SCHWARZ TYPE TOPOLOGICAL QUANTUM FIELD THEORIES

R. K. Kaul\(^\ast\), T. R. Govindarajan\(^\dagger\)
The Institute of Mathematical Sciences,
Chennai 600 113, India,

P. Ramadevi\(^\ddagger\)
Department of Physics, I I T Bombay,
Powai, Mumbai 400 076, India

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1 Introduction

Topological quantum field theories (TQFT) provide powerful tools to probe topology of manifolds, specifically in low dimensions\([1, 2, 3, 4, 5]\). This is achieved by incorporating very large gauge symmetries in the theory which lead to gauge invariant sectors with only topological degrees of freedom. These theories are of two types: (a) **Schwarz type** or **Chern-Simons type** and (b) **Witten type** or **Cohomological type**.

In Witten type topological field theories, action is a BRST exact form, so is the stress energy tensor \(T_{\mu\nu}\) so that their functional averages are zero \([4]\). The topological observables in these theories form cohomological classes. In four dimensions, such theories involving Yang-Mills gauge fields provide a field theoretic representation for Donaldson invariants.

On the other hand, Schwarz type TQFTs are described by local action functionals which are *explicitly* independent of metric \([2, 3]\). The examples of such theories are (i) Chern-Simons (CS) theories and (ii) BF theories.

The metric independence of the action \(S\) implies that stress energy tensor of a TQFT is zero: \(\frac{\delta S}{\delta g_{\mu\nu}} \equiv T_{\mu\nu} = 0\). There are no local propagating degrees of freedom; only degrees of freedom are topological. Expectation

\(\ast\)kaul@imsc.res.in
\(\dagger\)trg@imsc.res.in
\(\ddagger\)ramadevi@phy.iitb.ac.in
values of metric independent operators $W$ are also independent of the metric:

\[ \delta g_{\mu\nu} = 0. \]

Three dimensional Chern-Simons theories are of particular interest, for these provide a framework for the study of knots and links in any three-manifold. It was A.S. Schwarz who first conjectured [2] that the well known Jones polynomial may be related to Chern-Simons theory. In his famous paper, E. Witten[3] not only demonstrated this connection, but also set up a general field theoretic framework to study topological properties of knots and links in any arbitrary three-manifold. In addition this framework provides a method of obtaining some new manifold invariants. Chern-Simons theory is also known to describe gravity in three dimensional spacetime [6].

$BF$ theories in three dimensions also provide a field theoretic description of topological properties of knots and links. These theories with bilinear action in fields in addition can be defined in higher dimensions. In particular in $D = 4$, $BF$ theory, besides describing two-dimensional generalizations of knots and links, also provides a field theoretic interpretation of Donaldson invariants. This provides a connection of these theories with Witten-type TQFTs of Yang-Mills gauge fields.

Chern-Simons theories in three complex dimensions described in terms of holomorphic 1-forms have also been constructed. Such a theory on Calabi-Yau spaces can also be interpreted as a string theory in terms of a Witten-type topological field theory of a sigma model coupled to gravity[7]. The observables in this framework are the cohomological classes on the moduli space of Riemann surfaces. Such holomorphic generalizations of BF theories have also been developed [8].

In the following we shall survey three dimensional Chern-Simons theory as a description of knots/links, indicate how manifold invariants can be constructed from invariants for framed links, and also its application to three dimensional gravity. This will be followed by a brief discussion of BF theories in three and four dimensions.

2 Three-dimensional Chern-Simons theory with gauge group $U(1)$

The simplest Schwarz type topological field theory is the $U(1)$ Chern-Simons theory described by the action:

\[ S_{CS} = -\frac{1}{8\pi} \int_{\mathcal{M}} A \, dA \quad (1) \]

where $A$ is a connection one-form $A = A_\mu \, dx^\mu$ and $\mathcal{M}$ is the three manifold, which we shall take to be $S^3$ for the discussion below. The action has no dependence on the metric. Besides being the $U(1)$ gauge invariant, it is also general coordinate invariant.
In quantum CS field theory, we are interested in the functional averages of the gauge invariant and metric independent functionals $W[A]$: 

$$\langle W[A] \rangle = \frac{1}{Z} \int [DA] W[A] \exp \{ikS_{CS}\}, \quad Z = \int [DA] \exp \{ikS_{CS}\}$$ (2)

This theory captures some of the simple, but interesting, topological properties of knots and links in three dimensions. For a knot $K$, we associate a knot operator $\oint_K A$ which is gauge invariant and also does not depend on the metric of the three-manifold. Then for a link made of two knots $K_1$ and $K_2$, we have the loop correlation function $\langle \oint_{K_1} A \oint_{K_2} A \rangle$ which can be evaluated in terms of two-point correlator $\langle A_\mu(x)A_\nu(y) \rangle$ in $R^3$ (with flat metric). This correlator in Lorentz gauge ($\partial_\mu A^\mu = 0$) is:

$$\langle A_\mu(x)A_\nu(y) \rangle = \frac{i}{k} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3}$$ (3)

so that for two distinct knots $K_1$ and $K_2$

$$\langle \oint_{K_1} A \oint_{K_2} A \rangle = \frac{4\pi i}{k} \mathcal{L}(K_1, K_2)$$ (4)

where

$$\mathcal{L}(K_1, K_2) = \frac{1}{4\pi} \oint_{K_1} \oint_{K_2} dx^\mu dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3}$$ (5)

This integral is the well known topological invariant called Gauss linking number[9] of two distinct closed curves. It is an integer measuring the number of times one knot $K_1$ goes through the other knot $K_2$. Linking number does not depend on the location, size or shape of the knots. In electrodynamics, it has the physical interpretation of work done to move a monopole around a knot while electric current runs through the other knot[10].

Abelian CS theory also provides a field theoretic representation for another topological quantity called self-linking number, also known as framing number, of the knot. It is related to the functional average of $\langle \oint_K A \oint_K A \rangle$ where the two loop integrals are over the same knot. The coincidence singularity is avoided by a topological loop-splitting regularization. For a knot $K$ given by $x^\mu(s)$ parametrized along the length of the knot by $s$, we associate another closed curve $K_f$ given by $y^\mu(s) = x^\mu(s) + \epsilon n^\mu(s)$ where $\epsilon$ is a small parameter and $n^\mu(s)$ is a principal normal to the curve at $s$. The coincidence limit is then obtained at the end by taking the limit $\epsilon \to 0$. Such a limiting procedure is called framing and knot $K_f$ is the frame of knot $K$. The linking number of the knot $K$ and its frame $K_f$ is the self-linking number of the knot:

$$SL(K, n^\mu) = \frac{1}{4\pi} \oint y^\nu = x^\nu + \epsilon n^\nu \ d x^\mu \ d y^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3}$$ (6)
Hence coincidence two loop correlator is

\[
\langle \oint_K A \oint_K A \rangle = \frac{4\pi i}{k} S\mathcal{L}(K, n^\mu) 
\]

(7)

Notice that the self-linking number of a knot is independent of the regularization parameter \(\epsilon\), but does depend on the topological character of the normal vector field \(n^\mu(s)\). It is also related to two geometric quantities called twist \(T(K)\) and writhe \(w(K)\) through a theorem due to Calugareanu:

\[
S\mathcal{L}(K) = T(K) + \omega(K) 
\]

(8)

where

\[
T(K) = \frac{1}{2\pi} \oint_K ds \varepsilon_{\mu\nu\rho} \frac{dx^\mu}{ds} n^\nu \frac{dx^\rho}{ds} 
\]

(9)

\[
\omega(K) = \frac{1}{4\pi} \oint_K dt \oint_K ds \varepsilon_{\mu\nu\rho} \frac{de^\mu}{ds} \frac{de^\nu}{dt} e^\rho 
\]

(10)

Here

\[
e^\mu(s, t) = \frac{y^\mu(t) - y^\mu(s)}{|y(t) - y(s)|}
\]

is a unit map from \(K \otimes K \rightarrow S^2\) and \(n^\mu(s)\) is a normal unit vector field. \(T(K)\) and \(\omega(K)\) are not in general integers and represent the amount of twist and coiling of the knot. These are not topological invariants but their sum, self-linking number, is indeed always an integer and a topological invariant. This result has found interesting application in the studies of the action of enzymes on circular DNA[11].

3 Non-Abelian Chern-Simons theories

Non-Abelian CS theories provide far more information about the topological properties of the manifolds as well as knots and links in them.

Non-abelian CS theory in a three manifold \(\mathcal{M}\) (which as in last section we take to be \(S^3\)) is described by the action functional

\[
S_{CS} = \frac{1}{4\pi} \int_{\mathcal{M}} \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) 
\]

(11)

where \(A\) is a gauge field one-form which takes its value in the Lie algebra \(\mathcal{L}\mathcal{G}\) of a compact semi-simple Lie group \(\mathcal{G}\). For example, we may take this group to be \(SU(N)\) and \(A = A^a T^a\) where \(T^a\) is the fundamental \(N\) dimensional representation with \(\text{tr} T^a T^b = -1/2 \delta^{ab}\). As before the action functional is metric independent and general coordinate invariant. Under homotopically non-trivial gauge transformations this action is not
invariant, but changes by an amount $2\pi n$ where integers $n$ are the winding numbers characterizing the gauge transformations which fall in homotopic classes given by $\Pi_3(\mathcal{G}) = \mathbb{Z}$ for a compact semi-simple group $\mathcal{G}$. (For a general manifold $\mathcal{M}$, this change in action is characterised by the homotopic class of maps $\mathcal{M} \rightarrow \mathcal{G}$.) However, for quantum theory what is relevant is $\exp[ikS_{CS}]$ which is invariant even under homotopically non-trivial gauge transformations provided the coupling $k$ takes integer values $[12]$. So for integer $k$, the quantum field theory we discuss here is gauge invariant.

The topological operators are the Wilson loop operators for an oriented knot $K$:

$$W_R[K] = tr P \exp \oint_K A_R$$

where $A_R = A^a T^a_R$ with $T^a_R$ as the representation matrices of a finite dimensional representation $R$ of the $\mathcal{L}\mathcal{G}$. $P$ stands for the path ordering of the exponential. The observable Wilson link operator for a link $L = \bigcup_1^n K_i$, carrying representations $R_i$ on the respective component knots, is

$$W_{R_1R_2\ldots R_n}[L] = \prod_1^n W_{R_i}[K_i]$$

Expectation values of these operators are the functional averages:

$$V_{R_1,R_2\ldots R_n}[L] = \left[ \frac{\int [DA] W_{R_1R_2\ldots R_n}[L] \exp[ikS_{CS}]}{\int [DA] \exp[ikS_{CS}]} \right]$$

The measure $[DA]$ has to be metric independent. These expectation values depend only on the isotopy of the link $L$ and also on the set of the representations $\{R_i\}$. These can be evaluated in principle nonperturbatively. For example when $\mathcal{L}\mathcal{G} = su(N)$ and each the component knot of the links carries the fundamental $N$ dimensional representation, the Wilson link expectation values satisfy a recursion relation involving three link diagrams which are identical except for one crossing where they differ as over crossing ($L_+$), under crossing ($L_-$) and no crossing ($L_0$) as shown in the figure below.

\[
\begin{align*}
W_\mathcal{L} & \quad W_\mathcal{L}_0 \quad W_\mathcal{L}_-
\end{align*}
\]

The expectation values of these links are related as $[3]$:

$$q^{\frac{N}{2}} V_N[L_+] - q^{-\frac{N}{2}} V_N[L_-] = \left( q^{\frac{1}{2}} - q^{-\frac{1}{2}} \right) V_N[L_0]$$

where $q = \exp \left( \frac{2\pi i}{k + \frac{1}{2}} \right)$. This is precisely the well known skein relation for the HOMFLY polynomial. The famous Jones one-variable polynomial
(whose two-variable generalization is the HOMFLY polynomial), correspond
to the case of spin-1/2 representation of SU(2) CS theory:

\[ V_2[L] = \text{Jones Polynomial}[L] \] (16)

upto an overall normalization. These skein relations are sufficient to find
recursively all the expectation values of links with only fundamental rep-
resentation on the components. To obtain link invariants for any other
representation more general methods have to be developed. A complete and
explicit solution of the Chern-Simons field theory is thus obtained. One
such method has been presented in ref.[13]. The method makes use of the
following important theorem:

**Theorem:** Chern-Simons theory on a three-manifold \( \mathcal{M} \) with boundary \( \Sigma \)
is described by a WZNW (Wess-Zumino-Novikov-Witten) conformal field
theory on the boundary.

Using the same identification the functional average for Wilson lines
ending at \( n \) points on the boundary \( \Sigma \) is obtained from WZNW field theory
on the boundary with \( n \) punctures carrying representations \( R_i \):

We can represent CS functional integral as a vector in the Hilbert space \( \mathcal{H} \)
associated with the \( n \)-point vacuum expectation values of the primary fields
in WZNW conformal field theory on the boundary \( \Sigma \)[3]. Next, to obtain a
complete and explicit nonperturbative solution of the CS theory, the theory
of knots and links and their connection to braids is invoked [13, 14].

### 3.1 Knots/links and their connection to braids

Braids have an intimate connection with knots and links. This connection
is summarized as follows:

1. An \( n \)-braid is a collection of non intersecting strands connecting \( n \)
points on a horizontal rod to \( n \) points on another horizontal rod below
2. We associate representations $R_i$ of the group with the strands as their colors. We also put an orientation on each strand. When all the representations are identical and also all strands are oriented in the same direction, we get ordinary braids, otherwise we get colored oriented braids.

3. The colored oriented braids form a groupoid where product of the different braids is obtained by joining them with both colors and orientations matching on the joined strands. Unoriented monochromatic braids form a group.

4. A knot/link can be formed from a given braid by a process called plattting. We connect adjacent strands namely the $(2i+1)^{th}$ strand to $2i^{th}$ strand carrying the same color and opposite orientations in both the rods of an even-strand braid as shown in figure (a) below.

![Diagram of plattting process](a)

There is a theorem due to Birman which states that all colored oriented knots/links can be obtained through plattting [15]. This construction is not unique.

5. There is another construction associated with braids which relates them to knots and links. We obtain a closure of a braid by connecting the ends of the first, second, third, ... strands from above to those of the respective first, second, third, ... strands from below as shown in the figure (b) above. There is theorem due to Alexander[15] which states that any knot or link can be obtained as a closure of a braid, though again not uniquely.

### 3.2 Link invariants

This connection of braids to knots and links can be used to construct link invariants, say in $S^3$, from the Chern-Simons theories. To do so, from the three-manifold $S^3$ two non intersecting 3-balls are removed to obtain a manifold with two $S^2$ boundaries. Then we can arrange $2n$ Wilson line of, say
SU(\(N\)) Chern-Simons theory, as a 2\(n\)-strand oriented braid carrying representations \(R_i\) in this manifold. The CS functional integral over this manifold is a state in the tensor product of the Hilbert spaces \(\mathcal{H}_1 \otimes \mathcal{H}_2\) associated with the conformal field theory on the two boundaries. The two boundaries have 2\(n\) punctures carrying the set of representations \{\(R_i\)\} and \{\(R'_j\)\} respectively, the two sets being permutations of each other. This state can be expanded in terms of some convenient basis given by the conformal blocks for the 2\(n\)-point correlation functions of the \(SU(N)_k\) WZNW conformal field theory. The duality of these correlation functions represents the transformation between different bases for the Hilbert space. Their monodromy properties allow us to write down representations of the braid generators. Since an arbitrary braid is just a word in terms of these generators, this construction provides us a matrix representation \(B(\{R_i\}, \{R'_j\})\) for the colored-oriented braid in the manifold with two \(S^2\) boundaries. Then we plat this braid by gluing two balls \(B_1\) and \(B_2\) with Wilson lines as shown in the figure:

\[
\langle \psi(\{R_j\}) | B(\{R_i\}, \{R'_j\}) | \psi(\{R'_j\}) \rangle
\]

Each of the two caps again represents a state \(|\psi(\{R_j\})\rangle\) in the Hilbert space associated with the conformal field theory on the punctured boundary \(S^2\). Platting of the braid then simply is the matrix element of the braid representation \(B(\{R_i\}, \{R'_j\})\) with respect to these states \(|\psi(\{R_i\})\rangle\) and \(|\psi(\{R'_j\})\rangle\) corresponding to the two caps \(B_1, B_2\). Thus for a link in \(S^3\) the invariant is given by the following proposition:

**Theorem:** The vacuum expectation value of Wilson loop operator of a link \(L\) constructed from platting of a colored oriented 2\(n\)-braid with representation \(B(\{R_i\}, \{R'_j\})\) is given by:

\[
V[L] = \langle \psi(\{R_i\}) | B(\{R_i\}, \{R'_j\}) | \psi(\{R'_j\}) \rangle
\]

For detailed proof of this theorem for gauge group \(SU(2)\) see ref. [13]. This theorem can be used to calculate the invariant for any arbitrary link. For an unknot \(U\) carrying a fundamental \(N\) dimensional representation in an \(SU(N)\) CS theory, the knot invariant is:

\[
V_N[U] = [N], \quad \text{where} \quad [N] = \frac{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}
\]
The Wilson link expectation values calculated this way depend on the regularization, i.e., the definition of framing used in defining the coincident loop correlators. One such regularization usually used is the standard framing where the frame for every knot is so chosen that its self-linking number is zero.

The procedure outlined here has been used for explicit computations of knot/link invariants\cite{13, 14}. This has led to answers to several questions of knot theory\cite{16}. One such question relates to distinguishing chirality of knots. For example, among the knots in knot tables\cite{17}, the knots 9_{42} and 10_{71} and their respective mirror images are not distinguished by any older polynomial invariants including Jones and HOMFLY polynomials. But expectation value of the Wilson loop operator with spin 3/2 in SU(2) CS theory does distinguish them. Though knot invariants obtained from CS theories are sensitive to the chirality of many knots, yet they do not distinguish chirality of all knots. Inspite of the success of CS theory in its application to knot theory it fails to distinguish a class of links known as mutant links \cite{16}. A mutant link is obtained by removing a portion of weaving pattern in a link which is rotated about any one of three orthogonal axes by an amount $\pi$ and then glued back in to the link.

The CS invariants of knots and links can also be used to construct special three-manifold invariants. Hence CS theory provides an important tool to study these.

4 Three-manifold invariants from Chern-Simons theory

Classification of three dimensional manifolds is one of the challenging problems in mathematics. Interestingly, different three-manifolds can be constructed through a procedure called surgery of framed knots and links in three-sphere $S^3$ (Lickorish-Wallace theorem)\cite{18, 5}. However, the surgery construction relating the framed knots and links to the corresponding three-manifold is not unique. That is, there are many framed knots and links which give the same manifold. However rules of this equivalence are known: these are called Kirby moves\cite{19}.

Classification of three-manifolds would involve finding a method of associating a quantity with the manifold obtained by surgery on the corresponding framed knot/link on $S^3$. If the Kirby moves on the framed knot/link leave this quantity unchanged, then it is a three-manifold invariant. Knot/Link invariants of non-abelian CS theories provide a method of finding such three-manifold invariants \cite{5, 20}. Equivalently, this procedure gives an algebraic meaning to the surgery construction of three-manifolds.
4.1 Surgery of framed knots/links and Kirby moves

As discussed earlier, frame of a knot $K$ is an associated closed curve $K_f$ going along the length of the knot wrapping around it certain number of times. Self-linking number (also called framing number) is equal to the linking number of the knot with its frame. There are several ways of fixing this framing. The standard framing is one in which the frame number of the knot, that is, the linking number of the knot and its frame is zero. On the other hand, vertical framing is obtained by choosing the frame vertically above the knot projected on to a plane. In such a frame, the framing number of a knot is the same as its crossing number. In constructing the three-manifold invariants from Chern-Simons theories, we need vertical framing. The framing number may be denoted by writing the integer by the side of knot. We denote a framed $r$-component link by $[L, f]$ where framing $f = (n(1), n(2) ... n(r))$ is a set of integers denoting the framing number of component knots $K_1, K_2 ... K_r$ in the link $L$.

According to the Lickorish-Wallace theorem, surgery over links with vertical framing in $S^3$ yields all the three-manifolds. This surgery is performed in the following way.

Take a framed $r$-component link $[L, f]$ in $S^3$. Thicken the component knots $K_1, K_2 ... K_r$ such that the solid tubes $N_1, N_2 ... N_r$ so obtained are non-intersecting. Then the compliment $S^3 - (N_1 + N_2 ... + N_r)$ will have $r$ toral boundaries. On the $i^{th}$ toral boundary, we imagine an appropriate curve winding $n(i)$ times around the meridian and once along the longitude. Perform a modular transformation so that this curve bounds a disc. This construction is done with each of the toral boundaries. The tubes $N_1, N_2 ... N_r$ are then glued back in to the respective gaps. This surgery thus yields a new three-manifold. This construction is not unique. The rules of equivalence for surgery on framed knots/links in $S^3$ are the two Kirby moves[19].

**Kirby move I**: Take an arbitrary $r$-component framed link $[L, f]$ in $S^3$ and consider a curve $C$ with framing number $+1$ going around the unlinked strands of $L$ as shown in Figure 6(a). We refer to this $(r + 1)$-component link as $H[X]$, where $X$ represents a weaving pattern of the strands. Kirby move I constitutes of twisting the disc enclosed by $C$ in the clock-wise direction from below by an amount $2\pi$. This twisting thereby introduces new crossings between the curve $C$ and the strands enclosed by it. Then the curve $C$ is removed giving us a new $r$-component link $U[X]$ as shown in Figure 6(b). The framing numbers $n'(i)$ of the component knots in link $U[X]$ are related to the framing number $n(i)$ of the framed link $[L, f]$ as $n'(i) = n(i) - (\mathcal{L}(K_i, C))^2$, where $\mathcal{L}(K_i, C)$ is the linking number of knot $K_i$ and closed curve $C$. The surgery of the two framed links in the figure below will give the same three manifold.
Inverse Kirby move I involves removal of a curve $C$ with framing number $-1$ (instead of $+1$) after making one complete anti-clockwise twist from below on the disc enclosed by $C$. In the process the unlinked strands get twisted in the anti-clockwise direction leading to changed framing numbers $n'(i) = n(i) + (\mathcal{L}(K_i, C))^2$ of the component knots $K_i$.

**Kirby move II:** This move consists of removing a disjoint unknot $C$ with framing $-1$ from framed link $[L, f]$ without changing the rest of the link as shown in the figure below. The surgery of two framed links in this figure give the same three-manifold.

Inverse Kirby move II involves removal of a disjoint unknot with framing $+1$ (instead of $-1$) from a framed link.

### 4.2 Three-manifold invariants

Now a three-manifold invariant can be constructed by an appropriate combination of the invariants of framed links in such a way that this algebraic expression is unchanged under the Kirby moves I and II. What we need for this purpose are the invariants for links with vertical framing in $S^3$.

Let $\mathcal{M}$ be the three manifold obtained from surgery of a $r$-component framed link $[L, f]$ in $S^3$. Then a three-manifold invariant $\hat{f}^{(G)}[\mathcal{M}]$ for $\mathcal{M}$ is given as a linear combination of the framed link invariants $V^{(G)}_{R_1, R_2, \ldots, R_r}[L, f]$, with representations $R_1, R_2, \ldots, R_r$ living on component knots, obtained from
Chern-Simons theory based on any compact semi-simple group $\mathcal{G}$ [5, 20]:

$$\hat{F}(\mathcal{G})[\mathcal{M}] = \alpha^{-\sigma[L,f]} \sum_{R_1, R_2, \ldots, R_r} \left( \prod_{i=1}^{r} \mu_{R_i} \right) V_{R_1, R_2, \ldots, R_r}^{(\mathcal{G})}[L,f], \quad (19)$$

Here $\sigma[L,f]$ is the signature of the linking matrix and

$$\mu_{R_i} = S_{0R_i}, \quad \alpha = e^{ie\pi c/4}, \quad (20)$$

where $c$ is the central charge of the associated WZNW conformal field theory and $S_{0R_i}$ denotes the matrix element of the modular matrix $S$. General $S$-matrix elements for any compact group are given by [21]

$$S_{R_1R_2} = (-i)^{\frac{d-2}{2}|L_\omega/L|^{\frac{1}{2}} (k + C_v)^{-\frac{1}{2}} \sum_{\omega \in W} \epsilon(\omega) \exp\left( \frac{-2\pi i}{k + C_v}(\omega(\Lambda_{R_1} + \rho), \Lambda_{R_2} + \rho) \right) \quad (21)$$

where $W$ denotes the Weyl group and its elements $\omega$ are words constructed using the generator $s_{\alpha_i}$ — that is, $\omega = \prod_i s_{\alpha_i}$ and $\epsilon(\omega) = (-1)^{\ell(\omega)}$ with $\ell(\omega)$ as length of the word. Here $\Lambda_{R_i}$'s denotes the highest weights of the representations $R_i$'s and $\rho$ is the Weyl vector. The action of the Weyl generator $s_\alpha$ on a weight $\Lambda_R$ is: $s_\alpha(\Lambda_R) = \Lambda_R - 2\alpha(\Lambda_R, \alpha)\alpha$, and $|L_\omega/L|$ is the ratio of weight and co-root lattices (equal to the determinant of the cartan matrix for simply laced algebras).

It is important to stress that the expression $\hat{F}(\mathcal{G})[\mathcal{M}]$ can be shown to be unchanged under both Kirby moves I and II (for detailed proof, see refs. [5, 20]). Notice that for every compact gauge group, we have a new three-manifold invariant.

**Few examples of three-manifolds**

We now list the algebraic expressions of this invariant calculated explicitly from formula in Eq. [19] for a few three-manifolds in the table below. We have indicated the framed links in $S^3$ which on surgery yield the corresponding three-manifolds $\mathcal{M}$. $\mathcal{L}[p,q]$ stands for Lens spaces of the type $(p,q)$ and $C_R$ is the quadratic Casimir invariant for representation $R$ of the Lie algebra of the gauge group $\mathcal{G}$.
For a Chern-Simons theory in a manifold $\mathcal{M}$, the partition function is also an invariant characterizing the three-manifold $\mathcal{M}$. This has been calculated for several manifolds by different methods [22]. From the expressions for the invariant $\hat{F}^{(G)}[\mathcal{M}]$ for various manifolds listed above, it appears that this invariant is related to the Chern-Simons partition function $Z^{(G)}[\mathcal{M}]$ as follows:

$$\hat{F}^{(G)}[\mathcal{M}] = \frac{Z^{(G)}[\mathcal{M}]}{S_{00}}$$

(22)

For the gauge group $SU(2)$, this relationship has been established in ref. [23]. So the method of constructing three-manifold invariants above can also be used to calculate the partition function of Chern-Simons theories.

### 5 3D gravity and CS theory

Three dimensional Chern-Simons theory also provides a description of gravity. The 3D gravity action including the cosmological constant $\Lambda = \pm 1/\ell^2$ is:

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3 x \sqrt{-g} (R - 2\Lambda)$$

(23)

$G$ is the Newton’s constant, $g_{\mu\nu}$ is the metric on the three manifold $\mathcal{M}$ and $R$ is scalar curvature. These theories with cosmological constant were first discussed in ref. [24]. The Einstein’s field equations are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0$$

(24)

In three dimensions the solutions of these equations have a constant positive (negative) curvature if $\Lambda$ is positive (negative). It is also well known that there are no dynamical degrees of freedom for gravity in dimensions $D \leq 3$; it is indeed described by topological field theories. The gravity action above can be rewritten as a Chern-Simons gauge theory [6] in first order formulation. For triads $e^a_\mu$ and spin connection $\omega^a_\mu$ of Euclidean gravity, we define one-forms $e^a_\mu T^a dx^\mu$, $\omega^a_\mu T^a dx^\mu$, which have values in the
Lie-algebra of SU(2) whose generators are $T^a = i\sigma^a/2$ with $\sigma^a$ as three Pauli matrices. In terms of these we define two gauge field one-forms $A$ and $\bar{A}$ as:

$$A = \left(\frac{ie}{\ell} + \omega\right), \quad \bar{A} = \left(\frac{ie}{\ell} - \omega\right)$$

Then the Euclidean gravity action in terms of $A$ and $\bar{A}$ is:

$$S = kS_{CS}[A] - kS_{CS}[\bar{A}]$$

(25)

where the CS coupling constant $k = \ell/(4G)$ for negative cosmological constant $\Lambda = -1/\ell^2$. The gauge group for this theory is $SL(2,C)$. Infinitesimal diffeomorphisms are described by field dependent gauge transformations. The corresponding gauge group for Minkowski gravity with negative cosmological constant $\Lambda$ is $SO(2,R) \otimes SO(2,R)$. For positive $\Lambda$ one gets $SO(3,1)$ and $SO(4)$ for Minkowski and Euclidean metrics respectively. For $\Lambda = 0$ we have $ISO(2,1)$ ($ISO(3)$) as the gauge group for Minkowski (Euclidean) gravity. Hence the sign of cosmological constant determines the gauge group of the CS theory.

The identification of 3D gravity with CS theory can be used with some advantage to find the partition function for a black hole in 3D gravity with negative cosmological constant. This in turn yields an expression for entropy of the black hole.

### 5.1 BTZ black hole and its partition function

Only for negative $\Lambda$ we have a black hole solution of the Einstein’s equations. This solution, known as the BTZ black hole\[25\], in Euclidean gravity is given by the metric:

$$ds_E^2 = \left(-M + \frac{r^2}{\ell^2} - \frac{J^2}{4r^2}\right) dt^2 + \left(-M + \frac{r^2}{\ell^2} - \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2(d\theta - \frac{J}{2r}d\tau)^2$$

(26)

It is specified by two parameters $M$ and $J$ (the mass and the angular momentum). By a coordinate transformation this metric can be rewritten as:

$$ds_E^2 = \frac{l^2}{z^2}(dx^2 + dy^2 + dz^2), \quad z > 0$$

This is the 3D upper half hyperbolic space and can be rewritten using spherical polar coordinates as:

$$ds_E^2 = \frac{l^2}{R^2 \sin^2 \chi}(dR^2 + R^2 d\theta^2 + R^2 \sin^2 \theta d\chi^2)$$

(26)
We have the identifications \((R, \theta, \chi) \sim (R \exp\{2\pi r_+/l\}, \theta + \{2\pi r_-/l\}, \chi)\) where \(r_+\) and \(r_-\) are the outer and inner horizon radii respectively. It is clear from this identification that the metric topologically corresponds to a solid torus with a boundary. The functional integral over this manifold represents a state in the Hilbert space specified by the mass and the angular momentum. It is the micro canonical ensemble partition function and its logarithm is entropy of the back hole.

To evaluate this partition function, the connection one-form is kept at a constant value on the toroidal boundary through a gauge transformation. We define local coordinates on the torus boundary \(z = x + \tau y\) such that \(\int_a dz = 1\), \(\int_b dz = \tau\), where \(a\) \((b)\) stands for the contractible \((\)non-contractible\) cycle of the solid torus and \(\tau = \tau_1 + i\tau_2\) is the modular parameter of the boundary torus. Then the connection describing the black hole is:

\[
A = \left(\frac{-i\pi}{\tau_2} \frac{\bar{u}}{\tau_2} d\bar{z} + \frac{i\pi}{\tau_2} \frac{u}{\tau_2} dz\right) T_3 \tag{27}
\]

where \(u\) and \(\bar{u}\) are canonically conjugate with the commutation relation given by:

\[
[u, \bar{u}] = \frac{2\tau_2}{\pi(k + 2)} \tag{28}
\]

These are related to black hole parameters through the holonomies of the gauge field \(A\) around the contractible and non-contractible cycles:

\[
u = -\frac{i}{2\pi} \left(-i\Theta\tau + \frac{2\pi(r_+ + i|r_-|)}{l}\right), \]

\[
\bar{u} = -\frac{i}{2\pi} \left(-i\Theta\bar{\tau} + \frac{2\pi(r_+ + i|r_-|)}{l}\right) \tag{29}
\]

For a classical black hole solution \(\Theta = 2\pi\).

For a fixed value of connection, namely \(u\), the functional integral is described by a state \(\psi_0\) with no Wilson line in the bulk. The states with Wilson line carrying spin \(j/2\) are given by [26]:

\[
\psi_j(u, \tau) = \exp\left\{\frac{\pi k}{4\tau_2} u^2\right\} \chi_j(u, \tau) \tag{30}
\]

where \(\chi_j\) are the Weyl-Kac characters for affine \(\text{su}(2)\):

\[
\chi_j(u, \tau) = \frac{\Theta_{j+1}^{(k+2)}(u, \tau, 0) - \Theta_{j-1}^{(k+2)}(u, \tau, 0)}{\Theta_1^2(u, \tau, 0) - \Theta_{-1}^2(u, \tau, 0)}
\]

where \(\Theta\) functions are defined by:

\[
\Theta_{\mu}^{k}(u, \tau, 0) = \sum_{n \in \mathbb{Z}} \exp\left\{2\pi ik \left[ (n + \frac{\mu}{2k})^2 \tau + (n + \frac{\mu}{2k})u \right] \right\}
\]
Given the collection of states $\psi_j$ we write the partition function by choosing an appropriate ensemble by fixing the mass and angular momentum. This black hole partition function is:

$$Z_{BH} = \int d\mu(\tau, \bar{\tau}) \left| \sum_{j=0}^{k} (\psi_j(0, \tau))^* \psi_j(u, \tau) \right|^2$$  \hspace{1cm} (31)

where modular invariant measure is $d\mu(\tau, \bar{\tau}) = d\tau d\bar{\tau}/\tau_+^2$. This integral can be worked out for large black hole mass and zero angular momentum in saddle point approximation. The computation is done in ref. [27] and yields:

$$Z_{BH} = \frac{l^2}{r_+^2} \sqrt{\frac{8r_+G}{\pi l^2}} \exp\left(\frac{2\pi r_+}{4G}\right) + \cdots$$  \hspace{1cm} (32)

This gives not only the leading Bekenstein-Hawking behavior of the black hole entropy $S$ but also a sub leading logarithmic term:

$$S = \ln Z_{BH} = \frac{2\pi r_+}{4G} - \frac{3}{2} \ln \frac{2\pi r_+}{4G} + \cdots$$  \hspace{1cm} (33)

This is an interesting application of CS theory to 3D gravity. In fact three dimensional CS theory also has application in the study of black holes in four dimensional gravity: the boundary degrees of freedom of a black hole in 4D are also described by an SU(2) Chern-Simons theory [28, 29]. This allows a calculation of the degrees of freedom of, for example, Schwarzschild black hole. For large area black holes this in turn results in an expression for the entropy which, besides a Bekenstein-Hawking area term, also has a logarithmic area correction with same coefficient $-3/2$ as above[29]. This suggests a universal, dimension independent, nature of the these logarithmic corrections[30].

6 Topological BF theory

There is another class of Schwarz type topological field theories which are known as BF theories [31, 32]. These are defined using a connection one-form $A$ and a $(D - 2)$-form $B$ with values in the Lie algebra of a compact group $G$. The advantage of this class of theories is that these can be defined in arbitrary dimensions. While in three dimensions these theories, like three-dimensional Chern-Simons theories, yield a description of the topological invariants of knots and links, higher dimensional BF theories give topological properties of higher dimensional knots (imbedded manifolds of codimension 2) living in these higher dimensional manifolds.

The action for a BF theory is given by:

$$S = \int_{\mathcal{M}} tr (B \wedge F)$$  \hspace{1cm} (34)
where $F = dA + A \wedge A$ and now $\mathcal{M}$ is a $D$-dimensional manifold. We can add a cosmological term to this action which for $D = 3$ is

$$S_{\text{cos}, \kappa} = \frac{\kappa^2}{3} \int_{\mathcal{M}} \text{tr} (B \wedge B \wedge B)$$

(35)

and for $D = 4$ has a form

$$S_{\text{cos}, \kappa} = \frac{\kappa}{2} \int_{\mathcal{M}} \text{tr} (B \wedge B)$$

(36)

The name cosmological term for these comes from the fact in 3D gravity, (which can also be cast as a BF theory) this is exactly how such a term is written in terms of the triads.

But unlike CS theory and BF theory in three dimensions, it is difficult to define gauge invariant observables for higher dimensional BF theories. However the perturbative expansion of Wilson loop expectation values of CS theory which term by term corresponds to topological invariants known as Vassiliev invariants[33] has a generalization for BF theory in any dimensions.

In the following, we shall briefly discuss $D = 3, 4$ theories.

6.1 BF theory in $D = 3$

The simplest case is the BF theory based on $U(1)$ gauge group. Like in the $U(1)$ Chern-Simons theory, Abelian BF theory of two field one-forms $A$ and $B$ also provides a field theoretic characterization of linking and self-linking numbers of knots. The correlator $\langle \oint_{K_1} A \oint_{K_2} B \rangle$ for two distinct knots $K_1$ and $K_2$ can easily be seen to be related to the linking number of the two knots: $\langle \oint_{K_1} A \oint_{K_2} B \rangle = i\mathcal{L}(K_1, K_2)$. On the other hand self-linking number of a knot is given by $\langle \oint_{K} A \oint_{K} B \rangle = i\mathcal{L}(K)$, where coincident loop correlator is defined by loop splitting with the help of the framing knot $K_f$ as discussed earlier in the context of $U(1)$ CS theory.

The action for the non-Abelian BF theory including the cosmological term $S_{\text{BF}, \kappa} = S + S_{\text{cos}, \kappa}$ is interestingly related to three-dimensional CS action[31]:

$$S_{\text{CS}}(A + \kappa B) - S_{\text{CS}}(A - \kappa B) = \frac{\kappa}{\pi} S_{\text{BF}, \kappa}, \quad \frac{d}{d\kappa} S_{\text{CS}}(A + \kappa B) \bigg|_{\kappa = 0} = \frac{1}{2\pi} S_{\text{BF}, 0}$$

(37)

(38)

For BF theory with cosmological term, the topological operators for a link are constructed in terms of the operators associated with the component knots $K$: $W[A \pm \kappa B, K] = \text{tr} P \text{exp} \oint_{K} (A \pm \kappa B)$. Functional averages of these operators give knot/link invariants like in CS theory.
For the theory without the cosmological term, $\kappa = 0$, the topological operator associated with a knot $K$ is:

$$\left[ \frac{d}{d\kappa} \text{tr} P \exp \oint_K (A + \kappa B) \right]_{\kappa=0} = \text{tr} \oint_K P \left( \exp \int_0^y A \right) B(y) \left( \exp \int_y^x A \right)$$

These are related to perturbative expansion of CS theory which in turn is related to Vassiliev invariants[33]. Notice that BF theories also provide a description of gravity in three dimensions; compare Eq. 25 and Eq. 37.

Next consider the following CS functional integrals for a knot $K$:

$$Z_{CS}[M, K, k] = \int [DA] \exp (ikS_{CS}) \text{tr} P \exp \oint_K A$$

$$Z_{CS}[M, k] = \int [DA] \exp (ikS_{CS})$$

and the BF functional integrals:

$$Z_{BF,\kappa}[M, K, f] = \int [DADB] \exp (ifS_{BF,\kappa}) \text{tr} P \exp \oint_K (A + \kappa B)$$

$$Z_{BF,\kappa}[M, f] = \int [DADB] \exp (ifS_{BF,\kappa})$$

The partition functions of the BF and CS theories are related to each other through following relation:

$$Z_{CS}[M, k]Z_{CS}[M, k] = Z_{BF,\kappa}[M, f]$$

where $f = \kappa k/\pi$. This relates the BF theory partition function to Turaev-Viro invariant[34] which is the square of the CS partition function, $|Z_{CS}|^2$.

In addition the knot functional integrals in the two theories are related as:

$$Z_{CS}[M, K, k]/Z_{CS}[M, k] = Z_{BF,\kappa}[M, K, f]/Z_{BF,\kappa}[M, f].$$

This relates the knot invariants of CS theory with those of BF theory.

It is of interest to note that BF theory in $D = 3$ without cosmological constant provides the classic Alexander-Conway polynomial invariant for knots[31].

### 6.2 BF theory in $D = 4$

As mentioned earlier the BF theory can be defined in all dimensions and in particular in four dimensions. The action $S_{BF,\kappa} = S + S_{\cos,\kappa}$ in four dimensions has an interesting relation to the well known Chern-Weil form $Q_2 = \int_M \text{tr} F \wedge F$:

$$Q_2(F + \kappa B) - Q_2(F) = 2\kappa S_{BF,\kappa}$$

$$\left[ \frac{d}{d\kappa} Q_2(F + \kappa B) \right]_{\kappa=0} = 2S_{BF,0}$$
The natural geometrical setting for this case is principal bundle in the space of paths and loops. The gauge fields $A$ and $B$ are collectively connections on such a principal bundle. The observables are generalized Wilson loops obtained as trace of holonomies in the space of loops. When such observables do not involve the $B$ fields they are expected to be related to Donaldson invariants. This provides a connection of the four dimensional BF theories with the four dimensional Witten-type topological gauge field theories[4]. On the other hand, observables involving both $A$ and $B$ fields are associated with embedding of 2-surfaces in the 4-manifold as a generalization of knot theory to higher dimensions. Invariants characterizing these higher dimensional knots are obtained in theories without the cosmological term ($κ = 0$). These are generalizations of Vassiliev invariants of knots in three dimensions. We refer to the literature for further references[32].

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**References**

[1] M. Atiyah: *The Geometry and Physics of Knots*, Cambridge University Press (1989).

[2] A.S. Schwarz: Lett. Math. Phys 2 (1978) 247;
A.S. Schwarz: New topological invariants in the theory of quantised fields, Baku International Conference (1987).

[3] E. Witten: Commun. Math. Phys. 121 (1989) 351.

[4] E. Witten: Commun. Math. Phys. 117 (1988) 353.

[5] R.K. Kaul: Chern-Simons theory, knot invariants, vertex models and three-manifold invariants, hep-th/9804122, in *Frontiers of field theory, quantum gravity and strings* (Volume 227 in Horizons in World Physics), eds. R.K. Kaul et al, NOVA Science Publishers, New York (1999);
R. K. Kaul: Topological quantum field theories - a meeting ground for physicists and mathematicians, hep-th/9907119, in *Quantum field theory: a 20th century profile*, ed. A. N. Mitra, Indian National Science Academy, N. Delhi(2000).

[6] A. Achúcaro and P.K. Townsend: Phys. Letts 180 (1986) 89;
E. Witten: Nucl. Phys. B311 (1988) 46;
S. Carlip: *Quantum gravity in 2+1 dimensions*, Cambridge Monographs on Mathematical Physics (2003).
[7] E. Witten: Prog. Math. 133 (1995) 637.
[8] A.D. Popov: Phys. Letts. B 473 (2000) 65.
[9] C.F. Gauss: Werke Vol V, Göttingen, Königliche Gesellschaft der Wissenschaften, (1833) 605, Note of January 22.
[10] J.C. Maxwell: A treatise on Electricity and Magnetism, Oxford, Clarendon, England, (1873).
[11] F.H.C. Crick: Proc. Natl. Acad. Sciences, USA, 73 (1971) 2639.
[12] S. Deser, R. Jackiw and S. Templeton: Phys. Rev. Lett. 48 (1982) 975; S. Deser, R. Jackiw and S. Templeton: Ann. Phys. 140 (1982) 372.
[13] R.K. Kaul: Complete solution of SU(2) Chern-Simons theory, hep-th/9212129;
R.K. Kaul: Commun. Math. Phys. 162 (1994) 289.
[14] P. Ramadevi, T.R. Govindarajan and R.K. Kaul: Nucl. Phys. B402 (1993) 548.
[15] J.S. Birman: Braids, links and mapping class groups, Annals of Mathematics Studies, Princeton University Press (1975).
[16] P. Ramadevi, T.R. Govindarajan, R.K. Kaul: Mod. Phys. Letts. A9 (1994) 3205;
P. Ramadevi, T.R. Govindarajan, R.K. Kaul: Mod. Phys. Letts. A10 (1995) 1635.
[17] D. Rolfsen: Knots and links, Publish or Perish, Berkeley (1976).
[18] A.D. Wallace: Canad. J. Math. 12 (1960) 503;
W.B.R. Lickorish: Annal of Math. 76 (1962) 531.
[19] R. Kirby: Invent. Math. 45(1978)35;
R. Fenn and C. Rourke: Topology 18 (1979) 1.
[20] R.K. Kaul and P. Ramadevi: Commun. Math. Phys. 217 (2001) 295.
[21] P. Di Francesco, P. Mathieu and D. Senechal: Conformal Field Theory, Graduate Texts in Contemporary Physics, eds. J.L. Birman, J.W. Lynn, M.P. Silverman, H.E. Stanley, Mikhail Voloshin, Springer-Verlag (1997).
[22] L.C. Jeffrey: Commun. Math. Phys. 147 (1992) 563;
S. Kalyan Rama and S. Sen: Mod. Phys. Letts. A7 (1992) 2065.
[23] P. Ramadevi and Swatee Naik: Commun. Math. Phys. 209 (2000) 29.
[24] S. Deser and R. Jackiw: Ann. Phys. 153 (1984) 405.

[25] M. Bañados, C. Teitelboim and J. Zanelli: Phys. Rev. Letts 69 (1992) 1849.

[26] J.M. Labastida and A.V. Ramallo: Phys. Letts. B227 (1989) 92.

[27] T.R. Govindarajan, R.K. Kaul and V. Suneeta: Class. Quant. Grav. 18 (2001) 2877.

[28] L. Smolin: J. Math. Phys. 36 (1995) 6417;
   C. Rovelli: Phys. Rev. Letts. 77 (1996) 3288;
   A. Ashtekar, J. Baez, A. Corichi and K. Krasnov: Phys. Rev. Letts. 80 (1998) 904.

[29] R.K. Kaul and P. Majumdar: Phys. Rev. Letts. 84 (2000) 5255.

[30] S. Carlip: Class. Quant. Grav. 17 (2000) 4175.

[31] M. Blau and G. Thomson: Ann. Phys. 205 (1991) 130;
   A S Cattaneo, P. Cotta-Ramusino, J. Fröhlich and M. Martellini: Jour. Math. Phys. 36 (1995) 6137;
   A.S. Cattaneo: Commun. Math. Phys. 189 (1997) 795.

[32] A.S. Cattaneo, P. Cotta-Ramusino and C.A. Rossi: Lett. Math. Phys. 51 (2000) 301;
   A.S. Cattaneo and C.A. Rossi: Commun. Math. Phys. 221 (2001) 591.

[33] V. A. Vassiliev: Cohomology of Knot Spaces, in Theory of Singularities and Its Applications ed. V. I. Arnold, Providence, RI: Amer. Math. Soc., (1990) 23.

[34] V.G. Turaev and O.Y. Viro: Topology 31 (1992) 865.