LIE ALGEBRA GENERATED BY LOCALLY NILPOTENT DERIVATIONS ON DANIELEWSKI SURFACES

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Abstract. We give a full description of the Lie algebra generated by locally nilpotent derivations (short LNDs) on smooth Danielewski surfaces \( D_p \) given by \( xy = p(z) \). In case \( \deg(p) \geq 3 \) it turns out to be not the whole Lie algebra \( V\text{F}_\omega_{alg}(D_p) \) of volume preserving algebraic vector fields, thus answering a question posed by Lind and the first author. Also we show algebraic volume density property (short AVDP) for a certain homology plane, a homogeneous space of the form \( SL_2(\mathbb{C})/N \), where \( N \) is the normalizer of the maximal torus and another related example. At the end of the paper we show by example that for the group of holomorphic automorphisms of a Stein manifold (endowed with \( c.o. \) topology) the connected component and the path-connected component of the identity may not coincide.

1. Introduction

In this paper we study (using algebraic methods) the holomorphic automorphism group \( \text{Aut}_{hol}(D_p) \) of Danielewski surfaces of the form \( D_p = \{ xy = p(z) \} \). These surfaces are an object of intensive studies in affine algebraic geometry, see e.g. \cite{4}, \cite{5}, \cite{6}, \cite{7}, \cite{8}, \cite{9}, \cite{10}, \cite{12}, \cite{20}, \cite{21} and \cite{22}.

The study of these surfaces from the complex analytic point of view started in the paper of Kaliman and Kutzschebauch \cite{13}, where they proved the so called density property, or for short DP. This is a remarkable property, discovered in 1990’s by Andersén and Lempert \cite{1}, \cite{2} for Euclidean spaces, that to a great extend compensates for the lack of partition of unity for holomorphic automorphisms. The terminology was was introduced later by Varolin \cite{25}: A Stein manifold \( X \) has DP if the Lie algebra generated by completely integrable holomorphic vector fields is dense (in the compact-open topology) in the space of all holomorphic vector fields on \( X \). In the presence of DP one can construct global holomorphic automorphisms of \( X \) with prescribed local properties. More precisely, any local phase flow on a Runge domain in \( X \) can be approximated by global automorphisms. Needless to say that this lead to remarkable consequences (see surveys \cite{15}, \cite{23}).

If \( X \) is equipped with a holomorphic volume form \( \omega \) (i.e. \( \omega \) is a nowhere vanishing top holomorphic differential form) then one can ask whether a similar approximation holds for automorphisms and phase flows preserving \( \omega \), so called volume preserving automorphisms. Under a mild additional assumption the answer is yes in the presence of the volume density property (VDP) which means that the Lie algebra generated by completely integrable holomorphic vector fields of \( \omega \)-divergence zero is dense in the space of all holomorphic vector fields of \( \omega \)-divergence zero. Danielewski surfaces carry a unique nondegenerate algebraic 2-form \( \omega \) and we will concentrate on the group \( \text{Aut}_{\omega}^{\text{hol}}(D_p) \) of volume preserving holomorphic automorphisms.

The following definitions are due to Varolin and Kaliman, Kutzschebauch
Definition 1.1. We say that $X$ has the algebraic volume density property (ADP) if the Lie algebra $\text{Lie}_{\text{alg}}(X)$ generated by the set $\text{IVF}(X)$ of completely integrable algebraic vector fields coincide with the space $\text{AVF}(X)$ of all algebraic vector fields on $X$. Similarly in the presence of $\omega$ we can speak about the algebraic volume density property (AVDP) that means the equality $\text{Lie}_{\omega}(X) = \text{AVF}_{\omega}(X)$ for analogous objects (that is, all participating vector fields have $\omega$-divergence zero; say $\text{Lie}_{\omega}(X)$ is generated by $\text{IVF}_{\omega}(X)$).

It is worth mentioning that ADP and AVDP imply DP and VDP respectively (where the second implication is not that obvious) and in particular all remarkable consequences for complex analysis on $X$.

The study of holomorphic automorphisms of Danielewski surfaces was continued by Lind, and the first author in [19] where shear and overshear automorphisms were introduced, generalizing this notion introduced by Rudin and Rosay from Euclidean spaces to Danielewski surfaces. Shears are volume preserving automorphisms whereas overshears are not. Note that the algebraic shear vector fields are (up to coordinate change) exactly the LNDs (see theorem 2.15). Generalizing the results of Andersén and Lempert it was proved in [19] that

On a Danielewski surface the group generated by shears and overshears is dense in the path connected component of the group $\text{Aut}_{\text{hol}}(D_p)$ of holomorphic automorphisms with respect to the compact-open topology.

From the proof of DP in [13] it follows that the group generated by shears, overshears and hyperbolic automorphisms is dense in $\text{Aut}_{\text{hol}}(D_p)$. The point in the above result was not to use hyperbolic automorphisms. The corresponding generalization of Andersén and Lempert’s result in the volume preserving case, namely the question whether the group generated by shears is dense in the group $\text{Aut}_{\omega}(D_p)$ of volume preserving holomorphic automorphisms with respect to the compact-open topology, remained an unsolved question (see [19] problem 5.1).

In the present paper we solve the ”infinitesimal version” of this question to the negative. Namely we prove that the algebraic shear vector fields do not generate the Lie algebra $\text{VF}_{\omega}(D_p)$ of algebraic volume preserving vector fields if the degree of the defining polynomial $p$ is at least 3. More precisely we prove the following statement:

**Corollary** (see Corollary 3.15). For $p \in \mathbb{C}[z]$ with degree $n \geq 3$ the Lie algebra generated by holomorphic shear fields is not dense in the Lie algebra of holomorphic volume preserving vector fields.

If the degree is 2 or 1 we prove that the algebraic shear vector fields do generate the Lie algebra $\text{VF}_{\omega}(D_p)$ of algebraic volume preserving vector fields. If the degree is 1, the Danielewski surface is biholomorphic to $\mathbb{C}^2$ and we recover exactly the Andersén-Lempert result. Our main result is

**Theorem** (see Theorem 3.26). A volume preserving vector field $\Theta$ on the Danielewski surface $D_p$ is a Lie combination of LNDs if and only if its corresponding function with $i_\omega \Theta = df$ is of the form (modulo constant)

$$f(x, y, z) = \sum_{i=1}^{k} a_{ij} x^i z^j + \sum_{i=1}^{t} b_{ij} y^i z^j + (pq)'(z)$$

for a polynomial $q \in \mathbb{C}[z]$.

In the "positive" cases of degree 1 and 2 the proof of the main theorem of Andersén-Lempert theory implies the density of the group generated by shears in the (path connected component of the) group $\text{Aut}_{\omega}(D_p)$ of volume preserving
holomorphic automorphisms, whereas in the "negative" cases degree \( \geq 3 \) we cannot conclude that the the group generated by shears is not dense in the group \( \text{Aut}_{\text{hol}}^\omega(D_p) \) of volume preserving holomorphic automorphisms. Here we are lacking a quantity attached to an automorphism which is zero for all shear automorphisms but nonzero for the hyperbolic automorphisms \( H_f \) whose function \( f \) is not the second derivative of a function divisible by the defining polynomial \( p \).

The results of our paper are also interesting in connection with the following open problem formulated in [3]:

Does a flexible affine algebraic manifold equipped with an algebraic volume form have the algebraic volume density property?

Remember that an affine algebraic manifold is called flexible if the LNDs on it generate the tangent space at every point. By Proposition 2.5 this is true for \( D_p \).

Even thought \( D_p \) has the volume density property the Lie algebra generated by LNDs in not the Lie algebra \( \text{VF}_{\text{alg}}^\omega(D_p) \). The additional hyperbolic fields (algebraic \( \mathbb{C}^* \)-actions) are needed to get all of \( \text{VF}_{\text{alg}}^\omega(D_p) \). Thus we do not have a counterexample to the above problem, but near to a counterexample: We have an example where the LNDs span the tangent space at each point and at the same time do not generate the Lie algebra of volume preserving algebraic vector fields.

The paper is organized as follows. In section 2 we recall some known facts for Danielewski surfaces and give certain proofs in order to make the paper self contained. We believe that some of these proofs are new.

In section 3 we explain how volume preserving vector fields can be related to functions on the Danielewski surface and how this relation works with respect to Lie bracket. This is a new method, which is afterwards used to prove our main result, the characterization of the Lie algebra generated by LNDs on Danielewski surfaces.

On the way we use our method based on the duality between volume preserving vector fields and functions to prove (version of) the algebraic volume density property for \( D = \text{Sl}_2(\mathbb{C})/N \), where \( N \) is the normalizer of the maximal torus \( N \cong \mathbb{C}^* \times \mathbb{Z}_2 \). The importance of this lies in the fact that the methods (compatible pairs of globally integrable fields) for proving AVDP recently developed by KALIMAN and the first author do not work for this particular homogeneous space as explained in [10]. Also we prove AVDP for \( (D \times \mathbb{C}^*)/\mathbb{Z}_2 \) where \( \mathbb{Z}_2 \) acts diagonally. This is a good exercise, since the proof given in [16] is using very abstract methods. Comparing our calculations to that proof let one feel the strength of the method of semi-compatible vector fields developed in [16]. We end our paper with an example (not related to Danielewski surfaces) which shows that the connected component and the path connected component of the holomorphic automorphism group of a manifold (endowed with compact open topology) may differ, thus answering a question posed in [19].

2. Danielewski Surface

Let \( p \in \mathbb{C}[z] \) be a polynomial with simple zeros. The variety given by \( D_p = \{(x, y, z) \in \mathbb{C}^3 \mid xy = p(z)\} \) is called Danielewski surface. Since \( p \) has only simple zeros \( D_p \) is the preimage of a regular value and hence a complex manifold. Often it is useful to work in one of the two charts \( \mathbb{C}^* \times \mathbb{C} \to D_p : (x, z) \mapsto (x, \frac{p(z)}{y}, z) \) and \( (y, z) \mapsto (\frac{p(z)}{y}, y, z) \), which cover all points of \( D_p \) with \( x \neq 0 \) respective \( y \neq 0 \). An important fact is that every regular function \( f \in \mathbb{C}[D_p] \) can be written uniquely as

\[
f(x, y, z) = \sum_{i=0}^{k} a_{ij} x^i y^j z^l + \sum_{j=0}^{l} b_{ij} y^j z^l + \sum_{i=0}^{m} c_{ij} z^j
\]  

(1)
by substituting $xy = p(z)$ successively. As proven in [14] there is an algebraic
volume form $\omega$ on $D_p$, which is unique up to a constant. In the local charts
from before it is given by $\omega = \frac{dx}{y} \wedge dz$ and $\omega = -\frac{dy}{x} \wedge dz$ respectively. Here comes the
first well-known fact.

**Proposition 2.1.** The Danielewski surfaces $D_p$ are simply connected and we have
$H^2(D_p, \mathbb{C}) \cong \mathbb{C}^{\deg(p)-1}$.

**Proof.** It is possible to construct a strong deformation retraction onto a bouquet
of $(\deg(p) - 1)$ 2-spheres connecting the zeros of $p$. First choose a smooth curve
$\gamma : [0,1] \to \mathbb{C}_z \subset D_p$ in the z-plane connecting the zeros of $p$ and then retract
$D_p$ onto the spheres around the segments of the path between the zeros. Let
$\rho_t : [0,1] \times \mathbb{C}_z \to \mathbb{C}_z$ be a strong deformation retraction onto $\gamma$. We use this
retraction to define the strong deformation retraction

$$R_t : D_p \to \{(x, y, z) \in D_p : z \in \gamma\} : (x, y, z) \mapsto \left(\frac{p(\rho_t(z))}{p(z)} x, y, \rho_t(z)\right).$$

Additionally we define a strong deformation retraction $H_t$ from $\{(x, y, z) \in D_p : z \in \gamma\}$ onto a bouquet of 2-spheres.

$$H_t(x, y, z) := \left(\frac{p(z)}{t|p(z)|^{1/2} \frac{1}{|p(z)|} + (1-t)y}, t|p(z)|^{1/2} \frac{y}{|y|} + (1-t)y, z\right)$$

for $p(z) \neq 0$ and $|y| \geq |p(z)|^{1/2}$ and

$$H_t(x, y, z) := \left(\frac{p(z)}{t|p(z)|^{1/2} \frac{x}{|x|} + (1-t)x}, \frac{p(z)}{t|p(z)|^{1/2} \frac{1}{|p(z)|} + (1-t)x}, z\right)$$

for $p(z) \neq 0$ and $|x| \geq |p(z)|^{1/2}$. When $p(z) = 0$ then either $x = 0$ or $y = 0$ (or both). In this case choose

$$H_t(x, y, z) := (0, (1-t)y, z) \quad \text{or} \quad H_t(x, y, z) := ((1-t)x, 0, z).$$

The composition of $R_t$ and $H_t$ is the desired strong deformation retraction from $D_p$
to the bouquet of 2-spheres, therefore $D_p$ is simply connected and has $H^2(D_p, \mathbb{C}) \cong \mathbb{C}^{\deg(p)-1}$.

\[\square\]

2.1. **Vector fields on the Danielewski surface.** Let us begin with two equivalent
definitions of locally nilpotent derivations:

**Definition 2.2.** A globally integrable vector field $\Theta$ is a *locally nilpotent derivation
(LND)* iff its flow $\psi^t$ is an algebraic $\mathbb{C}^+$-action, i.e. $t \mapsto \psi^t$ is an algebraic map.
Equivalently a vector field $\Theta$ is a LND whenever for all $f \in \mathbb{C}[D_p]$ there is an integer
$N$ such that $\Theta^N(f) = \Theta \circ \ldots \circ \Theta(f) = 0$. For the equivalence of these definitions see [11] p.31. The subgroup of $\text{Aut}_{\text{alg}}(D_p)$ generated by flows from LNDs is called the *special automorphism group* $\text{SAut}_{\text{alg}}(D_p)$.

**Definition 2.3.** The algebraic vector fields of the Danielewski surface $D_p$

$$SF^x_i := p'(z)x^i \frac{\partial}{\partial y} + x^{i+1} \frac{\partial}{\partial z},$$

$$SF^y_i := p'(z)y^i \frac{\partial}{\partial x} + y^{i+1} \frac{\partial}{\partial z}$$

are called *shear fields* for all $i \in \mathbb{N}_0$ and the vector fields

$$HF_f := f(z)\left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\right)$$

are called *hyperbolic fields* for all $f \in \mathbb{C}[z]$. 
The vector fields above are globally integrable and volume preserving, their flows are:

\[ \phi_1^i : (x, y, z) \mapsto (x, \frac{p(z + tx^{i+1})}{x}, z + tx^{i+1}), \]

\[ \phi_2^i : (x, y, z) \mapsto (\frac{p(z + ty^{i+1})}{y}, y, z + ty^{i+1}), \]

\[ \phi_3^i : (x, y, z) \mapsto (e^{tf(z)}x, e^{-tf(z)}y, z). \]

Note that \( p(\frac{z + tx^{i+1}}{x}) = p(z) + tx^i(...) = y + tx^i(...). \) This shows that the shear fields are locally nilpotent derivations and the hyperbolic fields are not. For \( t = 1 \) this automorphisms are called \( x \)-resp \( y \)-shear automorphisms (short: shears) respectively hyperbolic automorphisms.

Recall the following definition from [3].

**Definition 2.4.** \( M \) is said to be flexible iff the LND-vector fields span the tangent space in all points of \( M \). For properties of flexible manifolds see [3].

**Proposition 2.5** ([17]). The Danielewski surface is flexible.

**Proof.** The two following LND-vector fields span the tangent space in every point of \( D_p \) where \( p'(z) \neq 0 \).

\[ p'(z) \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad p'(z) \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}. \]

For the points with \( p'(z) = 0 \) we add the following vector fields. Let \( \alpha_k(x, y, z) = (x, \frac{p(z-kx)}{x}, z-kx) \) then the with \( \alpha_k \) conjugated shear fields do the job (see the remark below)

\[ \alpha_k^i(SF^{y}_0) = p'(z+kx) \frac{\partial}{\partial x} + \frac{p(z+kx)p'(z) - p'(z+kx)p(z) - kxp(z+kx)p'(z) \frac{\partial}{\partial y}}{x^2} + \left( -kp'(z+kx) + \frac{p(z+kx)}{x} \right) \frac{\partial}{\partial z}. \]

Assume \( p' \) has \( n \) zeros, then the fields \( \alpha_k^i(SF^y_0) \) for \( k = 1, \ldots, n \) together with the two shear fields from above will span the tangent space at any point.

**Remark 2.6.** Given a vector field \( \Theta \) and a holomorphic automorphism \( \phi : M \to M \) then the vector field conjugated by \( \phi \) is given by \( (\phi^*\Theta)_p := ((D\phi^{-1})\Theta)_{\phi(p)}. \)

The vector field \( \phi^*\Theta \) is globally integrable whenever \( \Theta \) is it. Its flow is \( \phi^*\psi \) where \( \psi \) is the flow of \( \Theta. \) In particular a LND conjugated by an algebraic automorphism is an LND again. The interior product for a k-form \( \omega \) is \( i_{(\phi^*\Theta)}\omega = \phi^*(i_{\Theta}(\phi^{-1}\omega))) \), in particular when \( \omega \) is invariant under \( \phi \) then \( i_{(\phi^*\Theta)}\omega = \phi^*(i_{\Theta}\omega). \)

2.2. The (Special) Automorphism Group. The goal of this subsection is to see that the LNDs are exactly the shear fields and shear fields conjugated with shear automorphisms. This result is not new [22], in order to make the paper self contained we give a proof (which to our knowledge is new).

We begin with the description of the algebraic automorphism group \( \text{Aut}_{\text{alg}}(D_p). \) The following theorem is due to Makar-Limanov, he stated it in the end of the paper [20] without proving it.
Theorem 2.7 ([20]). Let $\text{deg}(p) \geq 3$ and let $p$ be generic in the following sense: No affine automorphism $\alpha$ of $\mathbb{C}$ permutes the roots of $p$. Then the group of all algebraic automorphisms is $\text{Aut}_{\text{alg}}(D_p) = G_0 \rtimes (H \rtimes J)$ where $G_0 = G_x \ast G_y$ is the free product of the subgroups $G_x$ (resp. $G_y$) generated by the $x$- (resp. $y$-) shear automorphisms, $H$ is the subgroup of algebraic hyperbolic automorphisms and $J$ is the subgroup consisting of the identity and $I(x,y,z) = (y,x,z)$ is the involution.

In the non generic case denote by $\Gamma$ the group of affine automorphisms of $\mathbb{C}$ permuting the roots of $p$, i.e. $p \circ \gamma = a_0 p(z)$, where $a_0$ is a root of unity (depending on $\gamma$). $\Gamma$ induces a group of automorphisms of $D_p$, which we denote by $\tilde{\Gamma}$. In this case we denote by $J$ the group generated by $\tilde{\Gamma}$ and $I$, then we have again $\text{Aut}_{\text{alg}}(D_p) = G_0 \rtimes (H \rtimes J)$ with $G_0$ and $H$ as above.

We will give a proof using the following main theorem in [20].

Theorem 2.8 ([20]). Let $\text{deg}(p) \geq 3$ and let $p$ be generic as above, then the group of all algebraic automorphisms of $D_p$, is generated by the following automorphisms:

$x$-shears: $\Delta_f(x,y,z) = \left( x, \frac{2(x + x f(x))}{x}, z + x f(x) \right)$ for $f \in \mathbb{C}[z]$

Hyperbolic rotations: $H_{\lambda}(x,y,z) = (\lambda x, \lambda^{-1} y, z)$ for $\lambda \in \mathbb{C}^*$

Involution: $I(x,y,z) = (y,x,z)$.

Note that $y$-shears are exactly the automorphisms of the form $I \Delta f I$.

In the non generic case or if $\text{deg}(p) = 2$ one has to add (the finite group) $\tilde{\Gamma}$ of automorphisms coming from symmetries of $p$:

$\gamma(x,y,z) = (x, a_0 y, \gamma(z))$, where $\gamma(z) = a_0 z + b$ is such that $p \circ \gamma(z) = a_0 p(z)$

Proof. see [20] \qed

Lemma 2.9. For $\text{deg}(p) \geq 3$ a nontrivial composition of $x$- and $y$-shears will never have a $z$-coordinate of the form $az + b$.

Proof. Since composition of $x$- (resp. $y$-) shears are $x$- (resp. $y$-) shears again, $G_x$ and $G_y$ are subgroups and we can assume that the composition is written in a reduced way (i.e. alternating $x$- and $y$-shears). For instance take an element $\Delta_{f_1} \Delta_{f_2} \cdots \Delta_{f_n} \Delta_{f_1}$ (the letter $\{x,y\}$ denotes whether it is a $x$- or a $y$-shear). Denote the image of $(x,y,z) = (x_0, y_0, z_0)$ after the first $i$ shears by $(x_i, y_i, z_i)$ e.g. for $i$ odd we get

$$x_i = x_{i-1}, \quad y_i = \frac{p(z_i - 1 + x_{i-1}f_1(x_{i-1}))}{x_{i-1}} \quad \text{and} \quad z_i = z_{i-1} + x_{i-1}f_1(x_{i-1}).$$

Since $y = \frac{p(x)}{x}$ we can see the elements $x_i, y_i, z_i$ as unique elements in $\mathbb{C}[x, x^{-1}, z]$ and therefore it makes sense to speak of the $x$-degree of such an element. The goal is to see that $z_i$ has a strictly positive $x$-degree for $i > 0$ and is therefore not of the form $az + b$. After the first shear $z_1 = z + x f_1(x)$ is obviously of positive $x$-degree, more precisely it has degree $\text{deg}(f_1) + 1$. Composing inductively with the proceeding shear automorphism a term $x_i f_{i+1}(x_i)$ or $y_i f_{i+1}(y_i)$ will be added. If we can see that the $x$-degree of a such term is always bigger than all previous ones then the claim is proven. Indeed the $x$-degree of $y_i f_{i+1}(y_i)$ is $\text{deg}(y_i) (\text{deg}(f_{i+1}) + 1) \geq \text{deg}(y_i) = \text{deg}(p) (\text{deg}(z_i - 1 + x_{i-1}f_1(x_{i-1})) - \text{deg}(x_{i-1}) \text{ which is by induction $\text{deg}(p) \text{deg}(x_{i-1}) (\text{deg}(f_1) + 1) - \text{deg}(x_{i-1}) = \text{deg}(x_{i-1}) (\text{deg}(p) (\text{deg}(f_1) + 1) - 1 > \text{deg}(x_{i-1}) (\text{deg}(f_1) + 1) = \text{deg}(x_{i-1} f_1(x_{i-1})).$} The inequality from the second last step follows from the fact that $\text{deg}(p) \geq 3$. The same calculation holds for $x_i f_{i+1}(x_i)$. And if our arbitrary elements starts with a $y$-shear then of course the same calculation holds when we exchange $x$ and $y$. \qed
Proof of theorem 2.7. To see that $\text{Aut}_{\text{alg}}(D_p) = G_0 \times (H \rtimes J)$ in the generic case it is necessary to verify several things. First we see that $\text{Aut}_{\text{alg}}(D_p) = G_0 \rtimes H_0$ where $G_0$ is the group generated by automorphisms of the form $\Delta_f$ and $I\Delta_f I$ and $H_0$ is generated by automorphisms $H_\lambda$ and $I$. $G_0$ is indeed normal since $I\Delta_f I$ and $IH_\lambda I = H_\lambda$ and $H^{-1}\Delta_f H_\lambda = \Delta_{\lambda f(\lambda)} \lambda \in G_0$. Since $IH_\lambda = H_\lambda^{-1} I$ we have $h^{-1}gh \in G_0$ for all elements $h \in H_0$ and $g \in G_0$. By the theorem above it is clear that $G_0$ and $H_0$ generate $\text{Aut}_{\text{alg}}(D_p)$ so the last thing to check is that the intersection is trivial. We observe that all elements of $H_0$ fix the $z$-coordinate but no nontrivial element from $G_0$ does by the previous lemma. Take a look at the surjective homomorphism $G_x \times G_y \to G_0$ sending a word to its interpretation in the group, to see that it is injective it is sufficient to see that the identity map can’t be written as a nontrivial composition of shear automorphisms, but this is clear since a nontrivial composition of shears never fixes the third component. To finish the proof we have to see that $H_0$, the subgroup generated by hyperbolic rotations and the involution is $H \rtimes J$, but this is clear since $IH_\lambda I = H_\lambda^{-1} I$ and therefore the subgroup $H$ generated by hyperbolic rotations is normal and $I$ is orientation reversing and therefore not part of the hyperbolic rotations. The statement in the non generic case is easy to see as well.

Here are some consequences of the theorem, remember that all of them hold just for $\deg(p) \geq 3$.

Remark 2.10. In the generic case the group of algebraic volume preserving automorphisms is therefore $\text{Aut}_{\text{alg}}(D_p) = G_0 \rtimes H$, since shears and hyperbolic automorphisms are volume preserving and the involution is volume reversing. The (non trivial) elements of $\Gamma$ from the non generic case multiply the volume form by a (non zero) root of unity, so the group can be bigger since it is possible to get an order two volume preserving automorphism of the form $I \circ \gamma$ with $\gamma \in \Gamma$. In this case the group of volume algebraic volume preserving automorphisms is $G_0 \rtimes (H \rtimes \mathbb{Z}_2)$.

Proposition 2.11. The group of special automorphisms $\text{SAut}_{\text{alg}}(D_p)$ (i.e. the group generated all algebraic $\mathbb{C}^+$-actions) is the group $G_\gamma \cong G_x \times G_y$ generated by the shear automorphisms.

Proof. Take any algebraic one parameter subgroup $\psi : \mathbb{C} \to \text{Aut}_{\text{alg}}(D_p)$. Since we have the projection homomorphism $\text{Aut}_{\text{alg}}(D_p) = G_0 \times (H \rtimes J) \to H \rtimes J$ we get an induced algebraic one parameter subgroup on $H \rtimes J$ and hence on its connected component $H$ the subgroup of hyperbolic rotations, but this subgroup has to be trivial since one parameter subgroups in $H$ can never be algebraic $\mathbb{C}^+$ action. Hence $\psi$ has its image in the shear automorphisms.

Lemma 2.12. A smooth one parameter subgroup $\psi : \mathbb{C} \to G_x \times G_y$ is conjugated to a one parameter subgroup $\psi^i$ either in $G_x$ or in $G_y$.

In order to prove this lemma we need some facts about free groups. Recall that for two groups $G$ and $H$ any element $g$ in $G \rtimes H$ has a unique reduced form with length denoted by $l(g)$.

Theorem 2.13. A subgroup $K$ of $G \rtimes H$ is conjugated to a subgroup in either $G$ or $H$ if and only if $\sup(l(k); k \in K) < \infty$.

Proof. See [21] theorem 8 p.36.

The following lemma is well known, see e.g. [18], for making the paper more self contained we give the proof.

Lemma 2.14. (1) Every element in $G \rtimes H$ is conjugated either to an element in either $G$ or $H$ or to an element of even length $> 0$. 

(2) Two commuting elements of $G \ast H$ with length $> 0$ have either both even or both odd length.

Proof. (1) Whenever an element has odd length its first and last letter belongs to the same group then after conjugating with the inverse of one of those letters it is either of even length or the length descends by 2 and we can proceed by induction.
(2) Take an element $a$ with even length $n$ and an element $b$ with odd length $m$, then either $l(ab) = m + n$ and $l(ba) < m + n$ or $l(ab) < m + n$ and $l(ba) = m + n$ and hence they cannot commute. \qed

Proof of lemma \ref{lem:2.12} First we show that for all $a$ and $b$ there is a unique $G$ such that $G$ is conjugate to a subgroup of $G$ in $G$. Since $a$ and $b$ commute, $G$ is isomorphic to $G \ast G$. Assume that this is not the case, then $a$ and $b$ commute, and therefore $G$ is conjugate to a subgroup of $G$ in $G$. Take any converging sequence of $x$-shears $\Delta f$, $\eta = (\eta_1, \eta_2, \eta_3)$. So we know that $(\eta + f_n(x))_\eta$ converges point-wise hence $f_n(z)$ converges, say to $f(z)$. Then $\eta_1(x, y, z) = x$ and $\eta_2(x, y, z) = z + f(x)$, since $\eta$ is algebraic $f$ is a polynomial and therefore $\eta = \Delta f$ is an $x$-shear. \qed

Theorem 2.15 \cite{[22]}. The LNDs of the Danielewski surface $D_p$ for $\deg(p) \geq 3$ are exactly the shear fields and the shear fields conjugated by compositions of shear automorphisms.

Proof. An algebraic $\mathbb{C}^+$-action $\psi : \mathbb{C} \to \Aut_{\text{alg}}(D_p)$ is by Proposition \ref{prop:2.11} and Lemma \ref{lem:2.12} conjugated to a one parameter subgroup in $G_x$ or $G_y$. \qed

3. Lie Combinations of Shear Fields

In this chapter we will understand which algebraic volume preserving vector fields of the Danielewski surface can be written as a Lie combination of the shear fields \cite{[23]}. The main tool for the description will be the 1-forms $i_\Theta \omega$ for volume preserving vector fields. Recall that the interior product $i_\Theta : \Omega^{k+1}(M) \to \Omega^k(M)$ is given by $i_\Theta \omega(\Theta_1, \ldots, \Theta_k) := \omega(\Theta) \Theta_1, \ldots, \Theta_k$. We will also use the Lie derivative of a differential form $\mu$ with respect to a vector field $\Theta$, which is given by $L_\Theta \mu = \frac{d}{dt} \psi(t \mu) |_{t=0}$ or the Cartan formula $L_\Theta \mu = (d \circ i_\Theta + i_\Theta \circ d)(\mu)$. The formula $i_{\Theta_1, \Theta_2} \mu = L_{\Theta_1}(i_{\Theta_2} \mu) - i_{\Theta_2}(L_{\Theta_1} \mu)$ gives a link between the interior product and the Lie derivative. Another useful formula $L_\Theta d\mu = dL_\Theta \mu$ is a direct consequence of the Cartan formula.
3.1. The Lie algebra generated by shear fields is a proper subalgebra of $\text{VF}_{\text{alg}}^\omega(D_p)$. From now on we will use the one-one correspondence between algebraic volume preserving vector fields and polynomial functions modulo constants on $D_p$. For every volume preserving vector field $\Theta$ holds $L_\omega(\Theta) = di_\Theta \omega + i_\Theta d\omega = 0$. Since $d\omega = 0$ the 1-form $i_\Theta \omega$ is closed and therefore exact (because $D_p$ is simply connected), hence when $\Theta$ is algebraic then $i_\Theta \omega = df$ for some regular $f \in \mathbb{C}[D_p]$. This defines a bijection between algebraic volume preserving vector fields and polynomial functions modulo constants. The following lemma gives the corresponding functions to the shear fields and hyperbolic vector fields.

**Lemma 3.1.** For $i \in \mathbb{N}_0$ holds:

$$i_{SF_x^i} \omega = \frac{dx^{i+1}}{i+1}, \quad i_{SF_y^i} \omega = \frac{dy^{i+1}}{i+1}, \quad i_{HF_z^i} \omega = \frac{dz^{i+1}}{i+1}.$$

**Proof.**

$$i_{SF_x^i} \omega(\Theta) = \omega(SF_x^i, \Theta) = \frac{1}{x} (dx(SF_x^i) d\omega(\Theta) - dx(\Theta) dz(SF_x^i)) = \frac{1}{x} (x^{i+1} dz(\Theta)) = -x^i dx(\Theta) = \frac{dx^{i+1}}{i+1}.$$

$$i_{SF_y^i} \omega(\Theta) = \omega(SF_y^i, \Theta) = \frac{1}{y} (dy(SF_y^i) dz(\Theta) - dy(\Theta) dz(SF_y^i)) = \frac{1}{y} (-y^{i+1} dy(\Theta)) = y^i dy(\Theta) = \frac{dy^{i+1}}{i+1}.$$

$$i_{HF_z^i} \omega(\Theta) = \omega(HF_z^i, \Theta) = \frac{1}{x} (dx(HF_z^i) dz(\Theta) - dx(\Theta) dz(HF_z^i)) = \frac{1}{x} (z^i x dz) = x^i dz(\Theta) = \frac{dz^{i+1}}{i+1}.$$

We need to know how to calculate the Lie bracket on the level of functions. An easy calculation shows the following lemma.

**Lemma 3.2.** Let $\Theta$ be a volume preserving vector field with $i_\Theta \omega = df$ and $\Psi$ another volume preserving vector field, then

$$i_{[\Psi, \Theta]} \omega = L_\Psi (i_\Theta \omega) - i_\Theta (L_\Psi (\omega)) = L_\Psi (df) = dL_\Psi (f).$$

The previous facts allow us to reprove the fact from [16] that $D_p$ has the volume density property.

**Theorem 3.3.** The Danielewski surface $D_p$ with the volume form $\omega$ satisfies the algebraic volume density property, in fact every algebraic volume preserving vector field is a Lie combination of shear fields and hyperbolic fields. Precisely: Every volume preserving vector field is a linear combination of vector fields $SF_x^i$, $SF_y^i$, $HF_z^i$, $[SF_x^i, HF_z^i]$ and $[SF_y^i, HF_z^i]$ for $i \in \mathbb{N}_0$ and polynomials $f \in \mathbb{C}[z]$. 

Proof. We have to find a Lie combination $A$ of shear fields and hyperbolic fields for every polynomial function $f$ on $D_p$ such that $i_A\omega = df$ holds. It is sufficient to find the corresponding Lie combination for the monomials $x^i, y^j, z^i, x^i z^j$ and $y^j z^j$ for all $i, j > 0$, but the first three are already covered by lemma 3.1. The corresponding vector fields of the last two monomials are $[SF_{x^i-1}, HF_{x^j}]$ and $[SF_{y^j-1}, HF_{z^j}]$, indeed:

$$i_{[SF_{x^i-1}, HF_{x^j}]}\omega = dL_{SF_{x^i-1}}(\frac{z^{j+1}}{j+1}) = dx^i \frac{1}{j+1}(j+1)z^j = dx^i z^j$$

A similarly calculation shows $i_{[SF_{y^j-1}, HF_{z^j}]}\omega = dy^j z^j$. \hfill $\square$

Now we have developed the method to show AVDP for the cases mentioned in the introduction. Let $D$ be the quotient of $SL_2(\mathbb{C})$ by the normalizer of the maximal torus. Consider $G = SL_2(\mathbb{C})$ as a subvariety of $\mathbb{C}^4_{a_1, a_2, b_1, b_2}$ given by $a_1 b_1 - a_2 b_1 = 1$, i.e. matrices

$$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

are elements of $G$. Let $T \simeq \mathbb{C}^*$ be the torus consisting of the diagonal elements and $N$ be the normalizer of $T$ in $SL_2$. That is, $N/T \simeq \mathbb{Z}_2$ where the matrix

$$A_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in N$$

generates the nontrivial coset of $N/T$.

Lemma 3.4. The variety $D = G/T$ is isomorphic to the hypersurface $xy = z^2 - 1$ in $\mathbb{C}_{x,y,z}$ such that the $\mathbb{Z}_2$-action is given by $(x, y, z) \rightarrow (-x, -y, -z)$.

Proof. Note that the ring of $T$-invariant regular functions on $G$ is generated by $x = a_1 b_1, y = a_2 b_2, v = a_1 b_2$, and $z = a_2 b_1$ where $v = z + 1$. Hence $X$ is isomorphic to the hypersurface $xy = z(z + 1)$ in $\mathbb{C}^3_{x,y,z}$. After a linear isomorphism of $\mathbb{C}^3$ we get the desired form. The formula for the $\mathbb{Z}_2$-action (induced by multiplication by $A_0$) is also a straightforward computation. \hfill $\square$

Definition 3.5. Let $X$ be an affine algebraic manifold equipped with an algebraic volume form $\omega$. Suppose a finite group $\Gamma$ acts freely and algebraically on $X$. We say that $X$ has the $\Gamma$-AVDP if the Lie algebra generated by $\Gamma$-invariant completely integrable volume preserving algebraic vector fields on $X$ is equal to the Lie algebra of all $\Gamma$-invariant volume preserving algebraic vector fields on $X$.

Theorem 3.6. The Danielewski surface $D$ has $\mathbb{Z}_2$-AVDP.

Proof. We proceed as in the proof of the previous theorem. The volume form $\omega$ is $\mathbb{Z}_2$ anti-invariant, i.e., $\sigma^*\omega = -\omega$. Thus using the invariant globally integrable fields $SF_{2n}, SF_{y}^{2n}, HF_{2n}$ $n \geq 0$ we have to produce all anti-invariant monomials $x^i$, $y^j$, $z^i$ for odd $i$ and $z^i x^j$, $z^i y^j$ for $i, j \geq 1, i + j \geq 3$ and odd. The first three are again covered by lemma 3.1 for even $i$.

For the other monomials we have to use the exact form of the the defining polynomial $p(z) = z^2 - 1$. We obtain the monomials $z^i x^j$ by induction on $i$. The monomials $z^i y^j$ are then obtained analogously.

Starting the induction with $i = 1$ consider

$$i_{[SF_{y}^{2n}, SF_{x^j}]}\omega = dSF_{y}^{2n}(\frac{x^{2k+1}}{2k+1}) = d\left((2z \frac{\partial}{\partial x} + y \frac{\partial}{\partial z})(-\frac{x^{2k+1}}{2k+1})\right) = -2dx z^{2k+1}$$

Suppose by induction hypothesis that all monomials $z^m x^n, m + n$ odd for $m \leq i$ are obtained. To produce a monomial $z^{i+1} x^j$ use the Lie bracket of $SF_{y}^{2n}$ with the field corresponding to the monomial $z^{i+1} x^j$ (which by induction hypothesis is obtained). We obtain the polynomial
theorem states that we have seen above. For a vector field $\Theta$ we again look at the corresponding form $i\Theta \omega$.

Lemma 3.8. The vector space (instead of Lie algebra) spanned by globally integrable $\mathbb{Z}_2$-invariant algebraic vector fields on $D$ is equal to all $\mathbb{Z}_2$ invariant algebraic vector fields. Also the vector space spanned by globally integrable $\mathbb{Z}_2$-anti-invariant algebraic vector fields on $D$ is equal to all $\mathbb{Z}_2$ anti-invariant algebraic vector fields.

This follows from the fact that in the above proof one uses Lie brackets of LNDs and maximally one other (hyperbolic) globally integrable field and the following general fact which holds on any affine algebraic manifold.

Remark 3.7. The vector space (instead of Lie algebra) spanned by globally integrable $\mathbb{Z}_2$-invariant algebraic vector fields on $D$ is equal to all $\mathbb{Z}_2$ invariant algebraic vector fields. Also the vector space spanned by globally integrable $\mathbb{Z}_2$-anti-invariant algebraic vector fields on $D$ is equal to all $\mathbb{Z}_2$ anti-invariant algebraic vector fields.

This follows from the fact that in the above proof one uses Lie brackets of LNDs and maximally one other (hyperbolic) globally integrable field and the following general fact which holds on any affine algebraic manifold.

Lemma 3.8. If $\Theta$ is an LND and $\Psi$ a finite sum of globally integrable algebraic vector field, then the Lie bracket $[\Theta, \Psi]$ is contained in the span of globally integrable algebraic vector fields. In particular the vector space spanned by LNDs is equal to the Lie algebra generated by LNDs.

Proof. Let $\phi_t$ denote the flow of $\Theta$ (which is an algebraic $\mathbb{C}$-action). Then the set $A = \{(\phi_t)^*(\Psi)\}$ is contained in a finite dimensional subspace of AVF and thus its span is closed (see Lemma 3.25). Since global integrability is preserved when applying an automorphisms, all fields in $A$ are in the span of globally integrable fields. Moreover the definition

$$[\Theta, \Psi] = \lim_{t \to 0} \frac{1}{t} (\phi_t)^*(\Psi) - \Psi$$

shows that the bracket $[\Theta, \Psi]$ is in the closure of the span of $A$, thus in the span.

Now the other example: Let $X = D \times \mathbb{C}^*$ equipped with the volume form $\omega_0 = \omega \times \frac{d\theta}{\theta}$ and $\mathbb{Z}_2$-action generated by $(x, y, z, \theta) \mapsto (-x, -y, -z, -\theta)$. The next theorem states that $X$ has $\mathbb{Z}_2$-AVDP, the proof technique is very close the technique we have seen above. For a vector field $\Theta$ we again look at the corresponding form $i\Theta \omega_0$ which is in this situation an anti-invariant closed 2-form. In order to find all those forms we need to find all anti-invariant exact 2-forms and additionally for each cohomology class one representative.

Theorem 3.9 (16). The manifold $X$ has $\mathbb{Z}_2$-AVDP.

Proof. The volume form $\omega_0$ is anti-invariant. We wish to find all anti-invariant closed 2-forms $\alpha$ on $X$ as $i_\chi \omega_0$ where $\chi$ is a Lie combination of invariant completely integrable fields on $X$. By Proposition 2.1 $H^2(D, \mathbb{C}) = \mathbb{C}$ and it is easy to check that the volume form $\omega$ represents the nontrivial class. By Künneth formula and $H^1(D, \mathbb{C}) = 0$ we have that $H^2(X, \mathbb{C})$ is isomorphic to $\mathbb{C}$ and $\omega$ (considered as a 2-form on $X$) is a generator. Remark that $\omega = i_\theta \omega_0$. Thus subtracting the completely integrable volume preserving invariant field $\theta \frac{d\theta}{\theta}$ from a given field $\chi$ we can assume that the form $\alpha$ is exact. It remains to construct all anti-invariant

\[ (2 \frac{\partial}{\partial x} + \frac{\partial}{\partial z})(z^i x^j z^{i+1}) = 2 z^{i+1} (j+1) x^j + iz^i-1 y x^{j+1} = 2 z^{i+1} (j+1) x^j + iz^i-1 (z^2-1) x^j = (2j + 2 + i) z^{i+1} x^j - iz^i-1 x^j \]

The monomial $z^{i-1} x^j$ is already obtained by induction hypothesis, thus the induction step is completed.

We do not get constant functions, they are not needed since they correspond to the zero field.

\[ \square \]

Remark 3.7. The vector space (instead of Lie algebra) spanned by globally integrable $\mathbb{Z}_2$-invariant algebraic vector fields on $D$ is equal to all $\mathbb{Z}_2$ invariant algebraic vector fields. Also the vector space spanned by globally integrable $\mathbb{Z}_2$-anti-invariant algebraic vector fields on $D$ is equal to all $\mathbb{Z}_2$ anti-invariant algebraic vector fields.

This follows from the fact that in the above proof one uses Lie brackets of LNDs and maximally one other (hyperbolic) globally integrable field and the following general fact which holds on any affine algebraic manifold.

Lemma 3.8. If $\Theta$ is an LND and $\Psi$ a finite sum of globally integrable algebraic vector field, then the Lie bracket $[\Theta, \Psi]$ is contained in the span of globally integrable algebraic vector fields. In particular the vector space spanned by LNDs is equal to the Lie algebra generated by LNDs.

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Theorem 3.9 (16). The manifold $X$ has $\mathbb{Z}_2$-AVDP.

Proof. The volume form $\omega_0$ is anti-invariant. We wish to find all anti-invariant closed 2-forms $\alpha$ on $X$ as $i_\chi \omega_0$ where $\chi$ is a Lie combination of invariant completely integrable fields on $X$. By Proposition 2.1 $H^2(D, \mathbb{C}) = \mathbb{C}$ and it is easy to check that the volume form $\omega$ represents the nontrivial class. By Künneth formula and $H^1(D, \mathbb{C}) = 0$ we have that $H^2(X, \mathbb{C})$ is isomorphic to $\mathbb{C}$ and $\omega$ (considered as a 2-form on $X$) is a generator. Remark that $\omega = i_\theta \omega_0$. Thus subtracting the completely integrable volume preserving invariant field $\theta \frac{d\theta}{\theta}$ from a given field $\chi$ we can assume that the form $\alpha$ is exact. It remains to construct all anti-invariant

\[ (2 \frac{\partial}{\partial x} + \frac{\partial}{\partial z})(z^i x^j z^{i+1}) = 2 z^{i+1} (j+1) x^j + iz^i-1 y x^{j+1} = 2 z^{i+1} (j+1) x^j + iz^i-1 (z^2-1) x^j = (2j + 2 + i) z^{i+1} x^j - iz^i-1 x^j \]
1-forms $\beta$ in the expression $d\beta = i_\omega \omega$ where $\chi$ is a Lie combination of invariant completely integrable fields on $X$. Of course we have to find all 1-forms $\beta$ up to closed ones, since these correspond to the zero vector field.

Since the restrictions of the 1-forms $dx$, $dy$ and $dz$ from $\mathbb{C}^3$ to the tangent space of $D$ generate the cotangent space of $D$ at any point, all 1-forms on $X$ can be written as

$$\beta = \sum_{n=-N}^{N} f_n(X)\theta^n dx + \sum_{n=-N}^{N} g_n(X)\theta^n dy + \sum_{n=-N}^{N} h_n(X)\theta^n dz + \sum_{n=-N}^{N} j_n(X)\theta^n d\theta$$

where $X = (x, y, z)$ and $f_n, g_n, h_n, j_n$ are regular functions on $D$ which are invariant if $n$ is even and anti-invariant if $n$ is odd. Of course this representation of a 1-form on $X$ is not unique, the relation $xdy + ydx = 2zd\theta$ holds, but this is irrelevant for our proof.

We begin by constructing all summands of the fourth sum. First consider the case of even $n$. The proof is analogous to the proof of the preceding theorem. The monomial forms $x^i\theta^0 d\theta$, $i$ even, you construct by inner product of the invariant completely integrable field $\theta^{n+1}SF_{x,-1}^0$ with $\omega_0$, $y^i\theta^n d\theta$ comes from $\theta^{n+1}SF_{y,-1}^0$ $i$ even, and $z^i\theta^n d\theta$ comes from the invariant field $\theta^{n+1}HF_{z,-1}$ $i$ even. Now use inductively Lie brackets with the invariant field $SF_0^0$ to obtain out of the form $x^i\theta^n d\theta$ the forms $x^{i'-1}z^{i''}\theta^m d\theta$, $x^{-1}z^2\theta^n d\theta$ and so on thus obtaining all 1-forms $x^kz^l\theta^n d\theta$ for $k + l$ even. The forms $y^2z^l\theta^n d\theta$, $k + l$ even, are obtained analogously. Now consider the case $n$ odd. Start with the monomial forms $x^i\theta^n d\theta$, $i$ odd, you constructed by inner product of the invariant completely integrable field $\theta^{n+1}SF_{y,-1}$ with $\omega_0$, all the rest goes analogously. We thus have constructed all anti-invariant 1-forms $\sum_{n=-N}^{N} j_n(X)\theta^n d\theta$, except for $j_n = constant$, but the forms $\theta^n d\theta$ are closed and therefore corresponding to the zero field.

In order to produce the summand in the first sum we introduce the invariant globally integrable volume preserving vector field $V = x\theta(x\partial/\partial x - y\partial/\partial y - \theta\partial/\partial \theta)$ and take the Lie bracket with the vector field corresponding to the 1-form $f(X)\theta^n d\theta$ (say $n$ even and $f$ invariant). This produces the 1-form

$$L_V(f(X)\theta^n d\theta) = V(f(X)\theta^n d\theta) - f(X)\theta^n (d(x\theta^2)) = (\ldots)d\theta - f(X)\theta^{n+2}dz$$

and therefore we get together with the above all 1-forms of the form $f(X)\theta^n d\theta$ where $n$ is even and $f$ invariant and similarly the ones with $n$ odd and $f$ anti-invariant. In the identical way we get all 1-forms $f(X)\theta^n d\theta$ by taking the invariant vector field $W = y\theta(x\partial/\partial x - y\partial/\partial y + \theta\partial/\partial \theta)$ instead. The invariant vector fields $xz\theta/\partial \theta$, $yz\theta/\partial \theta$ and $z^2\theta/\partial \theta$ will help to construct all forms of the form $f(X)\theta^n d\theta$. Indeed the calculations

$$L_{x\theta}g(X)\theta^n d\theta = (\ldots)d\theta + (\ldots)dx + x f(X)\theta^{n+1}dz,$$

$$L_{y\theta}g(X)\theta^n d\theta = (\ldots)d\theta + (\ldots)dy + y f(X)\theta^{n+1}dz,$$

$$L_{z\theta}g(X)\theta^n d\theta = (\ldots)d\theta + 2zf(X)\theta^{n+1}dz,$$

show that we get all 1-forms of the form $g(X)\theta^n dz$ where $g(X)$ is either a multiple of $x$, $y$ or $z$. Hence allowing linear combinations only the constant term $\theta^n dz$ is missing. But since the form $\theta^n d\theta + n\theta^{n+1}zd\theta$ is closed the corresponding vector field also corresponds to $-n\theta^{n+1}zd\theta$ and hence is already obtained. □

The question we like to investigate in the remaining part of the paper is, whether the hyperbolic vector fields are needed in the proof of Theorem 3.3 or if the shear fields could be enough. In the following section it is shown that the Lie algebra generated by the shear fields doesn’t contain all the hyperbolic fields. Here are
some preliminaries. The proof of the first fact is an easy consequence of the Jacobi
identity.

**Lemma 3.10.** Let $M$ be a set of vector fields, then the Lie algebra $\text{Lie}(M)$ generated
by $M$ is spanned (as a vector space) by elements of the form $[A_n, ... [A_2, [A_1, A_0] ... ]]$ with $A_i \in M$.

In order to study which polynomials correspond to Lie combinations of shear
fields it is therefore necessary to study functions of the type $i[A_n, ... [A_2, [A_1, A_0] ... ]]\omega$
where $A_i$ are shear fields.

**Lemma 3.11.** Let $A_i$ be shear fields for $0 \leq i \leq n$, then the polynomial $f$ with
$i[A_n, ... [A_2, [A_1, A_0] ... ]]\omega = df$ is of type (a) $x^j q(z)$, (b) $y^j q(z)$ or (c) $q(z)$ for some $j > 0$
and some polynomial $q \in \mathbb{C}[z]$.

**Proof.** For $n = 0$ the claim holds due theorem 3.1
(a) If $f = x^j q(z)$, lemma 3.12 shows
\[
\int_{SF_n^k} [A_n, ... [A_2, [A_1, A_0] ... ]]\omega = dx^j q(z),
\]
hence the polynomial is again of the type (a). Furthermore:
\[
\int_{SF_n^k} [A_n, ... [A_2, [A_1, A_0] ... ]]\omega = dx^j q(z)
\]
\[
= dp(z)j^k x^{j-1} q(z) + y^{k+1} x^j q(z)
\]
\[
= dy^k x^{j-1}(j^p(z) q(z) + yq(z))
\]
\[
= dy^k x^{j-1}(j^p(z) q(z) + p(z)q(z)).
\]

After substituting $xy = p(z)$ this polynomial is also of the type (a),(b) or (c),
depending weather $k < j - 1$, $k > j - 1$ or $k = j - 1$. Similarly it holds that:
(b) If $f = y^j q(z)$ then
\[
\int_{SF_n^k} [A_n, ... [A_2, [A_1, A_0] ... ]]\omega = dy^j q(z)
\]
\[
= dy^j q(z)
\]
\[
= dy^j q(z),
\]
\[
= dy^j q(z).
\]
(c) If $f = q(z)$ then
\[
\int_{SF_n^k} [A_n, ... [A_2, [A_1, A_0] ... ]]\omega = dq(z)
\]
\[
= dq(z)
\]
\[
= dq(z).
\]

**Lemma 3.12.** If the case (c) in lemma 3.11 occurs, that is $i[A_n, ... [A_2, [A_1, A_0] ... ]]\omega = df$
for $A_i$ shear fields, and in addition $f = f(z)$ for some polynomial in $z$, then
$f(z) = (p(z)q(z))^j$ for some polynomial $q$ in $z$.

**Proof.** Consider the vector field $[A_{n-1}, ... [A_2, [A_1, A_0] ... ]]$. Due to lemma 3.11 exactly
one of the following cases occurs:
\[
i[A_{n-1}, ... [A_2, [A_1, A_0] ... ]]\omega = \begin{cases} \frac{dx^j q(z)}{a} \\
\frac{dy^j q(z)}{b} \\
\frac{dq(z)}{c} \end{cases}
\]
for some $j > 0$ and some $q \in \mathbb{C}[z]$. If $A_n = SF_n^k$ for some $k \in \mathbb{N}_0$, then together
with the calculation in the proof of lemma 3.11 one gets:
\[
i[A_n, ... [A_{n-1}, ... [A_2, [A_1, A_0] ... ]]]\omega = df = \begin{cases} \frac{dx^{j+k} q(z)}{a} \\
\frac{dy^{j+k} q(z) + p(z)q(z)}{b} \\
\frac{dx^{j+k+1} q(z)}{c} \end{cases}
\]
Since $f$ is a polynomial in $z$ all cases except (b) with $k = j - 1$ can be excluded. Therefore $f = x^k y^{j-1}(jp'(z)q(z) + p(z)q'(z)) = p(z)^k((k+1)p'(z)q(z) + p(z)q'(z)) = (p(z)^{k+1}q(z))'$ for some $q \in \mathbb{C}[z]$. The identical consideration works for $A_n = SF^n$.

**Remark 3.13.** If one chose in the last step $k = j^2 - 1$ instead of $k = j - 1$ for some $i \in \mathbb{N}$, the polynomial in the end of the calculation would have been $x^i((pq)' = f(z)$ (respectively $y^i f(z)$). Hence if $f(z)$ corresponds to a Lie combination of shear fields, then so does the polynomial $x^i f(z)$ (respectively $y^i f(z)$). By permuting $SF^n_x$ and $SF^n_y$ the corresponding polynomial switches the sign and $x$ and $y$ get permuted, hence both $x^i f(z)$ and $y^i f(z)$ correspond to Lie combinations of shear fields.

**Corollary 3.14.** If a hyperbolic vector field is a Lie combination of shear fields then it is of the form $HF_{(pq)'}$ for some $q \in \mathbb{C}[z]$. In particular if $p \in \mathbb{C}[z]$ with degree $n \geq 3$, then the hyperbolic vector fields $HF_z$ with $i < n - 2$ are not Lie combinations of shear fields.

In addition we can make the following observation:

**Corollary 3.15.** For $p \in \mathbb{C}[z]$ with degree $n \geq 3$ the Lie algebra generated by holomorphic shear fields is not dense in the Lie algebra of holomorphic volume preserving vector fields.

**Proof.** Formula (1) for a regular function $f$ on $D_p$ can be viewed as a Laurent expansion of the restriction of $f$ to the open subset $x \neq 0 \cong \mathbb{C}^* \times \mathbb{C}$ with respect to the variable $x \in \mathbb{C}^*$ with coefficients being functions of $z$. Analogously any holomorphic function $g$ on $D_p$ has such a Laurent expansion with coefficients holomorphic functions in $z$

$$g = \sum_{i=1}^{\infty} a_i(z)x^i.$$

We have established that the regular function $f$ corresponding under $i_\omega \omega = df$ to an algebraic vector field $\Theta$ which is a Lie combination of algebraic shear fields satisfy the special condition $a_0(z) = (hp)'$, i.e., the absolute term $a_0(z)$ (which is unique associated to $\Theta$ up to a constant) is the derivative of a function divisible by the defining polynomial $p$. The condition that a function $g$ on $\mathbb{C}^* \times \mathbb{C}$ has an absolute term which is up to a constant the derivative of a function divisible by the defining polynomial $p$ is closed in c.-o. topology. More explicitly, let $z_1, \ldots, z_n$ be the distinct simple zeros of $p$, then the condition is equivalent to the equality of all the expressions

$$(z_j - z_i) \int_{z_1}^{z_j} \int_{|x| = 1} g(x, z) \frac{dz \wedge dx}{x} \quad j = 2, 3, \ldots, n.$$ 

Since holomorphic shear fields are limits (in c.-o. topology) of algebraic shear fields the holomorphic function corresponding to a Lie combination of holomorphic shear fields has an absolute term of the same form. Thus for $p$ with degree $\geq 3$ the Lie algebra generated by holomorphic shear fields is contained in the closed proper subset of the Lie algebra of holomorphic volume preserving vector fields defined by the above condition on the absolute term.

**3.2. Description of the Lie Algebra generated by Shear Fields.** After negating the question whether every volume preserving vector field is a Lie combination of shear fields, in this section it will be investigated which vector fields exactly are Lie combination of such ones. Concretely all of the volume preserving vector fields whose absolute term of the corresponding function is of the special
form described in Lemma \ref{11}, \ref{12} are a Lie combination of shear fields. This proof is following the same concept developed in \cite{19} to prove the fact that the shear fields and another class of (non volume preserving) vector fields called overshear fields do generate the Lie algebra of algebraic vector fields of $D_p$.

Lemma 3.16. The following equalities hold:

\begin{align*}
(1) & \quad i_{[S_{F_k}^x, S_{F_k}^y]} \omega = d(p^1 p') \\
(2) & \quad i_{[S_{F_k}^x, S_{F_k}^y]} \omega = d(p^p p')' \\
(3) & \quad i_{[S_{F_k}^x, S_{F_k}^y]} \omega = d(x^{i_1 + i_2 + \ldots + i_k} f^{(k-1)}) \\
(4) & \quad i_{[H_{F_k}, \ldots, [S_{F_k}^x, H_{F_k}], \ldots]} \omega = (i + 1)^{k-1} d(x^{i+1} f_1 f_2 \ldots f_k).
\end{align*}

Proof. The following calculations are according to theorem \ref{3.1} and lemma \ref{3.2}.

\begin{align*}
(1) & \quad i_{[S_{F_k}^x, S_{F_k}^y]} \omega = dL_{S_{F_k}^x} \left( \frac{y^{i+1}}{i+1} \right) \\
& \quad = d(p')(z)x^{i}y'. \\
(2) & \quad i_{[S_{F_k}^x, S_{F_k}^y]} \omega = d(y p')(z) \\
& \quad = d(p'(z)p'(z) + x y p'(z))' \\
& \quad = d(p'(z)(z))'.
\end{align*}

\begin{align*}
(3) & \quad i_{[S_{F_k}^x, S_{F_k}^y]} \omega = d(\eta_{11} f) \\
& \quad = dL_{S_{F_k}^x} (x^{i_1} f) \\
& \quad = x^{i_1 + i_2} f' \\
& \quad = dL_{S_{F_k}^x} (x^{i_1 + i_2 + \ldots + i_k} f^{(k-1)}) \\
& \quad = (i + 1)^{k-1} d(x^{i+1} f_1 f_2 \ldots f_k).
\end{align*}

\begin{align*}
(4) & \quad i_{[S_{F_k}^x, S_{F_k}^y]} \omega = d(x^{i+1} f_1) \\
& \quad = dL_{S_{F_k}^x} (x^{i+1} f_1) \\
& \quad = (i + 1)^{k-1} d(x^{i+1} f_1 \ldots f_{k-1}) \\
& \quad = (i + 1)^{k-1} d(x^{i+1} f_1 \ldots f_{k-1}) \\
& \quad = (i + 1)^{k-1} d(x^{i+1} f_1 \ldots f_{k-1}).
\end{align*}

\begin{align*}
& \quad i_{[H_{F_k}, \ldots, [S_{F_k}^x, H_{F_k}], \ldots]} \omega = (i + 1)^{k-1} d(x^{i+1} f_1 \ldots f_{k-1}) \\
& \quad = d((i + 1)^{k-1} d(x^{i+1} f_1 \ldots f_{k-1}).
\end{align*}

\[ \square \]

Corollary 3.17. The previous lemma shows:

\begin{align*}
(5) & \quad [S_{F_k}^x, S_{F_k}^y] = H_{F_{(p^1 p')}} \\
(6) & \quad [S_{F_k}^x, S_{F_k}^y] = H_{F_{(p^p p')}} \\
(7) & \quad [S_{F_k}^y, [S_{F_k}^x, [S_{F_k}^x, H_{F_k}], \ldots]] = H_{F_{(p^1 + \ldots + i_k + 1)^{k-1}}}. \\
(8) & \quad [S_{F_k}^y, [H_{F_k}, \ldots, [S_{F_k}^x, H_{F_k}], \ldots]] = H_{F_{(i+1)^{k-1}}}.
\end{align*}

Lemma 3.18. Let $n = \deg(p)$, then for every $q \in \mathbb{C}[z]$ the vector field $H_{F_{(p^q p')}}$ is a Lie combination of shear fields.

Proof. In a first step one observes that every polynomial $x^{n} q$ corresponds to a Lie combination of shear fields. Truly due to lemma \ref{3.10} the polynomials $x^{n} f^{(k)}$ for $k = 0, \ldots, (n-1)$ correspond to a Lie combination of shear fields, if $H_{F_k}$ was
already such a combination. According to corollary 3.17 it is possible to choose for \( f \) the polynomials \( p''(a), (pp')', (p^2p')', \ldots \) (i.e. polynomials of degree \( n-2, 2n-2, 3n-2, \ldots \)). Therefore after differentiating up to \( n \) times there is a polynomial for every degree and hence they build a basis for \( \mathbb{C}[z] \) and every polynomial \( q \in \mathbb{C}[z] \) can be substituted in \( x^aq \). After taking the Lie bracket with the shear field \( SF_{n-1} \) the vector field becomes \( HF_{(pq)''} \).

Let \( n = \deg p \) and \( W \subset \mathbb{C}[z] \) be a vector space with

\[
\begin{align*}
(i) & \quad (p')'' \in W & \forall i \in \mathbb{N} \\
(ii) & \quad (pp')' \in W \\
(iii) & \quad (p^2p')' \in W & \forall q \in \mathbb{C}[z] \\
(iv) & \quad f_1, \ldots, f_k \in W \Longrightarrow (p f_1, \ldots, f_k)'' \in W & \forall k \in \mathbb{N}.
\end{align*}
\]

Now the goal is to show that \( W \) contains all polynomials of the type \( (pq)'' \). Since the vector space of all \( f \) with \( HF \) a Lie combination of shear fields is a vector space with properties (i)-(iv), every vector field \( HF_{(pq)''} \) would be a such combination.

In a first step it is shown that the algebra \( A_W = \text{span}\{f_1, \ldots, f_k : f_i \in W, 1 \leq i \leq k \in \mathbb{N}\} \) generated by \( W \) is equal to \( \mathbb{C}[z] \). Then it is allowed to substitute all polynomials in (iv) and hence the claim is proven.

**Lemma 3.19.** There is no element \( a \in \mathbb{C} \) such that \( f(a) = 0 \) for all \( f \in A_W \).

**Proof.** Suppose there is such an \( a \), then \( p''(a) = 0 \) and \( p(a)p''(a) + p'(a)^2 = 0 \) ((i) with \( i = 1 \) and \( i = 2 \) would hold, and hence \( p'(a) = 0 \). Since \( p \) has no double zero point it follows that \( p(a) \neq 0 \). Due to (iii) \( (p^aq)''(a) = (p^{a+2})q + 2(p^aq)'q' + p^aq'''(a) = 0 \) holds for all \( q \in \mathbb{C}[z] \). The first summand vanishes due to (i), the second due to \( p'(a) = 0 \), therefore it remains \( p^aq'''(a) = 0 \). So it would be true that \( q''(a) = 0 \) for all \( q \in \mathbb{C}[z] \) which is clearly a contradiction.

**Lemma 3.20.** There is no element \( a \in \mathbb{C} \) such that \( f'(a) = 0 \) for all \( f \in A_W \).

**Proof.** Suppose there is such an \( a \). (i) with \( i = 1, 2, 3 \) shows that \( (p^2)''(a) = 0 \), \( (p^3)''(a) = 2(p(a)p''(a) + 3p'(a)p'''(a)) = 0 \) and \( 0 = (p^3)''(a) = 3(p(a)^2p'''(a) + 6p(a)p'(a)p'''(a) + 2p'(a)^3) \). The second equation shows that \( p'(a)p''(a) = 0 \) and therefore due to the third equation we have \( p'(a) = 0 \) and hence \( p(a) \neq 0 \). Furthermore (iii) shows \( (p^aq)'''(a) = (p^{a+2})(z)q + 3(p^aq)''(z) \mid_{z=a} = 0 \) and since the first summand vanishes \( (p^aq)''(a) = 0 \) remains. Altogether we have \( (p^aq)'''(a) = ((p^aq)''(a) + 3(p^aq)'q' + p^aq''(a)) = p(a)^aq'''(a) = 0 \) or \( q''(a) = 0 \) for all \( q \in \mathbb{C}[z] \), what is again a contradiction.

**Lemma 3.21.** There are no elements \( a \neq b \in \mathbb{C} \), such that \( f(a) = f(b) \) for all \( f \in A_W \).

**Proof.** Suppose there are two such elements \( a, b \in \mathbb{C} \). (iv) shows that \( (p^iz)'' \mid_{z=a} = (p^iz)'' \mid_{z=b} \) for all \( i \in \mathbb{N}_0 \). Since \( (p^iz)'' = (p^iz)'z + 2i(p^iz)z^{-1} + i(i-1)p^iz^2 \) one gets the system of linear equations, which summarizes the equations for \( i = 0, \ldots, 5 \):
\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
a & b & 1 & 1 & 0 & 0 \\
a^2 & b^2 & 2a & 2b & 2 & 2 \\
a^3 & b^3 & 3a^2 & 3b^2 & 6a & 6b \\
a^4 & b^4 & 4a^3 & 4b^3 & 12a^2 & 12b^2 \\
a^5 & b^5 & 5a^4 & 5b^4 & 20a^3 & 20b^3 \\
\end{pmatrix}
\begin{pmatrix}
(p^n)'(a) \\
-(p^n)'(b) \\
2(p^n)'(a) \\
-2(p^n)'(b) \\
(p^n)(a) \\
-(p^n)(b) \\
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

The determinant of this matrix is \(4(a - b)^9\) and therefore nonzero for \(a \neq b\) and hence it is shown that the coefficient vector is the zero vector and in particular \(p(a) = p(b) = 0\) and therefore \(p'(a) \neq 0 \neq p'(b)\).

Due to (ii) we have \((pp)'(a) = p(a)p''(a) + 3p'(a)p''(a) = p(b)p''(b) + p'(b)p''(b)\) and since \(p(a) = p(b) = 0\) and \(p''(a) = p''(b)\) (due to (i)) \(p'(a) = p'(b)\) holds. With (iv) \((k = 1)\) follows \((pp)'(a) = p(a)p''(a) + 2p'(b)p''(b) + p'(b)^2 = p(b)p''(b) + 2p'(b)p''(b) + p'(b)^2\) and hence \(p''(a) = p''(b)\). Using (iv) inductively one gets that \(p^{(l)}(a) = p^{(l)}(b)\) for all \(l\). Indeed a simple calculation shows that:

\[
W \ni P := \left(\sum_{i_1 + \cdots + i_{j+1} = 2j+2} \alpha_{ij} \cdot p^{(i_1)} \cdots p^{(i_{j+1})}\right)''
\]

with \(a_I \in \mathbb{N}\). After inserting \(a\) (resp. \(b\)) all summands with \(i_1 = 0\) vanish due to \(p(a) = p(b) = 0\). Assume that \(p^{(l)}(a) = p^{(l)}(b)\) for all \(l \leq j + 1\), so all the summands with \(i_{j+1} \leq j + 1\) have on both sides of the equation \(P(a) = P(b)\) the same value and hence vanish as well. For this reason only the equation \(\alpha_{j}p^{(j)}(a)/p^{(j+2)}(a) = \alpha_{j}p^{(j)}(b)/p^{(j+2)}(b)\) remains and it follows inductively that \(p^{(l)}(a) = p^{(l)}(b)\) for all \(l\).

This is a contradiction since the \((n - 1)\)-st derivative of a polynomial of degree \(n\) is a polynomial of degree one with a nonzero slope. \(\square\)

**Proposition 3.22.** The algebra \(A_W\) generated by \(W\) is equal to \(\mathbb{C}[z]\).

**Proof.** The previous two lemmas show that there is a \(k \in \mathbb{N}\) and polynomials \(q_1, \ldots, q_k \in A_W\) such that the map

\[
F : \mathbb{C} \to \mathbb{C}^k : \quad z \mapsto (q_1(z), \ldots, q_k(z))
\]

is an injective and immersive embedding. To achieve injectivity take the ideal in \(\mathbb{C}[x, y]\) generated by the polynomials \(q(x) - q(y)\) with \(q \in A_W\) which is finite generated by polynomials \(q_1(x) - q_1(y), \ldots, q_k(x) - q_k(y)\). Now we see that there are no \(c_1 \neq c_2\) such that \(q_i(c_1) = q_i(c_2)\) for all \(i\), otherwise we would have \(q(c_1) = q(c_2)\) for all \(q \in A_W\) which is not possible due to lemma 3.21. To guarantee immersivity we add for each cusp singularity (finite number!) a polynomial \(q \in A_W\) whose derivative doesn’t vanish at this point (lemma 8.20).

Now take any polynomial function \(g\) on \(\mathbb{C}\) and regard it as a regular function on the by \(F\) embedded \(\mathbb{C}\) in \(\mathbb{C}^k\). This function expands to a regular function \(G\) on \(\mathbb{C}^n\) hence \(G = a_0 + \sum_i a_iz_i^1 \cdots z_i^k\). So if we pull back \(G\) we get \(g(z) = G(F(z)) = a_0 + \sum_i a_iq_i(z)^{i_1} \cdots q_k(z)^{i_k}\) so the algebra generated by \(q_1, \ldots, q_k\) and constants is \(\mathbb{C}[z]\). Now the algebra generated by \(W\) is \(\mathbb{C}[z]\) or a subspace with codimension 1 and an ideal hence in the second case \(W\) is a principle ideal generated by a polynomial \((z - a)\). But this case can’t occur since \(a\) would be a common root of all elements of \(W\) what is impossible (lemma 8.19). \(\square\)
Now we know that a hyperbolic field $HF_f$ is a Lie combination of shear fields if and only if $f = (pq)^n$ for some polynomial $q$. In theorem 3.23, it was shown that every volume preserving vector field is a linear combination of the vector fields $SF_{x^i}$, $SF_{y^i}$, $HF_f$, $[SF_{x^i}, HF_f]$ and $[SF_{y^i}, HF_f]$ for $i \in \mathbb{N}_0$ and polynomials $f \in \mathbb{C}[z]$. To understand which vector fields are Lie combinations of shear fields, it remains to study the vector fields $[SF_{x^i}, HF_f]$ and $[SF_{y^i}, HF_f]$. 

**Proposition 3.23.** All the vector fields $[SF_{x^i}, HF_f]$ and $[SF_{y^i}, HF_f]$ for $i \in \mathbb{N}_0$ and polynomials $f \in \mathbb{C}[z]$ are Lie combinations of shear fields.

**Proof.** Since $i_{[SF_{x^i}, HF_f]} \omega = dx^i f$ it suffices to see that the polynomial $x^i f(z)$ corresponds for every $i \in \mathbb{N}$ and every $f \in \mathbb{C}[z]$ to a Lie combination of shear fields. In the proof of lemma 3.18 we saw that this is true for $i \geq n = \deg p$. So we already have every $x^n z^j$ for $j \in \mathbb{N}$. If one takes the Lie bracket with the vector field $SF_{y^i}$ one gets with the calculation in the proof of lemma 3.18 (a) the polynomial $(i+1)p'(z)z^j + jp(z)z^{j-1}$ for $i = n - 1$. Since every polynomial $(p(z)z^j)'$ corresponds to a Lie combination of shear fields, so does the polynomial $x'(p(z)z^j) = x'(p'(z)z^j + jp(z)z^{j-1})$ (due to remark 3.13). After a suitable linear combination of this two polynomials it follows that $x'p(z)z^j$ and $x'p(z)z^{j-1}$ correspond to a Lie combination of shear fields for all $j$. Therefore every $x^i f(z)$ with $f(z) \in (p) \cup (p') \subset \mathbb{C}[z]$ belongs to a Lie combination. Since $p$ and $p'$ have no common zeros it is true that $(p) \cup (p') = \mathbb{C}[z]$ and the claim is shown for every $i > n - 1$. Repeat the same procedure for $i = n - 2, \ldots, 1$ and the claim is shown for every $i \in \mathbb{N}$.

Now we have to make the final step allowing not only shear fields but also LNDs in our Lie combination. Since LNDs are shears conjugated by compositions of shear automorphisms (see theorem 2.13) the following lemma will do the job.

**Lemma 3.24.** Let $\phi : D_p \to D_p$ be a shear automorphism and let $\Theta$ be a Lie combination of shear fields. Then $\phi^* \Theta$ is a Lie combination of shear fields.

The proof of this lemma follows immediately from the following general fact.

**Lemma 3.25.** Let $\Theta$ be an LND with flow $\phi_t$ and $\Psi$ any algebraic vector field. Then for any fixed $t$ the vector field $(\phi_t)^* (\Psi)$ is contained in the Lie algebra generated by $\Theta$ and $\Psi$.

**Proof.** Since $\Theta$ is an LND the Taylor expansion of $(\phi_t)^* (\Psi)$ with respect to the variable $t$ around $t_0 = 0$

$$(\phi_t)^* (\Psi) = \Psi + t[\Theta, \Psi] + \frac{1}{2} t^2 [\Theta, [\Theta, \Psi]] + \ldots + \frac{1}{n!} t^n [\Theta, [\Theta, \ldots [\Theta, \Psi]] \ldots]$$

is a polynomial in $t$. This implies the claim. 

Thus we can now proof the main result.

**Theorem 3.26.** A volume preserving vector field $\Theta$ on the Danielewski surface $D_p$ is a Lie combination of LNDs if and only if its corresponding function with $i_\Theta \omega = df$ is of the form (modulo constant)

$$f(x, y, z) = \sum_{i=1}^{k} a_{i_1} x^i z^j + \sum_{j=0}^{l} b_{i_2} y^i z^j + (pq)'(z)$$

for a polynomial $q \in \mathbb{C}[z]$. 

Proof. By Proposition 3.22 together with Lemma 3.16 (4) and Proposition 3.23 the Lie algebra generated by shear fields consists exactly of those volume preserving fields described in the theorem. By Theorem 2.15 any LNDS is conjugated to a shear field $S$ by an automorphism $\psi$ which is a finite composition of shear automorphisms $\psi = \alpha_m \circ \ldots \circ \alpha_1$. Thus by Lemma 3.24 $\Theta = \psi^*S = \alpha_1^* (\ldots \alpha_{m-1}^* (\alpha_m^* S) \ldots)$ is contained in the Lie algebra generated by shear fields.

4. AN EXAMPLE CONCERNING THE TOPOLOGY OF THE HOLomorphic AUTOMORPHISM GROUP

The natural topology in complex analysis is compact-open topology. For the holomorphic automorphism group of a Stein manifold $X$ consider a monotonically increasing sequence of compacts $K_1 \subset K_2 \subset \ldots$ in $X$ such that $\bigcup_i K_i = X$ and a closed imbedding $\iota : X \hookrightarrow \mathbb{C}^m$. For every continuous map $\varphi : X \to \mathbb{C}^m$ denote by $||\varphi||\iota$ the standard norm of the restriction of $\varphi$ to $K_i$. Let $d$ be the metric on the space $\text{Aut}_{\text{hol}}(X)$ of holomorphic automorphisms of $X$ given by the formula

$$d(\Phi, \Psi) = \sum_{i=1}^{\infty} 2^{-i}(\min(||\Phi - \Psi||\iota, 1) + \min(||\Phi^{-1} - \Psi^{-1}||\iota, 1))$$

where automorphisms $\Phi, \Psi \in \text{Aut}_{\text{hol}}(X)$ are viewed as continuous maps from $X$ to $\mathbb{C}^m$. This metric makes $\text{Aut}_{\text{hol}}(X)$ a complete metric space.

In [19] the question (see question (5.2) there) was raised whether the connected components and the path-connected components of the group $\text{Aut}_{\text{hol}}(X)$ coincide. In that paper an example is given for the diffeomorphism group of a differentiable manifold equipped with compact open topology, such that connected components and path-connected components differ.

Here we give an example that the same phenomenon occurs in the holomorphic case as well:

Consider the Stein manifold $X = (\mathbb{C} \setminus \mathbb{N}) \times \mathbb{C}^\ast$. By Runge Theorem there is a sequence $f_i, i = 1, 2, \ldots$ of nowhere vanishing holomorphic functions on $\mathbb{C} \setminus \mathbb{N}$ with the properties

1. $|f_i(z) - 1| < \frac{1}{i + 100}$ for $|z| \leq i + 0.4$,
2. $f_i$ extends holomorphically to the point $i + 1$ and has a zero there.

Define a sequence of holomorphic automorphisms $\alpha_i \in \text{Aut}_{\text{hol}}(X)$ by $\alpha_i(z, w) = (z, w f_i(z))$. By property (1) the sequence $\alpha_i$ converges to the identity uniformly on compacts and thus $\alpha_i$ is in the connected component of the identity for sufficiently big $i$. On the other hand by property (2) each $\alpha_i$ acts non trivially on the fundamental group of $X$, thus non of them is contained in the path-connected component of the identity.

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