Counting Integral Points in Polytopes via Numerical Analysis of Contour Integration

Hiroshi Hirai, Ryunosuke Oshiro, and Ken’ichi Tanaka
Department of Mathematical Informatics, Graduate School of Information Science and Technology, The University of Tokyo, Tokyo, 113-8656, Japan.

hirai@mist.i.u-tokyo.ac.jp
ryunosuke_oshiro@mist.i.u-tokyo.ac.jp
kenichiro@mist.i.u-tokyo.ac.jp

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Abstract

In this paper, we address the problem of counting integer points in a rational polytope described by $P(y) = \{ x \in \mathbb{R}^m : Ax = y, x \geq 0 \}$, where $A$ is an $n \times m$ integer matrix and $y$ is an $n$-dimensional integer vector. We study the Z-transformation approach initiated by Brion-Vergne, Beck, and Lasserre-Zeron from the numerical analysis point of view, and obtain a new algorithm on this problem: If $A$ is nonnegative, then the number of integer points in $P(y)$ can be computed in $O(\text{poly}(n, m, \|y\|_{\infty})(\|y\|_{\infty} + 1)^n)$ time and $O(\text{poly}(n, m, \|y\|_{\infty}))$ space. This improves, in terms of space complexity, a naive DP algorithm with $O((\|y\|_{\infty} + 1)^n)$-size DP table. Our result is based on the standard error analysis to the numerical contour integration for the inverse Z-transform, and establish a new type of an inclusion-exclusion formula for integer points in $P(y)$.

We apply our result to hypergraph $b$-matching, and obtain a $O(\text{poly}(n, m, \|b\|_{\infty})(\|b\|_{\infty} + 1)^{(1-1/k)n})$ time algorithm for counting $b$-matchings in a $k$-partite hypergraph with $n$ vertices and $m$ hyperedges. This result is viewed as a $b$-matching generalization of the classical result by Ryser for $k = 2$ and its multipartite extension by Björklund-Husfeldt.

Keywords: Integer points in polytopes, counting algorithm, Z-transformation, numerical integration, trapezoidal rule

1 Introduction

Counting integer points in polytopes is a fundamental problem. There are numerous applications in various areas of mathematical science, and fascinating mathematics behind; see e.g., [2, 4]. This problem is computationally intractable, i.e., it is $\#P$-hard [18]. Approximate counting as well as exact counting under fixed parameter settings has been
rich sources for developments in the theory of algorithms and computational complexity. A seminal work by Barvinok [1] showed that there is a polynomial time algorithm to count integer points in rational polytope $P$ when the dimension $d$ of $P$ is fixed. His algorithm computes a certain “compact” expression (Brion-Lawrence formula) of the generation function $g_P(z) := \sum_{a \in P \cap \mathbb{Z}^d} z_1^{a_1} z_2^{a_2} \cdots z_d^{a_d}$ for multivariate indeterminate $z = (z_1, z_2, \ldots, z_d)$. This (extremely difficult) algorithm is now implemented in computer package Latte [15], and provides a useful tool to the study of geometric combinatorics.

A “dual” generating-function approach was initiated by Brion-Vergne [8], Beck [3], and Lasserre-Zeron [13 14]; see [12]. Suppose now that the input polytope $P = P(y)$ is given by

$$P(y) = \{ x \in \mathbb{R}^m : Ax = y, x \geq 0 \},$$

for $n \times m$ integer matrix $A$ and $n$-dimensional vector $y \in \mathbb{Z}^n$. Let $f_A(y) := |P(y) \cap \mathbb{Z}^m|$, and consider the Z-transform $\hat{f}_A(z) := \sum_{y \in \mathbb{Z}^n} f_A(y) z_1^{y_1} z_2^{y_2} \cdots z_n^{y_n}$. Brion-Vergne [8] showed that $\hat{f}_A$ admits a very simple closed formula $\hat{f}(z) = \prod_{k=1}^m 1/(1 - z_1^{A_{1k}} z_2^{A_{2k}} \cdots z_n^{A_{nk}})$, and that the wanted $f_A(y)$ is recovered by the inverse Z-transformation, which is a multi-dimensional contour integration of $\hat{f}_A$. This reduces the counting problem to the residue computation of $\hat{f}_A$. By this approach, Lasserre-Zeron [13 14] developed an $O((n + 1)^{m-n}\Lambda)$-time algorithm to count integer points in $P(y)$, where $\Lambda$ is a function of matrix $A$.

In this paper, we study the contour integration of the inverse Z-transformation from the numerical analysis point of view, and obtain a new algorithm to count integer points for an important class of polytopes. Our main result is as follows.

**Theorem 1.** Suppose that $A$ is nonnegative. For $y \in \mathbb{Z}^n$, the number of integer points in the polytope $P(y)$ can be computed in $O(\text{poly}(n, m, \|y\|_\infty)(\|y\|_\infty + 1)^n)$ time and $O(\text{poly}(n, m, \|y\|_\infty))$ space.

Notice that there is a simple DP algorithm with the same time complexity: For $k = 1, 2, \ldots, m$, consider the matrix $A^k$ consisting of the first $k$ columns of $A$, and the number $N^k(z)$ of integer points of polyhedron $\{ x \in \mathbb{R}^k : A^k x = z, x \geq 0 \}$. By the nonnegativity of $A$, integers $N^{k+1}(z)$ for $0 \leq z \leq y$ are obtained from $N^k(z)$ for $0 \leq z \leq y$ in $O(\text{poly}(n, \|y\|_\infty)(\|y\|_\infty + 1)^n)$ time. The resulting DP algorithm, however, requires an $O(\|y\|_\infty + 1)^n)$-space for the DP table. Thus our result is regarded as an improvement in terms of space complexity.

Our technique for proving Theorem 1 is as follows. Instead of the residue computation of $\hat{f}_A$, we apply the numerical integration to the inverse Z-transform of $\hat{f}_A$, where we use the trapezoidal rule, a basic and popular method of numerical integration. By the standard error analysis of the trapezoidal rule [17], we obtain an error estimate with respect to the number $N$ of sampled points of the numerical integration and contour radius $r$. Interestingly this estimate gives rise to a new inclusion-exclusion type formula for $f_A(y)$, and brings the algorithm in Theorem 1 which is quite simple and is easier to be implemented. A notable feature of our inclusion-exclusion is to use the cancellation structure of trigonometric function $\exp(2\pi ik/N)$ in the complex plane $\mathbb{C}$. This extends the usual inclusion-exclusion based on the cancellation of 1 and $-1$ in $\mathbb{R}$. Our algorithm computes the number of integer points in the expression $\sum_{k=0}^{N-1} a_k \exp(2\pi ik/N)$ for $a_k \in \mathbb{Q}$, and recovers the “true” value by algebraic computation on the group ring of cyclic group $\mathbb{Z}/N\mathbb{Z}$, which avoids numerical computation of $\exp(2\pi ik/N)$. 

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Our result is applicable to packing-type polytopes, which ubiquitously arise from graph theory and combinatorial mathematics. Consider the particular case where $A$ is 0-1 valued and $y$ is the all-one vector $1$. Then $f_A(1) = |P(1) \cap \mathbb{Z}^n|$ is the number of perfect matchings in the hypergraph corresponding to $A$. Exact exponential time counting algorithms for matchings have been intensively studied in recent years [5, 6, 9, 11]. The formula of hypergraph matching derived from our result is viewed as a variant of that given by Björklund-Husfeldt [6]. The time complexity $O^*(2^n)$ matches that of their algorithm. By exploiting special properties, Björklund-Husfeldt [6] improve the time complexity to $O^*(2^{(1-1/k)n})$ for $k$-partite hypergraphs. This result is viewed as a multipartite extension of the classical result of Ryser [16] for bipartite matching. We also exploit a special structure of our formula in $k$-partite hypergraphs, and prove the following $b$-matching generalization of the results of Ryser and Björklund-Husfeldt.

**Theorem 2.** Let $\mathcal{H} = (V, \mathcal{E})$ be a $k$-partite hypergraph, and let $b : V \rightarrow \mathbb{Z}_+$. The number of perfect $b$-matchings in $\mathcal{H}$ can be computed in $O(\text{poly}(|V|, |\mathcal{E}|, \|b\|_\infty)(\|b\|_\infty + 1)^{(1-1/k)|V|})$ time and $O(\text{poly}(|V|, |\mathcal{E}|, \|b\|_\infty))$ space.

Counting perfect $b$-matchings of a $k$-partite graph (i.e., $k$-dimensional $b$-matchings) has many applications in a wide range of mathematical sciences that include combinatorics, representation theory, and statistics; see e.g., [10]. For example, counting multiway contingency tables with prescribed margins, an important problem for statistical analysis on contingency tables, is nothing but $k$-dimensional $b$-matching counting.

The rest of this paper is organized as follows. In Section 2 we set up basic notation, and introduce Z-transformation, its inverse, and approximate inverse Z-transformation obtained by numerical integration. In Section 3 we present our algorithm to prove the main theorem. In Section 4 we discuss hypergraph matching and prove Theorem 2 in a further generalized form.

## 2 Preliminaries

### 2.1 Notation

Let $\mathbb{R}_{>0}$ denote the set of positive real numbers. Let $\mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers. For a matrix $A \in \mathbb{Z}^{n \times m}$, let $A_k$ denote the $k$-th column vector of the matrix. For an integer vector $y = (y_1, \ldots, y_n) \top \in \mathbb{Z}^n$ and a complex vector $z = (z_1, \ldots, z_n) \top \in \mathbb{C}^n$, define $z^y \in \mathbb{C}$ by

$$z^y := z_1^{y_1} \cdots z_n^{y_n}.$$ 

For a positive integer $N > 0$, define $\omega_N : \{0, 1, 2, \ldots, N-1\} \rightarrow \mathbb{C}$ by

$$\omega_N(h) := \exp \frac{2\pi i}{N} h \quad (h \in \{0, 1, 2, \ldots, N-1\}).$$

The following relation is well-known:

$$\sum_{j=0}^{N-1} \omega_N(kj) = \begin{cases} N & \text{if } k = 0 \mod N, \\ 0 & \text{otherwise,} \end{cases} \quad (k \in \mathbb{Z}). \quad (1)$$
For a function $f : D \to \mathbb{C}$ and $z \in D^n$, let $f(z)$ denote $(f(z_1), f(z_2), \ldots, f(z_n))^\top$, such as

$$
\exp(z) = (\exp(z_1), \exp(z_2), \ldots, \exp(z_n))^\top,
\ln z = (\ln z_1, \ldots, \ln z_n)^\top,
\omega_N(z) = (\omega_N(z_1), \ldots, \omega_N(z_n))^\top.
$$

The symbol $\bot$ is meant as “undefined.” We use $\bot$ when the function value is defined via integration or infinite summation, possibly not converging.

### 2.2 Z-transformation

For a function $f : \mathbb{Z}^n \to \mathbb{R}$, define the Z-transform $\hat{f} : \mathbb{C}^n \to \mathbb{C} \cup \{\bot\}$ of $f$ by

$$
\hat{f}(z) := \sum_{y \in \mathbb{Z}^n} f(y) z^{-y} \quad (z \in \mathbb{C}^n).
$$

The inverse of the Z-transformation is given as follows. For a function $g : \mathbb{C}^n \to \mathbb{C} \cup \{\bot\}$ and $r > 0$, define $I_r[g] : \mathbb{Z}^n \to \mathbb{R} \cup \{\bot\}$ by

$$
I_r[g](y) := \frac{1}{(2\pi i)^n} \oint_{|z_1|=r} \cdots \oint_{|z_n|=r} g(z) z^{y-1} dz_1 \cdots dz_n
= \int_{[0,1]} \cdots \int_{[0,1]} g(r \exp(2\pi it)) r^{1^\top} y \exp(2\pi it^\top y) dt_1 \cdots dt_n \quad (y \in \mathbb{Z}^n),
$$

where we change variables by $z_k = r \exp(2\pi it_k)$ in (2). Under an appropriate condition on $f$ and $r$, map $g \mapsto I_r[g]$ is actually the inverse of the Z-transformation:

$$
I_r[\hat{f}] = f.
$$

We do not go into details under which conditions (3) holds. Instead, we consider an approximate inverse Z-transform by the numerical integration applied to (2). Here we use the trapezoidal rule, which is a basic and popular method of numerical integration; see e.g., [7]. For a positive integer $N > 0$ (the number of points in the numerical integration), define $I_{N,r}[g] : \mathbb{Z}^n \to \mathbb{C} \cup \{\bot\}$ by

$$
I_{N,r}[g](y) := \frac{1}{N^n} \sum_{j \in \{0,1,\ldots,N-1\}^n} g(r \omega_N(j)) r^{1^\top} y \omega_N(j^\top y) \quad (y \in \mathbb{Z}^n).
$$

Recall notation $\omega_N(j) := \exp(2\pi ij/N) = (\exp(2\pi j_1/N), \exp(2\pi j_2/N), \ldots, \exp(2\pi j_n/N))$. Our counting algorithm is based on $I_{N,r}$.

### 3 Counting integral points in a polytope

Let $A$ be an $n \times m$ integral matrix. We assume that there is no nonzero nonnegative vector $x \in \mathbb{Z}_+^m \setminus \{0\}$ with $Ax = 0$. This assumption ensures that the polytope $\{x \in \mathbb{R}^m : Ax = y, x \geq 0\}$ is bounded for every $y \in \mathbb{Z}^n$. In the case where $A$ is a nonnegative matrix, this assumption is equivalent to the property that each column of $A$ has at least one nonzero entry.
As mentioned in the introduction, define function $f_A : \mathbb{Z}^n \to \mathbb{Z}$ by
\[ f_A (y) := \{ x \in \mathbb{Z}^m : Ax = y, x \geq 0 \} \quad (y \in \mathbb{Z}^n). \]

Our starting point is the following formula for $\hat{f}_A$.

**Theorem 3** ([3, 8, 13]). 1. For $z \in \mathbb{C}^n$ with $A^\top \ln |z| > 0$, the Z-transform $\hat{f}_A(z)$ is given by
\[ \hat{f}_A (z) = \sum_{h \in \mathbb{Z}_{\geq 0}^n} z^{-Ah} = \prod_{k=1}^m \frac{1}{1 - z^{-A_k}}, \]
where the series absolutely converges.

2. For $s \in \mathbb{R}_{\geq 0}^n$ with $A^\top \ln s > 0$, it holds
\[ f_A (y) = \frac{1}{(2\pi i)^n} \oint_{|z|=s_1} \cdots \oint_{|z|=s_n} \hat{f}_A (z) z^{-1} \, dz_1 \cdots dz_n. \]

We establish an approximate version of the above theorem as follows.

**Theorem 4.** Suppose that $A$ is nonnegative. For $y \in \mathbb{Z}_{\geq 0}^n$, $r > 1$, and $N := \|y\|_\infty + 1$, it holds
\[ I_{N,r}[\hat{f}_A] (y) = \frac{1}{N^n} \sum_{h \in \mathbb{Z}_{\geq 0}^n} \sum_{j \in \{0,1,\ldots,N-1\}^n} r^{1^\top(y-Ah)} \omega_N(j^\top(y-Ah)) \]
\[ = f_A (y) + \sum_{k=1}^m r^{-Nk} |\{ x \in \mathbb{Z}_{\geq 0}^m : y - Ax \geq 0, 1^\top(y-Ax) = Nk \}|. \]

**Proof.** From the assumption that $A$ is nonnegative and each column of $A$ has nonzero entry, for $z \in \mathbb{C}^n$ with $|z_1| = |z_2| = \cdots = |z_n| = r > 1$, it holds $A^\top \ln |z| = A^\top \ln r > 0$. By the previous theorem, the Z-transform $\hat{f}_A(z)$ is given by
\[ \hat{f}_A (z) = \sum_{h \in \mathbb{Z}_{\geq 0}^n} z^{-Ah}. \]

Substituting this expression to $I_{N,r}[\hat{f}_A](y)$ (in (41)), we have
\[ I_{N,r}[\hat{f}_A] (y) = \frac{1}{N^n} \sum_{j \in \{0,1,\ldots,N-1\}^n} \sum_{h \in \mathbb{Z}_{\geq 0}^n} r^{1^\top(y-Ah)} \omega_N(j^\top(y-Ah)) \]
\[ = \frac{1}{N^n} \sum_{h \in \mathbb{Z}_{\geq 0}^n} r^{1^\top(y-Ah)} \sum_{j \in \{0,1,\ldots,N-1\}^n} \omega_N(j^\top(y-Ah)) \]
\[ = \frac{1}{N^n} \sum_{h \in \mathbb{Z}_{\geq 0}^n} r^{1^\top(y-Ah)} m \prod_{l=1}^{N-1} \sum_{j_l=0}^{N_l-1} \omega_N(j_l(y_l - (Ah)_l)), \]
where the summations are interchangeable, thanks to the absolute convergence (by $r > 1$). By the relation (41), $\prod_{l=1}^m \sum_{j_l=0}^{N_l-1} \omega_N(j_l(y_l - (Ah)_l))$ is $N^n$ if $Ah - y \in N\mathbb{Z}_{\geq 0}^n$, and zero otherwise. Notice that $(Ah - y)_l \leq -N$ cannot occur since $\|y\|_\infty < N$. Thus we have
\[ I_{N,r}[\hat{f}_A] (y) = \sum_{u \in \mathbb{Z}_{\geq 0}^n} r^{-N1^\top u} |\{ x \in \mathbb{Z}_{\geq 0}^m : y - Ax = Nu \}|. \]

Gathering $u \in \mathbb{Z}_{\geq 0}^n$ with $1^\top u = k$, we obtain (43). \qed
This proof is inspired by the standard argument to derive the exponential convergence of the trapezoidal rule applied to periodic functions, where the the cancellation technique using \(f\) in the proof is known as aliasing in numerical analysis; see \cite{17} Theorem 2.1.

Notice that \(g(k) := |\{x \in \mathbb{Z}_{\geq 0}^n : y - Ax \geq 0, 1^\top (y - Ax) = k\}|\) is a variant of the Ehrhart (quasi)polynomial, and is bounded by a polynomial in \(k\) with degree \(m - n\). Hence the series in \(f\) actually absolutely converges for \(r > 1\).

**Corollary 5.** Suppose that \(A\) is nonnegative. Let \(y \in \mathbb{Z}_{\geq 0}^n\) and \(N := \|y\|_\infty + 1\). Then \(f_A(y)\) is equal to the coefficient of \(r^{-1} y\) in

\[
\frac{1}{N^n} \sum_{j \in \{0, 1, \ldots, N-1\}^n} \sum_{h \in \{0, 1, 2, \ldots, N-1\}^m} r^{-1} A h \omega_N(j^\top (y - Ah)) \tag{7}
\]

\[
= \frac{1}{N^n} \sum_{j \in \{0, 1, \ldots, N-1\}^n} \omega_N(j^\top y) \prod_{l=1}^m N-1 \sum_{h_l=0} r^{-1} A h_l \omega_N(-j^\top A_l h_l). \tag{8}
\]

Here we regard \(r\) as an indeterminate.

**Proof.** In the formula (5), if \(h \in \mathbb{Z}_{\geq 0}^n\) has \(h_l \geq N\), then \((Ah)_l > y_l\) for some \(l\), and \(h\) does not contribute to the constant term \(f_A(y)\). The claim follows from this fact and the observation \(\sum_{h \in \{0, 1, \ldots, N-1\}^m} r^{-1} A h \omega_N(-j^\top A h) = \prod_{l=1}^m N-1 \sum_{h_l=0} r^{-1} A h_l \omega_N(-j^\top A_l h_l). \)

Our goal is to compute the coefficient of \(r^{-1} y\) in \(f_A(y)\). Instead of numerical computation of the trigonometric function \(\omega_N\), we develop an algebraic algorithm. Let \(\mathbb{Q}[\mathbb{Z}/\mathbb{N}Z]\) denote the group ring of the cyclic group \(\mathbb{Z}/\mathbb{N}Z = \{0, 1, 2, \ldots, N - 1\}\) of order \(N\). Namely, \(\mathbb{Q}[\mathbb{Z}/\mathbb{N}Z]\) consists of polynomials with variable \(s\), rational coefficients, and degree at most \(N - 1\), in which the multiplication rule is given by \(s^l \cdot s'^r = s^{l+r} \mod N\). Consider the bivariate polynomial ring \(\mathbb{Q}[\mathbb{Z}/\mathbb{N}Z][t]\) with variables \(s, t\). Then \(p(s, t) \mapsto p(\omega_N(1), t)\) is a ring homomorphism from \(\mathbb{Q}[\mathbb{Z}/\mathbb{N}Z][t]\) to \(\mathbb{C}[t]\). Letting \(t = r^{-1}\) and \(s = \omega_N(1)\) in (7), we obtain a polynomial in \(\mathbb{Q}[\mathbb{Z}/\mathbb{N}Z][t]:\)

\[
\frac{1}{N^n} \sum_{j \in \{0, 1, \ldots, N-1\}^n} \sum_{h \in \{0, 1, \ldots, N-1\}^m} t A h \omega_N s^j(y - Ah). \tag{9}
\]

Let \(\overline{f}_A(s) \in \mathbb{Q}[\mathbb{Z}/\mathbb{N}Z]\) denote the coefficient of \(t^1 y\) in (9). Then it holds

\[
\overline{f}_A(\omega_N(1)) = f_A(y).
\]

Our algorithm first computes \(\overline{f}_A(s)\), and then computes \(\overline{f}_A(\omega_N(1)) = f_A(y)\).

**Lemma 6.** \(\overline{f}_A(s) = a_0 + a_1 s + \cdots + a_{N-1} s^{N-1}\) can be computed in \(O(\text{poly}(m, n, N)N^n)\) time and \(O(\text{poly}(n, m, N))\) space.

**Proof.** Let \(d := 1^\top y\). From (8), we first consider the computation of the coefficient \(b_j(s)\) of \(t^d\) in

\[
\prod_{l=1}^m N-1 \sum_{h_l=0} t A h_l s - j^\top A h_l \tag{10}
\]
for fixed $j$. It suffices to compute the above polynomial \( f \) modulo \( (t^{d+1}) \), which can be written as
\[
\prod_{l=1}^{m} s^k_l + s^k_1 t^1 + \cdots + s^k_d t^d \mod (t^{d+1}).
\] (11)
for some $k_l \in \{0, 1, 2, \ldots, N-1\}$. The integers $k_l$ are obtained in $O(\text{poly}(n, m, N))$ time by computing $\mathbf{1}^T A_l h_l (\leq d)$ and $j^T A_l h_l (\text{mod} N)$ for $l = 1, 2, \ldots, m$, $h_l = 0, 1, 2, \ldots, N - 1$. Expand \( f \) to the form \( \alpha_0(s) + \alpha_1(s) t^1 + \cdots + \alpha_d(s) t^d \), and obtain $b_i(s) = \alpha_d(s)$. This computation is done by the multiplication of $m$ polynomials with degree $d$ modulo \( (t^{d+1}) \), where their coefficients are polynomials with degree $N - 1$ in multiplication rule $s^i s^j = s^{i+j} \mod N$. Now \( f_A(s) \) is the sum of $s^j v b_j(s)$ over $j \in \{0, 1, \ldots, N-1\}^n$ divided by $N^n$. Thus \( f_A(s) \) is obtained in $O(\text{poly}(n, m, N)N^n)$ arithmetic operations (over $\mathbb{Z}$).

Finally we estimate the bit-size required for the computation. It suffices to estimate the size of $b_j(s)$, which is the sum of at most \( (d+m-1) \) terms of form $s^k$. Then the coefficients of $b_j(s)$ have bit-length $O(\text{poly}(n, m, N))$. Thus the required bit-size is at most $\text{poly}(n, m, N)n \log N$.

Next we consider how to compute \( f_A(\omega_N(1)) = f_A(y) \) from \( f_A(s) \).

**Lemma 7.** $f_A(s)$ is written as
\[
f_A(s) = f_A(y) + \sum_i K_i (1 + s^i + s^{2i} + \cdots + s^{N-1}).
\]
where the sum is taken over divisors $i < N$ of $N$ with some coefficient $K_i \in \mathbb{Q}$.

**Proof.** Regard $\{0, 1, \ldots, N-1\}^n$ as $(\mathbb{Z}/N\mathbb{Z})^n$. Then the map $\varphi_h : (\mathbb{Z}/N\mathbb{Z})^n \rightarrow \mathbb{Z}/N\mathbb{Z}$ defined by $j \mapsto j^T(y - Ah) \mod N$ is a group homomorphism. Therefore the image of $\varphi_h$ is the cyclic group $n_h \mathbb{Z}/N\mathbb{Z}$ for some divisor $n_h$ of $N$. Also the number of inverse images of each $k \in \{0, 1, \ldots, N/n_h - 1\}$ is given by $J_h := |\ker \varphi_h|$. Then, for $h \in \{0, 1, \ldots, N-1\}^m$ with $1^T A h = 1^T y$, it holds
\[
\sum_{j \in \{0, 1, \ldots, N-1\}^n} t^1^T A h s^j^T(y - Ah) = t^1^T y \sum_{j \in \{0, 1, \ldots, N-1\}^n} s^j^T(y - Ah) = t^1^T y J_h (1 + s^{n_h} + s^{2n_h} + \cdots + s^{N-n_h}).
\]

Thus we have
\[
f_A(s) = \sum_{h \in \{0, 1, \ldots, N-1\}^m} J_h (1 + s^{n_h} + s^{2n_h} + \cdots + s^{N-n_h})
\]
\[
= \sum_{i \text{ divisor of } N} K_i (1 + s^i + s^{2i} + \cdots + s^{N-1}),
\]
where $K_i$ is the sum of $J_h$ over $h \in (\mathbb{Z}/N\mathbb{Z})^n$ such that the image of $\varphi_h$ is $i\mathbb{Z}/N\mathbb{Z}$. Notice $K_N = f_A(y)$.

According to this lemma, we obtain a simple algorithm to compute $f_A(y)$ from $f_A(s)$ as follows.

0: Let $f_A(s) = a_0 + a_1 s + \cdots + a_{N-1} s^{N-1}$.  

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1: If \( a_i = 0 \) for all \( i > 0 \), then output \( a_0 = f_A(y) \); stop

2: Choose the minimum index \( i > 0 \) with \( a_i \neq 0 \). Let \( a_j \leftarrow a_j - a_i \) for each index \( j \) that is the multiple of the index \( i \), and go to step 1.

The correctness of the algorithm is clear from the above lemma: The chosen index \( i \) in step 2 is a divisor of \( N \) with \( a_i = K_i \). Hence the algorithm computes \( f_A(s) - \sum_i K_i(1 + s^1 + \cdots + s^{N-i}) \). After at most \( N \) iterations, the algorithm terminates and outputs the correct answer \( f_A(y) \).

Example 8. Consider the following matrix \( A \) and vector \( y \):

\[
A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.
\]

Then the polytope \( \{ x \in \mathbb{R}^3 : Ax = y, x \geq 0 \} \) has three integer points:

\[
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.
\]

Let us count the integer points according to Corollary 5. The coefficient \( f_A(s) \) of \( t^{12} \) in

\[
\frac{1}{6^2} \sum_{j_1=0}^{5} \sum_{j_2=0}^{5} s^{j_1+3j_2} \left( \sum_{h_1=0}^{5} t^{2h_1} s^{(-j_1-j_2)h_1} \right)^2 \left( \sum_{h_2=0}^{5} t^{4h_2} s^{(-3j_1-j_2)h_2} \right)
\]

is

\[
\frac{1}{6^2} \sum_{j_1=0}^{5} \sum_{j_2=0}^{5} s^{j_1+3j_2} \left( 5s^{4j_1-4j_2} + 3s^{5j_1-3j_2} + s^{-6j_1-j_2} \right)
\]

\[
= \frac{1}{36} \sum_{j_1=0}^{5} \sum_{j_2=0}^{5} \left( 5s^{j_1-j_2} + 3 + s^{-j_1+2j_2} \right)
\]

\[
= \frac{1}{36} \left( 120 + 12s + 12s^2 + 12s^3 + 12s^4 + 12s^5 \right) = 3 + \frac{1}{3} \left( 1 + s + s^2 + s^3 + s^4 + s^5 \right).
\]

Therefore we obtain \( f_A(\omega_6(1)) = 3 = f_A(y) \).

Example 9. Consider the following matrix \( A \) and vector \( y \):

\[
A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 7 \\ 5 \end{pmatrix}.
\]

Then the polytope \( \{ x \in \mathbb{R}^3 : Ax = y, x \geq 0 \} \) has only one integer point:

\[
\begin{pmatrix} 3 \\ 2 \end{pmatrix}.
\]

Consider the coefficient \( f_A(s) \) of \( t^{12} \) in

\[
\frac{1}{8^2} \sum_{j_1=0}^{7} \sum_{j_2=0}^{7} s^{7j_1+5j_2} \left( \sum_{h_1=0}^{7} t^{3h_1} s^{(-2j_1-j_2)h_1} \right) \left( \sum_{h_2=0}^{7} t^{3h_2} s^{(-j_1-2j_2)h_2} \right).
\]
Accordingly, vector $h_\alpha$ of columns such that the corresponding hyperedge meets $\{A_1, \ldots, A_n\}$ regarding as a block matrix follows from:

\[
\frac{1}{64} \sum_{j_1=0}^7 \sum_{j_2=0}^7 s^{j_1+j_2} (s^{-8j_1-4j_2} + s^{-7j_1-5j_2} + s^{-6j_1-6j_2} + s^{-5j_1-7j_2} + s^{-4j_1-8j_2})
\]

Thus we get:

\[
\frac{1}{64} \left( 104 + 24s + 40s^2 + 24s^3 + 40s^4 + 24s^5 + 40s^6 + 24s^7 \right)
\]

\[
= 1 + \frac{3}{8} \left( 1 + s + s^2 + s^3 + s^4 + s^5 + s^6 + s^7 \right) + \frac{1}{4} \left( 1 + s^2 + s^4 + s^6 \right).
\]

Thus we obtain $\overline{f_A}(\omega_8(1)) = 1 = f_A(y)$.

## 4 Hypergraph Matching

We next show Theorem 2 in a generalized form. Let $A$ be an $n \times m$ nonnegative integer matrix. For each column index $l \in \{1, 2, \ldots, m\}$, consider subset $F_l$ consisting of row indices $k \in \{1, 2, \ldots, n\}$ with $A_{kl} > 0$. Let $\mathcal{H}(A)$ denote the hypergraph on vertex set $\{1, 2, \ldots, n\}$ and hyperedge set $\{F_l : l = 1, 2, \ldots, m\}$. By a stable set of $\mathcal{H}(A)$ we mean a vertex subset $S \subseteq \{1, 2, \ldots, n\}$ such that every hyperedge meets at most one vertex in $S$.

**Theorem 10.** Suppose that we are given a stable set $S$ of $\mathcal{H}(A)$. For $y \in \mathbb{Z}^n$, we can compute $f_A(y)$ in $O(\text{poly}(n, m, \|y\|_{\infty})(\|y\|_{\infty} + 1)^{n-|S|})$ time and $O(\text{poly}(n, m, \|y\|_{\infty}))$ space.

**Proof.** Let $N = \|y\|_{\infty} + 1$, $d := 1^T y$, and $\nu := |S|$. As before, it suffices to compute $f_A(y)$ modulo $(t^{d+1})$. We show that (11) admits the following factorization:

\[
\sum_j \sum_h t^{1^T Ah} s^T (y - Ah) = \sum_{j' \in \{0, 1, \ldots, N-1\}^{n-\nu}} F_0(j') F_1(j') \cdots F_\nu(j'),
\]

(12)

where each $F_\alpha(j')$ is computable modulo $(t^{d+1})$ in $O(\text{poly}(n, m, N))$ time and space.

By arranging indices of $A$, we can assume that $S = \{1, 2, \ldots, \nu\}$, and that $A$ is regarded as a block matrix $A = (A^0 A^1 A^2 \cdots A^\nu)$, where $A_\alpha$ ($\alpha = 1, 2, \ldots, \nu$) consists of columns such that the corresponding hyperedge meets $\alpha \in S$ ($\alpha = 1, 2, \ldots, \nu$). Accordingly, vector $h \in \{0, 1, 2 \ldots N-1\}^m$ is also partitioned as

\[
h = \begin{pmatrix}
    h^0 \\
h^1 \\
    \vdots \\
h^{\nu}
\end{pmatrix}, \quad Ah = A^0 h^0 + A^1 h^1 + \cdots + A^\nu h^{\nu}.
\]

We suppose that an $n-\nu$-dimensional vector $j' \in \{0, 1, 2, \ldots, N-1\}^{n-\nu}$ is embedded to $\{0, 1, 2, \ldots, N-1\}^n$ by filling 0 to the first $\nu$ components. Each $j \in \{0, 1, \ldots, N-1\}^n$
is uniquely represented as \( j = j' + \sum_{\alpha=1}^{n} j_{\alpha} e_{\alpha} \) for \( j' \in \{0, 1, 2, \ldots, N-1\}^{n-\nu} \), where \( e_{\alpha} \) is the \( \alpha \)-th unit vector. Then we have

\[
\begin{align*}
    j^T y &= j'^T y + j_1 y_1 + \cdots + j_\nu y_\nu, \\
    j^T A^\alpha h^\alpha &= \begin{cases} 
        j'^T A^0 h^0 & \text{if } \alpha = 0, \\
        (j' + j_\alpha e_\alpha)^T A^\alpha h^\alpha & \text{if } \alpha > 0.
    \end{cases}
\end{align*}
\]

Define \( G_0(j', h^0) \) and \( G_\alpha(j', j_\alpha, h^\alpha) \) \( (\alpha = 1, 2, \ldots, \nu) \) by

\[
\begin{align*}
    G_0(j', h^0) &= t^{1^T A^0 h^0} s^{-j'^T A^0 h^0}, \\
    G_\alpha(j', j_\alpha, h^\alpha) &= t^{1^T A^\alpha h^\alpha} s^{-(j' + j_\alpha e_\alpha)^T A^\alpha h^\alpha}.
\end{align*}
\]

Then

\[
t^{1^T A h_s - j'^T A h} = G_0(j', h^0) G_1(j', j_1, h^1) \cdots G_\nu(j', j_\nu, h^\nu).
\]

From \( \sum_j \sum_h = \sum_{j'} \sum_{j_1} \cdots \sum_{j_\nu} \sum_{h_0} \cdots \sum_{h_\nu} \), we see that the left hand side of (12) is equal to

\[
\sum_{j' \in \{0, 1, \ldots, N-1\}^{n-\nu}} s^{j'^T y} \left( \sum_{h_0} G_0(j', h^0) \right) \prod_{\alpha=1}^{\nu} \sum_{j_\alpha = 0}^{N-1} s^{j_\alpha y_\alpha} \sum_{h_\alpha} G_\alpha(j', j_\alpha, h^\alpha),
\]

where \( h_\alpha \) ranges over \( \{0, 1, \ldots, N-1\}^{\nu_\alpha} \) and \( \nu_\alpha \) is the dimension of \( h_\alpha \). Now \( G_\alpha \) is a form of \( t_{a^T h^\alpha} s^{b^T h^\alpha} \), and hence \( \sum_{h_\alpha} G_\alpha \) is factorized as \( \prod_{l=1}^{\nu_\alpha} \sum_{k=0}^{N-1} t_{a^T k} s^k \) (as in (7)). Thus each \( \sum_{h_\alpha} G_\alpha \) is computable in \( O(\text{poly}(n, m, N)) \) time, and we have the desired expression (12).

A hypergraph \( \mathcal{H} = (V, E) \) is said to be \( k \)-partite if there is a partition of the vertex set into \( k \) nonempty subsets \( S_1, S_2, \ldots, S_k \) such that each hyperedge meets at most one vertex in each \( S_i \). Clearly some \( S_i \) has cardinality at least \( |V|/k \). Hence we obtain a generalization of Theorem 2.

**Corollary 11.** Suppose that \( \mathcal{H}(A) \) is \( k \)-partite and the partition is given. Then we can compute \( f_A(y) \) in \( O(\text{poly}(n, m, \|y\|_\infty)(\|y\|_\infty + 1)^{(1-1/k)n}) \) time and \( O(\text{poly}(n, m, \|y\|_\infty)) \) space.

Finally we note a combinatorial inclusion-exclusion formula for the number of perfect matchings derived from our formula (Corollary 5). A perfect matching in a hypergraph is a subset \( M \) of hyperedges such that each vertex belongs to exactly one hyperedge in \( M \).

**Corollary 12.** Let \( \mathcal{H} = (V, E) \) be a hypergraph. The number of perfect matchings in \( \mathcal{H} \) is equal to the coefficient of \( t^{|V|} \) in

\[
\frac{1}{2^{|V|}} \sum_{U \subseteq V} (-1)^{|U|} \prod_{F \in \mathcal{E}} \left( 1 + t^{|F|} (-1)^{|F \cap U|} \right).
\]

**Proof.** Let \( A \in \{0, 1\}^{|V| \times |E|} \) be the adjacency matrix of the hypergraph \( \mathcal{H} = (V, E) \). Then a perfect matching is exactly a solution \( x \) of \( Ax = 1 \). Apply Corollary 5 with \( N = 2 \). Then \( \omega_2(k) = (-1)^k \), and \( f_A(1) \) is equal to the coefficient of \( t^{|V|} \) in

\[
\frac{1}{2^{|V|}} \sum_{j \in \{0, 1\}^{|V|}} (-1)^{j^T} \prod_{l=1}^{|E|} \left( 1 + t^{-1^T A_l} (-1)^{-j^T A_l} \right).
\]
Identify \( j \in \{0, 1 \}^{|V|} \) with a subset \( U \subseteq V \). Then \( j^\top 1 = |U|, 1^\top A_l = |F| \) and \( j^\top A_l = |U \cap F| \), where \( F \) is the hyperedge corresponding to \( l \)-th column \( A_l \) of \( A \). Thus we have the formula.

A hypergraph is said to be \( \ell \)-uniform if each hyperedge has cardinality \( \ell \). A 2-uniform hypergraph is exactly a simple undirected graph. In the case of a uniform hypergraph, the coefficient of \( t^{|V|} \) in (12) is the following simple expression.

**Corollary 13.** Let \( \mathcal{H} = (V, \mathcal{E}) \) be an \( \ell \)-uniform hypergraph. The number of perfect matchings in \( \mathcal{H} \) is equal to

\[
\frac{1}{2^n} \sum_{U \subseteq V} (-1)^{|U|/\ell} \sum_{i=0}^{|V|/\ell} (-1)^i \binom{|\mathcal{E}_{U, \text{odd}}|}{i} \binom{|\mathcal{E} \setminus \mathcal{E}_{U, \text{odd}}|}{|V|/\ell - i},
\]

where \( \mathcal{E}_{U, \text{odd}} \) denotes the subsets of \( \mathcal{E} \) consisting of \( F \) with \( |F \cap U| \) odd.

**Proof.** The formula in Corollary 12 becomes

\[
\frac{1}{2^n} \sum_{U \subseteq V} (-1)^{|U|/\ell} \prod_{F \in \mathcal{E}_{U, \text{odd}}} (1 - t^\ell) \prod_{G \in \mathcal{E} \setminus \mathcal{E}_{U, \text{odd}}} (1 + t^\ell).
\]

By evaluating the coefficient of \( t^{|V|} \), we obtain the formula. \( \square \)

A similar inclusion-exclusion formula for hypergraph matching is given in [6]. In the case of a graph, i.e., \( \ell = 2 \), the formula (13) becomes

\[
\frac{1}{2^n} \sum_{U \subseteq V} (-1)^{|U|/2} \sum_{i=0}^{|V|/2} (-1)^i \binom{|\delta U|}{i} \binom{|\mathcal{E} \setminus \delta U|}{|V|/2 - i},
\]

where \( \delta U \) denotes the set of edges for which exactly one of ends belongs to \( U \).

## 5 Concluding Remarks

In this paper, we presented a new algorithm for counting integer points in polytopes. Our original attempt was to count integer points by computing the inverse \( Z \)-transformation directly by numerical integration in floating-point arithmetic. Although this approach did not work well, the theoretical analysis on the error estimate brought a new inclusion-exclusion formula for integer points, on which our algorithm is built.

We end this paper with some open problems and future work:

- We employed the trapezoidal rule for the numerical integration of the inverse \( Z \)-transformation. Can other (more sophisticated) methods of numerical integration and their error analysis lead to a better algorithm for integer point counting?

- For exact counting of perfect matchings in general \( n \)-vertex graphs, the current fastest (polynomial space) algorithms are \( O^*(2^{n/2}) \)-time algorithms by Björklund [5] and Cygan and Pilipczuk [9]. This time complexity matches one by Ryser for bipartite graphs.
For counting $b$-matchings in general graphs, our algorithm in Theorem 1 brings an $O^*(\text{poly}(b)(\|b\|_\infty + 1)^n)$-time algorithm, which is improved to $O^*(\text{poly}(b)(\|b\|_\infty + 1)^{n/2})$-time one for bipartite graphs (Theorem 2). So a natural question is: Can we design a polynomial space $O^*(\text{poly}(b)(\|b\|_\infty + 1)^{n/2})$-time algorithm for counting $b$-matchings in general graphs?

- Our algorithm is simple, and is not difficult to be implemented. Implementing our algorithm, evaluating its performance compared with LattE [15] (and other lattice counting problems), and incorporating heuristics for speeding up deserve interesting future research.

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