Quantum Fourier analysis

Arthur Jaffe, Chunlan Jiang, Zhengwei Liu, Yunxiang Ren, and Jinsong Wu

Quantum Fourier analysis is a subject that combines an algebraic Fourier transform (pictorial in the case of subfactor theory) with analytic estimates. This provides interesting tools to investigate phenomena such as quantum symmetry. We establish bounds on the quantum Fourier transform $\hat{\xi}$, as a map between suitably defined $L^p$ spaces, leading to an uncertainty principle for relative entropy. We cite several applications of quantum Fourier analysis in subfactor theory, in category theory, and in quantum information. We suggest a topological inequality, and we outline several open problems.

In this paper, we explore quantum Fourier analysis (QFA), a subject revolving around the study of Fourier analysis of quantum symmetries. The discovery of such symmetries emerged in the 1970s and has flourished ever since. It represents a major advance both in mathematics and in physics, as well as in the relation between these two subjects. Thus, QFA adds an extra dimension to the 200-year-old subject of classical Fourier analysis (CFA), analyzing the Fourier transform $F$.

CFA led to insights into and to solutions of problems in almost every field of mathematics, including partial differential equations, probability theory, number theory, representation theory, topology, geometry, etc. It ultimately led to the categorization of Fourier duality (1, 2). The Hausdorff–Young inequality is a bound on the norm $M_g = \|F\|_{L^p \to L^q}$, where $q = p/(p - 1)$. Hirschman discovered that differentiating $M_g$ gives an uncertainty principle for the Shannon entropy, generalizing the well-known Heisenberg principle. He and Everett conjectured the optimal inequality (3, 4). Deep and beautiful proofs were found (5–7).

Classical hypercontractivity states $\|e^{-tH}\|_{L^2 \to L^q} \leq 1$, where $H$ is a simple harmonic oscillator Hamiltonian with unit angular frequency and $e^{tH} \geq 1 - t \geq 1$ (8–10). The classical Hausdorff–Young inequality is a consequence of $\|F\|_{L^p \to L^q} \leq 1$, where $F = e^{itH/2}$. Further inequalities can be found in many papers such as refs. 11–18, suggesting, in retrospect, a bridge from CFA to QFA.

1. QFA

A quantum Fourier transform $\hat{\xi}$ defines Fourier duality between quantum symmetries, which could be analytic, algebraic, geometric, topological, and categorical. The quantum symmetries could be finite or infinite, discrete or continuous, commutative or noncommutative. In certain contexts $\hat{\xi}$ can be defined pictorially—as in the picture language program (19). QFA is the study of structures involving $\hat{\xi}$.

It is possible to estimate various norms $\|\hat{\xi}\|_{L^p \to L^q}$ as transformations between noncommutative $L^p$ spaces, and results in refs. 20 and 21 represent early breakthroughs in the application and formulation of QFA. As QFA is more sophisticated than CFA, these subjects have differences as well as similarities; we explore them both.

Let us consider an example of similarities and differences. In CFA, the extremizers of the Hausdorff–Young inequality (and many others) are Gaussians. In QFA, on subfactors, the Hausdorff–Young also holds. The extremizers are bispies of biprojections. A biprojection is a projection whose quantum Fourier transform is a multiple of a projection (22); so the behavior under Fourier transformation of the extremizers in QFA on subfactors are similar to those in CFA, while their algebraic properties differ.

In this paper, we give a unified view of QFA. We establish a “relative” inequality between pictures that yields an uncertainty principle for relative entropy. We propose a universal quantum inequality, namely Eq. 9, that unifies many other quantum inequalities. This is similar to the way the Brascamp–Lieb inequality unites Young’s inequality, Hölder inequality, and others, in CFA. Throughout the paper, we cite applications of QFA. Finally, in section 9, we state some general goals for the future and some open questions.

QFA reveals insight and intrinsic structure, as well as relations between fusion rings, fusion categories, and subfactors. We show how the “Schur product property” provides a powerful obstruction to distinguish mathematical objects. QFA also provides an approach to quantum entanglement, uncertainty relations, and other problems in quantum information. We are certain that QFA will lead to other advances in many different fields.

2. QFA on Fusion Rings

Let us start with fusion rings, as introduced by Lusztig (23); this is an interesting quantum symmetry beyond groups. See ref. 24 for further results and references. A fusion ring $\mathbb{A}$ is a ring that is free as a $\mathbb{Z}$-module, with a basis $\{x_1, x_2, \ldots, x_m\}$, $m \in \mathbb{N}$, with $x_1 = 1$, and such that

**Significance**

Classical Fourier analysis, discovered over 200 years ago, remains a cornerstone in understanding almost every field of pure mathematics. Its applications in physics range from classical electromagnetism to the formulation of quantum theory. It gives insights into chemistry, engineering, and information science, and it underlies the theory of communication. Quantum Fourier analysis extends this perspective. It yields insights and inequalities associated with uncertainty principles for quantum symmetries. In this paper, we introduce this mathematical subject, we show how it can solve some theoretical problems, and we give some applications to quantum physics with bounds on entropy and the analysis of quantum entanglement. We believe that quantum Fourier analysis, now in its infancy, will have significant future impact.

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3. QFA on Subfactors

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The constant $\delta$ is the square root of the Jones index. The extremizers of these inequalities have nine different characterizations. In particular, the red line $1/p + 1/q = 1$, for $1/2 \leq 1/p, 1/q \leq 1$ corresponds to the quantum Hausdorff–Young inequality. Moreover, all of the other quantum inequalities, such as quantum Young’s inequality, in Theorem 2.1 have been proved for subfactors planar algebras in ref. 21.

4. QFA on Unitary Fusion Categories

The QFA on subfactors also applies for unitary fusion categories through the quantum double construction, see e.g., ref. 33. Let $\mathcal{C}$ be a unitary fusion category and $I = \{X_1, X_2, \ldots, X_m\}$ be the set of simple objects. There is a Frobenius algebra $\gamma$ in $\mathcal{C} \otimes \mathcal{C}$ whose object is $\bigoplus_{i=1}^m X_i \otimes X_i$. Following the quantum double construction, we obtain an irreducible subfactor planar algebra, such that $\mathcal{A} = \mathcal{D}_{2, +} = \text{hom}_{\mathcal{C} \otimes \mathcal{C}}(\gamma)$ and $\mathcal{B} = \mathcal{D}_{2, -} = \text{hom}_{\mathcal{C} \otimes \mathcal{C}}(\gamma \otimes \gamma)$. Applying QFA to $\mathcal{A}$ on this subfactor, we obtain inequalities on the Grothendieck ring of unitary fusion categories as stated in Theorem 2.1. Applying QFA to $\mathcal{B}$, we obtain inequalities on the dual of the Grothendieck ring, which turn out to be highly nontrivial.

Application: Analytic Obstructions. It is important to determine whether a fusion ring can be the Grothendieck ring of a unitary fusion category. QFA provides powerful analytic obstructions to the unitary categorification of fusion rings. The quantum inequalities in Theorem 2.1 holds on the dual of Grothendieck rings. However, they may not necessarily hold on the dual of fusion rings, thereby providing analytic obstructions of the unitary categorification of fusion rings. Even the Schur product property on the dual of a fusion ring, $0 \leq x + y$ if $0 \leq x, y \in \mathcal{B}$, gives by itself a surprisingly efficient analytic obstruction:

**Theorem 4.1 (25).** If a fusion ring can be unitarily categorified, then the Schur product property holds on its dual.

There are 34 examples in the classification of simple integral fusion rings up to rank 8 and Frobenius–Perron dimension less than 3,780. Four of them are group-like. Methods based on previously known analytic, algebraic, and number theoretic obstructions did not determine whether the remaining 30 could be unitarily categorified. As a consequence of the Schur-product obstruction, 28 out of the 30 have no unitary categorization, as shown in ref. 25.

**Example:** Let us recall one example from ref. 25 to illustrate this obstruction. Let $\Lambda$ be the rank-7 simple integral fusion ring with the following seven fusion matrices, equal to

$$
\begin{align*}
&\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
&\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
&\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\end{align*}
$$

The eigenvalue table of these matrices (where $\zeta^7 = 1$) is:

$$
\begin{align*}
&\begin{pmatrix}
5 & -1 & -\zeta & -\zeta^6 & -\zeta^2 & -\zeta^4 & 0 \\
5 & -1 & -\zeta^3 & -\zeta^6 & -\zeta & -\zeta^4 & 0 \\
5 & -1 & -\zeta^3 & -\zeta & -\zeta^2 & -\zeta^4 & 0 \\
6 & 0 & -1 & -1 & -1 & 1 & 1 \\
7 & 1 & 0 & 0 & 0 & 0 & -3 \\
7 & 1 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix},
\end{align*}
$$

The first column is the Perron–Frobenius dimension of the seven simple objects. Take $X = x_1 + x_5 - 3x_6 + 2x_7$, then $X = X^* = X^2/15$.

The Schur product property on $\mathcal{B}$, equivalent to a dual version of Eq. 4, yields (with $x = y = z = X$) that

$$
d((XX^*) \odot (XX^*) \odot (XX^*)) \geq 0.
$$

However, it follows directly that Eq. 7 is false in this case, as

$$
\frac{1^3}{1} + \frac{0^3}{5} + \frac{0^3}{5} + \frac{0^3}{5} + \frac{1^3}{6} + \frac{(-3)^3}{7} + \frac{2^3}{7} = -\frac{65}{42} < 0.
$$

Therefore, the fusion ring $\Lambda$ cannot be unitarily categorized.

5. QFA on Locally Compact Quantum Groups

The previous results focus on finite quantum symmetry, such as fusion rings and finite-index subfactors. One might ask whether QFA can be established for infinite quantum symmetry. The answer is “yes”; there are results on infinite-dimensional Kac algebras and locally compact quantum groups. This relies on the theory of noncommutative $L^p$ spaces (e.g., ref. 34).

We recall the definition of the Fourier transform on locally compact quantum groups, of which the Fourier transform on Kac algebra is a special case (35). Let $G$ be a locally compact quantum group and $\varphi$ the left Haar weight. Suppose $W$ is the multiplicative unitary, $\phi$ is a normal semifinite faithful weight on the commutant of $L^\infty(G)$, and $\hat{d}$ is a normal semifinite faithful weight on the commutant of $L^\infty(\hat{G})$. Let $d = \frac{\hat{d}}{\hat{d} - d_{ad}}$, $\hat{d}$ be the Connes’ spatial derivatives, and let $L^p(\phi), L^p(\hat{\phi})$ be Hilsum’s space for any $1 \leq p < \infty$. The Fourier transform $\mathcal{F}_p: L^p(\phi) \rightarrow L^p(\hat{\phi})$, for $1 \leq p \leq 2$, and $1/p + 1/q = 1$, is defined by

$$
\mathcal{F}_p(xd^{1/p}) = (\varphi \otimes \chi)(W(x \otimes 1))d^{1/q}, \forall x \in T^2_\phi.
$$

Here, $\mathcal{F}_p \subseteq \mathcal{A}_\phi \cap \mathcal{A}_\phi^*$ is the space of elements analytic with respect to $\varphi$. Even the definition of the convolution on locally compact quantum groups is nontrivial.

The quantum inequalities in Theorem 2.1 on these infinite quantum symmetries have been partially studied in refs. 36–40. The quantum uncertainty principle $QU$-2 in Theorem 2.1 becomes a continuous family of inequalities on locally compact quantum groups (40).

6. Surface Algebras and a Universal Inequality

Surface Algebras. Many inequalities in CFA have not been axiomatized in a pictorial framework. Z.L. introduced surface algebras in ref. 41, formalizing the extension of planar algebras from two-dimensional (2D) to three-dimensional (3D) space.
outlined in ref. 42. Surface algebras are an extensive framework to capture additional pictorial features of Fourier analysis.

For any subfactor planar algebra, the actions of planar tangles can be further extended to the actions of surface tangles. (The arrow in planar diagrams corresponds to the $\delta$ sign in planar algebras. The clockwise/anticlockwise orientation of the arrow indicates the input/output disc in surface algebras.) One can represent Fourier transform, multiplication, and convolution as the action of the following surface tangles in the 3D space:

Using 3D pictures, one can consider the Fourier duality for surface tangles with multiple inputs and outputs. The 3D formalism has provided deep insights into quantum information, algebraic identities, and various other connections with physics.

One can consider a finite-dimensional Kac algebra $A$ as $\mathcal{A}$ and its dual $\hat{K}$ as $\mathcal{B}$, with a Fourier transform from $K$ to $\hat{K}$ defined analogously to section 5. The pair of Kac algebras $K$ and $\hat{K}$ can be understood as $\mathcal{A}$ and $\mathcal{B}$ for the surface algebra. The comultiplication is given by the picture:

The Hopf-axiom that the comultiplication is an algebraic homomorphism reduces to the string-genus relation of surface tangles given in equation 17 of ref. 42.

**A Universal Inequality.** In a subfactor planar/surface algebra $\mathcal{P}$, the Fourier transform, the multiplication, and the convolution can be realized by planar/surface tangles. In general, a surface tangle is a multilinear map on $\mathcal{T}_{X_n} \otimes \mathcal{T}_{Y_n}$. Now, we give a pictorial inequality in the quantum case, motivated by the classical Brascamp–Lieb inequality. We replace the dual of the linear transformation in the quantum case, motivated by the classical Hausdorff–Young inequality. We replace the dual of the linear transformation by a surface tangle. In general, a surface tangle is a multilinear map on $\mathcal{T}_{X_n} \otimes \mathcal{T}_{Y_n}$. Now, we give a pictorial inequality in the quantum case, motivated by the classical Brascamp–Lieb inequality. We replace the dual of the linear transformation by a surface tangle. In general, a surface tangle is a multilinear map on $\mathcal{T}_{X_n} \otimes \mathcal{T}_{Y_n}$. Now, we give a pictorial inequality in the quantum case, motivated by the classical Brascamp–Lieb inequality. We replace the dual of the linear transformation by a surface tangle. In general, a surface tangle is a multilinear map on $\mathcal{T}_{X_n} \otimes \mathcal{T}_{Y_n}$.

For those familiar with the Quon language (42), we can consider the pictorial inequalities whose $T_j$s are surface tangles with braided charged strings. In particular, if all of the inputs and outputs are 2-boxes, corresponding to qudits, then $T_j$ can be any Clifford transformation on qudits. These Clifford transformations can be considered as a quantum analog of the dual of a linear transformation $B_j : \mathbb{Q}^k \rightarrow \mathbb{Q}^k$. Considering the action on density matrices, the $n$-qudit Clifford gates on Pauli matrices are symplectic transformations on $2n$-dimensional symplectic spaces over $\mathbb{Q}_d$.

**7. Relative Inequalities, Entropy, and Uncertainty**

Here, we present a relative, quantum, Hausdorff–Young inequality. This leads to a relative, quantum, entropic uncertainty principle.

Let $\mathcal{P}_\bullet$ be an irreducible subfactor planar algebra with the Markov trace $tr$. Let $\varphi$ (resp. $\psi$) be a faithful state on $\mathcal{P}_{2,+}$ (resp. $\mathcal{P}_{2,-}$). Let $D_\varphi$ (resp. $D_\psi$) be the density operator of $\varphi$ (resp. $\psi$), namely, $\varphi(\cdot) = tr(D_\varphi \cdot)$. Now, we define a Fourier transform $\widehat{\mathcal{F}}_{p,\varphi,\psi} : L^p(\mathcal{P}_{2,+}, tr) \rightarrow L^q(\mathcal{P}_{2,-}, tr)$ for $1 \leq p \leq 2$, $q = p/(p - 1)$ as

$$\widehat{\mathcal{F}}_{p,\varphi,\psi}(x D_\psi^{1/p}) = \mathcal{F}(x D_\psi^{1/p})^{1/q - 1/2}.$$  \[10\]

This Fourier transform is represented pictorially as follows,

$$\cdot \longrightarrow \cdot.$$  \[11\]

From Plancherel’s theorem for $\mathcal{F}$, we infer Plancherel’s theorem for $\widehat{\mathcal{F}}_{p,\varphi,\psi}$,

$$\|\widehat{\mathcal{F}}_{p,\varphi,\psi}(x D_\psi^{1/p})\|_2 = \|\mathcal{F}(x D_\psi^{1/p})\|_2 = \|x D_\psi^{1/p}\|_2.$$  \[12\]

**Theorem 7.1 (Relative, quantum Hausdorff–Young inequality).** Let $\mathcal{P}_\bullet$ be an irreducible subfactor planar algebra and let $\varphi, \psi$ be faithful states on $\mathcal{P}_{2,\pm}$. Then for any $x \in \mathcal{P}_{2,\pm}$, $1 \leq p \leq 2$, and dual $2 \leq q = p/(p - 1)$, we have

$$\|\widehat{\mathcal{F}}_{p,\varphi,\psi}(x D_\psi^{1/p})\|_q \leq K_{p,\varphi,\psi} \|x D_\psi^{1/p}\|_p.$$  \[13\]

Here, $K_{p,\varphi,\psi} = \delta^{-2/p} \|D_\psi^{1/q - 1/2}\|_\infty \|D_\varphi^{-1/p}\|_1$. Pictorially,

$$\begin{array}{c}
\end{array}$$

where

$$\begin{array}{c}
\end{array}$$

The first inequality is a consequence of the quantum Hölder inequality. The second inequality is a consequence of the quantum Hausdorff–Young inequality (theorem 4.8 in ref. 21). Here, the constant is $K = \sup_{\|y\|_p = 1} K(y)$, with $K(y)$ equal to the following picture:

$$\begin{array}{c}
\end{array}$$

Jaffe et al.
To obtain the first inequality, we use the quantum Young inequality (theorem 4.13 in ref. 21). To obtain the second inequality, we use the quantum Hörder inequality.

Relative Entropy. We formulate relative entropy (RE) and the corresponding relative entropic quantum (REQ) uncertainty principle. For two positive functionals \( \omega, \varphi \) on \( \mathcal{P}_{2,+} \), recall that the relative entropy (43) is

\[
S(\omega|\varphi) = \text{tr}(D_\omega(\log D_\omega - \log D_\varphi)).
\]

The Relative Entropic Quantum Uncertainty Principle. For a positive functional \( \omega \) on \( \mathcal{P}_{2,+} \), define \( \tilde{\omega} \) as the positive functional on \( \mathcal{P}_{2,-} \) given by the density matrix

\[
D_\omega = |\tilde{\omega}(D_{1/2})|^2.
\]

It follows that \( \tilde{\omega}(1) = \omega(1) \). If \( \omega \) is a state, so is \( \tilde{\omega} \).

Theorem 7.2 (REQ Uncertainty Principle). Let \( \mathcal{P}_* \) be an irreducible subfactor planar algebra and \( \varphi, \psi \) be faithful positive functionals on \( \mathcal{P}_{2,\pm} \). Then, for any state \( \omega \) on \( \mathcal{P}_{2,+} \),

\[
S(\omega|\varphi) + S(\omega|\psi) \leq \log \|D_\omega^{-1}\|_{\infty} - 1 \frac{1}{\delta^2} S(\tilde{\omega} | \psi) - 2 \log \delta.
\]

Proof. Note that \( \tilde{\omega}_p,\varphi(D_{1/2}D_{1/2}^{p-1/2}) = \tilde{\omega}(D_{1/2}^{p-1/2})D_{1/2}^{p-1/2} \), as \( \|AB\|_p = \|A\|_p \|B\|_p \), using Eq. 15, we infer that \( \tilde{\omega},\varphi(D_{1/2}D_{1/2}^{p-1/2}) \) and \( D_{1/2}D_{1/2}^{p-1/2} \) have the same \( n \) q norms. Define the function \( f(p) \) as a picture, where \( q = p/(p - 1) \), and where \( K_{p,\varphi,\psi} \) is defined in Eq. 14.

The picture \( f(p) \) is negative for \( 1 \leq p \leq 2 \), by Theorem 7.1. Also \( f(2) = 0 \) by Plancheur’s theorem, so the left derivative \( f'(2) > 0 \). Then, Theorem 7.2 is a consequence of the expressions for the derivatives in the following lemma.

Lemma 7.3. For any positive functional \( \omega \) on \( \mathcal{P}_{2,+} \), we have

\[
\frac{d}{dp} \left| \begin{array}{c|c}
\begin{array}{c|c|c|c}
\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array} & \\
\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array} \right|_{p=2} = \frac{1}{4} \frac{1}{\delta^2} S(\tilde{\omega} | \psi) - \frac{1}{4} \frac{1}{\delta^2} S(\omega | \varphi),
\]

\[
\frac{d}{dp} \left| \begin{array}{c|c}
\begin{array}{c|c|c|c}
\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array} & \\
\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array} \right|_{p=2} = \frac{1}{4} \frac{1}{\delta^2} S(\tilde{\omega} | \psi) + \frac{1}{4} \frac{1}{\delta^2} S(\omega | \varphi),
\]

\[
\frac{d}{dp} \left| \begin{array}{c|c}
\begin{array}{c|c|c|c}
\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array} & \\
\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array} \right|_{p=2} = \frac{1}{2} \log \delta - \frac{1}{4} \frac{1}{\delta^2} \log \|D_\omega^{-1}\|_{\infty} + \frac{1}{48} \frac{1}{\delta^2} S(\omega | \varphi).
\]

8. QFA and Quantum Entanglement

Here, we use pictures as in refs. 30 and 44, which do not require shading. The Fourier transform of a multiple of the projection onto the zero-vector for the group \( \mathbb{Z}_d \), namely \( d^{1/2} |0\rangle \langle 0| \), is the identity.

\[
\mathfrak{S}(\bigcirc \bigcirc) = \mathfrak{S}(\bigcirc \bigcirc) = d^{-1/2} \sum_{k \in \mathbb{Z}_d} \sum_{\xi \in \mathbb{C}^d} \frac{k}{k} \cdot
\]

One can identify a linear transformation \( T \) on \( \mathbb{C}^d \) as a vector \( \mathfrak{T} \) in \( \mathbb{C}^2 \), namely

\[
\begin{pmatrix}
\frac{1}{2} \\
1
\end{pmatrix} \longleftrightarrow \begin{pmatrix}
\frac{1}{2} \\
-1
\end{pmatrix}
\]

Identifying Eq. 16 in this way gives the illustration of how the Fourier transform \( \mathfrak{S} \) acts on product states. In particular,

\[
\mathfrak{S}(\bigcirc \bigcirc) = \mathfrak{S}(\bigcirc \bigcirc) = d^{1/2} |\text{Max}|.
\]

If \( d = 2 \), then \( k = -k \in \mathbb{Z}_d \), and \( |\text{Max}| \) is the usual Bell state.

A similar picture and vector \( |\text{Max}| \) exists for \( n \)-qudits (44). This vector \( |\text{Max}| \) generalizes the classical Bell state for \( d = 2 \) to qubits to a maximally entangled state for \( n \)-qudits of order \( d \). Furthermore, one can apply \( \mathfrak{S} \) to any product basis state; one thereby obtains a Max basis (of maximally entangled states). These are related to a corresponding \( n \)-qudit Greenberger–Horne–Zeilinger (GHZ) basis (45). In section 4 in ref. 19, one finds expressions for the various \( |\text{Max}|_n \) basis states, as well as their relation to, and expressions for, the \( |\text{GHZ}|_n \) basis states. Also see ref. 42 for a Quon interpretation.

The quantum uncertainty principle QUP-1 in Theorem 2.1 for subfactors gives a lower bound for the entanglement entropy. The product ground state has minimal entanglement entropy. Each Max state has maximal entanglement entropy. In addition, we obtain a relative entropic uncertainty principle for quantum entanglement by applying Theorem 7.2.

9. Some Future Directions and Goals

We propose a few specific questions but first cite some general directions that appear ripe for the development of QFA:

- Establish additive combinatorics for quantum symmetries, such as for unitary modular tensor categories.
- Establish a general theory for \( \mathfrak{S}(\mathcal{P}_{n,\pm}) \), where \( \mathfrak{S}^2 = 1 \).
- Understand Fourier analysis for infinite quantum symmetries within a pictorial framework, such as surface algebras.
- Seek further applications of QFA to quantum information.

Questions on the Universal Inequality. There are three central problems for the classical Brascamp–Lieb inequality on \( \mathbb{R} \):

1) Can one find for which tuples of linear maps the best constant is finite?
2) Can the best constant be achieved? If so, it is proved in ref. 46 that there exist Gaussian extremizers.
3) Are all extremizers Gaussian?

Since all \( \mathcal{P}_{n,\pm}, n \in \mathbb{N} \), are finite-dimensional, the best constant \( C \) of the universal inequality is finite and the extremizer exists by the compactness. We ask the following questions for the universal inequality:

**Question 9.1.** In which case is the best constant achieved by tensor product of \( n \)-projections (the natural generalization of biprojections)?

**Question 9.2.** If all of the inputs belong to \( \mathcal{P}_{2,\pm} \), are all of the extremizers bishifts of biprojections?

**Question 9.3 (Finite abelian groups).** What are the best constants of the universal inequality on finite abelian groups?

Questions on Subfactor Planar Algebras. Suppose \( \mathcal{P}_{*,\pm} \) is an irreducible subfactor planar algebra.

**Conjecture 9.4.** For any \( \varepsilon > 0 \), there exists \( \varepsilon' \) such that if \( x \in \mathcal{P}_{2,\pm} \),\( |x - P||2 < \varepsilon' \), and \( \|\mathfrak{S}(x) - \lambda Q\|2 < \varepsilon' \), for some
projections $P$, $Q$ and constant $\lambda$, then there is a biprojection $B$, such that $\|x - B\| < \varepsilon$.

**Question 9.5.** Can one characterize the extremizers for the uncertainty principles on $n$-boxes, for $n \geq 3$?

**Block Renormalization Map and Quantum Central Limit Theorem.**

The block map $B_{\lambda}$ is a composition of convolution and multiplication,

$$B_{\lambda} (f) = \delta_{x} + \frac{(1-\lambda)}{\|x\|_{\infty}^{2}} \lambda$$

The limit points of the iteration of the block map are all biprojections for finite-index, irreducible subfactors (47). We regard this result as a quantum 2D central limit theorem.

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1. T. Tannaka, "Über den Dualitätssatz der nichtkommutativen topologischen Gruppen" in [German]. Tohoku Math. J., 45, 1–12 (1939).
2. M. G. Krein, A principle of duality for a bicommutant and block algebra. Dokl. Akad. Nauk. SSR, 69, 725–728 (1949).
3. I. Hirschman, A note on entropy. Am. J. Math., 79, 152–156 (1957).
4. H. Everett III, The many-worlds interpretation of quantum mechanics: The theory of the universal wave function. http://inspirehep.net/record/1358211/files/ dissertation.pdf. Accessed April 20, 2020.
5. W. Beckner, Inequalities in Fourier analysis. Ann. Math. 102, 159–182 (1975).
6. I. Białynicki-Birula, J. Mycielski, Uncertainty relations for information entropy in wave mechanics. Commun. Math. Phys. 44, 129–132 (1975).
7. H. Bostergard, E. Lieb, Best constants in Young’s inequality, its converse, and its generalization to more than three functions. Adv. Math. 20, 151–173 (1976).
8. E. Nelson, “A quartic interaction in two dimensions” in Mathematical Theory of Elementary Particles, R. Goodman, I. Segal, Eds. (MIT Press, Cambridge, MA, 1966), pp. 69–73.
9. J. Glimm, Boson Fields with nonlinear self-interaction in two dimensions. Commun. Math. Phys. 8, 12–25 (1968).
10. E. Nelson, The free Markoff field. J. Funct. Anal. 12, 211–227 (1973).
11. P. Federbush, Partially alternate derivation of a result of Nelson. J. Math. Phys. 10, 50–52 (1969).
12. L. Gross, Hypercontractivity and logarithmic Sobolev inequalities for the Clifford–Dirichlet form. Duke Math. J. 43, 383–396 (1975).
13. E. Carlen, E. Lieb, Optimal hypercontractivity for Fermi fields and related non-commutative integration inequalities. Commun. Math. Phys. 155, 27–46 (1992).
14. E. M. Stein, T. S. Murphy, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals (Princeton Mathematics Series, Princeton University Press, 1993), vol. 43.
15. S. Janson, On complex hypercontractivity. J. Funct. Anal. 151, 270–280 (1997).
16. É. Ricard, Q. Xu, A noncommutative martingale convexity inequality. Ann. Probab. 44, 867–882 (2016).
17. M. Junge, C. Palazuelos, J. Parcet, M. Perrin, É. Ricard, Hypercontractivity for free products. Ann. Sci. École Norm. Sup. 48, 861–889 (2015).
18. S. Beigi, C. King, Hypercontractivity and the logarithmic Sobolev inequality for the completely bounded norm. J. Math. Phys. 57, 015206 (2016).
19. A. Jaffe, Z. Liu, Mathematical picture language program. Proc. Natl. Acad. Sci. U.S.A. 115, 81–86 (2018).
20. Z. Liu, Exchange relation planar algebras of small rank. Trans. Am. Math. Soc. 368, 8303–8348 (2016).
21. C. Jiang, Z. Liu, J. Wu, Noncommutative uncertainty principles. J. Funct. Anal. 270, 264–311 (2016).
22. D. Bisch, A note on intermediate subfactors. Pac. J. Math. 163, 201–216 (1994).
23. G. Lustig, Leading coefficients of character values of Hecke algebras. Proc. Sym. Pure Math. 47, 235–262 (1987).
24. P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, “Tensor categories" in Mathematical Surveys and Monographs (American Mathematical Society, 2015), vol. 205.
25. Z. Liu, S. Palcoux, J. Wu, Fusion bidegrees and Fourier analysis. arXiv:1910:12059 (26 October 2019).
26. V. Jones, Index for subfactors. Invent. Math. 72, 1–25 (1983).
27. D. Evans, Y. Kawahigashi, Quantum Symmetries on Operator Algebras (Clarendon Press, Oxford, UK, 1998).
28. V. Jones, Planar algebras, I. arXiv:math/9909207v1 (4 September 1999).
29. M. Attiah, Topological quantum field theory. Publ. Math. IHÉS. 68, 175–186 (1988).
30. A. Jaffe, Z. Liu, Planar para algebras and reflection positivity. Commun. Math. Phys. 352, 95–137 (2017).
31. A. Jaffe, Z. Liu, Reflection positivity and Levin–Wen models. arXiv:1901.10662 (8 April 2019).
32. Z. Liu, J. Wu, Non-commutative Renyi entropic uncertainty principles. arXiv:1904.04292 (30 January 2019).
33. M. Muger, From subfactors to categories and topology II: The quantum double of tensor categories and subfactors. J. Pure Appl. Algebra 180, 159–219 (2003).
34. G. Pisier, Q. Xu, “Non-commutative $L^p$-spaces” in Handbook of the Geometry of Banach Spaces (Elsevier Science, 2003), vol. 2.
35. J. Kustermans, S. Vaes, Locally compact quantum groups. Ann. Sci. École Norm. Super. 48, 339–397 (2005).
36. T. Cooney, A Hausdorff-Young inequality for locally compact quantum groups. Int. J. Math. 21, 1619–1632 (2010).
37. M. Caspers, The $L^p$-Fourier transform on locally compact quantum groups. J. Operat. Theor. 68, 161–193 (2013).
38. Z. Liu, J. Wu, Uncertainty principles for Kac algebras. J. Math. Phys. 58, 052102 (2017).
39. Z. Liu, S. Wang, J. Wu, Young’s inequality for locally compact quantum groups. J. Operat. Theor. 77, 109–131 (2017).
40. C. Jiang, Z. Liu, J. Wu, Uncertainty principles for locally compact quantum groups. J. Funct. Anal. 274, 2399–2445 (2018).
41. Z. Liu, Quon language: Surface algebras and Fourier duality. Commun. Math. Phys. 366, 865–894 (2019).
42. Z. Liu, A. Wozniakowski, A. Jaffe, Quon 3D language for quantum information. Proc. Natl. Acad. Sci. U.S.A. 114, 2497–2502 (2017).
43. H. Araki, Relative entropy of states of von Neumann Algebras. Publ. Res. Inst. Math. Sci. 11, 809–833 (1975).
44. A. Jaffe, Z. Liu, A. Wozniakowski, Holographic software for quantum networks. Sci. China Math. 61, 593–626 (2018).
45. D. M. Greenberger, M. A. Horne, A. Zeilinger, “Going beyond Bell’s theorem” in Bell’s Theorem, Quantum Theory, and Conceptions of the Universe, M. Kafkas, Ed. (Fundamental Theories of Physics, Springer, Heidelberg, Germany, 1989), vol. 37.
46. J. Bennett, A. Carbery, M. Christ, T. Tao, The Brascamp–Lieb inequalities: Finiteness, structure and extremals. Geometr. Funct. Anal. 17, 1343–1415 (2008).
47. C. Jiang, Z. Liu, J. Wu, Block maps and Fourier analysis. Sci. China Math. 62, 1585–1614 (2019).

**Conjecture 9.6.** For any $f \in L^\infty (\mathbb{R}^n) \cap L^1 (\mathbb{R}^n) \cap L^2 (\mathbb{R}^n)$, $f$ converges either to 0 or to a Gaussian function, under the action of the iteration of the block map $B$.

**Conjecture 9.7.** For any $f \in L^\infty (\mathbb{R}^n) \cap L^1 (\mathbb{R}^n) \cap L^2 (\mathbb{R}^n)$, the Hirschman–Beckner entropy decreases under the action of the block map $B$.

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