$\frac{1}{N}$ Expansion and Particle Spectrum in Induced QCD

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Abstract

We study the $\frac{1}{N}$ expansion in the recently proposed model of the lattice gauge theory induced by heavy scalar field in adjoint representation. In the first approximation the fluctuations of the density of eigenvalues of the scalar field are Gaussian, so that the scalar glueball spectrum is defined from the corresponding linear wave equation.
1 Introduction

The problem of solving large $N$ QCD seemed hopeless until recently some new approach was taken[4, 5]. Instead of trying to solve it starting from the asymptotic freedom, we suggested to start much earlier, at such scales, that there are no gluons yet, but rather some gluon constituents.

The idea is that one could then use the freedom of choosing any constituents and any form of their interaction, as long as the effective theory at large scales would be still QCD. Clearly the gauge symmetry should be there from the start, but there need not be the Yang-Mills (or Wilson, in the lattice formulation) term in the bare Action. This term should be induced in an asymptotically free region, as result of integrating out heavy constituents.

This idea is quite old. Perhaps Sakharov was the first one to suggest the induced Gravity. After that the idea was unsuccessfully implemented by the author (see the review paper[1]) in the fermionic string model of QCD. The heavy constituents were introduced as 2-dimensional Majorana fermions at the world sheet of string. Formally, the Schwinger-Dyson equations of the large $N$ QCD were satisfied, but one could not make any sense out of this model.

As we see it now, after ten more years of study of the string theory, the real problem comes from the string. Apparently, there is no such thing as a noncritical string in any number $D > 1$ of external dimensions. The interaction at the world sheet must be essentially nonlocal, and/or it should involve some extrinsic geometry. The idea of string realization of the Lorentz group was a beautiful one, and it still has chance to work in Gravity, but in QCD I personally do not believe in this idea any more. Apparently, the coordinates could only be arguments of fields, rather then fields in QCD.

Coming back to the induced QCD, the idea of Kazakov and myself was to take heavy scalar field in adjoint representation of the gauge group as a constituent field. Naively, if one just integrates this field out, one induces the correct Yang-Mills action, with positive coefficient in front, scaling correctly at large $N$. By the way, the simpler choice of the fundamental representation would yield too large induced gauge coupling, so it could not induce an asymptotically free QCD at large $N$. The large ($\sim N$) number of fundamental scalars would do it, but this model is hard to solve.

The correct computation of induced gauge coupling is not so trivial, as there are feedback effects from induced hard gluons, and those from effective scalar quartic interaction. If one starts from asymptotically free region, and goes to smaller scales, there would be two conflicting effects. First, there would be antiscreening of gauge coupling from the gluons, which would tend to decrease the gauge coupling further. Second, there would be screening from the scalars, which is numerically smaller, then antiscreening, so that alone it cannot stop the logarithmic decrease of effective gauge coupling. The desired infinite gauge coupling at lattice scales could be generated by scalar quartic interaction, which is known to grow without limits at small scales. At least we know, that asymptotic freedom is not consistent with scalar quartic interactions. At larger scales, the scalars’ decouple, being heavy, so that true asymptotic freedom is left intact. So, the scalars effectively cut off the asymptotic freedom at the scale of their mass, and move the effective coupling towards another ”confinement” region at lattice scales.
What happens in this model at small scales is beyond the reach of the perturbative field theory. However, if we take the particular nonperturbative theory, we have a chance to adjust the parameters so, as to visit the perturbative region on the way from small to physical scales. In fact, it is hard to imagine how could we miss the asymptotic freedom, if we would get nontrivial theory in four dimensions. The common belief is that there is only one gauge theory in four dimensions: the asymptotically free, quark confining QCD.

The unique property of the scalar field in adjoint representation is that corresponding lattice gauge model without the Wilson term is soluble at large $N$, for arbitrary scalar potential. The one link integrals\cite{5} over the gauge fields $U_\mu$ reduce at large $N$ to certain functional determinant, which was analyzed in previous works\cite{2,3}. In particular, in the second work\cite{3}, we found some coupled integral equations, exactly soluble by means of the Riemann-Hilbert method.

The gauge field integration produces effective action, which depends only on the density of eigenvalues of the scalar field. One could therefore, integrate out the unitary matrices, which diagonalize the scalar field, or simply take the gauge where this field is diagonal. The corresponding Faddeev-Popov Jacobian is nothing but the Vandermonde determinant, which produces in effective Action the term, quadratic in the density of eigenvalues.

At $N = \infty$ this density does not fluctuate: it is given by the solution of the classical equations, minimizing this effective Action. By some historical reasons it was difficult for everybody, authors included, to accept this simple and natural scenario. We were misled by large $N$ matrix models of 2D Quantum Gravity, where the Itzykson-Zuber determinant was used in a different context. There, it reduced to the Slater determinant, thus revealing the hidden fermionic degrees of freedom. In higher dimensions there is no such thing as fermionization of the Bosonic problem, neither could there be a spatially inhomogeneous vacuum. So, there seems to be no alternative to the space independent vacuum density of the eigenvalues of the scalar field, which we call the Master Field.

Still, the technical problem of reexamining the old $C = 1$ matrix model with the new representation of the Itzykson-Zuber determinant was pending, so the author (unpublished) checked within the strong coupling expansion, that the old fermionic solution can also be correctly reproduced by the spatially independent Master Field. Recently, this problem was completely solved\cite{4}, so that there is no longer any contradiction between the fermions and the Master Field.

The classical solutions for the density come in two phases: the strong coupling phase, with density continuous at the origin, and the weak coupling phase, where there is a finite gap in the spectrum. The critical phenomena near the transition point are much richer then in the usual one-matrix models. The critical indices, computed in\cite{3} are transcendental numbers at $D > 2$, in particular, the density grows as $\rho(\lambda) \sim \lambda^\alpha$ and physical mass scale $m^2 \sim \lambda^{\alpha-1}$ where $\cos \pi \alpha = -\frac{D}{3D-2}, \alpha > 1$.

The scaling laws in the four-dimensional theory are amazing, but they do not contradict the asymptotic freedom, as there is extra renormalization from the lattice scales to the region of the asymptotic freedom. In fact, the induced gauge coupling $\beta_0 = \frac{1}{N^2 c}$ is expected to come out as some negative number times logarithm of the scalar mass in the lattice units, according to the RG analysis at the small scale region. Comparing this with the usual RG
There is, however, one unsolved problem in the induced QCD, that of the spurious local Abelian symmetry. It is well known, that the lattice theory in adjoint representation has extra local $Z_N$ symmetry. This issue was analyzed a long time ago\cite{7,8}, and it was noted, that unbroken symmetry locally confines quarks so that they cannot move separately even at the lattice scales. It was suggested in the recent work\cite{6}, that this symmetry breaks spontaneously at the transition point, as it does in the mean field approximation\cite{7,8}. In any case, this is the problem for the higher $\frac{1}{N}$ approximations, as the center of the gauge group is negligible at $N = \infty$.

The present paper is devoted to the next $\frac{1}{N}$ correction in the scalar sector of the strong coupling phase of the induced QCD.

In Section 2 we develop the general method of $\frac{1}{N}$ expansion, and find exact integral equations for the Gaussian fluctuations of the density.

In Section 3 we go to the local limit, using the same method, as before, and we find another Riemann-Hilbert problem, soluble in the scaling limit.

In the Appendix we present the computation of the second variation of the Itzykson-Zuber determinant.

\section{$\frac{1}{N}$ Expansion}

At infinite $N$ there are no fluctuations of the density of eigenvalues of the scalar field, and therefore, no interesting physics. These fluctuations show up in the first order of the $\frac{1}{N}$ expansion. There are two methods for the systematic $\frac{1}{N}$ expansion: the Schwinger-Dyson equations and the saddle point method.

For our present purposes the saddle point method appears to work better, so we elaborate it here, using the modification suggested by G.Parisi (unpublished). The idea is to change the integration variables from the eigenvalues to the corresponding density

$$\rho(\lambda) = \frac{1}{N} \sum_j \delta(\lambda - \lambda_j)$$  \hspace{1cm} (1)

The change of variables goes as follows

$$\int d^N \lambda \propto \int \mathcal{D}[\rho] J[\rho]$$  \hspace{1cm} (2)

where

$$J[\rho] = \int \mathcal{D}\epsilon \exp \left( iN \int d\lambda \epsilon(\lambda) \rho(\lambda) \right) \int d^N \lambda \exp \left( -i \sum_j \epsilon(\lambda_j) \right)$$  \hspace{1cm} (3)

We neglect everywhere the constant normalization factors, as we are interested only in averages.
It is convenient to shift $\epsilon(\lambda)$ by $i \ln \bar{\rho}(\lambda)$, and then single out the translational mode $\epsilon = \text{const}$ which yields the density normalization condition

$$\delta \left( 1 - \int d\lambda \rho(\lambda) \right)$$

(4)

The translational mode can be eliminated by the background gauge condition

$$\delta \left( \int d\lambda \bar{\rho}(\lambda) \epsilon(\lambda) \right)$$

(5)

with the trivial constant Jacobian, which does not require ghosts. After simple transformations we find

$$J[\rho] = \delta \left( 1 - \int d\lambda \rho(\lambda) \right) \exp \left( - N \int d\lambda \rho(\lambda) \ln \bar{\rho}(\lambda) \right) \tilde{J}[\rho]$$

(6)

where

$$\tilde{J}[\rho] = \int D\epsilon \delta \left( \int d\lambda \bar{\rho}(\lambda) \epsilon(\lambda) \right) \exp \left( - \frac{N}{2} \int d\lambda \bar{\rho}(\lambda) \epsilon^2(\lambda) \right) \exp N \left( i \int d\lambda \epsilon(\lambda) \rho(\lambda) + \frac{1}{2} \int d\lambda \bar{\rho}(\lambda) \epsilon^2(\lambda) + \ln \left( \int d\lambda \bar{\rho}(\lambda) \exp(-i\epsilon(\lambda)) \right) \right)$$

(7)

Now we could expand the last exponent in $\epsilon$, keeping in mind, that density fluctuations $\rho - \bar{\rho} \sim \frac{1}{N}$ and $\epsilon \sim N^{-\frac{1}{2}}$. The resulting expansion of $\tilde{J}$ is straightforward, this is one dimensional perturbation theory with propagator

$$\langle \epsilon(\lambda) \epsilon(\lambda') \rangle = \frac{1}{N} \left( \frac{\delta(\lambda - \lambda')}{\bar{\rho}(\lambda)} - 1 \right)$$

(8)

In the leading order we could set $\tilde{J} = 1 + O\left(\frac{1}{N}\right)$, which leaves us with extra term

$$N \int d\lambda \rho(\lambda) \ln \bar{\rho}(\lambda)$$

(9)

in effective Action. This term is local and linear in density, so it effectively shifts the scalar field potential, without altering the quadratic form of the second variations.

This quadratic form can be computed exactly, using the technique of the previous work[3]. Let us outline this computation. First let us rederive the saddle point equation with the new technique. The normalization condition would be satisfied identically for

$$\rho(\lambda) = \bar{\rho}(\lambda) - \frac{\psi'(\lambda)}{N}$$

(10)

where $\psi(\lambda)$ vanishes at the endpoints of the support of $\bar{\rho}(\lambda)$. The first variation with respect to $\psi(\lambda)$ is equivalent to differentiation $\frac{d}{d\lambda}$ and setting $\lambda_i \rightarrow \lambda$, as one can readily check.

The effective Action for the $x$-dependent density $\rho_x$ reads

$$S_{\text{eff}}[\rho] =$$

$$- N^2 \sum_x \int d\lambda \rho_x(\lambda) \int d\lambda' \rho_x(\lambda') \ln |\lambda - \lambda'| + N^2 \sum_x \int d\lambda \rho_x(\lambda) U(\lambda) - \sum_{<xy>} \ln I[\rho_x, \rho_y]$$

$$+ \sum_x \left( N \int d\lambda \rho_x(\lambda) \ln \bar{\rho}(\lambda) + \ln \tilde{J} \right)$$

(11)
where the first term comes from the Vandermonde determinant, the second one from the scalar field potential, the third one from the Itzykson-Zuber determinant, and the remaining terms - from the above Jacobian. These last terms would contribute only in the higher orders of the $\frac{1}{N}$ expansion.

The background density of eigenvalues is determined at $N = \infty$ from the saddle point equation

$$\frac{\delta S_{\text{eff}}[\rho]}{\delta \psi(\lambda)} = \frac{1}{N} \frac{d}{d\lambda} \frac{\delta S_{\text{eff}}[\rho]}{\delta \rho(\lambda)} = 0$$

This equation is the same, as before

$$F(\lambda) = -2\mathfrak{R}V'(\lambda) + U'(\lambda)$$

where

$$V'(\lambda) = \int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'}$$

and the function $F(\lambda)$ represents the logarithmic derivative of the Itzykson-Zuber determinant. This function for arbitrary density satisfies the set of Schwinger-Dyson equations, which were derived and studied in the previous work. Substituting the classical formula for $F(\lambda)$ in these equations, we arrived at the Master Field Equation (MFE)

$$R(\lambda) = \frac{1}{2D} U'(\lambda) + \frac{D-1}{D} \mathfrak{R} V'(\lambda)$$

$$\mathfrak{R} V'(\lambda) = \int \frac{d\lambda'}{\pi} \arctan \frac{\pi \rho(\lambda')}{\lambda - R(\lambda')}$$

Our objective now is to compute the second variation of the effective Action. The potential term does not contribute, being linear in density, the Vandermonde determinant yields a local term

$$-\varphi \int d\lambda \psi(\lambda) \int d\lambda' \psi(\lambda') \frac{1}{(\lambda - \lambda')^2}. \tag{17}$$

The Itzykson–Zuber determinant yields two terms, the local one being

$$-D \int d\lambda \psi(\lambda) \int d\lambda' \psi(\lambda') \sigma(\lambda, \lambda') \tag{18}$$

with

$$\sigma(\lambda, \lambda') = \frac{d}{d\lambda'} \frac{\delta F(\lambda)}{\delta \rho(\lambda')} \tag{19}$$

and the nonlocal one

$$-\frac{1}{2} \int d\lambda \psi_x(\lambda) \int d\lambda' \sum_{\mu=-D}^D \psi_{x+\mu}(\lambda') \eta(\lambda, \lambda') \tag{20}$$

\footnote{In the rest of the paper we denote the background density as $\rho$ rather then $\bar{\rho}$.}
\[ \eta(\lambda, \lambda') = \frac{d}{d\lambda'} \frac{\delta F(\lambda)}{\delta \rho_{\lambda' \mu}(\lambda')} \]  

(21)

Collecting all the finite terms, and going to the local limit, we find the standard form of effective action of continuum field theory

\[ S_{\text{eff}} = \int d^Dx \mathcal{L}(x) \]  

(22)

with effective Lagrangean

\[ \mathcal{L}(x) = \int d\lambda \int d\lambda' \frac{1}{2} \eta(\lambda, \lambda') \partial_\mu \psi(x, \lambda) \partial_\mu \psi(x, \lambda') \]

\[ -\left( \frac{1}{(\lambda' - \lambda)^2} + D (\sigma(\lambda, \lambda') + \eta(\lambda, \lambda')) \right) \psi(x, \lambda) \psi(x, \lambda') \]

\[ + \int d\lambda \psi(x, \lambda) \frac{\rho'(\lambda)}{\rho(\lambda)} \]  

(23)

The linear term shifts the vacuum average of the density by

\[ \delta \rho(\lambda) = -\frac{1}{N} \psi_0'(\lambda) \]  

(24)

where \( \psi_0 \) satisfies the equation

\[ \int d\lambda' \psi_0(\lambda') \left( \frac{1}{(\lambda' - \lambda)^2} + D (\sigma(\lambda, \lambda') + \eta(\lambda, \lambda')) \right) = \frac{\rho'(\lambda)}{2\rho(\lambda)} \]  

(25)

The particle spectrum is described by the wave equation for the fluctuations \( \psi - \psi_0 = Y(\lambda)e^{iP_\lambda x} \)

\[ 0 = \int d\lambda' Y(\lambda') \left( \frac{1}{(\lambda' - \lambda)^2} + D\sigma(\lambda, \lambda') + (D - \frac{1}{2}P^2)\eta(\lambda, \lambda') \right) \]  

(26)

In the Appendix we study the perturbed Schwinger-Dyson equation for the second variations and we find the following linear integral equations for the kernels \( \sigma, \eta \)

\[ \int d\lambda' \mathcal{K}(\lambda_0, \lambda') \eta(\lambda', \lambda'') = \frac{1}{(\lambda_0 - \lambda'')^2} \]  

(27)

\[ \int d\lambda' \mathcal{K}(\lambda_0, \lambda') \sigma(\lambda', \lambda'') = \int d\lambda' \mathcal{K}(\lambda_0, \lambda') \left( -\frac{1}{(\lambda' - \lambda'')^2} + \frac{1}{G(\lambda_0, \lambda')} \frac{d}{d\lambda''} G(\lambda_0, \lambda'') \right) \]  

(28)

where

\[ \mathcal{K}(\lambda_0, \lambda') = \frac{\rho(\lambda')}{(\lambda_0 - R(\lambda'))^2 + \pi^2 \rho^2(\lambda')} \]  

(29)

and

\[ G(\lambda_0, \lambda') = \frac{1}{\lambda_0 - R(\lambda')} \Re \exp \left( \int \frac{d\lambda''}{\pi(\lambda'' - \lambda')} \arctan \frac{\pi \rho(\lambda'')}{\lambda_0 - R(\lambda'')} \right) \]  

(30)
Now we eliminate the unknown functions $\sigma, \eta$ by integrating the wave equation over $\lambda$ with the weight $\mathcal{K}(\lambda_0, \lambda)$ which yields
\[
D \int \frac{d\lambda}{(\lambda_0 - \lambda)^2} \frac{\psi_0(\lambda')}{(\lambda_0 - \lambda')}^2 - \int \frac{d\lambda}{(\lambda_0 - \lambda)^2} \left( \frac{D - 1}{(\lambda - \lambda')^2} - \frac{D}{G(\lambda_0, \lambda')} \frac{d}{d\lambda'} G(\lambda_0, \lambda') \right) = \int d\lambda \mathcal{K}(\lambda_0, \lambda) \frac{\rho(\lambda)}{2\rho(\lambda)}
\]
(31)

These equations should be solved together with the MFE.

All above equations were exact in a sense that we did not go to the local limit, except for replacing the lattice derivatives by local ones \(^2\), neither did we assume anything about the support of eigenvalues. The same equations hold for the weak coupling phase, where there is a gap at the origin. In this paper we study the strong coupling phase where there is no such gap.

### 3 Local Wave Equation

Let us consider the local limit, when the density at the origin vanishes. The analysis of the Master Field Equation\(^3\) shows, that the scaling solution, independent of the ultraviolet cutoff, requires, that $\rho(\lambda)$ and $r(\lambda) = R(\lambda) - \lambda$ vanish as some power of the cutoff, so that
\[
\rho(\nu) \sim r(\nu) \ll \nu
\]
(33)

In this case the MFE can be expanded in $\rho, r$ as follows
\[
(\rho(\lambda)r(\lambda))'' = \varphi \int_{-\infty}^{\infty} d\nu \frac{\rho^2(\nu)}{(\lambda - \nu)^3}
\]
(34)
\[
r(\lambda) = \frac{1}{2D} u'(\lambda) + \frac{D - 1}{D} \Re V'(\lambda)
\]

where renormalized potential $u'(\nu)$ starts from the linear mass term $m^2 \nu$. The higher order terms are irrelevant in the scaling region, but they might be important for the full MFE, to provide required cancellations of the regular terms in the scaling region. These regular terms represent substruction terms in dispersion relations. There is no need for these terms in the full MFE at the lattice, but in the local theory there are ultraviolet divergencies.

This convenient method of elimination of regular terms is to introduce two analytic functions
\[
\mathcal{P}(z) = \frac{u'(z)}{2(1 - D)} - V'(z)
\]
(35)

\(^2\)One could restore the lattice theory by replacing $\frac{1}{2} P^2$ by $2 \sum_{\mu} \sin^2 \frac{1}{2} P_{\mu}$.
\[ Q(z) = \text{regular terms} + \pi \int_{-\infty}^{\infty} d\mu \frac{\rho^2(\mu)}{\mu - z} \] (36)

with the symmetry property

\[ \mathcal{P}(-\bar{z}) = -\bar{\mathcal{P}}(z); \quad Q(-\bar{z}) = -\bar{Q}(z) \] (37)

and note that at \( \Im z \to +0 \) by construction

\[ \Im Q = (\Im P)^2. \] (38)

On the other hand, in virtue of the above equation for density

\[ \Re Q = \frac{1 - D}{D} \Im (\mathcal{P}^2) \] (39)

This is the nonlinear Riemann-Hilbert problem.

The key identity, used in derivation of local MFE is the following one

\[ \int d\lambda K(\lambda, \lambda')A(\lambda') \to A(\lambda) - (r(\lambda)A(\lambda))' + \rho \int \frac{d\lambda' A(\lambda')}{(\lambda - \lambda')^2} + O(\rho^2) \] (40)

where the first two terms come from the small region \( \lambda' - \lambda \sim \rho(\lambda) \), and the last one comes from the region \( \lambda' \sim \lambda \). The integral in the small region can be reduced to the residue at the complex pole of the kernel. To be more precise, the \( O(\rho^2) \) correction to this formula was also used in derivation of the local MFE, where the linear terms cancel, but we do not need this correction below.

The expansion of the function \( G(\lambda, \phi) \) up to linear terms in \( \rho \) reads

\[ G(\lambda, \phi) \to \frac{1}{(\lambda - \phi + r(\phi))} \left( 1 + \rho \int \frac{d\nu \rho(\nu)}{(\phi - \nu)(\lambda - \nu)} \right) \] (41)

Substituting this into the wave equation (32), we find variety of terms. To reduce them, it is convenient to introduce the analytic function

\[ \mathcal{F}(z) = \int \frac{d\phi \rho(\phi)}{\phi - z} \varphi \int \frac{d\phi' Y(\phi')}{(\phi - \phi')^2} \] (42)

such that

\[ \Im \mathcal{F}(\phi + i0) = \pi \rho(\phi) \varphi \int \frac{d\phi' Y(\phi')}{(\phi - \phi')^2} \] (43)

The function \( Y(\phi) \) can be reconstructed from dispersion relation

\[ \varphi \int d\phi' Y'(\phi') \frac{1}{\phi' - \phi} = \frac{\Im \mathcal{F}(\phi + i0)}{\pi \rho(\phi)} \] (44)

\[ Y(\phi) = \int d\nu \frac{\Im \mathcal{F}(\nu + i0)}{\pi^3 \rho(\nu)} \ln |\phi - \nu| \] (45)
Let us now turn to the wave equation. After some algebra, we reduce it to the following boundary problem:

\[
2\pi \rho(\phi) \Re F'(\phi) + \left( -P^2 + 2 + 2D\Re V''(\phi) \right) \Im F(\phi) + 2(D-1)\rho(\phi) \left( \frac{r(\phi) \Im F(\phi)}{\rho(\phi)} \right)' = 0 \quad (46)
\]

Let us now solve these problems in the scaling limit. The local MFE allows the scaling solution, in proper units:

\[
\pi \rho(\lambda) = \lambda^\alpha; \quad \alpha = 1 + \frac{1}{\pi} \arccos \frac{D}{3D-2} \quad (47)
\]

Various branches of the arccosine correspond to various fixed points of the theory. This scaling form holds for \( \lambda \gg m^\gamma \) where \( \gamma = \frac{2}{\alpha-1} \), and \( m \) is the physical mass scale.

In this case the functions \( V'(\lambda), r(\lambda) \) are given by:

\[
\Re V'(\lambda) \rightarrow -\frac{m_1^2 + 2}{2D} \lambda - \lambda^\alpha \tan \frac{\pi}{2} \alpha
\]

\[
r(\lambda) \rightarrow -\lambda^\alpha \frac{D-1}{D} \tan \frac{\pi}{2} \alpha \quad (48)
\]

The relation between the mass \( m_1^2 \) and the scalar potential is as follows:

\[
m_0^2 \equiv U''(0) = 2D + u''(0) = 2D + 2 - \frac{2}{D} - \frac{D-1}{D} m_1^2 \quad (50)
\]

so that the critical point is at

\[
m_{0,c}^2 = \frac{2D^2 + D - 1}{D} \quad (51)
\]

In this paper we shall find the solution at \( \phi \gg m_1^\gamma \). Then we are left with the scaling terms, so that the scaling Ansatz for \( F \) could be used:

\[
F(z) = i^s (-iz)^\epsilon; \quad s = \{0, 1\} \quad (52)
\]

The discrete parameter \( s \) corresponds to the parity:

\[
F(-z) = (-1)^s \bar{F}(z) \quad (53)
\]

and the ratio of real and imaginary parts follows from symmetry and analyticity. Substituting this Ansatz into the equation, we find the equation for the index \( \epsilon \)

\[
0 = g_s(\epsilon) \equiv \tan \left( \frac{\pi}{2} \alpha \right) \left( D\alpha + \frac{(D-1)^2}{D} \epsilon \right) + \epsilon \cot \left( \frac{\pi}{2} \alpha \right) (\epsilon - s) \quad (54)
\]

There are infinitely many solutions \( \epsilon_{s,n} \) which grow asymptotically linearly with the principal quantum number \( n \).

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3 The real and imaginary parts at real \( \phi \) are understood, as usual limits from the upper half plane.
In general, there are also powerlike corrections to the density, so that the solution is not so elementary. The complete computation of the spectrum requires an exact solution of the nonlinear Riemann-Hilbert problem for the density. This is the subject of the next paper.

One important thing is clear right away: this is not a free field theory at $D > 1$, neither we see any pathology at $D > 1$, like those of the Liouville theory. This is confining theory, but not quite the string. In general, there are infinitely many masses, and they grow with quantum numbers, as one would expect for QCD.

4 Discussion

The puzzle still remains unsolved. Is this QCD?

At least this is the theory of scalar particles, confined by strong interaction with gluon field. Would gluon field decouple, like it does in one dimension, we would get trivial free particle spectrum. Instead, we are going to get infinitely many particles, with growing masses, stable at infinite $N$. We expect the decays to show up in the next $\frac{1}{N}$ approximations.

In QCD, we would like to get the mesons, from confined $\bar{q}q$ pair, in addition to pure glueballs. However, the quarks are locally confined in this model, unless the $Z_N$ symmetry would break, spontaneously or otherwise. This seems to be the most urgent thing to do.

Also, we would like to see the perturbative QCD. This can be checked already in the scalar sector. If there is admixture of glueballs to the scalar branch of the spectrum, then the correlation function of fluctuations of the scalar eigenvalue density should have logarithmic singularity at $P^2 \to \infty$, times some power of $P^2$. This singularity in perturbative QCD comes from the two gluon exchange, via the $F^2_{\mu\nu}$ operator. Here it must come out as something like

$$\sum_n \frac{1}{P^2 + \text{const} n} \sim \ln P^2$$

(55)

due to the divergencies of the spectral sums.

Everything seems to be set up for the comparison of induced QCD with perturbative QCD. Let us work hard and keep the fingers crossed.

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### A Second Variation of Itzykson-Zuber Integral

In the previous paper we derived the following coupled set of linear integral equations for functions $G_\lambda(\nu), F(\nu)$:

\[
\int d\mu \rho_1(\mu) \frac{G_\lambda(\mu) - G_\lambda(\nu)}{\mu - \nu} = -1 + (\lambda - F(\nu))G_\lambda(\nu) \tag{56}
\]

\[
\int d\mu \rho_1(\mu) G_\lambda(\mu) = \int d\nu \frac{\rho_2(\nu)}{\lambda - \nu} \tag{57}
\]

where $F(\nu)$ is the first variation of the logarithm of the Itzykson-Zuber integral (see the section 2.). The densities of eigenvalues at two endpoints of the link were denoted as $\rho_1$ and $\rho_2$. In order to find the second variations, we should vary this equation with respect to $\rho_1$ and $\rho_2$.

Let us compute nonlocal variation $\eta$ first, as it is technically simpler. In this Appendix we denote the eigenvalues as $\phi, \mu$ and $\lambda$. We are going to vary $F, G$ with respect to $\rho_2$. These variations

\[
\eta(\psi, \phi) = \frac{d}{d\psi} \frac{\delta F(\phi)}{\delta \rho_2(\psi)}; \quad g_\lambda(\psi, \phi) = \frac{d}{d\psi} \frac{\delta G_\lambda(\phi)}{\delta \rho_2(\psi)} \tag{58}
\]

satisfy the differentiated version of above equations (56), (57):

\[
\eta(\psi, \nu) G_\lambda(\nu) = (\lambda - F(\nu))g_\lambda(\psi, \nu) - \int d\mu \rho(\mu) \frac{g_\lambda(\psi, \mu) - g_\lambda(\psi, \nu)}{\mu - \nu} \tag{59}
\]

\[
\int d\mu \rho(\mu) g_\lambda(\psi, \mu) = \frac{1}{(\lambda - \psi)^2} \tag{60}
\]

As in [8], we have to introduce the auxiliary analytic function

\[
A_{\lambda, \psi}(z) = \int d\mu \frac{\rho(\mu) g_\lambda(\psi, \mu)}{z - \mu} \tag{61}
\]

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4 In this Appendix we treat $\lambda$ as subscript rather than as argument of $G$. 

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in the upper half plane of \( z \) and we find from above equations the following linear relations for the corresponding boundary values

\[ A_{\lambda, \psi}(z + i0) (\lambda - R(z) - i\pi \rho(z)) - A_{\lambda, b}(z - i0) (\lambda - R(z) + i\pi \rho(z)) = -2\pi i \rho(z) G_\lambda(z) \eta(\psi, z) \] (62)

At infinity this function decreases as \( \frac{1}{z} \) with coefficient in front known from (60)

\[ A_{\lambda, \psi}(z) \to \frac{1}{z} \frac{1}{(\lambda - \psi)^2} \] (63)

The solution of the boundary problem is straightforward. We use the previous solution of the homogeneous problem

\[ G_\lambda(\mu) = \frac{1}{\lambda - R(\mu)} \Re T_\lambda(\mu + i0) \] (64)

\[ \frac{T_\lambda(\nu + i0)}{T_\lambda(\nu - i0)} = \frac{\lambda - R(\nu) + i\pi \rho(\nu)}{\lambda - R(\nu) - i\pi \rho(\nu)} \] (65)

\[ T_\lambda(z) = \exp \left( \int \frac{d\nu}{\pi(\nu - z)} \arctan \frac{\pi \rho(\nu)}{\lambda - R(\nu)} \right) \] (66)

and we find

\[ A_{\lambda, \psi}(z) = T_\lambda(z) B_{\lambda, \psi}(z) \] (67)

where the new function \( B_{\lambda, \psi}(z) \) which decreases at infinity precisely as \( A_{\lambda, \psi}(z) \) is to be reconstructed from imaginary part

\[ \Im B_{\lambda, \psi}(z + i0) = \frac{-\pi \rho(z) G_\lambda(z) \eta(\psi, z)}{T_\lambda(z + i0) (\lambda - R(z) - i\pi \rho(z))} \] (68)

This expression can be simplified in virtue of above equations as follows

\[ \Im B_{\lambda, \psi}(z + i0) = \frac{-\pi \rho(z) \eta(\psi, z)}{(\lambda - R(z))^2 + \pi^2 \rho^2(z)} \] (69)

so that the resulting Cauchy integral reads

\[ B_{\lambda, \psi}(z) = \int d\nu \frac{\rho(\nu) \eta(\psi, \nu)}{(\lambda - R(\nu))^2 + \pi^2 \rho^2(\nu)} \frac{1}{z - \nu} \] (70)

Finally, comparing the asymptotic behavior of this integral with known asymptotics \( T_\lambda(\infty) = 1 \) and required condition (63) for \( A_{\lambda, \psi}(z) \) we find linear integral equation for \( \eta(\psi, \phi) \)

\[ \int d\phi \frac{\rho(\phi) \eta(\psi, \phi)}{(\lambda - R(\phi))^2 + \pi^2 \rho^2(\phi)} = \frac{1}{(\lambda - \psi)^2} \] (71)

Let us now turn to the second \( \phi \) derivative, and differentiate the above equations for the first derivative. We shall use the same notation \( g_\lambda(\phi', \phi) \) for derivatives of the \( G \) function

\[ \sigma(\phi', \phi) = \frac{d}{d\phi'} \frac{\delta F(\phi)}{\delta \rho_1(\phi')} \quad g_\lambda(\phi', \phi) = \frac{d}{d\phi'} \frac{\delta G_\lambda(\phi)}{\delta \rho_1(\phi')} \] (72)
The differentiated equations read

\[
\int d\mu \rho(\mu) \frac{g_\lambda(\phi', \mu) - g_\lambda(\phi', \phi)}{-\mu + \phi} + (\lambda - F(\phi))g_\lambda(\phi', \phi) = \sigma(\phi', \phi)G_\lambda(\phi) + \frac{d}{d\phi'} \frac{G_\lambda(\phi') - G_\lambda(\phi)}{(\phi' - \phi)}
\]

\[
\int d\mu \rho(\mu)g_\lambda(\phi', \mu) = 0
\]

This equation differs from the previous one only by the right side. Repeating the same steps as before we find

\[
\int d\mu \rho(\mu)g_\lambda(\phi', \mu) = \mathcal{T}_\lambda(z) \int d\phi \frac{\rho(\phi)}{z - \phi} \left( \sigma(\phi', \phi) + \frac{1}{G_\lambda(\phi)} \frac{d}{d\phi'} \frac{G_\lambda(\phi') - G_\lambda(\phi)}{(\phi' - \phi)} \right)
\]

Finally, setting \( z \to \infty \) in above formula and using the equation (74) we find equation for \( \sigma(\phi', \phi) \)

\[
0 = \int d\phi \rho(\phi) \frac{\sigma(\phi', \phi) + \frac{1}{G_\lambda(\phi)} \frac{d}{d\phi'} \frac{G_\lambda(\phi') - G_\lambda(\phi)}{(\phi' - \phi)}}{(\lambda - R(\phi))^2 + \pi^2 \rho^2(\phi)}
\]