Scattering theory for the Dirac equation in Schwarzschild-Anti-de Sitter space-time

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Abstract

We show asymptotic completeness for linear massive Dirac fields on the Schwarzschild-Anti-de Sitter spacetime. The proof is based on a Mourre estimate. We also construct an asymptotic velocity for this field.

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1 Introduction

The aim of this paper is to show asymptotic completeness for the massive Dirac equation on Anti-de Sitter Schwarzschild space-time.

When studying a physical system for which the dynamics is described by a Hamiltonian, one of the fundamental properties we want to prove is asymptotic completeness. Roughly speaking, it states that, for large time, our dynamics behaves, modulo possible eigenvalues, like a well-understood dynamics described by what we call a free hamiltonian.

The first asymptotic completeness results in General Relativity were obtained by J. Dimock and B. Kay in 1986 and 1987 ([15], [16], [17]) for classical and quantum scalar fields. This study was pursued in the 1990’s by A. Bachelot for classical fields. He obtains scattering theories for Maxwell in 1991 [2] and Klein-Gordon in 1994 [3]. After that, J-P. Nicolas obtained a scattering theory for massless Dirac fields in 1995 [33] and F. Melnyk obtained a complete scattering for massive charged Dirac fields [34] in 2003. In all these works, the authors used trace class perturbation methods. On the other hand, new techniques, using Mourre estimates, were applied to the wave equation on Schwarzschild space-time in 1992 by S. De Bièvre, P. Hislop and I.M Sigal [13]. Using this method, a complete scattering theory for the wave equation on stationary asymptotically flat space-times was obtained by D. Häfner in 2001 [23] and D. Häfner and J-P. Nicolas obtained a scattering theory for massless Dirac fields outside slowly rotating Kerr black holes in 2004 [25], making use of a positive conserved quantity which exists for Dirac equation and not for Klein-Gordon equation. In 2004, T. Daudé obtains a scattering theory for Dirac fields on Reissner-Nordström black holes [12] and on Kerr-Newman black holes in [11]. Using an integral representation for the Dirac propagator, D. Batic gives a new approach to the time-dependent scattering for massive Dirac fields on the Kerr metric in 2007. Recently, V. Georgescu, C. Gérard and D. Häfner obtained an asymptotic completeness result for the Klein-Gordon equation in the De-Sitter Kerr black hole, see [21]. See also M. Dafermos, G. Holzegel and I. Rodnianski for scattering results for the Einstein equations [10].

In our work, we are concerned with problems that arise from the Anti-de Sitter background. Indeed, the Schwarzschild Anti-de Sitter space-time is a solution of the Einstein vacuum equations with cosmological constant $-\Lambda < 0$ containing a spherically symmetric black hole. This space-time has a non-trivial causality. In fact, it is not globally hyperbolic, that is to say, Cauchy data defined on a slice $\{t = \text{constant}\} \times r_{SAdS}, +\infty \times S^2$ (where $r_{SAdS}$ correspond to the horizon) do not uniquely determine the evolution of the field in all the space-time. So, first of all, there’s a difficulty in defining the dynamics. This is due to the fact that, when studying the geodesics in Boyer-Lindquist coordinates, null geodesics can reach timelike infinity in finite time. This suggests that we will need to put asymptotic conditions as $r \to +\infty$ in order to determine the dynamics uniquely. This problem was first studied by Breitenlohner and Freedman ([21], [22]) for scalar fields. They showed that the need to put boundary conditions depends on the comparison between the mass of the fields and the cosmological constant and discovered two critical valued known as B-F bounds. More recently, A. Bachelot ([4]) showed a similar bound for the Dirac equation in the Anti-de Sitter space-time using spectral approach. This approach uses the fact that, in an appropriate coordinate system, the equation can be written $i\partial_t \psi = iH_m \psi$ with $H_m$ independent of $t$. We thus have to construct self-adjoint extension of $H_m$. In order to put the right boundary condition, we need to understand the asymptotic behaviour of the states in the natural domain of $H_m$. This kind of method was also used by Ishibashi and Wald ([31], [32]) for integer spin fields.

Using other techniques, there has been some recent advances concerning scalar fields. We first mention the works of G. Holzegel and J. Smulevici who proved, using vectorfield methods, a result of asymptotic stability of Schwarzschild-AdS with respect to spherically symmetric perturbations thanks to an exponential decay rate of the local energy [28]. However, looking at solutions of the linear wave equation on the Schwarzschild-AdS black hole
with arbitrary angular momentum $l$, resonances with imaginary part $e^{-C_l}$ appear (see [20] for details) and the local energy only decays logarithmically. The same phenomenon appear in Kerr-AdS, see [27]. Thus Kerr-AdS is supposed to be unstable. In these papers, it was supposed that Dirichlet boundary condition holds. More recently, G. Holzegel and C.M. Warnick consider other boundary conditions for the wave equation on asymptotically AdS black hole [29]. This includes some boundary conditions considered in the context of AdS-CFT correspondence. This correspondence was also in mind of A. Bachelot in his paper about the Klein-Gordon equation in $\text{AdS}^5$ [5] and of A. Enciso and N. Kamran when they study the Klein-Gordon equation in $\text{AdS}^5 \times Y^{p,q}$ where $Y^{p,q}$ is a Sasaki-Einstein 5-manifold [19].

We now present our results. We denote the natural domain of $H_m$ by

$$D(H_m) = \{ \phi \in \mathcal{H}; \ H_m \phi \in \mathcal{H} \},$$

and obtain:

**Proposition 1.1.** For $2ml \geq 1$, the operator $H_m$ is self-adjoint on $D(H_m)$.

For the case $2ml < 1$, we need to put MIT boundary conditions. This defines an operator $H_m^{\text{MIT}}$ with natural domain $D(H_m^{\text{MIT}})$. Then we obtain:

**Proposition 1.2.** The operator $H_m^{\text{MIT}}$ is self-adjoint on $D(H_m^{\text{MIT}})$.

The Cauchy problem is then well-posed by Stone’s theorem.

We then turn our attention to the scattering theory. By means of a Mourre estimate, we are able to prove velocity estimates. We then introduce the comparison operator $H_c = i\gamma_0 \gamma_1 \partial_x$ with domain $D(H_c) = \{ \phi \in H_{s,n}; \ H_c \phi \in H_{s,n}, \ \phi_1(0) = -\phi_3(0), \ \phi_2(0) = \phi_4(0) \}$. Making use of the velocity estimates, we obtain the following asymptotic completeness result:

**Theorem 1.3 (Asymptotic completeness).** For all $m > 0$ and all $\varphi \in \mathcal{H}$, the limits:

$$\lim_{t \to \infty} e^{itH_c} e^{-itH_m} \varphi$$

exist. If we denote these limits by $\Omega \varphi$ and $W \varphi$ respectively, then we have $\Omega^* = W$.

We eventually study the asymptotic velocity. We will say that $B = s - C_\infty - \lim_{n \to \infty} B_n$ if, for all $J \in C_\infty(\mathbb{R})$, we have $J(B) = s - \lim_{n \to \infty} J(B_n)$ (where $C_\infty(\mathbb{R})$ is the set of continuous functions which go to 0 at $\pm \infty$). Then, we obtain the following:

**Theorem 1.4 (Asymptotic velocity for $H_m$).** Let $J \in C_\infty(\mathbb{R})$. Then, for all $m > 0$, the limit:

$$s - \lim_{t \to \infty} e^{itH_m} J \left( \frac{A}{t} \right) e^{-itH_m}$$

exist. Moreover, if $J(0) = 1$, then

$$s - \lim_{R \to \infty} \left( s - \lim_{t \to \infty} e^{itH_m} J \left( \frac{A}{Rt} \right) e^{-itH_m} \right) = \mathbb{1}. \quad (1.4)$$

If we define

$$s - C_\infty - \lim_{t \to \infty} e^{itH_m} \frac{A}{t} e^{-itH_m} =: P_m^+, \quad (1.5)$$

then self-adjoint operator $P_m^+$ is densely defined and commute with $H_m$. The operator $P_m^+$ is called the asymptotic velocity.
The paper is organized as follows.

In section 2, we present the Schwarzschild-AdS geometry and, due to the lack of global hyperbolicity, the fact that radial null geodesics go to infinity in finite time. Using the Newman-Penrose formalism, we then obtain the Dirac equation on this space-time and give a spectral formulation of this equation for a coordinate system \((t, x, \theta, \varphi)\) where the horizon corresponds to \(x\) goes to \(-\infty\) and the Anti-de Sitter infinity corresponds to \(x = 0\). We eventually generalize this equation by giving asymptotic behaviours of the potentials and we ensure that the Dirac equation in Schwarzschild-AdS space-time is part of our generalization. In the rest of the paper, we will work with this generalization.

In section 3, we obtain the self-adjointness of our operator for all \(m > 0\). First, we present the spinoidal spherical harmonics and then we use this tool to decompose our operator (in fact, we diagonalize the Dirac operator on the sphere) which leads us to a \(1+1\) dimensional problem for the operator now denoted \(H_{s,n}^m\). Then we study the states in the natural domain \(D(H_{s,n}^m) = \{ \varphi \in H_{s,n} | H_{s,n}^m \varphi \in H_{s,n} \}\). The problem is coming from the Anti-de Sitter infinity where the potential behaves badly. Nevertheless, the potential behaves like in the result of A. Bachelot on Anti-de Sitter space. After a unitary transform we can use his result. In this way, we see that the states behave well when \(2ml \geq 1\) but it degenerates at \(0\) when \(2ml < 1\). When \(2ml \geq 1\), we prove that our operator is essentially self-adjoint on \(C^0_\infty([-\infty,0])\) and, using an elliptic estimate and a Hardy-type inequality, we give a precise description of the domain. In the case \(2ml < 1\), we need to put a boundary condition to obtain the self-adjointness of our operator. In this paper, we have chosen the MIT boundary condition. This allows us to solve the Cauchy problem. We finally prove the absence of eigenvalues for this operator.

In section 4, we prove a compactness result. We use an approximation of our resolvent separating the behaviour close to the black hole horizon and close to \(x = 0\). We then obtain that \(f(x)(H_{s,n}^m - \lambda)^{-1}\) is compact if \(f\) goes to \(0\) at the horizon and has a finite limit at \(x = 0\).

In section 5, we obtain a Mourre estimate for \(H_{s,n}^m\) using \(A = \Gamma_1 x\), where \(\Gamma_1\) is the matrix \(\text{diag}(1, -1, -1, 1)\), as conjugate operator. In section 6, we obtain propagation estimates. First, making use of the Mourre estimate and of an abstract result about minimal velocity estimates, we prove that the minimal velocity is 1. Then, using a standard observable and a general result which uses Heisenberg derivative to obtain velocity estimates, we prove that the maximal velocity is also 1.

In section 7, we are now able to prove asymptotic completeness for our hamiltonian. This result is first proved for fixed harmonics and then we prove that we can sum over all harmonics. It is proved by making use of the two velocity estimates and a similar reasoning as in the propagation estimates.

In section 8, we first prove the existence of the asymptotic velocity for \(H_c\) and then deduce the same result for \(H_m\) using the wave operators. We see that the asymptotic velocity operator is the identity.

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2 The Schwarzschild Anti-de Sitter space-time and the Dirac equation

In this section, we present the Schwarzschild Anti-de Sitter space-time and give the coordinate system that we will working with in the rest of the paper. We see that, along radial null geodesic, we go to infinity in finite Boyer-Lindquist time which is a problem in order to define uniquely our propagation. Then we formulate the Dirac equation as a system of partial differential equation which is derived from the two spinor component expression of
this equation by use of the Newman-Penrose formalism. We finally give a generalization of our equation by just considering potential that have the same asymptotic behaviour as in the case of the Schwarzschild Anti-de Sitter space-time. We finally give a generalization of our equation by just considering potential that have the same asymptotic behaviour as in the case of the Schwarzschild Anti-de Sitter space-time.

2.1 The Schwarzschild Anti-de Sitter space-time

Let $\Lambda < 0$. We define $l^2 = \frac{3\Lambda}{2}$. We note $M$ the black hole mass.

In Boyer-Lindquist coordinates, the Schwarzschild-Anti-de Sitter metric is given by:

$$g_{\mu\nu} = \left(1 - \frac{2M}{r} + \frac{r^2}{l^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{r^2}{l^2}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(2.1) We note $F(r) = 1 - \frac{2M}{r} + \frac{r^2}{l^2}$. Using the Cardan method, we can see that $F$ admits two complex conjugate roots and one real root $r = r_{SAdS}$, the expression is given below.

We deduce that the singularities of the metric are at $r = 0$ and $r = r_{SAdS} = p_+ + p_-$. where $p_{\pm} = \left(M^2 \pm \left(M^2 l^4 + \frac{6}{l^2}\right)^{1/2}\right)^{1/2}$.

(See [28])

In order to have a better understanding of this geometry, we study the outgoing (respectively ingoing) radial null geodesics (that is to say for which $\frac{dt}{dr} > 0$ (respectively $\frac{dt}{dr} < 0$)). Using the form of the metric we can see that along such geodesics, we have:

$$\frac{dt}{dr} = \pm F(r)^{-1}.$$ (2.2)

We thus introduce a new coordinate $r_*$ such that $t - r_*$ (respectively $t + r_*$) is constant along outgoing (respectively ingoing) radial null geodesics, that is to say such that

$$\frac{dr_*}{dr} = F(r)^{-1}.$$ (2.3)

The coordinate system $(t, r_*, \theta, \phi)$ is called Regge-Wheeler coordinates. $r_*$ is given by:

$$r_*(r) = \ln \left(\frac{r - r_{SAdS}}{r - r_{SAdS}}\right)^{\alpha_1} \left(r^2 - r_{SAdS}^2 + r_{SAdS}^2 + l^2 - \frac{2r}{M}\right)$$

$$+ \frac{l^2 (3r_{SAdS}^2 + 2l^2)}{(3r_{SAdS}^2 + l^2) (3r_{SAdS}^2 + 4l^2)^2} \arctan \left(\frac{2r + r_{SAdS}}{(3r_{SAdS}^2 + 4l^2)^{1/2}}\right).$$ (2.4)

where:

$$\alpha_1 = \frac{r_{SAdS}}{3r_{SAdS}^2 + l^2} = \frac{1}{2\kappa}.$$ (2.5)

For a static black hole (which is the case here), $\kappa$ is the limit at the horizon (that is to say at $r = r_{SAdS}$) of the force we need to exercise in order to keep an object of mass one at rest at infinity. We call it the surface gravity.

We note:

$$C = \frac{l^2 (3r_{SAdS}^2 + 2l^2)}{(3r_{SAdS}^2 + l^2) (3r_{SAdS}^2 + 4l^2)^{1/2}}.$$ (2.6)

We thus obtain $\lim_{r \to r_{SAdS}} r_*(r) = -\infty$ and $\lim_{r \to \infty} r_*(r) = C \frac{r}{2}$. We will consider the coordinate $x = r_* - C \frac{r}{2}$ rather than $r_*$. We thus have:

$$\lim_{r \to r_{SAdS}} x(r) = -\infty$$ (2.7)

$$\lim_{r \to \infty} x(r) = 0.$$ (2.8)

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This limit proves to us that, along radial null geodesic, a particle goes to timelike infinity in finite Boyer-Lindquist time (recall that along these geodesic, \( t - r_* \) and \( t + r_* \) are constants). This geometric property will be a major issue in our problem. This implies that our space-time is not globally hyperbolic, so that we cannot use the result of Leray concerning the global existence of solution of hyperbolic equations. A similar situation has been encountered by A.Bachelot in his article \([4]\) concerning the Dirac equation on Anti-de Sitter space-time. We expect to do a similar study concerning the self-adjoint extension.

2.2 The Dirac equation on Schwarzschild Anti-de Sitter space-time

In the two components spinor notation, the Dirac equation takes the following form:

\[
\begin{align*}
\nabla^{AA'} \phi_A &= \mu \chi_A' \\
\nabla^{AA'} \chi_A &= \mu \phi_A
\end{align*}
\]

or also

\[
\begin{align*}
\nabla_{AA'} \phi_A &= -\mu \chi_A' \\
\nabla_{AA'} \chi_A &= -\mu \phi_A
\end{align*}
\]

where \( \nabla_{AA'} \) is the Levi-Civita connection, \( \phi^A \) is a two-spinor, \( \mu = \frac{\sqrt{2}}{m} \) and \( m \geq 0 \) is the mass of the field.

Thanks to Newman-Penrose formalism, we can obtain the equation in the form of a system of partial differential equations. In this formalism, we introduce a null tetrad \((l^a, n^a, m^a, \bar{m}^a)\), that is

\[
l_a n^a = m_a \bar{m}^a = l_a m^a = n_a \bar{n}^a = 0,
\]

which is a basis of the complexified of the tangent space. We’ll say that the tetrad is normalized if:

\[
l_a n^a = 1 \quad m_a \bar{m}^a = -1.
\]

The two vectors \( l^a \) and \( n^a \) correspond to the directions along which the light goes to infinity (we can choose \( l^a \) as an outgoing null vector and \( n^a \) as an ingoing null vector). The vector \( m^a \) admits bounded integral curves. The vectors \( m^a \) and \( \bar{m}^a \) will generate rotations. In our case, we will consider:

\[
\begin{align*}
l^a \partial x_a &= \frac{1}{\sqrt{2}} F(r)^{-\frac{1}{2}} (\partial_t + \partial_x) \\
n^a \partial x_a &= \frac{1}{\sqrt{2}} F(r)^{-\frac{1}{2}} (\partial_t - \partial_x) \\
m^a \partial x_a &= \frac{1}{\sqrt{2r}} \left( \partial_{\theta} - \frac{i}{\sin \theta} \partial_{\varphi} \right) \\
\bar{m}^a \partial x_a &= \frac{1}{\sqrt{2r}} \left( \partial_{\theta} + \frac{i}{\sin \theta} \partial_{\varphi} \right).
\end{align*}
\]

We remark that this tetrad is normalized and since \( t \pm x \) is constant along null geodesics, the vector \( l^a \partial x_a \) and \( n^a \partial x_a \) are null. Moreover, using the equation of radial null geodesics with \( \lambda \) as our affine parameter, we deduce that \( \frac{dt}{dr} = \frac{d\lambda}{dr} = F(r)^{-1} \) which gives us an outgoing real null vector. We see as well that \( m^a \) is linked to rotations. We give the associated dual
Using this tetrad, it is then possible to decompose the covariant derivative in directional derivatives along these directions. We introduce the following symbols:

vectors:

\[
l_a dx^n = \frac{1}{\sqrt{2}} F(r)^{\frac{1}{2}} (dt - dx)
\]

\[
n_a dx^n = \frac{1}{\sqrt{2}} F(r)^{\frac{1}{2}} (dt + dx)
\]

\[
m_a dx^n = \frac{r}{\sqrt{2}} (-d\theta + i \sin(\theta) d\varphi)
\]

\[
m_m dx^n = \frac{r}{\sqrt{2}} (-d\theta - i \sin(\theta) d\varphi).
\]

Using this tetrad, it is then possible to decompose the covariant derivative in directional derivatives along these directions. We introduce the following symbols:

\[
D = \ell^a \nabla_a, \quad D^n = n^a \nabla_a, \quad \delta = m^n \nabla_a, \quad \delta' = \bar{m}^a \nabla_a.
\]

We have twelve spin coefficients that are defined by the following expressions:

\[
k = m^a Dl_a, \quad \rho = m^a \delta l_a, \quad \sigma = m^a \delta l_a, \quad \tau = m^a \delta l_a,
\]

\[
\epsilon = \frac{1}{2} \left( n^a Dl_a + m^a D\bar{n}_a \right), \quad \alpha = \frac{1}{2} \left( n^a \delta l_a + m^a \delta \bar{m}_a \right),
\]

\[
\beta = \frac{1}{2} \left( n^a \delta l_a + m^a \delta \bar{m}_a \right), \quad \gamma = \frac{1}{2} \left( n^a \delta l_a + m^a \delta \bar{m}_a \right),
\]

\[
\pi = -m^a D\bar{n}_a, \quad \lambda = -m^a \delta \bar{n}_a, \quad \mu = -m^a \delta \bar{n}_a, \quad \nu = -m^a \delta \bar{m}_a,
\]

where \( \hat{k} \) is the spin coefficient usually denoted \( k \), since \( k \) is the surface gravity in our convention. We can now give the equation (2.10) as a system of partial differential equations.

These equations act on the components of the spinor \( \phi^A, \chi^\Lambda \) in a normalized spinorial basis \( (\varrho^A, \epsilon^A) \) (that is such that \( o_A \epsilon^A = 1 \)). To choose our spinorial basis, we use the null tetrad we just defined. Indeed, we can define the spinorial basis \( (\varrho^A, \epsilon^A) \), uniquely up to an overall sign, using the following conditions:

\[
o^A \varrho^B = \ell^A, \quad \epsilon^A \epsilon^A = n^A, \quad o^A \epsilon^A = m^A, \quad \epsilon^A \varrho^A = \bar{m}^A, \quad o_A \epsilon^A = 1.
\]

The dual basis is given by:

\[
c_0^A = -\epsilon_A; \quad c_A = o_A. \quad (2.13)
\]

Let \( \phi^0, \phi^1, \chi^0, \chi^1 \) such that \( \phi^A = \phi^0 \varrho^A + \phi^1 \epsilon^A \) and \( \chi^A = \chi^0 \varrho^A + \chi^1 \epsilon^A \) where \( (\varrho^A, \epsilon^A) \) is the conjugate basis of \( (o^A, \epsilon^A) \).

The components of \( \phi_A \) and \( \chi_A \) are respectively:

\[
\phi_0 = -\phi^1, \quad \phi_1 = \phi^0
\]

\[
\chi^0 = -\chi^1, \quad \chi^1 = \chi^0.
\]

We obtain the following system of partial differential equations:

\[
\begin{align*}
\ell^a \partial_x \phi_1 - m^a \partial_x \phi_0 + (\epsilon - \rho) \phi_1 - (\pi - \alpha) \phi_0 &= m^a \chi^1 \\
m^a \partial_x \phi_0 - n^a \partial_x \phi_0 + (\sigma - \delta) \phi_1 - (\mu - \gamma) \phi_0 &= -m^a \chi^0 \\
m^a \partial_x \chi^0 - n^a \partial_x \chi^0 + (\epsilon - \bar{\rho}) \chi^0 + (\bar{\pi} - \bar{\alpha}) \chi^0 &= m^a \phi_0 \\
m^a \partial_x \chi^1 + n^a \partial_x \chi^1 + (\bar{\sigma} - \bar{\delta}) \chi^1 + (\bar{\mu} - \bar{\gamma}) \chi^1 &= -m^a \phi_0.
\end{align*}
\]

Using the following 4-component spinor:

\[
\psi = \begin{pmatrix}
\phi_0 \\
\phi_1 \\
\chi^0 \\
\chi^1
\end{pmatrix}, \quad (2.15)
\]

8
we obtain:
\[
\left( \partial_t + \gamma^0 \gamma^1 \left( F(r) \partial_r + \frac{F(r)}{r} + \frac{F'(r)}{4} \right) + \frac{F(r)^2}{r} B_{S^2} + i m \gamma^0 F(r) \right) \psi = 0.
\]
(2.16)

where \( m \) is the mass of the field and \( B_{S^2} \) is the Dirac operator on the sphere. In the coordinate system given by \((\theta, \varphi) \in [0; 2\pi] \times [0; \pi]\), we obtain: \( B_{S^2} = \gamma^0 \gamma^2 (\partial_\theta + \frac{1}{r} \cot \theta) + \gamma^0 \gamma^3 \frac{1}{r} \partial_\varphi \) where singularities appear, but we just have to change our chart in this case.

We will now work in these coordinates. Recall that Dirac matrices \( \gamma^\mu \), \( 0 \leq \mu \leq 3 \), unique up to unitary transform, are given by the following relations:
\[
\gamma^0 = \gamma^0 \quad \gamma^j = -\gamma^j, \quad 1 \leq j \leq 3 \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu \nu} 1, \quad 0 \leq \mu, \nu \leq 3.
\]
(2.17)

In our representation, the matrices take the form:
\[
\gamma^0 = i \left( \begin{array}{cc} 0 & \sigma^0 \\ -\sigma^0 & 0 \end{array} \right), \quad \gamma^k = i \left( \begin{array}{cc} 0 & \sigma^k \\ \sigma^k & 0 \end{array} \right), \quad k = 1, 2, 3
\]
(2.18)

where the Pauli matrices are given by:
\[
\sigma^0 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad \sigma^1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma^2 = \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right), \quad \sigma^3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -i \end{array} \right).
\]
(2.19)

We thus obtain:
\[
\gamma^0 \gamma^1 = \left( \begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right) \quad ; \quad \gamma^0 \gamma^2 = \left( \begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) ;
\]
\[
\gamma^0 \gamma^3 = \left( \begin{array}{cccc} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{array} \right) \quad ; \quad \gamma^0 = \left( \begin{array}{cccc} 0 & 0 & i & 0 \\ 0 & 0 & i & 0 \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{array} \right).
\]
(2.20)

We introduce the matrix:
\[
\gamma^5 = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3
\]
(2.21)

which satisfies the relations:
\[
\gamma^5 \gamma^\mu + \gamma^\mu \gamma^5 = 0, \quad 0 \leq \mu \leq 3.
\]
(2.22)

We introduce the change of spinor \( \tilde{\psi}(t, r, \theta, \varphi) = r F(r) \frac{d}{dr} \psi(t, r, \theta, \varphi) \) and obtain the following equation:
\[
\left( \partial_t + \gamma^0 \gamma^1 (r F(r) \partial_r + \frac{F(r)}{r} B_{S^2} + i m \gamma^0 F(r) \right) \tilde{\psi} = 0.
\]

Then we set \( \phi(t, x, \theta, \varphi) := \tilde{\psi}(t, r, \theta, \varphi) \), this gives:
\[
\left( \partial_t + \gamma^0 \gamma^1 \partial_x + \frac{F(r)}{r} B_{S^2} + i m \gamma^0 F(r) \right) \phi = 0.
\]

Finally, the equation takes the form:
\[
\partial_t \phi = i \gamma^0 \gamma^1 \partial_x + i \frac{F(r)}{r} B_{S^2} - m \gamma^0 F(r) \phi
\]
(2.23)
and we set:
\[
H_m = i \gamma^0 \gamma^1 \partial_x + i \frac{F(r)}{r} \partial_{\mathcal{E}^2} - m \gamma^0 F(r) \hat{\mathcal{E}}.
\]  
(2.24)

We introduce the Hilbert space:
\[
\mathcal{H} := [L^2([-\infty, 0]_x \times S^2, dx d\omega)]^4.
\]  
(2.25)

### 2.3 Generalization

Let \( q \in \mathbb{R} \) and \( n \in \mathbb{N} \), and define the spaces \( T^{q,n} \) by:
\[
T^{q,n} = \left\{ f \in C^\infty([-\infty, 0]) \mid \forall \alpha \in \mathbb{N}, |\partial_x^\alpha f(x)| \lesssim \begin{cases} e^{qx}, & \text{when } x \to -\infty \\ (-x)^n, & \text{when } x \to 0 \end{cases} \right\}.
\]  
(2.26)

We consider two smooth functions \( A_0, B_0 \) such that:
\[
A_0 = \begin{cases} 0 & \text{if } x \leq -2 \\ \frac{1}{x} & \text{if } x \geq -1 \end{cases}, \\
B_0 = \begin{cases} 0 & \text{if } x \leq -2 \\ -x & \text{if } x \geq -1 \end{cases}.
\]

We will consider the following operator:
\[
H_m = \Gamma^1 D_x + A(x) D_{\mathcal{E}^2} - m \gamma^0 B(x)
\]  
(2.27)

where \( m \) is the mass of the field and:
\[
\begin{align*}
A - A_0 & \in T_{\beta,2}^1, \\
B - B_0 & \in T_{\beta,1}^1
\end{align*}
\]  
(2.28)

where \( \beta, \beta \) are positive numbers. We also recall that \( \Gamma^1 = -\gamma^0 \gamma^1 = \text{diag}(1, -1, -1, 1) \) and \( D_x = \frac{i}{\gamma} \partial_x \).

We then check that the Schwarzschild Anti-de Sitter case enters in our abstract model. We introduce the constant:
\[
D_4 = e^{-2\kappa C \arctan \left( \frac{\sqrt{3} r_{SAdS}}{(\sqrt{3} r_{SAdS})^2 + 4\kappa^2} \right)} + C \pi e^\kappa.
\]

Then for \( x \) going to \(-\infty\), we have:
\[
r - r_{SAdS} = (3r_{SAdS}^2 + l^2)^\frac{1}{2} D_4 e^{2\kappa x} - C_1 D_4^2 e^{4\kappa x} + o(e^{4\kappa x})
\]  
(2.30)

where \( C_1 \) is a constant. Moreover, for \( x \) in a neighbourhood of 0, we have:
\[
r = -\frac{l^2}{x} + \frac{1}{3} (x) + o(-x).
\]  
(2.31)

So, for \( x \) going to \(-\infty\), we obtain:
\[
F(r)\frac{1}{r} = \frac{(3r_{SAdS}^2 + l^2)^\frac{1}{2} D_4^\frac{1}{2}}{r_{SAdS}^2} e^{\kappa x} + C_2 e^{3\kappa x} + o(e^{3\kappa x}),
\]  
(2.32)

\[
F(r)\frac{1}{r} = \frac{(3r_{SAdS}^2 + l^2)^\frac{1}{2} D_4^\frac{1}{2}}{r_{SAdS}^2} e^{\kappa x} + C_3 e^{3\kappa x} + o(e^{3\kappa x}).
\]  
(2.33)
where \( C_2, C_3 \) are constants. Then, in a neighbourhood of 0, we have:

\[
F(r)^{\frac{1}{2}} = -\frac{l}{x} - \frac{x}{6l} + o(x) \\
F(r)^{\frac{1}{2}} = -\frac{l}{x} + \frac{x^2}{2l^2} + o(x^2).
\]

We thus obtain:

\[
\begin{align*}
\frac{F(r)^{\frac{1}{2}}}{r} - A_0 & \in T^{n,2} \\
F(r)^{\frac{1}{2}} - B_0 & \in T^{n,1}.
\end{align*}
\]

(2.34)

This proves that our model is, in some sense, a generalization of the Schwarzschild Anti-de Sitter model.

3 Study of the hamiltonian

In this section, we first present the spinoidal spherical harmonics. This allows us to reduce our problem to the study of a 1 + 1 dimensional equation with a new hamiltonian denoted \( H_{s,n} \). We then use the fact that, at AdS infinity, the potential looks like the one considered by A. Bachelot in [4]. By means of a unitary transform and a cut-off near AdS infinity, we are able to make use of his result and obtain the asymptotic behaviour of the elements in the natural domain of our operator. As in [4], the need or not to put a boundary condition is linked to the comparison between the mass of the field and the cosmological constant. For \( 2ml \geq 1 \) (where \( m \) is the mass of the field and \( l \) is linked to the cosmological constant), there’s no need to put boundary conditions. When \( 2ml < 1 \), we consider the MIT boundary condition in order to determine the dynamics uniquely. We then prove the self-adjointness of our operators. Using an elliptic inequality, we are able to give the domain of our operator for \( 2ml > 1 \). Using Stone’s theorem, we can solve the Cauchy problem for our equation. At last, we give a proof of the absence of eigenvalue for all \( m > 0 \) which will be useful for the propagation estimates.

3.1 Description of the domain

3.1.1 The spinoidal spherical harmonics

In the rest of this paper, we will often make use of spinoidal spherical harmonics (we can refer to [4] for a more complete presentation of these harmonics) which will permit us to decompose \( \mathcal{H} \) as follows:

\[
\mathcal{H} = \bigoplus_{(s,n) \in I} \mathcal{H}_{s,n}
\]

(3.1)

where:

\[
I := \left\{ (s,n); \ s \in \mathbb{N} + \frac{1}{2}, \ n \in \mathbb{Z} + \frac{1}{2}, \ s - |n| \in \mathbb{N} \right\}
\]

(3.2)

and:

\[
\mathcal{H}_{s,n} = (L^2(x, dx))^4 \otimes Y_{s,n}
\]

(3.3)
where $Y_{s,n}$ is the space spanned by:

\[
\begin{pmatrix}
T_{\frac{s}{2},n} \\
T_{\frac{s-1}{2},n} \\
T_{\frac{s+1}{2},n} \\
T_{\frac{s+2}{2},n}
\end{pmatrix}.
\] (3.4)

Let us first remark that $T_{\frac{s}{2},n}(\theta, 2\pi) = -T_{\frac{s}{2},n}(\theta, 0) \neq 0$. Consequently, these functions are not smooth on the sphere $S^2$. Moreover, these functions satisfy the following relations:

\[
\frac{\partial}{\partial \theta} T_{\frac{s}{2},n} + \frac{1}{2 \tan \theta} T_{\frac{s}{2},n} = \pm \left( s + \frac{1}{2} \right) T_{\frac{s}{2},n},
\] (3.5)

\[
\frac{\partial}{\partial \phi} T_{\frac{s}{2},n} = -\frac{in}{2} T_{\frac{s}{2},n}.
\] (3.6)

We can decompose every function $f \in L^2(S^2)$ as follows:

\[
f(\theta, \phi) = \sum_{(s,n) \in I} u_{s,n}(f) T_{\frac{s}{2},n}(\theta, \phi), \quad u_{s,n}(f) \in \mathbb{C}.
\]

(3.7)

We can refer to [4] for a more complete presentation of these spaces. Nevertheless, we give some properties of these spaces which could be useful. We have:

\[
d \geq 0 \implies W^d_{\pm} = \left\{ f \in L^2(S^2) \mid ||f||_{W^d_{\pm}} < \infty \right\},
\]

and the dual space of $W^d_{\pm}$ can be identified with $W^{-d}_{\pm}$:

\[
d \in \mathbb{R}, \quad \left( W^d_{\pm} \right)' = W^{-d}_{\pm}.
\]

Moreover, $C^0_\alpha ([0, \pi] \times [0, 2\pi])$ is included in $W^d_{\pm}$.

In correspondence with the decomposition (3.8), we introduce the Hilbert spaces:

\[
W^d = W^d_+ \times W^d_+ \times W^d_+ \times W^d_+
\] (3.8)

equipped with the norm:

\[
||\Phi||_{W^d}^2 = \sum_{j=1}^{4} \sum_{(s,n) \in I} \left( s + \frac{1}{2} \right)^{2d} |u_{j,n}|^2
\] (3.9)
where:

\[
\Phi (\theta, \varphi) = \sum_{(s,n) \in I} \begin{pmatrix}
    u_{1(n)T_s}^{1,2,n}(\theta, \varphi) \\
    u_{2(n)T_s}^{1,2,n}(\theta, \varphi) \\
    u_{3(n)T_s}^{1,2,n}(\theta, \varphi) \\
    u_{4(n)T_s}^{1,2,n}(\theta, \varphi)
\end{pmatrix}.
\]

### 3.1.2 A result due to A.Bachelot

We recall a result obtained by A.Bachelot (see [4]). In his article, the hamiltonian considered was of the following form:

\[
H_{\text{B m}} = i\gamma_0^B \gamma_1^B \left( F_B(r) \partial_r + \frac{F_B'(r)}{r} + \frac{F_B(r)}{4} \right) + \frac{F_B(r)\gamma_2^B}{r} - m\gamma_0^B F_B(r) \gamma_3^B
\]

in \((r, \theta, \varphi)\) coordinates where \(F_B(r) = 1 + \frac{r^2}{l^2}\). Using a change of spinor and a change of coordinates, he obtains:

\[
H_{\text{B m}} := i\gamma_0^B \gamma_1^B \left( \frac{\partial}{\partial \zeta} + \frac{i}{\sin \zeta} \left[ \gamma_0^B \gamma_2^B \left( \frac{\partial}{\partial \theta} + \frac{1}{2 \tan \theta} \right) + \frac{1}{\sin \theta} \gamma_0^B \gamma_3^B \frac{\partial}{\partial \varphi} \right] - \frac{m}{\cos \zeta} \right)
\]

where the domain is given by:

\[
D(H_{\text{B m}}) := \{ \Phi \in L^2; H_{\text{B m}} \Phi \in L^2 \}.
\]

Here, we have \(m = \tilde{m} \sqrt{\frac{3}{2}}\) with \(\tilde{m}\) the mass of the field and \(\Lambda\) the cosmological constant. Moreover, he defines the space \(L^2\) by

\[
L^2 := L^2 ([0, \frac{\pi}{2}] \times [0, \pi] \times [0, 2\pi]) \times [0, \sin \theta d\zeta d\theta d\varphi].
\]

At last, we recall that the Dirac matrices \(\gamma_B, \gamma_{1B}, \gamma_{2B}, \gamma_{3B}\) take the form:

\[
\gamma_B = \begin{pmatrix} I_0 & 0 \\ 0 & -I_0 \end{pmatrix}, \quad \gamma_{kB} = \begin{pmatrix} 0 & \sigma_k^B \\ -\sigma_k^B & 0 \end{pmatrix}, \quad k = 1, 2, 3
\]

where the Pauli matrices are given by:

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1^B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2^B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3^B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]

The result is then the following (see Theorem V.1 in [4]):

**Theorem 3.1.** For all \(\Phi \in D(H_{\text{B m}})\), we have:

\[
\Phi \in C^0 \left( [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \right), \quad \|\Phi(\zeta,)\|_{W^{\frac{1}{2}}} = O(\sqrt{\zeta}), \quad \zeta \to 0,
\]

and for \(m > 0\), we have

\[
\int_0^\frac{\pi}{2} \|\Phi(\zeta,)\|_{W^{\frac{1}{2}}}^2 \frac{d\zeta}{\sin \zeta} \leq \|H_m \Phi\|_{L^2}^2.
\]

For \(m > \frac{1}{2}\), we have

\[
\|\Phi(\zeta,)\|_{L^2} = O \left( \sqrt{\frac{\pi}{2} - \zeta} \right), \quad \zeta \to \frac{\pi}{2}.
\]
For $m = \frac{1}{2}$, we have

$$||\Phi(\zeta, \cdot)||_{L^2(S^2)} = O \left( \sqrt{\frac{\pi}{2}} \ln \left( \frac{\pi}{2} - \zeta \right) \right), \quad \zeta \to \frac{\pi}{2}.$$  \hspace{1cm} (3.18)

For $0 < m < \frac{1}{2}$, there exist functions $\psi_- \in W^m_{\frac{3}{2}}, \chi_- \in W^m_{\frac{5}{2}}, \psi_+, \chi_+ \in L^2(S^2)$ and $\phi \in C^0 \left( [0, \frac{\pi}{2}]; L^2(S^2; C^4) \right)$ satisfying

$$\Phi(\zeta, \theta, \varphi) = \left( \frac{\pi}{2} - \zeta \right)^{-m} \begin{pmatrix}
\psi_-(\theta, \varphi) \\
\chi_-(\theta, \varphi) \\
-i\psi_- (\theta, \varphi) \\
i\chi_- (\theta, \varphi)
\end{pmatrix} + \left( \frac{\pi}{2} - \zeta \right)^m \begin{pmatrix}
\psi_+ (\theta, \varphi) \\
\chi_+ (\theta, \varphi) \\
+i\psi_+ (\theta, \varphi) \\
-i\chi_+ (\theta, \varphi)
\end{pmatrix} + \phi(\zeta, \theta, \varphi),$$  \hspace{1cm} (3.19)

$$||\phi(\zeta, \cdot)||_{L^2(S^2)} = o \left( \sqrt{\frac{\pi}{2}} - \zeta \right), \quad x \to \frac{\pi}{2}. \hspace{1cm} (3.20)$$

Conversely, for all $\psi_- \in W^m_{\frac{3}{2}}, \chi_- \in W^m_{\frac{5}{2}}, \psi_+ \in W^m_{\frac{3}{2}}, \chi_+ \in W^m_{\frac{1}{2}}$ there exist $\Phi \in D(H_m^N)$ satisfying (3.19) and (3.20).

For $m = 0$, we have

$$\Phi \in C^0 \left( [0, \frac{\pi}{2}] ; W^{\frac{3}{2}} \right). \hspace{1cm} (3.21)$$

Remark. This result concerning the asymptotic behaviour of elements in the domain of the operator $H_m^N$ is first proved for fixed harmonics (i.e fixed $(s, n) \in I$). In the next sections, we will often make use of the result obtained for fixed harmonics.

This result is in fact due to the $L^2$ condition put to the states in the natural domain of our operator. When the mass is sufficiently large, the term $\left( \frac{\pi}{2} - \zeta \right)^{-m}$ in (3.19) is not in $L^2$ so it cannot appear in the development of the states near $\frac{\pi}{2}$. The states are thus going to 0 at $\frac{\pi}{2}$ and this indicates that we do not need to put boundary conditions to obtain the self-adjointness of this operator and well-posedness of the Cauchy problem.

Unfortunately, for a mass too small compared to the cosmological constant, we see that the term $\left( \frac{\pi}{2} - \zeta \right)^{-m}$ in (3.19) is in $L^2$ so the states are not going to 0 at $\frac{\pi}{2}$ which is problematic for the symmetry of our operator. We thus need to put boundary conditions to get rid of this term and solve the Cauchy problem.

### 3.1.3 Unitary transform of $H_m$

Let us introduce the following domains:

- If $m \geq \frac{1}{2}$:

  $$D(H_m) = \{ \phi \in \mathcal{H}; \ H_m \phi \in \mathcal{H} \}.$$  \hspace{1cm} (3.22)

- If $m < \frac{1}{2}$, we consider the operator equipped with the domain whose elements satisfy a generalized MIT-bag condition (where $\alpha \in \mathbb{R}$ is called the Chiral angle and $\gamma^5 = -i\gamma^\alpha \gamma^\beta \gamma^\gamma$ (see [1])):

  $$D(H_m) = \{ \phi \in \mathcal{H}; \ H_m \phi \in \mathcal{H}, \ \left\| \left( \gamma^1 + ie^{i\alpha} \gamma^5 \right) \phi \right\|_2 = o \left( \sqrt{\frac{\pi}{2}} \right), \ x \to 0 \}. \hspace{1cm} (3.23)$$
First, we’ll try to remove $\alpha$ in the case $m < \frac{1}{2}$. Indeed, the matrix $e^{i\alpha \gamma^5}$ is unitary and we have the following commutation relation:

$$e^{i\alpha \gamma^5} \gamma^1 = \gamma^1 e^{-i\alpha \gamma^5}. \quad (3.24)$$

Then, our boundary condition becomes:

$$\left\| \left( \gamma^1 + i e^{i\alpha \gamma^5} \right) \phi \right\|^2_2 = \left\| e^{-i \gamma^5} \left( \gamma^1 + i e^{i\alpha \gamma^5} \right) \phi \right\|^2_2 = \left\| \left( \gamma^1 + i \right) e^{i\gamma^5} \phi \right\|^2_2. \quad (3.25)$$

We introduce the following operator:

$$H_\alpha^m = e^{i\gamma^5} H_m e^{-i\gamma^5}. \quad (3.26)$$

We see that, for $m < \frac{1}{2}$, $\phi \in D(H_m)$ if and only if $e^{i\gamma^5} \phi \in D(H_\alpha^m)$ where:

$$D(H_\alpha^m) = \{ \phi \in \mathcal{H}; \ H_m \phi \in \mathcal{H}, \ \left\| \left( \gamma^1 + i \right) \phi \right\|^2_2 = o \left( \sqrt{-x} \right), \ x \to 0 \}. \quad (3.27)$$

So we can restrict to the case $\alpha = 0$. In the case $\alpha \neq 0$, we realize the unitary transform and remark that the proof of self-adjointness doesn’t change. In the following, we’ll suppose that $\alpha = 0$.

We will modify our hamiltonian in order to exploit the result of A.Bachelot. We take:

$$\tilde{t} = -t$$

as a new time variable (we will continue to denote it by $t$) which allows us to obtain the following equation:

$$\partial_t \phi = i (\tilde{H}_m) \phi. \quad (3.28)$$

Let:

$$\tilde{H}_m = \gamma_0^\mu P^{-1}(-H_m) P \gamma_0^\mu \quad (3.29)$$

where:

$$P = \frac{1}{\sqrt{2}} e^{i \pi} \left( \begin{array}{cc} \text{Id} & \text{Id} \\ -\text{Id} & \text{Id} \end{array} \right), \quad \text{Det}(P) = 1$$

$$P^{-1} = \frac{1}{\sqrt{2}} e^{-i \pi} \left( \begin{array}{cc} \text{Id} & \text{Id} \\ -\text{Id} & -\text{Id} \end{array} \right)$$

$$\gamma_0^\mu = \left( \begin{array}{cc} 0 & \text{Id} \\ \text{Id} & 0 \end{array} \right),$$

and Id is the identity matrix of order 2.

The matrix $P$ satisfies the following relations:

$$\gamma^0 = P \gamma_0^\mu P^{-1}; \quad \gamma^j = -P \gamma_0^\mu P^{-1}, \quad 1 \leq j \leq 3. \quad (3.30)$$

where the Dirac matrices are defined by (4.13) and (2.18). The matrix $\gamma_0^\mu$ satisfies:

$$\gamma_0^\mu \gamma_0^\nu = -\gamma_0^\nu \gamma_0^\mu, \quad \forall \ 0 \leq \mu \leq 3. \quad (3.31)$$

We obtain:

$$\tilde{H}_m = i \gamma_0^0 \gamma_0^1 \partial_x + i \gamma_0^0 \gamma_0^2 A(x) \left( \partial_x + \frac{1}{2} \cot \theta \right) + i \gamma_0^0 \gamma_0^3 A(x) \frac{1}{\sin \theta} \partial_x - m \gamma_0^0 \gamma_0 B(x). \quad (3.32)$$

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3.1.4 Asymptotic behaviour of elements of the domain

Using equations (2.16) and (3.10), we see that we have the same expression for the two operators. When \( r \gg r_{SAdS} \), we also remark that the functions \( F \) and \( F_B \) have the same behaviour.

We introduce the projection \( P_s,n \) from \( \mathcal{H} \) to the subspace \( \mathcal{H}_{s,n} \) and the operators \( \tilde{H}_{m}^{s,n,B} = \tilde{H}_{m}|_{\mathcal{H}_{s,n}} \), \( H_{m}^{s,n,B} = H_{m}|_{\mathcal{H}_{s,n}} \) for \((s,n) \in I\). We denote:

\[
\psi_{s,n} = P_{s,n}(\psi) = \begin{pmatrix}
\psi_{1,n}^s(x)T_{B_{s,n}}^+ \\
\psi_{2,n}(x)T_{B_{s,n}}^- \\
\psi_{3,n}(x)T_{B_{s,n}}^- \\
\psi_{4,n}(x) \end{pmatrix}.
\]

Furthermore, the domain of \( H_{m}^{s,n,B} \) is given by:

- If \( m \geq \frac{1}{r} \):
  \[
  D \left( H_{m}^{s,n,B} \right) = \left\{ \varphi_{s,n} \in \mathcal{H}_{s,n}; H_{m}^{s,n,B} \varphi_{s,n} \in \mathcal{H}_{s,n} \right\} \quad (3.31)
  \]

- If \( m < \frac{1}{r} \):
  \[
  D \left( H_{m}^{s,n,B} \right) = \left\{ \varphi_{s,n} \in \mathcal{H}_{s,n}; H_{m}^{s,n,B} \varphi_{s,n} \in \mathcal{H}_{s,n}, \right\}
  ||(\gamma_B + i)\varphi_{s,n}(x,-)||_{\mathcal{Y}_B} = o \left( \sqrt{-x} \right), \quad x \to 0 \} .
  \quad (3.32)
  
We then have the:

Lemma 3.2. Let \( \psi \in D \left( \tilde{H}_{m} \right) \) and \( \chi \in C_0^\infty \left( [-2\epsilon, 0] \right) \) such that \( \chi = 1 \) on \([-\epsilon, 0]\) with \( \epsilon > 0 \). Then \( \chi \psi \in D \left( H_{m}^{B} \right) \).

Proof. The operator obtained by A. Bachelot in [1] is the following:

\[
H_{m}^{B} = i\gamma_0^B \gamma_B F_B \left( r \right) \left( \partial_r + \frac{1}{r} + \frac{F_B^\prime \left( r \right)}{4F_B \left( r \right)} \right) + \frac{F_B \left( r \right)}{r} D_{S^2} - m \gamma_0^B F_B \left( r \right) \quad (3.33)
\]

where \( F_B \left( r \right) = 1 + \frac{r^2}{4} \). This operator has the same form as in (2.16). Moreover, when \( r \gg r_{SAdS} \), \( F_B \) and \( F \) have the same behaviour (\( F \) is defined by \( F \left( r \right) = 1 + \frac{r^2}{4} - \frac{2M}{r} \)). So, we make the change of variable \( r \to x \) where \( \frac{1}{r} = F \left( r \right)^{-1} \) where \( F \) is defined on \( ]r_{SAdS}, +\infty[ \).

\[
H_{m}^{B} = i\gamma_0^B \gamma_B \left( x \right) \partial_x + i\gamma_0^B \gamma_B \left( \frac{F \left( r \right)}{r} + \frac{F^\prime \left( r \right)}{4} \right) + \frac{3M}{2r^2} + A_B \left( x \right) D_{S^2} - m \gamma_0^B B_B \left( x \right) \quad (3.34)
\]

where \( r \) is understood as a function of \( x \) and:

\[
\begin{align*}
g \left( x \right) & = 1 + \frac{2M}{l^4} \left( -x \right)^3 + o \left( \left( -x \right)^3 \right) \\
A_B \left( x \right) & = \frac{1}{l} + \frac{1}{2l^3} \left( -x \right)^2 + o \left( \left( -x \right)^2 \right) \\
B_B \left( x \right) & = \frac{1}{l} + \frac{1}{l^3} \left( -x \right) + o \left( -x \right) \\
F \left( r \right) & = \frac{1}{r - x} + \frac{2}{3l^2} \left( -x \right) + o \left( -x \right) \\
F^\prime \left( r \right) & = \frac{2}{r - x} - \frac{2}{3l^2} \left( -x \right) + o \left( -x \right)
\end{align*}
\]
when \( x \) goes to 0. Since \( P_{s,n}(\chi \psi) = \chi \psi_{s,n} \), we have:

\[
H_{mn}^{s,n}P_{s,n}(\chi \psi) = g(x)\tilde{H}_{mn}^{s,n}P_{s,n}(\chi \psi) + \left( H_{mn}^{s,n}P_{s,n}(\chi \psi) - g(x)\tilde{H}_{mn}^{s,n}P_{s,n}(\chi \psi) \right)
\]

\[
= g(x)\tilde{H}_{mn}^{s,n}P_{s,n}(\chi \psi) + i\gamma_B \gamma_\psi \frac{F(r)}{r} \left( F(r) + \frac{F'(r)}{4} \right) (1 - g(x)) \psi_{s,n} + \frac{3M}{2r} \psi_{s,n} + \gamma_B (g_B(x) - g(x)) A(x) \left( s + \frac{1}{2} \right) \psi_{s,n}
\]

\[
- m\gamma_B (B_B(x) - g(x) B(x)) \psi_{s,n}
\]

(3.35)

Since \( \psi \in D(\tilde{H}_m) \), \( g \) is bounded in a neighborhood of 0 and \( \chi \in C_0^\infty (-1,1) \), the first term is in \( L^2(x, dx) \). Using the behaviour at 0 of \( g \), we obtain:

\[
A_B(x) - g(x) A(x) = o(-x)
\]

\[
B_B(x) - g(x) B(x) = o(-x)
\]

\[
\left( \frac{F(r)}{r} + \frac{F'(r)}{4} \right) (1 - g(x)) = o(-x)
\]

near 0. We deduce that:

\[
H_{mn}^{s,n}P_{s,n}(\chi \psi) \in \mathcal{H}_{s,n}
\]

(3.36)

In particular, \( \chi \psi_{s,n} \in D(H_{mn}^{s,n}P_{s,n}) \).

To be able to sum over \((s,n)\), we need to know that \((s + \frac{1}{2})^2 \| \psi_{s,n} \|^2_{L^2(x,dx)}\) is summable.

Since \( \psi \in D(\tilde{H}_m) \), there exists \( f \in \mathcal{H} \) such that:

\[
\tilde{H}_m \psi = f
\]

(3.37)

where \( f \) admits the decomposition:

\[
f = \sum_{(s,n) \in I} \begin{pmatrix} f_{1,n}(x) T^{s,n}_{+} (\theta, \phi) \\ f_{2,n}(x) T^{s,n}_{+} (\theta, \phi) \\ f_{3,n}(x) T^{s,n}_{-} (\theta, \phi) \\ f_{4,n}(x) T^{s,n}_{-} (\theta, \phi) \end{pmatrix}
\]

We obtain four differential equations and then we multiply them by \( \psi_{j,n}^s \) for \( j = 1, \ldots, 4 \) such that:

\[
i \chi \psi_{1,n}^s (\chi \psi_{3,n}^s)' + \left( s + \frac{1}{2} \right) A(x) |\chi \psi_{4,n}^s|^2 - B(x) \psi_{4,n}^s \chi \psi_{1,n}^s = \overline{\psi_{4,n}^s} f_{1,n}^s,
\]

\[
i \chi \psi_{2,n}^s (\chi \psi_{4,n}^s)' + \left( s + \frac{1}{2} \right) A(x) |\chi \psi_{3,n}^s|^2 - B(x) \psi_{3,n}^s \chi \psi_{2,n}^s = \overline{\psi_{3,n}^s} f_{2,n}^s,
\]

\[
i \chi \psi_{3,n}^s (\chi \psi_{1,n}^s)' + \left( s + \frac{1}{2} \right) A(x) |\chi \psi_{2,n}^s|^2 + B(x) \psi_{2,n}^s \chi \psi_{1,n}^s = \overline{\psi_{2,n}^s} f_{3,n}^s,
\]

\[
i \chi \psi_{4,n}^s (\chi \psi_{3,n}^s)' + \left( s + \frac{1}{2} \right) A(x) |\chi \psi_{1,n}^s|^2 + B(x) \psi_{1,n}^s \chi \psi_{3,n}^s = \overline{\psi_{1,n}^s} f_{4,n}^s.
\]

Adding these equations and taking the real part, we obtain:

\[
\frac{d}{dx} \Re \left( \chi \psi_{1,n}^s \chi \psi_{2,n}^s + \chi \psi_{3,n}^s \chi \psi_{4,n}^s \right) + \left( s + \frac{1}{2} \right) A(x) \sum_{j=1}^{4} |\psi_{j,n}^s|^2 = \Re \left( \chi \psi_{4,n}^s f_{1,n}^s + \chi \psi_{3,n}^s f_{2,n}^s + \chi \psi_{2,n}^s f_{3,n}^s + \chi \psi_{1,n}^s f_{4,n}^s \right).
\]

(3.38)
Using that:
\[
\lim_{x \to 0} \Im \left( \psi_1^{s,n}_x \psi_2^{s,n}_x + \psi_3^{s,n}_x \psi_4^{s,n}_x \right) = 0. 
\]  
(3.39)

and that \( \psi_j^{s,n} \) is 0 at 1 for all \( j = 1, \cdots, 4 \), we obtain:
\[
\left( s + \frac{1}{2} \right) \int_{-\frac{1}{2}}^{0} A(x) \sum_{j=1}^{4} |\psi_j^{s,n}|^2 \, dx = \int_{-\frac{1}{2}}^{0} \Re \left( \psi_1^{s,n} f_1^{s,n} + \psi_2^{s,n} f_2^{s,n} + \psi_3^{s,n} f_3^{s,n} + \psi_4^{s,n} f_4^{s,n} \right) \, dx.
\]

After some calculations, this gives:
\[
\left( s + \frac{1}{2} \right)^2 \int_{-\frac{1}{2}}^{0} (2A(x) - 1) \sum_{j=1}^{4} |\psi_j^{s,n}|^2 \, dx \leq \int_{-\frac{1}{2}}^{0} \sum_{j=1}^{4} |f_j^{s,n}|^2 \, dx.
\]

Using the asymptotic behaviour of \( A \) (see (3.45)), we can prove that \( 2A(x) - 1 \geq 1 \) on the support of \( \chi \) (for \( \epsilon \) sufficiently small). Finally, we obtain:
\[
\left( s + \frac{1}{2} \right)^2 \int_{-\frac{1}{2}}^{0} \sum_{j=1}^{4} |\psi_j^{s,n}|^2 \, dx \leq C \int_{-\frac{1}{2}}^{0} \sum_{j=1}^{4} |f_j^{s,n}|^2 \, dx.
\]

and the right hand side is summable because \( f \in \mathcal{H} \). This gives the lemma. Q.E.D.

We can now apply Theorem 5.1 to \( \psi \) and obtain the asymptotic behaviour of \( \psi \):

**Proposition 3.3.** If \( m > \frac{1}{27} \), we have:
\[
\| \psi(x, \cdot) \|_{L^2(S^2)} = O \left( \sqrt{x} \right), \quad x \to 0.
\]  
(3.41)

If \( m = \frac{1}{27} \), we have:
\[
\| \psi(x, \cdot) \|_{L^2(S^2)} = O \left( \sqrt{\ln(x)} \right), \quad x \to 0.
\]  
(3.42)

If \( 0 < m < \frac{1}{27} \), there exists functions \( \psi_- \in W^{\frac{1}{2}}_+, \chi_- \in W^{\frac{1}{2}}_+, \psi_+ \in L^2(S^2) \) and \( \phi \in C^0 \left( [0, \infty) \right) \cap L^2(S^2; C^0) \) satisfying
\[
\psi(x, \theta, \phi) = (-x)^{-\frac{1}{2}} \begin{pmatrix} \psi_- (x, \theta, \phi) \\ -x \psi_- (x, \theta, \phi) \\ i\chi_- (x, \theta, \phi) \end{pmatrix} + (-x)^{\frac{1}{2}} \begin{pmatrix} \psi_+ (x, \theta, \phi) \\ \chi_+ (x, \theta, \phi) \\ -i\chi_+ (x, \theta, \phi) \end{pmatrix}
\]  
\[+ \phi(x, \theta, \phi), \quad (3.43)\]
\[
\| \phi(x, \cdot) \|_{L^2(S^2)} = o \left( \sqrt{x} \right), \quad x \to 0.
\]  
(3.44)

Conversely, for all \( \psi_- \in W^{\frac{1}{2}+m}_+, \chi_- \in W^{\frac{1}{2}+m}_+, \psi_+ \in W^{\frac{1}{2}-m}_+, \chi_+ \in W^{\frac{1}{2}-m}_+ \), there exists \( \psi \in D(H_m) \) satisfying \( 3.13 \) and \( 3.14 \).

**Remark.** By restriction to \( H_{a,n} \), we obtain the same result for \( s, n \) fixed. Moreover, if \( \varphi_{s,n} \in D \left( H^{m}_{a,n} \right) \), then \( \varphi_{s,n} \in H^1 \left( [-\infty, -c] \right) \) where \( c > 0 \) is an arbitrary constant. We conclude that \( \varphi_{s,n} \in C^0 \left( [-\infty, -c] \right) \cap L^2 \left( [-\infty, -c] \right) \) and:
\[
\| \varphi_{s,n} (x, \cdot) \|_{W^0} \to 0, \quad x \to -\infty.
\]  
(3.45)
3.1.5 Description of the domain

We now give a description of the domain of $H_m$ for fixed $(s, n) \in I$. Recall that $H_m$ and $\bar{H}_m$ are linked by a unitary transform, so it does not change the norm of the observables. We obtain:

- If $2ml \geq 1$, then
  \[ D(H_m^{s,n}) = \{ \psi_{s,n} \in \mathcal{H}_{s,n}; \ H_m^{s,n}\psi_{s,n} \in \mathcal{H}_{s,n} \} \]  
  \hspace{1cm} (3.46)

- If $2ml < 1$, then
  \[ D(H_m^{s,n}) = \{ \psi_{s,n} \in \mathcal{H}_{s,n}; \ H_m^{s,n}\psi_{s,n} \in \mathcal{H}_{s,n}, \ \| (\gamma^1 + i) \psi_{s,n}\|_{\mathcal{H}_{s,n}} = o(\sqrt{-x}), \ x \to 0 \} \]
  \hspace{1cm} (3.47)

3.2 Self-adjointness for fixed harmonic

In this section, $s$ and $n$ are fixed.

3.2.1 The case $2ml \geq 1$

**Lemma 3.4 (Elliptic estimate).** We suppose that $2ml > 1$. Then, there exists a constant $C > 0$ such that, for all $\varphi \in C_0^\infty([-\infty, 0])$, we have:

\[ \| -iD_x \varphi \|^2 \leq C \left( \| H_m^{s,n} \varphi \|^2 + \| \varphi \|^2 \right) \]  
  \hspace{1cm} (3.48)

**Proof.** We write $D_x = -i\partial_x$ and $\Gamma^1 = -\gamma^0\gamma^1$. Recall that:

\[ H_m^{s,n} = \Gamma^1 D_x + \left( s + \frac{1}{2} \right) A(x) \gamma^0 \gamma^2 - mB(x) \gamma^0. \]

We will often denote $V(x) = \left( s + \frac{1}{2} \right) A(x) \gamma^0 \gamma^2 - mB(x) \gamma^0$. Choose a partition of unity $\chi_1, \chi_2$ such that $\chi_1 + \chi_2 = 1$, $\text{supp}(\chi_1) \subseteq [-\infty, -\epsilon]$ and $\text{supp}(\chi_2) \subseteq [-\epsilon, 0]$ and $\chi_1 = 1$ on $[0, +\infty)$, $\epsilon > 0$ sufficiently small so that, if $\gamma_0^5$ and $P$ are unitary matrices defined as in (A.27), $\gamma_0^5 P^{-1} \chi_2 \varphi \in D(H_m^{s,n})$ when $\varphi \in D(H_m^{s,n})$ (it is possible by lemma 3.2). Recall that $m$ is the mass of the field and $l$ correspond to the cosmological constant. Using equation III.32 in theorem III.1.4 of [3], (3.27) and (3.34), we obtain:

\[ \| D_x (\gamma_0^5 P^{-1} \chi_2 \varphi) \| \leq C_{m,l} \| H_m^{s,n} (\gamma_0^5 P^{-1} \chi_2 \varphi) \| \]
\[ \leq C_{m,l} \left( \left\| (H_m^{s,n} - g(x) \bar{H}_m^{s,n}) (\gamma_0^5 P^{-1} \chi_2 \varphi) \right\| + \left\| g(x) \bar{H}_m^{s,n} (\gamma_0^5 P^{-1} \chi_2 \varphi) \right\| \right) \]
\[ \leq C_{m,l} \| \chi_2 \varphi \|^2 + C_{m,l} \| g(x) \gamma_0^5 P^{-1} (-H_m^{s,n}) P \gamma_0^5 (\gamma_0^5 P^{-1} \chi_2 \varphi) \| \]
\[ \leq C_{m,l} \| g(x) H_m^{s,n} (\chi_2 \varphi) \| + C_{m,l} \| \chi_2 \varphi \| , \]

where $C_{m,l}$ and $\tilde{C}_{m,l}$ are constants depending on $m$ and $l$. Since $\gamma_0^5 P^{-1}$ is unitary and commute with $D_x$ and $g$ is bounded near 0, we obtain:

\[ \| D_x (\chi_2 \varphi) \| \leq C_{m,l,s} \| H_m^{s,n} (\chi_2 \varphi) \| + \tilde{C}_{m,l} \| \chi_2 \varphi \|. \]  
  \hspace{1cm} (3.49)
On the other hand, we have:

\[ ||D_x (\chi_1 \varphi)|| = ||\Gamma^1 D_x (\chi_1 \varphi)|| \leq ||(\Gamma^1 D_x - H_{m,n}^s) (\chi_1 \varphi)|| + ||H_{m,n}^s (\chi_1 \varphi)|| \leq ||H_{m,n}^s (\chi_1 \varphi)|| + C_{V,\epsilon} ||\varphi|| \]

where \( C_{V,\epsilon} \) is a constant depending on the bound for \( V \) on \( ]-\infty, -\epsilon[ \). Since \( \chi_1, \chi_2 \) commute with \( V \), we obtain:

\[ ||D_x \varphi||^2 \leq C \left ( ||D_x (\chi_1 \varphi)||^2 + ||D_x (\chi_2 \varphi)||^2 \right ) \leq C \left ( ||H_{m,n}^s (\chi_1 \varphi)||^2 + ||H_{m,n}^s (\chi_2 \varphi)||^2 \right ) + C' ||\varphi||^2 \]

\[ \leq C \left ( ||\chi_1 H_{m,n}^s \varphi||^2 + ||\chi_2 H_{m,n}^s \varphi||^2 \right ) + \bar{C} \left ( ||(-i\Gamma^1 \chi_1) \varphi||^2 + ||(-i\Gamma^1 \chi_2) \varphi||^2 \right ) + C' ||\varphi||^2 \]

The constants may change from line to line in these inequalities. This gives the result.

Q.E.D

Proposition 3.5. For \( 2ml \geq 1 \), the operator \( \tilde{H}_{m,n}^s \) is essentially self-adjoint on \( C_0^\infty (]-\infty, 0[ ) \). Moreover, if \( 2ml > 1 \), the domain of this operator is given by \( H_0^1 (]-\infty, 0[ ) \).

Proof. Recall that:

\[ \tilde{H}_{m,n}^s = i\gamma_B^0 \gamma_B \partial_x + \gamma_B^0 \gamma_B \left ( s + \frac{1}{2} \right ) A(x) - m\gamma_B^0 B(x) \]

with domain \( D \left ( \tilde{H}_{m,n}^s \right ) = \{ \psi_{s,n} \in H_{s,n}; \tilde{H}_{m,n}^s \psi_{s,n} \in H_{s,n} \} \) and if \( \psi_{s,n} \in D \left ( \tilde{H}_{m,n}^s \right ) \), then we have:

- If \( 2ml > 1 \) then

\[ ||\psi_{s,n}(x,\cdot)||_{L^2(\mathbb{R}^2)} = O \left ( \sqrt{(-x)} \right ), \quad x \to 0 \quad (3.50) \]

- If \( 2ml = 1 \), then

\[ ||\psi_{s,n}(x,\cdot)||_{L^2(\mathbb{R}^2)} = O \left ( \sqrt{x \ln(-x)} \right ), \quad x \to 0. \quad (3.51) \]

And, in these two cases, we have:

\[ ||\psi_{s,n}||_{H^{\infty}} \to 0, \quad x \to -\infty. \quad (3.52) \]

Let us prove that \( \tilde{H}_{m,n}^s \) is symmetric on its domain. We remark that \( (\gamma_B^0 \gamma_B^2)^* = \gamma_B^0 \gamma_B^2 \) and \( (\gamma_B^0 \gamma_B^2)^* = \gamma_B^0 \gamma_B^2 \), then we can prove:

\[ \left \langle \gamma_B^0 \gamma_B^2 A(x) \left ( s + \frac{1}{2} \right ) \psi_{s,n}, \psi_{s,n} \right \rangle_{H_{s,n}} = \left \langle \phi_{s,n}, (\gamma_B^0 \gamma_B^2) A(x) \left ( s + \frac{1}{2} \right ) \psi_{s,n} \right \rangle_{H_{s,n}} \]

and:

\[ \left \langle \gamma_B^0 B(x) \phi_{s,n}, \psi_{s,n} \right \rangle_{H_{s,n}} = \left \langle \phi_{s,n}, \gamma_B^0 B(x) \psi_{s,n} \right \rangle_{H_{s,n}}. \]

So, in the calculation of \( \left \langle \tilde{H}_{m,n}^s \phi_{s,n}, \psi_{s,n} \right \rangle_{H_{s,n}} - \left \langle \phi_{s,n}, \tilde{H}_{m,n}^s \psi_{s,n} \right \rangle_{H_{s,n}} \), there is only the boundary term due to integration by parts. Using (3.50), this gives the symmetry of our operator on its domain.

We then use the same trick as in [3]. Let us consider a new operator \( H \) with the same expression as \( \tilde{H}_{m,n}^s \) but defined on \( D(H) = C_0^\infty (]-\infty, 0[ ) \). Then \( H^* \) is \( \tilde{H}_{m,n}^s \) with domain \( H_{s,n} \).
Recall that if \( \phi_\pm \in \ker (H^* \pm i I_d) \). Then, using the symmetry of \( \hat{H}_{m}^{n} \), we have:

\[
0 = \left( \hat{H}_{m}^{n} \phi_\pm , \phi_\pm \right) - \left( \phi_\pm , \hat{H}_{m}^{n} \phi_\pm \right) = (H^* \phi_\pm , \phi_\pm ) - (\phi_\pm , H^* \phi_\pm ) = (\mp i \phi_\pm , \phi_\pm ) - (\phi_\pm , \mp i \phi_\pm ) = \mp 2 i \| \phi_\pm \|_{H_{m}^{n}}^2 . \tag{3.53}
\]

We conclude that \( \phi_\pm = 0 \). This proves that \( \hat{H}_{m}^{n} \) is essentially self-adjoint on \( C_0^\infty ([−\infty , 0]) \).

For the last part, using the last lemma, we see that for \( 2 ml > 1 \), we have: \( D(H_{m}^{n}) \subset H_0^1 ([−\infty , 0]) \). Moreover, we have \( H_{m}^{n} = i \gamma^\theta \gamma^\varphi \partial_x + V(x) \) where:

\[
V(x) = \gamma^\theta \gamma^\varphi \left( s + \frac{1}{2} \right) A(x) - m \gamma^\theta B(x). \tag{3.54}
\]

with:

\[
A(x) = \begin{cases} 
\frac{1}{2} + x^2 + o(x^2) & \text{in a neighbourhood of } 0 \\
C_4 e^{i \theta x} + o(e^{i \theta x}) & x \to -\infty
\end{cases}
\]
\[
B(x) = \begin{cases} 
\frac{x}{2} - \frac{1}{2} x + o(x) & \text{in a neighbourhood of } 0 \\
C_B e^{i \varphi x} + o(e^{i \varphi x}) & x \to -\infty.
\end{cases}
\]

Using the fact that \( B \) and \( B \) have the same behaviour when \( x \to 0 \) and the unitary transform, we can use the proof of Theorem 3.I.4 in [1] to prove a Hardy type inequality of the form:

\[
\| B \chi_x \varphi \| \leq C (\| \varphi \| + \| - i \partial_x \varphi \|). \tag{3.55}
\]

Using the fact that \( A \) is bounded, we have a similar estimate for \( V \). Thus \( H_0^1 \subset D(H_{m}^{n}) \). This proves the proposition.

Q.E.D

### 3.2.2 The case \( 2 ml < 1 \)

Recall that if \( 0 < 2 ml < 1 \), then, for all \( \psi_{s,n} \in D(H_{m}^{n}) \), there exists functions \( \psi_\pm \in W^+_\infty \), \( \chi_\pm \in W^{1,2}_\infty \), \( \psi_\pm \in L^2(S^2) \) and \( \phi \in C^0 ([0, \frac{\pi}{2}] ; L^2(S^2, C^4)) \) satisfying:

\[
\psi_{s,n}(x, \theta, \varphi) = (-x)^{-ml} \begin{pmatrix} \psi^\pm_{s,n}(\theta, \varphi) \\ \chi^\pm_{s,n}(\theta, \varphi) \\ -i \psi^\pm_{s,n}(\theta, \varphi) \\ i \chi^\pm_{s,n}(\theta, \varphi) \end{pmatrix} + (-x)^{-ml} \begin{pmatrix} \psi^\pm_{s,n}(\theta, \varphi) \\ \chi^\pm_{s,n}(\theta, \varphi) \\ i \psi^\pm_{s,n}(\theta, \varphi) \\ -i \chi^\pm_{s,n}(\theta, \varphi) \end{pmatrix}
\]

\[+ \phi^\pm_{s,n}(x, \theta, \varphi), \tag{3.56}\]

where:

\[
\| \phi^\pm_{s,n}(x, \theta, \varphi) \|_{V_{00}} = o \left( \sqrt{(-x)} \right), \quad x \to 0. \tag{3.57}
\]

We denote by \( \hat{R}_{m,n}^{MIT} \) the operator \( \hat{R}_{m,n}^{n} \) with domain:

\[
D(\hat{R}_{m,n}^{MIT}) \equiv \{ \psi_{s,n} \in \mathcal{H}_{s,n}; \hat{R}_{m,n}^{n} \psi_{s,n} \in \mathcal{H}_{s,n}, \| (\gamma_B^I + i) \psi_{s,n} \| = o \left( \sqrt{-x} \right) \}
\]

\[
= \{ \psi_{s,n} \in \mathcal{H}_{s,n}; \hat{R}_{m,n}^{n} \psi_{s,n} \in \mathcal{H}_{s,n}, \psi^\pm_{s,n} = \chi^\pm_{s,n} = 0 \}. \tag{3.58}
\]

Indeed, this is a result that can be found in the discussion after proposition V.I.2 in [1]. We have the:
Proposition 3.6. The operator $\hat{H}_{m,n}^{MIT}$ is self-adjoint on $D(\hat{H}_m^{MIT})$.

Proof. Let $\phi_{s,n}, \psi_{s,n} \in D(\hat{H}_m^{MIT})$. As in the proof of proposition 3.5 when calculating

$$\left\langle \hat{H}_m^{MIT} \phi_{s,n}, \psi_{s,n} \right\rangle_{H_{s,n}} - \left\langle \phi, \hat{H}_m^{MIT} \psi \right\rangle_{H_{s,n}},$$

only boundary values of $\phi_{s,n}, \psi_{s,n}$ are left. Using that

$$\phi_{s,n}(x, \theta, \varphi) = (-x)^{-ml} \begin{pmatrix} \phi_{-n}^s(\theta, \varphi) \\ \xi_{-n}^s(\theta, \varphi) \\ -i\phi_{-n}^s(\theta, \varphi) \\ i\xi_{-n}^s(\theta, \varphi) \end{pmatrix} + \varphi_n^s(x, \theta, \varphi),$$

$$\psi_{s,n}(x, \theta, \varphi) = (-x)^{-ml} \begin{pmatrix} \psi_{-n}^s(\theta, \varphi) \\ \chi_{-n}^s(\theta, \varphi) \\ -i\psi_{-n}^s(\theta, \varphi) \\ i\chi_{-n}^s(\theta, \varphi) \end{pmatrix} + \sigma_n^s(x, \theta, \varphi),$$

with:

$$\|\varphi_{s,n}\|_{L^2(S^2)} = o\left(\sqrt{(-x)}\right), \ x \to 0,$n

$$\|\sigma_{s,n}\|_{L^2(S^2)} = o\left(\sqrt{(-x)}\right), \ x \to 0,$n

we can calculate these boundary values in a neighbourhood of 0:

$$\left\langle \phi_{s,n}(x, \cdot), \gamma_B^0 \gamma_B^1 \psi_{s,n}(x, \cdot) \right\rangle_{W^\theta_0} = (-x)^{-ml} \begin{pmatrix} \phi_{-n}^s(x, \cdot) \\ \xi_{-n}^s(x, \cdot) \\ -i\phi_{-n}^s(x, \cdot) \\ i\xi_{-n}^s(x, \cdot) \end{pmatrix} + (-x)^{-ml} \begin{pmatrix} \psi_{-n}^s(x, \cdot) \\ \chi_{-n}^s(x, \cdot) \\ -i\psi_{-n}^s(x, \cdot) \\ i\chi_{-n}^s(x, \cdot) \end{pmatrix} + \langle \varphi_n^s(x, \cdot), \sigma_n^s(x, \cdot) \rangle_{W^\theta_0}. \quad (3.59)$$

Indeed, the presence of $\gamma_B^0 \gamma_B^1$ arranges the terms such that

$$\left\langle (-x)^{-ml} \begin{pmatrix} \phi_{-n}^s \\ \xi_{-n}^s \\ -i\phi_{-n}^s \\ i\xi_{-n}^s \end{pmatrix}, (-x)^{-ml} \begin{pmatrix} \psi_{-n}^s \\ \chi_{-n}^s \\ -i\psi_{-n}^s \\ i\chi_{-n}^s \end{pmatrix} \right\rangle_{W^\theta_0} = 0.$$n

Using the behaviour at 0 of $\varphi_{s,n}, \sigma_{s,n}$ and at $-\infty$ of $\phi_{s,n}, \psi_{s,n}$, we deduce that:

$$\lim_{x \to 0^+} \langle \phi_{s,n}(x, \cdot), \gamma_B^0 \gamma_B^1 \psi_{s,n}(x, \cdot) \rangle_{W^\theta_0} = \lim_{x \to -\infty} \langle \phi_{s,n}(x, \cdot), \gamma_B^0 \gamma_B^1 \psi_{s,n}(x, \cdot) \rangle_{W^\theta_0} = 0. \quad (3.60)$$

So $\hat{H}_m^{MIT}$ is symmetric.

Let $\psi_{s,n} \in D(\hat{H}_m^{MIT,*})$. Then, since $D(\hat{H}_m^{MIT,*}) \subset D(\hat{H}_m^{MIT})$, $\psi$ admits a decomposition, in a neighbourhood of 0, of the form:

$$\psi_{s,n}(x, \theta, \varphi) = (-x)^{-ml} \begin{pmatrix} \psi_{-n}^s(\theta, \varphi) \\ \chi_{-n}^s(\theta, \varphi) \\ -i\psi_{-n}^s(\theta, \varphi) \\ i\chi_{-n}^s(\theta, \varphi) \end{pmatrix} + (-x)^{-ml} \begin{pmatrix} \psi_{-n}^s(\theta, \varphi) \\ \chi_{-n}^s(\theta, \varphi) \\ -i\psi_{-n}^s(\theta, \varphi) \\ i\chi_{-n}^s(\theta, \varphi) \end{pmatrix} + \sigma_n^s(x, \theta, \varphi),$$

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where:
\[ \|\sigma_{s,n}\|_{L^2(S^2)} = o(\sqrt{x}), \quad x \to 0. \]

Moreover, \( \tilde{H}_{m,n}^{\text{MIT}} = \tilde{H}_{m}^{s,n} \) on \( D\left( \tilde{H}_{m,n}^{\text{MIT}} \right) \) using distributions. Using the equality:
\[
\left\langle \tilde{H}_{m,n}^{\text{MIT}}, \phi_{s,n}, \psi_{s,n} \right\rangle - \left\langle \phi_{s,n}, \tilde{H}_{m,n}^{\text{MIT}}, \psi_{s,n} \right\rangle = 0,
\]
for all \( \phi_{s,n} \in D\left( \tilde{H}_{m,n}^{\text{MIT}} \right) \) and \( \psi_{s,n} \in D\left( \tilde{H}_{m,n}^{\text{MIT}} \right) \), we deduce that:
\[
\lim_{s \to 0} \left( (-x)^{-m} \begin{pmatrix} \phi_{s,n} \\ \xi_{s,n} \\ -i\phi_{s,n} \\ i\xi_{s,n} \end{pmatrix}, (-x)^{m} \begin{pmatrix} \psi_{s,n} \\ \chi_{s,n} \\ i\psi_{s,n} \\ -i\chi_{s,n} \end{pmatrix} \right) = 0,
\]
by the last calculation. In other words, we have:
\[
2 \left( \begin{pmatrix} \phi_{s,n} \\ \xi_{s,n} \end{pmatrix}, \begin{pmatrix} \psi_{s,n} \\ \chi_{s,n} \end{pmatrix} \right) = 0. \tag{3.61}
\]

But, for all \( \phi_{s,n}, \xi_{s,n} \in C_0^{\infty}(Y_s,n) \), we can find \( \phi \in D(\tilde{H}_{m,n}^{\text{MIT}}) \) admitting these components as coordinates. Thus \( \psi_{s,n} = \chi_{s,n} = 0 \). We conclude that \( D\left( \tilde{H}_{m,n}^{s,n,\text{MIT}} \right) \subset D\left( \tilde{H}_{m,n}^{s,n,\text{MIT}} \right) \) and that \( \tilde{H}_{m,n}^{s,n,\text{MIT}} \) is self-adjoint on his domain.

Q.E.D

### 3.3 Self-adjointness of \( \tilde{H}_m \)

#### 3.3.1 The case \( 2ml \geq 1 \)

We equip \( \tilde{H}_m \) with the domain:
\[
D(\tilde{H}_m) = \left\{ u \in \mathcal{H}; \tilde{H}_m^{\text{MIT}} u \in \mathcal{H} \right\} = \left\{ \sum_{(s,n) \in I} \begin{pmatrix} u_{1,n} T_{s,n}^x \\ u_{2,n} T_{s,n}^y \\ u_{3,n} T_{s,n}^z \\ u_{4,n} T_{s,n}^x \end{pmatrix}; \quad \forall (s,n) \in I, \quad u_{s,n} \in L^2([-\infty,0], dx), \right\}
\]
\[
\tilde{H}_{m,n}^{s,n} = \begin{pmatrix} u_{1,n} T_{s,n}^x \\ u_{2,n} T_{s,n}^y \\ u_{3,n} T_{s,n}^z \\ u_{4,n} T_{s,n}^x \end{pmatrix} \in L^2, \quad \sum_{(s,n) \in I} \left\| \begin{pmatrix} u_{1,n} T_{s,n}^x \\ u_{2,n} T_{s,n}^y \\ u_{3,n} T_{s,n}^z \\ u_{4,n} T_{s,n}^x \end{pmatrix} \right\|_{L^2}^2 < \infty \right\}. \tag{3.62}
\]

We then have:

**Proposition 3.7.** Suppose that \( 2ml \geq 1 \). Then the operator \( \tilde{H}_m \) is self-adjoint on its domain.

**Proof.** \( \tilde{H}_m \) is symmetric. Indeed, let \( \varphi, \psi \in D\left( \tilde{H}_m \right) \). We can write
\[
\varphi = \sum_{(s,n) \in I} \varphi_{s,n}; \quad \psi = \sum_{(s,n) \in I} \psi_{s,n},
\]

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Then:
\[
\langle \tilde{H}_m \varphi, \psi \rangle = \sum_{(s,n) \in I} \langle \tilde{H}_m^{s,n} \varphi_{s,n}, \psi_{s,n} \rangle = \sum_{(s,n) \in I} \langle \varphi_{s,n}, \tilde{H}_m^{s,n} \psi_{s,n} \rangle
\]
\[
= \langle \varphi, \tilde{H}_m \psi \rangle
\]
(3.63)
since \(\tilde{H}_m^{s,n}\) is symmetric. We can prove that \(\tilde{H}_m\) is closed in the same way. Let \(x = \sum_{(s,n) \in I} x_{s,n} \in \mathcal{H}\). Since \(\tilde{H}_m^{s,n}\) is self-adjoint, there exists \(y_{s,n} \in D \left( \tilde{H}_m^{s,n} \right)\) such that \((\tilde{H}_m \pm i)y_{s,n} = (\tilde{H}_m^{s,n} \pm i)y_{s,n} = x_{s,n}\). Thus \(x = \sum_{(s,n) \in I} (\tilde{H}_m \pm i)y_{s,n}\) where \(y = \sum_{(s,n) \in I} y_{s,n} \in D \left( \tilde{H}_m \right)\):
\[
\sum_{(s,n) \in I} \| \tilde{H}_m^{s,n} y_{s,n} \|^2 + \| y_{s,n} \|^2 = \sum_{(s,n) \in I} \| (\tilde{H}_m^{s,n} \pm i) y_{s,n} \|^2 = \sum_{(s,n) \in I} \| x_{s,n} \|^2 < \infty.
\]
Consequently, \((y_{s,n})_{(s,n) \in I}\) is summable and \(x \in \text{Im}(\tilde{H}_m \pm i)\) so \(\text{Im}(\tilde{H}_m \pm i) = \mathcal{H}\) and \(\tilde{H}_m\) is self-adjoint.

Q.E.D

3.3.2 The case \(2ml < 1\)

Let us denote \(\tilde{H}_m^{\text{MIT}}\) the operator \(\tilde{H}_m\) with domain:
\[
D \left( \tilde{H}_m^{\text{MIT}} \right) = \left\{ u \in \mathcal{H} : \tilde{H}_m^{\text{MIT}} u \in \mathcal{H}, \left\| (\gamma_B^1 + i) u (x, \cdot) \right\|_{L^2(\mathbb{R}^n)}^2 = o \left( \sqrt{-x} \right), x \to 0 \right\}
\]
\[
= \left\{ \sum_{(s,n) \in I} \begin{pmatrix} u_{1,n}^s T_{1,n}^{s \pm} \\ u_{2,n}^s T_{2,n}^{s \pm} \\ u_{3,n}^s T_{3,n}^{s \pm} \\ u_{4,n}^s T_{4,n}^{s \pm} \end{pmatrix} \right\}, \quad \forall (s,n) \in I, \quad u_{s,n}^s \in L^2 (] \to \infty, 0[ , dx),
\]
\[
\tilde{H}_m^{s,n} \in L^2, \sum_{(s,n) \in I} \left\| (\tilde{H}_m^{s,n} \pm i) \begin{pmatrix} u_{1,n}^s T_{1,n}^{s \pm} \\ u_{2,n}^s T_{2,n}^{s \pm} \\ u_{3,n}^s T_{3,n}^{s \pm} \\ u_{4,n}^s T_{4,n}^{s \pm} \end{pmatrix} \right\| \to \infty
\]
\[
\sum_{(s,n) \in I} \left\| (\gamma_B^1 + i) \begin{pmatrix} u_{1,n}^s T_{1,n}^{s \pm} \\ u_{2,n}^s T_{2,n}^{s \pm} \\ u_{3,n}^s T_{3,n}^{s \pm} \\ u_{4,n}^s T_{4,n}^{s \pm} \end{pmatrix} \right\|_{L^2}^2 = o \left( \sqrt{-x} \right), x \to 0 \}
\]
(3.64)

Proposition 3.8. Suppose that \(2ml < 1\). Then the operator \(\tilde{H}_m^{\text{MIT}}\) is self-adjoint with domain \(D \left( \tilde{H}_m^{\text{MIT}} \right)\).

Proof. Let us remark that, if \(\sum_{(s,n) \in I} \left\| (\gamma_B^1 + i) \begin{pmatrix} u_{1,n}^s T_{1,n}^{s \pm} \\ u_{2,n}^s T_{2,n}^{s \pm} \\ u_{3,n}^s T_{3,n}^{s \pm} \\ u_{4,n}^s T_{4,n}^{s \pm} \end{pmatrix} \right\| \left\| \begin{pmatrix} u_{1,n}^s T_{1,n}^{s \pm} \\ u_{2,n}^s T_{2,n}^{s \pm} \\ u_{3,n}^s T_{3,n}^{s \pm} \\ u_{4,n}^s T_{4,n}^{s \pm} \end{pmatrix} \right\| = o \left( \sqrt{-x} \right), x \to 0\) then
\[
\left\| (\gamma_B^1 + i) \begin{pmatrix} u_{1,n}^s T_{1,n}^{s \pm} \\ u_{2,n}^s T_{2,n}^{s \pm} \\ u_{3,n}^s T_{3,n}^{s \pm} \\ u_{4,n}^s T_{4,n}^{s \pm} \end{pmatrix} \right\|_{L^2}^2 = o \left( \sqrt{-x} \right), x \to 0 \text{ for all } (s,n) \in I.
\]
We can now...
prove, as in the proof of proposition 3.7, that \( \tilde{H}_m^{MIT} \) is symmetric on its domain. Show that \( \tilde{H}_m^{MIT} \) is closed will require more effort. Choose a sequence \( (\varphi_j)_{j \in \mathbb{N}} \) of elements of \( D \left( \tilde{H}_m^{MIT} \right) \) such that \( \varphi_j \to \varphi \) and \( \tilde{H}_m^{MIT} \varphi_j \to \psi \) where \( \varphi, \psi \in \mathcal{H} \) and the convergence is understood in the norm of \( \mathcal{H} \). Using distributions, we have \( \tilde{H}_m^{MIT} \varphi = \psi \in \mathcal{H} \) and we have to show that \( \varphi \) satisfies the boundary condition. We can write:

\[
\varphi_j = \sum_{(s,n) \in I} \varphi_j^{s,n}, \quad \varphi = \sum_{(s,n) \in I} \varphi^{s,n}, \quad \psi = \sum_{(s,n) \in I} \psi^{s,n}, \tag{3.65}
\]

and we obtain:

\[
\varphi_j^{s,n} \to \varphi^{s,n}; \quad \tilde{H}_m^{s,n,MIT} \varphi_j^{s,n} \to \psi^{s,n}
\]

in the norm of \( \mathcal{H}_{s,n} \). Thus, \( \varphi^{s,n} \in D \left( \tilde{H}_m^{s,n,MIT} \right) \) since \( \tilde{H}_m^{s,n,MIT} \) is closed and \( \varphi^{s,n} \) admits a decomposition of the form:

\[
\varphi^{s,n}(x,\theta,\varphi) = (-x)^{-mt} \begin{pmatrix}
\phi^s_{s,n}(\theta,\varphi) \\
\xi^s_{s,n}(\theta,\varphi) \\
- i \phi^s_{s,n}(\theta,\varphi) \\
i \xi^s_{s,n}(\theta,\varphi)
\end{pmatrix} + \phi^s_n(x,\theta,\varphi)
\]

where the functions \( \phi^s_n = \begin{pmatrix}
\phi^1_n \\
\phi^2_n \\
\phi^3_n \\
\phi^4_n
\end{pmatrix} \) satisfy

\[
\sum_{(s,n) \in I} \left( ||\phi^1_{s,n}||_{L^2(\mathbb{R})}^2 + ||\phi^2_{s,n}||_{L^2(\mathbb{R})}^2 + ||\phi^3_{s,n}||_{L^2(\mathbb{R})}^2 + ||\phi^4_{s,n}||_{L^2(\mathbb{R})}^2 \right) = o(-x)
\]

when \( x \) goes to 0, using the proof of theorem V.1 in [1] and the fact that \( \varphi \) is in the natural domain of \( \tilde{H}_m \). Using that

\[
(\gamma_1^1 + i) \varphi^{s,n} = \begin{pmatrix}
i \phi^1_{s,n} + \phi^3_{s,n} \\
i \phi^2_{s,n} - \phi^4_{s,n} \\
- \phi^1_{s,n} + i \phi^3_{s,n} \\
\phi^2_{s,n} + i \phi^4_{s,n}
\end{pmatrix},
\]

we have:

\[
\left( ||(\gamma_1^1 + i) \varphi^{s,n}||_{L^2(\mathbb{R})}^2 \right) = \sum_{(s,n) \in I} \left( ||\phi^1_{s,n}||_{L^2(\mathbb{R})}^2 + ||\phi^2_{s,n}||_{L^2(\mathbb{R})}^2 + ||\phi^3_{s,n}||_{L^2(\mathbb{R})}^2 + ||\phi^4_{s,n}||_{L^2(\mathbb{R})}^2 \right) = o(-x)
\]

which proves that the boundary condition is fulfilled and that the operator \( \tilde{H}_m^{MIT} \) is closed. To prove the self-adjointness of \( \tilde{H}_m^{MIT} \), we follow the same argument as in proposition 3.7 where we have to prove that \( y = \sum_{(s,n) \in I} y_{s,n} \in D \left( \tilde{H}_m^{MIT} \right) \). The only difference is that the boundary condition has to be fulfilled. Since \( y_{s,n} \in D \left( \tilde{H}_m^{s,n,MIT} \right) \), we can decompose \( y_{s,n} \) as for \( \varphi^{s,n} \) just above. A similar argument shows that \( y \) satisfies the boundary condition. Thus \( \tilde{H}_m^{MIT} \) is self-adjoint on \( D \left( \tilde{H}_m^{MIT} \right) \).

Q.E.D

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3.3.3 Self-adjointness of $H_m$

Recall that the domain of $H_m$ is:
- If $2ml \geq 1$:
  \[ D(H_m) = \{ \phi \in \mathcal{H}; \ H_m\phi \in \mathcal{H} \}. \]
- If $0 < m < \frac{1}{2}$, we consider this operator equipped with a domain where elements satisfy a MIT-bag boundary condition. We will denote $H_m^{MIT}$ the operator $H_m$ with domain:
  \[ D(H_m^{MIT}) = \{ \phi \in \mathcal{H}; \ H_m\phi \in \mathcal{H}, \ \| (\gamma^1 + i) \phi(x,.) \|_{L^2(S^2)} = o(\sqrt{-x}), \ x \to 0 \}. \]

Moreover, recall that $\tilde{H}_m = \gamma^5_B P^{-1} (-H_m) P \gamma^5_B$ where $\gamma^5_B$ and $P$ are unitary matrices. We thus obtain the following theorem:

**Theorem 3.9.**
- For all $m \geq \frac{1}{2}$, the operator $H_m$ with domain $D(H_m)$ is self-adjoint.
- For all $m < \frac{1}{2}$, the operator $H_m^{MIT}$ with domain $D(H_m^{MIT})$ is self-adjoint.

**Proof.** We have:
\[ H_m = P \gamma^5_B \left(-\tilde{H}_m\right) \gamma^5_B P^{-1}. \]
This is clear that $\psi \in D(H_m)$ if and only if $\gamma^5_B P^{-1} \psi \in D\left(\tilde{H}_m\right)$ for $m \geq \frac{1}{2}$. Moreover, recall that $\gamma^1 = -P \gamma^1_B P^{-1}$ and $\gamma^1_B \gamma^5_B = -\gamma^5_B \gamma^1_B$ using (3.28) and (3.29). We then obtain:
\[
\| (\gamma^1 + i) \psi \| = \| (-P \gamma^1_B P^{-1} + i) P \gamma^5_B \gamma^5_B P^{-1} \psi \|
\]
\[
= \| P \gamma^5_B (\gamma^1 + i) \gamma^5_B P^{-1} \psi \|
\]
\[
= \| (\gamma^1_B + i) \gamma^5_B P^{-1} \psi \|. \]

Thus $\psi \in D(H_m)$ if and only if $\gamma^5_B P^{-1} \psi \in D\left(\tilde{H}_m\right)$ for all $m > 0$. Since these operators are unitary equivalent and domains correspond, this ensure that $H_m$ is self-adjoint equipped with the convenient domain. Q.E.D

3.4 The Cauchy problem

We can now prove the:

**Theorem 3.10.** Let $\psi_0 \in \mathcal{H}$, there exists a unique solution $\psi$ to the equation:
\[ \partial_t \psi = i H_m \psi \quad (3.67) \]
such that
\[ \psi \in C^0(\mathbb{R}; \mathcal{H}) \quad (3.68) \]
and satisfying:
\[ \psi(t = 0,.) = \psi_0(.) \quad (3.69) \]
\[ \forall t \in \mathbb{R}, \ |\psi(t,.)|_\mathcal{H} = |\psi_0(.)|_\mathcal{H}. \quad (3.70) \]

**Proof.** Since $H_m$ is self-adjoint on its domain, this follows from Stone theorem. Q.E.D
3.5 Absence of eigenvalues

**Proposition 3.11.** For all \( m > 0 \), the Dirac operator \( H_m \), defined in (2.27), does not admit any real eigenvalues.

**Proof.** Let us first show the absence of eigenvalues for \( H_m^n \) for all \( m > 0 \) and all \( \{s,n\} \in I \).

Since \( H_m^n \) is self-adjoint on its domain, the eigenvalues (if they exist) are all real. So, suppose that there exists \( \lambda \in \mathbb{R} \) and \( \varphi \in D(H_m^n) \) such that \( H_m^n \varphi = \lambda \varphi \).

We define:

\[
w(x) = e^{i\lambda \gamma^1} \varphi(x)
\]

such that

\[
w'(x) = i\lambda \gamma^0 \varphi(x) + e^{i\lambda \gamma^1} x \varphi'(x).
\]

But, we have:

\[H_m^n \varphi - \lambda \varphi = 0 \Leftrightarrow i\gamma^0 \gamma^1 \varphi'(x) = (\lambda - V(x)) \varphi(x) \Leftrightarrow \varphi'(x) = i\gamma^0 \gamma^1 (V(x) - \lambda) \varphi(x)\]

where \( V(x) = \gamma^0 \gamma^2 A(x) \left(s + \frac{1}{2}\right) - m \gamma^0 B(x)\).

So, we obtain:

\[
w'(x) = i\gamma^0 \gamma^1 e^{i\lambda \gamma^1} x \varphi(x) = 0
\]

Thus \( \lambda = 0 \) and all \( \varphi \) satisfies the same estimate, so:

\[
\lim_{x \to -\infty} \int_{-\infty}^x |W(t)| \, dt < \infty
\]

exists and is finished. As a consequence, we have:

\[
\lim_{T \to -\infty} \int_{-\infty}^T W(t) \, dt w(T) = 0.
\]

We then deduce that \( w(x) = 0 \) for all \( x < 0 \) which implies that:

\[
\forall x < 0, \quad \varphi(x) = 0.
\]

Consequently, \( H_m^n \) admits no eigenvalues.

We can now consider \( H_m \). If \( \lambda \in \mathbb{R} \) is an eigenvalue of \( H_m \), then there exists \( \varphi \in D(H_m) \) such that \( (H_m - \lambda) \varphi = 0 \). Using the decomposition of \( \varphi \) in spherical harmonics, we obtain:

\[
\sum_{\{s,n\} \in I} (H_m^n - \lambda) \varphi_{s,n} = 0.
\]

Consequently, if \( \varphi \) is non zero, there exists \( \{s,n\} \in I \) such that \( \varphi_{s,n} \neq 0 \) and \( \varphi_{s,n} \) satisfies \( (H_m^n - \lambda) \varphi_{s,n} = 0 \). This is impossible since \( H_m^n \) does not admit eigenvalues. Thus \( \varphi \) is identically 0. We deduce that \( H_m \) does not admit any eigenvalue for all \( m > 0 \).

Q.E.D
4 Compactness results

The purpose of this section is to prove that, for a well chosen function \( f \), the operator \( f(x) (H_{m}^{n} - z)^{-1} \) is compact for all \( z \in \rho(H_{m}^{n}) \). We will make use of this result for proving Mourre estimates in the following section. The key point here for the Mourre estimate is that \( f \) only admits a finite limit at 0.

This result is proved by separating our operator in two operators denoted \( H_{+} \) and \( H_{-} \). The operator \( H_{+} \) has a potential which behaves like the one in \( H_{m}^{n} \) for \( x \) close to 0 and is extended so that the potential becomes confining. Hence the resolvent of this operator is itself compact. For \( H_{-} \), we preserve the behaviour near the horizon of the black hole and extend it so that it decreases to 0 at 0. By extending the states and the potential, we are thus able to view the resolvent as the restriction of a resolvent for an operator defined on the entire line. For this last resolvent, we are able to use standard results about Hilbert-Schmidt operators.

We now enter into the details. We have:

\[
H_{m}^{n} = \Gamma^{1} D_{x} + (s + \frac{1}{2}) A(x) \gamma_{0} \gamma^{2} - m \gamma_{0} B(x),
\]

(4.1)

where \( A \) and \( B \) behave like:

\[
A - A_{0} \in T^{n,2}
\]

\[
B - B_{0} \in T^{n,1}
\]

with \( \kappa, \bar{\kappa} > 0 \). Moreover, \( \Gamma^{1} = -\gamma_{0} \gamma^{1} \) where the expression of \( \gamma_{0} \gamma^{1} \) is given in (3.20). The main result of this section is:

**Proposition 4.1.** Let \( f \in C([-\infty, 0]) \) such that \( f \) goes to 0 at \(-\infty\). Let \( z \in \rho(H_{m}^{n}) \) where \( \rho(H_{m}^{n}) \) is the resolvent set of \( H_{m}^{n} \). Then the operator \( f(x) (H_{m}^{n} - z)^{-1} \) is compact on \( \mathcal{H} \) for all \( m > 0 \).

4.1 Asymptotic operators

4.1.1 Operator \( H_{+} \)

Let us first introduce the operator \( H_{+} = i \gamma_{0} \gamma^{1} \partial_{x} \) where \( \gamma_{0} \gamma^{1} = \text{diag}(-1, 1, 1, -1) \). We can thus prove the:

**Proposition 4.2.** The operator \( H_{+} = i \gamma_{0} \gamma^{1} \partial_{x} \) is self-adjoint on the domain defined by:

\[
D(H_{+}) = \{ \varphi \in \mathcal{H}_{s,n}; H_{+} \varphi \in \mathcal{H}_{s,n}, \varphi_{1}(0) = -\varphi_{3}(0), \varphi_{2}(0) = \varphi_{4}(0) \}
\]

**Proof.** Since \( D(H_{+}) \subset H^{1}([-\infty, 0]) \subset C^{0}([-\infty, 0]) \), we can deduce that the elements of \( D(H_{+}) \) go to 0 at \(-\infty\) and from the boundary conditions, we deduce the symmetry of \( H_{+} \). Moreover, taking a sequence \( (\varphi_{n})_{n \in \mathbb{N}} \) of elements of \( D(H_{+}) \) such that \( H_{+} \varphi_{n} \rightarrow \varphi \) and \( H_{+} \varphi_{n} \rightarrow \psi \) where \( \varphi, \psi \in \mathcal{H}_{s,n} \), we can prove, by a distribution argument, that \( H_{+} \varphi = \psi \in \mathcal{H}_{s,n} \). Consequently, \( \varphi_{n} \rightarrow \varphi \) in \( H^{1}([-\infty, 0]) \). So, writing \( \varphi_{n} = \begin{pmatrix} \varphi_{1n} \\ \varphi_{2n} \\ \varphi_{3n} \\ \varphi_{4n} \end{pmatrix} \), we obtain:

\[
\varphi_{n}^{i}(0) \rightarrow \varphi(0)
\]

for all \( i \in \{1, \cdots, 4\} \) by the second inclusion above. Thus \( \varphi \in D(H_{+}) \) and \( H_{+} \) is closed.

On the other hand, since \( C^{\infty}_{0}([-\infty, 0]) \subset D(H_{+}) \), we have:

\[
\langle \psi, H_{+} \varphi \rangle = \langle H_{+} \psi, \varphi \rangle
\]

(4.2)
in distributional sense, for all $\varphi \in C_0^\infty$ and all $\psi \in D(H^*_c)$. So, $H^*_c = H_c$ on $D(H^*_c)$.

Let $\psi \in \ker (H^*_c + i)$. Then the components of $\psi$ satisfy:

$$\partial_x \psi_1 = \psi_1 - \partial_x \psi_2 = \psi_2 - \partial_x \psi_3 = \psi_3 \partial_x \psi_4 = \psi_4.$$ 

Since $x \to e^{-x}$ is not in $L^2([-\infty, 0])$, we obtain:

$$\ker (H^*_c + i) = \text{vect} \left\{ \begin{pmatrix} e^x \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^x \end{pmatrix} \right\} \cap D(H^*_c).$$

But, if $\psi \in D(H^*_c)$, then, for all $\varphi \in D(H_c)$, we have:

$$\langle H_c \varphi, \psi \rangle - \langle \varphi, H^*_c \psi \rangle = 0,$$

where:

$$\langle H_c \varphi, \psi \rangle - \langle \varphi, H^*_c \psi \rangle = \lim_{x \to 0} \left( -i \varphi_1 (x) \overline{\psi_1 (x)} + i \varphi_2 (x) \overline{\psi_2 (x)} + i \varphi_3 (x) \overline{\psi_3 (x)} - i \varphi_4 (x) \overline{\psi_4 (x)} \right).$$

We can choose $\varphi$ such that $\varphi_1 (0) \neq 0$. Indeed, we can for example consider the function $\chi \in C^\infty$ such that $\chi \equiv 1$ on $[-1, 0]$ and is equal to $0$ on $]-\infty, -2]$. We can take $\varphi = \begin{pmatrix} \chi \\ 0 \\ 0 \end{pmatrix}$.

Then, we see that $\begin{pmatrix} e^x \\ 0 \\ 0 \end{pmatrix} \not\in D(H^*_c)$. Thus $\ker (H^*_c + i) = \{0\}$. We also have $\ker (H^*_c - i) = \{0\}$. This shows that $H_c$ is self-adjoint on $D(H_c)$.

Q.E.D

Now, let us define the operator $H_-$ by:

$$H_- = H_c + V_-(x)$$

where

$$V_-(x) = \begin{cases} x \operatorname{Id} \\ \sigma^0 \gamma^2 A(x) \left( s + \frac{1}{2} \right) - m \gamma^0 B(x) \end{cases}, \quad \text{for } x \geq d \quad \text{for } x \leq c.$$ (4.3)

with $c, d$ two negative constants such that $c \leq d$. We remark that $V_-$ is bounded on $\mathbb{R}^\ast_+$. Using Kato-Rellich theorem, we obtain:

**Corollary 4.3.** The operator $H_-$ equipped with $D(H_c)$ is self-adjoint.

**Remark.** Note that the potential of $H_-$ equals the potential of $H_{m,n}^*$ for $x$ negative and $|x|$ large.
4.1.2 Operator $H_+$

Let us define the operator $H_+$ by:

$$H_+ = \Gamma^1 D_x + V_+(x)$$  \hspace{1cm} (4.4)

where

$$V_+(x) = \begin{cases} \gamma^0 + i A(x) (s + \frac{1}{2}) - m s B(x) & \text{for } x \geq b. \\ -x^2 \gamma^0 & \text{for } x \leq a. \end{cases}$$  \hspace{1cm} (4.5)

This time, the potential behaves like the potential in $H_+^n$ at 0 and increases at $-\infty$. We then have a confining potential. This type of potential has been encountered in the article of A. Bachelot [4]. For proving the self-adjointness of his operator, he uses the method we have recovered for proving the self-adjointness of our operator $H_m$. We just indicate the different stages of the proof.

We introduce the domain:

$$D(H_+) = \left\{ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \in L^2(\mathbb{R}^+, \mathbb{C}^4); H_+ \varphi \in L^2(\mathbb{R}^+) \right\}$$

if $2ml < 1$ and we remove the boundary condition for $2ml \geq 1$. In the following proof of compactness of $(H_+ - z)^{-1}$, we obtain estimates that allow us to prove the symmetry of this operator for $ml \geq \frac{1}{2}$ and then the essential self-adjointness on $C^\infty_0([-\infty, 0])$. Indeed, we can realize a unitary transform of our operator:

$$H_+ \to \gamma^5 P^{-1} H_+ P \gamma^5 : = \tilde{H}_+.$$  \hspace{1cm} (4.6)

We remark that, choosing $\chi \in C^\infty_0([-1, 0])$ equal to 1 on $[-\frac{1}{2}, 0]$, for all $\varphi$ in the domain of $H_+$, $\chi \varphi$ is in the domain of $H_+^m$, which is given by (4.10), and we obtain the asymptotic behaviour of $\varphi$. This allows us to conclude in the case $ml \geq \frac{1}{2}$.

If $ml < \frac{1}{2}$, we introduce the MIT boundary condition and a suitable partition of unity in order to separate the behaviour at 0 from the one at $-\infty$. We then obtain:

**Proposition 4.4.** The operator $H_+$ equipped with $D(H_+)$ is self-adjoint.

4.2 Compactness of $f(x) (H_- - z)^{-1}$

**Lemma 4.5.** Let $f$ be a continuous function on $[-\infty, 0]$ such that $\lim_{x \to -\infty} f(x) = 0$. Then the operator $f(\cdot) (H_- - z)^{-1}$ is compact.

**Proof.** Let $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \in D(H_-)$ and $g = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = (H_- - z) \varphi$ be defined on $]-\infty, 0[$. We will extend these functions to $\mathbb{R}$ in the following way:

$$\tilde{\varphi}_1(x) = \begin{cases} \varphi_1(x) & \text{if } x \leq 0, \\ -\varphi_3(-x) & \text{if } x \geq 0 \end{cases}; \tilde{\varphi}_2(x) = \begin{cases} \varphi_2(x) & \text{if } x \leq 0, \\ \varphi_4(-x) & \text{if } x \geq 0 \end{cases}$$

$$\tilde{\varphi}_3(x) = \begin{cases} \varphi_3(x) & \text{if } x \leq 0, \\ -\varphi_1(-x) & \text{if } x \geq 0 \end{cases}; \tilde{\varphi}_4(x) = \begin{cases} \varphi_4(x) & \text{if } x \leq 0, \\ \varphi_2(-x) & \text{if } x \geq 0. \end{cases}$$
The components are thus in $H^1(\mathbb{R})$. We also extend $g$ into $\tilde{g} \in [L^2(\mathbb{R})]^4$ in the same way. Here, we have put $\tilde{H}^e$ for the operator with the same formula as $H_e$ but acting on functions defined on $\mathbb{R}$. We then see:

\[
\left(\tilde{H}^e - z\right)\tilde{\varphi} = \begin{pmatrix} (-i\partial_x - z)\varphi_1(x) \\ (i\partial_x - z)\varphi_2(x) \\ (-i\partial_x - z)\varphi_3(x) \\ (-i\partial_x - z)\varphi_4(x) \end{pmatrix} = \begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \\ g_4(x) \end{pmatrix}
\]

if $x \leq 0$;

\[
\tilde{H}^e \hat{\varphi} = \begin{pmatrix} (-i\partial_x - z - \varphi_3(-x)) \\ (i\partial_x - z - \varphi_4(-x)) \\ (-i\partial_x - z - \varphi_1(-x)) \\ (-i\partial_x - z - \varphi_2(-x)) \end{pmatrix} = \begin{pmatrix} -g_3(x) \\ g_4(x) \\ -g_1(x) \\ g_2(x) \end{pmatrix}
\]

so $(H_e - z)\varphi = g$ if and only if $\left(\tilde{H}^e - z\right)\tilde{\varphi} = \tilde{g}$ for all $z$ in the resolvent set of $H_e$.

Let $f$ be a continuous function which tends to 0 at $-\infty$ and has finite limit at 0. We consider a sequence $(g_n)_{n \in \mathbb{N}} \in (L^2(\mathbb{R}_+))^4$ such that $g_n \to 0$ and we want to prove that $f(x)(H_e - z)^{-1}g_n$ goes to 0 strongly in $L^2$. We introduce $u_n = (H_e - z)^{-1}g_n$ and extend $g_n$ and $u_n$ into $\tilde{g}_n$ and $\tilde{u}_n$ as before. Consequently, $\tilde{g}_n \to 0$ in $L^2(\mathbb{R})$ and $\tilde{u}_n = \left(\tilde{H}^e - z\right)^{-1}\tilde{g}_n$.

Since $x \to (x - z)^{-1} \in L^\infty$ and $|x - z|^{-1} \to 0$, we deduce, using a consequence of theorem IX.29 in [37], that:

\[
\tilde{f}(x) \left(\tilde{R}^e - z\right)^{-1}\tilde{g}_n \xrightarrow{n \to \infty} 0,
\]

where $\tilde{f}$ is the extension of $f$ by symmetry on $\mathbb{R}_+$. Therefore, we have:

\[
1_{[-\infty,0]}(x)\tilde{f}(x) \left(\tilde{H}^e - z\right)^{-1}\tilde{g}_n = 1_{[-\infty,0]}(x)f(x)\tilde{u}_n = f(x)u_n = f(x)(H_e - z)^{-1}g_n,
\]

where $1_{[-\infty,0]}$ is the characteristic function of $[-\infty,0]$. So $f(x)(H_e - z)^{-1}g_n \xrightarrow{n \to \infty} 0$ and the operator $f(x)(H_e - z)^{-1}$ is compact.

We then use the identity:

\[
f(x)(H_e - z)^{-1}g_n = -f(x)(H_e - z)^{-1}V_-(x)(H_e - z)^{-1} + f(x)(H_e - z)^{-1}
\]

Since $V_-$ goes to 0 at $-\infty$ and 0, we deduce that $(H_e - z)^{-1} - (H_e - z)^{-1}$ is compact and consequently that $f(x)(H_e - z)^{-1}$ is also compact.

Q.E.D

### 4.3 Compactness of $(H_+ - z)^{-1}$

**Lemma 4.6.** The operator $(H_+ - z)^{-1}$ is compact.

**Proof.** We follow the proof of the compactness result in [3]. Let us show that the set:

\[ K = \{ \varphi \in D(H_+) : \| \varphi \| + \| H_+ \varphi \| \leq 1 \} \tag{4.7} \]

is compact. We consider a sequence $(\varphi_n)_{n \in \mathbb{N}} \in K^\mathbb{N}$. Using the Banach-Alaoglu theorem and distributions, we obtain the existence of a sub-sequence $(\varphi_{n'})$ such that:

\[
\varphi_{n'} \xrightarrow{n' \to \infty} \varphi
\]

\[
f_{n'} = H_+ \varphi_{n'} \xrightarrow{n' \to \infty} H_+ \varphi := f.
\]

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We then study the equation: \[ H_+ \varphi_\nu = f_\nu \]

where \( \varphi_\nu, f_\nu \in L^2(]-\infty, 0[) \). Let:

\[
W(x) = \begin{cases} 
  mB(x) = -\frac{m^4}{x^2} + T(x), & \text{for } x \geq b. \\
  x^2, & \text{for } x \leq a.
\end{cases}
\]

where:

\[
T(x) = \frac{m T_0}{x} + o(x),
\]

\[
g^1_\nu \begin{cases} 
  (s + \frac{1}{2}) A(x) \varphi^2_\nu + f^1_\nu, & \text{for } x \geq b, \\
  f^1_\nu, & \text{for } x \leq a.
\end{cases}
\]

\[
g^2_\nu \begin{cases} 
  (s + \frac{1}{2}) A(x) \varphi^3_\nu + f^2_\nu, & \text{for } x \geq b, \\
  f^2_\nu, & \text{for } x \leq a.
\end{cases}
\]

\[
g^3_\nu \begin{cases} 
  -(s + \frac{1}{2}) A(x) \varphi^3_\nu + f^3_\nu, & \text{for } x \geq b, \\
  f^3_\nu, & \text{for } x \leq a.
\end{cases}
\]

where \( f_\nu = \left( \begin{array}{c} f^1_\nu \\ f^2_\nu \\ f^3_\nu \end{array} \right) \) and \( \varphi_\nu = \left( \begin{array}{c} \varphi^1_\nu \\ \varphi^2_\nu \\ \varphi^3_\nu \end{array} \right) \). We set \( g_\nu = \left( \begin{array}{c} g^1_\nu \\ g^2_\nu \\ g^3_\nu \end{array} \right) \). Since \( A \) is bounded, we see that \( g_\nu \in L^2(]-\infty, 0[) \) and that \( g_\nu \to g \) where \( g \) is defined in the same way. We thus obtain four differential equations:

\[
\begin{aligned}
\partial_t (\varphi^1_\nu + \varphi^3_\nu) + W(x) (\varphi^1_\nu + \varphi^3_\nu) &= i (g^1_\nu - g^3_\nu) \\
\partial_t (\varphi^2_\nu - \varphi^1_\nu) + W(x) (\varphi^2_\nu - \varphi^1_\nu) &= -i (g^2_\nu + g^3_\nu) \\
\partial_t (\varphi^3_\nu + \varphi^1_\nu) - W(x) (\varphi^3_\nu + \varphi^1_\nu) &= i (g^1_\nu + g^2_\nu) \\
\partial_t (\varphi^2_\nu - \varphi^3_\nu) - W(x) (\varphi^2_\nu - \varphi^3_\nu) &= -i (g^2_\nu - g^3_\nu)
\end{aligned}
\]

The solutions are given by the formulas:

\[
(\varphi^1_\nu + \varphi^3_\nu) (x) = \lambda^1_0 e^{-\int_{-\infty}^{x} W(u)du} + i \int_{-\infty}^{x} (g^1_\nu - g^3_\nu) e^{\int_{-\infty}^{u} W(u)du - \int_{-\infty}^{x} W(u)du} \, du \, dt \tag{4.9}
\]

\[
(\varphi^2_\nu - \varphi^1_\nu) (x) = \lambda^2_0 e^{-\int_{-\infty}^{x} W(u)du} - i \int_{-\infty}^{x} (g^2_\nu + g^4_\nu) e^{\int_{-\infty}^{u} W(u)du - \int_{-\infty}^{x} W(u)du} \, du \, dt \tag{4.10}
\]

\[
(\varphi^3_\nu + \varphi^1_\nu) (x) = \lambda^3_0 e^{\int_{-\infty}^{x} W(u)du} + i \int_{0}^{x} (g^1_\nu + g^3_\nu) e^{-\int_{0}^{u} W(u)du + \int_{-\infty}^{x} W(u)du} \, du \, dt \tag{4.11}
\]

\[
(\varphi^2_\nu - \varphi^3_\nu) (x) = \lambda^4_0 e^{\int_{-\infty}^{x} W(u)du} + i \int_{0}^{x} (g^2_\nu - g^4_\nu) e^{-\int_{0}^{u} W(u)du + \int_{-\infty}^{x} W(u)du} \, du \, dt \tag{4.12}
\]

where \( \lambda^j_0, \ j = 1, \ldots, 4 \), are constants.

Proof of the pointwise convergence of the integral terms.

We have:

\[
\int_{-1}^{x} W(u)du = \begin{cases} 
  -ml \ln(-x) + \int_{-1}^{x} T(u)du, & \text{for } x \geq b, \\
  x^2 - x^3 + \int_{-1}^{a} W(u)du, & \text{for } x \leq a.
\end{cases}
\]

where:

\[
\left| \int_{-1}^{x} T(u)du \right| \leq \int_{-1}^{x} |T(u)|du \leq C_0 \int_{-1}^{x} -udu \leq C \tag{4.14}
\]

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on $[b; 0]$ (with $C_0, C$ two positive constants). We obtain:

$$
e^{-f_{-1}^x W(u) du} = \begin{cases} \left( -\frac{x}{2} \right)^{m_l} e^{-f_{-1}^x T(u) du}, & \text{for } x \geq b. \\
C_1 e^{\frac{x^2}{2}}, & \text{for } x \leq a.
\end{cases}$$

where $C_1 = e^{-\frac{a^2}{2} + f_{-1}^a W(u) du}$ and $C_2 = e^{\frac{a^2}{2} - f_{-1}^a W(u) du}$ are positive constants. We thus see that $e^{-f_{-1}^x W(u) du}$ is square integrable on $[-\infty, x]$ and that $e^{-f_{-1}^x W(u) du}$ is square integrable on $[x, 0]$. Consequently, since $g_\nu$ is weakly convergent, we deduce that, when $\nu \to \infty$, we have the following:

$$\int_{-\infty}^{x} (g_\nu - g_\nu^3) e^{f_{-1}^x W(u) du - f_{-1}^x W(u) du} dt \to \int_{-\infty}^{x} (g^3 - g^3) e^{f_{-1}^x W(u) du - f_{-1}^x W(u) du} dt \quad (4.15)$$

$$\int_{0}^{x} (g_\nu^4 + g_\nu^2) e^{-f_{-1}^x W(u) du + f_{-1}^x W(u) du} dt \to \int_{0}^{x} (g^4 + g^2) e^{-f_{-1}^x W(u) du + f_{-1}^x W(u) du} dt \quad (4.16)$$

Majorations of integral terms by $L^2$ functions independent of $\nu$.

In the following, we will only treat $(\varphi_\nu^4 + \varphi_\nu^2)$ and $(\varphi_\nu^2 - \varphi_\nu^2)$. $(\varphi_\nu^4 + \varphi_\nu^2)$ and $(\varphi_\nu^2 - \varphi_\nu^2)$ can be treated in the same way. We have:

1. First, using Cauchy-Schwarz inequality and that $g_\nu^4 + g_\nu^2$ is bounded in $L^2$ since it is weakly convergent, we obtain:

$$\left| \int_{0}^{x} (g_\nu^4 + g_\nu^2) e^{f_{-1}^x W(u) du} dt \right|^2 \leq \int_{0}^{x} e^{-2 f_{-1}^x W(u) du + 2 f_{-1}^x W(u) du} dt \quad (4.17)$$

Therefore, we prove that the right hand side is integrable:

i) If $x > b$, we have:

$$\left| \int_{0}^{x} e^{-2 f_{-1}^x W(u) du + 2 f_{-1}^x W(u) du} dt \right| \leq e^{2C} \left| \int_{x}^{0} \left( \frac{1}{x} \right)^{2m_l} (-1)^{2m_l} d\nu \right| = e^{2C \frac{-x}{1 + 2m_l}}$$

using the expression of $W$ and (4.14). So $\left| \int_{0}^{x} e^{-2 f_{-1}^x W(u) du + 2 f_{-1}^x W(u) du} dt \right|$ is integrable on $[b, 0]$. 

ii) If $a \leq x \leq b$, we integrate a smooth function, so we obtain an integrable function.

iii) If $x \leq a$, we have:

$$\left| \int_{x}^{a} e^{f_{-1}^x W(u) du} dt \right| = (C_1)^2 e^{\frac{a^2}{2}} \left( \int_{x}^{a} (C_2)^2 e^{-\frac{2a^2}{2}} d\nu + \int_{a}^{0} e^{-f_{-1}^x W(u) du} dt \right).$$

The function $(C_1)^2 e^{\frac{a^2}{2}} \left( \int_{x}^{a} e^{-2 f_{-1}^x W(u) du} dt \right)$ is integrable on $[-\infty, a]$ and:

$$\int_{x}^{a} e^{-\frac{2a^2}{2}} d\nu \leq -\frac{1}{2\nu^2} e^{-\frac{a^2}{2}} + \frac{1}{ax^2} e^{-\frac{a^2}{2}} - \frac{1}{a} \int_{x}^{a} e^{-\frac{2a^2}{2}} d\nu,$$

by integration by parts. Choosing $a$ such that $1 + \frac{1}{ax^2} > 0$, we obtain:

$$(C_1)^2 e^{\frac{a^2}{2}} \int_{x}^{a} (C_2)^2 e^{-2 f_{-1}^x W(u) du} dt \leq (C_1)^2 (C_2)^2 \frac{e^{\frac{2a^2}{2}} \left( \frac{1}{ax^2} e^{-\frac{a^2}{2}} + \frac{1}{ax^2} \right)}{1 + \frac{1}{ax^2}}$$

where the right hand side is integrable on $[-\infty, a]$ and goes to 0 at $-\infty$. 

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b) Secondly, it remains:

\[
\left| \int_{-\infty}^{x} (g_1 - g_3^3) e^{f_1^2 W(u)du} dt \right|^2 \leq \int_{-\infty}^{x} e^{2(2 + 1 W(u)du)} dt.
\]

using the same argument as above. We study the integrability of the right hand side:

i) If \( x \geq b \), we have:

\[
\int_{-\infty}^{x} e^{2(2 + 1 W(u)du)} dt = (x)_{e^{2(2 + 1 W(u)du)}}^b_{-\infty} e^{2(2 + 1 W(u)du)} dt + (x)_{e^{2(2 + 1 W(u)du)}}^x_{-\infty} \left( \frac{1}{2} e^{2f_1^2 T(u)du} dt \right.
\]

Using (4.14), we see that \( e^{-2(2 + 1 W(u)du)} \) and \( e^{2f_1^2 T(u)du} \) are bounded on \([b; 0]\).

Moreover, \((x)_{e^{2(2 + 1 W(u)du)}}^b_{-\infty} e^{2f_1^2 T(u)du} dt\) is integrable on \([b, 0]\) and if \( ml \neq \frac{1}{2} \),

\[
(x)_{e^{2(2 + 1 W(u)du)}}^x_{-\infty} \left( \frac{1}{2} e^{2f_1^2 T(u)du} dt \right) = \frac{x}{1 - 2ml} - (x)_{e^{2(2 + 1 W(u)du)}}^{b-2ml}_{-\infty} \frac{1}{1 - 2ml}.
\]

If \( ml = \frac{1}{2} \), then:

\[
(x)_{e^{2(2 + 1 W(u)du)}}^x_{-\infty} \left( \frac{1}{2} e^{2f_1^2 T(u)du} dt \right) = (x)_{[-\ln(-t)]^x_b} \left( \frac{1}{2} e^{2f_1^2 T(u)du} dt \right)
\]

so \((x)_{e^{2(2 + 1 W(u)du)}}^x_{-\infty} \left( \frac{1}{2} e^{2f_1^2 T(u)du} dt \right)\) is integrable on \([b, 0]\) in the two cases.

ii) If \( a \leq x \leq b \), we obtain a smooth function which is integrable.

iii) If \( x \leq a \), we have:

\[
\int_{-\infty}^{x} e^{2(2 + 1 W(u)du)} dt = C_2^2 C_1^2 e^{-2f_1^2 x} \left( \left( \frac{1}{2f_1^2 x} \right) \right)_{-\infty}^{x} + \int_{-\infty}^{x} \frac{1}{e^{2f_1^2 x} dt} \leq C_2^2 C_1^2 \frac{1}{2f_1^2}
\]

since \( \int_{-\infty}^{x} \frac{1}{e^{2f_1^2 x} dt} \leq 0 \).

This ends the proof of the integrability.

Convergence in \( L^2 \) of integral terms.
We can use the dominate convergence theorem to obtain:

a) \( \int_{0}^{x} (g_1 + g_3^3) e^{-f_1^2 W(u)du + f_1^2 W(u)du} dt \xrightarrow{\nu \to \infty} \int_{0}^{x} (g_1 + g_3^3) e^{-f_1^2 W(u)du + f_1^2 W(u)du} dt. \) (4.18)

b) \( \int_{0}^{x} (g_1 - g_3^3) e^{-f_1^2 W(u)du + f_1^2 W(u)du} dt \xrightarrow{\nu \to \infty} \int_{0}^{x} (g_1 - g_3^3) e^{-f_1^2 W(u)du + f_1^2 W(u)du} dt. \) (4.19)

Study of the sequences \( \lambda_i^n, i = 1, \ldots, 4 \).

a) Let us study the convergence of \( \lambda_i^n \) in (4.11) (we can do the same for \( \lambda_i^1 \)).

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- If $mL < \frac{1}{\gamma}$, we can write:

$$
\| \lambda^3_\nu - \lambda^3 \|_{L^2}^2 \leq 
\left\langle \left( |\varphi^\nu_\lambda - \varphi^\nu_1| - \int_0^\nu (g^\nu_\nu + g^\nu_0) e^{-\int_0^\nu L^\nu_{s,n} W(u) du + \int_0^\nu L^\nu_{s,n} W(u) du} dt \right), e^{\int_0^\nu L^\nu_{s,n} W(u) du} \right\rangle_{L^2}
- \left\langle \left( |\varphi^\nu - \varphi^\nu| - \int_0^\nu (g^1 + g^3) e^{-\int_0^\nu L^\nu_{s,n} W(u) du + \int_0^\nu L^\nu_{s,n} W(u) du} dt \right), e^{\int_0^\nu L^\nu_{s,n} W(u) du} \right\rangle_{L^2}
\xrightarrow{\nu \to \infty} 0,
$$

using that $e^{\int_0^\nu L^\nu_{s,n} W(u) du} \in L^2$, $\varphi_\nu \to \varphi$ and [1.18]. We deduce that $\lambda^3_\nu \to \lambda^3$.

- If $mL \geq \frac{1}{\gamma}$, $e^{\int_0^\nu L^\nu_{s,n} W(u) du} \notin L^2$ and $\lambda^3_\nu = 0$.

b) We then study the convergence of $\lambda^3_\nu$ and $\lambda^2_\nu$.

Since $\varphi^3_\nu + \varphi^3_\nu \in L^2$, $e^{-\int_0^\nu L^\nu_{s,n} W(u) du} \notin L^2$ and the other terms are in $L^2$, we deduce that $\lambda^3_\nu = \lambda^2_\nu = 0$ for all $\nu \in \mathbb{N}$.

Convergence in $L^2$ of the sequences $\varphi^1_\nu - \varphi^1_\nu$, $\varphi^2_\nu - \varphi^2_\nu$, $\varphi^3_\nu - \varphi^3_\nu$, $\varphi^4_\nu - \varphi^4_\nu, \varphi^5_\nu - \varphi^5_\nu, \varphi^6_\nu - \varphi^6_\nu$.

We saw that $\varphi^1_\nu - \varphi^1_\nu$ is pointwise convergent and $|\varphi^1_\nu - \varphi^1_\nu|^2$ is bounded by an integrable function independent of $\nu$. Using the dominate convergence theorem, we deduce that $\varphi^1_\nu - \varphi^1_\nu \xrightarrow{\nu \to \infty} \varphi^1 - \varphi^1$. We also have: $\varphi^2_\nu + \varphi^2_\nu \xrightarrow{\nu \to \infty} \varphi^2 + \varphi^2$, $\varphi^3_\nu \xrightarrow{\nu \to \infty} \varphi^3$ and $\varphi^4_\nu \xrightarrow{\nu \to \infty} \varphi^4$. Thus, the sequence $(\varphi^1_\nu)_{\nu \in \mathbb{N}}$ admits a converging sub-sequence which proves that $K$ is compact.

Consequently, $(H_+ + i)^{-1}$ is compact and so is $(H_+ - z)^{-1}$ for all $z \in \rho(H_+)$ using a resolvent identity.

Q.E.D

4.4 Proof of proposition 4.1

Proof. Let $j_-, j_+ \in C^\infty$ such that $j_+^2 + j_-^2 = 1$, supp($j_-$) $\subset ]-\infty, c]$ and supp($j_+$) $\subset ]0, \infty]$. We define:

$$Q(z) = j_-(x) (H_- - z)^{-1} j_-(x) + j_+(x) (H_+ - z)^{-1} j_+(x).$$

Since $H^-_m - z = H_+ - z$ on $] - \infty, c]$ and $H^+_m - z = H_+ - z$ on $]0, \infty]$, we have:

$$(H^-_m - z) Q(z) = 1 - w(z)$$

where:

$w(z) = - \left( [(H^-_m - z), j_-(x)] (H_- - z)^{-1} j_-(x) + [(H^+_m - z), j_+(x)] (H_+ - z)^{-1} j_+(x) \right).$

Since $[(H^-_m - z), j_-(x)] = i r_0 \gamma_1 j_-(x)$ and $[(H^+_m - z), j_+(x)] = i r_0 \gamma_1 j_+(x)$ and $j_-, j_+$ have compact support, we deduce that $w(z)$ is compact for all $z \in \rho(H)$ using the last two sections. Moreover, $w : \rho(H) \to L(L^2)$ is analytic.

Let us show that $1 - w(z)$ is invertible for some $z$.

Since $j_-, j_+$ are bounded by constants which we denote respectively $c_1, c_2$ and $j_-, j_+$ are bounded by 1, we have:

$$\|w(z) \varphi\|_2 \leq \|i r_0 \gamma_1 j_-(x) (H_- - z)^{-1} j_-(x) \|_2 + \|i r_0 \gamma_1 j_+(x) (H_+ - z)^{-1} j_+(x) \varphi\|_2 \leq \frac{c_1 + c_2}{\|3z\|} \|\varphi\|_2,$$

for all $\varphi \in L^2$. We then choose $z$ such that the imaginary part satisfies $\frac{\|3z\|}{c_1 + c_2} < 1$. Therefore, $1 - w(z)$ is invertible. Using the analytic Fredholm theorem, we have that $1 - w(z)$ is invertible.
for all $z \in \rho(H) \setminus S$ where $S$ is a discrete set without accumulation points.

For these $z$, we deduce that:

$$ (H_{m,n}^z - z)^{-1} = Q(z) (1 - w(z))^{-1}. $$  \hfill (4.20)

Let $f$ be a continuous function going to 0 at $-\infty$ and admitting a finite limit at 0. Then $f(z)Q(z)$ is compact. Thus for $z \in \rho(H) \setminus S$, $f(z)(H_{m,n}^z - z)^{-1}$ is compact. If $s \in S$, we can choose a sequence $(z_n)_{n \in \mathbb{N}} \in (\rho(H_{m,n}^z) \setminus S)^\cap$ such that $z_n \to s$. Then, by analyticity of $z \to (H_{m,n}^z - z)^{-1}$, we have:

$$ (H_{m,n}^z - z_n)^{-1} \to (H_{m,n}^z - s)^{-1}. $$

Consequently, $f(x)(H_{m,n}^z - s)^{-1}$ is compact for all $s \in S$. We deduce that $f(x)(H_{m,n}^z - z)^{-1}$ is compact for all $z \in \rho(H_{m,n}^z)$.

Q.E.D

5 Mourre estimates

5.1 Mourre theory

We recall here some facts about Mourre theory. We say that the pair $(A, H)$ satisfies Mourre conditions if

$$ D(A) \cap D(H) \text{ is dense in } D(H) $$  \hfill (5.1)

$$ e^{itA} \text{ preserves } D(H) \text{ for } t > 0, \sup_{|t| \leq 1} \|He^{itA}u\| < \infty, \forall u \in D(H). $$  \hfill (5.2)

$$ [iH, A] \text{ defined as quadratic form on } D(H) \cap D(A) \text{ extend to a bounded operator from } D(H) \text{ into } \mathcal{H}. $$  \hfill (5.3)

Mourre conditions are stronger than $C^1(A)$ regularity. We recall the definition of $C^k(A)$:

**Definition 5.1.** We say that $H \in C^k(A)$ if there exists $z \in \mathbb{C} \setminus \sigma(H)$ such that

$$ \mathbb{R} \ni t \mapsto e^{itA} (z - H)^{-1} e^{-itA} $$  \hfill (5.4)

is $C^k$ for the strong topology of $\mathcal{L}(\mathcal{H})$.

We then have the following lemma (see [1 Proposition 5.1.2, Theorem 6.3.4]):

**Lemma 5.2.** Suppose that $(H, A)$ satisfies the Mourre conditions. Then $H \in C^1(A)$.

We also recall a lemma concerning the $C^2(A)$ regularity:

**Lemma 5.3.** Suppose that $H \in C^1(A)$ and that the commutator $[iA, H]$ extends to a bounded operator from $D(H)$ into $\mathcal{H}$. We denote $[iA, H]_0$ this extension. If, in addition, the commutator $[iA, [iA, H]]_0$ defined as a quadratic form on $D(A) \cap D(H)$ extends to a bounded operator from $D(H)$ into $D(H)^*$, then $H \in C^2(A)$.  

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5.2 Mourre estimate for Dirac operator in Schwarzschild-AdS space-time

We will use the following conjugate operator:

\[ A = \Gamma x \]  

(5.5)

where \( \Gamma = -\gamma^0\gamma^1 = \text{diag}(1, -1, -1, 1) \). We equip this operator with the domain:

\[ D(A) = \{ \varphi \in \mathcal{H}_{s,n}; \ A\varphi \in \mathcal{H}_{s,n} \}. \]  

(5.6)

The operator \( A \) is self-adjoint on its domain. Indeed, it is clearly symmetric. Moreover,

\[ D(A^*) = \{ \psi \in \mathcal{H}_{s,n}; \ \exists \chi \in \mathcal{H}_{s,n}, \langle \psi, A\varphi \rangle = \langle \chi, \varphi \rangle \ \forall \varphi \in D(A) \}. \]

But, for all \( \varphi \in C_0^\infty \subset \mathcal{H}_{s,n} \), we see that

\[ \langle \psi, A\varphi \rangle = \langle A\psi, \varphi \rangle = \langle \chi, \varphi \rangle. \]

Since \( C_0^\infty \) is dense in \( \mathcal{H}_{s,n} \), we deduce that \( \chi = A\psi \) so that \( D(A^*) = D(A) \).

**Lemma 5.4.** For all \( m > 0 \), the pair \((H^{n,n}_m, A)\) satisfies the Mourre conditions. Consequently, \( H^{n,n}_m \subset C(A) \)

**Proof.** Let us check (5.1):

**Case 2ml < 1:**

Let \( \chi \) be a \( C^\infty \) function such that \( \chi = 1 \) on \([-1, 0] \), \( \supp \chi \subset [-2, 0] \). We set \( \chi_k(x) = \chi \left( \frac{x}{k} \right) \) for all \( k \in \mathbb{N} \setminus \{0\} \) such that \( \supp \chi_k(x) = 1 \) on \([-k, 0] \). We have \( \chi_k(x) = \frac{1}{k} \chi \left( \frac{x}{k} \right) \) so that it is bounded. Using these facts, we see that \( \chi_k\varphi \in D(A) \cap D(H^{n,n}_m) \) if \( \varphi \in D(H^{n,n}_m) \).

We now show that \( \chi_k\varphi \xrightarrow{k \to \infty} \varphi \) for the norm: \( \|\varphi\|_{H^{n,n}_m} = \|\varphi\|_{\mathcal{H}_{s,n}} + \|H^{n,n}_m\varphi\|_{\mathcal{H}_{s,n}} \).

By dominate convergence theorem we have \( \chi_k\varphi \xrightarrow{k \to \infty} \varphi \). Moreover, \( |\chi_k'(x)| \leq \frac{C}{k} \) where \( C = \sup_{x \in [-\infty, 0]} |\chi'(x)| \). So:

\[ \left\| H^{n,n}_m\varphi - H^{n,n}_m\chi_k\varphi \right\| \leq \frac{C}{k} \left\| \varphi \right\| + \left\| H^{n,n}_m\varphi - \chi_k H^{n,n}_m\varphi \right\|. \]

which gives the desired result when \( k \) goes to infinity for \( \varphi \in D(H^{n,n}_m) \). We deduce (5.1).

We denote \( D(H^{n,n}_m) = \{ \chi_k\varphi; \ \varphi \in D(H^{n,n}_m), \ k \in \mathbb{N} \setminus \{0\} \} \).

**Case 2ml \geq 1:**

In this case, \( C_0^\infty \subset [-\infty, 0] \) is a subset of \( D(A) \cap D(H^{n,n}_m) \) and is dense in \( D(H^{n,n}_m) \) so (5.1) is proved.

Let us check (5.2):

For all \( s > 0 \),

\[ e^{itA} = \text{diag}(e^{itx}, e^{-itx}, e^{-itx}, e^{itx}). \]

Let \( \varphi \in D(H^{n,n}_m) \), then:

\[ -e^{itA}\varphi \in \mathcal{H}_{s,n}. \]

- \( H^{n,n}_m e^{itA} \varphi = e^{itA} H^{n,n}_m \varphi + te^{itA} \varphi \). So \( H^{n,n}_m e^{itA} \varphi \in \mathcal{H}_{s,n} \) and \( \sup_{|t| \leq 1} \|H^{n,n}_m e^{itA} \varphi\| < \infty \).

We need to check the boundary condition in the case 2ml < 1. We have:

\[ \left\| \langle \gamma^1 + 1 \rangle e^{itA} \varphi(x, \cdot) \right\|_{W_0^0} = \left\| \begin{pmatrix} ie^{itx}\varphi_1 + ie^{-itx}\varphi_3 \\ ie^{itx}\varphi_2 - ie^{-itx}\varphi_4 \\ ie^{itx}\varphi_1 + ie^{-itx}\varphi_3 \\ -ie^{-itx}\varphi_2 + ie^{itx}\varphi_4 \end{pmatrix} \right\|_{L^2(\mathbb{S}^2)}. \]
Let us, for example, consider the term: \( \|e^{itx}\varphi_1 + ie^{-itx}\varphi_2\|_{L^2(S^2)} \) in a neighbourhood of zero. Using Taylor expansion of \( e^{itx} \) and \( e^{-itx} \), we have

\[
\|e^{itx}\varphi_1 + ie^{-itx}\varphi_2\|_{L^2(S^2)} \leq \|i(\varphi_1(x,.) + \varphi_2(x,.))\|_{L^2(S^2)} - 2x \left( \|\varphi_1(x,.)\|_{L^2(S^2)} + \|\varphi_2(x,.)\|_{L^2(S^2)} \right).
\]

Since \( \varphi \in D(H_{m}^{n}) \), there exists functions \( \psi_{-} \in W^{\frac{1}{2}}, \chi_{-} \in W^{\frac{1}{2}}_{*}, \psi_{+}, \chi_{+} \in L^{2}(S^2) \) and \( \phi \in C^{0}([0, \frac{1}{2}] ; L^{2}(S^2; C^{1})) \) satisfying:

\[
\psi_{s,n} = (-x^{-ml}) \begin{pmatrix}
\psi_{s,n}^{+}(\theta, \varphi) \\
\chi_{s,n}^{+}(\theta, \varphi) \\
i\xi_{s,n}^{+}(\theta, \varphi)
\end{pmatrix} + \phi_{s}(r_{*}, \theta, \varphi)
\]

where:

\[
\|\phi_{s}(r_{*}, \theta, \varphi)\|_{W^{0}} = o \left( \sqrt{-x} \right), \quad x \to 0.
\]

We thus obtain:

\[
-2x \|\varphi_1(x,.)\|_{L^2(S^2)} \leq 2(-x)^{-1-ml} \left\| \begin{pmatrix}
\psi_{s,n}^{+}(\theta, \varphi) \\
\chi_{s,n}^{+}(\theta, \varphi) \\
i\xi_{s,n}^{+}(\theta, \varphi)
\end{pmatrix} \right\|_{L^2(S^2)} - 2x \left( o \left( \sqrt{-x} \right) \right).
\]

Since \( 1 - ml > \frac{1}{2} \) when \( ml < \frac{1}{2} \), we have that \( -2x \|\varphi_1(x,.)\|_{L^2(S^2)} = o \left( \sqrt{-x} \right) \). Since \( \varphi \in D(H_{m}^{n}) \), we deduce that the right hand side of this inequality is \( o \left( \sqrt{-x} \right) \) and so is the left hand side. We can do the same for other components. This proves (5.2).

Let us check (5.3):

First, we see that \( xA(x) \) and \( xB(x) \) are bounded functions on \( (-\infty, 0) \]. Let \( u, v \in D(H_{m}^{n}) \). In the case \( 2ml < 1 \) and \( u, v \in C_{0}^{\infty}(-\infty, 0) \) in the case \( 2ml \geq 1 \), we have:

\[
[H_{m}^{n}, iA] \{u, v\} = \left\{ u + 2ixA(x)\gamma^{1}u + 2imxB(x)\gamma^{1}u, v \right\}.
\]

This shows that:

\[
\|H_{m}^{n}, iA\{u, v\}\| \leq C_{1} \|u\|_{H_{s,n}} \|v\|_{H_{s,n}}
\]

for some constant \( C_{1} \) and consequently, (5.3) is satisfied.

We deduce that the Mourre conditions are satisfied by the pair \( (H_{m}^{n}, A) \). So \( H_{m}^{n} \in C^{1}(A) \).

We then have the following:

**Proposition 5.5.** Recall that \( A = \Gamma x \). Let \( I \subset \mathbb{R} \) be a compact non-empty interval. Then, for all \( m > 0 \), we have:

\[
1_{I}(H_{m}^{n}) [H_{m}^{n}, iA] 1_{I}(H_{m}^{n}) \geq 1_{I}^{2}(H_{m}^{n}) + 1_{I}(H_{m}^{n}) K 1_{I}(H_{m}^{n})
\]

(5.8)

where \( 1_{I} \) is the characteristic function of \( I \) and \( K \) is a compact operator.

**Proof.** We remark that \( xA(x) \to 0 \), that \( xB(x) \to 0 \) and that \( xB(x) \to -l \) using the asymptotic behaviour of \( A \) and \( B \) described in (5.2) and (5.4). We obtain:

\[
[H_{m}^{n}, iA] \geq \text{Id} - (2s + 1)xA(x)\gamma^{1}\gamma^{1} - 2mxB(x)\gamma^{1}.
\]

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Consider a compact non-empty interval $I \subset \mathbb{R}$ and $\bar{I}$ a compact interval strictly containing $I$. Let $\varsigma \in C_0^\infty (\bar{I})$ such that $\varsigma \equiv 1$ on $I$. Using Lemma 4.1 and the asymptotic behaviour of $A$ and $B$, we know that:

$$\varsigma (H_n^{s,m}) (- (2s + 1) \gamma^2 \gamma^1 - 2mx \gamma^1) \varsigma (H_m^{s,n})$$

is compact and we denote it by $K$. Consequently:

$$\varsigma (H_n^{s,m}) [H_n^{s,m}, iA] \varsigma (H_n^{s,m}) \geq \varsigma^2 (H_m^{s,n}) + K. \quad (5.9)$$

Multiplying both sides by $\mathbb{1}_I (H_n^{s,m})$, this gives the desired result since $\mathbb{1}_I \varsigma = \mathbb{1}_I$. Q.E.D

Using the absence of eigenvalues, we deduce the following corollary:

**Corollary 5.6.** For all $m > 0$ and all $\lambda \in \mathbb{R}$, there exists a compact non-empty interval $I' \subset \mathbb{R}$ containing $\lambda$ such that:

$$\mathbb{1}_{I'} (H_n^{s,m}) [H_n^{s,m}, iA] \mathbb{1}_{I'} (H_n^{s,m}) \geq (1 - \epsilon) \mathbb{1}_{I'}^2 (H_n^{s,m}) \quad (5.10)$$

for all $0 < \epsilon < 1$. Recall that $\mathbb{1}_{I'}$ is the characteristic function of $I'$.

**Proof.** We saw that, for a compact non-empty interval $I \subset \mathbb{R}$ containing $\lambda$, we have the estimate:

$$\mathbb{1}_I (H_n^{s,m}) [H_n^{s,m}, iA] \mathbb{1}_I (H_n^{s,m}) \geq \mathbb{1}_I^2 (H_n^{s,m}) + \mathbb{1}_I (H_n^{s,m}) K \mathbb{1}_I (H_n^{s,m}).$$

Let $I' \subset I$ such that $\lambda \in I'$. We can multiply both sides by $\mathbb{1}_{I'} (H_n^{s,m})$ to obtain the same inequality with $I$ replaced by $I'$. Since $\lambda$ is not an eigenvalue of $H_n^{s,m}$, $\mathbb{1}_{I'} (H_n^{s,m})$ tends strongly to 0 when the size of $I'$ decreases. Then $\mathbb{1}_{I'} (H_n^{s,m}) K \mathbb{1}_{I'} (H_n^{s,m})$ goes to 0 in the operator norm ($K$ is compact). We can thus choose $I'$ sufficiently small such that:

$$\mathbb{1}_{I'} (H_n^{s,m}) [H_n^{s,m}, iA] \mathbb{1}_{I'} (H_n^{s,m}) \geq (1 - \epsilon) \mathbb{1}_{I'}^2 (H_n^{s,m})$$

for all $0 < \epsilon < 1$. Q.E.D

# 6 Propagation estimates

In this section, we first present abstract results about propagation estimates and the minimal velocity estimate. Then, we apply this to prove that our minimal velocity is 1 and our maximal velocity is also 1. This will be useful in the proof of asymptotic completeness.

## 6.1 Abstract propagation estimates

We present the abstract theory of propagation estimates. Proofs can be found in [14].

Consider a Hilbert space $\mathcal{H}$ and $(H, D (H))$ a self-adjoint operator on $\mathcal{H}$. Let $\Phi (t)$ be a $C^1$ uniformly bounded function with values in $\mathcal{L} (\mathcal{H})$ defined on $\mathbb{R}^+$. We define the Heisenberg derivative of $\Phi$ by:

$$D \Phi (t) := \frac{d}{dt} \Phi (t) + i [H, \Phi (t)].$$
Lemma 6.1. Lemma B.4.1, B.4.2] Let \( \Phi (t) \) be a \( C^1 \) uniformly bounded function with values in \( \mathcal{L}(\mathcal{H}) \) and defined on \( \mathbb{R}^+ \).

i) If there exists measurable functions with values in \( \mathcal{L}(\mathcal{H}) \) \( B_i (t) \), \( i = 1, \cdots, n \) with

\[
\mathbb{D} \Phi (t) \geq C_0 B^* (t) B (t) - \sum_{i=1}^{n} B^*_i (t) \left| B_i (t) \right|
\]

such that for all \( i \in \{1, \cdots, n\} \)

\[
\int_{1}^{\infty} \left\| B_i (t) e^{-it\mathcal{H}} u \right\|^2 dt \leq C \left\| u \right\|^2, \quad \forall u \in \mathcal{H}
\]

then there exists a constant \( C_1 > 0 \) such that

\[
\int_{1}^{\infty} \left\| B (t) e^{-it\mathcal{H}} u \right\|^2 dt \leq C_1 \left\| u \right\|^2, \quad \forall u \in \mathcal{H}.
\]

ii) Suppose that the function \( \Phi \) satisfies

\[
\left| \left\langle \psi_2, \mathbb{D} \Phi (t) \psi_1 \right\rangle \right| \leq \sum_{i=1}^{n} \left\| B_{2,i} (t) \psi_2 \right\| \left\| B_{1,i} (t) \psi_1 \right\|,
\]

with

\[
\int_{1}^{\infty} \left\| B_{2,i} (t) e^{-it\mathcal{H}} u \right\|^2 dt \leq C_1 \left\| u \right\|^2, \quad \forall u \in \mathcal{H}
\]

and

\[
\int_{1}^{\infty} \left\| B_{1,i} (t) e^{-it\mathcal{H}} u \right\|^2 dt \leq C_1 \left\| u \right\|^2, \quad \forall u \in \mathcal{D},
\]

where \( \mathcal{D} \) is a dense subset of \( \mathcal{H} \). Then the limit

\[
s - \lim_{t \to -\infty} e^{it\mathcal{H}} \Phi (t) e^{-it\mathcal{H}} = 0.
\]

6.1.2 Abstract minimal velocity estimates

Proposition 6.2. Proposition A.1] Let \( H \in C^{1+\epsilon} (A) \). Let \( \Delta \) be an interval such that

\[
1_{\Delta} (H) \left| [H, iA] 1_{\Delta} (H) \right| \geq \epsilon_0 1_{\Delta} (H).
\]

Then, for all \( g \in C_0^\infty (\mathbb{R}) \), \( \text{supp} \, g \subset (-\infty, c_0) \) and for \( f \in C_0^\infty (\Delta) \), we have

\[
\int_{1}^{\infty} \left\| g \left( \frac{A}{t} \right) f (H) e^{-it\mathcal{H}} u \right\|^2 dt \leq C \left\| u \right\|^2, \quad \forall u \in \mathcal{H},
\]

\[
s - \lim_{t \to -\infty} g \left( \frac{A}{t} \right) f (H) e^{-it\mathcal{H}} = 0.
\]

6.2 Propagation estimates for Dirac operator in Schwarzschild Anti-de Sitter space-time

We have seen that \( [H_{m,n}, iA] \) admits a bounded extension from \( D(A) \cap D(H_{m,n}) \) to \( D(H_{m,n}) \). We denote this extension by \( [H_{m,n}, iA]_0 \). We have:

\[
[H_{m,n}, iA]_0 = 4 \left( s + \frac{1}{2} \right) x^2 A (x) \gamma^2 \gamma^0 + m x^2 B (x) \gamma^0
\]

so \( [H_{m,n}, iA]_0, iA \) extends to a bounded operator to \( D(H_{m,n}) \) with values in \( \mathcal{H}_{m,n} \). Using lemma 5.3 we deduce that \( H \in C^2 (A) \). Using Mourre estimates and a partition of unity argument, this gives:
Proposition 6.3. For all \( m > 0 \), \( g \in C_0^\infty (\mathbb{R}) \), \( \text{supp} (g) \subset (-\infty, 1-\delta) \) and \( f \in C_0^\infty (\mathbb{R}) \), we have:

\[
\int_1^{\infty} \left\| g \left( \frac{A}{t} \right) f (H_m^{s,n}) e^{-itH_m^{s,n}} u \right\|^2 \frac{dt}{t} \leq C \| u \|^2, \quad \forall u \in \mathcal{H}_{s,n}, \tag{6.2}
\]

\[
s - \lim_{t \to \infty} g \left( \frac{A}{t} \right) e^{-itH_m^{s,n}} u = 0.
\tag{6.3}
\]

Proof of proposition 6.3. Using the corollary after proposition 3.11, we obtain

\[
\int_1^{\infty} \left\| g \left( \frac{A}{t} \right) f (H_m^{s,n}) e^{-itH_m^{s,n}} u \right\|^2 \frac{dt}{t} \leq C \| u \|^2, \quad \forall u \in \mathcal{H}_{s,n},
\]

\[
\int_1^{\infty} g \left( \frac{A}{t} \right) f (H_m^{s,n}) e^{-itH_m^{s,n}} u \right\|^2 \frac{dt}{t} \leq C \| u \|^2, \quad \forall u \in \mathcal{H}_{s,n},
\]

\[
s - \lim_{t \to \infty} g \left( \frac{A}{t} \right) f (H_m^{s,n}) e^{-itH_m^{s,n}} u = 0,
\]

for \( f \in C_0^\infty (I) \) by the abstract velocity estimate. For \( f \in C_0^\infty (\mathbb{R}) \), we can cover \( \text{supp} (f) \) by a finite number of intervals \( I_1, \ldots, I_n \) where a Mourre estimate holds. Then, we consider a partition of unity subordinate to this cover \( \eta_1, \ldots, \eta_n \) and we note \( f_i = \eta_i f \) for all \( i = 1, \ldots, n \). Then:

\[
\int_1^{\infty} \left\| g \left( \frac{A}{t} \right) f (H_m^{s,n}) e^{-itH_m^{s,n}} u \right\|^2 \frac{dt}{t} \leq C_n \| u \|^2, \quad \forall u \in \mathcal{H}_{s,n},
\]

and:

\[
s - \lim_{t \to \infty} g \left( \frac{A}{t} \right) f (H_m^{s,n}) e^{-itH_m^{s,n}} u = \sum_{i=1}^{n} s - \lim_{t \to \infty} g \left( \frac{A}{t} \right) f_i (H_m^{s,n}) e^{-itH_m^{s,n}} u = 0.
\]

Thanks to a density argument, we obtain the desired limit. Q.E.D

Proposition 6.3 allows us to obtain:

Lemma 6.4. Let \( J_- \in C^\infty \) such that \( \text{supp} (J_-) \subset ]-\infty, 1-\epsilon[ \) and \( J_- (x) = 1 \) for all \( x \in ]-\infty, 1-2\epsilon[ \) and \( \text{supp} (\chi) \subset C_0^\infty \). Then, for all \( m > 0 \), we have:

\[
\int_1^{\infty} \left\| J_- \left( \frac{A}{t} \right) \chi (H_m^{s,n}) e^{-itH_m^{s,n}} u \right\|^2 \frac{dt}{t} \leq C \| u \|^2, \quad \forall u \in \mathcal{H}_{s,n}, \tag{6.4}
\]

\[
\lim_{t \to \infty} J_- \left( \frac{A}{t} \right) e^{-itH_m^{s,n}} u = 0, \quad \forall u \in \mathcal{H}_{s,n}. \tag{6.5}
\]

Proof. 1) Let \( \theta_1, \theta_2 \in C^\infty \) such that \( \text{supp} (\theta_1) \subset ]-\infty, -1-\epsilon[ \), \( \text{supp} (\theta_2) \subset ]1-\epsilon, 1-\epsilon[ \) and \( \theta_1 + \theta_2 = 1 \). Then, using triangular inequality and minimal velocity estimates, we only need to prove the integral estimate for \( \theta_1 J_- \).

So suppose that \( K \in C^\infty \) such that \( \text{supp} (K) \subset ]-\infty, -1-\frac{\epsilon}{2}[ \) and \( K (x) = 1 \) for all \( x \in ]-\infty, -1-\epsilon[ \). We define \( F (s) = \int_s^{\infty} K^2 (t) dt \) and

\[
\Phi (t) = \chi (H_m^{s,n}) F \left( \frac{A}{t} \right) \chi (H_m^{s,n})
\]

such that \( \Phi \) is \( C^1 \) uniformly bounded. We have:

\[
D \Phi (t) = \frac{1}{t} \chi (H_m^{s,n}) \frac{A}{t} K^2 \left( \frac{A}{t} \right) \chi (H_m^{s,n}) + i \chi (H_m^{s,n}) \left[ H_m^{s,n}, F \left( \frac{A}{t} \right) \right] \chi (H_m^{s,n}).
\]

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where

\[ H_{m,n}^s, F \left( \frac{A}{t} \right) = \frac{1}{t} K^2 \left( \frac{A}{t} \right) + \left( s + \frac{1}{2} \right) A \left( x \left( F \left( -\frac{x}{t} \right) - F \left( \frac{x}{t} \right) \right) \right) \gamma^1 \gamma^2 \]

with

\[ \left| F \left( -\frac{x}{t} \right) - F \left( \frac{x}{t} \right) \right| \leq \frac{2x}{t} \sup_{y \in \left[ -\frac{1}{t}, \frac{1}{t} \right]} K^2(y). \]

This supremum is non zero if \( x \leq (-1 - \frac{2}{t}) t \) which gives us:

\[ \sup_{y \in \left[ -\frac{1}{t}, \frac{1}{t} \right]} K^2(y) \leq 1 \{ x \leq (-1 - \frac{2}{t}) t \}, \]

where \( I \) is the characteristic function and \( \sup_{y \in \left[ -\frac{1}{t}, \frac{1}{t} \right]} K^2(y) \) is thought as a function depending on the variables \( x \) and \( t \). We know that for \( x \) sufficiently small, the functions \( A \) and \( B \) are exponentially decaying. If we fix \( T \) sufficiently large, then, for all \( t \geq T \), we have:

\[ \left| A \left( x \left( F \left( -\frac{x}{t} \right) - F \left( \frac{x}{t} \right) \right) \right) \right| \leq \frac{2x}{t} \left( x \leq (-1 - \frac{2}{t}) t \right) \]

\[ \leq C \frac{1}{t^2} \left( x \leq (-1 - \frac{2}{t}) t \right) \]

because \( e^x \leq \frac{1}{1-t} \) for \( x \) sufficiently small. We can do the same thing with \( B \). We obtain:

\[ -\Phi(t) = \frac{1}{t} \chi \left( H_{m,n}^s \right) \left( 1 - \frac{A}{t} \right) K^2 \left( \frac{A}{t} \right) \chi \left( H_{m,n}^s \right) + O \left( t^{-2} \right) \]

\[ \geq \frac{2 + \frac{\delta}{t} \chi \left( H_{m,n}^s \right) K^2 \left( \frac{A}{t} \right) \chi \left( H_{m,n}^s \right) + O \left( t^{-2} \right). \]

since \( \frac{\delta}{t} \leq -1 - \frac{2}{t} \) on the support of \( K^2 \). This shows that:

\[ \int_1^\infty \left\| K \left( \frac{A}{t} \right) \chi \left( H_{m,n}^s \right) e^{-itH_{m,n}^s} u \right\|^2 dt \leq C \| u \|^2 \quad (6.6) \]

for all \( u \in \mathcal{H}_{s,n} \). Thus, we have shown that the first term is integrable which prove the first statement of the lemma.

2) We next set:

\[ \Phi(t) = \chi \left( H_{m,n}^s \right) J^2 \left( \frac{A}{t} \right) \chi \left( H_{m,n}^s \right). \]

So, we have:

\[ \Phi(t) = \chi \left( H_{m,n}^s \right) J^2 \left( \frac{A}{t} \right) \chi \left( H_{m,n}^s \right) + O \left( t^{-2} \right) \]

where \( \text{supp} \left( J_- J_- \right) \subset \left[ 1 - 2\epsilon, 1 - \epsilon \right] \) so it is integrable by minimal velocity estimates. This gives

\[ \lim_{t \to \infty} e^{itH_{m,n}^s} \chi \left( H_{m,n}^s \right) J^2 \left( \frac{A}{t} \right) \chi \left( H_{m,n}^s \right) e^{-itH_{m,n}^s} u = 0, \forall u \in \mathcal{H}_{s,n}. \]

Using the last lemma, we obtain the desired limit by a density argument. Q.E.D

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Proposition 6.5. Let $g \in C^\infty$ such that $\text{supp}(g) \subset |1 + \epsilon, \infty|$ with $\epsilon > 0$ and such that $g(x) = 1$ for all $x \in [1 + 2\epsilon, \infty]$. Let $\zeta \in C_0^\infty(\mathbb{R})$. Then, for all $m > 0$, we have:

$$\int_1^\infty \left\| g \left( \frac{A}{t} \right) e^{-itH_{m,n}^s} \zeta(H_{m,n}^s) u \right\|^2 \frac{dt}{t} \leq C \|u\|^2, \quad \forall u \in \mathcal{H}_{s,n} \quad (6.7)$$

and

$$s - \lim_{t \to \infty} g \left( \frac{A}{t} \right) e^{-itH_{m,n}^s} = 0. \quad (6.8)$$

**Proof of the proposition.** Let $J \in C^\infty(\mathbb{R})$ such that $\text{supp}(J) \subset (1 + \epsilon, +\infty)$ with $\epsilon > 0$ and $J(x) = 1$ for all $x \in [1 + 2\epsilon, +\infty]$. Let $\zeta \in C_0^\infty(\mathbb{R})$. We define

$$F(s) = \int_{-\infty}^s J^2(u) \, du$$

and

$$\Phi(t) = \zeta(H_{m,n}^s) F \left( \frac{A}{t} \right) \zeta(H_{m,n}^s)$$

so that $\Phi$ is $C^1$ uniformly bounded. As in the last proof, we calculate the Heisenberg derivative of $\Phi$ and thanks to the support of $J$, we obtain:

$$-\partial_t \Phi(t) \geq \frac{A}{t} \zeta(H_{m,n}^s) J^2 \left( \frac{A}{t} \right) \zeta(H_{m,n}^s) + \zeta(H_{m,n}^s) \left( i \left( s + \frac{1}{2} \right) A(x) \left( F \left( \frac{-x}{t} \right) - F \left( \frac{x}{t} \right) \right) \gamma^2 + \Im B(x) \left( F \left( \frac{-x}{t} \right) - F \left( \frac{x}{t} \right) \right) \gamma^1 \right) \zeta(H_{m,n}^s), \quad (6.9)$$

and we have:

$$\left| F \left( \frac{-x}{t} \right) - F \left( \frac{x}{t} \right) \right| \leq \frac{2x}{t} \sup_{y \in \left[ \frac{x}{t}, \frac{A}{t} \right]} J^2(y) 1_{\{1 + \epsilon < \frac{x}{t}\}}$$

where $1$ means the characteristic function. Using the exponential decay of $A$ and $B$, we obtain:

$$\zeta(H_{m,n}^s) \left( i \left( s + \frac{1}{2} \right) A(x) \left( F \left( \frac{-x}{t} \right) - F \left( \frac{x}{t} \right) \right) \gamma^2 + \Im B(x) \left( F \left( \frac{-x}{t} \right) - F \left( \frac{x}{t} \right) \right) \gamma^1 \right) \zeta(H_{m,n}^s) = O \left( e^{-\frac{t}{\epsilon}} \right), \quad (6.10)$$

for $t$ sufficiently large. We deduce that:

$$\int_1^\infty \left\| g \left( \frac{A}{t} \right) e^{-itH_{m,n}^s} \zeta(H_{m,n}^s) u \right\|^2 \frac{dt}{t} \leq C \|u\|^2, \quad \forall u \in \mathcal{H}_{s,n}. \quad (6.11)$$

Next, we use:

$$\Phi(t) = \zeta(H_{m,n}^s) J^2 \left( \frac{A}{t} \right) \zeta(H_{m,n}^s),$$

and obtain:

$$\partial_t \Phi(t) = \frac{2t}{A} \zeta(H_{m,n}^s) \frac{-A}{t} J \left( \frac{A}{t} \right) J^\prime \left( \frac{A}{t} \right) \zeta(H_{m,n}^s) + \frac{2t}{A} \zeta(H_{m,n}^s) J \left( \frac{A}{t} \right) J^\prime \left( \frac{A}{t} \right) \zeta(H_{m,n}^s)$$

$$+ \zeta(H_{m,n}^s) \left( i \left( s + \frac{1}{2} \right) A(x) \left( J^2 \left( \frac{-x}{t} \right) - J^2 \left( \frac{x}{t} \right) \right) \gamma^2 + \Im B(x) \left( J^2 \left( \frac{-x}{t} \right) - J^2 \left( \frac{x}{t} \right) \right) \gamma^1 \right) \zeta(H_{m,n}^s).$$

The first two terms are integrable due to the support of $J$ and $(6.11)$. The last two are also integrable using the support of $J$. Consequently:

$$s - \lim_{t \to \infty} J \left( \frac{A}{t} \right) e^{-itH_{m,n}^s} \zeta(H_{m,n}^s)$$

exists and is zero by $(6.11)$. By a density argument, we obtain the last statement of the proposition.

Q.E.D

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7 Asymptotic completeness

7.1 Comparison operator

Our comparison operator will be \( H_c \) defined by:

\[
H_c = i \gamma^0 \gamma^1 \partial_x
\]

where \( \gamma^0 \gamma^1 = \text{diag}(-1, 1, 1, -1) \) and with domain:

\[
D(H_c) = \{ \varphi \in H_{s,n}; H_c \varphi \in H_{s,n}, \varphi_1(0) = -\varphi_3(0), \varphi_2(0) = \varphi_4(0) \}
\]

Using proposition 4.2, this is a self-adjoint operator on its domain.

7.2 Asymptotic completeness result for Dirac operator in the Schwarzschild-AdS space-time

Recall that \( A = \Gamma x \) where \( \Gamma = -\gamma^0 \gamma^1 \). We have:

**Theorem 7.1** (Asymptotic completeness for fixed harmonics). For all \( m > 0 \) and all \( \varphi \in H_{s,n} \), the limits

\[
\lim_{t \to \infty} e^{itH_c} e^{-itH_{m,n}} \varphi
\]

exist. If we denote them by:

\[
\Omega_{s,n} \varphi = \lim_{t \to \infty} e^{itH_c} e^{-itH_{m,n}} \varphi \quad \text{(7.5)}
\]

\[
W_{s,n} \varphi = \lim_{t \to \infty} e^{itH_{m,n}} e^{-itH_c} \varphi \quad \text{(7.6)}
\]

for all \( \varphi \in H_{s,n} \), we have \( \Omega_{s,n}^* = W_{s,n} \).

**Proof.** Let \( J_-, J_0, J_+ \in C^\infty \) such that \( J_- + J_0 + J_+ = 1 \), the supports of \( J_- \), \( J_+ \) are as in 6.5 and 6.4 and \( J_0 = 1 \) on \([1-\epsilon, 1+\epsilon]\), \( \text{supp}(J_0) \subset [1-2\epsilon, 1+2\epsilon] \) with \( \epsilon > 0 \). Using proposition 6.5 and lemma 6.4, it suffices to prove that, for all \( \varphi \in H_{s,n} \), the limit:

\[
\lim_{t \to \infty} e^{itH_c} J_0 \left( \frac{A}{t} \right) e^{-itH_{m,n}} \varphi
\]

exists. We remark that \( J_0 \left( \frac{A}{t} \right) \neq 0 \) if and only if \( x \geq (1-2\epsilon) t > 0 \). Since \( x < 0 \), \( J_0 \left( \frac{A}{t} \right) = 0 \), for all \( t > 0 \) and \( x < 0 \). We thus have:

\[
J_0 \left( \frac{A}{t} \right) = J_0 \left( \frac{-x}{t} \right) M_0
\]

where \( M_0 = \text{diag}(0, 1, 1, 0) \). We then define:

\[
\Phi(t) = \chi(H_c) J_0 \left( \frac{A}{t} \right) \chi(H_{m,n}^{s,n})
\]

and we have:

\[
\mathbb{D}\Phi(t) = \frac{d}{dt} \Phi(t) + i (H_c \Phi(t) - \Phi(t) H_{m,n}^{s,n})
\]

Denoting \( V(x) = (s + \frac{1}{2}) A(x) \gamma^0 \gamma^2 - m B(x) \gamma^0 \), we have:

\[
\mathbb{D}\Phi(t) = \frac{2}{t} \chi(H_c) \left( \frac{x}{t} + 1 \right) (J_0 J_0) \left( \frac{-x}{t} \right) M_0 \chi(H_{m,n}^{s,n}) - i \chi(H_c) J_0 \left( \frac{-x}{t} \right) M_0 V(x) \chi(H_{m,n}^{s,n})
\]

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On the support of $J_0^* J_0$, we have $\frac{x}{t} + 1 \leq 2e$. Moreover, $J_0 \left( \frac{x}{t} \right) \neq 0$ if and only if $- (1 + 2e) t \leq x \leq -(1 - 2e) t$. Since $A, B$ are exponentially decreasing at $-\infty$, we obtain:

$$
\mathbb{D}(t) \leq \frac{4e}{T} \chi \left( H_0 \right) \left( J_0^* J_0 \right) \left( H_{x,n}^* \right) + O \left( t^{-2} \right).
$$

Using the support of $J_0^* J_0$, minimal and maximal velocity estimates, the right hand side is integrable. Hence the limit exists. We can show that the second limit exists in the same way. Finally, for all $t > 0$ and $\varphi, \psi \in H_{s,n}$, we have $\left\langle e^{itH_c} e^{-itH_{x,n}^*} \varphi, \psi \right\rangle = \left\langle \varphi, e^{itH_{x,n}^*} e^{-itH_c} \psi \right\rangle$ which proves the last statement.

Therefore, we obtain:

**Theorem 7.2 (Asymptotic completeness).** For all $m > 0$ and all $\varphi \in H$, the limits:

$$
\lim_{t \to \infty} e^{itH_c} e^{-itH_m} \varphi
$$

(7.7)

$$
\lim_{t \to \infty} e^{itH_m} e^{-itH_c} \varphi
$$

(7.8)

exist. If we denote these limits by $\Omega_\varphi$ and $W\varphi$ respectively, we have $\Omega^* = W$.

**Proof.** We can decompose $\varphi = \sum_{(s,n) \in I} \varphi_{s,n}$, where $\varphi_{s,n} \in H_{s,n}$ and $\sum_{(s,n) \in I} \| \varphi_{s,n} \|^2_{H_{s,n}} < \infty$.

We have:

$$
e^{itH_c} e^{-itH_m} \varphi = \sum_{(s,n) \in I} e^{itH_c} e^{-itH_{x,n}^*} \varphi_{s,n}.
$$

Since $\lim_{t \to \infty} e^{itH_c} e^{-itH_{x,n}^*} \varphi_{s,n} = \Omega_{s,n} \varphi_{s,n}$ exists for all $(s,n) \in I$ and $e^{itH_c} e^{-itH_{x,n}^*}$ is unitary, we deduce, using dominate convergence theorem, that the limit in the theorem exists. We can do the same for the other limit. The last statement follows as in the last proof.  

Q.E.D

8 Asymptotic velocity

8.1 Abstract theory

In this section, we follow the appendix B.2 in [14]. We consider a sequence $(B_n)_{n \in \mathbb{N}}$ of vectors of self-adjoint operator which commutes in a Hilbert space $H$. More precisely:

$$
B_n = (B_1^n, \ldots, B_m^n), \quad [B_i^n, B_j^n] = 0, \quad 0 \leq i, j \leq m, \quad n = 1, 2, \ldots.
$$

We have the following proposition:

**Proposition 8.1.** Suppose that, for all $g \in C_\infty (\mathbb{R}^m)$, there exists

$$
s - \lim_{n \to \infty} g (B_n).
$$

(8.1)

Then there exists a unique vector of self-adjoint operators

$$
B = (B_1^1, \ldots, B_m^1)
$$

(8.2)

such that (5.1) is equal to $g (B)$. $B$ is densely defined if, for some $g \in C_\infty (\mathbb{R}^m)$ such that $g (0) = 1$, we have:

$$
s - \lim_{R \to \infty} \left( s - \lim_{t \to \infty} \right) \left( R^{-1} B_n \right) = 1.
$$

(8.3)

We then define:
Definition 8.2. Under the hypotheses of the preceding proposition, we will write:

\[ B = s - C_\infty - \lim_{n \to \infty} B_n. \]  

(8.4)

If the limit is a operator norm limit, then we will write

\[ B = C_\infty - \lim_{n \to \infty} B_n. \]  

(8.5)

8.2 Asymptotic velocity for \( H_c \)

We have:

Theorem 8.3 (Asymptotic velocity for \( H_c \)). Let \( J \in C_\infty (\mathbb{R}) \). Then the limit:

\[ s - \lim_{t \to \infty} e^{itH_c} J \left( \frac{A}{t} \right) e^{-itH_c} \]  

exists and is equal to \( J (1) \mathbb{I} \) where \( \mathbb{I} \) is the identity here. Moreover, if \( J (0) = 1 \), then

\[ s - \lim_{R \to \infty} \left( s - \lim_{t \to \infty} e^{itH_c} J \left( \frac{A}{t} \right) e^{-itH_c} \right) = \mathbb{I}. \]  

(8.7)

If we define

\[ s - C_\infty - \lim_{t \to \infty} e^{itH_c} \frac{A}{t} e^{-itH_c} =: P_c^+, \]  

(8.8)

then the self-adjoint operator \( P_c^+ \) is densely defined and it commutes with \( H_c \). \( P_c^+ \) is called the asymptotic velocity.

Proof. Recall that \( A = -\gamma_0 \gamma^1 x \) where \( -\gamma_0 \gamma^1 = \text{diag}(1,-1,-1,1) \). Thus, for \( J \in C_\infty (\mathbb{R}) \), we have \( J (\mp 1) = \text{diag}(J (\mp 1), J (\mp 1), J (\mp 1), J (\mp 1)) \). Moreover, we have \( H_c = ic \gamma^1 \partial_x \). Let \( \psi^0 \in D (H_c) \), we wish to solve the equation

\[ \partial_t \psi (t,x) = iH_c \psi (t,x), \]

\[ \psi (0,.) = \psi^0 (.) = (\psi_1^0 (.), \psi_2^0 (.), \psi_3^0 (.), \psi_4^0 (.)). \]

where \( iH_c = \text{diag}(1,-1,-1,1) \partial_x \). We will prove that the formula:

\[ \psi (t,x) = e^{itH_c} \psi^0 \]

\[ = \begin{pmatrix} \psi_1^0 (x+t) \mathbb{I}_{\mathbb{R}^+} (x+t) - \psi_2^0 (- (x+t)) \mathbb{I}_{\mathbb{R}^+} (x+t) \\ \psi_2^0 (x-t) \mathbb{I}_{\mathbb{R}^-} (x-t) + \psi_4^0 (-x+t) \mathbb{I}_{\mathbb{R}^+} (x-t) \\ \psi_3^0 (x-t) \mathbb{I}_{\mathbb{R}^-} (x-t) - \psi_1^0 (- (x+t)) \mathbb{I}_{\mathbb{R}^+} (x-t) \\ \psi_4^0 (x+t) \mathbb{I}_{\mathbb{R}^-} (x-t) + \psi_2^0 (- (x+t)) \mathbb{I}_{\mathbb{R}^+} (x+t) \end{pmatrix}. \]

gives an explicit solution for this problem. Since \( x < 0 \) in our case, \( \mathbb{I}_{\mathbb{R}^+} (x-t) = 0 \) for all \( t > 0 \), but we need this term for the group property of this solution.

We first prove that our formula gives a solution of the desired equation. Indeed, for all \( t > 0 \), we see that \( \psi_3 (t,0) = \psi_3^0 (-t) \) and \( \psi_1 (t,0) = -\psi_1^0 (-t) \) since \( \mathbb{I}_{\mathbb{R}^-} (t) = 0 \) for \( t > 0 \). Thus \( \psi_3 (t,0) = -\psi_1 (t,0) \). On the other hand, we have \( \psi_2 (t,0) = \psi_2^0 (-t) \) and \( \psi_4 (t,0) = \psi_4^0 (-t) \) which gives us \( \psi_2 (t,0) = \psi_4 (t,0) \). The boundary conditions are thus satisfied. It remains to prove that it satisfies the equation. For the first component of our formula, in the sense of distribution, we obtain:

\[ \partial_t \psi_1 (t,x) = \psi_0^0 (x+t) \mathbb{I}_{\mathbb{R}^-} (x+t) + \psi_1^0 (x+t) (-1) \delta_0 (x+t) \]

\[ + \psi_3^0 (- (x+t)) \mathbb{I}_{\mathbb{R}^+} (x+t) - \psi_0^0 (- (x+t)) (1) \delta_0 (x+t) \]

\[ = \psi_0^0 (x+t) \mathbb{I}_{\mathbb{R}^-} (x+t) + \psi_1^0 (x+t) (-1) \mathbb{I}_{\mathbb{R}^+} (x+t) - \psi_0^0 (0) - \psi_0^0 (0) \]

\[ = \psi_0^0 (x+t) \mathbb{I}_{\mathbb{R}^-} (x+t) + \psi_0^0 (x+t) (-1) \mathbb{I}_{\mathbb{R}^+} (x+t) \]

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using the boundary condition for the last equality. We also have:

\[ \partial_t \psi_0^0 (t, x) = \psi_0^0' (x + t) \mathbb{I}_{\mathbb{R}_-} (x + t) + \psi_0^0 (x + t) (1 - \delta_0(x + t)) \]

\[ + \psi_0^0' (- (x + t)) \mathbb{I}_{\mathbb{R}_+} (x + t) - \psi_0^0 (- (x + t)) (1) \delta_0(x + t) \]

\[ = \psi_0^0' (x + t) \mathbb{I}_{\mathbb{R}_-} (x + t) + \psi_0^0' (- (x + t)) \mathbb{I}_{\mathbb{R}_+} (x + t) \]

which gives \( \partial_t \psi_1 (t, x) = \partial_x \psi_1 (t, x) \). For the second and third components, \( \mathbb{I}_{\mathbb{R}_-} (x - t) \) is constant so its derivative is 0. We have

\[ \partial_t \psi_2 (t, x) = \partial_t \psi_2^0 (x - t) = - \psi_2^0' (x - t), \]

\[ \partial_x \psi_2 (t, x) = \partial_x \psi_2^0 (x - t) = \psi_2^0' (x - t). \]

So \( \partial_t \psi_2 (t, x) = - \partial_x \psi_2 (t, x) \). We can do the same manipulation for the third component which gives \( \partial_t \psi_3 (t, x) = - \partial_x \psi_3 (t, x) \). For the fourth component, we obtain:

\[ \partial_t \psi_4 (t, x) = \psi_0^0' (x + t) \mathbb{I}_{\mathbb{R}_-} (x + t) + \psi_0^0 (x + t) (1 - \delta_0(x + t)) \]

\[ - \psi_2^0' (- (x + t)) \mathbb{I}_{\mathbb{R}_+} (x + t) + \psi_2^0 (- (x + t)) (1) \delta_0(x + t) \]

\[ = \psi_0^0' (x + t) \mathbb{I}_{\mathbb{R}_-} (x + t) - \psi_2^0' (- (x + t)) + \psi_2^0 (0) - \psi_2^0 (0) \]

\[ = \psi_0^0' (x + t) \mathbb{I}_{\mathbb{R}_-} (x + t) - \psi_2^0' (- (x + t)) \]

using the boundary condition. We have the same for \( \partial_x \psi_4 (t, x) \) so that \( \partial_t \psi_4 (t, x) = \partial_x \psi_4 (t, x) \). So \( \partial_t \psi (t, x) = i H \psi (t, x) \) in the sense of distribution. Since \( \psi^0 \in D(H) \), the derivatives are, in fact, well defined in \( \mathcal{H}_{n,n} \) and the equality is satisfied in \( \mathcal{H}_{n,n} \). This gives the desired result.

We then turn our attention to the asymptotic velocity. We have:

\[ J \left( \frac{A}{t} \right) e^{-itH} \psi^0 = \begin{pmatrix} J (\frac{1}{t}) (\psi_1^0 (x - t) \mathbb{I}_{\mathbb{R}_-} (x - t) - \psi_0^0 (- (x + t)) \mathbb{I}_{\mathbb{R}_+} (x - t)) \\ J (\frac{1}{t}) (\psi_2^0 (x - t) \mathbb{I}_{\mathbb{R}_-} (x - t) + \psi_2^0 (- (x + t)) \mathbb{I}_{\mathbb{R}_+} (x - t)) \\ J (\frac{1}{t}) (\psi_3^0 (x - t) \mathbb{I}_{\mathbb{R}_-} (x - t) - \psi_3^0 (- (x + t)) \mathbb{I}_{\mathbb{R}_+} (x - t)) \\ J (\frac{1}{t}) (\psi_4^0 (x - t) \mathbb{I}_{\mathbb{R}_-} (x - t) + \psi_4^0 (- (x + t)) \mathbb{I}_{\mathbb{R}_+} (x - t)) \end{pmatrix} \]

Thus:

\[ e^{itH} J \left( \frac{A}{t} \right) e^{-itH} \psi^0 = \begin{pmatrix} J \left( \frac{1}{t} + 1 \right) (\psi_1^0 (x) \mathbb{I}_{\mathbb{R}_-} (x) \mathbb{I}_{\mathbb{R}_-} (x + t) + \psi_1^0 (x) \mathbb{I}_{\mathbb{R}_+} (- (x + t)) \mathbb{I}_{\mathbb{R}_+} (x + t)) \\ J \left( \frac{1}{t} + 1 \right) (\psi_2^0 (x) \mathbb{I}_{\mathbb{R}_-} (x) \mathbb{I}_{\mathbb{R}_-} (x + t) - \psi_2^0 (x) \mathbb{I}_{\mathbb{R}_+} (- (x + t)) \mathbb{I}_{\mathbb{R}_+} (x + t)) \\ J \left( \frac{1}{t} + 1 \right) (\psi_3^0 (x) \mathbb{I}_{\mathbb{R}_-} (x) \mathbb{I}_{\mathbb{R}_-} (x + t) - \psi_3^0 (x) \mathbb{I}_{\mathbb{R}_+} (- (x + t)) \mathbb{I}_{\mathbb{R}_+} (x + t)) \\ J \left( \frac{1}{t} + 1 \right) (\psi_4^0 (x) \mathbb{I}_{\mathbb{R}_-} (x) \mathbb{I}_{\mathbb{R}_-} (x + t) + \psi_4^0 (x) \mathbb{I}_{\mathbb{R}_+} (- (x + t)) \mathbb{I}_{\mathbb{R}_+} (x + t)) \end{pmatrix} \]

This last term converges pointwise to \( J (1) \psi^0 (x) \) as \( t \to \infty \). Since \( J, \mathbb{I}_{\mathbb{R}_-}, \mathbb{I}_{\mathbb{R}_+}, \mathbb{I}_{\mathbb{R}_-} \) are bounded and \( \psi^0 \in \mathcal{H}_{n,n} \), we can use the dominate convergence theorem of Lebesgue to conclude that

\[ \lim_{t \to \infty} e^{itH} J \left( \frac{A}{t} \right) e^{-itH} \psi^0 = J (1) \psi^0. \]

If \( J \in C_\infty (\mathbb{R}) \) with \( J (0) = 1 \), then

\[ \lim_{t \to \infty} e^{itH} J \left( \frac{A}{Rt} \right) e^{-itH} \psi^0 = J \left( \frac{1}{R} \right) \psi^0. \]

and the last term goes to \( J (0) \psi^0 = \psi^0 \). So

\[ s - \lim_{R \to \infty} \left( s - \lim_{t \to \infty} e^{itH} J \left( \frac{A}{Rt} \right) e^{-itH} \right) = 1. \]

The last part of the theorem follows from the abstract theory. Q.E.D
We can know study the spectrum of \( P_c^+ \):

**Proposition 8.4.** \( \sigma(P_c^+) = \{1\} \)

**Proof.** Let \( J \in C_\infty(\mathbb{R}) \) such that \( J(1) = 0 \). We can approach \( J \) by a sequence \((J_n)_{n \in \mathbb{N}}\) of \( C_\infty(\mathbb{R}) \) functions which are zero in a neighbourhood of 1 in \( L^\infty \). Using a density argument, we can suppose that \( J \in C_\infty(\mathbb{R}) \) and \( J \) is zero in a neighbourhood of 1. Using minimal and maximal velocity estimates, we obtain:

\[
J(P_c^+) = \lim_{t \to \infty} e^{itH_c} J \left( \frac{A}{t} \right) e^{-itH_c} = 0
\]  

(8.9)

Now, if we have \( J(1) \neq 0 \), we can suppose that \( J \in C_\infty(\mathbb{R}) \) is constant, non zero, in a neighbourhood of 1. Then, for all \( \varphi \in \mathcal{H} \), we have:

\[
J(P_c^+) \varphi - J(1) \varphi = \lim_{t \to \infty} e^{itH_c} \left( J \left( \frac{A}{t} \right) - J(1) \right) e^{-itH_c} \varphi.
\]

Since \( J(x) - J(1) \) is zero in a neighbourhood of 1, we obtain \( J(P_c^+) \varphi = J(1) \varphi \neq 0 \). This finished the proof. Q.E.D

We then have the following immediate consequence:

**Corollary 8.5.** \( P_c^+ = 1 \)

### 8.3 Asymptotic velocity for \( H_m \)

We have:

**Theorem 8.6** (Asymptotic velocity for \( H_m \)). Let \( J \in C_\infty(\mathbb{R}) \). Then, for all \( m > 0 \), the limit:

\[
s - \lim_{t \to \infty} e^{itH_m} J \left( \frac{A}{t} \right) e^{-itH_m}
\]  

exists. Moreover, if \( J(0) = 1 \), then

\[
s - \lim_{R \to \infty} \left( s - \lim_{t \to \infty} e^{itH_m} J \left( \frac{A}{Rt} \right) e^{-itH_m} \right) = 1
\]  

(8.11)

If we define

\[
s - C_\infty \lim_{t \to \infty} \frac{A}{t} e^{-itH_m} =: P_m^+
\]  

(8.12)

then the self-adjoint operator \( P_m^+ \) is densely defined and commutes with \( H_m \). The operator \( P_m^+ \) is called the asymptotic velocity.

**Proof.** We can write

\[
e^{itH_m} J \left( \frac{A}{t} \right) e^{-itH_m} = e^{itH_m} e^{itH_c} e^{-itH_m} J \left( \frac{A}{t} \right) e^{-itH_c} e^{itH_m}
\]

Using uniform boundedness of our operators and the identity:

\[
e^{itH_m} e^{-itH_c} e^{itH_c} J \left( \frac{A}{t} \right) e^{-itH_c} e^{itH_m} \varphi = W J(P_c^+) \Omega \varphi
\]

\[
e^{itH_m} e^{-itH_c} e^{itH_c} J \left( \frac{A}{t} \right) e^{itH_c} e^{-itH_m} \varphi - W J(P_c^+) \Omega \varphi
\]

\[
e^{itH_m} e^{-itH_c} e^{itH_c} J \left( \frac{A}{t} \right) e^{itH_c} e^{-itH_m} \varphi - J(P_c^+) \Omega \varphi + \left( e^{itH_m} e^{-itH_c} - W \right) J(P_c^+) \Omega \varphi
\]

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for all $\varphi \in H$, this limit is equal to $WJ(P_c^+)\Omega$ where $W,\Omega$ are defined in theorems. We can use the same argument for the second limit and the existence of $P_m^+$ follows by the abstract theory and we have:

$$J(P_m^+) = WJ(P_c^+)\Omega$$

(8.13)

Q.E.D

We deduce:

**Proposition 8.7.** For all $m > 0$, $\sigma(P_m^+) = \{1\}$

*Proof.* Using the last proof, we have:

$$J(P_m^+) = WJ(P_c^+)\Omega$$

for all $J \in C^\infty(R)$ where $\Omega, W$ are unitary and $\Omega^{-1} = W$. Thus $J(P_m^+)$ and $J(P_c^+)$ are unitary equivalent which gives the desired result. Q.E.D

We then have the following consequence:

**Corollary 8.8.** For all $m > 0$, $P_m^+ = 1$.

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