Since January 2020 Elsevier has created a COVID-19 resource centre with free information in English and Mandarin on the novel coronavirus COVID-19. The COVID-19 resource centre is hosted on Elsevier Connect, the company's public news and information website.

Elsevier hereby grants permission to make all its COVID-19-related research that is available on the COVID-19 resource centre - including this research content - immediately available in PubMed Central and other publicly funded repositories, such as the WHO COVID database with rights for unrestricted research re-use and analyses in any form or by any means with acknowledgement of the original source. These permissions are granted for free by Elsevier for as long as the COVID-19 resource centre remains active.
Stochastic COVID-19 SEIQ epidemic model with time-delay

Amir Khan a,b, Rukhsar Ikram c, Anwarud Din d, Usa Wannasingha Humphries a,e, Ali Akgul e

a Department of Mathematics, Faculty of Science, King Mongkuts University of Technology, Thonburi (KMUTT), 126 Pracha Uthai Road, Bang Mod, Thung Khr, Bangkok 10140, Thailand
b Department of Mathematics and Statistics, University of Swat, kpk, Pakistan
c Department of Mathematics, Qurtuba University of Science and Information Technology, Hayatabad Peshawar, Pakistan
d Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275, PR China
e Sırt University, Art and Science Faculty of Science, Department of Mathematics, TR-56100 Sırt, Turkey

A R T I C L E   I N F O

Keywords:
Stochastic model
Time delay
Brownian motion
Stochastic stability
Numerical simulations

A B S T R A C T

In this work, we consider an epidemic model for corona-virus (COVID-19) with random perturbations as well as time delay, composed of four different classes of susceptible population, the exposed population, the infectious population and the quarantine population. We investigate the proposed problem for the derivation of at least one and unique solution in the positive feasible region of non-local solution. For one stationary ergodic distribution, the necessary result of existence is developed by applying the Lyapunov function in the sense of delay-stochastic approach and the condition for the extinction of the disease is also established. Our obtained results show that the effect of Brownian motion and noise terms on the transmission of the epidemic is very high. If the noise is large the infection may decrease or vanish. For validation of our obtained scheme, the results for all the classes of the problem have been numerically simulated.

Introduction

Epidemiology deals with various epidemic models or problems for investigation of different outbreaks using the available data from medical sciences. The importance of this area may be seen by its interest flourishing from day to day. Therefore, many mathematical models were established in the past, like SI, SIR, H1N1, HBV, SIS model, SARS, SIER model, H5N1 etc. as may be seen in [1,2]. These all problems were formulated mathematically to provide some realistic predictions and the society gains information about the diseases which is helpful for stable society and stable health [3–6]. The needful and necessary issues are the stability and preventing of various diseases in the societies of human population. Due to this each and every biological infection or disease is converted to mathematical model as soon as possible and the field of mathematical epidemiology were established for such formulation. After The first attempt of Mckendrick and Kermack [7, 8] the said models were highly analyzed for controlling of different diseases. Using this gate way and basic concepts, different researchers analyzed the epidemic models of SEIS, SEIRS, SIRS, vaccinated models and delay-models by including different parameter for vaccination and delay-time [9–18].

As in the case of covid-19, spreading of disease have a direct relation with the quarantine of human population. Commonly, we have two types of quarantine, one is susceptible quarantine and second is infected quarantine. In our work we take the infected quarantine which means that those people will be quarantined if they are infected. Using this idea Chen et al. [19] constructed the epidemic model of covid-19 as follows

\[
\begin{align*}
\frac{dS}{dt} &= A - \beta S(t)I(t) - \mu S(t) + \epsilon I(t), \\
\frac{dE}{dt} &= \beta S(t)I(t) - (\mu + \sigma_1 + \gamma_1)E(t), \\
\frac{dI}{dt} &= \epsilon E(t) - (\mu + a + c + \sigma_2 + \gamma_2)I(t), \\
\frac{dQ}{dt} &= \sigma_1 E(t) + \sigma_2 I(t) - (\mu + a + \gamma_3)Q.
\end{align*}
\]

(1)

here $S(t)$ is healthy class, $E(t)$ is expose to infection class, $I(t)$ is infectious class and $Q(t)$ is the quarantine class at the any time ($t$) respectively. The explanation of all the parameters used in the proposed model are given in Table 1.

In the mathematical modeling of a biological phenomenon the stochastic differential equation models are more suitable than the deterministic one, because it can provide an additional degree of realism in comparison to their deterministic counterparts. Stochastic models produce more valuable output as compared to the deterministic ones because running a stochastic model several times, we can build up a distribution of the predicted outcomes, e.g., the number of infected classes at time $t$. On the other hand, a deterministic model will just give a single predicted value [20–22].

* Corresponding author.
E-mail address: usa.wan@kmutt.ac.th (U.W. Humphries).

https://doi.org/10.1016/j.rinp.2021.104775

Received 8 August 2021; Received in revised form 21 August 2021; Accepted 28 August 2021
Available online 23 September 2021
2211-3797/© 2021 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).
Various infectious models are unable to provide complete information about the concerned disease because of less time or lack of observation about the disease. Therefore, in [23], these models may be more informative if we investigate it after all the symptoms appeared in human bodies or wait up to incubation duration. This incubation period is called time-delay which will be very helpful for more realistic consideration of the others problems which have no time delay. Time delays are considered as a natural element of the dynamic process of economics, biology, epidemiology, ecology, mechanics and physiology. In the recent time some scholars worked on the delay models. Wu and Bai in [26] deals with non-linear incidence stationary waves for healthy, infectious and recovered (SIR) disease model. Based on temporary immunity, Liu et al. in [27], deals with asymptotic characteristics time delay stochastic SIR disease model. They also deals in [28] with non-local attraction and presence of the time delay scaling-free networking SIRS disease problem. Therefore our consideration will also deals with inclusion of delay-time representing the duration of incubation for the full symptoms appearance in Chen et al. [19] model. We will perturb our problem by an external factor of environmental noise or brownian motion and by changing the given parameters.

We added the latent delay into system 1, by keeping the above assumption to deal with time delay problem as follows;

\[ dS = [A - \beta SI(t - r) - \mu S(t) + cI(t)]dt + v_1 S\,dB_1(t), \]

\[ dE = [\beta SI(t - r) - (\mu + c + \sigma + \gamma_1)E(t)]dt + v_2 E\,dB_2(t), \]

\[ dI = [-\sigma E(t) - (\mu + a + c + \sigma_2 + \gamma_2)I(t)]dt + v_3 I\,dB_3(t), \]

\[ dQ = [\sigma E(t) + \sigma_1 I(t) - (\mu + a + \gamma Q)Q]dt + v_4 Q\,dB_4(t). \]

Here \( B_i(t); i = 1, 2, 3, 4 \) are the free Brownian or noise motions. \( v_i^2; i = 1, 2, 3, 4 \) and \( v_4 > 0 \) are the intensities of the environmental external white noises, having the initial approximation as follows:

\[ S(\theta) = \phi_S(\theta), \quad E(\theta) = \phi_E(\theta), \]

\[ I(\theta) = \phi_I(\theta), \quad Q(\theta) = \phi_Q(\theta), \]

\[ -\tau \leq \theta \leq 0, \]

\[ \phi_i(\theta) \in C, i = 1, 2, 3, 4. \]

Here \( C \) is the class of existing integral of Lebesgue operator through \([-\tau, 0] \) to \( \mathbb{R}_+^4 \).

The purpose of our article is to analyze the dynamical properties of the root of the system before disease occurrence and checking of at least one repeating solutions greater than zero of the considered stochastic COVID-19 epidemic model with time delay.

The paper is organized by the following sections. In the second Section 'Qualitative Analysis of positive solution' the solution of (2) is derived to be positive and having upper and lower bounds in a feasible region, which is not changing. Also, maintenance of the considered problem and necessary for reducing the infection are studied. The valid results for the dynamical behaviors of the stationary distribution are achieved in Section 'Existence of ergodic stationary distribution'. In Section 'Qualitative Analysis of positive solution', the results for extinction of stochastic model is provided. Based on this, in Section 'Extinction', we draw our obtained scheme by numerical simulation for stochastic-stability. The last Section 'Numerical simulations for stochastic stability', is the inclusion of some remarks related to conclusion and future work.

**Qualitative analysis of positive solution**

For investigation of dynamical behavior of SDE (2), we have to prove the problem (2) has one non-local solution in the feasible region. This can be achieved that if the coefficients of system (2) are fulfilling the growth and Lipschitzian conditions then their will exist one positive solution. So this can be very easy and we omit it. Next for positive and non-local solution we have to go ahead and use the techniques of Lyapunov operator [29–31].

**Theorem 1.** Problem (2) has one root greater then zero \( S(t), E(t), I(t), Q(t) \) on \( t \geq -\tau \), and the solution lie in \( \mathbb{R}_+^4 \) for the subjected initial approximation (3) with occurrence chance of one.

**Proof.** For given initial approximation \( (S(0), E(0), I(0), Q(0)) \), the coefficients of problem (2) fulfilling the local Lipschitzian result, so model (2) has one local root \( (S(t), E(t), I(t), Q(t)) \) on \( t \in [-\tau, \tau] \) a.s., here \( \tau \) shows the time taken for infection [29–31].

Our objective is to prove that the solution is non-local i.e. \( \tau = \infty \) a.s. Take \( n_0 \geq 1 \) be largely; \( S(0), E(0), I(0), Q(0); \text{and } Q(0) \theta \in [-\theta, 0) \in \left[ \frac{1}{n_0}, n_0 \right) \), \( \forall n \geq n_0, n \in \mathbb{N} \) give the definition time break as

\[ \tau_n = \inf \left\{ t \in [-\tau, \tau] : \max(S(t), E(t), I(t), Q(t)) \leq \frac{1}{n} \text{ or max}(S(t), E(t), I(t), Q(t)) \geq n \right\}. \]

consider \( min\theta = \infty \) (\( \theta \) is void set). As, \( \tau_n \) is growing as \( n \rightarrow \infty \). Consider \( \tau = \lim_{n \rightarrow \infty} \tau_n \), then \( \tau_n \leq \tau \) a.s. Hence, we have to prove that \( \tau = \infty \) a.s, then \( \tau = \infty \) a.s and \( (S(t), E(t), I(t), Q(t)) \in \mathbb{R}_+^4 \) a.s \( \forall t \geq -\tau. \) If \( i \in (0, 1) \) and \( \Theta > 0 \), hence \( S(\theta,t)\), \( E(\theta,t) \), \( I(\theta,t) \), \( Q(\theta,t) \) are bounded functions. Therefore, we can assume that if \( \tau_n \leq \tau \) a.s, then \( \exists \) an integer \( n_1 \geq n_0 \); where \( P(\tau \geq \tau_n) \geq 1 - n \geq n_1. \)

We take a \( C^2 \)-operator \( V : \mathbb{R}_+^4 \rightarrow \mathbb{R} \) as given:

\[ V(S, E, I, Q) = (S - a - a \ln S) + (E - 1 - \ln E) + (I - 1 - \ln I) \]

\[ + (Q - 1 - \ln Q) + \int_{-\tau}^t a \beta I(s) - r \,ds, \]

here \( a > 0 \) which will be evaluated. Applying Itô’s, we get

\[ dV = LV dt + v_1(S-a)dB_1(t) + v_2(E-1)dB_2(t) + v_3(I-1)dB_3(t) + v_4(Q-1)dB_4(t). \]

where

\[ LV = \left[ (1 - \frac{a}{S}) \beta S I(t - r) - \mu S(t) + c I(t) + (1 - \frac{E}{E}) \beta E(t) I(t) + (1 - \frac{I}{I}) \beta I(t) + (1 - \frac{Q}{Q}) \beta Q(t) \right] + \frac{1}{2} (v_1^2 + v_2^2 + v_3^2 + v_4^2) + a \beta I(t) - a \beta I(t - r). \]

\[ \leq A + a \sigma + c \sigma_1 + c \sigma_2 + c \sigma_3 + c \sigma_4 + 2a + c + \sigma_2 + \sigma_3 + \sigma_4 + \frac{(a \beta - (a + \gamma \beta)) I(t - r)}{2} \]

\[ \leq A + \mu(a + 3c + \sigma_1 + \gamma + 2a + c + \sigma_2 + \gamma_2 + \gamma_3 + \frac{a \beta^2 + a \beta^2 + a \beta^2 + a \beta^2}{2} := M. \]
here $M > 0$ which is free of $S(t), E(t), I(t),$ and $Q(t)$. Hence, 
\[
dV(S, E, I, Q) = M dt + v_1(S - a)dB_1(t) + v_2(E - 1)dB_2(t) + v_3(1 - L)dB_3(t)
\]
Upon integration (6) from 0 to $t_n \wedge \tilde{T}$ and then consider the expected value on both sides, we have
\[
EV \left( S(t_n \wedge \tilde{T}), E(t_n \wedge \tilde{T}), I(t_n \wedge \tilde{T}), Q(t_n \wedge \tilde{T}) \right) 
\leq EV \left( S(0), E(0), I(0), Q(0) \right) + M \tilde{T}.
\]  
(7)

Let $\Omega_n = \{ t_n \leq \tilde{T} \}$, for $n \geq n_1$ and in view of (4), we obtain $P(\Omega_n) \geq \epsilon$ such that, for every $\omega \in \Omega_n$, there is at least one of $S(\omega, t), E(\omega, t), I(\omega, t)$, or $Q(\omega, t)$ equating either $n$ or $\frac{1}{n}$ as
\[
V \left( S(\omega, t), E(\omega, t), I(\omega, t), Q(\omega, t) \right) \geq (n - 1 - \ln n) \left( \frac{1}{n} - 1 - \ln \frac{1}{n} \right).
\]  
(8)

According to (7), we get
\[
EV(S(0), E(0), I(0), Q(0)) + M \tilde{T} = \infty,
\]
which contradicts the supposition. So we conclude that $t_\infty = a.e.$ a.s., hence the required result is proved. $\square$

Existence of ergodic stationary distribution

Now in this section, we make a proper Lyapunov operator in sense of stochastic to deal with the existence of a one ergodic stationary division of the positive root to model (2) (32). Firstly, take $X(t)$ is a regular time-homogeneous “Markov process” in $\mathbb{R}^d$ shown by the SDE
\[
dx(t) = \sum_{i=1}^{d} g_i(t, X(t))dt + f(X(t) - t), X(t), tdt.
\]  
(11)

The diffusion matrix is
\[
A(x) = (a_{ij}(x)), \quad a_{ij}(x) = \sum_{r=1}^{d} g_r'(x)g_r'(x).
\]

Lemma 1. The process of Markov $X(t)$ has one ergodic stationary division $\pi(.)$

\begin{itemize}
  \item[(i)] A bounded pre-domain $U \subset \mathbb{R}^d$ with continuous boundary $\Gamma$, and $Q \ni K > 0$; $\sum_{i,j=1}^{d} a_{ij}(x)c_i c_j \geq K(c_i)^2, x \in U, i \in \mathbb{R}^d$.
  \item[(ii)] A non-negative $\mathcal{C}$ operator $\hat{V}; L \hat{V}$ is less than zero for any $\mathbb{R}^d \setminus U$.
\end{itemize}

Take the basic reproduction number of the model in the stochastic approach as follows:
\[
R_0 = \frac{\hat{\mu} \hat{e}}{\hat{\nu} \hat{a}^2}
\]  
(12)

where $\hat{\mu} = \mu + v_1^2; \quad \hat{e} = \mu + \epsilon + \sigma_1 + \gamma_1 + v_2^2; \quad \hat{a} = \mu + a + c + \sigma_2 + \gamma_2 + v_3^2$ and $\hat{\nu} = \mu + a + \gamma_3 + \frac{v_4^2}{2}$.

Theorem 2. Assume that $R_0 > 1$ and $\mu - \frac{v_1^2 + v_2^2 + v_4^2}{2} > 0$, then for $(S(0), E(0), I(0), Q(0)) \in \mathbb{R}_+^4$, model (2) has one ergodic stationary division $\pi(.)$.

Proof. Firstly, we have to validated the conditions (i) and (ii) of Lemma 1. To derive result (i), the diffusion matrix is:
\[
A = \begin{pmatrix}
  v_1^2S^2 & 0 & 0 & 0 \\
  0 & v_2^2E^2 & 0 & 0 \\
  0 & 0 & v_3^2I^2 & 0 \\
  0 & 0 & 0 & v_4^2Q^2
\end{pmatrix}
\]

The matrix $A$ well be positive-definite on any compact subset of $\mathbb{R}_+^4$, and result (i) of Lemma 1 is fulfilling.

Further, we derive condition (ii). Take $C^2$-operator $V : \mathbb{R}_+^4 \to \mathbb{R}$ as given:
\[
V(S, E, I, Q) = N \left( -\ln S_c - \ln E_c - \ln I - \ln Q + \beta \int_i^{i+s} I(\tau) \, d\tau \right) - \ln S + \beta \int_i^{i+s} I(\tau) \, d\tau - \ln Q + \frac{1}{\rho + 1} (S + E + I + Q)^{\rho+1}
\]  
(13)

where $c_1 = \frac{\hat{\mu} \hat{e}}{\hat{\nu} \hat{a}^2}$, $c_2 = \frac{\hat{\mu} \hat{e}}{\hat{\nu} \hat{a}^2}$, and $c_3 = \frac{\hat{\mu} \hat{e}}{\hat{\nu} \hat{a}^2}$. Note that $V(S, E, I, Q)$ is not only defined on each point, but also goes to $+\infty$ as $(S, E, I, Q)$ goes to the limit of $\mathbb{R}_+^4$ and $\| (S, E, I, Q) \| \to \infty$. Hence, we have a small point $(S(0), E(0), I(0), Q(0))$ in the domain of $\mathbb{R}_+^4$. We also take a $C^2$-operator $\hat{V} : \mathbb{R}_+^4 \to \mathbb{R}_+$ as given:
\[
\hat{V}(S, E, I, Q) = N \left( -\ln S_c - \ln E_c - \ln I - \ln Q + \hat{\beta} \int_i^{i+s} I(\tau) \, d\tau \right) - \ln S + \hat{\beta} \int_i^{i+s} I(\tau) \, d\tau - \ln Q + \frac{1}{\rho + 1} (S + E + I + Q)^{\rho+1}
\]  
(14)

where $(S, E, I, Q) \in (\frac{1}{n}, n) \times (\frac{1}{n}, n) \times (\frac{1}{n}, n)$ and $n > 1$ is a so larger integer, $V_i = -c_i \ln S - c_i \ln E - c_i \ln I - \ln Q + \beta \int_i^{i+s} I(\tau) \, d\tau, V_i = -\ln S + \beta \int_i^{i+s} I(\tau) \, d\tau, V_i = -\ln E + \beta \int_i^{i+s} I(\tau) \, d\tau, V_i = -\ln I + \beta \int_i^{i+s} I(\tau) \, d\tau, V_i = -\ln Q + \beta \int_i^{i+s} I(\tau) \, d\tau, V_i = \hat{\beta} \int_i^{i+s} I(\tau) \, d\tau - \ln Q + \frac{1}{\rho + 1} (S + E + I + Q)^{\rho+1}$, $\rho > 1$, fulfilling $\mu - \frac{v_1^2 + v_2^2 + v_4^2}{2} > 0$, and $N > 0$ is a so larger value fulfilling condition $-N\hat{\beta} + R \leq -2$, here $\hat{\mu} = \frac{\hat{\mu} \hat{e}}{\hat{\nu} \hat{a}^2} - (\mu + a + \gamma_3 + \frac{v_4^2}{2}) > 0$.

\[
R = \sup_{(S, E, I, Q) \in \mathbb{R}_+^4} \left( \frac{\mu - \frac{1}{2} (v_1^2 + v_2^2 + v_4^2)}{2} \right)^{\rho+1}
\]

\[
t + 3\mu + \epsilon + \sigma_1 + \gamma_1 + a + \gamma_3 + B + \frac{v_1^2}{2} + \frac{v_2^2}{2} + \frac{v_3^2}{2}
\]

and
\[
B = \sup_{(S, E, I, Q) \in \mathbb{R}_+^4} \left( A(S + E + I + Q)^{\rho} - \frac{1}{\rho + 1} (S + E + I + Q)^{\rho+1} \right) < \infty.
\]

(15)

(16)
Applying Itô’s formula to $V_1$, we have

\[
LV_1 = -\frac{c_1 A}{S} + \mu c_1 - \frac{c_2}{S} + \frac{c_2^2}{2} - \frac{c_2 \beta S}{E} + c_2 (\mu + \epsilon + \sigma_1 + \gamma_1) + \frac{c_3^2}{2} \\
- \frac{c_1 E}{I} + c_3 (\mu + \alpha + c + \gamma_2) + \frac{c_3^1}{2} - \frac{\sigma_1 E}{Q} - \frac{\sigma_2 I}{Q} + (\mu + \alpha + \gamma_3) \\
+ \frac{c_3^2}{2} + c_1 \beta I(t) \\
\leq -3 \sqrt{A} c_1 c_2 + c_1 \left( \mu + \frac{c_2^2}{2} \right) + c_2 \left( \mu + \epsilon + \sigma_1 + \gamma_1 + \frac{c_3^2}{2} \right) \\
c_1 \left( \mu + \alpha + c + \gamma_2 + \frac{c_3^1}{2} \right) + \left( \mu + \alpha + \gamma_1 + \frac{c_3^2}{2} \right) + c_1 \beta I(t) \\
\leq -\frac{A \beta c_1 c_2 c_3}{c_1 \beta \alpha} + \left( \mu + \alpha + \gamma_2 + \frac{c_3^1}{2} \right) + c_1 \beta I(t) \\
= -\delta + c_1 \beta I(t).
\]

Similarly, we can get

\[
LV_2 = -\frac{A}{S} + \mu - \frac{c_1}{S} + \beta I + \frac{c_2^2}{2} \\
LV_3 = -\frac{\beta S I (-\tau)}{E} + (\mu + \epsilon + \sigma_1 + \gamma_1) + \frac{c_3^2}{2} \\
LV_4 = -\frac{\sigma_1 E}{Q} - \frac{\sigma_2 I}{Q} + (\mu + \alpha + \gamma_2) + \frac{c_3^2}{2}
\]

In the set $\mathbb{R}^4_+ \setminus D$, take obeying the conditions as follows:

\[
-\frac{A}{\xi} + P \leq -1 \\
-\sigma_1 \xi + P \leq -1 \\
-\sigma_1 \xi + N c_1 \beta \xi^2 + R \leq -1 \\
-\frac{\sigma_2 c_2^2}{\xi^3} + P \leq -1 \\
-\frac{1}{4} \left[ -\frac{\rho}{2} (v_1^2 + v_2^2 + v_3^2) \right] \frac{1}{\xi^{2 + 7}} + P \leq -1 \\
-\frac{1}{4} \left[ -\frac{\rho}{2} (v_1^2 + v_2^2 + v_3^2) \right] \frac{1}{\xi^{2 + 7}} + P \leq -1 \\
-\frac{1}{4} \left[ -\frac{\rho}{2} (v_1^2 + v_2^2 + v_3^2) \right] \frac{1}{\xi^{2 + 7}} + P \leq -1
\]

Where

\[
P = \sup_{(S,E,I,Q) \in \mathbb{R}^4_+ \setminus D} \left\{ N c_1 \beta I - 4 \left[ -\frac{\rho}{2} (v_1^2 + v_2^2 + v_3^2) \right] I^{\mu+1} + 3 \mu + \epsilon + \sigma_i + \gamma_i + \alpha + \gamma_3 + B + \frac{c_1^2}{2} + \frac{c_2^2}{2} + \frac{c_3^2}{2} \right\}
\]

We need to show that $\bar{L} \bar{V} \leq -1$ for any $(S,E,I,Q) \in \mathbb{R}^4_+ \setminus D$, and $\mathbb{R}^4_+ \setminus D = \bigcup_{j=1}^{n} D_j$, where

\[
D_1 = \left\{ (S,E,I,Q) \in \mathbb{R}^4_+ ; 0 < S < \xi \right\}; \\
D_2 = \left\{ (S,E,I,Q) \in \mathbb{R}^4_+ ; 0 < E < \xi \right\}; \\
D_3 = \left\{ (S,E,I,Q) \in \mathbb{R}^4_+ ; 0 < I < \xi^2, E \geq \xi \right\}; \\
D_4 = \left\{ (S,E,I,Q) \in \mathbb{R}^4_+ ; 0 < Q < \xi^3, I \geq \xi^2 \right\}; \\
D_5 = \left\{ (S,E,I,Q) \in \mathbb{R}^4_+ ; S \geq \frac{1}{\xi} \right\}; \\
D_6 = \left\{ (S,E,I,Q) \in \mathbb{R}^4_+ ; E \geq \frac{1}{\xi} \right\}; \\
D_7 = \left\{ (S,E,I,Q) \in \mathbb{R}^4_+ ; I > \frac{1}{\xi^2} \right\}; \\
D_8 = \left\{ (S,E,I,Q) \in \mathbb{R}^4_+ ; Q > \frac{1}{\xi^3} \right\};
\]

Case 1. For any $(S,E,I,Q) \in D_1$, we obtain

\[
\bar{L} \bar{V} \leq -A + N c_1 \beta I - \frac{1}{4} \left[ -\frac{\rho}{2} (v_1^2 + v_2^2 + v_3^2) \right] I^{\mu+1} + 3 \mu + \epsilon + \sigma_i + \gamma_i + \alpha + \gamma_3 \\
+ B + \frac{c_1^2}{2} + \frac{c_2^2}{2} + \frac{c_3^2}{2} \leq -\frac{A}{\xi} + P \leq -1
\]

For $\xi > 0$, define a bounded closed set

\[
D = \left\{ (S,E,I,Q) \in \mathbb{R}^4_+ ; 0 \leq S \leq \frac{1}{\xi}, \xi \leq E \leq \xi, \xi^2 \leq I \leq \frac{1}{\xi^2}, \xi^3 \leq Q \leq \frac{1}{\xi^3} \right\}
\]

which is obtained from (23). Therefore, $\bar{L} \bar{V} \leq -1$ for any $(S,E,I,Q) \in D_1$. 

\[
\text{(23)} \\
\text{(24)} \\
\text{(25)} \\
\text{(26)} \\
\text{(27)} \\
\text{(28)} \\
\text{(29)}
\]
Case 2. For any \( (S, E, I, Q) \in D_2 \), we obtain
\[
L\tilde{V} \leq -\frac{\sigma_I E}{Q} + Nc_I \beta I - \frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] I_{\text{err}}^1 + 3\mu + \epsilon + \sigma_I + \gamma_I + \alpha + \gamma_3 \\
+ B + \frac{v_2^I v_2^I + v_3^I v_3^I}{2} \\
\leq -\frac{\sigma_I E}{Q} + P \\
\leq -\sigma^c V + P \leq -1.
\]
which follows from (24). Therefore, \( L\tilde{V} \leq -1 \) for any \( (S, E, I, Q) \in D_2 \).

Case 3. For any \( (S, E, I, Q) \in D_3 \), we obtain
\[
L\tilde{V} \leq -\frac{\sigma_E}{Q} v_2 + Nc_I \beta I - \frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] I_{\text{err}}^1 + 3\mu + \epsilon + \sigma_I + \gamma_I + \alpha + \gamma_3 \\
- \frac{\sigma_I E}{Q} + B + \frac{v_2^I v_2^I + v_3^I v_3^I}{2} \\
\leq -\frac{\sigma_I E}{Q} + P \\
\leq -\sigma^c V + P \leq -1.
\]
which is obtained from (25). Therefore, \( L\tilde{V} \leq -1 \) for any \( (S, E, I, Q) \in D_3 \).

Case 4. For any \( (S, E, I, Q) \in D_4 \), we obtain
\[
L\tilde{V} \leq -\frac{\sigma_I}{Q} + Nc_I \beta I - \frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] I_{\text{err}}^1 + 3\mu + \epsilon + \sigma_I + \gamma_I + \alpha + \gamma_3 \\
+ B + \frac{v_2^I v_2^I + v_3^I v_3^I}{2} \\
\leq -\frac{\sigma_I}{Q} + P \\
\leq -\sigma I + P \leq -1.
\]
which is obtained from (26). Therefore, \( L\tilde{V} \leq -1 \) for any \( (S, E, I, Q) \in D_4 \).

Case 5. For any \( (S, E, I, Q) \in D_5 \), we obtain
\[
L\tilde{V} \leq -\frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] S_{\text{err}} + Nc_I \beta I \\
- \frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] I_{\text{err}}^1 + 3\mu + \epsilon + \sigma_I + \gamma_I + \alpha + \gamma_3 \\
+ B + \frac{v_2^I v_2^I + v_3^I v_3^I}{2} \\
\leq -\frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] S_{\text{err}} + P \\
\leq -\frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] \frac{1}{2} S_{\text{err}} + P \leq -1.
\]
which is obtained from (27). Therefore, \( L\tilde{V} \leq -1 \) for any \( (S, E, I, Q) \in D_5 \).

Case 6. For any \( (S, E, I, Q) \in D_6 \), we obtain
\[
L\tilde{V} \leq -\frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] E_{\text{err}} + Nc_I \beta I \\
- \frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] I_{\text{err}}^1 + 3\mu + \epsilon + \sigma_I + \gamma_I + \alpha + \gamma_3 \\
+ B + \frac{v_2^I v_2^I + v_3^I v_3^I}{2} \\
\leq -\frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] E_{\text{err}} + P \\
\leq -\frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] \frac{1}{2} E_{\text{err}} + P \leq -1.
\]
which is obtained from (28). Therefore, \( L\tilde{V} \leq -1 \) for any \( (S, E, I, Q) \in D_6 \).

Case 7. For any \( (S, E, I, Q) \in D_7 \), we obtain
\[
L\tilde{V} \leq -\frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] Q_{\text{err}} + Nc_I \beta I \\
- \frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] I_{\text{err}}^1 + 3\mu + \epsilon + \sigma_I + \gamma_I + \alpha + \gamma_3 \\
+ B + \frac{v_2^I v_2^I + v_3^I v_3^I}{2} \\
\leq -\frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] Q_{\text{err}} + P \\
\leq -\frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] \frac{1}{2} Q_{\text{err}} + P \leq -1.
\]
which is obtained from (29). Therefore, \( L\tilde{V} \leq -1 \) for any \( (S, E, I, Q) \in D_7 \).

Case 8. For any \( (S, E, I, Q) \in D_8 \), we obtain
\[
L\tilde{V} \leq -\frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] M_{\text{err}} + Nc_I \beta I \\
- \frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] I_{\text{err}}^1 + 3\mu + \epsilon + \sigma_I + \gamma_I + \alpha + \gamma_3 \\
+ B + \frac{v_2^I v_2^I + v_3^I v_3^I}{2} \\
\leq -\frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] M_{\text{err}} + P \\
\leq -\frac{1}{4} \left[ \mu - \frac{\beta}{2} (v_2^E v_2^S v_3^I v_3^S) \right] \frac{1}{2} M_{\text{err}} + P \leq -1.
\]
this implies from (29). Hence, \( L\tilde{V} \leq -1 \) at any \( (S, E, I, Q) \in D_8 \). Consequently, result (ii) of Lemma 1 satisfied. So, we concluded that model (2) have one stationary distribution \( \mu(.) \). □

Extinction

For the vanishing or reducing of the epidemic, just see the lemmas as follows.

Lemma 2. Let \( M = \{ M_t \}_{t \geq 0} \) be a real valued define local martingale reduced at \( t = 0 \). Then
\[
\lim_{t \to \infty} \langle M, M \rangle_t = \infty \quad \text{a.s.} \quad \Rightarrow \quad \lim_{t \to \infty} \frac{M_t}{M(0)} = 0 \quad \text{a.s.,}
\]
and also
\[
\lim_{t \to \infty} \sup_{0 \leq t \leq 1} \frac{M_t}{M(0)} < \infty \quad \text{a.s.} \quad \Rightarrow \quad \lim_{t \to \infty} \frac{M_t}{M(0)} = 0 \quad \text{a.s.,}
\]
here \( \langle M, M \rangle_t \) shows the quadratic variants of \( M \).

Lemma 3. Take \( (S(t), E(t), I(t), Q(t)) \) be root of (2) with \( (S(0), E(0), I(0), Q(0)) \in \mathbb{R}_+^4 \), then
\[
\lim_{t \to \infty} \frac{S(t)}{E(0)} = 0, \quad \lim_{t \to \infty} \frac{E(t)}{I(0)} = 0, \quad \lim_{t \to \infty} \frac{I(t)}{Q(0)} = 0, \quad \lim_{t \to \infty} \frac{Q(t)}{M(0)} = 0 \quad \text{a.s.,}
\]
Furthermore, if \( \mu > \frac{v_2^E v_2^S v_3^I v_3^S}{\sqrt{2}} \), then
\[ \lim_{t \to \infty} \frac{\sigma L_t}{t} = 0, \quad \lim_{t \to \infty} \frac{E L_t}{t} = 0, \quad \lim_{t \to \infty} \frac{Q L_t}{t} = 0, \quad \text{a.s.} \]

**Theorem 3.** If \( R_0^\prime < 1 \) and \( \mu > \frac{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}{2} \), then root of (2) fulfilling the given as:

\[ \limsup_{t \to \infty} \frac{1}{t} \ln(\sigma_E E + I(\mu + a + c + \sigma_2 + \gamma_2)Q) \]

\[ \leq \beta - \frac{1}{2(\sigma_2)^2} \left( \sigma_2^2 + \gamma_2^2 + \frac{\gamma_1^2}{4} + \frac{\gamma_3^2}{4} \right) \left( \frac{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}{2} \right) \]

\[ \land (\mu + a + c + \sigma_2 + \gamma_2)^2(\alpha + \gamma_3 + \frac{\gamma_2^2}{4}) < 0 \]

and \( \lim_{t \to \infty} S = 1 \) a.s.

**Proof.** Take \( U(t) = \sigma_E E + I(\mu + a + c + \sigma_2 + \gamma_2)Q \), applying Itô’s expression, we obtain

\[
\frac{d\ln U(t)}{t} = \left\{ \sigma_E E + I(\mu + a + c + \sigma_2 + \gamma_2)Q \times [\sigma_2 S I(t - r) - \sigma_2(\mu + \gamma_2)E + \sigma_1(\mu + a + c + \gamma_2)E] - \gamma_1 + \gamma_2(\mu + a + c + \sigma_2 + \gamma_2)Q \right\} dt \]

\[ - \sigma_2^2 \left( \gamma_2^2 + \frac{\gamma_1^2}{4} + \frac{\gamma_3^2}{4} \right) \left( \frac{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}{2} \right) \]

\[ \land (\mu + a + c + \sigma_2 + \gamma_2)^2(\alpha + \gamma_3 + \frac{\gamma_2^2}{4}) < 0 \]

Taking Integral of (41) we get

\[
\psi_1(t) = \frac{1}{\mu} \left( \frac{1}{t} (S(0) + E(0) + I(0) + Q(0)) - \frac{1}{t} (S(t) + E(t) + I(t) + Q(t)) \right) \]

\[ - \gamma_1 + \gamma_2(\mu + a + c + \gamma_2)Q \]

\[ \leq \beta - \frac{1}{2(\sigma_2)^2} \left( \gamma_2^2 + \gamma_1^2 + \gamma_3^2 \right) \left( \frac{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}{2} \right) \]

\[ \land (\mu + a + c + \sigma_2 + \gamma_2)^2(\alpha + \gamma_3 + \frac{\gamma_2^2}{4}) < 0 \]

Using Lemmas 2 and 3, we get \( \lim_{t \to \infty} \psi_1(t) = 0 \) a.s. Consequently, (43) implies

\[
\limsup_{t \to \infty} (S + E + I + Q) = \frac{A}{\mu} \quad \text{a.s.} \]

**Numerical simulations for stochastic stability**

Now, we provide numerical simulations for the illustration and validation of our obtained theoretical scheme. For this, we apply stochastic iterative techniques of fourth order Runge Kutta method and to obtain
Results in Physics 30 (2021) 104775

Among the average and vanishing, which may be proved from the stochastic stability and for optimality, we have to mention the values of the parameters and the noises intensities values from Table 2 (Set A). So we considered the stochastic system has a one stationary distribution. For stochastic stability of system (2), we consider parameter values from Table 2 (Set B) and some parameters for optimal control.

Now we deals with the numerical approximation and biological features of system (2) and its corresponding deterministic version.

To bring out numerical simulation for investigating the dynamics of stochastic stability and for optimality, we have to mention the values of the parameters given in the problem (2) and some parameters for optimal control.

Next, in Theorem 2 an ergodic techniques is used to prove that the stochastic system has a one stationary distribution. For stochastic system (2), we consider parameter values from Table 2 (Set B) and compute $R_0^s > 1$, so, Theorem 2 is fulfilled. As seen in Fig. 2, the infection of problem (2) will lie in average which validate the results of Theorem 2, hence implies that problem (2) lie an ergodic stationary distribution. Observe 1000 attempts at $t = 200$, then compute average value, Theorem 2 implies that system (2) lie an ergodic stationary distribution as Fig. 3 explained this situation.

Example 1 (Stochastic Disease-Free Dynamical Behavior). We take the parameter values from Table 2 (Set A). So we compute the reproduction free equilibrium point showing global asymptotic stability. The disease may die as can be seen in Fig. 1.

Now for the corresponding deterministic approach the epidemic

the following discretization-transformation of model (2),

$$
S_{i+1} = S_i + \left[ A - \beta S_i I_i(t - \tau) - \mu S_i(t) + c t I_i(t) \right] \Delta t \\
+ v_1 S_i \sqrt{\Delta \xi_{1i}} + \frac{v_1^2}{2} E_i(\xi_{1i}^2 - 1) \Delta t \\
E_{i+1} = E_i + \left[ \beta S_i(t) I_i(t - \tau) - (\mu + \epsilon + \sigma_1 + \gamma_1) E_i(t) \right] \Delta t \\
+ v_2 E_i \sqrt{\Delta \xi_{2i}} + \frac{v_2^2}{2} E_i(\xi_{2i}^2 - 1) \Delta t \\
I_{i+1} = I_i + \left[ \epsilon E_i(t) - (\mu + \alpha + \epsilon + \sigma_2 + \gamma_2) I_i(t) \right] \Delta t \\
+ v_3 I_i \sqrt{\Delta \xi_{3i}} + \frac{v_3^2}{2} Y_i(\xi_{3i}^2 - 1) \Delta t \\
Q_{i+1} = Q_i + \left[ \sigma_1 E_i(t) + \sigma_2 I_i(t) - (\mu + \alpha + \gamma_3) Q_i(t) \right] \Delta t \\
+ v_4 Q_i \sqrt{\Delta \xi_{4i}} + \frac{v_4^2}{2} Z_i(\xi_{4i}^2 - 1) \Delta t
$$

Where $\xi_{kj}(k = 1, 2, 3, 4)$, are four free Gaussian general variables having $N(0, 1)$ and time-increment $\Delta t > 0$.

$$
\begin{array}{cccc}
\text{Parameters} & \text{Set A} & \text{References} & \text{Set B} & \text{References} \\
A & 0.9 & \text{Considered} & 5 & \text{Considered} \\
\gamma_1 & 0.02 & \text{Considered} & 0.02 & \text{Considered} \\
\gamma_2 & 0.03 & \text{Considered} & 0.02 & \text{Considered} \\
\gamma_3 & 0.02 & \text{Considered} & 0.02 & \text{Considered} \\
\beta & 0.95 & \text{Considered} & 5 & \text{Considered} \\
\mu & 0.02 & \text{Considered} & 0.05 & \text{Considered} \\
\sigma_1 & 0.05 & \text{Considered} & 0.5 & \text{Considered} \\
\sigma_2 & 0.5 & \text{Considered} & 0.5 & \text{Considered} \\
a & 0.02 & \text{Considered} & 0.2 & \text{Considered} \\
\epsilon & 0.03 & \text{Considered} & 0.2 & \text{Considered} \\
\tau & 1 & \text{Considered} & 1 & \text{Considered} \\
c & 0.05 & \text{Considered} & 0.02 & \text{Considered} \\
v_1 & 0.2 & \text{Considered} & 0.6 & \text{Considered} \\
v_2 & 0.3 & \text{Considered} & 0.9 & \text{Considered} \\
v_3 & 0.6 & \text{Considered} & 0.7 & \text{Considered} \\
v_4 & 0.4 & \text{Considered} & 0.11 & \text{Considered} \\
\kappa(0) & 10 & \text{Considered} & 10 & \text{Considered} \\
E(0) & 5 & \text{Considered} & 5 & \text{Considered} \\
I(0) & 2 & \text{Considered} & 2 & \text{Considered} \\
Q(0) & 10 & \text{Considered} & 10 & \text{Considered}
\end{array}
$$

Fig. 1. Trajectories of stochastic SEIQ model (2) and its corresponding deterministic version.
number $R_0^s < 1$ and by Theorem 3 the root of system (2) may fulfilled

$$\lim_{t \to \infty} \sup \frac{\log E(t)}{t} \leq 0, \text{ a.s.}$$

and

$$\lim_{t \to \infty} \sup \frac{\log I(t)}{t} \leq 0, \text{ a.s.}$$

Hence this implies that the epidemic will vanish from the community as in Fig. 1 shows that the numerical-simulation verify our scheme.

**Example 2 (Stochastic Endemic Dynamical Behavior).** By same fashion we take the parameter values from Table 2 (Set B). We prove $R_0^s > 1$, and by Theorem 2, the disease will lie or stable, and provide simulation to show our results in Fig. 2. Theorem 2 implies that system (2) has one stationary distribution which is verified by Fig. 2.
Conclusion

In the concluding remarks of this manuscript, we have established a SEIQ disease system having time-delay for the new-strain coronavirus COVID-19 in the sense of stochastic approach. The stochastically taken agents are treated in the system as white Gaussian noises because of external environment variances. We have derived some sufficient results for maintenance and reduction of disease in the average of the epidemic. The system has one stationary distribution which is ergodic for small intensities of white noises. Lastly, the numerical representation through various plotting is given for verification of our obtained results.

The researchers know that the stochastic SEIQ system is the try to know epidemiological properties of COVID-19. The system gives new scenes into epidemiological conditions when the environment noises (perturbations) and crossing-immunity are taken in the COVID-19 disease systems. The composition of white noises and time-delay, in the infectious systems, have more effect on the lying and vanishing of the infection and complicated the dynamical behavior of the system. The work of this paper may be analyzed by including controlling variables like vaccination and other treatment type actions.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

“This research was supported by King Mongkut’s University of Technology Thonburi’s Postdoctoral Fellowship, Thailand”.

References

[1] Kermack William Ogilvy, McKendrick Anderson G. A contribution to the mathematical theory of epidemics. Proc R Soc A 1927;115(772):700–21.
[2] Billiard L, Dayananda PWA. A multi-stage compartmental model for HIV-infected individuals: I–waiting time approach. Math Biosci 2014;249:92–101.
[3] Pongsan Puntani, Tang I. Dynamics of a new strain of the H1N1 influenza virus incorporating the effects of repetitive contacts. Comput Math Methods Med 2014;2014:14.
[4] Din Anwarud, Li Yongjin. Stationary distribution extinction and optimal control for the stochastic hepatitis B epidemic model under real statistical data and its fractal–fractional Atangana-Baleanu order model. Phys Scr 2021;96(7):074005.
[5] Naheed Afia, Singh Manmohan, Lucy David. Numerical study of SARS epidemic model with the inclusion of diffusion in the system. Appl Math Comput 2014;229:480–98.
[6] Upadhyay Ranjit Kumar, Kumari Nitu, Rao Viree Hari. Modeling the spread of bird flu and predicting outbreak diversity. Nonlinear Anal RWA 2008;9(4):1638–48.
[7] Din Anwarud, Li Yongjin. The complex dynamics of hepatitis B infected individuals with optimal control. J Syst Sci Complexity 2020;33:1–23.
[8] Kermack William Ogilvy, McKendrick Anderson G. Contributions to the mathematical theory of epidemics. II.—The problem of endemicity. Proc R Soc A 1932;138(834):55–83.
[9] Danane Jaouad, Allali Karam, Hammouch Zakia. Mathematical analysis of a fractional differential model of HBV infection with antibody immune response. Chaos Solitons Fractals 2020;136:109787.
[10] Atangana Abdon, Koca Ilknur. Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order. Chaos Solitons Fractals 2016;89:447–54.
[11] Ullah Saif, Khan Muhammad Altaf, Farooq Muhammad. A new fractional model for the dynamics of the hepatitis B virus using the Caputo-Fabrizio derivative. Eur Phys J Plus 2018;133(6):1–14.
[12] Kamal Shah, Sadaswy Aby, Al-Rabaiah Hussam, Baleanu Dumitru. On a new conceptual mathematical model dealing the current novel coronavirus-19 infectious disease. Results Phys 2020;19:103836.
[13] Edmunds WJ, Medley GF, Nokes DJ. The transmission dynamics and control of hepatitis B virus in the Gambia. Stat Med 1996;15(20):2215–33.
[14] Khan Amir, Zarin Rahat, Hussain Ghulam, Usman Awalu Hamisu, Humphries Usa Wannasingha, Gomez-Aguilar JF. Modeling and sensitivity analysis of HBV epidemic model with convex incidence rate. Results Phys 2021;22:103836.
[15] Din A, Li Y. Controlling heroin addiction via age-structured modeling. Adv Difference Equ 2020;2020:2020.
[16] Khan G Zaman. Classification of different hepatitis b infected individuals with saturated incidence rate. Springerplus 2016;5.
[17] Atangana Abdon, Koca Ilknur. Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order. Chaos Solitons Fractals 2016;89:447–54.
[18] Din Anwarud, Li Yongjin. Lévy noise impact on a stochastic hepatitis B epidemic model with the inclusion of diffusion in the system. Appl Math Comput 2016;89:447–54.
[19] Chen Xiangyong, Cao Jinde, Jin Zhen. Pattern formation of an epidemic model with non-monotone incidence rate. Adv Difference Equ 2020;521:2020.
[20] Din Anwarud, Li Yongjin. Stationary distribution extinction and optimal control for the stochastic hepatitis B epidemic model with partial immunity. Phys Scr 2021;96(7):074005.
[21] Atangana Abdon, Araz Sedağret. A stochastic SACR epidemic model for HBV transmission. J Biol Dyn 2020;14(1):788–801.
[22] Khan Amir, Hussain Ghulam, Zahir Mostafa, Zaman Gul, Humphries Usa Wannasingha. A stochastic SAGC epidemic model for HBV transmission. J Biol Dyn 2020;14(1):788–801.
[23] Li Jing, Sun Gui-Quan, Jin Zhen. Pattern formation of an epidemic model with time delay. Physica A 2014;403:105–9.
[24] Chen Chi Noe, Avila Vales Eric, Almeida Gerardo Garcia. Analysis of a HBV model with diffusion and time delay. J Appl Math 2012;2012.
[25] Li, et al. Delayed hepatitis B epidemic model with stochastic analysis. Chaos Solitons Fractals 2021.
[26] Bai Zhenguo, Wu Shi-Liang. Traveling waves in a delayed SIR epidemic model with nonlinear incidence. Appl Math Comput 2015;263:221–32.
[27] Liu Qun, Chen Qingmei, Jiang Daqing. The threshold of a stochastic delayed SIR epidemic model with temporary immunity. Physica A 2016;450:115–25.
[28] Jinhu Li, Teng Zhidong. Bifurcations of an SIRS model with generalized non-monotone incidence rate. Adv Difference Equ 2018;2018(1):1–21.
[29] Yongjin Khan, Li Hassan, Tahir Asaf, Khan, Wajahat, Ali Khan. Mathematical analysis of dengue stochastic epidemic model. Results Phys 2020;20.
[30] Din Anwarud, Li Yongjin. Stationary distribution extinction and optimal control for the stochastic hepatitis B epidemic model with partial immunity. Phys Scr 2021;96(7):074005.
[31] Khan Tahir, Khan Amir, Zaman Gul. The extinction and persistence of the stochastic hepatitis B epidemic model. Chaos Solitons Fractals 2018;108:123–8.
[32] Khasminskii RZ. Stochastic Stability of Differential Equations (Book). Alphen aan den Rijn, Netherlands, Sijthoff and Noordhoff (Monographs and textbooks on Mechanics of Solids and Fluids. Mechanics: Analysis, vol. 7, 1980.)