THE COTORSION PAIR GENERATED BY THE CLASS OF FLAT MITTAG-LEFFLER MODULES

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ABSTRACT. Let $R$ be a ring and denote by $\mathcal{FM}$ the class of all flat and Mittag-Leffler left $R$-modules. In [4] it is proved that, if $R$ is countable, the orthogonal class of $\mathcal{FM}$ consists of all cotorsion modules. In this note we extend this result to the class of all rings $R$ satisfying that each flat left $R$-module is filtered by totally ordered direct limits of projective modules. This class of rings contains all countable, left perfect and discrete valuation domains. Moreover, assuming that there do not exist inaccessible cardinals, we obtain that, over these rings, all flat left $R$-modules have finite projective dimension.

INTRODUCTION

Let $R$ be an associative ring with unit. A left $R$-module $M$ is said to be Mittag-Leffler if for each family $\{M_i : i \in I\}$ of right $R$-modules, the canonical morphism from $\left( \prod_{i \in I} M_i \right) \otimes_R M$ to $\prod_{i \in I} M_i \otimes_R M$ is monic. Mittag-Leffler modules were introduced in [22] and, in the last few years, they have been brought into play in different settings, such as to solve the Baer splitting problem raised by Kaplansky in 1962 in [16] (see [1]), or in proving that each tilting class is determined by a class of finitely presented modules (see [3]). Moreover, in [7, p. 266], Drinfeld has proposed quasi-coherent sheaves whose sections at affine open sets are flat and Mittag-Leffler modules as the appropriate objects defining infinite-dimensional vector bundles on a scheme (these sheaves are called Drinfeld vector bundles in [11]). Since then, many authors have studied the homological and homotopical properties of the class of all flat and Mittag-Leffler modules.

In this paper we are interested in the cotorsion pair generated by the class $\mathcal{FM}$ of all flat and Mittag-Leffler left $R$-modules, that is, with $\perp (\mathcal{FM})$ (where, for each class of left $R$-modules $\mathcal{X}$, $\perp \mathcal{X}$ and $\mathcal{X}^\perp$ are the left and right orthogonal classes of $\mathcal{X}$ with respect to the $\text{Ext}$-functor). In [4] it is proved that if $R$ is a countable ring, this cotorsion pair is precisely the Enochs cotorsion pair, i.e., the one generated by the flat modules. The key of the proof is Theorem 6 (a) which states that, for any ring, any countable totally ordered direct limit of flat Mittag-Leffler left $R$-modules belongs to the double orthogonal class $\perp (\mathcal{FM})$ (this result extends [24, Theorem 3.8], which gives another proof using the Singular Cardinal Hypothesis). In this paper we prove that this result is true for each
(not necessarily countable) totally ordered direct limit of flat Mittag-Leffler left $R$-modules by extending the construction given in [25]. As a consequence, we obtain that $(\mathcal{F}, \mathcal{M}^\perp)$ is the Enochs cotorsion pair for those rings in which each flat left $R$-module is filtered by totally ordered direct limits of projective modules.

This last condition is treated in Section 3. The Govorov-Lazard theorem states that each flat module is a direct limit of projective modules, see [13] and [18]. But, when is each flat module a totally ordered direct limit of projective modules? That is, if $F$ is a flat left $R$-module, when does there exist a direct system of projective modules, $(P_i, f_{ij})_{i,j \in I}$, such that $I$ is totally ordered and $F \cong \text{lim} P_i$? Moreover generally: which are the rings for which each flat module is filtered by totally ordered direct limits of projective modules? We devote Section 3 to study this class of rings, which contains all countable, left perfect and discrete valuation rings. We prove that there exist rings not satisfying this property (Example 3.2) and, assuming that there do not exist inaccessible cardinals, we prove that, for any ring $R$ satisfying this property, there exists a natural number $n$ such that $R$ is left $n$-perfect, that is, each flat left $R$-module has projective dimension less than or equal to $n$ (Corollary 3.6).

1. PRELIMINARIES

For any set $A$ we shall denote by $|A|$ its cardinality. Given a map $f : A \to B$ and a subset $A' \subseteq A$, $f \upharpoonright A'$ will be the restriction of $f$ to $A'$. For any cardinal $\lambda$, we shall denote its cofinality by $\text{cf}(\lambda)$ and by $\lambda^+$ the next cardinal of $\lambda$. An infinite cardinal $\lambda$ is said to be inaccessible if $\lambda$ is limit and regular. The existence of inaccessible cardinals cannot be proved in ZFC; moreover, the existence of inaccessible cardinals is relatively consistent with ZFC.

We fix, for the rest of the paper, a non necessarily commutative ring with identity $R$; module will mean left $R$-module. Given any family of modules, $\{M_i : i \in I\}$, and $x \in \prod_{i \in I} M_i$, we shall denote by $x(i)$ the $i$th-coordinate of $x$ for each $i \in I$ and by $\text{supp}(x) = \{i \in I : x(i) \neq 0\}$. If $\lambda$ is an infinite cardinal, we shall denote by $\prod^\lambda_{i \in I} M_i$ the $\lambda$-product of the family $\{M_i : i \in I\}$, that is,

$$\left\{ x \in \prod_{i \in I} M_i : |\text{supp}(x)| < \lambda \right\}.$$

Let $\mathcal{X}$ be a class of modules, $(X_i, f_{ij})_{i,j \in I}$ a direct of system of modules belonging to $\mathcal{X}$ and $M$ its direct limit. If $I$ is totally ordered (resp. well ordered), we shall say that $M$ is a totally ordered (resp. well ordered) direct limit of modules belonging to $\mathcal{X}$. A $\mathcal{X}$-filtration of a module $N$ is a continuous chain of submodules of $N$, $\{N_\alpha : \alpha < \kappa\}$, where $\kappa$ is an ordinal, such that $N = \bigcup_{\alpha < \kappa} N_\alpha$ and $\frac{N_{\alpha+1}}{N_\alpha} \in \mathcal{X}$ for each $\alpha < \kappa$. In this case, we shall say that $N$ is $\mathcal{X}$-filtered. Concerning continuous chains of direct summands, we shall use the following well known lemma (see, for instance, [22] Lemme (3.1.2)) or the proof of [13] Lemma 3.3):

**Lemma 1.1.** Let $M$ be a module, $\lambda$ an infinite regular cardinal and $\{M_\alpha : \alpha < \lambda\}$ a continuous chain of submodules of $M$ such that $M_0 = 0$ and $M = \bigcup_{\alpha < \lambda} M_\alpha$. Suppose that, for each $\alpha < \lambda$, there exists a submodule $N_\alpha \leq M$ such that $M_{\alpha+1} = M_\alpha \oplus N_\alpha$. Then $M = \bigoplus_{\alpha < \lambda} N_\alpha$. 


We shall denote by $\text{Sum}(\mathcal{X})$ the class consisting of all direct sums of modules in $\mathcal{X}$ and by $\perp^1 \mathcal{X}$ the corresponding Ext-orthogonal classes, i.e.,

$$\mathcal{X}^\perp = \{ Y \in R\text{-Mod} : \text{Ext}^1_R(X, Y) = 0 \text{ } \forall X \in \mathcal{X} \}$$

and

$$\perp^1 \mathcal{X} = \{ Y \in R\text{-Mod} : \text{Ext}^1_R(Y, X) = 0 \text{ } \forall X \in \mathcal{X} \}.$$ A cotorsion pair in $R\text{-Mod}$ is a pair of classes of modules, $(\mathcal{F}, \mathcal{C})$, such that $\mathcal{F} = \perp \mathcal{C}$ and $\mathcal{C} = \mathcal{F}^\perp$. The pair of classes $(\perp^1(\mathcal{X}^\perp), \mathcal{X}^\perp)$ is a cotorsion pair which is called the cotorsion pair generated by $\mathcal{X}$.

Let $\mathcal{Y}$ be a class of right $R$-modules. A left $R$-module $M$ is said to be $\mathcal{Y}$-Mittag-Leffler if for each family $\{ Y_i : i \in I \}$ of right $R$-modules belonging to $\mathcal{Y}$, the canonical morphism from $(\prod_{i \in I} Y_i) \otimes_R M$ to $\prod_{i \in I} Y_i \otimes_R M$ is monic. We shall denote by $MY$ the class of all $\mathcal{Y}$-Mittag-Leffler left $R$-modules and by $FM^Y$ the class of all flat and $\mathcal{Y}$-Mittag-Leffler left $R$-modules. When $\mathcal{Y} = \text{Mod-} R$, we shall call $\mathcal{Y}$-Mittag-Leffler modules simply Mittag-Leffler modules.

In [15, Theorem 2.6], $\mathcal{Y}$-Mittag-Leffler modules are characterized in terms of a local property. We shall use the following more general definition, which was stated in [6, Definition 2.2].

**Definition 1.2.** Let $\kappa$ be an infinite regular cardinal. We shall say that a module $M$ is $(\kappa, \mathcal{X})$-free if it has a $(\kappa, \mathcal{X})$-dense system of submodules, that is, a direct family of submodules of $M$, $\mathcal{S}$, satisfying:

1. $\mathcal{S} \subseteq \mathcal{X}$;
2. $\mathcal{S}$ is closed under well ordered ascending chains of length smaller than $\kappa$,
3. every subset of $M$ of cardinality smaller than $\kappa$ is contained in an element of $\mathcal{S}$.

**Remark 1.3.** A similar definition is given in [25, Definition 2.1]. Note that in our definition the modules in the dense system need not be $< \kappa$-generated.

Given an infinite regular cardinal $\kappa$, we shall say that $\mathcal{X}$ is closed under $\kappa$-free modules if each $(\kappa, \mathcal{X})$-free module belongs to $\mathcal{X}$. The relationship between this local free property and $\mathcal{Y}$-Mittag-Leffler, which was proved in [15, Theorem 2.6], is that the class of $\mathcal{Y}$-Mittag-Leffler is closed under $\aleph_1$-free modules.

**Theorem 1.4.** The classes $M^Y$ and $FM^Y$ are closed under $\aleph_1$-free modules.

2. **Calculating the Double Orthogonal Class**

We fix, for the rest of this section, an infinite regular cardinal $\lambda$, a well ordered system, $(F_\alpha, g_{\alpha\beta})_{\alpha, \beta < \lambda}$, and an infinite cardinal $\kappa$ greater or equal than $\lambda$. We shall denote by $F$ the direct limit of the direct system. Extending the construction of [25, §3], we are going to construct an $(\aleph_1, \mathcal{G})$-free module associated to the direct system and the cardinal $\kappa$, where $\mathcal{G} = \text{Sum}(\{ F_\alpha : \alpha < \lambda \})$.

We shall denote by $T_\kappa^{< \lambda}$ the set of all sequences of length smaller than $\lambda$ consisting of elements of $\kappa$, that is,

$$T_\kappa^{< \lambda} = \{ \tau : \mu \rightarrow \kappa | \mu < \lambda \}.$$ Then $T_\kappa^{< \lambda}$ is a tree whose set of branches, $\text{Br}(T_\kappa^{< \lambda})$, can be identified with the set of all sequences $\nu : \lambda \rightarrow \kappa$. Given any $\tau : \mu \rightarrow \kappa$, with $\mu < \kappa$, we shall denote by $\ell(\tau)$ its length; that is, $\ell(\tau) = \mu$. 

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For each \( \tau \in T_{\kappa}^{<\lambda} \), denote by \( D_\tau = F(\tau) \) and let \( P = \prod_{\tau \in T_{\kappa}^{<\lambda}} D_\tau \). Given \( \nu \in \text{Br}(T_{\kappa}^{<\lambda}) \) and \( \alpha < \lambda \), we define the morphism \( d_{\alpha\nu} : F_\alpha \to P \) as follows: for any \( x \in F_\alpha \), let \( d_{\alpha\nu}(x) \) be the element of \( P \) such that:

- \( [d_{\alpha\nu}(x)](\nu | \alpha) = x \);
- \( [d_{\alpha\nu}(x)](\nu | \beta) = g_{\alpha\beta}(x) \), for each \( \beta < \lambda \) with \( \beta > \alpha \), and
- \( [d_{\alpha\nu}(x)](\nu | \gamma) = 0 \), for each \( \gamma \in T_{\kappa}^{<\lambda} \) with \( \gamma \neq \nu | \gamma \) for each \( \alpha \leq \gamma < \lambda \).

Denote by \( Y_{\alpha\nu} \) the image of \( d_{\alpha\nu} \) and note that, since \( d_{\alpha\nu} \) is monic, \( Y_{\alpha\nu} \cong F_\alpha \).

Moreover, let \( X_{\alpha\nu} = \sum_{\gamma < \alpha} Y_{\gamma\nu} \). Finally, set \( X_\nu = \sum_{\alpha < \lambda} X_{\alpha\nu} \), \( L = \sum_{\nu \in \text{Br}(T_{\kappa}^{<\lambda})} X_\nu \) and \( D = L \cap \prod_{\tau \in T_{\kappa}^{<\lambda}} D_\tau \). The following two lemmas state the basic properties of these modules. They can be proved in a similar fashion to [25, Lemma 3.5 and 3.6].

**Lemma 2.1.**

1. For each \( \nu \in \text{Br}(T_{\kappa}^{<\lambda}) \), \( X_\nu = \bigoplus_{\alpha < \lambda} Y_{\alpha\nu} \cong \bigoplus_{\alpha < \lambda} F_\alpha \).
2. \( \frac{L}{D} \cong F^{(\text{Br}(T_{\kappa}^{<\lambda}))} \).
3. \( L \) is \((81, G)\)-free, where \( G = \text{Sum}(\{F_\alpha : \alpha < \lambda\})\).

**Proof.** (1) By definition, \( X_{\alpha\nu +1} = X_{\alpha\nu} + Y_{\nu\alpha} \) for each \( \alpha < \lambda \). We claim that this sum is direct. Then the result follows from Lemma 1.1, since \( X_\nu = \bigcup_{\alpha < \lambda} X_{\alpha\nu} \).

Let \( \alpha < \lambda \) and \( x \in X_{\alpha\nu} \cap Y_{\nu\alpha} \). Since \( x \in X_{\alpha\nu} \), there exist a finite sequence of ordinals smaller than \( \alpha \), \( \gamma_1, \ldots, \gamma_n \) and elements \( x_i \in F_{\gamma_i} \) for each \( i \in \{1, \ldots, n\} \) such that \( x = \sum_{i=1}^{n} d_{\nu\gamma_i}(x_i) \). Then, the element in position \( \nu | \gamma_i \) of \( x \) is \( x_i \).

Since \( x \in Y_{\nu\alpha} \), this element has to be zero. Thus \( x = \sum_{i=2}^{n} d_{\nu\gamma_i}(x_i) \). Proceeding recursively in this way, we can prove that \( x_2 = x_3 = \cdots = x_n = 0 \) and, consequently, \( x = 0 \).

(2) First of all, we prove that \( \frac{L}{D} = \bigoplus_{\nu \in \text{Br}(T_{\kappa}^{<\lambda})} \frac{X_\nu + D}{D} \). We only have to see that the family \( \{ \frac{X_\nu + D}{D} : \nu \in \text{Br}(T_{\kappa}^{<\lambda}) \} \) is independent. Let \( \nu_0, \nu_1, \ldots, \nu_n \in \text{Br}(T_{\kappa}^{<\lambda}) \) be distinct sequences. Then there exist \( \alpha < \lambda \) such that \( \nu_i | \gamma \neq \nu_j | \gamma \) for each \( i, j \in \{0, \ldots, n\} \) distinct and \( \gamma \geq \alpha \). Let \( x \in \frac{X_{\nu_0} + D}{D} \cap \sum_{i=1}^{n} \frac{X_{\nu_i} + D}{D} \) and let \( y \in X_{\nu_0} \) and \( z \in \sum_{i=1}^{n} X_{\nu_i} \) be such that \( x = y + D = z + D \). Then \( y - z \in D \). Since \( (y-z)(\nu_0 | \gamma) = y(\nu_0 | \gamma) \) for each \( \gamma \geq \alpha \), there exists \( \beta < \lambda \) such that \( y(\nu_0 | \gamma) = 0 \) for each \( \gamma \geq \beta \). This means that \( y \in D \) and, consequently, \( x = y + D = 0 \).

Now we prove that for each \( \nu \in \text{Br}(T_{\kappa}^{<\lambda}) \), \( F \cong \frac{X_\nu + D}{D} \). Define, for each \( \alpha < \lambda \), \( f_\alpha : F_\alpha \to \frac{X_\nu + D}{D} \) by \( f_\alpha(x) = d_{\alpha\nu}(x) + D \) for each \( x \in F_\alpha \). Then the family \( \{f_\alpha : F_\alpha \to \frac{X_\nu + D}{D}\}_{\alpha < \lambda} \) is a direct system of morphisms from \( (F_\alpha, g_{\alpha\beta})_{\alpha, \beta < \lambda} \) since, for each pair of ordinals \( \alpha < \beta \), \( f_\beta g_{\alpha\beta}(x) - f_\alpha(x) \) has support smaller than \( \lambda \) and, consequently, belongs to \( D \). By the universal property of the direct limit, the direct system of morphisms, \( \{f_\alpha : F_\alpha \to \frac{X_\nu + D}{D}\}_{\alpha < \lambda} \), induces a homomorphism \( f : F \to \frac{X_\nu + D}{D} \) that is easily checked to be injective and which is onto by the definition of \( X_\nu \).

The conclusion of the proof is that

\[
\frac{L}{D} = \bigoplus_{\nu \in \text{Br}(T_{\kappa}^{<\lambda})} \frac{X_\nu + D}{D} \cong F^{(\text{Br}(T_{\kappa}^{<\lambda}))},
\]

and we are done.

(3) Consider the family of submodules

\[
S = \{ \sum_{\nu \in \Gamma} X_\nu : \Gamma \subseteq \text{Br}(T_{\kappa}^{<\lambda}) \text{ with } |\Gamma| \text{ countable} \}.
\]
We are going to see that \( S \) is a \((N_1, \mathcal{G})\)-dense system.

Clearly, each countable subset of \( L \) is contained in some submodule of \( S \); and \( S \) is closed under unions of countable well ordered chains.

It remains to prove that each module in \( S \) belongs to \( \mathcal{G} \), i.e., that is a direct sum of modules in \( \{ F_\alpha : \alpha < \lambda \} \). First of all observe that, by \ref{1.1}, \( X_\nu \) can be written as a direct sum of \( Y \)'s for each \( \nu \in \text{Br}(T_\nu^{<\lambda}) \). Take \( \Gamma \subseteq \text{Br}(T_\nu^{<\lambda}) \) countable, write \( \Gamma = \{ \nu_\nu : n < \omega \} \) and denote \( Z = \sum_{n<\omega} X_{\nu_\nu} \). Define \( Z_n = \sum_{n \leq n} X_{\nu_\nu} \) for each \( n < \omega \). We claim that, for each \( n < \omega \), there exists an ordinal \( \beta_n < \lambda \) such that \( Z_{n+1} = Z_n \bigoplus \left( \bigoplus_{\delta > \beta_n} Y_{\nu_{n+1}^{\delta}} \right) \).

Let \( n < \omega \). It is easy to find an ordinal \( \beta_n < \lambda \) and a natural number \( k < n + 1 \) satisfying:

- \( \nu_{n+1} \upharpoonright \beta_n = \nu_k \upharpoonright \beta_n \),
- \( \nu_{n+1} \upharpoonright \alpha \neq \nu_n \upharpoonright \alpha \) for each \( \alpha > \beta_n \) and \( m < n + 1 \).

In order to see that \( Z_{n+1} = Z_n \bigoplus \left( \bigoplus_{\delta > \beta_n} Y_{\nu_{n+1}^{\delta}} \right) \), first note that \( Z_n \cap X_{\nu_{n+1}} \leq \prod_{\delta \leq \beta_n} D_{\nu_{n+1}^{\delta}} \) by election of \( \beta_n \), and that \( \bigoplus_{\delta > \beta_n} Y_{\nu_{n+1}^{\delta}} \leq \prod_{\delta > \beta_n} D_{\nu_{n+1}^{\delta}} \); this implies that \( Z_n \cap \left( \bigoplus_{\delta > \beta_n} Y_{\nu_{n+1}^{\delta}} \right) = 0 \). In order to see that \( Z_{n+1} = Z_n + \left( \bigoplus_{\delta > \beta_n} Y_{\nu_{n+1}^{\delta}} \right) \), we only have to prove that \( X_{\nu_{n+1}} \leq Z_n + \left( \bigoplus_{\delta > \beta_n} Y_{\nu_{n+1}^{\delta}} \right) \). Given \( \gamma < \lambda \) and \( x \in F_\gamma \), \( d_{\nu_{n+1}^\gamma}(x) \) trivially belongs to \( Z_n + \left( \bigoplus_{\delta > \beta_n} Y_{\nu_{n+1}^{\delta}} \right) \) if \( \gamma > \beta_n \).

If \( \gamma \leq \beta_n \) then, since \( \nu_{n+1} \upharpoonright \beta_n = \nu_k \upharpoonright \beta_n \), we have that

\[
d_{\nu_{n+1}^\gamma}(x) - d_{\nu_{n+1}^{\beta_n+1}}(g_{\gamma^{\beta_n+1}}(x)) = d_{\nu_{k}^{\gamma}}(x) - d_{\nu_{k}^{\beta_n+1}}(g_{\gamma^{\beta_n+1}}(x)),
\]

which implies that \( d_{\nu_{n+1}^\gamma}(x) \) belongs to \( Z_n + \left( \bigoplus_{\delta > \beta_n} Y_{\nu_{n+1}^{\delta}} \right) \). This proves the claim.

Now, using the claim and Lemma \ref{1.1} we get that

\[
Z = \bigoplus_{n<\omega} \left( \bigoplus_{\delta > \beta_n} Y_{\nu_n^{\delta}} \right)
\]

where \( \beta_{n-1} < \lambda \). This finishes the proof. \( \square \)

**Remark 2.2.** We shall call \( L \) the \( \aleph_1 \)-free module associated to the direct system \((F_\alpha, g_\alpha \beta)_{\alpha < \beta < \lambda} \) and the infinite cardinal \( \kappa \).

Let \( \mathcal{X} \) be a class of left \( R \)-modules. We are going to prove that if \( \mathcal{X} \) is closed under direct sums and \( \aleph_1 \)-free modules, then \( ^+ (\mathcal{X}^+) \) contains all totally ordered direct limits of modules in \( \mathcal{X} \). We begin with a lemma concerning infinite combinatorics:

**Lemma 2.3.** Let \( \mu \) be an infinite cardinal and \( \xi \) an infinite regular cardinal. Then there exists a cardinal \( \kappa \) such that \( \kappa > \mu \), \( \kappa^\xi = \kappa \) for each cardinal \( \eta < \xi \) and \( \kappa^\xi = 2^\kappa \).

**Proof.** Simply take \( \kappa \) to be the supremum of the increasing sequence of cardinals \( \{ \beta_\alpha : \alpha < \xi \} \) defined recursively as follows: \( \beta_0 = \mu \); \( \beta_{\alpha+1} = 2^{\beta_\alpha} \) for each \( \alpha < \xi \), and \( \beta_\alpha = \sup \{ \beta_\gamma : \gamma < \alpha \} \) for each \( \beta < \xi \) limit. \( \square \)

Now, reasoning as in the last part of the proof of \( \square \) Theorem 6 \( \square \) we get:
Theorem 2.4. Let $X$ be a class of modules closed under direct sums and $\aleph_1$-free modules. Then the class $\perp^+(X^\perp)$ contains all totally ordered direct limits of modules in $X$.

Proof. Let $(F_i, f_{ij})_i$ be a direct system of modules in $X$ such that $I$ is totally ordered and denote by $F$ its direct limit. By [26] §3, Theorem 36], there exists a well ordered subset $J$ of $I$ which is cofinal in $I$. We may assume that $J$ is a cardinal, say $\kappa$. Now, if $\text{cf}(\kappa) = \lambda$, there exists a cofinal subset $K$ of $I$ of size $\lambda$. In conclusion, there exists a well ordered system, $(G_{\alpha, \beta})_{\alpha, \beta < \lambda}$, such that $G_{\alpha} \in X$ for each $\alpha < \lambda$ and $F = \lim_{\alpha < \lambda} G_{\alpha}$.

Now suppose, in order to get a contradiction, that $F \notin \perp^+(X^\perp)$. Then there exists $X \in X^\perp$ with $\text{Ext}^1_R(F, X) \neq 0$. Take, using the preceding lemma, an infinite regular cardinal $\kappa$ satisfying $|X| \leq 2^\kappa$, $\kappa^\eta = \kappa$ for each cardinal $\eta < \lambda$ and $\kappa^\lambda = 2^\kappa$. Let $L$ be the locally $\aleph_1$-free module associated to the direct system $(G_{\alpha, \beta})_{\alpha < \beta < \lambda}$ and $\kappa$. Since $X$ is closed under direct sums and $\aleph_1$-free modules, $L \in X$. Applying $\text{Ext}^1_R(\cdot, X)$ to the short exact sequence

$$0 \to D \to L \to \text{Ext}^1_R(\text{Br}(T_{\leq \kappa}^\perp), X) \to 0$$

we get an epimorphism $\text{Hom}(D, X) \to \text{Ext}^1_R(F(\text{Br}(T_{\leq \kappa}^\perp)), X)$, which implies that $|\text{Hom}(D, X)| \leq |\text{Ext}^1_R(F(\text{Br}(T_{\leq \kappa}^\perp)), X)|$. But this cannot be true since $|\text{Hom}(D, X)| \leq 2^\kappa$ (as $|D| = \kappa$) and $|\text{Ext}^1_R(F(\text{Br}(T_{\leq \kappa}^\perp)), X)| \geq 2^{2^\kappa}$. \qed

If we specialize Theorem 2.4 to the class of flat a Mittag-Leffler modules, we obtain the following extension of [4, Theorem 6].

Corollary 2.5. Let $Y$ be any class of right $R$-modules. Then the class $\perp^+(\mathcal{M}^\perp)$ (resp. $\perp^+(\mathcal{FM}^\perp)$) contains all totally ordered direct limits of modules belonging to $\mathcal{M}^\perp$ (resp. $\mathcal{FM}^\perp$).

Proof. Follows from theorems 2.4 and 1.4 \qed

Finally, we describe the cotorsion pair generated by the class of all flat Mittag-Leffler modules for those rings in which all flat modules are filtered by totally ordered direct limits of projective modules. Of course, this class of rings contains all rings in which each flat module is a totally ordered direct limit of projective modules. Moreover, left perfect rings, discrete valuation rings and countable rings belong to this class, see Example 3.3. Section 3 is devoted to study this class of rings.

Corollary 2.6. Suppose that each flat left $R$-module is filtered by totally ordered direct limits of projective modules. Then the cotorsion pair generated by $\mathcal{FM}$ is the Enochs cotorsion pair $(\mathcal{F}, \mathcal{C})$, where $\mathcal{F}$ is the class of all flat modules and $\mathcal{C}$ is the class of all cotorsion modules.

Proof. By the preceding theorem and Theorem 1.4, each totally ordered direct limit of projective modules belongs to $\perp^+(\mathcal{FM}^\perp)$. Since $\perp^+(\mathcal{FM}^\perp)$ is closed under filtrations by [3] Theorem 1.2], the hypothesis implies that $\mathcal{F} \subseteq \perp^+(\mathcal{FM}^\perp)$. But this means that $\mathcal{F} = \perp^+(\mathcal{FM}^\perp)$. \qed

Theorem 2.4 is more general and allows us to prove the analogous of the preceding result for Mittag-Leffler modules. In this case, we determine the class $\perp^+(\mathcal{M}^\perp)$
for rings in which each module is filtered by totally ordered direct limits of pure projective modules.

**Corollary 2.7.** Suppose that each left $R$-module is filtered by totally ordered direct limits of pure projective modules. Then the cotorsion pair generated by $\mathcal{M}$ is $(R\text{-Mod}, \mathcal{I})$, where $\mathcal{I}$ is the class of all injective modules.

**Proof.** Again by the preceding theorem and Theorem 1.4, each totally ordered direct limit of pure projective modules belongs to $\perp (\mathcal{M}^1)$. Since $\perp (\mathcal{M}^1)$ is closed under filtrations by [13, Theorem 1.2], the hypothesis implies that $R\text{-Mod} \subseteq \perp (\mathcal{F}, \mathcal{M}^1)$. But this means that $R\text{-Mod} = \perp (\mathcal{F}, \mathcal{M}^1)$. □

Left pure semisimple rings trivially satisfy the hypothesis of the preceding corollary. Moreover, $\mathbb{Z}$, by [12, Corollary 18.4], and, more generally, discrete valuation domains satisfy the hypothesis of the corollary (this follows from the proof of Example 5.3 where it is proved that each module over a discrete valuation ring is the union of a countable chain of modules that are direct sums of cyclics; but direct sums of cyclics over a discrete valuation ring are pure-projective).

3. **TOTALLY ORDERED DIRECT LIMITS OF PROJECTIVE MODULES**

In this section we turn to when all flat modules are filtered by totally ordered direct limits of projective modules. It is not known which rings satisfy this property; even, it is not known which rings satisfy that flat modules are totally ordered direct limits of projectives. We begin with an example of a flat module which is not a totally ordered direct limit of projective modules. We shall use the following result which was communicated by J. Šaroch:

**Proposition 3.1.** Let $M$ be a $\aleph_\omega$-presented flat module which is a totally ordered direct limit of projective modules. Then $M$ has finite projective dimension.

**Proof.** By [23, §3, Theorem 36] there exists a well ordered system of projective modules, $(P_\alpha, f_{\alpha\beta})_{\alpha<\beta<\kappa}$, whose direct limit is $M$. Denote by $f_\alpha : P_\alpha \to M$ the direct limit canonical map. If $\text{cf}(\kappa) < \aleph_\omega$ and $\Gamma$ is a cofinal subset of $\kappa$ of cardinality $\text{cf}(\kappa)$, the direct limit of the system $(P_\alpha, f_{\alpha\beta})_{\alpha, \beta \in \Gamma}$ is $M$. By [19, Theorem 2.3], $M$ has finite projective dimension.

So suppose that $\text{cf}(\kappa) > \aleph_\omega$. We are going to construct a countable chain $\beta_0 < \beta_1 < \cdots$ in $\kappa$ and $\leq \aleph_\omega$-generated direct summands $S_n$ of $P_{\beta_n}$ satisfying $f_{\beta_n}(S_0) = M$, $f_{\beta_n\beta_{n+1}}(S_n) \leq S_{n+1}$ and $\text{Ker} f_{\beta_n\beta_{n+1}} \cap S_n = \text{Ker} f_{\beta_n} \cap S_n$ for each $n < \omega$. With this construction made, setting $h_{n+1} = f_{\beta_n\beta_{n+1}} | S_n$ for each $n < \omega$ (and $h_{nm}$ the corresponding compositions for each $n < m < \omega$), we get a direct system of projective modules, $(S_n, h_{nm})_{n<m<\omega}$, whose direct limit is $M$. Then, again by [19, Theorem 2.3], $M$ has finite projective dimension.

So, it only remains to make the construction; we shall proceed by induction on $n$. Suppose that $n = 0$. Since $M$ is $\aleph_\omega$-generated and $\text{cf}(\kappa) > \aleph_\omega$, there exists $\beta_0 < \kappa$ such that $f_{\beta_0}(P_{\beta_0}) = M$. Since $P_{\beta_0}$ is a direct sum of countably generated modules and $M$ is $\aleph_\omega$-generated, there exists a $\aleph_\omega$-generated direct summand $S_0$ of $P_{\beta_0}$ such that $f_{\beta_0}(S_0) = M$.

Now assume that we have made the construction for some $n < \omega$, and let us do it for $n+1$. Since $M$ is $\aleph_\omega$-generated and $f_{\beta_n}(S_n) = M$, $\text{Ker} f_{\beta_n} \cap S_n$ has a
generated set $X$ of cardinality less or equal than $\aleph_\omega$. Now, for each $x \in X$, as $f_{\beta_n}(x) = 0$, there exists $\beta_n < \kappa$ with $\beta_n \geq \beta_n$ such that $f_{\beta_n}(x) = 0$. Then, taking $\beta_{n+1}$ the supremum of $\{\beta_x : x \in X\}$, we get an ordinal smaller than $\kappa$ (remember: $\text{cf}(\kappa) > \aleph_\omega$) with the property that $\ker f_{\beta_n,\beta_{n+1}} \cap S_n = \ker f_{\beta_n} \cap S_n$. Finally, in order to finish the inductive step, take $S_{n+1}$ an $\aleph_\omega$-generated direct summand of $P_{\beta_{n+1}}$ containing $f_{\beta_n,\beta_{n+1}}(S_n)$.

Now we can give the announced example:

**Example 3.2.** Let $R$ be the boolean algebra freely generated (as a boolean algebra) by a set of $X$ of cardinality $\aleph_\omega$. Then, by [20] pp. 43, $X$ is a nice set of idempotents in the sense of [20]. The left ideal $I_X$ generated by $X$ is flat, as $R$ is von-Neumann regular, $\aleph_\omega$-presented (look at the projective presentation of $I_X$ constructed in [20] p. 642) and has infinite projective dimension by [20] Theorem A. By the preceding result, $I_X$ cannot be a totally ordered direct limit of projective modules.

Now we give examples of rings satisfying that flat modules are filtered by totally ordered direct limits of projectives.

**Examples 3.3.** (1) The following rings satisfy that all flat modules are totally ordered direct limits of projective modules:

(a) Left perfect rings.
(b) $\mathbb{Z}$.
(c) Discrete valuation domains.

(2) If $R$ is countable, the every flat module is filtered by totally ordered direct limits of projective modules.

As a consequence of Corollary 2.7. in each of these rings the cotorsion pair generated by the flat and Mittag Leffler modules is the Enochs cotorsion pair.

**Proof.** We begin with the proof of (1). Part (a) is trivial; (b) follows from [12] Corollary 18.4. Part (c) can be proved in a similar way as [12] Proposition 18.3 and Corollary 18.4 with some arithmetic modifications; we sketch the proof for completeness.

Suppose that $R$ is a discrete valuation domain with prime element $p$. Recall that for each $r \in R$ there exists $n \in \omega$ and units $u, v \in R$ such that $r = up^n = p^nv$; consequently, there exists $s \in R$ such that $pr = sp$. In what follows, if $M$ is a left $R$-module, $\{m_i : i \in I\}$ is a subset of $M$ and $\sum_{i \in I} r_im_i$ is a linear combination, the set $\{i \in I : r_i \neq 0\}$ is suppose to be finite. We refer to [17] for facts about discrete valuation rings.

First of all, we prove that if $M$ is a left $R$-module such that $pM$ is a direct sum of cyclic modules, then $M$ is a direct sum of cyclic modules too. Suppose that $pM = \bigoplus_{i \in I} Rm_i$ for a family of elements $\{n_i : i \in I\} \subseteq pM$ and write, for each $i \in I$, $n_i = pm_i$ for some $m_i \in M$. Then $S' = \{m_i : i \in I\}$ is an independent set since, if we take a linear combination $\sum_{i \in I} r_im_i$ which is equal to zero, then $\sum_{i \in I} pr_im_i = 0$. But, for each $i \in I$, $pr_i = sp_i$ for some $s_i \in R$, and, consequently, $\sum_{i \in I} n_im_i = 0$; since $\{n_i : i \in I\}$ is independent, $s_i = 0$ for each $i \in I$, which implies that $r_i = 0$ for each $i \in I$. Now, extend this independent set to a maximal one, say $S = \{m_i : i \in I\} \cup \{n_j : j \in J\}$. Then $S$ generates $M$ since, given any $m \in M$, $pm = \sum_{i \in I} r_ipm_i$ for a family $\{r_i : i \in I\} \subseteq R$; writing, for each $i \in I$, $r_ip = ps_i$ for some $s_i \in R$, we get that $p(m - \sum_{i \in I} s_im_i) = 0$ and, consequently, $m - \sum_{i \in I} s_im_i$ belongs to the socle of $M$, which is contained in the submodule generated by $S$. Then $m$ is generated by $S$ and we are done.
Now, define the chain of submodules of $M$, $\{M_n : n < \omega\}$, recursively as follows: $M_0$ is the submodule generated by a maximal independent subset of $M$; if $M_n$ has been defined for some $n < \omega$, set $M_{n+1} = \{m \in M : pm \in M_n\}$. Note that, for each $n < \omega$, $M_n$ is a direct sum of cyclics by the previous result. It remains to prove that $M = \bigcup_{n<\omega} M_n$. Suppose that this is not true and let $m \in M - \bigcup_{n<\omega} M_n$; then $p^\alpha m \notin M_0$ for each $n < \omega$, which implies that $S \cup \{m\}$ is linearly independent, since for any linear combination, $\sum_{i \in I} r_i m_i + rm$, if $\sum_{i \in I} r_i m_i + rm = 0$, writing $r = up^\alpha m$ for some unit $u \in R$ and $n < \omega$, we get $p^\alpha m = \sum_{i \in I} -ur_i m_i \in M_0$, which is a contradiction.

In order to derive (2) note that, as it is proved in [5] Proposition 7.4.3], if $R$ is countable then every flat module is filtered by countably generated flat modules. Since the ring is countable, these modules are actually countably presented. By [21] Theorem], each countably presented flat module is the direct limit of a countable system of projective modules; but it is easy to see that each countable direct poset has a well ordered cofinal subset. The conclusion is that each flat module is filtered by totally ordered direct limits of projective modules.

Now if we assume the hypothesis that there do not exist inaccessible cardinals, we can prove that all totally ordered direct limits of projective modules have finite projective dimension. We start with a theorem in ZFC.

**Theorem 3.4.** Let $M$ be a flat module. Suppose that $M$ is the direct limit of a well ordered system of projective modules, $(P_\alpha, f_{\alpha\beta})_{\alpha<\beta<\lambda}$, for some cardinal $\lambda$ smaller than the first inaccessible cardinal. Then $M$ has finite projective dimension.

**Proof.** We shall prove that for each infinite cardinal $\lambda$ smaller than the first inaccessible cardinal, there exists a natural number $n_\lambda$ such that the direct limit of each direct system $(P_\alpha, f_{\alpha\beta})_{\alpha<\beta<\mu}$, with $\mu \leq \lambda$, consisting of projective modules has projective dimension less or equal than $n_\lambda$. We shall proceed inductively on $\lambda$.

If $\lambda = \aleph_0$ the result follows from [19] Theorem 2.3] (actually from [5]). Suppose that $\lambda > \aleph_0$ and that we have proven the claim for each direct system of cardinality less than $\lambda$; let $(P_\alpha, f_{\alpha\beta})_{\alpha<\beta<\lambda}$ be a direct system consisting of projective modules. If $\lambda$ is a direct limit cardinal, $\text{cf}(\lambda)$ is smaller than $\lambda$ by hypothesis. Since $\lambda$ has a cofinal subset of cardinality $\text{cf}(\lambda)$, the induction hypothesis implies that $\lim P_\alpha$ has projective dimension less or equal than $n_{\text{cf}(\lambda)}$. Thus, the result is proven taken $n_\lambda = n_{\text{cf}(\lambda)}$.

It remains to prove case $\lambda$ successor. Assume that $\lambda = \mu^+$ for some infinite cardinal $\mu$. Denote, for each ordinal $\alpha < \lambda$, by $G_\alpha = \lim_{\gamma < \alpha} P_\gamma$ and, for any other ordinal $\beta > \alpha$, by $g_{\alpha\beta} : G_\alpha \to G_\beta$ the induced map by the structural maps of the corresponding direct limits. We obtain, in this way, a direct system $(G_\alpha, g_{\alpha\beta})_{\alpha<\beta<\lambda}$ whose direct limit is $\lim P_\alpha$ and such that the projective dimension of $G_\alpha$ is less or equal than $n_\mu$ (by induction hypothesis). Now consider the canonical presentation of $\lim P_\alpha$,

$$(1) \quad 0 \longrightarrow X \longrightarrow \bigoplus_{\alpha<\lambda} P_\alpha \longrightarrow \lim P_\alpha \longrightarrow 0$$

and, for each $\alpha < \lambda$, the corresponding one of $G_\alpha$,

$$(2) \quad 0 \longrightarrow X_\alpha \longrightarrow \bigoplus_{\gamma<\alpha} P_\gamma \longrightarrow G_\alpha \longrightarrow 0$$
It is very easy to see that $X = \bigcup_{\alpha < \lambda} X_\alpha$ and that this union is continuous. Moreover, since $X_\alpha$ has projective dimension less or equal than $n_\mu - 1$, $X$ has projective dimension less or equal than $n_\mu$ by \cite{[19]} Corollary 0.3. Then, looking at the short exact sequence \cite{[11]}, we conclude that $\lim P_\alpha$ has projective dimension less or equal than $n_\mu + 1$. Thus, the result is proved setting $n_\lambda = n_\mu + 1$. □

As a consequence of this result, if we assume that there do not exist inaccessible cardinals we get:

**Corollary 3.5.** Assume that there do not exist inaccessible cardinals. Then each flat module which is a totally ordered direct limit of projective modules has finite projective dimension.

**Proof.** Simply note that each totally ordered set has a cofinal well ordered subset (see \cite{[23]} §3, Theorem 36)). Then the proof of the preceding theorem works for this one. □

Recall that the ring $R$ (see \cite{[10]}) is said to be $n$-perfect if each flat module has projective dimension less or equal than $n$. As a consequence of the last result we get:

**Corollary 3.6.** Assume that there do not exist inaccessible cardinals. If each flat module is filtered by totally ordered direct limits of projective modules, then there exists a natural number $n$ such that $R$ is left $n$-perfect.

**Proof.** By the preceding corollary, each module which is a totally ordered direct limit of projective modules has finite projective dimension. By the hypotesis and Auslander’s Theorem (\cite{[2]} Proposition 3)), each flat module has finite projective dimension. Now, it is easy to see that there exists a natural number $n$ such that each flat module has projective dimension less or equal than $n$: simply use \cite{[6]} Lemma 3.9 taking $\mathcal{X}$ the class of all projective modules and $\mathcal{Y}$ the class of all totally ordered direct limits of projective modules, which is clearly closed under countably direct sums. □

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