A Computational Framework for the Mixing Times in the QBD Processes with Infinitely-Many Levels

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Abstract
In this paper, we develop some matrix Poisson’s equations satisfied by the mean and variance of the mixing time in an irreducible positive-recurrent discrete-time Markov chain with infinitely-many levels, and provide a computational framework for the solution to the matrix Poisson’s equations by means of the UL-type of $RG$-factorization as well as the generalized inverses. In an important special case: the level-dependent QBD processes, we provide a detailed computation for the mean and variance of the mixing time. Based on this, we give new highlight on computation of the mixing time in the block-structured Markov chains with infinitely-many levels through the matrix-analytic method.

Keywords: Mixing time; block-structured Markov chain; QBD process; Poisson’s equation; $RG$-factorization; generalized inverse; matrix-analytic method.

1 Introduction
In probability theory, the mixing time of a Markov chain is the time until the Markov chain is close to its steady state distribution. Up to now, the mixing time has been given many important applications to, for example, perturbation analysis, Poisson’s equations, coupling, spectral gap, random walks on graphs, and Markov chain Monte Carlo algorithms. During the last two decades considerable attention has been paid to studying
the mixing times in the Markov chains with finite states. Readers may refer to, such as, Aldous et al [3], Lovász and Winkler [36], Hunter [19, 20, 21, 23], Cao et al [10], Cao and Chen [9], and Li and Liu [31]. At the same time, the mixing times in Markov chains and random walks are studied in five excellent books by Aldous [1], Aldous and Fill [2], Montenegro and Tetali [40], Cao [8], and Levin et al [27]. It is worthwhile to note that for the mixing times in the Markov chains with infinite states, the available works are still few in the literature up to now.

The generalized inverses play an important role in the study of the mixing times in the Markov chains with finite states. Readers may refer to a book by Kemeny and Snell [25], and three survey papers by [38] and Hunter [17, 18]. Specifically, Hunter [17, 18] indicated how to apply the generalized inverses to computing the mean and variance of the mixing time. On the other hand, the fundamental matrix plays a key role in the theory of Markov chains, e.g., see Kemeny and Snell [25], Hunter [17], Neuts [42], Heyman and O’Leary [16], and da Silva Soares and Latouche [12]. Also, the kemeny’s constant has been an interesting topic for many years, readers may refer to Hunter [22] and Catral et al [11] for more details.

The two types of $RG$-factorizations have been a key method in performance computation of stochastic models, e.g., see Li [28] for a systematical analysis. Vigon [46] discussed the LU-factorization and Wiener-Hopf factorization in Markov chains, and further provided new highlight on some useful relations among the $RG$-factorizations, LU-factorization and Wiener-Hopf factorization. From a viewpoint of applications, the $RG$-factorizations have been applied to dealing with performance computation in a variety of stochastic models. Important examples include quasi-stationary distributions by Li and Zhao [32, 33], stochastic functionals by Li and Cao [30], tail probabilities by Li [29], repairable systems by Ruiz-Castro et al [45], computer networks by Wang et al [48, 47], manufacturing systems by Li et al [34] and Liu et al [35].

The QBD process is an important example in Markov chains, and provides a useful mathematical tool for studying stochastic models such as queueing systems, manufacturing systems, communication networks and transportation systems. Readers may refer to Chapter 3 of Neuts [41], Bright and Taylor [7], Ramaswami [44], Latouche and Ramaswami [26], and Li and Cao [30].

The Poisson’s equation frequently occur in the analysis of, such as, Markov chains, Markov decision processes and queueing systems. Important examples include the waiting
time of a queue by Asmussen and Bladt [4], Glynn [14] and Bladt [5]; perturbation analysis in Markov chains by Cao et al [10], Li and Liu [31], Cao [8] and Dendievel et al [13]; Markov decision processes by Cao [8] and Makowski and Shwartz [37]. For a comprehensive analysis of the Poisson’s equation, readers may refer to Nummelin [43], Meyn and Tweedie [39], Glynn [15], Makowski and Shwartz [37].

The main contributions of this paper are twofold. The first one is to develop some matrix Poisson’s equations in the study of mixing times of Markov chains with either finite states or infinite states. When the Markov chain has finite states, the generalized inverses are always utilized to study the mean and variance of the mixing time, e.g., see [19, 20, 21, 23]. However, for a Markov chain with infinite states, the available works for the solution to the matrix Poisson’s equations are few (e.g., see Nummelin [43], Cao and Chen [9], Makowski and Shwartz [37], Li and Liu [31] and Dendievel et al [13]), but the mean and variance of the mixing time have not yet been studied in the literature up to now. This motivates us in this paper to apply the UL-type of $R G$-factorization as well as the generalized inverses to setting up a computational framework for solving the matrix Poisson’s equations, which can lead to a detailed analysis for the mixing times in the Markov chains with infinite states. The second contribution of this paper is to provide a systematical discussion on the matrix Poisson’s equations satisfied by the mean and variance of the mixing time. Our main results are that the $R$, $U$- and $G$-measures are used in expressions both for the solution to the matrix Poisson’s equations and for the mean and variance of the mixing time. Note that some effective algorithms have been developed for computing the $R$, $U$- and $G$-measures in the QBD processes (e.g., see Bright and Taylor [6, 7]), thus this paper provides effectively numerical computation for the mean and variance of the mixing time through the matrix-analytic method. On the other hand, the first passage time described in this paper has a general block-structured probability meaning, which is different from the first passage times in the $R$, $U$- and $G$-measures by means of the taboo probability (e.g., see Neuts [41, 42] and Li [28]). Based on this, we develop an interesting and new research line on which the $R$, $U$- and $G$-measures are used to be able to deal with the general first passage time, the mixing time and the matrix Poisson’s equations in the Markov chains with infinite states.

The remainder of this paper is organized as follows. In Section 2, for a general block-structured Markov chain we develop some matrix Poisson’s equations satisfied by the
means and variances of the first passage time and of the mixing time. Furthermore, we apply the UL-type of RG-factorization as well as the generalized inverse to providing a computational procedure for solving the matrix Poisson’s equations. In Section 3, we simply review the UL-type of RG-factorization of the QBD process, and provide explicit expressions for the two key matrices \((I - RU)^{-1}\) and \((I - GL)^{-1}\) which are given by the R- and G-measures, respectively. For the level-dependent QBD process, Section 4 computes the means of the first passage time and of the mixing time, while Section 5 discusses the variances of the first passage time and of the mixing time. Some remarks and conclusions are given in the final section.

2 The Matrix Poisson’s Equations

In this section, for a general block-structured Markov chain with infinitely-many levels, we develop useful matrix Poisson’s equations satisfied by the means and variances of the first passage time and of the mixing time. Furthermore, we apply the UL-type of RG-factorization as well as the generalized inverse to providing a computational procedure for solving these matrix Poisson’s equations by means of the R- and G-measures.

We consider a general discrete-time block-structured Markov chain \(\{(X_n, J_n) : n \geq 0\}\) whose transition probability matrix is given by

\[
P = \begin{pmatrix}
    P_{0,0} & P_{0,1} & P_{0,2} & \cdots \\
    P_{1,0} & P_{1,1} & P_{1,2} & \cdots \\
    P_{2,0} & P_{2,1} & P_{2,2} & \cdots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

where \(X_n\) is the level process and \(J_n\) is the phase process, the size of the block \(P_{k,k}\) is \(m_k\) and the sizes of all the other blocks can be determined accordingly. We assume that the Markov chain \(P\) is irreducible and positive recurrent. In this case, the matrix \(P\) is stochastic, that is, \(Pe = e\), \(e\) is a column vector of ones with a suitable size. The stationary probability vector of the Markov chain \(P\) is partitioned accordingly into vectors \(\pi = (\pi_0, \pi_1, \pi_2, \ldots)\), where the size of the vector \(\pi_k\) is \(m_k\) for \(k \geq 0\).

When the Markov chain \(\{(X_n, J_n) : n \geq 0\}\) is irreducible and positive recurrent, it is clear that

\[
\lim_{n \to \infty} P \{(X_n, J_n) = (k, j) \mid (X_0, J_0) = (l, i)\} = \pi_{k,j},
\]
which is independent of the initial state \((X_0, J_0) = (l, i)\). If for some \(r \geq 0\), \(P \{(X_r, J_r) = (k, j)\} = \pi_{k,j}\), then for \(n \geq r\)
\[
P \{(X_n, J_n) = (k, j)\} = \pi_{k,j}.
\]

### 2.1 The mean of the mixing time

Let \(T_{l,i; k,j}\) be the first passage time of the Markov chain \(\{(X_n, J_n) : n \geq 0\}\) from state \((l, i)\) to state \((k, j)\), that is,
\[
T_{l,i; k,j} = \min \{n \geq 1 : (X_n, J_n) = (k, j) \mid (X_0, J_0) = (l, i)\}.
\]
Specifically, \(T_{k,j; k,j}\) is the first return time of the Markov chain \(\{(X_n, J_n) : n \geq 0\}\) to state \((k, j)\).

We write
\[
M_{l,i; k,j} = E[T_{l,i; k,j}],
\]
\[
M_{l,i,k} = \begin{pmatrix}
M_{l,1,k,1} & \cdots & M_{l,1,k,m_k} \\
M_{l,2,k,1} & \cdots & M_{l,2,k,m_k} \\
\vdots & \vdots & \vdots \\
M_{l,m_i,k,1} & \cdots & M_{l,m_i,k,m_k}
\end{pmatrix}
\]
and
\[
M = \begin{pmatrix}
M_{0,0} & M_{0,1} & M_{0,2} & \cdots \\
M_{1,0} & M_{1,1} & M_{1,2} & \cdots \\
M_{2,0} & M_{2,1} & M_{2,2} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Let \(E = ee^T\), where \(A^T\) denotes the transpose of the matrix \(A\). We write
\[
\text{diag}(\pi_k) = \text{diag}(\pi_{k,1}, \pi_{k,2}, \ldots, \pi_{k,m_k}), \quad k \geq 0,
\]
and
\[
\text{diag}(\pi) = \text{diag}(\text{diag}(\pi_0), \text{diag}(\pi_1), \text{diag}(\pi_2), \ldots).
\]

Then it follows from Theorem 4.4.4 in Kemeny and Snell [25] or (2.1) in Hunter [19] that
\[
(I - P) M = E - P [\text{diag}(\pi)]^{-1}.
\]

When the Markov chain \(P\) is at steady-state, the \((\sum_{i=0}^{k-1} m_i + j)\)th entry of the column vector \(Me\) is the mean of stationary first passage time to state \((k, j)\), hence we have
\[
\tau = \pi Me
\]
is the mean of stationary first passage time in the Markov chain \( \{(X_n, J_n) : n \geq 0\} \).

Let \( T \) be the mixing time of the Markov chain \( \{(X_n, J_n) : n \geq 0\} \), and \( Y \) a random variable whose probability distribution is the stationary probability vector \( \pi = (\pi_0, \pi_1, \pi_2, \ldots) \).

Then

\[
T = \min \{n : (X_n, J_n) = Y\},
\]

and the Markov chain \( \{(X_n, J_n) : n \geq 0\} \) reaches stationary or achieves mixing at time \( T \).

We define the mean of the mixing time as

\[
L_{l, i; k, j} = E[T \mid (X_0, J_0) = (l, i), (X_T, J_T) = (k, j)].
\]

Let

\[
L_{l, k} = \begin{pmatrix}
L_{l, 1; k, 1} & L_{l, 1; k, 2} & \cdots & L_{l, 1; k, m_k} \\
L_{l, 2; k, 1} & L_{l, 2; k, 2} & \cdots & L_{l, 2; k, m_k} \\
\vdots & \vdots & \ddots & \vdots \\
L_{l, m_l; k, 1} & L_{l, m_l; k, 2} & \cdots & L_{l, m_l; k, m_k}
\end{pmatrix}
\]

and

\[
L = \begin{pmatrix}
L_{0, 0} & L_{0, 1} & L_{0, 2} & \cdots \\
L_{1, 0} & L_{1, 1} & L_{1, 2} & \cdots \\
L_{2, 0} & L_{2, 1} & L_{2, 2} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

From (2.6) in Hunter \[19\]

\[
L_{l, i; k, j} = M_{l, i; k, j} \pi_{k, j}.
\]

we obtain

\[
L = M \text{diag} (\pi).
\]  \hspace{1cm} (2)

Using \((1)\) and \((2)\), we obtain

\[
(I - P) L = e \pi - P.
\]  \hspace{1cm} (3)

We write

\[
\eta_{k, j} = E[T \mid (X_0, J_0) = (k, j)],
\]

\[
\eta_k = (\eta_{k, 1}, \eta_{k, 2}, \ldots, \eta_{k, m_k})
\]

and

\[
\eta = (\eta_0, \eta_1, \eta_2, \eta_3, \ldots).
\]
It follows from Theorem 2.1 in Hunter [19] that
\[ \eta^T = M\pi^T, \] (4)

Using (4) and (2), we obtain
\[ \eta^T = Le, \]
which, together with (3), follows
\[ (I - P)\eta^T = 0. \] (5)

2.2 The variance of the mixing time

Let
\[ M^{(2)}_{l,i,k,j} = E\left[ T^{2}_{l,i;k,j} \right]. \]

We write
\[
M^{(2)}_{l,k} = \begin{pmatrix}
M^{(2)}_{l,1,k,1} & M^{(2)}_{l,1,k,2} & \cdots & M^{(2)}_{l,1,k,m_k} \\
M^{(2)}_{l,2,k,1} & M^{(2)}_{l,2,k,2} & \cdots & M^{(2)}_{l,2,k,m_k} \\
\vdots & \vdots & \ddots & \vdots \\
M^{(2)}_{l,m_l;k,1} & M^{(2)}_{l,m_l;k,2} & \cdots & M^{(2)}_{l,m_l;k,m_k}
\end{pmatrix}
\]

and
\[
M^{(2)} = \begin{pmatrix}
M^{(2)}_{0,0} & M^{(2)}_{0,1} & M^{(2)}_{0,2} & \cdots \\
M^{(2)}_{1,0} & M^{(2)}_{1,1} & M^{(2)}_{1,2} & \cdots \\
M^{(2)}_{2,0} & M^{(2)}_{2,1} & M^{(2)}_{2,2} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Using (2.7) in Hunter [20], we obtain
\[
(I - P)M^{(2)} = E + P \left\{ 2M - [\text{diag}(\pi)]^{-1} [I + 2(e\pi M)_d]\right\}, \] (6)

where \((e\pi M)_d\) is a diagonal matrix whose diagonal entries are given by the diagonal entries of the matrix \(e\pi M\).

Let
\[
L^{(2)}_{l,i,k,j} = E\left[ T^2 \mid (X_0, J_0) = (l, i), (X_T, J_T) = (k, j) \right].
\]

Then
\[
L^{(2)}_{l,i,k,j} = M^{(2)}_{l,i;k,j} \pi_{k,j}. \] (7)
We write

\[
L_{i,k}^{(2)} = \begin{pmatrix}
L_{i,1,k,1}^{(2)} & L_{i,1,k,2}^{(2)} & \cdots & L_{i,1,k,m_k}^{(2)} \\
L_{i,2,k,1}^{(2)} & L_{i,2,k,2}^{(2)} & \cdots & L_{i,2,k,m_k}^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
L_{i,m_i,k,1}^{(2)} & L_{i,m_i,k,2}^{(2)} & \cdots & L_{i,m_i,k,m_k}^{(2)}
\end{pmatrix}
\]

and

\[
L^{(2)} = \begin{pmatrix}
L_{0,0}^{(2)} & L_{0,1}^{(2)} & L_{0,2}^{(2)} & \cdots \\
L_{1,0}^{(2)} & L_{1,1}^{(2)} & L_{1,2}^{(2)} & \cdots \\
L_{2,0}^{(2)} & L_{2,1}^{(2)} & L_{2,2}^{(2)} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

It is easy to see from (7) that

\[
L^{(2)} = M^{(2)} \text{diag} \left( \pi \right).
\]  \hfill (8)

It follows from (6) that

\[
(I - P) L^{(2)} = e^{\pi} + P \left\{ 2M - \left[ \text{diag} \left( \pi \right) \right]^{-1} \left[ I + 2 \left( e^{\pi} M \right) \right] \right\} \text{diag} \left( \pi \right).
\]  \hfill (9)

Set

\[
\eta_{k,j}^{(2)} = E \left[ T^2 \mid (X_0, J_0) = (k, j) \right],
\]

\[
\eta_k^{(2)} = \left( \eta_{k,1}^{(2)}, \eta_{k,2}^{(2)}, \ldots, \eta_{k,m_k}^{(2)} \right)
\]

and

\[
\eta^{(2)} = \left( \eta_0^{(2)}, \eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}, \ldots \right).
\]

It follows from Theorem 1.1 in Hunter [20] that

\[
\left( \eta^{(2)} \right)^T = M^{(2)} \pi^T,
\]  \hfill (10)

Based on (6) and (10), we obtain

\[
(I - P) \left( \eta^{(2)} \right)^T = e + P \left\{ 2M - \left[ \text{diag} \left( \pi \right) \right]^{-1} \left[ I + 2 \left( e^{\pi} M \right) \right] \right\} \pi^T.
\]  \hfill (11)

Let

\[
v_{k,j}^{(2)} = \text{Var} \left[ T \mid (X_0, J_0) = (k, j) \right],
\]

\[
v_k^{(2)} = \left( v_{k,1}^{(2)}, v_{k,2}^{(2)}, \ldots, v_{k,m_k}^{(2)} \right)
\]

and

\[
V^{(2)} = \left( v_0^{(2)}, v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots \right).
\]

8
Then
\[
\left( V^{(2)} \right)^T = \left( \eta^{(2)} \right)^T - \eta^T \circ \eta^T
\]  
(12)

where \( A \circ B \) denotes the Hadamard Product of the two matrices \( A \) and \( B \), that is, \( A \circ B = (a_{i,j}b_{i,j}) \) if \( A = (a_{i,j}) \) and \( B = (b_{i,j}) \). Let \( v^{(2)} = \text{Var}(T) \). Then
\[
v^{(2)} = \pi \left( V^{(2)} \right)^T = \pi \left( \eta^{(2)} \right)^T - \pi \left( \eta^T \circ \eta^T \right),
\]  
(13)

which is the variance of the mixing time when the Markov chain \( P \) is at steady state.

It is worthwhile to note that the above matrix Poisson’s equations are a direct generalization of those in Hunter [20] both from the finite states to the infinite states, and from the scale entries to the block entries. Therefore, the matrix Poisson’s equations developed here are more general than those in Hunter [20], and they are useful and interesting in the study of stochastic models through the matrix-analytic method.

2.3 The Poisson’s equation

In the UL-type of \( RG \)-factorization
\[
I - P = (I - R_U)(I - \Psi_D)(I - G_L),
\]
where \( \Psi_D = \text{diag}(U_0, U_1, U_2, \ldots) \) and \( U_0 \) is the transition probability matrix of the censoring Markov chain to level 0. If the Markov chain \( P \) is irreducible and recurrent, then the censoring Markov chain \( U_0 \) is irreducible and positive recurrent, and \( \text{rank}(U_0) = m_0 - 1 \). Let \( v_0 \) be the stationary probability vector of the censoring Markov chain \( U_0 \). Then \( v_0(I - U_0) = 0 \).

Now, we deal with the Poisson’s equation:
\[
(I - U_0)x = g,
\]  
(14)

where \( g \) is a given column vector of size \( m_0 \). Hence it follows from (14) that \( v_0g = 0 \). This shows that the given vector \( g \) must satisfy the condition \( v_0g = 0 \) if there exists one solution to the Poisson’s equation (14).

If there exists a matrix \( V \) such that \((I - U_0)V(I - U_0) = (I - U_0)\), then the matrix \( V \) is called a generalized inverse of the matrix \( I - U_0 \).

From Theorem 3.3 in Hunter [17], we know that the matrix \( I - U_0 + tu \) is invertible and \((I - U_0 + tu)^{-1}\) is a generalized inverse of the matrix \( I - U_0 \), where \( t \) and \( u \) are two
arbitrary vectors such that \( v_0 t \neq 0 \) and \( u e \neq 0 \). It is clear that \( t \) and \( u \) are the column and row vectors, respectively. Furthermore, the matrix

\[
V = [I - U_0 + tu]^{-1} + ef + hv_0
\]  

(15)

is a generalized inverse of the matrix \( I - U_0 \), where \( f \) and \( h \) are two arbitrary vectors. Clearly, \( f \) and \( h \) are the row and column vectors, respectively. It is worthwhile to note that any generalized inverse of the matrix \( I - U_0 \) can be expressed by (15) with the four vectors: \( t, u, f \) and \( h \). Specifically, when \( t = e, u = v_0, f = 0 \) and \( h = 0 \), the matrix

\[
Z = [I - U_0 + ev_0]^{-1}
\]  

(16)

is called the Kemeny and Snell’s fundamental matrix.

It is easy to check that

\[
u [I - U_0 + tu]^{-1} = \frac{v_0}{v_0 t}
\]

and

\[
[I - U_0 + tu]^{-1} t = \frac{e}{ue}.
\]

Specifically, we have

\[
v_0 [I - U_0 + ev_0]^{-1} = v_0
\]

and

\[
[I - U_0 + ev_0]^{-1} e = e.
\]

Let \( V \) be any generalized inverse of the matrix \( I - U_0 \), given by (15). Then the Poisson’s equation \((I - U_0) x = g\) has a solution if and only if

\[
(I - U_0) V g = g.
\]

In this case, we have

\[
x = V g + [I - V (I - U_0)] \Theta,
\]  

(17)

where \( \Theta \) is an arbitrary column vector. Specifically, we have an important solution as follows:

\[
x = [I - U_0 + ev_0]^{-1} g + ce,
\]  

(18)

where \( c \) is an arbitrary constant.
Now, we give some useful observation on the solution (18) as follows:

\[(I - U_0) x = (I - U_0)[I - U_0 + ev_0]^{-1} g + (I - U_0) ce = g - ev_0g = g,\]

since \((I - U_0)e = 0\) and the basic condition \(v_0g = 0\).

Because of the fact that \(\text{rank}(U_0) = m_0 - 1\), it seems to have a better understanding for the only one free parameter \(c\) in the solution (18). On the contrary, the solution (17) have more free parameters through the five vectors \(t, u, f, h\) and \(\Theta\).

In the rest of this subsection, we consider a matrix Poisson’s equation

\[(I - U_0) X = G, \quad (19)\]

where \(G\) is a given matrix of size \(m_0\). To solve the matrix Poisson’s equation, we write

\[X = (x_1, x_2, \ldots, x_{m_0})\]

and

\[G = (g_1, g_2, \ldots, g_{m_0}).\]

Then the matrix Poisson’s equation is written as the \(m_0\) Poisson’s equations as follows:

\[(I - U_0) x_k = g_k, \quad 1 \leq k \leq m_0.\]

It is easy to see from (18) and (19) that

\[X = [I - U_0 + ev_0]^{-1} G + ec, \quad (20)\]

where \(c = (c_1, c_2, \ldots, c_{m_0})\) are an arbitrary row vector. It is seen that the solution (20) contains the most basic \(m_0\) free parameters in the vector \(c = (c_1, c_2, \ldots, c_{m_0})\). Note that the solution (20) will be useful in the remainder of this paper.

### 2.4 A computational procedure

From the above matrix Poisson’s equations (e.g., (1) and (6)), it is seen that we need to solve a general matrix Poisson’s equation as follows:

\[(I - P) A = B, \quad (21)\]
where \( B \) is a given matrix with \( \pi B = 0 \). From Chapter 2 in Li [28], the UL-type of \( RG \)-factorization for the general Markov chain \( P \) is given by

\[
I - P = (I - R_U)(I - \Psi_D)(I - G_L).
\]

This gives

\[
(I - R_U)(I - \Psi_D)(I - G_L)A = B,
\]

which follows

\[
(I - \Psi_D)(I - G_L)A = (I - R_U)^{-1}B,
\]

Let

\[
X = (I - G_L)A.
\] (22)

Then

\[
(I - \Psi_D)X = (I - R_U)^{-1}B.
\] (23)

We write

\[
I - \Psi_D = \text{diag} (I - U_0, I - \Phi_D),
\]

\[
\Phi_D = \text{diag} (U_1, U_2, U_3, \ldots);
\]

\[
X = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix},
\]

where \( X_0 \) is a matrix with the first \( m_0 \) row vectors of the matrix \( X \);

\[
(I - R_U)^{-1}B = \begin{pmatrix} C_0 \\ C_1 \end{pmatrix},
\] (24)

where \( C_0 \) is a matrix with the first \( m_0 \) row vectors of the matrix \( (I - R_U)^{-1}B \). It follows from (22) that

\[
(I - U_0)X_0 = C_0
\] (25)

and

\[
(I - \Phi_D)X_1 = C_1.
\] (26)

Note that the matrix \( I - U_k \) is invertible for \( k \geq 1 \), the matrix \( I - \Phi_D \) is invertible. Thus it follows from (26) that

\[
X_1 = (I - \Phi_D)^{-1}C_1.
\] (27)
Note that the matrix $I - U_0$ is singular, thus it follows from (25) and (20) that
\[ X_0 = ZC_0 + ec_0, \tag{28} \]
where $c_0$ is an arbitrary row vector.

Based on (27) and (28), we obtain
\[
X = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} = \begin{pmatrix} ZC_0 + ec_0 \\ (I - \Phi_D)^{-1}C_1 \end{pmatrix}. \tag{29}
\]

It follows from (22) that
\[
(I - G_L)A = \begin{pmatrix} ZC_0 + ec_0 \\ (I - \Phi_D)^{-1}C_1 \end{pmatrix}.
\]

Thus we obtain
\[
A = (I - G_L)^{-1} \begin{pmatrix} ZC_0 + ec_0 \\ (I - \Phi_D)^{-1}C_1 \end{pmatrix}. \tag{30}
\]

From (30) and (24), it is seen that a key for the solution to the matrix Poisson’s equation (21) is that the two matrices $(I - R_U)^{-1}$ and $(I - G_L)^{-1}$ can have the explicit expressions by means of the $R$- and $G$-measures. However, Chapter 2 and Appendix B in Li [28] indicated that $(I - R_U)^{-1}$ and $(I - G_L)^{-1}$ can explicitly be expressed only in two special cases: The QBD processes, and Markov chains of GI/G/1 type.

Note that the first passage time in the matrix $M$ is different from the first passage times in the $R$- and $G$-measures, because the $R$- and $G$-measures are defined by the taboo probability, e.g., see Chapter 2 in Li [28] and Neuts [41, 42]. Therefore, this paper provides new highlight on the block-structured Markov chains, including the QBD processes, and the Markov chains of GI/M/1 type and of M/G/1 type.

In the remainder of this paper, we will consider an important case: the QBD processes, and derive the means and variances of the first passage time and of the mixing time.

## 3 The QBD Processes

In this section, we consider an irreducible discrete-time level-dependent QBD process with infinitely-many levels, and simply review the UL-type of $RG$-factorization of the QBD process. Specifically, we provide explicit expressions for the two key matrices $(I - R_U)^{-1}$ and $(I - G_L)^{-1}$ through the $R$- and $G$-measures, respectively.
We consider an irreducible discrete-time level-dependent QBD process \( \{(X_n, J_n), n \geq 0\} \) whose transition probability matrix is given by

\[
P = \begin{pmatrix}
A_1^{(0)} & A_0^{(0)} \\
A_2^{(1)} & A_1^{(1)} & A_0^{(1)} \\
A_2^{(2)} & A_1^{(2)} & A_0^{(2)} \\
& \ddots & \ddots \\
& & & \\
\end{pmatrix},
\]

(31)

where the size of the matrix \( A_1^{(k)} \) is \( m_k \) for \( k \geq 0 \), \( A_i^{(0)} \geq 0 \) for \( i = 0, 1 \), \( A_j^{(k)} \geq 0 \) for \( j = 0, 1, 2 \) and \( k \geq 1 \), \( A_0^{(0)}e + A_1^{(0)}e = e \) and \( A_0^{(k)}e + A_1^{(k)}e + A_2^{(k)}e = e \) for \( k \geq 1 \).

Let the matrix sequences \( \{R_l : l \geq 0\} \) and \( \{G_k : k \geq 1\} \) be the minimal nonnegative solutions to the systems of matrix Poisson’s equations

\[
A_1^{(l)} + R_l A_1^{(l+1)} + R_l R_{l+1} A_2^{(l+2)} = R_l, \quad l \geq 0,
\]

(32)

and

\[
A_0^{(k)} G_{k+1} G_k + A_1^{(k)} G_k + A_2^{(k)} = G_k, \quad k \geq 1,
\]

(33)

respectively. Then the \( U \)-measure \( \{U_l : l \geq 0\} \) is given by

\[
U_l = A_1^{(l)} + R_l A_2^{(l+1)} = A_1^{(l)} + A_0^{(l)} G_{l+1}, \quad l \geq 0,
\]

(34)

For \( k \geq 1 \), the matrix \( I - U_k \) is invertible; while the matrix \( I - U_0 \) is singular, and the censoring Markov chain \( U_0 \) is irreducible and positive only if the QBD process \( P \) is irreducible and recurrent.

For the QBD process with infinitely-many levels given in (31), the UL-type of \( RG \)-factorization is given by

\[
I - P = (I - R_U) (I - \Psi_D) (I - G_L),
\]

(35)

where

\[
R_U = \begin{pmatrix}
0 & R_0 \\
0 & R_1 \\
0 & R_2 \\
& \ddots & \ddots \\
\end{pmatrix},
\]

\[
\Psi_D = \text{diag} (U_0, U_1, U_2, U_3, \ldots)
\]
and

\[
G_L = \begin{pmatrix}
0 & & & \\
G_1 & 0 & & \\
& G_2 & 0 & \\
& & G_3 & 0 \\
& & & \ddots \\
\end{pmatrix}
\]

Note that the \( RG \)-factorization can be given only if the QBD process is irreducible.

Let

\[
X_k^{(l)} = R_l R_{l+1} R_{l+2} \cdots R_{l+k-1}, \quad k \geq 1, l \geq 0,
\]

and

\[
Y_k^{(l)} = G_l G_{l-1} G_{l-2} \cdots G_{l-k+1}, \quad l \geq k \geq 1.
\]

Then

\[
(I - R_U)^{-1} = \begin{pmatrix}
I & X_1^{(0)} & X_2^{(0)} & X_3^{(0)} & \cdots \\
I & X_1^{(1)} & X_2^{(1)} & & \\
I & & & \ddots \\
\end{pmatrix}
\]

and

\[
(I - G_L)^{-1} = \begin{pmatrix}
I & & & \\
Y_1^{(1)} & I & & \\
Y_2^{(2)} & Y_1^{(2)} & I & \\
Y_3^{(3)} & Y_2^{(3)} & Y_1^{(3)} & I \\
& & & \ddots \\
\end{pmatrix}
\]

If the QBD process is level-independent, then

\[R_k = R, \quad k \geq 1,
\]

and

\[G_l = G, \quad l \geq 2.
\]
In this case, we obtain

\[(I - R_U)^{-1} = \begin{pmatrix}
I & R_0 & R_0R & R_0R^2 & \cdots \\
I & R & R^2 & \cdots \\
I & R & \cdots \\
I & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}\] (40)

and

\[(I - G_L)^{-1} = \begin{pmatrix}
I & & & & \\
G_1 & I & & & \\
GG_1 & G & I & & \\
G^2G_1 & G^2 & G & I & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.\] (41)

It is worthwhile to note that we can obtain the solution to the matrix Poisson’s equation (21) once the two key matrices \((I - R_U)^{-1}\) and \((I - G_L)^{-1}\) are expressed explicitly by means of the \(R\)-measure \(\{R_l : l \geq 0\}\) and the \(G\)-measure \(\{G_k : k \geq 1\}\), respectively. It is seen from (38) to (41) that the first passage time as well as the mixing time in the QBD processes can be given a detailed analysis.

In the remainder of this paper, we assume that the QBD process is irreducible and positive recurrent. It follows from Li and Cao [31] or Li [28] that

\[\pi_0 = \varphi v_0\] (42)

and

\[\pi_k = \varphi v_0 R_0 R_1 \cdots R_{k-1}, \quad k \geq 1,\] (43)

where \(v_0\) is the stationary probability vector of the censored Markov chain \(U_0 = A_1^{(0)} + R_0 A_2^{(1)}\) to level 0, and the normalized constant \(\varphi\) is given by

\[\varphi = \frac{1}{1 + \sum_{k=0}^{\infty} v_0 R_0 R_1 \cdots R_k e}.\]

### 4 The Mean of the Mixing Time

In this section, for the QBD process we apply the UL-type of \(RG\)-factorization as well as the generalized inverse to computing the means of the first passage time and of the mixing
time. Note that they can be expressed by means of the \( R \)- and \( G \)-measures through the computational procedure given in Subsection 2.4.

### 4.1 The mean of the first passage time

To compute the matrix \( M \), we need to solve the equation (1). Using the UL-type of \( RG \)-factorization, we obtain

\[
(I - R_U) (I - \Psi_D) (I - G_L) M = E - P [\text{diag} (\pi)]^{-1},
\]

which follows

\[
(I - \Psi_D) (I - G_L) M = (I - R_U)^{-1} \left\{ E - P [\text{diag} (\pi)]^{-1} \right\}. \quad (44)
\]

Let

\[
X = (I - G_L) M. \quad (45)
\]

Then

\[
(I - \Psi_D) X = (I - R_U)^{-1} \left\{ E - P [\text{diag} (\pi)]^{-1} \right\}. \quad (46)
\]

We write

\[
X = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix},
\]

where \( X_0 \) is a matrix with the first \( m_0 \) row vectors of the matrix \( X \);

\[
(I - R_U)^{-1} \left\{ E - P [\text{diag} (\pi)]^{-1} \right\} = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix},
\]

where

\[
F_0 = (F_{0,0}, F_{0,1}, F_{0,2}, \ldots), \quad F_1 = \begin{pmatrix} F_{1,0} & F_{1,1} & F_{1,2} & \cdots \\ F_{2,0} & F_{2,1} & F_{2,2} & \cdots \\ F_{3,0} & F_{3,1} & F_{3,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},
\]

\[
F_{i,j} = \begin{cases} E_{i,j} + \sum_{k=1}^{\infty} X_k^{(i)} E_{k+i,j} - X_{j-1}^{(i)} [\text{diag} (\pi_j)]^{-1}, & j \geq i + 1, i \geq 0, \\
E_{i,j} + \sum_{k=1}^{\infty} X_k^{(i)} E_{k+i,j} - \left[ A_1^{(i)} + X_1^{(i)} A_2^{(i+1)} \right] [\text{diag} (\pi_j)]^{-1}, & j = i, i \geq 0, \\
E_{i,j} + \sum_{k=1}^{\infty} X_k^{(i)} E_{k+i,j} - A_2^{(i)} [\text{diag} (\pi_j)]^{-1}, & j = i - 1, i \geq 1, \\
E_{i,j} + \sum_{k=1}^{\infty} X_k^{(i)} E_{k+i,j}, & j \leq i - 2, i \geq 2. \end{cases}
\]
It follows from (46) that
\[(I - U_0)X_0 = F_0\] (47)
and
\[(I - \Phi_D)X_1 = F_1.\] (48)

It follows from (48) that
\[X_1 = (I - \Phi_D)^{-1} F_1,\] (49)
and from (47) that
\[X_0 = ZF_0 + ec_0,\] (50)
where \(c_0 = (c_{0,0}, c_{0,1}, c_{0,2}, \ldots)\) is an arbitrary row vector.

Based on (49) and (50), we obtain
\[X = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} = \begin{pmatrix} ZF_0 + ec_0 \\ (I - \Phi_D)^{-1} F_1 \end{pmatrix}.\] (51)

It follows from (45) that
\[(I - G_L)M = \begin{pmatrix} ZF_0 + ec_0 \\ (I - \Phi_D)^{-1} F_1 \end{pmatrix}.\]

Thus we obtain
\[M = (I - G_L)^{-1} \begin{pmatrix} ZF_0 + ec_0 \\ (I - \Phi_D)^{-1} F_1 \end{pmatrix}.\]

This gives
\[M_{i,j} = \begin{cases} ZF_0 + ec_{0,j}, & i = 0, j \geq 0, \\ Y_i^{(i)}(ZF_0 + ec_{0,j}) + (I - U_i)^{-1} F_{i,j} + \sum_{k=1}^{i-1} Y_{i-k}^{(i)}(I - U_k)^{-1} F_{k,j}, & i \geq 1, j \geq 0. \end{cases}\]

4.2 The mean of the mixing time

From (2), we have \(L = M\text{diag}(\pi)\). Once the matrix \(M\) is given in Subsection 4.1, it is clear that the matrix \(L\) is obtained by \(M\text{diag}(\pi)\).

On the other hand, the matrix \(L\) can be solved by the matrix equation (3) of itself. It may be necessary to simply provide a simple outline for the solution to Equation (3).

To compute the matrix \(L\) from (3), using the UL-type of \(RG\)-factorization, we obtain
\[(I - R_U)(I - \Psi_D)(I - G_L)L = e\pi - P,\]
which follows
\[ (I - \Psi_D)(I - G_L)L = (I - R_U)^{-1}(e\pi - P). \] (52)

Let
\[ Y = (I - G_L)L. \] (53)

Then it follows from (52) that
\[ (I - \Psi_D)Y = (I - R_U)^{-1}(e\pi - P). \] (54)

We write
\[ Y = \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix}, \]
where \( Y_0 \) is a matrix with the first \( m_0 \) row vectors of the matrix \( Y \);
\[ (I - R_U)^{-1}(e\pi - P) = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}, \]
where
\[ H_0 = (H_{0,0}, H_{0,1}, H_{0,2}, \ldots), \]
\[ H_1 = \begin{pmatrix} H_{1,0} & H_{1,1} & H_{1,2} & \cdots \\ H_{2,0} & H_{2,1} & H_{2,2} & \cdots \\ H_{3,0} & H_{3,1} & H_{3,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \]
\[ H_{i,j} = \begin{cases} e\pi_j + \sum_{k=1}^{8} X_k^{(i)} e\pi_j - X_j^{(i)}, & j \geq i + 1, i \geq 0, \\ e\pi_j + \sum_{k=1}^{8} X_k^{(i)} e\pi_j - \left[ A_1^{(i)} + X_1^{(i)} A_2^{(i+1)} \right], & j = i, i \geq 0, \\ e\pi_j + \sum_{k=1}^{8} X_k^{(i)} e\pi_j - A_2^{(i)}, & j = i - 1, i \geq 1, \\ e\pi_j + \sum_{k=1}^{8} X_k^{(i)} e\pi_j, & j \leq i - 2, i \geq 2. \end{cases} \]

It follows from (54) that
\[ (I - U_0)Y_0 = H_0 \] (55)
and
\[ (I - \Phi_D)Y_1 = H_1. \] (56)

It follows from (56) that
\[ Y_1 = (I - \Phi_D)^{-1}H_1, \] (57)
and from (55) that
\[ Y_0 = ZH_0 + ec_0, \] (58)
where \( c_0 = (c_{0,0}, c_{0,1}, c_{0,2}, \ldots) \) is an arbitrary row vector.

Based on (57) and (58), we obtain
\[
Y = \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} = \begin{pmatrix} ZH_0 + ec_0 \\ (I - \Phi D)^{-1} H_1 \end{pmatrix}.
\]

It follows from (53) that
\[
(I - G_L) L = \begin{pmatrix} ZH_0 + ec_0 \\ (I - \Phi D)^{-1} H_1 \end{pmatrix}.
\]

Thus we obtain
\[
L = (I - G_L)^{-1} \begin{pmatrix} ZH_0 + ec_0 \\ (I - \Phi D)^{-1} H_1 \end{pmatrix},
\]
where
\[
L_{i,j} = \begin{cases} ZH_{0,j} + ec_{0,j}, & i = 0, j \geq 0, \\ Y^{(i)}_i (ZH_{0,j} + ec_{0,j}) + (I - U_i)^{-1} H_{i,j} + \sum_{k=1}^{i-1} Y^{(i)}_{i-k} (I - U_k)^{-1} H_{k,j}, & i \geq 1, j \geq 0. \end{cases}
\]

4.3 The generalized Kemeny’s constant

In this subsection, we generalize the Kemeny’s constant (e.g., see Hunter [22]) of the Markov chains from the finite state space to the infinite state space, where a key for such a generalization is to apply the censoring technique and the UL-type of \( RG \)-factorization.

If the QBD \( P \) is irreducible and positive recurrent, then the equation \((I - P)x = 0\) exists the unique, up to multiplication by a positive constant, solution: \( x = \gamma e \), where \( \gamma \) is positive constant. Note that such a \( \gamma \) is called the Kemeny’s constant in the study of Markov chains with finite states. Here, we extend the Kemeny’s constant to that in a Markov chain with infinite states. That is, the vector equation (5) is shown to have the unique solution
\[ \eta^T = \eta e, \] (59)
and \( \eta \) is called the generalized Kemeny’s constant of the Markov chain with infinite states, including the QBD process. Using the UL-type of \( RG \)-factorization and the censoring
technique, we establish a useful relation between the Kemeny’s constant (for the finite states) and the generalized Kemeny’s constant (for the infinite states).

Using the UL-type of $RG$-factorization, it follows from (5) that

$$(I - R_U) (I - \Psi_D) (I - G_L) \eta^T = 0,$$

which follows

$$(I - \Psi_D) (I - G_L) \eta^T = 0. \quad (60)$$

Let

$$\xi^T = (I - G_L) \eta^T. \quad (61)$$

Then it follows from (60) that

$$(I - \Psi_D) \xi^T = 0. \quad (62)$$

We write

$$\eta^T = \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \vdots \end{pmatrix}, \quad \xi^T = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \vdots \end{pmatrix}.$$ 

It follows from (62) that

$$(I - U_0) \xi_0 = 0 \quad (63)$$

and for $k \geq 1$

$$(I - U_k) \xi_k = 0. \quad (64)$$

Since the matrix $I - U_k$ is invertible for $k \geq 1$, it is clear from (64) that $\xi_k = 0$ for $k \geq 1$. Note that the Markov chain $U_0$ is irreducible and positive recurrent, thus we obtain

$$\xi_0 = \delta e,$$

where $\delta$ is the Kemeny’s constant of the censoring Markov chain $U_0$. Thus we obtain

$$\xi^T = \begin{pmatrix} \delta e \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \quad (65)$$
It follows from (61) and (65) that

\[
(I - G_L) \eta^T = \begin{pmatrix}
\delta e \\
0 \\
0 \\
\vdots
\end{pmatrix}.
\]

This gives

\[
\eta^T = (I - G_L)^{-1} \begin{pmatrix}
\delta e \\
0 \\
0 \\
\vdots
\end{pmatrix} = \begin{pmatrix}
I & 0 & 0 & \cdots \\
Y_1^{(1)} & I & 0 & \cdots \\
Y_2^{(2)} & Y_1^{(2)} & I & \cdots \\
Y_3^{(3)} & Y_2^{(3)} & Y_1^{(3)} & I & \cdots \\
& \vdots & \vdots & \ddots & \vdots
\end{pmatrix} \begin{pmatrix}
\delta e \\
0 \\
0 \\
\vdots
\end{pmatrix} = \begin{pmatrix}
\delta e \\
e \\
e \\
\vdots
\end{pmatrix} = \delta e,
\]

since \( Y_k^{(k)} e = e \) for \( k \geq 1 \) according to the fact that \( G_k e = e \) for \( k \geq 1 \) if the QBD process is irreducible and positive recurrent, e.g., see Li [28]. Note that \( \eta^T = \eta e \) and \( \eta^T = \delta e \), we have \( \eta = \delta \). It follows from (2.19) in Hunter [19] that

\[
\eta = \delta = \text{tr} (Z) = \text{tr} \left( (I - U_0 + e v_0)^{-1} \right).
\]  

(66)

Now, we further apply the censoring technique to computing the generalized Kemeny’s constant \( \eta \) of the QBD process. Let

\[
U_0 = \begin{pmatrix}
P_{1,1} & P_{1,2} \\
P_{2,1} & P_{2,2}
\end{pmatrix},
\]

where the sizes of the two matrices \( P_{1,1} \) and \( P_{2,2} \) are 2 and \( m_0 - 2 \), respectively. Then we can obtain a new censoring chain

\[
U_0^{(1,2)} = P_{1,1} + P_{1,2} (I - P_{2,2})^{-1} P_{2,1} = \begin{pmatrix}
1 - a & a \\
b & 1 - b
\end{pmatrix},
\]

22
where \(0 < a, b \leq 1\), since the censoring Markov chain \(U^{(1,2)}_0\) is irreducible and positive recurrent. Let \(\delta_{1,2}\) be the Kemeny’s constant of the Markov chain \(U^{(1,2)}_0\). Then it follows from (3.1) in Hunter [19] that

\[
\eta = \delta_{1,2} = 1 + \frac{1}{a + b}.
\]  

(67)

It is seen from (66) and (67) that the censoring technique and the UL-type of \(RG\)-factorization can be applied to highlight the Kemeny’s constant of the Markov chains from the finite state space to the infinite state space, and also provide an effective algorithm for computing the generalized Kemeny’s constant in the Markov chains with infinite states.

## 5 The Variance of the Mixing Time

In this section, for the QBD process we apply the UL-type of \(RG\)-factorization as well as the generalized inverse to computing the variances of the first passage time and of the mixing time. Note that the variances can be expressed by means of the \(R\)- and \(G\)-measures through the computational procedure given in Subsection 2.4.

### 5.1 The variance of the first passage time

To compute the matrix \(M^{(2)}\) given by the first passage time, we need to solve the equation (6). Using the UL-type of \(RG\)-factorization, we obtain

\[
(I - R_U) (I - \Psi_D) (I - G_L) M^{(2)} = \left\{ E + P \left\{ 2M - \left[ \text{diag} (\pi) \right]^{-1} [I + 2 (e\pi M)_d] \right\} \right\}.
\]

which follows

\[
(I - \Psi_D) (I - G_L) M^{(2)} = (I - R_U)^{-1} \left\{ E + P \left\{ 2M - \left[ \text{diag} (\pi) \right]^{-1} [I + 2 (e\pi M)_d] \right\} \right\}.
\]

(68)

Let

\[
X = (I - G_L) M^{(2)}.
\]

(69)

Then

\[
(I - \Psi_D) X = (I - R_U)^{-1} \left\{ E + P \left\{ 2M - \left[ \text{diag} (\pi) \right]^{-1} [I + 2 (e\pi M)_d] \right\} \right\}.
\]

(70)

We write

\[
X = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix}.
\]
where $X_0$ is a matrix with the first $m_0$ row vectors of the matrix $X$;

$$(I - R_U)^{-1}\left\{ E + P \left\{ 2M - \text{diag}(\pi) \right\}^{-1} \left[ I + 2(e\pi M)\right] \right\} = \begin{pmatrix} S_0 \\ S_1 \end{pmatrix},$$

where

$$S_0 = (S_{0,0}, S_{0,1}, S_{0,2}, \ldots),$$

$$S_1 = \begin{pmatrix} S_{1,0} & S_{1,1} & S_{1,2} & \cdots \\ S_{2,0} & S_{2,1} & S_{2,2} & \cdots \\ S_{3,0} & S_{3,1} & S_{3,2} & \cdots \\ \vdots & \vdots & \vdots & \vdotsof{5} \end{pmatrix},$$

$$S_{i,j} = E_{i,j} + \Gamma_{i,j} + \sum_{k=1}^{\infty} X_k^{(i)} (E_{k+i,j} + \Gamma_{k+i,j}), \ i \geq 0, j \geq 0,$$

$$\Gamma_{i,j} = \begin{cases} 2\Lambda_{i,j} - A_0^{(i)} [\text{diag}(\pi)]^{-1} \left[ I + 2 \left( \sum_{i=0}^{\infty} e\pi_i M_{i,j} \right) \right] , & j = i + 1, i \geq 0, \\ 2\Lambda_{i,j} - A_1^{(i)} [\text{diag}(\pi)]^{-1} \left[ I + 2 \left( \sum_{i=0}^{\infty} e\pi_i M_{i,j} \right) \right] , & j = i, i \geq 0, \\ 2\Lambda_{i,j} - A_2^{(i)} [\text{diag}(\pi)]^{-1} \left[ I + 2 \left( \sum_{i=0}^{\infty} e\pi_i M_{i,j} \right) \right] , & j = i - 1, i \geq 1, \\ 2\Lambda_{i,j}, & \text{otherwise}, \end{cases}$$

$$\Lambda_{i,j} = \begin{cases} A_0^{(i)} M_{1,j} + A_1^{(i)} M_{0,j}, & i = 0, j \geq 0, \\ A_0^{(i)} M_{i+1,j} + A_1^{(i)} M_{i,j} + A_2^{(i)} M_{i-1,j}, & i \geq 1, j \geq 0. \end{cases}$$

It follows from (70) that

$$(I - U_0) X_0 = S_0 \tag{71}$$

and

$$(I - \Phi_D) X_1 = S_1. \tag{72}$$

It follows from (72) that

$$X_1 = (I - \Phi_D)^{-1} S_1, \tag{73}$$

and from (71) that

$$X_0 = Z S_0 + ec_0, \tag{74}$$

where $c_0$ is an arbitrary row vector.
Based on (73) and (74), we obtain

\[
X = \begin{pmatrix}
X_0 \\
X_1
\end{pmatrix} = \begin{pmatrix}
ZS_0 + ec_0 \\
(I - \Phi_D)^{-1} S_1
\end{pmatrix}.
\]

It follows from (69) that

\[
(I - G_L) M^{(2)} = \begin{pmatrix}
ZS_0 + ec_0 \\
(I - \Phi_D)^{-1} S_1
\end{pmatrix}.
\]

Thus we obtain

\[
M^{(2)} = (I - G_L)^{-1} \begin{pmatrix}
ZS_0 + ec_0 \\
(I - \Phi_D)^{-1} S_1
\end{pmatrix},
\]

where

\[
M^{(2)}_{i,j} = \begin{cases}
ZS_{0,j} + ec_{0,j}, & i = 0, j \geq 0, \\
Y^{(i)}_i (ZS_{0,j} + ec_{0,j}) + (I - U_i)^{-1} S_{i,j} + \sum_{k=1}^{i-1} Y^{(i)}_i (I - U_k)^{-1} S_{k,j}, & i \geq 1, j \geq 0.
\end{cases}
\]

5.2 The variance of the mixing time

From (8), we have \( L^{(2)} = M^{(2)} \text{diag}(\pi) \). Once the matrix \( M^{(2)} \) is given in Subsection 5.1, it is clear that the matrix \( L^{(2)} \) is obtained by \( M^{(2)} \text{diag}(\pi) \).

On the other hand, the matrix \( L^{(2)} \) can be solved by the matrix equation (9) of itself. It may be necessary to simply provide the outline of solution to Equation (9).

To compute the matrix \( L^{(2)} \) given by the mixing time, we need to solve the equation (9). Using the UL-type of \( RG \)-factorization, we obtain

\[
(I - \Psi_D) (I - G_L) L^{(2)} = (I - R_U)^{-1} \left\{ e\pi + P \left( 2M - [\text{diag}(\pi)]^{-1} [I + 2 (e\pi M)] \right) \text{diag}(\pi) \right\}.
\]  

Let

\[
Y = (I - G_L) L^{(2)}.
\]

Then it follows from (75) that

\[
(I - \Psi_D) Y = (I - R_U)^{-1} \left\{ e\pi + P \left( 2M - [\text{diag}(\pi)]^{-1} [I + 2 (e\pi M)] \right) \text{diag}(\pi) \right\}.
\]

We write

\[
Y = \begin{pmatrix}
Y_0 \\
Y_1
\end{pmatrix}.
\]
where $Y_0$ is a matrix with the first $m_0$ row vectors of the matrix $Y$;

$$(I - RU)^{-1} \left\{ e\pi + P \left\{ 2M - \text{diag} (\pi) \right\}^{-1} [I + 2(e\pi M_d)] \right\} \text{diag} (\pi) = \begin{pmatrix} T_0 \\ T_1 \end{pmatrix},$$

where

$$T_0 = (T_{0,0}, T_{0,1}, T_{0,2}, \ldots), \quad T_1 = \begin{pmatrix} T_{1,0} & T_{1,1} & T_{1,2} & \cdots \\ T_{2,0} & T_{2,1} & T_{2,2} & \cdots \\ T_{3,0} & T_{3,1} & T_{3,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and for $i \geq 0, j \geq 0$

$$T_{i,j} = e\pi_j + \Gamma_{i,j} \text{diag} (\pi_j) + \sum_{k=1}^{\infty} X_k^{(i)} [e\pi_j + \Gamma_{k+i,j} \text{diag} (\pi_j)].$$

It follows from (77) that

$$(I - U_0) Y_0 = T_0$$

(78)

and

$$(I - \Phi D) Y_1 = T_1.$$ 

(79)

It follows from (79) that

$$Y_1 = (I - \Phi D)^{-1} T_1,$$ 

(80)

and from (78) that

$$Y_0 = Z T_0 + e c_0,$$ 

(81)

where $c_0 = (c_{0,0}, c_{0,1}, c_{0,2}, \ldots)$ is an arbitrary row vector.

Based on (80) and (81), we obtain

$$Y = \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} = \begin{pmatrix} Z T_0 + e c_0 \\ (I - \Phi D)^{-1} T_1 \end{pmatrix}.$$ 

It follows from (76) that

$$(I - G_L) L^{(2)} = \begin{pmatrix} Z T_0 + e c_0 \\ (I - \Phi D)^{-1} T_1 \end{pmatrix}.$$ 

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Thus we obtain

\[ \mathbf{L}^{(2)} = (I - G_L)^{-1} \left( \frac{Z \mathbf{T}_0 + \epsilon c_0}{(I - \Phi_D)^{-1} \mathbf{T}_1} \right), \]

where

\[ L_{i,j}^{(2)} = \begin{cases} 
Z_{T,j}^0 + \epsilon c_{0,j}, & i = 0, j \geq 0, \\
Y_i^{(i)} (Z_{T,j}^0 + \epsilon c_{0,j}) + (I - U_i)^{-1} T_{i,j} + \sum_{k=1}^{i-1} Y_{i-k}^{(i)} (I - U_k)^{-1} T_{k,j}, & i \geq 1, j \geq 0.
\end{cases} \]

In the rest of this section, we compute \((\eta^{(2)})^T\). Note that

\[ (\eta^{(2)})^T = \mathbf{L}^{(2)} \mathbf{e} = \mathbf{M}^{(2)} \pi^T, \]

hence it is clear that the vector \((\eta^{(2)})^T\) is obtained by \(\mathbf{M}^{(2)} \pi^T\) once the matrix \(\mathbf{M}^{(2)}\) is given in Subsection 5.1.

On the other hand, the vector \((\eta^{(2)})^T\) can be solved by the matrix equation (11) of itself. It may be necessary to simply provide the outline of solution to Equation (11).

Now, we solve the equation (11). Using the UL-type of RG-factorization, we obtain

\[ (I - \Psi_D) (I - G_L) \left( \eta^{(2)} \right)^T = (I - R_U)^{-1} \left\{ e + P \left\{ 2 \mathbf{M} - [\text{diag} (\pi)]^{-1} [I + 2 (\epsilon \pi \mathbf{M})_d] \right\} \pi^T \right\}. \]  

Let

\[ \mathcal{R} = (I - G_L) \left( \eta^{(2)} \right)^T. \]  

Then it follows from (82) that

\[ (I - \Psi_D) \mathcal{R} = (I - R_U)^{-1} \left\{ e + P \left\{ 2 \mathbf{M} - [\text{diag} (\pi)]^{-1} [I + 2 (\epsilon \pi \mathbf{M})_d] \right\} \pi^T \right\}. \]

We write

\[ \mathcal{R} = \begin{pmatrix} \mathcal{R}_0 \\ \mathcal{R}_1 \end{pmatrix}, \]

where \(\mathcal{R}_0\) is a vector with the first \(m_0\) entries of the vector \(\mathcal{R}\);

\[ (I - R_U)^{-1} \left\{ e + P \left\{ 2 \mathbf{M} - [\text{diag} (\pi)]^{-1} [I + 2 (\epsilon \pi \mathbf{M})_d] \right\} \pi^T \right\} = \begin{pmatrix} \mathbf{W}_0 \\ \mathbf{W}_1 \end{pmatrix} \]

\[ \mathbf{W}_1 = \begin{pmatrix} W_{1,1} \\ W_{2,1} \\ W_{3,1} \\ \vdots \end{pmatrix}, \quad \mathbf{W}_0 \overset{\text{def}}{=} \mathbf{W}_{0,1}, \]

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and for $i \geq 0$

$$W_{i,1} = e + \sum_{j=0}^{\infty} \Gamma_{i,j} \pi_j + \sum_{k=1}^{\infty} X_k^{(i)} \left[ e + \sum_{j=0}^{\infty} \Gamma_{k+i,j} \pi_j \right].$$

It follows from (84) that

$$(I - U_0) \mathcal{R}_0 = W_0$$ (85)

and

$$(I - \Phi_D) \mathcal{R}_1 = W_1.$$ (86)

It follows from (86) that

$$\mathcal{R}_1 = (I - \Phi_D)^{-1} W_1,$$ (87)

and from (85) that

$$\mathcal{R}_0 = ZW_0 + ec_0,$$ (88)

where $c_0$ is an arbitrary constant.

Based on (87) and (88), we obtain

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}_0 \\ \mathcal{R}_1 \end{pmatrix} = \begin{pmatrix} ZW_0 + ec_0 \\ (I - \Phi_D)^{-1} W_1 \end{pmatrix}.$$ (89)

It follows from (83) that

$$(I - G_L) \left( \eta^{(2)} \right)^T = \begin{pmatrix} ZW_0 + ec_0 \\ (I - \Phi_D)^{-1} W_1 \end{pmatrix}.$$ (90)

Thus we obtain

$$\left( \eta^{(2)} \right)^T = (I - G_L)^{-1} \begin{pmatrix} ZW_0 + ec_0 \\ (I - \Phi_D)^{-1} W_1 \end{pmatrix},$$

where

$$\left( \eta^{(2)}_i \right)^T = \begin{cases} ZW_{0,1} + ec_0, & i = 0, \\ Y_i^{(i)} (ZW_{0,1} + ec_0) + (I - U_i)^{-1} W_{i,1} + \sum_{k=1}^{i-1} Y_{i-k}^{(i)} (I - U_k)^{-1} W_{k,1}, & i \geq 1. \end{cases}$$

Finally, we obtain

$$\left( \eta^{(2)} \right)^T = \left( \eta^{(2)} \right)^T - \eta^2 e.$$ (91)
\( v^{(2)} = \pi \left( V^{(2)} \right)^T = \pi \left( \eta^{(2)} \right)^T - \eta^2 \\
= -\pi^2 + \pi_0 (ZW_{0,1} + \epsilon e_0) + \sum_{i=1}^{\infty} \pi_i \left[ Y_i^{(i)} (ZW_{0,1} + \epsilon e_0) \\
+ (I - U_i)^{-1} W_{i,1} + \sum_{k=1}^{i-1} Y_{i-k}^{(i)} (I - U_k)^{-1} W_{k,1} \right] \).

6 Concluding Remarks

In this paper, we develop some matrix Poisson’s equations satisfied by the mean and variance of the mixing time in an irreducible positive-recurrent discrete-time Markov chain with infinitely-many levels, and provide a computational framework for the solution to the matrix Poisson’s equations by means of the UL-type of RG-factorization as well as the generalized inverses. In an important special case: the level-dependent QBD Processes, we provide a detailed computation for the mean and variance of the mixing time through the matrix-analytic method.

The results of this paper can be applied to performance computation of stochastic models such as a discrete-time MAP/PH/c queue and a discrete-time MAP/PH/1 retrial queue by means of the mixing time. Our future work in this direction contains several different research lines:

1. Provide algorithms for computing the solution to the matrix Poisson’s equations,
2. analyzing performance measures of practical stochastic models, and
3. extend the method of this paper, which is based on the UL-type of RG-factorization as well as the generalized inverses, to study more general Markov models including, Markov chains of GI/G/1 type, level-dependent Markov chains of M/G/1 type and of GI/M/1 type, continuous-time block-structured Markov chains, and Markov renewal processes, e.g., see Li [28] and Hunter [17].

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