**Supersymmetry and d-Wave Superconductivity**

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Motivated by a recent development in the field theory of the fractional quantum Hall effect, we propose a supersymmetric field theoretical model of quantum critical d-wave and \(d + id\)-wave superconductors. New concept is a composite particle with the supercharge which is formed by electron (hole) and supersymmetric collective configurations of spin and charge. Quantum critical d-wave superconductor is characterized as the condensate of these composite particles.

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**Introduction.** High \(T_c\) superconductor [1] is a strongly correlated system exhibiting strange behaviors which cannot be captured by the Fermi liquid theory where quasiparticles are screened enough to be weakly bound. A key basis of strong correlation is low-dimensionality: the transport property is dominated by the \(\text{CuO}_2\) planes between insulating layers. The most crucial consequence of low dimensionality is the strong Coulomb interaction. In two dimensions, the Coulomb interaction remains much stronger than in three dimensions because the constraint to the plane prevents the electrons (holes) to be screened enough. This is actually the essential cause of the fractional quantum Hall effect (FQHE). There, instead of screening, electrons non-perturbatively capture a vortex, forming composite fermions. This mechanism enables electrons to recoil each other by acquiring non-zero relative angular momentum, whereby the Coulomb energy is lowered. Although FQHE takes place in a strong magnetic field, this logic is by no means restricted to such a situation.

Besides the Coulomb interaction, antiferromagnetic (AF) interaction is believed to be important in high \(T_c\) superconductor. The parent compound of high \(T_c\) superconductor is an insulator with the long-range antiferromagnetic order. Only a small amount of doping is necessary to destroy the order. Doping introduces holes on the \(\text{CuO}_2\) planes which become the charge carrier. From the logic above, these holes should capture a vortex to lower the Coulomb energy. However, as these holes also have spin, it seems sensible to consider that the captured vortex also has the flux of spin (spin flux). Such a flux is carried by spin-texture. This kind of topological doping idea has been considered in some studies [2, 3]. It naturally explains the fact that a small portion of doping is enough to destroy the AF order since such a topological charge can flip the spins globally. Also, in the AF background, to avoid to flip many spins, the configuration with spin flux is expected to have a tendency to form a pair or stripe. In Ref. [2], the capturing mechanism of spin flux is formulated as an abelian Chern-Simons (CS) gauge theory interacting with a complex boson which is akin to the field theory of FQHE. In this model, superconducting phase is realized as the paired condensate of composite fermionic particles with the spin flux. The pairing symmetry is \(p + ip\)-wave. On the FQHE side, this model corresponds to the Halperin 331 state [4].

Various experiments on high \(T_c\) superconductor confirm that the actual pairing symmetry is \(d\)-wave. This pairing symmetry leads to gapless quasiparticles residing at the nodes of the pair wavefunction. Recent photoemission [5] and THz conductivity experiments [6] have indicated an anomalously short lifetime for the fermionic quasiparticles near the nodes. A natural explanation below \(T_c\) is due to a proximity to a quantum phase transition [10] to some other state [11]. The global symmetry and field theoretical considerations [11] restrict the state to be the \(d + id\) state [12]. Under the existence of such a quantum phase transition, the finite temperature property of high \(T_c\) superconductor may be dictated by the quantum critical point. In particular, high temperature may be sustained by the quantum critical point.

It is tempting to ask whether it is possible to synthesize the topological idea with the quantum phase transition. Because the \(d + id\) state is \(T\) violating, the spin flux attachment via Chern-Simons gauge theory may be playing a role. However \(d\)-wave pairing symmetry is outside the reach of abelian Chern-Simons theory. On the FQHE side, an example of \(d\)-wave paired FQH state is known as the Haldane-Rezayi (HR) state [13] which was originally proposed to solve the \(\nu = 5/2\) puzzle [14]. The HR state is quantum critical [15] with a neutral massless excitation of spin 1/2. Recently a supersymmetric quantum field theory of the HR state has been formulated [16]. It is a certain 2+1 dimensional super Chern-Simons gauge theory. In this theory, the topological charge dual to spin is not just a spin flux, but a super flux with an additional fermionic flux associated with the supercharges. These fluxes lead to the quantum critical \(d\)-wave pairing of composite fermions. Quantum criticality is tied to the local supersymmetry breaking in this perspective.

In this paper, we extend these concepts to a model of high \(T_c\) superconductor by extending the spin flux attachment to the super flux attachment. Namely we consider a picture that holes introduced in the \(\text{CuO}_2\) plane by doping will capture a supersymmetric vortex with the super flux. Somewhat puzzling appearance of two abelian gauge symmetries for the HR state noted in [16] turns out to have a natural physical meaning. The \(d + id\) state will be characterized as the condensate of composite particle with the flux associated with these gauge symmetries.
The Theory. For convenience, we will call the charge carrier as electron hereafter. We first present a preliminary observation on the ground state (GS) wave function \( \Psi_{GS}\) of 2N electrons in two dimensions with the Coulomb interaction. The vortex capturing mechanism described above results in the zero point structure of \( \Psi_{GS} \). The most convenient way to express the structure is through Operator Product Expansion (OPE). We assume that \( \Psi_{GS} \) has a following OPE,

\[
\lim_{x_1 \to x_2} \Psi_{GS}(x_1, x_2, \cdots) \sim |x_1 - x_2|^l \Phi_{12}^{GS} + \cdots
\]

where \( l \) is a nonnegative integer determined only by the quantum numbers of particles 1, 2 and \( \Phi_{12}^{GS} \) is a function which does not have a zero point at \( x_1 \to x_2 \). We also assume that \( \Psi_{GS} \) has a factorization into the spin and the charge factors \( \Psi_{GS} = \Psi_{spin} \Psi_{charge} \) where \( \Psi_{charge} \) is non-singular function while \( \Psi_{spin} \) can have a polynomial singularity. The zeroes of \( \Psi_{charge} \) and the singularity of \( \Psi_{spin} \) are constrained by the condition of \( \Psi_{GS} \) to stand for the electron system.

With these assumptions we construct a field theory where electrons capture a super flux. For this end, we will directly construct the model with all the required symmetries rather than take a map of singular gauge transformation. Thus, we consider a lagrangian of the form \( L = L_{spin} + L_{charge} \). \( L_{charge} \) is the lagrangian for the charge sector with the Coulomb interaction. We consider a theory in which \( L_{spin} \) has the supersymmetry of Ref. [16]. Let us give the supersymmetry algebra. It has four generators \( P^\mu, Q, \bar{Q}, \mu = 0, 1 \), satisfying following commutation relations

\[
[P^\mu, P^\nu] = 0, \quad [Q, Q] = 2\gamma_\mu P^\mu, \quad [P^\mu, Q] = 2i\gamma^\mu \bar{Q},
\]

where we set \( \gamma^0 = \gamma^1 = -\gamma_0 = \gamma_1 = 1 \). Other commutators are all set to zero. These commutation relations satisfy the Jacobi identity and form a graded Lie algebra. We may define an invariant metric by the trace

\[
\text{Tr}(P^\mu P^\nu) = -\frac{1}{2} \eta^\mu^\nu, \quad \text{Tr}(Q \bar{Q}) = \frac{i}{2}.
\]

In [3], supercharges appear unsymmetrically. To treat them in a symmetric way, we may take a basis in spin by \( Q^+ = \frac{1}{\sqrt{2}}(Q + \bar{Q}), \quad Q^- = \frac{1}{\sqrt{2}}(Q - \bar{Q}) \). Similar enlarged gauge symmetry has been known in supergravity [17] and also studied in the relation between CS theory and superstring theories [18]. Physical meaning of [2] is that \( Q \) is the sum of the supercharges for spins, \( \bar{Q} \) is the asymmetry between them, and \( P^\mu \) expresses the effect of the Coulomb interaction on the paired supercharges [19].

The one-form gauge field has the expansion

\[
A = iA^\mu P_\mu + \psi Q - \bar{\psi} \bar{Q},
\]

where \( A^\mu \) is a U(1) gauge field, \( \psi \) and \( \bar{\psi} \) are fermionic gauge fields. We will couple a non-relativistic matter to these gauge fields. We consider the minimal supermultiplet which arises from a parametrization of supergroup \( Z = e^X \) where \( X = i\phi_\mu P^\mu + \frac{1}{\sqrt{2}}\lambda Q - \frac{1}{\sqrt{2}}\bar{\lambda} \bar{Q} \). Here \( \phi_\mu \) is a real boson, \( \lambda \) and \( \bar{\lambda} \) are real fermions. The construction based on real bosons instead of complex bosons means that we deal with a vortex field from the start. Locally supersymmetric lagrangian is given as nonrelativistic chiral lagrangian:

\[
L_{spin} = 2i\text{Tr}(XZ - D_1 Z) - \frac{1}{m} \text{Tr}(Z^{-1} D_1 Z - D_1 Z) + A_\mu J^\mu_{top} + V(Z) + L_{CS},
\]

where \( D_1 = \partial_1 - A_1, D_i = \partial_i - A_i. \) The first term in \( L_{spin} \) is fixed by the requirement for \( \phi_\mu \) to have the ordinary non-relativistic kinetic term with mass \( m \).

\[
J^\mu_{top} = \frac{1}{2} \epsilon^{lmn} \partial_m \partial_n \phi^\mu \text{ is the topological charge originated from the topology of the target space (a torus } T^2). \quad V(Z) \text{ is an invariant potential for } Z. \quad L_{CS} \text{ is the CS term for } A\]

\[
L_{CS} = \frac{1}{4\pi} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) = \frac{1}{4\pi} \left( \frac{1}{2} A^\mu \wedge dA_\mu + i\bar{\psi} d\psi + i\psi \wedge \gamma^\mu \psi \wedge A_\mu \right).
\]

The lagrangian \( L_{spin} \) has the local supersymmetry \( Z \to U^{-1}(x) Z U(x) \). In components, \( L_{spin} \) becomes

\[
L_{spin} = \phi^\mu (i\partial_\mu + \frac{\Delta}{2m}) \phi_\mu + i\bar{\lambda}(i\partial_\mu + \frac{\Delta}{2m}) \lambda - 2\sqrt{2} \lambda \gamma^\mu \phi_\mu \psi
\]

\[
- \frac{1}{m} \{ i\partial_\mu \lambda \gamma_\mu A^\mu_\nu + \sqrt{2}(\partial_\mu \lambda \gamma^\nu \phi_\nu + \lambda \gamma^\nu \partial^\nu \phi_\nu) \psi \}
\]

\[
+ A_\mu J^\mu_{top} + V(Z) + L_{CS}.
\]

In the low-energy limit, matter fields and collective excitations are reduced to the Wilson lines. For a Wilson lines, their current is a sum of delta functions at their lines, \( J = \sum_{i=1}^n j^i T_\nu \delta(C_i) \) where \( T_\nu \) represents the generators of the gauge group. Then the physics in the low-energy limit is described by the action

\[
S_{spin} = \frac{1}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \int \text{Tr} (A \wedge J).
\]

In this form, the states in this system can be exactly studied through the CS/CFT relation [20]. The Hilbert space of the Chern-Simons theory [8] is the space of \( c = -2 \) conformal blocks of the \( c = -2 + 2 \) conformal field theory on the cylinder at the infinity [10]. The \( c = -2 \) part is realized by the symplectic fermions \( \psi^a \), while the \( c = 2 \) part is realized by two bosons \( \phi^\mu \) which together form a complex boson \( \phi^a \). This CFT has a Parisi-Soulsat symmetry \( Q^a_\mu \) as well as two U(1) symmetries \( P^0_\mu \) and \( P^1_\mu \) (\( g \) stands for “global”). Configurations of the system are classified by their behavior at the infinity which is determined by the action of \( Q^a_\mu, P^0, P^1 \) and U(1) symmetry of \( L_{charge} \). We can identify them by the fermions and the charges and denote them as \( (\psi^a, q^0, q^1, q_\mu) \). Here \( a \) and \( s \) stand for...
the spin and the holonomy for the fermion respectively. We denote \( s = \text{R} \) for the trivial holonomy and \( s = \text{NS} \) for the holonomy \( e^{i\pi} \).

Let us now describe phases in this system. The charge part is implicit but satisfies the conditions described above.

**Quantum Critical d-Wave.** We first consider the ground state of 2N electrons which is obtained as a paired collection of \( N (\psi_{\text{R}}^\dagger, 1, -1, 1) \) and \( N (\psi_{\text{R}}^\dagger, -1, 1, 1) \). The charge part \( \Psi_{\text{charge}} \) has zero points of degree 2 for this state. From the CS/CFT relation of Ref. [4], \( \Psi_{\text{spin}} \) can be explicitly calculated as a conformal block of corresponding primary fields. It is given by

\[
\Psi_{\text{spin}} = \langle \partial \theta^\dagger e^{z^\dagger} (z_1^\dagger) \partial \theta e^{-z^\dagger} (z_2^\dagger) \cdots \partial \theta^e e^{-\bar{z}^\dagger} (z_{2N}^\dagger) \rangle = \det \left( \frac{1}{(z_i^\dagger - z_j^\dagger)^2} \right).
\]

This state is spin-singlet for the holonomy doesn’t vanish in the configuration of one species of composite particle. The supersymmetry is divided into two parts each of which vanishes in the configuration of one species of composite particle. A collection of one species breaks half of the supersymmetry. When both species are present, both parts of the supersymmetry are broken.

Let us next consider quasiparticles. For the paired state, the elementary quasiparticle will appear with magnetic flux halved. Such quasiparticle also generates a non-trivial holonomy \( e^{i\pi} \) for \( A^\text{d} \) or \( \psi^a \). As shown in Ref. [21], the modular invariance of the CFT admits the existence of such a configuration only for \( \psi^a \). Thus the elementary quasiparticle and the elementary quasihole are characterized as the species \((\psi^a_{NS}, 0, 0, -\frac{1}{2})\), \((\psi^a_{NS}, 0, 0, \frac{1}{2})\). In the CFT, these excitations are created by the twist field \( \sigma^a \) for \( \theta^a \) (with the charge part).

We also note that half of the supersymmetry which doesn’t vanish in the configuration of one species of composite particle produce a fermionic zero mode in the configuration. The configuration with a fermionic zero mode gives a neutral massless quasiparticle with spin 1/2 which is generated by the logarithmic field in the CFT. This is an evidence of quantum criticality of the state as argued for the HR state [15].

**d + id Wave.** We next consider the ground state of 2N electrons which is obtained as a paired collection of \( N (0, 1, -1, 1) \) and \( N (0, -1, 1, 1) \). In this state electrons do not carry a fermionic charge. Accordingly the fermionic gauge fields are decoupled in this state. These Wilson lines correspond to \( \partial \varphi^a e^{z^\dagger}, \partial \varphi^a e^{-z^\dagger} \) in the CFT respectively (not \( e^{z^\dagger}, e^{-z^\dagger} \) since they have conformal dimension zero). This is justified by the property of these fields that they become the extending fields which gives a consistent \( c = 2 \) rational conformal field theory with the modular invariance. The charge part \( \Psi_{\text{charge}} \) has a holomorphic zero points of odd integer degree for this state.

\( \Psi_{\text{spin}} \) is explicitly given by

\[
\Psi_{\text{spin}} = \langle \partial \varphi^a e^{z^\dagger} (z_1^\dagger) \partial \varphi^a e^{-z^\dagger} (z_2^\dagger) \cdots \partial \varphi^a e^{-\bar{z}^\dagger} (z_{2N}^\dagger) \rangle = \text{per} \left( \frac{1}{(z_i^\dagger - z_j^\dagger)^2} \right).
\]

This pairing is spin-triplet d-wave the form of which is known in the context of FQHE [13]. In this state, two U(1) symmetries are spontaneously broken. Quasiparticles in this system has a half flux for the combined gauge field \( \gamma^a A^\mu \). In the corresponding \( c = 2 \) CFT, it is created by the twist field for the complex boson \( \varphi^a \). This state is gapped without massless neutral excitation. Thus it is actually \( d + id \)-wave. For the FQHE side, the emergence of this state in the theory suggests the existence of “spin-triplet hierarchy” on the HR state [22].

**Quantum Critical p-Wave.** We next consider the state of 2N electrons which is obtained as a paired collection of \( N (\psi_{\text{R}}^\dagger, 1, 0, 1) \) and \( N (\psi_{\text{R}}^\dagger, -1, 0, 1) \). The charge part \( \Psi_{\text{charge}} \) has holomorphic zero points of odd integer degree for this state. The CFT for the spin part is given by the \( c = -1 \) \( \beta - \gamma \) ghost system [23] with an extension on the zero modes [21]. \( \Psi_{\text{spin}} \) can be explicitly calculated as a conformal block of \( \beta - \gamma \)

\[
\Psi_{\text{spin}} = \langle \beta (z_1^\dagger) \gamma (z_2^\dagger) \cdots \beta (z_{2N-1}^\dagger) \gamma (z_{2N}^\dagger) \rangle = \langle \partial \theta^\dagger e^{\sigma^a} (z_1^\dagger) \partial \theta e^{-\sigma^a} (z_2^\dagger) \cdots \partial \theta e^{-\sigma^a} (z_{2N}^\dagger) \rangle = \text{per} \left( \frac{1}{(z_i^\dagger - z_j^\dagger)^2} \right).
\]

This state is spin-singlet p-wave. The FQH counterpart of this state is the permanent state forming a hierarchy of the HR state [22]. As in the quantum critical d-wave state, a collection of one species breaks half of the supersymmetry. When both species are present, both parts of the supersymmetry are broken. Accordingly this state also has neutral massless excitations created by logarithmic field in the CFT. This is the sign of the quantum criticality as was argued for the quantum critical d-wave state.

Quasiparticle in this state generates a non-trivial holonomy \( e^{i\pi} \) for \( \psi^a \). The elementary quasiparticle and the quasihole are characterized as the species \((\psi^a_{NS}, 0, 0, -\frac{1}{2}), (\psi^a_{NS}, 0, 0, \frac{1}{2})\). As in the quantum critical d-wave state, these excitations are created by the twist field \( \sigma^a \) for \( \theta^a \) (with a charge part).

The CFT of this state and the CFT for the \( d + id \) state have the same partition function up to the inclusion of a half flux [21]. It suggests a close relation between them.

**Phase Diagram.** In Ref. [1], the phase diagram shown in Fig. 1 is proposed for the d-wave superconductor. It is argued that the d-wave superconductor is at a proximity to the quantum phase transition to the \( d + id \) superconductor. In Fig the parameter \( r \) can be the doping parameter, the inverse of the strength of the external magnetic field and so on.
Let us compare this picture with our theory. The $d+i d$ superconductor and the quantum critical point are identified with the $d+i d$-wave state and the quantum critical $d$-wave state respectively. Thus neutral massless quasiparticles at the 'quantum critical soup' arises from the supersymmetry breaking.

In our theory, time-reversal symmetry is implicitly broken, thus non $T$-violating $d$-wave superconductor is out of reach. One possibility is to consider it as the quantum critical $d$-wave state disturbed by the underlying square lattice symmetry. This line of argument has been given for the $p$-wave case in Ref. [3].

Our theory predicts the existence of another quantum critical point at which the quantum critical $p$-wave state emerges. Almost equivalent CFTs for the $d+i d$ and the quantum critical $p$-wave states suggest that the $d+i d$ state exposes to the quantum critical point. By interpreting $r$ as the inverse of the strength of the external magnetic field, quantum phase transitions from $d$-wave $\rightarrow$ quantum critical $d \rightarrow d+i d \rightarrow$ quantum critical $p$ is likely to be induced by increasing the magnetic field strength (Fig.8). An intriguing aspect of this picture is a possible direct transition between two quantum critical states, which is compared to the spin-singlet FQH hierarchy [21,16]. As noted in Ref. [16], the inverse transition from quantum $p$-wave to quantum $d$-wave is achieved by capturing of the vortex with the unit flux for $A^1$, which is the same as the $p+i p$-wave pairing [8].

Another intriguing aspect on the quantum critical $p$-wave state is its relevance to the pseudo-gap state. This arises when we interpret $r$ as the hole concentration. To relate the quantum critical $p$-wave state to the pseudogap state, we must consider the quantum critical $p$-wave state without the charge condensation (Fig.3). It means that the condensation occurs only for the spin-part. This identification seems natural when one takes into account the symmetric staggered flux in $\Psi_{\text{spin}}$ of (11) (the permanent factor) is what is expected for the spin-texture picture of doped holes in the AF background.

Discussions. In this paper, we proposed a supersymmetric field theory which naturally embodies conjectured phases of the $d$-wave superconductor. This model has rich theoretical contents rooted in deep aspects of topological quantum field theory [19], supersymmetry [17,18] and conformal field theory [23]. Although we concentrate on condensed phases, it is also possible to consider the normal state, providing a realization of the spin-charge separation [24] in a new way.

The most important question we didn’t address in this paper is the high critical temperature $T_c$. We’d like to consider it as a property of the field theory at the quantum critical point. In our model, the key property is the local supersymmetry of [8] and the emergence of fermionic gauge field associated with it. By the strong suppression through the fermionic gauge field, $\Psi_{\text{GS}}$ at the quantum critical $d$-wave state has no zero points between the paired electrons. The same mechanism should modify the temperature dependence of the effective potential of the charges. Thus we suggest a study on the beta function of the potential for future work.

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FIG. 1. Schematic Phase diagram recently advocated by Vojta et al.

FIG. 2. Modified Schematic Phase diagram 1

FIG. 3. Modified Schematic Phase diagram 2