Liouville perturbation theory for Laughlin state and Coulomb gas

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Received 11 February 2021, revised 24 June 2021
Accepted for publication 14 July 2021
Published 29 July 2021

Abstract
We consider the generating functional (logarithm of the normalization factor) of the Laughlin state on a sphere, in the limit of a large number of particles \(N\). The problem is reformulated in terms of a perturbative expansion of a 2d QFT, resembling the Liouville field theory. We develop an analog of the Liouville loop perturbation theory, which allows us to quantitatively study the generating functional for an arbitrary smooth metric and an inhomogeneous magnetic field beyond the leading orders in large \(N\).

Keywords: Laughlin state, Liouville theory, quantum Hall effect

1. Introduction

In order to study the Laughlin wave function [35] analytically it is customary to define it on a compact Riemann surface. The convenience of this approach, pioneered in [22, 24, 51] is that one does not need to impose the boundary conditions, which may break holomorphy of the wave function in the bulk. Another advantage stems from the fact that Riemann surfaces are equipped with various discrete and continuous parameter spaces, such as the genus, moduli, Riemannian metric and the response to a change in these parameters allows one to extract analytically a wealth of physical information, such as the Hall conductance, Hall viscosity, bulk central charge, corrections to the static structure factors, etc [3, 4, 8–11, 13, 15–17, 21, 23, 26, 27, 29–32, 34, 37, 41–45, 48, 49, 54]. Adiabatic geometric transport of the vector bundles of quantum Hall states over these parameter spaces gives rise to the Chern–Simons description of the quantum Hall effect, for recent developments see [1, 7, 19, 20, 31].

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After the holomorphic part of the Laughlin state on a compact surface has been defined, the main problem is to compute its $L^2$ normalization, in the limit where the number of particles $N$ goes to infinity. This problem is closely related to the computation of the partition function of the Coulomb gas at integer values of inverse temperature $\beta$, at least in the case of the spherical topology.

In this paper we study the normalization factor of the Laughlin state $Z[W]$, equation (2.2) on the sphere as a functional of the background metric and inhomogeneous magnetic field, at large $N$. The logarithm of the normalization factor can be used to compute the density–density correlation functions and for this reason it is also called the generating functional. In the case of the integer quantum Hall state ($\beta = 1$), the large $N$ expansion of the generating functional can be extracted rigorously [29, 31] from the Bergman kernel expansion for high powers of the magnetic line bundle [38, 52, 57]. In the fractional case $\beta > 1$, the leading terms of the generating functional can be computed [9, 13] using the $\beta$-ensemble Ward identity method of Zabrodin–Wiegmann [53] or with the help of the free field construction of Moore–Read [39]. The rigorous proof of the large $N$ expansion of the generating functional is available up to the order $N$ due to [36], see also [6]. For the rigorous approach to the Ward identities we refer the reader to [2] and for the work on structure factors in the Coulomb gas we refer to [28].

The goal of this paper is twofold. On the one hand, we would like to systematically incorporate the case of inhomogeneous magnetic field $B$. This is an important issue since the corresponding contributions to the generating functional are not small (for example the leading term is of order $B \log B \sim N \log N$). We derive the corresponding terms up to the order $N^0$ and compare them to the result conjectured in [10, equation (130)]. On the other hand both in the Ward identity and free field methods it is hard to control the subleading corrections of the large $N$ expansion. In reference [13] the terms up to the order $N^0$ of the generating functional were derived, and remainder term for the large $N$ expansion was written as a path integral in an interacting 2d QFT, but only general arguments were given as to why this term is expected to be small, and local, at large $N$. In this paper we develop a perturbative scheme inspired by the semiclassical analysis of the Liouville theory [55, 56] in order to compute the order $1/N$ contribution to the generating functional.

Let us now describe our main result. From general principles we expect the following formula for the large $N$ expansion of the generating functional up to the order $1/N$,

$$
\log Z = -\frac{1}{2\pi i} \pi \int_{(S^2)^2} B \Delta^{-1} B - \frac{1}{2\pi i} \pi \int_{(S^2)^2} B \Delta^{-1} R
+ \frac{1}{2\pi i} \frac{c_H}{48} \int_{(S^2)^2} R \Delta^{-1} R - \frac{1}{2\pi i} \int_{S^2} \sqrt{g} d^2 z \left[ \frac{2 - \beta}{2\beta} B \log B 
+ \left( \frac{1}{24} - \frac{(\beta - 2)(\beta - 2s)}{8\beta} \right) R \log B 
+ \frac{c}{48} (\log B) \Delta \log B \right]
- \frac{1}{2\pi i} \int_{S^2} \sqrt{g} d^2 z \left( c_1(\beta)(\Delta \log B)^2 B^{-1} 
+ c_1(\beta) R (\Delta \log B) B^{-1} + c_1(\beta) R^2 B^{-1} \right) + O(N^{-2}).
$$

(1.1)

First let us describe how terms in this expansion scale with $N$. The magnetic field is assumed to be large $B \sim N$ while the scalar curvature $R$ of the background metric is of order $N^0$. In this sense the terms in the first line here are of orders $N^2, N$ and $N^0$ respectively. The terms in the
second line are $N \log N$, $\log N$ and $N^0$. Terms in the last line are subleading and vanish when $N \to \infty$.

The first line in (1.1) contains so-called anomalous terms, which were computed in [9, 13, 29]. The coefficients of the anomalous terms are

$$\sigma_H = 1/\beta, \quad \varsigma_H = \frac{1}{4} - \frac{s}{2\beta}, \quad c_H = 1 - \frac{3(\beta - 2s)^2}{\beta},$$ (1.2)

respectively the Hall conductance, the Hall viscosity and the (bulk) Hall central charge. Parameter $s$ is the so-called gravitational spin, introduced according to (2.4). Since in this paper we will not be concerned with the anomalous terms, we refer to [13, 14] for the proper definitions of the double integrals and regularized Green functions of the scalar Laplacian $\Delta g^{-1}$ in equation (1.1). The second line involves the terms which are single integrals of functions of the scalar curvature, the magnetic field and their derivatives. Interestingly, the value of the parameter $c$ here coincides with the value of the bulk central charge $c_H$ at $s = 1$

$$c = 1 - \frac{3(\beta - 2s)^2}{\beta}. \quad (1.3)$$

This value was also conjectured to be relevant to Liouville theory in [50].

Finally, the last line in (1.1) contains the terms which are of order $N^{-1}$ and smaller. We expect these terms to be given by local integrals of the densities depending on $B, R$ and their derivatives, with the expansion being essentially the derivative expansion. The form of the terms in the last line is constrained by the diffeomorphism invariance of the generating functional. The coefficients $c_1(\beta), c'_1(\beta), c''_1(\beta)$ are real-valued functions of $\beta$ whose value is only known at $\beta = 1$, for constant magnetic field [29]. In this paper we use our Liouville perturbation theory in order to compute these coefficients as the Laurent series around $\beta = 0$. We establish that their principal part is a simple pole in $\beta$ and compute several terms in the small-$\beta$ expansion perturbatively. Diffeomorphism invariance implies that only one of the numeric coefficients at this order is independent, which we take to be $c_1(\beta)$, while the other two are related to it by

$$c'_1(\beta) = 2c_1(\beta) + e^{1 - \frac{s}{48}}, \quad (1.4)$$

$$c''_1(\beta) = c_1(\beta) + \left(1 - s\right) \left(3\beta^2 - (10 + 3s)\beta + 6(s + 1)\right) \frac{1}{48 \beta}. \quad (1.5)$$

Laurent expansion of $c_1(\beta)$ takes the following form

$$c_1(\beta) = -\frac{1}{8\beta} + \frac{101}{480} + \beta \left( l_2 - \frac{5}{48} \right) + \beta^2 \left( l_3 + \frac{1}{64} \right) + O(\beta^3) \quad (1.6)$$

with numeric coefficients $l_2 \approx -1.69 \times 10^{-3}$ and $l_3 \approx -0.28 \times 10^{-3}$ arising from multi-dimensional integrals involving Bessel functions. We furthermore argue that the coefficients in the higher order terms of the large $N$ expansion are also given by the Laurent series with simple poles at $\beta = 0$.

While this is not our main goal, our method can also be used to compute the generating function in the regime $N \to \infty$ and $\beta \to 0$ with $\beta < CN^{-\epsilon}$, where $0 < \epsilon \leq 1$. The case of $\epsilon = 1$ corresponds to the generalized Gibbs ensemble in the probabilistic approach to Kähler–Einstein metrics [5].
The paper is organized as follows. In section 2 we define the generating functional and introduce the notations relevant for equation (1.1). In section 3 we recall the relation between the Laughlin state on curved backgrounds and free fields. We then define an interacting QFT that can be used to compute the generating functional. In section 4 we make our proposal more precise by emphasizing similarity of our model with the Liouville field theory (LFT). We find a convenient reparametrization of the path integral that naturally separates the leading order terms of the generating functional from the subleading ones. In section 5 we develop a perturbation theory which in principle allows to compute the subleading terms as the Laurent expansions in $\beta$. We focus on the first non-trivial coefficient and compute it up to three loop orders. In section 6 we review our results and outline directions for future work.

2. Generating functional

2.1. Definition

The main object in the present paper is the generating functional defined as the logarithm

$$F[W] = \log Z[W]$$

of the $N$-fold integral over a sphere with respect to the volume form $d\nu(z) = \sqrt{g} d^2z$

$$Z[W] = \int_{(S^2)^N} \prod_i d\nu(z_i) |\psi(z_1, \ldots, z_N)|^2$$

of the mod squared of the Laughlin wave function $\psi$ at filling fraction $\beta^{-1}$ with $\beta \in \mathbb{Z}_+$,

$$\psi(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^\beta \prod_i e^{\frac{i}{2} W(z_i, \bar{z}_i)}.$$  

Here the locally-defined potential function $W(z, \bar{z})$

$$W(z, \bar{z}) = K(z, \bar{z}) - s \log \sqrt{g}(z, \bar{z})$$

is the sum of the potential function $K$ for the magnetic field $B = -\frac{i}{2} \Delta g K$, which is strictly positive $B > 0$, and the logarithm of the metric where the coefficient $s \in \frac{\beta}{\gamma} \mathbb{Z}$ is the so-called gravitational spin of the Laughlin state. For the fully filled Laughlin ground state the quantized flux of magnetic field is related to the number of particles as follows

$$N_\beta = \frac{1}{2\pi} \int_{S^2} B \sqrt{g} d^2z = \beta(N-1) + 2s.$$  

The generating functional encodes the response of the ground state to external electromagnetic and gravitational fields and contains a wealth of information about the state. For instance, in the spherical geometry one application of the generating functional is to compute the density–density correlation functions [53].

2.2. Structure of the large $N$ expansion

Here we start with a reminder of what is known about the large $N$ expansion of the generating functional and introduce several conventions and notations along the way. First, we note that one can think of $F[W]$ as a functional of $W$, or more precisely of $W + \log \sqrt{g}$, as is apparent...
from the definition. On the other hand it is useful to think of it as a functional of the magnetic potential \( K \) and the metric \( g \), as well as their derivatives—the magnetic field \( B \), scalar curvature \( R \) etc. The generating functional naturally splits (see e.g. [32] for a review) into the anomalous \( \mathcal{F}_A \) and exact parts \( \mathcal{F}_E \)

\[
F[W] = \mathcal{F}_A[g, B] + \mathcal{F}_E[g, B].
\] (2.6)

The anomalous part can be formally written as a non-local expression

\[
\mathcal{F}_A[g, B] = -\frac{1}{2\pi} \sigma_H \int_{\partial \Sigma^2} B \Delta_g^{-1} B - \frac{1}{2\pi} 2^{\frac{3s}{2}} \int_{\partial \Sigma^2} B \Delta_g^{-1} R + \frac{c_H}{96\pi} \int_{\partial \Sigma^2} R \Delta_g^{-1} R. \tag{2.7}
\]

This is a schematic formula, for details of the definition we refer to [13, 32]. This expression is a combination of gauge, gravitational and mixed gauge-gravitational anomaly functionals and the corresponding coefficients, depending on \( \beta \) and \( s \), encode the Hall conductance \( \sigma_H = 1/\beta \), the Hall central charge \( c_H = 1 - \frac{3\beta - 2s^2}{\beta} \) and the third coefficient \( s_H = \frac{1}{4} - \frac{1}{12} \beta \), related to Hall viscosity.

According to (2.5) the magnetic field \( B \) is of order \( N \) for \( N \) large, while the curvature \( R \) is of order one, hence anomalous part contains the terms of order \( N^2, N^1 \) and \( N^0 \) of the large \( N \) expansion. In turn, the exact part \( \mathcal{F}_E \) is a local (single-integral) gradient expansion in terms of the scalar curvature and magnetic field with the leading terms given by

\[
\mathcal{F}_E[g, B] = -\frac{1}{2\pi} \int_{\partial \Sigma^2} \sqrt{g} d^2z \left[ \left( \frac{1}{\beta} - \frac{1}{2} \right) B \log B + \left( \left( \frac{1}{\beta} - \frac{1}{2} \right) \beta - \frac{2s}{4} + \frac{1}{24} \right) R \log B - \left( \frac{1}{4\beta} - \frac{13}{48} + \frac{\beta}{16} \right) (\log B) \Delta_g \log B \right] + \mathcal{R}_N, \tag{2.8}
\]

where \( \mathcal{R}_N \) is of order \( O(N^{-1}) \) This equation was proven rigorously at \( \beta = 1 \) in [31] (and [29] for constant magnetic field) and conjectured for any \( \beta \) in [9, equation (130)]. The difficult part is to control the remainder terms \( \mathcal{R}_N \) well enough to prove that they are subleading. We note that both formulas (2.7) and (2.8) are written as functionals that depend separately on the magnetic field and spin. However their sum can in fact be recombined into a functional of \( W + \log \sqrt{g} \) alone.

A word about conventions. As exemplified by equation (2.6) throughout the paper we use calligraphic symbols like \( \mathcal{F}_E \) for functionals that depend on magnetic field and metric separately and reserve plain symbols like \( F[W] \) for true functionals of potential \( W \). Note also a complementary notation for the functionals of \( W \) discussed around (2.18). In section 3 we propose a path integral formulation of the generating functional, which produces the leading order terms of the large \( N \) expansion in a way that unifies (2.7) and (2.8).

2.3. Notation

2.3.1. Differential geometry. We will assume that the metric in conformal coordinates is \( ds^2 = 2g_{zz} \, dz \, d\bar{z} \) and introduce the scalar Laplacian and the scalar curvature

\[
\Delta_g = \frac{4}{\sqrt{g}} \partial_z \partial_{\bar{z}}, \quad R = -\Delta_g \log \sqrt{g}. \tag{2.9}
\]
with $\sqrt{g} = 2g_{zz}$. Conventions are such that $\int_{S^2} \sqrt{g} \, d^2z \, R = 4\pi \chi$ with $\chi$ being the Euler characteristic ($\chi = 2$ for a sphere).

We will frequently consider a pair of metrics related by a Weyl rescaling with some conformal factor $\Omega(z, \bar{z}) > 0$. In this case we will write $g_\Omega$ for the rescaled metric with $(g_\Omega)_{zz} = \Omega g_{zz}, \quad \sqrt{g_\Omega} = \Omega \sqrt{g}$ (2.10)

and also

$$
\Delta_\Omega = \Omega^{-1} \Delta_g, \quad R_\Omega = \Omega^{-1} \left( R - \Delta_g \log \Omega \right).
$$

We normalize delta-functions on a curved sphere with respect to the volume element

$$
\int_{S^2} \sqrt{g} \, d^2z \, \delta(z, w) = 1.
$$

2.3.2. Effective magnetic field. We will refer to the following combination of magnetic field and scalar curvature

$$
H = B + \frac{1 - s}{2} R = - \frac{1}{2} \Delta_g (W + \log \sqrt{g}).
$$

as the effective magnetic field. The flux of $H$ is

$$
N_H = \frac{1}{2\pi} \int_{S^2} \sqrt{g_\Omega} \, d^2z \, H = \beta(N - 1) + 2.
$$

Since $H > 0$, we can introduce a fictitious metric $g_H$ as is suggested by (2.10)

$$
\sqrt{g_H} = H \sqrt{g},
$$

and its normalized version

$$
\sqrt{g_H} = h \sqrt{g}, \quad h = \frac{H}{N_H},
$$

which has a fixed area in the limit $N \to \infty$

$$
\frac{1}{2\pi} \int_{S^2} \sqrt{g_H} \, d^2z \, h = \frac{1}{2\pi} \int_{S^2} \sqrt{g_\Omega} \, d^2z = 1.
$$

Note that since $\sqrt{g} \Delta_g$ is in fact independent of the metric the combination $\sqrt{g} H$ is a function of $W + \log \sqrt{g}$ alone. Later we will mostly think of the generating functional as the functional of $g_H = N_H g_{\Omega}$ and with a slight abuse of notation write

$$
F[W] = F[g_H].
$$

3. Laughlin state and 2d QFT

3.1. Laughlin state from free QFT

Consider a free scalar field theory with action

$$
S_0[\phi, g, B] = \frac{1}{4\pi} \int_{S^2} \sqrt{g} \, d^2z \left[ -\phi \Delta_g \phi + i \sqrt{\frac{8}{\beta}} \left( B + \frac{\beta - 2s}{4} R \right) \phi \right].
$$

(3.1)
Laughlin’s wavefunction (2.3) can be interpreted as a correlation function of the electron operators [39],

\[ V_e(z, \bar{z}) = e^{i \sqrt{2} \beta \phi(z, \bar{z})} \]  

(3.2)

in this theory. Precisely [13],

\[ |\psi(z_1, \ldots, z_N)|^2 = e^{\mathcal{F}_\phi \left[ g, B \right]} \langle \prod_i V_e(z_i, \bar{z}_i) \rangle = e^{\mathcal{F}_\phi \left[ g, B \right]} \int Dg \phi e^{-S_0[\phi, g, B]} \prod_i e^{i \sqrt{2} \beta \phi(z_i, \bar{z}_i)}. \]  

(3.3)

Note that the correlation function here is not normalized.

The scalar field is assumed to be compact \( \phi \equiv \phi + \frac{2\pi}{r_c} \) with the radius of the circle \( r_c = \sqrt{\frac{\beta}{2}} \). The electron operators are invariant under shifts of \( \phi \) by multiples of \( 2\pi r_c \). As can be seen by separating the zero-mode integration in (3.3) this correlation function is only well-defined (invariant under \( \phi \to \phi + 2\pi r_c \)) and non-vanishing if the magnetic field has flux (2.5). Conversely, given a magnetic field with flux (2.5) only the correlation function with \( N \) electron operators is non-zero.

Substituting (3.3) in (2.1) and canceling the common anomaly factors one finds

\[ e^{\mathcal{F}_\phi \left[ g, B \right]} = \int_{S^2} \prod_i \sqrt{g} d^2 z_i \left\langle \prod_i V_e(z_i, \bar{z}_i) \right\rangle = \left\langle \left( \int_{S^2} \sqrt{g} d^2 z \prod_i V_e(z, \bar{z}) \right)^N \right\rangle. \]  

(3.4)

The holomorphic part of Laughlin’s wavefunction can be identified with conformal block of a free boson CFT. We emphasize that theory (3.1) is strictly speaking not a CFT as the presence of the magnetic field breaks the conformal invariance. This distinction might not seem significant at the level of free fields but it becomes essential in the interacting theory that we will now introduce.

### 3.2. Generating functional from interacting QFT

Let us define a modified version of the free action (3.1)

\[ S[\phi, g, B] = \frac{1}{4\pi} \int_{S^2} \sqrt{g} d^2 z \left( -\phi \Delta_g \phi + i \sqrt{\frac{8}{\beta}} \left( B + \frac{\beta - 2s}{4R} \right) \phi + 4\pi \mu e^{i \sqrt{2} \beta \phi} \right), \]  

(3.5)

which can be thought of as a perturbation by the electron operator

\[ S = S_0 + \mu \int_{S^2} \sqrt{g} d^2 z V_e(z, \bar{z}). \]  

(3.6)

Now consider the partition function of this theory and formally expand it in powers of \( \mu \)

\[ \int Dg \phi e^{-S[\phi, g, B]} = \sum_{n=0}^{\infty} \frac{(-\mu)^n}{n!} \int Dg \phi e^{-S_0[\phi, g, B]} \left( \int_{S^2} \sqrt{g} d^2 z V_e(z, \bar{z}) \right)^n. \]  

(3.7)
The right-hand side here is a sum of the integrated correlators in the free theory. In fact, only one term with \( n = N \) satisfies the charge neutrality and is non-vanishing. Comparing to (3.4) we find that the partition function of interacting theory (3.5) with properly adjusted flux of the magnetic field (2.5) computes the generating functional of the Laughlin state

\[
\mathcal{F}_E[g,B] = \int \mathcal{D} \phi \, e^{-S[\phi,g,B]}. \tag{3.8}
\]

This observation is the starting point for most of our further developments. Note that we have absorbed an inessential constant \((-\mu)^N/N!\) into \( \mathcal{F}_E \) and we will keep this convention in the following.

4. Leading terms from Liouville field theory

4.1. Relation to LFT

Any non-trivial QFT is only tentatively defined by its classical action. To shape our proposal we will draw on the similarity between (3.5) and the LFT defined by action

\[
S_L[\phi, g] = \frac{1}{4\pi} \int_S \sqrt{g} \, d^2z \left( -\phi \Delta_g \phi + Q R \phi + 4\pi \mu e^{2b\phi} \right), \tag{4.1}
\]

where \( Q = b + b^{-1} \). In fact one finds

\[
S[\phi, g, B] = S_L[\phi, g] - \frac{1}{2\pi b} \int_S \sqrt{g} \, d^2z \, H \phi \tag{4.2}
\]

if we identify

\[
b = i \sqrt{\frac{\beta}{2}}, \tag{4.3}
\]

and we will comment on the choice of imaginary \( b \) below. To facilitate comparison with LFT in the current section we always use \( b \) instead of \( \beta \). We will return to the conventional FQHE parameter \( \beta \) in section 5.

LFT is known to be a conformal field theory with central charge

\[
c = 1 + 6Q^2. \tag{4.4}
\]

In particular this implies that under a Weyl rescaling of the metric \( g \rightarrow g_\Omega \) its partition function

\[
Z_L[g] = \int \mathcal{D} \phi \, e^{-S_L[\phi,g,B]} \tag{4.5}
\]

transforms as

\[
Z_L[g_\Omega] = Z_L[g] e^{A[g,\Omega]}. \tag{4.6}
\]

Here \( A[g,\Omega] \) is the Polyakov Liouville functional

\[
A[g,\Omega] = -\frac{1}{96\pi} \int_S \sqrt{g} \, d^2z \left( \log \Omega \Delta_g \log \Omega - 2R \log \Omega \right). \tag{4.7}
\]
To obtain (4.6) one uses the transformation properties of the path integral measure and the action under a Weyl rescaling

$$\frac{D_{g\Omega} \phi}{D_g \phi} = e^{A[\phi, \Omega]}, \quad S_L[\phi, g\Omega] = S_L[\phi + \frac{Q}{2} \log \Omega, g] - 6Q^2A[g, \Omega].$$  \hspace{1cm} (4.8)

To derive the transformation law for the action one needs the following relation

$$[e^{2b\phi}]_{g\Omega} = e^{b \log \Omega} [e^{2b\phi}]_g,$$  \hspace{1cm} (4.9)

which is a part of the quantization prescription of the classical theory (4.1). The interaction exponent is assumed to be normal-ordered implying that the short-distance singularities are regulated with a reference to the length scale introduced by metric $g$. When the metric is Weyl rescaled the interaction term transforms non-trivially (4.9).

We now emphasize several distinctions between LFT and our model (4.2). Note that in the last step of deriving (4.6) from (4.8) one needs to shift the path integral variable $\phi \to \phi - Q^2 \log \Omega$. This is possible literally when $Q \in \mathbb{R}$ (i.e. $c > 25$) and $\phi$ is not compact. This is the best studied regime of LFT. In contrast, the case that is directly relevant for our discussion (4.3) is $c \leq 1$ with $Q^2 \in \mathbb{Q}$. By definition LFT is still conformal and obeys (4.6) but its description in terms of the path integral can be more subtle [25]. However, precisely when $Q^2$ is rational the Liouville theory is believed to be essentially similar to generic Re $c > 1$ and in particular obtainable by the analytic continuation from that region [47]. This suggests that the relevant path integral allows for complex deformations of the integration contour. We will assume this to hold also for the path integral with the magnetic field (4.2). An additional subtlety is that in (4.2) the free boson is compact and so the zero mode integration is over a finite range. This seems to prevent us from making constant shifts in $\phi$. However, the result of such a shift on the generating functional (3.8) is to rescale the parameter $\mu$ which simply leads to a constant shift in $F_E$ and is not important. As we will see shortly, the assumptions we have made are justified a posteriori as they allow to correctly reproduce the leading order terms of the generating functional.

Finally, we stress that the true power of LFT is grounded in conformal symmetry. In this sense although the modification by the additional background term in (4.2) might seem mild it actually breaks the conformal invariance and is therefore essential. Nevertheless, the analogy with LFT will be a very useful reference point for our further developments.

4.2. Generating functional: anomalies and leading terms

As the first application of our proposal we will derive the leading terms in the large $N$ expansion of the generating functional. We begin with the path-integral representation (3.8) for the exact part of the generating functional $F_E$ and make the Weyl rescaling from metric $g$ to $g_H$ (2.15).

As we will see this transformation naturally decouples the leading terms in the generating functional from the $O(N^{-1})$ tail. Using relation to the LFT action (4.2) and the transformation property of the latter (4.8) it follows

$$S[\phi - \frac{Q}{2} \log H, g, B] = S[\phi, g_H] + 6Q^2A[g, H],$$  \hspace{1cm} (4.10)

where we introduced a new action $S$ that only depends on $g_H$

$$S[\phi, g_H] = S_L[\phi, g_H] - \frac{N_H}{2\pi b} \int \sqrt{g_H} d^2 \phi.$$

(4.11)
Shifting the path integral variable $\phi \to \phi - \frac{Q}{2} \log H$, accounting for the measure anomaly (4.8) and an additional contribution from the second term in (4.11) one finds

$$\mathcal{F}_E[g, B] = \tilde{\mathcal{F}}_E[g, B] + F_E[g_H]$$

with

$$\tilde{\mathcal{F}}_E[g, B] = -cA[g, H] + \frac{Q}{4\pi b} \int_{S^2} \sqrt{g} \, d^2z \, H \log H$$

$$= \frac{Q}{4\pi b} \int_{S^2} \sqrt{g} \, d^2z \, H \log H + \frac{c}{96\pi} \int_{S^2} \sqrt{g} \, d^2z \, \left( \log H \left( \Delta_g \log H \right) - 2R \log H \right).$$

Here $c$ is the central charge (4.4) and $F_E[g_H]$ is the generating functional associated with action (4.11) (explicitly defined later in (5.2)).

Passing from the original metric $g$ to metric $g_H$ in the path integral which leads to decomposition (4.12) is very useful. First, using definition of $H$ (2.13) one can check that $\tilde{\mathcal{F}}_E$ reproduces the leading exact terms of the generating functional expansion (2.8)

$$\tilde{\mathcal{F}}_E[g, B] = \mathcal{F}_E[g, B] + O(N^{-1}).$$

The remaining part of the exact term $F_E[g_H]$ in (4.12) is then expected to capture the remainder terms $R_N$ in (2.8) and be subleading. We will argue that this is indeed the case in the next section using path integral representation for $F_E[g_H]$.

Here we would also like to note that $\tilde{\mathcal{F}}_E$ can be neatly combined with the anomaly terms (2.7) into a function of $g_H$ alone

$$F_A[g_H] = \mathcal{F}_A[g, B] + \tilde{\mathcal{F}}_E[g, B]$$

$$= \frac{N_H}{4\pi b^2} \int_{S^2} \Delta_h^{-1} - \frac{N_H Q}{4\pi b} \int_{S^2} \Delta_h^{-1} R_h + \frac{c}{96\pi} \int_{S^2} R_h \Delta_h^{-1} R_h.$$ (4.15)

Here we used a normalized metric $g_h = g_H/N_H$ (2.16) (with unit area) to make the order of terms manifest. To put it differently, expression (4.15) reproduces both the standard from of the anomaly terms (2.7) and the leading orders of the exact part (2.8). As a bonus this formula also nicely interprets coefficients in (2.8) in terms of Liouville central charge $c$. We used this observation in (1.1).

5. Subleading corrections

5.1. Setup and conventions

In the previous section we have found that there is a natural split of the generating functional

$$F[g_H] = F_A[g_H] + F_E[g_H].$$

where $F_A[g_H]$ defined by (4.15) accounts for all leading terms of the large $N$ expansion (both anomalous and exact). The rest $F_E[g_H]$ is then expected to capture the remainder terms $R_N$ in (2.8) and vanish in the large $N$ limit. It is the purpose of the current section to explain why $F_E[g_H]$ is indeed subleading with respect to $F_A[g_H]$ and how it could be computed in our approach.
Because $F_{EL}[g_H]$ does not depend on the magnetic field $B$ and the metric $g$ separately, but only through a combination $g_H = N_H g_h$ (2.15) we can greatly lighten the notations in the current section by dropping the subscript $h$ in $g_h$ and write simply
\[
g \equiv g_h, \quad F[g_H] \equiv F[g, N] \quad \text{(in this section)}.\]

In the final results (see equation (6.2)) we will recall that $g$ is not the actual physical metric but a shorthand for $g_h$ and that dependence on the proper metric and magnetic fields are recovered by explicit substitution $g \to g_h$ in all metric-dependent quantities like $\Delta_g$ or $R$.

With these conventions in place we are now ready to give a path integral representation for $F_{EL}[g, N] \equiv F_{EL}[g_H]$
\[
e^{F_{EL}[x,N]} = \int \mathcal{D}g \phi \; e^{-S[\phi,g,N]} 
= \int \mathcal{D}g \phi \; \exp \left[ -\frac{1}{4\pi} \int_{S^2} \sqrt{g} \; d^2z \left( -\phi \Delta_g \phi + i\sqrt{\frac{2}{\beta}} M \phi + 4\pi \mu \rho \sqrt{2} \phi \right) \right], \quad (5.2)\]
where for later convenience we introduced
\[
M = 2N_H + \frac{\beta - 2}{2} R \quad \text{(5.3)}\]
and also switched back to $\beta$ from the Liouville $b$ via (4.3).

5.2. Gradient expansion

The large $N$ expansion of the exact part of the generating functional is believed to be a local gradient expansion in $R$ and its derivatives, with general form fixed by the dif-invariance
\[
F_{EL}[g, N] = \frac{c_{\perp}(\beta)N_H}{2\pi} \int_{S^2} \sqrt{g} \; d^2z + \frac{c_0(\beta)}{2\pi} \int_{S^2} \sqrt{g} \; d^2z \; R + \frac{c_1(\beta)}{2\pi N_H} \int_{S^2} \sqrt{g} \; d^2z \; R^2 
+ \frac{1}{2\pi N_H^2} \int_{S^2} \sqrt{g} \; d^2z \left( c_2(\beta)R^3 + c'_2(\beta)\Delta_g R + O(N_H^{-1}) \right). \quad (5.4)\]

Note that the integrals weighted by $c_{\perp}(\beta)$ and $c_0(\beta)$ describe the total area and the Euler characteristic of the surface respectively. They thus do not exhibit any interesting dependence on the metric and we will simply omit them in what follows. The first non-trivial term in (5.4) is thus
\[
F_{EL}[g, N] \simeq \frac{c_1(\beta)}{2\pi N_H} \int_{S^2} \sqrt{g} \; d^2z \; R^2 + O(N_H^{-2}) \quad \text{(5.5)}\]
with $\simeq$ implying that the two trivial leading terms are omitted.

Computing coefficients $c_1(\beta)$ seems to be a rather non-trivial task. Presently they are only known at $\beta = 1$ when the generating functional admits a determinantal representation and can be related to the Bergman kernel. Several orders in expansion of the full generating functional at $\beta = 1, s = 0$ in homogeneous magnetic field were found in [29, equation (6.6)]. Here we quote the result at $\beta = 1^4$
\[
F[g, N] \simeq -\frac{5}{384\pi N} \int_{S^2} \sqrt{g} \; d^2z \; R^2 + O(N^{-2}). \quad (5.6)\]

\(^4\)Note that normalization of the scalar curvature used in [29] is related to ours by a factor $1/2$. 

\[\text{}\]
To deduce the value of $c_1(\beta = 1)$ one must subtract the contribution of (4.13) at this order. Substituting $H = N - 1 + \frac{R}{2}$ which follows from (2.13) at $\beta = 1, s = 0$ gives

$$\tilde{F}_E[g, B] \simeq -\frac{1}{96\pi N} \int_{S^2} \sqrt{g} \, d^2z \, R^2 + O(N^{-2})$$

(5.7)

and hence

$$c_1(\beta = 1) = -\frac{1}{192}$$

(5.8)

in our conventions here.

### 5.3. Semiclassical expansion

Given that $N_H$ is a large parameter in the action (5.2) it may be tempting to treat the path integral semiclassically. However, this large parameter effectively only couples to the zero mode and hence does not suppress quantum fluctuations. In the LFT the semiclassical regime instead is $\beta \to 0$. Our strategy is to compute the gradient expansion to the order $1/N$ from the LFT perturbation theory, compare it to the expected form (5.4) and extract the information about the coefficients $c_n(\beta)$ in perturbation theory around $\beta = 0$.

First, let us introduce a rescaled scalar field $\varphi = \sqrt{2\beta} \phi$ and rewrite (5.2) accordingly

$$e^{F_{\text{eff}}[g, N]} = \int \mathcal{D}g \varphi \exp \left[ -\frac{1}{8\pi \beta} \int_{S^2} \sqrt{g} \, d^2z \left( -\varphi \Delta_g \varphi + 2iM \varphi + 8\pi \mu \beta e^{\varphi} \right) \right].$$

(5.9)

This representation suggests to consider small $\beta$ limit at fixed $M$. Note that for the constant scalar curvature function $M$ (5.3) is also a constant and therefore defined by its flux. As follows from (2.14)

$$\frac{1}{2\pi} \int_{S^2} \sqrt{g} \, d^2z \, M = 2\beta N.$$  

(5.10)

Thus we are naturally led to consider the perturbative expansion in small $\beta$ and then large $M$, the latter turns out to correspond to the gradient expansion. While this limit can be interesting on its own, our goal here is purely practical since this is the regime where perturbative calculations are possible. From the saddle-point arguments it follows that the coefficients $c_n(\beta)$ are given by Laurent expansions in $\beta$ with the principle part being by a simple pole,

$$c_n(\beta) = \sum_{k=1}^{\infty} c_{nk} \beta^k.$$  

(5.11)

In what follows we will focus on computing coefficient $c_1(\beta)$ of the first non-trivial term in (5.4). Up to three loop orders we obtain a result of the form

$$c_1(\beta) = l_0(\beta) + l_1(\beta) + \beta l_2 + \beta^2 l_3 + O(\beta^3),$$  

(5.12)

where $l_0(\beta)$ comes from the saddle point action, $l_1(\beta)$ from the one-loop determinant, and $l_2, l_3$ from the two- and three-loop diagrams. We note in advance that in our perturbation theory the saddle point and the one loop contributions mix together several orders of $\beta$ while higher loops contribute to $c_1(\beta)$ monomials with a power of $\beta$ determined by the number of loops. There is no mixing of orders if $M$ is taken as a large expansion parameter instead of $N_H$ but in order to extract $c_1(\beta)$ one needs to substitute (5.3) and re-expand in $N_H$, introducing the mixing.
Let us adopt one more technical simplification. Since our goal is to fix the coefficient in front of the $R^2$ term in expansion (5.4) we can perform the calculation assuming constant round metric $R \equiv \text{const}$. However both the term of interest in (5.4) and some intermediate steps in our computations are valid for arbitrary $R$ up to subleading derivative corrections. We choose to emphasize this when possible by writing $\int_{S^2} \sqrt{g} \, d^2 z R^2$ instead of $2\pi R^2$ (recall that $g \equiv g_h$ has area $2\pi$ (2.16)).

5.4. Saddle point

The classical equation of motion for the action in (5.9) is

$$4\pi \mu \beta e^{i\varphi_s} = -M + \Delta g \varphi_s. \quad (5.13)$$

Recall that we are interested in the case of constant $M$ which allows for a constant solution

$$\varphi_s = -i \log \frac{-M}{4\pi \mu \beta}. \quad (5.14)$$

For $\varphi_s$ to be real the argument of the logarithm must be a pure phase. This can always be achieved by an appropriate choice of $\mu$ which we assume. The action evaluated on the classical solution reads

$$S[\varphi_s, g, N] = \frac{1}{8\pi \beta} \int_{S^2} \sqrt{g} \, d^2 z \left[ 2M \log \frac{-M}{4\pi \mu \beta} + 2M \right]. \quad (5.15)$$

To extract the contribution of the saddle point to $c_1(\beta)$ we substitute (5.3) and expand in $N_H$

$$S[\varphi_s, g, N] = \text{const.} - N \log 4\pi \mu \beta - \frac{l_0(\beta)}{2\pi N_H} \int_{S^2} \sqrt{g} \, d^2 z R^2 + O(N_H^{-2}), \quad (5.16)$$

with

$$l_0(\beta) = -\frac{(\beta - 2)^2}{32\beta}. \quad (5.17)$$

We have kept dependence on $\mu$ in the classical action to emphasize that $e^{\frac{2h}{\beta}} \propto e^{-S[\varphi_s]} \propto \mu^N$ consistently with (3.8).

5.5. Quantum fluctuations

The fluctuations about the classical solution $\chi = \varphi - \varphi_s$ are described by the following action

$$S[\chi] = \frac{1}{8\pi \beta} \int_{S^2} \sqrt{g} \, d^2 z \left[ -\chi \Delta_g \chi - 2M \left( e^{i\chi} - i\chi - 1 \right) \right]. \quad (5.18)$$

We will use a standard perturbation theory expanding (5.18) around the minimum of the potential and treating higher order terms perturbatively

$$S[\chi] = S_2[\chi] + \sum_{n=3}^{\infty} \frac{g_n}{n!} \int_{S^2} \sqrt{g} \, d^2 z \chi^n, \quad g_n = M \frac{(i)^{n+2}}{4\pi \beta}, \quad (5.19)$$

where the quadratic part is

$$S_2[\chi] = \frac{1}{8\pi \beta} \int_{S^2} \sqrt{g} \, d^2 z \left[ -\chi \Delta_g \chi + M \chi^2 \right]. \quad (5.20)$$
Let us introduce the propagator of this Gaussian field by
\[ G_M(z, w) = \frac{1}{2\beta} \langle \chi(z)\chi(w) \rangle. \]  
(5.21)

The normalization is chosen so that \( G_M(z, w) \) is \( \beta \)-independent. The propagator solves equation
\[ (-\Delta_g + M)G_M(z, w) = 2\pi\delta(z, w). \]  
(5.22)

We will need the expansion of the propagator near the diagonal which can be obtained for example from the off-diagonal expansion of the heat kernel (see for instance [18])
\[ G_M(z, w) = A_0K_0 \left( d\sqrt{M} \right) + A_1d^2K_1 \left( d\sqrt{M} \right) + A_2d^4K_2 \left( d\sqrt{M} \right) + O(d^3). \]  
(5.23)

Here \( d = d_r(z, w) \) is the geodesic distance between points with coordinates \( z, w \), and \( A_i \) are heat kernel coefficients. For constant scalar curvature they are given by
\[ A_0 = 1 + \frac{R}{24}d^2 + \frac{R^2}{640}d^4 + O(R^3), \quad A_1 = \frac{R}{6} + \frac{R^2}{120}d^2 + O(R^3), \quad A_2 = \frac{R^2}{60} + O(R^3). \]  
(5.24)

Finally, the modified Bessel functions of the second kind \( K_0, K_1, K_2 \) can be defined by
\[ K_0(r) = \int_0^\infty \frac{ds}{2s}e^{-\frac{s^2}{2}}, \quad K_1(r) = -K_0'(r), \quad K_2(r) = 2K_0''(r) - K_0(r). \]  
(5.25)

Function \( K_0(r) \) has a logarithmic singularity at the origin (\( \gamma_E \approx 0.577 \) is Euler’s constant)
\[ K_0(r) = -\log r + \gamma_E - \log 2 + O(r) \]  
(5.26)

and decays exponentially as \( r \to \infty \) as is expected from the propagator of a massive field.

We will compute the loop integrals in the coordinate space. UV-divergences then appear as singularities of the propagator at coincident points. Following [56] we regularize them by introducing the regularized propagator
\[ G_M^\beta(z) = \lim_{w \to z} (G_M(z, w) + \log d(z, w)). \]  
(5.27)

With this prescription all loop diagrams are finite. Explicitly the regularized propagator is
\[ G_M^\beta(z) = -\frac{1}{2} \log M + \log 2 - \gamma_E + \frac{R}{12M} + \frac{R^2}{120M^2} + O(M^{-3}). \]  
(5.28)

5.6. One loop

We are now in a position to carry out the loop computations. The one-loop determinant \( W \) is given by the partition function of the Gaussian field
\[ e^{W_{[g,N]}} = \int D\chi \exp \left( -\frac{1}{8\pi\beta} \int_S \sqrt{g} d^2z \left[ -\chi \Delta_g \chi + M\chi^2 \right] \right), \]  
(5.29)
To compute it we note that
\[
\frac{\partial W_{[g, N]}}{\partial M} = \frac{1}{8\pi^2} \int_{S^2} \sqrt{g} \, d^2z \langle \chi^2(z) \rangle = \frac{1}{4\pi} \int_{S^2} \sqrt{g} \, d^2z \, G_M^R(z)
\]
(5.30)
and hence \( W \) can be found by integrating the regularized propagator. The result is
\[
W_{[g, N]} = -\frac{1}{4\pi} \int_{S^2} \sqrt{g} \, d^2z \left[ -\frac{1}{2} M \log M + \frac{1}{12} R \log M \right. \\
+ \left( \frac{1}{2} + \log 2 - \gamma_E \right) M - \frac{1}{120} R^2 \right] + O(N_H^{-2}).
\]
(5.31)
Substituting (5.3) and expanding in \( N_H \) we find
\[
W_{[g, N]} \approx \frac{l_1(\beta)}{2\pi N_H} \int_{S^2} \sqrt{g} \, d^2z \, R^2 + O(N_H^{-2})
\]
(5.32)
with
\[
l_1(\beta) = \frac{(\beta - 2)(3\beta - 8)}{192} + \frac{1}{480}.
\]
(5.33)
5.7 Two loops
We now consider the two loop contributions. There are three connected diagrams at this order
\[
D_2^{(1)} = -\frac{1}{4\pi} \frac{\beta}{6\pi} \int_{(g^2)^2} \sqrt{g} \, d^2z \sqrt{g} \, d^2w \, M^2 \left[ G_M(z, w) \right]^3
\]
(5.34)
\[
D_2^{(2)} = -\frac{1}{4\pi} \frac{\beta}{4\pi} \int_{(g^2)^3} \sqrt{g} \, d^2z \sqrt{g} \, d^2w \, M^2 \left[ G_M^R(z) G_M(z, w) G_M^R(w) \right]
\]
(5.35)
\[
D_2^{(3)} = \frac{1}{4\pi} \frac{\beta}{2} \int_{S^2} \sqrt{g} \, d^2z \, M \left[ G_M^R(z) \right]^2
\]
(5.36)
At large \( M \) all integrals of this type localize near diagonals of the propagators and in the large \( N_H \) expansion acquire a local form of equation (5.4). In fact, one can formally solve equation (5.22) in the large \( M \) expansion as follows
\[
G_M(z, w) = \frac{2\pi}{M - \Delta} \delta(z, w) = \frac{2\pi}{M} \delta(z, w) + \frac{2\pi}{M^2} \Delta \delta(z, w) + \ldots.
\]
(5.37)
This expansion in terms of delta-functions emphasizes the localization of the integrals. Technically (5.37) gives a sufficient approximation if the propagator is to be integrated against
smooth at $z = w$ functions. We can use it to show that diagrams (5.35) and (5.36) in fact cancel each other. Indeed, substituting (5.37) in (5.35) gives

$$D_2^{(2)} = -\frac{1}{4\pi} \frac{\beta}{2} \int_{S^2} \sqrt{g} \, d^2 z \, M \left[ G_M^R(z) \right]^2 - \frac{1}{4\pi} \frac{\beta}{2} \int \sqrt{g} \, d^2 z \, G_M^R(z) \Delta_g G_M^R(z) + O(N_H^2).$$

(5.38)

The first term here precisely cancels (5.36) while all other involve derivatives of $G_M^R$ and vanish at constant curvature.

Thus the two-loop contribution comes entirely from (5.34). Formal expansion (5.37) is not applicable to (5.34) as it produces singularities of the type $\delta(0)$. We will take a more direct approach instead. Let us fix $z$ in (5.34) and perform an integration over $w$. We assume that coordinate system $w$ is chosen so that the metric has the form

$$g_{\bar{w}w} = \frac{1}{(1 + R^8 r^2)^2} = 1 - \frac{R}{4} r^2 + \frac{3R^2}{64} r^4 + O(R^3),$$

(5.39)

where $r = |w - z|$. Expansion of the geodesic distance in these coordinates reads

$$d(z, w) = r - \frac{R}{24} r^3 + \frac{R^2}{320} r^5 + O(R^3).$$

(5.40)

Substituting this expansion into (5.23) one finds

$$G_M(z, w) = K_0(\rho) + \frac{R}{24 M} \left( \rho^2 K_0(\rho) + \rho(2 + \rho^2) K_1(\rho) \right)$$

$$+ \frac{R^2}{5760 M^2} \left( \rho^2(24 + 9 \rho^2 + 5 \rho^4) K_0(\rho) - 3 \rho(\rho^4 - 8 \rho^2 - 16) K_1(\rho) \right) + O(N_H^{-3}),$$

(5.41)

with $\rho = r\sqrt{M}$.

Finally, substituting (5.41) into (5.34) one finds $D_2^{(1)} \simeq \frac{dl_2}{2\pi N_H^2} \int \sqrt{g} \, d^2 z \, R^2 + O(N_H^{-2})$ with

$$l_2 = -\frac{1}{23040} \int_0^{\infty} dr r^2 K_0(r) \left[ 10(r^2 + 2)^2 K_1(r)^2 \right.$$}

$$+ r \left( 5r^4 + 49r^2 + 24 \right) K_0(r)^2$$

$$+ \left( -43r^4 - 56r^2 + 48 \right) K_1(r) K_0(r) \left].

(5.42)

Numerically evaluating (5.42) gives

$$l_2 \approx -1.688 \times 10^{-3}.$$

(5.43)

5 We note that as a non-trivial check for the correctness of these expansions one can verify that they satisfy equation $\partial_\nu G_M(z, \nu') = -\frac{1}{\pi} \int \sqrt{g} \, d^2 z \, G_M(z, w) G_M(w, \nu')$ which can be derived by differentiating (5.22) with respect to $M$. 
5.8. Three loops

There are fifteen connected three-loop diagrams. However, all diagrams involving regularized propagators (i.e., self-loops) cancel up to order $O(N_H^{-2})$ by the same mechanism that canceled (5.35) and (5.36). The following four diagrams remain

\[ D_3^{(1)} = \frac{\beta}{48\pi^2} \int M^2 \left[ G_{12}^4 \right] \]  

(5.44)

\[ D_3^{(2)} = -\frac{\beta^2}{16\pi^3} \int M^3 \left[ G_{13}^2 G_{13}^2 G_{23} \right] \]  

(5.45)

\[ D_3^{(3)} = \frac{\beta^2}{64\pi^4} \int M^4 \left[ G_{12}^2 G_{13}^2 G_{24} G_{34} \right] \]  

(5.46)

\[ D_3^{(4)} = \frac{\beta^2}{96\pi^4} \int M^4 \left[ G_{12} G_{13} G_{14} G_{23} G_{24} G_{34} \right] \]  

(5.47)

where for notational clarity we abbreviated $G_{i,j} = G_M(z_i, z_j)$ and omitted the integration measures. Introduce also coefficients $f_i^{(k)}$ by

\[ D_3^{(k)} \simeq \frac{\beta^2 f_i^{(k)}}{2\pi N_H} \int \sqrt{g} \, d^2 z \, R^2 + O(N_H^{-2}). \]  

(5.48)

Evaluation of the diagram (5.44) reduces to a one-dimensional integral similar to (5.34) and gives numerically (quoted to three significant digits)

\[ f_3^{(1)} \approx 1.45 \times 10^{-3}. \]  

(5.49)

In contrast, other diagrams result in multidimensional integrals which are harder to treat numerically with high precision. Running multiple Monte-Carlo evaluations we can reliably (although without guaranteed precision) fix them within relative accuracy of about 1%

\[ f_3^{(2)} \approx -4.82 \times 10^{-3}, \quad f_3^{(3)} \approx 2.16 \times 10^{-3}, \quad f_3^{(4)} \approx 0.93 \times 10^{-3}. \]  

(5.50)
Combining all contributions together gives
\[ l_3 \approx -0.28 \times 10^{-3}. \]  
(5.51)

Putting all loop contributions together we find the following result for \( c_1(\beta) \)
\[ c_1(\beta) = -\frac{1}{8\beta} + \frac{101}{480} + \beta \left( l_2 - \frac{5}{48} \right) + \beta^2 \left( l_1 + \frac{1}{64} \right) + O(\beta^3). \]  
(5.52)

Rational coefficients in this expression come from the saddle-point and the one-loop determinant while the coefficients \( l_2, l_3 \) arise from the calculation of higher loop diagrams.

6. Summary and outlook

We have studied the generating functional for the Laughlin state \( Z[W] \) on a sphere with arbitrary smooth metric \( g \) and magnetic field \( B \) which is given by a Coulomb-type integral (2.2) and in particular its large \( N \) expansion. We emphasized that the generating functional does not depend on \( g \) and \( B \) separately, but only on their combination \( g_H \) and the large \( N \) expansion is most economically written in these terms. For convenience of the reader we reproduce here the relevant definitions
\[ H = B + \frac{1 - s}{2} R, \quad N_H = \frac{1}{2\pi} \int_{S^2} \sqrt{g} \, d^2z \quad H = \beta(N - 1) + 2, \]  
(6.1)

and then \( g_H = gH \). To make the scaling of terms with \( N \) explicit we instead use \( g_h = gh \) with \( h = H/N_H \) (area of \( g_h \) is normalized to \( 2\pi \)). The large \( N \) expansion of the generating functional then is
\[ \log Z[W] = -\frac{N_h^2}{2\pi\beta} \int_{S^2} \Delta_h^{-1} \, - \frac{N_H}{2\pi} \, \frac{\beta - 2}{2\beta} \int_{S^2} \Delta_h^{-1} R_h \]  
\[ + \frac{c}{96\pi} \int_{S^2} R_h \Delta_h^{-1} R_h \]  
\[ + c_1(\beta) \frac{1}{2\pi N_H} \times \int_{S^2} \sqrt{g_h} \, d^2z \, R_h^2 + O(N_H^{-2}). \]  
(6.2)

Coefficient \( c \) here is given by
\[ c = 1 - 3 \left( \frac{\beta - 2}{\beta} \right)^2. \]  
(6.3)

It corresponds to the central charge of the associated Liouville theory and appears similarly to the Polyakov effective action. The main technical effort of the present paper was to compute the coefficient of the subleading term \( c_1(\beta) \) using QFT perturbation theory. The result is a Laurent expansion at \( \beta = 0 \) explicitly written in (5.52).

In order to pass from the effective metric \( g_h \) to the standard representation of the generating functional in terms of magnetic field and metric one needs to re-express all quantities associated
with metric $g_h$ in (6.2) in these terms, e.g. $g_h = g h, R_h = h^{-1} (R - \Delta_h)$ etc. The result reads

$$\log Z(W) = -\frac{1}{2\pi} \sigma_{H} \int \left( S^2 \right)^2 B \Delta_g \log B + \frac{1}{2\pi} \sigma_{H} \int \left( S^2 \right)^2 R \Delta_g \log R$$

$$+ \frac{1}{2\pi} \int_{S^2} \sqrt{g} d^2 z \left( \frac{2 - \beta}{2\beta} B \log B + \left( \frac{1}{24} - \frac{\beta - 2)(\beta - 2s)}{8\beta} \right) \right)$$

$$\times R \log B + \frac{c}{48} \left( \log B \Delta_g \log B \right)$$

$$+ \frac{1}{2\pi} \int_{S^2} \sqrt{g} d^2 z \left( c_1(\beta)(\Delta_g \log B)^2 B^{-1} \right)$$

$$+ c'_1(\beta) R(\Delta_g \log B)^2 B^{-1} + c''_1(\beta) R^2 B^{-1} + O(N^{-2}).$$

(6.4)

Here the coefficients $c', c''$ are related to $c_1$ as

$$c'_1(\beta) = 2c_1(\beta) + \frac{1 - s}{48},$$

(6.5)

$$c''_1(\beta) = c_1(\beta) + \frac{(1 - s)(3\beta^2 - 10 + 3s)\beta + 6(s + 1)}{48\beta}.$$  

(6.6)

Let us now discuss possible future directions and put forward some curious coincidences and speculations. A key relation that we conjectured is a connection between the generating functional of the Laughlin state and the Liouville-like QFT partition function (3.8). Although in the present paper we have focused on the generating functional for the Laughlin state on a sphere the basic idea seems to apply much more broadly. For one, both the Laughlin state and the LFT can be defined on higher genus surfaces [33] and/or in presence of boundaries. It is natural to expect that in these cases a similar connection exists.

Another interesting problem would be to address the screening hypothesis [26] for the quasiholes in our approach. Laughlin’s state with quasiholes can be obtained as a correlation function in the free theory (3.1) with insertion of the operators

$$V_q(z, \bar{z}) = e^{\sqrt{Z} \delta_0(z, \bar{z})}.$$  

(6.7)

The quantum-mechanical norm of such state is then computed by correlation function of the quasihole operators in the interacting theory (3.5). On a homogeneous background the screening hypothesis implies that such correlation functions are independent of the quasihole positions, when they are sufficiently separated. One could try to approach this problem perturbatively in $\beta$. Note that in the semiclassical limit $\beta \to 0$ the quasihole operators scale precisely in such a way that their insertion into the path integral (3.5) modifies the classical equation (5.13) by introduction of the source terms. Thus, for example at the lowest order in $\beta$ the problem of quasihole screening would be reduced to studying the properties of the relevant PDE with source terms.

More ambitiously, we could think of generalizing our model to other FQHE states. The first natural candidate is the non-abelian Moore–Read (Pfaffian) state, which is essentially

\[\text{Quasiholes are ‘heavy’ in the jargon of LFT.}\]
the Laughlin state multiplied by the correlation function of free fermions. Remarkably, the interacting theory corresponding to overlaps of Moore–Read states \[12\] appears to be the \(\mathcal{N} = 1\) supersymmetric Liouville theory \[40\] (of course, also deformed by the magnetic field).

Unfortunately, we were only able to make quantitative analysis of the path integral in perturbation theory at small \(\beta\). This case is not directly relevant for the physics of the QHE where \(\beta \in \mathbb{Z}_+\). Let us mention though a curious fact that if we compare our \(c_1(\beta)\) \((5.52)\) computed up to three loops to the known exact numerical value at \(\beta = 1\) \((5.8)\) we find an agreement with 98% accuracy. It is thus possible that the perturbative series in \(\beta\) converge quickly even away from \(\beta = 0\) but further study is needed to test this guess.

Anyway it would be extremely desirable to find alternative ways to work with the functional integral \((3.8)\) beyond perturbative expansion in \(\beta\). One possible approach here would be to take the relation to the Liouville theory even more seriously and reframe the problem. Instead of computing the partition function of the LFT with the conformal symmetry broken by the magnetic field we can consider a correlation function of non-local operator representing the magnetic field \(\sim e^{\int \mathcal{B}_0}\) in the genuine LFT. This is not immediately useful because conformal symmetry is not as powerful for non-local operators as it is for local ones, but perhaps with a proper discretization of the magnetic field one can put this picture to work. An additional motivation to formulate the problem in terms of a CFT comes from an observation that with central charge \((6.3)\) the quasi-hole operator \((6.7)\) has dimension \(\Delta(V_q) = \frac{3}{2} - \frac{1}{2\beta}\). It is therefore a candidate for a degenerate operator \(V_{2,1}\) which has many remarkable properties \([46]\).

Acknowledgments

We thank Sylvain Ribault for useful discussions. The work of NN is partly funded by DFG projects CRC/TRR 191 and SFB/TRR 183. The work of SK has benefitted from support provided by the University of Strasbourg Institute for Advanced Study (USIAS) for a Fellowship, within the French national programme ‘Investment for the future’ (IdEx-Unistra), by the University of Strasbourg IdEx program and by the RFBR grant 18-01-00926.

Data availability statement

No new data were created or analysed in this study.

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