BEST APPROXIMATION OF ORBITS IN ITERATED FUNCTION SYSTEMS

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(Communicated by Sylvain Crovisier)

ABSTRACT. Let $\Phi = \{\phi_i : i \in \Lambda\}$ be an iterated function system on a compact metric space $(X,d)$, where the index set $\Lambda = \{1,\ldots,l\}$ with $l \geq 2$, or $\Lambda = \{1,2,\ldots\}$. We denote by $J$ the attractor of $\Phi$, and by $D$ the subset of points possessing multiple codings. For any $x \in J \setminus D$, there is a unique integer sequence $\{\omega_n(x)\}_{n \geq 1} \subset \Lambda^\mathbb{N}$, called the digit sequence of $x$, such that

$$\{x\} = \bigcap_n \phi_{\omega_1(x)} \circ \cdots \circ \phi_{\omega_n(x)}(X).$$

In this case we write $x = [\omega_1(x),\omega_2(x),\ldots]$. For $x,y \in J \setminus D$, we define the shortest distance function $M_n(x,y)$ as

$$M_n(x,y) = \max \{k \in \mathbb{N} : \omega_{i+1}(x) = \omega_{i+1}(y), \ldots, \omega_{i+k}(x) = \omega_{i+k}(y)\}$$

for some $0 \leq i \leq n-k$, which counts the run length of the longest same block among the first $n$ digits of $(x,y)$.

In this paper, we are concerned with the asymptotic behaviour of $M_n(x,y)$ as $n$ tends to $\infty$. We calculate the Hausdorff dimensions of the exceptional sets arising from the shortest distance function. As applications, we study the exceptional sets in several concrete systems such as continued fractions system, Lüroth system, $N$-ary system, and triadic Cantor system.

1. Introduction. Ergodic theory is the study of the long-term behaviour of a measure-preserving system. One of the classical results is the Poincaré recurrence theorem, which states that almost all points in a prescribed set of positive measure are infinitely recurrent. In a metric space $(X,d)$, the theorem implies that

$$\liminf_{n \to \infty} d(T^n(x),x) = 0$$

for $\mu$-almost all $x \in X$. The theorem provides a qualitative rather than quantitative description of the long-term behaviour, and the quantitative behaviour of Poincaré recurrence has been studied by Boshernitzan [1], Ornstein & Weiss [20] and Barreira & Saussol [3]. Other notions, such as the recurrence time, hitting time, etc., have

2020 Mathematics Subject Classification. Primary: 11K55; Secondary: 37F35, 28A80.

Key words and phrases. Best approximation, Shortest distance function, Iterated function systems.

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been introduced to study the quantitative nature of recurrence [9, 21, 22, 23, 26, 30]. Among others, let us mention the shrinking target problem introduced by Hill & Velani [11]. Let \( \psi : \mathbb{N} \to \mathbb{R}^+ \) be a positive function such that \( \psi(n) \to 0 \) as \( n \to \infty \), \( \{z_n\}_{n \geq 1} \) be a sequence of points in \( X \). Define the “well approximable” set

\[
W_{\{z_n\}}(T, \psi) = \{ x \in X : d(T^n(x), z_n) < \psi(n) \text{ for infinitely many } n \in \mathbb{N} \}.
\]

It follows that the orbits \( \{T^n(x)\}_{n \geq 1} \) can be well approximated by the sequence \( \{z_n\}_{n \geq 1} \). For an expanding rational map of a Riemann sphere acting on its Julia set, Hill & Velani [11, 12] estimated the Hausdorff dimension of the set \( W_{\{z_n\}}(T, \psi) \). Li, Wang, Wu & Xu [16] computed the dimension of \( W_{\{z_n\}}(T, \psi) \) in the dynamical system of continued fractions. Bugeaud & Wang [4] studied the case of \( \beta \)-dynamical systems. The metric result has been obtained by Chernov & Kleinbock [6]. They proved that the set \( W_{\{z_n\}}(T, \psi) \) has null \( \mu \)-measure if \( \sum_{n \geq 1} \mu(B_n) < \infty \), and has full measure if

\[
\sum_{n \geq 1} \mu(B_n) = \infty \quad \text{with} \quad \sum_{1 \leq n \leq m \leq N} R_{n,m} \leq C \sum_{1 \leq n \leq N} \mu(B_n),
\]

where \( B_n := B(z_n, \psi(n)) \), and \( R_{n,m} := |\mu(T^{-n}B_n \cap T^{-m}B_m) - \mu(B_n)\mu(B_m)| \) is the decay of the correlations. For more information about the shrinking target problems, the readers are referred to [8, 11, 12, 24] and the references therein.

Taking \( z_n = T^n(y) \) (\( n \geq 1 \)) for a fixed \( y \in X \), we would like to investigate the quantitative properties of the distance between \( \{T^n(x)\}_{n \geq 1} \) and \( \{T^n(y)\}_{n \geq 1} \).

We study the problem in the framework of iterated function systems. Let \( (X, d) \) be a compact metric space. Let \( \Phi = \{\phi_i : i \in \Lambda\} \) be a collection of injective mappings of \( X \), where \( \Lambda = \{1, 2, \ldots, l\} \) (\( l \geq 2 \)) or \( \Lambda = \{1, 2, \ldots\} \). We call \( \Phi \) a uniformly contractive iterated function system (IFS for short) if there exists \( 0 < \rho < 1 \) such that for any \( i \in \Lambda \) and \( x, y \in X \),

\[
d(\phi_i(x), \phi_i(y)) \leq \rho \cdot d(x, y).
\]

The limit set associated with an IFS \( \{\phi_i : i \in \Lambda\} \) can be defined as the image of the coding space. Let \( \Lambda^\ast = \cup_{n \geq 0} \Lambda^n \), the space of finite words, and let \( \Lambda^\infty \) be the collection of all infinite words over \( \Lambda \). For \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \Lambda^n \), we set \( \phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n} \). For \( \omega \in \Lambda^\ast \cup \Lambda^\infty \) and \( n \geq 1 \), we denote by \( \omega_n \) the word \( (\omega_1, \omega_2, \ldots, \omega_n) \) (when \( \omega \) is a finite word, \( n \leq |\omega| \) is required, where \( |\omega| \) denotes the length of \( \omega \)). For \( \omega \in \Lambda^\infty \), \( \{\phi_{\omega_n}(X)\}_{n \geq 1} \) is a nested sequence of compact sets, whose intersection is a singleton; the point in the intersection will be denoted by \( \pi(\omega) \). The limit set (or the attractor) of the IFS is

\[
J = \pi(\Lambda^\infty) = \bigcup_{\omega \in \Lambda^\infty} \bigcap_{n \geq 1} \phi_{\omega_n}(X) = \bigcap_{n \geq 1} \bigcup_{\omega \in \Lambda^n} \phi_\omega(X).
\]

The map \( \pi : \Lambda^\infty \to J \) is called the coding map, which is surjective but is not necessarily injective. Any \( \omega = (\omega_1, \omega_2, \ldots) \in \Lambda^\infty \) such that \( x = \pi(\omega) \) is called a coding of \( x \in J \), and a point in \( J \) may have multiple codings. Let \( D = \{x \in J : x \text{ has multiple codings}\} \). Hence, for any \( x \in J \setminus D \), there is a unique coding \( (\omega_1(x), \omega_2(x), \ldots) \) such that \( \pi(\omega) = x \), whence we write

\[
x = [\omega_1(x), \omega_2(x), \ldots].
\]

For \( x = [\omega_1(x), \omega_2(x), \omega_3(x), \ldots] \in J \setminus D \), we define

\[
T x = T([\omega_1(x), \omega_2(x), \omega_3(x), \ldots]) = [\omega_2(x), \omega_3(x), \ldots].
\]
For \( n \geq 1 \) and \((\omega_1, \ldots, \omega_n) \in \Lambda^n\), we call
\[
J_n(\omega_1, \ldots, \omega_n) = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n}(X)
\]
an \( n \)-th order cylinder. For \( x \in J\setminus D\), let \( J_n(x) \) denote the unique \( n \)-th cylinder that contains \( x \), i.e., \( J_n(x) = J_n(\omega_1(x), \ldots, \omega_n(x)) \). The \( \sigma \)-algebra over \( J \) generated by all \( n \)-th order cylinders are denoted as \( C^n_n \).

For \( x = [\omega_1(x), \omega_2(x), \ldots] \), \( y = [\omega_1(y), \omega_2(y), \ldots] \in J\setminus D\), we define the shortest distance function \( M_n(x, y) \) as
\[
M_n(x, y) = \max \{ k \in \mathbb{N} : \omega_{i+1}(x) = \omega_{i+1}(y), \ldots, \omega_{i+k}(x) = \omega_{i+k}(y) \}
\]
for some \( 0 \leq i \leq n - k \), which counts the longest run length of the longest same block among the first \( n \) digits of \((x, y) \in J\setminus D \times J\setminus D \).

**Remark 1.** (1) \( M_n(x, y) = \max \{ k \in \mathbb{N} : T^i(x) \in J_k(T^i(y)) \} \) for some \( 0 \leq i \leq n - k \);

(2) We define a metric \( d \) on \( J\setminus D \): if \( x = [\omega_1, \omega_2, \ldots], y = [\nu_1, \nu_2, \ldots] \in J\setminus D \), the distance of \( x \) and \( y \) is defined as
\[
d(x, y) = \exp \{ -\inf \{ i \geq 1 : \omega_i \neq \nu_i \} \}.
\]
Then \( M_n(x, y) \leq -\log \min_{1 \leq i \leq n} d(T^i(x), T^i(y)) \).

Let \( \mu \) be a \( T \)-invariant complete Borel probability measure \( \mu \) on \( J \). We then have a measure-preserving system \((J, \mu, T)\), and \( \mu \)-almost every point \( x \in J \) has a unique coding. For this system, we define the lower Renyi entropy \( H_\ast \) to be
\[
H_\ast = \liminf_{n \to \infty} -\frac{1}{n} \log \sum_{(\omega_1, \ldots, \omega_n) \in \Lambda^n} \mu(J_n(\omega_1, \ldots, \omega_n))^2,
\]
and define the upper Renyi entropy \( H^\ast \) similarly. When \( H_\ast \) and \( H^\ast \) coincide, we say that the Renyi entropy exists and denote the common value as \( H \).

We say a system \((J, \mu, T)\) is \( \psi \)-mixing if for any \( U \in C^n_n \), \( V \in C^m_m \ (m, n \in \mathbb{N}) \),
\[
|\mu(U \cap T^{-(n+k)}V) - \mu(U)\mu(V)| \leq \psi(k)\mu(U)\mu(V), \tag{1}
\]
where \( \psi : \mathbb{N} \to \mathbb{R}^+ \) vanishes at infinity. The system \((J, \mu, T)\) is said to be \( \psi \)-mixing with an exponential decay if moreover the function \( \psi \) in \( (1) \) satisfies
\[
\limsup_{n \to \infty} \frac{\psi(n + 1)}{\psi(n)} < 1.
\]
We mention several examples which are \( \psi \)-mixing with an exponential decay: continued fractions system, Luroth system, N-ary system, triadic Cantor system (see Section 5).

With the notation above, we state our main results.

**Theorem 1.1.** Let \((J, \mu, T)\) be a \( \psi \)-mixing system with an exponential decay.
(1) For \( \mu \otimes \mu \)-almost all \((x, y) \in J \times J\), we have
\[
\limsup_{n \to \infty} \frac{M_n(x, y)}{\log n} = \frac{1}{H_\ast}.
\]
(2) For \( \mu \otimes \mu \)-almost all \((x, y) \in J \times J\), we have
\[
\liminf_{n \to \infty} \frac{M_n(x, y)}{\log n} = \frac{1}{H^\ast}.
\]
If the Rényi entropy exists, then for \( \mu \otimes \mu \)-almost all \((x, y) \in J \times J\),

\[
\lim_{n \to \infty} \frac{M_n(x, y)}{\log n} = \frac{1}{H}.
\]

Here we use the convention that \( \frac{1}{0} = \infty \) and \( \frac{1}{\infty} = 0 \).

**Remark 2.** Fix a point \( y_0 = [\omega_1(y_0), \omega_2(y_0), \ldots] \in J \setminus D \). For \( x = [\omega_1(x), \omega_2(x), \ldots] \), we define a new quantity \( M_n(x, y_0) \) similar to \( M_n(x, y) \) as follows:

\[
M_n(x, y_0) = \max\{k \in \mathbb{N} : \omega_{i+1}(x) = \omega_1(y_0), \ldots, \omega_{i+k}(x) = \omega_k(y_0) \}
\]

for some \( 0 \leq i \leq n - k \).

We call \( M_n(x, y_0) \) the maximal hitting depth of \( x \) to \( y_0 \), which reflects the degree how the trajectories of \( x \) can approach \( y_0 \). We define the lower local entropy of \( y_0 \) to be \( D_\mu(y_0) = \liminf_{n \to \infty} \frac{1}{n} \log \mu(J_n(y_0)) \) and define the upper local entropy \( D_\mu(y_0) \) similarly. By using the same method as in proving Theorem 1.1, we have that for \( \mu \)-almost all \( x \in J \),

\[
\limsup_{n \to \infty} \frac{M_n(x, y_0)}{\log n} = \frac{1}{D_\mu(y_0)}, \quad \liminf_{n \to \infty} \frac{M_n(x, y_0)}{\log n} = \frac{1}{D_\mu(y_0)}.
\]

By Theorem 1.1 and Remark 1(2), we have that for \( \mu \otimes \mu \)-almost all \((x, y) \in J \times J\),

\[
\limsup_{n \to \infty} \frac{-\log \min_{1 \leq i \leq n} d(T^i(x), T^i(y))}{\log n} \geq \frac{1}{H^*},
\]

and

\[
\liminf_{n \to \infty} \frac{-\log \min_{1 \leq i \leq n} d(T^i(x), T^i(y))}{\log n} \geq \frac{1}{H^*}.
\]

The existence of Rényi entropy has been proved only for a few special measures: Bernoulli measures, Markov measures and, more generally, Gibbs measures with Hölder continuous potentials \( \varphi \) (see [2, 13, 27]).

It is natural to study the Hausdorff dimensions of exceptional sets of points violating the metric properties in Theorem 1.1. Let \( \Phi \) be a conformal IFS on \( X = [0, 1] \) (see Section 2 for the formal definition). Due to the OSC in \([0, 1]\), there are at most countably many points \( x \in J \) having multiple codings. A countable set is negligible in the sense of the Hausdorff dimension, so as far as Hausdorff dimensions are concerned, we may assume that every point in \( J \) has a unique coding. Let \( \varphi : \mathbb{N} \to (0, \infty) \) be a function. For \( 0 \leq \alpha \leq \beta \leq \infty \), we define the set

\[
E_{\alpha, \beta}^\varphi = \left\{ (x, y) \in J \times J : \liminf_{n \to \infty} \frac{M_n(x, y)}{\varphi(n)} = \alpha, \limsup_{n \to \infty} \frac{M_n(x, y)}{\varphi(n)} = \beta \right\}.
\]

**Theorem 1.2.** Let \( \Phi = \{ \phi_i : [0, 1] \to [0, 1], i \in \Lambda \} \) be a conformal IFS that satisfies \( \phi_i'(x) > 0 \) or \( \phi_i'(x) < 0 \) for all \( i \in \Lambda \). Let \( \varphi : \mathbb{N} \to (0, \infty) \) be an increasing function satisfying \( \lim_{n \to \infty} \varphi(n) = \infty \) and \( \lim_{n \to \infty} \varphi(n + 1) - \varphi(n) = 0 \). Then for \( 0 \leq \alpha \leq \beta \leq \infty \) we have

\[
2 \dim_H J \leq \dim_H E_{\alpha, \beta}^\varphi \leq \dim_H J + \dim_P J.
\]

Throughout this paper, \( \dim_H \) and \( \dim_P \) denote the Hausdorff dimension and the packing dimension respectively.

**Corollary 1.** Under the hypotheses of Theorem 1.2, if moreover \( \Lambda \) is finite, then

\[
\dim_H E_{\alpha, \beta}^\varphi = 2 \dim_H J.
\]
In fact, if \( \Lambda \) is finite, then \( \dim_H J = \dim_P J \) (see [18]). We remark that if \( \Lambda \) is infinite, \( \dim_H J = \dim_P J \) does not hold in general.

Our paper is organised as follows. In Section 2 we introduce the conformal IFSs and cite some lemmas that will be used later. We then prove Theorems 1.1 and 1.2 in Sections 3 and 4 respectively. In Section 5 we study the asymptotic behaviour of \( M_n(x, y) \) and compute the Hausdorff dimensions of the exceptional sets in the continued fractions system, Lüroth system, \( N \)-ary system, and triadic Cantor system by applying Theorems 1.1 and 1.2.

2. Preliminaries. This section is devoted to recalling the definition of conformal IFSs and introducing some properties for conformal IFSs. For more information on estimating the Hausdorff dimension of a fractal set (see [7]).

Definition 2.1. We call IFS \( \Phi = \{ \phi_i : [0, 1] \to [0, 1], i \in \Lambda \} \) a conformal IFS if the following conditions are satisfied.

1. (OSC) For \( i \in \Lambda, \phi_i((0, 1)) \subset (0, 1) \) and for \( i, j \in \Lambda, i \neq j, \phi_i((0, 1)) \cap \phi_j((0, 1)) = \emptyset \).
2. There exists an open connected set \( V \subset \mathbb{R} \) such that \( [0, 1] \subset V \) and all maps \( \phi_i \) extend to \( C^1 \)-diffeomorphisms of \( V \) into \( V \).
3. (BDP) There exists \( K \geq 1 \) such that for any \( x, y \in V \) and \( \omega \in \Lambda^* \)
   \[ |\phi_i^\omega(x)| \leq K |\phi_i^\omega(y)|. \]

The topological pressure function \( P(t) \) for a conformal IFS \( \Phi \) is defined as
\[
P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sup_{\omega \in \Lambda^n} \sup_{x \in [0, 1]} |\phi_i^\omega(x)|^t.
\]

Mauldin & Urbański [18] proved that \( P(t) \) is non-increasing on \([0, \infty)\). They also established a continuity property of a conformal system with respect to its finite subsystems and showed the relationship between the topological pressure function \( P(t) \) and the Hausdorff dimension of attractor \( J \). Let \( F \) be a finite subset of \( \Lambda \), and let
\[
J_F = \{ x \in J : \text{there exists a coding } \omega \in F^\Lambda \text{ such that } x = \pi(\omega) \}.
\]

Lemma 2.2 ([18]).
\[
\dim_H J = \sup \{ \dim_H J_F : F \text{ is a finite subset of } \Lambda \} = \inf \{ t : P(t) \leq 0 \}.
\]

BDP property (3) in Definition 2.1 provides an estimation on the diameter of a cylinder.

Proposition 1 ([25]). For \( (\omega_1, \ldots, \omega_n) \in \Lambda^n \) with \( n \geq 1 \), \( \xi \in [0, 1] \), we have:

1. The diameter of \( J_n(\omega_1, \ldots, \omega_n) \) satisfies
   \[ K^{-1} |\phi_i^\omega(\omega_1, \ldots, \omega_n)(\xi)| \leq |J_n(\omega_1, \ldots, \omega_n)| \leq K |\phi_i^\omega(\omega_1, \ldots, \omega_n)(\xi)|. \]
2. For every \( 1 \leq k \leq n \),
   \[ K^{-1} \leq \frac{|J_n(\omega_1, \ldots, \omega_n)|}{|J_k(\omega_1, \ldots, \omega_k)| \cdot |J_{n-k}(\omega_{k+1}, \ldots, \omega_n)|} \leq K. \]
3. \[ |J_n(\omega_1, \ldots, \omega_n)| \leq \rho^n. \]

The following result can be obtained directly from the \( \psi \)-mixing property (1).
Lemma 2.3. Product formulae. With the convention of lemmas. Throughout Proof of Theorem 1.1. 3.

Lemma 2.4. For Proposition 2.

\[ \mu \otimes \mu((U \times U') \cap (T \times T)^{-k}(V \times V')) \leq (1 + \psi(k))^2 \mu \otimes \mu(U \times U') \cdot \mu \otimes \mu(V \times V'). \]

Proof. It is readily checked that

\[
\begin{align*}
\mu \otimes \mu((U \times U') \cap (T \times T)^{-k}(V \times V')) \\
= \mu \otimes \mu((U \cap T^{-k}V) \times (U' \cap T^{-k}V')) \\
= \mu(U \cap T^{-k}V) \cdot \mu(U' \cap T^{-k}V') \\
\leq (1 + \psi(k))\mu(U) \mu(V) \cdot (1 + \psi(k))\mu(U') \mu(V) \\
\leq (1 + \psi(k))^2 \mu \otimes \mu(U \times U') \cdot \mu \otimes \mu(V \times V').
\end{align*}
\]

We cite two properties of Hausdorff dimension, namely Hölder property and Product formulae.

Lemma 2.3 ([7]). Let \( E \subseteq \mathbb{R} \). If \( f : E \to \mathbb{R} \) is \( \eta \)-Hölder, i.e., there exists a constant \( C > 0 \) such that for all \( x, y \in E \),

\[ |f(x) - f(y)| \leq C|x - y|^{\eta}, \]

then

\[ \dim_H f(E) \leq \frac{1}{\eta} \dim_H E. \]

Lemma 2.4 ([7]). If \( E \subseteq \mathbb{R}^d, F \subseteq \mathbb{R}^n \), then

\[ \dim_H E + \dim_H F \leq \dim_H (E \times F) \leq \dim_H E + \dim_H F. \]

We end this section by listing some notation of words.

For \( n, m \geq 1 \), let \( \omega = (\omega_1, \ldots, \omega_n) \in B \subseteq \Lambda^n, \tau = (\tau_1, \ldots, \tau_m) \in C \subseteq \Lambda^m \),

- \( \omega \tau = (\omega_1, \ldots, \omega_n, \tau_1, \ldots, \tau_m) \);
- \( BC = \{ \omega \tau : \omega \in B, \tau \in C \} \);
- \( B^n = B \ldots B \) \( m \) times

3. Proof of Theorem 1.1. We divide the proof of Theorem 1.1 into a sequence of lemmas. Throughout \( \lfloor x \rceil \) denotes the integer part of the real number \( x \), and \( X \ll Y \) (or \( Y \gg X \)) means that \( X \leq CY \) for some absolute constant \( C > 0 \).

Lemma 3.1. Let \( (J, \mu, T) \) be a measure-preserving system. Then for almost all \( (x, y) \in J \times J \), we have

\[ \limsup_{n \to \infty} \frac{M_n(x, y)}{\log n} \leq \frac{1}{H_*}. \]

Proof. With the convention \( \frac{1}{\eta} = \infty \), we only need to consider the case \( H_* > 0 \).

Fix \( s_1 < s_2 < H_* \). By the definition of the \( H_* \), we have that

\[ \sum_{(\omega_1, \ldots, \omega_n) \in \Lambda^n} \mu(J_n(\omega_1, \ldots, \omega_n))^2 < \exp \left\{ -\frac{s_1 + s_2}{2} n \right\} \]  

for \( n \) large enough. Let \( u_n = \lfloor \frac{\log{n}}{s_1} \rfloor \). For \( k \geq 1 \), we have that

\[ \mu \otimes \mu(\{(x, y) \in J \times J : M_n(x, y) > u_n\}) \]
\[
\leq \sum_{k=1}^{\infty} \mu \otimes \mu \left( \{(x, y) \in J \times J : M_n(x, y) = k\} \right)
\]
\[
\leq \sum_{k=1}^{\infty} \sum_{j=0}^{n-k} \mu \otimes \mu \left( \{(x, y) \in J \times J : \omega_{j+1}(x) = \omega_{j+1}(y), \ldots, \omega_{j+k}(x) = \omega_{j+k}(y)\} \right).
\]

Since \(T\) is measure-preserving, we deduce that
\[
\mu \otimes \mu \left( \{(x, y) \in J \times J : M_n(x, y) > u_n\} \right)
\]
\[
\leq n \sum_{k=1}^{\infty} \mu \otimes \mu \left( \{(x, y) \in J \times J : \omega_1(x) = \omega_1(y), \ldots, \omega_k(x) = \omega_k(y)\} \right)
\]
\[
= n \sum_{k=1}^{\infty} \sum_{(\omega_1, \ldots, \omega_k) \in \Lambda^k} \mu(J_k(\omega_1, \ldots, \omega_k))^2
\]
\[
\ll n \cdot \exp \left\{ -\frac{s_1 + s_2}{2} u_n \right\} \text{ (by (5))}
\]
\[
\ll n^{\frac{s_2 + s_3}{s_1}}.
\]

Taking \(n_k = 2^k\) for \(k \geq 1\), we readily check that
\[
\sum_{k=1}^{\infty} \mu \otimes \mu \left( \{(x, y) \in J \times J : M_{n_k}(x, y) > u_{n_k}\} \right) < \infty. \quad (6)
\]

Combining (6) with the Borel-Cantelli lemma, we have that for almost all \((x, y) \in J \times J,\)
\[
\limsup_{k \to \infty} \frac{M_{n_k}(x, y)}{\log n_k} \leq \frac{1}{s_1}.
\]

The monotonicity of \(M_n(x, y)\) yields that
\[
\limsup_{n \to \infty} \frac{M_n(x, y)}{\log n} \leq \limsup_{k \to \infty} \frac{M_{2^k+1}(x, y)}{\log 2^{k+1}} = \limsup_{k \to \infty} \frac{M_{2^k+1}(x, y)}{\log 2^k} \leq \frac{1}{s_1}.
\]

Therefore, by the arbitrariness of \(s_1\), we have that
\[
\limsup_{n \to \infty} \frac{M_n(x, y)}{\log n} \leq \frac{1}{H_*}.
\]

\[\square\]

**Lemma 3.2.** Let \((J, \mu, T)\) be a \(\psi\)-mixing system with an exponential decay. Then for almost all \((x, y) \in J \times J,\) we have
\[
\limsup_{n \to \infty} \frac{M_n(x, y)}{\log n} \geq \frac{1}{H_*}.
\]

**Proof.** We only need to consider the case \(H^* < \infty\). By the definition of \(H_*\), we can choose an integer sequence \(\{u_k\}_{k \geq 1}\) such that
\[
\lim_{k \to \infty} -\log \sum_{(\omega_1, \ldots, \omega_{u_k}) \in \Lambda_{u_k}} \mu(J_{u_k}(\omega_1, \ldots, \omega_{u_k}))^2 \frac{1}{u_k} = H_*.
\]

For any \(s > H_*\), we have
\[
\sum_{(\omega_1, \ldots, \omega_{u_k}) \in \Lambda^{u_k}} \mu(J_{u_k}(\omega_1, \ldots, \omega_{u_k}))^2 > \exp \left\{ -\frac{s + H_*}{2} u_k \right\} \quad (7)
\]
for $k$ large enough.

Taking $n_k = \lfloor e^{u_k} \rfloor$ and $l_k = \lfloor \frac{n_k}{u_k} \rfloor$ for $k \geq 1$, we claim that

$$\mu \otimes \mu \left( \{ (x, y) \in J \times J : M_{n_k}(x, y) < u_k \text{ for infinitely many } k \in \mathbb{N} \} \right) = 0.$$ 

If the claim is proved, for almost all $(x, y) \in J \times J$, we have that

$$\limsup_{n \to \infty} \frac{M_n(x, y)}{\log n} \geq \frac{1}{s},$$

which completes the proof of the lemma by the arbitrariness of $s$.

We proceed to show the claim. For $n > m \geq 1$, set

$$M_{[m,n]}(x, y) = \max \{ k : \omega_{i+1}(x) = \omega_{i+1}(y), \ldots, \omega_{i+k}(x) = \omega_{i+k}(y) \}$$

for some $m - 1 \leq i \leq n - k$.

Writing $(x, y) \in J \times J : M_n(x, y) < k$ by $\{ M_n(x, y) < k \}$ for brevity, we have that

$$\{ M_{n_k}(x, y) < u_k \} \subset \{ M_{[u_k^2+1, u_k^2+u_k]}(x, y) < u_k, 0 \leq i < l_k \} \cap \{ M_{u_k}(x, y) < u_k \} \cap (T \times T)^{-u_k^2} \{ M_{[u_k^2+1, u_k^2+u_k]}(x, y) < u_k, 0 \leq i < l_k - 1 \}.$$

By Proposition 2, we deduce that

$$\mu \otimes \mu \left( \{ M_{n_k}(x, y) < u_k \} \right)$$

$$\leq \mu \otimes \mu \left( \{ M_{u_k}(x, y) < u_k \} \right) \cap (T \times T)^{-u_k^2} \{ M_{[u_k^2+1, u_k^2+u_k]}(x, y) < u_k, 0 \leq i < l_k - 1 \}$$

$$\leq \mu \otimes \mu \left( \{ M_{u_k}(x, y) < u_k \} \right) \cdot (1 + \psi(u_k^2 - u_k))^{2l_k}.$$

The exponential decay implies that $\psi(u_k^2 - u_k)^{2l_k} \to 0$ as $k \to 0$. So,

$$\mu \otimes \mu \left( \{ M_{n_k}(x, y) < u_k \} \right)$$

$$\leq \left( 1 - \sum_{(\omega_1, \ldots, \omega_{u_k}) \in \Lambda^{u_k}} \mu(J_{u_k}(\omega_1, \ldots, \omega_{u_k}))^2 \right) l_k (1 + \psi(u_k^2 - u_k))^{2l_k}$$

$$\ll \exp \left\{ -\frac{n_k}{u_k} \cdot \frac{e^{u_k} - u_k}{u_k} \right\} \cdot \exp \left\{ \psi(u_k^2 - u_k) \frac{2n_k}{u_k} \right\} \text{ (by (7))}$$

$$\ll \exp \left\{ -\frac{e^{u_k} - u_k}{u_k} \right\}.$$

As a result, we have

$$\sum_{k=1}^{\infty} \mu \otimes \mu \left( \{ (x, y) \in J \times J : M_{n_k}(x, y) < u_k \} \right) \ll \infty,$$

as desired. $\Box$

We study the behaviour of $\liminf_{n \to \infty} \frac{M_n(x, y)}{\log n}$ in the following two lemmas.

**Lemma 3.3.** Let $(J, \mu, T)$ be a $\psi$-mixing system with an exponential decay. Then for almost all $(x, y) \in J \times J$, we have

$$\liminf_{n \to \infty} \frac{M_n(x, y)}{\log n} \geq \frac{1}{H^*}.$$
**Proof.** We may assume $H^* < \infty$. For $s > H^*$, we have that
\[
\sum_{(\omega_1, \ldots, \omega_n) \in \Lambda^n} \mu(J_n(\omega_1, \ldots, \omega_n))^2 > \exp\left\{-\frac{s + H^*}{2} n\right\}
\]
for $n$ large enough. Let $u_n = \lfloor \log n s \rfloor$ and $l_n = \lfloor u_n^{2n} \rfloor$. As in the proof of Lemma 3.2, we deduce that
\[
\mu \otimes \mu(J_n(x, y) < u_n) \leq \mu \otimes \mu(J_{u_n}(x, y) < u_n)^l_n (1 + \psi(u_n^2 - u_n))^{2l_n}
\]
and
\[
\mu \otimes \mu(J_n(x, y) < u_n) \ll \exp\left\{-\frac{n - H^*}{u_n^2}\right\}.
\]
Thus, we have
\[
\sum_{n=1}^{\infty} \mu \otimes \mu(J_n(x, y) < u_n) < \infty.
\]
As a result, we have that for almost all $(x, y) \in J \times J$,
\[
\liminf_{n \to \infty} \frac{M_n(x, y)}{\log n} \geq \frac{1}{s}.
\]

**Lemma 3.4.** Let $(J, \mu, T)$ be a measure-preserving system. Then for almost all $(x, y) \in J \times J$, we have
\[
\liminf_{n \to \infty} \frac{M_n(x, y)}{\log n} \leq \frac{1}{H^*}.
\]

**Proof.** Likewise, we assume that $H^* > 0$. Let $\{u_k\}_{k \geq 1}$ be a subsequence of $\mathbb{N}$ such that
\[
\lim_{k \to \infty} -\log \sum_{(\omega_1, \ldots, \omega_{u_k}) \in \Lambda_{u_k}} \mu(J_{u_k}(\omega_1, \ldots, \omega_{u_k}))^2 = H^*.
\]
For $s_1 < s_2 < H^*$, we have that
\[
\sum_{(\omega_1, \ldots, \omega_{u_k}) \in \Lambda_{u_k}} \mu(J_{u_k}(\omega_1, \ldots, \omega_{u_k}))^2 < \exp\left\{-\frac{s_1 + s_2}{2} u_k\right\}
\]
for $k$ large enough. Letting $n_k = \lfloor e^{s_1 u_k} \rfloor$, we have that
\[
\mu \otimes \mu(J_{n_k}(x, y) \geq u_k) \leq \mu \otimes \mu(J_{n_k}(x, y) = i) \leq \sum_{i=1}^{n_k} \mu \otimes \mu(J_{n_k}(x, y) = i)
\]
\[
\leq \sum_{i=1}^{n_k} \mu \otimes \mu\left(\bigcup_{j=0}^{n_k-1} (x, y) : \omega_{j+1}(x) = \omega_{j+1}(y), \ldots, \omega_{j+i}(x) = \omega_{j+i}(y), \omega_{j+i+1}(x) \neq \omega_{j+i+1}(y) \right) \cup \{(x, y) : \omega_{n_k-i}(x) = \omega_{n_k-i+1}(y) \}
\]
\[
= \omega_{n_k-i+1}(y), \ldots, \omega_{n_k}(x) = \omega_{n_k}(y) \} \cap (J \times J).
\]
inequality (3) repeatedly, we deduce that

\[ \mu \otimes \mu \{(x, y): \omega_1(x) = \omega_1(y), \ldots, \omega_i(x) = \omega_i(y), \omega_{i+1}(x) \neq \omega_{i+1}(y) \} \]

\[ + \mu \otimes \mu \{(x, y): \omega_1(x) \neq \omega_1(y), \omega_2(x) = \omega_2(y), \ldots, \omega_{i+1}(x) = \omega_{i+1}(y) \} \]

\leq 2n_k \cdot \# \{(x, y) \in J \times J: \omega_1(x) = \omega_1(y), \ldots, \omega_{u_k}(x) = \omega_{u_k}(y) \}

\leq 2n_k - \frac{2n_k - 1}{s_1}.

Then we can pick a subsequence \( \{u_{k_i}\}_{i \geq 1} \) from \( \{u_k\}_{k \geq 1} \) such that

\[ \sum_{i=1}^{\infty} \mu \otimes \mu \{(x, y) \in J \times J: M_{u_{k_i}}(x, y) \geq u_{k_i} \} < \infty, \]

and thus for almost all \((x, y) \in J \times J\),

\[ \liminf_{n \to \infty} \frac{M_n(x, y)}{\log n} < \frac{1}{s_1}. \]

\[ \square \]

4. Proof of Theorem 1.2. We begin the proof of Theorem 1.2 with two key lemmas.

Let \( J \) be the attractor of the 1-dimensional conformal IFS

\[ \{\phi_i: [0, 1] \to [0, 1], \ i \in \Lambda\}. \]

Let \( 1 < M \leq \# \Lambda \) be a positive integer and

\[ J_M = \{x \in J: 1 \leq \omega_n(x) \leq M, \ n \geq 1\}, \]

where \( \# \) means the cardinality of a set. Then \( J_M \) is the attractor of the finite IFS

\[ \{\phi_i: [0, 1] \to [0, 1], 1 \leq i \leq M\}. \]

For \( 1 \leq c \leq M, p > 1 \), set

\[ J_{M,c} = \{x \in J_M: \omega_{kp+1}(x) = c, \ k \geq 0\}. \]

The Hausdorff dimension of \( J_M \) can be approximated by that of \( J_{M,c} \).

**Lemma 4.1.** Let \( \Phi = \{\phi_i: [0, 1] \to [0, 1], \ i \in \Lambda\} \) be a conformal IFS. For \( \epsilon > 0 \),

\[ 1 \leq c \leq M, \text{ there exists a positive integer } p_0 \text{ such that} \]

\[ \dim_H J_{M,c} \geq \dim_H J_M - \epsilon \]

for all \( p \geq p_0 \).

**Proof.** We write

\[ I_{M,c} = \{(\omega_1, \ldots, \omega_p) \in \Lambda^p: \omega_1 = c, 1 \leq \omega_j \leq M \text{ for } 1 < j \leq p\}. \]

By Lemma 2.2, we only need to show that

\[ P(\dim_H J_M - \epsilon) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{(\omega_1, \ldots, \omega_p) \in I_{M,c}} \sup_{x \in [0, 1]} |\phi'_{(\omega_1, \ldots, \omega_p)}(x)|^{\dim_H J_M - \epsilon} \geq 0. \]

We take a block \((\omega_1, \ldots, \omega_p)\) in \( I_{M,c} \), that is, \((\omega_{kp+1}, \ldots, \omega_{(k+1)p}) \in I_{M,c}\) for any \( 0 \leq k < n \). From \((\omega_1, \ldots, \omega_p)\) we delete all the \((kp+1)\)-st terms for \( 0 \leq k < n \), and denote by \((\tilde{\omega}_1, \ldots, \tilde{\omega}_p)\) the caused block (which is of length \( n(p-1) \)). Applying inequality (3) repeatedly; we deduce that

\[ |J_{np}(\omega_1, \ldots, \omega_p)| \geq \left( \frac{1}{K} \right)^{3n-2} |J_1(c)|^n |J_{np-n}(\tilde{\omega}_1, \ldots, \tilde{\omega}_p)|. \]
By (2) and (4) we have that
\[ P(\dim_H J_M - \epsilon) \]

\[ = \lim_{n \to \infty} \frac{1}{n} \log \sum_{(\omega_1, \ldots, \omega_n) \in \mathcal{I}_{M,p,c}^n} \left| \sup_{x \in [0,1]} |\phi'_{(\omega_1, \ldots, \omega_n)}(x)| \right|^{\dim_H J_M - \frac{\epsilon}{2}} \]

\[ \geq \lim_{n \to \infty} \frac{1}{n} \log \sum_{(\omega_1, \ldots, \omega_n) \in \mathcal{I}_{M,p,c}^n} \left( \frac{1}{\rho} \right) \left( \frac{K^{2-3n} \log(\alpha_0)}{\rho} \right)^{\dim_H J_M - \frac{\epsilon}{2}} \]

This estimation yields that
\[ P(\dim_H J_M - \epsilon) \geq \]

\[ \lim_{n \to \infty} \frac{1}{n} \log \sum_{(\omega_1, \ldots, \omega_n) \in \mathcal{I}_{M,p,c}^n} \left( \frac{1}{\rho} \right) \left( K^{2-3n} \rho^{-n} \log(\alpha_0) \right)^{\dim_H J_M - \frac{\epsilon}{2}} \]

\[ \geq \frac{1}{2} p \log \frac{1}{\rho} + \left( \dim_H J_M - \frac{\epsilon}{2} \right) \log(\alpha_0) \]

\[ + \lim_{n \to \infty} \frac{1}{n} \log \sum_{1 \leq \nu_1, \ldots, \nu_{n-1} \leq M} \left| J_{n-1}(\nu_1, \ldots, \nu_{n-1}) \right|^{\dim_H J_M - \frac{\epsilon}{2}} \quad \text{(by (8))} \]

\[ \geq \frac{1}{2} p \log \frac{1}{\rho} - \log \left( K^{3 \log(\alpha_0)} \right) \]

where the last inequality follows from that \( \log(\alpha_0) < 0 \) and
\[ \lim_{n \to \infty} \frac{1}{n} \log \sum_{1 \leq \nu_1, \ldots, \nu_{n-1} \leq M} \left| J_{n-1}(\nu_1, \ldots, \nu_{n-1}) \right|^{\dim_H J_M - \frac{\epsilon}{2}} \geq 0. \]

Writing \( \alpha_0 = \sup \{|\phi'_i(x)|, x \in [0,1]| \} \) and \( p_0 = \frac{2\left( \log K^{4-\log(\alpha_0)} \right)}{\epsilon \log \frac{1}{\rho}} \), we obtain that when \( p \geq p_0 \),
\[ P(\dim_H J_M - \epsilon) \geq 0. \]

\[ \square \]

Let \( \{k_n\}_{n \geq 1} \) be an infinite subsequence of \( \mathbb{N} \), and \( \{B_n\}_{n \geq 1} \) be a sequence of finite blocks over \( \{1,2,\ldots,M\} \), say \( B_n = (a^n_1, \ldots, a^n_{b_n}) \). For \( x \in J_{M,p,c} \), we write its coding as
\[ (\omega_1, \ldots, \omega_{k_1}, \omega_{k_1+1}, \ldots, \omega_{k_2}, \omega_{k_2+1}, \ldots). \]

We insert the blocks \( \{B_n\}_{n \geq 1} \) after the positions \( \{k_n\}_{n \geq 1} \) in the coding of \( x \), and obtain a new sequence
\[ (\omega_1, \ldots, \omega_{k_1}, a^1_1, \ldots, a^1_{b_1}, \omega_{k_1+1}, \ldots, \omega_{k_2}, a^2_1, \ldots, a^2_{b_2}, \omega_{k_2+1}, \ldots). \]

There exists a unique point \( y := \tau(x) \in [0,1] \), which possesses the above sequence as its coding. In this way, we define a mapping \( \tau : J_{M,p,c} \to [0,1] \), and we denote the range by \( J_{M,p,c}(\{k_n\}, \{B_n\}) \), i.e., \( J_{M,p,c}(\{k_n\}, \{B_n\}) = \tau(J_{M,p,c}). \)

**Lemma 4.2.** Let \( \Phi = \{\phi_i : [0,1] \to [0,1], i \in \Lambda\} \) be a conformal IFS such that \( \phi'_i(x) > 0 \) or \( \phi'_i(x) < 0 \) for all \( i \in \Lambda \). If
\[ \lim_{n \to \infty} \frac{b_1 + b_2 + \cdots + b_n}{k_n} = 0, \]
\[ \tag{9} \]
where \( b_n = |B_n| \) is the length of \( B_n \), then

\[
\dim_H J_{M,p,c} = \dim_H J_{M,p,c}(\{k_n\}, \{B_n\}).
\]

**Proof.** Obviously, the mapping \( \tau \) establishes a one-to-one correspondence between \( J_{M,p,c} \) and \( J_{M,p,c}(\{k_n\}, \{B_n\}) \). We write \( \varpi \) for the inverse of \( \tau \). Write

\[
\gamma = \min_{1 \leq i \leq M} \sup \{ |\varphi_i'(x)|, x \in [0, 1] \}.
\]

By OSC, the intervals \( \varphi_1((0, 1)), \varphi_2((0, 1)), \ldots, \varphi_M((0, 1)) \) are pairwise disjoint. Reindexing them if necessary, we may assume these intervals are arranged from left to right.

**Step 1.** We show \( \dim_H J_{M,p,c}(\{k_n\}, \{B_n\}) \geq \dim_H J_{M,p,c} \) by establishing the \((1 - \epsilon)\)-Hölder property of the mapping \( \varpi \) for any \( \epsilon \in (0, 1) \), since by Lemma 2.3, the Hölder property implies that

\[
\dim_H J_{M,p,c}(\{k_n\}, \{B_n\}) \geq (1 - \epsilon) \dim_H J_{M,p,c}.
\]

By (9), we can choose \( l_0 \) large enough such that for any \( l > l_0 \),

\[
\left( \frac{1}{\rho} \right)^{kl} \geq \left( \frac{K^4}{\gamma} \right)^{b_1 + \ldots + b_l}.
\]  

(10)

Take \( x_1, x_2 \in J_{M,p,c}(\{k_n\}, \{B_n\}) \) with \( x_1 < x_2 \). There is a greatest integer \( n \) such that \( x_1 \) and \( x_2 \) belong to a common cylinder of order \( n \). And an integer \( l \) exists such that

\[
k_l + b_1 + \ldots + b_l \leq n < k_{l+1} + b_1 + \ldots + b_{l+1}.
\]

There is no loss of generality in assuming that \( l \geq l_0 \). We consider two cases.

Case 1: \( \varphi_i'(x) > 0 \) for all \( i \in \Lambda \).

From the construction of \( J_{M,p,c}(\{k_n\}, \{B_n\}) \), we know that \( x_1 \) and \( x_2 \) are separated by the cylinder

\[
J_{n+p}(\omega_1(x_1), \ldots, \omega_{n+p-1}(x_1), M) \text{ or } J_{n+p}(\omega_1(x_2), \ldots, \omega_{n+p-1}(x_2), 1),
\]

and thus

\[
|x_1 - x_2| \geq \min \{|J_{n+p}(\omega_1(x_1), \ldots, \omega_{n+p-1}(x_1), M)|, |J_{n+p}(\omega_1(x_2), \ldots, \omega_{n+p-1}(x_2), 1)|\} \geq K^{-2p\gamma^p}|J_n(\omega_1(x_1), \omega_2(x_1), \ldots, \omega_n(x_1))|,
\]

where the second inequality follows from (2) and (3).

On the other hand, we know that

\[
\varpi(x_1), \varpi(x_2) \in J_{n-(b_1+\ldots+b_l)}(c_1, c_2, \ldots, c_n-(b_1+\ldots+b_l)),
\]

where the block \((c_1, c_2, \ldots, c_n-(b_1+\ldots+b_l))\) is obtained by deleting the blocks \( \{B_n\}_{n=1}^l \) from \((\omega_1(x_1), \ldots, \omega_n(x_1))\). By Proposition 1 and (10), we deduce that

\[
|\varpi(x_1) - \varpi(x_2)| \leq |J_{n-(b_1+\ldots+b_l)}(c_1, c_2, \ldots, c_n-(b_1+\ldots+b_l))| \leq \left( \frac{K^4}{\gamma} \right)^{b_1 + \ldots + b_l} |J_n(\omega_1(x_1), \omega_2(x_1), \ldots, \omega_n(x_1))| \leq \left( \frac{1}{\rho} \right)^{n\rho} |J_n(\omega_1(x_1), \omega_2(x_1), \ldots, \omega_n(x_1))| \ll |x_1 - x_2|^{1-\epsilon}.
\]

Case 2: \( \varphi_i'(x) < 0 \) for all \( i \in \Lambda \).
In this case, we note that the gap of \(x_1\) and \(x_2\) is larger than the length of the cylinder
\[
J_{p+1}(\omega_1(x_1), \ldots, \omega_{p+1}(x_1), 1) \text{ or } J_{p+1}(\omega_1(x_1), \ldots, \omega_{p+1}(x_1), M), \quad i = 1, 2.
\]
And the similar arguments as in Case 1 apply.

**Step 2.** We establish the \((1 - \epsilon)\)-Hölder property of \(\tau\) to obtain
\[
\dim_{H} J_{M, p,c}(\{k_n\}, \{B_n\}) \leq \dim_{H} J_{M, p,c}.
\]
For \(y_1, y_2 \in J_{M, p,c}\), let \(x_1, x_2\) be the corresponding points in \(J_{M, p,c}(\{k_n\}, \{B_n\})\) respectively. Let \(n\) be the smallest integer such that \(\omega_{n+1}(y_1) \neq \omega_{n+1}(y_2)\), and let \(L\) be an integer such that \(k_L \leq n < k_{L+1}\). By (9), we may assume that
\[
(K^3 \rho)^{b_1 + b_2 + \ldots + b_L} \leq \left(\frac{1}{\rho}\right)^{k_L \epsilon}.
\]
By the choice of \(n\), we have
\[
|y_1 - y_2| \geq K^{-2p} \gamma^p |J_n(\omega_1(y_1), \omega_2(y_1), \ldots, \omega_n(y_1))|.
\]
By the definition of \(\tau\), \(x_1\) and \(x_2\) have a common prefix up to the position \(n + b_1 + b_2 + \ldots + b_L\). By Proposition 1, we deduce that
\[
|x_1 - x_2| \leq |J_{n+b_1+b_2+\ldots+b_L}(\omega_1(x_1), \omega_2(x_1), \ldots, \omega_{n+L}(x_1))| \\
\quad \leq (K^3 \rho)^{b_1 + b_2 + \ldots + b_L} |J_n(\omega_1(y_1), \omega_2(y_1), \ldots, \omega_n(y_1))| \\
\quad \leq \left(\frac{1}{\rho}\right)^{n\epsilon} |J_n(\omega_1(y_1), \omega_2(y_1), \ldots, \omega_n(y_1))| \\
\quad \ll |y_1 - y_2|^{-\epsilon}.
\]

We are now in a position to prove Theorem 1.2. The upper bound estimation is obvious by Lemma 2.4, and the lower bound estimation relies on Lemmas 4.1 & 4.2 and constructing cantor-like subsets with Hausdorff dimensions approaching to \(2 \dim_{H} J\), and the proof will be divided into several cases according to the values of \(\alpha\) and \(\beta\). We provide a detailed proof for the case \(0 < \alpha < \beta < +\infty\), and only sketches of proofs for the remaining cases.

**Case 1:** \(0 < \alpha < \beta < +\infty\).

Since \(\lim_{n \to \infty} \varphi(n) = \infty\) and \(\lim_{n \to \infty} \varphi(n+1) - \varphi(n) = 0\), we may assume that for all \(n \geq 1\),
\[
\beta \varphi(n) \geq 1
\]
and
\[
\varphi(n+1) - \varphi(n) < 1.
\]
We choose two integer sequences \(\{n_k\}_{k \geq 0}\) and \(\{m_k\}_{k \geq 0}\) satisfying that
\[
n_0 \geq 1, \left(\frac{\beta}{\alpha}\right)^k - 1 \leq \varphi(n_k) < \left(\frac{\beta}{\alpha}\right)^k, m_k = \lfloor \beta \varphi(n_k) \rfloor.
\]
It is clear that
\[
\lim_{k \to \infty} \frac{\varphi(n_{k+1})}{\varphi(n_k)} = \frac{\beta}{\alpha}.
\]
(11)

Note that
\[
\varphi(n_k + m_k) = \varphi(n_k) + \varphi(n_k + 1) - \varphi(n_k) + \ldots + \varphi(n_k + m_k) - \varphi(n_k + m_k - 1)
\]
\[ \leq \varphi(n_k) + m_k \max_{n_k \leq n < n_k + m_k} \{ \varphi(n + 1) - \varphi(n) \}. \]

Since \( \varphi \) is increasing, we obtain that
\[
\lim_{k \to \infty} \frac{\varphi(n_k + m_k)}{\varphi(n_k)} = 1. \tag{12}
\]

We readily check that
\[
\lim_{k \to \infty} \frac{m_k}{n_{k+1} - n_k} = 0, \tag{13}
\]

and thus we may assume without loss of generality that \( n_{k+1} - n_k > m_k \) for all \( k \geq 0 \).

Now we construct Cantor-type subsets of \( J_M \). To this end, for \( k \geq 0 \), we write
\[
n_k + 1 - n_k = m_k \cdot \iota_k + \theta_k,
\]

where
\[
\iota_k = \left\lfloor \frac{n_k + 1 - n_k}{m_k} \right\rfloor, 0 \leq \theta_k < m_k.
\]

We then define a position set \( P \). By just fixing digits on all positions in \( P \), we obtain the Cantor-type subsets. The position set is defined as
\[
P = \bigcup_{k \geq 0} \{ n_k, n_k + 1, n_k + 2, \ldots, n_k + m_k, n_k + 2m_k, n_k + 3m_k, \ldots, n_k + \iota_km_k \}.
\]

The digits on the positions in \( P \) are given by \( a_{n_k} = 1, a_{n_k+1} = \ldots = a_{n_k+m_k-1} = M, a_{n_k+m_k} = a_{n_k+2m_k} = \ldots = a_{n_k+\iota_km_k} = 1 \), and \( b_n = M \) for \( n \in P \). The Cantor-type subsets are defined as
\[
E_M(\{n_k\}, \{m_k\}) = \bigcap_{n \geq 1} \bigcup_{(\varepsilon_1, \ldots, \varepsilon_n) \in D_n} J_n(\varepsilon_1, \ldots, \varepsilon_n),
\]
\[
F_M(\{n_k\}, \{m_k\}) = \bigcap_{n \geq 1} \bigcup_{(\varepsilon_1, \ldots, \varepsilon_n) \in D'_n} J_n(\varepsilon_1, \ldots, \varepsilon_n),
\]

where
\[
D_n = \{ (\varepsilon_1, \ldots, \varepsilon_n) \in \{1,2,\ldots,M\}^n : \varepsilon_k = a_k \text{ if } k \in P \},
\]
\[
D'_n = \{ (\varepsilon_1, \ldots, \varepsilon_n) \in \{1,2,\ldots,M\}^n : \varepsilon_k = b_k \text{ if } k \in P \}.
\]

We make two claims.

**Claim 1:** \( E_M(\{n_k\}, \{m_k\}) \times F_M(\{n_k\}, \{m_k\}) \subset E_{\alpha,\beta}^\varphi \).

Fix \((x, y) \in E_M(\{n_k\}, \{m_k\}) \times F_M(\{n_k\}, \{m_k\}) \), and \( n_k \leq n < n_{k+1} \).

1. if \( n_k \leq n < n_k + m_k \), then \( m_{k-1} - 1 \leq M_n(x, y) \leq m_k - 1 \);
2. if \( n_k + m_k \leq n < n_{k+1} \), then \( M_n(x, y) = m_k - 1 \).

From (11), (12) we deduce that
\[
\liminf_{n \to \infty} \frac{M_n(x,y)}{\varphi(n)} \geq \min \left\{ \lim_{k \to \infty} \frac{m_{k-1} - 1}{\varphi(n_k + m_k)}, \lim_{k \to \infty} \frac{m_k - 1}{\varphi(n_k + 1)} \right\} = \alpha
\]

and
\[
\limsup_{n \to \infty} \frac{M_n(x,y)}{\varphi(n)} \leq \max \left\{ \lim_{k \to \infty} \frac{m_{k-1} - 1}{\varphi(n_k)}, \lim_{k \to \infty} \frac{m_k - 1}{\varphi(n_k + m_k)} \right\} = \beta.
\]
On the other hand, since $M_{n+1}(x,y) = M_{n+m}(x,y) = m - 1$, it follows from (11), (12) again that
\[
\lim_{k \to \infty} \frac{M_{n+1}(x,y)}{\varphi(n+1)} = \lim_{k \to \infty} \frac{m - 1}{\varphi(n+1)} = \alpha
\]
and
\[
\lim_{k \to \infty} \frac{M_{n+m}(x,y)}{\varphi(n+m)} = \lim_{k \to \infty} \frac{m - 1}{\varphi(n+m)} = \beta.
\]
Therefore,
\[
\liminf_{n \to \infty} \frac{M_n(x,y)}{\varphi(n)} = \alpha \quad \text{and} \quad \limsup_{n \to \infty} \frac{M_n(x,y)}{\varphi(n)} = \beta.
\]

**Claim 2:** The density of $P \subset \mathbb{N}$ is zero.

1. If $n_k \leq n < n_k + m_k$, then
   \[
   |\{i \leq n, i \in P\}| = \sum_{j=0}^{k-1} (m_j + \iota_j) + n - n_k + 1.
   \]

Writing $N_k = \cup_{k \geq 0} \{n_k, n_k + 1, \ldots, n_k + m_k\}$, we have
\[
\limsup_{n \in N_k, n \to \infty} \frac{|\{i \leq n, i \in P\}|}{n} \leq \limsup_{k \to \infty} \left( \frac{\sum_{j=0}^{k-1} (m_j + \iota_j)}{n_k} + \frac{m_k}{n_k} \right)
\]
\[
\leq \limsup_{k \to \infty} \frac{\sum_{j=0}^{k-1} (m_j + \iota_j)}{n_k} + \limsup_{k \to \infty} \frac{m_k}{n_k}
\]
\[
\leq \limsup_{k \to \infty} \frac{m_{k-1} + \iota_{k-1}}{n_k - n_k - 1} = 0,
\]
where the third inequality follows from the Stolz-Cesàro theorem, and the last one holds by (13).

2. If $n_k + l m_k \leq n < n_k + (l+1)m_k$ for some $0 < l < \iota_k$, then
   \[
   |\{i \leq n, i \in P\}| = \sum_{j=0}^{k-1} (m_j + \iota_j) + m_k + l.
   \]

3. If $n_k + \iota_k m_k \leq n < n_{k+1}$, then
   \[
   |\{i \leq n, i \in P\}| = \sum_{j=0}^{k} (m_j + \iota_j).
   \]

Claim 2 follows from these estimations.

For $j \geq 0$, $1 \leq t \leq \iota_j$, we put
\[
k_{\iota_0 + \ldots + \iota_{j-1} + t} = n_j + (t - 1)m_j - \sum_{i=0}^{j-1} (m_i + \iota_i) - 1,
\]
\[
B_{\iota_0 + \ldots + \iota_{j-1} + t} = \begin{cases} \begin{array}{l} (1 \ldots M) 1, \quad t = 1; \\ m_{j-1} \quad (1), \quad 1 < t \leq \iota_j. \end{array} \end{cases}
\]
and $B'_n = (M \ldots M)$ for any $n \geq 1$ (so $|B_n|$ and $|B'_n|$ have the same length).
For $p > 1$ and $c \in \{1, 2, \ldots, M\}$, we can check that
\[ J_{M,p,c}(\{k_n\}, \{B_n\}) \subset E_M(\{n_k\}, \{m_k\}), \quad J_{M,p,c}(\{k_n\}, \{B'_n\}) \subset F_M(\{n_k\}, \{m_k\}), \]
and thus by Claim 1 we get that
\[ J_{M,p,c}(\{k_n\}, \{B_n\}) \times J_{M,p,c}(\{k_n\}, \{B'_n\}) \subset E_{\alpha,\beta}^p. \]
By Lemmas 2.4, 4.2 and Claim 2,
\[ \dim_H E_{\alpha,\beta}^p \geq \dim_H (J_{M,p,c}(\{k_n\}, \{B_n\}) \times J_{M,p,c}(\{k_n\}, \{B'_n\})) \geq 2 \dim_H J_{M,p,c}. \]
Letting $p \to \infty$ we obtain by Lemma 4.1 that $\dim_H E_{\alpha,\beta}^p \geq 2 \dim_H J_M$. On account of Lemma 2.2 we complete the proof for this case.

Similar arguments apply to the remaining cases. In the following, we only give the constructions for the proper sequences $\{n_k\}_{k \geq 0}$ and $\{m_k\}_{k \geq 0}$. Let
\[ J = \left( \sup_{k \geq n} \{ \varphi(k) - \varphi(k) \} \right)^{1/\alpha}. \]

Since $\varphi(n)$ is increasing and $\varphi(n+1) - \varphi(n) \to 0$ as $n \to \infty$, it follows that $\eta(n) \to \infty$ as $n \to \infty$.

\[ \text{CASE 2: } 0 < \alpha = \beta < \infty. \]
\[ n_{k+1} = n_k + [\eta(n_k)\varphi(n_k)], \quad m_k = [\alpha\varphi(n_k)] \quad \text{for } k \geq 0. \]

\[ \text{CASE 3: } \alpha = 0, 0 < \beta < \infty. \]
\[ n_k = \min \left\{ n \geq 1 : 2^k - 1 \leq \varphi(n) < 2^k \right\}, \quad m_k = [\beta\varphi(n_k)] \quad \text{for } k \geq 0. \]

\[ \text{CASE 4: } \alpha = 0, \beta = \infty. \]
\[ n_{k+1} = \min \left\{ n \geq 1 : n_k\varphi(n) - 1 \leq \varphi(n) < n_k\varphi(n_k) \right\}, \quad m_k = [\sqrt{n_k\varphi(n_k)}] \quad \text{for } k \geq 0. \]

\[ \text{CASE 5: } 0 < \alpha < \beta < \infty. \]
\[ n_{k+1} = \min \left\{ n \geq 1 : \eta(n_k)\varphi(n) - 1 \leq \varphi(n) < \eta(n_k)\varphi(n_k) \right\}, \quad m_k = [\alpha\eta(n_k)\varphi(n_k)] \quad \text{for } k \geq 0. \]

\[ \text{CASE 6: } \alpha = \beta = \infty. \]
\[ n_{k+1} = n_k + [\eta(n_k)\varphi(n_k)], \quad m_k = [\sqrt{\eta(n_k)\varphi(n_k)}] \quad \text{for } k \geq 0. \]

\[ \text{CASE 7: } \alpha = \beta = 0. \]
\[ n_{k+1} = (n_k)^2, \quad m_k = [\sqrt{\varphi(n_k)}] \quad \text{for } k \geq 0. \]

5. Applications.

5.1. Best approximation in continued fractions system. Let $([0,1), T)$ be the continued fractions system, where the Gauss transformation $T$ is defined as
\[ T(x) = \begin{cases} \frac{1}{x} - [\frac{1}{x}], & x \in (0,1), \\ 0, & x = 0. \end{cases} \]
Then, every real number $x \in [0,1)$ can be expanded as the continued fraction expansion
\[ x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \cdots}} = [a_1(x), a_2(x), \ldots]. \]
The digits $a_1(x) = \lfloor \frac{1}{x} \rfloor$ and $a_n(x) = a_1(T^{n-1}(x))$ \((n \geq 2)\) are called the partial quotients of \(x\). For \(n \geq 1\) and \((a_1, \ldots, a_n) \in \mathbb{N}^n\), we call \(I_n(a_1, \ldots, a_n) = \{x \in [0, 1) : a_1(x) = a_1, \ldots, a_n(x) = a_n\} \) an \(n\)-th cylinder. For more information on the continued fraction expansion, the readers are referred to [5, 10, 15, 28].

The corresponding IFS consists of \(\phi_i(x) = \frac{1}{i+x} \) \((i \in \mathbb{N})\).

It is worth pointing out that the system is not uniformly contractive, since \(\phi_1'(0) = -1\); this can be overcome by considering the system \(\{\phi_{i_1i_2} : i_1, i_2 \in \mathbb{N}\}\) which is uniformly contractive.

It is well known (see for example [29]) that Gauss transformation \(T\) is measure-preserving and ergodic with respect to the Gauss measure \(G\), where \(G\) is defined as
\[
dG = \frac{1}{\log 2} \frac{1}{x+1} dx.
\]

**Lemma 5.1.** The Rényi entropy \(H\) of the continued fractions system exists.

**Proof.** Denoting
\[b_n = \log \sum_{(a_1, \ldots, a_n) \in \mathbb{N}^n} G(I_n(a_1, \ldots, a_n))^2,\]
we check that
\[b_{n+m} \leq b_n + b_m + C\]
for some constant \(C\). So the Rényi entropy \(H = \lim_{n \to \infty} \frac{b_n}{n}\) exists. \(\square\)

**Theorem 5.2.** Let \(([0, 1), T)\) be the continued fractions system with the Gauss measure \(G\). Then
\[G \otimes G \left( \left\{(x, y) \in [0, 1) \times [0, 1) : \lim_{n \to \infty} \frac{M_n(x, y)}{\log n} = \frac{1}{H} \right\} \right) = 1\]
and for \(0 \leq \alpha \leq \beta \leq \infty\),
\[\dim_H E_{\alpha, \beta} = 2.\]

5.2. **Best approximation in Lüroth system.** Let \(([0, 1), T)\) be the Lüroth expansion system, where \(T\) is the Lüroth transformation given by
\[T(x) = \begin{cases} n(n+1)x - n, & x \in \left[\frac{1}{n+1}, \frac{1}{n}\right), \\ 0, & x = 0. \end{cases}\]
Then every \(x \in [0, 1)\) has a Lüroth expansion
\[x = \frac{1}{d_1(x)} + \sum_{n \geq 2} \frac{1}{d_1(x)(d_1(x) - 1) \ldots (d_n(x) - 1)d_n(x)},\]
where the digits \(d_1(x) = \lfloor \frac{1}{x} \rfloor + 1\) and \(d_n(x) = d_1(T^{n-1}(x))\) for \(n \geq 2\).

The corresponding IFS is given by
\[\{ \phi_i(x) = \frac{1}{i(i-1)} x + \frac{1}{i} : i \geq 2 \} \]
Jager & de Vroedt [14] proved that the transformation \(T\) is invariant and ergodic with respect to the Lebesgue measure \(\mathcal{L}\). The Rényi entropy
\[H = -\log \sum_{n=2}^{\infty} \left( \frac{1}{n(n-1)} \right)^2.\]
Theorem 5.3. Let \((0, 1), T\) be the Lüroth expansion system. We have
\[
\mathcal{L} \otimes \mathcal{L}\left( \left\{ (x, y) \in [0, 1) \times [0, 1): \lim_{n \to \infty} \frac{M_n(x, y)}{\log n} = \frac{1}{H} \right\} \right) = 1,
\]
and
\[
\dim_H E_{\alpha, \beta}^\varphi = 2, \quad 0 \leq \alpha \leq \beta \leq \infty.
\]

5.3. Best approximation in N-ary system. Let \(N \geq 2\) be an integer and \(T\) be the \(N\)-adic transformation defined by \(T(x) = Nx - \lfloor Nx \rfloor\) for \(x \in [0, 1]\). It is well known that every \(x\) admits a unique non-terminating \(N\)-ary expansion
\[
x = \frac{\varepsilon_1(x)}{N} + \frac{\varepsilon_2(x)}{N^2} + \ldots = [\varepsilon_1(x), \varepsilon_2(x), \ldots],
\]
where \(\varepsilon_k(x) \in \{0, \ldots, N-1\}\). The IFS is
\[
\left\{ \phi_i(x) = \frac{x + i}{N}: i \in \{0, \ldots, N-1\} \right\},
\]
and the Rényi entropy \(H = \log N\).

Theorem 5.4. ([17]) Let \((0, 1), T\) be the \(N\)-ary expansion system. We have
\[
\mathcal{L} \otimes \mathcal{L}\left( \left\{ (x, y) \in [0, 1) \times [0, 1): \lim_{n \to \infty} \frac{M_n(x, y)}{\log n} = \frac{1}{\log N} \right\} \right) = 1
\]
and
\[
\dim_H E_{\alpha, \beta}^\varphi = 2, \quad 0 \leq \alpha \leq \beta \leq \infty.
\]

5.4. Best approximation in triadic Cantor system. The triadic Cantor set \(C\) is an attractor of the finite IFS
\[
\left\{ \phi_1(x) = \frac{x}{3}, \ \phi_2(x) = \frac{x + 2}{3} \right\}.
\]
Define the corresponding map \(T: C \to C\) as \(Tx = 3x \pmod{1}\). Let \(\mu\) be the Cantor measure which is the restriction to \(C\) of \(s\)-dimensional Hausdorff measure, with \(s = \dim_H C = \frac{\log 2}{\log 3}\). Then the Rényi entropy \(H = \log 2\).

Theorem 5.5. Let \((C, T)\) be the triadic Cantor system and \(\mu\) be the Cantor measure. We have
\[
\mu \otimes \mu\left( \left\{ (x, y) \in [0, 1] \times [0, 1): \lim_{n \to \infty} \frac{M_n(x, y)}{\log n} = \frac{1}{\log 2} \right\} \right) = 1
\]
and
\[
\dim_H E_{\alpha, \beta}^\varphi = \frac{2\log 2}{\log 3}, \quad 0 \leq \alpha \leq \beta \leq \infty.
\]

Acknowledgments. We would like to thanks for reviewers’ comments and suggestions. Those comments and suggestions are all valuable and very helpful for revising and improving our paper.
REFERENCES

[1] M. D. Boshernitzan, Quantitative recurrence results, Invent. Math., 113 (1993), 617–631.
[2] D. Bessis, G. Paladin, G. Turchetti and S. Vaienti, Generalized dimensions, entropies, and Liapunov exponents from the pressure function for strange sets, J. Statist. Phys., 51 (1988), 109–134.
[3] L. Barreira and B. Saussol, Hausdorff dimension of measures via Poincaré recurrence, Comm. Math. Phys., 219 (2001), 443–463.
[4] Y. Bugeaud and B.-W. Wang, Distribution of full cylinders and the Diophantine properties of the orbits in β-expansions, J. Fractal Geom., 1 (2014), 221–241.
[5] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, Ergodic Theory, Springer-Verlag, New York, 1982.
[6] N. Chernov and D. Kleinbock, Dynamical Borel-Cantelli lemmas for Gibbs measures, Israel J. Math., 122 (2001), 1–27.
[7] K. Falconer, Fractal Geometry, Mathematical Foundations and Applications, 3rd edition, John Wiley & Sons, Ltd., Chichester, 2014.
[8] J. L. Fernández, M. V. Melián and D. Pestana, Quantitative recurrence properties of expanding maps, preprint, arXiv:math/0703222.
[9] S. Galatolo, Dimension and hitting time in rapidly mixing systems, Math. Res. Lett., 14 (2007), 797–805.
[10] G. H. Hardy and E. M. Wright, An Introduction to The Theory of Numbers, 5th edition, The Clarendon Press, Oxford University Press, New York, 1979.
[11] R. Hill and S. L. Velani, The ergodic theory of shrinking targets, Invent. Math., 119 (1995), 175–198.
[12] R. Hill and S. L. Velani, Metric Diophantine approximation in Julia sets of expanding rational maps, Inst. Hautes Études Sci. Publ. Math., (1997), 193–216.
[13] N. Haydn and S. L. Vaienti, The Rényi entropy function and the large deviation of short return times, Ergodic Theory Dynam. Systems., 30 (2010), 159–179.
[14] H. Jager and C. de Vroedt, Lüroth series and their ergodic properties, Nederl. Akad. Wetensch. Proc. Ser., 31 (1969), 31–42.
[15] A. Ya. Khintchine, Continued Fractions, Translated by Peter WynnP. Noordhoff, Ltd., Groningen 1963.
[16] B. Li, B. W. Wang, J. Wu and J. Xu, The shrinking target problem in the dynamical system of continued fractions, Proc. London Math. Soc., 108 (2014), 159–186.
[17] J. Li and X. Yang, On longest matching consecutive subsequence, Int. J. Number Theory., 15 (2019), 1745–1758.
[18] R. D. Mauldin and M. Urbański, Dimensions and measures in infinite iterated function systems, Proc. London Math. Soc., 73 (1996), 105–154.
[19] R. D. Mauldin and M. Urbański, Conformal iterated function systems with applications to the geometry of continued fractions, Trans. Amer. Math. Soc., 351 (1999), 4995–5025.
[20] D. S. Ornstein and B. Weiss, Entropy and data compression schemes, IEEE Trans. Inform. Theory., 39 (1993), 78–83.
[21] L. Peng, On the hitting depth in the dynamical system of continued fractions, Chaos. Solitons. Fractals., 69 (2014), 22–30.
[22] L. Peng, B. Tan and B. W. Wang, Quantitative Poincaré recurrence in continued fraction dynamical system, Sci. China Math., 55 (2012), 131–140.
[23] B. Saussol, Recurrence rate in rapidly mixing dynamical systems, Discrete Contin. Dyn. Syst., 15 (2006), 259–267.
[24] B. O. Stratmann and M. Urbański, Jarník, Julia: a Diophantine analysis for geometrically finite Kleinian groups with parabolic elements, Math. Scand., 91 (2002), 27–54.
[25] S. Seuret and B.-W. Wang, Quantitative recurrence properties in conformal iterated function systems, Adv. Math., 280 (2015), 472–505.
[26] B. Tan and B.-W. Wang, Quantitative recurrence properties for beta-dynamical system, Adv. Math., 228 (2011), 2071–2097.
[27] F. Takens and E. Verbitski, Generalized entropies: Rényi and correlation integral approach, Nonlinearity., 11 (1998), 771–782.
[29] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, 79. Springer-Verlag, New York-Berlin, 1982.

[30] Q.-L. Zhou, Dimensions of recurrent sets in $β$-symbolic dynamics, *J. Math. Anal. Appl.*, 472 (2019), 1762–1776.

Received October 2020; revised December 2020.

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