Admissible Hom-Novikov-Poisson and Hom-Gelfand-Dorfman color Hom-algebras

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Abstract

The main feature of color Hom-algebras is that the identities defining the structures are twisted by even linear maps. The purpose of this paper is to introduce and give some constructions of admissible Hom-Novikov-Poisson color Hom-algebras and Hom-Gelfand-Dorfman color Hom-algebras. Their bimodules and matched pairs are defined and the relevant properties and theorems are given. Also, the connections between Hom-Novikov-Poisson color Hom-algebras and Hom-Gelfand-Dorfman color Hom-algebras is proved. Furthermore, we show that the class of admissible Hom-Novikov-Poisson color Hom-algebras is closed under tensor product.

1 Introduction

A Novikov algebra has a binary operation such that the associator is left-symmetric and that the right multiplication operators commute. Novikov algebras play a major role in the studies of Hamiltonian operators and Poisson brackets of hydrodynamic type [15,28,29,32–34]. The left-symmetry of the associator implies that every Novikov algebra is Lie admissible, i.e., the commutator bracket \([x,y] = xy - yx\) gives it a Lie algebra structure.
Poisson algebras are used in many fields in mathematics and physics. In mathematics, Poisson algebras play a fundamental role in Poisson geometry [76], quantum groups [21, 27], and deformation of commutative associative algebras [35]. In physics, Poisson algebras are a major part of deformation quantization [43], Hamiltonian mechanics [4], and topological field theories [63]. Poisson-like structures are also used in the study of vertex operator algebras [31].

The theory of Hom-algebras has been initiated in [37, 54, 55] motivated by quasi-deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras. Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced first by Hartwig, Larsson and Silvestrov in [37] where a general approach to discretization of Lie algebras of vector fields using general twisted derivations (σ-derivations) and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed. The general quasi-Lie algebras, containing the quasi-Hom-Lie algebras and Hom-Lie algebras as subclasses, as well their graded color generalization, the color quasi-Lie algebras including color quasi-Hom-Lie algebras, color Hom-Lie algebras and their special subclasses the quasi-Hom-Lie superalgebras and Hom-Lie superalgebras, have been first introduced in [37, 53–56, 69]. Subsequently, various classes of Hom-Lie admissible algebras have been considered in [45]. In particular, in [45], the Hom-associative algebras have been introduced and shown to be Hom-Lie admissible, that is leading to Hom-Lie algebras using commutator map as new product, and in this sense constituting a natural generalization of associative algebras as Lie admissible algebras leading to Lie algebras using commutator map. Furthermore, in [45], more general G-Hom-associative algebras including Hom-associative algebras, Hom-Vinberg algebras (Hom-left symmetric algebras), Hom-pre-Lie algebras (Hom-right symmetric algebras), and some other Hom-algebra structures, generalizing G-associative algebras, Vinberg and pre-Lie algebras respectively, have been introduced and shown to be Hom-Lie admissible, meaning that for these classes of Hom-algebras, the operation of taking commutator leads to Hom-Lie algebras as well. Also, flexible Hom-algebras have been introduced, connections to Hom-algebra generalizations of derivations and of adjoint maps have been noticed, and some low-dimensional Hom-Lie algebras have been described. In Hom-algebra structures, defining algebra identities are twisted by linear maps. Since the pioneering works [37, 45, 53–56], Hom-algebra structures have developed in a popular broad area with increasing number of publications in various directions. Hom-algebra structures include their classical counterparts and open new broad possibilities for deformations, extensions to Hom-algebra structures of representations, homology, cohomology and formal deformations, Hom-modules and Hom-bimodules, Hom-Lie admissible Hom-coalgebras, Hom-coalgebras, Hom-bialgebras, Hom-Hopf algebras, L-modules, L-comodules and Hom-Lie quasi-bialgebras, n-ary generalizations of BiHom-Lie algebras and BiHom-associative algebras and generalized derivations, Rota-Baxter operators, Hom-dendriform color Hom-algebras, Rota-Baxter bisystems and covariant bialgebras, Rota-Baxter coystems, coquasitriangular mixed bialgebras, coassociative Yang-Baxter pairs, coassociative Yang-Baxter equation and generalizations of Rota-Baxter systems and algebras, curved O-operator systems and their connections with tridendriform systems and pre-Lie algebras, BiHom-algebras, BiHom-Frobenius algebras and double constructions, infinitesimal BiHom-bialgebras and Hom-dendriform D-bialgebras, Hom-algebras have been considered [2, 3, 6, 9, 10, 12–14, 16–18, 20, 25, 26, 30, 36, 38–42,
In [84] the author initiated the study of a twisted generalization of Novikov algebras, called Hom-Novikov algebras. A Hom-Novikov algebra $A$ has a binary operation $\cdot$ and a linear self-map $\alpha$, and it satisfies some $\alpha$-twisted versions of the defining identities of a Novikov algebra. In [84] several constructions of Hom-Novikov algebras were given and some low dimensional Hom-Novikov algebras were classified. Further, Hom-Poisson algebras were defined in [46] by Makhlouf and Silvestrov. It is shown in [46] that Hom-Poisson algebras play the same role in the deformation of commutative Hom-associative algebras as Poisson algebras do in the deformation of commutative associative algebras.

In this paper, we introduce and obtain some results on construction of admissible Hom-Novikov-Poisson color Hom-algebras and Hom-Gelfand-Dorfman color Hom-algebras. Their bimodules and matched pairs are defined and the relevant properties and theorems are obtained. We also show that the class of admissible Hom-Novikov-Poisson color Hom-algebras are closed under tensor product. In Section 2, we introduce the notions of bimodules and matched pairs of Hom-associative color Hom-algebras, Hom-Novikov color Hom-algebras and Hom-Lie color Hom-algebras in which we give some results and some examples. In Section 3, we establish the notions of admissible Hom-Novikov-Poisson color Hom-algebras and we give some explicit constructions. Their bimodule and matched pair are defined and their related relevant properties are also given. Finally, we show that the much larger class of admissible Hom-Novikov-Poisson color Hom-algebras is also closed under tensor products. In Section 4, we introduce the notions of Hom-Gelfand-Dorfman color Hom-algebras and we discuss some basic properties and examples of these objects. Moreover, we characterize the representations and matched pairs of Hom-Gelfand-Dorfman color Hom-algebras and provide some key constructions.

2 Preliminaries and some results

Throughout the article, we assume that all linear spaces are over an algebraically closed field $\mathbb{K}$ of characteristic 0, and denote by $\mathbb{K}^* = \mathbb{K}\setminus\{0\}$ the group of invertible elements of $\mathbb{K}$ with respect to the multiplication in $\mathbb{K}$.

In this section, we introduce the notions of bimodules and matched pairs of Hom-associative color Hom-algebras, Hom-Novikov color Hom-algebras and Hom-Lie color Hom-algebras in which we give some results and examples.

Let $\Gamma$ be an abelian group. A linear space $V$ is said to be $\Gamma$-graded, if there is a family $(V_\gamma)_{\gamma \in \Gamma}$ of vector subspace of $V$ such that

$$V = \bigoplus_{\gamma \in \Gamma} V_\gamma.$$ 

An element $x \in V$ is said to be homogeneous of degree $\gamma \in \Gamma$ if $x \in V_\gamma, \gamma \in \Gamma$, and in this case, $\gamma$ is called the degree of $x$. In the sequel, we will denote the set of all the homogeneous elements of $V$ by $\mathcal{H}(V)$. As usual, we denote by $\overline{\tau}$ the degree of an element $x \in V$. Thus each homogeneous element $x \in V$ determines a unique group element $\overline{\tau} \in \Gamma$ by $x \in V_{\overline{\tau}}$. Thus, when no confusion occur, we can drop ”$-” in notation of degree for convenience of exposition.

Let $V = \bigoplus_{\gamma \in \Gamma} V_\gamma$ and $V' = \bigoplus_{\gamma \in \Gamma} V'_\gamma$ be two $\Gamma$-graded linear spaces. A linear mapping $f : V \rightarrow V'$ is said to be homogeneous of degree $\nu \in \Gamma$ if $f(V_\gamma) \subseteq V'_{\gamma+\nu}$ for
all $\gamma \in \Gamma$. If in addition $f$ is homogeneous of degree zero, i.e. $f(V_{\gamma}) \subseteq V'_{\gamma}$ holds for any $\gamma \in \Gamma$, then $f$ is said to be even.

An algebra $\mathcal{A}$ is said to be $\Gamma$-graded if its underlying linear space is $\Gamma$-graded, i.e. $\mathcal{A} = \bigoplus_{\gamma \in \Gamma} \mathcal{A}_{\gamma}$, and if, furthermore $\mathcal{A}_{\gamma} \mathcal{A}_{\gamma'} \subseteq \mathcal{A}_{\gamma + \gamma'}$, for all $\gamma, \gamma' \in \Gamma$. It is easy to see that if $\mathcal{A}$ has a unit element $e$, it follows that $e \in \mathcal{A}_{0}$. A subalgebra of $\mathcal{A}$ is said to be $\Gamma$-graded if it is $\Gamma$-graded as a subspace of $\mathcal{A}$. Let $\mathcal{A}'$ be another $\Gamma$-graded algebra. A homomorphism $f : \mathcal{A} \to \mathcal{A}'$ of $\Gamma$-graded algebras is by definition a homomorphism of the algebra $\mathcal{A}$ into the algebra $\mathcal{A}'$, which is, in addition an even mapping.

**Definition 2.1** ([7, 22, 23, 50, 59, 64–66, 71]). Let $\mathbb{K}$ be a field and $\Gamma$ be an abelian group. A map $\varepsilon : \Gamma \times \Gamma \to \mathbb{K}^*$ is called a commutation factor on $\Gamma$ if the following identities hold, for all $a, b, c$ in $\Gamma$

1. $\varepsilon(a, b) \varepsilon(b, a) = 1$,
2. $\varepsilon(a, b + c) = \varepsilon(a, b) \varepsilon(a, c)$,
3. $\varepsilon(a + b, c) = \varepsilon(a, c) \varepsilon(b, c)$.

The definition above implies, in particular, the following relations

$$\varepsilon(a, 0) = \varepsilon(0, a) = 1, \ varepsilon(a, a) = \pm 1, \text{ for all } a \in \Gamma.$$ 

If $x$ and $x'$ are two homogeneous elements of degree $\gamma$ and $\gamma'$ respectively and $\varepsilon$ is a skew-symmetric bicharacter, then we shorten the notation by writing $\varepsilon(x, x')$ instead of $\varepsilon(\gamma, \gamma')$ since the degree of every homogeneous element is unique.

**Remark 2.2.** Let $A$ and $V$ be two $\Gamma$-graded linear spaces such that

$$A \oplus V = \bigoplus_{\gamma \in \Gamma} (A \oplus V)_{\gamma} = \bigoplus_{\gamma \in \Gamma} (A_{\gamma} \oplus V_{\gamma}),$$

then, for all $X_1 = x_1 + v_1 \in A_{\gamma_1} \oplus V_{\gamma_1}, X_2 = x_2 + v_2 \in A_{\gamma_2} \oplus V_{\gamma_2}$ we have

$$\varepsilon(x_1, x_2) = \varepsilon(x_1, v_2) = \varepsilon(v_1, x_2) = \varepsilon(v_1, v_2) = \varepsilon(X_1, X_2).$$

**Example 2.3.** Some standard examples of skew-symmetric bicharacters are:

1. $\Gamma = \mathbb{Z}_2, \ \varepsilon(i, j) = (-1)^{ij}$,
2. $\Gamma = \mathbb{Z}^n_2, \ \varepsilon((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n)) := (-1)^{\alpha_1 \beta_1 + \cdots + \alpha_n \beta_n}$,
3. $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2, \ \varepsilon((i_1, i_2), (j_1, j_2)) = (-1)^{i_1 j_2 - i_2 j_1}$,
4. $\Gamma = \mathbb{Z} \times \mathbb{Z}, \ \varepsilon((i_1, i_2), (j_1, j_2)) = (-1)^{(i_1 + i_2)(j_1 + j_2)}$.

**Definition 2.4.** A color Hom-algebra or a Hom-color algebra, $(A, \cdot, \varepsilon, \alpha)$ is a $\Gamma$-graded linear space $A$ equipped with even bilinear multiplication $\cdot$, even twisting map $\alpha$ and commutation factor $\varepsilon$.

**Definition 2.5.** A derivation of degree $d \in \Gamma$ on a color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$ is a linear map $D : A \to A$ such that for any $x, y \in \mathcal{H}(A)$,

$$D(x \cdot y) = D(x) \cdot y + \varepsilon(d, x)x \cdot D(y).$$

In particular, an even derivation $D : A \to A$ is a derivation of degree zero, i.e., for all $x, y \in \mathcal{H}(A)$,

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y).$$
2.1 $\varepsilon$-Commutative Hom-associative color Hom-algebras

**Definition 2.6** ([80]). A Hom-associative color Hom-algebra is a color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$ satisfying for $x, y, z \in H(A)$,

$$ \alpha s_A(x, y, z) := \alpha(x) \cdot (y \cdot z) - (x \cdot y) \cdot \alpha(z) = 0. \quad \text{(Hom-associativity)} \quad (2.1) $$

If in addition, for any $x, y \in H(A)$,

$$ x \cdot y = \varepsilon(x, y)y \cdot x, \quad (2.2) $$

then $(A, \cdot, \varepsilon, \alpha)$ is said to be a $\varepsilon$-commutative Hom-associative color Hom-algebra.

**Example 2.7.** Let $A = A_0 \oplus A_1 = \langle e_1 \rangle \oplus \langle e_2, e_3 \rangle$ be a 3-dimensional superspace. Then $A$ is a $\varepsilon$-commutative Hom-associative color Hom-algebra with

- the bicharacter: $\varepsilon(i, j) = (-1)^{ij}$,
- the multiplication: $e_1 \cdot e_2 = e_2 \cdot e_1 = -2e_3$,
- the even linear map $\alpha : A \to A$ defined by $\alpha(e_1) = \sqrt{2} e_1$, $\alpha(e_2) = e_3 - e_2$, $\alpha(e_3) = e_3$.

In the following, we introduce the notion of bimodule of $\varepsilon$-commutative Hom-associative color Hom-algebra.

**Definition 2.8.** Let $(A, \cdot, \varepsilon, \alpha)$ be an $\varepsilon$-commutative Hom-associative color Hom-algebra, $(V, \beta)$ be a pair consisting of $\Gamma$-graded linear space $V$ and an even linear map $\beta : V \to V$, and $s : A \to \text{End}(V)$ be an even linear map. The triple $(s, \beta, V)$ is called a bimodule of $(A, \cdot, \varepsilon, \alpha)$ if for all $x, y \in H(A), v \in H(V)$,

$$ s(x \cdot y)\beta(v) = s(\alpha(x))s(y)v. \quad (2.3) $$

**Proposition 2.9.** Let $(s, \beta, V)$ is a bimodule of a $\varepsilon$-commutative Hom-associative color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$. Then the direct sum of $\Gamma$-graded linear spaces,

$$ A \oplus V = \bigoplus_{\gamma \in \Gamma} (A \oplus V)_\gamma = \bigoplus_{\gamma \in \Gamma} (A_\gamma \oplus V_\gamma), $$

is turned into a $\varepsilon$-commutative Hom-associative color Hom-algebra by defining multiplication and the twisting map in $A \oplus V$ for all $X_1 = x_1 + v_1 \in A_{\gamma_1} \oplus V_{\gamma_1}, X_2 = x_2 + v_2 \in A_{\gamma_2} \oplus V_{\gamma_2}$ by

$$ (x_1 + v_1) \ast (x_2 + v_2) = x_1 \cdot x_2 + (s(x_1)v_2 + \varepsilon(v_1, x_2)s(x_2)v_1), \quad (2.4) $$

$$ (\alpha \ast \beta)(x_1 + v_1) = \alpha(x_1) + \beta(v_1). \quad (2.5) $$

**Proof.** We prove the commutativity and Hom-associativity in $A \oplus V$. For all elements $X_i = x_i + v_i \in A_{\gamma_i} \oplus V_{\gamma_i}, i = 1, 2, 3$,

$$ X_1 \ast X_2 = (x_1 + v_1) \ast (x_2 + v_2) $$

$$ = x_1 \cdot x_2 + (s(x_1)v_2 + \varepsilon(v_1, x_2)s(x_2)v_1) $$

$$ = \varepsilon(x_1, x_2)x_2 \cdot x_1 + (\varepsilon(v_1, x_2)s(x_2)v_1 + s(x_1)v_2) $$

$$ = \varepsilon(X_1, X_2)x_2 \cdot x_1 + (\varepsilon(X_1, X_2)s(x_2)v_1 + s(x_1)v_2) $$

$$ = \varepsilon(x_1, x_2)x_2 \cdot x_1 + \varepsilon(v_1, x_2)s(x_2)v_1 + s(x_1)v_2 $$

$$ = \varepsilon(x_1, x_2)x_2 \cdot x_1 + (\varepsilon(v_1, x_2)s(x_2)v_1 + s(x_1)v_2) $$

$$ = \varepsilon(x_1, x_2)x_2 \cdot x_1 + \varepsilon(v_1, x_2)s(x_2)v_1 + s(x_1)v_2 $$

$$ = \varepsilon(x_1, x_2)x_2 \cdot x_1 + (\varepsilon(v_1, x_2)s(x_2)v_1 + s(x_1)v_2) $$

$$ = \varepsilon(x_1, x_2)x_2 \cdot x_1 + \varepsilon(v_1, x_2)s(x_2)v_1 + s(x_1)v_2 $$

$$ = \varepsilon(x_1, x_2)x_2 \cdot x_1 + (\varepsilon(v_1, x_2)s(x_2)v_1 + s(x_1)v_2) $$

$$ = \varepsilon(x_1, x_2)x_2 \cdot x_1 + \varepsilon(v_1, x_2)s(x_2)v_1 + s(x_1)v_2 $$

$$ = \varepsilon(x_1, x_2)x_2 \cdot x_1 + (\varepsilon(v_1, x_2)s(x_2)v_1 + s(x_1)v_2) $$

$$ = \varepsilon(x_1, x_2)x_2 \cdot x_1 + \varepsilon(v_1, x_2)s(x_2)v_1 + s(x_1)v_2 $$
\[ = \varepsilon(X_1, X_2)(x_2 \cdot x_1 + s(x_2)v_1 + \varepsilon(X_2, X_1)s(x_1)v_2) \]
\[ = \varepsilon(X_1, X_2)(x_2 \cdot x_1 + s(x_2)v_1 + \varepsilon(v_2, x_1)s(x_1)v_2) \]
\[ = \varepsilon(X_1, X_2)(x_2 + v_2) \ast (x_1 + v_1) \]
\[ = \varepsilon(X_1, X_2)X_2 \ast X_1, \]
\[ (X_1 \ast X_2) \ast (\alpha + \beta)X_3 = (\alpha + \beta)X_1 \ast (X_2 \ast X_3) \]
\[ = ((x_1 + v_1) \ast (x_2 + v_2)) \ast (\alpha + \beta)(x_3 + v_3) \]
\[ = (x_1 \cdot x_2 + s(x_1)v_2 + \varepsilon(v_1, x_2)s(x_2)v_1) \ast (\alpha(x_3) + \beta(v_3)) \]
\[ = (x_1 \cdot x_2) \cdot \alpha(x_3) + s(x_1 \cdot x_2)\beta(v_3) + \varepsilon(x_1 + x_2, x_3)s(\alpha(x_3))s(x_1)v_2 \]
\[ + \varepsilon(x_1 + x_2, x_3)\varepsilon(x_1, x_2)s(\alpha(x_3))s(x_2)v_1 - \alpha(x_1) \cdot (x_2 \cdot x_3) \]
\[ - s(\alpha(x_1))s(x_2)v_3 - \varepsilon(x_2, x_3)s(\alpha(x_1))s(x_3)v_2 \]
\[ - \varepsilon(v_1, x_2 + x_3)s(x_2 \cdot x_3)\beta(v_1) \]
\[ = 0 \quad \text{by (2.1)} \]
\[ + \left( s(x_1 \cdot x_2)\beta(v_3) - s(\alpha(x_1))s(x_2)v_3 \right) \]
\[ = 0 \quad \text{by (2.3)} \]
\[ + \varepsilon(x_2, x_3)\left( \varepsilon(x_1, x_3)s(\alpha(x_3))s(x_1)v_2 - s(\alpha(x_1))s(x_3)v_2 \right) \]
\[ = 0 \quad \text{by (2.2) and (2.3)} \]
\[ + \varepsilon(x_1, x_2 + x_3)\left( \varepsilon(x_2, x_3)s(\alpha(x_3))s(x_2)v_1 - s(x_2 \cdot x_3)\beta(v_1) \right) = 0. \]

The \( \varepsilon \)-commutative Hom-associative color Hom-algebra constructed in Proposition 2.9 is denoted by \((A \oplus V, \ast, \varepsilon, \alpha + \beta)\) or \(A \ltimes_{s, \alpha, \beta} V\).

**Example 2.10.** Let \((A, \cdot, \varepsilon, \alpha)\) be a \(\varepsilon\)-commutative Hom-associative color Hom-algebra. Then, \((S, \varepsilon, \alpha)\) with \(S(x)y = x \cdot y\) for all \(x, y \in \mathcal{H}(A)\), is a bimodule of \((A, \cdot, \varepsilon, \alpha)\) called the regular bimodule of \((A, \cdot, \varepsilon, \alpha)\).

In the following, we introduce the notion of matched pair of \(\varepsilon\)-commutative Hom-associative color Hom-algebras.

**Proposition 2.11.** Let \((A, \cdot_A, \varepsilon, \alpha)\) and \((B, \cdot_B, \varepsilon, \beta)\) be \(\varepsilon\)-commutative Hom-associative color Hom-algebras. Suppose that there are even linear maps \(s_A : A \to \text{End}(B)\) and \(s_B : B \to \text{End}(A)\) such that \((s_A, \beta, B)\) is a bimodule of \(A\), and \((s_B, \alpha, A)\) is a bimodule of \(B\), satisfying, for any \(x, y \in \mathcal{H}(A), a, b \in \mathcal{H}(B)\), the following conditions:

\[
\varepsilon(b, x)\beta(a) \cdot_B (s_A(x)b) + \varepsilon(a, b + x)s_A(s_B(b)x)\beta(a) = \varepsilon(a + b, x)s_A(\alpha(x))(a \cdot_B b), \quad (2.6)
\]
\[
\beta(a) \cdot_B (s_A(x)b) + \varepsilon(a, x + b)\varepsilon(x, b)s_A(s_B(b)x)\beta(a) = \varepsilon(a, x)s_A(x)a \cdot_B \beta(b) + s_A(s_B(a)x)\beta(b), \quad (2.7)
\]
Next, we prove the Hom-associativity condition:

\begin{equation}
\varepsilon(y, a)\alpha(x) \cdot_A (s_B(a)y) + \varepsilon(x, y + a)s_B(s_A(y)a)\alpha(x) \\
= \varepsilon(x + y, a)s_B(\beta(a))(x \cdot_A y), \tag{2.8}
\end{equation}

\begin{equation}
\alpha(x) \cdot_A (s_B(a)y) + \varepsilon(x, a + y)\varepsilon(a, y)s_B(s_A(y)a)\alpha(x) \\
= \varepsilon(x, a)s_B(a)x \cdot_A \alpha(y) + s_B(s_A(x)a)\alpha(y). \tag{2.9}
\end{equation}

Then, \((A, B, l_A, r_A, \beta, l_B, r_B, \alpha)\) is called a matched pair of \(\varepsilon\)-commutative Hom-associative color Hom-algebras. In this case, there is a \(\varepsilon\)-commutative Hom-associative color Hom-algebra structure on the direct sum of the underlying \(\Gamma\)-graded linear spaces of \(A\) and \(B\),

\[
A \oplus B = \bigoplus_{\gamma \in \Gamma}(A \oplus B)_{\gamma} = \bigoplus_{\gamma \in \Gamma}(A_{\gamma} \oplus B_{\gamma}),
\]

given for all \(x + a \in A_{\gamma_1} \oplus B_{\gamma_1}, y + b \in A_{\gamma_2} \oplus B_{\gamma_2}\) by

\[
(x + a) \cdot (y + b) = (x \cdot_A y + s_B(a)y + \varepsilon(x, b)s_B(b)x) \\
+ (a \cdot_B b + s_A(x)b + \varepsilon(a, y)s_A(y)a); \tag{2.10}
\]

\[
(\alpha \oplus \beta)(x + a) = \alpha(x) + \beta(a). \tag{2.11}
\]

**Proof.** Let \(X = x + a \in A_{\gamma_1} \oplus B_{\gamma_1}, Y = y + b \in A_{\gamma_2} \oplus B_{\gamma_2}, Z = z + c \in A_{\gamma_3} \oplus B_{\gamma_3}\). First, we prove the commutativity condition:

\[
X \cdot Y - \varepsilon(X, Y)Y \cdot X = (x + a) \cdot (y + b) - \varepsilon(X, Y)(y + b) \cdot (x + a) \\
= x \cdot_A y + s_B(a)y + \varepsilon(x, b)s_B(b)x + a \cdot_B b + s_A(x)b + \varepsilon(a, y)s_A(y)a \\
- \varepsilon(X, Y)(y \cdot_A x + s_B(b)x + \varepsilon(y, a)s_B(a)y + b \cdot_B a + s_A(y)a + \varepsilon(b, x)s_A(x)b) \\
= (x \cdot_A y - \varepsilon(X, Y)y \cdot_A x) + (a \cdot_B b - \varepsilon(X, Y)b \cdot_B a) \\
+ (s_B(a)y - \varepsilon(X, Y)e(y, a)s_B(a)y) + (\varepsilon(x, b)s_B(b)x - \varepsilon(X, Y)s_B(b)x) \\
+ (s_A(x)b - \varepsilon(X, Y)e(b, x)s_A(x)b) + (\varepsilon(a, y)s_A(y)a - \varepsilon(X, Y)s_A(y)a) = 0.
\]

(using Remark 2.2 and (2.2))

Next, we prove the Hom-associativity condition:

\[
(X \cdot Y) \cdot (\alpha + \beta)Z = ((x + a) \cdot (y + b)) \cdot (\alpha + \beta)(z + c)
\]

(using Remark 2.2)

\[
= \left( (x \cdot_A y + s_B(a)y + \varepsilon(x, y)s_B(b)x) \\
+ (a \cdot_B b + s_A(x)b + \varepsilon(x, y)s_A(y)a) \right) \cdot (\alpha(z) + \beta(c)) \\
= \left( x \cdot_A y + s_B(a)y \cdot_A \alpha(z) + s_B(a)y \cdot_A \alpha(z) + \varepsilon(x, y)s_B(b)x \cdot_A \alpha(z) \\
+ s_B(a) \cdot_B b \cdot_A \alpha(z) + s_B(s_A(x)b) \cdot_A \alpha(z) + \varepsilon(x, y)s_B(s_A(y)a) \cdot_A \alpha(z) \\
+ \varepsilon(x + y, z) \left( s_B(\beta(c))(x \cdot_A y) + s_B(\beta(c))s_B(a)y \\
+ \varepsilon(x, y)s_B(\beta(c))s_B(b)x \right) \\
+ (a \cdot_B b \cdot_B \beta(c) + s_A(x)b \cdot_B \beta(c) + \varepsilon(x, y)s_A(y)a \cdot_B \beta(c)) \\
+ s_A(x \cdot_A y) \beta(c) + s_A(s_B(a)y) \beta(c) + \varepsilon(x, y)s_A(s_B(b)x) \beta(c) \\
+ \varepsilon(x + y, z) \left( s_A(\alpha(z))(a \cdot_B b) + s_A(\alpha(z))s_A(x)b + \varepsilon(x, y)s_A(\alpha(z))s_A(y)a \right)
\]

\]


Remark 2.13 \(\text{when the twisting linear map is the identity map.} \)
So, Novikov color Hom-algebras are a special case of Hom-Novikov color Hom-algebras.

2.2 On Hom-Novikov color Hom-algebras

\[A \vartriangleleft \vartriangleright B,\]

This \((\text{using Remark 2.2})\)

\[
\begin{align*}
\alpha \cdot (y \cdot z) &+ \alpha (x) \cdot A s_B(b)z + \alpha(y, z)\alpha(x) \cdot A s_B(c)y \\
+ s_B(\beta(a))(y \cdot A z) + s_B(\beta(a))s_B(b)z + \varepsilon(y, c)s_B(\beta(a))s_B(c)y \\
+ \varepsilon(x, y + z)\left(s_B(b \cdot B c)\alpha(x) + s_B(s_A(y)c)\alpha(x) \\
+ \varepsilon(b, z)s_B(s_A(z)b)\alpha(x)\right) + \beta(a) \cdot B (b \cdot B c) + \beta(a) \cdot B s_A(y)c \\
+ \varepsilon(b, z)\beta(a) \cdot B s_A(z)b + s_A(\alpha(x))(b \cdot B c) + s_A(\alpha(x))s_A(y)c \\
+ \varepsilon(x, z)s_A(\alpha(x))s_A(z)b + \varepsilon(x, y + z)\left(s_A(y \cdot A z)\beta(a) \\
+ s_A(s_B(b)z)\beta(a) + \varepsilon(y, z)s_A(s_B(c)y)\beta(a)\right).
\end{align*}
\]

Using (2.6)-(2.9) and that \((s_A, \beta, B)\) and \((s_B, \alpha, A)\) are bimodules of \((A, \cdot_A, \varepsilon, \alpha)\) and \((B, \cdot_B, \varepsilon, \beta)\), respectively, we derive that \((A \oplus B, \cdot, \varepsilon, \alpha + \beta)\) is \(\varepsilon\)-commutative Hom-associative color Hom-algebra. This completes the proof. \(\square\)

This \(\varepsilon\)-commutative Hom-associative color Hom-algebra, constructed in Proposition 2.11, is denoted by \((A \vartriangleright\left\triangleleft B, \cdot, \varepsilon, \alpha + \beta\right)\) or \(A \vartriangleleft\vartriangleright B\).

2.2 On Hom-Novikov color Hom-algebras

Definition 2.12 [8]). A color Hom-algebra \((A, \cdot, \varepsilon, \alpha)\) is called a Hom-Novikov color Hom-algebra if the following identities are satisfied for all \(x, y, z \in \mathcal{H}(A)\):

\[
\begin{align*}
(x \cdot y) \cdot A \alpha(z) - \alpha(x) \cdot (y \cdot z) &\varepsilon(x, y)((y \cdot x) \cdot A \alpha(z) - \alpha(y) \cdot (x \cdot z)), \quad (2.12) \\
(x \cdot y) \cdot A \alpha(z) &\varepsilon(y, z)(x \cdot z) \cdot A \alpha(y). \quad (2.13)
\end{align*}
\]

Remark 2.13. If \(\alpha = id_A\) in Definition 2.12, we recover a Novikov color Hom-algebra.
So, Novikov color Hom-algebras are a special case of Hom-Novikov color Hom-algebras when the twisting linear map is the identity map.
Examples 2.14 (Hom-Novikov color Hom-algebras). Here there are some examples of Hom-Novikov color Hom-algebras.

(i) Any $\varepsilon$-commutative Hom-associative color Hom-algebra is a Hom-Novikov color Hom-algebra.

(ii) Let $A = A_0 \oplus A_1 = \langle e_1, e_2 \rangle \oplus \langle e_3, e_4 \rangle$ be a 4-dimensional superspace. Then $A$ is a Hom-Novikov color Hom-algebra with

the bicharacter $\varepsilon(i, j) = (-1)^{ij}$,

the multiplication:

$e_1 \cdot e_1 = \lambda_1 e_2, \quad e_1 \cdot e_3 = \lambda_2 e_4,$

$e_3 \cdot e_3 = \lambda_3 e_2, \quad e_3 \cdot e_1 = \lambda_4 e_4,$

$\lambda_i \in \mathbb{K}$

the even linear map $\alpha : A \to A$ defined by $\alpha(e_1) = -e_1, \quad \alpha(e_2) = e_1 - e_2,$

$\alpha(e_3) = e_4, \quad \alpha(e_4) = e_3 + 2e_4.$

Proposition 2.15 ([8]). Let $A = (A, \cdot, \varepsilon)$ be a Novikov color Hom-algebra and $\alpha : A \to A$ be a Novikov color Hom-algebra morphism. Define $\cdot_\alpha : A \times A \to A$ for all $x, y \in \mathcal{H}(A)$, by $x \cdot_\alpha y = \alpha(x \cdot y)$. Then, $A_\alpha = (A, \cdot_\alpha, \varepsilon, \alpha)$ is a Hom-Novikov color Hom-algebra called the $\alpha$-twist or Yau twist of $(A, \cdot, \varepsilon)$.

In the following we introduce the notions of bimodule and matched pair of Novikov color Hom-algebras.

Definition 2.16. Let $(A, \cdot, \varepsilon, \alpha)$ be a Hom-Novikov color Hom-algebra, $(V, \beta)$ is a pair of $\Gamma$-graded linear space $V$ and an even linear map $\beta : V \to V$. Let $l, r : A \to \text{End}(V)$ be two even linear maps. The quadruple $(l, r, \beta, V)$ is called a bimodule of $(A, \cdot, \varepsilon, \alpha)$ if for all $x, y \in \mathcal{H}(A), v \in \mathcal{H}(V)$,

\begin{align*}
  l(x \cdot y)\beta(v) - l(\alpha(x))l(y)v &= \varepsilon(x, y)(l(y \cdot x)\beta(v) - l(\alpha(y))l(x)v), \quad (2.14) \\
  r(\alpha(y))l(x)v - l(\alpha(x))r(y)v &= \varepsilon(x, v)(r(\alpha(y))r(x)v - r(x \cdot y)\beta(v)), \quad (2.15) \\
  r(\alpha(y))r(x)v - r(y \cdot x)\beta(v) &= \varepsilon(v, x)(r(\alpha(y))l(x)v - l(\alpha(x))r(y)v), \quad (2.16) \\
  l(x \cdot y)\beta(v) &= \varepsilon(y, v)r(\alpha(y))l(x)v, \quad (2.17) \\
  r(\alpha(y))l(x)v &= \varepsilon(v, y)r(\alpha(x))l(y)v, \quad (2.18) \\
  r(\alpha(y))r(x)v &= \varepsilon(x, y)r(\alpha(x))r(y)v. \quad (2.19)
\end{align*}

Proposition 2.17. Let $(l, r, \beta, V)$ is a bimodule of a Hom-Novikov color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$. Then the direct sum of $\Gamma$-graded linear spaces,

\[ A \oplus V = \bigoplus_{\gamma \in \Gamma} (A \oplus V)_\gamma = \bigoplus_{\gamma \in \Gamma} (A_\gamma \oplus V_\gamma), \]

is turned into a Hom-Novikov color Hom-algebra by defining multiplication in $A \oplus V$ for all $X_1 = x_1 + v_1 \in A_{\gamma_1} \oplus V_{\gamma_1}, X_2 = x_2 + v_2 \in A_{\gamma_2} \oplus V_{\gamma_2}$ by

\begin{align*}
  (x_1 + v_1) * (x_2 + v_2) &= x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1), \quad (2.20) \\
  (\alpha \oplus \beta)(x_1 + v_1) &= \alpha(x_1) + \beta(v_1). \quad (2.21)
\end{align*}

Proof. We prove the axioms (2.12) and (2.13) in $A \oplus V$. For all $X_i = x_i + v_i \in A_{\gamma_i} \oplus V_{\gamma_i}, i \in \{1; 2; 3\}$,

\begin{align*}
  (X_1 * X_2) * (\alpha + \beta)X_3 - (\alpha + \beta)X_1 * (X_2 * X_3)
\end{align*}
\(- \varepsilon(X_1, X_2)((X_2 \ast X_1) \ast (\alpha + \beta)X_3 - (\alpha + \beta)X_2 \ast (X_1 \ast X_3))
\)

\[
= \left( (x_1 + v_1) \ast (x_2 + v_2) \right) \ast (\alpha + \beta)(x_3 + v_3) \\
- (\alpha + \beta)(x_1 + v_1) \ast ((x_2 + v_2) \ast (x_3 + v_3)) \\
- \varepsilon(X_1, X_2) \left( ((x_2 + v_2) \ast (x_1 + v_1)) \ast (\alpha + \beta)(x_3 + v_3) \\
- (\alpha + \beta)(x_2 + v_2) \ast ((x_1 + v_1) \ast (x_3 + v_3)) \right)
\]

\[
= (1 \cdot x_2 + l(x_1)v_2 + r(x_2)v_1) \ast (\alpha(x_2) + \beta(v_3)) \\
- (\alpha(x_1) + \beta(v_1))(x_2 \cdot x_3 + l(x_2)v_3 + r(x_3)v_2) \\
- \varepsilon(x_1, x_2) \left( (x_2 \cdot x_1 + l(x_2)v_1 + r(x_1)v_2) \ast (\alpha(x_3) + \beta(v_3)) \\
- (\alpha(x_2) + \beta(v_2)) \ast (1 \cdot x_3 + l(x_1)v_3 + r(x_3)v_1) \right)
\]

\[
= (x_1 \cdot x_2) \cdot \alpha(x_3) + l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))(l(x_1)v_2 + r(\alpha(x_3))r(x_2)v_1) \\
- \alpha(\alpha(x_1) \cdot x_3) - l(\alpha(x_1))(l(x_2)v_3 - l(\alpha(x_1))r(x_3)v_2 - r(\alpha(x_2) \cdot x_3)\beta(v_1)) \\
- \varepsilon(x_1, x_2) \left( (x_2 \cdot x_1) \cdot \alpha(x_3) + l(x_2 \cdot x_1)\beta(v_3) + r(\alpha(x_3))(l(x_2)v_3 - l(\alpha(x_2))r(x_3)v_1 \\
+ r(\alpha(x_3))r(x_1)v_2 - \alpha(\alpha(x_2)(x_1 \cdot x_3) - l(\alpha(x_2))l(x_1)v_3 - l(\alpha(x_2))r(x_3)v_1 \\
- r(\alpha(x_2) \cdot x_3)\beta(v_2) \right)
\]

\[
= (x_1 \cdot x_2) \cdot \alpha(x_3) - \alpha(x_1)(x_2 \cdot x_3) - \varepsilon(x_1, x_2)((x_2 \cdot x_1) \cdot \alpha(x_3) - \alpha(x_2) \cdot (x_1 \cdot x_3))
\]

\[
= 0 \text{ by (2.12) in } A
\]

\[
+ (l(x_1 \cdot x_2)\beta(v_3) - l(\alpha(x_1))(l(x_2)v_3 - \varepsilon(x_1, x_2)(l(x_2 \cdot x_1)\beta(v_3) - l(\alpha(x_2))l(x_1)v_3))
\]

\[
= 0 \text{ by (2.14)}
\]

\[
+ (r(\alpha(x_3))(l(x_1)v_2 - l(\alpha(x_1))r(x_3)v_2 - \varepsilon(x_1, x_2)(r(\alpha(x_3))r(x_1)v_2 - r(x_1 \cdot x_3)\beta(v_2))))
\]

\[
= 0 \text{ by (2.15)}
\]

\[
+ (r(\alpha(x_3))r(x_2)v_1 - r(x_2 \cdot x_3)\beta(v_1) - \varepsilon(x_1, x_2)(r(\alpha(x_3))l(x_2)v_1 - l(\alpha(x_2))r(x_3)v_1))
\]

\[
= 0 \text{ by (2.16)}
\]

\[
(X_1 \ast X_2) \ast (\alpha + \beta)X_3 - \varepsilon(X_2, X_3)(X_1 \ast X_3) \ast (\alpha + \beta)X_2
\]

\[
= ((x_1 + v_1) \ast (x_2 + v_2)) \ast (\alpha + \beta)(x_3 + v_3) \\
- \varepsilon(X_2, X_3)((x_1 + v_1) \ast (x_3 + v_3)) \ast (\alpha + \beta)(x_2 + v_2) \\
= (x_1 \cdot x_2 + l(x_1)v_2 + r(x_2)v_1) \ast (\alpha(x_3) + \beta(v_3)) \\
- \varepsilon(x_2, x_3)(x_1 \cdot x_3 + l(x_1)v_3 + r(x_3)v_1) \ast (\alpha(x_2) + \beta(v_2)) \\
= (x_1 \cdot x_2) \cdot \alpha(x_3) + l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))(l(x_1)v_2 + r(\alpha(x_3))r(x_2)v_1) \\
- \varepsilon(x_2, x_3)((x_1 \cdot x_3) \cdot \alpha(x_2) + l(x_1 \cdot x_3)\beta(v_2) + r(\alpha(x_2))(l(x_1)v_3 + r(\alpha(x_2))r(x_3)v_1)
\]

\[
= (x_1 \cdot x_2) \cdot \alpha(x_3) - (x_1 \cdot x_3) \cdot \alpha(x_2)) + (l(x_1 \cdot x_2)\beta(v_3) - \varepsilon(x_2, x_3)(r(\alpha(x_2))l(x_1)v_3)
\]

\[
= 0 \text{ by (2.13) in } A
\]

\[
= 0 \text{ by (2.17)}
\]

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Then, direct sum algebras. In this case, there is a Hom-Novikov color Hom-algebra structure on the linear spaces of \( A \perp \perp B \), we denote this Hom-Novikov color Hom-algebra by \( (A \perp \perp B, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \).

**Example 2.18.** Let \((A, \cdot, \cdot, \cdot, \cdot)\) be a Hom-Novikov color Hom-algebra. Then \((L, R, \alpha, A)\) is a bimodule of \((A, \cdot, \cdot, \cdot, \cdot)\), where \(L(x)y = x \cdot y\) and \(R(x)y = y \cdot x\) for all \(x, y \in H(A)\), called the regular bimodule of \((A, \cdot, \cdot, \cdot, \cdot)\).

**Proposition 2.19.** Let \((A, \cdot, \cdot, \cdot, \cdot)\) and \((B, \cdot, \cdot, \cdot, \cdot)\) be two Hom-Novikov color Hom-algebras. Suppose there are even linear maps \(l_A, r_A : A \to \text{End}(B)\) and \(l_B, r_B : B \to \text{End}(A)\) such that the quadruple \((l_A, r_A, \beta, B)\) is a bimodule of \(A\), and \((l_B, r_B, \alpha, A)\) is a bimodule of \(B\), satisfying, for any \(x, y \in H(A), a, b \in H(B)\), the following conditions:

\[
\begin{align*}
 r_A(\alpha(x))(a \cdot_B b) &- \beta(a) \cdot_B (r_A(x)b) - r_A(l_B(b)x)\beta(a) \\
&= \varepsilon(a, b)(r_A(\alpha(x))(b \cdot_B a) - \beta(b) \cdot_B (r_A(x)a) - r_A(l_B(a)x)\beta(b)), \\
(r_A(x)a) \cdot_B \beta(b) + l_A(l_B(a)x)\beta(b) - \beta(a) \cdot_B (l_A(x)b) - r_A(l_B(b)x)\beta(a) \\
&= \varepsilon(x, a)((l_A(x)a) \cdot_B \beta(b) + l_A(r_B(a)x)\beta(b) - l_A(\alpha(x))(a \cdot_B b)), \\
(l_A(x)a) \cdot_B \beta(b) - l_A(r_B(a)x)\beta(b) - l_A(\alpha(x))(a \cdot_B b) \\
&= \varepsilon(x, a)((r_A(x)a) \cdot_B \beta(b) + l_A(l_B(a)x)\beta(b) - \beta(a) \cdot_B (l_A(x)b) \\
&- r_A(l_B(b)x)\beta(b)), \\
\end{align*}
\]

\[(2.22)\]

\[
\begin{align*}
 r_B(\beta(a))(x \cdot_A y) - \alpha(x) \cdot_A (r_B(a)y) - r_B(l_A(y)a)\alpha(x) \\
&= \varepsilon(x, y)(r_B(\beta(a))(y \cdot_A x) - \alpha(y) \cdot_A (r_B(a)x) - r_B(l_A(x)a)\alpha(y)), \\
(r_B(a)x) \cdot_A \alpha(y) + l_B(l_A(x)a)\alpha(y) - \alpha(x) \cdot_A (l_B(a)y) - r_B(r_A(y)a)\alpha(x) \\
&= \varepsilon(x, a)((l_B(x)a) \cdot_A \alpha(y) + l_B(r_A(a)x)\alpha(y) - l_B(\beta(a))(x \cdot_A y)), \\
(l_B(a)x) \cdot_A \alpha(y) - l_B(r_A(x)a)\alpha(y) - l_B(\beta(a))(x \cdot_A y) \\
&= \varepsilon(x, a)((r_B(a)x) \cdot_A \alpha(y) + l_B(l_A(x)a)\alpha(y) - \alpha(x) \cdot_A (l_B(a)y) \\
&- r_B(r_A(y)a)\alpha(x)). \\
\end{align*}
\]

\[(2.25)\]

\[(2.26)\]

\[(2.27)\]

Then, \((A, B, l_A, r_A, \beta, l_B, r_B, \alpha)\) is called a matched pair of Hom-Novikov color Hom-algebras. In this case, there is a Hom-Novikov color Hom-algebra structure on the direct sum \(A \oplus B = \bigoplus_{\gamma \in \Gamma} (A \oplus B)_{\gamma} = \bigoplus_{\gamma \in \Gamma} (A_{\gamma} \oplus B_{\gamma})\), of the underlying \(\Gamma\)-graded linear spaces of \(A\) and \(B\) given for all \(x + a \in A_{\gamma_1} \oplus B_{\gamma_1}, y + b \in A_{\gamma_2} \oplus B_{\gamma_2}\) by

\[
\begin{align*}
(x + a) \cdot (y + b) &= (x \cdot_A y + l_B(a)y + r_B(b)x) + (a \cdot_B b + l_A(x)b + r_A(y)a), \\
(\alpha \oplus \beta)(x + a) &= \alpha(x) + \beta(a). \\
\end{align*}
\]

\[(2.28)\]

We denote this Hom-Novikov color Hom-algebra by \((A \bowtie B, \cdot, \cdot, \cdot, \cdot)\) or \(A \bowtie B_{l_A, r_A, \beta, l_B, r_B, \alpha}\).

### 2.3 On Hom-Lie color Hom-algebras

**Definition 2.20 (\([1, 19, 24, 55, 56, 69, 80]\)).** A Hom-Lie color Hom-algebra is a quadruple \((A, [\cdot, \cdot], \cdot, \cdot)\) consisting of a \(\Gamma\)-graded vector space \(A\), a bi-character \(\cdot, \cdot\), an even bilinear
mapping $[\cdot, \cdot] : A \times A \to A$, (i.e. $[A_a, A_b] \subseteq A_{a+b}$, for all $a, b \in \Gamma$) and an even homomorphism $\alpha : A \to A$ such that for homogeneous elements $x, y, z \in A$ we have

$$[x, y] = -\varepsilon(x, y)[y, x], \quad \varepsilon\text{-skew symmetry,} \quad (2.29)$$

$$\circlearrowleft_{x, y, z} \varepsilon(z, x)[\alpha(x), [y, z]] = 0, \quad \varepsilon\text{-Hom-Jacobi identity} \quad (2.30)$$

where $\circlearrowleft_{x, y, z}$ denotes the cyclic sum over $(x, y, z)$.

**Remark 2.21.** Hom-Lie color Hom-algebras contain ordinary Lie color algebras, Lie superalgebras and Lie algebras, as well as Hom-Lie superalgebras and Hom-Lie algebras for specific choices of the twisting map, grading group and commutation factor.

(i) When $\alpha = id_A$, one recovers Lie color algebras, and in particular if the grading group is $\mathbb{Z}_2$ and the commutation factor is defined as $\varepsilon(i, j) = (-1)^{ij}$ for all $i, j \in \mathbb{Z}$, then one gets Lie superalgebras. [7, 22, 23, 50, 59, 64–66].

(ii) When $\alpha = id_A$ and $A$ is trivially graded, by the group with one element, we recover Lie algebras.

(iii) When $A$ is trivially graded, while $\alpha$ is an arbitrary linear map, we recover Hom-Lie algebras [37, 45, 54–56], and if $A$ is graded by the group of two elements $\mathbb{Z}_2$, while $\alpha$ is an arbitrary even linear map, and the commutation factor is defined as $\varepsilon(i, j) = (-1)^{ij}$ for all $i, j \in \mathbb{Z}_2$, then we get Hom-Lie superalgebras.

**Proposition 2.22** ([8]). Let $(A, \cdot, \varepsilon, \alpha)$ be a Hom-Novikov color Hom-algebra. Then, there exists a Hom-Lie color algebra structure on $A$ given for all $x, y \in H(A)$ by

$$[x, y] = x \cdot y - \varepsilon(x, y)y \cdot x. \quad (2.31)$$

**Example 2.23.** Let $A = A_0 \oplus A_1 = \langle e_1 \rangle \oplus \langle e_2, e_3 \rangle$ be a 3-dimensional superspace. The quintuple $(A, \cdot, \varepsilon, \alpha)$ is a Hom-Novikov color Hom-algebra with

- the multiplication: $e_1 \cdot e_2 = e_3, \quad e_2 \cdot e_1 = -e_3$,
- the bicharacter: $\varepsilon(i, j) = (-1)^{ij}$,
- the even linear map $\alpha : A \to A$ defined by $\alpha(e_1) = -2e_1, \quad \alpha(e_2) = e_3, \quad \alpha(e_3) = e_2 - e_3$.

Therefore, by Proposition 2.22, $(A, [\cdot, \cdot], \varepsilon, \alpha)$ is a Hom-Lie color Hom-algebra with

$$[e_1, e_2] = -[e_2, e_1] = 2e_3.$$

**Proposition 2.24** ([1]). Let $A = (A, [\cdot, \cdot], \varepsilon)$ be a Hom-Lie color Hom-algebra and $\alpha : A \to A$ be a Hom-Lie color Hom-algebras morphism. Define $[\cdot, \cdot]_\alpha : A \times A \to A$ for all $x, y \in A$, by $[x, y]_\alpha = \alpha([x, y])$. Then, $A_\alpha = (A, [\cdot, \cdot]_\alpha, \varepsilon, \alpha)$ is a Hom-Lie color Hom-algebra called the $\alpha$-twist or Yau twist of $(A, [\cdot, \cdot], \varepsilon)$.

**Definition 2.25.** Let $(A, [\cdot, \cdot], \varepsilon, \alpha)$ be a Hom-Lie color Hom-algebra, $(V, \beta)$ is a pair of $\Gamma$-graded linear space $V$ and an even linear map $\beta : V \to V$. Let $\rho : A \to \text{End}(V)$ an even linear map. The triple $(\rho, \beta, V)$ is called a representation of $(A, [\cdot, \cdot], \varepsilon, \alpha)$ if for all $x, y \in H(A), v \in H(V)$,

$$\rho([x, y])\beta(v) = \rho(\alpha(x))\rho(y)v - \varepsilon(x, y)\rho(\alpha(y))\rho(x)v. \quad (2.32)$$
Proposition 2.26. Let $(\rho, \beta, V)$ is a representation of a Hom-Lie color Hom-algebra $(A, [\cdot, \cdot], \varepsilon, \alpha)$. Then the direct sum of $\Gamma$-graded linear spaces,

$$A \oplus V = \bigoplus_{\gamma \in \Gamma} (A \oplus V)_\gamma = \bigoplus_{\gamma \in \Gamma} (A_\gamma \oplus V_\gamma),$$

is turned into a Hom-Lie color Hom-algebra by defining multiplication in $A \oplus V$ for all $X_1 = x_1 + v_1 \in A_{\gamma_1} \oplus V_{\gamma_1}, X_2 = x_2 + v_2 \in A_{\gamma_2} \oplus V_{\gamma_2}$ by

\begin{align*}
[x_1 + v_1, x_2 + v_2] &= [x_1, x_2] + \rho(x_1)v_2 - \varepsilon(v_1, x_2)\rho(x_2)v_1, \\
(\alpha \oplus \beta)(x_1 + v_1) &= \alpha(x_1) + \beta(v_1).
\end{align*}

(2.33)

(2.34)

The Hom-Lie color Hom-algebra constructed in previous Proposition is denoted by $(A \oplus V, [\cdot, \cdot], \varepsilon, \alpha + \beta)$ or $A \ltimes_{\alpha, \beta} V$.

Example 2.27. Let $(A, [\cdot, \cdot], \varepsilon, \alpha)$ be a Hom-Lie algebra. Then $(ad, \alpha, A)$ is a representation of $(A, [\cdot, \cdot], \varepsilon, \alpha)$, where $ad(x)y = [x, y]$ for all $x, y \in \mathcal{H}(A)$, called the adjoint representation of $(A, [\cdot, \cdot], \varepsilon, \alpha)$.

Now, we introduce the notion of matched pair of Hom-Lie color Hom-algebra

Proposition 2.28. Suppose that $(A, [\cdot, \cdot], \varepsilon, \alpha)$ and $(B, [\cdot, \cdot], \varepsilon, \beta)$ are Hom-Lie color Hom-algebras, and there are even linear maps $\rho_A : A \to \text{End}(B)$ and $\rho_B : B \to \text{End}(A)$ such that $(\rho_A, \beta, B)$ is a representation of $A$ and $(\rho_B, \alpha, A)$ is a representation of $B$ satisfying for any $x, y \in \mathcal{H}(A), a, b \in \mathcal{H}(B)$,

\begin{align*}
\varepsilon(x, a)(\rho_A(\rho_B(a)x)\beta(b) - [\beta(a), \rho_A(x)]B) + \varepsilon(a + x, b)([\beta(b), \rho_A(x)]A - \rho_A(\rho_B(b)x)\beta(a)) + \rho_A(\alpha(x))(a, b)_B = 0, \\
\varepsilon(a, x)(\rho_B(\rho_A(x)a)\alpha(y) - [\alpha(x), \rho_B(a)y]A) + \varepsilon(x + a, y)([\alpha(y), \rho_B(a)x]A - \rho_B(\rho_A(y)a)\alpha(x)) + \rho_B(\beta(a))(x, y)_A = 0.
\end{align*}

(2.35)

Then, $(A, B, \rho_A, \beta, \rho_B, \alpha)$ is called a matched pair of Hom-Lie color Hom-algebras. In this case, there is a Hom-Lie color Hom-algebra structure on the linear space of the underlying $\Gamma$-graded linear spaces of $A$ and $B$,

$$A \oplus B = \bigoplus_{\gamma \in \Gamma} (A \oplus B)_\gamma = \bigoplus_{\gamma \in \Gamma} (A_\gamma \oplus B_\gamma),$$

given for all $x + a \in A_{\gamma_1} \oplus B_{\gamma_1}, y + b \in A_{\gamma_2} \oplus B_{\gamma_2}$ by

\begin{align*}
[x + a, y + b] &= [x, y]A + \rho_A(x)b - \varepsilon(a, y)\rho_A(y)a + [a, b]_B + \rho_B(a)y - \varepsilon(x, b)\rho_B(b)x, \\
(\alpha \oplus \beta)(x + a) &= \alpha(x) + \beta(a).
\end{align*}

(2.36)

(2.37)

3 Admissible Hom-Novikov-Poisson color Hom-algebras

In this Section, we recall the main result of Hom-Novikov-Poisson color Hom-algebras in [5] and we introduce their notions of bimodules and matched pairs. Next, we introduce the definition of admissible Hom-Novikov-Poisson color Hom-algebras and we give some explicit constructions. Finally, we show that the much larger class of admissible Hom-Novikov-Poisson color Hom-algebras is also closed under tensor products.
3.1 Constructions and bimodules of (admissible) Hom-Novikov-Poisson color Hom-algebras

**Definition 3.1** ([5]). Hom-Novikov-Poisson color Hom-algebras are defined as quintuples \((A, \cdot, \circ, \varepsilon, \alpha)\) consisting of an \(\varepsilon\)-commutative Hom-associative color Hom-algebra \((A, \cdot, \varepsilon, \alpha)\) and a Hom-Novikov color Hom-algebra \((A, \circ, \varepsilon, \alpha)\) obeying for \(x, y, z \in \mathcal{H}(A)\),
\[
(x \cdot y) \circ \alpha(z) = \varepsilon(y, z)(x \circ z) \cdot \alpha(y),
\]
\[
(x \circ y) \cdot \alpha(z) - \alpha(x) \circ (y \cdot z) = \varepsilon(x, y)((y \circ x) \cdot \alpha(z) - \alpha(y) \circ (x \cdot z)).
\]

A Hom-Novikov-Poisson color Hom-algebra is called multiplicative if the linear map \(\alpha : A \to A\) is multiplicative with respect to \(\cdot\) and \(\circ\), that is, for all \(x, y \in \mathcal{H}(A)\),
\[
\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y), \quad \alpha(x \circ y) = \alpha(x) \circ \alpha(y).
\]

**Remark 3.2.** Hom-Novikov-Poisson color Hom-algebras contain Novikov-Poisson color Hom-algebras, Hom-Novikov-Poisson algebras and Novikov-Poisson algebras for special choices of the twisting map and grading group.

(i) When \(\alpha = \text{id}\), we get Novikov-Poisson color Hom-algebra.

(ii) When \(\Gamma = \{e\}\) and \(\alpha \neq \text{id}\), we get Hom-Novikov-Poisson algebra [85].

(iii) When \(\Gamma = \{e\}\) and \(\alpha = \text{id}\), we get Novikov-Poisson algebra [77, 78].

**Example 3.3.** Let \(A = A_0 \oplus A_1 = \langle e_1, e_2 \rangle \oplus \langle e_3, e_4 \rangle\) be a 4-dimensional superspace. Then \((A, \cdot, \circ, \varepsilon, \alpha)\) is a Hom-Novikov-Poisson color Hom-algebra with

the bicharacter \(\varepsilon(i, j) = (-1)^{ij}\),

the multiplications: \(e_2 \cdot e_2 = \lambda_1 e_1, \ e_2 \cdot e_4 = e_4 \cdot e_2 = \lambda_2 e_3, \ \lambda_i \in \mathbb{K}\),
\[
e_2 \circ e_4 = \mu_2 e_3, \quad e_4 \circ e_2 = \mu_3 e_3, \quad e_4 \circ e_4 = \mu_4 e_1, \quad \mu_i \in \mathbb{K},
\]

the even linear map \(\alpha : A \to A\) defined by \(\alpha(e_1) = 2e_1, \ \alpha(e_2) = e_2 - e_1, \ \alpha(e_3) = -e_4, \ \alpha(e_4) = e_3\).

**Definition 3.4.** Let \((A, \cdot, \circ, \alpha)\) and \((A', \cdot', \circ', \alpha')\) be Hom-Novikov-Poisson color Hom-algebras. A linear map of degree zero \(f : A \to A'\) is a Hom-Novikov-Poisson color Hom-algebra morphism if
\[
\cdot' \circ (f \otimes f) = f \circ \cdot, \quad \circ' \circ (f \otimes f) = f \circ \circ \text{ and } f \circ \alpha = \alpha' \circ f.
\]

**Theorem 3.5** ([5]). Let \(A = (A, \cdot, \circ, \varepsilon)\) be a Hom-Novikov-Poisson color Hom-algebra and \(\alpha : A \to A\) be a Hom-Novikov-Poisson color Hom-algebras morphism. Define \(\cdot_\alpha, \circ_\alpha : A \times A \to A\) for all \(x, y \in \mathcal{H}(A)\), by \(x \cdot_\alpha y = \alpha(x \cdot y)\) and \(x \circ_\alpha y = \alpha(x \circ y)\). Then, \(A_\alpha = (A_\alpha = (A_\alpha = (A_{\alpha}, \cdot_\alpha, \circ_\alpha, \varepsilon, \alpha))\) is a Hom-Novikov-Poisson color Hom-algebra called the \(\alpha\)-twist or Yau twist of \((A, \cdot, \circ, \varepsilon)\).

**Definition 3.6.** Let \((A, \cdot, \circ, \varepsilon, \alpha)\) be a Hom-Novikov-Poisson color Hom-algebra. A bimodule of \((A, \cdot, \circ, \varepsilon, \alpha)\) is a quintuple \((s, l, r, \beta, V)\) such that \((s, \beta, V)\) is a bimodule of the \(\varepsilon\)-commutative Hom-associative color Hom-algebra \((A, \cdot, \varepsilon, \alpha)\) and \((l, r, \beta, V)\) is
a bimodule of the Hom-Novikov color Hom-algebra \((A, \diamond, \varepsilon, \alpha)\) and satisfying, for all \(x, y \in \mathcal{H}(A), v \in \mathcal{H}(V)\),

\[
\begin{align*}
  l(x \cdot y)\beta(v) &= \varepsilon(x, y)s(\alpha(y))l(x)v, \\
  r(\alpha(y))s(x)v &= \varepsilon(v, y)s(x \diamond y)\beta(v), \\
  r(\alpha(y))s(x)v &= s(\alpha(x))r(y)v, \\
  s(x \diamond y)\beta(v) - l(\alpha(x))s(y)v &= \varepsilon(x, y)(s(y \diamond x)\beta(v) - l(\alpha(y))s(x)v), \\
  \varepsilon(x + v, y)(s(\alpha(y))l(x)v - \varepsilon(x, v)s(\alpha(y))r(x)v) &= \varepsilon(v, y)(l(\alpha(x))s(y)v - \varepsilon(x, v)r(x \cdot y)\beta(v)).
\end{align*}
\]

(3.3)

(3.4)

(3.5)

(3.6)

(3.7)

**Proposition 3.7.** Let \(A \oplus V = \bigoplus_{\gamma \in \Gamma} (A \oplus V)_\gamma = \bigoplus_{\gamma \in \Gamma} (A_\gamma \oplus V_\gamma)\), the direct sum of \(\Gamma\)-graded linear spaces. Then, \((A \oplus V, \cdot', \diamond', \varepsilon, \alpha + \beta)\) is a Hom-Novikov-Poisson color Hom-algebra, where \((A \oplus V, \cdot', \varepsilon, \alpha + \beta)\) is the semi-direct product \(\varepsilon\)-commutative Hom-associative color Hom-algebra \(A \ltimes s, \alpha, \beta V\) and \((A \oplus V, \diamond', \varepsilon, \alpha + \beta)\) is the semi-direct product Hom-Novikov color Hom-algebra \(A \ltimes t, r, \alpha, \beta V\).

**Proof.** Let \((A, \cdot, \diamond, \varepsilon, \alpha)\) be a Hom-Novikov-Poisson color Hom-algebra, and \((s, t, r, \beta, V)\) be a bimodule. By Proposition 2.9 and Proposition 2.17, \((A \oplus V, \cdot', \varepsilon, \alpha + \beta)\) is a \(\varepsilon\)-commutative Hom-associative color Hom-algebra, and \((A \oplus V, \diamond', \varepsilon, \alpha + \beta)\) is a Hom-Novikov color Hom-algebra respectively. Now, we show that the compatibility conditions (3.1)-(3.2) are satisfied. For all \(X_i = x_i + v_i \in A_{\gamma_i} \oplus V_{\nu_i}, i = 1, 2, 3\) we have

\[
\begin{align*}
(X_1 \cdot' X_2) \diamond' (\alpha + \beta)X_3 - \varepsilon(X_2, X_3)(X_1 \diamond' X_3) \cdot' (\alpha + \beta)X_2 \\
&= ((x_1 + v_1) \cdot' (x_2 + v_2)) \diamond' (\alpha + \beta)(x_3 + v_3) \\
&\quad - \varepsilon(x_2, x_3)((x_1 + v_1) \diamond' (x_3 + v_3)) \cdot' (\alpha + \beta)(x_2 + v_2) \\
&= (x_1 \cdot x_2 + s(x_1)v_2 + \varepsilon(v_1, x_2)s(x_2)v_1) \diamond' (\alpha(x_3) + \beta(v_3)) \\
&\quad - \varepsilon(x_2, x_3)((x_1 \diamond x_3 + l(x_1)v_3 + r(x_3)v_1) \cdot' (\alpha(x_2) + \beta(v_2)) \\
&= (x_1 \cdot x_3) \diamond' l(x_1 \cdot x_2) + s(\alpha(x_2))s(x_2)v_3 + r(\alpha(x_3))s(x_1)v_2 \\
&\quad + \varepsilon(v_1, x_2)r(\alpha(x_3))s(x_2)v_1 - \varepsilon(x_2, x_3)(x_1 \diamond x_3) \cdot' l(x_1)v_3 + r(\alpha(x_3))s(x_1)v_2 \\
&\quad - \varepsilon(x_1, x_2)(s(\alpha(x_2))l(x_1)v_3 + s(\alpha(x_2))r(x_3)v_1) \\
&\quad = 0 \text{ by (3.1)} \\
&\quad + \left(l(x_1 \cdot x_2)\beta(v_3) - \varepsilon(x_1, x_2)s(\alpha(x_2))l(x_1)v_3\right) \\
&\quad = 0 \text{ by (3.3)} \\
&\quad + \left(r(\alpha(x_3))s(x_1)v_2 - \varepsilon(x_2, x_3)s(x_1 \diamond x_3)\beta(v_2)\right) \\
&\quad = 0 \text{ by (3.4) and Remark 2.2} \\
&\quad + \varepsilon(x_1, x_2)\left(r(\alpha(x_3))s(x_2)v_1 - s(\alpha(x_2))r(x_3)v_1\right) = 0, \\
&\quad = 0 \text{ by (3.5)} \\
(X_1 \diamond' X_2) \cdot' (\alpha + \beta)X_3 - (\alpha + \beta)X_1 \diamond' (X_2 \cdot' X_3)
\end{align*}
\]
\[-\varepsilon(X_1, X_2)((X_2 \circ' X_1) \cdot (\alpha + \beta)X_3 - (\alpha + \beta)X_2 \circ' (X_1 \cdot X_3))\]
\[
= ((x_1 + v_1) \circ' (x_2 + v_2)) \cdot (\alpha + \beta)(x_3 + v_3)
- (\alpha + \beta)(x_1 + v_1) \circ' ((x_2 + v_2) \cdot (x_3 + v_3))
- \varepsilon(X_1, X_2)((x_2 + v_2) \circ' (x_1 + v_1)) \cdot (\alpha + \beta)(x_3 + v_3)
- (\alpha + \beta)(x_2 + v_2) \circ' ((x_1 + v_1) \cdot (x_3 + v_3))
\]
\[
= (x_1 \circ x_2)\alpha(x_3) + s(x_1 \circ x_2)\beta(v_3) + \varepsilon(x_1 + x_2, v_3)s(\alpha(x_3))l(x_1)v_2
+ \varepsilon(x_1 + x_2, v_3)s(\alpha(x_3))r(x_2)v_1 - \alpha(x_1) \circ (x_2 \cdot x_3)
- l(\alpha(x_1))s(x_2)v_3 + \varepsilon(v_2, x_3)l(\alpha(x_1))s(x_3)v_2 + r(x_2 \cdot x_3)\beta(v_1)
- \varepsilon(x_1, x_2)(x_2 \circ x_1) \cdot \alpha(x_3) + s(x_2 \circ x_1)\beta(v_3) + \varepsilon(x_1 + x_2, v_3)s(\alpha(x_3))l(x_2)v_1
+ \varepsilon(x_1 + x_2, v_3)s(\alpha(x_3))r(x_1)v_2 - \alpha(x_2) \circ (x_1 \cdot x_3) - l(\alpha(x_2))s(x_1)v_3
+ \varepsilon(v_1, x_3)l(\alpha(x_2))s(x_3)v_1 + r(x_1 \cdot x_3)\beta(v_2))
\]
\[
= \left(\varepsilon(x_1, x_2)((x_2 \circ x_1) \cdot \alpha(x_3) - \alpha(x_2) \circ (x_1 \cdot x_3))\right)
- \varepsilon(x_1, x_2)(\alpha(x_1) \circ (x_2 \cdot x_3))
\]
\[
= 0 \text{ by (3.2)}
\]
\[
+ \left(\varepsilon(x_1, x_2)\alpha(x_3) - l(\alpha(x_1))s(x_2)v_3
- \varepsilon(x_1, x_2)s(x_2 \circ x_1)\beta(v_3) - l(\alpha(x_2))s(x_1)v_3\right)
= 0 \text{ by (3.6)}
\]
\[
+ \left(\varepsilon(x_1 + x_2, v_3)l(x_1)v_2 - \varepsilon(x_1, x_2)s(\alpha(x_3))r(x_1)v_2
- \varepsilon(x_1, x_2)r(x_1 \cdot x_3)\beta(v_2) + \varepsilon(x_2, v_3)l(\alpha(x_1))s(x_3)v_2\right)
= 0 \text{ by (3.7) and Remark 2.2}
\]
\[
- \varepsilon(x_1, x_2)\left(\varepsilon(x_1 + x_2, v_3)s(\alpha(x_3))l(\alpha(x_2))v_1 - \varepsilon(x_2, x_1)s(\alpha(x_3))r(x_2)v_1
- \varepsilon(x_1, x_3)l(\alpha(x_2))s(x_3)v_1 + \varepsilon(x_2, x_1)r(x_2 \cdot x_3)\beta(v_1)\right)
= 0 \text{ by (3.7) and Remark 2.2}
\]

Hence, \((A \oplus V, \cdot, [\cdot, \cdot], \varepsilon, \alpha + \beta)\) is a Hom-Novikov-Poisson color Hom-algebra.

The Hom-Novikov-Poisson color Hom-algebra constructed in previous Proposition is denoted by \(A \bowtie s, l, r, \alpha, \beta, V\).

**Examples 3.8.** (i) Let \((A, \cdot, \circ, \varepsilon, \alpha)\) be a Hom-Novikov-Poisson color Hom-algebra, and let \(S(x)y = x \cdot y = \varepsilon(x, y)y \cdot x, L(x)y = x \circ y\) and \(R(x, y) = y \circ x\), for all \(x, y \in \mathcal{H}(A)\). Then, \((S, L, R, \alpha, A)\) is a bimodule of \((A, \cdot, \circ, \varepsilon, \alpha)\), called the regular bimodule of \((A, \cdot, \circ, \varepsilon, \alpha)\).
Theorem 3.9. Suppose that $A = (A, \cdot, \circ, \varepsilon, \alpha)$ is a Hom-Novikov-Poisson color Hom-algebras morphism, then $(s, l, r, \beta, A')$ becomes a bimodule of $A$ via $f$, that is, for all $(x, y) \in \mathcal{H}(A) \times \mathcal{H}(A')$, $s(x)y = f(x) \cdot_2 y$, $l(x)y = f(x) \circ_2 y$, $r(x)y = y \circ_2 f(x)$.

Then, $(A, B, s_A, l_A, r_A, \beta, s_B, l_B, r_B, \alpha)$ is called a matched pair of the Hom-Novikov-Poisson color Hom-algebras. In this case, on the direct sum $A \oplus B$ of the underlying linear spaces of $A$ and $B$, there is a Hom-Novikov-Poisson color Hom-algebra structure.
which is given for any \( x + a \in A_{\gamma_1} \oplus B_{\gamma_1}, \ y + b \in A_{\gamma_2} \oplus B_{\gamma_2} \) by

\[
(x + a) \cdot (y + b) = x \cdot_A y + (s_A(x)b + \varepsilon(a, y)s_A(y)a) + a \cdot_B b + (s_B(a)y + \varepsilon(x, b)s_B(b)x),
\]

\[ (x + a) \circ (y + b) = x \circ_A y + (I_A(x)b + r_A(y)a) + a \circ_B b + (I_B(a)y + r_B(b)x). \]

**Proof.** By Proposition 2.11 and Proposition 2.19, we deduce that \((A \oplus B, \cdot, \varepsilon, \alpha + \beta)\) is a \(\varepsilon\)-commutative Hom-associative color Hom-algebra and \((A \oplus B, \circ, \alpha + \beta)\) is a Hom-Novikov color Hom-algebra. Now, the rest, it is easy (in a similar way as for Proposition 3.11) to verify the compatibility conditions are satisfied. \(\Box\)

**Definition 3.10.** A transposed Hom-Poisson color Hom-algebra is defined as a quintuple \((A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)\), where \((A, \cdot, \varepsilon, \alpha)\) is a \(\varepsilon\)-commutative Hom-associative color Hom-algebra and \((A, [\cdot, \cdot], \varepsilon, \alpha)\) is a Hom-Lie color Hom-algebra, satisfying the transposed Hom-\(\varepsilon\)-Leibniz identity for \(x, y, z \in \mathcal{H}(A)\),

\[
2\alpha(z) \cdot [x, y] = [z \cdot x, \alpha(y)] + \varepsilon(z, x)[\alpha(x), z \cdot y].
\]

**Proposition 3.11.** Let \((A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)\) be a multiplicative transposed Hom-Poission color Hom-algebra. Then the following identities hold for all \(h, x, y, z \in \mathcal{H}(A)\),

\[
\circ_{x,y,z} \varepsilon(z, x)\alpha(x) \cdot [y, z] = 0,
\]

\[
\circ_{x,y,z} \varepsilon(z, x)[\alpha(h) \cdot [x, y], \alpha^2(z)] = 0,
\]

\[
\circ_{x,y,z} \varepsilon(z, x)[\alpha(h) \cdot \alpha(x), [\alpha(y), \alpha(z)]] = 0,
\]

\[
\circ_{x,y,z} \varepsilon(z, x)[\alpha(h), \alpha(x)] \cdot [\alpha(y), \alpha(z)] = 0.
\]

**Proof.**

**Proof of (3.22):** Let \(x, y, z \in \mathcal{H}(A)\). By the transposed Hom-\(\varepsilon\)-Leibniz identity,

\[
\circ_{x,y,z} \varepsilon(z, x)\left(2\alpha(x) \cdot [y, z]\right) = \circ_{x,y,z} \varepsilon(z, x)\left([x \cdot y, \alpha(z)] + \varepsilon(x, y)[\alpha(y), x \cdot z]\right)
\]

\[
= \circ_{x,y,z} \varepsilon(z, x)\left([x \cdot y, \alpha(z)] + \varepsilon(x + y, z)[z \cdot x, \alpha(y)]\right)
\]

\[
= \circ_{x,y,z} \left(\varepsilon(z, x)[x \cdot y, \alpha(z)] - \varepsilon(y, z)[z \cdot x, \alpha(y)]\right) = 0.
\]

Then we have (3.22).

**Proof of (3.23):** Let \(x, y, z, h \in \mathcal{H}(A)\). First, by (3.21), we have

\[
\circ_{x,y,z} \varepsilon(z, x)\left(2\alpha^2(h)[[x, y], \alpha(z)]\right) = \circ_{x,y,z} \varepsilon(z, x)\left(\varepsilon(h, x + y)[\alpha([x, y]), \alpha(h \cdot z)]\right).
\]

Applying the Hom-Jacobi identity of the above equality, we obtain

\[
\circ_{x,y,z} \varepsilon(z, x)\left([\alpha(h), [x, y], \alpha^2(z)]\right)
\]

\[
+ \circ_{x,y,z} \varepsilon(z, x)\left(\varepsilon(h, x + y)[\alpha([x, y]), \alpha(h \cdot z)]\right) = 0.
\]

Next, by the Hom-Jacobi identity, we have

\[
\varepsilon(h, x + y)[[\alpha(x), \alpha(y)], \alpha(h \cdot z)] + \varepsilon(x + y, z)[[h \cdot z, \alpha(x)], \alpha^2(y)]
\]
Taking the difference between the two equations (3.23) and (3.26) we obtain

\[
\varepsilon(h, y)\varepsilon(x, y + z)[[\alpha(y), h \cdot z], \alpha^2(x)] = 0,
\]

and by (3.21) we have

\[
\begin{align*}
\varepsilon(h, y)\varepsilon(x, y + z)[[\alpha(y), h \cdot z], \alpha^2(x)] &= 2\varepsilon(x, y + z)[\alpha(h) \cdot [y, z], \alpha^2(x)], \\
- \varepsilon(x, y + z)[[h \cdot y, \alpha(z)], \alpha^2(x)] &= 0.
\end{align*}
\]

Thus we obtain

\[
\begin{align*}
\varepsilon(h, x + y)[[\alpha(x), \alpha(y)], \alpha(h \cdot z)] + \varepsilon(x + y, z)[[h \cdot z, \alpha(x)], \alpha^2(y)]
\end{align*}
\]

\[
+ \varepsilon(x, y + z)(2[\alpha(h) \cdot [y, z], \alpha^2(x)] - \varepsilon(x, y + z)[[h \cdot y, \alpha(z)], \alpha^2(x)]) = 0.
\]

Similarly, we have

\[
\begin{align*}
\circ_{x,y,z} \varepsilon(z, x)\left(\varepsilon(h, x + y)[[\alpha(x), \alpha(y)], \alpha(h \cdot z)] + \varepsilon(x + y, z)[[h \cdot z, \alpha(x)], \alpha^2(y)]
\end{align*}
\]

\[
+ \varepsilon(x, y + z)(2[\alpha(h) \cdot [y, z], \alpha^2(x)] - \varepsilon(x, y + z)[[h \cdot y, \alpha(z)], \alpha^2(x)]) = 0.
\]

Taking the above sum, we obtain

\[
\begin{align*}
\circ_{x,y,z} \varepsilon(z, x)\left(\varepsilon(h, x + y)[[\alpha(x), \alpha(y)], \alpha(h \cdot z)]
\end{align*}
\]

\[
+ \circ_{x,y,z} \varepsilon(z, x)\left(2[\alpha(h) \cdot [x, y], \alpha^2(x)]\right) = 0. \tag{3.27}
\]

Finally, taking the difference between the two equations (3.26) and (3.27) we obtain (3.28).

**Proof of (3.24):** Taking the difference between the two equations (3.23) and (3.26) we obtain (3.24).

**Proof of (3.25):** Let \(x, y, z, h \in H(A)\). By (3.21) we have

\[
\begin{align*}
\circ_{x,y,z} \varepsilon(z, x)\left(2\varepsilon(h, x + y)[[\alpha(x), \alpha(y)], \alpha(h \cdot z)]
\end{align*}
\]

\[
= \circ_{x,y,z} \varepsilon(z, x)\left([\alpha(h) \cdot [x, y], \alpha^2(z)] + \varepsilon(x + y, z)[\alpha^2(h), \alpha(z) \cdot [x, y]]\right).
\]

Applying equations (3.22) and (3.23) to above equality, we obtain (3.25). \(\square\)

**Proposition 3.12.** Let \((A, \cdot, \varepsilon, \alpha)\) be a Hom-Novikov-Poisson color Hom-algebra. Define

\[
[x, y] = x \circ y - \varepsilon(x, y)y \circ x, \quad \forall x, y \in H(A). \tag{3.28}
\]

Then \((A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)\) is a Hom-transposed-Poisson color Hom-algebra.

**Proof.** By definition, we have \((A, \cdot, \varepsilon, \alpha)\) is a \(\varepsilon\)-commutative Hom-associative color Hom-algebra and by Proposition 2.22, \((A, [\cdot, \cdot], \varepsilon, \alpha)\) is a Hom-Lie color Hom-algebra. Now, we show that the \(\varepsilon\)-transposed Hom-Leibniz identity is satisfied. For any \(x, y, z \in H(A)\) we have

\[
\begin{align*}
\varepsilon(z, x)[x \cdot z, \alpha(y)] + \varepsilon(z, x + y)[\alpha(x), y \cdot z] - 2\alpha(z) \cdot [x, y]
\end{align*}
\]

\[
= \varepsilon(z, x)\left((x \cdot z) \circ \alpha(y) - \varepsilon(x + z, y)\alpha(y) \circ (x \cdot z)\right)
\]

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Hence the conclusion holds.

Example 3.13. Let $A = A_0 \oplus A_1 =\langle e_1, e_2 > \oplus < e_3, e_4 >$ be a 4-dimensional superspace. The quintuple $(A, \cdot, \circ, \varepsilon, \alpha)$ is a Hom-Novikov-Poisson color Hom-algebra with

the bicharacter: $\varepsilon(i,j) = (-1)^{ij}$,

the multiplications: $e_2 \cdot e_2 = e_1, \quad e_2 \cdot e_4 = e_4 \cdot e_2 = e_3, \quad e_2 \cdot e_4 = -e_3, \quad e_4 \cdot e_2 = e_3, \quad e_4 \cdot e_4 = 2e_1$,

the even linear map $\alpha : A \to A$ defined by $\alpha(e_1) = 2e_1, \quad \alpha(e_2) = -e_2, \quad \alpha(e_3) = -e_4, \quad \alpha(e_4) = e_3$.

Therefore, using the Proposition 3.12, $(A, \cdot, \circ, \varepsilon, \alpha)$ is a transposed Hom-Poisson color Hom-algebra with

$[e_2, e_4] = [e_4, e_2] = -2e_3, \quad [e_4, e_4] = 4e_1$.

Definition 3.14 ([11]). A Hom-Poisson color Hom-algebra is defined as a quintuple $(A, \cdot, \circ, \varepsilon, \alpha)$ such that $(A, \cdot, \circ, \varepsilon, \alpha)$ is a $\varepsilon$-commutative Hom-associative color Hom-algebra, and $(A, \cdot, \circ, \varepsilon, \alpha)$ is a Hom-Lie color Hom-algebra, satisfying for all $x, y, z$ in $\mathcal{H}(A)$, the Hom-$\varepsilon$-Leibniz identity,

$$[\alpha(x), y \cdot z] = \varepsilon(x, y)\alpha(y) \cdot [x, z] + \varepsilon(x + y, z)\alpha(z) \cdot [x, y]. \quad (3.29)$$

Condition (3.29), expressing the compatibility between the multiplication and the Poisson bracket, can be reformulated equivalently as

$$[x \cdot y, \alpha(z)] = \varepsilon(y, z)\alpha(y) \cdot [x, z] + \alpha(y) \cdot [x, z] + \alpha(x) \cdot [y, z]. \quad (3.30)$$

Definition 3.15. Let $(A, \cdot, \circ, \varepsilon, \alpha)$ be Hom-Novikov-Poisson color Hom-algebra. Then $A$ is called admissible if $(A, \cdot, \circ, \varepsilon, \alpha)$ is a Hom-Poisson color Hom-algebra, where

$$[x, y] = x \cdot y - \varepsilon(x, y)y \cdot x. \quad (3.31)$$

Lemma 3.16. Let $(A, \cdot, \circ, \varepsilon, \alpha)$ be a Hom-Novikov-Poisson color Hom-algebra. Then for any $x, y, z \in \mathcal{H}(A)$,

$$(x \cdot y) \circ \alpha(z) = \alpha(x) \circ (y \cdot z). \quad (3.32)$$

Proof. For all $x, y, z \in \mathcal{H}(A)$, we have

$$((A, \cdot, \alpha) \text{ is } \varepsilon-\text{commutative})$$

$$(x \cdot y) \circ \alpha(z) = \varepsilon(x, y)(y \cdot x) \circ \alpha(z)$$
Define \( \alpha \) be an admissible if and only if \( \alpha : \mathcal{H}(A) \rightarrow A \) for all \( x, y \in \mathcal{H}(A) \), by \( x \cdot y = \alpha(x \cdot y) \). Then \( \mathcal{A}_\alpha = (\mathcal{A}, \cdot, \ast, \delta, \varepsilon, \alpha) \) is an admissible Hom-Novikov-Poisson color Hom-algebra called the \( \alpha \)-twist or Yau twist of \((A, \cdot, \ast, \varepsilon, \alpha)\).

The following result gives a necessary and sufficient condition under which a Hom-Novikov-Poisson color Hom-algebra is admissible.

**Theorem 3.17.** Let \((A, \cdot, \circ, \varepsilon, \alpha)\) be Hom-Novikov-Poisson color Hom-algebra. Then \( A \) is an admissible if and only if
\[
\text{as}_\alpha^{I}(x, y, z) = (x \cdot y) \circ \alpha(z) - \alpha(x) \circ (y \cdot z) = 0. \tag{3.33}
\]

**Proof.** By definition, \((A, \cdot, \circ, \varepsilon, \alpha)\) is an \( \varepsilon \)-commutative Hom-associative color Hom-algebra and by Proposition 2.22, \((A, [\cdot, \cdot], \varepsilon, \alpha)\) is a Hom-Lie color Hom-algebra. Therefore,
\[
\text{as}_\alpha^{I}(x, y, z) = (x \cdot y) \circ \alpha(z) - \alpha(x) \circ (y \cdot z) = 0,
\]

Then \( A \) satisfies the Hom-\( \varepsilon \)-Leibniz-identity if and only if \( \alpha(y) \circ (x \cdot z) = (y \cdot x) \circ \alpha(z) \).

**Example 3.18.** Let \( A = A_0 \oplus A_1 = \langle e_1, e_2 \rangle \oplus \langle e_3, e_4 \rangle \) be a 4-dimensional superspace. The quintuple \((A, \cdot, \circ, \varepsilon, \alpha)\) is a Hom-Novikov-Poisson color Hom-algebra with

the bicharacter: \( \varepsilon(i, j) = (-1)^{|ij|} \),

the multiplications: \( e_2 \cdot e_2 = \lambda_1 e_1, \quad e_2 \cdot e_4 = e_4 \cdot e_2 = \lambda_2 e_3, \quad \lambda_i \in \mathbb{K} \)
\[
e_2 \circ e_2 = \mu_1 e_1, \quad e_4 \circ e_2 = \mu_2 e_3, \quad e_4 \circ e_4 = \mu_3 e_1, \quad \mu_i \in \mathbb{K}
\]

the even linear map \( \alpha : A \rightarrow A \) defined by \( \alpha(e_1) = 2e_1 - e_2, \quad \alpha(e_2) = e_1, \quad \alpha(e_3) = -e_4, \quad \alpha(e_4) = e_3 - e_4 \).

Then by Theorem 3.17, the Hom-Novikov-Poisson color Hom-algebra \((A, \cdot, \circ, \varepsilon, \alpha)\) is admissible.

**Theorem 3.19.** Let \( \mathcal{A} = (A, \cdot, \circ, \varepsilon) \) be an admissible Novikov-Poisson color Hom-algebra and \( \alpha : A \rightarrow A \) be an admissible Novikov-Poisson color Hom-algebras morphism. Define \( \cdot_\alpha, \circ_\alpha : A \times A \rightarrow A \) for all \( x, y \in \mathcal{H}(A) \), by \( x \cdot_\alpha y = \alpha(x \cdot y) \) and \( x \circ_\alpha y = \alpha(x \circ y) \). Then, \( \mathcal{A}_\alpha = (A, \cdot_\alpha, \circ_\alpha, \varepsilon, \alpha) \) is an admissible Hom-Novikov-Poisson color Hom-algebra called the \( \alpha \)-twist or Yau twist of \((A, \cdot, \circ, \varepsilon)\).
Proof. By Theorem 3.4, $A_\alpha$ is a Hom-Novikov-Poisson color Hom-algebra. Moreover, the left Hom-associators in $A$ and $A_\alpha$ are related as

$$\text{as}^l_{A_\alpha}(x,y,z) = \alpha^2 \text{as}^l_A(x,y,z) \quad \text{for all } x,y,z \in \mathcal{H}(A).$$

Since $A$ is left Hom-associative by Theorem 3.17, it follows that so is $A_\alpha$. Therefore, by Theorem 3.17 again $A_\alpha$ is admissible. 

\begin{corollary}
If $A = (A, \cdot, \circ, \varepsilon, \alpha)$ is a multiplicative admissible Novikov-Poisson color Hom-algebra, then for any $n \in \mathbb{N}^*$,

(i) The $n$th derived admissible Novikov-Poisson color Hom-algebra of type 1 of $A$ is defined by

$$A^n_1 = (A, \cdot^{(n)} = \alpha^n \circ \cdot, \circ^{(n)} = \alpha^n \circ \varepsilon, \alpha^{n+1}).$$

(ii) The $n$th derived admissible Novikov-Poisson color Hom-algebra of type 2 of $A$ is defined by

$$A^n_2 = (A, \cdot^{(2n-1)} = \alpha^{2n-1} \circ \cdot, \circ^{(2n-1)} = \alpha^{2n-1} \circ \varepsilon, \alpha^{2n}).$$

\end{corollary}

\begin{example}
Let $A = A_0 \oplus A_1 = \langle e_1, e_2 \rangle \oplus \langle e_3, e_4 \rangle$ be a 4-dimensional superspace. There is a multiplicative admissible Hom-Novikov-Poisson color Hom-algebra $(A, \cdot, \circ, \varepsilon, \alpha)$ with the bicharacter, $\varepsilon(i,j) = (-1)^{ij}$, and the multiplications tables for a basis $\{e_1, e_2, e_3, e_4\}$:

|        | $e_1$ | $e_2$ | $e_3$ | $e_4$ | \(\circ\) | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|--------|------|------|------|------|----------|------|------|------|------|
| $e_1$  | 0    | 0    | 0    | 0    | $e_1$    | 0    | 0    | 0    | 0    |
| $e_2$  | 0    | $e_1$| 0    | 4$e_3$| $e_2$    | 0    | 4$e_3$| 0    | 4$e_3$|
| $e_3$  | 0    | 0    | 0    | 0    | $e_3$    | 0    | 0    | 0    | 0    |
| $e_4$  | 0    | 4$e_3$| 0    | 0    | $e_4$    | 0    | 0    | 0    | $e_1$|

$\alpha(e_1) = 4e_1, \quad \alpha(e_2) = -2e_2, \quad \alpha(e_3) = e_3, \quad \alpha(e_4) = -2e_4.$

Then there are admissible Hom-Novikov-Poisson color Hom-algebras $A^n_1$ and $A^n_2$ with multiplications tables respectively:

|        | $e_1$ | $e_2$ | $e_3$ | $e_4$ | \(\cdot^{(n)}\) | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|--------|------|------|------|------|-----------------|------|------|------|------|
| $e_1$  | 0    | 0    | 0    | 0    | $e_1$          | 0    | 0    | 0    | 0    |
| $e_2$  | 0    | $2^n e_1$| 0    | 4$e_3$| $e_2$          | 0    | 4$e_3$| 0    | 4$e_3$|
| $e_3$  | 0    | 0    | 0    | 0    | $e_3$          | 0    | 0    | 0    | 0    |
| $e_4$  | 0    | 4$e_3$| 0    | 0    | $e_4$          | 0    | 0    | 0    | 2$2^n e_1$|

$\alpha^{n+1}(e_1) = 4^{n+1} e_1, \quad \alpha^{n+1}(e_3) = e_3, \quad \alpha^{n+1}(e_4) = (-2)^{n+1} e_4.$

|        | $e_1$ | $e_2$ | $e_3$ | $e_4$ | \(\cdot^{(2n-1)}\) | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|--------|------|------|------|------|-----------------|------|------|------|------|
| $e_1$  | 0    | 0    | 0    | 0    | $e_1$          | 0    | 0    | 0    | 0    |
| $e_2$  | 0    | $2^{2(2n-1)} e_1$| 0    | 4$e_3$| $e_2$          | 0    | 4$e_3$| 0    | 4$e_3$|
| $e_3$  | 0    | 0    | 0    | 0    | $e_3$          | 0    | 0    | 0    | 0    |
| $e_4$  | 0    | 4$e_3$| 0    | 0    | $e_4$          | 0    | 0    | 0    | 2$2^{2(2n-1)} e_1$|

$\alpha^{2n}(e_1) = 4^{2n} e_1, \quad \alpha^{2n}(e_2) = 2^{2n} e_2, \quad \alpha^{2n}(e_3) = e_3, \quad \alpha^{2n}(e_4) = 2^{2n} e_4.$

\end{example}
3.2 Tensor products of admissible Hom-Novikov-Poisson color Hom-algebras

Now, we show that the much larger class of admissible Hom-Novikov-Poisson color Hom-algebras is also closed under tensor products.

**Theorem 3.22.** Let \((A_1, \cdot_1, \circ_1, \epsilon, \alpha_1)\) and \((A_2, \cdot_2, \circ_2, \epsilon, \alpha_2)\) be admissible Hom-Novikov-Poisson color Hom-algebras and let \(A = A_1 \otimes A_2\). Define the operations \(\cdot : A \to A\) and \(\circ : A \otimes A \to A\) by the following formulae for all \(x_i, y_i \in \mathcal{H}(A_i), i \in \{1, 2\}\),

\[
\alpha = \alpha_1 \otimes \alpha_2,
\]

\[
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = \epsilon(x_2, y_1) (x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2),
\]

\[
(x_1 \otimes x_2) \circ (y_1 \otimes y_2) = \epsilon(x_2, y_1) \left( (x_1 \circ_1 y_1) \otimes (x_2 \circ_2 y_2) + (x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2) \right),
\]

Then \((A, \cdot, \circ, \epsilon, \alpha)\) is an admissible Hom-Novikov-Poisson color Hom-algebra.

**Proof.** Pick \(x = x_1 \otimes x_2, y = y_1 \otimes y_2\) and \(z = z_1 \otimes z_2\) homogeneous elements in \(A\).

**Step 1:** We show that \((A, \cdot, \epsilon, \alpha)\) is \(\epsilon\)-commutative Hom-associative-color Hom-algebra:

\[
x \cdot y = (x_1 \otimes x_2) \cdot (y_1 \otimes y_2)
= \epsilon(x_2, y_1) (x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2)
= \epsilon(x_2, y_1) \epsilon(x_1, y_1) \epsilon(x_2, y_2) (y_1 \cdot_1 x_1) \otimes (y_2 \cdot_2 x_2)
= \epsilon(x_1 + x_2, y_1 + y_2) \epsilon(y_2, x_1) (y_1 \cdot_1 x_1) \otimes (y_2 \cdot_2 x_2)
= \epsilon(x_1 + x_2, y_1 + y_2) (y_1 \otimes y_2) \cdot (x_1 \otimes x_2)
= \epsilon(x, y) y \cdot x,
\]

\[
(x \cdot y) \cdot \alpha(z) = ((x_1 \otimes x_2) \cdot (y_1 \otimes y_2)) \cdot (\alpha_1 \otimes \alpha_2)(z_1 \otimes z_2)
= \left( \epsilon(x_2, y_1) \cdot (x_1 \cdot_1 y_1) \otimes (x_2 \otimes_2 y_2) \right) \cdot (\alpha_1 \otimes \alpha_2)(z_1 \otimes z_2)
= \epsilon(x_2 + y_2, z_1) \epsilon(x_2, y_1) \left( (x_1 \cdot_1 y_1) \cdot_1 \alpha_1(z_1) \otimes (x_2 \cdot_2 y_2) \cdot_2 \alpha_2(z_2) \right)
= \epsilon(x_2, y_1 + z_1) \epsilon(y_2, z_1) \left( \alpha_1(x_1) \cdot_1 (y_1 \cdot_1 z_1) \otimes \alpha_2(x_2) \cdot_2 (y_2 \cdot_2 z_2) \right)
= (\alpha_1 \otimes \alpha_2)(x_1 \otimes x_2) \cdot ((y_1 \otimes y_2) \cdot (z_1 \otimes z_2))
= \alpha(x \cdot (y \cdot z)).
\]

Hence, \((A_1 \otimes A_2, \cdot, \epsilon, \alpha)\) is a \(\epsilon\)-commutative Hom-associative color Hom-algebra.

**Step 2:** We show that \((A, \circ, \epsilon, \alpha)\) is Hom-Novikov-color Hom-algebra.

\[
(x \circ y) \circ \alpha(z) - \alpha(x) \circ (y \circ z) - \epsilon(x, y) (y \circ x) \circ \alpha(z) - \alpha(y) \circ (x \circ z)
= ((x_1 \otimes x_2) \circ (y_1 \otimes y_2)) \circ \alpha(z_1 \otimes z_2) - (\alpha_1 \otimes \alpha_2)(x_1 \otimes x_2) \circ ((y_1 \otimes y_2) \circ (z_1 \otimes z_2))
- \epsilon(x_1 + x_2, y_1 + y_2) \left( ((y_1 \otimes y_2) \circ (x_1 \otimes x_2)) \circ (\alpha_1 \otimes \alpha_2)(z_1 \otimes z_2) 
- (\alpha_1 \otimes \alpha_2)(y_1 \otimes y_2) \circ ((x_1 \otimes x_2) \circ (z_1 \otimes z_2)) \right)
= \epsilon(x_2, y_1) ((x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2) + (x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2)) \circ (\alpha_1 \otimes \alpha_2)(z_1 \otimes z_2)
- \epsilon(y_2, z_1) (\alpha_1 \otimes \alpha_2)(x_1 \otimes x_2) \circ ((y_1 \cdot_1 z_1) \otimes (y_2 \cdot_2 z_2) + (y_1 \cdot_1 z_1) \otimes (y_2 \cdot_2 z_2))
- \epsilon(x_1 + x_2, y_1 + y_2) \left( \epsilon(y_2, x_1) ((y_1 \cdot_1 x_1) \otimes (y_2 \cdot_2 x_2) + (y_1 \cdot_1 x_1) \otimes (y_2 \cdot_2 x_2)) \circ 
\]
\[
(\alpha_1 \otimes \alpha_2)(z_1 \otimes z_2) - \varepsilon(x_2, z_1)(\alpha_1 \otimes \alpha_2)(y_1 \otimes y_2) \circ ((x_1 \cdot \varepsilon_1) \otimes (x_2 \cdot \varepsilon_2))
\]

\[
= \varepsilon(x_1 + x_2, y_1 + y_2) \times
\]

\[
\begin{array}{ccccccc}
A_1 & + & A_2 & + & A_3 & + & A_4 & + & A_5 & + & A_6 & + & A_7 & + & A_8 \\
(y_1 \cdot x_1) & \otimes & (x_2 \cdot y_1) & \otimes & (x_2 \cdot x_2) & \otimes & (x_2 \cdot y_2) & \otimes & (x_2 \cdot y_2) & \otimes & (x_2 \cdot y_2) & \otimes & (x_2 \cdot y_2) & \otimes & (x_2 \cdot y_2) & \otimes & (x_2 \cdot y_2)
\end{array}
\]

Furthermore, we have

\[
(A_1 + A_5) - \varepsilon(x_1, y_1) \varepsilon(x_2, y_2)(B_1 + B_5)
\]

(by (2.1) and (2.2))

\[
= \text{[expression]}
\]

\[
(A_4 + A_8) - \varepsilon(x_1, y_1) \varepsilon(x_2, y_2)(B_4 + B_8)
\]

(by (2.1) and (2.2))

\[
= \text{[expression]}
\]

\[
(A_2 + A_7) - \varepsilon(x_1, y_1) \varepsilon(x_2, y_2)(B_2 + B_7)
\]

(by (2.1), (3.1) and (2.2))
\[
= \left[ (x_1 \cdot y_1) \cdot \alpha_1(z_1) - \alpha_1(x_1) \cdot (y_1 \cdot z_1) \right] \\
\quad - \varepsilon(x_1, y_1) \left[ (y_1 \cdot x_1) \cdot \alpha_1(z_1) - \alpha_1(y_1) \cdot (x_1 \cdot z_1) \right] \otimes (x_2 \cdot y_2) \otimes \alpha_2(z_2)
\]
(by (3.2)) \quad = 0,

\[(A_3 + A_4) - \varepsilon(x_1, y_1)\varepsilon(x_2, y_2)(B_3 + B_6)
\]
(by (2.1), (3.1) and (2.2))

\[
= (x_1 \cdot y_1) \cdot (x_2 \cdot y_2) \cdot (y_2 \cdot z_2) - \varepsilon(x_2, y_2) \left[ (y_2 \cdot x_2) \cdot \alpha_2(z_2) - \alpha_2(y_2) \cdot (x_2 \cdot z_2) \right]
\]
(by (3.2)) \quad = 0.

Then, we obtain

\[
(x \circ y) \circ \alpha(z) - \alpha(x) \circ (y \circ z) - \varepsilon(x, y) \left[ (y \circ x) \circ \alpha(z) - \alpha(y) \circ (x \circ z) \right] = 0.
\]

\[
(x \circ y) \circ \alpha(z) - \varepsilon(y, z) \left[ (x \circ z) \circ \alpha(y) \right]
\]

\[
= \left[ (x_1 \cdot y_1) \cdot (x_2 \cdot y_2) \cdot \alpha_1(z_1) \otimes (x_2 \cdot y_2) \cdot \alpha_2(z_2) \right] \\
\quad + \varepsilon(y_1 + y_2, z_1 + z_2) \left[ \left( \varepsilon(x_2, y_2) \cdot (x_1 \cdot y_1) \cdot (x_2 \cdot y_2) \right) \otimes \alpha_1(z_1) \otimes (y_1 \otimes y_2) \right]
\]

\[
= \left[ (x_1 \cdot y_1) \cdot (x_2 \cdot y_2) \cdot \alpha_1(z_1) \otimes (x_2 \cdot y_2) \cdot \alpha_2(z_2) \right] \\
\quad + \varepsilon(y_1 + y_2, z_1 + z_2) \left[ \left( \varepsilon(x_2, y_2) \cdot (x_1 \cdot y_1) \cdot (x_2 \cdot y_2) \right) \otimes \alpha_1(z_1) \otimes (y_1 \otimes y_2) \right]
\]

\[
= \left[ (x_1 \cdot y_1) \cdot (x_2 \cdot y_2) \cdot \alpha_1(z_1) \otimes (x_2 \cdot y_2) \cdot \alpha_2(z_2) \right] \\
\quad + \varepsilon(y_1 + y_2, z_1 + z_2) \left[ \left( \varepsilon(x_2, y_2) \cdot (x_1 \cdot y_1) \cdot (x_2 \cdot y_2) \right) \otimes \alpha_1(z_1) \otimes (y_1 \otimes y_2) \right]
\]

Furthermore, we have

\[
\varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)C_1 - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1)D_1
\]
(by (2.2))

\[
= \varepsilon(x_2, y_1 + y_2)\varepsilon(x_2 + y_2, z_1) \times \\
\quad \left[ (x_1 \cdot y_1) \cdot \alpha_1(z_1) \otimes (y_2 \cdot x_2) \cdot \alpha_2(z_2) \right] \\
\quad - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2 + z_2, y_1 + y_2)\varepsilon(x_2, z_1) \times \\
\quad \left[ (x_1 \cdot y_1) \cdot \alpha_1(y_1) \otimes \alpha_2(y_2) \cdot (x_2 \cdot z_2) \right]
\]
(by (2.1))
\[
\begin{align*}
&= \varepsilon(x_2, y_1 + y_2)\varepsilon(x_2 + y_2, z_1) \times \\
&\quad \left[(x_1 \cdot 1 y_1) \cdot 1 \alpha_1(z_1) - \varepsilon(y_1, z_1)(x_1 \cdot 1 z_1) \otimes \alpha_2(y_1) \cdot 2 (x_2 \cdot 2 z_2)
\right] \\
&\quad \otimes \alpha_2(y_2) \cdot 2 (x_2 \cdot 2 z_2)
\end{align*}
\]
(by (2.13))  
\[
= 0.
\]
\[
\varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)C_4
= \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1)D_4
\]
(by (2.2))  
\[
= \varepsilon(x_2, y_2)\varepsilon(x_2 + y_2, z_1) \times \\
\quad \left[(y_1 \cdot 1 x_1) \cdot 1 \alpha_1(z_1) \otimes \left((x_2 \cdot 2 y_1) \cdot 1 \alpha_1(z_2)\right)
\right]
\]
(by (2.1))  
\[
= \varepsilon(x_1 + x_2, y_1)\varepsilon(x_2 + y_2, z_1) \times \\
\quad \left[(y_1 \cdot 1 x_1) \cdot 1 \alpha_1(z_1) \left[(x_2 \cdot 2 y_2) \otimes \alpha_2(z_2)
\right] - \varepsilon(y_2, z_2)(x_2 \cdot 2 z_2) \otimes \alpha_2(y_2)
\right]
\]
(by (2.13))  
\[
= 0.
\]
\[
\varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)C_2
= \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1)D_3
\]
(by (3.1))  
\[
= \varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1) \left[\varepsilon(y_1, z_1)(x_1 \cdot 1 y_1) \cdot 1 \alpha_1(z_1) \otimes (x_2 \cdot 2 y_2) \otimes \alpha_2(z_2)
\right]
\]
(by (3.1))  
\[
= \varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1) \left[\varepsilon(y_1, z_1)(x_1 \cdot 1 y_1) \cdot 1 \alpha_1(z_1) \otimes (x_2 \cdot 2 y_2) \cdot 2 \alpha_2(z_2)
\right]
\]
(by (3.1))  
\[
= 0.
\]
Then, we obtain  
\[(x \cdot y) \diamond \alpha(z) - \varepsilon(y, z)((x \cdot z) \diamond \alpha(y) = 0.
\]
Hence, \((A_1 \otimes A_2, \diamond, \varepsilon, \alpha)\) is a Hom-Novikov color Hom-algebra.

**Step 3:** We show that the compatibility conditions of Hom-Novikov-Poisson color Hom-algebras are satisfied

\[
(x \cdot y) \diamond \alpha(z) - \varepsilon(y, z)(x \cdot z) \cdot \alpha(y)
\]
\[
= \left((x_1 \otimes x_2) \cdot (y_1 \otimes y_2)\right) \diamond (\alpha_1 \otimes \alpha_2)(z_1 \otimes z_2)
\]
\[- \varepsilon(y_1 + y_2, z_1 + z_2)(\alpha_1 \otimes \alpha_2)(y_1 \otimes y_2)
\]
\[
= \varepsilon(x_2, y_1)\left(\left(x_1 \cdot 1 y_1\right) \diamond (x_2 \cdot 2 y_2)\right) \diamond (\alpha_1 \otimes \alpha_2)(z_1 \otimes z_2)
\]
\[- \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)(x_1 \cdot 1 y_1) \diamond (x_2 \cdot 2 z_2)
\]
\[
+ \left(x_1 \cdot 1 z_1\right) \diamond (x_2 \cdot 2 z_2)\right) \diamond (\alpha_1 \otimes \alpha_2)(y_1 \otimes y_2)
\]
\[
= \varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1) \times \\
\left(\left((x_1 \cdot 1 y_1) \cdot 1 \alpha_1(z_1)\right) \otimes \left((x_2 \cdot 2 y_2) \cdot 2 \alpha_2(z_2)\right)
\right)
\]
\[
\text{E}_1
\]
26
\[
\left( (x_1 \cdot y_1) \cdot \alpha_1(z_1) \right) \otimes \left( (x_2 \cdot y_2) \circ_2 \alpha_2(z_2) \right) \\
E_2
\]
\[
- \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1 \varepsilon(x_2 + z_2, y_1) \times \\
\left( (x_1 \cdot z_1) \cdot \alpha_1(y_1) \right) \otimes \left( (x_2 \cdot z_2) \cdot \alpha_2(y_2) \right) \\
F_1
\]
\[
+ \left( (x_1 \cdot z_1) \cdot \alpha_1(y_1) \right) \otimes \left( (x_2 \cdot z_2) \cdot \alpha_2(y_2) \right) . \\
F_2
\]

Furthermore, we have
\[
\varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)E_1 - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1)F_1
\]
(by (2.2))
\[
= \varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)\varepsilon(x_2, y_2)\left( (x_1 \cdot y_1) \circ_1 \alpha_1(z_1) \right) \otimes \left( (y_2 \cdot x_2) \cdot \alpha_2(z_2) \right) \\
- \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1 + y_2) \times \\
\left( (x_1 \cdot z_1) \cdot \alpha_1(y_1) \right) \otimes \left( \alpha_2(y_2) \cdot \alpha_2(y_2) \cdot (x_2 \cdot z_2) \right)
\]
(by (2.1))
\[
= \varepsilon(x_2, y_1 + y_2)\varepsilon(x_2 + y_2, z_1) \times \\
\left( (x_1 \cdot y_1) \cdot \alpha_1(z_1) - \varepsilon(y_1, z_1)(x_1 \cdot z_1) \cdot \alpha_1(y_1) \right) \otimes \alpha_2(y_2) \cdot \alpha_2(y_2) (x_2 \cdot z_2)
\]
(by (3.1))
\[
= 0,
\]
\[
\varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)E_2 - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1)F_2
\]
(by (2.2))
\[
= \varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)\varepsilon(x_1 + y_1, z_1) \times \\
\left( \alpha_1(z_1) \cdot (x_1 \cdot y_1) \right) \otimes \left( (x_2 \cdot y_2) \circ_2 \alpha_2(z_2) \right) \\
- \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_1 + x_2, z_1)\varepsilon(x_2 + z_2, y_1) \times \\
\left( (z_1 \cdot x_1) \cdot \alpha_1(y_1) \right) \otimes \left( (x_2 \cdot z_2) \circ_2 \alpha_2(y_2) \right)
\]
(by (2.1))
\[
= \varepsilon(x_2, y_1)\varepsilon(x_1 + x_2 + y_1 + y_2, z_1) \times \\
\alpha_1(z_1) \cdot (x_1 \cdot y_1) \otimes \left( (x_2 \cdot y_2) \circ_2 \alpha_2(z_2) - \varepsilon(y_2, z_2)(x_2 \cdot z_2) \cdot \alpha_2(y_2) \right)
\]
(by (3.1))
\[
= 0.
\]

Then, \((x \cdot y) \cdot \alpha(z) - \varepsilon(y, z)(x \cdot z) \cdot \alpha(y) = 0\). Similarly,
\[
(x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot \alpha(y \cdot z) = \varepsilon(x, y) ((y \cdot x) \cdot \alpha(z) - \alpha(y) \cdot (x \cdot z)).
\]

Hence, \((A_1 \otimes A_2, \cdot, \circ, \varepsilon, \alpha)\) is a Hom-Novikov-Poisson color Hom-algebra.

**Step 4:** We show that the Equation (3.33) is satisfied:
\[
\alpha(x) \cdot (y \circ z) = \alpha(x \otimes x_2) \cdot \varepsilon(y_2, z_1) \left( (y_1 \cdot z_1) \otimes (y_2 \cdot z_2) + (y_1 \cdot z_1) \otimes (y_2 \cdot z_2) \right)
\]
\[
= \varepsilon(y_2, z_1)\varepsilon(x_2, y_1 + z_1) \left( (\alpha(x_1) \cdot (y_1 \cdot z_1)) \otimes (\alpha(x_2) \cdot (y_2 \cdot z_2)) \right) \\
+ \left( (\alpha(x_1) \cdot (y_1 \cdot z_1)) \otimes (\alpha(x_2) \cdot (y_2 \cdot z_2)) \right)
\]
\[
\alpha(x) \cdot (y \cdot z) = (\alpha(x_1) \otimes \alpha(x_2)) \cdot ((y_1 \otimes y_2) \cdot (z_1 \otimes z_2))
\]
Now, using Theorem 3.17 we conclude that \( \alpha(x) \cdot (y \circ z) = \alpha(x) \circ (y \cdot z) \). Therefore \( \text{asd}_A(x, y, z) = 0 \) and hence, \( (A = A_1 \otimes A_2, \cdot, \circ, \varepsilon, \alpha) \) is an admissible Hom-Novikov-Poisson color Hom-algebra.

By taking in Theorem 3.22, \( \alpha_1 = \text{id}_{A_1} \) and \( \alpha_2 = \text{id}_{A_2} \), we have the following result.

**Corollary 3.23.** Let \( (A_1, \cdot_1, \circ_1, \varepsilon_1) \) and \( (A_2, \cdot_2, \circ_2, \varepsilon_2) \) be admissible Novikov-Poisson color Hom-algebras and let \( A = A_1 \otimes A_2 \). Define the operations \( \cdot, \circ : A \otimes A \to A \) by the following formulae for \( x_i, y_i \in H(A_i), \ i \in \{1; 2\}, \)

\[
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = \varepsilon(x_2, y_1)(x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2),
\]
\[
(x_1 \otimes x_2) \circ (y_1 \otimes y_2) = \varepsilon(x_2, y_1)\big((x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2) + (x_1 \cdot_1 y_1) \otimes (x_2 \circ_2 y_2)\big).
\]

Then \( (A, \cdot, \circ, \varepsilon) \) is an admissible Novikov-Poisson color Hom-algebra.

By taking in Theorem 3.22, \( \Gamma = \{\varepsilon\} \), we recover the following result.

**Corollary 3.24 (65).** Let \( (A_1, \cdot_1, \circ_1, \alpha_1) \) and \( (A_2, \cdot_2, \circ_2, \alpha_2) \) be admissible Hom-Novikov-Poisson algebras and let \( A = A_1 \otimes A_2 \). Define the operations \( \alpha : A \to A \) and \( \cdot, \circ : A \otimes A \to A \) by the following formulae for \( x_i, y_i \in A_i, \ i \in \{1; 2\}, \)

\[
\alpha = \alpha_1 \otimes \alpha_2,
\]
\[
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = (x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2),
\]
\[
(x_1 \otimes x_2) \circ (y_1 \otimes y_2) = \big((x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2) + (x_1 \cdot_1 y_1) \otimes (x_2 \circ_2 y_2)\big).
\]

Then \( (A, \cdot, \circ, \alpha) \) is an admissible Hom-Novikov-Poisson algebra.

By taking in Theorem 3.22, \( \alpha_1 = \text{id}_{A_1}, \alpha_2 = \text{id}_{A_2} \) and \( \Gamma = \{\varepsilon\} \), we have the following result.

**Corollary 3.25 (67).** Let \( (A_1, \cdot_1, \circ_1) \) and \( (A_2, \cdot_2, \circ_2) \) be admissible Novikov-Poisson algebras and let \( A = A_1 \otimes A_2 \). Define the operations \( \cdot, \circ : A \otimes A \to A \) by the following formulae for \( x_i, y_i \in A_i, \ i \in \{1; 2\}, \)

\[
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = (x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2),
\]
\[
(x_1 \otimes x_2) \circ (y_1 \otimes y_2) = \big((x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2) + (x_1 \cdot_1 y_1) \otimes (x_2 \circ_2 y_2)\big).
\]

Then \( (A, \cdot, \circ) \) is an admissible Novikov-Poisson algebra.

### 4 Hom-Gelfand-Dorfman color Hom-algebras

In this section, our goals are to introduce Hom-Gelfand-Dorfman color Hom-algebras and to discuss some basic properties and examples of these objects. Moreover we characterize the representation of Hom-Gelfand-Dorfman color Hom-algebras and provide some key constructions.
**Definition 4.1.** A Gelfand-Dorfman color Hom-algebra is a quadruple \((A, \cdot, [,], \varepsilon)\) such that \((A, \cdot, \varepsilon)\) is a Novikov color Hom-algebra and \((A, [,], \varepsilon)\) is a Lie color Hom-algebra satisfying for all \(x, y, z \in \mathcal{H}(A)\), the following compatibility condition:

\[
y \cdot [x, z] = \varepsilon(y, x)[x, y \cdot z] - \varepsilon(x + y, z)[y, x] + [y, x] \cdot z - \varepsilon(x, z)[y, z] \cdot x.
\] (4.1)

**Definition 4.2.** A Hom-Gelfand-Dorfman color Hom-algebra is defined as a quintuple \((A, ', [,], \varepsilon, \alpha)\) such that \((A, \cdot, \varepsilon, \alpha)\) is a Hom-Novikov color Hom-algebra and \((A, [,], \varepsilon, \alpha)\) is a Hom-Lie color Hom-algebra satisfying for all \(x, y, z \in \mathcal{H}(A)\), the following compatibility condition:

\[
\alpha(y) \cdot [x, z] = \varepsilon(y, x)[\alpha(x), y \cdot z] - \varepsilon(x + y, z)[\alpha(z), y \cdot x] + [y, x] \cdot \alpha(z) - \varepsilon(x, z)[y, z] \cdot \alpha(x).
\] (4.2)

A Hom-Gelfand-Dorfman color Hom-algebra is called multiplicative if the even linear map \(\alpha : A \to A\) is multiplicative with respect to \(\cdot\) and \([,]\), that is, for all \(x, y \in \mathcal{H}(A)\),

\[
\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y), \quad \alpha([x, y]) = [\alpha(x), \alpha(y)].
\]

**Remark 4.3.** Hom-Gelfand-Dorfman color Hom-algebras contain the Gelfand-Dorfman algebras and the Hom-Gelfand-Dorfman Hom-algebras for special choices of grading group and the twisting map.

(i) When \(\Gamma = \{e\}\) and \(\alpha = id\), we get Gelfand-Dorfman algebra [34, 79].

(ii) When \(\Gamma = \{e\}\) and \(\alpha \neq id\), we get Hom-Gelfand-Dorfman Hom-algebra [88].

**Example 4.4.** Let \(\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2\) be an abelian group and \(A\) be a 4-dimensional \(\Gamma\)-graded linear space with one-dimensional homogeneous subspaces

\[
A_{(0,0)} = \langle e_1 \rangle, \quad A_{(0,1)} = \langle e_2 \rangle, \quad A_{(1,0)} = \langle e_3 \rangle, \quad A_{(1,1)} = \langle e_4 \rangle.
\]

Then \((A, ', [,], \varepsilon, \alpha)\) is a Hom-Gelfand-Dorfman color Hom-algebra with

- the bicharacter: \(\varepsilon((i_1, i_2), (j_1, j_2)) = (-1)^{ij_1i_2j_2}\),
- the multiplication: \(e_2 \cdot e_3 = \lambda_1 e_4, \quad e_3 \cdot e_2 = \lambda_2 e_4, \quad e_3 \cdot e_3 = \lambda_3 e_1, \quad \lambda_i \in \mathbb{K}\),
- the bracket: \([e_2, e_2] = \mu_1 e_4, \quad [e_3, e_2] = \mu_2 e_4, \quad \mu_i \in \mathbb{K}\),
- the even linear map \(\alpha : A \to A\) given by \(\alpha(e_1) = -e_1, \quad \alpha(e_2) = 2e_2, \quad \alpha(e_3) = -2e_3, \quad \alpha(e_4) = e_4\).

**Definition 4.5.** Let \((A, \cdot, [,], \varepsilon, \alpha)\) and \((A', \cdot', [,], \varepsilon', \alpha')\) be Hom-Gelfand-Dorfman color Hom-algebras. A linear map of degree zero \(f : A \to A'\) is a Hom-Gelfand-Dorfman color Hom-algebra morphism if

\[
\cdot' \circ (f \otimes f) = f \circ \cdot', \quad [,] \circ (f \otimes f) = f \circ [,], \quad f \circ \alpha = \alpha' \circ f.
\]

**Proposition 4.6.** Let \((A, \cdot, \varepsilon, \alpha)\) be a Hom-Novikov color Hom-algebra. For all \(x, y \in \mathcal{H}(A)\), let

\[
[x, y] = x \cdot y - \varepsilon(x, y)y \cdot x.
\] (4.3)

Then \((A, \cdot, [,], \varepsilon, \alpha)\) is a Hom-Gelfand-Dorfman color Hom-algebra.
Proof. Let \((A, \cdot, \varepsilon, \alpha)\) be a Hom-Novikov color Hom-algebra. By Proposition \ref{proposition2.22}, \((A, [-, -], \varepsilon, \alpha)\) is a Hom-Lie color Hom-algebra. Now, we show that the compatibility condition \((4.2)\) is satisfied. For any \(x, y, z \in \mathcal{H}(A)\) we have

\[
\begin{align*}
\alpha(y) \cdot [x, z] &- \varepsilon(y, x)[\alpha(x), y \cdot z] + \varepsilon(x + y, z)[\alpha(z), y \cdot x] - [y, x] \cdot \alpha(z) \\
+ \varepsilon(x, z)[y, z] \cdot \alpha(x) \\
&= \alpha(y)(x \cdot z - \varepsilon(x, y + z)(y \cdot z) \cdot \alpha(x)) + \varepsilon(x + y, z)(\alpha(z) \cdot (y \cdot x) \\
&- \varepsilon(z, y + x)(y \cdot x) \cdot \alpha(z)) - (y \cdot x - \varepsilon(y, x)x \cdot y)) \cdot \alpha(z) + \varepsilon(x, z)(y \cdot z) \\
&- \varepsilon(y, z)z \cdot y \cdot \alpha(x) \\
&= \alpha(y) \cdot (x \cdot z) - \varepsilon(x, z)\alpha(y) \cdot (z \cdot x) - \varepsilon(y, x)\alpha(x) \cdot (y \cdot z) + \varepsilon(x, z)(y \cdot z) \cdot \alpha(x) \\
&+ \varepsilon(x + y, z)\alpha(z) \cdot (y \cdot x) - (y \cdot x) \cdot \alpha(z) - (y \cdot x) \cdot \alpha(z) + \varepsilon(y, x)(x \cdot y) \cdot \alpha(z) \\
&+ \varepsilon(x, z)(y \cdot z) \cdot \alpha(x) - \varepsilon(x + y, z)(x \cdot y) \cdot \alpha(x) + \varepsilon(y, x)(x \cdot y) \cdot \alpha(z) \\
&= \begin{cases} 
\alpha(y) \cdot (x \cdot z) - (y \cdot x) \cdot \alpha(z) - \varepsilon(y, x)(\alpha(x) \cdot (y \cdot z) - (x \cdot y) \cdot \alpha(z)) \\
\end{cases}
\end{align*}
\]

= 0 by \((2.12)\)

\[
+ \varepsilon(x, z) \begin{cases} 
(y \cdot z) \cdot \alpha(x) - \alpha(y) \cdot (z \cdot x) - \varepsilon(y, z)(\alpha(z) \cdot (y \cdot x) - (z \cdot y) \cdot \alpha(x)) \\
\end{cases}
\]

= 0 by \((2.12)\)

\[
- \begin{cases} 
(y \cdot x) \cdot \alpha(z) - \varepsilon(x, z)(y \cdot z) \cdot \alpha(x) \\
\end{cases}
= 0.
\]

Hence, \((A, \cdot, [-, -], \varepsilon, \alpha)\) is a Hom-Gelfand-Dorfman color Hom-algebra. \(\square\)

**Definition 4.7.** Let \((A, \cdot, [-, -], \varepsilon, \alpha)\) be a Hom-Gelfand-Dorfman color Hom-algebra. A \(\Gamma\)-graded subspace \(H\) of \(A\) is called

(i) color Hom-subalgebra of \((A, \cdot, [-, -], \varepsilon, \alpha)\) if

\[\alpha(H) \subseteq H, \ H \cdot H \subseteq H, \ [H, H] \subseteq H,\]

(ii) color Hom-ideal of \((A, \cdot, [-, -], \varepsilon, \alpha)\) if

\[\alpha(H) \subseteq H, \ A \cdot H \subseteq H, \ H \cdot A \subseteq H, \ [A, H] \subseteq H.\]

**Proposition 4.8.** Let \((A, \cdot, [-, -], \varepsilon, \alpha)\) a Hom-Gelfand-Dorfman color Hom-algebra and \(I\) a color Hom-ideal of \((A, \cdot, [-, -], \varepsilon, \alpha)\). Then \((A/I, \cdot, \{\cdot, \cdot\}, \varepsilon, \alpha)\) is a Hom-Gelfand-Dorfman color Hom-algebra where \(\overline{x} \cdot \overline{y} = \overline{x \cdot y}, \ \{\overline{x}, \overline{y}\} = \overline{[x, y], \varepsilon(\overline{x}) = \alpha(x)\) and \(\varepsilon(\overline{x}, \overline{y}) = \varepsilon(x, y)\), for all \(\overline{x}, \overline{y} \in \mathcal{H}(A/I)\).

**Proof.** It follows from a straightforward computation. \(\square\)

**Proposition 4.9.** Any transposed Hom-Poisson color Hom-algebra is a Hom-Gelfand-Dorfman color Hom-algebra.

**Proof.** Let \((A, \cdot, [-, -], \varepsilon, \alpha)\) be a transposed Hom-Poisson color Hom-algebra. By definition \((A, \cdot, \varepsilon, \alpha)\) be a \(\varepsilon\)-commutative Hom-associative color Hom-algebra, then \((A, \cdot, \varepsilon, \alpha)\) is a Hom-Novikov color Hom-algebra and \((A, [-, -], \varepsilon, \alpha)\) is a Hom-Lie color Hom-algebra.
Now, we show that the compatibility condition (4.2) is satisfied. For any \( x, y, z \in \mathcal{H}(A) \) we have
\[
\alpha(y) \cdot [x, z] - \varepsilon(y, x)[\alpha(x), y \cdot z] + \varepsilon(x + y, z)[\alpha(z), y \cdot x] - [y, x] \cdot \alpha(z) + \varepsilon(x + y, z)[\alpha(z), y \cdot x]
\]
\[
= \alpha(y) \cdot (x \cdot z - \varepsilon(y, x)z \cdot x) - \varepsilon(y, x)(\alpha(x) \cdot (y \cdot z) - \varepsilon(x, y + z)(y \cdot z) \cdot \alpha(x))
\]
\[
- \varepsilon(x + y, z)(\alpha(z) \cdot (y \cdot x) - \varepsilon(z, y + x)(y \cdot x) \cdot \alpha(z))
\]
\[
+ \varepsilon(x + y, z)\alpha(z) \cdot (y \cdot x) - (y \cdot x) \cdot \alpha(z) - (y \cdot x) \cdot \alpha(z) + \varepsilon(y, x)(x \cdot y) \cdot \alpha(z)
\]
\[
+ \varepsilon(x, y)(y \cdot z) \cdot \alpha(x) - \varepsilon(x + y, z)(z \cdot y) \cdot \alpha(x)
\]
\[
= \alpha(y) \cdot (x \cdot z - (y \cdot x) \cdot \alpha(z) - \varepsilon(x, z)(\alpha(y) \cdot (z \cdot x) - (y \cdot z) \cdot \alpha(x))
\]
\[
= 0 \text{ by (2.1)}
\]
\[
- \varepsilon(y, x)\alpha(x) \cdot (y \cdot z) - (x \cdot y) \cdot \alpha(z)
\]
\[
+ \varepsilon(x + y, z)(\alpha(z) \cdot (y \cdot x) - (z \cdot y) \cdot \alpha(x))
\]
\[
= 0 \text{ by (2.1)}
\]
\[
+ \varepsilon(x, y)(y \cdot z) \cdot \alpha(x) - (y \cdot x) \cdot \alpha(z)
\]
\[
= 0 \text{ by (2.1) and (2.2)}
\]

which completes the proof.

\section*{Lemma 4.10.} \cite{8} Let \( (A, \cdot, \varepsilon, \alpha) \) be a \( \varepsilon \)-commutative Hom-associative color Hom-algebra with an even derivation \( D \) such that \( \alpha \circ D = D \circ \alpha \). Define
\[
x \diamond y = x \cdot D(y), \text{ for all } x, y \in \mathcal{H}(A).
\]

Then \( (A, \diamond, \varepsilon, \alpha) \) is a Hom-Novikov color Hom-algebra.

\section*{Theorem 4.11.} Let \( (A, \cdot, [\cdot, \cdot], \varepsilon, \alpha) \) be a Hom-Poisson color Hom-algebra with an even derivation \( D \) relative to the both products. Define a new operation \( \diamond \) on \( A \) in the following way:
\[
x \diamond y = x \cdot D(y).
\]

Then \( (A, \diamond, [\cdot, \cdot], \varepsilon, \alpha) \) is a Hom-Gelfand-Dorfman color Hom-algebra.

\section*{Proof.} By Lemma 4.10, \( (A, \diamond, \varepsilon, \alpha) \) is a Hom-Novikov color Hom-algebra and by definition of Hom-Poisson color Hom-algebra, \( (A, [\cdot, \cdot], \varepsilon, \alpha) \) is a Hom-Lie color Hom-algebra. Now, we show that the compatibility condition (4.2) is satisfied. For any \( x, y, z \in \mathcal{H}(A) \),
\[
\alpha(y) \circ [x, z] - \varepsilon(y, x)[\alpha(x), y \circ z] + \varepsilon(x + y, z)[\alpha(z), y \circ x]
\]
\[
- [y, x] \circ \alpha(z) + \varepsilon(x, y + z)[\alpha(z), y \circ x]
\]
\[
\text{(using (4.4))}
\]
\[
= \alpha(y) \cdot D([x, z]) - \varepsilon(y, x)[\alpha(x), y \cdot D(z)] + \varepsilon(x + y, z)[\alpha(z), y \cdot D(x)]
\]
\[
- [y, x] \cdot D(\alpha(z)) + \varepsilon(x, z)[y, z] \cdot D(\alpha(x))
\]
\[
\text{(D is derivation)}
\]
\[
= \alpha(y) \cdot [D(x), z] + \alpha(y) \cdot [x, D(z)] - \varepsilon(y, x)[\alpha(x), y \cdot D(z)]
\]
For all algebra respectively. Now, we show that the compatibility condition (4.2) is satisfied.

\[ \begin{align*}
+ \varepsilon(x + y, z) &\left[ \alpha(z), y \cdot D(x) \right] - [y, x] \cdot \alpha(D(x)) \\
+ \varepsilon(x, z) [y, z] \cdot \alpha(D(x)) \\
= - \varepsilon(y, x) \left( [\alpha(x), y \cdot D(z)] - \varepsilon(x, y) \alpha(y) \cdot [x, D(z)] - \varepsilon(y + x, z) \alpha(D(x)) \cdot [x, y] \right) \\
&= 0 \text{ by (3.30)} \\
+ \varepsilon(x + y, z) &\left[ \alpha(z), y \cdot D(x) \right] - \varepsilon(z, y) \alpha(y) \cdot [z, D(x)] - \alpha(D(x)) \cdot [z, y] \\
&= 0 \text{ by (3.30)}
\end{align*} \]

Let us call a Hom-Gelfand-Dorfman color Hom-algebra \((A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)\) special if it can be embedded into a differential Hom-Poisson color Hom-algebra with operations \([\cdot, \cdot]\) and \(\cdot\) given by (4.5).

**Definition 4.12.** Let \((A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)\) be a Hom-Gelfand-Dorfman color Hom-algebra. A representation of \((A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)\) is a quintuple \((l, r, \rho, \beta, V)\) such that \((l, r, \beta, V)\) is a bimodule of the Hom-Novikov color Hom-algebra \((A, \cdot, \varepsilon, \alpha)\) and \((\rho, \beta, V)\) is a representation of the Hom-Lie color Hom-algebra \((A, [\cdot, \cdot], \varepsilon, \alpha)\) satisfying, for all \(x, y \in \mathcal{H}(A), v \in \mathcal{H}(V)\),

\[
\begin{align*}
\ell(\alpha(y))\rho(x)v &= \rho(y \cdot x)\beta(v) + \varepsilon(y, x)\rho(\alpha(x))\ell(y)v \\
&- \varepsilon(x, v)r(\alpha(x))\rho(y)v + l([y, x])\beta(v), \\
r([x, y])\beta(v) &= \varepsilon(v, x)\rho(\alpha(x))r(y)v - r(\alpha(y))\rho(x)v \\
&+ \varepsilon(x + v, y)(r(\alpha(x))\rho(y)v - r(\alpha(y))r(x)v). 
\end{align*}
\]

**Proposition 4.13.** Let \((A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)\) be a Hom-Gelfand-Dorfman color Hom-algebra, and let \((l, r, \rho, \beta, V)\) its representation. Then, \((A \oplus V, \cdot', [\cdot, \cdot]', \varepsilon, \alpha + \beta)\) is a Hom-Gelfand-Dorfman color Hom-algebra, where \((A \oplus V, \cdot', [\cdot, \cdot]', \varepsilon, \alpha + \beta)\) is the semi-direct product Hom-Novikov color Hom-algebra \(A \ltimes_{l, r, \alpha, \beta} V\), and \((A \oplus V, [\cdot, \cdot]', \varepsilon, \alpha + \beta)\) is the semi-direct product Hom-Lie color Hom-algebra \(A \ltimes_{\rho, \alpha, \beta} V\).

**Proof.** Let \((l, r, \rho, \beta, V)\) be a representation of a Hom-Gelfand-Dorfman color Hom-algebra \((A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)\). By Proposition 2.17 and Proposition 2.26, \((A \oplus V, \cdot', [\cdot, \cdot]', \varepsilon, \alpha + \beta)\) is a Hom-Novikov color Hom-algebra, and \((A \oplus V, [\cdot, \cdot]', \varepsilon, \alpha + \beta)\) is a Hom-Lie color Hom-algebra respectively. Now, we show that the compatibility condition (4.2) is satisfied. For all \(X_i = x_i + v_i \in A_{\alpha_i} \oplus V_{\gamma_i}, \ i = 1, 2, 3\) we have

\[
\begin{align*}
(\alpha + \beta)(x_2 + v_2) &\ast [x_1 + v_1, x_3 + v_3]' \\
- \varepsilon(x_2 + v_2, x_1 + v_1) &\left[ (\alpha + \beta)(x_1 + v_1), (x_2 + v_2)' (x_3 + v_3) \right]' \\
+ \varepsilon(x_1 + x_2, x_3) &\left[ (\alpha + \beta)(x_3 + v_3), (x_2 + v_2)' (x_1 + v_1) \right]' \\
- [(x_2 + v_2), (x_1 + v_1)]' &\left[ (\alpha + \beta)(x_3 + v_3) \right]' \\
+ \varepsilon(x_1, x_3) &\left[ (x_2 + v_2), (x_3 + v_3) \right]' (\alpha + \beta)(x_1 + v_1) \\
&\ast (\alpha(\beta) + \beta(v_2))' (x_1, x_3) + \rho(x_1) v_3 - \varepsilon(x_1, x_3) \rho(\beta(x_3) v_1) \\
- \varepsilon(x_2, x_1) &\left[ \alpha(x_1) + \beta(v_1), x_2 \cdot x_3 + l(x_2) v_3 + r(x_3) v_2' \right]' \\
+ \varepsilon(x_1 + x_2, x_3) &\left[ \alpha(x_3) + \beta(v_3), x_2 \cdot x_1 + l(x_2) v_1 + r(x_1) v_2' \right]' \\
- [(x_2, x_1) + \rho(x_2) v_1 - \varepsilon(x_2, x_1) \rho(x_1) v_2]' &\left( \alpha(x_3) + \beta(v_3) \right)
\end{align*}
\]

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Hence, \((A \oplus V, \cdot, [, ], \varepsilon, \alpha + \beta)\) is a Hom-Gelfand-Dorfman color Hom-algebra.

**Examples 4.14.** Some important examples of representations of Hom-Gelfand-Dorfman color Hom-algebras can be constructed as follows.

(i) Let \((A, \cdot, [, ], \varepsilon, \alpha)\) be a Hom-Gelfand-Dorfman color Hom-algebra. If

\[
L(a)b = a \cdot b, \quad R(a)b = b \cdot a, \quad ad(a)b = [a, b] = -\varepsilon(a, b)[b, a], \quad \forall a, b \in \mathcal{H}(A),
\]
then \((L, R, ad, \alpha, A)\) is a representation of \((A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)\).

(ii) If \(f : \mathcal{A} = (A, \cdot, [\cdot, \cdot], \varepsilon, \alpha) \to (A', \cdot, [\cdot, \cdot], \varepsilon, \beta)\) is a Hom-Gelfand-Dorfman color Hom-algebras morphism, then \((l, r, \rho, \beta, A')\) becomes a representation of \(\mathcal{A}\) via \(f\), that is, for all \((x, y) \in \mathcal{H}(A) \times \mathcal{H}(A')\),

\[ l(x)y = f(x) \cdot y, \quad r(x)y = y \cdot f(x), \quad \rho(x)y = [f(x), y]_2. \]

**Theorem 4.15.** Let \(\mathcal{A} = (A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)\) and \(\mathcal{B} = (B, \cdot, [\cdot, \cdot], \varepsilon, \beta)\) be Hom-Gelfand-Dorfman color Hom-algebras. Suppose that there are such even linear maps \(l_A, r_A, \rho_A : A \to \text{End}(B)\) and \(l_B, r_B, \rho_B : B \to \text{End}(A)\) that \(A \triangleright_B B\) is a matched pair of Hom-Lie color Hom-algebras, and \(A \triangleright_B B\) is a matched pair of Hom-Novikov color Hom-algebras, and for all \(x, y \in \mathcal{H}(A), a, b \in \mathcal{H}(B),\) the following equalities hold:

\begin{align*}
l_A(\alpha(x))(a, b)_B &= r_A(\alpha(x))([b, a]) \\
r_A(\rho_B(a)x)\beta(b) - r_A(\alpha(x))([b, a]) \\
&= \varepsilon(a, x)(\beta(b) \cdot B \rho_A(x)a - \rho_A(l_B(b)x)\beta(a) \\
&- r_A(\rho_B(b)x)\beta(a)) - \varepsilon(a + b, x)(\rho_A(\alpha(x))(b \cdot B a) \\
&- \rho_A(x)b \cdot B (\beta(a) + B \varepsilon(b, a)]\beta(a), r_A(x)b|_B),
\end{align*}

\begin{align*}
l_B(\beta(a))(x, y)_B &= r_B(\beta(a))([y, x]) \\
r_B(\rho_A(x)\alpha(y)) - r_B(\beta(a))([y, x]) \\
&= \varepsilon(x, a)(\alpha(y) \cdot A \rho_B(a)x - \rho_B(l_A(y)a)\alpha(x) \\
&- r_B(\rho_A(y)a)\alpha(x)) - \varepsilon(x + y, a)(\rho_B(\beta(a))(y \cdot A x) \\
&- \rho_B(a)y \cdot A \alpha(x) + \varepsilon(y, x)[\alpha(x), r_B(a)y]|_A),
\end{align*}

\begin{align*}
l_B(\beta(a))(x, y)_A &= r_B(\beta(a))([x, y]) \\
r_B(\rho_A(x)\alpha(y)) - r_B(\beta(a))([x, y]) \\
&= \varepsilon(x, a)([\alpha(x), l_B(a)y]|_A - r_B(\rho_B(\alpha(x)a)\beta(y)) \\
&+ \varepsilon(x, y)(\rho_B(\alpha(x)a)\alpha(x) - \rho_B(a)y \cdot A \alpha(x)) \\
&+ \varepsilon(x + y, a)(l_B(\rho_B(y)a)\alpha(x) - [\alpha(y), l_B(a)x]|_A).
\end{align*}

Then, \((A, B, l_A, r_A, \rho_A, B, r_B, \rho_B, \alpha)\) is called a matched pair of the Hom-Gelfand-Dorfman color Hom-algebras. In this case, on the direct sum \(A \oplus B\) of the underlying linear spaces of \(A\) and \(B\), there is a Hom-Gelfand-Dorfman color Hom-algebra structure which is given for any \(x + a \in A_{\gamma_1} \oplus B_{\gamma_1}, y + b \in A_{\gamma_2} \oplus B_{\gamma_2}\) by

\begin{align*}
(x + a) \cdot (y + b) &= x \cdot A y + (s_A(x)b + \varepsilon(a, y)s_A(y)a) \\
&+ a \cdot B b + (s_B(a)y + \varepsilon(x, b)s_B(b)x), \quad (4.12) \\
[x + a, y + b] &= [x, y]_A + (\rho_A(x)b - \rho_A(y)a) \\
&+ [a, b]_B + (\rho_B(a)y - \rho_B(b)x).
\end{align*}

**Proof.** By Proposition 2.19 and Proposition 2.28, we deduce that \((A \oplus B, \cdot, \varepsilon, \alpha + \beta)\) is a Hom-Novikov color Hom-algebra and \((A \oplus B, [\cdot, \cdot], \alpha + \beta)\) is a Hom-Lie color Hom-algebra. Now, the rest, it is easy (in a similar way as for Proposition 2.11) to verify the compatibility condition satisfied. \(\Box\)
Taking the color $\varepsilon$-commutator in a Hom-Novikov-Poisson color Hom-algebra, we obtain the following result.

**Theorem 4.16.** Let $(A, \cdot , \diamond , \varepsilon , \alpha)$ be a Hom-Novikov-Poisson color Hom-algebra, and

$$[x, y] = x \cdot y - \varepsilon(x, y) y \diamond x, \quad \forall x, y \in \mathcal{H}(A). \quad (4.14)$$

Then $(A, \cdot , [\cdot , \cdot], \varepsilon , \alpha)$ is a Hom-Gelfand-Dorfman color Hom-algebra.

**Proof.** By definition $(A, \cdot , \varepsilon , \alpha)$ is a $\varepsilon$-commutative Hom-associative color Hom-algebra. Then $(A, \cdot , \varepsilon , \alpha)$ is a Hom-Novikov color Hom-algebra. Moreover, by Proposition 2.22, $(A, [\cdot , \cdot], \varepsilon , \alpha)$ is a Hom-Lie color Hom-algebra. Now, we show that the compatibility condition (4.2) is satisfied. For any $x, y, z \in \mathcal{H}(A)$,

$$\alpha(y) \cdot [x, z] - \varepsilon(y, x)[\alpha(x), y \cdot z] + \varepsilon(x + y, z)[\alpha(z), y \cdot x]$$

$$(\text{using (4.14)})$$

$$= \alpha(y) \cdot (x \diamond z - \varepsilon(x, z) z \diamond x) - \varepsilon(y, x)(\alpha(x) \diamond (y \cdot z)$$

$$\quad - \varepsilon(z, y + x)(y \cdot z) \diamond \alpha(x)) + \varepsilon(x + y, z)(\alpha(z) \diamond (y \cdot x)$$

$$\quad - \varepsilon(x + y, z)(y \cdot z) \diamond \alpha(x) + \varepsilon(y, z)(y \cdot z) - \varepsilon(x, y)(y \cdot z) \diamond \alpha(z)$$

$$\quad + \varepsilon(x, y)(y \cdot z) \diamond \alpha(x) + \varepsilon(y, z)(y \cdot z) \diamond \alpha(x)$$

$$= \varepsilon(y, x)((x \diamond y) \cdot \alpha(z) - \alpha(x) \diamond (y \cdot z) - \varepsilon(y, x)(y \diamond x) \cdot \alpha(z) - \alpha(y) \cdot (x \diamond z))$$

$$= 0 \text{ by (3.2)}$$

$$- \varepsilon(x + y, z) \left((z \diamond y) \cdot \alpha(x) - \alpha(z) \diamond (y \cdot x) - \varepsilon(z, y)(y \diamond z) \cdot \alpha(x) - \alpha(y) \cdot (z \diamond x)\right)$$

$$= 0 \text{ by (3.2)}$$

$$- \left((y \cdot x) \diamond \alpha(z) - \varepsilon(x, z)(y \diamond z) \cdot \alpha(x)\right) = 0.$$

\[\square\]

**Example 4.17.** Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ be an abelian group and $A$ be a 4-dimensional $\Gamma$-graded linear space defined by $A_{(0,0)} = < e_1 >$, $A_{(0,1)} = < e_2 >$, $A_{(1,0)} = < e_3 >$ and $A_{(1,1)} = < e_4 >$. The quintuple $(A, \cdot , \diamond , \varepsilon , \alpha)$ is a Hom-Novikov-Poisson color Hom-algebra with

- the bicharacter: $\varepsilon((i_1, i_2), (j_1, j_2)) = (-1)^{i_1 j_1 + i_2 j_2}$,
- the multiplication $\cdot$: $e_2 \cdot e_3 = e_3 \cdot e_2 = \mu e_4$, $\mu \in \mathbb{K}$,
- the multiplication $\diamond$: $e_2 \diamond e_3 = \lambda_1 e_4$, $e_3 \diamond e_2 = \lambda_2 e_4$, $e_3 \diamond e_3 = \lambda_3 e_1$, $\lambda_i \in \mathbb{K}$,
- the even linear map $\alpha: A \rightarrow A$ given by: $\alpha(e_1) = 2e_1$, $\alpha(e_2) = -e_2$, $\alpha(e_3) = -e_3$, $\alpha(e_4) = -2e_4$.

Therefore, $(A, \cdot , [\cdot , \cdot], \varepsilon , \alpha)$ is a Hom-Gelfand-Dorfman color Hom-algebra with

$$[e_2, e_3] = -[e_3, e_2] = (\lambda_1 - \lambda_2)e_4, \quad [e_3, e_3] = 2\lambda_3 e_3.$$
Lemma 4.18 ([5]). Let \((A, \cdot, \varepsilon, \alpha)\) be a \(\varepsilon\)-commutative Hom-associative color Hom-algebra and \(D\) be an even derivation. Define a bilinear operation \(\circ\) on \(A\) by

\[ x \circ y = x \cdot D(y), \quad \forall x, y \in \mathcal{H}(A). \tag{4.15} \]

Then \((A, \cdot, \circ, \varepsilon, \alpha)\) is a Hom-Novikov-Poisson color Hom-algebras.

Combining Theorem 4.16 and Lemma 4.18 leads to the following corollary.

Corollary 4.19. Let \((A, \cdot, \varepsilon, \alpha)\) be a \(\varepsilon\)-commutative Hom-associative color Hom-algebra and \(D\) be an even derivation. Then \((A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)\) is a Hom-Gelfand-Dorfman color Hom-algebra, where for all \(x, y \in \mathcal{H}(A)\),

\[ [x, y] = x \cdot D(y) - \varepsilon(x, y)y \cdot D(x). \tag{4.16} \]

Next theorem provides a procedure for construction of the Hom-Gelfand-Dorfman color Hom-algebras from Gelfand-Dorfman color Hom-algebras and morphisms.

Theorem 4.20. Let \(A = (A, \cdot, [\cdot, \cdot], \varepsilon)\) be a Gelfand-Dorfman color Hom-algebra and \(\alpha : A \rightarrow A\) be a Gelfand-Dorfman color Hom-algebras morphism. Define \(\cdot_{\alpha}, [\cdot, \cdot]_{\alpha} : A \times A \rightarrow A\) for all \(x, y \in \mathcal{H}(A)\), by \(x \cdot_{\alpha} y = \alpha(x \cdot y)\) and \([x, y]_{\alpha} = \alpha([x, y])\). Then, \(A_{\alpha} = (A_{\alpha} = A, \cdot_{\alpha}, [\cdot, \cdot]_{\alpha}, \varepsilon, \alpha)\) is a Hom-Gelfand-Dorfman color Hom-algebra called the \(\alpha\)-twist or Yau twist of \((A, \cdot, [\cdot, \cdot], \varepsilon)\). Moreover, assume that \(A' = (A', \cdot', [\cdot', \cdot'], \varepsilon)\) is another Hom-Gelfand-Dorfman color Hom-algebra and \(\alpha' : A' \rightarrow A'\) is a Hom-Gelfand-Dorfman color Hom-algebras morphism. Let \(f : A \rightarrow A'\) be a Hom-Gelfand-Dorfman color Hom-algebras morphism satisfying \(f \circ \alpha = \alpha' \circ f\). Then, \(f : A_{\alpha} \rightarrow A'_{\alpha}\) is a Hom-Gelfand-Dorfman color Hom-algebras morphism.

Proof. Being a Gelfand-Dorfman color Hom-algebras morphism, \(\alpha : A \rightarrow A\) is an even linear map which is multiplicative with respect to \(\cdot\) and \([\cdot, \cdot]\), that is,

\[ \forall x, y \in \mathcal{H}(A) : \quad \alpha(x \cdot y) = \alpha(x) \cdot \alpha(y), \quad \alpha([x, y]) = [\alpha(x), \alpha(y)]. \]

The equality (4.2) in \(A_{\alpha}\) is proved as follows:

\[
\alpha(y) \cdot_{\alpha} [x, z]_{\alpha} = \alpha(y) \cdot_{\alpha} \alpha([x, z]) = \alpha(\alpha(y) \cdot \alpha([x, z])) \\
(\alpha \text{ morphism}) \\
= \alpha^2(y) \cdot \alpha^2([x, z]) = \alpha^2(y) : [\alpha^2(x), \alpha^2(z)] \\
(A \text{ is a Hom-G. D. color alg}) \\
= \varepsilon(y, x)[\alpha^2(x), \alpha^2(y) \cdot \alpha^2(z)] - \varepsilon(x + y, z)[\alpha^2(z), \alpha^2(y) \cdot \alpha^2(x)] \\
+ [\alpha^2(y), \alpha^2(x)] \cdot \alpha^2(z) - \varepsilon(x, z)[\alpha^2(y), \alpha^2(z)] \cdot \alpha^2(x) \\
(\alpha \text{ morphism}) \\
= \varepsilon(y, x)[\alpha^2(x), \alpha(y \cdot_{\alpha} z)] - \varepsilon(x + y, z)[\alpha^2(z), \alpha(y \cdot_{\alpha} x)] \\
+ \alpha([\alpha(y), \alpha(x)]) \cdot \alpha^2(z) - \varepsilon(x, z)\alpha([y, z]_{\alpha}) \cdot \alpha^2(x) \\
= \varepsilon(y, x)[\alpha(x), y \cdot_{\alpha} z]_{\alpha} - \varepsilon(x + y, z)[\alpha(z), y \cdot_{\alpha} x]_{\alpha} \\
+ [y, x]_{\alpha} \cdot_{\alpha} \alpha(z) - \varepsilon(x, z)[y, z]_{\alpha} \cdot_{\alpha} \alpha(x). \]

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The second assertion follows from
\[ f(x \cdot y) = f(\alpha(x \cdot y)) = \alpha'(f(x \cdot y)) = \alpha'(f(x) \cdot f(y)) = f(x) \cdot f(y), \]
\[ f([x, y]_\alpha) = f(\alpha([x, y])) = \alpha'(f([x, y])) = \alpha'(\{f(x), f(y)\}_\alpha) = [f(x), f(y)]_\alpha. \]

**Corollary 4.21.** If \( A = (A, \cdot, [\cdot, \cdot], \varepsilon, \alpha) \) is a multiplicative Hom-Gelfand-Dorfman color algebra, then for any \( n \in \mathbb{N}^* \),

(i) The \( n \)th derived Hom-Gelfand-Dorfman color Hom-algebra of type 1 of \( A \) is defined by
\[ A_1^n = (A, \cdot^{(n)}, [\cdot, \cdot^{(n)}] = \alpha^n \circ [\cdot, \cdot] \circ \alpha^{n+1}). \]

(ii) The \( n \)th derived Hom-Gelfand-Dorfman color Hom-algebra of type 2 of \( A \) is defined by
\[ A_2^n = (A, \cdot^{(n-1)} = \alpha^{2n-1} \circ [\cdot, \cdot]^{(n-1)} = \alpha^{2n-1} \circ [\cdot, \cdot] \circ \alpha^{2n}). \]

**Proof.** Apply Theorem 4.20 with \( \alpha' = \alpha^n \) and \( \alpha' = \alpha^{2n-1} \) respectively. □

**Example 4.22.** Let \( \Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( A \) be a 4-dimensional \( \Gamma \)-graded linear space with \( A_{(0,0)} = \langle e_1 \rangle, A_{(0,1)} = \langle e_2 \rangle, A_{(1,0)} = \langle e_3 \rangle, A_{(1,1)} = \langle e_4 \rangle \). Then there is a multiplicative admissible Hom-Gelfand-Dorfman color Hom-algebra \( (A, \cdot, [\cdot, \cdot], \varepsilon, \alpha) \) with the bicharacter \( \varepsilon((i_1, i_2), (j_1, j_2)) = (-1)^{i_1j_1+i_2j_2}, \) and the multiplications tables for a basis \( \{e_1, e_2, e_3, e_4\} \):

| \cdot | e_1 | e_2 | e_3 | e_4 |
|---|---|---|---|---|
| e_1 | 0 0 0 0 | e_2 | 0 0 2e_4 0 | e_3 | 0 2e_4 e_1 0 | e_4 | 0 0 0 0 |
| e_2 | 0 0 2e_4 0 | e_3 | 0 2e_4 e_1 0 | e_4 | 0 0 0 0 |
| e_3 | 0 2e_4 e_1 0 | e_4 | 0 0 0 0 |

Then there are Hom-Gelfand-Dorfman color Hom-algebras \( A_1^n \) and \( A_2^n \) with multiplications tables respectively:

| \cdot^{(n)} | e_1 | e_2 | e_3 | e_4 |
|---|---|---|---|---|
| e_1 | 0 0 0 0 | e_2 | 0 0 2^n e_4 0 | e_3 | 0 2^n e_4 e_1 0 | e_4 | 0 0 0 0 |
| e_2 | 0 0 2^n e_4 0 | e_3 | 0 2^n e_4 e_1 0 | e_4 | 0 0 0 0 |
| e_3 | 0 2^n e_4 e_1 0 | e_4 | 0 0 0 0 |

| \cdot^{(n-1)} | e_1 | e_2 | e_3 | e_4 |
|---|---|---|---|---|
| e_1 | 0 0 0 0 | e_2 | 0 0 2^{n+1} e_4 0 | e_3 | 0 2^{n+1} e_4 e_1 0 | e_4 | 0 0 0 0 |
| e_2 | 0 0 2^{n+1} e_4 0 | e_3 | 0 2^{n+1} e_4 e_1 0 | e_4 | 0 0 0 0 |
| e_3 | 0 2^{n+1} e_4 e_1 0 | e_4 | 0 0 0 0 |

\[ \alpha^n(e_1) = e_1, \quad \alpha^n(e_2) = 2^n e_2, \quad \alpha^n(e_3) = e_3, \quad \alpha^n(e_4) = 2^n e_4. \]
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