Universality in spectral statistics of “open” quantum graphs.

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(Dated: May 11, 2014)

The unitary evolution maps in closed chaotic quantum graphs are known to have universal spectral correlations, as predicted by random matrix theory. In chaotic graphs with absorption the quantum maps become non-unitary. We show that their spectral statistics exhibit universality at the “soft” edges of the spectrum. The same spectral behavior is observed in many classical non-unitary ensembles of random matrices with rotationally invariant measures.

PACS numbers: 05.45.M, 02.10.Ox, 03.65.Sq

Since 1970’s a significant attention of physics community has been attracted to particularities of the energy spectrum of quantum systems with chaotic behavior in the classical limit. In 1984 Bohigas, Giannoni and Schmit (BGS) conjectured \(^1\) that the spectral fluctuations of closed chaotic Hamiltonian systems are universal and coincide with those of one of three canonical random matrix ensembles (RME). Based on the semiclassical considerations the validity of BGS conjecture has been established by now on the physical level of rigor \(^2\,\text{3}\). On the other hand, many theoretical and experimental studies have focused on open chaotic systems whose wave dynamics are described by non-unitary evolution operators. Such an opening may occur due to various physical phenomena: attaching external leads to quantum dots, the dissipation through the ohmic losses, partial reflection of microwaves at the boundaries of dielectric microcavities etc. It is of great interest to know whether (and under which conditions) open chaotic systems exhibit universal properties. So far, the majority of studies in this respect have been restricted to regimes of “weak” opening, where the mean dwell time of the particle in the system growth in the semiclassical limit. For instance, transport properties of quantum dots with a finite number of open channels have been shown to be universal and agree with the random matrix theory predictions \(^4\). The main goal of the present paper is establishing a new form of spectral universality for systems with “strong” opening when the dwell time remains finite in the semiclassical limit.

In this work we focus on the model of quantum graphs with broken time reversal symmetry. Quantum graphs were proposed as a paradigm for the study of compact \(^5\) and scattering \(^6\) quantum chaotic systems. They were also studied experimentally in the presence of absorption \(^7\). Let us briefly describe a standard construction of quantum graphs with \(V\) vertices connected by \(B\) bonds, see e.g., \(^8\) for details. At bonds \(b = 1, \ldots, B\) the waves \(\psi_b\) satisfy the free Schrödinger equation \((-i\hbar \partial_x + \mathcal{A}_b) \psi_b(x_b) = k^2 \psi_b(x_b)\), where \(x_b \in [0, L_b]\) measures the distance along the bond \(b\) and \(\mathcal{A}_b\) is a constant vector potential introduced to break the time-reversal symmetry. The corresponding general solution is a superposition of two plane waves propagating in opposite directions, \(\psi_b(x_b) = e^{-i \mathcal{A}_b x_b} (e^{i k x_b} a_b^+ + e^{-i k x_b} a_b^-)\). The constants \(a_b^\pm\) for different bonds are then connected in the vertices by means of the scattering matrices \(\sigma_v\), \(v = 1, \ldots, V\). To proceed further one introduces the associated directed graph \(\Gamma\) with the double number \(N = 2B\) of bonds \((b, b')\) carrying waves with the positive and negative momenta separately, such that \(L_b \equiv L_{b'}, \mathcal{A}_b = -\mathcal{A}_{b'}\). The complete spectral information is carried by the \(N \times N\) quantum map \(U(k) = S A(k)\), where the scattering matrix \(S\) depends on the graph’s structure and \(\sigma_v\)’s, \(\mathcal{A}_b\)’s, while dependence on the energy \(k^2\) is entirely stored in the diagonal part \(\Lambda(k) = \text{diag}\{e^{i k b_j}\}, j \in \{b, b'\}\). Note that \(S\) also fixes the matrix of “classical evolution” \(F\) on \(\Gamma\), whose elements \(F_{ij} = |S_{ij}|^2\) specify classical transition probabilities between bonds of the graph. The spectrum \(\{k_n\}\) of the system is provided by solutions of the secular equation \(\text{det}(I - U(k_n)) = 0\). If all \(\sigma_v\) are unitary and \(\mathcal{A}_b\) are real then the resulting quantum map \(U\) is unitary and the system’s spectrum is real. It is possible to open the system by either attaching external leads to the graph or by introducing absorption at bonds (or vertices) violating the aforementioned conditions. In any such case the resulting (internal) scattering matrix \(S\), as well as \(U\), are not unitary anymore and we will colloquially refer to \(\Gamma\) as open quantum graph.

An appealing feature of quantum graphs is the exact trace formula connecting the density of states \(d(k) = \sum_n \delta(k - k_n)\) with the traces of the quantum map \(U(k)\). As a result, the two-point spectral correlation function can be expressed as the discrete Fourier transform of the spectral form factor \(\langle \text{Tr} U(k) \rangle_k\), where \(\langle \cdot \rangle_k \equiv \lim_{K \to \infty} \frac{1}{K} \int_0^K d\lambda \langle \cdot \rangle\) is the average over the wave number. Furthermore, as was shown in \(^9\), for graphs with rationally independent bond lengths (which we assume through the paper) the average over \(k\) can be traded for the averages over independent parameters \(k L_b, b = 1, \ldots, B\). Therefore, the spectral correlations in individual quantum graphs can be found by solving the same problem for the ensemble of matrices \(U_\phi \equiv S A_\phi\), \(A_\phi = \text{diag}\{e^{i \phi_b}\}\), \(\phi_b = -\phi_{b'}\), where averages \(\langle \cdot \rangle_\phi\) are
taken over the flat probability measure \( \nu(\phi) = \prod_{k=1}^{N/2} d\phi_k \).

The case of graphs with unitary \( S \) has been analyzed by both semiclassical [10] and supersymmetry methods [12]. It was demonstrated that under certain condition on the gap of the classical evolution spectrum, BGS conjecture holds i.e., depending on the presence or absence of time reversal symmetry the ensemble of \( U_\phi \) has the same spectral statistics as either Gaussian Unitary or Gaussian Orthogonal matrices. For “strongly” open quantum graphs the eigenvalues \( \{z_k\}_{k=1}^N \) of \( U_\phi \) are not confined to the unit circle, but rather distributed isotropically over the complex plane with the mean distance of the order \( 1/\sqrt{N} \). (The isotropy follows immediately from the invariance of \( \nu(\phi) \) under the rotation \( \phi_k \to \phi_k + \phi \) \( k = 1, \ldots, N \).) Typically, with the increase of graph’s dimension \( z_k \)'s become more and more concentrated in an annulus whose boundaries are referred to as inner (resp. outer) spectral edge. As we show below, a spectral universality holds at the \( 1/\sqrt{N} \) neighborhood of these edges. For the sake of simplicity of exposition we formulate the result for the outer edge and then discuss its extension to the inner edge.

Main result. Let \( \Gamma \) be an infinite sequence of open quantum graphs with \( S(N)A_\phi, F(N) \) being their \( N \times N \) quantum and classical evolution, respectively. For the matrix \( F(N) \) we denote by \( \lambda, \bar{X}, \chi \) the largest eigenvalue and the corresponding left (resp. right) eigenvectors normalized by \( \bar{X} = N \). We will consider the spectrum \( \{z_k\}_{k=1}^N \) of rescaled quantum propagator \( S\Lambda_\phi, S = \frac{1}{\sqrt{\chi}}S(N) \) in the limit \( N \to \infty \) under the conditions:

(i) Large spectral gap of \( F \equiv F(N)/\lambda \): The next to the largest eigenvalue \( \lambda_2 \) satisfies \( 1 - |\lambda_2| = O(N^{-\kappa}), \kappa < \frac{1}{2} \);

(ii) Strong non-unitarity of \( S \): The parameter \( \mu(N) = \frac{1}{N} \text{Tr} (SXS^\dagger X^2 - (X\bar{X})^2) \) has a strictly positive limit \( \mu = \lim_{N \to \infty} \mu(N) > 0 \), where \( X, (\bar{X}) \) is the diagonal matrix constructed on \( \chi(\bar{X}) \);

Assuming (i, ii) hold, the spectral density \( \rho(z) = \frac{1}{N} \text{Tr} S^\dagger \delta(z - z_k) \phi_\phi \) is a function of \( r = |z| \) only and \( \rho(r) = 2\pi \nu \rho(r) \) has the universal form at \( 1/\sqrt{N} \) vicinity of the edge \( |z| = 1 \):

\[
\rho \left( 1 - \frac{s}{\sqrt{N}} \right) = \frac{1}{\mu} \left( 2 - \text{erfc} \left( \frac{s}{\sqrt{2N}} \right) \right) + O(N^{-\frac{3}{2} + \kappa}), \quad (1)
\]

in particular \( \rho(1) = \mu^{-1} + o(N^0) \). The form-factor \( K(n) = \frac{1}{\mu}(\text{Tr}(SA)^n)\phi \) demonstrates the universal asymptotics:

\[
\sqrt{N}K(n) = \frac{2}{\mu^2} \sinh \left( \frac{\mu t^2}{2} \right) + O(N^{-\frac{3}{2} + \kappa}), \quad (2)
\]

in the limit where \( t = n/\sqrt{N} \) is fixed and \( N \to \infty \).

Few remarks are in order. 1) Note, that the spectral density of the Ginibre unitary ensemble [13], and of other “strongly” non-unitary ensembles [14,15] with rotationally invariant measures demonstrate the same soft edge universal form (1). Only the scaling parameter \( \mu \) depends on the specifics of these ensembles. 2) Because of the difference in the mean level distance between eigenvalues, the semiclassical limit \( n \sim \sqrt{N}, N \to \infty \) considered here differs from the one in the unitary case \( n \sim N, N \to \infty \). 3) The asymptotics for the inner edge can be established by considering inverse matrices \( (S(N))^{-1} \Lambda_\phi^* \) whose spectrum \( \{z_k^{-1}\}_{k=1}^N \) has the density \( \rho'(r) = r^{-2}\rho(1/r) \). This inversion maps the inner edge to the outer and (1,2) become applicable to \( \rho' \) with the parameter \( \mu \) being defined by the matrices \( S^{-1} \). 4) By eq. (1) the outer and the inner edges of non-rescaled quantum maps \( S(N)L_\phi \) are given by \( \lambda \) and \( 1/\sqrt{\lambda} \), where \( \lambda, \chi \) are the highest eigenvalues of the “classical” maps \( |S(N)|_{i,j}^2, |(S(N))^{-1}|_{i,j}^2 \). If \( \lambda = \infty \) (e.g., \( S \) is not invertible) then the inner edge does not exist. 5) The condition (i) on the spectral gap of the classical map is analogous to Tanner’s condition [17] in the unitary case. The difference between \( \kappa < 1/2 \) (non-unitary) and \( \kappa < 1 \) (unitary) is due to the different time scales involved. It holds for many important classes of graphs, see e.g., [8] and examples below. The condition (ii) implies strong non-unitarity of \( S \). If, for instance, the number of open channels in a scattering graph is fixed then \( \lim_{N \to \infty} \mu(N) = 0 \) and (ii) is violated. Note that \( \mu(N) \geq 0 \) always and \( \mu(N) \equiv 0 \) if \( S \) is unitary.

Comparison with numerics. Before turning to the derivation of eqs. (1,2), let us consider several examples. (A) “Connectivity” graphs. For a graph \( \Gamma \) take \( S \) be its connectivity matrix i.e., each element of \( S \) is either 0 or 1. This choice is of a special interest due to the connection with the problem of length degeneracies in metric graphs [13]. For the case of \( d \)-regular graphs it is straightforward to see that \( \lambda = d, \chi = \chi = 1 \) implying \( \mu = d - 1 \). The comparison of (1,2) with numerics for such a graph is shown on fig. 4a,b. (B) “Doubly stochastic” graphs, fig. 4a. Let \( S \) be such that \( F \) is a doubly stochastic matrix i.e., \( \sum_i F_{i,j} = \sum_j F_{i,j} = 1 \). This can be achieved, for instance, by taking \( a_{i,j} = |u_{i,j}|^2 \), where \( u \) are arbitrary unitary matrices. It is known that these matrices almost surely satisfy the required spectral gap
FIG. 2: The (non-rescaled) spectral densities of quantum maps $S^{(N)} \Lambda_{\phi}$ v.s. asymptotics (1) (solid blue lines) for: (a) “Doubly stochastic” 10-regular graphs with $N = 1000$. Each vertex matrix $\sigma^v$ is fixed by $10 \times 10$ random unitary matrix. (b) “Damped” De Bruijn graph with $N = 2^7$ and $D_{i,j} = \delta_{i,j} f(2\pi/N) + f(x) = 3.2 + \sin(\pi x) + \sin(2\pi x) + \sin(3\pi x)$. The parameters are $\mu = 0.1542$, $\lambda_1 = 12.6578$ for the outer edge, and $\mu' = 0.3133$, $\lambda_1' = 0.1952$ for the inner edge. The inner $\sqrt{1/\lambda_1}$ and outer radii $\sqrt{\lambda_1}$ are depicted by vertical solid (red) left and right lines. The (red) line in the middle shows the mean value of log $f(x)$, where $\rho$ clusters at $N \to \infty$.

Condition (10). As in the previous example the highest eigenvector $\chi$ of $F$ is uniform, while $\lambda = 1$. This gives $\mu = \frac{1}{N} \text{Tr} (SS^T)^2 - 1$ (compare with the result of [13] for RME with unitary invariant measures). (C) “Damped” quantum maps were suggested in [19] as toy models for open quantum systems. They are represented as products $U_M \cdot D$, where $N \times N$ unitary matrix $U_M$ is a quantization of a classical map $M$ and “smooth” diagonal matrix $D$ introduces “absorption”. Here we checked a particular case of Walsh quantized baker’s map whose quantization for $N = 2^p$ can be written as $U^{(p)} \Lambda_{\phi}$, where $U_{i,j}^{(p)} = \frac{1}{\sqrt{2}} (\delta_{i,2j-1 \mod N} - \delta_{i,2j+N})$ and $\phi$ is arbitrary, see [20]. The matrix $S^{(p)} = U^{(p)} D$ can in turn be interpreted as the scattering matrix for the De Bruijn graph with an absorption. We compared spectral density of matrices $S^{(p)} \Lambda_{\phi}$ with (1) and found good agreement for both inner and outer edges, see fig. 2. Note that in this case the edge distribution does not “converge”, since, as numerics shows, the parameter $\mu^{(N)}$ slowly grows with $N$. This observation agrees with the phenomenon of eigenvalue clustering near a “typical” value found in [10].

Derivation of eqs. (12). By the definition the form-factor $K(n)$ can be represented as the double sum over $n$-periodic trajectories $\gamma$ of the graph:

$$\frac{N}{n} K(n) = \langle \sum_{\gamma} A_{\gamma} e^{(n,\gamma,\phi)} \rangle_{\phi} = \sum_{\gamma,\gamma'} A_{\gamma} A_{\gamma'}^* \delta_{n,\gamma',n,\gamma} \quad (3)$$

with $n_\gamma$ being an integer-valued $N$ dimensional vector, whose elements $n_\gamma$ indicate the number of times $\gamma$ visits the bond $b$ ($\sum_{\gamma} n_\gamma = n$). The amplitudes $A_{\gamma}$ are products of the matrix elements $S_{ij}$ taken along the path $\gamma$ and include the multiplicity factors which are 1 for prime periodic orbits. Following the standard semiclassical prescription [22] we analyze first the “diagonal”, $\gamma = \gamma'$, contribution in (3). Leaving out only prime periodic orbits and assuming long trajectory limit ($n \sim \sqrt{N} \gg 1$):

$$\sum_{i_1 \ldots i_n} |S_{i_1i_2}|^2 \ldots |S_{i_ni_1}|^2 = \text{Tr} F^n = 1 + O(N^{-\frac{1}{4}+\kappa})$$

where we used the condition (i) on the spectral gap of $F$.

To calculate the next contribution one takes into account pairs of self-crossing trajectories. In each pair the partners $\gamma, \gamma'$ posses the same vector $n_\gamma = n_{\gamma'}$, but traverse the bonds in different order, see fig. 3. Note, that because of broken time reversal symmetry only trajectories with an even number of encounters make contribution into (3). Furthermore, since $n$ is set to be of the same order as $\sqrt{N}$ only encounters with 2 entering and 2 exiting loops should be considered. In a sharp contrast with the unitary case (where the relevant scale is $n \sim N$), here the diagrams with $l$-encounters for $l > 2$ contribute to the subleading order $O(N^0)$ only.

The contribution from trajectories with 2$m$ encounters can be split into a product of three factors coming from: encounters $N_{\text{enc}}$, loops $N_{\text{loop}}$ connecting them and the combinatorics $N_{\text{comb}}$. The latter takes into account all possible reconnections of loops and encounters i.e., the number of different diagrams. For the diagrams with 2-encounters it is known [21] to be $N_{\text{comb}} = \frac{4^m (4m)!}{2^m (2m+1)!}$. Given 2$m$ encounters there are $\frac{4^m (4m)!}{2^m (2m+1)!}$ (to the leading order of $n$) choices to fix the lengths $\ell_{i_1}, \ldots, \ell_{i_m}$ of the loops connecting them such that the total length is fixed $\sum_{i=1}^{4m} \ell_i = n - 2 \sum_{i=1}^{4m} (k_i - 1)$, where $k_i$ is the length (i.e., the number of vertices) of the $i$-th encounter. The contribution from all possible paths of the length $\ell \gg 1$ connecting $j$th and $i$th bonds is given by $\sum_{i_1 \ldots i_{4m}} F_{i_{i_1}} \ldots F_{i_{i_{4m}}} = \chi_i \chi_j + O(N^{-\frac{1}{4}+\kappa})$. This yields for the total contribution from 4$m$ loops with fixed entering and exiting bonds:

$$N_{\text{loop}} = \frac{n^{4m}}{(4m)!} \left( \prod_{r=1}^{2m} \chi_{i_r} \chi_{j_r} \right) + O(N^{-\frac{1}{4}+\kappa})$$

Given that incoming $(i_1, i_2)$ and outgoing $(j_1, j_2)$ bonds of an encounter are fixed, the total contribution from all possible paths connecting them is

$$N_{\text{enc}}^{(1)} = (1 - \delta_{i_1, i_2})(1 - \delta_{j_1, j_2}) S_{i_1j_1} S_{i_2j_2} S_{j_1i_1} S_{j_2i_2}.$$
\[ N_{\text{enc}}^{(k)} = \sum_{i,j} (1 - \delta_{i_1,i_2})(1 - \delta_{j_1,j_2}) F_{i_1,i} F_{i_2,i} [Q^k]_{ij} F_{j_1,j} F_{j_2,j}, \]

for encounters of the lengths \( k = 1 \) (containing a single vertex) and \( k > 1 \), respectively. Here \( Q_{ij} = F_{ij}^2 \). Combining these expressions with the factors \( \tilde{x}_{i_1} \tilde{x}_{i_2} \chi_{j_1} \chi_{j_2} \) from \( N_{\text{loop}} \) and taking the sum over the indices gives for each encounter of the length \( k \):

\[
\begin{align*}
    k = 1 : & \sum_{j_1,j_2,i_1,i_2} \tilde{x}_{i_1} \tilde{x}_{i_2} \chi_{j_1} \chi_{j_2} N_{\text{enc}}^{(1)} \\
    & = \text{Tr} \left[ (S_{\chi} S^\dagger \chi)(S_{\chi} S^\dagger \chi)^2 - 2(S_{\chi} \chi)^2 Q^2 \right]; \\
    k > 1 : & \sum_{j_1,j_2,i_1,i_2} \tilde{x}_{i_1} \tilde{x}_{i_2} \chi_{j_1} \chi_{j_2} N_{\text{enc}}^{(k)} \\
    & = \text{Tr} \left[ \tilde{x}^2 Q^k \chi^2 - 2\tilde{x}^2 Q^{k+1} \chi^2 + \tilde{x}^2 Q^{k+2} \chi^2 \right].
\end{align*}
\]

After summing up over all \( k \), taking into account \( N_{\text{comb}} \) and the remaining combinatorial factor from \( N_{\text{loop}} \) we arrive at

\[ K(n) = \frac{n}{N} \sum_{m=0}^{\infty} \frac{n^{4m} \mu^{2m}}{(2N)^{2m}(2m+1)!} + O(N^{-\frac{4}{3} + \epsilon}), \quad (4) \]

which is the Taylor expansion of eq. (2). Finally, the spectral density can be restored through the relationship

\[ \rho(r) = \frac{1}{\pi^2 r^2} \lim_{\epsilon \to 0} \text{Im} R_L(r^{-1}), \quad R_L(r) = \sum_{n=1}^{\infty} (re^{2\pi \epsilon})^n K(n). \]

by substituting (2) and transforming the sum into integral. Applying the saddle point approximation to this integral in the regime \( n \sim \sqrt{N} \) results in eq. (1).

Formally eqs. (2) [1] can be also derived using the supersymmetry approach of [12]. To this end the function \( R_L(r) \) is represented as the integral over supersymmetric “fields”. The result then follows by leaving out only zero-mass mode. Contrary to the unitary case, however, even in the best case scenario of graphs with finite gaps the contribution of massive modes cannot be discounted on the basis of a rough estimation suggested in [12]. We defer the detailed discussion of the supersymmetry approach to a later publication.

In conclusion, we have shown that the spectral density and the form factor of the quantum map for strongly open quantum graphs show the universal behavior at the edges of the spectrum at the scales of mean distance between eigenvalues. We conjecture that higher order spectral correlations exhibit similar universality as well. In a sense our result can be seen as an extension of the well-established universality for closed quantum graphs. From the semiclassical point of view a strongly non-unitary case differs in the time scales involved: \( \sqrt{N} \) rather than \( N \). This results in the exclusion of all diagrams with \( l \)-encounters for \( l > 2 \). In a transitional case with weak unitarity breaking, where \( \mu^{(N)} \sim N^{-1} \), these diagrams must be actually included since they contribute to the same order (in \( n/N \)) as (4).

Acknowledgments: Financial support by SFB/TR12 and DFG research grant Gu 1208/1-1 is gratefully acknowledged.

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