Delay-induced stability switches in an SIRS epidemic model with saturated incidence rate and temporary immunity

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Abstract. This work considers a time-delayed SIRS epidemic model with temporary immunity and nonlinear incidence rate, where the susceptible host population satisfies the logistic equation and the incidence rate is of saturated form with the susceptible. The time delay represents a period of temporary immunity where disease-recovered individuals return to the susceptible class after a fixed period of time. By analyzing the associated characteristic equation with delay-dependent coefficients and regarding the time lag as the bifurcation parameter, the local stability of the endemic equilibrium is investigated and sufficient conditions for the occurrence of stability switches via Hopf bifurcations are obtained. It is shown that the delay parameter can induce a finite number of stability switches before completely destabilizing the system. Numerical simulations are carried out to illustrate theoretical results.

1. Introduction

Delay differential equations (DDE) have been successfully utilized to model various infectious diseases. Incorporating time delays in epidemic models considers the fact that the transmission dynamic behavior of a disease at time \( t \) depends not only on the state of time \( t \) but also on the state of previous times. Delays have been used to account the effect of disease characteristics such as latent period, course of infection, period of immunity and disease awareness (see, for example, [1-19] and the references cited therein).

The basic elements of an SIRS epidemic model are the three epidemiological classes: susceptibles \( S \), the number of individuals who are susceptible to infection; infectives \( I \), the number of individuals who have the disease and able to spread the infection; and the removed class \( R \), the number of individuals who have been infected and then removed with immunity. In 1992, Mena-Lorcat and Hethcote [20] considered five SIRS epidemiological models for populations of varying size, one of which is governed by the following system of ordinary differential equations (ODE)

\[
\begin{align*}
\dot{S}(t) &= \lambda - \mu S(t) - \beta S(t)I(t) + \delta R(t) \\
\dot{I}(t) &= \beta S(t)I(t) - \mu I(t) - \alpha I(t) - \gamma I(t) \\
\dot{R}(t) &= \gamma I(t) - \mu R(t) - \delta R(t)
\end{align*}
\]

(1)

where \( \lambda \) is the recruitment rate of the population, \( \mu \) is the natural death rate due to causes unrelated to the infection, \( \beta \) is the transmission coefficient of the disease, \( \alpha \) is the disease-related death rate, \( \gamma \) is the recovery rate of infected hosts, and \( \delta \) is the rate at which recovered hosts lose their immunity and become susceptible again.
In model (1), the temporary immunity time is distributed exponentially as $e^{-\delta t}$ with mean time of $1/\delta$. This means that the period of temporary immunity for each individual ranges from zero (no immunity) to infinite (permanent immunity). If we assume that the duration of temporary immunity is a constant $\tau = 1/\delta$ and no individuals in the removed class will return to the susceptible class until time $t > \tau$, then we obtain the SIRS model of Brauer et al. [2] governed by the following ODE-DDE system:

$$\begin{align*}
\dot{S}(t) &= \Lambda - \mu S(t) - \beta S(t)I(t), \\
\dot{I}(t) &= \beta S(t)I(t) - (\mu + \alpha + \gamma)I(t), \\
\dot{R}(t) &= \gamma I(t) - \mu R(t),
\end{align*}$$

$$\begin{align*}
\dot{S}(t) &= \Lambda - \mu S(t) - \beta S(t)I(t) + \gamma e^{-\mu t} I(t - \tau), \\
\dot{I}(t) &= \beta S(t)I(t) - (\mu + \alpha + \gamma)I(t), \\
\dot{R}(t) &= \gamma I(t) - \mu R(t) - \gamma e^{-\mu t} I(t - \tau),
\end{align*}$$

$$0 \leq t \leq \tau,$n

$$t > \tau.$$

In the previous model, the immunity gained by experiencing the disease is only temporary, that is, the disease-recovered individuals who are still alive after the period of temporary immunity $\tau$ will return to the susceptible class. The delay-dependent coefficient $0 < e^{-\mu t} \leq 1$ is the probability that the recovered individuals survive during the time period $t \leq \tau$. Hence, the term $\gamma e^{-\mu t} I(t - \tau)$ reflects the fact that a recovered individual has survived from natural death in the removed class before becoming susceptible again to the disease. The authors of the research articles [1-4] incorporated the term $\gamma e^{-\mu t} I(t - \tau)$ in the study of their respective models.

The model problem of this article is a modification of the second model. From a practical point of view, we consider the nonlinear incidence rate $\beta SI / (1 + \sigma S)$ introduced by May and Anderson [21] which is saturated with the susceptibles. The parameter $\sigma$ is the saturation factor that measures the inhibitory effect from the behavioral changes of the susceptible individuals when their number increases. Several scholars [7-9, 12, 20] utilized the saturated incidence rate $\beta SI / (1 + \sigma S)$ in their respective models. It is also assumed that in the absence of infection, the susceptible host population follows the logistic growth $rS(1 - S/K)$ with carrying capacity $K$ and a specific growth rate $r$. We propose the generalized SIRS epidemiological model as follows:

$$\begin{align*}
\dot{S}(t) &= rS(t) \left(1 - \frac{S(t)}{K}\right) - \frac{\beta S(t)I(t)}{1 + \sigma S(t)}, \\
\dot{I}(t) &= \frac{\beta S(t)I(t)}{1 + \sigma S(t)} - (\mu + \alpha + \gamma)I(t), \\
\dot{R}(t) &= \gamma I(t) - \mu R(t),
\end{align*}$$

$$0 \leq t \leq \tau,$n

$$\begin{align*}
\dot{S}(t) &= rS(t) \left(1 - \frac{S(t)}{K}\right) - \frac{\beta S(t)I(t)}{1 + \sigma S(t)} + \gamma e^{-\mu t} I(t - \tau), \\
\dot{I}(t) &= \frac{\beta S(t)I(t)}{1 + \sigma S(t)} - (\mu + \alpha + \gamma)I(t), \\
\dot{R}(t) &= \gamma I(t) - \mu R(t) - \gamma e^{-\mu t} I(t - \tau),
\end{align*}$$

$$t > \tau.$$

We add here that if $\tau = 0$, then the model reduces to an SIS model and if $\tau = \infty$, we obtain the SIR models (ODE version) in [8, 9]. Furthermore, Xu et al. [4] studied a similar problem by considering the nonlinear incidence rate $\beta SI / (1 + v I)$ which is saturated with the infectives [22]. The saturation level $\nu$ measures the inhibitory effect from the crowding effect of infected individuals. This time, in order to account the psychological behaviors of the members of the susceptible population, we analyze the dynamics of SIRS model (3)-(4).
2. Main results
The initial condition for model (3)-(4) takes the form
\[ S(0) > 0, \quad I(0) > 0, \quad R(0) \geq 0. \] (5)
For each initial condition (5), ordinary differential equation system (3) has a unique solution \((S(t), I(t), R(t))\) for \(t \in [0, \tau]\), which is exactly the initial history function of delayed system (4). It is easy to show that solutions remain non-negative for all \(t > 0\).

2.1. Linearized analysis
The dynamics of epidemic model (3)-(4) are mainly determined by the first two equations of delayed system (4)
\[
\begin{align*}
\dot{S}(t) &= rS(t) \left(1 - \frac{S(t)}{K}\right) - \frac{\beta S(t) I(t)}{1 + \sigma S(t)} + \gamma e^{-\mu t} I(t - \tau), \\
\dot{I}(t) &= \frac{\beta S(t) I(t)}{1 + \sigma S(t)} - (\mu + \alpha + \gamma) I(t)
\end{align*}
\] (6)
because the first two equations in system (4) do not depend on the third equation, and therefore this equation can be omitted without loss of generality. We analyze subsystem (6) throughout the remainder of this section.

**Theorem 1.** For the model system (6), there always exist disease-free equilibria \(E_0 = (0,0)\) and \(E_1 = (K,0)\). If
\[ R_0 := \frac{K[\beta - \sigma(\mu + \alpha + \gamma)]}{\mu + \alpha + \gamma} > 1, \] (7)
there exists an endemic equilibrium \(E_* = (\bar{S}, \bar{I})\), where
\[ (\bar{S}, \bar{I}) = \left(\frac{K}{R_0}, \frac{r \bar{S}^2}{K(\mu + \alpha + \gamma - \gamma e^{\mu t})(R_0 - 1)}\right). \] (8)

Let \(X(t) = [S(t), I(t)]^T\). Then the linearized system corresponding to (6) around an equilibrium \(E_* = (S^*, I^*)\) is
\[ \dot{X}(t) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} X(t) + \begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix} X(t - \tau) \] (9)
where
\[ a_1 = r - \frac{2r S^*}{K} - \frac{\beta I^*}{(1 + \sigma S^*)^2}, \quad a_2 = -\frac{\beta S^*}{1 + \sigma S^*}, \quad a_3 = \frac{\beta I^*}{(1 + \sigma S^*)^2}, \quad a_4 = \frac{\beta S^*}{1 + \sigma S^*} - (\mu + \alpha + \gamma), \quad b_1 = \gamma e^{-\mu t}, \]
and with corresponding characteristic equation
\[ D(\lambda, \tau) := \lambda I - \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + e^{-\lambda \tau} \begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix} \] = 0. \] (10)
By the stability theory of delayed differential equations, the equilibrium \(E_*\) is locally asymptotically stable if all characteristic roots of (10) have negative real parts and is unstable if there is one characteristic root which has a positive real part [23, 24].

2.2. Stability of \(E_0\) and \(E_1\)
Using (10), the characteristic equation at the disease-free equilibrium \(E_0 = (0,0)\) reduces to
\[ (\lambda - r)(\lambda + (\mu + \alpha + \gamma)) = 0 \]
whose roots are \(r > 0\) and \(-(\mu + \alpha + \gamma) < 0\). Thus, equilibrium \(E_0\) is always an unstable saddle. On the other hand, by using the value of \(R_0\) in (7) the characteristic equation of the linearized system around \(E_1\) is obtained to be
\[(\lambda + r) \left( \lambda - \mu + \alpha + \gamma \frac{1}{1 + \sigma K} (R_0 - 1) \right) = 0.\]

From this, we see that the disease-free equilibrium \(E_i\) is locally asymptotically stable when \(R_0 < 1\) and unstable when \(R_0 > 1\). Moreover, if \(R_0 = 1\), then \(\lambda = 0\) is a simple characteristic root. Thus, \(E_i\) is linearly neutrally stable if \(R_0 = 1\). The following theorem summarizes our results.

**Theorem 2.** For model system (6), the disease-free equilibrium \(E_i = (K, 0)\) is locally asymptotically stable if \(R_0 < 1\), linearly neutrally stable if \(R_0 = 1\) and unstable if \(R_0 > 1\).

### 2.3. Stability and existence of Hopf bifurcation at \(E_i\)

We now investigate the stability of the endemic equilibrium \(E_e = (\bar{S}, \bar{T})\). Note that a necessary and sufficient condition for the existence of the endemic equilibrium \(E_e\) is the condition \(R_0 > 1\) must hold. So from here on out, we always assume that \(R_0 > 1\). Using the values of \(\bar{S}\), \(\bar{T}\) and \(R_0\) defined in (8) and (7), we derive the following expressions

\[
R_0 = \frac{K}{S}, \quad \frac{\beta S}{1 + \sigma S} = \mu + \alpha + \gamma \quad \text{and} \quad \frac{\beta T}{1 + \sigma S} = \frac{r(\mu + \alpha + \gamma)}{\mu + \alpha + \gamma - \gamma e^{-\mu \tau}} \left(1 - \frac{1}{R_0}\right). 
\]

Thus, the characteristic equation of the linear system around \(E_e\) is obtained to be

\[
\bar{D}(\lambda, \tau) := [\lambda^2 + p(\tau)\lambda + d(\tau)] + [c(\tau)]e^{\lambda \tau} = 0
\]

(11)

where

\[
p(\tau) = a + b(\tau), \quad d(\tau) = (\mu + \alpha + \gamma)b(\tau), \quad c(\tau) = -\gamma b(\tau)e^{-\mu \tau},
\]

\[
b(\tau) = \frac{r(\mu + \alpha + \gamma)}{(1 + \sigma S)(\mu + \alpha + \gamma - \gamma e^{-\mu \tau})} \left(1 - \frac{1}{R_0}\right), \quad a = r \left(\frac{2}{R_0} - 1\right). 
\]

For characteristic equation (11), observe that the delay \(\tau\) does not only appear in terms with \(e^{\lambda \tau}\), but also in several places. The reason behind this is that our model system (6) involves delay-dependent parameters. Time-delayed systems with delay-independent parameters are simpler to study and most of the time, one can easily compute the critical delay values at which Hopf bifurcations occur (see, for example, [24]). Since biological models with delay-dependent parameters are now frequently arising in the literature, Beretta and Kuang [25] in 2002 contributed a geometric method to effectively study the difficult characteristic equations emerging from model systems involving delay dependent parameters. They combined graphical information with analytical work to show that the stability of a given equilibrium solution is simply determined by the graph of some functions of the time delay which can be expressed explicitly and thus can be easily depicted by popular graphing softwares. In practice though, one cannot compute explicitly these critical delay values at which Hopf bifurcations take place. We apply their technique here since it is applicable to our problem.

In order to apply the geometric criterion due to Beretta and Kuang [25], we need to verify the following properties for all \(\tau \geq 0\):

- (A1) \(\lambda = 0\) is not a root of \(\bar{D}(\lambda, \tau) = 0\), that is, \(P(0, \tau) + Q(0, \tau) = d(\tau) + c(\tau) \neq 0\).
- (A2) If \(\lambda = i\omega, \omega \in \mathbb{R}\) , then \(P(i\omega, \tau) + Q(i\omega, \tau) \neq 0\).
- (A3) \(\limsup |Q(\lambda, \tau)/P(\lambda, \tau)|: |\lambda| \to \infty, Re\lambda \geq 0 < 1\)
- (A4) \(F(\omega, \tau) := |P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2\) has at most a finite number of real zeros.
- (A5) Each positive root \(\omega(\tau)\) of \(F(\omega, \tau) = 0\) is continuous and differentiable with respect to \(\tau\) whenever it exists.

Note that \(d(\tau) + c(\tau) = (\mu + \alpha + \gamma - \gamma e^{-\mu \tau})b(\tau) > 0\). Thus, \(\lambda = 0\) is not a root of \(D(\lambda, \tau)\). (A2) is obviously true because \(P(i\omega, \tau) + Q(i\omega, \tau) = -\omega^2 + i\omega^2 + d(\tau) + c(\tau) \neq 0\). From (11), we know that
\[ Q(\lambda, \tau) / P(\lambda, \tau) \rightarrow 0 \text{ as } |\lambda| \rightarrow +\infty. \] Thus, (A3) follows immediately. Solving for \( F(\lambda, \omega) \), we obtain
\[ F(\lambda, \omega) = \omega^4 + (p^2(\tau) - 2d(\tau))\omega^2 + (d^2(\tau) - c^2(\tau)). \] It is obvious that (A4) is satisfied and by the Implicit Function Theorem, (A5) is also satisfied. Therefore, all properties (A1)-(A5) are satisfied.

If \( \tau = 0 \) is a characteristic root, then equation (11) reduces to \( \lambda^2 + p(0)\lambda + d(0) + e(0) = 0. \) Since \( d(0) + e(0) \) is always positive, we know that from the Routh-Hurwitz criteria that all of its roots have negative real part if and only if \( p(0) > 0. \) Hence, the equilibrium \( E_\tau \) is locally asymptotically stable for \( \tau = 0 \) if and only if \( p(0) > 0. \) Assume now that \( \lambda = i\omega(\omega > 0) \) is a root of (11). Splitting the real and imaginary parts yields,
\[ \sin \omega\tau = \frac{p(\tau)\omega}{c(\tau)}, \quad \cos \omega\tau = \frac{\omega^2 - d(\tau)}{c(\tau)}. \] If we can find \((\tau, \omega)\) satisfying (12), then (11) will have a pair of pure imaginary roots \( \pm i\omega \) at \( \tau. \) Squaring both sides in (12), adding, and rearranging gives
\[ F(\omega, \tau) := \omega^4 + (p^2(\tau) - 2d(\tau))\omega^2 + d^2(\tau) - c^2(\tau) = 0. \] Using the quadratic formula, the potential positive roots of equation (13) are obtained to be
\[ \omega_0(\tau) = \left( \frac{1}{2}(-p^2(\tau) + 2d(\tau)) \pm \frac{1}{2}\sqrt{(-p^2(\tau) + 2d(\tau))^2 - 4(d^2(\tau) - c^2(\tau))} \right)^{1/2}. \] It is easy to see that the coefficient \( d^2(\tau) - c^2(\tau) \) of equation (13) is always positive for all \( \tau \geq 0 \) from their values defined in (11). Because of this, we can only consider the following two cases for the solutions of (13). If \(-p^2(\tau) + 2d(\tau) < 0 \) for all \( \tau \geq 0, \) then there are no positive \( \omega(\tau) \) solutions of equation (13). On the other hand, if \(( -p^2(\tau) + 2d(\tau))^2 - 4(d^2(\tau) - c^2(\tau)) > 0 \) and \(-p^2(\tau) + 2d(\tau) > 0 \) for all \( \tau \) in some interval, then (13) has two distinct positive roots \( \omega_1(\tau) \) and \( \omega_2(\tau) \) whose values are given in (14). Assume now that \( R_1, R_2, r_1, \) and \( r_2 \) have values defined by
\[ R_1 = 1 + \frac{1 + \sigma S}{2 + \sigma S}, \quad R_2 = 1 + \frac{(\mu + \alpha)(1 + \sigma S)}{(\mu + \alpha + \gamma)(1 + \sigma S)}, \] \[ r_1 = -\frac{1}{2\mu} \ln \left( \frac{4r(\mu + \alpha + \gamma)}{\gamma^2(\mu + \alpha)(1 + \sigma S)} \left( 1 - \frac{1}{R_0} \right) \right) \] where \( \frac{4r(\mu + \alpha + \gamma)}{\gamma^2(\mu + \alpha)(1 + \sigma S)} \left( 1 - \frac{1}{R_0} \right) < 1, \) \[ r_2 = -\frac{1}{2\mu} \ln \left( \frac{4r(\mu + \alpha + \gamma)(1 + \sigma S)}{\gamma^2 \left( \frac{2}{R_0} - 1 \right)} \right) \] where \( \frac{4r(\mu + \alpha + \gamma)(1 + \sigma S)}{\gamma^2 \left( \frac{2}{R_0} - 1 \right)} < 1. \)

Let us consider the following hypotheses:
(H1) \( R_1 \leq R_0 \leq 2 \) and \( \tau \in [0, \tau_1], \)
(H2) \( 1 < R_0 \leq R_2 \) and \( \tau \in [0, \tau_1]. \)
Define \( I := [0, \tau] \) where \( \tau = \tau_1 \) if (H1) holds true or \( \tau = \tau_2, \) if (H2) holds true. If (H1) or (H2) is satisfied, then we can show using the techniques established in [26] that both \(( -p^2(\tau) + 2d(\tau))^2 - 4(d^2(\tau) - c^2(\tau)) \) and \(-p^2(\tau) + 2d(\tau) \) are positive for all \( \tau \in I. \) We thus have the following result.

**Lemma 3.** If either (H1) or (H2) is satisfied, then both \( \omega_1(\tau) \) and \( \omega_2(\tau) \) are positive on \( I. \) Furthermore, the endemic equilibrium \( E_\tau \) is locally asymptotically stable at \( \tau = 0 \) whenever (H1) or (H2) holds.

Now, for any \( \tau \in I, \) we can define the angles \( \theta_1(\tau) \in (0, 2\pi) \) as the solutions of (12), that is,
\[
\sin \theta_n(\tau) = \frac{p(\tau)\omega_n(\tau)}{c(\tau)}, \quad \cos \theta_n(\tau) = \frac{\omega_n^2(\tau) - d(\tau)}{c(\tau)}.
\]

We note here that the plus or minus signs are used consistently on both sides of any equations consisting them. Indeed, we can define the functions \(\theta_n : I \to \mathbb{R}\) by

\[
\theta_n(\tau) = \begin{cases} 
\pi + \arctan \left( \frac{p(\tau)\omega_n(\tau)}{\omega_n^2(\tau) - d(\tau)} \right) & \text{if } \omega_n^2(\tau) - d(\tau) < 0, \\
2\pi + \arctan \left( \frac{p(\tau)\omega_n(\tau)}{\omega_n^2(\tau) - d(\tau)} \right) & \text{if } \omega_n^2(\tau) - d(\tau) > 0.
\end{cases}
\]

The relation between the arguments \(\theta_n(\tau)\) and \(\omega_n(\tau)\tau\) for \(\tau \in I\) must be

\[
\omega_n(\tau)\tau = \theta_n(\tau) + 2n\pi, \quad n = 0, 1, 2, \ldots.
\]

Hence, we can introduce the sequence of maps \(S^+_n : I \to \mathbb{R}\) defined by

\[
S^+_n(\tau) = \tau - \frac{\theta_n(\tau) + 2n\pi}{\omega_n(\tau)} \quad \text{and} \quad S^-_n(\tau) = \tau - \frac{\theta_n(\tau) + 2n\pi}{\omega_n(\tau)}.
\]

It is clear that the functions are continuous and differentiable on \(I\) by the assumptions. Thus, we give the following theorem which is due to Beretta and Kuang [25].

**Theorem 4.** If there exists an integer \(n \geq 0\) such that \(S^+_n(\tau) = 0\) or \(S^-_n(\tau) = 0\) at some \(\tau = \tau^*\), then characteristic equation (11) has a pair of simple pure imaginary roots \(\pm \alpha(\tau^*)\) where \(\alpha(\tau^*) = \omega_n(\tau^*)\) if \(S^+_n(\tau^*) = 0\) and \(\alpha(\tau^*) = \omega_n(\tau^*)\) if \(S^-_n(\tau^*) = 0\). If \(\alpha(\tau^*) = \omega_n(\tau^*)\), this pair of simple conjugate pure imaginary roots crosses the imaginary axis from left to right if \(\delta_+(\tau^*) > 0\) and crosses the imaginary axis from right to left if \(\delta_-(\tau^*) < 0\), where

\[
\delta_+(\tau^*) := \text{sign} \left\{ \frac{d\text{Re} \lambda}{d\tau} \bigg|_{\tau=\tau^*} \right\} = \text{sign} \left( \frac{dS^+_n(\tau)}{d\tau} \bigg|_{\tau=\tau^*} \right).
\]

On the other hand, if \(\alpha(\tau^*) = \omega_n(\tau^*)\), this pair of simple conjugate pure imaginary roots crosses the imaginary axis from left to right if \(\delta_-(\tau^*) > 0\) and crosses the imaginary axis from right to left if \(\delta_-(\tau^*) < 0\), where

\[
\delta_-(\tau^*) := \text{sign} \left( \frac{d\text{Re} \lambda}{d\tau} \bigg|_{\tau=\tau^*} \right) = -\text{sign} \left( \frac{dS^-_n(\tau)}{d\tau} \bigg|_{\tau=\tau^*} \right).
\]

**Corollary 5.** If such \(\tau^*\) in Theorem 4 exists with \(\delta_+(\tau^*) \neq 0\) (depending on whether \(\alpha(\tau^*) = \omega_n(\tau^*)\)), then model system (6) undergoes Hopf bifurcation at the endemic equilibrium \(E_s\) when \(\tau = \tau^*\).

3. Numerical simulations

For numerical simulations, let us consider the delayed SIRS model (3)-(4) with the following set of parameters \((r, K, \beta, \sigma, \mu, \alpha, \gamma) = (0.1, 0.1, 0.02, 0.005, 0.001, 0.465, 0.5)\) for such parameters, Theorem 4 can be used since we have \(1 < R_0 = 1.569 < 1.570 = R_s < 2\) and \(r \in I = [0, \tau^*_1] = [0, 62]\). In other words, hypothesis (H1) holds and stability switches for the equilibrium \(E_s\) may depend on all real roots of \(S^+_n(\tau)\) and \(S^-_n(\tau)\) in \(I = [0, 62]\). Figure 1 shows the graphs of \(S^+_n(\tau)\) and \(S^-_n(\tau)\) on \(I\). The functions \(S^+_n(\tau)\) and \(S^-_n(\tau)\) both have zeros given by \(\tau^*_1 = 15.32\) and \(\tau^*_0 = 32.48\), respectively. Figure 1 also shows that \(S^+_n(\tau)\) has a zero \(\tau^*_1 = 39.73\). From Theorem 4, we have \(\delta_+(\tau^*_0) = +1\),
\[ \delta_0(\tau) = -1 \text{ and } \delta_1(\tau) = +1. \] Thus, Hopf bifurcations occur at these values of \( \tau \). The occurrence of stability switches (via Hopf bifurcations) at \( \tau_0, \tau_1^- \text{ and } \tau_1^+ \) are illustrated in Figure 2 using DDE-Bifftool, which is a numerical continuation and bifurcation analysis tool developed by Engelborghs et al. [27, 28]. The green and red represents the stable and unstable parts of the branches, respectively as \( \tau \) varies.

\[ \begin{align*}
\text{Figure 1.} \quad & \text{The functions } S_\delta^+(\tau) \text{ (red)} \text{ and } S_\delta^-(\tau) \text{ (blue) are plotted on } I. \\
\text{Figure 2.} \quad & \text{Stability switching and emergence of periodic solutions at Hopf bifurcation points (*)}. 
\end{align*} \]

4. Conclusion
In this paper, we formulated and studied a delayed SIRS epidemic model with a fixed period of temporary immunity and nonlinear incidence rate which is saturated with the susceptible host population. We investigated the local stability of the endemic equilibrium from the point of view of stability switches via Hopf bifurcations by regarding the time lag \( \tau \) as bifurcation parameter. Under certain restrictions on the non-delay parameters, we obtained explicit intervals at which critical delay values exist. We have shown that stability switching via Hopf bifurcation occurs at these critical values of \( \tau \). Lastly, we supported our theoretical results with some numerical examples.

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