Rationally Connected Varieties and Loop Spaces*

László Lempert†
Department of Mathematics
Purdue University
West Lafayette, IN 47907-2067, USA

Endre Szabó
Rényi Institute of the Hungarian Academy of Sciences
1364 Budapest
PO Box 127

Abstract

We consider rationally connected complex projective manifolds $M$ and show that their loop spaces—inefinite dimensional complex manifolds—have properties similar to those of $M$. Furthermore, we give a finite dimensional application concerning holomorphic vector bundles over rationally connected complex projective manifolds.

0 Introduction

Let $M$ be a complex manifold and $r = 0, 1, \ldots, \infty$. The space $C^r(S^1, M)$ of $r$ times continuously differentiable maps $x : S^1 \to M$, the (free) $C^r$
loop space of \( M \), carries a natural complex manifold structure, locally biholomorphic to open subsets of Banach (\( r < \infty \)) resp. Fréchet (\( r = \infty \)) spaces, see [L2]. The same is true of “generalized loop spaces”—or mapping spaces—\( C^r(V, M) \), where \( V \) is a compact \( C^r \) manifold, possibly with boundary; when \( r = 0 \), \( V \) can be just a compact Hausdorff space. A very general question is how complex analytical and geometrical properties of \( M \) and its loop spaces are related.

Our contribution to this problem mainly concerns rational connectivity. A complex projective manifold is a complex manifold, biholomorphic to a connected submanifold of some projective space \( \mathbb{P}^n(\mathbb{C}) \). Such a manifold \( M \) is called rationally connected if it contains rational curves (= holomorphic images of \( \mathbb{P}^1(\mathbb{C}) \)) through any finite collection of its points. This is equivalent to requiring that for a nonempty open \( U \subset M \times M \) and any \((p, q) \in U\) there should be a rational curve through \( p \) and \( q \). For the theory of rationally connected varieties see [AK, Kl, KMM].

Projective spaces, Grassmannians, and in general complex projective manifolds birational to projective spaces are rationally connected. In a sense rationally connected manifolds are the simplest manifolds; at the same time, general complex projective manifolds can be studied through rationally connected ones by the device of maximally rationally connected fibrations [Kl, Theorem IV.5.4].

Here is a brief description of the results presented in this paper. For more complete formulations and for background the reader is referred to Section 1. First we prove that loop spaces \( C^r(S^1, M) \) of rationally connected complex projective manifolds \( M \) contain plenty of rational curves, but in some other mapping spaces \( C^r(V, M) \) rational curves are rare. Then we shall discuss holomorphic functions and, more generally, holomorphic tensor fields. Extending earlier results of Dineen–Mellon and the first author [DM,L2] we show that on mapping spaces of rationally connected complex projective manifolds \( M \) holomorphic functions are locally constant, and the same is true on submanifolds of \( C^r(V, M) \) consisting of so called based maps. This infinite dimensional result has the finite dimensional corollary that holomorphic linear connections on vector bundles \( E \to M \) are trivial.

Next we consider the “trivial” component of \( C^r(V, M) \) consisting of contractible maps. We show that the constancy of holomorphic functions on this component already follows once we know \( M \) is compact, connected, and all contravariant symmetric holomorphic tensor fields on \( M \) (of positive weight) vanish (i.e., the only holomorphic sec-
tion the symmetric powers of \( T^*M \) admit is the zero section). This property is probably weaker than rational connectivity. In fact, rational connectivity implies that holomorphic contravariant tensor fields, symmetric or not, vanish, see [AK, Theorem 30]; and conjecturally the converse, “Castelnuovo’s criterion”, is also true for complex projective manifolds. Finally we prove that for compact connected \( M \), if on \( M \) all contravariant holomorphic tensor fields of positive weight vanish then the same holds on the trivial component of \( C^r(V,M) \).

1 Background and results

1.1. Mapping spaces. Fix \( r = 0, 1, \ldots, \infty \) and a compact manifold \( V \) of class \( C^r \), possibly with boundary (or just a compact Hausdorff space, when \( r = 0 \)). We start by quickly describing the complex manifold structure on the mapping space \( X = C^r(V,M) \) of a finite dimensional complex manifold \( M \). For generalities on infinite dimensional complex manifolds, see [L1, Section 2]. We need

**Lemma 1.1.** There are an open neighborhood \( D \subset M \times M \) of the diagonal and a \( C^\infty \) diffeomorphism \( F \) between \( D \) and a neighborhood of the zero section in \( TM \) with the following properties. Setting \( D^w = \{ z \in M : (z,w) \in D \} \) and \( F^w = F(\cdot,w) \), for all \( w \in M \) we have

(a) \( F^w \) maps \( D^w \) biholomorphically on a convex subset of \( T_w M \);
(b) \( F^w(w) \in T_w M \) is the zero vector;
(c) \( dF^w(w) : T_w D^w = T_w M \to T_w M \) is the identity.

(In (c) we have identified a tangent space to the vector space \( T_w M \) with the vector space itself.)

**Proof.** When \( M \) is a convex open subset of some \( \mathbb{C}^n \) so that \( TM \) is identified with \( \mathbb{C}^n \times M \), one can take \( D = M \times M \) and \( F(z,w) = (z - w, w) \). A general \( M \) being locally biholomorphic to convex open subsets of \( \mathbb{C}^n \), one obtains a covering of \( M \) by open sets \( W \) and \( C^\infty \) maps \( F_W : W \times W \to TW \) that satisfy (a, b, c) for \( w \in W \) (with \( D^w \) replaced by \( W \)). If \( \{ \chi_W \}_W \) is a corresponding \( C^\infty \) partition of unity on \( M \), one can take as \( F(z,w) \) the restriction of

\[
\sum_W \chi_W(w) F_W(z,w)
\]

to an appropriate neighborhood of the diagonal.
Given $D$ and $F$, define the complex structure on $X = C^r(V,M)$ as follows. A coordinate neighborhood of $y \in X$ consists of those $x \in X$ for which $(x(t), y(t)) \in D$ for all $t \in V$. This neighborhood is mapped to an open subset of $C^r(y^*TM)$, the space of $C^r$ sections of the induced bundle $y^*TM \to V$, by the map

$$\varphi_y: x \mapsto \xi, \quad \xi(t) = F(x(t), y(t)).$$

It is straightforward that the local charts $\varphi_y$ are holomorphically related and so define a complex manifold structure on $C^r(V,M)$; this structure is independent of the choice of $D$ and $F$.

The above construction is slightly simpler than the one in [L2, Section 2]. Its drawback is that it does not generalize to infinite dimensional manifolds $M$, that may not admit $C^\infty$ partitions of unity. By contrast, the construction in [L2] does generalize, since it uses partitions of unity on $V$ only.

A closed $A \subset V$ and $x_0 \in X$ determine a subspace of “based” maps. Denoting the $r$-jet of $x$ by $j^rx$, the subspace in question is

$$(1.1) \quad Z = C^r_{A,x_0}(V,M) = \{ x \in X : j^rx|A = j^r x_0|A \},$$

a complex submanifold of $X$. As explained in [L2, Sections 2,3], for $x \in X$ the tangent space $T_xX$ is naturally isomorphic to $C^r(x^*TM)$; if $x \in Z$, under this isomorphism $T_xZ \subset T_xX$ corresponds to

$$(1.2) \quad C^r_A(x^*TM) = \{ \xi \in C^r(x^*TM) : j^r \xi|A = 0 \}.$$

Up to this point $TX$, $TZ$ are real vector bundles. However, as in finite dimensions, the local charts endow the real tangent bundles of $X$ and $Z$ with the structure of a locally trivial holomorphic vector bundle, and we shall always regard $TX$ and $TZ$ as such.

1.2. Rational connectivity. While our principal interest is in complex manifolds, we will have to deal with projective (or quasiprojective) varieties defined over fields other than $\mathbb{C}$ as well. Then we shall use the language of algebraic geometry, in particular the topology implied will be Zariski’s. If $M$ is a variety defined over a field $k$, we write $M(k)$ for its points over $k$. When $k$ is algebraically closed we shall ignore the difference between $M$ and $M(k)$, so, for instance, a smooth projective variety $M$ over $\mathbb{C}$ will be thought of as a complex projective manifold determined by $M(\mathbb{C})$. 

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Definition 1.2. Let $M$ be a smooth projective variety defined over a field $k$ of characteristic 0. When $k$ is algebraically closed and uncountable, $M$ is rationally connected if there is a morphism $f : \mathbb{P}^1 \to M$ defined over $k$ (i.e., a rational curve) such that the induced subbundle $f^*TM$ is ample: $f^*TM \cong \bigoplus O_{\mathbb{P}^1}(d_j)$, with all $d_j > 0$. In general, $M$ is rationally connected if it is such when considered over some (and then over an arbitrary) uncountable algebraically closed field $K \supset k$.

Over a field of positive characteristic the above property defines the so called separably rationally connected varieties. When $k = \mathbb{C}$, rational connectivity is equivalent to requiring that there be a nonempty open $U \subset M(\mathbb{C}) \times M(\mathbb{C})$ such that for $(p,q) \in U$ there is a rational curve through $p$ and $q$; and also to requiring that through any finite collection of points in $M(\mathbb{C})$ there be a rational curve. For all this, see [AK, Definition–Theorem 29].

1.3. Rational connectivity of loop spaces.

Theorem 1.3. Let $M$ be a rationally connected complex projective manifold and $V$ a one real dimensional manifold. Then the space $C^n(V,M)$ is rationally connected in the sense that for any $n \in \mathbb{N}$ there is a dense open $O \subset C^n(V,M^n)$ such that through any $n$–tuple of maps $(x_1, \ldots, x_n) \in O$ there is a rational curve in $C^n(V,M)$.

Taking $V = S^1$ we see that $M$ must be simply connected, a result first proved by Campana [C].

The theorem would not hold for higher dimensional $V$. First, the space $C^n(V,M)$ may be disconnected, which precludes rational connectivity. But even within components rational curves will be scarce, typically. Let us call the component of $C^n(V,M)$ containing constant maps the trivial component.

Theorem 1.4. If $V$ is a closed connected surface and $h : \mathbb{P}^1(\mathbb{C}) \to C^n(V,\mathbb{P}^1(\mathbb{C}))$ holomorphic then $h$ is constant or else maps into the trivial component of $C^n(V,\mathbb{P}^1(\mathbb{C}))$.

We do not know whether in the trivial component generic $n$–tuples can be connected with rational curves; but at least a nonempty open set of $n$–tuples can be. We shall not prove this, but it follows along the lines of Section 2 (with Lemma 2.2 slightly modified), even for arbitrary $V$ and the mapping space $C^n(V,M)$ of a rationally connected $M$ instead of $\mathbb{P}^1(\mathbb{C})$. 5
1.4. Holomorphic functions. A simple consequence of Theorem 1.3 is that on loop spaces of rationally connected complex projective manifolds holomorphic functions are constant. It turns out that this generalizes to spaces $Z$ of based maps, see (1.1), even though a typical $Z$ will not contain compact subvarieties, let alone rational curves according to [L2, Theorem 3.4].

**Theorem 1.5.** If $M$ is a rationally connected complex projective manifold, $A \subset V$ is closed, and $x_0 \in C^r(V, M)$, then holomorphic functions $C^r_{A,x_0}(V, M) \to \mathbb{C}$ are locally constant.

The case when $M$ is a projective space was known earlier, see [DM, Theorems 7, 11] for $r = 0$ and [L2, Theorem 4.2] in general. The theorem has the following

**Corollary 1.6.** If a holomorphic vector bundle (possibly with fibers Banach spaces) $E \to M$ over a rationally connected complex projective manifold admits a holomorphic (linear) connection then both $E$ and the connection are trivial.

In fact, in section 4 we shall prove a rather more general result. However, something far more general may also be true that has nothing to do with rational connectivity. We conjecture that Corollary 1.6 is true for all simply connected compact Kähler manifolds $M$.—As Kollár noted, when $E$ has finite rank and $M$ is not only rationally connected but Fano, Corollary 1.6 immediately follows from [AW, Proposition 1.2]. Holomorphic connections have been studied first by Atiyah [A]. Among other things he proved a Lefschetz type theorem on generic hyperplanes, and completely classified holomorphic connections over Riemann surfaces in terms of the fundamental group.

1.5. Holomorphic tensor fields. All tensor fields will be contravariant, without explicit mentioning. Over a finite dimensional manifold $N$ these tensor fields are sections of tensor powers of $T^*N$. To avoid, in the infinite dimensional case, dealing with the ambiguous notion of tensor product of Banach or Fréchet spaces/bundles, we simply define a holomorphic tensor field on a complex manifold $N$ as a holomorphic function

$$g: T^j N = TN \oplus \ldots \oplus TN \to \mathbb{C},$$

multilinear on each fiber. The integer $j = 0, 1, \ldots$ is the weight of the tensor field; a tensor field of weight 0 is just a holomorphic function on $N$. If $g$ is symmetric on the fibers we speak of a symmetric tensor.
field. Examples of symmetric holomorphic tensor fields are the zero fields (for all weights) and the constant fields of weight 0. We call these fields trivial, and we shall be interested in manifolds \( M \) on which all holomorphic tensor fields are trivial (this implies \( M \) is connected). As said earlier, rationally connected complex projective manifolds are of this kind.

In the next theorem \( M \) can be a complex manifold locally biholomorphic to open sets in Banach spaces.

**Theorem 1.7.** Let \( M \) be a complex manifold and \( Y \subset C^r(V,M) \) a connected neighborhood of the space of constant maps.

(a) If on \( M \) all symmetric holomorphic tensor fields are trivial then holomorphic functions on \( Y \) are constant.

(b) If on \( M \) all holomorphic tensor fields are trivial then the same holds on \( Y \).

As already said, for complex projective manifolds having only trivial holomorphic tensor fields is conjecturally equivalent to rational connectivity. According to Theorems 1.3, 1.7, both properties are inherited by loop spaces, a fact we consider a mild additional evidence in favor of the conjecture.

## 2 Rational connectivity of loop spaces

In this section we shall prove Theorem 1.3. The key is the following.

**Lemma 2.1.** Let \( k \) be a field of characteristic zero and \( M \) a smooth, rationally connected projective variety over \( k \). Given distinct points \( p_1, \ldots, p_n \in \mathbb{P}^1 \) defined over \( k \), there are a smooth variety \( W \) and a morphism \( f: \mathbb{P}^1 \times W \to M \) over \( k \) such that the map

\[
\varphi = (f(p_\nu, \cdot))_{\nu=1}^n: W \to M^n
\]

is a surjective submersion on a dense open \( U \subset M^n \) and its fibers are irreducible.

In the proof we will have to extend the field \( k \). Since after a field extension our varieties might become reducible, and so cease to be varieties, we shall work in the larger category of schemes of finite type over a field. When dealing with a base field other than \( k \), we will indicate it in the subscript. Note that the notion of direct product
of varieties (or schemes) depends on the base-field. For a good introduction to the language of schemes, and a detailed explanation of the basic properties of families and their fibers we refer to chapters II/1–II/3 of [Hs]. Below we summarize what will be needed for the proof of Lemma 2.1.

**Definition 2.2.** All our schemes are assumed to be schemes of finite type without explicitly writing so.

**Definition 2.3 (Dominant families).** Let \( k \) be a field of characteristic 0 and \( N \) a variety over \( k \). A dominant family over \( N \) is a morphism \( A \to N \) from a \( k \)-scheme to \( N \) with dense image. The fiber of this family over a closed point \( p \in N \) is just the inverse image of \( p \in N \), a \( k \)-scheme. A generically defined morphism between two dominant families \( \alpha : A \to N \) and \( \beta : B \to N \) is an equivalence class of pairs \((U, \phi)\), where \( U \subseteq N \) is an open dense subset and \( \phi : \alpha^{-1}(U) \to B \) is a morphism such that \( \beta \circ \phi = \alpha \mid \alpha^{-1}(U) \) (i.e. \( \phi \) acts fiber-wise). Two pairs \((U, \phi)\) and \((V, \psi)\) are considered equivalent, if the restrictions of \( \phi \) and \( \psi \) to the subset \( \alpha^{-1}(U \cap V) \) are equal. Thus the dominant families over \( N \) form a category, whose morphisms are the generically defined morphisms.

Restricting a dominant family to a family over a dense open \( P \subset N \) defines an equivalence of categories. So, in the following discussion we shall assume that our \( N \) is an affine variety with coordinate ring \( \mathcal{N} \). Let \( K \) denote the field of rational functions on \( N \), the quotient field of \( \mathcal{N} \).

If \( A \subset N \times \mathbb{A}^n_k \) is a closed subscheme for some \( n \), where \( \mathbb{A}^n_k \) denotes the \( n \)-dimensional affine space over \( k \), and the restriction \( A \to N \) of the projection \( N \times \mathbb{A}^n_k \to N \) is dominant, we call the family \( A \to N \) affine. Then \( A \) is simply defined via the vanishing of certain \( n \)-variable polynomials whose coefficients are regular functions on \( N \), i.e. elements from \( \mathcal{N} \). Each dominant family is the union of finitely many affine dominant families. A standard method for dealing with families is first to study the situation for affine families, and then to glue them together.

Let \( B \subset N \times \mathbb{A}^m_k \) be another affine dominant family, and let \((U, \phi) : A \to B\) represent a generically defined morphism between them. Again, we may shrink \( U \) to an affine subset. Then \( \phi \) is simply given via \( m \) coordinates, each coordinate an \( n \)-variable polynomial whose coefficients are regular functions on \( U \), i.e., elements of \( K \).
A and B were defined via polynomial equations with coefficients in \( \mathcal{N} \). Now we think of them as polynomials with coefficients in \( K \), and define the generic fibers \( A_{\text{gen}} \subset \mathbb{A}^n_K \) and \( B_{\text{gen}} \subset \mathbb{A}^m_K \) via the same equations. These are affine \( K \)-schemes. Moreover, the \( m \) polynomials we used to define \((U, \phi)\) also have coefficients in \( K \). Hence they give us a morphism \( \phi_{\text{gen}} : A_{\text{gen}} \to B_{\text{gen}} \), and we call this the generic fiber of \((U, \phi)\). It is easy to check that taking generic fibers is a functor from the category of affine dominant families defined over \( \mathcal{N} \) into the category of affine \( K \)-schemes (in particular, it does not depend on the chosen affine embeddings).

There is also a functor in the opposite direction obtained as follows: Consider an arbitrary affine \( K \)-scheme defined via finitely many polynomial equations with coefficients in \( K \). Let \( D \in \mathcal{N} \) be a common denominator for these coefficients and let \( N_D \subset \mathcal{N} \) denote the complement of the zero set of \( D \). Then the coefficients are regular functions on \( N_D \), hence our \( K \)-scheme is isomorphic to the generic fiber of the affine dominant family over \( N_D \) defined via the same equations. A similar argument proves that each morphism of affine \( K \)-schemes is just the generic fiber of a generically defined morphism of affine dominant families. Hence taking generic fiber is an equivalence of categories.

The construction of generic fibers in affine dominant families being canonical, one can easily extend it to arbitrary dominant families, by simply gluing together the affine pieces. Before we state the next theorem, we recall a definition:

**Definition 2.4.** Let \( X \) be any \( k \)-scheme, and \( K \supset k \) a field extension. Then \( X \) is defined via gluing together certain affine subsets, the affine pieces are given via polynomial equations with coefficients in \( k \). If we consider the same equations over \( K \), we get affine \( K \)-schemes. These affine pieces glue together the same way as the original affine \( k \)-schemes to produce a \( K \)-scheme \( X_K \).

1. \( X_K \) will denote the \( K \)-scheme we obtain.

2. If \( p \in Y \subset X \) is a \( k \)-point and a \( k \)-subscheme, then the above procedure gives us a \( K \)-point and a \( K \)-subscheme \( p_K \in Y_K \subset X_K \).

3. We say that \( X \) is geometrically irreducible if \( X_K \) is irreducible for all field extensions \( K \supset k \).

We note that an open dense subset of a scheme \( X \) is geometrically irreducible precisely when \( X \) is.
Theorem 2.5 (Families and their fibers). Let $k$ be a field of characteristic zero, $N$ a variety over $k$, and $K$ the field of rational functions on $N$.

1. Each dominant family $A \rightarrow N$ has a generic fiber $A_{gen}$, which is a $K$-scheme. Each generically defined morphism between families $A \rightarrow N$ and $B \rightarrow N$ has a generic fiber $A_{gen} \rightarrow B_{gen}$.

2. Taking generic fiber is a functor from the category of dominant families of varieties over $N$ to the category of $K$-schemes. It is an equivalence of categories.

3. For a $k$-scheme $X$ the projection $X \times N \rightarrow N$ is called the trivial family with fiber $X$. It's generic fiber is $X_K$. Under the above equivalence of categories the subschemes $Y_{gen} \subset X_K$ correspond to families of subschemes of $X$ parameterized by $N$, i.e. subschemes $Y \subset X \times N$ which dominate $N$; and $K$-points $q_{gen} \in X_K$ correspond to families of points of $X$ parameterized by $N$, i.e. subvarieties $q \subset X \times N$ which project birationally to $N$. Furthermore, $q_{gen} \in Y_{gen}$ if and only if after restricting to some dense Zariski open subset $U \subset N$ the second family contains the first: $q|_U \subset Y|_U$. For a $k$-point and a $k$-subscheme $q' \in Y' \subset X$ the constant families $q = q' \times N$, $Y = Y' \times N$ correspond to the $K$-point $q_{gen} = q'_K \in X_K$ and $K$-scheme $Y_{gen} = Y_K \subset X_K$.

4. If $X \rightarrow N$ and $Y \rightarrow N$ are dominant families, then the fiber product $X \times_N Y \rightarrow N$ is again a dominant family. Moreover, the corresponding $K$-scheme is $(X \times_N Y)_{gen} \simeq X_{gen} \times_K Y_{gen}$, where $\times_K$ stands for the direct product of $K$-schemes.

5. If $f: M \rightarrow N$ is a dominant morphism of $k$-varieties with function fields $L \supseteq K$ and $X \rightarrow N$ is a dominant family, then the pullback family is the fiber product $X \times_M M \rightarrow M$, a dominant family over $M$ whose generic fiber is isomorphic to the $L$-scheme $(X_{gen})_L$. Moreover, the fiber of the pullback family at any $k$-point $p \in M$ is naturally isomorphic to the fiber of the original family at $f(p)$.

6. If the generic fiber of a dominant family is geometrically irreducible then almost all fibers are irreducible, hence connected in the Zariski topology.
Sketch of proof. We already discussed the first two statements; \( \textbf{3} \) and \( \textbf{4} \) follow easily from \( \textbf{1} \) \( \textbf{2} \) and the definitions. For a detailed explanation of the basic properties of families and their fibers we refer to chapters II/1–II/3 of \( \textbf{Hs} \). We give a short proof for \( \textbf{6} \) here. Supposing that \( \alpha : A \to N \) is a dominant family over \( k \) such that \( A_{\text{gen}} \) is geometrically irreducible, we shall show that its fibers are geometrically irreducible over a dense Zariski open subset of \( N \). Our \( \alpha \) is defined over a finitely generated subfield of \( k \), so we may assume that \( k \) is finitely generated, hence a subfield of \( \mathbb{C} \), the field of complex numbers. Then we may extend \( k \) to \( \mathbb{C} \) (reducible schemes stay reducible after field extensions), so from now on \( k = \mathbb{C} \). We may replace \( A \) with its reduced scheme structure. Since the regular part of \( A \) intersects almost all fibers in a dense Zariski open subset, we may assume that \( A \) is a smooth variety.

Next we pick an irreducible subvariety \( S \subset A \) of dimension \( \dim N \) such that the restricted map \( \alpha|_S : S \to N \) is dominant (if \( N \) is at least one-dimensional then we may simply intersect \( A \) with \( \dim A - \dim N \) hyperplanes in general position, if \( N \) is a point then take any closed point for \( S \)). We shall think of \( S \) as a multiple valued section of \( A \to N \). By the algebraic version of Sard's lemma (III/10.7 in \( \textbf{Hs} \)) we can find a dense, Zariski open subset \( U \subset N \) such that the restrictions \( \alpha^{-1}(U) \to U \) and \( (\alpha|_S)^{-1}(U) \to N \) are submersions, and further shrinking \( U \) we can achieve that the second map is finite, hence a covering space in the topological sense. For simplicity we replace \( N \) with \( U \), so from now on \( \alpha \) is a submersion, and the restriction \( S \to N \) is a finite covering space. We may do this since \( U \) is dense, hence we kept almost all fibers of the original family.

Let \( \alpha_S : A_S \to S \) denote the pullback of the family \( \alpha \) via \( S \to N \). The virtue of this family is that it has a section \( s : S \to A_S \) defined as \( s(p) = (p, p) \in A \times_N S \). In each fiber of \( \alpha_S \) we consider the connected component which intersects this section, and let \( A^*_S \subset A \) be their union. It is easy to see, that \( A^*_S \) is a connected component of \( A_S \) (in the classical topology, hence also in the Zariski topology). On the other hand, each component of \( A_S \) surjects onto \( A \), hence surjects onto \( N \), and also on \( S \) (since proper subvarieties of \( S \) may not surject onto \( N \)). Hence the irreducible components of \( A_S \) are in one-to-one correspondence with the irreducible components of its generic fiber. But this generic fiber can be obtained from \( A_{\text{gen}} \) by extending the base-field \( K \) to the function field of \( S \) (see \( \textbf{5} \) of this theorem), so it is irreducible by assumption. Hence \( A_S \) must also be irreducible.
Therefore $A_S = A^*_S$, and all fibers of $\alpha_S$ are irreducible. But the fibers of $\alpha_S$ are the same as the fibers of $\alpha$ (just they appear many times), we proved our claim.

**Proof of Lemma 2.1.** At the price of replacing $M$ by $\mathbb{P}^3 \times M$ it can be assumed that $\dim(M) \geq 3$. Our plan is to study the family of all $n$-tuples of points on $M$. This family is parameterized by $M^n$, and it is given by $n$ families of points of $M$: 

$$
\begin{array}{cccc}
q_\nu & \leftrightarrow & M \times M^n \\
\gamma_\nu & & \downarrow & (\nu = 1, 2, \ldots n) \\
M^n & \rightleftharpoons & M^n
\end{array}
$$

where the left hand side is 

$$q_\nu = \{(m; m_1, m_2, \ldots, m_n) \in M \times M^n : m = m_\nu\} \simeq M^n$$

and 

$$\gamma_\nu(m_\nu; m_1, m_2, \ldots, m_n) = (m_1, m_2, \ldots, m_n).$$

Taking generic fibers we get an $n$-tuple of $K$-points 

$$q_{1,\text{gen}}, q_{2,\text{gen}}, \ldots, q_{n,\text{gen}} \in M_K.$$

Our next goal is to find a smooth, geometrically irreducible $K$-variety $V$ together with a family $g : \mathbb{P}^1_K \times_K V \to M_K$ of smooth rational curves, each passing through all of the points $q_{1,\text{gen}}, q_{2,\text{gen}}, \ldots, q_{n,\text{gen}}$. Since $M$ is rationally connected, so is $M_K$, cf. Definition 1.2. By Theorem 16 of [KS], over $K$ there is a family of smooth rational curves in $M_K$, parameterized by a smooth, geometrically irreducible $K$-variety $W$, such that each curve passes through our $n$ points. But this theorem gives the rational curves as subvarieties of $M_K$, so our $V$ will be a $\text{PGL}_K(2)$-bundle over $W$ to account for all parameterizations. (Note here that although we found an entire family of curves, $V$ might not have any point defined over $K$, so we cannot easily get a single rational curve defined over $K$.)

By the smoothness assumption each rational curve $g(\cdot, v)$ passes through all $q_{\nu,\text{gen}}$ exactly once, whence there are $K$-morphisms 

$$\sigma_\nu : V \to \mathbb{P}^1_K, \quad \nu = 1, 2, \ldots n$$

such that $g(\sigma_\nu(v), v) = q_{\nu,\text{gen}}$ for all $v \in V$. Indeed, for each $\nu$ the projection 

$$S = \left\{(s, v) \in \mathbb{P}^1_K \times_K V : g(s, v) = q_{\nu,\text{gen}}\right\} \to V$$
is a bijective morphism between smooth varieties. Its inverse is rational by Sard’s lemma, [Hs, III/10.7], hence regular by Zariski’s Main Theorem, [Hs, V/5.2]. Composing this inverse with the projection $S \to \mathbb{P}^1_K$ we obtain $\sigma_\nu$. By shrinking $V$ to a dense open subset we may achieve that $\sigma_\nu(v)$ is never $\infty \in \mathbb{P}^1_K$.

Next we can think of the given points $p_\nu \in \mathbb{P}^1_K$ as points $p_{\nu,K} \in \mathbb{P}^1_K$ with the same homogeneous coordinates (see Definition 2.4). We can compose $g(\cdot, v)$ with appropriate interpolating polynomials—or rational functions if some $p_\nu = \infty$—to replace $\sigma_\nu(v)$ with $p_\nu$ and arrive at the situation

\begin{equation}
(2.2) \quad g(p_{\nu,K}, v) = q_{\nu, \text{gen}}, \quad v \in V, \; \nu = 1, 2, \ldots n.
\end{equation}

The rational curves $g(\cdot, v)$ may no longer be smooth, but this will not matter.

The $K$-variety $V$ is the generic fiber of a dominant family $\phi : V^* \to M^n$ whose fibers are reduced (since $V$ is reduced). By Theorem 2.5 we may assume that each fiber of $\phi$ is irreducible (we simply shrink $V^*$ to a dense open subset). Upon further shrinking, by Sard’s lemma (III/10.7 in [Hs]) we may also assume that $\phi$ is a submersion. We use again Theorem 2.5 and see that $\mathbb{P}^1_K \times_K V$ is the generic fiber of the dominant family $(\mathbb{P}^1 \times M^n) \times_{M^n} V^* \simeq \mathbb{P}^1 \times V^* \to M^n$ and $g$ is the generic fiber of a generically defined morphism of families:

\[
\begin{array}{ccc}
\mathbb{P}^1 \times V^* & \xrightarrow{g^*} & M \times M^n \\
\downarrow \phi \circ \text{pr}_{V^*} & & \downarrow \\
M^n & \xrightarrow{=} & M^n \\
\end{array}
\]

Since $g^*$ is a rational map into a projective variety, it extends (uniquely) to the complement of a codimension 2 subvariety of $\mathbb{P}^1 \times V^*$. Hence we may shrink $V^*$ further, and achieve that $g^*$ is an everywhere defined morphism (see the paragraph about fundamental points on page 50 of [M]). By Theorem 2.5, equation (2.2) implies that

\[g^*(p_\nu, v) = \gamma^{-1}_\nu(\phi(v)) = (\phi(v)_{\nu}; \phi(v)) \in M \times M^n,\]

where for $v \in V^*$, $\phi(v)_{\nu} \in M$ denotes the $\nu$-th coordinate of $\phi(v) \in M^n$, $\nu = 1, \ldots, n$. Now we set $W = V^*$, and $f = \text{pr}_M \circ g^* : \mathbb{P}^1 \times V^* \to M$, the first component of $g^*$. Then $f(p_\nu, v) = \phi(v)_{\nu}$, hence

\[
(f(p_\nu, \cdot))_{\nu=1}^n = \phi : W^* \to M^n
\]
is a submersion, and its fibers are irreducible. This proves the lemma.

Next we need a result from differential geometry. Let $V$ be a one dimensional compact manifold.

**Lemma 2.2.** Let $\varphi: W \to U$ be a surjective $C^\infty$ submersion between finite dimensional $C^\infty$ differential manifolds, whose fibers are connected. For any $r$ and $y \in C^r(V, U)$ there is such an $\eta \in C^r(V, W)$ that $\varphi \circ \eta = y$.

**Proof.** First observe that any compact subset $C$ of a fiber $\varphi^{-1}(u)$ has an open neighborhood $W_0 \subset W$ such that $\varphi|_{W_0}$ is a trivial fiber bundle with connected fibers. To verify this we can assume $U = \mathbb{R}^m$ and $u = 0$. A partition of unity argument gives a connection on $W$, i.e. a subbundle $H \subset TW$ complementary to the tangent spaces of the fibers of $\varphi$. Fix a relatively compact, connected open neighborhood $G \subset \varphi^{-1}(u)$ of $C$. Connect an arbitrary $v \in \mathbb{R}^m$ with $0 \in \mathbb{R}^m$ by a curve $\gamma$ consisting of $m$ segments, the $\mu$'th segment parallel to the $\mu$'th coordinate axis. If $v$ is in a sufficiently small neighborhood $U_0 \subset \mathbb{R}^m$ of $u$ and $c \in G$ then $\gamma$ can be uniquely lifted to a piecewise smooth curve $\Gamma$, tangent to $H$ and starting at $c$. Let $\psi(c, v)$ denote the endpoint of $\Gamma$. Then $\psi$ is a fiberwise diffeomorphism of $G \times U_0$ on an open neighborhood $W_0$ of $C$, as claimed.

It follows that there are closed arcs $A_1, \ldots, A_n$ covering $V$ and $C^r$ maps $\eta_\nu: \tilde{A}_\nu \to W$ such that $\varphi \circ \eta_\nu = y$. We show that there is a $C^r$ map $\tilde{\eta}: A_1 \cup A_2 \to W$ such that $\varphi \circ \tilde{\eta} = y$. Indeed, $A_1 \cap A_2$ is empty or consists of one or two components. In the first case $\tilde{\eta} = \eta_\nu$ on $A_\nu$, $\nu = 1, 2$, will do. Otherwise choose points $b_i$ from each component of $A_1 \cap A_2$; thus $A_1 \cup A_2 \setminus \{b_i\}$ is the disjoint union of two arcs $\tilde{A}_\nu \subset A_\nu$, $\nu = 1, 2$. Using the neighborhoods of $C = C_i = \{\eta_1(b_i), \eta_2(b_i)\}$ from our initial observation, it is straightforward to construct the required $\tilde{\eta}$; it will agree with $\eta_\nu$ on $\tilde{A}_\nu$, away from a small neighborhood of $b_i$.

Now one can continue in the same spirit, fusing more and more arcs, eventually to obtain the $\eta$ of the lemma.

**Proof of Theorem 1.3.** Fix distinct $p_1, \ldots, p_n \in \mathbb{P}^1(\mathbb{C})$ and apply Lemma 2.1, with $k = \mathbb{C}$. We obtain a holomorphic map $f: \mathbb{P}^1(\mathbb{C}) \times W \to M$ of complex manifolds so that 

$$
\varphi = (f(p_\nu, \cdot))_{\nu=1}^n: W \to M^n
$$

is a surjective submersion on a Zariski dense open $U \subset M^n$, with irreducible, hence connected fibers (see [M, 4.16 Corollary]). Since
the complement of $U$ is of real codimension 2 in $M^n$, $O = C^r(V, U)$ is dense in $C^r(V, M^n)$. Given $x_1, \ldots, x_n \in C^r(V, M)$ such that $y = (x_1, \ldots, x_n) \in O$, there is a holomorphic map $h: \mathbb{P}^1(\mathbb{C}) \to C^r(V, M)$ with $h(p_\nu) = x_\nu$, $\nu = 1, \ldots, n$. Indeed, using Lemma 2.2 one finds $\eta: V \to W$ such that $\varphi \circ \eta = y$. Setting
\begin{equation*}
F(p, t) = f(p, \eta(t)), \quad p \in \mathbb{P}^1(\mathbb{C}), \ t \in V,
\end{equation*}
the map $h$ given by $h(p) = F(p, \cdot)$ will do.

### 3 The Proof of Theorem 1.4

Fix $d \in \mathbb{N}$ and consider the space of pairs of complex polynomials
\begin{equation*}
\left\{ \left( \sum_{j=0}^{d} \alpha_j u_j^d, \sum_{j=0}^{d} \beta_j u_j^d \right) : (\alpha_d, \beta_d) \neq (0, 0) \right\},
\end{equation*}
a complex manifold on which the group $\mathbb{C}^*$ of nonzero complex numbers acts holomorphically and freely by coefficientwise multiplication. Denote the quotient manifold by $E$, and by $\pi: E \to \mathbb{P}^1(\mathbb{C})$ the projection
\begin{equation*}
\pi\left( \sum_{j=0}^{d} \alpha_j u_j^d : \sum_{j=0}^{d} \beta_j u_j^d \right) = (\alpha_d : \beta_d).
\end{equation*}
Thus $E$ is a locally trivial fiber bundle. Let $\Delta \subset E$ be the discriminant set, corresponding to pairs of polynomials with a common zero.

**Lemma 3.1.** If $V$ is a closed surface and $\psi: V \to E \setminus \Delta$ is continuous then $\pi \circ \psi: V \to \mathbb{P}^1(\mathbb{C})$ is homotopic to a constant.

**Proof.** First observe that the fiber maps
\begin{equation*}
H_s\left( \sum_{j=0}^{d} \alpha_j u_j^2 : \sum_{j=0}^{d} \beta_j u_j^2 \right) = \left( \alpha_d u^d + s \sum_{j=0}^{d-1} \alpha_j u^2 : \beta_d u^d + s \sum_{j=0}^{d-1} \beta_j u^2 \right),
\end{equation*}
$1 \geq s \geq 0$, deform $E$ on the image of a section. It follows that any section is a homotopy inverse of $\pi$, in particular the section
\begin{equation*}
\sigma(x_0: x_1) = (x_0 u^d - x_1: x_1 u^d - x_0), \quad (x_0: x_1) \in \mathbb{P}(\mathbb{C}).
\end{equation*}
Next let $L \to E$ denote the holomorphic line bundle determined by the hypersurface $\Delta$. Thus $L$ has a holomorphic section that vanishes precisely on $\Delta$; in particular $\psi^*L$ is trivial. One checks that the graph
of \( \sigma \) intersects \( \Delta \) in two points, whence the holomorphic line bundle \( \sigma^*L \to \mathbb{P}^1(\mathbb{C}) \) has a holomorphic section with two zeros. It follows that \( \sigma^*L \) is not (even topologically) trivial. On the other hand, since \( \sigma \circ \pi \simeq \text{id}_E \),

\[(\pi \circ \psi)^*\sigma^*L = (\sigma \circ \pi \circ \psi)^*L \approx \psi^*L,\]

is trivial. Comparing Chern classes we find that \( \pi \circ \psi \) must induce the zero map \( H^2(\mathbb{P}^1(\mathbb{C})) \to H^2(V) \), and so \( \pi \circ \psi \) is homotopic to a constant by Hopf’s theorem, see [S, Chapter 8, Section 1].

**Proof of Theorem 1.4.** Define \( g \in C^r(\mathbb{P}^1(\mathbb{C}) \times V, \mathbb{P}^1(\mathbb{C})) \) by

\[g(s,t) = h(s)(t), \quad s \in \mathbb{P}^1(\mathbb{C}), \ t \in V.\]

For each \( t \in V \) \( g(\cdot, t) \) is a holomorphic map \( \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \), whose degree \( d \) is independent of \( t \); we assume \( h \) is nonconstant so that \( d > 0 \).

A degree \( d \) self map of \( \mathbb{P}^1(\mathbb{C}) \) is a rational function

\[
\sum_{j=0}^{d} \frac{\alpha_j s^j}{\sum_{j=0}^{d} \beta_j s^j}, \quad s \in \mathbb{C} \subset \mathbb{P}^1(\mathbb{C}),
\]

with coprime numerator and denominator, and \((\alpha_d, \beta_d) \neq (0, 0)\). Since numerator and denominator are determined up to a common factor \( \lambda \in \mathbb{C}^* \), degree \( d \) maps correspond to points in \( E \setminus \Delta \), and \( h \) induces a \( C^r \) map \( \psi: V \to E \setminus \Delta \). By Lemma 3.1 \( h(\infty) = \pi \circ \psi \in C^r(V, \mathbb{P}^1(\mathbb{C})) \) is homotopically trivial, and therefore so are all maps \( h(s), s \in \mathbb{P}^1(\mathbb{C}), \) q.e.d.

### 4 Holomorphic Functions on the Manifold of Based Loops

In this section we shall consider a rationally connected complex projective manifold \( M \), the space \( C^r_{A,x_0}(V, M) = Z \) of based maps, \( A \subset V \), \( x_0 \in C^r(V, M) \), and we shall show that complex valued holomorphic functions on \( Z \) are locally constant, Theorem 1.5. We shall also derive Corollary 1.6, in a more general form.

**Lemma 4.1.** Given \( p \in M \) and \( v \in T_p M \), there are a neighborhood \( U \) of \( p \) and a holomorphic map \( \varphi: \mathbb{P}^1(\mathbb{C}) \times U \to M \) such that

\[\varphi(\infty, \cdot) = \text{id}_U \quad \text{and} \quad \varphi_* T_{(\infty,p)}(\mathbb{P}^1(\mathbb{C}) \times \{p\}) \ni v.\]
Proof. By [AK, Definition–Theorem 29] there is a holomorphic map \( \varphi_0 : \mathbb{P}^1(\mathbb{C}) \to M \) such that \( \varphi_0(\infty) = p \) and the induced bundle \( \varphi_0^*TM \) is ample. We shall obtain \( \varphi \) by deforming \( \varphi_0 \). Let \( \zeta \) denote the zero section of \( T\mathbb{P}^1(\mathbb{C}) \). By deformation theory, there are a pointed complex manifold \((W, o)\) and a holomorphic map \( \psi: \mathbb{P}^1(\mathbb{C}) \times W \to M \) such that \( \psi(\cdot, o) = \varphi_0 \) and

\[
(4.1) \quad T_o W \ni \omega \mapsto \psi_* (\zeta, \omega) \in \mathcal{O}(\varphi_0^*TM)
\]

is an isomorphism; here \((\zeta, \omega)\) is a section of \( T(\mathbb{P}^1(\mathbb{C}) \times W) \) over \( \mathbb{P}^1(\mathbb{C}) \times \{o\} \). This follows from [Kd], that studies deformations of submanifolds of a given manifold. To make the connection with deformations of maps \( \mathbb{P}^1(\mathbb{C}) \to M \) needed here, observe that deformations of \( \varphi_0 \) correspond to deformations of the graph of \( \varphi_0 \) as a submanifold of \( \mathbb{P}^1(\mathbb{C}) \times M \).

Since \( \varphi_0^*TM \) is spanned by global sections, (4.1) shows \( \psi(\infty, \cdot) \) is a submersion near \( o \in W \). Shrinking \( W \) we can therefore arrange that \( \psi(\infty, \cdot) \) is everywhere submersive, and so

\[
W_\infty = \{ w \in W : \psi(\infty, w) = p \}
\]

is a submanifold. Under the isomorphism (4.1) \( T_0 W_\infty \subset T_0 W \) corresponds to

\[
\mathcal{O}_\infty(\varphi_0^*TM) = \{ \sigma \in \mathcal{O}(\varphi_0^*TM) : \sigma(\infty) = 0 \}.
\]

We let \( s \) denote the standard complex coordinate on \( \mathbb{C} \subset \mathbb{P}^1(\mathbb{C}) \), and for \( \sigma \in \mathcal{O}_\infty(\varphi_0^*TM) \) define

\[
\sigma'(\infty) = \lim_{s \to \infty} s \sigma(s) \in T_pM.
\]

As \( \varphi_0^*TM \) is ample, the map \( \sigma \mapsto \sigma'(\infty) \) is onto.

Note that the vector field \( s^2 \partial / \partial s \) extends to all of \( \mathbb{P}^1(\mathbb{C}) \). With \( \varphi_w = \psi(\cdot, w) \) consider the map

\[
(4.2) \quad W_\infty \ni w \mapsto \varphi_w *(s^2 \partial / \partial s|_{s=\infty}) \in T_pM.
\]

We shall assume that (4.2) maps \( o \in W_\infty \) to 0, which can always be arranged upon replacing \( \varphi_0(s) \) by \( \varphi_0(s^2) \). One then computes, say in local coordinates, that the differential of the map (4.2) is the composition of (4.1) with the map

\[
\mathcal{O}_\infty(\varphi_0^*TM) \ni \sigma \mapsto \sigma'(\infty) \in T_pM \approx T_0(T_pM).
\]
It follows that (4.2) is a submersion near \( o \); hence \( v \in \varphi_w^* T_\infty \mathbb{P}^1(\mathbb{C}) \) for some \( w \in W_\infty \). In some neighborhood \( U \subset M \) of \( p \) the submersion \( \psi(\infty, \cdot) \) has a holomorphic right inverse \( \rho: U \to W \) with \( \rho(p) = w \); then

\[
\varphi(s, u) = \psi(s, \rho(u)), \quad s \in \mathbb{P}^1(\mathbb{C}), \; u \in U
\]
defines the map sought.

**Proof of Theorem 1.5.** Let \( f: Z \to \mathbb{C} \) be holomorphic; we have to prove \( df(\xi) = 0 \) for all \( x \in Z \) and \( \xi \in T_x Z \). Fix \( x \). Given \( \tau \in V \) and nonzero \( v \in T_{x(\tau)} M \), construct \( U \) and \( \varphi \) as in Lemma 4.1, and with a sufficiently small neighborhood \( B \subset V \) of \( \tau \) define a \( C^r \) map

\[
\Phi: \mathbb{P}^1(\mathbb{C}) \times B \ni (s, t) \mapsto \varphi(s, x(t)) \in M,
\]
holomorphic in \( s \). Note that

\[
\Phi(\infty, t) = x(t), \quad t \in B.
\]
We take \( B \) compact and (when \( r \geq 1 \)) a \( C^r \) manifold with boundary. We also arrange that \( \Phi^t = \Phi(\cdot, t) \) is an immersion near \( \infty \), when \( t \in B \).

First suppose that \( \xi \in T_x Z \approx C^r_A(x^* TM) \) is supported in the (relative) interior of \( B \), and

\[
(4.3) \quad \xi(t) \in \Phi^t_* T_\infty \mathbb{P}^1(\mathbb{C}), \quad \text{for all } t \in B.
\]
To show that \( df(\xi) = 0 \), consider the map \( \nu: C^r_{(A \cap B) \cup \partial B, \infty}(B; \mathbb{P}^1(\mathbb{C})) \to Z \),

\[
\nu(y)(t) = \begin{cases} 
\Phi(y(t), t), & \text{if } t \in B \\
x(t), & \text{if } t \in V \setminus B
\end{cases}
\]
By [L2, Propositions 2.3, 3.1] \( \nu \) is holomorphic, and so is

\[
f \circ \nu: C^r_{(A \cap B) \cup \partial B, \infty}(B; \mathbb{P}^1(\mathbb{C})) \to \mathbb{C}.
\]
Therefore \( f \circ \nu \) is locally constant by [L2, Theorem 4.2]; for the case \( r = 0 \), see the earlier [DM]. Now \( \xi \) is in the range of \( \nu^* \); indeed, \( \xi = \nu^* \eta \), if \( \eta(t) \in T_\infty \mathbb{P}^1(\mathbb{C}) \) is defined by \( \eta(t) = 0 \) when \( t \in V \setminus B \) and \( \Phi^t_* \eta(t) = \xi(t) \) when \( t \in B \), cf. (4.3). It follows that

\[
df(\xi) = df(\nu)(\eta) = 0.
\]

Next choose a basis \( v = v_1, \ldots, v_m \) of \( T_{x(\tau)} M \) and construct corresponding maps

\[
\Phi_1 = \Phi, \Phi_2, \ldots, \Phi_m: \mathbb{P}^1(\mathbb{C}) \times B \to M.
\]
If $B$ is sufficiently small then

$$T_{x(t)}M = \bigoplus_j \Phi_{j,t}^* T_\infty \mathbb{P}^1(\mathbb{C}) \quad t \in B.$$ 

For each $j$, if $\xi_j \in T_x Z \approx C_A^\omega(x^*TM)$ has support in int $B$ and satisfies (4.3), with $j$ appended, then $df(\xi_j) = 0$. Since any $\xi \in C_A^\omega(x^*TM)$ supported in int $B$ is the sum of such $\xi_j$'s, we conclude each $\tau \in V$ has a neighborhood $B$ so that $df(\xi) = 0$ when supp $\xi \subset$ int $B$. But then a partition of unity gives $df(\xi) = 0$ for all $\xi \in T_x Z$, as needed.

We shall apply Theorem 1.5 to study holomorphic connections in the following setting. Let $\pi: E \to N$ be a holomorphic map of complex manifolds locally biholomorphic to open subsets of Banach spaces. Assume $\pi$ is a submersion, i.e. $\pi_*(e): T_e E \to T_{\pi(e)} N$ is surjective for all $e \in E$. A holomorphic connection on $E$ (or on $\pi$) is a holomorphic subbundle $D \subset TE$ such that $D_e$ is complementary to Ker $\pi_*(e), e \in E$. The connection is complete if curves in $N$ can be lifted to horizontal curves in $E$, i.e., for any $x \in C^1([0,1],N)$ and $e \in \pi^{-1}(x(0))$ there is a $y \in C^1([0,1],E)$ such that $y(0) = e, \pi \circ y = x$, and $y'(t) \in D_{y(t)}$ for all $0 \leq t \leq 1$. The lift is unique by the uniqueness theorem for ODE’s. For example, linear connections on Banach bundles and $G$–invariant connections on principal $G$ bundles—$G$ a Banach–Lie group—are complete.

The simplest example of a connection is on a trivial bundle $\pi: E = F \times N \to N$, with $D_{(f,n)} = T_{(f,n)}(\{f\} \times N)$. Connections isomorphic to such a connection are called trivial. Corollary 1.6 follows from

**Theorem 4.2.** Let $M$ be a rationally connected smooth complex projective manifold, $E$ a complex manifold locally biholomorphic to open subsets of Banach spaces, and $\pi: E \to M$ a holomorphic submersion such that on each fiber holomorphic functions separate points. If $\pi$ admits a complete holomorphic connection $D$ then the connection is trivial.

**Proof.** The mapping space $C^1([0,1], E)$ has a natural structure of a complex manifold—the construction in [L2, Section 2] carries over to Banach manifolds. Horizontal lift defines a map $\Lambda$ of the manifold

$$\{(e,x) \in E \times C^1([0,1], M): \pi(e) = x(0)\}$$

into $C^1([0,1], E)$. This map is holomorphic. To see this, note that for $(e,x)$ in a small neighborhood of a fixed $(e_0,x_0)$, and for small
Finding $y = \Lambda(e,x)$ over the interval $[0, \tau]$ amounts to solving an ODE. Doing this by the standard iterative scheme of Picard–Lindelöf (see [Hm, p. 8]) shows the local lift $y|[0, \tau] \in C^1([0, \tau], E)$ depends holomorphically on $(e,x)$. Since the full lift $y$ is obtained by concatenating local lifts, $\Lambda$ is indeed holomorphic. It is also equivariant with respect to reparametrizations: if $\sigma: [0, 1] \to [0, 1]$ is a $C^1$ map, $\sigma(0) = 0$, then

\begin{equation}
\Lambda(e,x) \circ \sigma = \Lambda(e,x \circ \sigma), \quad x \in C^1([0, 1], M).
\end{equation}

With fixed $p \in M$ and variable $q \in M$ consider

\begin{equation*}
Y = \{x \in C^1([0, 1], M): x(0) = p\}, \quad Y_q = \{x \in Y: x(1) = q\}, \quad \text{and}
\end{equation*}

\begin{equation*}
Z_q = \{x \in Y_q: x'(0) \in T_pM \text{ and } x'(1) \in T_qM \text{ are both zero}\},
\end{equation*}

connected manifolds since $M$ is simply connected by [C], or by our Subsection 1.3. Therefore Theorem 1.5 implies that $\mathbb{C}$–valued holomorphic functions on $Z_q$ are constant. In particular, for any $e \in \pi^{-1}(p)$ and holomorphic function $h: \pi^{-1}(q) \to \mathbb{C}$, $h(\Lambda(e,x)(1))$ is independent of $x \in Z_q$. Since holomorphic functions separate points of $\pi^{-1}(q)$, $\Lambda(e,x)(1)$ itself is independent of $x \in Z_q$. It follows from (4.4) that $\Lambda(e,x)(1)$ is even independent of $x \in Y_q$ (take e.g. $\sigma(t) = 3t^2 - 2t^3$, then $x \circ \sigma \in Z_q$), and so there is a holomorphic map $\Psi: \pi^{-1}(p) \times M \to E$ such that

\begin{equation}
\Lambda(e,x)(1) = \Psi(e,x(1)).
\end{equation}

One checks that $\Psi$ is biholomorphic and maps $\pi^{-1}(p) \times \{q\}$ to $\pi^{-1}(q)$, $q \in M$.

To conclude, note that with $\tau \in [0, 1]$ and $\sigma(t) = \tau t$ (4.4), (4.5) imply

\begin{equation*}
\Lambda(e,x)(\tau) = \Psi(e,x(\tau)),
\end{equation*}

i.e. $\Psi$ maps curves $(e, x)$ to horizontal curves in $E$. It follows that the induced connection $\Psi^{-1}_*D$ on the bundle $\pi^{-1}(p) \times M \to M$ is trivial, hence so is $D$.

## 5 Holomorphic Tensor Fields

To prove Theorem 1.7 we first discuss the notion of order of vanishing. Let $Y$ be a complex manifold, locally biholomorphic to open sets in
Banach or even Fréchet spaces, \( y \in Y \), and \( f : Y \to \mathbb{C} \) holomorphic. We say that \( f \) vanishes at \( y \) to order \( n \) if for arbitrary \( 0 \leq k < n \) and vector fields \( v_1, \ldots, v_k \) on \( Y \), holomorphic near \( y \)

\[
(v_1 v_2 \ldots v_k f)(y) = 0.
\]

If \( Y \) is connected and \( f \) vanishes at \( y \) to all orders then \( f \equiv 0 \).

Suppose \( f \) vanishes at \( y \) to order \( n \). To see if it vanishes to order \( n + 1 \), one is led to consider holomorphic vector fields \( v_1, \ldots, v_n \) in a neighborhood of \( y \) and

\[
(v_1 v_2 \ldots v_n f)(y).
\]

Observe first that (5.1) is independent of the order in which the vector fields are applied (since e.g.

\[
v_2 v_1 v_3 \ldots v_n f = v_1 v_2 \ldots v_n f - [v_1, v_2] v_3 \ldots v_n f = v_1 v_2 \ldots v_n f
\]

at \( y \); next that (5.1) vanishes if some \( v_i \) vanishes at \( y \) (since this is clearly so if \( v_1(y) = 0 \)). It follows that (5.1) depends only on the values that the \( v_i \) take at \( y \), and so (5.1) induces a symmetric \( n \)-linear map

\[
d^n f : T^n_Y = T_y Y \oplus \ldots \oplus T_y T \to \mathbb{C}.
\]

**Proof of Theorem 1.7(a).** Constant maps \( V \to M \) form a submanifold of \( Y \), biholomorphic to \( M \); we shall simply denote this manifold by \( M \subset Y \). If \( f : Y \to \mathbb{C} \) is holomorphic then by assumption \( f|M \) is constant. At the price of subtracting this constant from \( f \) we can assume \( f \) vanishes at each point of \( M \) to first order. We shall prove by induction it vanishes at each \( p \in M \) to arbitrary order.

Suppose \( f \) is already known to vanish to order \( n \geq 1 \) at each \( p \in M \), so that the differentials \( d^n f(p) \) are defined on \( T^n p Y \). We want to show \( d^n f(p) = 0 \), i.e.,

\[
d^n f(p)(\eta_1, \ldots, \eta_n) = 0, \quad \eta_i \in T_p Y, \quad p \in M.
\]

Note that by Subsection 1.1 \( T_p Y \) is naturally isomorphic to \( C^r(V, T_p M) \). With fixed \( \varphi_1, \ldots, \varphi_n \in C^r(V, \mathbb{C}) \) define a homomorphism \( \Phi_n : T^n M \to T^n Y|M \) of holomorphic vector bundles

\[
\Phi_n : T^n M \ni (\xi_1, \ldots, \xi_n) \mapsto (\varphi_1 \xi_1, \ldots, \varphi_n \xi_n) \in T^n Y|M.
\]

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the pullback of $d^n f$ by $\Phi^n$ is a symmetric holomorphic tensor field on $M$, of weight $n \geq 1$, hence vanishes. Therefore (5.2) holds when each $\eta_i$ is of form $\varphi_i \xi_i$, and also when each $\eta_i$ is a linear combination of such tangent vectors. When $\dim T_p M < \infty$, linear combinations

$$\sum_{j=1}^k \varphi^{(j)}(\xi^{(j)}), \quad k \in \mathbb{N}, \quad \varphi^{(j)} \in C^r(V, \mathbb{C}), \quad \xi^{(j)} \in T_p M,$$

constitute all of $C^r(V, T_p M)$, and in general a dense subspace; whence indeed $d^n f(p) = 0$, $p \in M$. This means $f$ vanishes to order $n + 1$ along $M$, hence to all orders, and therefore $f = 0$ on $Y$.

For the rest of Theorem 1.7 we first extend the notions of vanishing order and higher differentials to tensor fields. Let now $f$ be a holomorphic tensor field of weight $j$ on the manifold $Y$. We say that $f$ vanishes at $y \in Y$ to order $n \geq 0$ if for all $0 \leq k < n$ and holomorphic vector fields $v_1, \ldots, v_k, w_1, \ldots, w_j$, defined near $y$

$$v_1 \ldots v_k f(w_1, \ldots, w_j) = 0 \quad \text{at } y.$$

Note that vanishing to order 0 is automatic. Suppose $f$ does vanish to order $n$. As before,

(5.4) $$\left( v_1 \ldots v_n f(w_1, \ldots, w_j) \right)(y)$$

is symmetric in the $v_l$, and for fixed $w_i$, depends only on the values $v_l(y)$.

**Proposition 5.1.** If some $w_i$ vanishes at $y$ then (5.4) vanishes.

**Proof.** First observe that if $F$ is a Fréchet space, $h$ an $F$ valued holomorphic function defined in a neighborhood of 0 in some $\mathbb{C}^q$, and $h(0) = 0$, then there are holomorphic functions $h_1, \ldots, h_q$ such that

$$h(z_1, \ldots, z_q) = \sum_{s=1}^q z_s h_s(z_1, \ldots, z_q)$$

in a neighborhood of 0. Indeed,

$$h(z) = \int_0^1 \frac{d}{d\lambda} h(\lambda z) d\lambda = \sum_s z_s \int_0^1 \frac{\partial h}{\partial z_s}(\lambda z) d\lambda.$$

Now suppose, for concreteness, that $w_1(y) = 0$. Since as far as the $v_l$ are concerned, (5.4) depends only on $v_l(y)$, we can assume that all $v_l$
are tangent to a finite, say $q$, dimensional submanifold $Q \subset Y$ passing through $y$. After a local trivialization of $TY$ the above observation gives holomorphic functions (local coordinates) $\zeta_1, \ldots, \zeta_q$ on $Q$ and holomorphic sections $h_1, \ldots, h_q$ of $TY\mid Q$ near $y$, such that

$$\zeta_1(y) = \cdots = \zeta_q(y) = 0 \quad \text{and} \quad w_1\mid Q = \sum_s \zeta_s h_s.$$ 

Then Leibniz’s rule implies

$$v_1 \cdots v_n f(w_1, \ldots, w_j) = \sum_s v_1 \cdots v_n \{ \zeta_s f(h_s, w_2, \ldots, w_\nu) \} = 0$$

at $y$.

It follows that (5.4) depends only on ($f$ and) the values that $v_1, \ldots, w_j$ take at $y$; therefore (5.4) induces a multilinear map

$$d^n f(y): T_{y}^{n+j}Y \to \mathbb{C}.$$ 

Proof of Theorem 1.7(b). Assume now $f$ is a holomorphic tensor field of weight $j \geq 1$ on $Y$. Suppose we know $f$ vanishes at all $p \in M$ to order $n \geq 0$. As before, a choice of $\varphi_1, \ldots, \varphi_{n+j} \in C^\infty(V, \mathbb{C})$ defines a homomorphism $\Phi_{n+j}: T^{n+j}M \to T^{n+j}Y\mid M$, cf. (5.3). The pullback of $d^n f$ by $\Phi_{n+j}$ is a holomorphic tensor field on $M$, hence 0; from which it follows, as earlier, that $d^n f(p) = 0$, $p \in M$. Thus $f$ vanishes to order $n + 1$ along $M$, so to all orders. This implies $f = 0$ on $Y$ as claimed.

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