N=2 local and N=4 nonlocal reductions of supersymmetric KP hierarchy in N=2 superspace

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Abstract
A $N = 4$ supersymmetric matrix KP hierarchy is proposed and a wide class of its reductions which are characterized by a finite number of fields are described. This class includes the one-dimensional reduction of the two-dimensional $N = (2|2)$ superconformal Toda lattice hierarchy possessing the $N = 4$ supersymmetry – the $N = 4$ Toda chain hierarchy – which may be relevant in the construction of supersymmetric matrix models. The Lax pair representations of the bosonic and fermionic flows, corresponding local and nonlocal Hamiltonians, finite and infinite discrete symmetries, the first two Hamiltonian structures and the recursion operator connecting all evolution equations and the Hamiltonian structures of the $N = 4$ Toda chain hierarchy are constructed in explicit form. Its secondary reduction to the $N = 2$ supersymmetric $\alpha = -2$ KdV hierarchy is discussed.

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1 Introduction

Recently an infinite class of bosonic and fermionic solutions of the symmetry equation of the two-dimensional $N = (1|1)$ superconformal Toda lattice equation was constructed in [1, 2]. These solutions generate bosonic and fermionic flows of the $N = (1|1)$ supersymmetric Toda lattice hierarchy in the same way as their bosonic counterparts – solutions of the symmetry equation of the Toda equation – produce the flows of the bosonic Toda lattice hierarchy. The symmetry equation is a complicated nonlinear functional equation, and its general solution is not known. In this respect one cannot exclude that beside the solutions found in [1, 2] there are other solutions of the symmetry equation which could be responsible for additional flows of the hierarchy. Moreover, most probably such solutions do exist. Indeed, if one remembers that the $N = (1|1)$ superconformal Toda lattice equation actually admits an $N = (2|2)$ superconformal symmetry [3] one can suppose that its one-dimensional reduction – supersymmetric Toda chain equation – possesses a global $N = 4$ supersymmetry. In this respect it seems interesting to develop the Lax-pair representation method for the supersymmetric Toda chain hierarchy which is still unknown, aiming at the description of as many of its flows as possible. This is the main goal of the present paper.

Using a dressing formalism, we construct the Lax-pair description of the supersymmetric Toda chain hierarchy and observe new series of both fermionic and bosonic flows at our knowledge unknown before. We show that it represents a reduction of the supersymmetric KP hierarchy in the $N = 2$ superspace which actually possesses an $N = 4$ supersymmetry. We also derive its infinite-dimensional algebra of flows and demonstrate that it indeed contains a global $N = 4$ supersymmetry algebra including its $gl(2)$ automorphisms. Due to this reason we call this hierarchy the $N = 4$ supersymmetric Toda chain hierarchy. The Lax operators that we propose possess a peculiarity which make them different from all other Lax operators applied earlier to the description of integrable supersymmetric systems in the $N = 2$ superspace (for recent papers, see [4]–[25] and references therein): they are not invariant with respect to the $U(1)$ automorphism of the $N = 2$ supersymmetry. As a byproduct we propose an infinite family of new reductions of the well known supersymmetric KP hierarchy in the $N = 2$ superspace which lead to $U(1)$ non-invariant Lax operators.

The paper is organized as follows. In section 2 we develop tools for the Lax-pair description of integrable systems with $N = 4$ supersymmetry. Thus, in the framework of the dressing approach we give the description of the supersymmetric KP hierarchy in the $N = 2$ superspace and demonstrate that it actually is $N = 4$ supersymmetric. In section 3 we construct a zero-curvature representation for the $N = (2|2)$ superconformal Toda lattice equation and use it to derive the Lax operator generating the bosonic flows of the supersymmetric Toda chain hierarchy. Then, in section 4 we use this Lax operator to construct a consistent reduction of all other flows of the $N = 4$ supersymmetric KP hierarchy preserving its algebra structure, and show that the reduced hierarchy – supersymmetric Toda chain – possesses an $N = 4$ supersymmetry. In section 5 we discuss its finite and infinite discrete symmetries, and use them to obtain new Lax operators. In section 6 we construct its local and nonlocal Hamiltonians, the first two Hamiltonian structures and the recursion operator. In section 7 we obtain the $U(1)$–automorphism transformations for all the flows and derive a new Lax operator. In section 8 we establish a relationship between the $N = 4$ Toda chain hierarchy and $N = 2, \alpha = -2$ KdV hierarchy. In section 9 we propose a generalization of the $N = 4$ KP and Toda chain hierarchies in the matrix case. An appendix contains a realization of the algebra of flows of the $N = 4$ supersymmetric KP hierarchy.
2 N=4 supersymmetric KP hierarchy

In this section we discuss the hierarchy which is usually called the \( N = 2 \) supersymmetric KP hierarchy and focus on the remarkable fact that it actually possesses an \( N = 4 \) supersymmetry algebra.

Our starting object is the \( N = 2 \) supersymmetric dressing operator \( W \)

\[
W \equiv 1 + \sum_{n=1}^{\infty} (w_n(0) + w_n^{(+)} D_+ + w_n^{(-)} D_-) \partial^{-n},
\]

where all the functions \( w_n \) involved into \( W \) are \( N = 2 \) superfields in the superspace \( Z \equiv (z, \theta^+, \theta^-) \), and \( D_\pm \) are fermionic covariant derivatives which together with the supersymmetry generators \( Q_\pm \) form the algebra

\[
\{D_\pm, D_\pm\} = +2 \partial, \quad \{Q_\pm, Q_\pm\} = -2 \partial
\]

with the standard superspace realization:

\[
D_\pm \equiv \frac{\partial}{\partial \theta^\pm} + \theta^\pm \partial, \quad Q_\pm \equiv \frac{\partial}{\partial \theta^\pm} - \theta^\pm \partial.
\]

Our aim is to construct a maximal set of consistent Sato equations for the dressing operator \( W \) which are the flows of the extended supersymmetric KP hierarchy in the \( N = 2 \) superspace.

Keeping this in mind, we first construct all linearly independent linear differential operators defined in the \( N = 2 \) superspace which are then dressed by the operator \( W \). As a result we obtain the operators

\[
L^\pm_i \equiv W D^i_\pm W^{-1}, \quad M^\pm_i \equiv W Q^i_\pm W^{-1}, \quad N^\pm_i \equiv W \frac{1}{2} [\theta^\pm, D_\pm] \partial^i W^{-1},
\]

\[
N_i \equiv -W (\theta^+ D_- + \theta^- D_+) \partial^i W^{-1}, \quad \overline{N}_i \equiv W \frac{1}{2} (\theta^- (D_+ + Q_+) - \theta^+ (D_- + Q_+)) \partial^i W^{-1}
\]

with the obvious properties:

\[
L^\pm_i \equiv (L^\pm_i)^\dagger, \quad M^\pm_i \equiv (M^\pm_i)^\dagger, \quad L^\pm_{2i} \equiv L_{2i} \equiv (-1)^i M^\pm_{2i} \equiv (-1)^i M^\pm_{2i}
\]

which will be useful in what follows.

Second, using the operators (4) we construct the consistent Sato equations for \( W \)

\[
\frac{\partial}{\partial t_i} W = -(L^\pm_{2i})_- W, \quad U^\pm_i W = -(N^\pm_i)_- W,
\]

\[
U_i W = -(N_i)_- W, \quad \overline{U}_i W = -(\overline{N}_i)_- W,
\]

\[
D^\pm_i W = -(L^\pm_{2i-1})_- W, \quad Q^\pm_i W = -(M^\pm_{2i-1})_- W
\]

where the subscript \(-\) (\(+\)) denote the purely pseudo-differential (differential) part of the operator. The bosonic (fermionic) evolution derivatives \( \{\frac{\partial}{\partial t_i}, U^\pm_i, U_i, \overline{U}_i\} \) ( \( \{D^\pm_i, Q^\pm_i\} \) ) generating bosonic (fermionic) flows of the hierarchy under consideration have the following length dimensions:

\[
[\frac{\partial}{\partial t_i}] = [U^\pm_i] = [U_i] = [\overline{U}_i] = -l, \quad [D^\pm_i] = [Q^\pm_i] = -l + \frac{1}{2}. \tag{7}
\]

\(^{1}\)Hereafter, we explicitly present only non-zero brackets.
We would like to stress that the flows \{\partial_{\theta_l}, D_l^\pm\} form the hierarchy which is usually called the \(N = 2\) supersymmetric KP hierarchy. Therefore, the flows \{U_l^\pm, U_l, \overline{U}_l, Q_l^\pm\} when added to the \(N = 2\) KP hierarchy, produce an extended hierarchy possessing a richer algebra structure.

In order to understand deeper what is the extended hierarchy we have in fact obtained, let us calculate its algebra of flows. One can use a supersymmetric generalization \cite{26,27} of the Radul map \cite{28} which is a homomorphism between the algebra of flows we are looking for and the algebra of the operators \(L_l^\pm, M_l^\pm, N_l^\pm, N_l^i, N_l^f\), and \(\nabla_l\) \cite{4}. The resulting algebra is:

\[
\{D_k^\pm, D_l^\pm\} = -2 \frac{\partial}{\partial t_{k+l-1}}, \quad \{Q_k^\pm, Q_l^\pm\} = +2 \frac{\partial}{\partial t_{k+l-1}},
\]

and its realization is given in the appendix.

A simple inspection of the superalgebra \(\{8-10\}\) shows that the flows \(\partial_{\theta_l}, U_0^\pm, U_0, \overline{U}_0, D_1^\pm\) and \(Q_1^\pm\) form the \(N = 4\) supersymmetry algebra including its \(gl(2)\) automorphisms. This invariance algebra forms a finite–dimensional subalgebra of the superalgebra \(\{8-10\}\). Due to this reason the extended hierarchy can be called the \(N = 4\) supersymmetric KP hierarchy.

To close this section let us only mention that a similar approach with respect to the \(N = 1\) supersymmetric dressing operator was developed in \cite{27}, and as a result \(N = 2\) supersymmetric flows were derived. In \cite{27} the corresponding supersymmetric hierarchy was called the maximal SKP hierarchy, and it includes both Manin-Radul \cite{29} and Mulase-Rabin \cite{30,31} hierarchies.

3 Hint for the \(N=4\) KP reduction: zero-curvature representation of the \(N = (2|2)\) superconformal Toda lattice

In this section we derive the Lax operator generating the bosonic flows of the supersymmetric Toda chain hierarchy using the zero-curvature representation of the \(N = (2|2)\) superconformal Toda lattice equation.

One starts from the two-dimensional \(N = (2|2)\) superconformal Toda lattice equation

\[
D_- D_+ \ln b_i = b_{i+1} - b_{i-1}
\]

written in terms of the bosonic \(N = (1|1)\) superfield \(b_i \equiv b_i(z^+, \theta^+; z^-, \theta^-)\) defined on the lattice, \(i \in \mathbb{Z}\), and \(D_\pm\) are the \(N = 1\) supersymmetric fermionic derivatives in the right and left chiral subspaces \((z^\pm, \theta^\pm)\),

\[
\{D_\pm, D_\pm\} = +2 \partial_\pm, \quad D_\pm \equiv \frac{\partial}{\partial \theta^\pm} + \theta^\pm \partial_\pm.
\]

Eq. (11) can be rewritten in the form of a system of two equations \cite{11}

\[
D_- f_i = b_i + b_{i+1}, \quad D_+ \ln b_i = f_i - f_{i-1}
\]
which admits the zero-curvature representation
\[ \{ D_+ - A_{\theta^+}, D_- - A_{\theta^-} \} = 0 \] (14)
with the fermionic connections
\[
(A_{\theta^+})_{ij} \equiv f_i \delta_{i,j} + \delta_{i,j-1}, \quad (A_{\theta^-})_{ij} \equiv -b_i \delta_{i,j+1},
\] (15)
where \( f_i \equiv f_i(z^+, \theta^+; z^-, \theta^-) \) is a fermionic \( N = (1|1) \) lattice superfield. One can define the bosonic connections \( A_{z^\pm} \) by
\[
\partial_+ + A_{z^+} \equiv (D_+ - A_{\theta^+})^2, \quad \partial_- + A_{z^-} \equiv (D_- - A_{\theta^-})^2.
\] (16)
They explicitly read
\[
(A_{z^-})_{ij} \equiv D_- b_i \delta_{i,j+1} - b_i b_{i+1} \delta_{i,j+2}, \quad (A_{z^+})_{ij} \equiv -D_+ f_i \delta_{i,j} + (f_i - f_{i+1}) \delta_{i,j-1} - \delta_{i,j-2}
\] (17)
and due to (14) obviously satisfy the zero-curvature condition
\[ [\partial_- + A_{z^-}, \partial_+ + A_{z^+}] = 0 \] (18)
which is a consistency condition for the following linear system:
\[
(\partial_- + A_{z^-}) \Psi = \lambda \Psi, \quad \quad (\partial_+ + A_{z^+}) \Psi = 0,
\] (19) (20)
where \( \Psi \equiv \Psi_i \) is the lattice wave function and \( \lambda \) is a spectral parameter. Taking into account the first relation of eqs. (16), equation (20) can equivalently be rewritten in the following form:
\[ (D_+ - A_{\theta^+}) \Psi = 0. \] (21)
The linear system (19), (21) is a key object in our consideration.

Now, let us consider a consistent reduction to a one-dimensional subspace by setting
\[ \partial_- = \partial_+ \equiv \partial \] (22)
which leads to the supersymmetric Toda chain. Then, in order to derive the Lax operator we are looking for, we follow a trick proposed in [32] and express each lattice function entering the spectral equation (19) in terms of lattice functions defined at the single lattice point \( i \) using eqs. (13) and (21). Taking into account also the reduction condition (22), we obtain the following resulting spectral equation
\[ (D_- + \frac{1}{D_+ - f_i})^2 \Psi_i = \lambda \Psi_i. \] (23)
For each fixed value of \( i \), it represents the spectral equation of the differential supersymmetric Toda chain hierarchy, i.e. the hierarchy of equations involving only the superfields \( b_i, f_i \) at a single lattice point. The discrete lattice shift (i.e., the system of eqs. (13)) when added to the differential hierarchy, generates the discrete supersymmetric Toda chain hierarchy [2]. Thus,
the discrete hierarchy appears as a collection of an infinite number of isomorphic differential hierarchies [32].

It is well known that a spectral equation is just an equation for a Lax operator. For a fixed value of \( i \) one can omit the lattice index in the spectral equation (23), and it is obvious that the operator

\[
L = (D_+ + \frac{1}{D_+ - f})^2
\]  

(24)

entering eq. (23) is the Lax operator which is responsible for the bosonic flows of the differential hierarchy. It can be simplified in the new superfield basis \( \{v_i, u_i\} \) defined as

\[
b_i \equiv u_i v_i, \quad f_i \equiv D_+ \ln v_i
\]  

(25)

in which the system (13) takes the form of the \( N = (1|1) \) supersymmetric generalization of the Darboux transformation [1]

\[
u_{i+1} = \frac{1}{v_i}, \quad D_- D_+ \ln v_i = u_{i+1} v_{i+1} + u_i v_i,
\]  

(26)

and the Lax operator (24) is

\[
L = (D_+ + v D_-^{-1} u)^2.
\]  

(27)

This Lax operator will be used in the next section to construct a consistent reduction of all other flows of the \( N = 4 \) supersymmetric KP hierarchy.

### 4 Reduction: bosonic and fermionic flows

In this section we consider a reduction of the \( N = 4 \) supersymmetric KP hierarchy which is inspired by the Lax operator \( L \) (27) and preserves its algebra of flows [8-10].

Keeping in mind the results of the previous section and equation (27), let us introduce the following constraint on the operator \( L^-_1 \) [4]

\[
L^-_1 = \mathcal{L} \equiv D_- + v D_+^{-1} u,
\]  

(28)

where \( \mathcal{L} \) is the square root of the operator \( L \) in eq. (27). The operator \( \mathcal{L} \) possesses the following important properties:

\[
(\mathcal{L}^{2(l-1)})_- = \sum_{k=0}^{2l-3} (\mathcal{L}^{2l-3-k} v) D_+^{-1} ( (\mathcal{L}^k)^T u), \quad l = 0, 1, 2...,
\]  

(29)

\[
(\mathcal{L}^{2l-1})_- = \sum_{k=0}^{2l-1} (\mathcal{L}^{2l-1-k} v) D_+^{-1} ( (\mathcal{L}^k)^T u) + \sum_{k=0}^{2l-3} (-1)^k (\mathcal{L}^{2l-3-k} v) D_+^{-1} D_- ( (\mathcal{L}^k)^T u)
\]  

(30)

\[\text{Let us recall the operator conjugation rules: } D_+^T = -D_-, \quad (OP)^T = (-1)^{d_O d_P} P^T O^T, \quad \text{where } O \text{ (P) is an arbitrary operator with the Grassmann parity } d_O \text{ (d_P), and } d_O = 0 \text{ (d_P = 1) for bosonic (fermionic) operators } O. \text{ All other rules can be derived using these. Hereafter, we use the notation } (\mathcal{O}f) \text{ for an operator } \mathcal{O} \text{ acting only on a function } f \text{ inside the brackets.}\]
which can be proved by induction in a similar way as analogous formulae in the bosonic \cite{33} and $N = 1$ supersymmetric \cite{34} cases. As one can see from eq. (28), even powers of $L$ contain only the operator $D_+$ (and not $D_-$) and are in some sense $N = 1$ like. Moreover, formula (28) coincides with an analogous formula in the $N = 1$ supersymmetric case \cite{34}.

Substituting the expression (4) for $L^-_1$ in terms of the dressing operator $W$ (1) into constraint (28), it becomes

$$WD_-W^{-1} = D_- + vD_+^{-1}u$$  \hspace{1cm} (31)

and gives an equation for $W$ which can be solved iteratively with the following unique solution $\mathcal{W}$:

$$\mathcal{W} \equiv W(w_n^{(-)} = 0, w_n^{(1)} = 0) \equiv 1 + \sum_{n=1}^{\infty} (w_n^{(0)} + w_n^{(+)D_+}) \partial^{-n},$$

$$w_1^{(+)D_+} = -D_-^{-1}(uv), \quad w_1^{(0)} = -D_-^{-1}(vD_+u + uvD_+^{-1}(uv)),$$

$$w_2^{(+)D_+} = -D_-^{-1}(vu' + uvD_+D_-^{-1}(uv)) + (D_-^{-1}(uv))D_-^{-1}(vD_+u + uvD_+^{-1}(uv)), \quad \ldots$$  \hspace{1cm} (32)

Replacing $W$ by $\mathcal{W}$ in eqs. (4) one can obtain the reduced operators $M_i^\pm, N_i^\pm, N_l$ and $\overline{N}_l$ as well. As an example, we present a few terms of the series in $D_+^{-1}$ in the case of $L^+_1$,

$$L_1^+ \equiv WD_+W^{-1} = D_+ + 2w_1^{(+)D_+} - (D_+w_1^{(+)D_+})D_+^{-1} - ((D_+w_1^{(0)}) - 2w_2^{(+)D_+} + w_1^{(+)D_+}D_+^{-1}$$

$$+ 2w_1^{(0)}D_+^{-2} - ((D_+(w_2^{(0)} - w_1^{(0)})) - (D_+(w_1^{(0)})^2))D_+^{-3} + \ldots$$  \hspace{1cm} (33)

where the functions $w_n^{(0)}$ and $w_n^{(+)D_+}$ are defined in eqs. (32). The following obvious relations:

$$M_1^- = L_1^- - 2\theta_-L_2^-, \quad M_1^+ = L_1^+ - 2\mathcal{W}\theta_+\mathcal{W}^{-1}L_2^+, \quad N_1^- = \theta_-L_2^{-1} - \frac{1}{2}L_2^-, \quad N_1^+ = \mathcal{W}\theta_+\mathcal{W}^{-1}L_2^{+1} - \frac{1}{2}L_2^{+}$$  \hspace{1cm} (34)

together with the relations (3) are useful at calculations.

The most complicated task is to construct a consistent set of Sato equations for the reduced $\mathcal{W}$ generalizing the unreduced equations (1) and preserving their algebra structure \cite{33,10}. A priori, it is completely unclear whether such equations exist at all. Moreover, there are still no algorithmic methods to construct them. Recently, a similar task was carried out in \cite{34} for some reductions of the Manin-Radul $N = 1$ supersymmetric KP hierarchy \cite{23}, and we use some of the ideas developed there. We succeeded in this construction only for the reduced $\frac{\partial}{\partial t_l}, U_i^\pm, D_i^\pm,$ and $Q_i^\pm$ flows. Nevertheless, at the end of this section we propose a heuristic construction which allows the remaining $U_l$ and $\overline{U}_l$ flows of the reduced $N = 4$ KP hierarchy to be restored as well.

The resulting Sato equations have the following form:

$$\frac{\partial}{\partial t_l}\mathcal{W} = -(L_2^{-1})_\mathcal{W},$$

$$U^+_l\mathcal{W} = -(N^+_l)_\mathcal{W}, \quad U^-_l\mathcal{W} = -((N^-_l) - \tilde{N}^-_{2l+1})\mathcal{W},$$

$$D^+_l\mathcal{W} = -(L^+_l)\mathcal{W}, \quad D^-_l\mathcal{W} = -((L^-_{2l-1}) - \tilde{L}^-_{2l-1})\mathcal{W},$$

$$Q^+_l\mathcal{W} = -(M^+_l)_\mathcal{W}, \quad Q^-_l\mathcal{W} = -((M^-_{2l-1}) - M^-_{2l-1})\mathcal{W},$$  \hspace{1cm} (35)

where new operators $\tilde{L}^-_{2l-1}, \tilde{M}^-_{2l-1}$ and $\tilde{N}^-_{2l-1}$ have been introduced,
where the equations (39–40) have been used.
Using eqs. (43) for the bosonic and fermionic flows, we present for illustration the first few:

\[ \frac{\partial}{\partial t_0} \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} +v \\ -u \end{pmatrix}, \quad \frac{\partial}{\partial t_1} \begin{pmatrix} v \\ u \end{pmatrix} = \partial \begin{pmatrix} v \\ u \end{pmatrix}, \]

\[ \frac{\partial}{\partial t_2} v = +v'' + 2uv(D_+D_-v) - 2(D_+D_-v^2)u - v^2(D_+D_-u) - 2v(uw)^2, \]

\[ \frac{\partial}{\partial t_2} u = -u'' + 2uv(D_+D_-u) - 2(D_+D_-u^2)v - u^2(D_+D_-v) - 2u(uw)^2, \]

\( D_1^\pm v = -D_\pm v \pm 2vD_\mp^{-1}(uv), \quad D_1^\pm u = -D_\pm u \mp 2uD_\mp^{-1}(uv), \)

\( D_2^\pm v = -D_\pm v' \pm 2v'D_\mp^{-1}(uv) \pm (D_\pm v)D_\mp^{-1}D_\pm(1)(uv) \pm vD_\mp^{-1}[uv' + (D_\pm v)D_\pm u], \)

\( D_2^\pm u = +D_\pm u' \pm 2u'D_\mp^{-1}(uv) \pm (D_\pm u)D_\mp^{-1}D_\pm(1)(uv) \pm uD_\mp^{-1}[uv' + (D_\pm u)D_\pm v], \)

\[ Q_1^\pm \begin{pmatrix} v \\ u \end{pmatrix} = Q^\pm \begin{pmatrix} v \\ u \end{pmatrix}, \]

\[ U_0^\pm v = \frac{1}{2} v - \theta^\pm (D_\pm v \mp 2vD_\mp^{-1}(uv)), \quad U_0^\pm u = \frac{1}{2} u - \theta^\pm (D_\pm u \pm 2uD_\mp^{-1}(uv)). \]

The flows (44–46) reproduce the one-dimensional reduction of the flows of the two-dimensional N = (1|1) superconformal Toda lattice hierarchy derived in [1, 2].

As it was already announced at the beginning of this section, now we would like to briefly discuss the construction of the remaining two series of the N = 4 KP flows, i.e. \( U_k \) and \( \overline{U}_k \), for the reduced hierarchy. This construction is based on the very simple and obvious observation that the first two commutation relations of the subalgebra (9) of the algebra (8–10) can be used to construct the flows \( U_k \) and \( \overline{U}_k \) for \( k = 1, 2, \ldots \) in terms of the flows \( U_k^\pm \) already known and presented in eqs. (43) as well as the flows \( U_0 \) and \( \overline{U}_0 \)

\[ U_0 v = -\theta^+(D_-v + 2vD_+^{-1}(uv)) - \theta^-(D_+v - 2vD_-^{-1}(uv)), \]

\[ U_0 u = -\theta^+(D_-u - 2uD_+^{-1}(uv)) - \theta^-(D_+u + 2uD_-^{-1}(uv)), \]

\[ \overline{U}_0 \begin{pmatrix} v \\ u \end{pmatrix} = \frac{1}{2} \left( \theta^-(D_+ + Q_-) - \theta^+(D_- + Q_+ \right) \begin{pmatrix} v \\ u \end{pmatrix} \]

which were derived by brute force in order to satisfy the algebra (8–10). Therefore, due to existence of the flows \( U_0 \) and \( \overline{U}_0 \) (48–49), the flows \( U_k \) and \( \overline{U}_k \) for \( k = 1, 2, \ldots \) can indeed be obtained, and they explicitly read:

\[ U_k = -\left[ \overline{U}_0, U_k^+ \right], \quad \overline{U}_k = \left[ U_0, U_k^+ \right]. \]

We would like to close this section with a few remarks.

First, a simple inspection of eqs. (43) shows that the reduced flows differ from the N = 4 KP flows in that almost all flows except the bosonic flows \( \frac{\partial}{\partial t_i} \) are nonlocal (for an example, see eqs. (43), (47–48) ). Nonlocal fermionic flows of supersymmetric KdV, GNLS and Toda type systems were discussed also earlier in [33, 36, 37, 38, 39, 40, 41, 42] (see also the quite recent paper [39]).

\(^{3}\)We have rescaled some evolution derivatives to simplify the presentation of some formulæ.
Second, the flows \( \{ \frac{\partial}{\partial t^1}, U_0^\pm, U_0, \mathcal{U}_0, D_1^\pm \} \) forming the \( N = 4 \) supersymmetry algebra are non-locally and non-linearly realized in terms of the initial superfields \( v \) and \( u \). However, there exists another superfield basis \( \{ \hat{v}, \hat{u} \} \), defined as

\[
\{v, \ u\} \implies \{ \hat{v} \equiv v \exp[-D_+^1 D_-^1(uv)], \ \hat{u} \equiv u \exp[+D_+^1 D_-^1(uv)] \},
\]

which localizes and linearizes the \( N = 4 \) supersymmetry realization which becomes now

\[
\frac{\partial}{\partial t^1} = \partial, \ D_1^\pm = -D_{\pm}, \ Q_1^\pm = Q_{\pm}, \ U_0^\pm = \frac{1}{2} [D_{\pm}, \theta^\pm], \ U_0 = - (\theta^+ D_- + \theta^- D_+) - \frac{1}{2} (\theta^- (D_+ + Q_+ - \theta^+ (D_+ + Q_-)).
\]

However, in this basis the even flows \( \frac{\partial}{\partial t^l} \) for \( l \geq 2 \) are nonlocal.

Third, the flows \( Q_l^\pm \) for \( l \geq 2 \) as well as the flows \( U_l^\pm, U_l \) and \( \mathcal{U}_l \) in eqs. (13) and (50) are obtained at our knowledge for the first time together with the Lax-pair representation of the whole hierarchy.

Let us finally stress that, contrary to all known Lax operators used before in the literature for the description of integrable supersymmetric systems in the \( N = 2 \) superspace, the Lax operators proposed in this section are not invariant with respect to the \( U(1) \)-automorphism of the \( N = 2 \) supersymmetry. We will return in section 7 to the discussion of consequences of this important difference.

5 Discrete symmetries, Darboux-Bäcklund transformations and solutions

In this section we discuss finite and infinite discrete symmetries of the reduced hierarchy, and use them to construct its solutions and new Lax operators.

Direct verification shows that the flows (14-15) admit the four involutions:

\[
(v, u)^* = i(u, v), \ (z, \theta^\pm)^* = (z, \theta^\mp), \\
(t_p, U_p^\pm, U_p, \mathcal{U}_p, D_p^\pm, Q_p^\pm)^* = (-1)^{p-1} (t_p, -U_p^\pm, -U_p, -\mathcal{U}_p, D_p^\pm, Q_p^\pm),
\]

\[
(v, u)^\dagger = (u, v), \ (z, \theta^\pm)^\dagger = (z, \theta^\mp), \\
(t_p, U_p^\pm, U_p, \mathcal{U}_p, D_p^\pm, Q_p^\pm)^\dagger = (-1)^{p-1} (t_p, -U_p^\pm, -U_p, \mathcal{U}_p, D_p^\pm, Q_p^\pm),
\]

\[
(v, u)^* = (v, u), \ (z, \theta^\pm)^* = (z, \pm \theta^\mp), \\
(t_p, U_p^\pm, U_p, \mathcal{U}_p, D_p^\pm, Q_p^\pm)^* = (t_p, U_p^\mp, -U_p, \mathcal{U}_p, \pm D_p^\pm, \pm Q_p^\mp),
\]

\[
v^* = v \exp[(\theta^+ (D_-^1 - Q_-^1) - \theta^- (D_+^1 - Q_+^1))(uv)], \\
u^* = u \exp[(\theta^+ (D_-^1 - Q_-^1) - \theta^- (D_+^1 - Q_+^1))(uv)], \\
(z, \theta^\pm)^* = -(z, \theta^\mp), \ (t_p, U_p^\pm, U_p, \mathcal{U}_p, D_p^\pm, Q_p^\pm)^* = (-1)^{p} (t_p, U_p^\pm, U_p, \mathcal{U}_p, -Q_p^\pm, -D_p^\pm)
\]

which are consistent with their algebra (2), (8-10).
It is a simple exercise to check that all the flows (39–42) (or (43)) also possess the involution (52), using the following involution property of the dressing operator $\mathcal{W}$:

$$\mathcal{W}^* = (\mathcal{W}^{-1})^T$$

resulting from eq. (51) and its consequences

$$(L^\pm_t)^* = (-1)^{j(t-1)}(L^\pm_t)^T,$$  

$$(M^\pm_t)^* = (-1)^{j(t-1)}(M^\pm_t)^T,$$  

$$(N^\pm_t)^* = (-1)^{j(t-1)}(N^\pm_t)^T,$$  

$$\tilde{L}_{2l-1}^* = (-1)^j(\tilde{L}_{2l-1})^T,$$  

$$M_{2l-1}^* = (-1)^j(M_{2l-1})^T,$$  

$$\tilde{N}_{2l-1}^* = (-1)^j(\tilde{N}_{2l-1})^T$$

for the operators entering eqs. (39–42). As regards the involutions (54–56), we do not have a direct proof that they are symmetries of all flows due to the complicated transformation properties of the dressing operator. Fortunately, a simple proof can be given using the recurrence relations (58), to be derived later. We return to this question in section 6 (see the paragraph after eq. (59)).

Beside involutions (33–35), flows (44–49) possess the infinite-dimensional group of discrete Darboux transformations (28) [1, 2]

$$(v, u)^\dagger = (v(D_-D_+ \ln v - uv), \frac{1}{v}), \quad (z, \theta^\pm)^\dagger = (z, \theta^\pm),$$

$$(t_p, U^\pm_p, U_p, U_p, D^\pm_p, Q_p^\pm)^\dagger = (t_p, -U^\pm_p, -U_p, U_p, -D^\pm_p, Q_p^\pm)$$

which may be written on the Lax operator $\mathcal{L}$ (28) as:

$$\mathcal{L}^{(4)} = -\mathcal{T}\mathcal{L}\mathcal{T}^{-1}, \quad \mathcal{T} \equiv vD_+v^{-1}.$$  

Applying involutions (33–35) and the discrete group (39) to the Lax operator $\mathcal{L}$ (28) one can derive other consistent Lax operators

$$\mathcal{L}^* = D_- - uD_+^{-1}v \equiv -\mathcal{L}^T,$$  

$$\mathcal{L}^\dagger = D_+ + uD_-^{-1}v,$$  

$$\mathcal{L}^* = -D_+ + vD_-^{-1}u \equiv (\mathcal{L}^\dagger)^T,$$  

$$\mathcal{L}^* = -Q_- + v^*Q_+^{-1}u^*,$$

$$\mathcal{L}^{(j+1)} = D_- + v^{(j+1)}D_+^{-1}u^{(j+1)} \equiv -\mathcal{T}^{(j)}\mathcal{L}^{(j)}\mathcal{T}^{(j)}^{-1}, \quad \mathcal{T}^{(j)} \equiv v^{(j)}D_+v^{(j)}^{-1}$$

generating isomorphic flows. $\mathcal{L}^{(j)}$ is obtained from $\mathcal{L}$ by applying $j$ times the discrete transformation (50), e.g. $\mathcal{L}^{(3)} \equiv ((\mathcal{L}^\dagger)^\dagger)^3, \mathcal{L}^{(0)} \equiv \mathcal{L}$.

One can construct an infinite class of solutions of the reduced hierarchy under consideration generalizing the results obtained in [3] for the bosonic and fermionic flows $\frac{\partial}{\partial \tau_k}$ and $D^\pm_k$. These results are based on the discrete Darboux symmetry (24), and we present their generalization with brief comments referring to [3] for details.

The simplest solution of the hierarchy is:

$$u = 0,$$

then the bosonic and fermionic flows for the remaining superfield $v \equiv -\tau_0$ are linear and have the following form:

$$\frac{\partial}{\partial \tau_k} \tau_0 = \partial^k \tau_0, \quad D^\pm_k \tau_0 = -D^\pm \partial^{k-1} \tau_0, \quad Q^\pm_k \tau_0 = Q^\pm \partial^{k-1} \tau_0,$$

---

4For the reduced Manin-Radul $N = 1$ supersymmetric KP hierarchy the Darboux-Bäcklund transformations were discussed in [3] (see also references there).
\[ U_k^\pm \tau_0 = \frac{1}{2} [D_\pm, \theta^\pm] \partial^k \tau_0, \quad U_k \tau_0 = - (\theta^+ D_- + \theta^- D_+) \partial^k \tau_0, \]
\[ \Upsilon_k \tau_0 = \frac{1}{2} (\theta^- (D_+ + Q_+) - \theta^+ (D_- + Q_-)) \partial^k \tau_0. \quad (65) \]

To derive these equations it is only necessary to take into account the length dimensions \([7]\) of the evolution derivatives, their algebra \([8, 10]\) and the invariance of all flows \([39–42]\) with respect to the following \(U(1)\) transformations

\[ (v, u) \implies (\exp(+i\beta) v, \exp(-i\beta) u) \quad (66) \]

( consequently, only linear equations for \(v\) are admissible at \(u = 0\) ) which is obvious due to the invariance of the reduction constraint \([31]\). \(\beta\) is an arbitrary parameter.

Here, in order to simplify formulae, we restrict the analysis of the whole hierarchy to the case when only the flows \(\{\partial_\tau, D_\pm^k, Q_\pm^k\}\) are considered. Then, using the realization \((A.1)\) of the appendix the solution of eqs. \((64)\) is

\[ \tau_0 = \int d\lambda \, d\eta_+ \, d\eta_- \, \varphi(\lambda, \eta_+, \eta_-) \times \exp\left\{ x \lambda - \sum_{\alpha = \pm} \eta_\alpha \theta^\alpha \sum_{k=1}^{\infty} \left[ t_k + \sum_{\alpha = \pm} \left( \eta_\alpha (\theta_k^\alpha - \rho_k^\alpha) \lambda^{1-\alpha} + \theta^\alpha (\rho_k^\alpha - \theta_k^\alpha) - \theta_k^\alpha \rho_k^\alpha \lambda^{n-\alpha} \right) \right] \lambda^k \right\} \quad (67) \]

where \(\varphi\) is an arbitrary function of the bosonic (\(\lambda\)) and fermionic (\(\eta_\pm\)) spectral parameters with length dimensions

\[ [\lambda] = -1, \quad [\eta_\pm] = -\frac{1}{2}. \quad (68) \]

Applying the discrete group \((59)\) to the solution constructed \(\{u = 0, \, v = -\tau_0\}\), an infinite class of new solutions of the hierarchy is generated through an obvious iterative procedure \([3]\)

\[ v^{(2j+1)\parallel} = +(-1)^j \frac{\tau_{2j}}{\tau_{2j+1}}, \quad u^{(2j+1)\parallel} = +(-1)^j \frac{\tau_{2j+1}}{\tau_{2j}}, \quad (69) \]

where the \(\tau_j\) are\[5\]

\[ \tau_{2j} = \text{sdet}\left( \begin{array}{cc} \partial^{p+q+} \tau_0 & \partial^{p+q+m} D_- \tau_0 \\ \partial^{k+q} D_+ \tau_0 & \partial^{k+q+m} D_+ \tau_0 \end{array} \right)_{0 \leq p, q \leq j}, \quad 0 \leq k, m \leq j-1, \]
\[ \tau_{2j+1} = \text{sdet}\left( \begin{array}{cc} \partial^{p+q+} \tau_0 & \partial^{p+q+m} D_- \tau_0 \\ \partial^{k+q} D_+ \tau_0 & \partial^{k+q+m} D_+ \tau_0 \end{array} \right)_{0 \leq p, q \leq j}, \quad 0 \leq k, m \leq j. \quad (70) \]

6 Hamiltonian structure

In this section we construct local and nonlocal Hamiltonians, the first two Hamiltonian structures and the recursion operator of the reduced hierarchy.

\[ \text{sdet} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \equiv \text{det}(A - BD^{-1}C)(\text{det}D)^{-1}. \]

5 The superdeterminant is defined as \(\text{sdet} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \equiv \text{det}(A - BD^{-1}C)(\text{det}D)^{-1}. \)
Let us first present our notations for the $N = 2$ superspace measure and delta function
\[ dZ \equiv dzd\theta^+d\theta^-, \quad \delta^{N=2}(Z) \equiv \theta^+\theta^-\delta(z), \]
respectively, as well as the realization of the inverse derivatives
\[ D_\pm^{-1} \equiv D_\pm\partial_\pm^{-1}, \quad Q_\pm^{-1} \equiv -Q_\pm\partial_\pm^{-1}, \quad \partial_z^{-1} \equiv \frac{1}{2}\int_{-\infty}^{+\infty} dx\epsilon(z-x), \]
\[ \epsilon(z-x) = -\epsilon(x-z) \equiv 1, \quad \text{if} \quad z > x \]
which we use in what follows. We also use the correspondence:
\[ \frac{\partial}{\partial\tau^\dagger} \equiv \{ \partial_t^\dagger, U^\pm_\dagger, U^\dagger_\pm, \partial_z^\dagger \} \leftrightarrow \mathcal{H}_l^\alpha \equiv \{ \mathcal{H}_l^U_\dagger, \mathcal{H}_l^U, \mathcal{H}_l^D_\dagger, \mathcal{H}_l^D \} \]
between the evolution derivatives $\frac{\partial}{\partial\tau^\dagger}$ and Hamiltonian densities $\mathcal{H}_l^\alpha$. Consequently, the last ones have the length dimensions:
\[ [\mathcal{H}_l^U] = [\mathcal{H}_l^{U_\dagger}] = [\mathcal{H}_l^D] = [\mathcal{H}_l^{D_\dagger}] = -l, \quad [\mathcal{H}_l^{D^\dagger} \equiv [\mathcal{H}_l^{D_\dagger} \equiv -l + \frac{1}{2}. \]
The length dimensions of the Hamiltonian $\mathcal{H}_l^\alpha$,
\[ H_l^\alpha \equiv \int dZ\mathcal{H}_l^\alpha, \]
and its density $\mathcal{H}_l^\alpha$ coincide because the $N = 2$ superspace measure has zero length dimension, $[dZ] = 0$.

Let us next discuss a proper definition of the residue for the class of pseudo-differential operators generated by powers of the Lax operators proposed in section 4. As we already mentioned, even powers of our Lax operator $\mathcal{L}$ (28) look like pure $N = 1$ supersymmetric operators, since they do not contain the operator $D_-$ (see paragraph after eq. (30)). Due to this reason it seems reasonable to suppose that the residue have to coincide with the residue usually used for an $N = 1$ supersymmetric pseudo-differential operator, i.e. with the coefficient of the operator $D_-^{-1}$. Remembering that this coefficient, when integrated over the $N = 1$ supersymmetric measure $dZ_+ \equiv dzd\theta_+$, has to be actually $N = 2$ supersymmetric as the hierarchy we started with, it should necessarily admit a representation in the form of the operator $D_-^{-1}$ acting on a functional $F(z, \theta_+, \theta_-)$ of the superfields $v$ and $u$ of the hierarchy. Indeed, in this case the equality $\int dZ_+(D_-F)(z, \theta_+, \theta_-) \equiv \int dZF(z, \theta_+, \theta_-)$ is obviously valid just due to the standard definition of the $N = 2$ superspace integral and, consequently, as a result we obtain an $N = 2$ supersymmetric object. Taking into account these heuristic arguments we define the residue of a pseudo-differential operator $\Psi$ with respect to the fermionic covariant derivative $D_+$ according to the rule:
\[ \Psi \equiv \ldots + (D_-res(\Psi))D_+^{-1} + \ldots \]
which will also be justified a posteriori. Then, bosonic and fermionic Hamiltonian densities can be defined as:
\[ \mathcal{H}_l^t \equiv res(L_2^t), \]
\[ \mathcal{H}_l^{U_\dagger} \equiv res(N^+_l), \quad \mathcal{H}_l^{U_\dagger} \equiv res(N^-_l - \tilde{N}_{2l+1}^{-}), \]
\[ \mathcal{H}_i^+ \equiv \text{res}(L_{2l-1}^+), \quad \mathcal{H}_i^- \equiv \text{res}(L_{2l-1}^- - L_{2l-1}^-), \quad (79) \]

\[ \widetilde{\mathcal{H}}_i^+ \equiv \text{res}(M_{2l-1}^+), \quad \widetilde{\mathcal{H}}_i^- \equiv \text{res}(M_{2l-1}^- - M_{2l-1}^-). \quad (80) \]

Let us underline that all the operators generating the flows (35) are included into these formulae.

Using these formulae and the relations (29–30) one can derive the general formulae for the Hamiltonians \( H_i^t, H_i^U, H_i^- \) and \( \widetilde{H}_i^- \) in terms of the Lax operator \( \mathcal{L} \) [28] modulo inessential factors [6].

\[ H_i^t = \int dZ D_{-1}^{2l-3} \sum_{k=0}^{2l-3} (-1)^k \left( \mathcal{L}^{2l-3-k} v \right) \left( \mathcal{L}^k T_u \right), \]
\[ H_i^U = \int dZ D_{-1}^{2l-3} \sum_{k=0}^{2l} (-1)^k \left( \mathcal{L}^{2l-3-k} v \right) \left( \mathcal{L}^k T_u \right), \]
\[ H_i^- = \int dZ D_{-1}^{2l-1} \sum_{k=0}^{2l} (-1)^k \left( \mathcal{L}^{2l-1-k} v \right) \left( \mathcal{L}^k T_u \right), \]
\[ \widetilde{H}_i^- = \int dZ D_{-1}^{2l-1} \sum_{k=0}^{2l-1} (-1)^k \left( \mathcal{L}^{2l-1-k} v \right) \left( \mathcal{L}^k T_u \right). \quad (81) \]

We present, for example, the following explicit expressions for the first few bosonic,

\[ H_1^t = \int dZ u v, \quad H_2^t = \int dZ u v', \]
\[ H_3^t = \int dZ \left[ uv'' + vu(D_+ D_- v) - v(D_+ D_- u) \right] + \frac{2}{3} (uv)^3, \quad (82) \]
\[ H_1^U^\mp = \int dZ \left[ \frac{1}{2} uv - \theta^\mp (v D_\mp u \mp v D_\pm^{-1} (uv)) \right], \quad (83) \]
\[ H_1^U = \int dZ \left[ \theta^+ (v D_+ u - uv D_+^{-1} (uv)) + \theta^- (v D_- u + uv D_-^{-1} (uv)) \right], \]
\[ H_1^U^- = \int dZ \frac{1}{2} u \left[ \theta^- (D_+ + Q_+) - \theta^+ (D_- + Q_-) \right] v, \quad (84) \]

and fermionic,

\[ H_1^\mp = \widetilde{H}_1^\mp = \int dZ D_\mp^{-1} (uv), \quad (85) \]
\[ H_2^D^\mp = \frac{3}{2} H_2^D^\mp + \frac{1}{2} H_2^Q^\mp, \quad \widetilde{H}_2^D^\mp = \frac{1}{2} H_2^D^\mp + \frac{3}{2} H_2^Q^\mp, \]
\[ H_2^D^\mp = \int dZ \left[ \mp v D_\pm u + uv D_\pm^{-1} (uv) \right], \quad H_2^Q^\mp = \int dZ v Q_\pm u, \]
\[ H_3^D^\mp = \int dZ \left[ \mp v D_\pm u' + 2vu' D_\pm^{-1} (uv) + v(D_\pm u) D_\pm^{-1} D_\mp (uv) \right], \quad (86) \]

---

6Let us recall that Hamiltonian densities are defined up to terms which are fermionic or bosonic total derivatives of an arbitrary functional \( f(Z) \) of the initial superfields which, however, should satisfy the following constraint: \( f(+\infty, \theta^+, \theta^-) - f(-\infty, \theta^+, \theta^-) = 0 \).
Hamiltonians\(^7\). The Hamiltonians \(H^U_T\) and \(H^T_T\) (84) were found out by hand so that they are conserved quantities with respect to the flows \(\frac{\partial}{\partial t_i}\) (14). In eqs. (86) the Hamiltonians derived from eqs. (73) (81) are presented as a combination of the linearly independent Hamiltonians of the basis (73).

Let us stress that the Hamiltonians in eqs. (77) (80) are only conjectured to be conserved under the bosonic flows \(\frac{\partial}{\partial t_i}\) (33). This conjecture was checked for a few of them under the explicit flows (44).

It is well known that a bi-Hamiltonian system of evolution equations can be represented as:

\[
\frac{\partial}{\partial \tau_T}(\begin{array}{c} v \\ u \end{array}) = J_1 \begin{pmatrix} \delta/\delta v \\ \delta/\delta u \end{pmatrix} H^a_{i+1} = J_2 \begin{pmatrix} \delta/\delta v \\ \delta/\delta u \end{pmatrix} H^a_i,
\]

(87)

where \(J_1\) and \(J_2\) are the first and second Hamiltonian structures. In terms of these the Poisson brackets of the superfields \(v\) and \(u\) are given by the formula:

\[
\{ \begin{array}{c} v(Z_1) \\ u(Z_1) \end{array} \} \circ \{ \begin{array}{c} v(Z_2) \\ u(Z_2) \end{array} \} = J_1(Z_1)\delta^{N=2}(Z_1 - Z_2).
\]

(88)

Using the flows (44) (49) and Hamiltonians (82) (86), we have found the first and second Hamiltonian structures.

\[
J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

(89)

and second

\[
J_2 = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},
\]

\[
J_{11} = +vD_1^{-1}vD_1 - (D_vD_1^{-1}v - 2vD_1^{-1}uvD_1^{-1}v)
- vD_1^{-1}vD_1 + (D_1v)D_1^{-1}v - 2vD_1^{-1}uvD_1^{-1}v,
\]

\[
J_{22} = +uD_1^{-1}uD_1 - (D_1v)D_1^{-1}u - 2uD_1^{-1}uvD_1^{-1}u
- uD_1^{-1}uD_1 + (D_1v)D_1^{-1}u - 2uD_1^{-1}uvD_1^{-1}u,
\]

\[
J_{12} = \partial + \{D_1vD_1^{-1}u\} + 2vD_1^{-1}uvD_1^{-1}u - \{D_1vD_1^{-1}u\} + 2vD_1^{-1}uvD_1^{-1}u,
\]

\[
J_{21} = \partial + \{D_1vD_1^{-1}u\} + 2vD_1^{-1}uvD_1^{-1}u - \{D_1vD_1^{-1}u\} + 2vD_1^{-1}uvD_1^{-1}u
\]

(90)

Hamiltonian structures of the hierarchy.

The Jacobi identities for the first Hamiltonian structure \(J_1\) (83) are obviously satisfied. The second Hamiltonian structure \(J_2\) (90) is a very complicated, nonlinear and nonlocal algebra, and due to this reason it is a very nontrivial, technical task to verify its Jacobi identities. It becomes a simpler linear algebra in terms of the original Toda-lattice superfields \(b\) and \(f\) (25)

\[ b \equiv uv, \quad f \equiv D_1 \ln v. \]

(91)

The corresponding Hamiltonian structures \(J_1^{(b,f)}\) and \(J_2^{(b,f)}\) can be expressed via \(J_1\) and \(J_2\) (89) (90) by the following standard relation:

\[
J_1^{(b,f)} = F J_1 F_T, \quad F \equiv \begin{pmatrix} u & v \\ D_1v & 0 \end{pmatrix},
\]

(92)

\(^7\)When deriving eqs. (83) (86) we integrated by parts and made essential use of realizations (72) for the inverse derivatives and of the relationship \(Q_\pm \equiv D_\pm - 2\theta^\pm \partial\). We also used the following definition of the superspace integral: \(\int dZ f(Z) \equiv \int dz (D_+ D_- f)(z, 0, 0)\).
Thus, one concludes that the recursion operator from the compatible pair of Hamiltonian structures \((40)\). One finds:

\[
J_1^{(b,f)} = \begin{pmatrix} 0 & D_+ \\ D_+ & 0 \end{pmatrix}, \quad J_2^{(b,f)} = \begin{pmatrix} J_{11}^{(b,f)} & J_{12}^{(b,f)} \\ J_{21}^{(b,f)} & J_{22}^{(b,f)} \end{pmatrix},
\]

\[
J_{11}^{(b,f)} \equiv +\partial b + b\partial, \\
J_{12}^{(b,f)} \equiv -\partial D_+ + D_- b + D_+ b D_+ D_-^{-1} + (D_+ f) D_+ , \\
J_{21}^{(b,f)} \equiv +\partial D_+ + b D_- - D_+ D_-^{-1} b D_+ + D_+ (D_+ f), \\
J_{22}^{(b,f)} \equiv -2 D_- D_+ + 2 b - 2 (D_- f) + [(D_+ f), D_+ D_-^{-1}] - 2 D_+ D_-^{-1} b D_+ D_-^{-1} .
\] (93)

The Jacobi identities for \(J_2^{(b,f)} (93)\) are still very complicated, and we have only verified the consistency of the Jacobi identities \(\{b, b, f\}\) and \(\{b, f, f\}\). Taking into account that \(J_2\) correctly reproduces all the flows explicitly derived, it is natural to expect that the most complicated, remaining Jacobi identity \(\{f, f, f\}\) is satisfied as well, but we cannot present a proof here. One more argument in the favour of this expectation is given by the secondary reduction of \(J_2^{(b,f)}\) (see section 8) which coincides with the \(N = 2\) superconformal algebra \((126)\).

Using equations (83), (87) and (89), we obtain, for example, the 0-th fermionic flow,

\[
D_0^+ v = v\theta^+, \quad D_0^+ u = -u\theta^+ .
\] (94)

Knowledge of the first and second Hamiltonian structures allows us to construct the recursion operator of the hierarchy using the following general rule:

\[
R = J_2 J_1^{-1} \equiv \left( \begin{array}{cc} J_{12} & -J_{11} \\ J_{22} & -J_{21} \end{array} \right), \quad \frac{\partial}{\partial \tau_{i+1}^l} \left( \begin{array}{c} v \\ u \end{array} \right) = R \frac{\partial}{\partial \tau_i^l} \left( \begin{array}{c} v \\ u \end{array} \right), \quad J_{i+1} = R^l J_i .
\] (95)

The Hamiltonian structures \(J_1^{(b,f)}\) and \(J_2^{(b,f)} (93)\) (and, consequently the original Hamiltonian structures \(J_1\) and \(J_2 (89) (90)\) ) are obviously compatible: the deformation of the superfield \(f \Rightarrow f + \gamma \theta^+\), where \(\gamma\) is an arbitrary parameter, transforms \(J_2^{(b,f)}\) into the Hamiltonian structure

\[
J_2^{(b,f + \gamma \theta^+)} = J_2^{(b,f)} + \gamma J_1^{(b,f)} .
\] (97)

Thus, one concludes that the recursion operator \(R (93)\) is hereditary as the operator obtained from the compatible pair of Hamiltonian structures \(10)\).

Applying formulae (93) we obtain the following recurrence relations for the flows:

\[
\frac{\partial}{\partial \tau_{i+1}^l} v = + \frac{\partial}{\partial \tau_i^l} v' + \left( -1 \right)^{d_r^a} v D_+^{-1} \frac{\partial}{\partial \tau_i^l} H_2^l D_-^{-1} + [(D_- v) + v(D_+^{-1} u v)] D_+^{-1} \frac{\partial}{\partial \tau_i^l} H_1^l ,
\]

\[
\frac{\partial}{\partial \tau_{i+1}^l} u = - \frac{\partial}{\partial \tau_i^l} u' - \left( -1 \right)^{d_r^a} u D_+^{-1} \frac{\partial}{\partial \tau_i^l} H_2^l D_-^{-1} - [(D_+ u) + u(D_-^{-1} u v)] D_-^{-1} \frac{\partial}{\partial \tau_i^l} H_1^l ,
\]

\[
\frac{\partial}{\partial \tau_{i+1}^l} v = -2 u D_+^{-1} \frac{\partial}{\partial \tau_i^l} H_1^l ,
\] (98)

\(\text{The Jacobi identity} \{b, b, b\} \text{ is satisfied because the restriction of the Poisson brackets to the superfield} b \text{ forms the classical Virasoro algebra.}\)
where \(d_{+a}\) is the Grassmann parity of the evolution derivative \(\frac{\partial}{\partial \tau}\) and

\[
\mathcal{H}_1^t \equiv uv, \quad \mathcal{H}_2^{D^\pm} \equiv \mp vD_\mp u + uvD_\mp^{-1}(uv)
\]

are the densities of the Hamiltonians \(H_1^t\) (82) and \(H_2^{D^\pm}\) (86), respectively.

Taking into account the involution properties

\[
- (\mathcal{H}_1^t)^* = (\mathcal{H}_1^t)^\dagger = (\mathcal{H}_1^t)^\star = \mathcal{H}_1^t,
\]

\[
(\mathcal{H}_2^{D^\pm})^* = \pm D_\mp \mathcal{H}_1^t + \mathcal{H}_2^{D^\pm}, \quad (\mathcal{H}_2^{D^\pm})^\dagger = \mp D_\mp \mathcal{H}_1^t + \mathcal{H}_2^{D^\pm}, \quad (\mathcal{H}_2^{D^\pm})^\star = \pm \mathcal{H}_2^{D^\pm}
\]

of the Hamiltonian densities \(\mathcal{H}_1^t\) and \(\mathcal{H}_2^{D^\pm}\) (99), one can verify that the recurrence relations (98) possess the involutions (53–56). Together with the fact already verified in section 5 that the first flows (44–49) also admit these involutions, one can conclude that all the flows of the hierarchy under consideration admit them as well.

Using eqs. (98), we obtain, for example, the 3rd bosonic flow

\[
\begin{align*}
\frac{\partial}{\partial t_3} v &= v''' - 3(D_+ v)'(D_- uv) + 3(D_- v)'(D_+ uv) - 3v'(D_+ u)(D_- v) \\
&\quad + 3v'(D_- u)(D_+ v) - 6uv'(D_+ D_- u) + 6(uv)^2 v', \\
\frac{\partial}{\partial t_3} u &= u''' - 3(D_- u)'(D_+ uv) + 3(D_+ u)'(D_- uv) - 3u'(D_- v)(D_+ u) \\
&\quad + 3u'(D_+ v)(D_- u) - 6uu'(D_+ D_- v) + 6(uv)^2 u',
\end{align*}
\]

which coincides with the corresponding flow that can be derived from the Lax-pair representation (39).

Let us end this section with the remark that due to the local nature of bosonic flows (39) and the relation

\[
\mathcal{H}_1^{D^\pm} = D_\mp^{-1}\mathcal{H}_1^t
\]

(see eqs. (82) and (83)), the evolution equations for \(\mathcal{H}_1^t\) with respect to the bosonic times \(t_j\) should admit the following very special, important representation:

\[
\frac{\partial}{\partial t_j} \mathcal{H}_1^t = h_1^{(1)} + D_\pm D_\mp h_1^{(2)},
\]

where \(h_1^{(1)}\) and \(h_1^{(2)}\) are some local functions. We have explicitly checked it for the first few values of \(l\) and observed that

\[
h_1^{(1)} = \mathcal{H}_1^t, \quad h_1^{(2)} = \mathcal{H}_1^t \mathcal{H}_{l-1}^t
\]

modulo an inessential factor and total derivatives. We also analyzed a similar construction of fermionic flows (45) and derived the following formula:

\[
D_\pm \mathcal{H}_1^t = -\mathcal{H}_1^{D^\pm}'.
\]

These formulae are conjectured to be valid for any value of \(l\), consequently the flows (98) allow the bosonic and fermionic Hamiltonian densities \(\mathcal{H}_1^t\) (77) and \(\mathcal{H}_1^{D^\pm}\) (79) to be constructed using eqs. (103–105), i.e. almost all information about Hamiltonians and flows is encoded in the recursion operator (95).
7 \(U(1)\)-symmetric basis

In this section we introduce a new basis of the algebra of flows (8–10), find \(U(1)\)–automorphism transformations for all reduced flows, and applying these transformations derive a new Lax operator.

It is instructive to introduce the new basis

\[
\{D_{\pm}, Q_{\pm}, D_{l}^{\pm}, Q_{l}^{\pm}, u, v\} \rightarrow \{D, \overline{D}, Q, \overline{Q}, D_{l}, Q_{l}, \overline{Q}_{l}, \frac{1}{\sqrt{t}} u, \frac{1}{\sqrt{t}} v\},
\]

\[
D \equiv \frac{1}{\sqrt{2}}(D_{-} + iD_{+}), \quad \overline{D} \equiv \frac{1}{\sqrt{2}}(D_{-} - iD_{+}),
\]

\[
Q \equiv \frac{1}{\sqrt{2}}(Q_{-} + iQ_{+}), \quad \overline{Q} \equiv \frac{1}{\sqrt{2}}(Q_{-} - iQ_{+}),
\]

\[
D_{k} \equiv \frac{1}{\sqrt{2}}(D_{k}^{-} + iD_{k}^{+}), \quad \overline{D}_{k} \equiv \frac{1}{\sqrt{2}}(D_{k}^{-} - iD_{k}^{+}),
\]

\[
Q_{k} \equiv \frac{1}{\sqrt{2}}(Q_{k}^{-} + iQ_{k}^{+}), \quad \overline{Q}_{k} \equiv \frac{1}{\sqrt{2}}(Q_{k}^{-} - iQ_{k}^{+})
\]

(107)

in the algebras (3) and (8) which now become:

\[
\{D, \overline{D}\} = +2\partial, \quad \{Q, \overline{Q}\} = -2\partial,
\]

\[
\{D_{k}, \overline{D}_{k}\} = -2 \frac{\partial}{\partial t_{k+1}}, \quad \{Q_{k}, \overline{Q}_{k}\} = +2 \frac{\partial}{\partial t_{k+1}},
\]

(108, 109)

respectively, where \(i\) is the imaginary unity. Then, the first bosonic and fermionic flows from eqs. (14–16) and recurrence relations (18) become

\[
\frac{\partial}{\partial t_{1}} \begin{pmatrix} v \\ u \end{pmatrix} = \partial \begin{pmatrix} v \\ u \end{pmatrix}, \quad Q_{1} \begin{pmatrix} v \\ u \end{pmatrix} = Q \begin{pmatrix} v \\ u \end{pmatrix}, \quad \overline{Q}_{1} \begin{pmatrix} v \\ u \end{pmatrix} = \overline{Q} \begin{pmatrix} v \\ u \end{pmatrix},
\]

\[
D_{1} v = -D v - 2\partial^{-1} D(uv), \quad D_{1} u = -D u + 2u\partial^{-1} D(uv),
\]

\[
\overline{D}_{1} v = -\overline{D} v + 2\partial^{-1} \overline{D}(uv), \quad \overline{D}_{1} u = -\overline{D} u - 2u\partial^{-1} \overline{D}(uv),
\]

(110)

\[
\frac{\partial}{\partial t_{1+1}} v = + \frac{\partial}{\partial t} v',
\]

\[
+ (-1)^{d_{u}} [v\partial^{-1} D \frac{\partial}{\partial \sigma_{l}} (vD_{l} u - uv\partial^{-1} D(uv)) - v\partial^{-1} \overline{D} \frac{\partial}{\partial \sigma_{l}} (vD_{l} u + uv\partial^{-1} D(uv))]
\]

\[
- [(\overline{D} v) + v(\partial^{-1} \overline{D} uv)]\partial^{-1} D \frac{\partial}{\partial \sigma_{l}} (uv) + [(D v) - v(\partial^{-1} D uv)]\partial^{-1} \overline{D} \frac{\partial}{\partial \sigma_{l}} (uv),
\]

\[
\frac{\partial}{\partial t_{1+1}} u = - \frac{\partial}{\partial t} u',
\]

\[
- (-1)^{d_{u}} [u\partial^{-1} \overline{D} \frac{\partial}{\partial \sigma_{l}} (uD_{l} v - uv\partial^{-1} D(uv)) - u\partial^{-1} D \frac{\partial}{\partial \sigma_{l}} (u\overline{D} v + uv\partial^{-1} \overline{D}(uv))]
\]

\[
+ [(D u) + u(\partial^{-1} D uv)]\partial^{-1} \overline{D} \frac{\partial}{\partial \sigma_{l}} (uv) - [(\overline{D} u) - u(\partial^{-1} \overline{D} uv)]\partial^{-1} D \frac{\partial}{\partial \sigma_{l}} (uv),
\]

(111)

respectively, and their simple inspection shows that they admit the \(U(1)\) automorphism of the \(N = 2\) supersymmetry algebras (108)–(109) hidden in the former basis,

\[
\left(\frac{\partial}{\partial t_{l}}\right) \rightarrow \left(\frac{\partial}{\partial t_{l}}\right),
\]

\[
(D, \quad Q, \quad D_{l}, \quad Q_{l}) \quad \rightarrow \quad \exp (+i\phi) (D, \quad Q, \quad D_{l}, \quad Q_{l}),
\]

\[
(\overline{D}, \quad \overline{Q}, \quad \overline{D}_{l}, \quad \overline{Q}_{l}) \quad \rightarrow \quad \exp (-i\phi) (\overline{D}, \quad \overline{Q}, \quad \overline{D}_{l}, \quad \overline{Q}_{l}),
\]

(112)
where \( \phi \) is an arbitrary parameter. Consequently, all higher flows admit this automorphism as well. Without going into additional technical details let us also present the corresponding \( U(1) \)-transformations for the remaining bosonic flows of the hierarchy

\[
\begin{align*}
(U_i^+ + U_i^-) &\quad \Rightarrow (U_i^+ + U_i^-), \\
(U_i^+ - U_i^- \pm iU_i) &\quad \Rightarrow \exp(\mp 2i\phi) (U_i^+ - U_i^- \pm iU_i) .
\end{align*}
\]  

(113)

As we already mentioned at the end of section 4 the Lax operator \( L^{(28)} \) is not invariant with respect to \( U(1) \) transformations, therefore applying these to \( L \) one can derive the one-parameter family of consistent Lax operators

\[
L \quad \Rightarrow \quad L^\phi = \cos \phi \ L + \sin \phi \ L^\star
\]

(114)

which generate isomorphic flows, where \( L^\star \) is defined in eq. (114).

For completeness, we present also the explicit expressions of the first and second Hamiltonian structures in the special superfield basis

\[
\{u, v\} \quad \Rightarrow \quad \{b \equiv iuv, \quad \tilde{b} \equiv (\ln v)'\},
\]

(115)

where they possess both a manifest \( N = 2 \) supersymmetry and form linear superalgebras,

\[
J_1^{(b, \tilde{b})} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad J_2^{(b, \tilde{b})} = \begin{pmatrix} J_{11}^{(b, \tilde{b})} & J_{12}^{(b, \tilde{b})} \\ J_{21}^{(b, \tilde{b})} & J_{22}^{(b, \tilde{b})} \end{pmatrix},
\]

\[
J_{11}^{(b, \tilde{b})} \equiv + \partial b + b \partial, \\
J_{12}^{(b, \tilde{b})} \equiv - \partial^2 - \nabla b \partial + \nabla \partial b + b \partial, \\
J_{21}^{(b, \tilde{b})} \equiv + \partial^2 + \nabla b \partial - \nabla \partial b + \partial b, \\
J_{22}^{(b, \tilde{b})} \equiv + [D, \nabla \partial] \partial - 2D \nabla \partial - 2 \nabla b \partial + (D \tilde{b}) \nabla b - (\nabla \tilde{b}) D - ([D, \nabla \partial] \partial^{-1} \tilde{b}) \partial .
\]

(116)

The last superalgebra is minimally nonlocal: it contains only a single term with an inverse derivative in the \( J_{22}^{(b, \tilde{b})} \) component. It seems that there is no superfield basis where it is possible to avoid this nonlocality. It would be interesting to clarify its origin in the framework of Hamiltonian reduction of affine superalgebras, but this question is still unclear now.

8 Secondary reduction: the \( N = 2 \) supersymmetric \( \alpha = -2 \) KdV hierarchy

In this section, by means of a secondary reduction, we establish a relationship between the \( N = 4 \) Toda chain and \( N = 2, \alpha = -2 \) KdV hierarchies.

Let us study the secondary reduction of the \( N = 4 \) Toda chain hierarchy considered in the preceding sections. With this aim, we impose the following secondary constraint on the Lax operator \( \mathcal{L} \) (28):

\[
\mathcal{L}^T = D_+ \mathcal{L} D_+^{-1}
\]

(117)

\footnote{See also refs. \cite{1, 12, 37}, where the similar reduction of the Manin-Radul \cite{29} and Mulase-Rabin \cite{30, 31} \( N = 1 \) supersymmetric KP and KdV hierarchies has been discussed.}
which can easily be resolved in terms of the superfield $v$ entering $\mathcal{L}$,

$$v = 1.$$  

(118)

Then, the reduced Lax operator $\mathcal{L}^{\text{red}}$ becomes

$$\mathcal{L}^{\text{red}} = D_+ + D_+^{-1}u.$$  

(119)

The condition (117) by means of eq. (31) induces the secondary constraint

$$(\mathcal{W}^{-1})^T = D_+ \mathcal{W} D_+^{-1}$$  

(120)

on the dressing operator $\mathcal{W}$ (32) which in turn induces the following secondary constraints on the operators $L^\pm_1, M^\pm_1, N^\pm_1$ (4):

$$(L^\pm_2)^T = (-1)^l D_+ L^\pm_2 D_+^{-1},$$
$$(L^\pm_{2l-1})^T = (-1)^{l-l+\frac{1}{2}} D_+ L^\pm_{2l-1} D_+^{-1},$$
$$(M^\pm_{2l-1})^T = (-1)^{l-1} D_+ M^\pm_{2l-1} D_+^{-1},$$
$$(N^-_l)^T = (-1)^{l+1} D_+ (N^-_l - L^-_2) D_+^{-1},$$
$$(N^+_l)^T = (-1)^{l} D_+ N^+_l D_+^{-1}$$  

(121)

which are identically satisfied if constraint (117) (or (120)) is imposed. The following important consequences obviously result from eqs. (121):

$$(L^\pm_{2(2k-1)})_0 = (L^+_2(2k-1))_0 = (M^+_2(2k-1))_0 = (N^+_2(2k-1))_0 = 0, \quad k = 1, 2, \ldots, 122$$

where the subscript 0 refers to the constant part of the operators. Consequently, the following equations:

$$((L^\pm_{2(2k-1)})_0 + 1) = ((L^+_2(2k-1))_0 + 1) = ((M^+_2(2k-1))_0 + 1) = ((N^+_2(2k-1))_0 + 1) = 0$$  

(123)

are identically satisfied as well. Using these relations, the involution (53) and the algebra structure (8 10) we are led to the conclusion that only half of the flows (13) are consistent with the reduction (117 118) and these flows are

$$\{ \frac{\partial}{\partial x_{2k-1}}, U_{2k-1}, U_{2k}, D_{2k}, Q_{2k-1}^\pm \}.$$  

(124)

In order to understand deeper what kind of reduced hierarchy we have in fact derived, let us analyze its Hamiltonian structure via Hamiltonian reduction of the first and second Hamiltonian structures (89) and (90), respectively, of the original hierarchy we started with. It is easier to reduce the less complicated expressions (93) in terms of the superfields $b, f$ (11). In this basis constraint (118) becomes

$$f = 0,$$  

(125)

and on the constraint shell the superfield $b$ coincides with the superfield $u$.

Let us start with the first Hamiltonian structure $J_1^{(b,f)}$ (93). In this case constraint (125) is a gauge constraint, and the gauge can be fixed by the condition $b = 0$. Thus, as a result a trivial reduced Hamiltonian structure is generated.

In the case of the second Hamiltonian structure $J_2^{(b,f)}$ (93), constraint (125) is second class, and we can use the Dirac brackets in order to obtain the second Hamiltonian structure of the reduced system. The result is

$$J_{11}^{(\text{Dirac})} = J_{11}^{(u,0)} - J_{12}^{(u,0)} J_{22}^{(u,0)} - 1 J_{21}^{(u,0)} \equiv \frac{1}{2} (\partial D_+ D_- - D_- u D_+ - D_+ u D_+ + 2\partial u + 2u\partial).$$  

(126)
where the relations
\[ J^{(b,0)}_{12} D_+ = \frac{1}{2} D_- J^{(b,0)}_{22} D_- = D_- J^{(b,0)}_{21} = -\partial D_+ D_- + D_- b D_+ + D_+ b D_+ , \] (127)
which can easily be read from eqs. (133), have been exploited.

From eq. (129) we see that the second Hamiltonian structure of the reduced hierarchy coincides with the \( N = 2 \) superconformal algebra, and from this remarkable fact one can conclude that the reduced hierarchy should reproduce one of the three existing \( N = 2 \) supersymmetric KdV hierarchies [4]. In order to establish which, let us derive the third flow that the reduced hierarchy should reproduce one of the three existing hierarchies.

Here, \( \omega \equiv \omega_n = \sum_{n=1}^{\infty} \omega_n \equiv 0 \) and fermionic superfields for \( n = \ldots, n \) are rectangular matrix-valued superfields, respectively, and \( I \equiv \delta_{A,B} \equiv \delta_{1,1} \) is the unity matrix. In (130) a matrix product is understood, for example \((uv)_{AB} \equiv \sum_{n=1}^{n+m} v_{Aa} u_{aB} \). The matrix entries are bosonic superfields for \( a = \ldots, n \) and fermionic superfields for \( a = n+1, \ldots, n+m \), i.e., \( v_{Aa} u_{bB} = (-1)^{d_a d_b} v_{Aa} u_{bB} \), where \( d_a \) and \( d_b \) are the Grassmann parities of the matrix elements \( v_{Aa} \) and \( u_{bB} \), respectively, \( d_a = 1 \) (fermionic) for bosonic (bosonic) entries. The grading chosen guarantees that the Lax operator \( L_1^- \) is Grassmann odd [20].

The detailed analysis of the corresponding hierarchies is however out of the scope of the present paper and will be discussed elsewhere. Nevertheless, let us only present a few first nontrivial bosonic and fermionic flows in the noncommutative, matrix case (compare with the abelian flows [14, 43]):
\[ \frac{\partial}{\partial t} v = +v'' + 2(D_-, vD_+ v) + 2v(uv)^2, \]
\[ D_+^v v = -D_+ v + 2(D_-^v I u) v, \]
\[ D_-^v v = -D_- v - 2vD_+^{-1}(uv), \]
\[ \frac{\partial}{\partial t} u = -u'' - 2(D_+, uD_- v) u - 2(uv)^2 u, \]
\[ D_+^u u = -D_+ u - 2u D_-^{-1}(uv), \]
\[ D_-^u u = -D_- u + 2I(D_+^{-1}uv) u \] (131)
which are derived using Lax-pair representations \([39-40]\) with \(L^-_{1} (130)\) and

\[(L^+_1)_+ = ID_+ - 2(D_-^{-1}(vIu)), \quad (132)\]

and the matrix \(I\) is defined as

\[I ≡ (-1)^d_a \delta_{ab}. \quad (133)\]

To close this section let us only remark that for the particular case when the index \(A = 1\) the matrix reduced Lax operator \((130)\) becomes the scalar operator generating the reduced hierarchy with the \(n + m\) pairs of the scalar superfields \(v_a, u_a\). In the very particular case when the indices \(A = 1, a = 1\) and \(n = 1, m = 0\) the Lax operator \((130)\) reproduces the Lax operator \((28)\) of the \(N = 4\) supersymmetric Toda chain hierarchy.

10 Conclusion

In this paper we have defined the \(N = 4\) generalization of the \(N = 2\) supersymmetric KP hierarchy and derived its flows and their algebra in the framework of the dressing approach. Then we have analyzed the possibility of viewing the supersymmetric Toda chain hierarchy as a reduction of the \(N = 4\) KP hierarchy and restored all \(N = 4\) KP reduced flows including the flows of the \(N = 4\) supersymmetry. Due to this it is called the \(N = 4\) supersymmetric Toda chain hierarchy. Furthermore we have exhibited its finite and infinite discrete symmetries and using them derived its solutions and new Lax operators generating isomorphic flows. Then we have explicitly calculated its first two Hamiltonian structures and recursion operator connecting all its systems of evolution equations and Hamiltonian structures. Then we have established its secondary reduction to the \(N = 2\) supersymmetric \(\alpha = -2\) KdV hierarchy. Finally we have proposed a matrix generalization of the \(N = 4\) supersymmetric KP hierarchy and described an infinite family of its reductions characterized by a finite number of superfields.

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In eqs. (A.2) we have introduced the functions

$$D^\pm_k = \frac{\partial}{\partial \theta_k^\pm} - \sum_{l=1}^{\infty} \theta_l^\pm \frac{\partial}{\partial t_{k+l-1}}, \quad Q^\pm_k = \frac{\partial}{\partial \rho_k^\pm} + \sum_{l=1}^{\infty} \rho_l^\pm \frac{\partial}{\partial t_{k+l-1}}, \quad (A.1)$$

$$U^\pm_k = \frac{\partial}{\partial h_k^\pm} - \sum_{l=1}^{\infty} \left( \theta_l^\pm \frac{\partial}{\partial \rho_{k+l}^\pm} + \rho_l^\pm \frac{\partial}{\partial \theta_{k+l}^\pm} \right),$$

$$U_k = \sum_{l=1}^{\infty} \left\{ i(\bar{u}_{l-1} \frac{\partial}{\partial h_{k+l-1}} - u_{l-1} \frac{\partial}{\partial \bar{h}_{k+l-1}}) + \frac{1}{2} \left( \frac{\partial}{\partial h_0} \bar{u}_{l-1} - \frac{\partial}{\partial \bar{h}_0} u_{l-1} \right) \left( \frac{\partial}{\partial h_{k+l-1}} - \frac{\partial}{\partial \bar{h}_{k+l-1}} \right) \right\} + \theta_l^+ \frac{\partial}{\partial \rho_{k+l}^+} - \theta_l^- \frac{\partial}{\partial \rho_{k+l}^-} + \rho_l^+ \frac{\partial}{\partial \theta_{k+l}^+} - \rho_l^- \frac{\partial}{\partial \theta_{k+l}^-} \right\},$$

$$U_k = - \sum_{l=1}^{\infty} \left\{ u_{l-1} \frac{\partial}{\partial h_{k+l-1}} + u_{l-1} \frac{\partial}{\partial \bar{h}_{k+l-1}} - i \left( \frac{\partial}{\partial h_0} \bar{u}_{l-1} - \frac{\partial}{\partial \bar{h}_0} u_{l-1} \right) \left( \frac{\partial}{\partial h_{k+l-1}} - \frac{\partial}{\partial \bar{h}_{k+l-1}} \right) \right\} - \theta_l^+ \frac{\partial}{\partial \theta_{k+l}^+} - \theta_l^- \frac{\partial}{\partial \theta_{k+l}^-} - \rho_l^+ \frac{\partial}{\partial \rho_{k+l}^+} + \rho_l^- \frac{\partial}{\partial \rho_{k+l}^-} \right\}, \quad (A.2)$$

where $t_k, h_k^\pm, h_k, \bar{h}_k (\theta_k^\pm, \rho_k^\pm)$ are bosonic (fermionic) abelian evolution times with dimensions

$$[t_k] = [h_k^+] = [h_k] = [\bar{h}_k] = k, \quad [\theta_k^\pm] = [\rho_k^\pm] = k - \frac{1}{2}, \quad (A.3)$$

In eqs. (A.2) we have introduced the functions

$$\bar{u}_l \equiv \frac{1}{l!} \frac{\partial^l}{\partial \sigma^l} \left\{ (1 + \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sigma^{k+n} h_k \bar{h}_n) \exp \left[ +i \sum_{m=0}^{\infty} \sigma^m (h_m^+ - h_m^-) \right] \right\} \big|_{\sigma = 0},$$

$$u_l \equiv \frac{1}{l!} \frac{\partial^l}{\partial \sigma^l} \left\{ (1 + \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sigma^{k+n} h_k \bar{h}_n) \exp \left[ -i \sum_{m=0}^{\infty} \sigma^m (h_m^+ - h_m^-) \right] \right\} \big|_{\sigma = 0}, \quad l = 0, 1, \ldots$$

$$\bar{u}_l \equiv 0, \quad u_l \equiv 0, \quad l < 0 \quad (A.4)$$

possessing the following properties:

$$\frac{\partial}{\partial x_k} \bar{u}_l \equiv \frac{\partial}{\partial x_0} \bar{u}_{l-k}, \quad \frac{\partial}{\partial x_k} u_l \equiv \frac{\partial}{\partial x_0} u_{l-k}, \quad \frac{\partial}{\partial h_0} \bar{u}_l \equiv \pm i \bar{u}_l, \quad \frac{\partial}{\partial h_0} u_l \equiv \mp i u_l \quad (A.5)$$

which together with the formula

$$\frac{\partial^l}{\partial \sigma^l} (ab) \equiv \sum_{k=0}^{l} \frac{l!}{(l-k)!k!} \left( \frac{\partial^k}{\partial \sigma^k} a \right) \left( \frac{\partial^{l-k}}{\partial \sigma^{l-k}} b \right) \quad (A.6)$$

can be used to check that the generators (A.1, A.2) indeed satisfy the algebra (8–10). Here, $x_k$ is any evolution time from the set $\{h_k^\pm, h_k, \bar{h}_k\}$ and $a, b$ are arbitrary functions of some argument $\sigma$. 

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