A BIEBERBACH THEOREM FOR CRYSTALLOGRAPHIC GROUP EXTENSIONS

JOHN G. RATCLIFFE AND STEVEN T. TSCHANTZ

ABSTRACT. In this paper we prove that for each dimension \( n \) there are only finitely many isomorphism classes of pairs of groups \((\Gamma, N)\) such that \( \Gamma \) is an \( n \)-dimensional crystallographic group and \( N \) is a normal subgroup of \( \Gamma \) such that \( \Gamma/N \) is a crystallographic group.

1. Introduction

An \( n \)-dimensional crystallographic group (\( n \)-space group) is a discrete group \( \Gamma \) of isometries of Euclidean \( n \)-space \( E^n \) whose orbit space \( E^n/\Gamma \) is compact. The \( 3 \)-space groups are the symmetry groups of crystalline structures, and so are of fundamental importance in the science of crystallography.

In response to Hilbert’s Problem 18, L. Bieberbach [2] proved that for each dimension \( n \) there are only finitely many isomorphism classes of \( n \)-space groups. In this paper we prove a relative version of Bieberbach’s theorem. We prove that for each dimension \( n \) there are only finitely many isomorphism classes of pairs of groups \((\Gamma, N)\) such that \( \Gamma \) is an \( n \)-space group and \( N \) is a normal subgroup of \( \Gamma \) such that \( \Gamma/N \) is a space group.

Our relative Bieberbach theorem has a geometric interpretation in the theory of flat orbifolds. By Theorems 7, 8, and 10 of [8] the isomorphism classes of pairs of groups \((\Gamma, N)\) such that \( \Gamma \) is an \( n \)-space group and \( N \) is a normal subgroup of \( \Gamma \) such that \( \Gamma/N \) is a space group correspond to the affine equivalence classes of geometric orbifold fibrations of compact, connected, flat \( n \)-orbifolds. Therefore, our relative Bieberbach theorem is equivalent to the theorem that for each dimension \( n \) there are only finitely many affine equivalence classes of geometric orbifold fibrations of compact, connected, flat \( n \)-orbifolds. This is known for \( n = 3 \) by the work of Conway-Friedrichs-Huson-Thurston[4] and Ratcliffe-Tschantz [8].

We now outline the proof of our relative Bieberbach theorem. Let \( m \) be a positive integer less than \( n \). Let \( M \) be an \( m \)-space group and let \( \Delta \) be an \((n - m)\)-space group. Let \( \text{Iso}(\Delta, M) \) be the set of isomorphism classes of pairs \((\Gamma, N)\) where \( N \) is a normal subgroup of an \( n \)-space group \( \Gamma \) such that \( \Gamma \) is isomorphic to \( M \) and \( \Gamma/N \) is isomorphic to \( \Delta \). As there are only finitely many isomorphism classes of the groups \( \Delta \) and \( M \) by Bieberbach’s theorem [2], it suffices to prove that \( \text{Iso}(\Delta, M) \) is finite.

Next, we define a set \( \text{Out}(\Delta, M) \) in terms of \( \text{Out}(\Delta) \) and \( \text{Out}(M) \). That the set \( \text{Out}(\Delta, M) \) is finite follows easily from a theorem of Baues and Grunewald [1] that the outer automorphism group of a crystallographic group is an arithmetic group. We define a function \( \omega : \text{Iso}(\Delta, M) \rightarrow \text{Out}(\Delta, M) \). We prove that \( \text{Iso}(\Delta, M) \) is finite by showing that the fibers of \( \omega \) are finite by a cohomology of groups argument.
2. Normal Subgroups of Space Groups

A map \( \phi : E^n \rightarrow E^n \) is an isometry of \( E^n \) if and only if there is an \( a \in E^n \) and an \( A \in O(n) \) such that \( \phi(x) = a + Ax \) for each \( x \in E^n \). We shall write \( \phi = a + A \).

In particular, every translation \( \tau = a + I \) is an isometry of \( E^n \).

Let \( \Gamma \) be an \( n \)-space group. Define \( \eta : \Gamma \rightarrow O(n) \) by \( \eta(a + A) = A \). Then \( \eta \) is a homomorphism whose kernel is the group \( T \) of translations in \( \Gamma \). The image of \( \eta \) is a finite group \( \Pi \) called the point group of \( \Gamma \).

Let \( H \) be a subgroup of an \( n \)-space group \( \Gamma \). Define the \textit{span} of \( H \) by the formula

\[
\text{Span}(H) = \text{Span}\{a \in E^n : a + I \in H\}.
\]

Note that \( \text{Span}(H) \) is a vector subspace of \( E^n \). Let \( V^\perp \) denote the orthogonal complement of \( V \) in \( E^n \).

**Theorem 1.** (Theorem 2 [8]) Let \( N \) be a normal subgroup of an \( n \)-space group \( \Gamma \), and let \( V = \text{Span}(N) \).

1. If \( b + B \in \Gamma \), then \( BV = V \).
2. If \( a + A \in N \), then \( a \in V \) and \( V^\perp \subseteq \text{Fix}(A) \).
3. The group \( N \) acts effectively on each coset \( V + x \) of \( V \) in \( E^n \) as a space group of isometries of \( V + x \).

Let \( \Gamma \) be an \( n \)-space group. The \textit{dimension} of \( \Gamma \) is \( n \). If \( N \) is a normal subgroup of \( \Gamma \), then \( N \) is a \( m \)-space group with \( m = \dim(\text{Span}(N)) \) by Theorem 1(3).

**Definition:** Let \( N \) be a normal subgroup \( N \) of an \( n \)-space group \( \Gamma \), and let \( V = \text{Span}(N) \). Then \( N \) is said to be a \textit{complete normal subgroup} of \( \Gamma \) if

\[
N = \{a + A \in \Gamma : a \in V \text{ and } V^\perp \subseteq \text{Fix}(A)\}.
\]

**Lemma 1.** (Lemma 1 [8]) Let \( N \) be a complete normal subgroup of an \( n \)-space group \( \Gamma \), and let \( V = \text{Span}(N) \). Then \( \Gamma/N \) acts effectively as a space group of isometries of \( E^n/V \) by the formula \( (N(b + B))(V + x) = V + b + Bx \).

**Remark 1.** A normal subgroup \( N \) of a space group \( \Gamma \) is complete precisely when \( \Gamma/N \) is a space group by Theorem 5 of [8].

Let \( N \) be a complete normal subgroup of an \( n \)-space group \( \Gamma \), let \( V = \text{Span}(N) \), and let \( V^\perp \) be the orthogonal complement of \( V \) in \( E^n \). Let \( \gamma \in \Gamma \). Then \( \gamma = b + B \) with \( b \in E^n \) and \( B \in O(n) \). Write \( b = \tilde{b} + b' \) with \( \tilde{b} \in V \) and \( b' \in V^\perp \). Let \( \overline{B} \) and \( B' \) be the orthogonal transformations of \( V \) and \( V^\perp \), respectively, obtained by restricting \( B \). Let \( \overline{\gamma} = \tilde{b} + \overline{B} \) and \( \gamma' = b' + B' \). Then \( \overline{\gamma} \) and \( \gamma' \) are isometries of \( V \) and \( V^\perp \), respectively.

Euclidean \( n \)-space \( E^n \) decomposes as the Cartesian product \( E^n = V \times V^\perp \). Let \( x \in E^n \). Write \( x = v + w \) with \( v \in V \) and \( w \in V^\perp \). Then

\[
(b + B)x = b + Bx = \tilde{b} + b' + Bv + Bw = (\tilde{b} + \overline{B}v) + (b' + B'w).
\]

Hence the action of \( \Gamma \) on \( E^n \) corresponds to the diagonal action of \( \Gamma \) on \( V \times V^\perp \) defined by the formula

\[
\gamma(v, w) = (\overline{\gamma}v, \gamma'w).
\]

Here \( \Gamma \) acts on both \( V \) and \( V^\perp \) via isometries. The kernel of the corresponding homomorphism from \( \Gamma \) to \( \text{Isom}(V) \) is the group

\[
K = \{b + B \in \Gamma : b \in V^\perp \text{ and } V \subseteq \text{Fix}(B)\}.
\]
We call $K$ the kernel of the action of $\Gamma$ on $V$. The group $K$ is a normal subgroup of $\Gamma$. The action of $\Gamma$ on $V$ induces an effective action of $\Gamma/K$ on $V$ via isometries. The group $\Gamma/K$ acts on $V$ as a discrete group of isometries if and only if $\Gamma/K$ is a finite group by Theorem 3(4) of [10].

The group $N$ is the kernel of the action of $\Gamma$ on $V^\perp$, and so the action of $\Gamma$ on $V^\perp$ induces an effective action of $\Gamma/N$ on $V^\perp$ via isometries. Orthogonal projection from $E^n$ to $V^\perp$ induces an isometry from $E^n/V$ to $V^\perp$. Hence $\Gamma/N$ acts on $V^\perp$ as a space group of isometries by Lemma 1.

Let $\overline{\Gamma} = \{\overline{\gamma} : \gamma \in \Gamma\}$. If $\gamma \in \Gamma$, then $(\overline{\gamma})^{-1} = \overline{\gamma^{-1}}$, and if $\gamma_1, \gamma_2 \in \Gamma$, then $\overline{\gamma_1 \gamma_2} = \overline{\gamma_1} \overline{\gamma_2}$. Hence $\overline{\Gamma}$ is a subgroup of $\text{Isom}(V)$. The map $B : \Gamma \to \overline{\Gamma}$ defined by $B(\gamma) = \overline{\gamma}$ is an epimorphism with kernel $K$. The group $\overline{\Gamma}$ is a discrete subgroup of $\text{Isom}(V)$ if and only if $\Gamma/N$ is finite by Theorem 3(4) of [10].

Let $\Gamma' = \{\gamma' : \gamma \in \Gamma\}$. If $\gamma \in \Gamma$, then $(\gamma')^{-1} = (\gamma^{-1})'$, and if $\gamma_1, \gamma_2 \in \Gamma$, then $\gamma_1' \gamma_2' = (\gamma_1 \gamma_2)'$. Hence $\Gamma'$ is a subgroup of $\text{Isom}(V^\perp)$. The map $P' : \Gamma \to \Gamma'$ defined by $P'(\gamma) = \gamma'$ is epimorphism with kernel $N$, since $N$ is a complete normal subgroup of $\Gamma$. Hence $P'$ induces an isomorphism $P : \Gamma/N \to \Gamma'$ defined by $P(N\gamma) = \gamma'$.

The group $\Gamma'$ is a space group of isometries of $V^\perp$ with $V^\perp/\Gamma' = V^\perp/(\Gamma/N)$.

Let $\overline{\mathcal{N}} = \{\overline{\nu} : \nu \in \mathcal{N}\}$. Then $\overline{\mathcal{N}}$ is a subgroup of $\text{Isom}(V)$. The map $B : \mathcal{N} \to \overline{\mathcal{N}}$ defined by $B(\nu) = \overline{\nu}$ is an isomorphism. The group $\overline{\mathcal{N}}$ is a space group of isometries of $V$ with $V/\overline{\mathcal{N}} = V/N$.

The action of $\Gamma$ on $V$ induces an action of $\Gamma/N$ on $V/N$ defined by 

$$(N\gamma)(N\nu) = N\overline{\gamma} \overline{\nu}.$$ 

The action of $\Gamma/N$ on $V/N$ determines a homomorphism 

$$\Xi : \Gamma/N \to \text{Isom}(V/N)$$ 

defined by $\Xi(N\gamma)(N\nu) = N\overline{\gamma} \overline{\nu}$, where $\overline{\gamma} : V/N \to V/N$ is defined by $\overline{\gamma}(N\nu) = N\overline{\nu}$.

**Theorem 2.** Let $M$ be an $m$-space group, let $\Delta$ be an $(n-m)$-space group, and let $\Theta : \Delta \to \text{Isom}(E^m/M)$ be a homomorphism. Identify $E^n$ with $E^m \times E^{n-m}$, and extend $M$ to a subgroup $N$ of $\text{Isom}(E^n)$ such that the point group of $N$ acts trivially on $(E^m)^\perp = E^{n-m}$. Then there exists a unique $n$-space group $\Gamma$ containing $N$ as a complete normal subgroup such that $\Gamma' = \Delta$, and if $\Xi : \Gamma/N \to \text{Isom}(E^m/N)$ is the homomorphism induced by the action of $\Gamma/N$ on $E^m/N$, then $\Xi = \Theta P$ where $P : \Gamma/N \to \Gamma'$ is the isomorphism defined by $P(N\gamma) = \gamma'$ for each $\gamma \in \Gamma$.

**Proof.** Let $\delta \in \Delta$. By Lemma 1 of [9], there exists $\hat{\delta} \in N_{E}(M)$ such that $\hat{\Theta}(\delta) = \Theta(\delta)$. The isometry $\hat{\delta}$ is unique up to multiplication by an element of $M$. Let $\delta = \hat{\delta} + D$, with $\hat{\delta} \in E^{n-m}$ and $D \in O(n-m)$, and let $\delta = \hat{\delta} + \tilde{D}$, with $\tilde{D} \in E^m$ and $\tilde{D} \in O(m)$. Let $\tilde{\delta} = \hat{\delta} + D$. Then $\overline{\delta} \overline{\delta}' = \delta$. Then $\hat{\delta}$ is an isometry of $E^n$ such that $\delta = \hat{\delta}$ and $\overline{\delta}' = \delta$. The isometry $\hat{\delta}$ is unique up to multiplication by an element of $N$. We have that 

$$\hat{\delta}^{-1} = -\tilde{D}^{-1} \tilde{\delta} + \tilde{D}^{-1} = -\tilde{D}^{-1} \hat{\delta} + D^{-1} \tilde{D} \times D^{-1},$$

and so $\overline{\hat{\delta}}^{-1} = (\hat{\delta})^{-1}$ and $(\hat{\delta})^{-1}' = \delta^{-1}$. We have that 

$$\delta N \delta^{-1} = \delta \hat{\delta} N \hat{\delta}^{-1} = \hat{\delta} M \hat{\delta}^{-1} = M \overline{N}$$

and 

$$(\delta N \hat{\delta}^{-1})' = \hat{\delta} N'(\hat{\delta}^{-1})' = \delta I'(\delta^{-1})' = \delta \{I'\} \delta^{-1} = \{I'\}.$$
Therefore $\hat{\delta}N\hat{\delta}^{-1} = N$.

Let $\Gamma$ be the subgroup of $\text{Isom}(E^n)$ generated by $N \cup \{\delta : \delta \in \Delta\}$. Then $\Gamma$ contains $N$ as a normal subgroup, and the point group of $\Gamma$ leaves $E^m$ invariant. Suppose $\gamma \in \Gamma$. Then there exists $\nu \in N$ and $\delta_1, \ldots, \delta_k \in \Delta$ and $\epsilon_1, \ldots, \epsilon_k \in \{\pm 1\}$ such that $\gamma = \nu\delta_1^\epsilon_1 \cdots \delta_k^\epsilon_k$. Then we have

$$\gamma' = \nu'(\delta_1')^\epsilon_1 \cdots (\delta_k')^\epsilon_k = \delta_1^\epsilon_1 \cdots \delta_k^\epsilon_k.$$ 

Hence $\Gamma' = \Delta$, and we have an epimorphism $P' : \Gamma \to \Delta$ defined by $P'(\gamma) = \gamma'$. The group $N$ is in the kernel of $P'$, and so $P'$ induces an epimorphism $P : \Gamma/N \to \Delta$ defined by $P(N\gamma) = \gamma'$. Suppose $P(N\gamma) = 1'$. By Lemma 1 of [9], we have that

$$P(N\gamma) = 1' \implies \gamma' = 1' \implies \delta_1^\epsilon_1 \cdots \delta_k^\epsilon_k = 1' \implies \Theta(\delta_1)^\epsilon_1 \cdots \Theta(\delta_k)^\epsilon_k = \overline{T},$$

$$\implies (\delta_1)^\epsilon_1 \cdots (\delta_k)^\epsilon_k = \overline{T},$$

$$\implies M(\delta_1)^\epsilon_1 \cdots M(\delta_k)^\epsilon_k = M$$

$$\implies M\overline{\delta_1}^\epsilon_1 \cdots \overline{\delta_k}^\epsilon_k = M \implies M\overline{\gamma} = M \implies \overline{\gamma} \in \Delta.$$

As $\gamma' = 1'$ and $\overline{\gamma} \in \Delta$, we have that $\gamma \in N$. Thus $P$ is an isomorphism.

We next show that $\Gamma$ acts discontinuously on $E^n$. Let $C$ be a compact subset of $E^n$. Let $K$ and $L$ be the orthogonal projections of $C$ into $E^m$ and $E^{n-m}$, respectively. Then $C \subseteq K \times L$. As $\Delta$ acts discontinuously on $E^{n-m}$, there exists only finitely many elements $\delta_1, \ldots, \delta_k$ of $\Delta$ such that $L \cap \delta_i L \neq \emptyset$ for each $i$. Let $\gamma_i = \delta_i$ for $i = 1, \ldots, k$. The set $K_i = K \cap \gamma_i(K)$ is compact for each $i = 1, \ldots, k$. As $N$ acts discontinuously on $E^m$, there is a finite subset $F_i$ of $N$ such that $K_i \cap \nu K_i \neq \emptyset$ for each $i$. Hence $K \cap \nu K \neq \emptyset$. Let $\nu_i \in F_i$ for each $i$ and $\nu \in N$. Then $L \cap \gamma' L \neq \emptyset$, and so $\gamma' = \delta_i$ for some $i$. Hence $\gamma = \nu\gamma_i$ for some $\nu \in N$. Now we have that $K \cap \gamma_i K \neq \emptyset$, and so $K \cap \nu \gamma_i K \neq \emptyset$. Hence $\nu \in F_i$. Therefore $\gamma \in F$. Thus $\Gamma$ acts discontinuously on $E^n$. Therefore $\Gamma$ is a discrete subgroup of $\text{Isom}(E^n)$ by Theorem 5.3.5 of [7].

Let $D_M$ and $D_\Delta$ be fundamental domains for $M$ in $E^m$ and $\Delta$ in $E^{n-m}$, respectively. Then their topological closures $\overline{D_M}$ and $\overline{D_\Delta}$ are compact sets. Let $x \in E^n$. Write $x = \overline{x} + x'$ with $\overline{x} \in E^m$ and $x' \in E^{n-m}$. Then there exist $\delta \in \Delta$ such that $\delta x' \in \overline{D_\Delta}$, and there exist $\nu \in N$ such that $\nu\delta \overline{x} \in \overline{D_M}$. We have that

$$\nu\delta x = \nu\delta \overline{x} + \nu\delta x' = \nu\delta \overline{x} + \delta x' \in \overline{D_M} \times \overline{D_\Delta}.$$ 

Hence the quotient map $\pi : E^n \to E^n/\Gamma$ maps the compact set $\overline{D_M} \times \overline{D_\Delta}$ onto $E^n/\Gamma$. Therefore $E^n/\Gamma$ is compact. Thus $\Gamma$ is an $n$-space group.

Let $\Xi : \Gamma/N \to \text{Isom}(E^n/N)$ be the homomorphism induced by the action of $\Gamma/N$ on $E^n/N$. Let $\gamma \in \Gamma$. Then there exists $\delta \in \Delta$ such that $N\gamma = N\delta$. Then $\gamma' = (\delta)^\epsilon = \delta$, and we have that $\Xi = \Theta P$, since

$$\Xi(N\gamma) = \Xi(N\delta) = (\delta)_* = \delta_* = \Theta(\delta) = \Theta(\gamma') = \Theta P(N\gamma).$$

Suppose $\gamma$ is an isometry of $E^n$ such that $\gamma N\gamma^{-1} = N$ and $\gamma' \in \Delta$ and $\tau_* = \Theta(\gamma')$. Then $\gamma' \in \Gamma$. Now $\tau_* = \gamma_*'$, and so $\overline{\tau} = \overline{\gamma'}$ for some $\nu \in N$ by Lemma 1 of [9]. Then $\gamma = \nu\gamma'$. Hence $\gamma \in \Gamma$. Thus $\Gamma$ is the unique $n$-space group that contains $N$ as a complete normal subgroup such that $\Gamma' = \Delta$ and $\Xi = \Theta P$. \qed
3. Isomorphisms of Pairs of Space Groups

An affinity $\alpha$ of $E^n$ is a map $\alpha: E^n \to E^n$ for which there is an element $a \in E^n$ and a matrix $A \in \text{GL}(n, \mathbb{R})$ such that $\alpha(x) = a + Ax$ for all $x \in E^n$. We write simply $\alpha = a + A$. The set $\text{Aff}(E^n)$ of all affinities of $E^n$ is a group that contains $\text{Isom}(E^n)$ as a subgroup.

Let $N_i$ be a complete normal subgroup of an $n$-space group $\Gamma_i$ for $i = 1, 2$. We want to know when $(\Gamma_1, N_1)$ is isomorphic to $(\Gamma_2, N_2)$, that is, when there is an isomorphism $\zeta: \Gamma_1 \to \Gamma_2$ such that $\zeta(N_1) = N_2$. By a Theorem of Bieberbach an isomorphism $\zeta: \Gamma_1 \to \Gamma_2$ is equal to conjugation by an affinity $\phi$ of $E^n$. In this section, we determine necessary and sufficient conditions such that there exists an affinity $\phi$ of $E^n$ such that $\phi(\Gamma_1, N_1)\phi^{-1} = (\Gamma_2, N_2)$.

Let $\phi$ be an affinity of $E^n$ such that $\phi(\Gamma_1, N_1)\phi^{-1} = (\Gamma_2, N_2)$. Write $\phi = c + C$ with $c \in E^n$ and $C \in \text{GL}(n, \mathbb{R})$. Let $V_i = \text{Span}(N_i)$, for $i = 1, 2$. Let $a + I \in N_1$. Then $\phi(a + I)\phi^{-1} = Ca + I$. Hence $CV_1 \subseteq V_2$. Let $\overline{C}: V_1 \to V_2$ be the linear transformation obtained by restricting $C$. Let $\overline{C}': V_1^{\perp} \to V_2$ and $C' : V_1^{\perp} \to V_2^{\perp}$ be the linear transformations obtained by restricting $C$ to $V_1^{\perp}$ followed by the orthogonal projections to $V_2$ and $V_2^{\perp}$, respectively. Write $c = \tau + c'$ with $\tau \in V_2$ and $c' \in V_2^{\perp}$. Let $\overline{\phi}: V_1 \to V_2$ and $\phi': V_1^{\perp} \to V_2^{\perp}$ be the affine transformations defined by $\overline{\phi} = \tau + \overline{C}$ and $\phi' = c' + C'$.

**Lemma 2.** Let $N_i$ be a complete normal subgroup of an $n$-space group $\Gamma_i$, with $V_i = \text{Span}(N_i)$, for $i = 1, 2$. Let $\phi = c + C$ be an affinity of $E^n$ such that $\phi(\Gamma_1, N_1)\phi^{-1} = (\Gamma_2, N_2)$. Then $\overline{C}$ and $C'$ are invertible, with $\overline{C}^{-1} = \overline{C}^{-1}$ and $(C')^{-1} = (C^{-1})'$, and $(\overline{C}^{-1})' = -\overline{C}^{-1}\overline{C}'(C')^{-1}$. If $b + B \in \Gamma_1$, then $\overline{C}B' = CBC^{-1}C'$. Moreover $\overline{C}V_1^{\perp} \subseteq \text{Span}(Z(N_2))$.

**Proof.** We have that

$$\dim V_1 = \dim N_1 = \dim N_2 = \dim V_2.$$ 

Let $y \in E^n$ and write $y = \overline{\tau} + y'$ with $\overline{\tau} \in V_2$ and $y' \in V_2^{\perp}$. Then

$$\overline{\tau} = CC^{-1}(\overline{\tau}) = \overline{CC^{-1}(\tau)},$$

and so $\overline{C}$ is invertible with $\overline{C}^{-1} = \overline{C}^{-1}$. We have that

$$y' = CC^{-1}y' = C((C^{-1})'(y') + (C^{-1})'(y')) = C(C^{-1})'(y') + C(C^{-1})'(y') = C(C^{-1})'(y') + \overline{C}(C^{-1})'(y') + C'(C^{-1})'(y').$$

Hence $C'$ is invertible, with $(C')^{-1} = (C^{-1})'$, and $\overline{C}(C^{-1})' + \overline{C}'(C^{-1})' = 0$. Therefore $(C^{-1})' = -\overline{C}^{-1}\overline{C}'(C')^{-1}$. 

Let $\gamma = b + B \in \Gamma_1$. Then $\phi \gamma \phi^{-1} \in \Gamma_2$. Let $w \in V_2^\perp$. Then we have

\[
\begin{align*}
 w &= CBC^{-1}w \\
 &= CB((C^{-1})'(w) + (C^{-1})'(w)) \\
 &= C(B(C^{-1})'(w) + B'(C^{-1})'(w)) \\
 &= CB(C^{-1})'(w) + CB'(C^{-1})'(w) \\
 &= CB(C^{-1})'(w) + CB'(C^{-1})'(w) + C'B'(C^{-1})'(w).
\end{align*}
\]

As $\phi \gamma \phi^{-1}(w) \in V_2^\perp$, we have that

\[
CB(C^{-1})'(w) + CB'(C^{-1})' = 0.
\]

Therefore

\[
CB(C^{-1})'(w) + CB'(C^{-1})' = 0.
\]

Hence $CB' = CB(C^{-1})'$. 

Now suppose $\gamma \in N_1$. Then $\gamma = I'$ by Theorem 1(2). Hence $BC^{-1}C = CB(C^{-1})'$. 

By Lemma 5 of [9], we deduce that $CB^{-1}CBV_1^\perp \subseteq \text{Span}(Z(N_1))$. As $\phi Z(N_1) \phi^{-1} = Z(N_2)$, we have that $CBV_1^\perp \subseteq \text{Span}(Z(N_2))$. 

**Theorem 3.** Let $N_i$ be a complete normal subgroup of an $n$-space group $\Gamma_i$, with $V_i = \text{Span}(N_i)$ for $i = 1, 2$. Let $\Xi_i : \Gamma_i/N_i \to \text{Isom}(V_i/N_i)$ be the homomorphism induced by the action of $\Gamma_i/N_i$ on $V_i/N_i$ for $i = 1, 2$. Let $\alpha : V_1 \to V_2$, and $\beta : V_1^\perp \to V_2^\perp$ be affinities such that $\alpha N_1 \alpha^{-1} = N_2$ and $\beta \Gamma_1 \beta^{-1} = \Gamma_2$. 

Let $\alpha_* : V_1/N_1 \to V_2/N_2$ be the affinity defined by $\alpha_*(N_1 v) = N_2 \alpha(v)$, and let $\beta_* : \text{Aff}(V_1/N_1) \to \text{Aff}(V_2/N_2)$ be the isomorphism defined by $\alpha_*(\chi) = \beta_* \chi$. 

Let $\gamma = b + B \in \Gamma_1$, then $DB' = CB^{-1}C D$. 

Let $\Xi_i : \Gamma_i/N_i \to K_i$ be the isomorphism defined by $P_1(N_i \gamma) = \gamma'$ for each $i = 1, 2$, and let $p_1 : \Gamma_1/N_1 \to V_1^\perp$ be defined by $p_1(N_1 (b + B)) = b'$. 

Let $K_i$ be the connected component of the identity of the Lie group $\text{Isom}(V_i/N_i)$ for each $i = 1, 2$. 

Note that $K_i = \{v + T \} \subseteq \text{Span}(Z(N_i))$ by Theorem 1 of [9]. Then the following are equivalent:

1. There exists an affinity $\phi = c + C$ of $E^n$ such that $\phi(\Gamma_1, N_1) \phi^{-1} = (\Gamma_2, N_2)$, with $\phi = \alpha$, and $\phi' = \beta$, and $\phi = CB^{-1}$. 

2. We have that

\[
\Xi_2 P_2^{-1} \beta_* P_1 = (Dp_1)_* \alpha_2 \Xi_1
\]

with $(Dp_1)_* : \Gamma_1/N_1 \to K_2$ a crossed homomorphism defined by

\[
(Dp_1)_*(N_1 (b + B)) = (Db' + T)_*
\]

and $\Gamma_1/N_1$ acting on $K_2$ by $N_1 (b + B) (v + T)_* = (CB^{-1} v + T)_*$ for each $b + B \in \Gamma_1$ and $v \in \text{Span}(Z(N_2))$.

3. We have that

\[
\alpha_2^{-1} \Xi_2 P_2^{-1} \beta_* P_1 = (C^{-1} Dp_1)_* \Xi_1
\]

with $(C^{-1} Dp_1)_* : \Gamma_1/N_1 \to K_1$ a crossed homomorphism defined by

\[
(C^{-1} Dp_1)_*(N_1 (b + B)) = (C^{-1} D(b') + T)_*.
\]
and $\Gamma_1/N_1$ acting on $K_1$ by $N_1(b+B)(v+T)_* = (\overline{B}v+T)_*$ for each $b+B \in \Gamma_1$ and $v \in \operatorname{Span}(Z(N_1))$.

**Proof.** Suppose there exists an affinity $\phi = c + C$ of $E^n$ such that $\phi(\Gamma_1, N_1)\phi^{-1} = (\Gamma_2, N_2)$, with $\overline{c} = \alpha$, and $\phi' = \beta$, and $\overline{C} = D$. Let $\gamma = b + B \in \Gamma_1$. Then we have

$$\phi\gamma\phi^{-1} = (c + C)(b + B)(c + C)^{-1} = Cb + (I - CBC^{-1})c + CBC^{-1}.$$

Hence we have

$$\overline{\phi\gamma\phi^{-1}} = \overline{c} + \overline{C}\overline{b}' + (I - CBC^{-1})\overline{c} + CBC^{-1}$$

$$= (\overline{C}\overline{b}' + \overline{T})(\overline{c} + \overline{C})(\overline{b} + B)(\overline{c} + \overline{C})^{-1}$$

$$= (D\overline{b}' + \overline{T})\overline{c} + D(C^{-1}c + C^{-1}b') + D^{-1} + D^{-1}$$

and

$$\phi\gamma\phi^{-1}' = C'\overline{b}' + (I' - C'B'(C')^{-1})c' + C'B'(C')^{-1}$$

$$= (c' + C')(b' + B')(c' + C')^{-1}$$

$$= \phi'\gamma'(\phi')^{-1} = \beta\gamma'\beta^{-1}.$$

Observe that

$$\Xi_2P_2^{-1}\beta P_1(\gamma \gamma) = \Xi_2P_2^{-1}\beta(\gamma')$$

$$= \Xi_2P_2^{-1}(\beta\gamma')(\gamma')$$

$$= \Xi_2P_2^{-1}(\phi\gamma\phi^{-1})$$

$$= \Xi_2P_2^{-1}(\phi\gamma\phi^{-1})$$

$$= \Xi_2P_2^{-1}(\phi\gamma\phi^{-1})$$

and

$$\phi(x) = \alpha(x) + \beta(x')$$

for each $x \in E^n$, where $x = x + x'$ with $x \in V_1$ and $x' \in V_1^\perp$. Then $\phi$ is an affinity of $E^n$, with

$$\phi(y) = \overline{y} - \overline{C}^{-1}D\beta^{-1}(y') + \beta^{-1}(y')$$

for each $y \in E^n$ where $y = \overline{y} + y'$ with $\overline{y} \in V_2$ and $y' \in V_2^\perp$.

Write $\gamma = \gamma' + C'$ with $\gamma' \in V_1^\perp$ and $C' : V_1^\perp \to V_2^\perp$ a linear isomorphism. Write $\phi = c + C$ with $c \in E^n$ and $C$ a linear isomorphism of $E^n$. Then $c = \overline{c} + c'$ and $C = \overline{C} + C$ for each $x \in E^n$ with $\overline{x} \in V_1$ and $x' \in V_1^\perp$. We have that $\overline{\phi} = \alpha$, and $\phi' = \beta$ and $\overline{C} = D$. We also have $\phi^{-1} = -C^{-1}c + C^{-1}$ and

$$\overline{\phi^{-1}}y = \overline{C}^{-1}y - \overline{C}^{-1}D(C')^{-1}y' + (C')^{-1}y'$$

for all $y \in E^n$ with $\overline{y} \in V_2$ and $y' \in V_2^\perp$.

Let $\gamma \in \Gamma_1$. Write $\gamma = b + B$ with $b \in E^n$ and $B \in O(n)$. Then we have

$$\phi\gamma\phi^{-1} = (c + C)(b + B)(c + C)^{-1} = Cb + (I - CBC^{-1})c + CBC^{-1}.$$
Let \( y \in E^n \). Write \( y = \overline{y} + y' \) with \( \overline{y} \in V_2 \) and \( y' \in V_2^\perp \). Then we have that

\[
\begin{align*}
CBC^{-1} &= CBC^{-1} \overline{y} - C^{-1}D(C')^{-1}y' + (C')^{-1}y' \\
&= C(BC^{-1} \overline{y} - BC^{-1}D(C')^{-1}y' + B'(C')^{-1}y' \\
&= \frac{CBC^{-1}}{\overline{y}} - CB \frac{BC^{-1}}{\overline{y}} D(C')^{-1}y' + DB'(C')^{-1}y' + C'B'(C')^{-1}y' \\
&= \frac{CBC^{-1}}{\overline{y}} + C'B'(C')^{-1}y'.
\end{align*}
\]

Hence \( CBC^{-1} = \overline{CBC^{-1}} \times C'B'(C')^{-1} \) as a linear isomorphism of \( E^n = V_2 \times V_2^\perp \).

Moreover, we have

\[
\begin{align*}
\overline{\phi \gamma \phi^{-1}} &= CB + CB' + (C - CB)\overline{\gamma} + CBC^{-1} \\
&= (CB' + C')\overline{\gamma} + \overline{\gamma} + CB + CB^{-1} \\
&= (DB' + T)\overline{\phi \gamma \phi^{-1}} = (DB' + T)\alpha \gamma \alpha^{-1},
\end{align*}
\]

and

\[
\begin{align*}
(\phi \gamma \phi^{-1})' &= C'B' + (I - C'B'(C')^{-1})c' + C'B'(C')^{-1} \\
&= (c' + C')\overline{c} + B'(c' + c') \\
&= \phi' \gamma' \phi^{-1} = \beta \gamma \beta^{-1}.
\end{align*}
\]

As \( \Xi_2 \mathcal{P}_2^{-1} \beta \mathcal{P}_1 = (Dp_1)_\ast \mathcal{P}_2^{-1} \), we have that \( (\alpha \gamma \alpha^{-1})_\ast \) is an isometry of \( V_2/N_2 \). By Lemmas 1 and 7 of [7], we have that \( \alpha \gamma \alpha^{-1} \) is an isometry of \( V_2 \). Hence \( CBC^{-1} \) is an orthogonal transformation of \( V_2 \).

As \( \beta \Gamma_1 \beta^{-1} = \Gamma_2 \), we have that \( C'B'(C')^{-1} \) is an orthogonal transformation of \( V_2^\perp \). Hence \( CBC^{-1} = \overline{CBC^{-1}} \times C'B'(C')^{-1} \) is an orthogonal transformation of \( E^n = V_2 \times V_2^\perp \). Therefore \( \phi \gamma \phi^{-1} \) is an isometry of \( E^n \) for each \( \gamma \in \Gamma_1 \).

As \( \Gamma_1 \) acts discontinuously on \( E^n \) and \( \phi \) is a homeomorphism of \( E^n \), we have that \( \phi \Gamma_1 \phi^{-1} \) acts discontinuously on \( E^n \). Therefore \( \phi \Gamma_1 \phi^{-1} \) is a discrete subgroup of \( \text{Isom}(E^n) \) by Theorem 5.3.5 of [7]. Now \( \phi \) induces a homeomorphism \( \phi_\ast : E^n / \Gamma_1 \to E^n / \phi \Gamma_1 \phi^{-1} \) defined by \( \phi_\ast (\Gamma_1 x) = \phi \Gamma_1 \phi^{-1} \phi(x) \). Hence \( E^n / \phi \Gamma_1 \phi^{-1} \) is compact. Therefore \( \phi \Gamma_1 \phi^{-1} \) is a \( n \)-space group.

Now \( \phi_\ast : \Gamma_1 \to \phi \Gamma_1 \phi^{-1} \), defined by \( \phi_\ast (\gamma) = \phi \gamma \phi^{-1} \), is an isomorphism that maps the normal subgroup \( N_1 \) to the normal subgroup \( \phi N_1 \phi^{-1} \) of \( \phi \Gamma_1 \phi^{-1} \), and \( \phi \Gamma_1 \phi^{-1} / \phi N_1 \phi^{-1} \) is isomorphic to \( \Gamma_1 / N_1 \). Hence \( \phi \Gamma_1 \phi^{-1} / \phi N_1 \phi^{-1} \) is a space group. Therefore \( \phi N_1 \phi^{-1} \) is a complete normal subgroup of \( \phi \Gamma_1 \phi^{-1} \) by Theorem 5 of [8].

Now suppose \( \nu = a + A \in N_1 \). Then \( a' = 0 \) and \( A' = A' \), and so \( \nu' = I' \). Hence \( \phi \nu \phi^{-1} = \alpha \gamma \alpha^{-1} \) and \( (\phi \nu \phi^{-1})' = I' \). As \( \alpha \gamma \alpha^{-1} = N_2 \), we have that \( \phi N_1 \phi^{-1} = N_2 \).

Moreover, as \( \beta \Gamma_1 \beta^{-1} = \Gamma_2 \), we have that \( (\phi \Gamma_1 \phi^{-1})' = \Gamma_2 \).

Let \( \Xi : \phi \Gamma_1 \phi^{-1} / N_2 \to \text{Isom}(V_2 / N_2) \) be the homomorphism induced by the action of \( \phi \Gamma_1 \phi^{-1} / N_2 \) on \( V_2 / N_2 \). Let \( \gamma = b + B \in \Gamma_1 \), and let \( P : \phi \Gamma_1 \phi^{-1} / N_2 \to \Gamma_2 \) be the
isomorphism defined by \( P(N_2 \phi \gamma \phi^{-1}) = (\phi \gamma \phi^{-1})' \). Then we have that
\[
\Xi P^{-1}(\beta \gamma' \beta^{-1}) = \Xi P^{-1}((\phi \gamma \phi^{-1})')
\]
\[
= \Xi(N_2 \phi \gamma \phi^{-1})
\]
\[
= (P(N_2 \phi \gamma \phi^{-1})_*
\]
\[
= ((\beta \gamma' \beta^{-1})_* = (Db' + T) \alpha \gamma \alpha^{-1} \gamma_*
\]
\[
= (Db' + T)_* \alpha \gamma \alpha^{-1} \gamma_*$
\]
\[
= (Dp_1(N_1(b + B)))_*, \alpha_2(\gamma_*)
\]
\[
= (Dp_1(N_1 \gamma))_*, \alpha_2(\Xi_1(N_1 \gamma))
\]
\[
= \Xi_2 P^{-1}_2 \beta_2 P_1(N_1 \gamma) = \Xi_2 P^{-1}_2(\beta \gamma' \beta^{-1}).
\]

Hence we have that \( \Xi^{-1} P^{-1} = \Xi_2 P^{-1}_2 \). Therefore \( \phi \Gamma_1 \phi^{-1} = \Gamma_2 \) by Theorem 2. Thus \( \phi(N_1, N_1 \phi^{-1}) = (N_2, N_2) \).

Let \( \gamma = b + B \) and \( \gamma_1 = b_1 + B_1 \) be elements of \( \Gamma_1 \). Then we have that
\[
(Dp_1)_* (N_1 \gamma N_1 \gamma_1) = (Dp_1)_* (N_1(b + B b_1 + B B_1))
\]
\[
= (D(b + B b_1) + B T)_*,
\]
\[
= (D(b' + B b'_1) + T)_*,
\]
\[
= (Db' + DB b'_1 + T)_*,
\]
\[
= (Db' + CBC^{-1} Db'_1 + T)_*,
\]
\[
= (Db' + T)_* (CBC^{-1} Db'_1 + T)_*,
\]
\[
= (Db' + T)_* (N_1(b + B))(Db'_1 + T)_*,
\]
\[
= (Dp_1)_* (N_1 \gamma)(Dp_1)_* (N_1 \gamma_1).
\]

Therefore \( (Dp_1)_* : \Gamma_1 / N_1 \rightarrow K_2 \) is a crossed homomorphism. Thus statements (1) and (2) are equivalent.

The equation \( \Xi_2 P^{-1}_2 \beta_2 P_1 = (Dp_1)_* \alpha_2 \Xi_1 \) is equivalent to the equation
\[
\alpha_2^{-1} \Xi_2 P^{-1}_2 \beta_2 P_1 = \alpha_2^{-1} (Dp_1)_* \alpha_2 \Xi_1.
\]

Observe that
\[
\alpha_2^{-1} (Dp_1)_* \alpha_2 \Xi_1 (N_1 \gamma) = \alpha_2^{-1} ((Dp_1)_* (N_1 \gamma) \alpha_2 \Xi_1 (N_1 \gamma))
\]
\[
= \alpha_2^{-1} ((Db' + T)_* \alpha_2 \Xi_1)
\]
\[
= \alpha_2^{-1} ((Db' + T)_* \alpha_2 \Xi_1 \alpha^{-1})
\]
\[
= \alpha_2^{-1} ((Db' + T)_* \alpha_2 \Xi_1 \alpha^{-1}) \alpha_*
\]
\[
= (\alpha^{-1} (Db' + T)_* \alpha_2 \Xi_1)
\]
\[
= (C^{-1} Db' + T)_* \Xi_1 (N_1 \gamma)
\]
\[
= (C^{-1} Dp_1)_* (N_1 \gamma) \Xi_1 (N_1 \gamma).
\]

Hence we have that \( \alpha_2^{-1} (Dp_1)_* \alpha_2 \Xi_1 = (C^{-1} Dp_1)_* \Xi_1 \). By the same argument as with \( (Dp_1)_* : \Gamma_1 / N_1 \rightarrow K_2 \), we have that \( (C^{-1} Dp_1)_* : \Gamma_1 / N_1 \rightarrow K_1 \) is a crossed homomorphism. Thus (2) and (3) are equivalent. \( \square \)
4. Outer Automorphism Groups of Space Groups

Through this section, let \( m \) be a positive integer less than \( n \). Let \( M \) be an \( m \)-space group and let \( \Delta \) be an \( (n - m) \)-space group.

**Definition:** Define \( \text{Iso}(\Delta, M) \) to be the set of isomorphism classes of pairs \((\Gamma, N)\) where \( N \) is a complete normal subgroup of an \( n \)-space group \( \Gamma \) such that \( N \) is isomorphic to \( M \) and \( \Gamma/N \) is isomorphic to \( \Delta \). We denote the isomorphism class of a pair \((\Gamma, N)\) by \([\Gamma, N]\).

Let \( N \) be a complete normal subgroup of an \( n \)-space group \( \Gamma \), and let \( \text{Out}_E(N) \) be the Euclidean outer automorphism group of \( N \) defined in §4 of [9]. The group \( \text{Out}_E(N) \) is finite by Theorem 2 of [9]. The action of \( \Gamma \) on \( N \) by conjugation induces a homomorphism

\[
\mathcal{O}: \Gamma/N \to \text{Out}_E(N)
\]
defined by \( \mathcal{O}(N\gamma) = \gamma_*\text{Inn}(N) \) where \( \gamma_*(\nu) = \gamma
\nu\gamma^{-1} \) for each \( \gamma \in \Gamma \) and \( \nu \in N \). Let \( \alpha : N_1 \to N_2 \) be an isomorphism. Then \( \alpha \) induces an isomorphism

\[
\alpha_\#: \text{Out}(N_1) \to \text{Out}(N_2)
\]
defined by \( \alpha_\#(\zeta\text{Inn}(N_1)) = \alpha\zeta\alpha^{-1}\text{Inn}(N_2) \) for each \( \zeta \in \text{Aut}(N_1) \).

**Lemma 3.** Let \( N_i \) be a complete normal subgroup of an \( n \)-space group \( \Gamma_i \) for \( i = 1, 2 \). Let \( \mathcal{O}_i : \Gamma_i/N_i \to \text{Out}_E(N_i) \) be the homomorphism induced by the action of \( \Gamma_i \) on \( N_i \) by conjugation for \( i = 1, 2 \), and let \( \alpha : N_1 \to N_2 \) and \( \phi : \Gamma_1 \to \Gamma_2 \) and \( \beta : \Gamma_1/N_1 \to \Gamma_2/N_2 \) be isomorphisms such that the following diagram commutes

\[
\begin{array}{ccc}
1 & \to & N_1 & \to & \Gamma_1 & \to & \Gamma_1/N_1 & \to & 1 \\
\downarrow & & \downarrow \alpha & & \downarrow \phi & & \downarrow \beta \\
1 & \to & N_2 & \to & \Gamma_2 & \to & \Gamma_2/N_2 & \to & 1,
\end{array}
\]

where the horizontal maps are inclusions and projections, then \( \mathcal{O}_2 = \alpha_\#\mathcal{O}_1\beta^{-1} \).

**Proof.** Let \( \gamma \in \Gamma_1 \). Then we have that

\[
\mathcal{O}_2(N_2\phi(\gamma)) = \phi(\gamma)_*\text{Inn}(N_2),
\]

whereas

\[
\alpha_\#\mathcal{O}_1\beta^{-1}(N_2\phi(\gamma)) = \alpha_\#\mathcal{O}_1(N_1\gamma) = \alpha_\#(\gamma_*\text{Inn}(N_1)) = \alpha\gamma_*\alpha^{-1}\text{Inn}(N_2).
\]

If \( \nu \in N \), then

\[
\alpha\gamma_*\alpha^{-1}(\nu) = \alpha\gamma_*\alpha^{-1}(\nu) = \alpha(\gamma\alpha^{-1}(\nu)\gamma^{-1}) = \phi(\gamma)\nu\phi(\gamma)^{-1} = \phi(\gamma)_*(\nu).
\]

Hence \( \alpha\gamma_*\alpha^{-1} = \phi(\gamma)_* \). Therefore \( \mathcal{O}_2 = \alpha_\#\mathcal{O}_1\beta^{-1} \). \( \square \)

**Definition:** Define \( \text{Hom}_f(\Delta, \text{Out}(M)) \) to be the set of all homomorphisms from \( \Delta \) to \( \text{Out}(M) \) that have finite image.

The group \( \text{Out}(M) \) acts on the left of \( \text{Hom}_f(\Delta, \text{Out}(M)) \) by conjugation, that is, if \( g \in \text{Out}(M) \) and \( \eta \in \text{Hom}_f(\Delta, \text{Out}(M)) \), then \( g\eta = g_*\eta \) where \( g_* : \text{Out}(M) \to \)
\[\text{Out}(M)\] is defined by \(g \cdot (h) = ghg^{-1}\). Let \(\text{Out}(M) \setminus \text{Hom}_f(\Delta, \text{Out}(M))\) be the set of \(\text{Out}(M)\)-orbits. The group \(\text{Aut}(\Delta)\) acts on the right of \(\text{Hom}_f(\Delta, \text{Out}(M))\) by composition of homomorphisms. If \(\beta \in \text{Aut}(\Delta)\) and \(\eta \in \text{Hom}_f(\Delta, \text{Out}(M))\) and \(g \in \text{Out}(M)\), then
\[(g\eta)\beta = (g\ast \eta)\beta = g_{\ast} \eta \beta = g(\eta \beta).
Hence \(\text{Aut}(\Delta)\) acts on the right of \(\text{Out}(M) \setminus \text{Hom}_f(\Delta, \text{Out}(M))\) by
\[(\text{Out}(M)\eta)\beta = \text{Out}(M)(\eta \beta).
Let \(\delta, \epsilon \in \Delta\) and \(\eta \in \text{Hom}_f(\Delta, \text{Out}(M))\). Then we have that
\[\eta \delta \epsilon \eta^{-1} = \eta(\delta \epsilon \delta^{-1}) = \eta(\delta) \eta(\epsilon) \eta^{-1} = \eta(\delta) \cdot \eta(\epsilon) = (\eta(\delta) \eta)(\epsilon).
Hence \(\eta \delta \epsilon = \eta(\delta) \eta\). Therefore \(\text{Inn}(\Delta)\) acts trivially on \(\text{Out}(M) \setminus \text{Hom}_f(\Delta, \text{Out}(M))\).
Hence \(\text{Out}(\Delta)\) acts on the right of \(\text{Out}(M) \setminus \text{Hom}_f(\Delta, \text{Out}(M))\) by
\[(\text{Out}(M)\eta)(\beta \text{Inn}(\Delta)) = \text{Out}(M)(\eta \beta).
\textbf{Definition:} Define the set \(\text{Out}(\Delta, M)\) by the formula
\[\text{Out}(\Delta, M) = (\text{Out}(M) \setminus \text{Hom}_f(\Delta, \text{Out}(M)))/\text{Out}(\Delta).
If \(\eta \in \text{Hom}_f(\Delta, \text{Out}(M))\), let \([\eta] = (\text{Out}(M)\eta)\text{Out}(\Delta)\) be the element of \(\text{Out}(\Delta, M)\) determined by \(\eta\).

Let \((\Gamma, N)\) be a pair such that \([\Gamma, N] \in \text{Iso}(\Delta, M)\). Let \(O : \Gamma/N \to \text{Out}_E(N)\) be the homomorphism induced by the action of \(\Gamma\) on \(N\) by conjugation. Let \(\alpha : N \to M\) and \(\beta : \Delta \to \Gamma/N\) be isomorphisms. Then \(\alpha \# O \beta \in \text{Hom}_f(\Delta, \text{Out}(M))\).
Let \(\alpha' : N \to M\) and \(\beta' : \Delta \to \Gamma/N\) are isomorphisms. Observe that
\[\alpha' \# O \beta' = \alpha'_\# \alpha^{-1}_\# O \beta \beta^{-1} \beta' = (\alpha' \alpha^{-1})_\# O \beta (\beta^{-1} \beta') = (\alpha' \alpha^{-1})_\# \text{Inn}(M)_\# (\alpha \# O \beta (\beta^{-1} \beta')) = (\alpha' \alpha^{-1})_\# \text{Inn}(M)_\# (\alpha \# O \beta (\beta^{-1} \beta')).
Hence \([\alpha \# O \beta]\) in \(\text{Out}(\Delta, M)\) does not depend on the choice of \(\alpha\) and \(\beta\), and so \((\Gamma, N)\) determines the element \([\alpha \# O \beta]\) of \(\text{Out}(\Delta, M)\) independent of the choice of \(\alpha\) and \(\beta\).

Suppose \([\Gamma_i, N_i] \in \text{Iso}(\Delta, M)\) for \(i = 1, 2\), and \(\phi : (\Gamma_1, N_1) \to (\Gamma_2, N_2)\) is an isomorphism of pairs. Let \(\alpha : N_1 \to N_2\) be the isomorphism obtained by restricting \(\phi\), and let \(\beta : \Gamma_1/N_1 \to \Gamma_2/N_2\) be the isomorphism induced by \(\phi\). Let \(O_i : \Gamma_i/N_i \to \text{Out}_E(N_i)\) be the homomorphism induced by the action of \(\Gamma_i\) on \(N_i\) by conjugation for \(i = 1, 2\). Then \(O_2 = \alpha \# O_1 \beta^{-1}\) by Lemma 17. Let \(\alpha_1 : N_1 \to M\) and \(\beta_1 : \Delta \to \Gamma_1/N_1\) be isomorphisms. Let \(\alpha_2 = \alpha_1 \alpha^{-1}\) and \(\beta_2 = \beta \beta_1\). Then we have
\[(\alpha_2)_\# O_2 \beta_2 = (\alpha_1 \alpha^{-1})_\# \alpha \# O_1 \beta^{-1} \beta_2 = (\alpha_1)_\# O_1 \beta_1.
Hence \((\Gamma_1, N_1)\) and \((\Gamma_2, N_2)\) determine the same element of \(\text{Out}(\Delta, M)\). Therefore there is a function
\[\omega : \text{Iso}(\Delta, M) \to \text{Out}(\Delta, M)\]
defined by \(\omega([\Gamma, N]) = [\alpha \# O \beta]\) for any choice of isomorphisms \(\alpha : N \to M\) and \(\beta : \Delta \to \Gamma/N\).

\textbf{Lemma 4.} The set \(\text{Out}(\Delta, M)\) is finite.
Proof: The group Out(M) is arithmetic by Theorem 1.1 of [1]. Hence Out(M) has only finitely many conjugacy classes of finite subgroups, cf. §5 of [3]. As $\Delta$ is finitely generated there are only finitely many homomorphisms from $\Delta$ to a finite group $G$. Therefore Out(M)$\cdot$Hom$_f(\Delta, \text{Out}(M))$ is finite. Hence Out($\Delta$, $M$) is finite.

5. Fiber Cohomology Classes

Consider $\omega : \text{Iso}(\Delta, M) \to \text{Out}(\Delta, M)$. Suppose $[\Gamma_1, N_1]$ and $[\Gamma_2, N_2]$ are in the same fiber of $\omega$. We want to define a class $[\Gamma_1, N_1; \Gamma_2, N_2; \alpha, \beta]$ in the cohomology group $H^1(\Gamma_1/N_1, K_1)$ where $\Gamma_1/N_1$ acts on $K_1$ by $N_1(b + B)(v + T)_* = (\overline{B}v + \overline{T})_*$.

Let $\alpha_i : N_i \to M$ and $\beta_i : \Delta \to \Gamma_i/N_i$ be isomorphisms for $i = 1, 2$. Let $O_i : \Gamma_i/N_i \to \text{Out}(N_i)$ be the homomorphism induced by the action of $\Gamma_i$ on $N_i$ by conjugation for $i = 1, 2$. As $\omega([\Gamma_1, N_1]) = \omega([\Gamma_2, N_2])$, we have that $[(\alpha_1)\# O_1\beta_1] = [(\alpha_2)\# O_2\beta_2]$. Then there exists $\alpha_0$ in Aut($M$) and $\beta_0$ in Aut($\Delta$) such that $(\alpha_1)\# O_1\beta_1 = (\alpha_0)\# (\alpha_2)\# O_2\beta_2\beta_0$. We have that

$$O_1 = (\alpha_1^{-1}\alpha_0\alpha_2)\# O_2\beta_2\beta_0\beta_1^{-1}.$$

Let $\alpha : N_1 \to N_2$ be the isomorphism $\alpha_0^{-1}\alpha_1$, and let $\beta : \Gamma_1/N_1 \to \Gamma_2/N_2$ be the isomorphism $\beta_0\beta_1^{-1}$. Then $O_1 = \alpha\# O_2\beta$. Now $\alpha$ induces an isomorphism $\overline{\alpha} : \overline{N}_1 \to \overline{N}_2$ defined by $\overline{\alpha}(\overline{\nu}) = \alpha(\nu)$ for each $\nu$ in $N_1$. Let $V_i = \text{Span}(N_i)$ for $i = 1, 2$, and let $\overline{\alpha} : V_1 \to V_2$ be an affinity such that $\overline{\alpha}(\overline{N}_1\overline{\alpha}^{-1} = \overline{N}_2$ and $\overline{\alpha}_* = \overline{\alpha}$, that is, $\overline{\alpha}_*(\overline{\nu}) = \overline{\alpha}(\nu)$ for each $\nu$ in $N_1$.

Let $\Xi : \Gamma_i/N_i \to \text{Isom}(V_i/N_i)$ be the homomorphism induced by the action of $\Gamma_i/N_i$ on $V_i/N_i$ for $i = 1, 2$. Let $\Omega_i : \text{Isom}(V_i/N_i) \to \text{Out}(\Delta)$ be defined so that $\Omega_i(\zeta) = \text{Im}(N_i)\zeta$, where $\zeta$ is an isometry of $V_i$ that lifts $\zeta$ and $\zeta_*$ is the automorphism of $N_i$ defined by $\zeta_*(\nu) = \zeta_\nu\zeta_\nu^{-1}$ for $i = 1, 2$. Then we have that $\Omega_i\Xi_i = O_i$ for $i = 1, 2$. By Lemma 10 of [9], we have that

$$\Omega_1\Xi_1 = \alpha_0^{-1}\Omega_2\Xi_2 = \Omega_1(\alpha_0^{-1}\Xi_2\beta).$$

Let $\phi : \Gamma_1/N_1 \to \text{Aff}(V_1/N_1)$ and $\psi : \Gamma_1/N_1 \to \text{Aff}(V_1/N_1)$ be the homomorphisms defined by $\phi = \alpha_0^{-1}\Xi_2\beta$ and $\psi = \Xi_1$. Then we have that $\phi(g)\psi(g)^{-1}$ is in $K_1$ for each $g$ in $\Gamma_1/N_1$ by Theorem 3 of [3]. As $\psi$ takes values in Isom($V_1/N_1$) and $K_1$ is a subgroup of Isom($V_1/N_1$), we have that $\phi$ takes values in Isom($V_1/N_1$).

Let $g, h$ be in $\Gamma_1/N_1$, then we have that

$$\phi(gh)\psi(gh)^{-1} = \phi(g)\phi(h)(\psi(g)\psi(h))^{-1} = \phi(g)\phi(h)\psi(h)^{-1}\psi(g)^{-1} = \phi(g)g^{-1}\psi(g)\phi(h)\psi(h)^{-1}\psi(g)^{-1} = \phi(g)g^{-1}\psi(g)\phi(h)\psi(h)\psi(h)^{-1}.$$

with

$$\psi(N_1(b + B))_*(v + T) = (\overline{B} + \overline{T})_*(\overline{B} + \overline{T})^{-1} = (\overline{B}v + \overline{T})_*.$$

Hence the function $\phi\psi^{-1} : \Gamma_1/N_1 \to K_1$ is a crossed homomorphism, and so determines a class $[\Gamma_1, N_1; \Gamma_2, N_2; \alpha, \beta]$ in $H^1(\Gamma_1/N_1, K_1)$, cf. p. 105 of [3].

Let $[\Gamma, N]$ be a class in Isom($\Delta, M$), and let $V = \text{Span}(N)$. Let $C$ be the centralizer of $N$ in Aff($V$). By Lemmas 6 and 8 of [3], we have that

$$C = \{v + T : v \in \text{Span}(Z(N))\}.$$
The group $\Gamma/N$ acts on $\mathcal{C}$ by $(N(b + B))(v + \mathbf{T}) = \overline{B}v + \mathbf{T}$ and $\Gamma/N$ acts on $Z(N)$ by $(N(b + B))(u + I) = Bu + I$. We have a short exact sequence of $(\Gamma/N)$-modules

$$0 \to Z(N) \xrightarrow{\iota} \mathcal{C} \xrightarrow{\kappa} K \to 0$$

where $\iota(u + I) = u + \mathbf{T}$ and $\kappa(v + \mathbf{T}) = (v + \mathbf{T})$.

Let $T$ be the group of translations of $\Gamma$. Then $TN/N$ is a normal subgroup of $\Gamma/N$ of finite index and $TN/N$ is a subgroup of the group of translations of $\Gamma/N$ of finite index by Theorem 16 of [8]. The group $\Gamma/N$ acts on the abelian group $TN/N$ by $(N(b + B))(a + I) = N(Ba + I)$, and so $TN/N$ is a $(\Gamma/N)$-module. Moreover the group $TN/N$ acts trivially on $\mathcal{C}$.

**Lemma 5.** Let $f : \Gamma/N \to \mathcal{C}$ be a crossed homomorphism, and let $f_{res} : TN/N \to \mathcal{C}$ be the restriction of $f$. Then $f_{res}$ is a homomorphism of $(\Gamma/N)$-modules and the class of $f$ in $H^1(\Gamma/N, \mathcal{C})$ is completely determined by $f_{res}$.

**Proof.** According to [6], p. 354, we have an exact sequence of homomorphisms

$$H^1(\Gamma/TN, \mathcal{C}) \xrightarrow{inf} H^1(\Gamma/N, \mathcal{C}) \xrightarrow{res} H^1(TN/N, \mathcal{C})^{\Gamma/N} \to H^2(\Gamma/TN, \mathcal{C}).$$

The group $\Gamma/TN$ is finite and $\mathcal{C}$ is a torsion-free, divisible, abelian group, and so $H^i(\Gamma/TN, \mathcal{C}) = 0$ for $i = 1, 2$ by Corollary IV.5.4 of [6]. Hence

$$res : H^1(\Gamma/N, \mathcal{C}) \to H^1(TN/N, \mathcal{C})^{\Gamma/N}$$

is an isomorphism. Here $res([f]) = [f_{res}]$. By the Universal Coefficients Theorem (p. 77 of [6]), we have that

$$H^1(TN/N, \mathcal{C})^{\Gamma/N} = \text{Hom}(TN/N, \mathcal{C})^{\Gamma/N}.$$

Here $\Gamma/N$ acts on $\text{Hom}(TN/N, \mathcal{C})$ by $((N\gamma)h)(x) = (N\gamma)h(N\gamma^{-1}x)$ for each $\gamma \in \Gamma$, homomorphism $h : TN/N \to \mathcal{C}$, and element $x \in TN/N$. Therefore, we have that

$$\text{Hom}(TN/N, \mathcal{C})^{\Gamma/N} = \text{Hom}_{\Gamma/N}(TN/N, \mathcal{C}).$$

Hence $[f_{res}] = \{f_{res}\}$ and $f_{res}$ is a homomorphism of $(\Gamma/N)$-modules. Therefore the class of $f$ in $H^1(\Gamma/N, \mathcal{C})$ is completely determined by $f_{res}$. □

**Lemma 6.** Suppose that $\omega([\Gamma_1, N_1]) = \omega([\Gamma_2, N_2])$ with $O_1 = \alpha_2^{-1}O_2 \beta$. If the class $[\Gamma_1, N_1 ; \Gamma_2, N_2 ; \alpha, \beta]$ is in the image of $\kappa_* : H^1(\Gamma_1/N_1, \mathcal{C}_1) \to H^1(\Gamma_1/N_1, \mathcal{K}_1)$, then $[\Gamma_1, N_1] = [\Gamma_2, N_2]$.

**Proof.** Suppose that $[\Gamma_1, N_1 ; \Gamma_2, N_2 ; \alpha, \beta]$ is in the image of $\kappa_* : H^1(\Gamma_1/N_1, \mathcal{C}_1) \to H^1(\Gamma_1/N_1, \mathcal{K}_1)$. Then there is a crossed homomorphism $f : \Gamma_1/N_1 \to \mathcal{C}_1$ such that $\kappa_*([f]) = [\Gamma_1, N_1 ; \Gamma_2, N_2 ; \alpha, \beta]$.

By Lemma 5, the cohomology class $[f]$ is completely determined by the restriction $(\Gamma_1/N_1)$-module homomorphism $f_{res} : T_1N_1/N_1 \to C_1$.

To simplify notation, replace $\Gamma_1/N_1$ with $\Gamma'_1$. Then $T_1N_1/N_1$ corresponds to $T'_1 = \{b' + I' : b + I \in T_1\}$. Moreover $\Gamma_1/N_1$ acts on $T'_1$ by $(N_1(b + B))(b' + I') = B'b' + I'$ and $f_{res}$ corresponds to a homomorphism of $(\Gamma_1/N_1)$-modules $f'_{res} : T'_1 \to C_1$.

The group $T_1N_1/N_1$ has finite index in the group of translations of $\Gamma_1/N_1$, and so $T'_1$ has finite index in the group of translations of $\Gamma'_1$. Hence $\{b' : b + I \in T_1\}$ contains a basis of the vector space $V_1^\perp$. Therefore $f'_{res} : T'_1 \to C_1$ induces a linear transformation $L : V_1^\perp \to \text{Span}(Z(N_1))$ such that $f'_{res}(b' + I') = L(b') + \mathbf{T}$ for each $b + I$ in $T_1$ and if $b + B$ is in $\Gamma_1$, then $LB' = \overline{B}L$. 


Consider the function $h : \Gamma_1/N_1 \to C_1$ defined by $h(N_1(b + B)) = L(b') + \overline{T}$. Then $h$ is a crossed homomorphism. If $b + I$ is in $T_1$, then

$$h_{res}(N_1(b + I)) = L(b') + \overline{T} = f'_{res}(b' + I') = f_{res}(N_1(b + I)).$$

Hence $h_{res} = f_{res}$. Therefore $[h] = [f]$ in $H^1(\Gamma_1/N_1, C_1)$ by Lemma 5.

Let $\tilde{\alpha} : V_1 \to V_2$ be the affinity defined above and write $\tilde{\alpha} = \overline{\tau} + \overline{C}$ with $\tau \in V_2$ and $\overline{C} : V_1 \to V_2$ a linear isomorphism. Define a linear transformation $D : V_1^+ \to \text{Span}(N_2)_2$ by $D = C \overline{L}$. If $b + B \in \Gamma_1$, then $DB' = C\overline{BC}'^{-1}D$. Let $p_1 : \Gamma_1/N_1 \to V_1^+$ be the crossed homomorphism defined by $p_1(N_1(b + B)) = b'$.

Then $h(N_1\gamma) = \overline{C}^{-1}Dp_1(N_1\gamma) + \overline{T}$. Observe that $\kappa_*([h]) = [h_*]$ where $h_*$ is defined by $h_*(N_1\gamma) = (h(N_1\gamma))_*$. Thus $h_* = (\overline{C}^{-1}Dp_1)_*$. as defined in Theorem 3(3).

Let $v$ be in $\text{Span}(Z(N_1))$, and let $f_v : \Gamma_1/N_1 \to K_1$ be the principal crossed homomorphism determined by $(v + I)_*$. Then we have that

$$f_v(N_1(b + B)) = (N_1(b + B))(v + \overline{T})_* = (\overline{B}v + \overline{T})_* = (\overline{B}v - v + \overline{T})_*.$$

Now we have that $[h_*] = [\Gamma_1, N_1; \Gamma_2, N_2; \alpha, \beta]$ in $H^1(\Gamma_1/N_1, K_1)$. Hence there exists $v$ in $\text{Span}(Z(N_1))$ such that

$$(\tilde{\alpha}_v^{-1} \Xi_2 \beta)\Xi_1^{-1}f_v = (\overline{C}^{-1}Dp_1)_*.$$

Let $\tilde{\alpha}_v : V_1 \to V_2$ be the affinity defined by $\tilde{\alpha}_v = \tilde{\alpha}(v + \overline{T})$. Then $\tilde{\alpha}_vN_1\tilde{\alpha}_v^{-1} = N_2$ and $(\tilde{\alpha}_v)_* = \alpha_*$. By Lemma 6 of ([9], and so $(\tilde{\alpha}_v)_* = \alpha_*$. Observe that

$$((\tilde{\alpha}_v^{-1} \Xi_2 \beta)\Xi_1^{-1}f_v)(N_1(b + B))$$

$$= (\tilde{\alpha}_v^{-1} \Xi_2 \beta)(N_1(b + B))\Xi_1^{-1}(N_1(b + B))f_v(N_1(b + B))$$

$$= \tilde{\alpha}_v^{-1}(\Xi_2 \beta)(N_1(b + B))\tilde{\alpha}_v(\overline{B} + \overline{T})(\overline{B}v - v + \overline{T})_*$$

$$= \tilde{\alpha}_v^{-1}(\Xi_2 \beta)(N_1(b + B))\tilde{\alpha}_v(\overline{B} + \overline{T})_*(v - \overline{B}v + \overline{T})_*^{-1}$$

$$= \tilde{\alpha}_v^{-1}(\Xi_2 \beta)(N_1(b + B))\tilde{\alpha}_v(v - \overline{B}v + \overline{T})_*^{-1}$$

$$= \tilde{\alpha}_v^{-1}(\Xi_2 \beta)(N_1(b + B))\tilde{\alpha}_v((v + T)(\overline{B} + \overline{T})_*(v - \overline{B}v + \overline{T})_*^{-1}$$

$$= (\tilde{\alpha}_v)_*^{-1}(\Xi_2 \beta)(N_1(b + B))\tilde{\alpha}_v(\overline{B} + \overline{T})_*(\overline{B} + \overline{T})_*^{-1}$$

Hence we have

$$((\tilde{\alpha}_v)_v^{-1} \Xi_2 \beta)\Xi_1^{-1}f_v = ((\tilde{\alpha}_v)_v^{-1} \Xi_2 \beta)\Xi_1^{-1}.$$

Thus we have that

$$(\tilde{\alpha}_v)_v^{-1} \Xi_2 \beta = (\overline{C}^{-1}Dp_1)_* \Xi_1.$$

Let $P_i : \Gamma_i/N_i \to \Gamma_i'$ be the isomorphism defined by $P_i(N_i\gamma) = \gamma'$ for each $i = 1, 2$. Let $\beta' : \Gamma_1' \to \Gamma_2'$ be the isomorphism so that $P_2^{-1}\beta'P_1 = \beta$. Let $\hat{\beta} : V_1^+ \to V_2^+$ be an affinity such that $\hat{\beta}^{-1} = \hat{\beta}^{-1}$ and $\hat{\beta}_* = \beta_*$, that is, and $\hat{\beta}'\hat{\beta}^{-1} = (\beta'\gamma')$ for each $\gamma$ in $\Gamma_1$. Then we have that

$$(\tilde{\alpha}_v)_v^{-1} \Xi_2 P_2^{-1} \hat{\beta}_* P_1 = (\overline{C}^{-1}Dp_1)_* \Xi_1.$$

Therefore there exists an affinity $\phi = c + C$ of $E^n$ such that $\phi(\Gamma_1, N_1)\phi^{-1} = (\Gamma_2, N_2)$ with $\phi = \tilde{\alpha}_v$, $\phi' = \hat{\beta}$, and $\overline{C}' = D$ by Theorem 3. Thus $[\Gamma_1, N_1] = [\Gamma_2, N_2]$. 

\[\square\]
Lemma 7. Suppose that $\omega([\Gamma, N]) = \omega([\Gamma_1, N_1])$ with $O = (\alpha_1)^{-1}O_1\beta_1$ and that $\omega([\Gamma, N]) = \omega([\Gamma_2, N_2])$ with $O = (\alpha_2)^{-1}O_2\beta_2$. If $[\Gamma, N; \Gamma_1, N_1; \alpha_i, \beta_i]$ for $i = 1, 2$ are in the same coset of the image of $\kappa_* : H^1(\Gamma/N, C) \to H^1(\Gamma/N, K)$ in $H^1(\Gamma/N, K)$, then $[\Gamma_1, N_1] = [\Gamma_2, N_2]$.

Proof. Let $b + B \in \Gamma$, and let $b_1 + B_1$ be an element of $\Gamma_1$ such that

$$N_1(b_1 + B_1) = \beta_1(N(b + B)).$$

As $O = (\alpha_1)^{-1}O_1\beta_1$, we have that

$$(b + B)_*\text{Inn}(N) = (\alpha_1)^{-1}O_1\beta_1(N(b + B))$$

= $$(\alpha_1)^{-1}O_1(\beta_1(N_1(b + B_1)))$$

= $$(\alpha_1)^{-1}((b_1 + B_1)_*\text{Inn}(N_1))$$

= $$(\alpha_1)^{-1}(b_1 + B_1)_*\alpha_1\text{Inn}(N)$$

= $$(\tilde{\alpha}_1)^{-1}(b_1 + B_1)_*(\tilde{\alpha}_1)\text{Inn}(N)$$

= $$(\tilde{\alpha}_1 + \overline{\alpha}_1)^{-1}(b_1 + B_1)_*(\tilde{\alpha}_1 + \overline{\alpha}_1)\text{Inn}(N).$$

The action of $\Gamma/N$ on $C$ is given by $N(b + B)(u + T) = \overline{B}u + T$ and is determined by the action of $\Gamma/N$ on $Z(N)$ induced by conjugation. Now $\text{Inn}(N)$ acts trivially on $Z(N)$, and so the last computation implies that $\overline{B} = \overline{\tilde{C}_1} \cdot \tilde{C}_1$ on $\text{Span}(Z(N))$. Hence the pair of isomorphisms

$$\pi = (\beta_1 : \Gamma/N \to \Gamma_1/N_1, (\tilde{\alpha}_1)_1^{-1} : C_1 \to C)$$

is a change of groups isomorphism in the sense of [R] p. 108. Therefore we have an isomorphism $\pi^* : H^1(\Gamma_1/N_1, C_1) \to H^1(\Gamma/N, C)$ defined so that if $f_1 : \Gamma_1/N_1 \to C_1$ is a crossed homomorphism, then $\pi^*[f_1] = [\pi^*f_1]$ where $\pi^*f_1 : \Gamma/N \to C$ is the crossed homomorphism defined by $\pi^*f_1(x) = (\tilde{\alpha}_1)^{-1}(f_1(\beta_1(x))).$

Likewise the pair of isomorphisms

$$\varpi = (\beta_1 : \Gamma/N \to \Gamma_1/N_1, (\tilde{\alpha}_1)_2^{-1} : K_1 \to K)$$

is a change of groups isomorphism which induces an isomorphism $\varpi^*$ such that the following diagram commutes

$$\begin{array}{ccc}
H^1(\Gamma_1/N_1, C_1) & \xrightarrow{\pi^*} & H^1(\Gamma/N, C) \\
\downarrow (\kappa_1)_* & & \downarrow \kappa_* \\
H^1(\Gamma_1/N_1, K_1) & \xrightarrow{\varpi^*} & H^1(\Gamma/N, K).
\end{array}$$

Now we have that $\omega([\Gamma_1, N_1]) = \omega([\Gamma_2, N_2])$ with $O_1 = (\alpha_2\alpha_1^{-1})^\#O_2\beta_2\beta_1^{-1}$. Moreover, we have that

$$[\Gamma_1, N_1; \Gamma_2, N_2; \alpha_2\alpha_1^{-1}, \beta_2\beta_1^{-1}] = [((\tilde{\alpha}_2\tilde{\alpha}_1)^{-1})_{2^{-1}}\Xi_2\beta_2\beta_1^{-1}][\Xi_1^{-1}].$$

For $i = 1, 2$, we have that

$$[\Gamma, N; \Gamma_i, N_i; \alpha_i, \beta_i] = [((\tilde{\alpha}_i)^{-1}\Xi_i\tilde{\beta}_i)\Xi_1^{-1}].$$
Let $\gamma \in \Gamma$. Observe that
\[
((\tilde{\alpha}_2)^{-1}\Xi_2\beta_2\Xi^{-1})((\tilde{\alpha}_1)^{-1}\Xi_1\beta_1\Xi^{-1})(N\gamma)
= \frac{((\tilde{\alpha}_2)^{-1}\Xi_2\beta_2)(N\gamma)\Xi(N\gamma)^{-1}\Xi(N\gamma)((\tilde{\alpha}_1)^{-1}\Xi_1\beta_1)(N\gamma))}{((\tilde{\alpha}_2)^{-1}\Xi_2\beta_2)(N\gamma)(\alpha_1)^{-1}(\Xi_1\beta_1)(N\gamma))^{-1}}
= \frac{((\tilde{\alpha}_2)^{-1}\Xi_2\beta_2)(N\gamma)(\alpha_1)^{-1}(\Xi_1\beta_1)(N\gamma))}{(\alpha_1)^{-1}(\tilde{\alpha}_2\alpha_1^{-1})^{-1}\Xi_2\beta_2\Xi^{-1}(\beta_1)(N\gamma))}\Xi_1\beta_1(N\gamma))^{-1}
= \frac{\omega^\ast(((\tilde{\alpha}_2\alpha_1^{-1})^{-1}\Xi_2\beta_2\Xi^{-1}(\beta_1)(N\gamma))\Xi_1\beta_1(N\gamma))}{\Xi_1\beta_1(N\gamma)}
.
\]

Hence we have that
\[
\omega^\ast(\Gamma_1, N_1; \Gamma_2, N_2; \alpha_2\alpha_1^{-1}, \beta_2\beta_1^{-1}) = [\Gamma, N; \Gamma_2, N_2; \alpha_2, \beta_2][\Gamma, N_1; \alpha_1, \beta_1]^{-1}.
\]
The right-hand side of the above equation is in the image of $\kappa_\ast : H^1(\Gamma/N, \mathbb{C}) \to H^1(\Gamma/N, \mathbb{K})$. Therefore $[\Gamma_1, N_1; \Gamma_2, N_2; \alpha_2\alpha_1^{-1}, \beta_2\beta_1^{-1}]$ is in the image of $(\kappa_1)_\ast : H^1(\Gamma_1/N_1, \mathbb{C}_1) \to H^1(\Gamma_1/N_1, \mathbb{K}_1)$. Hence $[\Gamma_1, N_1] = [\Gamma_2, N_2]$ by Lemma 6.

6. The relative Bieberbach Theorem

Let $[\Gamma, N]$ be a class in Isom($\Delta, M$), and let $T$ be the group of translations of $\Gamma$. Then $\Gamma/TN$ is a finite group, since $\Gamma/T$ is finite. The group $TN/N$ acts trivially on $Z(N)$, $TN/N, \mathbb{C}, \mathbb{K}$, and so the action of $\Gamma/N$ on $Z(N)$, $TN/N, \mathbb{C}, \mathbb{K}$ induces an action of $\Gamma/TN$ on $Z(N)$, $TN/N, \mathbb{C}, \mathbb{K}$ making $Z(N)$, $TN/N, \mathbb{C}, \mathbb{K}$ into $(\Gamma/TN)$-modules.

**Lemma 8.** The group $H^1(\Gamma/TN, \mathbb{K})$ is finite.

**Proof.** The short exact sequence $0 \to Z(N) \to C \to K \to 0$ of $(\Gamma/TN)$-modules induces an exact sequence of cohomology groups
\[
H^1(\Gamma/TN, \mathbb{C}) \to H^1(\Gamma/TN, \mathbb{K}) \to H^2(\Gamma/TN, Z(N)) \to H^2(\Gamma/TN, \mathbb{C})
.
\]
As explained in the proof of Lemma 5, the outside groups are trivial, and so $H^1(\Gamma/TN, \mathbb{K})$ is isomorphic to $H^2(\Gamma/TN, Z(N))$. The group $H^2(\Gamma/TN, Z(N))$ is a torsion group by Proposition IV.5.3 of [6]. As $Z(N)$ is a free abelian group of finite rank, the group $H^2(\Gamma/TN, Z(N))$ is finitely generated. Hence $H^2(\Gamma/TN, Z(N))$ is finite, and so $H^1(\Gamma/TN, \mathbb{K})$ is finite.

**Lemma 9.** The cokernel of $\kappa_\ast : H^1(TN/N, \mathbb{C})^{\Gamma/TN} \to H^1(TN/N, \mathbb{K})^{\Gamma/TN}$ is finite.

**Proof.** By the Universal Coefficients Theorem, we have that
\[
H^1(TN/N, \mathbb{C})^{\Gamma/TN} = \text{Hom}(TN/N, \mathbb{C})^{\Gamma/TN} = \text{Hom}_{\Gamma/TN}(TN/N, \mathbb{C}),
\]
\[
H^1(TN/N, \mathbb{K})^{\Gamma/TN} = \text{Hom}(TN/N, \mathbb{K})^{\Gamma/TN} = \text{Hom}_{\Gamma/TN}(TN/N, \mathbb{K}).
\]
The short exact sequence $0 \to Z(N) \to C \to K \to 0$ induces an exact sequence
\[
0 \to \text{Hom}(TN/N, Z(N)) \to \text{Hom}(TN/N, C) \to \text{Hom}(TN/N, K) \to \text{Ext}(TN/N, Z(N))
\]
by Theorem III.3.4 of [6] (with $R = \mathbb{Z}$). We have that $\text{Ext}(TN/N, Z(N)) = 0$ by Theorems I.6.3 and III.3.5 of [6], since $TN/N$ is a free abelian group. Hence we have a short exact sequence of $(\Gamma/TN)$-modules
\[
0 \to \text{Hom}(TN/N, Z(N)) \to \text{Hom}(TN/N, C) \to \text{Hom}(TN/N, K) \to 0.
\]
Hence we have an exact sequence of cohomology groups
\[ H^0(\Gamma/\TN, \Hom(\TN/N, C)) \to H^0(\Gamma/\TN, \Hom(\TN/N, K)) \to H^1(\Gamma/\TN, \Hom(\TN/N, Z(N))), \]
which is equivalent to an exact sequence
\[ \Hom_{\Gamma/\TN}(\TN/N, C) \to \Hom_{\Gamma/\TN}(\TN/N, K) \to H^1(\Gamma/\TN, \Hom(\TN/N, Z(N))). \]
The group \( H^1(\Gamma/\TN, \Hom(\TN/N, Z(N))) \) is finite, since \( \Hom(\TN/N, Z(N)) \) is a free abelian group of finite rank. Hence the cokernel of \( \Hom_{\Gamma/\TN}(\TN/N, C) \to \Hom_{\Gamma/\TN}(\TN/N, K) \) is finite. Therefore the cokernel of \( \kappa_* : H^1(\TN/N, C)_{\Gamma/\TN} \to H^1(\TN/N, K)_{\Gamma/\TN} \) is finite.

**Lemma 10.** The cokernel of \( \kappa_* : H^1(\Gamma/N, C) \to H^1(\Gamma/N, K) \) is finite.

*Proof.* We have a short exact sequence \( 1 \to \TN/N \to \Gamma/N \to \Gamma/\TN \to 1 \), and so we have a commutative diagram with horizontal exact sequences (cf. p. 354 of [5])
\[
\begin{array}{cccc}
0 & = H^1(\Gamma/\TN, C) & \to H^1(\Gamma/N, C) & \to H^1(\TN/N, C)_{\Gamma/\TN} & \to H^2(\Gamma/\TN, C) = 0 \\
\downarrow & \alpha & \downarrow & \beta & \downarrow \\
0 & \to H^1(\Gamma/\TN, K) & \to H^1(\Gamma/N, K) & \to H^1(\TN/N, K)_{\Gamma/\TN} & \to H^2(\Gamma/\TN, K)
\end{array}
\]
where the homomorphisms \( \alpha, \beta, \gamma \) are induced by \( \kappa : C \to K \). By the snake lemma (Lemma III.5.1 of [5]), we have an exact sequence
\[ \text{coker}(\alpha) \to \text{coker}(\beta) \to \text{coker}(\gamma). \]
Now \( \text{coker}(\alpha) = H^1(\Gamma/\TN, K) \) is finite by Lemma 8, and \( \text{coker}(\gamma) \) is finite by Lemma 9. Hence \( \text{coker}(\beta) \) is finite. Thus the cokernel of \( \kappa_* : H^1(\Gamma/N, C) \to H^1(\Gamma/N, K) \) is finite. \( \square \)

**Theorem 4.** For each dimension \( n \), there are only finitely many isomorphism classes of pairs of groups \((\Gamma, N)\) such that \( \Gamma \) is an \( n \)-space group and \( N \) is a normal subgroup of \( \Gamma \) such that \( \Gamma/N \) is a space group.

*Proof.* Let \( m \) be a positive integer less than \( n \). Let \( M \) be an \( m \)-space group and let \( \Delta \) be an \((n - m)\)-space group. Let \( \text{Iso}(\Delta, M) \) be the set of isomorphism classes of pairs \((\Gamma, N)\) where \( N \) is a normal subgroup of an \( n \)-space group \( \Gamma \) such that \( N \) is isomorphic to \( M \) and \( \Gamma/N \) is isomorphic to \( \Delta \). As there are only finitely many isomorphism classes of the groups \( \Delta \) and \( M \) by Bieberbach’s theorem [2], it suffices to prove that \( \text{Iso}(\Delta, M) \) is finite.

In §4, we defined a function \( \omega : \text{Iso}(\Delta, M) \to \text{Out}(\Delta, M) \) with \( \text{Out}(\Delta, M) \) finite by Lemma 4. The fibers of \( \omega \) are finite by Lemmas 7 and 10. Therefore \( \text{Iso}(\Delta, M) \) is finite. \( \square \)

In view of Theorem 10 of [5], Theorem 4 is equivalent to the following theorem.

**Theorem 5.** For each dimension \( n \), there are only finitely many affine equivalence classes of geometric orbifold fibrations of compact, connected, flat \( n \)-orbifolds.
References

[1] O. Baues and F. Grunewald, Automorphism groups of polycyclic-by-finite groups and arithmetic groups, Publ. Math. Inst. Hautes Études Sci. 104 (2006), 213-268.
[2] L. Bieberbach, Über die Bewegungsgruppen der Euklidischen Räume II, Math. Ann. 72 (1912), 400-412.
[3] A. Borel, Arithmetic properties of linear algebraic groups, Proc. Int. Congress Math., Stockholm (1992), 10-22.
[4] J. H. Conway, O. D. Friedrichs, D. H. Huson, W. P. Thurston, On three-dimensional space groups, Beiträge Algebra Geom. 42 (2001), 475-507.
[5] P. J. Hilton and U. Stammbach, A Course in Homological Algebra, Graduate Texts in Math., vol. 4, Springer-Verlag, New York, Heidelberg, and Berlin, 1971.
[6] S. Mac Lane, Homology, Springer-Verlag, New York, 1967.
[7] J.G. Ratcliffe, Foundations of Hyperbolic Manifolds, Second Edition, Graduate Texts in Math., vol. 149, Springer-Verlag, Berlin, Heidelberg, and New York, 2006.
[8] J. G. Ratcliffe and S. T. Tschantz, Fibered orbifolds and crystallographic groups, Algebr. Geom. Topol. 10 (2010), 1627-1664.
[9] J. G. Ratcliffe and S. T. Tschantz, On the isometry group of a compact flat orbifold, Geom. Dedicata 177 (2015), 43-60.
[10] J. G. Ratcliffe and S. T. Tschantz, The Calabi Construction for Compact Flat Orbifolds, Topology Appl. 178 (2014), 87-106.

Department of Mathematics, Vanderbilt University, Nashville, TN 37240

E-mail address: j.g.ratcliffe@vanderbilt.edu