A PARTIAL DATA PROBLEM IN LINEAR ELASTICITY

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Abstract. We discuss the determination of the Lamé parameters of an elastic material by the means of boundary measurements. We will combine previous results of Eskin–Ralston and Isakov to prove inverse results in the case of bounded domains with partial data. Moreover, we generalise these results to infinite cylinders.

1. Introduction

Given an elastic body, we want to determine the Lamé parameters $\lambda$ and $\mu$ by measuring the “deformation-to-stress” map $\Lambda$ on the boundary. Mathematically, $\Lambda$ is the Dirichlet-to-Neumann operator. This problem is related to the inverse problem of electrical impedance tomography (EIT), where one considers the equation $\nabla \cdot (q\nabla u) = 0$ with bounded and positive potential $q$ on a bounded domain $\Omega$ with sufficiently regular boundary $\Gamma = \partial \Omega$. Here, the Dirichlet-to-Neumann operator $\Lambda_q$ is viewed as the “voltage-to-current” map.

In the pioneering works of Calderón [2] and Sylvester–Uhlmann [18], the authors constructed so-called complex geometric optics solutions (we refer to the survey [19] for more details). If on a bounded domain with potentials $q_1, q_2$ satisfy $\Lambda_{q_1} = \Lambda_{q_2}$ on the boundary, then it was shown in [18] that $q_1 = q_2$. There are also results for unbounded domains, for instance the slab was treated by Li–Uhlmann [11].

In applications, it is usually not possible to measure $\Lambda$ on the whole boundary. Therefore, it is natural to ask whether it suffices to know equality of $\Lambda_{q_1}$ and $\Lambda_{q_2}$ on a small part $\Gamma_0$ of the boundary $\Gamma$. Kenig–Sjöstrand–Uhlmann [9] showed that for the Schrödinger operator this is true if the Dirichlet-to-Neumann operator is measured in $\Gamma_0$ and $\Gamma$ is “convex enough” with respect to $\Gamma_0$ (again we refer to [19] and references therein). This assumption can be relaxed if one strengthens the assumptions on the $\Gamma_0$: Isakov [7] proved that if $\Gamma_0$ is part of a plane, then a reflection argument proves uniqueness.

In the case of the elasticity operator $L_j$ with Lamé parameters $\lambda_j$ and $\mu_j$, the problem is considerably more involved and even global uniqueness with full data is still open. Under an a-priori smallness assumption on the parameters $\mu_j$, the inverse problem was solved by Nakamura–Uhlmann [13,14,16] and Eskin–Ralston [3]. The principal symbol of the elasticity operator is not equal to the principal symbol of the matrix Laplacian, thus one cannot directly apply the method of complex geometric optics solutions. In [3], the authors considered an auxiliary equation, from which solutions of the elasticity system could be deduced (see also [16]). This approach leads to a $\bar{\partial}$-problem, which in turn was solved by Eskin [4].

We would like to thank Gunther Uhlmann for helpful suggestions.
For the partial data problem for the 3-dimensional elasticity-operator, we are only aware of the results of Imanuvilov–Uhlmann–Yamamoto \[6\], which operated under the assumption that \(\mu_1\) and \(\mu_2\) are constant and \(\lambda_1 = \lambda_2\) on \(\Gamma_0\).

In the present article we will use a reflection argument to extend the results of \[3, Theorem 2 and Theorem 3\] to the partial data problem (Theorem 2.1 and Theorem 2.2), which relaxes the assumptions on \(\mu_1\) and \(\mu_2\) compared to \[6\]. Furthermore, we show that the results are also true for infinite cylinders (Theorem 2.3 and Theorem 2.4).

The paper is structured as follows: In Section 2 we recall the definition of the elasticity operator and state our results. We define the Dirichlet-to-Neumann operator for bounded domains in Section 3. Section 4 is concerned with the construction of complex geometric optics solutions. In Section 5 we parametrize the complex geometric optics solutions to prove the main results in the case of bounded domains. We prove the main theorems for the case of an infinite cylinder in Section 6 by reducing them to the previous case of bounded domains. The paper ends with an appendix on the appropriate radiation conditions for the elasticity operator on an infinite cylinder.

### 2. Statement of the Problem

Let \(\Omega \subseteq \mathbb{R}^2 \times \mathbb{R}_{>0}\) be a domain with smooth boundary \(\Gamma := \partial \Omega\) and set \(\Gamma_1 := \Gamma \cap (\mathbb{R}^2 \times \mathbb{R}_{>0})\) and \(\Gamma_0 = \Gamma \setminus \overline{\Gamma_1}\). We assume that \(\Gamma_0 \neq \emptyset\) and that it has Lipschitz boundary.

![Bounded domain](image)

**Figure 1.** Bounded domain

In what follows, we will distinguish two inhomogeneous, isotropic elastic media with Lamé parameters \(\mu_j, \lambda_j \in C^\infty(\Omega), j = 1, 2\), which satisfy

\[
\mu_j(x), 3\lambda_j(x) + 2\mu_j(x) > 0 \text{ for all } x \in \overline{\Omega}.
\]

Since the proof of the main result will be based on a reflection argument, our main assumptions on the coefficients are the following:

**Assumption.** There exists an open set \(U \subset \mathbb{R}^3\) with \(\overline{\Omega} \subset U\) and functions \(\tilde{\lambda}_j, \tilde{\mu}_j \in C^\infty(U)\) such that \(\tilde{\lambda}_j|_{\overline{\Omega}} = \lambda_j\) and \(\tilde{\mu}_j|_{\overline{\Omega}} = \mu_j\) with the property that

\[
\partial_{x_3}^k \tilde{\mu}_j(x) = 0, \quad \partial_{x_3}^k \tilde{\lambda}_j(x) = 0 \quad \text{for all odd } k \in \mathbb{N}
\]

and \(x \in U \cap (\mathbb{R}^2 \times \{0\})\).
We denote by \( u : \Omega \to \mathbb{C}^3 \) the displacement field of the elastic material. The strain tensor of \( u \) is given by the matrix
\[
\varepsilon(u) := \frac{1}{2} \left( \partial_j u_k + \partial_k u_j \right)_{j,k} = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right),
\]
where \( \nabla u \) denotes the Jacobian of \( u \), and the stress is defined by the following expression
\[
\sigma^{(i)}(u) := 2\mu_i \varepsilon(u) + \lambda_i \text{div}(u) \mathbb{1}, \quad i = 1, 2.
\]
Here \( \mathbb{1} \) is the three-dimensional unit matrix. Then the corresponding Lamé operator reads as
\[
L^{(i)} u := \nabla \cdot \sigma^{(i)}(u),
\]
where we put \( (\nabla \cdot A)_j = \sum_k \partial_k A_{jk} \) for a matrix-valued function \( A = (A_{jk})_{jk} \in C^1(\Omega, \mathbb{C}^{3 \times 3}) \).

To obtain a well-posed problem, we have to impose suitable boundary conditions. The outward boundary forces are given by
\[
B^{(i)} u := \sigma^{(i)}(u)n,
\]
where \( n \) is the outward unit normal vector at the boundary \( \Gamma \). Moreover, we define by
\[
B_n^{(i)} u := \langle n, B^{(i)} u \rangle n, \quad B_T^{(i)} u := B^{(i)} u - B_n^{(i)} u,
\]
the corresponding projection in the normal direction respectively tangential direction. Likewise we define for the displacement field \( u \):
\[
u_n := \langle n, u \rangle n, \quad u_T := u - u_n.
\]
Then we will distinguish between different operators on the boundary: in addition to \( u = (u_n, u_T)^T \) and \( B^{(i)} u = (B_n^{(i)} u, B_T^{(i)} u)^T \), we define
\[
C^{(i)} u := \begin{pmatrix} B_n^{(i)} u \\ u_T \end{pmatrix}, \quad D^{(i)} u := \begin{pmatrix} -u_n \\ B_T^{(i)} u \end{pmatrix}.
\]
Then, for suitable \( g : \Gamma_1 \to \mathbb{C}^3 \), we consider the following boundary value problem
\[
\begin{cases}
L^{(i)} u = 0 \text{ in } \Omega \\
u = g \text{ on } \Gamma_1 \\
C^{(i)} u = 0 \text{ on } \Gamma_0 \quad \text{or} \quad D^{(i)} u = 0 \text{ on } \Gamma_0
\end{cases}
\]
and we define the corresponding Dirichlet-to-Neumann operator in the present case by
\[
\Lambda^{(i)} g := B^{(i)} u|_{\Gamma_1}.
\]
For the precise definition of the Dirichlet-to-Neumann operator in the case of bounded domains we refer to Section 3. We note that for infinite cylinders (see Section 6) we have to add a corresponding radiation condition such that the problem is well-posed.

Moreover, the homogeneous boundary conditions may be interpreted as follows:
\[
C^{(i)} u = 0 \text{ on } \Gamma_0 \quad \text{(Simply supported conditions)},
\]
\[
D^{(i)} u = 0 \text{ on } \Gamma_0 \quad \text{(Soft clamped boundary conditions)}.
\]
Then we obtain the following results:
Bounded domains. Let $\Omega$ be bounded. As in [3, Theorem 2] we have:

**Theorem 2.1.** Let $\Lambda^{(1)} = \Lambda^{(2)}$ and $\|f\|^2 := \int_{\Omega} e^{2\tau|x|^2} |f(x)|^2 \, dx$. Then for $\tau \gg 0$, we have
\[
\|\lambda_1 + \mu_1 - (\lambda_2 + \mu_2)\|_\tau \leq C \tau^{-1} \|\mu_1 - \mu_2\|_\tau,
\]
for a constant $C > 0$ independent of $\tau$.

Note that this implies uniqueness if we already know that either $\mu_1 = \mu_2$ or $\lambda_1 = \lambda_2$. Moreover, we have the following result, which corresponds to [3, Theorem 3].

**Theorem 2.2.** Let $b_j = \mu_j(\lambda_j + \mu_j)(2\lambda_j + 4\mu_j)^{-1}$, $j = 1, 2$, and
\[
Z := \{ \theta \in \mathbb{C}^3 \setminus \{0\} : \theta \cdot \theta = 0\}.
\]
Then for all $\theta \in Z$, we have
\[
b_1^{1/2}(\theta \cdot \partial_x)^2(b_1^{-1/2}) - b_1^{1/2}(\theta \cdot \partial_x)^2(b_1^{-1/2}) = b_2^{1/2}(\theta \cdot \partial_x)^2(b_2^{-1/2}) - b_2^{1/2}(\theta \cdot \partial_x)^2(b_2^{-1/2}).
\]

We refer to [3] for the corresponding implications, which result from the above Theorem.

Infinite cylinders. In this case we additionally assume that there exists a set $G \subseteq \mathbb{R}^2$ with smooth boundary $\partial G$ and a real number $R > 0$ such that
\[
\Omega \cap (\mathbb{R}^3 \setminus [-R, R]^3) = \mathbb{R} \times G.
\]
Moreover, there shall exist a compact set $K \subseteq \overline{\Omega}$ such that
\[
\begin{cases}
\mu_1(x) = \mu_2(x) =: \mu(x_2, x_3) & \text{for } x = (x_1, x_2, x_3) \in \overline{\Omega} \setminus K \\
\lambda_1(x) = \lambda_2(x) =: \lambda(x_2, x_3) & \text{for } x = (x_1, x_2, x_3) \in \overline{\Omega} \setminus K.
\end{cases}
\]
Then we have:

**Theorem 2.3.** Theorem 2.1 and Theorem 2.2 also hold in the case of infinite cylinders.

For $\Gamma_0 = \emptyset$ we additionally obtain:

**Theorem 2.4.** There exists a constant $\varepsilon > 0$ (depending only on $K$) such that, if $\|\nabla \mu_i\|_{C^2(K)} < \varepsilon$, then $\Lambda^{(1)} = \Lambda^{(2)}$ implies that $(\mu_1, \lambda_1) = (\mu_2, \lambda_2)$.

3. Definition of the Dirichlet-to-Neumann Operator

We briefly recall the definition of the Dirichlet-to-Neumann operator in the case that $\Omega$ is bounded. In what follows we will consider the boundary value problem
\[
\begin{aligned}
L^{(i)} u &= 0 \text{ in } \Omega \\
u &= g \text{ on } \Gamma_1 \\
C^{(i)} u &= 0 \text{ on } \Gamma_0,
\end{aligned}
\]
where assumptions on $g : \Gamma_1 \rightarrow \mathbb{C}^3$ will arise later. Note that we will only treat the case $C^{(i)} u = 0$, the case $D^{(i)} u = 0$ will follow likewise. Using Green’s formula, we obtain
\[
\int_{\Omega} \left< \sigma^{(i)}(u), \varepsilon(v) \right>_{\mathbb{C}^3 \times \mathbb{C}^3} \, dx = \int_{\Omega} L^{(i)} u \mathbf{\tau} \, dx + \int_{\Gamma} B^{(i)} u \mathbf{\tau} \, ds,
\]
where $\langle \cdot, \cdot \rangle_{C^3 \times C^3}$ denotes the standard sesquilinear product on $C^3$. We observe that the equation (7) may be rewritten as follows: If $u$ is a solution of (7), then for all $v \in H^1(\Omega; C^3)$ with $v = 0$ on $\Gamma_1$ and $v \tau = 0$ on $\Gamma_0$, we have that
\begin{equation}
\int_{\Omega} \langle \sigma^{(i)}(u), \varepsilon(v) \rangle_{C^3 \times C^3} \, dx = 0.
\end{equation}
For the sake of completeness we will consider the well-posedness of the boundary value problem (7) and recall the corresponding definition of the Sobolev spaces. Let $TT$ be the tangent bundle of $\Gamma$. By means of local charts we may define for $s \geq 0$ the corresponding Sobolev spaces $H^s(\Gamma; TT)$ and by duality the spaces $H^{-s}(\Gamma; TT) := H^s(\Gamma; TT)^*$. For a relatively open subset $U \subseteq \Gamma$ and $s \geq 0$ we put
\begin{align}
H^s(U; TU) & := \{ G|_U : G \in H^s(\Gamma; TT) \}, \\
H^s_{00}(U; TU) & := \{ g \in H^s(\Gamma; TT) : \text{supp}(g) \subseteq \overline{U} \},
\end{align}
as well as
\begin{equation}
H^{-s}(U; TU) := H^s_{00}(U; TU)^*, \quad \text{and} \quad H^{-s}_{00}(U; TU) := H^s(U; TU)^*.
\end{equation}
Note that that Formulae (10) and (11) hold for $s < 0$, after the correct interpretation of the right-hand sides, cf. [12, Theorem 3.30]. In a similar way, we may define $H^s(U; T^\perp U)$ and $H^s_{00}(U; T^\perp U)$, where $T^\perp U$ is the normal bundle, as well as $H^s(U; C^3)$, $H^s_{00}(U; C^3)$. Note that for arbitrary $s \in \mathbb{R}$,
\begin{align}
H^s(U; C^3) & = H^s(U; T^\perp U) \oplus H^s(U; TU), \\
H^s_{00}(U; C^3) & = H^s_{00}(U; T^\perp U) \oplus H^s_{00}(U; TU).
\end{align}
Considering again the boundary value problem (7) the Fredholm property reads as follows:

**Lemma 3.1.** Let $g \in H^{1/2}(\Gamma_1; T^\perp \Gamma_1) \oplus H^{1/2}_{00}(\Gamma_1; TT_1)$. Then the boundary value problem (7) is uniquely solvable if and only if corresponding homogeneous problem is uniquely solvable.

The proof is similar to the proof of [12, Theorem 4.11]. Moreover, a simple calculation shows that (7) with $g = 0$ is uniquely solvable. Hence, (7) has a unique solution $u \in H^1(\Omega; C^3)$. Next we define the corresponding Dirichlet-to-Neumann operator. We observe that if $u \in H^1(\Omega; C^3)$ with $L^{(i)}u = 0$ in $\Omega$, then we may define $B^{(i)}u \in H^{-1/2}(\Gamma; C^3)$ by Formula (8), i.e., we have for all $v \in H^1(\Omega; C^3)$ that
\begin{equation}
\langle B^{(i)}u, v|_\Gamma \rangle = \int_{\Omega} \langle \sigma^{(i)}(u), \varepsilon(v) \rangle_{C^3 \times C^3} \, dx.
\end{equation}
Finally, we set
\begin{equation}
\Lambda^{(i)}g := B^{(i)}u|_{\Gamma_1},
\end{equation}
where $u \in H^1(\Omega; C^3)$ solves (7). From the construction it is clear that the Dirichlet-to-Neumann operator $\Lambda^{(i)}$ has the following mapping properties
\begin{equation}
\Lambda^{(i)} : H^{1/2}(\Gamma_1; T^\perp \Gamma_1) \oplus H^{1/2}_{00}(\Gamma_1; TT_1) \to H^{-1/2}_{00}(\Gamma_1; T^\perp \Gamma_1) \oplus H^{-1/2}(\Gamma_1; TT_1).
\end{equation}
As a next step we consider on $H^1(\Omega, \mathbb{C}^3)$ the sesquilinear form

$$H_\Omega(u_1, u_2) := \int_\Omega \{ \lambda_2 - \lambda_1 \} \text{div}(u_1)\text{div}(u_2) + 2\{\mu_2 - \mu_1\} \langle \varepsilon(u_1), \varepsilon(u_2) \rangle_{\mathbb{C}^3 \times \mathbb{C}^3} \, dx,$$

which will be crucial in what follows. In the case that the Dirichlet-to-Neumann maps are equal we obtain the following result.

**Lemma 3.2.** If $\Lambda^{(1)} = \Lambda^{(2)}$, then we have $H_\Omega(u_1, u_2) = 0$ for all $u_1, u_2 \in H^1(\Omega; \mathbb{C}^3)$ that satisfy

$$\begin{cases}
L^{(i)} u_i = 0 & \text{in } \Omega \\
C^{(i)} u_i = 0 & \text{on } \Gamma_0.
\end{cases}$$

**Proof.** Using the previous notations we have

$$H_\Omega(u_1, u_2) = \int_\Omega \langle \sigma_2(u_1) - \sigma_1(u_1), \varepsilon(u_2) \rangle \, dx.$$ 

Let $u_1, u_2$ be given as above. Then we choose $v \in H^1(\Omega; \mathbb{C}^3)$ such that

$$\begin{cases}
L^{(2)} v = 0 & \text{in } \Omega \\
v = u_1 & \text{on } \Gamma_1 \\
C^{(2)} v = 0 & \text{on } \Gamma_0.
\end{cases}$$

As $u_1 - v = 0$ on $\Gamma_1$ and $(u_1 - v)^\tau = 0$ on $\Gamma_0$ we obtain from the variational formulation that

$$0 = \int_\Omega \langle \sigma_2(u_1 - v), \varepsilon(u_2) \rangle_{\mathbb{C}^3 \times \mathbb{C}^3} \, dx.$$ 

Now $\Lambda^{(1)} = \Lambda^{(2)}$ implies

$$\int_\Omega \langle \sigma_2(v), \varepsilon(u_2) \rangle_{\mathbb{C}^3 \times \mathbb{C}^3} \, dx = \langle B^{(2)} v, u_2 |\Gamma \rangle = \langle B^{(1)} u_1, u_2 |\Gamma \rangle = \int_\Omega \langle \sigma_1(u_1), \varepsilon(u_2) \rangle_{\mathbb{C}^3 \times \mathbb{C}^3} \, dx,$$

which proves the assertion. \hfill $\square$

### 4. Construction of Complex Geometric Optics Solutions

In what follows we will construct suitable solutions $u_i$ of $L^{(i)} u_i = 0$ in $\Omega$ and $C^{(i)} u_i = 0$ on $\Gamma_0$. The method is well-known for $\Gamma_0 = \emptyset$ however in the case of partial data, the main difficulty relies on the additional boundary condition. To this end we use the ideas in [7] and use a reflection argument along the axis $x_3 = 0$.

For $x = (x_1, x_2, x_3)$ we put $x^\vee := (x_1, x_2, -x_3)$. By the assumption on the Lamé coefficients, we may extend the functions $\mu_i$ and $\lambda_i$ evenly to $\Omega^{x_3} := \Omega \cup \Gamma_0 \cup \Omega^\vee$, where

$$\Omega^\vee := \{ x \in \mathbb{R}^3 : x^\vee \in \Omega \}.$$ 

Moreover, we define for $w \in L^2(\Omega^{x_3}; \mathbb{C}^3)$ the function

$$w^\vee(x) := w(x^\vee).$$
Thus, the functions \( \phi \mid C \) belong to \( \mathcal{H}_{1×2}(\Omega; \mathbb{C}^3) \). Let \( v \) be such that \( u \) satisfies the following statements:

In the general case \( u \) satisfies

Then the restriction \( u = w|_\Omega \) satisfies

\( \| u \|_{\mathcal{H}_{1×2}(\Omega; \mathbb{C}^3)} \leq C \). We have to show that \( C = \mathcal{H}_{1×2}(\Omega; \mathbb{C}^3) \) is a solution of the variational formulation. Let \( v \in H^1(\Omega; \mathbb{C}^3) \) with \( \nu = 0 \) on \( \Gamma_1 \) and \( \nu_r = 0 \) on \( \Gamma_0 \). We have to show that

To this end we note that we may approximate \( v = (v_1, v_2, v_3)^T \) in \( H^1(\Omega; \mathbb{C}^3) \) by functions \( \tilde{\phi}_k = (\phi_{1,k}, \phi_{2,k}, \phi_{3,k})^T \) such that \( \text{supp}(\phi_{1,k}), \text{supp}(\phi_{2,k}) \subseteq \Omega \) and \( \text{supp}(\phi_{3,k}) \subseteq \Omega \cup \Gamma_0 \).

Thus, the functions

belong to \( C^\infty_\Gamma(\Omega; \mathbb{C}^3) \) and we have

\[
\int_\Omega \langle \sigma(\nu), \varepsilon(v) \rangle_{\mathbb{C}^3 \times \mathbb{C}^3} \, dx = \lim_{k \to \infty} \frac{1}{2} \int_{\Omega \times \Omega} \langle \sigma(\nu), \varepsilon(\tilde{\phi}_k) \rangle_{\mathbb{C}^3 \times \mathbb{C}^3} \, dx = \lim_{k \to \infty} \frac{1}{2} \langle L(v), \tilde{\phi}_k \rangle = 0.
\]
This implies the assertion. \( \square \)

**Proof of Proposition 4.1.** Set \( u_i := w_i - w_i^\gamma \) on \( \Omega \). Then the functions \( u_i \) solve the boundary value problem (7) and by Lemma 3.2 we have that \( H_\Omega(u_1, u_2) = 0 \). Writing this in terms of \( w_1, w_2 \) yields

\[
0 = H_\Omega(w_1|_\Omega, w_2|_\Omega) + H_\Omega(w_1^\gamma|_\Omega, w_2^\gamma|_\Omega) - H_\Omega(w_1^\gamma|_\Omega, w_2|_\Omega) - H_\Omega(w_1^\gamma|_\Omega, w_2^\gamma|_\Omega) - H_\Omega(w_1|_\Omega, w_2^\gamma|_\Omega).
\]

The assertion follows from (12) as

\[
H_\Omega(w_1|_\Omega, w_2|_\Omega) + H_\Omega(w_1^\gamma|_\Omega, w_2^\gamma|_\Omega) = H_{\Omega \times 2}(w_1, w_2)
\]

\[
H_\Omega(w_1^\gamma|_\Omega, w_2|_\Omega) + H_\Omega(w_1|_\Omega, w_2^\gamma|_\Omega) = H_{\Omega \times 2}(w_1, w_2^\gamma).
\]

\( \square \)

Note that in general, \( H_{\Omega \times 2}(w_1, w_2) \) will not to vanish for solutions \( w_1, w_2 \) of \( L^{(i)}w_i = 0 \) on \( \Omega^{x^2} \). However we will construct a family of complex geometric optics solutions such that \( H_{\Omega \times 2}(w_1, w_2) \) vanishes asymptotically to infinite order, cf. Proposition 4.1. We briefly want to recall the construction of these functions. We note that the original construction is more involved. Following [18] for the Laplacian does not directly apply as the principal symbol of the elasticity operator

\[
\sigma^2(L) = -(\lambda + \mu)\xi\xi^T - \mu\|\xi\|^2 \mathbf{1}.
\]

is not of diagonal type. Moreover, due to the matrix structure of the operator the construction is more involved. Following [1, 3], we use the ansatz

\[
w := \mu^{-1/2}h + \mu^{-1}\nabla f - f\nabla\mu^{-1},
\]

where \( w \) satisfies \( Lw = 0 \) if

\[
M\begin{pmatrix} h \\ f \end{pmatrix} := \Delta \begin{pmatrix} h \\ f \end{pmatrix} + V_1(x) \begin{pmatrix} \nabla f \\ \nabla \cdot h \end{pmatrix} + V_0(x) \begin{pmatrix} h \\ f \end{pmatrix} = 0.
\]

Here, \( V_0 \) is a smooth function depending on \( \mu \) and \( \lambda \),

\[
V_1(x) = \begin{pmatrix} -2\mu^{1/2}\nabla^2\mu^{-1} & -\mu^{-1}\nabla\mu \\ \lambda + \mu & \lambda + \mu \end{pmatrix}.
\]

Note that the principal symbol of \( M \) equals \( -\|\xi\|^2 \cdot \mathbf{1} \), where \( \mathbf{1} \) is the identity on \( \mathbb{C}^4 \). Let

\[
Z := \{\theta \in \mathbb{C}^3 \setminus \{0\} : \theta \cdot \theta = 0\}.
\]

In what follows we construct solutions of (14) of the form \( (h, f) = e^{i\zeta x}(r, s), \zeta \in Z \).

As a first step we extend \( \mu \) and \( \lambda \) smoothly to some ball \( B_R := \{x \in \mathbb{R}^3 : |x| < R\} \), which contains \( \Omega^{x^2} \). The main assumption on the coefficients ensure that this is always possible. We denote by \( \psi \in C_\infty^\infty(\mathbb{R}^3) \) a cut-off function with \( \text{supp}(\psi) \subset B_R \) and \( \psi|_{\Omega^{x^2}} = 1 \). Then any solution \( (r, s) \) of the differential equation

\[
0 = M_\zeta\begin{pmatrix} r \\ s \end{pmatrix} := \Delta_\zeta\begin{pmatrix} r \\ s \end{pmatrix} + \psi(x)V_1(x)\begin{pmatrix} i\zeta s + \nabla s \\ i\zeta r + \text{div}(r) \end{pmatrix} + \psi(x)V_0(x)\begin{pmatrix} r \\ s \end{pmatrix},
\]

satisfies (14).
on $\Omega \times 2$ will give a solution $(h, f)$ of (14). Here $\Delta \zeta = \Delta + 2i\zeta \cdot \partial_x$. Choose $\theta \in \mathbb{Z}$ and $\ell_0 \in \mathbb{R}^3$ such that $\langle \ell_0, \theta \rangle = 0$. We define

$$
\zeta(\tau) := \frac{1}{2} \ell_0 + \tau \theta + \tau \rho(\tau) \Re \theta,
$$

where

$$(17) \quad \rho(\tau) := \left(1 - \frac{|\ell_0|^2}{4\tau^2}\right)^{1/2} - 1 = -\frac{|\ell_0|^2}{4\tau^2} + \mathcal{O}(\tau^{-4}).$$

For the sake of simplicity we often write $\zeta$ instead of $\zeta(\tau)$. The remaining steps are well-known, see e.g. [3]. We define $A : B_R \to (\mathbb{R}^{4\times4})^3$ by

$$A(x) \cdot \zeta \begin{pmatrix} r \\ s \end{pmatrix} := \psi(x) V_1(x) \begin{pmatrix} \zeta_s \\ \zeta \cdot r \end{pmatrix}$$

and let

$$\ell(\tau) := 2(\zeta(\tau) - \tau \theta) = \ell_0 + 2\tau \rho(\tau) \Re \theta.$$ 

Note that

$$M_\zeta = M + i(2\zeta \cdot \partial_x + A \cdot \zeta) = M_\frac{1}{2}\ell(\tau) + i\tau(2\theta \cdot \partial_x + A \cdot \theta).$$

To find a solution of $M_\zeta v = 0$, we make the formal ansatz

$$v = v_0 + v_1 + \ldots + v_n + \ldots,$$

and seek to construct $v_n$ such that $\|v_n\|_{H^k(B_R; \mathbb{C}^4)} = \mathcal{O}(\tau^{-n})$ as $\tau \to \infty$. To achieve this we look for solutions to

$$i\tau(2\theta \cdot \partial_x + A \cdot \theta)v_0 = 0 \quad (18)$$

$$i\tau(2\theta \cdot \partial_x + A \cdot \theta)v_n = -\psi M_\frac{1}{2}\ell(\tau)v_{n-1}, \quad n \geq 1. \quad (19)$$

From [4, Theorem 2.1] we have:

**Lemma 4.3.** There exists a matrix $C_0 = C_0(x, \theta)$ depending smoothly on $x$ and $\theta$, which satisfies

$$(20) \quad 2(\theta \cdot \partial_x)C_0 + A \cdot \theta C_0 = 0.$$ 

Moreover, we may choose $C_0$ such that it is invertible for all $(x, \theta) \in B_r \times Z$ and $C_0$ is homogeneous of degree 0 in $\theta$.

In particular, if $g(z)$ is any vector of polynomials in the complex variable $z$, then we may choose

$$v_0 = C_0(x, \theta)g(x \cdot \theta).$$

Note that the operator $\theta \cdot \partial_x$ is closely related to the $\bar{\partial}$-operator. In fact, for $\theta = (1, i, 0)^T$, we have that $\theta \cdot \partial_x = \partial_x + i\partial_{x_2}$. Denote by $\hat{f}$ the Fourier transform of $f$ and let

$$\textbf{\Pi}_\theta f(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix\eta} \hat{f}(\eta) i\eta \cdot \theta d\theta.$$
be the Fourier multiplier with symbol \( \eta \mapsto (i\eta \cdot \theta)^{-1} \). Then \( \Pi_\theta \) is the inverse of the differential operator \( \theta \cdot \partial_x \), i.e.

\[
\Pi_\theta (\theta \cdot \partial_x f)(x) = f(x),
\]

\[
\theta \cdot \partial_x (\Pi_\theta g)(x) = g(x)
\]

for \( f, g \in C^\infty_c(\mathbb{R}^3) \). Thus, we may put

\[
v_n := -(i\tau)^{-1} C_0(x, \theta) \Pi_\theta \left( C_0^{-1}(x, \theta) \psi(x) M_{\frac{1}{2}e(\tau)} v_{n-1} \right)
\]

and \( v_n \) will satisfy (19). In particular we obtain

\[
M_\zeta(v_0 + \ldots + v_n) = \left\{ M\frac{1}{2}e(\tau)^{-1} + i\tau(2\theta \cdot \partial_x + A \cdot \theta) \right\}(v_0 + \ldots v_n) + M'_\zeta v_n
\]

(21)

Note that \( 1 - \psi \) vanishes on \( \Omega \times 2 \). Moreover, comparing the orders in \( \tau \) and using that \( \Pi_\theta : H^{s}_{\text{comp}}(\mathbb{R}^3) \rightarrow H^{s}_{\text{loc}}(\mathbb{R}^3) \),

(22)

we obtain for any \( k \in \mathbb{N} \) that

\[
\|v_n\|_{H^k(B_R; \mathbb{C}^4)} = O(\tau^{-n}) \quad \text{as} \quad \tau \to \infty.
\]

Then we obtain as in [3] (cf. also [4, Sect. 3] and [16]):

**Lemma 4.4.** For large \( \tau > 0 \) there exists \( v_e \in H^k(B_R; \mathbb{C}^4) \) which satisfies

\[
M_\zeta v_e = -\psi M\frac{1}{2}e(\tau) v_n \quad \text{in} \quad B_R
\]

and \( \|v_e\|_{H^k(B_R; \mathbb{C}^4)} = O(\tau^{-n-1}) \) as \( \tau \to \infty \).

Let \( v_e \) be chosen as above. Setting

\[
v := v_0 + v_1 + \ldots + v_n + v_e
\]

we obtain that

\[
M_\zeta v = (1 - \psi) \sum_{i=0}^{n} M\frac{1}{2}e(\tau) v_i,
\]

and in particular \( M_\zeta v = 0 \) in \( \Omega \times 2 \). As a next step we use these complex geometric optics solutions to extract information from Proposition 4.1. To this end we choose

\[
\zeta(\tau) := \frac{1}{2} \ell_0 + \tau \theta + \tau \rho(\tau) \Re \theta,
\]

and

\[
\zeta'(\tau) := -\frac{1}{2} \ell_0 + \tau \theta + \tau \rho(\tau) \Re \theta.
\]

Then we obtain solutions \( v^{(i)} := (r^{(i)}, s^{(i)})^T \in C^\infty(\Omega \times 2; \mathbb{C}^4) \) such that

\[
M^{(1)}_\zeta v^{(1)} = 0 \quad \text{and} \quad M^{(2)}_\zeta v^{(2)} = 0 \quad \text{in} \quad \Omega \times 2.
\]
Here \( M_\gamma^{(i)} \), \( \gamma \in \mathbb{C}^3 \), \( i = 1, 2 \), is the corresponding operator corresponding to the pair \((\lambda^{(i)}, \mu^{(i)})\). Let

\[
\begin{align*}
    w_1 &= \mu_1^{-1/2} e^{i\gamma x} \mathcal{P}_1(1) + \mu_1^{-1} \nabla \left( e^{i\gamma x} s(1) \right) - \left( e^{i\gamma' x} s(1) \right) \nabla \mu_2^{-1}, \\
    w_2 &= \mu_2^{-1/2} e^{i\gamma' x} \mathcal{P}_2(2) + \mu_2^{-1} \nabla \left( e^{i\gamma' x} s(2) \right) - \left( e^{i\gamma' x} s(2) \right) \nabla \mu_2^{-1}.
\end{align*}
\]

Then \( L^{(i)}w_1 = 0 \) in \( \Omega \times 2 \) and by Proposition 4.1, we have

\[
H_{\Omega \times 2}(w_1, w_2) = H_{\Omega \times 2}(w_1, w_2').
\]

If we set

\[
Z_0 = \{ \theta \in Z : \text{Im} \theta_3 = 0 \},
\]

then we obtain the following result:

**Lemma 4.5.** Let \( \theta \in Z_0 \) and \( \ell_0 \in \mathbb{R}^3 \) such that \( \langle \theta, \ell_0 \rangle = 0 \) and \( \text{Re} \theta_3 \neq 0 \). Then

\[
H_{\Omega \times 2}(w_1, w_2) = \mathcal{O}(\tau^{-\infty}), \quad \text{as } \tau \to \infty.
\]

**Proof.** By the previous considerations, it suffices to show that

\[
H_{\Omega \times 2}(w_1, w_2') = \mathcal{O}(\tau^{-\infty}).
\]

Consider the functions \( p_1, q_1 \) defined by

\[
\begin{align*}
    p_1 &= e^{-i\gamma x} \text{div}(w_1), \\
    q_1 &= e^{-i\gamma x} e(w_1), \\
    p_2 &= e^{-i\gamma' x} \text{div}(w_2), \\
    q_2 &= e^{i\gamma' x} e(w_2).
\end{align*}
\]

From (23) and Lemma 4.4, we obtain for any \( k \in \mathbb{N} \) that

\[
\|p_1 p_2\|_{H^k(\Omega \times 2; \mathbb{C})} = \mathcal{O}(\tau^4), \quad \|q_1 q_2\|_{C(\Omega \times 2; \mathbb{C})} = \mathcal{O}(\tau^4).
\]

Putting

\[
F(x) := \{ \lambda_2 - \lambda_1 \} p_1(x) p_2(x') + 2 \{ \mu_2 - \mu_1 \} q_1(x) J q_2(x') J C_3 x C_3,
\]

we have

\[
H_{\Omega \times 2}(w_1, w_2') = \int_{\Omega \times 2} e^{i(\gamma - \gamma') x} F(x) \, dx.
\]

Thus, by partial integration for \( \Omega \) and \( \Omega' \) we obtain

\[
H_{\Omega \times 2}(w_1, w_2') = -\frac{1}{2i(\text{Re} \theta_3 + \mathcal{O}(1))} \int_{\Omega \times 2} e^{i(\gamma - \gamma') x} \partial_3 F(x) \, dx
\]

\[
+ \int_{\partial \Omega} e^{i(\gamma - \gamma') x} n_3 \partial_3 F(x) \, dx + \int_{\partial_3 (\Omega')} e^{i(\gamma - \gamma') x} n_3 \partial_3 F(x) \, dx.
\]

Using [15] we have that \( \mu_1 = \mu_2 \) and \( \lambda_1 = \lambda_2 \) on \( \Gamma_1 \) to infinite order. Then the symmetry assumptions imply that the boundary integrals vanish. By induction we have

\[
H_{\Omega \times 2}(w_1, w_2') = \left( \frac{1}{2i(\text{Re} \theta_3 + \mathcal{O}(1))} \right)^n \int_{\mathbb{R}^3} e^{i(\gamma - \gamma') x} \partial_3^n F(x) \, dx = \mathcal{O}(\tau^{-\infty}).
\]

\[ \square \]
5. Proof of Theorem 2.1 and Theorem 2.2

5.1. The Leading Order Term of the Form $H$. As a next step we want to apply the results by Eskin and Ralston. To this end we set
\[ a_i := (\theta \cdot \partial_x)^2 \mu_i \mu_i^{-1}, \quad b_i := \frac{\mu_i}{2} \cdot \frac{\lambda_i + \mu_i}{\lambda_i + 2\mu_i} \]
as well as
\[
\begin{pmatrix}
R_0^{(i)} \\
S_0^{(i)}
\end{pmatrix}(x, \theta) := \begin{pmatrix}
\mu_i^{-1} \theta^T & 0 \\
0 & 1
\end{pmatrix} C_0^{(i)}(x, \theta) g_i(x \cdot \theta), \quad i = 1, 2.
\]

Here $C_0^{(i)}(x, \theta)$ is chosen as in Lemma 4.3 and $g_i(z)$ is a vector of polynomials in $z$. By the same calculation as in [3], we obtain:

Lemma 5.1 (cf. [3, Lemma 2.1]). We have $H_{\Omega \times 2}(w_1, w_2) = \tau^2 H_2 + O(\tau)$, where
\[
H_2 = \int_{\Omega \times 2} e^{i\theta_0 \cdot x} \left( R_0^{(2)}(x, \theta), S_0^{(2)}(x, \theta) \right) V(x, \theta) \begin{pmatrix} R_0^{(1)} \n S_0^{(1)} \end{pmatrix}(x, \theta) \, dx,
\]
and
\[
V(x, \theta) := \left( \frac{(\lambda_1 + \mu_1 - (\lambda_2 + \mu_2))}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)} \mu_1 \mu_2}{2(\mu_2 \mu_1 - \mu_1 \mu_2)(\theta \cdot \partial_x) b_1}{2(\mu_2 - \mu_1)(b_1 a_1 + b_2 a_2)} \right).
\]

Using Lemma 4.5 we immediately obtain:

Corollary 5.2. Let $\theta \in Z_0$ and $\ell_0 \in \mathbb{R}^3$ such that $\langle \theta, \ell_0 \rangle = 0$. If $\text{Re}(\theta_0) \neq 0$, then
\[
0 = \int_{\Omega \times 2} e^{i\theta_0 \cdot x} \left( R_0^{(2)}, S_0^{(2)} \right)(x, \theta) V(x, \theta) \begin{pmatrix} R_0^{(1)} \n S_0^{(1)} \end{pmatrix}(x, \theta) \, dx.
\]

By continuity, the assertion also holds true for $\text{Re}(\theta_3) = 0$. The remaining part of the proof may be deduced as in [3], taking into account that we have to assume that $\text{Im}(\theta_3) = 0$. For the sake of completeness we want to recall the main steps. We note that $(R_0^{(i)}, S_0^{(i)})^T$ may be written as
\[
\begin{pmatrix}
R_0^{(i)} \\
S_0^{(i)}
\end{pmatrix}(x, \theta) = \tilde{C}^{(i)}(x, \theta) \tilde{g}_i(x \cdot \theta),
\]
where $\tilde{g}_i(z)$ is a 2-vector of polynomials and $\tilde{C}^{(i)} = \tilde{C}^{(i)}(x, \theta)$ satisfies the equation the equation
\[
(\theta \cdot \partial_x) \tilde{C}^{(i)} = \tilde{A}^{(i)} \tilde{C}^{(i)}.
\] Here $\tilde{A}^{(i)} = \tilde{A}^{(i)}(x, \theta)$ is given by
\[
\tilde{A}^{(i)} = \begin{pmatrix} 0 & -a_i \\
 b_i & 0 \end{pmatrix}.
\]

Again, we apply [4, Theorem 2.1] to obtain a solution $\tilde{C}^{(i)}(x, \theta)$ of (27) defined for $|\text{Re} \theta| = |\text{Im} \theta| = 1$ and that is invertible for $x < |R|$. 

\[ \]
We observe that we can smoothly extend $\tilde{C}^{(i)}$ to $\theta \in Z$ by requiring the homogeneity
\[
\tilde{C}^{(i)}(x, r\theta) = \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} \tilde{C}^{(i)}(x, \theta), \quad r > 0.
\]

Then we may consider in Equation (25) solutions of (27) instead of (20) and we obtain:

**Proposition 5.3** ([3, Sect. 3]). *Let $\tilde{C}^{(i)}(x, \theta)$ be a solution of (27), which is invertible for $|x| < R$ and let $\tilde{g}_i$ be any 2-vectors of polynomials. Then we have
\[
0 = \int_{\Omega^2} e^{i\ell_0 \cdot x} \tilde{g}_2(\theta \cdot x) (\tilde{C}^{(2)})^*(x, \theta) V(x, \theta) \tilde{C}^{(1)}(x, \theta) \tilde{g}(x, \theta) dx,
\]
where $\theta \in Z_0$, $\ell_0 \in \mathbb{R}^3$ are chosen such that $\langle \theta, \ell_0 \rangle = 0$.*

5.2. **Parametrizing the Complex Geometric Optics Solutions.** Since
\[
\left( \begin{array}{c|c|c} \Re \theta & \Im \theta & \ell_0 \\ \hline \Re \theta & \Im \theta & |\ell_0| \end{array} \right)
\]
forms an orthonormal basis, we change the coordinates into this basis. To this end, write $y = (y_1, y_2, y_3)$ with $y_1 = (\Re \theta / |\Re \theta|) \cdot x$, $y_2 = (\Im \theta / |\Im \theta|) \cdot x$, and $y_3 = (\ell_0 / |\ell_0|) \cdot x$.

We extend $V(\cdot, \theta)$ by zero to $\mathbb{R}^3$ and by the assumptions on the Lamé parameters $\lambda_j$ and $\mu_j$, this extension is smooth. We obtain from Proposition 5.3 by applying the Fourier transform that for any 2-vectors of polynomials $g_1, g_2$ and $\theta \in Z_0$,
\[
0 = \int_{\mathbb{R}^2} g_2(\theta \cdot y) (\tilde{C}_2)^*(y, \bar{\theta}) V(y, \theta) \tilde{C}_1(y, \theta) g_1(\theta \cdot y) dy_1 dy_2, \quad y_3 \in \mathbb{R}.
\]

Recall that $\Pi_0$ is the Fourier multiplier with $\eta \mapsto (i \eta \cdot \theta)^{-1}$. Set
\[
B_0(y, \theta) = (\tilde{C}_2)^*(y, \bar{\theta}) V(y, \theta) \tilde{C}_1(y, \theta),
\]
\[
B(y, \theta) = \Pi_0 B_0(y, \theta),
\]
\[
B'(y, \theta) = ((\tilde{C}_2)^*(y, \bar{\theta}))^{-1} B(y, \theta) (\tilde{C}_1(y, \theta))^{-1}.
\]

The reason for defining $B'$ is that it has a particular simple form:

**Proposition 5.4.** *There exist functions $b_{12}, b_{21}, b_{22}$ on $B_R$ such that for all $\theta \in Z_0$,
\[
B'(x, \theta) = \begin{pmatrix} 0 & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \cdot \theta \end{pmatrix}.
\]

The main result of this section is the following:

**Proposition 5.5.** *For all $\theta \in Z$ and $x \in \Omega$, we have the following set of equations
\[
\left\{ \begin{array}{l}
-b_2 b_{21} - b_1 b_{12} = v_{11} \\
\theta \cdot \partial_x b_{12} - b_2 b_{22} \cdot \theta = v_{12} \\
\theta \cdot \partial_x b_{21} - b_1 b_{22} \cdot \theta = v_{21} \\
\theta \cdot \partial_x b_{22} + b_2 b_{12} + b_1 b_{21} = v_{22},
\end{array} \right.
\]
where the $v_{jk}$ are the entries of the matrix $V = V(x, \theta)$ as in Lemma 5.1.*
As in [3], we have to parametrize the space $Z_0$ to extract information from equation (28). Let
\[\xi(t) = ((1/2)(t - t^{-1}), (i/2)(t + t^{-1}), 1)^T.\]

Then $Z_0 = \{r\xi(t) : r \in \mathbb{R}_+, t \in \mathbb{C} \setminus \{0\}\}$.

From the homogeneity of $V$ and $C$, we can calculate the homogeneity of the matrix entries of $B'$. We obtain:

**Lemma 5.6.** The coefficients $B'_{ij}(x, \theta)$ are positive homogeneous of degree $i + j - 3$ in $\theta$,
\[B'_{ij}(x, r \theta) = r^{i+j-3}B'_{ij}(x, \theta), \quad r > 0.\]

**Lemma 5.7.** $\tilde{\partial}_t B'(y, r\xi(t)) = 0$ for $t \in \mathbb{C} \setminus \{0\}$.

**Proof.** By homogeneity, we may assume that $r = 1$. We begin by calculating $\tilde{\partial}_t B_0(y, \xi(t))$.

For this, we set
\[M_1(y, t) = \tilde{C}^{(1)}(y, \xi(t))^{-1}\tilde{\partial}_t \tilde{C}^{(1)}(y, \xi(t)),\]
\[M_2(y, t) = (\tilde{C}^{(2)}(y, \xi(t))^{-1})^* \tilde{\partial}_t (\tilde{C}^{(2)}(y, \xi(t))^*),\]
then obtain
\[
\tilde{\partial}_t B_0(y, \xi(t)) = \tilde{\partial}_t (\tilde{C}^{(2)}(y, \xi(t))^*) V(y, \xi(t)) \tilde{C}^{(1)}(y, \xi(t)) \\
+ \tilde{C}^{(2)}(y, \xi(t))^* \tilde{\partial}_t V(y, \xi(t)) \tilde{C}^{(1)}(y, \xi(t)) \\
+ \tilde{C}^{(2)}(y, \xi(t))^* V(y, \xi(t)) \tilde{\partial}_t \tilde{C}^{(1)}(y, \xi(t)) \\
= M_2(y, t) B_0(y, \xi(t)) + B_0(y, \xi(t))^* M_1(y, t).
\]

The second summand vanishes because $V(y, \xi(t))$ is holomorphic in $t \in \mathbb{C} \setminus \{0\}$.

If we choose $g_1$ and $g_2$ the basis vectors, then it follows from (28) that
\[
\int_{\mathbb{R}^2} B_0(y, \theta) dy_1 dy_2 = 0, \quad y_1 \in \mathbb{R}
\]
and together with Lemma 6.1 from Eskin [4] this implies that $(\tilde{\partial}_t P_{\xi(t)}) B_0(y, \xi(t)) = 0$ for $t \in \mathbb{C} \setminus \{0\}$ and hence
\[
\tilde{\partial}_t B(y, \xi(t)) = (\tilde{\partial}_t P_{\xi(t)}) B_0(y, \xi(t)) + P_{\xi(t)} (\tilde{\partial}_t B_0(y, \xi(t)) \\
= P_{\xi(t)} (M_2(y, t) B_0(y, \xi(t)) + B_0(y, \xi(t))^* M_1(y, t)).
\]

Applying Lemma 6.1 again, we have that $M_2(y, t) P_{\xi(t)} B_0(y, \xi(t)) = P_{\xi(t)} M_2(y, t) B_0(x, \xi(t))$ and thus
\[
\tilde{\partial}_t B(y, \xi(t)) = M_2(y, t) B(x, \xi(t)) + B(y, \xi(t))^* M_1(y, t).
\]
Finally, we calculate
\[
\tilde{\partial}_t B'(y, \xi(t)) = \tilde{\partial}_t \left( \left( \tilde{\partial}^{(2)}(y, \xi(t)) \right)^{-1} B(y, \xi(t)) \tilde{\partial}^{(1)}(y, \xi(t)) \right)^{-1}
\]
\[
= \tilde{\partial}_t \left( \left( \tilde{\partial}^{(2)}(y, \xi(t)) \right)^{-1} B(y, \xi(t)) \tilde{\partial}^{(1)}(y, \xi(t)) \right)^{-1}
+ \left( \left( \tilde{\partial}^{(2)}(y, \xi(t)) \right)^{-1} \tilde{\partial}_t B(y, \xi(t)) \tilde{\partial}^{(1)}(y, \xi(t)) \right)^{-1}
+ \left( \left( \tilde{\partial}^{(2)}(y, \xi(t)) \right)^{-1} \tilde{\partial}_t \tilde{\partial}^{(1)}(y, \xi(t)) \right)^{-1}
\]
\[
= -\left( \left( \tilde{\partial}^{(2)}(y, \xi(t)) \right)^{-1} M_2(x, t) B(y, \xi(t)) \tilde{\partial}^{(1)}(y, \xi(t)) \right)^{-1}
+ \left( \left( \tilde{\partial}^{(2)}(y, \xi(t)) \right)^{-1} M_2(y, t) B(y, \xi(t)) + B(y, \xi(t)) M_1(y, t) \tilde{\partial}^{(1)}(y, \xi(t)) \right)^{-1}
- \left( \left( \tilde{\partial}^{(2)}(y, \xi(t)) \right)^{-1} B(y, \xi(t)) M_1(y, t) \tilde{\partial}^{(1)}(y, \xi(t)) \right)^{-1}
\]
\[
= 0.
\]
\[\square\]

**Proof of Proposition 5.4.** Write
\[
B'(x, \theta) = \begin{pmatrix} B'_{11}(x, \theta) & B'_{12}(x, \theta) \\ B'_{21}(x, \theta) & B'_{22}(x, \theta) \end{pmatrix}.
\]

For \(i, j \in \{1, 2\}\), we have by continuity of \(B'(x, \theta)\) that
\[
\sup_{|\theta|=1} |B'_{ij}(x, \theta)| < \infty
\]
for \(\theta \in Z_0\).

First, we consider the entries \(B'_{12}\) and \(B'_{21}\). From Lemma 5.6 it follows that
\[
|B'_{12}(x, \xi(t))| = |B'_{12}(x, \xi(t))/|\xi(t)|| < C < \infty,
\]
\[
|B'_{21}(x, \xi(t))| = |B'_{21}(x, \xi(t))/|\xi(t)|| < C' < \infty.
\]

By Lemma 5.7 these are therefore bounded entire functions in \(t\), hence Liouville’s theorem implies that \(B'_{12}\) and \(B'_{21}\) are constant in \(\xi(t)\).

For \(B'_{11}\) we have
\[
|B'_{11}(x, \xi(t))| = |B'_{11}(x, \xi(t))/|\xi(t)|| \cdot |\xi(t)|^{-1}.
\]

Therefore, \(B'_{11}(x, \xi(t))\) is entire and tends to 0 for \(|t| \to \infty\) and \(|t| \to 0\). This implies that \(B'_{11}(x, \xi(t)) = 0\).

By the same arguments, we obtain for \(B'_{22}(x, \xi(t))\) that
\[
B'_{22}(x, \xi(t)) = B'_{22,-1}(x)t^{-1} + B'_{22,0}(x) + B'_{22,1}(x)t.
\]

Setting
\[
b'_{22}(x) = \begin{pmatrix} B'_{22,1}(x) - B'_{22,-1}(x) \\ -i(B'_{22,1}(x) + B'_{22,-1}(x)) \\ B'_{22,0}(x) \end{pmatrix},
\]
we obtain \(B'(x, \theta) = b_{22}(x) \cdot \theta\).
\[\square\]
Proof of Proposition 5.5. Applying $\theta \cdot \partial_x$ to (31), we have that

$$\theta \cdot \partial_x B'(x, \theta) + (A^{(2)})^* B'(x, \theta) + B'(x, \theta) A^{(1)} = V(x, \theta).$$

Using Proposition 5.4 implies the claim for $x \in B_R$ and $\theta \in \mathbb{Z}_0$.

In the case that $\text{Im} \theta_3 \neq 0$, we observe that the first equality is independent of $\theta$, the second and third are linear in $\theta$ and the last one is quadratic in $\theta$. Hence, we can multiply by a complex number to reduce the case $\text{Im} \theta_3 \neq 0$ to $\text{Im} \theta_3 = 0$. □

6. The Case of Infinite Cylinders

Let as before $\Omega \subseteq \mathbb{R}^2 \times \mathbb{R}_{>0}$. We recall the corresponding assumptions. There exists $G \subseteq \mathbb{R}^2$

$$\Omega \cap (\mathbb{R}^3 \setminus [-R, R]^3) = (\mathbb{R} \setminus [-R, R]) \times G$$

and we suppose that there exists a compact set $K \subseteq \overline{\Omega}$ such that

$$\begin{cases}
\mu_1(x) = \mu_2(x) =: \mu(x_2, x_3) & \text{for } x = (x_1, x_2, x_3) \in \overline{\Omega} \setminus K \\
\lambda_1(x) = \lambda_2(x) =: \lambda(x_2, x_3) & \text{for } x = (x_1, x_2, x_3) \in \overline{\Omega} \setminus K.
\end{cases}$$

Moreover, for the sake of simplicity, we additionally assume that $\Gamma_1 = \partial \Omega \cap (\mathbb{R}^2 \times \{0\})$ is bounded.

![Figure 2. The set-up](image)

Let $k \in \mathbb{R}$ be fixed. We consider the boundary value problem

$$\begin{cases}
(L^{(i)} - k^2)u = f & \text{in } \Omega \\
u = g & \text{on } \Gamma_1 \\
C^{(i)}u = 0 & \text{on } \Gamma_0,
\end{cases}$$

for suitable $f : \Omega \to \mathbb{C}^3$ and $g : \Gamma_1 \to \mathbb{C}^3$. In order to ensure unique solvability, we have to impose the correct radiation conditions and consider exponentially weighted Sobolev spaces. We define for $\beta \in \mathbb{R}$ and $s \in \mathbb{R}$ the spaces

$$H^s_\beta(\Omega; \mathbb{C}^3) := \left\{ f \in H^s_{\text{loc}}(\Omega; \mathbb{C}^3) : e^{-\beta |x_1|} \chi(x_1)f(x) \in H^s(\Omega; \mathbb{C}^3) \right\},$$

$$H^s_\beta(\partial \Omega; \mathbb{C}^3) := \left\{ g \in H^s_{\text{loc}}(\partial \Omega; \mathbb{C}^3) : e^{-\beta |x_1|} \chi(x_1)g(x) \in H^s(\partial \Omega; \mathbb{C}^3) \right\}.$$
Here $\chi \in C^\infty(\mathbb{R})$ shall satisfy $\chi = 0$ in some neighbourhood of 0 and $\chi = 1$ outside some compact set. For $s = 0$ we write $H^0_\beta(\Omega; C^3) = L^2_\beta(\Omega; C^3)$. Note that

$$H^s_\beta(\partial \Omega; C^3) = H^{-s}_\beta(\partial \Omega; C^3)^*.$$ 

Without any major effort we may generalise the definitions in Section 3 and we obtain the spaces

$$H^s_\beta(\Gamma_0; TT_0), H^s_{00,\beta}(\Gamma_0, TT_0) \quad \text{and} \quad H^s_\beta(\Gamma_0; T^1 \Gamma_0), H^s_{00,\beta}(\Gamma_0, T^1 \Gamma_0).$$

Now choose $\beta > 0$ sufficiently small. A function $u \in H^1_\beta(\Omega; C^3)$ will be called outgoing if it satisfies a given asymptotic behaviour on the cylindrical ends. More precisely there exist functions $U_{j,+} \in H^2_{12}(\Omega; C^3)$, $j = 1, \ldots, n$ such that

$$(L(i) - k^2)U_{j,+} = 0, \quad \text{in } \Omega \setminus \tilde{K}$$

$$U_{j,+} = 0, \quad \text{on } \partial \Omega,$$

for a suitable compact set $\tilde{K} \subseteq \overline{\Omega}$ and $i = 1, 2$. By definition we assume for an outgoing function $H^1_\beta(\Omega; C^3)$ to have the following asymptotic behaviour

$$u - \sum_{j=1}^n c_j U_{j,+} \in H^1_{-\beta}(\Omega; C^3)$$

for coefficients $c_j \in \mathbb{C}$, $j = 1, \ldots, n$. A similar assertion holds true for incoming functions, where we will replace $U_{j,+}$ by incoming waves $U_{j,-}$. The precise definition is given in the appendix.

In what follows we assume that the boundary value problem (33) with $f = 0$ and $g = 0$ has a unique outgoing solution and a unique incoming solution, namely $u = 0$. Then we obtain the following result:

**Lemma 6.1.** Let $f \in H^1_\beta(\Omega; C^3)^*$ and $g \in H^{1/2}_{-\beta}(\Gamma_1; T^1 \Gamma_1) \oplus H^{1/2}_{00,-\beta}(\Gamma_1; TT_1)$. Then the boundary value problem (33) has a unique outgoing and a unique incoming solution in $H^1_\beta(\Omega; C^3)$.

The proof follows from the above assumption together with ideas in the appendix. Here we note that the boundary value problem will always be considered in its variational form. In particular, for $g = (g_n, g_r)$ the solution $u$ of (33) shall satisfy

$$\begin{cases}
  u_n = g_n & \text{on } \Gamma_1 \\
  u_r = g_r & \text{on } \Gamma_1 \\
  u_r = 0 & \text{on } \Gamma_0,
\end{cases}$$

as well as

$$\int_\Omega (\sigma^{(i)}(u), \varepsilon(v))_{C^3 \times C^3} \, dx = k^2 \int_\Omega u \, \bar{v} \, dx + \int_\Omega f \, \bar{v} \, dx,$$

for every $v \in H^{1/2}_{-\beta}(\Omega; C^3)$ such that $v = 0$ on $\Gamma_1$ and $v_r = 0$ on $\Gamma_0$.

Next we introduce the corresponding Dirichlet-to-Neumann operator. To this end let $g \in H^{1/2}_{-\beta}(\Gamma_1; T^1 \Gamma_1) \oplus H^{1/2}_{00,-\beta}(\Gamma_1; TT_1)$ and $f = 0$. Let $u \in H^1_\beta(\Omega; C^3)$ be the unique outgoing solution of the associated boundary value problem of (33). The weak normal
derivative of $u$, $B^{(i)}u \in H^{-1/2}_\beta(\partial \Omega; C^3)$, is again defined by Green’s formula (8), i.e., we have for all $v \in H^1_{-\beta}(\Gamma; C^3)$ that

$$\langle B^{(i)}u, v \rangle_{\Gamma} = \int_\Omega \langle \sigma_i(u), \varepsilon(v) \rangle_{C^3 \times C^3} dx - k^2 \int_\Omega u \overline{v} \, dx.$$  

We define as above

$$\Lambda^{(i)} g := B^{(i)}u|\Gamma_1$$

and obtain an operator

$$\Lambda^{(i)} : H^{-1/2}_{\beta}(\Gamma_1; T^\perp \Gamma_1) \oplus H^{1/2}_{00,-\beta}(\Gamma_1; TT_1) \to H^{-1/2}_{00,\beta}(\Gamma_1; T^\perp \Gamma_1) \oplus H^{-1/2}_{\beta}(\Gamma_1; TT_1).$$

As a next step we consider again the quadratic form

$$H_{\Omega}(u_1, u_2) := \int_\Omega \{ \lambda_2 - \lambda_1 \} \text{div}(u_1)\overline{\text{div}(u_2)} + 2\{ \mu_2 - \mu_1 \} \langle \varepsilon(u_1), \varepsilon(u_2) \rangle_{C^3 \times C^3} \, dx.$$  

Note that the $\mu_i$ and $\lambda_i$ differ only on the compact set $K$, and thus, we have

$$H_{\Omega}(u_1, u_2) = H_K(u_1|K, u_2|K) = \int_K \{ \lambda_2 - \lambda_1 \} \text{div}(u_1)\overline{\text{div}(u_2)} + 2\{ \mu_2 - \mu_1 \} \langle \varepsilon(u_1), \varepsilon(u_2) \rangle_{C^3 \times C^3} \, dx.$$  

The following theorem permits us to apply directly the results of the previous sections. Then Theorem 2.3 follows directly. Furthermore, 2.4 follows as in [3].

**Theorem 6.2.** If the Dirichlet-to-Neumann operators are equal, $\Lambda^{(1)} = \Lambda^{(2)}$, then we have $H_K(u_1, u_2) = 0$ for solutions $u_i \in H^1(K; C^3)$ of

$$\begin{cases} 
(L^{(i)} - k^2)u_i = 0 \text{ in } K \\
C^{(i)}u_i = 0 \text{ on } \Gamma_0.
\end{cases}$$

(37)

The remaining part of the section is devoted to the proof of Theorem 6.2. Its proof is mainly based on a corresponding Runge approximation theorem.

**Theorem 6.3.** Let $u \in H^1(K; C^3)$ be solution of (37) with $i = 1$. Then for every $\varepsilon > 0$ there exists an outgoing solution $u_\varepsilon \in H^1_\beta(\Omega; C^3)$ of

$$\begin{cases} 
(L^{(1)} - k^2)u_\varepsilon = 0 \text{ in } \Omega \\
C^{(1)}u_\varepsilon = 0 \text{ on } \Gamma_0,
\end{cases}$$

(38)

such that

$$\|u - u_\varepsilon\|_{H^1(K; C^3)} < \varepsilon.$$

**Proof.** This follows from a Hahn-Banach type argument as in [11, Lemma 3.3]. Let $f \in H^1(K; C^3)^*$ be an arbitrary functional with $f(u|K) = 0$ for all outgoing $u \in H^1_\beta(\Omega; C^3)$ that satisfy (38). The assertion follows from the Hahn-Banach theorem, if we show that $f(v) = 0$ for all $v \in H^1(K; C^3)$ such that $(L^{(1)} - k^2)v = 0$ in $\Omega$ and $C^{(1)}v = 0$ on $\Gamma_0$. 
To this end we extend $f$ to a functional $\tilde{f} \in H^1_\beta(\Omega; C^3)^*$ by putting $\tilde{f}(u) = f(u|_K)$. From Lemma 6.1 there exists an incoming solution $v_0 \in H^1_\beta(\Omega; C^3)$ such that

$$\begin{cases} (L^{(1)} - k^2)v_0 = \tilde{f} \text{ in } \Omega \\ v_0 = 0 \text{ on } \Gamma_1 \\ C^{(1)}v_0 = 0 \text{ on } \Gamma_0. \end{cases}$$

Note that $\tilde{f} = 0$ on $\Omega \setminus K$, and thus, from local elliptic regularity theorems we obtain $v_0 \in C^\infty(\overline{\Omega} \setminus K; C^3)$.

We show that $v_0 = 0$ on $\overline{\Omega} \setminus K$. For this let $g \in C^\infty_c(\partial \Omega; C^3)$ and consider the outgoing function $w_g \in H^1_\beta(\Omega; C^3)$, which satisfies

$$\begin{cases} (L^{(1)} - k^2)w_g = 0 \text{ in } \Omega \\ w_g = g \text{ on } \Gamma_1 \\ C^{(1)}w_g = 0 \text{ on } \Gamma_0. \end{cases}$$

If we assume that $w_g \in H^{-1}_\beta(\Omega; C^3)$, then the variational formulation (36) implies

$$0 = \int_{\Omega} \langle \sigma(w_g), \varepsilon(v_0) \rangle \, dx - k^2 \int_{\Omega} w_g \overline{v_0} \, dx.$$  

However taking into account the choice of the radiation condition (cf. the appendix) the equality holds true for all $w_g$. Likewise we obtain

$$0 = \int_{\Omega} \langle \sigma(v_0), \varepsilon(w_g) \rangle \, dx - k^2 \int_{\Omega} v_0 \overline{w_g} \, dx$$

$$= \tilde{f}(w_g) + \langle B^{(1)}v_0, w_g|_{\partial \Omega}\rangle_{\partial \Omega}$$

$$= \langle B^{(1)}v_0|_{\Gamma_1}, g|_{\Gamma_1}\rangle.$$

Since $g$ was chosen arbitrarily we have $B^{(1)}v_0 = 0$ on $\Gamma_1$. Hence, as $(L^{(1)} - k^2)v_0 = 0$ on $\Omega \setminus K$ we have also $v_0 = 0$ on $\overline{\Omega} \setminus K$ by unique continuation, cf. [1]. As a particular consequence we obtain $v_0 = 0$ on $\partial K \setminus \Gamma_0$.

Now let $u \in H^1(K; C^3)$ such that $(L^{(1)} - k^2)u = 0$ and $C^{(1)}u = 0$. Then the variational formulation applied to $u$ gives us that

$$0 = \int_{K} \langle \sigma(u), \varepsilon(v_0) \rangle \, dx - k^2 \int_{K} v_0 \overline{u} \, dx.$$  

As a next step we choose a sequence $\phi_n \in C^\infty_c(\overline{\Omega}, C^3)$ with $\phi_n|_K \rightarrow u \in H^1(K; C^3)$ and $\phi_n = 0$ on $\Gamma_0$. Then the previous considerations imply

$$0 = \int_{K} \langle \sigma(v_0), \varepsilon(u) \rangle \, dx - k^2 \int_{K} v_0 \overline{u} \, dx$$

$$= \lim_{n \rightarrow \infty} \int_{\Omega} \langle \sigma(v_0), \varepsilon(\phi_n) \rangle \, dx - k^2 \int_{\Omega} v_0 \overline{\phi_n} \, dx$$

$$= \lim_{n \rightarrow \infty} \tilde{f}(\phi_n) + \langle B^{(1)}v_0|_{\Gamma_1}, \phi_n|_{\Gamma_1}\rangle$$

$$= \lim_{n \rightarrow \infty} \tilde{f}(\phi_n) = f(u).$$
Now, we can prove the main result of this section:

**Proof of Theorem 6.2.** Using Theorem 6.3, it suffices to prove the following: If \( u_2 \) is a solution of (37) and \( v_1 \in H^1(\Omega; \mathbb{C}^3) \) an outgoing solution of (38), then we have that \( H_K(v_1|_K, u_2) = 0 \).

We choose \( v \in H^1(\Omega; \mathbb{C}^3) \) outgoing such that

\[
\begin{cases}
(L^{(2)} - k^2)v = 0 & \text{in } \Omega \\
v = v_1|_{\Gamma_1} & \text{on } \Gamma_1 \\
C^{(2)}v = 0 & \text{on } \Gamma_0.
\end{cases}
\]

Then we have \( B^{(1)}v_1|_{\Gamma_1} = B^{(2)}v|_{\Gamma_1} \). Since \( L^{(1)} = L^{(2)} = L \) on \( \Omega \setminus K \) and \( B^{(1)} = B^{(2)} = B \) on \( \partial\Omega \setminus K \) we have for \( w := v_1 - v \) that

\[
(L - k^2)w = 0 \quad \text{on } \Omega \setminus K, \quad w|_{\partial\Omega \setminus K} = 0, \quad Bw|_{\partial\Omega \setminus K} = 0.
\]

The unique continuation theorem assures that \( w = 0 \) on \( \Omega \setminus K \), cf. [1]. In particular we have \( w|_{\partial K \setminus \Gamma_0} = 0 \) and \( w_\tau = 0 \) on \( \Gamma_0 \) and we obtain with \( \text{supp } u_2 \subset K \) as before

\[
0 = \int_K \langle \sigma_2(w), \varepsilon(u_2) \rangle_{\mathbb{C}^3 \times \mathbb{C}^3} \, dx - k^2 \int_K w \overline{u_2} \, dx
= \int_K \langle \sigma_2(v_1), \varepsilon(u_2) \rangle_{\mathbb{C}^3 \times \mathbb{C}^3} \, dx - k^2 \int_K v_1 \overline{u_2} \, dx - \langle B^{(2)}v, u_2 \rangle_{\partial K}.
\]

We note that \( B^{(1)}v_1 = B^{(2)}v \) on \( \partial\Omega \setminus \Gamma_0 \) and due to local regularity theorems we may assume that \( w \in H^2 \) in a neighbourhood of \( \partial K \cap \Omega \) such that \( B^{(1)}v_1 = B^{(2)}v \) on \( \partial K \cap \Omega \). This implies:

\[
\int_K \langle \sigma_2(v_1), \varepsilon(u_2) \rangle_{\mathbb{C}^3 \times \mathbb{C}^3} \, dx = \langle B^{(2)}v, u_2 \rangle_{\partial K} + k^2 \int_K v_1 \overline{u_2} \, dx
= \langle B^{(1)}v_1, u_2 \rangle_{\partial K} + k^2 \int_K v_1 \overline{u_2} \, dx
= \int_K \langle \sigma_1(v_1), \varepsilon(u_2) \rangle_{\mathbb{C}^3 \times \mathbb{C}^3} \, dx.
\]

In particular we have

\[
H_K(v_1|_K, u_2) = 0.
\]

**Appendix A. Radiation conditions**

For the sake of simplicity we assume that \( \Gamma_0 = \emptyset \). The general case follows likewise. In order to define the notion of incoming and outgoing functions, we follow the approach in [8, 17]. To this end we consider the operator \( L = L(x_2, x_3, \partial x_1, \partial x_2, \partial x_3) \) corresponding
to the Lamé coefficients at infinity, \( \lambda \) and \( \mu \) are given as in (32). For \( k \in \mathbb{R} \) we are concerned with the following family of boundary value problems

\[
\mathfrak{A}(\xi) : H^2(G; \mathbb{C}^3) \to \mathbb{C}^3, \quad \mathfrak{A}(\xi)u = \left( (L(x_2, x_3, i\xi, \partial_{x_2}, \partial_{x_3}) - k^2)u \right)_{\mid \partial G}.
\]

It is well-known that for every \( \xi \in \mathbb{C} \) the operator \( \mathfrak{A}(\xi) \) is a Fredholm operator with vanishing index. Indeed, for the last assertion we observe that for real \( \xi \in \mathbb{R} \) the adjoint boundary value problem coincides with the original one. Moreover, \( \mathfrak{A}(\xi) \) is invertible for sufficiently large \( \xi \in \mathbb{R} \), and thus, it is invertible for all \( \xi \in \mathbb{C} \) with the possible exception of a discrete subset, cf. [5] We put

\[ \Xi := \{ \xi \in \mathbb{C} : \ker \mathfrak{A}(\xi) \neq \{0\} \}. \]

The elements of \( \Xi \) are called characteristic values of the operator pencil \( \mathfrak{A} \). We consider the boundary value problem

\[
\begin{aligned}
(39) \\
\left\{ \begin{array}{l}
(L^{(i)} - k^2)u = f \text{ in } \Omega \\
u = g \text{ on } \partial \Omega,
\end{array} \right.
\end{aligned}
\]

where now \( f \in H^{1-\beta}_\beta(\Omega; \mathbb{C}^3)^* \) and \( g \in H^{1/2}_\beta(\partial \Omega; \mathbb{C}^3) \). The Fredholm property reads as follows.

**Lemma A.1.** Let \( \beta \in \mathbb{R} \) such that \( \Xi \cap (\mathbb{R} \pm i\beta) = \emptyset \). Then either one of the following assertions hold true:

1. The homogeneous problem \((f = 0 \text{ and } g = 0)\) has the unique solution \( u = 0 \).
Then (39) has a unique solution \( u \in H^1_\beta(\Omega; \mathbb{C}^3) \).

2. There exists \( n \) solutions \( U_j \in H^1_\beta(\Omega; \mathbb{C}^3) \) of the homogeneous problems. Then (39) is solvable if and only if

\[
\langle U_j, f \rangle + (B^{(i)} U_j, g)_{\partial \Omega} = 0, \quad \text{for } j = 1, \ldots, n.
\]

The proof follows as in [10, Corollary 3.4.2]. For the case of the mixed problem this has to be combined with the ideas in the proof of [12, Theorem 4.10].

As a next step we want to consider the asymptotic behaviour of solutions of the boundary value problem (39). For \( \xi_0 \in \Xi \), we call \( v_0, \ldots, v_\ell \in H^2(G) \setminus \{0\} \) a Jordan chain of length \( \ell + 1 \) (associated with \( v_0 \)) for \( \mathfrak{A} \), if

\[
0 = \sum_{q=0}^{j} \frac{1}{q!} \frac{d^q}{d \xi^q} \mathfrak{A}(\xi) \big|_{\xi = \xi_0} v_{j-q}, \quad \text{for all } j = 0, \ldots, \ell.
\]

Note that in particular we have that \( v_0 \in \ker \mathfrak{A}(\xi_0) \).

If \( v_0, \ldots, v_\ell \) is a Jordan-chain, then we define the functions

\[
V_j(x_1, x_2, x_3) = e^{i\xi_0 x_1} \sum_{q=0}^{j} \frac{(it)^q}{q!} v_{j-q}(x_2, x_3), \quad j = 0, \ldots, \ell.
\]
These functions satisfy the equation \((L - k^2)V_j = 0\) in \(\mathbb{R} \times G\) and \(V_j|_{\mathbb{R} \times \partial G} = 0\). In what follows we assume that \(\beta > 0\) is chosen such that
\[
\Xi \cap \{ \xi \in \mathbb{C} : |\text{Im}(\xi)| < \beta \} \subseteq \mathbb{R}.
\]

We note that if \(\Xi \cap \mathbb{R} = \emptyset\), then we do not need to impose any radiation conditions. Assume that \(\Xi \cap \mathbb{R} \neq \emptyset\) and denote by \(N\) the total multiplicity of the characteristic values in \(\Xi \cap \mathbb{R}\). Note that \(N\) is even. As before we obtain functions \(V_j, j = 1, \ldots, N\). Let \(\chi_+ \in C^\infty(\mathbb{R})\) be chosen such that \(\chi_+(t) = 0\) for \(t \leq R\) and \(\chi_+(t) = 1\) for \(t \geq R + 1\), where \(R > 0\) is sufficiently large. We put \(\chi_-(x_1) := \chi_+(x_1)\). Then we have the following result:

**Lemma A.2** (see e.g. in [17]). Let \(f \in H^1_{\beta}(\Omega; \mathbb{C}^3)^*\) and \(g \in H^{1/2}_{-\beta}(\partial \Omega; \mathbb{C}^3)\) and let \(u \in H^{1,\beta}(\Omega; \mathbb{C}^3)\) be a solution of (39). Then there exists \(\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N \in \mathbb{C}\) such that
\[
u - \chi_+ \sum_{j=1}^N \alpha_j V_j + \chi_- \sum_{j=1}^N \beta_j V_j \in H^1_{-\beta}(\Omega; \mathbb{C}^3).
\]

This leads us to consider the space
\[
\mathbb{D} := \text{span}\{\chi_+ V_j : j = 1, \ldots, N\} + \text{span}\{\chi_- V_j : j = 1, \ldots, N\}.
\]

Let
\[
q(u, v) := \int_{\Omega} L_i u \bar{v} \, dx - \int_{\Omega} \bar{L}_i v \, dx + \int_{\partial \Omega} B_i u \bar{v} \, ds - \int_{\partial \Omega} u \bar{B}_i v \, ds, \quad u, v \in \mathbb{D}.
\]

Then we may choose a basis \(U_{1,+}, \ldots, U_{N,+}, U_{1,-}, \ldots, U_{N,-}\) of \(\mathbb{D}\) such that
\[
q(U_{j,+}, U_{j,+}) = i \delta_{jk}, \quad q(U_{j,-}, U_{k,-}) = -i \delta_{jk}, \quad q(U_{j,+}, U_{k,-}) = 0.
\]

Then a function \(u \in H^{1,\beta}_{\Omega}(\mathbb{C}^3)\) will be called outgoing if there are coefficients \(c_j \in \mathbb{C}\) such that
\[
u - \sum_{j=1}^n c_j U_{j,+} \in H^1_{-\beta}(\Omega; \mathbb{C}^3)
\]

Analogously, \(u\) is called incoming if \(u - \sum_{j=1}^n c_j U_{j,-} \in H^1_{-\beta}(\Omega; \mathbb{C}^3)\). Then Lemma 6.1 will be a consequence of the Fredholm property in [8, Theorem 2.2].

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