Quantum-critical scaling of fidelity in 2D BCS-like models

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Abstract

We study the feasibility of the quantum-fidelity approach in studies of quantum-critical points – an alternative to standard correlation-function approach, beyond 1D. For this purpose we consider 2D models of interacting spinful fermions, examples of the so-called pairing or BCS-like models, which originate from a 2D model of d-wave superconductivity proposed by Sachdev. First, due to the exact diagonalizability of the considered models in any dimensionality $D$, exact ground-state phase diagrams, with several kinds of quantum-critical points, are constructed. Closed-form analytic expressions for ground-state two-point correlation functions, with analytic expressions for their asymptotic behavior at large distances and in neighborhoods of quantum-critical points, are provided. In particular, explicit expressions for direction-dependent correlation lengths and the values of direction-dependent universal critical indices $\nu$, that characterize the divergence of correlation lengths on approaching critical points, are given. Then, due to the derived analytic expressions for the ground-state quantum fidelity, critical scaling of fidelity is analyzed numerically with great accuracy, not only for small systems but also for macroscopic ones, together with the crossover region between them. The obtained results are discussed in the light of quantum-critical scaling theory of quantum fidelity.

1 Introduction

In recent years, quantum phase transitions and quantum-critical phenomena constitute a subject of great interest and vigorous studies in condensed matter physics. Both, experimental and theoretical developments point out to the crucial role that quantum phase transitions play in physics of frequently studied high-$T_c$ superconductors, rare-earth magnetic systems, heavy-fermion systems or two-dimensional electrons liquids exhibiting fractional quantum Hall effect [1], [2]. Quantum-critical phenomena have been also observed in exotic systems as magnetic quasicrystals [3] and in artificial systems of ultracold atoms in optical lattices [4]. The so-called classical, thermal phase transitions originate from thermal fluctuations, a competition of internal energy and entropy, and are mathematically manifested as singularities in temperature and other thermodynamic parameters of various thermodynamic functions,
and such characteristics of correlation functions as the correlation length, at nonzero temperatures. In contrast, quantum phase transitions originate from purely quantum fluctuations and are mathematically manifested as singularities in system parameters of the ground-state energy density, which is also the zero-temperature limit of the internal energy density. Naturally, singularities of thermodynamic functions appear only in the thermodynamic limit. The importance of quantum phase transitions for physics and the related wide interest in such transitions stems from the fact that, while a quantum phase transition is exhibited by ground states, hence often termed a zero-temperature phenomenon, its existence in a system exerts a great impact on the behavior of that system also at nonzero temperatures. A quantum-critical point gives rise to the so-called quantum-critical region, which extends at nonzero temperatures, in some cases up to unexpectedly high temperatures [5], [6].

Theoretically, quantum phase transitions can be studied in quite complex quantum systems by qualitative and approximate methods, or in relatively simple but exactly solvable models by means of analytic methods and high-accuracy numerical calculations [1]. Naturally, for the purpose of testing and illustrating general or new ideas the second route is most suitable. Traditionally, this route involves studying the eigenvalue problem of a Hamiltonian, the ground state and excitation gaps, determining quantum-critical points and symmetries, constructing local-order parameters, calculating two-point correlation functions and their asymptotic behavior at large distances and in vicinities of quantum-critical points, with correlation lengths and the universal critical indices $\nu$ that characterize the divergence of correlation lengths on approaching critical points. Carrying out such a programme is a hard task, which has been accomplished only in a few one-dimensional models. Among those models, there are quantum spin chains as the isotropic and anisotropic XY models in an external transverse magnetic field, including their extremely anisotropic version—the Ising model [1]. Only in one dimension those models are equivalent to lattice gases of spinless fermions, which can exactly be diagonalized, and exact results concerning the phase diagram, quantum-critical points, correlation functions and dynamics have been obtained (concerning XY model see [7], [8], [9], concerning the Ising model see [10], and for both models [11]). Needless to say that parallel results for a higher-dimensional model are desirable; this is the first motivation of our investigations presented in this paper.

In the last decade, fresh ideas coming from quantum-information science entered the field of quantum phase transitions. One of them is the so-called quantum-fidelity method. Using this method it is possible to locate critical points [12], [13], [14] and to determine the correlation lengths and universal critical indices $\nu$. This is achieved by studying (typically numerically) critical scaling properties, with respect to the size of the system and the parameters of the underlying Hamiltonian, of the so-called quantum fidelity of two ground states in a vicinity of a critical point [15], [16]. To extract the index $\nu$, the scaling laws of quantum fidelity, derived by renormalization group arguments [15], [19], [20], [16] are needed. The task of determining scaling properties of the quantum fidelity should be much easier than the task of calculating large-distance behavior of two-point correlation functions. The most comprehensive results concerning the feasibility of the fidelity approach have been obtained for one-dimensional quantum spin systems in a perpendicular magnetic field [15] (the case of Ising model), [16] (the case of XY model). These results are very promising: except a vicinity of a multicritical point, the fidelity approach works fine. In order to verify the effectiveness of this approach in dimensions higher than one, we need an at least two-dimensional exactly solvable model, whose ground states, quantum-critical points, correlation lengths and critical indices in their vicinities, and analytic expressions for fidelity are known; this is the second motivation of our investigations reported in this paper.

To go beyond the one-dimensional case, we consider lattice fermion models which origi-
nate from the two-dimensional model of d-wave superconductivity proposed by Sachdev [17] (see also [1]), which are spinful BCS-like models. General, mathematical considerations of some classes of such models, but without specifying hopping intensities or coupling constants, which therefore do not reach such subtleties as quantum-critical points or critical behavior of correlation functions, can be found in [21], [22]. For translation-invariant hopping intensities and coupling constants the considered models are exactly diagonalizable in any dimension. Consequently, it is possible to derive analytical formulae for correlation functions of finite systems and then in the thermodynamic limit, where boundary conditions play no role. To limit further the great variety of possible models, we restrict the hopping intensities to nearest neighbors while the dimensionality is set to $D=2$. The underlying lattice is chosen as a square one while the hopping intensities – invariant under rotations by $\pi/2$. Similarly, we require the interactions of our systems not to extend beyond nearest neighbors and to be either invariant under rotations by $\pi/2$ (the symmetric model) or to change sign after such a rotation (the antisymmetric model).

The general plan of the paper is as follows. In section 2 we define the two models studied in this paper, the symmetric model and the antisymmetric one, give closed-form formulae for two basic two-point correlation functions of those models. In sections and subsections that follow we limit our considerations to one of those correlation functions – an off-diagonal matrix element of the ground-state one-body reduced density operator. The next section 3 presents the quantum fidelity method of investigating quantum-critical points; provides the formulae for fidelity in the models considered in the paper and the known quantum-critical scaling laws obeyed by fidelity sufficiently close to quantum-critical points. Then, in sections 4 and 5 we present our analytic results concerning the behavior of the correlation function at sufficiently large distances and sufficiently close to the quantum-critical points exhibited by our models, and numerical results concerning critical scaling of fidelity sufficiently close to quantum-critical points, for the symmetric and antisymmetric models. Finally, in section 6 we summarize our results and draw conclusions.

2 The models, their ground-states and ground-state correlation functions

We consider a $D$-dimensional spinful fermion model, given by the Hamiltonian,

$$H = \sum_{l,i,\sigma} \left[ \frac{t}{2} \left( a_{l,\sigma}^\dagger a_{l+e_i,\sigma} + \text{h.c.} \right) - \frac{\mu}{D} a_{l,\sigma}^\dagger a_{l,\sigma} - \frac{J}{2} \left( \sigma \Delta_i a_{l,\sigma}^\dagger a_{l+e_i,-\sigma} + \text{h.c.} \right) \right],$$

(1)

where $a_{l,\sigma}^\dagger$, $a_{l,\sigma}$ stand for creation and annihilation operators, respectively, of a spin 1/2 fermion, whose spin projection on a quantization axis is $\sigma = \pm 1$ in units of $\hbar/2$, in a state localized at site $l=(l_1, \ldots, l_D)$ of a $D$-dimensional hypercubic lattice. The edge of the lattice in the direction given by the unit vector $e_i$, $i=1, \ldots, D$, whose $m$-th component is $\delta_{i,m}$, consists of $L_i$ equidistant sites, labeled by $l_i=0, 1, \ldots, L_i - 1$, $l_j=0$ for $j \neq i$. In all the considerations that refer to finite systems, special boundary conditions, specified below, are chosen. The sums over $l, i$ in (1) amount to the sum over pairs of nearest neighbors, with each pair counted once. The real and positive parameter $t$ is the nearest-neighbor hopping intensity, $\mu$ – the chemical potential, $J$ – the coupling constant of the gauge-symmetry breaking interaction, and $\Delta_i, i=1, \ldots, D$, stand for direction-dependent, in general complex, dimensionless constants. Naturally, we can express the parameters $\mu$ and $J$ in units of $t$, while the lengths of the underlying lattice in units of the lattice constant, preserving the
original notation. We emphasize that in distinction to \[17\], \[1\], where Hamiltonian (1) was derived, \(\Delta_i\) are constants independent of \(\mu\) and \(J\). We note that Hamiltonian (1) is not gauge invariant unless \(J=0\). It is also not hole-particle invariant unless \(\mu=0\) and \(\Delta_i\), \(i=1,\ldots,D\), are real. The latter condition can be assumed to hold without any loss of generality, since Hamiltonian (1) with any complex \(\Delta\) is unitarily equivalent to that with \(\Delta\) replaced by \(|\Delta|\).

Imposing, independently in each direction \(e_i\), \(i=1,\ldots,D\), periodic or antiperiodic boundary conditions, Hamiltonian (1) can be simplified by passing from the site-localized to the plane-wave basis labeled by suitable wave vectors (quasimomenta) \(k=(k_1,\ldots,k_D)\),

\[
H = \sum_{k,\sigma} \varepsilon_k c_{k,\sigma}^\dagger c_{k,\sigma} - J \sum_{k,i} \cos k_i \left( \Delta_i c_{k,\uparrow}^\dagger c_{-k,\downarrow} + \text{h.c.} \right),
\]

where \(\varepsilon_k\) stands for the dispersion relation of the hopping term,

\[
\varepsilon_k = \sum_i \cos k_i - \mu,
\]

with \(k_i=2\pi[l_i-(L_i-1)/2]/L_i\) in the case of periodic boundary condition with an odd \(L_i\), and \(k_i=\pi[2l_i-L_i+1]/L_i\) in the case of antiperiodic boundary condition with an even \(L_i\).

Formally, Hamiltonian (2) differs from the well-known BCS Hamiltonian of s-wave superconductivity by the presence of \(\cos k_i\) factors in the gauge-symmetry breaking term. Such Hamiltonians can readily be diagonalized by means of the Bogoliubov transformation. The dispersion relation of quasi-particles reads

\[
E_k + \sum_{k} (\varepsilon_k - E_k),
\]

where \(\sum_{k} (\varepsilon_k - E_k)\) is the ground-state energy, and \(E_k\), given by

\[
E_k = \sqrt{\varepsilon_k^2 + \left| J \sum_i \Delta_i \cos k_i \right|^2},
\]

are the single quasi-particle energies. For a suitable choice of boundary conditions specified above, as long as our system is finite the excitation energies \(E_k\) remain strictly positive: \(E_k > 0\) for all values of \(k\), and this is assumed to hold in the sequel. Specifically, in the two-dimensional models discussed in the sections that follow, we impose periodic boundary conditions along one axis and antiperiodic ones along the orthogonal axis.

The Hamiltonian (1) preserves parity; therefore without any loss of generality we can restrict the state-space to the subspace of even number of fermions. In this subspace, the state \(|0\rangle_{qp}\) – the quasi-particle vacuum of an unspecified (but even) number of fermions, defined by

\[
|0\rangle_{qp} = \prod_{k} (u_k + v_k c_{k,\uparrow}^\dagger c_{-k,\downarrow}^\dagger) |0\rangle,
\]

where \(|0\rangle\) is the fermion vacuum, with \(u_k\) real and positive,

\[
u_k = \sqrt{\frac{1}{2} \left( 1 + \frac{\varepsilon_k}{E_k} \right)},
\]

and, in general, complex \(v_k\),

\[
|v_k| = \sqrt{\frac{1}{2} \left( 1 - \frac{\varepsilon_k}{E_k} \right)}, \quad \arg v_k = \arg \left( J \sum_i \Delta_i \cos k_i \right),
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\]
Two kinds of paths will be considered: µ-paths and σ-paths. Two-point correlation functions can be chosen as follows:

\[ q_p(0|a_{0,\sigma}^+a_{r,\sigma}|0)_{qp} \quad \text{and} \quad q_p(0|a_{0,\sigma}a_{r,-\sigma}|0)_{qp}, \]

with some \( \sigma \). The first correlation function, \( q_p(0|a_{0,\sigma}^+a_{r,\sigma}|0)_{qp} \), is gauge and spin-flip invariant; for \( r \neq 0 \) it represents diagonal matrix elements of the ground-state one-body reduced density operator, and amounts to

\[ q_p(0|a_{0,\sigma}^+a_{r,\sigma}|0)_{qp} = -\frac{1}{2L^D} \sum_k \frac{\varepsilon_k}{E_k} \exp ikr, \]

which, upon using the invariance of \( \varepsilon_k \) and \( E_k \) with respect to reflections of \( k \) in coordinate axes, in the thermodynamic limit becomes

\[ \lim_{L \to \infty} q_p(0|a_{0,\sigma}^+a_{r,\sigma}|0)_{qp} \equiv G(r) = -\frac{1}{2\pi^D} \int_{0 \leq k_j \leq \pi} dk \frac{\varepsilon_k}{E_k} \prod_{j=1}^{D} \cos k_j r_j. \]

Choosing the spin projection \( \sigma = +1 \), the second correlation function, measuring the degree of gauge-symmetry breaking, amounts to

\[ q_p(0|a_{0,+}a_{r,-}|0)_{qp} = -\frac{1}{2L^D} \sum_k J \sum_i \Delta_i \cos k_i \left[ \frac{\varepsilon_k}{E_k} \right] \exp ikr, \]

which, by the above arguments, in the thermodynamic limit becomes

\[ \lim_{L \to \infty} q_p(0|a_{0,+}a_{r,-}|0)_{qp} \equiv h(r) = -\frac{1}{2\pi^D} \int_{0 \leq k_j \leq \pi} dk \frac{J \sum_i \Delta_i \cos k_i}{E_k} \prod_{j=1}^{D} \cos k_j r_j. \]

Both the above defined two-point correlation functions are used to define the order parameters in the ground-state phase diagrams presented in the sections that follow. However, analytic results will be given only for the gauge-invariant correlation function \( G(r) \), defined in (11). For a fixed lattice direction and the parameters \( \Delta_i \), \( G(r) \) depends on three parameters: \( |r| \) - the distance between the two points of the correlation function, the chemical potential \( \mu \) and the coupling constant \( J \). The ground-state phase diagrams are presented in the \( (\mu, J) \)-plane, in particular the quantum-critical points of the considered models are uniquely defined by pairs \( (\mu, J) \). We shall be interested here only in the so-called doubly-asymptotic behavior of \( G(r) \) (for more comprehensive results see [25]) that is, for a fixed but sufficiently large \( |r| \), \( (\mu, J) \)-points approach a quantum-critical point along a specific path. Two kinds of paths will be considered: \( \mu \)-paths that are parallel to the \( \mu \)-axis and \( J \)-paths that are parallel to the \( J \)-axis.
In gapped phases, a decay with increasing $|r|$ of $G(r)$ is dominated by an exponential factor, \( \exp(-|r|/\xi) \), which defines the correlation length $\xi$. If additionally $(\mu, J)$-points approach a quantum-critical point, i.e. the distance $\delta$ between them tends to zero, then $\xi$ diverges as $\delta^{-\nu}$, which in turn defines a universal critical index $\nu$ associated with a particular quantum-critical point.

All the above $D$-dimensional expressions can be adapted to the 2D case by setting $\Delta_i = k_i = r_i = 0$ for $i > 2$. Due to the freedom in choosing the relation between the parameters $\Delta_1$ and $\Delta_2$, formula (11) represents a great variety of models. In this paper we limit our considerations to two cases only. Namely, the symmetric case, with the interaction term invariant under rotations by $\pi/2$, where $\Delta_1 = \Delta_2 = \Delta$, and the antisymmetric case, with the interaction term that changes sign under a rotation by $\pi/2$, where $\Delta_1 = - \Delta_2 = \Delta$. We note that in both cases the correlation functions of our systems are invariant not only with respect to lattice translations but also with respect to rotations by $\pi/2$.

As compared to the one-dimensional case, a novel feature of two-dimensional models is that the two-point correlation function $G(r)$ depends not only on the distance $|r|$ but also on the direction of $r$. Expressing $r$ by its Cartesian coordinates, $r=(r_1, r_2)$, we can parameterize directions by the ratio $r_1/r_2 \equiv n$. Then, for a given critical point, we can expect $n$-dependent doubly-asymptotic behaviors of correlations. Unfortunately, the analytic asymptotic formulae for $G(r)$ in offdiagonal directions, which we have been able to obtain, apply only to points $r$ such that $n \geq n_0 > 1$ or, by symmetry, $n \leq n_0^{-1} < 1$, that is for offdiagonal directions which form a sufficiently small angle with the axial directions. Therefore, the asymptotic formulae in the diagonal direction have been derived separately [25]. These formulae define $n$-dependent correlation lengths $\xi^{(\pm)}_{\text{offdiag}}$ in offdiagonal directions satisfying the conditions specified above and the correlations length $\xi^{(\pm)}_{\text{diag}}$ in the diagonal direction, where the superscript plus refers to the symmetric model and minus – to the antisymmetric one. Interestingly, our analytical and numerical results show that, for each critical point of the symmetric or the antisymmetric model, there are only two kinds of universal critical indices $\nu$: $\nu_{\text{offdiag}}$ for all offdiagonal directions and $\nu_{\text{diag}}$ for the diagonal direction.

In all the considerations below that refer to the symmetric or the antisymmetric model, we make the identification $J|\Delta| \equiv J$.

3 The ground-state fidelity and quantum-critical scaling laws

Let $\lambda$ be a vector whose components are those parameters of the considered system’s Hamiltonian that drive a quantum phase transition, and $e$ - a unit vector in the space of those parameters. Then, on varying parameter $\delta$ the vectors $\lambda + \delta e$ scan a neighborhood of $\lambda$ along direction $e$. For given $\lambda$ and $\delta$, the ground-state fidelity at $\lambda$ in direction $e$, $\mathcal{F}_e(\lambda, \delta)$, is the absolute value of the overlap of the ground states $|\lambda \pm \delta e\rangle$ at the points $\lambda \pm \delta e$,

$$
\mathcal{F}_e(\lambda, \delta) = |\langle \lambda - \delta e | \lambda + \delta e \rangle|.
$$

A list of general, system independent, properties of $\mathcal{F}_e(\lambda, \delta)$ can be found in [23]. The transition point of a continuous quantum phase transition, i.e. a quantum-critical point, denoted $\lambda_c$, is characterized by the power-law divergence of the correlation length $\xi(\lambda)$, as the quantum-critical point is approached: $\xi(\lambda) \sim |\lambda - \lambda_c|^{-\nu}$, with $\nu$ being one of universal characteristics of a critical point. Alternatively, $\lambda_c$ can be defined as the point where the gap between the ground-state energy and the energy of the lowest excited state vanishes.
In reference [12] it was demonstrated that the quantum-critical point \( \lambda_c \) can be identified as the minimum of fidelity, as \( \lambda \) is varied. However, the fidelity approach seeks an answer to a more general question: does the behavior of quantum fidelity in a neighborhood of a quantum-critical point encode not only the location of that point but also some universal properties of the underlying quantum phase transition? First results, pointing towards a positive answer to the raised question by providing some finite-size critical scaling of fidelity, have been obtained by Venuti and Zanardi [18].

According to finite-size scaling theories, the properties of a system are close to those at the thermodynamic limit, we say the system is macroscopic, if the linear size of the system, \( L \), is much greater than \( \xi(\lambda) \). In the opposite limit we observe the so called ”small-system” properties. Concerning finite-size scaling properties of fidelity \( \mathcal{F}_e(\lambda, \delta) \), it is expected that the characteristic length of the system, that differentiates between small and macroscopic system, is given by \( \tilde{\xi}_e(\lambda, \delta) \), which is the smaller of the two correlation lengths \( \xi(\lambda \pm \delta e) \). In other words, the crossover between small-system and macroscopic-system properties occurs, when the effective linear size of a system, \( L/\tilde{\xi}_e(\lambda, \delta) \), satisfies the crossover condition:

\[
L/\tilde{\xi}_e(\lambda, \delta) \sim 1.
\]

(15)

There are numerous papers devoted to critical scaling of small-system fidelity, see [23], [20], and references quoted there. Typically, small-system fidelity can be Taylor-expanded in \( \delta \),

\[
\mathcal{F}_e(\lambda, \delta) = 1 - \frac{\delta^2}{2\chi_e(\lambda)} + \ldots,
\]

(16)

where the first order term vanishes because of the symmetry of fidelity in \( \delta \) at zero. The coefficient of the second order term, \( \chi_e(\lambda) \), is known as the fidelity susceptibility. One expects some universal scaling properties of \( \chi_e(\lambda) \), provided \( \lambda \) is sufficiently close to a quantum-critical point \( \lambda_c \), where the correlation length diverges: \( \xi(\lambda_c \pm \delta e) \sim |\delta|^{-\nu} \). Fairly general, model-independent, arguments provide us with finite-size scaling of the fidelity susceptibility at \( \lambda_c \) [19], [20]:

\[
\chi_e(\lambda_c) \sim L^{2/\nu},
\]

(17)

or equivalently, in the small-system regime

\[
-\ln \mathcal{F}_e(\lambda_c, \delta) \sim \delta^2 L^{2/\nu}.
\]

(18)

Let us note here that in vicinities of some quantum-critical points fidelity oscillates on varying \( L \), with an amplitude that is particularly large, close to one, in the small-system regime [16], [24]. In such cases, the fidelity susceptibility is not well defined. However, the small-system scaling law [18] may still hold but in a generalized sense [24]. Specifically, it is the envelope of the minima of \( -\ln \mathcal{F}_e(\lambda_c, \delta) \) that scales according to [18].

In the macroscopic-system regime, quantum phase transitions have been studied by means of the so called fidelity per site, a quantity whose logarithm is equal to \( N^{-1} \ln \mathcal{F}_e(\lambda, \delta) \), where \( N = L^D \) is the number of sites in a \( D \)-dimensional system [13], [14]. However, critical scaling of a macroscopic-system fidelity has been considered only very recently by Rams and Damski [15], [16]. These authors have found that, while for small-systems the fidelity scaling is totally insensitive to the way the critical point \( \lambda_c \) is approached by the points \( \lambda \pm \delta e \) (i.e. for instance, whether they are located on one side of the critical point or on the opposite sides), in the case of macroscopic-system the way of approaching the critical point matters. To make this explicit, Rams and Damski substituted \( \lambda_c + c\delta e \) for \( \lambda \). By choosing the value of the parameter \( c \), the above mentioned location of the two points \( \lambda_c + c\delta e \pm \delta e \) with respect


\[ -\ln F_e(\lambda_c + c\delta e, \delta) \sim \delta^2 L^{2/\nu}. \]  

(19)

In contrast to the small-system case, the fidelity scaling law for macroscopic systems, derived by Rams and Damski [15], makes the dependence on parameter \( c \) explicit. Provided that the thermodynamic limit of \( N^{-1}\ln F_e(\lambda, \delta) \) does exist, it reads

\[ -\ln F_e(\lambda_c + c\delta e, \delta) \sim |\delta|^{D\nu} N A_e(c), \]  

(20)

where \( A_e(c) \) is the scaling function.

It should be emphasized that the small system scaling law (19) as well as the macroscopic system scaling law (20) have been derived, using critical-scaling theory arguments, under two conditions. The first one is that there is only one characteristic length scale in the underlying system, which discriminates between small systems and macroscopic systems. This characteristic length is identified with the effective correlation length \( \tilde{\xi}_e(\lambda_c, \delta) \). The second one is that the strict inequality \( D\nu < 2 \) holds true [19], [20], [15]. It is worth to mention that if this condition is satisfied, then in the small-system regime \( \chi_e(\lambda_c) \) or \(-\ln F_e(\lambda_c, \delta)\), formulae (17), (18), respectively, scale with system’s linear size in a superextensive way.

Let \( |0\rangle_{qp} \) and \( |0\rangle_{qp} \) be two ground states, the first one for pairs \((\mu, J)\), and the functions \( \varepsilon_k, E_k \), the second one for pairs \((\tilde{\mu}, \tilde{J})\), and the functions \( \tilde{\varepsilon}_k, \tilde{E}_k \). As a result of the product structure of the ground states, the quantum fidelity for these states has also a product structure,

\[ |_{qp\rangle_{0\rangle_{qp}} = \prod_k \left| (u_k \tilde{u}_k + |v_k| \tilde{v}_k) \exp i(\arg \tilde{v}_k - \arg v_k) \right|. \]  

(21)

After using (17), (18) the fidelity of two ground states (6) assumes the form

\[ |_{qp\rangle_{0\rangle_{qp}} = \prod_{k_1 > 0, k_2 > 0} f^2(k), \quad f(k) = \frac{1}{2} \left( 1 + \frac{\varepsilon_k \tilde{\varepsilon}_k + J \tilde{J} (\cos k_1 \pm \cos k_2)^2}{E_k \tilde{E}_k} \right), \]  

(22)

where the sum of cosine functions has to be taken in the case of the symmetric model and the difference – in the case of the antisymmetric model.

Let us adapt the general notation introduced in the beginning of this section to the considered models. As the location of critical points is uniquely determined by pairs \((\mu, J)\), we set \( \lambda \equiv (\mu, J) \), hence \( |\lambda\rangle \equiv |0\rangle_{qp} \). Then, in formula (22) for fidelity, the functions \( \varepsilon_k, E_k \), given by (3) and (5), respectively, are calculated at \( \lambda_c + (c - 1)\delta e \), while \( \tilde{\varepsilon}_k \) and \( \tilde{E}_k \) – at \( \lambda_c + (c + 1)\delta e \). Finally, we set \( |_{qp\rangle_{0\rangle_{qp}} \equiv F_e(\lambda_c, \delta) \).

Considering quantum-critical scaling of fidelity, we shall study numerically the sum

\[ \sum_{k_1 > 0, k_2 > 0} -2 \ln f(k_1, k_2) \equiv -\ln F_e(\lambda_c, \delta), \]  

(23)

as a function of parameter \( \delta \) for fixed system size \( N \), or vice versa, in neighborhoods of various critical points, in small- and macroscopic-system regimes. We recall that the values of \( k_1 \) are obtained with periodic boundary conditions and the values of \( k_2 \) with antiperiodic ones or vice versa. In all the considered cases the function \( \ln f(k) \) is either continuous in the whole square \([0, \pi]^2\) or it has an integrable singularity at some \( k \) (a discontinuity or
a logarithmic divergence). Therefore, in all the considered cases the limit $N \to \infty$ of the Riemann sum corresponding to (23) does exist,

$$\lim_{N \to \infty} -N^{-1} \ln \mathcal{F}_e(\lambda_c, \delta) = \frac{1}{2\pi^2} \int_{[0,\pi]^2} dk_1 dk_2 \left( -\ln f(k_1, k_2) \right).$$

(24)

Consequently, for given sufficiently small $\delta$ and sufficiently large $N$

$$-\ln \mathcal{F}_e(\lambda_c, \delta) \approx \frac{N}{2\pi^2} \int_{[0,\pi]^2} dk_1 dk_2 \left( -\ln f(k_1, k_2) \right)$$

(25)

approximately, that is in a macroscopic-system regime $-\ln \mathcal{F}_e(\lambda_c, \delta)$ scales with the system size as $N$.

Any study of critical scaling involves specifying a critical region, that is a critical point and its neighborhood. The quantum-critical points considered in our paper are displayed in Figs. 1 and 6. As for neighborhoods, we have chosen line neighborhoods, each one specified by a unit vector $e$ and a range of parameter $\delta$, which are scanned by vectors $\lambda_c + (c - 1)\delta e$ and $\lambda_c + (c + 1)\delta e$ on varying $\delta$. Without any loss of generality only $\delta > 0$ is considered.

Our aim in the sequel is to confront the predictions of quantum-critical scaling theory for fidelity with exact results, to find out limitations and advantages of fidelity approach in investigations of quantum-critical points in dimensions $D > 1$. To the best of our knowledge, this task has never been carried out. We note that the case $D = 1$ has been extensively studied in [15], [16], [24]. We would like to answer such questions as: to what extent the quantum fidelity is useful for determining the universal critical exponent $\nu$?, is it a simpler alternative to calculating the large-distance asymptotic behavior of a two-point correlation function?

For this purpose, we consider different kinds of critical points exhibited by the considered systems and calculate $-\ln \mathcal{F}_e(\lambda_c, \delta)$ as a function of $\delta$, keeping the linear size $L$ fixed or vice versa. Then, we make an attempt to determine the intervals of $\delta$ or $L$, where $-\ln \mathcal{F}_e(\lambda_c, \delta)$ obeys a power law. After that, we try to identify the regime of small system, the one of macroscopic system, and the characteristic length, or lengths, that discriminates between the small- and macroscopic-system regimes – the location and extent of the crossover regime. We note here that according to our results concerning the correlation length, summarized in sections that follow, for a given linear size of the system, we move towards the regime of small system by decreasing sufficiently $|\delta|$, and to the regime of macroscopic system – by increasing it sufficiently.

While the two-point correlation function $G(r)$, in particular its large-distance asymptotic behavior, and consequently the defined above effective correlation length $\xi_e(\lambda_c, \delta)$ depends on lattice directions, fidelity $\mathcal{F}_e(\lambda_c, \delta)$ does not even “know” what a lattice direction is. This fact rises the interesting question of the relation between the mentioned above characteristic length, or lengths, and direction-dependent $\xi_e(\lambda_c, \delta)$.

Finally, via the scaling laws (19) and (20), we calculate the values of $\nu$ and compare them with the known exact results displayed in the phase diagrams.

### 4 The case of symmetric model

We can distinguish four ground-state phases labeled by two order parameters, $O_1$ and $O_2$, defined as

$$O_1 = G(0) - \frac{1}{2}, \quad O_2 = -\Delta h(1, 0).$$

(26)
Figure 1: Phase diagram of the symmetric two-dimensional system in the \((\mu, J)\)-plane. The set of quantum-critical points consists of the \(J\)-axis and the closed interval \([-2, 2]\) of \(\mu\)-axis – thick lines. Those lines constitute also phase boundaries of the four phases, labeled by the order parameters \(\mathcal{O}_1\) and \(\mathcal{O}_2\). Double arrows indicate the types of neighborhoods of critical points, in which the asymptotic behaviors of \(G(r)\) are studied, except neighborhoods of the multicritical point. The universal critical indices \(\nu\) in those neighborhoods, whose values are given by the arrows, depend in general on lattice direction, whether it is diagonal \((\nu_{\text{diag}})\) or offdiagonal \((\nu_{\text{offdiag}})\).

The quantum-critical points of the symmetric two-dimensional system are located at the \(J\)-axis and in the closed interval \([-2, 2]\) of the \(\mu\)-axis. There are two critical end points \((\pm 2, 0)\) and a multicritical point \((0, 0)\). The ground-state phase diagram of the symmetric two-dimensional system is shown in Fig. 1. In all the analytic asymptotic formulae presented below, a vicinity of the multicritical point \((0, 0)\) is excluded. In particular, for \(\mu \to 0\) the \(J\)-coordinates of \(\mu\)-paths have to be away from zero; analogous condition applies to \(J\)-paths. The case of multicritical point will be discussed separately.

In the stripe \(|\mu| \leq 2\) of the \((\mu, J)\)-plane, but excluding the \(\mu=0\) and \(J=0\) lines, the large-distance asymptotic behavior of \(G(r)\) is:

\[
G(r', r) \approx -\text{sgn}(\mu) \frac{1}{2\pi} \left( \frac{J^2}{1 + J^2} \right)^{1/4} \frac{\exp(-r'/\xi^{(+)})}{r'} \cos(\theta r' + \phi),
\]

in the diagonal direction, and

\[
G(r_1, r_2) \approx \frac{C_r}{2\pi} \left( \frac{\mu^2 J^2}{1 + J^2} \right)^{1/4} \left( \frac{1 + n^2}{n^2} \right)^{1/2} \frac{\exp(-r/\xi^{(+)})}{r} \cos(r \theta_{\text{offdiag}} + \phi),
\]

with

\[
r = r_1 \sqrt{(1 + n^2)/n^2}, \quad C_r = \begin{cases} 1, & \text{if } \mu > 0, \\ (-1)^{(r_1 + r_2 + 1)}, & \text{if } \mu < 0, \end{cases}
\]

\[
\frac{1}{\xi^{(+)}} = \left( \frac{n^2}{1 + n^2} \right)^{1/2} \left( \frac{1}{\xi_1} + \frac{1}{n^2 \xi_2} \right),
\]

\(10\)
and,
\[ \theta_{\text{offdiag}} = \left( \frac{n^2}{1 + n^2} \right)^{1/2} \left( \theta_1 + \frac{1}{n^2} \theta_2 \right), \]
provided that the points \((r_1, r_2)\) become remote from the origin along a ray \(r_1/r_2=n=\text{const}\).

We recall that by symmetry the large-distance asymptotic behavior of \(G(\mathbf{r})\) in offdiagonal directions is the same for \(n \geq n_0 > 1\) and for \(n \leq n_0^{-1} < 1\) (specifically we found that one can choose \(n_0=3\)). However, the above formulae that refer to offdiagonal directions hold only for \(n \geq n_0 > 1\). The formulae \((27)\) and \((28)\) define the diagonal, \(\xi_{\text{diag}}^{(+)} = \sqrt{2\xi^{(+)}}\), and offdiagonal, \(\xi_{\text{offdiag}}^{(+)}\), correlation lengths, respectively. The parameters \(\xi^{(+)}\), \(\theta\), \(\xi_1\), \(\xi_2\), \(\theta_1\) and \(\theta_2\) will be expressed by \(\mu\) and \(J\) in vicinities of critical points in subsections that follow. Thus, together with \((27)\) and \((28)\), the behavior of \(G(\mathbf{r})\) in doubly-asymptotic regions will be specified.

### 4.1 Critical points at the line \(\mu=0\)

Specifically, in this subsection we consider \((\mu, J)\)-points approaching along a \(\mu\)-path a point belonging to any one of the two half lines of quantum-critical points, given by \(\mu=0\) and \(|J| \geq J_0 > 0\), for some \(J_0\). Then, in terms of \(\mu\) and \(J\) the parameters determining the large-distance asymptotic behavior of \(G(\mathbf{r})\) in \((27)\) and \((28)\) are given by

\[ \frac{1}{\xi^{(+)}} \approx \frac{|\mu J|}{1 + J^2}, \quad \theta \approx \pi - \frac{|\mu|}{1 + J^2}, \]

\[ \frac{1}{\xi_1} \approx \sqrt{\frac{2|\mu|}{\sqrt{1 + J^2}}} \sin \left( \frac{1}{2} \arctan |J| \right) = -\frac{2}{\xi_2}, \quad \theta_1 \approx \pi - 2\theta_2, \]

\[ \theta_2 \approx \sqrt{\frac{|\mu|}{2\sqrt{1 + J^2}}} \cos \left( \frac{1}{2} \arctan |J| \right). \]

In the left and right panels of Fig. 2 showing the plots of \(-\ln \mathcal{F}_{(1,0)}((0,1),\delta)\) versus the system’s linear size \(L\) or the deviation from the critical point \(\delta\), where \(\delta \equiv \mu\), respectively, two crossover regions are well visible. In the left panel, for fixed \(\delta=10^{-3}\), on increasing \(L\), well below the effective correlation length \(\xi_{(1,0)}((0,1),\delta)\) in the axial direction (denoted \(\xi_{\text{axial}}(\delta)\)), we observe that \(-\ln \mathcal{F}_{(1,0)}((0,1),\delta)\) scales as \(L^4\). Then, above \(\xi_{\text{axial}}(\delta)\) but below the effective correlation length \(\tilde{\xi}_{(1,0)}((0,1),\delta)\) in the diagonal direction (denoted \(\tilde{\xi}_{\text{diag}}(\delta)\)), a non-power-like scaling with \(L\) is observed. Finally, above \(\tilde{\xi}_{\text{diag}}(\delta)\) we observe \(L^2\) scaling. With the two effective correlation lengths, \(\xi_{\text{axial}}(\delta)\) and \(\tilde{\xi}_{\text{diag}}(\delta)\), we can associate two effective deviations \(\delta_{\text{axial}}(L)\) and \(\delta_{\text{diag}}(L)\), defined as the solutions of the equations \(\tilde{\xi}_{\text{axial}}(\delta)=L\) and \(\tilde{\xi}_{\text{diag}}(\delta)=L\), respectively. In the right panel, for fixed \(L=10^4\), on increasing \(\delta\), well below the effective deviation \(\delta_{\text{axial}}(L)\) in one of the axial directions, we observe \(\delta^2\) scaling, then between \(\delta_{\text{axial}}(L)\) and \(\delta_{\text{diag}}(L)\), and above \(\delta_{\text{diag}}(L)\), \(-\ln \mathcal{F}_{(1,0)}((0,1),\delta)\) exhibits an anomalous (non-power-like) scaling.

Let \(\xi_n(\delta)\) stand for the effective correlation length \(\tilde{\xi}_{(1,0)}((0,1),\delta)\) in direction given by \(n>1\) and let \(n_2>n_1>1\). From formulae \((30)\), \((32)\), and \((33)\) (see also Figs. 15, 16 and comments in section 5 in \cite{25}), one easily infers that for sufficiently small \(\delta\) the following inequalities hold true:

\[ \xi_{\text{axial}}(\delta) < \xi_{n_2}(\delta) < \xi_{n_1}(\delta) < \tilde{\xi}_{\text{diag}}(\delta), \]
provided \(n_1>n_0>1\); moreover, \(\xi_{\text{axial}}(\delta) = \lim_{n \rightarrow \infty} \xi_n(\delta)\). However, our numerical results presented in \cite{25} support the hypothesis that inequalities \((35)\) hold true for any \(n>1\), and then
\[ \xi_{\text{diag}}(\delta) = \lim_{n \to 1} \tilde{\xi}_n(\delta) \]. Let \( \tilde{\delta}_n(L) \) be the effective deviation in direction \( n \), i.e. the solution of the equation \( \tilde{\xi}_n(\delta) = L \) for some sufficiently large \( L \). Since \( \tilde{\xi}_n(\delta) \) is a decreasing function of \( \delta \), inequalities between effective correlation lengths in different directions imply analogous inequalities between the corresponding effective deviations:

\[
\tilde{\delta}_\text{axial}(L) < \tilde{\delta}_n(1) < \tilde{\delta}_{n-1}(L) < \tilde{\delta}_\text{diag}(L),
\]

for sufficiently large \( L \), where \( \tilde{\delta}_\text{axial}(L) = \lim_{n \to \infty} \tilde{\delta}_n(L) \), \( \tilde{\delta}_\text{diag}(L) = \lim_{n \to 1} \delta_n(L) \), and vice versa. From (35) one concludes that for sufficiently small deviation \( \delta \)

\[
\tilde{\xi}_\text{axial}(\delta) = \inf_{n>1} \tilde{\xi}_n(\delta) \equiv \tilde{\xi}_\text{lower}(\delta) \quad \text{and} \quad \tilde{\xi}_{\text{diag}}(\delta) = \sup_{n>1} \tilde{\xi}_n(\delta) \equiv \tilde{\xi}_\text{upper}(\delta),
\]

where we defined the lower effective correlation length \( \tilde{\xi}_\text{lower}(\delta) \) and the upper effective correlation length \( \tilde{\xi}_\text{upper}(\delta) \). Analogously, for sufficiently large \( L \), (36) implies that

\[
\tilde{\delta}_\text{axial}(L) = \inf_{n>1} \tilde{\delta}_n(L) \equiv \tilde{\delta}_\text{lower}(L) \quad \text{and} \quad \tilde{\delta}_\text{diag}(L) = \sup_{n>1} \tilde{\delta}_n(L) \equiv \tilde{\delta}_\text{upper}(L),
\]

where we defined the lower effective deviation \( \tilde{\delta}_\text{lower}(L) \) and the upper effective deviation \( \tilde{\delta}_\text{upper}(L) \).

Our numerical results, in particular those displayed in Fig. 2, show that there are two characteristic lengths, \( \tilde{\xi}_\text{lower}(\delta) \) and \( \tilde{\xi}_\text{upper}(\delta) \), or equivalently two characteristic deviations from the critical point, \( \tilde{\delta}_\text{lower}(L) \) and \( \tilde{\delta}_\text{upper}(L) \), that mark the crossover regions in the behavior of \(-\ln F_{(1,0)}((0,1),\delta)\) versus \( L \) or \( \delta \), respectively. We can naturally identify the range of sufficiently large but smaller than \( \tilde{\xi}_\text{lower}(\delta) \) values of \( L \), or the range of sufficiently large but smaller than \( \tilde{\delta}_\text{lower}(L) \) values of \( \delta \), as the small-system regime. Then, the ranges of \( L \) and \( \delta \), given by double inequalities \( \xi_{\text{lower}}(\delta)<L<\xi_{\text{upper}}(\delta) \) and \( \tilde{\delta}_{\text{lower}}(L)<\delta<\tilde{\delta}_{\text{upper}}(L) \) can be identified as the mesoscopic-system regimes. Finally, the ranges of \( L \) and \( \delta \) above \( \xi_{\text{upper}}(\delta) \) or \( \tilde{\delta}_{\text{upper}}(L) \), respectively – as the macroscopic-system regimes.

Remarkably, in spite of the fact that the fidelity does not depend explicitly on spatial directions \( n \), a fingerprint of the dependence of correlation properties on spatial directions can be seen in the behavior of the fidelity as a function of \( L \) or \( \delta \).
Having defined the characteristic lengths, $\xi_{\text{lower}}(\delta)$ and $\xi_{\text{upper}}(\delta)$, and characteristic deviations from the critical point, $\delta_{\text{lower}}(L)$ and $\delta_{\text{upper}}(L)$, which mark the crossover regions in the behavior of fidelity, and then the regimes of small- and macroscopic-system, we can describe our numerical results, from the perspective of the scaling laws (19), (20). We note that the divergence of $\xi_{\text{lower}}(\delta)=\xi_{\text{axial}}(\delta)$ as $\delta\to0$ is characterized by $\nu_{\text{offdiag}}=1/2$, which satisfies the condition $D\nu<2$. Then, in the small-system regime of $L<\xi_{\text{lower}}(\delta)$ the $L^4$ scaling of fidelity is consistent with (19) for $\nu=1/2$, which matches $\nu_{\text{offdiag}}$. In the macroscopic-system regime of $L>\xi_{\text{upper}}(\delta)$ we observe the standard (see section 3) $L^2$ scaling of fidelity. On the other hand the scaling of fidelity with respect to $\delta$ does not provide any information about the exponent $\nu$. In the small-system regime of $\delta<\delta_{\text{lower}}(L)$ we observe the standard (see section 3) $\delta^2$ scaling of fidelity, while in the macroscopic-system regime of $\delta>\delta_{\text{upper}}(L)$ the fidelity behaves in an anomalous, non-power-law way, which does not allow for estimating $\nu$ via (20). We note however, that $\tilde{\xi}_{\text{upper}}(\delta)$, which is used to calculate $\tilde{\delta}_{\text{upper}}(L)$, diverges as $\delta^{-1}$ if $\delta\to0$ ($\nu_{\text{diag}}=1$), which violates the condition $D\nu<2$.

4.2 Critical points in the intervals $J=0$, $0<|\mu|<2$

In $J$-path neighborhoods of the critical points at the line segments $J=0$ and $0<|\mu|<2$,

$$\frac{1}{\xi^{(+)}} \approx \frac{2|\mu|J}{\sqrt{4-\mu^2}}, \quad \theta \approx 2\arccos \frac{|\mu|}{2}. \quad (39)$$

$$\frac{1}{\xi_1} \approx |J|\sqrt{\frac{|\mu|}{2-|\mu|}}, \quad \theta_1 \approx \pi - 2 \arcsin \sqrt{\frac{|\mu|}{2}} + \frac{1}{2} \sqrt{\frac{3-|\mu|}{2-|\mu|}} - \frac{|\mu|}{2} |J|^2, \quad (40)$$

$$\frac{1}{\xi_2} \approx \frac{|\mu| - 1}{2} \frac{1}{\xi_1}, \quad \theta_2 \approx \frac{1}{2} \sqrt{\frac{|\mu|}{2-|\mu|}}, \quad (41)$$

for sufficiently small $|J|$.

In those neighborhoods, the effective correlation lengths $\tilde{\xi}(\delta)$, as functions of $\delta \equiv J$ and $n$, share the properties that were used in the previous subsection to define the lower and upper effective correlation lengths, $\tilde{\xi}_{\text{lower}}(\delta)$ and $\tilde{\xi}_{\text{upper}}(\delta)$. Therefore, we can adopt the same definitions of these lengths, the associated effective deviations, $\tilde{\delta}_{\text{lower}}(L)$ and $\tilde{\delta}_{\text{upper}}(L)$, and the small- and macroscopic-system regimes. In the special case of $\mu=1$, all the hierarchy of correlation lengths and the associated deviations collapses, $\tilde{\xi}_{\text{lower}}(\delta) \approx \tilde{\xi}_{\text{upper}}(\delta)$ for sufficiently small $\delta$, and $\tilde{\delta}_{\text{lower}}(L) \approx \tilde{\delta}_{\text{upper}}(L)$ for sufficiently large $L$. The critical exponents characterizing the divergence of $\tilde{\xi}_{\text{lower}}(\delta)$ and $\tilde{\xi}_{\text{upper}}(\delta)$ coincide, $\nu_{\text{offdiag}}=\nu_{\text{diag}}=1$, and violate the condition $D\nu<2$. In Fig. 3 we show an example of the behavior of fidelity in neighborhoods of the considered critical points. In the left panel, the plot of $-\ln F_{(0,1)}((1,0),10^{-6})$ versus $L$ exhibits pronounced oscillations, whose amplitude is particularly large for $L<\xi_{\text{sl}}$ and decreases with increasing $L$. We remark that the nature of those oscillations has been revealed in [16]. As we mentioned in section 3 (see also [24]) on presenting small-system scaling law, in case of oscillating fidelity one should consider scaling in the generalized sense. That is, in such a case it is the envelope of the minima of $-\ln F_{\text{sl}}(\lambda, \delta)$ that is the right quantity whose scaling should be studied. In the left panel, one crossover region separating different power-law behaviors, in the generalized sense, marked by some $\xi_{\text{sl}}(\delta)$, is visible. However, $\xi_{\text{sl}}$ does not match $\tilde{\xi}_{\text{lower}}(\delta) \approx \tilde{\xi}_{\text{upper}}(\delta)$, whose values are a lot larger than $\xi_{\text{sl}}(\delta)$. If we naively identify
the regime of $L<\xi_{sl}$ as the small-system regime, then the observed $L^4$-scaling of fidelity implies, via (19), $\nu=1/2$, which does not match the exact value 1 of this exponent. Moreover, $L^4$-scaling of $-\ln F_\epsilon(\lambda, \delta)$, consequently superextensivity, interestingly occurs despite the fact that the condition of superextensive behavior, which follows from critical-scaling theory, $D\nu<2$, is violated. For $L>\xi_{sl}$, identified as the macroscopic-system regime we observe the standard $L^2$-scaling. In the right panel, the plot of $-\ln F_\epsilon(1, 0, \delta)$ versus $\delta$ exhibits also one crossover region, separating different power-law behaviors, marked by some $\delta_{sl}$, which is the solution of the equation $\xi_{sl}(\delta)=L$. The power law (20) applied for $\delta$ well above $\delta_{sl}$ gives the incorrect result $\nu=1/2$. We note that similar results hold for other values of $\mu$, $|\mu|\neq 1$. The only difference is that $\xi_{lower}(\delta)<\xi_{upper}(\delta)$, but still $\xi_{lower}(\delta)\gg\xi_{sl}(\delta)$; thus, $\xi_{lower}(\delta)$ does not locate the crossover region correctly.

### 4.3 The end critical points $J=0$, $|\mu|=2$

If $(\mu, J)$-points approach along the $J$-path one of the two end critical points, $|\mu|=2$ and $J=0$, 

$$\frac{1}{\xi(\pm)} \approx 2|J|^{1/2}, \quad \theta \approx 2|J|^{1/2},$$  \hspace{1cm} (42)  

$$\frac{1}{\xi_1} \approx \sqrt{2|J|} = \frac{2}{\xi_2}, \quad \theta_1 \approx \sqrt{2|J|} = 2\theta_2,$$  \hspace{1cm} (43)  

for sufficiently small $J$.

In a $J$-path vicinity of one of the end critical points $J=0$ and $|\mu|=2$, setting $\delta=J$, we can adopt the same definitions of $\xi_{lower}(\delta)$ and $\xi_{upper}(\delta)$ of the small-, mesoscopic- and macroscopic-system regimes as in subsections 4.1 and 4.2. However, using (42) and (43) one verifies easily that, for sufficiently small $J=\delta$, $\xi_{axial}(\delta) \approx \xi_n(\delta) \approx \xi_{diag}(\delta)$, hence $\xi_{lower}(\delta) \approx \xi_{upper}(\delta)$. Therefore, there is no mesoscopic-system regime, there is only one characteristic

![Figure 3: (Color online) The symmetric model. Plots of $-\ln F_{(0,1)}((1, 0), \delta)$ for three values of $c$: $c=0$ – red line, $c=1$ – green line, $c=2$ – blue line. Left panel: plots of $-\ln F_{(0,1)}((1, 0), 10^{-6})$ versus $L$. Right panel: plots of $-\ln F_{(0,1)}((1, 0), \delta)$ versus $\delta$ for $L=10^3$. Both plots are in doubly logarithmic scale and the black dashed-dotted straight lines indicate the power-law scaling. The variable $L$ in the left panel changes by 4 up to 10,000, but above 10,000 the formula $L \rightarrow 4[1.05L/4]$ is used, where $\lceil \cdot \rceil$ denotes rounding up to the nearest integer (ceiling function). The latter way of sampling $L$ reveals the $L^2$ behaviour for sufficiently large $L$, where for computational as well as presentation reasons, continuing to change $L$ by 4 is no longer feasible. This formula has an advantage of evenly spacing values of $L$ in logarithmic scale but fails to capture properly the character of the oscillations for sufficiently small $L$.](image-url)
to perform; see also discussions of multicritical points in [16], [24]. Apparently, multicritical points are not amenable to the kind of scaling analysis that we try to perform, via (19), giving, via (19), \( \nu \) \( \lambda > L > \xi \). One perspective of the fidelity scaling laws. We observe the standard scaling for \( \delta > \delta \). Critical scaling theory, identify the region of \( L < \xi \) \( \xi \) marked by \( \xi \). Numerically calculated effective correlation lengths \( \tilde{\xi} \) against \( L \). \( \tau \) \( \lambda > \xi \). Two calculated exponents, \( \mu = 4.4 \) The multicritical point \( \mu = 0 = J \). In this case we cannot provide any analytic results concerning the correlation length. All the results referring to this case have been obtained numerically. The two calculated exponents, \( \nu_{\text{offdiag}} = 3/2, \nu_{\text{diag}} = 2 \), violate the condition \( D \nu < 2 \). Figure 4: (Color online) The symmetric model. Plots of \( -\ln F_{(0,1)}((2, 0), \delta) \) for three values of \( c \): \( c = 0 \) – red line, \( c = 1 \) – green line, \( c = 2 \) – blue line. Left panel: plots of \( -\ln F_{(0,1)}((2, 0), 10^{-6}) \) versus \( L \). Right panel: plots of \( -\ln F_{(0,1)}((2, 0), \delta) \) versus \( \delta \) for \( L = 10^3 \). Both plots are in doubly logarithmic scale and the black dashed-dotted straight lines indicate the power-law scaling. For more details see the text. 4.4 The multicritical point \( \mu = 0 = J \)
Figure 5: (Color online) The symmetric model. Plots of $-\ln F_{(1,1)}((0,0), \delta)$ for three values of $c$: $c=0$ – red line, $c=1$ – green line, $c=2$ – blue line. Left panel: plots of $-\ln F_{(1,1)}((0,0), 10^{-4})$ versus $L$. The calculated effective correlation lengths $\xi_{\text{axial}}(\delta)$ and $\xi_{\text{diag}}(\delta)$ are too large to be depicted in the figure. Right panel: plots of $-\ln F_{(1,1)}((0,0), \delta)$ versus $\delta$ for $L = 10^4$. The effective deviations $\tilde{\delta}_{\text{axial}}(L)$ and $\tilde{\delta}_{\text{diag}}(L)$ are well above $\delta_{mm}$. Both plots are in doubly logarithmic scale and the black dashed-dotted straight lines indicate the power-law scaling. For more details see the text.

5 The case of antisymmetric model

It is worth to mention that the Hamiltonian of the antisymmetric model can be obtained as a mean-field approximation to

$$\sum_{k,\sigma} \varepsilon_k c_{k,\sigma}^\dagger c_{k,\sigma} + J \sum_{l,i} S_l S_{l+e_i}, \quad (44)$$

where $S_l$ stands for the spin operator of a spin 1/2 fermion at site $l$ of the underlying lattice (for details of the notation see section 2). The parameters $\Delta_i$ of the mean-field Hamiltonian are no longer free parameters but are given implicitly by those solutions of the equations

$$\Delta_i = -\langle a_{l,\uparrow} a_{l+e_i,\downarrow} - a_{l,\downarrow} a_{l+e_i,\uparrow} \rangle, \quad (45)$$

where the brackets denote the Gibbs average with the mean-field Hamiltonian, that minimize the ground-state energy – physical solutions. When the underlying lattice is a square lattice, it turns out that the physical solutions satisfy the condition $\Delta_1 = -\Delta_2$, which corresponds to the so called $d_{x^2-y^2}$ pairing in theory of $d$-wave superconductivity.

In distinction to the previously considered case of the symmetric two-dimensional model, where the critical points are located at straight, intersecting lines, the quantum-critical points of the antisymmetric two-dimensional model fill up the stripe that extends between the two lines $|\mu|=2$, see the phase diagram in Fig. 6. Therefore, there are only two doubly-asymptotic regions of interest, where analytic formulae for $G(\mathbf{r})$ can be derived, namely those where $(\mu, J)$-points, with $|\mu|>2$ and $|J|$ not too close to zero, approach along a $\mu$-path a point belonging to one of the lines $|\mu|=2$. In these regions the asymptotic behavior of $G(\mathbf{r})$ in the diagonal direction is

$$G(\mathbf{r}', \mathbf{r}') \approx -\text{sgn}(\mu) \frac{J}{4\pi \xi(-)} \frac{\exp(-r'/\xi(-))}{r'}, \quad (46)$$

where

$$\frac{1}{\xi(-)} \approx 2\sqrt{\frac{|\mu|-2}{1+J^2}}, \quad (47)$$
that is, in the considered doubly-asymptotic region, the correlation length in the diagonal direction amounts to $\xi_{\text{diag}} = \sqrt{2}\xi^{(-)}$. Then, in offdiagonal directions we obtained the asymptotic formula

$$G(r_1, r_2) \approx -\frac{C_r}{2\pi} \left( \frac{\mu^2 J^2}{1 + J^2} \right)^{1/4} \left( \frac{1 + n^2}{n^2} \right)^{1/2} \exp \left( -\frac{r/\xi_{\text{offdiag}}}{r} \right) \cos(r\theta_{\text{offdiag}}^{(-)} + \phi^{(-)}),$$

(48)

with

$$\frac{1}{\xi_{\text{offdiag}}^{(-)}} = \left( \frac{n^2}{1 + n^2} \right)^{1/2} \left( \frac{1}{\xi_1^{(-)}} \right) + \frac{1}{n^2} \frac{1}{\xi_2^{(-)}},$$

(49)

and

$$\theta_{\text{offdiag}}^{(-)} = \left( \frac{n^2}{1 + n^2} \right)^{1/2} \left( \theta_1^{(-)} \right) + \frac{1}{n^2} \theta_2^{(-)},$$

(50)

provided that the points $(r_1, r_2)$ become remote from the origin along a ray $r_1/r_2 = n = \text{const}$.

Sufficiently close to the lines $|\mu| = 2$, the values of $\xi_1^{(-)}$, $\xi_2^{(-)}$, $\theta_1^{(-)}$, $\theta_2^{(-)}$, and $\phi^{(-)}$, which determine the offdiagonal correlation length $\xi_{\text{offdiag}}^{(-)}$ in a direction $n$ are given as follows:

$$\frac{1}{\xi_1^{(-)}} \approx \sqrt{\frac{1 + J^2}{1 + J^2}} \sqrt{|\mu| - 2}, \quad \theta_1^{(-)} \approx \sqrt{\frac{1 + J^2 - 1}{1 + J^2}} \sqrt{|\mu| - 2}, \quad \phi^{(-)} = \frac{\pi}{4},$$

(51)

$$\frac{1}{\xi_2^{(-)}} \approx -\left( \frac{1}{2} - \frac{1}{\sqrt{1 + J^2}} \right) \frac{1}{\xi_1^{(-)}},$$

(52)

$$\theta_2^{(-)} \approx -\left( \frac{1}{2} + \frac{1}{\sqrt{1 + J^2}} \right) \theta_1^{(-)}.$$

(53)

For the considered in this section critical points $\delta \equiv \mu$. In distinction to the symmetric model, the hierarchy of direction-dependent effective correlation lengths is not uniform in $J$ \[25\]. However, for $J > J_0$, with $J_0 \approx 1/4$, it is the same as in the symmetric case and is given in \[35\]. Consequently, for $J > J_0$ the lower and upper effective correlation lengths and...
Figure 7: (Color online) The antisymmetric model. Plots of \(-\ln F_{(1,0)}((2,1),\delta)\) for three values of \(c\): \(c=0\) – red line, \(c=1\) – green line, \(c=2\) – blue line. Left panel: plots of \(-\ln F_{(1,0)}((2,1),10^{-6})\) versus \(L\). Right panel: plots of \(-\ln F_{(1,0)}((2,1),\delta)\) versus \(\delta\) for \(L = 10^3\). Both plots are in doubly logarithmic scale and the black dashed-dotted straight lines indicate the power-law scaling. For more details see the text.

The associated lower and upper deviations from the critical point are given by (37) and (38), respectively. The critical exponents are direction independent, \(\nu_{\text{offdiag}}=\nu_{\text{diag}}=1/2\), and satisfy the condition \(D\nu<2\). In Fig. 7 we show plots of fidelity versus \(L\) and \(\delta\) in the particular case of \(J=1\). Since in the scale of Fig. 7 \(\bar{\xi}_{\text{lower}}(\delta)\) and \(\bar{\xi}_{\text{upper}}(\delta)\) are quite close to each other, hence \(\bar{\delta}_{\text{lower}}(L)\) and \(\bar{\delta}_{\text{upper}}(L)\) are close as well, one can hardly distinguish two crossover regions, so the mesoscopic-system regime between them is very narrow. We can say that there is just one crossover region whose position is given by, say, \(\bar{\xi}_{\text{lower}}(\delta)\) (left panel) or by \(\bar{\delta}_{\text{lower}}(L)\) (right panel). In the small-system regime of \(L<\bar{\xi}_{\text{lower}}(\delta)\) fidelity scales as \(L^4\), which via (19) gives \(\nu = 1/2\) in agreement with the exact value, while in the small-system regime of \(\delta<\bar{\delta}_{\text{lower}}(L)\) – the standard \(\delta^2\) scaling is observed. Then, in the macroscopic-system regime of \(L>\bar{\xi}_{\text{lower}}(\delta)\) the standard scaling \(L^2\) holds, while in the macroscopic-system regime of \(\delta>\bar{\delta}_{\text{lower}}(L)\) fidelity scales as \(\delta\), which via (20) reproduces again the exact value \(\nu=1/2\). Thus, in the case of antisymmetric model the predictions of critical scaling theory of fidelity match very well the exact results.

6 Summary

The standard way of characterizing quantum-critical points is to study the large-distance asymptotic behavior of two-point correlation functions, which is typically a hard task, irrespectively whether carried out numerically or analytically. Such studies provide the correlation lengths \(\xi\) and, in neighborhoods of quantum-critical points, the corresponding critical exponents \(\nu\). The novelty of 2D models, as compared to 1D ones, is that the correlation lengths and the corresponding critical exponents \(\nu\) may depend on direction. We show that they indeed do depend on lattice directions by providing analytic formulae for the large-distance asymptotic behavior of two-point correlation functions, the correlation lengths, and the values of the corresponding exponents \(\nu\) (a more elaborate analysis can be found in [25]). Moreover, in many studied instances the values of \(\nu\) are so large that the crucial in quantum-critical-scaling condition \(D\nu<2\), that is in our case \(\nu<1\), is violated.

The quantum-fidelity approach makes an attempt to extract similar characteristics of quantum-critical points, by studying the asymptotic behavior of quantum fidelity \(F_e(\lambda, \delta)\), defined in [3], as a quantum-critical point is approached. At least for small systems, that is those whose linear size is well below the correlation length, this task is typically more easy
to carry out than the standard one. Therefore, the quantum-fidelity approach would be an appealing alternative approach to the standard one, if we were able to extract $\xi$ and $\nu$ by studying scaling properties of $\mathcal{F}_e(\lambda, \delta)$.

It has been demonstrated, by studying examples involving one-dimensional models, that this is indeed the case. Specifically, quantum-critical points of a few exactly-solvable one-dimensional systems, so that calculations of fidelity could have been carried out not only for small systems but also for macroscopic ones, have been studied from the fidelity perspective. The list of those models includes: an Ising chain in a transverse magnetic field [15], an anisotropic XY chain in a transverse magnetic field [19], and the 1D version of the model considered in this paper [24]. We note that for all quantum-critical points exhibited by those models the condition $D\nu<2$ is satisfied. Then, for those quantum-critical points that are characterized by a unique correlation length, that is excluding the multicritical points, it has been shown that in critical neighborhoods in $\mathcal{F}_e(\lambda_c, \delta)$ (or its envelope in case of oscillations) scales according to a power law, in agreement with scaling laws (19) – for small systems and (20) – for macroscopic ones. The regimes of small- and macroscopic-systems, parameterized by system’s linear size $L$ (or a distance $\delta$ from a critical point), are separated by a crossover region whose location determines, as we call it, the effective correlation length $\xi_e(\lambda_c, \delta)$ (or the effective deviation $\delta(L)$ being the solution to $\xi_e(\lambda_c, \delta)=L$). The latter quantity amounts to the smaller one from the two correlation lengths corresponding to two ground states used to calculate fidelity.

The major question we have attempted to answer in this paper reads: to what extent the scenario sketched above holds in 2D systems? In particular, what are the characteristic lengths (deviations from a critical point) that mark crossover regions in case of direction-dependent effective correlation lengths?

Consider first ordinary quantum-critical points characterized by the unique correlation length. The examples studied in previous sections show that if condition $D\nu<2$ holds true in any direction, then fidelity exhibits two crossover regions. One of them separates the small-system regime from a mesoscopic regime. Its position is given by the lower effective correlation length $\xi_{\text{lower}}(\delta)$ (or the lower effective deviation $\delta_{\text{lower}}(L)$); in the small-system regime, that is well below $\xi_{\text{lower}}(\delta)$ (or $\delta_{\text{lower}}(L)$), scaling law (19) is satisfied. The second crossover region separates the mesoscopic regime from the macroscopic-system regime and is located by the upper effective correlation length $\xi_{\text{upper}}(\delta)$ (or the upper effective deviation $\delta_{\text{upper}}(L)$); in the macroscopic-system regime, that is well above $\xi_{\text{upper}}(\delta)$ (or $\delta_{\text{upper}}(L)$), scaling law (20) holds true.

Then, if condition $D\nu<2$ is satisfied only for $\nu$ corresponding to $\xi_{\text{lower}}(\delta)$ (or to $\xi_{\text{upper}}(\delta)$), then only small-system scaling (19) (macroscopic-system scaling (20)) is obeyed by $\ln \mathcal{F}_e(\lambda_c, \delta)$ for sufficiently large $L$ or $\delta$ but well below $\xi_{\text{lower}}(\delta)$ or $\delta_{\text{lower}}(L)$, respectively (for $L$ or $\delta$ well above $\xi_{\text{upper}}(\delta)$ or $\delta_{\text{upper}}(L)$).

After that, if condition $D\nu<2$ is violated in any direction, then $\ln \mathcal{F}_e(\lambda_c, \delta)$ may scale, in the generalized sense, according to a power law, and exhibit some crossovers, but neither the positions of crossovers are given by some $\xi_e(\delta)$ nor the scaling law (19) – for sufficiently small $L$ or $\delta$ and (20) – for sufficiently large $L$ or $\delta$ is satisfied. In this case, of note is also the superextensive behaviour of $-\ln \mathcal{F}_e(\lambda_c, \delta)$, see section 4.1 which seems to defy the scaling arguments, according to which should not be the case [18 [19 [20]. This curious phenomenon perhaps merits some further investigations into the properties of fidelity itself.

Finally, in a critical neighborhood of a multicritical point, the behavior of $\ln \mathcal{F}_e(\lambda_c, \delta)$ is as in the cases, where $D\nu<2$ is violated in any direction.
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