SPECTRAL RESULTS FOR MIXED PROBLEMS
AND FRACTIONAL ELLIPTIC OPERATORS

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Abstract. One purpose of the paper is to show Weyl type spectral asymptotic formulas for
pseudodifferential operators $P_a$ of order $2a$, with type and factorization index $a \in \mathbb{R}_+$ when
restricted to a compact set with smooth boundary. The $P_a$ include fractional powers of the
Laplace operator and of variable-coefficient strongly elliptic differential operators. Also the
regularity of eigenfunctions is described.

The other purpose is to improve the knowledge of realizations $A_{\chi, \Sigma^+}$ in $L^2(\Omega)$ of mixed
problems for second-order strongly elliptic symmetric differential operators $A$ on a bounded
smooth set $\Omega \subset \mathbb{R}^n$. Here the boundary $\partial \Omega = \Sigma$ is portioned smoothly into $\Sigma = \Sigma^- \cup \Sigma^+$,
the Dirichlet condition $\gamma_0 u = 0$ is imposed on $\Sigma^-$, and a Neumann or Robin condition $\chi u = 0$
is imposed on $\Sigma^+$. It is shown that the Dirichlet-to-Neumann operator $P_{\gamma, \chi}$ is principally
of type $\frac{1}{2}$ with factorization index $\frac{1}{2}$, relative to $\Sigma^+$. The above theory allows a detailed
description of $D(A_{\chi, \Sigma^+})$ with singular elements outside of $\mathbb{P}^2(\Omega)$, and leads to a spectral
asymptotic formula for the Krein resolvent difference $A_{\chi, \Sigma^+}^{-1} - A_{\gamma}^{-1}$.

Introduction

This paper has two parts. After a chapter with preliminaries, we establish in the
first part (Chapter 2) spectral asymptotic formulas of Weyl type for general Dirichlet
realizations of pseudodifferential operators ($\psi$do’s) of type $a > 0$, as defined in Grubb
[G13,14], and discuss the regularity of eigenfunctions.

In the second part (Chapter 3) we consider mixed boundary value problems for second-
order symmetric strongly elliptic differential operators, characterize the domain, and find
the spectral asymptotics of the Krein term (the difference of the resolvent from the Dirichlet
resolvent) in general variable-coefficient situations, extending the result of [G11a] for the
principally Laplacian case. This includes showing that the relevant Dirichlet-to-Neumann
operator fits into the calculus of the first part.

On Chapter 2: A typical example of the $\psi$do’s $P_a$ of type $a > 0$ and order $2a$ that
we treat is the $a$’th power of the Laplacian $(-\Delta)^a$ on $\mathbb{R}^n$, which is currently of great

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resolvent formula; Dirichlet-to-Neumann operator; boundary spectral asymptotics.
interest in probability and finance, mathematical physics and geometry. Also powers of variable coefficient-operators and much more general \( \psi \)dos are included. For the Dirichlet realization \( P_{a, \text{Dir}} \) on a bounded open set \( \Omega \subset \mathbb{R}^n \), spectral studies have mainly been aimed at the fractional Laplacian \((-\Delta)^a\). In the case of \((-\Delta)^a\), a Weyl asymptotic formula was shown already by Blumenthal and Getoor in [BG59]; recently a refined asymptotic formula was shown by Frank and Geisinger [FG11], and Geisinger gave an extension to certain other constant-coefficient operators [Ge14]. The exact domain \( D(P_{a, \text{Dir}}) \) has not been well described for \( a \geq \frac{1}{2} \), except in integer cases where the operator belongs to the calculus of Boutet de Monvel [B71]. Based on a recently published systematic theory [G13] of \( \psi \)do’s of type \( \mu \in \mathbb{C} \) (where those in the Boutet de Monvel calculus are of type 0), it is now possible to describe domains and parametrices of operators \( D(P_{a, \text{Dir}}) \) in an exact way, when \( \Omega \) is smooth. We analyse the sequence of eigenvalues \( \lambda_j \) (singular values \( s_j \) when the operator is not selfadjoint), showing that a Weyl asymptotic formula holds in general:

\[
s_j(P_{a, \text{Dir}}) \sim C(P_a, \Omega) j^{2a/n} \quad \text{for} \quad j \to \infty;
\]

moreover we show that the possible eigenfunctions are in \( d^a C^{2a}(\Omega) \) (in \( d^a C^{2a-\varepsilon}(\Omega) \) if \( 2a \in \mathbb{N} \)), where \( d(x) \sim \text{dist}(x, \partial \Omega) \). The results are generalized to operators \( P \) of order \( m = a + b \) with type and factorization index \( a, b \in \mathbb{R}_+ \).

On Chapter 3: The detailed knowledge of \( \psi \)do’s of type \( a \) has an application to the classical “mixed” boundary value problems for a second-order strongly elliptic symmetric differential operator \( A \) on a smooth bounded set \( \Omega \subset \mathbb{R}^n \). Here the boundary condition jumps from a Dirichlet to a Neumann (or Robin) condition at the interface of a smooth partition \( \Sigma = \Sigma^- \cup \Sigma^+ \) of the boundary \( \Sigma = \partial \Omega \); it is also called the Zaremba problem when \( A \) is the Laplacian. The \( L_2 \)-realization \( A_{\chi, \Sigma^+} \) it defines is less regular than standard realizations such as the Dirichlet realization \( A_{\gamma} \), but the domain has just been somewhat abstractly described; it is contained in \( \overline{H}^{3-\varepsilon}(\Omega) \) only (observed by Shamir [S68]), whereas \( D(A_{\gamma}) \subset \overline{H}^2(\Omega) \). The resolvent difference \( M = A_{\chi, \Sigma^+}^{-1} - A_{\gamma}^{-1} \) was shown by Birman [B62] to have eigenvalues satisfying \( \mu_j(M) = O(j^{-2/(n-1)}) \). The present author studied \( A_{\chi, \Sigma^+} \) from the point of view of extension theory for elliptic operators in [G11a] (to which we refer for more references to the literature); here we obtained the asymptotic estimate

\[
\mu_j(M) \sim c(M) j^{-2/(n-1)} \quad \text{for} \quad j \to \infty,
\]

in the case where \( A \) is principally Laplacian. This was drawing on the theories of Vishik and Eskin [E81] and Birman and Solomyak [BS77], and other pseudodifferential methods.

We now show that the Dirichlet-to-Neumann operator \( P_{\gamma, \chi} \) of order 1 on \( \Sigma \) associated with \( A \) is principally of type \( \frac{1}{2} \) with factorization index \( \frac{1}{2} \) relative to \( \Sigma^+ \). In the formulas connected with the mixed problem, \( P_{\gamma, \chi} \) enters as truncated to \( \Sigma^+ \). Therefore we can now use the detailed information on type \( \frac{1}{2} \) \( \psi \)do’s to describe the domain of \( A_{\chi, \Sigma^+} \) more precisely, showing how functions \( \notin \overline{H}^3(\Omega) \) occur. Moreover, using Chapter 2 we can extend the spectral asymptotic formula (0.2) to the general case where \( A \) has variable coefficients.

1. Preliminaries

The notations of [G13,G14b] will be used; we shall just give a brief summary here.
We consider a Riemannian \( n \)-dimensional \( C^\infty \) manifold \( \Omega_1 \) (it can be \( \mathbb{R}^n \)) and an embedded smooth \( n \)-dimensional manifold \( \Omega \) with boundary \( \partial \Omega \) and interior \( \Omega \). For \( \Omega_1 = \mathbb{R}^n \), \( \Omega \) can be \( \mathbb{R}^n_\pm = \{ x \in \mathbb{R}^n \mid x_n \geq 0 \} \); here \( (x_1, \ldots, x_{n-1}) = x' \). In the general manifold case, \( \Omega \) is taken compact. For \( \xi \in \mathbb{R}^n \), we denote \( (1 + |\xi|^2)^{\frac{1}{2}} = \langle \xi \rangle \). Restriction from \( \mathbb{R}^n \) to \( \mathbb{R}^n_\pm \) resp. \( \mathbb{R}^n_\mp \) (or from \( \Omega_1 \) to \( \Omega \) resp. \( \overline{\Omega} \)) is denoted \( r^+ \) resp. \( r^- \), extension by zero from \( \mathbb{R}^n_\pm \) to \( \mathbb{R}^n \) (or from \( \Omega \) resp. \( \overline{\Omega} \) to \( \Omega_1 \)) is denoted \( e^\pm \). In Chapter 3, the notation is used for a smooth subset \( \Sigma_+ \) of an \((n-1)\)-dimensional manifold \( \Sigma \).

A pseudodifferential operator \((\psi \text{do}) P \) on \( \mathbb{R}^n \) is defined from a symbol \( p(x, \xi) \) on \( \mathbb{R}^n \times \mathbb{R}^n \) by

\[
Pu = p(x, D)u = \text{OP}(p(x, \xi))u = (2\pi)^{-n} \int e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi = \mathcal{F}^{-1}_{\xi \rightarrow x}(p(x, \xi) \hat{u}(\xi));
\]

where \( \mathcal{F} \) is the Fourier transform \( \mathcal{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) dx \). The symbol \( p \) is assumed to be such that \( \partial_x^\alpha \partial_\xi^\beta p(x, \xi) \) is \( O(\langle \xi \rangle^{-|\alpha|}) \) for all \( \alpha, \beta \), for some \( r \in \mathbb{R} \) (defining the symbol class \( S^r_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \)); then it has order \( r \). The definition of \( P \) is carried over to manifolds by use of local coordinates; there are many textbooks (e.g. [G09]) describing this and other rules for operations with \( P \), e.g. composition rules. When \( P \) is a \( \psi \text{do} \) on \( \mathbb{R}^n \) or \( \Omega_1 \), \( P_+ = r^+ P e^+ \) denotes its truncation to \( \mathbb{R}^n_+ \) resp. \( \Omega \).

Let \( 1 < p < \infty \) (with \( 1 + 1/p' = 1 - 1/p \)), then we define for \( s \in \mathbb{R} \) the Bessel-potential spaces

\[
H^s_p(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{u}) \in L_p(\mathbb{R}^n) \},
\]

(1.2)

\[
\dot{H}^s_p(\mathbb{R}^n_+) = \{ u \in H^s_p(\mathbb{R}^n_+) \mid \text{supp } u \subset \mathbb{R}^n_+ \},
\]

\[
H^s_p(\mathbb{R}^n_-) = \{ u \in \mathcal{D}'(\mathbb{R}^n_-) \mid u = r^+ U \text{ for some } U \in H^s_p(\mathbb{R}^n) \};
\]

here \( \text{supp } u \) denotes the support of \( u \). For \( \Omega \) compact \( \subset \Omega_1 \), the definition extends to define \( H^s_p(\overline{\Omega}) \) and \( H^s_p(\Omega) \) by use of a finite system of local coordinates. When \( p = 2 \), we get the standard \( L^2 \)-Sobolev spaces, here the lower index \( 2 \) is usually omitted. (These and other spaces are thoroughly described in Triebel’s book [T95]. He writes \( \tilde{H} \) instead of \( H \); the present notation stems from Hörmander’s works.) We also need the Hölder spaces \( C^t \) for \( t \in \mathbb{R}_+ \setminus \mathbb{N} \); when \( t \in \mathbb{N}_0 \), \( C^t \) stands for functions with continuous derivatives up to order \( t \). \( C^t(\overline{\Omega}) \) denotes the \( C^t \)-functions on \( \Omega_1 \) supported in \( \overline{\Omega} \). Occasionally, we shall also formulate results in the Hölder-Zygmund spaces \( C^t_s \) for \( t \geq 0 \), that allow some statements to be valid for all \( t \); they equal \( C^t \) when \( t \not\in \mathbb{N}_0 \) and contain \( C^t \) in the integer cases (more details in [G14b]). The conventions \( \bigcup_{\varepsilon > 0} H^{s+\varepsilon}_p = H^{s+0}_p \), \( \bigcap_{\varepsilon > 0} H^{s-\varepsilon}_p = H^{s-0}_p \), defined in a similar way for the other scales of spaces, will sometimes be used.

A \( \psi \text{do} \) \( P \) is called classical (or polyhomogeneous) when the symbol \( p \) has an asymptotic expansion \( p(x, \xi) \sim \sum_{j \in \mathbb{N}_0} p_j(x, \xi) \) with \( p_j \) homogeneous in \( \xi \) of degree \( m - j \) for all \( j \). Then \( P \) has order \( m \). One can even allow \( m \) to be complex; then \( p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \); the operator and symbol are still said to be of order \( m \).

Here there is an additional definition: \( P \) satisfies the \( \mu \)-transmission condition (in short: is of type \( \mu \)) for some \( \mu \in \mathbb{C} \) when, in local coordinates,

\[
\partial_x^\beta \partial_\xi^\alpha p_j(x, -N) = e^{\pi i (m-2\mu-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x, N),
\]

(1.3)
for all $x \in \partial \Omega$, all $j, \alpha, \beta$, where $N$ denotes the interior normal to $\partial \Omega$ at $x$. The implications of the $\mu$-transmission property were a main subject of [G13,G14b]; the mapping properties for such operators in $C^\infty$-based spaces were shown in Hörmander [H85], Sect. 18.2.

A special role in the theory is played by the order-reducing operators. There is a simple definition of operators $\Xi_\pm^\mu$ on $\mathbb{R}^n$

$$\Xi_\pm^\mu = \text{OP}((\langle \xi \rangle \pm i\xi_n)^\mu);$$

they preserve support in $\mathbb{R}_\pm^n$, respectively. Here the functions $(\langle \xi \rangle \pm i\xi_n)^\mu$ do not satisfy all the estimates required for the class $S^{\text{Re} \mu}(\mathbb{R}^n \times \mathbb{R}^n)$, but the operators are useful for some purposes. There is a more refined choice $\Lambda_\pm^\mu$ that does satisfy all the estimates, and there is a definition $\Lambda^\mu(\pm)$ in the manifold situation. These operators define homeomorphisms for all $s \in \mathbb{R}$ such as

$$\begin{align*}
\Lambda^\mu_+ : \dot{H}^s_p(\Omega) &\cong \hat{H}^{s-\text{Re} \mu}_p(\Omega), \\
\Lambda^\mu_- : \overline{H}^s_p(\Omega) &\cong \overline{H}^{s-\text{Re} \mu}_p(\Omega);
\end{align*}$$

(1.4)

here $\Lambda^\mu_{-+}$ is short for $r^+\Lambda^\mu_{-}e^+$, suitably extended to large negative $s$ (cf. Rem. 1.1 and Th. 1.3 in [G13]).

The following special spaces introduced by Hörmander are particularly adapted to $\mu$-transmission operators $P$:

$$\begin{align*}
H^\mu_p(s)(\mathbb{R}_+^n) &= \Xi_\mu^+ e^+ \overline{H}^{s-\text{Re} \mu}_p(\mathbb{R}_+^n), \quad s > \text{Re} \mu - 1/p', \\
H^\mu_p(s)(\overline{\Omega}) &= \Lambda^\mu_+ e^+ \overline{H}^{s-\text{Re} \mu}_p(\overline{\Omega}), \quad s > \text{Re} \mu - 1/p', \\
\mathcal{E}_\mu(\overline{\Omega}) &= e^+ \{ u(x) = d(x)^\mu v(x) \mid v \in C^\infty(\overline{\Omega}) \};
\end{align*}$$

namely, $r^+P$ (of order $m$) maps them into $\overline{H}^{s-\text{Re} m}_p(\mathbb{R}_+^n)$, $\overline{H}^{s-\text{Re} m}_p(\Omega)$ resp. $C^\infty(\overline{\Omega})$ (cf. [G13] Sections 1.3, 2, 4), and they appear as domains of elliptic realizations of $P$. In the third line, $\text{Re} \mu > -1$ (for other $\mu$, cf. [G13]) and $d(x)$ is a $C^\infty$-function positive on $\Omega$ and vanishing to order 1 at $\partial \Omega$, e.g. $d(x) = \text{dist}(x, \partial \Omega)$ near $\partial \Omega$. One has that $H^\mu_p(s)(\overline{\Omega}) \supset \dot{H}^s_p(\overline{\Omega})$, and that the distributions are locally in $H^s_p$ on $\Omega$, but at the boundary they in general have a singular behavior. More details are given in [G13, G14b].

2. Spectral results for Dirichlet realizations of type $a$ operators

2.1 Dirichlet realizations of type $a$ operators.

Consider a Riemannian $n$-dimensional $C^\infty$ manifold $\Omega_1$ ($n \geq 2$) and an embedded compact $n$-dimensional $C^\infty$-manifold $\overline{\Omega}$ with boundary $\partial \Omega$ and interior $\Omega$. We consider an elliptic pseudodifferential operator on $\Omega_1$ with the following properties explained in detail in [G13]:

**Assumption 2.1.** Let $a \in \mathbb{R}_+$. $P_a$ is a classical elliptic pdo on $\Omega_1$ of order $2a$, which relative to $\Omega$ satisfies the a-transmission condition and has the factorization index $a$.

For example, $P_a$ can be the $a$-th power of a strongly elliptic second-order differential operator on $\Omega_1$, in particular $(-\Delta)^a$, or it can be the $a/m$-th power of a properly elliptic
differential operator of even order $2m$, but also other operators are allowed. (We call such operators “fractional elliptic”, because they share important properties with the fractional Laplacian.)

As in [G13], we define the Dirichlet realization $P_{a,\text{Dir}}$ in $L_2(\Omega)$ as the operator acting like $r^+P_a$ with domain

$$D(P_{a,\text{Dir}}) = \{ u \in \dot{H}^a(\Omega) \mid r^+P_a u \in L_2(\Omega) \}. \tag{2.1}$$

Then according to [G13], Sections 4-5,

$$D(P_{a,\text{Dir}}) = H^{a(2a)}(\Omega) = \Lambda^{(-a)}_+ e^+\mathcal{H}^a(\Omega). \tag{2.2}$$

We recall from [G13]:

**Lemma 2.2.** For $1 < p < \infty$, $s > a - 1/p'$, the spaces $H^a_p(\Omega)$ satisfy:

$$H^a_p(s)(\Omega) = \Lambda^{(-a)}_+ e^+\mathcal{H}^a_p(\Omega) \begin{cases} = \dot{H}^s_p(\Omega), & \text{if } s - a \in [1 - 1/p', 1/p], \\ \subset \dot{H}^{s-0}_p(\Omega), & \text{if } s = a + 1/p, \\ \subset d^a e^+\mathcal{H}^{s-a}_p(\Omega) + \dot{H}_p^{s}(\Omega), & \text{if } s - a - 1/p \in \mathbb{R}_+ \setminus \mathbb{N}, \\ \subset d^a e^+\mathcal{H}^{s-a}_p(\Omega) + \dot{H}_p^{s-0}(\Omega), & \text{if } s - a - 1/p \in \mathbb{N}. \end{cases} \tag{2.3}$$

Moreover,

$$H^a_p(s)(\Omega) \subset \dot{H}^a_p(\Omega), \text{ when } s - a \geq 0. \tag{2.4}$$

**Proof.** The equalities in (2.3) come from the definition of $H^a_p(s)(\Omega)$, and the inclusions are special cases of [G13] Th. 5.4. For the last statement, we note that when $s - a \geq 0$, $e^+\mathcal{H}^{s-a}_p(\Omega) \subset e^+L_p(\Omega)$, which is mapped into $\dot{H}^a_p(\Omega)$ by $\Lambda^{(-a)}_+$. □

In the case where $P_a$ is strongly elliptic, i.e., the principal symbol $p_{a,0}(x,\xi)$ satisfies

$$\text{Re } p_{a,0}(x,\xi) \geq c|\xi|^{2a},$$

with $c > 0$, we can describe $D(P_{a,\text{Dir}})$ in a different way:

Modifying $\Omega_1$ at a distance from $\overline{\Omega}$ if necessary, we can assume $\Omega_1$ to be compact without boundary. Then it is well-known that $P_a$ satisfies a Gårding inequality for $u \in C^\infty(\Omega_1)$:

$$\text{Re } (P_a u, u)_{L_2(\Omega_1)} \geq c_0 \| u \|^2_{H^a(\Omega_1)} - k \| u \|^2_{L^2(\Omega_1)}, \tag{2.5}$$

with $c_0 > 0$, $k \in \mathbb{R}$ (cf. e.g. [G09], Ch. 7), besides the inequality

$$\| (P_a u, v)_{L_2(\Omega_1)} \| \leq C \| u \|_{H^a(\Omega_1)} \| v \|_{H^a(\Omega_1)}.$$

(In the case of $(-\Delta)^a$ on $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, there is a slightly different formulation: For general $P_a$ one would here require $x$-estimates of the symbol to be uniform on the noncompact
Let \( \lambda \in \mathbb{C} \) denote an eigenvalue of the operator \( -\Delta + a \) and we can go on with solution results in Hölder spaces; this will be done further below.

**Theorem 2.3.** Let \( P_a \) satisfy Assumption 2.1.

If \( 0 \) is an eigenvalue of \( P_{a, \text{Dir}} \), its associated eigenfunctions are in \( \mathcal{E}_a(\overline{\Omega}) \).

When \( a \in \mathbb{R}_+ \setminus \mathbb{N} \), then the eigenfunctions \( u \) of \( P_{a, \text{Dir}} \) associated with nonzero eigenvalues \( \lambda \) lie in \( d^a C^{2a}(\overline{\Omega}) \) if \( 2a \notin \mathbb{N} \), in \( d^a C^{2a-\varepsilon}(\overline{\Omega}) \) (any \( \varepsilon > 0 \)) if \( 2a \in \mathbb{N} \); they are also in \( C^\infty(\Omega) \).

When \( a \in \mathbb{N} \), the eigenfunctions \( u \) of \( P_{a, \text{Dir}} \) associated with an eigenvalue \( \lambda \) lie in \( \{ u \in C^\infty(\overline{\Omega}) \mid \gamma_0 u = \gamma_1 u = \cdots = \gamma_{a-1} u = 0 \} \) (equal to \( \mathcal{E}_a(\overline{\Omega}) \) in this case).

**Proof.** (In some of the formulas here, the extension by zero \( e^+ \) is tacitly understood.) When \( \lambda \) is an eigenvalue, the associated eigenfunctions \( u \) are nontrivial solutions of

\[
(2.6) \quad r^+ P_a u = \lambda u.
\]

If \( \lambda = 0 \), then \( u \in \mathcal{E}_a(\overline{\Omega}) \), since the right-hand side in (2.6) is in \( C^\infty(\overline{\Omega}) \), and we can apply [G13] Th. 4.4.

Now let \( \lambda \neq 0 \). When \( a \in \mathbb{N} \), we are in a well-known standard elliptic case (as treated e.g. in [G96], Sect. 1.7); the eigenfunctions are in \( C^\infty(\Omega) \) as well as in \( \mathcal{E}_a(\overline{\Omega}) \), and \( \mathcal{E}_a(\overline{\Omega}) \) is the described subset of \( C^\infty(\Omega) \).

Next, consider the case \( a \in \mathbb{R}_+ \setminus \mathbb{N} \).

To begin with, we know that \( u \in \dot{H}^a(\overline{\Omega}) \) (from (2.1)). We shall use the well-known general embedding properties for \( p, p_1 \in ]1, \infty[ \):

\[
(2.7) \quad \dot{H}^a_p(\overline{\Omega}) \subset e^+ L_{p_1}(\Omega), \quad \text{when} \quad \frac{p}{p_1} \geq \frac{p}{p} - a, \quad \dot{H}^a_p(\overline{\Omega}) \subset \dot{C}^d(\overline{\Omega}) \quad \text{when} \quad a > \frac{p}{p_1}.
\]

If \( a > \frac{p}{2} \), we have already that \( \dot{H}^a(\overline{\Omega}) \subset \dot{C}^d(\overline{\Omega}) \), so (2.6) has right-hand side in \( C^d(\overline{\Omega}) \), and we can go on with solution results in Hölder spaces; this will be done further below.
If \( a \leq \frac{n}{2} \), we make a finite number of iterative steps to reach the information \( u \in C^0(\Omega) \), as follows: Define \( p_0, p_1, p_2, \ldots \), with \( p_0 = 2 \) and \( q_j = \frac{n}{p_j} \) for all the relevant \( j \), such that

\[
q_j = q_{j-1} - a \quad \text{for} \quad j \geq 1.
\]

This means that \( q_j = q_0 - ja \); we stop the sequence at \( j_0 \) the first time we reach a \( q_{j_0} \leq 0 \). As a first step, we note that \( u \in \dot{H}^a(\Omega) \subset e^+L_{p_1}(\Omega) \) implies \( u \in H^a_{p_1}(\Omega) \) by [G13] Th. 4.4, and then by (2.4), \( u \in \dot{H}^a_{p_1}(\Omega) \). In the next step we use the embedding \( \dot{H}^a_{p_1}(\Omega) \subset e^+L_{p_2}(\Omega) \) to conclude in a similar way that \( u \in \dot{H}^a_{p_2}(\Omega) \), and so on. If \( q_{j_0} < 0 \), we have that \( \frac{n}{p_{j_0}} < a \), so \( u \in \dot{H}^a_{p_{j_0}}(\Omega) \subset C^0(\Omega) \). If \( q_{j_0} = 0 \), the corresponding \( p_{j_0} \) would be \( +\infty \), and we see at least that \( u \in e^+L_p(\Omega) \) for any large \( p \); then one step more gives that \( u \in \dot{C}^0(\Omega) \).

The rest of the argument relies on Hölder estimates, as in [G13], Section 7, or still more efficiently by [G14b], Section 3. By the regularity results there,

\[
\begin{align*}
\dot{u} &\in C^0(\Omega) \quad \Longrightarrow \quad \dot{u} \in C^{(2a)}(\Omega) \subset e^+d^aC^a(\Omega) + \dot{C}^{2a-0}(\Omega) \subset e^+C^a(\Omega).
\end{align*}
\]

Next, \( u \in C^a(\Omega) \) implies

\[
\begin{align*}
u \in C^{(3a)}(\Omega) \subset e^+d^aC^{2a}(\Omega) + \dot{C}^{3a}(-\varepsilon)(\Omega) \subset e^+d^aC^{2a}(-\varepsilon)(\Omega)
\end{align*}
\]

where \(-\varepsilon\) is active if \( 2a \in \mathbb{N} \). Moreover, by the ellipticity of \( P_a - \lambda \) on \( \Omega_1 \), \( u \) is \( C^\infty \) on the interior \( \Omega \). \( \square \)

The fact that an eigenfunction in \( \dot{H}^a(\Omega) \) is in \( L_\infty(\Omega) \) was shown for \( P_a = (-\Delta)^a \) with \( 0 < a < 1 \) by Servadei and Valdinoci [SV13] by a completely different method.

**Remark 2.4.** For \( P_a = (-\Delta)^a \) it has been shown by Ros-Oton and Serra (see [RS14]) that an eigenfunction \( u \) cannot have \( u/d^a \) vanishing identically on \( \partial \Omega \). This implies that the regularity of \( u \) cannot be improved all the way up to \( \mathcal{E}_a(\Omega) \), when \( \lambda \neq 0 \), \( a \in \mathbb{R}_+ \setminus \mathbb{N} \). For if \( u \) were in \( \mathcal{E}_a(\Omega) \), it would also lie in \( C^\infty(\Omega) \) (since \( r^+P_a u = \lambda u \) would lie there). Now it is easily checked that \( C^\infty(\Omega) \cap \mathcal{E}_a(\Omega) = \dot{C}^\infty(\Omega) \) when \( a \in \mathbb{R}_+ \setminus \mathbb{N} \), where the functions vanish to order \( \infty \) at the boundary. In particular, \( u/d^a \) would be zero on \( \partial \Omega \), contradicting \( u \neq 0 \).

The theorem extends without difficulty to operators of order \( m = a + b \) considered in \( \dot{H}^a_{-}\)-spaces:

**Theorem 2.5.** Let \( P \) be of type \( a > 0 \) with factorization index \( a \), and of order \( m = a + b \), \( b > 0 \). Let \( 1 < p < \infty \), and define \( P_{\text{Dir}} \) as the operator from \( H^a_p(\Omega) \) to \( L_p(\Omega) \) acting like \( r^+P \). If \( 0 \) is an eigenvalue, the associated eigenfunctions are in \( \mathcal{E}_a(\Omega) \). If \( \lambda \neq 0 \) is an eigenvalue, the associated eigenfunctions are in \( d^aC^m(\Omega) \) (in \( d^aC^{m-\varepsilon}(\Omega) \) if \( m \) is integer).

**Proof.** The zero eigenfunctions are solutions with a \( C^\infty \) right-hand side, hence lie in \( \mathcal{E}_a(\Omega) \) by [G13] Th. 4.4.

Now let \( u \) be an eigenfunction associated with an eigenvalue \( \lambda \neq 0 \). In view of (2.4), we have \( u \in \dot{H}_p^a(\Omega) \). Using (2.7), we find by application of the regularity result of [G13]
Th. 4.4, by a finite number of iterative steps as in the proof of Theorem 2.3, that \( u \in \dot{H}^a_{p_1}, \dot{H}^a_{p_2}, \ldots \) with increasing \( p_j \)'s, until we reach \( u \in C^0(\Omega) \).

Now we can apply the Hölder results from [G13], [G14b]; this goes most efficiently by [G14b] Th. 3.2 \( \infty \) and Th. 3.3 for Hölder-Zygmund spaces:

\[
(2.9) \quad r^+ Pu \in C^a_s(\Omega) \implies u \in C^a(m+t)(\Omega) \subset d^a e^{t} \mathcal{C}^{m+t-a}(\Omega) + C^{m+t}(\Omega),
\]

\( t \geq 0, \) where \((-\varepsilon)\) is active if \( m + t - a \in \mathbb{N} \).

If \( b > a \), there are two steps:

\[
u \in C_0^a(\Omega) \implies u \in C^a(a+b)(\Omega) \subset e^{-a} \mathcal{C}^{m+b}(\Omega) + C^a(a+b)(\Omega) \subset e^{t} \mathcal{C}^{m+b}(\Omega).
\]

Next, \( u \in \mathcal{C}^{a+b}(\Omega) \) implies

\[
u \in C^a(m+a)(\Omega) \subset 2 \mathcal{C}^m(\Omega) + \mathcal{C}^a(-\varepsilon)(\Omega) \subset e^{t} \mathcal{C}^m(\Omega),
\]

where \((-\varepsilon)\) is active if \( m \in \mathbb{N} \).

If \( b \leq a \), we need a finite number of steps, such as:

\[
u \in C_0^a(\Omega) \implies u \in C^a(a+b)(\Omega) \subset e^{t} \mathcal{C}^{m+b}(\Omega) + C^a(a+b)(\Omega) \subset e^{t} \mathcal{C}^{m+b}(\Omega),
\]

where we use that \( a + b - \varepsilon > b \) for small \( \varepsilon \). Next, \( u \in \mathcal{C}^b(\Omega) \) implies

\[
u \in C^a(a+b)(\Omega) \subset e^{t} \mathcal{C}^{2b}(\Omega) + C^a(a+b)(\Omega) \subset e^{t} \mathcal{C}^{2b}(\Omega),
\]

where we use that \( a + 2b - \varepsilon > \min\{2b, a\} \) for small \( \varepsilon \). If \( 2b \geq a \), we end the proof as above. If not, we estimate again, now arriving at the exponent \( \min\{3b, a\} \), etc., continuing until we reach \( kb \geq a \); then the proof is completed as above. \( \square \)

2.3 Spectral asymptotics.

We shall now study spectral asymptotic estimates for our operators. We first recall some notation and basic rules.

As in [G84] we denote by \( \mathcal{C}_p(H, H_1) \) the \( p \)-th Schatten class consisting of the compact operators \( B \) from a Hilbert space \( H \) to another \( H_1 \) such that \( (s_j(B))_{j \in \mathbb{N}} \in \ell^p(\mathbb{N}) \). Here the \( s \)-numbers, or singular values, are defined as \( s_j(B) = \mu_j(B^*B)^{\frac{1}{2}} \), where \( \mu_j(B^*B) \) denotes the \( j \)-th positive eigenvalue of \( B^*B \), arranged nonincreasingly and repeated according to multiplicities. The so-called weak Schatten class consists of the compact operators \( B \) such that

\[
s_j(B) \leq C j^{-1/p} \text{ for all } j; \text{ we set } N_p(B) = \sup_{j \in \mathbb{N}} s_j(B) j^{1/p}.
\]

The notation \( \mathcal{S}_{(p)}(H, H_1) \) was used in [G84] for this space; instead we here use the name \( \mathcal{S}_{p,\infty}(H, H_1) \) (as in [G14a] and in other works). The indication \( (H, H_1) \) is replaced by \( (H) \) if \( H = H_1 \); it can be omitted when it is clear from the context. One has that \( \mathcal{S}_{p,\infty} \subset \mathcal{C}_{p+\varepsilon} \) for any \( \varepsilon > 0 \). They are linear spaces.
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We recall (cf. e.g. [G84] for details and references) that $N_p(B)$ is a quasinorm on $\mathcal{G}_{p,\infty}$, with a good control over the behavior under summation. Recall also that

\begin{equation}
\mathcal{G}_{p,\infty} \cdot \mathcal{G}_{q,\infty} \subset \mathcal{G}_{r,\infty}, \text{ where } r^{-1} = p^{-1} + q^{-1},
\end{equation}

and

\begin{equation}
s_j(B^*) = s_j(B), \quad s_j(EBF) \leq \|E\|s_j(B)\|F\|,
\end{equation}

when $E: H_1 \to H_3$ and $F: H_2 \to H$ are bounded linear maps between Hilbert spaces.

Moreover, we recall that when $\Xi$ is a bounded open subset of $\mathbb{R}^m$ and reasonably regular, or is a compact smooth $m$-dimensional manifold with boundary, then the injection $H^t(\Xi) \hookrightarrow L_2(\Xi)$ is in $\mathcal{G}_{m/t,\infty}$ when $t > 0$. It follows that when $B$ is a linear operator in $L_2(\Xi)$ that is bounded from $L_2(\Xi)$ to $H^t(\Xi)$, then $B \in \mathcal{G}_{m/t,\infty}$, with

\begin{equation}
N_{m/t}(B) \leq C\|B\|_{L(L_2(\Xi),H^t(\Xi))}.
\end{equation}

Recall also the Weyl-Ky Fan perturbation result:

\begin{equation}
s_j(B)j^{1/p} \to C_0, \quad s_j(B')j^{1/p} \to 0 \implies s_j(B + B')j^{1/p} \to C_0, \text{ for } j \to \infty.
\end{equation}

We shall moreover use Laptev’s result [L81]: When $P$ is a classical $\psi$do of order $t < 0$ on a closed $m$-dimensional manifold $\Xi_1$ with a smooth subset $\Xi$, $m \geq 2$, then

\begin{equation}
1_{\Xi_1 \setminus \Xi}P1_{\Xi} \in \mathcal{G}_{(m-1)/t,\infty};
\end{equation}

in fact it has a Weyl-type asymptotic formula of that order.

Results on the spectral behavior of compositions of $\psi$do’s of negative order interspersed with functions with jumps were shown in [G11b], see in particular Th. 4.3 there. We need to supply this result with a statement allowing a zero-order factor of the form of a sum of a pseudodifferential and a singular Green operator (in the Boutet de Monvel calculus); as functions with jumps we here just take $1_{\Omega}$.

**Theorem 2.6.** Let $M_\Omega$ be an operator on $\overline{\Omega}$ composed of $l \geq 1$ factors $P_{j,+}$ formed of classical pseudodifferential operators $P_j$ on $\Omega_1$ of negative orders $-t_j$ and truncated to $\overline{\Omega}$, $j = 1, \ldots, l$, and one factor $Q_+ + G$ (placed somewhere between them), where $Q$ is classical of order 0 and $G$ is a singular Green operator on $\overline{\Omega}$ of order and class 0:

\begin{equation}
M_\Omega = P_{1,+} \ldots P_{l_0,+}(Q_+ + G)P_{l_0+1,+} \ldots P_{l,+}.
\end{equation}

Let $t = t_1 + \cdots + t_l$, and let $m(x, \xi)$ be the product of the principal $\psi$do symbols on $\Omega_1$:

$$m(x, \xi) = p_{1,0}(x, \xi) \cdots q_0(x, \xi) \cdots p_{l,0}(x, \xi).$$

Then $M_\Omega$ has the spectral behavior:

\begin{equation}
s_j(M_\Omega)j^{t/n} \to c(M_\Omega)^{t/n} \text{ for } j \to \infty,
\end{equation}

where $c$ is a constant.
where

\begin{equation}
(2.17) \quad c(M_\Omega) = \frac{1}{n(2\pi)^n} \int_\Omega \int_{|\xi|=1} (m(x,\xi)^* m(x,\xi))^{n/2t} d\omega(\xi) dx
\end{equation}

**Proof.** By Th. 4.3 of [G11b] with interspersed functions of the form $1_\Omega$, the statement holds if $Q = 1$ and $G = 0$, so the new thing is to include nontrivial cases of $Q$ and $G$. We can assume that $l_0 \geq 1$. For the contribution from $Q$ we write

\begin{equation}
(2.18) \quad P_{l_0}Q^+ = r^+ P_{l_0}e^+ r^+ Q e^+ = r^+ P_{l_0}Q e^+ - r^+ P_{l_0} e^- r^- Q e^+.
\end{equation}

Here $P_{l_0}Q$ is a \(\psi\)-do of order $-t_{l_0} < 0$ with principal symbol $p_{l_0,0}q_0$, and when $r^+ P_{l_0}Q e^+$ is taken into the original expression, we get an operator of the type treated by Th. 4.3 of [G11b],

\begin{equation}
(2.19) \quad P_{l_0}Q^+ + P_{l_0+1} + \ldots + P_{l_+},
\end{equation}

for which the statement (2.16), (2.17) holds. For the other term in (2.18), we use that $r^+ P_{l_0}e^-$ is the type of operator covered by the theorem of Laptev [L81] (cf. (2.14)), belonging to $\mathfrak{S}_{(n-1)/t_{l_0},\infty}$, and $r^{-} Q e^{+}$ is bounded in $L_2$, so in view of the rules (2.10) and (2.11) for compositions, the full expression with this term inserted is in $\mathfrak{S}_{n/(t+\theta),\infty}$ for a certain $\theta > 0$. The spectral asymptotic estimate obtained for the term (2.19) is preserved when we add this term of a better weak Schatten class, in view of (2.13).

The contribution from $G$ will likewise be shown to be in a better weak Schatten class that the main $\psi$-do term; this requires a deeper effort. Actually, the strategy can be copied from some proofs in [G14a], as follows: Consider first the composition of $G$ with just one operator:

\[ M = P_+ G, \]

where $P$ is of order $-t < 0$. In local coordinates, we can extend Th. 4.1 in [G14a] to this operator, writing

\[ \psi P_+ G \psi_1 = \sum_{k \in \mathbb{N}_0} \psi P_+ K_k \Phi_k^* \psi_1 = \sum_{k \in \mathbb{N}_0} \psi P_+ \zeta K_k \Phi_k^* \psi_1 + \sum_{k \in \mathbb{N}_0} \psi P_+ K_k (1 - \zeta) \Phi_k^* \psi_1, \]

with Poisson and trace operators $K_k$ and $\Phi_k^*$ as explained in [G14a], and letting $P_+ K_k$ play the role of $K_k$ in the proof there. Here $(\psi P_+ K_k \zeta)^*$ is bounded from $L_2(B_{R,+})$ to $\overline{\mathcal{H}}'(B'_{R'})$ for a large $R'$, hence lies in $\mathfrak{S}_{(n-1)/t,\infty}$ (by the property of the injection of $\overline{\mathcal{H}}'(B'_{R'})$ into $L_2(B'_{R'})$, $B'_{R'} = \{x' \in \mathbb{R}^{n-1} \mid |x'| < R'\}$). The proof that the full series $P_+ G$ lies in $\mathfrak{S}_{(n-1)/t,\infty}$ goes as in [G14a] (using also that the terms with $1 - \zeta$ have a smoothing component, and that the series is rapidly convergent). Moreover, Cor. 4.2 there shows how the result is extended to the manifold situation.

When there are several factors in $M$, we need only use that $P_{j,+} \in \mathfrak{S}_{n/j,\infty}$ for the other factors and apply the product rule (2.10), and we end with the information that the full product is in $\mathfrak{S}_{n/(t+\theta),\infty}$ for some $\theta > 0$, so that the spectral asymptotics remains as that of (2.19), when this term is added on. □

The result extends easily to matrix-formed operators.

Now we can show a spectral asymptotic estimate for $P_{a,\text{Dir}}$.  


Theorem 2.7. Let $P_a$ satisfy Assumption 2.1. Assume that $P_{a,\text{Dir}}$ is invertible, or more generally that $P_{a,\text{Dir}} + c$ is invertible from $D(P_{a,\text{Dir}})$ to $L_2(\Omega)$ for some $c \in \mathbb{C}$ (this holds if $P_a$ is strongly elliptic).

The singular values $s_j(P_{a,\text{Dir}})$ (eigenvalues of $(P_{a,\text{Dir}}^* P_{a,\text{Dir}})^{1/2}$) have the asymptotic behavior:

\begin{equation}
(2.20) \quad s_j(P_{a,\text{Dir}}) = C(P_{a,\text{Dir}}) j^{2a/n} + o(j^{2a/n}), \text{ for } j \to \infty,
\end{equation}

where $C(P_{a,\text{Dir}}) = C'(P_{a,\text{Dir}})^{-2a/n}$, defined from the principal symbol $p_{a,0}(x, \xi)$ by:

\begin{equation}
(2.21) \quad C'(P_{a,\text{Dir}}) = \frac{1}{n(2\pi)^n} \int_{\Omega} \int_{|\xi|=1} |p_{a,0}(x, \xi)|^{-n/2a} d\omega(\xi) dx.
\end{equation}

Proof. By Th. 4.4 of [G13], $P_{a,\text{Dir}}$, acting like $r^+P_a$, has a parametrix of order $-2a$,

\begin{equation}
(2.22) \quad R = \Lambda_{+,+}^{(-a)}(\tilde{Q} + G)\Lambda_{-,+}^{(-a)} = r^+\Lambda_{+,+}^{(-a)}e^+(r^+\tilde{Q}e^+ + G)r^+\Lambda_{-,+}^{(-a)}e^+;
\end{equation}

in the last expression, we have written the restriction- and extension-operators out in detail. In comparison with the formula for $R$ in [G13], Th. 4.4, we have moreover placed an $r^+$ in front, which is allowed since $R$ maps into a space of functions supported in $\overline{\Omega}$.

The singular Green operator component $G$ was missing in some preliminary versions of [G13].) The operator is of the form treated in Theorem 2.6, which gives the asymptotic behavior of the $s$-numbers of $R$:

\begin{equation}
(2.23) \quad s_j(R) j^{2a/n} \to c(R)^{2a/n} \text{ for } j \to \infty;
\end{equation}

here $c(R) = C'(P_{a,\text{Dir}})$ defined in (2.21), since the principal symbol of $\Lambda_{+,+}^{(-a)}\tilde{Q}\Lambda_{-,+}^{(-a)}$ is the inverse of the principal symbol of $P_a$.

That $R$ is parametrix of $r^+P_a = P_{a,\text{Dir}}$ implies that

\begin{equation}
(2.24) \quad P_{a,\text{Dir}}R = I - S_1, \text{ where } S_1: L_2(\Omega) \to C_\infty(\overline{\Omega}).
\end{equation}

Consider the case where $P_{a,\text{Dir}}$ is invertible; it is clearly compact since it maps $L_2(\Omega)$ into $\mathcal{H}^a(\overline{\Omega})$. It follows from (2.24) that

\begin{equation}
P_{a,\text{Dir}}^{-1} = P_{a,\text{Dir}}^{-1}(P_{a,\text{Dir}}R + S_1) = R + S_2, \quad S_2 = P_{a,\text{Dir}}^{-1}S_1,
\end{equation}

where $P_{a,\text{Dir}}^{-1} \in \mathcal{G}_{n/a,\infty}$ (since it maps $L_2(\Omega)$ into $\mathcal{H}^a(\overline{\Omega})$), and $S_1 \in \bigcap_{p>0} \mathcal{G}_{p,\infty}$, so $S_2 \in \bigcap_{p>0} \mathcal{G}_{p,\infty}$ by (2.10). By (2.13), the spectral asymptotic formula (2.23) for $R$ will therefore imply the same spectral asymptotic formula for $P_{a,\text{Dir}}^{-1}$, so

\begin{equation}
s_j(P_{a,\text{Dir}}^{-1}) j^{2a/n} \to C'(P_{a,\text{Dir}}^{-1})^{2a/n}.
\end{equation}

The asymptotic formula can also be written as the formula (2.20) for the $s$-numbers of $P_{a,\text{Dir}}$. 

\begin{align*}
\text{SPECTRAL RESULTS 11}
\end{align*}
If instead $P_{a,\text{Dir}} + c$ is invertible, we can write

$$(P_{a,\text{Dir}} + c)R = I - S_1 + cR,$$

with $S_1$ as in (2.24), and hence

$$(P_{a,\text{Dir}} + c)^{-1} = (P_{a,\text{Dir}} + c)^{-1}((P_{a,\text{Dir}} + c)R + S_1 - cR)$$

$$= R + (P_{a,\text{Dir}} + c)^{-1}S_1 - c(P_{a,\text{Dir}} + c)^{-1}R.$$

Here $(P_{a,\text{Dir}} + c)^{-1}S_1 \in \bigcap_{p>0} \mathcal{S}_{p,\infty}$ and $c(P_{a,\text{Dir}} + c)^{-1}R \in \mathcal{S}_{n/3a,\infty}$, since $(P_{a,\text{Dir}} + c)^{-1} \in \mathcal{S}_{n/a,\infty}$, and $R \in \mathcal{S}_{n/2a,\infty}$ in view of its spectral behavior shown above. Thus $(P_{a,\text{Dir}} + c)^{-1}$ is a perturbation of $R$ by operators in better weak Schatten classes, and the desired spectral results follow for $(P_{a,\text{Dir}} + c)^{-1}$ and it inverse $P_{a,\text{Dir}} + c$. \(\square\)

When $P_{a,\text{Dir}}$ is selfadjoint $\geq 0$, its eigenvalue sequence $\lambda_j, j \in \mathbb{N}$, coincides with the sequence of $s_j$-values, and Theorem 2.7 gives an asymptotic estimate of the eigenvalues.

In this case, the asymptotic estimate extends to arbitrary open sets $\Omega$ (assumed bounded when $\Omega = \mathbb{R}^n$), with the Dirichlet realization defined by Friedrichs extension of $r^+P_a$ from $C_0^\infty(\Omega)$, since the eigenvalues can be characterized by the minimax principle, which gives a monotonicity property in terms of nested open sets.

As mentioned in the introduction, the estimate (2.20) was shown for the case $P_a = (-\Delta)^a$ by Blumenthal and Getoor in [BG59]. In this case, a two-terms asymptotic formula for the $N$th average of eigenvalues as $N \to \infty$ was obtained by Frank and Geisinger in [FG11], and Geisinger extended the estimate (2.20) to a larger class of constant-coefficient $\psi$do’s in [Ge14].

**Remark 2.8.** Theorem 2.7 extends straightforwardly to Dirichlet realizations of operators $P$ as in Theorem 2.5; in the proof, the factor $\Lambda^{(-a)}$ is replaced by $\Lambda^{(-b)}$, and $2a$ in the asymptotic formula is replaced by $m = a + b$.

## 3. Mixed problems for second-order symmetric strongly elliptic differential operators

### 3.1 The Krein resolvent formula.

We shall now apply the knowledge of the operators of type $\frac{1}{2}$ to the mixed boundary value problem for second-order elliptic differential operators. The setting is the following:

On a bounded $C^\infty$-smooth open subset $\Omega$ of $\mathbb{R}^n$ with boundary $\partial \Omega = \Sigma$ we consider a second-order symmetric differential operator with real coefficients in $C^\infty(\overline{\Omega})$:

(3.1)  

$$Au = -\sum_{j,k=1}^{n} a_{jk}(x)\partial_{kj}u + a_0(x)u,$$

here $a_{jk} = a_{kj}$ for all $j, k$. $A$ is assumed strongly elliptic, i.e., $\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2$ for $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$, with $c_0 > 0$. We denote as usual $u|_\Sigma = \gamma_0 u$, and consider moreover the conormal derivative $\nu$ and a Robin variant $\chi$ (both are Neumann-type boundary operators)

(3.2)  

$$\nu u = \sum_{j,k=1}^{n} n_j \gamma_0 (a_{jk}\partial_k u), \quad \chi u = \nu u - \sigma \gamma_0 u;$$
here $\vec{n} = (n_1, \ldots, n_m)$ denotes the interior unit normal to the boundary, and $\sigma$ is a real $C^\infty$-function on $\Sigma$. With $\Sigma_+$ denoting a closed $C^\infty$-subset of $\Sigma$, we define $L_2(\Omega)$-realizations $A_\gamma$ and $A_{\chi, \Sigma_+}$ of $A$ determined respectively by the boundary conditions:

$$
\begin{align*}
\gamma_0 u &= 0 \text{ on } \Sigma, \text{ the Dirichlet condition}, \\
\chi u &= 0 \text{ on } \Sigma_+, \gamma_0 u = 0 \text{ on } \Sigma \setminus \Sigma_+, \text{ a mixed condition.}
\end{align*}
$$

It is accounted for in [G11a] that with the domains defined more precisely by

$$
\begin{align*}
D(A_\gamma) &= \{ u \in \overline{H}^2(\Omega) \mid \gamma_0 u = 0 \}, \\
D(A_{\chi, \Sigma_+}) &= \{ u \in \overline{H}^1(\Omega) \cap D(A_{\max}) \mid \gamma_0 u \in \overline{H}^\frac{1}{2}(\Sigma_+), \chi u = 0 \text{ on } \Sigma_+ \},
\end{align*}
$$

where $A_{\max}$ denotes the operator acting like $A$ with domain $D(A_{\max}) = \{ u \in L_2(\Omega) \mid Au \in L_2(\Omega) \}$, the operators $A_\gamma$ and $A_{\chi, \Sigma_+}$ are selfadjoint lower bounded. We can and shall assume that a sufficiently large constant has been added to $A$ such that both operators have a positive lower bound.

Let

$$
X = \dot{H}^{-\frac{1}{2}}(\Sigma_+); \text{ then } X^* = \overline{H}^\frac{1}{2}(\Sigma_+),
$$

with respect to a duality consistent with the $L_2$-scalar product on $\Sigma_+$. The injection $i_X: X \hookrightarrow H^{-\frac{1}{2}}(\Sigma)$ can be viewed as an “extension by zero” $e^+$ (often tacitly understood), and its adjoint $(i_X)^*: H^\frac{1}{2}(\Sigma) \to \overline{H}^\frac{1}{2}(\Sigma_+)$ is the restriction $r^+$.

Recalling that $\gamma_0$ defines a homeomorphism from $Z = \ker(A_{\max}) = \{ u \in L_2(\Omega) \mid Au = 0 \}$ to $H^{-\frac{1}{2}}(\Sigma)$ with inverse $K_\gamma$ (a Poisson operator), we define

$$
V = K_\gamma X, \quad \gamma_V: V \xrightarrow{\sim} X;
$$

where $V$ is a closed subspace of $Z$ (both closed in the $L_2(\Omega)$-norm), and $\gamma_V$ denotes the restriction of $\gamma_0$ to $V$. Note that $\gamma_V^{-1}$ acts like $K_\gamma$ on $X$; it is also denoted $K_{\gamma, X}$ in [G11a]. We denote by $i_V$ the injection of $V$ into $Z$, its adjoint is the orthogonal projection $\text{pr}_V$ of $Z$ onto $V$. Let us moreover introduce the relevant Dirichlet-to-Neumann operators

$$
P_{\gamma, \nu} = \nu K_\gamma, \quad P_{\gamma, \chi} = \chi K_\gamma = P_{\gamma, \nu} - \sigma;
$$

they are pseudodifferential operators of order 1 on $\Sigma$, both formally selfadjoint.

The following Krein resolvent formula was shown in [G11a], Sect. 4.1:

**Proposition 3.1.** For the realizations of $A$ defined above,

$$
A_{\chi, \Sigma_+}^{-1} - A_\gamma^{-1} = i V \gamma_V^{-1} L^{-1} (\gamma_V^{-1})^* \text{pr}_V.
$$

Here $L$ is the (selfadjoint invertible) operator from $X$ to $X^*$ acting like $-r^+ P_{\gamma, \chi} e^+$ and with domain

$$
D(L) = \gamma_0 D(A_{\chi, \Sigma_+}).
$$

It was shown in [G11a] that $D(L) \subset \dot{H}^{1-\varepsilon}(\Sigma_+)$ for all $\varepsilon > 0$, but that the inclusion does not hold with $\varepsilon = 0$.

Since $L$ acts like $-P_{\gamma, \chi,+}$ and is surjective onto $\overline{H}^\frac{1}{2}(\Sigma_+)$, we also have

$$
D(L) = \{ \varphi \in \dot{H}^{1-\varepsilon}(\Sigma_+) \mid r^+ P_{\gamma, \chi} \varphi \in \overline{H}^\frac{1}{2}(\Sigma_+) \}.
$$

Below we shall improve the knowledge of the domain by setting $P_{\gamma, \chi}$ in relation to the types of operators studied in Chapter 2.
3.2 Structure of the Dirichlet-to-Neumann operator.

To study the symbol of $P_{\gamma,\chi}$ we consider the operators in a neighborhood $U$ of a point $x_0 \in \partial \Omega = \Sigma$, where local coordinates $x = (x_1, \ldots, x_n) = (x', x_n)$ are chosen such that $U \cap \Omega = \{(x', x_n) \mid x' \in B_1, 0 < x_n < 1\}$ and $U \cap \partial \Omega = \{(x', x_n) \mid x' \in B_1, x_n = 0\}; B_1 = \{x' \in \mathbb{R}^{n-1} \mid |\xi'| < 1\}$. In these coordinates, the principal symbol of $A$ at the boundary is a polynomial

$$a(x', 0, \xi) = \sum_{j,k=1}^{n} a_{jk}(x', 0)\xi_j \xi_k = a_{nn}(x', 0)\xi_n^2 + 2b(x', \xi')\xi_n + c(x', \xi'),$$

with

$$a_{ij}(x', 0) = a_{ji}(x', 0)$$

the coefficients are real with $a_{jk} = a_{kj}$. We often write $(x', 0)$ as $x'$. Since $A$ is strongly elliptic, $a(x', \xi', \xi_n) > 0$ when $\xi' \neq 0$, so the polynomial $a(x', \xi', \lambda)$ in $\lambda$ has no real roots when $\xi' \neq 0$. When we set

$$a'(x', \xi') = a_{nn}(x')c(x', \xi') - b(x', \xi')^2 = \sum_{j,k=1}^{n-1} a_{jk}'(x')\xi_j \xi_k,$$

we therefore have that $a'(x', \xi') > 0$ for $\xi' \in \mathbb{R}^{n-1} \setminus 0$. The roots of $a(x', \xi', \lambda)$ equal $\lambda_{\pm} = a_{nn}^{-1}(\xi_n^2) = a_{nn}^{-1}(-b \pm i\kappa_0)$, lying respectively in $\mathbb{C}_\pm = \{\lambda \in \mathbb{C} \mid \text{Im } \lambda \geq 0\}$, where we have set

$$\kappa_0(x', \xi') = a'(x', \xi')^{1/2} > 0.$$

Denote

$$\kappa_{\pm}(x', \xi') = \mp i\lambda_{\pm} = a_{nn}^{-1}(\kappa_0 \pm ib);$$

then $a$ has the factorization

$$a(x', \xi', \xi_n) = a_{nn}(x')(\kappa_{+}(x', \xi') + i\xi_n)(\kappa_{-}(x', \xi') - i\xi_n),$$

where $\kappa_{+}$ and $\kappa_{-}$ both have positive real part ($= \kappa_0$). This plays a role in standard investigations of boundary problems. We go on to study the Dirichlet-to-Neumann operators.

The principal symbol-kernel $\tilde{k}_{\gamma}(x', x_n, \xi')$ of $K_{\gamma}$ is the solution operator for the semihomogeneous model problem (with $\varphi$ given in $\mathbb{C}$):

$$a(x', \xi', D_n)u(x_n) = 0 \text{ on } \mathbb{R}_+, \quad u(0) = \varphi;$$

it is seen from (3.13) that the solution in $L_2(\mathbb{R}_+)$ is $\varphi e^{-\kappa_+ x_n}$, so

$$\tilde{k}_{\gamma}(x', x_n, \xi') = e^{-\kappa_+ x_n}.$$
Then the principal symbol of $P_{\gamma,\nu}$ is

$$p_{\gamma,\nu}(x', \xi')_1 = \gamma_0(a_{\alpha n} \partial_{x_n} + \sum_{k=1}^{n-1} a_{nk} i \xi_k) e^{\kappa_+ x_n} = -a_{\alpha n} \kappa_+ + \sum_{k=1}^{n-1} a_{nk} i \xi_k$$

$$= -a_{\alpha n}(-i) a_{\alpha n}^{-1}(-b + i \kappa_0) + ib = -\kappa_0.$$ 

Since $P_{\gamma,\chi} = P_{\gamma,\nu} - \sigma$ with $\sigma$ of order 0, $P_{\gamma,\chi}$ likewise has the principal symbol $-\kappa_0$.

The important fact that we observe here is that $\kappa_0(x', \xi')$ is even in $\xi'$;

$$\kappa_0(x', -\xi') = \kappa_0(x', \xi'), \quad \text{with} \quad \partial_x^\alpha \partial_{\xi'}^\beta \kappa_0(x', -\xi') = (-1)^{|\alpha|} \partial_x^\alpha \partial_{\xi'}^\beta \kappa_0(x', \xi')$$

for all $\alpha, \beta$.

(since $c(x', \xi')$ and $b(x', \xi')^2$ are clearly even in $\xi'$). Since $\kappa_0$ is homogeneous of degree 1, it therefore has the $\frac{1}{2}$-transmission property with respect to any smooth subset of $B_1$, satisfying (1.4) with $m = 1$, $\mu = \frac{1}{2}$.

Moreover, we shall show that it has factorization index $\frac{1}{2}$ with respect to any smooth subset of $B_1$: We can take the subset as $B_{1,+} = \{ x' \in \mathbb{R}^{n-1} \mid |x'| < 1, x_{n-1} > 0 \}$, with $(x_1, \ldots, x_{n-2})$ denoted $x''$. Now apply the same procedure as above to the polynomial $a'(x'', 0, \xi') = \kappa_0(x'', 0, \xi'', \varepsilon_{n-1})^2$ in $\varepsilon_{n-1}$. It has a factorization analogously to (3.13):

$$\kappa_0(x'', 0, \xi')^2 = a'_{n-1,n-1}(x'')(\kappa'_+(x'', \xi'') + i \varepsilon_{n-1})(\kappa'_-(x'', \xi'') - i \varepsilon_{n-1}),$$

where $a'_{n-1,n-1} > 0$ and $\kappa'_\pm$ have positive real part; here $\kappa'_\pm = \mp i \lambda'_\pm$, where $\lambda'_\pm$ are the roots of $a'(x'', 0, \xi'', \lambda)$ lying in $\mathbb{C}_\pm$, respectively. It follows that

$$\kappa_0(x'', 0, \xi') = a'_{n-1,n-1}(x'')^{\frac{1}{2}}(\kappa'_+(x'', \xi'') + i \varepsilon_{n-1})^{\frac{1}{2}}(\kappa'_-(x'', \xi'') - i \varepsilon_{n-1})^{\frac{1}{2}},$$

where $(\kappa'_+(x'', \xi'') + i \varepsilon_{n-1})^{\frac{1}{2}}$ extends analytically in $\varepsilon_{n-1}$ into $\mathbb{C}_-$ and $(\kappa'_-(x'', \xi'') - i \varepsilon_{n-1})^{\frac{1}{2}}$ extends analytically in $\varepsilon_{n-1}$ into $\mathbb{C}_+$ (in short, are a “plus-symbol” resp. a “minus-symbol”, cf. [E81], [G13]).

Carrying the information back to $\Omega$ and $\Sigma = \partial \Omega$, we have obtained:

**Theorem 3.2.** The principal symbol of the Dirichlet-to-Neumann operator $P_{\gamma,\chi}$ equals $-\kappa_0(x', \xi')$ (expressed in local coordinates in (3.10)-(3.11)), negative and elliptic of order 1. For any smooth subset $\Sigma_+ \subset \Sigma$, $\kappa_0$ is of type $\frac{1}{2}$ and has factorization index $\frac{1}{2}$ relative to $\Sigma_+$. An explicit factorization in local coordinates is given in (3.16).

### 3.3 Precisions on $L$ and $L^{-1}$.

Define $L_1$ to be a $\psi$do on $\Sigma$ with symbol $\kappa_0(x', \xi')$, and let $L_0 = -P_{\gamma,\chi} - L_1$. Then since $L$ acts like $-P_{\gamma,\chi,+}$, it acts like $L_{1,+} + L_{0,+}$:

$$L \varphi = L_{1,+} \varphi + L_{0,+} \varphi, \quad \text{for} \ \varphi \in D(L).$$

Here $L_1$, classical of order 1, is principally equal to $-P_{\gamma,\chi}$ and $-P_{\gamma,\nu}$, whereas the operator $L_0$ is a classical $\psi$do of order 0, containing both the local term $\sigma$ and the nonlocal difference between $P_{\gamma,\nu}$ and its principal part.
As shown in Theorem 3.2, $L_1$ is of type $\frac{1}{2}$ and has factorization index $\frac{1}{2}$ relative to $\Sigma_+$. Here $L_{1,+}$, when considered on $\dot{H}^{1-\varepsilon}(\Sigma_+)$, identifies with the operator $r^+ L_1$ in the homogeneous Dirichlet problem for $L_1$, going from $\dot{H}^{1-\varepsilon}(\Sigma_+)$ to $\dot{H}^{-\varepsilon}(\Sigma_+)$. It has according to [G13] Th. 4.4 a parametrix $R: \overline{\mathcal{H}}^{s-1}(\Sigma_+) \to \dot{H}^{\frac{s}{2}}(\Sigma_+)$ for $s > \frac{1}{2}$; here $\dot{H}^{\frac{s}{2}}(\Sigma_+) = \dot{H}^{s}(\Sigma_+)$ for $\frac{1}{2} < s < 1$, cf. (2.3), and $R$ is of the form

\begin{equation}
R = \Lambda_{\frac{1}{2}+}(\tilde{Q} + G)\Lambda_{\frac{1}{2}},
\end{equation}

with a $\psi$do $\tilde{Q}$ of order and type 0 and a singular Green operator $G$ of order and class 0. The parametrix property implies that

\begin{align}
L_{1,+}R &= I - S_1, & S_1: \overline{\mathcal{H}}^{1}(\Sigma_+) \to C^\infty(\Sigma_+), & \text{for } t > -\frac{1}{2}, \\
RL_{1,+} &= I - S_2, & S_2: \dot{H}^{1+t}(\Sigma_+) \to \mathcal{E}_{\frac{t}{2}}(\Sigma_+), & \text{for } -\frac{1}{2} < t < 0,
\end{align}

\begin{align}
S_3 &= L^{-1} - R = L^{-1}(L_{1,+}R + S_1) - L^{-1}(L_{1,+} + L_{0,+})R = L^{-1}S_1 - L^{-1}L_{0,+}R.
\end{align}

Some properties of $L^{-1}$ can be obtained by considerations similar to those in [G11a]:

**Proposition 3.3.** The operator $L^{-1}: X^* \to X$ extends to an operator $M_0$ that maps continuously

$$M_0: \overline{\mathcal{H}}^{s}(\Sigma_+) \to \dot{H}^{s+\frac{1}{2}-\varepsilon}(\Sigma_+) \text{ for } -1 < s \leq \frac{1}{2}, \text{ any } \varepsilon > 0.$$ 

In particular, the closure of $L^{-1}$ in $L_2(\Sigma_+)$ is a continuous operator from $L_2(\Sigma_+)$ to $\dot{H}^{\frac{1}{2}-\varepsilon}(\Sigma_+)$. The operators $L^{-1}$ and $M_0$ have the same eigenfunctions (for nonzero eigenvalues); they belong to $D(L)$.

**Proof.** We already know from [G11a] (cf. (3.9)) that $L^{-1}$ is continuous from $X^* = \overline{\mathcal{H}}^{\frac{1}{2}}(\Sigma_+)$ to $\dot{H}^{1-\varepsilon}(\Sigma_+)$. Then it has an adjoint $M_0$ (with respect to dualities consistent with the $L_2(\Sigma_+)$-scalar product) that is continuous from $\overline{\mathcal{H}}^{-1+\varepsilon}(\Sigma_+)$ to $\dot{H}^{-\frac{1}{2}}(\Sigma_+)$. But since $L^{-1}$ is known to be selfadjoint (from $X^*$ to $X$, consistently with the $L_2$-scalar product), $M_0$ must be an extension of $L^{-1}$. Now the asserted continuity for $-1 < s \leq \frac{1}{2}$ follows by interpolation. For $s = 0$ this shows the mapping property of the $L_2$-closure.

When $\varphi$ is a distribution in $\overline{\mathcal{H}}^{-1+\varepsilon}(\Sigma_+)$ such that $M_0\varphi = \lambda\varphi$ for some $\lambda \neq 0$, then since $M_0\varphi \in \overline{\mathcal{H}}^{-\frac{1}{2}+\varepsilon}(\Sigma_+) = \dot{H}^{-\frac{1}{2}+\varepsilon}(\Sigma_+), \varphi$ lies there. Next, it follows that $M_0\varphi \in \overline{\mathcal{H}}^{1}(\Sigma_+) = \dot{H}^{1}(\Sigma_+)$, and hence $\varphi$ also lies there. Finally, we conclude that $M_0\varphi \in \overline{\mathcal{H}}^{\frac{1}{2}+\varepsilon}(\Sigma_+)$, so that $\varphi$ also lies there. Here $M_0$ coincides with $L^{-1}$. □

We can now find exact information on the domain of $L$:
Theorem 3.4. \(L^{-1}\) maps \(\overline{H}^{\frac{1}{2}}(\Sigma^+_0)\) onto \(H^{\frac{1}{2}(\frac{2}{3})}(\Sigma_+).\) In other words, the domain of \(L\) satisfies:

\[
D(L) = H^{\frac{1}{2}(\frac{2}{3})}(\Sigma_+) = \Lambda_+^{(-\frac{1}{2})}e^+\overline{H}^{\frac{1}{2}}(\Sigma^+_0),
\]

which is contained in \(d^\frac{1}{2}e^+H^{\frac{1}{2}}(\Sigma^+_0) + \dot{H}^{\frac{2}{3}}(\Sigma_+).\)

Proof. It is seen from the second line in (3.19) that \(S_3 = L^{-1} - R\) is also described by

\[
S_3 = (RL_{1,+} + S_2)L^{-1} - R(L_{1,+} + L_{0,+})L^{-1} = S_2L^{-1} - RL_{0,+}L^{-1}.
\]

Here \(S_2L^{-1}\) maps \(\overline{H}^{\frac{1}{2}}(\Sigma^+_0)\) into \(E^{\frac{1}{2}}_1(\Sigma_+)\) in view of (3.19). For the other term, we note that \(L_{0,+}\) maps \(\dot{H}^{1-\epsilon}(\Sigma_+)\) into \(\overline{H}^{1-\epsilon}(\Sigma^+_0),\) since an extension by zero is understood, and \(R\) maps the latter space into \(H^{\frac{1}{2}(2-\epsilon)}(\Sigma_+).\) Thus \(S_3\) maps \(\overline{H}^{\frac{1}{2}}(\Sigma^+_0)\) into \(H^{\frac{1}{2}(2-\epsilon)}(\Sigma_+).\) Since \(R\) maps \(\overline{H}^{\frac{1}{2}}(\Sigma^+_0)\) into \(H^{\frac{1}{2}(2-\epsilon)}(\Sigma_+),\) it follows that \(L^{-1}\) maps \(\overline{H}^{\frac{1}{2}}(\Sigma^+_0)\) into \(H^{\frac{1}{2}(\frac{2}{3})}(\Sigma_+).\) Thus \(D(L) \subset H^{\frac{1}{2}(\frac{2}{3})}(\Sigma_+).\)

The opposite inclusion also holds, since \(r^+L_1\) maps \(H^{\frac{1}{2}(\frac{2}{3})}(\Sigma_+)\) into \(\overline{H}^{\frac{1}{2}}(\Sigma^+_0),\) and \(H^{\frac{1}{2}(\frac{2}{3})}(\Sigma_+) \subset \overline{H}^{\frac{1}{2}}(\Omega)\) by Lemma 2.4, which \(r^+L_0\) maps into \(\overline{H}^{\frac{1}{2}}(\Sigma^+_0).\)

This shows the identity. The last statement follows from (2.3). \(\square\)

Remark 3.5. By this information we can explain more precisely in which way \(D(L),\) known to be contained in \(\dot{H}^{1-\epsilon}(\Sigma_+),\) reaches outside of \(\dot{H}^{1}(\Sigma_+),\) namely by certain non-trivial elements of \(d^\frac{1}{2}e^+\overline{H}^{\frac{1}{2}}(\Sigma^+_0)\) (lying in \(H^{\frac{1}{2}(\frac{2}{3})}(\Sigma_+)\)).

Consider the spaces in local coordinates, where \(\Sigma\) and \(\Sigma_+\) are replaced by \(\mathbb{R}^{n-1}\) and \(\mathbb{R}^{n-1}_+.\) As a typical element of \(x^{\frac{1}{2}}_n\) \(e^+\overline{H}^{\frac{1}{2}}(\mathbb{R}^{n-1}_+)\) lying in \(H^{\frac{1}{2}(\frac{2}{3})}(\mathbb{R}^{n-1}_+)\), we can take

\[
\varphi(x') = c x^{\frac{1}{2}}_n K_0 \psi, \quad c = \Gamma(\frac{3}{2})^{-1},
\]

where \(\psi(x'') \in H^{\frac{1}{2}}(\mathbb{R}^{n-2}).\) Here \(K_0\) is the Poisson operator from \(H^{\frac{1}{2}}(\mathbb{R}^{n-2})\) to \(\overline{H}^{\frac{1}{2}}(\mathbb{R}^{n-1}_+)\) solving

\[
(1 - \Delta)\zeta(x') = 0 \quad \text{on} \quad \mathbb{R}^{n-1}_+, \quad \gamma_0 \zeta = \psi \quad \text{on} \quad \mathbb{R}^{n-2},
\]

namely

\[
\zeta = K_0 \psi = F_{\xi'-x'}((\xi'') + i \xi_{n-1})^{-1} \hat{\psi}(\xi'') = F_{\xi'-x'}(e^{-\xi''}x_{n-1} \hat{\psi}(\xi''),
\]

and \(\varphi(x') = c x^{\frac{1}{2}}_n \zeta(x').\)

To verify that \(\varphi(x') \in H^{\frac{1}{2}(\frac{2}{3})}(\mathbb{R}^{n-1}_+),\) we recall from [G13], Sect. 5, that the special boundary operator \(\gamma_{\frac{4}{3},0} H^{\frac{1}{2}(\frac{2}{3})}(\mathbb{R}^{n-1}_+) \to H^{\frac{1}{2}}(\mathbb{R}^{n-2})\) defined there satisfies

\[
\gamma_{\frac{4}{3},0} \varphi = c^{-1} \gamma_0 (x^{\frac{1}{2}}_n \varphi(x')) = \gamma_0 \Xi^{\frac{4}{3}}_+ \varphi, \quad \text{with} \quad \Xi^\mu_+ = \text{OP}((\xi'') + i \xi_{n-1})^\mu,
\]
and has the right inverse $K_{\frac{1}{2},0}^+$, where
\[
\varphi = K_{\frac{1}{2},0}^+ \psi = \Xi_+^{-\frac{1}{2}} e^+ K_0 \psi = c x_{n-1}^\frac{1}{2} K_0 \psi,
\]
cf. [G13], Cor. 5.3, and the analysis in Th. 5.4 there.

Now $\varphi$ defined by (3.23) is not in $\dot{H}^1$ (nor in $\overline{H}^1$) near $x_{n-1} = 0$, since
\[
\partial_{x_{n-1}} \varphi(x') = \frac{1}{2} x_{n-1}^{-\frac{1}{2}} \zeta(x') + x_{n-1}^\frac{1}{2} \partial_{x_{n-1}} \zeta(x'),
\]
where $x_{n-1}^\frac{1}{2} \partial_{x_{n-1}} \zeta(x')$ is clearly $L_2$-integrable over $\mathbb{R}^{n-2} \times [0,1]$, but $x_{n-1}^{-\frac{1}{2}} \zeta(x')$ is not so:
\[
\begin{align*}
\int_{\mathbb{R}^{n-2}} \int_{0 < x_{n-1} < 1} |x_{n-1}^{-\frac{1}{2}} - \zeta|^2 dx_{n-1} dx'' &= (2\pi)^{2-n} \lim_{\delta \to 0} \int_{\mathbb{R}^{n-2}} \int_{\delta < x_{n-1} < 1} x_{n-1}^{-1} e^{-2\langle \xi'' \rangle x_{n-1}} |\hat{\psi}(\xi'')|^2 dx_{n-1} d\xi'' \\
&\geq (2\pi)^{2-n} \lim_{\delta \to 0} \int_{\mathbb{R}^{n-2}} \int_{\delta < x_{n-1} < 1} x_{n-1}^{-1} e^{-2\langle \xi'' \rangle} |\hat{\psi}(\xi'')|^2 dx_{n-1} d\xi'' \\
&= (2\pi)^{2-n} \lim_{\delta \to 0} |\log \delta| \int_{\mathbb{R}^{n-2}} e^{-2\langle \xi'' \rangle} |\hat{\psi}(\xi'')|^2 d\xi'' = +\infty,
\end{align*}
\]
when $\psi \neq 0$. (It does not help to take $\psi$ very smooth.)

We consequently have for $D(A_{\chi,\Sigma_+})$:

**Corollary 3.6.** The domain of $A_{\chi,\Sigma_+}$ satisfies
\[
D(A_{\chi,\Sigma_+}) \subset D(A_\gamma) + K_\gamma \dot{H}^{\frac{1}{2}}(\Sigma_+)^2 \subset \overline{H}^2(\Omega) + K_\gamma(e^+ d(x')^\frac{1}{2} \overline{H}^1(\Sigma_+))
\]
(where we recall that $e^+$ denotes the extension from $\Sigma_+$ by zero on $\Sigma_-$, and $d(x')$ is a $C^\infty$-function on $\Sigma_+$ proportional to $\text{dist}(x', \partial \Sigma_+) \text{ near } \partial \Sigma_+$).

All elements of $K_\gamma \dot{H}^{\frac{1}{2}}(\Sigma_+)$ are reached from $D(A_{\chi,\Sigma_+})$.

Nontrivial elements of $K_\gamma(e^+ d(x')^\frac{1}{2} \overline{H}^1(\Sigma_+))$ are reached, that are not in $K_\gamma \dot{H}^1(\Sigma_+)$, nor in $K_\gamma(e^+ \overline{H}^1(\Sigma_+))$ (as in Remark 3.5), hence not in $\overline{H}^{\frac{3}{2}}(\Omega)$.

*Proof.* It is known from [G68], Th. II.1.2 that
\[
D(A_{\chi,\Sigma_+}) \subset D(A_\gamma) + D(T) = D(A_\gamma) + K_\gamma D(L),
\]
when we use that $A_\gamma = A_\beta$ and $K_\gamma D(L) = D(T)$ with the notation used there. Here all elements of $D(T)$ are reached, in the sense that for any $z \in D(T)$ there is a $v \in D(A_\gamma)$ such that $u = v + z \in D(A_{\chi,\Sigma_+})$. Since $D(L) = \dot{H}^{\frac{3}{2}}(\Sigma_+)$, this shows the first inclusion in (3.25) and the first statement afterwards.

For the remaining part we use the last information in Theorem 3.4. Since $K_\gamma \dot{H}^{\frac{3}{2}} \subset \overline{H}^2(\Omega)$, this implies the second inclusion in (3.25). Remark 3.5 shows how nontrivial nonsmooth elements occur. \qed
3.4 The spectrum of the Krein term.

The spectral asymptotic behavior of the Krein term

\[ M = A_{-1}^{-1} - A^{-1} = i_V \gamma_V^{-1} L^{-1} (\gamma_V^{-1})^* \Pr_V \]  

will now be determined. We assume \( n \geq 3 \) in this section since applications on \( \Sigma \) of Laptev's result quoted in (2.14) requires the dimension \( m \) to be \( \geq 2 \), i.e., \( n - 1 \geq 2 \). It is used to show that some cut-off terms have a better asymptotic behavior than the one we are aiming for, hence can be disregarded. (We believe that there are ways to handle the case \( n - 1 = 1 \), either by establishing weaker versions of (2.14), or by using the variable-coefficient factorization of the principal symbol of \( L \), but we refrain from making an effort here. The case \( n = 2 \) was included in [G11a] for \( A \) principally Laplacian.)

First we study the spectrum of the factor \( L^{-1} \).

**Theorem 3.7.** \( S_3 \) belongs to \( \mathcal{G}_{(n-1)/(4-\varepsilon),\infty} \), and \( L^{-1} \) belongs to \( \mathcal{G}_{n-1,\infty} \) (when the operators are extended to \( L_2(\Sigma_+) \) by closure).

The eigenvalues of \( L^{-1} \) have the asymptotic behavior:

\[ \mu_j(L^{-1}) j^{1/(n-1)} \to c(L) \frac{1}{j^{(n-1)}} \text{ for } j \to \infty, \]

where

\[ c(L) = \frac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma_+} \int_{|\xi'| = 1} \kappa_0(x', \xi')^{-1} d\omega(\xi')dx'. \]

**Proof.** Recall that \( L^{-1} \) acts as follows:

\[ L^{-1} = R + S_3 = A_{-1,\Sigma}(-\frac{1}{2})(\tilde{Q}_+ + G)A_{-1,\Sigma}(-\frac{1}{2}) + S_3, \]

cf. (3.18). By application of Theorem 2.6 to \( R \) we find that the singular values \( s_j(R) \) behave as in (3.27)–(3.28), where the constant is as in (3.28) since the principal pseudodifferential symbol of \( R \) is \( \kappa_0^{-1} \). In particular, \( R \in \mathcal{G}_{n-1,\infty} \).

Since the closure of \( L^{-1} \) maps \( L_2(\Sigma_+) \) continuously into \( \dot{H}^{\frac{1}{2}+\varepsilon}(\Sigma_+) \) by Proposition 3.3, it belongs to \( \mathcal{G}_{(n-1)/(4-\varepsilon),\infty} \). Moreover (cf. (3.19)), \( S_1 \in \bigcap_{\tau>0} \mathcal{G}_{\tau,\infty} \), and \( L_{0,+} \) is bounded in \( L_2(\Sigma_+) \). Then \( L^{-1} S_1 \) is in \( \bigcap_{\tau>0} \mathcal{G}_{\tau,\infty} \), and \( L^{-1} L_{0,+} R \in \mathcal{G}_{(n-1)/(4-\varepsilon),\infty} \). \( \mathcal{G}_{n-1,\infty} \subset \mathcal{G}_{(n-1)/(4-\varepsilon),\infty} \) by the rule (2.10), using that \( S_1 \) and \( L_{0,+} R \) map into spaces where \( L^{-1} \) coincides with its \( L_2 \)-closure. Therefore by (3.20),

\[ S_3 \in \mathcal{G}_{(n-1)/(4-\varepsilon),\infty}. \]

Now since \( L^{-1} \) acts like \( R + S_3 \), its closure is in \( \mathcal{G}_{n-1,\infty} \). This show the first statement in the theorem.

The last statement follows, since \( S_3 \) is of a better Schatten class than \( R \), so that (2.13) implies that the \( L_2 \)-closure of \( L^{-1} \) has the same asymptotic behavior of singular values as \( R \). Since \( L^{-1} \) is symmetric in \( L_2 \), the \( L_2 \)-closure is selfadjoint, so its singular values are eigenvalues; they are consistent with the eigenvalues of \( L^{-1} \) by Proposition 3.3. \( \square \)
We now turn to the Krein term \( M \) recalled in (3.26). Proceeding as in [G11a] Sect. 5.4, we have for the eigenvalues:

\[
\mu_j(M) = \mu_j(iV^* \gamma^{-1} L^{-1}(\gamma^{-1})^* \text{pr}_V) = \mu_j(L^{-1}(\gamma^{-1})^* \gamma^{-1}) = \mu_j(L^{-1} P_{1,+}),
\]

where \( P_1 = K^* K_\gamma \) is a selfadjoint nonnegative invertible elliptic \( \psi \)do of order \(-1\); in view of (3.14) it has principal symbol \((\kappa_+ + \bar{\kappa}_+)^{-1} = a_{\gamma n}(2\kappa_0)^{-1}\). Let \( P_2 = P_1^\perp \), then we continue the calculation as follows:

\[
\mu_j(M) = \mu_j(L^{-1} r^+ P_2 P_2 e^+) = \mu_j(P_2 e^+ L^{-1} r^+ P_2) = \mu_j(r^+ P_2 e^+ L^{-1} r^+ P_2 e^+ + S_4),
\]

where \( S_4 \) is a sum of three terms, each one a product of \( \psi \)do’s and cutoff functions of a total order \(-2\), and each containing a factor either \( r^+ P_2 e^+ \) or \( r^+ P_2 e^- \) (or both). To the terms in \( S_4 \) we can apply (2.14) together with product rules, concluding that they are in \( \mathcal{S}_{(n-1)/(2+\theta),\infty} \) for some \( \theta > 0 \).

The operator (cf. (3.25))

\[
M_1 = r^+ P_2 e^+ L^{-1} r^+ P_2 = P_{2,+} \Lambda^{-\frac{1}{2}}_+(\bar{Q}_+ + G) \Lambda^{-\frac{1}{2}}_+ P_{2,+} + P_{2,+} S_3 P_{2,+}
\]

is selfadjoint nonnegative, so its eigenvalues \( \mu_j \) coincide with the \( s \)-values. We can apply Theorem 2.6 to the first term, obtaining a spectral asymptotic formula (2.16)–(2.17) with \( t/n \) replaced by \( 2/(n-1) \); then the addition of the second term which lies in a better weak Schatten class \( \mathcal{S}_{(n-1)/(2+\theta),\infty} \) preserves the formulas.

Finally \( M \) (likewise selfadjoint nonnegative) differs from \( M_1 \) by the operator \( S_4 \) in a better weak Schatten class, so the spectral asymptotic formula carries over to this operator.

Hereby we obtain the theorem:

**Theorem 3.8.** The eigenvalues of \( M = A_{\gamma, \Sigma^+}^{-1} - A_{\gamma}^{-1} \) have the asymptotic behavior:

\[
\mu_j(M) j^{2/(n-1)} \to c(M)^{2/(n-1)} \text{ for } j \to \infty,
\]

where

\[
c(M) = \frac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma^+} \int_{|\xi'|=1} \left( \frac{a_{\gamma n}(\xi')}{2\kappa_0(\xi^\prime,\xi')^2} \right)^{(n-1)/2} d\omega(\xi^\prime) d\xi^\prime.
\]

**Proof.** It remains to account for the value of the constant \( c(M) \). It follows, since \( P_2^2 = P_1 \) has principal symbol \( a_{\gamma n}(2\kappa_0)^{-1} \) and the \( \psi \)do part of \( L^{-1} \) has principal symbol \( \kappa_0^{-1} \). □

**Remark 3.9.** We take the opportunity to recall two corrections to [G11a] (already mentioned in [G11b]): Page 351, line 4 from below, delete “\( H^{\frac{1}{2}}(\Sigma_+^\circ) \subset \)”, replace “\( H^1(\Sigma) \)” by “\( L_2(\Sigma) \)”. Page 361, line 4, replace “(Th. 3.3)” by “(Th. 4.3)”. 

SPECTRAL RESULTS

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