We present a novel solution to the problem of localizing magnetoencephalography (MEG) and electroencephalography (EEG) brain signals. The solution is sequential and iterative, and is based on minimizing the least-squares criterion by the Alternating Projection algorithm. Results from simulated and experimental MEG data from a human subject demonstrated robust performance, with consistently superior localization accuracy than scanning methods belonging to the beamformer and multiple-signal classification (MUSIC) families. Importantly, the proposed solution is more robust to forward model errors resulting from head rotations and translations, with a significant advantage in highly correlated sources.

Index Terms— source localization, MEG, EEG, forward model errors, least squares, alternating projection.

1. INTRODUCTION

Brain signals obtained by an array of magnetic or electric sensors using MEG or EEG offer the potential to investigate the complex spatiotemporal activity of the human brain. Localization of MEG and EEG sources has gained much interest in recent years since it can reveal the origins of neural signals during both normal and pathological brain function [1].

Mathematically, the localization problem can be cast as finding the location and moment of the set of dipoles whose field best matches the MEG/EEG measurements in a least-squares (LS) sense [2]. In this paper we focus on dipole fitting and scanning methods, which solve for a small parsimonious set of dipoles and avoid the ill-posedness associated with imaging methods, such as minimum-norm [3].

The dipole fitting methods solve the optimization problem directly using techniques such as gradient descent, Nedler-Mead simplex algorithm, multistart, genetic algorithm, and simulated annealing [4][5]. However, these techniques remain unpopular due to either convergence to sub-optimal solutions or high computational complexity.

The scanning methods use a localizer function and find the dipole positions by searching for the local maxima of the localizer function throughout a discrete grid representing the source space [6]. The most common scanning methods are beamformers [7][8] and MUSIC [2], both widely used for bioelectromagnetic source localization. Yet, when the correlation between the sources is significant, these methods result in partial or complete cancellation of the correlated sources and to distorted estimate of the time courses. The scanning methods are categorized as non-recursive or recursive. The original Beamformer [7] and MUSIC [2] are non-recursive, and the localization of multiple sources require the identification of the largest local maxima in the localizer function. Some multi-source variants are also non-recursive (e.g. [9][12]), and they either use brute-force optimization, assuming that the approximate locations of the sources have been identified a priori, or still require the identification of the largest local maxima in the localizer function. To overcome these limitations, recursive counterparts of the non-recursive methods which search for one source at a time, such as RAP-MUSIC [13], Truncated RAP-MUSIC [14], and RAP Beamformer [1], have been developed. While recursive methods generally perform better than their non-recursive counterparts, they still suffer from several limitations, including limited performance, the need for high signal-to-noise ratio (SNR), or inaccurate estimation as correlation values increase.

This paper presents a novel solution which overcomes these limitations. Our starting point is the LS estimation criterion for the dipole fitting problem. This criterion yields a multi-source nonlinear and nonconvex minimization problem, making it very challenging to avoid being trapped in undesirable local minima [15]. To overcome this challenge, we propose the use of the Alternating Projection (AP) algorithm [16]. The AP algorithm transforms the multi-source problem to an iterative process involving only single-source maximization problems, which are computationally much simpler. Moreover, the algorithm has a very effective initialization scheme which is the key to its good convergence. Here we demonstrate the robustness of the AP algorithm with MEG data, but the method is directly applicable to EEG.

2. PROBLEM FORMULATION

In this section we briefly review the notations used to describe measurement data, forward matrix, and sources, and formulate the problem of estimating current dipoles. Consider an array of $M$ MEG or EEG sensors that measures data from a
finite number \( Q \) of equivalent current dipole (ECD) sources emitting signals \( \{ s_q(t) \}_{q=1}^Q \) at locations \( \{ p_q \}_{q=1}^Q \). Under these assumptions, the \( M \times 1 \) vector of the received signals by the array is given by:

\[
y(t) = \sum_{q=1}^Q I(p_q)s_q(t) + n(t),
\]

where \( I(p_q) \) is the topography of the dipole at location \( p_q \) and \( n(t) \) is the additive noise. The topography \( I(p_q) \), is given by:

\[
I(p_q) = L(p_q)q,
\]

where \( L(p_q) \) is the \( M \times 3 \) forward matrix at location \( p_q \) and \( q \) is the \( 3 \times 1 \) vector of the orientation of the ECD source. Depending on the problem, the orientation \( q \) may be known, referred to as fixed-oriented dipole, or it may be unknown, referred to as freely-oriented dipole. Assuming that the array is sampled \( N \) times at \( t_1, ..., t_N \), the matrix \( Y \) of the sampled signals can be expressed as:

\[
Y = A(P)S + N,
\]

where \( Y \) is the \( M \times N \) matrix of the received signals:

\[
Y = [y(t_1), ..., y(t_N)],
\]

\( A(P) \) is the \( M \times Q \) mixing matrix of the topography vectors at the \( Q \) locations \( P = [p_1, ..., p_Q] \):

\[
A(P) = [I(p_1), ..., I(p_Q)],
\]

\( S \) is the \( Q \times N \) matrix of the sources:

\[
S = [s(t_1), ..., s(t_N)],
\]

with \( s(t) = [s_1(t), ..., s_Q(t)]^T \), and \( N \) is the \( M \times N \) matrix of noise. We further make the following assumptions regarding the emitted signals and the propagation model:

A1: The number of sources \( Q \) is known and \( Q < M \).

A2: The emitted signals are unknown and arbitrarily correlated, including the case that a subset of the sources or all of them are synchronous.

A3: The forward matrix \( L(p) \) is known for every location \( p \) (computed by the forward model).

A4: Every \( Q \) topography vectors \( \{ I(p_i) \}_{i=1}^Q \) are linearly independent, i.e., \( \text{rank} A(P) = Q \).

We can now state the problem of localization of brain signals as follows: Given the received data \( Y \), estimate the \( Q \) locations of the sources \( \{ p_q \}_{q=1}^Q \).

3. THE ALTERNATING PROJECTION SOLUTION

We next present the solution for fixed-oriented dipoles. The least-squares criterion for this problem, from (3), is given by

\[
\{ \hat{P}, \hat{S} \} = \arg \min_{P,S} \| Y - A(P)S \|_F^2.
\]

where subscript \( F \) denotes the Frobenius norm. To solve this minimization problem, we first eliminate the unknown signal matrix \( S \) by expressing it in terms of \( P \). To this end, we equate the derivative of (7) to zero with respect to \( S \), and solve for \( S \):

\[
\hat{S} = (A(P)^T A(P))^{-1} A(P)^T Y.
\]

Defining \( \Pi_{A(P)} \) the projection matrix on the column span of \( A(P) \):

\[
\Pi_{A(P)} = A(P)(A(P)^T A(P))^{-1} A(P)^T,
\]

equations (7), (8), and (9), using the rotation property of the trace operator and the idempotence property of the projection operator, yield:

\[
\hat{P} = \arg \min_{P} \| (I - \Pi_{A(P)}) Y \|_F^2
\]

\[
= \arg \max_{P} \| \Pi_{A(P)} Y \|_F^2
\]

\[
= \arg \max_{P} \text{tr}(\Pi_{A(P)} C \Pi_{A(P)})
\]

\[
= \arg \max_{P} \text{tr}(\Pi_{A(P)}) C
\]

where \( \text{tr}(\cdot) \) denotes the trace operator, \( C \) is the data covariance matrix \( C = YY^T \), \( I \) denotes the identity matrix.

This is a nonlinear and nonconvex \( 3Q \)-dimensional maximization problem. The AP algorithm [16] solves this problem by transforming it to a sequential and iterative process involving only a maximization over a single location at a time. The transformation is based on the projection-matrix decomposition formula. Let \( B \) and \( D \) be two matrices with the same number of rows, and let \( \Pi_{[B,D]} \) denote the projection-matrix onto the column span of the augmented matrix \([B, D] \). Then:

\[
\Pi_{[B,D]} = \Pi_{B} + \Pi_{B} \Pi_{[B,D]} \Pi_{B}.
\]

where \( \Pi_{B} \perp \) is the projection onto the orthogonal complement of the span of \( B \), given by \( \Pi_{B} \perp = (I - \Pi_{B}) \).

The AP algorithm exploits this decomposition to transform the multidimensional maximization [13] into a sequential and iterative process involving a maximization over only a single source at a time, with all the other sources held fixed at their pre-estimated values. More specifically, let \( j + 1 \) denote the current iteration number, and let \( q \) denote the current source to be estimated (\( q \) is sequenced from 1 to \( Q \) in every iteration). The other sources are held fixed at their pre-estimated values: \( \{ \hat{p}_{i}^{(j)} \}_{i=1}^{j} \), which have been pre-estimated in the current iteration, and \( \{ \hat{p}_{i}^{(j)} \}_{i=q+1}^{Q} \), which have been pre-estimated in the previous iteration. With this notation, let \( \hat{P}_{q}^{(j)} \) denote the \( M \times (Q - 1) \) matrix of the topographies corresponding to these values (note that the \( q \)-th topography is excluded), given by:

\[
\hat{A}(\hat{P}_{q}^{(j)}) = [I(\hat{p}_{1}^{(j)}), ..., I(\hat{p}_{q-1}^{(j)}), I(\hat{p}_{q+1}^{(j)}), ..., I(\hat{p}_{Q}^{(j)})].
\]

By the projection matrix decomposition [14], we have:

\[
\Pi_{[\hat{A}(\hat{P}_{q}^{(j)}), I(p_q)]} = \Pi_{\hat{A}(\hat{P}_{q}^{(j)})} + \Pi_{\Pi_{\hat{A}(\hat{P}_{q}^{(j)})} I(p_q)},
\]
Substituting (16) into (13), and ignoring the contribution of the first term since it is not a function of \( p_q \), we get:

\[
\hat{p}^{(j+1)}_q = \arg \max_{p_q} \text{tr}(\Pi_{A_i}^{\perp} I(p_q) C), \tag{17}
\]
or alternatively,

\[
\hat{p}^{(j+1)}_q = \arg \max_{p_q} \text{tr}(\Pi_{Q_i}^{\perp} I(p_q) C), \tag{18}
\]

where

\[
Q_i^{(j)} = \Pi_{A_i}^{\perp}, \tag{19}
\]
i.e., \( Q_i^{(j)} \) is a projection matrix that projects out all but the \( q \)-th source at the \( j \)-th iteration. Using the properties of the projection and trace operators we can rewrite (18) as:

\[
\hat{p}^{(j+1)}_q = \arg \max_{p_q} \frac{I^T(p_q) Q_i^{(j)} C Q_i^{(j)} I(p_q)}{I^T(p_q) Q_i^{(j)} I(p_q)}. \tag{20}
\]

The maximization is done by an exhaustive search over a discrete grid representing the source space.

The initialization of the algorithm, which is critical for its good convergence, is very straightforward. First we solve (13) for a single source, yielding

\[
\hat{p}^{(0)}_1 = \arg \max_{p_1} \frac{I^T(p_1) C I(p_1)}{I^T(p_1) I(p_1)}, \tag{21}
\]

Then, we add one source at a time and solve for the \( q \)-th source, \( q = 2, ..., Q \), yielding

\[
\hat{p}^{(0)}_q = \arg \max_{p_q} \frac{I^T(p_q) Q_i^{(0)} C Q_i^{(0)} I(p_q)}{I^T(p_q) Q_i^{(0)} I(p_q)}, \tag{22}
\]

where \( Q_i^{(0)} \) is the projection matrix that projects out the previously estimated \( q - 1 \) sources:

\[
Q_i^{(0)} = (I - \Pi_{A_i}^{\perp}), \tag{23}
\]

with \( A(\hat{p}^{(0)}_q) \) being the \( M \times (q - 1) \) matrix given by

\[
A(\hat{p}^{(0)}_q) = [I(\hat{p}^{(0)}_1), ..., I(\hat{p}^{(0)}_{q-1})]. \tag{24}
\]

Once the initial locations of the \( Q \) sources have been estimated, subsequent iterations of the algorithm, described by (22), refine the estimate. The iterations continue till the localization refinement from one iteration to the next is below a pre-defined threshold. Note that the algorithm climbs the peak of (13) along lines parallel to \( p_1, ..., p_Q \), with the climb rate depending on the structure of the cost function in the proximity of the peak. Since a maximization is performed at every step, the value of the maximized function cannot decrease. As a result, the algorithm is bound to converge to a local maximum which may not necessarily be the global one. Yet, as evidenced in the simulation results, the above initialization procedure provides good convergence.

Last, it can be shown that for freely-oriented dipoles, (20) and (2) lead to a generalized eigenvalue problem.
Fig. 1: Localization error with varying inter-source correlation ($\rho$) in the presence of head model registration errors: (1) 1 mm x-axis translation posterior. (2) 2 mm x-axis translation posterior. (3) 1° x-axis rotation right tilt. (4) 2° x-axis rotation right tilt. (5) 1 mm z-axis translation upward. (6) 2 mm z-axis translation upward. (7) 1° y-axis rotation upward. (8) 2° y-axis rotation upward. (9) 1 mm y-axis translation right. (10) 2 mm y-axis translation right.

Fig. 2: Localization performance for multisensory MEG human data. (a) Ground truth dipoles from a human experiment presenting somatosensory and auditory stimuli. Somatosensory and auditory evoked responses were then combined with different numbers of trials to assess localization performance in multisensory data: (b,c) AP method with 50 and 20 trials, respectively; (d,e) RAP-MUSIC with 50 and 20 trials, respectively. Results are shown in left and right lateral views of semi-transparent cortex. (f) Localization error averaged across the three dipoles.

100 ms and 40 ms respectively (the time of peak response), to yield multimodal data. Data were combined with 50 trials (SNR = 4.31 dB) or 20 trials (SNR = 0.33 dB) to assess the performance of the AP and RAP-MUSIC methods at different SNR levels.

The AP method localized the three dipoles in close agreement with the ground truth (Fig. 2b,c,f). In contrast, the RAP-MUSIC method yielded approximate results and was highly variable with the number of trials (Fig. 2d,e,f).

5. CONCLUSIONS

We have presented a new sequential and iterative solution to the localization of MEG/EEG signals based on minimization of the LS criterion by the AP algorithm. Our simulation and real data results demonstrated that the AP algorithm performs robustly for multiple sources with wide range of inter-source correlation values. Taken together, our work demonstrated the high performance of the AP algorithm in localizing MEG sources. It also revealed the importance of iterating through all sources multiple times until convergence instead of recursively scanning for sources and terminating the scan after the last source is found, which is the approach typically adopted by existing scanning methods. The results show the clear superiority of the AP solution in the presence of forward model errors resulting from head rotations and translations.

The analysis tools used in the current study are available at https://alternatingprojection.github.io/.

6. COMPLIANCE WITH ETHICAL STANDARDS

This study was performed in line with the principles of the Declaration of Helsinki. The human participant gave a written informed consent, and the study was approved by the IRB of the Massachusetts Institute of Technology.
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