A DYNAMICAL SYSTEM METHOD FOR SOLVING
THE SPLIT CONVEX FEASIBILITY PROBLEM

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(Communicated by Christian Clason)

Abstract. In this paper a dynamical system model is proposed for solving
the split convex feasibility problem. Under mild conditions, it is shown that
the proposed dynamical system globally converges to a solution of the split
convex feasibility problem. An exponential convergence is obtained provided
that the bounded linear regularity property is satisfied. The validity and tran-
sient behavior of the dynamical system is demonstrated by several numerical
examples. The method proposed in this paper can be regarded as not only a
continuous version but also an interior version of the known CQ-method for
solving the split convex feasibility problem.

1. Introduction. The split convex feasibility problem (SCFP) due to Censor and
Elfving [16] is to find $x \in \mathbb{R}^n$ such that

$$ x \in C \text{ and } Ax \in Q, \quad (1) $$

where $C \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ are two nonempty closed convex sets, and $A \subset \mathbb{R}^{m \times n}$
is a real matrix. SCFP arises in many applications, such as the phase retrieval
problems [16], signal processing [13], image reconstruction [28, 35], the intensity-
modulated radiation therapy [15], and the gene regulatory network inference arising
in systems biology [44]. A more general model is the following prototypical split
inverse problem (SIP) introduced by Censor et al. [17]:

Find $x \in X$ that $x$ solves $IP_1$ such that $y = Ax \in Y$ solves $IP_2$,

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2020 Mathematics Subject Classification. Primary: 65K05, 65K10, 90C25; Secondary: 47H09,
65L09.

Key words and phrases. Split convex feasibility problem, dynamical system, CQ-algorithm,
stability and convergence rate.

This work was partially supported by the National Science Foundation of China (No. 11471230)
and the Scientific Research Foundation of the Education Department of Sichuan Province (No.
16ZA0213).

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where $X$ and $Y$ are two spaces, $A : X \to Y$ is a bounded linear operator, $\text{IP}_1$ is an inverse problem formulated in the space $X$ and $\text{IP}_2$ is another inverse problem formulated in the space $Y$. $\text{SCFP}$ is the first instance of a SIP in which two inverse problems are convex feasible problems. In fact, many models of inverse problems can be cast in SIP by choosing different inverse problems for $\text{IP}_1$ and $\text{IP}_2$. For details, we refer the reader to [17]. In the literature, many numerical algorithms have been proposed for solving $\text{SCFP}$ (see e.g., [41, 38, 18, 2, 44, 19, 37, 25]). Among them, a classical algorithm for $\text{SCFP}$ is the known CQ-algorithm proposed by Byrne [13, 14] which is defined by: for given $x_0$,

$$x_{n+1} = P_C[x_n - \gamma A^T(I - P_Q)Ax_n], \quad n = 0, 1, 2, \cdots,$$

(2)

where $\gamma$ is a suitable positive constant, $A^T$ is the transpose of $A$, $I$ is the identical operator, $P_C$ and $P_Q$ are the orthogonal projections onto $C$ and $Q$, respectively. Byrne's CQ-algorithm is an effective and popular method for solving $\text{SCFP}$ because it involves only easily calculated orthogonal projections and it requires no matrix inversions. It is worth mentioning that the CQ-algorithm is actually a projected gradient method applied to the following constrained minimization problem:

$$\min_{x \in C} \frac{1}{2} \|Ax - P_QAx\|^2,$$

which encapsulates the split convex feasibility problem ($\text{SCFP}$). Among the existing works on CQ-algorithm and its variations for $\text{SCFP}$, a great deal of attention has been paid to the convergence of algorithms, but only a few papers deal with the convergence rate. Under the bounded linear regularity condition, Wang et al. [44] studied the linear convergence rate of the varying stepsize CQ-algorithm for $\text{SCFP}$. Qu et al. [37] proposed a Newton projection method (which can also be reviewed as a variation of CQ-algorithm) for $\text{SCFP}$ and analyzed the convergence properties of the method. Especially, under some assumptions which are necessary for the case where the projection operator is nonsmooth, Qu et al. [37] proved that the sequence of iterates generated by the algorithm superlinearly converges to a regular solution of $\text{SCFP}$.

On the other hand, dynamical system approaches were proposed by scholars for solving optimization problems, Wardropian user equilibria, variational inequality problems and inclusion problems. Pyne [36] first proposed the dynamical system approach for solving the linear programming problem on an electronic analogue computer. Xia and Wang [47, 46] presented neural networks for solving linear and quadratic programming problems. Attouch et al. [3] proposed a gradient-projection dynamical system for solving constrained optimization problems. Effati et al. [21] proposed a projection neural network for solving convex programming problems. Friesz et al. [24] presented a dynamical system model for calculating static Wardropian user equilibria on congested networks with fully general demand and cost structure. Xia and Wang [48] proposed a projection dynamical system for solving the variational inequality problem. Attouch and Svaiter [5] proposed a continuous dynamical Newton-like approach for solving the monotone inclusion problem. Abbas et al. [1] proposed a dynamical system for solving the inclusion problem with the sum of a convex subdifferential and a monotone cocoercive operator. For more results on dynamical system approaches, we refer the reader to [4, 10, 23, 26, 32, 34, 33, 49, 51] and the references therein.
Motivated by the above works, in this paper, we attempt to propose a dynamical system model for solving SCFP. In order to present our model, we rewrite CQ-algorithm (2) as
\[ x_{n+1} - x_n = PC[x_n - \gamma A^T(I - P_Q)Ax_n] - x_n. \]
By bringing stepsize \( h_n \) into the above equation, we get
\[ \frac{x_{n+1} - x_n}{h_n} = PC[x_n - \gamma A^T(I - P_Q)Ax_n] - x_n. \]
(3)
Set
\[ \mathcal{U} = I - \gamma A^T(I - P_Q)A. \]
Based on (3), we propose the following dynamical system for solving SCFP:
\[ \frac{dx(t)}{dt} = \lambda\{PC \circ \mathcal{U}x(t) - x(t)\}, \]
where \( \lambda > 0 \) is a scaling factor. By the above arguments, system (4) can be regarded as a continuous version of CQ-algorithm (2).

Comparison with the CQ-algorithm, system (4) enjoys advantages and much stronger properties. In general, the sequence generated by the CQ-algorithm (2) is not in the interior \( \text{int} C \) of \( C \) even if the initial point \( x_0 \in \text{int} C \). We will see that the trajectory of system (4) with the initial point being an interior always lies in the interior of \( C \); see Remark 4. So system (4) can be considered as not only a continuous version of the CQ-algorithm (2) but also an interior version. As known, the explicit discretizing scheme for system (4) may lead to the CQ-algorithm. In fact, we can also obtain other numerical algorithms for (SCFP) by using different discretizing schemes. For example, taking a fixed time step size \( h > 0 \) and setting \( t_n = nh, x_n = x(t_n) \), the central difference scheme for system (4) gives:
\[ x_{n+1} = 2h\lambda PC[x_n - \gamma A^T(I - P_Q)Ax_n] - 2h\lambda x_n + x_{n-1}. \]
Furthermore, system (4) has the advantage that it can be solved by different methods such as the finite difference methods and the finite element method. This will be seen in Section 5.

The remainder of this paper is organized as follows. In Section 2, we recall some concepts and preliminary results. In Section 3, we investigate SCFP by proving that system (4) globally converges to a solution of SCFP. We further obtain an exponential convergence under the bounded linear regularity condition. In Section 4, we apply results of Section 3 to the linear equation problem. In Section 5, numerical examples are reported to illustrate the effectiveness and performance of the dynamical system approach. Finally, we end the paper with a conclusion.

2. Preliminaries. In this section, we recall some concepts and results which will be used in the sequel.

**Definition 2.1.** (See [8]) Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a mapping.
(a) \( T \) is called \( k \)-Lipschitz continuous on \( \mathbb{R}^n \) if
\[ \|T(x) - T(y)\| \leq k\|x - y\|, \quad \forall x, y \in \mathbb{R}^n, \]
and \( T \) is called nonexpansive if it is 1-Lipschitz continuous.
(b) \( T \) is called \( r \)-inverse strongly monotone on \( \mathbb{R}^n \) if there exists \( r > 0 \) such that
\[ \langle T(x) - T(y), x - y \rangle \geq r\|T(x) - T(y)\|^2, \quad \forall x, y \in \mathbb{R}^n. \]
\( T \) is called firmly nonexpansive if it is 1-inverse strongly monotone.
T : \mathbb{R}^n \rightarrow \mathbb{R}^n is called an \( \alpha \)-averaged mapping if it can be written as
\[
T = (1 - \alpha)I + \alpha S,
\]
where \( \alpha \in (0, 1) \), \( I \) denotes the identity mapping, and \( S : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a nonexpansive mapping.

**Remark 1.**
(i) \( T \) is firmly nonexpansive if and only if
\[
\|T(x) - T(y)\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in \mathbb{R}^n.
\]
(ii) \( T \) is nonexpansive if it is averaged or firmly nonexpansive.
(iii) \( T \) is firmly nonexpansive if and only if \( I - T \) is firmly nonexpansive.
(iv) \( P_C \) and \( I - P_C \) are both firmly nonexpansive, i.e.,
\[
\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in \mathbb{R}^n
\]
and
\[
\langle (I - P_C)x - (I - P_C)y, x - y \rangle \geq \|(I - P_C)x - (I - P_C)y\|^2, \quad \forall x, y \in \mathbb{R}^n.
\]

**Lemma 2.2.** (13) If \( \gamma \in (0, 2/\|A\|^2) \), then \( U : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is an averaged mapping.

**Lemma 2.3.** (See [6, 13]) Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a function defined by \( f(x) = \frac{1}{2}\|Ax - P_QAx\|^2 \). Then
(a) \( f \) is convex and differentiable with \( \nabla f = AT(I - P_Q)A \).
(b) \( \nabla f \) is \( \frac{1}{\|A\|^2} \)-inverse strongly monotone.

**Lemma 2.4.** (See [14, 38]) Suppose that the solution set \( \Lambda \) of SCFP is nonempty. Let \( x^* \in \mathbb{R}^n \). Then the following statements are equivalent:
(i) \( x^* \) solves the SCFP.
(ii) \( P_C Ux^* = x^* \).
(iii) \( x^* \in C \) and \( Ux^* = x^* \).
(iv) \( x^* \in C \) and \( Ax^* = P_QAx^* \).

**Definition 2.5.** (See [44]) Suppose the solution set \( \Lambda \) of SCFP is nonempty. We say that SCFP satisfies the bounded linear regularity property if, for any \( r > 0 \) such that \( \Lambda \cap B(0, r) \neq \emptyset \), there exists \( \kappa_r > 0 \) such that
\[
\text{dist}(x, \Lambda) \leq \kappa_r \text{dist}(Ax, Q), \quad \forall x \in C \cap B(0, r),
\]
where \( B(x, r) \) and \( \overline{B(x, r)} \) denote the open and closed ball with center at \( x \) and radius \( r \), respectively.

Remark that regularity-type conditions have been widely used to analyze the convergence rates of many algorithms; see [7, 12, 30, 44] and references therein.

**Definition 2.6.** (See [42, 46]) Let \( \Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n \). Consider the following dynamical system
\[
\frac{dx(t)}{dt} = \Phi(x(t)).
\]
(a) \( x^* \in \mathbb{R}^n \) is called an equilibrium point for (7) if \( \Phi(x^*) = 0 \).
(b) An equilibrium point \( x^* \) of (7) is called stable if, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for every \( x_0 \in B(x^*, \delta) \), the solution \( x(t) \) of system (7) with initial point \( x(t_0) = x_0 \) exists and is contained in \( B(x^*, \varepsilon) \) for all \( t \geq t_0 \), where \( B(x^*, \varepsilon) \) denotes the open ball with center at \( x^* \) and radius \( \varepsilon \).
Lemma 2.8. (See [40, 42]) An equilibrium point $x^*$ of system (7) is stable if and only if there exists a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}$ for the dynamical system (7) at $x^*$, which is defined as: for some neighborhood $B(x^*)$ of $x^*$, $V(x)$ satisfies:

(i) $V(x) > 0$ if $x \neq x^*$ and $V(x^*) = 0$;
(ii) $V(x)$ has continuous first order partial derivatives;
(iii) its time derivative along any state trajectory $x(t)$ of dynamical system (7) is non-positive, i.e., $\frac{dV(x(t))}{dt} \leq 0$.

Lemma 2.9. (See [50, 42]) Assume that $\Phi(x)$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Then for arbitrary $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$ there exists a local solution $x(t)$ of (7) satisfying $x(t_0) = x_0, t \in [t_0, \tau)$ for some $\tau > t_0$. If furthermore $\Phi(x)$ is locally Lipschitz continuous at $x_0$ then the solution is unique, and if $\Phi(x)$ is Lipschitz continuous on $\mathbb{R}^n$ then $\tau$ can be extended to $\infty$.

Remark 2. By (a) of Definition 2.6 and Lemma 2.4, the solution set $\Lambda$ of SCFP coincides with the set of all the equilibrium points for system (4).

Lemma 2.7. (See [40, 42]) An equilibrium point $x^*$ of system (7) is stable if and only if there exists a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}$ for the dynamical system (7) at $x^*$, which is defined as: for some neighborhood $B(x^*)$ of $x^*$, $V(x)$ satisfies:

(i) $V(x) > 0$ if $x \neq x^*$ and $V(x^*) = 0$;
(ii) $V(x)$ has continuous first order partial derivatives;
(iii) its time derivative along any state trajectory $x(t)$ of dynamical system (7) is non-positive, i.e., $\frac{dV(x(t))}{dt} \leq 0$.

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Lemma 2.9. (See [27]) Let $t_0 \in \mathbb{R}$ and $h : (t_0, +\infty) \to \mathbb{R}^n$ be a uniformly continuous function, e.g., for every real $\epsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in \mathbb{R}^n$ with $\|x-y\| < \delta$, we have that $\|h(x)-h(y)\| < \epsilon$. If $\int_{t_0}^{+\infty} \|h(s)\| ds < +\infty$, then $\lim_{t \to +\infty} h(t) = 0$.

3. Convergence and stability. In this section, we shall analyze convergence of a solution of dynamical system (7), and then derive some convergence rate. Before that, we first prove the existence and uniqueness of solution. In what follows, we always suppose that $\gamma \in (0, 2/\|A\|^2)$.

Theorem 3.1. For arbitrary initial point $x_0 \in \mathbb{R}^n$, there exists a unique solution $x(t)$ of system (4) with $x(t_0) = x_0$ in the global time interval $t \in [t_0, \infty)$.

Proof. Set $T = P_C \circ U - I$. Then (4) can be written as

$$\frac{dx(t)}{dt} = \lambda T(x(t)).$$

(9)
By Lemma 2.2 and (ii) of Remark 1, \( T \) is nonexpansive. We claim that \( T \) is Lipschitz continuous. Indeed, for any \( x, y \in \mathbb{R}^n \), it follows that

\[
\| T(x) - T(y) \| = \|(P_C \circ U - I)x - (P_C \circ U - I)y\| \\
\leq \| P_C \circ Ux - P_C \circ Uy \| + \| x - y \| \\
\leq \| Ux - Uy \| + \| x - y \| \\
\leq 2\| x - y \| .
\]

By Lemma 2.8, there exists a unique solution \( x(t) \) of (4) satisfying \( x(t_0) = x_0 \) on \([t_0, +\infty)\).

In the following theorem, we discuss the stability of system (4) in the sense of Lyapunov and the convergence of \( \| (I - P_Q)Ax(t) \| \) and \( \| P_C \circ Ux(t) - x(t) \| \).

**Theorem 3.2.** For arbitrary initial point \( x_0 \in \mathbb{R}^n \), let \( x(t) \) be the unique solution of dynamical system (4) with \( x(t_0) = x_0 \). Then, for every equilibrium point \( x^* \in \Lambda \), the following statements hold:

(i) the system (4) is stable at \( x^* \);
(ii) \( \int_{t_0}^{+\infty} \| (I - P_Q)Ax(s) \| ds < +\infty \);
(iii) \( \int_{t_0}^{+\infty} \| P_C \circ Ux(s) - x(s) \| ds < +\infty \);
(iv) the trajectory \( x(t) \) is Lipschitz continuous on \([t_0, +\infty)\);
(v) \( \lim_{t \to +\infty} \| (I - P_Q)Ax(t) \| = 0 \);
(vi) \( \lim_{t \to +\infty} \| P_C \circ Ux(t) - x(t) \| = 0 \).

**Proof.** Define \( V : \mathbb{R}^n \to \mathbb{R} \) by

\[
V(x) = \frac{\gamma}{2} \| (I - P_Q)Ax \|^2 + \frac{1}{2} \| x - x^* \|^2 , \quad \forall x \in \mathbb{R}.
\]

By Lemma 2.3, \( V(x) \) is differentiable and

\[
\nabla V(x) = \gamma A^T (I - P_Q)Ax + x - x^* .
\]

(10)

Since \( x^* \in \Lambda \), by Lemma 2.4 we have

\[
(I - P_Q)Ax^* = 0 , \quad P_C \circ Ux^* = x^* \quad \text{and} \quad Ux^* = x^* .
\]

(11)

For any initial point \( x_0 \in \mathbb{R}^n \), by Theorem 3.1, there exists a unique solution \( x(t) \) of system (4) with \( x(t_0) = x_0 \). We claim that \( \frac{dV(x(t))}{dt} \leq 0 \). Indeed, from (4), (10)
and (11) we have
\[
\frac{dV(x(t))}{dt} = \langle \gamma A^T(I - P_Q)Ax(t) + x(t) - x^*, \frac{dx(t)}{dt} \rangle
\]
\[
= \langle (I - U)x(t) - (I - U)x^* + x(t) - x^*, \\
\lambda P_C \circ Ux(t) - \lambda P_C \circ Ux^* - \lambda x(t) + \lambda x^* \rangle
\]
\[
= \lambda\langle (I - U)x(t) - (I - U)x^*, \\
P_C \circ Ux(t) - P_C \circ Ux^* \rangle
\]
\[
- \lambda\langle (I - U)x(t) - (I - U)x^*, x(t) - x^* \rangle
\]
\[
+ \lambda\langle x(t) - x^*, P_C \circ Ux(t) - P_C \circ Ux^* \rangle - \lambda\|x(t) - x^*\|^2
\]
\[
= \lambda\langle x(t) - x^*, P_C \circ Ux(t) - P_C \circ Ux^* \rangle
\]
\[
- \lambda\langle Ux(t) - Ux^*, P_C \circ Ux(t) - P_C \circ Ux^* \rangle
\]
\[
- \lambda\langle A^T(I - P_Q)Ax(t) - A^T(I - P_Q)Ax^*, x(t) - x^* \rangle
\]
\[
+ \lambda\langle x(t) - x^*, P_C \circ Ux(t) - P_C \circ Ux^* \rangle - \lambda\|x(t) - x^*\|^2
\]
\[
= -\lambda\langle Ux(t) - Ux^*, P_C \circ Ux(t) - P_C \circ Ux^* \rangle
\]
\[
+ 2\lambda\langle x(t) - x^*, P_C \circ Ux(t) - P_C \circ Ux^* \rangle - \lambda\|x(t) - x^*\|^2
\]
\[
- \lambda\gamma\langle (I - P_Q)Ax(t) - (I - P_Q)Ax^*, Ax(t) - Ax^* \rangle.
\]
\[
(12)
\]
It follows from (5), (6), (10) and (12) that
\[
\frac{dV(x(t))}{dt} \leq -\lambda\|P_C \circ Ux(t) - P_C \circ Ux^*\|^2 + 2\lambda\langle x(t) - x^*, P_C \circ Ux(t) - P_C \circ Ux^* \rangle
\]
\[
- \lambda\|x(t) - x^*\|^2 - \lambda\gamma\|(I - P_Q)Ax(t) - (I - P_Q)Ax^*\|^2
\]
\[
= -\lambda\|P_C \circ Ux(t) - P_C \circ Ux^*\|^2 + 2\lambda\langle x(t) - x^*, P_C \circ Ux(t) - P_C \circ Ux^* \rangle
\]
\[
- \lambda\|x(t) - x^*\|^2 - \lambda\gamma\|(I - P_Q)Ax(t)\|^2
\]
\[
= -\lambda\|x(t) - P_C \circ Ux(t)\|^2 - \lambda\gamma\|(I - P_Q)Ax(t)\|^2
\]
\[
(13)
\]
\[
\leq 0.
\]
By Lemma 2.7, the equilibrium point \(x^*\) of system (4) is stable in the sense of Lyapunov. Next, we verify the statements (ii) and (iii). Since \(V(x(t))\) is nonincreasing as \(t \to +\infty\) and bounded (it is nonnegative), we deduce that \(\lim_{t \to +\infty} V(x(t))\) exists and the trajectory \(x(t)\) is bounded. Integrate (13) from \(t_0\) to \(+\infty\) to get
\[
\lim_{t \to +\infty} V(x(t)) + \lambda\gamma \int_{t_0}^{+\infty} \|(I - P_Q)Ax(s)\|ds + \lambda \int_{t_0}^{+\infty} \|P_C \circ Ux(s) - x(s)\|ds
\]
\[
\leq V(x(t_0)),
\]
which implies
\[
\int_{t_0}^{+\infty} \|(I - P_Q)Ax(s)\|ds < +\infty
\]
and
\[
\int_{t_0}^{+\infty} \|P_C \circ Ux(s) - x(s)\|ds < +\infty.
\]
So (ii) and (iii) hold. Now we proof (iv). Integrating (9) from \(t_1\) to \(t_2\) and using mean value theorem, there exists \(\tilde{t} \in [t_1, t_2]\) such that
\[
\|x(t_1) - x(t_2)\| \leq \|T(x(\tilde{t}))\|(t_2 - t_1).
\]
This along with the bounded of \( x(t) \) and the Lipschitz property of \( T \) yield the assertion (iv). And finally we shall show that (v) and (vi) are true. It is clear that \( I - P_Q \) is Lipschitz continuous and \( A \) is a bounded linear operator. Combining with (iv) we have that \( (I - P_Q)A_x \) is Lipschitz continuous. Therefore, the assertions (v) and (vi) are true by Lemma 2.9.

In Theorem 3.2, we have studied the stability of system (4). In fact, we can further obtain the global convergence of system (4) and a convergence rate \( o(\frac{1}{\sqrt{t}}) \) of \( \| P_C \circ U(t) - x(t) \| \).

**Theorem 3.3.** Let \( x(t) \) be the solution of system (4) with \( x(t_0) = x_0 \in \mathbb{R}^n \). Then the following statements are true:

(i) \( x(t) \) converges to some equilibrium point of system (4) as \( t \to \infty \).

(ii) \( \| P_C \circ U(t) - x(t) \| = o(\frac{1}{\sqrt{t}}) \).

**Proof.** Recall that \( U = I - \gamma A^T (I - P_Q)A \) and \( \frac{dx(t)}{dt} = \lambda [P_C \circ U(x(t)) - x(t)] \). To prove (i), we divide the proof into two steps.

**Step 1:** We shall show that \( \| x - x^* \| \) is nonincreasing on \([0, +\infty)\), for every \( x^* \in \Lambda \).

Set

\[
W(x) = \frac{1}{2} \| x - x^* \|^2. \tag{14}
\]

By the classical derivation chain rule, we have

\[
\frac{dW(x(t))}{dt} = \langle x(t) - x^*, \frac{dx(t)}{dt} \rangle
= \langle x(t) - x^*, \lambda P_C \circ U(x(t)) - \lambda x(t) \rangle
= \frac{1}{2} \| x(t) - x^* + \lambda P_C \circ U(x(t)) - \lambda x(t) \|^2
- \frac{1}{2} \| x(t) - x^* \|^2 - \frac{\lambda^2}{2} \| P_C \circ U(x(t)) - x(t) \|^2
= \frac{1}{2} \| (1 - \lambda) (x(t) - x^*) + \lambda (P_C \circ U(x(t)) - x^*) \|^2
- \frac{1}{2} \| x(t) - x^* \|^2 - \frac{\lambda^2}{2} \| P_C \circ U(x(t)) - x(t) \|^2
= \frac{1 - \lambda}{2} \| x(t) - x^* \|^2 + \frac{\lambda^2}{2} \| P_C \circ U(x(t)) - x^* \|^2
- \frac{\lambda - \lambda^2}{4} \| x(t) - P_C \circ U(x(t)) \|^2 - \frac{1}{2} \| x(t) - x^* \|^2
- \frac{\lambda^2}{2} \| P_C \circ U(x(t)) - x(t) \|^2
\leq \frac{-\lambda^2 + \frac{\lambda}{2} \| P_C \circ U(x(t)) - x(t) \|^2}{4}, \tag{15}
\]

where the fifth equality is from (8). So \( \| x - x^* \| \) is nonincreasing on \([0, +\infty)\), for every \( x^* \in \Lambda \).

**Step 2.** We shall show that \( \lim_{t \to +\infty} \text{dist}(x(t), \Lambda) = 0 \). It is worth noting that \( \| x(t) - x^* \| \) is nonincreasing and \( x(t) \) is bounded on \([0, +\infty)\). Let \( \hat{x} \) be a cluster of \( x(t) \) and \( t_n \to +\infty \) such that \( x(t_n) \to \hat{x} \). From (vi) in Theorem 3.2, we have \( P_C \circ U \hat{x} = \hat{x} \). By Lemma 2.4, \( \hat{x} \in \Lambda \). By replacing \( x^* \) with \( \hat{x} \) in (14), we get that

\[
\lim_{t \to +\infty} \| x(t) - \hat{x} \| \text{ exists. So } x(t) \to \hat{x} \text{ as } t \to +\infty.
\]
In order to prove (ii), we first shall show that \( \|P_C \circ U(x(t)) - x(t)\| \) is nonincreasing on \([t_0, +\infty)\). By the Rademacher’s theorem (see Theorem 3.16 in [22]) and the differentiability of \(x(t)\), we know that \(P_C \circ U(x(t))\) is differentiable almost everywhere with respect to \(t\). Since \(P_C \circ U\) is nonexpansive, by (b) of Remark 1 of [11], we get that

\[
\left\| \frac{dP_C \circ U(x(t))}{dt} \right\| \leq \left\| \frac{dx(t)}{dt} \right\| \tag{16}
\]

holds almost everywhere. Set \( G(x(t)) = \frac{1}{2}\|P_C \circ U(x(t)) - x(t)\|^2 \). Deriving \(G(x(t))\) with respect to time, from (4) and (16) we have that

\[
\frac{dG(x(t))}{dt} = (P_C \circ U(x(t)) - x(t), \frac{dP_C \circ U(x(t))}{dt} - \frac{dx(t)}{dt}) - (P_C \circ U(x(t)) - x(t), \frac{dx(t)}{dt})
\]

\[
\leq -\lambda \|P_C \circ U(x(t)) - x(t)\|^2 + \|P_C \circ U(x(t)) - x(t)\| \cdot \left\| \frac{dx(t)}{dt} \right\|
\]

\[
\leq -\lambda \|P_C \circ U(x(t)) - x(t)\|^2 + \lambda \|P_C \circ U(x(t)) - x(t)\|^2
\]

\[
= 0
\]

holds almost everywhere. This together with the continuity of \(P_C \circ U(x(t))\) implies that \( \|P_C \circ U(x(t)) - x(t)\| \) is nonincreasing on \([0, +\infty)\).

Then using the monotonicity of \(G(x(t))\) and the conclusion (vi) in Theorem 3.2, we obtain

\[
\frac{t}{2} \|P_C \circ U(x(t)) - x(t)\|^2 \leq \int_{\frac{t}{2}}^{t} \|P_C \circ U(x(s)) - x(s)\|^2 ds
\]

\[
= \int_{t_0}^{t} \|P_C \circ U(x(s)) - x(s)\|^2 ds
\]

\[
- \int_{t_0}^{\frac{t}{2}} \|P_C \circ U(x(s)) - x(s)\|^2 ds
\]

\[
\rightarrow 0 \quad (\text{as } t \rightarrow \infty).
\]

As a consequence,

\[
\|P_C \circ U(x(t)) - x(t)\| = o\left(\frac{1}{\sqrt{t}}\right).
\]

\[\square\]

Remark 3. An abstract dynamical system governed by a non-expansive mapping \(T\) instead of \(P_C \circ U\) has been already considered in Bot and Csetnek [11]. As pointed out by one reviewer that: (a) (iii) and (vi) of Theorem 3.2, and Theorem 3.3 can follow directly from Theorem 6 and Theorem 10 of Bot and Csetnek [11] (although our method is different from the one in [11]); (b) The dynamics proposed in [10] for a structured convex minimization problem is also able to solve the split convex feasibility problem.

It is worth mentioning that Bot and Csetnek [11] did not consider the exponential convergence of the dynamical system. Next, we further derive the exponential convergence of the trajectory of system (4) with an initial point \(x_0 \in C\) when the bounded linear regularity property holds. To do this, we need the following lemma.
Lemma 3.4. Let \( f : [0, \infty) \to \mathbb{R}^n \) be a continuous mapping such that \( f(t) \in C \) for all \( t \geq t_0 \) and let \( x(t) \) be a trajectory of the following dynamical system:

\[
\frac{dx(t)}{dt} = \lambda(f(t) - x(t)), \quad x(t_0) = x_0 \in C,
\]

where \( \lambda > 0 \) and \( C \) is a nonempty, closed and convex set. Then we have \( x(t) \in C \) for all \( t \geq t_0 \).

Proof. From (17) we get

\[
e^{\lambda(t-t_0)} \frac{dx(t)}{dt} = \lambda e^{\lambda(t-t_0)} f(t) - \lambda e^{\lambda(t-t_0)} x(t).
\]

Integrating it from \( t_0 \) to \( t \), we have

\[
e^{\lambda(t-t_0)} x(t) = x_0 + \lambda \int_{t_0}^{t} e^{\lambda(\tau-t_0)} f(\tau) d\tau.
\]

This yields

\[
x(t) = e^{-\lambda(t-t_0)} x_0 + \lambda e^{-\lambda(t-t_0)} \int_{t_0}^{t} e^{\lambda(\tau-t_0)} f(\tau) d\tau
\]

\[
= e^{-\lambda(t-t_0)} x_0 + (1 - e^{-\lambda(t-t_0)}) \frac{\lambda}{e^{\lambda(t-t_0)} - 1} \int_{t_0}^{t} e^{\lambda(\tau-t_0)} f(\tau) d\tau. \quad (18)
\]

We claim that

\[
g(t) := \frac{\lambda}{e^{\lambda(t-t_0)} - 1} \int_{t_0}^{t} e^{\lambda(\tau-t_0)} f(\tau) d\tau \in C, \quad \forall t \geq t_0.
\]

Indeed, let \([t_i, t_{i+1}], i = 0, 1, \ldots, N\) be any tagged partition of \([t_0, t]\), \( \Delta := \max\{\triangle_i : i = 1, 2, \ldots, N\} \) be the mesh of such a tagged partition with \( \triangle_i \) being the width of sub-interval \([t_i, t_{i+1}]\). Let \( t_i \in [t_i, t_{i+1}] \). By the definition of integral we get

\[
g(t) = \frac{\lambda}{e^{\lambda(t-t_0)} - 1} \lim_{\Delta \to 0} \sum_{i=0}^{N} f(t_i) e^{\lambda(\tau_i-t_0)} \triangle_i
\]

\[
= \frac{\lambda}{e^{\lambda(t-t_0)} - 1} \lim_{\Delta \to 0} \left( \sum_{i=0}^{N} e^{\lambda(\tau_i-t_0)} \triangle_i \right) \sum_{i=0}^{N} \left( \sum_{i=0}^{N} e^{\lambda(\tau_i-t_0)} \triangle_i \right) f(t_i)
\]

\[
= \frac{\lambda}{e^{\lambda(t-t_0)} - 1} \lim_{\Delta \to 0} \left( \sum_{i=0}^{N} e^{\lambda(\tau_i-t_0)} \triangle_i \right) \lim_{\Delta \to 0} \left( \sum_{i=0}^{N} e^{\lambda(\tau_i-t_0)} \triangle_i \right) f(t_i)
\]

\[
= \frac{\lambda}{e^{\lambda(t-t_0)} - 1} \int_{t_0}^{t} e^{\lambda(\tau-t_0)} d\tau \lim_{\Delta \to 0} \sum_{i=0}^{N} \left( \sum_{i=0}^{N} e^{\lambda(\tau_i-t_0)} \triangle_i \right) f(t_i)
\]

\[
= \lim_{\Delta \to 0} \sum_{i=0}^{N} \frac{e^{\lambda \tau_i} \triangle_i}{\sum_{i=0}^{N} e^{\lambda \tau_i} \triangle_i} f(t_i) \in C.
\]

This together with (18) implies that \( x(t) \in C \) for all \( t \geq t_0 \). \( \square \)

Remark 4. The well-known methods for solving approximatively system (4) are finite difference methods. Taking time step size \( \Delta t_n > 0 \), the explicit difference scheme for system (4) gives:

\[
x_{n+1} = \lambda \Delta t_n P_C [x_n - \gamma A^T (I - P_Q) Ax_n] + (1 - \lambda \Delta t_n)x_n. \quad (19)
\]
For $\lambda \Delta t_n = 1$, the CQ-algorithm (2) is recovered in (19). Comparison with the CQ-algorithm (2), system (4) enjoys much stronger property. In general, the sequence generated by the CQ-algorithm (2) is not in the interior int $C$ (respectively, relative interior ri $C$) of $C$ even if the initial point $x_0 \in \text{int} C$ (respectively, ri $C$) while system (4) is an interior method as soon as the initial point is strictly feasible. Indeed, by the argument (18) of Lemma 3.4, we have $x(t) \in \text{int} C$ (respectively, ri $C$) for all $t \geq t_0$ if $x_0 \in \text{int} C$ (respectively, ri $C$). In fact, the discrete approximation (19) with $\lambda \Delta t_n < 1$ also enjoys this interior property.

Now, with Lemma 3.4 in hand, we derive the exponential convergence of the trajectory of system (4).

**Theorem 3.5.** Let $x(t)$ be the solution of system (4) with $x(t_0) = x_0 \in C$ and $\bar{x} \in \Lambda$ such that $x(t) \to \bar{x}$ as $t \to \infty$ (This is guaranteed by Theorem 3.3). Suppose that SCFP satisfies the bounded linear regularity property. Then there exist $\bar{t} \geq t_0$ and $\beta_0 > 0$ such that

$$
\text{dist}(x(t), \Lambda) \leq e^{-\beta_0(t-\bar{t})} \text{dist}(x_0, \Lambda), \quad \forall t \geq \bar{t}.
$$

**Proof.** Set $f(t) = P_C \circ Ux(t)$. Clearly, $f$ is continuous and $f(t) \in C$ for all $t \in t_0$. By Lemma 3.4, $x(t) \in C$ for all $t \geq t_0$. Define

$$
W(t) = \frac{1}{2} \text{dist}^2(x(t), \Lambda) = \frac{1}{2} \|x(t) - P_\Lambda x(t)\|^2.
$$

By Lemma 2.4, we have $P_C \circ U(P_\Lambda x(t) = P_\Lambda x(t)$. Deriving $W(x(t))$ with respect to time $t$, we obtain

$$
\frac{dW(t)}{dt} = \langle x(t) - P_\Lambda x(t), \frac{dx(t)}{dt} \rangle
$$

$$
= \langle x(t) - P_\Lambda x(t), \lambda P_C \circ Ux(t) - \lambda x(t) \rangle
$$

$$
= \lambda \langle x(t) - P_\Lambda x(t), P_C \circ Ux(t) - P_C \circ U(P_\Lambda x(t)) + P_\Lambda x(t) - x(t) \rangle
$$

$$
= \lambda \langle x(t) - P_\Lambda x(t), P_C \circ Ux(t) - P_C \circ U(P_\Lambda x(t)) \rangle - \lambda \|x(t) - P_\Lambda x(t)\|^2
$$

$$
\leq \frac{\lambda}{2} \|x(t) - P_\Lambda x(t)\|^2 + \frac{\lambda}{2} \|P_C \circ Ux(t)
$$

$$
\leq \frac{\lambda}{2} \|x(t) - P_\Lambda x(t)\|^2 + \frac{\lambda}{2} \|Ux(t) - U(P_\Lambda x(t))\|^2,
$$

(20)

where the first equality follows from (a) of Lemma 2.3. Recalling that $U = I - \gamma A^T(I - P_Q)A$, we get

$$
\frac{\lambda}{2} \|Ux(t) - U(P_\Lambda x(t))\|^2
$$

$$
= \frac{\lambda}{2} \|x(t) - P_\Lambda x(t) - \gamma A^T(I - P_Q)A x(t) - \gamma A^T(I - P_Q)A(P_\Lambda x(t))\|^2
$$

$$
= \frac{\lambda}{2} \|x(t) - P_\Lambda x(t)\|^2 + \frac{\lambda}{2} \|\gamma A^T(I - P_Q)A x(t) - \gamma A^T(I - P_Q)A(P_\Lambda x(t))\|^2
$$

$$
- \lambda \langle x(t) - P_\Lambda x(t), \gamma A^T(I - P_Q)A x(t) - \gamma A^T(I - P_Q)A(P_\Lambda x(t)) \rangle.
$$

(21)

We now estimate the last two terms of (21). By (iii) of Lemma 2.4 and the definition of $L$, we get that

$$
\frac{\lambda}{2} \|\gamma A^T(I - P_Q)A x(t) - \gamma A^T(I - P_Q)A(P_\Lambda x(t))\|^2 \leq \frac{\lambda^2 L}{2} \|I - P_Q A x(t)\|^2.
$$
From (6) and (iv) of Lemma 2.4 we have that
\[ -\lambda \langle x(t) - P_\Lambda x(t) , \gamma A^T (I - P_Q) Ax(t) - \gamma A^T (I - P_Q) A (P_\Lambda x(t)) \rangle \]
\[ = -\lambda \gamma \langle Ax(t) - A (P_\Lambda x(t)) , (I - P_Q) Ax(t) - (I - P_Q) A (P_\Lambda x(t)) \rangle \]
\[ \leq -\lambda \gamma \|(I - P_Q) Ax(t)\|^2. \]
Substituting these two inequalities into (21), we have
\[ \frac{\lambda}{2} \|U x(t) - U (P_\Lambda x(t))\|^2 \leq \frac{\lambda}{2} \|x(t) - P_\Lambda x(t)\|^2 - \frac{\lambda \gamma (2 - \gamma L)}{2} \|(I - P_Q) Ax(t)\|^2. \]
This together with (20) yields that
\[ \frac{dW(t)}{dt} \leq -\lambda \gamma \frac{(2 - \gamma L)}{2} \|(I - P_Q) Ax(t)\|^2. \] (22)
Since \( x(t) \in C \) for all \( t \geq t_0 \), \( x(t) \to \bar{x} \) as \( t \to \infty \) and SCFP satisfies the bounded linear regularity property, there exist \( \kappa > 0 \) and \( \bar{t} \geq t_0 \) such that
\[ \text{dist} (x(t), \Lambda) \leq \kappa \cdot \|(I - P_Q) Ax(t)\|, \quad \forall t \geq \bar{t}. \] (23)
It follows from (22) and (23) that
\[ \frac{dW(t)}{dt} \leq -\beta W(t) \] (24)
with
\[ \beta = \frac{\lambda \gamma (2 - \gamma L)}{\kappa^2} > 0. \]
Let
\[ \bar{W}(t) = e^{\beta t} W(t) = \frac{1}{2} e^{\beta t} \text{dist}^2 (x(t), \Lambda). \] (25)
It follows that
\[ \frac{d\bar{W}(t)}{dt} = \beta e^{\beta t} W(t) + e^{\beta t} \frac{dW(t)}{dt} \]
\[ \leq \beta e^{\beta t} W(t) - \beta e^{\beta t} W(t) \]
\[ = 0, \] (26)
which implies that \( \bar{W}(t) \) is nonincreasing on \([\bar{t}, \infty)\). From (25) and (26) we get
\[ e^{\beta t} W(t) - e^{\beta \bar{t}} W(\bar{t}) = \frac{1}{2} e^{\beta t} \text{dist}^2 (x(t), \Lambda) - \frac{1}{2} e^{\beta \bar{t}} \text{dist}^2 (x(\bar{t}), \Lambda) \leq 0. \]
This yields
\[ \text{dist}^2 (x(t), \Lambda) \leq e^{-\beta (t-\bar{t})} \text{dist}^2 (x(\bar{t}), \Lambda). \]
By (22), \( \text{dist}^2 (x(t), \Lambda) \) is nonincreasing on \([t_0, \infty)\). This together with the last inequality implies that
\[ \text{dist} (x(t), \Lambda) \leq e^{-\beta_0 (t-\bar{t})} \text{dist} (x(\bar{t}), \Lambda), \quad \forall t \geq \bar{t} \]
with \( \beta_0 = \frac{1}{2} \beta > 0. \)

Remark 5. In some sense, Theorem 3.5 can be viewed as a continuous form of Theorem 2.3 of [44] where the linear convergence of a varying stepsize CQ algorithm was established under the bounded linear regularity property.
4. Application to the linear equation problem. When $C = \mathbb{R}^n$ and $Q = \{b\}$, the split feasible problem (1) reduces to the following linear equation problem (LEP): find $x \in \mathbb{R}^n$ such that

$$Ax = b.$$  \hfill (27)

In this case, $P_C(I - \gamma A^T(I - P_Q)A)(x) = x + \lambda \gamma A^T(b - Ax)$, and CQ-algorithm algorithm (2) reduces to the following Landweber method [31]:

$$x_{n+1} = x_n + \lambda \gamma A^T(b - Ax_n), \quad n = 0, 1, 2, \ldots,$$  \hfill (28)

for $x_0$ arbitrary. As a consequence, system (4) reduces to the following dynamical system:

$$\frac{dx(t)}{dt} = \lambda \gamma A^T(b - Ax(t)).$$  \hfill (29)

Naturally, the Landweber method (28) can be regarded as an explicit discrete-time form of system (29). When $A$ is a matrix with full column rank, an implicit discrete-time scheme of system (29) yields the Piccard algorithm for (27): where $\rho > 0$. As an application of results in Section 3, we solve LEP (27) via system (29). In what follows, we always suppose that the solution set $E$ of (27) is nonempty.

**Theorem 4.1.** For any $x_0 \in \mathbb{R}^n$, there exists a unique solution $x(t)$ of system (29) with $x(t_0) = x_0 \in \mathbb{R}^n$ in the global time interval $t \in [t_0, \infty)$.  

**Proof.** It is a direct consequence of Theorem 3.1. \hfill \square

**Theorem 4.2.** For arbitrary initial point $x_0 \in \mathbb{R}^n$, let $x(t)$ be the unique solution of dynamical system (29) with $x(t_0) = x_0$. Then the following the following statements hold:

(i) for every equilibrium point $x^* \in E$ of system (29) is stable;

(ii) $\int_{t_0}^{+\infty} \|(Ax(s) - b)\| ds < +\infty$;

(iii) the trajectory $x(t)$ is Lipschitz continuous on $[t_0, +\infty)$;

(iv) $\lim_{t \to +\infty} \|Ax(t) - b\| = 0$.

**Proof.** It is a direct consequence of Theorem 3.2. \hfill \square

**Theorem 4.3.** Let $x(t)$ be the solution of system (29) with $x(t_0) = x_0 \in \mathbb{R}^n$. Then the following statements are true:

(i) $x(t)$ converges to some equilibrium point of system (29) as $t \to \infty$.

(ii) $\|A^T(Ax(t) - b)\| = o(\frac{1}{\sqrt{t}})$.

**Proof.** Part (i) is a direct consequence of (i) of Theorem 3.3. Part (ii) follows from (ii) of Theorem 3.3 and the following fact:

$$\|P_C \circ Ux(t) - x(t)\| = \|\gamma A^T(Ax(t) - b)\|.$$  \hfill \square

**Theorem 4.4.** Let $x(t)$ be the solution of system (29) with $x(t_0) = x_0 \in \mathbb{R}^n$ and $\bar{x} \in E$ such that $x(t) \to \bar{x}$ as $t \to \infty$ (This is guaranteed by Theorem 4.3). Then there exist $t \geq t_0$ and $\beta_0 > 0$ such that

$$\text{dist}(x(t), E) \leq e^{-\beta_0(t-t_0)} \text{dist}(x_0, E), \quad \forall t \geq \bar{t}.$$
Proof. By a fundamental theorem given by Hoffman [29] (see also Robinson [39]), there exists a constant $\kappa > 0$ such that
\[
\text{dist} (x, E) \leq \kappa \|Ax - b\|, \quad \forall x \in \mathbb{R}^n.
\]
This implies that the bounded linear regularity property holds since
\[
(I - P_Q)A(x) = Ax - b, \quad \forall x \in \mathbb{R}^n.
\]
So the conclusion follows from Theorem 3.5. \hfill \Box

5. Simulation examples. In this section, we illustrate the validity of the proposed dynamical system approaches for solving SCFP (1) by several numerical examples. The simulations are conducted in Matlab (version 9.3.0.713579) R2017b. All the numerical procedures are performed on a personal Lenovo computer with Inter(R) Core(TM) i7-4600U, CORES 2.69GHz and RAM 8.00GB.

Example 1. (See [2, Example 5.2]) Consider the SCFP defined in (1) with
\[
A = \begin{bmatrix}
2 & -1 & 3 \\
4 & 2 & 5 \\
2 & 0 & 2
\end{bmatrix}, \quad C = \{x \in \mathbb{R}^3 \mid x_1 + x_2^2 + 2x_3 \leq 0\} \quad \text{and} \quad Q = \{y \in \mathbb{R}^3 \mid y_1^2 + y_2 - y_3 \leq 0\}.
\]

We use dynamical system (4) to solve Example 1. Take $\gamma = \frac{1}{\|A\|^2}$. Figure 1 describes the asymptotical behavior of dynamical system (4) with $x_0 = [1, 1, 1]^T$ and $\lambda = 5$, where ode45 is exploited to solve (4). It is shown in Figure 1 that the trajectory $x(t)$ of dynamic system (4) converges to $x^* = [0.2488, 0.0831, -0.2969]^T \in \Lambda$.

Figure 2 describes the asymptotical behavior of dynamical system (4) with $x_0 = [1, 2, 3]^T$ and $\lambda = 0.2$, where the central difference method is exploited to solve (4) approximately. It is shown in Figure 2 that the trajectory $x(t)$ of dynamic system (4) converges to $x^* = [-0.5534, 0.0841, 0.2257]^T \in \Lambda$. Figure 3 describes the transient behavior of dynamical system (4) with $x_0 = [1, 1, 1]^T$ and different $\lambda$ where the explicit difference method is exploited to solve (4) approximately.

Example 2. (See [45, Example 1]) We consider SCFP (1) as following: the matrix $A = (a_{i,j})_{2 \times 2}$ and $a_{i,j} \in (0, 1)$ are generated randomly, the nonempty closed convex set $C = \{u \in \mathbb{R}^2 \mid c(u) \leq 0\}$ and $Q = \{v \in \mathbb{R}^2 \mid q(v) \leq 0\}$, where
\[
c(u) = -u_1 + u_2^2
\]
and
\[
q(v) = v_1 + v_2^2
\]
for all $u = (u_1, u_2)^T \in \mathbb{R}^2$ and $v = (v_1, v_2)^T \in \mathbb{R}^2$.

Take $\gamma = \frac{1}{\|A\|^2}$ and $\lambda = \frac{1}{3}$. Figure 4 describes the transient behavior of dynamical system (4) with $x_0 = [5, -4]$ and $A = \begin{bmatrix}
0.6324 & 0.2785 \\
0.0975 & 0.5469
\end{bmatrix}$ generated randomly. The trajectory $x(t)$ converges to $x^* = [0.0959, -0.3094]^T \in \Lambda$.

Figure 5 describes the transient behavior of dynamical system (4) with $x_0 = [-10, 4]$ and $A = \begin{bmatrix}
0.9572 & 0.8013 \\
0.4854 & 0.1419
\end{bmatrix}$ generated randomly. The trajectory $x(t)$ converges to $\hat{x} = [0, 0]^T \in \Lambda$.

Here, the explicit difference method is exploited to solve approximately dynamical system (4).
Example 3. Consider LEP defined in (27) with $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 2 & 1 & 2 \end{bmatrix}$ and $b = (1, 4, 5)^T$.

By means of algebraic calculation, it is easy to verify that $x^* = [9, -1, -6]^T$ is a unique solution for Example 3. Set $\gamma = \frac{1}{\|A\|_2}$ and $\lambda = 35$ that satisfy the operating conditions. Figure 6, where the explicit difference method is exploited, displays that the trajectories of dynamical system (4) with 20 random initial points all globally converge to $x^*$. Figure 7 and Figure 8 depict the asymptotical behavior the trajectory of dynamical system (4) with initial points $x_0 = [-3, 5, 2]^T$ where the finite element method with $\lambda = 1$ and the Piccard algorithm with $\rho = \frac{1}{5\|A\|_2}$ are exploited to solve dynamical system (4), respectively.

Example 4. In this example we use our dynamical system approach for solving the problem of recovering a noisy sparse signal from a limited number of sampling which was considered in [25, Subsection 5.1]. Let $x \in \mathbb{R}^n$ be a $K-$sparse signal and $K << n$. The sampling matrix $A \in \mathbb{R}^{m \times n}$ is stimulated by standard Gaussian distribution and vector $b = Ax + e$ with $e$ being an additional noisy. The task is to recover the signal $x$ from the data $b$. This problem can be modeled as the following LASSO problem [43]:

$$
\min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - b\|^2_2 \\
\text{s.t. } \|x\|_1 \leq t,
$$

where $t > 0$ is a given constant. Taking $C = \{x : \|x\|_1 \leq t\}$ and $Q = \{b\}$, the above LASSO problem is transformed into SCFP(1). So we can solve the problem via dynamical system (4).

In the implementation of dynamical system (4) we choose the following set of parameters: the initial point $x_0 \in \mathbb{R}^n$ of dynamical system (4) and the matrix $A \in \mathbb{R}^{m \times n}$ are generated randomly. The true signal $x \in \mathbb{R}^n$ with $K$ non-zero elements is generated from uniform distribution in the interval $[-2, 2]$ and vector $b = Ax$. Take $m = 2^{13}, n = 2^{13}, t = K = 50, \gamma = \frac{1}{\|A\|_2}$ and $\lambda = 5$. Figure 9 describes the asymptotical behavior of dynamical system (4) where ode45 is exploited to solve (4). It is shown in Figure 9 that the trajectory $x(t)$ of dynamic system (4) converges. Figure 10 depicts the recovery of the true signal by the proposed dynamical system approach. Figure 11 displays the objective function value against the time for the LASSO problem solved through the dynamical system (4) with different choices of $\lambda$.

6. Conclusion. In this paper we propose a projection dynamical system for solving SCFP. We demonstrate that the trajectory of the projection dynamical system is globally convergent to a solution of SCFP. It is also shown that the fixed point residual enjoys the $o(1/t^\beta)$ convergence rate. Under the bounded linear regularity condition, we further obtain an exponential convergence for the trajectory of the the projection dynamical system. The method proposed in this paper can be regarded as not only a continuous version but also an interior version of the known CQ-method for solving the split convex feasibility problem.
Figure 1. The transient behavior of dynamical system (4) with initial points $x_0 = [1, 1, 1]^T$ in Example 1 via ode45.

Figure 2. The transient behavior of dynamical system (4) with initial points $x_0 = [1, 2, 3]^T$ in Example 1 via the central difference method.

Acknowledgments. The authors would like to thank two anonymous referees and the handling editor for their helpful comments and suggestions which have led to the improvement of the early version of this paper.
Figure 3. The transient behavior of dynamical system (4) with different $\lambda$ in Example 1 via the explicit difference method.

Figure 4. The transient behavior of dynamical system (4) with initial points $x_0 = [5, -4]^T$ in Example 2 via the explicit difference method.

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Figure 6. The transient behavior of dynamical system (29) with 20 random initial points in Example 3 via the explicit difference method.
Figure 7. The transient behavior of dynamical system (29) with $x_0 = [-3, 5, 2]^T$ in Example 3 via the finite element method method.

Figure 8. The transient behavior of dynamical system (29) with $x_0 = [-3, 5, 2]^T$ in Example 3 via the Piccard algorithm.

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Figure 10. The recovered sparse signal versus the true $50-$sparse signal in Example 4.

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Received August 2019; revised March 2020.

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