Adaptive Output Consensus For Multi-Agent Systems With An Uncertain Leader

Shimin Wang and Xiangyu Meng

Abstract—In this technical note, the leader-following output consensus problem is studied for a class of multi-agent systems subject to an uncertain leader system. The leader system is described by the sum of sinusoids with unknown amplitudes, frequencies and phases. A distributed adaptive observer is established for each agent to estimate the state of the leader even if the agent cannot directly access the output signal of the leader. It is shown that if the output signal of the leader is sufficiently rich, all agents can learn its frequencies based on the distributed adaptive observer. Utilizing the estimated state of the leader and its output frequencies, a distributed adaptive control law is synthesized for each agent to solve the leader-following output consensus problem.

Index Terms—Output consensus, adaptive observer, multi-agent systems, distributed control, parameter estimation.

I. INTRODUCTION

MULTI-AGENT coordination attracts gradually increasing research interests and many problems of it have been extensively studied, such as leaderless consensus problem [1-4], leader-following consensus problem [5-10], formation control problem [11, 12] and distributed optimization problem [13-15]. The parameters of the leader play a key role in the design of distributed control laws for solving the cooperative control problem of a multi-agent system. For example, reference [16] used the system matrix of the leader to design an internal model and solved the robust cooperative output regulation problem for linear uncertain multi-agent systems. Besides, the demands of the parameters of the leader in the leader-following problem give rise to the distributed parameter estimation problem for multi agent systems. If each agent in the network could access the signal of the leader, we could design a traditional adaptive observer for each agent as proposed in [17-24] to handle this parameter estimation problem. As communication constraints exist in the network, the followers in a multi-agent system can be classified into two groups. The first group consists of the agents which can access the exogenous signal for parameter estimation, and the second group consists of the rest of the followers. Since the followers in the second group cannot access the exogenous signal for designing the traditional adaptive observer, the distributed parameter estimation problem of a multi-agent system cannot be handled by the classical approach proposed in [17-24].

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Reference [25] firstly proposed a distributed observer to solve the leader-following problem under the assumption that each agent knows the system matrix of the leader. The distributed observer for a nonlinear leader with known parameters was investigated in [8]. Then, the assumption that the parameters of the leader are known by each follower is relaxed in [5] and it is assumed that only the followers who are the children of the leader know the parameters of the leader. In fact, [5] used the communication network to spread the parameters of the leader to design the distributed observer. Some efforts have been made to solve the leader-following consensus problem by removing the dynamics of the leader, such as [7, 26-28]. However, the convergence parameter is not analyzed. In practice, a leader system may contains some unknown parameters and the state or the output is available for only some followers. To deal with this situation, references [29, 30] further developed an adaptive dynamic compensator that can estimate both state and system matrix of the leader using the local information only. Such a distributed dynamic compensator is called full state distributed adaptive observer for an uncertain leader system. It was shown that, as long as the state of the leader is persistently exciting, the estimated parameters asymptotically converge to the actual value of the parameters of the leader as time goes to infinity.

In many practical applications, neither the state nor the parameters of the leader system are known by any followers and only the output of the leader system is available to some followers. For example, a signal can be generated by the sum of sinusoids with unknown amplitudes, frequencies and phases. To cater for this scenario, in this paper, we will first establish an distributed adaptive observer for a linear leader system which consists of the sum of several sinusoids with unknown amplitudes, frequencies and phases. The distributed adaptive observer of each followers can estimate the state of the leader through the communication network of the system without knowing the frequencies of the leader. We further show that if the output signal of the leader is sufficiently rich, the proposed distributed adaptive observer can asymptotically learn the frequencies of the output signal of the leader. Based on the observer, we synthesize an distributed adaptive control law for solving the leader-following output consensus problem subject to an uncertain leader system. Our result significantly enhances the results in [5, 26-29] by allow the output signal of the leader to be a sum of multi-tone sinusoidal signals with unknown amplitudes, frequencies, and phases.

The rest of this paper is organized as follows: In Section [11] we formulate the output consensus problem. Section [III] is devoted to the design of distributed adaptive observers and
distributed observer based controller. In Section IV detailed analysis are given to show the effectiveness of the proposed control strategy for solving the leader-following output consensus problem with an unknown leader system. A simulation example is given in Section VI. Finally, we conclude this paper in Section VII.

II. PROBLEM FORMULATION

A. Multi-agent network

As in [5], the multi-agent system is composed of a leader and N followers. The network topology of the multi-agent system is described by a graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) with \( \mathcal{V} = \{0, \ldots, N\} \) and \( \mathcal{E} \subseteq [\mathcal{V}]^2 \), which is the 2-element subsets of \( \mathcal{V} \). Here node 0 is associated with the leader and node \( i \) is associated with follower \( i \) for \( i = 1, \ldots, N \). For \( i = 0, 1, \ldots, N \), \( j = 1, \ldots, N \), \((i,j) \in \mathcal{E}\) if and only if agent \( j \) can receive information from agent \( i \). Let \( \mathcal{N}_i = \{j|(j,i) \in \mathcal{E}\} \) denote the neighborhood set of agent \( i \). Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) denote the induced subgraph of \( \mathcal{G} \) with \( \mathcal{V} = \{1, \ldots, N\} \). Assume that \( \mathcal{G} \) contains a spanning arborescence with node 0 as the root and \( \mathcal{G} \) is an undirected graph. Let \( \Delta \) be the Laplacian matrix of the graph \( \mathcal{G} \), and \( H \) is obtained by deleting the first row and column of \( \Delta \). Then \( H \) is a positive definite symmetric matrix [51]. Let \( \lambda_n \) and \( \lambda_1 \) be the largest and smallest eigenvalues of \( H \), respectively. More details of the graph theory can be found in [22].

B. Leader dynamics

The output \( y_0(t) \in \mathbb{R} \) of the leader is generated by the following exosystem:

\[
\begin{align*}
\dot{x}_{0,2k-1}(t) &= x_{0,2k}(t), & k = 1, \ldots, l \\
\dot{x}_{0,2k}(t) &= -\omega_k^2x_{0,2k-1}(t), & k = 1, \ldots, l \\
y_0(t) &= x_{0,1}(t) + x_{0,3}(t) + \cdots + x_{0,2l-1}(t) 
\end{align*}
\]

(1)

where \( x_0(t) = \text{col}\,(x_{0,1}(t), \ldots, x_{0,2l}(t)) \) is the state of the leader with the initial conditions \( x_{0,k}(0) = 0 \) for \( k = 1, \ldots, 2l \). The output \( y_0(t) \) in (1) can be rewritten as:

\[
y_0(t) = \sum_{k=1}^{l} \varphi_k \sin(\omega_k t + \psi_k),
\]

(2)

where \( \varphi_k > 0, \psi_k \) and \( \omega_k > 0 \) for \( k = 1, \ldots, l \) are unknown amplitudes, phases and frequencies, respectively. Assume that \( \omega_k < \bar{\omega} \) for \( k = 1, \ldots, l \), where \( \bar{\omega} \) is a known upper bound. Also assume that the output \( y_0 \) is sufficiently rich of order \( 2l \), that is, it consists of at least \( l \) distinct frequencies [20].

C. Follower dynamics

The dynamics of follower \( i \) are described by the following single-input and single-output system:

\[
\begin{align*}
\dot{x}_{i,s}(t) &= x_{i,s+1}(t), & s = 1, \ldots, r - 1, \\
\dot{x}_{i,r}(t) &= u_i(t), \\
y_i(t) &= x_{i,1}(t)
\end{align*}
\]

(3a), (3b), (3c)

for \( i = 1, \ldots, N \), where \( x_i = \text{col}\,(x_{i,1}, \ldots, x_{i,r}) \in \mathbb{R}^r \) is the state vector, \( u_i \in \mathbb{R} \) is the control input, and \( y_i \in \mathbb{R} \) is the output.

D. Output consensus problem

The output consensus problem considered in this paper is formulated as follows.

**Problem 1** (Leader-following Output Consensus Problem). Given a multi-agent network \( \mathcal{G} \) with the leader dynamics (1) and the follower dynamics (3), find a control law \( u_i \) for \( i = 1, \ldots, N \) such that \( x_i(t) \) is bounded for all \( t \geq 0 \) and

\[
\lim_{t \to \infty} (y_i(t) - y_0(t)) = 0
\]

for any initial conditions \( x_0(0) \) and \( x_i(0) \), and for \( i = 1, \ldots, N \).

**Remark 1.** Problem [7] was studied in [5, 8, 25] using a distributed observer approach under the assumption that the parameters of the leader are known exactly. However, this assumption is restricted in practical applications. We consider the case when the output signal \( y_0(t) \) of the leader in (2) is a sum of \( l \) sinusoids with unknown amplitudes, frequencies and phases, and it is available only to some followers. Distributed adaptive observers proposed in references [5, 8, 25, 27, 29] require that some followers know the full state of the leader. Therefore, they cannot be applied in this case. The result in this paper will allow us to deal with multi-tone sinusoidal signals with arbitrary unknown amplitudes, initial phases and frequencies based on only the output of the leader.

III. OUTPUT BASED DISTRIBUTED ADAPTIVE CONTROL DESIGN

This section is devoted to engineers who are not interested in the theoretical analysis.

A. Distributed Adaptive Observer

Choose an arbitrary vector \( a = \text{col}\,(a_1, a_2, \ldots, a_{2l-1}) \) such that all the roots of the polynomial equation

\[
\lambda^{2l-1} + a_1 \lambda^{2l-2} + \cdots + a_{2l-2} \lambda + a_{2l-1} = 0,
\]

have negative-real parts. The matrices \( A \in \mathbb{R}^{(2l-1) \times (2l-1)} \) and \( B \in \mathbb{R}^{2l-1} \) are defined as

\[
A = \begin{bmatrix}
-a_1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-a_{2l-2} & 0 & \cdots & 1 \\
-a_{2l-1} & 0 & \cdots & 0 \\
\end{bmatrix},
B = \begin{bmatrix}
a_2 - a_1 a_1 \\
a_3 - a_2 a_1 \\
\vdots \\
a_{2l-1} - a_{2l-2} a_1 \\
a_{2l-1} - a_{2l-2} a_1 \\
\end{bmatrix}.
\]

(4a)

Let \( P \in \mathbb{R}^{(2l-1) \times (2l-1)} \) and \( Q \in \mathbb{R}^{(2l-1) \times (2l-1)} \) be the solutions of the following Lyapunov equations

\[
PA + A^T P + 3I_{2l-1} = 0, \quad (4a)
\]

\[
AQ + Q A^T + 3I_{2l-1} = 0. \quad (4b)
\]

Since all eigenvalues of \( A \) have negative real parts, the above Lyapunov equations have unique positive definite symmetric solutions \( P \) and \( Q \) according to Lyapunov Theorem. Denote \( \lambda_P \) and \( \lambda_Q \) are the maximum eigenvalues of \( P \) and \( Q \), respectively. Finally, let us choose a constant \( \mu \) such that

\[
\mu \geq \frac{1}{2\lambda_1^2} \frac{\lambda_P^2}{2 + N\pi^2}\lambda_n^2,
\]

(5)
where \( \pi \) will be introduced later.

With the above notations, we are ready to introduce the distributed adaptive observer for agent \( i \):

\[
\begin{align*}
\dot{\hat{\theta}}_i &= \gamma F \hat{x}_i \sum_{j \in \mathcal{N}_i} (\hat{y}_j - \hat{y}_i) \\
\dot{\hat{y}}_i &= E^T \hat{a}_i + \alpha_1 \hat{y}_i + \chi_i^T F^T \hat{\theta}_i + \mu \sum_{j \in \mathcal{N}_i} (\hat{y}_j - \hat{y}_i)
\end{align*}
\]

(6a, 6b, 6c, 6d)

where the observer states \( \hat{\theta}_i \in \mathbb{R}^{2l-1}, \hat{x}_i \in \mathbb{R}^{2l-1}, \hat{\theta}_i \in \mathbb{R}, \hat{y}_i \in \mathbb{R} \) for \( i = 1, \ldots, N \), \( \hat{y}_0 = y_0 \), \( \gamma \) is a positive constant relating to the adaptation gain, \( E = \text{col} (1, 0_{1 \times 2l-2}) \) and \( F = \text{block diag} ((I_{l-1} \otimes [-1, 0]), -1) \in \mathbb{R}^{l \times 2l-1} \).

**B. Distributed Observer Based Controller**

Define functions \( f_i : \mathbb{R}^{2l} \times \mathbb{R}^l \mapsto \mathbb{R} \) for \( i = 1, \ldots, r + 1 \) as:

\[
\begin{align*}
 f_i (x, y) &= C x \\
 f_{s+1} (x, y) &= \frac{\partial f_s (x, y)}{\partial x} g(y)x, \quad s = 1, \ldots, r,
\end{align*}
\]

(7a, 7b)

where \( C = [1 \ 0_{1 \times 2l-1}] \) and \( g(y) \) is a matrix function \( g : \mathbb{R}^l \mapsto \mathbb{R}^{2l \times 2l} \) defined as

\[
g(y) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-1 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]

(8)

Then, let us find a matrix \( L \) such that \( M - LC \) is a Hurwitz matrix, where \( M \) is defined as

\[
M = \begin{bmatrix}
0_{2l-1 \times 1} & I_{2l-1} \\
0 & 0_{1 \times 2l-1}
\end{bmatrix}.
\]

(9)

Finally, choose \( \alpha_1, \ldots, \alpha_r \) such that the roots of the polynomial equation

\[
\lambda^r + \alpha_r \lambda^{r-1} + \cdots + \alpha_2 \lambda + \alpha_1 = 0
\]

have negative-real parts.

The distributed observer based controller for agent \( i \) is given as follows:

\[
\begin{align*}
\dot{\hat{v}}_i &= M \dot{\hat{y}}_i - L (C \hat{y}_i - \hat{y}_i) - \sum_{k=1}^l \hat{\theta}_{i,k} E_{2k} \hat{y}_i, \\
\dot{u}_i &= \sum_{s=1}^{r+1} \alpha_s f_s (\hat{v}_i, \hat{\theta}_i) - \sum_{s=1}^r \alpha_s x_{i,s},
\end{align*}
\]

(10a, 10b)

where \( \alpha_{r+1} = 1 \), and \( E_{2k} = \text{col} (0_{1 \times 2k-1}, 1, 0_{1 \times 2l-2k}) \).

**IV. SOLVABILITY ANALYSIS**

Let us review some results in [33] on how to estimate the output \( y_0(t) \) for a single-agent system. As a byproduct, the result in [33] also shows how to estimate the unknown frequencies \( \omega_k \) for \( k = 1, \ldots, l \).

The characteristic polynomial of system (2) is

\[
\prod_{k=1}^l (s^2 + \omega_k^2) = s^{2l} + \sum_{k=1}^l \omega_k^2 s^{2(n-1)} + \cdots + \prod_{k=1}^l \omega_k^2
\]

where \( \theta = \text{col} (\theta_1, \ldots, \theta_l) \) is an invertible reparametrization of the \( l \) unknown parameters \( \omega_1^2, \ldots, \omega_l^2 \). Let \( \pi = \max_{\omega_k \leq \omega, k = 1, \ldots, l} || \theta \|| \). Then, there exists a realization in the following observable canonical form

\[
\dot{v} = g(\theta)v = M v - \sum_{k=1}^l \theta_k E_{2k} y_0
\]

(11a)

\[
y_0 = Cv
\]

(11b)

where the matrix function \( g(\cdot) \) is defined in [33]. Following the same steps as in [33], (11) can be transformed into the following form

\[
\dot{\hat{v}} = A \hat{v} + B y_0,
\]

(12a)

\[
\dot{\hat{\chi}} = A^{T} \hat{\chi} + E \hat{y}_0,
\]

(12b)

\[
\dot{\hat{y}}_0 = E^{T} \hat{\eta} + a_1 \hat{y}_0 + \chi^T F^T \hat{\theta},
\]

(12c)

where \( \eta = \text{col} (\eta_1, \eta_2, \ldots, \eta_{2l-1}) \in \mathbb{R}^{2l-1} \).

The adaptive observer for the system (12) proposed in [33] is given as follows:

\[
\dot{\hat{\theta}} = \gamma F \hat{\chi} (y_0 - \hat{y}_0),
\]

(13a)

\[
\dot{\hat{y}}_0 = E^T \hat{\eta} + a_1 \hat{y}_0 + \chi^T F^T \hat{\theta}.
\]

(13b)

**Definition 1.** [34] A bounded piecewise continuous function \( f : [0, +\infty) \mapsto \mathbb{R}^n \) is said to be persistently exciting (PE) if there exist positive constants \( c, T_0 \) such that

\[
\frac{1}{T_0} \int_t^{t+T_0} f(s) f^T (s) ds \geq c I_n, \quad \forall t \geq 0
\]

According to [33], if \( F \chi(t) \) satisfies the persistency of excitation condition, then the states \( \theta(t) - \hat{\theta}(t) \) and \( \hat{y}(t) - \hat{y}(t) \) are bounded and tend to zero as \( t \) goes to infinity for any initial conditions.

**Remark 2.** The signals \( \eta \) and \( \chi \) in (13) are generated by the system (12a) and (12b) with \( y_0 \) as the input, respectively. However, there exist some followers who cannot access the output signal of the leader \( y_0 \) due to the communication constraints induced by \( \mathcal{G} \) which implies they could not generate the signals \( \eta \) and \( \chi \) by the system (12a) and (12b). Thus, the distributed parameter estimation problem cannot be solved by the traditional adaptive observer (13). Furthermore, the output consensus problem cannot be solved by an observer based control law using the adaptive observer (13) for those agents who do not have access to \( y_0 \).
A. Output estimation error analysis

Let \( y_i = \hat{y}_i - y_n, \hat{y}_i = \hat{y}_i - \eta, \chi_i = \hat{\chi}_i - \chi, z_i = \sum_{j \in \mathcal{N}} (\hat{y}_j - y_j) \) and \( \theta_i = \bar{\theta} - \theta \). Then, based on (6), (12) and (13), we can obtain:

\[
\begin{align*}
\dot{\hat{y}} &= A\hat{y} + \theta \hat{y}, \quad (14a) \\
\dot{\chi} &= A^T \chi + E\hat{y}, \quad (14b) \\
\dot{\bar{\theta}} &= \chi^T F \dot{\chi} \chi, \quad (14c) \\
\dot{\bar{y}} &= E^T \hat{y} + a_1 \bar{y} + \chi^T F^T \theta + \chi^T F^T \bar{\theta} + \mu \bar{z}, \quad (14d)
\end{align*}
\]

Define \( \bar{y} = \text{col}(\bar{y}_1, \ldots, \bar{y}_N) \) and \( z = \text{col}(z_1, \ldots, z_N) \). Then, we have the following relation:

\[
z = -H \bar{y}. \tag{15}
\]

It can be shown that (14) can be written in the following compact form:

\[
\begin{align*}
\dot{\bar{y}} &= (I_N \otimes A) \bar{y} + (I_N \otimes B) \bar{y} \tag{16a} \\
\dot{\chi} &= (I_N \otimes A^T) \chi + (I_N \otimes E) \bar{y} \tag{16b} \\
\dot{\bar{\theta}} &= -\gamma (I_N \otimes F) \chi \bar{H} \bar{y} \tag{16c} \\
\dot{\bar{y}} &= (a_1 I_N - \mu H) \bar{y} + (I_N \otimes E^T) \hat{\eta} \\
&\quad + \chi^T \left[ I_N \otimes (F^T \theta) \right] + \chi^T \left[ I_N \otimes F^T \right] \bar{\theta} \tag{16d}
\end{align*}
\]

where \( \otimes \) denotes the Kronecker product of matrices, \( I_N \in \mathbb{R}^{N} \) is the vector of all ones, and

\[
\begin{align*}
\bar{y} &= \text{col}(\bar{y}_1, \ldots, \bar{y}_N), \quad \chi_d = \text{blockdiag}(\chi_1, \ldots, \chi_N), \\
\bar{\theta} &= \text{col}(\bar{\theta}_1, \ldots, \bar{\theta}_N), \quad \chi_d = \text{blockdiag}(\chi_1, \ldots, \chi_N), \\
\hat{\chi} &= \text{col}(\hat{\chi}_1, \ldots, \hat{\chi}_N).
\end{align*}
\]

We now ready to establish our main technical lemmas.

**Lemma 1.** Consider systems (2), (6), (12) and (16). For any \( \hat{y}(0), \chi(0), \bar{\theta}(0) \) and \( \bar{y}(0), \) if \( \gamma > 0 \) and \( \mu \) satisfies (5), then \( \hat{y}(t), \chi(t) \) and \( \bar{\theta}(t) \) are bounded for all \( t \geq 0 \) and satisfy

\[
\begin{align*}
\lim_{t \to \infty} \bar{y}(t) &= 0 \tag{17} \\
\lim_{t \to \infty} \chi(t) &= 0 \tag{18} \\
\lim_{t \to \infty} \bar{\theta}(t) &= 0 \tag{19} \\
\lim_{t \to \infty} \chi_d(t) \left( I_N \otimes F^T \right) \bar{\theta} &= 0. \tag{20}
\end{align*}
\]

**Proof.** Consider the following Lyapunov function candidate for (16):

\[
V = \bar{y}^T H \bar{y} + \hat{\eta}^T (I_N \otimes P) \bar{y} + \chi^T (I_N \otimes Q) \chi + \frac{1}{\gamma} \bar{y}^T \bar{\theta}, \tag{21}
\]

where \( H \) is defined in Section II-A, \( P \) and \( Q \) are defined in (4). Differentiating (21) along the trajectory of (16) gives

\[
\begin{align*}
\dot{V} &= 2\bar{y}^T H \dot{\bar{y}} + \hat{\eta}^T (I_N \otimes P) \dot{\bar{y}} + 2\chi^T (I_N \otimes Q) \dot{\chi} + \frac{2}{\gamma} \bar{y}^T \dot{\bar{\theta}} \\
&= 2\bar{y}^T \left[ H (I_N \otimes E^T) \hat{\eta} + a_1 H \bar{y} + H \chi_d \left[ I_N \otimes (F^T \theta) \right] + \chi^T \left[ I_N \otimes F^T \right] \bar{\theta} - \mu H^2 \bar{y} ight] \\
&\quad + 2\hat{\eta}^T \left[ (I_N \otimes PA) \hat{\eta} + (I_N \otimes PB) \bar{y} \right] \\
&\quad + 2\chi^T \left[ (I_N \otimes QA^T) \hat{\chi} + (I_N \otimes QE) \bar{y} \right] - 2\bar{\theta}^T (I_N \otimes F) \chi_d H \bar{y}.
\end{align*}
\]

It’s easy to verify that

\[
\begin{align*}
\bar{y}^T H \chi_d^T (I_N \otimes F^T) \bar{\theta} &= \left[ \bar{y}^T H \chi_d^T (I_N \otimes F^T) \bar{\theta} \right]^T \\
&= \theta^T (I_N \otimes F) \chi_d H \bar{y}. \tag{23}
\end{align*}
\]

Then, we have

\[
\dot{V} = 2a_1 \bar{y}^T H \bar{y} - 2\gamma \bar{y}^T \bar{y} \\
+ \hat{\eta}^T (I_N \otimes (PA + A^T P)) \hat{\eta} \\
+ \chi^T (I_N \otimes (QA^T + AQ)) \chi \\
+ 2\bar{\theta}^T (I_N \otimes E^T) \hat{\eta} + 2\bar{\theta}^T H \chi_d^T \left[ I_N \otimes (F^T \theta) \right] \\
+ 2\chi^T (I_N \otimes QB) \bar{y} + 2\chi^T (I_N \otimes QE) \bar{y} \\
+ 2\bar{\theta}^T \chi_d^T (I_N \otimes F^T) \bar{\theta} - 2\bar{\theta}^T (I_N \otimes F) \chi_d H \bar{y}. \tag{22}
\]

From (4), we have

\[
\begin{align*}
\dot{V} &= 2(1 - \mu) \bar{y}^T H \bar{y} + 2a_1 \bar{y}^T H \bar{y} + ||\bar{y}||^2 \bar{y}^2 \\
&\quad + \left( ||P||^2 ||B||^2 + ||H||^2 ||\theta||^2 \right) \bar{y}^2 \\
&\quad - ||\bar{y}||^2 - ||\chi||^2 \\
&\quad \leq 2 \left( a_1 \lambda_1 + 2a_1^2 \lambda_2^2 + 2\lambda_2^2 ||B||^2 + \lambda_2^2 \right) \bar{y}^2 - ||\bar{y}||^2 - ||\chi||^2.
\end{align*}
\]

From (5), we have

\[
\dot{V} \leq -||\bar{y}||^2 - ||\hat{\eta}||^2 - ||\chi||^2 \leq 0.
\]

Since \( V \) is positive definite and \( \dot{V} \) is negative semi-definite, \( V \) is bounded, which means \( \bar{y}, \chi, \hat{\eta}, \hat{\chi} \) and \( \bar{\theta} \) are all bounded. From (16), \( \bar{y}, \hat{\eta}, \chi \) and \( \bar{\theta} \) are bounded, which implies that \( \dot{V} \) is bounded. According to Barbalat’s Lemma in (34), we have

\[
\lim_{t \to \infty} \dot{V}(t) = 0,
\]

which implies (17), (18) and (19). Thus, by (15), we have

\[
\lim_{t \to \infty} z(t) = 0.
\]
which together with (16c) yielding $\lim_{t \to \infty} \dot{\theta} = 0$. To show (20), differentiating $\dot{y}$ gives,

$$
\dot{y} = (I_N \otimes E^T) \dot{y} + a_1 \dot{y} + \frac{\dot{\chi}_T}{\chi} (E_N \otimes (F^T \theta)) - \mu H \dot{y} + \frac{\dot{\chi}_T}{\chi} (I_N \otimes F^T) \dot{\theta} + \frac{\dot{\chi}_d}{\chi} (I_N \otimes F^T) \dot{\theta}.
$$

(25)

We have shown that $\dot{\eta}_i$, $\dot{y}_i$, $\dot{\chi}_i$, $\dot{\theta}$, $\dot{\chi}_d$, and $\dot{\theta}$ are all bounded. Thus, $\dot{y}$ is bounded. By using Barbalat’s Lemma in [34] again, we have $\lim_{t \to \infty} \dot{y}(t) = 0$, which together with (17), (18) and (19) implies $\lim_{t \to \infty} \dot{\chi}_d(t) (I_N \otimes F^T) \dot{\theta}(t) = 0$.

Lemma 1 does not guarantee $\lim_{t \to \infty} \dot{\theta}(t) = 0$. It is possible to make $\lim_{t \to \infty} \dot{\theta}(t) = 0$ if the signal $(I_N \otimes F) \dot{\chi}_d(t)$ is persistently exciting. We also need the following result which is taken from Lemma 2.4 of [35].

**Lemma 2.** Consider a continuously differentiable function $g : [0, +\infty) \rightarrow \mathbb{R}^n$ and a bounded piecewise continuous function $f : [0, +\infty) \rightarrow \mathbb{R}^n$, which satisfy $\lim_{t \to \infty} g(t) f(t) = 0$. Then $\lim_{t \to \infty} g(t) = 0$ holds under the following conditions:

(i) $\lim_{t \to \infty} g(t) = 0$

(ii) $f(t)$ is persistently exciting.

**Proof.** (i): For the system (12b), $A^T \in \mathbb{R}^{(2l-1) \times (2l-1)}$ is a stable matrix, and the input $y_0(t)$ is sufficiently rich of order 2l. Following the same steps as in [33], and using Theorem 2.7.2 in [26], it is shown in [33] $F(\chi)$ is persistently exciting. From (18) in Lemma 1 we have

$$
\lim_{t \to \infty} \dot{\chi}_d(t) = 0,
$$

Then, by Lemma 3.2 in [29], we will have $F(\chi_i)(t)$ is PE for $i = 1, \ldots, N$.

(ii): From (20) in Lemma 1 we have

$$
\lim_{t \to \infty} \dot{\chi}_i(t) F^T \dot{\theta}(t) = 0.
$$

Since $F(\chi_i)(t)$ is PE, we have $\lim_{t \to \infty} \dot{\theta}_i(t) = 0$ for $i = 1, \ldots, N$ according to Lemma 2.

**B. State estimation error analysis**

We will show that $\hat{v}_i$ in (10b) is an estimate $v$ in (11). Let $\hat{v}_i = \hat{v}_i - v$. Then we have the following equations

$$
\dot{\hat{v}}_i = (M - LC) \hat{v}_i + L \dot{y}_i - \sum_{k=1}^{l} \dot{\theta}_{i,k} E_{2k} \dot{y}_i - \theta_k E_{2k} \dot{y}_i
$$

$$
= (M - LC) \hat{v}_i + L \dot{y}_i
$$

$$
- \sum_{i=1}^{l} \dot{\theta}_{i,k} E_{2k} \dot{y}_i = (M - LC) \hat{v}_i + L \dot{y}_i
$$

for $i = 1, \ldots, N$. 

**Lemma 4.** Consider the systems (2), (6), (10a) and (12). For any $\hat{y}_i(z \chi)(0)$, $(\hat{v}_i(0)$, $\theta(0)$, if $\gamma > 0$ and $\mu$ satisfies (3), then

$$
\lim_{t \to \infty} \hat{v}_i(t) = 0, \quad i = 1, \ldots, N.
$$

**Proof.** By Lemma 1 we have both $\hat{\theta}_i(t)$ and $\hat{y}_i(t)$ converge to zero as $t \to \infty$, for $i = 1, \ldots, N$. As $M - LC$ is a Hurwitz matrix, the system (26) could be viewed as a stable system with $\sum_{k=1}^{l} (\hat{\theta}_{i,k} E_{2k} \hat{y}_i + \theta_k E_{2k} \hat{y}_i)$ and $L \tilde{y}_i$ as the inputs. The inputs are bounded for all $t \geq 0$ and tend to zero as $t \to \infty$. We conclude that $\hat{v}_i(t)$ will decay to zero as $t \to \infty$, for $i = 1, \ldots, N$.

**C. Output consensus analysis**

The dynamics of the leader system (11) are different from the dynamics of the follower systems (3). Motivated by the output regulation theory in [37], [38], the regulator equations associated with follower $i$ in (3) and the leader system in (11) are defined by (7).

Let $f(v, \theta) = \text{col}(f_1(v, \theta), f_2(v, \theta), \cdots, f_r(v, \theta))$. In order to solve the problem, we perform the following coordinate transformation:

$$
\tilde{x}_i = x_i - f(v, \theta),
$$

where $\tilde{x}_i = \text{col}(\tilde{x}_{i,1}, \cdots, \tilde{x}_{i,r})$ for $i = 1, \ldots, N$. Then, we can obtain the following error system

$$
\dot{\tilde{x}}_i = \tilde{x}_{i,s+1}, \quad s = 1, \ldots, r - 1,
$$

$$
\dot{\tilde{x}}_i = u_i - f_{r+1}(v, \theta).
$$

Let us first assume that $\theta$ and $v$ are known by each follower. Under the control law (10b):

$$
\dot{\tilde{x}}_i = \sum_{s=1}^{r+1} \alpha_s f_s(v, \theta) - \sum_{s=1}^{r} \alpha_s \tilde{x}_{i,s} - f_{r+1}(v, \theta).
$$

the augmented system becomes

$$
\dot{\tilde{x}}_i = \Phi \tilde{x}_i,
$$

where

$$
\Phi = \begin{bmatrix}
0 & I_{r-1} \\
-\alpha_1 & -\alpha_2 & \cdots & -\alpha_r
\end{bmatrix}.
$$

The system (29) is exponentially stable due to the selection of $\alpha_1, \cdots, \alpha_r$, which implies

$$
\lim_{t \to \infty} \tilde{x}_{i,1}(t) = \lim_{t \to \infty} (y_i(t) - y_0(t)) = 0,
$$

for $i = 1, \ldots, N$. Thus, the control law (10) solves Problem 1 if $\theta$ and $v$ are known by each follower.

**Remark 3.** In our problem formulation, $\theta$ and $v$ are unknown to any follower. Only the output of the leader is available to some followers. The distributed control law proposed in [3] is not applicable here.
Thus we have \( \lim_{t \to \infty} v_i, \theta_i = 0 \) and for any \( \hat{\eta}(0), \hat{\chi}(0), \theta(0), \hat{y}(0), \hat{v}_i(0), x_i(0) \) and for \( i = 1, \ldots, N \) if \( \gamma > 0 \) and \( \mu \) satisfies (5), Problem [2] is solved by the distributed adaptive observer [6] and the distributed control law (10).

**Proof.** Substituting (10b) into the error system (27b) yields the following system

\[
\dot{x}_i = \Phi \hat{x}_i + g_i(t), \quad i = 1, \ldots, N,
\]

where \( \Phi \) is defined in the form (30), and

\[
g_i(t) = D \sum_{k=1}^{r+1} \alpha_k \left( f_k(\hat{v}_i, \hat{\theta}_i) - f_k(v, \theta) \right)
\]

where \( D = \text{col}(0_{r-1 \times 1}, 1) \). For \( i = 1, \ldots, N \), we can rewrite \( g_i(t) \) as

\[
g_i(t) = D \sum_{k=1}^{r+1} \alpha_k \left( f_k(\hat{v}_i + v, \hat{\theta}_i + \theta) - f_k(v, \theta) \right)
\]

By Lemma [1] Lemma [3] and Lemma [4] for any \( \hat{\eta}(0), \hat{\chi}(0), \dot{\hat{y}}(0), \hat{v}(0), \theta(0) \), if \( \gamma > 0 \) and \( \mu \) satisfies (5), \( \hat{v}(t) \) and \( \hat{\theta}(t) \) are bounded for all \( t \geq 0 \) and satisfy

\[
\lim_{t \to \infty} \hat{v}_i(t) = 0, \quad \lim_{t \to \infty} \hat{\theta}_i(t) = 0.
\]

Thus we have \( \lim_{t \to \infty} g_i(t) = 0 \), for \( i = 1, \ldots, N \). As \( \Phi \) is a Hurwitz matrix, the system (31) can be viewed as a stable system with \( g_i(t) \) as the input. Since this input is bounded for all \( t \geq 0 \) and tends to zero as \( t \to \infty \), we can conclude that \( \lim_{t \to \infty} \hat{x}_i(t) = 0 \) which implies

\[
\lim_{t \to \infty} \hat{x}_{i,1}(t) = 0,
\]

for \( i = 1, \ldots, N \). Therefore, Problem [1] is solved.

\[\square\]

V. SIMULATION

Consider a multi-agent systems with five followers, and the underlying communication topology is shown in Fig. [1]. The output signal of the leader is generated by the following system

\[
y_0 = \varphi_1 \sin(\omega_1 t + \psi_1) + \varphi_2 \sin(\omega_2 t + \psi_2),
\]

where \( \varphi_1, \varphi_2 \) are arbitrary unknown positive real numbers, \( \omega_1 \neq \omega_2 \) with \( \omega_1, \omega_2 \in (0, 1], \psi_1 \) and \( \psi_2 \) are arbitrary unknown real numbers.

The dynamics of the followers are given below:

\[
\dot{x}_{i,2} = u_i, \quad y_i = x_{i,1},
\]

for \( i = 1, \ldots, 5 \). The output signal (33) of the leader can be generated by the following linear model of order four

\[
\dot{v} = M v - \theta_1 E_2 y_0 - \theta_2 E_4 y_0 , \quad y_0 = C x,
\]

(35)

where \( \theta_1 = \omega_1^2 + \omega_2^2 \) and \( \theta_2 = \omega_1^2 \omega_2^2 \) being an invertible reparametrization of the unknown positive frequencies, \( E_2 = \text{col} (0, 1, 0, 0) \) and \( E_4 = \text{col} (0, 0, 0, 1) \). Thus \( \pi = \max_{\omega_1, \omega_2 \leq 1} \| \theta \| = \sqrt{\pi} \).

Choose the vector \( \alpha = \text{col} (1, 2, 0.7) \). The matrices \( P \) and \( Q \) can be found by solving Lyapunov equations (4)

\[
P = \begin{bmatrix}
5.1099 & 3.6099 & 2.1429 \\
3.6099 & 11.6868 & 5.7198 \\
2.1429 & 5.7198 & 6.8126
\end{bmatrix}
\]

Figure 1. Communication topology \( \hat{G} \)

Figure 2. Trajectory of \( e_i, i = 1, \ldots, 5 \)

Figure 3. Trajectory of \( y_i, i = 0, 1, \ldots, 5 \)
θ distributed adaptive observer directly provides the estimate of can estimate the unknown parameters θ is sufficiently rich of order of each agent, for errors et to the origin as time (between 0 γ law can be designed in the form of (10) with µ ϕ ̂ Figure 5. Trajectory of \|\hat{\theta}_i\|, i = 1, \ldots, 5.

Hence, we have μ ≥ 2982 from \(3\). The distributed control law can be designed in the form of \(10\) with μ = 3 \times 10^3, γ = 8 \times 10^3, α_1 = 4, α_2 = 4 and \(L = \text{col}(5, 19, 15, 14)\). Simulation is conducted with the following initial conditions: ϕ_1 = 1, ϕ_2 = 1, ψ_1 = 0, ψ_2 = 0, \(\chi_i(0) = 0\), \(\eta_i(0) = 0\), \(\chi_i(0) = 0\), and \(x_i(0)\), \(y_i(0)\), \(\hat{\theta}_i(0)\) are randomly chosen \((-2, 2)\), i = 1, \ldots, 5. Figure 2 shows the tracking errors \(e_i(t) = y_i(t) - y_0(t)\) for i = 1, \ldots, 5, which all converge to the origin as time t \to \infty. Figure 3 shows the output \(y_i(t)\) of each agent, for i = 0, \ldots, 5.

The actual value of ω is ω = col(1, 0.5). Hence the signal y_0 is is sufficiently rich of order 4. Thus, our distributed observer can estimate the unknown parameters θ asymptotically. The distributed adaptive observer directly provides the estimate of \(θ = (\theta_1, \theta_2)\) which are related to the estimates of \(\omega_1\) and \(\omega_2\). Figure 4 shows the trajectories of \(∥\hat{\theta}_i(t)∥, i = 1, \ldots, 5\), which all converge to the origin as time t \to \infty. Figure 5 shows the trajectories of \(\omega_i(t), i = 1, \ldots, 5\), which, as expected, all converge to the actual value of ω.

VI. Conclusions

In this paper, we have studied the leader-following output consensus problem of multi-agent systems subject an unknown leader system, which consists of the sum of l sinusoids with unknown amplitudes, frequencies and phases. This distributed adaptive observer are established for all followers to estimate both the parameters and the states of the leader through the communication network of the system. Based on this distributed adaptive observers, we have further synthesized a distributed adaptive control law for solving the output consensus problem. The leader-following output consensus problem is solved even if none of the followers knows the parameters of the leader system.

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