Copulas from Order Statistics

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Abstract

A new class of copulas based on order statistics was introduced by Baker (2008). Here, further properties of the bivariate and multivariate copulas are described, such as that of likelihood ratio dominance (LRD), and further bivariate copulas are introduced that generalize the earlier work. One of the new copulas is an integral of a product of Bessel functions of imaginary argument, and can attain the Fréchet bound. The use of these copulas for fitting data is described, and illustrated with examples. It was found empirically that the multivariate copulas previously proposed are not flexible enough to be generally useful in data fitting, and further development is needed in this area.

Key words: Copulas; Order-statistics; Bessel function; random numbers.

1 Introduction

The use of copulas has become popular for modeling multivariate data. Initially, the marginal distributions are fitted, using the vast range of univariate models available, and the dependence between variables is then modeled using a copula. This approach is sometimes easier than seeking a ‘natural’ multivariate distribution derived from a probabilistic model, because there may be no suitable multivariate distribution with the required marginals.

Baker [3] gave a class of bivariate and multivariate copulas based on order-statistics, and this work seeks to ‘dig deeper’. The ordering and some other properties of these copulas are derived here, and the copulas are generalized into further copulas in the bivariate case. Further experience is also gained in fitting distributions derived from the bivariate and multivariate copulas to data.

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First, we briefly recapitulate the essential concept of the earlier work. Many topics covered there, such as the connection with Bernstein polynomials \cite{13} and the Farlie-Gumbel-Morgenstern (FGM) distribution \cite{8}, are not repeated here.

The derivation of the class of distributions of interest is most easily done by considering the generation of correlated random variables from independent random variables $X$ and $Y$ with distribution functions $F(x), G(y)$ and pdfs (where defined) $f(x), g(y)$. If $n$ sets of random variables $X_1 \ldots X_n$ and $Y_1 \ldots Y_n$ are sorted into order statistics $X_{(1)} \ldots X_{(n)}$ and $Y_{(1)} \ldots Y_{(n)}$, they can be paired off as $(X_{(1)}, Y_{(1)}), \ldots, (X_{(n)}, Y_{(n)})$, and one such pair randomly selected. This scheme yields a pair of dependent (positively correlated) random variables, the Spearman (grade) correlation between them being $(n-1)/(n+1)$.

The marginal distributions of $X$ and $Y$ are still $F(x), G(y)$ respectively, because a randomly chosen order-statistic from a distribution is just a random variable from that distribution. We term the resulting bivariate distribution the ‘bivariate distribution of order $n$’. The procedure also works in the general multivariate case, when $n$ sets of $p$ random variables can be similarly grouped.

In addition to random variable pairs being selected as described, to be ‘in phase’, they can be chosen to be in antiphase, by pairing $X_{(k)}$ with $Y_{(n+1-k)}$, so giving a negative grade correlation of $-(n-1)/(n+1)$. This is not discussed further; to model negative correlations one simply replaces $G(y)$ by $1 - G(y)$ in the formulas.

Baker \cite{3} obtained a one-parameter family of bivariate copulas for a given $n$ by randomly choosing a pair from $(X_{(1)}, Y_{(1)}), \ldots, (X_{(n)}, Y_{(n)})$ with probability $q$, and a pair from $(X_1, Y_1), \ldots, (X_n, Y_n)$ with probability $1 - q$; these random variables have grade correlation $q(n-1)/(n+1)$. Equivalently, with probability $1 - q$ we can choose $X$ and $Y$ randomly and independently from their $n$ order statistics $X_{(1)} \ldots X_{(n)}$ etc; we could then say that $X$ and $Y$ are chosen from independent cycles. The resulting distributions are a mixture of the distribution of order $n$ and the independent distribution. We term them ‘mixture distributions of order $n$’. The device of taking mixtures of copulas will be used later to derive new copulas.

Some mathematical preliminaries are necessary: the distribution function $F_{k,n}(x)$ of the $k$th of $n$ order statistics is given by

$$F_{k,n}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j} \quad (1)$$
The corresponding pdf if it exists is
\[ f_{k,n}(x) = n\binom{n-1}{k-1} F(x)^{k-1}(1 - F(x))^{n-k} f(x), \quad (2) \]
and the bivariate distribution function of a random order-statistic pair is
\[ H^{(n)}(x, y) = n^{-1} \sum_{k=1}^{n} F_{k,n}(x) G_{k,n}(y). \quad (3) \]

The mixture distribution of order \( n \) then has distribution function
\[ H(x, y) = (1 - q) F(x) G(y) + q n^{-1} \sum_{k=1}^{n} F_{k,n}(x) G_{k,n}(y) \quad (4) \]
\[ = (1 - q) H^{(1)}(x, y) + q H^{(n)}(x, y), \]
and where applicable, pdf
\[ h(x, y) = (1 - q) f(x) g(y) + q n^{-1} \sum_{k=1}^{n} f_{k,n}(x) g_{k,n}(y). \quad (5) \]

Note that the copula in fact has one continuous and one discrete parameter.

There is no need to pair corresponding order statistics: in the most general case the pair \((X(i), Y(j))\) can be chosen with probability \(r_{ij}\), so that
\[ H(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} F_{i,n}(x) G_{j,n}(y), \quad (6) \]
where for the correct marginal distributions, we must have
\[ \sum_{i=1}^{n} r_{ij} = \sum_{j=1}^{n} r_{ij} = 1/n. \quad (7) \]

The matrix \(nr\) is doubly stochastic, and \(r\) has \((n-1)^2\) independent elements. Equation (4) corresponds to the choice \(r_{ij} = (1 - q)/n^2 + (q/n)\delta_{ij}\).

The Birkhoff-von Neumann theorem \([4]\) states that the set of doubly stochastic matrices of order \( n \) is the convex hull of the set of permutation matrices of order \( n \), and that the extreme points of the set are the permutation matrices. Here, we can view (6) as a mixture of the \(n!\) possible pairings of the \(X\) and \(Y\) order statistics. The pairing chosen in (4) can achieve the largest grade correlation for a given \(n\), so the corresponding copula is of special interest.

When considering copulas rather than bivariate distribution functions, it is convenient to define analogously to (1) the distribution functions of order
statistics of \( u = F(x), v = G(y) \) as

\[
Q_{i,n}(u) = \sum_{l=i}^{n} B_{l,n}(u),
\]

where \( B_{i,n}(u) = \binom{n}{i} u^i (1 - u)^{n-i} \) is a Bernstein polynomial; see Lorenz [13] for their mathematical description. Then the copula corresponding to (3) (the copula of order \( n \)) is

\[
C(u, v) = n^{-1} \sum_{k=1}^{n} Q_{k,n}(u) Q_{k,n}(v)
\]

(8)

and for completeness, the mixture copula is

\[
C(u, v) = (1 - q) uv + q n^{-1} \sum_{k=1}^{n} Q_{k,n}(u) Q_{k,n}(v).
\]

(9)

The material presented so far, except for the nomenclature and the remarks on the Birkhoff-von Neumann theorem, was given in [3]. The essence of the earlier paper was the derivation of the bivariate distribution of order \( n \) (3) and its mixture with a distribution of order 1 to obtain the mixture distribution of order \( n \). This distribution allows arbitrary correlations, and the corresponding copula (9) has the unusual feature of possessing one continuous and one discrete parameter.

From this point on, new results are presented. The mixture copula (9) is of interest in itself, and some further properties of it are derived, such as its ordering properties. It is however convenient to start from the more basic copula of order \( n \) (8) both when deriving properties of (9) and when deriving further copulas. In the next section, some further properties of the bivariate copulas (8) and (9) are given.

2 Properties of the bivariate copula

2.1 Dependence Orderings

The strongest ordering property is positive likelihood ratio dependence (LRD), where \( h(x_1, y_1) h(x_2, y_2) > h(x_1, y_2) h(x_2, y_1) \) when \( x_2 > x_1, y_2 > y_1 \). The LRD property implies all other quadrant dependence properties (Nelsen, 2006 [12]). The FGM distribution is known to be LRD (e.g. Drouet-Mari and Kotz, 2001 [8]) and it is proved here that (3) is LRD. The general distribution (6) is not.
Writing for brevity \( F(x) = F_1, 1 - F(x) = S_1, 1 - G(y) = T_1 \) etc, from (3) we have that

\[
\frac{h(x_1, y_1)h(x_2, y_2) - h(x_1, y_2)h(x_2, y_1)}{f(x_1)f(x_2)g(y_1)g(y_2)} = n^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \begin{array}{c} n-1 \\ i-1 \end{array} \right) \left( \begin{array}{c} n-1 \\ j-1 \end{array} \right) A_{ij},
\]

where

\[
A_{ij} = \{(F_1G_1)^{i-1}(S_1T_1)^{n-i}(F_2G_2)^{j-1}(S_2T_2)^{n-j} - (F_1G_2)^{i-1}(S_1T_2)^{n-i}(F_2G_1)^{j-1}(S_2T_1)^{n-j}\}. \]

Since \( A_{ii} = 0 \), the right-hand side can be rewritten as \( n^2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left( \begin{array}{c} n-1 \\ i-1 \end{array} \right) \left( \begin{array}{c} n-1 \\ j-1 \end{array} \right) (A_{ij} + A_{ji}) \). This can be factored into

\[
A_{ij} + A_{ji} = (F_1F_2G_1G_2)^{i-1}(S_1S_2T_1T_2)^{n-i}\{(F_2/S_2)^{j-i} - (F_1/S_1)^{j-i}\}\{(G_2/T_2)^{j-i} - (G_1/T_1)^{j-i}\}. \tag{10}
\]

Since \( F_2 > F_1, G_2 > G_1 \), then \( F_2/S_2 > F_1/S_1, G_2/T_2 > G_1/T_1 \). As \( j - i \geq 1 \), each bracket of (10) is positive, and the LRD property follows. It follows straightforwardly that mixture distributions derived from (3) such as (4) are also LRD.

### 2.2 Measures of Association

The calculation of Kendall’s tau for (8) and (9) is given in Baker [3]. Blomqvist’s medial coefficient or beta is another widely used measure of association, given by \( \beta = 4C(1/2, 1/2) - 1 \). From (3),

\[
\beta = (4/n)(1/2)^{2n} \sum_{k=1}^{n} \sum_{i=k}^{n} \left( \begin{array}{c} n \\ i \end{array} \right)^2 - 1.
\]

This does not simplify much; it can also be written

\[
\beta = (1/2)^{2n-2}\left\{ \left( \begin{array}{c} 2n-1 \\ n-1 \end{array} \right) + 2 \sum_{i=1}^{n} \left( \begin{array}{c} n-1 \\ i-1 \end{array} \right) \sum_{j>i}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) \right\} - 1.
\]

There is an additional factor of \( q \) for the copula (9).

Note that at the median \((\tilde{x}, \tilde{y})\), \( F(\tilde{x}) = G(\tilde{y}) = 1/2 \), and the pdf from (3) takes a simple form because the series can then be summed, to give

\[
h(\tilde{x}, \tilde{y}) = f(\tilde{x})g(\tilde{y})(1/2)^{2(n-1)n}\left( \begin{array}{c} 2n-2 \\ n-1 \end{array} \right).
\]

Since \( \left( \begin{array}{c} 2m \\ m \end{array} \right) < 2^m/\sqrt{\pi m} \), we have that \( h(\tilde{x}, \tilde{y}) < f(\tilde{x})g(\tilde{y})n/\sqrt{\pi(n-1)} \).
Gini’s gamma is a coefficient of association that can be expressed as \( \gamma = 4 \int_0^1 \{ C(u, u) + C(u, 1-u) \} \, du - 2 \) (Nelsen, [12]). For (8) this gives after some algebra

\[
\gamma = \frac{4}{n(2n+1)} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \min(i, j) + \min(i, n-j) \right) \binom{n}{i} \binom{n}{j} \left( \frac{2n}{i+j} \right)^{-1} \right\} - 2.
\]

There is again an additional factor of \( q \) for the copula (9).

Note that the Schweizer-Wolff sigma defined as \( \sigma = \frac{12}{n} \int \int |C(u, v) - uv| \, du \, dv \) is numerically identical to Spearman’s rho for (4), because it possesses the PQD (positive quadrant dependence) property \( C(u, v) > uv \) as a consequence of the LRD property.

These dependence measures are shown in figure 1 plotted against \( n \). Dependence increases with \( n \) and the Fréchet bound is attained as \( n \to \infty \).

The coefficient of tail dependence (e.g. Joe [9]) is defined in general as \( \lambda = \lim_{p \to 0} \{ \Pr(y > y^*|x > x^*) \} \), where \( F(x^*) = 1 - p, F(y^*) = 1 - p \). From its definition, \( 0 \leq \lambda \leq 1 \), and \( \lim_{p \to 0} \{ \Pr(y > y^*|x > x^*) = \lim_{p \to 0} \{ \Pr(x > x^*|y > y^*) \}. \) The distribution (3) can be shown after some algebra to yield \( \lambda = 0 \), so that the random variables are asymptotically independent. This property also holds for all finite mixture distributions.

### 2.3 Symmetry

The copula (8) and its mixtures possess the radial or reflective symmetry \( C(1-u, 1-v) = 1-u-v + C(u, v) \), also seen for example in the Frank and Plackett copulas. All existing copulas seem to have the simpler symmetry property \( C(v, u) = C(u, v) \). Asymmetry between \( X \) and \( Y \) is usually handled by using different marginal distributions \( F(x) \) and \( G(y) \), but there is no reason why the copula itself should be symmetric. Asymmetry can not occur for Archimedean copulas, for which \( C(v, u) = C(u, v) \) since \( \varphi(C(u, v)) = \varphi(u) + \varphi(v) \), where \( \varphi \) is a (decreasing) function. Yet, as order statistics of \( Y \) can be paired with any permutation of order statistics of \( X \), and still give marginal distributions \( F(x), G(y) \), it is easy to construct copulas which do not have this symmetry.

For example, when \( n = 3 \), the copula

\[
C(u, v) = (1/3) \{ Q_{1,3}(u)Q_{2,3}(v) + Q_{2,3}(u)Q_{3,3}(v) + Q_{3,3}(u)Q_{1,3}(v) \}
\]

which can be written
where \(F \) and \(G\) are distribution functions. The denominator can be much larger than \((1 - F)(1 - G)\), so that the correlation between the random variables can decrease the hazard in the tail. In the right-hand tail, where \(A(x) = 1 - F(x) \ll 1\), \(B(x) = 1 - G(x) \ll 1\), only the \(j = n\) term survives from (11), for all \(k\). It follows that for \(q > 0\), we have

\[
z(x, y) \sim \frac{f(x)g(y)\{nq + (1 - q)\}}{q\{(1 - F(x)) + (1 - G(y))\}}.
\]

As the denominator can be much larger than \((1 - F(x))(1 - G(y))\), the correlation between the random variables can decrease the hazard in the tail.

The pdf from (3) can be written as a hypergeometric function:

\[
\frac{h(x, y)}{f(x)g(y)} = n(F(x)G(y))^{n-1} \frac{\binom{1 - F(x)}{1 - n} - \frac{(1 - F(x))(1 - G(y))}{F(x)G(y)}}{F(x)G(y)}.
\]

Median regression is the curve \(y = \tilde{y}(x)\), where \(\Pr(Y \leq \tilde{y}|X = x) = 1/2\). This does not take any simple form for these distributions.

Finally, the pdf from (4) is proportional to the probability \(p_{00}(2n)\) that a random walk in the plane returns to its start point after \(2n\) steps. Given probabilities \(p_1, p_2\) of moving left or right, and probabilities \(q_1, q_2\) of moving up or down, so that \(p_1 + p_2 + q_1 + q_2 = 1\), we have that

\[
p_{00}(2n) = \sum_{k=0}^{n} \frac{(2n)!}{k!(n-k)!} (p_1p_2)^k (q_1q_2)^{n-k},
\]

so that

\[
\frac{h(x, y)}{f(x)g(y)} = 2^{2n}n!(n-1)! \frac{(2n)!}{p_{00}(2n-2)},
\]

where \(FG = 4p_1p_2\), \((1 - F)(1 - G) = 4q_1q_2\), e.g. \(p_1 = F/2, p_2 = G/2, q_1 = (1 - F)/2, q_2 = (1 - G)/2\). At the median where \(F(x) = G(y) = 1/2\), the
random walk is symmetric, with probability 1/4 of moving in any direction.

3 Further Bivariate models

Having derived new properties of the copulas (8) and (9) introduced earlier, we now seek to generalize them into bivariate models that could be useful for fitting to data. The most general form of the bivariate model (6) could be fitted directly to data for low $n$, and has $(n-1)^2$ parameters. From the definition of the grade correlation, we have (Nelsen, 2006 [12])

$$\rho_s = 12E(F(x)G(y)) - 3 = 12 \int \int h(x, y)F(x)G(y) \, dx \, dy - 3.$$  

Generalizing the proof in Baker [3], this gives

$$\rho_s = \frac{12}{(n+1)^2} \sum_{i=1}^{n} \sum_{j=1}^{n} ij r_{ij} - 3 = \frac{12}{(n+1)^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (i - (n+1)/2)(j - (n+1)/2) r_{ij}.$$  

(11)

In the most general case, this model can reproduce any copula with increasing accuracy as $n \to \infty$, and all models of lower order $m < n$ can be written as (6) for some choice of $r$. However, this does not help in the construction of simple models with few parameters, which is our aim. We first discuss several possible approaches, before introducing what seems the most useful new copula, the ‘Bessel function copula’.

3.1 Generalized bivariate models

One possibility for reducing the number of model parameters from $(n-1)^2$ would be to modify the scheme for generating correlated random numbers given in the introduction, by first generating a uniformly-distributed random number $U$. Then a random variable is chosen as the order statistic number $\lfloor mU \rfloor + 1$ of $m$ from $F(x)$, and number $\lfloor nU \rfloor + 1$ of $n$ from $G(y)$, where $\lfloor \rfloor$ is the ‘floor’ function. This allows the two random variables to be chosen from different orders of order statistic. The resulting model is a special case of (6) where $n \geq m$. Unfortunately, the resulting distributions, characterised by two discrete parameters, are not mathematically tractable.

We therefore seek instead to obtain models with only a few parameters by generalizing [3]. A more general model arises from pairing order statistics only within some range or ranges; for example, suppose only the 1st to $m_1$th and $m_2$th to $n$th order statistics pair, and the remainder associate randomly.
Then
\[ H(x, y) = n^{-1} \sum_{k=1}^{m_1} F_{k,n}(x)G_{k,n}(y) + n^{-1} \sum_{k=n-m_2}^{n} F_{k,n}(x)G_{k,n}(y) + \frac{\sum_{i=m_1+1}^{n-m_2-1} \sum_{j=m_1+1}^{n-m_2-1} F_{i,n}(x)G_{j,n}(y)}{n(n - m_1 - m_2)}. \]

From (11) it follows that
\[ \rho_s = \frac{n - 1}{n + 1} + \frac{2m_1(m_1 + 1)(2m_1 + 1) - 2(n - m_2 - 1)(n - m_2)(2n - 2m_2 - 1)}{n(n + 1)^2} + \frac{3(n + m_1 - m_2 - 1)^2(n - m_1 - m_2)}{n(n + 1)^2}. \]

This allows a distribution with three discrete parameters where the random variables correlate strongly only in one or both tails.

Another way to generate models that are more general than (3) is to form a finite mixture distribution
\[ H(x, y) = \sum_{i=1}^{n} w_i H^{(i)}(x, y), \]
where \( \sum_{i=1}^{n} w_i = 1 \). This of course can be expressed as a special case of (6). The Spearman correlation is simply
\[ \rho_s = \sum_{i=1}^{n} w_i (i - 1)/(i + 1). \] (12)

3.2 The Bessel Function Copula

A new distribution can be derived by taking an infinite mixture of models. This gives copulas indexed by one parameter, if the mixing distribution is a 1-parameter distribution. In general the pdf is
\[ h(x, y) = \sum_{n=1}^{\infty} n w_n \sum_{k=1}^{n} \left( \frac{n - 1}{k - 1} \right)^2 F(x)^{k-1}(1-F(x))^{n-k}G(y)^{k-1}(1-G(y))^{n-k}f(x)g(y). \] (13)

Rearranging,
\[ \frac{h(x, y)}{f(x)g(y)} = \sum_{k=1}^{\infty} (F(x)G(y))^{k-1} \frac{1}{(k - 1)!^2} \sum_{n=k}^{\infty} n w_n (n - 1)!^2 \left\{ (1 - F(x))(1 - G(y)) \right\}^{n-k}. \]
An interesting distribution arises on taking

\[ w_n = \frac{\theta^{n-1/2}}{(n-1)!n!I_1(2\theta^{1/2})}, \]  

(14)

where \( n > 0, \theta > 0, \) and \( I \) denotes the Bessel function of imaginary argument. This is a special case of 2-parameter discrete Bessel function distribution first described by Pitman and Yor [15] and later by Yuan and Kalbfleisch [18,10]. Then from the series expansion of the Bessel function, we have that

\[ \frac{h(x,y)}{f(x)g(y)} = \frac{\theta^{1/2}}{I_1(2\theta^{1/2})} I_0(2(F(x)G(y)\theta)^{1/2}) I_0(2\{(1 - F(x))(1 - G(y))\theta\}^{1/2}). \]  

(15)

The copula is

\[ C(u,v) = \frac{\theta^{1/2}}{I_1(2\theta^{1/2})} \int_0^u \int_0^v I_0(2(2wz)^{1/2}) I_0(2(1-w)(1-z))^{1/2} \, dw \, dz. \]  

(16)

Figures 2 and 3 illustrate the copula as scatterplots, for \( \theta = 250 \) and \( \theta = 5000 \) respectively. Here a randomly generated sample of size 1000 was generated from the joint distribution with copula \( C \) and uniform marginals. This copula is the first one known to the author that requires special functions; all others require only exponentials, logarithms, and powers.

The Spearman correlation is calculated from (12) and (14) as

\[ \rho_s = \sum_{n=1}^{\infty} \frac{\theta^{n-1/2}}{n!(n-1)!I_1(2\theta^{1/2})} \frac{n-1}{n+1} = \sum_{n=0}^{\infty} \frac{\theta^{n+1/2}}{n!(n+3)!I_1(2\theta^{1/2})} = \frac{I_3(2\theta^{1/2})}{I_1(2\theta^{1/2})}, \]  

(17)

To obtain negative correlations, one sets \( G(y) \to 1 - G(y) \).

As \( \theta \to 0 \), (15) gives \( h(x,y) \to f(x)g(y) \). As \( z \to \infty \), since \( I_\nu(z) \to \exp(z)/\sqrt{2\pi z} \), we have that as \( \theta \to \infty \)

\[ \frac{h(x,y)}{f(x)g(y)} \sim \frac{\exp(-2\theta^{1/2}T(x,y))}{2(\pi)^{1/2}\theta^{1/4}\{F(x)(1 - F(x))\}^{1/4}\{G(y)(1 - G(y))\}^{1/4}}, \]

where \( T(x,y) = (F^{1/2}(x) - G^{1/2}(y))^2 + ((1 - F(x))^{1/2} - (1 - G(y))^{1/2})^2 \). This shows that \( h(x,y) \to 0 \) if \( F(x) \neq G(y) \). Hence as \( \theta \to \infty \) the distribution attains the Fréchet bound. From (17) as \( \theta \to \infty \), we have that \( \rho_s \to 1 \), so the grade correlation approaches unity, as it must.

As this copula is not a finite mixture of the copula \( \xi \), the coefficient of tail dependence could be nonzero. However, using the reflection symmetry of the copula, we have that the coefficient of right (and left) tail dependence is \( \lim_{p \to 0} C(p,p)/p \), where the double integral in (16) is \( O(p^2) \). The coefficient of
tail dependence is thus still zero, except of course in the limit as the Fréchet-
Hoeffding bound is approached as \( \theta \to \infty \).

Random variables from this copula can be derived by generating \( N \) from the
discrete Bessel distribution, as described by Devroye \[7\], and then randomly
selecting one of the \( N \) order-statistic pairs. This is how figures 2 and 3 were
generated. Here, \( N \) was generated using the inverse probability method. This
general strategy would be efficient if many random numbers were required,
when unused order statistic pairs could be stored and used in preference to
generating fresh ones. It also requires only generation of random numbers from
the marginal distributions, and does not require the use of the inverse proba-
bility transformation on these distributions. The alternative method, of gener-
ating \( U \) and then generating \( V \) from the conditional distribution \( \partial C(u, v)/\partial u \)
is not recommended as it is computationally more time consuming.

One can also take the weight

\[
w_n = \frac{\theta^{n-1}}{(n-1)!^2 I_0(2\theta^{1/2})}, \tag{18}
\]

another special case of the Bessel function distribution. After summing the
series, this yields the more complex form

\[
h(x, y) = \frac{AI_1(2A)I_0(2B) + BI_0(2A)I_1(2B) + I_0(2A)I_0(2B)}{I_0(2\theta^{1/2})},
\]

where \( A = \{F(x)G(y)\theta\}^{1/2} \), \( B = \{(1 - F(x))(1 - G(y))\theta\}^{1/2} \). The Spearman
correlation from (12) is

\[
\rho_s = \frac{2\theta^{-1/2}I_3(2\theta^{1/2}) + I_4(2\theta^{1/2})}{I_0(2\theta^{1/2})}.
\]

The copula is in general similar to (10) but slightly more complex.

Still other choices can be made for \( w_n \), but these lead to pdfs that are much
less tractable, being infinite sums of hypergeometric functions. The Spearman
correlations however are more tractable; for example taking a displaced Pois-
son distribution for the weights \( w_n = \theta^{n-1} \exp(-\theta)/(n-1)! \), the Spearman
correlation may be shown to be

\[
\rho_s = 1 - 2\theta^{-1} + 2\theta^{-2}(1 - \exp(-\theta)).
\]
4 Bivariate Data Fitting Example

4.1 Australian Institute of Sports data

A dataset from the Australian Institute of Sport is used as an example. This is given in Cook and Weisberg [5] and has been used as a testbed for new distributions by Azzalini and others [21]. Here, percentage body fat and weight of 102 male athletes were used. Figure 4 shows the skew distribution of percentage body fat. The distribution of weight (not shown) was also slightly skew.

A suitable univariate model for the marginal distributions was chosen as the lagged normal distribution [6], where the random variable \( X = Z + Y \), where \( Z \) is Gaussian, and \( Y \) is exponential. In fact, taking \( X = Z + Y_1 - Y_2 \) gives a distribution that can be skew in either direction and long-tailed to either or both left and right. Taking the normal mean as \( \xi \) and standard deviation \( \beta \), and the exponential means as \( \alpha_1 \) and \( \alpha_2 \), the pdf is

\[
f(x) = \frac{1}{\alpha_1 + \alpha_2} \left[ \exp \left\{ \frac{1}{2} (\beta/\alpha_1)^2 - (x - \xi)/\alpha_1 \right\} \Phi \left( \frac{x - \xi}{\beta} - \frac{\beta}{\alpha_1} \right) \\
+ \exp \left\{ \frac{1}{2} (\beta/\alpha_2)^2 + (x - \xi)/\alpha_2 \right\} \Phi \left( -\frac{x - \xi}{\beta - \beta/\alpha_2} \right) \right],
\]

(19)

where \( \Phi \) is the normal distribution function. The distribution function \( F(x) \) is

\[
F(x) = \Phi \left( \frac{x - \xi}{\beta} \right) + \frac{1}{\alpha_1 + \alpha_2} \left[ -\alpha_1 \exp \left\{ \frac{1}{2} (\beta/\alpha_1)^2 - (x - \xi)/\alpha_1 \right\} \Phi \left( \frac{x - \xi}{\beta} - \frac{\beta}{\alpha_1} \right) \\
+ \alpha_2 \exp \left\{ \frac{1}{2} (\beta/\alpha_2)^2 + (x - \xi)/\alpha_2 \right\} \Phi \left( -\frac{x - \xi}{\beta - \beta/\alpha_2} \right) \right].
\]

(20)

The mean \( \mu = \xi + \alpha_1 - \alpha_2 \), variance \( \sigma^2 = \beta^2 + \alpha_1^2 + \alpha_2^2 \), skewness \( \gamma = 2(\alpha_1^2 - \alpha_2^2)/\sigma^3 \), and kurtosis \( \kappa = 6(\alpha_1^4 + \alpha_2^4)/\sigma^4 \). Since \( \Phi \) is a well-known special function, and the distribution function can be written as a function of \( \Phi \), and also the moments can be written down, this distribution is quite an attractive choice for fitting data that depart from normality, and are not heavy tailed. The easy computation of the distribution function makes it particularly attractive for use in fitting multivariate distributions via copulas. Care is needed in computing the pdf and distribution function when \( \alpha_1/\beta \) or \( \alpha_2/\beta \) are small. One can then use the asymptotic expansion for \( \Phi(z) \),

\[
\Phi(z) = \frac{\exp(-z^2/2)}{(2\pi)^{1/2}} \left\{ -1/z + 1/z^3 - 3/z^5 + 3 \times 5/z^7 - \cdots \right\},
\]
which avoids the rounding errors implicit in taking the product of very large
and very small quantities. The symmetric form of this distribution, with \( \alpha_1 = \alpha_2 \), is described in Johnson, Kotz and Balakrishnan [11], vol. 2, chap. 24.

The percentage body fat could be fitted by maximum likelihood to a lagged
normal distribution, where only the right tail was needed, so that \( \alpha_2 = 0 \). Figure 4 shows the fitted curve, with Azzalini’s skew normal distribution [1] also fitted. Both distributions fitted satisfactorily, according to the Kolmogorov
test, although better fits can be achieved at the expense of using more parameters; there is even a suggestion of bimodality in the data. This is possible, as the sample comprises athletes from a variety of different sports. Weight and
height look normal, but weight has a lower AIC (Akaike Information Criterion)
if fitted to a lagged normal, and this was done.

The bivariate pdf (5) with \( n = 10 \) fitted the data with a log-likelihood of
\( \ell = -607.54 \) and a weight \( q = 0.78 \). The observed Spearman and Pearson
correlations were 0.613 and 0.581, and the predictions from the model were
0.640 and 0.576. The Bessel function pdf (15) also fitted satisfactorily, with
\(-\ell = 606.47\), and predicted Spearman correlation of 0.65, Pearson correlation
0.565. The fitted value of \( \theta \) was \( \hat{\theta} = 23.7 \). For comparison, the Azzalini
bivariate distribution [1] fitted with \(-\ell = 612.22\), with the same number (7)
of parameters. The point here is that the Bessel function copula (16) per-
forms satisfactorily, as does the whole copula-based methodology of modeling
the marginal distributions individually, and gluing them together with a cop-
ula. One can obtain good fits to the data, without forcing both the marginal
distributions to be of the same form. There is then the freedom to vary the
marginal modeling, for example by fitting a bimodal distribution in figure 4,
which option is not available on fitting a standard multivariate distribution.

5 Multivariate models

Consider the multivariate generalization of the models presented so far. This
topic was only briefly touched on in [3], and the results here are new.

Denote the \( i \)th of \( p \) random variables by \( X_i \), and denote the corresponding
distribution functions, pdfs, and distribution functions of the \( k \)th of \( n \) or-
der order statistics by \( F^{(i)}(x_i), f^{(i)}(x_i) \) and \( F_{k,n}^{(i)}(x_i) \) respectively. The most general
multivariate model of order \( n \) would be

\[
H(x) = \sum_{k_1, \ldots, k_p} \prod_{i=1}^{p} F_{k_i,n}^{(i)}(x_i) r_{k_1,\ldots,k_p},
\]

where \( r_{k_1,\ldots,k_p} \geq 0 \) and \( \sum_{k_j \neq m} r_{k_1,\ldots,k_p} = 1/n \) for all \( m \).
This would have \((n - 1)^p\) parameters. To reduce this number, one could consider only models in which the random variables are in phase (or in antiphase, for negative correlations) in cycles of length \(n\). Variables could all be in the same cycle, or some could be in independent cycles. For example, with 5 variables, two could be paired in one cycle, two in another independent cycle, and the fifth variable be in a cycle of its own. To generate random numbers from such a distribution, one could compute the \(n\) order statistics for the 5 variables, and then one random number would decide which order statistic was to be taken for variables 1 and 2, another independent random number would select the third and fourth random variable pair, and a third independent random choice would select the fifth random variable from among its \(n\) order statistics. Clearly, random numbers for variables in such single cycles could be more efficiently generated by simply choosing a random variable from the appropriate marginal distribution.

The number of models \(a_p\) is the number of ways \(p\) distinguishable objects (random variables) fit into \(p\) or fewer identical boxes (cycles). This is given by the recursion relation

\[
a_p = \sum_{j=0}^{p-1} \binom{p-1}{j} a_{p-1-j},
\]

with \(a_0 = 1\) (Tucker, 17). This may be derived by considering the addition of the \(p\)th object. It must occur in a box containing \(0 \leq j \leq p - 1\) other objects, where the \(j\) other objects can be chosen in \(\binom{p-1}{j}\) ways, and the remaining \(p - 1 - j\) other objects in the other boxes can be arranged in \(a_{p-1-j}\) ways. The recursion relation follows, and the number of mixing parameters for a mixture model is \(a_p - 1\). Table 1 shows the number of models resulting; the number grows faster than exponentially with \(p\). The table also shows the number of parameters \(2^p - p - 1\) for the subset of models obtained by simply including or excluding random variables from one common cycle. This simple scheme gives distributions whose marginals allow differing Spearman correlations, and is feasible up to dimensions of 5 or 6, beyond which the number of model parameters becomes excessive. The multivariate distribution function \(H(x)\) can be written

\[
H(x) = \sum_{S=1}^{2^p} w_S \prod_{i \in S} F^{(i)}(x_i) \{n^{-1} \sum_{k=1}^{n} \prod_{j \in S} F^{(j)}_{k,n}(x_{ij})\},
\]

where the sets \(S\) run through all possible subsets of the \(p\) random variables, and where \(\sum_{S=1}^{2^p} w_S = 1\). The model parameters can be estimated in the same way as described earlier for bivariate mixture models.
6 Multivariate Data Fitting Examples

With the notation
\[ T_{ij}(x_i, x_j) = \frac{1}{n} \sum_{k=1}^{n} F_{k,n}^{(i)}(x_i) F_{k,n}^{(j)}(x_j) \]

etc, the trivariate case of (21) can be written
\[ H(x) = w_0 F^{(1)}(x_1) F^{(2)}(x_2) F^{(3)}(x_3) + w_1 F^{(1)}(x_1) T_{23}(x_2, x_3) + w_2 F^{(2)}(x_2) T_{13}(x_1, x_3) + w_3 F^{(3)}(x_3) T_{12}(x_1, x_2) + w_4 T_{123}(x_1, x_2, x_3), \tag{22} \]

where \( w_0 + w_1 + w_2 + w_3 + w_4 = 1. \)

This model was fitted to the percentage of body fat, weight, and height for the Australian Institute of Sport data from Cook and Weisberg [5].

After fitting the three marginal distributions by maximum likelihood, using the lagged normal distribution, the trivariate distribution was fitted by maximum likelihood to find the four parameters \( w_1 \ldots w_4, \) keeping the marginal distributions fixed. Subsequently, allowing the parameters of the marginal distributions to float increased the log-likelihood \( \ell \) by only a very small amount.

Fitting weights that sum to unity poses a computational problem. The simple solution adopted was to use parameters \( v_0 \) to \( v_4, \) fixing one parameter \( v_4 = 0, \) and then the \( w_i = \exp(v_i) / \sum_{j=0}^{4} \exp(v_j) \) sum to unity, while the 4 free parameters can take any value on the real line. Any of the \( v_j \) can be set to 0, but the choice is best altered if the term chosen fits to very small weight \( w_j, \) as then all the other \( v_j \) become huge.

The observed and predicted Pearson and Spearman correlations are given in table 2 showing fairly good agreement. A value of \( n = 12 \) was used, but the results are not very sensitive to this, as long as \( n \) is large enough to allow the highest correlation. The five fitted weights in (22) were respectively 0.0003, 0.435, 0.0112, 0.284, 0.270. Interestingly, the independence (first) term is not needed. A common Spearman correlation of \((11/13) \times 0.27\) derives from the last term in (22), and the correlation between weight and height is then boosted by the second term, while that between body fat and weight is boosted by the fourth term.

The trivariate fit by this copula fits just slightly worse than the 3-dimensional Azzalini model, in terms of log-likelihood, even although the marginal fits are slightly better. The Azzalini model gave \( \ell = -931.3, \) while this model gave \( \ell = -932.6. \) Unfortunately, this is the ‘little rift within the lute’ that limits the usefulness of these multivariate models. Although they can accommodate large and variable correlations, they cannot fit an arbitrary correlation matrix. This
was seen much more clearly on moving to a quadrivariate example, taken from a study by Penrose [14] and available online via Statlib, in which percentage body fat, weight, height and abdominal circumference were fitted, for a sample of 252 men.

The quadrivariate model in a terse notation, writing e.g. \( F^{(1)}(x_1) = T_1 \), was

\[
H(x) = w_0 T_1 T_2 T_3 T_4 \\
+ w_1 T_1 T_2 T_3 + w_2 T_1 T_3 T_4 + w_3 T_1 T_4 T_23 + w_4 T_2 T_3 T_14 + w_5 T_2 T_4 T_13 + w_6 T_3 T_4 T_{12} \\
+ w_7 T_1 T_{234} + w_8 T_2 T_{134} + w_9 T_3 T_{124} + w_{10} T_4 T_{123} \\
+ w_{11} T_{1234} + w_{12} T_1 T_2 T_3 + w_{13} T_1 T_3 T_4 + w_{14} T_1 T_4 T_{23}.
\]

Although the lagged normal distribution gave satisfactory marginal fits, the quadrivariate model fitted with 14 parameters gave \(-\ell = 3352\), compared with the quadrivariate Azzalini distribution fit of \(-\ell = 3184\), and most of the weights fitted as zero. It was clear that the fitted correlations were in general too small. Hence the usefulness of these multivariate distributions seems limited.

7 Conclusions

Following the introduction of a new copula in Baker [3], it became clear to the author that its properties had not been fully enumerated, and also that it was possible to derive further copulas by generalizing it. Further, only a few bivariate distributions had been fitted to data, and there was no practical experience at all with fitting multivariate distributions.

In this paper, several ways of extending the class of copulas have been given. Perhaps the most promising one is to make the order \( n \) a random variable from the discrete Bessel distribution. This leads to the ‘Bessel function’ copula (16), the only copula in the author’s experience that requires a special function for its expression. This copula is indexed by one parameter \( \theta \). Like the Frank, Clayton and Plackett copulas, it contains the independence case \( C(u, v) = uv \), and can attain the Fréchet bound as \( \theta \to \infty \). Negative correlations are dealt with by e.g. setting \( G(y) \to 1 - G(y) \). The use of this copula has been illustrated by fitting it to the Australian Institute of Sport dataset [5]. The fact that the copula must be written either as a double integral, or as a series expansion is a drawback, but in fitting to data by likelihood-based methods, the crucial requirement is that the pdf must be easily computable. This pdf is easy to compute, given the widespread existence of routines to compute the special functions \( I_0 \) and \( I_1 \). It is also not difficult to generate random variables. This copula is by the way not Archimedean; Archimedean copulas must be
associative, but computations showed a difference between $C(u, C(v, w))$ and $C(C(u, v), w)$ (lack of associativity) of up to about 2%.

The properties of the bivariate copulas have been further explored. The most significant is probably that the original copula in (1) and its mixtures possess the LRD (likelihood ratio dominance) ordering property. This property is therefore also possessed by the Bessel function copula.

The properties of the analogous multivariate copulas have also been studied, but here results are less positive. They do have some flexibility; marginal distributions need not have identical parameters, and high correlations can be accommodated. Although the hitherto untried process of fitting trivariate and quadrivariate models to data by maximum-likelihood estimation proved entirely feasible, it seems that despite their many parameters these distributions can not reproduce an arbitrary correlation matrix. The use of these distributions for $p > 2$ is therefore problematical. They may however prove to be a starting point for the development of more useful distributions.
Fig. 1. Measures of association as a function of the order \( n \) of the distribution for the distribution of equation (3). The key gives the curves from top to bottom.

Fig. 2. The Bessel function copula with \( \theta = 250 \).
Fig. 3. The Bessel function copula with $\theta = 5000$.

Fig. 4. Fits of the lagged normal and Azzalini distributions to the percentage of body fat for 102 male athletes (Australian Institute of Sport data).
Table 1
Numbers of parameters to be estimated for two classes of multivariate model, and the number of correlations. The models, from left to right, are the single cycle model, and the multicycle model.

| Dimension | p | Params, model 1 | Params, model 2 | p(p − 1)/2 |
|-----------|---|----------------|----------------|-------------|
| 2         | 1 | 1              | 1              | 1           |
| 3         | 4 | 4              | 4              | 3           |
| 4         | 11| 14             | 6              | 6           |
| 5         | 26| 51             | 10             | 10          |

Table 2
Observed and predicted Pearson correlations (ρ) and Spearman correlations (ρ_s) on fitting the trivariate model in equation 22 with lagged normal marginals.

| Variables          | Obs. ρ | Pred. ρ | Obs. ρ_s | Pred. ρ_s |
|--------------------|--------|---------|----------|-----------|
| % Body fat & Weight| 0.581  | 0.412   | 0.613    | 0.468     |
| % Body fat & Height| 0.192  | 0.199   | 0.237    | 0.237     |
| Height & Weight    | 0.666  | 0.596   | 0.677    | 0.596     |
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