Swanson Hamiltonian: non-PT-symmetry phase

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Abstract

In this work, we study the non-Hermitian Swanson Hamiltonian, particularly the non-parity-time symmetry phase. We use the formalism of Gel’fand triplet to construct the generalized eigenfunctions and the corresponding spectrum. Depending on the region of the parameter model space, we show that the Swanson Hamiltonian represents different physical systems, i.e. parabolic barrier, negative mass oscillators. We also discussed the presence of Exceptional Points of infinite order.

Keywords: parity time-symmetry Swanson Hamiltonian, Gel’fand triplet, exceptional points, phase transitions

1. Introduction

The study of non-Hermitian Parity-time (PT) reversal symmetry Hamiltonians was proposed in the pioneering work of Bender and Boettcher [1]. A parametric family of Hamiltonians can be obtained by varying the different variables of the corresponding systems. The essential feature of a PT-symmetric Hamiltonian is the existence of certain values of the parameters at which the spectrum is real and the Hamiltonian is similar to a Hermitian one [2–7]. For other values of the parameters of the model, the spectrum contains complex-conjugate pairs of eigenvalues and the corresponding eigenfunctions are no longer PT-symmetric. In the boundary of both regions of the model space, two or more eigenvalues and their corresponding eigenstates can be coalescent. This set of parameters are called exceptional points [8–20]. The time evolution of a given initial state under the action of these hamiltonians strongly depends on the characteristics of the spectrum, particularly at EPs [21–30], where the exponential law is generally not valid.

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The Swanson model has been introduced in [31] as an example of a Hamiltonian that obeys PT-symmetry. It admits real eigenvalues for a well-defined region of the parameter model space. The similarity between the Swanson Hamiltonian and the harmonic oscillator as well as the dynamic of observables in the PT-symmetry region have been extensively analysed [32–46]. Among the extensions of the Swanson model we can include super-symmetry realizations [38, 47]. Another approach to the Swanson model comes from investigating different q-deformation boson algebras [52–55]. As an example, in [52] different representations of deformed canonical variables are presented. In this work, the effect that produces the modification of the generalized canonical variables commutator, over the region of PT-symmetry and exceptional points, is analysed.

In [53], the authors studied the particular case of the Swanson Hamiltonian using the generalization of the Milne quantization. In [54] the Swanson Hamiltonian is discussed in the framework the formalism of generalized pseudo bosons. The authors of [54], by employing generalized Bogoliubov transformation, present a mapping of the Swanson model to a standard bosons Hamiltonian. In [55] the Swanson model is obtained as the quadratic limit of a deformed general Hamiltonian constructed from a non-standard oscillator algebra.

However, to our knowledge, much less has been investigated in the region of PT-broken symmetry. In this line, the authors of [56] have proposed different extensions of the Swanson model. They have described the continuous spectrum of the different generalizations by analyzing the corresponding similar Hermitian hamiltonians. More recently, the authors of [57] have studied the solutions of the time-dependent Swanson Hamiltonian. By applying the formalism presented in [58], their work includes the construction of a time-dependent metric to compute the time evolution of the observables of the system. A new proposal has been presented in [59], where the authors construct time-dependent metrics by point transformations. In this work, the construction of non-Hermitian invariants for the Swanson model and the implementation of Dyson maps is analysed. From another point of view, the authors of [60] have made use of the Darboux transformation to provide solutions to the Swanson model for a particular set of parameters.

In this work, we study the non-Hermitian Swanson Hamiltonian, particularly the non-PT symmetry phase. The work is organized as follows. In section 2 we analyse the different model space regions. In subsection 2.1 we discuss the use of the formalism of rigged Hilbert space to construct the generalized eigenfunctions and the corresponding spectrum of the model. In subsection 2.2 we present our results for each region of the model parameter space. In section 3 we formalize the calculation of mean values of observables and their time evolution. Conclusions are drawn in section 4.

2. Formalism

Let us start with the Hamiltonian of the squeezed harmonic oscillator [61–63]. It is given by

$$H = \hbar \omega \left(a^\dagger a + \frac{1}{2}\right) + \hbar \alpha (a^2 + a^\dagger 2).$$  (1)

In equation (1) $\hbar \omega$ represents the energy scale of the harmonic oscillator sector of the Hamiltonian, while $\hbar \alpha$ is the coupling constant of the quadratic interaction. Both $\omega$ and $\alpha$ take real values. As it is well-known [61–63], the relevance of the Hamiltonian of equation (1) is related to the study of the Heisenberg uncertainty relations for the momentum and position operators. The eigenvalues and eigenfunctions of the Hamiltonian of equation (1), for $|\alpha| < |\omega|/2$, can...
be obtained analytically, and the lowest eigenstate is a squeezed state, that is a state which minimizes the variance of the momentum operator $\hat{p}$ by increasing the variance of the coordinate operator $\hat{x}$, or vice versa.

In this work, we shall consider the Hamiltonian proposed in [31], which is a non-Hermitian generalization of the Hamiltonian of equation (1). It reads

$$H(\omega, \alpha, \beta) = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right) + \hbar \alpha a^2 + \hbar \beta a^4. \quad (2)$$

In equation (2) the energy scale is fixed by the real energy $\hbar \omega$, where $\omega$ can be interpreted as the frequency of a harmonic oscillator of mass $m_0$ and restoring parameter $k_0$, $\omega = (k_0/m_0)^{1/2}$. The quantities $\hbar \alpha$ and $\hbar \beta$ stand for the coupling strengths of Swanson model, $\alpha, \beta \in \mathbb{R}$. We can write $a$ and $a^\dagger$ in terms of the coordinate operator, $\hat{x}$, and of the momentum operator, $\hat{p}$:

$$a = \frac{1}{\sqrt{2}} \left( \frac{\hat{x}}{b_0} + \frac{\hbar}{\hbar} \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left( \frac{\hat{x}}{b_0} - \frac{\hbar}{\hbar} \right), \quad (3)$$

where, if we assume that the harmonic oscillator sector of the Hamiltonian of equation (1) describes the dynamics of a particle of mass $m_0$, the real parameter $b_0 = (\hbar/(m_0 \omega))^{1/2}$ represents the characteristic length of the system. Thus, the Hamiltonian in (2) reads

$$H(\omega, \alpha, \beta) = \frac{1}{2} \hbar \omega \left( \frac{\hat{x}^2}{b_0} + \frac{b_0 \hat{p}^2}{\hbar} \right) + \frac{1}{2} \hbar (\alpha + \beta) \left( \frac{\hat{x}^2}{b_0} - \frac{b_0 \hat{p}^2}{\hbar} \right)
+ \frac{1}{2} \hbar (\alpha - \beta) \left( 2 \frac{1}{\hbar} \hat{p}^2 + 1 \right). \quad (4)$$

To verify that the Hamiltonian of equation (4) is invariant under PT-symmetry, let us review the transformation properties of the position operator, $\hat{x}$, and momentum, $\hat{p}$. Under parity transformation, $\hat{x}$ and $\hat{p}$ transform as $\mathcal{P} \hat{x} \mathcal{P}^{-1} = -\hat{x}$, and $\mathcal{P} \hat{p} \mathcal{P}^{-1} = -\hat{p}$. Whereas, under time reversal the operators are transformed as $\mathcal{T} \hat{x} \mathcal{T}^{-1} = \hat{x}$, and $\mathcal{T} \hat{p} \mathcal{T}^{-1} = -\hat{p}$, and, as $\mathcal{T}$ is an anti-unitary operator, a complex number $\alpha$ is transformed as $\mathcal{T} \alpha \mathcal{T}^{-1} = -\alpha^*$. So that $\mathcal{PT} H(\omega, \alpha, \beta) \mathcal{T}^{-1} \mathcal{P}^{-1} = H(\omega, \alpha, \beta)$.

On the other hand, the Hermitian conjugate operator $H_\dagger$, is given by $H_\dagger(\omega, \alpha, \beta) = H(\omega, \beta, \alpha)$.

Let us introduce, for $\omega \neq \alpha + \beta$, a new set of complementary operators $\hat{P}$ and $\hat{X}$, namely:

$$\hat{P} = \left( \hat{p} + i \hbar \frac{\alpha - \beta}{(\omega - \alpha - \beta) b_0^2} \hat{x} \right), \quad \hat{X} = \hat{x}, \quad (5)$$

$\hat{P}$ and $\hat{X}$ obey the usual commutation relation $[\hat{X}, \hat{P}] = i \hbar$.

In terms of $\hat{P}$ and $\hat{X}$, the Hamiltonian of equation (4) can be written as

$$H = \frac{1}{2m} \hat{P}^2 + \frac{k}{2} \hat{X}^2, \quad (6)$$

with $k = m \Omega^2$ and

$$\Omega = \Omega(\omega, \alpha, \beta) = \sqrt{\omega^2 - 4 \alpha \beta} = |\Omega| e^{\kappa}, \quad (7)$$
The spectrum and the eigenfunctions of $H$ depend on the sign of the functions $m(\omega, \alpha, \beta, b_0)$, $\Omega^2(\omega, \alpha, \beta, b_0)$.

There are four possible regions in the parameter model space. If $m > 0$ and $\Omega^2 > 0$ the Hamiltonian is similar to the usual harmonic oscillator, region I. The case $m > 0$ and $\Omega^2 < 0$ corresponds to a parabolic barrier, region II \cite{64–67}. While the case $m < 0$ and $\Omega^2 > 0$ can be interpreted as a harmonic oscillator with negative mass, region III \cite{68–73}. Finally, the case $m < 0$ and $\Omega^2 < 0$ can be interpreted as a parabolic barrier for a system with negative mass, region IV. The case $k(\omega, \alpha, \beta, b_0) = 0$ corresponds to a free particle. It can be observed that $\Omega^2$ is a continuous function, while $m$ has a discontinuity at the plane $\alpha/\omega + \beta/\omega - 1 = 0$.

The different regions are displayed in figure 1 in terms of the adimensional coupling constants $\alpha/\omega$ and $\beta/\omega$.

2.1. Gel'fand triplet

In general, neither $H$ nor $H_c$ has eigenfunctions in $L^2(\mathbb{R})$. To overcome this problem, and to be able to compute the mean value of observables, we shall use the Gel'fand triplet \cite{74, 75}. Let us briefly review the essentials of the formalism.

To describe a quantum system, we need a Hausdorff vector space with a convex topology and a scalar product, $(\Psi, \tau)$. The Hilbert space with the topology $\tau_H$, $(\mathcal{H}, \tau_H)$, is the completion of $(\Psi, \tau)$. Let us define another completion of $(\Psi, \tau)$, with a finer topology $\tau_{\Phi}$, $(\Phi, \tau_{\Phi})$, so that $\Phi \subset \mathcal{H} \Rightarrow \Phi^* \subset \Phi^*$. Here, $\Phi^*$ is the dual space of $\Phi$, $\Phi^* = \{ F | F : \Phi \rightarrow \mathbb{C}, F(\psi) = \langle \psi | \phi \rangle \}$. 

![Figure 1](https://via.placeholder.com/150)

Figure 1. Different regions of the parameter space in terms of the sign of $m(\omega, \alpha, \beta, b_0)$ and $\Omega^2(\omega, \alpha, \beta, b_0)$. 

$$m = m(\omega, \alpha, \beta, b_0) = \frac{\hbar}{(\omega - \alpha - \beta)b_0}$$

(8)
v ∈ Φ}. Also, we shall introduce the antidual space of Φ, Φ×, that is the space of the antilinear functionals on Φ, Φ× = \{G|G : Φ → ℂ, G(v) = \langle v|G \rangle, \ v ∈ Φ\}. Along these lines, we obtain the Gel’fand triplet

\[ \Phi \subset \mathcal{H} \subset Φ^×. \] (9)

We shall denote by H× and Hc× the extension of the operators H and Hc on Φ×. The eigenfunction of H× and Hc× are functionals of Φ×.

In Φ×, the stationary Schrödinger equation for H× and Hc× can be written as

\[ H×\tilde{φ}(x) = E\tilde{φ}(x), \quad \text{and} \quad Hc×\overline{ψ}(x) = E\overline{ψ}(x). \] (10)

For ω − α − β ≠ 0 we shall introduce the Hermitian invertible linear operator [76]

\[ Υ = e^{αω + β}, \] (11)

so that

\[ ΥH×Υ^{-1} = h×, \quad Υ^{-1}Hc×Υ = h×, \] (12)

with

\[ h×φ(x) = -\frac{\hbar^2}{2m} \frac{d^2φ(x)}{dx^2} + \frac{1}{2}kx^2φ(x) = Eφ(x). \] (13)

The eigenfunctions of H× and Hc× are related to those of h× by \( \tilde{φ}(x) = Υ^{-1}φ(x) \) and \( \overline{ψ}(x) = Υ\overline{φ}(x) \), respectively.

Equation (13) can be interpreted as the Schrödinger equations corresponding to a potential of the form

\[ u(x) = \frac{1}{2}kx^2 = \frac{1}{2}mΩ^2x^2. \] (14)

From equation (12), it can be observed that H× is a pseudo-Hermitian Hamiltonian, namely \( UH× = Hc×U \), with \( U = Υ^2 \) Hermitian and invertible [76–78]. Thus, we can introduce a new inner product, \( \langle .|.| \rangle_U \), in terms of the positive define operator U

\[ \langle .|.| \rangle_U : Φ× × Φ× → ℂ, \quad \langle \tilde{φ}_\nu|\tilde{φ}_\nu' \rangle_U = \langle φ_\nu|U\tilde{φ}_\nu' \rangle. \] (15)

It can be observed that the set \( \{\overline{ψ}_\nu,\tilde{φ}_\nu\} \) is bi-orthogonal:

\[ \langle \tilde{φ}_\nu|\overline{ψ}_\nu' \rangle_U = \langle \overline{ψ}_\nu|\tilde{φ}_\nu' \rangle = \langle φ_\nu|φ_\nu' \rangle = δ_{\nuν'}, \] (16)

and the identity operator can be written as

\[ 1 = \sum_\nu \langle \tilde{φ}_\nu|\overline{ψ}_\nu \rangle. \] (17)

In this scheme, given a pseudo-Hermitian operator [79–81]

\[ \hat{Q} = Υ^{-1}\hat{q}Υ, \] (18)
spectrumin the different regions. Consequently, the spectrum and the eigenfunctions are given by

\[ E_n = E_{cn} = \hbar \sqrt{\omega^2 - 4\alpha \beta} \left( n + \frac{1}{2} \right), \]
\[ \tilde{\psi}_n(x) = e^{-\omega x^2/2 \sigma_0} \sqrt{2^{-n} \sigma \frac{\lambda}{2 \sigma_0}^n} \frac{1}{\sqrt{\pi n!}} \frac{1}{b_0} H_n \left( \frac{x}{b_0} \right), \]
\[ \overline{\psi}_n(x) = e^{-\omega x^2/2 \sigma_0} \sqrt{2^{-n} \sigma \frac{\lambda}{2 \sigma_0}^n} \frac{1}{\sqrt{\pi n!}} \frac{1}{b_0} H_n \left( \frac{x}{b_0} \right), \]

with \( \sigma = \sqrt{\frac{m(\Omega)}{\hbar}} b_0, \Omega = \sqrt{\omega^2 - 4\alpha \beta}, \) and \( H_n(x) \) the Hermite polynomial. Notice that we can interpret \( b_0/\sigma \) as the modified characteristic length due to the interaction system.

In terms of the eigenfunctions of \( H^+ \) and \( H^- \), the bi-orthogonality relation is given by

\[ \int_{-\infty}^{\infty} \psi_m(x) \tilde{\psi}_n(x) dx = \delta_{mn} \]

and the completeness relation is

\[ \sum_n \psi_n(x) \tilde{\psi}_n(x') = \delta(x - x'). \]

2.2.2. Region III. In region III, the eigenvalues take real values \( (\Omega^2 > 0) \) and the effective mass \( m \) is negative. As shown in appendix A, the problem of a harmonic oscillator with effective negative mass can be seen as the problem of a positive mass with inverted spectrum.

We can define \( \sigma = \sqrt{\frac{|m|\Omega}{\hbar}} b_0 \in \mathbb{R}, \) as in the previous case. So that

\[ E_n = E_{cn} = -\hbar \sqrt{\omega^2 - 4\alpha \beta} \left( n + \frac{1}{2} \right), \]
\[ \tilde{\phi}_n(x) = e^{-\omega x^2/2 \sigma_0} \sqrt{2^{-n} \sigma \frac{\lambda}{2 \sigma_0}^n} \frac{1}{\sqrt{\pi n!}} \frac{1}{b_0} H_n \left( \frac{x}{b_0} \right), \]
\[ \overline{\psi}_n(x) = e^{-\omega x^2/2 \sigma_0} \sqrt{2^{-n} \sigma \frac{\lambda}{2 \sigma_0}^n} \frac{1}{\sqrt{\pi n!}} \frac{1}{b_0} H_n \left( \frac{x}{b_0} \right), \]

and the bi-orthogonality and completeness relations are the same as in region I.

The fact that the spectrum of the particle with effective negative mass is inverted respect to that of the particle with positive mass, implies that the energy is lowered by increasing the
amplitude of the oscillations. One of the first proposals to take advantage of this instability was introduced by Glauber [68]. In [68] Glauber presented a theoretical quantum amplifier by coupling a harmonic oscillator with negative mass to a thermal bath at zero temperature. Recently, the authors of [73] reported the experimental realization of the negative effective mass in an atomic gas driven by optical cavity backaction, showing the coherent amplification of a correlated mode due to the negative-mass instability. The negative mass oscillator was achieved by the collective spin of an atomic gas of $^{87}$Ru polarized opposite to an external magnetic field.

2.2.3. Region II. In this region, $m > 0$ and $\Omega^2 < 0$. The potential $u(x)$ of equation (14) corresponds to that of a parabolic barrier [64–67]. Both Hamiltonians, $H^x$ and $H^c$, display continuous spectrum as well as discrete complex conjugate pairs of eigenvalues.

In appendix A we present the construction of generalized eigenfunctions and the corresponding eigenvalues. Let us summarize the results as follows.

The generalized eigenfunctions of $H^x$, $\tilde{\phi}^\pm_n(x)$, with eigenvalues

$$E^\pm_n = \pm i\hbar|\Omega| \left( n + \frac{1}{2} \right)$$

are given by

$$\tilde{\phi}^\pm_n(x) = e^{\frac{\alpha - \beta}{2} \omega - \frac{1}{2} x^2} \phi^\pm_n(x), \quad (24)$$

while, the generalized eigenfunctions of $H^c$, $\psi^\pm_n(x)$, with eigenvalues

$$E^\pm_{cn} = \mp i\hbar|\Omega| \left( n + \frac{1}{2} \right)$$

are given by

$$\psi^\pm_n(x) = e^{\frac{\alpha - \beta}{2} \omega - \frac{1}{2} x^2} \phi^\mp_n(x). \quad (25)$$

Being

$$\phi^\pm_n(x) = \frac{\sqrt{i\sigma}}{b_0} \frac{1}{2\pi n!} e^{-\frac{i\sigma^2}{2b_0^2}} H_n \left( \frac{\sqrt{i\sigma}}{b_0} x \right), \quad \phi^\mp_n(x) = \phi^\pm_n(x)^*, \quad (26)$$

with $\sigma = \sqrt{\frac{\text{Im}\Omega}{\hbar} b_0}$.

The bi-orthogonality relation reads

$$\int_{-\infty}^{\infty} \overline{\psi^s_n(x)} \tilde{\phi}^s_m(x) = \delta_{nm}, \quad (27)$$

and the completeness relation is given by

$$\sum_{n=0}^{\infty} \overline{\psi^s_n(x)} \phi^s_n(x') = \delta(x - x'). \quad (28)$$
The generalized eigenfunctions associated to the continuous spectrum, \( E \in (-\infty, +\infty) \), are given by

\[
\tilde{\phi}_\pm^E(x) = e^{\frac{\alpha - \beta}{2\hbar_0}} \phi_\pm^E(x), \quad \tilde{\psi}_\pm^E(x) = e^{\frac{-\alpha - \beta}{2\hbar_0}} \phi_\pm^E(x),
\]

with \( \nu = \frac{i}{\hbar_0} \frac{E}{|\Omega|} - \frac{1}{2} \) and

\[
\phi_\pm^E(x) = C \Gamma(\nu + 1)D_{-\nu - 1} \left( \mp \sqrt{2i\hbar_0} \frac{x}{b_0} \right).
\]

where \( D_{-\nu - 1}(w) \) are the parabolic cylinder functions and the normalization constant takes the value

\[
C = \frac{\pi^{\nu/2}}{(2\pi)^{1/4}}.
\]

The bi-orthogonality and the completeness relation can be written as

\[
\int_{-\infty}^{\infty} (\bar{\tilde{\psi}}_\pm^E(x))^* \phi_\pm^E(x) dx = \delta(E - E'),
\]

\[
\sum_{x, x'} \int_{-\infty}^{\infty} (\bar{\tilde{\psi}}_\pm^E(x))^* \phi_\pm^E(x) dE = \delta(x - x').
\]

To complete the analysis of this region we have to discuss the analytical properties of the previous solutions.

It is easy to see that the poles of \( \phi_\pm^E(x) \) are those of \( \Gamma(\nu + 1) \), that is: \( -n = \nu + 1 \), with \( n \in \mathbb{N} \) and \( E_n = \pm \hbar|\Omega| (n + \frac{1}{2}) \).

In the other hand, the poles of \( (\phi_\pm^E(x))^* \) are those of \( \Gamma(-\nu) \), that is: \( n = \nu \), with \( n \in \mathbb{N} \) and \( E_n = \pm \hbar|\Omega| (n + \frac{1}{2}) \). As shown in appendix A, we shall introduce \( \eta_\pm^E(x) = (\phi_\pm^E(x))^* \), which are eigenfunctions of \( h^* \) obtained by replacing \( E \leftrightarrow -E \).

In the coordinate representation, we have \( \phi_\pm^E(x) = (x|\phi_\pm^E) \) and \( \eta_\pm^E(x) = (x|\eta_\pm^E) \). Consequently:

\[
|x\rangle = C \sum_{x, x'} \int_{-\infty}^{\infty} (\eta_\pm^E(x) \langle \phi_\pm^E | + \phi_\pm^E(x) \langle \eta_\pm^E |) dE,
\]

so that \( \langle x|x'\rangle = \delta(x - x') \).

A function \( \xi(x) = \langle \zeta|x\rangle \), can be written as

\[
\xi(x) = \xi^+(x) + \xi^-(x),
\]

\[
\xi^+(x) = C \sum_{x, x'} \int_{-\infty}^{\infty} \eta_\pm^E(x) \langle \zeta|\phi_\pm^E \rangle dE, \quad \xi^-(x) = C \sum_{x, x'} \int_{-\infty}^{\infty} \phi_\pm^E(x) \langle \zeta|\eta_\pm^E \rangle dE.
\]
Let us take $\Phi$ of equation (9) as the space of Hardy class function [83, 84]. We shall define $\mathcal{H}_+$ as the Hardy class functions in the upper half-plane $\mathbb{C}^+$, that is the set of the analytic functions, $f(z)$, in $\mathbb{C}^+$ so that

$$\int_{-\infty}^{\infty} |f(x + iy)|^2 \, dx < \infty.$$ 

In the same way, $\mathcal{H}_-$ is the set of the Hardy class functions in the lower half-plane, $\mathbb{C}^-$. It should be noticed that an $H_\pm$ function is completely determined by its value on $\mathbb{R}$. We shall define

$$\Xi_- = \{ \xi \in \Phi | f(E) = \langle \xi | \eta^{E}_- \rangle \in \mathcal{H}_- \},$$

$$\Xi_+ = \{ \xi \in \Phi | f(E) = \langle \xi | \phi^{E}_+ \rangle \in \mathcal{H}_+ \}. \tag{34}$$

Following the lines of [65, 66], we can expand functions $\xi^-(x) \in \Xi_-$ and $\xi^+(x) \in \Xi^+$ as

$$\xi^-(x) = \sum_{\pm} \int_{-\infty}^{\infty} dE \phi^{E}_\pm(x) \langle \phi^{E}_\pm | \xi^- \rangle = \sum_{n=0}^{\infty} \phi^+_n(x) \langle \phi^+_n | \xi^- \rangle,$$

$$\xi^+(x) = \sum_{\pm} \int_{-\infty}^{\infty} dE \eta^{E}_\pm(x) \langle \eta^{E}_\pm | \xi^+ \rangle = \sum_{n=0}^{\infty} \phi^-_n(x) \langle \phi^-_n | \xi^+ \rangle, \tag{35}$$

respectively. In the same manner, we can construct the following spectral resolution for $h^\times$:

$$h^\times = \sum_{\pm} \int_{-\infty}^{\infty} dE \langle \phi^{E}_\pm | \xi^\times \rangle = \sum_{n=0}^{\infty} E^-_n \langle \xi^-_n | \phi^+_n \rangle,$$ 

on $\Xi_-$, and

$$h^\times = \sum_{\pm} \int_{-\infty}^{\infty} dE \langle \eta^{E}_\pm | \xi^\times \rangle = \sum_{n=0}^{\infty} E^+_n \langle \xi^+_n | \phi^-_n \rangle, \tag{36}$$

on $\Xi_+$. Consequently, $H^\times = \Upsilon^{-1} h^\times \Upsilon$ and $H^\times = \Upsilon h^\times \Upsilon^{-1}$:

$$H^\times = \sum_{\pm} \int_{-\infty}^{\infty} dE \left( \langle \phi^{E}_\pm | \bar{\phi}^{E}_\pm \rangle | E^-_n \rangle + \langle \eta^{E}_\pm | \bar{\eta}^{E}_\pm \rangle | E^+_n \rangle \right)$$

$$= \sum_{n=0}^{\infty} \left( E^-_n \langle \bar{\phi}^-_n | \phi^-_n \rangle + E^+_n \langle \phi^+_n | \bar{\phi}^+_n \rangle \right), \tag{38}$$

and

$$H^\times = \sum_{\pm} \int_{-\infty}^{\infty} dE \left( \langle \phi^{E}_\pm | \bar{\phi}^{E}_\pm \rangle | E^-_n \rangle + \langle \eta^{E}_\pm | \bar{\eta}^{E}_\pm \rangle | E^+_n \rangle \right)$$

$$= \sum_{n=0}^{\infty} \left( E^-_n \langle \bar{\phi}^-_n | \phi^-_n \rangle + E^+_n \langle \phi^+_n | \bar{\phi}^+_n \rangle \right). \tag{39}$$
2.2.4. Region IV. Region IV, the effective mass takes negative values, \( m < 0 \), as well as \( \Omega^2 < 0 \). The results are similar to the ones of region II by replacing \( E \leftrightarrow -E \), as proved in appendix A.

2.2.5. Boundary I-II and III-IV. In both boundaries, I-II and III-IV, \( \Omega \) takes the value \( \Omega = 0 \). When \( \Omega = 0 \) and \( \omega - (\alpha + \beta) \neq 0 \), the problem reduces to that of a free particle of energy \( E \). As shown in appendix A, the generalized eigenfunctions can be written as

\[
\tilde{\phi}(x) = (A e^{i k x} + B e^{-i k x}) e^{\frac{-\Omega}{2} \frac{x^2}{b_0^2}},
\]

\[
\tilde{\psi}(x) = (A e^{i k x} + B e^{-i k x}) e^{\frac{-\Omega}{2} \frac{x^2}{b_0^2}},
\]

with \( k = \sqrt{\frac{2 E}{\hbar (\omega - \alpha - \beta)}} \).

2.2.6. Exceptional points. As pointed out in [52], at the boundary I-II and III-IV we observe the presence of EPs. At these points, the discrete eigenvalues and the eigenfunctions of region I and II, and of region III and IV, are coalescent. At each EP the energy of the state converges to \( E = 0 \) for all values of \( n \), and from both sides of the boundary the eigenfunctions converge to

\[
\tilde{\phi}(x) = (c_1 x + c_0) e^{\frac{\alpha + \beta}{2} \frac{x^2}{b_0^2}}, \quad \tilde{\psi}(x) = (d_1 x + d_0) e^{\frac{\beta - \alpha}{2} \frac{x^2}{b_0^2}},
\]

\( E = 0 \).

Thus, at the boundary of regions I-II and III-IV, the spectrum consists of EPs of infinite order [85–88], with \( E = 0 \), which resides within the continuum spectrum [89]. The details are given in appendix C.

2.2.7. Boundary I-III. To study the boundary between regions I and III, we have to look at the Hamiltonian of equation (4). If \( \omega - (\alpha + \beta) = 0 \), it reads

\[
H^\times(\theta) = \hbar (\alpha + \beta) \left( \frac{\hat{x}}{b_0} \right)^2 + \hbar \frac{(\alpha - \beta)}{2} \left( 2 \frac{\hat{x}}{\hbar} \frac{\hat{p}}{\hbar} + 1 \right),
\]

and its adjoint is given by

\[
H_\times^*(\theta) = \hbar (\alpha + \beta) \left( \frac{x}{b_0} \right)^2 + \hbar \frac{(\beta - \alpha)}{2} \left( 2 \frac{\hat{x}}{\hbar} \frac{\hat{p}}{\hbar} + 1 \right).
\]

We shall introduce a new similarity transformation by defining the operator

\[
\tau = e^{\alpha + \beta \frac{x^2}{2b_0^2}},
\]

it results

\[
\tau H^\times \tau^{-1} = \hbar^\times, \quad \tau^{-1} H_\times^* \tau = -\hbar^\times.
\]
with
\begin{equation}
\hbar^* \phi(x) = \frac{\hbar(\alpha - \beta)}{2} \left( 2x \frac{d}{dx} + 1 \right) \phi(x) = E\phi(x). \tag{46}
\end{equation}

It is straightforward to see that $H^\times$ and $H^c_\times$ are anti-pseudo-Hermitian \cite{90} at the boundary, that is $H^\times = -S^{-1}H^c_\times S$, with $S = \tau^2$.

The spectrum of Hamiltonian of equation (46) is real. The generalized eigenfunctions of $H^\times$ and $H^c_\times$ are given by
\begin{align*}
\tilde{\phi}_n^+(x) &= \frac{1}{\sqrt{n!}} e^{-\frac{\alpha + \beta}{2}x^2} X^n, \quad \tilde{E}_n^+ = E_n, \\
\tilde{\phi}_n^-(x) &= \frac{(-1)^n}{\sqrt{n!}} e^{-\frac{\alpha - \beta}{2}x^2} \delta^{(n)}(x), \quad \tilde{E}_n^- = -E_n. \tag{47}
\end{align*}

While for $H^c_\times$ the corresponding generalized eigenfunctions are:
\begin{align*}
\tilde{\psi}_n^+(x) &= \frac{1}{\sqrt{n!}} e^{-\frac{\alpha - \beta}{2}x^2} X^n, \quad \tilde{E}_n^+ = -E_n \\
\tilde{\psi}_n^-(x) &= \frac{(-1)^n}{\sqrt{n!}} e^{-\frac{\alpha + \beta}{2}x^2} \delta^{(n)}(x), \quad \tilde{E}_n^- = E_n. \tag{48}
\end{align*}

with $E_n = \hbar(\alpha - \beta) \left( n + \frac{1}{2} \right)$.

Thus, the bi-orthogonality relations are given by
\begin{equation}
\int_{-\infty}^{\infty} \tilde{\psi}_m^+(x) \tilde{\phi}_n^-(x) dx = \delta_{mn}. \tag{46}
\end{equation}

The details are presented in appendix A.

The eigenfunctions with positive spectrum of the boundary between regions I-III can be obtained by a limit procedure \cite{85, 86} from the eigenfunctions of region I. In the same form, the eigenfunctions corresponding to negative values of the spectrum can be obtained from the eigenfunctions of region III. The details are presented in appendix B.

3. Time evolution of observables

Let us discuss in first place regions I and III. In these cases, the spectrum of $H^\times$ is real and discrete.

We shall compute the mean values of the pseudo-Hermitian operators associated to $\hat{p}$ and $\hat{x}$. According to equation (18), they read
\begin{align*}
\hat{P} &= \Upsilon^{-1} \hat{p} \Upsilon = \hat{p} + i\hbar \frac{\alpha - \beta}{(\omega - \alpha - \beta)\hbar_0} \hat{\alpha}, \\
\hat{X} &= \Upsilon^{-1} \hat{x} \Upsilon = \hat{x}. \tag{49}
\end{align*}
It is not difficult to prove that the mean value of the operators $\hat{X}$ and $\hat{P}$ of equation (49) between different states, $\langle \tilde{\phi}_n \rangle$, obeys

$$
\langle \tilde{\phi}_m | \hat{U} | \tilde{\phi}_n \rangle = \int_{-\infty}^{\infty} \tilde{\psi}^*_n(x) \hat{X} \tilde{\phi}_n(x) \, dx = \frac{\hbar b_0}{\sqrt{2}} \left( \sqrt{\nu + 1} \delta_{m,n+1} + \sqrt{\nu} \delta_{m,n-1} \right),
$$

$$
\langle \tilde{\phi}_m | \hat{P} | \tilde{\phi}_n \rangle = \int_{-\infty}^{\infty} \tilde{\psi}^*_n(x) \hat{P} \tilde{\phi}_n(x) \, dx = \frac{\hbar}{\sqrt{2 \hbar b_0}} \left( \sqrt{\nu + 1} \delta_{m,n+1} - \sqrt{\nu} \delta_{m,n-1} \right),
$$

$$
\langle \tilde{\phi}_m | \hat{P}^2 | \tilde{\phi}_n \rangle = \int_{-\infty}^{\infty} \tilde{\psi}^*_n(x) \hat{P}^2 \tilde{\phi}_n(x) \, dx = -\frac{\hbar^2}{\hbar b_0} \left( \sqrt{(n+2)(n+1)} \delta_{m,n+2} - (2n+1) \delta_{m,n} + \sqrt{n(n-1)} \delta_{m,n-2} \right),
$$

with $b_0 = b_0/\sigma$.

The time evolution of a given initial state $|I(0)\rangle$, $|I(0)\rangle = \sum_n \phi_n |\tilde{\phi}_n\rangle$, such that $\langle I(0) | I(0) \rangle_U = 1$, is given by

$$
\langle I(t) | \tilde{O} | I(t) \rangle_U = \langle I(0) | e^{-i\hat{H}t} U \tilde{O} e^{-i\hat{H}t} | I(0) \rangle = \sum_{nm} \phi^*_n \phi_m e^{-i\hbar \Omega_{mn} t} \langle \phi_n | \tilde{O} \hat{\phi}_m \rangle \langle \tilde{\phi}_m | \tilde{\phi}_n \rangle.
$$

In regions II and IV, the spectrum of $\hat{H}^+$ consists of real continuous eigenvalues and discrete complex conjugate pairs of eigenvalues, the resonant and anti-resonant states. Resonant states are eigenstates which decay (grow) exponentially in time for $t > 0$ $(t < 0)$, while anti-resonant states are eigenstates which decay (grow) exponentially for $t < 0$ $(t > 0)$. As pointed before, resonant (anti-resonant) states of the system with $m > 0$ behaves as anti-resonant (resonant) states for the system of $m < 0$.

If $U(t) = e^{-i\hat{H}t}$ is the operator for the time evolution in $\mathcal{H}$, $U(t)^* = e^{i\hat{H}t}$ is the operator for the time evolution due to $\hat{H}$ in $\Phi^*$ [66], and $U_c(t)$ is $e^{i\hat{H}t}$ for $H^c$, respectively. Thus $U(t)^* |\tilde{\phi}_n\rangle = e^{\pm \hbar \Omega(n+\frac{1}{2}) \delta_{m,n}} |\tilde{\phi}_n\rangle$, and $U_c(t)^* |\tilde{\phi}_n\rangle = e^{\pm \hbar \Omega(n+\frac{1}{2}) \delta_{m,n}} |\tilde{\phi}_m\rangle$.

Let us adopt a general initial state, $|I\rangle$, and use the identity

$$
1 = \sum_n (\langle \tilde{\phi}^+_n | \tilde{\phi}^+_n \rangle + | \tilde{\phi}^-_n \rangle \langle \tilde{\phi}^-_n |) = \sum_n (\langle \tilde{\psi}^+_n | \tilde{\psi}^+_n \rangle + | \tilde{\psi}^-_n \rangle \langle \tilde{\psi}^-_n |)
$$

Then, the initial state can be split as

$$
|\tilde{I}\rangle = |\tilde{I}^+\rangle + |\tilde{I}^-\rangle,
$$

$$
|\tilde{I}^-\rangle = \sum_n |\tilde{\phi}^-_n \rangle \langle \tilde{\psi}^-_n | I\rangle,
$$

$$
|\tilde{I}^+\rangle = \sum_n |\tilde{\phi}^+_n \rangle \langle \tilde{\psi}^+_n | I\rangle,
$$

(53)
consequently

\[ \tilde{I}(t) = \tilde{I}^+(t) + \tilde{I}^-(t), \]

\[ \tilde{I}^+(t) = \sum_{n} e^{i/(\hbar)(n+1/2)} (\bar{\varphi}_n)(\bar{\varphi}_n^+), \quad t < 0, \]

\[ \tilde{I}^-(t) = \sum_{n} e^{-i/(\hbar)(n+1/2)} (\bar{\varphi}_n)(\bar{\varphi}_n^+), \quad t \geq 0. \quad (54) \]

This result is consistent with extending the contour of the integrals of equation (35) to the lower and upper plane, respectively [66].

The survival probability can be expressed as

\[ \tilde{I}^\pm(t)_{U} = \langle \tilde{I}^\pm(0)|\tilde{I}^\pm(t) \rangle, \quad (55) \]

while the mean value, as a function of the time, of a given operator \( \hat{O} \) reads

\[ \langle \tilde{I}^\pm(t)|\hat{O}|\tilde{I}^\pm(t) \rangle_{U} = \langle \tilde{I}^\pm(t)|\hat{O}|\tilde{I}^\pm(t) \rangle. \quad (56) \]

Given a particular problem [91–94], we may deform the contour of integration of equation (35) in a different way so to include some poles of both the upper and the lower plane [25]. An alternative approach for the study of the time evolution is given in [25] by using the Green function formalism.

4. Conclusions

In this work, we have studied the non-Hermitian Swanson Hamiltonian, both in the PT-symmetry phase and the non-PT-symmetry phase. We have proved that the Hamiltonian is pseudo-Hermitian by introducing a Hermitian invertible similarity transformation. As a result, we have mapped the Swanson model to different physical systems depending on the adopted values for the parameters \( \alpha/\omega \) and \( \beta/\omega \). We have classified regions and their boundaries. We have shown that region I corresponds to the usual harmonic oscillator, region III to a harmonic oscillator with negative mass, region II represents a parabolic barrier, and region IV a parabolic barrier for a particle with negative mass. We have used the formalism of the Gel’fand triplet to construct the generalized eigenfunctions and the corresponding spectrum in each region. We have shown that it is possible to construct metric operators in the rigged Hilbert space. Also, we have proved that we can establish a bi-orthogonality relation among the generalized functions of \( H^\times \) and \( H^\times_c \). In the same line, we have formalized the computation of the mean value of observables and the time evolution of the system in the different regions of the model space. Also, we have verified that the boundary between the regions I-II and III-IV is formed by exceptional points of infinite order embedded in a continuum spectrum [85, 86]. An interesting feature results from the study of the boundary between the regions I and III, \( H^\times \) and of \( H^\times_c \) are anti-pseudo-Hermitian [90].

Work is in progress concerning the computation of the evolution of different initial states as a function of time for different regions of the model space. Particularly, in the boundary between regions I-III where \( H^\times \) and \( H^\times_c \) are anti-pseudo-Hermitian [90], and between regions I-II and III-IV, where the continuum spectrum includes the presence of exceptional points with energy \( E = 0 \) [23, 24].
Acknowledgments

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A

We shall follow the works of [64–66] to construct the generalized eigenfunctions in the rigged Hilbert space. Let us briefly review the essentials of the procedure.

After performing the similarity transformations of equation (45), for \( \omega - \alpha - \beta \neq 0 \), the eigenvalue problem for both Hamiltonians, \( H^x \) and its adjoint \( H^x_c \), can be reduce to find the spectrum and the generalized eigenfunctions of

\[
\mathcal{H}_x \phi(x) = -\frac{\hbar^2}{2m} \frac{d^2 \phi(x)}{dx^2} + \frac{1}{2} m\Omega^2 x^2 \phi(x) = E \phi(x). \tag{A.1}
\]

**A.1. Regions I and III**

In regions I and III, the parameter \( \Omega \) takes real values, \( \Omega^2 > 0 \). In region I, the parameter \( m \) takes positive values, \( m = |m| \), while in region III we have \( m = -|m| \). Taken this fact into account, we can write the previous equation as

\[
-\frac{\hbar}{2|m|\Omega} \frac{d^2 \phi(x)}{dx^2} + \frac{1}{2} \frac{|m|\Omega}{\hbar} x^2 \phi(x) = \frac{\varepsilon}{\hbar\Omega} \phi(x), \tag{A.2}
\]

with \( \varepsilon = E \) in region I, and \( \varepsilon = -E \) in region III.

For a review on the solutions in different representations the reader is kindly referred to [82].

**A.2. Boundary I-III**

In the boundary between regions I and III, we have to find the generalized eigenfunctions of \( h^x \)

\[
\mathcal{H}_x \phi(x) = \frac{h(\alpha - \beta)}{2} \left( 2x \frac{d}{dx} + 1 \right) \phi(x) = E \phi(x). \tag{A.3}
\]

It is straightforward to verify that the solutions are

\[
\phi_n^+ (x) = \frac{x^n}{\sqrt{n!}}, \quad E_n = h(\alpha - \beta) \left( n + \frac{1}{2} \right),
\]

\[
\phi_n^- (x) = (-1)^n \frac{\delta^{(n)}(x)}{\sqrt{n!}}, \quad E_n = -h(\alpha - \beta) \left( n + \frac{1}{2} \right). \tag{A.4}
\]
A.3. Regions II and IV

In regions II and IV, we have $\Omega^2 < 0$. Furthermore, in region II $m = |m|$, while in region IV we have $m = -|m|$. Taken this fact into account, we can write the previous equation as

$$\frac{\hbar}{2|m|\Omega} \frac{d^2\phi(x)}{dx^2} + \frac{1}{2} \frac{|m|\Omega}{\hbar} x^2 \phi(x) = \frac{\varepsilon}{\hbar\Omega} \phi(x), \quad (A.5)$$

with $\varepsilon = E$ in region II, and $\varepsilon = -E$ in region IV.

Let us introduce the parameter $\sigma = \sqrt{|m|\Omega}/\hbar$, with $\Omega = \pm i|m|$. We shall take $z = e^{i\pi/4} |\sigma| x$, and define the new set of operators

$$\hat{u}^\dagger = \frac{\hat{z} - i\hat{p}_x}{\sqrt{2}}, \quad \hat{v} = \frac{\hat{z} + i\hat{p}_x}{\sqrt{2}}, \quad (A.6)$$

with $[\hat{v}, \hat{u}^\dagger] = 1$. To study the spectrum of the Hamiltonian of equation (A.5), we shall split it as follows:

$$\frac{i\hbar|\Omega|}{2} \left( \frac{\hat{u}^\dagger \hat{v} + 1}{2} \right) \phi^+ = \varepsilon \phi^+, \quad (A.7)$$
$$\frac{i\hbar|\Omega|}{2} \left( \frac{\hat{v}^\dagger \hat{u} + 1}{2} \right) \phi^- = \varepsilon \phi^- \quad (A.8)$$

We can proceed as before. It results

$$\hat{v} |\phi^+_n\rangle = 0, \quad \varepsilon^+_n = i\frac{\hbar|\Omega|}{2}, \quad |\phi^+_n\rangle = \frac{\hat{u}^\dagger^n}{\sqrt{n!}} |\phi^+_0\rangle, \quad \varepsilon^+_n = i\frac{\hbar|\Omega|}{2} [n], \quad (A.9)$$

and

$$\hat{u} |\phi^-_n\rangle = 0, \quad \varepsilon^-_n = -i\frac{\hbar|\Omega|}{2}, \quad |\phi^-_n\rangle = \frac{\hat{v}^\dagger^n}{\sqrt{n!}} |\phi^-_0\rangle, \quad \varepsilon^-_n = -i\frac{\hbar|\Omega|}{2} [n]. \quad (A.10)$$

It is straightforward to prove that

$$\langle \phi^+_m | \phi^+_n \rangle = \delta_{mn}, \quad (A.11)$$

and

The eigenfunctions in the $z$-representation can be obtained by constructing the generating functions

$$G^+(t, z) = \sum_n \phi^+_n(z) e^{\frac{t}{\sqrt{n!}}} = \langle z | e^{i\hat{u}^\dagger t} | \phi^+_0 \rangle, \quad (A.12)$$
$$G^-(t, z) = \sum_n \phi^-_n(z) e^{\frac{t}{\sqrt{n!}}} = \langle z | e^{i\hat{v}^\dagger t} | \phi^-_0 \rangle.$$
that is

\[ G^+(t, z) = \pi^{-1/4} e^{-\frac{z^2}{4} + 2z \sqrt{t}} \left( \frac{1}{2} \right)^2 \]

\[ G^-(t, z) = G^+(t, z^*) \quad (A.13) \]

Using the generating function for the Hermite polynomials \( H_n(z) \):

\[ e^{2z - z^2} = \sum_n H_n(z) \frac{z^n}{n!} \quad (A.14) \]

we obtain

\[ \phi_n^+(z) = \frac{1}{\sqrt{n! \sqrt{2}}} H_n(z), \quad \phi_n^-(z) = (\phi_n^+(z))^* \quad (A.15) \]

In this representation, the relations of completeness and orthogonality are given by:

\[ \sum_{n=0}^{\infty} ((\phi_n^+(z))^* \phi_n^+(z')) + (\phi_n^+(z))^* \phi_n^-(z')) = \delta(z - z'), \]

\[ \int_{-\infty}^{\infty} (\phi_n^+(z))^* \phi_m^+(z) dz = \delta_{nm} \quad (A.16) \]

In the representation of coordinates

\[ \langle z \rangle = (2\pi)^{1/4} \sum_n \left( \phi_n^+(z) | \phi_n^+ \rangle + \phi_n^-(z) | \phi_n^- \rangle \right) \quad (A.17) \]

with

\[ 1 = \int_{-\infty}^{\infty} |z\rangle \langle z| dz, \langle z' | z \rangle = \delta(z - z') \quad (A.18) \]

The momentum representation can be constructed in the usual form by performing the Fourier transform of \( \phi_n^+(z) \):

\[ F[\phi_n^+(p)] = \int_{-\infty}^{\infty} e^{ipz} \phi_n^+(z) dz = \mathbf{1}^n \phi_n^+(p), \quad (A.19) \]

so that

\[ |p\rangle = (2\pi)^{1/4} \sum_n \left( \phi_n^-(p) | \phi_n^+ \rangle + \phi_n^+(p) | \phi_n^- \rangle \right) \quad (A.20) \]

and

\[ 1 = \int_{-\infty}^{\infty} |p\rangle \langle p| dp, \langle p | p' \rangle = \delta(p - p') \quad (A.21) \]
To deal with the continuous spectrum, the Hamiltonian of equation (A.7) can be written as

\[ u \frac{d\phi(u)}{du} = \nu \phi(u), \tag{A.22} \]

with \( \nu = -i\lambda - \frac{1}{2} \) and \( \lambda = \frac{E}{\hbar|\Omega|} \). The solutions of equation (A.22) on \( \Phi \) are the generalized functions

\[ \phi^\nu(u) = u^\nu, \tag{A.23} \]

with

\[
\begin{align*}
    u^\nu_+ &= \begin{cases} 
    u^\nu & u \geq 0 \\
    0 & u < 0 
    \end{cases} \\
    u^\nu_- &= \begin{cases} 
    0 & u \geq 0 \\
    |u|^\nu & u < 0 
    \end{cases}
\end{align*}
\] (A.24)

To obtain the coordinate representation we shall use the framework of the bilateral Mellin transformation [95], which is a generalization of the expansion in series of Taylor.

The generating function, \( G(t, z) = \pi^{-1/4} e^{-z^2/4} t^{i\lambda - 1/2} e^{-it\lambda}, \) can be written as

\[ G(t, z) = \frac{1}{\sqrt{2\pi}} \sum_s \int_{\mathbb{R}} \phi^s_{\lambda}(z) t^{i\lambda + 1/2} d\lambda. \tag{A.25} \]

By inverting the previous equation and using that [95]

\[ \frac{1}{2\pi} \sum_s \int_{\mathbb{R}} t^{i\lambda + 1/2} e^{-it\lambda} dt = \delta(\lambda - \lambda')\delta_{s/s'}, \tag{A.26} \]

we obtain

\[
\begin{align*}
    \phi^s_{\lambda}(z) &= C \int_0^\infty dt \ t^{i\lambda + 1/2} G(t, z) = C \Gamma \left( -i\lambda + \frac{1}{2} \right) D_{i\lambda - \frac{1}{2}}(-z\sqrt{2}) \\
    \phi^s_{\lambda}(z)' &= C \int_0^\infty dt \ t^{-i\lambda + 1/2} G(-t, z) = C \Gamma \left( -i\lambda + \frac{1}{2} \right) D_{i\lambda - \frac{1}{2}}(\sqrt{2}z). \tag{A.27}
\end{align*}
\]

Moreover [65]

\[
\begin{align*}
    &\int_{-\infty}^{\infty} (\phi^E_{\pm}(z))' \phi^E_{\pm}(z') dz = \delta(E - E') \\
    &\sum_{s = \pm} \int_{-\infty}^{\infty} (\phi^E_{s}(z))' \phi^E_{s}(z') dE = \delta(z - z'). \tag{A.28}
\end{align*}
\]
There is also a set of solutions corresponding to the Hamiltonian of equation (A.7), that is
\[ v \frac{d\phi(v)}{dv} = \lambda \phi(v), \tag{A.29} \]
with \( \lambda = i \frac{\bar{h}}{\hbar} - \frac{1}{2} = \nu' = -(\nu + 1) \). That is, we change \( \nu \to -(\nu + 1) \) and \( E \to -E \), so that \( \hbar|\phi\rangle = E|\phi\rangle \to \hbar|\eta\rangle = -E|\eta\rangle \). Thus the solutions of equation (A.29) can be given in terms of the solutions of equation (A.22) as
\[ \eta^E_\pm(z) = (\phi^E_\pm(z))^*. \tag{A.30} \]
We can express equation (A.28) as
\[ \int_{-\infty}^{\infty} \eta^E_\pm(z) \phi^E_\pm(z) dz = \delta(E - E') \]
\[ \int_{-\infty}^{\infty} \left( \eta^E_+(z) \phi^E_+(z') + \phi^E_+(z) \eta^E_-(z') \right) dE = \delta(z - z'). \tag{A.31} \]
In the coordinate representation, we have \( \phi^E_\pm(z) = \langle z|\phi^E_\pm \rangle \) and \( \eta^E_\pm(z) = \langle z|\eta^E_\pm \rangle = . \) Consequently:
\[ |z\rangle = \frac{1}{2\sqrt{\pi}} \sum_{j=\pm} \left( \eta^E_j(z) \phi^E_j \right) , \tag{A.32} \]
so that \( \langle z|z'\rangle = \delta(z - z') \).

A.4. Boundaries I-II and III-IV
In both boundaries, I-II and III-IV, \( \Omega \) takes the value \( \Omega = 0 \). When \( \Omega = 0 \) and \( \omega - (\alpha + \beta) \neq 0 \), the problem reduces to that of a free particle of energy \( E \), that is
\[ -\frac{\hbar^2}{2m} \frac{d^2\phi(x)}{dx^2} = E \phi(x), \tag{A.33} \]
so that the generalized eigenfunction can be written as \( \phi(x) = A e^{ikx} + B e^{-ikx} \), with \( k = \sqrt{\frac{2E}{\hbar^2} - \omega - \beta \nu_0^2} \).

For the case \( E = 0 \), we have another solution, \( \phi(x) = c_0 + c_1 x \).

Appendix B

The eigenfunctions and the spectrum in the boundary between regions I-III can be obtained in a limit procedure from region I and region III [85, 86]. To see this, consider the Hamiltonian of equation (4) replacing \( \omega - \alpha - \beta \) by \( \varepsilon \):
\[ H^\varepsilon(\varepsilon, \alpha, \beta) = \frac{1}{2} \hbar (\varepsilon + 2(\alpha + \beta)) \left( \frac{\dot{x}}{b_0} \right)^2 + \frac{1}{2} \hbar \varepsilon \left( \frac{\dot{b}_0 \dot{p}}{\hbar} \right)^2 + \hbar \frac{\alpha - \beta}{2} \left( 2 \frac{1}{\hbar} \dot{p} + 1 \right). \tag{B.1} \]
Its generalized eigenfunctions are given by
\[ \tilde{\phi}_{n,\varepsilon}(x) = e^{\frac{x^2}{2\varepsilon}} e^{-\frac{x^2}{2\varepsilon}} H_n \left( \frac{x}{\sqrt{\varepsilon}} \right), \] (B.2)
with \( r(\varepsilon) = \sqrt{(\alpha - \beta)^2 + 2\varepsilon(\alpha + \beta) + \varepsilon^2} \). As shown before, the solutions for \( H^{\pm}(\varepsilon, \alpha, \beta)\tilde{\phi}_{n,\varepsilon}(x) = E_{n,\varepsilon}\phi_n(x) \), with \( E_{n,\varepsilon} = (n + \frac{1}{2}) \cdot r(\varepsilon) \).

Let us further introduce a new parameter \( G \) defined by \( G = \frac{\alpha \varepsilon}{\tau} \). The limit \( \varepsilon \rightarrow 0 \) should be replaced by the limit \( G \rightarrow \infty \). In terms of \( G \):
\[ \varepsilon_{\pm} = \frac{(\alpha + \beta)}{G^2 - 1} \pm \frac{\sqrt{4\alpha\beta + G^2(\alpha - \beta)^2}}{G^2 - 1}. \] (B.3)

For \( \varepsilon = \varepsilon_{+} \), we have
\[ \tilde{\phi}^{(1)}_{n, G}(x) = C \tau^{-1} e^{-\frac{x^2}{2\varepsilon}} \left( \sqrt{1 + \frac{4\alpha\beta}{G^2(\alpha - \beta)^2}} \right) H_n \left( \frac{x}{\sqrt{\varepsilon}} \sqrt{G} \right), \] (B.4)
where \( C = \frac{2^{n}G^{n/2}}{\sqrt{\varepsilon}} \). Notice that, when \( G \rightarrow \infty \), we obtain
\[ \tilde{\phi}^{(1)}_{n, G}(x) \rightarrow \tau^{-1} \frac{x^n}{\sqrt{n!}}. \] (B.5)

Now, we shall take \( \varepsilon = \varepsilon_{-} \). Then \( \tilde{\phi}_{n,\varepsilon}(x) \) can be expressed in terms of \( G \) as
\[ \tilde{\phi}^{(2)}_{n, G}(x) = C \tau^{-1} e^{-\frac{x^2}{2\varepsilon}} \left( \sqrt{1 + \frac{4\alpha\beta}{G^2(\alpha - \beta)^2}} \right) H_n \left( \sqrt{G} x \right), \] (B.6)
we shall call
\[ \phi^{(2)}_{n, G}(x) = \tau \tilde{\phi}^{(2)}_{n, G}(x). \] (B.7)

It can be proved that
\[ \mathcal{F}[\phi^{(1)}_{n, G}](\omega) = q_0(G)\phi^{(2)}_{n, G} \left( \frac{i(\alpha - \beta)}{\sqrt{\alpha \beta}} \omega \right), \] (B.8)
where
\[ q_0(G) = \frac{2(\sqrt{\alpha \beta})^n}{(\alpha - \beta)^n} \frac{1}{G^{n+\frac{1}{2}} \left( 1 - \sqrt{1 + \frac{4\alpha\beta}{G^2(\alpha - \beta)^2}} \right)^{n+\frac{1}{2}}}. \] (B.9)

Taking \( a = \frac{\alpha \varepsilon}{\varepsilon_{+}} \) and using the property of the Fourier transform \( \mathcal{F}(f(at))(\omega) = \frac{1}{|a|} \mathcal{F}(f(t))(\frac{\omega}{a}) \), we obtain
\[ q_0(G)\phi^{(2)}_{n, G}(w) = \mathcal{F}[\phi^{(1)}_{n, G}](w/a) = |a| \left( \frac{1}{|a|} \mathcal{F}[\phi^{(1)}_{n, G}(x)](w/a) \right). \]
\[ q_0(G)\phi^{(2)}_{n, G}(w) \rightarrow |a| \left( \frac{1}{|a|} \mathcal{F} \left( \frac{x^n}{\sqrt{n!}} \right) \left( \frac{w}{a} \right) \right) = e_n (\frac{(-1)^n \delta^{(n)}(w)}{\sqrt{n!}}) \]
The solution to the problem are given by

\[ \psi_n(x) = \begin{cases} \bar{\phi}_n(x), & n, \epsilon > 0; \\ \bar{\phi}_n(x), & n, \epsilon < 0. \end{cases} \]

The same study can be made for \( H^c \), so that

\[ \bar{\phi}^{(1)}_{n,G}(x) \rightarrow \bar{\psi}^+_n(x), \quad \bar{\phi}^{(2)}_{n,G}(x) \rightarrow \bar{\psi}^+_n(x). \]

The eigenfunctions of region I, with discrete positive spectrum, have as punctual limit the eigenfunctions of the boundary I-III with positive eigenvalues. In the same way, the eigenfunctions of region III, with discrete negative spectrum, have as punctual limit the eigenfunctions of the boundary I-III with negative eigenvalues.

**Appendix C**

To study the coalescence of the discrete eigenvalues and eigenfunctions between regions I-II and III-IV, we have to look at the Hamiltonian of equation (4). If \( \omega^2 - 4\alpha\beta = 0 \), we have

\[ H^c = \frac{1}{2} \left( \frac{w + 2\beta}{8\beta} \right) \hat{\phi}^2 + \frac{1}{2} \left( \frac{w - 2\beta}{8\beta} \right) \left( \frac{\hat{p}}{\hbar} \right)^2 + \frac{\hbar}{8\beta} \left( \frac{w^2 - 4\beta^2}{8\beta} \right) \left( 2\hat{\beta} \frac{\hat{p}}{\hbar} + 1 \right), \quad (C.1) \]

and

\[ H^c = \frac{1}{2} \left( \frac{w + 2\beta}{8\beta} \right) \hat{\phi}^2 + \frac{1}{2} \left( \frac{w - 2\beta}{8\beta} \right) \left( \frac{\hat{p}}{\hbar} \right)^2 - \frac{\hbar}{8\beta} \left( \frac{w^2 - 4\beta^2}{8\beta} \right) \left( 2\hat{\beta} \frac{\hat{p}}{\hbar} + 1 \right). \quad (C.2) \]

The discrete eigenvalues, \( E_n \rightarrow 0 \) when \( \omega^2 - 4\alpha\beta \rightarrow 0 \). We shall look for the eigenfunctions of \( H^c \) and \( H^c \) for \( E = 0 \):

\[ H^c \bar{\phi}(x) = 0, \quad H^c \bar{\psi}(x) = 0. \quad (C.3) \]

The solutions to the problem are given by

\[ \bar{\phi}(x) = (c_1 x + c_0) e^{-\frac{\sqrt{\omega^2 + 2\beta^2}}{\hbar} x}, \quad \bar{\psi}(x) = (d_1 x + d_0) e^{-\frac{\sqrt{\omega^2 + 2\beta^2}}{\hbar} x}. \quad (C.4) \]

The behaviour of the eigenfunctions at the border between I-II and III-IV can be obtained by using a limit procedure [85,86] from the discrete solutions in each region.

In region I, we shall take \( \omega^2 - 4\alpha\beta = \epsilon^2 \) and we shall solve the equation

\[ H^c \psi_{n,e}(x) = E_{n,e} \psi_{n,e}(x). \quad (C.5) \]

We obtain

\[ \bar{\phi}_{n,e}(x) = N_0(\epsilon) e^{-\frac{4\beta^2 - \omega^2 - \epsilon^2}{2\hbar^2} x^2} H_n(x \sigma(\epsilon)), \quad E_{n,e} = \hbar \sqrt{n + \frac{1}{2}}, \quad (C.6) \]

with \( \sigma(\epsilon) = \sqrt{\frac{4\beta^2}{\omega^2 + 4\hbar^2 - 4\epsilon^2}} \) and

\[ N_0^2(\epsilon) \sim \frac{1}{2} \left( \beta - (-1)^n \right) \epsilon + (-1)^n + 1 P_{\text{even}}(\epsilon). \quad (C.7) \]
where \( P_{g(n)}(\varepsilon) = \sum_{k=0}^{g(n)} k \varepsilon^k \), \( g(n) = \frac{1}{2}(-1 + (-1)^n + 2n) \) and \( t_0 \sim (\omega^2 - 4\beta^2)\varepsilon^{n(n)}. \) Notice that
\[
\mathcal{N}_{2n}(\varepsilon) \sim \varepsilon^0, \quad \mathcal{N}_{2n+1}(\varepsilon) \sim \varepsilon^{-1/2},
\]
then
\[
\tilde{\phi}_{2n}(x) \to e^{-\frac{x^2(n+2\beta)}{4n(2\beta)^2}}, \quad \tilde{\phi}_{2n+1}(x) \to xe^{-\frac{x^2(n+2\beta)}{4n(2\beta)^2}},
\]
so that finally:
\[
\tilde{\psi}_{2n}(x) \to (c_0 + c_1 x)e^{-\frac{x^2(n+2\beta)}{4n(2\beta)^2}} , \quad \tilde{\psi}_{2n+1}(x) \to (d_0 + d_1 x) e^{-\frac{x^2(n+2\beta)}{4n(2\beta)^2}},
\]
\( E_{n,\varepsilon} \to 0. \quad \text{(C.8)} \)

Let us procedure in the same form in region II. As \( \Omega^2 < 0, \) we shall take \( \omega^2 - 4\alpha\beta = -\varepsilon^2, \)
so that the discrete spectrum its given by \( E_{n,\varepsilon} = \pm i\hbar \sqrt{\varepsilon} \left(n + \frac{1}{2}\right). \) The eigenfunctions, in terms of \( \varepsilon, \) are given by
\[
\tilde{\phi}_{n,\varepsilon}^\pm (x) = \mathcal{N}_{n}^\pm (\varepsilon) e^{\frac{4\beta^2 - \gamma^2(1 + \gamma^2)}{(\omega^2 + 4\beta^2)(\omega^2 - 4\beta^2)} \frac{\varepsilon^2}{4n(\omega^2 - 4\beta^2)} H_n \left(e^{\pm \pi/4} x \sigma(\varepsilon)\right)},
\]
\[
E_{n,\varepsilon} = \pm i\hbar \varepsilon \left(n + \frac{1}{2}\right),
\]
\[
\tilde{\psi}_{n,\varepsilon}^\pm (x) = \mathcal{N}_{n}^\pm (\varepsilon) e^{\frac{4\beta^2 - \gamma^2(1 + \gamma^2)}{(\omega^2 + 4\beta^2)(\omega^2 - 4\beta^2)} \frac{\varepsilon^2}{4n(\omega^2 - 4\beta^2)} H_n \left(e^{\pm \pi/4} x \sigma(\varepsilon)\right)},
\]
\[
E_{n,\varepsilon} = \mp i\hbar \varepsilon \left(n + \frac{1}{2}\right). \quad \text{(C.10)}
\]

In this case, normalization constants are given by
\[
(\mathcal{N}_{n}^\pm)^2(\varepsilon) \sim \frac{1}{2} \left(\sqrt{\varepsilon} \right)^2 \left(\frac{2\beta + \sqrt{\varepsilon}}{\sqrt{\varepsilon} - \sqrt{\varepsilon - 1}}\right)^n \left(\frac{2\beta - \sqrt{\varepsilon}}{\sqrt{\varepsilon} + \sqrt{\varepsilon - 1}}\right)^n P_{g(n)}(\varepsilon),
\]
where \( P_{g(n)}(\varepsilon) \) is a polynomial with order \( g(n) = \frac{1}{2}(-1 + (-1)^n + 2n) \) in \( \varepsilon \) and independent term \( t_0 \sim (\omega^2 - 4\beta^2)\varepsilon^{n(n)}. \) As before
\[
(\mathcal{N}_{n}^\pm)^2(\varepsilon) \sim \varepsilon^0, \quad (\mathcal{N}_{n}^\pm)^2(\varepsilon) \sim \varepsilon^{-1/4},
\]
and
\[
\tilde{\phi}_{2n}(x) \to e^{\frac{x^2(n+2\beta)}{4n(2\beta)^2}}, \quad \tilde{\phi}_{2n+1}(x) \to xe^{\frac{x^2(n+2\beta)}{4n(2\beta)^2}},
\]
so that finally:
\[
\tilde{\psi}_{2n}(x) \to (c_0 + c_1 x)e^{\frac{x^2(n+2\beta)}{4n(2\beta)^2}} , \quad \tilde{\psi}_{2n+1}(x) \to (d_0 + d_1 x) e^{\frac{x^2(n+2\beta)}{4n(2\beta)^2}},
\]
\( E_{n,\varepsilon} \to 0. \quad \text{(C.12)} \)

Clearly, from equations (C.9) and (C.12), it can be concluded that the boundary I-II includes EPs.
The procedure we have applied to the limit of regions I and II can be implemented in the limit between regions III and IV. It is straightforward to prove that the boundary III-IV, also includes EPs.

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References

[1] Bender C M and Boettcher S 1998 Phys. Rev. Lett. 80 5243
[2] Bender C M, Berry M V and Mandilara A 2002 J. Phys. A: Math. Gen. 35 L467
[3] Bender C M, Berntson B K, Parker D and Samuel E 2013 Am. J. Phys. 81 173
[4] Bender C, Gianfreda M, Ozdemir S K, Peng B and Yang L 2013 Phys. Rev. A 88 062111
[5] Beygi A, Klevansky S P and Bender C M 2015 Phys. Rev. A 91 062101
[6] Wei Z and Bender C M 2020 J. Phys. A: Math. Theor. 53 375302
[7] Bender C M and Jones H F 2008 J. Phys. A: Math. Theor. 41 244406
[8] Álvarez G A, Danieli E P, Levstein P R and Pastawski H M 2006 J. Chem. Phys. 124 194507
[9] Guitart J M and Plyushchay S M 2017 J. High Energy Phys. JHEP12(2017)061
[10] Correa F, Jakubsy V and Plyushchay S M 2015 Phys. Rev. A 91 062101
[11] Beygi A, Klevansky S P and Bender C M 2015 Phys. Rev. A 91 062101
[12] Wen Z and Bender C M 2020 J. Phys. A: Math. Theor. 53 375302
[13] Bender C M and Jones H F 2008 J. Phys. A: Math. Theor. 41 244406
[14] Worhonzon illuo M, Abbasi M, Joglekar Y N and Murch K W 2019 Nat. Phys. 15 1232
[15] Pick A, Silberstein S, Moiseyev N and Bar-Gill N 2019 Phys. Rev. Res. 1 013015
[16] Yoshida T, Peters R, Kawakami N and Hatsugai Y 2019 Phys. Rev. B 100 041409
[17] Ashida Y, Furukawa S and Ueda M 2017 Nat. Commun. 8 15791
[18] Nakagawa M, Kawakami N and Ueda M 2021 Phys. Rev. Lett. 126 114040
[19] Znojil M 2018 Phys. Rev. A 93 093402
[20] Znojil M and Borisov D I 2020 Nucl. Phys. B 957 115064
[21] Garmon S and Ordonez G 2017 J. Math. Phys. 58 062101
[22] Kanki K, Garmon S, Tanaka S and Petrosky T 2017 J. Math. Phys. 58 092101
[23] Garmon S, Noba K, Ordonez G and Segal D 2019 Phys. Rev. A 99 010102
[24] Dunham Y, Kanki K, Garmon S, Tanak S and Ordonez G 2019 Phys. Rev. A 103 043513
[25] Hatano N and Ordonez G 2014 J. Math. Phys. 55 122116
[26] Garmon S, Gianfreda M and Hatano N 2015 Phys. Rev. A 92 022125
[27] Ordonez G and Hatano N 2017 J. Phys. A: Math. Theor. 50 405304
[28] Hatano N and Ordonez G 2019 Entropy 21 380
[29] Ramírez R and Reboiro M 2019 J. Math. Phys. 60 012106
[30] Ramírez R, Reboiro M and Tielas D 2020 Eur. Phys. J. D 74 193
[31] Swanson M S 2004 J. Math. Phys. 45 585
[32] Ahmed Z 2002 Phys. Lett. A 294 287
[33] Ahmed Z, Ghosh D and Nathan J A 2015 Phys. Lett. A 379 1639
[34] Musumbo D P, Geyer H B and Heiss W D 2007 J. Phys. A: Math. Theor. 40 F75
[35] Quense C 2007 J. Phys. A: Math. Theor. 40 F745
[36] Sinha A and Roychoudhury R 2002 Phys. Lett. A 301 163
[37] Sinha A and Roy P 2007 J. Phys. A: Math. Theor. 40 10599
[38] Sinha A and Roy P 2008 J. Phys. A: Math. Theor. 41 335306
[39] Jones H F 2005 J. Phys. A: Math. Gen. 38 1741
[40] Yesilatas O 2011 J. Phys. A: Math. Theor. 44 305305
[41] Assis P E G and Fring A 2008 J. Phys. A: Math. Theor. 41 244001
[42] Midya B, Dube P P and Roychoudhury R 2011 J. Phys. A: Math. Theor. 44 062001
[43] Mostafazadeh A 2008 J. Phys. A: Math. Theor. 41 244017
[44] Mostafazadeh A 2010 Int. J. Geom. Methods Mod. Phys. 07 1191
[45] Znojil M 2009 J. Math. Phys. 50 122105
[46] Bagchi B and Fring A 2009 Phys. Lett. A 373 4307
[47] Bagchi B and Marquette I 2015 Phys. Lett. A 379 1584
[48] Znojil M 2009 Phys. Lett. A 259 220
[49] Znojil M 2016 Mod. Phys. Lett. A 31 1650195
[50] Znojil M and Rika F 2019 Mod. Phys. Lett. A 34 1950085
[51] Znojil M 2020 Sci. Rep. 10 18523
[52] Dey S, Fring A and Khantoul B 2013 J. Phys. A: Math. Theor. 46 335304
[53] Dey S, Fring A and Gouba L 2015 J. Phys. A: Math. Theor. 48 40FT01
[54] Bagarello F and Fring A 2017 Int. J. Mod. Phys. B 31 1750085
[55] Ramírez R and Reboiro M 2016 Phys. Lett. A 380 1117
[56] Sinha A and Roy P 2009 J. Phys. A: Math. Theor. 42 052002
[57] Dey S, Fring A and Khantoul B 2013 J. Phys. A: Math. Theor. 46 335304
[58] Fring A and Moussa M H Y 2016 Phys. Rev. A 94 042128
[59] Fring A and Moussa M H Y 2016 Phys. Rev. A 93 042114
[60] Fring A and Tenney R 2021 Phys. Lett. A 410 127548
[61] Inzunza L and Plyushchay M 2021 arXiv:2104.08351v2[hep-th]
[62] Zhang Y-Z 2013 J. Phys. A: Math. Theor. 46 455302
[63] Lo C F 2014 J. Phys. A: Math. Theor. 47 078001
[64] Chruściński D 2003 J. Math. Phys. 44 3718
[65] Chruściński D 2004 J. Math. Phys. 45 841
[66] Marucci G and Conti C 2016 Phys. Rev. A 94 052136
[67] Bermudez D and Fernández C. D J 2013 Ann. Phys., NY 333 290
[68] Glauber R J 1986 Ann. New York Acad. Sci. 480 336
[69] Kryuchkov S I, Suslov S K and Vega-Guzmán J M 2013 J. Phys. B: At. Mol. Opt. Phys. 46 104007
[70] Di Mei F, Caramazza P, Pierangeli D, Di Domenico G, Ilan H, Agranat A J, Di Porto P and DelRe E 2016 Phys. Rev. Lett. 116 153902
[71] Mochizuki K, Hatano N, Feinberg J and Obuse H 2020 Phys. Rev. E 102 012101
[72] Ahmed Z 2012 J. Phys. A: Math. Theor. 45 032004
[73] Mostafazadeh A 2009 Phys. Rev. Lett. 102 220402
[74] Simón M A, Buendía A, Kiely A, Mostafazadeh A and Muga J G 2019 Phys. Rev. A 99 052110
[75] Wolf K B 1979 Integral Transform in Science and Engineering (New York: Plenum)