The phenomena of Nature are modelled by inequalities.
The equations are an accident.

Abstract

Several different problems make the study of the so called Lyapunov type inequalities of great interest, both in pure and applied mathematics. Although the original historical motivation was the study of the stability properties of the Hill equation (which applies to many problems in physics and engineering), other questions that arise in systems at resonance, crystallography, isoperimetric problems, Rayleigh type quotients, etc. lead to the study of $L_p$ Lyapunov inequalities ($1 \leq p \leq \infty$) for differential equations. In this work we review some recent results on these kinds of questions which can be formulated as optimal control problems. In the case of Ordinary Differential Equations, we consider periodic and antiperiodic boundary conditions at higher eigenvalues. Then, we establish Lyapunov inequalities for systems of equations. For Partial Differential Equations on a domain $\Omega \subset \mathbb{R}^N$, we consider the Laplace equation with Neumann or Dirichlet boundary conditions. It is proved that the relation between the quantities $p$ and $N/2$ plays a crucial role in order to obtain nontrivial $L_p$ Lyapunov type inequalities (which are called Sobolev inequalities by many authors). One of the main applications of Lyapunov inequalities is its use in the study of nonlinear resonant problems. To this respect, combining the linear results with Schauder fixed point theorem, we show some new results about the existence and uniqueness of solutions for resonant nonlinear problems for ODE or PDE, both in the scalar case and in the case of systems of equations.

Key words: Lyapunov inequalities, boundary value problems, resonance, stability, ordinary differential equations, partial differential equations.

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1 Introduction

The Hill equation

$$u''(t) + a(t)u(t) = 0, \ t \in \mathbb{R}$$

(1)

where the function $a$ satisfies

$$a : \mathbb{R} \to \mathbb{R} \text{ is } T \text{ – periodic and } a \in L^1(0, T),$$

(2)

models many phenomena in applied sciences ([17], [21], [31]). In a broader sense, Hill’s equation connotes the class of homogeneous, linear, second order differential equations with real and periodic coefficients. In the preface to the book on Hill’s equation, by Magnus and Winkler [31], we can read:

“There exist hundreds of applications of Hill’s equation to problems in engineering and physics, including problems in mechanics, astronomy, the theory of electric circuits, of the electric conductivity of metals, of the cyclotron and new applications are continually discovered”.

In what follows, we denote by $L_T(\mathbb{R}, \mathbb{R})$ the set of functions $a(\cdot)$ satisfying (2).

The study of stability properties of (1) is of special interest. Whenever all solutions of (1) are bounded, we say that (1) is stable; otherwise we say that it is unstable. Floquet theory assures that such stability properties depend on characteristic multipliers, which are defined using any fundamental matrix of the given equation. This important theoretical result requires the knowledge of all solutions of (1).

In early twentieth century Lyapunov proved ([30], [31]) that if

$$0 < a, \int_0^T a(t) \, dt \leq \frac{4}{T},$$

(3)

then (1) is stable. Here, for $c, d \in L^1(0, T)$, we write $c < d$ if $c(t) \leq d(t)$ for a.e. $t \in [0, T]$ and $c(t) < d(t)$ on a set of positive measure.

Lyapunov’s result is remarkable, among others, for two main reasons: first, one can check (3) directly from the equation (1) and second, it is optimal in the following sense ([31]): for any $\varepsilon \in \mathbb{R}^+$, there are unstable differential equations (1) with $a$ satisfying (2), for which

$$0 < a, \int_0^T a(t) \, dt \leq \frac{4}{T} + \varepsilon$$

Condition (3) has been generalized in several ways ([2], [25]). More recently the authors provide in [38] optimal stability criteria by using $L^p$ norms of $a^+$, $1 \leq p \leq \infty$.

The parametric equation

$$u''(t) + (\mu + a(t))u(t) = 0, \ \mu \in \mathbb{R},$$

(4)

plays a fundamental role in the study of the stability properties of the Hill equation (1). In fact, if $\lambda_i(a), \ i \in \mathbb{N} \cup \{0\}$ and $\tilde{\lambda}_i(a), \ i \in \mathbb{N}$, denote, respectively, the