NOTE ON THE NON-PRESERVATION OF DEPTH

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Abstract. Let $K$ be a local field of characteristic $p$. We consider the local Langlands correspondence for tori, and construct examples for which depth is not preserved.

1. Introduction

Let $K$ be a local non-archimedean field. Let $T = R_{L/K} \mathbb{G}_m$ be an induced torus, when $L$ is a finite separable extension of $K$. The LLC (Local Langlands Correspondence) for tori induces an isomorphism

$$\lambda_T : \text{Hom}(T(K), \mathbb{C}^\times) \cong H^1(W_K, T^\vee)$$

where $W_K$ is the Weil group of $K$ and $T^\vee$ is the complex dual torus, see [MP] and [Yu].

For background material on depth, see [ABPS]. Concerning depth-preservation, we have the theorem of Yu [Yu, §7.10]: In the LLC for tori, if $T$ splits over a tamely ramified extension, then we have

$$\text{dep}(\chi) = \text{dep}(\lambda_T(\chi)).$$

Mishra and Patanayak [MP] have recently constructed, in characteristic 0, an explicit example of a wildly ramified torus for which depth is not preserved under LLC for all positive depth characters.

We produce explicit examples, in characteristic $p$, of wildly ramified tori for which depth is not preserved under LLC for all positive depth characters. In particular, let $K$ be a local field of characteristic 2, and let $L/K$ be a totally ramified quadratic extension: there are countably many of these, with ramification breaks given by $m = 1, 3, 5, 7, \ldots$. In the LLC for tori, the depth, for all positive depth characters, is not preserved.

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2. On depth

Let $K$ be a local field of characteristic $p$. Let $\mathfrak{o}$ be the ring of integers in $K$ and $p \subset \mathfrak{o}$ the maximal ideal. Let $\varphi(x) = x^p - x$.

Let $\overline{K} = K/\varphi(K)$. Let $D \neq \mathfrak{o}$ be an $\mathbb{F}_p$-line in $\overline{K}$, $m$ the integer such that $D \subset \overline{p}^{-m}$ but $D \not\subset \overline{p}^{-m+1}$; we know that $m > 0$ and prime to $p$. Fix an element $a \in p^{-m}$ whose image generates $D$, let $\alpha$ be a root of $T^p - T - a$ (in an algebraic closure of $K$), and let $L = K(\alpha) = K(\varphi^{-1}(D))$. See [Da, §6].
The extension $L/K$ is totally (and wildly) ramified. The unique ramification break of the degree $p$ cyclic extension $L/K$ occurs at $m$, see [Da, §6]. Set $T = L^\times$, then $T$ is a wildly ramified torus.

**Theorem 2.1.** Let $K$ be a local field of characteristic $p$, let $L = K(\sqrt[p]{1}(D))$ as above, let $\chi$ be any character of $T$ of positive depth. In the local Langlands correspondence for tori, the depth of the character $\chi$ is not preserved.

**Proof.** We have the elegant recent formula of Mishra and Patanayak [MP]:

\[ \varphi_{L/K}(e \cdot \text{dep}_T(\chi)) = \text{dep}_W(\lambda_T(\chi)) \]

where $\varphi_{L/K}$ is the Hasse-Herbrand function, $\chi$ is a character of $T$, $\text{dep}_T(\chi)$ is the depth of $\chi$, $\text{dep}_W(\lambda_T(\chi))$ is the depth of the Langlands parameter $\lambda_T(\chi)$, and $e = e(L/K)$ is the ramification index. If $u$ is a real number $\geq -1$, $G_u$ denotes the ramification group $G_i$, where $i$ is the smallest integer $\geq u$. Then the Hasse-Herbrand function is

\[ \varphi_{L/K}(u) = \int_0^u \frac{1}{(G_0 : G_t)} dt. \]

We will write $d := \text{dep}_T(\chi)$.

**First case.** We suppose that $d > m/p$. We apply the formula (1):

\[
\begin{align*}
\text{dep}_W(\lambda_T(\chi)) &= \varphi_{L/K}(e \cdot \text{dep}_T(\chi)) \\
&= \varphi_{L/K}(pd) \\
&= \int_0^{pd} \frac{1}{(G_0 : G_t)} dt \\
&= \int_0^m 1 dt + \int_m^{pd} \frac{1}{p} dt \\
&= m + (pd - m)/p \\
&= d + m(1 - 1/p) \\
&> \text{dep}_T(\chi)
\end{align*}
\]

**Second case.** We suppose that $0 < d \leq m/p$. We have

\[
\begin{align*}
\text{dep}_W(\lambda_T(\chi)) &= \varphi_{L/K}(e \cdot \text{dep}_T(\chi)) \\
&= \varphi_{L/K}(pd) \\
&= \int_0^{pd} \frac{1}{(G_0 : G_t)} dt \\
&= pd \\
&> \text{dep}_T(\chi)
\end{align*}
\]

\[ \square \]

**Corollary 2.2.** Let $K$ be a local field of characteristic 2, and let $L/K$ be a totally ramified quadratic extension: there are countably many of these, with ramification breaks given by $m = 1, 3, 5, 7, \ldots$. In the LLC for tori, the depth, for all positive depth characters, is not preserved.
References

[ABPS] A.-M. Aubert, P. Baum, R.J. Plymen, M. Solleveld, Depth and the local Langlands correspondence, Arbeitstagung Bonn 2013, Progress in Math., Birkhauser 2016, arxiv.org/abs/1311.1606

[Da] C.S. Dalawat, Further remarks on local discriminants, J. Ramanujan Math. Soc., (4) 25 (2010) 393–417.

[MP] M. Mishra, B. Patanayak, A note on depth preservation, J. Ramanujan Math. Soc., to appear.

[Yu] Jiu-Kang Yu. On the local Langlands correspondence for tori. In Ottawa lectures on admissible representations of reductive $p$-adic groups, volume 26 of Fields Institute Monograph, pages 177–183. Amer. Math. Soc., Providence, RI, 2009.

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