Approximation Algorithms for Scheduling under Non-Uniform Machine-Dependent Delays

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Abstract

We consider the problem of scheduling precedence-constrained jobs on heterogenous machines in the presence of non-uniform machine-dependent communication delays. We are given as input a set of $n$ unit size precedence-constrained jobs, and a set of $m$ related machines each with size $m_i$ (machine $i$ can execute at most $m_i$ jobs at any time). Each machine $i$ also has an associated in-delay $\rho_i^{\text{in}}$ and out-delay $\rho_i^{\text{out}}$. For any job $v$, machine $i$, and time $t$, we say that $v$ is available to machine $i$ at time $t$ if $v$ is completed on $i$ before time $t$ or on any machine $j$ before time $t - (\rho_i^{\text{in}} + \rho_j^{\text{out}})$. If job $v$ is scheduled at time $t$ on machine $i$, then all of its predecessors must be available to $i$ by time $t$. The objective is to construct a schedule that minimizes makespan, which is the maximum completion time over all jobs.

We consider schedules which allow duplication of jobs as well as schedules which do not. When duplication is allowed, we provide an asymptotic $\text{polylog}(n)$-approximation algorithm; it is a true approximation if the makespan also accounts for the time to communicate the jobs to the machines and for the time to communicate the results out. For no-duplication schedules, we also obtain an asymptotic $\text{polylog}(n)$-approximation via a reduction to the case with duplication, and a true $\text{polylog}(n)$-approximation for symmetric delays ($\rho_i^{\text{in}} = \rho_i^{\text{out}}$ for all machines $i$). These results represent the first polylogarithmic approximation algorithms for scheduling with non-uniform communication delays.

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1 Introduction

With the increasing scale and complexity of scientific and data-intensive computations, it is often necessary to process workloads with many dependent jobs on a network of heterogeneous computing devices with varying computing capabilities and communication delays. For instance, the training and evaluation of neural network models, which are iterated directed acyclic graphs (DAGs), is often distributed over diverse devices such as CPUs, GPUs, or other specialized hardware. This process, commonly referred to as device placement, has gained significant interest [15, 17, 29, 30]. Similarly, many scientific workflows are best modeled as DAGs, and the underlying high-performance computing system as a heterogeneous networked distributed system with communication delays [2, 39, 43].

Optimization problems associated with scheduling under communication delays have been studied extensively, but provably good approximation bounds are few and several challenging open problems remain [1, 3, 11, 19, 23, 31, 32, 34–36, 38]. With a communication delay, scheduling a DAG of uniform size jobs on identical machines is already NP-hard [36, 38], and several inapproximability results are known [3, 19]. However, the field is still underexplored and scheduling under communication delay was listed as one of the top ten open problems in scheduling surveys [4, 40]. While there has been progress on approximation algorithms for the case of uniform communication delays [13, 23, 26, 28], little is known for more general delay models.

This paper considers the problem of scheduling precedence-constrained jobs on machines connected by a network with non-uniform communication delays. In general, the delay incurred in communication between two machines could vary with the machines as well as with the data being communicated, which in turn may depend on the jobs being executed on the machines. For many applications, however, simpler models suffice. For instance, job-pairwise delays model scenarios where the delay incurred in the communication between two jobs running on two different machines is a function primarily of the two jobs. This is suitable when the communication is data-intensive. On the other hand, machine-pairwise delays, where the delay is only a function of the pair of machines, model scenarios where network latency is the dominant cost of communication. Recent work in [12] presents hardness results for the job-pairwise model, providing preliminary evidence that obtaining sub-polynomial approximation factors may be intractable.

1.1 Our contributions

Our central contribution is that for an important class of machine-pairwise delays, which we call machine-dependent delays, polylogarithmic approximations are attainable in polynomial time. Under machine-dependent delays, each machine \(i\) has an in-delay \(\rho_{i}^{\text{in}}\) and out-delay \(\rho_{i}^{\text{out}}\), and the time taken to communicate a result from \(i\) to \(j\) is \(\rho_{i}^{\text{out}} + \rho_{j}^{\text{in}}\). This model, illustrated in Figure 1, is especially suitable for environments where data exchange between jobs occurs via the cloud, an increasingly common mode of operation in modern distributed systems [25, 27, 44]: \(\rho_{i}^{\text{in}}\) and \(\rho_{i}^{\text{out}}\) represent the cloud download and upload times, respectively, for machine \(i\). Figure 2 situates the machine-dependent delay model and other variants we study in this paper within the landscape of communication delay models.

![Figure 1: Communicating a result from \(i\) to \(j\) takes \(\rho_{i}^{\text{out}} + \rho_{j}^{\text{in}}\) time.](image)

**Definition 1.1. (Machine-Dependent Precedence-constrained Scheduling MDPS)** We are given as input a directed acyclic graph (DAG), which models a set of \(n\) unit size jobs with precedence relations among them, and a set of \(m\) machines. Each machine has a number \(m_{i}\) of processors and a speed \(s_{i}\) at which each processor processes jobs. For any job \(v\), machine \(i\), and time \(t\), we say that \(v\) is available to machine \(i\) at time \(t\) if \(v\) is completed on \(i\) before time \(t\) or on any machine \(j\) before time \(t - (\rho_{i}^{\text{in}} + \rho_{j}^{\text{out}})\). If job \(v\) is scheduled at time \(t\) on machine \(i\), then all of its predecessors must be available to \(i\) by time \(t\). For any schedule \(\sigma\), let \(\Delta_{\sigma}\) denote the makespan of \(\sigma\), which is the maximum completion time over all jobs, and let...
\( \rho_\sigma \) denote \( \max\{\rho_{i_{\text{in}}} + \rho_{i_{\text{out}}}\} \) over all machines \( i \) on which a job is executed in \( \sigma \). The objective of MDPS is to construct a schedule that minimizes makespan.

**General Delays**

Delays can depend on features of machines, features of jobs, or features of the schedule

- **Machine Pairwise**
  - Delay from machine \( i \) to \( j \) is a function of \( (i, j) \)
- **Job Pairwise**
  - Delay from job \( u \) to \( v \) is a function of \( (u, v) \) [12]

- **General Metric**
  - Delay from \( i \) to \( j \) is given by a metric over the machines
- **Machine Dependent**
  - Delay from \( i \) to \( j \) is \( \rho_{i_{\text{out}}} + \rho_{j_{\text{in}}} \)
- **Symmetric Machine Dependent**
  - The delay from \( i \) to \( j \) is \( \rho_i + \rho_j \)
- **Single Job**
  - Delay from job \( v \) to any successor is a function of \( v \)
- **Uniform**
  - Fixed delay \( \rho \) [13, 14, 23, 26, 28]

**Figure 2**: Selection of scheduling models with communication delays. \( a \rightarrow b \) indicates that \( b \) is a special case of \( a \). We prove individual results for Machine Dependent delays and Symmetric Machine Dependent delays. Citations reflect results relevant to approximation algorithms.

We present the first approximation algorithms for scheduling under non-uniform communication delays. In the presence of delays, a natural approach to hide latency and reduce makespan is to duplicate some jobs (for instance, a job that is a predecessor of many other jobs) [1, 35]. We consider MDPS for both schedules that allow duplication (which we assume by default) and those that do not. Our main result is a polylogarithmic asymptotic approximation for MDPS when duplication is allowed.

**Theorem 1 (Polylogarithmic asymptotic approximation for MDPS).** There exists a polynomial time algorithm for MDPS that produces a schedule with makespan at most \( \text{polylog}(n) \cdot (\Delta_\sigma + \rho_\sigma) \), for any schedule \( \sigma \).

We emphasize that if the makespan of any MDPS schedule includes the delays incurred in distributing the problem instance and collecting the output of the jobs, then the algorithm of Theorem 1 is, in fact, a true polylogarithmic approximation for makespan. (From a practical standpoint, in order to account for the time incurred to distribute the jobs and collect the results, it is natural to include in the makespan the in- and out-delays of every machine used in the schedule.)

The exact approximation factor in Theorem 1 depends on the non-uniformity of the particular delay model. For the most general machine-dependent delays model, our proof achieves a \( O(\log^{10} n) \) bound. We obtain improved bounds when any of the three defining machine parameters—size, speed, and delay—are uniform. In particular, for the special cases where any two of these parameters are uniform (e.g., when all machines have the same size and speed but different delays), we obtain an \( O(\log^4 n) \) bound. We note that despite some uniformity, these special cases can model certain two-level non-uniform network hierarchies with processors at the leaves, zero or low delays at the first level, and high delays at the second level.

We next consider the problem of designing schedules for MDPS that do not allow duplication, and obtain a polylogarithmic asymptotic approximation via a reduction to scheduling with duplication.
There exists a polynomial time algorithm that produces a no-duplication schedule whose makespan is at most \(\text{polylog}(n)(\Delta_{\sigma} + \rho_{\sigma})\) for any no-duplication schedule \(\sigma\).

Our final result is a true polylogarithmic approximation for no-duplication schedules under symmetric delays. To achieve this result, we present an approximation algorithm to estimate if the makespan of an optimal no-duplication schedule is at least the delay of any given machine; this enables us to identify machines that cannot communicate in the desired schedule.

### Theorem 3 (Polylogarithmic approximation for MDPS with symmetric delays and no duplication)

If \(\rho_{i}^{\text{in}} = \rho_{i}^{\text{out}}\) for all \(i\), then there exists a polynomial time \(\text{polylog}(n)\)-approximation algorithm for MDPS with no duplication.

### 1.2 Our techniques

Our proof of Theorem 1 relies on a framework consisting of a carefully crafted linear programming relaxation and a series of reductions that derive results for MDPS from results for uniform machines. While each individual component of the framework is simple or builds on prior work, taken together they offer a flexible recipe for designing approximation algorithms for scheduling DAGs on a distributed system of heterogeneous machines with non-uniform delays. Given the hardness conjectures of [12] for the job-pairwise setting, we find it surprising that a fairly general machine-pairwise model is tractable.

The main components of our MDPS algorithm of Theorem 1 are outlined in Figure 3. First, we observe that any instance of MDPS can be reduced to an instance of MDPS in which all out-delays are 0, meaning that in the new instance delays depend only on the machine receiving the data. This reduction is helpful in the second component, which uses a standard method to partition the machines into groups of uniform machines, i.e. each machine in a group can be treated as having the same in-delay, speed, and size (to within a constant factor). The final approximation factor we establish grows as \(K^{3}\), where \(K\) is the number of groups and depends on the extent of non-uniformity among the machines. We bound \(K\) by \(O(\log^{3} n)\) in the general case when the speeds, sizes, and delays of machines are non-uniform. We emphasize that, even with the machines partitioned in this way, an optimal schedule must still judiciously distribute jobs among the groups depending on the structure of the DAG and the particular machine parameters.

The third component of our algorithm solves an LP relaxation for MDPS aimed at finding a placement of the jobs (with possible duplication) on the groups. Building on the approach of [28], our LP uses job placement, duplication, and completion time variables, and incorporates non-uniform speeds, sizes, and communication delays. One challenge is bounding the amount of duplication allowed within a communication phase of a particular group. To this end, we incorporate constraints that capture the optimal makespan for scheduling the duplicated jobs on uniform machines. This strategy points to the interesting possibility of capturing more complex processor structure, such as might be modelled with a multi-level tree hierarchy.

The fourth component rounds an optimal LP solution to an integer solution by placing each job on the group for which the job’s LP mass is maximized. A benefit of this simple rounding is that it accommodates many different machine properties as long as the number of groups can be kept small. Finally, we construct a schedule using the integer LP solution. This subroutine divides the set of jobs assigned to each group into phases and constructs a schedule for each phase by invoking a schedule for the uniform machines case, appending each schedule to the existing schedule for the entire instance.

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1We note that the corresponding problem for duplication schedules is a min-max partitioning variant of the Minimum \(k\)-Union problem and related to the Min-Max Hypergraph \(k\)-Partitioning problem, both of which have been shown to be Densest-\(k\)-Subgraph-hard [7, 9]; this might suggest a similar hardness result for deriving a true approximation for MDPS with duplication.
Reduce to Problem with No Out-Delays  |  Partition machines into groups of identical machines  |  Solve LP defined over groups  |  Assign each job to group where its LP mass is largest  |  Using job assignments, construct schedule by concatenating schedules for each group phase-by-phase

Figure 3: Framework for MDPS asymptotic approximation

The proof of Theorem 2, which extends an asymptototic polylogarithmic approximation to no-duplication schedules for general machine-dependent delays, follows from the structure of the schedule designed in Theorem 1 and a general reduction in [28] from duplication to no-duplication schedules in the uniform delay case. Avoiding the additive delay penalty of Theorem 2 to achieve the true approximation of Theorem 3 is more difficult. In the symmetric delay setting, we can distinguish those machines whose delay is low enough to communicate with other machines from those machines with high delay. One of the central challenges is then to distribute jobs among the high-delay machines. We overcome this difficulty by revising the LP in the framework of Theorem 1 to partition the jobs among low- and high-delay machines, and rounding the corresponding solutions separately. We note that the design of no-duplication schedules via a reduction to duplication schedules incurs a loss in approximation factor of an additional polylogarithmic factor. While this may not be desirable in a practical implementation, our results demonstrate the flexibility of the approach and highlight its potential for more general delay models.

1.3 Related work

Precedence constrained scheduling. The problem of scheduling precedence-constrained jobs was initiated in the classic work of Graham who gave a constant approximation algorithm for uniform machines [16]. Jaffe presented an \(O(\sqrt{m})\) makespan approximation for the case with related machines [20]. This was improved upon by Chudak and Shmoys who gave an \(O(\log m)\) approximation [10], then used the work of Hall, Schulz, Shmoys, and Wein [18] and Queyranne and Sviridenko [37] to generalize the result to an \(O(\log m)\) approximation for weighted completion time. Chekuri and Bender [8] proved the same bound as Chudak and Shmoys using a combinatorial algorithm. In subsequent work, Li improved the approximation factor to \(O(\log m / \log \log m)\) [24]. The problem of scheduling precedence-constrained jobs is hard to approximate even for identical machines, where the constant depends on complexity assumptions [5, 21, 41]. Also, Bazzi and Norouzi-Fard [6] showed a close connection between structural hardness for \(k\)-partite graph and scheduling with precedence constraints.

Precedence constrained scheduling under communication delays. Scheduling under communication delays has been studied extensively [35, 38, 42]. For unit size jobs, identical machines, and unit delay, a \((7/3)\)-approximation is given in [32], and [19] proves the NP-hardness of achieving better than a \(5/4\)-approximation. Other hardness results are given in [3, 36, 38]. More recently, Davies, Kulkarni, Rothvoss, Tarnawski, and Zhang [13] give an \(O(\log \rho \log m)\) approximation in the identical machine setting using an LP approach based on Sherali-Adams hierarchy, which is extended to include related machines in [14]. Concurrently, Maiti, Rajaraman, Stalfa, Svitkina, and Vijayaraghavan [28] provide a polylogarithmic approximation for uniform communication delay with related machines as a reduction from scheduling with duplication. The algorithm of [28] is combinatorial in the case with identical machines.

Davies, Kulkarni, Rothvoss, Sandeep, Tarnawski, and Zhang [12] consider the problem of scheduling precedence-constrained jobs on uniform machine in the presence of non-uniform, job-pairwise communication delays. That is, if \(u \prec v\) and \(u\) and \(v\) are scheduled on different machines, then the time between their executions is at least \(\rho_u, v\). The authors reduce to this problem from Unique-machines Precedence-Constrained Scheduling (UMPS) in which there is no communication delay, but for each job there is some particular machine on which that job must be placed. The authors show that UMPS is hard to approximate
to within a logarithmic factor by a straightforward reduction from job-shop scheduling, and conjecture that UMPS is hard to approximate within a polynomial factor.

**Precedence constrained scheduling under communication delays with job duplication.** Using duplication with communication delay first studied by Papadimitriou and Yannakakis [35], who give a 2-approximation for DAG scheduling with unbounded processors and fixed delay. Improved bounds for infinite machines are given in [1,11,33,34]. Approximation algorithms are given by Munier and Hanen [31,32] for special cases in which the fixed delay is very small or very large, or the DAG restricted to a tree. The first bounds for a bounded number of machines are given by Lepere and Rapine [23] who prove an asymptotic $O(\log \rho / \log \log \rho)$ approximation. Recent work has extended their framework to other settings: [28] uses duplication to achieve an $O(\log \rho \log m / \log \log \rho)$ approximation for a bounded number of related machines, and Liu, Purohit, Svitkina, Vee, and Wang [26] improve on the runtime of [23] to a near linear time algorithm with uniform delay and identical machines.

### 1.4 Discussion and open problems

Our results indicate several directions for further work. First, we conjecture that our results extend easily to the setting with non-uniform job sizes. We believe the only barriers to such a result are the technical difficulties of tracking the completion times of very large jobs that continue executing long after they are placed on a machine. Also, while our approximation ratios are the first polylogarithmic guarantees for scheduling under non-uniform delays, we have not attempted to optimize logarithmic factors. There are obvious avenues for small reductions in our ratio, e.g. the technique used in [23] to reduce the ratio by a factor of $\log \log \rho$. Further reduction, however, may require more novel ideas. Additionally, in the setting without duplication, we incur even more logarithmic factors owing to our reduction to scheduling with duplication. These factors may be reduced by using a more direct method, possibly extending the LP-hierarchy style approach taken in [13,14].

Aside from improvements to our current results, our techniques suggest possible avenues to solve related non-uniform delay scheduling problems. Our incorporation of parallel processors allows our results to apply to a two-level machine hierarchy, where machines are the leaves and the delay between machines is a function of their lowest common ancestor. We would like to explore extensions of our framework to constant-depth hierarchies and tree metrics. More generally, the problem of scheduling under arbitrary machine metric delays remains wide open.

Finally, we believe there are useful analogs to these machine-dependent delay models in the job-pairwise regime. A job $v$ with in-delay $\rho_{in}^v$ and out-delay $\rho_{out}^v$ has the natural interpretation of the data required to execute a job, and the data produced by a job. A job tree hierarchy could model the shared libraries required to execute certain jobs: jobs in different subtrees require different resources to execute, and downloading these additional resources incurs a delay. Given the hardness conjectures of [12], alternate delay models such as these may be necessary to achieve provably good bounds.

### 2 Model and preliminaries

**Jobs and machines.** Let $V$ denote a set of $n$ unit-size jobs, partially-ordered according to $\prec$. For any jobs $u$ and $v$ such that $u \prec v$, we refer to $u$ as a predecessor of $v$. Let $M$ denote a set of $m$ machines. We associate with each machine $i \in M$, an integer size $m_i \geq 1$ indicating the number of processors in $i$, an integer speed $s_i \geq 1$ for each processor of $i$, integer in-delay of $\rho_{in}^i$, and integer out-delay of $\rho_{out}^i$.

**Schedules.** A schedule $\sigma$ is an assignment of times to job-machine pairs. If $\sigma(v, i) = t$, this indicates that $v$ is executed on machine $i$ at time $t$ (if $v$ is not executed on $i$ then $\sigma(v, i)$ is undefined). By default, we allow schedules to duplicate jobs; so, for a given job $v$, $\sigma(v, i)$ and $\sigma(v, j)$ may be defined for distinct machines $i$ and $j$. In a no-duplication schedule, for each job $v$ there is exactly one machine $i$ such that $\sigma(v, i)$ is defined. Preemption is not allowed. For any job $v$, machine $i$ and time $t$, we say that $v$ is available to $i$ at time $t$ if $v$
is executed on \( i \) before time \( t \) or on any machine \( j \) before time \( t - (\rho^\text{in}_i + \rho^\text{out}_j) \). If \( v \) is scheduled at time \( t \) on machine \( i \), then all of its predecessors must be available to \( i \) by time \( t \). The number of jobs that can be executed on any \( i \) at any time \( t \) is at most \( m_i \). The time to process a single job on any machine \( i \) is \( 1/s_i \).

Given a set of jobs and machines as above, we let \( \text{MDPS} \) denote the problem of determining a schedule that minimizes the makespan, which is the maximum completion time of any job in the schedule.

**Reduction to in-delays.** We first show that \( \text{MDPS} \) reduces to the special case of \( \text{MDPS} \) with zero out-delays. This allows us to focus on \( \text{MDPS} \) with zero out-delays in the remainder of the paper.

**Lemma 2.1.** For any \( \alpha \geq 1 \), a polynomial-time \( \alpha \)-approximation algorithm for \( \text{MDPS} \) (resp., \( \text{MDPS} \) without duplication) in which all out-delays are 0 implies a polynomial-time \( \alpha \)-approximation for \( \text{MDPS} \) (resp., \( \text{MDPS} \) without duplication).

**Proof.** Let \( I \) be an instance of \( \text{MDPS} \). We prove the lemma by showing the existence of polynomial-time computable functions \( f \) and \( f_2 \) with the following properties: (a) \( f(I) \) is an instance of \( \text{MDPS} \) in which all out-delays are 0, (b) the optimal makespan of \( I \) is at least the optimal makespan of \( f(I) \), and (c) if \( \sigma \) is a schedule (resp., no-duplication schedule) of \( f(I) \) then \( f_2(\sigma) \) is a schedule of \( I \) (resp., no-duplication) with the same makespan. The existence of these functions is sufficient to prove the lemma. For a given schedule \( \sigma \) and arbitrary job \( v \) and machine \( i \), we define \( t_{v,i}^\sigma \) to be the execution time of \( v \) on machine \( i \) according to \( \sigma \). We first define the instance \( f(I) \) in which all out delays are 0. Since there are no out-delays, we use \( \rho_i \) to denote the in-delay of an arbitrary machine \( i \) in instance \( f(I) \); \( \rho^\text{in}_i \) and \( \rho^\text{out}_i \), respectively, denote the in- and out-delay of \( i \) in \( I \). \( f(I) \) is identical to \( I \) except that, for each machine \( i \), the out-delay of \( i \) is 0 and the in-delay \( \rho_i \) is equal to \( \rho^\text{in}_i + \rho^\text{out}_i \). It is easy to see that \( f \) is polynomial-time computable and has property (a).

We show that \( f \) has property (b). Consider a schedule \( \sigma \) of \( I \). We construct a schedule \( \sigma' \) of \( f(I) \) with the same makespan. For any job \( v \) and machine \( i \), if \( v \) is executed on \( i \) at time \( t \) according to \( \sigma \), then \( v \) is executed on \( i \) at time \( t + \rho^\text{out}_i \) in \( \sigma' \). It is easy to see that the makespan (the time at which the result of all jobs is available to the central machine) is the same for both \( \sigma \) and \( \sigma' \). It is also easy to see that the ordering of jobs on a single machine is maintained in \( \sigma' \). So we must only show that the communication constraints are satisfied. Consider two jobs \( u \) and \( v \) such that \( u < v \), \( u \) is executed on \( i \) and \( v \) is executed on \( j \neq i \), and \( t_{v,j}^\sigma - t_{u,i}^\sigma \geq \rho^\text{out}_i + \rho^\text{in}_j \). Then \( t_{v,j}^\sigma - t_{u,i}^\sigma = (t_{v,j}^\sigma + \rho^\text{out}_j) - (t_{u,i}^\sigma + \rho^\text{in}_j) \geq \rho^\text{out}_j + \rho^\text{in}_j + \rho^\text{out}_i - \rho^\text{in}_i = \rho^\text{in}_i + \rho^\text{out}_i = \rho_j \), implying that \( \sigma' \) is a valid schedule of \( f(I) \). Therefore, property (b) holds.

We now define \( f_2 \), which uses similar reasoning as above. Consider a schedule \( \sigma \) of \( f(I) \). We define \( f_2(\sigma) \) to be schedule \( \sigma' \), constructed as follows. For any job \( v \) and machine \( i \), if \( v \) is executed on \( i \) at time \( t \) according to \( \sigma \), then \( v \) is executed on \( i \) at time \( t - \rho^\text{out}_i \) according to \( \sigma' \). Note that the definition ensures that if \( \sigma \) is a no-duplication schedule, then so is \( \sigma' \). Again, it is easy to see that \( f_2 \) is polynomial-time computable, that the makespan is the same for both schedules, and that precedence constraints are maintained on individual machines. So consider two jobs \( u \) and \( v \) such that \( u < v \), \( u \) is executed on \( i \) and \( v \) is executed on \( j \neq i \), and \( t_{v,j}^\sigma - t_{u,i}^\sigma \geq \rho_j = \rho^\text{in}_j + \rho^\text{out}_j \). Then \( t_{v,j}^\sigma - t_{u,i}^\sigma = (t_{v,j}^\sigma - \rho^\text{out}_j) - (t_{u,i}^\sigma - \rho^\text{out}_j) \geq \rho^\text{in}_j + \rho^\text{out}_j - \rho^\text{out}_j + \rho^\text{out}_j = \rho^\text{in}_i + \rho^\text{out}_i \), implying that \( \sigma' \) is a valid schedule of \( I \). Therefore, property (c) holds.

### 3 A polylogarithmic approximation algorithm for MDPS

In this section, we present an asymptotic approximation algorithm for \( \text{MDPS} \). Following Lemma 2.1, we focus on the setting where \( \rho^\text{out}_i = 0 \) for all machines \( i \). Since there are no out-delays, we use \( \rho_i \) to denote the in-delay \( \rho^\text{in}_i \) of machine \( i \). In Section 3.1, we first organize the machines according to their size, speed, and delay. Section 3.2 presents a linear programming relaxation, assuming that the machines are organized into groups so that all machines in a group have the same size, speed, and delay. Section 3.3 presents a procedure for rounding any fractional solution to the linear program. Finally, Section 3.4 presents a combinatorial algorithm that converts an integer solution to the linear program of Section 3.2 to a schedule.
3.1 Partitioning machines into groups

We first partition machines into groups according to delay, speed, and size. Group \((\ell_1, \ell_2, \ell_3)\) consists of those machines \(i\) such that \(2^{\ell_1-1} \leq \rho_i < 2^{\ell_1}, 2^{\ell_2} \leq m_i \leq 2^{\ell_2+1}, \) and \(2^{\ell_3} \leq s_i \leq 2^{\ell_3+1}.\) We define a new set of machines \(M'\). For every \(i \in M,\) if \(i \in \langle \ell_1, \ell_2, \ell_3 \rangle,\) we define \(i' \in M'\) such that \(\rho_i = 2^{\ell_1}, m_i = 2^{\ell_2},\) and \(s_i = 2^{\ell_3}.\) For the remainder of the section, we work with the machine set \(M'.\) This ensures that every group consists of machines with identical delay, speed, and size, which will simplify the exposition of the algorithm and its proofs. Our reduction to identical machine groups is justified in the following lemma.

Lemma 3.1. The optimal makespan over the machine set \(M'\) is no more than a factor of 12 greater than the optimal solution over \(M.\)

Proof. Consider any schedule \(\sigma\) on the machine set \(M.\) We first show that increasing the delay of each machine by a factor of 2 increases the makespan of the schedule by at most a factor of 2. Let \(\sigma(v, i)\) be the completion time of \(v\) on machine \(i\) according to \(\sigma\) (and undefined if \(v\) is not executed on \(i\)). We define \(\sigma'\) such that \(\sigma'(v, i) = 2 \cdot \sigma(v, i)\) (undefined if \(\sigma(v, i)\) is undefined). It is easy to see that \(\sigma'\) maintains the precedence ordering of jobs, and that the time between the executions of any two jobs has been doubled. Therefore, \(\sigma'\) is a valid schedule with all communication delays doubled, and with the makespan doubled.

Next, we show that reducing the size of all machines by a factor of 2 increases the makespan by at most a factor of 3. Let \(t_u\) be the completion time of \(v\) according to \(\sigma.\) We define \(\sigma'\) as follows. For each machine \(i,\) we arbitrarily order \(i's\) processors \(1, \ldots, m_i.\) If a job \(v\) is executed on processor \([m_i/2] + p\) of machine \(i\) at time \(t,\) then we place it on processor \(p\) of machine \(i\) at time \(t' = 2t + 1.\) If a job \(v\) is not executed on an eliminated processor, then it is placed on the same processor and executed at time \(t' = 2t.\) It is easy to see that the number of \(i's\) processors used is at most half. Suppose that for some pair of jobs \(u, v\) we have \(t_u - t_v \geq d.\) Then \(t'_u - t'_v \geq 2t_u - (2t_u + 1) = 2(t_u - t_v) - 1 \geq 2d - 1 \geq d.\) This shows that all delay and precedence constraints are obeyed and the makespan of \(\sigma'\) is at most 3 times the original.

Finally, we show that reducing the speed of each machine by a factor of 2 increases the makespan by at most a factor of 2. Again, by setting \(\sigma'(v, i) = 2 \cdot \sigma(v, i)\) (where \(\sigma(v, i)\) is defined) we see that the time between any two jobs is increased by a factor of 2 and the makespan is increased by a factor of 2. Therefore, if the speed of each machine is halved, the schedule will remain valid.

We order the groups arbitrarily \(\langle 1 \rangle, \langle 2 \rangle, \ldots, \langle K \rangle\) and designate the delay, size, and speed of all machines in group \(\langle k \rangle\) as \(\bar{\rho}_k, \bar{m}_k,\) and \(\bar{s}_k,\) respectively. This yields \(K = \log \max_k \{\bar{\rho}_k\} \cdot \log \max_k \{\bar{s}_k \bar{m}_k\}\) groups. Note that if, for any machine \(i,\) we have \(s_i \geq n,\) then we can construct a schedule with makespan \(1 \leq \Delta + \rho_\sigma,\) for any \(\sigma,\) by placing all jobs on \(i.\) So we can assume \(\max_k \{\bar{s}_k\} \leq n.\) Also, suppose \(m_i \geq n\) for any \(i.\) Then, given a schedule \(\sigma,\) we can construct the exact same schedule on a machine set where \(m_i = n.\) So we can assume \(\max_k \{\bar{m}_k\} \leq n.\) We can also assume that \(\max_k \{\bar{\rho}_k\} \leq n\) since, if we ever needed a machine with delay greater than \(n\) we could schedule everything on a single machine in less time.

Therefore, in the most general case, we have \(K \leq (\log n)^3\) groups. Our approximation ratio accrues several factors of \(K,\) so we draw attention to special cases in which \(K\) is reduced. The reasoning above implies that if any one of the parameters is uniform across all machines, then we have \((\log n)^2\) groups, and if any two of the parameters are uniform we have \(\log n\) groups. We emphasize in particular the case with \(\log n\) groups where all machine sizes and speeds are uniform and the delay is variable across the machines.

3.2 The linear program

In this section, we design a linear program \(LP_\alpha,\) parametrized by \(\alpha \geq 1,\) for \(MDPS.\) Following Section 3.1, we assume that the machines are organized in groups, where each group \(\langle k \rangle\) is composed of machines that have identical sizes, speeds, and delays.
\[ 2D \hat{s}_k \bar{m}_k \cdot |\langle k \rangle| \geq \sum_v (x_{v,k} + y_{v,k}) \quad \forall k \quad (1) \]
\[ D \geq \hat{\rho}_k \cdot x_{v,k} \quad \forall v, k \quad (2) \]
\[ C_v \geq C_u + \hat{\rho}_k \cdot (x_{v,k} - z_{u,v,k}) \quad \forall u, v, k : u \prec v \quad (3) \]
\[ \alpha \cdot \hat{s}_k \bar{\rho}_k \bar{m}_k \geq \sum_u z_{u,v,k} \quad \forall v, k \quad (4) \]
\[ C^u_{w,k} \geq C^v_{w,k} + \frac{x_{u,v,k}}{s_k} \quad \forall u, u', v, k : u' \prec u \quad (5) \]
\[ \alpha \cdot \bar{\rho}_k \geq C^v_{u,k} \quad \forall u, v, k \quad (6) \]
\[ C_v \geq C_u + \sum_k \frac{x_{v,k}}{s_k} \quad \forall u, v : u \prec v \quad (7) \]
\[ \sum_k x_{v,k} = 1 \quad \forall v \quad (9) \]
\[ D \geq C_v \quad \forall v \quad (8) \]
\[ C^u_{u,k} \geq 0 \quad \forall u, v, k \quad (11) \]
\[ y_{u,k} \geq z_{u,v,k} \quad \forall u, v, k \quad (12) \]
\[ z_{u,v,k} \geq 0 \quad \forall u, v, k \quad (13) \]
\[ z_{u,v,k} \geq 1 \quad \forall u, v, k \quad (14) \]

Variables. \( D \) represents the makespan of the schedule. For each job \( v \), \( C_v \) represents the earliest completion time of \( v \). For each job \( v \) and group \( \langle k \rangle \), \( x_{v,k} \) indicates whether or not \( v \) is first executed on a machine in group \( \langle k \rangle \). For each \( \langle k \rangle \) and pair of jobs \( u, v \) such that \( u \prec v \), \( z_{u,v,k} \) indicates whether \( v \) is first executed on a machine in group \( \langle k \rangle \) and the earliest execution of \( u \) is less than \( \hat{\rho}_k \) time before the execution of \( v \). Intuitively, \( z_{u,v,k} \) indicates whether there must be a copy of \( u \) executed on the same machine that first executes \( v \). For each job \( v \) and group \( \langle k \rangle \), \( y_{v,k} \) indicates whether or not \( z_{u,v,k} = 1 \) for some \( u \); that is, whether or not some copy of \( u \) must be placed on group \( \langle k \rangle \). Finally, for each job \( v \) and machine group \( \langle k \rangle \), we can think of the variables \( \{C^u_{w,k}\}_{u \in V} \) as giving the completion times for a schedule of jobs \( u \) such that \( z_{u,v,k} = 1 \) on a single machine of group \( \langle k \rangle \). Note that constraints (8, 10 - 14) guarantee that all variables are non-negative.

Makespan (1, 2, 8). Constraint (1) states that the makespan is at least the load on any single group. The factor of 2 is a result of the fact that a single job \( v \) may be counted twice on a group \( \langle k \rangle \): once by \( x_{v,k} \) and once by \( y_{v,k} \). Constraint (2) states that we require at least one round of communication to any machine that executes even a single job. This allows us to minimize the additive term in our asymptotic bound. Constraint (8) states that the makespan is at least the maximum completion time of any job.

Machine-Dependent Communication delays (3). Constraint (3) states that the earliest completion time of \( v \) must be at least \( \hat{\rho}_k \) after the earliest completion time of any predecessor \( u \) if \( v \) is first executed on a machine in group \( \langle k \rangle \) and no copy of \( u \) is duplicated on the same machine as \( v \).

Duplication (4 - 6). This set of constraints governs the amount of duplication allowed for any given job. Fix any job \( v \) and group \( \langle k \rangle \), and suppose that \( v \) first executes on some machine \( i \in \langle k \rangle \) at time \( t \). In order to execute \( v \) on \( i \) at this time, we may need to duplicate some of its predecessors. Note that only those predecessors that do not complete on any machine by time \( t - \hat{\rho}_k \) need to be duplicated on \( i \): any predecessors executed earlier are available to \( i \) by time \( t \), regardless of where they are executed. Let \( U \) be the set of predecessors of \( v \) executed on \( i \) after time \( t - \hat{\rho}_k \). The duplication constraints ensure that we can schedule \( U \) in at most \( \hat{\rho}_k \) time on \( i \). Intuitively, these constraints capture the load and chain length of the set \( U \), via Graham’s list scheduling theorem [16], give tight bounds for no-delay, uniform machine scheduling. Constraint (4) states that the total load that \( U \) places on group \( \langle k \rangle \) is no more than \( \hat{\rho}_k \). Constraints (5) and (6) together guarantee that the maximum chain length in \( U \) is at most \( \hat{\rho}_k \).

The remaining constraints enforce standard scheduling conditions. Constraint (7) states that the completion time of \( v \) is at least the completion time of any of its predecessors, and constraint (9) ensures that every job is executed on some group. Constraints (9) and (12) guarantee that \( z_{u,v,k} \leq 1 \) for all \( u, v, k \). This is an important feature of the LP, since a large \( z \)-value could be used to disproportionately reduce the delay between two jobs in constraint (3).
Lemma 3.2. \textit{(LP$_1$ is a valid relaxation)} The minimum of $D$ for LP$_1$ is $\leq (\Delta_\sigma + \rho_\sigma)$, for any schedule $\sigma$.

\textbf{Proof}. Consider an arbitrary schedule $\sigma$. We assume the schedule is \textit{minimal} in the sense that a job is executed at most once on any machine and no job is executed on any machine after the result of that job is available to the machine. It is easy to see that if $\sigma$ is not minimal, there exists a minimal schedule $\sigma'$ such that $\Delta_\sigma' \leq \Delta_\sigma$. We give a solution to the LP in which $D = \Delta_\sigma + \rho_\sigma$.

**LP solution.** For each job $v$, set $C_v$ to be the earliest completion time of $v$ in $\sigma$. Set $x_{u,v,k} = 1$ if $v$ first completes on a machine in group $\langle k \rangle$ (0 otherwise). For $u, v, k$, set $z_{u,v,k} = 1$ if $u \prec v$, $x_{v,k} = 1$, and $C_v - C_u < \bar{\rho}_k$ (0 otherwise). Set $y_{u,v} = \max_v \{z_{u,v,k}\}$. Fix a job $v$ and group $(k)$. Let $U = \{u : z_{u,v,k} = 1\}$. If $U = \emptyset$, then set $C_{u,v,k} = 0$ for all $u$. Otherwise, suppose $u \in U$. Then, by assignment of $z_{u,v,k}$, the earliest completion of $v$ is on some machine $i \in \langle k \rangle$. Since the schedule is minimal, we can infer that exactly one copy of $u$ is executed on $i$ as well. Let $t^*_u$ be the completion time of $u$ on $i$. Set $C_{u,v,k} = \bar{\rho}_k - (C_k - t^*_u)$. Now suppose $u \not\in U$. If there is no $u' \in U$ such that $u' \prec u$, then set $C_{u,v,k} = 0$. Otherwise, set $C_{u,v,k} = \max_{u' \prec u} \{C_{u,v,k}'\}$.

**Feasibility.** We now establish that the solution defined is feasible. Constraints (7 - 14) are easy to verify. We first establish constraints (1) and (2). Consider constraint (1) for fixed group $(k)$. Both $\sum_v x_{u,v,k}$ and $\sum_v y_{v,k}$ are upper bound by the total load $L$ on $(k)$. The constraint follows from $D \geq \Delta_\sigma \leq L/|\langle k \rangle| \cdot \hat{s}_k \bar{m}_k$. For fixed $v, k$, constraint (2) holds by assignment of $D$ and definition of $\rho_\sigma$.

Consider constraint (3) for fixed $u, v, k$ where $u \prec v$. Let $X = x_{u,v,k}$ and let $Z = z_{u,v,k}$. If $(X, Z) = (0, 0), (0, 1)$, or $(1, 1)$ then the constraint follows from constraint (7). Suppose $(X, Z) = (1, 0)$. By assignment of $z_{u,v,k}$ we can infer that $C_v - C_u \geq \bar{\rho}_k$, which shows the constraint is satisfied.

To show that constraints (4 - 6) are satisfied, consider fixed $v, k$ and let $U = \{u : z_{u,v,k} = 1\}$. If $v$ is not executed on any machine in $(k)$ (i.e. $x_{v,k} = 0$) then $U = \emptyset$ and the constraints are trivially satisfied by the fact that $C_{u,v,k} = 0$ for all $u$. So suppose that $x_{v,k} = 1$.

**Claim 3.2.1.** \textit{The completion times given by $\{C_{u,v,k}\}_{u \in U}$ give a valid schedule of $U$ on any machine in $(k)$.}

\textbf{Proof}. Suppose the schedule were not valid for a given machine $i \in \langle k \rangle$. By assignment of the $C_{u,v,k}$ variables, there are at most $\bar{m}_k$ jobs scheduled on $i$ in any time step. So the completion times fail to give a schedule only because there is some $u \prec u'$ such that $t^*_u \geq t^*_{u'}$. By the fact that the schedule is minimal, the result of $u$ is not available to $i$ by time $t^*_u$, and so is not available to $i$ by time $t^*_{u'}$. Therefore, the original schedule is not valid, which contradicts our supposition.

We proceed to show that (4 - 6) are satisfied using the same fixed $v, k$ and the same definition of $U$. First consider constraint (6) for fixed $u$. Let $u^* = \arg \max_{u \in U} \{C_{u,v,k}'\}$. By assignment, we have $C_{u,v,k} \geq C_{u^*,k}$. By definition, we have $C_v = t^*_u$ and $t^*_u > t^*_{u'}$. Therefore, $C_{u,v,k} \leq C_{u^*,k} = \bar{\rho}_k - (C_k - t^*_u) = \bar{\rho}_k - (t^*_v - t^*_{u'}) \leq \bar{\rho}_k$, which shows the constraint is satisfied. Consider constraint (4). Let $\tilde{D}$ be the optimal max completion time of scheduling just $U$ on a machine in $(k)$. Then $\sum_u z_{u,v,k} = |U| \leq \tilde{D} \cdot \hat{s}_k \bar{m}_k \leq \max_{u \in U} \{C_{u,v,k}'\} \cdot \hat{s}_k \bar{m}_k$ by the claim above. By constraint (6), we have that $\max_{u \in U} \{C_{u,v,k}'\} \leq \bar{\rho}_k$, which entails that the constraint is satisfied. Consider constraint (5) for fixed $u, u'$ such that $u' \prec u$. If $z_{u,v,k} = 0$, then the constraint holds by the fact that $C_{u,v,k} = \max_{u \prec u} \{C_{u,v,k}'\}$. If $z_{u,v,k} = 1$ then the constraint holds by the claim above.

**Definition 3.3.** $(C, D, x, y, z)$ is an integer solution to LP$_{\alpha}$ if all values of $x, y, z$ are either 0 or 1.

3.3 Deriving an integer solution to the linear program

Let LP$_1$ be defined over groups $\langle 1 \rangle, \langle 2 \rangle, \ldots, \langle K \rangle$. Given a solution $(\tilde{C}, \tilde{D}, \tilde{x}, \tilde{y}, \tilde{z})$ to LP$_1$, we construct an integer solution $(C, D, x, y, z)$ to LP$_{2K}$ as follows. For each $v, k$, set $x_{v,k} = 1$ if $k = \max_k \{\tilde{x}_{v,k}\}$ (if there is more than one maximizing $k$, arbitrarily select one); set to 0 otherwise. Set $z_{u,v,k} = 1$ if $x_{v,k} = 1$
and \( \hat{z}_{u,v,k} \geq 1/(2K) \); set to 0 otherwise. For all \( u, k \), \( y_{u,k} = \max_u \{ z_{u,v,k} \} \). Set \( C_v = 2K \cdot \hat{C}_v \). Set \( C_u^{i,k} = 2K \cdot \hat{C}_u^{i,k} \). Set \( D = 2K \cdot \hat{D} \).

**Lemma 3.4.** If \((\hat{C}, \hat{D}, \hat{x}, \hat{y}, \hat{z})\) is a valid solution to LP\(_1\), then \((C, D, x, y, z)\) is a valid solution to LP\(_{2K}\).

**Proof.** By constraint (9), \( \sum_k \hat{x}_{v,k} \) is at least 1, so \( \max_k \{ \hat{x}_{v,k} \} \) is at least 1/\( K \). Therefore, \( x_{v,k} \leq K \hat{x}_{v,k} \) for all \( v \) and \( k \). Also, \( z_{u,v,k} \leq 2K \hat{z}_{u,v,k} \) for any \( u, v, k \) by definition. By the setting of \( C_v \) for all \( v, C_u^{i,k} \) for all \( u, v, k, y_{u,k} \) for all \( v, k, D \), it follows that constraints (2, 4-14) of LP\(_1\) imply the respective constraints of LP\(_{2K}\). We first establish constraint (1). For any fixed group \( \langle k \rangle \),

\[
4K \hat{D} m_k s_k \cdot |\langle k \rangle| \geq 2K \sum_v (\hat{x}_{v,k} + \hat{y}_{v,k}) \geq \sum_v (\hat{x}_{v,k} + \max_u \{ \hat{z}_{v,u,k} \}) \geq \sum_v (x_{v,k} + y_{v,k}) \text{ by definition of } y_{v,k}
\]

which entails constraint (1) by \( D = 2K \hat{D} \). It remains to establish constraint (3) for fixed \( u, v, k \). We consider two cases. If \( \hat{x}_{v,k} < 1/K \), then \( x_{v,k} = 0 \), so the constraint is trivially true in LP\(_{2K}\). Otherwise, \( x_{v,k} - z_{u,v,k} \) equals \( 2 - z_{u,v,k} \), which is 1 only if \( x_{v,k} - z_{u,v,k} \) is at least \( 1/(2K) \). This establishes constraint (3) of LP\(_{2K}\) and completes the proof of the lemma.

**Lemma 3.5.** \( D \leq 24 \cdot K(\Delta_\sigma + \rho_\sigma) \) for any schedule \( \sigma \).

**Proof.** Lemma 3.1 shows that our grouping of machines does not increase the value of the LP by more than a factor of 12. Therefore, by Lemma 3.2, \( D = 2K \hat{D} \leq 24 \cdot K(\Delta_\sigma + \rho_\sigma) \) for any schedule \( \sigma \).

### 3.4 Computing a schedule given an integer solution to the LP

Suppose we are given a partition of \( M \) into \( K \) groups such that group \( \langle k \rangle \) is composed of identical machines (i.e., for all \( i, j \in \langle k \rangle \), \( s_i = s_j \), \( m_i = m_j \), and \( \rho_i = \rho_j \)). We are also given a valid integer solution \( (C, D, x, y, z) \) to LP\(_\sigma\) defined over \( \langle 1 \rangle, \ldots, \langle K \rangle \). In this section, we show that we can construct a schedule that achieves an approximation for MDPS in terms of \( \alpha \) and \( K \). The combinatorial subroutine that constructs the schedule is defined in Algorithm 1.

We define the Uniform Delay Precedence-constrained Scheduling (UDPS) problem to be identical to MDPS except that, for all \( i, j \), \( m_i = m_j \), \( s_i = s_j \), and \( \rho_i = \rho_j \). In Appendix A, we give an algorithm UDPS-Solver for UDPS.

---

**Algorithm 1:** Multiprocessor In-Delay Scheduling with Duplication

\[
\text{Init: } \forall v, i, \sigma(v, i) \leftarrow \bot; \ T \leftarrow 0
\]

1. while \( T \leq \max_v \{ C_v \} \) do
   
   2.   forall groups \( \langle k \rangle \) do
      
      3.       if for some integer \( d \), \( T = d \bar{\rho}_k \) then
         
         4.           \( V_{k,d} \leftarrow \{ v : x_{v,k} = 1 \text{ and } T \leq C_v < T + \bar{\rho}_k \} \)
         
         5.           \( U_{k,d} \leftarrow \{ u : \exists v \in V_{k,d}, u < v \text{ and } T \leq C_u < T + \bar{\rho}_k \} \)
         
         6.           \( \sigma' \leftarrow \text{UDPS-Solver on } (V_{k,d} \cup U_{k,d}, \langle k \rangle) \)
         
         7.           \( \forall v, i, \text{ if } \sigma'(v, i) \neq \bot \text{ then } \sigma(v, i) \leftarrow \sigma'(v, i) + \max_u \{ \sigma(u, j) \} + \bar{\rho}_k \)
      
      8.       \( T \leftarrow T + 1 \)
We now describe Algorithm 1 informally. The subroutine takes as input the integer variable assignments $(C, D, x, y, z)$ and initializes an empty schedule $\sigma$ and global parameter $T$ to 0. For a fixed value of $T$, we check each group $\langle k \rangle$ to see whether there is some integer $d$ such that $T = d\bar{\rho}_k$. If so, we define $V_{k,d}$ to be the set of jobs $v$ such that $x_{u,v} = 1$ and $T \leq C_v < T + \bar{\rho}_k$. We define $\bar{U}_{v,k}$ to be the set of jobs $u$ such that, for some job $v \in V_{k,d}$, we have $u \prec v$ and $T \leq C_u \leq t + \bar{\rho}_k$. We then run UDPS-Solver on jobs $V_{k,d} \cup \bar{U}_{k,d}$ using machines in $\langle k \rangle$ and append the resulting schedule to $\sigma$. Once all groups $\langle k \rangle$ have been checked, we increment $T$ and repeat until all jobs are scheduled. The structure of the schedule produced by Algorithm 1 is depicted in Figure 4.

**Lemma 3.6.** Let $U$ be a set of jobs such that for any $v \in U$ the number of predecessors of $v$ in $U$ (i.e., $|\{u \prec v \cap U\}|$) is at most $\alpha \bar{\rho}_k \bar{m}_k \bar{s}_k$, and the longest chain in $U$ has length at most $\alpha \bar{\rho}_k$. Then given as input the set $U$ of jobs and the set $\langle k \rangle$ of identical machines, UDPS-Solver produces, in polynomial time, a valid schedule with makespan at most $3 \log(\alpha \bar{\rho}_k \bar{m}_k \bar{s}_k) \bar{\rho}_k + \frac{2|U|}{\bar{m}_k \bar{s}_k}$.

For a proof of Lemma 3.6, see Appendix A.

**Lemma 3.7.** Algorithm 1 outputs a valid schedule in polynomial time.

**Proof.** It is easy to see that the algorithm runs in polynomial time, and Lemma 3.6 entails that precedence constraints are obeyed on each machine. Consider a fixed $v, k, d$ such that $v \in V_{k,d}$. By line 7, we insert a communication phase of length $\bar{\rho}_k$ before appending any schedule on group $\langle k \rangle$. So, by the time Algorithm 1 executes any job in $V_{k,d}$, every job $v$ such that $C_u < d\bar{\rho}_k$ is available to every machine in group $\langle k \rangle$. So the only predecessors of $v$ left to execute are those jobs in $\bar{U}_{k,d}$. Therefore, all communication constraints are satisfied.

**Claim 3.8.1.** For any $u, k, d$, if $u$ is in $U_{k,d}$ then for some $v$ in $V_{k,d}$ we have $z_{u,v,k} = 1$.

**Proof.** Fix a job $u$. If $u$ is in $U_{k,d}$, then by the definition of $U_{k,d}$, there exists a job $v$ in $V_{k,d}$ such that $u \prec v$ and $C_u$ is in $[T, T + \bar{\rho}_k)$, where $T = d\bar{\rho}_k$. By the definition of $V_{k,d}$, $x_{v,k}$ is 1 and $C_v$ is also in $[T, T + \bar{\rho}_k)$. Therefore, $C_v - C_u < \bar{\rho}_k$. By constraint (3), $C_v \geq C_u + \bar{\rho}_k(1 - z_{u,v,k})$, implying that $z_{u,v,k}$ cannot equal 0. Since $z_{u,v,k}$ is either 0 or 1, we have $z_{u,v,k} = 1$.

**Claim 3.8.2.** For any $v, k, d$, if $v \in V_{k,d}$ then $|\{u \prec v : u \in V_{k,d} \cup U_{k,d}\}|$ is at most $\alpha \bar{\rho}_k \bar{m}_k \bar{s}_k$ and the longest chain in $U_{k,d}$ has length at most $\alpha \bar{\rho}_k \bar{s}_k$.

**Proof.** Fix $v$ and consider any $u$ in $V_{k,d} \cup U_{k,d}$ such that $u \prec v$. By Claim 3.8.1, $z_{u,v,k} = 1$. Therefore, $|\{u \prec v : u \in V_{k,d} \cup U_{k,d}\}|$ equals the right-hand side of constraint (4), and hence at most $\alpha \bar{\rho}_k \bar{m}_k \bar{s}_k$. Let $u_1 \prec u_2 \prec \ldots \prec u_{\ell}$ denote a longest chain in $U_{k,d}$. By Claim 3.8.1, $z_{u_i,v,k}$ equals 1 for $1 \leq i \leq \ell$. By constraint (11), $C_{u_1}^{w,k} \geq 0$. By a repeated application of constraint (5), we obtain $C_{u_i}^{w,k} \leq \ell$. By constraint (6), $\ell \leq \alpha \bar{\rho}_k \bar{s}_k$, and hence the longest chain in $U_{k,d}$ has length at most $\alpha \bar{\rho}_k \bar{s}_k$.

**Claim 3.8.3.** For any $k$, $|\bigcup_d V_{k,d} \cup U_{k,d}| \leq D \cdot |\langle k \rangle| \cdot \bar{m}_k \bar{s}_k$.

**Proof.** Fix a $k$. For any $v$ in $V_{k,d}$ we have $x_{v,k} = 1$ by the definition of $V_{k,d}$. Consider any $u$ in $U_{k,d}$. By Claim 3.8.1, there exists a $v$ in $V_{k,d}$ such that $z_{u,v,k}$ equals 1; fix such a $v$. By constraint (13), $y_{u,k} = 1$. Thus, $|\bigcup_d V_{k,d} \cup U_{k,d}|$ is at most the right-hand side of constraint (1), which is at most $D \cdot |\langle k \rangle| \cdot \bar{m}_k \bar{s}_k$. \qed
So, by Lemma 3.6 and Claim 3.8.2, the total time spent executing jobs on a single group is upper bounded by

$$\sum_d \left( 3\log(\alpha \bar{\rho}_k \bar{m}_k \hat{s}_k) \alpha \bar{\rho}_k + \frac{2 \cdot |V_{k,d} \cup U_{k,d}|}{|\langle k \rangle| \cdot \bar{m}_k \hat{s}_k} \right) = 3D \alpha \bar{\rho}_k \log(\alpha \bar{\rho}_k \bar{m}_k \hat{s}_k) = \frac{2 \sum_d |V_{k,d} \cup U_{k,d}|}{|\langle k \rangle| \cdot \bar{m}_k \hat{s}_k}$$

where the second line follows by Claim 3.8.3 and the fact that $D/\bar{\rho}_k \geq 1$. Summing over all $k$, the total time in the schedule is at most $5KD\alpha \log(\alpha \rho m)$. The lemma follows from the fact that $D \leq \beta \cdot (\Delta_\sigma + \rho_\sigma)$ and $\bar{\rho}_k \bar{m}_k \hat{s}_k \leq n^3$.

**Theorem 1.** There exists a polynomial time algorithm that produces a schedule with makespan $O(K^3 \log n \cdot (\Delta_\sigma + \rho_\sigma))$ for any schedule $\sigma$.

**Proof.** Lemma 3.4 entails that $(C, D, x, y, z)$ is a valid solution to LP$_{2K}$. Lemma 3.5 entails that $D \leq 24 \cdot K(\Delta_\sigma + \rho_\sigma)$. With $\alpha = 2K$ and $\beta = 24K$, Lemma 3.8 entails that the makespan of our schedule is at most $15K \cdot 2K \cdot 24K \cdot \log(2Kn) \cdot (\Delta_\sigma + \rho_\sigma)$. 

For the most general case with variable delays, sizes, and speeds, we have $K = (\log n)^3$ groups, yielding an $O((\log n)^3 \cdot (\Delta_\sigma + \rho_\sigma))$ approximation for any schedule $\sigma$. For special cases in which any one parameter is uniform across all machines, we have $K = (\log n)^2$ groups, yielding an $O((\log n)^2 \cdot (\Delta_\sigma + \rho_\sigma))$ approximation for any schedule $\sigma$. For special cases in which any two parameters are uniform across all machines, we have $K = (\log n)$ groups, yielding an $O((\log n)^4 \cdot (\Delta_\sigma + \rho_\sigma))$ approximation for any schedule $\sigma$.

### 4 Approximation algorithms for no-duplication schedules

In this section, we consider no-duplication schedules and prove Theorem 2 in Section 4.1 and Theorem 3 in Section 4.2.
4.1 Machine-dependent delays

Our algorithm for no-duplication schedules first runs Algorithm 1 for the problem, which produces a schedule (with possible duplications) having a structure illustrated in Figure 4. By line 7 of Algorithm 1, the schedule $\sigma_{k,d}$ for each $V_{k,d}$ includes a delay of $\hat{\rho}_k$ at the start. We next convert the schedule $\sigma_{k,d}$, for each $V_{k,d}$, to a no-duplication schedule using the following lemma from [28].

**Lemma 4.1** (Restatement of Theorem 3 from [28]). Given a UDPS instance where the delay is $\rho$ and a schedule of length $D \geq \rho$, there is a polynomial time algorithm that computes a no-duplication schedule for the instance with length $O((\log m)(\log n)^2 D)$.

If the no-duplication schedule thus computed for $V_{k,d}$ does not have an initial delay $\hat{\rho}_k$, then we introduce a delay of $\hat{\rho}_k$ at the start of the schedule. Let $\hat{\sigma}_{v,k}$ denote the no-duplication schedule thus computed for $V_{k,d}$. We concatenate the $\hat{\sigma}_{v,k}$ in the order specified in Figure 4. By construction, there is no duplication of jobs within any $\hat{\sigma}_{v,k}$. To ensure that the final schedule has no duplication, we keep the first occurrence of each job in the schedule, and prune all duplicate occurrences. We now show that no precedence constraints are violated.

**Lemma 4.2.** In the above algorithm, all precedence constraints are satisfied.

**Proof.** Suppose $u \prec v$. Pruning of a duplicate copy of $u$ can only cause a precedence violation if the copy of $u$ that $v$ waits for is pruned, so we assume $u$ is pruned. For each copy of $v$ scheduled on some machine $i$, either some copy of $u$ is also scheduled on $i$, or it needs to wait $\rho_i$ time to wait for those completed on other machines to be transmitted. In the former case, $u$ will only be pruned if it is not the first occurrence. This can only happen if the first occurrence of $u$ is in an earlier $V_{k,d}$, since our subroutine guarantees that there is no duplication within a $V_{k,d}$. Due to our in-delay model, as long as a later occurrence of $u$ can be transmitted in time, so can all its previous occurrences. Therefore a copy of $u$ will be available to $v$. In the latter case, either $u$ is within the same $V_{k,d}$, or is in a previous one. In either case, the earlier occurrences of $u$ have to be in earlier $V_{k,d}$s, and will be available to $v$. So the pruning of this copy of $u$ will not cause a precedence violation.

**Theorem 2.** There exists a polynomial time algorithm that produces a no-duplication schedule whose makespan is at most $\text{polylog}(n)(\Delta_{\sigma} + \rho_{\sigma})$) for any schedule $\sigma$.

**Proof.** Lemma 4.2 establishes the feasibility of the schedule. By definition, each job is scheduled exactly once. It remains to bound the approximation ratio. Let $\tau_{k,d}$ be the makespan of the schedule of $V_{k,d}$ given by the subroutine we use on line 6 of Algorithm 1, and $\tau'_{k,d}$ be the makespan of the de-duplicated version. Then Lemma 4.1 shows that $\tau'_{k,d} \leq O((\log n_{k,d})^2(\log m)(\tau_{k,d} + \rho_k))$, where $n_{k,d}$ is the number of jobs in $V_{k,d}$, which is upper bounded by $n$. There are at most $n$ jobs to schedule, so having more than $n$ machines does not help us, therefore we can assume $m \leq n$ without loss of generality. Algorithm 2 guarantees that there is at least an in-delay at the start of $\tau_{k,d}$, so $\tau_{k,d} \geq \rho_k$, therefore $\tau'_{k,d} \leq O((\log n_{k,d})^3 \tau_{k,d})$. Therefore, the total makespan

$$\sum_{k,d} \tau'_{k,d} \leq \sum_{k,d} O((\log n)^3 \tau_{k,d}) \leq O((\log n)^3) \cdot \sum_{k,d} \tau_{k,d} = O(\text{polylog}(n)\Delta_{\sigma} + \rho_{\sigma})$$

**4.2 Symmetric delays**

In this section, we establish a true polylogarithmic approximation for the no-duplication model if the delays are symmetric, i.e., $\rho_i^\text{in} = \rho_i^\text{out}$. The additive term of delay in Theorems 1 and 2 comes from Algorithm 2,
Lemma 4.3. There exists a polynomial time algorithm such that given \( T \), either gives a schedule of length \( O(\text{polylog}(n)T) \), or asserts that \( \text{OPT} > T \).

Let \( S \) (resp., \( S' \)) be the set of machines that have delay at most (resp., larger than) \( T \). Let \( G \) denote the DAG defined by the set of jobs and its precedence constraints. Let \( G' \) denote the undirected graph obtained from \( G \) by removing directions from all edges. Under symmetric delays, we can afford no communication in either direction with machines in \( S' \), so in the absence of duplication both \( \text{OPT} \) and our algorithm will only schedule full connected components of \( G' \) on \( S' \). A priori, it is unclear how to distribute the connected components among \( S \) and \( S' \). We revise the LP framework of Section 3 so that it guides this distribution while yielding a fractional placement without duplication on \( S' \) and a fractional placement with duplication on \( S \). We then show how to design a full no-duplication schedule from these fractional placements.

For each connected component \( d \) and each machine \( i \in S' \), we define an LP variable \( X_{d,i} \), indicating whether the component is scheduled on machine \( i \). Therefore for the completion of jobs, we replace constraint (9) with

\[
\sum_k x_{v,k} + \sum_{i \in S'} X_{d,i} = 1, \quad \forall d, \forall v \in d
\]  

(15)

Use \( w(d) \) to denote the total number of jobs in connected component \( d \), and \( L(d) \) the length of its critical path. We also need to add the following constraints:

\[
D \geq \sum_d X_{d,i} \cdot w(d) \frac{1}{m_i s_i}, \quad \forall i
\]

(16)

\[
0 \leq X_{d,i} \leq 1, \quad \forall d, i
\]

(17)

\[
X_{d,i} = 0, \quad \forall d, \text{ if } l(d) > s_i \cdot T
\]

(18)

\[
X_{d,i} = 0, \quad \forall d, \text{ if } w(d) > m_i \cdot s_i \cdot T
\]

(19)

After solving the LP, if \( D > T \), we know \( \text{OPT} \) cannot have a schedule less than \( T \), since a no-duplication schedule is also a valid with-duplication schedule, and satisfies the revised LP. If \( D \leq T \), we proceed to round and give a schedule. Let \( \delta_d = \sum_i X_{d,i} \). For all component \( d \) such that \( \delta_d \geq \frac{1}{2} \), we assign them to the long delay machines \( S' \). To be more precise, we increase their \( X_{d,i} \) variables by a factor of \( 1/\delta_d \), and set the corresponding \( x_{v,k}, z_{u,v,k} \) to be zero. Since \( \delta_d \geq 1/2 \), we have the corresponding constraints.

\[
\sum_{i \in S'} X_{d,i} = 1, \quad \forall d
\]

(20)

\[
2D \geq \sum_d X_{d,i} \cdot \frac{w(d)}{m_i s_i}, \quad \forall i
\]

(21)

For all other components with \( \delta_d < \frac{1}{2} \), set their \( X_{d,i} = 0 \). For every job \( v \) in component \( d \), scale its corresponding \( x_{v,k}, y_{v,k}, z_{u,v,k}, C_v, C_u, k \) up, by a factor of \( 1/(1 - \delta_d) \). In addition, we also replace \( \alpha \) with \( 2\alpha \), \( D \) with \( 2D \), since \( 2 \) is an upper bound of \( 1/(1 - \delta_d) \). It is not hard to check that such that all constraints (1) to (8) and (10) to (14) holds, and constraint (15) degrades to constraint (9). We have thus established that \( (C, D, X, x, y, z) \) as defined above satisfies the LP.
Now we need to further the LP variables into a solution. For jobs assigned to $S$, we use the algorithm of Section 4.1, and obtain a schedule of length $\text{polylog}(n)T$. So we only need to round the non-zero $X_{d,i}$ into an integral solution and get a schedule of connected components (and their jobs) on $S'$ within time polylog of $T$, and we actually upper bound it with $2T + 2D = O(T)$.

We first focus on the load $w(d)$. Viewing each component as a job, the problem of assigning components to machines is exactly minimizing makespan while scheduling general length jobs on related machines. For this problem, we use a classic result due to Lenstra, Shmoys, and Tardos, restated here for completeness [22].

**Lemma 4.4 (Restatement of Theorem 1 from [22]).** Let $P = (p_{i,j}) \in \mathbb{Z}_+^{m \times n}, (d_1, \ldots, d_m) \in \mathbb{Z}_+^m$, and $t \in \mathbb{Z}_+$. Let $J_i(t)$ denote the set of jobs that require no more than $t$ time units on machine $i$, and let $M_j(t)$ denote the set of machines that can process job $j$ in no more than $t$ time units. Consider a decision version of scheduling problem where for each machine $i$ there is a deadline $d_i$ and where we are further constrained to schedule jobs so that each uses processing time at most $t$; we wish to decide if there is a feasible schedule.

If the linear program

$$
\sum_{i \in M_j(t)} x_{ij} = 1, \quad j = 1, \ldots, n
$$

$$
\sum_{j \in J_i(t)} p_{ij} x_{ij} \leq d_i, \quad i = 1, \ldots, m
$$

$$
x_{ij} \geq 0, \quad \forall j \in J_i(t), i = 1, \ldots, m
$$

has a feasible solution, then any vertex $\bar{x}$ of this polytope can be rounded to a feasible solution $\bar{x}$ of the integer program

$$
\sum_{i \in M_j(t)} x_{ij} = 1, \quad j = 1, \ldots, n
$$

$$
\sum_{j \in J_i(t)} p_{ij} x_{ij} \leq d_i + t, \quad i = 1, \ldots, m
$$

$$
x_{ij} \in \{0, 1\}, \quad \forall j \in J_i(t), i = 1, \ldots, m
$$

It is not hard to see that if we view components and items, and set $p_{i,d} = w_d/(m_is_i)$, the constraints are exactly constraints (20) and (21). Therefore we can round them into an integer solution such that,

$$
\sum_d X_{d,i} \cdot w(d) m_is_i \leq T + \frac{1}{\delta_d} \cdot D, \quad \forall i
$$

(22)

In other words, for machine $i \in S'$, the total number of jobs on it is at most $m_is_i(T + 2D)$. Since each DAG has critical path length $T$, using Graham’s list scheduling, we can find a schedule for machine $i$ in time $\frac{m_is_i(T + 2D)}{m_is_i} + T = 2T + 2D$, completing the proof of Lemma 4.3 and Theorem 3.

## A Algorithm for Uniform Machines

In this section, we show the existence of an algorithm which achieves provably good bounds for scheduling on uniform machines with fixed communication delay. The algorithm presented here is a generalization of the algorithm in [23], the main difference being our incorporation of parallel processors for each machine.

Recall that UDPS is identical to MDPS with $\rho_i^{\text{out}} = 0$ for all $i$, and with $\rho_i = \rho_j$, $m_i = m_j$, and $s_i = s_j$ for all $i, j$. We define UDPS-Solver as Algorithm 2. The algorithm takes as input a set of jobs $U$ and a group $\langle k \rangle$ of identical machines with delay $\tilde{\rho}_k$, speed $\tilde{s}_k$, and size $\tilde{m}_k$. We are guaranteed that the length of the longest chain in $U$ is at most $\alpha\tilde{\rho}_k\tilde{s}_k$ and that, for any job $v \in U$, the number of jobs $u \in U$ such that $u \prec v$ is at most $\alpha\tilde{\rho}_k\tilde{m}_k\tilde{s}_k$. 

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Algorithm 2: UDPS-Solver

Data: set of jobs $U$, set of identical machines $\langle k \rangle$
Init: $t \leftarrow 0$

1 while $U \neq \emptyset$ do
2 \hspace{1em} \forall i \in \langle k \rangle, V_{i,t} \leftarrow \emptyset
3 \hspace{1em} forall jobs $v \in U$ do
4 \hspace{2em} $U_{v,t} \leftarrow \{ u \in U : u < v \}$
5 \hspace{2em} $U_{v,t}^{\text{dup}} \leftarrow U_{v,t} \cap \{ u \in U : u < v \text{ and } \exists i, u \in V_{i,t} \}$
6 \hspace{2em} if $|U_{v,t}| \geq 2 \cdot |U_{v,t}^{\text{dup}}|$ then
7 \hspace{3em} $i \leftarrow \arg \min_{j \in \langle k \rangle} \{|V_{j,t}|\}$
8 \hspace{3em} $V_{i,t} \leftarrow V_{i,t} \cup U_{v,t}$
9 \hspace{1em} \forall $i \in \langle k \rangle$, list schedule $V_{i,t}$ on $i$ starting at time $t$
10 \hspace{1em} $U \leftarrow U \setminus (\bigcup_i V_{i,t})$
11 \hspace{1em} $t \leftarrow \bar{\rho}_k + \max \{ t, \max_v \{ \text{completion time of } v \} \}$

Lemma A.1. Algorithm 2 produces a valid schedule of $U$ on $\langle k \rangle$ in polynomial time.

Proof. For fixed $i, t$, the list scheduling subroutine guarantees that precedence constraints are maintained in placing $V_{i,t}$. So we must only show that precedence and communication constraints are obeyed across different values of $t$. For a fixed value $t^*$, line 10 entails that all jobs in $\bigcup_i V_{i,t}$ for $t < t^*$ have been removed from $U$. This entails that all precedence constraints are obeyed in scheduling $V_{i,t^*}$. The update at line 11 ensure that no jobs are executed for $\bar{\rho}_k$ before scheduling $V_{i,t^*}$. Since there is no communication necessary for scheduling $\bigcup_i V_{i,t^*}$, this entails that communication constraints are obeyed.

Lemma A.2. For any $t$, $\sum_i |V_{i,t}| \leq 2 \cdot |\bigcup_i V_{i,t}|$.

Proof. Consider a fixed value of $t$. For any machine $i$, let $V_{i,t,0}$ be the value of $V_{i,t}$ after initialization at line 2. Also, for all machines $i$, let $V_{i,t,\ell}$ be the value of $V_{i,t}$ after the $\ell$th time some set $V_{j,t}$ is updated. We prove the lemma by induction on $\ell$.

The lemma holds trivially for $\ell = 0$ since $\sum_i |V_{i,t,0}| = |\bigcup_i V_{i,t,0}| = 0$. Suppose the lemma holds up until the $\ell$th update of any $V_{i,t}$, and consider the $(\ell+1)$th update. Let $U'$ be the set of jobs placed during the update. The condition at line 6 entails that at least half the jobs in $U$ are not in $\bigcup_i V_{i,t,\ell}$. Therefore,

$$|\bigcup_i V_{i,t,\ell+1}| = |U \cup \bigcup_i V_{i,t,\ell}| = |U| + |\bigcup_i V_{i,t,\ell}| - |\bigcup_i V_{i,t,\ell}|$$

$$\geq |U| + \frac{|U|}{2} - \frac{|U|}{2} \quad \text{by supposition}$$

$$\geq \frac{|U|}{2} + \frac{\sum_i |V_{i,t,\ell}|}{2} \geq \frac{1}{2} \sum_i |V_{i,t,\ell+1}| \quad \text{by IH}$$

This proves the lemma.

Lemma A.3. If $t_1$ and $t_2$ are two consecutive values of $T$, then $t_2 \leq t_1 + \frac{2 \cdot |\bigcup_i V_{i,t}|}{|\langle k \rangle| \cdot m_k s_k} + 3 \alpha \bar{\rho}_k$.

Proof. Consider a fixed value of $t$. If there are no jobs added to $V_{i,t}$ for any $i$, then $t_2 = t_1 + \bar{\rho}_k$ by line 11 and the lemma holds. So suppose that $\max_v \{ \text{completion time of } v \} > t_1$ on execution of line 11. At the
where the last line follows from the fact that

\[ \bigcup_i \{ \text{completion time of } v \text{ on } i^* \} - t_1 \]

by line 11

\[ \leq \hat{\rho}_k + t_1 + \frac{|V_{i,t}|}{m_k s_k} \]

by Graham [16]

\[ \leq \hat{\rho}_k + t_1 + \frac{|V_{i,t}|}{s_k \bar{m}} + \alpha \hat{\rho}_k \]

by supposition

\[ \leq \hat{\rho}_k + t_1 + \min_i \left( \frac{|V_{i,t}|}{m_k s_k} \right) + \frac{\alpha \hat{\rho}_k m_k s_k}{m_k s_k} + \alpha \hat{\rho}_k \]

by supposition and line 7

\[ \leq t_1 + \frac{\sum_i |V_{i,t}|}{|\langle k \rangle| \cdot m_k s_k} + \hat{\rho}_k (2\alpha + 1) \leq t_1 + \frac{2 \cdot \sum_i |V_{i,t}|}{|\langle k \rangle| \cdot m_k s_k} + \hat{\rho}_k (2\alpha + 1) \]

by Lemma A.2

The appeal to line 7 invokes the fact that the algorithm always chooses the least loaded machine when placing new jobs. Since every job has a maximum of \( \alpha \hat{\rho}_k m_k s_k \) predecessors in \( U \), by supposition, the difference \( |V_{i,t}| - |V_{j,t}| \) is always maintained to be less than \( \alpha \hat{\rho}_k m_k s_k \), for any \( i, j \). This proves the lemma. \( \square \)

**Lemma A.4 (Restatement of Lemma 3.6).** Let \( U \) be a set of jobs such that for any \( v \in U \) the number of predecessors of \( v \) in \( U \) (i.e., \(|\{ u \prec v \} \cap U\| \)) is at most \( \alpha \hat{\rho}_k m_k s_k \), and the longest chain in \( U \) has length at most \( \alpha \hat{\rho}_k \). Then given as input the set \( U \) of jobs and the set \( \langle k \rangle \) of identical machines, UDP-Solver produces, in polynomial time, a valid schedule with makespan less than \( 3 \log(\alpha \hat{\rho}_k m_k s_k) + 2 |\langle k \rangle| \frac{|V_{i,t}|}{m_k s_k} + \hat{\rho}_k \).

**Proof.** By the condition of line 6, each time the variable \( t \) increments we can infer that all unscheduled jobs have their number of unscheduled predecessors halved. By our supposition, this entails \( t \) can be incremented only \( \log(\alpha \hat{\rho}_k m_k s_k) \) times before all jobs are scheduled. Let \( t_\ell \) be the \( \ell \)th value of \( t \), for \( \ell = 0, 1, 2, \ldots, \log(\alpha \hat{\rho}_k m_k) \).

**Claim A.4.1.** For any \( \ell \), we have \( t_{\ell+1} \leq 3\ell \alpha \hat{\rho}_k + \frac{2 \cdot \sum_i |V_{i,t_\ell}|}{|\langle k \rangle| \cdot m_k} + \hat{\rho}_k \)

**Proof.** We prove the claim by induction on \( \ell \). For \( \ell = 0 \), the claim holds since the initial value of \( t \) is 0. Suppose the claim holds up to \( \ell \). Then

\[ t_{\ell+1} \leq t_\ell + \frac{2 \cdot \sum_i |V_{i,t_\ell}|}{|\langle k \rangle| \cdot m_k} + 3\alpha \hat{\rho}_k \]

by Lemma A.3

\[ \leq 3\ell \alpha \hat{\rho}_k + \frac{2 \cdot \sum_i |V_{i,t_\ell}|}{|\langle k \rangle| \cdot m_k} + 2 \cdot \frac{2 \cdot \sum_i |V_{i,t_\ell}|}{|\langle k \rangle| \cdot m_k} + 3\alpha \hat{\rho}_k \]

by I.H.

\[ \leq 3\alpha \hat{\rho}_k (\ell + 1) + \frac{2 \cdot \sum_i |V_{i,t_\ell}|}{|\langle k \rangle| \cdot m_k} \]

where the last line follows from the fact that \( \bigcup_i V_{i,t_\ell} \) and \( \bigcup_i V_{i,t_{\ell-1}} \) have no members in common. \( \square \)

The claim is sufficient to establish the lemma. \( \square \)

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