Practical stability of time-varying positive systems with time delay

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Funding information
Shandong Provincial Natural Science Foundation, China, Grant/Award Number: ZR2017JL028

Abstract
This paper considers the problem of practical stability for time-varying positive systems with time delay. For both non-linear time-varying positive systems and linear time-varying positive (LTVP) systems with time delay, sufficient conditions on practical stability are derived by the max-separable Lyapunov-Krasovskii (L–K) functional method. Then based on the obtained results, an effective method for designing a desired controller in terms of state feedback and state feedback with input delay is established for LTVP systems with time delay. At the end, numerical examples are presented to illustrate the effectiveness of our results.

1 | INTRODUCTION

As a class of common dynamical systems, positive systems [1, 2] involve variables that are limited to take non-negative values for all time. Such positive parameters and variables in natural systems and artificial systems are easy to see, for instance, absolute temperature, probability, material concentration, population density. In other words, positive systems are defined on a positive cone instead of the whole vector space. Therefore, traditional methods for general systems are no longer applicable to the analysis and control of positive systems, or they may increase the conservatism when they are directly used for positive systems. This characteristic makes the analysis and control of positive systems full of challenges and significance.

Time delays are frequent and inevitable in reality, such as network latency. Therefore, the stability analysis of dynamical systems with time delay is of practical importance. First of all, because of the existence of time delays, information transmission between system modules are not instantaneous, which account for imprecise system models. More seriously, time delays influence the performance, and bring about oscillation, overshoot even instability. Therefore, a lot of interesting results have been studied and found for time-delay systems, such as asymptotical stability [3–5], stochastic stability [6–8], robust stability [9–11] and finite-time stability [12–14]. For uncertain stochastic system, time-varying state and input delays, interval time-varying delays and neutral delay are, respectively, discussed in [31–33]. Some elegant and delay-dependent conclusion are presented by Lyapunov stability theory and stochastic analysis technique and linear matrix inequality (LMI) technique. Especially for asymptotical stability of positive systems with time delays, an important result which has already been proved in [15, 16, 24], is that positive systems with bounded time delays is asymptotically stable if and only if the corresponding delay free positive system is asymptotically stable.

In fact, in many real-world applications, the systems may be asymptotic unstable, but stay nearby a state with an acceptable fluctuation. For example, rocket launchers have a trajectory that is mathematically considered unstable. However, the vibration effect of rocket system in rocket launcher is satisfactory. Many problems can be explained in this way, including the temperature retention of chemical reactions within a certain range and the flight of cosmic devices between two points. Practical stability which was introduced by LaSalle and Lefschetz first in 1961 in [17] can deal with this situation well. In [18, 19, 21], corresponding criteria are provided for several non-linear systems by the practical stability of comparison system. Generally speaking, a proper comparison system is proposed firstly and this comparison system is practically stable. Then the considered non-linear system is practically stable by comparison principle. This is an usual and effective method. For fractional-order positive discrete-time linear systems in [20], some new conditions on practical stability are presented and numerically checking method of practical stability based on some new
conditions is given. For discrete-time switched affine systems discussed in [28], novel state feedback stability conditions based on Lyapunov-Metzler inequalities are proposed. The problem of practically stabilizing a sampled-data switched systems with quantization and delay [29] are addressed using the sampled and quantized state feedback under the influence of time-varying delay. Some interesting conditions on practical stability are given by the combination increment rate of the Lyapunov function and the total mismatch time. And in [30], a novel controller for DC/DC converters is designed by Lyapunov function strategy and practical stability.

In [3, 24, 26], exponential stability and asymptotical stability are considered for non-linear positive systems. It can be found that, if there exists a vector \( r > 0 \) such that \( f(v) + g(v) < 0 \), non-linear time-invariant positive system will be globally asymptotically stable for all bounded time delays. For practical stability, the condition \( f(v) + g(v) < 0 \) on asymptotical stability should be generalized and the effects of time delay on stability should be clearer. In [18, 19, 21], practical stability are considered by comparison principle, the essential characteristics of the system are ignored. On the other hand, time delay is not included in [18–20]. Thus, whether the time delay will affect the practical stability of the system needs carefully considerations.

From references mentioned above, the stability of linear time-invariant (LTI) systems can be solved well as long as the system matrix is known. Nevertheless, there is no significant relationship between stability and eigenvalues of system matrix for LTV systems. The stability of LTV systems heavily relies on the state transition matrix that is difficult to compute. And non-linear systems are more complex than linear systems. So, what about practical stability of non-linear positive systems and linear time-varying positive (LTVP) systems with time delays? However, to the best of our knowledge, little progresses have been got on practical stability of positive systems with time delays. On the other hand, as a immensely popular method for time-delay systems, no sharp characterization of an eligible Lyapunov-Krasovskii (L–K) functional exists to date which can be applied to both non-linear and linear time-varying systems.

The key point of this article is the study of the relationship between time delay and practical stability for time-varying positive (LTVP) systems with time delays. For linear cases, time-varying systems with input delay and state delay can be stabilized further. In this paper, associated with the max-separable Lyapunov function [22, 23, 34], a max-separable L–K functional including all complete information about time-delay systems is introduced which is not only for LTVP systems but also for non-linear positive systems with time delays. Inspired by this and pertinent literatures, we will make the following contributions:

1) Using the max-separable L–K functional, sufficient conditions on practical stability will be derived for non-linear positive system with time delay, when the vector fields are homogeneous and cooperative. The obtained criterion on practical stability can be extended to the relevant cases with multiple time delays. More importantly, the obtained condition can be used for asymptotical and exponential stability of non-linear positive systems when \( \gamma(t) < 0 \). And the influence of time delay on stability is clear.

2) For the LTVP systems, sufficient conditions on practical stability will be given for single time delay and multiple time delays as well via the same method. In the same way, the obtained criterion can be used for asymptotical and exponential stability of linear positive systems when \( \gamma(t) < 0 \).

3) Based on the obtained practical stability results, the issues on practical stabilization for LTVP systems with time delay will be addressed. State feedback control will be discussed in two cases: state feedback without input delay and state feedback with input delay. And sufficient conditions on practical stabilization will be derived in the meantime.

The organization of this paper is as follows. In Section 2, some basic conceptions and preliminaries that are essential for the development of our results will be introduced. Section 3 and Section 4 are main parts which provide sufficient criteria on practical stability for non-linear positive systems and LTVP systems with time delays, respectively. We also provide important generalizations for multiple delays systems. In Section 5, we will discuss the practical stabilization of LTVP systems with time delay in terms of state feedback without input delay and state feedback with input delay. Numerical examples are presented in Section 6 to verify the validity of our results. The final Section 7 concludes.

### 2.1 NOTATIONS AND PRELIMINARY RESULTS

Throughout this paper, the set of all non-negative real numbers are denoted by \( \mathbb{R}_+ \); \( \mathbb{R}_n \) and \( \mathbb{R}_n^{\infty} \) denote \( n \) dimensional real space and \( n \times m \) dimensional real matrix space, respectively. The set of \( n \times m \) non-negative matrices are denoted by \( \mathbb{R}_{n \times m}^+ \). \( \mathbb{R}_n^+ \setminus \{0\} \) means the set \( \mathbb{R}_n^+ \) minus the Zero vector set \( \{0\} \). For \( x \in \mathbb{R}_n^+ \), \( x_i \) denotes the \( i \)-th coordinate of \( x \) for \( i \in I_n = \{1, 2, ..., n\} \). If \( A \) and \( A_{ij} \in \mathbb{R}_n^{\infty} \), \( A_{ij} \) is the \( i \)-th row of \( A \) and \( A_{ij} \), respectively. For vectors \( x, y \in \mathbb{R}_m^+ \), we write: \( x \geq y \) (\( x \leq y \)) if \( x_i \geq y_i \) (\( x_i \leq y_i \)) for all \( i \in I_n \); \( x > y \) (\( x < y \)) if \( x_i > y_i \) (\( x_i < y_i \)) and \( x \neq y \); \( x \geq y \) (\( x < y \)) if \( x_i > y_i \) (\( x_i < y_i \)) for all \( i \in I_n \). Especially, for \( x, y \in \mathbb{R}_1^+ \), \( x \geq y \) if \( x - y \geq 0 \). \( 1_n \) denotes the column vector whose components are all one. \( A^T \) denotes the transpose of \( A \). \( I \) is identity matrix with appropriate dimensions. \( C([0, \infty), \mathbb{R}_n) \) denotes the space of real-valued continuous functions on interval \( J \) taking values in \( \mathbb{R}_n^+ \). For \( y(t) \in C([0, \infty), \mathbb{R}_n) \), \( y(t) = y^+(t) + y^-(t) \), where \( y^+(t) := \begin{cases} y(t), & \text{when } y(t) \geq 0; \\ 0, & \text{when } y(t) < 0. \end{cases} \)

\( y^-(t) := \begin{cases} y(t), & \text{when } y(t) \leq 0; \\ 0, & \text{when } y(t) > 0. \end{cases} \)

Given a vector \( v \in \mathbb{R}_n^+ \), the weighted \( l_\infty \) norm is defined by \( \|v\|_\infty = \max_{i \in I_n} \{v_i\} \).

Consider the following non-linear time-delay system

\[
\begin{cases}
\dot{x}(t) = f(t, x(t)) + g(t, x(t - \tau)), & t \geq 0, \\
x(t) = \varphi(t), & t \in [-\tau, 0],
\end{cases}
\]
where \( f, g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \) are system vector fields with \( f(t, 0) = g(t, 0) = 0, \forall t \geq 0 \). \( \tau \) is a constant delay satisfying \( \tau \geq 0 \), and \( \varphi(\cdot) \in C([-\tau, 0], \mathbb{R}^n) \) is the vector-valued function specifying the initial state of the system. \( x(t) = x(t, \varphi) \) denotes the solution of system (1) with initial condition \( \varphi(\cdot) \). We denote \( x_t = x(t + \theta), \theta \in [-\tau, 0] \). System (1) is said to be positive if for every non-negative initial condition \( \varphi(\cdot) \), the corresponding state trajectory preserves in the non-negative quadrant, that is \( x(t) \in \mathbb{R}_{+}^n, \forall t \geq 0 \).

**Definition 1.** [3] A vector field \( f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \) is homogeneous of degree \( \alpha \) if for all \( x \in \mathbb{R}^n \) and all real constant \( \lambda > 0 \) satisfy \( f(t, \lambda x) = \lambda^\alpha f(t, x), \forall t \in [0, \infty) \).

In this paper, only the case \( \alpha = 1 \) will be taken into consideration for simplicity. At this case, \( f \) is referred to as homogeneous of degree one.

**Definition 2.** [3] A continuous vector field \( f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \), which is \( C^1 \) on \( \mathbb{R}_+^n \setminus \{0\} \), is cooperative when the Jacobian matrix \( \langle \partial f/\partial x \rangle(t, a) \) is Metzler for any \( t \geq 0 \) and all \( a \in \mathbb{R}^n_+ \setminus \{0\} \).

**Definition 3.** [26] A continuous vector field \( g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \) is order preserving on \( \mathbb{R}^n_+ \), if for any \( x, y \in \mathbb{R}^n_+, x \geq y \) implies \( g(t, x) \geq g(t, y), \forall t \geq 0 \).

### 3 Practical Stability of Non-linear Positive Systems with Time Delays

In this section, the effect of time delays on practical stability will be studied for non-linear positive systems. First, the definition of practical stability will be introduced which can fully reflect the positiveness of each component of the state.

**Definition 4.** Let positive number pair \( (\lambda, \rho), 0 < \lambda < \rho \), and positive vector \( \tau > 0 \) be given. The positive system (1) with time delay \( \tau \) is said to be **practically stable** with respect to \( (\lambda, \rho, \tau) \), if for any initial condition \( \varphi(\cdot) \leq \lambda \psi, \psi \in [-\tau, 0] \) implies \( x(t) \leq \rho \psi \) for any \( t \geq 0 \).

**Assumption 1.** [26]

(1) \( f \) is continuous and continuously differentiable, cooperative and homogeneous of degree one on \( \mathbb{R}^n \);

(2) \( g \) is continuous and continuously differentiable, homogeneous of degree one and order preserving on \( \mathbb{R}^n_+ \).

Assumption 1 implies that \( f \) and \( g \) are globally Lipschitz on \( \mathbb{R}^n \). Since cooperativeness of \( f \) and order preserving of \( g \), system (1) is positive (proof see [3]). The following theorem is our first pivotal conclusion, which presents a sufficient condition on practical stability for system (1).

**Theorem 1.** Let the positive number pair \( (\lambda, \rho), 0 < \lambda < \rho \) and \( \tau > 0 \) be given with initial condition \( \varphi(t) \leq \lambda \psi, \forall \psi \in [-\tau, 0] \). For system (1) under Assumption 1, suppose that there exist \( \gamma \in C([0, \infty), \mathbb{R}) \) such that

(i) \( f(t, \psi) + g(t + \tau, \psi) \leq \gamma(t) \psi, \forall \psi \geq 0 \);

(ii) \( \int_0^t \gamma^+(s) \psi(s) ds \leq \ln \frac{\rho}{\lambda(1 + r_{\max})}, \forall t \geq 0 \), where \( r_{\max} \) denotes the maximum, \( \max_{\psi \in I_{:, \psi} \in [0, \tau]} \gamma^+(\psi) \).

Then, system (1) is strongly and practically stable with respect to \( (\lambda, \rho, \tau) \).

**Proof.** Let the positive number pair \( (\lambda, \rho), 0 < \lambda < \rho \) and \( \tau > 0 \) be given, \( \varphi(t) \) denote the solution of system (1) with initial data \( \varphi(\cdot) \leq \lambda \psi, \forall \psi \in [-\tau, 0] \). Under Assumption 1, the non-linear time-delay system (1) is positive ([24]). Consider the max-separable L–K functional \( V(t, \varphi) \) given by

\[
V(t, \varphi) = \max_{a \in I_{a}} \left\{ -\int_{0}^{t} g(s + \tau, x(s)) ds \right\}. 
\]

From \( \varphi(t, 0) = 0 \) and condition (2) in Assumption 1, obviously, vector field \( g \) is positive and

\[
\max_{a \in I_{a}} \left\{ \int_{0}^{t} g(s + \tau, x(s)) ds \right\} \leq V(t, \varphi). \tag{2}
\]

When \( t = 0 \), the initial condition of \( V(t, \varphi) \),

\[
V(0, \psi) = \max_{a \in I_{a}} \left\{ \int_{0}^{t} g(s + \tau, x(s)) ds \right\}. 
\]

Since \( g \) is homogeneous of degree one and initial state \( \varphi(\cdot) \leq \lambda \psi \) for \( t \in [-\tau, 0] \),

\[
V(0, \psi) \leq \max_{a \in I_{a}} \left\{ \lambda + \lambda \int_{0}^{t} g(s + \tau, \psi(s)) ds \right\}. 
\]

Let \( s_{1} = s + \tau \) for \( s \in [-\tau, 0], s_{1} \in [0, \tau] \). And let the maximum \( \max_{a \in I_{a}} \left\{ \int_{0}^{t} g(s + \tau, \psi(s)) ds \right\} \) be denoted by \( r_{max} \). Therefore,

\[
V(0, \psi) \leq \lambda + \lambda \tau r_{max}.
\]

Let subscript \( m \) indicate the index which makes the upper-right Dini derivative of \( V(t, \varphi) \) maximal. Then, the upper-right Dini...
where $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $g_k : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ for $k \in I_p$ with $p$ be a positive integer. $\tau_k$ is time delay with $0 \leq \tau_k \leq \tau_{\text{max}}$. $\varphi(\cdot) \in C([-\tau_{\text{max}}, 0], \mathbb{R}_+^n)$ is the vector-valued function specifying the initial state of system (3).

**Theorem 2.** Let the positive number pair $(\lambda, \rho)$, $0 < \lambda < \rho$ and $v > 0$ be given with initial condition $\varphi(t) \leq \lambda v \forall t \in [-\tau_{\text{max}}, 0]$. For system (3) under Assumption 1 with $g$ replaced by $g_k$ $(k \in I_p)$, suppose that there exist $\gamma(t) \in C([0, \infty), \mathbb{R})$ such that

(i) $f(t, v) + \sum_{k=1}^p g_k(t + \tau_k, v) \leq \gamma(t) v, \forall t \geq 0$;

(ii) $\int_0^\tau t \gamma(t) ds \leq \ln \frac{\rho}{\lambda (1 + \tau \max_{i \in I_p} G_{i,\text{max}})}, \forall \tau \geq 0$, where $G_{i,\text{max}}$

denotes the maximum, $\max_{i \in I_p, t \in [0, \tau_{\text{max}}]} \left\{ \sum_{k=1}^p g_k(t, v) \right\}$.

Then, system (3) is positive and practically stable with respect to $(\lambda, \rho, v)$.

**Proof.** Similarly to the proof of Theorem 1, when the max-separable L–K functional $V(t, x(t))$ is taken as

$$V(t, x(t)) = \sup_{i \in I_p} \left\{ \frac{x_i(t)}{v_i} \right\},$$

the proof can be got. Here, it is omitted. \qed

## 4 PRACTICAL STABILITY OF LTVP SYSTEMS WITH TIME DELAY

In this section, practical stability will be considered for LTVP systems with single delay or multiple delays, respectively. First, let us consider the LTVP system with single delay

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) + A_d(t)x(t - \tau), & t \geq 0, \\
x(t) &= \varphi(t), & t \in [-\tau, 0]
\end{aligned}$$

(4)

where $x(t) \in \mathbb{R}^n$ is state vector, $A_d(t) \in \mathbb{R}^{n \times n}$ and $A(t) \in \mathbb{R}^{n \times n}$ are Lebesgue measurable locally essentially bounded functions of time $t$. $\tau$ is the time delay with $\tau \geq 0$ and $\varphi(\cdot) \in C([-\tau, 0], \mathbb{R}_+^n)$ is the vector-valued function specifying the initial state of system (4). For the positivity and practical stability study of system (4), we need the following definition and lemma.

**Definition 5.** [26] A matrix $A(t) = [a_{ij}(t)] \in \mathbb{R}^{n \times n}$ is called a Metzler matrix on $[0, \infty)$, if its off-diagonal entries are non-negative, that is, $a_{ij}(t) \geq 0$ for $i, j \in I_n, i \neq j, t \geq 0$.

**Lemma 1.** [25] System (4) is positive if and only if $A(t)$ is Metzler and $A_d(t) \geq 0$ on $[0, \infty)$.

Based on the above, the sufficient criterion on practical stability of system (4) can be drawn as follows.
Lemma 1. Because system (4) is a special case of system (1) with $\tau = 0$, we have $\gamma(t) = \gamma^*(t) = 0$ for $t \geq 0$.

Proof. The positivity of system (4) can be got from the Lemma 1. Because system (4) is a special case of system (1) with $f(t, x(t)) = A(t)x(t)$ and $g(t, x(t)) = A_g(t)x(t - \tau)$, when taking max-separable L–K functional

$$V'(t, x_i) = \max_{\rho \in L_{i}} \left\{ \frac{\sum_{j=1, j \neq i}^n (a_{ij}(t) + a_{dd,j}(t + \tau)) v_j}{v_i} \right\},$$

the initial condition of $V'(t, x_i)$ for $t = 0$,

$$V(0, x_0) = \max_{\rho \in L_{i}} \left\{ \frac{\sum_{j=1}^n (a_{ii}(t) + a_{dd,i}(t + \tau)) v_j}{v_i} \right\}.$$

Since $x(t) = \varphi(t) \leq \lambda v$ for $t \in [-\tau, 0]$,

$$V(0, x_0) \leq \max_{\rho \in L_{i}} \left\{ \lambda + \lambda \tau H_{\max} \right\}.$$

Let $s_1 = s + \tau$ for $s \in [-\tau, 0]$, $s_1 \in [0, \tau]$ and the maximum $\max_{\rho \in L_{i}} \left\{ \frac{\sum_{j=1}^n (a_{ii}(t) + a_{dd,i}(t + \tau)) v_j}{v_i} \right\}$ is denoted by $H_{\max}$. Therefore,

$$V(0, x_0) \leq \lambda + \lambda \tau H_{\max}.$$

Let the subscript $m$ indicates the index which makes the upper-right Dini derivative of $V'(t, x_i)$ maximal.

$$D^+V'(t, x_i) = \frac{\sum_{j=1}^n (a_{ij}(t) + a_{dd,j}(t + \tau)) v_j}{v_i} \frac{\sum_{j=1}^n (a_{ij}(t) + a_{dd,j}(t + \tau)) x_j}{v_i}$$

$$= \frac{\sum_{j=1}^n (a_{ij}(t) + a_{dd,j}(t + \tau)) x_j}{v_i}$$

$$= \frac{\sum_{j=1}^n (a_{ij}(t) + a_{dd,j}(t + \tau)) x_j}{v_i}$$

$$= \frac{\sum_{j=1}^n (a_{ij}(t) + a_{dd,j}(t + \tau)) x_j}{v_i}$$

From conditions (i) and (ii) of Theorem 3, similar to the proof of Theorem 1, it can be got that system (4) is practically stable.

Similarly, considering the LTV system with multiple delays in the form of

$$\dot{x}(t) = A(t)x(t) + \sum_{k=1}^n A_{dk}(t)x(t - \tau_k), \quad t > 0,$$

$$x(t) = \varphi(t), \quad t \in [-\tau_{\max}],$$

where $A(t)$ is Metzler and $\sum_{k=1}^n A_{dk}(t) \geq 0$ are Lebesgue measurable locally essentially bounded functions of time $t, 0 \leq \tau_k \leq \tau_{\max}, k \in I_p$, we have the sufficient conditions on practical stability shown as in the following theorem. The proof of it is also omitted here.

Theorem 4. Let the positive number pair $(\lambda, \rho)$, $0 < \lambda < \rho$ and $\tau > 0$ be given with $\varphi(t) \leq \lambda v$ for $t \in [-\tau_{\max}, 0]$. For system (6) with $A(t)$ Metzler and $\sum_{k=1}^n A_{dk}(t) \geq 0, \forall t \in [0, \infty)$, suppose that there exist $\gamma(t) \in C([0, \infty), R)$ such that

(i) $a_{ij}(t) + a_{dd,j}(t + \tau) \geq \gamma(t)$ for $t \geq 0$ and $i, j \in I$,;

(ii) $\int_0^\tau \gamma^+(s) ds \leq -\frac{\rho}{\lambda(1 + \tau_{\max})}$, $\forall t \geq 0$, where $H_{\max}$ denote the maximum

$$\max_{\rho \in L_{i}} \left\{ \frac{\sum_{j=1}^n (a_{ii}(t) + a_{dd,i}(t + \tau)) v_j}{v_i} \right\}.$$

Then, system (6) is positive and practically stable with respect to $(\lambda, \rho, \tau)$.

Proof. Similar to the proof of Theorem 3, when the max-separable L–K functional $V'(t, x_i)$ is taken as

$$V'(t, x_i) = \max_{\rho \in L_{i}} \left\{ \frac{\sum_{j=1}^n (a_{ij}(t) + a_{dd,j}(t + \tau)) x_j}{v_i} \right\},$$

the proof can be got. Here, it is omitted.
5 PRACTICAL STABILIZATION OF LTVP SYSTEMS WITH TIME DELAY

In this section, we will study the issue of practical stabilization of LTVP systems with time delay in two cases: (1) state feedback without input delay; (2) state feedback with input delay.

Consider the linear control system given by
\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + A_d(t)x(t-\tau) + B(t)u(t), & t > 0, \\
x(t) &= \varphi(t), & t \in [-\tau, 0],
\end{align*}
\]
where input matrix \( B(t) = [B_i(t)]^{n \times 1} \in \mathbb{R}^n \) and control input \( u(t) \geq 0 \).

5.1 State feedback without input delay

In this case, take state feedback controller \( u(t) = K(t)x(t) \), where \( (K(t))^T \in \mathbb{R}^n \). The resultant closed-loop system is shown as
\[
\begin{align*}
\dot{x}(t) &= (A(t) + B(t)K(t))x(t) + A_d(t)x(t-\tau), & t > 0, \\
x(t) &= \varphi(t), & t \in [-\tau, 0].
\end{align*}
\]

Theorem 5. Let the positive number pair \((\lambda, \rho), 0 < \lambda < \rho\) and \(v \in \mathbb{R}^n_+\) be given with \( \varphi(t) \leq \lambda v \) for \( t \in [-\tau, 0] \). For system (8) with \( A_d(t) \geq 0 \), suppose that there exist \( \gamma(t) \in C([0, \infty), \mathbb{R}) \), constant \( \zeta > 0 \), vectors \( \omega_1(t), \omega_2(t) \) and \( \zeta(t) \in \mathbb{R}^n_+ \), such that for all \( t \geq 0 \)

\[
A(t) (\omega_1^T(t)\omega_2(t)) + B(t)\zeta^T(t) + \zeta I \geq 0,
\]
\[
\begin{align*}
\left( a_{ii}(t) + \frac{B_i(t)\zeta_i(t)}{\omega_1^T(t)\omega_2(t)} + a_{di}(t + \tau) + \sum_{j=1,j \neq i}^n (a_{ij}(t) + \frac{B_j(t)\zeta_j(t)}{\omega_1^T(t)\omega_2(t)}) \right) v_i &< \gamma(t), \\
\int_0^t \gamma^+(s)ds &\leq \ln \frac{\rho}{\lambda(1 + \tau H_{\text{max}})},
\end{align*}
\]
where \( a_{ij}(t), a_{di}(t) \) are the \( ij \)-th entries of \( A(t) \) and \( A_d(t) \), respectively, and \( \zeta_i(t) \) is the \( i \)-th entry of \( \zeta(t) \), \( i, j \in I_n, H_{\text{max}} \) denotes the maximum
\[
\max_{i \in I_n, j \in [0, t]} \frac{|A_{kj}(t)|}{v_i}.
\]
Then, system (7) under the state feedback
\[
u(t) = \frac{\zeta^T(t)}{\omega_1^T(t)\omega_2(t)} x(t)
\]
is positive and practically stable with respect to \((\lambda, \rho, v)\).

Proof. Since \( \omega_1(t) \geq 0, \omega_2(t) \geq 0 \) and \( \zeta > 0 \), \( \omega_1^T(t)\omega_2(t) \geq 0 \), \( \omega_1^T(t)\omega_2(t) \geq 0 \). Form (9),

\[
A(t) + B(t)\zeta^T(t) + \frac{\zeta}{\omega_1^T(t)\omega_2(t)} I \geq 0.
\]
Combined with (12), we have

\[
A(t) + B(t)K(t) + \frac{\zeta}{\omega_1^T(t)\omega_2(t)} I \geq 0,
\]
that is, \( A(t) + B(t)K(t) \) is Metzler. Due to \( A_d(t) \geq 0 \), and from Lemma 1, system (8) is positive.

Similar to the proof of Theorem 1, taking the max-separable L-K functional as

\[
V(t, x_t) = \max_{i \in I_n} \left\{ x_i(t) + \int_{t-\tau}^t A_{ai}(s + \tau)\zeta(s)ds \right\},
\]
with the initial condition of \( V(t, x_t) \) for \( t = 0 \),

\[
V(0, x_0) = \max_{i \in I_n} \left\{ x_i(0) + \int_{t=0}^{\tau} A_{ai}(s + \tau)\zeta(s)ds \right\}.
\]

By the upper-right Dini derivative of \( V(t, x_t) \) and the conditions (9)–(11), the practically stabilization of (7) can be got under the state feedback controller (12). The concrete proof process is omitted here.

Remark 2. In Theorem 5, \( \omega_1^T(t)\omega_2(t) \in \mathbb{R}_+ \). The whole process of verifying the practical stability of the closed-loop system (8) can be divided into three steps:

Step 1. For a given closed-loop system (8), \( A(t), A_d(t) \) and \( B(t) \) is known. Parameters \((\lambda, \rho, v), \tau, \omega_1(t) \) and \( \omega_2(t) \) need to be given first which directly affect the feasibility and reliability of the conclusion. From (11), choose \( \gamma(t) \) and \( \gamma^+(t) \) appropriately.

Step 2. Find feasible solutions \( \zeta(t) \) and \( \zeta^+ \) by (9). Otherwise, return to Step 1.

Step 3. Check (10). If (10) holds, \( \zeta(t) \) is solved, that is, feedback gain matrix \( K(t) \) is admissible and system (8) is practically stable with respect to \((\lambda, \rho, v)\). Otherwise, return to Step 2.

5.2 State feedback with input delay

In this case, for simplicity, the system matrix \( A_d(t) \in \text{system (7)} \) is taken as 0. Taking state feedback controller with constant
input delay τ, system (7) can be described as
\[
\begin{cases}
\dot{x}(t) = A(t)x(t) + B(t)u(t - \tau), & t \geq 0, \\
x(t) = \varphi(t), & t \in [-\tau, 0].
\end{cases}
\]
Taking the state feedback controller \( u(t) = K(t)x(t) \) still, the resultant closed-loop system is
\[
\begin{cases}
\dot{x}(t) = A(t)x(t) + B(t)K(t - \tau)x(t - \tau), & t \geq 0, \\
x(t) = \varphi(t), & t \in [-\tau, 0].
\end{cases}
\]

Theorem 6. Let the positive number pair \((\lambda, \rho), 0 < \lambda < \rho \) and \( v > 0 \) be given with \( \varphi(t) \leq \lambda v \) for \( t \in [-\tau, 0] \). For system (14) with \( A(t) \) is Metzler, suppose that there exist \( \gamma(t) \in C([0, \infty), \mathbb{R}) \), vectors \( \omega_1(t), \omega_2(t) \) and \( z(t) \in \mathbb{R}^{n \times 1} \), such that for all \( t \geq 0 \)
\[
B(t)\frac{\gamma^T(t - \tau)}{\omega_1^T(t - \tau)\omega_2(t - \tau)} \geq 0,
\]
\[
\left( a_{ij}(t) + \frac{B_i(t + \tau)z_j(t)}{\omega_1^T(t)\omega_2(t)} \right)v_j + \sum_{j=1, j \neq i}^n \left( a_{ij}(t) + \frac{B_i(t + \tau)z_j(t)}{\omega_1^T(t)\omega_2(t)} \right)v_j < \gamma(t),
\]
\[
\int_0^t \gamma^+(s)ds \leq \ln \frac{\rho}{\lambda (1 + \tau H_{\max})}.
\]

where \( a_{ij}(t) \) is the \( i \)th entry of \( A(t) \), \( z_j(t) \) is the \( i \)th entry of \( z(t) \), \( i, j \in I_n \) and \( H_{\max} \) denotes the maximum, \( \max_{\omega_1^T(0) \in [-\tau, 0]} B_{i(n+1)j}z_{j(n+1)} \).

Then, for system (13), taking the state feedback controller as in (12), the resultant closed-loop system (14) is positive and practically stable with respect to \((\lambda, \rho, v)\).

Proof. From (15), \( B(t)(t - \tau) \geq 0 \). Since \( A(t) \) is Metzler, system (14) is positive from Lemma 1. Taking the max-separable L-K functional as
\[
V(t, x_t) = \max_{\omega_1 \in I_n} \left\{ x_t(t) + \int_{-\tau}^t B(t + \tau)K(t)x(s)ds \right\},
\]
with the initial condition of \( V(t, x_t) \) for \( t = 0 \),
\[
V(0, x_0) = \max_{\omega_1 \in I_n} \left\{ x_0(0) + \int_0^0 B(t + s)K(t)x(s)ds \right\}.
\]
By the upper-right Dini derivative of \( V(t, x_t) \) and conditions (15)–(17), the practical stabilization of (13) can be got under the state feedback controller (12). The concrete proof process is omitted here.

Remark 3. Using the linear programming approach, state feedback controllers are designed in [27] for switched linear time-invariant positive systems. Under the controllers, the closed-loop systems are positive and asymptotically stable under certain average dwell time switching signals. Here, using the similar method, for LTV systems with time delay, the gain matrix of state feedback controller is designed by \( K(t) \). The resultant closed-loop system is proved to be positive and practically stable.

6 SIMULATION EXAMPLE

In this section, three simulation examples are presented to demonstrate the effectiveness of our main results.

Example 1. Consider the continuous-time non-linear dynamic system with time delay given by (1) with
\[
\begin{align*}
\dot{x}(t) &= \begin{pmatrix} -1.5x_1 + 0.7x_2 + \frac{e^{-0.6t}}{x_1^3 + x_2^3} \\ -1.01x_2 + 0.1x_1 \end{pmatrix}, \\
g(t) &= \begin{pmatrix} \frac{e^{0.6t}}{x_1^3 + x_2^3} \\ x_1 + 0.1x_2 \end{pmatrix}.
\end{align*}
\]

and \( \tau = 0.03 \).

It is easy to know that the Assumption 1 holds and system (18) is positive. Let the positive number pair \((\lambda, \rho) = (1, 2)\) and positive vector \( v = (1, 1)^T \). There exists \( \gamma(t) = 0.6e^{-\tau} \) such that \( f(t, 1_2) + g(t + 0.03, 1_2) \leq \gamma(t)1_2 = (0.6e^{-\tau}, 0.6e^{-\tau})^T \). Then by straightforward computation, \( \int_0^t \gamma^+(s)ds = 0.6(1 - e^{-\tau}) < 0.66624 \). So, conditions (i) and (ii) in Theorem 1 hold. Therefore, system (1) with (18) is practically stable with respect to \((1, 2, (1, 1)^T)\). The simulation curves of \((x_1, x_2)^T\) with initial condition \( \varphi(t) = (0.9, 1)^T \leq \lambda v \) are shown in Figure 1. From Figure 1, it can be seen that, the state trajectories preserve in the positive orthant, and \( x(t) \leq (2, 2)^T = \rho v \).

Example 2. Consider LTV systems with time delay given by (4) with
\[
\begin{align*}
A(t) &= \begin{pmatrix} -0.9e^{-3\sin t} & 0.9e^{-3\sin t} \\ 2.3e^{-0.5\sin t} & -3.9e^{-0.5\sin t} \end{pmatrix}, \\
A_d(t) &= \begin{pmatrix} 3e^{-\lambda t} & 0 \\ 0 & 3e^{-\lambda t} \end{pmatrix}, \text{ and } \tau = 0.03.
\end{align*}
\]

Since \( A(t) \) is Metzler and \( A_d(t) \geq 0 \), system (4) with (19) is positive from Lemma 1. Let the positive number pair \((\lambda, \rho) = (0.6, 2.5)\) and positive vector \( v = (1, 1)^T \) be given. There exist \( \gamma(t) = 3e^{-2.5t} \) such that \( \max_{0 \leq t \leq 2 \lambda} |a_{ii}(t) + a_{di}(t +\)
In this part, we consider the practical stabilization of LTV systems with time delay given by (8) with

\[
A(t) = \begin{pmatrix}
-0.9 & 0.25e^{-3t} \\
-0.1e^{-t} & 0
\end{pmatrix},
\quad A_d(t) = \begin{pmatrix}
0.8e^{-2t} & 0 \\
0 & 0
\end{pmatrix},
\quad B(t) = \begin{pmatrix}
e^{-t} \\
e^{-t}
\end{pmatrix},
\quad \tau = 0.03.
\]

For the given positive number pair \((\lambda, \varphi) = (0.5, 1.5)\) and vector \(v = (1, 1)^T\). There exist \(\omega_1(t) = (e^{-2t}, e^{-2t})^T\), \(\omega_2(t) = (1, 1)^T\), \(\xi(t) = (0.8e^{-2t}, 0.6e^{-3t})^T\), \(\gamma(t) = 0.9e^{-t}\), \(\zeta = 4\) such that \(K(t) = \frac{e^{T}(t)}{\omega_1(t)\omega_2(t)} = (0.4, 0.3e^{-2})\) and

\[
A(t) + B(t)K(t) + \frac{\zeta}{\omega_1(t)\omega_2(t)} \mathbb{1} = \begin{pmatrix}
0.4e^{-t} - 0.9 & 0.55e^{-3t} \\
0.3e^{-t} & 0.3e^{-3t}
\end{pmatrix} + \begin{pmatrix}
2e^{2t} & 0 \\
0 & 2e^{2t}
\end{pmatrix} \succeq 0.
\]

It is easy to see that \(A(t) + B(t)K(t)\) is a Metzler matrix and \(A_d(t) \succeq 0\). The closed-loop system of (8) is positive from Lemma 1. For \(\gamma(t) = 0.9e^{-t}\), \(\max_{j=1,2} \frac{B_j(t)\xi(t)}{\omega_1(t)\omega_2(t)} + a_{ji}(t + \tau) + 0.03) \leq \lambda e^{\rho t} \leq e^{\rho t} \leq 1.5e^{-t} < 1.5e^{-t} < \ln \frac{1.5}{0.512} = 1.075\). Obviously, condition (10) in Theorem 5 holds. By a straightforward computation, \(\lambda(1 + \tau H_{\max}) = \lambda(1 + 0.03 \times 0.8) = 0.512\). And taking \(\gamma(t) = 0.9e^{-t}\), \(\int_0^{\infty} 0.9e^{-t} dt = 0.9(1 - e^{-t}) < \ln \frac{1.5}{0.512} = 1.075\). Therefore, system (8) with (20) is practically stabilizable with respect to \((0.5, 1.5, (1, 1)^T)\).

Example 3. In this part, we consider the practical stabilization of LTV systems with time delay given by (8) with

\[
A(t) = \begin{pmatrix}
-0.9 & 0.25e^{-3t} \\
-0.1e^{-t} & 0
\end{pmatrix},
\quad A_d(t) = \begin{pmatrix}
0.8e^{-2t} & 0 \\
0 & 0
\end{pmatrix},
\quad B(t) = \begin{pmatrix}
e^{-t} \\
e^{-t}
\end{pmatrix},
\quad \tau = 0.03.
\]

7  |  CONCLUSION

Practical stability of positive system with time delay is considered in this paper. To be specific, sufficient conditions on practically stable are obtained severally for non-linear positive systems with single and multiple time delays by the max-separable
L–K functional method. With the same method, sufficient conditions for LTVP systems with single and multiple time delays are provided as well. In addition, practical stabilization of LTVP systems in terms of state feedback with and without input delay are discussed, and sufficient conditions on practical stabilizable systems are derived, respectively. Simulations are presented to demonstrate the effectiveness of our results.

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