The catalytic Ornstein-Uhlenbeck Process with Superprocess Catalyst

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Abstract

The main objective of this work is to study a natural class of catalytic Ornstein-Uhlenbeck (O-U) processes with a measure-valued random catalyst, for example, super-Brownian motion. We relate this to the class of affine processes that provides a unified setting in which to view Ornstein-Uhlenbeck processes, superprocesses, and Ornstein-Uhlenbeck processes with superprocess catalyst. We then review some basic properties of super-Brownian motion which we need and introduce the Ornstein-Uhlenbeck process with catalyst given by a superprocess. The main results are the affine characterization of the characteristic functional-Laplace transform of the joint catalytic O-U process and catalyst process and the identification of basic properties of the quenched and annealed versions of these processes.

Key concepts: Stochastic partial differential equations, superprocesses, measure-valued processes, affine processes, catalytic Ornstein-Uhlenbeck processes, moment measures, characteristic Laplace functional, quenched and annealed processes in a random medium.

1 Introduction

Beginning with the work of K. Itô there has been extensive study of the class of infinite dimensional Ornstein-Uhlenbeck processes (e.g. [Ito-1]). In the finite dimensional case O-U processes belong to the family of affine processes (e.g. Duffie et al [Du 03]). Other affine processes such Cox-Ingersoll-Ross and Heston processes arise in financial modelling and a characterization of the general finite dimensional affine processes is known (cf. [K-R]). The main topic of this paper is the notion of catalytic infinite dimensional O-U processes which give examples of infinite dimensional affine process. It was established in [PAD 12] that catalytic O-U processes can arise as fluctuation limits of super-Brownian motion in a super-Brownian catalytic medium. These processes involve measure-valued catalysts and the resulting catalytic O-U processes have as state spaces a class of Sobolev spaces. The annealed versions also give examples of infinite dimensional non-Gaussian random fields.

2 The catalytic Ornstein-Uhlenbeck

The name Ornstein-Uhlenbeck process, was originally given to the process described by the stochastic differential equation:
\[ dX_t = \theta(a - X_t)dt + \sigma dB_t \]

where \( \theta > 0, a \in \mathbb{R}, \sigma > 0 \) are parameters, \( X_t \in \mathbb{R} \) and \( B_t \) is Brownian motion. The corresponding infinite dimensional analogue has developed into what is now known as the generalized Ornstein-Uhlenbeck process:

\[ dX_t = AX_t dt + dW_t. \]

Here \( X_t \) takes values in some Hilbert space \( H; A \in \mathcal{L}(H) \) and \( \{W_t\}_{t \geq 0} \) is a Hilbert-space-valued Wiener process. One important case is the cylindrical Wiener process whose distributional derivative: \( \partial^2 W_t / \partial x \partial t \) is space-time white noise.

In this paper, we will consider the Generalized Ornstein-Uhlenbeck (OU) process in catalytic media, that is, a process that satisfies a stochastic evolution equation of the form:

\[ dX(x,t) = AX(x,t) + dW_\mu(x,t) \quad (2.1) \]

where \( A \) and \( X \) are as before, but this time, \( \mu = \{\mu_t\}_{t \geq 0} \) is a measure-valued function of time and \( W_\mu \) is a Wiener process based on \( \mu(t) \), i.e. \( W \) defines a random set function such that for sets \( A \in \mathcal{E} \):

(i) \( W_\mu(A \times [0,t]) \) is a random variable with law \( N(0, \int_0^t \mu_s(A)ds) \).

(ii) if \( A \cap B = \emptyset \) then \( W_\mu(A \times [0,t]) \) and \( W_\mu(B \times [0,t]) \) are independent and \( W_\mu((A \cup B) \times [0,t]) = W_\mu(A \times [0,t]) + W_\mu(B \times [0,t]). \)

As it will be seen later, \( \mu_t \) will play the role of the catalyst. For the rest of the discussion, we will assume that \( \mu_t \) is a measure-valued Markov process, for example, the super-Brownian motion (SBM).

As a simple example, consider the case of a randomly moving atom \( \mu_t = \delta_{B_t} \) where \( B_t \) is a Brownian motion in \( \mathbb{R}^d \) starting at the origin. In the case of a random catalyst there are two processes to consider. The first is the solution of the perturbed heat equation conditioned on a given realization of the catalyst process - this is called the quenched case. The second is the process with probability law obtained by averaging the laws of the perturbed heat equation with respect to the law of the catalytic process - this is called the annealed case.

We will now determine the behavior of the annealed process and show that it also depends on the dimension as expressed in the following:

\textbf{Theorem 2.1.} Let \( X(t,x) \) be the solution of \( (2.1) \) where \( \mu_t = \delta_{B_t} \) with \( B_t \) a Brownian motion in \( \mathbb{R}^d \) and with \( A = \frac{1}{2} \Delta \). Then \( X \) is given by:

\[ X(t,x) = \int_0^t \int_{\mathbb{R}^d} p(t-s,x,y)W_{\delta_{B(s)}}(dy,ds) \quad (2.2) \]

then, the annealed variance of \( X(t,x) \) is given by:

\[ E[\text{Var} X(t,x)] = \begin{cases} 
1/4 & d = 1, \quad x = 0 \\
< \infty & d = 1, \quad x \neq 0 \\
\infty & d \geq 2 
\end{cases} \]

Proof. The second moments are computed as follows:

\[ E X^2(t,x) = E \int_0^t \frac{1}{2\pi(t-s)} \exp \left( -\frac{\|x - B(s)\|^2}{(t-s)} \right) ds. \]
In the case \( d = 1 \), \( x = 0 \) the expectation in the last integral can be computed using the Laplace transform \( M_X \) of the \( \chi^2_1 \) distribution as:

\[
\int_0^t \frac{1}{2\pi(t-s)} \mathbb{E} \exp \left( -\frac{B^2(s)}{t-s} \right) ds
\]

with:

\[
\mathbb{E} \exp \left( -\frac{B^2(s)}{t-s} \right) = \frac{1}{2\pi} \mathbb{E} \exp \left( -\frac{s}{t-s} B^2(s) \right) = \frac{1}{2\pi} M_X \left( -\frac{s}{t-s} \right) = \frac{1}{2\pi} \left( \frac{t}{s} + t + s \right)^{1/2}
\]

A trigonometric substitution shows:

\[
\mathbb{E}X^2(t,0) = \frac{1}{2\pi} \int_0^t \frac{1}{(t^2-s^2)^{1/2}} ds = \frac{1}{4}.
\]

For \( x \neq 0 \) and \( d = 1 \), using a spatial shift we have

\[
\mathbb{E}X^2(t,x) = \mathbb{E} \int_0^t \frac{1}{2\pi(t-s)} \exp \left( -\frac{(B(s)-x)^2}{(t-s)} \right) ds
\]

\[
= \int_0^t \frac{1}{2\pi(t-s) (2\pi s)^{1/2}} \int e^{-\frac{(y-x)^2}{2s}} e^{-\frac{y^2}{2s}} dy ds 
\]

\[
\leq \int_0^t \frac{1}{2\pi(t-s) (2\pi s)^{1/2}} \int e^{-\frac{(y-x)^2}{2s}} dy ds 
\]

\[
\leq C \int_0^t \frac{1}{s(t-s)^{1/2}} ds < \infty.
\]

When \( d \geq 2 \) and the perturbation is \( \delta_B(t) \) as before, then \( \|B(s)\|^2 \) is distributed as \( \chi^2_d \), and its Laplace transform is:

\[
M_x(t) = \left[ \frac{1}{1-2t} \right]^{d/2}
\]

So:

\[
\mathbb{E}X^2(t,0) = \int_0^t \frac{ds}{(t^2-s^2)^{d/2}}
\]

Which is infinite when \( d \geq 2 \) at \( x = 0 \). A modified calculation shows this is true everywhere.

**Remark 2.1.** Note that the annealed process is not Gaussian since a similar calculation can show that \( E(X^4(t,x)) \neq 3(E(X^2(t,x)))^2 \).

### 3 Affine Processes and Semigroups

In recent years affine processes have raised a lot of interest, due to their rich mathematical structure, as well as to their wide range of applications in branching processes, Ornstein-Uhlenbeck processes and mathematical finance.

In a general affine processes are the class of stochastic processes for which, the logarithm of the characteristic function of its transition semigroup has the form \( \langle x, \psi(t,u) \rangle + \phi(t,u) \).

Important finite dimensional examples of such processes are the following SDE’s:
1. **Ornstein-Uhlenbeck process**: \( z(t) \) satisfies the Langevin type equation

\[
dz(t) = (b - \beta z(t))dt + \sqrt{2}\sigma dB(t)
\]

This is also known as a Vasieck model for interest rates in mathematical finance.

2. **Continuous state branching immigration process**: \( y(t) \geq 0 \) satisfies

\[
dy(t) = (b - \beta z(t))dt + \sigma \sqrt{2y(t)} dB(t)
\]

with branching rate \( \sigma^2 \), linear decay rate \( \beta \) and immigration rate \( b \). This is also called the Cox-Ingersoll-Ross (CIR) model in mathematical finance.

3. **The Heston Model (HM)**: Also of interest in mathematical finance, it assumes that \( S_t \) the price of an asset is given by the stochastic differential equation:

\[
dS_t = \mu S_t dt + \sqrt{V_t} S_t dB^1_t
\]

where, in turn, \( V_t \) the instantaneous volatility, is a CIR process determined by:

\[
dV_t = \kappa (\theta - V_t) dt + \sigma \sqrt{V_t} (\rho dB^1_t + \sqrt{1-\rho^2} dB^2_t)
\]

and \( dB^1_t, dB^2_t \) are Brownian motions with correlation \( \rho \), \( \theta \) is the long-run mean, \( \kappa \) the rate of reversion and \( \sigma \) the variance.

4. **A continuous affine diffusion process** in \( \mathbb{R}^2 \): \( r(t) = a_1 y(t) + a_2 z(t) + b \) where

\[
dy(t) = (b_1 - \beta_{11} y(t)) dt + \sigma_{11} \sqrt{2y(t)} dB_1(t) + \sigma_{12} \sqrt{2y(t)} dB_2(t)
\]

\[
dz(t) = (b_2 - \beta_{21} y(t) - \beta_{22} z(t)) dt + \sigma_{21} \sqrt{2y(t)} dB_1(t) + \sigma_{22} \sqrt{2y(t)} dB_2(t) + \sqrt{2a} dB_0(t)
\]

where \( B_0, B_1, B_2 \) are independent Brownian motions.

It can be seen that the general affine semigroup can be constructed as the convolution of a *homogeneous* semigroup (one in which \( \phi = 0 \)) with a *skew convolution semigroup* which corresponds to the constant term \( \phi(t,u) \).

**Definition 3.1.** A transition semigroup \( (Q(t))_{t \geq 0} \) with state space \( D \) is called a homogeneous affine semigroup (HA-semigroup) if for each \( t \geq 0 \) there exists a continuous complex-valued function \( \psi(t,\cdot) := (\psi_1(t,\cdot),\psi_2(t,\cdot)) \) on \( U = \mathbb{C}_- \times (i\mathbb{R}) \) with \( \mathbb{C}_- = \{ a + ib : a \in \mathbb{R}_-, b \in \mathbb{R} \} \) such that:

\[
\int_D \exp\{u,\xi\} Q(t,x,d\xi) = \exp\{ (x, \psi(t,u)) \}, \quad x \in D, u \in U.
\]  

(3.3)

The HA-semigroup \( (Q(t))_{t \geq 0} \) given above is regular; that is, it is stochastically continuous and the derivative \( \psi'_t(0,u) \) exists for all \( u \in U \) and is continuous at \( u = 0 \) (see [K-R]).

**Definition 3.2.** A transition semigroup \( (P(t))_{t \geq 0} \) on \( D \) is called a (general) affine semigroup with the HA-semigroup \( (Q(t))_{t \geq 0} \) if its characteristic function has the representation

\[
\int_D \exp\{u,\xi\} P(t,x,d\xi) = \exp\{ (x, \psi(t,u)) + \phi(t,u) \}, \quad x \in D, u \in U
\]  

(3.4)

where \( \psi(t,\cdot) \) is given in the above definition and \( \phi(t,\cdot) \) is a continuous function on \( U \) satisfying \( \phi(t,0) = 0 \).

Further properties of the affine processes can be found in ([Du 03], [DaLi 06] and [K-R]). We will see that the class of catalytic Ornstein-Uhlenbeck processes we introduce in the next section forms a new class of infinite dimensional affine processes.
4 A brief review on super-processes and their properties

Given a measure $\mu$ on $\mathbb{R}^d$ and $f \in \mathcal{B}(\mathbb{R}^d)$, denote by $\langle \mu, f \rangle := \int_{\mathbb{R}^d} f \, d\mu$, let $M_p(\mathbb{R}^d)$ be the set of finite measures on $\mathbb{R}^d$ and let $C^d(\mathbb{R}^d)_+$ denote the continuous and positive functions, we also define:

$$C_p(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) : \|f(x) \cdot |x|^p\|_\infty < \infty, p > 0\}$$

$$M_p(\mathbb{R}^d) = \{\mu \in M(\mathbb{R}^d) : (1 + |x|^p)^{-1} \, d\mu(x) \text{ is a finite measure}\}$$

**Definition 4.1.** The $(\alpha, d, \beta)$-superprocess $Z_t$ is the measure-valued process, whose Laplace functional is given by:

$$\mathbb{E}_\mu[\exp(-\langle \psi, Z_t \rangle)] = \exp(-\langle U_t \psi, \mu \rangle) \quad \mu \in M_p(\mathbb{R}^d), \, \psi \in C^d(\mathbb{R}^d)_+$$

where $\mu = Z_0$, and $U_t$ is the nonlinear continuous semigroup given by the the mild solution of the evolution equation:

$$\dot{u}(t) = \Delta_\alpha u(t) - u(t)^{1+\beta}, \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 1$$

$$u(0) = \psi, \quad \psi \in D(\Delta_\alpha)_+.$$  \hspace{1cm} (4.5)

Here $\Delta_\alpha$ is the generator of the $\alpha$-symmetric stable process, given by:

$$\Delta_\alpha u(x) = A(d, \alpha) \int_{\mathbb{R}^d} \frac{u(x+y) - u(x)}{|y|^{d+\alpha}} \, dy$$

$$A(d, \alpha) = \pi^{2\alpha-d}/\Gamma((d-2\alpha)/2)/\Gamma(\alpha)$$

Then, $u(t)$ satisfies the non-linear integral equation:

$$u(t) = U_t \psi - \int_0^t U_{t-s}(u^{1+\beta}(s)) \, ds.$$

This class of measure-valued processes was first introduced in [Wat 68], an up-to-date exposition of these processes is given in [Li 11]. It can be verified that the process with the above Laplace functional is a finite measure-valued Markov process with sample paths in $D(\mathbb{R}_+, M_p(\mathbb{R}^d))$. The special case $\alpha = 2$, $\beta = 1$ is called super-Brownian motion (SBM) and this has sample paths in $C(\mathbb{R}_+, M_p(\mathbb{R}^d))$

Consider the following differential operator on $M_p(\mathbb{R}^d)$:

$$LF(\mu) = \frac{1}{2} \int_{\mathbb{R}^d} \mu(dx) \frac{\delta^2 F}{\delta \mu(x)^2} + \int_{\mathbb{R}^d} \mu(dx) \Delta \left( \frac{\delta F}{\delta \mu} \right)(x)$$  \hspace{1cm} (4.6)

Here the differentiation of $F$ is defined by

$$\frac{\delta F}{\delta \mu(x)} = \lim_{\epsilon \downarrow 0} (F(\mu + \epsilon \delta_x) - F(\mu))/\epsilon$$

where $\delta_x$ denotes the Dirac measure at $x$. The domain $\mathcal{D}(L)$ of $L$ will be chosen a class containing such functions $F(\mu) = f(\langle \mu, \phi_1 \rangle, \ldots, \langle \mu, \phi_n \rangle)$ with smooth functions $\phi_1, \ldots, \phi_n$ defined on $\mathbb{R}^d$ having compact support and a bounded smooth function $f$ on $\mathbb{R}^d$. As usual $\langle \mu, \phi \rangle := \int \phi(x) \mu(dx)$.

The super-Brownian motion is also characterized as the unique solution to the martingale problem given by $(L, \mathcal{D}(L))$. The process is defined on the probability space $(\Omega, \mathcal{F}, P_\mu, \{Z_t\}_{t \geq 0})$ and $P_\mu(Z(0) = \mu) = 1, \quad P_\mu(Z \in C([0, \infty), M_p(\mathbb{R}^d))) = 1$.

An important property of super-Brownian motion is the compact support property discovered by Iscoe [Is 88], that is, the closed support $S(Z_t)$ is compact if $S(Z_0)$ is compact.
5 Catalytic OU with the \((\alpha, d, \beta)\)-superprocess as catalyst

5.1 Formulation of the process

The main object of this section is the catalytic OU process given by the solution of

\[
dX(t, x) = \frac{1}{2} \Delta X(t, x) dt + W_{Z_t}(dt, dx), \quad X_0(t, x) \equiv 0
\]

where \(Z_t\) is the \((\alpha, d, \beta)\)-superprocess, in the following discussion, we will assume that the catalyst and the white noise are independent processes.

In order to study this process we first note that conditioned on the process \(\{Z_t\}\), the process \(X\) is Gaussian. We next determine the second moment structure of this Gaussian process.

**Proposition 5.1.** The variance of the process \(X(t)\) given by (5.7), is:

\[
\mathbb{E}X^2(t, x) = \int_0^t \int_{\mathbb{R}^d} p^2(t - s, x, u) Z_s(du) ds
\]

and its covariance by:

\[
\mathbb{E}X(t, x)X(t, y) = \int_0^t \int_{\mathbb{R}^d} p(t - s, x, u)p(t - s, y, u) Z_s(du) ds
\]

**Proof.** In order to compute the covariance of \(X(t)\); recall that the solution of (5.7) is given by the stochastic convolution:

\[
X(t, x) = \int_0^t \int_{\mathbb{R}^d} p(t - s, x, u) W_{Z_s}(ds, du)
\]

From which, the covariance is computed as:

\[
\mathbb{E}X^2(t, x) = \int_0^t \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} p(t - r, x, u)p(t - s, x, w) \mathbb{E}[W_{Z_r}(dr, du)W_{Z_s}(ds, dw)]
\]

The covariance measure is defined by:

\[
\text{Cov}_{W_Z}(dr, ds; du, dw) \quad \triangleq \quad \mathbb{E}[W_{Z_r}(dr, du)W_{Z_s}(ds, dw)]
\]

\[
= \delta_s(r) dr \delta_u(w) Z_r(dw)
\]

The second equality is due to the fact that \(W_{Z_r}\) is a white noise perturbation and has the property of independent increments in time and space.

5.2 Affine structure of the catalytic OU-process

We will compute now the characteristic functional of the annealed process \(\mathbb{E}[\exp(i \langle \phi, X_t \rangle)]\), where by definition:

\[
\langle \phi, X_t \rangle \triangleq \int_{\mathbb{R}^d} X(t, x) \phi(x) dx, \quad \phi \in C(\mathbb{R}^d)
\]

which is well-defined since \(X_t\) is a random field and also the characteristic functional-Laplace functional of the joint process \(\{X(t), Z(t)\}\). For this, we need the following definitions and properties, for further details see [Is 86].
Definition 5.1. Given the $(\alpha, d, \beta)$-superprocess $Z_t$, we define the weighted occupation time process $Y_t$ by

$$<\psi, Y_t> = \int_0^t <\psi, Z_s> ds \quad \psi \in C_p(\mathbb{R}^d).$$

Remark 5.1. The definition coincides with the intuitive interpretation of $Y_t$ as the measure-valued process satisfying:

$$Y_t(B) = \int_0^t Z_s(B) ds, \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d)$$

Theorem 5.2. It can be shown ([Is 86]) that, given $\mu \in M_p(\mathbb{R}^d)$ and $\phi, \psi \in C_p(\mathbb{R}^d)_+$, and $p < d + \alpha$ then the joint process $[Z_t, Y_t]$ has the following Laplace functional:

$$E_{\mu}[\exp(-<\psi, Z_t> - \int_0^t <\phi, Y_s> ds)] = \exp[-<U^\phi t \psi, \mu>], \quad t \geq 0,$$

where $U^\phi t$ is the strongly continuous semigroup associated with the evolution equation:

$$\begin{align*}
\dot{u}(t) &= \Delta_\alpha u(t) - u(t)^{1+\beta} + \phi \\
u(0) &= \psi
\end{align*} \quad (5.8)$$

A similar expression will be derived when $\phi$ is a function of time, but before, we need the following:

Definition 5.2. A deterministic non-autonomous Cauchy problem, is given by:

$$(\text{NACP}) \quad \begin{cases}
\dot{u}(t) = A(t)u(t) + f(t) & 0 \leq s \leq t \\
u(s) = x
\end{cases}$$

where $A(t)$ is a linear operator which depends on $t$.

Similar to the autonomous case, the solution is given in terms of a two parameter family of operators $U(t, s)$, which is called the propagator or the evolution system of the problem (NACP), with the following properties:

- $U(t, t) = I$, \quad $U(t, r)U(r, s) = U(t, s)$.
- $(t, s) \rightarrow U(t, s)$ is strongly continuous for $0 \leq s \leq t \leq T$.
- The solution of (NACP) is given by:

$$u(t) = U(t, s)x + \int_s^t U(t, r)f(r)dr \quad (5.9)$$

- in the autonomous case the propagator is equivalent to the semigroup $U(t, s) = T_{t-s}$.

Theorem 5.3. Let $\mu \in M_p(\mathbb{R}^d)$, and $\Phi : \mathbb{R}_+^1 \rightarrow C_p(\mathbb{R}^d)_+$ be right continuous and piecewise continuous such that for each $t > 0$ there is a $k > 0$ such that $\Phi(s) \leq k \cdot (1 + |x|^p)^{-1}$ for all $s \in [0, t]$. Then

$$E_{\mu}\left[\exp\left(-\langle\Psi, Z_t\rangle - \int_0^t \langle\Phi(s), Z_s\rangle ds\right)\right] = \exp\left(-\langle U_{t, \xi_0}^\Phi \Psi, \mu\rangle\right)$$
where \( U_{t,t_0}^\Phi \) is the non-linear propagator generated by the operator \( Au(t) = \Delta \alpha u(t) - u^{1+\beta}(t) + \Phi(t) \), that is, \( u(t) = U_{t,t_0}^\Phi \Psi \) satisfies the evolution equation:

\[
\dot{u}(s) = \Delta \alpha u(s) - u(s)^{1+\beta} + \Phi(s), \quad t_0 \leq s \leq t
\]

\[u(t_0) = \Psi \geq 0.\] (5.10)

**Proof.** The existence of \( U_{t,t_0}^\Phi \) follows from the fact that \( \Phi \) is a Lipschitz perturbation of the maximal monotone non-linear operator \( \Delta \alpha (\cdot) - u^{1+\beta}(\cdot) \) (refer to [Br] pp. 27).

Note that the solution \( u(t) \) depends continuously on \( \Phi \), to see this, denote by \( V_{\alpha}^t \) the subgroup generated by \( \Delta \alpha \), then the solution \( u(t) \) can be written as:

\[u(t) = V_t(\Psi) + \int_0^t V_{\alpha,t-s}(\Phi(s) - u^{1+\beta}(s)) \, ds\]

from which the continuity follows.

So we will assume first that \( \Phi \) is a step function defined on the partition of \([0,t] \) given by \( 0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = t \) and \( \Phi(t) = \phi_k \) on \([s_{k-1}, s_k] \), \( k = 1, \cdots, N \), we will denote the step function by \( \Phi_N \).

Note also that the integral:

\[\langle \Phi(t), Y_t \rangle = \int_0^t \langle \Phi(s), Z_s \rangle \, ds, \quad \Phi : \mathbb{R}_+^1 \to C_p(\mathbb{R}^d)_+\]

can be taken a.s. in the sense of Riemann since we have enough regularity on the paths of \( Z_t \).

Taking a Riemann sum approximation:

\[
E_\mu \left( \exp \left[ -\langle \Psi, Z_t \rangle - \int_0^t \langle \Phi(s), Z_s \rangle \, ds \right] \right)
\]

\[= \lim_{N \to \infty} E_\mu \left( \exp \left[ -\sum_{k=1}^{N-1} \left( \phi_k, Z_{s_k} \right) \frac{t}{N} - \left( \frac{t}{N} \Phi_N + \Psi, Z_t \right) \right] \right).\]

Denote by \( U_t^\Phi \) and \( U_t^{\Phi_N} \) the non-linear semigroups generated by \( \Delta \alpha u(t) - u^{1+\beta}(t) + \Phi(t) \) and \( \Delta \alpha u(t) - u^{1+\beta}(t) + \Phi_N(t) \) respectively. Conditioning and using the Markov property of \( Z_t \) we can calculate:

\[
E_\mu \left( \exp \left[ -\langle \Psi, Z_t \rangle - \int_0^t \langle \Phi(s), Z_s \rangle \, ds \right] \right)
\]

\[= \lim_{N \to \infty} E_\mu \left( \exp \left[ -\langle \Psi, Z_t \rangle - \sum_{k=1}^{N-1} \left( \phi_k, Z_{s_k} \right) \frac{t}{N} - \int_{s_{k-1}}^{s_k} \langle \phi_k, Z_s \rangle \, ds \right] \right)
\]

\[= \lim_{N \to \infty} E_\mu \left( \exp \left[ -\langle \Psi, Z_t \rangle - \sum_{k=1}^{N-1} \int_{s_{k-1}}^{s_k} \langle \phi_k, Z_s \rangle \, ds - \int_{s_{N-1}}^{s_N} \langle \phi_N, Z_s \rangle \, ds \right] \mid \sigma(Z_s), 0 \leq s \leq s_{N-1} \right)\]
\[
\begin{align*}
&= \lim_{N \to \infty} E_\mu \left( \exp \left( - \sum_{k=1}^{N-1} \int_{s_{k-1}}^{s_k} \langle \phi_k, Z_s \rangle \, ds \right) \right) \mathbb{E}_\mu \left[ \exp(- \langle \Psi, Z_t \rangle - \int_0^t \langle \phi_N, Z_s \rangle \, ds) \right] \\
&\quad \mid \sigma(Z_s), 0 \leq s \leq s_{N-1} \\
&= \lim_{N \to \infty} E_\mu \left( \exp \left( - \sum_{k=1}^{N-1} \int_{s_{k-1}}^{s_k} \langle \phi_k, Z_s \rangle \, ds \right) \right) E_{s_{N-1}} \left[ \exp(- \langle \Psi, Z_t \rangle - \int_0^t \langle \phi_N, Z_s \rangle \, ds) \right] \\
&= \lim_{N \to \infty} E_\mu \left( \exp \left( - \sum_{k=1}^{N-1} \int_{s_{k-1}}^{s_k} \langle \phi_k, Z_s \rangle \, ds \right) \right) \left( \exp(- \langle U_{s_{N-1}}^{\Psi} \Psi, Z_{s_{N-1}} \rangle) \right) \\
&= \lim_{N \to \infty} E_\mu \left( \exp \left[ - \langle U_{s_{N-1}}^{\Psi} \Psi, Z_{s_{N-1}} \rangle \right) \right] \\
&= \lim_{N \to \infty} \exp \left[ - \langle U_{s_{N-1}}^{\Psi} \Psi, Z_{s_{N-1}} \rangle \right] \\
&= \exp \left[ - \langle U_{s_{N-1}}^{\Psi} \Psi, Z_{s_{N-1}} \rangle \right] \\
\end{align*}
\]

where \( U_{s_{N-1}}^{\Phi} \Psi \) is the mild solution of the equation:

\[
\dot{u}(s) = \Delta_\alpha u(s) - u(s)^{1+\beta} + \phi_{N-1}, \quad s_{N-1} \leq s \leq s_N
\]

\[
u(s_{N-1}) = \Psi.
\]

Performing this step \( N \) times, one obtains:

\[
\begin{align*}
&\mathbb{E}_\mu \left( \exp \left[ - \langle \Psi, Z_t \rangle - \int_0^t \langle \phi(s), Z_s \rangle \, ds \right) \right] \\
&= \lim_{N \to \infty} \mathbb{E}_\mu \left( \exp \left[ - \langle U_{s_{N-1}}^{\Psi}, U_{s_{N-2}}^{\Psi}, \ldots, U_0^{\Psi}, Z_{s_{N-1}} \rangle \right) \right) \\
&= \lim_{N \to \infty} \mathbb{E}_\mu \left( \exp \left[ - \langle U_{s_{N-1}}^{\Psi} \Psi, Z_{s_{N-1}} \rangle \right) \right] \\
&= \lim_{N \to \infty} \exp \left[ - \langle U_{s_{N-1}}^{\Psi} \Psi, Z_{s_{N-1}} \rangle \right] \\
&= \exp \left[ - \langle U_{s_{N-1}}^{\Psi} \Psi, Z_{s_{N-1}} \rangle \right]
\end{align*}
\]

where in the fourth line \( U_0^{\Psi} = 1 \). The last equality follows because the solution depends continuously on \( \Phi \) as noted above and the result follows by taking the limit as \( N \to \infty \).

Now let \( X_t \) be the solution of:

\[
dX(t, x) = \frac{1}{2} \Delta X(t, x) dt + W Z_i (dt, dx) \quad X(0, x) \equiv 0
\]

(5.11)

with values in the space of Schwartz distributions on \( \mathbb{R}^d \). The previous result will allow us to compute the characteristic-Laplace functional \( \mathbb{E}[\exp(i \langle \phi, X_t \rangle - \langle \lambda, Z_t \rangle)] \) of the pair \([X_t, Z_t]\).

**Theorem 5.4.** The characteristic-Laplace functional of the joint process \([X_t, Z_t]\) is given by:

\[
\mathbb{E}_\mu [\exp(i \langle \phi, X_t \rangle - \langle \lambda, Z_t \rangle)] = \exp(- \langle u(t, \phi, \lambda), \mu \rangle)
\]

(5.12)
where $u(t, \phi, \lambda)$ is the solution of the equation:

$$\frac{\partial u(s, x)}{\partial s} = \Delta_\nu u(s, x) - u^{1+\beta}(s, x) + G_\phi^2(t-s, x), \quad 0 \leq s \leq t$$

$$u(0, x) = \lambda$$

(5.13)

with $G_\phi$ defined as:

$$G_\phi(t, s, z) = \int_{\mathbb{R}^d} p(t-s, x, z) \phi(x) dx$$

**Proof.** Denote by $\langle \cdot, \cdot \rangle$ the standard inner product of $L^2$ and recalling the following property of the Gaussian processes:

$$E[\exp(i \langle \phi, X_t \rangle)] = \exp(-\text{Var}[\langle \phi, X_t \rangle])$$

We get

$$[\langle \phi, X_t \rangle^2] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} X(t, x) X(t, y) \phi(x) \phi(y) \, dx \, dy$$

Hence:

$$\text{Var}[\langle \phi, X_t \rangle] = E[\langle \phi, X_t \rangle^2] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x) E[X(t, x) X(t, y)] \phi(y) \, dx \, dy$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x) \Gamma_t(x, y) \phi(y) \, dx \, dy$$

where by definition:

$$\Gamma_t(x, y) = E[X(t, x) X(t, y)] = \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) p(t-s, y, z) Z_s(dz) \, ds$$

So:

$$\text{Var}[\langle \phi, X_t \rangle | Z_t] = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x) p(t-s, x, z) p(t-s, y, z) \phi(y) Z_s(dz) \, dx \, dy \, ds$$

$$= \int_0^t \int_{\mathbb{R}^d} g_\phi(t-s, z) Z_s(dz) \, ds$$

In the last line:

$$g_\phi(t-s, z) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(t-s, x, z) p(t-s, y, z) \phi(x) \phi(y) \, dx \, dy$$

$$= \left[ \int_{\mathbb{R}^d} p(t-s, x, z) \phi(x) \, dx \right]^2$$

Note that the function:

$$G_\phi(t-s, z) \equiv \int_{\mathbb{R}^d} p(t-s, x, z) \phi(x) \, dx$$

considered as a function of $s$, satisfies the backward heat equation:

$$\frac{\partial}{\partial s} G_\phi(t-s, z) + \frac{1}{2} \Delta G_\phi(t-s, z) = 0, \quad 0 \leq s \leq t$$
with final condition:

\[ G_\phi(t) = \phi(x) \]

So:

\[ \mathbb{E}[\exp(i \langle \phi, X_t \rangle)] = \exp \left[ - \int_0^t < G_\phi^2(t-s,z), Z_s(dz) > \, ds \right] \]

where \( G_\phi^2 \) is given by the above expression. With this, assuming \( Z_0 = \mu \) we have the following expression for the Laplacian of the joint process \([X_t, Z_t]\\):

\[
\mathbb{E}_\mu[\exp(i \langle \phi, X_t \rangle - \langle \lambda, Z_t \rangle)] \\
= \mathbb{E}_\mu[\mathbb{E}_\mu[\exp(i \langle \phi, X_t \rangle - \langle \lambda, Z_t \rangle)|\sigma(Z_s), 0 \leq s \leq t]] \\
= \mathbb{E}_\mu[\exp(- \langle \lambda, Z_t \rangle)\mathbb{E}_\mu[\exp(i \langle \phi, X_t \rangle)|\sigma(Z_s), 0 \leq s \leq t]] \quad \text{(measurability)} \\
= \mathbb{E}_\mu \left[ \exp \left( - \int_0^t \int_{\mathbb{R}^d} G_\phi^2(t-s,z)Z_s(dz) \, ds - \langle \lambda, Z_t \rangle \right) \right] \\
= \mathbb{E}_\mu \left[ \exp \left( - \int_0^t \langle G_\phi^2(t-s,z), Z_s \rangle \, ds - \lambda \langle 1, Z_t \rangle \right) \right] \quad \text{(by definition)} \\
= \exp(- \langle u(t), \mu \rangle) \quad \text{(by Theorem 5.3)}
\]

and the result follows after renaming the variables. \(\square\)

**Remark 5.2.** We can generalize this to include a second term

\[
dX(t,x) = \frac{1}{2} \Delta X(t,x)dt + W_{Z_t}(dt,dx) + b(t,x)W_2(dt,dx), \quad X(0,x) = \chi(x) \quad (5.14)
\]

where \( Z_t \) is the \((\alpha, d, \beta)\)-superprocess, and \( W_2(dt,dx) \) is space-time white noise. In this case the characteristic-Laplace functional has the form

\[
\mathbb{E}_{\mu, \chi}[\exp(i \langle \phi, X_t \rangle - \langle \lambda, Z_t \rangle)] = \exp \left( - \langle u(t, \phi, \lambda), \mu \rangle - \int \int q(t,x_1,x_2)\phi(x_1)\phi(x_2)dx_1dx_2 + i \int \int p(t,x,y)\chi(x)\phi(y)dxdy \right). \quad (5.15)
\]

This is an infinite dimensional analogue of the general affine property defined in (3.4).

**Remark 5.3.** If in Theorem 5.4, \( \mathbb{R}^d \) is replaced by a finite set \( E \) and \( \frac{1}{2} \Delta \) and \( \Delta_\alpha \) are replaced by the generators of Markov chains on \( E \), then the analogous characterization of the characteristic-Laplace functional remains true and describes a class of finite dimensional (multivariate) affine processes.

### 6 Some properties of the quenched and annealed catalytic O-U processes

In this section we formulate some basic properties of the catalytic O-U process in the case in which the catalyst is a super-Brownian motion, that is \( \alpha = 2 \) and \( \beta = 1 \). For processes in a random catalytic medium
a distinction has to be made between the *quenched* result above, which gives the process conditioned on \( Z \) and the *annealed* case in which the process is a compound stochastic process. In the quenched case the process is Gaussian. The corresponding annealed case leads to a *non-Gaussian* process. These two cases will of course require different formulations. For the quenched case, properties of \( X_t \) are obtained for a.e. realization. On the other hand for the annealed case we obtain results on the *annealed laws* \( P^Z(X(\cdot) \in A) = \int_{C([0,1])} P_\mu(dZ) P^Z(X(\cdot) \in A) \), where \( P^Z(X(\cdot) \in A) \) denotes the probability law for the super-Brownian motion \( \{X_t\} \) in the catalyst \( \{Z(\cdot) \in C([0,\infty),C([0,1]))\} \).

### 6.1 The quenched catalytic OU process

In order to exhibit the role of the dimension of the underling space we now formulate and prove a result for the quenched catalytic O-U process on the set \([0,1]^d \subseteq \mathbb{R}^d\).

**Theorem 6.1.** In dimension \( d \geq 1 \) and with \( Z_0 \in M_F(\mathbb{R}^d) \), consider the initial value problem:

\[
\begin{cases}
  dX(t,x) = AX(t,x)dt + \int_{\mathbb{R}^d} W_Z(t, dx, dt) & t \geq 0, \ x \in [0,1]^d \\
  X(0,x) = 0 & x \in \partial [0,1]^d
\end{cases}
\]

where \( A = \frac{1}{2} \Delta \) on \([0,1]^d\) with Dirichlet boundary conditions. Then for almost every realization of \( \{Z_t\} \), \( X(t) \) has \( \beta \)-Hölder continuous paths in \( H_{-n} \) for any \( n > d/2 \) and \( \beta < \frac{1}{2} \), where \( H_{-n} \) is the space of distributions on \([0,1]\) defined below in subsection 7.1.

### 6.2 The annealed O-U process with super-Brownian catalyst (\( \alpha = 2, \beta = 1 \)):

**State space and sample path continuity**

Now, we apply the techniques developed above to determine some basic properties of the annealed O-U process \( X(\cdot) \) including identification of the state space, sample path continuity and distribution properties of the random field defined by \( X(t) \) for \( t > 0 \).

**Theorem 6.2.** Let \( X(t) \) be the solution of the stochastic equation:

\[
(CE) \quad \begin{cases}
  dX(t,x) = \Delta X(t,x)dt + \int_{\mathbb{R}^d} W_Z(t, dx, dt) & t \geq 0, \ x \in \mathbb{R} \\
  X(0,x) = 0 & x \in \mathbb{R}
\end{cases}
\]

(i) For \( t > 0 \), \( X(t) \) is a zero-mean non-Gaussian leptokurtic random field.

(ii) if \( Z_0 = \delta_0(x) \), then the annealed process \( X(t) \) satisfies

\[
\mathbb{E}\|X(t)\|_2^4 < \infty
\]

and has continuous paths in \( L^2(\mathbb{R}) \)
7 Proofs

In this section we give the proofs of the results formulated in Section 6.

7.1 Proof of Theorem 6.1.

Proof. We first introduce the dual Hilbert spaces $H_n, H_{-n}$. Let $\Delta$ denote the Laplacian on $[0,1]^d$ with Dirichlet boundary conditions. Then $\Delta$ has a CONS of smooth eigenfunctions $\{\phi_n\}$ with eigenvalues $\{\lambda_n\}$ which satisfy $\sum_j (1 + \lambda_j)^{-p} < \infty$ if $p > d/2$, (see [Kr 99]).

Let $E_0$ be the set of $f$ of the form $f(x) = \sum_{j=1}^{N} c_j \phi_j(x)$, where the $c_j$ are constants. For each integer $n$, positive or negative, define the space $H_n = \{ f \in H_0 : \|f\|_n < \infty \}$ where the norm is given by:

$$\|f\|_n^2 = \sum_j (1 + \lambda_j)^n c_j^2.$$

The $H_{-n}$ is defined to be the dual Hilbert space corresponding to $H_n$.

The solution of (6.1) is given by:

$$X(t,x) = \int_0^t \int_{[0,1]^d} e^{-\lambda_k (t-s)} \phi_k(x) \phi_k(y) W_z(dy,ds)$$

Let

$$A_k(t) = \int_0^t \int_{[0,1]^d} e^{-\lambda_k (t-s)} \phi_k(y) W_z(dy,ds)$$

Then

$$X(t,x) = \sum_{k \geq 1} \phi_k(x) A_k(t)$$

we will show that $X(t)$ has continuous paths on the space $H_n$, which is isomorphic to the set of formal eigenfunction series

$$f = \sum_{k=1}^{\infty} a_k \phi_k$$

for which

$$\|f\|_n = \sum_{k=1}^{\infty} a_k^2 (1 + \lambda_k)^n < \infty$$

Fix some $T > 0$, we first find a bound for $\mathbb{E} \left\{ \sup_{t \leq T} A_k^2(t) \right\}$, let $V_k(t) = \int_0^t \int_{[0,1]^d} \phi_k(x) W(Z_s(dx), ds)$, integrating by parts in the stochastic integral, we obtain:

$$A_k(t) = \int_0^t e^{-\lambda_k (t-s)} dV_k(s) = V_k - \int_0^t \lambda_k e^{-\lambda_k (t-s)} V_k(s) ds$$

Thus:

$$\sup_{t \leq T} |A_k(t)| \leq \sup_{t \leq T} |V_k(t)| (1 + \int_0^t \lambda_k e^{\lambda_k (t-s)} ds) \leq 2 \sup_{t \leq T} |V_k(t)|$$

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Hence:
\[
\mathbb{E} \left\{ \sup_{t \leq T} A_k^2(t) \right\} \leq 4 \mathbb{E} \left\{ \sup_{t \leq T} V_k^2(t) \right\} \\
\leq 16 \mathbb{E} \left\{ V_k^2(T) \right\} \quad \text{(Doob’s inequality)} \\
= 16 \int_{[0,1]^d} \int_0^T \phi_k^2(x) Z_s(dx) ds \leq 16 T Z_T([0,1]^d) = CT.
\]

Therefore:
\[
\mathbb{E} \left( \sum_{k \geq 1} \sup_{t \leq T} A_k^2(t) (1 + \lambda_k)^{-n} \right) \leq CT \sum_{k \geq 1} (1 + \lambda_k)^{-n}.
\]

Using now:
\[
\sum_{k \geq 1} (1 + \lambda_k)^{-p} < \infty \quad \text{if } p > d/2
\]
then (7.16) is finite if \(n > d/2\) and clearly:
\[
E[\|X(t)\|_{L^2}^2] = \sum_{k \geq 1} [A_k^2(t)](1 + \lambda_k)^{-n} < \infty \quad (7.17)
\]
and hence \(X(t) \in H_{-n}\) a.s.. Moreover, if \(s > 0\),
\[
\|X(t + s) - X(t)\|_{L^2}^2 = \sum E[(A_k(t + s) - A_k(t))^2](1 + \lambda_k)^{-n} \leq Cs.
\]
Then since conditioned on \(Z\), \(X(t)\) is Gaussian, (7.17) together with [DpZ 92], Proposition 3.15, implies that there is a \(\beta\)-Hölder continuous version with \(\beta < \frac{1}{2}\).

\[ \square \]

### 7.2 Second moment measures of SBM

In this section we evaluate the second moment measures of SBM, \(\mathbb{E}(Z_1(dx)Z_2(dy))\), which are needed to compute \(\mathbb{E}[\|Z\|_{L^2}^4]\) in the the annealed catalytic OU process.

To accomplish this, given any two random measures \(Z_1\) and \(Z_2\), then it is easy to verify that:
\[
\mathbb{E}(Z_1(A), Z_2(A)), \quad A \in \mathcal{B}(\mathbb{R}^d) \\
\mathbb{E}(Z_1(A) \cdot Z_2(B)), \quad A, B \in \mathcal{B}(\mathbb{R}^d)
\]
are well defined measures which will be called the first and second moment measures. Similarly, given the measure-valued process \(\{Z_t\}_{t \geq 0}\) one can define the n-th moment measure of n given random variables denoted by:
\[
\mathcal{M}_{t_1, \ldots, t_n}(dx_1, \ldots, dx_n) = \mathbb{E}(Z_{t_1}(dx_1)Z_{t_2}(dx_2) \cdots Z_{t_n}(dx_n)).
\]

The main result of this section, is the following:

**Proposition 7.1.** The super-Brownian motion \(Z_t\) in \(\mathbb{R}^1\) has the following first and second moment measures:

(i) if \(Z_0 = dx\), \(\mathcal{M}_t(dx) = dx\) the Lebesgue measure.

(ii) if \(Z_0 = \delta_0(x)\), then:
\[
\mathcal{M}_t(dx) = p(t, x) \cdot dx
\]
(iii) if \( Z_0 = dx \):

\[
M_t(dx_1 dx_2) = \left( \int_0^t p(2s, x_1, x_2) ds \right) \cdot dx_1 dx_2
\]

(7.18)

(iv) if \( Z_0 = \delta_0(x) \):

\[
M_t(dx_1 dx_2) = \left( \int_0^t \int_R p(t-s, 0, y)p(2s, x_1, x_2) dy ds \right) \cdot dx_1 dx_2
\]

(7.19)

where \( dx, dx_1 dx_2 \) denote the Lebesgue measures on \( \mathbb{R} \) and \( \mathbb{R}^2 \), respectively.

(v) if \( t_1 < t_2 \) and \( Z_0 = dx \):

\[
M_{t_1,t_2}(dx_1 dx_2) = \left( \int_0^{t_1} p(2s + t_2 - t_1, x_1, x_2) ds \right) \cdot dx_1 dx_2
\]

(7.20)

(vi) if \( t_1 < t_2 \) and \( Z_0 = \delta_0(x) \):

\[
M_{t_1,t_2}(dx_1 dx_2) = \left( \int_0^{t_1} \int_R p(t_1 - s, 0, y)p(s, y, x_1)p(t_2 - t_1 + s, y, x_2) dyds \right) \cdot dx_1 dx_2
\]

(7.21)

where \( dx, dx_1 dx_2 \) denote the Lebesgue measures on \( \mathbb{R} \) and \( \mathbb{R}^2 \), respectively.

Proof. Consider SBM \( Z_t \) with \( Z_0 = \mu \). Then for \( \lambda \in \mathbb{R}_+ \) and \( \phi \in C_+(\mathbb{R}^d) \):

\[
\mathbb{E}_\mu[\exp(-\lambda \langle \phi, Z_t \rangle)] = \exp[-\langle u(\lambda, t), \mu \rangle] = \exp(-F(\lambda))
\]

where, \( u(\lambda, t) \) satisfies the equation:

\[
\dot{u}(t) = \Delta u(t) - u^2(t)
\]

\[
u(\lambda, 0) = \lambda \phi
\]

Then the first moment is computed according to

\[
\mathbb{E}\left( \int \phi(x) Z_t(dx) \right) = \left[ \frac{\partial e^{F(\lambda)}}{\partial \lambda} \right]_{\lambda=0} = F'(0)
\]

since \( F(0) = 0 \). \( F(\lambda) \) will be developed as a Taylor series below. First rewrite (7.22) as a Volterra integral equation of the second kind, namely:

\[
u(t) + \int_0^t T_{t-s}(u^2(s)) \, ds = T_t(\lambda \phi)
\]

whose solution is given by its Neumann series:

\[
u(t) = T_t(\lambda \phi) + \sum_{k=1}^{n} (-1)^k \tau^k (T_t(\lambda \phi))
\]

(7.23)

where the operators \( T_t \) and \( \tau \) are defined by:
\[ T_i(\lambda \phi(x)) = \lambda \int_{\mathbb{R}} p(t, x, y) \phi(y) \, dy \]

\[ T(T_i(\lambda \phi)) = -\int_0^t T_{t-s}[(T_s(\lambda \phi))^2] \, ds - \lambda^2 \int_0^t T_{t-s}[(T_s)^2] \, ds \]

\[ = -\lambda^2 \int_0^t \int_{\mathbb{R}} p(t-s, x, y) \left( \int_{\mathbb{R}} p(s, y, z) \phi(z) \, dz \right)^2 \, dy \, ds. \tag{7.24} \]

We obtain \( F(\lambda) = \sum_{n=1}^{\infty} (-1)^{n+1} c_n \lambda^n \), in particular:

\[ c_1 = \int_{\mathbb{R}} T_i \phi \mu(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} p(t, x, y) \phi(y) \, dy \, \mu(dx) \tag{7.25} \]

\[ c_2 = \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} p(t-s, x, y) \left( \int_{\mathbb{R}} p(s, y, z) \phi(z) \, dz \right)^2 \, dy \, ds \, \mu(dx) \]

\[ = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} p(t-s, x, y) p(s, y, z_1) p(s, y, z_1) \phi(z_1) \phi(z_2) \, dz_1 \, dz_2 \, dy \, \mu(dx) \, ds \tag{7.26} \]

\[ c_4 = \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} p(t-s, x, w) \left[ \int_0^s \int_{\mathbb{R}} p(s-s_1, w, y) \left( \int_{\mathbb{R}} p(s, y, z) \phi(z) \, dz \right)^2 \, dy \, ds_1 \right]^2 \, ds \, dw \, \mu(dx) \tag{7.27} \]

and so on.

Taking now \( d = 1, Z_0 = dx \), a given Borel set \( A \in \mathcal{B}(\mathbb{R}) \) and \( \phi = I_A \), we obtain from (7.26):

\[ \mathbb{E} \langle A, Z_t \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} p(t, x, y) \phi(y) \, dy \, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} p(t, x, y) \, dx \, I_A(y) \, dy = \int_A dy \]

i.e. \( \mathbb{E}\{Z_t(dx)\} \) is the Lebesgue measure.

The covariance measure \( \mathbb{E}(Z_t(dx_1)Z_t(dx_2)) \) can now be computed applying the same procedure to \( \phi = \lambda_1 I_{A_1} + \lambda_2 I_{A_2} \) in which case we conclude that \( -\langle u(t), \mu \rangle = F(\lambda_1, \lambda_2) \) is again a Taylor series in \( \lambda_1 \), and \( \lambda_2 \), with constant coefficient equal to zero, and only the coefficient of \( \lambda_1 \lambda_2 \) has to be computed:

\[ T(T_i \phi) = -\int_0^t T_{t-s}[(T_s)^2] \, ds \]

\[ = -\int_0^t \left[ T_{t-s} \lambda_1^2 (T_s I_{A_1})^2 + 2 \lambda_1 \lambda_2 T_s I_{A_1} T_s I_{A_2} + \lambda_2^2 (T_s I_{A_2})^2 \right] \, ds \]

\[ = -\lambda_1^2 \int_0^t T_{t-s}[(T_s I_{A_1})^2] \, ds - 2 \lambda_1 \lambda_2 \int_0^t T_{t-s} [T_s I_{A_1} \cdot T_s I_{A_2}] \, ds \]

\[ - \lambda_2^2 \int_0^t T_{t-s}[(T_s I_{A_2})^2] \, ds \]  \tag{7.28}\]

So:

\[ \mathbb{E}(Z_t(A_1)Z_t(A_2)) = \frac{1}{2} \left[ \frac{\partial^2 F(\lambda_1, \lambda_2)}{\partial \lambda_1 \partial \lambda_2} \right]_{\lambda_1=\lambda_2=0} \]

\[ = \int_0^t \int_{\mathbb{R}} T_{t-s} [T_s I_{A_1} \cdot T_s I_{A_2}] \, \mu(dx) \, ds. \tag{7.29} \]
In order to obtain the density for the measure \( E(Z_t(dx_1)Z_t(dx_2)) \), we write in detail the last integral as follows:

\[
\int^t_0 \int_R \int_R p(t-s,x,y) \left( \int_{A_1} p(s,y,z_1)dz_1 \cdot \int_{A_2} p(s,y,z_2)dz_2 \right) \mu(dx) \, ds \, dy
\]

applying Fubini yields:

\[
\int_{A_1} \int_{A_2} \int^t_0 \int_R p(t-s,x,y)p(s,y,z_1)p(s,y,z_2) \, dy \, \mu(dx) \, ds \, dz_1 \, dz_2
\]

from which, we conclude that \( E(Z_t(dx_1)Z_t(dx_2)) \) has the following density w.r.t. Lebesgue measure:

\[
\int^t_0 \int_R p(t-s,x,y) \mu(dx) \, ds
\]  

(7.30)

assuming \( \mu(dx) \) is the Lebesgue measure in \( \mathbb{R} \) and \( dx_1 \cdot dx_2 \) is the Lebesgue measure in \( \mathbb{R}^2 \) one obtains the second moment measure \( \mathcal{M}_t(dx_1dx_2) \) defined as:

\[
\mathcal{M}_t(dx_1dx_2) = \left( \int^t_0 p(2s,x_1,x_2)ds \right) \cdot dx_1 \, dx_2
\]  

(7.31)

For the case \( t_1 \neq t_2 \), assume \( t_1 < t_2 \) then:

\[
\mathbb{E} \left[ \int \phi_1 Z_{t_1} \cdot \int \phi_2 Z_{t_2} \right]
= \mathbb{E} \left[ \int \phi_1 Z_{t_1} \cdot \int \phi_2 Z_{t_2} \right] \left\{ \sigma(Z_r) : 0 \leq r \leq t_1 \right\} \\
= \mathbb{E} \left[ \int \phi_1 Z_{t_1} \cdot \mathbb{E} \left[ \int \phi_2 Z_{t_2} | \mathcal{F}_{Z_{t_1}} \right] \right] \\
= \mathbb{E} \left[ \int \phi_1 Z_{t_1} \cdot T_{t_2-t_1} \left( \int \phi_2 Z_{t_1} \right) \right] \\
= \mathbb{E} \left[ \int \phi_1 Z_{t_1} \cdot T_{t_2-t_1} (\phi_2) Z_{t_1} \right] \\
= \mathbb{E} \left[ \langle \phi_1, Z_{t_1} \rangle \langle T_{t_2-t_1} (\phi_2), Z_{t_1} \rangle \right]
\]

so, replacing in (7.20) \( I_{A_2} \) by \( T_{t_2-t_1} (I_{A_2}) \) and performing the same analysis, we can define following measure on \( \mathbb{R}^2 \)

\[
\mathcal{M}_{t_1t_2}(dx_1dx_2) = \left( \int^t_0 p(2s + t_2 - t_1, x_1, x_2)ds \right) \cdot dx_1 \, dx_2.
\]

Finally, note that if \( Z_0 = \delta_0(x) \), then (7.30) yields:

\[
\mathcal{M}_{t_1t_2}(dx_1dx_2) = \left( \int^t_0 \int_R p(t_1 - s, 0, y)p(s,y,x_1)p(t_2 - t_1 + s, y, x_2) \, dy \, ds \right) \cdot dx_1 \, dx_2
\]

\[\square\]

**Remark 7.1.** The above procedure can also be used for any \( \alpha \in (0,2], \beta = 1 \), any dimension \( d \geq 1 \) and any \( C_0 \) semigroup \( S_t \) with probability transition function \( p(t,x,y) \).
7.3 Proof of Theorem 6.2

We can now use the results of the last subsection to determine properties of the solutions of (5.7).

(i) It follows from the form of the characteristic function (given by (5.12) with $\beta = 1$ in (5.13)) that the log of the characteristic function of $\langle \lambda \phi, X_t \rangle$ is not a quadratic in $\lambda$ and in fact the fourth cumulant is positive so that the random field is leptokurtic.

(ii) Recall that

$$X(t, x) = \int_0^t \int_{\mathbb{R}} p(t-s, x, y)W_{Z_s}(ds, dy)$$

Lemma 7.2.

$$E[\|X_t\|_{L^2}^4] = CE \left[ \int_0^t \int_{\mathbb{R}} Z_s(dy) \sqrt{t-s} ds \right]^2$$

$$+ 2 \int_{\mathcal{D}_2} p^2(2t-s_1-s_2, y_1, y_2)E(Z_{s_1}(dy_1)Z_{s_2}(dy_2)) ds_1 ds_2.$$  \hspace{1cm} (7.32)

Proof. Using the shorthand $p = p(t-s, x, y), dW^s = W_{Z_s}(ds, dy); p_i(x) = p(t-s_i, x, y_i), p_{ij} = p(t-s_i, x_j, y_i), dW^{s_i} = W_{Z_{s_i}}(ds_i, dy_i)$, for $i = 1, \cdots, j = 1, 2$ as well as $\mathcal{D}_1 = [0,t] \times \mathbb{R}, \mathcal{D}_2 = [0,t]^2 \times \mathbb{R}^2, \mathcal{D}_4 = [0, t]^4 \times \mathbb{R}^4$, and noting that $p_{ij} = p_i(x_j)$, we first compute:

$$E[\|X_t\|_{L^2}^4 \mid Z_s, 0 \leq s \leq t] = E \left[ \int_{\mathbb{R}} \left( \int_0^t \int_{\mathbb{R}} p(t-s, x, y)W_{Z_s}(ds, dy) \right)^2 dx \right]^2$$

$$= E \left[ \int_\mathcal{D}_1 \left( \int_{\mathbb{R}} p_i(x) \mathbb{I} dx \right)^2 \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p_i(x) \mathbb{I} dx \right)^2 \mathbb{I} dx \right] \right]$$

$$= E \left[ \int_\mathcal{D}_1 \left( \int_{\mathbb{R}} p_1(x_1) \mathbb{I} dx \right) \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p_2(x_2) \mathbb{I} dx \right)^2 \mathbb{I} dx \right] \right]$$

$$\cdot \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p_3(x_3) \mathbb{I} dx \right) \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p_4(x_4) \mathbb{I} dx \right)^2 \mathbb{I} dx \right] \right]$$

$$= E \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p_{11}(x_1) \mathbb{I} dx \right) \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p_{12}(x_2) \mathbb{I} dx \right) \mathbb{I} dx \right] \right]$$

$$\cdot \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p_{13}(x_3) \mathbb{I} dx \right) \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p_{14}(x_4) \mathbb{I} dx \right) \mathbb{I} dx \right] \right]$$

$$= E \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p_{12}(x_2) \mathbb{I} dx \right) \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p_{13}(x_3) \mathbb{I} dx \right) \mathbb{I} dx \right] \right]$$

$$\cdot \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p_{14}(x_4) \mathbb{I} dx \right) \mathbb{I} dx \right]$$

$$= \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3.$$

Each one of the above integrals can be evaluated using the independence of the increments of the Wiener process, according to the following cases:

Case 1: $s_1 = s_2 \wedge s_3 = s_4$, then: $p_{11}p_{21} = p_{11}^2(x_1), p_{32}p_{42} = p_{32}^2(x_2), \text{ and } E(dW_t^s dW_z^{s_2} dW_t^{s_4} dW_t^{s_4}) = Z_{s_1}(dy_1)ds_1Z_{s_3}(dy_3)ds_3$, hence:
\[
\mathbb{I}_1 = \int_R^2 \int_D^2 p^2(x_1)p^2(x_2) Z_{s_1}(dy_1)Z_{s_2}(dy_2)ds_1 ds_2 dW_1 dW_2 \\
= \left[ \int_R^2 \int_D^2 p^2(x_1) Z_{s_1}(dy_1)ds_1 dx_1 \right] \left[ \int_R^2 \int_D^2 p^2(x_2) Z_{s_2}(dy_2)ds_2 dx_2 \right] \\
= \left[ \int_R^2 \int_D^2 p^2 Z_s(dy)dsx \right]^2 \\
= \left[ \int_0^t \int_R^2 \int_R^2 p^2 dx Z_s(dy)ds \right]^2 \\
= C \left[ \int_0^t \int_R \frac{Z_s(dy)}{\sqrt{t-s}} ds \right]^2 \\
+ 2 \int_{D_2} p^2(2t - s_1 - s_2, y_1, y_2) Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2
\]

Case 2: \( s_1 = s_3 \land s_2 = s_4 \), then: \( p_{11}p_{32} = p_{11}p_{12} = p_1(x_1)p_1(x_2), p_{21}p_{42} = p_{21}p_{22} = p_2(x_1)p_2(x_2) \), and
\[
\mathbb{E}(dW^{s_1}dW^{s_2}dW^{s_3}dW^{s_4}) = Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2 , \text{ hence:}
\]
\[
\mathbb{I}_2 = \int_R^2 \int_D^2 p(x_1)p_2(x_2)p_1(x_2)p_2(x_1) Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2 dx_1 dx_2 \\
= \int_{D_2} \left( \int_R p(x_1)p_2(x_1) dx_1 \cdot \int_R p(x_2)p_2(x_2) dx_2 \right) Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2 dx_1 \\
= \int_{D_2} p^2(2t - s_1 - s_2, y_1, y_2) Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2
\]

Case 3: \( s_1 = s_3 \land s_2 = s_3 \), then: \( p_{11}p_{42} = p_{11}p_{12} = p_1(x_1)p_1(x_2), p_{21}p_{32} = p_{21}p_{22} = p_2(x_1)p_2(x_2) \), and
\[
\mathbb{E}(dW^{s_1}dW^{s_2}dW^{s_3}dW^{s_3}) = Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2, \text{ and we get the same result as before, namely:}
\]
\[
\mathbb{I}_3 = \int_{D_2} p^2(2t - s_1 - s_2, y_1, y_2) Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2
\]

Putting everything together, yields:

\[
\mathbb{E}[\|X_t\|_{L_2}^4 | Z_s, 0 \leq s \leq t] \\
= C \left[ \int_0^t \int_R \frac{Z_s(dy)}{\sqrt{t-s}} ds \right]^2 + 2 \int_{D_2} p^2(2t - s_1 - s_2, y_1, y_2) Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2)ds_2
\]

(7.33)

So:

\[
\mathbb{E}[\|X_t\|_{L_2}^4] = \mathbb{E} \left( \mathbb{E}[\|X_t\|_{L_2}^4 | Z_s, 0 \leq s \leq t] \right) \\
= CE \left[ \int_0^t \int_R \frac{Z_s(dy)}{\sqrt{t-s}} ds \right]^2 + 2 \int_{D_2} p^2(2t - s_1 - s_2, y_1, y_2) Z_{s_1}(dy_1)ds_1 Z_{s_2}(dy_2) ds_1 ds_2.
\]

(7.34)

We now return to the proof of Theorem 6.2
Proof. The first term on the right of (7.32) is evaluated below

\[ \mathbb{E} \left[ \frac{Z_s(dx)}{\sqrt{t-s}} \right]^2 = \]

\[ = \mathbb{E} \left[ \int_0^t \int_0^t \frac{1}{\sqrt{(t-s_1)(t-s_2)}} \int \int Z_{s_1}(dx_1)Z_{s_2}(dx_2) \ ds_1 \ ds_2 \right] \]

\[ = \int_0^t \int_0^t \left( \frac{1}{(t-s_1)(t-s_2)} \right)^{1/2} \int \int \mathbb{E}[Z_{s_1}(dx_1)Z_{s_2}(dx_2)] \ ds_1 \ ds_2 \]

\[ + \int_0^t \int_0^t \left( \frac{1}{(t-s_1)(t-s_2)} \right)^{1/2} \int \int \mathbb{E}[Z_{s_1}(dx_1)Z_{s_2}(dx_2)] \ ds_1 \ ds_2 \]

(7.35)

Assume now \( Z_0 = \delta_0(x) \), and \( s_1 \leq s_2 \), using (7.21), one obtains:

\[ \int \int \mathbb{E}[Z_{s_1}(dx_1)Z_{s_2}(dx_2)] = \int_0^{s_1} \int_0^t p(s_1-s,0,y) \ dy \ ds = \int_0^{s_1} \ ds = s_1 \]

and similarly, for \( s_2 \leq s_1 \):

\[ \int \int \mathbb{E}[Z_{s_1}(dx_1)Z_{s_2}(dx_2)] = s_2 \]

so that (7.35) can be written as:

\[ \mathbb{E} \left[ \int_0^t \int_0^t \frac{Z_s(dx)}{\sqrt{t-s}} \right]^2 = \int_0^t \int_0^{s_2} \frac{s_1 ds_1 s_2 ds_2}{(t-s_1)(t-s_2)}^{1/2} + \int_0^t \int_0^{s_2} \frac{s_2 ds_1 ds_2}{(t-s_1)(t-s_2)}^{1/2} \]

the first of the above integrals one can be directly done:

\[ \frac{1}{(t-s_1)^{1/2}} \int_0^{s_2} \frac{s_1 ds_1}{(t-s_1)^{1/2}} = \frac{1}{(t-s_2)^{1/2}} \left( \frac{4}{3} t^{3/2} + \frac{2}{3} (t-s_2)^{3/2} - 2t(t-s_2)^{1/2} \right) \]

\[ \leq \frac{1}{(t-s_2)^{1/2}} \left( \frac{4}{3} t^{3/2} + \frac{2}{3} (t-s_2)^{3/2} \right) \]

\[ = \frac{4}{3} t^{3/2} + \frac{2}{3} \]

Hence:

\[ \int_0^t \int_0^{s_2} \frac{s_1 ds_1 s_2 ds_2}{(t-s_1)(t-s_2)}^{1/2} \leq \frac{4}{3} t^2 + \frac{2}{3} t^2 = 2t^2 \]

similarly, for the second integral:

\[ \frac{s_2}{(t-s_2)^{1/2}} \int_0^t \frac{ds_1}{(t-s_1)^{1/2}} = s_2 \]

and:

\[ \int_0^t \int_0^{s_2} \frac{s_2 ds_1 ds_2}{(t-s_1)(t-s_2)}^{1/2} \leq \frac{t^2}{2}. \]

Both inequalities together imply:

\[ \mathbb{E} \left[ \int_0^t \int_0^t \frac{Z_s(dx)}{\sqrt{t-s}} \right]^2 \leq C_1 t^2 \]

(7.36)

Similarly, we estimate below the second integral of (7.32):
\[
\int_{D_2} p^2(2t - s_1 - s_2, x_1, x_2) \mathbb{E}(Z_{s_1}(dx_1)Z_{s_2}(dx_2)) \, ds_1ds_2 = \\
= \int_0^t \int_0^{s_2} \left( \int_\mathbb{R} \int_\mathbb{R} p^2(2t - s_1 - s_2, x_1, x_2) \mathbb{E}(Z_{s_1}(dx_1)Z_{s_2}(dx_2)) \right) \, ds_1ds_2 \\
+ \int_0^t \int_0^{s_2} \left( \int_\mathbb{R} \int_\mathbb{R} p^2(2t - s_1 - s_2, x_1, x_2) \mathbb{E}(Z_{s_1}(dx_1)Z_{s_2}(dx_2)) \right) \, ds_1ds_2
\]

when \(0 \leq s_1 \leq s_2\). Using (7.21) with \(0 \leq s \leq s_1 \leq s_2 \leq t\), we obtain the following estimate for:

\[
\mathbb{E}\left[\mathbb{I}_1(t, s_1, s_2)\right] = \int_\mathbb{R} \int_\mathbb{R} p^2(2t - s_1 - s_2, x_1, x_2) \mathbb{E}(Z_{s_1}(dx_1)Z_{s_2}(dx_2)) \\
= \frac{\sqrt{2}}{\sqrt{2t - s_1 - s_2}} \int_0^{s_2} \frac{1}{\sqrt{(2t - s_1 - s_2) + 4s}} \, ds \\
\leq \frac{\sqrt{2}}{\sqrt{2t - s_1 - s_2}} \int_0^{s_2} \frac{1}{4s} \, ds \\
\leq C_1 \sqrt{\frac{s_2}{2t - s_1 - s_2}}
\]

so, we obtain:

\[
\mathbb{I}_1 = \int_0^t \int_0^{s_2} \mathbb{E}\left[\mathbb{I}_1(t, s_1, s_2)\right] \, ds_1ds_2 \leq C_1 \int_0^t \int_0^{s_2} \sqrt{\frac{s_2}{2t - s_1 - s_2}} \, ds_1 \, ds_2 \\
= C_1 \int_0^t \sqrt{s_2} \left[2\sqrt{2t - s_2} - 2\sqrt{2t - 2s_2}\right] \, ds_2 \tag{7.37} \\
\leq C_2 \int_0^t \sqrt{s_2} \sqrt{t} \, ds_2 \leq C_3 t^2.
\]

When \(0 \leq s_2 \leq s_1\), using (7.21) with \(0 \leq s \leq s_2 \leq s_1 \leq t\), and following the same steps above for the integral:

\[
\mathbb{I}_2 \leq C_1 \sqrt{\frac{t}{2t - s_1 - s_2}}
\]

therefore:

\[
\mathbb{I}_2 = \int_0^t \int_0^{s_2} \mathbb{I}_2(t, s_1, s_2) \, ds_1ds_2 \leq C_4 \int_0^t \int_0^{s_2} \sqrt{\frac{t}{2t - s_1 - s_2}} \, ds_1 \, ds_2 \\
= C_5 \int_0^t \sqrt{t} \left[2\sqrt{2t - s_2} - 2\sqrt{t - s_2}\right] \, ds_2 \tag{7.38} \\
\leq C_6 \int_0^t \sqrt{t} \sqrt{t} \, ds_2 = C_6 t^2.
\]

Gathering (7.36), (7.37) and (7.38) yields:

\[
\mathbb{E}[\|X_t\|_{L_2}^4] \leq Ct^2.
\]

In the same way we can obtain the estimates

\[
\mathbb{E}[\|X_t - X_s\|_{L_2}^4] \leq C(t - s)^2.
\]

for the increments which shows the continuity of the paths using Kolmogorov’s criteria.
8 Comments and Open Problems

1. Our results correspond to the analogue of the Heston model (HM) with $\rho = 0$. The general case when $\rho \neq 0$ would require additional techniques and is left as an open problem.

2. The study of the properties of the annealed case for arbitrary $\alpha, \beta \neq 1$ and for the more general continuous state branching involves a catalyst with infinite second moments and is left as an open problem.

3. It would be interesting to determine the state space for annealed process in $\mathbb{R}^d$ with $d > 1$.

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