LONG-TIME SOLVABILITY IN BESOV SPACES FOR THE INVISCID 3D-BOUSSINESQ-CORIOLIS EQUATIONS

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Abstract. We investigate the long-time solvability in Besov spaces of the initial value problem for the inviscid 3D-Boussinesq equations with Coriolis force. First we prove a local existence and uniqueness result with critical and supercritical regularity and existence-time $T$ uniform with respect to the rotation speed $Ω$. Afterwards, we show a blow-up criterion of BKM type, estimates for arbitrarily large $T$, and then obtain the long-time existence and uniqueness of solutions for arbitrary initial data, provided that $Ω$ is large enough.

1. Introduction. We analyse the long-time solvability in the full inviscid case for the Boussinesq-Coriolis system in the whole $\mathbb{R}^3$ which describes the evolution of an incompressible fluid under the effects of convection and Earth’s rotation. In its complete form, including thermal diffusivity and kinematic viscosity, that system reads as

$$\begin{align*}
    \partial_t u + Ω e_3 \times u + \nabla p + (u \cdot \nabla)u - \nu \Delta u &= g θ e_3 \text{ in } \mathbb{R}^3 \times (0, \infty), \\
    \partial_t θ + (u \cdot \nabla)θ - \kappa \Delta θ &= 0 \text{ in } \mathbb{R}^3 \times (0, \infty), \\
    \nabla \cdot u &= 0 \text{ in } \mathbb{R}^3 \times (0, \infty), \\
    u|_{t=0} = u_0, \quad θ|_{t=0} = θ_0 \text{ in } \mathbb{R}^3,
\end{align*}$$

(1.1)

where $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ is the velocity field, $θ = θ(x, t)$ is the temperature and $p = p(x, t)$ stands for the pressure. The initial velocity $u_0 = (u_{0,1}(x), u_{0,2}(x), u_{0,3}(x))$ satisfies the condition $\nabla \cdot u_0 = 0$ and $θ_0 = θ_0(x)$ is the initial temperature. The kinematic viscosity is represented by $ν ≥ 0$, the thermal diffusivity is denoted by $κ ≥ 0$ and $g$ is the gravity. The Coriolis parameter $Ω$.

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$\Omega \in \mathbb{R}$ plays the role of twice the rotation speed around the vector unit $e_3 = (0, 0, 1)$. The term $g\theta e_3$ represents the buoyancy force that arises from the Boussinesq approximation in which density variations are proportional to temperature variations and assumed to exist only in the gravitational term. The reader is referred to the book [30] for more details about the model (1.1).

Among the problems in partial differential equations that are of great interest to the mathematical community, there are the equations in fluid mechanics, especially those dealing with atmospheric and oceanographic phenomena where heat transfer and rotation effects play an important role. Such is the case of the system (1.1) and related models on which different theoretical studies have been carried out in the last decades, analyzing various aspects such as the local and global well-posedness, regularity and stability of solutions, blow-up criteria, among others. In what follows, we will briefly review some results about these aspects.

We begin by commenting on results about the global well-posedness for the 2D-dimensional Boussinesq equations (that is, for $\Omega = 0$ in (1.1)). In $L^p$ spaces, Cannon and Di Benedetto [9] obtained the global well-posedness in the totally viscous case. Later, Hou and Li [24] showed the global well-posedness to the partially viscous Boussinesq system ($\nu > 0$ and $\kappa = 0$) for initial data $u_0 \in H^m$ and $\theta_0 \in H^{m-1}$ with $m \geq 3$. For $\kappa = 0$ and $\nu > 0$ or $\kappa > 0$ and $\nu = 0$, Chae [11] proved the global well-posedness in $H^m$ with $m > 2$ integer. Next, Abidi and Hmidi [1] obtained the global well-posedness for $\kappa = 0$ and $\nu > 0$ with $u_0 \in B^{1-\frac{1}{r}}_{\infty,1} \cap L^r$ and $\theta_0 \in B^{0}_{2,1}$, and for $\kappa > 0$ and $\nu = 0$, Hmidi and Keraani [22] showed the global well-posedness with $u_0 \in B^{1+\frac{1}{r}}_{p,1}$ and $\theta_0 \in B^{1+\frac{1}{r}}_{p,1} \cap L^r$ for $1 < p < \infty$ and $2 < r < \infty$. For the case $\kappa > 0$ and $\nu = 0$, Danchin and Paicu [17] obtained the global well-posedness with initial data $u_0 \in L^2$, $\theta_0 \in L^2 \cap B^{1-1}_{\infty,1}$ and initial vortex $\omega_0 \in L^r \cap L^\infty$, $2 \leq r < \infty$. In the inviscid case, Yuan [37] and Liu et al. [29] showed the local existence and uniqueness of solutions in Besov spaces $B^s_{p,q}(\mathbb{R}^2)$ in the supercritical case $s > 1 + \frac{2}{p}$, $1 < p < \infty$, $1 \leq q \leq \infty$, and critical case $s = 1 + \frac{2}{p}$, $1 < p < \infty$ and $q = 1$. Wan and Chen [34] obtained the global well-posedness of the inviscid Boussinesq equations for the gravitational constant $g$ sufficiently large with initial data in Sobolev spaces.

For the 3D-Boussinesq equations (i.e., (1.1) with $\Omega = 0$), the global smoothness of solutions is a challenging open problem for any viscosity case, that is inviscid, partially and totally viscous cases. Some results about the local and global well-posedness and blow-up criteria have been obtained in different function spaces for this system. In particular, for the viscous case (i.e., $\nu > 0$ and $\kappa > 0$), see, e.g., the works [2, 16, 18, 20, 23] and their references. More precisely, Ferreira and Villamizar-Roa [20] showed well-posedness and asymptotic self-similarity of small solutions in the framework of weak-$L^p$ spaces; Hmidi and Rousset [23] proved the global well-posedness with axisymmetric initial data in Sobolev spaces; de Almeida and Ferreira [2] considered homogeneous critical Morrey spaces and proved results on well-posedness, self-similarity and large-time behavior of small solutions. Deng and Cui [18] showed the well-posedness of small solutions with initial data belonging to Besov spaces of negative regularity index $s = -1$; and Dai et al. [16] established a blow-up criterion of weak solutions in terms of the pressure in the homogeneous Besov spaces $B^s_{\infty,\infty}(\mathbb{R}^3)$. We also refer the reader to the references contained in the papers mentioned in the present paragraph.

In the partially viscous case (i.e., $\nu = 0$ and $\kappa > 0$ or $\nu > 0$ and $\kappa = 0$), Danchin and Paicu [17] stated the global existence of finite energy weak solutions in $\mathbb{R}^n$. 
and the global well-posedness for \( n \geq 3 \) with small initial data \((u_0, \theta_0)\) satisfying 
\( u_0 \in \dot{B}^{s_0-\frac{2}{p}+\frac{n}{q}}_{p,1}(\mathbb{R}^n) \cap L^{n,\infty}(\mathbb{R}^n) \) and \( \theta_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n) \cap L^{\frac{n}{s}}(\mathbb{R}^n) \) for \( n \leq p \leq \infty \). Ye [36] obtained for initial data in \( H^{3}(\mathbb{R}^3) \times H^{3}(\mathbb{R}^3) \) a blow-up criterion using the homogeneous Besov space \( \dot{B}^{\frac{3}{p}+\frac{n}{q}-1}_{p,\infty}(\mathbb{R}^3) \) with \( \frac{3}{p} + \frac{n}{q} \leq 2 \) and \((p,q) \neq (\infty,\infty)\) for 
\( 1 < p,q \leq \infty \). Fu and Cai [21] proved that if the pressure \( p \) is in \( L^{\frac{2}{p}-\varepsilon}(0,T; \dot{B}^s_{p,\infty}) \) for \( r = \pm 1 \), smooth solutions can be continuously extended.

In the inviscid case (i.e., \( \nu = \kappa = 0 \)), Cui et al. [15] showed the local well-posedness and a blow-up criterion in Hölder spaces in \( \mathbb{R}^n \) for \( n \geq 2 \); Xiang and Yan [35] proved the well-posedness in the Triebel-Lizorkin-Lorentz spaces \( F^{s,r}_{p,q}(\mathbb{R}^n) \) for 
\( 1 < r \leq p < \infty \), \( 1 < q < \infty \) and 
\( s > 1 + \frac{n}{p} \), and also obtained a blow-up criterion in the Triebel-Lizorkin space \( F^{s,\infty}_{p,\infty}(\mathbb{R}^n) \). Bie et al. [8] obtained the local well-posedness and a blow-up criteria in Besov-Morrey spaces \( N^s_{p,q,r}(\mathbb{R}^3) \) for the supercritical case \( s > 1 + \frac{n}{p} \), \( 1 \leq q \leq \infty \), \( 1 \leq r \leq \infty \), and the critical case 
\( s = 1 + \frac{n}{p} \), \( 1 < q \leq \infty \), \( r = 1 \).

In the rotating case (i.e., \( \Omega \neq 0 \)) and with an additional stratification term 
\( -\mathcal{N}^2 u_3 \) in the R.H.S of the temperature equation in (1.1), results about global well-posedness in Sobolev spaces have been obtained in the viscous case (\( \nu > 0 \) and \( \kappa > 0 \)), see e.g. [6, 12, 13, 14, 25, 26] and references therein. These results require some conditions such as small initial data, sufficiently large stratification parameter \( \mathcal{N} \), sufficiently large \( \Omega \) (w.r.t the initial data norm), or a mixture thereof. Still considering stratification effects, Sun and Yang [32] obtained a result in the homogeneous hybrid Besov space \( B^{s_2-\frac{1}{2}}_{2,p,\mathcal{N}} \) whose norm depends on the parameter \( \mathcal{N} > 0 \). For the viscous stably stratified 3D Boussinesq equations (i.e., \( \Omega = 0 \), \( \mathcal{N} > 0 \), and \( \nu, \kappa \neq 0 \)), Lee and Takada [28] proved global well-posedness in the framework of Sobolev spaces with \( \mathcal{N} \) sufficiently large w.r.t initial data, after obtaining sharp dispersive estimate for the linear propagator related to the stable stratification. Recently, in the rotating viscous case of (1.1) (i.e., \( \Omega \neq 0, \nu, \kappa \neq 0 \), and without the additional stratification term \( -\mathcal{N}^2 u_3 \)), Sun, Liu and Yang [31] showed global well-posedness with \( \Omega \) sufficiently large w.r.t the norm of \((u_0,\theta_0)\) in the Besov spaces \( \dot{B}^{s_2}_{p,q} \times \dot{B}^{s_2}_{p,q} \), where \( 3/2 < p < 2 \), \( 1 \leq q \leq \infty \), 
\( s_1 = -1 + 3/p + 2/\delta \), 
\( s_2 = -1 + 3/p' + 2/\rho \), \( \delta \in (2,\infty) \) and \( \rho \in (1,\delta) \). It is worth mentioning that the first results of long-time solvability for Euler equations and global well-posedness for Navier-Stokes equations, under high rotation speed, were obtained by Babin, Mahalov and Nicolaenko [4, 5] by considering the framework of Sobolev spaces and periodic boundary conditions.

In view of the previous results, it is natural to wonder about the long-time solvability (i.e., arbitrarily large existence-time) of system (1.1) in the non-viscous (partially or inviscid) cases and without stratification effects. Inspired by the papers for Euler [10, 38] and Euler-Coriolis [3, 27] equations, we show long-time solvability of (1.1) in the inviscid case \( \kappa = \nu = 0 \) for initial data in the Besov space \( \dot{B}^{s_2}_{2,q} \times \dot{B}^{s_2}_{2,q} \), with critical and supercritical values of the regularity \( s \), \( 1 \leq q \leq \infty \) and [52] sufficiently large w.r.t to the initial data and existence-time \( T \). The critical regularity sense is that used in the context of Euler equations since we are working in the totally inviscid case. To be more precise, we consider the inviscid 3D-Boussinesq-Coriolis system
Theorem 1.1. Let \(\Omega\) be a bounded domain in \(\mathbb{R}^3\) with sufficiently smooth boundary. Then, the following assertions hold:

(i) (Local solvability) Let \((u_0, \theta_0) \in B^s_{2,q}(\mathbb{R}^3) \times B^s_{2,q}(\mathbb{R}^3)\) satisfy \(\nabla \cdot u_0 = 0\), then there exists \(T > 0\) depending only on \(\|u_0\|_{B^s_{2,q}}\) and \(\|\theta_0\|_{B^s_{2,q}}\), such that (1.2) possesses a unique solution \((u, \theta) \in C([0, T]; B^s_{2,q}(\mathbb{R}^3) \times B^s_{2,q}(\mathbb{R}^3))\), for all \(\Omega \in \mathbb{R}\).

(ii) (Long-time solvability) Let \(0 < T < \infty\) and \((u_0, \theta_0) \in B^{s+1}_{2,q}(\mathbb{R}^3) \times B^{s+1}_{2,q}(\mathbb{R}^3)\) satisfy \(\nabla \cdot u_0 = 0\), then there exists \(\Omega_0 = \Omega_0(T, \|u_0\|_{B^{s+1}_{2,q}}, \|\theta_0\|_{B^{s+1}_{2,q}}) > 0\) such that (1.2) possesses a unique solution \((u, \theta) \in C([0, T]; B^{s+1}_{2,q}(\mathbb{R}^3) \times B^{s+1}_{2,q}(\mathbb{R}^3))\) provided that \(|\Omega| \geq \Omega_0\).

Let us briefly discuss about some technical aspects in the proof of the theorem. To show item (i), we use an approximate linear iteration system similar to the one that Zhou [38] used to study the Euler equations. Here the sequence \((u_n, \theta_n)_{n \in \mathbb{N}}\) satisfying this approximate system converges to the unique solution of the system (1.2). Afterwards, in order to show item (ii) we prove a criterion of blow-up and employ a Strichartz estimate due to Koh et al. [27].

This paper is organized as follows. Section 2 is devoted to some preliminaries about Besov spaces together with product and commutator estimates in these spaces, as well as projection operators linked to the Coriolis term. The main result is proved in Section 3: first we prove the local-in-time solvability corresponding to item (i) on \([0, T]\) for some \(T > 0\) independent of \(\Omega\) (see subsection 3.1); second, we obtain a blow-up criterion to (1.1) in subsection 3.2; and finally we conclude with the proof of global solvability in subsection 3.3.

2. Preliminaries. In this section we give some preliminaries about Besov spaces. The reader is referred to the book [7] for further details on these spaces.

We begin by recalling the Schwartz class of rapidly decreasing smooth functions \(\mathcal{S}(\mathbb{R}^3)\) and the space of tempered distributions \(\mathcal{S}'(\mathbb{R}^3)\). For \(f \in \mathcal{S}',\) we denote the Fourier transform of \(f\) by \(\hat{f}\).

Next take a function \(\phi_0 \in \mathcal{S}(\mathbb{R}^3)\) such that \(0 \leq \hat{\phi}_0(\xi) \leq 1\) for all \(\xi \in \mathbb{R}^3\), \(\text{supp } \hat{\phi}_0 \subset \{\xi \in \mathbb{R}^3 : 2^{-1} < |\xi| < 2\}\) and

\[
\sum_{j \in \mathbb{Z}} \hat{\phi}_j(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^3 \setminus \{0\},
\]
where \( \phi_j(x) := 2^3j \phi_0(2^j x) \). For each \( k \in \mathbb{Z} \), consider the function \( S_k \in S \) defined in Fourier variables as

\[
\hat{S}_k(\xi) = 1 - \sum_{j \geq k+1} \hat{\phi}_j(\xi), \text{ for all } \xi \in \mathbb{R}^3. \tag{2.2}
\]

The Littlewood-Paley operator \( \Delta_j \) is defined by

\[
\Delta_j f := \phi_j * f \text{ for } f \in S'(\mathbb{R}^3).
\]

Let \( \mathcal{P} \) stand for the set of polynomials with 3 variables. For the indexes \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \), we define the homogeneous Besov space as

\[
\dot{B}^s_{p,q}(\mathbb{R}^3) = \left\{ f \in S'(\mathbb{R}^3)/\mathcal{P}; \|f\|_{\dot{B}^s_{p,q}} = \|\{2^{sj} |\Delta_j f|\|_{L^p}\}_{j \in \mathbb{Z}}\|_{L^q(\mathbb{Z})} < \infty \right\}
\]

and its inhomogeneous version as

\[
B^s_{p,q}(\mathbb{R}^3) = \left\{ f \in S'(\mathbb{R}^3); \|f\|_{B^s_{p,q}} = \|S_0 * f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}} < \infty \right\},
\]

where \( S_0 \) is given by (2.2). The pairs \( (\dot{B}^s_{p,q}, \|\cdot\|_{B^s_{p,q}}) \) and \( (B^s_{p,q}, \|\cdot\|_{B^s_{p,q}}) \) are Banach spaces.

Now we recall the Bernstein inequality.

**Lemma 2.1.** Let \( 1 \leq p \leq \infty \) and \( j \in \mathbb{Z} \). Assume also that \( f \in L^p \) satisfy \( \operatorname{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^3 : 2^{j-2} \leq |\xi| < 2^j \} \). Then, there exists a positive constant \( C = C(k) \) such that

\[
C^{-1} 2^{jk} \|f\|_{L^p} \leq \|D^k f\|_{L^p} \leq C2^{jk} \|f\|_{L^p}.
\]

Using (2.1)-(2.2), it is not difficult to see the equivalence of norms

\[\|f\|_{B^s_{p,q}} \sim \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}, \text{ for } s > 0.\]

Moreover, Bernstein inequality implies the equivalence

\[\|D^k f\|_{B^s_{p,q}} \sim \|f\|_{B^{s+k}_{p,q}}. \tag{2.3}\]

For \( s > 3/p \) with \( 1 \leq p, q \leq \infty \) or \( s = 3/p \) with \( 1 \leq p \leq \infty \) and \( q = 1 \), we have the continuous embedding (see, e.g., [7, p.153-154 and 163])

\[B^s_{p,q}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3). \tag{2.4}\]

In the lemma below we recall some product estimates in the Besov setting (see [10]).

**Lemma 2.2.** Let \( s > 0 \), \( 1 \leq p, q \leq \infty \), \( 1 \leq p_1, p_2 \leq \infty \) and \( 1 \leq r_1, r_2 \leq \infty \) satisfy \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2} \). There exists a positive universal constant \( C \) such that

\[
\|fg\|_{B^{s}_{p,q}} \leq C(\|f\|_{B^{s}_{p_1,q}} \|g\|_{L^{r_2}} + \|g\|_{B^{s}_{p_2,q}} \|f\|_{L^{r_2}}) \tag{2.5}
\]

\[
\|fg\|_{B^{s}_{p,q}} \leq C(\|f\|_{B^{s}_{p_1,q}} \|g\|_{L^{r_2}} + \|g\|_{B^{s}_{p_2,q}} \|f\|_{L^{r_2}}). \tag{2.6}
\]

We finish this section with estimates in the framework of Besov spaces for the commutator \( [v \cdot \nabla, \Delta_j]u = v \cdot \nabla (\Delta_j u) - \Delta_j (v \cdot \nabla u) \) (see [10, 33]).

**Lemma 2.3.** Let \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \).

(i) Let \( s > 0 \), \( v \in \dot{B}^s_{p_1,q}(\mathbb{R}^n) \) with \( \nabla v \in L^\infty(\mathbb{R}^n) \) and \( \nabla \cdot v = 0 \), and \( \theta \in \dot{B}^s_{p_2,q}(\mathbb{R}^n) \) with \( \nabla \theta \in L^\infty(\mathbb{R}^n) \). Then, there exists a universal constant \( C > 0 \) such that

\[
\left( \sum_{j \in \mathbb{Z}} 2^{sjq} \|\{v \cdot \nabla, \Delta_j \theta\}\|_{L^p} \right)^{1/q} \leq C \left( \|\nabla v\|_{L^\infty} \|\theta\|_{\dot{B}^s_{p_2,q}} + \|\nabla \theta\|_{L^\infty} \|v\|_{\dot{B}^s_{p_1,q}} \right). \tag{2.7}
\]
Let $s > -1$, $v \in \dot{B}^{s+1}_{p,q}(\mathbb{R}^n)$ with $\nabla v \in L^\infty(\mathbb{R}^n)$ and $\nabla \cdot v = 0$, and $\theta \in \dot{B}^s_{p,q}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, there exists a universal constant $C > 0$ such that

$$
\left( \sum_{j \in \mathbb{Z}} 2^{sj} \| v \cdot \nabla, \Delta_j \theta \|_{L^p}^2 \right)^{1/q} \leq C \left( \| \nabla v \|_{L^\infty} \| \theta \|_{\dot{B}^s_{p,q}} + \| \theta \|_{L^\infty} \| v \|_{\dot{B}^s_{p,q}} \right).
$$

3. Local solvability. We consider the following linear approximation scheme for (1.2)

$$
\begin{cases}
\partial_t u_{n+1} + \Omega F e_3 \times u_{n+1} + F(u_n \cdot \nabla) u_{n+1} = g \partial \theta_{n+1} e_3 \text{ in } (0, \infty) \times \mathbb{R}^3, \\
\partial_t \theta_{n+1} + (u_n \cdot \nabla) \theta_{n+1} = 0 \text{ in } (0, \infty) \times \mathbb{R}^3, \\
\nabla \cdot u_{n+1} = \nabla \cdot u_n = 0 \text{ in } (0, \infty) \times \mathbb{R}^3, \\
u_1|_{t=0} = S_2 u_0, \ u_{n+1}|_{t=0} = S_{n+2} u_0, \ \theta_1|_{t=0} = S_2 \theta_0, \ \theta_{n+1}|_{t=0} = S_{n+2} \theta_0.
\end{cases}
$$

(3.1)

Next, in order to build the local solution we are going to obtain error estimates for the sequence $(u_n, \theta_n)_{n \in \mathbb{N}}$.

**Uniform estimates.** First, we deal with estimates for $u_n$ and $\theta_n$ in the norm $\| \cdot \|_{\dot{B}^s_{p,q}}$. For that, we apply the operator $\Delta_j$ in the system (3.1), and then, we take the $L^2$-norm product with $\Delta_j u_{n+1}$ and $\Delta_j \theta_{n+1}$ to the first and second equation, respectively, to get

$$
\begin{align*}
\left\langle \Delta_j \partial_t u_{n+1}, \Delta_j u_{n+1} \right\rangle_{L^2} &= \left\langle [u_n \cdot \nabla, \Delta_j] u_{n+1}, \Delta_j u_{n+1} \right\rangle_{L^2} + \left\langle g \Delta_j \theta_{n+1} e_3, \Delta_j u_{n+1} \right\rangle_{L^2}, \\
\left\langle \Delta_j \partial_t \theta_{n+1}, \Delta_j \theta_{n+1} \right\rangle_{L^2} &= \left\langle [u_n \cdot \nabla, \Delta_j] \theta_{n+1}, \Delta_j \theta_{n+1} \right\rangle_{L^2}.
\end{align*}
$$

Here, we have used the skew-symmetry of $e_3 \times$ and the condition $\nabla \cdot \Delta_j u_n = 0$. Thus, by the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \| \Delta_j u_{n+1} \|_{L^2}^2 \leq \| u_n \cdot \nabla, \Delta_j \|_{L^2} \| \Delta_j u_{n+1} \|_{L^2} + g \| \Delta_j \theta_{n+1} \|_{L^2} \| \Delta_j u_{n+1} \|_{L^2}, \\
&\frac{1}{2} \frac{d}{dt} \| \Delta_j \theta_{n+1} \|_{L^2}^2 \leq \| u_n \cdot \nabla, \Delta_j \|_{L^2} \| \theta_{n+1} \|_{L^2} \| \Delta_j \theta_{n+1} \|_{L^2}.
\end{aligned}
$$

Integrating over $(0, t)$ we have that

$$
\begin{align*}
\| \Delta_j u_{n+1}(t) \|_{L^2} \leq & \| \Delta_j u_{n+1}(0) \|_{L^2} + \int_0^t \| u_n \cdot \nabla, \Delta_j \|_{L^2} \| \Delta_j u_{n+1}(\tau) \|_{L^2} \, d\tau, \\
&+ g \int_0^t \| \Delta_j \theta_{n+1}(\tau) \|_{L^2} \, d\tau, \\
\| \Delta_j \theta_{n+1}(t) \|_{L^2} \leq & \| \Delta_j \theta_{n+1}(0) \|_{L^2} + \int_0^t \| u_n \cdot \nabla, \Delta_j \|_{L^2} \| \theta_{n+1}(\tau) \|_{L^2} \, d\tau.
\end{align*}
$$

Multiplying by $2^{sj}$ and applying the $l^q$-norm, we arrive at the estimate

$$
\begin{align*}
\| u_{n+1}(t) \|_{\dot{B}^s_{p,q}} \leq & \| u_{n+1}(0) \|_{\dot{B}^s_{p,q}} + \int_0^t \left( \sum_{j \in \mathbb{Z}} 2^{sj} \| u_n(\tau) \cdot \nabla, \Delta_j \|_{L^2}^q \right)^{\frac{1}{q}} \, d\tau.
\end{align*}
$$
Using the inequality, we obtain the skew-symmetry of \( u \). Now, we deal with estimates for \( \theta_n \). Since \( \parallel \theta_n(0) \parallel_{B^s_{2,q}} \) and \( \parallel \theta_n(t) \parallel_{B^s_{2,q}} \) increase, it follows that
\[
\parallel u_{n+1}(t) \parallel_{B^s_{2,q}} \leq \parallel u_{n+1}(0) \parallel_{B^s_{2,q}} + C \int_0^t \parallel u_n(\tau) \parallel_{B^s_{2,q}} \parallel u_{n+1}(\tau) \parallel_{B^s_{2,q}} \, d\tau \\
+ g \int_0^t \parallel \theta_{n+1}(\tau) \parallel_{B^s_{2,q}} \, d\tau.
\]
(3.2)

Now, we deal with estimates for \( u_n \) and \( \theta_n \) in the norm \( \parallel \cdot \parallel_{L^2} \). Taking the \( L^2 \)-product with \( u_{n+1} \) and \( \theta_{n+1} \) to the first and second equation in (3.1), respectively, and using the skew-symmetry of \( c_3 \times \), the condition \( \nabla \cdot \Delta_j u_n = 0 \) and the Cauchy-Schwarz inequality, we obtain
\[
\parallel u_{n+1}(t) \parallel_{L^2} \leq \parallel u_{n+1}(0) \parallel_{L^2} + g \int_0^t \parallel \theta_{n+1}(\tau) \parallel_{L^2} \, d\tau,
\]
(3.3)

Combining (3.2) and (3.3), the resulting estimate is
\[
\parallel u_{n+1}(t) \parallel_{B^s_{2,q}} \leq \parallel u_{n+1}(0) \parallel_{B^s_{2,q}} + C \int_0^t \parallel u_n(\tau) \parallel_{B^s_{2,q}} \parallel u_{n+1}(\tau) \parallel_{B^s_{2,q}} \, d\tau \\
+ g \int_0^t \parallel \theta_{n+1}(\tau) \parallel_{B^s_{2,q}} \, d\tau,
\]
\[
\parallel \theta_{n+1}(t) \parallel_{B^s_{2,q}} \leq \parallel \theta_{n+1}(0) \parallel_{B^s_{2,q}} + C \int_0^t \parallel u_n(\tau) \parallel_{B^s_{2,q}} \parallel \theta_{n+1}(\tau) \parallel_{B^s_{2,q}} \, d\tau.
\]

Since \( \parallel u_{n+1}(0) \parallel_{B^s_{2,q}} \leq C \parallel u_0 \parallel_{B^s_{2,q}} \) and \( \parallel \theta_{n+1}(0) \parallel_{B^s_{2,q}} \leq C \parallel \theta_0 \parallel_{B^s_{2,q}} \), by the Gronwall inequality there exist positive constants \( C_0 \) and \( C_1 \) such that
\[
\parallel u_{n+1}(t) \parallel_{B^s_{2,q}} \leq \left( C_0 \parallel u_0 \parallel_{B^s_{2,q}} + g \int_0^t \parallel \theta_{n+1}(\tau) \parallel_{B^s_{2,q}} \, d\tau \right) \times \\
\times \exp \left( C_1 \int_0^t \parallel u_n(\tau) \parallel_{B^s_{2,q}} \, d\tau \right),
\]
(3.4)

\[
\parallel \theta_{n+1}(t) \parallel_{B^s_{2,q}} \leq C_0 \parallel \theta_0 \parallel_{B^s_{2,q}} \exp \left( C_1 \int_0^t \parallel u_n(\tau) \parallel_{B^s_{2,q}} \, d\tau \right).
\]

We are going to prove that there exist \( T > 0 \) and positive constants \( P \) and \( Q \) such that
\[
\parallel u_{n+1}(t) \parallel_{B^s_{2,q}} \leq P \parallel u_0 \parallel_{B^s_{2,q}} \quad \text{and} \quad \parallel \theta_{n+1}(t) \parallel_{B^s_{2,q}} \leq Q \parallel \theta_0 \parallel_{B^s_{2,q}},
\]
(3.5)

for all \( 0 \leq t \leq T \) and \( n \in \mathbb{N}_0 \).

For \( n = 0 \) and from (3.4), there exists \( Q > 0 \) such that
\[
\parallel \theta_1(t) \parallel_{B^s_{2,q}} \leq C_0 \parallel \theta_0 \parallel_{B^s_{2,q}} \exp \left( C_1 t \parallel u_0 \parallel_{B^s_{2,q}} \right) \leq Q \parallel \theta_0 \parallel_{B^s_{2,q}},
\]

for all \( 0 \leq t \leq T \).
for $0 \leq t \leq T_1$, where
\[ T_1 := \frac{\ln(Q/C_0)}{C_1\|u_0\|_{B^2_{2,q}}} . \]
And for $u_1$ and using (3.4), there exists $P > 0$ such that
\[ \|u_1(t)\|_{B^2_{2,q}} \leq \left[ C_0\|u_0\|_{B^2_{2,q}} + g \int_0^t \|\theta_1(\tau)\|_{B^2_{2,q}} \, d\tau \right] \exp \left( C_1 \int_0^t \|u_0\|_{B^2_{2,q}} \, d\tau \right) ; \]
\[ \leq \left[ C_0\|u_0\|_{B^2_{2,q}} + Qg\|\theta_0\|_{B^2_{2,q}} t \right] \exp \left( C_1 t\|u_0\|_{B^2_{2,q}} \right) \]
\[ \leq \|u_0\|_{B^2_{2,q}} \exp \left( C_0 + \left\{ Qg\|\theta_0\|_{B^2_{2,q}} + C_1\|u_0\|_{B^2_{2,q}} \right\} t \right) \]
\[ \leq P\|u_0\|_{B^2_{2,q}} \]
for $0 \leq t \leq \min\{T_1, T_2\}$, where
\[ T_2 := \frac{\log(P)}{Qg\|\theta_0\|_{B^2_{2,q}} + C_1\|u_0\|_{B^2_{2,q}}} . \]
Proceeding similarly for $n = 1$, it follows that
\[ \|\theta_2(t)\|_{B^2_{2,q}} \leq C_0\|\theta_0\|_{B^2_{2,q}} \exp \left( C_1 \int_0^t \|u_1(\tau)\|_{B^2_{2,q}} \, d\tau \right) \]
\[ \leq C_0\|\theta_0\|_{B^2_{2,q}} \exp \left( C_1 P\|u_0\|_{B^2_{2,q}} t \right) \]
\[ \leq Q\|\theta_0\|_{B^2_{2,q}} , \]
for $0 \leq t \leq \min\{T_1, T_2, T_3\}$, where
\[ T_3 := \frac{\log(Q)}{C_1 P\|u_0\|_{B^2_{2,q}}} . \]
And for $u_2$, we have that
\[ \|u_2(t)\|_{B^2_{2,q}} \leq \left[ C_0\|u_0\|_{B^2_{2,q}} + g \int_0^t \|\theta_2(\tau)\|_{B^2_{2,q}} \, d\tau \right] \exp \left( C_1 \int_0^t \|u_1(\tau)\|_{B^2_{2,q}} \, d\tau \right) \]
\[ \leq \left[ C_0\|u_0\|_{B^2_{2,q}} + Qg\|\theta_0\|_{B^2_{2,q}} t \right] \exp \left( C_1 P\|u_0\|_{B^2_{2,q}} \right) \]
\[ \leq \|u_0\|_{B^2_{2,q}} \exp \left( C_0 + \left\{ Qg\|\theta_0\|_{B^2_{2,q}} + C_1\|u_0\|_{B^2_{2,q}} \right\} t \right) \]
\[ \leq P\|u_0\|_{B^2_{2,q}} , \]
for $0 \leq t \leq \min\{T_1, T_2, T_3, T_4\}$, where
\[ T_4 := \frac{\log(Q)}{Qg\|\theta_0\|_{B^2_{2,q}} + C_1 P\|u_0\|_{B^2_{2,q}}} . \]
Denoting $T' := \min\{T_1, T_2, T_3, T_4\}$ and making the same accounts as in the case $n = 2$, we obtain that
\[ \|\theta_3(t)\|_{B^2_{2,q}} \leq Q\|\theta_0\|_{B^2_{2,q}} \quad \text{and} \quad \|u_3(t)\|_{B^2_{2,q}} \leq P\|u_0\|_{B^2_{2,q}} \quad \text{for all} \ 0 \leq t \leq T' . \]
Thus, proceeding inductively we get (3.5) for all $n \in \mathbb{N}_0$.

**Convergence of the approximation scheme.** Now, we show that the sequence $\{u_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^\infty(0, T; B^2_{2,q}^{-1}(\mathbb{R}^3))$ for some $T \leq T'$. So,
we consider the difference between the systems \( (AP_{n+1}) \) and \( (AP_n) \), and denoting 
\[ u_{n+1} := u_{n+1} - u_n \]  
and \( \theta_{n+1} := \theta_{n+1} - \theta_n \), we obtain the following system

\[
\begin{align*}
\partial_t u_{n+1} + \Omega e_3 \times u_{n+1} + \mathbb{P}(u_{n} \cdot \nabla) u_{n+1} + \mathbb{P}(u_{n-1} \cdot \nabla) u_{n+1} &= g \mathbb{P} \theta_{n+1} e_3, \\
\partial_t \theta_{n+1} + (u_{n} \cdot \nabla) \theta_{n+1} + (u_{n-1} \cdot \nabla) \theta_{n+1} &= 0, \\
\nabla \cdot u_{n+1} &= \nabla \cdot u_n = \nabla \cdot u_{n-1} = 0, \\
\theta_{n+1}|_{t=0} &= \Delta_{n+1} u_0, \quad \theta_{n+1}|_{t=0} = \Delta_{n+1} \theta_0.
\end{align*}
\]  
(3.6)

Applying both the Leray-Helmholtz projector \( \mathbb{P} \) and the operator \( \Delta_j \) to the above system, we get

\[
\begin{align*}
\langle \partial_t \Delta_j u_{n+1}, \Delta_j u_{n+1} \rangle_{L^2} &= -\langle \Delta_j (u_{n} \cdot \nabla) u_{n+1}, \Delta_j u_{n+1} \rangle_{L^2} \\
&\quad + \langle (u_{n-1} \cdot \nabla, \Delta_j u_{n+1}, \Delta_j u_{n+1} \rangle_{L^2} + \langle g \Delta_j \theta_{n+1} e_3, \Delta_j u_{n+1} \rangle_{L^2}, \\
\langle \partial_t \Delta_j \theta_{n+1}, \Delta_j \theta_{n+1} \rangle_{L^2} &= -\langle \Delta_j (u_{n} \cdot \nabla) \theta_{n+1}, \Delta_j \theta_{n+1} \rangle_{L^2} \\
&\quad + \langle (u_{n-1} \cdot \nabla, \Delta_j \theta_{n+1}, \Delta_j \theta_{n+1} \rangle_{L^2}.
\end{align*}
\]

Here, we have used the skew-symmetry of \( e_3 \times \) and the condition \( \nabla \cdot u_n = \nabla \cdot u_{n-1} = 0 \). By the Cauchy-Schwarz inequality, it follows that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \Delta_j \overline{u_{n+1}} \|_{L^2}^2 &\leq \left( \| \Delta_j (u_{n} \cdot \nabla) u_{n+1} \|_{L^2} + \| u_{n-1} \cdot \nabla, \Delta_j \overline{u_{n+1}} \|_{L^2} \right) \| \Delta_j \overline{u_{n+1}} \|_{L^2}, \\
\frac{1}{2} \frac{d}{dt} \| \Delta_j \overline{\theta_{n+1}} \|_{L^2}^2 &\leq \left( \| \Delta_j (u_{n} \cdot \nabla) \theta_{n+1} \|_{L^2} + \| u_{n-1} \cdot \nabla, \Delta_j \overline{\theta_{n+1}} \|_{L^2} \right) \| \Delta_j \overline{\theta_{n+1}} \|_{L^2},
\end{align*}
\]

and then

\[
\begin{align*}
\frac{d}{dt} \| \Delta_j \overline{u_{n+1}} \|_{L^2} &\leq \| \Delta_j (u_{n} \cdot \nabla) u_{n+1} \|_{L^2} + \| u_{n-1} \cdot \nabla, \Delta_j \overline{u_{n+1}} \|_{L^2} + g \| \Delta_j \overline{\theta_{n+1}} \|_{L^2}, \\
\frac{d}{dt} \| \Delta_j \overline{\theta_{n+1}} \|_{L^2} &\leq \| \Delta_j (u_{n} \cdot \nabla) \theta_{n+1} \|_{L^2} + \| u_{n-1} \cdot \nabla, \Delta_j \overline{\theta_{n+1}} \|_{L^2}.
\end{align*}
\]

Integrating over \((0, t)\), we arrive at the estimates

\[
\| \Delta_j \overline{u_{n+1}}(t) \|_{L^2} \leq \| \Delta_j \overline{u_{n+1}}(0) \|_{L^2} + \int_0^t \| \Delta_j (u_{n}(\tau) \cdot \nabla) u_{n+1}(\tau) \|_{L^2} \, d\tau \\
+ \int_0^t \| u_{n-1}(\tau) \cdot \nabla, \Delta_j \overline{u_{n+1}}(\tau) \|_{L^2} \, d\tau + g \int_0^t \| \Delta_j \overline{\theta_{n+1}}(\tau) \|_{L^2} \, d\tau,
\]

\[
\| \Delta_j \overline{\theta_{n+1}}(t) \|_{L^2} \leq \| \Delta_j \overline{\theta_{n+1}}(0) \|_{L^2} + \int_0^t \| \Delta_j (u_{n}(\tau) \cdot \nabla) \theta_{n+1}(\tau) \|_{L^2} \, d\tau \\
+ \int_0^t \| u_{n-1}(\tau) \cdot \nabla, \Delta_j \overline{\theta_{n+1}}(\tau) \|_{L^2} \, d\tau.
\]

Multiplying by \(2^{(s-1)j}\) and applying the \( l^q(Z)\)-norm, we obtain

\[
\| \overline{u_{n+1}}(t) \|_{\dot{B}^{s-1}_{2,q}} \leq \| \overline{u_{n+1}}(0) \|_{\dot{B}^{s-1}_{2,q}} \\
+ \int_0^t \| (u_{n}(\tau) \cdot \nabla) u_{n+1}(\tau) \|_{\dot{B}^{s-1}_{2,q}} \, d\tau + g \int_0^t \| \overline{\theta_{n+1}}(\tau) \|_{\dot{B}^{s-1}_{2,q}} \, d\tau
\]
second equation in (3.6), respectively, to get

From (3.7) and (3.8), we arrive at the estimates

\[ \|\theta_{n+1}(t)\|_{B^s_{2,q}} \leq \|\theta_{n+1}(0)\|_{B^s_{2,q}} + \int_0^t \|\nabla\theta_{n+1}(\tau)\|_{B^s_{2,q}} d\tau \]

\[ + \int_0^t \left( \sum_{j\in\mathbb{Z}} 2^{(s-1)jq} \|u_{n-1}(\tau) \cdot \nabla, \Delta_j |u_{n+1}(\tau)\|_{L^2}^{q} \right)^{\frac{1}{q}} d\tau. \]

Next, we can take the \(L^2\)-norm product with \(\theta_{n+1}\) and \(\bar{\theta}_{n+1}\) to the first and the second equation in (3.6), respectively, to get

\[ \langle \partial_t u_{n+1}, u_{n+1}\rangle_{L^2} + \langle \nabla u_{n+1}, u_{n+1}\rangle_{L^2} = \langle g\theta_{n+1}e_3, u_{n+1}\rangle_{L^2}, \]

\[ \langle \partial_t \theta_{n+1}, \theta_{n+1}\rangle_{L^2} + \langle \nabla \theta_{n+1}, \theta_{n+1}\rangle_{L^2} = 0. \]

Cauchy-Schwarz inequality leads us to

\[ \frac{d}{dt} \|\theta_{n+1}\|_{L^2} \leq \|\nabla\theta_{n+1}\|_{L^2} + c \|\theta_{n+1}\|_{L^2}, \]

\[ \frac{d}{dt} \|\bar{\theta}_{n+1}\|_{L^2} \leq \|\nabla\theta_{n+1}\|_{L^2}. \]

Integrating over \((0,t)\), we have that

\[ \|\theta_{n+1}(t)\|_{L^2} \leq \|\theta_{n+1}(0)\|_{L^2} + \int_0^t \|\nabla\theta_{n+1}(\tau)\|_{L^2} d\tau, \]

\[ \|\bar{\theta}_{n+1}(t)\|_{L^2} \leq \|\bar{\theta}_{n+1}(0)\|_{L^2} + \int_0^t \|\nabla\theta_{n+1}(\tau)\|_{L^2} d\tau. \]

From (3.7) and (3.8), we arrive at the estimates

\[ \|\theta_{n+1}(t)\|_{B^s_{2,q}} \leq \|\theta_{n+1}(0)\|_{B^s_{2,q}} \]

\[ + \int_0^t \|\nabla\theta_{n+1}(\tau)\|_{B^s_{2,q}} d\tau + c \int_0^t \|\theta_{n+1}(\tau)\|_{B^s_{2,q}} d\tau \]

\[ + \int_0^t \left( \sum_{j\in\mathbb{Z}} 2^{(s-1)jq} \|u_{n-1}(\tau) \cdot \nabla, \Delta_j |u_{n+1}(\tau)\|_{L^2}^{q} \right)^{\frac{1}{q}} d\tau. \]

\[ \|\bar{\theta}_{n+1}(t)\|_{B^s_{2,q}} \leq \|\bar{\theta}_{n+1}(0)\|_{B^s_{2,q}} + \int_0^t \|\nabla\theta_{n+1}(\tau)\|_{B^s_{2,q}} d\tau \]

\[ + \int_0^t \left( \sum_{j\in\mathbb{Z}} 2^{(s-1)jq} \|u_{n-1}(\tau) \cdot \nabla, \Delta_j |\bar{\theta}_{n+1}(\tau)\|_{L^2}^{q} \right)^{\frac{1}{q}} d\tau. \]

In view of the embedding (2.4) and equivalence (2.3), we have that

\[ \|\nabla f\|_{L^\infty} \leq C\|\nabla f\|_{B^{s-1}_{p,q}} \leq C\|f\|_{B^s_{p,q}}, \]

provided that the indexes \(s, p, q\) satisfy \(s > 3/p + 1\) with \(1 \leq p, q \leq \infty\) or \(s = 3/p + 1\) with \(1 \leq p \leq \infty\) and \(q = 1\).
Thus, using Lemma 2.2, Lemma 2.3, (2.4) and (3.9), we obtain
\[
\|\bar{u}_{n+1}(t)\|_{L_2^1} \leq \|\bar{u}_{n+1}(0)\|_{L_2^1} + C \int_0^t \|\bar{u}_n(\tau)\|_{L_2^1} \|\bar{u}_{n+1}(\tau)\|_{L_2^1} d\tau + C \int_0^t \|u_{n-1}(\tau)\|_{B_{2,q}^{s-1}} \|\bar{u}_{n+1}(\tau)\|_{B_{2,q}^{s-1}} d\tau + g \int_0^t \|\bar{u}_{n+1}(\tau)\|_{B_{2,q}^{s-1}} d\tau,
\]
\[
\|\bar{u}_{n+1}(t)\|_{B_{2,q}^{s-1}} \leq \|\bar{u}_{n+1}(0)\|_{B_{2,q}^{s-1}} + C \int_0^t \|u_{n-1}(\tau)\|_{B_{2,q}^{s-1}} \|\theta_{n+1}(\tau)\|_{B_{2,q}^{s-1}} d\tau + C \int_0^t \|u_{n-1}(\tau)\|_{B_{2,q}^{s-1}} \|\bar{u}_{n+1}(\tau)\|_{B_{2,q}^{s-1}} d\tau.
\]
Since \(\bar{u}_{n+1}(0) = \Delta_{n+1} u_0\), \(\bar{u}_{n+1}(0) = \Delta_{n+1} \theta_0\) and from (3.5), it follows that
\[
\|\bar{u}_{n+1}\|_{L^\infty(0,T;B_{2,q}^{s-1})} \leq C_0 2^{-n} \|u_0\|_{B_{2,q}^{s-1}} + CP \|u_0\|_{B_{2,q}^{s-1}} T \|\bar{u}_n\|_{L^\infty(0,T;B_{2,q}^{s-1})} + C \int_0^t \|\bar{u}_n(\tau)\|_{L^\infty(0,T;B_{2,q}^{s-1})} \|\bar{u}_{n+1}(\tau)\|_{L^\infty(0,T;B_{2,q}^{s-1})} d\tau \leq C \|\theta_0\|_{B_{2,q}^{s-1}} T' \|\bar{u}\|_{L^\infty(0,T;B_{2,q}^{s-1})},
\]
\[(3.10)\]
we get
\[
\|\bar{u}_{n+1}\|_{L^\infty(0,T';B_{2,q}^{s-1})} \leq \frac{4}{3} C 2^{-n} \|\theta_0\|_{B_{2,q}^{s-1}} + \frac{1}{3} \|\bar{u}_n\|_{L^\infty(0,T';B_{2,q}^{s-1})}.
\]
(3.11)
Substituting (3.11) in the first inequality of (3.10), we obtain
\[
\|\bar{u}_{n+1}\|_{L^\infty(0,T'';B_{2,q}^{s-1})} \leq \frac{4}{3} C 2^{-n} \|u_0\|_{B_{2,q}^{s-1}} + \frac{1}{3} \|\bar{u}_n\|_{L^\infty(0,T'';B_{2,q}^{s-1})} + \frac{4}{9} CT'' 2^{-n} \|\theta_0\|_{B_{2,q}^{s-1}}.
\]
Using the previous inequality and proceeding iteratively, we arrive at the following inequality
\[
\|\bar{u}_{n+1}\|_{L^\infty(0,T'';B_{2,q}^{s-1})} \leq C_0 4 \|u_0\|_{B_{2,q}^{s-1}} \sum_{k=0}^{n-1} \left( \frac{2(1 + T'' g)}{3} \right)^k + \frac{4}{9} CT'' 2^{-n} \|\theta_0\|_{B_{2,q}^{s-1}} \sum_{k=0}^{n-1} \left( \frac{2(1 + T'' g)}{3} \right)^k.
\]
Then, taking \(T > 0\) such that \(0 < T < \min\{T'', \frac{1}{2g}\}\), it follows that
\[
\sum_{n=1}^\infty \|\bar{u}_{n+1}\|_{L^\infty(0,T;B_{2,q}^{s-1})} < \infty,
\]
and thus
\[
\|\bar{u}_{n+1}\|_{L^\infty(0,T;B_{2,q}^{s-1})} \to 0 \quad \text{and} \quad \|\bar{u}_{n+1}\|_{L^\infty(0,T;B_{2,q}^{s-1})} \to 0, \quad \text{as} \quad n \to \infty.
\]
Therefore there exists \((u, \theta) \in C([0, T]; B_{2,q}^{s-1}(\mathbb{R}^3)) \times C([0, T]; B_{2,q}^{s-1}(\mathbb{R}^3))\) solution to the system (1.2), which is obtained by taking the limit of the sequence \((u_n, \theta_n)_{n \in \mathbb{N}}\). Moreover, from equations it holds that \((u, \theta) \in C([0, T]; B_{2,q}^s(\mathbb{R}^3)) \times C([0, T]; B_{2,q}^s(\mathbb{R}^3))\).

**Uniqueness.** Let \((u_i, \theta_i) \in C([0, T]; B_{2,q}^s(\mathbb{R}^3)) \times C([0, T]; B_{2,q}^s(\mathbb{R}^3)), i = 1, 2,\) be two solutions to (1.2). Doing \(u := u_1 - u_2\) and \(\theta := \theta_1 - \theta_2,\) we have that \(u\) and \(\theta\) satisfy the following system

\[
\begin{aligned}
\partial_t u + \mathbb{P}(u \cdot \nabla) u &= -\mathbb{P}(u \cdot \nabla) u_2 + g\theta e_3, \\
\partial_t \theta + (u_1 \cdot \nabla) \theta &= - (u \cdot \nabla) \theta_2, \\
\nabla \cdot u &= 0, \\
u|_{t=0} = 0, \quad \theta|_{t=0} = 0.
\end{aligned}
\]

(3.12)

Multiplying by \(u\) and \(\theta\) in the first and second equations in the system (3.12), and integrating over \(\mathbb{R}^3,\) respectively, we obtain

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2} &= -(u \cdot \nabla)u_2, \\
\|\theta\|^2_{L^2} &= -(u \cdot \nabla)\theta_2.
\end{aligned}
\]

By the Cauchy-Schwarz inequality, it holds

\[
\begin{aligned}
\frac{d}{dt} \|u\|_{L^2} &\leq \|(u \cdot \nabla)u_2\|_{L^2}, \\
\frac{d}{dt} \|\theta\|_{L^2} &\leq \|(u \cdot \nabla)\theta_2\|_{L^2}.
\end{aligned}
\]

Thus, integrating over \((0, t)\) and using (3.9), we arrive at

\[
\begin{aligned}
\|u(t)\|_{L^2} &\leq C\|u_2\|_{L^\infty(0, T; B_{2,q}^s)} \int_0^t \|u(\tau)\|_{L^2} \, d\tau + g \int_0^t \|\theta(\tau)\|_{L^2} \, d\tau, \\
\|\theta(t)\|_{L^2} &\leq C\|\theta_2\|_{L^\infty(0, T; B_{2,q}^s)} \int_0^t \|u(\tau)\|_{L^2} \, d\tau.
\end{aligned}
\]

(3.13)

By substituting the second inequality in the first inequality of (3.13), we get

\[
\begin{aligned}
\|u(t)\|_{L^2} &\leq C\|u_2\|_{L^\infty(0, T; B_{2,q}^s)} \int_0^t \|u(\tau)\|_{L^2} \, d\tau \\
&+ Cg \|\theta_2\|_{L^\infty(0, T; B_{2,q}^s)} T \int_0^t \|u(\tau)\|_{L^2} \, d\tau \\
&\leq C\|u_2\|_{L^\infty(0, T; B_{2,q}^s)} + g \|\theta_2\|_{L^\infty(0, T; B_{2,q}^s)} T \int_0^t \|u(\tau)\|_{L^2} \, d\tau.
\end{aligned}
\]

Using the Gronwall inequality, it follows that \(\|u(t)\|_{L^2} = 0\) for all \(t \in [0, T],\) and then \(u_1 \equiv u_2.\) Also, from (3.13), we have that \(\theta_1 \equiv \theta_2.\) This concludes the proof of the uniqueness of solutions to (1.2).

3.2. **Blow-up criterion.** The subject of this subsection is a blow-up criterion of BKM type.

**Proposition 3.1.** Let \(s\) and \(q\) satisfy \(s > 3/2 + 1\) if \(1 \leq q \leq \infty\) or \(s = 3/2 + 1\) if \(q = 1,\) and let \((u_0, \theta_0) \in B_{2,q}^s(\mathbb{R}^3) \times B_{2,q}^s(\mathbb{R}^3)\) be such that \(\nabla \cdot u_0 = 0.\) If \((u, \theta) \in C([0, T); B_{2,q}^s(\mathbb{R}^3) \times B_{2,q}^s(\mathbb{R}^3))\) is a solution for (1.2), then \(u\) can be extended to
[0, T') for some T' > T with (u, θ) ∈ C([0, T'); B^s_{2,q}(\mathbb{R}^3) × B^s_{2,q}(\mathbb{R}^3)) provided that 
∫_0^T ||\nabla u(t)||_{L^\infty} dt < \infty.

Proof. First, we obtain an estimate for u in B^{s}_{2,q}(\mathbb{R}^3) depending only of s, q, 
||u_0||_{B^s_{2,q}}, ||\theta_0||_{B^s_{2,q}} and ∫_0^T ||\nabla u(t)||_{L^\infty} dt. For that, we apply ∆_j in the first and 
the second equations in (1.2), and we take the L^2-inner product with ∆_j u and ∆_j θ, respectively, to get

\frac{1}{2} \frac{d}{dt} ||\Delta_j u||_{L^2}^2 + (\Delta_j (u \cdot \nabla) u, \Delta_j u)_{L^2} = \langle g \Delta_j \theta e_3, \Delta_j u \rangle_{L^2},

\frac{1}{2} \frac{d}{dt} ||\Delta_j \theta||_{L^2}^2 + (\Delta_j (u \cdot \nabla) \theta, \Delta_j \theta)_{L^2} = 0,

where here we have used the skew-symmetry of operator e_3 × and the condition \nabla \cdot u = 0. Now, using the Cauchy-Schwartz inequality and recalling the definition 
of commutator [\cdot, \cdot], we have that

\frac{d}{dt} ||\Delta_j u||_{L^2} \leq ||u \cdot \nabla, \Delta_j u||_{L^2} + g ||\Delta_j \theta||_{L^2},

\frac{d}{dt} ||\Delta_j \theta||_{L^2} \leq ||u \cdot \nabla, \Delta_j \theta||_{L^2}.

Next, we integrate over (0, t), multiply by 2^j and apply the l^1(\mathbb{Z})-norm in order to obtain

||u(t)||_{B^s_{2,q}} \leq ||u_0||_{B^s_{2,q}} + \int_0^t \left( \sum_{j \in \mathbb{Z}} 2^{jsq} ||[u(\tau) \cdot \nabla, \Delta_j]u(\tau)||^q_{L^2} \right)^{\frac{1}{q}} d\tau

+ g \int_0^t ||\theta(\tau)||_{B^s_{2,q}} d\tau,

||\theta(t)||_{B^s_{2,q}} \leq ||\theta_0||_{B^s_{2,q}} + \int_0^t \left( \sum_{j \in \mathbb{Z}} 2^{jsq} ||[u(\tau) \cdot \nabla, \Delta_j]\theta(\tau)||^q_{L^2} \right)^{\frac{1}{q}} d\tau.

By Lemma 2.3 and the inclusion B^s_{2,q}(\mathbb{R}^3) ⊆ \hat{B}^s_{2,q}(\mathbb{R}^3), we have that

||u(t)||_{\hat{B}^s_{2,q}} \leq ||u_0||_{\hat{B}^s_{2,q}} + C \int_0^t ||u(\tau)||_{B^s_{2,q}} ||\nabla u(\tau)||_{L^\infty} d\tau + g \int_0^t ||\theta(\tau)||_{B^s_{2,q}} d\tau,

||\theta(t)||_{\hat{B}^s_{2,q}} \leq ||\theta_0||_{\hat{B}^s_{2,q}} + C \int_0^t ||\theta(\tau)||_{B^s_{2,q}} ||\nabla u(\tau)||_{L^\infty} + ||u(\tau)||_{B^s_{2,q}} ||\nabla \theta(\tau)||_{L^\infty} d\tau.

(3.14)

For the L^2-norm, we can proceed similarly as in (3.8) to obtain

||u(t)||_{L^2} \leq ||u_0||_{L^2} + g \int_0^t ||\theta(\tau)||_{L^2} d\tau,

||\theta(t)||_{L^2} \leq ||\theta_0||_{L^2}.

(3.15)

Combining (3.14) and (3.15), it follows that

||u(t)||_{\hat{B}^s_{2,q}} \leq ||u_0||_{\hat{B}^s_{2,q}} + C \int_0^t ||u(\tau)||_{\hat{B}^s_{2,q}} ||\nabla u(\tau)||_{L^\infty} d\tau + g \int_0^t ||\theta(\tau)||_{\hat{B}^s_{2,q}} d\tau,

||\theta(t)||_{\hat{B}^s_{2,q}} \leq ||\theta_0||_{\hat{B}^s_{2,q}} + C \int_0^t ||\theta(\tau)||_{\hat{B}^s_{2,q}} ||\nabla u(\tau)||_{L^\infty} + ||u(\tau)||_{\hat{B}^s_{2,q}} ||\nabla \theta(\tau)||_{L^\infty} d\tau.
Now Gronwall inequality, (3.9) and (3.5) yield
\[
\|\theta(t)\|_{B_{2,q}^s} \\
\leq \left( \|\theta_0\|_{B_{2,q}^s} + C \int_0^t \|u(\tau)\|_{B_{2,q}^s} \|\nabla \theta(\tau)\|_{L^\infty} \, d\tau \right) \exp \left( C \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right) \\
\leq \left( \|\theta_0\|_{B_{2,q}^s} + CPQ \|u_0\|_{B_{2,q}^s} \|\theta_0\|_{B_{2,q}^s} T \right) \exp \left( C \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right),
\]
(3.16)
and then we get the estimate
\[
\|u(t)\|_{B_{2,q}^s} \\
\leq \left[ \|u_0\|_{B_{2,q}^s} + g \int_0^t \|\theta(\tau)\|_{B_{2,q}^s} \, d\tau \right] \exp \left( C \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right) \\
\leq \left[ \|u_0\|_{B_{2,q}^s} + g \|\theta_0\|_{B_{2,q}^s} (1 + CPQT \|u_0\|_{B_{2,q}^s}) T \exp \left( C \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right) \right] \times \\
\times \exp \left( C \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right),
\]
(3.17)
for all \( t \in [0,T) \). Since the existence time \( T > 0 \) depends only on \( s, q \) and initial data norms \( \|u_0\|_{B_{2,q}^s} \) and \( \|\theta_0\|_{B_{2,q}^s} \), by standard arguments \( u \) can be continued to \([0, T')\) for some \( T' > T \), whenever \( \int_0^T \|\nabla u(t)\|_{L^\infty} \, dt < \infty \). \( \square \)

**Remark 3.2.** Under the assumptions of Proposition 3.1, if \( T = T_\ast \) is the maximal existence-time then
\[
\int_0^{T_\ast} \|\nabla u(t)\|_{L^\infty} \, dt = \infty.
\]

3.3. **Long-time solvability.** Let \((u, \theta) \in C([0,T_\ast); B_{2,q}^{s+1}(\mathbb{R}^3) \times B_{2,q}^{s+1}(\mathbb{R}^3))\) be the solution to (1.2) with initial data \((u_0, \theta_0) \in B_{2,q}^{s+1}(\mathbb{R}^3) \times B_{2,q}^{s+1}(\mathbb{R}^3)\) satisfying the condition \( \nabla \cdot u_0 = 0 \) and maximal existence time \( T_\ast > 0 \).

Next recall the projection operators \( P_{\pm} : L^2(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)^3 \) defined by (see [19, 27])
\[
P_{\pm}v := \frac{1}{2} \left( \mathbb{P}v \pm i \frac{D}{|D|} \times v \right),
\]
where \((\frac{D}{|D|} \times v)(\xi) := \frac{\xi}{|\xi|} \times \hat{v}(\xi)\), which satisfy \( P_{\pm} = P_{\pm}^\ast \). Moreover, if \( \nabla \cdot v = 0 \) we have that \( v = P_{\pm}v + P_{\mp}v, \mathbb{P}(e_3 \times v) = -i \frac{D}{|D|} (P_{\pm}v - P_{\mp}v) \), \( P_{\pm}P_{\mp} = P_{\pm} \), and \( P_{\pm}P_{\mp} = 0 \).

Applying \( P_{\pm} \) to the first equation of (1.2), we arrive at
\[
\begin{cases}
\partial_t P_{\pm} u + i\Omega \frac{D}{|D|} P_{\pm} u + P_{\pm} (u \cdot \nabla) u = gP_{\pm} \theta e_3, \\
P_{\pm} u \big|_{t=0} = P_{\pm} u_0.
\end{cases}
\]
Denoting $A_\pm := \pm \Omega \frac{D_3}{|D|}$ and using Duhamel’s principle, we get the following equality
\[
P_\pm u(t) = e^{\pm \Omega t \frac{D_2}{|D|}} P_\pm u_0 - \int_0^t e^{\pm \Omega (t-\tau) \frac{D_2}{|D|}} P_\pm (u(\tau) \cdot \nabla) u(\tau) \, d\tau + g \int_0^t e^{\pm \Omega (t-\tau) \frac{D_2}{|D|}} P_\pm \theta(\tau) e_3 \, d\tau.
\]  
(3.18)

Now, we recall the Strichartz estimates of [27] which asserts that if $2 \leq r, \theta \leq \infty$ with $(r, \theta) \neq (2, \infty)$ and $\frac{1}{r} + \frac{1}{\theta} \leq \frac{1}{2}$, then
\[
\|e^{\pm it \frac{D_2}{|D|}} f\|_{L^r([0,\infty);L^\theta)} \leq C\|f\|_{L^2}.
\]  
(3.19)

Considering $2 < r < \infty$ and using a scaling argument in (3.19), we get that there exists a constant $C = C(r) > 0$ such that
\[
\|\Delta_j e^{\pm it \frac{D_2}{|D|}} f\|_{L^r([0,\infty);L^\theta)} \leq C 2^{\frac{j}{2}} |\Omega|^{-\frac{1}{2}} \|\Delta_j f\|_{L^2},
\]  
(3.20)

for all $j \in \mathbb{Z}$ and $\Omega \in \mathbb{R} \setminus \{0\}$. Here we divide the proof in two cases depending on the value of $s$, namely, the cases $s = 5/2$ with $q = 1$ and $s > 5/2$ with $1 \leq q \leq \infty$. For the first case, we derive an estimate in $B^{1}_{\infty,1}$ for $u$. Since $u = P_+ u + P_- u$, it suffices to obtain an estimate in $B^{1}_{\infty,1}$ for $P_+ u$ and $P_- u$. By (3.19) and (3.20), we remark that
\[
\|e^{\pm it \frac{D_2}{|D|}} P_\pm u_0\|_{L^r([0,\infty);B^{1}_{\infty,1})} \leq \sum_{j \in \mathbb{Z}} 2^j \|\Delta_j e^{\pm it \frac{D_2}{|D|}} P_\pm u_0\|_{L^r([0,\infty);L^\infty)}
\]
\[
\leq C|\Omega|^{-\frac{1}{2}} \sum_{j \in \mathbb{Z}} 2^j 2^{\frac{j}{2}} \|\Delta_j P_\pm u_0\|_{L^2}
\]
\[
= C|\Omega|^{-\frac{1}{2}} \|u_0\|_{B^{1/2}_{2,1}}.
\]

Thus,
\[
\|e^{\pm it \frac{D_2}{|D|}} P_\pm u_0\|_{L^r([0,\infty);B^{1}_{\infty,1})} \leq C|\Omega|^{-\frac{1}{2}} \|u_0\|_{B^{1/2}_{2,1}}.
\]

Similarly, we have that
\[
\left\| \int_0^t e^{\pm \Omega (t-\tau) \frac{D_2}{|D|}} P_\pm (u(\tau) \cdot \nabla) u(\tau) \, d\tau \right\|_{L^r([0,T];B^{1}_{\infty,1})}
\leq C|\Omega|^{-\frac{1}{2}} \int_0^T \|P_\pm (u(\tau) \cdot \nabla) u(\tau)\|_{B^{1/2}_{2,1}} \, d\tau,
\]
\[
\left\| \int_0^t e^{\pm \Omega (t-\tau) \frac{D_2}{|D|}} P_\pm \theta(\tau) e_3 \, d\tau \right\|_{L^r([0,T];B^{1}_{\infty,1})} \leq C|\Omega|^{-\frac{1}{2}} \int_0^T \|P_\pm \theta(\tau)\|_{B^{1/2}_{2,1}} \, d\tau.
\]

Therefore, for all $0 < T < T_*$, it holds that
\[
\|u\|_{L^r([0,T];B^{1}_{\infty,1})} \leq C|\Omega|^{-\frac{1}{2}} \left( \|u_0\|_{B^{1/2}_{2,1}} + \int_0^T \|u(\tau) \cdot \nabla) u(\tau)\|_{B^{1/2}_{2,1}} \, d\tau + g \int_0^T \|\theta(\tau)\|_{B^{1/2}_{2,1}} \, d\tau \right).
\]
Using (3.16), it follows that
\[
\|u\|_{L^r(0,T;B^1_{\infty,1})} \leq C|\Omega|^{-\frac{1}{2}} \left( \|u_0\|_{B^5_{2,1}} + \int_0^T \| (u(\tau) \cdot \nabla) u(\tau) \|_{B^5_{2,1}} \, d\tau \right.
\]
\[
+ g\|\theta_0\|_{B^5_{2,1}} \int_0^T \exp \left( C \int_0^T \| \nabla u(\tau') \|_{L^\infty} \, d\tau' \right) \, d\tau \bigg) \]
\[
\leq C|\Omega|^{-\frac{1}{2}} \left( \|u_0\|_{B^7_{2,1}} + \int_0^T \| u(\tau) \|_{B^7_{2,1}}^2 \, d\tau \right.
\]
\[
+ g\|\theta_0\|_{B^7_{2,1}} \int_0^T \exp (CU(\tau)) \, d\tau \bigg) \]
\[
\leq C|\Omega|^{-\frac{1}{2}} \left( \|u_0\|_{B^7_{2,1}} + \int_0^T \| u(\tau) \|_{B^7_{2,1}}^2 \, d\tau \right.
\]
\[
+ g\|\theta_0\|_{B^7_{2,1}} \int_0^T \exp (CU(\tau)) \, d\tau \bigg) \]
\[
\leq C|\Omega|^{-\frac{1}{2}} \left[ \|u_0\|_{B^7_{2,1}}^2 + g\|\theta_0\|_{B^7_{2,1}} T \left( 1 + CPQT \|u_0\|_{B^7_{2,1}} \exp (CU(\tau)) \right) \right]^2 \times
\]
\[
\bigg( \frac{\exp (2CU(\tau))}{\|u_0\|_{B^7_{2,1}}^2} \bigg) \bigg). \]

Hence, there exist positive constants $C_2$ and $C_3$ (independent of $\Omega$) such that
\[
U(t) \leq C_2 t^{1-\frac{1}{2}} |\Omega|^{-\frac{1}{2}} \|u_0\|_{B^7_{2,1}} \left( 1 + \frac{g\|\theta_0\|_{B^7_{2,1}}}{\|u_0\|_{B^7_{2,1}}} t \exp (C_3 U(t)) \right) +
\]
\[
+ t \left[ \|u_0\|_{B^7_{2,1}} + g\|\theta_0\|_{B^7_{2,1}} T \left( 1 + CPQT \|u_0\|_{B^7_{2,1}} \exp (C_3 U(t)) \right) \right]^2 \times
\]
\[
\bigg( \frac{\exp (2CU(\tau))}{\|u_0\|_{B^7_{2,1}}^2} \bigg) \bigg).
that, we assume by contradiction that
\[ U(\hat{T}_s) \leq C_T T_1^{-\frac{1}{2}} \| u_0 \|_{B^{7/2}_{2,1}} \]
and its supremum \( \hat{T}_s = \sup C_T \), we are going to show that \( \hat{T}_s < \min \{ T, T_s \} \). Thus we can select \( \hat{T}_s \) so that \( T_s < \hat{T}_s < \min \{ T, T_s \} \), and then, since \( u \in C([0, \hat{T}_s]; B^{7/2}_{2,1}(\mathbb{R}^3)) \), it follows that \( U(t) \) is uniformly continuous on \([0, \hat{T}_s] \) and
\[ U(\hat{T}_s) \leq C_T T_1^{-\frac{1}{2}} \| u_0 \|_{B^{7/2}_{2,1}}. \]
Choosing \( |\Omega| > 0 \) such that
\[ |\Omega| \geq 2 \left( 1 + \frac{g \| u_0 \|_{B^{7/2}_{2,1}}}{T \exp \left( C_T \| u_0 \|_{B^{7/2}_{2,1}} \right)} + T \left[ \| u_0 \|_{B^{7/2}_{2,1}} + \left( \frac{g \| u_0 \|_{B^{7/2}_{2,1}}}{T \exp \left( C_T \| u_0 \|_{B^{7/2}_{2,1}} \right)} \right)^2 \right] \times \left( 1 + \frac{g \| u_0 \|_{B^{7/2}_{2,1}}}{T \exp \left( C_T \| u_0 \|_{B^{7/2}_{2,1}} \right)} \right) \right), \]
and by (3.22), (3.23) and (3.24), we obtain
\[ U(\hat{T}_s) \leq C_T (\hat{T}_s)^{-\frac{1}{2}} |\Omega|^{-\frac{1}{2}} \| u_0 \|_{B^{7/2}_{2,1}} \left( 1 + \frac{g \| u_0 \|_{B^{7/2}_{2,1}}}{T \exp \left( C_T \| u_0 \|_{B^{7/2}_{2,1}} \right)} + \hat{T}_s \left[ \| u_0 \|_{B^{7/2}_{2,1}} + \left( \frac{g \| u_0 \|_{B^{7/2}_{2,1}}}{T \exp \left( C_T \| u_0 \|_{B^{7/2}_{2,1}} \right)} \right)^2 \right] \times \left( 1 + \frac{g \| u_0 \|_{B^{7/2}_{2,1}}}{T \exp \left( C_T \| u_0 \|_{B^{7/2}_{2,1}} \right)} \right) \right) \]
\[ \leq C_T T_1^{-\frac{1}{2}} |\Omega|^{-\frac{1}{2}} \| u_0 \|_{B^{7/2}_{2,1}} \left( 1 + \frac{g \| u_0 \|_{B^{7/2}_{2,1}}}{T \exp \left( C_T \| u_0 \|_{B^{7/2}_{2,1}} \right)} + \hat{T}_s \left[ \| u_0 \|_{B^{7/2}_{2,1}} + \left( \frac{g \| u_0 \|_{B^{7/2}_{2,1}}}{T \exp \left( C_T \| u_0 \|_{B^{7/2}_{2,1}} \right)} \right)^2 \right] \times \left( 1 + \frac{g \| u_0 \|_{B^{7/2}_{2,1}}}{T \exp \left( C_T \| u_0 \|_{B^{7/2}_{2,1}} \right)} \right) \right) \]
\[ \leq \frac{1}{2} C_T T_1^{-\frac{1}{2}} \| u_0 \|_{B^{7/2}_{2,1}}. \]
The latter estimate contradicts the definition of \( \hat{T}_s \), since we can choose \( L \in (\hat{T}_s, \hat{T}) \) such that \( U(L) \leq C_T T_1^{-\frac{1}{2}} \| u_0 \|_{B^{7/2}_{2,1}} \). Therefore, with \( \Omega \) satisfying (3.24), it follows that \( \hat{T}_s = \min \{ T, T_s \} \). If \( T_s < T \), it holds that \( T_s = \hat{T}_s = \sup C_T \) and
\[ U(t) = \int_0^t \| \nabla u(\tau) \|_{L^\infty} \, d\tau \leq C_T T_1^{-\frac{1}{2}} \| u_0 \|_{B^{7/2}_{2,1}} < \infty, \quad \text{for all} \quad t \in [0, T_s). \]
From the blow-up criterion, it follows the result for the case \( s = 5/2 \) with \( q = 1 \).
Now we turn to the case \( s > 5/2 \) with \( 1 \leq q \leq \infty \). In this case, we can choose \( \alpha = \alpha(s) \in (0, 1) \) such that \( s \geq 5/2 + \alpha \). Also, for each \( 1 \leq q \leq \infty \) we consider \( 2 < r \leq \infty \) such that \( q \leq r \). We are going to obtain an estimate in \( B^{L+\alpha}_{2,1,\infty} \) for the solution \( u \). Due to the equality \( u = P_+ u + P_- u \), it is suffices to derive an estimate in

for some $B_{\infty,\infty}$ for $P_{\pm}u$ and $P_{\mp}u$. By the embedding $l^q \hookrightarrow l^\infty$, Minkowski inequality ($q \leq r$) and (3.20), note that

$$
\|e^{\pm i\Omega t} \frac{\partial}{\partial t} P_{\pm} u_0\|_{L^r(0,\infty; B_{\infty,\infty}^{1+\alpha})} = \left\| \sup_{j \in \mathbb{Z}} 2^{(1+\alpha)j} \| \Delta_j e^{\pm i\Omega t} \frac{\partial}{\partial t} P_{\pm} u_0 \|_{L^\infty} \right\|_{L^1_t(0,\infty)} \leq \left\| \left( \sum_{j \in \mathbb{Z}} 2^{(1+\alpha)j} \| \Delta_j e^{\pm i\Omega t} \frac{\partial}{\partial t} P_{\pm} u_0 \|_{L^\infty}^q \right)^{\frac{1}{q}} \right\|_{L^1_t(0,\infty)} \leq \left( \sum_{j \in \mathbb{Z}} 2^{(1+\alpha)j} \| \Delta_j P_{\pm} u_0 \|_{L^r(0,\infty; L^\infty)}^q \right)^{\frac{1}{q}} \leq C|\Omega|^{-\frac{1}{q}} \| P_{\pm} u_0 \|_{B_{2,q}^{\frac{2}{2}+\alpha}} \leq C|\Omega|^{-\frac{1}{q}} \| P_{\pm} u_0 \|_{L^r(0,\infty; B_{\infty,\infty}^{1+\alpha})}.
$$

Therefore, there exists a positive constant $C = C(r, \alpha)$ such that

$$
\|e^{\pm i\Omega t} \frac{\partial}{\partial t} P_{\pm} u_0\|_{L^r(0,\infty; B_{\infty,\infty}^{1+\alpha})} \leq C|\Omega|^{-\frac{1}{q}} \| P_{\pm} u_0 \|_{B_{2,q}^{\frac{2}{2}+\alpha}}.
$$

Using similar arguments for the other terms in (3.18), it follows that

$$
\left\| \int_0^t e^{\pm i\Omega(t-\tau)} \frac{\partial}{\partial \tau} P_{\pm} (u(\tau) \cdot \nabla) u(\tau) \, d\tau \right\|_{L^r(0,T; B_{\infty,\infty}^{1+\alpha})} \leq C|\Omega|^{-\frac{1}{q}} \int_0^T \| P_{\pm} (u(\tau) \cdot \nabla) u(\tau) \|_{B_{2,q}^{\frac{2}{2}+\alpha}} \, d\tau,
$$

and

$$
\left\| \int_0^t e^{\pm i\Omega(t-\tau)} \frac{\partial}{\partial \tau} P_{\pm} \theta(\tau)e_3 \, d\tau \right\|_{L^r(0,T; B_{\infty,\infty}^{1+\alpha})} \leq C|\Omega|^{-\frac{1}{q}} \int_0^T \| P_{\pm} \theta(\tau) \|_{B_{2,q}^{\frac{2}{2}+\alpha}} \, d\tau,
$$

for some $C = C(r, \alpha) > 0$. Then, for any $0 < T < T_*$, we have that

$$
\|u\|_{L^r(0,T; B_{\infty,\infty}^{1+\alpha})} \leq C|\Omega|^{-\frac{1}{q}} \left( \|u_0\|_{B_{2,q}^{\frac{2}{2}+\alpha}} + \int_0^T \| (u(\tau) \cdot \nabla) u(\tau) \|_{B_{2,q}^{\frac{2}{2}+\alpha}} \, d\tau \right) + g \int_0^T \| \theta(\tau) \|_{B_{2,q}^{\frac{2}{2}+\alpha}} \, d\tau.
$$

We recall the definition of $U(t)$ and use (3.17) and (3.16) in order to get

$$
\|u\|_{L^r(0,T; B_{\infty,\infty}^{1+\alpha})} \leq C|\Omega|^{-\frac{1}{q}} \left( \|u_0\|_{B_{2,q}^{\frac{2}{2}+1}} + g\|\theta_0\|_{B_{2,q}^{\frac{2}{2}+1}} \int_0^T \exp(CU(\tau)) \, d\tau + \|u_0\|_{B_{2,q}^{\frac{2}{2}+1}} \int_0^t \left[ \|u_0\|_{B_{2,q}^{\frac{2}{2}+1}} + g\|\theta_0\|_{B_{2,q}^{\frac{2}{2}+1}} T(1 + CPQT\|u_0\|_{B_{2,q}^{\frac{2}{2}+1}}) \times \exp(CU(\tau)) \right]^2 \exp(2CU(\tau)) \, d\tau \right).
$$
Now, proceeding as in the previous case, we obtain that
\[
U(t) \leq C|\Omega|^{-\frac{1}{4}}\|u_0\|_{B_{2,q}^{r+1}} \left(1 + g\frac{\|\theta_0\|_{B_{2,q}^{r+1}}}{\|u_0\|_{B_{2,q}^{r+1}}} \int_0^t \exp(CU(\tau)) \, d\tau + \right. \\
+ \left. \int_0^t \left(\|u_0\|_{B_{2,q}^{r+1}} + g\|\theta_0\|_{B_{2,q}^{r+1}} T(1 + CPQT\|u_0\|_{B_{2,q}^{r+1}}) \times \right. \\
\times \exp(CU(\tau)) \right) \frac{2\exp(2CU(\tau))}{\|u_0\|_{B_{2,q}^{r+1}}} \, d\tau, \right)
\]
for all \( t \in [0, T_s) \). Then, there exist \( C_4 \) and \( C_5 \) (independent of \( \Omega \)) such that
\[
U(t) \leq C_4|\Omega|^{-\frac{1}{4}}\|u_0\|_{B_{2,q}^{r+1}} \left(1 + g\frac{\|\theta_0\|_{B_{2,q}^{r+1}}}{\|u_0\|_{B_{2,q}^{r+1}}} \int_0^t \exp(C_5U(\tau)) \, d\tau + \right. \\
+ \left. \int_0^t \left(\|u_0\|_{B_{2,q}^{r+1}} + g\|\theta_0\|_{B_{2,q}^{r+1}} T(1 + CPQT\|u_0\|_{B_{2,q}^{r+1}}) \times \right. \\
\times \exp(C_5U(\tau)) \right) \frac{2\exp(2C_5U(\tau))}{\|u_0\|_{B_{2,q}^{r+1}}} \, d\tau, \right)
\]
for all \( t \in [0, T_s) \). As before, we also define
\[
D_T = \{ t \in [0, T] \cap [0, T_s) : U(t) \leq C_T T^{1-\frac{1}{4}}\|u_0\|_{B_{2,q}^{r+1}} \}, \quad \tilde{T}_s = \sup C_T.
\]
We are going to prove that \( \tilde{T}_s = \min\{T, T_s\} \) by contradiction. In fact, assuming that \( \tilde{T}_s < \min\{T, T_s\} \), we can take \( \tilde{T} \) so that \( \tilde{T}_s < \tilde{T} < \min\{T, T_s\} \). Since \( u \in C([0, \tilde{T}] ; B_{2,q}^{r+1}(\mathbb{R}^3)) \) we have the uniform continuity of \( U(t) \) on \([0, \tilde{T}] \) and
\[
U(\tilde{T}_s) \leq C_T T^{1-\frac{1}{4}}\|u_0\|_{B_{2,q}^{r+1}}. \tag{3.26}
\]
Considering \(|\Omega| > 0 \) such that
\[
|\Omega|^{\frac{1}{4}} \geq 2 \left(1 + g\frac{\|\theta_0\|_{B_{2,q}^{r+1}}}{\|u_0\|_{B_{2,q}^{r+1}}} T \exp(C_4 C_T T^{1-\frac{1}{4}}\|u_0\|_{B_{2,q}^{r+1}}) + T\left[\|u_0\|_{B_{2,q}^{r+1}} + \\
+ g\|\theta_0\|_{B_{2,q}^{r+1}} T(1 + CPQT\|u_0\|_{B_{2,q}^{r+1}}) \exp(C_4 C_T T^{1-\frac{1}{4}}\|u_0\|_{B_{2,q}^{r+1}}) \right]^2 \times \right. \\
\times \frac{2\exp(2C_4 C_T T^{1-\frac{1}{4}}\|u_0\|_{B_{2,q}^{r+1}}^2)}{\|u_0\|_{B_{2,q}^{r+1}}^2}, \right)
\]
and using (3.25), (3.26) and (3.27), we get
\[
U(\tilde{T}_s) \leq \frac{1}{2} C_4 T^{1-\frac{1}{4}}\|u_0\|_{B_{2,q}^{r+1}}.
\]
Therefore, we can choose \( S \in (\tilde{T}_s, \tilde{T}) \) such that \( U(S) \leq C_4 T^{1-\frac{1}{4}}\|u_0\|_{B_{2,q}^{r+1}} \) contradicting the definition of \( \tilde{T}_s \), which implies that \( \tilde{T}_s = \min\{T, T_s\} \), provided that \( \Omega \) is as in (3.27). If \( T_s < T \), it follows that \( T_s = \tilde{T}_s = \sup D_T \) and
\[
U(t) = \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \leq C_T T^{1-\frac{1}{4}}\|u_0\|_{B_{2,q}^{r+1}} < \infty, \quad \text{for all} \quad t \in [0, T_s).
\]
The blow-up criterion concludes the proof for the present case.
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