The Enumeration of Permutations Avoiding 3124 and 4312

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We find the generating function for the class of all permutations that avoid
the patterns 3124 and 4312 by showing that it is an inflation of the union
of two geometric grid classes.

1. Preliminaries

We consider permutations in one-line notation so that a permutation of length \( n \) is treated as a linear ordering of the symbols of \( \{1, 2, \ldots, n\} \). For permutations \( \tau \) and \( \pi \) with lengths \( k \) and \( n \) respectively, we write that \( \tau \leq \pi \) (or that “\( \pi \) contains \( \tau \)” if there is a set of indices \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \) such that the sequence \( \pi(i_1), \pi(i_2), \ldots, \pi(i_k) \) is in the same relative order as \( \tau \). For example, \( 4312 \leq 4756231 \) because the entries 7523 in 4756231 have the same relative order as 4312.

A permutation class is a set of permutations that is closed downward under this order, i.e., if \( C \) is a permutation class and if \( \pi \in C \) and \( \tau \leq \pi \), then \( \tau \in C \). We can describe a permutation class by specifying a list of permutations that it avoids. As an example, the set of all strictly increasing permutations may be denoted Av(21). This paper studies the class of permutations that avoid both 3124 and 4312, denoted Av(3124, 4312). In particular, we derive its generating function \( \sum a_n x^n \), where \( a_n \) is the number of permutations of length \( n \) in this class.

A permutation class is called a 2×4 class if the minimal elements in the set of permutations not in the class consist exactly of two permutations of length four. Up to symmetries of the permutation containment order — the group generated by the symmetries reverse, complement, and group-theoretic inverse — there are 56 different 2×4 classes. Some of these have the same enumeration; it has been shown that there are precisely 38 different enumerations for the 2×4 classes \([6, 13, 14, 15, 16]\). This paper will bring the number of different enumerations that have been found to 27. See Wikipedia \([22]\) for a list of currently known enumerations.

Albert, Atkinson, and Vatter \([4]\) enumerated three of the 2×4 classes by studying inflations of geometric grid classes. We show that Av(3124, 4312) is also amenable to the same techniques despite its significantly more complicated structure. Furthermore, in Section 9 we show that these methods cannot be applied to any other unenumerated 2×4 classes.
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2. Simple Permutations and Inflations

An interval of a permutation is a nonempty set of consecutive indices \( \{i, i+1, \ldots, i+k\} \) such that the entries \( \{\pi(i), \pi(i+1), \ldots, \pi(i+k)\} \) form a set of consecutive integers. A permutation of length \( n \) is said to be simple if its only intervals are those of length 1 and \( n \). In Figure 1, the permutation 31468572 is not simple because it contains a nontrivial interval at the indices \( \{4, 5, 6, 7\} \) with entries \( \{5, 6, 7, 8\} \), and the permutation 63814725 is simple because it contains no nontrivial intervals. We say that the permutations 1, 12, and 21 are simple. We use \( \text{Simples}(\mathcal{C}) \) to denote the set of simple permutations of a class \( \mathcal{C} \).

Simple permutations are the foundation from which we can build all other permutations. The inflation of a permutation \( \pi \) of length \( n \) by a sequence of nonempty permutations \( \tau_1, \ldots, \tau_n \), denoted \( \pi[\tau_1, \ldots, \tau_n] \), is the permutation that results from taking each entry \( \pi(i) \) and replacing it with an interval that is order-isomorphic to \( \tau_i \) such that the intervals themselves are order-isomorphic to \( \pi \). For example

\[
3142[231, 21, 123, 1] = 564 21 789 3.
\]

Inflations of the permutations 12 and 21 are given their own names. The sum \( \sigma \oplus \tau \) is defined to be the inflation \( 12[\sigma, \tau] \), while the skew sum \( \sigma \ominus \tau \) is defined to be the inflation \( 21[\sigma, \tau] \). A permutation \( \pi \) is said to be sum decomposable if \( \pi = \sigma \oplus \tau \) for some \( \sigma \) and \( \tau \); otherwise \( \pi \) is said to be sum indecomposable. Analogously, a permutation \( \pi \) is said to be skew decomposable if \( \pi = \sigma \ominus \tau \) for some \( \sigma \) and \( \tau \), and is otherwise skew indecomposable.

Each element of a permutation class can be expressed as an inflation of a unique simple permutation in the class, as given by the following lemma.

**Lemma 2.1** (Albert and Atkinson [2]). For every permutation \( \pi \) there is a unique simple permutation \( \sigma \) such that \( \pi = \sigma[\tau_1, \ldots, \tau_k] \). When \( \sigma \neq 12 \) and \( \sigma \neq 21 \), the intervals of \( \pi \) that correspond to \( \tau_1, \ldots, \tau_k \) are uniquely determined. When \( \sigma = 12 \) (respectively, \( \sigma = 21 \)), the intervals are unique so long as we require the first of the two intervals to be sum (respectively, skew) indecomposable.

3. Geometric Grid Classes

Let \( M \) be a matrix whose entries are from the set \( \{-1, 0, +1, \bullet\} \). To be consistent with plots of permutations, we index matrices with Cartesian coordinates: first by column (left to right) and then by row (bottom to top), starting at 1. Define the standard figure of \( M \) to be the drawing on the Cartesian plane that consists of:

![Figure 1](image-url)
Figure 2: The permutation 827546319 is in the class of permutations that can be drawn on an X, denoted Geom \( \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \). The permutation 578219364 is in the class of permutations that can be drawn on a diamond, denoted Geom \( \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \).

- the line segment from \((k - 1, \ell - 1)\) to \((k, \ell)\), if \(M_{k,\ell} = +1\),
- the line segment from \((k - 1, \ell)\) to \((k, \ell - 1)\), if \(M_{k,\ell} = -1\), and
- the point \((k - \frac{1}{2}, \ell - \frac{1}{2})\), if \(M_{k,\ell} = \bullet\).

Informally, each entry of the matrix becomes a cell that is either empty or contains an increasing line segment, a decreasing line segment, or a single point, depending on the corresponding matrix entry.

A permutation class \(C\) is a geometric grid class if there exists a matrix \(M\) such that every permutation in \(C\) can be drawn on the standard figure of \(M\) by placing entries anywhere on the line segments, with at most one entry placed in each “•” cell. In this case, we say that \(C = \text{Geom}(M)\). Some well-studied geometric grid classes are, for example, those permutations that can be drawn on an \(X\) [20, 21] and those permutations that can be drawn on a diamond [9, 21]. See Figure 2.

A consistent orientation of a geometric grid class is a way of assigning a direction to each line segment in the standard figure such that in any row either all arrows point upward or all arrows point downward and in any column either all arrows point leftward or all arrows point rightward. The base point of a cell is the beginning endpoint of its directed line segment, if it has one.

Figure 3 shows the three geometric grid classes studied in this paper\(^1\), along with the consistent orientations that we use for each. It also shows the cell alphabet for each class, which is described below.

Let \(C = \text{Geom}(M)\) be a geometric grid class, and let \(\Sigma\) be a set (an alphabet) that contains one letter for each nonempty cell of \(M\), called the cell alphabet. As usual with languages, let \(\Sigma^*\) be the set of words formed by the alphabet \(\Sigma\). Let \(w = w_1w_2\cdots w_n \in \Sigma^*\). We now describe how to map this word to a permutation in \(C\).

Pick distances \(0 < d_1 < d_2 < \cdots < d_n < 1\). The actual values of these distances doesn’t matter, so long as no two are equal. For each letter \(w_i\) in the word \(w\), if the cell corresponding to \(w_i\) has a directed line segment, then place a permutation entry in cell \(w_i\) on the line segment at (infinity-norm) distance \(d_i\) from the base point of that cell. If the cell corresponding to \(w_i\) has just a single point, place the permutation entry at this point. There can be at most one entry per point. The result is a permutation

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\(^1\)Computation suggests, although we have not proved, that \(G_1 = \text{Av}(3124, 4123, 4312, 4312, 21435, 21534, 32541)\), \(G_2 = \text{Av}(3124, 4132, 4312, 21354, 21435, 21543, 31542)\), and \(G_3 = \text{Av}(3124, 4123, 4123, 4231, 4312, 21354, 21435, 21534, 21543, 31542, 32541)\). It would then follow that \(G_1 \cup G_2 = \text{Av}(3124, 4132, 21435, 51243, 51324, 51342, 52431, 213645, 216354, 216453, 316452, 324651, 326541, 426531)\).
drawn on the standard figure which we can read by labeling the entries in ascending order from bottom to top and then recording these labels from left to right.

Consider the leftmost geometric grid class in Figure 3. The cell alphabet is $\Sigma = \{a, b, c, d\}$. Let $w = bacdbd \in \Sigma^*$. Then (using, say, distances that are evenly spaced), the placement of entries onto the standard figure is as follows.

Numbering the entries in ascending order from bottom to top and then recording these entries from left to right gives the permutation 234165. Given a geometric grid class $C$, the map described above is a surjection $\varphi : \Sigma^* \rightarrow C$.

Geometric grid classes are especially tractable because for any given geometric grid class $C$ we can construct a regular language $L$ such that there is a length-preserving bijection $L \rightarrow C$. It is well-known that the generating function for the words in any regular language (by length) is rational. (See, for instance, Flajolet and Sedgewick [10, Section I.4 and Appendix A.7].) This fact is crucial in the proof of the following theorem.

**Theorem 3.1** (Albert, Atkinson, Bouvel, Ruskuc, and Vatter [3, Theorem 8.1]). Every geometric grid class has a rational generating function.

Furthermore, this result extends to certain subsets of permutations in a geometrically grid class.

**Theorem 3.2** (Albert, Atkinson, Bouvel, Ruskuc, and Vatter [3, Theorem 9.1]). The simple, sum indecomposable, and skew indecomposable permutations in every geometric grid class each have rational generating functions.

In [3], the authors describe the two obstacles which must be overcome in order to restrict the domain of $\varphi$ to some regular language $L \subseteq \Sigma^*$ such that $\varphi|_L$ is a bijection.

1. There are pairs of cells whose letters “commute”, resulting in two words that map to the same permutation. These are the pairs that share neither a row nor a column. In the example above, since the cells $b$ and $d$ commute, the words $bacdbd$ and $bacbd$ also map to the same permutation.

Such words have their entries in the same cells, but the entries are shifted around within each cell.
(2) We can sometimes move entries between cells to produce the same word. In the example above, we can move the first three entries corresponding to \( \text{bac} \) all into cell \( a \) to yield the same permutation, i.e., \( \varphi(\text{bacddbd}) = \varphi(\text{aaaddbd}) \).

The first obstacle is easily dealt with: for each pair of commuting cells, we pick a preferred order, e.g., \( bd \) instead of \( db \), and we forbid all words that have any occurrence of a non-preferred order. The second issue is more delicate. In [3], a non-constructive proof is given to show that eliminating such duplicate words leaves a regular language. Since we can’t use this proof to actually construct the regular language, we are required to find each of our regular languages by hand. Then, using the Automata [8] package of GAP [11], we recover the rational generating function that counts each regular language. The code used to generate the results in this paper can be found on the author’s website\(^2\).

In Section 4 we show that the simple permutations of \( \text{Av}(3124, 4312) \) are exactly the simple permutations in the union of \( G_1 \) and \( G_2 \). In Sections 5, 6, and 7 we construct the regular languages which are in bijection with all permutations and the simple permutations in each class \( G_1 \), \( G_2 \), and \( G_3 \). Then, we determine exactly how the simple permutations in each class may be inflated to yield permutations still in \( \text{Av}(3124, 4312) \).

In Section 8 we use this information along with Sage [19] (and its embedded libraries GMP [12], Maxima [17], and GiNaC [5]) to compute the rational generating function for \( \text{Simples}(\text{Av}(3124, 4312)) \) and the algebraic generating function for \( \text{Av}(3124, 4312) \).

Lastly, in Section 9 we explore the applicability of this method to other permutation classes.

4. The Simple Permutations of \( \text{Av}(3124, 4312) \)

Consider the three geometric grid classes \( G_1 \), \( G_2 \), and \( G_3 \) whose standard figures are shown in Figure 3. It is clear that \( G_3 \) is the intersection \( G_1 \cap G_2 \). This will be useful when we count the permutations in \( \text{Av}(3124, 4312) \); those that arise as inflations of simple permutations that lie in both \( G_1 \) and \( G_2 \) will be double-counted, and hence in order to compensate, those that arise as inflations of simple permutations that lie in \( G_3 \) must be subtracted.

Whereas simple permutation of the classes in Albert, Atkinson, and Vatter [4] were each contained in a single geometric grid class, the class studied here has simple permutations contained in the union of two geometric grid classes. This fact considerably lengthens both the proof of our upcoming Theorem 4.1 and the subsequent analysis.

In the following arguments, we make frequent use of Albert’s PermLab application [1] to help us determine valid permutation configurations. We use permutation diagrams, comprised of a permutation plotted on top of a grid of cells. A cell is white if we are allowed to insert a new entry into that cell without creating an occurrence of 3124 or 4312. A cell is shaded dark gray if insertion into that cell would create a forbidden pattern, i.e., a 3124 or 4312. A cell is shaded light gray if we have specifically forbidden insertion into that cell as part of an argument, e.g., if we assume a particular entry is the maximal entry, then we can forbid insertion into all cells above it.

In order to talk about certain regions in a permutation diagram, we define the rectangular hull of a set \( S \) of points to be the smallest axis-parallel rectangle in the plane that contains all points of \( S \). In particular, the rectangular hull of \( S \) frequently contains additional points not in \( S \). In our case, the points are entries in a permutation diagram.

\(^2\)At the time of this publication, the author’s website is located at http://www.math.ufl.edu/~jaypantone.
Theorem 4.1. The simple permutations of $\text{Av}(3124, 4312)$ coincide with the simple permutations of $G_1 \cup G_2$.

Proof. It is clear by inspection that the permutations 3124 and 4312 cannot be drawn on the standard figures of $G_1$ or $G_2$. Therefore, $\text{Simples}(G_1 \cup G_2) \subseteq \text{Simples}(\text{Av}(3124, 4312))$. The reverse inclusion is much harder to show. Though the ideas in the proof are not particularly deep, we need to consider many cases.

Let $\pi \in \text{Simples}(\text{Av}(3124, 4312))$ have length $n$. Let $\pi_L$ be the entries to the left of $n$ and let $\pi_R$ be the entries to the right of $n$, as in Figure 4. In order to avoid both 3124 and 4312 patterns, we must have $\pi_L, \pi_R \in \text{Av}(312)$.

Case 1: Assume $\pi_R$ does not contain the pattern 132. Then, $\pi_R \in \text{Av}(132, 312)$ which implies that $\pi_R$ is a wedge permutation of the shape shown in Figure 5. Since $\pi_R$ is nonempty, we know that the last entry $\pi(n)$ lies in the $\pi_R$ part of $\pi$.

First assume that $\pi_R$ contains only the entry $\pi(n)$. Then, in order for $\pi$ to be simple, the rectangular hull of $n$ and $\pi(n)$ must be split to the left in the cell marked $A$ in Figure 6(a). We claim that any entries in cell $A$ must be increasing. To see this, assume there is a descent (a 21 pattern) in cell $A$ and choose the ‘2’ to be the topmost possible entry and the ‘1’ to be the bottommost possible entry for the chosen 2. This gives the diagram in Figure 6(b). The rectangular hull of this 21 pattern must be split to the left. Assume the separating entry is as far to the left as possible. Then, as seen in Figure 6(c), there exists an interval of length 3 that is impossible to split. Hence, any entries in cell $A$ must be increasing. The argument that we just made relating to the unsplittable 21 pattern will be used many times in this proof. For its remainder, we will simply refer to an “unsplittable 21 pattern” to mean that if we choose the ‘2’ to be as high as possible and the ‘1’ to be as low as possible, and then choose a separating entry as far to the left as possible, we get an interval of length 3 that cannot be split.

In addition to being increasing, cell $A$ must also be nonempty. Consider an entry in cell $A$ that is as low as possible, yielding Figure 6(d). If cell $B$ has any entry, then we get an unsplittable 21 pattern. If cell $C$ has any descent, then we again get an unsplittable 21 pattern. If cell $D$ has an ascent, then there is a 3124 pattern. Hence, $B$ is empty, $C$ is increasing, and $D$ is decreasing. This leaves us with the diagram in Figure 6(e). It is clear that any permutation drawn on this figure lies in both $G_1$ and $G_2$. This concludes the analysis of the case in which $\pi_R$ has exactly one entry.

Now suppose instead that $\pi_R$ has more than one entry, but is strictly decreasing. We then have the diagram in Figure 6(f). The rectangular hull of $\pi_R$ must be split with an entry to the left. Choose the leftmost possible entry to get the diagram in Figure 6(g). If cell $A$ contains an entry, then there is an interval that cannot be split. Hence, cell $A$ is empty. Cells $B$ and $C$ together must be decreasing to avoid a 3124 pattern. To avoid an unsplittable 21 pattern, cell $D$ must be increasing. For the same reason, cell $E$ must also be increasing. Thus, we have the situation in Figure 6(h).
Now, the rectangular hull of \( n \) and the entry to its immediate right must be split to the left. The result is Figure 6(i). Cells \( F, G, H, \) and \( I \) together form one decreasing block. Cell \( K \) must be empty in order to avoid an unsplittable interval. Consider cell \( J \). If there were a 312 pattern in cell \( J \), then there would be a 3124 pattern. If there were a 213 pattern in cell \( J \), then there would be an unsplittable interval. Hence, the permutation inside cell \( J \) must be an element of \( \text{Av}(213, 312) \), which is the class of permutations that can be drawn on the shape \( \wedge \) (a wedge permutation with the point at the top). See Figure 6(j). In order to be able to draw such a permutation onto the standard figure of \( G_1 \), we can’t have both a descent from the wedge and a descent in the big decreasing block below it. If we had two such descents, then there would be an unsplittable interval, violating simplicity. Therefore, in this case (when \( \pi_R \) is strictly decreasing), \( \pi \) is an element of \( \text{Simples}(G_1) \).

Next, we must rule out a case that cannot occur. We know that \( \pi_R \) has the shape of a wedge permutation as shown in Figure 5. We now show that if \( \pi_R \) is not strictly decreasing then its rightmost entry, \( \pi(n) \), lies on the top part of the wedge permutation, not the bottom part. Assume otherwise. Then, we have the picture in Figure 7(a), where the rightmost entry is \( \pi(n) \), the leftmost entry is \( n \), and the middle entry is some other entry that must be in \( \pi_R \), which we’ve picked to be the last entry on the upper part of the wedge.

Continuing the analysis of this case, since \( \pi_R \) is not strictly decreasing, there must be some other entry in \( \pi_R \) that is not in the decreasing block. Choose the leftmost possible entry, as in Figure 7(b). If cell \( L \) were empty, then we would have \( \pi(n) = 1 \), which contradicts the simplicity of \( \pi \). As such, cell \( L \) must be nonempty. Then, the rectangular hull of the middle two entries in Figure 7(b) must be split as far to the left as possible, yielding Figure 7(c). The rectangular hull of the 4 middle entries shown in the figure cannot be split; there are four possible boxes where we can begin to try to split it, but we quickly see that any entry in each of these boxes just makes the interval bigger while not offering any path that will lead to splitting the interval. Therefore, this case is impossible.

Hence, we can now assume that \( \pi(n) \) is in the upper part of the wedge. Therefore, we have the permutation diagram shown in Figure 7(d), in which the first entry is \( n \) and the second entry is \( \pi(n) \). The top three cells in Figure 7(d) are shaded because we’re assuming that first entry is \( n \), and so there
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(a) L
(b)  
(c)  
(d) 

Figure 7: Permutation diagrams corresponding to steps in the proof of Theorem 4.1.

is no entry greater than it. The other three cells are shaded because the second entry is \( \pi(n) \) (so it is the rightmost entry of \( \pi \)), and it’s in the increasing part of \( \pi_R \) (so it is as high as possible). Now assume that \( \pi_R \) has at least two entries, and let the second entry be as low as possible. Thus, we have the diagram in Figure 7(e).

If \( E \) has a descent, then there is a 4312 pattern. If \( F \) is empty, then \( G \) must have an entry, and this creates an unsplittable 21 pattern. Hence \( E \) is increasing and there must be an entry in \( F \). Select the leftmost possible entry. See Figure 7(f).

To avoid an unsplittable 21 pattern, cell \( H \) must be increasing. Furthermore, any entry in \( H \) has no entry above it and to its left, since there would then be a 3124 pattern. This implies that cell \( H \) is actually empty because otherwise \( \pi(1) = 1 \) and in this case \( \pi \) would not be simple. The cells \( I \) and \( J \) together must be decreasing to avoid a 3124 pattern. Cell \( K \) must be increasing to avoid an unsplittable 21 pattern. Hence, we have the diagram in Figure 7(g). Suppose that cell \( L \) has some entry. Place it as far to the left as possible. This gives us the diagram in Figure 7(h). Any permutation drawn on this figure must lie in \( G_2 \).

Now suppose instead that cell \( L \) is actually empty. This yields the diagram in Figure 8(a). If cell \( M \) is empty, we get the diagram in Figure 8(b). Any permutation drawn on Figure 8(b) must lie in \( G_2 \). So, suppose instead that cell \( M \) has an entry, and place it as high as possible. This gives us Figure 8(c). Cell \( N \) must be empty to avoid an unsplittable 21 pattern. Additionally, any entry of \( O \) must be to the left of any entry in \( P \). By the usual monotonicity arguments, we get the diagram in Figure 8(d). Any permutation drawn on this figure lies in the class \( G_2 \). This completes the first case.

Case 2: Assume \( \pi_R \) contains a 132 pattern. Let the ‘1’ be the bottommost possible entry, let the ‘3’ be the topmost possible entry for the chosen ‘1’, and let the ‘2’ be the rightmost possible entry for the chosen ‘1’ and ‘3’. This yields the diagram in Figure 8(e). Any entry in cell \( B \) leads to an unsplittable 21 pattern. Hence, there must be an entry in cell \( A \) to split the rectangular hull of the last two entries shown in the figure. Pick this entry to be as far to the left as possible. To avoid 3124 and 4312 patterns, we have the monotone conditions shown in Figure 8(f).
We now show that cell $E$ is empty. Assume it has some entry. Then, the rectangular hull of the leftmost five entries of $\pi$ shown in Figure 8(g) must be split in order for $\pi$ to be simple. There is only one place (cell $F$) where we could possibly have a separating element. However, there can be no entry that is both to the right of an entry in $F$ and below the entry of height 2 in Figure 8(g). Therefore, the interval can never be split. Hence, cell $E$ is empty.

Lastly, any entries in the cells $C$ and $D$ together must be increasing, since any descent contained in these two cells would form an unsplittable 21 pattern. So, we have shown that in this case, $\pi$ can be drawn on the diagram in Figure 8(h). Thus, $\pi$ can be drawn on the standard figure of $G_1$.

This completes the proof that $\text{Simples}(\text{Av}(3124, 4312)) = \text{Simples}(G_1 \cup G_2)$.

5. The Regular Language and Inflations of $G_1$

The standard figure for $G_1$ is shown in Figure 3, along with the directional arrows corresponding to a consistent orientation.

Albert, Atkinson, and Vatter [4] determined the regular languages that are in bijection with $G_1$ and $\text{Simples}(G_1)$. We repeat this derivation here because it is a good introduction to the following two sections.

Recall the standard notation for regular languages: if $x$ is a letter (or a set of letters), then “$x^*$” means zero or more occurrences of $x$ and “$x^+$” means one or more occurrences of $x$.

As discussed in Section 3, there are two impediments to the bijectivity of the map $\varphi : \Sigma^* \rightarrow G_1$. Firstly, we prevent duplicate words that arise as a result of commuting pairs of cells. In this geometric grid class, the set of commuting pairs is $\{(a, c), (a, d), (b, d)\}$. Therefore, to prevent these duplicate words, we forbid all occurrences of $ca$, $da$, and $db$. 

Figure 8: Permutation diagrams corresponding to steps in the proof of Theorem 4.1.
Next, we must prevent duplicate words that arise from moving some entry to a different cell. Among all such duplicate words, we choose to prefer the word that has the most entries in the first column, then the most entries in the second column, and then the most entries in the first row.

We define \( L_1 \) to be the regular language consisting of all words \( \Sigma^* \), with the following restrictions.

- As above, we forbid all words that contain \( ca, da, \) or \( db \).
- If a word begins with a \( b \), then the corresponding entry could be moved to cell \( a \). Hence, we forbid all words that begin with a \( b \).
- If a word begins with \( a^*c \), then entry corresponding to the \( c \) could be moved into cell \( a \). Hence, we forbid all words that begin with \( a^*c \).
- If a word ends with a \( d \), then the entry corresponding to the \( d \) could be moved into cell \( c \). Hence, we forbid all words that end with \( d \).
- If a word starts with a \( d \), has no \( c \), and has no other \( d \), then the entry corresponding to the \( d \) could be moved into cell \( c \). Hence, we forbid all words of the form \( d\{a, b\}^* \).
- If a word is of the form \( a^*\{c, d\}^* \), then all entries corresponding to \( c \) and \( d \) could be moved into cells \( a \) and \( b \). Hence, we forbid all words of this form.

Then, \( L_1 \) is in (length-preserving) bijection with the geometric grid class \( G_1 \). We can compute the multivariate generating function for this regular language, in which each letter \( a, b, c, d \) is represented by a variable \( x_a, x_b, x_c, x_d \) and the coefficient of the term \( x_a^{p_1} x_b^{p_2} x_c^{p_3} x_d^{p_4} \) is the number of words in \( L_1 \) that have \( p_1 \) occurrences of the letter \( a \), \( p_2 \) occurrences of the letter \( b \), etc. This multivariate generating function is

\[
\frac{x_a - x_a^2 - 2x_a x_c - 2x_a x_d + 2x_a^2 + 2x_a^2 x_c + x_b x_c x_d - x_a^2 x_c^2 - 2x_a^2 x_c x_d - x_a^2 x_d^2 - x_b x_c x_d^2}{(1 - x_a)(1 - x_c - x_d)(1 - x_a - x_b - x_c - x_d + x_a x_c + x_a x_d + x_b x_d)}
\]

We can find the univariate generating function for \( G_1 \) by setting all four variables to \( x \). The univariate generating function is:

\[
\frac{x - 4x^2 + 5x^3}{(1 - x)(1 - 2x)(1 - 3x)} = x + 2x^2 + 6x^3 + 20x^4 + 66x^5 + 212x^6 + 666x^7 + \cdots ,
\]

which is sequence A083323 in the OEIS [18], and has closed form \( 3^{n-1} - 2^{n-1} + 1 \) for \( n \geq 1 \).

We will now restrict \( L_1 \) to a new regular language \( S_1 \) that is in bijection with \( \text{Simples}(G_1) \). Permutations that are not simple arise due to either repeated letters (an interval in one cell) or one of the shaded regions involving two or more cells shown in Figure 9. So, we make the following restrictions.

- We exclude any words that contain consecutive occurrences of any letter: \( aa, bb, \) or \( dd \).
- To avoid intervals of the first type, we forbid all words that begin with two or more occurrences of \( \{a, b, c\} \).
- To avoid intervals of the second type, we forbid all words that end with \( ca^*da^* \) or \( d\{a, b\}^*ca^* \).
- To avoid intervals of the third type, we require that the last \( a \) is followed by a \( b \), i.e., we forbid all words that end with \( a\{c, d\}^* \).
Lastly, we explicitly forbid the word \textit{dcb}, which does not correspond to a simple permutation but is not forbidden by any of the previous rules.

From these rules, we can find the multivariate generating function of $S_1$:

\[
S_1(x_a, x_b, x_c, x_d) = \frac{x_b x_c x_d (1 + x_b) (x_a + x_c + x_a x_c + x_c x_d)}{1 - x_a x_b - x_b x_c - x_c x_d - x_a x_b x_c - x_b x_c x_d}.
\]

In particular, this shows that every simple permutation of $G_1$ has at least one entry in each of cells $b$, $c$, and $d$. Note that this multivariate generating function excludes the permutation of length 1 and both permutations of length 2. This is because we don’t need to consider inflations of the permutation 1 and we will handle inflations of 12 and 21 separately.

We can see that the univariate generating function of Simples($G_1$) of length at least 4 is thus:

\[
M_1(x) = \frac{2x^4}{1 - 2x} = 2x^4 + 4x^5 + 8x^6 + 16x^7 + 32x^8 + 64x^9 + \cdots,
\]

which is sequence A000079 in the OEIS [18], and has closed form $2^{n-3}$ for $n \geq 4$.

Lastly, we must determine the ways in which we can inflate a simple permutation of $G_1$ to yield a permutation in Av(3124, 4312). For the remainder of the paper, we define the two functions

\[
m = \frac{x}{1 - x} = x + x^2 + x^3 + x^4 + \cdots
\]

to be the generating function for nonempty increasing (or decreasing) permutations, and

\[
c = \frac{1 - 2x - \sqrt{1 - 4x}}{2x} = x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + \cdots
\]

to be the generating function for nonempty permutations in Av(312), which are counted by the Catalan numbers. Additionally, we will let $f$ denote the generating function for Av(3124, 4312).

In order to find the allowed inflations of a simple permutation in $G_1$, we need to split the letter $c$ into two letters $c_1$ and $c_2$. A $c$ is a $c_1$ if there is any $b$ following it, and a $c_2$ otherwise. By simplicity, there is at most one $c_2$. Additionally, we must split Simples($G_1$) into two types of permutations. Let a permutation $\pi \in \text{Simples}(G_1)$ be \textit{Type A} if its corresponding word meets the following criteria:

\begin{itemize}
  \item there is no $c$ after any $b$,
  \item there is no $d$ after any $c$,
\end{itemize}
the first \( b \) is before any \( a \).

There is one simple permutation of each odd length in Simples(\( G_1 \)) that meets these criteria. By construction, there is no \( c_2 \) in any Type A permutation; every \( c \) is a \( c_1 \).

It follows that the multivariate generating function for the Type A permutations in Simples(\( G_1 \)) is

\[
S_{1,1} (x_a, x_b, x_{c_1}, x_{c_2}, x_d) = \frac{x_a x_b x_{c_1} x_{c_2}}{1 - x_a x_b}.
\]

The remainder of the simple permutations of length at least 4 in \( G_1 \) (we will call them Type B) thus have multivariate generating function

\[
S_{1,2} = S_1 - S_{1,1}.
\]

In Type A permutations, we can inflate any entry in cell \( a \) by any permutation in Av(312). We can inflate any entries in cell \( b \) by any decreasing permutation except for the first entry in cell \( b \), which can be inflated by any permutation in Av(312). Any entry in cell \( c \) (which must be a \( c_1 \)) may be inflated by any increasing permutation. Entries in cell \( d \) may be inflated by permutations in Av(312). Therefore, the generating function for inflations of simple permutations of Type A that are still in Av(312, 4312) is

\[
\frac{c}{m} \cdot S_{1,1} (c, m, m, 0, c).
\]

In Type B permutations, we consider two cases. In both cases, entries in cell \( a \) can be inflated by permutations in Av(312), entries in cell \( b \) can be inflated by decreasing permutations, and the \( c_1 \) entries in cell \( c \) can be inflated by increasing permutations. If a \( c_2 \) entry in cell \( c \) is inflated only by an increasing permutation, then the first entry in cell \( d \) may be inflated by a permutation in Av(312) while all other entries in cell \( d \) can only be inflated by decreasing permutations. Otherwise, if the \( c_2 \) entry is inflated by a permutation with a descent (i.e., a permutation in Av(312, 4312) \( \setminus \) Av(21)), then this forces all entries in cell \( d \) to be inflated by only decreasing permutations.

Define \( S_{1,3} = S_{1,2} - \left( S_{1,2} \mid x_{c_2} = 0 \right) \), so that \( S_{1,3} \) is the multivariate generating function of Type B permutations that contain a \( c_2 \) entry. Now, the generating function for inflations of simple permutations of Type B that are still in Av(312, 4312) is

\[
\frac{c}{m} \cdot S_{1,2} (c, m, m, m, m) + S_{1,3} (c, m, m, f - m, m).
\]

Combining the above results, the (univariate) generating function for the inflations of simple permutations of length at least 4 from \( G_1 \) that lie in Av(312, 4312) is

\[
I_1 (x) = \frac{c}{m} \cdot S_{1,1} (c, m, m, 0, c) + \frac{c}{m} \cdot S_{1,2} (c, m, m, m, m) + S_{1,3} (c, m, m, f - m, m).
\]

6. The Regular Language and Inflations of \( G_2 \)

The standard figure for \( G_2 \) is shown in Figure 3, along with the directional arrows corresponding to a consistent orientation. In this geometric grid class, the set of commuting pairs is

\[
\{(a, b), (a, c), (a, d), (a, f), (b, d), (b, e), (b, f), (c, e), (c, f), (d, e), (d, f)\}.
\]
We note that the three letters \(a, e,\) and \(f\) commute with the three letters \(b, c,\) and \(d\). So, we forbid all words in which any \(b, c,\) or \(d\) comes before any \(a, e,\) or \(f\). This handles all commuting pairs except for \((a, f)\) and \((b, d)\), so we further forbid all words containing either \(fa\) or \(db\).

Next, we must prevent duplicate words that arise from moving some entry to a different cell. Among all such duplicate words, we prefer the word that has the most entries in the first column, then the second column, then the third column, then the first row, then the second row, and then the third row.

We define \(L_2\) to be the regular language consisting of all words \(\{a, b, c, d, e, f\}^\ast\), with the following restrictions.

- As above, we forbid all words that contain the factor \(\{b, c, d\}\{a, e, f\}\) as well as all words containing either \(fa\) or \(db\).
- If a word begins with \(e\), then the corresponding entry can be moved to cell \(c\). Hence, we forbid all words that start with \(e\).
- We can move an entry in cell \(d\) to cell \(b\) if the \(d\) has no \(c\) or \(f\) before it and no \(f\) after it. Hence, we forbid all words of the form \(\{a, b, d, e\} \ast d \{a, b, c, d, e\}^\ast\).
- If a word has no \(f\), then any entry in cell \(b\) can be moved to cell \(a\) (by also moving some entries from cell \(c\) to cell \(e\) and some entries from cell \(d\) to cell \(f\), as needed). Hence, we forbid words of the form \(\{a, b, c, d, e\} \ast b \{a, b, c, d, e\}^\ast\).
- Consider a word that ends in the form \(b\{a, e, f\} \ast d \{a, b, c, e, f\}^\ast\). Then, the entry in cell \(b\) can be moved into cell \(a\) (by also moving some entries in cell \(c\) to cell \(e\) as needed). Hence, we forbid all words that end with \(b\{a, e, f\} \ast d \{a, b, c, e, f\}^\ast\).
- If a word has an \(f\) which has no \(b, c,\) or \(e\) before it and no \(b\) or \(c\) after it, then the first \(f\) can be moved to cell \(c\). Hence, we forbid words of the form \(\{a, d, f\} \ast f \{a, d, e, f\}^\ast\).
- If a word starts with the prefix \(\{a, e, f\} \ast c\), then the entry in cell \(c\) can be moved to cell \(b\). Hence, we forbid words that start with \(\{a, e, f\} \ast c\).

The language \(L_2\) is in (length-preserving) bijection with the geometric grid class \(G_2\). We can compute the multivariate generating function for this regular language, but it is too long to display here. We find the univariate generating function for \(G_2\) by setting all six variables to \(x\). The univariate generating function is

\[
\frac{x - 7x^2 + 19x^3 - 22x^4 + 9x^5 - x^6}{(1 - x)(1 - 2x)(1 - 3x + x^2)^2} = x + 2x^2 + 6x^3 + 21x^4 + 73x^5 + 244x^6 + 786x^7 + \cdots,
\]

sequence \textbf{A226431} in the \textbf{OEIS} [18].

We now restrict \(L_2\) to a new regular language \(S_2\) that is in bijection with \(\text{Simples}(G_2)\) (excluding the permutations 1, 12, and 21). Permutations that are not simple arise due to either repeated letters (an interval in one cell) or one of the shaded regions involving two or more cells shown in Figure 10. So, we make the following restrictions.

- We exclude any words that contain consecutive occurrences of any letter: \(aa, bb, cc, dd, ee,\) or \(ff\).
- To avoid intervals of the first type, we require that words do not end with \(a\{a, b, c, d, f\}^\ast\).
Figure 10: The five regions that are not already forbidden by previous rules in which an interval containing entries from at least two cells can occur.

- To avoid intervals of the second type, we require that words do not end with \( e \{b, c, d, e\}^* \).
- To avoid intervals of the third type, we forbid words of the form \( \{a, b, c, e, f\}^* b \{a, b, c, e, f\} c \).
- To avoid intervals of the fourth type, we must exclude any words that begin with \( \{a, b, c, e, f\}^* b \{a, b, c, e, f\} c \).
- To avoid intervals of the last type, we forbid all words of the form \( \{a, d, f\}^* f \{a, d, e, f\} c \{a, c, e, f\}^* \).

From these rules, we can find the multivariate generating function of \( S_2 \):

\[
M_2(x) = \frac{x_d x_f (1 + x_e) (x_a x_e + x_c x_d + x_b x_e x_d - x_a x_b x_e x_d - x_a x_b x_c x_d x_e)}{(1 - x_b x_c - x_c x_d - x_b x_c x_d)(1 - x_a x_e - x_c x_f - x_a x_c x_f)}.
\]

The univariate generating function of Simples(\( \mathcal{G}_2 \)) of length at least 4 is thus

\[
M_2(x) = \frac{x^4 (1 - x)^2 (2 + x)}{(1 - x - x^2)^2} = 2x^4 + 3x^5 + 7x^6 + 13x^7 + 25x^8 + 46x^9 + 84x^{10} + 151x^{11} + \cdots
\]

This is sequence A226432 in the OEIS [18]. It is the sum of the Fibonacci sequence with the self-convolved Fibonacci sequence, i.e., if \( F_n \) is the \( n \)th Fibonacci number (with \( F_1 = F_2 = 1 \)), then

\[
[x^n] M_2(x) = F_{n-3} + \sum_{i=1}^{n-3} F_i F_{n-i-2}.
\]

In order to find the allowed inflations of a simple permutation in \( \mathcal{G}_2 \), we need to split the letters \( c \) and \( d \) each into two letters. We say that a \( c \) is a \( c_2 \) if there is no \( b \) or \( c \) before it and it does not simultaneously have a \( d \) both before and after it. It is a \( c_1 \) otherwise. We say that a \( d \) is a \( d_2 \) if the word has no \( e \), the word has at most one \( f \), and the \( d \) has no \( c \) after it. It is a \( d_1 \) otherwise. By simplicity, a word can have at most one \( d_2 \). The multivariate generating function for \( S_2 \) with these new letters will be denoted \( S_2(x_a, x_b, x_{c_1}, x_{c_2}, x_d, x_{d_1}, x_{d_2}, x_e, x_f) \).

We handle three separate cases. In all cases, entries corresponding to \( a, b, \) and \( c_2 \) can be inflated by any permutation in Av(312), while entries corresponding to \( c_1 \) and \( e \) can be inflated by decreasing permutations and entries corresponding to \( d_1 \) can be inflated by increasing permutations. The three cases below specify how entries corresponding to \( d_2 \) and \( f \) may be inflated.
In the case that the word has no $d_2$, all entries in cell $f$ can be inflated by permutations in $\text{Av}(312)$. The multivariate generating function for the words in $S_2$ that have no $d_2$ is defined to be $S_{2,1} = S_2 \big|_{x_{d_2} = 0}$.

In the case that the word has a $d_2$ and this $d_2$ is inflated by an increasing permutation, entries in cell $f$ may be inflated by any permutation in $\text{Av}(312)$. If this $d_2$ contains a descent (i.e., is inflated by a permutation in $\text{Av}(3124, 4312) \setminus \text{Av}(12)$), then entries in cell $f$ may only be inflated by decreasing permutations. The multivariate generating function for the words in $S_2$ that have a $d_2$ is $S_{2,2} = S_2 - S_{2,1}$.

Combining the above results, the (univariate) generating function for the inflations of simple permutations of length at least 4 in $G_2$ is

$$I_2(x) = S_{2,1}(c, c, m, c, m, 0, m, c) + S_{2,2}(c, c, m, c, m, m, m, c) + S_{2,2}(c, c, m, c, m, f - m, m, m).$$

7. The Regular Language and Inflations of $G_3$

The standard figure for $G_3$ is shown in Figure 3, along with the directional arrows corresponding to a consistent orientation. The set of commuting pairs is $\{(a, c), (a, d), (b, d), (c, d)\}$. Thus, we forbid words that contain $ca$, $da$, $db$, or $dc$.

Next, we must prevent duplicate words that arise from moving some entry to a different cell. Among all such duplicate words, we prefer the word that has the most entries in the first column, then the second column, then the first row, and then the second row.

We construct $L_3$ to be the regular language consisting of all words $\{a, b, c, d\}^*$, with the following restrictions.

- To conform to the definition of a “•” entry in a geometric grid class, we forbid all words that contain more than one $d$.
- As above, to avoid duplicate permutations due to commuting pairs, we forbid words that contain $ca, da, db, or dc$.
- If a word starts with $b$, then the corresponding entry could be moved to cell $a$. Hence, we forbid words that begin with $b$.
- If a word has no $d$ and starts with $a^*c$, then the entry corresponding to the $c$ could be moved into cell $a$. Thus, we forbid words of the form $a^*c\{a, b, c\}^*$.
- If a word has no $b$ and at least one $c$ or $d$, then the entry corresponding to a $c$ could be moved to cell $a$ or the entry corresponding to a $d$ could be moved to cell $b$. Thus, we forbid all words of the form $\{a, c, d\}^*\{c, d\}\{a, c, d\}^*$.
- If a word has no $c$ and has a $d$, then the entry corresponding to the $d$ can be moved into cell $c$. Thus, we forbid words of the form $\{a, b, d\}^*d\{a, b, d\}^*$.

The language $L_3$ is then in (length-preserving) bijection with the geometric grid class $G_3$. We can compute the multivariate generating function for this regular language, but again it is far too long to
display here. We can find the univariate generating function for $G_3$ by setting all four variables to $x$. The univariate generating function is:

$$x - 5x^2 + 10x^3 - 8x^4 + x^6 \over (1 - x)^2(1 - 2x)(1 - 3x + x^2) = x + 2x^2 + 6x^3 + 19x^4 + 56x^5 + 157x^6 + 428x^7 + 1149x^8 + \cdots,$$

sequence A226433 in the OEIS [18].

We will now restrict $L_3$ to a new regular language $S_3$ that is in bijection with Simples($G_3$). Permutations that are not simple arise due to either repeated letters (an interval in one cell) or one of the shaded regions involving two or more cells shown in Figure 11. So, we make the following restrictions.

- We exclude any words that contain consecutive occurrences of any letter: $aa$, $bb$, $cc$. (We already can’t have $dd$.)
- To prevent intervals of the first type, we require that there is a $b$ after the last $a$. Hence, we forbid words that end in $a\{c, d\}^*$.
- To prevent intervals of the second type, we forbid words that begin with $ab$.
- Lastly, we explicitly forbid the word $cbd$, which does not correspond to a simple permutation but is not forbidden by any previous rule.

From these rules, we can find the multivariate generating function of $S_3$:

$$x_bx_cx_d (1 + x_b)(x_a + x_c + x_a x_c) \over 1 - x_a x_b - x_b x_c - x_a x_b x_c.$$ 

The univariate generating function of Simples($G_3$) of length at least 4 is thus

$$M_3(x) = \frac{2x^4 + x^5}{1 - x - x^2} = 2x^4 + 3x^5 + 5x^6 + 8x^7 + 13x^8 + 21x^9 + 34x^{10} + 55x^{11} + \cdots,$$

which is sequence A000045 in the OEIS [18].

In order to find the allowed inflations of a simple permutation in $G_3$, we need to split the letters $b$ and $c$ each into two letters. A $b$ is a $b_2$ if there is no $a$, $b$, or $c_2$ before it and no $c$ after it. It is a $b_1$ otherwise. There is at most one $b_2$. A $c$ is a $c_2$ if there is no $b$ after it. It is a $c_1$ otherwise. By simplicity, a word
has at most one \( c_2 \). The multivariate generating function for \( S_3 \) using these new letters will be denoted \( S_3(x_a, x_b_1, x_b_2, x_c_1, x_c_2, x_d) \).

We handle three separate cases. In all cases, entries corresponding to \( a \) and \( b_2 \) can be inflated by any permutation in \( \text{Av}(312) \), while entries corresponding to \( b_1 \) can be inflated by decreasing permutations and entries corresponding to \( c_1 \) can be inflated by increasing permutations. The three cases below specify how entries corresponding to \( c_2 \) and \( d \) may be inflated.

In the case that the word has no \( c_2 \), an entry in cell \( d \) can be inflated by permutations in \( \text{Av}(312) \). The multivariate generating function for the words in \( S_3 \) that have no \( c_2 \) is defined to be \( S_3, 1 = S_3 \mid x_{c_2} = 0 \).

If the word has a \( c_2 \) and that \( c_2 \) is inflated by an increasing permutation, then an entry in cell \( d \) can be inflated by permutations in \( \text{Av}(312) \). If the word has a \( c_2 \) and that \( c_2 \) is inflated by a permutation containing a descent (i.e., a permutation in \( \text{Av}(3124, 4312) \setminus \text{Av}(21) \)), then an entry in cell \( d \) can only be inflated by decreasing permutations. The multivariate generating function for the words in \( S_3 \) that have a \( c_2 \) is \( S_3, 2 = S_3 - S_3, 1 \).

Combining the above results, the (univariate) generating function for the inflations of simple permutations of length at least 4 in \( G_3 \) is

\[
I_3(x) = S_{3, 1}(c, m, c, m, 0, c) + S_{3, 2}(c, m, c, m, m, c) + S_{3, 2}(c, m, c, m, f - m, m).
\]

8. Computing the Generating Function of \( \text{Av}(3124, 4312) \)

We proved earlier that \( \text{Simples}(\text{Av}(3124, 4312)) = \text{Simples}(G_1 \cup G_2) \). Therefore, we can count the simple permutations in \( \text{Av}(3124, 4312) \) with the results from previous sections. We start by counting the permutation of length 1, the two permutations of length 2, the simple permutations in \( G_1 \), and the simple permutations in \( G_2 \). However, this double-counts the simple permutations that lie in both \( G_1 \) and \( G_2 \). Since \( G_1 \cap G_2 = G_3 \), we correct for this subtracting the generating function for the simple permutations in \( G_3 \). From this we see that the generating function for the simple permutations in \( \text{Av}(3124, 4312) \) is

\[
S(x) = x + 2x^2 + (M_1(x) + M_2(x) - M_3(x)) \\
= \frac{x - 2x^2 - 5x^3 + 12x^4 + x^5 - 8x^6 - 3x^7}{(1 - 2x)(1 - x - x^2)^4} \\
= x + 2x^2 + 2x^4 + 4x^5 + 10x^6 + 21x^7 + 44x^8 + 89x^9 + \cdots,
\]

sequence A226430 in the OEIS [18]. For \( n \geq 4 \), we have

\[
[x^n] S(x) = 2^{n-3} - F_{n-2} + \sum_{i=1}^{n-3} F_i F_{n-i-2}
\]

(where \( F_n \) is the \( n \)th Fibonacci number, with \( F_1 = F_2 = 1 \)).

The previous three sections have detailed the allowed inflations of simple permutations of length at least 4. By Lemma 2.1, it remains to determine the inflations of the permutations 12 and 21. To
assure uniqueness, we require that the first component in the inflations of 12 (respectively, 21) be sum indecomposable (respectively, skew indecomposable).

Let \( \pi = \sigma \oplus \tau \in \text{Av}(3124, 4312) \) be sum decomposable. We must have \( \sigma \in \text{Av}(312) \) and for uniqueness, we assume that \( \sigma \) itself is sum indecomposable. For this, we use the notation \( \sigma \in \text{Av}_{\oplus}(312) \). Then, \( \tau \) can be any permutation in \( \text{Av}(3124, 4312) \). Every permutation in \( \text{Av}_{\oplus}(312) \) is of the form \( \alpha \oplus 1 \) for \( \alpha \in \text{Av}(312) \). Therefore, the class \( \text{Av}_{\oplus}(312) \) is enumerated by the shifted Catalan numbers, which have the generating function \( xc + x \). Now we see that the sum decomposable permutations in \( \text{Av}(3124, 4312) \) are equal to

\[
\text{Av}_{\oplus}(312) \oplus \text{Av}(3124, 4312)
\]

and are enumerated by

\[
f_{\oplus} = (xc + x)f.
\]

Let \( \pi = \sigma \oplus \tau \in \text{Av}(3124, 4312) \) be skew decomposable. There are two possibilities. If \( \sigma \) is increasing, then we must have \( \tau \in \text{Av}(312) \). Otherwise, if \( \sigma \in \text{Av}_{\oplus}(3124, 3124) \) has a descent, then we must have \( \tau \in \text{Av}(12) \). Hence, the skew decomposable permutations in \( \text{Av}(3124, 4312) \) are equal to

\[
(\text{Av}(21) \oplus \text{Av}(312)) \cup \left( \text{Av}_{\oplus}(3124, 4312) \oplus \text{Av}(12) \right).
\]

We enumerate this class by adding the enumerations of each component of the union and then subtracting the intersection of the two parts (which is \( \text{Av}(21) \oplus \text{Av}(12) \)). Let \( f_{\ominus} \) be the generating function for the skew decomposable permutations in \( \text{Av}(3124, 4312) \). Then, by the above reasoning,

\[
f_{\ominus} = mc + (f - f_{\oplus}) m - m^2
\]

which has the solution

\[
f_{\ominus} = \frac{m(f + c - m)}{1 + m}.
\]

The permutation class \( \text{Av}(3124, 4312) \) contains the single permutation of length 1, the sum and skew decomposable permutations, and the inflations of simple permutations of length at least 4. Therefore, the generating function \( f \) of \( \text{Av}(3124, 4312) \) satisfies the equation

\[
f = x + (xc + x)f + \frac{m(f + c - m)}{1 + m} + (I_1 + I_2 - I_3).
\]

We solve this for \( f \) to find that

\[
f(x) = \frac{(8x^5 - 16x^4 + 28x^3 - 26x^2 + 9x - 1) + \sqrt{1 - 4x^2} (2x^4 - 8x^3 + 14x^2 - 7x + 1)}{2x^2(1 - 6x + 9x^2 - 4x^3)}
\]

The first few terms of the expansion of \( f(x) \) are

\[
f(x) = x + 2x^2 + 6x^3 + 22x^4 + 88x^5 + 363x^6 + 1507x^7 + 6241x^8 + 25721x^9 + 105485x^{10} + \cdots
\]

sequence A165534 in the OEIS [18].

3We may now compute the number of sum decomposable and skew decomposable permutations in \( \text{Av}(3124, 4312) \):

\[
f_{\oplus} = (xc + x)f = x^2 + 3x^3 + 10x^4 + 37x^5 + 146x^6 + 595x^7 + 2456x^8 + 10167x^9 + \cdots
\]

(sequence A226434 in the OEIS [18]) and

\[
f_{\ominus} = \frac{m(f + c - m)}{1 + m} = x^2 + 3x^3 + 10x^4 + 35x^5 + 129x^6 + 494x^7 + 1935x^8 + 7670x^9 + \cdots
\]

(sequence A228769 in the OEIS [18]). Similarly we can now enumerate the sum and skew indecomposable permutations \( f_{\oplus} = f - f_{\ominus} \) and \( f_{\ominus} = f - f_{\oplus} \).
9. Applicability to Other $2\times4$ Classes

One may wonder whether these methods apply to any of the eleven remaining $2\times4$ classes that have not yet been enumerated. The key property of $Av(3124, 4312)$ that makes these arguments possible is that its simple permutations lie in a geometric grid class. It is easy to see that a permutation class is not geometrically griddable if it contains either arbitrarily long sums of the permutation 21 or arbitrarily long skew sums of the permutation 12.

To this end, we show that each of the remaining $2\times4$ classes contains a family of simple permutations that contains arbitrarily long sums of 21 or skew sums of 12. A natural candidate is the family of increasing oscillations. The increasing oscillating sequence is the infinite sequence

$$4, 1, 6, 3, 8, 5, \ldots, 2k, 2k+1, \ldots,$$

plotted in Figure 12. An increasing oscillation is any simple permutation that is contained in the increasing oscillating sequence.

Brignall, Ruskuc, and Vatter [7] showed that the class of all permutations contained in all but finitely many increasing oscillations is $Av(321, 2341, 3412, 4123)$. To show that a class $C = Av(B)$ contains the family of all increasing oscillations, we need to show that $Av(321, 2341, 3412, 4123) \subseteq C$. This amounts to checking that each $\beta \in B$ contains some permutation in $\{321, 2341, 3412, 4123\}$. The following nine $2\times4$ classes contain the family of increasing oscillations and hence their simple permutations are not geometrically griddable:

\begin{align*}
Av(3214, 4231), & \quad Av(1432, 4213), \quad Av(3214, 4312), \quad Av(4231, 4321), \quad Av(3421, 4213) \\
Av(3214, 4321), & \quad Av(4123, 4321), \quad Av(3412, 4123), \quad Av(4123, 4312).
\end{align*}

This leaves only the two classes $Av(2143, 4213)$ and $Av(2413, 3412)$, which contain neither the family of increasing oscillations nor the analogously defined family of decreasing oscillations.

Consider instead the family of simple permutations depicted in Figure 13. No permutation in this family contains either a 2143 or a 4213 pattern. Hence this infinite family, which cannot be geometrically gridded, lies in $Av(2143, 4213)$. The last class in question, $Av(2413, 3412)$, does not contain this infinite family, but it is symmetric (by a $90^\circ$ rotation) to $Av(2143, 2413)$ which does contain this infinite family.

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