Integer and Rational Solutions to Polynomial Equations

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Abstract

A formalism is given to count integer and rational solutions to polynomial equations with rational coefficients. These polynomials $P(x)$ are parameterized by three integers, labeling an elliptic curve. The counting of the rational solutions to $y^2 = P(x)$ is facilitated by another elliptic curve with integral coefficients. The problem of counting is described by two elliptic curves and a map between them.
The integer and rational solutions to polynomial equations with rational coefficients is relevant to many areas of physics, mathematics, and applied science. The counting of these solutions seems problematic without direct solving of the equations. One conjecture relates the expansion of certain Hecke modular forms (L-series) around unity to the cardinality of the rational solutions to $P(x) = y^2$. A direct counting of the solutions, and their characterizations is more interesting and relevant.

In this work, all rational polynomial equations are examined in one context. The counting of solutions $p/q$ for a given $q$ to the equation $P(x) = m/n$ is formulated in this work. Several functions $f_i$ are required to analyze the complete set of polynomial equations. The symmetry and uniformizations of the counting problem is described by the general functional form of these functions, which consist of a map from one elliptic curve to another. The maps are not given here; however, a determination would certainly be of interest for both practical applications of the counting and for formal extensions of algebra [1].

As an example of the counting problem, consider the expansion of the free energy of a certain statistical model as described in [2] (with related work in [3]),

$$\sum_{x=1}^{M} \exp \left( - \sum_{i=1}^{N} b_i x^i \tau \right) = \sum_{p} \Delta(p)e^{-p\tau}.$$  \hspace{1cm} (1)

The statistical free energy in (1) involves a one parameter sum over integers $x$, which admit a $\tau = 1/k_b T$ expansion at small temperature. The expansion is facilitated by the integer solutions to

$$\sum_{i=1}^{N} b_i x^i = P(m) = p,$$  \hspace{1cm} (2)

and a counting of the number of solutions $m$. The formalism of the model in (1) specifying the $b_i$ coefficients (which are in general rational) guarantees that for integer $x$ the value of $P(x)$ is an integer. The number of solutions at level $p$ to the equation in (2) defines the function $\Delta(p)$. The function $\Delta(p)$ is defined by its values at $M$ points, and hence can also be characterized by a polynomial of degree $M$. These polynomials in (2) together with their solutions define quasi-modular and modular forms via the expansion in (1).

The example describes one purpose of the counting of the divisors of the rational polynomials, and how the countings of $m$ to $P(m) = p$ are described by a polynomial
\( \Delta(p) \). The polynomial aspect to \( \Delta(p) \) is examined in further detail here, albeit in a different context and generalized to the set of all polynomials with rational coefficients.

**Counting of Solutions**

Algebraic equations \( P(x) = y^2 \) are examined, with the degree of \( P(x) \) and integer \( n \). Rational solutions of the form \( x = s/t \) and \( y^2 = p/q \) are considered. In general \( y^2 \) may be taken to \( y^d \), generating a further restriction on the \( y \) values.

The general polynomial equation

\[
P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 ,
\]

may be specified by three integers, in the case of rational coefficients \( a_i = p_i/q_i \). For example, first specify a base \( m \) for \( B \). Then specify two polynomials parameterizing the numerators and denominators of the coefficients pertaining to \( P(x) \) via,

\[
N = \prod_{j=1}^{n} (x - p_i) \quad D = \prod_{j=1}^{n} (x - q_i) .
\]

The base \( x \) must be chosen larger than any of the coefficients \( \prod_i p_i \) and \( \prod_i q_i \). The expansion of the numbers \( M \) and \( N \) in the base \( x \) follows from,

\[
N = \sum_{k=1}^{n} N_k x^k \quad D = \sum_{k=1}^{n} D_k x^k ,
\]

with \( N_k \) and \( D_k \) smaller than the integer \( x \). In this base, the numbers \( N \) and \( D \) uniquely parameterized the coefficients \( N_k \) and \( D_k \).

These three numbers \( B, N, \) and \( D \) parameterize the polynomial \( P(x) \). One may group these numbers into an algebraic elliptic curve \( E \),

\[
x^3 + Bx^2 + Nx + D = y^2 ,
\]

or

\[
Bx^2 + Nx + D = y^2 ,
\]

or into a 3-tuple \((B, N, D)\). The elliptic curve is parameterized up to relabeling of the three numbers.
As a comment, the parameterization of the rational polynomials is not unique. Another decomposition is to write the polynomial in terms of a product and an integer, in base $B$: \( a \prod (x - c_j) + M = N + M \), which requires three numbers \((x, N, M)\). The one used here seems natural, and the former is related transcendentally.

To illustrate the procedure a counting map is generated for integer solutions to \( P(x) = p/q \). The question to be examined is how many rational solutions \( x = s/t \), i.e. divisors, there are to this equation.

First rationalize the polynomial equation by multiplying the equation by the denominators of the coefficients \( t^n q \prod_{i=1}^{n} b_i \). This generates a polynomial equation in \( s \), \( P_t(s) = p \), containing integer coefficients.

The question to be examined is how many integer solutions there are to this equation \( P_t(s) = p \). Label the count as \( C_{t}^{p,q}(s) \), valid in a range of values \( s = q_1 \) to \( s = q_2 \). This count \( C_{t}^{p,q}(s) \) is specified by another polynomial,

\[
Q_{t}(x) = b_L x^L + b_{L-1} x^{L-1} + \ldots b_0 = 0 , \tag{8}
\]

with \( x \) ranging from \( q_1 \) to \( q_2 \) for example. In principle, the values \( q_1 \) and \( q_2 \) may be taken to \(-\infty \) and \( \infty \), as a limit of the two integers; this results in infinite degree (or an analytic function, possessing a Taylor series expansion with the coefficients \( b_j \)).

By definition, the polynomial in (8) counts the solutions \( x = s/t \) to the polynomial equation \( P(x) = p/q \), denoted by \( C_{t}^{p,q}(s) \) in the range \( s = q_1 \) to \( s = q_2 \). This polynomial \( Q(x) \) is also specified by an associated elliptic curve \( E_C \), or a 3-tuple,

\[
B_C x^2 + N_C x + D_C = 0 . \tag{9}
\]

The three integers \( B_C \), \( N_C \), and \( D_C \) label the curve.

Given the elliptic curve \( E_C \), the polynomial \( Q(x) \) may be reconstructed by expanding the numbers \( N_C \) and \( D_C \) in the base \( B_C \). In this manner the elliptic curve counts the solutions \( C_{t}^{p,q}(s) \) in the range \( q_1 \) to \( q_2 \).

The map required is a transformation of the two curves from

\[
B x^2 + N x + D = y^2 \quad \rightarrow \quad B_C x^2 + N_C x + D_C = y^2 , \tag{10}
\]

or \( x^3 + B_i x^2 + N_i x + D_i = y^2 \) (as with all of the other three tuple forms of the curves previously described). This is an \( \text{SL}(2,\mathbb{Z})_w \) transformation parameterized for example
by three numbers, or a complex rational number \( f_{r/g} + if_{h/g} \). Denote the map by \( M_{t}^{p,q;q_1q_2} \).

All of the information of the rational count \( x = s/t \) to the polynomial equation \( P(x) = p/q \) is encoded in the map between the elliptic curves. These are essentially fibrations of the polynomial equations, e.g. the elliptic curves over the Riemann surfaces \( P(x) = y^2 \) or \( P(x) = y^d \). The question is then to do determine \((B_C, N_C, D_C)\) from \((B, N, C)\) specified by \( t \) and \( q_1, q_2 \), and \( p \) and \( q \).

The simplest limit is \( q_1 = -\infty, q_2 = \infty \), and \( p = q = 0 \). In this case, there is a one parameter map \( M_t \) labeled by the denominator of the solution \( x = s/t \), and the function \( C_t(s) \) counts the \( x \)-solutions. The most general mapping contains the five parameters, possibly a function of the curve parameters \( B, N, D \).

An interesting question is when the transformation from \( E \) to \( E_C \) depends on the parameters \( B, N, D \), and how it depends on these transformations. Obviously there are relations between rational solutions \( x = s/t \) to polynomials with rational coefficients \( P(x) = p/q \) and integer solutions \( x = m \) to polynomials \( P_Z(x) = p \) with integer coefficients. The various mappings should break into characteristic classes, perhaps via representation of \( \text{SL}(2,\mathbb{Z}) \).

It would be interesting to find a differential operator with solutions that generate the maps \( M \) from \( E \) to \( E_C \), or a recursive construction of the coefficients pertaining to the maps. This is possibly obtained via topological properties of the manifolds that describe the solution space, as found in [4].

The counting of the rational points on the elliptic and hyperelliptic surfaces is an outstanding problem pertinent to various areas of algebra and geometry. The approach here is more general than finding the cardinality, i.e. infinite of finite, of the rational points. This means that the maps would find all sets of points \( x = s/t \) for \( p/q \) at given values of \( t \) and \( p/q \), as opposed to whether the number of solutions is infinite.

The three functions via the torus map are generated via the polynomials,

\[
B_C(B, N, D) = \sum_{i,j,k} \beta_{i,j,k}^{BC} B^i N^j D^k \tag{11}
\]

\[
N_C(B, N, D) = \sum_{i,j,k} \beta_{i,j,k}^{NC} B^i N^j D^k \tag{12}
\]

\[
D_C(B, N, D) = \sum_{i,j,k} \beta_{i,j,k}^{DC} B^i N^j D^k \tag{13}
\]
in which the coefficients are rational numbers. The map is polynomial in nature because the numbers $B_C$, $N_C$ and $D_C$ are integral. These maps also define three manifolds via the polynomial $y^2 = B_C$, etc... These special manifolds define the counting of the rational solutions to the polynomial equations. The definitions of the elliptic parameters requires a specification of the base, in general away from base 2. So there is an integral ambiguity parameterized by $Z$ of the maps.

Due to the three-dimensional interpretation of the function describing the map between the two torii, it is natural to inquire as to the form of the moduli space parameterizing the equivalent geometries; meaning which polynomials $P(z) = p/q$ have the same zero set solutions in the rational numbers (either in an entire context or partially). This is not examined in the text here.

Due to the fact that all of the number solution counting is obtained from the torus map, encoded in the three functions in (13). It is possible that the numbers $\beta_{i,j,k}$ have some special number theoretic significance, either recursive or differential geometric (mentioned that this is possibly obtained via topological properties of the associated manifold in [4], with related background in [5]), and that the homology (i.e. the intersection matrix) of the 3-manifolds defined by (13) is also significant. This significance should shed light on a proof of the coefficients and also on the geometric properties of the distributions of the countings.

The maps are not generated here, however, there are constraints imposed on their construction. The solution to the maps is expected to be transcendental, and possibly stochastic, due to the parameterizations of the relevant numbers; there may be further parameterizations which are useful for determinations of the mappings. Uniformization of the curves is interesting.
References

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