Abstract. We introduce a notion of an algebra of generalized pseudo-differential operators and prove that a spectral triple is regular if and only if it admits an algebra of generalized pseudo-differential operators. We also provide a self-contained proof of the fact that the product of regular spectral triples is regular.

0. Introduction

In their work on the local index theorem [CM95], Connes and Moscovici introduced the notion of a regular spectral triple. Later, Higson showed that a spectral triple is regular if and only if a certain algebra constructed from the spectral triple is what he calls an algebra of generalized differential operators [Hig04, Hig06]. See Theorem 2.4 for the precise statement.

Motivated by their work we introduce a notion of an algebra of generalized pseudo-differential operators; it clarifies the role of the regularity condition and serves as a convenient framework to study index theory via complex powers. I should note that most of the main ideas in this paper can be traced back to [CM95, Appendix B] or [Hig06, Subsection 4.5], but for the convenience of the reader we tried to remain self-contained.

Now we describe the content of the paper. After some preliminaries in Section 1 we recall Higson’s notion of an algebra of generalized differential operators in Section 2. In Section 3 we develop the notion of an algebra of generalized pseudo-differential operators. As usual, there are two versions – operators of all orders and operators of order at most zero. Extending Higson’s result, we show that a spectral triple is regular if and only if it admits an algebra of generalized pseudo-differential operators (see Theorem 4.1).

Finally, we show that the product of regular spectral triples is again regular (see Theorem 4.4). This is a folklore and it is implicitly contained in or follows from results in [CM95, Hig04, Hig06]. But no direct reference seems to exist in the literature.

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1. Preliminaries

1.1. Spectral Triples. The following definition of a regular spectral triple is due to Connes and Moscovici \cite{CM95}.

**Definition 1.1.** A spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) consists of an associative algebra \(\mathcal{A}\), a graded Hilbert space \(\mathcal{H}\) equipped with an even representation of \(\mathcal{A}\) and a densely defined, self-adjoint, odd operator \(\mathcal{D}\) such that

1. \(\mathcal{A}\) is contained in the domain of the derivation \([\mathcal{D},-]\), that is, if \(\text{dom}(\mathcal{D})\) is the domain of \(\mathcal{D}\), then any \(a \in \mathcal{A}\) satisfies \(a \cdot \text{dom}(\mathcal{D}) \subseteq \text{dom}(\mathcal{D})\) and the commutator \([\mathcal{D}, a] : \text{dom}(\mathcal{D}) \to \mathcal{H}\) extends by continuity to a bounded operator on \(\mathcal{H}\) and
2. the operators \(a \cdot (\mathcal{D} \pm i)^{-1}\) and \((\mathcal{D} \pm i)^{-1} \cdot a\) are compact for any \(a \in \mathcal{A}\).

We say that a spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) is regular, if

3. the space \(\mathcal{A} + [\mathcal{D}, \mathcal{A}]\) is contained in the smooth domain of the derivation \(\delta(T) := [||\mathcal{D}, T|] \text{ on } \mathcal{L}(\mathcal{H})\).

We remind the reader that the smooth domain of \(\delta\) is defined as follows. The domain \(\text{dom}(\delta)\) of \(\delta = [||\mathcal{D}, -|]\) is the set of \(T \in \mathcal{L}(\mathcal{H})\) such that

(i) \(T \cdot \text{dom}(|\mathcal{D}|) \subseteq \text{dom}(|\mathcal{D}|)\) and
(ii) \([|\mathcal{D}, T] \text{ extends to a bounded operator on } \mathcal{H}\).

For \(k \geq 2\), we define \(\text{dom}(\delta^k)\) inductively as

\[
\text{dom}(\delta^k) := \{ b \in \text{dom}(\delta) \mid \delta(b) \in \text{dom}(\delta^{k-1})\}.
\]

The smooth domain of \(\delta\) is \(\text{dom}^\infty(\delta) := \bigcap_{k=1}^\infty \text{dom}(\delta^k)\).

For simplicity, we do not consider any additional structure on \(\mathcal{A}\); even though in most natural examples \(\mathcal{A}\) have a topology or a norm, it will not play any role in our analysis.

Obviously, the following is the basic example that should be mentioned in any paper on spectral triples.

**Example 1.2.** Let \(M\) be a complete Riemannian manifold of even dimension and let \(S \to M\) be a complex spinor bundle. Let \(\mathcal{H} := L^2(M, S)\) denote the graded Hilbert space of \(L^2\)-sections of \(S\). Let \(\mathcal{D}\) be a Dirac-type operator acting on \(C^\infty_c(M, S)\). Then standard \(\Psi\)-DO theory implies that \(\mathcal{D}\) with domain \(C^\infty_c(M, S) \subset L^2(M, S)\) is essentially self-adjoint and

\[
(C^\infty_c(M), L^2(M, S), \mathcal{D}),
\]

is a regular spectral triple, where \(\mathcal{D}\) is the closure of \(\mathcal{D}\). See \cite{HR00}.

The material in the following two subsections are elementary and well-known. We include it for reference and the convenience of the reader.
1.2. Sobolev Spaces. Let $\Delta$ be an invertible, positive, self-adjoint operator on a Hilbert space $\mathcal{H}$.

**Definition 1.3.** The $\Delta$-Sobolev space of order $s \in \mathbb{R}$, denoted

$$W^s = W^s(\mathcal{H}, \Delta),$$

is the Hilbert space completion of $\text{dom}(\Delta^2)$ with respect to the inner product given by

$$\langle \xi, \eta \rangle_{W^s} := \langle \Delta^s \xi, \Delta^s \eta \rangle_{\mathcal{H}}$$

for $\xi, \eta \in \text{dom}(\Delta^2)$.

Note that the invertibility hypothesis guaranties that $\| \cdot \|_{W^s}$ is nondegenerate. The following is well-known.

**Lemma 1.4.** For $s \geq t$, we have a continuous inclusion

$$W^s \subseteq W^t.$$ Moreover, for $s \geq 0$, $\text{dom}(\Delta^2)$ is complete and thus $W^s = \text{dom}(\Delta^2)$. □

**Example 1.5.** If $\Delta$ is bounded, then nothing interesting happens: for any $s \in \mathbb{R}$, $W^s = \mathcal{H}$ and the inner products are different but equivalent.

**Example 1.6.** Suppose that $\Delta$ has compact resolvents. Then there exists a complete orthonormal basis $\{\xi_n\}$ of $\mathcal{H}$ consisting of eigenvectors: $\Delta \xi_n = \lambda_n \xi_n$ with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$. Hence the Sobolev spaces can be identified as the weighted $l^2$-space:

$$W^s = \left\{ (a_n)_{n=1}^\infty \mid \sum_n \lambda_n^s |a_n|^2 < \infty \right\}.$$

**Example 1.7** (Classical Sobolev Spaces). Let $M$ be a closed manifold and let $\Delta_0$ be a strictly positive order-two elliptic partial differential operator acting on the smooth functions on $M$. Let $\Delta = \overline{\Delta_0}$ be its closure. Then $\Delta$ is invertible and self-adjoint and has compact resolvents (cf. [HR00]). It follows from the basic estimate that as a Hilbert space $W^s$ is equivalent to the “standard” $s$-Sobolev space (cf. [Shu01, Proposition I.7.3]).

**Definition 1.8.** Let $\Delta$ be an invertible, positive, self-adjoint operator. The space of $\Delta$-smooth vectors is

$$W^\infty := \bigcap_{s \in \mathbb{R}} W^s = \bigcap_{n=0}^\infty W^{2n} = \bigcap_{n=0}^\infty \text{dom}(\Delta^n) \subseteq \mathcal{H}.$$ The space $W^\infty$ contains $1_{[0,M]}(\mathcal{H})$ for any $M > 0$, and thus $W^\infty$ is dense in $\text{dom}(\Delta^z)$ for any $z \in \mathbb{C}$. It follows that $W^\infty$ is dense in $W^s$ for any $s \in \mathbb{R}$. 

Lemma 1.9. Let $z \in \mathbb{C}$. For any $s \in \mathbb{R}$, the operator $\Delta^z_{W^\infty} : W^\infty \to \mathcal{H}$ extends to an isometry
\[(1.8) \quad \Delta^z : W^{s+2\Re(z)} \to W^s.\]
In particular, $\Delta^z(W^\infty) = W^\infty$.

Proof. First note that $\Delta^i \Im(z)$ is a unitary operator on $\mathcal{H}$. Therefore, for any $\xi \in W^\infty$,
\[(1.9) \quad ||\Delta^z \xi||_{W^s} = ||\Delta^{\Re(z)} + \frac{i}{2} \xi||_{\mathcal{H}} = ||\xi||_{W^{s+2\Re(z)}}.\]

Lemma 1.10. The space $W^\infty \subset \mathcal{H}$ is a common core for the operators $\Delta^z$, $z \in \mathbb{C}$, i.e. $\Delta^z$ is essentially self-adjoint on $W^\infty \subset \mathcal{H}$.

Proof. Any vector $\xi \in 1_{[0,M]}(\Delta)\mathcal{H} \subset W^\infty$ is an analytical vector for $\Delta^z$. Indeed, for any $\xi \in 1_{[0,M]}(\Delta)\mathcal{H}$,
\[(1.10) \quad ||(\Delta^z)^n \xi|| \leq M^{\Re(z)n}||\xi||\]
and consequently
\[(1.11) \quad \sum_{n=0}^{\infty} \frac{||(\Delta^z)^n \xi||}{n!} t^n < \infty\]
for any $t > 0$. Since $\Delta^z$ preserves $W^\infty$, all the analytical vectors for $\Delta^z$ are also analytical for the restriction $\Delta^z_{W^\infty}$ and applying Nelson’s theorem [RS75 Theorem X.39] to $\Delta^z_{W^\infty}$, we see that $\Delta^z$ is essentially self-adjoint on $W^\infty$.

Another convenient way to express the complex powers $\Delta^z$ is using the Cauchy integral formula: for $\Re(z) < 0$,
\[(1.12) \quad \Delta^z = \frac{1}{2\pi i} \int \lambda^z (\lambda - \Delta)^{-1} d\lambda,\]
where the integral is a contour integral along a downwards pointing vertical line in $\mathbb{C}$ which separates 0 from Spec($\Delta$). Let $\Delta \geq c > 0$ and let $s \in \mathbb{R}$. Then it follows from the spectral theorem that for any $\lambda \notin \text{Spec}(\Delta)$, the resolvent $(\lambda - \Delta)^{-1}$ is a bounded operator on the Sobolev spaces $W^s$ with norm at most $((\Re(\lambda) - c)^2 + \Im(\lambda)^2)^{-\frac{1}{2}}$. Hence for any $\Re(z) < 0$, the integral converges to a bounded operator on $W^s$.

More generally, we have the following.

Lemma 1.11. For $k \in \mathbb{N}$ and $\Re(z) < k$,
\[(1.13) \quad \left(\frac{z}{k}\right)^{\Delta^z-k} = \frac{1}{2\pi i} \int \lambda^z (\lambda - \Delta)^{-k-1} d\lambda \quad \text{in} \quad \mathcal{L}(W^s), \quad s \in \mathbb{R},\]
where the integral is a contour integral along a downwards pointing vertical line in $\mathbb{C}$ which separates 0 from Spec($\Delta$).

\[\text{Note that if } s > 0 \text{ the range actually shrinks to } W^s \subset \mathcal{H}.\]
1.3. **Operators of Finite Analytic Order.** We consider various classes of linear operators on $W^{\infty}$. The algebra of all linear operators $W^{\infty} \rightarrow W^{\infty}$ is denoted $\text{End}(W^{\infty})$.

**Example 1.12.** If an (unbounded) operator $P$ has domain $\text{dom}(P) \supseteq W^{\infty}$ and preserves $W^{\infty}$, i.e. $P(W^{\infty}) \subseteq W^{\infty}$, then the restriction $P|_{W^{\infty}}$ gives an element of $\text{End}(W^{\infty})$. We often write, simply, $P$ for $P|_{W^{\infty}}$.

**Definition 1.13.** We say that a linear operator $W^{s} \rightarrow W^{\infty}$ has analytic order at most $t \in \mathbb{R}$ if it extends by continuity to a bounded linear operator $W^{s+t} \rightarrow W^{s}$ for every $s \in \mathbb{R}$. We write

\[(1.14) \quad \text{Op}^{t} = \text{Op}^{t}(\Delta) = \text{Op}^{t}(H, \Delta)\]

for the class of operators of analytic order at most $t$ and define

\[(1.15) \quad \text{Op} = \text{Op}^{\infty} := \bigcup_{t} \text{Op}^{t} \quad \text{and} \quad \text{Op}^{-\infty} := \bigcap_{t} \text{Op}^{t} \quad \text{is a two-sided ideal.}\]

**Lemma 1.14.** Operators with finite analytic order form a filtered algebra:

(a) $\text{Op}^{s} \subseteq \text{Op}^{t}$ for $s \leq t$ and

(b) $\text{Op}^{s} \cdot \text{Op}^{t} \subseteq \text{Op}^{s+t}$.

In particular, $\text{Op}^{0} \subset \text{Op}$ is a subalgebra and $\text{Op}^{-\infty} \subset \text{Op}$ and $\text{Op}^{t} \subset \text{Op}^{0}$, $t \in [-\infty, 0)$ are two-sided ideals. □

Notice that operators with analytic order at most 0 extend, in particular, to bounded linear operators on $\mathcal{H} = W^{0}$ allowing us to identify $\text{Op}^{0}$ with a subalgebra of $\mathcal{L}(\mathcal{H})$, the algebra of bounded linear operators on $\mathcal{H}$.

**Example 1.15.** We see from Lemma 1.9 that the operator $\Delta^{z}$ belongs to $\text{Op}^{2 \Re(z)}$ for any $z \in \mathbb{C}$. It follows from the spectral theorem that for $z$, $w \in \mathbb{C}$

\[(1.16) \quad \Delta^{z} \cdot \Delta^{w} = \Delta^{z+w} \quad \text{in Op}.\]

**Lemma 1.16.** If $\Psi$ is a filtered subalgebra of Op such that $\Delta^{t}$ belongs to $\Psi^{t}$ for all $t \in \mathbb{R}$, then $\Psi^{0}$ is unital and

\[(1.17) \quad \Psi^{t} = \Delta^{t} \Psi^{0} = \Psi^{0} \Delta^{t}.\]

Conversely, if a unital subalgebra $\Psi^{0} \subset \text{Op}^{0}$ satisfies $\Delta^{t} \Psi^{0} = \Psi^{0} \Delta^{t}$ for all $t \in \mathbb{R}$, then $\Psi^{t} := \Delta^{t} \Psi^{0}$ defines a filtered subalgebra of Op such that $\Delta^{t}$ belongs to $\Psi^{t}$.

**Proof.** For the first statement: since $\Delta^{0} = 1$, $\Psi^{0}$ is unital. Moreover, for any $t \in \mathbb{R}$,

\[(1.18) \quad \Psi^{t} = \Delta^{t} \Delta^{-t} \psi = \Delta^{t - t} \psi \subseteq \Delta^{t} \Psi^{0} \subseteq \Psi^{t} \psi \subseteq \Psi^{t} \Psi^{0} \subseteq \Psi^{t}.\]

Similarly for the other side.
For the second statement: since $\Psi^0$ is unital,

\begin{equation}
\Psi^t \Psi^s = \Delta^s \Psi^0 \Delta^s \Psi^0 = \Delta^s \Delta^s \Psi^0 \Psi^0 = \Psi^{t+s} \quad \text{and}
\end{equation}

\begin{equation}
\Delta^s \text{ belongs to } \Psi^t.
\end{equation}

\[\square\]

**Corollary 1.17.** For any $t \in \mathbb{R}$,

\begin{equation}
\text{Op}^t = \Delta^s \text{Op}^0 = \text{Op}^0 \Delta^s.
\end{equation}

\[\square\]

**Corollary 1.18.** Let $\Delta$ be an invertible positive self-adjoint operator. Let $K, L^p$ and $L^{(p, \infty)}$ denote the ideal of compact, $p$-Schatten and $p$-Dixmier operators.

(a) If $\Delta^{-\frac{s}{2}} \in K$ then $\text{Op}^{-t} \subseteq K$ for $t > 0$.

(b) If $\Delta^{-\frac{s}{2}} \in L^p$, $p \geq 1$ then $\text{Op}^{-t} \subseteq L^{p/t}$ for $0 < t \leq p$.

(c) If $\Delta^{-\frac{s}{2}} \in L^{(p, \infty)}$, $p \geq 1$ then $\text{Op}^{-t} \subseteq L^{(p/t, \infty)}$ for $0 < t \leq p$.

**Proof.** Using the fact that $\text{Op}^{-t} \subseteq \Delta^{-\frac{s}{2}} L$, these follow immediately from well-known properties of the respective ideals. \[\square\]

## 2. Algebra of Generalized Differential Operators

In this section, we recall Higson’s notion of an algebra of generalized differential operators [Hig04, Hig06]. Let $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

**Definition 2.1** (Higson). An $\mathbb{N}$-filtered subalgebra $D \subseteq \text{Op}(\Delta)$ is called an algebra of generalized differential operators if $D$ is closed under the derivation $[\Delta, -]$ and satisfies

\begin{equation}
[\Delta, D^k] \subseteq D^{k+1}, \quad k \in \mathbb{N}.
\end{equation}

**Example 2.2.** Let $M$ be a complete Riemannian manifold and let $S \to M$ be a Hermitian vector bundle. Let $\Delta$ be the closure of a scalar Laplacian +1 (to ensure invertibility). Then the algebra $D = D_c(M, S)$ of compactly supported differential operators acting on the sections of $S$ is an example of an algebra of generalized differential operators. Note that in this example $\Delta$ is not an element of $D$. See [HR00].

**Example 2.3** (Polynomial Weyl Algebra). Consider the usual Lebesgue measure on $\mathbb{R}^n$ and let $\mathcal{H} := L^2(\mathbb{R}^n)$. The operator

\begin{equation}
1 + \sum_{i=1}^n \left( x_i^2 - \frac{\partial^2}{\partial x_i^2} \right)
\end{equation}

with domain $C_c^\infty(\mathbb{R}^n)$ of compactly supported smooth functions is called the **harmonic oscillator**. It is essentially self-adjoint and strictly positive.

Let $\Delta$ denote the closure. Then $\Delta$ is invertible positive self-adjoint operator with compact resolvent. Let $W = W_n$ be the algebra of polynomial
differential operators on $\mathbb{R}^n$ acting on $L^2(\mathbb{R}^n)$. As an algebra, it is generated by $x_1, \ldots, x_n$ and $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. We filter $\mathcal{W}$ by requiring that

$$\text{order}(x_i) := 1 \quad \text{and} \quad \text{order} \left( \frac{\partial}{\partial x_i} \right) := 1, \quad 1 \leq i \leq n.$$  

(The nonzero degree of $x_i$ compensates the noncompactness of $\mathbb{R}^n$.) Then $\mathcal{W}$ is an algebra of generalized differential operators and $\Delta$ is an element of $\mathcal{W}^2$. See [Shu01].

Now we relate generalized differential operators to regularity. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple and let $\Delta = \mathcal{D}^2 + 1$. Then it is clear that $\mathcal{D} \in \text{Op}^1(\Delta) \subseteq \text{End}(\mathcal{W}^\infty)$. Suppose that $\mathcal{A} \cdot \mathcal{W}^\infty \subseteq \mathcal{W}^\infty$. Define an $\mathbb{N}$-filtered algebra $\mathcal{D} \subseteq \text{End}(\mathcal{W}^\infty)$ inductively by

1. $\mathcal{D}^0 := \text{the subalgebra generated by } \mathcal{A} + [\mathcal{D}, \mathcal{A}] \subseteq \text{End}(\mathcal{W}^\infty)$ and
2. $\mathcal{D}^1 := \mathcal{D}^0 + [\Delta, \mathcal{D}^0] + \mathcal{D}^0[\Delta, \mathcal{D}^0] \subseteq \text{End}(\mathcal{W}^\infty)$ and
3. $\mathcal{D}^k := \mathcal{D}^{k-1} + \sum_{j=1}^{k-1} \mathcal{D}^j \cdot \mathcal{D}^{k-j} + [\Delta, \mathcal{D}^{k-1}] + \mathcal{D}^0[\Delta, \mathcal{D}^{k-1}] \subseteq \text{End}(\mathcal{W}^\infty)$ for $k \geq 2$.

**Theorem 2.4** ([Hig06 Theorem 4.26]). Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple and let $\Delta = \mathcal{D}^2 + 1$. Suppose that $\mathcal{A} \cdot \mathcal{W}^\infty \subseteq \mathcal{W}^\infty$. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is regular if and only if $\mathcal{D}^k \subseteq \text{Op}^k$, $k \geq 0$.

We will give a proof in Section 4 essentially rephrasing the arguments in [CM95, Hig06] in the language of algebra of generalized pseudo-differential operators, which we describe next.

### 3. Algebra of Generalized Pseudodifferential Operators

In this section, we define and study algebras of generalized pseudodifferential operators. Just as in the classical case, these provide a convenient framework to study index theoretic problems, see [Hig06].

#### 3.1. Operators of all orders.

**Definition 3.1.** An $\mathbb{R}$-filtered subalgebra $\Psi \subseteq \text{Op}(\Delta)$ is called an algebra of generalized pseudo-differential operators if $\Psi$ satisfies for $z \in \mathbb{C}$, $t \in \mathbb{R}$,

$$\Delta^z \Psi^t \subseteq \Psi^{\text{Re}(z)+t} \quad \text{and} \quad \Psi^t \Delta^z \subseteq \Psi^{\text{Re}(z)+t}$$

and

$$[\Delta^z, \Psi^t] \subseteq \Psi^{\text{Re}(z)+t-1}.$$  

**Remark 3.2.** If $\Delta^z$ belongs to $\Psi^{\text{Re}(z)}$ for $z \in \mathbb{C}$, then (3.1) is trivially satisfied and

$$\Psi^t = \Delta^z \Psi^0 = \Psi^0 \Delta^z, \quad t \in \mathbb{R},$$

by Lemma 1.16.

The following is the classical example.
Example 3.3 (Pseudodifferential Operators). Let $M$ be a closed manifold. Let $\Psi = \bigcup \Psi^t$ denote the $\mathbb{R}$-filtered algebra of \textit{pseudo-differential operators} on $M$ with scalar symbols and let $\Psi_{cl} \subset \Psi$ denote the filtered subalgebra of \textit{classical pseudo-differential operators} on $M$ (see for instance [Shu01]).

Let $\Delta$ be (the closure of) an invertible, positive, order two, partial-differential operator on $M$. Then we have inclusions of $\mathbb{R}$-filtered algebras

$$\Psi_{cl} \subset \Psi \subset \text{Op}(\Delta).$$

Moreover $\Delta^z$ belongs to $\Psi_{cl}^{\text{Re}(z)} \subset \Psi^{\text{Re}(z)}$, $z \in \mathbb{C}$, and since

$$[\Psi^s, \Psi^t] \subseteq \Psi^{s+t-1}, \quad s, t \in \mathbb{R},$$

we have

$$[\Delta^z, \Psi^t] \subseteq \Psi_{cl}^{\text{Re}(z)+t-1} \quad \text{and} \quad [\Delta^z, \Psi^t] \subseteq \Psi^{\text{Re}(z)+t-1}.$$  

Hence $\Psi_{cl}$ and $\Psi$ are algebras of generalized pseudo-differential operators (in view of Remark 3.2).

More generally, we could consider pseudo-differential operators acting on sections of a vector bundle. Then we need to assume that $\Delta$ has scalar principal symbol: the property (3.5) would not hold in general, but (3.6) would still hold.

For a calculus of pseudo-differential and classical pseudo-differential operators on $\mathbb{R}^n$, see [Shu01, Chapter IV] or [Fed96, Chapter 3].

Remark 3.4. Let $\Psi \subseteq \text{Op}$ be an algebra of generalized pseudo-differential operators. Then $\mathcal{D}^k := \Psi^k$, $k \in \mathbb{N}$, is clearly an algebra of generalized differential operators.

In the other direction, we have the following lemma.

Lemma 3.5 (cf. [Hig06, Proposition 4.31]). Let $\mathcal{D}$ be an algebra of generalized differential operators. Let $\Psi^t$ denote the space of linear combinations of operators $P \in \text{Op}$ such that for any $l \in \mathbb{R}$, $P$ may be decomposed as

$$P = X \Delta^{\frac{z-m}{2}} + Q$$

with $\text{Re}(z) \leq t$ and $X \in \mathcal{D}^m$, $m \in \mathbb{N}$ and $Q \in \text{Op}^l$. Then $\Psi$ is an algebra of generalized pseudo-differential operators.

First we prove an auxiliary lemma (cf. [CM95, Theorem B1], [Hig06, Lemma 4.20]).

Notation 3.6. In a filtered space, we write

$$P \sim \sum_{k \in \Lambda} P_k$$

if for any $l \in \mathbb{R}$, there exists a finite subset $F \subset \Lambda$ such that $P - \sum_{k \in F} P_k$ has order at most $l$. 
Lemma 3.7 (cf. [CM95, Theorem B1], [Hig06, Lemma 4.20]). Let \( z \in \mathbb{C} \) and let \( Y \in \mathcal{D} \). Let \( Y^{(0)} := Y \) and \( Y^{(k)} := [\Delta, Y^{(k-1)}] \), \( k \geq 1 \). Then
\[
\Delta^z Y \sim \sum_{k=0}^{\infty} \binom{z}{k} Y^{(k)} \Delta^{z-k}.
\]

Note that \( \text{order}(Y^{(k)}) \leq \text{order}(Y) + k \), so \( \text{order}(Y^{(k)} \Delta^{z-k}) \leq \text{order}(Y) + 2 \text{Re}(z) - k \).

**Proof.** Assume \( \text{Re}(z) < 0 \). Let \( R \) denote the resolvent \((\lambda - \Delta)^{-1} \in \text{Op}^{-2}\). Then it is easy to see that
\[
RY = YR + RY^{(1)} R.
\]

Hence, for any \( n \in \mathbb{N} \),
\[
RY = YR + Y^{(1)} R^2 + \cdots + Y^{(n)} R^{n+1} + RY^{(n+1)} R^{n+1}.
\]

Applying the Cauchy integral formula (Lemma 1.11), we see that
\[
\Delta^z Y = \sum_{k=0}^{n} \binom{z}{k} Y^{(k)} \Delta^{z-k} + \frac{1}{2\pi i} \int \lambda^z (\lambda - \Delta)^{-1} Y^{(n+1)} (\lambda - \Delta)^{-n-1} d\lambda.
\]

But the last integral converges absolutely in \( \mathcal{L}(W^{s+l}, W^s) \) for \( l = \text{order}(Y) + n + 1 - 2(n+1) \), hence has order at most \( \text{order}(Y) - n - 1 \).

For general \( z \in \mathbb{C} \), the identity
\[
\Delta^{z+1} Y = \Delta^z Y \Delta + \Delta^z Y^{(1)}
\]
allows one to reduce to the case \( \text{Re}(z) < 0 \). \(\square\)

**Proof of Lemma 3.5.** Clearly \( \Psi \) is a filtered subspace of \( \text{Op} \). It follows from Lemma 3.7 that \( \Psi \) is a subalgebra satisfying (3.1). For \( P = X \Delta \frac{w-m}{2} + Q \) in \( \Psi^t \) with \( \text{Re}(w) = t \) and \( X \in \mathcal{D}^m \) and \( \text{order}(Q) \) small, the expansion
\[
[\Delta^z, P] \sim \sum_{k=1}^{\infty} \binom{z}{k} X^{(k)} \Delta^{\frac{z+ w-m - 2k}{2}} + [\Delta^z, Q]
\]
shows that \( [\Delta^z, P] \in \Psi^{\text{Re}(z)+t-1} \), proving (3.2). \(\square\)

### 3.2. Operators of order at most zero.

**Definition 3.8.** A subalgebra \( \mathcal{B} \subseteq \text{Op}^0 \) is called an *algebra of generalized pseudo-differential operators of order at most zero* if \( \mathcal{B} \) is closed under the derivation \( \delta := [\Delta^z, -] \).

**Lemma 3.9** (cf. [CM95, Lemma B1]). Let \( \mathcal{B} = \text{dom}^\infty(\delta) \) denote the smooth domain of \( \delta \). Then \( \mathcal{B} \) is an algebra of generalized pseudo-differential operators of order at most zero.
Proof. We just need to show that $B$ is a subset of $\text{Op}^0$, since by construction $B$ is an algebra closed under $\delta$.

Let $s = \text{Re}(z) < 0$. Then for any $b \in B$ and $n \in \mathbb{N},$

$$\Delta^s b = \sum_{k=0}^{n} \binom{z}{k} \delta^k(b) \Delta^{\frac{s-k}{2}} + \frac{1}{2\pi i} \int \lambda^z (\lambda - \Delta^s)^{-1} \delta^{n+1}(b)(\lambda - \Delta^s)^{-n-1} d\lambda,$$

in $L(H)$, where the integral is a contour integral along a downwards pointing vertical line in $\mathbb{C}$ which separates 0 from Spec($\Delta^\frac{1}{2}$). The proof of Lemma 3.7 applied to $\Delta^\frac{1}{2}$ goes through ad verbatim.

The last integral gives a bounded operator in $L(H^0)$ so taking $n$ large enough we see that $||\Delta^s b\xi|| \leq C ||\Delta^s \xi||$, $\xi \in \text{dom}(\Delta^s)$, for some $C > 0$. Hence $b \cdot W^s \subseteq W^s$ for $s < 0$. For $s > 0$, the identity

$$\Delta^{s+1} b = \Delta^s b \Delta^\frac{1}{2} + \Delta^s \delta(b)$$

allows one to reduce to the case $s < 0$. Hence $B \subseteq \text{Op}^0$.

Remark 3.10. It is clear that if $\Psi \subseteq \text{Op}$ is an algebra of generalized pseudo-differential operators then $B := \Psi^0$ is an algebra of generalized pseudo-differential operators of order at most zero.

Conversely, we have the following.

Lemma 3.11 (cf. [Hig06, Lemma 4.27]). Let $B \subseteq \text{Op}^0$ be an algebra of generalized pseudo-differential operators of order at most zero. Then

$$\mathcal{D}^k := \sum_{j=0}^{k} \Delta^j B \subseteq \text{Op}^k, \quad k \in \mathbb{N},$$

is an algebra of generalized differential operators.

Proof. First note that since

$$\Delta^k b = \sum_{j=0}^{k} \binom{k}{j} \delta^j(b) \Delta^{\frac{k-j}{2}}, \quad k \in \mathbb{N},$$

we have

$$\Delta^k B = B \Delta^\frac{k}{2}, \quad k \in \mathbb{N}.$$  

Hence $\mathcal{D}$ is an $\mathbb{N}$-filtered subalgebra of Op (see the proof of Lemma 1.16). Then using the facts

$$\delta(\Delta^s B) = \Delta^s \delta(B) \subseteq \Delta^s B \quad \text{and}$$

$$[\Delta, P] = 2\Delta^s \delta(P) - \delta(\delta(P)),$$

we see that $\mathcal{D}$ is indeed an algebra of generalized differential operators. $\square$
Remark 3.12. Suppose that \( \mathcal{B} \) is an algebra of generalized pseudo-differential operators of order at most zero satisfying, for all \( t \in \mathbb{R} \),

\[
\Delta^it \in \mathcal{B} \quad \text{and} \quad \Delta^i \mathcal{B} \Delta^{-i} \subseteq \mathcal{B}.
\]

Then one can show that \( \Psi^t := \Delta^i \mathcal{B}, t \in \mathbb{R} \), is an algebra of generalized pseudo-differential operators, using Lemma 3.7. This is applicable to \( \mathcal{B} = \text{dom}^\infty(\delta) \) of Lemma 3.9.

4. Regularity of Spectral Triples

See [CM95, Appendix B], [Hig04, Theorem 3.25], [Hig06, Theorem 4.26].

Theorem 4.1. Let \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) be a spectral triple. Let \( \Delta := \mathcal{D}^2 + 1 \) and let \( \delta := [\Delta^\frac{1}{2}, -] \). Then the following conditions are equivalent:

1. The spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) is regular, that is, \( \mathcal{A} + [\mathcal{D}, \mathcal{A}] \) is contained in \( \text{dom}^\infty([|\mathcal{D}|, -]) \).
2. The set \( \mathcal{A} + [\mathcal{D}, \mathcal{A}] \) is contained in \( \text{dom}^\infty(\delta) \).
3. There exists an algebra of generalized pseudo-differential operators of order at most zero containing \( \mathcal{A} + [\mathcal{D}, \mathcal{A}] \).
4. There exists an algebra of generalized differential operators containing \( \mathcal{A} + [\mathcal{D}, \mathcal{A}] \) in degree 0.
5. There exists an algebra of generalized pseudo-differential operators containing \( \mathcal{A} + [\mathcal{D}, \mathcal{A}] \) in degree 0.

Proof. (1) \( \Leftrightarrow \) (2) is clear, since \( \Delta^\frac{1}{2} \) is a bounded perturbation of \( |\mathcal{D}| \). The implication (2) \( \Rightarrow \) (3) follows from Lemma 3.9.

To prove (3) \( \Rightarrow \) (2), suppose that \( \mathcal{B} \subseteq \text{Op}^0 \) is an algebra of generalized pseudo-differential operators of order at most zero. Then \( \mathcal{B} \) is contained in \( \text{dom}^\infty(\delta) \). Indeed, if \( b \in \mathcal{B} \subseteq \text{Op}^0 \), then \( b \cdot \text{dom}(\Delta^\frac{1}{2}) \subseteq \text{dom}(\Delta^\frac{1}{2}) \) and, being an operator of order 0, the commutator \( [\Delta^\frac{1}{2}, b] \in \mathcal{B} \subseteq \text{Op}^0 \) extends to a bounded operator on \( \mathcal{H} \). In other words, \( b \) belongs to \( \text{dom}(\delta) \). But \( \mathcal{B} \) is closed under \( \delta \), therefore \( \mathcal{B} \subseteq \text{dom}^\infty(\delta) \).

The implication (3) \( \Rightarrow \) (4) follows from Lemma 3.11 (1) \( \Rightarrow \) (5) follows from Lemma 3.10 and (5) \( \Rightarrow \) (3) follows from Remark 3.10 \( \square \).

As a corollary we obtain a proof of Theorem 2.4.

Proof of Theorem 2.4. If \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) is regular, then by Theorem 4.1 there exists an algebra of generalized differential operators \( \mathcal{E} \subseteq \text{Op} \). It is clear, by induction, that \( \mathcal{D}^k \subseteq \mathcal{E}^k \), \( k \in \mathbb{N} \). Thus \( \mathcal{D}^k \subseteq \text{Op}^k \). Conversely, if \( \mathcal{D}^k \subseteq \text{Op}^k \), \( k \in \mathbb{N} \), then \( \mathcal{D} \) is an algebra of generalized differential operators and by Theorem 4.1 \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) is regular \( \square \).

Example 4.2. (1) The commutative spectral triple of Example 1.2 is regular, because of Example 2.2 or Example 3.3.

(2) The spectral triple of [CM95] associated to a triangular structure is regular because the algebra of \( \Psi \)DO'-operators is an example of an algebra of generalized pseudo-differential operators.
Lemma 4.3. Let $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ be spectral triples. Let $\mathcal{A}_1 \otimes_{\text{alg}} \mathcal{A}_2$ denote the algebraic tensor product and let $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ denote the graded Hilbert space tensor product. Then the operator

\[(4.1) \quad \mathcal{D} := \mathcal{D}_1 \hat{\otimes} 1 + 1 \hat{\otimes} \mathcal{D}_2\]

with domain

\[(4.2) \quad \text{dom}(\mathcal{D}) := \text{dom}(\mathcal{D}_1) \hat{\otimes}_{\text{alg}} \text{dom}(\mathcal{D}_2) \subseteq \mathcal{H},\]

is essentially self-adjoint and $(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2, \mathcal{D})$ is a spectral triple.

Proof. See [Otg09].

We write $\mathcal{D}_1 \times \mathcal{D}_2$ for the closure $\mathcal{D}$ of $\mathcal{D}$ and call

\[(4.3) \quad (\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2, \mathcal{D}_1 \times \mathcal{D}_2),\]

the product of $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$.

The following theorem is implicitly contained in or follows from results in [CM95, Hig04, Hig06]. But no direct reference seems to exist in the literature.

Theorem 4.4. The product of regular spectral triples are again regular.

Proof. Let $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ be regular spectral triples and let

\[\mathcal{D}_i \subseteq \text{Op}(\mathcal{H}_i, \mathcal{D}_i^2 + 1)\]

be algebras of generalized differential operators for $(\mathcal{A}_i, \mathcal{H}_i, \mathcal{D}_i)$ respectively, i.e.

\[\mathcal{A}_i + [\mathcal{D}_i, \mathcal{A}_i] \subseteq \mathcal{D}_i^0 \quad \text{for } i = 1, 2.\]

We claim that $\mathcal{D}_1 \hat{\otimes} \mathcal{D}_2$, with the product filtering, is an algebra of generalized differential operators for the product $(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2, \mathcal{D}_1 \times \mathcal{D}_2)$. Then by Theorem 4.4 we see that the product is regular.

Let $\Delta_i = \mathcal{D}_i^2 + 1$ and $\Delta = (\mathcal{D}_1 \times \mathcal{D}_2)^2 + 1$. Then it is easy to check that

\[(4.4) \quad \text{Op}^s(\mathcal{H}_1, \Delta_1) \hat{\otimes} \text{Op}^t(\mathcal{H}_2, \Delta_2) \subseteq \text{Op}^{s+t}(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2, \Delta).\]

Hence $\mathcal{D}_1 \hat{\otimes} \mathcal{D}_2$ is a filtered subalgebra of $\text{Op}(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2, (\mathcal{D}_1 \times \mathcal{D}_2)^2 + 1)$. For $P_1 \in \mathcal{D}_1^k$, $P_2 \in \mathcal{D}_2^l$, we see that

\[(4.5) \quad [\Delta, P_1 \hat{\otimes} P_2] = [\Delta_1, P_1] \hat{\otimes} P_2 + P_1 \hat{\otimes} [\Delta_2, P_2],\]

belongs to $(\mathcal{D}_1 \hat{\otimes} \mathcal{D}_2)^{k+l-1}$ and thus $\mathcal{D}_1 \hat{\otimes} \mathcal{D}_2$ is an algebra of generalized differential operators. Finally, for $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$, clearly $a_1 \otimes a_2$ belongs to $(\mathcal{D}_1 \hat{\otimes} \mathcal{D}_2)^0$ and so does

\[(4.6) \quad [\mathcal{D}_1 \times \mathcal{D}_2, a_1 \otimes a_2] = [\mathcal{D}_1, a_1] \hat{\otimes} a_2 + a_1 \hat{\otimes} [\mathcal{D}_2, a_2].\]

This completes the proof.

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