Determining plane curve singularities from its polars

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Abstract

This paper addresses a very classical topic that goes back at least to Plücker: how to understand a plane curve singularity using its polar curves. Here, we explicitly construct the singular points of a plane curve singularity directly from the cluster of base points of its polars. In particular, we determine the equisingularity class (or topological equivalence class) of a germ of plane curve from the equisingularity class of generic polars and combinatorial data about the non-singular points shared by them.

1 Introduction

Polar germs are one of the main tools to analyze plane curve singularities, because they carry very deep analytical information on the singularity (see [18]). This holds still true for germs of hypersurfaces or even germs of analytic subsets of $\mathbb{C}^n$. There have been lots of efforts in the literature with the aim of distinguishing which of this information is in fact purely topological. Among these let us quote the works of Teissier [22, 23], Merle [19], Kuo and Lu [14], Lê and Teissier [17], Lê, Michel and Weber [15, 16], Casas-Alvero [3, 4], Gaffney [11], Delgado [7], García-Barroso [12], and García-Barroso and González-Pérez [13].

One of the main problems in this direction is recovering the cluster of singular points of a plane curve, or its topological equivalence class, from some invariant associated to the polar germs. The first steps towards the solution of this problem were settled more than twenty years ago by works of Teissier [22, 23] and Merle [19], which involved the use of the equisingularity (or topological equivalence) class of its generic polars. But it turns out that this analytic invariant carries not enough topological information about the singularity, and the problem remained open since then. As far as we are aware, the first and unique positive answer in this direction has been given by Casas-Alvero in his book [5], Theorem 8.6.4, where it is proved that the cluster of base points of the generic polars uniquely determines the singular points of the curve, although it is very non-constructive and nothing is said about the relation between both objects. It turns out that it is not enough to consider the singular points of a generic polar, since it is also necessary to take into account the non-singular points shared by generic polars (or by all polars, if we are considering the notion of “going virtually through a cluster” of infinitely near points, as it will be explained in Section 2.4). Recognizing the difference in difficulty, this could be interpreted as a sort

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of local version of the known, quite elementary fact in algebraic geometry that the proper singular points of plane projective algebraic curves are exactly the proper base points of its polar curves.

The aim of this work is to explicitly construct the singular points of a plane curve singularity directly from the cluster of base points of its polars. Namely, we present an algorithm which recovers the cluster of singular points from the cluster of base points of the polar germs. In particular, the algorithm applies to describe the equisingularity class (or topological equivalence class) of a germ of plane curve (by giving this information combinatorially encoded by means of an Enriques diagram) from the Enriques diagram which encodes the equisingularity class of a generic polar enlarged by some extra vertices representing the simple (non-singular) points shared by generic polars. As we will show, these extra vertices are only relevant for recovering the polar invariants (which are topological invariants of the singularity computable from the polar germs). Once the polar invariants are computed as a previous step in Lemma 4.3, our procedure shows in which way the equisingularity class (or the singular points) of generic polars determines the equisingularity class of the curves. Furthermore, our approach applies for any pair of polars in different tangents, regardless whether they are generic or even transverse ones (see Corollary 4.11). As an additional value, our algorithm gives a quite clear and neatly different proof of the crucial Casas-Alvero’s result about polars. We address the problem by reinterpretating it in terms of the theory of planar analytic morphisms, recently developed in [6], and a careful and ingenious use of these new techniques enables us to construct our new proof.

On the other side, starting from a different setting but falling on the same stream of recovering the equisingularity class of a germ of plane curve from invariants associated to polars, there is the recent work of García-Barroso. In [12], Th. 6.1 proves that the partial polar invariants of a plane curve $C$ and the multiplicities of the branches of $C$ determine the equisingularity type of $C$. Partial polar invariants are defined from the intersection multiplicity of each branch of $C$ with the branches of the polar of $C$. Hence, in order to have the partial polar invariants at the beginning, one needs to know some information about the topological type of $C$ (the number of branches, their multiplicity, and their intersection with each branch of the polar). Our work, instead, does not take for granted any knowledge of the original curve $C$, and the equisingularity type of $C$ is computed entirely from the polars.

This paper is structured as follows. In Section 2 we give a survey on the tools used all along the work, recalling definitions and facts about infinitely near points, polar germs of singular curves and germs of planar morphisms. The section is closed relating our problem about polar germs to the theory of planar analytic morphisms. Section 3 contains the technical results needed to solve the problem, which we believe are interesting on their own. It is divided into two parts, the first of which is devoted to the study of the growth of some rational invariants, $I_{\xi}(p)$, associated to the equisingularity class of the curve, independently of its polars, while the second one studies the relation between these topological invariants, the values $v_\xi(\xi)$ of the curve and some invariants, the multiplicities $n_p$ and the heights $m_p$, of the morphism associated to a generic polar. Finally, in Section 4 we develop the results which build up our algorithm and apply it to a paradigmatic example of Pham and to a more complicated curve, illustrating how the algorithm works.

## 2 Preliminaries and translation of the problem to a morphism

In this section we introduce the notations and concepts needed in the development of the results of this work. We start recalling some notions about infinitely near points, equisingularity of plane germs of curve and base points of linear systems, followed by some results relating them to polar germs. The last part of the section is devoted to expose a brief review of the theory of planar analytic morphisms developed by Casas-Alvero in [6], explaining how our problem fits in that context. For the sake of brevity, we have kept this section merely descriptive, and the reader is referred, for instance, to [5] Chapters 3, 4 and 6] and [6] for further details or proofs.

### 2.1 Infinitely near points.

From now on, suppose $O$ is a smooth point in a complex surface $S$, and denote by $O = O_{S,O}$ the local ring at $O$, i.e. the ring of germs holomorphic functions in a neighbourhood of $O$. We denote by $N_O$ the set of points infinitely near to $O$, which can be viewed as the disjoint union of all exceptional divisors obtained by successive blowing-ups above $O$. The points in $S$ will be called proper points in order to
distinguish them from the infinitely near ones. Given any \( p \in \mathcal{N}_O \), we denote by \( \pi_p : S_p :\rightarrow S \) the minimal composition of blowing-ups that realizes \( p \) as a proper point in a surface \( S_p \), and by \( E_p \) the exceptional divisor obtained by blowing up \( p \) in \( S_p \), which is also called its first neighbourhood. The set \( \mathcal{N}_O \) is naturally endowed with and order relation \( \preceq \) defined by \( p \preceq q \) (read \( p \) precedes \( q \)) if and only if \( q \in \mathcal{N}_p \).

Given a function \( f \in \mathcal{O} \), it defines a (germ of) curve \( \xi : f = 0 \) at \( O \), whose branches are the germs given by the factors of \( f \). The germ \( \xi \) is irreducible if and only if its equation is irreducible. In the sequel, we will implicitly assume that all the curves are reduced (i.e. they have no multiple branches). The multiplicity of \( \xi \) at \( O \), \( e_O(\xi) \), is defined to be the order of vanishing of the equation \( f \) at \( O \). From now on consider that \( \xi : f = 0 \) is a given curve at \( O \). For any \( p \in \mathcal{N}_O \) we denote by \( \xi_p : \pi_p^*f = 0 \) its total transform at \( p \), which contains a multiple of the exceptional divisor of \( \pi_p \). If we subtract these components we obtain the strict transform \( \xi_p \), which might be viewed as the closure of \( \pi_p^{-1}(\xi - \{O\}) \).

The multiplicity and the value of \( \xi \) at \( p \) are defined respectively as \( e_p(\xi) = e_p(\xi_p) \) and \( v_p(\xi) = e_p(\xi_p) \). We say that \( p \) lies on \( \xi \) if and only if \( e_p(\xi) > 0 \), and we denote by \( \mathcal{N}_O(\xi) \) the set of all such points. A point \( p \in \mathcal{N}_O(\xi) \) is simple (resp. multiple) if and only if \( e_p(\xi) = 1 \) (resp. \( e_p(\xi) > 1 \)). In the case \( \xi \) is irreducible, \( \mathcal{N}_O(\xi) \) is totally ordered and the sequence of multiplicities is non-increasing.

Given two germs of curve \( \xi, \zeta \), its intersection multiplicity at \( O \) can be computed by means of the Noether’s formula (see [5, Theorem 3.3.1]) as

\[
[\xi,\zeta]_O = \sum_{p \in \mathcal{N}_O(\xi) \cap \mathcal{N}_O(\zeta)} e_p(\xi)e_p(\zeta).
\] (1)

Given \( p \preceq q \) points infinitely near to \( O \), \( q \) is proximate to \( p \) (written \( q \rightarrow p \)) if and only if \( q \) lies on the exceptional divisor \( E_p \). A point \( p \) is free (resp. satellite) if it is proximate to exactly one point (resp. two points), and these are the only possibilities. Note that \( q \rightarrow p \) implies \( q \succeq p \), but not conversely. We also say that \( q \) is satellite of \( p \) (or \( p \)-satellite) if \( q \) is satellite and \( p \) is the last free point preceding \( q \). Proximity allows to establish the proximity equalities

\[
e_p(\xi) = \sum_{q \rightarrow p} e_q(\xi),
\] (2)

and the following relation between values and multiplicities

\[
v_p(\xi) = e_p(\xi) + \sum_{p \rightarrow q} v_p(\xi).
\] (3)

A point \( p \in \mathcal{N}_O(\xi) \) is singular (on \( \xi \)) if it is either multiple, or satellite, or precedes a satellite point lying on \( \xi \), and it is non-singular otherwise. Equivalently, \( p \) is non-singular if and only if it is free and there is no satellite point \( q > p \). The set of singular points of \( \xi \) weighted by the multiplicities or the values of \( \xi \) at them is denoted by \( S(\xi) \). Two curves \( \xi, \zeta \) are equisingular if it exists a bijection \( \varphi : S(\xi) \rightarrow S(\zeta) \) (called an equisingularity) preserving the natural order \( \preceq \), the multiplicities (or values) and the proximity relations. It is known that two such curves are equisingular if and only if they are topologically equivalent in a neighbourhood of \( O \) (seen as germs of topological subspaces of \( \mathbb{C}^2 = \mathbb{R}^4 \)). Thus, \( S(\xi) \) determines the topological class of (the embedding of) the curve \( \xi \).

The set of singular points of a curve is a special case of a (weighted) cluster. A cluster is a finite subset \( K \subset \mathcal{N}_O \) such that if \( p \in K \), then any other point \( q < p \) also belongs to \( K \). A weighted cluster \( K = (K, \nu) \) is a cluster \( K \) together with a function \( \nu : K \rightarrow \mathbb{Z} \). The number \( \nu_p = \nu(p) \) is the virtual multiplicity of \( p \) in \( K \). Two clusters \( K, K' \) are similar if there exists a bijection (similarity) \( \varphi : K \rightarrow K' \) preserving the ordering and the proximity. In the weighted case we also impose \( \varphi \) to preserve the virtual multiplicities.

A cluster can be represented by means of an Enriques diagram, which is a rooted tree whose vertices are identified with the points in \( K \) (the root corresponds to the origin \( O \)) and there is an edge between \( p \) and \( q \) if and only if \( p \) lies on the first neighbourhood of \( q \) or vice-versa. Moreover, the edges are drawn according to the following rules:

- If \( q \) is free, proximate to \( p \), the edge joining \( p \) and \( q \) is curved and if \( p \neq O \), it is tangent to the edge ending at \( p \).
If $p$ and $q$ (in the first neighbourhood of $p$ or $q$) have been represented, the rest of points proximate to $p$ in successive neighbourhoods of $q$ are represented on a straight half-line starting at $q$ and orthogonal to the edge ending at $q$.

In the weighted case, the vertices are labeled with their virtual multiplicities.

A curve $\xi$ goes through $O$ with virtual multiplicity $v_\Omega$ if $e_\Omega(\xi) \geq v_\Omega$, and in this case the virtual transform is $\xi = \xi - v_\Omega E_\Omega$. This definition can be extended inductively to any point $p \in K$ whenever the multiplicities of the successive virtual transforms are non-smaller than the virtual ones. In this case it is said that $\xi$ goes (virtually) through the cluster $K$. Moreover, if $e_\Omega(\xi) = v_p$ for all $p \in K$, it is said that $\xi$ goes through $K$ with effective multiplicities equal to the virtual ones. It might happen that there is no curve going through a given weighted cluster with effective multiplicities equal to the virtual ones, but when there exists such a curve the cluster is said to be consistent. Furthermore, if this is the case, there are curves going through $K$ with effective multiplicities equal to the virtual ones and missing any finite set of points not in $K$. Equivalently, $K$ is consistent if and only if $v_p \geq \sum_{q \rightarrow p} v_q$ for all $p \in K$, which resembles the proximity equalities (2). In this case, the difference $p_\rho = v_p - \sum_{q \rightarrow p} v_q$ is the excess of $K$ at $p$, and $p$ is dicritical if and only if $p_\rho > 0$. Finally, we say that $\xi$ goes sharply through $K$ if it goes through $K$ with virtual multiplicities equal to the virtual ones and furthermore it has no singular points outside $K$. All germs going sharply through a consistent cluster are reduced and equisingular (cf. [5, Proposition 4.2.6]), or more generally, germs going sharply through similar consistent clusters are equisingular. Moreover, if $\xi$ goes sharply through $K$ and $p \in K$, $\xi$ has exactly $p_\rho$ branches going through $p$ and whose point in the first neighbourhood of $p$ is free and does not belong to $K$.

**Definition 2.1.** Given $p \in N_\Omega$, we denote by $K(p)$ the (irreducible cluster) consisting of the points $q \leq p$ such that $p_\rho = v_p = 1$ and $p_\rho = 0$ for every $q < p$. Thus, germs going sharply through $K(p)$ are irreducible, with multiplicity one at $p$, and its (only) point in the first neighbourhood of $p$ is free and non-singular.

Based on the Noether’s formula, it is possible to define the intersection number of a weighted cluster with a curve, or even two clusters, as

$$[\xi, K] = \sum_{p \in K} v_p e_p(\xi) \quad \text{and} \quad [K, K'] = \sum_{p \in K \cap K'} v_p v'_p.$$ 

In particular, the self-intersection of a weighted cluster is defined as $K^2 = \sum_{p \in K} v_p^2$.

Most of the examples of weighted clusters arise as base points of linear families of curves. A linear family is a set of the form $L = \{ f : f = 0 | f \in F \} \setminus \{ 0 \}$, where $F \subset O$ is a linear subspace of the local ring at $O$. We say that $L$ is a linear system if $F$ is an ideal, and a pencil if $\dim F = 2$. A pencil of lines is a pencil of the form $L = \{ ax - by = 0 | [a, b] \in \mathbb{P}^1 \}$ for some local coordinates $x, y$ centered at $O$. A linear system is relevant if the ideal defining it is not the whole ring $O$. The fixed part of a linear family $L$ (resp. linear system, pencil) is the curve $\eta$ defined by the greatest common divisor $g$ of the equations in $F$. Dividing out by $g$ we obtain a new linear family $L'$ (resp. linear system, pencil) without fixed part, which is the variable part of $L$.

Assume now that $L$ is a neat linear system (i.e. relevant without fixed part) defined by the ideal $I \subseteq O$. The multiplicity of $L$ at $0$ is $e = e_O(L) = \min\{ e_O(\xi) | \xi \in L \}$. If $p \in E_0$ is any point in the first neighbourhood of $O$, the pull-back of functions induces an injective homomorphism of rings $\varphi_p : O \rightarrow O_{S_{p}, p}$. We consider the ideal generated by $z^{-e} \varphi_p(I)$ (where $z$ is any equation for $E_0$ near $p$) as a sort of proper transform of $I$ at $p$, which defines a linear system $L_p$, called the transform of $L$ at $p$. This definition can be extended to any $p \in N_\Omega$, as well as the multiplicity $e_p(L) = e_p(L_p)$. Since $L$ has no fixed part, the set of points such that $L_p$ is relevant is finite, and hence is a cluster (weighted by the multiplicities), called cluster of base points of $L$ and denoted by $BP(L)$.

This construction may be applied to any linear family $L$ without fixed part defined by $F$, just taking the linear system defined by the ideal spanned by $F$, and in particular to pencils without fixed part. All germs in the linear family go through $BP(L)$, and generic ones (i.e. those defined by the equations in some Zariski-open subset of $F$) go sharply through it, miss any fixed finite set of points not in $BP(L)$, and in particular are reduced and have the same equisingularity class. In the particular case of a pencil $P$, this means that all but finitely many germs in $P$ go sharply through $BP(P)$, are reduced and equisingular. Moreover, any two such germs share exactly the points in $BP(P)$, and the self-intersection $BP(P)^2$ coincides with the intersection of two distinct germs in $P$. 

4
2.2 Polar germs and its base points.

In this section we remind the basic definitions and facts about polar germs of curve. We will assume \( \xi : f = 0 \) is a non-empty, reduced, singular germ of curve at \( O \). A polar of \( \xi \) is any germ given by the vanishing of the jacobian determinant

\[
P_\xi(f) : \frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = 0
\]

with respect to some local coordinates \((x,y)\) at \( O \), where \( g \) defines a smooth germ \( \eta \) at \( O \). The equation \( \eta \) actually defines a curve unless \( \xi \) is a multiple of \( \eta \) (in this case the determinant vanishes identically), which we assume not to hold from now on. We might even suppose that \( \eta \) is not a component of \( \xi \), since in this case the polar is composed by \( \eta \) and the polar of \( \xi - \eta \). The set of polar curves obtained in this way does not depend on the coordinates, but it actually depends on the equation \( f \), and not only on the curve \( \xi \) itself. However, this is not a problem because we are interested in properties of the polar curves depending only on \( \xi \). A polar is transverse if the curve \( \eta \) is not tangent to \( \xi \). Polar germs may be treated intrinsically by means of the jacobian ideal, defined as \( J(\xi) = (f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \subset O \). This ideal does not depend on the choice of the equation \( f \) for \( \xi \), and carries very deep information about the singularity of \( \xi \). Indeed, it was shown by Mather and Yau in [18] that two germs \( \xi_1, \xi_2 \) are analytically equivalent if and only if the rings \( O/J(\xi_1) \) and \( O/J(\xi_2) \) are isomorphic.

The jacobian ideal defines a linear system \( J(\xi) \) called the jacobian system of \( \xi \). Although all the polars belong to the jacobian system, the converse is not true. However, every germ in the jacobian system of multiplicity \( e_\xi(x) - 1 \) is indeed a polar curve. If \( \xi \) is reduced and singular, its jacobian system is neat and hence its generic members are reduced and go sharply through its cluster of base points \( BP(J(\xi)) \) (hence they are equisingular and, furthermore, they share all their singular points). This motivates the following

Definition 2.2. Let \( \zeta \) be a polar of a reduced singular curve \( \xi \). We say that \( \zeta \) is topologically generic if it goes sharply through \( BP(J(\zeta)) \).

The cluster \( BP(J(\xi)) \) is difficult to compute from its definition, but it can be shown (cf. [5] Corollary 8.5.7] and [21]) that it coincides with the cluster of base points \( BP\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \) of the pencil spanned by the partial derivatives of any equation of \( \xi \). But base points of pencils are easy to compute (see for instance the algorithm in [2]). The cluster \( BP(J(\xi)) \) is deeply related to the cluster of singular points of \( \xi \). As a first result, it contains all the free singular points of \( \xi \), but the most striking result is the following

Theorem 2.3. ([22] Theorem 8.6.4]) Let \( \xi_1 \) and \( \xi_2 \) be germs of curve, both reduced and singular. Then

1. If \( BP(J(\xi_1)) = BP(J(\xi_2)) \), then \( S(\xi_1) = S(\xi_2) \).
2. If \( BP(J(\xi_1)) \) and \( BP(J(\xi_2)) \) are similar weighted clusters, then \( \xi_1 \) and \( \xi_2 \) are equisingular.

The proof of Casas-Alvero works in two steps. The first one is to recover the polar invariants (which will be introduced below), and the second step is a procedure involving a careful tracking of the Newton polygon of the iterated strict transforms of a generic polar under blowing up. However, the major drawback of this proof is that it throws no light on the connection between the singular points of both objects: germ of curve and generic polars.

Our aim is to give a precise description of the relation between the singular points of the curve and those of its generic polars. This will provide a new alternative proof of Theorem 2.3. As a previous step we will also recover the polar invariants, but in contrast, our algorithm will give a different proof of the second step, avoiding the use of the Newton polygon and the tracking of the polars after successive blowing-ups.

A classical tool to study the relation between a germ and its polar curves are the polar invariants, which were introduced by Teissier in [22] and are numerical topological invariants of \( \xi \) closely related to its (transverse) polar curves. A point \( p \in N_0(\xi) \) is a rupture point of \( \xi \) if either there are at least two free points on \( \xi \) in its first neighbourhood, or \( p \) is satellite and there is at least one free point on \( \xi \) in its first neighbourhood. Equivalently, \( p \) is a rupture point if and only if the total transform \( \xi_p \) has three different
tangents. We denote by $R(\xi)$ the set of rupture points of $\xi$. More generally, if $p \in N_0$ is a free point, $R^p(\xi)$ denotes the subset of rupture points of $\xi$ which are either equal to $p$ or $p$-satellite. Note that all rupture points are singular, and also all maximal singular points are rupture points.

For any $p \in N_0$, take $\gamma^p$ to be any irreducible germ of curve going through $p$ and whose point in the first neighbourhood of $p$ is free and does not lie on $\xi$, and define the rational number

$$I(p) = I_\xi(p) = \frac{[\xi, \gamma^p]}{e^p(\gamma^p)} = \frac{[\xi, K(p)]}{\nu(\xi, K(p))},$$

which is independent of the choice of $\gamma^p$ and will be called invariant quotient at $p$. The polar invariants of $\xi$ are the invariant quotients $I(q)$ at the rupture points $q \in R(\xi)$. Note that they (as well as the invariant quotients) can be computed from an Enriques diagram of $\xi$, and hence are topological invariants of $\xi$. In fact, it was shown by Merle in [19] that if $\xi$ is irreducible, its equisingularity class is determined by its multiplicity at $O$ and by its polar invariants. Polar invariants have an interesting topological meaning which was given by Lê, Michel and Weber in [10].

We have defined the polar invariants without any mention to polar germs. Its relation to polar germs is given by the next

**Proposition 2.4.** ([5, Theorems 6.11.5 and 6.11.8]) Let $\zeta = P_0(\xi)$ be a transverse polar of a non-empty reduced germ of curve $\xi$, and let $\gamma_1, \ldots, \gamma_l$ be the branches of $\zeta$. Then

$$\left\{ \frac{[\xi, \gamma_i]}{e^0(\gamma_i)} \right\}_{i=1,\ldots,l} = \{I(q)\}_{q \in R(\xi)}.$$

Furthermore, if $p \in N_0(\xi)$ is either $O$ or any free point lying on $\xi$, the set of quotients $\frac{[\xi, \gamma]}{e^0(\gamma)}$, for $\gamma$ a branch of $\zeta$ going through $p$ and missing all free points on $\xi$ after $p$, is just $\{I(q)\}_{q \in R^p(\xi)}$.

### 2.3 Planar analytic morphisms.

We end the preliminary material summarizing some definitions and results concerning germs of morphisms between surfaces which will be used along the paper. We now consider two points $O \in S, O' \in T$ lying on two smooth surfaces. A germ of morphism of surfaces at them is a morphism $\varphi : U \to V$ defined on some neighbourhoods of $O$ and $O'$, such that $\varphi(O) = O'$. We will assume that the morphism is dominant, i.e. its image is not contained in any curve through $O'$, or equivalently the pull-back morphism $\varphi^* : O_{T,O'} \to O_{S,O}$ is a monomorphism. Since the surfaces are smooth, we can attach two systems of coordinates $(x,y)$ and $(u,v)$ centered at $O$ and $O'$ respectively, obtaining isomorphisms $O_{S,O} \cong \mathbb{C}[x,y]$ and $O_{T,O'} \cong \mathbb{C}[u,v]$. Under this isomorphisms, we denote by $h \in \mathbb{C}[x,y]$ the initial form of any $h \in O_{S,O}$, and by $o_0(h) = \deg h$ its order (and analogously for $h' \in O_{T,O'}$).

The pull-back of germs at $O'$ is defined by pulling back equations, and the push-forward, or direct image, of germs at $O$ is defined on irreducible germs and then extended by linearity. For an irreducible germ $\gamma$ at $O$ its push-forward $\varphi_*(\gamma)$ is defined as the image curve $\gamma = \varphi(\gamma)$ counted with multiplicity equal to the degree of the restriction $\varphi_\gamma : \gamma \to \sigma$. With this definitions, it holds the projection formula

$$[\xi, \varphi^*(\zeta)]_O = [\varphi_*(\xi), \zeta]_{O'}$$

for all germs of curve $\xi$ at $O$ and $\zeta$ at $O'$.

Let $(f(x,y), g(x,y))$ be the expression of $\varphi$ in the coordinates fixed above. The multiplicity of $\varphi$ is defined as $e_\varphi(\varphi) = n = n_\varphi \in \{ o_\varphi(f), o_\varphi(g) \}$. Consider now the pencil $P = \{ \lambda f + \mu g = 0 \}$. Its fixed part $\Phi$ is the contracted germ of $\varphi$, defined by $h = \gcd(f,g)$. If both $\frac{f}{g}$ and $\frac{g}{f}$ are non-invertible, the variable part $P'$ is a pencil without fixed part whose cluster of base points is by definition the cluster of base points of $\varphi$, denoted $BP(\varphi)$. The multiplicity $e_p(\varphi)$ of $\varphi$ at any point $p \in N_0$ infinitely near to $O$ is defined as the sum of $e_p(\Phi)$ and the virtual multiplicity of $BP(\varphi)$ at $p$. A point $p$ is fundamental of $\varphi$ if $e_p(\varphi) > 0$. The multiplicity can alternatively be extended to any $p \in N_0$ as the multiplicity of the composition $\varphi_p = \varphi \circ \pi_p$, which is denoted by $e(\varphi_p)$ or $n_p$ if the morphism is clear from the context. These two possible generalizations of the notion of multiplicity correspond respectively to the multiplicities and the values of a curve at a point. Indeed, they verify the following formula (see [8, Proposition 13.1])

$$e(\varphi_p) = e_p(\varphi) + \sum_{p \to q} e(\varphi_q).$$ (7)
So far we have attached to \( \varphi \) a cluster of points infinitely near to \( O \). There is a natural way to construct a cluster of points at \( O' \): the trunk of \( \varphi \). Let \( \mathcal{L} = \{ l_\alpha : \alpha \in \mathbb{P}_k^1 \} \) be a pencil of lines at \( O \), and consider its direct images \( \{ \gamma_\alpha = \varphi_*(l_\alpha) \} \). All but finitely many of them may be parametrized as 
\[
(u(t), v(t)) = (t^n, \sum_{i \geq n} a_i t^i)
\]
where \( n = e_O(\varphi) \) and the \( a_i \) may depend on \( \alpha \). Indeed, since \( \varphi \) is supposed to be dominant, at least one of them will depend on \( \alpha \). Since the coefficients of a Puiseux series determine the position of the points (cf. [2, Chapter 5]), all but finitely many of the \( \gamma_\alpha \) share a finite number of points with the same multiplicities. This cluster is independent of the choice of the pencil of lines \( \mathcal{L} \), it is denoted by \( T = T(\varphi) \), and it is called the (main) trunk of \( \varphi \). The smallest integer \( m = m_O \) such that \( a_m \) is not constant is the height of the trunk. These definitions can be extended to any \( p \in \mathcal{N}_O \) by considering the morphism \( \varphi_p \) instead of \( \varphi \). In [3, Section 10] it is developed an algorithm to compute the trunk of any morphism from its expression in coordinates.

The last concept we want to recall is the jacobian germ or \( \varphi \). It is defined as the germ
\[
J(\varphi) : \frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = 0,
\]
which is a germ of curve at \( O \) (the determinant does not vanish identically because \( \varphi \) is dominant). Note that when \( g \) defines a smooth germ, the jacobian germ is a polar of \( \xi : f = 0 \). One of the main results of [2] gives an explicit formula to compute the multiplicities of the jacobian germ from the multiplicities and the heights of the trunks of the composites \( \varphi_p \).

**Proposition 2.5.** ([3, Theorem 14.1]) For any point \( p \in \mathcal{N} \), we have
\[
e_p(J(\varphi)) = \begin{cases} m + n - 2 & \text{if } p = O, \\ m_p + n_p - m_p' - n_p' - 1 & \text{if } p \text{ is free and proximate to } p', \\ m_p + n_p - m_p' - n_p' - m_p'' - n_p'' & \text{if } p \text{ is satellite and proximate to } p' \text{ and } p''. \end{cases}
\]

In particular, we will use the following

**Corollary 2.6.** ([3, Corollary 14.4]) If \( p \) is a non-fundamental point of \( \varphi \), then \( m_p = m_p' + e_p(J(\varphi)) + 1 \) if \( p \) is free proximate to \( p' \), and \( m_p = m_p' + m_p'' + e_p(J(\varphi)) \) if \( p \) is satellite proximate to \( p' \) and \( p'' \). In any case, \( m_p > m_p' \).

### 2.4 The problem.

Our aim is to give an explicit algorithm which computes the cluster \( S(\xi) \) of singular points of a singular and reduced germ of curve \( \xi \) from the cluster of base points of the jacobian system \( BP(J(\xi)) \). In particular, we shall obtain a new proof of Theorem [2.3]. To achieve this, we reinterpret the problem in terms of the theory of planar analytic morphisms as follows.

Let \( (x,y) \) be a system of coordinates in a neighbourhood \( U \) of \( O \), \( f \) an equation for the germ \( \xi \), and \( \eta : g = 0 \) a smooth germ at \( O \) such that the point on \( \eta \) in the first neighbourhood of \( O \) is not in \( BP(J(\xi)) \) and \( \zeta = P_g(f) : \frac{\partial(f,g)}{\partial(x,y)} = 0 \) is a topologically generic transverse polar of \( \xi \). Note that being topologically generic is a generic property, and being transverse excludes finitely many tangent directions at \( O \), so the existence of such a \( \eta \) is guaranteed. The key observation is that we can think of the polar \( \zeta \) as the jacobian germ of the morphism \( \varphi : U \to \mathbb{C}^2 \) defined as \( \varphi(x,y) = (f(x,y), g(x,y)) \).

Let us first study the fundamental points of \( \varphi \). Since we are assuming \( \zeta \) to be transverse, we know that \( f \) and \( g \) share no factors, so \( \varphi \) has no contracted germ. Thus the only fundamental points of \( \varphi \) are its base points \( BP(\varphi) = BP(\{ \xi_\lambda : \lambda f + \lambda 2g = 0 \}) \). Note that \( \xi_{[1,0]} = \xi \) and \( \xi_{[0,1]} = \eta \). We have \( e_O(\xi_\lambda) = 1 \) for \( \lambda \neq [1,0] \), and so \( \nu_O(BP(\varphi)) = 1 \). Since the cluster of base points of a pencil is consistent, this forces \( BP(\varphi) \) to be irreducible and to have only free points with virtual multiplicity one. Moreover, its self-intersection is \( BP(\varphi)^2 = e_O(\xi) \), so \( BP(\varphi) \) consists of \( e_O(\xi) \) points lying on \( \eta \). We have thus proved the following
Lemma 2.7. The fundamental points of \( \varphi \) are exactly the first \( eo(\xi) \) points in \( N_O(\eta) \). In particular, there are no fundamental points in \( BP(\mathcal{J}(\xi)) \) but the origin \( O \).

Combining this result with formula (1) and Corollary 2.6 we obtain the following

Lemma 2.8. If \( p \neq O \) is either a base point of \( \mathcal{J}(\xi) \) or a satellite of one of them (or more generally, it is not a fundamental point of \( \varphi \)), then

\[
n_p = \sum_{q \to p} n_q, \quad m_p = m_{p'} + e_p(\xi) + 1 \text{ if } p \text{ is free, and } m_p = m_{p'} + e_p(\xi) \text{ if } p \text{ is satellite},
\]

while for \( p = O \) we have \( n_O = 1 \) and \( m_O = e_O(\xi) = e_O(\xi) + 1 \).

3 Tracking the behaviour of the invariant quotients

In this section we develop the main technical results which describe the behaviour of the invariant quotients \( I_\xi(p) \) as \( p \) ranges over \( N_O \), as well as its relation to the values \( e_p(\xi) \) and the heights \( m_p \) associated to the morphism \( \varphi \) introduced above.

3.1 Growth of the invariant quotients

First of all, we need to introduce a new order relation in \( N_O \). Recall (Definition 2.1) that for any \( p \in N_O \), \( K(p) \) is the irreducible cluster whose last point is \( p \).

Definition 3.1. Let \( q_1 \neq q_2 \) be two points infinitely near to \( O \), equal to or satellite of the free points \( p_1 \) and \( p_2 \) respectively. We say that \( q_1 \) is smaller than \( q_2 \) (or \( q_2 \) is bigger than \( q_1 \)) if \( p_1 \leq p_2 \) (with the usual order) and \( \nu_{p_1}(K(q_1)) \leq \nu_{p_2}(K(q_2)) \). Obviously, we denote by \( q_1 \preceq q_2 \) the situation in which \( q_1 \prec q_2 \) or \( q_1 = q_2 \), and similarly for \( q_2 \succeq q_1 \).

We introduce also the following relation between points and irreducible curves.

Definition 3.2. Let \( \gamma \) be any irreducible germ, let \( p \) be any free point and let \( q \) be either \( p \) or a \( p \)-satellite point. We say that \( q \) is smaller than \( \gamma \) (or that \( \gamma \) is bigger than \( q \)) if \( p \in N_O(\gamma) \) (or equivalently \( e_p(\gamma) > 0 \)) and \( \nu_{p}(K(q)) < \nu_{p}(K(\gamma)) \). We denote it \( q \prec \gamma \).

Remark 3.3. 1. Notice that the quotient \( \frac{\nu_{p}(K(q))}{\nu_{p}(K(\gamma))} \) is equal to \( e_{p}(\gamma) \) for any irreducible curve \( \gamma \) going through \( q \) and whose point in the first neighbourhood of \( q \) is free but not necessarily non-singular (i.e. the curve \( \gamma \) does not need to go sharply through \( K(q) \)). In particular, these definitions can be equivalently stated replacing each \( K(q) \) by such a curve \( \gamma \).

2. In particular, if \( p' \) is the point which \( p \) is proximate to, the quotient \( \frac{\nu_{p}(K(q))}{\nu_{p}(K(\gamma))} \) is independent of the point \( q \geq p \), so the previous definitions may be stated in terms of \( \frac{\nu_{p'}(K(q))}{\nu_{p'}(K(\gamma))} \) instead of \( \frac{\nu_{p}(K(q))}{\nu_{p}(K(\gamma))} \).

Remark 3.4. These definitions are closely related to the theory of valuations. Indeed, each point \( q \in N_O \) determines a (divisorial) valuation \( \nu_q \) of the local ring \( O_{S,O} \), given by \( \nu_q(f) = \frac{|K(p)\{f^{-1}(0)\}|}{\nu_{p}(K(\gamma))} \).

These valuations are normalized in such a way that the minimum value in the maximal ideal \( m_{S,O} \) is 1. This normalization allows valuations to take non-integral values, while in [13] they are normalized to take always values in \( \mathbb{Z} \). In this setting, it can be easily checked that \( p < \gamma \) if and only if \( \nu_p \leq \nu_q \). Analogously, \( q \prec \gamma \) is equivalent to \( \nu_q \leq \nu_{\gamma} \). However, it is no longer true that \( q \prec \gamma \) if and only if \( q \geq \gamma \), because curve valuations are maximal in the set of normalized valuations, so they can never be smaller than a divisorial valuation. For further information, see [13] Chapter 8, [11] and [9] Chapter 1.

It is also worth noting that the ordering between infinitely near points coincides with the ordering of the exceptional divisors of the dual graph of a composition of blowing-ups. More precisely, let \( \pi : \bar{S} \to S \) be the composition of blowing up all the points in some cluster containing \( q_1 \) and \( q_2 \), and let \( \Gamma \) be the dual graph of the exceptional locus. Then \( q_1 \prec q_2 \) if and only if the vertex corresponding to \( E_n \) belongs to the minimal path from \( E_0 \) to \( E_2 \). These two approaches may be related following [9, Chapter 6], where it is proved that the dual graph of any resolution can be embedded as a subtree of the valuative tree.
The following lemmas summarize the main properties of the order relation \( \prec \).

**Lemma 3.5.** Let \( p \in \mathcal{N}_O \) be any free point different from \( O \), proximate to \( p' \). Then:

1. The satellite point \( q \) in the first neighbourhood of \( p \) satisfies \( p' \prec q \prec p \).
2. If \( q \) is a \( p \)-satellite point, the two satellite points \( q_1, q_2 \) in its first neighbourhood may be ordered as \( p' \prec q_1 \prec q_2 \prec p \). We will call \( q_1 \) (resp. \( q_2 \)) the first (resp. second) satellite of \( q \). Moreover, every \( p \)-satellite point \( q' \) infinitely near to \( q_1 \) (resp. \( q_2 \)) satisfies \( q' \prec q \) (resp. \( q' \succ q \)).

**Proof.** The proof follows easily from the relation between the set of \( p \)-satellite points in \( \mathcal{K}(q) \) and the expansion as a continued fraction of the quotient \( \frac{\nu_O(K(q))}{\nu_O(K(q))'} \), combined with some elementary properties of continued fractions (see for instance [1] Remark 2.1 and Lemma 3.5). \( \square \)

**Definition 3.6.** For future reference, the satellite point \( q \) in the first neighbourhood of a free point \( p \) will also be called the first satellite of \( p \).

It will also be useful to study how a satellite point is ordered with respect to the two points which it is proximate to.

**Lemma 3.7.** Let \( q \) be a satellite point, proximate to \( q_1 \) and \( q_2 \), and assume \( q_1 \prec q_2 \). Then

\[ q_1 \prec q \prec q_2. \]

**Proof.** The proof can be carried out using the same tools as in the proof of Lemma 3.5. \( \square \)

We now turn to the relation between the ordering \( \prec \) and the growth of the invariant quotients \( I_\xi(p) \).

**Proposition 3.8.** Let \( p \neq O \) be a free point proximate to \( p' \), let \( q_1 \) be a \( p \)-satellite point and let \( q_2 \succ q_1 \) be either \( p \) or another \( p \)-satellite point. Then the following inequalities hold:

\[ I_\xi(p') \overset{(a)}{\leq} I_\xi(q_1) \overset{(b)}{\leq} I_\xi(q_2). \]

Moreover, equality holds in (a) if and only if \( p \notin \mathcal{N}_O(\xi) \), and equality holds in (b) if and only if there is no branch \( \gamma \) of \( \xi \) such that \( q_1 \prec \gamma \) (bigger than \( q_1 \)). In particular, note that equality in (a) implies equality in (b).

**Proof.** For any infinitely near point \( q \in \mathcal{N}_O \), let \( \gamma^q \) be any irreducible curve going through \( q \) and having a free point in its first neighbourhood which does not lie on \( \xi \). The first inequality, as well as the characterization of equality, is easily obtained computing the intersections \([\xi, \gamma^q] \) and \([\xi, \gamma^q_i] \) with Noether’s Formula [1] and applying Remark 3.3.

For the second inequality, let \( \xi_1, \ldots, \xi_k \) be the branches of \( \xi \) and expand each \( I_\xi(q_i) \) as

\[ I_\xi(q_i) = \frac{[\xi, \gamma^q_i]}{e_O(\gamma^q_i)} = \sum_{j=1}^{k} \frac{[\xi_j, \gamma^q_i]}{e_O(\gamma^q_i)}. \]

For branches \( \xi_1 \) not going through \( p \), we have\( \frac{[\xi_1, \gamma^q_i]}{e_O(\gamma^q_i)} = \frac{[\xi_1, \gamma^q_i]}{e_O(\gamma^q_i)} \) again by Noether’s Formula and Remark 3.3.

For the rest of the branches, following Proposition 2.5 in [1] we can write

\[ \frac{[\xi_1, \gamma^q_i]}{e_O(\gamma^q_i)} = \sum_{q<p} \frac{e_q(\xi_j)^2}{e_O(\xi_j)} + e_p(\xi_j) \min \left\{ \frac{e_p(\xi_j)}{e_O(\xi_j)}, \frac{e_p(\gamma^q_i)}{e_O(\gamma^q_i)} \right\}, \]

and so we just need to take care of the minimum in the last summand. In the case \( \frac{e_p(\xi_j)}{e_O(\xi_j)} \leq \frac{e_p(\gamma^q_i)}{e_O(\gamma^q_i)} \) this minimum is the same for \( i = 1, 2 \), while in the opposite case (i.e. when \( \xi_1 \succ q_1 \)) the minimum for \( i = 1 \) is strictly smaller than for \( i = 2 \), giving strict inequality in (b) as wanted. \( \square \)
Corollary 3.9. Let \( p \neq O \) be a free point proximate to \( p' \), and let \( q \) be a \( p \)-satellite point. Then the following inequalities hold:
\[
I_\xi(p') \leq I_\xi(q) \leq I_\xi(p).
\]
Moreover, equality holds in (b) if and only if there is no branch of \( \xi \) bigger than \( q \). Furthermore, we have \( I_\xi(p') = I_\xi(p) \) (equality both in (a) and (b)) if and only if \( p \notin N_O(\xi) \).

In particular, if there is no free point on \( \xi \) in the first neighbourhood of \( p \) and \( q^* \) denotes the biggest \( p \)-satellite point on \( \xi \), then \( I_\xi(q) < I_\xi(q^*) = I_\xi(p') = I_\xi(p) \) for all \( p \)-satellites \( q < q' < q^* \). If otherwise there is a free point on \( \xi \) in the first neighbourhood of \( p \), the same is true taking \( p \) instead of \( q^* \).

Proof. Just take \( q_2 = p \) and \( q_1 = q \) in Proposition 3.8. □

Also next corollary follows immediately.

Corollary 3.10. If \( p \in N_O(\xi) \) is a free point, all the polar invariants \( I_\xi(q) \) associated to points \( q \in R^p(\xi) \) are different.

Unfortunately, these results are not precise enough for our purposes, so we need a more sophisticated result which deals with a particular case.

Proposition 3.11. Let \( \xi \) be a germ of curve at \( O \), let \( p \in N_O \) be any free point different from \( O \), proximate to \( p' \). Assume there is exactly one branch \( \gamma \) of \( \xi \) going through \( p \) and whose point in the first neighbourhood of \( p \) is free, and suppose in addition that this point is non-singular (on \( \xi \)). Assume also that there is at least another branch of \( \xi \) going through \( p \) (whose point in the first neighbourhood of \( p \) must be satellite) and let \( q \in N_O(\xi) \) be the biggest \( p \)-satellite rupture point on \( \xi \). Then
\[
\frac{[\gamma \cdot \gamma^p]}{e_0(\gamma^p)} - \frac{1}{e_0(\gamma^p)} < \frac{[\gamma \cdot \gamma^q]}{e_0(\gamma^q)}, \quad \text{or} \quad \frac{1}{e_0(\gamma^q)} > \frac{[\gamma \cdot \gamma^q]}{e_0(\gamma^q)} = e_\mu(\gamma) \left( \frac{e_\mu(\gamma^p)}{e_0(\gamma^p)} - \frac{e_\mu(\gamma^q)}{e_0(\gamma^q)} \right), \quad (11)
\]
where \( \gamma^p \) and \( \gamma^q \) are as in the proof of Proposition 3.8, \( \gamma^p \) going sharply through \( K(p) \), and the last equality is a consequence of Proposition 2.5 in [1]. Now, noting that both \( \gamma \) and \( \gamma^p \) go sharply through \( K(p) \), we get \( e_\mu(\gamma^p) = e_\mu(\gamma) = e_\mu(\gamma) = 1 \) and the inequality in (11) becomes obvious. □

Remark 3.12. Propositions 3.8 and 3.11 are generalizations of Proposition 7.6.8 in [3], extending it to points not necessarily lying on \( \xi \) and giving more precise descriptions of some cases. Similar results can be found also in [15], where they are developed to tackle the converse of our problem: to determine from the plane singular curve (or its equisingularity class) the behaviour of its generic polar.

3.2 Relating the invariant quotients to the morphism

We keep the notations of the previous section: \( \eta : g = 0 \) is any smooth germ at \( O \) such that \( \zeta : P_\eta(f) = 0 \) is a topologically generic transverse polar, and so goes sharply through \( BP(\mathcal{J}(\xi)) \). Furthermore, we consider \( \zeta \) as the jacobian germ of the morphism \( \varphi = (f, g) : S \rightarrow \mathbb{C}^2 \). We now wish to study the relation between the invariant quotients \( I_\xi(p) \), the values \( v_p(\xi) \) of \( \xi \) and the multiplicities \( n_p \) and heights \( m_p \) of the morphisms \( \varphi \circ \pi_p \) for the points in \( BP(\mathcal{J}(\xi)) \) or satellite of them (or more generally, for any \( p \in N_O \) such that \( K(p) \cap N_O(\eta) = \{O\} \), i.e. any irreducible germ going through \( p \) is not tangent to \( \eta \)). We begin with an easy

Lemma 3.13. If \( p \in N_O \) belongs to \( BP(\mathcal{J}(\xi)) \) or is satellite of such a point (or more generally, \( K(p) \cap N_O(\eta) = \{O\} \)), then \( [\xi, K(p)] = v_p(\xi) \) and \( v_0(K(p)) = v_p(\eta) = n_p \). In particular
\[
I_\xi(p) = \frac{v_p(\xi)}{n_p}.
\]

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Proof. The intersection number $[\xi, K(p)]$ equals $[\xi, \gamma^p]$ for any $\gamma^p$ going sharply through $p$ and missing any point on $\xi$ in the first neighbourhood of $p$, and this intersection turns out to be $v_p(\xi)$. Indeed, if $\pi_p : S_p \rightarrow S$ is the composition of blowing-ups giving rise to $p$, then $\gamma^p = \pi_p(l_p)$ for some smooth curve $l_p$ at $p$ non-tangent to $\xi_p$. Then, by the projection formula (4), we have

$$[\xi, \gamma^p] = [\xi, \pi_p(l_p)] = [\pi_p^*(\xi), l_p] = [\xi_p, l_p] = e_p(\xi_p) v_p(l_p) = v_p(\xi).$$

For the second part, the virtual multiplicity $v_0(K(p))$ may be written as the intersection $[\eta, K(p)]$ (because $K(p) \cap N_0(\eta) = \{\Omega\}$), and thus $v_0(K(p)) = v_0(\eta)$ by the same reason as above. But the values $v_0(\eta)$ also satisfy the recursive formula of Lemma 2.8 with the same initial value $n_0 = 1 = O(\eta)$, and hence $e_O(\gamma^p) = n_0 e_0(\gamma^p)$. The last equality is immediate.

We now focus on the relation between the values and the heights.

**Proposition 3.14.** Let $p \in N_0$ be a base point of $J(\xi)$ or satellite of one of them (or more generally, such that $K(p) \cap N_0(\eta) = \{\Omega\}$). Then the inequality $v_p(\xi) \leq m_p$ holds, with equality if and only if the total transforms $\xi_p$ and $\eta_p$ at $p$ have non-homothetical tangent cones (counting multiplicities, or equivalently, considered as divisors on $E_p$, the first neighbourhood of $p$).

**Proof.** The proof is based on the algorithm given in Section 10 of [6] to compute the trunk of a morphism. This algorithm produces a sequence of pencils whose clusters of base points have strictly increasing heights (the definition of the height of a trunk works for any multiple of an irreducible cluster). It is immediate to check that the cluster in the first step of this algorithm has height exactly $v_p(\xi) = o(\varphi^p(u))$, and that the algorithm stops after this first step if and only if the initial forms of $\varphi^p(u)$ and $\varphi^p(v)$ are non-homothetical, which is equivalent to the total transforms $\xi_p$ and $\eta_p$ at $p$ having non-homothetical tangent cones.

We are now ready to state the main results relating the values and the heights:

**Theorem 3.15.** Let $p \in N_0$ be a base point of $J(\xi)$ or satellite of one of them (or more generally, such that $K(p) \cap N_0(\eta) = \{\Omega\}$). Then $v_p(\xi) \leq m_p$, with equality if and only if

- either $p$ is free and there is a free point proximate to $p$ lying on $\xi$ (in particular, $p$ lies on $\xi$),
- or $p$ is satellite and there exists a branch of $\xi$ which goes through the point of which $p$ is satellite, and this branch is not smaller than $p$.

Equivalently, $v_p(\xi) < m_p$ if and only if all branches of $\xi$ going through the point of which $p$ is satellite are smaller than $p$.

**Proof.** Let us first consider the case $p$ free. By Proposition 3.14 we know that $v_p(\xi) = m_p$ if and only if the total transforms $\xi_p$ and $\eta_p$ have non-homothetical tangent cones. Since $p$ is free, it is proximate to a single point $p'$. Let $E_p'$ be the germ (at $p$) of the exceptional divisor of $\pi_p : S_p \rightarrow S$. By definition, $\xi_p = v_p(\xi) E_p' + \xi_p$, and by the hypothesis on $p$, $\eta_p = v_p(\eta) E_p'$. So, $\xi_p$ and $\eta_p$ have homothetical tangent cones if and only if every branch of $\xi_p$ is also tangent to $E_p'$, which means that there is no free point in the first neighbourhood of $p$ lying on $\xi$. So, $v_p(\xi) = m_p$ if and only if there is some free point in the first neighbourhood of $p$ lying on $\xi$, as wanted.

Now let us deal with the case $p$ satellite, proximate to two points $q$ and $q'$. Assume that $q < q'$, so that $q < p < q'$ by Lemma 3.7 and let $p'$ denote the point which $p$ is satellite of. By definition and the hypothesis on $p$ we have $\xi_p = v_q(\xi) E_q + v_{q'}(\xi) E_{q'} + \xi_p$ and $\eta_p = n_q E_q + n_{q'} E_{q'}$. Let $a_q$ (resp. $a_{q'}$) denote the multiplicity of $E_q$ (resp. $E_{q'}$) in the tangent cone of $\xi_p$. Then $\xi_p$ and $\eta_p$ have homothetical tangent cones if and only if every branch of $\xi_p$ is tangent to either $E_q$ or $E_{q'}$ (equivalently, $a_q + a_{q'} = e_p(\xi)$) and

$$\frac{v_q(\xi) + a_q}{n_q} = \frac{v_{q'}(\xi) + a_{q'}}{n_{q'}}.$$

So assume $\xi_p$ and $\eta_p$ have homothetical tangent cones, which by the previous proposition means that $v_p(\xi) < m_p$, and take $a = \frac{v_q(\xi) + a_q}{n_q} = \frac{v_{q'}(\xi) + a_{q'}}{n_{q'}}$. Then on the one hand we have

$$a = \frac{v_q(\xi) + a_q + v_{q'}(\xi) + a_{q'}}{n_q + n_{q'}} = \frac{v_q(\xi) + v_{q'}(\xi) + e_p(\xi)}{n_p} = \frac{v_p(\xi)}{n_p} = I_p(\xi).$$
and on the other hand
\[ \alpha = I_\xi(q) + \frac{a_q}{n_q} \geq I_\xi(q) \quad \text{and} \quad \alpha = I_\xi(q') + \frac{a_{q'}}{n_{q'}} \geq I_\xi(q'). \]

But we have assumed \( q < p < q' \), and thus by Proposition 3.8 we have \( I_\xi(q) \leq I_\xi(p) \leq I_\xi(q') \), which combined with the above equalities implies that \( I_\xi(p) = I_\xi(q') (= \alpha) \) and \( a_{q'} = 0 \). This in turn implies (by Proposition 3.8) that every branch of \( \xi \) going through \( p' \) is smaller than \( p \), as wanted, and that \( a_q = e_p(\xi) \).

It remains to prove that if \( \bar{\xi}_p \) and \( \bar{\eta}_p \) have non-homothetical tangent cones (i.e. \( v_p(\xi) = m_p \)), then there is some branch of \( \xi \) going through \( p' \) which is not smaller than \( p \). But this case only may occur either if \( a_q + a_{q'} < e_p(\xi) \) or if \( v_a(\xi) + a_{q'} \neq v_a(\xi) + a_{q'} \). In the former case there is a branch of \( \xi \) through \( p \) whose point in its first neighbourhood is free, and such a branch is not smaller than \( p \). In the latter case we can assume that \( a_q + a_{q'} = e_p(\xi) \) (for if not we are in the previous case) and then we have that the quotient \( I_\xi(p) = \frac{v_a(\xi) + a_{q'}}{n_{q'}} \) fits between \( I_\xi(q) + \frac{a_q}{n_q} = \frac{v_a(\xi) + a_a}{n_a} \) and \( I_\xi(q') + \frac{a_{q'}}{n_{q'}} = \frac{v_a(\xi) + a_{q'}}{n_{q'}} \). Since \( p < q' \) implies \( I_\xi(p) \leq I_\xi(q') \), we are in fact in the situation
\[ I_\xi(q) + \frac{a_q}{n_q} < I_\xi(p) < I_\xi(q') + \frac{a_{q'}}{n_{q'}}. \]

Now we have to consider in which cases the second inequality holds. If we already have \( I_\xi(p) < I_\xi(q') \), then by Proposition 3.8 there exists a branch of \( \xi \) going through \( p' \) and bigger than \( p \), as we want. If otherwise \( I_\xi(p) = I_\xi(q') \), then \( a_{q'} > 0 \) and there is at least one branch of \( \xi \) whose strict transform at \( p \) is tangent to \( E_{q'} \). We are done because this branch is bigger than \( p \).

**Corollary 3.16.** If \( p \) is a rupture point of \( \xi \), then \( v_p(\xi) = m_p \).

**Proof.** Since \( p \) is a rupture point of \( \xi \), there is at least one branch of \( \xi \) going through it and whose point in the first neighbourhood if free. Such a branch clearly goes through the point of which \( p \) is satellite and is not smaller than \( p \). Thus, we have \( v_p(\xi) = m_p \) in virtue of Theorem 3.15.

### 4 Recovering the singular points from the base points of the polars

This section presents the main result of this paper, namely the procedure which recovers the weighted cluster of singular points \( S(\xi) \) (of a singular reduced germ of curve \( \xi \)) directly from the weighted cluster \( BP(J(\xi)) \) of base points of the jacobian system of \( \xi \). This procedure uses only invariants computable from the Enriques diagram of \( BP(J(\xi)) \) (weighted with the virtual multiplicities) and hence one of the strengths of this procedure is that it applies also to obtain the topological class of \( \xi \) directly from the similarity class of \( BP(J(\xi)) \).

The weighted cluster \( S(\xi) \) will be recovered in two steps: first we will recover the underlying cluster of \( S(\xi) \) (i.e. its set of points), which is equivalent to recover the set of rupture points \( R(\xi) \), since the maximal singular points of \( \xi \) are rupture points. Second we will recover the values of \( \xi \) at its singular points, which is equivalent to recover the virtual multiplicities of \( S(\xi) \) by means of the formula (3).

#### 4.1 Recovering rupture points

In order to recover the set of rupture points \( R(\xi) \), and hence recover the whole set of singular points of \( \xi \), just from \( BP(J(\xi)) \), we will argue as follows. Let \( D \) be the set of dicritical points of \( BP(J(\xi)) \). First we will show that, to each \( d \in D \), we can associate a uniquely determined rupture point \( q_d \in R(\xi) \), which will be recognized because of the equality of invariant quotients \( I_\xi(q_d) = I_\xi(d) \); moreover we will see that any rupture point is associated to some dicritical point in this way (see forthcoming Proposition 4.1). This approach has two main difficulties to be overcome. On one side, in spite we are able to compute the polar invariants \( \{ I_\xi(d) \}_{d \in D} \) from \( BP(J(\xi)) \) (see Lemma 4.3), we have no way to know the invariant quotient
$I_\xi(p)$ for whatever $p$, and hence the possibility to check equality $I_\xi(p) = I_\xi(d)$ (in order to identify the rupture point $q_d$ associated to $d$) is beyond our scope. On the other side, if $q_d$ happens to be $p_d$-satellite, then $q_d$ does not necessarily belong to $BP(J(\xi))$; and, in spite we manage to characterize $p_d$ (which actually belongs to $BP(J(\xi))$) in terms of the invariants $n_{p_d}$ and $m_{p_d}$ (see forthcoming Proposition 4.3), there might be many $p_d$-satellite points $q$ with the same invariant quotient $I_\xi(q) = I_\xi(q_d)$. We will solve these inconveniences by a cunning use of the invariants $I_\xi(q)$, $n_q$ and $m_q$ (thanks to the properties developed in Section 3.2), since, as we will exhibit, the heights $m_q$ can distinguish between the $p_d$-satellite points $q$ when the invariants $I_\xi(q)$ cannot.

Next we will develop the results that justify our procedure, which will be detailed as an algorithm at the end of the section.

**Proposition 4.1.** Let $d \in D$ be a dicritical point of $BP(J(\xi))$, and suppose $p_d$ is the last free point lying both on $\xi$ and $K(d)$. Then there exists a unique rupture point $q_d \in R^p(\xi)$ such that $I_\xi(q_d) = I_\xi(d)$. Furthermore, $q_d \preceq d$.

Moreover, any rupture point is associated to some dicritical point in this way.

**Proof.** Let $\gamma$ be a branch of a topologically generic transverse polar $\zeta$ of $\xi$ going sharply through $K(d)$ (such a $\gamma$ exists because $d$ is a dicritical point of $BP(J(\xi))$ and $\zeta$ goes sharply through it). Then $p_d$ is the last free point lying both on $\xi$ and $\gamma$, and the existence of a $q_d \in R^p(\xi)$ satisfying $I_\xi(q_d) = I_\xi(d) = [\gamma, \xi]_0(\gamma)$ is guaranteed by Proposition 2.3. Moreover, Proposition 2.3 also says that for any rupture point $q$ there exists a branch $\gamma'$ (not necessarily unique) of $\zeta$ such that $I_\xi(q) = [\gamma', \xi]_0(\gamma')$, and that if $p$ is the point of which $q$ is satellite, then $p$ is the last free point lying both on $\xi$ and $\gamma'$. So it only remains to prove that the same branch $\gamma$ cannot work for several rupture points, which is equivalent to prove the uniqueness of $q_d$.

The case $p_d = O$ is quite easy, since $O$ has no $O$-satellite points, and thus $q_d = O$ is the only possibility.

For the rest of the proof assume $p_d \neq O$, and suppose that $q_1 < q_2$ are two rupture points of $\xi$ equal to or satellite of $p_d$ and such that $I_\xi(q_1) = I_\xi(q_2) = I_\xi(d)$. By Proposition 3.8 no branch of $\xi$ can be bigger than $q_1$. But since $q_2$ is a rupture point, there exists a branch of $\xi$ going through $q_2$ and having a free point in its first neighbourhood, and such a branch is clearly bigger than $q_1$, which leads to a contradiction. Therefore, there exists a unique $p_d$-satellite rupture point $q_d$ satisfying $I_\xi(q_d) = I_\xi(d)$.

In order to prove that $q_d \preceq \gamma$, which is equivalent to $q_d \preceq d$, note that we can consider $\frac{d}{\partial f} \frac{\partial f}{\partial x}$ as $I_\xi(q')$, where $q'$ is the last $p_d$-satellite point on $\gamma$ (because $p_d$ is the last free point lying both on $\gamma$ and $\xi$). Then $I_\xi(q_d) = I_\xi(q')$, and again by Proposition 3.8 we obtain that $q_d \preceq q'$, which implies $q_d \preceq \gamma$ by definition. 

**Corollary 4.2.** The number of rupture points of a reduced singular curve $\xi$ is bounded above by the number of dicritical points of $BP(J(\xi))$.

From now on, if $d \in D$ is a dicritical point of $BP(J(\xi))$, $p_d$ will denote the last free point lying both on $\xi$ and $K(d)$, and $q_d$ will stand for the rupture point associated to $d$ according to Proposition 4.1. Note that $q_d$ may be either equal to or satellite of $p_d$. As a particular case, if $O \in D$, then $d = O$ because it is the only point $\preceq O$. However, determining $q_d$ in the case $d \neq O$, which we assume from now on, is not so easy and needs some more work.

The first step to determine $q_d$ is to compute the polar invariant $I_\xi(q_d) = I_\xi(d) = \frac{[\text{KP}(d)]}{m_{\xi}(d)}$ from $BP(J(\xi))$, and we can do it thanks to the following

**Lemma 4.3.** If $d \in D$ is a dicritical point of $BP(J(\xi))$, then $I_\xi(d) = \frac{[\text{KP}(J(\xi))_K(d)]}{n_d} + 1$.

**Proof.** Let $\gamma$ be a branch of a topologically generic transverse polar $\zeta$ of $\xi$ going sharply through $K(d)$. So, proving the statement is equivalent to prove $I_\xi(q_d) = I_\xi(q) = \frac{[\text{KP}(J(\xi))_K]}{\text{m}_{\xi}(\gamma)} + 1$. By definition, there exists some equation $f$ of $\xi$ and some smooth germ $g = 0$ such that $\zeta$ is given by the equation $\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} = 0$. Up to change of coordinates, we may assume $g = x$, and thus $\zeta : \frac{\partial f}{\partial g} = 0$.

Since $BP(J(\xi)) = BP\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, all but finitely many germs $\zeta'$ of the pencil $\left\{\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = 0\right\}$ go sharply through $BP(J(\xi))$ and miss the first point lying on $\gamma$ and not in $BP(J(\xi))$. Then, for any such
Proposition 4.4. Let \( \zeta' \) be a dicritical point of \( \mathcal{B} P (J (\xi)) \) associated (according to Proposition 4.1) to the rupture point \( q_d \), which is either free and equal to or satellite of \( p_d \). Assume \( \zeta' \) is the last point \( \zeta' < d \) such that \(\frac{m_{\zeta'}}{n_{\zeta'}} < I_\xi (d)\) and its next point in \( K (d) \) is free. Then \( p_d \) is the next point of \( \zeta' \) in \( K (d) \).

Proof. Suppose \( p_d \) is proximate to \( p_d' \). Since \( q_d \leq d \), we must have \( p_d \leq d \), and hence \( p_d' < d \). Moreover, combining Proposition 3.8 and Theorem 3.15 we obtain that \( v_{p_d'} (\xi) = m_{p_d'} \) and

\[
I_\xi (p_d') = \frac{m_{p_d'}}{n_{p_d'}} < I_\xi (q_d).
\]

So, among all points \( \zeta' < d \) whose next point in \( K (d) \) is free, \( p_d' \) must satisfy \(\frac{m_{p_d'}}{n_{p_d'}} < I_\xi (d)\). We need to show that indeed \( p_d' \) is the last point with such property. Let \( O < p_1 < p_2 < \ldots < p_k \) be the free points in \( K (d) \), and for each \( i \geq 1 \) let \( p_i' \) the point to which \( p_i \) is proximate. Then \( p_i' + 1 \) is either equal to or satellite of \( p_i \), and hence Proposition 3.8 gives

\[
I_\xi (p_i') \leq I_\xi (p_i' + 1) \leq I_\xi (p_i) \quad \text{for all } 1 \leq i < k,
\]

where the inequality (a) is strict if and only if \( p_i' \leq p_i' \), since this is equivalent to \( p_i \) lying on \( \xi \). In particular, the sequence \( \{I_\xi (p_i')\}_{i=1}^k \) is strictly increasing up to \( p_d' \), and it becomes constant after that.

Suppose now to get a contradiction that \( p_r' = p_r \) for some \( 1 \leq r \leq k \), but that it is not the last \( p_i' \) such that \(\frac{m_{p_r'}}{n_{p_r'}} < I_\xi (d)\), i.e. assume \( r < k \) and \(\frac{m_{p_s'}}{n_{p_s'}} < I_\xi (d)\) for some \( r < s \leq k \). This implies that

\[
I_\xi (p_s') \leq \frac{m_{p_s'}}{n_{p_s'}} < I_\xi (d),
\]

but since \( p_r \) is the last free point lying both on \( \xi \) and \( K (d) \), it holds the equality \( I_\xi (p_s') = I_\xi (d) \), which leads to a contradiction and we are done.

Now that we have determined \( p_d \), it only remains to know which of its satellite points is \( q_d \). The problem is that there might be many points \( q \), equal to or satellite of \( p_d \), with the same invariant quotient \( I_\xi (q) = I_\xi (d) \). Moreover, although Corollary 3.10 implies that \( q_d \) is the smallest (by \( \prec \)) such point, there is no way to determine it explicitly from the last \( p_d \)-satellite point in \( K (d) \). Fortunately, the \( p_d \)-satellite points \( q \) bigger than \( q_d \) and with the same invariant are exactly the points for which \( v_q (\xi) < m_q \) (Theorem 3.15), and this fact enables us to solve this case. In other words, the heights \( m_q \) can distinguish between
the \( p_d \)-satellite points when the invariants \( I_d(q) \) cannot. This fact allows us to develop an algorithm which computes \( q_d \) just from the polar invariant \( I_d(d) \) and the already determined \( p_d \), by seeking the unique point \( q \) which is either equal to or satellite of \( p_d \) and for which the equality \( \frac{m_q}{n_q} = I_d(q_d) = I_d(d) \) holds. In fact, it computes step by step all the intermediate points \( p_d = q_0 < q_1 < \cdots < q_{k-1} < q_k = q_d \) (where \( q_i \) is in the first neighbourhood of \( q_{i-1} \)).

The procedure works as follows:

- **Start with** \( i = 0 \) and \( q_0 = p_d \).
- **While** \( \frac{m_{q_i}}{n_{q_i}} \neq I_d(d) \) do
  - If \( \frac{m_{q_i}}{n_{q_i}} > I_d(d) \) take \( q_{i+1} \) to be the first satellite of \( q_i \).
  - If \( \frac{m_{q_i}}{n_{q_i}} < I_d(d) \) take \( q_{i+1} \) to be the second satellite of \( q_i \).
- Increase \( i \) to \( i + 1 \).
- If \( \frac{m_{q_k}}{n_{q_k}} = I_d(d) \), end by taking \( k = i \) and \( q_d = q_k \).

**Theorem 4.5.** Keep the above notations. The above procedure ends after a finite number of steps, and actually computes the rupture point \( q_d \).

**Proof.** First of all, note that since \( q_0 \) is a rupture point, Corollary 3.10 implies that \( I_d(d) = I_d(q_0) = \frac{m_{q_0}}{n_{q_0}} \). Therefore, since there are finitely many points between \( p_d \) and \( q_d \), it is enough to check that each \( q_i \) actually precedes \( q_d \) and that if \( \frac{m_{q_i}}{n_{q_i}} = I_d(d) \) then \( q_i = q_d \).

To see that \( q_i \leq q_d \) for each \( i \) we use induction on \( i \). For \( i = 0 \), we have \( q_0 = p_d \), and hence \( q_0 = p_d \leq q_d \) by definition of \( p_d \). Now suppose we have reached the step \( i \) of the algorithm and we have to perform another step. This means that \( q_i \leq q_d \) and \( \frac{m_{q_i}}{n_{q_i}} \neq I_d(d) \). We know that in this case \( q_i < q_d \), and we claim that the point \( q_{i+1} \) computed by the algorithm still precedes \( q_d \). Indeed, since \( p_d \leq q_i < q_d \) and \( q_d \) is \( p_d \)-satellite, the point in the first neighbourhood of \( q_i \) preceding \( q_d \) must be satellite. Hence, it only remains to check that the choice made by the algorithm is the correct one.

- If \( \frac{m_{q_i}}{n_{q_i}} < I_d(d) \), then \( I_d(q_i) = \frac{v_{q_i}(\xi)}{n_{q_i}} < \frac{m_{q_i}}{n_{q_i}} < I_d(d) \) by Theorem 3.15. Therefore, by Lemma 3.3 and Proposition 3.8 the next point \( q_{i+1} \) must be the second satellite point, for if it was the first one the invariants \( I_d(q) \) would be strictly smaller than \( I_d(d) \) for every satellite \( q \geq q_{i+1} \).
- If \( \frac{m_{q_i}}{n_{q_i}} > I_d(d) \), then either \( I_d(d) < I_d(q_i) \leq \frac{m_{q_i}}{n_{q_i}} \) or \( I_d(q_i) \leq I_d(d) < \frac{m_{q_i}}{n_{q_i}} \). In the former case we apply Lemma 3.3 and Proposition 3.8 as above to see that \( q_{i+1} \) must be the first satellite point of \( q_i \). In the latter case we have that \( v_{q_i}(\xi) < m_{q_i} \), and hence by Theorem 3.15 every branch of \( \xi \) through \( p_d \) is smaller than \( q_i \). This implies in particular that \( q_d < q_i \), and thus by Lemma 3.3 \( q_d \) must be infinitely near to the first satellite of \( q_i \).

In any case, the algorithm is correct.

In order to complete the proof, we must check that the algorithm does not stop before reaching the point \( q_d \). That is, we have to show that if \( q \) is either \( p_d \) or any \( p_d \)-satellite point strictly preceding \( q_d \), then \( \frac{m_q}{n_q} \neq I_d(q_d) \).

- If \( q < q_d \), any branch of \( \xi \) going through \( q_d \) is bigger than \( q \). Then Proposition 3.8 implies that \( I_d(q) < I_d(d) \), and by Theorem 3.15 we also have that \( v_q(\xi) = m_q \). So \( I_d(q) = \frac{m_q}{n_q} < I_d(d) \) and in particular \( \frac{m_q}{n_q} \neq I_d(d) \).
- Consider now the case \( q > q_d \). Then, on the one hand Proposition 3.8 implies that \( I_d(d) \leq I_d(q) \), with equality if and only if every branch of \( \xi \) going through \( p_d \) is not bigger than \( q_d \). On the other hand, Theorem 3.15 says that \( v_q(\xi) \leq m_q \), and equality holds if and only if there is some branch of \( \xi \) not smaller than \( q \). Summarizing, we have \( I_d(d) \leq I_d(q) = \frac{m_q}{n_q} \), and having equality \( I_d(d) = \frac{m_q}{n_q} \) would imply (by Theorem 3.15) that there is some branch of \( \xi \) through \( p_d \) which is not smaller than \( q \). But such a branch would be bigger than \( q_d \), implying (by Proposition 3.8) that \( I_d(d) < I_d(q) \leq \frac{m_q}{n_q} \) and thus contradicting the equality \( I_d(d) = \frac{m_q}{n_q} \).
4.2 Recovering values

This section is devoted to explaining how the values of a curve $\xi$ at its singular points can be recovered from the invariants $m_p$ and $n_p$, provided the set of rupture points $R(\xi)$ (and hence the set of singular points $S(\xi)$) is already known (for example, if it has been determined as explained in the previous section).

Recall that from Lemma 3.13 we already know that $v_p(\xi) = n_p I_\xi(p)$ at any $p \in S(\xi)$, but that the difficulty lies on the computation of the invariant quotient $I(\xi)$.

We will consider the different cases ordered by increasing difficulty: we will start with the rupture points (which are the easiest ones), followed by the free singular points, and we will finish by considering the satellite points which are not rupture points (the most complicated case).

The easiest cases are rupture points, because Corollary 3.14 implies that $v_p(\xi) = m_p$ for any $p \in R(\xi)$.

Let us now consider $p \in S(\xi)$ a free singular point which is not a rupture point. By Theorem 3.15 we have the equality $v_p(\xi) = m_p$ if and only if there is a free point in the first neighbourhood of $p$ lying on $\xi$. In particular, if there is a free singular point in the first neighbourhood of $p$, we can also assert that $v_p(\xi) = m_p$. But what can we say if there are no free singular points in the first neighbourhood of $p$ and $p$ is not a rupture point? This situation is in which Proposition 3.11 plays an important role. So suppose that $p$ is a free singular point of $\xi$, but that it is not a rupture point and that there is no free singular point on $\xi$ in its first neighbourhood. This means that there is at most one free point lying on $\xi$ in the first neighbourhood of $p$, and if it exists, it is non-singular. If there is no such a point, then Corollary 3.9 implies that

$$v_p(\xi) = n_p I_\xi(p) = n_p I_\xi(q) = \frac{n_p}{n_q} v_q(\xi) = \frac{n_p}{n_q} m_q,$$

where $q$ is the biggest $p$-satellite point in $R(\xi)$. On the contrary, if there is one such point, Proposition 3.11 Lemma 3.15 and Corollary 3.16 give the inequalities

$$\frac{v_p(\xi) - 1}{n_p} < \frac{v_q(\xi)}{n_q} = \frac{m_q}{n_q} < \frac{v_p(\xi)}{n_p},$$

which are equivalent to

$$\frac{n_p}{n_q} m_q < v_p(\xi) < \frac{n_p}{n_q} m_q + 1,$$

where as before $q$ is the biggest $p$-satellite point in $R(\xi)$. Hence, in any case, $v_p(\xi)$ belongs to the real interval $\left[\frac{n_p}{n_q} m_q, \frac{n_p}{n_q} m_q + 1\right]$. But the width of this interval is one, so there is exactly one integer in it, and thus the value $v_p(\xi)$ is uniquely determined.

So far we have proved the following

**Proposition 4.6.** Let $p \in S(\xi)$ be a free singular point which is not a rupture point.

- If there is a free singular point in the first neighbourhood of $p$, then $v_p(\xi) = m_p$.
- Otherwise, let $q$ be the biggest point in $R^p(\xi)$ (which must be non-empty). Then $v_p(\xi)$ is the only integer in the interval $\left[\frac{n_p}{n_q} m_q, \frac{n_p}{n_q} m_q + 1\right]$.

Moreover, the equality $v_p(\xi) = \frac{n_p}{n_q} m_q$ holds if and only if there is no branch of $\xi$ going through $p$ and whose point in the first neighbourhood of $p$ is free.

It only remains to consider the case of satellite points $p \in S(\xi)$ which are not rupture points, and it is solved by the next

**Proposition 4.7.** Let $p \in S(\xi)$ be a satellite point of $\xi$ which is not a rupture point. Suppose moreover that $p$ is satellite of $p' \in S(\xi)$ and let $q$ be the biggest point in $R^{p'}(\xi)$. Then

$$v_p(\xi) = \begin{cases} \frac{n_p}{n_q} v_p'(\xi) & \text{if } p \succ q \text{ and } v_p'(\xi) = \frac{n_p}{n_q} m_q \text{ both hold,} \\ \frac{n_p}{m_p} & \text{otherwise.} \end{cases}$$
Proof. If \( p' = q \) is itself a rupture point, there exists a branch of \( \xi \) going through \( p' \) and having a free point in its first neighbourhood, and the same holds if otherwise \( p' \neq q \) but \( v_p(\xi) \neq \frac{n \omega}{n_q} m_q \) (by Proposition 4.10). Thus, in any case Theorem 3.15 implies that \( v_p(\xi) = m_p \frac{n \omega}{n_q} m_q \). Then there is no branch of \( \xi \) going through \( p' \) and having a free point in its first neighbourhood. If furthermore \( p < q \), Theorem 3.15 applies to give \( v_p(\xi) = m_p \) again, but if otherwise \( p > q \), Corollary 3.9 gives that

\[
v_p(\xi) = n_p I_\xi(p) = n_p I_\xi(p') = \frac{n_p}{n_{p'}} v_{p'}(\xi).
\]

As a consequence of the proof of Proposition 4.14 we infer the following result, which determines for which free points \( p \in S(\xi) \) (besides the rupture points) there exists some branch of \( \xi \) which goes through \( p \) and is non-singular after it.

**Corollary 4.8.** Let \( p \in S(\xi) \) be a free singular point. Then there is some branch of \( \xi \) non-smaller than \( p \) if and only if either \( p \) is itself a rupture point or \( v_p(\xi) \neq \frac{n \omega}{n_q} m_q \) (where \( q \) is the biggest \( p \)-satellite rupture point of \( \xi \)).

### 4.3 The algorithm

In this section, we present explicitly the algorithm which computes the cluster \( S(\xi) \) and the values \( v_p(\xi) \) for \( p \in S(\xi) \) from the weighted cluster \( BP(\mathcal{J}(\xi)) \) of base points of the polar germs of \( \xi \).

The algorithm:

- **Recovering the rupture points and the singular points.**
  1. Start with \( \mathcal{R} = \mathcal{S} = \emptyset \), and let \( \mathcal{D} \) be the set of dicritical points of \( BP(\mathcal{J}(\xi)) \).
  2. If \( O \in \mathcal{D} \), then set \( \mathcal{R} = \mathcal{S} = \{O\} \).
  3. For each \( d \in \mathcal{D} - \{O\} \) do:
     
     (a) Compute \( I = \left\lfloor \frac{BP(\mathcal{J}(\xi) \cup K(d))}{n_q} \right\rfloor + 1 \).
     
     (b) Find the last point \( p' < d \) such that \( \frac{m_{p'}}{n_q} < I \) and its next point \( p \) in \( K(d) \) is free.
     
     (c) Take \( i = 0 \) and \( q_0 = p \).
     
     (d) While \( \frac{m_{q_i}}{n_{q_i}} \neq I \) do
         
         - If \( \frac{m_{q_i}}{n_{q_i}} > I \), take \( q_{i+1} \) to be the first satellite of \( q_i \).
         
         - If \( \frac{m_{q_i}}{n_{q_i}} < I \), take \( q_{i+1} \) to be the second satellite of \( q_i \).
         
         Increase \( i \) to \( i + 1 \).
     
     (e) If \( \frac{m_{q_i}}{n_{q_i}} = I \), set \( \mathcal{R} = \mathcal{R} \cup \{q_i\} \) and \( \mathcal{S} = \mathcal{S} \cup \{q | q \leq q_i\} \).

- **Recovering values.**
  1. For each \( p \in \mathcal{R} \) set \( v_p = m_p \).
  2. For each free point \( p \in \mathcal{S} - \mathcal{R} \)
      
      - If there is a free point both in \( \mathcal{S} \) and in the first neighbourhood of \( p \), set \( v_p = m_p \).
      
      - Otherwise, let \( q \) be the biggest \( p \)-satellite point in \( \mathcal{R} \) and set \( v_p \) the only integer in the interval \( \left[ \frac{m_p}{n_q} m_q, \frac{n_p}{n_q} m_q + 1 \right] \).
  3. For each satellite point \( p \in \mathcal{S} - \mathcal{R} \), let \( p' \) be the free point of which \( p \) is satellite, and let \( q \) be the biggest point in \( \mathcal{R} \) which is either equal to or satellite of \( p' \).
      
      - If \( p > q \) and \( v_{p'} = \frac{n_{p'}}{n_{p'}} m_q \) both hold, set \( v_p = \frac{n_p}{n_{p'}} v_{p'} \).
– Otherwise, set \( v_p = m_p \).

From the previous results in this section, it is immediate to check the following

**Theorem 4.9.** The previous algorithm actually computes the set of rupture points and the cluster of singular points of \( \xi \), together with the values of \( \xi \) at these points. More precisely, at the end of the algorithm we have \( R(\xi) = R \), \( S(\xi) = S \) and \( v_p(\xi) = v_p \) for any \( p \in S(\xi) \).

**Remark 4.10.** This algorithm gives a proof of the first statement in Theorem 2.3. Furthermore, it is obvious that the algorithm yields similar clusters if it is applied to similar clusters, so in fact it also proves the second statement in Theorem 2.3, as we wanted.

By combining the algorithm given in this paper with the one in [2], which computes \( BP(\mathcal{J}(\xi)) = BP(\partial f/\partial x, \partial f/\partial y) \) from the partial derivatives of any equation of \( \xi \) we obtain the procedure which has been followed when dealing with the examples (see Section 4.4):

**Corollary 4.11.** The cluster of singular points \( S(\xi) \) of any reduced singular curve \( \xi : f = 0 \) is determined and may be computed from any two polars \( P_\gamma(f) \) and \( P_{\bar{\gamma}}(f) \), provided \( q_1 \) and \( q_2 \) have different tangents, regardless whether they are topologically generic or even transverse ones.

Given \( d \in D \) the algorithm yields its associated rupture point \( q_d \), once the free point \( p_d \) to which \( q_d \) is satellite is already found. However, there are some cases for which this task is quite straightforward: if there are two or more rupture points equal to or satellite of the same free point, then all but for the biggest rupture point (which is easily recognized as that having the greatest invariant quotient) are geometrically characterized as follows (recall that by Corollary 3.10 all the polar invariants associated to points in \( R^{pd}(\xi) \) are different).

**Proposition 4.12.** Let \( d \in D \) be a dicritical point of \( BP(\mathcal{J}(\xi)) \) with polar invariant \( I = I_\xi(d) \), and suppose \( p_d \) is the last free point lying both on \( \xi \) and \( K(d) \). Assume that there exists another dicritical point \( d' \in D \) for which \( p_{d'} = p_d \) but whose polar invariant \( I' = I_{\xi}(d') \) is greater than \( I \). Then \( q_d \) is the last \( p_d \)-satellite point in \( K(d) \).

**Proof.** Suppose the claim is false and let \( \bar{q}_d \) be the last \( p_d \)-satellite point in \( K(d) \). Proposition 4.1 implies that \( \bar{q}_d \succeq q_d \), and hence \( q_d \succeq \bar{q}_d \). Moreover, since \( p_d \) is the last free point lying both on \( \xi \) and \( K(d) \), we can take indistinctly \( \gamma_d \) or \( \gamma_{\bar{q}_d} \) and compute

\[
I_{\xi}(\bar{q}_d) = \frac{[\xi, \gamma_{\bar{q}_d}]}{e_0(\gamma_{\bar{q}_d})} = \frac{[\xi, \gamma_d]}{e_0(\gamma_d)} = I.
\]

If \( q_{d'} \) is the rupture point associated to \( d' \), we claim that \( q_{d'} \succeq q_d \). Indeed, if it is not the case, Proposition 3.8 would imply that \( I' = I_{\xi}(q_{d'}) \leq I_{\xi}(\bar{q}_d) = I \) contradicting our hypothesis. Therefore, there exists some branch of \( \xi \) bigger than \( q_d \), and then Proposition 3.8 again will give \( I_{\xi}(q_d) < I_{\xi}(\bar{q}_d) = I \), which contradicts that \( q_d \) is the rupture point associated to \( d' \).

Based on Proposition 4.12 we present an alternative version of the algorithm for the part of recovering the rupture and the singular points. This apparently longer version gives a more precise and geometrical description of some of the rupture points \( q_d \), for which also avoids the tedious task of performing the iterations in step (d).

**Proposition 4.13.** An alternative presentation of the algorithm is obtained by replacing the first part by

1. Start with \( R = S = \emptyset \), and let \( D \) be the set of dicritical points of \( BP(\mathcal{J}(\xi)) \).
2. If \( O \in D \), then set \( R = S = \{O\} \)
3. For each \( d \in D - \{O\} \) compute \( I_d = \frac{BP(\mathcal{J}(\xi), K(d))}{1} \), and order \( D - \{O\} = \{d_1, \ldots, d_k\} \) by descending order of \( I_d \) (i.e., \( I_{d_1} \geq I_{d_2} \geq \ldots \geq I_{d_k} \)).
4. For each \( j = 1, \ldots, k \) do:
(a) Find the last point \( p' < d_j \) such that \( \frac{m_{p'}}{n_{p'}} < I_{d_j} \) and its next point \( p \in \operatorname{K}(d_j) \) is free.

(b) If \( p \) has already appeared at this step, let \( q_j \) be the last \( p \)-satellite point in \( \operatorname{K}(d_j) \) and set \( R = R \cup \{q_j\} \) and \( S = S \cup \{q \mid q \leq q_j\} \). Then skip to the next \( j \).

(c) Otherwise, take \( i = 0 \) and \( q_0 = p \).

(d) While \( \frac{m_{q_i}}{n_{q_i}} \neq I_{d_j} \) do

- If \( \frac{m_{q_i}}{n_{q_i}} > I_{d_j} \), take \( q_{i+1} \) to be the first satellite of \( q_i \).
- If \( \frac{m_{q_i}}{n_{q_i}} < I_{d_j} \), take \( q_{i+1} \) to be the second satellite of \( q_i \).

Increase \( i \) to \( i + 1 \).

(e) If \( \frac{m_{q_i}}{n_{q_i}} = I_{d_j} \), set \( R = R \cup \{q_i\} \) and \( S = S \cup \{q \mid q \leq q_i\} \).

4.4 Examples

Let us illustrate through some examples the application of the algorithm for the computation of the weighted cluster of singular points \( S(\xi) \) of a singular reduced germ of curve \( \xi \) from the weighted cluster \( BP(J(\xi)) \) of base points of the jacobian system of \( \xi \). We work each example as follows: we start from an equation \( f \) of \( \xi \) and then we present our initial data, the cluster of base points \( BP(J(\xi)) \) which has been computed using the algorithm given in \([2]\) (this part will not be explained in any case). Then we apply to \( BP(J(\xi)) \) the algorithm of subsection \([4.3]\) to recover the cluster \( S(\xi) \) with the corresponding values, showing the invariants \( \frac{m_{q_i}}{n_{q_i}} \) computed and explaining how the algorithm works. At the end, it can be checked that our output coincides with the original \( \xi \).

For each example of singular curve \( \xi \), four Enriques diagrams will be shown: the first one shows the equisingularity class of the original curve \( \xi \). The second one contains the names of the singular points of \( \xi \) and the base points of \( J(\xi) \), where the dots in each square mean that there are as many free points as the number in the same square. The third diagram represents the cluster \( BP(J(\xi)) \) with its virtual multiplicities, and the fourth one shows the heights of the trunks \( m_p \) and the multiplicities \( n_p \) of the morphism \( \varphi_p \) for each \( p \in S(\xi) \cup BP(J(\xi)) \). We start with a pair of simple examples, which are classical in the literature about polars and were given by Pham \([20]\) in order to prove that the equisingularity class of a curve does not determine the equisingularity class of its topologically generic polars. Namely, the curve \( \xi \) of Example \([4.14]\) and that of Example \([4.15]\) are equisingular, while its topologically generic polars are not. Observe that nor are similar their respective clusters \( BP(J(\xi)) \), proving also that the reciprocal of Theorem \([2.9]\) does not hold.

**Example 4.14** (See figure [1]). Take \( \xi \) to be given by \( y^3 - x^{11} + ax^8y = 0 \), with \( a \neq 0 \). It is irreducible and has only one characteristic exponent: \( \frac{11}{3} \). The cluster \( BP(J(\xi)) \) is shown in figure [7] and hence topologically generic polars of \( \xi \) consist of two smooth branches sharing the points on \( \xi \) up to \( p_3 \), the point on \( \xi \) in the third neighbourhood of \( O \). Moreover, topologically generic polars of \( \xi \) share four further fixed free points after \( p_3 \), two on each branch.

Since \( O \notin D = \{p_8, p_9\} \), we start with \( R = S = \emptyset \). The polar invariants are

\[
I = I_{p_8} = I_{p_9} = \frac{[BP(J(\xi)),K(p_8)]}{n_{p_8}} + 1 = \frac{2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 1^2 + 1}{1} + 1 = 11.
\]

We start with \( p_8 \). The corresponding point \( p' \) is \( p_2 \), and thus the rupture point associated to \( p_8 \) is satellite of \( q_0 = p_3 \). Step 4(d) consists of the next two iterations:

- \( \frac{m_{q_1}}{n_{q_1}} = 12 > 11 = I \), so we take \( q_1 = p_4 \), the first satellite of \( p_3 \).
- \( \frac{m_{q_2}}{n_{q_2}} = \frac{21}{2} < 11 = I \), so we take \( q_2 = p_5 \), the second satellite of \( p_4 \).
of step 2 gives

Example 4.16 (See figures 3, 4, 5 and 6). It is not necessary to add any further point to $R_S$ of the other hand, since there are no free singular points in the first neighbourhood of singular points in the first neighbourhood of $O, p$. Finally, the third step of the second part applies to recover $v$. We must follow the second instance of step 3 and set (See figure 2)

Example 4.15 (See figure 2). Now consider the curve $\xi$ given by $y^3 - x^{11} = 0$. It is again irreducible with single characteristic exponent $\frac{11}{3}$, and hence it is equisingular to the curve in the previous example (in fact, it corresponds to take $\alpha = 0$ in the equation of Example 4.14). However, $BP(\mathcal{J}(\xi))$ is not equisingular to that in Example 4.14. In this case, topologically generic polars also consist of two smooth branches, but they share five points (instead of four, as happened in the previous example) and there are no more base points.

In this case there is only one dicritical point in $BP(\mathcal{J}(\xi))$: $p_6$, and its corresponding polar invariant is again

$1 = I_{p_6} = \frac{[BP(\mathcal{J}(\xi)) \cdot K(p_6)]}{n_{p_6}} + 1 = \frac{2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1}{1} + 1 = 11$.

Moreover, the point $p$ is again $p_3$, and hence the algorithm works as it does in example 4 (recovering both the rupture points and the values).

We expose now a more complicated example. After this example it will be clear that the computation of $S(\xi)$ by hand is much faster using the version of the algorithm of Proposition 4.13.

Example 4.16 (See figures 3, 4, 5 and 6). Consider the curve $\xi$ given by

$$(y^4 - \alpha_1 x^{11})(y^3 - \alpha_2 x^8)(y^9 - \alpha_3 x^{22})(y^{12} - \alpha_4 x^{30})(y^4 - \alpha_5 x^9) = 0,$$
where $\alpha_i \neq 0$ for $i = 1, \ldots, 5$. Thus $\xi$ consists of five branches $\gamma_1, \ldots, \gamma_5$ of characteristic exponents $\frac{4}{3}, \frac{8}{3}, \frac{29}{12}, \frac{29}{72}$ and $\frac{9}{7}$, respectively. From the Enriques diagram of figure 3 it is immediate that the set of dicritical points of $\xi$ is $R(\xi) = \{p_5, p_9, p_{10}, p_{11}\}$.

The weighted cluster $BP(\mathcal{J}(\xi))$ is represented in figure 3, where it can be seen that topologically generic polars of $\xi$ have six branches.

Let us apply the algorithm to this case step by step. It starts with $R = S = \emptyset$ and the ordered set of dicritical points $D = \{p_{12}, p_{14}, p_{19}, p_{20}, p_{21}, p_{22}\}$. The corresponding polar invariants are $I_{12} = I_{p_{12}} = 79$, $I_{14} = I_{p_{14}} = 79$, $I_{19} = I_{p_{19}} = \frac{230}{3}$, $I_{20} = I_{p_{20}} = \frac{694}{3}$, $I_{21} = I_{p_{21}} = \frac{236}{3}$, $I_{22} = I_{p_{22}} = 72$. Let us start step 4 of the first part of the algorithm.

- **Start with** $p_{12}$. We have $p' = p_1$ because $\frac{m_1}{n_{p_1}} = 64 < I_{12} = 79 \leq \frac{m_9}{n_{p_9}} = 80$, and then $p = p_2$. Since it is the first step, we must take $q_0 = p_2$ and perform 4 (d).
  - $-\frac{m_{10}}{n_{p_9}} = 80 > 79 = I_{12}$, and hence we take $q_1 = p_3$, the first satellite of $p_2$.
  - $-\frac{m_{14}}{n_{p_1}} = \frac{155}{2} < 79 = I_{12}$, and hence we take $q_2 = p_4$, the second satellite of $p_3$.
  - $-\frac{m_{12}}{n_{p_2}} = \frac{283}{3} < 79 = I_{12}$, and hence we take $q_3 = p_5$, the second satellite of $p_4$.

Since $\frac{m_9}{n_{p_9}} = 79 = I_{12}$, we stop and set $R = \{p_5\}$ and $S = \{O, p_1, \ldots, p_5\}$.

- **Take now** $p_{14}$. By the same reason as in the previous step, we have $p' = p_1$ and $p = p_2$. Moreover, the equality $I_{14} = I_{12} = 79$ implies that the procedure yields the same result as $p_{12}$, so we will omit it.

- **Take now the point** $p_{19}$. Since $\frac{m_{19}}{n_{p_9}} = 64 < I_{19} = \frac{236}{3} \leq \frac{m_{20}}{n_{p_9}} = \frac{236}{3}$, we again have $p' = p_1$, and hence $p = p_2$. But $p_2$ has already appeared as $p$ before, so the rupture point associated to $p_{19}$ is $p_4$, the last $p_2$-satellite point in $K(p_{19})$. Thus, we must set $R = \{p_4, p_5\}$ and $S = \{O, p_1, \ldots, p_5\}$.

- **Consider now** $p_{20}$. Here $p'$ is again $p_1$ because $\frac{m_{20}}{n_{p_{11}}} = 64 < I_{20} = \frac{694}{3} \leq \frac{m_{21}}{n_{p_{11}}} = \frac{694}{3}$. Therefore $p = p_2$, which has already appeared, and thus the rupture point associated to $p_{20}$ is the last $p_2$-satellite in $K(p_{20})$: $p_9$. This step finishes with $R = \{p_4, p_5, p_9\}$ and $S = \{O, p_1, \ldots, p_9\}$.

- **Take now the point** $p_{21}$. Since $\frac{m_{21}}{n_{p_{11}}} = 64 < I_{21} = \frac{230}{3} \leq \frac{m_{22}}{n_{p_{11}}} = \frac{230}{3}$, we have $p' = p_1$ and $p = p_2$. But since $p_2$ has already appeared, the rupture point associated to $p_{20}$ is $p_{11}$, the last $p_2$-satellite in $K(p_{22})$. We finish this step with $R = \{p_4, p_5, p_9, p_{11}\}$ and $S = \{O, p_1, \ldots, p_9, p_{11}\}$.
Figure 3: Singular points of $\xi$ with its multiplicities and values $(e_p(\xi), v_p(\xi))$.

Figure 4: Singular points of $\xi$ and base points of $J(\xi)$.

- We finally take $p_{22}$, the last dicritical point. We have again that $p' = p_1$ because $\frac{m_{p_1}}{n_{p_1}} = 64 < I_2 = 72 \leq \frac{m_{p_5}}{n_{p_5}} = \frac{288}{4}$, and hence $p = p_2$. But it has already appeared, and therefore the rupture point associated to $p_{22}$ is $p_{10}$.

Thus, the first part of the algorithm is completed and it yields $R = \{p_4, p_5, p_9, p_{10}, p_{11}\}$ and $S = \{O, p_1, \ldots, p_{11}\}$, which actually coincide with $R(\xi)$ and $S(\xi)$ respectively.

The second part begins recovering of the values at the rupture points: $v_{p_4} = m_{p_4} = 236$, $v_{p_9} = m_{p_9} = 316$, $v_{p_{10}} = 694$, $v_{p_{11}} = 288$, and $v_{p_{22}} = 920$.

Then it is the turn of the free singular non-rupture points. Firstly, $v_O = m_O = 32$ and $v_{p_1} = m_{p_1} = 64$ because $p_1$ and $p_2$ are free points in the first neighbourhood of $O$ and $p_1$ respectively. Secondly, $v_{p_2} = 79$ because it is the only integer in the interval

$$\left[ \frac{n_{p_2}}{n_{p_5}}, \frac{n_{p_2}}{n_{p_5}} m_{p_5} + 1 \right] = [79, 80]$$

(where $p_5$ is the biggest $p_2$-satellite rupture point).

Finally, we must consider the satellite non-rupture points, which are $p_3, p_6, p_7$ and $p_8$. But all these points are smaller than $p_5$, which is the biggest $p_2$-satellite rupture point. Therefore we must always apply the second instance of step 3, obtaining the equality $v_p = m_p$ for all these points. More explicitly, we have $v_{p_3} = 155$, $v_{p_6} = 223$, $v_{p_7} = 381$, and $v_{p_8} = 538$.

It is immediate to check that these values are the values of $\xi$ at its singular points, as claimed.
Figure 5: Base points of $\mathcal{J}(\xi)$ with its virtual multiplicities $\nu_p$.

Figure 6: $S(\xi) \cup BP(\mathcal{J}(\xi))$ with heights of the trunks and multiplicities of $\varphi_p$, $\left(\frac{m_p}{\nu_p}\right)$.
We close the paper with a last example. It is even more complicated than the previous one, since two of the branches of the curve have two characteristic exponents.

**Example 4.17.** (See figures 7, 8, 9, and 10)

Let \( \xi \) be the curve with branches \( \gamma_1, \ldots, \gamma_5 \) given by the Puiseux series \( s_1(x) = x^{3/4} + x^{27/10}, s_2(x) = x^{13/20} + x^{25/16}, s_3(x) = x^{1/4}, s_4(x) = x^{1/4}, \) and \( s_5(x) = x^{1/4} \). One possible equation for \( \xi \) is

\[
f = (y^3 - x^8)(y^4 - x^9)(y^7 - x^{16})
\]

\[
(16x^{21}y^3 - 4x^{32}y^5 - 16x^{31}y^3 + 56x^{22}y^2 - 16x^{43}y + x^{44} - x^{51})
\]

\[
(5x^{11}y^4 + 10x^{22}y^3 - 140x^{24}y^2 - 40x^{35}y^4 = 620x^{35}y^8 - 110x^{37}y^4 + 5x^{44}y^4 - 260x^{46}y^4 + 340x^{48}y^4 - 20x^{50}y^4 - x^{55} - 4x^{57} - 6x^{59} - 4x^{61} - x^{63}),
\]

and its Enriques’ diagram is shown in figure 7. It is immediate that the set of rupture points of \( \xi \) is \( R(\xi) = \{p_4, p_5, p_7, p_9, p_{13}, p_{14}\} \).

The representation of \( BP(\mathcal{J}(\xi)) \) in figure 8 shows in particular that topologically generic polars of \( \xi \) have seven branches. One of the branches is smooth, four of them have only one characteristic exponent, and the two remaining branches have two characteristic exponents. This example also shows that \( BP(\mathcal{J}(\xi)) \) may contain a lot of points which are simple on the topologically generic polars.

Now we run the algorithm. Step 1 sets \( \mathcal{S} = \emptyset, \mathcal{D} = \{p_{15}, p_{17}, p_{21}, p_{22}, p_{23}, p_{29}, p_{30}\} \), and since \( O \notin \mathcal{D} \) we go to step 3.

The polar invariants are \( I_{15} = I_{17} = 132, I_{21} = 129, I_{22} = \frac{799}{11}, I_{23} = 283 \), \( I_{29} = 544 \), and \( I_{30} = \frac{678}{7} \).

Hence, in step 4 we must process the critical points in the order \( p_{29}, p_{30}, p_{15}, p_{17}, p_{21}, p_{22}, p_{23} \).

- **Start with** \( p_{29} \). We have \( p' = p_9 \) and \( p = p_{10} \) because \( \frac{m_{12}}{n_{p_9}} = \frac{537}{49} < I_{29} = \frac{544}{4} \leq \frac{m_{12}}{n_{p_{10}}} = \frac{544}{4} = 136 \).

Since it is the first iteration, we take \( q_0 = p_{10} \) and perform \( 4(d) \).

\[
- \frac{m_{10}}{n_{p_9}} = 136 > \frac{544}{4} = I_{29}, \text{ so that we take } q_1 = p_{11}, \text{ the first satellite of } p_{10}.
\]

\[
- \frac{m_{13}}{n_{q_1}} = 1083 \frac{4}{11} < \frac{544}{4} = I_{29}, \text{ so } q_2 = p_{12}, \text{ the second satellite of } p_{11}.
\]

\[
- \frac{m_{14}}{n_{q_2}} = 547 \frac{3}{11} < \frac{544}{4} = I_{29}, \text{ and therefore } q_3 = p_{13}, \text{ the second satellite of } p_{12}.
\]

Since \( \frac{m_{12}}{n_{q_3}} = \frac{2172}{16} = I_{29} \), this first iteration finishes with \( \mathcal{R} = \{p_{13}\}, \mathcal{S} = \{O, p_1, \ldots, p_5, p_9, \ldots, p_{13}\} \).

- **Take now** \( p_{30} \). Since \( \frac{m_{12}}{n_{p_{10}}} = \frac{537}{49} < I_{30} = \frac{678}{7} \leq \frac{m_{12}}{n_{p_{14}}} = \frac{292}{13} \), we have \( p' = p_9 \) and \( p = p_{10} \). But \( p_{10} \) has already appeared as \( p \), and so the rupture point associated to \( p_{30} \) is \( p_{14} \), the last \( p_{10} \)-satellite point in \( K(p_{30}) \). Up to now we have \( \mathcal{R} = \{p_{13}, p_{14}\} \) and \( \mathcal{S} = \{O, p_1, \ldots, p_5, p_9, \ldots, p_{14}\} \).

- **Take now the point** \( p_{15} \). Since \( \frac{m_{12}}{n_{p_{17}}} = 100 < I_{15} = 132 \leq \frac{m_{12}}{n_{p_{21}}} = 133 \), we have \( p' = p_{1} \) and \( p = p_{2} \).

It is the first time \( p_{2} \) appears, so we must perform the iterations of \( 4(d) \) starting from \( q_0 = p = p_{2} \):

\[
- \frac{m_{10}}{n_{p_9}} = 133 > 132 = I_{15}, \text{ and hence we take } q_1 = p_{3}, \text{ the first satellite of } p_{2}.
\]

\[
- \frac{m_{13}}{n_{q_1}} = 245 \frac{2}{3} < 132 = I_{15}, \text{ and hence we take } q_2 = p_{4}, \text{ the second satellite of } p_{3}.
\]

\[
- \frac{m_{14}}{n_{q_2}} = 387 \frac{3}{11} < 132 = I_{15}, \text{ and hence we take } q_3 = p_{5}, \text{ the second satellite of } p_{4}.
\]

And now we stop because \( \frac{m_{12}}{n_{q_3}} = \frac{528}{11} = 132 = I_{15} \). We finish this step by setting \( \mathcal{R} = \{p_{5}, p_{13}, p_{14}\} \) and \( \mathcal{S} = \{O, p_1, \ldots, p_5, p_9, \ldots, p_{14}\} \).

- **The case of** \( p_{17} \) **is exactly the same of** \( p_{15} \), **so we omit it.**

- **Take now the point** \( p_{21} \). Since \( \frac{m_{12}}{n_{p_{21}}} = 100 < I_{21} = \frac{38}{3} \leq \frac{m_{12}}{n_{p_{22}}} = 133 \), we have \( p' = p_{1} \) and \( p = p_{2} \).

But \( p_{2} \) has already appeared, and hence we obtain that the rupture point associated to \( p_{21} \) is \( p_{4} \), the last \( p_{2} \)-satellite point in \( K(p_{21}) \). Therefore we have by the moment \( \mathcal{R} = \{p_{4}, p_{5}, p_{13}, p_{14}\} \) and \( \mathcal{S} = \{O, p_1, \ldots, p_5, p_9, \ldots, p_{14}\} \).
Next, because $p$ which is the biggest $p$ both by the same reason as above. In second place, both $R$ which actually coincide with singular points, as claimed.

Then we take care of the free singular non-rupture points, starting with

$$\{p_4, p_5, p_7, p_8, p_{13}, p_{14}\} \text{ and } S = \{O, p_1, \ldots, p_{14}\}.$$  

We finally take $p_{23}$, the last dicritical point. We have again $p' = p_1$ because $\frac{n_{p_2}}{n_{p_1}} = 100 < I_{23} = \frac{225}{2} \leq \frac{n_{p_2}}{n_{p_1}} = 133$, and hence $p = p_2$. But it has already appeared (three times), and therefore the rupture point associated to $p_{23}$ is $p_7$.

Thus, the first part of the algorithm finishes with $\mathcal{R} = \{p_4, p_5, p_7, p_8, p_{13}, p_{14}\}$ and $S = \{O, p_1, \ldots, p_{14}\}$, which actually coincide with $\mathcal{R}(\xi)$ and $S(\xi)$ respectively.

The second part begins recovering the values of the rupture points:

$$v_{p_4} = 387, \ v_{p_5} = 528, \ v_{p_7} = 450, \ v_{p_8} = 799, \ v_{p_{13}} = 2172, \ \text{and} \ v_{p_{14}} = 2712.$$  

Then we take care of the free singular non-rupture points, starting with

$$v_O = m_O = 50, \ v_{p_1} = m_{p_1} = 50 \ \text{and} \ v_{p_9} = m_{p_9} = 537$$  

because $p_1, p_2$ and $p_{10}$ are free singular points in the first neighbourhoods of $O, p_1$ and $p_9$ respectively. Next, 

$$v_{p_{2}} = 132 \ \text{and} \ v_{p_{10}} = 543$$  

because they are the only integers in the intervals

$$\left[\frac{n_{p_2}}{n_{p_3}m_{p_5}}, \frac{n_{p_2}}{n_{p_5}m_{p_3}} + 1\right] = [132, 133] \ \text{and} \ \left[\frac{n_{p_{10}}}{n_{p_{13}}m_{p_{12}}}, \frac{n_{p_{10}}}{n_{p_{12}}m_{p_{13}} + 1}\right] = [543, 544]$$  

respectively, and $p_5$ (resp. $p_{10}$) is the biggest $p_2$-satellite (resp. $p_{10}$-satellite) rupture point.

Finally, we must consider the satellite non-rupture points, which are $p_3, p_6, p_{11}$ and $p_{12}$. In first place, both $p_3$ and $p_6$ are smaller than $p_5$, the biggest $p_2$-satellite rupture point, and hence we have

$$v_{p_3} = m_{p_3} = 245 \ \text{and} \ v_{p_6} = m_{p_6} = 348$$  

because the second instance of step 3 applies. In second place, both $p_{11}$ and $p_{12}$ are smaller than $p_{13}$, which is the biggest $p_{10}$-satellite rupture point. Therefore we get

$$v_{p_{11}} = m_{p_{11}} = 1083 \ \text{and} \ v_{p_{12}} = m_{p_{12}} = 1628$$  

by the same reason as above.

As in all the other examples, it is immediate to check that these values are the values of $\xi$ at its singular points, as claimed.
Figure 8: Singular points of $\xi$ and base points of $\mathcal{J}(\xi)$.

Figure 9: Base points of $\mathcal{J}(\xi)$ with its virtual multiplicities $\nu_p$. 
Figure 10: $S(\xi) \cup BP(J(\xi))$ with heights of the trunks and multiplicities of $\varphi_p, \left(\frac{m_p}{n_p}\right)$.

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