Beyond Adiabatic Elimination in Topological Floquet Engineering

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Abstract

Manipulating and tailoring the intricate quasi-energy spectrum still remain a challenging issue in periodically-driven quantum systems. Here, we reported two typical spectral decompositions beyond the conventional adiabatic elimination, which enable to target the certain isolated quasi-energy bands by cutting-off the irrelevant far-off-resonant degrees of freedom. According to driving protocols in Floquet engineering, we classified the two non-adiabatic eliminations with Floquet driving frequency: namely, the quasi-adiabatic elimination, which non-adiabatically emerges at in-between driving frequency range, and the high-frequency-limited elimination, which requires rapidly-driven forces or modulations. Due to the failure of adiabatic condition, the protection mechanism of these driven eliminations is the first time to be demonstrated by selectively targeting the non-trivial isolated edge states via the bulk-edge correspondence, as a principle, from topological insulators and quantum field theories. Meanwhile, the near-field measurements using our well-fabricated photonic Floquet simulators correspondingly confirmed observation of these eliminations and their stroboscopic dynamics. The non-adiabatic eliminations for Floquet quasi-energy spectrum are eventually proposed to open up the profound possibilities for artificially-controlling driven systems in topological Floquet engineering that may have numerous promising applications ranging from condensed matters to advanced photonics.
Adiabatic elimination (AE) is a standard decomposition technique in quantum physics that allows one to get rid of the irrelevant states and produces the effective Hamiltonian for a relevant subspace in closed form. The underlying idea behind adiabatically eliminating certain degrees of freedom out of the dynamics under study has been successfully applied in many fields, such as N-level atomic physics [1-4], quantum optics [1, 5-7], nonlinear optics [7-9] and plasmonic multi-layer systems [10], but except for the generic periodically-driven systems [11,12]. Recently, the concept of Floquet engineering in those quantum or classical driven systems has gained more and more attentions because it also similarly allows one to engineer the out-of-equilibrium or synthetic properties of the effective Hamiltonian by introducing the driving protocols and approximations [11,12]. However, often the powerful adiabatic elimination technique fails inevitably due to the violation of adiabatic condition if naively extended into the driven system, especially for description of these emergent driven phenomena or dynamics [1,13,14]. Here we proposed a new decomposition technique that is capable of non-adiabatically eliminating the irrelevant degrees of freedom of quantum driven systems beyond the limitation of adiabatic theorem, and in other words, identifying and tailoring the isolated quasi-energy bands that we are potentially interested in.

For the specific time-periodic quantum systems with driven topological phases, we expected that the generic bulk-edge correspondence, as one of the fundamental principles and manifestations in topological insulators and quantum field theories [15-19], can lead to non-adiabatic driven eliminations. The correspondence in Floquet topological insulators connecting the complicated quasi-energy band and its isolated protected edge states predicts the existence of widely separated quasi-energy scales (i.e. the isolated edge states) which can be selected out as our primary achievement of non-adiabatic spectral tailoring. Roughly speaking, the non-trivial edge/surface states are topologically-protected by gaps emerging critically in Floquet topological insulators which excluded from all the bulk’s states of far-off-resonance at high energy levels and produced an effective reduced Hamiltonian in the gaps, and for these reasons, this approach enables us to eventually ripe the gapped quasi-energy bands out non-adiabatically. More specifically, the anomalous existence of those driven topological edge/surface states are able to be
isolated appreciably and engineered artificially using various periodic driving forces or modulations, and as a driving protocol of Floquet engineering that can be widely applied in various current rapidly-developing fields, such as quantum simulations and computation [4,29], ultra-cold atoms, spin systems and laser-induced atomic physics. [20-25] These Floquet engineered isolated states are exactly what we would like to target and select out from quasi-energy spectrum non-adiabatically through the principle of bulk-edge correspondence. Finally, the following procedure but not the only one to achieve our goal is to break the detergency of protected edge states with the finite-size reduction that would induces a small energy splitting by overlapping the edge states in the band gap. As a result, one can realize non-adiabatic driven eliminations by hybridizing the isolated protected edge/surface states selectively with extension of topological Floquet engineering in periodically-driven systems.

**Results and Discussion**

In this letter, we reported two typical non-adiabatic eliminations in periodically-driven systems, which termed as ‘quasi-adiabatic elimination’ (QAE) and ‘high-frequency-limit elimination’ (HFLE). These driven-based eliminations, altogether with the adiabatic elimination (AE) as depicted in Figure 1, are illustrated numerically and experimentally by using our photonic Floquet simulators composed of coupled microwave waveguides with well-designed curving profiles. As a warm-up contrast, Figure 1a shows the traditional AE for the case of four coupled straight waveguides arrays. The waveguides were fabricated using ultrathin corrugated metallic strips, which support spoof surface Plasmon polaritons (SPPs), [26,27] as single-waveguide-mode excitations at microwave frequencies that allows fabrication-flexibility and measurement-visualization to configure a versatile Floquet simulation platform with high propagation efficiency from microwave to THz frequencies. [28,29] In our array configuration, through the coupled-mode theory, the coupled waveguides can be accurately mapped into a tight-binding-approximated Schrodinger-like equation \((i\partial\psi/\partial z)=\mathcal{H}(z)\psi\), [28] with the guiding propagation direction \((z)\) mapping equivalently to the time \((t)\). Thus, the microwave-propagation of the straight coupled-waveguides can be given by
where \( \beta_0 \) is the propagation constant in the weak-guiding approximation, and \( \kappa_0 \pm \delta \kappa \) is the effective coupling strength between two adjacent waveguides and we assume \( \kappa_0, \delta \kappa > 0 \). For adiabatic decomposition for the straight waveguides, also as the adiabatic limit in Floquet engineering with the bending configuration, the elimination condition has been completely investigated which is \( |(\kappa_0 - \delta \kappa)/(\kappa_0 + \delta \kappa)| < 1 \), then as a result, the 4-level waveguide array can be reduced adiabatically into the effective two-level subsystem (see the Methods). The simulation with CST commercial software shows the two outer waveguides coupled effectively by eliminating the two inner waveguide modes out of the field propagation. Correspondingly, the array configuration parameters \( \kappa_0 = 0.042 \), \( \delta \kappa = 0.02 \) confirm the satisfaction of condition \( |(\kappa_0 - \delta \kappa)/(\kappa_0 + \delta \kappa)| = 1/3 \leq 1 \). Noted that other types of adiabatic decompositions (e.g., on-site potential) also can be realized via the fabrication of propagation constant configuration [5,6].
Figure 1: The coupled-waveguides demonstration of adiabatic elimination (AE), high-frequency-limited elimination (HFLE) and quasi-adiabatic elimination (QAE) of Floquet engineering. The AE (a) appears at adiabatic limit \((\Lambda \to \infty)\), the HFLE (b) exists at high-frequency limit \((\Lambda \to 0)\), and the QAE (c) only emerges at non-trivial driving transition regime in-between the two limits. The relevant simulation parameters are the curving period of array \(\Lambda\) and the coupling strength \(\kappa_0 \pm \delta\kappa(z)\).

In principle, from driving protocols in the language of Floquet engineering [11,12,28,30], three kinds of eliminations are classified in three different engineered regimes, correspondingly, the adiabatic limit at Floquet frequency \(\omega \to 0\), the high-frequency limit at \(\omega \to \infty\) and the quasi-adiabatic transition regime between the two opposite limits when the frequency is comparable to energy bandwidth (i.e., \(\omega \sim |\Delta_c|\)), respectively. The QAE and HFLE are shown in Fig.1c&1b, in which the propagating field patterns appear to be roughly similar to that of AE. Their mechanisms, however, are completely different with the fast-driving force or modulation \((\omega \neq 0)\) in which the Floquet systems are driven into non-adiabatic (off-) resonance regimes [11,12]. For driven coupled-waveguides systems
furnished with non-negligible periodic modulations, in weak-guiding approximation, the propagation-dependent Hamiltonian ($\mathcal{H}(z)$) is approximately obtained as

$$
\mathcal{H}(z) - \beta_0 \mathbf{1}_{4 \times 4} = \begin{pmatrix}
0 & \kappa_0 - \delta \kappa(z) & 0 & 0 \\
\kappa_0 - \delta \kappa(z) & 0 & \kappa_0 + \delta \kappa(z) & 0 \\
0 & \kappa_0 + \delta \kappa(z) & 0 & \kappa_0 - \delta \kappa(z) \\
0 & 0 & \kappa_0 - \delta \kappa(z) & 0
\end{pmatrix}. \quad (2)
$$

Practically, the bending profiles in our microwave waveguides setup result in the periodic-modulation of the coupling strength which approximately decomposed as the averaged central coupling constant $\kappa_0$ and the shifted staggered coupling strength in the guiding propagation $\delta \kappa(z) = \delta_0 + \delta \kappa \cos(2\pi z / \Lambda + \theta_0)$, where $\delta_0$ is the global staggered off-central shift, $\delta \kappa$ is the coupling amplitude with the bending-profiled period $\Lambda$ and the initial position (Floquet gauge) $\theta_0$ [11,12]. Note that the dependence of propagation constant on the waveguide curving is negligible and approximates identically to constant. Regarding of the given array configuration, we may introduce the ‘adiabatic-like’ condition for quasi-spectrum decomposition, which is

$$
\left| \frac{\kappa_0 - \Delta \kappa(z)}{\kappa_0 + \Delta \kappa(z)} \right| \ll 1, \quad (3)
$$

for any propagation position $z$ (or time $t$). The shifted staggered coupling strength $\delta \kappa(z)$ depends on driving frequency, and if this strong adiabatic-like condition (3) satisfies, for example at $\kappa_0 > \max \{ \delta \kappa(z) \}$ and $\delta \kappa \approx \delta_0$, thus the quasi-spectrum decomposition be always reduced similarly to AE condition. However, the rigorous argument of condition (3) is extremely severe that hardly reflects the general plentiful stroboscopic dynamics in driven systems. One way to slightly loosen the satisfaction of strong statement that corresponds to high-frequency universal behaviors is to consider the equivalent static Hamiltonian in high-frequency regimes. In other words, the coupling strength $\delta \kappa(z)$ can be replaced by the effective coupling $\Delta \kappa_{\text{eff}} = \lim_{\Lambda \to 0} \int_0^\Lambda \delta \kappa(z) dz + \delta_0$, resulting expectedly into the effective elimination condition $| (\kappa_0 - \Delta \kappa_{\text{eff}}) / (\kappa_0 + \Delta \kappa_{\text{eff}}) | < 1$. Figure 1b shows the high-
frequency-limited elimination where the simulation parameters were optimally setup \( \kappa_0=0.042, \delta_0=0.042, \delta\kappa=0.02 \) and the experimental measurement were conducted as shown in the Fig.4b. High-frequency-limited elimination (HFLE) only takes place and is well-mapped when Floquet driving \( \omega \to \infty \) (i.e. \( \Lambda=2\pi/\omega \to 0 \)), as we emphasized the effective Hamiltonian in high-frequency limit is universally equivalent to a non-driven case and thus the adiabatic decomposition for effective static system is always comparable satisfied.

Alternatively, we found an unexpected eliminated stroboscopic evolution at the case \( \kappa_0=0.042, \delta_0=0, \delta\kappa=0.02 \), as demonstrated in Fig.1c. The flow of microwave fields propagated along the outer waveguide but periodically-driven by the curving profile, and then coupled effectively to the other outer waveguide, just similar to the evaluation pattern of adiabatic elimination (Fig.1a). We called it “quasi-adiabatic elimination” (QAE), basically because this driven elimination occurs at certain driven frequencies between adiabatic and high-frequency limits, whose stroboscopic dynamic requires the proper driven protocol of array configuration and preparation of initial states. As we checked numerically here, not only the violation of strong argument (eq.3) since the staggered coupling strength \( \delta\kappa(z)=\delta\kappa\cos(2\pi z/\Lambda+\theta_0) \) changes its sign varying sinuously, but also the failure of the effective elimination-like condition in the high-frequency limit (i.e., \( |(\kappa_0-\Delta\kappa_{\text{eff}})/(|\kappa_0+\Delta\kappa_{\text{eff}}|)|=1, \Delta\kappa_{\text{eff}}=0 \)) are confirmed. Thus, compare to the effective non-driven explanation of HFLE in the conventional adiabatic condition, unfortunately, we are forced to develop a new decomposition mechanism for the underlying explanation of QAE.
Figure 2: The physical origins of quasi-adiabatic elimination (QAE) and high-frequency-limited elimination (HFLE) from the violation of degeneracy of π and zero-modes of topologically protected periodically-driven systems. Two typical cases with waveguide number $N=80$ and $N=4$ are compared. The non-adiabatic eliminations are expected at the conditions $1/3 < \omega/|\Delta_\epsilon| < 1$ and $\omega/|\Delta_\epsilon| > 1/2$, respectively. (e) The CST simulation from π mode to QAE by finite size-reduction from $N=8$ to $4$, respectively.

The quasi-energy spectrum is one of point of penetration and breach to explain the QAE and also the HFLE. We note that any stroboscopic evaluation may be decomposed into Floquet quasi-energy state, where the driving frequency plays a vital role in realizing the eliminations, so that we calculate its quasi-energy spectrum of QAE as a function of
driving frequency at the cases of waveguide number N=80 and 4 as shown in Figure 2. Fig.2 shows the quasi-energy spectrum (ε) of Floquet Hamiltonian $\mathcal{H}_t=i\partial_z-\mathcal{H}(z)$ as a function of Floquet driving frequency $\omega=2\pi/\Lambda$. The simulations were performed in our microwave waveguides arrays for the waveguide number from N=8 to 4 in Fig.2e. At the case of N=80, which is in fact the periodically-driven Su-Schrieffer-Heeger (SSH) model that has been wide studied [28,31-35]. This driven SSH model has been proposed theoretically to hold the anomalous topological phase, that is, $\pi$ modes as classified by symmetry-protections [28,34,35], which were confirmedly observed in our recent Floquet simulators, as shown in Fig.2e (N=8). The driven $\pi$ modes are isolated edge states that appears at degenerate positions ($\pm \pi/\Lambda$) in spectrum (Fig.2a), associated with driven frequency ($1/3<\omega/|\Delta\varepsilon|<1$) and well-separated from the other bulk states. The topological-protection with emergence of anomalous topological invariant in the driven-induced quasi-energy gap originates from the Floquet bulk-edge correspondence. Thus, as a result, these protected edge states are driving-dependent, and possible to engineer the quasi-adiabatic elimination among $\pi$ modes that is essential to its survival with violation of its degeneracy by edge-state hybridization.

Finite-size reduction (i.e., reduction of waveguide number in array) is the simplest procedure to violate the degeneracy and induce an energy splitting between the $\pi$ modes. We reduced the waveguide number to N=4 and plotted the corresponding quasi-energy band in Fig.2b and its stroboscopic dynamics in Fig.2e. Figure 2b shows the band splitting of the $\pm\pi$ modes that produces the QAE due to the finite-size effect. Now each split energy band has mixed the two $\pi$ modes and make them coupled, which remains the protection of QAE in quasi-energy gap. Strictly speaking, the size-induced energy splitting breaks the degeneracy of $\pi$ modes and destroys its topological invariant. But the size-induced gap is controllable smaller than the driven-induced gap from bulk-edge correspondence which implies the coupling between two $\pi$ modes hardly scattering into bulk states, and retains the effective exclusion of those irrelevant states. Fig.2e shows the transition from the $\pi$ modes degeneracy to QAE by CST simulations. For the high-frequency-limited elimination, we also explain the mechanism with topological zero-modes in high-frequency limit. Similarly, Fig.2c&d demonstrated the achievement of
HFLE by the finite-size reduction from the zero modes with splitting the degenerate edge states with inherit of the Floquet driving frequency $\omega/|\Delta\varepsilon|>1/2$.

Figure 3: Near-field measurements of Floquet gauge dependence of quasi-adiabatic elimination (QAE). The equivalence between input positions and Floquet gauge dependence from the symmetric analysis combined the sublattice and time-reverse symmetries with the case (a) equivalent to (c), as well as the case (b) to (d).

The hybridization of the protected edge states (i.e., $\pi$ modes and zero modes) is crucial to furnish QAE and HFLE with driven topological phases in Floquet engineering. In fact, the QAE and $\pi$ modes are two distinct physical phenomena. QAE shows the possibility
how to reduce the quasi-energy spectrum into only few relevant quasi-energy band as compared to adiabatic elimination, while the $\pi$ modes show one of driven topological invariants of Floquet systems in connection to static topological insulators. On the other hand, the band-coupling of anomalous $\pi$ modes that leads to the demonstration of QAE as a compelling application (Fig.2a&b), however, indicates the inherent relationship between the protection mechanism of QAE and the hybridization of $\pi$ modes. For generic quasi-energy spectrum with driving protocols, not merely in high-frequency limit, a unified method to target these interesting isolated quasi-states is achieved, but for the given non-trivial gently-driven cases a decomposition strategy is also found to realize QAE from the finite-size reduction of anomalous topological phases. It should be addressed here that those trivial Floquet systems without topological invariants still have the possibilities to engineer the QAE and HFLE, demanding more detailed theoretical investigation in tailoring the intricate quasi-energy spectrum, even though which is still missing.

One intriguing feature of QAE is Floquet gauge dependence as observed with our fabricated samplings and near-field measurement as shown in Fig.3. The parameters of waveguide array profiles follow the simulation setup (in Fig.1). The four cases are setup for testing different combined Floquet gauges and array inputs, where the two cases in Fig.3a&c show the pattern of QAE but the other two diffusive trivially in Fig.3b&d, respectively. For the initial field input from the first outer waveguide, the pattern of QAE emerges at gauge $\theta_0=0$ but disappears at $\theta_0=\pi$. Besides, if the second waveguide is excited as ejected as shown in Fig.3c, the pattern at $\theta_0=\pi$ also shows the QAE but on the contrary no feature of QAE at $\theta_0=0$. Due to the implicit symmetry $\mathcal{H}(0,-\kappa)=\mathcal{H}(\Lambda/2,\kappa)$ in the extended Hamiltonian-like (eq.3), we found that stroboscopic dynamics with input from the first waveguide at gauge $\theta_0=0$ is equivalent to that from the second waveguide at gauge $\theta_0=\pi$. In particular, the QAE is dependent to periodically-driven modulations with the bending profiles (period and gauges). The characteristic of gauge dependence and frequency dependence promises the QAE to be potentially engaged into novel optical switches or modulators, and non-adiabatic driven pumping and transport of light flow. In contrast, the HFLE is gauge-independent with the universal behavior at high frequency
limit. Figure 4 experimentally demonstrated the adiabatic elimination (AE) and high-frequency-limited elimination (HFLE). The fabricated designs in experiment followed the same parameters as suggested in Fig.1 for the input microwave field with frequency 17GHz. The stroboscopic pattern of HFLE showed the similar feature as that of AE that effectively propagated at the two outer waveguides.

![Near-field measurements of adiabatic elimination (a) and high-frequency-limited elimination (b) in two opposite driving limits.](image)

Figure 4: Near-field measurements of adiabatic elimination (a) and high-frequency-limited elimination (b) in two opposite driving limits.

Additionally, for the given fabricated array configuration for QAE, we observed the near-field distributions by ejecting initial microwave field with different frequencies from 18GHz to 10GHz, as demonstrated experimentally in Fig.5. The remarkable distinct field evolution features emerge unexpectedly with vanishing of QAE’s pattern by decreasing the input microwave frequency into the range (10-14GHz). The underlying reason is that the spoof SPP waveguide mode has not been well-excited by low-frequency microwave input [46] and the coupling strength are also affected to be reduced for field energy/intensity transfer between the coupled waveguides. Equivalently, the effective bandwidth of spoof waveguide array becomes smaller and thus the corresponding relative Floquet driving frequency exceeds out of the transition regime and instead into the high-frequency driving regime. The propagation pattern at low-frequency microwave in Fig.5
eventually appears to the high-frequency-limited behavior. Nevertheless, Figure 5 demonstrated the dependence of Floquet frequency for the QAE as the cheapest effective method to test, rather than changing the curving profiles which may draw more fabrication uncertainty and errors.

Figure 5: The near-field measurement with different input microwave frequencies from non-adiabatic transition regime into the high-frequency regime. The input frequency scans from 10GHz to 18GHz, with the excitation of the spoof SPP waveguide mode spreading on the given array configuration.

We would like to address the non-adiabatic resonant transition regime in-between the two opposite limits, requires more attentions, even with the failure to map Floquet driven systems equivalently into the static cases. Instead, one has to face the intricate and highly-risked quasi-energy spectrum in Floquet engineering. But fortunately, the driving frequency and gauge dependence of the in-between driven transition regime would exclusively yield the most intriguing and unexplored driven-based stroboscopic dynamics and evaluations as a by-product retreat. Beyond adiabatic elimination in our microwave Floquet simulators brings us a new research direction ‘Floquet photonics’ that enables to be applied into the advanced optical devices and integrated photonics. As a big picture of Floquet photonics, the stroboscopic dynamics of light/microwave flow fall into three relevant classified regimes including universal fluctuations [30] and quantum/adiabatic
pumping [17, 36,37] in adiabatic limit, anomalous topological phases [18,28,38,39], the Floquet gauge dependence [11,12,28] and Floquet phase transitions [18,39,40] in the transition regime, and universe high-frequency behaviors [11,12] and driven topological phases [18,22,41-45] in the high-frequency approximation regime.

Conclusions

In conclusion, we reported two non-adiabatic driven eliminations (i.e., quasi-adiabatic elimination and high-frequency-limited elimination) for exclusion of irrelevant degrees of freedom in Floquet engineering since the adiabatic condition is generally violated. We adapted the generalized bulk-edge correspondence to identify and tailor the topologically-protected edge states from quasi-energy spectrum, and engineered those protected modes to realize driven eliminations by finite-size reduction. Especially, the quasi-adiabatic elimination holds unique controllable features in protected Floquet engineering that depends on the driving frequency and Floquet gauges, while the high-frequency-limited elimination is protected by the fast-driving modulation or configuration. The non-adiabatic elimination mechanisms are capable of engineering the periodic-driven systems with or without topological invariants, and as a powerful and compelling tool, of analyzing the quasi-energy band-crossing and its stroboscopic dynamics in complicated but fruitful driving protocols.

Methods

Sample fabrication and Simulation. The sample was fabricated in a printed circuit board (F4BK) with dielectric constant 2.65, loss tangent 0.001, and thickness 0.2 mm, the copper film thickness is 0.018 mm. Take consideration for eliminating reflections in the experiment, we added an absorbing material with length thirty millimeters coating the end of the waveguide lattices.

Experimental measurement. We conducted the near-field measurements using a near-electric-field platform, which is composed of a Vector Network Analyzer (VNA) (E5063A), a monopole antenna as the detector, and a translation stages that can move in the x and y- directions automatically controlled by a stepper motor. The input port of the
plasmonic waveguide is connected to port 1 of the VNA through the SMA to feed the energy, and the end is adhered with absorber to eliminate reflections. To probe the vertical \((z)\) components of the electric fields, the monopole antenna is fixed on top of the plasmonic waveguide around 1.4 mm and connected to port 2 of the VNA for recording the propagation pattern.

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