Novel Symmetries in Vector Schwinger Model

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Abstract: We derive nilpotent and absolutely anticommuting (anti-)co-BRST symmetry transformations for the bosonized version of (1 + 1)- dimensional (2D) vector Schwinger model. These symmetry transformations turn out to be the analogue of co-exterior derivative of differential geometry as the total gauge-fixing term remains invariant under it. The exterior derivative is realized in terms of the (anti-)BRST symmetry transformations of the theory whereas the bosonic symmetries find their analogue in the Laplacian operator. The algebra obeyed by these symmetry transformations turns out to be exactly same as the algebra obeyed by the de Rham cohomological operators of differential geometry.

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1. Introduction

The Becchi-Rouet-Stora-Tyutin (BRST) formalism is one of the most intuitive approaches to quantize a gauge theory. In BRST formalism the unitarity and “quantum” gauge (i.e. BRST) invariance are respected together at any arbitrary order of perturbative computations. These (anti-)BRST symmetry transformations always satisfy the two sacrosanct properties: (i) the nilpotency of order two, and (ii) the absolute anticommutativity. The former property implies fermionic nature of the (anti-)BRST symmetry transformations whereas the latter one encodes the linear independence of these transformations (see, e.g. [1, 2, 3, 4]).

The Schwinger model, which describes the quantum electrodynamics in $(1 + 1)$-dimension with massless fermions, is a well-studied model as far as the two dimensional field theories are concerned (see, e.g. [5, 6, 7, 8, 9, 10, 11, 12, 13]). In this work we consider the $(1 + 1)$-dimensional bosonized version of vector Schwinger model (VSM). The VSM (as well as the chiral Schwinger model) can be obtained from a generalized version of Schwinger model (see, e.g. [9] for details). The VSM is an exactly solvable model and endowed with the first-class constraints, in the language of Dirac’s prescription for the classification of constrained systems, which makes it a gauge invariant model. The Hamiltonian and BRST formulations of this model have been discussed in [14].

As far as the framework of BRST formalism is concerned, it has been shown that any $p$-form (with $p = 1, 2, 3$) Abelian gauge theories in $D = 2p$ dimensions of spacetime are tractable models of Hodge theory [15, 16, 17, 18, 19], where the underlying theory is endowed with, in totally, six continuous symmetries [i.e. (anti-)BRST, (anti-)co-BRST, bosonic and ghost symmetries]. At this juncture, it is worthwhile to mention that the higher $p$-form ($p \geq 2$) fields appear in the excitations of the quantized versions of (super)strings and related extended objects (see, e.g. [20]). Moreover, at algebraic level, the 1D model of rigid rotor also provides a toy model for Hodge theory [21]. With the help of such kind of studies it has been proven that 2D Abelian 1-form gauge theory is a new model for topological field
theory [22], whereas 4D Abelian 2-form gauge theory provides an example of quasi-topological field theory [23]. Thus, these kind of studies play an important role from physical point of view. Furthermore, in the case of non-Abelian gauge theories, it has been shown that the 2D free non-Abelian 1-form gauge theory provides a model for Hodge theory [24].

The prime motivation towards the present investigation comes from one of our recent works [16] on the chiral Schwinger model (CSM) where we have shown that the modified version of 2D bosonized CSM is endowed with, in totality, six [i.e. (anti-)BRST, (anti-)co-BRST, bosonic and ghost] continuous symmetry transformations. Furthermore, this model has been shown to be a tractable field theoretic model for the Hodge theory where all the de Rham cohomological operators of differential geometry find their analogue in terms of the symmetry transformations (and their corresponding generators) of the underlying theory. Thus, keeping above in mind, it is worthwhile to investigate whether the bosonized version of 2D VSM have the similar kind of symmetry structure as that of the modified version of 2D CSM. We find, within the framework of BRST formalism, that both the above mentioned models have similar properties as far as the continuous symmetries and their algebraic structures are concerned.

To establish the existence of (anti-)co-BRST symmetries, in this model, is also important due to the following reasons. First, the (anti-)BRST and (anti-)co-BRST symmetries have completely different origins and realized in different ways (see, for details [17]). The different way of realization implies that the co-BRST symmetries can give different superselection sector from the BRST symmetries [25]. Second, the physical states of the underlying theory could be locally identified with those states that are both (anti-)BRST and (anti-)co-BRST invariant. Thus, for the direct cohomological description of the physical states of the system, only BRST charge (corresponding to the BRST symmetry) is not enough [26].

Our present paper is organized as follows. In the second section, we briefly discuss about the gauge symmetries, constrained structure and first-order formalism of the 2D bosonized version of VSM for the sake of completeness. The third section contains a discussion about the off-shell nilpotent and ab-
olutely anticommuting (anti-)BRST symmetry transformations. Our fourth section is devoted to the derivation of the off-shell nilpotent (anti-)co-BRST symmetry transformations. In the fifth section, we derive a bosonic symmetry transformations. Our sixth section includes the discussion on ghost and discrete symmetries of the theory. The algebraic structure obeyed by the above symmetry transformations and their connection with the de Rham cohomological operators of differential geometry is included in seventh section. Finally, in the last section, we make concluding remarks and point out some future directions.

2. Preliminaries: Gauge symmetries

We start with the following Lagrangian density of (1+1)–dimensional bosonized version of vector Schwinger model \[\mathcal{L}_{VSM}\]

\[
\mathcal{L}_{VSM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e \varepsilon^{\mu\nu} \partial_\mu \phi \, A_\nu + \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi \\
= \frac{1}{2} E^2 - e \varepsilon^{\mu\nu} \partial_\mu \phi \, A_\nu + \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi, \tag{1}
\]

where \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is field strength tensor for 1-form \((A^{(1)} = dx^\mu A_\mu)\) gauge field \(A_\mu\). In the above, the first term represents the kinetic energy term for the gauge field \(A_\mu\) and, in 2D, it has only electric field \(E\) as its existing component. Moreover, there are no propagating degrees of freedom left for \(A_\mu\) in 2D. The second term corresponds to the coupling of gauge field with massless bosonic field \(\phi\) (or equivalently fermionic field in 2D) where \(e\) is a coupling constant. The last term is the kinetic term for field \(\phi\). The canonical conjugate momenta, calculated from above Lagrangian density, are:

\[
\Pi_\phi = \dot{\phi} + e A_1, \quad \Pi^0 = 0, \quad \Pi^1 = E. \tag{2}
\]

It is clear from the above that \((\chi_1 :=)\Pi^0 \approx 0\) is a primary constraint on the theory and by demanding that the primary constraint should remain intact
with respect to time leads to the secondary constraint \( \chi_2 := E' - e \phi' \approx 0 \). It is straightforward to check that there are no further constraints in the theory \[13\]. It is turn out that, using Dirac’s prescription for the classification of constraints \[27, 28\], that the above mentioned constraints \( \chi_1 \) and \( \chi_2 \) are first-class in nature. This implies that the underlying theory is a gauge theory.

The canonical Hamiltonian density \( \mathcal{H}_c \), calculated from (1) and (2), has following structure:

\[
\mathcal{H}_c = \Pi_\phi \dot{\phi} + \Pi_0 \dot{A}_0 + E \dot{A}_1 - \mathcal{L}_{VSM}
\]

\[
= \frac{1}{2} (E^2 + \Pi_\phi^2 + \phi'^2 + e^2 A_1^2) + EA'_0 - e \Pi_\phi A_1 + e \phi' A_0.
\]

(3)

Therefore, the total Hamiltonian density \( \mathcal{H}_T \) can be given as

\[
\mathcal{H}_T = \mathcal{H}_c + \Pi_0 \lambda,
\]

(4)

where \( \lambda \) is a Lagrange multiplier and \( \Pi_0 \) is the primary constraint on the theory. Thus, the first-order Lagrangian density \( \mathcal{L}_F \), has following form \[14\]

\[
\mathcal{L}_F = \frac{1}{2} (E^2 - \Pi_\phi^2 - \phi'^2 - e^2 A_1^2) + \Pi_\phi \dot{\phi} + e \Pi_\phi A_1 - e \phi' A_0 + p_\lambda \dot{\lambda},
\]

(5)

here \( p_\lambda \) is canonically conjugate momenta to \( \lambda \). The above mentioned first-order Lagrangian density remains invariant under following infinitesimal gauge symmetry transformation \( \delta_g \)

\[
\delta_g A_0 = \dot{\beta}, \quad \delta_g A_1 = \beta', \quad \delta_g \Pi_\phi = e \beta', \quad \delta_g E = 0,
\]

\[
\delta_g \phi = 0, \quad \delta_g \lambda = 0, \quad \delta_g p_\lambda = 0,
\]

(6)

because \( \mathcal{L}_F \) goes to a total spacetime derivative, as

\[
\delta_g \mathcal{L}_F = \partial_\mu [e \beta \varepsilon^{\mu \nu} \partial_\nu \phi].
\]

(7)

Therefore, the corresponding action remains invariant and hence \[6\] are the symmetry transformations of the theory.

\[2\]We differ from the first-order Lagrangian density of \[14\] for the sake of brevity and algebraic convenience.
3. (Anti-)BRST symmetry transformations: Analogue of exterior derivative

The (anti-)BRST invariant first-order Lagrangian density, in its full blaze of glory, can be given as follows:

\[ L_b = \frac{1}{2} (E^2 - \Pi^2_\phi - \phi'^2 - e^2 A_1^2) + \Pi_\phi \dot{\phi} + e \Pi_\phi A_1 - e \phi' A_0 + p_\lambda \dot{\lambda} \]
\[ + b (\dot{A}_0 - A'_1) + \frac{b^2}{2} + \dot{\bar{C}} \dot{C} - \bar{C}' C', \]  

where \((\bar{C})^C\) are fermionic \([C^2 = \bar{C}^2 = 0, \bar{C} \bar{C} + \bar{C} C = 0]\)(anti-)ghost fields and \(b\) is the Nakanishi-Lautrup auxiliary field which is used to linearize the gauge fixing term. The following off-shell nilpotent (anti-)BRST symmetry transformations \((s_{(a)b})\)

\[ s_b A_0 = \dot{\bar{C}}, \quad s_b A_1 = C', \quad s_b C = 0, \quad s_b \bar{C} = b, \]
\[ s_b \Pi \phi = e C', \quad s_b E = 0, \quad s_b [b, \phi, p_\lambda, \lambda] = 0, \]
\[ s_{ab} A_0 = \dot{\bar{C}}, \quad s_{ab} A_1 = \bar{C}', \quad s_{ab} \bar{C} = 0, \quad s_{ab} C = -b, \]
\[ s_{ab} \Pi \phi = e \bar{C}', \quad s_{ab} E = 0, \quad s_{ab} [b, \phi, p_\lambda, \lambda] = 0, \] leave the Lagrangian density \((8)\) quasi-invariant as obvious from the expressions given below

\[ s_b L_b = \partial_\mu [e C \varepsilon^{\mu\nu} \partial_\nu \phi + b \partial_\mu \bar{C}], \quad s_{ab} L_{b} = \partial_\mu [e \bar{C} \varepsilon^{\mu\nu} \partial_\nu \phi + b \partial_\mu C]. \]

It is worthwhile to mention that the kinetic term (i.e. \(\frac{1}{2} E^2\)) remains invariant under (anti-)BRST symmetry transformations as it is evident from \((9)\) that \(s_{(a)b}[E] = 0\). This kinetic term (i.e. \(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} E^2\)) owes its origin to the exterior derivative \(d = dx^\mu \partial_\mu\) (with \(d^2 = 0\)) because the two-form \(B^{(2)} = \frac{1}{2!} (dx^\mu \wedge dx^\nu) F_{\mu\nu}\) defines it through \(B^{(2)} = dA^{(1)}\) where \(A^{(1)} = dx^\mu A_\mu\) introduces the gauge potential \(A_\mu\).

At this juncture, it is interesting to point out that the gauge fixing and ghost terms of above (anti-)BRST invariant Lagrangian density can be derived in the following standard fashion (modulo a total derivative), using \([29]\)

\[ s_b \left[ \bar{C} \left\{ (\partial \cdot A) + \frac{b}{2} \right\} \right] = b (\dot{A}_0 - A'_1) + \frac{b^2}{2} + \dot{\bar{C}} \dot{C} - \bar{C}' C'. \]
The above gauge fixing condition (11) can be, equivalently (modulo a total derivative), written as

\[ s_{ab} \left[ -C \left\{ (\partial \cdot A) + \frac{b}{2} \right\} \right] = b(\dot{A}_0 - A'_1) + \frac{b^2}{2} + \dot{C} \dot{C} - \ddot{C}' C' \equiv s_{ab} s_b \left[ \frac{A^\mu A_\mu}{2} + \frac{C \ddot{C}}{2} \right]. \]  

(12)

The relationship between (11) and (12) holds good because of the absolutely anticommuting nature (i.e. \( s_b s_{ab} + s_{ab} s_b = 0 \)) of the (anti-)BRST symmetry transformations (9). It is straightforward to check that \( (s_b s_{ab} + s_{ab} s_b) \Phi = 0 \), where \( \Phi(= A_\mu, C, \bar{C}, \phi, \lambda, b) \) is the generic field of (8).

4. (Anti-)co-BRST symmetry transformations: Analogue of co-exterior derivative

The (anti-)BRST invariant Lagrangian density \( L_b \) is also endowed with the off-shell nilpotent \( [s^2_{(a)d}] = 0 \) (anti-)co-BRST symmetry transformations \( s_{(a)d} \). For this purpose, we incorporate an auxiliary field \( \bar{b} \) to linearize the kinetic term as below:

\[
\mathcal{L}_d = \bar{b} E - \frac{1}{2}(\bar{b}^2 + \Pi_\phi^2 + \phi'^2 + e^2 A_1^2) + \Pi_\phi \dot{\phi} + e \Pi_\phi A_1 - e \phi' A_0 + p_\lambda \dot{\lambda} + b(A_0 - A'_1) + \frac{b^2}{2} + \dot{C} \dot{C} - \ddot{C}' C'.
\]  

(13)

The above Lagrangian density respects following off-shell nilpotent (anti-)co-BRST symmetry transformations

\[
s_{d} A_0 = -\bar{C}', \quad s_d A_1 = -\dot{C}, \quad s_d \Pi_\phi = -e \dot{C}, \quad s_d E = -\Box \bar{C}, \\
s_d C = \bar{b} - e \phi, \quad s_d \bar{C} = 0, \quad s_d [b, \bar{b}, \phi, p_\lambda, \lambda] = 0, \\
s_{ad} A_0 = -C', \quad s_{ad} A_1 = -\dot{C}, \quad s_{ad} \Pi_\phi = -e \dot{C}, \quad s_{ad} E = -\Box C, \\
s_{ad} \bar{C} = -\bar{b} - e \phi, \quad s_{ad} C = 0, \quad s_{ad} [b, \bar{b}, \phi, p_\lambda, \lambda] = 0,
\]  

(14)

which leaves (13) invariant because of the fact

\[
s_d \mathcal{L}_d = -\partial_\mu [\bar{b} \partial^\mu \bar{C}], \quad s_{ad} \mathcal{L}_d = -\partial_\mu [\bar{b} \partial^\mu C].
\]  

(15)
Therefore, the corresponding action remains invariant under \( s_{(a)d} \). At this stage, it is worthwhile to point out that the total gauge fixing term remains invariant under the (anti-)co-BRST symmetry transformations (i.e. \( s_{(a)d}[b(\dot{A}_0 - A'_1) + \frac{1}{2}b^2] = 0 \)). This gauge fixing term has its origin in the co-exterior derivative \( \delta = \pm \ast d\ast \) (with \( \delta^2 = 0 \)) of differential geometry as the operation of \( \delta \) on a one-form produces the gauge-fixing term [i.e. \( \delta A^{(1)} = (\partial \cdot A) \)]. Here \( \ast \) is the Hodge duality operation on the 2D spacetime manifold and the \( \pm \) sign is dictated by the dimensionality of the spacetime \([30, 31]\). Thus, the nilpotent (anti-)co-BRST symmetry transformations has its origin to the co-exterior derivative (\( \delta \)) of differential geometry.

Furthermore, it is straightforward to check that these (anti-)co-BRST symmetry transformations are absolutely anticommuting in nature [i.e. \( (s_d s_{ad} + s_{ad} s_d)\Phi = 0 \)] where \( \Phi \) is any generic field of the theory.

5. Bosonic symmetry transformations: Analogue of Laplacian operator

It is clear that the bosonized version of 2D VSM is endowed with four nilpotent (fermionic) symmetry transformations (i.e. \( s_{(a)b}, s_{(a)d} \)). In addition to that, the following infinitesimal version of bosonic symmetry (\( s_\omega = \{s_b, s_d\} \)) transformations (with \( s_\omega^2 \neq 0 \))

\[
\begin{align*}
  s_\omega A_0 &= -b' + \dot{b} - e\dot{\phi}, \\
  s_\omega A_1 &= -\dot{b} + \ddot{b} - e\phi', \\
  s_\omega E &= -\Box b, \\
  s_\omega \Pi_\phi &= -e(\dot{b} - \ddot{b} + e\phi'), \\
  s_\omega [C, \bar{C}, \phi, p_\lambda, \lambda, b, \bar{b}] &= 0,
\end{align*}
\]

also leaves the Lagrangian density \([13]\) quasi-invariant. It is explicitly given as follows

\[
  s_\omega \mathcal{L}_d = \partial_\mu [e \varepsilon^{\mu\nu} \dot{b} \partial_\nu \phi - e b \partial^\mu \phi].
\]

The other anticommutator (i.e. \( \{s_{ab}, s_{ad}\} = s_\tilde{\omega} \)) also produces a bosonic symmetry of the theory which is not independent of \( s_\omega \). Moreover, it is easy to check that \( (s_\omega + s_\tilde{\omega})\Phi = 0 \), where \( \Phi \) is any generic field of the theory. In summary, the following relationship is true

\[
  s_\omega = \{s_b, s_d\} = -\{s_{ab}, s_{ad}\} = -s_\tilde{\omega}.
\]
The noteworthy point is that the ghost term remains invariant under the bosonic symmetry transformations. This bosonic symmetry transformations find its analogue in terms of the Laplacian operator $(\Delta = \{d, \delta\})$ of differential geometry.

6. Ghost and discrete symmetries

The ghost number for the bosonic fields $A_0, A_1, \phi, b, \bar{b}, \lambda$ of the theory is equal to zero whereas the ghost number corresponding to the fermionic fields $C$ and $\bar{C}$ is equal to $\pm 1$. Thus, keeping above in mind, we define following ghost scale transformations:

$$
A_0 \rightarrow A_0, \quad A_1 \rightarrow A_1, \quad \phi \rightarrow \phi, \quad b \rightarrow b, \quad \bar{b} \rightarrow \bar{b},
$$

$$
\lambda \rightarrow \lambda, \quad p_\lambda \rightarrow p_\lambda, \quad \Pi_{\phi} \rightarrow \Pi_{\phi}, \quad C \rightarrow e^{+\Lambda} C, \quad \bar{C} \rightarrow e^{-\Lambda} \bar{C}.
$$

(19)

In the above, $\Lambda$ is global infinitesimal scale parameter and $\pm 1$ in the exponentials of $C$ and $\bar{C}$ corresponds to the ghost numbers. The infinitesimal version of the above mentioned ghost scale transformations ($s_g$) can be given as

$$
s_g A_0 = 0, \quad s_g A_1 = 0, \quad s_g \phi = 0, \quad s_g b = 0, \quad s_g \bar{b} = 0,
$$

$$
s_g \lambda = 0, \quad s_g p_\lambda = 0, \quad s_g \Pi_{\phi} = 0, \quad s_g C = C, \quad s_g \bar{C} = -\bar{C}.
$$

(20)

These are the symmetry transformations as Lagrangian density (13) remains invariant under $s_g$. Moreover, the ghost sector of Lagrangian density (13) is also endowed with the following discrete symmetry transformations

$$
C \rightarrow \pm i \bar{C}, \quad \bar{C} \rightarrow \pm i C.
$$

(21)

The above discrete symmetry transformations are useful in enabling us to obtain the anti-BRST symmetry transformations from the BRST symmetries and vice versa. Furthermore, the above transformation connects the co-BRST symmetry transformations to the anti-co-BRST symmetry transformations in the similar fashion as in the case of (anti-)BRST symmetry transformations.
7. Algebraic structures and physical relevance

It is clearly shown, in previous sections, that the bosonized version of 2D VSM is endowed with the (anti-)BRST \((s_{(a)b})\), (anti-)co-BRST \((s_{(a)d})\), a bosonic symmetry \((s_\omega)\) and ghost scale symmetry \((s_g)\) transformations. The operator form of these symmetry transformations obey the following algebra

\[
\begin{align*}
    s^2_{(a)b} &= 0, & s^2_{(a)d} &= 0, & \{s_b, s_{ab}\} &= 0, & \{s_d, s_{ad}\} &= 0, & s_\omega &= (s_b + s_d)^2, \\
    s_\omega &= \{s_b, s_d\} = -\{s_{ab}, s_{ad}\}, & [s_\omega, s_r] &= 0, & r &= b, ab, d, ad, g.
\end{align*}
\] (22)

These algebraic structure are exactly same as the algebra obeyed by the de Rham cohomological operators of differential geometry. The following algebra

\[
\begin{align*}
    d^2 &= 0, & \delta^2 &= 0, & \Delta &= \{d, \delta\} \equiv (d + \delta)^2, \\
    [\Delta, d] &= 0, & [\Delta, \delta] &= 0.
\end{align*}
\] (23)

is constituted by the de Rham cohomological operators, namely; exterior derivative \((d = dx^\mu \partial_\mu)\), the co-exterior derivative \((\delta = \pm \ast d\ast)\) and the Laplacian operator \((\Delta = (d + \delta)^2 = \{d, \delta\}\) of differential geometry. Here \(*\) is the Hodge duality operation on a manifold without boundary.

Thus, on a compact manifold, we have following two-to-one mapping from the symmetries of the theory to the cohomological operators of differential geometry: \((s_b, s_{ab}) \rightarrow d, (s_d, s_{ad}) \rightarrow \delta\) and \(\{s_b, s_d\} = -\{s_{ab}, s_{ad}\} \rightarrow \Delta\). Hence, in this way, we can precisely identify all the symmetry transformations of the theory with the de Rham cohomological operators of differential geometry where the latter are defined on a compact manifold without boundary (see, [30, 31, 32] for details).

8. Conclusions

The central theme of our present investigation was to obtain the off-shell nilpotent (anti-)co-BRST symmetry transformations \textit{together} with the usual (anti-)BRST symmetry transformations in the case of 2D bosonized version
of vector Schwinger model. We have accomplished this goal. In fact, we have explicitly shown that the 2D bosonized version of VSM is endowed with, in totality, six continuous symmetry transformations as listed in (9), (14), (16) and (20).

In our present model, the BRST symmetry transformations turn out to be the analogue of the exterior derivative of differential geometry as the kinetic term, having its origin to the exterior derivative, remains invariant under it. Similarly, the gauge fixing term, owing its origin to the co-exterior derivative, remains invariant under co-BRST symmetry transformations. Thus, co-exterior derivative can be realized in terms of co-BRST symmetries of the present theory. The anticommutator of BRST and co-BRST transformations produces a bosonic symmetry which is analogue of the Laplacian operator. It is the ghost terms of the theory which remain invariant under the bosonic symmetry transformations.

Finally, we have shown that, at the algebraic level, the above mentioned symmetry transformations follow the same algebra as the algebra obeyed by the de Rham cohomological operators of differential geometry. It would be a nice endeavor to find the analogue of the Hodge duality operation (∗) in terms of full discrete symmetries of the present theory which, in turn, enable us to prove this model to be a model for Hodge theory. This aspect is under investigation and our results will be reported in our future publications [33].

References

[1] C. Becchi, A. Rouet and R. Stora, *Phys. Lett. B* 32, 344 (1974)

[2] C. Becchi, A. Rouet and R. Stora, *Commun. Math. Phys.* 42, 127 (1975)

[3] C. Becchi, A. Rouet and R. Stora, *Ann. Phys. (N. Y.)* 98, 287 (1976)

[4] I. V. Tyutin, Lebedev Institute Preprint, Report No: FIAN-39 unpublished (1975)

[5] J. Schwinger, *Phys. Rev.* 128, 2425 (1962)
[6] A. Casher, J. Kougt and L. Susskind, *Phys. Rev. Lett.* 31, 792 (1973)

[7] A. Casher, J. Kougt and L. Susskind, *Phys. Rev. D* 10, 732 (1974)

[8] M. B. Halpern, *Phys. Rev. D* 13, 337 (1976)

[9] D. Boyanovsky, I. Schmidt and M. F. L. Golterman, *Ann. Phys.* 185, 111 (1988)

[10] R. Jackiw and R. Rajaraman, *Phys. Rev. Lett.* 54, 1219 (1985)

[11] R. Rajaraman, *Phys. Lett. B* 184, 369 (1987)

[12] N. K. Falk and G. Kramer, *Ann. Phys.* 176, 369 (1987)

[13] R. P. Malik, *Phys. Lett. B* 212, 445 (1988)

[14] U. Kulshreshtha, D. S. Kulshreshtha and H. J. W. Müller-Kirsten, *Helv. Phys. Acta.* 66, 752 (1993)

[15] S. Gupta and R. P. Malik, *Eur. Phys. J. C* 58, 517 (2008)

[16] S. Gupta, R. Kumar and R. P. Malik, *Eur. Phys. J. C* 65, 311 (2010)

[17] S. Krishna, A. Shukla and R. P. Malik, *Mod. Phys. Lett. A* 26, 2739 (2011)

[18] R. Kumar, S. Krishna, A. Shukla and R. P. Malik, *Eur. Phys. J. C* 72, 1980 (2012)

[19] R. Kumar, S. Krishna, A. Shukla and R. P. Malik, [arXiv:1203.5519](https://arxiv.org/abs/1203.5519) [hep-th]

[20] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory* Vols 1 and 2. (Cambridge: Cambridge University Press) (1987)

[21] S. Gupta and R. P. Malik, *Eur. Phys. J. C* 68, 325 (2010)

[22] R. P. Malik, *J. Phys. A: Math. Gen.* 41, 4167 (2001)

[23] R. P. Malik, *J. Phys. A: Math. Gen.* 36, 5095 (2003)
[24] R. P. Malik, *Mod. Phys. Lett.* A **14**, 1937 (1999)

[25] H. S. Yang and B. H. Lee, *J. Math. Phys.* **37**, 6106 (1996)

[26] D. McMullan, *Commun. Math. Phys.* **149**, 161 (1992)

[27] P. A. M. Dirac, *Lectures on Quantum Mechanics*. Belfer Graduate School of Science (New York: Yeshiva University Press) (1964)

[28] K. Sundermeyer, *Constrained Dynamics. Lecture Notes in Physics* (Berlin: Springer) Vol 169 (1982)

[29] D. Nemeschansky, C. Preitschopf and M. Weinstein, *Ann. Phys. (N.Y.)* **183**, 226 (1988)

[30] T. Eguchi, P. B. Gilkey and A. Hanson, *Phys. Rep.* **66**, 213 (1980)

[31] S. Mukhi and N. Mukunda, *Introduction to Topology, Differential Geometry and, Group Theory for Physicists* (New Delhi: Wiley Eastern) (1990)

[32] J. W. van Holten, *Phys. Rev. Lett.* **64**, 2863 (1990)

[33] S. Gupta, under preparation.