Forbidden Information

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To the memory of Andrei Kolmogorov,
In the 100th year since his birth (4/25/1903).

Abstract

Gödel Incompleteness Theorem leaves open a way around it, vaguely perceived for a long time but not clearly identified. (Thus, Gödel believed informal arguments can answer any math question.) Closing this loophole does not seem obvious and involves Kolmogorov complexity. (This is unrelated to, well studied before, complexity quantifications of the usual Gödel effects.) I consider extensions $U$ of the universal partial recursive predicate (or, say, Peano Arithmetic). I prove that any $U$ either leaves an $n$-bit input (statement) unresolved or contains nearly all information about the $n$-bit prefix of any r.e. real $\rho$ (which is $n$ bits for some $\rho$). I argue that creating significant information about a specific math sequence is impossible regardless of the methods used. Similar problems and answers apply to other unsolvability results for tasks allowing non-unique solutions, e.g., non-recursive tilings.

1 Introduction.

D. Hilbert asked if Peano Arithmetic (PA: consisting of logic and algebraic axioms and an infinite family of Induction Axioms) can be consistently extended to a complete theory. The question was somewhat vague since an obvious answer was “yes”: just add to PA axioms a maximal consistent set, clearly existing albeit hard to find.\footnote{K.Gödel formalized this question as existence, among such extensions, of recursively enumerable (r.e.) ones and gave it a negative answer. Its mathematical essence is the absence of total recursive extensions of universal partial recursive predicate.} K.Gödel formalized this question as existence, among such extensions, of recursively enumerable (r.e.) ones and gave it a negative answer. Its mathematical essence is the absence of total recursive extensions of universal partial recursive predicate.

This negative answer apparently was never accepted by Hilbert, and Gödel himself \cite{G61} had reservations: “Namely, it turns out that in the systematic establishment of the axioms of mathematics, new axioms, which do not follow by formal logic from those previously established, again and again become evident. It is not at all excluded by the negative results mentioned earlier that nevertheless every clearly posed mathematical yes-or-no question is solvable in this way. For it is just this becoming evident of more and more new axioms on the basis of the meaning of the primitive notions that a machine cannot imitate.”

As is well known, the absence of algorithmic solutions is no obstacle when the requirements do not make a solution unique. A notable example is generating strings of linear Kolmogorov complexity, e.g., those that cannot be compressed to half their length. Algorithms fail, but a set of dice does a perfect job!\footnote{I assume PA is consistent: another can of worms.}

\footnote{Note that no such problems arise for the weaker and less dramatic version of Gödel Theorem which makes the extension unique by requiring $\omega$-consistency (it means $\exists x A(x)$ cannot be proven if $A(x)$ can be refuted for each specific constant $x$). \cite{dLMSS} proved that no randomized algorithm has a positive probability to solve a task with a unique non-recursive solution. \cite{Barzdin} and \cite{Jockusch,Soare} observed that this is no longer the case when the solution is not unique, e.g., for such interesting tasks as listing infinite subsets of an immune co-r.e. set.}
Thus, while r.e. sets of axioms cannot complete PA, completion by other realistic means remained a possibility. In fact, it is easy to construct an r.e. theory $R$ that, like PA, allows no consistent completion with r.e. axiom sets. Yet, [Barzdin 69, Jockusch, Soare 72] showed that this theory (though not PA itself) allows a recursive set of pairs of axioms such that random choice of one in each pair assures such completion with probability 99%.

Of course, Gödel’s remark envisioned more sophisticated ways to choose axioms :-). However, the impossibility of a task can be formulated more generically. [Kolmogorov 65] defined a concept of mutual information in two finite strings. It can be refined and extended to infinite sequences, so that it satisfies conservation inequalities: cannot be increased by deterministic algorithms or in random processes or with any combinations of both. In fact, it seems reasonable to assume that no physically realizable process can increase information about a specific sequence.

In this framework one can ask if non-mechanical means could really enable the Hilbert-Gödel task of consistent completion for PA (as they do for the artificial system $R$ just mentioned). A negative answer follows from the existence of a specific sequence $\rho$ that has infinite mutual information with all total extensions of a universal partial recursive predicate. $\rho$ plays a role of a password: no substantial information about it can be guessed, no matter what methods are allowed.

Note that invoking Gödel’s name, does not mean my intent to consider widely discussed complexity aspects and implications of incompleteness theorem. In particular, I ignore complexity of completions of PA. Much of this was considered in the 60-s,\(^3\) but does not answer our question: are such completions really possible? Strings of any complexity are easy to generate.

There are other interesting situations with a similar gap between the proven result and its usual interpretation. Let me mention tiling, a cute task studied in many areas of CS, Math, Physics, etc. A tile is a unit size square with colored edges. A palette is a finite set of tiles with copies of which one can tile the plane so that the colors of adjacent tiles match. Classical papers by Berger, Meyers, and others constructed palettes $P$ that can tile an infinite plane, but only non-recursively, which is typically interpreted as an impossibility of tiling. That is, all programs $t(i, j) \in P$ have bounds on sizes $n$ of squares $S_n = \{i, j < n\}$ they tile so that $t(S_n)$ can appear on $P$-tiled planes. [Durand, Levin, Shen 01] pushes these results to the limit, with a palette for which $n < \|t\|$. Such palettes, thus, only allow tilings with square borders of maximal complexity. This stronger result, though, makes the standard interpretation suspicious: may these borders be just random, thus easy to tile with dice? Or could more sophisticated and yet realistic means work?

For some palettes this is, indeed, the case, but not for all. Like all co-r.e. sets, the set of planar tilings with any given palette has members with information about any specific sequence growing with radius $n$ as slowly as $\log n$. Still, this bound cannot be improved for some palettes. The same holds for complete extensions of universal p.r.p. and formal systems. Thus, Gödel Theorem is not really misleading. The proof does not seem to be obvious and is the main point of this article.

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\(^3\)The original publications, such as [Barzdin 68], gave the technical cores of the results and avoided discussion of straightforward implications for formal theories, Gödel Theorem, etc. These implications were discussed in talks. An example is [Kolmogorov 72] where also relevant names of the technical authors can be found. I do not provide the bibliography, since these results are only superficially related to the issues I address here.
2 Complexity Tools.

Conventions. Let \( \mathbb{R}, \mathbb{Q}, \mathbb{N}, B=\{0,1\}, S=B^*, \Omega=B^\mathbb{N} \) be, respectively, the sets of non-negative reals, rationals, integers, bits, finite, and infinite binary sequences (\( \mathbb{N}, S \) usually identified); \( x_{[n]} \) is the \( n \)-bit prefix and \( \|x\| \) is the bit-length of \( x \in S \). A real function \( f \) and its values are enumerable or r.e. (co-r.e. for \(-f\)) if its subgraph \( \{(x,q) : f(x) > q \in \mathbb{Q}\} \) is. Elementary (\( f \in \mathcal{E} \)) are functions \( f : \Omega \to \mathbb{Q} \) depending on a finite number of digits. Majorant is an r.e. function largest, up to a constant factor, among r.e. functions in its class.

2.1 Integers: Complexity, Randomness.

Let us define Kolmogorov complexity \( K(x) \) as \([-\log m(x)]\) where \( m : \mathbb{N} \to \mathbb{R} \) is the universal measure, i.e., a majorant r.e. function with \( \sum_x m(x) \leq 1 \). It was introduced in [ZL 70], and noted in [L 73, L 74, Gacs 74] to be a modification (restriction to self-delimiting codes) of the least length of binary programs for \( x \) defined in [Kolmogorov 65]. While technically different, \( m \) relies on intuition similar to that of [Solomonoff 64]. The proof of the existence of the largest function was a straightforward modification of proofs in [Solomonoff 64, Kolmogorov 65] which have been a keystone of the informational complexity theory.

For \( x \in \mathbb{N}, y \in \mathbb{N} \) or \( y \in \Omega \), similarly, \( m(\cdot) \) is the largest r.e. real function with \( \sum_x m(x|y) \leq 1 \);

\[ K(x|y) \leq -\log m(x|y) \] (and is the least length of self-delimiting programs transforming \( y \) into \( x \)).

[Kolmogorov 65] defines randomness deficiency \( d(x) \) of uniformly distributed \( x \) as \( \|x\| - K(x) \).

Our modified \( K \) allows extending this to other measures \( \mu \) on \( \mathbb{N} \). A \( \mu \)-test is \( f : \mathbb{N} \to \mathbb{R} \) with mean \( \mu(f) \leq 1 \) (and, thus, small values \( f(x) \) on randomly chosen \( x \)). For computable \( \mu \), a majorant r.e. test is \( m(x)/\mu(x) \). This suggests defining \( d(x|\mu) \) as \( |\log \mu(x)| - K(x) = |\log(m(x)/\mu(x))| \pm O(1) \).

2.2 Integers: Information.

In particular, \( x=(a,b) \) distributed with \( \mu=m \otimes m \), is a pair of two independent, but otherwise completely generic, finite objects. Then, \( I(a:b) \stackrel{df}{=} d((a,b)|m \otimes m) = K(a)+K(b) - K(a,b) \) is seen as deficiency of independence and also measures mutual information in \( a,b \). It was shown (see [ZL 70]) by Kolmogorov and Levin to be close (within \( \pm O(\log K(a,b)) \)) to the expression \( K(a)-K(a\|b) \) of [Kolmogorov 65]. Unlike this earlier expression (see [Gacs 74]), our \( I \) is symmetric and monotone: \( I(a:b) \leq I((a,a') : b) + O(1) \) (which will allow extending \( I \) to \( \Omega \)): it equals \( K(a) - K(a\|b,K(b)) \pm O(1) \) and satisfies the following Independence Conservation Inequalities [L 74, L 84]: For any computable transformation \( A \) and measure \( \mu \), and some family \( t_{a,b} \) of \( \mu \)-tests

\[ I(A(a):b) \leq I(a:b) + O(1), \quad I((a,w):b) \leq I(a:b) + \log t_{a,b}(w) + O(1). \]

(The \( O(1) \) error terms reflect the constant complexities of \( A, \mu \).) So, independence of \( a \) from \( b \) is preserved in random processes, in deterministic computations, their combinations, etc. These inequalities are not obvious (and false for the original 1965 expression \( I(a:b) = K(a) - K(a\|b) \)) even with \( A \), say, simply cutting off half of \( a \). An unexpected aspect of \( I \) is that \( x \) contains all information about \( k=K(x) \), \( I(x : k) = K(k) \pm O(1) \), despite \( K(k|x) \) being \( \sim \|k\| \) or \( \sim \log \|x\| \), in the worst case [Gacs 74]. One can view this as an “Occam Razor” effect: with no initial information about it, \( x \) is as hard to obtain as its simplest (\( k \)-bit) description.

All the above works as well for the \( I_z \) variation of \( I \) allowing all algorithms access to oracle \( z \).
2.3 Randomness and Information for Reals.

We now extend the above concepts to reals $\alpha \in \Omega$. This abstraction is often convenient (if not taken too far) for concealing $O(1)$ terms and other small mismatches in formulas for finite objects. A measure on $\Omega$ is a function $\mu(x)=\mu(x0)+\mu(x1)$, for $x\in S$. Its mean $\mu(f)$ is a linear functional on $E$: $\mu(f+g)=\mu(f)+\mu(g)$. It extends to other functions, as usual. $\mu$-tests are lower semi-continuous $f$, $\mu(f) \leq 1$; computable $\mu$ have a universal (i.e., majorant r.e.) $\mu$-test $T_\mu$. The deficiency of $\mu$-randomness is $d(\alpha|\mu) = \log T_\mu(\alpha)$. Martin-Lof $\mu$-random are those reals $\alpha$ with $\mathbf{d}(\alpha|\mu) < \infty$.

P. Martin-Lof noted that some random reals are definable in arithmetic. In fact, the smallest in $[0,1]$ real with the minimal deficiency of randomness (see [ZL 70], sec. 4.4) is random $\alpha_{[0,1]}$. Random are exactly those $\alpha$ that give a stronger than $\mathbf{I}(\alpha)$ expression for $\mathbf{K}(\alpha)$. Any random r.e. $\alpha$ is an expression for $\mathbf{K}(\alpha)$, see [ZL 70, Downey Hirschfeldt Nies 02]. Random are exactly those r.e. $\alpha$ that dominate all others (see [Kucera, Slaman 01]).

2.3 Randomness and Information for Reals: Stronger Version.

For $\alpha, \beta \in \mathbb{N}$, this equals our previous expression $\mathbf{K}(\alpha)+\mathbf{K}(\beta)-\mathbf{K}(\alpha, \beta) \pm O(1)$ since both satisfy the above Independence Conservation Inequalities. In fact, the above extension of $\mathbf{I}$ to $\Omega$ is the smallest satisfying the independence conservation. (The largest is described below.)

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[L 74] extends $\mathbf{I}$ to reals as $\mathbf{I}(\alpha : \beta) = \log \sum \mathbf{m}(\alpha|\beta) \mathbf{I}(i:j)$. (As always, we average in the linear scale and switch to the logarithmic scale for the final expression.)

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4Indeed, let $t_{a,b}(x)$ be 1 for $x < a < b$ and 0 elsewhere. Let an r.e. $T = \sum_i c_i t_{a_i, b_i}$ be an r.e. test with $T(X+Y) = \infty$, $\sum_i c_i (b_i-a_i) \leq 1$. Since $X, Y$ are r.e. , we may modify $T$ by blocking the intervals $(a_i, b_i)$ if $a_i \geq X+Y$ or shifting them by $y_i < Y$ to $(a_i-y_i, b_i-y_i)$ with $a_i-y_i < X$. Then $T$ remains an r.e. test with $T(X) = \infty$.

Conversely, let $Z, X \in [0,1]$ be enumerated as $Z = \sup z_i, X = \sup x_i, x_i' = x_i - x_{i-1} \geq 0$. Let $s_i = \max \{z_i, s_{i-1}+x_i' \}$ and $t_k(V)$ be 1 for $s_{i-1} < V < s_i-s_{i-1}+x_i'$ and 0 elsewhere. Then $2^n t_k$ and $T(V) = \sum_k (2^k/k^k) t_k$ are r.e. tests. Let $y_i = s_i-s_{i-1}$ and $y_i' = s_i-s_{i-1}+x_i'$. If $Z = \sup s_i$ for some $k$, then $Z = Y + Z/Y$ for some $i$, so $t_k(Z) = 1$ for all $k$ and $T(Z) = \infty$. Otherwise $s_{i-1} < Z < s_i-s_{i-1}+x_i'$ for some $i$, so $t_k(Z) = 1$ for all $k$ and $T(Z) = \infty$. 

Reals: Information. [L 74] extends $\mathbf{I}$ to reals as $\mathbf{I}(\alpha : \beta) = \log \sum \mathbf{m}(\alpha|\beta) \mathbf{I}(i:j)$. (As always, we average in the linear scale and switch to the logarithmic scale for the final expression.)

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3 Consistent Objects.

Consistency of theories and of other objects can be expressed as membership in co-r.e. sets of reals. It is convenient to define such sets via co-r.e. trees, i.e., infinite sets $T \subseteq S$ containing all prefixes and some extensions of each member. Let $\tilde{T}$ be the set of those $\omega \in \Omega$ with all prefixes in $T$. Some co-r.e. trees have only strings of linear Kolmogorov complexity. Contrast this with

**Proposition 1** For each $\beta \in \Omega$, each co-r.e. tree $T$ has $\alpha \in \tilde{T}$ with $I(\alpha_{[n]} : \beta) \leq 5 \log n + O(1)$.

**Lemma 1** For each co-r.e. tree $T$ there is a measure $\mu(x) = \mu(x0) + \mu(x1)$ with $\mu(\tilde{T}) > 1/2$, computable as $\mu(x) = G(x, \rho_{[\log \|x\|]})$ by an algorithm $G$ using $5 \log \|x\|$ digits of a hint $\rho \in \Omega$.

Lemma 1 implies Proposition 1. Indeed, algorithms can transform uniform distribution of inputs $\omega$ into any computable one; same holds for computations with oracle $\rho$. Consider an algorithm using $\rho$ to compute $\mu$ and transforming $\omega$ into a $\mu$-distributed $\alpha \in \tilde{T}$ with high probability. $I(\rho_{[\log \|x\|]} : \beta) = O(\log \|x\|)$; random $\omega$ cannot add information with high probability, and the algorithm cannot increase it either (due to conservation inequalities).

**Proof of Lemma 1:** $G$ uses $\rho$ to list all converging $k$-bit programs. As [Barzdin 68] noted, for this it needs just one of them, the slowest. $\rho$ can be any r.e. real with $K(\rho_{[n]}) = k = n - o(n)$, e.g., a random one. Programs that use $\rho_{[n]}$ waiting for enumeration of $\rho$’s lower bounds to exceed $\rho_{[n]}$, are slower than any programs $P$ of complexity $< k - 2 \log n$: otherwise $\rho_{[n]}$ can be generated from $P, n$.

$G$ computes $\mu$ recursively in slices $\mu_i(x)$ for $\|x\| = n = 2^i$, assuming $\mu_{i-1}$ already computed. It will approximate $T \cap B^n$ as $T_i = T_i(\rho)$ by limited co-enumeration and distribute $\mu_{i-1}(x)$ uniformly on all $n$-bit extensions of $x$ (which always exist in $T$).

Let $h_i$ be the Shannon entropy of $\mu_i$, with the fractional part rounded up to $2 \log 2i$ bits. Given $\mu_{i-1}$, shrinking $T_i$ lowers $h_i$. $G$ uses $\rho$ to compute the least possible $h_i$ and co-enumerates $T_i$ until it reaches this bound. Rounding $h_i$, leaves a fraction $f_i$ of $x \in T_i \setminus T$. Yet, $\sum_i f_i < 1/2$. ■

In particular, randomized algorithms can generate strings of length $\geq n$ of any co-r.e. tree $T$ with probability $1/k^2 n, k = \lceil \log n \rceil$ by guessing $k, \lceil \log (T \cap B^2) \rceil$.

3.1 Example: Tiling

An illustration is the tiling question from the introduction. [Durand, Levin, Shen 01] constructs a palette $P$ forcing, on each $P$-tiling, high complexity of all horizontal tile strings not crossing one specific column. The same construction works if complexity restriction is replaced with membership in any bi-tree, i.e., a (co-r.e.) tree containing all substrings (not only prefixes) of its members. To use it, we need to encode any co-r.e. tree $T$ as an equivalent bi-tree $T_2$.

Let $b(2^k(2l+1)) \equiv (l \mod 2)$. The pattern of $b(n)$ for $2^n$ consecutive $n$ determines the $a-2$ tail bits of $n$. Let $i$ double each bit of $i$ and alter the result’s first bit. If $n$ ends with $k$ followed by $\|k\|$, let $sf(n) \equiv k$. Let $\tilde{T}_1$ be a tree of sequences $\alpha : \mathbb{N} \to B^2$ such that $\alpha(n)=(b(n), t)$ and for some $s \in \tilde{T}$, whenever $sf(n)$ is defined, $t=s(sf(n))$. Let $T_2$ be a bi-tree of all segments of members of $\tilde{T}_1$. Each $n$-bit $T_2$-string represents the first $n/O(\|n\|^2)$ bits\(^5\) of a $T$-string $s$.

In particular, $T$ (and $T_2$) can force $s$ to be random i.e., have maximal complexity. This illustrates the point: all such tilings are highly non-recursive, yet easy to generate (with dice). They can be expressed in a formal system that allows trivial completions but no recursive ones.

\(^5\)Optimizing $\tilde{T}$ coding to $\|n\| = K(i)$ improves the overhead $\|n\|^2$ to $1/m(\|n\|)$ but cannot eliminate it.
4 The Taboo.

The above example does not show that all co-r.e. trees, such as tilings with an arbitrary palette, allow easily generated members. Proposition 1 sets a small but growing bound on the information needed for that, leaving open the question the article started with. It is resolved by the following observation central to this paper. We represent in $\Omega$ partial predicates as their graphs listed in arbitrary order. Let $u$ be a universal partial recursive predicate (p.r.p).

**Theorem 1** Let $\rho_{\sim n}$ be an $n+K(n)$ bit prefix of a random r.e. real $\rho$ and $U$ be a partial predicate that on $B^n$ is a total extension\(^6\) of $u$. Then $I(U : \rho_{\sim n}) \geq n - K(K(n)|n) - O(1)$.

(This almost suffices to compute $\rho_{\sim n}$ from the $B^n$-restriction $U_n$ of $U$ and $K(U_n)$.
Same holds for all r.e. $\rho$ since they reduce to same-length segments of any random one.
The $K(K(n)|n)$ decrement is small; yet [Gacs 74] ingeniously proves it $\sim \log \log n$, not $O(1)$.)

**Proof.** We define a p.r.p. $P : S \to B$ inductively on $B^n$. If $x=0^n$ or $P(x-1)$ is defined, let $m_{x,i}$ denote the combined universal measure $\sum_Q m(Q|n)$ of all total predicates $Q$ on $B^n$ that agree with $P$ on $[0^n, x-1]$ and $Q(x) \neq i \in B$. Then $P(x)$ enumerates lower bounds for $m_{x,i}$ until either exceeds $2^{-n}$ and yields $P(x)=i$, decreasing $\sum m_{x+1,i}$ by $>2^{-n}$. For some $x_n$, $m_{x_n,i} \leq 2^{-n}$ and $P$ diverges on $[x_n, 1^n]$ with $\sum m(Q|n) \leq 2/2^n$ for all total extensions $Q$ of $P$ on $B^n$.

For all such $Q$, this bound allows $2^n/O(1)$-fold increase of $m(Q|n)$,\(^7\) compared to just $m(Q|n) = m(K(n)|n)m(Q)/O(m(n))$. Now, $u(px)$, with a fixed $p$, computes $P(x)$, and $U(px)$ extends $P(x)$ on $B_n$ to a total $Q_n$, with $m(Q_n|U) = m(n)/O(1)$. Also, $x_n, \rho_{\sim n}, K(n) = \|\rho_{\sim n}\|-n$ are r.e., so can be computed from one of them whose enumeration ends latest. This could only be $\rho_{\sim n}$, being random and long enough to dominate in complexity (which computations cannot increase).

Thus, $m(Q_n|\rho_{\sim n}) = m(Q_n|x_n)/O(1) = 2^n m(K(n)|n)m(Q_n)/O(m(n))$. Then, $I(U : \rho_{\sim n}) \geq \log m(Q_n|\rho_{\sim n})m(Q_n|U)/m(Q_n)) \geq \log 2^n m(K(n)|n) - O(1) \geq n - K(K(n)|n) - O(1)$. \(\blacksquare\)

Since random strings contain $k$ bits of information about $\rho$ only with probability $2^{-k}$ and algorithms do not increase information (due to the Conservation Inequalities), Theorem 1 implies

**Corollary 1** The probability that a randomized algorithm computes on $B^n$ a total extension of $u$ is at most $O(2^{-n})/m(K(n)|n)$.
(Strengthening the o(1) bound of [Jockusch, Soare 72].)

(Thus, not all palettes, formal theories, etc. allow randomness-based tilings, completions, etc.)

Of course, nobody envisioned choosing fundamental Math axioms by coin flips. Yet, Theorem 1 supports a more general impossibility. Just like the usual interpretation of Gödel Theorem is a matter of accepting Church’s Thesis, judging if Theorem 1 makes the completion task impossible is a matter of accepting the Independence Postulate discussed in the appendix.

**Acknowledgments**

I am grateful to Robert Solovay, Alexander Shen, and Bruno Durand for insightful discussions and to Rod Downey, Denis Hirschfeldt, and Stephen Simpson for three references.

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\(^6\)One can weaken this total extension condition to being consistent with $u$ and defined on the specific input $px_n$ if the simple $I$ of [L 74] is strengthened to one of [L 84] and $M(\{0^n|x\}) \ldots P(x-1))$ replaces $m_{x,i}$ in $P$ defined below.

\(^7\)This also means $I_n(x_n : Q) = n \pm O(1)$ (and already implies Corollary 1, even simplifies its bound to $O(2^{-n})$). So, $K(x_n|(Q, K(Q|n))) = O(1)$ thus the $B_n$-restriction $u_n$ of $u$ can be computed from $U_n, K(U_n|n)$, if not from $U_n$ itself.
Appendix: The Independence Postulate.

**IP:** Let $X$ be a sequence defined with an $n$-bit mathematical statement (e.g., in PA or set theory). Suppose a sequence $Y$ can be located in the physical world with a $k$-bit instruction set. Then $I(X : Y) < k + n + c$, for some small absolute constant $c$.

(Note that $X$ and $Y$ can each have much more than $k + n + c$ bits of information.)

Thus, a (physical) sequence of all mathematical publications has little information about the (mathematical) sequence of all true statements of arithmetic. This is of little concern because the latter has, in turn, little information about the stock market (a physical sequence).

Of course, Kolmogorov information is not the only desirable commodity. Yet, IP has interesting applications [L 84]. It can be restated as a “finitary” version of the Church-Turing thesis (CT) by calling *recursive* those finite sequences with recursive descriptions nearly as short as any their “higher-level” math descriptions. IP postulates that only such recursive sequences exist in reality.

Let me add (in order of increasing relevance) some comparisons between IP and CT:

1. IP is stated with greater care than CT: Obviously not all strings we generate are algorithmic (non-communist election results better not be :-). Only mathematically defined strings need be algorithmic to be generatable. IP includes this math clause explicitly, CT rarely does.

2. IP is simpler, CT more abstract. All sequences we ever see are computable just by being finite: CT is useless for them! IP works equally well for finite and infinite sequences.

3. IP is easier to support: CT is usually stated with vague reasoning. IP has broad conservation laws to support it and a general intuition that target information cannot be increased.

4. IP is much stronger: CT prohibits only generating the target math sequence itself; IP bars all strings with any significant information about it.

One application is dousing Gödel’s hope cited in the Introduction, regardless of any realizable process of axiom selection. The argument is “inductive”. It seems, complicated processes we observe, can ultimately be *explained*, i.e., reduced to simpler ones. These reductions use deterministic models and random ones, but neither can *increase* the starting information about a target. The toolkit of our models may change (e.g., quantum amplitudes work somewhat differently than probabilities) but it is hard to expect new realistic primitives allowing such “information leaks”.

So, if *complicated* processes generate unlimited target information, so must do some *elementary* processes, that admit no further explanations (reductions to simpler processes). The existence of such elementary unexplainable information Sources cannot be ruled out. Yet Infidels :-) can postulate it away. Just like the impossibility of generating power from uniform heat, this is an unprovable postulate, supported by proven arguments.

Note that the above argument is based on Independence Conservation Inequalities (ICI) of [L 74, L 84]. They deal with generation of strings by deterministic algorithms or by random processes *from other strings*. If the preexisting string has no significant target information, neither will the generated one. And despite being intuitive, ICI are not technically trivial and should not be confused with the easy remark that randomized algorithms cannot generate *from scratch* information about math targets, such as $e.g.$, r.e. reals. (Math community never tried choosing their fundamental axioms this way :-). But the difficulty pays off, being essential for the inductive nature of the support ICI give to IP.
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