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M. Asorey

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Maximal Non-Abelian Gauges and Topology of Gauge Orbit Space

M. Asorey

Departamento de Física Teórica. Facultad de Ciencias
Universidad de Zaragoza. 50009 Zaragoza. Spain

Abstract

We introduce two maximal non-abelian gauge fixing conditions on the space of gauge orbits $\mathcal{M}$ for gauge theories over spaces with dimensions $d \leq 3$. The gauge fixings are complete in the sense that describe an open dense set $\mathcal{M}_0$ of the space of gauge orbits $\mathcal{M}$ and select one and only one gauge field per gauge orbit in $\mathcal{M}_0$. There are not Gribov copies or ambiguities in these gauges. $\mathcal{M}_0$ is a contractible manifold with trivial topology. The set of gauge orbits which are not described by the gauge conditions $\mathcal{M} \setminus \mathcal{M}_0$ is the boundary of $\mathcal{M}_0$ and encodes all non-trivial topological properties of the space of gauge orbits. The gauge fields configurations of this boundary $\mathcal{M} \setminus \mathcal{M}_0$ can be explicitly identified with non-abelian monopoles and they are shown to play a very relevant role in the non-perturbative behaviour of gauge theories in one, two and three space dimensions. It is conjectured that their role is also crucial for quark confinement in 3+1 dimensional gauge theories.
1. Introduction

One of the prominent features of non-abelian gauge theories is the highly non-trivial geometric and topological structure of the space of physically relevant classical gauge field configurations. Because of gauge invariance this space is the gauge orbits space $\mathcal{M} = \mathcal{A}/\mathcal{G}$, i.e., the space of gauge fields $\mathcal{A}$ modulo the group of gauge transformations $\mathcal{G}$. The first evidence of the non-trivial structure of $\mathcal{M}$ came from Gribov’s observation on the existence of ambiguities and possible incompleteness of Coulomb and Landau gauges in Yang-Mills theories [1]. The impossibility of a global gauge fixing was shown to be a consequence of the non-trivial topological structure of the space of gauge orbits [2]. However, the need of efficient gauge fixing is a requirement for analytic approaches to the dynamical behaviour of the theory both in the asymptotically free ultraviolet regime (perturbative) and in the confining infrared regime (non-perturbative).

Landau gauge and its $\alpha$-gauge generalizations played a leading role in the development of the perturbative renormalization program of quantum gauge theories because of its covariant linear and local characteristics [3]. Coulomb gauge was very useful in the formulation of the Hamiltonian approach [4].

The Gribov observation, however, points out the existence of a possible problem by under/over-counting the quantum fluctuations of classical gauge fields on these gauges. The problem does not affect perturbative calculations because the Gribov horizon and the Gribov copies give contributions of order $e^{-1/g^2}$ [5]. There are, however, perturbative effects like anomalies which are very sensitive to the global structure of the space of gauge orbits. In fact, anomalies provide the most direct evidence that the non-trivial topological structure of $\mathcal{M}$ turns out to be relevant for the quantum physics. The reason why anomalies which already appear in perturbation theory can unveil the non-trivial topological structure of the gauge orbit space is due to universality and non-renormalization theorems. They essentially establish that the anomaly structure is stable for all energy scales. In the ultraviolet regime they are determined by perturbative methods, but in the infrared regime they are extremely connected with the cohomology of the space of gauge orbits $\mathcal{M}$. Whether the topology of the orbit space is relevant or not for other non-perturbative effects like confinement requires a deeper study incorporating the contributions of gauge field configurations affected by the Gribov problem.

To circumvent this problem and include the non-perturbative effects associated to those configurations there are three alternatives:
i) Restrict the analysis to subsets of gauge fields satisfying gauge conditions which are free of ambiguities (no overcounting gauge orbits). For instance, consider several Coulomb/Landau gauges around different background fields. Then, consider a complete covering of the orbit space by means of those subsets and use a partition of unity to get the right contributions to the quantum fluctuations. Obviously, the procedure is rather cumbersome, but consistent. The BRST symmetry is also consistent in the overlap of those subsets of gauge fields satisfying different gauge conditions which means that it can be globally defined over the whole space of gauge orbits and guarantees the consistency of the whole procedure [5].

ii) Prove that any gauge orbit is described by few fields in a particular gauge condition and that the sum over all Gribov copies gives the right contribution up to a global factor associated to the volume of overcounted copies. Although a gauge condition with this property is hard to implement in the continuum, which is absolutely necessary to define a consistent quantum theory, in the lattice approach it works for some gauge conditions [6].

iii) Find a special set of gauge fields satisfying a single gauge condition such that their gauge orbits unambiguously describe an open dense subset of the space of gauge orbits. Then, the fluctuations associated to those gauge fields are enough to describe the quantum effects of the theory. In the Hamiltonian formalism the role of the remaining orbits re-emerge as boundary conditions on physical quantum states.

These ways of solving the Gribov problem can be illustrated with the following example. Let us consider the quantization of classical system on $\mathbb{R}^3$ with a kinetic term independent of the radial degree of freedom. The effective configuration space is a 2-dimensional sphere $S^2$. This space can be considered as the space of the orbits of the physical space $\mathbb{R}^3 - \{x_0\}$ under dilations, $S^2 = \mathbb{R}^3/\mathbb{R}^+$. The first method can be implemented by the choice of four planes intersecting in a cube around the origin of $\mathbb{R}^3$. The parametrizations of the dilation group orbits given by these four charts cover the whole $S^2$ sphere. The use of spherical coordinates provides a parametrization of the orbits in terms of angular variables by a simple chart that only excludes the two azimuthal points of $S^2$. This provides an example of gauge fixing of the third type. There are many other ways of parametrizing the 2-sphere not always directly connected with dilation orbits of $\mathbb{R}^3$ (e.g. by stereographic projection coordinates).

It is obvious that the third type of gauge is the most economic for the description of the dynamics. The aim of this paper is to find a gauge condition of this type for Yang-Mills theory.
In Landau gauge it has been proved that it is possible to find a subset of configurations (fundamental domain) which parametrize a dense set of orbits with respect to the $L^2$-norm of the space of gauge field configurations $A$ [7], although the construction of the fundamental domain is not achieved in a very explicit way. However, the main problem is that the $L^2$-norm is not relevant for the measure of quantum fluctuations. The leading contributions to the functional integral come from more singular configurations and when we consider a continuum regularization [8] the relevant contributions need to be smoother than those described by the $L^2$-norm to guarantee that the orbit space is a smooth manifold and that the functional measure is well defined globally on $\mathcal{M}$ [8],[9]. There not exists a generalization of the Semenov-Tyan-Shanskii-Franke result to guarantee the existence of a similar domain in Landau gauge for smooth ($C^\infty$) or Sobolev (with $k > 1 + d/2$) gauge fields for spaces of dimension $d > 1$ [10].

In this paper we introduce a different gauge fixing method which leads to a complete parametrization of a dense set of gauge orbits in the space of gauge fields not only in the $L^2$-norm but also in the $C^\infty$-smooth and Sobolev topologies of gauge fields for spaces with dimensions lower than four. The special configurations whose orbits are at the boundary of this domain have a peculiar dynamical behaviour which can be related to non-perturbative effects of the theory. Another relevant feature of this novel gauge fixing is that is very explicit and the fundamental domain can be identified without ambiguities.

Another type of gauge conditions which has also been extensively considered in the recent literature are the abelian gauges introduced by ’t Hooft some years ago [11]. They allow the identification of some classical configurations with magnetic monopoles and numerical simulations suggest that those configurations carrying a magnetic charge play a leading role in confinement mechanisms. Abelian gauge conditions involve partial gauge fixing: the gauge is fixed in the non-abelian modes and abelian modes are left without gauge fixing which allows the description of abelian magnetic monopoles. These gauge conditions are essentially defined by means of an auxiliary gauge covariant functional $\Phi(A)$. The dependence on the choice of this functional makes unclear the intrinsic physical role of the configurations carrying magnetic charges. The most popular abelian gauge, the maximal abelian gauge, which is well defined on the lattice formulation [12] is not necessarily complete in continuum space-times[11]. Another special gauge of this type is the Laplacian abelian gauge where the functional $\Phi(A)$ is chosen as the lowest eigenfunction of the covariant laplacian operator $\Delta_A$ [13]. This is not uniquely defined for gauge field configurations with degenerated ground states of $\Delta_A$. 

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The Maximal Non-Abelian Gauges that we introduce in this paper shares with abelian
gauges some features like the association of magnetic charges to gauge configurations but
has the advantage of being intrinsically defined. The magnetic charge of a gauge config-
uration is not a gauge fixing artifact like in the abelian gauges. On the other hand the
gauge conditions are uniquely defined without ambiguities and are complete. Although
the domain of the orbit space where the gauges are defined is contractible and thus topo-
logically trivial, the non-trivial nature of the whole orbit space is recovered when we add
the gauge orbits sitting at the boundary of the fundamental domain.

The outline of the paper is as follows. We review in Section 2 the main topological
properties of the orbit space of gauge fields. In Sections 3 and 5 we analyze the structure of
the orbit spaces of one and two dimensional gauge fields which turn out to be the essential
ingredients for the definition of the maximal non-abelian gauges. These are explicitly
introduced and analyzed in sections 4 and 6. The relation of non-abelian monopoles with
gauge configurations which lie beyond the boundary of the fundamental domain of maximal
non-abelian gauges is analyzed in section 7, where it is also discussed the relevance of
those configurations for different non-perturbative effects. We conclude in section 8 with
a summary of the main results.

2. Topological structure of the gauge orbit space $M$

Let us consider SU(N) gauge fields defined on a d-dimensional sphere $S^d$. The infinite
volume case requires a separate discussion and will be considered below. The action of the
group of gauge transformations $G$ in the space of gauge fields $A$ is not free even if we mod
out by the center of the group $Z_N$ which obviously does not act on $A$. This problem is
due to the existence of reducible gauge connections with smaller holonomy groups SU(N'),
$N' < N$. Their orbits have a larger isotopy group which generates singularities in the orbit
space. There are two ways of circumventing this problem. The first option consists in do not
consider those singular gauge orbits at all. Indeed, irreducible gauge fields define an open
dense subspace of $A$ and for any Borel measure on $M$ the reducible orbits will have zero
measure. However, important configurations belong to the class of reducible gauge fields
(e.g. classical vacua) and it is not very reasonable to exclude those fields from the physical
configuration space. We shall consider an alternative option which proceeds by considering
all gauge fields but modding out only by the group of pointed gauge transformations $G_0$,
that is the group of gauge transformations with reduce to identity for a fixed given point
of $S^d$. This group has no center and acts freely on $\mathcal{A}$. The corresponding quotient space $\mathcal{M} = \mathcal{A}/\mathcal{G}_0$ is also a smooth manifold. The only problem of this approach based on the pointed orbit space is that the full gauge group is a symmetry of the dynamics which implies that there is residual global symmetry which leads to the existence of zero modes in the propagators in perturbation theory. The problem is solved by the introduction of collective coordinates. From a non-perturbative point of view the analysis is completely consistent because the contribution of the fields associated to the residual symmetry is always finite and controlled by the volume of the gauge group $SU(N)$.

The fact that $\mathcal{A}$ is an affine space implies that its topology is trivial. However, the gauge group $\mathcal{G}_0$ exhibits a rather non-trivial topological structure. Since $\mathcal{A}(\mathcal{M}, \mathcal{G}_0)$ has a principal bundle structure and $\mathcal{A}$ is homotopically trivial the homotopy groups of the orbit space are given by

$$\pi_n(\mathcal{M}) = \pi_{n-1}(\mathcal{G}_0) = \pi_{n+d-1}(SU(N)).$$

In particular this means that the first non-trivial homotopy groups in two, three and four dimensions are $\pi_2(\mathcal{M}S^2) = \mathbb{Z}$, $\pi_1(\mathcal{M}S^3) = \mathbb{Z}$ and $\pi_0(\mathcal{M}S^4) = \mathbb{Z}$, respectively. In the last case, the different connected components of the orbit space correspond to classes of gauge fields defined on different principal bundles parametrized by the second Chern class $c_2$. This property of four dimensional space-times also holds for higher dimensional space-times, i.e. $\pi_0(\mathcal{M}) = \mathbb{Z}$. Of course every connected component of $\mathcal{M}$ also has non-trivial higher homotopy groups.

Because of the non-trivial structure of the bundle $\mathcal{A}(\mathcal{M}, \mathcal{G}_0)$ there are not continuous global sections, i.e. it is impossible to fix a continuous global gauge [2].

For the same reasons the cohomology groups of $\mathcal{M}$ are non-trivial. They are given by

$$H^n(\mathcal{M}, \mathbb{R}) = \mathbb{Z}r_n,$$

where $r_n$ is the coefficient of the $t^n$ term in the series

$$P(t) = (1-t^2)^{-1}(1-t^4)^{-1}\cdots(1-t^{2N-d})^{-1} \quad \text{for even } d$$

$$P(t) = (1+t)(1+t^3)\cdots(1+t^{2N-d}) \quad \text{for odd } d$$

(2.1)

Many of those cohomology groups are associated with gauge anomalies of quantum field theories with fermion fields.

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1 We consider (infinitely) differentiable gauge fields and gauge transformations, but similar results will hold for gauge fields in a high enough Sobolev class, $k > \frac{d}{2} + 1$
This highly non-trivial topological structure of the orbit space is accompanied by a non-trivial Riemannian geometric structure induced by the dynamics of gauge field theories [14]. One of the most relevant consequences of this structure is the absence of a standard particle interpretation of the energy spectrum. In pure scalar (or fermionic) field theories the configuration space of classical fields is a linear space $\mathcal{Q}$ and the quantum states are functionals of the space $L^2(\mathcal{Q},\mu)$ of square integrable functionals with respect to a probability measure $\mu$ of $\mathcal{Q}$ associated to the quantum vacuum. One relevant subspace of $L^2(\mathcal{Q})$ is the space of linear functionals $\mathcal{Q}^*$ which can be identified with the one particle states of the quantum theory. It is obvious that $L^2(\mathcal{Q},\mu)$ can be identified with Fock space $\bigoplus_{n=0}^{\infty} \mathcal{H}^n$ associated to $\mathcal{H} = \mathcal{Q}^*$. This identification makes natural the particle interpretation of the energy spectrum. In the case of gauge theories the physical states are functionals on the space $L^2(\mathcal{M},\mu)$ and because of the curved nature of $\mathcal{M}$ there is no analogue of the subspace of one particle states $\mathcal{Q}^*$. This does not exclude a particle interpretation of the spectrum but makes it less evident to hold for all energy scales.

Remark: In infinite volumes the gauge fixing problem becomes more subtle. Naively speaking there is no gauge fixing problem for $C^\infty$-smooth gauge fields. The reason being that the group of smooth $C^\infty$ gauge transformations is a contractible manifold with trivial topology. In that case Singer’s proof of topological obstruction to the existence of a global gauge fixing fails. Indeed, any given $C^\infty$-smooth pointed gauge transformation over $\mathbb{R}^d$, $\phi : \mathbb{R}^d \to SU(N)$, with $\phi(0) = \mathbb{I}$ can be homotopically contracted to the trivial transformation $\phi(x) = \mathbb{I}$ by the map

$$\phi(t, x) = \phi(e^{(1-1/t)}x), \quad t \in [0, 1],$$

which obviously is continuous and smooth and interpolates between $\phi(1, x) = \phi(x)$ and $\phi(0, x) = \mathbb{I}$.

However, in the quantum theory the relevant fields satisfy some regularity conditions at infinity. For instance, in the ultraviolet regularized theory the relevant fields belong to an appropriate Sobolev class which implies that they satisfy specific boundary conditions. This makes the gauge fixing problem relevant for the quantum field theory. In order not to prejudge the infinity volume behaviour we can choose an $S^{d-1}$ fibration of $\mathbb{R}^d$ given by radial spheres centered at the origin of $\mathbb{R}^d$. The gauge fields are parametrized by a family of gauge fields defined on the different $S^{d-1}$ radial shells of $\mathbb{R}^d$ and a family of (radial) Higgs fields defined on the same spherical shells. Quantum dynamics imposes additional conditions on the behaviour of the fields for infinite radius. Anyhow, this shows that the study of gauge fields on spheres is also relevant for the description of the infinite volume limit.
In two dimensions any principal $SU(N)$ bundle $P(S^2, SU(N))$ is trivial $P = S^2 \times SU(N)$. The two-dimensional sphere $S^2$ also has a natural complex structure. This makes possible to identify the space of $SU(N)$ gauge fields on $S^2$ with the space of $SL(N, \mathbb{C})$ holomorphic bundles on the trivial vector bundle $E(S^2, \mathbb{C}^N)$ with $E = S^2 \times \mathbb{C}^N$ [15].

A $SL(N, \mathbb{C})$ holomorphic bundle is a vector bundle structure with holomorphic transition functions, i.e. the transition functions of a holomorphic bundle $g_{ij}(x) \in SL(N, \mathbb{C})$ must satisfy, besides the compatibility conditions

$$g_{ii}(x) = I_N \quad \text{for } x \in U_i$$
$$g_{ij}(x) g_{jk}(x) g_{ki}(x) = I_N \quad \text{for } x \in U_i \cap U_j \cap U_k,$$

the holomorphic condition

$$\partial_z g_{ij}(x) = 0 \quad \text{for } x \in U_i \cap U_j.$$ 

Given a $SL(N, \mathbb{C})$ holomorphic bundle in $E$ there exists a unique $SU(N)$ gauge field $A \in \mathcal{A}$ whose covariant derivative operator $D = d_A$ satisfies $D_z \sigma = 0$ for any local holomorphic section $\sigma$ of $E$. We consider the trivial hermitean structure of $E = S^2 \times \mathbb{C}^N$ induced by the scalar product of $\mathbb{C}^N$.

In local coordinates $D_z = \partial_z + h^{-1} \partial_z h$, where $h : U \to SL(N, \mathbb{C})$ are the coordinates of a given local holomorphic frame in $E = S^2 \times \mathbb{C}^N$. Notice that the local expression does not depend on the choice of such a frame and only depends on the $SL(N, \mathbb{C})$ holomorphic bundle structure.

Conversely, given a $SU(N)$ gauge field $A$ there exists a unique $SL(N, \mathbb{C})$ holomorphic bundle structure on $E$ whose associated gauge field is $A$. This follows from the fact that the local sections $\sigma$ satisfying the condition $D_z \sigma = 0$ define a $SL(N, \mathbb{C})$ holomorphic bundle structure on $E$. It can be shown that the correspondence between $SU(N)$ gauge fields and $SL(N, \mathbb{C})$ holomorphic bundle structures is one-to-one in two dimensions.

In four dimensions there is a similar correspondence, but in such a case the corresponding gauge fields associated to holomorphic bundles must be selfdual. In two dimensions there is no constraint on the associated unitary connections.

The characterization of 2-dimensional gauge fields in terms of holomorphic bundles has been very useful for the resolution of various field theories like the O(3) sigma model
Holomorphic bundles which are related by a linear homomorphism of $E$ are said to be equivalent. In terms of the gauge field representation this induces the following equivalence relation

$$A^h = h^{-1}A_\zeta h + h^{-1}\partial_\zeta h,$$

(3.1)

given by the action of the group of linear complex automorphisms $G_C = \text{Maps}(S^2, SL(N, \mathbb{C}))$ on the space of gauge fields $A$. $G_C$ is twice larger than the group of ordinary gauge transformations $G = \text{Maps}(S^2, SU(N))$ because the automorphisms of $G_C$ correspond to complex gauge transformations, i.e. $G_C \approx G \times G$. For such reasons the orbit space of this larger symmetry group is smaller than that of $G$. In fact, since the complex gauge transformation involves the same number of local degrees of freedom that the two-dimensional gauge fields (two $su(N)$-valued scalar fields), this orbit space is expected to be a finite dimensional topological space. The only problem is that this orbit space $\mathfrak{m}$ has an stratified structure.

The different isomorphism classes of $SL(N, \mathbb{C})$ holomorphic bundle structures on $S^2$ were classified by Grothendieck and turn out to define a discrete moduli space. They are given by the following transition functions connecting two holomorphic patches defined on the north and south hemispheres

$$g_{ij}(x) = \begin{pmatrix} z_1^{n_1} & 0 & \cdots & 0 \\ 0 & z_2^{n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & z_N^{n_N} \end{pmatrix},$$

(3.2)

where $n_i \in \mathbb{Z}$ for all $i = 1, \cdots, N$ and $n_1 \geq n_2 \geq \cdots \geq n_N$ with $n_1 + n_2 + \cdots + n_N = 0$. The fact that the exponents $n_i$ are integers suggests that the corresponding gauge fields can carry a non-trivial non-abelian monopole structure as we will stress in section 7. The holomorphic bundle structures associated to non-trivial integers are called unstable holomorphic bundles according to Mumford’s classification. The main difference between the trivial class, where all $n_i = 0$, and the other classes of holomorphic structures which correspond to unstable bundles is that in the first case the transition function (3.2) is proportional to the identity, i.e. only one chart is required to describe the corresponding holomorphic bundle. Thus, all bundles of this type define a subset $C_0$ of the space of all holomorphic bundles which is in one to one correspondence with the space of connections $A_0$ of the form

$$A_\zeta = h^{-1}\partial_\zeta h, \quad h : S^2 \to SL(N, \mathbb{C}).$$

(3.3)
It is well known that $C_0$ is dense on $C$, thus, the space $\mathcal{A}_0$ of gauge fields of the type (3.3) is also dense in the space of all gauge fields $\mathcal{A}$. In consequence, the space of the ordinary gauge orbits of the fields in $\mathcal{A}_0$ define a dense submanifold $\mathcal{M}_0$ of the orbit space $\mathcal{M}$. In fact, the boundary of this space, $\mathcal{M}_* = \mathcal{M} \setminus \mathcal{M}_0$ which correspond to gauge fields associated to unstable bundles which cannot be globally written as (3.3) is a submanifold of $\mathcal{M}$ which has codimension two $[20]^2$.

Since the measure of the boundary $\mathcal{M} \setminus \mathcal{M}_0$ is zero with respect to any borelian measure one might be tempted to consider that from quantum viewpoint the effects of the fields which do not belong to $\mathcal{A}_0$ are negligible. From a functional integral point of view their contribution is certainly negligible but in the Hamiltonian formalism the fact that physical states must satisfy some boundary conditions just precisely at those field configurations has a deep significance for the low energy behaviour of the theory.

4. Maximal Non-abelian Holomorphic Gauge

One consequence of the previous analysis is that, for the gauge orbits of two-dimensional gauge fields in $\mathcal{M}_0$ which are associated to trivial holomorphic bundles, it is possible to find a global gauge fixing condition which is free of Gribov ambiguities. The basic idea is that these fields can be globally written as $A_z = h^{-1}\partial_z h$ in terms of a complex scalar field $h$ with values on $\text{SL}(N, \mathbb{C})$ and any matrix of $\text{SL}(N, \mathbb{C})$ can be split by the polar decomposition $h = HU$ as the product of a positive hermitean matrix $H \in SL^+(N, \mathbb{C})$ and a unitary matrix $U \in SU(N)$, both with unit determinant. $H$ can be identified with the positive square root of the positive operator $hh\dagger$ (i.e. $H = \sqrt{hh\dagger}$) and $U = H^{-1}h$.

Using such a decomposition any gauge field $A \in \mathcal{A}_0$ can be rewritten as

$$A_z = U\dagger(H^{-1}\partial_z H)U + U\dagger\partial_z U \tag{4.1}$$

which means that $A$ is gauge equivalent to the gauge field

$$A_z^U = H^{-1}\partial_z H, \tag{4.2}$$

because (4.1) implies that

$$A^U = UAU\dagger + UdU\dagger.$$  

$^2$ We shall consider two types of regularity conditions on the gauge fields. Either gauge fields in a Sobolev class $k > 1 + d/2$ or simply $C^\infty$-smooth gauge fields. Although the results hold for more general regularity conditions.
Therefore, the orbit space of gauge fields which are associated to trivial holomorphic bundles, \( \mathcal{M}^{S^2}_0 \), is in one-to-one correspondence with the space of pointed functionals from \( S^2 \) into the space of positive definite hermitean matrices with unit determinant, \( H : S^2 \to \mathfrak{M}^+ (N, \mathfrak{C}) \), i.e.

\[
\mathcal{M}^{S^2}_0 \equiv \text{Maps}_0 (S^2, \mathfrak{M}^+ (N, \mathfrak{C})),
\]

where the subscript indicates that the maps are trivial at the north pole \( x_n \) of \( S^2 \), \( H(x_n) = \mathbb{1} \).

The Maximal Non-Abelian Gauge is defined by the condition

\[
A_z = H^{-1} \partial_z H. \tag{4.3}
\]

Notice that all gauge fields of the form \( (4.3) \) are gauge inequivalent, thus the Maximal Non-Abelian Gauge is free of Gribov ambiguities. The gauge orbits of these fields under the group of pointed gauge transformations \( \mathcal{G}_0 \) fill the open dense subset \( \mathcal{A}_0 \) of the whole space \( \mathcal{A} \) of gauge fields over \( S^2 \) and the parametrization \( (4.3) \) is a complete gauge fixing condition for them. This implies that the domain of gauge orbits \( \mathcal{M}^{S^2}_0 \) parametrized by the trivial holomorphic gauge fields is dense in the whole orbit space \( \mathcal{M}_{S^2} \). In this sense the Maximal Holomorphic Gauge condition is complete. This also implies that \( \mathcal{M}^{S^2}_0 \) is contractible because the space of positive hermitean matrices with unit determinant is open and contractible. The cohomology groups of \( \mathcal{M}^{S^2}_0 \) are, thus, trivial \( H^n (\mathcal{M}^{S^2}_0, \mathbb{R}) = 0 \) for \( n > 0 \). This can be explicitly checked by showing that all the non-trivial closed forms of \( \mathcal{M}_{S^2} \) become pure differentials or exact on \( \mathcal{M}^{S^2}_0 \). The cohomology and homotopy of \( \mathcal{M}_{S^2} \) are encoded in the boundary \( \mathcal{M}^{S^2}_0 = \mathcal{M}^{S^2} \setminus \mathcal{M}^{S^2}_0 \) of \( \mathcal{M}^{S^2}_0 \). Moreover, the orbits belonging to this boundary can easily be identified because the gauge fields \( \mathcal{A} \) of those orbits satisfy the following equivalent properties [20],

i) the Dirac operator \( \bar{\partial}_A \) has zero modes.

ii) for any map \( h : S^2 \setminus \{x_n\} \to SL(N, \mathfrak{C}) \) such that \( A_z = h^{-1} \partial_z h \) the Wess-Zumino-Witten action \( S_{WZW}(h) \) diverges.

iii) the holomorphic bundle associated to \( \mathcal{A} \) is unstable according to Mumford’s definition

iv) the operator \( D_z \) has pointed zero modes on the adjoint bundle, i.e. there are global holomorphic sections of this bundle vanishing at one point of the sphere.
Moreover gauge fields of the boundary of $\mathcal{M}_0$ can be explicitly identified with well known gauge fields. Let us consider the abelian magnetic monopole in $SU(2)$ theory,

$$A_{z}^{\text{mon}} = \varphi_+^{-1} \partial_z \varphi_- = \varphi_+^{-1} \partial_z \varphi_+ = \frac{1}{(1 + |z|^2)^2} \left( \begin{array}{c} -z \\ z^2 \end{array} \right) \left( \begin{array}{c} -1 \\ -1 \\ \end{array} \right),$$

(4.4)

with

$$\varphi_- = \left( -\frac{z}{(1 + |z|^2)^{-1}} \right), \quad \varphi_+ = \left( -\frac{z}{(1 + |z|^2)^{-1}} \right).$$

It is obvious that $A^{\text{mon}}$ can never be globally expressed as $h^{-1} \partial_z h$. In fact, it can be shown that the transition function connecting two holomorphic patches associated to the north and south hemispheres is of the form

$$g_{12}(x) = \left( \begin{array}{cc} z & 0 \\ 0 & 1/z \end{array} \right),$$

which means that the corresponding configurations can be associated with monopoles of magnetic charges $\pm 1$. This shows that $[A^{\text{mon}}] \in \mathcal{M}^{S^2}_* = \mathcal{M}^{S^2} \setminus \mathcal{M}^{S^2}_0$. The traceless character of $\mathfrak{su}(N)$ means that properly speaking there is no net magnetic charge for $SU(N)$ gauge fields. However, for abelian $SU(N)$ gauge fields this vanishing condition can be satisfied in two different ways: either $A$ is made of elementary $U(1)$ magnetic monopoles with positive and negative magnetic charges which cancel each other out, or $A$ does not contain magnetic monopoles at all. The field (4.4) corresponds to the first type of fields. It is, thus, not inappropriate to consider these kind of configurations as non-abelian magnetic monopoles; and as we shall see later this notion can be extended for arbitrary non-abelian gauge fields.

There is another indication that the orbit of $A^{\text{mon}}$ belongs to $\mathcal{M}^{S^2}_*$. It is the existence of pointed holomorphic sections on the adjoint bundle. From the four independent holomorphic sections $\chi(\mu) ; \mu = 0,1,2,3$ of the adjoint bundle

$$\chi(3) = \frac{1}{1 + |z|^2} \left( \begin{array}{ccc} |z|^2 & 0 & 2z \\ 0 & 2z & 1 - |z|^2 \end{array} \right),$$

$$\chi(k) = \frac{z^k}{(1 + |z|^2)^2} \left( \begin{array}{ccc} z & 0 & -z^2 \\ 0 & 1 & -z \end{array} \right), \quad k = 0,1,2$$

only two vanish at $z = \infty$ ($\chi(0)$ and $\chi(1)$). Then, the group of pointed complex (chiral) gauge transformations which leave the holomorphic bundle associated to $A^{\text{mon}}$ invariant is two-dimensional, i.e.

$$\dim \{ h : S^2 \rightarrow SL(N, \mathbb{C}) , h(\infty) = I ; A_{\text{mon}}^h = A_{\text{mon}} \} = 2.$$
This means that the isotopy group of $G_0^\mathbb{C}$ for this bundle is two-dimensional and that the codimension of the corresponding $G_0^\mathbb{C}$-orbit has (real) codimension four in the space of all gauge fields $\mathcal{A}$. This is in contrast with the orbit generated by the full group of complex gauge transformations which has codimension two. The difference is covered by the the non-trivial $G_0^\mathbb{C}$ moduli space of dimension two of unstable bundles of monopoles. Notice that the bundles of this two dimensional moduli space are equivalent with respect to the group of complex gauge transformations $G^\mathbb{C}_I$.

Thus, the boundary of the domain $\mathcal{M}_0^{S^2}$ where the Maximal Non-Abelian Holomorphic gauge is well defined is the closure $\overline{\mathcal{M}}_1^{S^2}$ of the orbit space $\mathcal{M}_1^{S^2}$ of gauge fields which are holomorphically equivalent to the monopole $(4.4)$ by complex gauge transformations of $G^\mathbb{C}$. This makes possible to extend the gauge fixing condition to a larger domain $\mathcal{M}_0^{S^2} \cup \mathcal{M}_1^{S^2}$ in $\mathcal{M}^{S^2}$. The orbits of $\mathcal{M}_1^{S^2}$ can be parametrized in terms of maps $H : S^2 \rightarrow \mathcal{M}^+(N, \mathbb{C})$ by the gauge condition

$$A = H^{-1} A^{\text{mon}} H + H^{-1} dH. \quad (4.5)$$

Notice that $\chi(j), j = 0, 1, 2, 3$ generate one-parametric subgroups of the gauge group of complex gauge transformations which leave $A^{\text{mon}}$ invariant. Therefore the parametrization (4.5) is not unique unless we mod out by those subgroups. This fact also explains why $\mathcal{M}_1^{S^2}$ has codimension 2 in $\mathcal{M}^{S^2}$ whereas $\mathcal{M}_0^{S^2}$ had codimension 0. The procedure could be extended in similar way for higher magnetic monopole field configurations giving rise to parametrizations of larger subsets of gauge fields orbits. However, the different charts cover disjoint subsets of $\mathcal{M}^{S^2}$ with different codimensions, and they can not be considered as a single gauge condition.

The above construction of the Maximal Non-Abelian Holomorphic gauge condition can be generalized for higher dimensional spaces. For simplicity, we only consider gauge fields $A$ defined on a trivial bundle $S^d \times SU(N)$, although the extension of the results for more general cases is straightforward. In such a case a gauge field $A$ can be identified with a $\mathfrak{su}(N)$-valued one form over $S^d$.

Let us first introduce a very special coordinate system on $S^d$. If $x_j, j = 1, \cdots, d$ are the cartesian coordinates of $\mathbb{R}^d$, we define the angular coordinates $\varphi_j = 2 \arctan(x_j/2), j = 1, \cdots, d$ which compactify $\mathbb{R}^d$ into a torus $T^d$. Now, if we exclude the north pole $x_n$ of the sphere it can be identified with $\mathbb{R}^d$ by means of the stereographic projection $\pi_s$ from the north pole $x_n \in S^d$. Thus, the coordinates $\{\varphi_j \in (-\pi, \pi); j = 1, \cdots, d\}$ define a complete
set of orthogonal coordinates on $S^d \setminus \{x_n\}$. The pullback by the stereographic projection \( \pi_s \) of the hyperplane

\[
\pi^{-1}\{(\varphi_1 = \varphi, \varphi_2, \cdots, \varphi_d); \varphi_j \in [-\pi, \pi]; j = 2, \cdots, d\}
\]

of \( \mathbb{R}^d \) defined by the condition \( \varphi_1 = \varphi \in [-\pi, \pi] \) generates a \( S^d_{\varphi} \)-sphere in \( S^d \) which reduces at a single point \( x_n \) for \( \varphi = \pm \pi \).

Finally, we normalize the radius \( R_\varphi = \cos \varphi/2 \) of the different spheres \( S^d_{\varphi} \)-spheres to unit which means that the corresponding embedding \( j_\varphi \) of \( S^{d-1} \) into \( S^d \) becomes singular at \( \varphi = \pm \pi \). The pullback by \( j_\varphi \) of any gauge field \( A \) on \( S^d \) defines a loop of gauge fields on \( S^{d-1} \),

\[
A(\varphi) = j^*_\varphi A
\]

with the same gauge group \( SU(N) \). In the extreme cases \( \varphi = \pm \pi \) the induced gauge field becomes trivial \( A(\pm \pi) = 0 \).

On the same way, \( A \in \mathcal{A}^{S^d} \) defines a loop of Higgs fields

\[
\Phi(\varphi) = A_1(\varphi) = A(\partial_{\varphi_1})
\]

over \( S^{d-1} \) with values on \( su(N) \), the Lie algebra of \( SU(N) \). Let us denote by

\[
\mathcal{H}^{S^n} = \{ \phi: S^n \to \text{ad} \, P(S^n, SU(N)) \}
\]

the space of Higgs fields defined as sections of the adjoint bundle of a \( SU(N) \) principal bundle \( P(S^n, SU(N)) \) over the \( S^n \)-sphere. In our case, these bundles \( P_\varphi(S^{d-1}, SU(N)) \)
are trivial because the original bundle $P(S^d, SU(N))$ was assumed to be trivial. Then, $H^{S^n} = \{ \phi : S^n \to su(N) \}$. The above construction shows that there is map

$$A^{S^d} \rightarrow \text{Maps}_0 (S^1, H^{S^{d-1}} \times A^{S^{d-1}})$$

(4.9)

defined by the loops of fields

$$A \mapsto (A(\varphi), \Phi(\varphi)) \in A^{S^{d-1}} \times H^{S^{d-1}} \quad \varphi \in [-\pi, \pi]$$

which satisfy the boundary conditions

$$A(\pm \pi) = 0 \quad A(\pm \pi) = \Phi_0 \prod_{j=2}^{d} \cos \varphi_j / 2.$$ (4.10)

In fact, because the map (4.9) defines a one-to-one correspondence it is possible to reconstruct from the data $(A(\varphi), \Phi(\varphi)) \in A^{S^{d-1}} \times H^{S^{d-1}}$ the original $S^d$–gauge field $A$. Modding out by the group of gauge transformations we obtain a one to one correspondence

$$\mathcal{M}^{S^d} \leftrightarrow \text{Maps}_0 (S^1, H^{S^{d-1}} \times \mathcal{M}^{S^{d-1}})$$

(4.11)

between the orbit space $\mathcal{M}^{S^d}$ and the pointed loop space of $H^{S^{d-1}} \times \mathcal{M}^{S^{d-1}}$. Iterating this procedure we can establish the following sequence of one-to-one correspondences

$$\mathcal{M}^{S^d} \leftrightarrow \text{Maps}_0 (S^n, H^{S^{d-n}} \times H^{S^{d-n}} \times \cdots \times H^{S^{d-n}} \times \mathcal{M}^{S^{d-n}})$$

(4.12)

for any positive integer $n \leq d - 1$. Obviously, these new characterizations of the space of gauge orbits preserve the non-trivial topological structure of $\mathcal{M}^{S^d}$. In particular, it is straightforward to check that homotopy and cohomology groups are identical to those of $\mathcal{M}^{S^d}$ displayed in Section 2.

One of those characterizations offers an special interest for the construction of the maximal gauge. It is the characterization based in the 2-dimensional gauge fields orbit space $\mathcal{M}^{S^2}$, i.e.

$$\mathcal{M}^{S^d} \leftrightarrow \text{Maps}_0 (S^{d-2}, H^{S^2} \times H^{S^2} \times \cdots \times H^{S^2} \times \mathcal{M}^{S^2})$$

(4.13)

In particular, for $d = 3$ one extra Higgs field is enough to describe the gauge orbit space in terms of 2-dimensional gauge fields. In four dimensions the same construction requires
two Higgs fields. Notice, that since the Higgs sector is topologically trivial and all interesting topological properties are encoded in this picture by the structure of the space $\text{Maps}_0 (S^{d-2}, \mathcal{M}^{S^2})$.

For non-trivial bundles the generalization is obvious but then the boundary conditions (4.10) change in a way that the correspondences do not lead to pointed maps from $S^n$ into the space of gauge fields over lower dimensional spaces (4.9) but the description in terms pointed closed maps from $S^n$ into the orbits spaces (4.12) remains a one-to-one correspondence.

Now, we know from Section 3 the structure of $\mathcal{M}^{S^2}$, then, we can define in analogy with what was done there a Maximal Non-Abelian Holomorphic gauge for the gauge fields $[A] \in \mathcal{M}^{S^2}_0$ which corresponds to maps of (4.12) which entirely lie on $\mathcal{M}^{S^2}_0$, i.e.

$$\mathcal{M}^{S^d}_0 = \text{Maps}_0 (S^{d-2}, \mathcal{H}^{S^2} \times \mathcal{H}^{S^2} \times \cdots \times \mathcal{H}^{S^2} \times \mathcal{M}^{S^2}_0).$$

The gauge fields orbits in such a submanifold are characterized by means of the correspondence

$$A \mapsto (\Phi_1, \Phi_2, \cdots, \Phi_{d-2}, H^{-1} \partial_2 H)$$

in terms of a set of $d-1$ pointed functionals

$$(\Phi_1, \cdots, \Phi_{d-2}, H) \in [\text{Maps}_0 (S^{d-2}, \mathcal{H}^{S^2})]^{d-2} \times \text{Maps}_0 (S^2 \times S^{d-2}, \mathfrak{sm}^+(N, \mathbb{C}))$$

of $S^2 \times S^{d-2}$ with values in $\mathfrak{su}(N)$ and $\mathfrak{sm}^+(N, \mathbb{C})$, respectively.

This defines the Maximal Non-Abelian Holomorphic gauge condition for any dimension. It is obvious that fields satisfying the gauge condition (4.14) do not have Gribov copies under the group of pointed gauge transformations $\mathcal{G}_0 = \{ U \in \mathcal{G}; U(x_n) = \mathbb{I} \}$. From the definition it follows that $\mathcal{M}^{S^d}_0$ can be essentially parametrized in terms of a system of affine coordinates in $\text{Maps}_0 (S^2, \mathfrak{sm}^+(N, \mathbb{C}))$ and $[\mathcal{H}^{S^2}]^{d-2}$. This explicitly shows that the fundamental domain $\mathcal{M}^{S^d}_0$ of the Maximal Non-Abelian Holomorphic gauge is a contractible submanifold of the space of gauge orbits $\mathcal{M}^{S^d}$. The topologically non-trivial sector of $\mathcal{M}^{S^d}$ is essentially equivalent to $\text{Maps}_0 (S^2, \mathfrak{sm}^+(N, \mathbb{C})$ because of the triviality of the Higgs fields sector.

Now, the efficiency of the gauge fixing condition decreases as space–time dimension increases. In three dimensions $\mathcal{M}^{S^3}_0$ constitute an open dense set on $\mathcal{M}^{S^3}$, because $\mathcal{M}^{S^3} \setminus \mathcal{M}^{S^3}_0$ is a co-dimension two manifold in $\mathcal{M}^{S^3}$. Thus, the space of loops which do not reach the manifold $\mathcal{M}^{S^2} \setminus \mathcal{M}^{S^2}_0$ is open and dense in the space of all pointed loops which
means by the correspondence (4.13) that $\mathcal{M}_0^{S^2}$ is an open and dense submanifold of the whole space of orbits $\mathcal{M}^{S^2}$. In four dimensions, however, the space $\text{Maps}_0(S^2, \mathcal{M}_0^{S^2})$ is not dense in $\text{Maps}_0(S^2, \mathcal{M}^{S^2})$, because if a map reaches the dense submanifold $\mathcal{M}^{S^2} \setminus \mathcal{M}_0^{S^2}$, generically it cannot be transformed by an infinitesimal transformation into another one that does not intersect such a submanifold. In particular this implies that the gauge fixing (4.14) is not complete because there is an open set of gauge orbits which do not intersect the gauge condition slice. Moreover, if we consider a non-trivial bundle $P(S^d, SU(N))$ none of the gauge fields with non-trivial topological charge induce a map in $\text{Maps}_0(S^2, \mathcal{M}^{S^2})$ which does not intersects $\mathcal{M}^{S^2} \setminus \mathcal{M}_0^{S^2}$. The situation gets even worse for dimensions $d > 4$.

However, the fact that gauge condition (4.14) is complete for $d = 3$ is very important because the Yang-Mills theory in 4+1 dimensions can be described in the Hamiltonian formalism in terms of 3-dimensional gauge fields. Moreover, as will be analysed in section 7, the fact that there are many four-dimensional gauge configurations which can not be described by the maximal gauge condition might have relevant implications for the understanding of confinement in the dual superconductor picture [21][22].

On the other hand, the characterization of gauge fields orbits in a $d$ dimensional space-time in terms of pointed maps from an $S^{d-2}$ sphere into $[\mathcal{H}^{S^2}]^{d-2} \times \mathcal{M}^2$ is always one-to-one. The problem is that one would like to have a parametrization by affine coordinates of a maximal open subset in such a space. With the gauge fixing condition described above we only can achieve a complete parametrization on the cases $d \leq 3$. It is not, however, excluded the existence of another gauge in higher dimensions covering a larger subset of gauge orbit spaces. In fact, this is possible if we exclude the requirement of having a uniform parametrization by affine coordinates of the same dimensionality, which seems to be necessary to have a correct particle interpretation of the physical spectrum. Indeed, to describe fields beyond the horizon of $\mathcal{M}^{S^d}$ we have to consider maps in $\text{Maps}_0(S^{d-2}, \mathcal{M}^{S^2})$ which reach the stratum $\mathcal{M}_1^{S^2}$. But, since the orbits of $\mathcal{M}_1^{S^2}$ can also be parametrized by (4.5) a larger set of gauge field orbits over $S^d$ can be parametrized along the same lines. Since the codimension of $\mathcal{M}^{S^2} \setminus (\mathcal{M}_0^{S^2} \cup \mathcal{M}_1^{S^2})$ is six the set $\text{Maps}_0(S^{d-2}, \mathcal{M}_0^{S^2} \cup \mathcal{M}_1^{S^2})$ is not only an open subset of $\text{Maps}_0(S^{d-2}, \mathcal{M}^{S^2})$ but its boundary has codimension 8-d which implies that it is also dense for $d < 8$. The only problem is that the parametrization of the corresponding open dense domain in the orbit space $\mathcal{M}^{S^d}$ is achieved in terms of two sets of affine coordinates with different dimensions.

To some extend this analysis shows from a different perspective the reasons why it is impossible to cover the whole space of orbits in the non-abelian case. In principle, the
problem of the existence of obstructions to a complete gauge fixing looks like a technical problem, but the observed difference between the $d \leq 3$ and $d > 3$ cases might have a physical meaningful interpretation if as suggested below the confinement mechanism is related to this kind of topological obstruction. Notice that in $d \leq 3$, perturbation theory always leads to a confining potential which only monopoles in $d = 3$ improve a little bit from being logarithmic to become linear. We will see that those configurations belong to the boundary $\mathcal{M}_s^{d-1}$ of $\mathcal{M}_0^{d-1}$ which is the first indication that the configurations with lie beyond the boundary of $\mathcal{M}_0^{d-1}$ might play an special role in the confinement mechanism. In Section 7 we will see that these configurations do carry a kind of non-abelian magnetic charge and therefore might represent the intrinsic realization of the non-abelian monopoles which will drive the vacuum structure to that one suited for dual superconductor picture.

5. One Dimensional Gauge Fields

One could proceed further in the dimensional reduction mechanism to get another characterization of a contractible dense submanifold in the space of gauge orbits.

Iterating once more the above procedure we can establish one more one-to-one correspondence

$$\mathcal{M}_{S^d} \leftrightarrow \text{Maps}_0 (S^{d-1}, \mathcal{H}^1 \times \mathcal{H}^1 \times \cdots \times \mathcal{H}^1 \times \mathcal{M}^1).$$

The non-trivial topological sector is encoded in $\text{Maps}_0 (S^{d-1}, \mathcal{M}^1)$ which can be easily analysed because the pointed orbit space over the circle is isomorphic to the gauge group $SU(N)$ itself. It is easy to check that the topology of $\mathcal{M}$ does coincide with that of $\text{Maps}_0 (S^{d-1}, \mathcal{M}^1)$. In that parametrization is clear what the real obstruction to a global gauge fixing is: the topological structure of the gauge group $SU(N)$. The problem of finding a maximal gauge of the $S^d$-dimensional gauge theory is reduced to find a maximal gauge with affine coordinates in a contractible open dense subset $\mathcal{M}_0^{S^1}$ of the orbit space of one-dimensional gauge fields $\mathcal{M}^{S^1}$ following the same steps as in the case of maximal holomorphic gauge. It is then necessary to select a domain in the one dimensional gauge orbit space $\mathcal{M}^{S^1} \equiv SU(N)$ with a convenient parametrization by affine coordinates. This is equivalent to find an open dense set in $SU(N)$ with those properties.

One subset of unitary matrices which is contractible, open and dense in $\mathcal{M}^{S^1} = SU(N)$ is

$$\mathcal{M}_0^{S^1} = SU(N)_0 = \{ U \in SU(N); \text{dim ker}(U - e^{i \alpha} \mathbb{I}) > 1, |\alpha| < \frac{2\pi}{N} \}.$$
Its boundary

\[ \mathcal{M}_s^{S^1} = \mathcal{M}^{S^1} \setminus \mathcal{M}_0^{S^1} = SU(N)_s = \{ U \in SU(N); \text{If } \dim \ker(U - e^{i\alpha}I) > 1, |\alpha| \geq \frac{2\pi}{N} \}, \]

is a stratified space whose larger strata, given by the matrices of \( SU(N)_s \) with only double
degenerated spectrum, has codimension three. The natural system of affine coordinates for
\( \mathcal{M}_0^{S^1} \) is given by the exponential map in \( SU(N) \). The Lie algebra \( \mathfrak{su}(N) \) of \( SU(N) \) defined
by traceless \( N \times N \) hermitean matrices, is completely covered by the coadjoint orbits of
the Cartan subalgebra of traceless diagonal real matrices \( \mathfrak{h}_N \) of \( \mathfrak{su}(N) \)

\[ D = \begin{pmatrix}
\alpha_1 & \cdots & 0 & 0 & \cdots & 0 \\
. & \cdots & . & . & \cdots & . \\
0 & \cdots & \alpha_k & 0 & \cdots & 0 \\
0 & \cdots & 0 & \beta_1 & \cdots & 0 \\
. & \cdots & . & . & \cdots & . \\
0 & 0 & \cdots & 0 & \cdots & \beta_{N-k}
\end{pmatrix} \quad \text{tr}D = 0. \]

Let us consider the open dense subset \( \mathfrak{h}_0 \) in the Cartan subalgebra \( \mathfrak{h}_N \) of \( \mathfrak{su}(N) \) defined by

\[ \mathfrak{h}_0 = \left\{ D \in \mathfrak{h}_N; \text{with } \alpha_i \in [0, 2\pi), \beta_j \in (-2\pi, 0); \sum_{i=1}^k \alpha_i < 2\pi, \right. \\
\left. 2\pi + \sum_{i=1}^k \alpha_i + \beta_1 \geq 2\pi; \alpha_1 \leq \cdots \leq \alpha_k \leq 2\pi + \beta_1 \leq \cdots \leq 2\pi + \beta_{N-k}; \right. \\
\left. \text{and } \alpha_i < \frac{2\pi}{N} \text{ if } \alpha_i = \alpha_j, \text{ and } \beta_i > -\frac{2\pi}{N} \text{ if } \beta_i = \beta_j \right\} . \]

The subset \( \mathfrak{su}_0(N) \) defined by the coadjoint orbits of \( \mathfrak{h}_0 \),

\[ \mathfrak{su}_0(N) = \{ V \in \mathfrak{su}(N); V = U^\dagger DU, \text{for } D \in \mathfrak{h}_0, U \in SU(N) \}, \]

is a contractible open dense subset of \( \mathfrak{su}(N) \). It is trivial to see that the restriction of the
exponential map to \( \mathfrak{su}_0(N) \) establishes a one-to-one correspondence between \( \mathfrak{su}_0(N) \) and
\( \mathcal{M}_0^{S^1} \). This method provides a unique and unambiguous parametrization of the submanifold \( \mathcal{M}_0^{S^1} \) via the exponential map and shows that \( \mathcal{M}_0^{S^1} \) is a maximal contractible open
subset of the space of gauge orbits of one–dimensional gauge fields \( \mathcal{M}^{S^1} \). Coordinates of
\( \mathcal{M}_0^{S^1} \) can be defined by the parameters of diagonal matrices in \( \mathfrak{h}_0 \) and the angular variables of the coset \( SU(N)/T_D \) where the unitary matrices \( T_D \) leaving invariant the matrix
\( D \) have been modded out. The coordinates are not well defined for matrices of \( \mathcal{M}_0^{S^1} \) with
double degeneracy which have a larger isotopy group $T_D$, e.g for the identity $D = \mathbb{I} \in \mathfrak{su}(N)$ the isotopy group $T_D$ is the full group SU(N). This \textit{angular} coordinate system becomes singular for those particular matrices in a similar manner as the origin is singular for polar coordinates of the plane. However, it is possible to choose another set of affine coordinates which is non-singular in the subset $\mathfrak{su}(N)$ and provides a complete unambiguous affine coordinate system for $M_0^{S^1}$. The inclusion of gauge fields with degenerated eigenvalues in the fundamental domain $M_0^{S^1}$ is necessary to describe the classical vacua $A_{\text{vac}} = 0$ and define a maximal open dense set around it. The geometric discussion would be simpler if we exclude those configurations and restrict the whole construction to the exponential of the Weyl alcove of the Lie-algebra. The singularities associated to double degeneracies can be considered then as defects and a proper counting of the associated magnetic charges provides a description of the topological charge in terms of those magnetic charges [23]-[25], although it is not very satisfactory from a physically point of view to associate a magnetic charge to points where the gauge field vanishes. Our approach solves all those potential problems by the choice of the gauge orbit space of the group of pointed gauge transformations. In the particular case of $d = 1$ the gauge condition becomes equivalent to Landau gauge. Thus, the above construction explicitly shows the validity of the generalization of the Semenov-Tyanyan-Shanskii-Franke result for $C^\infty$-smooth gauge fields.

For U(N) gauge fields there exists a similar maximal open subset $M_0^{S^d}$. It is defined by $M_0^{S^d} = \{ U \in \text{U}(N); \det(U + I) = 0 \}$ and it is parametrized via exponential map by the set of hermitean matrices with eigenvalues $\alpha \in (-\pi, \pi)$. The boundary of $M_0^{S^d}$ has generically codimension one in the space of gauge orbits of one-dimensional U(N) gauge fields. This parametrization explicitly shows the nature of the topological obstruction to get a complete gauge fixing condition because in this case we have $\pi_1(M^{S^d}) = \pi_1(\text{U}(N)) = \mathbb{Z}$.

6. Maximal Non-abelian $\sigma$-gauge

Let us return to SU(N) gauge theories. Since $M_0^{S^d}$ has generically codimension three the subset $M_0^{S^d}$ of $M^{S^d}$, defined by the the gauge field orbits whose associated maps

$$M_0^{S^d} = \text{Maps}_0(S^{d-1}, \mathcal{H}^{S^1} \times \mathcal{H}^{S^1} \times \cdots \times \mathcal{H}^{S^1} \times M_0^{S^1}),$$

in the correspondence (5.1) do not intersect the set $M_0^{S^d}$, is open and dense in the space of all gauge orbits only for dimensions $d \leq 3$. This is in agreement with the behaviour pointed out in the Section 4 for the maximal holomorphic gauge. In this sense the obstructions to
the extension of both gauge conditions are compatible. However, they are not completely identical as the following example points out. Let us consider the abelian U(1) gauge theory over a two-dimensional sphere $S^2$. In that case we have

$$\mathcal{M}^{S^2} \leftrightarrow \text{Maps}_0 (S^1, \mathcal{H}^{S^1} \times \mathcal{M}^{S^1}),$$

which contains an open contractible subset given by the maps

$$\mathcal{M}^{S^2}_0 \equiv \text{Maps}_0 (S^1, \mathcal{H}^{S^1}_0 \times \mathcal{M}^{S^1}_0),$$

where $\mathcal{M}^{S^1}_0 = \mathcal{M}^{S^1} \setminus \mathcal{M}^{S^1}_*$ and $\mathcal{M}^{S^1}_* = \{-\pi\}$. In this way we get a gauge fixing for gauge fields associated to the maps which do not intersect $\mathcal{M}^{S^1}_*$, but this subset is not very large. For instance, it we consider U(1) gauge fields over $S^2$ with non-trivial magnetic charge ($c_1(A) \neq 0$) every associated map intersects $\mathcal{M}^{S^1}_0$. Moreover, in the zero magnetic charge sector there is a open set of maps which intersect $\mathcal{M}^{S^1}_0$ because $\mathcal{M}^{S^1}_*$ has codimension one in $\mathcal{M}^{S^1} \equiv U(1)$. Of course the maps described by the gauge condition have zero winding number but not all maps with winding number zero are in $\mathcal{M}^{S^2}_0$. This restriction occurs in spite of the fact that the whole set of maps with zero winding number is a contractible manifold. In fact, this set is topologically equivalent to the two-dimensional orbit space for trivial bundles $\mathcal{M}^{S^1}$ which is also a contractible manifold. It is this contractibility property what makes possible the application of the method described in the Section 4 and provides a complete gauge fixing condition for $\mathcal{M}^{S^2}_0$ (holomorphic). Any gauge field orbit in that space has one and only one representative of the form

$$A_z = H^{-1} \partial_z H$$

where $H : S^2 \to \mathbb{R}^+$ is any positive real function over $S^2$. In this sense the maximal holomorphic gauge goes beyond the maximal $\sigma$-gauge, i.e. $\mathcal{M}^{S^2}_0 (\sigma) \subset \mathcal{M}^{S^2}_0 (\text{hol}) \subset \mathcal{M}^{S^2}_0$.

One could think that something similar might occur for four dimensional gauge fields. Finding a better gauge in three dimensions defined over a subset of gauge orbits larger than $\mathcal{M}^{S^2}_0$ will allow to go beyond the non-dense subset $\mathcal{M}^{S^4}$ of $\mathcal{M}^{S^4}$. This requires the introduction of a gauge condition in three dimensions for a dense domain of gauge orbits with boundary of codimension higher than one. However, the fact that the topology of $\text{Maps}_0 (S^3, SU(N))$ is non-trivial even if they are restricted to maps with zero winding number indicates that such a possibility does not occur as in the abelian case, where the corresponding set was contractible. This feature does not exclude, however, the possibility

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of existence of a more efficient gauge condition in four-dimensions, which can be achieved following the lines indicated in Section 4 by adding to $\mathcal{M}_{S}^{S^{1}}$ the strata associated to double degenerated diagonal traceless matrices.

It has been shown in Ref. [26] that the maps in $\text{Maps}_{0}(S^{d-1}, \mathcal{H}^{S^{1}} \times \mathcal{H}^{S^{1}} \times \cdots \times \mathcal{H}^{S^{1}} \times \mathcal{M}^{S^{1}})$ associated to connections defined in non-trivial SU(N) bundles are generically non-contractible. In fact, in four dimensions ($d = 4$) the instanton number $c_{2}(A)$ equals the winding number of the corresponding map from $S^{3}$ into $\mathcal{M}^{S^{1}} = SU(N)[26]$. For symmetric gauge fields the corresponding maps may become degenerated but in any case they always intersect the submanifold $\mathcal{M}_{s}^{S^{1}}[20]$.

It is remarkable the analogy of the gauge condition analyzed in this section and the temporal gauge. It is well known that temporal gauges do not completely fix the gauge because time independent gauge transformations still transform the fields without leaving the temporal gauge fixing slice. This potential pathology does not occur in our approach because all the temporal Polyakov lines used to define the gauge condition intersect at the point $x_{n}$ which excludes the existence of Gribov copies under any kind of pointed gauge transformation. The remaining gauge freedom under global gauge transformations only involves a finite number of degrees of freedom which do not lead to any infrared divergence because the weight of the redundant copies of generic gauge fields is bounded by the finite volume of the gauge group $SU(N)$.

This characterization of $SU(N)$ pure gauge fields in terms of maps from $S^{d-1}$ into the gauge group $SU(N)$ is reminiscent of the low energy description of QCD in terms of Chiral models. This suggest that a gauge invariant description of the physical degrees of freedom of pure Yang-Mills theory can be achieved in similar terms. In that description glueballs can be naturally identified in terms of the $SU(N)$ sigma model variables. This opens a new avenue to the description of the low energy spectrum of pure gauge theories as an effective theory described by an $SU(N)$ sigma model.

7. Beyond the Horizon: Non-abelian Monopoles and Confinement

Although the main goal of this paper is to provide a complete gauge fixing for a dense set $\mathcal{A}_{0}$ of gauge fields $\mathcal{A}$ for $d < 4$, one might wonder whether the remaining gauge fields $\mathcal{A} \setminus \mathcal{A}_{0}$ have any physical relevance. Indeed, they have zero measure with respect to any borelian functional measure, but there are indications that they can play a very relevant
role in the non-perturbative dynamics of gauge field theories. In particular, for physical effects which depend on topological properties their contribution is of leading order. For instance, they are responsible for the inconsistency of pathological anomalous chiral theories. In the Hamiltonian formalism, their role is enhanced because of the boundary conditions that physical states must satisfy at $\mathcal{A} \setminus \mathcal{A}_0$, and we know that boundary conditions are very important for the low energy behaviour of any quantum theory. On the other hand, one might ask why a specific choice of coordinates on the orbit space would be more relevant than another. The answer comes from the fact that the Yang-Mills functional gives a different weight to the classical configurations either in the euclidean approach where it measures the contribution of the different classical configurations or in the Hamiltonian where it indicates which configurations are more relevant not only for the classical dynamics but also for the quantum theory. In both cases the leading configurations are the solutions of Yang-Mills equation, irrespectively of their stable or unstable character. In the canonical approach the later provide a measure of quantum tunneling and include sphalerons. In the covariant approach the unstable stationary configurations mark the interphases of the different instanton liquids. We shall see that most of these configurations lie at the boundary of the fundamental domain the Maximal Non-Abelian Gauge.

Another interesting problem which is related to those boundary configurations is confinement. In the dual superconducting scenario for confinement the basic ingredient is a monopole condensation. However, it is commonly accepted that there is no intrinsic definition of a magnetic monopole for SU(N) gauge fields. One of the appealing characteristics of abelian projection is that it allows to identify some configurations which carry in that representation a sort of magnetic charge [11]. Moreover, those configurations seem to play a key role in confinement when they are considered in the maximal abelian projection gauge. However, this concept of monopole is extremely gauge dependent.

We are now in a position of providing an intrinsic definition of such monopoles based in the above characterization of gauge fields. We will say that a $S^d$ gauge field configuration carries a non-abelian magnetic charge (i.e. it is made of monopoles) when the corresponding map from $S^{d-2}$ into $\mathcal{M} S^0$ intersects the orbits associated to unstable bundles $\mathcal{M} S^2 \setminus \mathcal{M}_0 S^2$. The interpretation of configurations of $\mathcal{M} S^2 \setminus \mathcal{M}_0 S^2$ as non-abelian monopoles is quite natural because the associated holomorphic bundles do admit abelian subbundles with non-trivial first Chern classes. The characterization is completely intrinsic it is only based on the complex structure of $S^2$. For higher dimensional gauge fields
the generalization is obvious if we assume that to detect a magnetic charge we need a sort of two-dimensional $S^2$ device to measure the magnetic flux leaving the enclosed domain which indicates the presence or absence of monopoles. This is what the above definition prescribes.

The only problematic aspect of this intrinsic concept of monopole is that it does not carry an extensive charge and that the charge is not localized. In fact it is a quite subtle concept of charge because for a given field configuration, a fixed $S^2$ sphere inside $S^d$ can enclose a 2-d monopole but a slight perturbation of it does not. However, this evanescent aspect has also some advantages: it is possible to attain gauge field configurations which describe a dense gas of monopoles and they become generic in four-dimensions. They correspond just to the configurations which make our gauge fixing incomplete. The fact that in 3-dimensions configurations without monopoles are generic whereas in 4-dimensions they are not, suggests that those configurations might play a relevant role in providing an effective linear potential at large distances. Notice that in 3-dimensions the change from the perturbative Coulombian logarithmic confining potential to the real linear potential is not so substantial and in fact can be achieved due to the mild contribution of point-like monopoles which are at the boundary of $\mathcal{M}_0$ [27]. A dilute gas of monopoles picture is enough to describe the phenomenon. In four dimensions our conjecture implies that the linear potential is likely to be associated with the fluctuations of those monopole configurations which seem to have wilder field interactions promoting the role of the configurations associated to a dense gas or liquid of monopoles.

In order to understand the role boundary/monopole configurations in quantum gauge theories let us analyze some related quantum effects.

In two-dimensional QCD it is known that the fermionic determinant vanishes for gauge field configurations of $\mathcal{M}^{S^2} \setminus \mathcal{M}_0^{S^2}$ [28]. In fact this was the first theory where it was realized the role of a holomorphic gauge fixing.

The physical relevance of those boundary configurations is not exclusive of 1+1 dimensional gauge systems. In 2+1 dimensional Yang-Mills theories on a finite volume, there exist static solutions of Yang-Mills equations which are critical points of the 2-dimensional Yang-Mills functional that is the effective potential of the 2+1-dimensional theory. They are called sphalerons and although they cannot be stable they are unstable in a minimal way. There is only a finite number of instability decaying modes. On the other hand the actual value of the Yang-Mills functional on those saddle point configurations marks the height of the potential barrier responsible of the existence of relevant non-perturbative
effects. These sphalerons were extensively analyzed by Atiyah and Bott for any Riemann surface and Lie group [15]. In our case the Riemann surface is the two-dimensional sphere $S^2$, and the Atiyah-Bott results show that in such a case all sphalerons belong to the boundary $\mathcal{M}^{S^2} \setminus \mathcal{M}_0^{S^2}$. In fact for each class of unstable bundles there exist only one different type of sphaleron solution. For SU(2) the first non-trivial solution is the abelian magnetic monopole (4.4).

Those configurations have leading role in the non-perturbative dynamics in the Hamiltonian approach. For instance, it is well known that in the abelian case (QED$_{2+1}$) when a compact lattice regularization is introduced the logarithmic perturbative Coulomb potential becomes linear by means of Debye screening of electric charges in a monopole gas [27] in a similar manner as vortices drive the Berezinskii-Kosterlitz-Thouless phase transition in the XY model [29] [30]. It is, therefore, natural to conjecture that the role of configurations sitting at the boundary $\mathcal{M}^{S^2} \setminus \mathcal{M}_0^{S^2}$ which have been identified with the non-abelian generalization of magnetic monopoles would play a similar role in the mechanism of quark confinement of the non-abelian theory. This provides a geometric setting for the ’t Hooft-Mandelstam scenario [21][22] in 2+1 dimensional gauge theories.

This conjecture is supported by the fact that those monopole-like boundary configurations become extremely suppressed in the vacuum state of topologically massive gauge theories [8] which is a non-confining medium. In fact, in those theories the vacuum functional exactly vanishes for such configurations [31]. The result follows from Ritz variational principle which establishes that the expectation value of the Hamiltonian on physical states has to be minimized by the quantum vacuum. The existence of vacuum nodes at gauge fields in the boundary of the maximal holomorphic gauge fixing condition is not only required for the minimization of the kinetic term, but also for that of the Yang-Mills potential term. Both terms, kinetic and potential, of the Hamiltonian conspire to force the vanishing of the vacuum functional on gauge fields which are on the complex gauge orbits of the monopoles (4.4), i.e. the whole boundary $\mathcal{M}^{S^2} \setminus \mathcal{M}_0^{S^2}$. Since for $k \neq 0$ the theory is not confining it is foreseeable to associate to the suppression of the fluctuations of those nodal configurations a leading role in the breaking mechanism of quark confinement. Conversely, it is conceivable that those fluctuations could play a relevant role in the mechanism of quark confinement when $k = 0$, where vacuum nodes are not expected to appear according to Feynman’s qualitative arguments [32].

In 3+1 dimensions we also have relevant configurations carrying intrinsic monopole charges. In a finite volume $S^3$–sphere there are sphaleron solutions of Yang-Mills equations
which measure the height of the potential barrier between classical vacua and, therefore, the transition temperature necessary for the appearance of direct coalescence between those vacua. They also serve as indicators of the relevance of non-perturbative contributions. For $SU(2)$ the sphaleron in stereographic coordinates reads

$$A^\text{ph}_{j} = \frac{4\rho}{(x^2 + 4\rho^2)^2} (4\rho \varepsilon^a_{jk} x^k + 2x^a x_j - [x^2 - 4\rho^2] \delta^a_j) \sigma_a,$$

where $\rho$ is the radius of the $S^3$ sphere. The unstable mode can be identified with the deformation under scale transformations.

Now, the sphaleron (7.1) defined on the sphere of radius $\rho$ induces in the Maximal Non-Abelian Holomorphic gauge a loop of 2-dimensional gauge fields $\{j^s_{\varphi} A, -\pi < \varphi \leq \pi\}$ on the $S^2$-sphere, which for $\varphi = 0$ becomes gauge equivalent to the abelian Dirac monopole gauge field (4.4), i.e. $j^s_{0} A = \Phi^{-1} A^{\text{monopole}} \Phi$, with $\Phi = \exp(-i\sigma_3 \pi/4)$. This means that $[j^s_{0} A] \in \mathcal{M}^{S^2} \setminus \mathcal{M}^{S^2}_0$, and, therefore, the sphaleron itself belongs to the boundary of the maximal domain of the maximal non-abelian holomorphic gauge, i.e. $[A^\text{ph}] \in \mathcal{M}^{S^2} \setminus \mathcal{M}^{S^2}_0$. Once more a relevant configuration for the non-perturbative behaviour of the theory belongs to the boundary of the maximal domain of the gauge condition.

Another very relevant property of sphalerons are that they give a very special value to the Chern-Simons functional

$$C_s(A^\text{ph}) = \frac{1}{4\pi} \int \text{tr} \left( A^\text{ph} \wedge dA^\text{ph} + \frac{2}{3} A^\text{ph} \wedge A^\text{ph} \wedge A^\text{ph} \right) = \pi.$$  

This property together with the parity behaviour of sphalerons and the fact that $[A^\text{ph}]$ belongs to the boundary of $\mathcal{M}^{S^2}_0$ implies that the vacuum state of Yang-Mills theory at $\theta = \pi$ vanishes for sphaleron gauge fields, i.e. $\psi_0(A^\text{ph}) = 0$ [33]. The same properties also imply that parity symmetry is not spontaneous broken and the vacuum state $\psi_0$ is parity even [33]. The absence of spontaneous breaking of parity for $\theta = 0$ [34] and $\theta = \pi$ is based on different physical arguments, but in both cases the configurations of the boundary of $\mathcal{M}^{S^2}_0$ play a leading role.

The existence of nodes in the theory at $\theta = \pi$ is in contrast with what happens at $\theta = 0$ where there are not such nodes [32]. Since the theory is expected to deconfin for $\theta = \pi$, the result suggest that those nodes might be again responsible for the confining properties of the vacuum in absence of $\theta$ term where the vacuum has no classical nodal configurations.

In 3+1 dimensions we also have instantons which are the main responsible of the non-perturbative contributions associated to tunnel effects between classical vacua. Their effect
seems to be very similar to that of monopoles in compact QED in three-dimensional spacetimes. However, their contribution to confinement does not seem to be crucial. Indeed, a standard argument due to Witten shows that their contribution is exponentially suppressed in the large N limit, whereas quark confinement is strengthened in that limit [35]. However, the instanton contribution is very relevant for the problem of chiral symmetry breaking in the presence of dynamical quarks [36]. Instantons with unit topological charge and structure group $SU(2)$ are given by

$$A_\mu = \frac{2\tau_{\mu\nu}(x - x_0)\nu}{(x - x_0)^2 + \rho^2} \quad (7.2)$$

in stereographic coordinates of the four dimensional sphere $S^4$. There are two collective coordinates which parametrize the moduli space of $k = 1$ $SU(2)$ instantons: the radius $\rho$ and its center $x_0$. The $\mathfrak{sl}(2,\mathbb{C})$ matrices $\tau_{\mu\nu} = i(\tau^j_\nu \tau^j_\mu - \delta_{\mu\nu})/2$, with $\tau = (-I, i\sigma)$ define a coupling between internal and external degrees of freedom.

In the maximal holomorphic picture they define a map $[j^* A]$ from $S^2$ into $\mathcal{M}^{S^2}$. It can be shown that $[j^* A]$ also reaches the boundary of the maximal domain $\mathcal{M}^{S^2} \setminus \mathcal{M}_0^{S^2}$ because the two-dimensional gauge field $j^* A_{0,0}$

$$\left(j^* A_{0,0}\right)_i = \frac{1}{2} \frac{\epsilon_{i j k} x^j \sigma_3}{1 + |x|^2}, \quad (7.3)$$

is gauge equivalent the abelian Dirac monopole (4.4) with unit magnetic charge in $\mathcal{M}^{S^2} \setminus \mathcal{M}_0^{S^2}$. Therefore, the instanton configuration also induces a non-trivial surface of two-dimensional gauge fields.

The same property holds for configurations with higher number of instantons. For instance, the gauge field configuration with two instantons symmetrically centered at $x_+ = (x_0, 0, 0, 0)$ and $x_- = (-x_0, 0, 0, 0)$ and one single scale $\rho$ reads

$$A_\mu = \mathfrak{T}_{\mu\nu} \partial^\nu \phi(x)$$

with

$$\phi(x) = \log \left(1 + \frac{\rho^2}{(x - x_+)^2} + \frac{\rho^2}{(x - x_-)^2}\right)$$

and $\mathfrak{T}_{\mu\nu} = i(\mathfrak{T}^j_{\nu} \tau^j_\mu - \delta_{\mu\nu})/2$, $\mathfrak{T} = (-I, -i\sigma)$.

The corresponding map $[j^* A] : S^2 \rightarrow \mathcal{M}^{S^2}$ induced by the maximal holomorphic gauge also reaches the boundary of the maximal domain $\mathcal{M}^{S^2} \setminus \mathcal{M}_0^{S^2}$ because the two-dimensional gauge field $j^* A_{0,0}$ is again the abelian Dirac monopole (4.4) with unit magnetic charge in $\mathcal{M}^{S^2} \setminus \mathcal{M}_0^{S^2}$ [20].
In fact it can be shown that this property is satisfied by any gauge field carrying a non-trivial topological charge. This is not surprising because there are topological reasons which imply that generic gauge fields with multi-instantons must induce at least one non-abelian monopole in some 2-dimensional spheres of $S^4$ [20]. More precisely, a generic (non-symmetric) SU(N) gauge field $A$ with non-trivial second Chern class $c_2(A) = k$ induces a map $[j^*A] : S^2 \to \mathcal{M}^{S^2}$ which belongs to the $k$ class of the second homotopy group of $\mathcal{M}$, $\pi_2(\mathcal{M}) = \mathbb{Z}$ [26]. Then, since the fundamental domain $\mathcal{M}_0^{S^2}$ is contractible and, thus, homotopically trivial the image of $[j^*A]$ cannot be completely contained in $\mathcal{M}_0^{S^2}$.

However, the connection with non-abelian monopoles is not exclusive of 4-dimensional gauge fields with non-trivial topological charge. As we have seen in previous sections there is a codimension zero sector of 4-dimensional gauge fields with trivial topological charge whose induced maps $[j^*A] : S^2 \to \mathcal{M}^{S^2}$ do not lie inside $\mathcal{M}_0^{S^2}$.

All these facts suggest that the obstruction to the extension of the domain of the non-abelian gauges described in this paper is based in physical grounds. We know by general topological arguments that a complete global gauge cannot exist but the special characteristics of the maximal non-abelian gauges point out in an intrinsic way which configurations are relevant for some low energy non-perturbative effects. The characterization of those configurations in terms of non-abelian magnetic monopoles provides a sound basis for a physical realization of the 't Hooft-Mandelstam confinement mechanism.

8. Conclusions

In infinite volume we can consider a similar construction based on a family of $S^{d-1}$ spheres centered at the origin of $\mathbb{R}^d$ and with radius $R$ varying from $R = 0$ to $R = \infty$. Finiteness of Yang-Mills potential implies that any gauge configuration $A$ with finite energy verifies that $j^*_R A$ is a pure gauge field for $R = 0$ and $R = \infty$, which allow to associate to any field configuration $A$ with finite potential energy on $\mathbb{R}^d$ one loop of gauge fields on $S^{d-1}$. Iterating this procedure as in Sections 4 and 6 leads to the construction of complete gauges for $d \leq 3$.

On the other hand the monopole identification also works for field configurations in infinite volumes, which gives an intrinsic physical meaning to the whole construction beyond the infrared limit.

\footnote{The result is to some extent dual of the descendent technique in the study of anomalies [37]}
In summary, the two new gauge conditions introduced in Sections 4 and 6 based on the special structure of the orbit spaces one and two-dimensional gauge fields are complete for gauge fields over spaces with dimensions lower than four in a maximal domain $\mathcal{M}_0$ which is open and dense in whole orbit space $\mathcal{M}$. The gauge conditions are free of Gribov ambiguities on $\mathcal{M}_0$.

The obstruction to completeness in four-dimensional spaces is related to configurations describing a dense gas or liquid of instantons which seem to be relevant for confinement.

One of the interesting features of these maximal non-abelian gauge conditions in lower dimensions is that the configurations sitting at the boundary of the maximal domain do play a very relevant role in non-perturbative physical effects like the existence of nodes in the vacuum functional of Topologically Massive Gauge Theory in 2+1 dimensions and Yang-Mills theory with $\theta = \pi$ in 3+1 dimensions. The disappearance of these nodes when the topological mass or the $\theta$ term vanish suggest that those configurations play a fundamental role in the confinement mechanism which is also activated when those terms vanish.

This characterization of gauge fields over spaces of arbitrary dimension in terms of one or two dimensional gauge fields provides an alternative description of the topological properties of the corresponding orbit space. Homotopy and cohomology classes of the orbit spaces $\mathcal{M}^{S^d}$ are faithfully described in terms of those of lower dimensional gauge fields, $\mathcal{M}^{S^2}$ and $\mathcal{M}^{S^1}$. This description also makes possible an intrinsic characterization of non-abelian monopoles for SU(N) gauge fields in terms of field configurations lying beyond the boundary of the fundamental domain of the maximal abelian gauges, which provides an appropriate geometric framework for the realization of the dual superconducting scenario for confinement. In fact, the characterization of gauge fields in terms of SU(N)-valued fields is closely related to the $\sigma$-model chiral description QCD at low energies. This opens a new perspective to the description of the low energy glueball spectrum of pure gauge theories as an effective theory described by an SU(N) sigma model.

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