Constraining the Generalized Uncertainty Principle: Black Hole Shadow M87* and Quasiperiodic Oscillations

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In this paper we study the effect of the Generalized Uncertainty Principle (GUP) on the shadow of GUP-modified Kerr black hole and the correspondence between the shadow radius and the real part of the quasinormal modes (QNMs). We find that the shadow curvature radius of the GUP-modified Kerr black hole is bigger compared to the Kerr vacuum solution and increases linearly monotonically with the increase of the GUP parameter. We then investigate the characteristic points of intrinsic curvature of the shadow from a topological point of view to calculate the angular size radii of the shadow. To this end, we have used the EHT data for the M87* black hole to constrain the upper limits of the GUP parameter. Finally, we have explored the connection between the shadow radius and the scalar/electromagnetic/gravitational QNMs. The GUP-modified Kerr black hole is also used to provide perfect curve fitting of the particle oscillation upper and lower frequencies to the observed frequencies for three microquasars and to restrict the values of the correction parameter in the metric of the modified black hole to very reasonable bounds.

I. INTRODUCTION

Numerous geometrical and mathematical investigations concerning the interior regions of black hole suggest that not only general relativity but also quantum mechanics (more generally, known laws of physics) break down at the singularity. However, an amalgamation of general relativity and quantum mechanics or quantum gravity can predict novel features of black hole near the Planck length scale. Since the Heisenberg uncertainty principle (HUP) is not valid in its apparent form in the strong gravity, this law of physics gets gravitational corrections to investigate physics near high energy or short distance scales. In literatures, there are several versions of GUP from the HUP which are based either from some model of quantum gravity such as string theory, loop quantum gravity or phenomenology [1]. The well-known HUP obeys the following rule between the position and momentum operators: \[ [x, p] = i\hbar, \]

while the generalizations are proposed to include linear and quadratic terms in momentum as follows:

\[ [x, p] = i\hbar(1 - 2\beta p + 4\beta^2 p^2) \]

A prediction using GUP considerations is that it can prevent the black hole evaporation completely near the Planck scale [4]. The backreaction effects involving the interaction of photons with electrons might stop the process of black hole evaporation as soon as the Planck mass is reached. The model proposes that the quantum gravity effects lead to the formation of black remnant [5, 6]. If the remnants exist in significant number in the universe than they can be a candidate of dark matter as well. The final remnant has a structure with a degenerate, extremal, horizon of radius of the order of the minimal length [7]. The Hawking temperature of the black hole horizons gets GUP corrections which are meaningful only when black hole mass is greater than Planck mass otherwise the temperature gets imaginary. Thus the maximum possible black hole temperature is associated with the Planck scale.

An implication of GUP is that the ADM mass of black hole is modified as \[ M = M + \beta M_p^2 / M^2, \]

where \( M \) is the bare mass of the black hole, \( M_p \) is the Planck mass while \( \beta > 0 \) is a constant [17]. The authors proposed and analyzed the GUP corrected Schwarzschild, Reissner-Nordstrom and Kerr spacetimes. They showed that for a given value of \( \beta \), there exists a critical charge and a critical spin above which the solutions bifurcate into sub-Planckian and super-Planckian phases, separated by a mass gap in which no black holes can form. In literature, an upper bound on the GUP parameter is obtained using the data of the shadow of M87* central black hole [9]. By relating the GUP parameter with the deviation from the circularity of M87* black hole shadow, it is shown that \( \beta < 10^{-90} \). To achieve this numerical value, the author employed a model of Kerr de Sitter black hole with a regular interior. Our objective is to constrain the parameter \( \beta \) appearing in the GUP corrected Kerr metric by making comparisons with the black hole shadow and quasi-periodic oscillations (QPOs).
The paper is structured as follows: In Sec. II, a brief review of GUP-modified Kerr black hole is presented. In Sec. III, we study the shadow of the GUP-modified Kerr black hole. In Sec. IV, we consider the observational constraint of the GUP. In Sec. V, we investigate the connection between quasinormal modes and the shadow radius. In Sec. VI, we study the quasi-periodic oscillations (QPOs) to constraint GUP. Finally in Sec. VII, we comment on our results. We choose the units $c = \hbar = G = 1$ and occasionally set Planck mass to unity.

II. GUP-MODIFICATION OF KERR BLACK HOLES

Recently the GUP-modified Kerr metric has been proposed in Ref. [17]. To do so, we can start with the Kerr line element [17]

$$ds^2 = -(1 - \frac{r_s}{\rho^2}) dt^2 + \frac{\Sigma^2}{\Delta} dr^2 - \frac{2r_s a \sin^2 \theta}{\Sigma^2} dt d\phi + \Sigma^2 d\theta^2 + \left( r^2 + a^2 \frac{r_s a^2}{\Sigma^2} \sin^2 \theta \right) \sin^2 \theta d\phi^2 \quad (1)$$

where $r$ is the spherical radial coordinate along with

$$r_s = 2M, \quad a = \frac{J}{M}, \quad \Sigma^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - r_s r + a^2. \quad (2)$$

Then to find the GUP corrected metric, we need to replace the black hole mass by the following transformation [17]

$$M \rightarrow M + \frac{\beta}{2M}. \quad (3)$$

In that case, we have the following relations

$$r_s = 2M \left(1 + \frac{\beta}{2M^2}\right) = 2M_\zeta^{-1}, \quad (4)$$

$$a = a \left(1 + \frac{\beta}{2M^2}\right)^{-1} = a\zeta, \quad (5)$$

$$\Delta = r^2 - 2M_\zeta^{-1} r + a^2\zeta^2, \quad (6)$$

where we have defined

$$\zeta = \left(1 + \frac{\beta}{2M^2}\right)^{-1}. \quad (7)$$

In order to find the contour of a black hole shadow, we need to separate the null geodesic equations in the general rotating spacetime metric (1) using the Hamilton-Jacobi equation given by

$$\frac{\partial S}{\partial \sigma} = -\frac{1}{2} \Gamma_{\mu\nu\rho} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}, \quad (8)$$

where $\sigma$ is the affine parameter, $S$ is the Jacobi action.

For this purpose we can express the action in terms of known constants of the motion as follows

$$S = \frac{1}{2} \mu^2 \sigma - Et + l\phi + S_r(r) + S_\theta(\theta), \quad (9)$$

in the last equation $\mu$ represents the mass of the test particle, $E = -\mu t$ gives the conserved energy and $l = p_\phi$ is the conserved angular momentum, respectively. After we take $\mu = 0$, one can obtain the following equations of motion for the light rays

$$\sum \frac{dt}{d\lambda} = a\zeta(l - a\zeta E \sin^2 \theta) + \frac{r^2 + a^2 \zeta^2}{\Delta} \left[E(r^2 + a^2 \zeta^2) - a\zeta l\right], \quad (10)$$

$$\sum \frac{dr}{d\lambda} = \pm \sqrt{\Delta}, \quad (11)$$

$$\sum \frac{\theta}{d\lambda} = \pm \sqrt{\zeta}, \quad (12)$$

$$\sum \frac{d\phi}{d\lambda} = \left( l \csc^2 \theta - a\zeta E \right) + \frac{a\zeta}{\Delta} \left[E(r^2 + a^2 \zeta^2) - a\zeta l\right], \quad (13)$$

where $\sigma$ is the affine parameter, $l$ is the angular momentum of the photon, $E$ is the energy of the photon and $K$ is the Carter constant. In addition we have introduced

$$\mathcal{R} = \left(a^2\zeta^2 E - a\zeta E + E r^2\right)^2 - \Delta \left[K + (l - a\zeta E)^2\right], \quad (14)$$

$$\Theta = \mathcal{K} - (l \csc \theta - a\zeta E \sin^2 \theta)^2 + (l - a\zeta E)^2. \quad (15)$$

Next, we can determine the size and shape the black hole shadow by using the unstable circular photon orbits given by the following conditions

$$\mathcal{R}(r) = 0, \quad \frac{d\mathcal{R}(r)}{dr} = 0, \quad \frac{d^2\mathcal{R}(r)}{dr^2} > 0. \quad (16)$$

By using this condition the circular orbit radius $r_{ph}$ of the photon can be obtained and the parameters $\zeta \equiv \frac{l}{E}$ and $\eta \equiv \frac{K}{E^2}$ can thus be expressed as where $X(r) = (r^2 + a^2\zeta^2)$, and $\Delta(r)$ is defined by Eq. (6), while $K$ is known as the Carter separation constant. From these conditions one can show that the motion of the photon can be determined by the following two impact parameters

$$\zeta = \frac{X_{ph}\Delta'_{ph} - 2\Delta_{ph}X'_{ph}}{a\zeta^2\Delta'_{ph}}, \quad (17)$$

$$\eta = \frac{4a^2\zeta^2X'^2_{ph}\Delta_{ph} - \left[(X_{ph} - a^2\zeta^2)\Delta'_{ph} - 2X'_{ph}\Delta_{ph}\right]^2}{a^2\zeta^2\Delta'_{ph}^2}. \quad (18)$$

One constraint for the value of the photon’s circular orbit radius is $\mathcal{R}(r_{ph}) > 0$. As we know, the shape of the shadow seen by an observer located at spatial infinity can be obtained from the geodesics of the photons and described by the celestial coordinates

$$x = -\zeta \csc \theta_0, \quad (19)$$

$$y = \pm \sqrt{\eta + a^2\zeta^2 \cos^2 \theta_0 - \zeta^2 \cot^2 \theta_0}. \quad (20)$$

In Fig. 2 we plot the shape of the shadow by varying the GUP parameter. In our plots we are going to use an inclination angle $\theta_0 = 17^\circ$. This angle has been obtained by studying the emission jet from the black hole (see [19]). We observe that an increase of the parameter...
The shape of shadow and the characteristic points of intrinsic curvature of the shadow (picture adopted from the Ref. [18]).

\[
R = \frac{64M^{1/2}(r_0^4 - a^2r_0^2 \cos^2 \theta_0)^{3/2}}{(r_0 - M)^3 \left[3(8r_0^4 - a^4 - 8a^2r_0^2) - 4a^2(6r_0^2 + a^2) \cos(2\theta_0) - a^4 \cos(4\theta_0)\right]},
\]

(17)

We can therefore introduce the GUP effects, if we use the scaling
\[
M \to M \zeta^{-1}, \ a \to a \zeta,
\]

(18)
to obtain the following result for the local curvature radius
\[
R = \frac{64M^{1/2} \zeta^{-1/2}(r_0^4 - a^2 \zeta^2 r_0^2 \cos^2 \theta_0)^{3/2}}{(r_0 - M \zeta^{-1})^3 \left[3(8r_0^4 - a^4 - 8a^2 \zeta^2 r_0^2) - 4a^2 \zeta^2 (6r_0^2 + a^2 \zeta^2) \cos(2\theta_0) - a^4 \zeta^4 \cos(4\theta_0)\right]},
\]

(19)

In what follows, we are going to use the last equation to evaluate the intrinsic curvature of the shadow in three characteristic points: D, R and T, respectively. In particular, we are going to compute the horizontal and the vertical angular size for these curvature radii of the shadow, noted as \(\Delta x\) and \(\Delta y\), respectively. In Fig. 3 and Fig. 4, we show the plots the local curvature and the angular size for these curvature radii of the shadow as a function of \(\beta\) by fixing \(\theta_0\) and \(a\). We see that the local curvature monotonically increases with the increase of \(\beta\).

### III. OBSERVATIONAL CONSTRAINTS

In this section we can use the reported angular size of the shadow in the M87* center detected by the EHT given by \(\theta_s = (42 \pm 3)\mu\text{as}\), along with the distance to M87 given by \(D = 16.8\) Mpc, and the mass of M87 central object \(M = 6.5 \times 10^8\) M\(_\odot\), to constrain the GUP parameter. To do so, we are going to consider two cases: 

a) In the first case, we identify the the diameter of the shadow in units of mass \(d_{\text{M87}}\) with the horizontal angular size \(\Delta x\).

b) In the second case, we identify the diameter of the shadow in units of mass \(d_{\text{M87}}\) with the vertical angular size \(\Delta y\). Using the above EHT result, we can write

\[
d_{\text{M87}} = \frac{D \theta_s}{M} = 11.0 \pm 1.5.
\]

(20)

This means that within 1\(\sigma\) confidence we have the interval \(9.5 \lesssim d_{\text{M87}} \lesssim 12.5\), whereas within 2\(\sigma\) uncertainties we have \(8 \lesssim d_{\text{M87}} \lesssim 14\) [18]. In Fig. 5 we show the regions of parameter space of the diameter of the shadow and the GUP parameter \(\beta\) for two cases. In the first case, within 1\(\sigma\) confidence, we find an upper limit of the GUP parameter \(\beta/M^2 \lesssim 0.77\). On the other hand, within 2\(\sigma\) confidence, we find an upper limit \(\beta/M^2 \lesssim 0.49\). In the second case, we can identify the diameter of the shadow in units of mass \(d_{\text{M87}}\) with the vertical angular size \(\Delta y\). Here we find similar results, namely within
1σ confidence, we find the upper limit to be $\beta/M^2 \lesssim 0.5$. On the other hand, within 2σ confidence, we find the interval $\beta/M^2 \lesssim 0.78$. Notice also that we rule out negative values of $\beta$. In other words, these results are in good agreement. This means that within 2σ, we have the upper and lower limit of the GUP parameter in terms of the interval $0 \lesssim \beta/M^2 \lesssim 0.78$. The GUP parameter has been estimated to be $\beta/M^2 < 0.78$, but measured in units of black hole mass squared. In order $\beta$ to be dimensionless, we need to restore the Planck mass $[M_P = 2.2 \times 10^{-5} \text{g}]$, this means $\beta \lesssim 0.78 \times M^2 / M_P^2$. For the M87* black hole mass if we take $6.5 \times 10^9 M_\odot$, we obtain a upper bound $\beta \lesssim 2.7 \times 10^9$. Since this quantity scales with the black hole mass, we see that as the black hole mass decreases, this bound should decrease as well. That’s why quantum systems are better in constraining GUP. For primordial black holes we can take the mass about $10^{15}$ g and obtain $\beta \lesssim 1.6 \times 10^{39}$. According to GUP, there should be some final size or so called remnant mass where the Hawking evaporation stops in such case the black hole mass is of the order of Planck mass, hence $\beta \lesssim 0.78$.

IV. CONNECTION BETWEEN SHADOW RADIUS AND QNMS

In this section we are going to investigate the correspondence between the radius of the black hole shadow and the real part of the QNMs frequency. Before that, let us recall that the real part of the the QNMs frequen-
FIG. 3: The plots the local curvatures of the shadow as a function of $\beta$ for the three points: R, D, and T. We have set $M = 1$, $a/M = 0.95$ and $\theta_0 = 17^0$.

cies is related to the angular velocity of the unstable null geodesic in the eikonal limit $[30]$

$$\omega_{\text{QNM}} = \Omega_c l - i \left( n + \frac{1}{2} \right) |\lambda|,$$  \hspace{1cm} (21)

here $\Omega_c$ is the angular velocity at the unstable null geodesic, and $\lambda$ is the Lyapunov exponent which determines the instability time scale of the orbits. Recently, one of the authors of this paper pointed out that the following relation relates the real part of the QNMs frequencies and the shadow radius (see for details $[40][42]$)

$$\omega_R = \lim_{l \gg 1} \frac{l}{R_s},$$  \hspace{1cm} (22)

which is precise only in the eikonal limit having large values of multipole number $l$. Here $R_s$ denotes the radius of the black hole shadow. Hence, we can quickly rewrite the expression (21) as

$$\omega_{\text{QNM}} = \lim_{l \gg 1} \frac{l}{R_s} - i \left( n + \frac{1}{2} \right) |\lambda|.$$  \hspace{1cm} (23)
The importance of this correspondence relies on the fact that the shadow radius represents an observable quantity which can be measured in astronomical measurements. Therefore, it provides an alternative way to evaluate the real part of the QNMs frequencies in terms of the black hole shadow radius. Sometimes the inverse problem may be more effective, namely we can determine the shadow radius once having the oscillating frequency of the black hole without involving the standard geodesic method.

This close connection could be understood from the fact that the gravitational waves can be treated as massless particles propagating along the last null unstable orbit and out to infinity. In in the eikonal limit, this correspondence suggest that scalar, electromagnetic, and the gravitational field perturbations have the same behavior. This property is of course lost in the limit of small $l$.

Given the fact that the relation (22) is very accurate only for large $l$, this relation can provide valuable information even for small $l$. To illustrate this fact, we can use the correspondence between the shadow radius of the black hole and the real part of QNMs to sub-leading regime to half of its value reported recently in Ref. [45]

$$\omega_R = R_s^{-1} \left( l + \frac{1}{2} \right). \quad (24)$$

From the last equation it is clear that at high angular mo-
\[ l = 1, n = 0 \quad l = 2, n = 0 \quad l = 3, n = 0 \quad l = 4, n = 0 \]

| \( \beta/M^2 \) | \( \omega_R \) | \( \omega_{\Re} \) | \( \omega_R \) | \( \omega_{\Re} \) | \( R_s/M \) |
|----------------|-------------|-------------|-------------|-------------|-------------|
| 0              | 0.288675    | 0.481125    | 0.673575    | 0.866025    | 3\( \sqrt{3} \) |
| 0.1            | 0.274929    | 0.458214    | 0.641500    | 0.824786    | 5.45596     |
| 0.2            | 0.262432    | 0.437387    | 0.612341    | 0.787296    | 5.71577     |
| 0.3            | 0.251022    | 0.418370    | 0.585718    | 0.753066    | 5.97558     |
| 0.4            | 0.240563    | 0.400938    | 0.561313    | 0.721688    | 6.23538     |

**TABLE I:** Numerical values for the shadow radius and the real part of QNMs obtained via Eq. (24).

![FIG. 6](image)

**FIG. 6:** Left panel: The real part of QNMs as a function of \( \beta \) for a given \( l \). Right panel: The real part of QNMs as a function of \( l \) for a fixed \( \beta \).

In the momentum regime, i.e. \( l >> 1 \), Eq. (22) is obtained. Again, we should keep in mind that the above correspondence is accurate in the eikonal regime, but sometimes it is still accurate even for small multipole number \( l \). The shadow radius for the our GUP corrected black hole in the static case reads

\[ R_s = 3\sqrt{3}M \left( 1 + \frac{\beta^2}{2M^2} \right), \]

thus we can find \( \omega_{\Re} \) from Eq. (24). In Table I we show the numerical values for the real part of QNMs obtained from the shadow radius. It is observed that the frequency decreases with the increase of \( \beta \). In what follows we shall compare the above results with the results obtained via the WKB method.

### A. Scalar field perturbations

Let us consider the equation of motion for a massless scalar field described by the Klein-Gordon equation in the background of the curve spacetime written as follows

\[ \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \Phi \right) = 0. \]  

(26)

Here \( \Phi \) represents the massless scalar field and it is a function of coordinates \( (t, r, \theta, \phi) \). We further consider an ansatz of the scalar field

\[ \Phi(t, r, \theta, \phi) = \sum_{l=0}^\infty e^{-i\omega t} \frac{\Psi_l(r)}{r} Y_{lm}(r, \theta), \]  

(27)

where \( e^{-i\omega t} \) represents the time evolution of the field and \( Y_{lm}(r, \theta) \) denotes the spherical harmonics function. Plunging the ansatz (27) into (26) and applying the separation of variables method, we obtain the standard Schrödinger-like wave equations

\[ \frac{d^2\Psi_l(r_s)}{dr_s^2} + \left( \omega^2 - V_s(r_s) \right) \Psi_l(r_s) = 0, \]  

(28)
where $\omega$ is the frequency of the perturbation and $r_*$ represents the tortoise coordinates having the relation

$$dr_* = \frac{dr}{f(r)} \Rightarrow r_* = \int \frac{dr}{f(r)}$$  \hspace{1cm} (29)$$

The advantage of using the tortoise coordinates here is to extend the range used in a survey of the QNMs. The tortoise coordinate is being mapped the semi-infinite region from the horizon to infinity into $(-\infty, +\infty)$ region. The effective potential in gives

$$V_s(r_*) = \left[ 1 - \frac{2M}{r} \left( 1 + \frac{\beta}{2M^2} \right) \right] \left[ \frac{l(l+1)}{r^2} + \frac{2M}{r^3} \left( 1 + \frac{\beta}{2M^2} \right) \right]$$  \hspace{1cm} (30)$$

where $l$ denotes the multipole number. On having the expression of the effective potential we can apply the WKB approach to compute the QNMs due to the scalar field perturbations. In the present work, we are going to consider the sixth-order WKB method developed by Konoplya [24]. Our results show that an increase of $\beta$ decreases the real part of QNMs (cf. Table II). This indicates that the scalar field perturbations in a GUP modified black hole oscillate more slowly compared to the Schwarzschild vacuum black holes.

### B. Electromagnetic field perturbations

In the case of the electromagnetic field perturbations we have the field equation

$$\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} (\partial_{\lambda} A_\nu - \partial_{\nu} A_\lambda) \right) = 0.$$  \hspace{1cm} (31)$$

Without going into details, after we substitute all related expressions into (31) and using the standard tortoise coordinate transformation $dr_* = dr/f(r)$ yielding the effective potential

$$V_E(r) = \left[ 1 - \frac{2M}{r} \left( 1 + \frac{\beta}{2M^2} \right) \right] \frac{l(l+1)}{r^2}.$$  \hspace{1cm} (32)$$

In Table III we show the numerical results obtained via WKB method. Similarly to the previous case, the real part of QNMs decreases.

### C. Gravitational field perturbations

Our final example will be the study of gravitational field perturbations. Before applying the WKB method, we recall that the general form of the perturbed metric can be written as

$$ds^2 = -e^{2\nu} dt^2 + e^{2\psi} (d\theta - \sigma dt - q_r dr - q_\theta d\theta)^2$$

$$+ e^{-2\mu_2} d^2 r + e^{-2\mu_3} d^2 \theta,$$  \hspace{1cm} (33)$$

with $\nu = e^{\mu_2} = f(r)$, $\psi = q_r = q_\theta = 0$ for non-perturbed case. The perturbations will lead to non-vanishing values of $\sigma, q_r, q_\theta$ and increments in $\nu, \mu_2, \mu_3, \psi$, which are corresponding to axial and polar perturbations, respectively. Here we shall consider the axial type ones. The perturbation equation reads

$$r^4 \frac{\partial}{\partial r} \left( \frac{f(r)}{r^2} \frac{\partial Q}{\partial r} \right) + \sin^2 \theta \left( \frac{1}{\sin^2 \theta} \frac{\partial Q}{\partial \theta} \right) - \frac{r^2}{f(r)} \frac{\partial^2 Q}{\partial t^2} = 0,$$  \hspace{1cm} (34)$$

where

$$Q(t, r, \theta) = e^{i\omega t} Q(r, \theta),$$

$$Q(r, \theta) = \bar{r}^2 f(r) \sin^2 \theta Q_{r, \theta},$$

$$Q_{r, \theta} = q_r, q_\theta.$$  \hspace{1cm} (35)$$

Further with $Q(r, \theta) = r \Psi(r) e^{-2/3}$, it can be reduced to Schrödinger wave-like equations:

$$\frac{d^2 \Psi}{dr_*^2} + [\omega^2 - V_G(r)] \Psi = 0, \hspace{0.5cm} dr_* = f(r) dr,$$  \hspace{1cm} (36)$$

for gravitational field $\Psi$. The effective potentials take the form as:

$$V_G(r) = \left[ 1 - \frac{2M}{r} \left( 1 + \frac{\beta}{2M^2} \right) \right] \frac{l(l+1)}{r^2}$$

$$- \frac{6M}{r^3} \left( 1 + \frac{\beta}{2M^2} \right).$$  \hspace{1cm} (37)$$

In Table IV, we show the effect of GUP parameter on QNMs frequencies. Note that the oscillation frequency $\omega$ depend on the black hole parameters by converting the frequencies calculated in geometrical units into kHz, one should multiply $\omega$ by $2\pi (5.142kHz)M_{\odot}/M$. For example, the first gravitational quasinormal mode frequency of a Schwarzschild black hole corresponds to the fundamental $n = 0$ quadrupole $l = 2$ mode and it is $\omega M \approx 0.3736 - 0.089 i$, where we measure in units if the black hole mass $M = 1$. For a black hole of 10 solar masses for $\beta/M^2 = \{0, 0.1, 0.2, 0.3, 0.4\}$ we have the oscillation frequency is $f = \{1.2, 1.16, 1.12, 1.08, 1.04\} kHz$. This shows that the oscillation frequency due to the GUP effect indeed decreases.
If we compare now the results obtained by means of the shadow radius and the WKB method we see that the agreement is very good even in the limit of small $l$. In fact, we get better agreement for scalar field perturbations than the electromagnetic perturbations. In the limit $l > > 1$, the agreement become exact, i.e. eikonal limit. This is very interesting result, having an experimental result for the shadow radius of a given black hole allows us to estimate the frequency of the gravitational waves if the black hole is perturbed. However, sometimes it may be useful to do the opposite; namely having an experimental result for a detected gravity wave, allows us to estimate the shadow radius of that black hole.

D. The case of asymptotically high overtones

It is interesting to point out the asymptotically high overtones which corresponds to the case of overtone number $n \to \infty$ or at least very large. This case can be used to explore a very interesting conjecture that links the quasinormal modes to black hole thermodynamics. For the Schwarzschild quasinormal modes, it has been found by Motl [34] that

$$\omega = \frac{\ln 3}{8\pi M} - \frac{i}{4M} \left( n + \frac{1}{2} \right).$$  \hspace{1cm} (38)

One can notice easily that the real part is also related to the Hawking temperature $\omega_R = \ln 3 \times T_H$. If we consider the scaling of the mass we obtain

$$\omega = \frac{\ln 3}{8\pi M} \left( 1 + \frac{\beta}{2M^2} \right)^{-1} - \frac{i}{4M} \left( 1 + \frac{\beta}{2M^2} \right)^{-1} \left( n + \frac{1}{2} \right).$$  \hspace{1cm} (39)

This expression can be used further to explore the effect of GUP and the connection between the quasinormal modes and the black hole thermodynamics. If the mass of a black hole is quantized, it was suggested by Bekenstein and Mukhanov [35] that this fact would lead to the quantization of the black hole area $\Delta A$, one expect the quantization to be effect by the GUP parameter.
V. QUASI-PERIODIC OSCILLATIONS

Introducing the relevant universal constants, the metric [1] takes the form

\[
\begin{align*}
g_{tt} &= -c^2\left(1 - \frac{2GMr}{c^2(r^2 + a^2\zeta^2 \cos^2 \theta)}\right), \\
g_{\phi\phi} &= \frac{2aGMr}{c(r^2 + a^2\zeta^2 \cos^2 \theta)}
\end{align*}
\]

where \( \zeta \) is expressed in terms of the dimensionless parameter \( \beta_0 \) as

\[\zeta = \frac{2}{2 + \beta_0} \quad \text{and} \quad \beta_0 \equiv \frac{\beta}{M^2}. \quad (41)\]

Recall that for the Kerr BH \( \zeta = 1 \) and \( \beta_0 = 0 \) and that \( \zeta(\beta_0) \) is a decreasing function of \( \beta_0 \).

For the numerical calculations to be carried out in this section, we take \( M_\odot = 1.9888 \times 10^{30} \) (solar mass), \( G = 6.673 \times 10^{-11} \) (gravitational constant), and \( c = 299792458 \) (speed of light in vacuum) all given in SI units. These same constants will be written explicitly in some subsequent formulas of this section.

\[
\frac{r}{r_g} \quad \text{(GRO J1655–40)}
\]

FIG. 7: Plot of the dimensionless radius of the circle \( u = r/r_g \) [51], where the 3/2 resonance occurs nearest the isco, versus the dimensionless parameter \( \beta_0 \) [41] for the microquasar GRO J1655–40 taking \( a_0 = 0.70 \) (which is the intermediate value [42]). There is another increasing \( u(\beta_0) \) branch but concave up (not shown in this plot) which provides higher values for \( u \). Such a branch is not favorable since we believe that the resonance occurs nearest the isco.

The point \((\beta_0, u) = (-0.254251, 4.1748)\), where the lower \( v_L = 300 \) Hz QPO and the upper \( v_U = 450 \) Hz QPO occur, is shown by the black spot.

In the power spectra of Fig. 3 of Ref. [50], we clearly see two peaks at 300 Hz and 450 Hz, representing, respectively, the possible occurrence of the lower \( v_L = 300 \) Hz quasi-periodic oscillation (QPO), and of the upper \( v_U = 450 \) Hz QPO from the Galactic microquasar GRO J1655-40. Similar peaks have been obtained for the microquasars XTE J1550-564 and GRS 1915+105 obeying the remarkable relation, \( v_U/v_L = 3/2 \) [51]. Some of the physical quantities of these three microquasars and their uncertainties are as follows [50] [52]:

- **GRO J1655-40**:
  \[
  \frac{M}{M_\odot} = 6.30 \pm 0.27, \quad \frac{a}{r_g} = 0.70 \pm 0.05,
  \quad v_U = 450 \pm 3 \text{ Hz}, \quad v_L = 300 \pm 5 \text{ Hz}, \quad (42)
  \]

- **XTE J1550-564**:
  \[
  \frac{M}{M_\odot} = 9.1 \pm 0.6, \quad \frac{a}{r_g} = 0.405 \pm 0.115,
  \quad v_U = 276 \pm 3 \text{ Hz}, \quad v_L = 184 \pm 5 \text{ Hz}, \quad (43)
  \]

- **GRS 1915+105**:
  \[
  \frac{M}{M_\odot} = 14.0 \pm 4.4, \quad \frac{a}{r_g} = 0.99 \pm 0.01,
  \quad v_U = 168 \pm 3 \text{ Hz}, \quad v_L = 113 \pm 5 \text{ Hz}, \quad (44)
  \]

where \( r_g \equiv GM/c^2 \).

These twin values of the QPOs are most certainly due to the phenomenon of resonance which occurs in the vicinity of the ISCO, where the in-falling particles perform radial and vertical oscillations around almost circular orbits. The local radial and vertical oscillations are denoted by \((\Omega_r, \Omega_\phi)\), respectively. These two oscillations couple generally non-linearly to yield resonances...
FIG. 9: Fitting the particle oscillation upper and lower frequencies to the observed frequencies (in Hz) for the microquasar XTE J1550–564 at the 3/2 resonance radius. In the black plots the microquasar is treated as a GUP-modified Kerr BH given by (40) with \( \beta_0 = -0.532673 \) (\( \zeta = 1.36302 \)). The upper black curve represents \( v_U = v_\theta \) and the lower black curve represents \( v_L = v_r \) with \( v_U/v_L = 3/2 \), and the green curve represent the mass error band as given in (45). The black curves cross the mass error bands ensuring a good curve fitting. In the red plots the microquasar is treated as a Kerr BH \( \beta_0 = 0 \) (\( \zeta = 1 \)). Since the red plots do not cross the mass error bands, a description of the astrophysical object by a Kerr BH fails to justify the occurrence of the 3/2 resonance.

in the power spectra [53, 54]. For the case of uncharged rotating BH (\( \Omega_r, \Omega_\theta \)) are given by [55] (see also [56, 57])

\[
\begin{align*}
\Omega^2_r &\equiv (\partial_r \Gamma^r_{ij} - 4 \Gamma^r_{jk} \Gamma^j_i) u^i u^j, \quad (i, j, k = t, \phi), \\
\Omega^2_\theta &\equiv (\partial_\theta \Gamma^\theta_{ij}) u^i u^j, \quad (i, j, k = t, \phi),
\end{align*}
\]

(45)

In obtaining these expressions we assumed that the main motion of the particle is circular in the equatorial plane (\( \theta = \pi/2 \)) where the particle exhibits radial and vertical oscillations. The circular motion is stable only if \( \Omega_r^2 > 0 \) and \( \Omega_\theta^2 > 0 \). In the equatorial plane the four-velocity vector of the particle has only two nonzero components \( u^\theta = (u^r, 0, 0, u^\phi) = u^t(1, 0, 0, \omega) \), where \( \omega = d\phi/dt \) is the angular velocity of the test particle. They are related by [53]

\[
\begin{align*}
\omega &= -\partial_t g_{t\phi} \pm \sqrt{(\partial_r g_{t\phi})^2 - \partial_r g_{tt} \partial_t g_{\phi\phi}}, \\
u^t &= \frac{c}{\sqrt{-(g_{tt} + 2\partial_t g_{t\phi} \omega + g_{\phi\phi} \omega^2)}}, \\
u^\phi &= \omega u^t.
\end{align*}
\]

(46)

In these expressions the summations extend over \( (t, \phi) \). It is understood that all the functions appearing in (45) and (46) are evaluated at \( \theta = \pi/2 \).

The locally measured frequencies (\( \Omega_r, \Omega_\theta \)) are related to the spatially-remote observer’s frequencies \( (v_r, v_\theta) \) by

\[
\begin{align*}
v_r &= \frac{1}{2\pi} \frac{1}{u^t} \Omega_r, \\
v_\theta &= \frac{1}{2\pi} \frac{1}{u^t} \Omega_\phi.
\end{align*}
\]

(47)

FIG. 10: Fitting the particle oscillation upper and lower frequencies to the observed frequencies (in Hz) for the microquasar GRS 1915+105 at the 3/2 resonance radius. In the black plots the microquasar is treated as a GUP-modified Kerr BH given by (40) with \( \beta_0 = 0.071347 \) (\( \zeta = 0.965555 \)). The upper black curve represents \( v_U = v_\theta \) and the lower black curve represents \( v_L = v_r \) with \( v_U/v_L = 3/2 \), and the green curve represent the mass error band as given in (45). The black curves cross the mass error bands well in the middle ensuring a good curve fitting. In the red plots the microquasar is treated as a Kerr BH \( \beta_0 = 0 \) (\( \zeta = 1 \)). The red plots also cross the mass error bands but at the rightmost points. A description of the astrophysical object by a Kerr BH does not provide a good curve fitting to justify the occurrence of the 3/2 resonance.

Using the form (40) of the metric and introducing the dimensionless parameters \( y \) and \( a_0 \) defined by

\[
u_r = \frac{c^3 (u^{3/2} - a_0 \sqrt{\zeta}) \sqrt{\zeta u^2 - 6u + 8a_0 \zeta^{3/2} \sqrt{u} - 3a_0^2 \zeta^{3/2}}}{2\pi GM(\mu^3 - a_0^2 \zeta^{3/2})u},
\]

(49)

\[
u_\theta = \frac{c^3 (u^{3/2} - a_0 \sqrt{\zeta}) \sqrt{\zeta u^2 - 4a_0 \zeta^{3/2} \sqrt{u} + 3a_0^2 \zeta^{3/2}}}{2\pi GM(\mu^3 - a_0^2 \zeta^{3/2})u}.
\]

Setting \( \zeta = 1 \) these expressions reduce to the corresponding expressions for the Kerr BH.

On confronting the observed ratio \( v_U/v_L = 3/2 \) most workers in this field appeal to parametric resonance to explain the observed ratio assuming that \( v_\theta/v_r = n/2 \) and \( n \in \mathbb{N}^+ \). In almost all applications of parametric resonance one considers the case \( n = 1 \) [58–61] where in this case \( v_r \) is the natural frequency of the system and \( v_\theta \) is the parametric excitation \( (T_\theta = 2T_r, \text{the corresponding periods}), \) that is, the vertical oscillations supply energy to the radial oscillations causing resonance [61]. However, since always \( v_\theta > v_r \), it is neither possible to have \( n = 1 \) nor \( n = 2 \) in the vicinity of the iso where...
it is thought that the resonance effects take place. The next allowed choice is thus $n = 3$ by which $v_r$ becomes the parametric excitation that supplies energy to the vertical oscillations. In this work we work with the ansatz $v_U = v_r, v_L = v_r$ along with $v_0/v_r = 3/2$ ($n = 3$). Equation (49) yields

\[ \left( \frac{v_U}{v_L} \right)^2 = \frac{\zeta u^2 - 4a_0^2 \zeta^{3/2} \sqrt{u} + 3a_0^2 \zeta^3}{\zeta u^2 - 6u + 8a_0^2 \zeta^{3/2} \sqrt{u} - 3a_0^2 \zeta^3} = \frac{9}{4} \]  

(50)

Expressing $u$, where the 3/2 resonance occurs nearest the isco, in terms of $(a_0, \zeta)$ we obtain

\[ u = \frac{1}{5 \zeta} \left[ 27 + Z - \sqrt{486 + 130a_0^2 \zeta^4 - \frac{9680a_0^2 \zeta^4}{Z} + \frac{\left( 14348907 - 5a_0^2 \zeta^4 \right) (4048137 - 1259725a_0^2 \zeta^4 - 54925a_0^2 \zeta^8 + 24200a_0^2 \zeta^6 \left( 12393 - 23378a_0^2 \zeta^4 + 10985a_0^2 \zeta^8 \right)^{1/3}}{Z} + 243 + 65a_0^2 \zeta^4 + \frac{Y}{X} + X. \right] \]  

(51)

For fixed $a_0$ this provides a relation between the dimensionless parameter $\beta_0$ (11) and the dimensionless radius of the circle $u$ where the 3/2 resonance occurs nearest the isco. For the microquasar GRO J1655-40 we take $a_0 = 0.70$, which is the intermediate value (42), and we plot $u = r/r_g$ in terms of $\beta_0$ as shown in Fig. 7 which an increasing function of $\beta_0$ and concave down. There is another increasing $u(\beta_0)$ branch but concave up (not shown in Fig. 7) which provides higher values for $u$. Such a branch is not favorable since we believe that the resonance occurs nearest the isco. For the other two microquasars (43) and (44) we have obtained similar increasing $u(\beta_0)$ functions. Keeping only one branch of $u(\beta_0)$, now to determine the unique values of $(\beta_0, u)$ corresponding to the occurrence of the upper $v_U$ and lower $v_L$ we solve the equation $v_U = \text{observed value}$ (or the equation $v_L = \text{observed value}$) for each microquasar, where the observed values of $(v_U, v_L)$ for the three microquasars are given in (42), (43) and (44). For the three microquasars taking $(M, a_0)$ to be the intermediate value given in (42), (43) and (44), we obtained

- GRO J1655-40: $\beta_0 = -0.254251, \quad u = 4.1748,$
- XTE J1550-564: $\beta_0 = -0.532673, \quad u = 4.54216,$
- GRS 1915+105: $\beta_0 = 0.071347, \quad u = 5.0168.$

Recall that isco is defined by $\Omega_i = \Omega_{\text{isco}} = 0$ and these values of $u$ are certainly higher than $u_{\text{isco}}$ (55) (see also (56)). In Fig. 7 corresponding to the microquasar GRO J1655-40, the point $(\beta_0, u) = (-0.254251, 4.1748)$, where the lower $v_L = 300$ Hz QPO and the upper $v_U = 450$ Hz QPO occur, is shown by the black spot.

Perfect curve fitting of the particle oscillation upper and lower frequencies to the observed frequencies (in Hz) for each microquasar are shown in Figs. 8[6] and 10.

In the black plots each microquasar is treated as a GUP-modified Kerr BH given by (40) and taking the coordinates of $(\beta_0, u)$ to be the values given in (52). Here $u$ is the dimensionless radius where the 3/2 resonance occurs. In the red plots the microquasar is treated as a Kerr BH $\beta_0 = 0$ ($\zeta = 1$). We see from these three figures that the black plots cross the mass error bands exactly in the middle point. For the microquasars GRO J1655-40 and XTE J1550-564, the red plots do not cross the mass error bands, hence a description of the astrophysical object by a Kerr BH fails to justify the occurrence of the 3/2 resonance. For the microquasar GRS 1915+105, however, the red plots also cross the mass error bands but at the rightmost points. Knowing that for the microquasar GRS 1915+105 the mass error band is the largest, this provides a mediocre curve fitting of the particle oscillation upper and lower frequencies to the observed frequencies.

Based on the previous analysis, we restrict the values of $\beta_0$ to lie between the smallest and largest values we obtained above:

\[-0.532673 \lesssim \beta_0 \lesssim 0.071347 \implies 0.965555 \lesssim \zeta \lesssim 1.36302.\]  

(53)

Using the last result, we can determine the upper bound of GUP restoring the Planck mass via $\beta \lesssim 0.071347 \times M^2/M_p^2$. If we take the mass $M = 14 \times M_\odot$, we find $\beta \lesssim 1.15 \times 10^{-7}$. Compared to the shadow case, here we obtain a better constraint for $\beta$.

VI. CONCLUSIONS

In this paper we have studied the effect of the Generalised Uncertainty Principle (GUP) on the shadow of GUP-modified Kerr black hole and the correspondence between the shadow radius and the real part of the quasinormal modes (QNMs). We have found that the shadow curvature radius of the GUP-modified Kerr black hole is bigger compared to the Kerr vacuum solution and increases linearly monotonically with the increase of the GUP parameter. Using the characteristic points of intrinsic curvature of the shadow or have calculated the angular size for these curvature radii of the shadow. Using the EHT data for the M87* black hole we have constraint the upper limit of the GUP parameter. Within $2\sigma$, we have the upper and lower limit of the GUP parameter in terms of the interval $0 \leq \beta/M^2 \lesssim 0.78$.

Finally, we have explored the connection between the shadow radius and the scalar/electromagnetic QNMs. It is argued that this correspondence works well even in the case of small $l$. This provides an interesting connection between the experimental data of the shadow and the detection of the gravity waves. Having the
shadow radius one can estimate the value of the QNMs frequency of a given black hole, or vice versa.

We have shown that describing the microquasars GRO J1655-40, XTE J1550-564 and GRS 1915+105 as GUP-modified Kerr BHs yields perfect curve fitting of the particle oscillation upper and lower frequencies to the observed frequencies provided we restrict the values of the correction dimensionless parameter $\beta_0$ by $-0.532673 \lesssim \beta_0 \lesssim 0.071347$. These are very reasonable bounds knowing that the metric of the GUP-modified Kerr BH is a correction of the Kerr one where $\beta_0$ should lie in the vicinity of zero.
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