Faster Queries on BWT-runs Compressed Indexes

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Abstract

Although a significant number of compressed indexes for highly repetitive strings have been proposed thus far, developing compressed indexes that support faster queries remains a challenge. Run-length Burrows-Wheeler transform (RLBWT) is a lossless data compression by a reversible permutation of an input string and run-length encoding, and it has become a popular research topic in string processing. Recently, Gagie et al. presented r-index, an efficient compressed index on RLBWT whose space usage does not depend on text length. In this paper, we present a new compressed index on RLBWT, which we call r-index-f, in which r-index is improved for faster locate queries. We introduce a novel division of RLBWT into blocks, which we call balanced BWT-sequence as follows: the RLBWT of a string is divided into several blocks, and a parent-child relationship between each pair of blocks is defined. In addition, we present a novel backward search algorithm on the balanced BWT-sequences, resulting in faster locate queries of r-index-f. We also present new algorithms for solving the queries of count query, extract query, decompression and prefix search on r-index-f.

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1 Introduction

A text index represents a string in a compressed format that supports locate queries (i.e., computing all the positions including a query on a string). Burrows-Wheeler transform (BWT) is a lossless data compression by a reversible permutation of an input string, and a wide variety of applications for the text index on BWT have been proposed (e.g., [19, 23, 13]). Highly repetitive string is a string including many repetitions. Examples include human genomes, version controlled documents and source code in repositories. A significant number of text indexes on various compressed formats for highly repetitive strings have been proposed thus far (e.g., SLP-index [6], LZ-indexes [4, 10], BT-indexes [5, 17]). For a large collection of highly repetitive strings, the most powerful and efficient is run-length BWT (RLBWT) [3], which is a BWT compressed by run-length encoding. Mäkinen et al. [15] presented the first text index named RLFM-index on RLBWT that solves locate queries by using a backward search algorithm in RLBWT. While the space usage of RLFM-index depends on string length, RLFM-index can solve locate queries in \( O(r + n/s) \) words of space and \( O((m + s \cdot \text{occ}) \log \sigma + (\log \log n)^2) \) time for length \( n \) of text \( T \), length \( m \) of query, number \( r \) of runs in RLBWT of \( T \) and alphabet size \( \sigma \), number \( \text{occ} \) of occurrences of a query in \( T \) and parameter \( s \). Recently, Gagie et al. [11] presented r-index that reduces the space usage of RLFM-index into a space not dependent on string length. The r-index can solve locate
queries more space-efficiently in only $O(r)$ words of space and $O((m + \text{occ}) \log \log w (\sigma + (n/r)))$ time for machine word size $w$.

In this paper, we present a new text index on RLBWT, which we call r-index-f, in which r-index is improved for faster locate queries. We introduce a novel division of RLBWT into blocks, which we call balanced BWT-sequence as follows: the RLBWT of a string is divided into several blocks, and a parent-child relationship between each pair of blocks is defined. In addition, we present a novel backward search algorithm on the balanced BWT-sequences, resulting in faster locate queries in $O(m \log \log w \sigma + \text{occ} \log \log w (n/r))$ time and $O(r)$ words of space. In addition, r-index-f can support the following fast queries on strings in its application, which are summarized in Table 1.

- **Count query**: r-index-f can return the number of occurrences of a query in an input string in $O(r)$ words of space and $O(m \log \log w \sigma)$ time that is not dependent on string length and the number of runs in RLBWT.

- **Extract query**: r-index-f can return substrings starting at a given position bookmarked beforehand in a string in $O(1)$ time per character and $O(r + b)$ words of space, where $b$ is the number of bookmarked positions. Solving extract queries is also known as the bookmarking problem [10, 11].

- **Decompression**: r-index-f decompresses the original string in $O(n)$ time and $O(r)$ words of space. The decompression is fastest among algorithms on compressed indexes in $O(r)$ words of space.

- **Prefix search**: r-index-f can return the strings in a set $D$ that include a query as their prefixes in $O(m + \text{occ'})$ time and $O(r')$ words of space, where $\text{occ'}$ is the number of output strings and $r'$ is the number of runs in the RLBWT of a string made by concatenating the strings in $D$.

This paper is organized as follows. In Section 2, we introduce several important notions used in this paper. In Section 3, we present the balanced BWT-sequence and r-index-f. Count and locate queries on r-index-f are also presented. Important properties of the balanced BWT-sequence are proven in Section 4. Section 5 presents the other queries that r-index-f can support.

## 2 Preliminaries

Let $\Sigma = \{1, 2, \ldots, \sigma\}$ be an ordered alphabet of size $\sigma$, $T$ be a string of length $n$ over $\Sigma$ and $|T|$ be the length of $T$. Let $T[i]$ be the $i$-th character of $T$ (i.e., $T = T[1], T[2], \ldots, T[n]$) and $T[i..j]$ be the substring of $T$ that begins at position $i$ and ends at position $j$. For two strings, $T$ and $P$, $T \prec P$ means that $T$ is lexicographically smaller than $P$. We assume that (i) $\sigma = n^{O(1)}$ and (ii) the last character of string $T$ is a special character $\$ not occurring on substring $T[1..n-1]$ and $\$ $\prec c$ holds for any character $c \in \Sigma \setminus \{\$\}. For two integers, $b$ and $e$ ($b \leq e$), interval $[b, e]$ is a set $\{b, b+1, \ldots, e\}$. $\text{Occ}(T, P)$ denotes all the occurrence positions of a string $P$ in a string $T$, i.e., $\text{Occ}(T, P) = \{i \mid i \in [1, n - |P| + 1] \text{ s.t. } P = T[i..(i + |P| - 1)]\}$.

Let $C_T$ be an array of size $\sigma$ such that $C_T[c]$ is the number of occurrences of characters lexicographically smaller than $c \in \Sigma$ in string $T$ i.e., $C_T[c] = |\{i \mid i \in [1, n] \text{ s.t. } T[i] \prec c\}|$.

Our computation model is a unit-cost word RAM with a machine word size of $w = \Omega(\log_2 n)$ bits. We evaluate the space complexity in terms of the number of machine words. A bitwise evaluation of space complexity can be obtained with a $\log_2 n$ multiplicative factor.

We use log base 2 throughout this paper if the logarithmic base is not indicated.
Table 1 Summary of methods of (i) locate and (ii) count queries on RLBWT, (iii) extract query (i.e., the bookmarking problem), (iv) decompression of BWT or RLBWT and (v) prefix search, where $n$ is the length of the input string $T$, $m$ is the length of a given string $P$, $\text{occ}$ is the number of all occurrences of $P$ in $T$, $\sigma$ is the alphabet size of $T$, $w = O(\log n)$ is the machine word size, $r$ is the number of runs in the RLBWT of $T$, $s$ is a parameter, $g$ is the size of a compressed grammar deriving $T$, $b$ is the number of input positions for the bookmarking problem, $D$ is a set of strings of total length $n$, $\text{occ}'$ is the number of the strings in $D$ such that each string has $P$ as a prefix and $r'$ is the number of runs in the RLBWT of a string made by concatenating the strings in $D$.

| (i) Locate query | Space (words) | Time |
|------------------|---------------|------|
| RLFM-index [15]  | $O((r + n/s))$ | $O((m + s \cdot \text{occ})\left(\frac{\log \log \sigma}{\log \log \sigma} + (\log \log n)^2\right))$ |
| r-index [11]     | $O(r)$        | $O((m + \text{occ}) \log \log_{\sigma}(\sigma + (n/r)))$ |
|                  | $O(r \log \log_{\sigma}(\sigma + (n/r)))$ | $O((m + \text{occ}) \log \log_{\sigma}(\sigma + (n/r)))$ |
|                  | $O(rw \log_{\sigma} \log n)$ | $O((m + \text{occ}) \log \log_{\sigma}(\sigma + (n/r)))$ |

| (ii) Count query | Space (words) | Time |
|------------------|---------------|------|
| RLFM-index [15]  | $O(r)$        | $O(m(\frac{\log \log \sigma}{\log \log \sigma} + (\log \log n)^2))$ |
| r-index [11]     | $O(r)$        | $O(m \log \log_{\sigma}(\sigma + (n/r)))$ |
|                  | $O(r \log \log_{\sigma}(\sigma + (n/r)))$ | $O(m \log \log_{\sigma}(\sigma + (n/r)))$ |
|                  | $O(rw \log_{\sigma} \log n)$ | $O(m \log \log_{\sigma}(\sigma + (n/r)))$ |

| (iii) Extract query | Space (words) | Time per character |
|---------------------|---------------|-------------------|
| Gagie et al. [10]   | $O(g + \log^2 n)$ | $O(1)$ |
| Cording et al. [2]  | $O((g + b) \max(1, \log^* g - \log^* (\frac{2}{3} - \frac{1}{3})))$ | $O(1)$ |

| (iv) Decompression | Space (words) | Time |
|--------------------|---------------|------|
| Lauuth and Lukovszki [14] | $O(n \log \log n + n \log \sigma/w)$ | $O(n)$ |
| Golynski et al. [2]   | $O(n \log \sigma/w)$ | $O(n \log \log \sigma)$ |
| Predecessor queries [2] | $O(r)$ | $O(n \log \log_{\sigma}(n/r))$ |

| (v) Prefix search | Space (words) | Time |
|------------------|---------------|------|
| Compact trie     | $(n \log \sigma/w + O(D))$ | $O(m + \text{occ})$ |
| Lauther and Lukovszki [14] | $(n \log \sigma/w + O(D))$ | $O(n \log \log \sigma/w + \log m + \log \log \sigma + \text{occ'})$ |
| Z-fast trie [1]   | $(n \log \sigma/w + O(D))$ | $O(n \log \log_{\sigma}(\sigma + (n/r)))$ |
| Packed c-trie [24] | $(n \log \sigma/w + O(D))$ | $O(m \log \log_{\sigma}(\sigma + (n/r)))$ |
| c-trie++ [22]     | $(n \log \sigma/w + O(D))$ | $O(m \log \log_{\sigma}(\sigma + (n/r)))$ |
| This study        | $O(r' + \log D)$ | $O(m + \text{occ})$ |

2.1 Predecessor, rank, count and locate queries

For an integer $x$ and a set $S$ of integers, a predecessor query $\text{pred}(S, x)$ returns the number of elements that are no more than $x$ in $S$ (i.e., $\text{pred}(S, x) = \{|y| \ y \in S \text{ s.t. } y \leq x\}$). Belazzougui and Navarro [2] proposed a predecessor data structure that solved the predecessor query for $x$ on $S$ in $O(\log \log \log u(|S|))$ time and with $O(|S|)$ words of space for the size $u$ of the universe of elements. Constructing predecessor data structures takes $O(|S| \log \log \log u(|S|))$ time and $O(|S|)$ words of space by processing $S$ [11].

A rank query $\text{rank}(T, c, i)$ on a string $T$ returns the number of occurrences of character $c$ in $T[1..i]$, i.e., $\text{rank}(T, c, i) = |\text{Occ}(T[1..i], c)|$. Belazzougui and Navarro [2] also proposed a rank data structure solving a rank query on $T$ in $O(\log \log \sigma)$ time and with $O(n)$ words of space. Constructing the rank data structure takes $O(n)$ time and $O(n)$ words of working space by processing $T$ [11].

A count query on string $T$ returns the number of occurrences of a given string $P$ in $T$, i.e., $|\text{Occ}(T, P)|$. Similarly, a locate query on string $T$ returns all the starting positions of $P$ in $T$, i.e., $\text{Occ}(T, P)$. 
2.2 Suffix array (SA), sa-interval and LF function

Suffix array (SA) [10] of a string \( T \) is an integer array \( SA \) of size \( n \) such that \( SA[i] \) stores the starting position of \( i \)-th suffix of \( T \) in lexicographical order. Formally, \( SA \) is a permutation of \([1, n]\) such that \( T[SA[1..n]] \prec \cdots \prec T[SA[n..n]] \) holds. Each value in \( SA \) is called sa-value.

Suffix array interval (sa-interval) of a string \( P \) is an interval \([b, c]\) on \([1, n]\) such that \( SA[b..c] \) represents all the occurrence positions of \( P \) in string \( T \); that is, for any integer \( p \in [1, n] \), \( T[p..p + |P| - 1] = P \) only if \( p \in \{SA[b], SA[b + 1], \ldots, SA[c]\} \).

LF is the function that returns the position with sa-value \( SA[i] - 1 \) on \( SA \) (i.e., \( SA[LF(i)] = SA[i] - 1 \)) for a given integer \( i \in [1, n] \) if \( SA[i] \neq 1 \); otherwise, it returns the position with sa-value \( n \) (i.e., \( SA[LF(i)] = n \)).

2.3 BWT and run-length BWT (RLBWT)

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
\text{i} & \text{SA} & \text{LF} & \text{F} & \text{L} & \text{i} & \text{SA} & \text{LF} & \text{F} & \text{L} \\
1 & 15 & 10 & S & baabaabaaba & b & 1 & 15 & 10 & S & baabaabaaba & b \\
2 & 7 & 11 & a & abababbaaba & b & 2 & 7 & 11 & a & abababbaaba & b \\
3 & 10 & 12 & a & abababababa & b & 3 & 10 & 12 & a & abababababa & b \\
4 & 2 & 13 & a & abababaababa & b & 4 & 2 & 13 & a & abababaababa & b \\
5 & 13 & 14 & b & baababaabaaba & b & 5 & 15 & 15 & b & baababaabaaba & b \\
6 & 5 & 15 & a & baababaababa & b & 6 & 5 & 15 & a & baababaababa & b \\
7 & 8 & 2 & a & baababbaabab & a & 7 & 8 & 2 & a & baababbaabab & a \\
8 & 11 & 3 & b & ababbaabab & a & 8 & 11 & 3 & b & ababbaabab & a \\
9 & 3 & 4 & a & ababbaabab & b & 9 & 3 & 4 & a & ababbaabab & b \\
10 & 14 & 5 & b & Shaabaababa & a & 10 & 14 & 5 & b & Shaabaababa & a \\
11 & 6 & 6 & b & aababaabab & a & 11 & 6 & 6 & b & aababaabab & a \\
12 & 9 & 7 & b & aabababababa & a & 12 & 9 & 7 & b & aabababababa & a \\
13 & 1 & 1 & b & aababaabab & b & 13 & 1 & 1 & b & aababaabab & b \\
14 & 12 & 8 & b & ababbaababa & a & 14 & 12 & 8 & b & ababbaababa & a \\
15 & 4 & 9 & b & ababbaabab & b & 15 & 4 & 9 & b & ababbaabab & b \\
\end{array}
\]

**Figure 1** Left figure illustrates the BWT, SA, LF function, \( F \) and the sorted circular strings of \( T = baababaababaaba \). Middle and right figures illustrate the two cases for toehold lemma (Lemma 2).

BWT [3] of a string \( T \) is an array \( L \) built by permutations of \( T \) as follows: (i) all the \( n \) rotations of \( T \) are sorted in the lexicographical order; (ii) \( L[i] \) for any \( i \in [1, n] \) is the last character at the \( i \)-th rotation in the sorted order. Similarly, \( F[i] \) for any \( i \in [1, n] \) is the first character at the \( i \)-th rotation in the sorted order. Formally, let \( L[i] = T[SA[LF(i)]] \) and \( F[i] = T[SA[i]] \) for any \( i \in [1, n] \). The left figure in Figure 1 illustrates the BWT, SA, LF function, \( F \) and \( L \) of a string.

BWT has the following two properties. First, for any integer \( i \in [1, n] \), \( LF(i) \) is equal to the number of characters that are lexicographically smaller than character \( L[i] \) plus the rank of \( L[i] \) on the BWT. Thus, \( LF(i) = C_L[c] + \text{rank}(L, c, i) \) holds for \( c = L[i] \). This is because \( LF(i) < LF(j) \) only if either of the following conditions holds: (i) \( L[i] < L[j] \) or (ii) \( L[i] = L[j] \) and \( i < j \) for any pair of integers \( i, j \in [1, n] \).

Second, for a character \( c \) and string \( P \), the starting and ending positions of the sa-interval of \( cP \) is respectively equal to the results obtained by applying the LF function to the first and last occurrences of character \( c \) on the sa-interval of \( P \) on BWT. This is because (i) \( P \) is a prefix of any suffix in the sa-interval of \( P \), and (ii) \( L[i] \) is the previous character of suffix \( T[SA[i..n]] \). Thus, the sa-interval of \( cP \) for any character \( c \) can be computed from the sa-interval of \( P \) by backward search as follows.
Lemma 1 (Backward search \[12\]). Let \([b, e]\) and \([b’, e’]\) be the sa-intervals of \(P\) and \(cP\) for string \(P\) and character \(c\), respectively. The following three statements hold: (i) \(L[i] = c\) for any integer \(i \in [b, e]\) only if \(L[i] = c\); (ii) \(b’ = C_L[c] + \text{rank}(L, c, b-1) + 1\) and \(e’ = C_L[c] + \text{rank}(L, c, e)\) if \([b’, e’]\) \(\neq \emptyset\) where \(\text{rank}(L, c, 0) = 0\); (iii) \(\text{rank}(L, c, e) - \text{rank}(L, c, b-1) = 0\) only if \([b’, e’]\) = \(\emptyset\).

The RLBWT of \(T\) is the BWT encoded by the run-length encoding, i.e., RLBWT is a partition of \(L\) into \(r\) substrings \(L_i^{\text{RL}} = L_1^{\text{RL}}, L_2^{\text{RL}}, \ldots, L_r^{\text{RL}}\) such that each substring \(L_i^{\text{RL}}\) is a maximal repetition of the same character in \(L\) (i.e., \(L_i^{\text{RL}}[1] \neq L_i^{\text{RL}}[1] \neq L_i^{\text{RL}}[1]\) and \(L_i^{\text{RL}}[1] = L_i^{\text{RL}}[2] = \cdots = L_i^{\text{RL}}[|L_i^{\text{RL}}|]\)). We call each \(L_i^{\text{RL}}\) run. RLBWT can be stored in \(2r\) words because we can represent each run in \(2\) words. Figure 1 shows the RLBWT of the string in the left table is \(L_i^{\text{RL}} = bbbbbbb, aaaaaaa, \$\), \(aa\).

Toehold lemma \[13\] enables us to compute all the occurrences of \(cP\) on string \(T\) using a sampled suffix array of size \(2r\) and the sa-interval of string \(P\) for any character \(c \in \Sigma\), which is formalized as follows.

Lemma 2 (Toehold lemma \[13\]). Let \([b, e]\) \((\neq \emptyset)\) and \([b’, e’]\) \((\neq \emptyset)\) be the sa-intervals of \(P\) and \(cP\) for string \(P\) and character \(c\), respectively. The following two cases hold: (i) If \([b, e] = [b’, e’]\), \(L[p] = c\) for any integer \(p \in [b, e]\); (ii) otherwise, there exists an integer \(q \in [1, r]\) such that \(L_i^{\text{RL}}[1] = c\) holds and \([b, e]\) contains the starting or ending position of \(L_q\) (i.e., \(1 + \sum_{x=1}^{q-1} |L_x^{\text{RL}}| \in [b, e]\) or \(\sum_{x=1}^{q} |L_x^{\text{RL}}| \in [b, e]\).

In case (i), any character on interval \([b, e]\) in the BWT represents an occurrence of \(cP\) (i.e., \(\text{SA}[\text{LF}(i)] \in \text{Occ}(T, cP)\) for any \(i \in [b, e]\)) because interval \([b, e]\) occurs on a run of character \(c\) in the RLBWT. Thus, we can compute an occurrence of \(cP\) if we know an occurrence \(p\) of \(P\) because \(p - 1\) is an occurrence of \(cP\).

In case (ii), the interval contains the starting or ending position of a run of character \(c\) because the interval on the BWT contains at least two distinct characters. Thus, we can compute at least one occurrence position of \(cP\) by storing sa-values on the starting and ending positions of runs in the RLBWT because \(p - 1\) is an occurrence of \(cP\) for sa-value \(p\) on the starting or ending position of a run of character \(c\) such that the position is contained in interval \([b, e]\). The middle and right figures in Figure 1 show an example of the two cases.

2.4 Data structure computing adjacent sa-values

Gagie et al. \[11\] proposed a data structure enabling computations of adjacent sa-values \(\text{SA}[i-1]\) and \(\text{SA}[i+1]\) from a given sa-value \(\text{SA}[i]\) in \(O(\log \log_m(n/r))\) time and \(O(r)\) words of space. Let \(\phi(x) = \text{SA}[i-1]\) for a given position \(i\) with sa-value \(x\) (i.e., \(\text{SA}[i] = x\)). Let \(\phi^{-1}\) be an inverse function of \(\phi\) that returns \(\text{SA}[i+1]\) for a given sa-value \(\text{SA}[i]\) (i.e., \(\phi^{-1}(\phi(\text{SA}[i])) = \text{SA}[i]\)). The data structure was used for supporting locate queries in \[11\].

3 r-index-f

In this section, we present r-index-f, a new text index on RLBWT supporting faster count and locate queries. Formally, we show the following theorem.

Theorem 3. There exists a text index of \(O(r)\) words supporting count and locate queries on a string \(T\) in \(O(|P| \log \log_m \sigma)\) and \(O(|P| \log \log_m \sigma + \text{occ} \log \log_m (n/r))\) time, respectively, where \(P\) is a given string with count or locate query and \(\text{occ} = |\text{Occ}(T, P)|\). We can construct the data structure in \(O(n + r \log r)\) time and \(O(r)\) words of working space by processing the RLBWT of \(T\).
r-index-f is built on the notion of a novel partition of the BWT of string \( T \), which we call a balanced BWT-sequence. The balanced BWT-sequence is introduced in Section 3.1. We present r-index-f and queries in Section 3.2. Due to space limitations, we present a construction algorithm of r-index-f in Appendix A.

### 3.1 Balanced BWT-sequence

We introduce an important notion of the balanced BWT-sequence. A BWT-sequence of a string \( T \) is a sequence of strings \( L' = L_1, L_2, \ldots, L_k \) satisfying the following two conditions: (i) the concatenation of the strings is equal to the BWT of \( T \) (i.e., \( L = L_1 L_2 \cdots L_k \)); (ii) each string \( L_i \) is a repetition of a character (i.e., \( L_i[1] = L_i[2] = \cdots = L_i[|L_i|] \)).

\[ L = \{ \ell^+_1, \ell^+_2, \ldots, \ell^+_k \} \]

is defined as the set of the starting positions of phrases in \( L' \), i.e., \( \ell^+_1 = 1 \) and \( \ell^+_i = \ell^+_{i-1} + |L_{i-1}| \) for \( i \in [2, k] \).

For the BWT \( \text{bbbbbbaaaaaa$saaa$} \) of string \( T = \text{baababaaabab} \), two examples of BWT-sequences are \( b^3, b^3, a^6, s, a^2 \) and \( b^2, b, b^2, b, a^6, s, a^2 \), where \( c^d \) denotes \( d \) repetitions of the same character \( c \). For any integer \( i \in [1, |L|] \), such that \( |\text{LF}(\ell^+_i) \| < |\text{LF}(\ell^+_2) \| < \cdots < |\text{LF}(\ell^+_k) \| \) holds. Then, \( F_i = L_{\lambda_i} \) holds for any integer \( i \in [1, k] \).

We call each string in \( L' \) and the F-sequence phrase. \( F^+ = \{ f^+_1, f^+_2, \ldots, f^+_k \} \) is defined as the set of the starting positions of phrases in \( F' \) (i.e., \( f^+_1 = |\text{LF}(\ell^+_1)| \)) is the starting position of phrase \( F_1 \).

\[ \text{Figure 2} \text{ Examples of a BWT-sequence and the corresponding F-sequence.} \]

A parent and children relationship between phrases can be defined. Children of the \( i \)-th phrase \( L_i \) are defined as a set of phrases such that the starting position of each phrase in the set is contained in interval \( |\text{LF}(\ell^+_i) \| + [|L_i| - 1] \), where interval \( |\text{LF}(\ell^+_i) \| + [|L_i| - 1] \) is equal to the interval of phrase \( F_i' \) corresponding to \( L_i \) by the LF function (i.e., \( |\text{LF}(\ell^+_i) \| + [|L_i| - 1] \)). We define \( \text{children}(L', i) \) as a set of the starting positions of the children of the \( i \)-th phrase \( L_i \) in BWT-sequence \( L' \) (i.e., \( \text{children}(L', i) = \{ \ell^+_j | j \in [1, k] \text{ s.t. } \ell^+_j \in [\text{LF}(\ell^+_i) \|, |L_i| - 1] \} \)). For \( L' = b^3, b^3, a^6, s, a^2 \) in Figure 2, \( \text{children}(L', 1) = \emptyset \), \( \text{children}(L', 2) = \{ 13, 14 \} \), \( \text{children}(L', 3) = \{ 4, 7 \} \), \( \text{children}(L', 4) = \{ 1 \} \) and \( \text{children}(L', 5) = \emptyset \).

A BWT-sequence \( L' \) of string \( T \) is balanced if (i) any phrase \( L_i \) has at most three children (i.e., \( |\text{children}(L', i)| \leq 3 \) for any integer \( i \in [1, k] \)), and (ii) the number of phrases in \( L' \) is at most \( 2r \) (i.e., \( k \leq 2r \)). The left BWT-sequence in Figure 2 is balanced because any phrase has at most two children and the size of the sequence is no more than \( 2r(= 8) \). On the other hand, the right BWT-sequence is not balanced because the fifth phrase in the BWT-sequence has four children.

There must exist at least one balanced BWT-sequence of any string as follows.

\[ \text{Theorem 4. } \text{The following two statements hold:} \ (i) \text{there exists a balanced BWT-sequence} \]
of any string $T$; and (ii) we can construct a balanced BWT-sequence of $T$ in $O(r \log r)$ time and $O(r)$ words of working space by processing the RLBWT of $T$.

Proof. See Section 2

### 3.2 Data structures and search algorithms

Our text index consists of the following six data structures:

(i) A balanced BWT-sequence $L' = L_1, L_2, \ldots, L_k$ of $T$;
(ii) Five arrays of size $k$ storing (1) $\ell_1^+, \ell_2^+, \ldots, \ell_k^+$ (2) $f_1^+, f_2^+, \ldots, f_k^+$, (3) $|F_1|, |F_2|, \ldots, |F_k|$, (4) $SA[f_1^+, \ldots, SA[f_k^+]$, and (5) $SA[f_1^++|F_1| - 1], \ldots, SA[f_k^++|F_k| - 1]$;
(iii) A rank data structure for a string $L_{\text{first}}$ such that each $i$-th character is the first character of phrase $L_i$ (i.e., $L_{\text{first}} = L_1[1], L_2[1], \ldots, L_k[1]$);
(iv) An array $D_{\text{index}}$ of size $k$ such that $D_{\text{index}}[i]$ stores the index of the phrase in $L'$ containing character $L[i]$ (i.e., $D_{\text{index}}[i] = \text{pred}(L^+, f_i^+)$);
(v) The data structure for the functions $\phi$ and $\phi^{-1}$ introduced in Section 2.4;
(vi) An array $V_2$ of size $\sigma$ such that $V_2[c]$ stores a 5-tuple $(C_{L_{\text{first}}}[c], b, e, \text{pred}(F^+, b), \text{pred}(F^+, e))$ for sa-interval $[b, e)$ of character $c$.

As shown in Theorem 4, the space usage of those data structures is $O(r)$ words in total, which we prove in Appendix A.

#### 3.2.1 Fast predecessor query on the BWT-sequence

**Figure 3** Two phrases $F_{\text{pred}(F^+, p)}$, $L_{\text{pred}(L^+, p)}$ for position $p$, and phrase $L_j$ for $j = D_{\text{index}}[\text{pred}(F^+, p)]$ (left). The sa-intervals of two strings $s$ and $cs$ on a balanced BWT-sequence (right).

Solving a predecessor query on set $L^+$ (i.e., $\text{pred}(L^+, p)$) is essential for count and locate queries. Thus, we present an algorithm that returns $\text{pred}(L^+, p)$ in constant time for a given position $p$ and $\text{pred}(F^+, p)$. A linear search can be adopted for solving predecessor queries by leveraging the balanced BWT-sequence. Phrases $L_1, L_{j+1}, \ldots, L_k$ are searched one by one for integer $j$ stored in $D_{\text{index}}[\text{pred}(F^+, p)]$ (i.e., $j = D_{\text{index}}[\text{pred}(F^+, p)]$), and phrase $L_{\text{pred}(L^+, p)}$ is found as a solution. This is made possible because $L_j$ contains the starting position of $F_{\text{pred}(F^+, p)}$ and position $p$ is contained in two phrases $F_{\text{pred}(F^+, p)}$ and
\(L_{\text{pred}(L^+, p)}\) (i.e., \(\text{pred}(L^+, p) \geq j\)). Computation complexity is \(O(\text{pred}(L^+, p) - j + 1)\) time by using the array storing the starting positions of phrases in \(L'\). This computation complexity can be bounded by \(O(1)\) (i.e., \(\text{pred}(L^+, p) - j = O(1)\)) because any phrase in a balanced BWT-sequence has at most three children. Formally, the following lemma holds.

**Lemma 5.** \(0 \leq \text{pred}(L^+, p) - j \leq 3\) for any integer \(p \in [1, n]\) and \(j = D_{\text{index}}[\text{pred}(F^+, p)]\).

An example of three phrases \(F_{\text{pred}(F^+, p)}, L_{\text{pred}(L^+, p)}\) and \(L_j\) is presented in the left figure in Figure 3.

In the remaining section, we present two new algorithms for count and locate queries using the predecessor query.

### 3.2.2 Count query

A count query computes the length of an sa-interval for a given query \(P\) using a backward search on \(L'\). Let \([b, e]\) be the sa-interval of string \(s\) and let \([b', e']\) be the sa-interval of string \(cs\) for character \(c\). Given a 5-tuple \((b, e, c, \text{pred}(F^+, b), \text{pred}(F^+, e))\), the backward search on \(L'\) returns a 4-tuple \((b', e', \text{pred}(F^+, b'), \text{pred}(F^+, e'))\). Our backward search algorithm consists of the following two steps: (i) compute \(\text{pred}(L^+, b)\) and \(\text{pred}(L^+, e)\) by Lemma 5 (ii) compute \((b', e', \text{pred}(F^+, b'), \text{pred}(F^+, e'))\) by rank queries on string \(L_{\text{first}}\).

The right figure in Figure 3 illustrates the relation between \(L_{\text{first}}\) and the first characters of strings \(F^\prime\). Similarly, \(\text{pred}(F^+, e')\) is equal to \(\text{rank}(L_{\text{first}}, c, \text{pred}(L^+, e)) + C_{\text{num}}[c]\). Thus, we compute \(\text{pred}(F^+, b')\) and \(\text{pred}(F^+, e')\) using two rank queries on \(L_{\text{first}}\). We also verify whether \([b', e']\) is empty because the above algorithm can return incorrect \(\text{pred}(F^+, b')\) and \(\text{pred}(F^+, e')\) if \([b', e']\) is empty. \([b', e']\) is empty only if substring \(L_{\text{first}}[\text{pred}(L^+, b) . \text{pred}(L^+, e)]\) does not contain character \(c\), and hence we complete the verification by two rank queries on \(L_{\text{first}}\).

Next, \(b'\) can be computed as \(f^+_{\text{pred}(F^+, b')}\) if \(L_{\text{first}}[\text{pred}(L^+, b)] \neq c\) because interval \([b, e]\) completely contains phrase \(L_j\) corresponding to \(F_{\text{pred}(F^+, b)}\) (i.e., \(j = \lambda_{\text{pred}(F^+, b)}\)). Otherwise, \(b'\) can be computed as \(f^+_{\text{pred}(F^+, b')} + (b - t^+_{\text{pred}(L^+, b)})\) because \(L_{\text{pred}(L^+, b)}\) corresponds to \(F_{\text{pred}(F^+, b')}\) and the \((b - t^+_{\text{pred}(L^+, b)} + 1)\)th character of \(L_{\text{pred}(L^+, b)}\) is the first occurrence of character \(c\) in \([b, e]\). Similarly, integer \(e'\) is computed. Thus, we can compute \(b'\) and \(e'\) in constant time after computing \(\text{pred}(F^+, b')\) and \(\text{pred}(F^+, e')\). Our backward search algorithm runs in \(O(\log \log w, \sigma)\) time. The following two lemmas hold.

**Lemma 6.** The following two statements hold: (i) \([b', e'] \neq \emptyset\) only if \(L_{\text{first}}[\text{pred}(L^+, b) . \text{pred}(L^+, e)]\) contains character \(c\) (i.e., \(\text{rank}(L_{\text{first}}, c, \text{pred}(L^+, e)) - \text{rank}(L_{\text{first}}, c, \text{pred}(L^+, b) - 1) \geq 1\); (ii) \(\text{pred}(F^+, b') = C_{\text{num}}[c] + \text{rank}(L_{\text{first}}, c, \text{pred}(L^+, b) - 1) + 1\) and \(\text{pred}(F^+, e') = C_{\text{num}}[c] + \text{rank}(L_{\text{first}}, c, \text{pred}(L^+, e))\) unless \([b', e'] \neq \emptyset\).

**Lemma 7.** Assume that \([b', e'] \neq \emptyset\). Then, \(b' = f^+_{\text{pred}(F^+, b')}\) if \(\text{pred}(L^+, b) [1] \neq c\); otherwise \(b' = f^+_{\text{pred}(F^+, b')} + (b - t^+_{\text{pred}(L^+, b)})\). Similarly, \(e' = f^+_{\text{pred}(F^+, e')}\) if \(\text{pred}(L^+, e) [1] \neq c\); otherwise \(e' = f^+_{\text{pred}(F^+, e')} + (e - t^+_{\text{pred}(L^+, e)})\).

The sa-interval of a given string \(P\) can be computed by executing the backward search \(|P| - 1\) times. Thus, a count query can be computed in \(O(|P| \log \log w, \sigma)\) time.
3.2.3 Locate query

A locate query enumerates all the sa-values in the sa-interval $[b, e]$ of $P$ using two functions, $\phi$ and $\phi^{-1}$. The sa-interval $[b, e]$ of $P$ is computed by the count query, and $\text{SA}[b, \ldots, e - 1]$ and $\text{SA}[b + 1, e]$ are recursively computed for position $\hat{p} \in [b, e]$ by applying $\phi$ and $\phi^{-1}$ to $\text{SA}[p]$, respectively.

Let $[b, e]$ be an sa-interval of string $P[t..|P|]$ and let $[b', e']$ be an sa-interval of string $cP[t..|P|]$ for an integer $t \in [2, |P|]$, where $c = P[t - 1]$. A basic idea of our algorithm is to compute $p'$ and $\text{SA}[p']$ for a given 7-tuple $(h, c, b', e', p, \text{SA}[p], \text{pred}(F^+, b'))$ by the toehold lemma in Lemma 2, where $p$ and $p'$ are two positions such that $p \in [b, e]$ and $p' \in [b', e']$. We compute $\hat{p}$ and $\text{SA}[\hat{p}]$ by executing the algorithm for each $t = |P|, |P| - 1, \ldots, 2$, which is explained next.

The toehold lemma ensures $\text{LF}(p) \in [b', e']$ and $\text{SA}[\text{LF}(p)] = \text{SA}[p] - 1$ if two sa-intervals are the same size (i.e., $|[b, e]| = |[b', e']|$). In addition, it ensures $\text{LF}(p) = b' + (p - b)$ because $L[b + i - 1]$ corresponds to $F[b' + i - 1]$ by the LF function for any integer $i \in [1, e - b + 1]$. If $|[b, e]| = |[b', e']|$, our algorithm returns $\text{LF}(p)$ and $\text{SA}[\text{LF}(p)]$. Otherwise, $[b', e']$ contains the starting or ending position of phrase $F_{\text{pred}(F^+, b')}$ because $[b, e]$ contains the starting or ending position of phrase $L_{y}$ corresponding to $F_{\text{pred}(F^+, b')}$ in this case, where $y = \text{pred}(F^+, b')$. If $[b, e]$ contains the starting position of $F_{\text{pred}(F^+, b')}$, our algorithm returns the starting position of $F_{\text{pred}(F^+, b')}$ and its sa-value; otherwise it returns the ending position of $F_{\text{pred}(F^+, b')}$ and its sa-value. Since the data structures store the sa-values on the starting and ending positions of any phrase in $F^+$, $p'$ and $\text{SA}[p']$ can be returned in constant time. The total running time for locate query is $O(|P| \log \sigma + oce \log \sigma (n/r))$. The following lemma holds.

Lemma 8 (Toehold lemma on BWT-sequence). Let $[b, e]$ and $[b', e']$ be the sa-intervals of $P$ and $cP$ for a string $P$ and character $c$, and let $p$ be an integer in $[b, e]$. Then, the following three statements hold: (i) $b' + (p - b) \in [b', e']$ and $\text{SA}[b' + (p - b)] = \text{SA}[p] - 1$ if $|[b, e]| = |[b', e']|$. (ii) $\text{pred}(F_{\text{pred}(F^+, b')}) \in [b', e']$ if $|[b, e]| \neq |[b', e']|$ and $b' = \text{pred}(F_{\text{pred}(F^+, b')})$; (iii) $\text{pred}(F_{\text{pred}(F^+, b')}) + |F_{\text{pred}(F^+, b')}| - 1 \in [b', e']$ otherwise.

4 Proofs for Theorem 4

In this section, we prove Theorem 4 by introducing a new notion named balanced RLBWT as a variant of a BWT-sequence. Due to a space limitation, a construction algorithm of the balanced RLBWT is presented in Appendix B. The construction algorithm runs in $O(n + r \log r)$ time and is used for proving the second statement in Theorem 4.

4 Balanced RLBWT

We introduce a split operation on BWT-sequences, which is used for defining the balanced RLBWT. For a given BWT-sequence $L' = L_1, L_2, \ldots, L_k$, split operation $\text{split}(L')$ splits a phrase $L_x = L^+_x, L^+_{x+1} + [L_x - 1]$ with at least four children (i.e., $x \in \{i | i \in [1, k] \text{ s.t. } |\text{children}(L', i)| \geq 4\}$) into two phrases $L_x[1..y - 1]$ and $L[y..|L_x|]$ with integer $y$ such that LF($\ell^+_x$ + $y$) equals to the starting position of the third child of phrase $L_x$.

Let $v$ be the smallest integer such that LF($\ell^+_{v+1}$) $\leq \ell^+_v$ holds. Integers $\ell^+_v$, $\ell^+_{v+1}$, and $\ell^+_{v+2}$ are the starting positions of the first, second and third children of phrase $L_x$, respectively. In addition, LF($\ell^+_x$ + $y$) $= \ell^+_{v+1}$ holds. split($L'$) is the BWT-sequence obtained by applying a split operation to $L'$, which is defined as follows: split($L'$) = $L_1, L_2, \ldots, L_{x-1}, L_x[1..y - 1], L_{x+1}, \ldots, L_k$ if $\{i | i \in [1, k] \text{ s.t. } |\text{children}(L', i)| \geq 4\}$ $\neq \emptyset$; split($L'$) = $L'$ otherwise.
Figure 4 illustrates the split operation of BWT-sequence $L' = b, c, c, b, a^3, c^2, s, c, c, a^3, b^4$. The split operation can split phrase $L_5$ or $L_{11}$ because both phrases have four children. In the example, the split operation splits $L_{11}$, and hence $x = 11$ and $y = 3$.

Balanced RLBWT balance($L^{RL}$) of a string $T$ is BWT-sequence computed by recursively applying the split operation to the RLBWT of $T$ (i.e., $L^{RL}$) until any phrase has at most three children in the obtained BWT-sequence. The algorithm for a BWT-sequence $L'$ is recursively presented as follows: (i) balance($L'$) = balance(split($L'$)) if split($L'$) $\neq L'$; (ii) balance($L'$) = $L'$ otherwise.

4.2 Proof of Theorem 4(i)

We show that the balanced RLBWT of a string $T$ is a balanced BWT-sequence of $T$, i.e., (i) any phrase in the balanced RLBWT has at most three children and (ii) the number of phrases in the balanced RLBWT is at most $2r$. The first statement holds because we apply split operations to the RLBWT until any phrase has at most three children. Let |split| be the number of the split operations applied to the RLBWT for constructing the balanced RLBWT. We prove the second statement using a property of the balanced RLBWT such that the number of the phrases in the balanced RLBWT is at least \(2|\text{split}|\) (i.e., balance($L^{RL}$) $\geq 2|\text{split}|$). Since the number of phrases in a given BWT-sequence increases by only one per split operation, balance($L^{RL}$) = |split| $+ r$ holds.

In addition, since equality balance($L^{RL}$) = |split| $+ r$ and inequality balance($L^{RL}$) $\geq 2|\text{split}|$ hold, split operations are conducted at most $r$ times. Thus, balance($L^{RL}$) = $r + |\text{split}|$ $\leq 2r$ holds, i.e., the second statement holds if balance($L^{RL}$) $\geq 2|\text{split}|$, which is proved below.

The number of phrases with at least two children in a given BWT-sequence increases by at least one per split operation, i.e., $\{i \mid i \in [1,|\text{split}(L')|]\}$ s.t. $\text{children}(\text{split}(L'), i) \geq 2 \} \geq \{i \mid i \in [1,|L'|]\}$ s.t. $\text{children}(L', i) \geq 2 \} + 1$ if split($L'$) $\neq L'$. This is because (i) the split operation removes a phrase from $L'$, (ii) it adds two new phrases with at least two children into $L'$ and (iii) it does not decrease the number of children of any phrase in $L'$ except for the removed phrase. Since the balanced RLBWT is constructed using |split| split operations, balance($L^{RL}$) $\geq 2|\text{split}|$ is obtained. Thus, Theorem 4(i) holds.

Let $x$ be the index of the split phrase and let $y$ be the split position in the split phrase by the split operation for BWT-sequence $L' = L_1, L_2, \ldots, L_k$ (i.e., split($L'$) = $L_1, L_2, \ldots, L_{x-1}, L_x[1..y-1], L_x[y..L_x[|L_x|]], L_{x+1}, \ldots, L_k$). Let map($i$) be the index of the phrase in split($L'$) corresponding to $L_i$ in $L'$ for an integer $i \in ([1, k]\setminus\{x\}$), i.e., map($i$) = $i$ if $i < x$; map($i$) = $i + 1$ otherwise. The following lemma holds.

**Lemma 9.** The following three statements hold:

(i) $|\text{children}(L', x)|, |\text{children}(\text{split}(L'), x)|, |\text{children}(\text{split}(L'), x + 1)| \geq 2$;

(ii) For an integer $i \in ([1, k]\setminus\{x\}$); $\text{children}(\text{split}(L'), \text{map}(i)) = \text{children}(L', i)$ if $\ell_x^+ + y \notin \{L_x[1], L_x[1] + [L_x[|L_x|] - 1]; \text{children}(\text{split}(L'), \text{map}(i)) = \text{children}(L', i) \cup \{\ell_x^+ + y\}$ otherwise.

(iii) $|\{i \mid i \in [1,|\text{split}(L')|]\}$ s.t. $\text{children}(\text{split}(L'), i) \geq 2 \} \geq \{i \mid i \in [1,|L'|]\}$ s.t. $\text{children}(L', i) \geq 2 \} + 1$ if split($L'$) $\neq L'$. 

Proof. See Appendix.

5 Applications

5.1 Extract query

Let string $T$ of length $n$ have $b$ marked positions $i_1, i_2, \ldots, i_b \in [1, n]$. An extract query (also known as the bookmarking problem) is to return substring $T[i_j, i_j + d]$ for a given integer $j \in [1, b]$ and $d \in [1, n - i_j + 1]$. We present a new data structure for solving extract queries in constant time per character by leveraging $r$-index-$L$. Our data structure solves extract queries on the reversed string of $T$ and returns substring $T[i_j - d + 1..i_j]$ in constant time per character.

We use the following data structures: (i) a balanced BWT-sequence of $T$; (ii) a data structure that returns pair $(\text{LF}(i), \text{pred}(L^+, \text{LF}(i)))$ in $O(r)$ words of space and in a constant time for a given integer $i$ and $\text{pred}(L^+, i)$, and the data structure uses the modified algorithm for predecessor query introduced in Section 3.2.1 to compute the output pair; (iii) the string $L_{\text{first}}$ introduced in Section 3.2; (iv) an array of size $b$ such that $j$-th element is a pair $(i'_j, \text{pred}(L^+, i'_j))$ for any integer $j \in [1, b]$, where $i'_j$ is the position with $\text{sa}$-value $i_j$ on the $\text{SA}$ of $T$ (i.e., $\text{SA}[i'_j] = i_j$). The space usage is $O(r + b)$ words in total.

Our algorithm for the extract query computes character $T[i_j - t]$ on the BWT of $T$ by recursively applying the LF function $t$ times to index $i'_j$ for any integer $t \in [0, d - 1]$, i.e., the algorithm outputs $L[\text{LF}_t(i'_j)]$ as $T[i_j - t]$. Here, $\text{LF}_t(i)$ is the LF function recursively applied to a given position $i$ $t$ times, i.e., $\text{LF}_t(i) = \text{LF}_{t-1}(\text{LF}(i))$ if $t \geq 1$; $\text{LF}_t(i) = i$ otherwise. Thus, $L[\text{LF}_t(i'_j)]$ can be computed as $T[i_j - t]$ because $L[i]$ corresponds to $T[\text{SA}[i] - 1]$ (i.e., $L[i] = T[\text{SA}[i] - 1]$) for any integer $i$, and $L[\text{LF}(i)]$ corresponds to $T[\text{SA}[i] - 2]$. Therefore, the output of our algorithm is equal to substring $T[i_j - d + 1..i_j]$.

Our algorithm consists of the following two steps: (i) compute $d - 1$ pairs $(\text{LF}_1(i'_j), \text{pred}(L^+, L_{\text{first}}(i'_j))), (\text{LF}_2(i'_j), \text{pred}(L^+, \text{LF}_1(i'_j))), \ldots, (\text{LF}_{d-1}(i'_j), \text{pred}(L^+, \text{LF}_{d-2}(i'_j)))$ by the second data structure; (ii) compute $L[\text{LF}_t(i'_j)]$ using $L[\text{LF}_t(i'_j)] = L_{\text{first}}[\text{pred}(L^+, \text{LF}_{t-1}(i'_j))]$ for any integer $t \in [0, d - 1]$. The running time is $O(d)$ in total. Formally, we obtain the following theorem.

Theorem 10. There exists a data structure of $O(r_{\text{rev}} + q)$ words that solves the bookmarking problem for string $T$ and $b$ positions $i_1, i_2, \ldots, i_b$ ($1 \leq i_1 < i_2 < \cdots < i_b \leq n$), where $r_{\text{rev}}$ is the number of runs in the RLBWT of the reversed string $T[n]T[n - 1] \cdots T[1]$ of $T$. The data structure can be constructed by processing the RLBWT and positions $i_1, i_2, \ldots, i_b$ in $O(n)$ time and $O(r_{\text{rev}} + b)$ words of space.

Proof. See Appendix.

5.2 Decompression of RLBWT

Theorem 10 with position 1 ensures that our data structure can return string $T[1..n]$ in $O(n)$ time (i.e., the data structure can recover the original string $T$ from the RLBWT of string $T$ in linear time to the length of $T$). The $O(n)$ time decompression is the fastest among other decompression algorithms on compressed indexes in $O(r)$ words of space. We obtain the following theorem.

Theorem 11. We can enumerate the characters of $T$ in the right-to-left order (i.e., $T[n], T[n - 1], \ldots, T[1]$) in $O(n)$ time and $O(r)$ words of space by processing the RLBWT of string $T$.

Proof. See Appendix.
5.3 Prefix search

Prefix search for a set of strings \( D = \{T_1, T_2, \ldots, T_d\} \) returns the indexes of the strings in \( D \) that include a given string \( P \) as their prefixes (i.e., \( \{i \mid i \in [1, d] \text{ s.t. } T_i[1..|P|] = P\} \)).

We construct a data structure supporting the prefix search by combining Theorem 10 with compact trie.

A compact trie for a set of strings \( D \) is a trie for \( D \) such that all unary paths are collapsed, and each node represents the string by concatenating labels on the path from the root to the node. For simplicity, we assume that any string in \( D \) is not a prefix of any other string, and hence each leaf in the compact trie represents a distinct string in \( D \).

Our data structure consists of the following two data structures. The first data structure is a compact trie for \( D \) without the strings on the edges where each internal node stores (i) the number of the leaves under the node, (ii) the pointers to the leftmost and rightmost leaves under the node and (iii) a perfect hash table storing the first character of the outgoing edges; each leaf stores the index of the string represented by the leaf and the pointer to the right leaf. The second data structure is the data structure presented in Theorem 10 and stores the strings on the edges. The data structure can be stored in \( O(r' + d) \) words for the number \( r' \) of runs in the RLBWT of the reversed string of a string \( T \) made by concatenating the strings in \( D \) (i.e., \( |Occ(T, T_i)| \geq 1 \) for any integer \( i \in [1, d] \)).

Let \( v \) be the lowest node such that \( P \) is a prefix of the string represented by the node. To answer a prefix search query, we traverse the path from the root to the lowest node \( v \), and output the indexes stored in the leaves under the lowest node. Therefore, the running time is \( O(|P| + occ') \), where \( occ' \) is the number of the leaves under the lowest node.

The data structure can also return the number of the leaves (i.e., the number of strings in \( D \) that include \( P \) as their prefixes) in \( O(|P|) \) time. Formally, we obtain the following theorem.

\[ \textbf{Theorem 12.} \text{ There exists a data structure that supports prefix search in } O(|P| + occ') \text{ time and } O(r' + d) \text{ words of space for a set of strings } D = \{T_1, T_2, \ldots, T_d\}. \text{ Here, } r' \text{ is the number of runs in the RLBWT of the reversed string of a string } T \text{ containing the strings in } D, \text{ and } occ' \text{ is the number of the strings in } D \text{ that include a given string } P \text{ as their prefixes. The data structure also returns the number of the strings in } D \text{ that include } P \text{ as their prefixes in } O(|P|) \text{ time.} \]

6 Conclusion

We presented r-index-f, a new text index that can support count and locate queries on RLBWT. R-index-f works in \( O(r) \) words of space and can support various fast queries on RLBWT, such as extract query, decompression and prefix search. We presented a decompression algorithm working in time linearly proportional to a text length and space linearly proportional to the number of runs in the RLBWT of the text. The decompression is fastest among algorithms on compressed indexes in \( O(r) \) words of space.

We introduced a balanced BWT-sequence (i.e., a novel division of RLBWT), and then we presented a faster backward search and LF function working in \( O(r) \) words of space by leveraging the balanced BWT-sequence. Both algorithms are general and applicable to various queries on RLBWT. Thus, an important future work is to develop algorithms for faster queries on RLBWT.
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Appendix A: Proofs for Section 3

Proof of Theorem 3 To prove Theorem 3 we use the following four lemmas.

Lemma 13. For a given BWT-sequence \( L' = L_1, L_2, \ldots, L_k \) for a string \( T \), we can compute permutation \( \lambda_1, \lambda_2, \ldots, \lambda_k \) in \( O(n) \) time and \( O(k) \) words of working space by processing \( L' \), where the permutation is introduced in Section 3.1.

Proof. We can construct the permutation by applying a stable sorting algorithm to the characters in string \( L_{\text{first}} \) because \( LF(i^+, \lambda_1) < LF(i^+, \lambda_2) \) only if either of the following conditions holds: (i) \( L[i^+] < L[i^+] \) or (ii) \( L[i^+] = L[i^+] \) and \( x < y \) for any pair of integers \( x, y \in [1, k] \) by the first property of BWT. We sort the characters in \( O(n) \) time by using a standard sorting algorithm if \( k < \frac{n}{\log n} \); otherwise we use an LSD radix sort with a bucket size of \( \frac{n}{\log n} \), i.e., we sort the characters with bucket sort in \( O(n/\log n + k) \) time by using the \( \log(n/\log n) \) bits starting at position \( (i-1)\log(n/\log n) \) for each step \( i = 1, 2, \ldots, \frac{n}{\log n} \). The space usage of the radix sort is \( O(n/\log n + k) = O(k) \) words and the running time is \( O(n) \) time in total because \( \frac{n}{\log n} \).

Lemma 14. For a string \( T \), there exists a data structure of \( O(r) \) words that returns \( LF(i) \) and \( L[i] \) in \( O(\log \log_w(n/r)) \) time for a given integer \( i \in [1, n] \). We can construct the data structure in \( O(n) \) time and \( O(r) \) words of working space by processing the RLBWT of \( T \) if \( r > \frac{n}{\log n} \).

Proof. Let \( L' \) be the BWT-sequence that is equal to the RLBWT of \( T \) (i.e., \( L_i = L_{\text{RL}}[i] \)) for any integer \( i \in [1, r] \). We use the following four data structures: (i) a string \( L_{\text{first}} \); (ii) two arrays storing \( \ell_1^+, \ell_2^+, \ldots, \ell_r^+ \) and \( f_1^+, f_2^+, \ldots, f_r^+ \); (iii) an array \( \Lambda \) storing the inverse permutation of \( \lambda_1, \lambda_2, \ldots, \lambda_r \) (i.e., \( \Lambda[\lambda_i] = i \) for any integer \( i \in [1, r] \)); (iv) the predecessor data structure for set \( L^+ \). The space usage is \( O(r) \) words in total.

\( L[i] \) is equal to \( L_{\text{first}}[\Lambda[\lambda_i]] \) for any integer \( i \in [1, n] \), and hence we can compute \( L[i] \) in \( O(\log \log_w(n/r)) \) time. Next, \( LF(i) = \ell_{\text{pred}(L^+, i)}^+ - \ell_{\text{pred}(L^+, i)}^+ \) holds because \( LF(i) = \ell_{\text{pred}(F^+, LF(i))}^+ - \ell_{\text{pred}(L^+, i)}^+ \) holds by the first property of BWT, and \( \Lambda[\text{pred}(L^+, i)] \) stores \( \text{pred}(F^+, LF(i)) \), i.e., \( \Lambda[\text{pred}(L^+, i)] \) corresponds to \( F_{\text{pred}(F^+, LF(i))} \) by the LF function. Thus, we can also compute \( LF(i) \) in \( O(\log \log_w(n/r)) \) time.

Next, we give a construction algorithm for the four data structures. We construct array \( \Lambda \) in \( O(n) \) time by using the permutation \( \lambda_1, \lambda_2, \ldots, \lambda_r \) obtained by Lemma 13. The other data structures can be constructed in \( O(r \log \log_w(n/r)) \) time by processing the permutation and RLBWT. Therefore, the construction time and working space for the four data structures are \( O(n + r \log \log_w(n/r)) = O(n) \) and \( O(r) \) words, respectively, if \( r > \frac{n}{\log n} \).

Lemma 15. Let \( L' \) be a balanced BWT-sequence of a string \( T \). Then, there exists a data structure of \( O(r) \) words that returns a 4-tuple \( (LF(i), \text{pred}(F^+, LF(i)), \text{pred}(L^+, LF(i)), \text{pred}(L_{\text{RL}}^+, LF(i))) \) in constant time for a given integer \( i \in [1, n] \) and \( \text{pred}(L^+, i) \), where \( L_{\text{RL}}^+ \) is the set of the starting positions of runs in the RLBWT of \( T \), i.e., \( L_{\text{RL}}^+ = \{1, 1 + \sum_{x=1}^1 |L_x^{|}, 1 + \sum_{x=2}^2 |L_x^{|}, \ldots, 1 + \sum_{x=n}^{n-1} |L_x^{|}\} \). The data structure can be constructed in \( O(n) \) time and \( O(r) \) words of working space by processing \( L' \).

Proof. We use the following four data structures: (i) three arrays of size \( k \) storing (1) \( |L_1|, |L_2|, \ldots, |L_k| \), (2) \( \ell_1^+, \ell_2^+, \ldots, \ell_k^+ \), and (3) \( f_1^+, f_2^+, \ldots, f_k^+ \); (ii) an array \( \Lambda \) of size \( k \) introduced in the proof of Lemma 14; (iii) an array \( D_{\text{index}} \) introduced in Section 3.2; (iv) an array \( D_{\text{RL}} \) of size \( k \) such that \( D_{\text{RL}}[i] \) stores the index of the run containing \( L_i \) for any
integer \( i \in [1,k] \), i.e., \( D_{RL}[i] = \text{pred}(L^+_{RL}, \ell'^+_i) \). The space usage is \( O(k) = O(r) \) words in total by Theorem 4.

\[ i - \ell'^+_i = \text{LF}(i) - f^+_{\lambda}(L^+_{RL}, i) \]

holds for any integer \( i \) (See the proof of Lemma 14), and hence we can compute \( \text{LF}(i) \) and \( \text{pred}(F^+, \text{LF}(i)) \) in constant time. Next, we can compute \( \text{pred}(L^+, \text{LF}(i)) \) in constant time for a given position \( \text{LF}(i) \) and \( \text{pred}(F^+, \text{LF}(i)) \) by the algorithm presented in Section 3.2.1. We can also compute \( \text{pred}(L^+_{RL}, \text{LF}(i)) \) in constant time because \( D_{RL}[\text{pred}(L^+_{RL}, \text{LF}(i))] = \text{pred}(L^+_{RL}, \text{LF}(i)) \). Therefore, we can compute the 4-tuple \( (\text{LF}(i), \text{pred}(F^+, \text{LF}(i)), \text{pred}(L^+, \text{LF}(i)), \text{pred}(L^+_{RL}, \text{LF}(i))) \) in constant time.

Next, we give a construction algorithm of the three data structures. We construct array \( \Lambda \) in \( O(n) \) time by processing the permutation \( \lambda_1, \lambda_2, \ldots, \lambda_k \) obtained by Lemma 13. The other data structures can be constructed in \( O(k) \) time by processing the permutation and \( L' \).

The construction time is \( O(n) \) time in total, and the working space is \( O(k) = O(r) \) words. Therefore, Lemma 15 holds.

\[ \text{Lemma 16.} \quad \text{We can construct the data structure of \( (r) \) words supporting \( \phi \) and \( \phi^{-1} \) in } O(\log \log_w(n/r)) \text{ time by processing the RLBTW of } T \text{ in } O(n) \text{ time and } O(r) \text{ words of working space.} \]

**Proof.** We prove Lemma 16 using the data structure for \( \phi \) and \( \phi^{-1} \) proposed by Gagie et al. [11]. Their data structure consists of (i) four arrays \( SA^- \), \( SA^+ \), \( DSA^- \), and \( DSA^+ \) of size \( r \) and (ii) two predecessor data structures for \( SA^- \) and \( SA^+ \). \( SA^- \) and \( SA^+ \) store positions in the starting and ending positions of runs in the RLBTW of \( T \), respectively. \( DSA^+ \) and \( DSA^- \) store the differences in adjacent \( sa \)-values at positions in two arrays \( SA^+ \) and \( SA^- \), respectively. Formally, let \( L' \) be the BWT-sequence of \( T \) such that \( L_i = L^+_{RL} \) for any integer \( i \in [1,r] \), i.e., \( L' \) is the RLBTW of \( T \). Let \( s_1, s_2, \ldots, s_r \) be the starting positions of runs in the RLBTW in increasing order of their \( sa \)-values, i.e., \( s_1, s_2, \ldots, s_r \) are the permutation on set \( L^+ \) such that \( SA[s_1] < SA[s_2] < \cdots < SA[s_r] \). Similarly, let \( e_1, e_2, \ldots, e_r \) be the ending positions of runs in the RLBTW in increasing order of their \( sa \)-values, i.e., \( e_1, e_2, \ldots, e_r \) are the permutation on set \( \{ \ell^+_1 + |L_1| - 1, \ell^+_2 + |L_2| - 1, \ldots, \ell^+_r + |L_r| - 1 \} \) such that \( SA[e_1] < SA[e_2] < \cdots < SA[e_r] \). Then, we define the four arrays in the following way:

- \( SA^- = SA[e_1], SA[e_2], \ldots, SA[e_r] \),
- \( SA^+ = SA[s_1], SA[s_2], \ldots, SA[s_r] \),
- \( DSA^- = SA[e_1] - SA[e_1 + 1], SA[e_2] - SA[e_2 + 1], \ldots, SA[e_r] - SA[e_r + 1] \), and
- \( DSA^+ = SA[s_1] - SA[s_1 - 1], SA[s_2] - SA[s_2 - 1], \ldots, SA[s_r] - SA[s_r - 1] \).

Here, let \( SA[n + 1] = SA[1] \) and \( SA[0] = SA[0] \). Then, the following lemma holds. Gagie et al. showed that for any integer \( i \), \( \phi^{-1}(i) = SA[i] - DSA^-[i - i] \) and \( \phi(i) = SA[i] - DSA^+[i + i] \) hold [11], where \( i = \text{pred}(SA^-[i], SA[i]) \) and \( i = \text{pred}(SA^+[i], SA[i]) \). Therefore, their data structure can compute \( \phi \) and \( \phi^{-1} \) in \( O(\log \log_w(n/r)) \) time by using two predecessor queries on \( SA^- \) and \( SA^+ \).

Next, we show that their data structure can be constructed in \( O(n) \) time and \( O(r) \) words of working space. To construct the data structure, we use two arrays \( D^+ \) and \( D^- \) of size \( r \) storing \( SA[\ell^+_1], SA[\ell^+_2], \ldots, SA[\ell^+_r] \) and \( SA[\ell^-_1 + |L_1| - 1], SA[\ell^-_2 + |L_2| - 1], \ldots, SA[\ell^-_r + |L_r| - 1] \), respectively. If \( r > \frac{n}{\log n} \), then we construct the two arrays by using the data structure of Lemma 14 and the predecessor data structure for set \( L^+ \). Let \( p_b \) be the position of the special character on the BWT \( L \) (i.e., \( L[p_b] = $ \)). Then, \( D^+[i'] = n - d \) only if \( \ell^+_i = \text{LF}_d(p_b) \) for any integer \( d \in [0, n - 1] \), where \( i = \text{pred}(L^+, \text{LF}_d(p_b)) \) and \( \text{LF}_d(i) \) is a recursive \( LF \) function applied \( d \) times to a given position \( i \). Similarly, \( D^-[i'] = n - d \) only if \( \ell^-_i + |L_i| - 1 = \text{LF}_d(p_b) \) for any integer \( d \in [0, n - 1] \). The data structure of Lemma 14 enumerates \( \text{LF}_0(p_b), \text{LF}_1(p_b), \ldots, \text{LF}_{n-1}(p_b) \) in \( O(n \log \log_w(n/r)) \) time, and hence we can
construct two arrays $D^+$ and $D^-$ using $n$ predecessor queries on $L^+$. The total running time is $O(n \log \log (n/r)) = O(n)$ and the working space is $O(r)$.

If $r \leq \frac{n}{\log n}$, then we construct the two arrays by using the data structure of Lemma 15. The data structure of Lemma 15 can be constructed in $O(r \log r + n)$ by processing the RLBWT of $T$, and the data structure can enumerate $n$ pairs $(LF_0(p_0), i'_0), (LF_1(p_0), i'_1), \ldots, (LF_{n-1}(p_0), i'_{n-1})$ in $O(n)$ time, where $i'_d = \text{pred}(L^+, LF_d(p_0))$ for any integer $d \in [0, n-1]$. Thus, we can construct two arrays $D^+$ and $D^-$ in $O(n + r \log r) = O(n)$ time by modifying the construction algorithm for the case $r \geq \frac{n}{\log n}$. Therefore, $D^+$ and $D^-$ can be constructed in $O(n)$ time and $O(r)$ words of working space for any RLBWT.

Next, we give an construction algorithm of the data structure for $\phi$ and $\phi^{-1}$ by processing two arrays $D^+$ and $D^-$. We can construct $\text{SA}^+$ by sorting $D^+$, and hence we sort $D^+$ by a standard sorting algorithm if $r \leq \frac{n}{\log n}$; otherwise we use the LSD radix sort used in the proof of Lemma 13. Thus, we can construct $\text{SA}^+$ in $O(n)$ time and $O(r)$ words of working space. Similarly, $\text{SA}^-$ can be constructed by sorting $D^-$ in $O(n)$ time and $O(r)$ words of working space. $\text{DSA}^+$ and $\text{DSA}^-$ also can be constructed in the same running time and working space by using four arrays $\text{SA}^+$, $\text{SA}^-$, $D^+$, and $D^-$. The two predecessor data structures can be constructed in $O(r \log \log \log \log (n/r)) = O(n)$ time and $O(r)$ working of working space by processing $\text{SA}^-$ and $\text{SA}^+$. Therefore, we obtain Lemma 16.

We show that our index uses $O(r)$ words. Our index requires $O(r + \sigma) = O(r)$ words if $\sigma \leq r$. Otherwise we map the characters in $T$ to an alphabet $\Sigma_{\text{new}} = \{1, 2, \ldots, \sigma'\}$ by a mapping function $\gamma$ before indexing our index, and our index stores deterministic dictionary storing deterministic dictionary [20] storing the mapping function, where $\sigma' \leq r$ is the number of the distinct characters in $T$. Here, $\gamma(c)$ returns the rank of a given character $c \in \Sigma$ in string $T$, i.e., $\gamma(c) = |\{c' \mid c' \in \Sigma' \text{ s.t. } c' \leq c\}|$ if $c \in \Sigma'$; otherwise $\gamma(c) = 0$, where $\Sigma'$ is the set of the distinct characters in $T$ (i.e., $\Sigma' = \{T[c] \mid i \in [1, n]\}$). In this case, we map each character of a given pattern $P$ to the corresponding character in alphabet $\Sigma_{\text{new}}$ by using the deterministic dictionary before answering to count or locate query. The answers to count and locate query are 0 and $\emptyset$, respectively, if a character of $P$ mapped to 0 because $T$ does not contain character 0. The deterministic dictionary can compute $\gamma(c)$ for a given character $c$ in constant time, and its space usage is $O(\sigma') = O(r)$ words. Therefore, the space usage of our index is $O(r)$ words in total for any alphabet size.

Next, we construct the six data structures listed in Section 3.2 as follows. (i) We construct balanced BWT-sequence $L'$ by using Theorem 14. (ii) We construct the five array in $O(n + r \log r)$ time by using the data structure of Lemma 15. (iii) We construct the rank data structure for string $L_{\text{first}}$ in $O(r)$ time by processing $L_{\text{first}}$. and $L_{\text{first}}$ can be constructed in $O(r)$ time by accessing $L_1[1], L_2[1], \ldots, L_k[1]$. (iv) We construct array $D_{\text{index}}$ in $O(r)$ time by processing two sequences $L'$ and the F-sequence in left-to-right order. (v) We construct the data structure for $\phi$ and $\phi^{-1}$ by Lemma 16. (vi) Recall that $V_{\Sigma}[c]$ stores a 5-tuple $(C_{\text{left}}[c], b_c, e_c, \text{pred}(F^+, b_c), \text{pred}(F^+, e_c))$, where $[b_c,e_c]$ is the sa-interval of character $c$. $C_{\text{left}}[c]$ is equal to $f^+_{\text{pred}(F^+, b_c)} - 1$. $F_{\text{pred}}(F^+, b_c)$ and $F_{\text{pred}}(F^+, e_c)$ are the first and last phrases consisting of character $c$, respectively. $b_c$ and $e_c$ are equal to the starting position of $F_{\text{pred}}(F^+, b_c)$ and the ending position of $F_{\text{pred}}(F^+, e_c)$, respectively. Thus, we can construct $V_{\Sigma}$ in $O(r)$ time by processing the F-sequence. Therefore, the construction time and working space are $O(n + r \log r)$ time and $O(r)$ words, respectively.

If $\sigma > r$, then we need to construct the deterministic dictionary storing mapping function $\gamma$ and map the characters in the given RLBWT of $T$ by the mapping function $\gamma$. The dictionary can compute $\gamma(c)$ for any character $c \in \Sigma$, and we can construct the dictionary in $O(\sigma'(\log \log \sigma')^2) = O(r \log r)$ time and $O(\sigma')$ words of working space [20] after sorting the
distinct characters in the RLBWT, where \( \sigma' \leq r \) is the number of distinct characters in \( T \). Therefore, the construction time and working space are \( O(n + r \log r) \) time and \( O(r) \) words, respectively, for any alphabet size. Finally, we obtain Theorem 9.

Appendix B: Proofs for Section 4

Proof of Lemma 9. (i) The \( x \)-th phrase in \( L' \) has at least four children unless \( \text{split}(L') \neq L' \). The \( x \)-th phrase in \( \text{split}(L') \) has the first and second children of the \( x \)-th phrase in \( L' \) as its children. Similarly, the \((x + 1)\)-th phrase in \( \text{split}(L') \) has the third and fourth children of the \( x \)-th phrase in \( L' \) as its children. (ii) \text{children}(L',i) = [\ell^+_i..\ell^+_i + |L_i| - 1] \cap L^+ and \text{children}(\text{split}(L'),\text{map}(i)) = [\ell^+_i..\ell^+_i + |L_i| - 1] \cap (L^+ \cup \{\ell^+_i + y\}) \) hold for an integer \( i \in ([1,k] \setminus \{x\}) \). Therefore, we obtain the second statement in Lemma 9. (iii) The third statement in Lemma 9 holds by the first and second statements.

Proof of Theorem 4 (ii).

In this section, we present a construction algorithm for the balanced RLBWT of \( T \) (i.e., balance(\( L^{RL} \)). Let \( L' \) be the RLBWT of \( T \) and \( k = |L'| \). The algorithm is built on the following three data structures: (i) two doubly-linked lists \( V_L \) and \( V_F \) of size \( k \). The \( i \)-th elements of \( V_L \) and \( V_F \) correspond to \( L_i \) and \( F_i \), respectively. The \( i \)-th element of \( V_L \) stores (a) integer \( \ell^+_i \), (b) character \( L_i[1] \), and (c) the pointer to \( j \)-th element in \( V_F \) such that its phrase corresponds to \( L_i \) (i.e., \( f^+_j = \text{LF}(\ell^+_i) \)). Similarly, the \( i \)-th element of \( V_F \) stores (a) integer \( f^+_i \), and (b) the pointer to \( j \)-th element in \( V_L \) such that its phrase corresponds to \( F_i \) (i.e., \( f^+_i = \text{LF}(\ell^+_i) \)). (ii) A stack \( \Upsilon \). The stack stores the pointers to the elements in \( V_L \) such that the phrase corresponding to each element has at least four children, and the pointers are stored in the stack in any order. (iii) Two self-adjusting balanced binary trees \( T_L \) and \( T_F \). \( T_L \) stores the pointers to \( k \) elements in \( V_L \) in the increasing order of the starting positions of their phrases, and the tree can return an element with phrase \( L_j \) containing the \( i \)-th character of the BWT of \( T \) (i.e., \( \ell^+_j \leq i \leq \ell^+_j + |L_j| - 1 \)) in \( O(\log k) \) time for a given integer \( i \). Similarly, \( T_F \) stores the pointers to \( k \) elements in \( V_F \) in the increasing order of the starting positions of their phrases, and the tree can return an element with phrase \( F_j \) containing the \( i \)-th character of the permutation \( F \) in \( O(\log k) \) time for a given integer \( i \). Therefore, the space usage is \( O(k) \) words in total.

The algorithm repeats split operations until any phrase has at most three children, and hence \( L' \) is equal to the balanced RLBWT after the algorithm is executed. Each split operation splits the top phrase in stack \( \Upsilon \) and removes the phrase from the stack. If the split operation creates phrases such that each phrase has at least four children and the phrase is not contained in the stack, then the algorithm pushes the phrases into the stack. Therefore, any phrase has at most three children in \( L' \) only if the stack is empty.

Formally, the algorithm repeats the following two steps until stack \( \Upsilon \) is empty. (i) Take the top phrase \( L_x \) from stack \( \Upsilon \) and split \( L_x \) into two new phrases \( L_x[1..y-1] \) and \( L_x[y..|L_x|] \). (ii) Update the data structures in accordance with \( L' \) changed by the split operation, i.e., the algorithm executes the following three steps. (a) Remove phrase \( L_x \) and the corresponding F-phrase from list \( V_L \) and \( V_F \), respectively, and insert the new phrases and their corresponding F-phrases into \( V_L \) and \( V_F \), respectively. (b) Similarly, remove phrase \( L_x \) and the corresponding F-phrase from list \( T_L \) and \( T_F \), respectively, and insert the new phrases and their corresponding F-phrases into \( V_L \) and \( V_F \), respectively. (c) Push the BWT-phrases into \( \Upsilon \) such that (1) each phrase has at least four children and (2) the phrase is not contained in \( \Upsilon \). The empty stack indicates that a split operation does not split the current BWT-sequence, and hence we obtain balance(\( L^{RL} \)) by reading list \( V_L \) in left-to-right order.
order after \( Y \) is empty.

Next, we show that the algorithm runs in \( O(\log k) \) time per step. In step (i), we need to compute integer \( y = \ell^+_{x} - f^+_{x} \), where \( F_x \) is the F-phrase corresponding to \( L_x \) (i.e., \( f^+_{x} = LF(\ell^+_{x}) \)) and \( L_0 \) is the third child of \( L_x \) (i.e., \( \ell^+_{x-3} < f^+_{x} < \ell^+_{x-2} < \ell^+_{x-1} < \ell^+_{x} \)). The algorithm computes \( y \) by the following steps. (a) Get \( f^+_{x} \) by using the pointer stored in \( L_x \) in \( V_L \). (b) Get \( L_{v-2} \) in \( V_L \) by searching for the phrase containing the \( f^+_{x} \)-th character in \( T_L \). (c) Get \( \ell^+_{x} \) by reading \( L_{v-2}, L_{v-1}, L_{v} \) in \( V_L \), and compute \( x' = \ell^+_{x} - f^+_{x} \). Therefore, the running time is \( O(\log k) \) in step (i).

The algorithm executes step (ii-a) in constant time because the element for \( L_x \) in \( V_L \) stores the pointer to the corresponding phrase in \( V_F \). Similarly, the algorithm executes step (ii-b) in \( O(\log k) \) time. In step (ii-c), we need to find the BWT-phrases where (1) each phrase has at least four children and (2) the phrase is not contained in \( Y \). The corresponding phrase contains \( \ell^+_{x} + y - 1 \). The algorithm obtains the F-phrase corresponding to \( L_x \) by searching for the phrase containing the \( (\ell^+_x + y - 1) \)-th character in \( T_F \), and it verifies whether each candidate has at least four children in \( O(\log k) \) time by using tree \( T_L \). The running time is also \( O(\log k) \) in step (ii), and hence the running time is \( O(r \log r) \) for the whole algorithm by \( k = O(r) \). The data structures for our construction algorithm can be constructed in \( O(r \log r) \) time by processing the RLBWT of \( T \). Finally, we obtain Theorem 10 (ii).

Appendix C: Proof of Theorem 10

We present two data structures of \( O(r) \) words. The first data structure returns \( T[i_j - d + 1..i_j] \) in constant time per character and the data structure can be constructed in \( O(n) \) time by processing the RLBWT if \( r \leq \frac{n}{\log n} \). The second data structure returns \( T[i_j - d + 1..i_j] \) in \( O(\log \log w(n/r)) \) per character, and the data structure can be constructed in \( O(n) \) time by processing the RLBWT if \( r > \frac{n}{\log n} \). Thus, we obtain Theorem 10 by combining these two data structures.

The first data structure is the data structure introduced in Section 5.1. We compute a pair \( (LF(i), \text{pred}(L^+, \text{LF}(i))) \) in constant time for a given integer \( i \) and \( \text{pred}(L^+, i) \) by the data structure of Lemma 15. The data structure of Lemma 15 can compute \( (i'_1, \text{pred}(L^+, i'_1)), (i'_2, \text{pred}(L^+, i'_2)), \ldots, (i'_b, \text{pred}(L^+, i'_b)) \) in \( O(n) \) time. Therefore, we can construct the first data structure in \( O(n + r \log r) = O(n) \) time if \( r \leq \frac{n}{\log n} \).

The second data structure consists of the data structure of Lemma 14 and an array storing \( i'_1, i'_2, \ldots, i'_b \). The data structure can return \( L[\text{LF}_0(i'_j)], L[\text{LF}_1(i'_j)], \ldots, L[\text{LF}_{d-1}(i'_j)] \) in \( O(d \log \log w(n/r)) \) time for any integer \( j \in [1, b] \). The running time is \( O(d \log \log w(n/r)) \) time. The data structure of Lemma 14 can compute \( i'_1, i'_2, \ldots, i'_b \) in \( O(n \log \log w(n/r)) \) time. Therefore, we can construct the second data structure in \( O(n \log \log w(n/r)) = O(n) \) time and the data structure can return \( T[i_j - d + 1..i_j] \) in \( O(d \log \log w(n/r)) = O(d) \) time if \( r > \frac{n}{\log n} \).