Classification of Simple Current Invariants

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ABSTRACT

We summarize recent work on the classification of modular invariant partition functions that can be obtained with simple currents in theories with a center \((\mathbb{Z}_p)^k\) with \(p\) prime. New empirical results for other centers are also presented. Our observation that the total number of invariants is monodromy-independent for \((\mathbb{Z}_p)^k\) appears to be true in general as well.
Despite many efforts, very little has actually been classified in rational conformal field theory (RCFT). The classification of all such theories still looks extremely difficult. Part of this program is the classification of all modular invariant partition functions (MIPF’s) for a given RCFT. This too appears to be quite hard, and has been solved completely in only a few cases (some free theories, SU(2) WZWN models at arbitrary level [1] and related cosets, and some theories with few enough primary fields to allow explicit computer calculations). In general, for any RCFT one would hope to list all matrices $M$ consisting of non-negative entries that commute with the modular transformation matrices $S$ and $T$, with $M_{00} = 1$ (where “0” denotes the identity). The matrix $M$ indicates the left-right pairing of the characters of the RCFT in the partition function.

Although the general problem is clearly quite difficult, it may well be possible to reduce our ignorance about MIPF’s to a few isolated “exceptional” cases using simple currents [2]. Simple currents are special primary fields whose fusion rules with any other field yield just one term. This means in particular that they have well-defined monodromy properties with all fields. One can use this to define a charge for any field with respect to any current, which is precisely the monodromy phase. This charge is conserved and hence corresponds to a symmetry of the conformal field theory. These symmetries form an Abelian group called the center of a RCFT. Non-trivial simple currents exist in many RCFT’s. First of all most WZWN models have them: except for one case ($E_8$ level 2), the center of these conformal field theories is isomorphic to the center of the corresponding Lie algebra [3]. Consequently, many coset theories have a non-trivial center as well. However, we are in no way restricted to such theories, since the entire discussion can be carried out abstractly, without detailed explicit knowledge of the modular transformation matrices $S$ and $T$.

The presence of simple currents in a RCFT usually implies the existence of extra off-diagonal MIPF’s. Essentially these are obtained by applying an orbifold-like procedure to the center. In explicitly known cases one usually gets most of the off-diagonal MIPF’s in that way. For example simple currents yield all the A and D invariants of $SU(2)$, and only the 3 E-invariants are missing. This situation is likely to improve if one considers tensor products of conformal field theories. The number of simple current invariants of a tensor product increases very rapidly with the number of factors in the product. This rapid increase is first of all due to the increase of the number of
subgroups of the center. This is not all, however. The number of distinct invariants is equal to the number of subgroups only if the center is $\mathbb{Z}_N$ [4], but is larger than that if the center consists of more than one factor. There are two reasons for this: first of all one gets new solutions from consecutive orbifold twists, and secondly there exist sometimes additional invariants that cannot be obtained by orbifolding [5].

On the other hand, the number of exceptional invariants (i.e. those that cannot be obtained with simple currents) probably does not grow so fast with the number of factors. If one uses conformal embeddings [6] as a guideline (following [7]) one would conclude that they will only appear sporadically (see e.g. [8]). Explicit calculations seem to confirm this [9]. In any case it is certainly true that most of the existing literature on constructing MIPF’s (mainly for the purpose of building new string theories) can be understood in terms of simple currents. It is therefore important to study simple current invariants, and, if possible, to classify them completely.

This has been the subject of [10] and [11]. These papers consider RCFT’s with any modular transformation matrices $S$ and $T$. The goal is to find all matrices $M$ satisfying the aforementioned conditions plus three additional ones. First of all we impose an additional closure condition on $M$: its non-vanishing matrix elements must correspond to a set of operators whose operator products close within this set. This condition must in any case be satisfied if $M$ is to represent a sensible CFT. Nevertheless, it is probably not necessary to impose it explicitly, since we believe that it follows from the other ones (although we have unfortunately not been able to derive it in general). A second additional condition is imposed to rule out certain pathologies, mostly related to fixed points. For details we refer to [11]. This condition is almost always satisfied. The third additional condition is the crucial one that makes the problem solvable. It states that $M_{ab} = 0$ if $a$ and $b$ do not lie on the same orbit of some simple current. Such an invariant will be referred to as a simple current invariant. All others are, by definition, exceptional (one may wish to sharpen this definition further by eliminating invariants obtained by charge conjugation).

The status of the classification of simple current invariants can be summarized as follows. The classification of solutions without extension of the chiral algebra (i.e. fusion rule automorphisms) was completed in [10]. All possible extensions are also easily classified: as one might expect they simply correspond to all possible sets of
integer spin currents that close under fusion. The remaining difficulty is to classify all possible ways of combining these extensions between the left- and right-moving sectors of the theory.

The latter problem has only been solved so far if the simple currents generate a center that does not contain factors $\mathbb{Z}_{p^n}$, $n > 1$, $p$ prime. In that case the answer is as follows. The simple currents fall into two classes: those that are local with respect to all simple currents in the theory (including themselves) are called type A currents, and all others are called type B currents. If the left algebra contains type A currents, then the same currents must be in the right algebra as well. On the other hand, type B currents may be combined freely between left and right, as long as the total number of currents in the left and right algebras is identical (of course both the left and right set of currents must form a valid integral spin extension of the chiral algebra).

The total number of modular invariants is then obtained by modifying all of these combinations of left and right algebras by all possible fusion rule automorphisms of the left CFT (or, equivalently, the right one, whose set of fusion rule automorphisms is isomorphic to that of the left CFT).

Now it turns out to be very interesting to count the total number of simple current invariants that one obtains for a given center. One would expect the result to depend not only on the Abelian group structure of the center, but also on the current-current monodromies, which for a center $(\mathbb{Z}_p)^k$ are parametrized by a $k \times k$ symmetric matrix with entries defined modulo $p$. However, even though the terms contributing to the sum depend on the current-current monodromies, the total does not. The total number of invariants $T$ is given by the simple universal formula

$$T = \prod_{l=0}^{k-1} (1 + p^l).$$

(if $p = 2$ this result holds for the effective center, obtained by removing currents whose spin is not an integer or a half-integer, and which do not contribute). This is a rather surprising result, since the detailed structure of the invariants, as well as, for example, the number of pure automorphisms do depend on the current-current monodromies in a complicated way.
A rather trivial example is $SU(3)$ level $k$. The center is $\mathbb{Z}_3$, and the formula predicts 2 invariants for any $k$, one of which is the diagonal invariant. There is indeed always a second, off-diagonal invariant, but if $k$ is a multiple of 3 it is an extension of the chiral algebra, whereas otherwise it is a fusion rule automorphism. (A similar situation arises for $SU(2)$, except for the additional complication that for odd levels the effective center is trivial.)

The formula for the total number of invariants was proved in [11] by computing a complicated sum over the number of invariants of each type, for each value of the monodromy matrix. Unfortunately this gives no insight into the origin of the universality. Clearly there is something very interesting to be understood here.

After completing [11] we have tried to investigate whether this result generalizes to arbitrary centers. To do so we need a conjecture regarding the complete classification of the MIPF’s for such a center. It is clear that a subset of the set of solutions is obtained by taking any allowed extension of the algebra (which can be enumerated straightforwardly) and multiplying the corresponding matrix $M$ by any fusion rule automorphism of the unextended theory (classified in [10]). For theories with a center $(\mathbb{Z}_p)^k$ this will indeed give the complete classification. The fusion rule automorphisms map any left algebra to any right algebra that can be paired with it. Furthermore they automatically provide the fusion rule automorphisms of the extended theory, since one can show that any such automorphism can be obtained from some fusion rule automorphism of the unextended theory whose action can be restricted to the representations of the extended algebra.

This last statement is not true if the center contains factors $\mathbb{Z}_{p^n}$, $n > 1$. For example $SU(9)$ level 3 (center $\mathbb{Z}_9$) has a modular invariant in which the extended algebra consists of 3 simple currents (including the identity). The representations of this algebra may either be paired diagonally or off-diagonally. The off-diagonal pairing represents a fusion rule automorphism of the extended theory, but there is no corresponding automorphism of the unextended theory. This suggest that we can try to generalize the construction described above by multiplying all matrices $M$ that extend the algebra from $\mathcal{A}$ to $\mathcal{A}_E$ not only by all fusion rule automorphisms of the theory with algebra $\mathcal{A}$, but also by those of the $\mathcal{A}_E$-theory, and even by the fusion rule automorphisms of any theory with an “intermediate” algebra $\mathcal{A}_I$, where $\mathcal{A} \subseteq \mathcal{A}_I \subseteq \mathcal{A}_E$. 
This will generate nothing new for centers \((\mathbb{Z}_p)^k\), but for more general centers it will. In particular the fusion rule automorphisms of the \(\mathcal{A}_f\)-theory may give rise to new left-right pairings of different algebras. Furthermore we can apply this procedure recursively, extending the algebra further in each step.

This procedure is in any case guaranteed to yield well-defined solutions to our problem, although they are certainly not all distinct. We conjecture that, after removing double counting, one does in fact obtain all the solutions in this way. In view of our results for centers \((\mathbb{Z}_p)^k\) it is interesting to compute now the total number of invariants for various choices of the monodromies. Using our conjecture, we find (for small centers and small values of \(p\)) that as before the total is independent of the monodromies. This is an important hint (though certainly not a proof) that our conjectured count of invariants might be correct, since it is rather unlikely that we would find monodromy-independence with an incorrect count of invariants. It also provides evidence for a second conjecture, namely that in general the total number of invariants will depend only on the Abelian group structure of the center, and not on the monodromies, as we have proved to be the case for \((\mathbb{Z}_p)^k\). The number of independent possibilities for the monodromies increases with \(n\) if the center contains factors \(\mathbb{Z}_{p^n}\), and therefore monodromy-independence becomes highly non-trivial for large \(n\).

Since the number of solutions increases very rapidly with the size of the center we have only been able to check these conjectures for a limited number of values of \(p\) and powers \(n\). The following formulas reproduce the numerical results we have obtained for the total number of invariants \(T\):

\[
\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m} (n \geq m) : \\
T = (m + 1)(n - m + 1)p^m + 2 \sum_{l=0}^{m-1} (l + 1)p^l
\]
\((\mathbb{Z}_p)^k \times \mathbb{Z}_p^n:\)

\[T = (1 + np^k) \prod_{l=0}^{k-1} (1 + p^l)\]

\(\mathbb{Z}_p \times (\mathbb{Z}_{p^2})^2:\)

\[T = 2(1 + p + 2p^2 + 3p^3 + 2p^4)\]

\((\mathbb{Z}_{p^2})^3:\)

\[T = 2 + 2p + 4p^2 + 6p^3 + 6p^4 + 4p^5 + 3p^6\]

It is difficult to see a general pattern emerge from these formulas and the one for \((\mathbb{Z}_p)^k\), but it would not be surprising if these formulas had appeared before in mathematics, in the context of some other problem. This would of course be very interesting to know, because it may give important information regarding the mathematical structure underlying these somewhat mysterious results. The methods we used for analyzing \((\mathbb{Z}_p)^k\) are too explicit to be useful for more general centers. Clearly some more powerful machinery is needed to continue from here, and to complete our project of classifying all simple current invariants for an arbitrary center. We hope that the empirical results listed above will provide some clues.

The aforementioned rapid increase of the number of invariants with the number of factors of the center is quite evident from these formulas. This increase is a reflection of the well-known fact that the number of RCFT’s representing string theories in four dimensions is very large: many of these string theories are in fact constructed out of tensor products of several conformal field theories. For a given combination of factors we are now able to produce a list of all the simple current invariants, which is likely to be a good approximation to the complete list of MIPF’s of such a theory. In view of the large number of solutions this is, however, not a practical way to find the string theories that might be of interest to physics. What one would hope to do instead is to develop an algebraic method for isolating those solutions that are of most interest, e.g. those with three families of quarks and leptons. This may not be as hopeless as it may seem, since it appears to be true that for a given tensor product all simple current invariants yield a number of families that is a multiple of some integer, which is usually larger than 3 \([12]\). (This has been observed for tensor products of \(N = 2\) minimal models yielding (2,2) and (2,1) string theories. The origin of this phenomenon or the extent to which it generalizes are unfortunately not understood.)
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