A CONCISE PROOF OF THE MULTIPLICATIVE ERGODIC THEOREM ON BANACH SPACES

CECILIA GONZÁLEZ-TOKMAN AND ANTHONY QUAS

Abstract. We give a streamlined proof of the multiplicative ergodic theorem for quasi-compact operators on Banach spaces with a separable dual.

1. Introduction

The multiplicative ergodic theorem (MET) is a very powerful result in ergodic theory establishing the existence of generalized eigenspaces for stationary compositions of linear operators. It is of great interest in many areas of mathematics, including analysis, geometry and applications. The MET was first established by Oseledets [9] in the context of matrix cocycles. The decomposition into generalized eigenspaces is called the Oseledets splitting.

After the original version, the MET was proved by a different method by Raghunathan [10]. The result was subsequently generalized to compact operators on Hilbert spaces by Ruelle [11]. Mañé [8] proved a version for compact operators on Banach spaces under some continuity assumptions on the base dynamics and the dependence of the operator on the base point. Thieullen [12] extended this to quasi-compact operators. Recently, Lian and Lu [7] proved a version in the context of linear operators on separable Banach spaces, in which the continuity assumption was relaxed to a measurability condition.

We give a streamlined alternative, hopefully simpler, proof of Lian and Lu’s result, under a mild additional assumption (the separability of the dual space, rather than the space itself). At the same time, we remove the assumption of injectivity of the operators. An important feature of the present approach is its constructive nature. Indeed, it provides a robust way of approximating the Oseledets splitting, following what could be considered a power method type strategy. This makes the work also relevant from an applications perspective.

The approach of this work is similar in spirit to that of Raghunathan, in that we primarily work with the ‘slow Oseledets spaces’. Mañé’s proof works hard to build the fast space, as do the subsequent works
based on Mañé’s template. These proofs rely on injectivity of the operators; some of them make use of natural extensions to extend the result to non-invertible operators – this was the strategy in [12], and it was also used by Doan in [1] to extend [7] to the non-invertible context. In contrast, we establish the non-invertible version first and recover the (semi-)invertible one, including the ‘fast spaces’, straightforwardly using duality. Another key simplifying feature of our method is that we prove measurability at the end of the proof, rather than working to ensure that all intermediate constructions are measurable.

While Raghunathan’s proof uses singular value decomposition and hence relies on the notion of orthogonality, we study instead collections of vectors with maximal volume growth. Another important difference with Raghunathan’s approach is that instead of dealing with the exterior algebra, we work with the Grassmannian. We claim this is more natural since subspaces correspond to rank one elements of the exterior algebra (those that can be expressed as $v_1 \wedge \ldots \wedge v_k$). In the Euclidean setting, rank one elements naturally appear as eigenvectors of $\Lambda^k(A^* A)$, but this does not seem to generalize to the Banach space case.

Section 2 contains volume calculations for bounded linear maps $T$ on a Banach space $X$. We establish an asymptotic equivalence between $k$-dimensional volume growth under $T$ and $T^*$, as well as other measures of volume growth.

Section 3 deals with random dynamical systems, or linear cocycles. The non-invertible Oseledets multiplicative ergodic theorem is proved under the assumptions that $X^*$ is separable and a quasi-compactness condition holds. Separability allows one to show all the necessary measurability conditions in an elementary fashion.

The recent semi-invertible version of the MET established in [4], building on [2, 3], and from which the invertible version of the MET can be recovered, is also presented here. This result is obtained via duality: complementary spaces to the Oseledets filtration arise as annihilators of the corresponding Oseledets filtration for the dual cocycle.

2. Volume calculations in Banach spaces

Let $X$ be a Banach space with norm $\| \cdot \|$. As usual, given a non-empty subset $A$ of $X$ and a point $x \in X$, we define $d(x, A) = \inf_{y \in A} d(x, y)$. We denote by $B_X$ and $S_X$ the unit ball and unit sphere in $X$, respectively. The linear span of a collection $C$ of vectors in $X$ will be denoted by $\text{lin}(C)$ with the convention that $\text{lin}(\emptyset) = \{ 0 \}$. The dual of $X$ will be denoted by $X^*$. 
We define the \textit{k-dimensional volume} of a collection of vectors \((v_1, \ldots, v_k)\) by

\[
\text{vol}_k(v_1, \ldots, v_k) = \prod_{i=1}^{k} d(v_i, \text{lin}\{v_j : j < i\}).
\]

It is easy to see that \(\text{vol}_k(\alpha_1 v_1, \ldots, \alpha_k v_k) = |\alpha_1| \ldots |\alpha_k| \text{vol}_k(v_1, \ldots, v_k)\). In the case where the normed space is Euclidean this notion corresponds with the standard notion of \(k\)-dimensional volume. Notice that \(\text{vol}_k(v_1, \ldots, v_k)\) is not generally invariant under permutation of the vectors.

Given a bounded linear map \(T\) from \(X\) to \(X\), we define \(d_k T(v_1, \ldots, v_k)\) to be \(\text{vol}_k(Tv_1, \ldots, Tv_k)\) and \(D_k T = \sup_{\|v_1\| = \ldots = \|v_k\| = 1} d_k T(v_1, \ldots, v_k)\).

**Lemma 1** (Submultiplicativity). Let \(T : X \to X\) and \(S : X \to X\) be linear maps. Then \(D_k(S \circ T) \leq D_k(S) D_k(T)\).

**Proof.** Let \(v_1, \ldots, v_k \in X\). Then one checks from the definition that for any collection of coefficients \((\alpha_{ij})_{j < i}\), the following holds

\[
(1) \quad \text{vol}_k(v_1, \ldots, v_k) = \text{vol}_k(v_1, v_2 - \alpha_{21} v_1, \ldots, v_k - \sum_{j < k} \alpha_{kj} v_j). 
\]

Since the linear spans in the definition of volume are finite-dimensional spaces, the minima are attained so that \(d_k T(v_1, \ldots, v_k) = \|T(v_1)\| \|T(v_2) - \alpha_{21} T(v_1)\| \ldots \|T(v_k) - \alpha_{k1} T(v_1) - \ldots - \alpha_{ kk-1} T(v_{k-1})\|\) for appropriate choices of \((\alpha_{ij})_{j < i}\).

Let \(w_j = v_j - \sum_{i < j} \alpha_{ji} v_i\) so that \(d_k T(v_1, \ldots, v_k) = \|T(w_1)\| \ldots \|T(w_k)\|\) and set \(u_j = T(w_j)/\|T(w_j)\|\). Using (1), we have

\[
d_k(S \circ T)(v_1, \ldots, v_k) = \text{vol}_k(ST(v_1), \ldots, ST(v_k)) \\
= \text{vol}_k(ST(w_1), \ldots, ST(w_k)) \\
= \|T(w_1)\| \ldots \|T(w_k)\| \text{vol}_k(S(u_1), \ldots, S(u_k)) \\
\leq d_k T(v_1, \ldots, v_k) D_k S.
\]

Taking a supremum over \(v_1, \ldots, v_k\) in the unit ball of \(X\), one obtains the bound \(D_k(S \circ T) \leq D_k(S) D_k(T)\) as required. \(\square\)

**Lemma 2.** Let \(T : X \to X\) be linear. Suppose that \(V\) is a \(k\)-dimensional subspace and \(\|Tx\| \geq M\|x\|\) for all \(x \in V\). Then \(D_k T \geq M^k\).

**Proof.** Let \(v_1, \ldots, v_k\) belong to \(V \cap S_X\) and satisfy \(d(v_j, \text{lin}\{v_i : i < j\}) = 1\). Then \(d_k T(v_1, \ldots, v_k) \geq M^k\). \(\square\)

We now proceed to compare volume estimates for a linear operator \(T\) and its dual. We introduce a third quantity to which we compare both \(D_k(T)\) and \(D_k(T^*)\). Given linear functionals \(\theta_1, \ldots, \theta_k \in X^*\) and
there exist positive constants $c_k$ and $C_k$ with the following property: For every bounded linear map $T$ from a Banach space $X$ to itself,

$$c_k D_k(T) \leq D_k(T^*) \leq C_k D_k(T).$$

**Proof.** The statement will follow from the following inequalities:

(2) \hspace{1cm} D_k(T) \leq E_k(T) \leq k! D_k(T)

(3) \hspace{1cm} D_k(T^*) \leq E_k(T) \leq k! D_k(T^*).

The second inequality of (2) is proved as follows: Let $x_1, \ldots, x_k$ and $\theta_1, \ldots, \theta_k$ all be of norm 1 in $X$ and $X^*$ respectively. Let $\alpha_j = d(Tx_j, \text{lin}(Tx_1, \ldots, Tx_{j-1}))$. Let $c_{j_1}^1, \ldots, c_{j_{i-1}}^j$ be chosen so that $\|Ty_j\| = \alpha_j$, where $y_j$ is defined by $y_j = x_j - (c_{j_1}^1 x_1 + \ldots + c_{j_{i-1}}^j x_{j-1})$. Note that $U'' = U((\theta_1), (y_j))$ may be obtained from $U = U((\theta_1), (x_j))$ by column operations that leave the determinant unchanged. Notice also that $|U''_{ij}| = |\theta_i(Ty_j)| \leq \alpha_j$. From the definition of a determinant, we see that $\det U = \det U'' \leq k! \alpha_1 \ldots \alpha_k$. This inequality holds for all choices of $\theta_i$ in the unit sphere of $X^*$. Now, maximizing over choices of $x_j$ in the unit sphere of $X$, we obtain the desired result.

The second inequality of (3) may be obtained analogously. We let $\beta_i = d(T^*\theta_i, \text{lin}(T^*\theta_1, \ldots, T^*\theta_i-1))$ and choose linear combinations $\phi_i$ of the $\theta_i$ for which the minimum is obtained. The matrix $U'' = U((\phi_i), (x_j))$ is obtained by row operations from $U$ and the $|U''_{ij}| = |\phi_i(Tx_j)| = |(T^*\phi_i)(x_j)| \leq \beta_i$.

To show the first inequality of (2), fix $x_1, \ldots, x_k$ of norm 1. As before, let $\alpha_j = d(Tx_j, \text{lin}(Tx_1, \ldots, Tx_{j-1}))$. By the Hahn-Banach theorem, there exist linear functionals $\phi_i(\theta_i k=1$ of norm 1 such that $\theta_i(x_i) = \alpha_i$ and $\theta_i(x_k) = 0$ for all $k < i$. Now

$$\det U((\theta_i), (x_j)) = \prod \alpha_i.$$

Maximizing over the choice of $(x_j)$, we obtain $E_k(T) \geq D_k(T)$ as required.

Finally, for the first inequality of (3), we argue as follows. Let $\epsilon > 0$ be arbitrary and let $\theta_1, \ldots, \theta_k$ belong to the unit sphere of $X^*$. We may assume that $T^*\theta_1, \ldots, T^*\theta_k$ are linearly independent – otherwise the inequality is trivial. Let $\phi_i = T^*\theta_i - \sum_{k<i} a_{ik} T^*\theta_k$ be such that $\|\phi_i\| = d(T^*\theta_i, \text{lin}(\{T^*\theta_k : k < i\}))$. We shall pick $x_1, \ldots, x_k$ inductively in such
a way that det((\phi_l(x_j))_{i,j< l}) is at least \( \prod_{i=1}^{l} (\|\phi_i\| - \epsilon) \) for each \( 1 \leq l \leq k \).

Suppose \( x_1, \ldots, x_{l-1} \) have been chosen. Then since det((\psi_i(x_j))_{i,j< l}) is non-zero, the rows span \( \mathbb{R}^{l-1} \). Hence there exist \( (b_i)_{i< l} \) such that \( \psi_l := \phi_l + \sum_{i< l} b_i \phi_i \) satisfies \( \psi_l(x_j) = 0 \) for all \( j < l \). By assumption, \( \|\psi_l\| \geq \|\phi_l\| \). Pick \( x_l \in S_X \) such that \( \psi_l(x_l) > \|\psi_l\| - \epsilon \). Then the matrix with a row for \( \psi_l \) and a column for \( x_l \) adjoined has determinant at least \( \prod_{i=1}^{l} (\|\phi_i\| - \epsilon) \). The corresponding matrix with \( \phi_l \) replacing \( \psi_l \) has the same determinant, completing the induction. Maximizing over the choice of \( (\theta_j)_{j\leq k} \), letting \( \epsilon \) shrink to 0, and observing that \( \det((\phi_l(x_j))_{i,j \leq k}) = \det((T^*\theta(x_j))_{i,j \leq k}) \) completes the proof.

\[
\square
\]

A fourth quantity that will play a crucial role in what follows is \( F_k(T) \), defined as

\[
F_k(T) = \sup_{\dim(V) = k} \inf_{v \in V \cap S_X} \|Tv\|.
\]

**Lemma 4** (Relation between determinants and \( F_k \)). Let \( T \) be a bounded linear map from a Banach space \( X \) to itself. Then

\[
E_{k-1}(T)F_k(T) \leq E_k(T) \leq k2^{k-1}E_{k-1}(T)F_k(T).
\]

**Proof.** We first show \( E_k(T) \leq k2^{k-1}E_{k-1}(T)F_k(T) \). We may assume \( E_k(T) > 0 \) as otherwise the inequality is trivial. Let \( \theta_1, \ldots, \theta_k \) be elements of the unit sphere of \( X^* \) and \( x_1, \ldots, x_k \) be elements of the unit sphere of \( X \). Let \( U \) be the matrix with entries \( \theta_i(Tx_j) \). Assume that \( \det U \neq 0 \). Since \( x_1, \ldots, x_k \) span a \( k \)-dimensional space, there exists a \( v = a_1x_1 + \ldots + a_kx_k \) of norm 1 such that \( \|Tv\| \leq F_k(T) \). By the triangle inequality, one of the \( a_i \)'s, say \( a_{j_0} \), must be at least \( \frac{1}{k} \). Let \( \tilde{x}_j = x_j \) for \( j \neq j_0 \) and \( \tilde{x}_{j_0} = v \) and set \( \tilde{U} \) to be the matrix with entries \( \theta_i(\tilde{x}_j) \).

By properties of determinants, we see \( |\det \tilde{U}| = a_{j_0} |\det U| \geq \frac{1}{k} |\det U| \). Next, there exists \( i_0 \) for which \( |\theta_{i_0}(v)| \) is maximal, this maximum not being 0 since \( |\det \tilde{U}| \) is positive. Let \( \tilde{\theta}_i = \theta_i - (\theta_{i_0}(v)/\theta_{i_0}(v))\theta_{i_0} \) for \( i \neq i_0 \) and \( \tilde{\theta}_{i_0} = \theta_{i_0} \), so that \( \|\tilde{\theta}_i\| \leq 2 \) and \( \tilde{\theta}_i(v) = 0 \) for \( i \neq i_0 \).

Now let \( \tilde{U}_{ij} = \tilde{\theta}_i(\tilde{x}_j) \), so that \( |\det \tilde{U}| \leq k |\det \tilde{U}| = k |\det \tilde{U}| \). Finally, the \( j_0 \)th column of \( \tilde{U} \) has a single non-zero entry that is at most \( \|Tv\| \leq F_k(T) \) in absolute value. The absolute value of the cofactor is \( |\det (\tilde{\theta}_i(\tilde{x}_j))_{i \neq i_0, j \neq j_0}| \leq 2^{k-1}E_{k-1}(T) \). Taking a supremum over choices of \( (\tilde{\theta}_i) \) and \( (x_j) \), we have shown \( E_k(T) \leq k2^{k-1}E_{k-1}(T)F_k(T) \).

For the other inequality, we may suppose that \( T \) has kernel of codimension at least \( k \), otherwise \( F_k(T) = 0 \) and there is nothing to prove. Let \( \theta_1, \ldots, \theta_{k-1} \) and \( x_1, \ldots, x_{k-1} \) be arbitrary. Let \( \Delta \) be the determinant of the matrix with entries \( \theta_i(Tx_j) \). Let \( V \) be a \( k \)-dimensional
subspace such that \(V \cap \ker T = \{0\}\). Let \(W = \text{lin}(Tx_1, \ldots, Tx_{k-1})\). Let \(z\) be a point in the unit sphere of \(T(V)\) such that \(d(z, W) = 1\) (the existence follows from a theorem of Gohberg and Krein, see Lemma 211 of [5]). Let \(v \in V \cap S_X\) be such that \(T(v)\) is a multiple of \(z\). Let \(\theta_k\) be a linear functional of norm \(1\) such that \(\theta_k|_W = 0\) and \(\theta_k(z) = 1\) and let \(x_k = v\). Now forming the \(k \times k\) matrix \((\theta_i(x_j))_{1 \leq i,j \leq k}\), we see the absolute value of the determinant is \(\Delta \cdot \theta_k(Tv) = \Delta \cdot ||Tv|| \geq \Delta \cdot \inf_{x \in V \cap S_X} ||Tx||\).

Taking suprema over choices of \(x\)'s, \(\theta\)'s and \(k\)-dimensional \(V\)'s, we see that \(E_k(T) \geq F_k(T)E_{k-1}(T)\) as required.

\[\square\]

**Corollary 5.** [of Lemmas 3 and 4] For each \(k > 0\), the quantities \(D_k(T), D_k(T^*), E_k(T)\) and \(\prod_{i \leq k} F_i(T)\) agree up to multiplicative factors that are independent of the bounded linear map \(T\) and the Banach space \(X\).

By definition, for each natural number \(k\), one can find sequences \((\theta_i)_{i \leq k}\) and \((x_j)_{j \leq k}\) such that \(\det(U(\theta_i, x_j)) \approx E_k(T)\). We now show that we can find infinite sequences \((\theta_i)\) and \((x_j)\) so that, for each \(k\), \(\det(U((\theta_i)_{i \leq k}, (x_j)_{j \leq k})) \approx E_k(T)\).

**Lemma 6** (Existence of consistent sequences). For any linear map \(T\), there exist \((\theta_i)_{i \geq 1}\) in \(S_X\), and \((x_j)_{j \geq 1}\) in \(S_X\) such that for all \(k\),

\[
\det \left( (\theta_i(Tx_j))_{1 \leq i,j \leq k} \right) \geq \frac{1}{2^k} \prod_{i \leq k} F_i(T); \quad \text{and} \quad \|Tx\| \geq 4^{-k} F_k(T) \|x\| \quad \text{for all} \quad x \in \text{lin}(x_1, \ldots, x_k).
\]

**Proof.** The proof is by induction: suppose \((\theta_i)_{i \leq k}\) and \((x_j)_{j < k}\) have been chosen and satisfy the desired inequalities at stage \(k - 1\). Then pick an arbitrary \(k\)-dimensional space \(V\) such that \(\|Tv\| \geq \frac{1}{2} F_k(T) \|v\|\) for all \(v \in V\). Using the result of Gohberg and Krein, let \(x_k \in V \cap S_X\) be such that \(d(Tx_k, \text{lin}(Tx_1, \ldots, Tx_{k-1})) = \|Tx_k\|.\) Finally choose \(\theta_k\) of norm \(1\) such that \(\theta_k(Tx_i) = 0\) for \(i < k\) and \(\theta_k(Tx_k) = \|Tx_k\|.\) The determinant inequality at stage \(k\) follows.

Let \(x = a_1x_1 + \ldots + a_kx_k\) be of norm \(1\). Then

\[
\|Tx\| \geq |a_k|d(Tx_k, \text{lin}(Tx_1, \ldots, Tx_{k-1})) \geq |a_k|\|Tx_k\| \geq |a_k|F_k(T)/2.
\]

Similarly,

\[
\|Tx\| \geq \left\| T(\sum_{j<k} a_jx_j) \right\| - |a_k|\|Tx_k\|.
\]

Averaging the inequalities, we get

\[
\|Tx\| \geq \frac{1}{2} \left\| T(\sum_{j<k} a_jx_j) \right\|.
\]
Lemma 7 (Lower bound on volume growth in a subspace of finite codimension). For any natural numbers $k > m$, there exists $C_k$ such that if $X$ is a Banach space, $T$ is a linear map of $X$ and $V$ is a closed subspace of $X$ of codimension $m$, then $D_k(T) \leq C_k D_m(T) D_{k-m}(T|_V)$.

Proof. Let $\epsilon > 0$. Let $P$ be a projection from $X$ to $V$ of norm at most $\sqrt{m} + \epsilon$ (such a projection exists by Corollary III.B.11 in the book of Wojtaszczyk [13]). Then, $\|1 - P\| \leq \sqrt{m} + \epsilon + 1$. Let $x_1, \ldots, x_k$ be a sequence of vectors in $X$ of norm 1. The proof of Lemma 3 shows that there exist $\psi_1, \ldots, \psi_k$ in $S_X^*$ such that $\det(\psi_i(x_j)) \geq d_k T(x_1, \ldots, x_k)$. Write $P_1$ for $P$ and $P_0$ for $1 - P$, which has $m$-dimensional range. There exists a choice $\epsilon_1, \ldots, \epsilon_k \in \{0, 1\}^k$ such that $\det(\psi_i(P_{\epsilon_i}x_j)) > 2^{-k} d_k T(x_1, \ldots, x_k)$, by multilinearity of the determinant. At most $m$ of the $\epsilon_j$ can be 0, as otherwise more than $m$ vectors lie in a common $m$-dimensional space, so that at least $k - m$ of them lie in $V$. Hence, there exist vectors $y_1, \ldots, y_m$ in $S_X$ and $y_{m+1}, \ldots, y_k$ in $S_X \cap V$ such that

$$|\det(\psi_i(y_j))| \geq (2(\sqrt{m} + \epsilon + 1))^{-k} d_k T(x_1, \ldots, x_k).$$

Using the proof of Lemma 3 again, we deduce that

$$d_q T(y_1, \ldots, y_m) d_{k-m} T(y_{m+1}, \ldots, y_k) \geq d_k T(y_1, \ldots, y_k) \geq (2(\sqrt{m} + \epsilon + 1))^{-k}/(k!) d_k T(x_1, \ldots, x_k).$$

This completes the proof. ☐

3. Random dynamical systems

A closed subspace $Y$ of $X$ is called complemented if there exists a closed subspace $Z$ such that $X$ is the direct sum of $Y$ and $Z$, written $X = Y \oplus Z$. That is, for every $x \in X$, there exist $y \in Y$ and $z \in Z$ such that $x = y + z$, and this decomposition is unique. The Grassmannian $G(X)$ is the set of closed complemented subspaces of $X$. We denote by $G^k(X)$ the collection of closed $k$-codimensional subspaces of $X$ (these are automatically complemented). We equip $G(X)$ with the metric $d(Y, Y') = d_H(Y \cap B_X, Y' \cap B_X)$ where $d_H$ denotes the Hausdorff distance. There is a natural map $\perp$ from $G(X)$ to $G(X^*)$ given by $Y^\perp = \{\theta \in X^*: \theta(y) = 0 \text{ for all } y \in Y\}$. Recall that if $X^*$ is separable, then $X$ is separable, as is $G^k(X)$.
In this section, we consider random dynamical systems. These consist of a tuple \( \mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L}) \), where \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space; \(\sigma\) is a measure preserving transformation of \(\Omega\); \(X\) is a separable Banach space; the generator \(\mathcal{L}: \Omega \to B(X, X)\) is strongly measurable (that is for fixed \(x \in X\), \(\omega \mapsto \mathcal{L}_\omega x\) is \((\mathcal{F}, \mathcal{B}_X)\)-measurable); and \(\log \|\mathcal{L}_\omega\|\) is integrable.

This gives rise to a cocycle of bounded linear operators \(\mathcal{L}^{(n)}_\omega\) on \(X\), defined by \(\mathcal{L}^{(n)}_\omega(x) = \mathcal{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}_\omega x\). We will consider \(\mathcal{F}\) and \(\mathbb{P}\) to be fixed, and thus refer to a random dynamical system as \(\mathcal{R} = (\Omega, \sigma, X, \mathcal{L})\). We say \(\mathcal{R}\) is ergodic whenever \(\sigma\) is ergodic.

When the base \(\sigma\) is invertible, we can also define the dual random dynamical system \(\mathcal{R}^* = (\Omega, \mathcal{F}, \mathbb{P}, \sigma^{-1}, X^*, \mathcal{L}^*)\), where \(X^*\) is the dual of \(X\) and \(\mathcal{L}^*(\omega, \theta) := (\mathcal{L}_{\sigma^{-1}\omega})^* \theta\). In this way, \(\theta(\mathcal{L}_\omega x) = \mathcal{L}^*_{\sigma\omega}(\theta(x))\) and, more generally, \(\mathcal{L}^{(n)}_{\sigma^n\omega}(\theta(x)) = \theta(\mathcal{L}^{(n)}_\omega x)\) for every \(x \in X, \theta \in X^*\). Thus, \(\mathcal{L}^{(n)}_{\sigma^n\omega} = (\mathcal{L}^{(n)}_\omega)^*\).

**Lemma 8** (Measurability I). Given a random dynamical system \(\mathcal{R} = (\Omega, \sigma, X, \mathcal{L})\) acting on a separable Banach space, the following functions are measurable:

- \(\omega \mapsto D_k(\mathcal{L}_\omega)\);
- \(\omega \mapsto \|\mathcal{L}_\omega\|\);
- \(\omega \mapsto \text{ic} (\mathcal{L}_\omega) := \inf \{ r > 0 : \mathcal{L}_\omega (B_X) \text{ can be covered by } \text{finitely many balls of radius } r \}\).

**Proof.** Fix a dense sequence \(x_1, x_2, \ldots\) in the unit ball of \(X\). By strong measurability, for each fixed \(x\), \(\omega \mapsto \|\mathcal{L}_\omega x\|\) is measurable. Then for each \(j_1, \ldots, j_k\), we have that \(f_{j_1, j_2, \ldots, j_{k-1}}(\omega) := \inf_{q_1, \ldots, q_{k-1} \in \mathbb{Q}} \|\mathcal{L}_\omega x_{j_1} - \sum_{1 \leq l < i} q_l \mathcal{L}_\omega x_{j_l}\|\) is measurable, so

\[
D_k(\mathcal{L}_\omega) = \sup_{j_1, \ldots, j_k} \prod_{1 \leq l \leq k} f_{j_l|j_1, \ldots, j_{l-1}}(\omega)
\]

is measurable. Similarly, \(\omega \mapsto \|\mathcal{L}_\omega\| = \sup_j \|\mathcal{L}_\omega x_j\|\) and \(\text{ic}(\mathcal{L}_\omega) = \lim_n \sup_j \inf_{k \leq n} \|\mathcal{L}_\omega(x_j) - (2\|\mathcal{L}_\omega\|)x_k\|\) are measurable. \(\square\)

**Remark 9** (Random dynamical systems with random domains). Let \(\mathcal{R} = (\Omega, \sigma, X, \mathcal{L})\) be a random dynamical system acting on a separable Banach space. Suppose there exists a measurable family of closed, finite codimensional subspaces \(\mathcal{V} = \{ V(\omega) \}_{\omega \in \Omega}\) which is equivariant under \(\mathcal{L}\) (that is, \(\mathcal{L}_\omega(V(\omega)) \subset V(\sigma\omega))\). Then, one can construct a new random dynamical system \(\mathcal{R}|_{\mathcal{V}}\) by restricting the generator \(\mathcal{L}\) to \(\mathcal{V}\). The results of this section apply to this more general type of random dynamical systems as well.
Indeed, the fact that the domains of $\mathcal{L}_\omega$ are the same is only used in Lemma 8, when a countable dense sequence in $S_X$ is chosen. In the case of the restriction $\mathcal{R}|_Y$, the choice of a sequence of measurable functions $\omega \mapsto u_j(\omega)$ so that $(u_j(\omega))_{j>0}$ is dense in $V(\omega) \cap S_X$ can be made as follows.

First, for fixed $v \in X, \omega \mapsto d(v,V(\omega))$ is a measurable function, as it is the composition of continuous and measurable functions. Fix a dense sequence $v_1,v_2,\ldots \in S_X$. Now for each $j$, set $u_0^j(\omega) = v_j$, and let $u_{j+1}^k(\omega) = v_l$, where $l = \min\{m : d(v_m,V(\omega) \cap S_X) \leq \frac{1}{4}d(u_k^j(\omega),V(\omega) \cap S_X) \}$ and $d(v_m,u_k^j(\omega)) \leq 2d(u_k^j(\omega),V(\omega) \cap S_X)$. For each $j$, this is a measurable convergent sequence and hence the limit point $u_\infty^j(\omega)$ is measurable, and belongs to $V(\omega)$. The sequence $(u_\infty^j(\omega))$ is dense in $V(\omega) \cap S_X$ because there are $v_j$ arbitrarily close to all points of $V(\omega) \cap S_X$.

One may also consider the quotient spaces $Q(\omega) := X/V(\omega)$, equipped with the quotient norm, $\|\bar{x}\|_{Q(\omega)} = d(x,V(\omega))$, and the induced $\bar{\mathcal{L}}_\omega : Q(\omega) \to Q(\sigma\omega)$. The measurability results of Lemma 8 also apply in this case. Indeed, if $v_1,v_2,\ldots$ is a countable dense sequence in $S_X$, and $\{u_i(\omega)\}_{i \in \mathbb{N}}$ is a sequence of measurable functions dense in $V(\omega) \cap S_X$ as constructed above, then measurability of $\omega \mapsto d(\mathcal{L}_\omega(v_i),V(\omega))$ is straightforward to establish, and hence $\omega \mapsto \|\bar{\mathcal{L}}_\omega\|$ is measurable.

The fact that $\omega \mapsto \|D_k(\bar{\mathcal{L}}_\omega)\|_{Q(\omega) \to Q(\sigma\omega)}$ is measurable follows similarly from Lemma 8.

When $\mathcal{R}$ is ergodic, Lemma 8 (or Remark 9) combined with Kingman’s sub-additive ergodic theorem ensures the existence of the maximal Lyapunov exponent of $\mathcal{R}$, defined by

$$\lambda(\mathcal{R}) := \lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{L}^{(n)}_\omega\|,$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. Similarly, using the fact that the index of compactness is also sub-multiplicative and bounded above by the norm, we have existence of the index of compactness of $\mathcal{R}$, defined by

$$\kappa(\mathcal{R}) := \lim_{n \to \infty} \frac{1}{n} \log \text{ic}(\mathcal{L}^{(n)}_\omega),$$

with the property that $\kappa(\mathcal{R}) \leq \lambda(\mathcal{R})$. Furthermore, we have the following.

**Lemma 10.** Given an ergodic random dynamical system $\mathcal{R}$, there exist constants $\Delta_k = \Delta_k(\mathcal{R})$ such that for almost every $\omega \in \Omega$,

$$\lim_{n \to \infty} \frac{1}{n} \log D_k(\mathcal{L}^{(n)}_\omega) = \Delta_k.$$
Furthermore, $\frac{1}{n} \log E_k(\mathcal{L}_n^0) \to \Delta_k$. Define $\Delta_0 = 0$ and let $\mu_k = \Delta_k - \Delta_{k-1}$ for each $k \geq 1$. Then, $\frac{1}{n} \log F_k(\mathcal{L}_n^0) \to \mu_k$.

Proof. The first claim follows from Kingman’s sub-additive ergodic theorem, via Lemma 8 (or Remark 9) and Lemma 1. The remaining two claims are consequences of Corollary 5.

The $\mu_k$’s of the previous lemma are called the Lyapunov exponents of $\mathcal{R}$. When $\mu_k > \kappa(\mathcal{R})$, $\mu_k$ is called an exceptional Lyapunov exponent.

**Theorem 11** (Lyapunov exponents and index of compactness). Let $\mathcal{R}$ be a random dynamical system with ergodic base acting on a separable Banach space $X$. Then

- $\mu_1 \geq \mu_2 \geq \ldots$;
- For any $\rho > \kappa(\mathcal{R})$, there are only finitely many exponents that exceed $\rho$;
- If $\sigma$ is invertible, the dual random dynamical system $\mathcal{R}^*$ and $\mathcal{R}$ have the same Lyapunov exponents.

Proof. That the $\mu_i$ are decreasing follows from Lemma 10 and the observation that $F_k(T) \leq F_{k-1}(T)$. That the system and its dual have the same exponents follows from Lemma 3 together with the simple result (in [2, Lemma 8.2]) that if $(f_n)$ is sub-additive and satisfies $f_n(\omega)/n \to A$ almost everywhere, then one has $f_n(\sigma^{-n}\omega)/n \to A$ also.

It remains to show that for $\rho > \kappa$, the system has at most finitely many exponents that exceed $\rho$. Let $\kappa < \alpha < \beta < \rho$. Since $\log \|\mathcal{L}_\omega\|$ is integrable, there exists a $0 < \delta < (\beta - \alpha)/2|\alpha|$ such that if $\mathbb{P}(E) < \delta$, then $\int_E \log \|\mathcal{L}_\omega\| d\mathbb{P}(\omega) < (\beta - \alpha)/2$. By the sub-additive ergodic theorem, there exists $L > 0$ such that $\mathbb{P}(\text{ic}(\mathcal{L}_\omega^{(L)}) \geq e^{\alpha L} < \delta/2$. If $\text{ic}(\mathcal{L}_\omega^{(L)}) < e^{\alpha L}$, then by definition, $\mathcal{L}_\omega^{(L)} B_X$ may be covered by finitely many balls of size $e^{\alpha L}$. Let $r$ be chosen large enough so that $\mathbb{P}(G) > 1 - \delta$, where $G$ (the good set) is defined by

$$G = \{\omega : \mathcal{L}_\omega^{(L)} B_X \text{ may be covered with } e^{rL} \text{ balls of size } e^{\alpha L}\}.$$  

We split the orbit of $\omega$ into blocks: if $\sigma^i\omega \in G$, then the block length is $L$; otherwise, if $\sigma^i\omega$ is bad, we take a block of length 1. Consider the following iterative process: start with a ball of radius 1. Then look at the current iterate of $\omega$, $\sigma^i\omega$. If it is good, replace the current crop of balls by $e^{rL}$ times as many balls of size $e^{\alpha L}$ times their current size. If the iterate of $\omega$ is bad, replace the balls by balls inflated by a factor of $\|\mathcal{L}_{\sigma^i\omega}\|$. In this way, one obtains a collection of balls that covers the push-forward of the unit ball.
We claim that for almost all \( \omega \), for sufficiently large \( N \), \( \mathcal{L}_\omega^{(N)}(B_X) \) is covered by at most \( e^{rN} \) balls of size \( e^{\beta N} \). For large \( N \), the combined inflation through the bad steps is less than \( e^{(\beta-\alpha)N/2} \). If \( \alpha \geq 0 \), then for large \( N \), through the good steps, the balls are inflated by a factor at most \( e^{\alpha N} \). If \( \alpha < 0 \), then combining the good blocks, the balls are scaled by a factor of \( e^{\alpha(1-\delta)N} < e^{(\alpha+\beta)N/2} \) or smaller. In both cases, we see that overall, balls are scaled by at most \( e^{\beta N} \). The splitting only takes place in the good blocks, and yields at most \( e^{rN} \) balls.

Now suppose that \( \mu_k > \rho \). For almost all \( \omega \), we have that for all large \( N \), we have \( D_k(\mathcal{L}_\omega^{(N)}) > e^{kN\rho} \). Fix such an \( N \), and suppose that \( x_1, \ldots, x_k \) have the property \( \prod_{i \leq k} D_i > e^{kN\rho} \) where

\[
D_i = d(\mathcal{L}_\omega^{(N)} x_i, \text{lin}(\{\mathcal{L}_\omega^{(N)} x_j : j < i\})).
\]

Let \( T_i = \{0, 1, \ldots, \lceil D_i/(2ke^{\beta N}) \rceil \} \) and notice that \( |T_i \times \cdots \times T_k| \geq e^{kN(\rho-\beta)/(2k)^k} \). For \( (j_1, \ldots, j_k) \in T_1 \times \cdots \times T_k \), define

\[
y_{j_1, \ldots, j_k} = \sum_{i=1}^k 2j_i e^{\beta N} \mathcal{L}_\omega^{(N)} x_i
\]

It is not hard to see that all of these points belong to the image of the unit ball of \( X \) under \( \mathcal{L}_\omega^{(N)} \). Further, from the definition of \( D_i \), one can check that these points are mutually separated by at least \( 2e^{\beta N} \), so that one requires at least \( e^{kN(\rho-\beta)/(2k)^k} \) balls to cover \( \mathcal{L}_\omega^{(N)}(B_X) \).

Hence we obtain

\[
e^{kN(\rho-\beta)/(2k)^k} \leq e^{rN}.
\]

Since this holds for all large \( N \), we deduce \( k \leq r/(\rho-\beta) \) as required. \( \square \)

**Lemma 12** (Measurability II). Suppose that \( X \) is a Banach space with separable dual. Suppose further that \( \mathcal{R} \) is an ergodic random dynamical system acting on \( X \).

Assume there exist \( \lambda' > \lambda \in \mathbb{R} \) and \( d \in \mathbb{N} \) such that for \( \mathbb{P} \)-almost every \( \omega \), there is a \( d \)-codimensional subspace \( V(\omega) \) of \( X \) such that for all \( v \in V(\omega) \), \( \limsup_{n \to \infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} v\| \leq \lambda \); and for \( v \not\in V(\omega) \) and arbitrary \( \epsilon > 0 \), \( \|\mathcal{L}_\omega^{(n)} v\| \geq d(v, V(\omega))e^{n(\lambda'-\epsilon)} \) for all large \( n \). Then \( \omega \mapsto V(\omega) \) is measurable.

**Proof.** Given \( V \in \mathcal{G}^d(X) \) and vectors \( w_1, \ldots, w_d \) such that \( V \oplus \text{lin}(w_1, \ldots, w_d) = X \), define a neighbourhood of \( V \) by

\[
N_{V, w_1, \ldots, w_d; \gamma} = \{ U \in \mathcal{G}^d : \|\text{Proj}_{\text{lin}(w_i) \oplus \text{lin}(\{w_j : j \neq i\})} v\| \leq \gamma \text{ for } 1 \leq i \leq d \}.
\]

Since \( \mathcal{G}^d(X) \) is separable, one can fix a countable collection of \( V \)'s dense in \( \mathcal{G}^d(X) \) and fix for each such \( V \) a complementary sequence of
w’s. By restricting to rational \( \gamma \)'s, we obtain a countable collection of neighbourhoods which generates the Borel \( \sigma \)-algebra on \( \mathcal{G}^d(X) \). (To see this, notice that each open set is the union of the neighbourhoods in the collection that it contains). Hence to show the desired measurability, it suffices to show that for each \( N = N_{V_1;w_1,...,w_d;\gamma}, \{ \omega : V(\omega) \in N \} \) is measurable.

Fix a dense set \( v_1, v_2,... \) in the unit sphere of \( V \). We claim that (provided \( \omega \) lies in the set of full measure on which the growth rates are \( \leq \lambda \) and \( \geq \lambda' \) and where the dimension is correct), \( V(\omega) \) lies in \( N \) if and only if the following condition holds:

For each rational \( \epsilon > 0 \), there is \( n_0 > 0 \) such that for each \( n \geq n_0 \), for each \( j \in \mathbb{N} \), there exist rational \( a_1^j, \ldots, a_d^j \) in \( [-\gamma,\gamma] \) such that

\[
\| \mathcal{L}^n_{\omega} (v_j - \sum_{i=1}^d a_i^j w_i) \| \leq e^{(\lambda + \epsilon)n}.
\]

Indeed, the ‘only if’ direction is straightforward. The ‘if’ direction follows from the fact that if \( V(\omega) \notin N \), then either (i) \( V(\omega) + \text{lin}\{w_j : 1 \leq j \leq d\} = Y \neq X \) and therefore there exists \( j \in \mathbb{N} \) such that \( d(v_j, Y) > 0 \); or (ii) there exists \( j \in \mathbb{N} \) such that \( v_j = v + \sum_{i=1}^d b_i w_i \), with \( v \in V(\omega) \) and \( |b_i| > \gamma \) for some \( 1 \leq k \leq d \). In the first case, for every \( a_1, \ldots, a_d \in [-\gamma,\gamma] \), \( d(v_j - \sum_{i=1}^d a_i w_i, V(\omega)) \geq d(v_j, Y) \). In the second case, there exists \( \eta > 0 \) such that for every \( a_1, \ldots, a_d \in [-\gamma,\gamma] \), \( d(v_j - \sum_{i=1}^d a_i w_i, V(\omega)) > \eta \). Hence, in both cases the condition above fails.

Since this condition is obtained by taking countable unions and intersections of measurable sets, the measurability of \( \{ \omega : V(\omega) \in N \} \) is demonstrated.

**Lemma 13.** Let \( \sigma \) be an ergodic measure-preserving transformation of a probability space \( (\Omega, \mathbb{P}) \). Let \( g \) be a non-negative measurable function and let \( h \geq 0 \) be integrable. Suppose further that \( g(\omega) \leq h(\omega) + g(\sigma(\omega)) \), \( \mathbb{P}\text{-a.e.} \). Then \( g \) is tempered; that is \( \lim_{n \to \infty} g(\sigma^n \omega)/n = 0 \), \( \mathbb{P}\text{-a.e.} \).

**Proof.** Let \( \epsilon > 0 \) and let \( K > \int h \). By the maximal ergodic theorem, \( B_1 = \{ \omega : h(\omega) + \ldots + h(\sigma^{n-1} \omega) < nK, \text{ for all } n \} \) has positive measure. Let \( M \) be such that \( B_2 := \{ \omega : g(\omega) < M \} \) has positive measure. As a consequence of the Birkhoff ergodic theorem, for any measurable set \( B \) with \( \mathbb{P}(B) > 0 \), for \( \mathbb{P}\text{-a.e.} \) \( \omega \), for all sufficiently large \( k \), there exists \( j \in [(1 + \epsilon)^k, (1 + \epsilon)^{k+1}] \) such that \( \sigma^j(\omega) \in B \). Now for \( \omega \in \Omega \), let \( k_0 \) be such that for all \( k \geq k_0 \), there exist \( j \in [(1 + \epsilon)^k, (1 + \epsilon)^{k+1}] \) such that \( \sigma^j \omega \in B_1 \) and \( j' \in [(1 + \epsilon)^{k+2}, (1 + \epsilon)^{k+3}] \) such that \( \sigma^{j'} \omega \in B_2 \). If \( n > (1 + \epsilon)^{k_0+1} \), then \( n \in [(1 + \epsilon)^{k+1}, (1 + \epsilon)^{k+2}] \) for some \( k \geq k_0 \). Let \( j \in [(1 + \epsilon)^k, (1 + \epsilon)^{k+1}] \) and \( j' \in [(1 + \epsilon)^{k+2}, (1 + \epsilon)^{k+3}] \) be as above. Then \( g(\sigma^n \omega) \leq \sum_{k=n}^{j'-1} h(\sigma^k \omega) + g(\sigma^{j'} \omega) \leq \sum_{k=j}^{j'-1} h(\sigma^k \omega) + g(\sigma^{j'} \omega) \leq \sum_{k=j}^{j'-1} h(\sigma^k \omega) + g(\sigma^{j'} \omega) \leq \int h(\omega) + g(\sigma(\omega))/n \leq \epsilon \).
\( K(j' - j) + M \), so that \( \limsup g(\sigma^n \omega)/n \leq 4\epsilon \). Since \( \epsilon \) is arbitrary, the conclusion follows. \( \square \)

**Theorem 14** (The Oseledets filtration). Let \( \mathcal{R} \) be an ergodic random dynamical system acting on a Banach space \( X \) with separable dual. Suppose that \( \kappa(\mathcal{R}) < \lambda(\mathcal{R}) \). Then there exist \( 1 \leq r \leq \infty \) and:

- a sequence of exceptional Lyapunov exponents \( \lambda(\mathcal{R}) = \lambda_1 > \lambda_2 > \ldots > \lambda_r > \kappa(\mathcal{R}) \);
- a sequence \( m_1, m_2, \ldots, m_r \) of positive integers; and
- an equivariant measurable filtration \( X = V_1(\omega) \supset V_2(\omega) \supset \ldots \supset V_r(\omega) \supset V_\infty(\omega) \)

such that for \( \mathbb{P}\text{-a.e. } \omega \), \( \text{codim} V_\ell(\omega) = m_1 + \cdots + m_{\ell - 1} \); for all \( v \in V_\ell(\omega) \setminus V_{\ell + 1}(\omega) \), one has \( \lim \frac{1}{n} \log \| \mathcal{L}^{(n)}_\omega v \| = \lambda_\ell \); and for \( v \in V_\infty(\omega) \), \( \limsup \frac{1}{n} \log \| \mathcal{L}^{(n)}_\omega v \| \leq \kappa(\mathcal{R}) \).

**Proof.** Let \( \mu_1 \geq \mu_2 \geq \ldots \) be as in Theorem 11. Let \( \lambda_1 > \lambda_2 > \ldots \) be the decreasing enumeration of the distinct \( \mu \)-values that exceed \( \kappa(\mathcal{R}) \) (if this is an infinite sequence, then Theorem 11 establishes that \( \lambda_i \to \kappa(\mathcal{R}) \)). The fact that \( \lambda(\mathcal{R}) = \lambda_1 \) is straightforward from the definitions. Let \( m_\ell \) be the number of times that \( \lambda_\ell \) occurs in the sequence \( (\mu_i) \) and let \( \mathcal{M}_\ell = m_1 + \ldots + m_\ell \), so that \( \mu_{\mathcal{M}_{\ell - 1}} = \lambda_{\ell - 1} \) and \( \mu_{\mathcal{M}_{\ell - 1}} = \lambda_\ell \).

We now turn to the construction of \( V_\ell(\omega) \). For a fixed \( \omega \), let the sequences \( (\theta^{(n)}_i)_{i \geq 1} \) and \( (x^{(n)}_j)_{j \geq 1} \) be as guaranteed by Lemma 6 for the operator \( \mathcal{L}^{(n)}_\omega \). We let \( V^{(n)}_\ell(\omega) \) be \( \text{lin}(\theta^{(n)}_1, \ldots, \theta^{(n)}_{\mathcal{M}_{\ell - 1}}, 1, \mathcal{M}_{\ell - 1}) \). Thus, \( X = V^{(n)}_\ell(\omega) \oplus Y^{(n)}_{\ell - 1}(\omega) \). All of these depend on the choice of \( \theta \)'s and \( x \)'s. No claim of uniqueness or measurability is made. The space \( V^{(n)}_\ell(\omega) \) is an approximate slow space.

The proof will go by the following steps:

(a) For almost all \( \omega \), for arbitrary \( \epsilon > 0 \) and for sufficiently large \( n \), \( \| \mathcal{L}^{(n)}_\omega x \| \leq e^{(\lambda_\ell + \epsilon)n} \| x \| \) for all \( x \in V^{(n)}_\ell(\omega) \);

(b) \( V^{(n)}_\ell(\omega) \) is a Cauchy sequence for almost all \( \omega \) – we define the limit to be \( V_\ell(\omega) \);

(c) The \( V_\ell(\omega) \) are equivariant: \( \mathcal{L}_\omega(V_\ell(\omega)) \subseteq V_\ell(\sigma(\omega)) \);

(d) If \( x \notin V_{\ell + 1}(\omega) \), then \( \| \mathcal{L}^{(n)}_\omega v \| > e^{(\lambda_\ell - \epsilon)n} d(v, V_{\ell + 1}(\omega)) \) for large \( n \).

(e) For all \( a > 0 \) and \( \epsilon > 0 \), there exists \( n_0 \) so that for all \( n \geq n_0 \) and all \( x \in S_x \) such that \( d(x, V_{\ell + 1}(\omega)) \geq a \), one has \( \| \mathcal{L}^{(n)}_\omega x \| \geq e^{(\lambda_\ell - \epsilon)n} \).

The remaining steps are proved by induction on \( \ell \).

(f) If \( x \in V_\ell(\omega) \), then \( \limsup \frac{1}{n} \log \| \mathcal{L}^{(n)}_\omega x \| \leq \lambda_\ell \);

\( ^{1} \)If \( r = \infty \), the conclusions must be replaced by: \( \lambda(\mathcal{R}) = \lambda_1 > \lambda_2 > \ldots > \kappa(\mathcal{R}) \), \( m_1, m_2, \ldots \in \mathbb{N} \) and \( X = V_1(\omega) \supset V_2(\omega) \supset \cdots \supset V_\infty(\omega) \).
(g) $\omega \mapsto V_\ell(\omega)$ is measurable;

(h) The restriction, $R_\ell$, of $R$ to $V_\ell(\omega)$ has the same exponents as $R$ with the initial $M_{\ell-1}$ exponents removed.

Proof of (a). Note that by construction
\[
\det \left( (\theta_i^{(n)}(L_\omega^{(n)} x_j^{(n)}))_{1 \leq i,j \leq M_{\ell-1}} \right) \geq 2^{-M_{\ell-1}} E_{M_{\ell-1}}(L_\omega^{(n)}).
\]

For an arbitrary $x \in V_\ell^{(n)} \cap S_X$, let $\phi \in S_{X^*}$ be such that $\phi(L_\omega^{(n)} x) = \|L_\omega^{(n)} x\|$. Then, adding a column for $x$ and a row for $\phi$ to the matrix $U((\theta_i^{(n)}), (x_j^{(n)}))_{1 \leq i,j \leq M_{\ell-1}}$, we see that the $x$ column has all 0 entries except for the $1 + M_{\ell-1}$-st (by definition of $V_\ell^{(n)}(\omega)$), and so we arrive at the bound (uniform over $x \in S_X \cap V_\ell^{(n)}$),
\[
2^{-M_{\ell-1}} E_{M_{\ell-1}}(L_\omega^{(n)}) \|L_\omega^{(n)} x\| \leq E_{1+M_{\ell-1}}(L_\omega^{(n)}).
\]

The conclusion follows from Lemma 10.

Proof of (b). Let us assume that $n_0$ is chosen large enough that for all $n \geq n_0$, the following conditions are satisfied: $\|L_\omega\|$, $\|L_{\sigma_{\ell}}\|$ are less than $e^{\epsilon n}$; $\|L_\omega^{(n)} x\| \leq e^{(\lambda_{\ell-1} + \epsilon)n} \|x\|$ for all $x \in V_\ell^{(n)}$; and $\|L_\omega^{(n)} x\| \geq e^{(\lambda_{\ell-1} - \epsilon)n} \|x\|$ for all $x \in Y_{\ell-1}^{(n)}(\omega)$ (using integrability of $\log \|L_\omega\|$; (a); and Lemma 6). Let $n \geq n_0$. Let $x \in V_\ell^{(n)}(\omega) \cap S_X$ and write $x = u + w$ where $u \in V_\ell^{(n+1)}(\omega)$ and $w \in Y_{\ell-1}^{(n+1)}(\omega)$. Now we have
\[
\|L_\omega^{(n+1)} x\| \leq e^{(\lambda_{\ell+2} \epsilon)n} \|L_{\sigma_{\ell}}\| \leq e^{(\lambda_{\ell+2} \epsilon)n}.
\]

We also have $\|u\| \leq 1 + \|w\|$, $\|L_\omega^{(n+1)} w\| \geq e^{(\lambda_{\ell-1} - \epsilon)(n+1)} \|w\|$ and $\|L_\omega^{(n+1)} u\| \leq e^{(\lambda_{\ell+2} \epsilon)(n+1)} (1 + \|w\|)$. Manipulation with the triangle inequality yields
\[
\|w\| \leq e^{-n(\lambda_{\ell-1} - \lambda_{\ell+2} \epsilon)}.
\]

Hence, each point in the unit sphere of $V_\ell^{(n)}(\omega)$ is exponentially close to $V_\ell^{(n+1)}(\omega)$. Since the two spaces have the same codimension, one obtains a similar inequality in the opposite direction (possibly with the constant increased by a factor of at most 2) - see Lemma 233 of [5]. This establishes that $V_\ell^{(n)}(\omega)$ is a Cauchy sequence.

Proof of (c). We argue essentially as in (b). For large $n$, we take $v \in V_\ell^{(n+1)}(\omega) \cap S_X$. We write $L_\omega(v)$ as $u + w$ with $u \in V_\ell^{(n)}(\sigma(\omega))$ and $w \in Y_{\ell-1}^{(n)}(\sigma(\omega))$. We have bounds of the form $\|L_\omega^{(n+1)} v\| \lesssim e^{\lambda n}$; $\|u\| \lesssim 1 + \|w\|$, $\|L_\omega^{(n)}(w)\| \lesssim e^{\lambda n} (1 + \|w\|)$ and $\|L_{\sigma}^{(n)} w\| \gtrsim e^{\lambda_{\ell-1} n} \|w\|$ (here $\lesssim$ means ‘is smaller up to sub-exponential factors’). Combining
the inequalities as before, one obtains a bound \( \|w\| \lesssim e^{-(\lambda_{\ell-1} - \lambda_\ell)n} \).

Taking a limit, we obtain \( \mathcal{L}_\omega V_\ell(\omega) \subset V_\ell(\sigma(\omega)) \) as required.

**Proof of (d).** Let \( x \not\in V_{\ell+1}(\omega) \), with \( \|x\| = 1 \). For large \( n \), if \( x \) is written as \( u_n + v_n \) with \( u_n \in V_{\ell+1}^{(n)}(\omega) \) and \( v_n \in Y_{\ell}^{(n)}(\omega) \), then \( \|v_n\| \geq \frac{1}{2} d(x, V_{\ell+1}(\omega)) \) and \( \|u_n\| \leq 1 + \|v_n\| \). By (6), \( \|\mathcal{L}_\omega^{(n)} u_n\| \leq e^{(\lambda_{\ell+1} + \epsilon)n} (1 + \|v_n\|) \) for large \( n \), while Lemma 6 gives \( \|\mathcal{L}_\omega^{(n)} v_n\| \geq e^{(\lambda_{\ell} - \epsilon)n} \|v_n\| \) for large \( n \). The conclusion follows. The proof of (e) is the same, using the uniformity in Lemma 6.

For the inductive part, notice that the case \( \ell = 1 \) is trivial. Suppose that the claims have been established for all \( k < \ell \).

**Proof of (f).** Let \( V_{2,\ell-1}(\omega) \) be the ‘\( V_2 \) space’ for the random dynamical system \( \mathcal{R}_{\ell-1} \), that is the original system restricted to \( V_{\ell-1}(\omega) \). Call the limit \( V_\ell(\omega) \). From (7) and the inductive step (h), we see that the distance between \( V_\ell(\omega) \) and \( V_{2,\ell-1}(\omega) \) is at most \( e^{-(\lambda_{\ell-1} - \lambda_{\ell-4})n} \) for sufficiently large \( n \). Given \( v \in V_\ell(\omega) \cap S_X \), it can be written as \( u + w \) with \( u \in V_{2,\ell-1}(\omega) \) and \( w \in V_{\ell-1}(\omega) \) with \( \|w\| \leq e^{-(\lambda_{\ell-1} - \lambda_{\ell-4})n} \). Then, \( \|\mathcal{L}_\omega^{(n)} u\| \lesssim e^{n\lambda_j} \) by (a) and (h) for \( \mathcal{R}_{\ell-1} \); and \( \|\mathcal{L}_\omega^{(n)} w\| \lesssim e^{-(\lambda_{\ell-1} - \lambda_{\ell-4})n} \cdot e^{\lambda_{\ell-1}n} \). So \( \|\mathcal{L}_\omega^{(n)} v\| \lesssim e^{\lambda_{\ell}n} \) by the triangle inequality. From (d), we deduce \( V_\ell(\omega) \subset V_\ell(\omega) \). Since, by (h) (applied to \( \mathcal{R}_{\ell-1} \)), \( V_\ell(\omega) \) and \( V_\ell'(\omega) \) have the same finite co-dimension as subspaces of \( V_{\ell-1}(\omega), V_\ell(\omega) = V_\ell'(\omega) \) and (f) follows.

**Proof of (g).** From (f) and (d), we see that \( V_\ell(\omega) = \{ v : \limsup \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} v\| \leq \lambda_j \} \) and the assumptions of Lemma 12 hold. Measurability of \( V_\ell(\omega) \) follows.

**Proof of (h).** Let \( W(\omega) = \{ w \in V_{\ell-1}(\omega) : \|w\| = 1, d(w, V_\ell(\omega)) \geq \frac{1}{2} \} \). Let \( \epsilon > 0 \). We claim that for sufficiently large \( n \),

\[
\text{(8)} \quad d(w', v') > e^{-\epsilon n} \text{ for all } w', v' \in S_X \cap \mathcal{L}_\omega^{(n)} W(\omega) \text{ and } v' \in V_\ell(\sigma^n_\omega).
\]

Let \( \delta < \frac{\epsilon}{4(\lambda_{\ell-1} - \lambda_{\ell})} \). Let \( g_\ell(\omega) = \sup_{p \in \mathbb{N}} e^{-p(\lambda_\ell + \delta)} \|\mathcal{L}_\omega^{(p)}|_{V_\ell(\omega)}\| \). Notice that \( \log^+ g_\ell(\omega) \leq \log^+ \|\mathcal{L}_\omega\| + \max(-\lambda_\ell - \delta, 0) + \log^+ g_\ell(\sigma(\omega)) \). Lemma 13 ensures that \( \lim_{n \to \infty} \frac{1}{n} \log^+ g_\ell(\sigma^n_\omega) = 0 \).

Then, there exists \( n_0(\omega) \) such that for \( p, n \geq n_0 \), one has

\[
\text{(9)} \quad \|\mathcal{L}_\omega^{(p)} z\| \leq \frac{1}{\delta} \exp(n \frac{\lambda_\ell}{2} + p(\lambda_{\ell-1} + \delta)) \|z\| \text{ for all } z \in V_{\ell-1}(\sigma^n_\omega);
\]

\[
\|\mathcal{L}_\omega^{(p)} v\| \leq \frac{1}{\delta} \exp(n \frac{\lambda_\ell}{2} + p(\lambda_{\ell} + \delta)) \|v\| \text{ for all } v \in V_\ell(\sigma^n_\omega).
\]

Additionally by (e), \( n_0 \) may be chosen so that

\[
\text{(10)} \quad e^{n(\lambda_{\ell-1} - \delta)} < \|\mathcal{L}_\omega^{(n)} w\| < e^{n(\lambda_{\ell-1} + \delta)} \text{ for all } w \in W(\omega) \text{ and } n \geq n_0.
\]
One checks, however, that by the choice of the two terms coming from (9) agree, giving $L_Q$ dynamical system to the equivariant family that combining (8) and (10), we see that the restriction of the random $L_\bar{\omega}$ satisfies for all sufficiently large $n$
setminus\text{e}^{-\epsilon n/2}e^{(p+n)(\lambda_{\ell-1}+\delta)}$.

One checks, however, that by the choice of $\delta$, this is smaller than $e^{(p+n)(\lambda_{\ell-1}-\delta)}$, contradicting (10). This establishes claim (8). Notice that combining (8) and (10), we see that the restriction of the random dynamical system to the equivariant family $Q_{\ell-1}(\omega) = V_{\ell-1}(\omega)/V_\ell(\omega)$ satisfies for all sufficiently large $n$,

$$\|L_\omega(n)\|_{Q_{\ell-1}(\sigma^n \omega)} \geq e^{(\lambda_{\ell-1} - \epsilon)n}\|\bar{w}\|_{Q_{\ell-1}(\omega)}$$

where $\bar{L}_\omega$ denotes the induced action of $L_\omega$ on $Q_{\ell-1}(\omega)$.

Set $\sigma = m_{\ell-1}$. To complete the proof of (h), let $n > n_0$ be arbitrary; let $v_1, \ldots, v_{k-m}$ be unit vectors in $V_\ell(\omega)$ and $w_1, \ldots, w_m$ be unit vectors in $V_{\ell-1}(\omega)$.

Then

$$d_k L_\omega(n)(v_1, \ldots, v_{k-m}, w_1, \ldots, w_m) \geq d_{k-m} L_\omega(n)(v_1, \ldots, v_{k-m}) d_m \bar{L}_\omega(n)(\bar{w}_1, \ldots, \bar{w}_{k-m}),$$

where $\bar{w}_i$ is $w_i + V_\ell(\omega)$. We therefore see

$$D_k L_\omega(n)|_{V_{\ell-1}(\omega)} \geq D_{k-m} L_\omega(n)|_{V_\ell(\omega)} \cdot D_m \bar{L}_\omega(n).$$

By (11) and Lemma 2, $D_m \bar{L}_\omega(n) \geq e^{\lambda_{\ell-1}mn}$.

This gives a matching upper bound for $D_{k-m} L_\omega(n)|_{V_\ell(\omega)}$ to the lower bound that we obtained in Lemma 7. Hence we deduce the first $k$ exponents of $R_{\ell-1}$ are $m = m_{\ell-1}$ repetitions of $\lambda_{\ell-1}$ followed by the first $k - m$ exponents of $R_\ell$, establishing (h).

The next corollary provides a splitting when the base $\sigma$ is invertible.

**Corollary 15** (The Oseledets splitting). Let $R$ be a random dynamical system as in Theorem 14. Suppose that the base $\sigma$ is invertible. Then there exist $1 \leq r \leq \infty$ and exceptional Lyapunov exponents and multiplicities as in Theorem 14. Furthermore, there is an equivariant measurable direct sum decomposition

$$X = Z_1(\omega) \oplus \cdots \oplus Z_r(\omega) \oplus V_\infty(\omega),$$

such that for $P$-a.e. $\omega$, $\dim Z_i(\omega) = m_i$ and $v \in Z_i(\omega) \setminus \{0\}$ implies

$$\lim_{n \to \infty} \frac{1}{n} \log \|L_\omega(n)v\| = \lambda_i; v \in V_\infty(\omega) \text{ implies } \limsup_{n \to \infty} \frac{1}{n} \log \|L_\omega(n)v\| \leq \kappa(R).$$
The deduction of this in general starting from a non-invertible filtration theorem, including uniqueness of the complement, appears in [4]. We give a simple proof here, based on duality, in the special case where \( X \) is reflexive. We make use of the following facts valid for reflexive Banach spaces. If \( X \) is reflexive and \( \Theta \) is a closed subspace of \( X^* \) of codimension \( k \), then its annihilator, \( \Theta^o \) is \( k \)-dimensional. Further if \( \theta \) is a bounded functional such that \( \theta|_{\Theta^o} = 0 \), then \( \theta \in \Theta \).

**Proof.** Let \( R^* \) be the dual random dynamical system to \( R \) as defined above. Applying Theorem 14 to \( R^* \), and recalling from Theorem 11 that the Lyapunov exponents and multiplicities of \( R \) and \( R^* \) coincide, yields a \( \sigma^{-1} \) equivariant measurable filtration \( X^* = V^*_1(\omega) \supset \cdots \supset V^*_r(\omega) \supset V^*_\infty(\omega) \), with the same codimensions as those of \( R \).

Let \( Y^*_\ell(\omega) = V^*_\ell(\omega)^\sigma \). Notice that \( \dim Y^*_\ell(\omega) = \mathcal{M}_{\ell-1} \). Since \( V^*_\ell(\omega) \) is measurable and \( (\cdot )^\sigma : \mathcal{G}(X^*) \rightarrow \mathcal{G}(X) \) is continuous [6, IV 2.2], then \( Y^*_\ell(\omega) \) is measurable. Also, for every \( \psi \in V^*_\ell(\sigma \omega) \), we have \( L^*_\sigma \psi = V^*_\ell(\omega) \) by equivariance of \( V^*_\ell(\cdot) \). Hence, for every \( y \in Y^*_\ell(\omega) \), \( 0 = L^*_\sigma \psi(y) = \psi(L^*_\sigma y) \). Thus, \( L^*_\sigma Y^*_\ell(\omega) \subset Y^*_\ell(\sigma \omega) \), yielding equivariance.

We define \( Z^*_\ell(\omega) = Y^*_{\ell+1}(\omega) \cap V^*_\ell(\omega) \). It remains to show that \( V^*_{\ell-1}(\omega) = V^*_\ell(\omega) \oplus Z^*_{\ell-1}(\omega) \), which would follow from showing \( V^*_\ell(\omega) \oplus Y^*_\ell(\omega) = X \).

Suppose this is not the case. Then, there exists \( w \in V^*_\ell(\omega) \cap Y^*_\ell(\omega) \cap S_X \).

Let \( \theta \in S_X \). be such that \( \theta(w) = 1 \). Let \( \bar{\theta} \) be the equivalence class of \( \theta \) in \( Q^*_{\ell-1}(\omega) = V^*_{\ell-1}(\omega)/V^*_\ell(\omega) \).

Since \( Q^*_{\ell-1}(\omega) \) is a finite-dimensional space, and \( L^*_\omega \) acts injectively on it by (11), we see \( L^*_\omega \) is bijective. Also the quantity \( C(\omega) = \inf_{\omega \in S_X} e^{-(\lambda r - c)n} \| \tilde{L}^*_\omega(w) \|_{Q^*_{\ell-1}(\omega)} \) is positive. Arguing as in Remark 9, we see that \( C(\omega) \) is measurable and hence exceeds some quantity \( c \) on a set of positive measure.

Let \( \tilde{\phi}_n \in Q^*_{\ell-1}(\sigma^n \omega) \) be such that \( L^*_\sigma(\phi_n) = \tilde{\phi}_n \). Then, \( \| L^*_\sigma(\phi_n) \| \geq C(\sigma^n \omega) e^{(\lambda r - c)n} \| \phi_n \| \). By ergodicity, there exist arbitrarily large values of \( n \) for which \( \| \phi_n \| \leq c^{-1} e^{-(\lambda r - c)n} \| \tilde{\phi} \| \). On the other hand, if \( \phi_n \in X^* \) is a representative of \( \tilde{\phi}_n \), then for every \( \psi \in V^*_\ell(\sigma^n \omega) \), one has \( L^*_\omega(w) \in Y^*_\ell(\sigma^n \omega) \), so that \( \psi(\tilde{L}^*_\omega(w)) = 0 \). Hence \( 1 = (\phi_n + \psi)^*(\tilde{L}^*_\omega(w)) \leq \| \phi_n + \psi \| e^{(\lambda r + c)n} \). Thus, \( \| \phi_n \| \geq e^{-(\lambda r + c)n} \), giving a contradiction. Hence, \( V^*_{\ell-1}(\omega) = V^*_\ell(\omega) \oplus Z^*_\ell(\omega) \) as required. \( \square \)

**Acknowledgments.** CGT acknowledges support from Australian Research Council Discovery Project DP110100068. AQ acknowledges support from the Canadian NSERC, and thanks the Universidade de São Paulo for the invitation to deliver a mini-course from which this work originated.
References

[1] T. S. Doan. Lyapunov Exponents for Random Dynamical Systems. PhD thesis, Fakultät Mathematik und Naturwissenschaften der Technischen Universität Dresden, 2009.

[2] G. Froyland, S. Lloyd, and A. Quas. Coherent structures and isolated spectrum for Perron-Frobenius cocycles. *Ergodic Theory Dynam. Systems*, 30:729–756, 2010.

[3] G. Froyland, S. Lloyd, and A. Quas. A semi-invertible Oseledets theorem with applications to transfer operator cocycles. *Discrete Contin. Dyn. Syst.*, 33(9):3835–3860, 2013.

[4] C. González-Tokman and A. Quas. A semi-invertible operator Oseledets theorem. *Ergodic Theory Dynam. Systems*, to appear.

[5] T. Kato. Perturbation theory for nullity, deficiency and other quantities of linear operators. *J. Analyse Math.*, 6:261–322, 1958.

[6] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[7] Z. Lian and K. Lu. Lyapunov exponents and invariant manifolds for random dynamical systems in a Banach space. *Mem. Amer. Math. Soc.*, 206(967):vi+106, 2010.

[8] R. Mañé. Lyapounov exponents and stable manifolds for compact transformations. In *Geometric dynamics (Rio de Janeiro, 1981)*, volume 1007 of *Lecture Notes in Math.*, pages 522–577. Springer, Berlin, 1983.

[9] V. I. Oseledec. A multiplicative ergodic theorem. Characteristic Lyapunov, exponents of dynamical systems. *Trudy Moskov. Mat. Obšč.*, 19:179–210, 1968.

[10] M. S. Raghunathan. A proof of Oseledec’s multiplicative ergodic theorem. *Israel J. Math.*, 32(4):356–362, 1979.

[11] D. Ruelle. Ergodic theory of differentiable dynamical systems. *Inst. Hautes Études Sci. Publ. Math.*, 50:27–58, 1979.

[12] P. Thieullen. Fibrés dynamiques asymptotiquement compacts. Exposants de Lyapounov. Entropie. Dimension. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 4(1):49–97, 1987.

[13] P. Wojtaszczyk. *Banach spaces for analysts*, volume 25 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1991.

(C González-Tokman) School of Mathematics and Statistics, University of New South Wales, Sydney, NSW, 2052, Australia

(Quas) Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada, V8W 3R4