ENTROPY APPROXIMATION VERSUS UNIQUENESS OF EQUILIBRIUM FOR A DENSE AFFINE SPACE OF CONTINUOUS FUNCTIONS

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Abstract. We show that for a \( \mathbb{Z}^l \)-action (or \( (\mathbb{N} \cup \{0\})^l \)-action) on a non-empty compact metrisable space \( \Omega \), the existence of an affine space dense in the set of continuous functions on \( \Omega \) constituted by elements admitting a unique equilibrium state implies that each invariant measure can be approximated weakly\( \ast \) and in entropy by a sequence of measures which are unique equilibrium states.

1. Introduction

Let \( \tau \) be an action of \( \mathbb{Z}^l \) (resp. \( (\mathbb{N} \cup \{0\})^l \)) on a non-empty compact metric space \( \Omega \) for some \( l \in \mathbb{N} \), let \( C(\Omega), \mathcal{M}(\Omega), \mathcal{M}^\tau(\Omega), h^\tau, P^\tau \) denote respectively the set of real-valued continuous functions on \( \Omega \) endowed with the uniform topology, Borel probability measures on \( \Omega \) endowed with the weak\( \ast \)-topology, \( \tau \)-invariant elements of \( \mathcal{M}(\Omega) \), measure-theoretic entropy and pressure maps ([21]). We assume that \( h^\tau \) is finite and upper semi-continuous.

Let \( (\nu_\alpha, t_\alpha) \) be a net where \( \nu_\alpha \) is a Borel probability measures on \( \mathcal{M}(\Omega) \), \( t_\alpha > 0 \) and \( (t_\alpha) \) converges to zero. Recall that \( (\nu_\alpha) \) is said to satisfy a large deviation principle in \( \mathcal{M}(\Omega) \) with powers \( (t_\alpha) \) if there exists a \([0, +\infty]\)-valued lower semi-continuous function \( I \) on \( \mathcal{M}(\Omega) \) such that
\[
\limsup_{\alpha} t_\alpha \log \nu_\alpha(F) \leq -\inf_{x \in F} I(x) \leq -\inf_{x \in G} I(x) \leq \liminf_{\alpha} t_\alpha \log \nu_\alpha(G)
\]
for every closed set \( F \subset \mathcal{M}(\Omega) \) and every open set \( G \subset \mathcal{M}(\Omega) \) with \( F \subset G \); such a function \( I \) is unique and called the rate function governing the large deviation principle. Let \( f \in C(\Omega) \) and assume that \( (\nu_\alpha, t_\alpha) \) fulfills
\[
\forall g \in C(\Omega), \quad \lim_{\alpha} t_\alpha \log \int_{\mathcal{M}(\Omega)} e^{t_\alpha^{-1} \int_\Omega g(\xi) \mu(d\xi)} \nu_\alpha(d\mu) = P^\tau(f + g) - P^\tau(f).
\]
There are two general conditions ensuring that \( (\nu_\alpha) \) satisfies a large deviation principle with powers \( (t_\alpha) \) and convex rate function: The first one is the existence of a vector space \( V \) dense in \( C(\Omega) \) such that \( f + g \) has a unique equilibrium state for all \( g \in V \) ([13], [9]); the second one, that we shall denote by (D) hereafter, is the following entropy approximation property: For each \( \mu \in \mathcal{M}^\tau(\Omega) \) there exists a net \( (\mu_i) \) in \( \mathcal{M}^\tau(\Omega) \) such that \( \lim_i \mu_i = \mu \).

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lim \( h^\tau (\mu_i) = h^\tau (\mu) \) and \( \mu_i \) is the unique equilibrium state for some \( g_i \in C(\Omega) \) (\cite{4}, Theorem 5.2); note that the first countability of \( M^\tau (\Omega) \) allows us to replace "net" by "sequence" in (D); since a measure on \( \Omega \) is the unique equilibrium state for some element in \( C(\Omega) \) if and only if it is ergodic (\cite{19}), one can also replace "unique equilibrium state for some \( g_i \in C(\Omega) \)" by "ergodic"; in particular, (D) implies that either \( M^\tau (\Omega) \) is a singleton or \( M^\tau (\Omega) \) is the Poulsen simplex, i.e. the set of ergodic states is dense in \( M^\tau (\Omega) \).

In this note, we prove that the first above condition implies the second one (Theorem 1); consequently, all the large deviation principles proved applying Kifer’s theorem (namely, Theorem 2.1 of \cite{13} or Theorem C of \cite{3} (as well as its generalization given by Remark B.2) can be proved using Theorem 5.2 of \cite{4}. We also provide two examples that can be proved using (D) but not with the first condition (Example 3.1.1, Example 3.1.2).

The results are given in Section 3. We recall below some basic definitions and discuss the main difference between both conditions; we also review some important cases where (D) has been used to prove a large deviation principle.

2. Preliminaries

2.1. Pressure and equilibrium state. Let \( (\Omega, \tau) \) be a dynamical system as in \cite{1} Order \( \mathbb{N}^l \) lexicographically. For each \( a \in \mathbb{N}^l \) we put \( \Lambda(a) = \{(x_1, \ldots, x_l) \in (\mathbb{N} \cup \{0\})^l : x_i < a_i, 1 \leq i \leq l \} \) and let \( |\Lambda(a)| \) denote the cardinality of \( \Lambda(a) \). For each \( \varepsilon > 0 \) and for each \( a \in \mathbb{N}^l \) let \( \Omega_{\varepsilon,a} \) be a maximal \( (\varepsilon, \Lambda(a)) \)-separated set. Recall that \( P^\tau (g) \) is defined for each \( g \in C(\Omega) \) by

\[
P^\tau (g) = \lim_{\varepsilon \to 0} \limsup_a \frac{1}{|\Lambda(a)|} \log \sum_{\xi \in \Omega_{\varepsilon,a}} e^{\sum_{x \in \Lambda(a)} g(\tau^x \xi)},
\]

and fulfills

\[
P^\tau (g) = \lim_{\varepsilon \to 0} \liminf_a \frac{1}{|\Lambda(a)|} \log \sum_{\xi \in \Omega_{\varepsilon,a}} e^{\sum_{x \in \Lambda(a)} g(\tau^x \xi)} = \sup_{\mu \in M^\tau (\Omega)} \{\mu(g) + h^\tau (\mu)\}. \tag{2}
\]

(\cite{21}, §6.7, §6.12 and Exercise 2 p. 119 for \( \mathbb{Z}^l \)-action, §6.18 for \( (\mathbb{N} \cup \{0\})^l \)-action). Since \( M^\tau (\Omega) \) is compact and \( h^\tau \) is finite and upper semi-continuous, the above supremum is a maximum, and each element realizing this maximum is called an equilibrium state for \( g \); the map \( P^\tau \) is finite convex and continuous on \( C(\Omega) \) (\cite{21}, §6.8 and §6.18).

2.2. Connection with large deviation theory. Each one of the two conditions stated in \cite{1} is nothing but the specialization in the dynamical setting of the corresponding well-known sufficient condition in large deviation theory in topological vector spaces that imply the large deviation principle (\cite{6}, \cite{5}); they appear virtually in one form or another when establishing a level-2 large deviation principle (i.e. in the space \( M(\Omega) \)). Regarding the first one, the reader is referred to \cite{3}, where Theorem C together with Remark B.2 provides a version of Theorem 2.1 of \cite{13} valid for general dynamical systems as in \cite{1} see also §1.3 and Remark 3.3 of \cite{3} for a discussion on the functional approach in large deviation theory in connection with dynamical systems, and specially with Corollary 4.6.14 of \cite{6}. With respect to (D) and particularly the connection with the general result involving exposed points in
large deviation theory in topological vector spaces (namely, Baldi’s theorem), see [4], where Theorem 5.2 establishes that (D) yields the large deviation principle for any net \((\nu_\alpha, t_\alpha)\) fulfilling (1).

2.3. **Advantage of (D) over the first condition.** A straightforward but important observation that differentiates the two above conditions is that the first one implies the uniqueness of equilibrium for \(f\), whereas (D) does not impose any condition on \(f\) (cf. "Important Remark" in §1.3 of [4]); this simple fact allows us to obtain for a dynamical system \((\Omega, \tau)\) as in [4] examples of large deviation principles that can be proved using (D) but that cannot be proved using the first condition: We just have to consider nets fulfilling (1) with \(f\) admitting several equilibrium states; Example 4.1 of [4] furnishes such an example when \((\Omega, \tau)\) is given by the iteration of a hyperbolic rational map; Theorem 5.7 of [4] provides other two examples when \((\Omega, \tau)\) is the multidimensional full shift.

- Example 4.1 and Theorem 5.7 (a) of [4] both concern nets \((\nu_{f,\alpha}, t_\alpha)\) of the form

\[
\nu_{f,\alpha} = \sum_{\xi \in \Omega_\alpha} e^{\sum_{x \in \Lambda_\alpha} f(\tau^x \xi)} \delta_{\sum_{x \in \Lambda_\alpha} \delta_{\tau^x(\xi)}}.
\]

and

\[
t_\alpha = |\Lambda_\alpha|^{-1},
\]

where \(f\) is an arbitrary element of \(C(\Omega)\), \((\Lambda_\alpha)\) a van Hove net of nonempty finite subsets of \(\mathbb{N}^l\) for some \(l \in \mathbb{N}\), \(|\Lambda_\alpha|\) the cardinality of \(\Lambda_\alpha\), and \(\Omega_\alpha\) a maximal \((\varepsilon, \Lambda_\alpha)\)-separated set for some \(\varepsilon\) small enough; the expansiveness property that holds in these examples makes (1) easy to establish; we present here the general case (Example 3.1.1).

- The example given in Theorem 5.7(b) of [4] deals with nets \((\nu_{f,\alpha}, t_\alpha)\) similar to the above case but where \(\alpha \in \mathbb{N}^l\), \(\Lambda_\alpha\) is the parallelepiped whose angles are determined by \(\alpha\), and \(\Omega_\alpha\) is the set of \(\alpha\)-periodic configurations; Example 3.1.2 extends this case to subshifts of finite type satisfying strong specification, recovering Theorem C of [7] in a very direct way.

2.4. **Basic examples.** It should be pointed out that in all the examples below, in addition to (D), the proofs appearing in the cited articles require highly technical and or theoretical (e.g. Shannon-Mac-Millan theorem) intermediate results, while once (D) is established, the large deviation principle follows by a straightforward application of Theorem 5.2 of [4]; this is in particular the case of Theorem A and Theorem C of [7], which follow from Theorem B of [7] together with Theorem 5.2 of [4] (cf. b) and c) below, respectively); the last case is detailed in Example 3.1.2.

a) In statistical mechanics, the use of (D) dates back to the proof of the large deviation principle for Gibbs random fields on \(\mathbb{Z}^l\) for some \(l \in \mathbb{N}\) ([4]), where the underlying dynamical system \((\Omega, \tau)\) is the \(l\)-dimensional full shift and the function \(f\) as in (1) is the local energy function associated with some translation invariant summable interaction \(\phi\); more precisely, given a van Hove sequence \((\Lambda_\alpha)\) of finite subsets of \(\mathbb{Z}^l\), one considers
the sequence \((ν_n, t_n)\), where \(t_n = |Λ_n|^{-1}\) and \(ν_n\) is the distribution of the field

\[
Ω \ni ξ \mapsto \frac{1}{|Λ_n|} \sum_{x \in Λ_n} δ_{τ x ξ}
\]

induced by an equilibrium state \(μφ\) for \(f\), i.e.

\[
∀ n ∈ \mathbb{N}, \quad ν_n(\cdot) = μ_{fφ}(\{ξ ∈ Ω : 1/|Λ_n| \sum_{x \in Λ_n} δ_{τ x ξ} ∈ \cdot\}). \tag{4}
\]

The fact that (D) holds for full shifts is known for a long time ([11], Lemma IV, 3.2); refinements have been given in [10] showing that the approximating measures may be chosen as full-supported equilibrium measures for local energy functions associated with invariant short-range interactions.

b) The foregoing is a particular case of a general result for \(\mathbb{Z}^l\)-actions with upper semi-continuous entropy: indeed, Theorem B of [7] asserts that if such a system satisfies the weak specification property, then (D) holds; in [7] the authors apply this result to prove the large deviation principle for the same sequence \((ν_n, t_n)\) as in (a) above, but in the more general case where \((Ω, τ)\) is a subshift of finite type satisfying weak specification ([7], Theorem A); it turns out that for these sequences, the weak specification also implies (1) (i.e. equation (2.23) of [7]). In fact, [7] deals with the more general van Hove net constituted by all finite subsets of \(\mathbb{Z}^l\) ordered by inclusion, case which can be reduced to the one of sequences (thanks to the weak specification).

c) In [7] the large deviation principle is also proved for subshifts of finite type satisfying strong specification and for the net \((ν_α, t_α)_{α ∈ \mathbb{N}^l}\) given for each \(α ∈ \mathbb{N}^l\) by

\[
ν_α = \frac{1}{|Per_α|} \sum_{ξ ∈ Per_α} δ_{Per_α} \sum_{ξ ∈ Λ(α)} δ_{τ x ξ},
\]

and

\[
t_α = |Λ(α)|^{-1},
\]

where \(Per_α\) denotes the set of \(α\)-periodic configurations ([7], Theorem C). The possible several equilibrium states for \(f = 0\) as an obstacle to the application of the first condition (i.e. results of [13]) has been noted by the authors who, instead, apply the large deviation principle obtained previously in the above case b) (namely, Theorem A of [7]).

d) In one dimensional dynamics, let us consider the system \((Ω, τ)\) constituted by the iteration of a rational map \(T\) of degree at least two ([1]); more precisely, \(Ω\) is the Julia set of \(T\) endowed with the induced chordal metric, and the action \(τ\) is defined by

\[
N ∪ \{0\} \ni n ↦ τ(n) = (T|_Ω)^n;
\]

such a system has a unique measure of maximal entropy ([15]); we assume furthermore that \(T\) fulfils a weak form of hyperbolicity, the so-called Topological Collet-Eckman (TCE) condition: There exists \(λ > 1\) such that every periodic point \(p ∈ Ω\) with period \(n\) satisfies

\[
|(T^n)'(p)| ≥ λ^n. \tag{5}
\]
(see Main Theorem of [20] for other equivalent definitions). It is known that (D) holds when \( T \) is hyperbolic (i.e. when (5) holds for all \( p \in \Omega \)) ([14], Theorem 8); in [14] the author uses (D) to prove the large deviation principle for the sequence \((\nu_n, n^{-1})\), where \( \nu_n \) is the distribution of the Birkhoff averages with respect to the measure of maximal entropy \( \mu_0 \), i.e.

\[
\forall n \in \mathbb{N}, \quad \nu_n(\cdot) = \mu_0\{\xi \in \Omega : \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\tau^k \xi} \in \cdot\}.
\]

e) The above example d) has been our starting point in [4] leading to the general Theorem 5.2. In the case of the multidimensional full shift, the example of Theorem 5.7(b) above mentioned in §2.3 generalizes also Theorem 3.5 and Theorem 4.2 of [16] in the case of a finite spin space with uniform single spin measure, by allowing \( f \) arbitrary in \( C(\Omega) \) (and not only \( f = 0 \)) (cf. Remark 5.8 of [4] for more details). Other examples are given in §4 of [4] (and in particular those considered in [17] and [18], where only the large deviation upper bounds are proved) but where \( f \) is assumed to have a unique equilibrium state and thus for which the large deviation principle can be proved as well using the first condition (namely, Theorem C together with Remark B.2 of [3]); see also Remarks 5.4 and 5.5 of [4].

3. Results

Our main result is the following.

**Theorem 1.** Let \((\Omega, \tau)\) be a dynamical system as in [11]. Let \( f \in C(\Omega) \). We assume that there exists a vector space \( V \) dense in \( C(\Omega) \) such that \( f + g \) has a unique equilibrium state for all \( g \in V \). Then, for each \( \mu \in M^\tau(\Omega) \) there exists a sequence \((\mu_n, g_n)\) in \( M^\tau(\Omega) \times V \) such that \( \lim_n \mu_n = \mu, \lim_n h^\tau(\mu_n) = h^\tau(\mu) \) and \( \mu_n \) is the equilibrium state for \( f + g_n \) for all \( n \in \mathbb{N} \).

The proof of Theorem 1 will be given after establishing a few lemmas.

**Lemma 1.** Let \( J \) and \( L \) be directed sets, let \( s \) be a real-valued function on \( J \times L \), let \( \wp \) denote the set \( J \times L \) pointwise directed, let \((s_i)_{i \in \wp}\) be the net in \( \mathbb{R} \) defined by putting \( s_i = s(j, u(j)) \) for all \( i = (j, u) \in \wp \). For each \( r \in \mathbb{R} \) we have

\[
\lim \sup_i s_i \leq r \iff \lim \sup_j \lim \sup_l s(j, l) \leq r;
\]

in particular,

\[
\lim_i s_i = r \iff \lim \inf_j \lim \inf_l s(j, l) = \lim \sup_j \lim \sup_l s(j, l) = r.
\]

**Proof.** Let \( \delta > 0 \). First assume that \( \lim \sup_i s_i \leq r \). There exists \( j_0 \in J \) and \( u_0 \in L \) such that \( s(j, u(j)) < r + \delta \) for all \((j, u)\) greater than or equal \((j_0, u_0)\). Suppose that \( \lim \sup_j \lim \sup_l s(j, l) > r + \delta \). There exists \((j_1, l_1)\) in \( J \times L \) with \( j_1 \) (resp. \( l_1 \)) greater than or equal \( j_0 \) (resp. \( u_0(j_1) \)) such that \( s(j_1, l_1) > r + \delta \). Putting \( u_1(j_1) = l_1 \) and
$u_1(j) = u_0(j)$ for all $j \in J \setminus \{l_1\}$, we get an element $(j_1, u_1) \in \varphi$ greater than or equal $(j_0, u_0)$ fulfilling $s(j_1, u_1(j_1)) > r + \delta$, which gives the contradiction. Therefore, we have $\limsup_j \limsup_l s(j, l) \leq r + \delta$ hence $\limsup_j \limsup_l s(j, l) \leq r$ since $\delta$ is arbitrary.

Assume now that $\limsup_j \limsup_l s(j, l) \leq r$. There exists $j_0 \in J$ and for each $j \in J$ greater than or equal $j_0$ there exists $u_0(j) \in L$ such that $s(j, l) < r + \delta$ for all $j$ and $l$ greater than or equal $j_0$ and $u_0(j)$, respectively. Putting $u_0(j) = u_0(j_0)$ for all $j$ lesser than $j_0$, we get an element $(j_0, u_0) \in \varphi$ such that $s(j_0, u(j_0)) < r + \delta$ for all $(j, u) \in \varphi$ greater than or equal $(j_0, u_0)$; therefore, $\limsup_j s_i \leq r + \delta$ hence $\limsup_j s_i \leq r$ since $\delta$ is arbitrary. The first assertion is proved; the second assertion is a direct consequence since $\liminf_i s_i \geq r$ if and only if $-\limsup_i -s_i \geq r$ if and only if $\limsup_i -s_i \leq -r$ if and only if $\limsup_j \limsup_l s(j, l) \leq -r$ if and only if $\liminf_j \liminf_l s(j, l) \leq -r$ if and only if $\liminf_j \liminf_l s(j, l) \geq r$ (where the third equivalence follows from the first assertion applied to the net $(-s_i)$ and the real $-r$).

Let $f \in C(\Omega)$. Let $Q$ be the map defined on $C(\Omega)$ by
\[
\forall g \in C(\Omega), \quad Q(g) = P^\tau(f + g) - P^\tau(f).
\]

**Lemma 2.** The function $Q$ is proper convex and continuous; its Fenchel-Legendre transform $Q^*$ has effective domain $\mathcal{M}^*(\Omega)$ and fulfills
\[
\forall \mu \in \mathcal{M}^*(\Omega), \quad Q^*(\mu) = P^\tau(f) - h^\tau(\mu) - \mu(f).
\]
In particular, $Q^*$ vanishes exactly on the set of equilibrium states for $f$.

**Proof.** Clearly, $Q$ is proper convex and continuous since $P^\tau$ and $\hat{f}$ are (cf. [21]). Let $\hat{\mathcal{M}}(\Omega)$ denote the space of signed Radon measures on $\Omega$ endowed with the weak-* topology. Putting
\[
U(\mu) = \left\{ \begin{array}{ll}
-\mu(f) - h^\tau(\mu) & \text{if } \mu \in \mathcal{M}^*(\Omega) \\
+\infty & \text{if } \mu \in \hat{\mathcal{M}}(\Omega) \setminus \mathcal{M}^*(\Omega),
\end{array} \right.
\]
we have
\[
P^\tau(f + g) = \sup_{\mathcal{M}^*(\Omega)} \{ \mu(f + g) + h^\tau(\mu) \} = \sup_{\mu \in \hat{\mathcal{M}}(\Omega)} \{ \mu(g) - U(\omega) \}.
\]
Since $h^\tau$ is bounded affine and upper semi-continuous, $U$ is proper convex and lower semi-continuous; consequently, we have $U = U^{**}$, i.e.
\[
\forall \mu \in \hat{\mathcal{M}}(\Omega), \quad U(\mu) = \sup_{g \in C(\Omega)} \{ \mu(g) - P^\tau(f + g) \} = \sup_{g \in C(\Omega)} \{ \mu(g) - P^\tau(f) - Q(g) \} = -P^\tau(f) + \sup_{g \in C(\Omega)} \{ \mu(g) - Q(g) \} = -P^\tau(f) + Q^*(\mu),
\]
which proves the lemma. \qed

For each $d \in \mathbb{N}$, each $S = (g_1, ..., g_d) \in C(\Omega)^d$ and each $t = (t_1, ..., t_d) \in \mathbb{R}^d$, let $tS$ denote the function $\sum_{i=1}^d t_ig_i$, put $L_S(t) = P^\tau(f + tS) - P^\tau(f)$, let $p_S : \mathcal{M}(\Omega) \to \mathbb{R}^d$ defined by $p_S(\mu) = (\mu(g_1), ..., \mu(g_d))$ for all $\mu \in \mathcal{M}(\Omega)$, and let $I_S : \mathbb{R}^d \to [0, +\infty]$ defined by
\[
I_S(x) = \left\{ \begin{array}{ll}
\inf\{Q^*(\mu) : \mu \in \mathcal{M}^*(\Omega), p_S(\mu) = x \} & \text{if } x \in p_S(\mathcal{M}^*(\Omega)) \\
+\infty & \text{otherwise};
\end{array} \right.
\]
note that since $\mathcal{M}^*(\Omega)$ is compact and $Q^*$ is lower semi-continuous and real-valued on $\mathcal{M}^*(\Omega)$ (cf. Lemma 2), for each $x \in p_S(\mathcal{M}^*(\Omega)$ there exists $\mu_x \in \mathcal{M}^*(\Omega)$ such that $I_S(x) = Q^*(\mu_x)$. Let $(d, S) \in \mathbb{N} \times C(\Omega)^d$.

**Lemma 3.** The function $I_S$ is proper convex and lower semi-continuous.

**Proof.** Let $x \in \mathbb{R}^d$, let $(x_i)$ be a net converging to $x$, and assume $\lim \inf I_S(x_i) < \delta$ for some real $\delta$. For some subnet $(x_j)$ we have eventually $I_S(x_j) < \delta$ and so $Q^*(\mu_j) < \delta$ for some $\mu_j \in \mathcal{M}^*(\Omega)$ satisfying $p_S(\mu_j) = x$. Let $(\mu_j')$ be a subnet of $(\mu_j)$ converging to some $\mu' \in \mathcal{M}^*(\Omega)$; note that $p_S(\mu') = x$. We have

$$I_S(x) \leq Q^*(\mu) \leq \lim \inf Q^*(\mu_j') < \delta,$$

which proves the lower semi-continuity of $I_S$. For each $(x_1, x_2, \beta) \in \mathbb{R}^{2d} \times [0, 1]$ we have

$$I_S(\beta x_1 + (1 - \beta)x_2) = \inf \{Q^*(\mu) : \mu \in \mathcal{M}^*(\Omega), p_S(\mu) = \beta x_1 + (1 - \beta)x_2\}$$

$$\leq \inf \{Q^*(\beta \mu_1 + (1 - \beta)\mu_2) : \mu_1 \in \mathcal{M}^*(\Omega), \mu_2 \in \mathcal{M}^*(\Omega), p_S(\mu_1) = x_1, p_S(\mu_2) = x_2\}$$

$$\leq \inf \{\beta Q^*(\mu_1) + (1 - \beta)Q^*(\mu_2) : \mu_1 \in \mathcal{M}^*(\Omega), \mu_2 \in \mathcal{M}^*(\Omega), p_S(\mu_1) = x_1, p_S(\mu_2) = x_2\}$$

$$= \beta I_S(x_1) + (1 - \beta)I_S(x_2),$$

hence $I_S$ is convex; $I_S$ is proper by the observation following its definition. □

**Lemma 4.** $I_S = L_S^*$.

**Proof.** Let $\langle \cdot, \cdot \rangle$ denote the scalar product in $\mathbb{R}^d$. Suppose $\sup_{x \in \mathbb{R}^d} \{\langle t, x \rangle - I_S(x)\} < L_S(t)$ for some $t \in \mathbb{R}^d$. Since

$$L_S(t) = Q(tS) = Q^{**}(tS) = \sup_{\mu \in \mathcal{M}^*(\Omega)} \{\langle t, p_S(\mu) \rangle - Q^*(\mu)\}$$

there exists $\mu \in \mathcal{M}^*(\Omega)$ such that $\sup_{x \in \mathbb{R}^d} \{\langle t, x \rangle - I_S(x)\} < \langle t, p_S(\mu) \rangle - Q^*(\mu)$, which gives the contradiction by taking $x = p_S(\mu)$ in the left hand side. Conversely, if $\sup_{x \in \mathbb{R}^d} \{\langle t, x \rangle - I_S(x)\} > L_S(t)$ for some $t \in \mathbb{R}^d$, then $\langle t, x \rangle - I_S(x) > \sup_{\mu \in \mathcal{M}^*(\Omega)} \{\langle t, p_S(\mu) \rangle - Q^*(\mu)\}$ for some $x = p_S(\mu)$ with $\mu \in \mathcal{M}^*(\Omega)$, which gives the contradiction. Therefore, we have $I_S^* = L_S$, which is equivalent to the conclusion since $I_S$ is convex proper and lower semi-continuous by Lemma 3. □

Let $\text{dom} \delta L_S^*$ denotes the set of $x \in \mathbb{R}^d$ for which the set $\delta L_S^*(x)$ of subgradients of $L_S^*$ at $x$ is nonempty.

**Lemma 5.** For each $(x, t) \in \text{dom} \delta L_S^* \times \delta L_S^*(x)$ we have $x = p_S(\mu)$ where $\mu$ is an equilibrium state for $f + tS$; moreover, $L_S^*(p_S(\mu)) = Q^*(\mu)$.

**Proof.** Let $(x, t) \in \text{dom} \delta L_S^* \times \delta L_S^*(x)$. Necessarily $x$ belongs to the effective domain of $L_S^*$, and so by Lemma 2 there exists $\mu \in \mathcal{M}^*(\Omega)$ such that $x = p_S(\mu)$ and $L_S^*(x) = Q^*(\mu)$. Consequently, we have

$$\langle t, x \rangle - L_S^*(x) = L_S(t) = Q(tS) = \langle t, p_S(\mu) \rangle - Q^*(\mu) = \mu(tS) - Q^*(\mu),$$

which proves the lemma. □
Proof of Theorem \[1\] Let $V$ be a vector space as in Theorem \[1\]. Since $C(\Omega)$ is second countable, $V$ contains a countable set $\{g_n : n \in \mathbb{N}\}$ dense in $C(\Omega)$; put $W = \text{span}\{g_n : n \in \mathbb{N}\}$. Let $\mu \in \mathcal{M}^\tau(\Omega)$. For each $n \in \mathbb{N}$ we put $S_n = (g_1, \ldots, g_n)$, $x_n = p_{S_n}(\mu)$; note that $x_n$ belongs to the effective domain of $L_{S_n}^\tau$ by Lemma \[4\]. For each $n \in \mathbb{N}$, Bröndsted-Rockafellar theorem (\[2\], Theorem 2) ensures the existence of a sequence $(x_{n,m})$ in $\delta L_{S_n}^\tau$ such that $\lim_{m} x_{n,m} = x_n$ and $\lim_{m} L_{S_n}^\tau(x_{n,m}) = L_{S_n}^\tau(x_n)$; by Lemma \[5\] we have $x_{n,m} = p_{S_n}(\mu_{n,m})$ and $L_{S_n}^\tau(x_{n,m}) = Q^\ast(\mu_{n,m})$, where $\mu_{n,m}$ is the unique equilibrium state for some $f + t_{n,m}S_n$, with $t_{n,m} \in \delta L_{S_n}^\tau(x_{n,m})$; therefore, we have

$$\forall(n, g) \in \mathbb{N} \times \text{span}\{g_1, \ldots, g_n\}, \quad \lim_{m} \mu_{n,m}(g) = \mu(g)$$

and

$$\forall n \in \mathbb{N}, \quad Q^\ast(\mu) = I_{S_n}(x_n) = L_{S_n}^\tau(x_n) = \lim_{m} Q^\ast(\mu_{n,m}).$$

Since each $g \in W$ belongs to $\text{span}\{g_1, \ldots, g_n\}$ for all $n$ large enough, \[6\] yields

$$\forall g \in W, \quad \lim_{n} \lim_{m} \mu_{n,m}(g) = \mu(g).$$

By considering the product set $\mathcal{F} = \mathbb{N} \times \mathbb{N}^\mathbb{N}$ pointwise directed, we obtain a net $(\mu_i)_{i \in \mathcal{F}}$ in $\mathcal{M}^\tau(\Omega)$ defined for each $i = (n, u) \in \mathcal{F}$ by $\mu_i = \mu_{n,u(n)}$; since $W$ is dense in $C(\Omega)$, \[8\] yields

$$\lim_i \mu_i = \mu$$

\[12\]. Theorem on Iterated Limits, p. 69). Let $s$ be the function defined on $\mathbb{N}^2$ by

$$s(n, m) = Q^\ast(\mu_{n,m});$$

note that $s$ is real-valued by Lemma \[2\]. The lower semi-continuity of $Q^\ast$ and \[9\] yield

$$\liminf_i Q^\ast(\mu_i) \geq Q^\ast(\mu).$$

Since $Q^\ast(\mu) \geq \limsup_i \lim_{n} Q^\ast(\mu_{n,m})$ by \[7\], we have

$$Q^\ast(\mu) \geq \limsup_i Q^\ast(\mu_i)$$

by Lemma \[1\]. From \[10\] and \[11\] we get $\lim Q^\ast(\mu_i) = Q^\ast(\mu)$, i.e. $\lim_i h^\tau(\mu_i) = h^\tau(\mu)$ by Lemma \[2\] which together with \[10\] shows that the net $(\mu_i, h^\tau(\mu_i))$ converges to $(\mu, h^\tau(\mu))$. Denoting $\mathcal{F}$ the subset of $\mathcal{M}^\tau(\Omega)$ constituted by the measures that are unique equilibrium states for some element in $f + W$, we have proved that the graph of $h^\tau|_{\mathcal{F}}$ is dense in the graph of $h^\tau$; the conclusion follows since the graph of $h^\tau$ is a first countable space. \[\square\]

3.1. Examples of large deviation principles as consequence of (D). In this section we present two examples where the large deviation principle is a direct consequence of (D), but where the first condition may not be fulfilled, and thus that cannot be proved using Theorem C and Remark B.2 of \[3\] neither with Theorem 2.1 of \[13\]. They are both obtained from a net $(t_{\alpha}, m_{\alpha})$ generating the pressure in the sense that

$$\forall g \in C(\Omega), \quad \lim_{\alpha} t_{\alpha} \log \int_{\mathcal{M}(\Omega)} e^{t_{\alpha}^{-1} \int_{\Omega} g(\xi)|\mu|d\xi} m_{\alpha}(d\mu) = P^\tau(g),$$

\[8\] HENRI COMMAN

\[1\]
which yields (11) after normalization and the choice of an arbitrary function \( f \in C(\Omega) \) (so that no vector space \( V \) can fulfill the first condition stated in (11) when \( f \) admits several equilibrium states); however, assuming (D), the large deviation principle follows as a straightforward application of Theorem 5.2 of [4].

3.1.1. Property (D) and maximal separated sets. We generalize Theorem 5.7(a) and Example 4.1 of [4] to any dynamical system \((\Omega, \tau)\) as in (11).

Let \( \varphi \) denote the product set \([0, +\infty[ \times \mathbb{N}^l \times \mathbb{N}^l\) pointwise directed, where \([0, +\infty[ \) (resp. \( \mathbb{N}, \mathbb{N}^l\)) is endowed with the inverse of the natural order on \( \mathbb{R}\) (resp. natural order, lexicographic order), i.e. \((\varepsilon, u) \in \varphi\) is less than or equal to \((\varepsilon', u') \in \varphi\) if \( \varepsilon \geq \varepsilon'\) and \( u(\delta)\) is lexicographically less than or equal to \( u'(\delta)\) for all \( \delta \in [0, +\infty[\) (cf. [12]). For each \( \alpha = (\varepsilon, u) \in \varphi \) we put \( \Lambda_\alpha = \Lambda(u(\varepsilon)), \Omega_\alpha = \Omega_{\varepsilon,u(\varepsilon)} \) and

\[
\forall f \in C(\Omega), \quad \nu_{f,\alpha}^\tau = \sum_{\xi \in \Omega_\alpha} \frac{e^{\sum_{\varepsilon \in \Lambda_\alpha} f(\tau^\varepsilon \xi)}}{e^{\sum_{\varepsilon \in \Lambda_\alpha} f(\tau^\varepsilon \xi)} \delta_{\tau^\varepsilon \xi}} \sum_{\varepsilon \in \Lambda_\alpha} \delta_{\tau^\varepsilon}\xi.
\]

**Proposition 1.** If (D) holds, then for each \( f \in C(\Omega) \) the net \( (\nu_{f,\alpha}^\tau) \) satisfies a large deviation principle in \( \mathcal{M}(\Omega) \) with powers \( (|\Lambda_\alpha|^{-1}) \) and rate function

\[
\mathcal{M}(\Omega) \ni \mu \mapsto \begin{cases} P^\tau(f) - \mu(f) - h^\tau(\mu) & \text{if } \mu \in \mathcal{M}^\tau(\Omega) \\ +\infty & \text{if } \mu \in \mathcal{M}(\Omega) \setminus \mathcal{M}^\tau(\Omega). \end{cases}
\]

**Proof.** For each \( g \in C(\Omega) \) let \( s_g \) be the real-valued map on \([0, +\infty[ \times \mathbb{N}^l\) defined by

\[
\forall (\varepsilon, a) \in [0, +\infty[ \times \mathbb{N}^l, \quad s_g(\varepsilon, a) = \frac{1}{|\Lambda(\varepsilon)|} \log \sum_{\xi \in \Omega_{\varepsilon,a}} e^{\sum_{\varepsilon \in \Lambda(\varepsilon)} g(\tau^\varepsilon \xi)}.
\]

Then, (2) together with Lemma (11) (applied with \( J = [0, +\infty[, L = \mathbb{N}^l \) and the above function \( s_g \) yields

\[
\forall g \in C(\Omega), \quad P^\tau(g) = \lim_{\alpha} \frac{1}{|\Lambda_\alpha|} \log \sum_{\xi \in \Omega_\alpha} e^{\sum_{\varepsilon \in \Lambda_\alpha} g(\tau^\varepsilon \xi)}
\]

hence

\[
\forall (f, g) \in C(\Omega)^2, \quad \lim_{\alpha} \frac{1}{|\Lambda_\alpha|} \log \int_{\mathcal{M}(\Omega)} e^{(t^\tau_\alpha)^{-1} \int g(\omega) \mu(\omega) \nu_{f,\alpha}^\tau(d\mu)} = P^\tau(f + g) - P^\tau(f)
\]

(i.e. (11) holds with \( (\nu_\alpha, t_\alpha) = (\nu_{f,\alpha}^\tau, |\Lambda_\alpha|^{-1})\); the conclusion follows from Theorem 5.2 of [4].

3.1.2. Subshifts of finite type and periodic points. The following example illustrates how direct is the use of (D) in order to get a large deviation principle in comparison with usual proofs: indeed, the conclusion of Proposition (2) when \( f = 0 \) is exactly Theorem C of [7] (note that the fact that we get at once the general case \( f \in C(\Omega) \) is just a bonus since it follows from the case \( f = 0 \) by a standard result in large deviation theory, cf. Appendix B in [8]).
Let $(\Omega, \tau)$ be a $l$-dimensional subshift of finite type satisfying strong specification (22, 7). For each $a \in \mathbb{N}^l$ we put

$$\forall f \in C(\Omega), \quad \nu_{f,a} = \sum_{\xi \in \text{Per}_a} e^{\sum_{x \in \Lambda(a)} f(\tau^x \xi)} \frac{1}{|\Lambda(a)|} \sum_{\xi' \in \text{Per}_a} e^{\sum_{x \in \Lambda(a)} f(\tau^x \xi')} \delta_{\sum_{x \in \Lambda(a)} \delta(\tau^x \xi)},$$

where $\text{Per}_a$ denote the set of $a$-periodic points.

**Proposition 2.** For each $f \in C(\Omega)$ the net $(\nu_{f,a})$ satisfies a large deviation principle in $\mathcal{M}(\Omega)$ with powers $(|\Lambda(a)|^{-1})$ and rate function

$$\mathcal{M}(\Omega) \ni \mu \mapsto \begin{cases} P^\tau(f) - \mu(f) - h^\tau(\mu) & \text{if } \mu \in \mathcal{M}^\tau(\Omega), \\ +\infty & \text{if } \mu \in \mathcal{M}(\Omega) \setminus \mathcal{M}^\tau(\Omega), \end{cases}$$

Proof. Theorem 2.2 of [22] yields

$$\forall g \in C(\Omega), \quad P^\tau(g) = \lim_{a \to \infty} \frac{1}{|\Lambda(a)|} \log \sum_{\xi \in \text{Per}_a} e^{\sum_{x \in \Lambda(a)} g(\tau^x \xi)}$$

hence

$$\forall (f, g) \in C(\Omega)^2, \quad \lim_{a \to \infty} \frac{1}{|\Lambda(a)|} \log \int_{\mathcal{M}(\Omega)} e^{(t_a)^{-1} \int_\Omega g(\omega) \mu(d\omega)} \nu_{f,a}(d\mu) = P^\tau(f + g) - P^\tau(f).$$

Since the strong specification implies the weak specification, and since $h^\tau$ is upper semi-continuous by expansiveness, (D) holds by Theorem B of [7]; the conclusion follows from Theorem 5.2 of [4]. \qed

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