Geometry, Coordinatization and Cardinality of the Rational Numbers from Physical Perspective

Kaushik Ghosh
Vivekananda College, 269 Diamond Harbour Road, Kolkata - 700063, India
E-mail: kghosh7211@gmail.com

Abstract. In this article, we will first discuss the completeness of real numbers in the context of an alternate definition of the straight line as a geometric continuum. According to this definition, points are not regarded as the basic constituents of a line segment and a line segment is considered to be a fundamental geometric object. This definition is in particular suitable to coordinatize different points on the straight line preserving the order properties of real numbers. Geometrically fundamental nature of line segments are required in physical theories like the string theory. We will construct a new topology suitable for this alternate definition of the straight line as a geometric continuum. We will discuss the cardinality of rational numbers in the later half of the article. We will first discuss what we do in an actual process of counting and define functions well-defined on the set of all positive integers. We will follow an alternate approach that depends on the Hausdorff topology of real numbers to demonstrate that the set of positive rationals can have a greater cardinality than the set of positive integers. This approach is more consistent with an actual act of counting. We will illustrate this aspect further using well-behaved functionals of convergent functions defined on the finite dimensional Cartesian products of the set of positive integers and non-negative integers. These are similar to the partition functions in statistical physics. This article indicates that the axiom of choice can be a better technique to prove theorems that use second-countability. This is important for the metrization theorems and physics of spacetime.

MSC: 51P05, 26A03, 58A05, 81T30, 83C56, 03E25
PACS: 02.40.-k, 02.40.Pc, 11.25.-w, 02.30.Lt, 02.30.Nw
Key-words: geometric continuum, linear continuum, strings, coordinatization, cardinality of rationals, axiom of choice

1. Introduction
In this article, we will discuss an alternate definition of geometric continuum like the straight line put forward by a few authors in recent years [1,2,3,4]. Here a continuum need not to be compact [5]. The term straight line refers to the one dimensional line in Euclidean geometry that can be coordinatized by the real numbers. The real numbers form a linear continuum [6]. We will state this definition shortly. In the conventional definition, where a straight line is considered to be a collection of points, we have to break the linear continuum structure of real numbers to coordinatize different points on the straight line [7]. The new definition states that points need not to be the basic constituents of a line segment and a line segment can be
considered to be a fundamental geometric object. This is consistent with microscopic physical theories where the elementary particles are considered to be extended objects without having any finer structure. One such theory is the string theory. This definition is also more consistent with the linear continuum structure of real numbers. We need not break the linear continuum structure of the real numbers to coordinatize the straight line. The linear continuum structure of the real numbers is required to define concepts like finite open intervals and finite limits in general in the set of real numbers. We will reformulate the completeness axiom of real numbers that is more consistent with the new definition of straight line.

We can generalize the above aspect to higher dimensional geometric continua. A geometric continuum of dimension \( n \) is not just a collection of geometric continua of dimension \( n - 1 \) and is a fundamental geometric object. This is important for the foundations of set theory and topology. We note that the intersection of two line segments like \( PQ \) and \( QR \) can be a point \( Q \) which is different from either set when we consider a line segment to be a fundamental geometric object. We also face problem to construct a topology. Considering the straight line only, we have two kinds of fundamental objects in it: geometrical points and line segments, and we have to construct open sets to have a convenient topology in the straight line. We solve this problem by assuming that the geometric continua of dimension one and zero form a universal set with a line segment being a geometric continuum of dimension one and a point being a geometric continuum of dimension zero. We can introduce familiar topologies in this universal set. The new definition of straight line as a geometric continuum is naturally expressed as a connected topological space in the topology given by geometric open intervals on the straight line. This topology coincides with a topology derived from a geometrical simple order relation present between different points on the straight line. This topology satisfy the Hausdorff and normality conditions [6] due to the fundamental nature of line segments and coincides with the standard topology of real numbers [6] through coordinatization. Note that to begin with there are now two kinds of 'points' present in the topological space. In this article, we refer Hausdorff condition to mean separability of geometric points while normality indicates separability of disjoint line segments. It is more natural to rename normality condition as also the Hausdorff condition in the straight line and we will discuss this aspect in a future article.

In section 3, we will discuss a new approach to find the cardinal property of the set of rational numbers. We will denote the set of all positive integers: \( \{1, 2, 3, \ldots\} \) by \( Z_+ \) and the set of all non-negative integers: \( \{0, 1, 2, 3, \ldots\} \) by \( \omega \). We will emphasize on what we do in an actual process of counting and discuss an improved description of injective functions defined on limit ordinals [8] like \( Z_+ \). In this article, two sets are said to be isomorphic if there is a bijection between the two. We first note that the equivalence class \( [Z_+] \) of sets isomorphic to \( Z_+ \) do not possess the order property \( O2 \) mentioned in the next section. We now consider two sets of points on the \( XY \)-plane given by: \( W_1 = \{(n, 0) | n \in Z_+\} \) and \( W_2 = \{(-n, 0) | n \in Z_+\} \). The sets gives two different representations of \( Z_+ \) and each is bijectively related with \( Z_+ \). They are also bijectively related with each other through a reflection about the origin. Thus \( Z_+ \), \( W_1 \) and \( W_2 \) have the same cardinality by the Schröder–Bernstein theorem [8] and are countable [6]. The discrete set \( W = W_1 \cup W_2 \) is the union of two mutually disjoint discrete sets \( W_1 \) and \( W_2 \) and is expected to have a greater cardinality than each of them unless we ignore the different bijections used to construct \( W_1 \) and \( W_2 \) from \( Z_+ \). The Schröder–Bernstein theorem is applicable even to uncountable sets like connected intervals of non-zero length from the set of real numbers \( R \). There exist bijections between any pair of these sets due to the order property \( O2 \) an they are uncountable. \( W \), \( W_1 \) and \( W_2 \) do not possess the order property \( O2 \) and we may have to construct a domain larger than each of \( W_1 \) and \( W_2 \) to define an injective function on \( W \). We next consider a convergent sequence of positive rationals \( \{r_n\}, n \in Z_+ \), isomorphic to \( Z_+ \) and having the same cardinality as that of \( Z_+ \) by the Schröder–Bernstein theorem [8]. This is the case with \( \{r_n = 3/2 + 1/n\}, n \in Z_+ \). The set \( G = \{1/2\} \cup \{r_n \in Z_+ \}, n \in Z_+ \), is
different from \( \{ r_n = 3/2 + 1/n \}, \ n \in Z_+ \), and is expected to have a greater cardinality than \( Z_+ \). We will discuss in details the problems that arise when we try to construct injective functions from \( W \) or \( G \) to \( Z_+ \). We will demonstrate that the set of positive rationals \( Q_+ \) have a greater cardinality than the set of positive integers \( Z_+ \). We will use Theorem 1-7.1. [6], the Hausdorff topology of the real numbers and convergent sequences of rational numbers isomorphic to \( Z_+ \) to demonstrate this. It follows that the set of rationals has a greater cardinality than the set of integers. This can give us a departure from the continuum hypothesis [6,8]. We find that the axiom of choice [6,8] can be a better technique to prove theorems that use second-countability [5]. This is important for the metrization theorems and is relevant to quantum gravity.

2. Geometric Continuum, Coordinatization and Hausdorff Condition in the Straight Line

In this section, we will confine our attention on the straight line which is a geometric continuum of one dimension. We will give a precise definition of geometric continuum shortly. Henceforth, we will denote the straight line by the symbol \( E^1 \) although we have not yet introduced any metric on the straight line. We now define a topology in \( E^1 \). For any two distinct points \( a, b \) on \( E^1 \), we can define a geometrical simple order relation between them that asserts that \( a < b \) if \( b \) lies at the right of \( a \). We choose a basis element as \((a,b)\): the open interval defined as the entire line interval between \( a \) and \( b \) excluding themselves with \( a < b \) in the geometrical simple order relation. We get the null set when \( a \) and \( b \) coincide. We get a topology on \( E^1 \) by adding the null set to the basis elements obtained above. We denote this topology by \( U \). The line segment \( ab \), also denoted by \( [a,b] \), is the the entire line interval between \( a \) and \( b \) that includes \( a \) and \( b \).

We can generate different basis elements by considering \( a,b \) to be a pair of indices that take values from the same index set \( K \). \( K \) should be large enough to give all possible geometric open intervals on \( E^1 \). It is not a trivial issue that \((a,b)\) can be expressed as an open interval from the set of real numbers \( R \) which fixes \( K = R = (−\infty, \infty) \). This is done through the completeness axiom of Dedekind which states that for any point on the number axis, a straight line, there corresponds a real number from \( R \) [9,10]. The real numbers include the rational and irrational numbers. We can then coordinatize different points on \( E^1 \) when we choose a suitable point on \( E^1 \) to be represented by zero and a particular point to be represented by one. The completeness axiom and the algebraic properties of real numbers as a field leads us to introduce the Euclidean metric on \( E^1 \) [6]. The distance between two points with coordinates \( x, y \) is given by \(|x - y|\). The set of real numbers \( R \) form an algebraic field that has an order relation \(<\) that satisfy the following order properties [6]:

O1. The order relation \(<\) has the least upper bound property. Every nonempty subset \( X \) of \( R \) that is bounded above has a least upper bound.

O2. If \( x < y \), then there exists an element \( z \) such that \( x < z \) and \( z < y \), where \( x, y, z \in R \).

Existence of coordinatization indicates that \( E^1 \) can possess the above simple order relation present in \( R \) with the properties:

O1. For any nonempty \( A \subset E^1 \), there is a least upper bound for \( x(A) \) when \( x(A) \) is bounded above.

O2. If \( x(a) < x(c) \) there is a point \( b \) between \( a \) and \( c \) such that \( x(a) < x(b) \) and \( x(b) < x(c) \).

Here \( x(a) \) denotes the coordinate of the point \( a \in E^1 \) and \( x(A) \) includes the coordinate for any
point that belongs to the subset $A$ of $E^1$. The order property of $R$ introduce an order topology in $E^1$. In $E^1$, the topology $\mathcal{U}$ can be now given by $(x(a), x(b))$. This topology and the order topology induced by $R$ through coordinatization coincide when we choose $x(a) < x(b)$ [6]. This topology also coincides with the standard topology of $R$ given by the open sets $(x, y)$, $x, y \in R$ and $x < y$. Note that we can use $(x(a), x(b))$ to denote open balls in the Euclidean metric topology through the midpoint property of Euclidean metric in the straight line. $(x(a), x(b))$ gives us an open ball centered at the midpoint of the interval. The linear order property $O2$ brings us closer to an alternate definition of $E^1$ as a geometric continuum through the following axiom [4]:

**AXIOM 2-1.** Points are not the basic constituents of a straight line. A line segment is not just a collection of points and is a fundamental geometric object.

The above axiom may be apparently counter intuitive since two straight lines intersect at a point. However, whatever may be the cardinality, a collection of zero-extension points can not give an extended and homogeneous object like a one dimensional straight line. The same is obviously valid for line segments. Also a homogeneous collection of points give us a single point since the extension of a point is zero. We can illustrate the last aspect in the following way:

We consider a homogeneous linear array of marbles touching each other. If we now shrink the volume of every marble keeping them in contact (so that the array is always homogeneous), we will get a single point when the volume of every marble is made to vanish.

Axiom 2-1 is consistent with the order property $O2$. We do not need adjacent points to construct a line. The notion of adjacent points was introduced not to have gaps in the straight line when we consider it to be merely a collection of points [7]. This notion contradicts $O2$ [7] and can not give an extended object like a line segment without spoiling homogeneity. Axiom 2-1 allows us to impose the order property $O2$ on the geometrical simple order relation present between any pair of distinct points in $E^1$ mentioned at the beginning of the present section where $a < b$ if $b$ lies at the right of $a$. Every pair of different points on $E^1$ are now separated by an open line interval, and we always have a point in between any such pair. Thus the geometrical simple order relation satisfies the order property $O2$. It is not difficult to impose the order property $O1$ on the geometrical simple order relation present in $E^1$. We can introduce the symbol $\infty$ to represent unboundedness of $E^1$ in this order relation. $E^1$ then includes all points $e$ with $-\infty < e < \infty$. This indicates that $E^1$ is unbounded in both directions similar to the real numbers. We can impose $O1$ to any proper subset of $E^1$ bounded in the geometric order relation. However, we note that $E^1$ is no longer merely a collection of points. We will elaborate this aspect shortly.

A few physical aspects of the defining Axiom 2-1 of geometric continuum is mentioned in [2]. Here we note that Axiom 2-1 is required in the string theory of theoretical physics [11]. In string theory, point particles are replaced by one dimensional strings as the most elementary building blocks of matter and radiation. We can not assume a string to be a collection of points, in which case it will not be a fundamental object and we will have to hypothesize a new interaction that will hold the points forming a string. Since string theory involves high energy physics of quantum gravity, it is unlikely that we will have such an interaction that will always bind the points forming a string at the high energy scale of quantum gravity. Similar discussions hold if we assume that elementary particles like electrons or quarks are not point particles and have a radius at least greater than their Schwarzschild radius. The Schwarzschild radius of an electron is approximately $1.35 \times 10^{-57} \text{m}$. Note that black holes do not have baryonic or leptonic numbers by the no-hair theorems [12], and it is unlikely that an electron has a leptonic number as well.
as a radius smaller than its Schwarzschild radius.

We can generalize Axiom 2-1 to higher dimensional geometric continua. A geometric continuum of dimension \( n \) is not just a collection of geometric continua of dimension \( n - 1 \) and is a fundamental geometric object. This is important for the foundations of set theory and topology. We note that the intersection of two line segments \([a, b]\) and \([b, c]\) is a point \( \{b\} \) which is different from either set when we consider a line segment to be a fundamental geometric object. We also face problems to construct a topology. We now have two kinds of fundamental objects in \( E^1 \): geometric points and line segments. We want to use open line intervals to construct a simple topology in \( E^1 \) [5]. We remove this problem by adapting the naive definition of sets: sets are collections of objects which are closed under the operations of set theoretic union and intersection. This definition allows us to introduce the null set. Thus, we assume that geometric continua of different dimensions form a universal set with a point being a geometric continuum of dimension zero. We represent this universal set by the symbol \( \mathcal{M} \). We confine our attention to only \( E^1 \) which is a subset of \( \mathcal{M} \) and try to construct a topology for itself. It is now possible to define set theoretic operations involving line segments and points since they are subsets of the same universal set. We construct a topology similar to \( \mathcal{U} \) by defining \((a, b)\) as \([a, b] - \{a\} = \{b\} = [a, b] \cap \{a\} \cup \{b\} \cup \{a\} \cap \{b\} \cup \{a\}. \) We represent this topology by a different symbol \( \mathcal{V} \), since \((a, b)\) is no longer a mere collection of points. \((a, b)\) is not also a collection of line segments and points. It is introduced to give a topology. We can not introduce \((a, b)\) as a set without defining \( \mathcal{M} \). We need not to introduce \((a, b)\) to do pure geometry. We can also use geometric points and line segments to give a topology [5]. We now derive a topology from the geometrical simple order relation. The basis elements of this topology consist of \( (1) \) the null set and the straight line, \( (2) \) for each point \( a \) on the line, the whole line interval at the left of \( a \) excluding \( a \), \( (3) \) open intervals \((a, b)\) for every pair of distinct points \( a, b \) on the line with \( a < b \) and \( (4) \) for every point \( b \) on the line, the whole line interval at the right of \( b \) excluding \( b \). Again we note that, we have used the geometric order relation between any pair of distinct points on \( E^1 \) to construct open sets that are not merely collections of points. We denote this topology by \( \mathcal{D} \). In this topology, we can not separate the straight line into two mutually disjoint open sets. Thus, Axiom 2-1 is consistent with the formal definition of connected topological spaces in the topology \( \mathcal{D} \). This topology coincides with the topology of geometric open intervals \( \mathcal{V} \). We can represent \( E^1 \) as \((−∞, ∞)\) in the topology \( \mathcal{V} \). In \( \mathcal{V} \) also, we can not separate \( E^1 \) into two disjoint open sets \( A \) and \( B \) such that \( E^1 = A \cup B \). We can consider the above conclusions as topological definition of the straight line as a geometric continuum that follows from Axiom 2-1.

We now proceed to construct a representation of the index set \( K \) that is consistent with Axiom 2-1. With Axiom 2-1 as the definition of a line segment, we can reformulate the axiom of completeness of the real numbers as follows:

**AXIOM 2-2.** Any segment on the straight line \( E^1 \) can be ascribed a real number greater than zero. The number can be either rational or irrational. We call this number to be the length of the segment. A point has zero length.

Note that, Axiom 2-2 refers to segments in \( E^1 \) and is not stated in terms of points. We assign the same length to \([a, b] \), \((a, b)\), \([a, b]\) and \((a, b)\). The length of a line segment is uniquely determined provided we choose a given line segment to be of unit length. We now choose a suitable point on \( E^1 \) as the origin (midpoint) and coordinatize any point on \( E^1 \) according to the length of the line segment joining the two with a particular line segment chosen to be of unit length. Points on the right side of the origin are assigned positive coordinates while those on the left have negative coordinates. With both \( E^1 \) and \( R \) being unbounded in their respective order relations, we can assume \( K = R \), where \( K \) is the index set mentioned before. We recover the standard topology of \( R \) when \( a, b \) in \((a, b)\) are replaced by the coordinates of \( a \) and \( b \) given
by \( x(a) \) and \( x(b) \) respectively. \((x(a), x(b)) \) now represents the open line interval \((a, b)\) on \( E^1 \). As before, we can consider it to be an open ball in the Euclidean metric topology. However, we no longer consider it to be merely a collection of geometric points. Thus, we can assume that the topology \( D \) derived from the geometric order relation present in \( E^1 \), the topology \( V \) of geometric open intervals in \( E^1 \), the order topology in \( R \) and the standard topology in \( R \) coincide through Axioms 2-1 and 2-2. Axiom 2-1 and the order property \( O2 \) of \( R \) give us a homeomorphism between \( E^1 \) and \( R \) through coordinatization. However, in this article we will continue to use different symbols for them to distinguish between geometry and algebra respectively.

It easily follows from Axiom 2-1 that \( E^1 \) is Hausdorff \((T_2)\) and normal \((T_4)\) in the topology \( V \) given by the geometric open intervals \((a, b)\). In this article, we refer Hausdorff condition to mean separability of geometric points by disjoint open sets while normality indicates separability of disjoint line segments by disjoint open sets. Since line segments are fundamental objects, every pair of different points on \( E^1 \) are always separated by an open line interval. They are contained within an open line interval and we can construct two disjoint open sets separating the pair by deleting any point within the line segment joining the two. The same is valid for two disjoint line segments \( PQ \) and \( RS \) with \( P < Q < R < S \) in the geometric order relation. \( PQ \) and \( RS \) closed sets and are contained within an open line interval. The points \( Q \) and \( R \) are always separated by an open line interval and we can construct two disjoint open sets separating \( PQ \) and \( RS \) by deleting any point in the line segment joining \( Q \) and \( R \). Thus, \( E^1 \) is Hausdorff and normal in the topology \( V \). However, it would be more appropriate to say that \( E^1 \) is a Hausdorff space if we consider the fundamental nature of geometric points and line segments. We do not follow this approach for the present article. In a coordinatization scheme, the Hausdorff and normality conditions are ensured by the completeness of real numbers and the order property \( O2 \) of \( R \). Given a pair of different points \( a \) and \( c \) of \( E^1 \) and a coordinatization where \( x(a) < x(c) \), \( O2 \) indicates that for any point \( b \) in between \( a \) and \( c \) there exists a number \( x(b) \) that coordinatize \( b \) such that \( x(a) < x(b) < x(c) \). \((x(a_\text{–}), x(b)), (x(b), x(c_\text{+})) \) with \( x(a_\text{–}) < x(a) \) and \( x(c) < x(c_\text{+}) \) are a pair of open sets that separate \( a \) and \( c \) respectively. Similar arguments can be used to establish normality condition by considering the coordinates \( x(Q) \) and \( x(R) \) of \( PQ \) and \( RS \). This is expected since \((a, b)\) and \((x(a), x(b))\) give the same topology as mentioned before. However, the Hausdorff condition in the topology \( U \) does not remain valid for every pair of points if we regard the straight line to be merely a collection of points. Look at the discussions given below Axiom 2-1 with reference to [7]. If we assume the existence of adjacent points \( e_1, e_2 \) on \( E^1 \), the sets \((u_1, e_2)\) and \((e_1, u_2)\) with \( u_1 < e_1 < e_2 < u_2 \), that separate \( e_1 \) and \( e_2 \) no longer remain open in \( U \). There is no open set containing \( e_1 \) that is contained in \((u_1, e_2)\). Similar aspect remains valid for \( e_2 \) and \((e_1, u_2)\). With this definition, \( E^1 \) is Hausdorff only in the discrete topology. Lastly, we note that \( x(a), x(b) \) of \((x(a), x(b))\) can be rational or irrational. We can not use the basis of open balls with rational radii centered at points with rational coordinates to give a Hausdorff topology in \( E^1 \). This is due to the existence of irrational numbers like \( \sqrt{2}, \pi \). The points with irrational coordinates should belong to one or other of these basis elements and hence can not be separated from every point with rational coordinate by open sets obtained from this covering. This is important for differential geometry.

3. Cardinality of the Rationals
In this subsection, we will discuss a new approach to find the cardinal property of the set of rationals. We will emphasize on what we do in an actual process of counting as in the case of statistical mechanics to show that the set of positive rationals \( Q_+ \) have a greater cardinality than the set of positive integers \( Z_+ \). We will use the Hausdorff topology \( V \) of \( E^1 \) which is same as the standard topology of \( R \) through coordinatization. We will call it the standard topology of \( E^1 \) in the following. We express \( R \) as \((-\infty, \infty)\) with the understanding that the limit \( x \to \infty \) means \( x \) can be increased indefinitely without any upper bound. It follows that \( R \) is an uncountable set.
and $E^3$ contains an uncountable number of line segments. The terminology used in this section mostly follow [6]. We use the symbol $|A|$ to express the cardinality of the set $A$. The cardinality of a set gives a measure of the amount of elements present in the set. In general, the cardinality of a set may not be given by ordinary positive integers. The cardinality of a finite set is given by a positive integer and the set is considered to be countable. We consider the set of positive integers $\mathbb{Z}_+$ to be countable. Any element of $\mathbb{Z}_+$ can give us the cardinality of a discrete set containing the same number of elements. $\mathbb{Z}_+$ is considered to be the largest countable set [6]. Any set with a greater cardinality is uncountable [6]. We do not include zero in $\mathbb{Z}_+$ because zero decides only the requirement of counting and is not an outcome of an actual act of counting. We now state Theorem 1-7.1., [6]:

**THEOREM 3-1.** Let $B$ be a nonempty set. Then the following are equivalent:

1. There is a surjective function $f: \mathbb{Z}_+ \to B$.
2. There is an injective function $g: B \to \mathbb{Z}_+$.
3. $B$ is countable.

The proof can be found in [6]. Isomorphic sets have the same cardinality by the Schröder – Bernstein theorem [8]. We now demonstrate that the set of rational numbers has a greater cardinality than the set of integers.

We first discuss the following comments regarding functions defined on $\mathbb{Z}_+$. These comments are consistent with the inductive character of $\mathbb{Z}_+$ and also with the characteristics expected of an actual act of counting mentioned in the following paragraph. We denote the set of first $N$ positive integers $\{1, 2, 3, \ldots, N\}$ by $\mathbb{Z}_N$, the set of first $N$ positive odd integers $\{1, 3, 5, \ldots, 2N - 1\}$ by $O_N$ and the set of first $N$ positive even integers $\{2, 4, 6, \ldots, 2N\}$ by $E_N$. We first note that $|Z^q_N| = |Z_N|^q$ for bounded values of $N$, where $Z^q_N$ is the $q$ -th order Cartesian product of $Z_N$. Picking the odd integers from a given $Z_N$ is not an injective function on $Z_N$ for any $N \in \mathbb{Z}_+$ apart from $N = 1$. Similar situation remains valid if we want to pick any proper subset of $\mathbb{Z}_+$ from $\mathbb{Z}_+$. These functions are similar to the projection operators. A different problem arises for injective functions like $2n - 1$ and $2n$, $n \in \mathbb{Z}_+$. To define these functions on $Z_N$, the range should be at least $Z_{2N}$ for all $N \in \mathbb{Z}_+$. Neither the range $Z_{2N}$ nor these functions are well-defined in the limit $N \to \infty$, i.e, when $N$ is increased indefinitely without any upper bound. This is because there can not exist any integer greater than $N$ in the limit $N \to \infty$, i.e, when $N$ is increased indefinitely without any upper bound. We need to consider the limit $N \to \infty$ due to the inductive character of $\mathbb{Z}_+$ containing the set of all positive integers. We also note that $\mathbb{Z}_+$ being the largest set of positive integers, we can not propose the existence of a set of positive integers larger than $\mathbb{Z}_+$. Thus, functions like $2n$ and $2n - 1$ are not defined on $\mathbb{Z}_+$ to $\mathbb{Z}_+$. The above discussions remain valid if we try to construct any injective function $f(n)$ on $\mathbb{Z}_+$ such that $f(n)$ is finite for finite values of $n$ but $f(n) > n$ as $n \to \infty$. This is the case with $2n - 1, 2n, m^n; m, n \in \mathbb{Z}_+$ and $m$ is finite. Such functions are required when we try to construct an injective function from a a discrete set $D$ to $\mathbb{Z}_+$ where $\mathbb{Z}_+$ is a proper subset of $D$ or is isomorphic to a proper subset of $D$. Note that the induction principle ceases to hold in the limit $n \to \infty$ because $n + 1$ does not exist in this limit. Similar aspect also remains valid for real variables. To illustrate, we consider two open intervals of $R$ given by $(-x, x)$ and $(-2x, 2x)$ with $x > 0$. $(-x, x) \subset (-2x, 2x)$ for finite or bounded values of $x$. However, in the limit $x \to \infty$, i.e, when $x$ is increased indefinitely without any upper bound, $(-x, x)$ coincides with $R$ and the function $f(x) = 2x$ ceases to exist for $x \to \infty$. Further illustrations can be found in [2].

We conclude that we can not express $\mathbb{Z}_+$ as $O_+ \cup E_+$ where $O_+ = \{x | x = 2n - 1, n \in \mathbb{Z}_+\}$ and $E_+ = \{x | x = 2n, n \in \mathbb{Z}_+\}$ since $O_+$ and $E_+$ do not exist. This is not unexpected since $Z_N \neq O_N \cup E_N$ for any finite $N \in \mathbb{Z}_+$. In particular, we can not construct an injective function from $\{1, 2\} \times \mathbb{Z}_+$ to $\mathbb{Z}_+$ by using functions like $m^n; m, n \in \mathbb{Z}_+$ and $m$ is finite. To illustrate,
to define an injective function like \( f[(k, n)] = 2^k 5^n, \) \( k = 1, 2; \) \( n \in \mathbb{Z}_+ \) on \( \{1, 2\} \times \mathbb{Z}_+ \) the range should be at least \( \mathbb{Z}_{25^n} \) with the limit \( n \to \infty \). The range and the function do not exist in this limit. The above aspects are not considered in some discussions that aim at demonstrating that \( \mathbb{Z}_+ \times \mathbb{Z}_+ \) and \( \mathbb{Q}_+ \) are countable [6,8]. In such demonstrations, functions like \( 2n - 1, 2n, \) and \( m^n \) that are well-defined from \( \mathbb{Z}_N \) to \( \mathbb{Z}_+ \) for finite values of \( N \) are inappropriately extended to \( \mathbb{Z}_+ \). In passing, we note that we can have an injective function from \( (-2x, 2x) \in \mathbb{R} \to (-x, x) \in \mathbb{R} \), both of which satisfy the order property \( O2 \). However, \( \mathbb{Z}_+ \times \mathbb{Z}_+ \) and \( \mathbb{Z}_+ \) are discrete collections of points. We can introduce an order relation in \( \mathbb{Z}_+ \) by using the order relation of \( \mathbb{R} \) and use the dictionary order relation in \( \mathbb{R} \times \mathbb{R} \) [6] to introduce an order relation in \( \mathbb{Z}_+ \times \mathbb{Z}_+ \). With these order relations, \( \mathbb{Z}_+ \) and \( \mathbb{Z}_+ \times \mathbb{Z}_+ \) do not have the order property \( O2 \).

We now mention the following comments. In an actual act of counting of the number of elements in a set \( A \), we first separate the elements of \( A \) and enumerate them in an increasing order using the elements of \( \mathbb{Z}_+ \) as \( 1, 2, 3,... \). The maximum integer in the above sequence without any gap gives us the cardinality of \( A \). Using other members of the equivalence class \( [\mathbb{Z}_+] \) of sets isomorphic to \( \mathbb{Z}_+ \) for enumeration is not much meaningful in an actual act of counting. We can use the elements of the set \( S_+ \) given by: \( s_n = f(n), n \in \mathbb{Z}_+; \) where \( f(n) \) is a well-defined injective function, to enumerate the elements of \( A \) but we have to use the index \( n \in \mathbb{Z}_+ \) of \( s_n \) and not \( s_n \in S_+ \) itself to express the number of elements present in \( A \). To illustrate further, we consider two towers of points in the plane given by: \( H_1 = (1, m), H_2 = (2, n), 1 \leq m \leq M, 1 \leq n \leq N; m, n, M, N \in \mathbb{Z}_+ \). We can enumerate the elements of \( H_1 \) using the elements of the set \( S_+ \). We can enumerate the elements of \( H_2 \) using the elements of \( T_+ \) defined as \( T_+ = -S_+ \). Thus, we can enumerate the elements of \( H_1 \) as: \( s_1, s_2, ..., s_M \), and the elements of \( H_2 \) as: \( -s_1, -s_2, ..., -s_N \). However, the cardinality of \( H_1 \) is \( M \), that of \( H_2 \) is \( N \) and the cardinality of \( H = H_1 \cup H_2 \) is given by \( M + N \in \mathbb{Z}_+ \). Note that, in this case we have a mapping from a proper subset of \( \{1, 2\} \times \mathbb{Z}_+ \) to \( \mathbb{Z}_+ \). \(|H|\) may not remain well-defined for all possible choices of \( H_1 \) and \( H_2 \). For example, we can consider \( H_1 \) to consist of all points with \( m \in \mathbb{Z}_+ \) and take \( N = 4 \) for \( H_2 \). We can easily construct a bijective function: \( h : H_1 \to \mathbb{Z}_+ \) defined as \( h[(1, m)] = m \). This indicates that \(|H_1| = |\mathbb{Z}_+|\), by the \textit{Schröder–Bernstein} theorem [6,8]. The cardinality of \( H \), given by: \(|H| = |H_1|+|H_2| = |\mathbb{Z}_+|+4\), does not correspond to any integer in \( \mathbb{Z}_+ \) and is greater than \(|\mathbb{Z}_+|\). This conclusion justifies that \( H \) contains points in addition to those present in \( H_1 \). Similar conclusion remains valid if we try to construct any injective function \( h \) from \( H \) to \( \mathbb{Z}_+ \). The problems with such constructions was discussed in the above paragraph. In the second example, \( H_1 \) is isomorphic to \( \mathbb{Z}_+ \) and should have the same cardinality as that of \( \mathbb{Z}_+ \). \( H \) should have a cardinality greater than \( H_1 \) unless \( H_2 \) is null. We conclude that \( H \) is uncountable in this case. The situation is similar to the cardinality of the sets \( W \) and \( G \) mentioned in the introduction. As a further illustration, we can use convergent functions from Cartesian products of \( \omega \) or \( \mathbb{Z}_+ \) to \( \mathbb{R} \) to measure the number of points present in such products. This is often done in statistical mechanics. Here we construct an example. We first construct the function \( g(n_1, n_2, ..., n_q) \) from the finite dimensional Cartesian product \( \omega^q \) to \( \mathbb{R} \) defined as:

\[
g(n_1, n_2, ..., n_q) = \exp(-\alpha \sum_i n_i) = \prod_i \exp(-\alpha n_i), \quad n_i \in \omega, \quad \alpha > 0
\]  

where the product is over the \( q \) copies of \( \omega \). We next construct the functional:

\[
Z = \sum_{n_1} ... \sum_{n_q} g(n_1, n_2, ..., n_q), \quad n_i \in \omega
\]

\[
\rightarrow \left( \frac{e^\alpha}{e^\alpha - 1} \right)^q
\]
Where we have first considered the partial sum of first \( n+1 \) terms of the series \( \sum_k \exp (-\alpha k) \), \( k \in \omega \). We express it as:

\[
S(n) = \frac{1}{1-r} - \frac{r^n}{1-r} + r^n, \quad r = \exp (-\alpha)
\]  

(3)

The sum over \( \omega \) is obtained by taking the limit \( n \to \infty \). The functions \( r^n : (0,1) \to R \) converge to 0 when \( n \to \infty \). Thus \( Z \) converges to \( (\frac{r^n}{e^n})^\omega \). We also consider the functional:

\[
E = -\frac{\partial}{\partial \alpha} \ln(Z) = \frac{q}{e^\alpha - 1}.
\]  

(4)

Both \( Z \) and \( E \) are defined as weighted sum over the complete \( \omega^q \) and gives a measure of \( |\omega^q| \) using the convergent function \( g \) from \( \omega^q \) to \( R \). Eqs.(2,4) indicate that \( |\omega^q| > |\omega| \), consistent with the discussions given in this paragraph. Similar conclusion remains valid for \( \omega^q \) when we consider the single particle partition function for a microscopic particle confined in a \( q \)-dimensional rigid box. In Eq.(2), we can restrict the sums over all \( n_i \) apart from \( n_1 \) to nonvanishing finite values. The resulting expression is larger than the case when all \( n_i \) apart from \( n_1 \) are vanishing. This indicates that the cardinality of the sets like \( H \) mentioned before is greater than \( Z_\omega \).

It is expected from the above discussions that \( Q_\omega \) has a greater cardinality than \( Z_\omega \). We also note that \( Q_\omega \) have the order property \( O2 \) but \( Z_\omega \) do not. We now prove the following theorem. We follow an approach similar to the proof of Theorem 3-6.5. [6]. In the proof of this theorem, we will consider a counting where the elements of a set are separated using the Hausdorff condition. We start by noting that we use a bijective function on \( Z_\omega \), now consider the set \( \{ a_n \} \) \( n \in Z_\omega \) isomorphic to \( Z_\omega \). Any other bijective function from \( Z_\omega \) to the sequence \( \{ a_n \} \) can be considered to be a permutation on the elements of \( Z_\omega \) because the index set of \( \{ a_n \} \) is \( Z_\omega \).

**THEOREM 3.2.** The set of positive rational numbers and the set of positive real numbers are uncountable.

*Proof.* We will first construct a set \( A = S \cup X \), where \( S \) is a sequence of positive rationals which is isomorphic to \( Z_\omega \) and \( X \) is another set of positive rationals whose elements are different from \( S \). We will consider \( X \) to be finite. Thereafter, we will show that \( |Z_\omega| < |A| \). It then follows that: \(|Z_\omega| < |Q_\omega| \).

We consider a nested family of closed intervals of \( E^1 \) given by \( [3/10,1/2], [3/100,1/2], [33/1000,1/2], ... \). We index these intervals as \( I_1, I_2, I_3, ... \), with \( I_n \) containing \( I_{n+1} \) and \( n \in Z_\omega \). We now consider a set of points on \( E^1 \) with coordinates \( y_n \in R, n \in Z_\omega \). For \( n \geq 2, y_n \) is the midpoint between the lower boundaries of \( I_{n-1} \) and \( I_n \) and has coordinate: \( y_n = \frac{1}{2}(\frac{x_{n-1}}{10^n} + \frac{x_n}{10^n}) = (1/2)(0.x_1...x_{n-1} + 0.x_1...x_n) \), where all \( x_i \) are 3 and \( y_1 = 3/20 = 0.15 \). All \( y_n \) are different and rational. The first few are given by \( 3/20, 63/200, 663/2000, 6663/20000 \).

By the Hausdorff property of the standard topology of \( E^1 \) or \( R \), the lower boundary of every \( I_n \) can be separated from the corresponding (in index) \( y_n \) by two disjoint open intervals and every \( I_n \) excludes all \( y_m \) with \( m \leq n \). We can construct the bijective function \( f: Z_\omega \to R \) by \( f(n) = y_n, n \in Z_\omega \). Each \( I_n \) excludes the corresponding point \( y_n \). We now consider the closed interval \( I_R = [5/12,1/2] \). \( 1/3 \) is the limit of the sequence \( \{ u_n = \frac{x_i}{10^n} \}, n \in Z_\omega \) and \( x_i = 3 \) for all \( i \in Z_\omega \). \( I_R \) is contained in the intersection of all \( I_n \) and contains the points with coordinates \( 5/12 \) and \( 1/2 \), but none of \( f(n) = y_n \) is contained in \( I_R \). We can replace \( N \) by the cardinal that represents the cardinality of \( Z_\omega \). We continue to use \( N \) with the note that \( N \not\in Z_\omega \). We now consider the set \( A = \{ y_1, y_2, ... \} \cup \{ y_{10} = 1/2 \} \). The function \( f \), although injective, is not a surjection of \( Z_\omega \) to \( A \). There can not exist such a surjection from \( Z_\omega \) to \( A \) since \( N \not\in Z_\omega \). Any one-to-one surjective (bijective) function \( g \) from \( Z_\omega \) to \( A \) can be considered to be a permutation on the elements of \( Z_\omega \) onto the index set of the elements of \( A \). This can be easily understood if
we compose with $g$ from the left the bijective function $h$ from $A$ to the index set of its elements where, $h(y_n) = f^{-1}(y_n) = n$, $n \in Z_+$ and $h(y_0 = 1/2) = \aleph_0$. Such a permutation is possible if the the index set of $A$ would have been $Z_+$. However, we can not permute the elements of $Z_+$ to generate an additional element so that they can correspond to the elements of a set containing one more element in addition to those present in $Z_+$. Thus, there can not exist a one-to-one surjective function from $Z_+$ to $A$. As mentioned before, it is important to keep in mind that there exists no injective function $f(n)$ from $Z_+$ to $Z_+$ such that $f(n)$ is finite for finite values of $n$ but $f(n) > n$ as $n \to \infty$. We can consider other examples of $S$ and $I_n$. We can take $I_1 = [1/4, 3/2], z_1 = 8/5 = 1.6$ and $I_n = [1/4, 1/2 + 1/n], z_n = \frac{1}{2} + \left(\frac{1}{2}\right)(\frac{1}{n-1} + \frac{1}{n})$ for $n \geq 2$ and $n \in Z_+$. We take $I_\infty = [1/4, 3/8]$. In this case, the point with coordinate $1/2$ is the limit of the sequence $\{1/2 + 1/n\}, n \in Z_+$. We replace $A$ by $B = \{z_1, z_2, \ldots\} \cup \{z_\infty = z_1 = 1/4\}$. Similar arguments as above show that there can not exist a one-to-one surjective (bijective) function from $Z_+$ to $B$.

We now show that there can not exist any surjective function from $Z_+$ to $A$. To elaborate, let there is a surjective function $p$ from $Z_+$ to $A$ which is not injective. Let $Y$ be the subset of $Z_+$ that is not mapped injectively to $A$. Let, $p(Y) = P; Y \subset Z_+, P \subset A$ and $|P| < |Y|$. $P$ is a proper subset of $A$ and $Y$ can not be $Z_+$ if $p$ has to be surjective. For any such $p$, we can always redefine the bijective function $f$ used in the previous paragraph to another bijective function $f' : Z_+ \to \{y_1, y_2, \ldots\}$ so that $f'(Z_+ - Y) \cap P = \emptyset$. We can not extend $p$ to a bijective function from $Z_+ - Y$ to $f'(Y) \cup \{y_n\} - P \cup f'(Z_+ - Y)$. The arguments are similar to those used above to prove the corresponding result for $Z_+$ and $A$. We should note that $f'(Y) \cup \{y_n\} - P$ is nonempty and is different from $f'(Z_+ - Y)$. They have different index sets. We now show that we can not have $Y = Z_+$ and $P = A$. In this case, we can restrict $p$ to construct a bijective function $p'$ from a suitable subset of $Z_+$ to $A$. $(p')^{-1}$ gives us an injective function from $A$ to $Z_+$. We exclude these examples of $p$ by showing that we can not have an injective function from $A$ to $Z_+$. Every injective function $q$ from $A$ to $Z_+$ can be considered to be a bijection $q'$ on the indices of $\{y_n\}, n \in Z_+$ and $y_n$ to the image set of $q$. We can use the bijective function $h^{-1}$ on the index set of $A$ and compose it with $q$ from the right $(=qh^{-1})$ to understand this. A bijection on the elements of a set can not give a lesser number of images. Since $\aleph_0 \notin Z_+$, it is not possible to have an injective function from $A$ to $Z_+$ even if the image set of $q$ coincides with $Z_+$. Similar conclusion is valid for the second example with $A$ replaced by $B$. Thus, there can not exist a surjective function from $Z_+$ to $A$ or $B$. There can exist a surjective function from the set $C$ to the set $D$ even if there may not exist any bijective function from $C$ to $D$ if both the sets have subsets that satisfy the order property $O2$. This is easy to show when the bijective functions are homeomorphisms. We can take $C = [0, 2]$ and $D = [0, 1] \cup \{2\}$. $Z_+, A$ and $B$ do not satisfy the order property $O2$. Similarly, we could have an injective function from $A$ to $Z_+$ if both $A$ and $Z_+$ would have satisfied the order property $O2$.

It follows from Theorem 3-1 that $A$ and $B$ are uncountable and $|Z_+| < |A|$ and $|Z_+| < |B|$. $A$ and $B$ are proper subsets of $Q_+$. Thus, there can not exist a surjective function from $Z_+$ to $Q_+$ and the later is uncountable. It is obvious that $R_+$ is also uncountable. Q.E.D.

An alternate proof of the above theorem using inductive procedure will be given later.

It is always safer to use the inductive procedure to resolve confusions regarding $|Z_+|$. To illustrate, the set $Z_+ - \{1\}$ can be obtained inductively from $Z_{N-1} - \{1\} = \{2, 3, \ldots, N\}, N \geq 2$. We consider $(Z_+ - \{1\}) \notin [Z_+], Z_+ - \{1\}$ can be considered to be the set of all positive integers excluding 1. $Z_+ - \{1\}$ is countable since any element of $Z_+ - \{1\}$ can represent the possible outcome of an act of counting on a countable set except the case when the number of elements to count is one or the set is isomorphic to $Z_+$. We consider the cardinality of $Z_+ - \{1\}$ to be given by the cardinal number $\varepsilon - 1$, where $\varepsilon$ is the cardinal number: $\varepsilon = |Z_+|$. $\varepsilon - 1$ is less than $\varepsilon$ although we can not represent the former by any positive integer. Similarly, the
cardinality of the set of all odd integers or even integers contained in $Z_+$ can be given by $\varepsilon/2$. Algebraic operations on cardinal numbers and non-negative integers can be interpreted as set theoretic operations on suitable members of equivalence classes whose cardinalities are given by the corresponding cardinal numbers and non-negative integers. The equivalence classes are collections of isomorphic sets. Thus, the cardinality of the set of integers $Z$ can be given by $2\varepsilon + 1$. Further discussions in this respect can be found in [2].

We now illustrate the significance of the inductive procedure to define $Z_+$ with a proof of the following lemma. A different version of this lemma is proved in the proof of Theorem 3-2. The proof of this lemma together with Theorem 3-1 gives us an alternate proof of Theorem 3-2.

LEMMA 3-1. There cannot exist a surjective function from $Z_+$ onto $A = \{1/2\} \cup Z_+$.

Proof. Here $1/2$ signifies an element different from all elements of $Z_+$. Let $S$ be a set whose elements are the members of a sequence $\{s_n\}, n \in Z_+, s_n > 1/2$ for all $n \in Z_+$ and $s_i \neq s_j$ for all $i, j \in Z_+$. There exists a bijective function $f : Z_+ \rightarrow S$ given by: $f(N) = s_N$. The collection of all such sets are bijectively related with one another. $Z_+$ is a particular member of this collection. We denote this collection of isomorphic sets by the class $[S]$. This is a subset of the equivalence class of all sets that are isomorphic with $Z_+$. We define $S_N$ to be the set $\{s_1, s_2, ..., s_N\}$, where $s_i \in S \in [S]$. Let $A_N = \{1/2\} \cup Z_N$ and $A = \{1/2\} \cup Z_+$. It is obvious that there cannot exist a surjective function from $S_1, S_2, S_3$ onto $A_1, A_2, A_3$ respectively for all $S \in [S]$. This follows since a function cannot assign more than one value to a single element of its domain. Suppose, there does not exist a surjective function from $S_N$ onto $A_N$ for all $S \in [S]$. We then show that there cannot exist a surjective function from $S_{N+1}$ onto $A_{N+1} = A_N \cup \{N + 1\}$ for any $S \in [S]$. For, if there exists such a function $f$ for an $S \in [S]$, we can have two cases. We may have $f(s_{N+1}) = N + 1 \in A_{N+1}$. Otherwise, we can always rearrange $S_{N+1}$ to have an $S_{N+1}'$ with $S' \in [S]$ so that $f(S'_{N+1}) = N + 1 \in A_{N+1}$. This can be done in the following way. Let $f(s_k) = N + 1 \in A_{N+1}, s_k \in S_{N+1}, k \neq N + 1$. We remove $s_k$ from $S_{N+1}$ and construct $S'_N$ where $s'_i = s_i, 1 \leq i < k$ and $s'_i = s_{i+1}, k \leq i \leq N$. We then add $s_k$ to $S'_N$ as $s'_{N+1}$ to obtain $S'_{N+1}$. We supplement further the rest of the terms $s_{N+2}, s_{N+3}, s_{N+4}, ...$ of $S$ to $S'_{N+1}$ to construct $S' \in [S]$. In either case, we cannot extend $f$ to be a surjective function on the rest of $S_{N+1}(S'_{N+1})$ onto the rest of $A_{N+1}$. Since, such a function does not exist by assumption. But the assumption holds for $N = 1, 2, 3$. Hence by induction, we cannot have a surjective function from $S(S')$ onto $A$. In the later case, existence of a surjective function from $S$ onto $A$ will give a surjective function from $S'$ onto $A$ since, a bijective function from $S'$ to $S$ followed by a surjective function from $S$ onto $A$ is a surjective function from $S'$ onto $A$. However, we have shown that the later cannot exist. Thus, there cannot exist a surjective function from $S$ onto $A$ in both the cases. This also holds for $Z_+ \in [S]$. Existence of a surjective function from $Z_+$ onto $A$ will give a surjective function from $S$ onto $A$ since, a bijective function from $S$ to $Z_+$ followed by a surjective function from $Z_+$ onto $A$ is a surjective function from $S$ onto $A$ and the later cannot exist. Q.E.D

It is easy to show that there cannot exist an injective function from $A$ to $Z_+$. [2].

It is now easy to show:

LEMMA 3-2. $E^1$ contains an uncountable number of line segments.

Proof. This follows from Axiom 2-2 and uncountability of $R$. 

It is easy to show that there cannot exist an injective function from $A$ to $Z_+$, [2].

It is now easy to show:

LEMMA 3-2. $E^1$ contains an uncountable number of line segments.

Proof. This follows from Axiom 2-2 and uncountability of $R$. 

It is easy to show that there cannot exist an injective function from $A$ to $Z_+$, [2].

It is now easy to show:

LEMMA 3-2. $E^1$ contains an uncountable number of line segments.

Proof. This follows from Axiom 2-2 and uncountability of $R$. 

It is easy to show that there cannot exist an injective function from $A$ to $Z_+$, [2].
4. Conclusion
As an important consequence of the present section, Theorem 3-2 indicates that the covering of \( R \) by the collection of open balls with rational radii centered at points with rational coordinates no longer gives a countable covering of \( R \). We can use \((r_1, r_2)\) to denote such open balls in the Euclidean metric topology through the midpoint property of Euclidean metric in the straight line. Here, \( r_1, r_2 \) are two rational numbers and \((r_1, r_2)\) gives us an open ball centered at the midpoint of the interval. This is important for the Lindelöf covering theorem in \( R \) which uses the above covering and we will have to construct a new countable covering to show that \( R \) is a Lindelöf space [13]. We will also have to change the proof of Heine-Borel theorem given in [13] accordingly. We will later try to construct an alternate proof of the Heine-Borel theorem by using the *axiom of choice* [5]. The present article can be significant for differential geometry and quantum gravity. General affine connections including non-metricity are pertinent to construct a theory of quantum gravity and can be useful to explain dark energy and inflation [14]. The corresponding mathematics relies heavily on second-countability [5,15]. Thus, it requires due attention if the real line itself is not second-countable. We can try to use the *axiom of choice* as an alternative to prove theorems that use countable basis [5]. We also note that \( R \) need not to be first-countable when the rationals are not countable.

5. References
[1] K. Ghosh 2021 Geometric Continuum, Coordinatization and Hausdorff Spaces *Preprint* https://hal.archives-ouvertes.fr/hal-03092015
[2] K. Ghosh 2006 A Few Comments on Classical Electrodynamics *Preprint* http://arxiv.org/physics/0605061
[3] M. Klien and D. Shadmi 2008 International Journal of Pure and Applied Mathematics 49 (3) 329
[4] K. Ghosh 2012 International Journal of Pure and Applied Mathematics 76 (2) 251
[5] John G. Hocking and Gail S. Young 1961 *Topology* (Dover Publications, Inc., New York)
[6] James R. Munkres 1994 *Topology A First Course* (Prentice-Hall of India Private Limited)
[7] L. Carroll 1956 *Continuity* (The World of Mathematics vol 4) ed James Newman (Dover Publications, Inc.)
[8] Paul J. Cohen 1994 *Set Theory and the Continuum Hypothesis* (Dover Publications, Inc.)
[9] R. Dedekind 1956 *Irrational Numbers* (The World of Mathematics vol 1) ed James Newman (Dover Publications, Inc.)
[10] R. Courant and F. John 1989 *Introduction to Calculus and Analysis* vol 1 (Springer-Verlag New York Inc.)
[11] J. Polchinski 1998 *String Theory* vol 1 (Cambridge University Press)
[12] C. W. Misner, K. S. Thorne and J. A. Wheeler 1970 *Gravitation* (W.H. Freeman and company, New York)
[13] Tom M. Apostol 1992 *Mathematical Analysis* (Narosa Publishing House)
[14] K. Ghosh 2019 Physics of the Dark Universe 26 100403 (2019)
[15] S. Kobayashi and K. Nomizu 1991 *Foundations of Differential Geometry* vol 1 (Wiley Classics Library)