabla u\|^2_{L^2(B_{\varepsilon R_3}(M^\varepsilon_k) \setminus \omega_k^\varepsilon)} \right), \quad k \in \mathbb{M}^\varepsilon_{R, 0}.
\]

By Lemma 3.1 we have a similar estimate:

\[
\|u^\varepsilon\|^2_{L^2(B_{\varepsilon R_3}(M^\varepsilon_k) \setminus \omega_k, \varepsilon)} \leq C \varepsilon^2 \eta^{-n+2} \kappa \|\nabla u\|^2_{L^2(B_{\varepsilon R_3}(M^\varepsilon_k) \setminus \omega_k^\varepsilon)}, \quad k \in \mathbb{M}^\varepsilon_{D}.
\]

Covering (2.12) yields that

\[
\|u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} \leq \sum_{k \in \mathbb{M}^\varepsilon_{R, 0} \cup \mathbb{M}^\varepsilon_{D}} \|u^\varepsilon\|^2_{L^2(B_{\varepsilon R_4}(M^\varepsilon_k) \setminus \theta^\varepsilon)}, \quad \text{ for } \theta^\varepsilon
\]

The sets \( B_{\varepsilon R_4}(M^\varepsilon_k) \setminus \theta^\varepsilon \) not necessarily coincide with \( B_{\varepsilon R_4}(M^\varepsilon_k) \setminus \omega_k^\varepsilon \) but can have a more complicated shape. The reason is that for a given \( k \), the ball \( B_{\varepsilon R_4}(M^\varepsilon_k) \) can have a non-zero intersection with other cavities \( \omega_j^\varepsilon \), \( j \neq k \), see Fig. 3. To overcome technical difficulties in local estimates related to possible shapes of the domains \( B_{\varepsilon R_4}(M^\varepsilon_k) \setminus \omega_k^\varepsilon \), we make an auxiliary continuation of the function \( u^\varepsilon \). For \( k \in \mathbb{M}^\varepsilon_D \) we simply continue the function \( u^\varepsilon \) by zero inside the cavities \( \omega_k^\varepsilon \):

\[
u^\varepsilon(x) := 0 \quad \text{in } \omega_k^\varepsilon, \quad k \in \mathbb{M}^\varepsilon_D.
\]

For \( k \in \mathbb{M}^\varepsilon_R \) the aforementioned continuation is made as follows:

\[
u^\varepsilon(\tau, s) := \langle u^\varepsilon \rangle + u^\perp_{\varepsilon} (-\tau, s) \chi_2(\tau \varepsilon^{-1} \eta^{-1}) \quad \text{for } x \in \omega_k^\varepsilon, \quad \text{dist}(x, \partial \omega_k^\varepsilon) \leq \varepsilon \eta \tau_0,
\]

\[
u^\varepsilon(\tau, s) := \langle u^\varepsilon \rangle \quad \text{for } x \in \omega_k^\varepsilon, \quad \text{dist}(x, \partial \omega_k^\varepsilon) > \varepsilon \eta \tau_0,
\]

with \( \langle u^\varepsilon \rangle \) and \( u^\perp_{\varepsilon} \) defined in (3.27) for the chosen \( k \) and \( \chi_2 \) is the cut-off function introduced in the proof of Lemma 3.2. In view of (3.27), the resulting continued
function belongs to $W^1_2(B_{eR_3}(M^e_k))$ for each $k \in \mathbb{M}^e$ and due to estimate (3.18) we have:

$$\|u_{e}\|_{L^2(\omega_{k}^e)}^2 \leq C \left( (\|u_{e}\|_{L^2(B_{eR_2}(M_k^e) \setminus \omega_{k}^e)}^2 + \|u_{\perp} \|_{L^2(B_{eR_2}(M_k^e) \setminus \omega_{k}^e)}^2) \right) \leq C \|u_{e}\|_{L^2(B_{eR_2}(M_k^e) \setminus \omega_{k}^e)}^2,$$

$$\|\nabla u_{e}\|_{L^2(\omega_{k}^e)}^2 \leq C \left( \varepsilon^{-2} \eta^{-2} \|u_{e}\|_{L^2(B_{eR_2}(M_k^e) \setminus \omega_{k}^e)}^2 \right) \leq C \|\nabla u_{e}\|_{L^2(B_{eR_2}(M_k^e) \setminus \omega_{k}^e)}^2,$$

$$\leq C \|\nabla u_{e}\|_{L^2(B_{eR_2}(M_k^e) \setminus \omega_{k}^e)}^2.$$

(3.34)

We also continue $u_{e}$ it by zero outside $\partial \Omega$. The continued function $u_{e}$ is defined on entire $\Omega$ and in a fixed neighbourhood of $\Omega$ and in view of the above estimates it satisfies

$$\|u_{e}\|_{L^2(\Omega)} \leq C \|u_{e}\|_{L^2(\Omega^c)}, \quad \|\nabla u_{e}\|_{L^2(\Omega)} \leq C \|\nabla u_{e}\|_{L^2(\Omega^c)}.$$

Hence, we can rewrite estimate (3.33) as

$$\|u_{e}\|_{L^2(\Omega^c)}^2 \leq \sum_{k \in \mathbb{M}^e_D \cup \mathbb{M}^e_{R,0}} \|u_{e}\|_{L^2(B_{eR_4}(M_k^e))}^2,$$

(3.35)

where for the balls $B_{eR_4}(M_k^e)$ intersecting the boundary of $\partial \Omega$ in the corresponding norms we mean the above made continuation of $u_{e}$ by zero outside $\Omega$.

For each $k \in \mathbb{M}^e_D \cup \mathbb{M}^e_{R,0}$ the function $u_{e}(x) \chi_5(|x - M_k^e|^{-1})$ belongs to $W^1_2(B_{eR_4}(M_k^e) \setminus \omega_{k}^e)$, vanishes on $B_{eR_2}(M_k^e)$ and coincides with $u_{e}$ in the $B_{eR_4}(M_k^e) \setminus B_{eR_5}(M_k^e)$. Hence, by Lemma 3.1 with $R_3$ replaced by $R_4$, $\eta = 1$ and $\omega_{k,\varepsilon}$ replaced by $B_{R_2}(0)$, for each $k \in \mathbb{M}^e_D \cup \mathbb{M}^e_{R,0}$ we have:

$$\|u_{e}\|_{L^2(B_{eR_4}(M_k^e) \setminus B_{eR_5}(M_k^e))}^2 \leq \|u_{e}\|_{L^2(B_{eR_4}(M_k^e) \setminus B_{eR_5}(M_k^e))}^2 \leq C \varepsilon^{-2} \|\nabla u_{e}\|_{L^2(B_{eR_4}(M_k^e) \setminus B_{eR_5}(M_k^e))}^2 \leq C \left( \varepsilon^{-2} \|\nabla u_{e}\|_{L^2(B_{eR_4}(M_k^e) \setminus B_{eR_5}(M_k^e))}^2 \right).$$

Hence, by (3.31), (3.32), (3.34),

$$\|u_{e}\|_{L^2(B_{eR_4}(M_k^e))}^2 \leq C \left( \varepsilon^{-n-1} \int_{\partial \omega_k^e} \alpha_{k}^e |u_{e}|^2 \, ds \right) \leq C \left( \varepsilon^{-n-2} \|\nabla u_{e}\|_{L^2(B_{eR_5}(M_k^e) \setminus \omega_{k}^e)}^2 \right), \quad k \in \mathbb{M}^e_{R,0},$$

$$\|u_{e}\|_{L^2(B_{eR_4}(M_k^e))}^2 \leq C \varepsilon^{-n-2} \|\nabla u_{e}\|_{L^2(B_{eR_5}(M_k^e) \setminus \omega_{k}^e)}^2, \quad k \in \mathbb{M}^e_D.$$
We sum up the obtained estimate over \( k \in \mathbb{M}_{R,0}^e \cup \mathbb{M}_D^e \) and by (3.35) we find:

\[
\| u_\varepsilon \|_{L^2(\Omega^e)}^2 \leq \sum_{k \in \mathbb{M}_{R,0}^e \cup \mathbb{M}_D^e} \| u_\varepsilon \|_{L^2(B_{\varepsilon R_k}(M_k^e))}^2 \leq C \varepsilon \eta^{-n+1} \sum_{k \in \mathbb{M}_{R,0}^e} \int \alpha_k^e |u_\varepsilon|^2 \, ds + C \varepsilon^2 \eta^{-n+2} \kappa \sum_{k \in \mathbb{M}_D^e \cup \mathbb{M}_{R,0}^e} \| \nabla u_\varepsilon \|_{L^2(B_{\varepsilon R_k}(M_k^e) \setminus \partial \Omega_\varepsilon^e)}^2.
\]

(3.36)

Given a point \( x \in \Omega^e \), let us estimate the number of balls \( B_{\varepsilon R_k}(M_k^e) \), \( k \in \mathbb{M}_{R,0}^e \cup \mathbb{M}_D^e \), containing this point; we denote this number by \( N_\varepsilon(x) \). It is clear that \( N_\varepsilon(x) \) is equal to the number of points \( M_k^e \), \( k \in \mathbb{M}_{R,0}^e \cup \mathbb{M}_D^e \), such that dist\((x, M_k^e) < \varepsilon R_k^e\). By inequality (2.9) in Assumption A1, the mutual distances between the points \( M_k^e \) is at least \( 2R_3^e \). Associating each point \( M_k^e \) with a \( n \)-dimensional cube having a side \( 2R_3^e \), we then conclude that \( N_\varepsilon(x) \) does not exceed the number of such cubes intersecting the ball \( B_{\varepsilon R_k}(x) \). All such cubes are located inside the ball of the bigger radius \( \varepsilon(R_4 + 2\sqrt{n}R_3^e) \) centered at \( x \). Comparing then their volumes, we get:

\[
N_\varepsilon(x) \leq \frac{\operatorname{mes} B_{\varepsilon(R_4 + 2\sqrt{n}R_3^e)}(0)}{(2\varepsilon R_3^e)^n} \leq \left( \frac{R_4}{2R_3^e} + \sqrt{n} \right)^n \operatorname{mes} B_1(0),
\]

and hence, the number \( N_\varepsilon(x) \) is bounded uniformly in \( \varepsilon \) and \( x \in \Omega^e \). This fact allows us to continue estimating in (3.36):

\[
\| u_\varepsilon \|_{L^2(\Omega^e)}^2 \leq C \left( \varepsilon \eta^{-n+1} \mu^{-1} + \varepsilon^2 \eta^{-n+2} \kappa \right) \left( \mu \sum_{k \in \mathbb{M}_{R,0}^e} (\alpha_k^e u_\varepsilon, u_\varepsilon)_{L^2(\partial \Omega_\varepsilon^e)} + \| \nabla u_\varepsilon \|_{L^2(\Omega^e)}^2 \right).
\]

(3.37)

Then the right hand of (2.26) obeys the inequality

\[
\| f \|_{L^2(\Omega^e)} \| u_\varepsilon \|_{L^2(\Omega^e)} \leq C \left( \varepsilon \eta^{-n+1} \mu^{-1} + \varepsilon^2 \eta^{-n+2} \kappa \right) \| f \|_{L^2(\Omega^e)}^2 + \frac{\mu}{2} \sum_{k \in \mathbb{M}_{R,0}^e} (\alpha_k^e u_\varepsilon, u_\varepsilon)_{L^2(\partial \Omega_\varepsilon^e)} + \frac{c_0}{8} \| \nabla u_\varepsilon \|_{L^2(\Omega^e)}^2.
\]

Substituting this estimate into (2.26), we finally get:

\[
\| u \|_{W^1_2(\Omega^e)}^2 + \mu(\varepsilon) \sum_{k \in \mathbb{M}_{R,0}^e} (\alpha_k^e u_\varepsilon, u_\varepsilon)_{L^2(\partial \Omega_\varepsilon^e)} \leq C \left( \varepsilon \eta^{-n+1} \mu^{-1} + \varepsilon^2 \eta^{-n+2} \kappa \right) \| f \|_{L^2(\Omega^e)}^2.
\]
Since $\alpha_k^\varepsilon$ are non-negative, the obtained estimate implies immediately (2.14) as well as

$$
\mu \sum_{k \in \mathbb{M}_{R,0}} (\alpha_k^\varepsilon u_\varepsilon, u_\varepsilon)_{L^2(\partial \Omega_1^\varepsilon)} \leq C (\varepsilon \eta^{-n+1} \mu^{-1} + \varepsilon^2 \eta^{-n+2} \varepsilon) \| f \|^2_{L^2(\Omega^\varepsilon)}.
$$

This estimate and (2.14), (3.37) then yield (2.15).

We proceed to proving that estimates (2.14), (2.15) are order sharp. A main idea is to construct particular examples of problem (2.6) such that the norms of its solutions have exactly the smallness order stated in estimates (2.14), (2.15). In all our examples we let $\Omega := \mathbb{R}^n$, $\mathcal{L} = -\Delta$, $a^\varepsilon(x, u) := u$. The perforation is assumed to be pure periodic, namely, the points $M^\varepsilon_k$ are chosen as $M^\varepsilon_k := 4\varepsilon k$, $k \in \mathbb{Z}^n$, $k = (k_1, \ldots, k_n)$. The domains $\omega^\varepsilon_k$ are supposed to be fixed and independent of $k$ and $\varepsilon$, and we simply let $\omega^\varepsilon_k := B_1(0)$. The cavities then are of the form $\omega^\varepsilon_k := \{ x : |x - 4\varepsilon k| < \varepsilon n\}$, $k \in \mathbb{Z}^n$. The Dirichlet condition is imposed on the boundaries of the cavities $\partial \omega^\varepsilon_k$ as $k_n \geq 1$, while the Robin condition is settled on the boundaries $\partial \omega^\varepsilon_k$ for $k_n \leq 0$. It is clear that under the made assumptions, problem (2.6) is uniquely solvable for all $f \in L^2(\Omega^\varepsilon)$ and $\lambda = 0$.

Suppose first that $\eta$ is fixed and is independent of $\varepsilon$. We denote $\Box := \{ \xi \in \mathbb{R}^n : |\xi_i| < 2, i = 1, \ldots, n\}$, where $\xi = (\xi_1, \ldots, \xi_n)$ are Cartesian coordinates in $\mathbb{R}^n$, and consider a boundary value problem

$$
-\Delta_\xi v_0 = 1 \quad \text{in} \quad \Box \setminus B_\eta(0), \quad v_0 = 0 \quad \text{on} \quad \partial B_\eta(0),
$$

$$
v_0 \bigg|_{\xi_i = -2} = v_0 \bigg|_{\xi_i = 2}, \quad \frac{\partial v_0}{\partial \xi_i} \bigg|_{\xi_i = -2} = \frac{\partial v_0}{\partial \xi_i} \bigg|_{\xi_i = 2}. \quad (3.38)
$$

Owing to the Dirichlet condition on $\partial B_\eta(0)$, this problem is obviously uniquely solvable and due to the standard smoothness improving theorems, the solution is infinitely differentiable in $\Box \setminus B_\eta(0)$. Hereinafter all solutions to various boundary value problems in $\Box \setminus B_\eta(0)$ are supposed to be $\Box$-periodically continued with the same notations for their continuations.

Let $f_D = f_D(x)$ be a non-zero infinitely differentiable real function with a compact support located in $\{ x : x_n > 5\}$. We introduce one more function:

$$
u_D^\varepsilon(x) := \varepsilon^2 v_0(x\varepsilon^{-1}) f_D(x).
$$

In view of the properties of the functions $v_0$ and $f_D$, the function $u_D^\varepsilon$ is infinitely differentiable, vanishes outside the support of $f_D$ and solves boundary value problem (2.6) with

$$
f = f_D - h_D^\varepsilon, \quad h_D^\varepsilon(x) := 2\varepsilon \nabla_\xi v_0(x\varepsilon^{-1}) \cdot \nabla f_D(x) + \varepsilon^2 v_0(x\varepsilon^{-1}) \Delta f_D.
$$

Our next step is to calculate $L^2$- and $W^1_2$-norms for the functions $u_D^\varepsilon$ and $h_D^\varepsilon$. In order to do it, we employ the following auxiliary lemma.
Lemma 3.8 Let \( v = v(\xi) \) be a \( \Box \)-periodic function belonging to \( L_1(\Box \setminus B_\eta(0)) \) and \( h = h(x) \) be a continuously differentiable compactly supported function defined on \( \mathbb{R}^n \). The identity holds:

\[
\int_{\Omega_1^\varepsilon} v(x \varepsilon^{-1}) h(x) \, dx = \frac{1}{4^n} \int_{\Box \setminus B_\eta(0)} v(\xi) \, d\xi \int_{\mathbb{R}^d} h(x) \, dx + \varepsilon V_\varepsilon,
\]

\[
|V_\varepsilon| \leq C (\text{mes supp } h + 1) \| v \|_{L_1(\Box \setminus B_\eta(0))} \max_{\mathbb{R}^d} |\nabla h|,
\]

where \( C \) is some constant independent of \( \varepsilon, \eta, h, \supp h \) and \( v \).

**Proof** We fix an arbitrary \( k \in 4 \mathbb{Z}^n \) and for \( x \in \varepsilon \Box + \varepsilon p \) by the Hadamard lemma we have:

\[
h(x) = h(\varepsilon k) + h_k(x),
\]

\[
h_k(x) = \sum_{i=1}^n (x_i - \varepsilon k_i) \int_0^1 \frac{\partial h}{\partial x_i}(\varepsilon k + t(x - \varepsilon k)) \, dt,
\]

\[
|x_i - \varepsilon k_i| \leq 2\varepsilon.
\]

Then by the Cauchy-Schwarz inequality we obtain:

\[
|h_k(x)| \leq 2\sqrt{n} \varepsilon \max_{\mathbb{R}^d} |\nabla h|, \quad x \in \varepsilon k + \varepsilon \Box.
\]

Hence,

\[
h(x) = h(\varepsilon k) + h_k(x) = \frac{1}{4^n \varepsilon^n} \int_{\varepsilon k + \varepsilon \Box} (h(x) - h_k(x)) \, dx + h_k(x)
\]

\[
= \frac{1}{4^n \varepsilon^n} \int_{\varepsilon k + \varepsilon \Box} h(x) \, dx + h^{(k)}(x),
\]

\[
h^{(k)}(x) := h(x) - \frac{1}{4^n \varepsilon^n} \int_{\varepsilon k + \varepsilon \Box} h(x) \, dx, \quad |h^{(k)}(x)| \leq 4\sqrt{n} \varepsilon \max_{\mathbb{R}^d} |\nabla h|. \quad (3.39)
\]
Employing the $\Box$-periodicity of the function $v$, we represent the integral in question as

$$
\int_{\Omega} v(x\varepsilon^{-1}) h(x) \, dx = \sum_{k \in 4\mathbb{Z}^n} \int_{\varrho k + \varepsilon \Box \cap \text{supp } h \neq \emptyset} v(x\varepsilon^{-1}) h(x) \, dx
$$

$$
= \sum_{k \in 4\mathbb{Z}^n} \frac{1}{4^n} \int_{\varepsilon \Box + k} h(x) \, dx \int_{\varepsilon (\Box \setminus B_\eta(0))} v(x\varepsilon^{-1}) \, dx
$$

$$
+ \sum_{p \in 4\mathbb{Z}^n} \int_{\varrho k + \varepsilon \Box \cap \text{supp } h \neq \emptyset} v(x\varepsilon^{-1}) h^{(k)}(x) \, dx.
$$

(3.40)

We estimate the second term in the right hand side of the obtained identity by means of inequality (3.39) and we arrive at the statement of the lemma.

We employ this lemma to calculate the following norms of the functions $u_D^\varepsilon$ and $f_D - h_D^\varepsilon$:

$$
\|u_D^\varepsilon\|_{L_2(\Omega')} = \frac{\varepsilon^4}{4^n} \|v_0\|_{L_2(\Box \setminus B_\eta(0))}^2 \|f_D\|_{L_2(\mathbb{R}^d)}^2 + O(\varepsilon^5),
$$

$$
\|\nabla u_D^\varepsilon\|_{L_2(\Omega')} = \frac{\varepsilon^2}{4^n} \|\nabla \xi v_0\|_{L_2(\Box \setminus B_\eta(0))}^2 \|f_D\|_{L_2(\mathbb{R}^d)}^2 + O(\varepsilon^3),
$$

$$
\|f_D - h_D^\varepsilon\|_{L_2(\Omega')} = \frac{1}{4^n} \|f_D\|_{L_2(\mathbb{R}^d)}^2 + O(\varepsilon^2).
$$

Hence,

$$
\frac{\|u_D^\varepsilon\|_{L_2(\Omega')}}{\|f_D - h_D^\varepsilon\|_{L_2(\Omega')}} = \varepsilon^2 \|v_0\|_{L_2(\Box \setminus B_\eta(0))} + O(\varepsilon^3),
$$

$$
\|\nabla u_D^\varepsilon\|_{L_2(\Omega')} = \varepsilon \|\nabla \xi v_0\|_{L_2(\Box \setminus B_\eta(0))} + O(\varepsilon^2).
$$

(3.41)

These identities show that as $\eta = \eta_0$, the terms $\varepsilon \eta^{-\frac{d}{2} + 1}(\varepsilon) \kappa(\varepsilon)$ and $\varepsilon^2 \eta^{-n+2}(\varepsilon) \kappa(\varepsilon)$ in (2.14), (2.15) are order sharp.

Let us prove that the other terms in (2.14), (2.15) are also order sharp as $\eta = \eta_0$.

We first of all observe that in this case

$$
\varepsilon \eta^{-n+1} \mu^{-1} = \varepsilon^2 \eta^{-n+1}(\varepsilon \mu)^{-1}
$$
\[
\begin{cases}
ono \text{as } \varepsilon \mu \to +\infty, \varepsilon \to +0, & O(\varepsilon^2) \quad \\
o \text{as } 0 < C_1 \leq \varepsilon \mu \leq C_2 \neq 0, \varepsilon \to +0, & C_1, C_2 = \text{const},
\end{cases}
\]

and this is why we need to prove the sharpness of the other terms in (2.14), (2.15) only in the case \( \varepsilon \mu \to +0 \) as \( \varepsilon \to +0 \).

We consider a boundary value problem

\[
-\Delta_\xi v_\mu = 1 \quad \text{in } \Box \setminus B_\eta(0), \quad \frac{\partial v_\mu}{\partial |\xi|} = \varepsilon \mu v_\mu \quad \text{on } \partial B_1(0),
\]

subject to the periodic boundary conditions as in (3.38). This problem is also uniquely solvable. Let us study the behavior of the function \( u_\mu \) as \( \varepsilon \mu \to +0 \). By \( v_1 \) we denote the solution to the problem

\[
-\Delta_\xi v_1 = 1 \quad \text{in } \Box \setminus B_\eta(0), \quad \frac{\partial v_1}{\partial |\xi|} = c_4 \quad \text{on } \partial B_1(0), \quad c_4 := \frac{\text{mes } \Box \setminus B_\eta(0)}{\text{mes } \partial B_\eta(0)},
\]

subject to the periodic boundary conditions as in (3.38). This problem is solvable and there is a unique solution satisfying the identity

\[
\int_{\partial B_\eta(0)} v_1 \, d\xi = 0.
\]

This solution is infinitely differentiable in \( \Box \setminus B_\eta(0) \). Using this solution, for each function \( u \in C^1(\Box \setminus B_\eta(0)) \) satisfying the periodic boundary conditions on the lateral boundaries of \( \Box \) we get the identity

\[
\int_{\Box \setminus B_\eta(0)} |u|^2 \, d\xi = \int_{\Box \setminus B_\eta(0)} |u|^2 \Delta_\xi v_1 \, d\xi = c_4 \int_{\partial B_\eta(0)} |u|^2 \, ds + \int_{\Box \setminus B_\eta(0)} \nabla_\xi v_1 \cdot \nabla_\xi |u|^2 \, d\xi
\]

and the estimate

\[
\|u\|^2_{L^2(\Box \setminus B_\eta(0))} \leq C \left( \|u\|^2_{L^2(\partial B_\eta(0))} + \|\nabla_\xi u\|^2_{L^2(\Box \setminus B_\eta(0))} \right),
\]

where \( C \) is some constant independent of \( u \). Since the space \( C^1(\Box \setminus B_\eta(0)) \) is dense in \( W^1_2(\Box \setminus B_\eta(0)) \), estimate (3.44) is also true for all \( u \in W^1_2(\Box \setminus B_\eta(0)) \).

We consider one more boundary value problem

\[
-\Delta_\xi v_2 = 0 \quad \text{in } \Box \setminus B_\eta(0), \quad \frac{\partial v_2}{\partial |\xi|} = v_1 \quad \text{on } \partial B_1(0),
\]

subject to the periodic boundary conditions as in (3.38). Thanks to condition (3.43), this problem is solvable and possesses a unique solution also obeying condition (3.43)
and being infinitely differentiable in $\Box \setminus B_\eta(0)$. We denote

$$\hat{v}_\mu := v_\mu - \frac{c_4}{\varepsilon \mu} - v_1 - \varepsilon \mu v_2.$$  

It is easy to see that the introduced function solves the boundary value problem

$$-\Delta_\xi \hat{v}_\mu = 0 \text{ in } \Box \setminus B_\eta(0), \quad \frac{\partial \hat{v}_\mu}{\partial |\xi|} = \varepsilon \mu v_\mu - \varepsilon^2 \mu^2 v_2 \text{ on } \partial B_1(0),$$

subject to the periodic boundary conditions as in (3.38). The integral identity corresponding to this problem with $\hat{v}_\mu$ as the test function reads:

$$\|\nabla \hat{v}_\mu\|_{L^2(\Box \setminus B_\eta(0))}^2 + \varepsilon \mu \|\hat{v}_\mu\|_{L^2(\Box \setminus B_\eta(0))}^2 = \varepsilon^2 \mu^2 (v_2, \hat{v}_\mu)_{L^2(B_\eta(0))}.$$  

Hence,

$$\|\nabla \hat{v}_\mu\|_{L^2(\Box \setminus B_\eta(0))}^2 + \varepsilon \mu \|\hat{v}_\mu\|_{L^2(\Box \setminus B_\eta(0))}^2 \leq \varepsilon^2 \mu^2 \|v_2\|_{L^2(B_\eta(0))} \|\hat{v}_\mu\|_{L^2(B_\eta(0))}.$$  

Then by inequality (3.44) we get:

$$\|\hat{v}_\mu\|_{L^2(\partial B_\eta(0))} \leq \varepsilon \mu \|v_2\|_{L^2(B_\eta(0))},$$  

$$\|\nabla \hat{v}_\mu\|_{L^2(\Box \setminus B_\eta(0))} \leq \varepsilon^2 \mu \|v_2\|_{L^2(B_\eta(0))},$$  

$$\|\hat{v}_\mu\|_{L^2(\Box \setminus B_\eta(0))} \leq C \varepsilon \mu \|v_2\|_{L^2(B_\eta(0))},$$

where $C$ is a constant independent of $\varepsilon$, $\mu$, $v_2$ and $\hat{v}_\mu$. These inequalities yield the following asymptotic representation for $v_\mu$:

$$v_\mu = \frac{c_4}{\varepsilon \mu} + v_1 + O(\varepsilon \mu), \quad \varepsilon \to +0,$$  

(3.45)  

in $W^1_2(\Box \setminus B_\eta(0))$-norm as $\varepsilon \mu \to +0$.

Let $f_R = f_R(x)$ be a non-zero infinitely differentiable real function with a compact support located in $\{x : x_n < -5\}$. We let:

$$u^\varepsilon_R(x) := \varepsilon^2 v_\mu(x \varepsilon^{-1}) f_R(x).$$

This function is infinitely differentiable, vanishes outside the support of $f_R$ and solves problem (2.6) in the considered particular case with the right hand side

$$f = f_R - h^\varepsilon_R, \quad h^\varepsilon_R(x) := 2\varepsilon \nabla_\xi v_\mu(x \varepsilon^{-1}) \cdot \nabla f_R(x) + \varepsilon^2 v_\mu(x \varepsilon^{-1}) \Delta f_R.$$
By Lemma 3.8 and representation (3.45) we get:

\[
\begin{align*}
\|u_R\|_{L^2(\Omega^\varepsilon)} &= c_4 \varepsilon \mu^{-1} \|f_R\|_{L^2(\mathbb{R}^n)} + O(\varepsilon^3), \\
\|f_R - h_R^\varepsilon\|_{L^2(\Omega^\varepsilon)} &= \|f_R\|_{L^2(\mathbb{R}^n)} + O(\varepsilon \mu^{-1}), \\
\|\nabla u_R\|_{L^2(\Omega^\varepsilon)} &= \|c_4 \varepsilon \mu^{-1} \nabla f_R + \varepsilon \nabla_{\xi} v_1 (\cdot \varepsilon^{-1}) f_R\|_{L^2(\Omega^\varepsilon)} + O(\varepsilon^2 \mu) \\
\end{align*}
\]

(3.46)
as \varepsilon \mu \to +0. It is easy to see that

\[
\begin{align*}
\|c_4 \varepsilon \mu^{-1} \nabla f_R + \varepsilon \nabla_{\xi} v_1 (\cdot \varepsilon^{-1}) f_R\|_{L^2(\Omega^\varepsilon)}^2 &= \varepsilon^2 \mu^{-1} \|c_4 \nabla f_R\|_{L^2(\Omega^\varepsilon)}^2 \\
&+ \varepsilon^2 \|\nabla_{\xi} v_1 (\cdot \varepsilon^{-1}) f_R\|_{L^2(\Omega^\varepsilon)}^2 \\
&+ \varepsilon^2 \mu^{-1} c_4 \int_{\Omega^\varepsilon} \nabla f_R^2 \cdot \nabla_{\xi} v_1 (\cdot \varepsilon^{-1}) \, dx \\
\end{align*}
\]

(3.47)

and in view of boundary value problem (3.42) we can integrate by parts as follows:

\[
\begin{align*}
\varepsilon^2 \mu^{-1} \int_{\Omega^\varepsilon} \nabla f_R^2 \cdot \nabla_{\xi} v_1 (\cdot \varepsilon^{-1}) \, dx \\
= - \varepsilon^2 \mu^{-1} \int_{\partial \Omega^\varepsilon} f_R^2 v_1 (\cdot \varepsilon^{-1}) \, ds - \varepsilon \mu^{-1} \int_{\Omega^\varepsilon} f_R^2 \Delta_{\xi} v_1 (\cdot \varepsilon^{-1}) \, dx \\
= \varepsilon \mu^{-1} \|f_R\|_{L^2(\Omega^\varepsilon)}^2 - \varepsilon^2 \mu^{-1} \int_{\partial \Omega^\varepsilon} f_R^2 v_1 (\cdot \varepsilon^{-1}) \, ds. \\
\end{align*}
\]

(3.48)

Employing (3.39) with \(h = f_R^2\) and proceeding as in (3.40), we find:

\[
\begin{align*}
\varepsilon^2 \mu^{-1} \int_{\partial \Omega^\varepsilon} f_R^2(x) v_1(x \varepsilon^{-1}) \, ds &= \varepsilon^2 \mu^{-1} \sum_{k \in 4Z^n} \int_{(sk + \varepsilon \Box) \cap \text{supp} h \neq \emptyset} f_R^2(x) v_1(x \varepsilon^{-1}) \, dx \\
&= \varepsilon^2 \mu^{-1} \sum_{k \in 4Z^n} \frac{1}{4^n \varepsilon^n} \int_{sk + \varepsilon \Box} f_R^2(x) \, dx \int_{sk + \varepsilon \Box} v_1(x \varepsilon^{-1}) \, ds + O(\varepsilon^2 \mu^{-1}) \\
\end{align*}
\]

(3.49)

and in view of (3.43) we see that

\[
\varepsilon^2 \mu^{-1} \int_{\partial \Omega^\varepsilon} f_R^2(x) v_1(x \varepsilon^{-1}) \, ds = O(\varepsilon^2 \mu^{-1}).
\]
Applying now Lemma 3.8, by (3.47), (3.48) we get:

\[ \| c_4 \varepsilon \mu^{-1} \nabla_x f_R + \varepsilon \nabla_x v_1 (\cdot, \varepsilon^{-1}) f_R \|_{L^2(\Omega')}^2 = \varepsilon \mu^{-1} \| f_R \|_{L^2(\mathbb{R}^n)}^2 + O(\varepsilon^2 \mu^{-1}). \]

This identity and (3.46) prove the order sharpness of the terms \( \varepsilon^{1/2} \eta^{-n/2 + 1} (\varepsilon) \mu^{-1} (\varepsilon) \) and \( \varepsilon \eta^{-n+1} (\varepsilon) \mu^{-1} (\varepsilon) \) in the right hand sides of (2.14), (2.15) as \( \eta \) is independent of \( \varepsilon \).

We proceed to the case \( \eta \rightarrow +0 \) as \( \varepsilon \rightarrow +0 \). We begin with an auxiliary problem

\[ -\Delta v_3 = 1 \quad \text{in} \quad \square \setminus \{0\}, \quad v_3 = G_n(|\xi|) + O(|\xi|^2), \quad \xi \rightarrow 0, \]

subject to the periodic boundary conditions as in (3.38), where

\[ G_n(t) := \frac{16}{2\pi} \ln t \quad \text{as} \quad n = 2, \quad G_n(t) := \frac{4^n t^{-n+2}}{(2n-1) \text{mes}_{n-1}(\partial B_1(0))} \quad \text{as} \quad n \geq 3, \]

where \( \text{mes}_{n-1}(\partial B_1(0)) \) is the \((n-1)\)-dimensional area of the sphere \( \partial B_1(0) \). Such problem is uniquely solvable and the solution is infinitely differentiable in \( \square \setminus \{0\} \).

By \( \chi_6 = \chi_6(t) \) we denote an infinitely differentiable cut-off function vanishing as \( t > 2 \) and equalling to one as \( t < 1 \). We let \( \chi_7(\xi) := \chi_6(|\xi| \eta^{-\frac{n}{2}}) \) and

\[ v_4(\xi) := \left(1 - \chi_7(\xi)\right) v_3(\xi) + G_n(|\xi|) \chi_7(\xi) - G_n(\eta), \quad v_5(\xi) := v_4(\xi) + (\varepsilon \mu)^{-1} G_n'(\eta), \]

The function \( v_4 \) satisfies the Dirichlet boundary condition on \( \partial B_\eta(0) \), while the function \( v_5 \) does the Robin condition:

\[ \frac{\partial v_5}{\partial |\xi|} = \varepsilon \mu v_5 \quad \text{on} \quad \partial B_\eta(0). \]

It is also straightforward to confirm that

\[ -\Delta_\xi v_4 = -\Delta_\xi v_5 = 1 + f_\eta(\xi) \quad \text{in} \quad \square \setminus B_\eta(0), \]

\[ f_\eta(\xi) := 2 \nabla_\xi \chi_7 \cdot \nabla(v_3 - G_n) + (v_3 - G_n) \Delta_\xi \chi_7 - \chi_7, \]

\[ \| f_\eta \|_{L^2(\square \setminus B_\eta(0))} = O(\eta^{\frac{n}{2}}), \]

\[ \| v_4 \|^2_{L^2(\square \setminus B_\eta(0))} = 4^n G_n^2(\eta) + O(\eta^{-n+4} \ln^2 \eta), \]

\[ \| v_5 \|^2_{L^2(\square \setminus B_\eta(0))} = 4^n \left((\varepsilon \mu)^{-1} G_n'(\eta) - G_n(\eta)\right)^2 + O(\eta^{-n+4} \ln^2 \eta), \]

\[ \| \nabla_\xi v_4 \|^2_{L^2(\square \setminus B_\eta(0))} = \| \nabla_\xi v_5 \|^2_{L^2(\square \setminus B_\eta(0))} = c_5 G_n(\eta) + O(\eta^{-n+3}), \]

where \( c_5 \) is some fixed non-zero constant independent of \( \eta \). We then let

\[ u_D(x) := \varepsilon^2 v_4(x \varepsilon^{-1}) f_D(x), \quad u_R(x) := \varepsilon^2 v_5(x \varepsilon^{-1}) f_R(x), \]

and we see immediately that both these functions solve problem (2.6) with the right hand sides.
\[
\begin{align*}
f(x) &= f_D(x) + h_D(x, \varepsilon), \\
h_D(x, \varepsilon) &= f_D(x) f_\eta(x \varepsilon^{-1}) - 2\varepsilon \nabla_\xi v_4(x \varepsilon^{-1}) \cdot \nabla f_D(x) - \varepsilon^2 v_4(x \varepsilon^{-1}) \Delta f_D(x), \\
f(x) &= f_R(x) + h_R(x, \varepsilon), h_R(x, \varepsilon) := f_R(x) f_\eta(x \varepsilon^{-1}) - 2\varepsilon \nabla_\xi v_5(x \varepsilon^{-1}) \cdot \nabla f_R(x) - \varepsilon^2 v_5(x \varepsilon^{-1}) \Delta f_R(x).
\end{align*}
\]

Applying Lemma 3.8 and using identities (3.50), we get:
\[
\begin{align*}
\|u_D\|^2_{L_2(\Omega^\varepsilon)} &= \varepsilon^4 G_n^2(\eta) \|f_D\|^2_{L_2(\mathbb{R}^d)} + O(\varepsilon^4 \eta^{-n+5} \ln^2 \eta + \varepsilon^5 G_n^2(\eta)), \\
\|u_R\|^2_{L_2(\Omega^\varepsilon)} &= \varepsilon^4 ((\varepsilon \mu)^{-1} G'_n(\eta) - G_n(\eta))^2 \|f_R\|^2_{L_2(\mathbb{R}^d)} \\
&\quad + O(\varepsilon^4 \eta^{-n+4} \ln^2 \eta + \varepsilon^5 ((\varepsilon \mu)^{-1} G'_n(\eta) - G_n(\eta))^2), \\
\|\nabla u_D\|^2_{L_2(\Omega^\varepsilon)} &= \frac{c_5}{4\varepsilon} \varepsilon^2 G_n(\eta) \|f_D\|^2_{L_2(\mathbb{R}^d)} + O(\varepsilon^2 \eta^{-n+3} + \varepsilon^3 G_n(\eta)), \\
\|h_D\|^2_{L_2(\Omega^\varepsilon)} &= O\left(\eta^2 + \varepsilon^2 |G_n(\eta)|\right), \\
\|h_R\|^2_{L_2(\Omega^\varepsilon)} &= O\left(\eta^2 + \varepsilon^2 G_n(\eta) + \varepsilon^2 \mu^{-2} \eta^{-2n+2}\right).
\end{align*}
\]

(3.51)

To understand the behavior of the norm \(\|\nabla u_R\|^2_{L_2(\Omega^\varepsilon)}\) as \(\varepsilon \to +0\), we first observe that in view of the mentioned boundary value problem for \(u_R\), it satisfies the integral identity
\[
\|\nabla u_R\|^2_{L_2(\Omega^\varepsilon)} = (f_R + h_R, u_R)_{L_2(\Omega^\varepsilon)} - \mu \|u_R\|^2_{L_2(\partial B_1(0))}
= (f_R + h_R, u_R)_{L_2(\Omega^\varepsilon)} - \varepsilon^2 \mu^{-1} (G'(\eta))^2 \int_{\partial B_1(0)} f_R^2(x) \, dx.
\]

It follows from (3.49) with \(v_1\) replaced by 1 that
\[
\varepsilon^2 \mu^{-1} (G'(\eta))^2 \int_{\partial B_1(0)} f_R^2(x) \, dx = \frac{\text{mes}_{n-1} B_1(0)}{4^n} \varepsilon \mu^{-1} (G'(\eta))^2 \eta^{n-1} \int_{\mathbb{R}^d} f_R^2 \, dx + O\left(\varepsilon^2 \mu^{-1} \eta^{-n+1}\right).
\]

Then by Lemma 3.8 we obtain:
\[
\begin{align*}
\|\nabla u_R\|^2_{L_2(\Omega^\varepsilon)} &= \frac{\varepsilon^2}{4^n} \|f_R\|^2_{L_2(\mathbb{R}^d)} \int_{\mathbb{R}^d \setminus B_1(0)} v_5(\xi) \, d\xi (1 + O(\varepsilon)) \\
&\quad - \frac{4^n \text{mes}_{n-1} B_1(0)}{\text{mes}_{n-1} \partial B_1(0)} \varepsilon \mu^{-1} \eta^{-n+1} \|f_R\|^2_{L_2(\mathbb{R}^d)} + O\left(\varepsilon^2 \mu^{-1} \eta^{-n+1}\right) \\
&= \left(\frac{4^n}{\text{mes}_{n-1} \partial B_1(0)} \left(1 - \frac{1}{n}\right) \varepsilon \mu^{-1} - \varepsilon^2 G_n(\eta)\right) \|f_R\|^2_{L_2(\mathbb{R}^d)} \\
&\quad + O\left(\varepsilon^2 \mu^{-1} \eta^{-n+1} + \varepsilon^2 |G_n(\eta)|\right).
\end{align*}
\]

Employing the obtained identity and (3.51) and proceeding as in (3.41), we see that estimates (2.14), (2.15) are order sharp as \(\eta \to +0\). The proof of Theorem 2.1 is complete.
3.3 Proof of Theorem 2.2

The proof follows the same lines as in the previous subsection and below we just list necessary modifications. Lemma 3.7 remains true and no changes are needed. Identity (3.25) and inequality (3.26) become

\[
\text{Re} \, \mathfrak{h}_A(u_\varepsilon, u_\varepsilon) + \sum_{k \in M_\varepsilon^D} \left( a_\varepsilon(\cdot,u_\varepsilon), u_\varepsilon \right)_{L^2(\partial \Omega^\varepsilon_k)} - \text{Re} \, \lambda \| u_\varepsilon \|^2_{L^2(\Omega^\varepsilon)} = \text{Re}(f, u_\varepsilon)_{L^2(\Omega^\varepsilon)},
\]

\[
c_0 \frac{1}{4} \| \nabla u_\varepsilon \|^2_{L^2(\Omega^\varepsilon)} + \| u_\varepsilon \|^2_{L^2(\Omega^\varepsilon)} \leq \| f \|_{L^2(\Omega^\varepsilon)}\| u_\varepsilon \|_{L^2(\Omega^\varepsilon)}. \quad (3.52)
\]

Inequality (3.33) remains true, just the sum is to be taken over \( k \in M_\varepsilon^D \). A next step is a modification of estimates (3.36), (3.37), which is true owing to Lemma 3.1:

\[
\| u_\varepsilon \|^2_{L^2(\Omega^\varepsilon)} \leq C \sum_{k \in M_\varepsilon^D} \| u_\varepsilon \|^2_{L^2(B_\varepsilon R_4(M_\varepsilon^k))} \\
\leq C \varepsilon^2 \eta^{-n+2} \varepsilon(\varepsilon) \sum_{k \in M_\varepsilon^D} \| \nabla u_\varepsilon \|^2_{L^2(B_\varepsilon R_4(M_\varepsilon^k))} \\
\leq C \varepsilon^2 \eta^{-n+2} \varepsilon(\varepsilon) \| \nabla u_\varepsilon \|^2_{L^2(\Omega^\varepsilon)} \quad (3.53)
\]

with some constants \( C \) independent of \( \varepsilon, \eta \) and \( u_\varepsilon \). This estimate and (3.52) then yields (2.17). Substituting this inequality into (3.53), we arrive at (2.18). The sharpness of estimates (2.17), (2.18) can be checked by means of the functions \( u_D \) introduced in the previous section. The proof is complete.

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Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

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