MAXIMAL OPERATORS ASSOCIATED WITH FOURIER MULTIPLIERS AND APPLICATIONS

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Abstract. In this paper, we introduce a criterion for maximal operators associated with Fourier multipliers to be bounded on $L^p(\mathbb{R}^d)$ for each $p \in (1, \infty)$. Noteworthy examples satisfying the criterion are multipliers of the Mikhlin type or limited decay which are not necessarily radial. To do so, we make use of modified square function estimates and bilinear interpolation. For the bilinear interpolation we introduce a function space $\Sigma^2(B)$ where $B$ denotes a Banach space of functions on $\mathbb{R}^d$, which is a variant of weighted Sobolev spaces. In result, we obtain convergence results for fractional half-wave equations and surface averages as well as the $L^p$ boundedness for the maximal operators.

1. Introduction

Maximal operators and Fourier multipliers are crucial objects in harmonic analysis due to their importance in harmonic analysis itself and vast applications in other areas of mathematics such as theory of partial differential equations. Maximal operators of our particular interest are given in terms of Fourier multipliers. One of the best examples of this kind could be the spherical maximal function

$$M_{\text{sph}}(f)(x) := \sup_{t > 0} \left| \int_{S^{d-1}} f(x - ty) d\sigma_{\text{sph}}(y) \right|,$$

where $d\sigma_{\text{sph}}$ denotes the spherical measure on $S^{d-1}$ induced by the Lebesgue measure. For $f \in \mathcal{S}(\mathbb{R}^d)$, we make use of the Fourier inversion formula to write

$$M_{\text{sph}}(f)(x) := \sup_{t > 0} \left| \int_{\mathbb{R}^d} e^{2\pi i \langle x, \xi \rangle} \widehat{d\sigma_{\text{sph}}}(t\xi) \widehat{f}(\xi) d\xi \right|,$$

which is the form of $\sup_{t > 0} \left| \langle m(\cdot), \hat{f}(\cdot) \rangle \right|$. Note that $\widehat{d\sigma_{\text{sph}}}$ is the Fourier transform of the surface-carried measure of $S^{d-1}$ and equals to $\frac{J_{d-1}(|\xi|)}{|\xi|^{d-1}}$ up to constant, where $J_\nu(r)$ is the Bessel function of order $\nu$. $M_{\text{sph}}$ is bounded on $L^p(\mathbb{R}^d)$ for $p > \frac{d}{d-1}$ when $d \geq 3$ due to E. M. Stein [19; 21] and for $d = 2$ by J. Bourgain [3].

In studies of $M_{\text{sph}}$, what has been mainly concerned is the relation between the $L^p$ boundedness and geometric aspects of $M_{\text{sph}}$. Generally speaking, geometric properties

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of \( \widehat{d\sigma_{\text{sph}}} \) are one of many examples which allow maximal operators associated with the Fourier multipliers to be bounded on \( L^p(\mathbb{R}^d) \). Thus, one can naturally question that for which conditions on the Fourier multipliers \( m \) \( M_m(f) := \sup_{t>0} \left| \left( m(t\cdot) \hat{f}(\cdot) \right) \right| \) is bounded on \( L^p \).

In [17], Rubio de Francia showed mapping properties of \( M_m \) when \( m \) satisfies the limited decay condition, i.e.,

\[
|\partial^\gamma m(\xi)| \leq C|\xi|^{-a} \text{ for a given } a > 0,
\]

which is a generalisation of geometric properties of \( \widehat{d\sigma_{\text{sph}}} \). Results in [17] are based on the following square function estimates:

**Theorem A** ([17], Lemma 4). Let \( m \) be a function of class \( C^s(\mathbb{R}^d) \) with \( s > d/2 \) and supported in \( \{1/2 < |\xi| < 2\} \), and let the \( g \)-function be given by

\[
G_m(f)(x) = \left( \int_0^\infty \left| \left( m(t\xi) \hat{f}(\xi) \right) \gamma(x) \right|^2 \frac{dt}{t} \right)^{1/2}.
\]

Then for \( \beta > d/2 \) we have

\[
\|G_m(f)\|_{L^1(\mathbb{R}^d)} \leq C_\beta \|m\|_{L^2(\mathbb{R}^d)} \|f\|_{H^1(\mathbb{R}^d)}.
\]

We also introduce results of Dappa and Trebels [6], which could be understood as a radial analogue of Theorem A.

**Theorem B** ([6], Theorem 1). Let \( \rho \in C([d/2+1](\mathbb{R}^d \setminus \{0\}) \) be a positive, \( A_t \)-homogeneous distance function. Let \( m \) be a measurable function on \( (0, \infty) \) which vanishes at infinity.

(1) If \( m \) satisfies

\[
\int_0^\infty t^{\lambda-1}|m(\lambda)(t)|dt \leq B,
\]

\[
\left( \int_0^\infty |t^{\lambda}m(\lambda)(t)|^2 \frac{dt}{t} \right)^{1/2} \leq B,
\]

for \( \lambda > d|\frac{1}{p} - \frac{1}{2}| \), then \( M_{m\circ \rho} \) is of strong type \((p,p), p \in (1, \infty), \) also \( M_{m\circ \rho} \) is of weak type \((1,1).\)

(2) If \( m \) satisfies (1.1) for \( \lambda > (d-1)\left(\frac{1}{2} - \frac{1}{p}\right) + 1 \), then \( M_{m\circ \rho} \) is of strong type \((p,p), 2 \leq p \leq \infty.\)

By \( A_t \)-homogeneous distance, we mean a continuous function on \( \mathbb{R}^d \) with

\[
\rho > 0, \quad \rho(A_t \xi) = t\rho(\xi), \quad \text{for all } t > 0, \xi \in \mathbb{R}^d,
\]

where \( A_t \) is a dilation matrix given by \( A_t := e^{P \log t} = \sum_{n=0}^{\infty} \frac{(\log t)^n}{n!} P^n \) for \( P = (p_{ij}) \) introduced in [20]. Since one of examples of \( m \circ \rho \) is a radial Fourier multiplier, our main result (Theorem 1.1) can be understood as a non-radial generalisation for \( m \circ \rho \). When \( m \) is radial, Theorem [13] is extended to spectral multipliers and we recommend [7, 13] and references therein.
In [5], Christ, Grafakos, Hozík and Seeger studied $M_m$ when $m$ is the Hörmander-Mikhlin multiplier.

**Theorem C** ([5], Corollary 1.3). Suppose $1 < p < \infty$, $r = \min\{p, 2\}$, and $\beta > d/r$. Suppose that

$$\|m(2^k \cdot \hat{\psi}(\cdot))\|_{L^p_{\beta}(\mathbb{R}^d)} \leq \omega(k), \quad k \in \mathbb{Z},$$

$$\omega^*(0) + \sum_{l=1}^{\infty} \frac{\omega^*(l)}{l} < \infty.$$  

Then $M_m$ is bounded on $L^p(\mathbb{R}^d)$. Note that $\omega^*(t) := \sup\{\lambda > 0 : \text{card}\{k : |\omega(k)| > \lambda\} > t\}$. Moreover, when $p = \infty M_m$ maps $L^\infty$ to $\text{BMO}$.

**Remark 1.** In [5] the authors also present a counter example which yields that being the Mikhlin multiplier is not sufficient for $m$ to guarantee the $L^p$ mapping property of $M_m$. In [10], Grafakos, Hozík and Seeger proved that the same result holds when the denominator $l$ in the summation of (1.2) is replaced into $l \sqrt{\log(l)}$ and $\alpha > d/p + 1/p'$ for $p \in (1, 2]$, $\alpha > d/2 + 1/p$ for $p > 2$. For more details in this direction, we recommend [5, 10] and references therein.

Considering historical results, one of novelties of this paper is that we obtain a low regularity result in terms of Theorem C with an operator norm derived from Theorems A and B. We should note that our result, Theorem 1.1, doesn’t yield an improvement for Theorem C. However, since Theorem 1.1 requires reduced regularity of $m$, so the regularity difference allows us to consider not only Mikhlin type multipliers but also certain singular Fourier multipliers introduced by Miyachi in [15] and Fourier multipliers of limited decay in [17]. To do so, we first modify square function estimates for $M_m$ by making use of fractional calculus and bilinear complex interpolation. For the bilinear interpolation, we consider the square function as a bilinear operator for $m$ and $f$. Also we introduce an appropriate function space for $m$ to apply the interpolation.

To state main theorem, we recall notations for maximal Fourier multiplier operators.

$$M_m(f)(x) := \sup_{t>0} |T_{m(t)}(f)(x)|,$$

$$T_m(f)(x) := \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} m(\xi) \hat{f}(\xi) d\xi.$$

We also make use of a function space $\Sigma^2(B)$ whose norm is given by

$$\|f\|_{\Sigma^2(B)} := \left( \sum_{j \in \mathbb{Z}} \|f(2^j \cdot \hat{\psi}(\cdot))\|_B^2 \right)^{1/2},$$

where $\text{supp}(\hat{\psi})$ is in $\{ \frac{1}{2} < |\xi| < 2 \}$ as defined in (2.1) and $B$ denotes a Banach space of functions on $\mathbb{R}^d$. Properties of $\Sigma^2(B)$ are introduced in Section 2. In most cases, we choose $B = L^p_s$ or $B^{s}_{p,q}$. We note that $B^{s}_{p,q}(\mathbb{R}^d)$ is a space of tempered distributions $f$ whose norm
is given by
\[ \|f\|_{B_{p,q}^s(\mathbb{R}^d)} := \|S_0 f\|_{L^p(\mathbb{R}^d)} + \left( \sum_{j=1}^{\infty} (2^{jd}\|\psi_j * f\|_{L^p(\mathbb{R}^d)})^q \right)^{1/q}, \quad S_0 f := \sum_{j \leq 0} \psi_j * f, \]

where \( \psi_j(\cdot) = 2^{jd}\psi(2^j\cdot) \). Then for \( p = q \) we let \( B_{p,p}^s(\mathbb{R}^d) = B_p^s(\mathbb{R}^d) \), and \( B_p^s(\mathbb{R}^d) \) satisfies the embedding property \( L^2(\mathbb{R}^d) \subseteq B_p^s(\mathbb{R}^d) \) for \( p \geq 2 \) and \( s \in \mathbb{R} \). Now we introduce our main theorem.

**Theorem 1.1.** Let \( p \in (1, \infty) \). If \( m \in \Sigma^2(B_{p_0}^s) \), \( \frac{1}{p_0} = \frac{1}{p} - \frac{1}{2} \) and \( s > d\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{1}{2} \), then we have
\[ \|\mathcal{M}_m(f)\|_{L^p(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(B_{p_0}^s)} \|f\|_{L^p(\mathbb{R}^d)}. \]

Note that \( B_{p_0}^s = B_{p_0,p_0}^s(\mathbb{R}^d) \) denotes the Besov space on \( \mathbb{R}^d \). Moreover, when \( p = 1 \) or \( p = \infty \), \( \mathcal{M}_m \) satisfies \( (H^1 \to L^1) \), \( (L^\infty \to BMO) \) estimates, respectively.

In proving Theorem 1.1, we first obtain a pointwise upper bound of \( \mathcal{M}_m \) by a Hilbert space valued operator \( T_m \), where \( \tilde{m} \) is a variant of \( m \) by using the fractional calculus. We note that the vector-valued operator satisfies \( (L^1 \to H^1) \), \( (BMO \to L^\infty) \) estimates due to [16]. Then we consider the vector-valued operator \( T_{\tilde{m}}(f) \) as a bilinear operator for \( m \) and \( f \), and make use of multilinear interpolation introduced by Calderón [4, paragraph 10.1]. We also introduce an appropriate function space for \( m \) and \( \tilde{m} \). The function space and the multilinear perspective allows us to apply the multilinear interpolation, which yields interpolation among conditions on \( m \) so that we obtain condition for \( \mathcal{M}_m \) to be bounded on each \( L^p(\mathbb{R}^d) \).

**Remark 2.** For \( m \) in Theorem 1.1, we can say that \( \|m\|_{\Sigma^2(C(\mathbb{R}^d))} < \infty \) since \( B_{p_0}^s(\mathbb{R}^d) \subset C(\mathbb{R}^d) \). Thus it follows that \( m(\xi) \to 0 \) as \( |\xi| \to 0 \) or \( |\xi| \to \infty \). If one considers \( m \) such that \( m(\xi) \not\to 0 \) as \( |\xi| \to 0 \), Theorem 1.1 still holds if \( m \) is of class \( C^\infty \) near the origin. Hence we have
\[ T^*(f) := \sup_{t > 0} \{ |m(t\cdot)\varphi_0(t\cdot)f(\cdot)| \} \lesssim \mathcal{M}_f, \]

where \( \varphi_0 \in C^\infty(\mathbb{R}^d), \varphi_0 \equiv 1 \) in \( B(0,1) \) and vanishes outside of \( B(0, 2) \). Although \( m \) is not smooth, if \( m \) is of class \( C^{(d/2)+2}(\mathbb{R}^d) \), then one can use [16] Theorem II.2.1 to obtain the \( L^p \) boundedness of \( T^* \) for \( 1 < p \leq \infty \).

The novelty of Theorem 1.1 is that we actually obtain maximal estimates for \( \mathcal{M}_m \) when \( m \) is not differentiable at the origin; the condition of Theorem 1.1 allows us to consider \( m \) whose derivatives blow up at the origin. For examples of these kinds, we present Propositions 1.5 and 1.7.

Even if \( m \) is not smooth and is not of class \( C^{(d/2)+2}(\mathbb{R}^d) \), \( L^p \) estimates for \( \mathcal{M}_m \) still follow from Theorem 1.1.
Corollary 1.2. Let $p \in (1, \infty)$, and $m$ be a function of class $B^s_{p,0}(\mathbb{R}^d)$ where $s > d\left|\frac{1}{p} - \frac{1}{2}\right| + \frac{1}{2}$. Then, for any fixed $\phi_0 \in C_c^\infty(\mathbb{R}^d)$ we have
\[
\|M_{m\phi_0}(f)\|_{L^p(\mathbb{R}^d)} \lesssim \|m\|_{B^s_{p,0}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}.
\]
Moreover, when $p = 1$ or $p = \infty$, $M_{m\phi_0}$ satisfies $(H^1 \to L^1)$, $(L^\infty \to BMO)$ estimates, respectively. In particular, if $m$ is a function of class $L^s_2$ with $s > \frac{d+1}{2}$ and has compact support, then $\|M_m\|_{L^p \to L^p}$ is bounded for any $p \in (1, \infty)$.

Due to $\frac{d+1}{2} < \left[\frac{d}{2}\right] + 2$, Corollary 1.2 improves the condition in Remark 2 that $m$ is of class $C(\frac{d}{2})^2(\mathbb{R}^d)$. To our best knowledge, Corollary 1.2 is the first improvement for $m$ supported near the origin since Rubio de Francia suggested the condition $m \in C(\frac{d}{2})^2(\mathbb{R}^d)$ in [17].

Now we introduce distinguishing features of Theorem 1.1 in terms of Theorem C in [6], the authors actually suggested a condition for non-radial $m$ under which $M_m$ is of strong type $(p, p)$ for $1 \in (1, \infty)$. We note that Theorem 1.1 requires $m$ to be regular in terms of $s > d\left|\frac{1}{p} - \frac{1}{2}\right|$ for $L^p$ boundedness for all $p \in (1, \infty)$, while conditions in [6] require $s > d$ for the same result. For results of radial $m$ in [6], Theorem 1.1 yields the following counter part:

Corollary 1.3. Let $m$ be a radial symmetric function given by $m(\xi) = h(|\xi|)$ and $s > d\left|\frac{1}{p} - \frac{1}{2}\right| + \frac{1}{2}$. Then we have
\[
\|M_m(f)\|_{L^p(\mathbb{R}^d)} \lesssim \|h\|_{\Sigma^2(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}.
\]

Corollary 1.3 is similar with the result of [6] in the radial symmetric case. We note that (1.3) maybe useful in some situations due to the norm equivalence such as Lemma 4.1 or Proposition 4.2.

We now investigate results of Theorem 1.1 and Theorem C which is [5] Corollary 1.3. In [5], Corollary 1.3 for $p, q \in (1, \infty)$ the condition
\[
(1.4) \quad \sum_{j \in \mathbb{Z}} \|m(2^j \cdot) \hat{\psi}(\cdot)\|_{L^q_j}^q < \infty, \quad r = \min\{p, 2\}, \quad s > \frac{d}{r}
\]

yields the $L^p$ boundedness of $M_m$. Recall that in Theorem 1.1, $M_m$ is bounded on $L^p(\mathbb{R}^d)$ whenever
\[
(1.5) \quad \sum_{j \in \mathbb{Z}} \|m(2^j \cdot) \hat{\psi}(\cdot)\|_{B^p_{p_0}}^2 < \infty, \quad \frac{1}{p_0} = \left|\frac{1}{p} - \frac{1}{2}\right|, \quad s > \frac{d}{\frac{1}{p} - \frac{1}{2}} + \frac{1}{2}.
\]

It should be noted that the quantities (1.4) and (1.5) cannot be directly compared. When $p \leq 2$, however, comparing the ranges of $s$ in (1.4) and (1.5), one can check a gain of regularity by $\frac{d+1}{2}$. Such gain of regularity allows us to consider non-Mikhlin type Fourier multipliers. For examples of non-Mikhlin type, we suggest singular Fourier multipliers in [19] or multipliers of limited decay in [17], which cannot be handled by the condition (1.4). We also present a convergence type result for those Fourier multipliers, which are Propositions 1.5 and 1.7.
On the other hand, for $p > 2d$ \eqref{1.3} is dominated by \eqref{1.2} with $q = 2$, $r = 2$, $\alpha > d/2$, so one can actually improve Theorem \ref{thm1.1} by interpolating $L^2$ result of this paper and $L^\infty \rightarrow \text{BMO}$ result of \cite{5}. We should remark that the interpolation yields
\begin{equation}
\|M_m\|_{L^p \rightarrow L^p} \lesssim \|m\|_{\Sigma^2(B_{p,q})}, \quad \text{for } s > d \left(\frac{1}{2} - \frac{1}{p} \right) + \frac{1}{p}, \quad \text{and } p \geq 2,
\end{equation}
where we adapt interpolation for vector-valued Hardy spaces and BMO in \cite{2}. Note that \eqref{1.6} yields a gain of regularity by $\frac{1}{2} - \frac{1}{p}$ compared to \eqref{1.5} whenever $p \geq 2$. Due to \eqref{1.6}, we remark that every results derived from Theorem \ref{thm1.1} can be improved when $p \geq 2$.

Now we introduce applications of Theorem \ref{thm1.1} whose proofs are given in Section 6. First of all, we define the notion of Fourier multipliers with \textit{slow decay}. A function $m$ is said to be a Fourier multiplier with slow decay if $m$ satisfies
\begin{equation}
|\partial^\gamma m(\xi)| \leq C|\xi|^{-\beta|\gamma|}, \quad \text{for some } \delta \in (0, 1).
\end{equation}
It can be directly checked that the slow decay condition \eqref{1.7} cannot be implied by the Mikhlin condition. We also give a non-trivial example of multipliers with slow decay. Let $m_\beta$ be a Mikhlin multiplier such that $|\partial^\gamma m_\beta(\xi)| \lesssim |\xi|^{-\beta|\gamma|}$ and $m_\beta$ vanishes near the origin. Then for $\alpha \in (0, 1)$ we define $m_{\alpha,\beta}$, $M_{\alpha,\beta}$ as following:
\begin{equation}
m_{\alpha,\beta}(\xi) := e^{i|\xi|^\alpha} m_\beta(\xi),
\end{equation}
\begin{equation}
M_{\alpha,\beta}(f)(x) := \sup_{t > 0} |T_{m_{\alpha,\beta}(t)}(f)(x)|.
\end{equation}
Then $m_{\alpha,\beta}$ is slowly decaying with $\delta = 1 - \alpha$ in \eqref{1.7}. In terms of Theorem \ref{thm1.3} one can see that $M_{\alpha,\beta}$ is bounded on $L^p(\mathbb{R}^d)$ for $\frac{1}{p} < \frac{\beta/\alpha}{d}$.

The proof of Corollary \ref{cor1.4} is given in Section 6. Here we present a brief reasoning for the range $\frac{1}{p} < \frac{\beta/\alpha}{d} + \frac{d-1}{2d}$. It is straight forward to see that $|\partial^\gamma m_{\alpha,\beta}(\xi)| \lesssim |\xi|^{-\beta+(1-\alpha)k}$ for $|\gamma| = k$. To apply Theorem \ref{thm1.1} we check $k = \beta + (1-\alpha)k$, so $k = \beta/\alpha$. Then we must have $k = \beta/\alpha > d(\frac{1}{2} - \frac{1}{2}) + \frac{1}{2}$, which gives $\frac{1}{p} < \frac{\beta/\alpha}{d} + \frac{d-1}{2d}$. In \cite{15}, one can find that for radial $m_{\alpha,\beta}$ the maximal operator $M_{\alpha,\beta}$ is bounded on $L^p(\mathbb{R}^d)$ for $p \in (1, \infty)$ when $\beta > \frac{da}{2}$ since the Fourier transform of $m_{\alpha,\beta}$ is in $L^1$ for those $\beta, \alpha$. In our case, for $M_{\alpha,\beta}$ which is not necessarily radial $M_{\alpha,\beta}$ is bounded on $L^p(\mathbb{R}^d)$ for $p \in (1, \infty)$ when $\beta > \frac{d\alpha}{2} + \frac{d}{2}$. One can understand the difference $\frac{d}{2}$ is a radial counter part.

As an application of Corollary \ref{cor1.4}, we can study $e^{-it(-\Delta)^{\alpha/2}}$, which is a solution of a half wave equation or fractional Schrödinger equations. Particularly, for $\alpha = 1$ one can relate $e^{-it(-\Delta)^{1/2}}$ to a Fourier intergral operator of degree 1, which is a central object in harmonic analysis. In studies on dispersive equations, the Fourier integral operators, and the related topics, $e^{-it(-\Delta)^{1/2}}(f)$ is considered as an extension operator $\mathcal{F}f(x, t)$, and $(L^2 \rightarrow L^q L^r_x)$ estimates of $\mathcal{F}f$ is one of main goals of studying Fourier integral operators and dispersive
equations. In this paper, we do not discuss the mixed norm estimates for $e^{-it(-\Delta)^{\frac{\alpha}{2}}}$, and for $\alpha = 1$ we recommend [11, 14] and references therein. Instead of the mixed norm estimates, we are interested in $L^p$ behavior of

$$U_{\alpha,\beta}(f)(x,t) = \frac{e^{-it(-\Delta)^{\frac{\alpha}{2}}} - I(f)(x)}{t^{\beta}}, \text{ as } t \to 0.$$  

**Proposition 1.5.** For $\alpha \in (0,1)$ and $\beta \in (1/2,1)$, $U_{\alpha,\beta}(f)(x,t) \to 0$ in $L^p(\mathbb{R}^d)$ as $t \to 0$, whenever $\frac{d-2\beta+1}{2d} < \frac{1}{p} < \frac{d+2\beta-1}{2d}$, and $f \in \dot{L}^p_{\alpha,\beta}(\mathbb{R}^d)$. In particular, under the same condition on $\alpha, \beta$ and $f$, we have for a.e. $x$

$$|e^{-it(-\Delta)^{\frac{\alpha}{2}}} f(x) - f(x)| = O(t^{\beta}), \quad t \to 0.$$  

Note that $f \in \dot{L}^p_{\alpha,\beta}$ is equivalent to $(1 \cdot |s \hat{f}|) \in L^p$.

Theorem 1.1 can be also applied to $\mathcal{M}_m$ when $m$ enjoys both Mikhlin’s condition and the limited decay. Let $a > 0$, $b > \frac{1}{2}$. We assume that $m_{a,b}$ is of class $L^2_{s, loc}$ for $s > \frac{d+1}{2}$ and satisfies that for given $p \in (1, \infty)$ and $j > 0$,

$$|m_{a,b}(2^{j} \cdot \hat{\psi}(\cdot))|_{L^p_s(\mathbb{R}^d)} \lesssim 2^{-j \min(a,b-s)}, \quad \frac{1}{p_0} = \left| \frac{1}{p} - \frac{1}{2} \right|, \quad s > d\left| \frac{1}{p} - \frac{1}{2} \right| + \frac{1}{2}.$$  

Under the condition (1.9) we have

**Corollary 1.6.** Let $m_{a,b}$ be a Fourier multiplier satisfying (1.9). Then $\mathcal{M}_{a,b}$ is bounded on $L^p(\mathbb{R}^d)$ for $\frac{d+1-2b}{2d} < \frac{1}{p} < \frac{d-1+2b}{2d}$.

The condition (1.9) can be understood in a two-fold manner. If $m$ satisfies

$$|m(2^j \cdot \hat{\psi}(\cdot))|_{L^p_s(\mathbb{R}^d)} \lesssim 2^{-ja}, \quad j > 0,$$

then $m$ can be regarded as a Mikhlin multiplier such that $|\partial^{\gamma} m(\xi)| \lesssim (1 + |\xi|)^{-a} |\xi|^{-|\gamma|}$ and it satisfies conditions of Theorem C which guarantees that $\mathcal{M}_m$ to be bounded on $L^p$ for certain $p$. On the other hand, if we have

$$|m(2^j \cdot \hat{\psi}(\cdot))|_{L^p_s(\mathbb{R}^d)} \lesssim 2^{-j(b-s)}, \quad j > 0,$$

then $m$ is a multiplier of the limited decay whose maximal estimates are introduced in [17]. It can be checked that (1.10) behaves worse than (1.11) for $s < b - a$ and (1.11) enjoys worse decay than that of (1.10) for $s > b - a$. Thus, (1.9) considers the worst case when the cases (1.10) and (1.11) are combined.

Using Corollary 1.6, we obtain a pointwise convergence type result. For $m \in C^{[\frac{d+1}{2d}]_+ - 1}_{loc}(\mathbb{R}^d)$ such that $|\partial^{\gamma} m(\xi)| \lesssim (1 + |\xi|)^{-a}$ for any multi-index $\gamma$, and $m(0) = 1$, we have

**Proposition 1.7.** For $\alpha \in (0,1)$ and $f \in \dot{L}^p_{\alpha}(\mathbb{R}^d)$ with $\frac{d+1-2(\alpha+\beta)}{2d} < \frac{1}{p} < \frac{d-1+2(\alpha+\beta)}{2d}$, we have

$$\lim_{t \to 0} \frac{f(x) - T_{m(t)} f(x)}{t^\alpha} = 0, \quad \text{for almost every } x \in \mathbb{R}^d.$$
A noteworthy example for $m$ in Proposition 1.2 is the Fourier transform of a surface-carried measure $\sigma$ on $S \subset \mathbb{R}^d$, where $S$ is a hypersurface of nonvanishing $d-1$ principal curvatures.

Notation. Let $A \lesssim B$ denote $A \leq CB$, where $C$ is independent of $A$ and $B$. $B(x, R)$ denotes a ball in $\mathbb{R}^d$ of radius $R$ centered at $x$.

2. Preliminaries

2.1. Function spaces. Choose $\phi \in \mathcal{S}(\mathbb{R}^d)$ such that $\hat{\phi} \equiv 1$ on $B(0,1)$ and $\hat{\phi} \equiv 0$ on $B(0,2)^c$. Define

$$\psi_j = \hat{\psi}(\cdot/2^j) = \hat{\phi}(\cdot/2^j) - \hat{\phi}(\cdot/2^{j-1}) \text{ for } j \in \mathbb{Z}. \tag{2.1}$$

Then we introduce function spaces appeared in our arguments. Let $\mathcal{B}$ be a Banach space continuously embedded in the space of distributions $\mathcal{D}'(\mathbb{R}^d)$. For $q \in (1, \infty)$ and $\theta \in \mathbb{R}$, we define normed spaces

$$\Sigma^q_\theta(\mathcal{B}) := \{ f \in \mathcal{D}'(\mathbb{R}^d \setminus \{0\}) : \|f\|_{\Sigma^q_\theta(\mathcal{B})} := \sum_{j \in \mathbb{Z}} 2^{\theta j} \|f(2^j \cdot)\hat{\psi}(\cdot)\|_B < \infty \},$$

$$l^q_\theta(\mathcal{B}) := \{ \{f_j\}_{j \in \mathbb{Z}} : f_j \in \mathcal{B}, \text{ and } \sum_{j \in \mathbb{Z}} 2^{\theta j} \|f_j\|_B < \infty \}. \tag{2.2}$$

Note that $\Sigma^q_\theta(\mathcal{B})$ is a Banach space where we additionally assume that

$$\|\hat{\phi} f\|_B \lesssim \|f\|_B \text{ and } \|f(2 \cdot)\|_B \simeq \|f\|_B. \tag{2.3}$$

Let $S$ and $R$ be mappings given by

$$S(f) = \{ f(2^j \cdot)\hat{\psi}(\cdot) \}_{j \in \mathbb{Z}} = \{ f_j \}_{j \in \mathbb{Z}},$$

$$R(\{f_j\}) = \sum_{j} (\hat{\psi}_{j-1} + \hat{\psi}_j + \hat{\psi}_{j+1})(\cdot) f_j(2^{-j} \cdot).$$

Then by (2.2) $S \in L(\Sigma^q_\theta(\mathcal{B}), l^q_\theta(\mathcal{B}))$ and $R \in L(l^q_\theta(\mathcal{B}), \Sigma^q_\theta(\mathcal{B}))$, respectively, where $L(\mathcal{A}, \mathcal{B})$ denotes a space of bounded linear maps from $\mathcal{A}$ to $\mathcal{B}$. Moreover, one can check that $R \circ S$ is an identity map of $Id_{\Sigma^q_\theta(\mathcal{B})}$. Hence by [22] Theorem 1.2.4 and 1.18.1 $\Sigma^q_\theta(\mathcal{B})$ is isomorphic to a closed subspace of $l^q_\theta(\mathcal{B})$ which is a reflexive Banach space when $\mathcal{B}$ is reflexive. If $\theta = 0$, then we simply write $\Sigma^q(\mathcal{B})$.

Let $(\mathcal{A}, \mathcal{B})$ be an interpolation couple of Banach spaces, $q = (1-\delta) q_0 + \delta q_1$, and $\theta = (1-\delta) \theta_0 + \delta \theta_1$. Then we have from [22] Theorem 1.2.4 and 1.18.1

$$[\Sigma^q_{\theta_0}(\mathcal{A}), \Sigma^q_{\theta_1}(\mathcal{B})[right]_\delta = \Sigma^q_{\theta}(\mathcal{A}, \mathcal{B})[right]_\delta. \tag{2.3}$$

For instance, if we choose $\mathcal{A} = L^2_{s_1}, \mathcal{B} = L^2_{s_2}, s_1, s_2 \geq 0$ with $\theta = 0, q = 2$, then we have by (2.3)

$$[\Sigma^2(L^2_{s_1}), \Sigma^2(L^2_{s_2})[right]_\delta = \Sigma^2(L^2_{s})[right], s = (1-\delta)s_1 + \delta s_2, \delta \in (0,1). \tag{2.4}$$
Another example for $\Sigma^q(B)$ is the weighted Sobolev space $H^{p,\gamma}_0$ introduced in \cite{12}, which equals $\Sigma^p_{\theta/p}(L^p_N)$. When $\gamma \in \mathbb{N} \cup \{0\}$ we have

\begin{equation}
H^{p,\gamma}_0(\mathbb{R}^d \setminus \{0\}) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|^{p,\gamma}_{H^{p,\gamma}_0(\mathbb{R}^d \setminus \{0\})} := \sum_{l=0}^\gamma \int_{\mathbb{R}^d} |D^l_x(f)(x)|^p |x|^{d+\theta-d} dx < \infty \}.
\end{equation}

Indeed, one can check that by the change of variables,

\[
\sum_{l=0}^\gamma \int_{\mathbb{R}^d} |D^l_x(f)(x)|^p |x|^{d+\theta-d} dx \simeq \sum_{l=0}^\gamma \sum_{j \in \mathbb{Z}} 2^{j(l+\theta-d)} \int_{\mathbb{R}^d} |D^l_x(f)(x)|^p |\hat{\psi}_j(x)|^p dx
\]

\[
= \sum_{l=0}^\gamma \sum_{j \in \mathbb{Z}} 2^{j\theta} \int_{\mathbb{R}^d} |D^l_x(f(2^j \cdot))(x)|^p |\hat{\psi}(x)|^p dx
\]

\[
\simeq \sum_{l=0}^\gamma \sum_{j \in \mathbb{Z}} 2^{j\theta} \int_{\mathbb{R}^d} |D^l_x(f(2^j \cdot)\hat{\psi}(\cdot))(x)|^p dx.
\]

Note that $|\cdot|^{-d}$ is just a scaling factor and for $\theta = 0$ it follows that $H^{p,\gamma}_0$-norm is scaling invariant. For more information of $H^{p,\gamma}_0$, we recommend \cite{12} and references therein.

\section{Fractional calculus.}

To study a maximal operator, classical square function argument makes use of the fundamental theorem of calculus with respect to $t > 0$. In doing so, one must take derivative with respect to $t$, which in turn yields loss of regularity of a symbol. Such loss may be optimized if we use a fractional analogue of the fundamental theorem.

We define the Riemann-Liouville integrals and fractional derivatives of order $\alpha \in (0,1)$. For all $t \in (0,\infty)$

\[
I^\alpha_{0+}f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,
\]

\[
D^\alpha_{0+}F(t) := \frac{d}{dt}(I^{1-\alpha}_{0+}F)(t)
\]

where $f$ is a locally integrable function on $[0, \infty)$ and $F$ satisfies $I^{1-\alpha}_{0+}F$ is absolutely continuous. For further use of $I^\alpha_{0+}$ and $D^\alpha_{0+}$, we introduce the following lemma which is crucial in the proof of Theorem \ref{thm:1.1}.

\textbf{Lemma 2.1.} Let $F \in C_{loc}([0,\infty)) \cap C_{loc}^{0,\beta}((0,\infty))$ for a fixed $\beta \in (0,1]$. Then for any $\alpha \in (0,\beta)$, $D^\alpha_{0+}F$ satisfies the following identity:

\begin{equation}
D^\alpha_{0+}F(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{F(t)}{t^\alpha} + \int_0^t \frac{F(t) - F(s)}{(t-s)^{1+\alpha}} ds \right).
\end{equation}
Moreover, we have

\[ F(t) = I_0^\alpha D_0^{\alpha} F(t) + \frac{t^\alpha}{\Gamma(\alpha)} F(0). \]

**Proof.** From the definition of \( I^{1-\alpha} F(t) \), for \( h > 0 \) we have

\[
\frac{I^{1-\alpha} F(t+h) - I^{1-\alpha} F(t)}{h} = \frac{1}{h} \int_t^{t+h} \left( (t+h-s)^{-\alpha} (F(s) - F(t)) \right) ds
\]

\[ + \frac{1}{h} \int_0^t \left( (t+h-s)^{-\alpha} - (t-s)^{-\alpha} \right) (F(s) - F(t)) ds
\]

\[ + \frac{1}{h} F(t) \left( \int_0^{t+h} (t+h-s)^{-\alpha} - \int_0^t (t-s)^{-\alpha} ds \right)
\]

\[ =: I_1 + I_2 + I_3. \]

Define

\[ A = \sup_{0 \leq s \leq 2t} |F(s)| \text{ and } B = \sup_{t/2 \leq s \leq 2t} \frac{|F(t) - F(s)|}{|t-s|^{\beta}}. \]

For \( 0 < h < t/2 \), by change of variables, we have

\[ |I_1| \leq \int_0^1 (hr)^{-\alpha} |F(t + h(1 - r)) - F(t)| ds \leq h^{\alpha-\alpha} B \int_0^1 r^{-\alpha} (1 - r)^{\beta} dr. \]

Therefore, \( I_1 \to 0 \) as \( h \to 0 \). For \( I_2 \), we make use of the fundamental theorem of calculus to obtain

\[ I_2 = -\alpha \int_0^t \int_0^1 (t-s+hr)^{-1-\alpha} dr (F(s) - F(t)) ds. \]

Note that

\[ \int_0^t \int_0^1 (t-s+hr)^{-1-\alpha} |F(s) - F(t)| dr ds \]

\[ \leq \int_0^t (t-s)^{-1-\alpha} |F(s) - F(t)| ds \]

\[ \lesssim At^{-\alpha} + Bt^{\beta-\alpha} < \infty. \]

Thus by the dominated convergence theorem, it follows that

\[ I_2 \to -\alpha \int_0^t (t-s)^{-1-\alpha} (F(s) - F(t)) ds \quad \text{as} \quad h \to 0. \]

For \( I_3 \), by direct calculation we obtain

\[ I_3 = \frac{F(t)}{1-\alpha} \times \frac{(t+h)^{1-\alpha} - t^{1-\alpha}}{h} \to \frac{F(t)}{t^\alpha}. \]
For \( \frac{I^{1-\alpha}F(t) - I^{1-\alpha}F(t-h)}{h} \) and \( h > 0 \), we have
\[
\frac{I^{1-\alpha}F(t) - I^{1-\alpha}F(t-h)}{h} = \frac{1}{h} \int_{t-h}^{t} ((t-s)^{-\alpha})(F(s) - F(t-h))ds \\
+ \frac{1}{h} \int_{0}^{t-h} ((t-s)^{-\alpha} - (t-h-s)^{-\alpha})(F(s) - F(t-h))ds \\
+ \frac{1}{h} F(t-h) \left( \int_{0}^{t} (t-s)^{-\alpha} - \int_{0}^{t-h} (t-h-s)^{-\alpha}ds \right).
\]
Then by the same argument it follows that for \( h > 0 \)
\[
\frac{I^{1-\alpha}F(t) - I^{1-\alpha}F(t-h)}{h} \to \frac{F(t)}{t^{\alpha}} + \int_{0}^{t} \frac{F(t) - F(s)}{(t-s)^{1+\alpha}}ds \quad \text{as} \quad h \to 0.
\]
Therefore (2.6) is proved.
From the assumption of \( F \) we obtain \( D_{0+}^{\alpha}F(t) \in L^{1}_{\text{loc}}([0,\infty)) \), and the equality (2.7) holds by [18, Theorem 2.4]. The lemma is proved.

3. Proof of Theorem 1.1

Let \( \varepsilon \in (0, \frac{1}{6}) \) be sufficiently small and \( s > d(\frac{1}{p} - \frac{1}{2}) + \frac{1}{2} + 3\varepsilon \). Since \( \Sigma^{2}(B_{r_{0}}^{s}) \subset \Sigma^{2}(C_{\text{loc}}^{0, \frac{1}{2} + 3\varepsilon}) \), we consider \( m \) as an element of \( C_{\text{loc}}^{0, \frac{1}{2} + 3\varepsilon}(\mathbb{R}^{d} \setminus \{0\}) \) and
\[
\|m(2^{j} \cdot \hat{\psi})\|_{C^{0, 1/2 + 3\varepsilon}} \to 0 \quad \text{as} \quad j \to -\infty.
\]
This implies that \( m(\xi) \to 0 \) as \( \xi \to 0 \). Thus we want to show that for such \( m \),
\[
\|\mathcal{M}_{m}(f)\|_{L^{p}(\mathbb{R}^{d})} \lesssim \|m\|_{\Sigma^{2}(B_{r_{0}}^{s})} \|f\|_{L^{p}(\mathbb{R}^{d})}.
\]
To do this we introduce a square function type estimate using fractional calculus. From Lemma 2.1, we obtain that
\[
m(t\xi) = \frac{1}{\Gamma(\frac{1}{2} - \varepsilon)} I^{\frac{1}{2} + \varepsilon}(\cdot)^{-\frac{1}{2} - \varepsilon} \tilde{m}(\cdot)(t),
\]
where
\[
\tilde{m}(t\xi) := m(t\xi) + (1/2 + \varepsilon) \int_{0}^{1} \frac{m(t\xi) - m(st\xi)}{(1-s)^{\frac{1}{2} + \varepsilon}}ds.
\]
Note that for fixed \( \xi \in \mathbb{R}^{d} \) with \( |\xi| \in (2^{j-1}, 2^{j+1}) \) for some \( j \in \mathbb{Z} \), we have
\[
|\tilde{m}(\xi)| \lesssim \|m\|_{L^{\infty}} + \left| \int_{0}^{1} \frac{m(\xi) - m(s\xi)}{(1-s)^{\frac{1}{2} + \varepsilon}}ds \right| \lesssim \|m\|_{L^{\infty}} + \int_{1/2}^{1} \frac{m(\xi) - m(s\xi)}{(1-s)^{\frac{1}{2} + \varepsilon}}ds.
\]
Note that \( \|m\|_{\infty} \lesssim \|m\|_{\Sigma(L^\infty)} \lesssim \|m\|_{\Sigma(C^{0,1/2+3\varepsilon})}. \) We also have
\[
\left| \int_{1/2}^t \frac{m(\xi) - m(s\xi)}{(1-s)^{\frac{1}{2}+\varepsilon}} ds \right| \lesssim |\xi|^{\frac{1}{2}+3\varepsilon} \sup_{1/2 < s < 1} \left| \frac{m(\xi) - m(s\xi)}{|\xi - s\xi|^{\frac{1}{2}+3\varepsilon}} \right| \\
\lesssim 2^{j(\frac{1}{2}+3\varepsilon)} \sum_{k=j-1}^{j+1} \left[ m\hat{m}_k \right]_{C^{0,1/2+3\varepsilon}} \\
\lesssim \sum_{k=j-1}^{j+1} \|m(2^k\cdot)^\wedge(\cdot)\|_{C^{0,1/2+3\varepsilon}} \lesssim \|m\|_{\Sigma^2(C^{0,1/2+3\varepsilon})} < \infty.
\]

Since we choose arbitrary \( \xi \in \mathbb{R}^d, \) \( \tilde{m} \) is bounded. By the boundedness of \( \tilde{m}, \) we make use of Fubini’s theorem to obtain

\[
T_{m(t)} f = \frac{1}{\Gamma(\frac{1}{2} - \varepsilon)} I^{1+\varepsilon} (t^{-\frac{1}{2} - \varepsilon} T_{\tilde{m}(t)} f).
\]

Thus it follows that
\[
|T_{m(t)} f|^2 = \left| \frac{1}{\Gamma(\frac{1}{2} - \varepsilon)} I^{1+\varepsilon} (t^{-\frac{1}{2} - \varepsilon} T_{\tilde{m}(t)} f) \right|^2 \\
= \left| \frac{1}{\Gamma(\frac{1}{2} - \varepsilon)} \int_0^t (t-s)^{-\frac{1}{2}+\varepsilon} s^{-\frac{1}{2}-\varepsilon} T_{\tilde{m}(s)} f ds \right|^2 \\
\lesssim \int_0^t (t-s)^{-1+2\varepsilon} s^{-2\varepsilon} ds \times \int_0^t |T_{\tilde{m}(s)} f|^2 \frac{ds}{s}.
\]

By taking supremum over \( t > 0 \) on (3.1), we obtain
\[
|\mathcal{M}_m f|^2 \lesssim \int_0^\infty |T_{\tilde{m}(t)} f|^2 \frac{dt}{t} =: G_{\tilde{m}}(f)^2.
\]

Thanks to (3.2), we obtain \( L^2(\mathbb{R}) \)-valued operator instead of supremum over \( t > 0 \) in price of \( \tilde{m}. \) Since we desire a condition of \( m \) for the \( L^p \) boundedness, the following mapping property of \( m \mapsto \tilde{m} \) is required:

**Lemma 3.1.** For \( \beta \geq 0 \)
\[
\sup_{\xi \neq 0} \left( \int_0^\infty |\tilde{m}(t\xi)|^2 \frac{dt}{t} \right)^{1/2} \lesssim \|m\|_{\Sigma^2(C^{0,1/2+3\varepsilon})},
\]
\[
\|\tilde{m}\|_{\Sigma^2(L^2(\mathbb{R}))} \lesssim \|m\|_{\Sigma^2(L^2_{\beta+1/2+3\varepsilon}(\mathbb{R}))}.
\]

Additionally, to perform bilinear interpolation, we need initial estimates for \( G_m \) in terms of the left hand sides of (3.3) and (3.4), which is Lemma 3.2.

**Lemma 3.2.** For the square function
\[
G_m(f) = \left( \int_0^\infty |T_{m(t)} f|^2 \frac{dt}{t} \right)^{1/2},
\]
we have
\[ \|G_m(f)\|_{L^2(\mathbb{R}^d)} \lesssim \sup_{\xi \neq 0} \left( \int_0^\infty |m(t\xi)|^{2} \frac{dt}{t} \right)^{1/2} \|f\|_{L^2(\mathbb{R}^d)}, \]
\[ \|G_m(f)\|_{L^1(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(L^2)} \|f\|_{H^1(\mathbb{R}^d)}, \]
\[ \|G_m(f)\|_{BMO(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(L^2)} \|f\|_{L^\infty(\mathbb{R}^d)}. \]

For further reasoning, we temporarily assume Lemmas 3.1 and 3.2. We present their proofs in Sections 4 and 5. Then by Lemmas 3.1 3.2 we conclude that
\[ \text{(3.5)} \quad \|G_{\tilde{m}}(f)\|_{L^2(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(C^{0,1/2+3\varepsilon})} \|f\|_{L^2(\mathbb{R}^d)}, \]
\[ \text{(3.6)} \quad \|G_{\tilde{m}}(f)\|_{L^1(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(L^2_{(d+1)/2+3\varepsilon})} \|f\|_{H^1(\mathbb{R}^d)}. \]

We interpolate (3.5), (3.6) in terms of Lemma 5.2 to obtain
\[ \|G_{\tilde{m}}(f)\|_{L^p(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(B_{p_0}^s)} \|f\|_{L^p(\mathbb{R}^d)}, \]
for \( p_0 = |1/p - 1/2|^{-1} \) and \( s \geq d(1/p - 1/2) + 1/2 + 3\varepsilon \). We present the details for the interpolation in subsection 5.2 and note that the interpolation is valid since we consider \( G_{\tilde{m}} \) as an \( L^2(\frac{dt}{t}) \)-valued bilinear operator \( B \) in the sense that
\[ B(m,f)(x,t) := T_{\tilde{m}(t\cdot)} f(x) \quad \text{and} \quad \|B(m,f)\|_{L^2(\frac{dt}{t})} = G_{\tilde{m}}(f). \]

Then by (3.2), we have
\[ \|M_m(f)\|_{L^p(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(B_{p_0}^s)} \|f\|_{L^p(\mathbb{R}^d)}. \]
This proves the theorem.

4. Proof of Lemma 3.1
4.1. Proof of (3.3). We first show that \( \sup_{t \neq 0} \left( \int_0^\infty |m(t\xi)|^{2} \frac{dt}{t} \right)^{1/2} \lesssim \|m\|_{\Sigma^2(C^{0,1/2+3\varepsilon})}. \)
Recall that
\[ \tilde{m}(\xi) = m(\xi) + (1/2 + \varepsilon) \int_0^1 (1 - s)^{-\frac{3}{2} - \varepsilon} (m(\xi) - m(s\xi)) ds, \]
where \( m \) vanishes near the origin. Then for \( m \) we have
\[ \sup_{t \neq 0} \int_0^\infty |m(t\xi)|^{2} \frac{dt}{t} \leq \sum_{j \in \mathbb{Z}} \sup_{\xi \neq 0} \int_0^{2^{j+1}/|\xi|} |m(t\xi)|^{2} \frac{dt}{t} \]
\[ \lesssim \sum_{j \in \mathbb{Z}} \|m(2^j \cdot)\tilde{\psi}\|_\infty^2 \leq \sum_{j \in \mathbb{Z}} \|m(2^j \cdot)\tilde{\psi}\|_{C^{0,1/2+3\varepsilon}(\mathbb{R}^d)}^2, \]
Thus, without loss of generality we set
\[ \text{(4.1)} \quad \tilde{m}(\xi) = \int_0^1 (1 - s)^{-\frac{3}{2} - \varepsilon} (m(\xi) - m(s\xi)) ds. \]
Due to
\[
|\tilde{m}(\xi)| \lesssim \int_0^{1/2} |m(\xi)| + |m(s\xi)| ds + \int_{1/2}^1 (1-s)^{-\frac{3}{2} - \varepsilon}|m(\xi) - m(s\xi)| ds,
\]
we have
\[
(4.2) \int_0^\infty |\tilde{m}(t\xi)|^2 \frac{dt}{t} \lesssim \int_0^\infty \left( |m(t\xi)|^2 + I(t\xi)^2 \right) \frac{dt}{t} = \sum_{j \in \mathbb{Z}} \int_0^{2^{|j|} - 1} \left( |m(2^j t\xi)|^2 + I(2^j t\xi)^2 \right) \frac{dt}{t}
\]
where
\[
I(\xi) = \int_0^{1/2} (1-s)^{-\frac{3}{2} - \varepsilon}|m(\xi) - m(s\xi)| ds.
\]

Put \(\hat{\eta} = \hat{\psi}_{-1} + \hat{\psi}_0 + \hat{\psi}_1\) which equals to 1 on \(\{\xi : 1/2 \leq |\xi| \leq 2\}\). For \(|\xi|^{-1} \leq t \leq 2|\xi|^{-1}\) and \(1/2 \leq s \leq 1\), we obtain
\[
\frac{1}{2} \leq |st\xi| \leq |t\xi| \leq 2.
\]
Therefore we have
\[
|m(2^j t\xi)| \leq \|m(2^j \cdot \hat{\eta})\|_{C^0,1/2+3\varepsilon}, \quad \frac{|m(2^j t\xi) - m(2^j s t\xi)|}{((1-s)t|\xi|)^{1/2+3\varepsilon}} \leq [m(2^j \cdot \hat{\eta})]_{C^0,1/2+3\varepsilon},
\]
which implies
\[
|m(2^j t\xi)| + I(2^j t\xi) = |m(2^j t\xi)| + (t|\xi|)^{1/2+3\varepsilon} \int_0^{1/2} (1-s)^{-1+2\varepsilon} \frac{|m(2^j t\xi) - m(2^j s t\xi)|}{((1-s)t|\xi|)^{1/2+3\varepsilon}} ds \lesssim \|m(2^j \cdot \hat{\eta})\|_{C^0,1/2+3\varepsilon}.
\]
By (4.2), (4.3), and that \(C^{0,1/2+3\varepsilon}\) satisfying the property (2.2), we have
\[
\int_0^\infty |\tilde{m}(t\xi)|^2 \frac{dt}{t} \lesssim \sum_j \|m(2^j \cdot \hat{\eta})\|_{C^0,1/2+3\varepsilon}^2 \lesssim \sum_j \|m(2^j \cdot \hat{\psi})\|_{C^0,1/2+3\varepsilon}^2.
\]

4.2. **Proof of (3.4).** Recall that for \(H_0^{p,\gamma}\) in (2.5) we have \(\Sigma^2(L^2_\beta) = H_0^{2,\beta}\). Since \(C^\infty_c(\mathbb{R}^d)\) is dense in every \(H_0^{p,\gamma}\) due to [12, Proposition 2.2.2], it suffices to show that the mapping \(m \mapsto \tilde{m}\) initially defined for \(m \in C^\infty_c(\mathbb{R}^d)\) can be bounded from \(\Sigma^2(L^2_{\beta+1/2+2\varepsilon})\) to \(\Sigma^2(L^2_\beta)\). Since \(\|m\|_{\Sigma^2(L^2_\beta)} \lesssim \|m\|_{\Sigma^2(L^2_{\beta+1/2+2\varepsilon})}\) we consider \(\tilde{m}\) as in (4.1). We actually prove the mapping property of \(m \mapsto \tilde{m}\) for \(\beta = 0\), which yields the case of \(\beta \in \mathbb{N}\). For real \(\beta\), we interpolate the results of \(\beta \in \mathbb{N}\) cases in terms of (2.4). We make use of the following
equivalent quantities to \( \|m\|_{\Sigma^2(L^2)} \), \( \|m\|_{\Sigma^2(L^2_{n+\alpha})} \) for \( n \in \mathbb{N} \cup \{0\} \) and \( \alpha \in (0, 1) \).

\begin{align*}
(4.4) \quad & \|m\|_{\Sigma^2(L^2_n)}^2 \lesssim \sum_{|\gamma| \leq n} \int_{\mathbb{R}^d} |\xi|^{|\gamma|} |\partial^\gamma m(\xi)|^2 \frac{d\xi}{|\xi|^d}, \\
(4.5) \quad & \|m\|_{\Sigma^2(L^2_{n+\alpha})}^2 \approx \|m\|_{\Sigma^2(L^2_n)} + \|D^\alpha m\|_{\Sigma^2(L^2_n)}, \\
(4.6) \quad & \|m\|_{\Sigma^2(L^2_n)} \approx \int_{\mathbb{R}^d} |m(\xi)|^2 |\xi|^{2n-d} d\xi + \int_{\mathbb{R}^d} \left( \int_{|y-\xi| \leq \frac{|\xi|}{2}} \frac{|m(y) - m(\xi)|^2}{|y-\xi|^{d+2\alpha}} dy \right) \frac{d\xi}{|\xi|^{d-2\alpha-2n}}.
\end{align*}

(4.4) and (4.5) are given in Lemma 4.1 and (4.6) is a special case of Proposition 4.2 in Subsection 4.4. Since \( m \in \Sigma^2(L^2_{n+\alpha}) \subset \Sigma^2(L^2_n) \), without loss of any generality we put

\[ \tilde{m}(\xi) = \int_0^1 (1-s)^{-\frac{d}{2} - \varepsilon} (m(\xi) - m(s\xi)) ds \]

and let

\[ \int_0^1 (\cdots) = \int_0^{1-\varepsilon} (\cdots) + \int_{1-\varepsilon}^1 (\cdots) = I + II \]

for some \( \varepsilon > 0 \) chosen later. Then we consider \( I \) and \( II \) separately. For \( I \), we have

\[ \|I\|_{\Sigma^2(L^2)} \leq \int_0^{1-\varepsilon} (1-s)^{-\frac{d}{2} - \varepsilon} \left( \|m\|_{\Sigma^2(L^2_n)} + \|m(s)\|_{\Sigma^2(L^2_n)} \right) ds. \]

Since the right hand side of (4.4) is scaling invariant, it follows that

\[ \|I\|_{\Sigma^2(L^2_n)} \leq 2\varepsilon^{-\frac{d}{2}} \|m\|_{\Sigma^2(L^2_n)}. \]

Thus it remains to handle \( II \)-term. For \( II \), we claim that if \( s \in (1-\varepsilon, 1) \), then

\[ \|m(s) - m(s)\|_{\Sigma^2(L^2_n)} \lesssim (1-s)^{1/2 + 2\varepsilon} \|m\|_{\Sigma^2(L^2_{n+1/2+2\varepsilon})}, \]

which is given in the next subsection. By (4.7), we can show that

\[ \|II\|_{\Sigma^2(L^2_n)} \lesssim \delta^{1-\varepsilon} \|m\|_{\Sigma^2(L^2_{n+1/2+2\varepsilon})}. \]

Thus we have shown that

\[ (4.8) \quad \|\tilde{m}\|_{\Sigma^2(L^2_n)} \lesssim \|m\|_{\Sigma^2(L^2_{n+1/2+2\varepsilon})}. \]

The inequality (4.8) means that \( m \in \Sigma^2(L^2_{1/2+2\varepsilon}) \) implies \( \tilde{m} \in \Sigma^2(L^2) \). We want to generalize this statement to the case of \( m \in \Sigma^2(L^2_{n+1/2+2\varepsilon}) \) and \( \tilde{m} \in \Sigma^2(L^2_n) \).

To do so, note that by (4.4) and (4.5) we have

\[ \|m\|_{\Sigma^2(L^2_{n+1/2+2\varepsilon})} \approx \sum_{|\gamma| \leq n} \|\xi|^{\gamma} \partial^\gamma m(\xi)\|_{\Sigma^2(L^2_{1/2+2\varepsilon})}, \]

\[ \|\tilde{m}\|_{\Sigma^2(L^2_n)} \approx \sum_{|\gamma| \leq n} \|\xi|^{\gamma} \partial^\gamma \tilde{m}(\xi)\|_{\Sigma^2(L^2_n)}. \]
Also, one can observe that \( (|\xi|^\gamma \partial^\gamma \tilde{m}(\xi)) \sim |\xi|^\gamma \partial^\gamma \tilde{m}(\xi) \). Then from (4.8) we have
\[
\| \tilde{m} \|_{L^2(S_\gamma^{n+1/2+\varepsilon})} \leq \sum_{|\gamma| \leq n} \left\| |\xi|^\gamma \partial^\gamma \tilde{m}(\xi) \right\|_{L^2(S_\gamma^n)} \approx \sum_{|\gamma| \leq n} \left\| |\xi|^\gamma \partial^\gamma \tilde{m}(\xi) \right\|_{L^2(S_\gamma^n)}.
\]
Thus we have shown that \( m \in L^2(S_\gamma^{n+1/2+\varepsilon}) \) implies \( \tilde{m} \in L^2(S_\gamma^n) \) for \( n = 0, 1, 2, \cdots \).

### 4.3. Proof of the claim (4.7)

Recall that \( \| m(\cdot) - m(s\cdot) \|_{L^2(S_\gamma^n)}^2 \) is
\[
\int_{\mathbb{R}^d} |m(\xi) - m(s\xi)|^2 \frac{d\xi}{|\xi|^d}.
\]

Put \( \delta = \frac{1}{8} \). For a fixed \( \xi \in \mathbb{R}^d \setminus \{0\} \), put
\[
B_{1,\xi} = \{ y \in \mathbb{R}^d : |y - \xi| < \delta(1-s)|\xi| \},
B_{2,\xi} = \{ z \in \mathbb{R}^d : |z - s\xi| < \delta(1-s)|s\xi| \},
\]
and make use of the triangle inequality
\[
|m(\xi) - m(s\xi)| \leq |m(\xi) - m(y)| + |m(y) - m(z)| + |m(z) - m(s\xi)|
\]
to obtain
\[
\| m(\cdot) - m(s\cdot) \|_{L^2(S_\gamma^n)}^2 \leq \int_{\mathbb{R}^d} \int_{B_{2,\xi}} \int_{B_{1,\xi}} \left( |m(\xi) - m(y)|^2 + |m(y) - m(z)|^2 + |m(z) - m(s\xi)|^2 \right) \frac{dy}{|B_{1,\xi}|} \frac{dz}{|B_{2,\xi}|} \frac{d\xi}{|\xi|^d} =: II_1 + II_2 + II_3.
\]

We deal with \( II_1, II_2, \) and \( II_3, \) separately.
For $II_1$, since $y \in B_{1,\xi}$ satisfies $|y - \xi| < \delta(1 - s)|\xi|$, we have

$$II_1 = \int_{\mathbb{R}^d} \int_{B_{1,\xi}} |m(\xi) - m(y)|^2 \frac{dy}{|B_{1,\xi}|} \frac{d\xi}{|\xi|^d}$$

$$\simeq \int_{\mathbb{R}^d} \int_{B_{1,\xi}} (\delta(1 - s)|\xi|)^{d} \frac{dy}{|\xi|^d}$$

$$= \delta^{-d} \int_{\mathbb{R}^d} \left( \int_{B_{1,\xi}} \frac{|m(\xi) - m(y)|^2}{(1 - s)|\xi|^d} dy \right) ((1 - s)|\xi|)^{1+4\varepsilon} \frac{d\xi}{|\xi|^d}$$

$$\leq \delta^{1+4\varepsilon}(1 - s)^{1+4\varepsilon} \int_{\mathbb{R}^d} \int_{|y - \xi| < \delta|\xi|} \frac{|m(\xi) - m(y)|^2}{\xi - y} dy \frac{d\xi}{|\xi|^{d-1-4\varepsilon}}.$$  

Since $\delta < 1/2$, it follows that

$$II_1 \lesssim (1 - s)^{1+4\varepsilon} \|m\|^2_{L^2(\mathbb{R}^d \times 2 \mathbb{R}^d)}.$$  

For $II_3$, we perform the argument of the case $II_1$ to obtain

$$II_3 = \int_{\mathbb{R}^d} \int_{B_{2,\xi}} |m(z) - m(s\xi)|^2 \frac{dz}{|B_{2,\xi}|} \frac{d\xi}{|\xi|^d}$$

$$\simeq \int_{\mathbb{R}^d} \int_{B_{2,\xi}} (\delta(1 - s)|s\xi|)^{d} \frac{dz}{|\xi|^d}$$

$$\leq \delta^{1+4\varepsilon}(1 - s)^{1+4\varepsilon} \int_{\mathbb{R}^d} \int_{|z - \xi| < \delta|\xi|} \frac{|m(z) - m(\xi)|^2}{z - \xi} dz \frac{d\xi}{|\xi|^{d-1-4\varepsilon}}.$$  

Again, since $\delta < 1/2$ we have

$$II_3 \lesssim (1 - s)^{1+4\varepsilon} \|m\|^2_{L^2(\mathbb{R}^d \times 2 \mathbb{R}^d)}.$$  

Lastly, we consider the term $II_2$. In this case, we need to handle both $y$- and $z$-integrals. To do this, define a set $E$ for $s \in (1 - \delta, 1)$.

$$E := \{(y, z, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d : |y - \xi| \leq \delta(1 - s)|\xi|, |z - s\xi| \leq \delta(1 - s)|s\xi|\}.$$  

Observe that on $E$ we have $|y - \xi| < \delta(1 - s)|\xi| \leq \delta^2|\xi|$, which yields

$$|y| < 2|\xi|, \quad |\xi| < 2|y|.$$  

Moreover,

$$|z - y| \leq |z - s\xi| + (1 - s)|\xi| + |\xi - y| \leq 2(1 - s)|\xi| \leq 2\delta|\xi| \leq 4\delta|y|, \quad s > 1 - \delta.$$
Now we can estimate $II_2$.

$$II_2 = \int_{\mathbb{R}^d} \int_{B_{2,\xi}} \int_{B_{1,\xi}} |m(y) - m(z)|^2 \frac{dy \, dz \, d\xi}{|B_{1,\xi}| \cdot |B_{2,\xi}| \cdot |\xi|^d}$$

$$\simeq \int \int \int_{E \setminus \{0\}} \frac{|m(y) - m(z)|^2}{(1 - s)^{2d} s^d |\xi|^d} \, dy \, dz \, d\xi$$

$$\lesssim (1 - s)^{-d + 1 + 4\epsilon} \int \int \int_{E \setminus \{0\}} \frac{|m(y) - m(z)|^2}{|y - z|^{d + 1 + 4\epsilon} |y|^{2d - 1 - 4\epsilon}} \, dy \, dz \, d\xi,$$

where we use $|\xi| \simeq |y|$, $|y - z| < 2(1 - s)|\xi|$, and $s > 1 - \delta$ in the last inequality. Since $\delta = \frac{1}{8}$, if we apply the change of variables in terms of $E$, then $II_2$ is dominated by $\delta^{-2d}(1 - \delta)^{-d}$ times of the following quantity:

$$(1 - s)^{-d + 1 + 4\epsilon} \int \int \int_{y \in \mathbb{R}^d \setminus \{0\}} \frac{|m(y) - m(z)|^2}{|y - z|^{d + 1 + 4\epsilon} |y|^{2d - 1 - 4\epsilon}} \, dy \, dz \, d\xi$$

$$\lesssim (1 - s)^{-d + 1 + 4\epsilon} \int \int \int_{y \in \mathbb{R}^d \setminus \{0\}} \frac{|m(y) - m(z)|^2}{|y - z|^{d + 1 + 4\epsilon} |y|^{2d - 1 - 4\epsilon}} \, dy \, dz \, d\xi$$

$$= \delta^d \int \int \int_{y \in \mathbb{R}^d \setminus \{0\}} \frac{|m(y) - m(z)|^2}{|y - z|^{d + 1 + 4\epsilon} |y|^{d - 1 - 4\epsilon}} \, dy \, dz \, d\xi.$$

Since $4\delta = 1/2$ and Proposition 4.2, we have

$$(4.11) \quad II_2 \lesssim (1 - s)^{1 + 4\epsilon} \|m\|^2_{\Sigma^2(L_{1/2 + 2\epsilon})}.$$

Together with $(4.9)$, $(4.10)$, and $(4.11)$, we prove

$$\|m(\cdot) - m(s\cdot)\|_{\Sigma^2(L^2)} \lesssim (1 - s)^{\frac{1}{2} + 2\epsilon} \|m\|_{\Sigma^2(L_{1/2 + 2\epsilon})},$$

which is $(4.7)$. This proves the lemma.

### 4.4. Equivalent norm for $\Sigma^2(L^2_n)$. We show the equivalence of $(4.5)$. Recall that $\Sigma^2_\theta(B)$ is equipped with the norm given by

$$\|f\|_{\Sigma^2_\theta(B)} := \left( \sum_{j \in \mathbb{Z}} 2^{j\theta} \|f(2^j \cdot)\tilde{\psi}(\cdot)\|_{B}^2 \right)^{1/2}.$$

For $B = L^2_n$ or $L^2_{n+\alpha}$ with $\alpha \in (0, 1)$, the following equivalences are known:

**Lemma 4.1.** [12] Proposition 2.2.3, Theorem 4.1 Let $n \in \mathbb{N}_0$ and $\alpha \in (0, 1)$.

1. $\|m\|_{\Sigma^2_\theta(L^2_n)}^2 \simeq \sum_{k=0}^n \int_\Omega |D^k m|^2 |x|^{2\theta + 2k - d} \, dx$

2. $\|m\|_{\Sigma^2_{\theta}(L^2_{n+\alpha})} \simeq \|m\|_{\Sigma^2_{\theta}(L^2_{n})} + \|D^\alpha m\|_{\Sigma^2_{\theta+\alpha}(L^2_{n})}$

Then by making use of Lemma 4.1, we can obtain the equivalence $(4.6)$.

**Proposition 4.2.** For any $\alpha \in (0, 1)$, $p \in (1, \infty)$ and $\theta \in \mathbb{R}$,

$$\|m\|_{\Sigma^2_{\theta}(L^p_{n})}^2 \simeq \int_\Omega |m|^2 |x|^{2\theta - d} \, dx + \int_\Omega \left( \int_{|x-x'| < \frac{1}{4\theta}} \frac{|m(x) - m(y)|^2}{|x - y|^{d + 2\alpha}} \, dy \right) |x|^{2\alpha + 2\theta - d} \, dx$$
Proof. From the representation of Besov norm on $\mathbb{R}^d$ in [22] p.189, 190],
\[
\|m\|_{L^2_a}^2 \simeq \|m\|_{L^2}^2 + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|m(x) - m(y)|^2}{|x-y|^{d+2\alpha}} dxdy,
\]
we obtain
\[
\|m\|_{L^2_a}^2 \simeq \sum_{k \in \mathbb{Z}} 2^{2k\theta}\|m(2^k \cdot)\hat{\psi}\|_2^2 + \sum_{k \in \mathbb{Z}} 2^{2k\theta} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|m(2^k x)\hat{\psi}(x) - m(2^k y)\hat{\psi}(y)|^2}{|x-y|^{d+2\alpha}} dxdy
\]
\[=: I_0 + I_1.\]

By Lemma 4.1 we have
\[(4.12)\quad I_0 \simeq \int_{\mathbb{R}^d} |m|^2 |x|^{2\beta - d} dx.
\]

For $I_1$ we perform the change of variables $2^k x \rightarrow x$, $2^k y \rightarrow y$ so that
\[I_1 := \sum_{k \in \mathbb{Z}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|m(x)\hat{\psi}(2^{-k} x) - m(y)\hat{\psi}(2^{-k} y)|^2}{|x-y|^{d+2\alpha}} 2^{k(2\theta - d+2\alpha)} dxdy
\]
\[= \sum_{k \in \mathbb{Z}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x|^{\theta-d/2+\alpha} m(x)\eta(2^{-k} x) - |y|^{\theta-d/2+\alpha} m(y)\eta(2^{-k} y)|^2}{|x-y|^{d+2\alpha}} dxdy
\]
\[\lesssim \sum_{k \in \mathbb{Z}} \iint_{|x-y| \geq \frac{|x|}{2}} \frac{|\eta(2^{-k} x)|^2 |x|^{\theta-d/2+2\alpha} |m(x)|^2}{|x-y|^{d+2\alpha}} + \frac{|\eta(2^{-k} y)|^2 |y|^{\theta-d/2+2\alpha} |m(y)|^2}{|x-y|^{d+2\alpha}} dxdy
\]
\[+ \sum_{k \in \mathbb{Z}} \iint_{|x-y| < \frac{|x|}{2}} \frac{|\eta(2^{-k} x) - \eta(2^{-k} y)|^2}{|x-y|^{d+2\alpha}} |x|^{2\theta-d+2\alpha} |m(x)|^2 dxdy
\]
\[+ \sum_{k \in \mathbb{Z}} \iint_{|x-y| < \frac{|x|}{2}} |\eta(2^{-k} y)|^2 \frac{|x|^{\theta-d/2+\alpha} m(x) - |y|^{\theta-d/2+\alpha} m(y)|^2}{|x-y|^{d+2\alpha}} dxdy
\]
\[=: I_{1,1} + I_{1,2} + I_{1,3}
\]

where $\eta(x) = |x|^{-\theta+d/2-\alpha} \hat{\psi}(x)$. Note that we apply $|x|^{\theta-d/2+\alpha} m(x)\eta(2^{-k} y)$ for the last inequality when $|x-y| < \frac{|x|}{2}$.

To progress further, observe that for any $a \in \mathbb{R}$
\[(4.13)\quad \sum_{k \in \mathbb{Z}} 2^{ak} |\eta(2^{-k} x)|^2 \simeq |x|^a,
\]
\[(4.14)\quad \sum_{k \in \mathbb{Z}} 2^{ak} |(\nabla \eta)(2^{-k} x)|^2 \lesssim |x|^a.
\]
For $\alpha > 0$ we have
\[
\int_{|x-y| \geq \frac{|x|}{2}} |x-y|^{-d-2\alpha} \, dy \simeq |x|^{-2\alpha},
\]
\[
\int_{|x-y| \geq \frac{|x|}{2}} |x-y|^{-d-2\alpha} \, dx \leq \int_{|x| \geq |y|} |x|^{-d-2\alpha} \, dx \simeq |y|^{-2\alpha}.
\]
Then by (4.13) we can estimate $I_{1,1}$.

(4.15) \quad I_{1,1} \lesssim \int \int_{|x-y| \geq \frac{|x|}{2}} \frac{|x|^{2\theta-d+2\alpha} |m(x)|^2}{|x-y|^{d+2\alpha}} \, dy \, dx \lesssim \int_{\mathbb{R}^d} |m(x)|^2 |x|^{2\theta-d} \, dx \simeq I_0.

For $I_{1,2}$ we make use of (4.14) to obtain

(4.16) \quad \sum_k \left| \eta(2^{-k} x) - \eta(2^{-k} y) \right|^2 \leq \sum_k |x-y|^2 2^{-2k} \left( \int_0^1 |(\nabla \eta)(2^{-k}(1-r)x + 2^{-k}ry)| \, dr \right)^2 \\
\leq |x-y|^2 \int_0^1 \sum_k 2^{-2k} \left| (\nabla \eta)(2^{-k}(1-r)x + 2^{-k}ry) \right|^2 \, dr \\
\lesssim |x-y|^2 \int_0^1 |(1-r)x + ry|^{-2} \, dr \\
\lesssim |x-y|^2 |x|^{-2}

when $|x-y| < |x|/2$. By (4.16) we have

$$I_{1,2} \lesssim \int \int_{|x-y| < \frac{|x|}{2}} |x-y|^{-d+2(1-\alpha)} |m(x)|^2 |x|^{2\theta-d+2(\alpha-1)} \, dx \, dy.$$ 

Since
\[
\int_{|x-y| < \frac{|x|}{2}} |x-y|^{-d+2(1-\alpha)} \, dy \lesssim |x|^{2(1-\alpha)},
\]
it follows that

(4.17) \quad I_{1,2} \lesssim \int_{\mathbb{R}^d} |m(x)|^2 |x|^{2\theta-d} \, dx \simeq I_0.

Therefore, by (4.12), (4.15), (4.17) we have

(4.18) \quad \|m\|^2_{\Sigma^2_{\eta}(T_{\frac{2}{3}})} \lesssim \int_{\mathbb{R}^d} |m(x)|^2 |x|^{2\theta-d} \, dx + I_{1,3}.

For $I_{1,3}$ we apply (4.13) so that

$$I_{1,3} \simeq \int \int_{|x-y| < \frac{|x|}{2}} \frac{||x|^{\theta-d/2+\alpha} m(x) - |y|^{\theta-d/2+\alpha} m(y)|^2}{|x-y|^{d+2\alpha}} \, dx \, dy.$$
Then we have
\[
\int \int_{|x-y| < \frac{|x|}{2}} \frac{|x|^\theta - d/2 + \alpha m(x) - |y|^\theta - d/2 + \alpha m(y)|^2}{|x-y|^{d+2\alpha}} \, dx \, dy
\]
whenever
\[
|x|^\theta - d/2 + \alpha \int \frac{|x|^\theta - d/2 + \alpha m(x) - |y|^\theta - d/2 + \alpha m(y)|^2}{|x-y|^{d+2\alpha}} \, dx
\]
whenever $|x| - |y| \leq |x|/2$ so that
\[
\int_{|x-y| < \frac{|x|}{2}} \frac{|x|^\theta - d/2 + \alpha - |y|^\theta - d/2 + \alpha|^2}{|x-y|^{d+2\alpha}} \, dx \lesssim |y|^\theta - d - 2d - 2\alpha + 2 \int_{|x-y| < |y|} |x-y|^{-d-2\alpha} \, dx
\]
\[
\approx |y|^\theta - d - 2d - 2\alpha \times |y|^{-2\alpha} = |y|^{\theta - d}.
\]
Thus we have
\[
\int \int_{|x-y| < \frac{|x|}{2}} \frac{|x|^\theta - d/2 + \alpha - |y|^\theta - d/2 + \alpha|^2 |m(y)|^2}{|x-y|^{d+2\alpha}} \, dx \, dy \lesssim \int_{\mathbb{R}^d} |m(x)|^2 |x|^{\theta - d} \, dx.
\]
By (4.18), (4.20), (4.21), we have
\[
\|m\|^2_{L^2_{t,x}(L^2_{m})} \lesssim \int_{\mathbb{R}^d} |m(x)|^2 |x|^{\theta - d} \, dx + \int_{\mathbb{R}^d} \left( \int_{|y-x| < \frac{|x|}{2}} \frac{|m(x) - m(y)|^2}{|x-y|^{d+2\alpha}} \, dy \right) |x|- N \, dx
\]
which yields
\[
\int_{\mathbb{R}^d} \left( \int_{|y-x| < \frac{|x|}{2}} \frac{|m(x) - m(y)|^2}{|x-y|^{d+2\alpha}} \, dy \right) |x|^{2\alpha + \theta - d} \, dx
\]
\[
\lesssim \int_{|y-x| < \frac{|x|}{2}} \frac{|x|^\theta - d/2 + \alpha - |y|^\theta - d/2 + \alpha|^2 |m(y)|^2}{|x-y|^{d+2\alpha}} \, dx \, dy
\]
\[
\lesssim I_{1,3} + \int_{\mathbb{R}^d} \left( \int_{|y-x| < \frac{|x|}{2}} \frac{|x|^\theta - d/2 + \alpha - |y|^\theta - d/2 + \alpha|^2}{|x-y|^{d+2\alpha}} \, dx \right) |m(y)|^2 \, dy
\]
\[
\lesssim I_{1,3} + I_0.
\]
where the last inequality holds due to \(\text{(1.21)}\). Moreover, one can observe that
\[
|\langle x \rangle^N m(x) \eta(2^{-k} y) - |y|^N m(y) \eta(2^{-k} y)|^2 \\
\lesssim |\langle x \rangle^N m(x) |^2 |\eta(2^{-k} y) - \eta(2^{-k} x)|^2 + |\langle x \rangle^N m(x) \eta(2^{-k} x) - |y|^N m(y) \eta(2^{-k} y)|^2,
\]
which yields
\[
I_{1,3} \lesssim I_{1,2} + I_1.
\]

Since \(I_{1,2} \lesssim I_0\) and \(I_0 + I_1 \simeq \|m\|^2_{L^2(\mathbb{R}^d)}\), we have shown that
\[
\text{(4.23)} \quad \int_{\mathbb{R}^d} \left( \int_{|y-x| < \frac{|x|}{2}} \frac{|m(x) - m(y)|^2}{|x-y|^{d+2\alpha}} dy \right) |x|^{2\alpha+2\theta-d} dx \lesssim \|m\|^2_{L^2(\mathbb{R}^d)}.
\]

By \(\text{(4.23)}\), it follows that
\[
\int_{\mathbb{R}^d} |m(x)|^2 |x|^{2\theta-d} dx + \int_{\mathbb{R}^d} \left( \int_{|y-x| < \frac{|x|}{2}} \frac{|m(x) - m(y)|^2}{|x-y|^{d+2\alpha}} dy \right) |x|^{2\alpha+2\theta-d} dx \lesssim \|m\|^2_{L^2(\mathbb{R}^d)}.
\]

This proves the proposition. \(\square\)

5. Proof of Lemma 3.2 and bilinear interpolation

5.1. Proof of Lemma 3.2. We note that Lemma 3.2 is a modified version of the following result in \([17]\):

**Lemma 5.1 ([17], Lemma 4).** Let \(m\) be a function of class \(C^s(\mathbb{R}^d)\) with \(s > \frac{d}{2}\) and supported in \(\{1/2 < |\xi| < 2\}\), and let the \(g\)-function be given by
\[
G_m(f)(x) = \left( \int_0^\infty |(m(t\xi)\hat{f}(\xi))'(x)|^2 \frac{dt}{t} \right)^{1/2}.
\]

Then for \(\beta > d/2\) we have
\[
\|G_m(f)\|_{L^1(\mathbb{R}^d)} \leq C_\beta \|m\|_{L^2(\mathbb{R}^d)} \|f\|_{H^s(\mathbb{R}^d)}.
\]

In [17], Lemma 5.1 is proved by showing the following inequality due to the Calderón-Zygmund theory:
\[
\int_{|x| > 2|y|} \left( \int_0^\infty |K_t(x-y) - K_t(x)|^2 \frac{dt}{t} \right)^{1/2} dx \leq C_\beta \|m\|_{L^2(\mathbb{R}^d)}.
\]

To perform the Calderón-Zygmund theory, one needs to obtain initial \(L^2\)-estimates. For the \(L^2\) estimate, we make use of the Plancherel theorem to obtain
\[
\|G_m(f)\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \left( \int_0^\infty |m(t\xi)|^2 \frac{dt}{t} \right) d\xi
\]
\[
\leq \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} \|m(t\xi)|^2_{H^s(\mathbb{R}^d)} \|f\|^2_{L^2(\mathbb{R}^d)},
\]
\[
(5.2)
\]
where $\mathcal{H}(\mathbb{R}_+)$ denotes the Hilbert space $L^2(\mathbb{R}_+, \frac{dt}{t})$. Note that (5.2) yields the first assertion of Lemma 3.2. Moreover, let $2^k \leq |\xi_0| \leq 2^{k+1}$ for some $k \in \mathbb{Z}$. Then we have
\[
\|m(t\xi_0)\|^2_{\mathcal{H}(\mathbb{R}_+)} \leq \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} |m(t\xi_0)|^2 \frac{dt}{t}
\leq \sum_{j \in \mathbb{Z}} \sup_{t \in [2^j, 2^{j+1}]} |m(t2^k\xi_0')|^2
\leq \sum_{j \in \mathbb{Z}} \sup_{1 \leq |\xi'| \leq 4} |m(2^{j+k}\xi')|^2 \leq \sum_{j \in \mathbb{Z}} \|m(2^j \cdot \hat{\psi}(\cdot))\|_{L^2_\beta(\mathbb{R}^d)}^2,
\]
where $\xi_0 = 2^k \xi_0'$ and we apply the change of variable for the second inequality. Thus it follows that if $m \in \Sigma^2(L^\infty)$, then
\[
(5.3) \quad \|G_m(f)\|_{L^2(\mathbb{R}^d)} \leq \|m\|_{\Sigma^2(L^\infty)} \|f\|_{L^2(\mathbb{R}^d)}.
\]
Back to (5.1), observe that by Hörmander’s multiplier theorem in vector-valued setting, we have
\[
\int_{|x|>2|y|} \left( \int_0^\infty |K_t(x-y) - K_t(x)|^2 \frac{dt}{t} \right)^{1/2} dx \leq \sup_{j \in \mathbb{Z}} \|m(2^j t \xi) \hat{\psi}(\xi)\|_{L^2_\beta(\mathbb{R}_d; \mathcal{H}(\mathbb{R}_+))}.
\]
Then (5.1) can be accomplished by demonstrating the following inequality: For $k \in \mathbb{N}_0$ and $m(2^j \xi) = m(2^j \xi) \hat{\psi}(\xi)$
\[
\|m(2^j \cdot \hat{\psi}(\cdot))\|_{L^2_\beta(\mathbb{R}^d; \mathcal{H}(\mathbb{R}_+))}^2 = \sum_{l: |l| \leq k} \int_{\mathbb{R}^d} \|D^l_\xi (\mu(2^j \xi))\|_{\mathcal{H}}^2 d\xi
\leq N(k, d) \sum_{l: |l| \leq k} \int_{\mathbb{R}^d} D^l_\xi m(2^j \xi) \|D^l_\xi m(2^j \xi)\|_{\mathcal{H}}^2 \frac{dt}{t}
\leq \sum_{l: |l| \leq k} \int_{\mathbb{R}^d} \|D^l_\xi m(\xi)\|^2 \left( \int_{\|\xi\|=\frac{|l|}{2^j}}^{\|\xi\| \leq \frac{|l|}{2^j} - \frac{d}{2}} \frac{(2^j t)^2 |\xi|^2 - dt}{t} \right) d\xi
\leq N(k, d) \sum_{l: |l| \leq k} \int_{\mathbb{R}^d} \|D^l_\xi m(\xi)\|^2 |\xi|^2 |\xi|^{-d} d\xi
\leq N(k, d) \|m\|_{\Sigma^2(L^\infty)}^2.
\]
In (5.4) we choose $k \in \mathbb{N}_0$, so one can perform interpolation to obtain the following result:
\[
\sup_{j \in \mathbb{Z}} \|m(2^j \xi) \hat{\psi}(\xi)\|_{L^2_\beta(\mathbb{R}_d; \mathcal{H}(\mathbb{R}_+))} \lesssim \|m\|_{\Sigma^2(L^\infty)} \beta \geq 0.
\]
Thus we conclude that for any $\varepsilon > 0$ if $m \in \Sigma^2(L^2_{d/2+\varepsilon})$, then
\[
(5.5) \quad \|G_m(f)\|_{L^1(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(L^2_{d/2+\varepsilon})} \|f\|_{H^1(\mathbb{R}^d)}.
\]
For $L^\infty \rightarrow BMO$ estimates, it is well-known (by duality) that
\[ \|G_m\|_{L^\infty \rightarrow BMO} \lesssim \|G_m\|_{L^2 \rightarrow L^2} + \|G_m\|_{H^1 \rightarrow L^1}, \]
which is a special case of [16] Theorem 4.2. Since $\|G_m\|_{L^2 \rightarrow L^2} \lesssim \|m\|_{\Sigma^2(L^\infty)}$ by (5.3), we make use of $\|f\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}$ for $s > \frac{d}{2}$ and obtain
\[ \|m\|_{\Sigma^2(L^\infty)} \lesssim \|m\|_{\Sigma^2(L^2_d)}. \]
Thus together with $\|G_m\|_{H^1 \rightarrow L^1} \lesssim \|m\|_{\Sigma^2(L^2_d)}$ for $s > \frac{d}{2}$, we have
\[ (5.6) \quad \|G_m(f)\|_{BMO(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(L^2_{d/2+\epsilon})}\|f\|_{L^\infty(\mathbb{R}^d)}, \]
which proves Lemma 3.2 together with (5.2) and (5.5).

5.2. Interpolation argument. We introduce a complex interpolation result for multilinear operators due to Calderón [4].

Lemma 5.2. [4, paragraph 10.1] Let $L(x_1, \ldots, x_n)$ be a multilinear mapping defined for $x_i \in A_i \cap B_i, i = 1, \ldots, n$ with values in $A \cap B$ and such that
\[ \|L(x_1, x_2, \ldots, x_n)\|_A \leq M_0 \prod_{i=1}^n \|x_i\|_{A_i}, \]
\[ \|L(x_1, x_2, \ldots, x_n)\|_B \leq M_1 \prod_{i=1}^n \|x_i\|_{B_i}. \]
Then we have
\[ \|L(x_1, x_2, \ldots, x_n)\|_C \leq M_0^{1-\theta} M_1^\theta \prod_{i=1}^n \|x_i\|_{C_i}, \]
where $C = [A, B]\theta$, $C_i = [A_i, B_i]\theta$, $i = 1, \ldots, n$ and thus $L$ can be extended continuously to a multilinear mapping of $C_1 \times \ldots \times C_n$ into $C$.

To make use of Lemma 5.2 it suffices to check $x_i \in A_i \cap B_i$ for $i = 2, \ldots, n$. In our case, we want $f \in \mathcal{S}^\infty$, where
\[ \mathcal{S}^\infty := \{ u \in \mathcal{S}(\mathbb{R}^d) : \int x^\alpha u(x) dx = 0 \text{ for all multi-indices } \alpha \} \]
Then $\mathcal{S}^\infty$ is a dense subset of $H^p$ for $0 < p < \infty$ (p.128 in [21]) and $f$ can be an element of $L^2 \cap H^1$. For $(L^2, L^\infty)$-interpolation, we choose $f \in \mathcal{S}(\mathbb{R}^d)$ so that $f$ can be an element of $L^2 \cap L^\infty$.

Now we consider $G_m(f)(x)$ as an $L^2(\frac{d}{2})$-norm of a bilinear operator
\[ B(m,f)(x,t) := T_{m(t)} f(x). \]
The novelty of bilinear approach is that we do not need to construct an analytic family for $m$ to obtain desired interpolated norms of $m$. Then the strategy is to apply the multilinear
interpolation of Lemma 5.2 on the following bilinear estimates given by Lemma 3.1:

\[ \|B(m, f)\|_{L^2(\mathbb{R}^d, \mathcal{H})} \lesssim \sup_{\xi \neq 0} \left( \int_0^\infty |\tilde{m}(t\xi)| |\tilde{f}(t\xi)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(C^{0,1/2+3\epsilon})} \|f\|_{L^2(\mathbb{R}^d)}, \]

\[ \|B(m, f)\|_{L^1(\mathbb{R}^d, \mathcal{H})} \lesssim \|\tilde{m}\|_{\Sigma^2(L^2_0)} \|f\|_{H^1(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(L^2_0)} \|f\|_{H^1(\mathbb{R}^d)}. \]

Choose \( \theta = 2\left(\frac{1}{p} - \frac{1}{2}\right) \in (0, 1) \), then we will show that

\[ \left[ L^1(\mathbb{R}^d; \mathcal{H}), L^2(\mathbb{R}^d; \mathcal{H}) \right]_\theta = L^p(\mathbb{R}^d; \mathcal{H}), \tag{5.7} \]

\[ \left[ \Sigma^2(L^2_0), \Sigma^2(C^{0,1/2+3\epsilon}) \right]_\theta = \Sigma^2(B^s_{p_0}), \quad p_0 = \frac{2p}{2 - p}, \quad s = d\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{1}{2} + 3\epsilon, \tag{5.8} \]

\[ \left[ H^1(\mathbb{R}^d), L^2(\mathbb{R}^d) \right]_\theta = L^p(\mathbb{R}^d). \tag{5.9} \]

Note that the second interpolation follows from (2.3). Now, taking (5.7), (5.8), (5.9) for granted, by Lemma 5.2 we can show that

\[ \|B(m, f)\|_{L^p(\mathbb{R}^d, \mathcal{H})} \lesssim \|m\|_{\Sigma^2(B^s_{p_0})} \|f\|_{L^p(\mathbb{R}^d)}, \quad s \geq d\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{1}{2} + 3\epsilon. \]

When \( p \geq 2 \) we replace \( (H^1 \to L^1) \)-estimate into \( (L^\infty \to BMO) \)-estimate, which is (5.6).

That is,

\[ [BMO(\mathbb{R}^d; \mathcal{H}), L^2(\mathbb{R}^d; \mathcal{H})]_\theta = L^p(\mathbb{R}^d; \mathcal{H}), \]

\[ \left[ \Sigma^2(L^2_0), \Sigma^2(C^{0,1/2+3\epsilon}) \right]_\theta = \Sigma^2(B^s_{p_0}), \quad p_0 = \frac{2p}{2 - p}, \quad s = d\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{1}{2} + 3\epsilon, \]

\[ [L^\infty(\mathbb{R}^d), L^2(\mathbb{R}^d)]_\theta = L^p(\mathbb{R}^d). \]

Then by Lemma 5.2 we obtain

\[ \|B(m, f)\|_{L^p(\mathbb{R}^d, \mathcal{H})} \lesssim \|m\|_{\Sigma^2(B^s_{p_0})} \|f\|_{L^p(\mathbb{R}^d)}, \quad s \geq d\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{1}{2} + 3\epsilon, \]

which yields \( \frac{1}{p_0} = \frac{1}{p} - \frac{1}{2} \) and \( s \geq d\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{1}{2} + 3\epsilon \) in general.

For the Hardy space interpolation, it is given in [8] Section 5] that \( [H^1, L^p]_\theta = [L^1, L^p]_\theta \) for \( \theta \in (0, 1) \) and \( p > 1 \). For \( BMO(\mathbb{R}^d) \), it is well known in the literature such as [9] Theorem 3.4.7] that \( [BMO, L^p]_\theta = [L^\infty, L^p]_\theta \) for \( \theta \in (0, 1) \) and \( 1 \leq p < \infty \), and we refer to [2] for a vector-valued analogue. Therefore we end this section by verifying the vector-valued interpolation and (5.5).

For the interpolation in the vector-valued setting, it is known in [22] Theorem 1.18.4] and [2] that for an interpolation couple \( \{A_0, A_1\} \) and \( \theta, \gamma \in (0, 1) \)

\[ [L^1(A_0), L^2(A_1)]_\theta = L^p([A_0, A_1]_\theta), \quad \frac{1}{p} = 1 - \theta + \frac{\theta}{2}, \]

\[ [BMO(A_0), L^2(A_1)]_\gamma = L^q([A_0, A_1]_\gamma), \quad \frac{1}{q} = \frac{\gamma}{2}. \]

In our case, \( A_0 = A_1 = \mathcal{H} \) and \( \mathcal{H} \) is the Hilbert space. Thus it follows that

\[ [L^1(\mathcal{H}), L^2(\mathcal{H})]_\theta = L^p(\mathcal{H}) \quad \text{and} \quad [BMO(\mathcal{H}), L^2(\mathcal{H})]_\gamma = L^q(\mathcal{H}). \]
For (5.8) together with $\Sigma^2(L^2_{(d+1)/2+3\varepsilon}) = \Sigma^2(B^2_{2(d+1)/2+3\varepsilon})$, we perform the complex interpolation to obtain

$$[\Sigma^2(L^2_{(d+1)/2+3\varepsilon}), \Sigma^2(C^{0,1/2+3\varepsilon})]_\theta = [\Sigma^2(B^2_{2(d+1)/2+3\varepsilon}), \Sigma^2(B^2_{1/2+3\varepsilon})]_\theta$$

$$= \Sigma^2((B^2_{2(d+1)/2+3\varepsilon}, B^2_{1/2+3\varepsilon})_\theta).$$

By the interpolation of the Besov space, we have

$$\|M\|_{B^s_{p_0}} \leq \|M\|_{B^s_{p_0}} \leq \|\theta\|_{B^s_{p_0}}.$$

for $p_0 = 2p/2 - p$, $s = d(1/p - 1/2) + 1/2 + 3\varepsilon$. Interpolation procedure for (5.7) and (5.8) is given in Theorem 1.18.1 and Theorem 2.4.1 of [22] and Theorem 6.4.5 of [1].

6. APPLICATIONS OF THEOREM 1.7

In this section, we provide proofs of Corollaries 1.2, 1.3, 1.4, 1.6 and Propositions 1.5, 1.7.

6.1. Proof of Corollary 1.2. We use the following properties of $B^s_{p_0}$:

- Since $s > \frac{d}{p_0} + \frac{1}{2}$, $B^s_{p_0}(\mathbb{R}^d)$ is continuously embedded in $C^{0,1/2}(\mathbb{R}^d) = B^1_{\infty}(\mathbb{R}^d)$.
- For $f \in B^s_{p_0}(\mathbb{R}^d)$, $\|f\|_{B^s_{p_0}(\mathbb{R}^d)} \simeq \|f\|_{L^p_0(\mathbb{R}^d)} + \|f\|_{B^s_{p_0}(\mathbb{R}^d)}$, and $\|f(\lambda \cdot)\|_{B^s_{p_0}(\mathbb{R}^d)} \simeq \lambda^{s-d/p_0} \|f\|_{B^s_{p_0}(\mathbb{R}^d)}$. Here, $B^s_{p_0}$ is the homogeneous Besov space whose norm is given by

$$\|f\|_{B^s_{p_0}} = \left(\sum_{j \in \mathbb{Z}} (2^{js} \|\psi_j * f\|_{L^p_0(\mathbb{R}^d)})^{P_0}\right)^{1/P_0}.$$

The embedding $B^s_{p_0}(\mathbb{R}^d) \subset C^{0,1/2}(\mathbb{R}^d) = B^1_{\infty}(\mathbb{R}^d)$ implies that

$$|m(0)| + \frac{|m(\xi) - m(0)|}{|\xi|^{1/2}} \lesssim \|m\|_{B^s_{p_0}} \quad \text{for any } \xi \in \mathbb{R}^d \setminus \{0\}. $$

Let $\phi_0$ supported on $B(0, 2^N)$. Then denote $m_0 = m(0)\phi_0$ and $m_1 = m\phi_0 - m(0)\phi_0$. Since $\phi_0 \in C^\infty_c(\mathbb{R}^d)$, we obtain

$$\|m(0)\|_{B^s_{p_0}} \leq \|m\|_{B^s_{p_0}} \|\mathcal{M}(f)\|,$$

where $\mathcal{M}(f)$ is the Hardy-littlewood maximal function of $f$.

For $\mathcal{M}_{m_1}$, first we consider the case of $p \in (1, \infty)$. Observe that for $j \leq N$

$$\|m_1(2^j \cdot)\|_{L^p_0} \lesssim \left\|(m(2^j \cdot) - m(0))\hat{\psi}\right\|_{L^\infty} \lesssim 2^{j/2}\|m\|_{B^s_{p_0}},$$

and

$$\|m_1(2^j \cdot)\|_{B^s_{p_0}} \lesssim \|m_1(2^j \cdot)\|_{B^s_{p_0}} \lesssim 2^{j/2}\|m\|_{B^s_{p_0}} + \|m(0)\| \lesssim 2^{j/2}\|m\|_{B^s_{p_0}}.$$
which yield 
\[
\|m_1\|_{\Sigma^2(B_{p_0})}^2 \lesssim \sum_{j \leq N} \left( \|m_1(2^j \cdot)\hat{\psi}\|_{B_{p_0}} + \|m_1(2^j \cdot)\hat{\psi}\|_{B_{p_0}} \right)^2 \lesssim \left( \sum_{j \leq N} 2^j \right) \|m\|_{B_{p_0}}^2.
\]
Therefore it follows by Theorem 1.1 that 
\[
(6.2) \quad \|m_1(2^j \cdot)\hat{\psi}\|_{B_{p_0}} \lesssim \|m_1\|_{\Sigma^2(B_{p_0})} \|f\|_p \lesssim \|m\|_{B_{p_0}} \|f\|_p.
\]
By combining (6.1) and (6.2) we obtain that 
\[
\|m_1(2^j \cdot)\hat{\psi}\|_{B_{p_0}} \lesssim \|m_1\|_{\Sigma^2(B_{p_0})} \|f\|_p \lesssim \|m\|_{B_{p_0}} \|f\|_p.
\]
The case of \( p = 1 \) and \( p = \infty \) are proved by the same argument.

6.2. Proof of Corollary 1.3
Due to \( \mathcal{M}_m(f) \leq G_{\hat{m}}(f) \) in (6.2), it suffices to show 
\[
\|G_{\hat{m}}(f)\|_{L^p(\mathbb{R}^d)} \lesssim \|h\|_{\Sigma^2(L^2(\mathbb{R}^d)))} \|f\|_{L^p(\mathbb{R}^d)},
\]
where \( s > d\frac{1}{p} - \frac{1}{2} + \frac{1}{2} \) and \( m(\xi) = h(|\xi|) \). By direct calculation, we obtain 
\[
\tilde{m}(\xi) = \tilde{h}(|\xi|) \quad \text{where} \quad \tilde{h}(r) := h(r) + (1/2 + \varepsilon) \int_0^\infty (1 - s)^{-\frac{3}{2} - \varepsilon} (h(r) - h(sr)) \, ds.
\]
For the \( L^2 \) boundedness of \( G_{\hat{m}} \), note that 
\[
(6.3) \quad \int_0^\infty |\tilde{m}(t\xi)|^2 \frac{dt}{t} = \int_0^\infty |\tilde{h}(t\xi)|^2 \frac{dt}{t} = \int_0^\infty |\tilde{h}(t)|^2 \frac{dt}{t} \simeq \|\tilde{h}\|_{\Sigma^2(L^2(\mathbb{R})))}^2.
\]
Then by Lemma 3.1 we have 
\[
(6.4) \quad \|\tilde{h}\|_{\Sigma^2(L^2(\mathbb{R})))} \lesssim \|h\|_{\Sigma^2(L^2_{1/2+2\varepsilon}(\mathbb{R})))} \lesssim \|h\|_{\Sigma^2(L^2_{1/2+3\varepsilon}(\mathbb{R})))}^2.
\]
Making use of Lemma 4.1 we also have for a nonnegative integer \( k \) 
\[
\|m\|_{\Sigma^2(L^2_k)}^2 \simeq \sum_{l \leq k} \int_{\mathbb{R}^d} |m(\xi)|^2 |\xi|^{2l-2} \, d\xi \simeq \sum_{l \leq k} \int_0^\infty |h(r)|^2 r^{2l-2} \, dr \simeq \|h\|_{\Sigma^2(L^2_k(\mathbb{R})))}^2,
\]
which yields 
\[
\|m\|_{\Sigma^2(L^2_k)} \lesssim \|h\|_{\Sigma^2(L^2_k(\mathbb{R})))}
\]
for real \( s \geq 0 \). Thus by the equality of (5.2), (6.3) and (6.4), we have 
\[
(6.5) \quad \|G_{\hat{m}}(f)\|_{L^2(\mathbb{R}^d)} \lesssim \|h\|_{\Sigma^2(L^2_{1/2+3\varepsilon}(\mathbb{R})))} \|f\|_{L^2(\mathbb{R}^d)},
\]
and by Theorem 1.1 we have 
\[
(6.6) \quad \|G_{\hat{m}}(f)\|_{L^1(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(L^2_{1/2+3\varepsilon}(\mathbb{R})))} \|f\|_{H^1(\mathbb{R}^d)} \lesssim \|h\|_{\Sigma^2(L^2_{1/2+3\varepsilon}(\mathbb{R})))} \|f\|_{H^1(\mathbb{R}^d)}.
\]
Corollary 1.3 follows from applying Lemma 5.2 on (6.5), (6.6) and duality.
6.3. Proof of Corollary 1.4. Before applying Theorem 1.1 we recall embedding property of $B_{p,q}^{s}$. Since we use $B_{p_0,q_0}^{s}$ for $p_0 \geq 2$, $s \in (d\frac{1}{p} - \frac{1}{2} + \frac{1}{2}, \frac{1}{2})$, it follows that

$$L_{p_0}^{s} = F_{p_0,2}^{s} \subseteq F_{p_0,p_0}^{s} = B_{p_0,p_0}^{s}.$$  

Thus for $m_{\alpha,\beta}$ it suffices to check

$$\| m_{\alpha,\beta} \|_{\Sigma^2(L_{p_0}^{p}({\mathbb R}^{d}))} < \infty.$$

Moreover, $\ell^2$-summation over $\mathbb{Z}$ can be reduced to summation over non-negative integers since $m_{\alpha,\beta}$ vanishes near the origin. Therefore we have

$$\| m_{\alpha,\beta} \|_{\Sigma^2(B_{p_0,q_0}^{s}({\mathbb R}^{d})))} \leq \sum_{j \geq N} \| m_{\alpha,\beta}(2^j \cdot \hat{\psi}(\cdot)) \|_{L_{p_0}^{p}({\mathbb R}^{d})}$$

for some $N \in \mathbb{Z}$. To compute $L_{p_0}^{s}$-norm of $m_{\alpha,\beta}$, recall that $|\partial^\gamma m_{\alpha,\beta}(\xi)| \leq C|\xi|^{-\beta - |\gamma|(1 - \alpha)}$, which yields

$$\| m_{\alpha,\beta}(2^j \cdot \hat{\psi}(\cdot)) \|_{L_{p_0}^{p}({\mathbb R}^{d})} \leq C 2^{j + j(\beta + s(1 - \alpha))}.$$ 

Since we choose $\beta/\alpha > d |\xi|^{1/2} + \frac{1}{2}$, the summation in (6.8) is bounded and we have

$$\frac{\beta}{\alpha} > d \left| \frac{1}{p} - \frac{1}{2} \right| + \frac{1}{2}.$$ 

Note that (6.10) is equivalent to

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{2\beta - \alpha}{2\alpha d} = \frac{\beta/\alpha}{d} - \frac{1}{2d},$$

which proves Corollary 1.4.

6.4. Proof of Proposition 1.5. Note that the symbol of $U_{\alpha,\beta}$ is

$$\frac{e^{it|\xi|^{\alpha}} - 1}{t^\beta}.$$ 

We define

$$m_{\alpha,\beta}(t^{1/\alpha} \xi) = \frac{e^{it|\xi|^{\alpha}} - 1}{t^\beta|\xi|^{\alpha \beta}}$$

in the sense of (1.5).

Let $0 < \beta \leq 1$ and $|\xi| \leq 1$. By Taylor’s expansion we have

$$m_{\alpha,\beta}(\xi) = \sum_{k=1}^{\infty} \frac{i^k}{k!} |\xi|^{\alpha(k-\beta)} \quad \text{and} \quad m_{\alpha,\alpha}(\xi) = i + \sum_{k=2}^{\infty} \frac{i^k}{k!} |\xi|^{\alpha(k-1)}.$$ 

For each $k \in \mathbb{N}$

$$\| 2^j |\xi|^{\alpha(k-\beta)} \hat{\psi}(\xi) \|_{L_{p_0}^{p}({\mathbb R}^{d})} \lesssim 2^{j \alpha(k-\beta)} \| |\xi|^{\alpha(k-\beta)} \hat{\psi} \|_{L_{p_0}^{p}({\mathbb R}^{d})} \lesssim 2^{j \alpha(k-\beta) k^d}.$$
Then we have for $\beta \in (0, 1)$
\[
\left( \sum_{j \leq 0} \left\| m_{\alpha, \alpha \beta}(2^j \cdot) \hat{\psi}(\cdot) \right\|_{L^p_{\alpha}(\mathbb{R}^d)}^2 \right)^{1/2} \leq \sum_{j \leq 0} \left\| m_{\alpha, \alpha \beta}(2^j \cdot) \hat{\psi}(\cdot) \right\|_{L^p_{\alpha}(\mathbb{R}^d)} \leq \sum_{j \leq 0} \sum_{k=1}^{\infty} \frac{1}{k!} \left\| 2^j \xi^{\alpha(k-\beta)} \hat{\psi}(\xi) \right\|_{L^p_{\alpha}(\mathbb{R}^d)} \leq \sum_{j \leq 0} \sum_{k=1}^{\infty} \frac{1}{k!} \alpha j^{\alpha(k-\beta)} k^d s_k
\]
\[
\leq N \sum_{k=1}^{\infty} \frac{1}{k!} k^d s_k < \infty.
\]

For $\beta = 1$ since we have $m_{\alpha, \alpha \beta} \varphi_0 = i \varphi_0 + \sum_{k=2}^{\infty} \frac{1}{k^2} |\xi|^{\alpha(k-1)} \varphi_0$ and $\varphi_0$ is of class $C_c^\infty(\mathbb{R}^d)$, we only consider the second term.
\[
\left( \sum_{j \leq 0} \left\| \sum_{k=2}^{\infty} \frac{i^k}{k!} 2^j \xi^{\alpha(k-1)} \hat{\psi}(\xi) \right\|_{L^p_{\alpha}(\mathbb{R}^d)}^2 \right)^{1/2} \leq \sum_{j \leq 0} \sum_{k=2}^{\infty} \frac{1}{k!} \left\| 2^j \xi^{\alpha(k-1)} \hat{\psi}(\xi) \right\|_{L^p_{\alpha}(\mathbb{R}^d)} \leq \sum_{j \leq 0} \sum_{k=2}^{\infty} \frac{1}{k!} \alpha j^{\alpha(k-1)} k^d s_k \leq N \sum_{k=2}^{\infty} \frac{1}{k!} k^d s_k < \infty.
\]

For $|\xi| \geq 1$ we have
\[
|\partial^\gamma m_{\alpha, \alpha \beta}(\xi)| \lesssim |\xi|^{-\alpha \beta - (1-\alpha)\gamma},
\]
so $m_{\alpha, \alpha \beta}$ is a slowly decaying Fourier multiplier. Thus we make use of (6.7), (6.8), (6.9) to obtain
\[
\left\| m_{\alpha, \alpha \beta}(2^j \cdot) \hat{\psi}(\cdot) \right\|_{L^p_{\alpha}(\mathbb{R}^d)} \leq C 2^{j s} 2^{-j (\alpha \beta + s(1-\alpha))},
\]
which yields $\beta > d |\frac{1}{p} - \frac{1}{2}| + \frac{1}{2}$ by Theorem 1.1. Then it follows that
\[
\left\| \sup_{t > 0} |U_{\alpha, \alpha \beta} f(\cdot, t)| \right\|_{L^p(\mathbb{R}^d)} = \left\| \sup_{t > 0} |U_{\alpha, \alpha \beta} f(\cdot, t\alpha)| \right\|_{L^p(\mathbb{R}^d)} = \left\| \sup_{t > 0} |T_{m_{\alpha, \alpha \beta}(t \cdot)} ((-\Delta)^{\alpha \beta} f) \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p_{\alpha \beta}(\mathbb{R}^d)},
\]
(6.12)
whenever $\frac{1}{p} - \frac{1}{2} < \frac{\beta}{d} - \frac{1}{2}$. Together with $\beta \leq 1$, we have $\frac{d-2\beta+1}{2d} < \frac{1}{p} < \frac{d+2\beta-1}{2d}$. Finally, (6.12) yields that
\[
\lim_{t \to 0} \left\| U_{\alpha, \alpha \beta} f(\cdot, t) \right\|_{L^p(\mathbb{R}^d)} = \lim_{t \to 0} \left\| U_{\alpha, \alpha \beta} f(\cdot, t) \right\|_{L^p(\mathbb{R}^d)} = 0,
\]
since $\lim_{t \to 0} \frac{\epsilon_{\alpha \beta} - 1}{\epsilon_{\alpha \beta}} = 0$ for a constant $A$ and $\beta \in (0, 1)$. This proves the proposition.
6.5. **Proof of Corollary 1.6.** Since we assume that $m_{a,b} \in L^2_{(d+1)/2+\varepsilon, \text{loc}}$ for some $\varepsilon > 0$, it follows that $m\phi \in B^2_2(\mathbb{R}^d)$ for any $\phi \in C_0^\infty(\mathbb{R}^d)$. Thus by Corollary 1.2 it suffices to check
\begin{equation}
\sum_{j \geq 0} \| m_{a,b}(2^j \cdot) \hat{\psi}(\cdot) \|^2_{B^2_2(\mathbb{R}^d)} < \infty.
\end{equation}
To show (6.13), we make use of (6.7) and (1.9) to obtain
\begin{equation}
\sum_{j \geq 0} \| m_{a,b}(2^j \cdot) \hat{\psi}(\cdot) \|^2_{B^2_2(\mathbb{R}^d)} \leq \sum_{j \geq 0} \| m_{a,b}(2^j \cdot) \hat{\psi}(\cdot) \|^2_{L^p_0(\mathbb{R}^d)} \leq \sum_{j \geq 0} 2^{-j^2 \min(a,b-s)}.
\end{equation}
Since we choose $a, b > 0$ and $s > d|\frac{1}{p} - \frac{1}{2}| + \frac{1}{2}$, the RHS of (6.14) converges if $b - s > 0$. That is, we have
\[ b > s > d|\frac{1}{p} - \frac{1}{2}| + \frac{1}{2}, \]
which yields $\frac{d+1-2b}{2d} < \frac{1}{p} < \frac{d+1+2b}{2d}$. This proves the corollary.

6.6. **Proof of Proposition 1.7.** By the Fourier transform, we know that
\[ \frac{f(x) - T_{m(t)} f(x)}{t^\alpha} = \int_{\mathbb{R}^d} e^{i(x, \xi)} \frac{1 - m(t\xi)}{|t\xi|^\alpha} |\xi|^\alpha \hat{f}(\xi) d\xi. \]
We define $m_\alpha(\xi) := \frac{1 - m(\xi)}{|\xi|^\alpha}$. Since $|1 - m(\xi)| \lesssim |\xi|$ due to $m(0) = 1$ and the mean value theorem, $m_\alpha$ is well defined and $|m_\alpha(\xi)| \lesssim (1 + |\xi|)^{-\alpha}$. Also, due to $|\partial^\gamma m(\xi)| \lesssim (1 + |\xi|)^{-\beta}$ and $|1 - m(\xi)| \lesssim |\xi|$, it follows that for $|\xi| \leq 1$
\[ |\partial^\gamma m_\alpha(\xi)| \lesssim |\xi|^{1-|\gamma|+1} \]
Then for $|\xi| \leq 1$, we have for any $s > 0$
\[ \sum_{j \leq 0} \| m_\alpha(2^j \cdot) \hat{\psi}(\cdot) \|^2_{L^p_0(\mathbb{R}^d)} \leq \sum_{j \leq 0} 2^{-2j(\alpha-1)} < \infty. \]
For $|\xi| \geq 1$, it follows that for $1/p_0 = |1/p - 1/2|$ and $s > d|1/p - 1/2| + 1/2$ as in (1.9)
\[ \| m_\alpha(2^j \cdot) \hat{\psi}(\cdot) \|^2_{L^p_0(\mathbb{R}^d)} \lesssim 2^{-j\alpha} (1 + \| m(2^j \cdot) \hat{\psi}(\cdot) \|^2_{L^p_0(\mathbb{R}^d)}) \]
\[ \lesssim 2^{-j\alpha} + 2^{-j\alpha} 2^{-j(\beta-s)} \leq 2^{-j\alpha} \min(a, a+\beta-s). \]
Thus we can conclude that $m_\alpha$ is a Fourier multiplier of type $m_{\alpha, \alpha+\beta}$ given in (1.9). Therefore, Proposition 1.7 can be proved by Corollary 1.6.

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