Correlation effects in the presence of a spin bath

Álvaro Gómez-León
Instituto de Ciencia de Materiales, CSIC, Cantoblanco, Madrid E-28049, Spain

Tim Cox and Philip Stamp
Department of Physics and Astronomy, University of British Columbia,
6224 Agricultural Road, Vancouver, British Columbia, V6T 1Z1, Canada

(Dated: November 20, 2018)

In this work we analyze the effect of a bath of spins interacting with a quantum Ising model in terms of a hierarchy of correlations. We show that this formalism can be used with general spin systems and baths, and discuss the concrete case of a transverse Ising model coupled to a bath. We show how mean field results change when correlations are added and the properties of the quasiparticle excitations in terms of the density of states and self-energy.

Introduction: Spins systems are among the most studied models in physics, but their properties are also closely related with systems in many other fields of science [1–3]. These models are important in physics because they show collective effects and quantum phase transitions, and in addition, their importance in the development of quantum computers has played a central role during recent years [4]. Previous works on spin models mainly considered closed systems and studied the equilibrium phase diagrams, critical exponents, and more recently, the study of quench dynamics [5–7]. Less is known about these models in the presence of environments, specially for those dealing with localized modes, as it is known that they can produce radically different effects than the usual oscillator bath [8–10]. Also, as the coupling to spin environments is generally not weak [11], the calculations are generally non-perturbative and must be done carefully, which implies that in the case of quantum models, the effect of correlations can become quite important.

For practical purposes, the study of spin models has recently become more important due to the development of the first quantum simulators and quantum computers [10–12]. In this case, environments are ubiquitous and its interaction with the quantum computer during the computation time needs to be properly understood — e.g., the fate of the quantum critical point (QCP) during the quantum annealing process, in the presence of an environment. In the case of quantum computers made of flux qubits, spin environments can be produced by paramagnetic impurities and nuclear isotopes in the substrate among others, and even if they weakly couple, their effect in the dynamics can be non-trivial [13]. Also it has been shown that for superconducting qubits dielectric loss can dominate, and it is produced by effective two level systems [14].

To understand the physical effect of spin bath environments we study a generalized Ising model coupled to a set of two-level systems. We use the equation of motion (EOM) technique for the Green’s functions, and a decoupling scheme based on a hierarchical expansion in terms of correlations. This approach transforms the EOM into a Dyson’s equation for the Green’s function, with explicit non-perturbative expressions for the self-energy that include a full frequency dependence. Hence it can also be used to study quantum dynamics of excited states, as already shown in [12]. The lowest order solution agrees with mean-field (MF) treatments and can explain some general features of the effect of arbitrary baths of spins, as the solutions are solely characterized by the connectivity between central spins and bath spins (i.e., the three different coordination numbers that one can define for this model). Higher order corrections depend on the specific details of the bath, however, we numerically solve the self-consistent equations for some specific cases and demonstrate that quantum correlations tend to suppress the effect of the bath of spins which was previously obtained in MF calculations. Furthermore, we show that the entanglement between Ising and bath spins produces an extra broadening of the quasiparticle excitation spectra and a large shift in energy for the self-energy.

Method of the equations of motion: We consider a model with a central system and a bath. The central system is made of a set of interacting spins coupled to an external field \( \vec{B} \). Similarly, the bath is made of a set of spins as well, but we assume that these are non-interacting, although they can be affected by an external field \( \vec{B} \). The total Hamiltonian is \( H = H_S + H_B \), where \( H_S \) and \( H_B \) are given by (lowercase Greek indices specify the direction of the spin vector and sum over repeated indices is assumed):

\[
H_S = -\sum_i B_{\mu i} S_i^{\mu} - \sum_{i,j > i} V_{i,j}^{\mu,\nu} S_i^{\mu} S_j^{\nu} \\
H_B = -\sum_i B_{\mu i} I_i^{\mu} + \sum_{i,l} A_{i,l}^{\mu,\nu} S_i^{\mu} I_l^{\nu}
\]

Here \( H_S \) corresponds to a generalized Ising model with arbitrary interaction \( V_{i,j}^{\mu,\nu} \) and field \( \vec{B} \), and \( H_B \) corresponds to the bath Hamiltonian, which also contains arbitrary fields \( \vec{B} \) and it is coupled to the central system.
via $A^{\mu,\nu}_{n, l}$ (This can represent the hyperfine interaction or a more general type of coupling). For simplicity of the calculations we will work in the Majorana representation for half-integer spins $[12]$: 

$$S^\alpha_n = -\frac{i}{2} \epsilon_{\alpha \theta_1 \theta_2} \gamma^\alpha_n \phi^\theta_1 \phi^\theta_2 (3)$$

$$I^\alpha_n = -\frac{i}{2} \epsilon_{\alpha \theta_1 \theta_2} \gamma^\alpha_n \phi^\theta_1 \phi^\theta_2 (4)$$

$\eta^\alpha_n$ being a Majorana fermion at site $n$ that fulfills the usual anti-commutation relation for fermions, and in addition $\eta^2 = 1/2$. Similarly the $\gamma^\alpha_n$ refer to the Majorana fermions for the bath spins. Nevertheless, the final results will be presented in terms of physical observables such as the magnetization, as this local mapping preserves the angular momentum algebra and the rotational symmetry.

To study the properties of the system we define the double-time Green’s functions $[17]$:

$$G_{n, m}^{\alpha, \beta}(t, t') = -i\langle \eta^\alpha_n(t); \eta^\beta_m(t') \rangle (5)$$

$$P_{n, m}^{\alpha, \beta}(t, t') = -i\langle \gamma^\alpha_n(t); \gamma^\beta_m(t') \rangle (6)$$

where $;$ implies the Green’s function may be a time ordered, retarded or advanced Green’s function. They all fulfill the same equation of motion $\partial_t (\eta^\alpha_n; \eta^\beta_n) = \delta(t - t') \delta_{n, m} \delta_{\alpha, \beta} + i\langle [H, \eta^\alpha_n]; \eta^\beta_n \rangle$, but with different boundary conditions, ($H$ being expressed in the Majorana representation). These Green’s functions are related to the on-site magnetization via Eq.3. It is also worth mentioning that other choices of Green’s functions, in terms of standard spins, would give similar results $[18, 19]$.

For time-independent systems it is easier to work with Heisenberg’s equations of motion in frequency domain:

$$\omega G_{n, m}^{\alpha, \beta} = \delta_{n, m} \delta_{\alpha, \beta} + iB_{\mu} \epsilon_{\mu \alpha \beta} \epsilon_{\gamma \delta} \sum_{i \neq n} V_{i, n}^{\mu, \nu} G_{i, n, m}^{\gamma, \delta}$$

$$+ \frac{1}{2} \epsilon_{\mu \nu \gamma \delta} \sum_{i \neq n} A_{i, n}^{\mu, \nu} Y_{l, n, m}^{\gamma, \delta}$$

$$- \frac{1}{2} \epsilon_{\mu \nu \gamma \delta} \sum_{i \neq n} A_{i, n}^{\mu, \nu} Y_{l, n, m}^{\gamma, \delta}$$

where $-i\langle \eta^\alpha_i; \eta^\beta_i \rangle = \gamma^{\alpha, \beta}_{l, n, m}$ is the system-bath correlator and $G^{\alpha, \beta}_{l, n, m}$ the spin-spin correlator. Eq.4 and the equivalent equation for $P_{n, m}^{\alpha, \beta}(t, t')$ are the main equations analyzed in this work (both shown in the Appendix).

An important feature of many-body systems is the hierarchical structure of the EOM, where interaction terms produce higher-order Green’s functions. To deal with the hierarchy of equations one needs to devise a method to decouple the infinite set of equations into smaller blocks, which correctly captures the regime of interest. For that purpose we separate the Green’s functions into their correlated and uncorrelated parts, as e.g. $\langle \eta^\alpha_n \eta^\beta_m; \eta^\alpha_n \rangle = \langle \eta^\alpha_n \eta^\beta_m \rangle \langle \eta^\alpha_n; \eta^\beta_m \rangle + \langle \eta^\alpha_n \eta^\beta_m; \eta^\alpha_n \rangle C$.

Here, the uncorrelated part encodes the local properties of the system, while the correlated part $\langle \eta^\alpha_n \eta^\beta_m; \eta^\alpha_n \rangle C$ (defined as the difference between the total and the uncorrelated) encodes the properties that depend on non-local correlations between spins (i.e., many-body effects due to correlations between different spins). This separation is completely general and just transforms the initial EOM for the two-point function in two coupled equations, one for the uncorrelated and one for the correlated part.

Our approximation to close the system of equations will assume that higher correlated parts scale in some particular way with the parameters of the system, such that they can be neglected when they are small enough $[20, 21]$. Here we will organize terms in inverse powers of the coordination number $Z$. Doing this, it is possible to systematically include higher order corrections by simply adding higher order correlated parts.

It must be mentioned that the scaling of correlations converges very slowly in 1D (it goes as $\sim Z^{-1}$, with $Z = 2$), and that the correction to the critical point is typically overestimated by two-point correlations, which should be compensated by higher order correlation functions $[31]$. Thus, although the phase diagram would not be fully captured in 1D, it is still an interesting case, because in low dimensional systems quantum corrections are usually enhanced. This facilitates the study of quantum corrections due to the spin bath. For this reason we will consider the 1D case for the numerical results, but one must keep in mind that the equations are general for arbitrary dimension, and that the 1D results are qualitative, but useful to understand the role of quantum corrections to MF calculations ($Z^{-1}$ decoupling becomes more accurate in systems with large coordination numbers and long-range interactions). For accurate results in low dimensional cases, the present technique can be combined with other fermionization or bosonization techniques $[8]$, and straightforwardly applying the same Green’s function analysis.

Let us apply this decoupling scheme to Eq.7. The separation into correlated and uncorrelated parts leads to:

$$\omega G_{n, n}^{\alpha, \beta} = \delta_{\alpha, \beta} + i\epsilon_{\mu \alpha \beta} h^n_{\mu} (n) G^{\alpha, \beta}_{l, n}$$

$$+ \frac{1}{2} \epsilon_{\mu \nu \gamma \delta} \sum_{i \neq n} V_{i, n}^{\mu, \nu} G^{\gamma, \delta}_{l, n, n}$$

$$- \frac{1}{2} \epsilon_{\mu \nu \gamma \delta} \sum_{i \neq n} A_{i, n}^{\mu, \nu} Y^{\gamma, \delta}_{l, n, n}$$

where $h^n_{\mu}(n) = B_{\mu} + \sum_{i \neq n} V_{i, n}^{\mu, \nu} M_{i, n}^{\nu} - \sum_{i} A_{i, n}^{\mu, \nu} m_{i, n}^{\nu}$ and the calligraphic Green’s functions correspond to the correlated parts of the two-point correlators defined above. Notice that Eq.8 contains the bath magnetization $m_{i, n}^{\nu}$ so that the EOM for $P_{l, l}^{\alpha, \beta}$ is also required:

$$\omega P_{l, l}^{\alpha, \beta} = \delta_{\alpha, \beta} + i\epsilon_{\mu \alpha \beta} h^n_{\mu} (l) P^{\alpha, \beta}_{l, l}$$

\[9\]
where \( h_i^\alpha (l) = B_s - \sum_j A_i^{\alpha \beta} M_j^\beta \).

Finally, before we study some specific cases, we discuss a more compact notation that will be used for the rest of the manuscript. In general, one can rewrite the equations of motion in matrix form:

\[
[\omega - \mathbf{H}_s (n)] \hat{G}_{n,n} = \delta + \sum_{i\neq n} \hat{V}_{i,n} \cdot \hat{G}_{i,n} + \sum_l \hat{A}_{l,n} \cdot \hat{\gamma}_{i,n} \\
[\omega - \mathbf{H}_b (l)] \hat{P}_{l,l} = \delta + \sum_l \hat{A}_{l,l} \cdot \hat{R}_{l,l}
\]

where the uncorrelated Hamiltonians:

\[
\mathbf{H}_s = \begin{pmatrix} 0 & -i h^x_y & -i h^y_x \\ -i h^x_y & 0 & i h^x_z \\ -i h^y_x & -i h^x_z & 0 \end{pmatrix} \\
\mathbf{H}_b = \begin{pmatrix} 0 & i h^x_b & -i h^y_b \\ -i h^x_b & 0 & i h^y_b \\ i h^y_b & -i h^x_b & 0 \end{pmatrix}
\]

represent the propagation of a single particle excitation in a fixed background. If one defines the unperturbed solutions as \( \hat{g}_{n,n} = [\omega - \mathbf{H}_s (n)]^{-1} \) and \( \hat{p}_{l,l} = [\omega - \mathbf{H}_b (l)]^{-1} \) the EOM resembles a Dyson’s equation, where as we will show below, studying the EOM for the correlated parts allows to straightforwardly define a self-energy. This is due to the fact that the EOM for the correlated parts are always sourced by the uncorrelated ones, and therefore the solution can be expressed as \( \hat{G}_{n,n} \propto \hat{G}_{i,n} \cdot \hat{G}_{i,n} \).

**Solution neglecting correlations:** The simplest analysis for the model neglects correlations between different spins in Eq.\( 10 \). Then the solution for the Green’s functions are given by a simple matrix inversion. We find poles at \( \omega_s (n) = \sqrt{(\sum_{\alpha} h_{\alpha}^s (n))^2} \) and \( \omega_b (l) = \sqrt{(\sum_{\alpha} h_{\alpha}^b (l))^2} \). As the Green’s functions are functions of the local magnetization \( M^\alpha_s \) and \( m^\alpha_l \), they need to be determined self-consistently. Physically, this happens due to the non-linearities introduced by the interaction term. Defining the spectral function for the system \( J_{s,\alpha,\beta} (\omega) = i \left[ g_{n,n}^{\alpha,\beta}(\omega + i\epsilon) - g_{n,n}^{\alpha,\beta}(\omega - i\epsilon) \right] (\epsilon \hat{\tau} + 1)^{-1} \) and similarly for the bath \( J_{b,\alpha,\beta} (\omega) \), the self-consistency equation is obtained from the relation between the statistical average and the spectral function \( M_{\alpha} = -i \frac{\omega_{\alpha}}{2\pi} \int J_{s,\alpha,\beta} (\omega) \, d\omega / 2\pi [17] \). The previous solution for the Green’s functions, in combination with the solutions of these non-linear equations which characterize the local magnetizations \( M_{\alpha} \) and \( m_{\alpha} \), fully solve the system when correlations can be neglected. Furthermore, as in this uncorrelated case the Green’s functions display simple poles only, one can rewrite the integral over frequency as a sum over poles, and the final result for the self-consistency equations is:

\[
M_{\alpha} = \frac{h_{\alpha}^s (n)}{2\omega_s (n)} \tanh \left( \frac{\omega_s (n)}{2T} \right)
\]

The previous result is general for quantum spin systems coupled to a spin bath, if correlations are small enough that can be neglected. As a more specific case, lets consider the ferromagnetic quantum Ising model, longitudinally coupled to a bath of spins. In this case the poles are given by \( \omega_s = \sqrt{B_s^2 + (Z_s V^{zz} M_z - Z_B A^{zz} m_z)^2} \) and \( \omega_b = Z_{BS} A^{zz} M_z \) (here \( V^{zz} \) and \( A^{zz} \) correspond to the bare interaction between two neighboring Ising spins and between an Ising spin and a bath spin, respectively), and the self-consistency equations directly obtained from Eq.\( 13 \). Notice that we have assumed homogeneous magnetization to simplify the expressions, which holds if the system has unbroken translational symmetry. Importantly, when correlations are neglected, the physics is dominated by three coordination numbers or connectivities: \( Z_S \) corresponds to the number of spins which directly interact in the Ising model, \( Z_B \) to the number of bath spins that each Ising spin interacts with, and \( Z_{BS} \) to the number of Ising spins that each bath spin interacts with. The coordination number \( Z_S \) is well known in mean-field analysis of the Ising model, but with the addition of the bath of spins, one now needs two additional coordination numbers to fully characterize the bath at the mean-field level.

An obvious question now is how the different phase transitions are affected by the bath. To obtain the Curie temperature, \( T_c \) we fix \( B_c = 0 \) and expand to third order in powers of \( M_z \). The solution for \( M_z = 0 \) is found to be:

\[
T_c = \frac{Z_S V^{zz}}{8} + \sqrt{\left( \frac{Z_S V^{zz}}{8} \right)^2 + Z_B Z_{BS} \left( \frac{A}{4} \right)^2}
\]

It is easy to see that \( T_c (A \rightarrow 0) \rightarrow V_0 / 4 \), in agreement with the Curie-Weiss law for the Ising model. Similarly to find \( B_c \) we take the limit \( T \rightarrow 0 \) and expand in powers of \( M_z \):

\[
M_z \simeq \frac{Z_B A^{zz}}{4 \sqrt{B^2 + \left( \frac{A^{zz} V^{zz}}{2} \right)^2}} + \frac{B^2 Z_B V^{zz} M_z}{2 \left[ \frac{B^2 + (A^{zz}/4)^2}{2} \right]^{3/2}} + O (M_z^2)
\]

This shows that \( M_z (T = 0) = 0 \) is not a solution and the quantum phase transition (QPT) between the ferromagnetic and paramagnetic phases is blocked due to a remnant magnetization induced by the bath. This means, strictly speaking, that there is no QPT, although transverse terms in the interaction can recover the QCP [22]. It is important to notice that all three coordination numbers appear in the result for the critical temperature, while \( Z_{BS} \) is absent in the expression for the QPT point. Also it is worth mentioning that large corrections to \( T_c \) are expected due to correlations, but in this work we are mainly interested in the \( T = 0 \) quantum regime.
Effect of correlations: In general, quantum systems consist of a set of interacting particles, and although their interactions may be weak, the correlations generated by these interactions may still have important consequences. For example, it is well known that near classical and quantum phase transitions, many-body systems display macroscopic correlations, at length scales much larger than the typical scales present in the microscopic model. This is a particular case of emergence, where the system behaves in a different way than the particles in the underlying microscopic model. Therefore, it is clear that in some cases correlations are important, and results where correlations are neglected will be significantly affected. Another important case, present in low dimensional quantum systems, is when due to confinement, the role of quantum fluctuations is enhanced. A well known consequence of this is the absence of certain phase transitions in low dimensional models. Below we will show how one can generally introduce correlations in the previous approach for arbitrary spin models, and how doing this is equivalent to find an explicit expression for the self-energy.

To include the effect of correlations we need to solve the EOM for the correlated part of the Green’s function
\[ G^A_{in,n} = G_0^A_{in,n} - \langle \eta_i, \eta_n \rangle G_0^{\sigma,0}_{n,n} \]
This leads in general to couplings to higher order Green’s functions, but typically it is possible to neglect them at some order. Here we consider first order corrections only (i.e., terms proportional to \( Z^{-1} \) which characterize coherent flips of pairs of spins), and show that this captures important features that qualitatively describe the effect of the spin bath. In general the EOM for the correlated part can be written as a matrix equation of the form of integrals over continuum sets of poles. Then, one expects to find more complex structures than the simple poles obtained in the uncorrelated case. The remaining correlated parts \( \hat{Y} \) and \( \hat{R} \) also share the previously discussed properties, and all the EOM are explicitly calculated in the Supplemental Material.

Although the formal description of the previous solutions seems quite simple, a full solution can be a challenging task due to the self-consistency equations, and in general, one needs to make use of numerical methods. In this work we implement a recursive method that begins from the exact uncorrelated solution as a source, and iterates the self-consistency equations until full convergence. Nevertheless, it is also possible to obtain some analytical results by means of perturbative expansions and other approximation methods. In addition, we mention that the case of the classical Ising model (i.e., \( B_z = 0 \)) can be treated analytically, due to the presence of simple poles only [2].

Correlations in the Quantum Ising model with a spin bath: Now we add quantum correlations to the transverse Ising model, as the significance of this model for current quantum computing architectures is crucial (we consider a local spin bath with \( Z_{BS} = 1 \), but other choices are possible). In this case one begins by solving the EOM for the correlated parts of the Green’s functions, which can be done analytically, if higher order correlations are neglected. The main characteristic of these solutions is the appearance of new dispersive quasiparticles, the so called magnons, as poles of the correlated part of the Green’s functions at energy:
\[ \omega_k = \pm \sqrt{B_z^2 - B_z M_z V^{z,z}_k + \omega_0^2} \]
where
\[ \omega_0 = Z_S V^{z,z} M_z - Z_B A^{z,z} m_z \]
being the poles obtained in the uncorrelated case. In contrast with the the uncorrelated poles they are momentum-dependent, and in absence of the bath, the band gap closes at the Ising critical point \( B_c = M_z V^{z,z}_0 \). As previously mentioned, one can analytically obtain certain perturbative results, by for example adding correlations perturbatively to the MF solution. Doing this, one finds the expected result that the Ising bath is always fully polarized for \( T \ll V^{z,z}, A^{z,z} \) and \( B_z \), and that the QCP is lifted because the divergence in the spin-spin correlation function becomes smooth. At this order, system-bath and bath-bath correlations vanish, as they are proportional to \( \frac{1}{T} - m_z^2 \), and only correlations between the Ising spins are non-vanishing.

More interesting and rich is the non-perturbative solution obtained from solving the self-consistency equations, iterating until convergence is reached. In Fig. 1 we compare the phase diagram for the isolated 1D Quantum Ising model (blue) with the case where it is coupled to a bath of spins (yellow). As expected for the isolated case, two-body correlations tend to suppress the ferro
magnetic phase found in the MF solution (green). The presence of correlations modifies the predicted behavior near the QCP and the predicted critical exponents, which is specially clear for the critical exponent $\beta$, due to the different behavior of $M_z$ near $B_c$. More striking is that the effect of the bath spins is largely suppressed due to correlations between spins. Even for large values of $A^{x,z}$, which would produce important changes in the phase diagram of the MF solution, when correlations are included in the calculation one finds only small deviations from the case with no spin bath. In principle, when correlations are added, the QPT is still destroyed by the longitudinal coupling to the bath and the corresponding remnant field that it produces (numerically, when the spin bath is included, $M_z = 0$ does not seem to be a fixed point of the self-consistency equations anymore), however our results indicate that the remnant field is highly renormalized to a smaller value due to correlations. In addition, spin-spin, spin-bath and bath-bath correlations are, in general, all non-vanishing now.

To understand the fate of the quasiparticle excitations once correlations are added, we show in Fig.2 (full line) the longitudinal self-energy $\Sigma^{x,z}(\omega)$ and the density of states (DOS) $\rho_{y,y}(\omega)$ for the two-point function of the Majorana particle $G^{y,y}(\omega)$, when $B_x$ is close to the critical point of the Quantum Ising model with $A^{x,z} = 0$. In general the self-energy displays combinations of two types of excitations: i) single spin-flips with energy $\omega_z$, and ii) magnons with energy $\omega_b$. The single flip excitations appear at energies close to $\pm \omega_z$ and they are similar to the poles of the uncorrelated solution, with the exception that they are slightly shifted and broaden due to interactions. This means that the quasiparticle life-time is reduced and its effective mass renormalized. Processes involving two correlated spins can either leave the total energy unchanged, if each spin flips in opposition to the other, or increase/decrease the total energy by twice the energy of a single flip, if they flip in the same direction.

Furthermore, as these non-local excitations correspond to magnons with certain momenta, they form continuous bands of excitations around $\omega \sim \pm 2\omega_0$ and $\omega \sim 0$, as it is seen in Fig.2 (top). Similarly, the DOS shows that the initial MF pole at $\omega = \omega_0$ splits in two when correlations are included, giving rise to a band of excitations, and to the appearance of new bands around $\omega \sim \{0, \pm 2\omega_0\}$, as expected from the self-energy (The spectral weight for $\rho_{y,y}(\omega \approx 0)$ is very small).

When the bath of spins is present, it produces an Overhauser field on the Ising spins and blocks the QPT. This is observed as a finite shift in the excitation energies, proportional to the coupling strength $A^{x,z}$ and to the number of bath spins for each Ising spin $Z_B$ (i.e., to the strength of the Overhauser field). For the case with moderate coupling and small bath (dashed line in Fig.2), the self-energy is slightly shifted and broadened from the case without a bath, indicating a small but finite Overhauser field. On the other hand, when the bath coupling increases (dotted line in Fig.2), the shift becomes very large and more importantly, the width of the bands is largely reduced. The consequence of this effect on the DOS is clear, it transforms from a highly complicated band of excitations to a Lorentzian, due to the large energy separation between excitation peaks in the self-energy. Finally, these differences between the three cases are largely reduced for large $B_x$, because the longitudinal magnetization is very small in this regime.

**Conclusions:** We have shown how a large variety of spin models interacting with general spin baths can be treated within the same formalism, including arbitrary correlations in a very systematic way. As an example, we have discussed some properties of the transverse Ising model coupled to a local bath of spins, where each Ising spin is coupled to a set of $Z_B$ independent two-level systems. This case is interesting due to the radical changes produced by the longitudinal coupling on its critical behavior. We have found that a strong suppression of the Overhauser field is produced by two-point correlations, and the appearance of continuous bands of excitations in the self-energy, whose shift is proportional to the Overhauser field strength. Physically this means that the spin bath can strongly renormalize the effective mass of the spin excitations. We have also found that the life-time of the spin excitations is long for large Overhauser fields, as expected from decoherence experiments where the spin bath initial polarization can be controlled. Then, as it gets reduced, the quasiparticle width gets broadened, indicating the formation of a correlated state with the bath spins, which would affect the standard Ising critical point.

It would be very interesting to apply these method to more complicated spin models, as for example with transverse terms and dipolar interactions, or including more complex connectivity between bath and system (e.g., if a bath spin can interact with more than a single Ising spin, the results would appreciably change).
To conclude, effective models of spin systems interacting with localized modes are ubiquitous in experiments at low temperature. Some examples are: dangling bonds [23], nuclear spins [7, 24, 25], paramagnetic impurities [26, 27], and localized vibrational modes [28]. Therefore a theoretical approach that allows to systematically study them is very relevant for the field.

We would like to thank R. D. McKenzie for helpful discussions. This work was supported by the National Scientific and Engineering Research Council of Canada and A.G-L. acknowledges the Juan de la Cierva program.

[1] E. B.-N. P. L. Krapivsky, S. Redner, A Kinetic View of Statistical Physics, Cambridge University Press, 2010.
[2] S. Lloyd, J. Phys.: Conf. Ser. 302, 012037 (2011).
[3] H. G. Hiscock et al., Proceedings of the National Academy of Sciences (2016).
[4] A. Das and B. K. Chakrabarti, Rev. Mod. Phys. 80, 1061 (2008).
[5] K. Sengupta, S. Powell, and S. Sachdev, Phys. Rev. A 69, 053616 (2004).
[6] S. Sachdev, Quantum Phase Transitions, Cambridge University Press, 2 edition, 2011.
[7] H. M. Rennow et al., Science 308, 389 (2005).
Appendix A: Detailed calculations in the Majorana representation

For the calculation of the correlation functions we have decided to make use of the Majorana representation of spins. The reason is that it allows to directly apply the formalism of Fermionic Green’s functions, and the self-consistently equations are slightly simpler than using the initial spin representation.

The initial Hamiltonians in Eqs.1 and 2 of the main text transform to the next Majorana Hamiltonian when the transformations are applied (Eqs.3 and 4):

\[ H = \frac{i}{2} \epsilon_{\mu \theta_1 \theta_2} B_\mu \sum_{i} \eta_{i}^\dagger \eta_{i}^2 + \frac{1}{4} \epsilon_{\mu \theta_1 \theta_2 \epsilon_{\nu \theta_3 \theta_4}} \sum_{i,j \neq i} V_{i,j}^{\mu,\nu} \eta_i^\dagger \eta_i^2 \eta_j^\dagger \eta_j^2 + \frac{i}{2} \epsilon_{\mu \theta_1 \theta_2} B_\mu \sum_{i} \gamma_{i}^\dagger \gamma_{i}^2 - \frac{1}{4} \epsilon_{\mu \theta_1 \theta_2 \epsilon_{\nu \theta_3 \theta_4}} \sum_{i,j} A_{i,j}^{\mu,\nu} \gamma_i^\dagger \gamma_i^2 \gamma_j^\dagger \gamma_j^2 \]  \hspace{1cm} (A1)

Actually, this Hamiltonian is also interesting within the context of non-Fermi liquids and the SYK model, as it corresponds to interacting Majorana fermions.

For the two-point Majorana functions one has the general equations of motion:

\[ \omega G_{n,n}^{\alpha,\beta} = \delta_{\alpha,\beta} + i G_{n,n}^{\alpha,\beta} \sum_{\mu} B_{\mu} \epsilon_{\mu \alpha \theta} + G_{n,n}^{\alpha,\beta} \frac{1}{2} \sum_{\mu,\nu} \epsilon_{\nu \theta_1 \theta_2} \epsilon_{\mu \alpha \theta} \left( \sum_{i \neq n} V_{i,n}^{\mu,\nu} \langle \eta_i^\dagger \eta_i^2 \rangle - \sum_{l} A_{n,l}^{\mu,\nu} \gamma_l^\dagger \gamma_l^2 \right) \]  \hspace{1cm} (A2)

\[ \omega P_{t,t}^{\alpha,\beta} = \delta_{\alpha,\beta} + i \sum_{\mu} B_{\mu} \epsilon_{\mu \alpha \theta} P_{t,t}^{\alpha,\beta} - P_{t,t}^{\alpha,\beta} \frac{1}{2} \sum_{\mu,\nu} \epsilon_{\nu \theta_1 \theta_2} \epsilon_{\mu \alpha \theta} \sum_{l} A_{l,t}^{\mu,\nu} \gamma_l^\dagger \gamma_l^2 \]  \hspace{1cm} (A3)

\[ \omega R_{t,t}^{\alpha,\beta} = \delta_{\alpha,\beta} + i \sum_{\mu} B_{\mu} \epsilon_{\mu \alpha \theta} R_{t,t}^{\alpha,\beta} - R_{t,t}^{\alpha,\beta} \frac{1}{2} \sum_{\mu,\nu} \epsilon_{\nu \theta_1 \theta_2} \epsilon_{\mu \alpha \theta} \sum_{l} A_{l,t}^{\mu,\nu} \gamma_l^\dagger \gamma_l^2 \]  \hspace{1cm} (A4)

where $G_{n,m}^{\alpha,\beta}$ refers to the Ising spins and $P_{t,t}^{\alpha,\beta}$ to the bath spins propagator. Notice that in the previous equations of motion we have already separated the four-point functions into its correlated and uncorrelated parts, which is a general decomposition. Neglecting the correlated parts lead to the mean-field solutions analyzed in the first part of the manuscript, and since it is a straightforward calculation we will not discuss its details here.

More involved is the calculation of the correlated parts, which requires to calculate the equation of motion for the four-point function and then remove the equation of motion for its uncorrelated part. As an example, for the Ising spins correlation function one would have: $\partial_t G_{\mu \nu \rho \sigma}^{\theta_1 \theta_2} (t, t') = \partial_t G_{\mu \nu \rho \sigma}^{\theta_1 \theta_2} (t, t') - \langle \eta_{\mu \rho}^{\theta_1 \theta_2} \eta_{\nu \sigma}^{\theta_1 \theta_2} \rangle \partial_t G_{\mu \nu \rho \sigma}^{\theta_1 \theta_2} (t, t').$ Furthermore, the equations can be highly simplified by making use of the conservation laws for a steady state solution $\partial_t (\gamma_{\mu \nu}^{\theta_1 \theta_2} \eta_{\nu \sigma}^{\theta_1 \theta_2}) = 0$. By doing this, one arrives to the equations of motion for the correlated parts, which have terms of order $Z^{-n}$ or higher. Neglecting terms of order $Z^{-n}$ ($n > 1$) one finally arrives to the desired equations of motion including correlations between pairs of spins:

\[ \omega G_{\mu \nu \rho \sigma}^{\theta_1 \theta_2 \gamma_\alpha \beta} = -i \sum_{\mu} B_{\mu} \left( \epsilon_{\mu \theta_1 \theta_2} G_{\rho \sigma}^{\theta_1 \theta_2 \gamma_\alpha \beta} + \epsilon_{\mu \sigma_1 \theta_2} G_{\nu \rho}^{\theta_1 \theta_2 \gamma_\alpha \beta} + \epsilon_{\mu \theta_1 \sigma_2} G_{\nu \rho}^{\theta_1 \theta_2 \gamma_\alpha \beta} + \epsilon_{\mu \theta_1 \theta_2} G_{\nu \rho}^{\theta_1 \theta_2 \gamma_\alpha \beta} \right) \]  \hspace{1cm} (A5)

\[ + \frac{1}{2} \sum_{\mu,\nu} \epsilon_{\nu \theta_1 \theta_2} \sum_{k} A_{\mu,k}^{\mu,\nu} \epsilon_{\mu \theta_1 \theta_2} \left( \langle \gamma_{k}^{\theta_1 \theta_2} \eta_{k}^{\gamma_\alpha \beta} \rangle \right) + \sum_{i \neq n} V_{i,n}^{\mu,\nu} \langle \eta_i^\dagger \eta_i^2 \rangle \]  \hspace{1cm} (A6)

\[ + \frac{1}{2} \sum_{\mu,\nu} \epsilon_{\nu \theta_1 \theta_2} \sum_{k} A_{\mu,k}^{\mu,\nu} \epsilon_{\mu \theta_1 \theta_2} \left( \langle \gamma_{k}^{\theta_1 \theta_2} \eta_{k}^{\gamma_\alpha \beta} \rangle \right) + \sum_{i \neq n} V_{i,n}^{\mu,\nu} \langle \eta_i^\dagger \eta_i^2 \rangle \]  \hspace{1cm} (A7)

\[ - \frac{1}{2} \sum_{\mu,\nu} \epsilon_{\nu \theta_1 \theta_2} \sum_{i \neq n} V_{i,n}^{\mu,\nu} \langle \eta_i^\dagger \eta_i^2 \rangle \]  \hspace{1cm} (A8)

\[ - \frac{1}{2} \sum_{\mu,\nu} \epsilon_{\nu \theta_1 \theta_2} \sum_{i \neq n} V_{i,n}^{\mu,\nu} \langle \eta_i^\dagger \eta_i^2 \rangle \]  \hspace{1cm} (A9)
\[-\frac{1}{2} \sum_{\mu,\nu} \epsilon_{\mu} \theta_{\nu} \sum_{i \neq p, n} V_{i, n}^{\mu, \nu} \epsilon_{\nu} \alpha \left( \langle \eta_{p}^{\gamma_{1}, \eta_{p}^{\gamma_{2}}} \langle \eta_{i}^{\theta_{1}} \eta_{i}^{\theta_{2}} \rangle \rangle G_{\gamma_{1}, \gamma_{2}}^{\theta_{1}, \theta_{2}} + \langle \eta_{i}^{\theta_{1}} \eta_{i}^{\theta_{2}} \rangle \sigma_{\gamma_{1}, \gamma_{2}}^{\theta_{1}, \theta_{2}} \right) \]

\[-\frac{1}{2} \sum_{\mu, \nu} \epsilon_{\mu} \theta_{\nu} \sum_{i \neq p, n} V_{i, n}^{\mu, \nu} \left( \sum_{\mu, \nu} \epsilon_{\nu} \alpha \sigma_{\gamma_{1}, \gamma_{2}}^{\theta_{1}, \theta_{2}} \right) \left( C_{\gamma_{1}, \gamma_{2}}^{\theta_{1}, \theta_{2}} + \langle \eta_{i}^{\theta_{1}} \eta_{i}^{\theta_{2}} \rangle \sigma_{\gamma_{1}, \gamma_{2}}^{\theta_{1}, \theta_{2}} \right) \]

\[-\frac{1}{2} \sum_{\mu, \nu} \epsilon_{\mu} \theta_{\nu} \sum_{i \neq p, n} V_{i, n}^{\mu, \nu} \left( \langle \eta_{p}^{\gamma_{1}, \eta_{p}^{\gamma_{2}}} \eta_{i}^{\theta_{1}} \eta_{i}^{\theta_{2}} \rangle \sigma_{\gamma_{1}, \gamma_{2}}^{\theta_{1}, \theta_{2}} \right) \left( C_{\gamma_{1}, \gamma_{2}}^{\theta_{1}, \theta_{2}} - \langle \eta_{i}^{\theta_{1}} \eta_{i}^{\theta_{2}} \rangle \sigma_{\gamma_{1}, \gamma_{2}}^{\theta_{1}, \theta_{2}} \right) \]

\[
\omega_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}} = - \sum_{\mu} B_{\mu} \epsilon_{\mu} \alpha \gamma_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}} + \epsilon_{\mu} \sigma_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}} \gamma_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}}
\]

\[
\omega_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}} = - \sum_{\mu} B_{\mu} \epsilon_{\mu} \alpha \gamma_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}} + \epsilon_{\mu} \sigma_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}} \gamma_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}}
\]

\[
\omega_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}} = - \sum_{\mu} B_{\mu} \epsilon_{\mu} \alpha \gamma_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}} + \epsilon_{\mu} \sigma_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}} \gamma_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}}
\]

\[
\omega_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}} = - \sum_{\mu} B_{\mu} \epsilon_{\mu} \alpha \gamma_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}} + \epsilon_{\mu} \sigma_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}} \gamma_{\gamma_{1}, \gamma_{2}}^{\sigma_{1}, \sigma_{2}}
\]
\[ + \frac{1}{2} \sum_{\mu,\nu} \epsilon_{\mu_1 \theta_2} \sum_i A^\mu,\nu_{i,r} \epsilon_{\nu_0 \theta_2} \left( \langle \eta_{\theta_1} \eta_{\theta_2} \rangle P^\sigma_{r,l,l} + \langle \gamma_\sigma_{r} \gamma_\sigma_r \rangle R^\theta_{r,l,l} \right) \]
\[ + \frac{1}{2} \sum_{\mu,\nu} \epsilon_{\mu_1 \theta_2} \epsilon_{\nu_0 \theta_2} \sum_i A^\mu,\nu_{i,l} \left( \langle \eta_{\theta_1} \eta_{\theta_2} \gamma_\sigma_{r} \gamma_\sigma_r \rangle C_{l,l}^P + \langle \eta_{\theta_1} \eta_{\theta_2} \rangle P^\sigma_{r,l,l} \right) \]

with the addition of the exact equation:
\[ G_{n,n,n}^{x,y,z,\beta} (\omega) = \frac{i}{\omega} (\delta_{x,\beta} M_x + \delta_{y,\beta} M_y + \delta_{z,\beta} M_z) \quad (A5) \]

These equations are exactly solved for the case of the quantum Ising model in the main text, as a function of the two-point correlators \( G_{n,n}^{\alpha,\beta} \) and \( P_{l,l}^{\alpha,\beta} \). Then inserting them in the equation of motion for the two point correlators one can define a self-energy as the correction to the mean-field Hamiltonian, and finally solve the self-consistency equations numerically, until full convergence.