Spectrum of the totally asymmetric simple exclusion process—bulk eigenvalues

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Abstract
We consider the totally asymmetric simple exclusion process (TASEP) on a periodic one-dimensional lattice of \( L \) sites. Using the Bethe ansatz, we derive parametric formulas for the eigenvalues of its generator in the thermodynamic limit. This allows us to study the curve delimiting the edge of the spectrum in the complex plane. A functional integration over the eigenstates leads to an expression for the density of eigenvalues in the bulk of the spectrum. The density vanishes with an exponent \( 2/5 \) close to the eigenvalue 0.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Markov processes [1] form a class of mathematical models much studied in relation to non-equilibrium statistical physics. Their evolution in time is generated by an operator \( M \), the Markov matrix, whose non-diagonal entries represent the rates at which the state of the system changes from a given microstate to another.

For processes verifying the detailed balance condition, which forbids probability currents between the different microstates at equilibrium, the operator \( M \) is real symmetric, up to a similarity transformation. It implies that its eigenvalues are real numbers. An example is the Ising model with, e.g., Glauber dynamics. Processes that do not satisfy the detailed balance, on the other hand, generally have a complex spectrum. This is the case for the asymmetric simple exclusion process (ASEP) [2–9], which consists of classical hard-core particles hopping between nearest neighbour sites of a lattice with a preferred direction.

In one dimension, ASEP is known to be exactly solvable by means of the Bethe ansatz. This has allowed exact calculations of the gap of the spectrum [10–14] and of the fluctuations of the current, both in the infinite line setting [15–20] and on a finite lattice with either periodic [21–26] or open [27–29] boundary conditions. The exponents and scaling functions obtained in these articles are universal: they characterize not only driven diffusive systems [30, 31] far from equilibrium, to which ASEP belongs, but also interface growth models.
[32–34] and directed polymers in a random medium [34–37]. This forms the Kardar–Parisi–Zhang universality class [38–40].

We focus in this paper on the special case of ASEP with unidirectional hopping of the particles called the totally asymmetric simple exclusion process (TASEP). We consider the system with periodic boundary conditions, for which the number of microstates is finite: the spectrum is then a discrete set of points in the complex plane. The aim of the present paper is to obtain a large scale description of these points. We obtain explicit expressions for the curve delimiting the edge of the spectrum in the complex plane. This curve is singular near the eigenvalue 0, with an imaginary part scaling as the real part to the power \(5/3\). By a functional integration over the eigenstates, we also derive expressions for the density of eigenvalues. Near the eigenvalue 0, the density vanishes with an exponent \(2/5\).

The paper is organized as follows. In section 2, we recall a few known things about TASEP and present our main results. In section 3, we derive parametric expressions for the eigenvalues of the Markov matrix of TASEP in the thermodynamic limit, and apply this to the curve delimiting the edge of the spectrum in section 4. In section 5, we study the density of eigenvalues. Finally in section 6 we consider a generating function for the cumulants of the eigenvalues.

2. Totally asymmetric exclusion process on a ring

We consider in this article the TASEP with \(N\) particles on a periodic one-dimensional lattice of \(L\) sites (see figure 1). A site is either empty or occupied by one particle. We call \(\Omega\) the set of all microstates (or configurations), which has the cardinality \(|\Omega| = \binom{L}{N}\). A particle at site \(i\) can hop to site \(i + 1\) if the latter is empty. The hopping rate for any particle is equal to 1, i.e. each particle allowed to move has a probability \(\delta t\) in a small time interval \(\delta t\). In the following, we call \(\rho = N/L\) the density of particles.

2.1. Master equation

We call \(P_t(C)\) the probability to observe the system in the microstate \(C\) at time \(t\). The probabilities evolve in time by the master equation

\[
\frac{dP_t(C)}{dt} = \sum_{C' \not\in C} [w(C' \leftarrow C')P_t(C') - w(C' \leftarrow C)P_t(C)].
\]  

(1)

The rate \(w(C' \leftarrow C)\) is equal to 1 if it is possible to go from configuration \(C\) to configuration \(C'\) by moving one particle to the next site, and 0 otherwise.

The master equation (1) can be conveniently written as a matrix equation by defining the vector \(|P_t\rangle = \sum_{C \in \Omega} P_t(C)|C\rangle\) of the configuration space \(V\) with dimension \(|\Omega|\), where \(|C\rangle\) is the canonical vector of \(V\) corresponding to the configuration \(C\). Then, calling \(M\) the matrix with non-diagonal entries \(\langle C|M|C'\rangle = w(C' \leftarrow C)\) and diagonal \(\langle C|M|C\rangle = -\sum_{C' \not\in C} w(C' \leftarrow C)\), one has

\[
\frac{d}{dt}|P_t\rangle = M|P_t\rangle,
\]  

(2)

which is formally solved in terms of the time evolution operator \(e^{tM}\) as \(|P_t\rangle = e^{tM}|P_0\rangle\).

The graph of allowed transitions for TASEP presents an interesting cyclic structure with period \(L\): let us consider the observable \(X\) such that \(X(C)\) is the sum of the positions of the particles (we take a fixed arbitrary site to be the origin of positions). We see that each time a particle hops to the next site, \(X(C)\) increases of 1 modulo \(L\). It is then possible to split the configurations in \(L\) sectors according to the value of \(X\) modulo \(L\). The only allowed transitions
Figure 1. The TASEP with $N = 3$ particles on a periodic one-dimensional lattice with $L = 16$ sites. The particles hop to the nearest site in the clockwise direction with rate 1 provided that site is empty.

Figure 2. Graph of all allowed transitions between the configurations of TASEP with $N = 2$ particles on a periodic lattice of $L = 6$ sites.

between configurations are then transitions from configurations of a sector $r$ to configurations of sector $r + 1$ (modulo $L$), see figure 2. We emphasize that this cyclic structure is not a consequence of the periodic boundary conditions. Indeed, a similar cyclic structure exists for
ASEP on an open segment of $L$ sites connected to reservoirs of particles, with periodicity $L + 1$ instead of $L$.

2.2. Current fluctuations

Equal time observables, such as the density profile of particles in the system or the number of clusters of consecutive particles can be extracted directly from the knowledge of $|P_t\rangle$. Other observables, however, require the $P_t(C)$ for several values of the time. A much studied example is the current of particles and especially the fluctuations around its mean value.

We define the observable $Y_t$, which counts the total displacement of particles between time 0 and time $t$. Starting with $Y_0 = 0$, it is then updated by $Y_t \rightarrow Y_t + 1$ each time a particle hops anywhere in the system.

The joint probability $P_t(C, Y)$ to observe the system in the configuration $C$ with $Y_t = Y$ obeys the master equation

$$\frac{dP_t(C, Y)}{dt} = \sum_{C' \neq C} [w(C' \leftarrow C)P_t(C', Y - 1) - w(C' \leftarrow C)P_t(C, Y)].$$  (3)

It is convenient to introduce the quantity $F_t(C) = \sum_{Y = -\infty}^{\infty} e^{\gamma Y} P_t(C, Y)$. It verifies the deformed master equation [21]

$$\frac{dF_t(C)}{dt} = \sum_{C' \neq C} [e^{\gamma} w(C \leftarrow C')F_t(C') - w(C' \leftarrow C)F_t(C)].$$  (4)

Introducing the vector $|F_t\rangle = \sum_{C \in \Omega_1} F_t(C) |C\rangle$ and a deformation $M(\gamma)$ of the Markov matrix, one has

$$\frac{d}{dt}|F_t\rangle = M(\gamma)|F_t\rangle.$$  (5)

For $\gamma = 0$, $F_t$ reduces to $P_t$ and $M(0) = M$. In the following, we will be interested in the spectrum of the operators $M$ and $M(\gamma)$.

2.3. Mapping to a height model and continuous spectrum

It is well known that TASEP can be mapped to a model of growing interface [34]: for each occupied site $i$ of the system, we draw a portion of interface decreasing from height $h_i$ to $h_{i+1} = h_i - (1 - \rho)$, and for each empty site $i$, we draw a portion of interface increasing from height $h_i$ to $h_{i+1} = h_i + \rho$. The interface obtained is continuous and periodic, see figure 3. The dynamics of TASEP then implies that parallelograms (squares at half-filling $N = L/2$) deposit on local minima of the interface with rate 1.

Unlike TASEP, the set of microstates of the growth model is not finite since the total height is not bounded: there always exists a local minimum from which the interface can grow. It is possible to identify any microstate of the growth model by the corresponding configuration of the exclusion process and the total current $Y$ defined in section 2.2. Then, (3) can be interpreted as the master equation for the growth model, to which is associated the infinite dimensional Markov matrix

$$\mathcal{M} = M^{(+)} \otimes S + M^{(0)} \otimes 1,$$  (6)

where $M^{(0)}$ and $M^{(+)}$ are respectively the diagonal and non-diagonal part of $M$, and $S$ is the translation operator in $Y$ space, $S = \sum_{Y = -\infty}^{\infty} |Y\rangle \langle Y - 1|$. 
We would like to diagonalize $M$ in order to study its spectrum. We note that $M$ commutes with $S$. This implies that the eigenvectors $|\Psi\rangle$ of $M$ must be of the form

$$|\Psi\rangle = |\psi\rangle \otimes |\phi(\theta)\rangle,$$

where $|\psi\rangle$ is a vector in configuration space and

$$|\phi(\theta)\rangle = \frac{1}{\sqrt{2\pi}} \sum_{Y=-\infty}^{\infty} e^{-i\theta Y} |Y\rangle$$

the right eigenvector of $S$ with eigenvalue $e^{i\theta}$. The eigenvalue equation for $|\Psi\rangle$ can be written

$$E|\Psi\rangle = [(e^{i\theta} M^{(+)} + M^{(0)})|\psi\rangle] \otimes |\phi(\theta)\rangle = [M(i\theta)|\psi\rangle] \otimes |\phi(\theta)\rangle,$$

where $M(i\theta)$ is the deformation of the Markov matrix introduced in section 2.2 and $|\psi\rangle$ and $E$ an eigenvector and an eigenvalue of $M(i\theta)$. We finally find that the spectrum of $M$ is the reunion of the spectra of the $M(\gamma)$ with $|\gamma| = 1$. This spectrum is represented in figure 4 for $L = 8, N = 4$.

The spectrum of $M$ can also be constructed from the finite spectra obtained by counting the total current $Y$ modulo $KL$ with $K$ a positive integer, and taking the limit $K \to \infty$. The case $K = 1$ is the usual TASEP because of the cyclic structure of the graph of allowed transitions discussed at the end of section 2.1. The case $K = N$ corresponds to TASEP with distinguishable particles, restricted to a subspace with a given cyclic order of the particles since the particles cannot overtake each other.

We call $M^{(K)}(\gamma)$ the corresponding deformed Markov matrix. The $K|\Omega|$ configurations arrange themselves in $KL$ sectors according to the value of $Y$. Calling $P_r$ the projector on the $r$th sector and $M^{(K)}_{r,r+1}(\gamma) = e^{ly} P_{r+1} M^{(K)} P_r + P_r M^{(K)} P_r$, one has

$$M^{(K)}(\gamma) = \sum_{r=1}^{KL} M^{(K)}_{r,r+1}(\gamma).$$

Introducing $U = \sum_{r=1}^{KL} e^{ly} P_r$, one finds

$$U^{-1} M^{(K)}(\gamma) U = \sum_{r=1}^{KL} M^{(K)}_{r,r+1} + e^{ly} P_{KL} M^{(K)}_{KL} P_{KL} + P_{KL} M^{(K)}_{KL} P_{KL},$$
Figure 4. The spectrum of the Markov matrix of TASEP with four particles on eight sites. The black dots are the eigenvalues for undistinguishable particles, while the black + red (grey in printed version) dots are the eigenvalues for distinguishable particles. The curves are the eigenvalues of the corresponding interface growth model.

with $M^{(K)}_{r+1} = M^{(K)}_{r+1}(0)$. This implies that the spectrum of $M^{(K)}(γ)$ is invariant under the transformation $γ → γ + 2iπ/(KL)$. Starting from a given eigenvalue of $M^{(K)}(γ)$ and following the eigenvalue during the continuous change from $γ$ to $γ + 2iπ/(KL)$, one does not in general come back to the initial eigenvalue: we observe by numerical diagonalization that one goes from one eigenvalue to the next one anticlockwise on the same continuous curve in figure 4.

2.4. Large scale description of the spectrum

Typical eigenvalues of the Markov matrix $M$ scale proportionally with $L$. Dividing all the eigenvalues by $L$, the rescaled spectrum fills a region of the complex plane in the thermodynamic limit. We call $e(μ)$ a parametrization of the curve at the edge of this region. An exact representation of this curve, (54), (55), is obtained in section 4. We observe that $e(μ)$ is singular near $e = 0$, with the scaling

$$\text{Re } e \simeq -\frac{2^{2/3}3^{1/6}π^{2/3}}{10}|\text{Im } e|^{5/3}. \quad (12)$$

The density of eigenvalues $D(e)$ in the rescaled spectrum is also studied in section 5. It grows for large $L$ as

$$D(e) \sim e^{s(e)}. \quad (13)$$

The quantity $s(e)$ is computed exactly. For eigenvalues close to 0, but far from the edge of the spectrum (i.e. with $|\text{Im } e|^{5/3} \ll |\text{Re } e|$), one finds in particular

$$s(e) \simeq \xi (-\text{Re } e)^{2/5} \quad \text{with} \quad \xi \simeq 1.58600. \quad (14)$$
The exponent 2/5 is different from the exponent 1/3 obtained in appendix D for undistinguishable non-interacting particles hopping unidirectionally on a periodic one-dimensional lattice.

2.5. Trace of the time evolution operator and cumulants of the eigenvalues

We consider the quantity

$$Q(t) = \frac{1}{|\Omega|} \text{tr} e^{tM} = \frac{1}{|\Omega|} \sum_{\mathcal{C} \in \Omega} \langle \mathcal{C} | e^{tM} | \mathcal{C} \rangle.$$  \hspace{1cm} (15)

Here $\langle \mathcal{C} | e^{tM} | \mathcal{C} \rangle$ is the probability that the system is in the microstate $\mathcal{C}$ at time $t$ conditioned on the fact it was already in the microstate $\mathcal{C}$ at time 0. Since all configurations are equally probable in the stationary state of periodic TASEP [2], $Q(t)$ is simply the stationary probability that the system is in the same microstate at both times 0 and $t$.

2.5.1. Perturbative expansion. We consider more generally (15) with $M$ replaced by the deformation $M(\gamma)$, and define

$$f(t) = \frac{1}{L} \log \text{tr} e^{tM(\gamma)}.$$  \hspace{1cm} (16)

The quantity $f(t)$ can be seen as the generating function of the cumulants of the eigenvalues of $M(\gamma)$ (or more precisely the cumulants of the uniform probability distribution on the set of the eigenvalues). The moments $\mu_k = |\Omega|^{-1} L^{-k} \text{tr} M(\gamma)^k$ and the cumulants $c_k = L^{1-k} f^{(k)}(0)$ of the eigenvalues are indeed related from (16) by

$$\log \left( 1 + \sum_{k=1}^{\infty} \frac{\mu_k t^k}{k!} \right) = \sum_{k=1}^{\infty} \frac{c_k t^k}{k!}.$$  \hspace{1cm} (17)

The first cumulants are $c_1 = \mu_1 = f'(0)$, $c_2 = \mu_2 - \mu_1^2 = f''(0)/L$ and $c_3 = \mu_3 - 3\mu_1 \mu_2 + 2\mu_1^3 = f'''(0)/L^2$.

The moments and cumulants of the eigenvalues are independent of the deformation $\gamma$. Indeed, one can write

$$\text{tr} M(\gamma)^k = \sum_{C_1, \ldots, C_k} e^{\sum_{j=1}^{k} \langle C_j | M | C_j \rangle} \langle C_1 | M | C_2 \rangle \langle C_2 | M | C_3 \rangle \ldots \langle C_k | M | C_1 \rangle.$$  \hspace{1cm} (18)

Because of the cyclic structure of allowed transitions explained at the end of section 2.1, only $k$-tuples of configurations such that $\sum_{j=1}^{k} \langle C_j | C_{j+1} \rangle$ is divisible by $L$ contribute to (18). For $k < L$, it implies that $\text{tr} M(\gamma)^k$ cannot depend on $\gamma$. One has then

$$f(t) = \left( \frac{1}{L} \log \text{tr} e^{tM} + O(t^2) \right).$$  \hspace{1cm} (19)

It is possible to calculate directly the coefficients of the expansion near $t = 0$ of $f(t)$ by considering the case $\gamma \to -\infty$, for which the matrix $M(\gamma)$ becomes diagonal in configuration basis: $\langle C | M(\gamma) | C \rangle$ is equal to minus the number $m(C)$ of clusters of consecutive particles in the system. It implies

$$f(t) = \frac{1}{L} \sum_{C \in \Omega} e^{-t m(C)} = \frac{1}{L} \sum_{m=1}^{N} |\Omega(m)| \ e^{-t m}.$$  \hspace{1cm} (20)

The total number $|\Omega(m)|$ of configurations with $m$ clusters can be calculated in the following way: a configuration with $m$ clusters for which the last site is occupied can be described as $a(0) \geq 0$ particles followed by $b(1) > 0$ empty sites, $a(1) > 0$ particles, $\ldots$, $b(m) > 0$ empty sites, $a(m) = 0$, and vice versa. Therefore, $m(C)$ is a non-negative integer.

$$f(t) = \frac{1}{L} \sum_{m=1}^{N} |\Omega(m)| \ e^{-t m}.$$  \hspace{1cm} (20)
sites and \(a(m) > 0\) particles. The total number of such configurations is \(A_{m+1}(N+1)A_m(L-N)\) with

\[
A_m(r) = \sum_{b(1),\ldots,b(m)=1} \mathbb{1}_{b(1)+\cdots+b(m)=r} \equiv \oint \frac{dz}{2\pi i} (1-z)^{m-r} = \left( r - 1 \right)_{m-1}. \tag{21}
\]

From particle-hole symmetry, the number of configurations with \(m\) clusters for which the last site is empty is \(A_m(N)\). This implies

\[
|\Omega(m)| = \frac{mL}{N(L-N)} \binom{N}{m} \binom{L-N}{m}. \tag{22}
\]

A saddle point approximation of the sum over \(m\) in (20) finally gives

\[
f(t) = \rho \log \left( \frac{(1 - 2\rho) + 2(1 - \rho) e^t + \sqrt{1 + 4\rho(1 - \rho)(e^t - 1)}}{2\rho e^t} \right) + (1 - \rho) \log \left( \frac{-(1 - 2\rho) + 2(1 - \rho) e^t + \sqrt{1 + 4\rho(1 - \rho)(e^t - 1)}}{2(1 - \rho) e^t} \right). \tag{23}
\]

which simplifies at half-filling to

\[
f(t) = \log(1 + e^{-t/2}). \tag{24}
\]

The expressions (23) and (24) must be understood as an equality between the Taylor series.

In section 6, we consider again the quantity \(f(t)\) as a testing ground for the formulas derived from the Bethe ansatz in section 3 for the eigenvalues of \(M\) in the thermodynamic limit. We write the summation over the eigenvalues as an integral over a function \(\eta\) that index the eigenstates. After a saddle point calculation in the functional integral, we recover (23).

2.5.2. Finite \(t\). The total number of particles hopping during a finite time \(t\) is roughly proportional to the average number of clusters of consecutive particles in the system. For typical configurations, this number scales proportionally with the system size in the thermodynamic limit at a finite density of particles. Because of the cyclic structure with period \(L\) in the graph of allowed transitions described at the end of section 2.1, the quantity \(f(t)\) should have oscillations for finite times. The same reasoning also works for undistinguishable non-interacting particles. For distinguishable particles, on the other hand, a similar argument shows that \(f(t)\) should show oscillations on the scale \(t \sim L\), since the cyclic structure of the graph of allowed transitions has then period \(NL\): all the particles need to come back to their initial state.

The oscillations of \(f(t)\) are observed for TASEP from numerical diagonalization, see figure 5. For non-interacting particles, they are confirmed by a direct calculation in appendix D, see figure D2. In both cases, we observe that the oscillations of \(f(t)\) are not smooth: the function \(f(t)\) is defined piecewise. There exists in particular a time \(t(\gamma)\) such that \(f(t)\) is analytic (and independent of \(\gamma\)) for \(t\) between 0 and \(t(\gamma)\). We find \(t(\gamma) \simeq 1.7085 e^{-\gamma}\) for undistinguishable free particles. For TASEP at half-filling, figure 5 seems to indicate that \(t(0)\) is slightly larger than 1.

3. Parametric formulas for the eigenvalues

In this section, we derive an exact parametric expression, (35), (36), for all the eigenvalues of the Markov matrix of TASEP.
3.1. The Bethe ansatz

The deformed Markov matrix $M(\gamma)$ of TASEP is equivalent by similarity transformation to minus the Hamiltonian of a ferromagnetic XXZ spin chain with anisotropy $\Delta = \infty$ and twisted boundary conditions [4]. The integrability of TASEP is a consequence of this. The eigenfunctions of $M(\gamma)$ are given by the Bethe ansatz as [2–4]

$$\psi(x_1, \ldots, x_N) = \det \left( \frac{y_k}{y_j} \right)_{N-j}^{N \times j} \left( \frac{1-e^{-L\gamma}}{1-y_j} \right)_{j=1}^{N} \text{det}_{j=1}^{N} \left( y_k \right)_{k=1}^{N},$$

(25)

provided the quantities $y_j$, called Bethe roots, verify the Bethe equations

$$\frac{(1-y_j)^L}{y_j^N} = (-1)^{N-1} e^{-L\gamma} \prod_{k=1}^{N} \frac{1}{y_k}. \text{ (26)}$$

The Bethe equations have many different solutions, corresponding to the various eigenstates of $M(\gamma)$. The eigenvalue of $M(\gamma)$ corresponding to (25) is

$$E = \sum_{j=1}^{N} \frac{y_j}{1-y_j}. \text{ (27)}$$

**Remark 1.** The Bethe ansatz is usually written in terms of the variables $z_j = e^{y_j} = e^{y_j (1-y_j)}$ instead of the $y_j$. The eigenvectors are then linear combinations of plane waves with momenta $q_j$.

**Remark 2.** Proving the completeness of the Bethe ansatz for periodic TASEP is still an open problem, although one observes numerically for small systems that it does give all the eigenstates. The main difficulties consist in proving that the Bethe equations (26) have $|\Omega|$ solutions, and that all the eigenvectors generated form a basis of the configuration space of the model. Alternatively, the completeness would follow from a direct proof of the resolution of the identity, $I = \sum_{k=1}^{[\Omega]} |\psi_k\rangle \langle \psi_k|$. For periodic TASEP in a discrete time setting with parallel update, such a proof was given by Povolotsky and Priezzhev in [41].
3.2. Parametric solution of the Bethe equations

The Bethe equations (26) of TASEP have the particularity that the rhs does not depend specifically on \( y_j \), but is instead a symmetric function of all the \( y_k \). We write this rhs \((-1)^{N-1}/B^L\). This allows to us solve the Bethe equations in a parametric way, by first solving for each \( y_j \) the polynomial equation

\[
(1 - y_j)y_j^N = (-1)^{N-1}/B^L
\]

as a function of \( B \), and then solving a self-consistency equation for \( B \). This ‘decoupling property’ was already used in [10, 11, 13, 14] for the calculation of the gap, and in [21, 22] for the calculation of the eigenvalue of \( M(\gamma) \) with largest real part. The same property is also true for periodic TASEP in a discrete time setting with parallel update [41].

Taking the power \( 1/L \) of the Bethe equations, there must exist numbers \( k_j \), \( j = 1, \ldots, N \), integers if \( N \) is odd, half-integers if \( N \) is even, such that

\[
1 - y_j y_j^{\rho} = e^{2i\pi k_j/L}/B.
\]

We also define

\[
\ell = -\rho \log \rho - (1 - \rho) \log(1 - \rho).
\]

The Bethe equations (29) involve a function \( g \), defined as

\[
g : \mathbb{C} \setminus \mathbb{R}^- \to \mathbb{C} \setminus \{ (\exp^{i\pi/2}\exp^i, \infty) \cup [\exp^{i\pi/2}\exp^i, \infty) \}
\]

\[
y \mapsto \frac{1 - y}{\exp^\rho}.
\]

It turns out that this function is a bijection for \( 0 < \rho < 1 \), see appendix A. This is a key point, as it allows us to formally solve the Bethe equations as

\[
y_j = g^{-1}(e^{2i\pi k_j/L}/B).
\]

Then, the corresponding eigenvalue can be computed from (27), and \( B \) is fixed by solving (30). We assumed that \( e^{2i\pi k_j/L}/B \) does not belong to the cut of \( g^{-1} \). If this is not the case for some eigenstate, a continuity argument in the parameter \( \gamma \) should still allow to use (33).

Remarks. There are exactly \( |\Omega| \) ways to choose the \( k_j \) with the constraint \( 0 < k_1 < \cdots < k_N \leq L \). We observe numerically on small systems that all \( |\Omega| \) eigenstates are recovered with this constraint. A similar argument was given in [13], using instead a rewriting of the Bethe equations as a polynomial equation of degree \( L \) for the \( z_j \); the number of ways to choose \( N \) roots from this polynomial equation is also equal to \( |\Omega| \).

3.3. Functions \( \varphi \) and \( \psi \)

We define the rescaled eigenvalue \( e = E/L \) and the parameter \( b = \log B \). We introduce the functions

\[
\varphi(z) = \frac{g^{-1}(z)}{1 - g^{-1}(z)} \quad \text{and} \quad \psi(z) = \log g^{-1}(z).
\]
In terms of \( \varphi \) and \( \psi \), the parametric expression (27), (30) is rewritten as

\[
e = \frac{1}{L} \sum_{j=1}^{n} \varphi(e^{\frac{2\pi i j}{L}} - b)
\]

(35)

\[
b = \gamma + \frac{1}{L} \sum_{j=1}^{n} \psi(e^{\frac{2\pi i j}{L}} - b).
\]

(36)

From (34) and (32), the functions \( \varphi \) and \( \psi \) verify the relations

\[
\log z + \rho \log \varphi(z) + (1 - \rho) \log(1 + \varphi(z)) = 0
\]

(37)

\[
e^{\psi(z)} + z e^{\varphi(z)} = 1.
\]

(38)

They are also related by the two equations

\[
\varphi(z) = \frac{e^{\psi(z)}}{1 - e^{\psi(z)}}
\]

(39)

\[
z \varphi(z) = e^{(1 - \rho) \psi(z)}.
\]

(40)

The expansion near \( z = 0 \) of \( \varphi(z) \) and \( \psi(z) \) at arbitrary filling can be computed explicitly from (34) and the observation [21] that for any meromorphic function \( h \), \( h(g^{-1}(z)) \) can be written as an integral contour. One has

\[
h(g^{-1}(z)) = \oint_{\Gamma} \frac{dw}{2\pi i} \frac{h(g^{-1}(w))}{w - z} = \oint_{\Gamma} \frac{dy}{2\pi i} \frac{g'(y)h(y)}{g(y) - z}.
\]

(41)

where the contour \( \Gamma \) encloses \( g^{-1}(z) \) but none of the poles of \( h \), and does not cross the cut \( \mathbb{R}^- \) of \( g \). We will need to expand \( h(g^{-1}(z)) \) for small \( z \). In the previous expression, it is possible to expand the integrand near \( z = 0 \) as long as \( |g(y)| > |z| \) for all \( y \in \Gamma \). As shown in figure 6, it is possible to find such a contour \( \Gamma \) only if \( |z| < e^\ell \), otherwise the contour would have to go through the cut.

We use (41) for \( h(y) = \log y \) and expand for small \( z \) inside the integral. After computing the residues at \( y = 1 \) (which is always inside the contour, see figure 6), we obtain

\[
\psi(z) = \sum_{r=1}^{\infty} \left( \frac{\rho r}{\rho} \right)^{-1} z^r r^r.
\]

(42)

Using (41) again for \( h(y) = y/(1 - y) \), one also finds

\[
\varphi(z) = z^{-1} - (1 - \rho) \sum_{r=0}^{\infty} \left( \frac{\rho r}{r} \right)^{-1} z^r r^r + 1.
\]

(43)

The first term takes into account the pole at \( y = 1 \) of \( y/(1 - y) \), so that the contour \( \Gamma \) encloses both poles of the integrand \( y = 1 \) and \( y = g^{-1}(z) \).

At half-filling, the summation over \( r \) in (43) and (42) can be done explicitly. One finds

\[
\varphi(z) = \frac{-1}{2} + \frac{\sqrt{4 + z^2}}{2z}
\]

(44)

\[
\psi(z) = -2 \arcsinh(z/2) = -2 \log \left( \frac{z}{2} + \sqrt{1 + \frac{z^2}{4}} \right).
\]

(45)
which case \( b \simeq \gamma \). Besides, when \( \gamma = 0 \), numerical checks on small system seem to indicate that (36) always has a solution inside the radius of convergence of the series in \( e^{-b} \) for all the eigenstates, except the one with eigenvalue 0. For the latter, it seems that (36) does not have a solution \( b \in \mathbb{C} \), although formally \( b = -\infty \) is a solution of (36) for which (35) gives \( e = 0 \). We emphasize that for all other solutions of (36), even the ones for which the eigenvalue is close to 0 such as the gap, equation (36) seems to have a solution.

### 3.4. Thermodynamic limit

For large \( L \), both (35) and (36) become independent of the detailed structure of the \( k_j \), and only retain information about the density profile of the \( k_j \). For each eigenstate, we introduce a function \( \eta \) such that \( L \eta(u) \, du \) is the number of \( k_j \) in the interval \([Lu, L(u + du)]\). The function \( \eta \) then obeys the normalization

\[
\rho = \int_0^1 du \eta(u),
\]

while the eigenvalue (35) becomes

\[
e = \int_0^1 du \eta(u) \varphi(e^{2i\pi u - b}),
\]

and the equation for the parameter \( b \) (36) is rewritten as

\[
b = \gamma + \int_0^1 du \eta(u) \psi(e^{2i\pi u - b}).
\]

In sections 5 and 6, we will need to count the number of eigenstates corresponding to a given function \( \eta \) in order to sum over the eigenvalues. Since the number of ways to place \( L \eta(u) \, du \, k_j \)
in any interval \( L \) \( da \), Stirling’s formula implies that the total number of eigenstates corresponding to \( \eta \) is \( \Omega[\eta] \sim e^{Ls} \), where we defined an ‘entropy per site’

\[
s = - \int_0^1 da \ [\eta(a) \log \eta(a) + (1 - \eta(a)) \log(1 - \eta(a))]. \tag{49}
\]

4. Edge of the spectrum

In the thermodynamic limit, the rescaled eigenvalues \( e = E/L \) fill a bounded domain in the complex plane, see figure 9. We study in this section the boundary of this domain, called in the following the edge of the spectrum.

We observe numerically that the eigenvalues located at the edge of the spectrum correspond to eigenstates for which the \( k_j \) are consecutive numbers. There are \( L \) such possibilities, that we index with an integer \( m \) between 0 and \( L - 1 \). We write \( k_j = m + j - (N + 1)/2 \). The eigenvalue with largest real part corresponds to \( m = 0 \) for \( \gamma > 0 \). For \( \gamma = 0 \), however, we recall that there is no solution to (36) when \( m = 0 \). We will see that it corresponds to a singular point for the curve at the edge of the spectrum.

In the limit \( L \to \infty \), the corresponding density profile \( \eta(u) \) of the \( k_j \) is the function with period 1 such that

\[
\eta(u) = \begin{cases} 
1 & \text{for } \mu - \frac{\rho}{2} < u < \mu + \frac{\rho}{2} \\
0 & \text{otherwise},
\end{cases}
\tag{50}
\]

where we defined \( \mu = m/L \). With this choice of \( \eta \), writing explicitly the dependence in \( \mu \) of the eigenvalue and of the parameter \( b \), one finds

\[
e(\mu) = \int_{-\rho/2}^{\rho/2} du \ \phi(e^{2\pi i(u+\mu) - b(\mu)}) \tag{51}
\]

and

\[
b(\mu) = \int_{-\rho/2}^{\rho/2} du \ \psi(e^{2\pi i(u+\mu) - b(\mu)}). \tag{52}
\]

At half-filling, a nice parametric representation of the curve \( e(\mu) \) can be written by replacing \( b(\mu) \) by a new variable \( d(\mu) \) defined by

\[
d(\mu) = -i \arccos \left( \frac{e^{2\pi i - b(\mu)}}{2} \right). \tag{53}
\]

Taking the derivative with respect to \( \mu \) of the equation for \( b(\mu) \) and calculating explicitly the integrals, one obtains

\[
e(\mu) = -\frac{\tanh d(\mu) - d(\mu)}{2\pi} \tag{54}
\]

and

\[
d'(\mu) = \frac{\pi^2}{d(\mu) \tanh d(\mu)}. \tag{55}
\]

The initial condition for the differential equation (55) depends on the value of \( \gamma \). For \( \gamma = 0 \), since \( e(0) = 0 \) (largest eigenvalue of a Markov matrix), then one must have \( d(0) = 0 \), which corresponds to \( b(0) = -\log 2 \). This is a singular point for the differential equation (55), which is related to the fact that \( b = -\log 2 \) corresponds to the border of the region where the expansion of \( \phi(z) \) and \( \psi(z) \) for small \( z \) in section 3 is convergent.
Figure 7. Solutions of the differential equation (55) for small \( \mu \) with initial condition \( d(0) = 10^{-10} e^{i\theta} \), for \( \theta = \pi/15, 3\pi/15, \ldots, 14\pi/15 \). The initial values \( d(0) \) are represented on the circle. The arrows indicate the direction of the increment given by the derivative at \( \mu = 0 \), \( d'(\mu) \approx \pi^2/2d(\mu) \). The curves starting on the circle represent the solution of the differential equation (55) solved numerically, which is essentially undistinguishable from the solution of \( d'(\mu) = \pi^2/d(\mu)^2 \) for the values of \( d \) shown. The dotted lines represent the 3 solutions of (55) unstable under a perturbation near \( d = 0 \).

Expanding the differential equation at second order near \( \mu = 0 \) leads to three solutions. Inserting them in the equation for \( e(\mu) \), one finds that only the solution

\[
\frac{d(\mu)}{2} = \frac{\pi^2}{24} + d(\mu) \log(1 + e^{-2d(\mu)}) - \frac{1}{2} \text{Li}_2(-e^{-2d(\mu)}) = \pi^2 \mu - \pi^2 \left(1 + \left\lfloor \mu - \frac{1}{5} \right\rfloor\right).
\]

(58)

4.1. Dilogarithm

An alternative expression to (55) can be written by solving explicitly the differential equation for \( d \) in terms of a dilogarithm function, as

\[
\frac{d(\mu)^2}{2} - \frac{\pi^2}{24} + d(\mu) \log(1 + e^{-2d(\mu)}) - \frac{1}{2} \text{Li}_2(-e^{-2d(\mu)}) = \pi^2 \mu - \pi^2 \left(1 + \left\lfloor \mu - \frac{1}{5} \right\rfloor\right).
\]

(58)
Figure 8. Curve in the complex plane of $d(\mu)$ (left) and $-e^{-2d(\mu)}$ (right), from a numerical resolution of the differential equation (55). In both graphs, the two dots are the explicit values $-e^{-2d(\pm 1/5)} = (3 \pm \sqrt{5})/2$. In the graph on the left, the outer, red curve encloses the domain for which $\Re b > -\log 2$, for which the expansions (42) and (43) hold. In the graph on the right, the thick, red line corresponds to the cut of the dilogarithm $[1, \infty)$.

For $\mu$ in a neighbourhood of 0, the rhs of the previous equation is equal to $\pi^2\mu$. The second term in the rhs is however needed since $e^{-2d(\mu)}$ crosses the cut $[1, \infty)$ of the dilogarithm at $\mu = 1/5$, see figure 8. Indeed, inserting $d(1/5) = -\log((1 + \sqrt{5})/2) - i\pi/2$ and using

$$\text{Li}_2\left(\frac{3 + \sqrt{5}}{2}\right) = -\frac{11\pi^2}{15} - \log^2\left(-\frac{1 + \sqrt{5}}{2}\right),$$

one observes that the equation (58) is verified, with $e^{-2d(1/5)} = (3 + \sqrt{5})/2 > 1$. For $\mu = -1/5$, the solution is also explicit: using

$$\text{Li}_2\left(\frac{3 - \sqrt{5}}{2}\right) = \frac{\pi^2}{15} - \log^2\left(\frac{1 + \sqrt{5}}{2}\right),$$

one finds $d(-1/5) = \log((1 + \sqrt{5})/2) - i\pi/2$. Here however, $e^{-2d(-1/5)} = (3 - \sqrt{5})/2 < 1$ does not cross the cut of the dilogarithm.

4.2. The edge of the spectrum, scale $L^0$ (half-filling)

We observe in figure 9 that the edge of the spectrum shows $L$ small peaks. These peaks are a $1/L$ correction to the leading behaviour (54), (55). They are a consequence of the constraint that all the integers $k_j$ are different. This phenomenon does not happen for non-interacting particles, see figure D1.

In order to study this correction, one must go back to the exact expressions (35) and (36). Numerically, one observes that the $k_j$ that contribute to the edge are such that all the $k_j$ are consecutive except at most one of them. We will write $k_j = \mu L + j - (N + 1)/2$ for $j = 1, \ldots, N - 1$ and $k_N = (\mu + \nu)L$. On the scale studied here, $\mu$ can only take the values $1/L, 2/L, \ldots, 1$. The parameter $\nu$ verifies $0 \leq \nu \leq 1 - \rho$. We focus again on the half-filled case $\rho = 1/2$. 
Figure 9. The spectrum of TASEP with $N = 9$ particles on $L = 18$ sites. The small black dots are the eigenvalues $E$ of the Markov matrix divided by $L$. The black curve is the edge of the spectrum in the thermodynamic limit at half-filling. The red (grey in printed version) curve corresponds to the asymptotic expression (12) of the edge of the spectrum near the origin. The two big blue dots correspond to the explicit values $E/L = -\frac{1}{4} \pm \frac{1}{2}i\pi \left(\sqrt{5} - \log \frac{\sqrt{2e}}{2}\right)$. The first order correction to the edge from equation (61) is plotted in grey.

At leading order in $L$, one recovers (54), (55) using again the change of variable (53). Writing $e = e(\mu + 1/(2L)) + \delta e(\mu, \nu)$, one finds at the end of the calculation
\[
\delta e(\mu, \nu) = \frac{\tanh d(\mu)}{2d(\mu) L} \left( \frac{d(\mu) \sqrt{1 - e^{4\pi \nu} \cosh^2 d(\mu)}}{i e^{2\pi \nu} \sinh d(\mu)} - \text{arcsinh}[i e^{2\pi \nu} \cosh d(\mu)] \right),
\]
where $d(\mu)$ is the solution of (55).

From (58) and the relation $\text{Li}_2(-e^z) + \text{Li}_2(-e^{-z}) = -\pi^2/6 - z^2/2$, the function $d(\mu)$ verifies the symmetry relation $d(-\mu) = -d(\mu)$, where $\bar{\cdot}$ denotes complex conjugation. This implies $\delta e(-\mu, 1/2 - \nu) = \delta e(\mu, \nu)$. The latter symmetry is a consequence of the term $+1/(2L)$ in the definition of $\delta e$. The first-order correction (61) is plotted in figure 9 along with the exact spectrum for $L = 18, N = 9$.

5. Density of eigenstates

The total number of eigenstates for TASEP is $|\Omega| \sim e^{\ell L}$, with $\ell$ defined in terms of the density of particles $\rho$ in (31). In the bulk of the spectrum, the number of eigenstates with a rescaled eigenvalue $E/L$ close to a given $e$ is expected to be of the form $e^{s(e)}$. The function $s(e)$ is studied in this section.
5.1. Optimal function $\eta$

The density of eigenstates near the rescaled eigenvalue $e$ can be formally defined by the functional integral

$$D(e) = \int D\eta \frac{1}{\sqrt{\eta \rho \omega}} e^{\eta \rho \omega}.$$  \hfill (62)

We write explicitly the dependence in $\eta$ of $e$, $b$ and $s$ in this section. We want to maximize for $\eta$

$$s[\eta] + \lambda \left( \int_0^1 du \eta(u) - \rho \right) + \text{Re}(2\omega(e[\eta] - e)).$$  \hfill (63)

It gives an optimal function $\eta^*$, which depends on the two Lagrange multipliers $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{C}$. Those must then be set such that the constraints $\int_0^1 du \eta(u) = \rho$ and $e[\eta^*] = e$ are satisfied, and we can finally write (13) with $s(e) = s[\eta^*]$.

For given values of the Lagrange multipliers, writing the variation of $s[\eta]$, $b[\eta]$ and $e[\eta]$ for a small variation $\delta \eta$ of $\eta$ and using (46)–(48), we find

$$\delta s = -\int_0^1 du \delta \eta(u) \log \frac{\eta(u)}{1-\eta(u)},$$  \hfill (64)

$$\delta b = \frac{\int_0^1 du \delta \eta(u) \psi(e^{2\pi i u - b[\eta]})}{1 + \frac{1}{2\pi} \int_0^1 du \eta(u) \partial_u \psi(e^{2\pi i u - b[\eta]})},$$  \hfill (65)

and

$$\delta e = \int_0^1 du \delta \eta(u) \left[ \psi(e^{2\pi i u - b[\eta]}) + a[\eta] \psi(e^{2\pi i u - b[\eta]}) \right].$$  \hfill (66)

We have defined

$$a[\eta] = -\frac{1}{2\pi} \int_0^1 du \eta(u) \partial_u \psi(e^{2\pi i u - b[\eta]})$$  \hfill (67)

It implies that the optimal function $\eta^*$ verifies

$$-\log \frac{\eta^*(u)}{1-\eta^*(u)} + \lambda + \text{Re}[2\omega \psi(e^{2\pi i u - b[\eta^*]}) + 2a[\eta^*] \omega \psi(e^{2\pi i u - b[\eta^*]})] = 0.$$  \hfill (68)

The optimal function is then equal to

$$\eta^*(u) = \left(1 + e^{-\lambda - 2\text{Re}[\omega \psi(e^{2\pi i u - b[\eta^*]}) + 2a[\eta^*] \omega \psi(e^{2\pi i u - b[\eta^*]})]} \right)^{-1}.$$  \hfill (69)

This is a real function that satisfies $0 < \eta^*(u) < 1$ for all $u$. The expression (69) is reminiscent of a Fermi–Dirac distribution. On the other hand, the corresponding expression for undistinguishable non-interacting particles, (D.5), resembles a Bose–Einstein distributions: allowing some momenta to be equal gives a term $-1$ in the denominator of (D.5), while forbidding equal momenta gives the term $+1$ in the denominator of (69).

5.2. Contour integrals

For given values of the Lagrange multipliers $\lambda$ and $\omega$, the expression (69) is completely explicit except for the two unknown complex quantities $a[\eta^*]$ and $b[\eta^*]$, that must be determined self-consistently from (67) and (48). It is possible to simplify the problem a little by noticing that
we can replace \( a[r^+] \) and \( b[r^+] \) by two real quantities \( \alpha \) and \( \beta \). Indeed, the definitions (67) and (69) imply

\[
\text{Im}(a[r^+]) = \text{Im} \left[ -\frac{\omega}{2i\pi} \int_0^1 du \, \eta^*(u) \partial_u (\psi(e^{2i\pi u-b}) + a\psi(e^{2i\pi u-b})) \right] \\
= \frac{1}{4\pi} \int_0^1 du \, \frac{(\eta^*)(u)}{1 - \eta^*(u)} = 0.
\]

(70)

We define \( \alpha = a[r^+] \in \mathbb{R} \). The imaginary part of \( b[r^+] \) can be eliminated by a shift of \( u \) and a redefinition of \( \eta \): we introduce \( \sigma \) such that

\[
\sigma(e^{2i\pi u}) = \eta^*(u + \frac{\text{Im}(b)}{2\pi}).
\]

(71)

Defining \( \beta = \text{Re}(b[r^+]) \), one has

\[
\sigma(z) = (1 + e^{-\lambda - 2 \text{Re} \omega (e^{-\beta} e^{2i\pi z})})^{-1}.
\]

(72)

It is not possible to eliminate the quantity \( \beta \) by changing the contour of integration, since e.g. \( \text{Re}(\psi(e^{-\beta} e^{z})) \) is not an analytic function of \( z \). It can also be seen by writing

\[
2 \text{Re}(\psi(e^{-\beta} e^{z})) = \psi(e^{-\beta} e^{z}) + \psi(e^{-\beta} e^{-z})
\]

for \( \beta = 1 \).

The quantities \( s = s[r^+] \), \( \alpha \), \( \beta \) can be rewritten in terms of \( \sigma \) as

\[
s = -\frac{1}{2i\pi} \oint \frac{dz}{z} \sigma(z) \log \sigma(z) + (1 - \sigma(z)) \log(1 - \sigma(z)) \tag{73}
\]

\[
\beta = \text{Re} \gamma + \text{Re} \left[ \frac{1}{2i\pi} \oint \frac{dz}{z} \sigma(z) \psi(e^{-\beta} e^{z}) \right] \tag{74}
\]

and

\[
\alpha = -\frac{1}{2i\pi} \oint dz \sigma(z) \partial_z (\omega \psi(e^{-\beta} e^{z}) + \alpha \psi(e^{-\beta} e^{z})) \tag{75}
\]

All the contour integrals are over the circle of radius 1 and centre 0 in the complex plane. Similarly, the constraints for \( \rho \) and \( e[r^+] \) give

\[
\rho = \frac{1}{2i\pi} \oint \frac{dz}{z} \sigma(z) \tag{76}
\]

and

\[
e = \frac{1}{2i\pi} \oint \frac{dz}{z} \sigma(z) \psi(e^{-\beta} e^{z}) \tag{77}
\]

For given \( \rho \) and \( e \), one has to solve (76) and (77) in order to obtain \( \lambda \) and \( \omega \) in terms of them. Like for (75) and (74), it does not seem that these equations can be solved analytically in general. At half-filling, however, it is possible to show that

\[
\lambda = \text{Re} \omega \tag{78}
\]

leaving only the equations for \( \alpha \), \( \beta \) and \( \omega \) to be solved numerically. Indeed, for \( \rho = 1/2 \), one has the identities \( \psi(-z) = -1 - \psi(z) \) and \( \psi(-z) = -\psi(z) \). Setting \( \lambda = \text{Re} \omega \) in (72) then implies

\[
\sigma(z) + \sigma(-z) = 1,
\]

from which (76) follows at half-filling.

We note that if \( \omega \in \mathbb{R} \) then \( e \in \mathbb{R} \). This is a consequence of (77), \( \sigma(-z) = \sigma(z) \) and \( \text{Im}(\psi(-z)) = -\text{Im}(\psi(z)) \). By the definition (63) of the Lagrange multiplier \( \omega \), it implies that \( s \) is also the density of eigenvalues with a given real part when \( \omega \) is real.

The maximum of \( s \) is located at \( e = -\rho(1-\rho), s = \ell \). It corresponds to \( \omega = 0, \sigma(z) = \rho, \lambda = \log(\rho/(1-\rho)), \alpha = 0, \beta = \text{Re} \gamma \). Unlike free particles, the spectrum is not symmetric with respect to the maximum of \( s \).
In figure 10, $e$ is plotted for various values of $\omega$ at half-filling, along with the optimal function $\sigma(e^{2i\pi u})$. In figure 11, $s$ is plotted as a function of $e \in \mathbb{R}$ and compared with the density of real part of eigenvalues obtained from numerical diagonalization of the Markov matrix $M$ for $N = 9, L = 18$. The agreement is not very good for eigenvalues close to the edges. This is caused by the ‘arches’ at distance $\sim 1/L$ of the edge of the spectrum, which still contribute much for $N = 9, L = 18$, see figure 9.
5.3. Density of eigenvalues close to the origin ($\rho = 1/2$)

We consider the limit $\epsilon \to 0$ of $s(\epsilon)$ at half-filling for the Markov matrix (deformation $\gamma = 0$). It corresponds to $|\omega| \to \infty$ with $\arg \omega \to 0$.

In the limit $|\omega| \to \infty$, the optimal function $\sigma(e^{2i\pi u})$ approaches 1 if $u_1 < u < u_2$ and 0 if $u_2 < u < u_1 + 1$. The relation $\sigma(-z) = 1 - \sigma(z)$ at half-filling and the normalization condition (76) imply that $u_2 = u_1 + 1/2$. Furthermore, if we consider a scaling such that $\arg \omega \to 0$ when $|\omega| \to \infty$, one has $u_1 = -1/4$ and $u_2 = 1/4$, which is the same as what we had in section 4 for the edge of the spectrum near the eigenvalue 0.

In appendix B, we compute explicitly the large $|\omega|$ limit of the integrals (73), (74), (75) and (77) with $\lambda$ given by (78). We find two different regimes, depending on the respective scaling between the real and imaginary part of $\epsilon$.

The first regime, $\epsilon \to 0$ with $|\text{Im} e^{5/3} / \text{Re} \epsilon| \to 0$, corresponds to the central part of the spectrum, far from the edge. We find $\beta \simeq -\log 2 + \delta \beta |\omega|^{-2/3}$, where $\delta \beta$ is the solution of

$$
\frac{1}{6\pi} = \int_0^\infty \frac{\text{Im}[(1 + 2i\pi y)^{1/2}]}{1 + e^{\frac{\text{Im}[y]}{3\pi} \log(1 + (2i\pi y)^{3/2})}}.
$$

The real part of the eigenvalue $\epsilon$ and the 'entropy' $s$ are equal to

$$
\text{Re} \epsilon = \frac{4\sqrt{2} \delta \beta^{5/2}}{3|\omega|^{5/3}} \left[ \frac{1}{10\pi} - \int_0^\infty \frac{\text{Im}[(1 + 2i\pi x)^{3/2}]}{1 + e^{\frac{\text{Im}[x]}{3\pi} \log(1 + (2i\pi x)^{3/2})}} \right]
$$

and

$$
s = -5 |\omega| \text{Re} \epsilon.
$$

Solving (79) numerically, one finds $\delta \beta \simeq 0.706532$, which implies $\text{Re} \epsilon \simeq -0.147533 |\omega|^{-5/3}$, $s \simeq 0.737667 |\omega|^{-2/3}$ and (14).

The second regime, $\epsilon \to 0$ with $\text{Re} \epsilon$ and $|\text{Im} e^{5/3} / (\text{Re} \epsilon)|$ related by (12), corresponds to the edge of the spectrum. In this regime, one finds

$$
s(\epsilon) \simeq \frac{2^{4/5} \pi^{2/5}}{3^{3/5} 10^{1/10}} (-\text{Re} \epsilon)^{3/5} \sqrt{1 + \frac{2^{2/3} 3^{7/6} \pi^{2/3}}{10} \frac{|\text{Im} e^{5/3}|}{\text{Re} \epsilon}}.
$$

The crossover between the two regimes corresponds to $|\text{Im} e^{5/3} / (-\text{Re} \epsilon)|$ converging to a constant different from the $2^{2/3} 3^{7/6} \pi^{2/3} / 10$ characteristic of the edge. Explicit expressions are given in appendix B.

6. Trace of the time evolution operator

In this section, we study the quantity $f(t)$ defined in equation (16). This is another application of the formulas (47), (48) derived in section 3 for the eigenvalues.

6.1. Optimal function $\eta$

As in section 5 for the density of eigenvalues, one can write the summation over all eigenvalues as

$$
\text{tr} e^{M} = \sum_{[k_1, \ldots, k_N]} e^{E(k_1, \ldots, k_N)} \simeq \int D\eta \prod_0^1 du \delta(\eta(u) = \epsilon) e^{L(\eta[\epsilon] + t \epsilon[\eta])}.
$$

If the functional integral is dominated by the contribution of an optimal function $\eta^*$, one finds from (16) $f(t) = s[\eta^*] + t \epsilon[\eta^*]$. The function $\eta^*$ generally depends on $t$. The normalization (46)
of $\eta$ is enforced by the Lagrange multiplier
\begin{equation}
\lambda \left( -\rho + \int_0^1 \mathrm{d}u \eta(u) \right). \tag{84}
\end{equation}

The change in $s$, $b$ and $e$ from a variation $\delta \eta$ of $\eta$ is still given by (64), (65) and (66). The optimal function then verifies
\begin{equation}
-\log \eta^*(u) + \lambda + t (\phi(e^{2\pi u - \ln(\eta^*)}) + a[\eta^*] \psi(e^{2\pi u - \ln(\eta^*)})), \tag{85}
\end{equation}
with $a[\eta]$ still given by (67). We obtain
\begin{equation}
\eta^*(u) = (1 + e^{-\lambda - t (\phi(e^{2\pi u - \ln(\eta^*)}) + a[\eta^*] \psi(e^{2\pi u - \ln(\eta^*)}))})^{-1}. \tag{86}
\end{equation}

**Remark.** Unlike section 5, the optimal function $\eta^*(u)$ is not a real function. This means that the saddle point of the functional integral (83) lies in the complex plane. The function $\eta^*(\eta)$ can be recovered from the function $\eta^*$ of (69) by the choice $\omega = t$ and $\overline{\omega} = 0$, where $\overline{\cdot}$ denotes complex conjugation, and $\omega$ and $\overline{\omega}$ must be thought of as independent variables.

### 6.2. Contour integrals

The optimal function $\eta^*(u)$ is an analytic function of $z = e^{2\pi u - \ln(\eta^*)}$, one defines
\begin{equation}
\sigma(z) = \eta^*(u) = (1 + e^{-\lambda - t (\phi(z) + a[\eta^*] \psi(z))})^{-1}. \tag{87}
\end{equation}

Under the assumption that the contours of integration can be freely deformed from $|z| = e^{-\Re \ln(\eta^*)}$ to something independent of $b[\eta^*]$, we recover the equations (73) and (76) for $s \equiv s[^*]$ and $\rho$, but with $\sigma$ now given by (87) instead of (72). The equations for the quantities $e \equiv e[^*]$ and $a = a[^*]$ become
\begin{equation}
e = \frac{1}{2i\pi} \oint \frac{dz}{z} \sigma(z) \phi(z) \tag{88}
\end{equation}
and
\begin{equation}
a = -\oint \frac{dz}{2i\pi} \sigma(z) \partial_z (\phi(z) + a \psi(z)). \tag{89}
\end{equation}

Equation (87) implies
\begin{equation}
\phi(z) + a \psi(z) = -\frac{\lambda}{t} - \frac{1}{t} \log \frac{1 - \sigma(z)}{\sigma(z)}. \tag{90}
\end{equation}

Combining this with (89) gives
\begin{equation}
a = \oint \frac{dz}{2i\pi t} \partial_z \log(1 - \sigma(z)) = \frac{w}{t}, \tag{91}
\end{equation}
where $w \in \mathbb{Z}$ is the winding number of $1 - \sigma(z)$ around the origin. We assume in the following that $w = 0$, hence $a = 0$ and
\begin{equation}
\sigma(z) = (1 + e^{-\lambda - t \phi(z)})^{-1}. \tag{92}
\end{equation}

The property $a = 0$ is compatible, at least for small times, with numerical solutions of (76) with $a = 0$, see figure 12.

Unlike section 5, the expression (92) for $\sigma(z)$ allows us to calculate explicitly the Lagrange multiplier $\lambda$, the eigenvalue $e$, the ‘entropy’ $s$ and the function $f(t) = s + t e$. Details are given in appendix C. In the end, we recover (23).
6.3. Singularities of $\sigma(z)$

In the previous subsection, we used a deformation of the contour of integration in order to eliminate completely the parameter $b$. We implicitly assumed that this was possible without crossing singularities of $\sigma(z)$. We come back to this issue here, with the expression (92) for $\sigma(z)$ resulting from the assumption $a = 0$.

We focus on the half-filled case. From (44) and (C.7), the singularities of the function $\sigma$ consist in an essential singularity at $z = 0$, two cuts starting at $\pm 2i$, and an infinity of poles located at $\pm z_k, k \in \mathbb{N}$, with

$$z_k = \frac{2\mu}{\sqrt{t^2 + 4(2k + 1)^2\pi^2}}. \quad (93)$$

For $t = 0$, we observe that all the poles are located at the origin. For times $t < t(\gamma)$, where $t(\gamma)$ is the first non-analyticity of $f$ discussed at the end of section 2.5, the contour of integration in (73) and (88) then has to enclose all the poles, but must not cross the cuts. The circle of centre 0 and radius

$$r_t = 1 + t/\sqrt{t^2 + 4\pi^2} \quad (94)$$

is a possible contour.

We observed in section 2.5 that the function $f(t)$ is defined piecewise. The non-analyticity of $f(t)$ should be the sign of the presence of several saddle points competing in the functional integral (83). This is similar to what happens in the direct calculations of $f(t)$ for free particles in appendix D, with a simple integral instead of a functional integral. It is not completely clear, however, how several saddle points emerge from the functional integral (83) for TASEP. It might be due to a change in the contour of integration at $t = t(\gamma)$, with a new contour that does not enclose all the poles of $\sigma$. There could also be a transition in equation (89) from the solution $a = 0$ to another value due to a change in the winding number of $1 - \sigma(z)$ around 0.
7. Conclusion

Parametric expressions can be derived for all the eigenvalues of TASEP using the Bethe ansatz. These expressions allow a study of the large scale properties of the spectrum in the thermodynamic limit, in particular the curve marking the edge of the spectrum, the density of eigenvalues in the bulk of the spectrum and the generating function of the cumulants of the eigenvalues.

A natural extension of the present work would be to analyse the structure of the eigenvalues closer to the origin. Of particular interest are eigenvalues with a real part scaling as $L^{-3/2}$, which control the relaxation to the stationary state. Another goal would be to obtain asymptotic expressions for the scalar product between an eigenstate characterized by a density $\eta$ of the $k_j$ as in sections 5 and 6 and a microstate characterized by a density profile of particles. This would allow us to study physical quantities more interesting than the trace of the time evolution operator.

Another very interesting extension would be the case of the ASEP with partial asymmetry, where particles are allowed to hop in both directions, with an asymmetry parameter controlling the bias. It would be nice if it were possible to derive parametric expressions for the eigenvalues analogous to (35) and (36). A good starting point seems to be the quantum Wronskian formulation of the Bethe equations [26, 42], where a parameter analogous to the parameter $B$ we used here exists. It would allow us to study how the spectrum changes at the transitions between equilibrium and non-equilibrium.

Finally, it would be nice to understand how the approach used here to study the spectrum of TASEP relates to the thermodynamic Bethe ansatz [43]. The latter follows from the observation that, in the thermodynamic limit, Bethe roots accumulate on a curve in the complex plane. The density of the Bethe roots along this curve can usually be shown to be the solution of a nonlinear integral equation. We would like to understand whether the absence of nonlinear integral equations in our calculations is only due to the special decoupling (28) of the Bethe equations for TASEP.

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Appendix A. Bijection $g$

The function $g$ is defined in (32) on the whole complex plane minus the negative real axis ($-\infty, 0$], if one chooses the usual cut of the logarithm. It is convenient to use polar coordinates, writing $y = re^{i\theta}$ with $r > 0$ and $-\pi < \theta < \pi$. Then $g(y)$ is divergent in the limit of small and large values of $r$. One has

$$g(y) \simeq r^{-\rho} e^{-i\rho \theta} \quad \text{for} \quad r \to 0 \quad \text{(A.1)}$$

$$g(y) \simeq -r^{1-\rho} e^{i(1-\rho)\theta} \quad \text{for} \quad r \to \infty. \quad \text{(A.2)}$$

This implies $\arg g(y) \in (-\rho \pi, \rho \pi)$ for small $r$ and $\arg g(y) \in (-\pi, -\rho \pi) \cup (\rho \pi, \pi]$ for large $r$. The image of a point on the cut of $g$ is

$$g(-r \pm i0^\pm) = e^{\pm \pi \rho} \frac{1 + r}{r^\rho}. \quad \text{(A.3)}$$
**Figure A1.** Deformation of a grid by the function \( g \) (32), with \( \rho = 1/3 \). Left: a grid in polar coordinates, with angles \( \theta \) regularly spaced of \( 2\pi/10 \) and radii 0.0005, 0.002, 0.005, 0.02, 0.05, 0.2, 0.5, 1.0, 20, 40 coloured from blue for small radius to red for large radius. The thick red line corresponds to the cut of the function \( g \). Right: the image by the function \( g \) of the previous grid is drawn, using the same colours for a curve in the initial grid and its image by \( g \). The almost semi-circular curves on the left part of the complex plane are red, the ones on the right are blue. The two thick red lines correspond to the image of the cut by \( g \).

**Figure A2.** Deformation of a grid by the inverse function \( g^{-1} \) (32), with \( \rho = 1/3 \). Left: a grid in polar coordinates, with angles \( \theta \) regularly spaced of \( 2\pi/15 \) and radii 0.1 \( e^\ell \), 0.25 \( e^\ell \), 0.5 \( e^\ell \), 0.75 \( e^\ell \), \( 1e^\ell \), 1.1 \( e^\ell \), 1.5 \( e^\ell \) coloured from blue for small radius to red for large radius, with \( \ell \) defined in (31). The two thick red lines correspond to the cuts of the function \( g^{-1} \). Right: the image by the function \( g^{-1} \) of the previous grid, using the same colours for a curve in the initial grid and its image by \( g^{-1} \). The almost circular shapes of large radius and small radius around 0 are red. The small almost circular shapes around 1 are blue.

The derivative of \( g(y) \) with respect to \( \theta \) at this point verifies

\[
\frac{\partial g}{g} (-r \mp i 0^\pm) = \frac{i(1 - \rho)}{r + 1} \left[ r - \frac{\rho}{1 - \rho} \right].
\]
The case $\chi<\epsilon$ decomposed as a sum of 4 terms. Up to terms exponentially small in
where the functions $g$ and its inverse $g^{-1}$ are given in section 5.3 and appendix B.1.

A summary of the different scalings obtained is given in table B1. The scalings $c<0$ then follows from the invariance of the spectrum by complex conjugation.

This implies that the curves $\{ g(re^{i\theta}), \theta \in (-\pi, \pi) \}$ join the cuts in the image space orthogonally (which already follows from the local holomorphicity of $g$, as the image of the orthogonality of any circle of centre 0 with the negative real axis), from one side or the other depending on whether $c$ is smaller or larger than $\rho/(1-\rho)$.

When $y$ spans $(-\infty, 0] \pm 0i$, the image of the cut spans $e^{i\pi\rho}[e^\ell, \infty) \cup e^{-i\pi\rho}[e^\ell, \infty)$, with $\ell$ defined in equation (31). The bijective nature of $g$ is clearly seen in figures A1 and A2 where the functions $g$ and its inverse $g^{-1}$ are represented.

### Appendix B. Density of eigenvalues close to the origin

In this appendix, we perform the calculations related to the limit $e \to 0$ of the density of eigenvalues of the Markov matrix at half-filling. We compute the limit $|\omega| \to \infty$, arg $\omega \to 0$ of the integrals for the quantities $\alpha$ (75), $\beta$ (74), $e$ (77) and $s$ (73). The integrals must be decomposed as a sum of 4 terms. Up to terms exponentially small in $e^{-1} \gg 1$, one has

$$\int_0^1 du F[\sigma(e^{2i\pi u}), e^{2i\pi u}] \simeq \int_{-\frac{1}{2}}^{\frac{1}{2}-\epsilon} du F[1, e^{2i\pi u}] + \int_{\frac{1}{2}}^{\frac{1}{2}+\epsilon} du F[0, e^{2i\pi u}]$$

After a little rewriting, one obtains

$$\int_0^1 du F[\sigma(e^{2i\pi u}), e^{2i\pi u}] \simeq \int_{-\frac{1}{2}}^{\frac{1}{2}} du F[1, e^{2i\pi u}] + \int_{\frac{1}{2}}^{\frac{1}{2}} du F[0, e^{2i\pi u}]$$

For the four quantities $\alpha, \beta, e$ and $s$, the terms with $F[0, e^{2i\pi u}]$ vanishes, as well as the terms with $F[1, e^{2i\pi u}]$ for $s$.

In order to continue the calculations, several scalings need to be considered for $\text{Im } \omega$ when $|\omega| \to \infty$. We write $\omega = r + i\chi r'$ with $r > 0$, $\chi > 0$, $c < 1$, and will take the limit $r \to \infty$. The case $\chi < 0$ then follows from the invariance of the spectrum by complex conjugation. A summary of the different scalings obtained is given in table B1. The scalings $c = -2$, $c = \frac{1}{3}$, $c = 0$ are given in section 5.3 and appendix B.1.
−1, −1/2, 0, 1/6, 1/5, 1/4, 1/3, 1/2, 2/3, 3/4, 4/5 were checked by solving numerically (75), (74) for \( r = 100, 200, \ldots, 1000 \). We used the BST algorithm [44] in order to improve the convergence to \( r \to \infty \). For all the scalings studied, the relative errors in the numerical coefficients of \( \alpha, \beta, \text{Re} \epsilon, \text{Im} \epsilon \) and \( s \) obtained from the BST algorithm were smaller than \( 10^{-3} \) compared to the exact values.

B.1. Scaling \( c = 0 \)

Writing \( \alpha = r/4 + \delta \alpha/r^{1/3}, \beta = -\log 2 + \delta \beta/r^{2/3} \) and expanding the equations (74) and (75) respectively up to order \( r^{-1} \) and \( r^{-2/3} \) with \( \epsilon \sim r^{-2/3} \) in (B.2) give after straightforward, but rather tedious calculations (79) and

\[
\delta \alpha \left[ \frac{1}{2\pi} - \int_0^\infty dx \frac{\text{Im}[(1 + 2i\pi x)^{-1/2}]}{1 + e^{r(2\pi x)^{3/2} \text{Im}(1 + 2i\pi x)^{1/2})}} \right] = \frac{1}{3\pi} \int_0^\infty dx \frac{\text{Im}[(1 + 2i\pi x)^{1/2}]}{1 + e^{r(2\pi x)^{3/2} \text{Im}(1 + 2i\pi x)^{1/2})}}
\]

\[
+ \frac{\sqrt{2\delta \beta}}{4} \int_0^\infty dx \frac{\text{Re}[(1 + 2i\pi x)^{1/2}] \text{Re}[(1 + 2i\pi x)^{-1/2}] e^{r(2\pi x)^{3/2} \text{Im}(1 + 2i\pi x)^{1/2})}}{(1 + e^{r(2\pi x)^{3/2} \text{Im}(1 + 2i\pi x)^{1/2})})^2}.
\]

We used the expansion

\[
\int_{-1/4}^{1/4} du \psi(2 \xi e r^{-u}) = -\log 2 + e^{-4\sqrt{2}/3\pi} \xi^{3/2} + \frac{2\sqrt{2}}{15\pi} \xi^{5/2} + O(\xi^{7/2}).
\]

We observe that the equations (79) and (B.3) for \( \delta \beta \) and \( \delta \alpha \) decouple in the scaling \( c = 0 \), unlike the original equations (74) and (75) for \( \beta \) and \( \alpha \).

Expanding the equation (77) for \( \epsilon \), we find at leading order in \( r \) (80) and

\[
\text{Im} \epsilon = -\frac{4\delta \beta^2}{r^{1/3}} \int_0^\infty dx \frac{[\text{Re}[(1 + 2i\pi x)^{1/2}] e^{r(2\pi x)^{3/2} \text{Im}(1 + 2i\pi x)^{1/2})}]}{(1 + e^{r(2\pi x)^{3/2} \text{Im}(1 + 2i\pi x)^{1/2})})^2},
\]

while the equation (73) for \( s \) gives (81). Numerically, one has \( \delta \alpha \simeq -0.164 \, 473 + 0.107 \, 018 \, \chi^2 \) and \( \text{Im} \epsilon \simeq -0.224 \, 251 \, \chi \, r^{-4/3} \).

B.2. Scaling \( c = 1/3 \)

Writing \( \alpha = r/4 + \delta \alpha r^{1/3}, \beta = -\log 2 + \delta \beta r^{-2/3} \) and expanding the equations for \( \beta \) (74) and \( \alpha \) (75) respectively up to order \( r^{-1} \) and \( r^{-2/3} \) in (B.2) give after again long but straightforward calculations

\[
\frac{1}{3\pi} = \int_0^\infty dx \left[ \frac{\text{Im}[(1 + 2i\pi x)^{1/2}]}{1 + e^{r(2\pi x)^{3/2} \text{Im}(1 + 2i\pi x)^{1/2})}} + \frac{\text{Im}[(1 + 2i\pi x)^{1/2}]}{1 + e^{r(2\pi x)^{3/2} \text{Im}(1 + 2i\pi x)^{1/2})}} \right]
\]

and

\[
\delta \alpha \left( \frac{1}{2\pi} - \int_0^\infty dx \left[ \frac{\text{Im}[(1 + 2i\pi x)^{-1/2}]}{1 + e^{r(2\pi x)^{3/2} \text{Im}(1 + 2i\pi x)^{1/2})}} + \frac{\text{Im}[(1 + 2i\pi x)^{-1/2}]}{1 + e^{r(2\pi x)^{3/2} \text{Im}(1 + 2i\pi x)^{1/2})}} \right] \right) = \frac{-\chi}{4} \int_0^\infty dx \left[ \frac{\text{Re}[(1 + 2i\pi x)^{-1/2}]}{1 + e^{r(2\pi x)^{3/2} \text{Im}(1 + 2i\pi x)^{1/2})}} - \frac{\text{Re}[(1 + 2i\pi x)^{-1/2}]}{1 + e^{r(2\pi x)^{3/2} \text{Im}(1 + 2i\pi x)^{1/2})}} \right],
\]

with the definition

\[
\Phi_{\pm}(x) = \frac{(2\delta \beta)^{3/2}}{3} \text{Im}[(1 + 2i\pi x)^{3/2})] - \frac{4\sqrt{2}\delta \alpha \text{Im}[(1 + 2i\pi x)^{1/2})]}{\pm \sqrt{2}\delta \beta \chi \text{Re}[(1 + 2i\pi x)^{1/2})].
\]

\[\text{(B.8)}\]
We observe that the equations for $\delta \alpha$ and $\delta \beta$ are now coupled, unlike in the scaling $c = 0$. Similar calculations with the equations for $e$ (77) and $s$ (73) give at leading order in $r$
\[ \text{Re } e = \frac{2\sqrt{2} \delta \beta^{3/2}}{3r^{5/3}} \left( \frac{1}{5\pi} - \int_0^{\infty} dx \left[ \frac{\text{Im}[(1 + 2i\pi x)^{3/2}]}{1 + e^{\Phi_+(x)}} + \frac{\text{Im}[(1 + 2i\pi x)^{3/2}]}{1 + e^{\Phi_-(x)}} \right] \right), \]  
\[ \text{Im } e = \frac{\sqrt{2} \delta \beta^{3/2}}{r} \int_0^{\infty} dx \left[ \frac{\text{Re}(\delta \beta + 2i\pi x)^{1/2}}{1 + e^{\Phi_+(x)}} - \frac{\text{Re}(1 + 2i\pi x)^{1/2}}{1 + e^{\Phi_-(x)}} \right], \]  
and
\[ s = -5r \text{Re } e + 3\chi r^{1/3} \text{Im } e. \]  
In the limit $\chi \to 0$, we see that $\delta \beta$, $\text{Re } e$ and $s$ converge to their value in the scaling $c = 0$, while $\delta \alpha \simeq 0.107018 \chi^2$ and $\text{Im } e \simeq -0.224251 \chi / r$, with the same numerical constants as in the scaling $c = 0$.

The limit $\chi \to \infty$ is a bit more complicated. We check that $\delta \alpha \simeq \sqrt{3} \chi / 12$ and $\delta \beta \simeq \sqrt{3} \chi / 2$ in this limit. Indeed, with these values for $\delta \alpha$ and $\delta \beta$, we observe that $\Phi_+(\chi u) > 0$ for all $u > 0$, while $\Phi_-(\chi u) < 0$ when $u < 3/(4\pi)$ and $\Phi_-(\chi u) > 0$ otherwise. In the integrals of (B.6) and (B.7), the first term of the integrands gives a contribution exponentially small in $\chi$ to the integral, while the second term of the integrands contributes only for $v/\chi < 3/(4\pi)$, with $e^{\Phi_-(v)} \to 0$. Performing explicitly the integrals then shows that (B.6) and (B.7) are indeed verified. We also checked numerically that the solutions of (B.6) and (B.7) for finite $\chi$ seem to grow as $\delta \alpha \simeq \sqrt{3} \chi / 12$ and $\delta \beta \simeq \sqrt{3} \chi / 2$ for large $\chi$.

Similar calculations lead to $\text{Re } e \simeq -2^{1/2} 3^{3/4} \chi^{5/2} / (5\pi r^{5/3})$ and $\text{Im } e \simeq -2^{1/2} \chi^{3/2} / (3^{1/4} r \chi)$ in the limit $\chi \to \infty$. Using (B.11), it implies $s = 0$ on the scale $\chi^{5/2}$. Going beyond that requires some more work: one has to take into account the contribution of the integrals near $v = 3\chi / (4\pi)$, making the change of variables $v = 3\chi / (4\pi) + x^{-1} u$. Writing $\delta \alpha = \sqrt{3} \chi / 12 + \chi^{-2} \delta \alpha_2$ and $\delta \beta = \sqrt{3} \chi / 2 + \chi^{-2} \delta \beta_2$, one finds at the end of the calculation $\delta \alpha_2 \simeq -1/2^{144}$, $\delta \beta_2 \to 0$, $\text{Re } e \simeq -2^{1/2} 3^{3/4} \chi^{5/2} / (5\pi r^{5/3}) - \pi / (2^{1/2} 3^{3/4} \chi^{1/2} r^{5/3})$ and $\text{Im } e \simeq -2^{1/2} \chi^{3/2} / (3^{1/4} r \chi) + \pi / (2^{1/2} 3^{3/4} \chi^{1/2} r^{5/3})$. It finally leads to $s \sim \pi / (2^{1/2} 3^{3/4} \chi^{1/2} r^{5/3})$.

**B.3. SCALINGS $c < 0$ AND $0 < c < 1/3$**

The crossover scaling $c = 0$ is surrounded by the two regimes $c < 0$ and $0 < c < 1/3$. Similar calculations to the ones performed for the scalings $c = 0$ allow to compute the quantities $\alpha$, $\beta$, $\text{Re } e$, $\text{Im } e$ and $s$ at leading order in $r$.

We observe that the results found in the regime $c < 0$ are identical to the limit $\chi \to 0$ in the scaling $c = 0$. Similarly, the results found in the regime $0 < c < 1/3$ are identical to the limit $\chi \to \infty$ in the scaling $c = 0$.

Since the regimes $c < 0$ and $0 < c < 1/3$ differ only by the value of the parameter $\delta \alpha$, which is just an intermediate quantity needed for the calculations, one can for all purpose consider this as a unique regime, $c < 1/3$: the quantities of interest $e$ and $s$ then depend in a simple way on $r$, $\chi$ and $c$ in the whole regime.

**B.4. SCALING $1/3 < c < 1$**

The regime $1/3 < c < 1$ is much more complicated. There, one is lead to make a change of variable of the form $u = \pm 1/4 + d_1 r^{1-c} + d_2 r^{2(1-c)} + \cdots + d_m r^{m(1-c)} + v / r^{(c+1)/2}$ in the integrals, where the constants $d_j$ must be such that the argument of the exponential in $\sigma(u)$ is of order $r^0$. Using a similar rewriting to (B.2), but with $\pm 1/4$ replaced by
\[ \pm 1/4 + d_1/r^{1-c} + d_2/r^{2(1-c)} + \ldots + d_m/r^{m(1-c)}, \] the same kind of calculations as in the other scalings can in principle be done.

We checked only the specific case \( c = 2/3 \). There, making the change of variables \( u = \pm 1/4 + d_1/r^{1/3} + d_2/r^{2/3} + v/r^{5/6} \) in the integrals, we find that \( d_1 \) must be solution of the equation
\[ (3\chi - 4\pi d_1)\text{Re}[\delta \beta + 2i\pi d_1]^{1/2} = 2(\delta \beta - 6\delta \alpha) \text{Im}[\delta \beta + 2i\pi d_1]^{1/2}. \] (B.12)
while \( d_2 \) has a rather complicated (but completely explicit) expression in terms of \( \chi, \delta \alpha, \delta \beta \) and \( d_1 \). The equation for \( \beta \) at order \( r^{-1/2} \) then gives
\[ \text{Re}[\delta \beta + 2i\pi d_1]^{3/2} = 0, \] (B.13)
while the equation for \( \alpha \) at order \( r^{1/2} \) leads to
\[ 3\chi - 4\pi d_1 = 2i(\delta \beta - 6\delta \alpha). \] (B.14)

Gathering the last three equations, one finds \( d_1 = 3\chi/(4\pi) \), \( \delta \alpha = \sqrt{3}\chi/12 \) and \( \delta \beta = \sqrt{3}\chi/2 \).

The equation for \( d_2 \) then gives \( d_2 = -7\chi^2/(10\sqrt{3}\pi) \). From the equation for \( e \), one obtains
\[ \text{Re}\ e = -2^{1/2}3^{3/2}\chi^{5/2}/(5\pi r^{5/6}) \] and
\[ \text{Im}\ e = -2^{1/2}3^{3/2}(3^{1/4}\pi r^{1/2}), \] while the equation for \( s \) leads to \( s = \pi/(2^{1/2}3^{3/4}\chi^{1/2}r^{3/6}) \).

We observe that the numerical constants are the same as in the limit \( \chi \to \infty \) of the scaling \( c = 1/3 \). We conjecture that this is the case for all the scaling \( 1/3 < c < 1 \).

**Appendix C. Explicit calculations for the generating function \( f(t) \)**

In this appendix, we calculate explicitly the contour integrals in (76), (88) and (73), with \( \sigma(\omega) \) given by (92). In order to do this, we first prove two useful formulas, (C.1) and (C.3) for the exponential of the functions \( \psi \) (42) and \( \varphi \) (43).

**C.1. Formula for \( e^{i\psi(z)} \)**

From (34), one has \( e^{i\psi(z)} = (g^{-1}(z))^4 \). Using (41) with \( h(y) = y^4 \), the expansion near \( z = 0 \) leads to
\[ e^{i\psi(z)} = z \sum_{r=0}^{\infty} \binom{\rho r + x}{r} (\frac{-1)^r z^r}{\rho r + x}. \] (C.1)

**C.2. Formula for \( e^{-i\varphi(z)} \)**

Expanding the exponential in \( e^{-i\varphi(z)} \) and using (40), one finds
\[ e^{-i\varphi(z)} = 1 + \sum_{k=1}^{\infty} (-z)^{-k} \frac{x_k}{k!} e^{k(1-\rho)\psi(z)}. \] (C.2)

Equation (C.1) then leads to
\[ e^{-i\varphi(z)} = 1 + (1 - \rho) \sum_{k=1}^{\infty} \frac{x_k}{(k - 1)!} \sum_{r=0}^{\infty} \binom{\rho r + (1 - \rho) k}{r} \frac{(-1)^r z^r}{\rho r + (1 - \rho) k}. \] (C.3)

In particular, one has
\[ \int_{2\pi} e^{-i\varphi(z)} \frac{dz}{2\pi} = \rho + (1 - \rho) e^{i}. \] (C.4)
C.3. Exact expression for the Lagrange multiplier $\lambda$

The Lagrange multiplier $\lambda$ is fixed by the normalization condition (76) with $\sigma$ given by (92). One has

$$\rho = \sum_{r=0}^{\infty} (-1)^r (e^{-\lambda})^r \oint \frac{dz}{2i\pi} \frac{e^{-r\psi(z)}}{z}.$$  \hfill (C.5)

Using (C.4) with $x = rt$ implies

$$\rho = \rho_1 + e^{-\lambda} + \frac{1 - \rho}{1 + e^{-\lambda}}.$$ \hfill (C.6)

Solving for $\lambda$ finally gives

$$\lambda = \log \frac{2e^t}{1 - 2\rho + \sqrt{1 + 4\rho(1 - \rho)(e^t - 1)}}.$$ \hfill (C.7)

which simplifies at half-filling to $\lambda = t/2$. The sign + is chosen in front of the square root for continuity at $t = 0$, for which one has $\lambda = \log[\rho/(1 - \rho)]$.

C.4. Exact expression for the eigenvalue $e$

The expression (88) for the eigenvalue can be made completely explicit. From (92), one has

$$e = \sum_{r=0}^{\infty} (-1)^r (e^{-\lambda})^r \oint \frac{dz}{2i\pi} \frac{e^{-r\psi(z)}}{z}$$

$$= \oint \frac{dz}{2i\pi} \psi(z) + \partial_t \left[ \sum_{r=1}^{\infty} (-1)^{r-1} (e^{-\lambda})^r \oint \frac{dz}{2i\pi} \frac{e^{-r\psi(z)}}{z} \right].$$ \hfill (C.8)

Using (C.4) to compute the residue, one finds

$$e = -\frac{1 - \rho}{1 + e^{-\lambda}} = \frac{1 - \sqrt{1 + 4\rho(1 - \rho)(e^t - 1)}}{2(e^t - 1)}.$$ \hfill (C.9)

C.5. Exact expression for $s$

After a little rewriting, the expression (73) for $s$ becomes

$$s = \oint \frac{dz}{2i\pi z} [(1 - \sigma(z))(\lambda + t\psi(z)) + \log(1 + e^{-\lambda-\psi(z)})].$$ \hfill (C.10)

Using (76), (88) and (92), one has

$$s = \lambda(1 - \rho) - t(1 - \rho) - te + \sum_{r=1}^{\infty} \frac{(-1)^{r-1} e^{r\lambda}}{r} \oint \frac{dz}{2i\pi} \frac{e^{-r\psi(z)}}{z}.$$ \hfill (C.11)

Using (C.4) to compute the residue, one finds

$$s = -te + \rho \log(1 + e^{-\lambda}) + (1 - \rho) \log(1 + e^{\lambda-t}).$$ \hfill (C.12)
Appendix D. Free particles

In this appendix, we study a system of $N = \rho L$ non-interacting particles hopping to the nearest site on the right with rate 1 on a periodic lattice of $L$ sites. Unlike TASEP, there is no exclusion constraint. We will consider both the case of distinguishable particles and the case of undistinguishable particles.

Similarly to TASEP, we call $M(\gamma)$ the deformation of the Markov matrix which counts the current of particles. The action of $M(\gamma)$ on a microstate with particles at positions $x_1, \ldots, x_N$ is

$$M(\gamma)|x_1, \ldots, x_N\rangle = \sum_{j=1}^{N} (e^{\gamma} | \ldots, x_j + 1, \ldots, x_j, \ldots \rangle).$$

The $x_j$ need not be distinct. For distinguishable particles, the $j$th element of $|x_1, \ldots, x_N\rangle$ is the position of the $j$th particle, and the total number of microstates is $|\Omega_{\text{free}}^{d}| = L^N$. For undistinguishable particles, the positions are kept ordered as $x_1 \leq \ldots \leq x_N$, and the number of configurations is then $|\Omega_{\text{free}}^{u}| = \binom{L+N-1}{N}$. In both cases, the eigenvectors of $M(\gamma)$ are products of plane waves, and the eigenvalues are of the form

$$E = \sum_{j=1}^{N} (e^{\gamma} - 2\pi k_j/L - 1),$$

where each $k_j$ is an integer between 1 and $L$. For distinguishable particles, there is no further restriction on the $k_j$. For undistinguishable particles the $k_j$ must be ordered, $k_1 \leq \ldots \leq k_N$.

As in the case of TASEP, we define a density of eigenvalues $D(e)$ as in (62) and a quantity $f(t)$ as in (16). We calculate in this appendix $D(e)$ in the thermodynamic limit in the case of undistinguishable particles, and $f(t)$ for both distinguishable and undistinguishable particles.

### D.1. Density of eigenvalues for undistinguishable particles

Defining the density profile $\eta(u)$ of the $k_j$ as in section 3.4, the rescaled eigenvalue $e = E/L$ is expressed in terms of $\eta$ as

$$e[u] = \int_{0}^{1} du \eta(u)(e^{\gamma} - 2\pi u - 1).$$

The number $\Omega[u]$ of ways to place $L \eta(u)$ du $k_j$ in an interval of length $L$ du is $\binom{L+L \eta(u) du - 1}{L \eta(u) du}$. Stirling’s formula implies $\Omega[u] \sim e^{s[u]}$, where the ‘entropy’ $s[u]$ is

$$s[u] = \int_{0}^{1} du [-\eta(u) \log \eta(u) + (1 + \eta(u)) \log(1 + \eta(u))].$$

The difference with equation (49) for TASEP comes from the fact that several $k_j$ can be equal for non-interacting particles.

We introduce the two Lagrange multipliers $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{C}$ as in (63). The optimal function $\eta^*(u)$ that maximizes (63) with $s[u]$ given by (D.4) is

$$\eta^*(u) = (-1 + e^{-\lambda-2Re[\omega e^{2\pi i u} - 1]})^{-1}.$$  

Solving numerically (46) for several values of $\omega$ allows to plot $s(e)$, see figure D1. We are interested in the limit $e \to 0$, which corresponds to $\omega \to \infty$, $\gamma = 0$. We will only treat the case $e \in \mathbb{R}$, for which $\omega \in \mathbb{R}$. We first expand the denominator and the exponential in the expression (69) of $\eta^*$, as

$$\eta^*(u) = \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{k^j}{j!} e^{k(\lambda-2\omega)} (\omega e^{-2\pi i u} + \overline{\omega} e^{2\pi i u})^j,$$
Figure D1. Left: the spectrum of the Markov matrix $M$ for a system of $N = 6$ non-interacting undistinguishable particles on $L = 12$ sites. The black dots are the eigenvalues rescaled by a factor $1/L$. The black circle is the edge of the spectrum in the thermodynamic limit with $\rho = 1/2$. The red curve is the parabolic approximation of the circle near the origin. Right: the number of rescaled eigenvalues with a given real part $e$ for non-interacting undistinguishable particles at half-filling, as a function of $e$. The black curve corresponds to the expressions (D.4), (D.7) and (D.8). After summing over $j$, we find

$$\rho = \sum_{k=1}^{\infty} e^{k(\lambda - 2\omega)} I_0(2k\omega), \quad (D.7)$$

$$e = -\rho + \sum_{k=1}^{\infty} e^{k(\lambda - 2\omega)} I_1(2k\omega), \quad (D.8)$$

and

$$s = -\rho \lambda - 2\omega e + \sum_{k=1}^{\infty} \frac{e^{k(\lambda - 2\omega)}}{k} I_0(2k\omega). \quad (D.9)$$

In the previous expressions, $I_0$ and $I_1$ are modified Bessel functions of the first kind. Taking the asymptotics of the Bessel functions for large argument and summing over $j$ gives

$$\sqrt{2\pi} \rho \simeq \frac{\text{Li}_{1/2}(e^\lambda)}{\sqrt{2\omega}} + \frac{\text{Li}_{3/2}(e^\lambda)}{8(2\omega)^{3/2}} + \frac{9 \text{Li}_{5/2}(e^\lambda)}{128(2\omega)^{5/2}}, \quad (D.10)$$

$$\sqrt{2\pi} (e + \rho) \simeq \frac{\text{Li}_{1/2}(e^\lambda)}{\sqrt{2\omega}} - \frac{3\text{Li}_{3/2}(e^\lambda)}{8(2\omega)^{3/2}} - \frac{15 \text{Li}_{5/2}(e^\lambda)}{128(2\omega)^{5/2}}, \quad (D.11)$$

and

$$\sqrt{2\pi} (s + \rho \lambda + 2\omega e) \simeq \frac{\text{Li}_{3/2}(e^\lambda)}{\sqrt{2\omega}} + \frac{\text{Li}_{5/2}(e^\lambda)}{8(2\omega)^{3/2}} + \frac{9 \text{Li}_{7/2}(e^\lambda)}{128(2\omega)^{5/2}}. \quad (D.12)$$
In the limit $\omega \to \infty$, one has $\lambda \to 0^{-}$. After expanding the polylogarithms, we finally obtain

$$
s(e) \simeq \frac{3 \zeta(3/2)(-e)^{1/3}}{2\pi^{1/3}} - \frac{\pi^{1/3}(-e)^{2/3}}{\rho \zeta(3/2)^{2/3}} + \left( \frac{2\pi}{3\rho^{2}\zeta(3/2)} \right) e - \frac{\zeta(1/2)}{\rho^{2}\zeta(3/2)} - \frac{\zeta(5/2)}{4\zeta(3/2)} e,
$$

where $\zeta$ is Riemann zeta function. We observe that $s(e)$ vanishes for $e = 0$ with an exponent $1/3$. This exponent should not be confused with the exponent $1/2$ obtained from $\text{Re} E \simeq -2\pi^{2} \sum_{j=1}^{N} k^{2}_{j}/L^{2}$ for eigenvalues that do not scale proportionally with $L$.

**D.2. Function $f(t)$ for distinguishable particles**

From the definition (16) and the expression (D.2) for the eigenvalues, one has

$$
f(t) = \rho \log \left( \sum_{k=1}^{L} e^{i(t - 2\pi k/L - 1)} \right).
$$

For finite times, the sum becomes an integral in the large $L$ limit. After a rewriting as a contour integral, one finds

$$
f(t) = \rho \log L = \rho \log \left( \oint \frac{dz}{z} e^{i(t - 1)} \right) = -\rho t.
$$

We are also interested in $f(t)$ for times $t$ of order $L$. Expanding the exponential in (D.14) leads to

$$
f(\tau L) = \rho \log \left( \sum_{k=1}^{L} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{(-1)^{j-m} L^{j} e^{\gamma \rho} e^{-2\pi km/L}}{j!} \right).
$$

Exchanging the order of the summations over $j$ and $m$ allows to perform the summation over $j$. One finds

$$
f(\tau L) = \rho \log \left( \sum_{k=1}^{L} \sum_{m=0}^{\infty} \frac{L^{m} e^{-\tau L} e^{\gamma \rho} e^{-2\pi km/L}}{m!} \right).
$$
The summation over \( k \) is then done with the help of
\[
\sum_{k=1}^{L} e^{-2\pi ikm/L} = L \sum_{r=0}^{\infty} \delta_{m,rL},
\]
which leads to
\[
f(\tau L) = \rho \log L = \rho \log \left( \sum_{r=0}^{\infty} L^{rL} e^{-rL} e^{r\gamma L}/(rL)! \right). \tag{D.18}
\]
Using Stirling’s formula for \((rL)!\) (except for the term \( r = 0 \)) and extracting the leading term of the sum, one has
\[
\frac{f(\tau L)}{L} \simeq \max_{r \in \mathbb{N}} \rho(-\tau + r(1 + \log(e^{\gamma} r/r)))
\]
\[
- \frac{1}{2} \log(2\pi r(\tau)L) - \frac{1}{2} \frac{1}{12r(\tau)L^2}, \tag{D.20}
\]
with the convention \( r \log r = 0 \) for \( r = 0 \). In the second and third terms, \( r(\tau) \) is the \( r \) corresponding to the maximum in the first term. Defining \( \tau_0 = 0 \) and for \( r \in \mathbb{N}^* \)
\[
e^{\gamma} \tau_r = \frac{e^{-1}r}{(r-1)^{\gamma-1}}, \tag{D.21}
\]
one finally finds for \( \tau_r \leq \tau \leq \tau_{r+1} \)
\[
\lim_{L \to \infty} \frac{f(\tau L)}{L} = -\rho \tau + \rho \tau + \rho \tau \log \frac{e^{\gamma} \tau}{r}. \tag{D.22}
\]
For large \( r \), one has \( e^{\gamma} \tau_r \simeq r - 1/2 \), hence for large \( \tau \)
\[
\lim_{L \to \infty} \frac{f(\tau L)}{L} \simeq \rho (e^{\gamma} - 1) \tau - \frac{\rho (e^{\gamma} - [e^{\gamma} \tau])^2}{2e^{2\gamma}}, \tag{D.23}
\]
where \([e^{\gamma} \tau]\) is the integer closest to \( e^{\gamma} \tau \).

### D.3. Function \( f(t) \) for undistinguishable particles

From the definition (16) and the expression (D.2) for the eigenvalues, one has
\[
f(t) = \frac{1}{L} \log \left( \sum_{1 \leq k_1 \leq \ldots \leq k_N \leq L} \prod_{i=1}^{N} e^{i(\alpha - 2\pi r_i/k_i - 1)} \right). \tag{D.24}
\]
Expanding the exponential leads to
\[
f(t) = \frac{1}{L} \log \left( \sum_{1 \leq k_1 \leq \ldots \leq k_N \leq L} \prod_{i=1}^{N} \prod_{j=0}^{\infty} j \frac{(-1)^{j+m_j} e^{my_j} e^{-2\pi km_j/L}}{j!} \right). \tag{D.25}
\]
Exchanging the order of the summations over \( j \) and \( m \) allows to perform the summation over \( j \). One finds
\[
f(t) = \frac{1}{L} \log \left( \sum_{1 \leq k_1 \leq \ldots \leq k_N \leq L} \prod_{i=1}^{N} \sum_{m=0}^{\infty} t^m e^{-t} e^{my} e^{-2\pi km/L} m! \right). \tag{D.26}
\]
So far, the calculation parallels the one for distinguishable particles in the scaling \( t \sim L \). In order to perform the summation over the \( k_i \), we first use the relation
\[
S_N(L) = \sum_{1 \leq k_1 \leq \ldots \leq k_N \leq L} \prod_{i=1}^{N} f(k_i) = \int f \frac{dz}{2i\pi} \exp \left[ \sum_{a=1}^{\infty} \sum_{k=1}^{L} \frac{z^a f(k)^a}{a} \right]. \tag{D.27}
\]
The contour integral is over a contour enclosing 0. Equation (D.27) can be proved by considering the formal series in $z$

$$\sum_{N=0}^{\infty} z^N S_N(L) = \prod_{k=1}^{L} \frac{1}{1 - z f(k)}. \quad (D.28)$$

Equation (D.27) gives

$$f(t) = \frac{1}{L} \log \left( \oint \frac{dz}{2\pi i} \frac{e^{-Nt}}{z^{N+1}} \exp \left[ \sum_{a=1}^{\infty} \frac{z^a}{a} \sum_{k=1}^{L} \sum_{m_1, \ldots, m_a=0}^{\infty} \prod_{i=1}^{a} \frac{m_i^{m_i} e^{-2\pi k m_i/L}}{m_i!} \right] \right). \quad (D.29)$$

Using (D.18) to sum over $k$ leads to

$$f(t) = \frac{1}{L} \log \left( \oint \frac{dz}{2\pi i} \frac{e^{-Nt}}{z^{N+1}} \exp \left[ L \sum_{a=1}^{\infty} \frac{z^a}{a} \sum_{r=0}^{\infty} \sum_{m_1, \ldots, m_a=0}^{\infty} \delta_{rL} c_r \sum_{t=1}^{m_a} \frac{m_i^{m_i} e^{-t/m_i}}{m_i!} \right] \right). \quad (D.30)$$

The multinomial sum over the $m_i$ can be performed. One has

$$f(t) = \frac{1}{L} \log \left( \oint \frac{dz}{2\pi i} \frac{e^{-Nt}}{z^{N+1}} \exp \left[ L \sum_{r=0}^{\infty} \left( \frac{e^r t}{r!} \right)^L \right. \right). \quad (D.31)$$

The summation over $a$ can be done explicitly. For $r \geq 1$, it gives a polylogarithm. One finds

$$f(t) = \frac{1}{L} \log \left( \oint \frac{dz}{2\pi i} \frac{e^{-Nt}}{z^{N+1}} \exp \left[ L \sum_{r=0}^{\infty} \left( \frac{e^r t}{r!} \right)^L \right] \log \left. \frac{1 - e^{z/L}}{1 - z} \right) \right). \quad (D.32)$$

We deform the contour of integration so that it encloses the negative real axis. In the thermodynamic limit, it is then possible to use the asymptotics

$$\text{Li}_{i-1}(z) \simeq \Gamma(n + 1)(-\log z)^{-n-1} \quad (D.33)$$

for large $n$. It leads to

$$f(t) \simeq \frac{1}{L} \log \left( \oint \frac{dz}{2\pi i} \frac{e^{-Nt}}{z^{N+1}} \exp \left[ \sum_{r=1}^{\infty} \frac{1}{r} \left( -\frac{e^r t}{r L} \right)^L \right] \right). \quad (D.34)$$

Summing explicitly over $r$ gives

$$f(t) \simeq \frac{1}{L} \log \left( \oint \frac{dz}{2\pi i} \frac{e^{-Nt}}{z^{N+1}} \exp \left[ \sum_{r=0}^{\infty} \frac{1}{r} \left( -\frac{e^r t}{r L} \right)^L \right] \right). \quad (D.35)$$

Expanding the last factor of the denominator, we finally obtain

$$f(t) \simeq \frac{1}{L} \log \left( \sum_{r=0}^{\infty} \oint \frac{dz}{2\pi i} \frac{e^{-Nt}}{z^{N+1}} \left( 1 - \left( -\frac{e^r t}{r L} \right)^L \right) \right). \quad (D.36)$$

The thermodynamic limit of $f(t)$ is extracted by calculating the saddle point $z_r$ of the contour integral. One finds

$$f(t) \simeq \max_{r \in \mathbb{N}} \left( -\rho t - \rho \log z_r - \log(1 - z_r) + r \log \left( -\frac{e^r t}{r L \log z_r} \right) \right), \quad (D.37)$$

where $z_r$ verifies the equation

$$\frac{r}{\log z_r} = \frac{z_r}{1 - z_r} - \rho. \quad (D.38)$$

One has $z_0 = \rho/(1 + \rho)$. For $\rho = 1/2$, we find numerically $z_1 \simeq 0.085$, $z_2 \simeq 0.016$, $z_3 \simeq 0.0024$. We introduce times $t_r, r \in \mathbb{N}$ such that when $t_r < t < t_{r+1}$, the saddle point that
dominates (D.37) is $z_r$. The value of $t_r$ is determined by continuity of $f(t)$. One has $t_0 = 0$, and for $\rho = 1/2$, $e^{t_1} = 1.7085$, $e^{t_2} = 3.236$, $e^{t_3} = 5.040$.

For large $r$, $z_r$ decreases to 0 as $z_r \simeq e^{-r/\rho}$. This implies, for $t_r < t < t_{r+1}$,

$$f(t) \simeq -\rho t + r + r \log \frac{\rho e^t}{r}, \quad (D.39)$$

with $t_r$ given for large $r$ by

$$\rho e^{t_r} \simeq \frac{e^{-1/r}}{(r-1)^{r-1}} \simeq r - \frac{1}{2}. \quad (D.40)$$

For large times, we finally obtain

$$f(t) \simeq \rho (e^{t_r} - 1)t - \rho e^{t_r} \left( \frac{\rho e^{t_r} - \lfloor \rho e^{t_r} \rfloor^2}{2 \rho e^{t_r}} \right), \quad (D.41)$$

where $\lfloor \rho e^{t_r} \rfloor$ is the integer closest to $\rho e^{t_r}$. This expression is very similar to the one found for distinguishable particles on the scale $t \sim L$.

References

[1] Spitzer F 1970 Interaction of Markov processes Adv. Math. 5 246–90
[2] Derrida B 1998 An exactly soluble non-equilibrium system: the asymmetric simple exclusion process Phys. Rep. 301 65–83
[3] Schütz G M 2001 Exactly Solvable Models for Many-body Systems Far from Equilibrium (Phase Transitions and Critical Phenomena vol 19) (San Diego, CA: Academic)
[4] Golinelli O and Mallick K 2006 The asymmetric simple exclusion process: an integrable model for non-equilibrium statistical mechanics J. Phys. A: Math. Gen. 39 12679–705
[5] Derrida B 2007 Non-equilibrium steady states: fluctuations and large deviations of the density and of the current J. Stat. Mech. P07023
[6] Sasamoto T 2007 Fluctuations of the one-dimensional asymmetric exclusion process using random matrix techniques J. Stat. Mech. P07007
[7] Ferrari P L and Spohn H 2011 Random growth models The Oxford Handbook of Random Matrix Theory ed G Akemann, J Baik and P Di Francesco (Oxford: Oxford University Press)
[8] Mallick K 2011 Some exact results for the exclusion process J. Stat. Mech. P01024
[9] Chou T, Mallick K and Zia R K P 2011 Non-equilibrium statistical mechanics: from a paradigmatic model to biological transport Rep. Prog. Phys. 74 116601
[10] Gwa L-H and Spohn H 1992 Six-vertex model, roughened surfaces, and an asymmetric spin Hamiltonian Phys. Rev. Lett. 68 725–8
[11] Gwa L-H and Spohn H 1992 Bethe solution for the dynamical-scaling exponent of the noisy Burgers equation Phys. Rev. A 46 844–54
[12] Kim D 1995 Bethe ansatz solution for crossover scaling functions of the asymmetric XXZ chain and the Kardar–Parisi–Zhang-type growth model Phys. Rev. E 52 3512–24
[13] Golinelli O and Mallick K 2004 Bethe ansatz calculation of the spectral gap of the asymmetric exclusion process J. Phys. A: Math. Gen. 37 3321–31
[14] Golinelli O and Mallick K 2005 Spectral gap of the totally asymmetric exclusion process at arbitrary filling J. Phys. A: Math. Gen. 38 1419–25
[15] Johansson K 2000 Shape fluctuations and random matrices Commun. Math. Phys. 209 437–76
[16] Prihofer M and Spohn H 2002 Current fluctuations for the totally asymmetric simple exclusion process In and Out of Equilibrium: Probability with a Physics Flavor (Progress in Probability vol 51) (Boston, MA: Birkhäuser) pp 185–204
[17] Borodin A, Ferrari P L, Prihofer M and Sasamoto T 2007 Fluctuation properties of the TASEP with periodic initial configuration J. Stat. Phys. 129 1055–80
[18] Tracy C A and Widom H 2009 Total current fluctuations in the asymmetric simple exclusion process J. Math. Phys. 50 095204
[19] Sasamoto T and Spohn H 2010 The crossover regime for the weakly asymmetric simple exclusion process J. Stat. Phys. 140 209–31
[20] Amir G, Corwin I and Quastel J 2011 Probability distribution of the free energy of the continuum directed random polymer in 1 + 1 dimensions Commun. Pure Appl. Math. 64 466–537
[21] Derrida B and Lebowitz J L 1998 Exact large deviation function in the asymmetric exclusion process Phys. Rev. Lett. 80 209–13
[22] Derrida B and Appert C 1999 Universal large-deviation function of the Kardar–Parisi–Zhang equation in one dimension J. Stat. Phys. 94 1–30
[23] Prolhac S and Mallick K 2008 Current fluctuations in the exclusion process and Bethe ansatz J. Phys. A: Math. Theor. 41 175002
[24] Prolhac S 2008 Fluctuations and skewness of the current in the partially asymmetric exclusion process J. Phys. A: Math. Theor. 41 365003
[25] Prolhac S and Mallick K 2009 Cumulants of the current in a weakly asymmetric exclusion process J. Phys. A: Math. Theor. 42 175001
[26] Prolhac S 2010 Tree structures for the current fluctuations in the exclusion process J. Phys. A: Math. Theor. 43 105002
[27] Lazarescu A and Mallick K 2011 An exact formula for the statistics of the current in the TASEP with open boundaries J. Phys. A: Math. Theor. 44 315003
[28] Gorissen M, Lazarescu A, Mallick K and Vanderzande C 2012 Exact current statistics of the asymmetric simple exclusion process with open boundaries Phys. Rev. Lett. 109 170601
[29] Lazarescu A 2013 Matrix ansatz for the fluctuations of the current in the ASEP with open boundaries J. Phys. A: Math. Theor. 46 145003
[30] Katz S, Lebowitz J L and Spohn H 1984 Nonequilibrium steady states of stochastic lattice gas models of fast ion conductors J. Stat. Phys. 34 497–537
[31] Schmittmann B and Zia R K P 1998 Driven diffusive systems: an introduction and recent developments Phys. Rep. 301 45–64
[32] Barabási A L and Stanley H E 1995 Fractal Concepts in Surface Growth (Cambridge: Cambridge University Press)
[33] Mazenko P 1998 Fractals, Scaling and Growth Far from Equilibrium (Cambridge: Cambridge University Press)
[34] Halpin-Healy T and Zhang Y-C 1995 Kinetic roughening phenomena, stochastic growth, directed polymers and all that: aspects of multidisciplinary statistical mechanics Phys. Rep. 254 215–414
[35] Brunet E and Derrida B 2000 Probability distribution of the free energy of a directed polymer in a random medium Phys. Rev. E 61 6789–801
[36] Dotsenko V 2010 Bethe ansatz derivation of the Tracy–Widom distribution for one-dimensional directed polymers Europhys. Lett. 90 20003
[37] Calabrese P, Le Doussal P and Rosso A 2010 Free-energy distribution of the directed polymer at high temperature Europhys. Lett. 90 20002
[38] Kardar M, Parisi G and Zhang Y-C 1986 Dynamic scaling of growing interfaces Phys. Rev. Lett. 56 889–92
[39] Kriecherbauer T and Krug J 2010 A pedestrian’s view on interacting particle systems, KPZ universality and random matrices J. Phys. A: Math. Theor. 43 403001
[40] Sasamoto T and Spohn H 2010 The 1 + 1-dimensional Kardar–Parisi–Zhang equation and its universality class J. Stat. Mech. P11013
[41] Povolotsky A M and Priezzhev V B 2007 Determinant solution for the totally asymmetric exclusion process with parallel update: H. Ring geometry J. Stat. Mech. P08018
[42] Pronko G P and Stroganov Y G 1999 Bethe equations ‘on the wrong side of the equator’ J. Phys. A: Math. Gen. 32 2333–40
[43] Yang C N and Yang C P 1969 Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction J. Math. Phys. 10 1115–22
[44] Henkel M and Schütz G M 1988 Finite-lattice extrapolation algorithms J. Phys. A: Math. Gen. 21 2617–33