EXTREMAL METRICS FOR THE \( Q' \)-CURVATURE IN THREE DIMENSIONS

MÉTRIQUES EXTRÉMALES POUR LA \( Q' \)-COURBURE EN DIMENSION 3)

JEFFREY S. CASE, CHIN-YU HSIAO, AND PAUL YANG

Abstract. We construct contact forms with constant \( Q' \)-curvature on compact three-dimensional CR manifolds that admit a pseudo-Einstein contact form and satisfy some natural positivity conditions. These contact forms are obtained by minimizing the CR analogue of the \( H \)-functional from conformal geometry. Two crucial steps are to show that the \( P' \)-operator can be regarded as an elliptic pseudodifferential operator and to compute the leading order terms of the asymptotic expansion of the Green’s function for \( \sqrt{P'} \).

Résumé. On construit des formes de contact \( Q' \)-courbure constante sur les variétés de Cauchy–Riemann de dimension 3 qui admettent une pseudo-forme de contact d’Einstein et satisfont certaines conditions naturelles de positivité. Ces formes sont obtenues en minimisant l’analogue en CR-géométrie du \( H \)-fonctionelle en géométrie conforme. Cette construction repose sur deux étapes cruciales: On montre que le \( P' \)-opérateur peut être vu comme un opérateur pseudodifferentiel elliptique et on calcule les terms dominants du développement asymptotique de la forme de Green pour \( \sqrt{P'} \).

1. Introduction

On an even-dimensional manifold \((M^{2n}, g)\), the pair \((P, Q)\) of the (critical) GJMS operator \(P\) and the (critical) \(Q\)-curvature \(Q\) possesses many of the same properties of the pair \((-\Delta, K)\) on surfaces, where \(K\) is the Gauss curvature. For example, \(P\) is a conformally covariant formally self-adjoint operator with leading order term \((-\Delta)^{n/2}\) that annihilates constants \([13, 15]\) and \(Q\) is a Riemannian invariant with leading order term \(c_n(-\Delta)^{n-2} R\), where \(R\) is the scalar curvature, which transforms in a particularly simple way within a conformal class \([4]\): if \(\tilde{g} = e^{2u} g\), then

\[ e^{nu} \tilde{Q} = Q + Pu. \]

In particular, \(\int Q\) is conformally invariant on closed even-dimensional manifolds; indeed, it computes the Euler characteristic modulo integrals of pointwise conformal invariants \([1]\). It also follows that metrics of constant \(Q\)-curvature within a conformal class are in one-to-one correspondence with critical points of the functional

\[ II[u] = \int_M u Pu + 2 \int_M Qu - \frac{2}{n} \left( \int_M Q \right) \log \left( \frac{1}{\text{Vol}(M)} \int_M e^{nu} \right). \]

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This functional can always be minimized on the two-sphere \([21]\) and on four-manifolds with positive Yamabe constant and nonnegative Paneitz operator \([2, 9, 16]\), with important applications to logarithmic functional determinants \([5, 21]\) and sharp Onofri-type inequalities \([2]\). Due to the parallels between conformal and CR geometry, it is interesting to determine whether a similar pair exists in the latter setting.

Works by Graham and Lee \([14]\) and Hirachi \([17]\) identified CR analogues of the Paneitz operator and \(Q\)-curvature in dimension three. However, the kernel of the Paneitz operator contains the (generally infinite-dimensional) space \(P\) of CR pluriharmonic functions and the total \(Q\)-curvature is always zero. In particular, an Onofri-type inequality involving the CR Paneitz operator cannot be satisfied.

Branson, Fontana and Morpurgo overcame this latter issue on the CR spheres by introducing a formally self-adjoint operator \(P'\) which is CR covariant on CR pluriharmonic functions and in terms of which one has the sharp Onofri-type inequality

\[
\int_{S^{2n+1}} u P' u + 2 \int_M Q' u - \frac{2}{n} \int_M \left( \int_M Q' \right) \log \left( \frac{1}{\text{Vol}(M)} \int_M e^{nu} \right)
\]

for all \(u \in W^{n+1,2} \cap P\), where \(Q'\) is an explicit dimensional constant \([6]\). The construction of \(P'\) is analogous to the construction of the \(Q\)-curvature from the GJMS operators by analytic continuation in the dimension.

It was observed by the first- and third-named authors in dimension three \([8]\) and by Hirachi in general dimension \([18]\) that one can define the \(P'\)-operator on general pseudohermitian manifolds \((M^{2n+1}, T^1, 0, \theta)\). Roughly speaking, if \(P^{N+2}_{2n+2}\) is the CR GJMS operator of order \(2n+2\) on a \((2N+1)\)-dimensional manifold, one defines \(P'\) as the limit of \(\frac{2}{N-n} P^{N+2}_{2n+2}|\theta\) as \(N \to n\). This is made rigorous by explicit computation in dimension three \([8]\) and via the ambient metric in general dimension \([18]\). Regarded as a map from \(P\) to \(C^\infty(M)/P\perp\), the \(P'\)-operator is CR covariant: if \(\theta = e^\sigma \theta\), then \(e^{(n+1)\sigma} \tilde{P}' = P'\).

If \(\theta\) is a pseudo-Einstein contact form (cf. \([8, 18, 20]\)), then the \(P'\)-operator is formally self-adjoint and annihilates constants. Note that if \(M^{2n+1}\) is the boundary of a domain in \(C^{n+1}\), then the defining functions constructed by Fefferman \([11]\) induce pseudo-Einstein contact forms on \(M\). One can construct a pseudohermitian invariant \(Q'\) on pseudo-Einstein manifolds by formally considering the limit \(\frac{2}{N-n} P^{N+2}_{2n+2}(1)\) as \(N \to n\); this can be made rigorous by direct computation in dimension three \([8]\) and via the ambient metric in general dimension \([18]\). Regarded as \(C^\infty(M)/P\perp\)-valued, the \(Q'\)-curvature transforms linearly with a change of contact form: if \(\tilde{\theta} = e^\sigma \theta\) is also pseudo-Einstein, then

\[
(1.1) \quad e^{2(n+1)\sigma} \tilde{Q}' = Q' + P'\sigma.
\]

Since \(\tilde{\theta}\) is pseudo-Einstein if and only if \(\sigma \in P\) \([17, 20]\), this makes sense. It follows from the properties of \(P'\) that \(\int Q'\) is independent of the choice of pseudo-Einstein contact form. Direct computation on \(S^{2n+1}\) shows that it is a nontrivial invariant; indeed, in dimension three it is a nonzero multiple of the Burns–Epstein invariant \([8]\). In particular, the pair \((P', Q')\) on pseudo-Einstein manifolds has the same properties as the pair \((P, Q)\) on Riemannian manifolds.
Corollary 1.2. Let \( M^{2n+1}, T^{1,0}, \theta \) be a compact pseudo-Einstein manifold \( \mathcal{P} \) and \((1.1)\) imply that critical points of the functional \( II: \mathcal{P} \to \mathbb{R} \) defined by

\[
II[u] = \int_M u P' u + 2 \int_M Q' u - \frac{2}{n + 1} \left( \int_M Q' \right) \log \left( \frac{1}{\text{Vol}(M)} \int_M e^{(n+1)u} \right)
\]

are in one-to-one correspondence with pseudo-Einstein contact forms with constant \( Q' \)-curvature (still regarded as \( C^\infty(M)/\mathcal{P} \)-valued). The existence and classification of minimizers of the \( II \)-functional on the standard CR spheres was given by Branson, Fontana and Morpurgo [6]. In this note, we discuss the main ideas used by the authors to give criteria which guarantee that minimizers exist for the \( II \)-functional on a given pseudo-Einstein three-manifold [7].

**Theorem 1.1.** Let \( (M^3, T^{1,0}, \theta) \) be a compact, embeddable pseudo-Einstein three-manifold such that \( P' \geq 0 \) and \( \ker P' = \mathbb{R} \). Suppose additionally that

\[
\int_M Q' \theta \wedge d\theta < 16\pi^2
\]

Then there exists a function \( w \in \mathcal{P} \) which minimizes the \( II \)-functional. Moreover, the contact form \( \hat{\theta} := e^w \theta \) is such that \( \hat{Q}' \) is constant.

The assumptions on \( P' \) mean that the pairing \( (u, v) := \int u P' v \) defines a positive definite quadratic form on \( \mathcal{P} \). It is important to emphasize that the conclusion is that \( \hat{Q}' \) is constant as a \( C^\infty(M)/\mathcal{P} \)-valued invariant: a local formula for the \( Q' \)-curvature was given by Chanillo, Chiu and the third-named author [8], while we observe that, on \( S^1 \times S^2 \) with any of its locally spherical contact structures, there is no pseudo-Einstein contact form with \( Q' \) pointwise zero; see [7, Section 5].

As in the study of Riemannian four-manifolds (cf. [9, 16]), the hypotheses of Theorem 1.1 can be replaced by the nonnegativity of the pseudohermitian scalar curvature and of the CR Paneitz operator. Indeed, Chanillo, Chiu and the third-named author proved that these assumptions imply that \( (M^3, T^{1,0}) \) is embeddable [10]; the first- and third-named authors proved that these assumptions imply both that \( P' \geq 0 \) with \( \ker P' = \mathbb{R} \) and that \( \int Q' \leq 16\pi^2 \) with equality if and only if \( (M^3, T^{1,0}) \) is CR equivalent to the standard CR three-sphere [8]; and Branson, Fontana and Morpurgo showed that minimizers of the \( II \)-functional exist on the standard CR three-sphere [6].

**Corollary 1.2.** Let \( (M^3, T^{1,0}, \theta) \) be a compact pseudo-Einstein manifold with nonnegative scalar curvature and nonnegative \( \text{CR Paneitz operator} \). Then there exists a function \( w \in \mathcal{P} \) which minimizes the \( II \)-functional. Moreover, the contact form \( \hat{\theta} := e^w \theta \) is such that \( \hat{Q}' \) is constant.

2. **Sketch of the proof of Theorem 1.1**

The proof of Theorem 1.1 proceeds analogously to the proof of the corresponding result on four-dimensional Riemannian manifolds [9] with one important difference: \( P' \) is defined as a \( C^\infty(M)/\mathcal{P} \)-valued operator; in particular, it is a nonlocal operator. Let \( \tau: C^\infty(M) \to \mathcal{P} \) be the orthogonal projection with respect to the standard \( L^2 \)-inner product. A key observation is that the operator \( \overline{P}' := \tau P': \mathcal{P} \to \mathcal{P} \) is a self-adjoint elliptic pseudodifferential operator of order \( -2 \); see [7, Theorem 9.1]. This follows from the observation that, while the sub-Laplacian \( \Delta_b \) is subelliptic, the Toeplitz operator \( \tau \Delta_b \tau \) is a classical elliptic pseudodifferential operator of order
This is achieved by writing $\Delta_b = 2\Box_b + iT$, relating $\tau$ to the Szegő projector $S$, and using well-known properties of the latter operator (cf. [3, 19]).

Since $\int u P' v = \int u \overline{P} v$ for all $u, v \in P$, it follows that $\overline{P}$ is a nonnegative operator with $\ker \overline{P} = \mathbb{R}$. In particular, the positive square root $(\overline{P})^{1/2}$ of $\overline{P}$ is well-defined and such that $\ker (\overline{P})^{1/2}$. Using the pseudodifferential calculus and the fact that, as a local operator, $P'$ equals $\Delta^2_b$ plus lower order terms [8], we then observe that the Green’s function of $(\overline{P})^{1/2}$ is of the form $c\rho^{-2} + O(\rho^{-1-\varepsilon})$ for $\rho^4(z, t) = |z|^4 + t^2$ the Heisenberg pseudo-distance, $\varepsilon \in (0, 1)$, and $c$ the same constant as the computation on the three-sphere [6]; for a more precise statement, see [7, Theorem 1.3].

From this point, the remaining argument is fairly standard. The above fact about the Green’s function of $(\overline{P})^{1/2}$ allows us to apply the Adams-type theorem of Fontana and Morpurgo [12] to conclude that the former operator satisfies an Adams-type inequality with the same constant as on the standard CR three-sphere. This has two important effects. First, it implies that $H$-functional is coercive under the additional assumption $\int Q' < 16\pi^2$; see [7, Theorem 4.1]. Second, it implies that if $w \in W^{2,2} \cap P$ satisfies

$$\tau (P' w + Q' - \lambda \omega^2 w) = 0,$$

then $w \in C^\infty(M)$; see [7, Theorem 4.2]. The former assumption allows us to minimize $H$ within $W^{2,2} \cap P$ and the latter assumption yields the regularity of the minimizers. The final conclusion follows from the transformation formula (1.1) for the $Q'$-curvature.

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References

[1] S. Alexakis. The decomposition of global conformal invariants., vol. 182, Princeton University Press, Princeton, NJ, 2012.
[2] W. Beckner. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. Ann. of Math. (2), 138(1):213–242, 1993.
[3] L. Boutet de Monvel and J. Sjöstrand. Sur la singularité des noyaux de Bergman et de Szegő. In Journées: Équations aux dérivées partielles de Rennes (1975), Astérisque, No. 34–35. Soc. Math. France, Paris, 1976, pp. 123–164.
[4] T. P. Branson. Sharp inequalities, the functional determinant, and the complementary series. Trans. Amer. Math. Soc., 347(10):3671–3742, 1995.
[5] T. P. Branson, S.-Y. A. Chang, and P. C. Yang. Estimates and extremals for zeta function determinants on four-manifolds. Comm. Math. Phys., 149(2):241–262, 1992.
[6] T. P. Branson, L. Fontana, and C. Morpurgo. Moser-Trudinger and Beckner-Onofri’s inequalities on the CR sphere. Ann. of Math. (2), 177(1):1–52, 2013.
[7] J. S. Case, C.-Y. Hsiao, and P. C. Yang. Extremal metrics for the $Q'$-curvature in three dimensions. Preprint.
[8] J. S. Case and P. C. Yang. A Paneitz-type operator for CR pluriharmonic functions. Bull. Inst. Math. Acad. Sin. (N.S.), 8(3):285–322, 2013.
[9] S.-Y. A. Chang and P. C. Yang. Extremal metrics of zeta function determinants on 4-manifolds. Ann. of Math. (2), 142(1):171–212, 1995.
[10] S. Chanillo, H.-L. Chiu, and P. Yang. Embeddability for 3-dimensional Cauchy-Riemann manifolds and CR Yamabe invariants. Duke Math. J., 161(15):2909–2921, 2012.
[11] C. Fefferman. Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains. *Ann. of Math. (2)*, 103(2):395–416, 1976.

[12] L. Fontana and C. Morpurgo. Adams inequalities on measure spaces. *Adv. Math.*, 226(6):5066–5119, 2011.

[13] C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling. Conformally invariant powers of the Laplacian. I. Existence. *J. Lond. Math. Soc. (2)*, 46(3):557–565, 1992.

[14] C. R. Graham and J. M. Lee. Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains. *Duke Math. J.*, 57(3):697–720, 1988.

[15] C. R. Graham and M. Zworski. Scattering matrix in conformal geometry. *Invent. Math.*, 152(1):89–118, 2003.

[16] M. J. Gursky. The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE. *Comm. Math. Phys.*, 207(1):131–143, 1999.

[17] K. Hirachi. Scalar pseudo-Hermitian invariants and the Szegő kernel on three-dimensional CR manifolds. In *Complex geometry (Osaka, 1990)*, volume 143 of *Lecture Notes in Pure and Appl. Math.* Dekker, New York, 1993, pp. 67–76.

[18] K. Hirachi. $Q'$-prime curvature on CR manifolds. *Differential Geom. Appl.*, 33(suppl.):213–245, 2014.

[19] C.-Y. Hsiao. Projections in several complex variables. *Mém. Soc. Math. Fr. (N.S.)*, (123):131, 2010.

[20] J. M. Lee. Pseudo-Einstein structures on CR manifolds. *Amer. J. Math.*, 110(1):157–178, 1988.

[21] B. Osgood, R. Phillips, and P. Sarnak. Extremals of determinants of Laplacians. *J. Funct. Anal.*, 80(1):148–211, 1988.

Department of Mathematics, McAllister Building, The Pennsylvania State University, State College, PA 16802

E-mail address: jscase@psu.edu

Institute of Mathematics, Academia Sinica, 6F, Astronomy-Mathematics Building, No.1, Sec.4, Roosevelt Road, Taipei 10617, Taiwan

E-mail address: chsiao@math.sinica.edu.tw

Department of Mathematics, Princeton University, Princeton, NJ 08544

E-mail address: yang@math.princeton.edu