Rayleigh-Rayleigh Distribution: Properties and Applications

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Abstract. In this paper, a new compound distribution named Rayleigh-Rayleigh (Ra-Ra) is presented. Several structural statistical properties of new distribution containing explicit expressions for the r-th moments, characteristic function, quantile function, order statistics, Shannon and relative entropies, besides stress strength reliability were considered and studied. The unknown parameters of Ra-Ra distribution have been estimated under the maximum likelihood estimation method. Moreover, the Ra-Ra distribution is applied upon a simulation study and real data set in order to evaluate its utility and flexibility.

Keywords. Rayleigh distribution, Compound distribution, Statistical properties, Shannon and relative entropies, Stress strength.

1. Introduction
Practically, it is observed that most common distributions are not sufficiently flexible to accommodate various phenomena of nature. For this purpose, researchers have focused on the expansion of these distributions in order to create a more and more realistic and flexible model for modeling data. The generalized families have been established by Mudholkar et al. [1], Marshall and Olkin [2], and Gupta et al. [3]. Newly, a number of new-generation families can be found in, for example, Ristic and Balakrishnan [4], Alzaatreh et al. [5], Nadarajah et al. [6], Tahir et al. [7], Ahmad et al. [8] and Al-Noor et al. [9].

Among probability distributions, the Rayleigh distribution is one of the most commonly used distributions. The Rayleigh distribution introduced by Rayleigh in 1880 and it has appeared as a special case of the Weibull distribution. It plays a key role in modeling and analyzing life-time data such as project effort loading modeling, survival and reliability analysis, theory of communication, physical sciences, technology, diagnostic imaging, applied statistics and clinical research. With regard to this importance and the desire to give greater flexibility to this distribution, several researchers have developed extensive extensions to Rayleigh distribution for example, among others, Kundu and Raqab [10], Voda [11], Merovci [12][13], Merovci and Elbatal [14], Ateeq et al. [15]. In this paper, a new generation family of continuous distributions based on Rayleigh distribution is presented and suggested. This family is built on inspired by the composing two cumulative distribution functions together, say H and G, as follows.

Assume that \( G(x) \) and \( g(x) \) be any continuous baseline cumulative distribution and probability density functions (cdf and pdf) of \( x \) random variable \( X \), also assume respectively that \( H(x) \) and \( h(x) \) be the cdf and pdf of any \( [0, \infty) \) continuous distribution. The general formula of reliability function for this class named H - G is given by (see [9])

\[
R(x)_{H-G} = \int_0^{-\ln G(x)} h(x) \, dx = H(-\ln G(x))
\]  

(1)
The general formulas of cdf and its associated pdf are (see [9]):

\[ F(x)_{H-G} = 1 - R(x)_{H-G} = 1 - H(-\ln G(x)) \]

\[ f(x)_{H-G} = \frac{d}{dx} [F(x)_{H-G}] = \frac{g(x)}{G(x)} h(-\ln G(x)) \]

Based on the above general formulas, a new family named Rayleigh – G distributions along with one of its special cases "sub-model" named Rayleigh-Rayleigh distribution are proposed and offered. The remains of this article are established as follows: In section 2, a brief detail about the Rayleigh – G distributions is provided. Sections 3, 4 and 5 respectively address the new Rayleigh – Rayleigh distribution with its structural statistical properties and the maximum likelihood estimation of its parameters. Sections 6 address the numerical illustration via a simulation study and a real data set application. Finally, the conclusions are presented in section 7.

2. Rayleigh – G distributions

Let \( H(.) \) and \( h(.) \) that mentioned in (1), (2) and (3) be the cdf and pdf of Rayleigh distribution (see [14]) with positive scale parameter \( \theta \) as

\[ H(-\ln G(x); \theta) = 1 - e^{-\frac{\theta}{\varphi}(-\ln G(x))^\varphi} \]

\[ h(-\ln G(x); \theta) = \theta(-\ln G(x)) e^{-\frac{\theta}{\varphi}(-\ln G(x))^\varphi} \]

By substituting (4) and (5) in (2) and (3), the cdf and pdf of new family named Rayleigh – G (for short Ra – G) distributions will be

\[ F(x)_{Ra-G} = e^{-\frac{\theta}{\varphi}(-\ln G(x))^\varphi} \]

\[ f(x)_{Ra-G} = \frac{g(x)}{G(x)} \theta(-\ln G(x)) e^{-\frac{\theta}{\varphi}(-\ln G(x))^\varphi} \]

The pdf in (7) can be rewritten as an expanded formula, say \( f(x)_{Ra-G} \), that is relevant for obtaining the moments as one of the basic statistical properties when dealing with some particular cases. Thus, via using series expansion \( e^{-z} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} z^i \), we get

\[ f(x)_{Ra-G} = \frac{g(x)}{G(x)} \theta(-\ln G(x)) \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left[ \frac{\theta}{2} (-\ln G(x))^2 \right]^i \]

and then

\[ f(x)_{Ra-G} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{g^{i+1}}{2^i} \frac{g(x)}{G(x)} [-\ln G(x)]^{2i+1} \]

For \( i \geq 1 \), using special formula \([-\ln z]^a = \sum_{k,l=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{j+k+l} a}{a-j} C_k^{k-a} C_j^{k} C_l^{2i+k} P_{j,k} z^l \) (see [9]) we have

\[ [-\ln G(x)]^{2i+1} = \sum_{k,l=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{j+k+l} (2i + 1)}{2i + 1 - j} C_k^{k-2l-1} C_j^{k} C_l^{2i+k+1} P_{j,k} [G(x)]^j \]
Now the above formula \( f(x)_{Ra-g} \) will be
\[
f(x)_{Ra-g} = \frac{\varphi^2}{g_2/g_3 /g_2/g_3 /g_2/g_3 /g_2/g_3 /g_2/g_3 /g_2/g_9 /g_4 /g_8/g_3 /g_5} \]

Then the expansion formula \( f(x)_{Ra-g} \) for the pdf \( f(x)_{Ra-g} \) in (7) will be
\[
f(x)_{Ra-g} = \sum_{i,k,l=0}^{\infty} \sum_{j=0}^{K} \frac{(-1)^{i+j+k+l}}{2i!(2i + 1 - j)} C_i^{k-2i-1} C_j^{k-2i-1} C_k^{j} \cdot \theta^{j+1} g(x) |G(x)|^{l-1}
\]

where \( p_{j,k} = 1 \) for \( j \geq 0 \) and \( p_{j,k} = k^{-1} \sum_{n=1}^{K} \frac{(-1)^{n} (j+k-n)}{n+1} p_{j,k-n} \) for \( k = 1, 2, \ldots \).

3. Rayleigh – Rayleigh distribution

Suppose that \( G(x) \) and \( g(x) \) in (6), (7) and (8) be the cdf and pdf of \( Ra \) with positive parameter \( \lambda \) given respectively as
\[
G(x; \lambda) = 1 - e^{-\frac{\lambda}{2} x^2} \]
\[
g(x; \lambda) = \lambda x e^{-\frac{\lambda}{2} x^2}
\]

The cdf and pdf of new distribution named Rayleigh – Rayleigh (for short \( Ra – Ra \)) can be found according to (6) and (7) as
\[
F(x)_{Ra-Ra} = e^{-\theta x^2 \left( \ln \left( 1 - e^{-\frac{\lambda}{2} x^2} \right) \right)^2} \]
\[
f(x)_{Ra-Ra} = \frac{\theta \lambda x e^{-\frac{\lambda}{2} x^2}}{1 - e^{-\frac{\lambda}{2} x^2}} \left( \ln \left( 1 - e^{-\frac{\lambda}{2} x^2} \right) \right) e^{-\theta x^2 \left( \ln \left( 1 - e^{-\frac{\lambda}{2} x^2} \right) \right)^2} \]

The expansion formula of \( Ra – Ra \) density function can be found according to (8) as
\[
f(x)_{Ra-Ra} = \sum_{i,k,l=0}^{\infty} \sum_{j=0}^{K} \frac{(-1)^{i+j+k+l}}{2i!(2i + 1 - j)} C_i^{k-2i-1} C_j^{k-2i-1} C_k^{j} \cdot \theta^{j+1} \lambda x e^{-\frac{\lambda}{2} x^2} \left( 1 - e^{-\frac{\lambda}{2} x^2} \right)^{l-1}
\]

The \( Ra – Ra \) reliability and hazard rate functions can be found as
\[
F(x)_{Ra-Ra} = 1 - F(x)_{Ra-Ra} = 1 - e^{-\theta x^2 \left( \ln \left( 1 - e^{-\frac{\lambda}{2} x^2} \right) \right)^2}
\]
\[
D(x)_{Ra-Ra} = \frac{\theta \lambda x e^{-\frac{\lambda}{2} x^2} \ln \left( 1 - e^{-\frac{\lambda}{2} x^2} \right) e^{-\frac{\lambda}{2} x^2 \left( \ln \left( 1 - e^{-\frac{\lambda}{2} x^2} \right) \right)^2}}{\left( 1 - e^{-\frac{\lambda}{2} x^2} \right) \left( 1 - e^{-\frac{\lambda}{2} x^2 \left( \ln \left( 1 - e^{-\frac{\lambda}{2} x^2} \right) \right)^2} \right)}
\]

Figures 1 and 2 show the shapes of the cdf and pdf of \( Ra – Ra \) distribution under a variety of selected values of parameters.
4. Statistical properties of the $Ra - Ra$ distribution

In this section, the most necessary statistical properties of the $Ra - Ra$ distribution are given respectively as

4.1 The $r$-th moment: The $r$-th moment of $Ra - Ra$ distribution can be obtained as follows

Firstly recall $f(x)_{Ra-Ra}$ in (13) where $\left(1 - e^{-\frac{\lambda}{\beta} x^2}\right)^{l-1}$ have two cases $l - 1 > 0$ and $l - 1 < 0$ via using

$$(1 - z)^b = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \frac{\Gamma(b + l)}{\Gamma(b - l + 1)} z^l ; \; z < l, \; b > 0$$

$$(1 - z)^{-b} = \sum_{l=0}^{\infty} \frac{\Gamma(b + l)}{l! \Gamma(b)} z^l ; \; z < l, \; b > 0$$

where
\[ (1 - e^{-\frac{\lambda}{2} x^2} )^{l-1} = \begin{cases} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(l)}{\Gamma(l-m)} e^{-\frac{\lambda m}{2} x^2} & \text{for } l - 1 > 0 \\
\sum_{m=0}^{\infty} \frac{\Gamma(l-1+m)}{m! \Gamma(l-1)} e^{-\frac{\lambda m}{2} x^2} & \text{for } l - 1 < 0 \end{cases} \]

Now \( f(x)^{E}_{Ra-Ra} \) for \( l - 1 > 0 \) will be

\[
f(x)^{E}_{Ra-Ra} = \sum_{i,k,l=0}^{\infty} \sum_{j=0}^{K} \frac{(-1)^{i+j+k+l} (2i + 1)}{i! 2^i (2i + 1 - j) \Gamma(l)} c_k^{l-2i-1} c_j^{l+k+1} p_{j,k} \theta^{i+1} x e^{-\frac{\lambda}{2} x^2} e^{-\frac{\lambda m}{2} x^2} = \sum_{i,k,l,m = 0}^{\infty} \sum_{j=0}^{K} \frac{(-1)^{i+j+k+l+m} (2i + 1)}{i! m! 2^i (2i + 1 - j) \Gamma(l)} c_k^{l-2i-1} c_j^{l+k+1} p_{j,k} \theta^{i+1} x e^{-\frac{\lambda (m+1)}{2} x^2}
\]

and \( f(x)^{E}_{Ra-Ra} \) for \( l - 1 < 0 \) will be

\[
f(x)^{E}_{Ra-Ra} = \sum_{i,k,l=0}^{\infty} \sum_{j=0}^{K} \frac{(-1)^{i+j+k+l} (2i + 1)}{i! 2^i (2i + 1 - j) \Gamma(l)} c_k^{l-2i-1} c_j^{l+k+1} p_{j,k} \theta^{i+1} x e^{-\frac{\lambda}{2} x^2} e^{-\frac{\lambda m}{2} x^2} = \sum_{i,k,l,m = 0}^{\infty} \sum_{j=0}^{K} \frac{(-1)^{i+j+k+l+m} (2i + 1)}{i! m! 2^i (2i + 1 - j) \Gamma(l)} c_k^{l-2i-1} c_j^{l+k+1} p_{j,k} \theta^{i+1} x e^{-\frac{\lambda (m+1)}{2} x^2}
\]

The \( f(x)^{E}_{Ra-Ra} \) can be rewritten as

\[
f(x)^{E}_{Ra-Ra} = W \lambda x e^{-\frac{\lambda (m+1)}{2} x^2}
\]

where

\[
W = \begin{cases} \sum_{i,k,l,m = 0}^{\infty} \sum_{j=0}^{K} \frac{(-1)^{i+j+k+l+m} (2i + 1)}{i! m! 2^i (2i + 1 - j) \Gamma(l)} c_k^{l-2i-1} c_j^{l+k+1} p_{j,k} \theta^{i+1} x e^{-\frac{\lambda (m+1)}{2} x^2} & \text{for } l - 1 > 0 \\
\sum_{i,k,l,m = 0}^{\infty} \sum_{j=0}^{K} \frac{(-1)^{i+j+k+l+m} (2i + 1)}{i! m! 2^i (2i + 1 - j) \Gamma(l)} c_k^{l-2i-1} c_j^{l+k+1} p_{j,k} \theta^{i+1} x e^{-\frac{\lambda (m+1)}{2} x^2} & \text{for } l - 1 < 0 \end{cases}
\]

Now the \( r \)-th moment of \( Ra - Ra \) distribution can be obtained as
where \( \gamma_1 / \gamma_2 / \gamma_3 / \gamma_4 / \gamma_5 / \gamma_6 / \gamma_7 / \gamma_8 / \gamma_9 / \gamma_{10} / \gamma_{11} \) represents the \( r \)-th moment of distribution with parameter \( \gamma_1 / \gamma_2 / \gamma_3 / \gamma_4 / \gamma_5 / \gamma_6 / \gamma_7 / \gamma_8 / \gamma_9 / \gamma_{10} / \gamma_{11} \) i.e. \( \gamma_1 / \gamma_2 / \gamma_3 / \gamma_4 / \gamma_5 / \gamma_6 / \gamma_7 / \gamma_8 / \gamma_9 / \gamma_{10} / \gamma_{11} \) and \( \gamma_1 / \gamma_2 / \gamma_3 / \gamma_4 / \gamma_5 / \gamma_6 / \gamma_7 / \gamma_8 / \gamma_9 / \gamma_{10} / \gamma_{11} \) is the gamma function.

Thus the \( E(X^r)_{Ra-Ra} \) will be

\[
E(X^r)_{Ra-Ra} = W \frac{\frac{2}{r}}{(m+1)^{\frac{r+1}{2}}} \Gamma \left( 1 + \frac{r}{2} \right)
\]

(18)

where \( W \) as in (17) and \( \Gamma(\cdot) \) is the gamma function.

Additional properties of \( Ra - Ra \) distribution for instance the mean, variance, coefficients of kurtosis and skewness can be found with specific value of \( \gamma_1 / \gamma_2 / \gamma_3 / \gamma_4 / \gamma_5 / \gamma_6 / \gamma_7 / \gamma_8 / \gamma_9 / \gamma_{10} / \gamma_{11} \).

4.2 The characteristic function: The characteristic function of \( Ra - Ra \) distribution can be found by

\[
\varphi_X(t)_{Ra-Ra} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(X^r)_{Ra-Ra} = W \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{\frac{2}{r}}{(m+1)^{\frac{r+1}{2}}} \Gamma \left( 1 + \frac{r}{2} \right)
\]

(19)

where \( W \) as in (17) and \( \Gamma(\cdot) \) is the gamma function.

4.3 The quantile function: The quantile function of \( Ra - Ra \) random variable is defined as a solution of \( P(x \leq x(q)) = F(x(q))_{Ra-Ra} \) w.r.t. \( x(q) : x(q) > 0 \) and \( 0 < q < 1 \). Therefore it can be found via using (11) as

\[
x(q)_{Ra-Ra} = \left[ -\frac{2}{\lambda} \ln \left( 1 - e^{-\left( -\frac{2}{\lambda} \ln(q) \right)^2} \right) \right]^2
\]

(20)

The median of \( Ra - Ra \) random variable can be found via setting \( q = 0.5 \). A random variable \( X \) has the \( Ra - Ra \) distribution can be simulated by

\[
x = \left[ -\frac{2}{\lambda} \ln \left( 1 - e^{-\left( -\frac{2}{\lambda} \ln(u) \right)^2} \right) \right]^2
\]

(21)

where \( U \) has the standard uniform distribution.

4.4 Order statistics: Consider \( x_1, x_2, \ldots, x_n \) as a random sample of size \( n \) taken independently from \( Ra - Ra \) distribution. Let \( x_{1:n}, x_{2:n}, \ldots, x_{n:n} \) be the corresponding order statistics. Then the pdf of \( x_{k:n}, k \leq n \) can be found via the following standard formula statistics

\[
f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} \left( F(x) \right)^{k-1} (1 - F(x))^{n-k} f(x) ; 0 \leq x_{k:n} < \infty
\]

and then based on (11) and (12) we get
The joint pdf of order statistics can be found via the following standard formula

\[
f_{j,k,n}(x,y) = \frac{n!}{(j-1)! (k-j-1)! (n-k)!} (F(x))^{j-1} (1-F(y))^{k-j-1} \left( f(x) f(y) \right)^{n-k}
\]

and then

\[
f_{j,k,n}(x,y) = \frac{n!}{(j-1)! (k-j-1)! (n-k)!} \left( \frac{\theta^2 x^2 e^{-\frac{\lambda}{2} x^2}}{1-e^{-\frac{\lambda}{2} x^2}} \right)^{j-1} \left( \frac{\theta^2 y^2 e^{-\frac{\lambda}{2} y^2}}{1-e^{-\frac{\lambda}{2} y^2}} \right)^{k-j-1} \left( 1-e^{-\frac{\lambda}{2} x^2} \right)^{j-1} \left( 1-e^{-\frac{\lambda}{2} y^2} \right)^{k-j-1}
\]

4.5 Shannon Entropy: The Shannon entropy can be found via \( SH_{Ra-Ra} = - \int_{0}^{\infty} \ln(f(x)) f(x) dx \). Taking the natural logarithm of (12) we get

\[
\ln(f(x)) = \ln(\theta \lambda) + \ln(x) - \frac{\lambda}{2} x^2 - n \left( 1-e^{-\frac{\lambda}{2} x^2} \right) + n \left( \ln \left( 1-e^{-\frac{\lambda}{2} x^2} \right) \right) - \frac{\theta}{2} \left( \ln \left( 1-e^{-\frac{\lambda}{2} x^2} \right) \right)^2
\]

Now \( SH_{Ra-Ra} \) is given by

\[
SH_{Ra-Ra} = - \left\{ \ln(\theta \lambda) + E(\ln(X)) - \frac{\lambda}{2} E(X^2) - E \left( \ln(1-e^{-\frac{\lambda}{2} x^2}) \right) \right\} + E \left( \ln \left( 1-e^{-\frac{\lambda}{2} x^2} \right) \right) - \frac{\theta}{2} E \left( \left( \ln \left( 1-e^{-\frac{\lambda}{2} x^2} \right) \right)^2 \right)
\]

Now for \( E(\ln(X)) \) recall (16) we get

\[
E(\ln(X)) = \int_{0}^{\infty} \ln(x) f(x) dx = \int_{0}^{\infty} \ln(x) W_x e^{-\frac{\lambda (n+1)}{2} x^2} dx
\]
Let \( y = \frac{\lambda^{(n+1)}}{2} x^2 \rightarrow 1^2 - \frac{2y}{\lambda(m+1)} \rightarrow x = -\frac{2y}{\lambda(m+1)} \) and \( dx = -\frac{1}{\lambda(m+1)} dy \) then

\[
E(\ln(X)) = W \int_0^\infty \lambda \ln \left( \frac{2y}{\lambda(m+1)} \right) \left( \frac{2y}{\lambda(m+1)} \right)^{\frac{1}{2}} e^{-y} \left( \frac{1}{2\lambda(m+1)} \right)^{\frac{1}{2}} dy
\]

\[
= W \int_0^\infty \frac{1}{2} \ln \left( \frac{2y}{\lambda(m+1)} \right) \left( \frac{1}{m+1} \right) e^{-y} dy
\]

\[
= W \frac{1}{2(m+1)} \int_0^\infty \left( \ln \left( \frac{2}{\lambda(m+1)} \right) + \ln(y) \right) e^{-y} dy
\]

\[
= W \frac{1}{2(m+1)} \left[ \int_0^\infty e^{-y} dy + \int_0^\infty \ln(y) e^{-y} dy \right]
\]

Since \( \int_0^\infty e^{-y} dy = 1 \), and using \( \int_0^\infty \ln(z) e^{-mz} dz = m^{-s} \Gamma(s)(\psi(s) - \ln(m)) \) we get \( \int_0^\infty \ln(z) e^{-y} dy = \psi(1) \) so

\[
E(\ln(X)) = W \frac{1}{2(m+1)} \left[ \ln \left( \frac{2}{\lambda(m+1)} \right) + \psi(1) \right]
\]

For \( k \left( \ln \left( 1 - e^{-\frac{1}{2} x^2} \right) \right) \) using \( \ln(1 - z) = -\sum_{i=1}^\infty \frac{z^i}{i} \), \( |z| < 1 \) and \( e^{-z} \) we get

\[
\ln \left( 1 - e^{-\frac{1}{2} x^2} \right) = -\sum_{i=1}^\infty \frac{1}{i} e^{-\frac{1}{2} i x^2} = -\sum_{j=1}^\infty \sum_{j=1}^\infty \frac{(-1)^j}{j!} \left( \frac{a}{2} x^2 \right)^j = \sum_{i=1}^\infty \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j!} \left( \frac{a}{2} i x^2 \right)^j
\]

Then

\[
E \left( \ln \left( 1 - e^{-\frac{1}{2} x^2} \right) \right) = \sum_{i=1}^\infty \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j!} \left( \frac{a}{2} i x^2 \right)^j E \left( X^{2j} \right)
\]

where \( E(X^{2j}) \) as in (18) with \( r = 2j \).

For \( E \left( \ln \left( -\ln \left( 1 - e^{-\frac{1}{2} x^2} \right) \right) \right) \) using the above formula of \( \ln(1 - z) \), as well as using

\[
\ln z = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}(z-1)^{k+1}}{k+1}; \quad 0 < z \leq 2,
\]

\[
(a + b)^n = \sum_{k=0}^{n} C^n_k a^{n-k} b^k = \sum_{k=0}^{n} C^n_k a^k b^{n-k}; \quad n \geq 0, \quad C^n_k = \frac{n!}{k!(n-k)!} \]

is Binomial coefficients, and \( e^{-z} \) we get

\[
\ln \left( -\ln \left( 1 - e^{-\frac{1}{2} x^2} \right) \right) = \ln \left( \sum_{i=1}^\infty \frac{1}{i} e^{-\frac{1}{2} x^2} \right)
\]

\[
= \sum_{i=1}^\infty \frac{(-1)^{k+1}}{k+1} \left( \sum_{i=1}^\infty e^{-\frac{1}{2} i x^2} \right)^k + \ln \left( \sum_{i=1}^\infty \frac{1}{i} e^{-\frac{1}{2} x^2} \right)
\]
\[ E \left( \ln \left( 1 - e^{-\frac{1}{2} \chi^2} \right) \right) = \sum_{k,j=0}^{\infty} (\xi_k^j)^2 \sum_{i=0}^{\infty} \frac{\xi_k^j}{i!} \frac{(\lambda(2+k))^i}{2^i} \chi^{2i} \]

Then
\[ E \left( \ln \left( 1 - e^{-\frac{1}{2} \chi^2} \right) \right) = \frac{1}{2} \left( \sum_{k,j=0}^{\infty} (\xi_k^j)^2 \right) \sum_{i=0}^{\infty} \frac{\xi_k^j}{i!} \frac{(\lambda(2+k))^i}{2^i} \chi^{2i} \]

where \( \xi (\chi^2) \) as in (18) with \( r = 2j \).

For \( E \left( \left( 1 - e^{-\frac{1}{2} \chi^2} \right)^2 \right) \) using \( -\ln(1 - \alpha) = u \xi \sum_{k=0}^{\infty} \frac{\xi}{i!} \sum_{j=0}^{\infty} \frac{\xi^j}{j!} \frac{(\lambda(2+k))^i}{2^i} \chi^{2i} \) and \( e^{-\xi} \) we get
\[ \left( 1 - e^{-\frac{1}{2} \chi^2} \right)^2 = \frac{1}{2} \sum_{k,j=0}^{\infty} (\xi_k^j)^2 \sum_{i=0}^{\infty} \frac{\xi_k^j}{i!} \frac{(\lambda(2+k))^i}{2^i} \chi^{2i} \]

Then
\[ E \left( \left( 1 - e^{-\frac{1}{2} \chi^2} \right)^2 \right) = \frac{1}{2} \left( \sum_{k,j=0}^{\infty} (\xi_k^j)^2 \right) \sum_{i=0}^{\infty} \frac{\xi_k^j}{i!} \frac{(\lambda(2+k))^i}{2^i} \chi^{2i} \]

where \( \bar{E}(\chi^2) \) as in (18) with \( r = 2l \) and \( P_{j,k} = k^{-1} \sum_{m=1}^{l} [k - m(j + 1)] C_{n} P_{j,k} \) for \( \bar{E}(\chi^2) \) with \( P_{j,0} = 1 \) and \( C_k = (-1)^{k+1} (k+1)^{-1} \).

Now from (24) the \( R \alpha - R \alpha \) Shannon entropy can be found by
4.6 The Relative Entropy: The \( Ra - Ra \) relative entropy can be found via

\[
RE_{Ra-Ra} = \int_0^\infty \ln \left( \frac{f(x)_{Ra-Ra}}{f_1(x)_{Ra-Ra}} \right) f(x)_{Ra-Ra} \, dx.
\]

Taking the natural logarithm of the \( f(x)_{Ra-Ra} \) in (12) w.r.t. \( f_1(x)_{Ra-Ra} \) with parameters \( \theta_1, \lambda_1 \) we get

\[
\ln \left( \frac{f(x)_{Ra-Ra}}{f_1(x)_{Ra-Ra}} \right) = \ln(\theta) - \ln(\theta, \lambda) + \ln(x) - \ln(\lambda) - \frac{\lambda}{2}x^2 + \frac{\lambda_1}{2}x^2 - \ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right)
\]

\[
+ \ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right) + \ln \left( -\ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right) \right) - \ln \left( -\ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right) \right)
\]

\[
- \frac{\theta_1}{2} \left( -\ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right) \right)^2 + \frac{\theta_1}{2} \left( -\ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right) \right)^2
\]

Then \( Ra - Ra \) relative entropy will be

\[
RE_{Ra-Ra} = \ln \left( \frac{\theta_1}{\theta, \lambda} \right) - \left( \frac{\lambda_1}{\lambda} \right) E(X^2) - E \left( -\ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right) \right)
\]

\[
+ E \left( -\ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right) \right) + E \left( -\ln \left( -\ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right) \right) \right) - E \left( -\ln \left( -\ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right) \right) \right)
\]

\[
- \frac{\theta_1}{2} E \left( -\ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right) \right)^2 + \frac{\theta_1}{2} \left( -\ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right) \right)^2
\]

where

\( E(X^2) \) as in (18) with \( r = 2 \) and the extra expectations respectively as in (25), (26), (27), (28) with indicated parameters.

4.7 The stress strength: Let \( Y \) be the stress and \( X \) be the strength of \( Ra - Ra \) independent random variables with different parameters, then the stress strength \( SS_{Ra-Ra} = P(Y < X) \) can be found by

\[
SS_{Ra-Ra} = \int_0^\infty f_X(x)_{Ra-Ra} F_Y(x)_{Ra-Ra} \, dx
\]

where

\[
F_Y(x)_{Ra-Ra} = e^{-\frac{\theta_1}{2} \left( -\ln \left( 1 - e^{-\frac{\lambda_1}{2}x^2} \right) \right)^2}
\]

Using \( e^{-x^2} \), \( F_Y(x)_{Ra-Ra} \) can rewritten as
For \((\ln(1 - e^{-\frac{X^2}{2}}))^2\) using \((-\ln(1 - z))^a, e^{-z} and follows the previous similar steps of getting \((-\ln(1 - e^{-\frac{X^2}{2}}))^2\) we get

\[
(-\ln(1 - e^{-\frac{X^2}{2}}))^2 = 2i \Sigma_{k=0}^\infty C_k \sum_{j=0}^{\infty} (-1)^{i+k} \frac{2i}{2i-j} C_j \sum_{m=0}^{\infty} \frac{(\lambda(2i+j+k)}{2} x^m
\]

Then

\[
F_Y(x) = \Sigma_{k=0}^\infty \sum_{j=0}^{\infty} (-1)^{i+j+k+m} 2i \frac{(\theta_1)^i}{i! m! (2i-j)} C_k C_j \sum_{m=0}^{\infty} \frac{(\lambda(2i+j+k))}{2} x^m
\]

The Ra - Ra stress strength can be found based on (31) as

\[
S_{Ra-Ra} = \Sigma_{k=0}^\infty \sum_{j=0}^{\infty} (-1)^{i+j+k+m} 2i \frac{(\theta_1)^i}{i! m! (2i-j)} C_k C_j \sum_{m=0}^{\infty} \frac{(\lambda(2i+j+k))}{2} x^m
\]

5 Estimation of Ra-Ra parameters

The method of maximum likelihood estimation is considered here to estimate the parameters of \(Ra-Ra\) distribution with complete sample. Consider a complete \(Ra-Ra\) random sample, say \(x_1, x_2, \ldots, x_n\), with parameter vector \(\xi = (\theta, \lambda)^T\). The natural logarithm likelihood function \(\ell(\xi|x)\) in relation to (12) is

\[
\ell(\xi|x) = n \ln(\theta) + n \ln(\lambda) + \Sigma_{i=1}^n \ln(x_i) - \frac{\lambda}{2} \sum_{i=1}^n x_i^2 - \frac{\theta}{2} \sum_{i=1}^n \left(1 - e^{-\frac{X^2}{2}}\right)^2
\]

The maximum likelihood estimates (MLEs) of two parameters \((\theta, \lambda)\) can be found via solving the nonlinear system of natural logarithm likelihood equations

\[
\left(\frac{\partial\ell(\xi|x)}{\partial\theta}, \frac{\partial\ell(\xi|x)}{\partial\lambda}\right)^T = 0
\]

through iterative numerical techniques.
6 Numerical illustrations

Numerical illustrations is presented here to exhibit the abilities of Ra – Ra distribution via simulation study and application with real data set.

6.1 Simulation study

In this subsection a simulation study is carried out to exhibit the performances of the MLEs of Ra – Ra distribution. The steps of process are as follows

1. Generate an i.i.d. random samples follow Ra – Ra distribution. The number of replicated samples was made 1000 times each with sizes n = 25, 50, 100 and 200.
2. Select the initial “true” values of parameters to be as in Tables (1-3) with θ > λ, θ < λ and θ = λ.
3. Calculate Bias and RMSE “root mean squared error” where
   \[ \text{Bias}(\hat{\theta}) = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\theta} - \theta) \] and \[ \text{RMSE}(\hat{\theta}) = \sqrt{\frac{1}{1000} \sum_{i=1}^{1000} (\hat{\theta} - \theta)^2} \]
4. Repeat step 3 for other parameter λ.

The empirical results are shown in Tables (1-3). It clearly appears that RMSE values increase as the values of the parameters increasing and the RMSE values decrease as the sample size increases.

| Table 1. The Bias and RMSE of the Ra-Ra parameters estimation using MLE with 📊 |
| --- |
| n | Par. | Initial | Bias | RMSE | Initial | Bias | RMSE |
| --- | --- | --- | --- | --- | --- | --- | --- |
| 25 | 1 | 0.1164 | 0.3277 | 0.5 | 0.0758 | 0.2054 |
| | 3 | 1.1336 | 2.9564 | 1 | 0.2784 | 0.7177 |
| 50 | 1 | 0.0586 | 0.2066 | 0.5 | 0.0379 | 0.1282 |
| | 3 | 0.4818 | 1.4252 | 1 | 0.1218 | 0.3766 |
| 100 | 1 | 0.0288 | 0.1341 | 0.5 | 0.0182 | 0.0817 |
| | 3 | 0.2336 | 0.8214 | 1 | 0.0604 | 0.2261 |
| 200 | 1 | 0.0133 | 0.0929 | 0.5 | 0.0084 | 0.0562 |
| | 3 | 0.1040 | 0.5224 | 1 | 0.0266 | 0.1460 |

| Table 2. The Bias and RMSE of the Ra-Ra parameters estimation using MLE with 📊 |
| --- |
| n | Par. | Initial | Bias | RMSE | Initial | Bias | RMSE |
| --- | --- | --- | --- | --- | --- | --- | --- |
| 25 | 1 | 0.4593 | 1.2391 | 1 | 0.1960 | 0.5064 |
| | 3 | 0.2812 | 0.7206 | 0.5 | 0.1200 | 0.3059 |
| 50 | 1 | 0.2300 | 0.7751 | 1 | 0.0958 | 0.3084 |
| | 3 | 0.1243 | 0.3821 | 0.5 | 0.0530 | 0.1679 |
| 100 | 1 | 0.1094 | 0.4899 | 1 | 0.0449 | 0.1924 |
| | 3 | 0.0607 | 0.2265 | 0.5 | 0.0263 | 0.1016 |
| 200 | 1 | 0.0504 | 0.3388 | 1 | 0.0209 | 0.1317 |
| | 3 | 0.0268 | 0.1469 | 0.5 | 0.0116 | 0.0663 |
Table 3. The Bias and RMSE of the Ra-Ra parameters estimation using MLE with $g_{20}$.

| $n$ | Par. Initial Bias | RMSE Initial Bias | RMSE Initial Bias | RMSE Initial Bias |
|-----|-------------------|-------------------|-------------------|-------------------|
| 25  | 0.5               | 0.0975            | 0.2529            | 0.1559            | 0.4173 |
|     | 0.5               | 0.1193            | 0.3041            | 0.2875            | 0.7309 |
|     | 0.5               | 0.0480            | 0.1546            | 0.0771            | 0.2594 |
|     | 0.5               | 0.0532            | 0.1684            | 0.1251            | 0.3843 |
|     | 0.5               | 0.0224            | 0.0963            | 0.0362            | 0.1639 |
|     | 0.5               | 0.0263            | 0.1019            | 0.0603            | 0.2268 |
|     | 0.5               | 0.0104            | 0.0662            | 0.0167            | 0.1128 |
|     | 0.0116            | 0.0666            | 0.0267            | 0.1467            | 0.1018 |

6.2 Real data application

In this subsection, the application of real data set is analyzed to verify the flexibility of the proposed family. The Ra-Ra distribution has been compared with four distributions "Gamma Rayleigh (GaRa), Marshal Olkin Rayleigh (MORa), Truncated-Exponential Skew Symmetric Rayleigh (TESRa), and Rayleigh (Ra) distributions. For more details about the compared distributions see [2][4][6]. The R software has been used to compute the analytical measures "negative log-likelihood (NLL), Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Hanan and Quinn Information Criteria (HQIC)", and values of parameters estimation via MLE method.

The data set taken from Mathers et al. [16] representing the crude mortality rate (CMR) among people who inject drugs. The data set consist of the following observations:

2.01, 6.32, 3.52, 2.15, 5.42, 2.04, 2.77, 2.26, 1.95, 1.00, 2.45, 0.74, 0.98, 1.27, 2.77, 3.68, 1.18, 1.09, 1.60, 0.57, 3.33, 0.91, 7.14, 2.08, 3.85, 1.99, 7.76, 2.52, 1.57, 4.67, 4.22, 1.92, 1.59, 4.08, 2.02, 0.84, 6.85, 2.18, 2.04, 1.05, 2.91, 1.37, 2.43, 2.28, 3.74, 1.30, 1.59, 1.83, 3.85, 6.30, 4.83, 0.50, 3.40, 2.33, 4.25, 3.49, 2.12, 0.83, 0.54, 3.23, 4.50, 0.71, 0.48, 2.30, 7.73.

The fitting results for each of the fitted distributions are shown in Table 4.

Table 4. Results of fitting real data set.

| Distribution | NLL   | AIC   | CAIC  | BIC   | HQIC  | MLE    |
|--------------|-------|-------|-------|-------|-------|--------|
| Ra-Ra        | 118.4921 | 240.9842 | 241.1777 | 245.333 | 242.7000 | 1.252622 |
| GaRa         | 120.3703 | 244.7406 | 244.9342 | 249.0894 | 246.4565 | 2.248383 |
| MORa         | 119.0476 | 242.0952 | 242.2888 | 246.444 | 243.8111 | 2.274229 |
| TESRa        | 119.6947 | 243.3893 | 243.5829 | 247.7381 | 245.1052 | 2.510078 |
| Ra           | 123.6520 | 249.3039 | 249.3674 | 251.4783 | 250.1618 | 1.1864005 |

From Table 4 the newly proposed Ra – Ra distribution displays a precise good representation as it has the lowest values for the analytical measures NLL, AIC, CAIC, BIC, and HQIC. Furthermore, Figures 3 and 4 present respectively the histogram plot of the data set with the other compared distributions and the corresponding empirical cdf plot. The fit of Ra – Ra density is closer to the empirical histogram than the fits of other compared distributions.
Figure 3. Histogram plot of the dataset with other compared probability distributions

Figure 4. Empirical cdf of the dataset with other compared probability distributions

7. Conclusions

In this article, a new extended family to the Rayleigh distribution built on composing two cumulative distribution functions (cdfs) is adopted. This adoption leads to a new family of continuous distributions, named the Rayleigh - G distributions. General formulas for the basic functions of the new family are investigated. A special case "sub-model" of the new family, named Rayleigh - Rayleigh distribution is considered. The most essential statistical properties of the new distribution are investigated. There is a certain advantage of using the proposed distribution like its cdf has a closed-form. The parameters estimation via the maximum likelihood method is discussed. Numerical illustrations via simulation study and application with real data set are conducted. Through the simulation study, the proficiency and consistency of the maximum likelihood estimators (MLEs) of the proposed distribution are illustrated. In the practical application, the real data set taken from Mathers et al. [16] representing the crude mortality rate (CMR) among people who inject drugs. The proposed distribution "Rayleigh – Rayleigh" reveals better fits "more flexibility" to this real data set than the other compared distributions. This flexibility enables using Rayleigh – Rayleigh distribution in various application areas.


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