Gauge and space-time symmetry unification

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Abstract

Unification ideas suggest an integral treatment of fermion and boson spin and gauge-group degrees of freedom. Hence, a generalized quantum field equation, based on Dirac’s, is proposed and investigated which contains gauge and flavor symmetries, determines vector gauge field and fermion solution representations, and fixes their mode of interaction. The simplest extension of the theory with a 6-dimensional Clifford algebra predicts an $SU(2)_L \times U(1)$ symmetry, which is associated with the isospin and the hypercharge, their vector carriers, two-flavor charged and chargeless leptons, and scalar particles. A mass term produces breaking of the symmetry to an electromagnetic $U(1)$, and a Weinberg’s angle $\theta_W$ with $\sin^2(\theta_W) = .25$. A more realistic 8-d extension gives coupling constants of the respective groups $g = 1/\sqrt{2} \approx .707$ and $g' = 1/\sqrt{6} \approx .408$, with the same $\theta_W$.

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1 Introduction

Unification has proved to be a powerful assumption leading to new connections among previously considered unrelated phenomena. It is not an exaggeration to say that most substantial advances in the history of physics have been accompanied by the realization of links among facts originally appearing to be independent. The application of unification ideas differs though from one case to the other in the scope, methods, and results, and it is therefore difficult to characterize it uniquely by a single rule. Thus, the unification of known facts has sometimes led to the prediction of new phenomena, and these connections have been either experimental, theoretical, or both. It shall be useful instead to briefly review some highlights.

The concept of unification is linked to the very idea of science (or then philosophy) conceived by the early Greek philosophers, who in their research of nature sought unifying principles, although those they found were premature in their applicability. However, a perdurable idea from those times representing probably the most powerful tool in physics is that of assuming a mathematical structure behind physical phenomena, an idea ascribed to Pythagoras. In modern times Galileo helped to revive the idea of the universality of physical law in the cosmos by presenting supporting evidence (e.g. the shadows provoked by the sun in the moon). The principle of relativity is a related idea he discovered which assumes this universality for different inertial frames, putting powerful constraints on the possible allowed laws. Newton showed, with his new understanding of gravity, that the motion of cosmic and terrestrial bodies obeys the same laws, thus demonstrating for the first time a deep relation between phenomena in both expanses.

Electric and magnetic phenomena were considered separated until the XIX Century. With the work of Ampère and Faraday it was found experimentally that one leads to the other by changing the kinematic state of the charges involved. Maxwell carried out the formalization of this into a series of equations which provided a new
understanding of light as one of many possible waves with an electromagnetic origin, and traveling at the speed of light, a quantity that was predicted from the equations.

In this century, Einstein’s special relativity integrated Galileo’s relativity principle with Maxwell’s equations’ invariance into a new framework by dethroning time from its privileged use and putting it on a similar footing to space, while the speed of light was assumed constant in all reference frames. From here a series of new phenomena were predicted, as the equivalence between mass and energy. These ideas were expanded by linking gravity, matter, and space-time through general relativity (GR), a theory which assumes a geometrical framework. However, this was done only partially since in Einstein’s GR equations only the side describing gravity and space-time’s geometry has this interpretation while the other, containing the energy-momentum tensor has not necessarily this form, and waits for geometrization, if ever. GR predicts new phenomena as black holes, while in the Newtonian weak gravitational limit it produces small corrections.

Einstein attempted to unify gravity and electromagnetism, but he was not really successful; nevertheless, in the meantime Kaluza and Klein developed the idea of extending GR to more than 3+1 dimensions, relating an additional dimension to a vector potential which could be shown to describe the electromagnetic field. This feat has not led to more information, the prediction of new phenomena, or has been an option that has been possible to test but it does show that it is a viable possibility. Therefore, although it cannot be classified as a successful unification it retains the status of a useful working hypothesis that is actually applied in theories as supergravity or superstrings.

Quantum mechanics (QM) successfully accounted for the prevalence of particle and wave characteristics encountered in different experiments with the same objects, which is otherwise contradictory in a classical framework. This comprised a completely new feature for the constituents of nature, which were previously thought to
belong to separated classes presenting each kind of behavior. The introduction of Planck’s constant required by QM gives rise, when using Newton’s gravity constant and the speed of light, to fundamental values of mass, time, and position; this constitutes a unification in the sense that all measurable quantities can be related to these fundamental constants. A theory which would join together GR and QM should certainly use these.

One of the most beautiful examples of unification comes from Dirac who discovered a new type of equation that both satisfied the principles of special relativity and those of QM. Their marriage in this new setting provided a new understanding of the spin 1/2 degree of freedom, a variable previously postulated to account for various atomic phenomena and understood to be related to magnetic properties of fermions, but with an otherwise inscrutable origin. Dirac’s equation not only naturally gives rise to this variable but also predicts the electron’s magnetic moment with a relatively close accuracy.

In recent times the latest success in the application of unification ideas has been in relating the weak and electromagnetic interactions in the Weinberg-Salam model (Glashow, 1961, Salam, 1968, Weinberg, 1967), which considers them to originate in a gauge symmetry, although their respective groups $SU(2)_L \times U(1)$ assume totally different forms. Still, the theory succeeded in predicting parameters as the masses of the vectors carrying these interactions and the existence of neutral currents.

Many of today’s puzzles in fundamental aspects of physics are encountered in the current theory of elementary particles and fields, the standard model (SM), which involves many mysteries. Although it is quite successful in describing their behavior, its very construction requires input determined by phenomenology but which is otherwise ad hoc, and which consists of a large number of parameters. Worse, many aspects of this input still need justification. It is not clear why there are three generations of leptons and quarks nor the origin of their masses and the latter’s mixing
angles. Neither is clear what is the source of the parameters needed to describe the Higgs particle, which is as yet only a mathematical device to break the gauge symmetry and give masses to particles; indeed, we lack a more fundamental reason for the presence of a spin 0 particle. We also lack information on the origin of the gauge groups of the fundamental interactions $SU(3) \times SU(2)_L \times U(1)$, the origin of their coupling constants values, and the reason for the isospin force acting only on a given chirality, which leads to parity violation (Lee, Yang, 1956). However, in this case, very interesting connections have been obtained from grand unified models on both the forces and the values of coupling constants (Georgi, Glashow, 1974). These models assume a common origin for these forces’ gauge groups through the postulation of a group containing them as subgroups. Still, the overall picture hints at a missing piece of information on an underlying principle. It may be worth to return to unification ideas for a clue. In particular, we now concentrate on the current concepts of spin and space-time symmetries, invoked by the first, and from here we follow a possible connection path to gauge symmetries.

While QM offers a common description for the shared properties of bosons and fermions it still requires a specialized treatment for each to account for their differences. Thus, while the space-time description of the propagation of a fermion is similar to that of a boson this differs in the spin wave functions, which from quantum field theory (QFT) are known to have a determinant influence on their very different collective behavior. A unified theory describing both kind of particles should address the question of their spin. The only physical connection comes along through the vertex interaction which is determined uniquely by the gauge symmetry (e. g., in the electromagnetic case). Still, boson and fermion degrees of freedom are presently otherwise assumed independent from each other. In looking for a closer connection among them it is worth having in mind that the spin $1/2$ particle representation of the Lorentz group ($SO(3,1)$) is more fundamental than the vector one as the latter can be obtained from a tensor product of the first but not the other way around.
On another plane, the fact that a particle description requires both configuration (or momentum) and spin spaces leads in turn to the fact that it is only a combination of both types of corresponding generators that allows for invariance under Lorentz transformations, which makes them equally necessary. In this context, it is worth recalling the Kaluza-Klein idea and wonder whether there exists a connection of the forces of nature to extended spin spaces, instead of additional spatial dimensions. In a way, this idea underlay the attempt of Heisenberg and Condon to understand the difference between the proton and the neutron. Having in mind the similarity with spin, they assigned them with hindsight a doublet structure, calling it isotopic spin or iso-spin, a concept that evolved into the $SU(2)_L$ group underlying the modern treatment of the weak interactions.

In this paper we propose a new field equation, based on Dirac’s, which allows for a unified treatment of both boson and fermion spin degrees of freedom by making the solutions share the same solution space and at the same time which encompasses degrees of freedom which can be assigned to the gauge groups. The equation and the surrounding formalism are developed in a quantum mechanical relativistic framework, but some aspects of QFT will be touched. We will show the dimensionality of the solution space restricts both the possible solutions and the symmetries present, and that from these an interaction prescription emerges naturally among the field solutions. In particular, we obtain vertices and their coupling constants. We analyze the simplest extension to 5+1 and we find that an $SU(2)_L \times U(1)$ symmetry is predicted. The solutions will be hence related to physical fields. In Section II we study the 3+1 dimensional version of the new equation by considering its symmetries and a set of commuting operators characterizing the solutions. We also find and analyze their link to quantized fields. In Section III we present its bosons solutions, both at the massless and massive levels and in Section IV we study a particular reduction of the equation and its transformations leading to fermion solutions too. We argue that both versions of the field equation contain a gauge invariance. In section V we present
some conserved currents and, through them we find a link to a vertex interaction between a pair of spin 1/2 particles and a boson, which is implied in the formalism. In Section VI we generalize the equation to six dimensions using the 5+1 Clifford algebra and we analyze the embedded 4-d Clifford subalgebras, and corresponding symmetries. We show that for one subalgebra chain an $SU(2)_L \times U(1)$ symmetry is implied. In Section VII we present the massless solutions and link these symmetries to the isospin and hypercharge generators, respectively. In Section VIII we present the massive ones. In Section IX we link these solutions to physical fields in the SM, and obtain the fermion-vector couplings and coupling constants. In Section X we summarize this work, indicate its main results, and draw conclusions.

2 Generalized field equation from Dirac formalism

We search for a description of vectors and scalars as close as exists for fermions in order to be able to relate both representations. We also demand that the field equation which provides this description be enclosed in a variational principle framework. Indeed, these requirements are achieved by generalizing Dirac’s equation and extending its multiplet content. At this point we concentrate only on the free particle case and later on we will show how interactions are implied in this formalism. Then, instead of assuming the Dirac operator acts on a spinor (Dirac, 1947)

\[(i\partial_{\mu}\gamma^{\mu} - M)\psi = 0,\]  

(1)

where $\psi$ is the column vector with components $\psi_{\alpha}$, we assume it acts on a $4 \times 4$ matrix $\Psi$ with components $\Psi_{\alpha\beta}$ so that the equation becomes

\[(i\partial_{\mu}\gamma^{\mu} - M)\Psi = 0.\]  

(2)

The form of this equation implies all symmetry operators valid for the Dirac equation in eq. (1) (with its corresponding particular cases of massless and massive cases) will
be valid as well for it. The operators therefore satisfy the Poincaré algebra. There are other possible Lorentz-invariant terms that could enter eq. 2; further justification for the choice of the terms in this equation is related to a gauge symmetry described in Section IV.

We postulate all transformations and symmetry operations on the Dirac operator

\[ (i\partial_\mu \gamma^\mu - M) \rightarrow U(i\partial_\mu \gamma^\mu - M)U^{-1} \]

induce a corresponding transformation

\[ \Psi \rightarrow U\Psi U^\dagger. \]  

(3)

Here, the lhs \( U \) is fixed by the Dirac operator transformations but there is a liberty for the rhs term, whose choice will shortly prove its utility. With this assumption the elements of \( \Psi \), which can be expanded in terms of the tensor product of two spinors \( \sum_{i,j} a_{ij}|w_i\rangle\langle w_j| \), are expected to Lorentz transform as scalars, vectors and antisymmetric tensors. We will show below modified symmetry operators classify some solutions as fermions too.

The vector space spanned by the matrix solutions allows to define an algebra to which they belong and which is closed. By using the matrix product, if \( A, B \) are solutions, the new field

\[ C = AB, \]  

(4)

is another element of the algebra which may or may not be a solution, but lives in the same vector space. We find here a connection to QFT as we have an algebra of operator solutions. In fact, we will show the product among fields leads to interactions among them.

The quantum mechanical dot product of \( A, B \) is defined by

\[ \langle A|B \rangle = tr(A^\dagger B). \]  

(5)

A trace of over the coordinates is also implied. This definition satisfies the usual properties expected for a measure. The use of the product in eq. 4 implies the
number of terms entering the point product is not restrained and it may include more than two fields to be evaluated. Expectation values of operators or any matrix element with the overlap of two solutions can therefore be defined from here. An interpretation of these products requires also taking care of the Lorentz structure.

We note transformation 3 is also valid for hermitian conjugated fields \( \Psi^\dagger \), which satisfy the equation

\[
0 = \Psi^\dagger (\Psi^\dagger - M). \tag{6}
\]

We will extend our space of solutions by considering also combinations of fields \( A, B^\dagger \),

\[
A + B^\dagger, \tag{7}
\]

respectively satisfying eqs. 2 and 3. In fact, it is by taking account also of these fields, that we can span the function space on the 32-dimensional complex \( 4 \times 4 \) matrices.

**Conserved Operators**

We shall be interested in plane-wave solutions of the form

\[
\Psi_k^{(+)}(x) = u(k)e^{-ikx} \tag{8}
\]

\[
\Psi_k^{(-)}(x) = v(k)e^{ikx}, \tag{9}
\]

where \( k^\mu \) is the momentum four-vector \((E, k), k_0 = E\).

By putting eq. 4 into Hamiltonian form and using the plane-wave states of eqs. 8, 9 each spinor satisfies respectively the stationary equation

\[
\gamma_0 (\gamma \cdot k + M) u(k) = Eu(k) \tag{10}
\]

and

\[
\gamma_0 (-\gamma \cdot k + M) v(k) = -Ev(k), \tag{11}
\]
with \( \gamma = (\gamma^1, \gamma^2, \gamma^3) \). To classify the solutions we use the Hamiltonian

\[
H = \gamma_0(k \cdot \gamma + M)
\]  

(12)

and the Pauli-Lubansky vector

\[
W_\mu = -\frac{1}{2} \epsilon_{\muho\sigma} J^{\rho\sigma} p^\rho,
\]  

(13)

defined from the Lorentz-transformation generators

\[
J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu},
\]  

(14)

with the spin operators given by

\[
\sigma_{\mu\nu} = i \frac{1}{2} [\gamma_\mu, \gamma_\nu],
\]  

(15)

and momentum operator

\[
p^\mu = i \partial^\mu.
\]  

(16)

\( W_\mu \) is projected over the space-like four-vector \( n_k \), orthogonal to the momentum, of norm \(-1\) (the conventions for the norm \( g_{\mu\nu} \) are given in the appendix).

\[
n_k = \left( \frac{|k|}{M}, \frac{E_k}{M|k|} \right),
\]  

(17)

giving

\[
\frac{1}{M} W \cdot n_k = \Sigma \cdot \hat{k},
\]  

(18)

where

\[
\Sigma = \frac{1}{2} \gamma_5 \gamma_0 \gamma.
\]  

(19)

The definition in eq. (18) is valid both for the massless and the massive cases.

**Solutions as quantized fields**
Consistency of the definition \( \mathfrak{g} \) when applied to the generator of time translations, the Hamiltonian, implies formally that the energy should be obtained by taking the commutator
\[
[\gamma_0(-i\nabla \cdot \gamma + M), \Psi]
\]
This operation calls for a rule on the action of the derivative on the right. We will proceed heuristically here and apply the transformation rule \( p^\mu \to k^\mu \) on \( H \). We apply the same rule on the \( \frac{1}{M}W \cdot n_k \) operator, which leads to \( \Sigma \cdot \hat{k} \). This prescription is already taken into account in eqs. [12] and [18] and is as expected for spin-derivative operators acting on a tensor product space. We shall use this assignment for these operators which classify the solutions throughout this paper. As a bonus, we obtain that hermitian conjugates of negative energy solutions have positive energies with the opposite spin \( s \), just as occurs in QFT, which in turn reproduces hole theory. Indeed, assuming for the \( v(k) \) component of \( \Psi_k^{(-)}(x) \) in eq. [3],
\[
[H, v(k)] = -Ev(k)
\]
\[
[-\Sigma \cdot \hat{k}, v(k)] = sv(k)
\]
we find that for the hermitian conjugate wave function field \( v^\dagger(k)e^{-ikx} \), satisfying eq. [8],
\[
[H, v^\dagger(k)] = Ev^\dagger(k)
\]
\[
[-\Sigma \cdot \hat{k}, v^\dagger(k)] = -sv^\dagger(k).
\]
We expect a more formal justification of this operation will be given in the rigorous context of QFT. In addition, consistency with eq. [4] will require a choice of the normalization for agreement with the energy \( E \).

3 Vectors, scalars, and antisymmetric tensors

Massless solutions
The massless equation

\[ i \partial_\mu \gamma^\mu \Psi = 0 \]  

leads to the expressions for the operators in eqs. 12 and 18, assuming (from here and for all massless solutions, except when otherwise stated) that the space component of momentum \( k^\mu \) is along the \( \hat{z} \) direction,

\[ \Sigma \cdot \hat{k} = \frac{i}{2} \gamma_1 \gamma_2, \]  

\[ H/k_0 = \gamma_0 \gamma^3, \]  

where the former is the helicity operator and latter is the Hamiltonian divided by the energy.

The polarization components of the solutions of eq. 25, bilinear in the \( \gamma_s \), are given on Tables 1, 2. We set the coordinate dependence as

\[ \Psi_{ki}^{(+)}(x) = u_i(k)e^{-ikx} \]  

\[ \Psi_{ki}^{(-)}(x) = \tilde{u}_i(k)e^{-ikx}. \]

They are given together with their quantum numbers corresponding to the operators in eqs. 26, 27. The solutions are also eigenfunctions of these operators \( O \) in the simple form \( Ou_i(k) = \lambda u_i(k) \) and we present the eigenvalues \( \lambda \) too, where here and throughout the solutions are normalized as, e. g.,

\[ tr(\tilde{u}_i^\dagger(k)\tilde{u}_i(k)) = 1. \]  

Solutions \( u_{-1}(k), \tilde{u}_1(k) \) correspond to on-shell particles with transverse polarizations but opposite helicities, while the off-shell \( u_0(k), \tilde{u}_0(k) \) are polarized in the longitudinal-scalar directions. All these solutions correspond to waves propagating in the \( \hat{z} \) direction. The other terms propagate in the \( -\hat{z} \) direction, which is represented by four-vector \( \tilde{k}^\mu = k^\mu \), and are classified with the appropriate relations as eqs. 26 and 27.
Vector solutions $\gamma_0 \gamma^3 \frac{i}{2} \gamma_1 \gamma_2 \ [H/k_0, \ ] [\Sigma \cdot \hat{p}, ]$

$u_{-1}(k) = \frac{1}{4}(1 - \gamma_5)\gamma_0(\gamma_1 - i\gamma_2) \quad 1 -1/2 \quad 2 \quad -1$

$u_{-1}(\tilde{k}) = \frac{1}{4}(1 - \gamma_5)\gamma_0(\gamma_1 + i\gamma_2) \quad -1 \quad 1/2 \quad 2 \quad -1$

$u_0(k) = \frac{1}{4}(1 - \gamma_5)\gamma_0(\gamma_0 - \gamma_3) \quad 1 \quad -1/2 \quad 0 \quad 0$

$u_0(\tilde{k}) = \frac{1}{4}(1 - \gamma_5)\gamma_0(\gamma_0 + \gamma_3) \quad -1 \quad 1/2 \quad 0 \quad 0$

Table 1: V-A terms.

Vector solutions $\gamma_0 \gamma^3 \frac{i}{2} \gamma_1 \gamma_2 \ [H/k_0, \ ] [\Sigma \cdot \hat{p}, ]$

$\tilde{u}_1(k) = \frac{1}{4}(1 + \gamma_5)\gamma_0(\gamma_1 + i\gamma_2) \quad 1 \quad 1/2 \quad 2 \quad 1$

$\tilde{u}_1(\tilde{k}) = \frac{1}{4}(1 + \gamma_5)\gamma_0(\gamma_1 - i\gamma_2) \quad -1 \quad -1/2 \quad 2 \quad 1$

$\tilde{u}_0(k) = \frac{1}{4}(1 + \gamma_5)\gamma_0(\gamma_0 - \gamma_3) \quad 1 \quad 1/2 \quad 0 \quad 0$

$\tilde{u}_0(\tilde{k}) = \frac{1}{4}(1 + \gamma_5)\gamma_0(\gamma_0 + \gamma_3) \quad -1 \quad -1/2 \quad 0 \quad 0$

Table 2: V+A terms.

The coordinate dependence of these solutions is given by

$$\Psi^{(+)}_{ki}(x) = u_{i}(\tilde{k})e^{-ikx}. \quad (31)$$

These solutions do not represent independent polarization components as e.g. $u_{i}(\tilde{k})$ can be obtained by rotating the $u_{i}(k)$. The classification $V+A$ and $V-A$, consisting respectively of the $\tilde{u}_i$ and $u_i$ terms, corresponds to specifying the weight of vector and axial components, which is further clarified below. These two types of solutions are also characterized by the two vector spaces projected by $\frac{1}{2}(1 + \gamma_5)$ and $\frac{1}{2}(1 - \gamma_5)$ which they generate respectively but which they do not exhaust. We need
to consider the negative energy solutions

\[ \Psi_{ki}^-(x) = v_i(k)e^{ikx} \]  \hspace{1cm} (32)

and use their hermitian conjugates, which in fact generate other polarization components, in order to completely span the space. In the massless case we have negative energy solutions

\[ v_i(k) = u_i(k), \quad \tilde{v}_i(k) = \tilde{u}_i(k) \]  \hspace{1cm} (33)

and \( \tilde{k} \) terms, that is, with opposite helicities. The combinations of the type

\[ \frac{1}{\sqrt{2}}[\tilde{u}_i(k) \pm v_i(k)]e^{-ikx}, \quad \frac{1}{\sqrt{2}}[u_i(k) \pm v_i^\dagger(k)]e^{-ikx} \]

(\( v_i^\dagger(k) \equiv [v_i(k)]^\dagger \)) will be interpreted as vector solutions with varied polarizations.

The chirality operator \( \gamma_5 \) further characterizes these solutions as non-chiral since, using rule 3, it gives \( [\gamma_5, \Psi] = 0 \). The most general form of the solutions can be obtained by rotating and boosting these solutions through a Lorentz transformation, using \( J_{\mu\nu} \) in eq. 14.

Eq. 2 also satisfies the discrete invariances of time, and space inversion, and charge conjugation, expressed respectively by the operators

\[ T = i\gamma_1\gamma_3\mathcal{K}\mathcal{T} \]  \hspace{1cm} (33)

\[ P = \gamma_0\phi \]  \hspace{1cm} (34)

\[ C = i\gamma_2\mathcal{K} \]  \hspace{1cm} (35)

where \( \mathcal{K} \) is the complex conjugation operator \( \mathcal{K}i\mathcal{K} = -i \), \( \mathcal{T} \) changes \( t \to -t \), and \( \phi \) changes \( x \to -x \) and consequently \( p \to -p \); we use the Dirac representation for the \( \gamma_\mu \) matrices (see appendix). It is then possible to form combinations of the above solutions transforming as vectors and as axial vectors. For example, the combination

\[ \Psi_{k\hat{x}} = \frac{i}{2}[\tilde{u}_1(k) + u_{-1}(k) + \tilde{v}_1^\dagger(k) + v_{-1}^\dagger(k)]e^{-ikx} = \frac{i}{2}\gamma_0\gamma_1e^{-ikx} \]  \hspace{1cm} (36)

represents a vector particle linearly polarized along \( \hat{x} \), that is, it transforms into \(-\Psi_{k\hat{x}}(\tilde{x})\) under \( P \), with \( \tilde{x}_\mu = x^\mu \). In general

\[ A_\mu(x) = \frac{i}{2}\gamma_0\gamma_\mu e^{-ikx} \]  \hspace{1cm} (37)
(and the corresponding negative energy solution) transforms into $A^\mu(\tilde{x})$ under $P$, into $A^\mu(-\tilde{x})$ under $T$, and into $-A_\mu(-x)$ under $C$.

$$A_5^\mu(x) = \frac{i}{2}\gamma_5\gamma_0\gamma_\mu e^{-ikx} \tag{38}$$

transforms into $-A_5^\mu(\tilde{x})$ under $P$, into $A_5^\mu(-\tilde{x})$ under $T$, and into $A_5^\mu(-x)$ under $C$. The combination $A_\mu(x) + CA_\mu(x)C^\dagger$ transforms into minus itself under charge conjugation, as expected for a non-axial vector. Given the quantum numbers of $A_\mu(x)$ it becomes possible to relate it to the vector potential of an electromagnetic field. Indeed, similar mixtures of $\tilde{u}$, $\tilde{v}^\dagger$, $u$, and $v^\dagger$ solutions have been shown, under certain conditions, to satisfy Maxwell’s equations (Bargmann, Wigner, 1948).

The remaining eight degrees of freedom in the massless case are classified into six forming an antisymmetric tensor and two scalars, which as solutions appear mixed. The chirality $\gamma_5$ further divides them into left and right-handed. Their respective coordinate dependence is

$$\Psi_{ki}^{(-)}(x) = w_i(k)e^{-ikx} \tag{39}$$

$$\Psi_{ki}^{(+)}(x) = \tilde{w}_i(k)e^{-ikx} \tag{40}$$

(and corresponding definitions for $\tilde{k}$) and the explicit form of the matrix components together with their quantum numbers is shown on Tables 3, 4.

To see these terms have this interpretation we should apply transformation 3 with $U$ containing a Lorentz transformation, acting on $1\Psi = \gamma_0\gamma_0\Psi$, which leads to $U^\dagger\gamma_0 = \gamma_0U^{-1}$. Labeling the antisymmetric terms by

$$A_{\mu\nu} = \frac{1}{4}\gamma_0[\gamma_\mu, \gamma_\nu], \tag{41}$$

and the scalar and pseudoscalar terms by

$$\phi = \frac{1}{2}\gamma_0, \tag{42}$$

$$\phi_5 = \frac{1}{2}\gamma_0\gamma_5, \tag{43}$$

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**Scalars and antisymmetric tensors**  
\[ \gamma_0\gamma^3 \frac{i}{2}\gamma_1\gamma_2 \quad [H/k_0,] \quad [\Sigma \cdot \hat{p},] \]

- \[w_0(k) = \frac{1}{4}(1 - \gamma_5)(\gamma_0 + \gamma_3) \quad 1 \quad -1/2 \quad 2 \quad 0\]
- \[w_0(\tilde{k}) = \frac{1}{4}(1 - \gamma_5)(\gamma_0 - \gamma_3) \quad -1 \quad 1/2 \quad 2 \quad 0\]
- \[w_{-1}(k) = \frac{1}{4}(1 - \gamma_5)(\gamma_1 - i\gamma_2) \quad 1 \quad -1/2 \quad 0 \quad -1\]
- \[w_{-1}(\tilde{k}) = \frac{1}{4}(1 - \gamma_5)(\gamma_1 + i\gamma_2) \quad -1 \quad 1/2 \quad 0 \quad -1\]

Table 3: left-handed bosons.

**Scalars and antisymmetric tensors**  
\[ \gamma_0\gamma^3 \frac{i}{2}\gamma_1\gamma_2 \quad [H/k_0,] \quad [\Sigma \cdot \hat{p},] \]

- \[\tilde{w}_0(k) = \frac{1}{4}(1 + \gamma_5)(\gamma_0 + \gamma_3) \quad 1 \quad 1/2 \quad 2 \quad 0\]
- \[\tilde{w}_0(\tilde{k}) = \frac{1}{4}(1 + \gamma_5)(\gamma_0 - \gamma_3) \quad -1 \quad -1/2 \quad 2 \quad 0\]
- \[\tilde{w}_1(k) = \frac{1}{4}(1 + \gamma_5)(\gamma_1 + i\gamma_2) \quad 1 \quad 1/2 \quad 0 \quad 1\]
- \[\tilde{w}_1(\tilde{k}) = \frac{1}{4}(1 + \gamma_5)(\gamma_1 - i\gamma_2) \quad -1 \quad -1/2 \quad 0 \quad 1\]

Table 4: right-handed bosons.

The expressions on Tables 3, 4 can be written in terms of \(A_{\mu\nu}, \phi, \) and \(\phi_5,\) This requires also hermitian conjugates of negative energy solutions

\[
\Psi_{ki}^{(-)}(x) = z_i(k)e^{ikx} \quad (44)
\]
\[
\tilde{\Psi}_{ki}^{(-)}(x) = \tilde{z}_i(k)e^{ikx}, \quad (45)
\]

where \(z_i(k) = w_i(k), \tilde{z}_i(k) = \tilde{w}_i(k)\) (and \(\tilde{k}\) terms). While the scalar and pseudoscalar particles obtained have a straightforward interpretation as on-shell particles the antisymmetric solutions do not have a recognizable interpretation, given that their on-
shell components do not have transverse polarizations. A vector interpretation can be given using the identities

\[ \tilde{w}_i(k) = \frac{1}{2|k|} \tilde{k}u_{-i}(k), \quad \tilde{w}_i(\tilde{k}) = \frac{1}{2|\tilde{k}|} \tilde{\tilde{k}}u_{-i}(k), \]

(46)

\[ w_{-i}(k) = \frac{1}{2|k|} \tilde{k}u_i(k), \quad w_{-i}(\tilde{k}) = \frac{1}{2|\tilde{k}|} \tilde{\tilde{k}}u_i(k), \]

and similar expressions for negative energy solutions. The gauge symmetry discussed below suggests some of these solutions may be gauged out.

**Polarization vectors**

The solutions presented so far on Tables 1, 2 and 3, 4 are given in terms of components that are eigenstates of the helicity operator and are therefore components of spherical harmonic vectors. In general, we can show the solutions generate a quadrivector basis whose components can be given in a spherical or in a vector basis.

A set of corresponding polarization vectors \( \epsilon^{(\lambda)}(k) \) can be defined which coincide with the directions that some of the actual solutions in Tables 1, 2 and 3, 4 take. We define a unitary vector \( n \), along the time direction, that is, \( n^2 = 1 \). Assuming a general \( k \) we choose \( \epsilon^{(1)}(k) \), and \( \epsilon^{(2)}(k) \) in the transverse directions, orthogonal to \( k \) and \( n \), and \( \epsilon^{(\lambda)}(k) \cdot \epsilon^{(\lambda')} (k) = -\delta^{\lambda \lambda'} \). Then we pick \( \epsilon^{(3)}(k) \), the longitudinal vector, along the plane \( k-n \) and orthogonal to \( n \) and \( \epsilon^{(0)}(k) \), the scalar component, along \( n \). These vectors are orthogonal among themselves:

\[ \epsilon^{(\lambda)}(k) \cdot \epsilon^{(\lambda')} (k) = g^{\lambda \lambda'}. \]

(47)

In the case of the solutions \( \tilde{u}_i \) on Table 4, which propagate along \( \pm \hat{z} \), the polarization vectors in the spherical basis are

\[ \epsilon^{(1)}(k) = \tilde{u}_1(k), \]

(48)

\[ \epsilon^{(2)}(k) = \tilde{u}_1(\tilde{k}), \]

(49)

\[ \epsilon^{(3)}(k) = \frac{1}{\sqrt{2}}(\tilde{u}_0(\tilde{k}) - \tilde{u}_0(k)), \]

(50)
\[ e^{(0)}(k) = \frac{1}{\sqrt{2}}(\bar{u}_0(k) + \bar{\bar{u}}_0(k)) \]  

The associated vector form of the polarizations is given by \( \frac{1}{2\sqrt{2}}(1 + \gamma_5)\gamma_0\gamma_\mu \). In fact, the sixteen components of the four vectors \( e^{(\lambda)}(k) \) form a tensor which connect among the two bases. The components are obtained from

\[ e^{(\lambda)} = tr(e^{*(\lambda)}(k)\frac{1}{2\sqrt{2}}(1 + \gamma_5)\gamma_0\gamma_\mu), \]  

where we use the conjugate polarizations

\[ e^{*^{(1)}}(k) = \frac{1}{4}(1 + \gamma_5)(\gamma_1 - i\gamma_2)\gamma_0 \]  
\[ e^{*^{(2)}}(k) = \frac{1}{4}(1 + \gamma_5)(\gamma_1 + i\gamma_2)\gamma_0 \]  
\[ e^{*^{(3)}}(k) = \frac{1}{2\sqrt{2}}(1 + \gamma_5)\gamma_3\gamma_0 \]  
\[ e^{*^{(0)}}(k) = \frac{1}{2\sqrt{2}}(1 + \gamma_5). \]  

For the non-axial vectors of the form \[ \mathcal{B} \] the terms \( \frac{1}{2}\gamma_0\gamma_\mu \) constitute the the vector basis. Indeed, we can use the relation

\[ tr[(\gamma_0\gamma_\mu)(\gamma_0\gamma_\nu)] = tr[(\gamma^\mu\gamma_0)(\gamma_0\gamma_\nu)] = 4g^\mu_\nu \]  

in order to project precisely those components; namely, we should seek

\[ C^\mu_\Psi = tr(\frac{1}{2}\gamma^\mu\gamma_0\Psi) \]  

\( (\gamma_0 \text{ is included to account for the other } \gamma_0 \text{ factor that is included in the solutions}). \)

For solutions \( w_i, \bar{w}_i \) on Tables \[ \text{B} \] \[ \text{C} \] an orthonormal vector basis can be found in the vector interpretation of eq. \[ \text{D} \] which contains them. This is obtained by using, for example, the vectors

\[ b^\mu = i\gamma_0 \frac{1}{2\sqrt{-\Box}}(\gamma^\mu \varphi + \sqrt{2}\varphi^\mu) \]  
\[ b^{\mu*} = -i\frac{1}{2\sqrt{-\Box}}(\gamma^\mu \varphi - \sqrt{2}\varphi^\mu)\gamma_0 \]  

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which also satisfy
\[ tr(b^*_\mu b_\nu) = g_{\mu\nu}, \] (60)
as can be shown by using the relation \( tr(\gamma_\mu \partial\gamma_\nu \varphi) = 4(2\partial_\mu\partial_\nu - g_{\mu\nu}\Box) \). The presence of the \( \partial_\mu \) in these expressions uses the fact that we may generate a vector solution from a scalar by taking the derivative. Given that we have constructed solutions that satisfy Dirac’s equation \[ it \] it follows the solutions will also satisfy the Klein-Gordon equation. This means that when projecting the solutions on vectors \[ \gamma^\mu \gamma^\nu \] these can only be defined as a limiting case as the \( \frac{1}{\sqrt{-\Box}} \) operator is singular when applied on the solutions.

**Gauge invariance**

A clue for the interpretation of all massless solutions described so far is suggested by the fact that eq. \[ [\partial, a(x)] = \varphi a(x), \] is invariant to first order under a set of gauge transformations, that is, with local dependence, which implies some are spurious. In trying to generate this transformation we expect it to be unitary and Lorentz invariant. However, we can only present a transformation satisfying either property but not both together. However, we expect that it satisfy both properties when applied on the space of solutions. This is reminiscent of the QFT case.

We now consider the transformation \( U_G = e^{iG} \) with a similar sense to eq. \[ \gamma_\mu \varphi \] with generator \( G = \varphi a(x) \), and \( a(x) \) any real function, where we use the form \( H \rightarrow \bar{U}^\dagger HU \). When applying the corresponding infinitesimal transformation to the operator \( i\gamma_0 \varphi \) we need to consider only the commutator (or anticommutator, if we take \( i\varphi a(x) \) as the generator) with the Dirac operator, which contains
\[ [\varphi, a(x)]_\pm = \Box a(x) \pm \partial_\mu a(x)\partial_\nu \gamma^\mu \gamma^\nu. \] (61)
The (anti-)commutator with the operator \( a(x) \) \( \varphi \) gives
\[ [\varphi, a(x) \varphi]_\pm = \partial_\mu a(x)\partial_\nu \gamma^\mu \gamma^\nu \pm a(x)\Box. \] (62)
From these equations we see \( U_{G^a} = e^{iG^a} \), where

\[ G^a = \partial a(x) + a(x) \partial \] (63)

(or the transformation with generator \( \partial a(x) - a(x) \partial \)) will be invariant to first order provided \( \Box a(x) + 2\partial_{\mu}a(x)k^\mu = 0 \). Consequently, the symmetry is linked to the space of solutions. We also get a cancellation to second order in \( G^a \) if \( \Box a(x) = 0 \), and \( \partial_{\mu}a(x)k^\mu = 0 \) are satisfied. We note these conditions mean \( a(x) \) satisfies the massless Klein-Gordon equation in a reduced number of directions. Although \( U_{G^a} \) is a Lorentz-invariant operator it is not hermitian.

The term

\[ G_5^b = i[b(x)\gamma_5 \partial + \gamma_5 \partial b(x)] \] (64)

under the conditions \( \Box b(x) = 0 \) and \( \partial_{\mu}b(x)k^\mu = 0 \) is similarly the generator of another symmetry operator \( U_{G_5^b} = e^{iG_5^b} \) since

\[ [\partial, \gamma_5 \partial b(x)]_{\pm} = \gamma_5(-\Box b(x) \pm \partial_{\mu}b(x)\partial_{\nu}\gamma^\mu\gamma^\nu), \] (65)

and

\[ [\partial, b(x)\gamma_5 \partial]_{\pm} = \gamma_5(-\partial_{\mu}b(x)\partial_{\nu}\gamma^\mu\gamma^\nu \pm b(x)\Box). \] (66)

The case with different \( V, A \) contributions is considered below.

We have obtained two sets of local transformations restrained by the condition on the functions \( a(x) \) and \( b(x) \). We may understand this as a manifestation of a gauge symmetry, where we attribute the restriction to the choice of gauge. Indeed, we find a similarity with the gauge invariance of the electromagnetic field \( A_\mu \). The Lorentz gauge condition (Lorentz invariant) for it

\[ \partial^\mu A_\mu = 0 \] (67)

reduces Maxwell’s equations to

\[ \Box A_\mu = 0. \] (68)
In this case the gauge freedom is reduced to transformations $A_\mu \to A_\mu + \partial_\mu \phi$, where $\phi$ satisfies $\Box \phi = 0$. The fact that solutions on Tables 1, 2 and 3, 4 also satisfy the massless Klein-Gordon equation supports the interpretation of these solutions as vector particles satisfying Maxwell’s equation within the Lorentz gauge. In fact, these solutions resemble more the case of the quantized electromagnetic field in this gauge. It is easy to see that this set of solutions does not satisfy eq. 67. However, eq. 2 can be interpreted as implying that the Lorentz condition is satisfied in the mean, a condition required for the quantized electromagnetic field

$$\partial^\mu A_\mu |\psi\rangle = 0,$$  \hfill (69)

where $|\psi\rangle$ describes states from the electromagnetic field. To put eq. 2 in this form we only need to use combined solutions as obtained in eq. 36.

It is interesting that in our case eq. 69 is a condition that we derive and not one that we impose additionally from gauge fixing. We therefore obtain again a connection to QFT. The suggested gauge symmetry also would imply not all the solutions in the fields on Tables 3, 4 are independent, but rather that they could be obtained from the fields on Tables 1, 2 by a gauge transformation. The presence of a gauge symmetry places constraints on the choices of terms in a quantum relativistic equation, as happens in QFT. Thus, here we find a justification for the choice of Lorentz-invariant terms in eq. 2.

**Massive solutions**

In order to describe the solutions of eq. 2 with $M \neq 0$ we choose the rest frame so that they only have time dependence

$$\Psi_{k\bar{i}}^{(+)}(x) = U_i(k)e^{-iMt}$$  \hfill (70)

$$\Psi_{k\bar{i}}^{(-)}(x) = V_i(k)e^{iMt}.$$  \hfill (71)

The matrix components are classified by the eigenvalue of the parity operator $P$ into the $P = -1$ group on Table 3 and $P = 1$ group on Table 4 where the $\tilde{0}$ subscript
Massive bosons \( \gamma_0 \frac{i}{2} \gamma_1 \gamma_2 \) \([H/k_0, \ ] \ [\Sigma \cdot \hat{k}, \ ] \).

\[
U_1(M, 0) = \frac{1}{4}(1 + \gamma_0)(\gamma_1 + i\gamma_2) \quad 1 \quad 1/2 \quad 2 \quad 1
\]
\[
V_1(M, 0) = \frac{1}{4}(1 - \gamma_0)(\gamma_1 + i\gamma_2) \quad -1 \quad 1/2 \quad -2 \quad 1
\]
\[
U_{-1}(M, 0) = \frac{1}{4}(1 + \gamma_0)(\gamma_1 - i\gamma_2) \quad 1 \quad -1/2 \quad 2 \quad -1
\]
\[
V_{-1}(M, 0) = \frac{1}{4}(1 - \gamma_0)(\gamma_1 - i\gamma_2) \quad -1 \quad -1/2 \quad -2 \quad -1
\]
\[
U_0(M, 0) = \frac{1}{4}(1 + \gamma_0)(\gamma_5 - \gamma_3) \quad 1 \quad 1/2 \quad 2 \quad 0
\]
\[
V_0(M, 0) = \frac{1}{4}(1 - \gamma_0)(\gamma_5 + \gamma_3) \quad -1 \quad 1/2 \quad -2 \quad 0
\]
\[
U_\tilde{0}(M, 0) = \frac{1}{4}(1 + \gamma_0)(\gamma_5 + \gamma_3) \quad 1 \quad -1/2 \quad 0 \quad 2 \quad 0
\]
\[
V_\tilde{0}(M, 0) = \frac{1}{4}(1 - \gamma_0)(\gamma_5 - \gamma_3) \quad -1 \quad -1/2 \quad -2 \quad 0
\]

Table 5: Parity \( P = -1 \) massive bosons.

labels the solutions with negative eigenvalue of \( \frac{i}{2} \gamma_1 \gamma_2 \). The solutions are classified with the aid of the normalized mass operator \( H/M = \gamma_0 \) and the helicity \( \frac{i}{2} \gamma_1 \gamma_2 \) (this operator is obtained from the limiting case \( |k| \rightarrow 0 \) in eq. [18]). We note the solutions are mixed components of vector, antisymmetric and scalar components. We can also construct combinations with definite properties under the discrete transformations.

Thus we find vectors \( \gamma_0 \gamma_\mu \), axial vectors \( \gamma_5 \gamma_0 \gamma_\mu \), scalars \( \gamma_0 \), and pseudo-scalars \( \gamma_5 \gamma_0 \). We obtain that the vectors become massive and its longitudinal component becomes physical. Just as in the massless case, an orthogonal polarization basis can be defined.

For the pseudovector, its transverse and longitudinal components are not physical.

On the other hand, the condition that they all belong to a vector representation forces us to assume the antisymmetric and scalar terms are in fact derivatives as in the massless case. We note also there remain two vector components constructed
Massive bosons

\[ \gamma_0 \frac{i}{2} \gamma_1 \gamma_2 [H/k_0, ] [\Sigma \cdot \hat{k}, ] \]

\[
\begin{align*}
\bar{U}_1(M, 0) &= \frac{1}{4} \gamma_5 (1 - \gamma_0) (\gamma_1 + i \gamma_2) \quad 1 \quad 1/2 \quad 0 \quad 1 \\
\bar{V}_1(M, 0) &= \frac{1}{4} \gamma_5 (1 + \gamma_0) (\gamma_1 + i \gamma_2) \quad -1 \quad 1/2 \quad 0 \quad 1 \\
\bar{U}_{-1}(M, 0) &= \frac{1}{4} \gamma_5 (1 - \gamma_0) (\gamma_1 - i \gamma_2) \quad 1 \quad -1/2 \quad 0 \quad -1 \\
\bar{V}_{-1}(M, 0) &= \frac{1}{4} \gamma_5 (1 + \gamma_0) (\gamma_1 - i \gamma_2) \quad -1 \quad -1/2 \quad 0 \quad -1 \\
\bar{U}_0(M, 0) &= \frac{1}{4} \gamma_5 (1 - \gamma_0) (\gamma_5 + \gamma_3) \quad 1 \quad 1/2 \quad 0 \quad 0 \\
\bar{V}_0(M, 0) &= \frac{1}{4} \gamma_5 (1 + \gamma_0) (\gamma_5 - \gamma_3) \quad -1 \quad 1/2 \quad 0 \quad 0 \\
\bar{U}_0(M, 0) &= \frac{1}{4} \gamma_5 (1 - \gamma_0) (\gamma_5 - \gamma_3) \quad 1 \quad -1/2 \quad 0 \quad 0 \\
\bar{V}_0(M, 0) &= \frac{1}{4} \gamma_5 (1 + \gamma_0) (\gamma_5 + \gamma_3) \quad -1 \quad -1/2 \quad 0 \quad 0
\end{align*}
\]

Table 6: Parity \( P = 1 \) massive bosons.

without internal spin, that is, constructed from derivatives of scalar particles. This structure is reminiscent of the Higgs mechanism, in which massive vector fields absorb scalar degrees of freedom due to breaking of the symmetry.

4 Massless spin 1/2 particles

We now show it is possible to give a Lorentz transformation which describes fermions too. This is done more naturally in the context of a matrix equation of the type of eq. and whose solutions are bosons and fermions. This constitutes progress in the task of giving a unified description of these fields. Indeed, we obtain solutions that under Lorentz transformations of the form one of the sides transforms trivially, and therefore, we get spin 1/2 objects transforming as the (1/2, 0) or (0, 1/2) representations
of the Lorentz group.

The equation

\[(1 - \gamma_5)i\gamma_0\partial_\mu\gamma^\mu\Psi = 0 \quad (72)\]

has this type of solutions. The invariance algebra of this equation contains the Lorentz generator

\[J^{-\mu\nu} = \frac{1}{2}(1 - \gamma_5)J_{\mu\nu} = \frac{1}{2}(1 - \gamma_5)[i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\sigma_{\mu\nu}] \quad (73)\]

(and the other Poincaré generators). Among the solutions of eq. \((72)\) we have again the V-A vectors \(u_i\) on Table 1 which under the effect of \(\frac{1}{2}(1 - \gamma_5)\Sigma \cdot \hat{k}\) and \(H/k_0 = \frac{1}{2}(1 - \gamma_5)\gamma_0\gamma_3\) lead to the same quantum numbers.

The Dirac operator in eq. \((72)\) is defined on a \(2 \times 2\) matrix space; nevertheless, the solutions lie in the larger \(4 \times 4\) matrix space. It is precisely this structure which leads to a set of solutions classified as spin \(1/2\) particles under \(J^{-\mu\nu}\). Actually, we have as additional symmetry of eq. \((72)\) the group of linear complex transformations \(G(2, C)\) with eight components, generated by

\[f_{\mu\nu} = -\frac{i}{2}(1 + \gamma_5)\sigma_{\mu\nu}, \quad (74)\]

where \(\sigma_{\mu\nu}\) is given in eq. \((13)\). This means eq. \((72)\) has a freedom in the choice of the Lorentz transformation since, e.g., both \(J^{-\mu\nu}\) and \(J_{\mu\nu}\) are possible ones. The \(w_i\) terms on Table 3 are also a set of solutions of eq. \((72)\). However, their interpretation changes to fermions when using \(J^{-\mu\nu}\) to classify them. Clearly, the nature of the solutions depends on the Hamiltonian and the set of transformations that we choose to classify them. But once our choice is made, there is no ambiguity.

The unitary subgroups \(SU(2) \times U(1)\) of the \(G(2, C)\) symmetry operators in eq. \((74)\) imply we have two additional quantum numbers we can assign to the solutions. In consideration that this symmetry does not act on the vector solution part, and taking account of the known quantum numbers of fermions in nature we shall associate these
Left-handed spin $1/2$ particles

$$\frac{1}{2}(1 - \gamma_5)\gamma_0\gamma_3 \quad \frac{i}{4}(1 - \gamma_5)\gamma_1\gamma_2 \quad [f_{30}, \ ]$$

| Spin State | Operator | Flavor | Lepton Number |
|------------|----------|--------|---------------|
| $w_{-1/2}(k)$ | $\frac{1}{4}(1 - \gamma_5)(\gamma_0 + \gamma_3)$ | 1 | $-1/2$ |
| $\hat{w}_{-1/2}(k)$ | $\frac{1}{4}(1 - \gamma_5)(\gamma_1 + i\gamma_2)$ | $-1$ | $1/2$ |
| $\hat{w}_{-1/2}(\tilde{k})$ | $\frac{1}{4}(1 - \gamma_5)(\gamma_1 - i\gamma_2)$ | 1 | $-1/2$ |
| $\hat{w}_{-1/2}(\tilde{k})$ | $\frac{1}{4}(1 - \gamma_5)(\gamma_0 - \gamma_3)$ | $-1$ | $1/2$ |

Table 7: Massless fermions.

operators with flavor and lepton number respectively. The $SU(2)$ set of operators in eq. 74 leads to a flavor doublet. The $U(1)$ is in this case not independent from the chirality. Choosing among the generators of $SU(2)$ $f_{30}$ to classify the solutions of eq. 72, these are given on Table 7, where the hat is used to distinguish the flavor and in this case the product and commutator of $H$ and $\frac{1}{2}(1 - \gamma_5)\Sigma \cdot \hat{k}$ give the same results.

Eq. 39 can be used to obtain the full coordinate dependence. As in the Weyl equation we obtain solutions of a defined chirality or helicity. We have also negative energy solutions of the form $44$ whose hermitian conjugates are interpreted as right-handed antiparticles. The latter could have been obtained by departing from an equation with $V + A$ solutions. In order to have a Dirac fermion and a fermion mass we need to have an equation mixing both chirality solutions. These shall be obtained in Sections VII and VIII.

**Gauge invariance**

We prove eq. 72 has a gauge symmetry in the limiting case of $\alpha_+ \to 0$ in

$$[\alpha_+ \frac{1}{2}(1 + \gamma_5) + \alpha_- \frac{1}{2}(1 - \gamma_5)]i\gamma_0 \partial_\mu \gamma^\mu \Psi = 0. \quad (75)$$

Using eqs. 61, 62 and 63, 64, and commutation relations with $i\gamma_5\gamma_0 \partial_\mu \gamma^\mu$ it can be
shown

\[(1 - \gamma_5) \partial, (1 + \gamma_5) \partial b(x) \] = 2[\partial, \partial b(x)]_+ - 2\gamma_5[\partial, \partial b(x)]_+ \tag{76}

\[(1 + \gamma_5) \partial, (1 - \gamma_5) \partial b(x) \] = 2[\partial, \partial b(x)]_- + 2\gamma_5[\partial, \partial b(x)]_+ \tag{77}

\[(1 + \gamma_5) \partial, (1 - \gamma_5) \partial b(x) \] = 2[\partial, \partial b(x)]_- + 2\gamma_5[\partial, \partial b(x)]_+ \tag{78}

\[(1 + \gamma_5) \partial, (1 - \gamma_5) \partial b(x) \] = 2[\partial, \partial b(x)]_- + 2\gamma_5[\partial, \partial b(x)]_+ \tag{79}

The application of the symmetry generator

\[\bar{G}^b = [\alpha_-(1 + \gamma_5) + \alpha_+(1 - \gamma_5)]G^b \tag{80}\]

on eq. 75, where \(G^a\) is given in eq. 63, produces the terms in eqs. 76-79 and cancels the anticommutator contributions. Therefore, we obtain \(\bar{G}^b\) will be a generator of a gauge symmetry of eq. 75 in the sense explained before. In fact, in the limit \(\alpha_+ \to 0\) the symmetry is satisfied to first order since all other terms in \(e^{i\bar{G}^b}\) cancel. Unlike the case of eq. 73, we note only one gauge symmetry is allowed.

5 Currents and vertex interaction

From Noether’s theorem we have a conserved current for each continuous symmetry present in the system. We can construct, using eqs. 2, 6, bilinear current vector operators \(j_\mu\), based on eq. 4, and current vector expectation values \(tr(j_\mu)\), based on eq. 5, satisfying

\[\partial^\mu j_\mu = 0, \quad \partial^\mu tr(j_\mu) = 0. \tag{81}\]

The form of the \(j_\mu\) is similar to the currents in Dirac’s equation, given that some symmetries are shared by both Dirac equation and eqs. 2, 6. In the case of eqs. 2, 25, and 72 we also have the global symmetry \(\Psi \to e^{ia}\Psi\), where \(a\) is a real parameter. The corresponding current operator is

\[j_\mu = \Psi^\dagger \gamma_\mu \gamma_5 \Psi. \tag{82}\]
This symmetry implies conservation of number of particles, with the zero component of the current being positive definite so that it can be interpreted as a probability density. This component has already been considered when setting a normalization condition in eq. 30.

The symmetry $\Psi \rightarrow e^{i b \gamma_5} \Psi$, with $b$ real, is valid for the massless equations 25 and 72 and leads to the chirality current

$$j_\mu^5 = \Psi^\dagger \gamma_5 \gamma_0 \gamma_\mu \Psi.$$  \hspace{1cm} (83)

Expressions can be obtained also for the currents corresponding to the energy-momentum tensor and the generalized angular momentum which are equal to those obtained for the Dirac equations. It is these which underlie the classification of the solutions with $H$ and $\Sigma \cdot \hat{k}$. This partly justifies as well the classification done with the commutators of these operators and the solutions, given that they are also eigenfunctions under them.

The current operator corresponding to the gauge symmetry in eq. 75 (which overlaps with the above currents $j_\mu$, $j_\mu^5$) is given by

$$j_\mu^{\text{gau}} = \Psi^\dagger \frac{1}{2} (1 - \gamma_5) \gamma_0 \gamma_\mu \Psi$$  \hspace{1cm} (84)

(we take here the bra $\Psi'^\dagger$ possibly distinct from the ket $\Psi$). Comparison of the current $j_\mu^{\text{gau}}$ with the form of the field

$$A^-_\mu(k) = \frac{i}{2\sqrt{2}(1 - \gamma_5) \gamma_0 \gamma_\mu e^{-ikx}}$$  \hspace{1cm} (85)

derived from the $u_i$ terms on Table 3 strongly suggests a connection to the transition matrix element of the $A^-_\mu$ operator field between the two massless fermion solutions $\Psi'$ and $\Psi$

$$\langle \Psi' | A^-_\mu(k) | \Psi \rangle.$$  \hspace{1cm} (86)
Indeed, in QFT the minimal coupling $\mathcal{L} = g\gamma^\mu A^-\mu$ in the Lagrangian implies a vertex interaction which can lead to the expectation value of the form

$$g\frac{1}{2}(1 - \gamma_5)\gamma_0\gamma_\mu|\alpha\beta \rightarrow g\langle u(p_f, s_f)|\frac{1}{2}(1 - \gamma_5)\gamma_0\gamma_\mu|u(p_i, s_i)\rangle,$$

where $(p_{i,f}, s_{i,f})$ are the initial and final momenta and spins of the fermions, $k_\mu$ is the momentum of the vector field, and $g$ is the coupling constant. A consistent interpretation of eqs. 84-86 is possible along these lines by understanding $\langle \Psi'|A^-_\mu(k)|\Psi \rangle$ as an interaction with its assignment to the vertex in eq. 87 and the coupling constant $g = \frac{1}{\sqrt{2}}$.

(A more formal argument should take account of the exponential factor in eq. 85 which can be done in the context of QFT; it would lead, together with the space-time dependence of the fermion wave functions to Dirac’s delta $(2\pi)^4 \delta^4(k_i - p_f + p_i)$. Also, the substitution 87 is one of many ways to obtain a contribution in a perturbation expansion in terms of diagrams. Although $A^-_\mu(k)$ should be properly normalized as a field of units of [energy] it is enough for our argument to keep the polarization normalized).

6 **Lorentz (3, 1) Structure and Scalars from 6-d Clifford Algebra**

In previous sections we have derived a description of fermions and bosons through equations implied by the structure of the Clifford algebra in $d = 4$. Although the structure obtained is too simple to describe thoroughly aspects of the SM (for example, the model cannot include massive fermions) we have useful results which we would like to keep as the prediction of interactions in the form of vertices relating vectors and fermions, coupling constants, and in particular, hints of a description of isospin on left-handed particles. These features are expected to remain in higher dimensions, where we find a more elaborate structure.
The simplest generalization of the above model is to consider the six-dimensional Clifford algebra, (the \( d = 5 \) lives also in a \( 4 \times 4 \) space). This is composed of \( 64 \ 8 \times 8 \) matrices and it can be obtained as a tensor product of the original \( 4 \times 4 \) algebra and the \( 2 \times 2 \) matrices generated by the unit matrix \( 1_2 \) and the three Pauli matrices, \( \sigma_1, \sigma_2, \sigma_3 \). We will use a basis for the \( 8 \times 8 \) matrix space in which we can identify the underlying \( d = 4 \) components

\[
\gamma_0 \rightarrow \gamma'_0 = 1_2 \otimes \gamma_0, \ \gamma_1 \rightarrow \gamma'_1 = 1_2 \otimes \gamma_1, \ \gamma_2 \rightarrow \gamma'_2 = \sigma_1 \otimes \gamma_2, \ \gamma_3 \rightarrow \gamma'_3 = 1_2 \otimes \gamma_3. \tag{88}
\]

Then

\[
1_8, \ \sigma_1 \otimes 1_4, \ \gamma'_5 = \sigma_2 \otimes \gamma_2, \ \gamma'_6 = \sigma_3 \otimes \gamma_2 \tag{89}
\]

are 4-\( d \) scalars since they commute with the spin operators

\[
\sigma'_{\mu \nu} = \frac{i}{2} [\gamma'_\mu, \gamma'_\nu], \ \mu = 0, ..., 3, \ \nu = 0, ..., 3. \tag{90}
\]

In fact, the matrices \( \gamma'_\mu \) defined in eqs. (88) and (89) form the 6-\( d \) Clifford algebra

\[
\{ \gamma'_\mu, \gamma'_\nu \} = 2g_{\mu \nu}. \tag{91}
\]

As all \( \gamma_\mu \) are generalized to \( 8 \times 8 \) matrices through a tensor product \( \gamma_\mu \rightarrow 1_2 \otimes \gamma_\mu, \ \mu = 0, ..., 3 \), without danger of ambiguity we shall use a notation in which we now assume that \( \gamma_\mu \) represent \( 8 \times 8 \) matrices. We also use the quaternion-like notation for the representation of \( 1/2 \) and the Pauli matrices in the \( 8 \times 8 \) matrix space

\[
1_8 = 1_2 \otimes 1_4, \ I = \sigma_1 \otimes 1_4, \ J = \sigma_2 \otimes 1_4, \ K = \sigma_3 \otimes 1_4. \tag{92}
\]

The 4-\( d \) algebra will be written in terms of

\[
\gamma'_\mu = \gamma_\mu, \ \mu = 0, 1, 3 \ \gamma'_2 = I\gamma_2 \tag{93}
\]

and the scalars in eq. (89) (here in Hermitian form) in terms of

\[
1 = 1_8, \ I, \ i\gamma'_5 = iJ\gamma_2, \ i\gamma'_6 = iK\gamma_2. \tag{94}
\]
Because $I$, $J$, $K$ commute with $\gamma_2$, it is possible to omit the tensor product sign. A more explicit form of these matrices can be found in the appendix. Then, all 64 elements of the $8 \times 8$ algebra are obtained by multiplying the 16 elements of the 4-\textit{d} algebra generated by the terms in eqs. 93 and 94, and they can be written with this notation. Hence, it will be possible to identify every element constructed in this way in terms of the 4-\textit{d} Lorentz representation it belongs to.

The preceding definitions will also be applied for the assignment $\gamma_5 \to 1_2 \otimes \gamma_5 \equiv \gamma_5$. Then, besides the scalar elements of eq. 89 (or eq. 94), we have the scalars

$$\gamma_5, \ I\gamma_5, \ J\gamma_2\gamma_5, \ K\gamma_2\gamma_5.$$  (95)

From these, $I\gamma_5$ commutes with these and the scalar elements in eq. 94. Excluding it and the identity, the remaining six elements generate an $SO(4)$ algebra, or equivalently, an $SU(2) \times SU(2)$ algebra. The latter’s generators consist of the right-handed elements

$$\frac{1}{4}(1 + I\gamma_5)I, \ \frac{i}{4}(1 + I\gamma_5)J\gamma_2, \ \frac{i}{4}(1 + I\gamma_5)K\gamma_2,$$  (96)

and left-handed elements

$$I_1 = \frac{i}{4}(1 - I\gamma_5)J\gamma_2 \quad (97)$$

$$I_2 = -\frac{i}{4}(1 - I\gamma_5)K\gamma_2 \quad (98)$$

$$I_3 = -\frac{1}{4}(1 - I\gamma_5)I. \quad (99)$$

The eight form an $SU(2)_R \times SU(2)_L \times U(1) \times U(1)$, where the subscripts $L$ and $R$ are added accordingly (the normalization is chosen to fit $\frac{1}{2}\sigma_i$).

**Chain breaking of $d = 6$ Algebra**

The above symmetry operators immediately show a close connection to the actual symmetries observed in nature at the massless level, that is, the $SU(2)_L$ of isospin and $U(1)_Y$ of hypercharge groups. The eight scalars in eqs. 94 and 93, have a Cartan
algebra of dimension four, for which we can take the basis 1, \( I, \gamma_5, I\gamma_5 \). These operators can be arranged into the four projection operators

\[
\begin{align*}
P_{++} &= \frac{1}{4}(1 + I\gamma_5)(1 + I) \\
P_{+-} &= \frac{1}{4}(1 + I\gamma_5)(1 - I) \\
P_{-+} &= \frac{1}{4}(1 - I\gamma_5)(1 + I) \\
P_{--} &= \frac{1}{4}(1 - I\gamma_5)(1 - I)
\end{align*}
\]

which, when combined with the Dirac operator, create the general massless Lorentz-invariant equation

\[
(\alpha_{++} P_{++} + \alpha_{+-} P_{+-} + \alpha_{-+} P_{-+} + \alpha_{--} P_{--}) \gamma_0 (i\partial_\mu \gamma_\mu') \Psi = 0 \quad \mu = 0, \ldots, 3.
\]

We then have four different Lorentz-invariant degrees of freedom \( \alpha_{++}, \alpha_{+-}, \alpha_{-+}, \alpha_{--} \), for constructing a generalized equation. One or various vanishing coefficients lead to degrees of freedom disappearing from the spectrum. In fact, the choice of non-vanishing coefficients divides this equation into four classes. For each class we assume that all fields transform under the same Lorentz representation. Additional conditions on the coefficients might lead to more symmetries to appear. The different choices are as follows:

In class I, only one coefficient is non-vanishing, e.g. \( \alpha_{-+} \neq 0 \), and \( \alpha_{++} = \alpha_{+-} = \alpha_{--} = 0 \) (we will not consider the different four permutations of the \( \alpha_{ij} \) belonging to this class and others, which have similar properties). Without loss of generality here and in similar cases, we may assume \( \alpha_{-+} = 1 \). This type of equation is similar to eq. 72, except that in this case, in addition to the \( U(1) \) gauge symmetry generated by \( P_{-+} \), we have a flavor \( SU(6) \), whose elements are projected by \( P_{++} + P_{-+} + P_{--} \).

In class II, in which two \( \alpha_{ij} \) vanish, we have in general at least a \( U(1) \) gauge symmetry and a \( SU(4) \) flavor symmetry. Furthermore, we consider three possibilities
for choices of the $\alpha_{ij}$. In the case $\alpha_{-+} = \alpha_{--} \neq 0$ we have in particular a $U(1)_L \times SU(2)_L$ gauge symmetry. In this case, both fermions and vectors are obtained in the spectrum, but the fermions are all left-handed as for solutions on Table 4, and their antiparticles right-handed. The cases $\alpha_{++} \neq 0, \alpha_{-+} \neq 0$, or $\alpha_{++} \neq 0, \alpha_{--} \neq 0$ resemble eq. 73 and break any possible gauge $SU(2)$ symmetry.

For class III, where only one $\alpha_{ij} = 0$, we have in general an $SU(2)$ flavor symmetry, and three gauge $U(1)$s. In the case, $\alpha_{-+} = \alpha_{--}$ instead of one $U(1)$ we have also a gauge $SU(2)_L$ symmetry; by setting e.g. $\alpha_{-+} = \alpha_{+-}$ or $\alpha_{--} = \alpha_{+-}$ we get an equation which has a projection of the form of eq. 25, that is, with parity as a symmetry, a condition necessary to have a solution of the form of an electromagnetic field. The representations contain both vectors and fermions which are both left-handed and right-handed.

Finally, for class IV, in the case $\alpha_{++} = \alpha_{+-} = \alpha_{-+} = \alpha_{--}$ we have a gauge $U(2)_L \times U(2)_R$ and the representations only contain vectors of the type appearing on Tables 1, 2 and 3, 4. There is a possibility of finding a similar description to that of class III if we use a Lorentz transformation projected with $L = P_{+-} + P_{-+} + P_{--}$. This case will not be considered.

From the four choices described it is type III (or type IV under the condition stated) with $\alpha_{-+} = \alpha_{--} = \alpha_{+-}$ which can be parity conserving and which contains an $SU(2)_L$ symmetry. We shall associate this group with the isospin and one $U(1)$ with the hypercharge. This case is analyzed in detail in the following section.

7 Massless case: Type III spectrum, Unified $SU(2) \times U(1)$

We analyze the equation

$$iL\gamma_0 \partial^\mu \gamma^\prime_\mu \Psi = 0, \quad \mu = 0, ..., 3,$$

(105)
where we use the projection operator

\[ L = P_{++} + P_{--} + P_{+-} = \frac{3}{4} - \frac{1}{4}(I + \gamma_5 + I \gamma_5), \]  

which corresponds to the type III case with \( \alpha_{+-} = \alpha_{-+} = \alpha_{--} \). The equation is invariant under the set of Lorentz transformations

\[ J^L_{\mu\nu} = L[i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma'_{\mu\nu}], \]  

where \( \sigma'_{\mu\nu} \) is defined in eq. 90. The scalar symmetries are classified into flavor, with its generators projected by \( P_{++} \), an \( SU(2)_L \) gauge symmetry, and two other \( U(1) \) gauge, according to the arguments following eq. 75. We choose one generator of \( SU(2)_L \) to classify the solutions, say, \( I_3 \) in eq. 99, with eigenvalue \( I_{s3} \). The other two \( U(1) \) gauge generators, are chosen orthogonal to \( I_3 \) and will be taken

\[ Y = -1 + \frac{1}{2}(I + \gamma_5) \]  

and \( \tilde{\gamma}_5 = L\gamma_5 \). \( \tilde{\gamma}_5 \) is orthogonal to \( Y \) and \( I_3 \) in the sense of \( tr(\tilde{\gamma}_5 Y) = 0 \), and \( tr(\tilde{\gamma}_5 I_3) = 0 \). The choice in eq. 108 can be obtained from the demand that the operator lead to a gauge symmetry in the sense of eq. 80. Another justification for these definitions will be given later. Although we call it gauge we still need to prove that a chiral symmetry as the \( I_i \) actually is but we shall make this assumption.

There are also global symmetries which are related to particle number conservation. \( L \) is interpreted as the lepton number, whose quantum number we denote by \( l \). The Casimir of \( SU(2)_L \) \( I_1^2 + I_2^2 + I_3^2 \), with eigenvalue \( I_s(I_s + 1) \), is not an independent component, but a linear combination of \( \tilde{\gamma}_5 \) and \( Y \).

We present the fermion and boson solutions of eq. 105 which we classify according to the Hamiltonian and helicity projections, \( L\gamma_0 \gamma^3, \frac{i}{2}LI_1 \gamma_2 \), the generator \( I_3 \), and the quantum numbers, \( Y, I_s \). We also define the flavor as \( f_{30} = \frac{1}{2}(1 - L)\gamma_3 \gamma_0 \) and its eigenvalue \( f \).

**Spin 1/2 particles**
\[ L_{\gamma_0 \gamma^3} \frac{i}{2} L I \gamma_1 \gamma_2 \quad I_3 \]

\[
\left( \begin{array}{c}
\nu_{-1/2}(k) \\
l_{-1/2}(k)
\end{array} \right)_L = \left( \begin{array}{c}
\frac{1}{8}(1 - I \gamma_5)(J \gamma_2 - iK \gamma_2)(\gamma_0 + \gamma_3) \\
\frac{1}{8}(1 - I \gamma_5)(1 + I)(\gamma_0 + \gamma_3)
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\nu_{-1/2}(k) \\
l_{-1/2}(k)
\end{array} \right)_L = \left( \begin{array}{c}
\frac{1}{8}(1 - I \gamma_5)(J \gamma_2 - iK \gamma_2)(\gamma_1 + iI \gamma_2) \\
\frac{1}{8}(1 - I \gamma_5)(1 + I)(\gamma_1 + iI \gamma_2)
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\hat{\nu}_{-1/2}(k) \\
\hat{l}_{-1/2}(k)
\end{array} \right)_L = \left( \begin{array}{c}
\frac{1}{8}(1 - I \gamma_5)(J \gamma_2 - iK \gamma_2)(\gamma_1 - iI \gamma_2) \\
\frac{1}{8}(1 - I \gamma_5)(1 + I)(\gamma_1 - iI \gamma_2)
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\hat{\nu}_{-1/2}(k) \\
\hat{l}_{-1/2}(k)
\end{array} \right)_L = \left( \begin{array}{c}
\frac{1}{8}(1 - I \gamma_5)(J \gamma_2 - iK \gamma_2)(\gamma_0 - \gamma_3) \\
\frac{1}{8}(1 - I \gamma_5)(1 + I)(\gamma_0 - \gamma_3)
\end{array} \right)
\]

Table 8: \( l = 1, I_s = 1/2, Y = -1 \) massless fermion multiplets in 5+1 d.

The \( l = 1, I_s = 1/2, Y = -1 \) massless fermions are given on Table 8, where the subscript \(-1/2\) refers to the value of the helicity operator \( [\Sigma \cdot \mathbf{p} \cdot \mathbf{]}, \) so that we also present particles with opposite momentum, and the index \( L \) denotes (in a redundant way) the left-handed character of the solutions. The space-time dependence of these solutions can be obtained also following eq. 39. Negative energy solutions are obtained by changing the sign of the exponential and the hermitian conjugates of the latter are the antiparticle solutions.

These spin 1/2 particles belong to the fundamental representation of the non-abelian group \( SU(2)_L \) and are labeled also by the \( Y \) operator. In consideration of the quantum numbers of leptons in nature, it follows we can associate \( Y \) with the hypercharge and the \( I_i \) with the three generators of isospin. We also associate the two elements distinguished only by the \( f \) quantum number (and a hat) with a flavor doublet which we identify with any two lepton families among the three generations, e. g., the left-handed electron and muon and their neutrinos.
Right-handed massless spin $1/2$ particles

\[ L\gamma_0\gamma^3 \frac{i}{2} LI\gamma_1\gamma_2 \quad [f_{30}] \]

\[ l_{1/2R}(k) = \frac{1}{8}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)\gamma_0(\gamma_1 + iI\gamma_2) \quad 1 \quad \frac{1}{2} \quad \frac{1}{2} \]

\[ \tilde{l}_{1/2R}(\tilde{k}) = \frac{1}{8}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)\gamma_0(\gamma_0 + \gamma_3) \quad -1 \quad -\frac{1}{2} \quad 1/2 \]

\[ \hat{l}_{1/2R}(k) = \frac{1}{8}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)\gamma_0(\gamma_0 - \gamma_3) \quad 1 \quad 1/2 \quad -1/2 \]

\[ \hat{\tilde{l}}_{1/2R}(\tilde{k}) = \frac{1}{8}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)\gamma_0(\gamma_1 - iI\gamma_2) \quad -1 \quad -1/2 \quad -1/2 \]

Table 9: \( l = 1, I_s = 0, Y = -2 \) massless fermion multiplets in 5+1 d.

Another part of the spectrum has positive chirality, \( l = 1, I_s = 0, Y = -2 \) and is given on Table 9, where antiparticles can be obtained with the corresponding transformations, and as in previous cases, the solutions presented can be obtained from each other by a rotation. The quantum numbers correspond to right-handed charged leptons, as we will show, again in good correspondence with the SM.

Vectors

The pure vector solutions are similar to the \( u_i, \tilde{u}_i \) terms in Tables 1, 2. The isospin scalars can be separated into their \( V^+A \) and \( V^-A \) components (all have lepton number \( l = 0 \), as required). The first are given on Table 10. Comparing these solutions with those on Table 2, we see they differ by the substitution \( P_{+}^{+} \) and the projector \( P_{+-} \). Similarly, the \( V^-A \) terms can be obtained straightforwardly from Table 1 and the projector \( P_{-+} + P_{--} \). These are given on Table 11. Taking account of the normalization, a combination of the terms \( B_i \) and \( \tilde{B}_i \) can be taken which carries the hypercharge \( Y \) in eq. 108. We shall associate this combination with the \( B_\mu \) fields which carry the hypercharge in the Weinberg-Salam model (Glashow, 1961, Salam, 1968, Weinberg, 1967).

Three additional sets of solutions of the equation can be described in terms of the
Vector solutions

\[ H/k_0, \quad [\Sigma \cdot \hat{p}, \ ] \]

\[ \tilde{B}_1(k) = \frac{1}{8}(1 + I\gamma_5)(1 - I)\gamma_0(\gamma_1 + i I\gamma_2) \quad 2 \quad 1 \]

\[ \tilde{B}_1(\tilde{k}) = \frac{1}{8}(1 + I\gamma_5)(1 - I)\gamma_0(\gamma_1 - i I\gamma_2) \quad 2 \quad 1 \]

\[ \tilde{B}_0(k) = \frac{1}{8}(1 + I\gamma_5)(1 - I)\gamma_0(\gamma_0 - \gamma_3) \quad 0 \quad 0 \]

\[ \tilde{B}_0(\tilde{k}) = \frac{1}{8}(1 + I\gamma_5)(1 - I)\gamma_0(\gamma_0 + \gamma_3) \quad 0 \quad 0 \]

Table 10: \( I_s = 0, \ Y = 0, \ V+A \) vectors in 5+1 d.

fields \( B_i \) on Table 11 and the generators of \( SU(2)_L \) in eqs. 97-99, which are written in a spherical basis on Table 12. As these \( V-A \) vector solutions belong to the adjoint representation of group \( SU(2)_L, \ I_s = 1, \) we associate them with the fields \( W_{\mu}^{\pm}, W_{\mu}^{0} \) of the electroweak theory.

Scalars and antisymmetric tensors

The last part of the boson spectrum is composed of scalar and antisymmetric \( Y = -1 \) doublets (and antiparticles). The solutions are constructed similarly to the \( \tilde{w}_i \) components on Table 4 with the addition of the factors containing \( I, \ J\gamma_2, \ K\gamma_2, \) which account for the hypercharge and isospin quantum numbers. The corresponding \( Y = -1 \) doublets are given on Table 13. The same problems arise regarding the Lorentz interpretation of antisymmetric terms as for Tables 3, 4. The same procedure in extracting from these solutions vector and scalar components can be used. Again here there is a parallelism with the SM. A scalar particle appears in a doublet and we will see it is involved in giving masses to the particles. For this reason, we may associate this degree of freedom with the Higgs particle. We leave open the question of whether these mass terms can be obtained from a gauge transformation, although the form of the proposed gauge transformation here suggests it should be possible.
Vector solutions \([H/k_0, \Sigma \cdot \hat{p}, \Sigma] \]

\[
B_{-1}(k) = \frac{1}{4\sqrt{2}}(1 - I\gamma_5)\gamma_0(\gamma_1 - i\gamma_2) \quad 2 \quad -1
\]

\[
B_{-1}(\tilde{k}) = \frac{1}{4\sqrt{2}}(1 - I\gamma_5)\gamma_0(\gamma_1 + i\gamma_2) \quad 2 \quad -1
\]

\[
B_0(k) = \frac{1}{4\sqrt{2}}(1 - I\gamma_5)\gamma_0(\gamma_0 - \gamma_3) \quad 0 \quad 0
\]

\[
B_0(\tilde{k}) = \frac{1}{4\sqrt{2}}(1 - I\gamma_5)\gamma_0(\gamma_0 + \gamma_3) \quad 0 \quad 0
\]

Table 11: \(I_s = 0, Y = 0, V-A\) vectors in \(5+1\) d.

Isospin vector carriers

\[
W^+_i(k) = \frac{1}{\sqrt{2}}(J\gamma_2 - iK\gamma_2)B_i(k) \quad 1
\]

\[
W^0_i(k) = IB_i(k) \quad 0
\]

\[
W^-_i(k) = \frac{1}{\sqrt{2}}(J\gamma_2 + iK\gamma_2)B_i(k) \quad -1
\]

Table 12: Isospin triplet vector bosons in \(5+1\) d.

Summarizing, the positive energy solutions are the vectors \(B_i\) and \(\tilde{B}_i\) which amount to eight degrees of freedom, where we are taking account of both directions of momenta for given helicity. The isospin vectors \(W_i^{\pm,0}\) have twelve degrees of freedom and the antisymmetric tensors and scalars \(\tilde{n}_i\) and \(\tilde{v}_i\) eight, and with their antiparticles sixteen. These add up to thirty-six bosons. We have obtained massless spin \(1/2\) particles in a doublet and a singlet; these use four and two degrees of freedom respectively. Taking account of antiparticles and the two flavors we have twenty-four fermion degrees of freedom. Altogether, these add up to sixty degrees of freedom. The reason for not having sixty-four active is the four inert degrees of freedom projected by \(P_{++}\), which are not influenced by the Hamiltonian, projected by \(L\).
Scalars and antisymmetric tensors

\[
\begin{bmatrix}
[\bar{H}/k_0,] & [\bar{\Sigma} \cdot \bar{p},] & I_3
\end{bmatrix}
\]

\[
\begin{pmatrix}
\bar{n}_0(k) \\
\bar{\bar{n}}_0(k)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{8}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)(\gamma_0 + \gamma_3) \\
\frac{1}{8}(1 + I\gamma_5)(1 - I)(\gamma_0 + \gamma_3)
\end{pmatrix}
\]

2 0 1/2 -1/2

\[
\begin{pmatrix}
\bar{n}_0(k) \\
\bar{\bar{n}}_0(k)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{8}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)(\gamma_0 - \gamma_3) \\
\frac{1}{8}(1 + I\gamma_5)(1 - I)(\gamma_0 - \gamma_3)
\end{pmatrix}
\]

2 0 1/2 -1/2

\[
\begin{pmatrix}
\bar{n}_1(k) \\
\bar{\bar{n}}_1(k)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{8}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)(\gamma_1 + iI\gamma_2) \\
\frac{1}{8}(1 + I\gamma_5)(1 - I)(\gamma_1 + iI\gamma_2)
\end{pmatrix}
\]

0 1 1/2 -1/2

\[
\begin{pmatrix}
\bar{n}_1(k) \\
\bar{\bar{n}}_1(k)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{8}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)(\gamma_1 - iI\gamma_2) \\
\frac{1}{8}(1 + I\gamma_5)(1 - I)(\gamma_1 - iI\gamma_2)
\end{pmatrix}
\]

0 1 1/2 -1/2

Table 13: \( I_s = 1/2, \ Y = -1, \) boson chiral terms in 5+1 \( d. \)

8 Massive case: Symmetry breaking of \( SU(2) \times U(1) \)

In seeking a massive extension of eq. [107] we expect all the hermitian combinations of the scalar terms in eqs. [94] and [95], multiplied by \( \gamma_0 \) (in a Hamiltonian form of the equation), to be scalars with respect to the Lorentz transformation

\[
J'_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2}\sigma'_{\mu\nu},
\]

which just generalizes eq. [4]. However, if we also demand that they be scalars with respect to \( J^L_{\mu\nu} \) in eq. [107], then the choices are reduced to

\[
M_1 = \frac{M}{2}(1 - I),
\]

\[
M_2 = \frac{iM}{2}(\gamma_5 - I\gamma_5),
\]

\[
M_3 = -\frac{M}{2}J\gamma_2(1 + \gamma_5),
\]

\[
M_4 = \frac{M}{2}K\gamma_2(1 + \gamma_5),
\]

where \( M \) is the mass constant. Now, the only non-trivial scalar that commutes with all \( M_i \) terms is \( L. \) Nevertheless, if we relax this condition we obtain in addition that
precisely and only

\[ Q = I_3 + \frac{1}{2} Y \]  

(114)

commutes with \( M_3 \) and \( M_4 \) (\( Q' = I_3 - \frac{1}{2} Y \) commutes with \( M_1 \) and \( M_2 \)). As \( Q \) is the electric charge we deduce the electromagnetic \( U(1)_{em} \) remains a symmetry while the hypercharge and isospin are broken. We stress that \( Q \) is deduced, rather than being imposed, as the only additional symmetry consistent with massive terms. \( M_3 \) and \( M_4 \) do not commute among themselves, but rather, can be obtained from each other through a unitary transformation involving \( \gamma_5 \). We therefore choose one, \( M_3 \), to study the massive representations. We will show the equation

\[ (L\gamma_0 i \partial^\mu \gamma'_\mu - M_3 \gamma_0) \Psi = 0 \quad \mu = 0, ..., 3, \]  

(115)

gives rise to massive and massless fermions and vectors that are contained in the SM, at symmetry breaking.

**Vectors**

Despite the presence of a massive term we get a set of vector components which remain massless, as their product with the mass term \( M_3 \) (or \( M_4 \)) in eq. 115 vanishes. These are the combination of the massless and chargeless terms on Tables 11 and 12

\[ A_{Li} = \frac{1}{\sqrt{2}} (B_i - W_i^0). \]  

(116)

There are several parity operators for eq. 115 with the necessary properties. They differ by the square which leads to different projection operator combinations. The only one leading to non-trivial solutions acts on the same space as \( Q \) and is of the same rank. This is

\[ P = M_3 \gamma_0 \varphi, \]  

(117)

where \( \varphi \) is defined as for eq. 34 and \( M_3 \) is given in eq. 112.
The remaining bosons become massive. The massive chargeless solutions are a combination of the vectors $B_i, \tilde{B}_i$ on Tables 10, 11, the $W_i^0$ on Table 12, the $n_i$ bosons on Table 13, and their antiparticles $n_i$, with $n_0(k) = \tilde{n}_0^\dagger(k), n_0(\tilde{k}) = \tilde{n}_0(k), n_{-1}(k) = -\tilde{n}_1^\dagger(k), n_{-1}(\tilde{k}) = -\tilde{n}_1(k)$. We construct the latter using Table 8 and multiplying on the left by operators carrying the isospin and hypercharge, with phases as on Table 13. The solutions can be classified in two groups, depending on the value of the commutator $[M_3\gamma_0, \Psi]$, or equivalently, by the value of the operator $P$. When the commutator is zero we have the solutions on Table 14. The $k$ and $\tilde{k}$ arguments simply label the vector components (in the massless solutions) in terms of which the massive solutions are constructed. The non-zero terms for the commutator with $M_3\gamma_0$ are given on Table 15.

We have also a set of charged vector particles, constructed from the $W_{\mu}^\pm$ on Table 12 and the charged doublet components $\tilde{v}_i$ on Table 13, and their antiparticles. The $Q = -1$ components are given on Table 16, where the $0$ subscript labels the solution with negative eigenvalue of $\frac{1}{2}LI\gamma_1\gamma_2$. The positively charged terms can be obtained from $(W_{\mu}^-)^\dagger$.

**Spin 1/2 particles**

The application of the massive terms $M_3$ and $M_4$ in eqs. 112 and 113 to the $Y = -1, I_{s3} = 1/2, (\text{with } Q = 0)$, “neutrino” elements and their antiparticles gives zero, which implies the neutrinos remain massless. In addition, the neutrino and antineutrino solutions lack a right-handed and left-handed partner respectively to be able to form Dirac massive particles.

On the other hand, the massive term $M_3$ (or $M_4$) breaks the chiral symmetry mixing values of chirality and causing the charged fermions to acquire a mass. The lepton number $l = 1$, and charge $Q = -1$ wave functions are given on Table 17. The charge of these fermions leads to their association with any two of the negatively charged massive leptons $e^-, \mu^-, \tau^-$. 
\[ P_1(M, 0) = \frac{1}{2}(\tilde{B}_1(k) + \frac{1}{\sqrt{2}}(B_{-1}(\tilde{k}) + W^{-1}_0(\tilde{k}))) \]
\[ Q_1(M, 0) = \frac{1}{2}(\tilde{B}_1(k) + \frac{1}{\sqrt{2}}(B_{-1}(\tilde{k}) + W^{-1}_0(\tilde{k}))) \]
\[ P_{-1}(M, 0) = \frac{1}{2}(\tilde{B}_1(\tilde{k}) + \frac{1}{\sqrt{2}}(B_{-1}(k) + W^{-1}_0(k))) \]
\[ Q_{-1}(M, 0) = \frac{1}{2}(\tilde{B}_1(\tilde{k}) + \frac{1}{\sqrt{2}}(B_{-1}(k) + W^{-1}_0(k))) \]
\[ P_0(M, 0) = \frac{1}{2}(\tilde{B}_0(k) + \frac{1}{\sqrt{2}}(B_0(\tilde{k}) + W^0_0(\tilde{k}))) \]
\[ Q_0(M, 0) = \frac{1}{2}(\tilde{B}_0(k) + \frac{1}{\sqrt{2}}(B_0(\tilde{k}) + W^0_0(\tilde{k}))) \]

**Table 14:** \(P = 1\) massive bosons.
Massive bosons $M_3 \gamma_0/M \frac{i}{2} L I \gamma_2 [H/k_0, ] [\Sigma \cdot \hat{k}, ]$

\[
P_1(M, 0) = \frac{1}{2}(\tilde{B}_1(k) - \frac{1}{\sqrt{2}}(B_{-1}(\tilde{k}) + W^0_{-1}(\tilde{k}))) + \tilde{n}_1(k) + n_{-1}(\tilde{k}) \quad 1 \quad 1/2 \quad 2 \quad 1
\]

\[
\bar{Q}_1(M, 0) = \frac{1}{2}(\tilde{B}_1(k) - \frac{1}{\sqrt{2}}(B_{-1}(\tilde{k}) + W^0_{-1}(\tilde{k})) - \tilde{n}_1(k) - n_{-1}(\tilde{k}) \quad -1 \quad 1/2 \quad -2 \quad 1
\]

\[
P_{-1}(M, 0) = \frac{1}{2}(\tilde{B}_1(\tilde{k}) - \frac{1}{\sqrt{2}}(B_{-1}(k) + W^0_{-1}(k))) + \tilde{n}_1(\tilde{k}) + n_{-1}(k) \quad 1 \quad -1/2 \quad 2 \quad -1
\]

\[
\bar{Q}_{-1}(M, 0) = \frac{1}{2}(\tilde{B}_1(\tilde{k}) - \frac{1}{\sqrt{2}}(B_{-1}(k) + W^0_{-1}(k)) - \tilde{n}_1(\tilde{k}) - n_{-1}(k) \quad -1 \quad -1/2 \quad -2 \quad -1
\]

\[
P_0(M, 0) = \frac{1}{2}(\tilde{B}_0(k) - \frac{1}{\sqrt{2}}(B_{0}(\tilde{k}) + W^0_{0}(\tilde{k}))) + \tilde{n}_0(k) - n_{0}(\tilde{k}) \quad 1 \quad 1/2 \quad 2 \quad 0
\]

\[
\bar{Q}_0(M, 0) = \frac{1}{2}(\tilde{B}_0(k) - \frac{1}{\sqrt{2}}(B_{0}(\tilde{k}) + W^0_{0}(\tilde{k})) - \tilde{n}_0(\tilde{k}) - n_{0}(\tilde{k}) \quad -1 \quad 1/2 \quad -2 \quad 0
\]

\[
P_{\tilde{0}}(M, 0) = \frac{1}{2}(\tilde{B}_0(\tilde{k}) - \frac{1}{\sqrt{2}}(B_{0}(k) + W^0_{0}(k))) + \tilde{n}_0(\tilde{k}) - n_{0}(k) \quad 1 \quad -1/2 \quad 2 \quad 0
\]

\[
\bar{Q}_{\tilde{0}}(M, 0) = \frac{1}{2}(\tilde{B}_0(\tilde{k}) - \frac{1}{\sqrt{2}}(B_{0}(k) + W^0_{0}(k)) - \tilde{n}_0(\tilde{k}) + n_{0}(k) \quad -1 \quad -1/2 \quad -2 \quad 0
\]

**Table 15:** $P = -1$ massive bosons.
There remains to classify the vector fields obtained in the breaking of the $SU(2)_L \times U(1)_Y$ to the $Q$ symmetry, according to the discrete symmetries. The terms found, $A_{Li}$ in eq. [116] and $P_i, \bar{P}_i, Q_i, \bar{Q}_i$ on Tables [14], sum twenty degrees of freedom. Similar combinations as for the massive vector terms $U_i, \bar{U}_i, V_i, \bar{V}_i$ on Tables [3] can be taken to obtain terms with the necessary transformation properties.

The (normalized) vector components solutions of Tables [14], [15], which transform as a non-axial vector by $P$ in eq. [117] are given by

$$A_\mu = \frac{1}{2} Q \gamma_0 \gamma_\mu. \quad (118)$$
Charged massive spin 1/2 particles

\begin{align*}
u^-_{1/2} &= \frac{1}{\sqrt{2}} (l^-_{-1/2L}(k) - l^-_{1/2R}(k)) & 1 & 1/2 & 1/2 \\
v^+_{1/2} &= \frac{1}{\sqrt{2}} (l^+_{-1/2L}(k) + l^+_{1/2R}(k)) & -1 & 1/2 & 1/2 \\
u^-_{-1/2} &= \frac{1}{\sqrt{2}} (l^-_{-1/2L}(k) - l^-_{1/2R}(\tilde{k})) & 1 & -1/2 & 1/2 \\
v^+_{-1/2} &= \frac{1}{\sqrt{2}} (l^+_{-1/2L}(k) + l^+_{1/2R}(\tilde{k})) & -1 & -1/2 & 1/2 \\
\hat{u}^-_{1/2} &= \frac{1}{\sqrt{2}} (\hat{l}^-_{-1/2L}(k) - \hat{l}^-_{1/2R}(k)) & 1 & 1/2 & -1/2 \\
\hat{v}^+_{1/2} &= \frac{1}{\sqrt{2}} (\hat{l}^+_{-1/2L}(k) + \hat{l}^+_{1/2R}(\tilde{k})) & -1 & 1/2 & -1/2 \\
\hat{u}^-_{-1/2} &= \frac{1}{\sqrt{2}} (\hat{l}^-_{-1/2L}(k) - \hat{l}^-_{1/2R}(\tilde{k})) & 1 & -1/2 & -1/2 \\
\hat{v}^+_{-1/2} &= \frac{1}{\sqrt{2}} (\hat{l}^+_{-1/2L}(k) + \hat{l}^+_{1/2R}(\tilde{k})) & -1 & -1/2 & -1/2 \\
\end{align*}

Table 17: Charged $Q = -1$ massive fermions.

$A_\mu$ can be represented as a mixture of two chargeless and massless components. On the one hand, Tables 14, 15 contain the special combination, of the $B_i$ on Table 10, and the $\tilde{B}_i$ on Table 11, which form precisely

\[ B_\mu = \frac{1}{2\sqrt{3}} Y \gamma_0 \gamma_\mu, \]

that is, the hypercharge carriers. This gives another justification for the choice of $Y$ given in eq. 108, which is the operator giving the correct values for the hypercharge of fermions. Thus we obtain another argument needed to set $Y$ whose background is in the way we arrive at the expression for $Q$ in eq. 114. On the other hand, we can extract the chargeless vector components for the isospin triplet from Table 12

\[ W^0_\mu = I_3 \gamma_0 \gamma_\mu, \]
where $I_3$ is given in eq. 99.

From the expression for $Q$ and eqs. 118, 119, and 120 we easily obtain

$$A_\mu = \frac{1}{2} W_\mu^0 + \frac{\sqrt{3}}{2} B_\mu.$$  \hspace{1cm} (121)

The value of Weinberg’s angle $\theta_W$ is derived immediately from this equation by making an analogy with the new fields obtained in the SM, after application of the Higgs mechanism. The photon then has the form

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (g B_\mu + g' W_\mu^0),$$  \hspace{1cm} (122)

where $g$ and $g'$ are respectively the isospin and hypercharge coupling constants. We obtain $\frac{g'}{g} = \frac{1}{\sqrt{3}}$. As in the SM $\tan(\theta_W) = \frac{g'}{g}$ we find

$$\sin^2(\theta_W) = .25.$$  \hspace{1cm} (123)

The $Z_\mu$ field can be constructed by considering the orthogonal combination to $A_\mu$ in eq. 121

$$Z_\mu = \frac{\sqrt{3}}{2} W_\mu^0 - \frac{1}{2} B_\mu.$$  \hspace{1cm} (124)

We therefore find the $A_\mu$ and $Z_\mu$ span the vector components on Tables 14, 15.

The charged massive solutions can be related to the $W_\mu^\pm$ components in the SM. Although we obtained a difference in the masses of the $Z_\mu$ and $W_\mu^\pm$ this is not the corresponding to the one obtained in the SM. Also, the vector particle $A_\mu$ is massive. We attribute these differences to the fact that term $M_3 \gamma_0$ does not commute with the kinetic term in eq. 105 which is required to allow for simultaneously massive and massless solutions in the space projected by $Q$. Indeed, the space spanned by the vectors $A_{Li}$ in eq. 110 is annihilated by $M_3$. This fact allows them to be massless solutions.

**Coupling constants: Vector fermion-current vertices**
Following the steps which allow for a vertex interpretation of eq. 86 it is possible to derive the vertices describing the coupling of the fermions to the vectors obtained from the solutions. This information can be summarized through the Lagrangian density

\[
\mathcal{L} = \frac{g}{2\sqrt{2}} \left[ \nu^\dagger (1 - I\gamma_5)\gamma_0\gamma^\mu u W^+_\mu + hc \right] - \\
et\left[ \tan(\theta_W) (2l_R^\dagger \gamma_0\gamma^\mu l_R + \nu^\dagger \gamma_0\gamma^\mu \nu + l_L^\dagger \gamma_0\gamma^\mu l_L) + \right. \\
\cot(\theta_W) (\nu^\dagger \gamma_0\gamma^\mu \nu - l_L^\dagger \gamma_0\gamma^\mu l_L) \right] Z_\mu - \epsilon u^\dagger \gamma_0\gamma^\mu u A_\mu,
\]

where \( \nu, l_L \) are given on Table 8, \( l_R \) are given on Table 9 and \( u \) are given on Table 17. The electric charge is given by \( e = gg'/\sqrt{g'^2 + g^2} \).

In addition, the vertices give information on the coupling constants \( g' \) and \( g \), which cannot be extracted from eqs. as 121 or 124. Information on these can be obtained by calculating the overlap of the vectors with the corresponding fermion currents, which are given explicitly in eq. 125. This can be done more realistically by considering the 7 + 1 dimensional Clifford algebra where vector massless solutions become possible. The coupling constant \( g \) can be obtained from the coupling of the massive charged vectors \( W^+_{\mu\nu} \) in eq. 118 and the charged current, obtained from the neutrino and charged massive lepton wave functions, represented by the first term of eq. 125. It is

\[
g = 1/\sqrt{2} \approx .707.
\]

The coupling \( g' \) is deduced from the second term in eq. 125 to be

\[
g' = 1/\sqrt{6} \approx .408.
\]

It is a consistency feature of the theory that these values agree with Weinberg’s angle in eq. 123. (At 5+1 dimensions the also consistent values \( g = 1, g' = 1/\sqrt{3} \approx .577 \) are obtained). In addition, these values are to be compared with the experimentally measured ones at energies of the mass of the \( W \) particle, which is where the breakdown
of the $SU(2)_L \times U(1)_Y$ symmetry occurs. These are $g'_{\text{exp}} \approx .35$, $g_{\exp} \approx .65$, and $\sin^2(\theta_W \text{exp}) \approx .23$.

10 Summary and conclusions

In this work we have departed from a generalized Dirac equation whose solutions, with the rules we have postulated to interpret them, exhibit some similarity to quantum fields. We have first studied these in the framework of the 3+1 Clifford matrices. They comprise non-axial and axial vectors, spin zero particles, antisymmetric tensors, and, under a choice of the Lorentz generators, even spin 1/2 particles of a given chirality. We have investigated a gauge symmetry of the equation. A comparison among the different solutions is possible with the application of a generalized point product within the quantum mechanical framework of the equation. Through it the transition amplitude of a vector field and two fermions is a vertex, and hence, it is interpreted as an interaction. The coupling constant is then determined.

We have also investigated the simplest generalization of the equation which is in the context of a 5+1 dimensional Clifford algebra. By focusing on the 3+1 underlying structure we have obtained an $SU(2)_L \times U(1)$ symmetry. We get a boson and fermion set of solutions for the massless case. The addition of a mass term to the equation implies the breaking of the symmetry to a $U(1)_Q$, which can be interpreted as the gauge symmetry defining the electromagnetic interaction. We also obtain the field solutions, their spectrum, and some of the couplings among them. We have shown they exhibit a close similarity to the particles and coupling constants in the $SU(2)_L \times U(1)$ sector of the standard model at symmetry breaking.

The main result in this work has been to derive gauge interactions and the particle spectrum from an extended spin space, some of whose components transform under the usual Lorentz generators in 3+1 space-time. The gauge forces emerge as excitation determined by the symmetries permitted by the Clifford algebra in which
the 3+1 subalgebra is embedded. Thus, we find a relation between gauge and space-time symmetries. In the simplest Clifford algebra containing the 3+1 subalgebra, we have found a symmetry as large as $U(2)_L \times U(2)_R$ and we have shown we have only two choices for a model with a $SU(2)_L \times U(1)$ which contains both fermions and bosons. The $SU(2)_L \times U(1)$ symmetry group is consequently derived rather than being imposed. It is noteworthy that the chiral nature of the $SU(2)$ gauge interaction is predicted. The formalism also predicts gauge vector carriers which are as well generators lying in the adjoint representation of the group.

In general, a field theory is determined by the couplings among fields, which are defined at tree level. The power of field theory in describing nature stems from the possibility of using this simple description in perturbation theory to account for more complex behavior by considering repeated interactions. The values of the coupling constants are arbitrary and must be fixed by experiment. In our case, the very nature of the fermion and boson solutions defines the coupling at tree level. In fact, our theory determines the type of fields involved and the normalization restriction fixes the values of the coupling constants. It is the compositeness feature of the solutions, the fact that some may be constructed from the product of others, that determines their interaction. For example, the form of the spin 1/2 particle pair coupling to vectors and scalar particles is restrained by the symmetries of the theory. Thus, the restrictiveness in the choice of the representations in our theory constitutes also its asset.

In the model described in eq. we have obtained leptons with the correct gauge quantum numbers corresponding to a left-handed doublet of $SU(2)$ (that is, in the fundamental representation) with hypercharge $Y = -1$, and a singlet with $Y = -2$, which can be interpreted as massless neutrinos and charged leptons. These fields appear in doublets characterized by a conserved quantum number which does not affect interactions with vector bosons and which we have therefore associated
with flavor. The flavor doublets are a consequence of using a Hamiltonian which allows for a certain matrix solution space, although the size of the flavor multiplet can change in higher dimensional models. This may constitute a clue on the puzzle of generations. Furthermore, the fermions have a conserved lepton number. Thus the fermions obtained could be identified with any pairs of the particle set \( e, \nu_e \); \( \mu, \nu_\mu \); and \( \tau, \nu_\tau \).

We have also obtained a spinless boson doublet with \( Y = -1 \) which can be identified with a Higgs particle. It is interesting that this boson appears here as part of the solution representations and not put in by hand. We obtain that introducing a mass term into the equation, as represented by eq. 113, implies an additional interaction of the scalar particle which gives masses to some of the fields. We have shown the \( SU(2)_L \times U(1)_Y \) symmetry is broken to a \( U(1)_Q \) symmetry. This procedure goes further than the SM where the Higgs mechanism is a mathematical device to create massive terms, and which requires explicitly that \( U(1)_Q \) remain unbroken. In our case the presence of a mass term implies it is \( U(1)_Q \) the unbroken gauge symmetry in the real world.

We have obtained masses for the vector bosons different to those in the SM. We have ascribed this difference to the fact that the \( 6-d \) model does not allow for massless vector solutions, which is permitted in the next Clifford algebra at 7+1 dimensions. It is encouraging that the values obtained then for the coupling constants are within \( 7\% - 15\% \) of their values in the SM at electroweak breakdown. The accordance of the values of the coupling constants, vertices and particles described in the theory is further fortified by the fact that other reducible representations will reproduce only some aspects while others will change. For example, the trace, which fixes the interaction, is representation-dependent.

There remain several aspects to be studied. The argument leading to the quantization condition in eq. 113 implies the present equations carry on with them an
implied gauge-fixing. While we have shown in detail the extent to which this is true in the abelian case, we still have to prove this for the non-abelian case. Analogy with the non-abelian QFT description requires ghosts to satisfy unitarity. These are scalar objects lying in the adjoint representation which do not appear as physical particles. We speculate spurious degrees of freedom as the antisymmetric $n$ and $v$ could conform such a counterpart in extended theories. We have a different characterization of the spinless and $Z_\mu, W_\mu$ fields from the corresponding ones in the SM in relation to their discrete transformation properties, since we get different weights for their scalar or pseudoscalar, and $V, A$ contributions, respectively. As the $Z_\mu, W_\mu$ particles interact only weakly and this characteristics fit into their interaction scheme this aspect is, however, difficult to test. Furthermore, the interactions among vectors need further study. We attribute the mass of the lepton to be of the same magnitude as for the $W_\mu$ to the unified approach we use. The possibility of this theory providing information on the corrections to lepton masses is under investigation.

Finally, we have heuristically obtained fields which reproduce properties of quantized operators in a quantum field theory. The implication is that quantization is not derived as a condition on the fields but as a consequence of the definitions of the equations. This points out at a closer relation to quantization which should be researched with more detail in the future. Causality and unitarity requirements demand more study on related aspects as propagators, commutation relations, and how to include radiative corrections.

The presence of bosons and fermions solutions became possible from the use of bi-spinors as solution space. Further extensions with the use of more spin indices will allow for a description of spin 3/2 and spin 2 objects; this may point to a connection to gravity. This possibility can be used in turn to propose a new interpretation of the wave function, with the implication of a a closer connection of it to space-time. The development of this idea is done from another standpoint elsewhere (Besprosvany,
The similarity in the representation of the fields in this formalism and the operators which carry out a Lorentz transformation for the spin parts could imply a possible connection between these two. Thus, a Lorentz transformation could be considered not independent of the fields needed to perform it physically. On the other hand, just as the choice of gauge interaction is restricted by the Clifford algebras, we also find that the interactions restrain the possible type of space-time symmetry. In this way we obtain a possible clue on the origin of the number of dimensions of space-time. Thus, although its (3,1) structure is not predicted it is among the few which are consistent with a $SU(2)_L \times U(1)$ symmetry, in the 5+1 dimensional Clifford algebra.

The unified treatment of space-time and gauge symmetries proposed here has proven fruitful. The formalism presented has the quality of giving information on a set of representation solutions and their interactions by literally restricting them. Their agreement with aspects of the standard model makes the theory a plausible alternative, all the more that it assumes a rather conventional relativistic quantum mechanical framework, of proven simplicity and universality. Information on additional aspects of the standard model may be found with the application of the theory in extended spaces, making certainly worth its further study.

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Appendix

We set here the conventions for the Clifford algebras used in this work and we present explicitly the matrices generating it.
In 4-d we use the metric

\[ g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]  

(128)

The 4 \times 4 matrices in the paper are in the Dirac representation, and in order to define them we use the Pauli matrices \( \sigma_i, i = 1, 2, 3 \) and the 2 \times 2 unit matrix \( 1_2 \):

\[ \gamma_0 = \sigma_3 \otimes 1_2 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \]

so the vector \( \gamma = (\gamma^1, \gamma^2, \gamma^3) \) is given by

\[ \gamma = i\sigma_2 \otimes \sigma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}. \]

(130)

From here all other matrices can be defined. For example,

\[ \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1 \otimes 1_2 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}. \]

(131)

For the 6-d Clifford algebra we use

\[ g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \]

(132)

The definitions leading to eqs. [3] and [4] imply the 4-d vectors subset of 8 \times 8 matrices are given explicitly by

\[ \gamma'_\mu = 1_2 \otimes \gamma_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & \gamma_\mu \end{pmatrix}, \quad \mu = 0, 1, 3, \]

(133)

\[ \gamma'_2 = \sigma_1 \otimes \gamma_2 = \begin{pmatrix} 0 & \gamma_2 \\ \gamma_2 & 0 \end{pmatrix}, \]

(134)
and the 4-d scalars by

$$1_8 = 1_2 \otimes 1_4 = \begin{pmatrix} 1_4 & 0 \\ 0 & 1_4 \end{pmatrix},$$

$$I = \sigma_1 \otimes 1_4 = \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix},$$

$$i\gamma'_5 = iJ\gamma_2 = i\sigma_2 \otimes \gamma_2 = \begin{pmatrix} 0 & \gamma_2 \\ -\gamma_2 & 0 \end{pmatrix},$$

$$i\gamma'_6 = iK\gamma_2 = i\sigma_3 \otimes \gamma_2 = \begin{pmatrix} i\gamma_2 & 0 \\ 0 & -i\gamma_2 \end{pmatrix}.$$  

(135) 

(136) 

(137) 

(138)

All $8 \times 8$ matrices can be generated by products of these matrices. We use a notation for which the $\gamma'_\mu$ matrices are written in terms of the $\gamma_\mu$ matrices and from eq. 93 onwards the latter are assumed to be $8 \times 8$ matrices.

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