ON \((r, D, R)\)-BLASCHKE-SANTALÓ DIAGRAMS WITH REGULAR \(k\)-GON GAUGES.

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Abstract. We provide a study of Blaschke-Santaló diagrams for the inradius, diameter, and circumradius, measured with respect to different gauges. This contrasts previous works on those diagrams, which are all considered for euclidean measure. By proving several new inequalities and properties between these three functionals, we compute the intersection and the union over all possible gauges of those diagrams, showing that they coincide with the corresponding diagrams of a parallelotope and (in the planar case) a triangle, respectively. Further new extremal properties are derived considering the diagrams with respect to a regular pentagon or hexagon gauge.

1. Introduction

Let \(K^n\) be the set of convex bodies (i.e. convex and compact sets) in \(\mathbb{R}^n\) and \(K^n_0\) be the subset of \(K^n\) formed by centrally symmetric sets. For any \(K, C \in K^n\), let \(R(K, C)\) be the circumradius of \(K\) with respect to \(C\), i.e. the smallest value \(\lambda \geq 0\) such that a translate of \(K\) is contained in \(\lambda C\), and \(r(K, C)\) be the inradius of \(K\) with respect to \(C\), i.e. the largest value \(\lambda \geq 0\) such that a translate of \(K\) contains \(\lambda C\). The second set \(C\) is usually fulldimensional and therefore called the gauge the functionals are based on. Finally, let \(D(K, C)\) be the diameter of \(K\) with respect to \(C\), i.e. twice the maximal circumradius \(R(\{x, y\}, C)\) for some \(x, y \in K\).

The aim of this paper is to describe the range of values that the inradius, circumradius and diameter of \(K\) and \(C\) may achieve, for some fixed gauges \(C\), but also for varying \(K\) and \(C\). To do so, consider the mapping

\[
f : K^n \times K^n \to [0, 1]^2, \quad f(K, C) = \left( \frac{r(K, C)}{R(K, C)}, \frac{D(K, C)}{2R(K, C)} \right).
\]

The set \(f(K^n, C)\) is the well-known Blaschke-Santaló diagram for the inradius, circumradius and diameter with respect to \(C\), or \((r, D, R)\)-diagram for short. This naming honours two important mathematicians. Blaschke on the one side, who considered in 1916 the corresponding diagram for the volume, surface area and mean width of 3-dimensional convex bodies [5].

Key words and phrases. Convex sets; Blaschke-Santaló diagram; Geometric inequalities; Complete system of inequalities; Circumradius; Diameter; Inradius; Parallelotope; Simplex; \(k\)-gon; Optimal Containment; Jung constant.

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\(^1\)There are other generalizations of the diameter for general gauges, but this one is the most common.
Santaló [31] on the other side, who described in 1961 several such diagrams, involving three functionals out of area, perimeter, circumradius, inradius, diameter, and minimum width of planar convex sets, and who completely described $f(K^2, B_2)$, where $B_2 = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ denotes the euclidean unit ball. To do so Santaló proved the validity of the inequality

$$2R(K, B_2)(2R(K, B_2) + \sqrt{4R(K, B_2)^2 - D(K, B_2)^2})r(K, B_2) \geq D(K, B_2)^2 \sqrt{4R(K, B_2)^2 - D(K, B_2)^2}$$

for all $K \in \mathcal{K}^2$ and observed that, together with the well-known inequalities

$$D(K, B_2) \leq 2R(K, B_2)$$
$$r(K, B_2) + R(K, B_2) \leq D(K, B_2)$$
$$\sqrt{2(n + 1)}R(K, B_2) \leq \sqrt{n}D(K, B_2)$$

fully described the diagram $f(K^2, B_2)$ (see Figure 1).

When investigating such diagrams one typically discovers that lots of special classes of convex sets describe the boundaries and that they do helps us to understand even more about their speciality. Centrally symmetric, parallelotopes, constant width set, complete or reduced sets, or simplices are just examples of bodies filling different boundaries of those diagrams.

After Santaló’s paper, several authors gave full descriptions of 2-dimensional diagrams for planar sets [7, 14, 15, 16, 17, 18, 19, 22, 23, 26, 27, 29], see also [1, 2, 4], for higher dimensional sets [24, 25], or even 3-dimensional diagrams (see [8] and the incomplete description in [33]). But all these diagrams have been considered for euclidean spaces only.

However, many functionals can be naturally extended to Minkowski (or Banach) spaces, or even further to generalized Minkowski spaces, as we have done above in the case of the circumradius, inradius and diameter.
In our view, the most significant result of this paper is twofold, and we split it into two theorems, so that two aspects can be better understood. The first one fully describes the Blaschke-Santaló diagram $f(K^2, K^2)$.

**Theorem 1.1.** Let $K, C \in K^2$. Then

\[ D(K, C) \leq 2R(K, C) \]
\[ 4r(K, C) + 2R(K, C) \leq 3D(K, C) \]
\[ D(K, C) (2R(K, C) - D(K, C)) \leq 4r(K, C)D(K, C). \]

Moreover, those three inequalities completely describe the boundary of $f(K^2, K^2)$.

Notice that the first inequality is basic and well-known, while the third is completely new. The second is the 2-dimensional version of a new inequality that holds true for arbitrary dimensions, which is almost a direct consequence from a result in [9].

**Theorem 1.2.** Let $K, C \in K^n$. Then

\[ nr(K, C) + R(K, C) \leq \frac{n+1}{2} D(K, C). \]

Besides showing the validity of the two new inequalities to prove Theorem 1.1, we need to see that the boundaries they describe are filled. This is done by the second aspect of our main result.

**Theorem 1.3.** $f(K^2, K^2) = f(K^2, S)$ for every triangle $S$.

Notice that Theorem 1.3 shows that the boundary of $f(K^2, K^2)$ is completely described from investigating a single diagram, choosing a triangle as the gauge $C$.

Having in mind that $f(K^2, S)$ contains every other planar $(r, D, R)$-diagram, if $S$ is a triangle, one may wonder whether the same holds true in the case of Minkowski spaces: does there exist $C_0 \in K_0^2$ such that $f(K^2, K_0^2) = f(K^2, C_0)$? We will show that the right
candidate (up to affine transformations) to this covering property would be a regular hexagon \( H \). However, it finally turns out that the answer to this question in negative, which is a consequence of the following result, which in particular implies that \( f(K^2, H) \) cannot cover \( f(K^2, B_2) \).

**Theorem 1.4.** Let \( K \in K^2 \) and \( H \in K_0^2 \) be such that \( H \) is a regular hexagon. If \( D(K, H) < 2R(K, H) \), then \( r(K, H) \geq R(K, H)/4 \).

Let us mention at this point that it is well known that \( D(K, C) = 2R(K, C) \) whenever both, \( K, C \) are symmetric. Thus, replacing also \( K^n \) above by \( K_0^n \) in the first argument gives \( f(K_0^n, K_0^n) = f(K_0^n, C) = \text{conv} \left( \left\{ \left( \frac{1}{2}, \frac{1}{2} \right) \right\} \right) \) for all \( C \in K_0^n \). However, the next result shows that if we do not restrict the first argument this unusual 1-dimensional behaviour only occurs when the gauge \( C \) is a parallelotope.

**Theorem 1.5.** Let \( C \in K^n \). Then \( \text{conv} \left( \left\{ \left( \frac{1}{2}, \frac{1}{2} \right) \right\} \right) \subset f(K^n, C) \) and equality holds if and only if \( C \) is a parallelotope.

Essentially, this is a direct corollary from the well known characterization that paralleloptopes are exactly the class of sets with Helly dimension 1 and the theorem even stays true when replacing the inradius by any of the functionals area, perimeter, or (minimal) width. Anyway, since we are not aware of a published proof of the above fact, we will close this gap here.

Theorems 1.1, 1.4, and 1.5 show that Blaschke-Santaló diagrams of some \( k \)-gons, (regular) triangles, regular hexagons and squares) encode extreme behaviours. In the cases of full dimensional diagrams they share many common properties. For instance, restricting \( K \in K^2 \) to triangles whenever \( C \) is either the euclidean ball \( B_2 \), a triangle \( \Delta \), or a regular hexagon \( H \), the supremum over the choices of \( K \) of the ratio of the circumradius \( R(K, C) \) and the diameter \( D(K, C) \) (the so called Jung constant of the gauge \( C \)) is attained if and only if \( K \) is an equilateral triangle, i.e. a triangle whose edges all have the same length with respect to \( C \). Even so it is easy to argue that the leftmost point in the Jung-boundary must be reached by a triangle it is not always equilateral, as the regular pentagon \( C \) shows.

**Theorem 1.6.** Let \( K, P \in K^2 \) such that \( P \) is a regular pentagon. Then

\[
R(K, P) \leq \frac{1 + \sqrt{5}}{2} D(K, P).
\]

Moreover, there exist triangles \( T \) and \( T' \) attaining equality above, such that \( T \) is an isosceles triangle, with exactly two diametrical edges, whereas \( T' \) is an equilateral one.

The paper is organized as follows. In Section 2 we explain the notations and preliminary results needed along the paper, accompanied by two technical Lemmas. In Section 3 we prove Theorem 1.5 using a characterization of paralleloptopes and the fact that the intersection of \( (r, D, R) \)-diagrams equals the diagram of any single paralletope. Section 4 treats the main result of the paper, namely, the inequalities exposed in Theorems 1.1 and 1.2 and the fact that the union of all \( (r, D, R) \)-diagrams in the planar case equals the diagram of any single triangle in Theorem 1.3. Finally in Section 5 we show that when restricting to centrally
symmetric containers the union of diagrams is no longer given by a single diagram, where
the regular hexagon plays a crucial role (cf. Theorem 1.4). Moreover, we also show that the
\((r, D, R)\)-diagram of the regular pentagon has a Jung-extreme triangle which is not equilateral
(cf. Theorem 1.6).

2. Notation and preliminary results

For every \(K, C \in \mathcal{K}^n\) let \(K + C = \{ x + y : x \in K, y \in C \}\) be the Minkowski sum of
\(K\) and \(C\) and for every \(\lambda \in \mathbb{R}\) let \(\lambda K = \{ \lambda x : x \in K \}\) be the \(\lambda\)-dilatation of \(K\). We use
\(-K := (-1)K\) for short.

For every \(X \subset \mathbb{R}^n\), let \(\text{conv}(X)\), \(\text{aff}(X)\) and \(\text{pos}(X)\) be the convex, affine, and positive hull
of \(X\), respectively. Moreover, for any \(x, y \in \mathbb{R}^n\), let \([x, y] := \text{conv}(\{x, y\})\) be the line segment
with endpoints \(x\) and \(y\).

The circumradius is homogeneous of degree 1 and monotonically increasing on its first
entry, whereas it is homogeneous of degree \(-1\) and monotonically decreasing on its second
entry, i.e. for every \(K_1, K_2, C_1, C_2 \in \mathcal{K}^n\) with \(K_1 \subset K_2, C_2 \subset C_1\), and \(\lambda > 0\), we have
\[R(K_1, C_1) \subset R(K_2, C_1) \subset R(K_2, C_2)\quad\text{and}\quad R(\lambda K_1, C_1) = R(K_1, \frac{1}{\lambda} C_1) = \lambda R(K_1, C_1)\]

Notice that those and the remaining properties of the circumradius are inherited by the
inradius and the diameter. This can easily be seen due to the fact that \(r(K, C) = R(C, K)^{-1}\)
and \(D(K, C) = 2 \max_{x,y \in K} R(\{x, y\}, C)\) [12]. Notice also that \(D(K, C) = D([x, y], C)\) for
some extreme points \(x, y\) of \(K\), i.e. such that no line segment containing \(x\) or \(y\) on its interior
can be contained in \(K\) [20].

For every \(K \in \mathcal{K}^n\), let \(\text{bd}(K)\) be the boundary of \(K\) and if \(p \in \text{bd}(K)\) let \(N(K, p)\) be
the (outer) normal cone of \(K\) at \(p\), i.e. \(N(K, p) = \{ u \in \mathbb{R}^n : u^T(x - p) \leq 0 \text{ for all } x \in K \}\). Moreover, for every \(u \in \mathbb{R}^n\) let \(h(K, u)\) be the support function of \(K\) at \(u\), defined by \(h(K, u) := \sup\{x^Tu : x \in K\}\). Using the support function, it is well known that the diameter
can also be expressed by
\[D(K, C) = 2 \sup_{u \in \mathbb{R}^n \setminus \{0\}} \frac{h(K - C, u)}{h(C, u)}\]
for every \(K, C \in \mathcal{K}^n\) [12].

We say that \(K \subset^{opt} C\) if \(K \subset C\) and \(K \not\subset c + \lambda C\) for any \(\lambda \in (0, 1)\) and \(c \in \mathbb{R}^n\). The
situation in which \(K \subset^{opt} C\) is characterized by a touching condition between a finite amount
of boundary points of \(K\) and \(C\) [11, Theorem 2.3].

**Proposition 2.1.** Let \(K, C \in \mathcal{K}^n\) with \(K \subset C\). Then the following are equivalent:

(i) \(R(K, C) = 1\).

(ii) There exist \(p^1, \ldots, p^k \in K \cap \text{bd}(C)\), for some \(k \in \{2, \ldots, n + 1\}\), and \(w^j \in N(C, p^j)\),
\(j \in [k]\), such that \(0 \in \text{conv}\{w^j : j \in [k]\}\).

In particular, this means that the circumradius is affine invariant, i.e. for every affine
transformation \(A\) and \(K, C \in \mathcal{K}^n\) we have \(R(A(K), A(C)) = R(K, C)\). The same holds for
the inradius and diameter.
For the next lemma, one should recognize that using Caratheodory’s Theorem we can assume that the points $p^i$ as well as the outer normals $u^i$ in Proposition 2.1 can be chosen affinely independent.

**Lemma 2.2.** Let $K, C \in \mathcal{K}^n$ be such that $K \subset \mathcal{C}$. Then there exist an $t$-dimensional simplex $T \in \mathcal{K}^n$ and a generalized prism $S \in \mathcal{K}^n$ with $t$-dimensional simplicial base, such that $T \subset K \subset C \subset S$ for some $t \in [n]$, fulfilling

$$R(T, S) = 1, \quad r(T, S) \leq r(K, C), \quad \text{and} \quad D(T, S) \leq D(K, C).$$

Moreover, if $C = -C$, then we obtain the same conclusion within the chain of inclusions $T \subset K \subset C \subset S \cap (-S)$, i.e. replacing $S$ by $S \cap (-S)$.

**Proof.** Since $K \subset \mathcal{C}$ we know from Proposition 2.1 that there exist $p^1, \ldots, p^k \in K \cap \text{bd}(C)$, $k \in \{2, \ldots, n+1\}$, and $u^j \in N(C, p^j)$, $j \in [k]$, such that $0 \in \text{conv}(\{u^1, \ldots, u^k\})$. We immediately obtain the claimed result from defining $t := k-1$, $T := \text{conv}(\{p^1, \ldots, p^k\})$, and $S := \bigcap_{j=1}^k \{x \in \mathbb{R}^n : (u^j)^T x \leq (u^j)^T p^j\}$. Moreover, if $C = -C$ then $C \subset S$ directly implies $C \subset -S$, too, and therefore $C \subset S \cap (-S)$.

Just combining the definition of the diameter and the monotonicity of the circumradius we directly deduce that

$$D(K, C) \leq 2R(K, C). \quad (2)$$

Equality in (2) holds, for instance, whenever $K = (1-\lambda)[x, y] + \lambda C$, for some $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. Seeking for a reverse inequality to the one above, led to the Jung constants

$$j_C = \sup\{R(K, C)/D(K, C) : K \in \mathcal{K}^n\},$$

referring to Jung who showed that $j_{B^2} = \sqrt{n/(2(n+1))}$ [28]. Bohnenblust proved that

$$j_C \leq n/(n+1) \quad (3)$$

whenever $C \in \mathcal{K}_0^n$ [5]. Moreover, one has $R(K, C) = n/(n+1)D(K, C)$ if and only if $K$ is an $n$-dimensional simplex, i.e. the convex hull of $n+1$ affinely independent points, with barycenter 0 and $C$ fulfills

$$K - K \subset D(K, C)C \subset (n+1)(K \cap (-K)) \quad (4)$$

(see [10], Corollary 2.9]). More generally, we also know that $j_C \leq n/2$, with $2R(K, C) = nD(K, C)$ if $K = -C$ is a full-dimensional simplex [11].

Maybe, the first non trivial inequality relating all three functionals was the so called concentricity inequality (proven in [31] for the euclidean plane, in [34] for general euclidean spaces, and in [30] in the general case), which states that for any $K \in \mathcal{K}^n$ and $C \in \mathcal{K}_0^n$ we have

$$r(K, C) + R(K, C) \leq D(K, C). \quad (5)$$

For every $C \in \mathcal{K}^n$, the asymmetry measure of Minkowski $s(C)$, or Minkowski asymmetry for short, is the smallest $\lambda \geq 0$ such that $\lambda C$ contains some translate of $-C$, i.e. $s(C) = R(-C, C)$
such that \(0 \in \mathbb{N}\). It is well known that \(s(C) \geq 1\), with equality if and only if \(C \in \mathcal{K}_0^n\), and \(s(C) \leq n\), with equality if and only if \(C\) is a fulldimensional simplex.

Making use of the Minkowski asymmetry the concentricity inequality has been generalized for arbitrary \(C \in \mathcal{K}^n\): \(\text{(6)}\)

\[
s(C)r(K, C) + R(K, C) \leq \frac{s(C) + 1}{2} D(K, C).\]

In particular, when \(s(C) = n\), equality holds for sets of the form \(K = (1 - \lambda)(-C) + \lambda C\), \(\lambda \in [0, 1]\) (and, more generally, for constant width sets with respect to \(C\)).

In the Euclidean case it is shown in [8, Lemma 2.1] that \(f(\mathcal{K}^n, \mathbb{B}_n)\) is star-shaped with respect to the upper-right vertex \(f(\mathbb{B}_n, \mathbb{B}_n) = (1, 1)^T\). For practical purposes this means that these diagrams can be fully described by simply explaining the sets mapped onto its boundaries. The next lemma shows that the star-shapedness is still true when replacing \(\mathbb{B}_n\) by an arbitrary \(C \in \mathcal{K}^n\).

**Lemma 2.3.** Let \(K, C \in \mathcal{K}^n\). Then \(f((1 - \lambda)K + \lambda C, C) = (1 - \lambda)f(K, C) + \lambda f(C, C)\), for every \(\lambda \in [0, 1]\).

**Proof.** We start noticing that because of the translation and dilatation invariance of \(f\) we may assume without loss of generality that \(K \subset_{opt} C\) and doing so we have \((1 - \lambda)K + \lambda C \subset (1 - \lambda)C + \lambda C = C\). Using Proposition 2.1 there exist \(p^1, \ldots, p^k \in K \cap \text{bd}(C)\) and \(u^j \in N(C, p^j), j \in [k]\) for some \(k \in \{2, \ldots, n + 1\}\), such that \(0 \in \text{conv}\{u^1, \ldots, u^k\}\). Now, since \(p^j = (1 - \lambda)p^j + \lambda p^j \in \{(1 - \lambda)K + \lambda C\} \cap \text{bd}(C)\) we can again conclude from Proposition 2.1 that \(R((1 - \lambda)K + \lambda C, C) = 1\), for every \(\lambda \in [0, 1]\).

The same argument give us \(r((1 - \lambda)K + \lambda C, C) = (1 - \lambda)r(K, C) + \lambda\) and \(D((1 - \lambda)K + \lambda C, C) = (1 - \lambda)D(K, C) + 2\lambda\). Thus

\[
f((1 - \lambda)K + \lambda C, C) = \left((1 - \lambda)r(K, C) + \lambda, (1 - \lambda)\frac{D(K, C)}{2} + \lambda\right)\]

\[
= (1 - \lambda)f(K, C) + \lambda(1, 1),\]

concluding the proof. \(\square\)

3. The Helly-dimension and its meaning for the minimal diagram

The **Helly dimension** \(\text{him}(C)\) of a set \(C \in \mathcal{K}^n\) is defined as the smallest positive number \(k \in \mathbb{N}\) such that whenever we consider a set of indices \(I \subset \mathbb{N}\) with the property \(\bigcap_{i \in I}(x_i + C) \neq \emptyset\) for all \(J \subset I\) with \(|J| \leq k + 1\) and \(x_i \in \mathbb{R}^n\), it already follows that \(\bigcap_{i \in I}(x_i + C) \neq \emptyset\) (for more details on the Helly-dimension see [3, 32]). We say that a point \(p \in \text{bd}(C), C \in \mathcal{K}^n\), is **regular or smooth** if \(\text{dim}(N(C, p)) = 1\). In [13, Ch. IV] it is proven that the Helly dimension is equivalent to the **minimal dependence** \(\text{md}(C)\), which is the largest number \(k \in \mathbb{N}\) such that there exist regular points \(p^j \in \text{bd}(C), j \in [k + 1]\), and vectors \(u^j \in N(C, p^j), j \in [k + 1]\), such that \(0 \in \text{conv}\{u^j : j \in [k + 1]\}\) and such that for every \(I \subset [k + 1], |I| \leq k\), the vectors \(\{u^j : j \in I\}\) are linearly independent.
Finally, notice that Szökefalvi-Nagy \[32\] proved that if \( C \in \mathcal{K}^n \) then \( \text{him}(C) = 1 \) if and only if \( C \) is a parallelotope.

**Lemma 3.1.** Let \( C \in \mathcal{K}^n \). The following are equivalent:

1. \( C \) is a parallelotope.
2. \( D(K, C) = 2R(K, C) \) for every \( K \in \mathcal{K}^n \).

**Proof.** Let us first mention that we make use of the fact \( \text{him}(C) = \text{md}(C) = 1 \) if and only if \( C \) is a parallelotope and essentially prove that (ii) implies \( \text{md}(C) = 1 \), while \( C \) being parallelotope implies (i).

We start proving “(ii) \( \Rightarrow \) (i)” and assume that \( k = \text{md}(C) > 1 \). That means there exist smooth boundary points \( p^1, \ldots, p^{k+1} \) of \( C \) and \( u^j \in N(C, p^j) \), \( j \in [k + 1] \), such that \( 0 \in \text{conv}\{u^1, \ldots, u^{k+1}\} \) and since \( \text{md}(C) = k \) we know that \( \{u^1, \ldots, u^k\} \) is linearly independent for every choice \( 1 \leq j_1 < \cdots < j_k \leq k + 1 \).

Now, let \( K = \text{conv}\{p^1, \ldots, p^{k+1}\} \in \mathcal{K}^n \) and notice that by Proposition 2.1 we have \( K \subset^\text{opt} C \). Using (ii) and assuming without loss of generality that \( p^1, p^2 \) is a diametrical pair of \( K \), we conclude \( 2 = 2R(K, C) = D(K, C) = D[p^1, p^2, C] \). Hence we have \( R(p^1, p^2|C) = 1 \) and therefore \( [p^1, p^2] \subset^\text{opt} C \). Using the smoothness of \( p^1, p^2 \) and Proposition 2.1 again, we conclude \( 0 \in \text{conv}\{u^1, u^2\} \), which shows \( \text{md}(C) = k = 1 \).

We now show “(i) \( \Rightarrow \) (ii)” and remember that (i) implies that \( C \) is a parallelootope. After suitable translations and dilatations of \( K \) and \( C \) we may assume \( C = \bigcap_{i=1}^n \{x \in \mathbb{R}^n : |(v^i)^T x| \leq 1\} \) for some linearly independent vectors \( v^1, \ldots, v^n \in \mathbb{R}^n \) as well as \( K \subset^\text{opt} C \). By Proposition 2.1 there exist \( p^1, \ldots, p^{n+1} \in K \cap \text{bd}(C) \) and \( u^j \in N(C, p^j) \), \( j \in [n + 1] \), such that \( 0 \in \text{conv}\{u^1, \ldots, u^{n+1}\} \). To fulfill the latter the points \( p^1, \ldots, p^{n+1} \) have to touch a pair of opposing facets of \( C \). Otherwise (after changing signs of the \( v^i \), if necessary) we would have \( (v^i)^T p^j = 1 \) for all \( j \in [n + 1] \) and \( i \in [n] \), which directly implies \( u^j \in \text{pos}(\{v^1, \ldots, v^n\}) \) for all \( j \in [n + 1] \), contradicting \( 0 \in \text{conv}(\{u^1, \ldots, u^{n+1}\}) \).

Thus, we may assume \( (v^1)^T p^1 = -1 \) and \( (v^1)^T p^2 = 1 \). However, this implies by Proposition 2.1 that \( [p^1, p^2] \subset^\text{opt} C \) and therefore \( D(K, C) \geq 2R([p^1, p^2], C) = 2R(K, C) \geq D(K, C) \), which concludes the proof.  

**Proof of Theorem 1.3.** The first part is a direct consequence of \( f(K, C) = \{\frac{1}{i}\} \) if \( K \) is a segment and \( f(C, C) = \{\frac{1}{1}\} \) in combination with Lemma 2.3, while the second part follows directly from Lemma 3.1.

4. **The maximal diagram – proof of the main result**

We start noticing that Theorem 1.2 is a direct consequence of (6).

**Proof of Theorem 1.2.** Let \( g(x) := ax/(x + 1) + b/(x + 1) \) for some \( 0 \leq a \leq b \) and \( x \geq 1 \). Since \( g'(x) = (a - b)/(x + 1)^2 \leq 0 \) we see that \( g \) is non increasing. Now, using (6), the fact that \( 0 \leq r(K, C) \leq R(K, C) \), and that \( s(C) \leq n \), we conclude that

\[
\frac{n}{n+1} r(K, C) + \frac{1}{n+1} R(K, C) \leq \frac{s(C)}{s(C)+1} r(K, C) + \frac{1}{s(C)+1} R(K, C) \leq \frac{D(K, C)}{2}.
\]
Now, it only remains to prove the validity and tightness of the third inequality in Theorem 1.1. To do so we should first recognize that by dividing through the circumradius $R(K,C)$ this inequality can be rewritten as

\[ \frac{r(K,C)}{R(K,C)} \geq \frac{D(K,C)}{2R(K,C)} \left(1 - \frac{D(K,C)}{2R(K,C)}\right) \]

or simply $D(K,C)(2 - D(K,C)) \leq 4r(K,C)$ in case $R(K,C) = 1$.

First we show that certain triangles attain equality in that inequality.

**Lemma 4.1.** Let $S = \text{conv}(\{p^1, p^2, p^3\}) \in K^2$ be an equilateral triangle, with $\|p^i\| = 1$, $i = 1, 2, 3$, and $(p^i)^T p^j = -1/2$, $1 \leq i < j \leq 3$. Moreover, let

\[ T = \text{conv} \left( \left\{ \frac{1}{2}(p^1 + p^2), \frac{D}{2} p^1 + \left(1 - \frac{D}{2}\right) p^3, \frac{D}{2} p^2 + \left(1 - \frac{D}{2}\right) p^3 \right\} \right), \]

for some $D \in [1,2]$. Then $R(T,S) = 1$, $D(T,S) = D$, and

\[ r(T,S) = \frac{D(T,S)}{2} \left(1 - \frac{D(T,S)}{2}\right). \]

**Proof.** First of all, recognize that if $D = 2$ then $T = [p^1, p^2]$, which directly means that $D(T,S) = 2R(T,S) = 2$ and $r(T,S) = 0$, proving the claim in that case.

Hence we may suppose $D < 2$. In this case each vertex of $T$ touches each of the edges of $S$ in their relative interior, which by Proposition 2.1 directly implies $R(T,S) = 1$.

Now, observe that since $D \geq 1$ we have

\[ \frac{D}{2} p^i + \left(1 - \frac{D}{2}\right) p^3 \in \left[p^i, \frac{1}{2}(p^i + p^3)\right], \]

for $i = 1, 2$, hence implying that

\[ D \left( \left\{ \frac{D}{2} p^i + \left(1 - \frac{D}{2}\right) p^3, \frac{1}{2}(p^1 + p^2) \right\}, S \right) \leq D \left( \text{conv} \left( \left\{ p^i, \frac{1}{2}(p^i + p^3), \frac{1}{2}(p^1 + p^2) \right\} \right), S \right) \]

\[ = D \left( \left\{ p^i, \frac{1}{2}(p^1 + p^2) \right\}, S \right) \]

\[ = \frac{D\{p^1, p^2\}, S)}{2} = 1 \]

\[ \leq D = D \left( \left\{ \frac{D}{2} p^i + \left(1 - \frac{D}{2}\right) p^3 : j = 1, 2 \right\}, S \right) \]

for both $i = 1, 2$, thus showing that $D(T,S) = D$.

We now compute the vertices $z^i := c + r(T,S)p^i$, $i = 1, 2, 3$, of $c + r(T,S)S$ such that $c + r(T,S)S \subset T$, for some $c \in \mathbb{R}^2$. Without loss of generality, we may assume $p^1 = (\sqrt{3}/2, -1/2)^T$, $p^2 = (-\sqrt{3}/2, -1/2)^T$, and $p^3 = (0,1)^T$. The symmetry of $T$ and $S$ with respect to the vertical line passing through the origin implies that

\[ z^3 = \frac{1}{2} \left( \frac{D}{2} \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right)^T + \left(1 - \frac{D}{2}\right) (0,1)^T \right) + \frac{1}{2} \left( \frac{D}{2} \left( -\frac{\sqrt{3}}{2} - \frac{1}{2} \right)^T + \left(1 - \frac{D}{2}\right) (0,1)^T \right) \]

\[ = \left(0, 1 - \frac{3D}{4}\right)^T. \]
Let \( t \geq 0 \) be such that
\[
z^1 = z^3 + t(p^1 - p^3) = \left(0, 1 - \frac{3D}{4}\right)^T + t \left(\frac{\sqrt{3}}{2}, -\frac{3}{2}\right)^T = \left(\frac{\sqrt{3}t}{2}, 1 - \frac{3D}{4} - \frac{3t}{2}\right)^T.
\]
In particular, we get
\[
r(T, S) = \frac{t\|p^1 - p^3\|}{\|p^1 - p^3\|} = t.
\]
Notice, the line containing the vertices \((D/2)(\sqrt{3}/2, -1/2)^T + (1-D/2)(0, 1)^T\) and \((0, -1/2)^T\) of \(T\) is described (in \((x, y)\) coordinates) by the equation
\[
y = \frac{1 - \frac{3D}{4} + \frac{1}{2}}{\frac{\sqrt{3}}{2}} x - \frac{1}{2} = \frac{6 - 3D}{\sqrt{3D}} x - \frac{1}{2}.
\]
Since \(z^1\) has to fulfill this equation, we obtain
\[
1 - \frac{3D}{4} - \frac{3t}{2} = \frac{(6 - 3D)t}{2D} - \frac{1}{2} - \frac{3D}{4} - \frac{3t}{2} = \frac{(6 - 3D)t}{2D} - \frac{1}{2}
\]
which shows \(r(T, S) = t = D(2 - D)/4\), concluding the proof. \(\square\)

**Lemma 4.2.** Let \(T, S \in K^2\) be both triangles. Then
\[
r(T, S) \leq \frac{D(T, S)}{2R(T, S)} \left(1 - \frac{D(T, S)}{2R(T, S)}\right).
\]

**Proof.** Using the affine invariance of the radii, and since affine transformations of simplices are simplices, we can assume without loss of generality that \(S\) is an equilateral triangle centered at the origin. In particular, let \(S = \text{conv}\{p^1, p^2, p^3\}\) and \(T = \text{conv}\{q^1, q^2, q^3\}\), for some \(p^i, q^i \in \mathbb{R}^2\) with \(\|p^i\| = 1, i = 1, 2, 3\). Remember that \(r(T, S), D(T, S),\) and \(R(T, S)\) are all homogeneous of degree 1 on \(T\). Hence \((8)\) holds true for \(T\) if and only if \(8\) holds true for \(T/R(T, S)\). Thus we may assume that \(R(T, S) = 1\) and moreover, using an additional translation if necessary, that \(T \subset \text{opt} S\). Using Proposition 2.1 this again allows us to assume that each \(q^i\) belongs to the edge of \(S\) opposing \(p^i, i = 1, 2, 3\). Furthermore, since \(D(T, S)\) is attained by two extreme points, we can suppose that \(D(T, S) = D([q^1, q^2], S)\). Finally, without loss of generality, we may assume that \(\|q^2 - p^1\| \leq \|q^1 - p^2\|\). Keeping the equilaterality of \(S\) in mind this directly implies that if we choose \(t \in \mathbb{R}^2\), such that \(t + [q^1, q^2] \subset \text{opt} R([q^1, q^2], S)\) then \(t + q^2 = R([q^1, q^2], S)p^1\). Using this as well as the fact that \(-p^1\) is an outer normal of the edge \([p^2, p^3]\) of \(S\) we see that there exists a pair of parallel lines orthogonal to \(p^1\) that supports \(t + [q^1, q^2]\) as well as \(R([q^1, q^2], S)S\) which in particular implies
\[
2\frac{h([q^1 - q^2, q^2 - q^1], p^1)}{h(S - S, p^1)} = D([q^1, q^2], S) = D(T, S).
\]
In particular, \(T \subset \{x \in S : (p^1)^T x \leq (p^1)^T q^2\}\) and, using the symmetry of \(S\) with respect to the line through 0 in direction of \(p^3\), it must also be true that \(T \subset \{x \in S : x^T p^2 \leq (p^1)^T q^2\}\). Otherwise the width of \(T\) in direction of \(p^2\) would be greater than in direction of \(p^1\), contradicting that the latter is the maximal width.

For the next step, let \(c \in \mathbb{R}^2\) be such that \(c + r(T, S)S \subset T\). The idea now is that we replace \(q^1\) by the only point \(\bar{q}^1\) in \([p^2, p^3]\) such that \([\bar{q}^1, q^2]\) becomes parallel to \([p^1, p^2]\).
Using the intercept theorem we obtain $D([\bar{q}^1, q^2], S) = D([q^1, q^2], S)$ as well as $(p^1)^T q^2 = (p^2)^T \bar{q}^1 \geq (p^2)^T q^3$ and therefore $D([q^1, q^3], S) \leq D([q^1, q^3], S)$. Moreover, since $q^3 \in T \subset \{x \in S : (p^1)^T x \leq (p^1)^T q^2\}$ we have $(p_1)^T q_3 \leq (p_1)^T q_2$. Hence

$$d(\bar{q}^1, \text{aff}(q^2, q^3)) \leq d(q^1, \text{aff}(q^2, q^3)),$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance. Let $L := \text{aff}([c + r(T, S)p^2, c + r(T, S)p^3])$ and $\tilde{T} := \text{conv}\{\tilde{q}^1, q^2, q^3\}$. The inequality above shows that the length of the segment $T \cap L$ is not shorter than the length of the segment $\tilde{T} \cap L$. Moreover, since the distance from $q^3$ to $L$ is not longer than the distance from $q^2$ to $L$, we can conclude that we cannot find a triangle $\hat{c} + \rho S$ contained in $\tilde{T}$ with $\rho > r(T, S)$ (and $\rho = r(T, S)$ can only happen if $q^1 = \bar{q}^1$ or if $[q^2, q^3]$ is parallel to $[p^2, p^3]$). Thus, we have shown that $D(\tilde{T}, S) = D(T, S)$ and $r(\tilde{T}, S) \leq r(T, S)$, and evidently, we also have $R(\tilde{T}, S) = 1$.

Finally, if we consider any triangle $\tilde{T} := \text{conv}\{q^1, q^2, q^3\}$ with $\bar{q}^3$ in $[p^1, p^2]$, such that $\|\bar{q}^3 - p^1\| \geq \|q^2 - p^1\|$ and $\|\bar{q}^3 - p^2\| \geq \|q^1 - p^2\|$ we have $D(\tilde{T}, S) = D(T, S)$ for the same reasons as above and surely we also have $R(\tilde{T}, S) = 1$ again. Moreover, in case of $[q^1, q^2]$ parallel to $[p^1, p^2]$ we obtain again from the intercept theorem that $r(\tilde{T}, S) = r(T, S)$. Thus we may select $\bar{q}^3 := \frac{1}{2}(p^1 + p^2)$ and obtain from the above that the inradius of any triangle $T$ of circumradius 1 and diameter $D$ (all with respect to $S$), is at least as big as the inradius of the final triangle $\tilde{T}$ and it is an isosceles triangle by the choice of $q^3$. Thus we are in the situation described in Lemma 4.1. Hence

$$r(T, S) \geq r(\tilde{T}, S) = \frac{D(\tilde{T}, S)}{2} \left(1 - \frac{D(\tilde{T}, S)}{2}\right) = \frac{D(T, S)}{2} \left(1 - \frac{D(T, S)}{2}\right),$$

concluding the result.
Proof of Theorems 1.1 and 1.3. Since we already know the general validity of (1) and (2), we start proving the general validity of (7). After a suitable dilatation, we can assume that $R(K,C) = 1$ and in case $D(K,C) = 2$, the inequality is obviously true. Thus we may assume $D(K,C) < 2$ in the following. Let $T, S \subset K^2$ be as given in Lemma 2.2 such that $R(T, S) = 1$, $r(T, S) \leq r(K, C)$, and $D(T, S) \leq D(K, C)$. Recognize that $T$ is a segment and $S$ a rectangle if $\ell = 1$ and since they are both symmetric we immediately obtain $D(K, C) = 2$, contradicting our assumption.

Thus, we must have $\ell = 2$, i.e., $T$ and $S$ are both triangles.

For the sake of contradiction, let us assume that

$$r(K, C) < \frac{D(K, C)}{2} \left(1 - \frac{D(K, C)}{2}\right).$$

Let $g(y) = (y/2)(1 - y/2)$, $y \in (1, 2)$, and notice that $g'(y) = (2 - 2y)/4 < 0$. Thus $g(y)$ is a strictly decreasing function whenever $y \in (1, 2)$. Hence we would obtain

$$r(T, S) \leq r(K, C) < \frac{D(K, C)}{2} \left(1 - \frac{D(K, C)}{2}\right) \leq \frac{D(T, S)}{2} \left(1 - \frac{D(T, S)}{2}\right),$$

contradicting Lemma 4.2.

To finish the proof we know show that the boundaries described by the three inequalities are filled. First, notice that $(1 - \lambda)[x, y] + \lambda S$, $x, y \in \mathbb{R}^2$, attains equality in (2) for all $\lambda \in [0, 1]$ and in particular, the line segment $[x, y]$ and $S$ are the extreme cases $\lambda \in \{0, 1\}$. Second, we have $(1 - \lambda)(-S) + \lambda S$, $\lambda \in [0, 1]$, attains equality in (4), where the extreme cases are the triangle $S$ and its reverse $-S$. Finally, the family of triangles $T$ from Lemma 4.1 for some $D \in [1, 2]$, attains equality in (8) and their extreme cases are a line segment $[x, y]$ (when $D = 2$) and the triangle $-S$ (when $D = 1$).

This means that the points described by the equality cases of those three inequalities form a simple closed curve which contains $f(K^2, S)$ and for each boundary point, there is a convex body $K$ mapped onto it via $f(K, S)$. Moreover, let $T$ be such that $f(T, S)$ belongs to the boundary induced by Lemma 4.2. Due to the homogeneity of the first entry of $r$ and $D$ we have that

$$f(T, S) = \left(\frac{r(T, S)}{R(T, S)}, \frac{D(T, S)}{2R(T, S)}\right) = \left(r\left(\frac{T}{R(T, S)}, S\right), \frac{1}{2}D\left(\frac{T}{R(T, S)}\right)\right) = f\left(\frac{T}{R(T, S)}, S\right).$$

Hence we can replace $T$ by $T/R(T, S)$ and therefore assume that $R(T, S) = 1$. Finally, since $f(T, S)$ is linear in the first entry with respect to the Minkowski addition of $T$ and $S$ (see Lemma 2.3) and (2) and the inequality in Theorem 1.2 are linear with endpoint at $S$, thus the region contained between these three boundaries coincides with the diagram $f(K^2, S)$, since it is completely covered by the values

$$f((1 - \lambda)T + \lambda S, S) = (1 - \lambda)f(T, S) + \lambda(1, 1)^T, \quad \lambda \in [0, 1],$$

for every $D \in [1, 2]$, which concludes the proof of Theorem 1.1.

Theorem 1.3 now is a straightforward consequence of the fact that all the inequalities in Theorem 1.1 are already best possible if we choose $S$ to be a triangle. □
5. Importance of $k$-gons in the planar case

From the previous sections we learned that

\[(9) \quad \bigcup_{C \in \mathcal{K}^2} f(\mathcal{K}, C) = f(\mathcal{K}, \mathcal{K}^2) = f(\mathcal{K}, S) \quad \text{and} \quad \bigcap_{C \in \mathcal{K}^2} f(\mathcal{K}, C) = f(\mathcal{K}, P)\]

for some triangle $S$ and some parallelogram $P$. In view of the first in combination with Lemma \text{2.2} one may naturally conjecture that $\bigcup_{C \in \mathcal{K}^2} f(\mathcal{K}, C) = f(\mathcal{K}, C_0)$, for some $C_0 \in \mathcal{K}^2$. However, the symmetric case is a bit different as Theorem 1.4 shows.

\textbf{Proof of Theorem 1.4.} After a suitable translation and dilatation of $K$, let $K \subset \text{opt} H$ Now, by Proposition \text{2.1} there exist $p^i \in K \cap \text{bd}(H)$ and $u^i \in N(H, p^i)$ such that $0 \in \text{conv}(\{u^1, u^2, u^3\})$ and since $D(K, H) < 2R(K, H)$ we must have that the points $p^i$ belong to non-consecutive edge of $H$.

We now show that $r(K, H) \geq R(K, H)/4$. To do so, let $q^1, \ldots, q^6$ be the vertices of $H$ in clockwise order. Moreover, let us suppose that each $p^1$ belongs to $[q^2, q^3]$, $p^2$ to $[q^4, q^5]$, and $p^3$ to $[q^6, q^1]$. Notice that the length of

\[\text{conv}(\{q^1, q^2, \frac{1}{2}(q^4 + q^5)\}) \cap [q^3, q^6]\]

is half of the length of $[q^1, q^2]$. The same holds when considering the triangle of vertices $q^3, q^4, \frac{1}{2}(q^6 + q^1)$ and segment $[q^2, q^5]$ or the triangle of vertices $q^5, q^6, \frac{1}{2}(q^2 + q^3)$ and segment $[q^1, q^4]$. Therefore, those three triangles optimally contain a factor $\frac{1}{4}$ dilatation of $C$.

Now, assume that $c_1 + r_H \subset \text{opt} K$, for some $c_1 \in H$ and $r_1 < \frac{1}{4}$. In particular, this means that $c + rH \subset \text{opt} \text{conv}(\{p^1, p^2, p^3\})$, for some $r < 1/4$ and $c \in H$. Moreover, since $D(K, H) < 2R(K, H)$ we may assume without loss of generality, $p^1 \in (q^2, q^3)$, $p^2 \in (q^4, q^5)$, and $p^3 \in (q^6, q^1)$ and because of the symmetry group of $H$ that $c = tq^4 + su^1$, for some $t, s \geq 0$.

Now, let $L_2$ and $L_3$ be line segments supporting $c + rH$ with one endpoint at $p^2$ and $p^3$, respectively, such that $c + rH$ and $q^6$ are on the same halfspace induced by $L_2$, while $c + rH$ and $q^5$ are on the same halfspace induced by $L_3$ (cf. Figure 4). Notice that $c + rH$ cannot be contained in the halfspace determined by $[q^3, q^6]$ as otherwise, $L_2$ would not hit the boundary of $H$ again within $(q^2, q^3)$, contradicting that $c + rH$ is the inball of $\text{conv}(\{p^1, p^2, p^3\})$. The same reason implies that $c + rH$ cannot be contained in the halfspace determined by $[q^2, q^5]$.

In the remainder of the proof we want to show that $L_2$ and $L_3$ cannot both go through the same point $p^1 \in (q^2, q^3)$, a contradiction again.

Let $L'_2$ and $L'_3$ be line segments supporting $rH$ with endpoints at $q^6$ and $q^5$, respectively, such that $q^5$ and $rH$ belong to the same halfspace induced by $L'_2$, while $q^6$ and $rH$ belong to the same halfspace induced by $L'_3$. Since $r < 1/4$ this means that $L'_2$ intersects the boundary of $H$ at a point closer to $q^4$ (in terms of walking along the boundary of $H$) than $L'_3$. This last property will stay true on each of the next geometric steps of the proof.

Now, notice that replacing $rH$ by $tq^4 + rH$ and changing $L'_2$ and $L'_3$ accordingly, the two lines support $tq^4 + rH$ at $tq^4 + rq^4$ and $tq^4 + rq^4$, respectively. Since $[q^4, q^1]$ is parallel to $[q^2, q^6]$, we still have that $L'_2$ intersects the boundary of $H$ at a point closer to $q^4$ than $L'_3$. Again,
replacing \( tq^4 + rH \) by \( tq^4 + su^1 + rH = c + rH \), and changing \( L'_2 \) and \( L'_3 \) correspondingly, moves the intersection point of \( L'_2 \) with the boundary of \( H \) even closer to \( q^4 \) and the one of \( L'_3 \) even further away from \( q^4 \). Finally, replacing \( L'_2 \) by \( L_2 \) and \( L'_3 \) by \( L_3 \) again moves the intersection point with the boundary of \( H \) of the former closer to \( q^4 \) and the intersection point of the latter further away from \( q^4 \). Since at the beginning \( L'_2 \) and \( L'_3 \) could not intersect at a common boundary point of \( H \), we can conclude they can neither at the end. Hence \( L_2 \) and \( L_3 \) cannot intersect at any point of the boundary of \( H \), arriving at the desired contradiction. Thus we must have \( r(K, H) \geq R(K, H)/4 \). \( \square \)

**Remark 5.1.** Notice that if \( H \) is a regular hexagon, then (2) and (5) induce boundaries of the diagram \( f(K^2, H) \). Moreover, assuming that \( H = \text{conv}(\{q^1, \ldots, q^6\}) \), we conjecture that the family of isosceles triangles

\[
T_\lambda := \text{conv}\left\{ (1 - \lambda)q^1 + \lambda q^2, (1 - \lambda)q^4 + \lambda q^3, \frac{1}{2}(q^5 + q^6) \right\}
\]

induces the left boundary of the diagram \( f(K^2, H) \). Doing a detailed computation one can check that \( f(T_\lambda, H) = ((\lambda + 1)(2 - \lambda)/(4 + \lambda), (1 + \lambda)/2) \), or explicitly, that \( r(T_\lambda, H) = D(T_\lambda, H)(3 - D(T_\lambda, H))/(3 + D(T_\lambda, H)) \) (cf. Figure 5).

On the one hand, Theorem 1.4, together with the fact that there exist equilateral triangles \( T \) such that \( f(T, B_2) = (x, y) \) with \( y < 1 \) and \( x \) arbitrary close to 0, shows that the union \( f(K^2, K^2_0) = \bigcup_{C \in K^2_0} f(K^2, C) \) cannot be covered by \( f(K^2, H) \), \( H \) a regular hexagon. On the other hand, the equality case of [3] shows that \( (1/2, 3/4)^T \in f(K^2, C), C \in K^2_0 \), if and only if \( C \) is an affine transformation of a regular hexagon. Hence \( f(K^2, K^2_0) \) cannot be given by a single symmetric container.
Remark 5.2. With a bit more effort, we can also prove, by means of a new geometric property (that we will show in a forthcoming paper) together with Lemma 2.2, the non-obvious fact that

\[ f(\mathcal{K}^2, \mathcal{K}^2_0) = \bigcup_{S \text{ a triangle}} f(\mathcal{K}^2, S \cap (-S)). \]

We now turn to the case in which the diagram is considered with respect to a regular pentagon.

Proof of Theorem 1.6. Let \( p_1, \ldots, p_5 \) be the vertices of a regular pentagon \( P \) with \( \|p_i\| = 1 \), \( i = 1, \ldots, 5 \), and \( p^{ij} = (p^i + p^j)/2 \), \( 1 \leq i < j \leq 5 \). We start noticing that \( -P \subset s(P)P \), where we clearly have that \( s(P) = \|p^i\|/\|p^{34}\| = \sqrt{5} - 1 \). Notice also that since the diameter of a triangle is attained along an edge, selecting \( T := \text{conv}(\{p^{34}, p^{15}, p^{12}\}) \) we obtain

\[ D([p^{15}, p^{12}], P) < D([p^{34}, p^{15}], P) = D(-\frac{P}{s(P)}, P) = \frac{2}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{2}, \]

and so \( D(T, P) = (\sqrt{5} + 1)/2 \). Moreover, an analogous geometric argument shows that for any \( x \in [p^5, p^{15}] \) we have \( D([x, p^{34}], P) = (\sqrt{5} + 1)/2 \) too.

Now, let \( K \in \mathcal{K}^2 \). After a suitable dilatation and translation, we may assume that \( K \subset^{opt} P \). Using Proposition 2.1 there exist \( q^1, q^2, q^3 \in K \cap \text{bd}(P) \) such that, after reordering if needed, \( q^1 \in [p^3, p^4], q^2 \in [p^1, p^5], \) and \( q^3 \in [p^1, p^2] \).
Without loss of generality we may assume that \( q_1 \) is closer to \( p_3 \) than to \( p_4 \). Moreover, let \( x \in [p_1, p_5] \). Then
\[
D([x, q_1], P) \geq 2 \frac{h([x - q_1, q_1 - x], p_3)}{h(P - p_3, p_3)} \\
\geq 2 \frac{h([x - p_3^4, p_3^4 - x], p_3)}{h(P - p_3, p_3)} \\
\geq 2 \frac{h([p_3^4 - p_3^4, p_3^4 - p_3^1], p_3)}{h(P - p_3, p_3)} \geq \frac{\sqrt{5} + 1}{2},
\]
This shows that \( D(K, P) \geq D([q, q_1], P) \geq (\sqrt{5} + 1)/2 \), i.e.
\[
\frac{D(K, P)}{R(K, P)} \geq \frac{\sqrt{5} + 1}{2} = \frac{D(T, P)}{R(T, P)},
\]
concluding that \( j_p = (\sqrt{5} + 1)/2 \).

In order to finish the proof, let
\[
T_\lambda := \text{conv}(\{(1 - \lambda)p^5 + \lambda p^1, (1 - \lambda)p^2 + \lambda p^1, p_3^4\}),
\]
\( \lambda \in [0, 1/2] \) (which means \( T = T_{1/2} \)). From the computations above, we know that
\[
D([(1 - \lambda)p^5 + \lambda p^1, p_3^4], P) = D([(1 - \lambda)p^2 + \lambda p^1, p_3^4], P) = \frac{\sqrt{5} + 1}{2},
\]
for every \( \lambda \in [0, 1/2] \). Moreover, since
\[
D([(1 - \lambda)p^5 + \lambda p^1, (1 - \lambda)p^2 + \lambda p^1], P) \begin{cases}
< \frac{\sqrt{5} + 1}{2} & \text{if } \lambda = 1/2, \\
= 2 & \text{if } \lambda = 0,
\end{cases}
\]
by continuity there exists \( \lambda_0 \in (0, 1/2) \) such that \( D([(1 - \lambda_0)p^5 + \lambda_0 p^1, (1 - \lambda_0)p^2 + \lambda_0 p^1], P) = (\sqrt{5} + 1)/2 \). Such \( \lambda_0 \) fulfills in particular that \( ||p_3^4 - p_3^4|| = ||(1 - \lambda_0)p^5 + \lambda_0 p^1 - (1 - \lambda_0)p^2 - \lambda_0 p^1|| \), i.e. \( (1 - \lambda_0)||p_3^5 - p_2^5|| = ||p_3^4 - p_3^1|| \) and thus
\[
1 - \lambda_0 = \frac{||p_3^4 - p_3^1||}{||p_3^5 - p_2^5||} = \frac{||p_3^4 - p_3^1||}{||p_3^5 - p_3^1||} = \frac{D(T_1, P)}{2} = \frac{\sqrt{5} + 1}{4},
\]
from which \( \lambda_0 = (3 - \sqrt{5})/4 \). For such \( \lambda_0 \), define \( T' := T_{\lambda_0} \). Then all three edges of \( T' \) are diametrical, which means \( T' \) is equilateral and \( R(T', P)/D(T', P) = j_p \), which concludes the proof.

\begin{remark}
Notice that using the notation of Theorem 1.6, \( s(P) = \sqrt{5} - 1 \) together with \( \ref{5} \) gives
\[
(\sqrt{5} - 1)r(K, P) + R(K, P) \leq \frac{\sqrt{5}}{2} D(K, P)
\]
for every \( K \in \mathcal{K}^2 \) and equality holds in the above inequality for \( (1 - \lambda)(-P) + \lambda P, \lambda \in [0, 1] \) and especially for \( K = -P \) we have equality in \( \ref{5} \) for a set, which also achieves the Jung-constant \( j_P \). Moreover, we conjecture that the family filling the left boundary of the diagram \( f(K^2, P) \) is given by the isosceles triangles
\[
T_\lambda := \text{conv}(\{(1 - \lambda)p^1 + \lambda p^2, (1 - \lambda)p^1 + \lambda p^5, p_3^4\}),
\]
for $\lambda \in [0,1/2]$. After a lengthy computation, their image equals $f(T_\lambda, P) = (\lambda(1+\lambda\sqrt{5}-3), 1+\lambda\sqrt{5}-3\lambda)$. Explicitly, $r(T_\lambda, P) = 2\sqrt{5}-3\lambda(D(T_\lambda, P)/2 - 1)D(T_\lambda, P)/2$. Gathering this information, the Jung-constant of $P$ given in Theorem 1.6, 2, and 10, we obtain the (in one boundary conjectured) diagram $f(K^2, P)$ (cf. Figure 6).

**Remark 5.4.** Let us denote by $\mathcal{C}^k$ and $\mathcal{C}_0^k$ the families of all $k$-gons and all symmetric $k$-gons, respectively. Then we can find sequences $(C_m)_{m \in \mathbb{N}}$ in $\mathcal{C}^k$ or in $\mathcal{C}_0^k$ such that $C_m \to \mathcal{S}$ or $C_m \to \mathcal{S} \cap (-\mathcal{S})$, $(m \to \infty)$, respectively, for some triangle $\mathcal{S}$ and with respect to the Hausdorff metric. Doing so the continuity of the radii functionals together with Theorems 1.3 and 1.1 (see also Remark 5.2) would immediately imply that

$$f(K^2, \mathcal{K}^2) = \bigcup_{C \in \mathcal{C}^k} f(K^2, C) \quad \text{and} \quad f(K^2, \mathcal{K}_0^2) = \bigcup_{C \in \mathcal{C}_0^k} f(K^2, C),$$

for every $k \geq 3$ or $k \geq 6$, respectively. Moreover, in case $k \geq 4$ we could choose the sequence $(C_m)_{m \in \mathbb{N}}$ such that it converges against a parallelogram thus showing together with Theorem 1.5 that

$$\bigcap_{C \in \mathcal{C}^k} f(K^2, C) = \text{conv} \left( \{(0), (1)\} \right)$$

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