Interacting Intersections

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ABSTRACT

Intersecting $p$-branes can be viewed as higher-dimensional interpretations of multi-charge extremal $p$-branes, where some of the individual $p$-branes undergo diagonal dimensional oxidation, while the others oxidise vertically. Although the naive vertical oxidation of a single $p$-brane gives a continuum of $p$-branes, a more natural description arises if one considers a periodic array of $p$-branes in the higher dimension, implying a dependence on the compactification coordinates. This still reduces to the single lower-dimensional $p$-brane when viewed at distances large compared with the period. Applying the same logic to the multi-charge solutions, we are led to consider more general classes of intersecting $p$-brane solutions, again depending on the compactification coordinates, which turn out to be described by interacting functions rather than independent harmonic functions. These new solutions also provide a more satisfactory interpretation for the lower-dimensional multi-charge $p$-branes, which otherwise appear to be nothing more than the improbable coincidence of charge-centres of individual constituents with zero binding energy.

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1 Introduction

BPS saturated $p$-brane solitons in supergravities play an important role in non-perturbative string theory or M-theory, since the supersymmetry which these solutions preserve may protect them from non-perturbative quantum corrections in the theories. These $p$-brane solutions are described by harmonic functions on the space transverse to the world-volume of the $p$-brane. There are two different types of dimensional reduction that may be used to relate $p$-brane solutions of supergravity theories in different dimensions. The better known of these employs Killing symmetries of the $p$-brane solutions in $D$ dimensions to effect a simultaneous reduction on the world-volume and in the target space, yielding a $(p - 1)$-brane in $(D - 1)$ dimensions. This is the field-theoretic analogue of the double dimensional reduction procedure for $p$-brane actions [1], which is also applicable to supergravity solutions, causing “diagonal” movements on the $D$ versus $p$ “brane-scan.” Aside from a Weyl rescaling of the gravitational action, this diagonal reduction procedure does not change the asymptotic fall-off behaviour of a solution in the directions transverse to the world-volume. In other words, the harmonic functions that describe the $p$-branes in different dimensions are preserved under diagonal dimensional reduction.

The second type of reduction has been referred to in the literature as “constructing periodic arrays” [2, 3, 4]. This procedure employs the zero-force property of extremal $p$-brane solutions, which allows the existence of multi-centre solutions as well as single-centre ones. For example, let us consider an extremal 5-dimensional black hole solution, which is described by an harmonic function $H$ on the 4-dimensional transverse space. For a single black hole located at the origin, we have $H = 1 + \hat{Q} r^{-2}$. Owing to the no-force condition between extremal black holes we can construct multi-centre solutions, with, in particular, the black holes arrayed periodically along a line:

$$H = 1 + \sum_{k=-\infty}^{\infty} \frac{\hat{Q}}{r^2 + (z + 2\pi k R)^2}$$

(1)

$$= 1 + \frac{Q \sinh(r/R)}{2r (\cosh(r/R) - \cos(z/R))}.$$  

(2)

(The possibility of having a closed form for this summation was shown in [4].) Here, the 4-dimensional transverse space is described by three coordinates $y^m$, with $r^2 = y^m y^m$, together with the coordinate $z$ along the line of black holes. We have defined $Q$ in terms of the 5-dimensional charge $\hat{Q}$ by $Q = \hat{Q}/R$. If the internal coordinate $z$ has exactly the same $2\pi R$ period as the separation between the 5-dimensional black holes, then the solution (1) describes precisely a single black hole solution in a space with a periodically identified (i.e.
compactified) coordinate \( z \). The solution (1) or (2) cannot be dimensionally reduced on the \( z \) coordinate in all the space since it is \( z \)-dependent, contradicting the usual Kaluza-Klein requirement. However, for large values of \( r \), (or, equivalently, small values of the period \( 2\pi R \)), the \( z \)-dependence becomes negligible, and the harmonic function \( H \) tends to \( 1 + Q/(2r) \). (Note that \( Q \) therefore has the interpretation of being the 4-dimensional charge.) In other words, if we make a Fourier expansion of the solution (2) with respect to the compactified coordinate \( z \), the non-zero modes become insignificant in the regime \( r >> R \), and only the zero-mode survives. Thus the solution becomes effectively a 4-dimensional black hole.

To make this precise, we may perform the Fourier transformation of (2) explicitly. In an expansion that is convergent at large \( r \), it is not hard to see that this takes the form

\[
H = 1 + \frac{Q}{2r} \sum_m e^{-|m|r/R} e^{imz/R} = 1 + \frac{Q}{2r} \sum_{m>0} e^{-m r/R} \cos\left(\frac{m z}{R}\right),
\]

which shows that the \( z \)-dependent terms tend to zero exponentially at large \( r \). It is also of interest to consider a Fourier expansion that is convergent in the regime where \( r \) is small. It is easy to see that the appropriate expansion takes the form

\[
H = 1 - \frac{Q}{r} \sum_{m>0} \sinh\left(\frac{mr}{R}\right) \cos\left(\frac{m z}{R}\right).
\]

It is worth emphasising that in order to make the vertical Kaluza-Klein dimensional reduction rigorously exact, it is necessary to exploit the the no-force condition in the higher dimension in order first to construct a continuous uniform distribution of charge along the coordinate destined for Kaluza-Klein reduction \[3\]. Equivalently, this means choosing the relevant harmonic function \( H \) in the higher dimension to be independent of the intended compactification coordinate. Having then performed the Kaluza-Klein reduction, one can always choose to carry out the inverse procedure, and retrace the solution back to the higher dimension. This inverse of Kaluza-Klein reduction, which has therefore acquired the name “Kaluza-Klein oxidation,” leads back to the continuum solution in the higher dimension, where the harmonic function is independent of the compactification coordinate. Thus under this straightforward oxidation procedure, the vertical oxidation of a \( p \)-brane in the lower dimension yields a continuous line of \( p \)-branes in the higher dimension. By contrast diagonal oxidation, \( i.e. \) the inverse of a diagonal Kaluza-Klein reduction, takes a single lower-dimensional \( p \)-brane to a single higher-dimensional \( (p+1) \)-brane.

In general, let us consider a \( D \)-dimensional \( p \)-brane with \( d + 2 \) transverse dimensions.
The harmonic function for an isotropic solution satisfies

\[ \Box_{(d+2)} H = H'' + \frac{\tilde{d} + 1}{r} H' = 0, \]

(5)

where a prime denotes a derivative with respect to \( r = \sqrt{y^m y^n} \). The solution is therefore given by \( H = 1 + Q r^{-\tilde{d}} \). If the solution can be vertically oxidised to \( D + 1 \) dimensions, the higher dimensional harmonic function \( \tilde{H} \) will still retain this same form, \( \tilde{H} = H \), and satisfy the same equation (5), if we simply retrace the step of Kaluza-Klein dimensional reduction. However, we know that more general solutions for \( \tilde{H} \) are also possible in \( D + 1 \) dimensions, in which dependence on the extra coordinate \( z \) is also allowed. In general, the function \( \tilde{H} \) must satisfy the \((\tilde{d} + 3)\)-dimensional harmonic condition

\[ \Box_{(\tilde{d}+3)} \tilde{H} = \tilde{H}'' + \frac{\tilde{d} + 1}{r} \tilde{H}' + \ddot{\tilde{H}} = 0, \]

(6)

where a dot denotes a derivative with respect to the compactification coordinate \( z \). (We continue to assume that the function \( \tilde{H} \) depends on the lower-dimensional transverse coordinates \( y^m \) only through the isotropic distance \( r = \sqrt{y^m y^n} \).) There are special solutions to the harmonic equation (6) that include a single \( p \)-brane solution, with \( \tilde{H} \) of the form \( 1 + \tilde{Q} (r^2 + z^2)^{-(\tilde{d} + 1)/2} \), and also a periodic array of \( p \)-branes, of the form given in (5), appropriately generalised to a transverse space of the dimension \( \tilde{d} + 3 \). The most general solution can be obtained by Fourier expanding the dependence on the compactifying coordinate \( z \), and writing \( \tilde{H} = \sum_m f_m(r) e^{imz/R} \), where the functions \( f_m(r) \) will satisfy

\[ f_m'' + \frac{\tilde{d} + 1}{r} f_m' - m^2 f_m = 0. \]

(7)

The solution to this equation is expressed in terms of the modified Bessel functions \( K \) and \( I \),

\[ f_m = \frac{c_m}{r^{\tilde{d}/2}} K_{\tilde{d}/2}(\frac{|m|r}{R}) + \frac{\tilde{c}_m}{r^{\til{d}/2}} I_{\til{d}/2}(\frac{|m|r}{R}). \]

(8)

For a series solution that is nonsingular at large \( r \), we should choose \( \tilde{c}_m = 0 \) and retain only the \( K \) Bessel functions. Thus the non-zero modes, \( i.e. \ f_m(r) \) with \( m \neq 0 \), tend exponentially to zero for large \( r \). The zero-mode solution \( f_0 \) is precisely given by \( H \), the lower-dimensional harmonic function. The \( I \) Bessel functions are appropriate for an expansion valid at small values of \( r \). The example of the 5-dimensional black hole that we discussed previously fits into this general pattern, when we set \( \tilde{d} = 1 \). In particular, since we have

\[ K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}, \quad I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x, \]

(9)
we see that indeed the general form (8) encompasses the large-$r$ and small-$r$ expansions (3) and (4), with $c_m = Q \sqrt{m/(2\pi R)}$ or $\tilde{c}_m = -Q \sqrt{m\pi/(8R)}$ respectively.

We have seen that all configurations for the higher-dimensional solution can vertically reduced to lower-dimensional $p$-branes, as long as the space is compactified, and that we focus attention only on the regime where $r$ is large compared with the compactification radius. The construction of periodic arrays is not essential, as long as the non-zero modes for any configuration die off at large values of $r$. In this description, the Kaluza-Klein reduction is approximate, in that it can only be performed in the regime where $r$ is large. For smaller values of $r$, the internal dimensions, and hence the non-zero modes, become more and more important and the theory becomes essentially higher-dimensional. If one wishes to have a Kaluza-Klein reduction that is exact everywhere, the solution should be constructed using only the zero-mode, which can be achieved by taking a continuous “stack” of charges distributed uniformly around the internal direction.

To summarise, we may say that although the “naive” vertical oxidation of a single isotropic $p$-brane solution gives rise to a continuum of $p$-branes in the higher dimension, a more natural approach, from the point of view of solitons in string theory, is to consider a periodic array of $p$-branes in the higher dimension. Provided that one stays at distances large compared with the period, the solution admits a lower dimensional interpretation that approximates to the previously-discussed isotropic $p$-brane.

In our discussion so far, we have considered the dimensional reduction of a single $p$-brane. In general there exist multi-charge $p$-brane solutions, and the oxidation of such solutions will give rise to intersections in the higher dimensions. Multi-charge $p$-branes in $D \geq 2$ were classified in [7], and their higher dimensional interpretations as intersections were classified in [8, 7]. (See also a recent review on black holes and solitons.) To be precise, in higher dimensions the standard intersecting solutions describe intersections of $p$-branes whose charges are uniformly distributed on certain compactifying coordinates, such that the solution is independent of these coordinates. However, just as in the case of the single-charge $p$-branes we discussed previously, it is really more natural to look for solutions that describe periodic arrays in the higher dimension, rather than continuous distributions. In this paper we shall investigate this question, by studying how the solutions can depend on the compactifying coordinates (or relative transverse coordinates, if one considers only the higher dimensional theory). This will provide solutions for intersecting $p$-branes that go beyond those that are normally considered. As well as being more generic than the usual intersecting solutions, they also provide a resolution of a previously mysterious aspect of the
extremal multi-charge $p$-brane solutions. These are normally viewed as “bound states” of single-charge $p$-branes \[1\], with, however, zero binding energy. What is not normally clear, therefore, is why the multi-charge $p$-branes should have any particular physical significance in their own right, since there appears to be nothing to prevent the individual charge-centres from “drifting apart,” so that the configuration separates into its single-charge constituents. These multi-charge $p$-branes are the dimensional reductions of the standard intersecting solutions that involve continuous charge distributions in the higher dimension. If we instead consider the more generic intersections that we find in this paper, then the interpretation as lower-dimensional multi-charge $p$-branes is only approximate, in the regime where one is far away from the charge centres. The true solutions do not admit any “bound state at threshold” interpretation except as a large-distance approximation, and so the possibility of the charge centres drifting apart no longer arises.

Intersecting solutions of the more general kind we are considering here have also been discussed in \[1, 12, 13, 14, 15, 16\].

In section 2 we study pair-wise intersections, and we find that the form of the generalised solutions are similar to those obtained directly from Kaluza-Klein oxidation, except that now the harmonic functions generalise to become functions that “interact” with each other. We obtain general such solutions, discuss how they reduce to lower dimensions, and we comment on their physical significance. We also present summarising rules for how to generalise the usual intersecting solutions to the more general interacting ones. In section 3, we extend the discussion to multi-intersections, which can be constructed by requiring that the conditions for all possible pair-wise intersections be satisfied. In section 4, we study the supersymmetry of these generalised solutions. We conclude the paper in section 5.

In the subsequent discussion we follow the notation of \[17\] to denote the fields in $D$-dimensional maximal supergravity, which are obtained by dimensional reduction from the eleven-dimensional fields $g_{MN}$ and $A_{MNP}$:

\[
g_{MN} \rightarrow g_{MN}, \quad \vec{\phi}, \quad A_{1}^{(i)}, \quad A_{0}^{(ij)}, \quad A_{3} \rightarrow A_{3}, \quad A_{2}^{(i)}, \quad A_{1}^{(ij)}, \quad A_{0}^{(ijk)} \tag{10}
\]

where the superscript indices $i, j, k = 1, \ldots, 11 - D$ run over the $11 - D$ internal toroidally-compactified dimensions, starting from $i = 1$ for the step from $D = 11$ to $D = 10$, and the numerical subscripts denote the degrees of the differential forms. The bosonic Lagrangian in $D$ dimensions is given by \[17\]

\[
\mathcal{L} = eR - \frac{1}{4}e (\partial \vec{\phi})^2 - \frac{1}{35} e e^{\vec{\phi}} F_{(4)}^2 - \frac{1}{72} e \sum_{i} e^{\vec{a}} e^{\vec{a}} (F_{3}^{(i)})^2 - \frac{1}{4} e \sum_{i < j} e^{\vec{a}} e^{\vec{a}} (F_{2}^{(ij)})^2 \tag{11}
\]
\[-\frac{1}{4}\epsilon e \sum e^{\tilde{b}_i \phi} (F_2^{(i)})^2 - \frac{1}{2} e \sum e^{\tilde{a}_{ij} \phi} (F_1^{(ij)})^2 - \frac{1}{2} e \sum e^{\tilde{b}_{ij} \phi} (F_1^{(ij)})^2 + \mathcal{L}_{FFA},\]

where the “dilaton vectors” \(\tilde{a}, \tilde{a}_i, \tilde{a}_{ij}, \tilde{a}_{ijk}, \tilde{b}_i, \tilde{b}_{ij}\) are constants that characterise the couplings of the dilatonic scalars \(\tilde{\phi}\) to the various gauge fields. The full details of these vectors, the definitions of the field strengths, and the Wess-Zumino terms \(\mathcal{L}_{FFA}\) can be found in [17].

2 Pair-wise intersections

In this section we discuss general solution of pair-wise intersections, which are the dimensional oxidations of lower-dimensional \(p\)-branes. As in the case of the single-charge solution discussed in the introduction, if we simply use the usual Kaluza-Klein ansatz for the oxidation, the higher dimensional solution will necessarily be independent of the internal compactifying space. In the case where the oxidation is vertical, it describes a configuration where the higher-dimensional \(p\)-branes are uniformly distributed over the internal space. However in general the higher-dimensional solution can also have non-vanishing non-zero modes, in which the functions \(H\) acquire additional dependence on the internal coordinates, for which the non-zero-modes tend to zero in the regime where the lower-dimensional transverse radius is much larger than the compactification scale.

We shall first consider cases of double intersections arising from the diagonal oxidation of one extended object together with the vertical oxidation of the other extended object. Let us consider for example a two-charge black hole solution in \(D=4\), with the charges supported by the field strengths \(F_2^{(ij)}\) and \(F_2^{(k7)}\), where the indices \(i, j, k\) are all different, but chosen from the set \(\{1, 2, \ldots, 6\}\). (The index value \(i\) labels the internal coordinate \(z^i\) arising in the dimensional reduction from \(12-i\) to \(11-i\) dimensions.) The solution is given by

\[
d s_4^2 = -(H_1 H_2)^{-1/2} dt^2 + (H_1 H_2)^{1/2} dy^m dy^m, \quad \tilde{\phi} = \frac{1}{2} \tilde{a}_{ij} \log H_1 + \frac{1}{2} \tilde{a}_{k7} \log H_2, \quad F_2^{(ij)} = dt \wedge dH_1^{-1}, \quad F_2^{(k7)} = dt \wedge dH_2^{-1},
\]

where \(H_1\) and \(H_2\) are harmonic functions on the 3-dimensional transverse space \(y^m\), i.e. \(\Box H_1 = 0 = \Box H_2\), where \(\Box = \partial_m \partial_m\). Since \(\tilde{F}_2^{(ij)}\) and \(\tilde{F}_2^{(k7)}\) in \(D = 4\) arise from the dimensional reduction of \(\tilde{F}_2^{(ij)}\) and \(\tilde{F}_3^{(k)}\) in \(D = 5\), it follows that the oxidation of the solution (12) to \(D = 5\) gives a black hole (supported by \(\tilde{F}_2^{(ij)}\)) intersecting with a string (supported by \(\tilde{F}_3^{(k)}\)), viz.

\[
d s_5^2 = \hat{H}_1^{-2/3} \hat{H}_2^{-1/3} \left( -dt^2 + \hat{H}_1 \hat{H}_2 dy^m dy^m + \hat{H}_1 dz_7^2 \right),
\]
\[ \hat{\phi} = \frac{1}{2} \hat{a}_{ij} \log \hat{H}_1 + \frac{1}{2} \hat{a}_{k} \log \hat{H}_2, \]
\[ \hat{F}^{(ij)}_2 = dt \wedge d\hat{H}^{-1}_1, \quad \hat{F}^{(k)}_3 = dt \wedge dz_7 \wedge d\hat{H}_2^{-1}. \]

As in the example of the single-charge p-branes discussed in the Introduction, here too we may entertain the idea that the functions \( \hat{H}_1 \) and \( \hat{H}_2 \) in \( D = 5 \) need not simply be the same harmonic functions \( H_1 \) and \( H_2 \) of the 4-dimensional 2-charge solution, but might depend also on the compactification coordinate \( z_7 \). It is not so obvious in this 2-charge case, however, as to what form this additional coordinate dependence might take. Since the string component is diagonally oxidised from \( D = 4 \) to \( D = 5 \), it is natural to expect that \( \hat{H}_2 \) should continue to be an harmonic function in \( y^m \). The black hole component, on the other hand, is vertically oxidised to \( D = 5 \), and naively, one might expect that \( \hat{H}_1 \) could be an harmonic function on its entire transverse space \( \{ z_7, y^m \} \) in \( D = 5 \), as in the single-charge example. This would imply that there would be no force between the isotropic string and black hole. In fact it turns out that this naive guess does not work. We find indeed that more general solutions are in fact possible, in which \( \hat{H}_2 \) continues to depend only on the lower-dimensional transverse coordinates \( y^m \), while the function \( \hat{H}_1 \) is allowed to depend on \( z_7 \) as well as on \( y^m \). However \( \hat{H}_1 \) and \( \hat{H}_2 \) do not decouple. After calculations of some complexity, we find that the 5-dimensional equations of motion are all satisfied if \( \hat{H}_1 \) and \( \hat{H}_2 \) satisfy the equations

\[ \Box \hat{H}_2 = 0, \quad \Box \hat{H}_1 + \hat{H}_2 \ddot{\hat{H}}_1 = 0, \]

where \( \Box \) again denotes the Laplacian in the \( y^m \) coordinates, and a dot denotes a derivative with respect to \( z_7 \). Note that \( \hat{H}_2 \) is still a harmonic function on the 3-dimensional \( y^m \) space, just like \( H_2 \) in the lower dimension, but \( \hat{H}_1 \) is no longer a harmonic function in general. (This implies that in the presence of a string, there can be no static isotropic black hole.)

The metric, dilatons and field strengths are still given by (13), in terms of the new functions \( \hat{H}_1 \) and \( \hat{H}_2 \) that satisfy (14).

The equations in (14) admit two special solutions, in which \( \hat{H}_1 \) and \( \hat{H}_2 \) decouple, namely

1) \( \hat{H}_1 = H_1(y) \) and \( \hat{H}_2 = H_2(y) \) are both harmonic on the overall transverse space \( y^m \).

2) \( \hat{H}_1 = \hat{H}_1(z_7) \) and \( \hat{H}_2 = \hat{H}_2(y) \) are harmonic on \( z_7 \) and \( y^m \) respectively.

Note that the first special solution is independent of \( z_7 \), and the solution can be simply reduced to the \( D = 4 \) solution (14). Conversely, it is the solution in \( D = 5 \) that is
obtained simply by the standard oxidation of the 2-charge black hole in $D = 4$. The second special solution cannot be dimensionally reduced on the $z_7$ coordinate, on account of the $z_7$ dependence of $\hat{H}_1$. This type of special solution was also found in [8].

We are interested in finding more general classes of solutions to (14), where, however, the non-zero modes become negligible in the regime where the radius $r = \sqrt{y^m y^m}$ is large compared the radius of the compactifying coordinate. To construct such solutions, we consider the situation where $\hat{H}_2$ is still an isotropic harmonic function, i.e. $\hat{H}_2 = 1 + \hat{Q}_2/r$. In other words, we study the black hole configuration in the presence of a single string, which, without loss of generality, can be located at $y^m = 0$. We shall also take $\hat{H}_1$ to depend on $y^m$ isotropically through the coordinate $r$, but we allow it also to depend on $z_7$. (Note that since $\hat{H}_2$ is already assumed to be isotropic, this implies that only the $z_7$-independent part (i.e. the zero-mode) of $H_1$ could in any case naturally have any non-isotropic dependence on $y^m$.) It then follows from (14) that $\hat{H}_1$ satisfies the equation

$$\Box \hat{H}_1 + H_2 \dddot{\hat{H}}_1 = H_1'' + \frac{2}{r} H_1' + (1 + \frac{\hat{Q}_2}{r}) \dddot{\hat{H}}_1 = 0.$$  

(15)

It is worth emphasising again that for non-vanishing charge $\hat{Q}_2$, the function $\hat{H}_1$ is not an harmonic function on the 4-dimensional transverse space $\{z_7, y^m\}$. However, in the asymptotic regime where $r >> \hat{Q}_2$, the harmonic function $\hat{H}_2$ tends to unity, and then $\hat{H}_1$ becomes an harmonic function in its traverse space. Thus an observer at $r >> \hat{Q}_2$ would conclude that $\hat{H}_1$ is an harmonic function on its 4-dimensional transverse space $\{z_7, y^m\}$, which could be isotropic as if there existed an extremal 5-dimensional black hole. One can therefore construct a periodic array of such solutions aligned along the $z_7$ axis, for which the $z_7$-dependence will become negligible at large $r$.

An observer in the lower dimension who observed the black-hole configuration from large distances $r$ would conclude that there was a no-force condition between the two charge species, by virtue of the independent harmonic nature of the two functions $H_1$ and $H_2$. At large $r$, he would be unable to distinguish between the situation where the higher-dimensional functions $\hat{H}_1$ and $\hat{H}_2$ were also exactly harmonic, and the situation where they have the more general interacting form that we have been considering here. If they do interact, with $\hat{H}_1$ being a non-harmonic solution of the more general equation (13), then the no-force condition in the lower dimension, with its consequence that the two charge centres could separate freely, would be a pure artefact of the large-distance approximation.

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1Of course here, and for the similar equations we shall encounter later in the paper, more general solutions of the equations (14) are also possible, where we do not assume only isotropic dependence on the $y^m$ coordinates.
and a closer inspection of the small-\(r\) region near to the black hole would reveal a more complex higher-dimensional structure.

We shall now show that indeed such a more general configuration can also be dimensionally reduced to \(D = 4\), and give rise to (12), provided that we are in the regime \(r >> R_7\), where \(R_7\) is the radius of the compactification coordinate \(z_7\). To do this, we make a Fourier expansion of \(\hat{H}_1\) on \(z_7\), writing \(\hat{H}_1 = \sum_m f_m(r)e^{imz_7/R_7}\). The Fourier modes \(f_m(r)\) then satisfy
\[
 f''_m + \frac{2}{r}f'_m - \frac{m^2}{R^2}(1 + \frac{\dot{Q}_2}{r})f = 0 .
\]
(16)
The general solution to this equation is given by
\[
f_m = c_m U(1 + \frac{|m|\dot{Q}_2}{2R^2}, 2, \frac{2|m|r}{R^2}) e^{-|m|r/R_7} + \tilde{c}_m M(1 + \frac{|m|\dot{Q}_2}{2R^2}, 2, \frac{2|m|r}{R^2}) e^{-|m|r/R_7} ,
\]
where \(U(a, b, x)\) and \(M(a, b, x)\) denote Kummer’s confluent hypergeometric functions. We are interested for now in expansions that are non-singular at large \(r\), implying that we should take \(\tilde{c}_m = 0\) and retain only the \(U(a, b, x)\) functions. The non-zero modes \((f_m\) with \(m \neq 0\)) in fact tend to zero for large \(r/R\), and the zero-mode solution \(f_0\) is precisely given by \(H_1\), the lower-dimensional harmonic function. Thus we see that in the regime \(r >> R_7\), the solution of the non-linear equation of motion can also be dimensionally reduced, since its dependence on \(z_7\) becomes negligible, and it limits to the previous \(z_7\)-independent form.

Note that this description in terms of a Fourier expansion where the non-zero modes tend to zero at large \(r\) is precisely analogous to the description we gave for single \(p\)-branes in the Introduction. Indeed, if we set \(\hat{H}_2\) to be unity in the above discussion, the series (17) reduces precisely to the form of the expansion (8), for the particular case \(\tilde{d} = 1\).

As was observed in [10], the two-charge black hole solution (12) can be viewed as a bound state of two basic single-charge black-hole building blocks, however with zero binding energy. This latter feature implies that the two individual building blocks can move freely with respect to one another. This picture makes the existence of the two-charge black hole (12) in \(D = 4\) appear quite accidental, since the two charge centres can apparently freely drift apart. However, as we have shown, if we interpret such a solution from a higher-dimensional point of view, the non-interacting (harmonic) functions \(H_1\) and \(H_2\) in \(D = 4\) become functions \(\hat{H}_1\) and \(\hat{H}_2\) that in general do interact with each other, and hence the relative locations of the two charges are no longer arbitrary; they are located at the same point in the overall transverse space of the \(y^m\) coordinates, indicating that the individual components in the four-dimensional 2-charge solution are in some sense bound together and cannot drift apart. In fact the interpretation of the configuration as a bound state of two
basic objects is no longer appropriate for these more generic interacting solutions. (However, it is worth observing that there are no self-interactions of the individual \( \hat{H} \) functions, implying that there is no force between \( p \)-branes carried by the same field strength.) Of course there do also exist configurations in \( D = 5 \) where the two intersecting objects are genuinely independent. This happens when \( \hat{H}_1 \) and \( \hat{H}_2 \) are both taken to have one or other of the two special forms discussed previously. The first of these cases, where the functions have the coordinate dependences \( \hat{H}_1(y) \) and \( \hat{H}_2(y) \), implies that there is no force between a string and a black-hole configuration whose black hole charge is distributed uniformly along the string world-line. The second case, with \( \hat{H}_1(z_7) \) and \( \hat{H}_2(y) \), implies that there is no force between a string and a black-hole configuration whose charge is uniformly distributed over the entire space transverse to the string.

The above discussion applies to other pair-wise intersections as well. Let us look again at a similar example where upon oxidation, one extended object undergoes vertical oxidation, whilst the other undergoes diagonal oxidation. We shall consider an NS-NS dyonic string in \( D = 6 \), supported by the NS-NS 3-form field strength \( F_3^{(1)} \). When it oxidises to \( D = 10 \), it becomes the intersection of an NS-NS string and a 5-brane, with the solution in \( D = 10 \) given by

\[
\begin{align*}
  ds_{10}^2 &= \hat{H}_e^{-3/4} \hat{H}_m^{-1/4} \left(-dt^2 + dx_1^2 + H_e H_m \, dy^m dy^m + H_e \left(dz_2^2 + \ldots + dz_5^2\right)\right), \\
  \phi &= \frac{1}{2} \log(\hat{H}_e/\hat{H}_m), \\
  F_3^{(1)} &= dt \wedge dx_1 \wedge dH_e^{-1} + e^{-\phi} \ast (dt \wedge dx_1 \wedge d^4z \wedge dH_m^{-1}).
\end{align*}
\]

In the directly oxidised solution, the functions \( H_e \) and \( H_m \) depend only on the four transverse coordinates \( y^m \) of the original 6-dimensional solution. Again, however, we may consider more general types of solution in \( D = 10 \) where appropriate dependence on the internal compactification coordinates \( \vec{z} \) is also allowed. Specifically, we find that such more general solutions exist in \( D = 10 \), in which the metric, dilaton and field strength are still given in terms of \( \hat{H}_e \) and \( \hat{H}_m \) by (18), but now \( \hat{H}_e \) is allowed to depend on the internal coordinates \( \vec{z} \) as well as on \( y^m \). We find that the ten-dimensional equations of motion are all satisfied provided that the functions \( \hat{H}_e(y, z) \) and \( \hat{H}_m(y) \) satisfy

\[
\square_{\vec{g}} \hat{H}_m = 0, \quad \square_{\vec{g}} \hat{H}_e + \hat{H}_m \square_{\vec{z}} \hat{H}_e = 0,
\]

where \( \square_{\vec{g}} \) and \( \square_{\vec{z}} \) denote the flat-space Laplacians in the four \( y^m \) and the four \( z^i \) coordinates respectively.

Again these equations admit two special solutions analogous to those discussed in the previous example, namely a solution where \( \hat{H}_e \) is also independent of \( \vec{z} \), and so both functions
are simply harmonic in $\vec{y}$, and a solution where $\vec{H}_e$ depends only on $\vec{z}$, and the functions $\vec{H}_e$ and $\vec{H}_m$ are harmonic in $\vec{z}$ and in $\vec{y}$ respectively. The former corresponds to the simple oxidation of the $D = 6$ dyonic string, while the latter corresponds to a special kind of solution described in $\mathbb{R}$. We are interested instead in the general solutions that can be obtained by performing a Fourier expansion of $\vec{H}_e$ on the compactification coordinates, $\vec{H}_e(\vec{y}, \vec{z}) = \sum \vec{n}_f \vec{n}_f(\vec{y}) e^{i\vec{n} \cdot \vec{z}/R_i}$, where $R_i$ denotes the radius of the compactification circle for the coordinate $z^i$, and $\vec{n}$ denotes the set of integers $\{n_i\}$. The Fourier modes in this case satisfy

$$f''_{\vec{n}} + \frac{3}{r} f'_{\vec{n}} - \lambda^2_{\vec{n}} \frac{1 + Q_m}{r^2} f_{\vec{n}} = 0,$$

(20)

where we consider functions $\vec{H}_e(r, \vec{z})$ and $\vec{H}_m(r)$ that depend isotropically on $\vec{y}$, we take $H_m = 1 + Q_m/r^2$, and

$$\lambda_{\vec{n}} = \sqrt{\sum_i n_i^2/R_i^2}.$$

(21)

The solutions for the Fourier modes are then given in terms of modified Bessel functions by

$$f_{\vec{n}} = \frac{c_{\vec{n}}}{r} K_\alpha(\lambda_{\vec{n}} r) + \frac{\tilde{c}_{\vec{n}}}{r} I_\alpha(\lambda_{\vec{n}} r),$$

(22)

where $\alpha = \sqrt{1 + \lambda_{\vec{n}}^2 Q_m}$. In order to have expansions that are nonsingular at large $r$, we must take $\tilde{c}_{\vec{n}} = 0$, so that the solutions are given in terms of the $K$ Bessel functions only. It is easily verifiable that in the regime where $r \gg R_i$ for all $i$ the non-zero modes become negligible, and the solution effectively becomes the usual oxidation of the six-dimensional dyonic string.

In the above two examples, we considered cases where the internal space is the world-volume of one of the two extended objects. For a generic multi-charge $p$-brane, each individual object will undergo a vertical oxidation in general. For example, if we oxidise the above two examples further to $D = 11$, both extended objects in the intersection will undergo further vertical oxidations. In particular, if we oxidise the intersection of the 10-dimensional string and 5-brane further to $D = 11$, the 5-brane will now be vertically lifted, given to an intersection of membrane and 5-brane in $D = 11$:

$$ds^2_{11} = \tilde{H}_e^{-2/3} \tilde{H}_m^{-1/3} \left(-dt^2 + dx_1^2 + \tilde{H}_e \tilde{H}_m dy^m dy^m + \tilde{H}_m dz_1^2 + \tilde{H}_e (dz_2^2 + \cdots + dz_5^2)\right),$$

$$F_4 = dt \wedge dx_1 \wedge dz_1 \wedge d\tilde{H}_e^{-1} + *(dt \wedge dx_1 \wedge dz_2 \wedge \cdots \wedge dz_5 \wedge d\tilde{H}_m^{-1}).$$

In this case, the associated harmonic functions $\vec{H}_e \rightarrow \vec{H}_e$ and $\vec{H}_m \rightarrow \vec{H}_m$ will satisfy either

$$\Box_{\vec{y}} \vec{H}_m = 0, \quad \Box_{\vec{y}} \vec{H}_e + \vec{H}_m \Box_{\vec{z}} \vec{H}_e = 0,$$

(24)
or
\[
\Box_y \bar{H}_e = 0 , \quad \Box_y \bar{H}_m + \bar{H}_e \Box_z \bar{H}_m = 0 , \tag{25}
\]

If we oxidise the first example, of the 4-dimensional 2-charge black hole (12) to \(D = 11\), it becomes the intersection of two membranes. It is a special case of more general solutions that we can construct by starting from a 2-charge black-hole in \(D = 7\), supported by field strengths coming from the 4-form in \(D = 11\). Without loss of generality, we may consider the case using field strengths \(F_2^{(12)}\) and \(F_2^{(34)}\), with associated harmonic functions \(H_1(\vec{y})\) and \(H_2(\vec{y})\). We shall use the notation that indices \(a_1, b_1, \ldots\) run over 1 and 2, while indices \(a_2, b_2, \ldots\) run over 3 and 4. Having oxidised the seven-dimensional solution to \(D = 11\), we then generalise the harmonic functions to \(\bar{H}_1 = \bar{H}_1(\vec{y}, z^{a_2})\) and \(\bar{H}_2 = \bar{H}_2(\vec{y}, z^{b_2})\).\(^2\) We find that the eleven-dimensional configuration
\[
\begin{align*}
\text{ds}_{11}^2 &= (\bar{H}_1 \bar{H}_2)^{-2/3} (-dt^2 + \bar{H}_1 \bar{H}_2 dy^m dy^n + \bar{H}_1 (dz_3^2 + dz_4^2) + \bar{H}_2 (dz_1^2 + dz_2^2)) , \\
F_4 &= dt \wedge dz_1 \wedge dz_2 \wedge d\bar{H}_1^{-1} + dt \wedge dz_3 \wedge dz_4 \wedge d\bar{H}_2^{-1} .
\end{align*}
\tag{26}
\]
solves the supergravity equations of motion provided that the functions \(\bar{H}_1\) and \(\bar{H}_2\) satisfy
\[
\begin{align*}
\Box_y \bar{H}_1 + \bar{H}_2 \partial^2_{a_2} \bar{H}_1 &= 0 , \\
\Box_y \bar{H}_2 + \bar{H}_1 \partial^2_{a_1} \bar{H}_2 &= 0 ,
\end{align*}
\tag{27}
\]
and also the constraint
\[
\epsilon_{a_1 b_1} \partial_{b_1} \bar{H}_2 \partial_{a_2} \bar{H}_1 - \epsilon_{a_2 b_2} \partial_{a_1} \bar{H}_2 \partial_{b_2} \bar{H}_1 = 0 . \tag{28}
\]
This latter condition is equivalent to the constraint
\[
\partial_{a_1} \bar{H}_2 \partial_{a_2} \bar{H}_1 = 0 , \tag{29}
\]
which follows from (28) upon using the fact that \(\bar{H}_1\) and \(\bar{H}_2\) are real. Thus we have either \(\bar{H}_1 = \bar{H}_1(\vec{y})\) and \(\bar{H}_2 = \bar{H}_2(\vec{y}, z_1, z_2)\) or else \(\bar{H}_1 = \bar{H}_1(\vec{y}, z_3, z_4)\) and \(\bar{H}_2 = \bar{H}_2(\vec{y})\). Correspondingly, the remaining equations (27) become
\[
\begin{align*}
\Box_y \bar{H}_1 &= 0 , \\
\Box_y \bar{H}_2 + \bar{H}_1 \partial^2_{a_1} \bar{H}_2 &= 0 ,
\end{align*}
\tag{30}
\]
\(^2\)The general principle here, and in all other cases, governing the “extra” coordinate dependences that we consider for the \(\bar{H}\) functions is that we allow the function associated with each lower-dimensional single-charge \(p\)-brane component to depend on all its own transverse coordinates in the higher dimension, \textit{i.e.} on all the coordinates associated with vertical oxidation steps for that particular \(p\)-brane component (together with the original transverse coordinates in the lower dimension). Equivalently, we may state the conditions directly in the higher dimension, without reference to oxidation, as the assumption that each function in the intersection depends, \textit{a priori}, on all its transverse coordinates. Equations of motion, as we shall see below, may impose further restrictions.
To summarise, the spacetime of any double intersections of $p$-branes is split into overlapping world volume coordinates $x^\mu$, relative transverse coordinates $\vec{z}_1$ and $\vec{z}_2$, and overall transverse space coordinates $\vec{y}$. Assuming the two intersecting objects, described by functions $H_1$ and $H_2$, have world volumes $\{x^\mu, \vec{z}_1\}$ and $\{x^\mu, \vec{z}_2\}$ respectively, then these two functions satisfy

\[ \square_{\vec{y}} H_1 = 0, \quad \square_{\vec{y}} H_2 + H_1 \square_{\vec{z}_1} H_2 = 0, \quad (32) \]

or

\[ \square_{\vec{y}} H_2 = 0, \quad \square_{\vec{y}} H_1 + H_2 \square_{\vec{z}_2} H_1 = 0. \quad (33) \]

Thus we see that in general the functions $H_1$ and $H_2$ interact with each other. Two special cases arise, where either the two functions both depend only on the overall transverse space $\vec{y}$ (this is the usual kind of intersecting solution), or else one function depends on the relative transverse space while the other depends on the overall transverse space. This latter special case was found in [8]. In both cases, the $H_1$ and $H_2$ becomes independent harmonic functions in the relevant coordinates, implying a no-force condition. The first case describes $p$-branes with their charges uniformly distributed along each other’s relative world-volumes, while the second case describes one extended object with charges uniformly distributed along the overall transverse space of the other. Only in these two circumstances do the forces between the two objects cancel out. We can obtain more general solution to (32) or (33) by taking one of the functions to be harmonic in $\vec{y}$, and the other to depend on $\vec{y}$ and the appropriate $\vec{z}$ coordinates.

3 Multiple intersections

Multi-intersections in M-theory or string theories can be obtained from the dimensional oxidation of multi-charge $p$-brane solutions in lower dimensions. Having constructed the pair-wise intersections, the discussion for all multi-intersections follows straightforwardly. The only criterion for multiple intersections is that they should reduces to the known pair-wise solutions for all possible cases where all but two of the charges are set to zero.

The example we shall examine is the intersections arising from RN (Reissner-Nordstrøm) black holes in $D = 5$, which can be viewed as 3-charge black holes in the $D = 5$ maximal supergravity theory. There are a total of 45 possible field configurations that gives rise to such RN black holes in $D = 5$, which form a 45-dimensional representation of the Weyl group.
of $E_6$. We shall consider an example that uses the field strengths $\{ F_2^{(12)}, F_2^{(34)}, F_2^{(56)} \}$, for which the 5-dimensional black hole solution is given by

\begin{align}
 ds_5^2 &= -(H_1 H_2 H_3)^{-2/3} dt^2 + (H_1 H_2 H_3)^{1/3} dy^m dy^m, \\
 \check{\phi} &= \frac{1}{2} \check{a}_{12} \log H_1 + \frac{1}{2} \check{a}_{34} \log H_2 + \frac{1}{2} \check{a}_{56} \log H_3, \\
 F_2^{(12)} &= dt \wedge dH_1^{-1}, \quad F_2^{(34)} = dt \wedge dH_2^{-1}, \quad F_2^{(56)} = dt \wedge dH_3^{-1},
\end{align}

where $H_i$ are harmonic functions on 4-dimensional transverse space $y^m$. Let us first oxidise the solution to $D = 6$. The field strengths $\{ F_2^{(12)}, F_2^{(34)}, F_2^{(56)} \}$ in $D = 5$ are dimensionally reduced from the those $\{ \hat{F}_2^{(12)}, \hat{F}_2^{(34)}, \hat{F}_3^{(5)} \}$ in $D = 6$. The solution in $D = 6$ describes an intersection of 2-charge black hole with a string:

\begin{align}
 ds_6^2 &= (\hat{H}_1 \hat{H}_2)^{-3/4} H_3^{-1/2} (-dt^2 + \hat{H}_1 \hat{H}_2 \hat{H}_3 dy^m dy^m + \hat{H}_1 \hat{H}_2 dz_6^2), \\
 \hat{\check{\phi}} &= \frac{1}{2} \hat{a}_{12} \log \hat{H}_1 + \frac{1}{2} \hat{a}_{34} \log \hat{H}_2 + \frac{1}{2} \hat{a}_{56} \log \hat{H}_3, \\
 \hat{F}_2^{(12)} &= dt \wedge d\hat{H}_1^{-1}, \quad \hat{F}_2^{(34)} = dt \wedge d\hat{H}_2^{-1}, \quad \hat{F}_3^{(5)} = dt \wedge d\hat{H}_3^{-1}.
\end{align}

It is a solution provided that

\begin{align}
 \square_y \hat{H}_3 = 0, \quad \square_y \hat{H}_1 + H_3 \hat{\ddot{H}}_1 = 0, \quad \square_y \hat{H}_2 + H_3 \hat{\ddot{H}}_2 = 0.
\end{align}

Note that in $D = 6$, the solution should really be viewed as a double intersection of a string and a two-charge black hole. We are interested in particular a class of solution where $H_3$ is taken to be isotropic, i.e. $H_3 = 1 + Q_3/r^2$. Then $\hat{H}_i$ with $i = 1, 2$ can be Fourier expanded as $H_i = \sum_m f_m^i(r) e^{imz_6/R_6}$, where

\begin{align}
 (f_m^i)^\prime \prime + \frac{3}{r} (f_m^i)^\prime - \frac{m^2}{R_6^2} (1 + \frac{Q_3}{r^2}) f_m^i = 0.
\end{align}

This has the solution

\begin{align}
 f_m^i = \frac{\check{c}_m^i}{r} K_\alpha \left( \frac{mr}{R_6} \right) + \frac{\check{c}_m^i}{r} I_\alpha \left( \frac{mr}{R_6} \right),
\end{align}

where $\alpha = \sqrt{1 + m^2 \hat{Q}/R_6^2}$, and as usual we choose $\check{c}_m^i = 0$ and retain only the $K$ Bessel functions if we want expansions that are non-singular as $r$ tends to infinity.

It is important to note that the RN black hole has non-vanishing entropy $S \sim \sqrt{Q_1 Q_2 Q_3}$ despite the fact that it is extremal. This entropy can be understood from a six-dimensional viewpoint, where, for a suitable choice of the three supporting field strengths in $D = 5$, the solution becomes a “boosted” dyonic D-string \cite{13}. (This occurs when one of the three 2-form field strengths in $D = 5$ is chosen to be the Kaluza-Klein 2-form coming from the six-dimensional metric. Such a choice of three field strengths is related to any other valid
that the functions \( \tilde{H} \) run over 5 and 6. We find that (39) is then a solution of
\[ D D \]
now in
indices \( a \), ...
Under the straightforward oxidation, the three \( \tilde{H} \) functions would simply be the same as the functions \( \tilde{H} \) in \( D = 6 \). However, we can consider yet more general coordinate dependence
now in \( D = 11 \). Specifically, we may consider the following possibility:
\[ \tilde{H}_1 = \tilde{H}_1(\vec{y}, z^{a_2}, z^{a_3}) \], \hspace{1em} \tilde{H}_1 = \tilde{H}_1(\vec{y}, z^{a_1}, z^{a_3}) \], \hspace{1em} \tilde{H}_1 = \tilde{H}_1(\vec{y}, z^{a_1}, z^{a_2}) \],
where indices \( a_1, b_1, \ldots \) run over 1 and 2, indices \( a_2, b_2, \ldots \) run over 3 and 4, and \( a_3, b_3, \ldots \) run over 5 and 6. We find that (39) is then a solution of \( D = 11 \) supergravity, provided that the functions \( \tilde{H} \) satisfy the following conditions:
\[ \Box y \tilde{H}_1 + \tilde{H}_2 \partial_{a_2} \tilde{H}_1 + \tilde{H}_3 \partial_{a_3} \tilde{H}_1 = 0 \],
\[ \Box y \tilde{H}_2 + \tilde{H}_3 \partial_{a_3} \tilde{H}_2 + \tilde{H}_1 \partial_{a_1} \tilde{H}_2 = 0 \],
\[ \Box y \tilde{H}_3 + \tilde{H}_1 \partial_{a_1} \tilde{H}_3 + \tilde{H}_2 \partial_{a_3} \tilde{H}_3 = 0 \],
and in addition the constraints
\[ \epsilon_{a_1 b_1} \partial_{b_1} \tilde{H}_2 \partial_{a_2} \tilde{H}_1 - \epsilon_{a_2 b_2} \partial_{a_1} \tilde{H}_2 \partial_{b_2} \tilde{H}_1 = 0 \],
\[ \epsilon_{a_2 b_2} \partial_{b_2} \tilde{H}_3 \partial_{a_3} \tilde{H}_2 - \epsilon_{a_3 b_3} \partial_{a_2} \tilde{H}_3 \partial_{b_3} \tilde{H}_2 = 0 \],
\[ \epsilon_{a_3 b_3} \partial_{b_3} \tilde{H}_1 \partial_{a_1} \tilde{H}_3 - \epsilon_{a_1 b_1} \partial_{a_3} \tilde{H}_1 \partial_{b_1} \tilde{H}_3 = 0 \].
(These equations (41) and (42) can all be derived from the field equation for \( F_4 \) in \( D = 11 \); having imposed them, the Einstein equation is also satisfied.) The constraints (42) can be re-expressed in the simpler form
\[ \partial_{a_1} \tilde{H}_2 \partial_{a_2} \tilde{H}_1 = 0 \].
\[ \partial_{a_2} \tilde{H}_3 \partial_{a_3} \tilde{H}_2 = 0, \]
\[ \partial_{a_3} \tilde{H}_1 \partial_{a_1} \tilde{H}_3 = 0, \]

as a consequence of the fact that the \( \tilde{H} \) functions are real. Note that the equations (41) and the constraints (42) for three intersections are just the natural extensions of (27) and (28) for double intersections (27).

The conditions (43) impose restrictions, similar to those we have already encountered in our other examples, on the possible dependences of the \( \tilde{H} \) functions on the internal compactification coordinates. There are clearly a variety of ways in which the conditions (43) can be satisfied; provided this is done, then we will have 3-intersection solutions as long as the functions \( \tilde{H} \) satisfy now-familiar types of interacting equations, namely (41).

One example of an allowed choice of coordinate dependences is to take

\[ \tilde{H}_1 = \tilde{H}_1(\vec{y}, z_3, z_4), \quad \tilde{H}_2 = \tilde{H}_2(\vec{y}, z_5, z_6), \quad \tilde{H}_3 = \tilde{H}_3(\vec{y}, z_1, z_2), \]

leading to quite complicated interactions between the functions, as can be seen from (41).

Another inequivalent allowed choice for the coordinate dependences is

\[ \tilde{H}_1 = \tilde{H}_1(\vec{y}), \quad \tilde{H}_2 = \tilde{H}_2(\vec{y}, z_1, z_2), \quad \tilde{H}_3 = \tilde{H}_3(\vec{y}, z_1, z_2, z_3, z_4). \]

In this case, \( \tilde{H}_1 \) will satisfy the original harmonicity condition in \( \vec{y} \), while both \( \tilde{H}_2 \) and \( \tilde{H}_3 \) will satisfy more general equations in terms of Bessel functions.

Thus we note that the conditions for having triple-intersections follow from the conditions for all possible sub-combinations of pair-wise intersections. These rules generalise to higher numbers of intersections also.

4 Supersymmetry

The fractions of supersymmetry preserved by the various solutions we have been discussing here can be determined by standard means. In general, the simplest way to do this is first to oxidise the solution to \( D = 11 \), where the supersymmetry transformation rules are most easily analysed. Exploiting the fact that supersymmetry is preserved under Kaluza-Klein dimensional reduction, the supersymmetry fraction found in \( D = 11 \) will be the same as that in the lower dimension.

\footnote{While we were completing the writing of this paper, a preprint appeared that discusses some related ideas for generalised forms of intersecting p-brane solutions [20]. However, the configuration of three intersecting membranes in \( D = 11 \) proposed in [23], with \( \tilde{H}_1 = H_{11}(z^{a_1})H_{12}(z^{a_2})H_{13}(\vec{y}) \) and cyclic permutations, violates the constraints (42) or (43). Thus such configurations evidently conflict with the equations of motion.}
Our conclusions about the preserved fractions of supersymmetry are easily summarised:
for the generalised solutions we are considering in this paper, where certain of the functions
$H_i$ in an intersecting solution acquire additional dependence on certain of the internal
coordinates, the fractions of preserved supersymmetry are always the same as they were
in the corresponding “standard” solutions. Thus in particular, any generalised solution
involving two functions $H_i$ preserves $\frac{1}{4}$ of the supersymmetry, and any solution involving
three functions preserves $\frac{1}{8}$ of the supersymmetry.

We shall not present the complete details of the supersymmetry calculations for all
the solutions that we have obtained in this paper; instead, we shall simply give some of
the key steps. One necessary ingredient is the covariant derivative acting on spinors, i.e.
\[ D_\mu \psi = \partial_\mu \psi + \frac{1}{4} \omega_{\mu}^{ab} \Gamma_{ab} \psi. \]
It is useful to find the form of $D_\mu$ for a rather general class of
metrics, which we may represent in the form
\[ ds^2 = -e^{2u} dt^2 + e^{2v} dy^i dy^i + \sum_A e^{2w_A} dz^A dz^A, \] (46)
where the functions $u$, $v$ and $w^A$ depend on $y^i$ and $z^B$. After calculating the spin connection
in the obvious orthonormal basis $e^0 = e^u dt$, $e^i = e^v dy^i$, $e^A = e^{w_A} dz^A$, we find that the
tangent-space components of the spinor covariant derivative, $D_a = e^a_\mu D_\mu$, are given by
\[ D_0 = e^{-u} \frac{\partial}{\partial t} + \frac{1}{2} e^{-v} u_i \Gamma_{0i} + \frac{1}{2} e^{-w_A} u_A \Gamma_{0A}, \]
\[ D_A = e^{-w^A} \frac{\partial}{\partial z^A} + \frac{1}{2} e^{-v} w^A_i \Gamma_{Ai} + \frac{1}{2} e^{-w_B} w^B_i \Gamma_{AB}, \] (47)
\[ D_i = e^{-v} \frac{\partial}{\partial y^i} + \frac{1}{2} e^{-w^A} v_A \Gamma_{iA} + \frac{1}{2} e^{-v} v_j \Gamma_{ij}. \]

Here, the subscripts on the metric functions denote coordinate-space derivatives with respect
to the indicated coordinates, and the summation convention applies in the obvious places.
The supercovariant derivative $\hat{D}_a$, whose kernel is the space of Killing spinors associated
with unbroken supersymmetries, is then given in $D = 11$ by $\hat{D}_a = D_a + L_a$, where
\[ L_a = -\frac{1}{288} F_{bcde} \Gamma_a^{bcde} + \frac{1}{39} F_{abcd} \Gamma^{bcd}. \] (48)

It is now a straightforward exercise to substitute the intersecting solutions into $\hat{D}_a$, and
hence to solve for the Killing spinors. For example, for the 3-intersection solution (39), we
find that Killing spinors $\epsilon$ satisfying $\hat{D}_a \epsilon = 0$ exist provided that
\[ \Gamma_{12} \epsilon = \Gamma_{34} \epsilon = \Gamma_{56} \epsilon = \Gamma_0 \epsilon, \] (49)
and that $\epsilon = e^{u/2} \epsilon_0 = (H_1 H_2 H_3)^{-1/6} \epsilon_0$, where $\epsilon_0$ is a constant. Thus there are 4 inde-
pendent solutions, implying that the generalised triple-membrane intersections preserve $\frac{1}{8}$
of the supersymmetry. Note that the existence of the Killing spinors is assured once the equations of motion are satisfied.

5 Conclusions

Multi-charge $p$-brane solutions in lower dimensions can be oxidised to higher dimensions, where they become intersections of $p$-branes. To be precise, the direct Kaluza-Klein oxidation gives rise to intersections of $p$-branes for which the charges are uniformly distributed over the relative transverse directions. In this paper we have generalised these solutions, by allowing the original harmonic functions that depend only on the overall transverse space to be functions also of the relative transverse space. We have argued that these generalised solutions are the more appropriate ones to consider, since they are the analogues for multi-charge $p$-branes of the notion of a periodic array of higher-dimensional $p$-branes, as opposed to a continuum of higher-dimensional charge centres. We found that such generalised intersections between two isotropic $p$-branes are not possible. However, more general solutions do exist, for which the original independent lower-dimensional harmonic functions become more general functions that can couple to each other. Such solutions can themselves be dimensionally reduced, in the regime where the distance in the overall transverse space becomes much larger than the compactification size of the internal space. These dimensionally-reduced solutions are then like the standard lower-dimensional multi-charge solutions.

These results have the following significance. Firstly, they indicate that the lower-dimensional multi-charge solutions, which can be viewed in the lower dimension as “bound states at threshold” (i.e. with zero-binding energy), are not really to be viewed as combinations of distinct basic objects at all, owing to the interactions in the higher-dimension that are not observable by a lower-dimensional observer who is far away from the horizon. Secondly, since the lower-dimensional $p$-branes (including RN black holes) are approximate solutions that are valid for the observer far from the horizon, then as one approaches the horizon the compactification dimensions become more and more important, and the solution becomes essentially higher dimensional. In this case, the analysis for the entropy should include the contributions of the non-zero modes. Of course there can exist special configurations for which the non-zero modes are actually vanishing, in which case there will be genuine no-force conditions between the individual building blocks of the intersection. However these solutions, where the $H$ functions in the higher dimension depend only on the
overall transverse coordinates $\vec{y}$, are a set of measure zero in the space of all intersections. The more generic solutions that we have considered in this paper seem to avoid the problems of the apparent vulnerability of the usual multi-charge configurations to the “drifting apart” of their charge centres.

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References

[1] M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, *Superstrings in $D = 10$ from supermembranes in $D = 11*, Phys. Lett. **B191** (1987) 70.

[2] Multi-centre extremal static solutions have been considered in:
   D. Majumdar, Phys. Rev 72 (1947) 390;
   A. Papapetrou, Proc. R. Irish Acad. **A51** (1947) 191;
   G.W. Gibbons and S.W. Hawking, Phys. Lett. **B78** (1978) 430.

[3] R. Khuri, *A heterotic multi-monopole solution*, Nucl. Phys. **B387** (1992) 315: hep-th/9205081.

[4] J.P. Gauntlett, J.A. Harvey and J.T. Liu, *Magnetic monopoles in string theory*, Nucl. Phys. **B409** (1993) 363: hep-th/9211056.

[5] H. Lü, C.N. Pope and K.S. Stelle, *Vertical versus diagonal dimensional reduction for $p$-branes*, Nucl. Phys. **B481** (1996) 313: hep-th/9605082.

[6] G. Papadopoulos and P.K. Townsend, *Intersecting M-branes*, Phys. Lett. **B380** (1996) 273: hep-th/9604035;
   A.A Tseytlin, *Harmonic superpositions of M-brane*, Nucl. Phys. **B475** (1996) 149: hep-th/9604035.

[7] H. Lü, C.N. Pope, T.A. Tran and K.-W. Xu, *Classification of $p$-branes, NUTs, Waves and Intersections*, hep-th/9708055.

[8] E. Bergshoef, M. de Roo, E. Eyras, B. Janssen, J.P. van der Schaar, *Multiple intersections of D-branes and M-branes*, hep-th/9612093;
   M. de Roo, *Intersecting branes and supersymmetry*, hep-th/9703124.
E. Bergshoef, M. de Roo, E. Eyras, B. Janssen, J.P. van der Schaar, *Intersections involving monopoles and waves in eleven dimensions*, hep-th/9704120.

[9] D. Youm, *Black holes and solitons in string theory*, hep-th/9710046.

[10] J. Rahmfeld, *Extremal black holes as bound states*, Phys. Lett. **B372** (1996) 198: hep-th/9512089.

N. Khviengia, Z. Khviengia, H. Lü and C.N. Pope, *Intersecting M-branes and bound states*, Phys. Lett. **B388** (1996) 21: hep-th/9605077.

M.J. Duff and J. Rahmfeld, *Bound states of black holes and other p-branes*, Nucl. Phys. **B481** (1996) 332: hep-th/9605085.

[11] M. Cvetic and A.A. Tseytlin, *General class of BPS saturated dyonic black holes as exact superstring solutions*, Phys. Lett. **B366** (1996) 95: hep-th/9510097.

[12] M. Cvetic and A.A. Tseytlin, *General class of BPS saturated dyonic black holes*, Phys. Rev. **D53** (1996) 5619: hep-th/9512031.

[13] A.A. Tseytlin, *Extreme dyonic black holes in string theory*, Mod. Phys. Lett. **A11** (1996) 689: hep-th/9601177.

[14] G. Horowitz and D. Marolf, *Where is the information stored in black holes?*, Phys. Rev. **D55** (1997) 3654: hep-th/9610171.

[15] C.G. Callan, S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, *Absorption of fixed scalars and the D-brane approach to black holes*, Nucl. Phys. **B489** (1997) 65: hep-th/9610172.

[16] A.A. Tseytlin, *Composite BPS configurations of p-branes in 10 and 11 dimensions*, Class. Quantum Grav. **14** (1997) 2085: hep-th/9702163.

[17] H. Lü and C.N. Pope, *p-brane solitons in maximal supergravities*, Nucl. Phys. **B465** (1996) 127: hep-th/9512012.

[18] H. Lü, C.N. Pope and K.S. Stelle, *Weyl group invariance and p-brane multiplets*, Nucl. Phys. **B476** (1996) 89: hep-th/9602140.

[19] A. Strominger and C. Vafa, *Microscopic origin of Bekenstein-Hawking entropy*, Phys. Lett. **B379** (1996) 99: hep-th/9601023.

[20] M.G. Ivanov, *p-brane solutions and Beltrami-Laplace operator*, hep-th/9710110.