Multipole Moments of numerical spacetimes

George Pappas$^{1,2}$† and Theocharis A Apostolatos$^{1}$‡

$^1$ Section of Astrophysics, Astronomy, and Mechanics, Department of Physics, University of Athens, Panepistimiopolis Zografos GR15783, Athens, Greece
$^2$ Theoretical Astrophysics, IAAT, Eberhard Karls University of Tübingen, Tübingen 72076, Germany
E-mail: † georgios.pappas@guest.uni-tuebingen.de
E-mail: ‡ thapostol@phys.uoa.gr

Abstract. In this article we present some recent results on identifying correctly the relativistic multipole moments of numerically constructed spacetimes, and the consequences that this correction has on searching for appropriate analytic spacetimes that can approximate well the previously mentioned numerical spacetimes. We also present expressions that give the quadrupole and the spin octupole as functions of the spin parameter of a neutron star for various equations of state and in a range of masses for every equation of state used. These results are relevant for describing the exterior spacetime of rotating neutron stars that are made up of matter obeying realistic equations of state.

1. Introduction

The multipolar expansion of the gravitational field in Newtonian gravity is a straightforward exercise. In spherical coordinates one could express the field as an expansion in powers of the radial coordinate and the appropriate angular eigenfunctions of the Laplace operator. One could be tempted to straightforwardly apply the same procedure to General Relativity (GR), but in this case things are more complicated due to the non-linear nature of the theory (the principle of superposition does not apply in GR). The task of defining the multipole moments in GR was undertaken in the beginning of the 70s by Geroch and Hansen who defined the multipole moments of an asymptotically flat spacetime in the static and stationary case as tensors at infinity [1, 2]. Alternative definitions of the relativistic multipole moments were also given in the early 80s by Simon [3] and by Thorne [4] which were closer to the spirit of the Newtonian asymptotic expansion, where Simon’s moments end up to be the same moments as Geroch’s and Hansen’s, while Thorne’s moments are coordinate dependent and an appropriate choice of the coordinates should be made in order to read the usual Geroch-Hansen moments (for a review see [5]). Finally in 1989, Fodor et al. [6] developed an algorithm for computing the multipole moments of a spacetime that is additionally axially symmetric, taking advantage of the insightful Ernst-potential formalism, expressing thus the scalar (in this case) multipole moments in terms of the Ernst potential.

In the case of stationary and axially symmetric spacetimes, the relativistic multipole moments can fully characterize the spacetime as one can see in [6]. Thus the multipole moments are of interest as quantities that capture all the properties of a given spacetime. On the other hand, they are also of interest on experimental grounds, since they can be measured in gravitational
wave signals from gravitational inspirals. In particular, Ryan [7] related various observed properties of these inspirals to the multipole moments of the background spacetime. Specifically he gave relations that connected the multipole moments of a spacetime to precession frequencies of orbits on that spacetime that deviate slightly from being circular equatorial ones as well as to the number of cycles of the gravitational wave signal that is emitted by a low mass object inspiraling adiabatically into such a spacetime. These same expressions can be also used in the context of accretion discs and the quasi-periodic oscillations (QPOs) observed in the X-ray spectrum of low mass X-ray binaries (LMXRBs) [8], where one, assuming a relativistic precession model for the QPOs [9], can relate the QPO frequencies to the multipole moments of the central object [10]. The connection of the multipole moments to the structure and the physical properties of a compact object, like a neutron star, was first attempted by Laarakkers and Poisson, who related the multipole moments of a neutron star to the equation of state (EOS) for the matter in its interior [11].

Furthermore, the multipole moments can be used to construct analytic spacetimes that will have prescribed properties. Thus, in the case that we would like to have an analytic description of the spacetime around a neutron star with specific physical properties, we could use the appropriate multipole moments to construct such a spacetime following the procedure presented in [12, 13, 14, 15, 16] (the multipole moments can also be important in different approaches to the problem of constructing geometries for neutron stars, such as [17, 18]).

Working on the previously mentioned applications of multipole moments to analytic and numerical spacetimes, we came to realize that there is a problem with the way the multipole moments of numerically constructed spacetimes are identified from their asymptotic behavior. These issues were addressed and clarified in [19] where the correct expressions for the quadrupole and the spin octupole were derived using Ryan’s [7] algorithm for relating the gravitational-wave spectrum $\Delta E$ (the energy emitted per unit logarithmic frequency interval) of a test particle that is orbiting on a circular equatorial orbit in an asymptotically flat, stationary and axially symmetric spacetime, to the multipole moments of that spacetime. In [19] various consequences of correcting the calculation of the numerical multipole moments were also discussed, such as the effect this correction has on attempting to approximate the numerical spacetime exterior to neutron stars. In particular it was shown that when one uses an analytic spacetime that depends on a number of parameters that can be connected to the multipole moments of a numerical spacetime, such as the analytic spacetime of Manko et al. [20, 21], in order to approximate that numerical spacetime, the use of the corrected values for the multipole moments improved the performance of the analytic spacetime and resulted to a better fit to the numerical spacetime.

It was also shown, extending a previous result by Laarakkers and Poisson [11], that the reduced quadrupole, $q \equiv M_2/M^3$, and the reduced spin octupole, $s_3 \equiv S_3/M^4$, follow respectively quadratic and cubic dependance on the spin parameter, $j = J/M^2$, just as in the case of the Kerr spacetime where these moments are $-j^2$ and $-j^3$ respectively, with the difference being that the proportionality constant is larger than 1 for neutron stars.

In this work we briefly present these results, i.e., we give a short derivation of the correct expressions for the multipole moments and then present the resulting improvement in the fit of the numerical metric by the 3-parameter analytic metric of Manko et al. In addition we present some results from approximating the numerical metric with the 4-parameter two-soliton metric, that is extensively discussed in [16], that further strengthen our arguments in favor of using the corrected multipole moments as criteria for matching an analytic spacetime to a numerical one. Finally we give the proportionality constants between the quadrupole and the square of the spin parameter, $j^2$, and the spin octupole and the cube of the spin parameter, $j^3$, for different equations of state and for different masses. These coefficients are directly comparable to the coefficients of Laarakkers and Poisson for the quadrupole. All the numerical spacetimes are calculated using the RNS numerical code [22]. All the expressions are in units of $G = c = 1$. 
2. Asymptotic expansion of a metric and the multipole moments

The line element used for the spacetime for the interior and the exterior of a neutron star, which has symmetry with respect to time translations and rotations, i.e., it is stationary and axisymmetric, is usually written in quasi-isotropic coordinates in the form,

\[ ds^2 = -e^{2\nu} dt^2 + r^2 \sin^2 \theta B^2 e^{-2\nu} (d\phi - \omega dt)^2 + e^{2(\zeta - \nu)} (dr^2 + r^2 d\theta^2), \]  

where \( \nu, B, \omega, \) and \( \zeta \) are the four metric functions, all functions of the quasi-isotropic coordinates \((r, \theta)\), as it was introduced by Butterworth and Ipser [23]. From Einstein’s field equations, one can show that the asymptotic expansion of the metric functions takes the following form,

\[ \nu \sim \left\{ \frac{M}{r} + \frac{\tilde{B}_0 M}{3r^3} + \frac{J^2}{r^4} + \left[ -\frac{\tilde{B}_0^2}{5} + \frac{\tilde{B}_1^2}{15} - \frac{12J^2}{5r^5} \right] \frac{M}{r^5} + \ldots \right\} \right. + \left. \left\{ \tilde{\nu}_2 + \ldots \right\} P_2 + \ldots, \]  

\[ \omega \sim \left[ \frac{2J}{r^3} - \frac{6JM}{r^4} + \left( 8 - \frac{3\tilde{B}_0}{M^2} \right) \frac{6JM^2}{5r^5} + \ldots \right] \frac{dP_1}{d\mu} + \left[ \tilde{\omega}_2 + \ldots \right] \frac{dP_3}{d\mu} + \ldots, \]  

\[ B \sim \sqrt{\frac{\pi}{2}} \left[ \left( 1 + \frac{\tilde{B}_0}{r^2} \right) T_0^{1/2} \left( \frac{\pi}{2} \right)^{1/2} + \frac{\tilde{B}_2}{r^4} T_2^{1/2} + \ldots \right]. \]  

In the formulae above \( P_l \) are the Legendre polynomials expressed as functions of \( \mu = \cos \theta, T_1^{1/2} \) are the so called Gegenbauer polynomials (similar to the Legendre polynomials, also functions of \( \mu \)), and \( M, J \) are the first two multipole moments (the mass and the spin) of the spacetime. The rest of the coefficients are related to the higher multipole moments.

These metric functions can be rewritten using the following redefinitions,

\[ Be^{-\nu} = e^{\beta}, \quad \zeta = \nu + \alpha. \]

Then the combinations of \( \nu \) and \( \beta \),

\[ \gamma = \nu + \beta, \quad \rho = \nu - \beta, \]

along with \( \omega \) could again be expressed as power series in \( 1/r \) (this is done in [24, 25, 22]) in the same manner as in Eqs. (3-4):

\[ \rho = \sum_{n=0}^{\infty} \left( -2 \frac{M_{2n}}{r^{2n+1}} + \text{higher order} \right) P_{2n}(\mu), \]  

\[ \omega = \sum_{n=1}^{\infty} \left( -2 \frac{S_{2n-1}}{r^{2n-1}} + \text{higher order} \right) \frac{P_{2n-1}(\mu)}{\sin \theta}, \]  

\[ \gamma = \sum_{n=1}^{\infty} \left( \frac{D_{2n-1}}{r^{2n-2}} + \text{higher order} \right) \frac{\sin (2n - 1) \theta}{\sin \theta}. \]

By a simple comparison between the above expansion and the corresponding ones in Eqs. (3-4), one can see for example that \( M_2^* = -\tilde{\nu}_2 \) and \( S_3^* = \frac{2}{3}\tilde{\omega}_2 \). The coefficients \( M_{2n}^* \) and \( S_{2n-1}^* \) were mistakenly identified as the mass and current-mass moments, respectively, of the corresponding spacetime and it is exactly these quantities that the numerical code of Stergioulas and Friedman [22], RNS, provides.

In order to identify the correct multipole moments, i.e., the Geroch-Hansen multipole moments, of a spacetime given by the line element (11), we have used Ryan’s expression [7] that relates the multipole moments with the energy change per logarithmic interval of the rotational
frequency $\Delta \tilde{E} = -d\tilde{E}/d\log \Omega$ for circular equatorial orbits in a stationary and axisymmetric spacetime. Thus, for the line element of Eq. (1), we first expressed the orbital frequency of circular equatorial orbits,

$$\Omega = \frac{d\phi}{dt} = -g_{t\phi,r} + \frac{\sqrt{(g_{t\phi,r})^2 - g_{tt,r}g_{\phi\phi,r}}}{g_{\phi\phi,r}},$$

as a power series in $x = (M/r)^{1/2}$ and then inverted it to obtain,

$$x = v + \frac{v^3}{2} + \frac{ju^4}{3} + \frac{1}{24}(13 + 4b - 6q)v^5 + \frac{ju^6}{2} + \frac{(97 + 28b + 56j^2 + 144q)v^7}{144}$$

$$+ \frac{(373j + 292bj - 330jq - 270w_2)v^8}{360} + O(v^9),$$

where $v = (M\Omega)^{1/3}$, $j = \frac{I}{M^2}$, $q = \frac{\omega_2}{M^2}$, $w_2 = \frac{B_0}{M^2}$. Then we expanded $\tilde{E} = -\frac{g_{tt} - g_{t\phi}\Omega - \sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}{g_{\phi\phi}}$, which is the energy per unit mass for a specific circular orbit, as a power series in $x = (M/r)^{1/2}$ and substituted the previous expression for $x$. From the resulted expression we calculated $\Delta \tilde{E}$, from the equation,

$$\Delta \tilde{E} = -\frac{d\tilde{E}}{d\log \Omega} = -\frac{v}{3} \frac{d\tilde{E}}{dv},$$

and arrived to the expression,

$$\Delta \tilde{E} = \frac{v^2}{3} - \frac{v^4}{2} + \frac{20jv^5}{9} - \frac{(89 + 32b + 24q)v^6}{24} + \frac{(421 + 64b - 60q)j - 90w_2)v^7}{3}$$

$$- \frac{5(1439 + 896b - 256j^2 + 672q)v^8}{432} + \frac{(225 + 275j^2 + 70M_2)v^9}{9M^2} + \frac{(81S_1 + 6M_2j - 6S_1j)v^9}{M^2} + O(v^{10}).$$

If one compares the previous expression to the one produced by Ryan,

$$\Delta \tilde{E} = \frac{1}{3}v^2 - \frac{1}{2}v^4 + \frac{20}{9} \frac{S_1}{M^2} v^5 + \left(- \frac{27}{8} + \frac{M_2}{M^3}\right) v^6 + \frac{28}{3} \frac{S_1}{M^2} v^7$$

$$+ \left(- \frac{225}{16} + \frac{80S_1}{27M^2} + \frac{70}{9} \frac{M_2}{M^3}\right) v^8 + \left(\frac{81S_1}{2M^2} + 6 \frac{M_2}{M^3} - 6 \frac{S_3}{M^4}\right) v^9 + \ldots$$

where $S_1 = J$, one can see that from the coefficients of $v^5$ and $v^9$ terms of the two series, the following values for the quadrupole and the spin octupole can be obtained:

$$M_2^{GH} = -4 - \frac{4}{3} \left(\frac{1}{4} + b\right) M^3 = M_2^* - \frac{4}{3} \left(\frac{1}{4} + b\right) M^3,$$

$$S_3^{GH} = \frac{3}{2} \omega_2 - \frac{12}{5} \left(\frac{1}{4} + b\right) jM^4 = S_3^* - \frac{12}{5} \left(\frac{1}{4} + b\right) jM^4.$$

Henceforth we will omit the superscript $\text{GH}$, that indicates the Geroch-Hansen moments, in $M_2, S_3$ when we refer to the correct multipole moments.
Laarakkers and Poisson performed in [11] a similar calculation in order to identify the quadrupole of the metric (1), but their result was missing the last term in Eq. (16), i.e., the term with $(1/4 + b)$. That was because they erroneously assumed that the correct asymptotic behavior for the metric should be that of Schwarzschild, which has $B = 1 - M^2/4r^2$, which corresponds to $b = -1/4$. Thus they had a priori fixed the correcting last term to be zero. Under a more careful consideration though, there is no reason to fix the asymptotic behavior of a stationary and axially symmetric spacetime in this way. A counter example for this is the case of the Kerr spacetime, where one can see that in quasi-isotropic coordinates the metric function $B$ is given by,

$$B_{Kerr} = 1 - \left(\frac{M^2 - a^2}{4r^2}\right),$$

which corresponds to having $b_{Kerr} = -(1/4)(1 - j^2)$.

3. Improvement in approximating a numerical spacetime with analytic spacetimes due to the correction in the moments

We will now present the effect that the correction in the moments has when one attempts to use an analytic spacetime to approximate the exterior space time of a neutron star, that is calculated numerically, by matching the multipole moments of the analytic spacetime to the multipole moments of the numerical spacetime. Berti and Stergioulas [26] tried to match a three-parameter analytic solution, the solution of Manko et al. [20], to a wide diversity of uniformly rotating neutron-star models. Each analytic solution was constructed so that its first three multipole moments were equal to the corresponding moments (mass, spin and quadrupole) of the particular numerical neutron star, where these moments were read from the corresponding numerical metric. For their calculations, they used as moments the quantities $M^2_{2n}$ and $S^3_{2n-1}$. Their conclusion was that this type of analytic solution was quite good to describe the external metric of all kinds of fast rotating neutron stars. Since the specific metric cannot assume low values of quadrupole moment, the metric is not adequate to describe rotating neutron stars with rotation lower than some value. We have used the Manko et al. solution to demonstrate the effect of correcting the multipole moments, compared to the results by Berti and Stergioulas.

Since the Manko et al. solution is in the previously mentioned sense handicapped, and since it has only three free parameters which provide the freedom to explore the behavior of the metric by changing only the quadrupole, we have also used another analytic solution, the so called two-soliton solution [27], that can also be used to approximate the exterior spacetime of neutron stars [16] and has four parameters, which allow into play the spin octupole as well. Thus, with the freedom now to vary both the quadrupole and the spin octupole, we have performed comparisons between the two-soliton and specific numerical metrics in order to see which values of multipole moments give the best fit.

In both cases, for the Manko et al. and the two-soliton, we have found that when the parameters of the analytic metric are calculated using the correct values for the multipole moments, the fitting of the analytic metric to the corresponding numerical one is better. This result also supports our claim that the way that an analytic metric should be matched to the numerical metric is by identifying the corresponding multipole moments of the two spacetimes.

In Figure [1] we present a typical comparison between the analytic metrics that are calculated using the correct moments, i.e. $M_2, S_3$, and the analytic metrics calculated using the previously assumed moments. Specifically the figures show the relative difference between the numerical and the analytic metric $(g^a_{ij} - g^a_{ij})/g^a_{ij}$ as a function of coordinate radius for the Manko et al. analytic metric and the two-soliton analytic metric, calculated for the values $M^2_2, S_3$ and for the values $M_2, S_3$ (the value of the spin octupole is relevant only for the two-soliton spacetime which has 4 parameters). The red dashed curves correspond to the Manko et al. metric produced from
Figure 1. A typical log-log plot of the relative difference between the numerical and the analytic metric \((g_{ij}^n - g_{ij}^a)/g_{ij}^n\) for a specific numerical model (model #19 of EOS FPS of [19]). The left plot is for \(g_{tt}\) and the right one for \(g_{t\phi}\). The red dashed curves correspond to the Manko et al. metric produced from \(M^*_2\) and the blue dashed curves correspond to the Manko et al. produced from \(M_2\). The orange solid curves correspond to the two-soliton produced using the moments \(M^*_2, S^*_3\) and the green solid curves correspond to the two-soliton produced using the corrected moments \(M_2, S_3\).

\(M^*_2\) and the blue dashed curves correspond to the Manko et al. produced from \(M_2\). The orange solid curves correspond to the two-soliton produced using the moments \(M^*_2, S^*_3\) and the green solid curves correspond to the two-soliton produced using the corrected moments \(M_2, S_3\). The left plot in Figure 1 shows the relative difference for the \(g_{tt}\) metric component while the right plot shows the relative difference for the \(g_{t\phi}\) metric components.

In order to further test the effect that varying the moments \(M_2\) and \(S_3\) has on the performance of the analytic metric to approximate the numerical metric, we have used the two-soliton metric to investigate the parameter space of the variation of the moments. As a measure of the ability of the analytic metric to approximate the numerical metric we have used the quantity, that we call, “overall mismatch” between the analytic and the numerical metric functions, that is defined (see [19]) as

\[
\sigma_{ij} = \left[ \int_{R_S}^{\infty} (g_{ij}^n - g_{ij}^a)^2 \, dr \right]^{1/2},
\]

where \(R_S\) is the radius \(r\) at the surface of the star. For each numerical model of the various models of uniformly rotating neutron stars that we have constructed, we formed a set of two-soliton spacetimes that have the same mass \(M\) and angular momentum \(J\) with the numerical model, but the quadrupoles and the current octupoles take the values \(M_2^{(a)} = M_2(1 - dM_2)\) and \(S_3^{(a)} = S_3(1 - dS_3)\) respectively, where the \(dM_2\) and \(dS_3\) take various values. The quantities \(dM_2\) and \(dS_3\) denote the fractional differences of the corresponding moments used in the calculation of each two-soliton spacetime from the moments of the numerical spacetime. For each one of the two-soliton spacetimes belonging to the previously mentioned set, we calculated the “overall mismatch” between the analytic and the numerical metric functions producing contour plots of the “overall mismatch” on the space of \(dM_2\) and \(dS_3\) for the different numerical models (one can find further discussion on this in [16]). Examples of these contour plots are shown in Figure 2.

These contour plots show that the choice of multipole moments for an analytic spacetime that give the best approximation to the numerical spacetime, are those that correspond to the moments given by the equations (16,17), i.e., those with \((dM_2, dS_3) \simeq (0, 0)\).
4. Relation between the multipole moments and the spin parameter $j$

The connection of the higher moments of a compact object with its spin parameter $j$ was first attempted by Laarakkers and Poisson [11]. Specifically what they showed was that the reduced quadrupole moment of neutron stars constructed using realistic equations of state was proportional to the square of the spin parameter and the proportionality constant depended on the mass of the neutron star and the equation of state, i.e.,

$$ q \simeq -a(M,\text{EOS})j^2. \quad (20) $$

In [19] it was shown that this relation is true, even after the correction of the quadrupole moment because the correcting factor $(1/4 + b)$ is proportional to $j^2$, which is strikingly similar to the way that this factor behaves in the case of the Kerr spacetime. A similar result was shown to apply for the spin octupole as well, i.e., the spin octupole has a dependence on the cube of the spin parameter of the form,

$$ s_3 \simeq -\beta(M,\text{EOS})j^3. \quad (21) $$

The calculations performed in [19], were performed using evolutionary sequences, so these results were not directly comparable to the results of Laarakkers and Poisson. Here we perform these fits for the reduced quadrupole and the reduced spin octupole as functions of $j^2$ and $j^3$ respectively and calculate the coefficients $a$ and $\beta$ for neutron star models of constant mass and varying rotation. The results are presented in Table 1.

These results are comparable to those presented by Laarakkers and Poisson in Table VII of [11]. So, we can compare the quadrupole coefficients for the equations of state FPS and L that are given both here and in [11]. We should first note that there is a typo in Table VII that...
Table 1. The table gives the coefficients $a, \beta$ of $q = -aj^2$ and $s_3 = -\beta j^3$ for various equations of state ranging from soft (on the left) to stiff (on the right) and for different masses in the range of $0.9 - 2.1M_\odot$. More details on the equations of state used can be found in [25, 26].

| $M/M_\odot$ | EOS FPS | EOS APR | EOS AU | EOS L |
|-------------|---------|---------|-------|-------|
| 0.9 | 7.80 | 6.50 | 5.56 | 4.07 |
| 1.0 | 16.2 | 13.5 | 11.3 | 9.31 |
| 1.1 | 8.23 | 6.86 | 6.03 | 4.47 |
| 1.2 | 17.2 | 14.4 | 12.3 | 8.03 |
| 1.3 | 9.08 | 7.82 | 6.69 | 4.47 |
| 1.4 | 23.8 | 16.6 | 14.0 | 5.05 |
| 1.5 | 19.6 | 20.1 | 12.2 | 5.85 |
| 1.6 | 14.1 | 7.73 | 14.3 | 6.81 |
| 1.7 | 10.9 | 6.48 | 16.5 | 5.98 |
| 1.8 | 30.3 | 12.2 | 10.3 | 5.13 |
| 1.9 | 26.4 | 12.4 | 10.1 | 4.59 |
| 2.0 | 23.2 | 12.4 | 10.1 | 4.59 |
| 2.1 | 18.2 | 12.4 | 10.1 | 4.59 |

Table 2. The table gives the relative differences in the coefficients $a$ for the quadrupole between the results presented in Table 1 and in Table VII of [11] for the equations of state FPS and L. The relative differences are given as %.

| $M/M_\odot$ | EOS FPS | EOS L |
|-------------|---------|-------|
| 1.0 | -0.26 | -0.82 |
| 1.2 | -2.56 | -0.75 |
| 1.4 | -3.00 | -2.63 |
| 1.6 | -6.91 | -3.69 |
| 1.8 | -11.65 | -3.16 |

5. Conclusions
In this work we have briefly presented some recent results (see [19]) regarding the correct identification of the multipole moments of a spacetime given in quasi-isotropic coordinates,
with application to the evaluation of the multipole moments of numerical spacetimes. We have also briefly discussed the effect that this correction has on using the multipole moments as parameters for constructing parameterized analytic spacetimes that approximate numerical spacetimes around compact objects. Specifically we showed that when the correct moments are taken into account, an analytic spacetime gives a better approximation of the numerical spacetime and further more, for the case of the two-soliton spacetime, the best approximation is the one that implements the corrected quadrupole and spin octupole. Finally we have discussed the fact that the quadrupole and the spin octupole seem to be proportional to the square of the spin parameter, $j^2$, and the cube of the spin parameter, $j^3$, respectively, and we have presented the proportionality constants for various equations of state and various masses.

We hope that the coefficients presented in Table 1 will be useful to anyone that would like to construct parameterized analytic spacetimes for the exterior of neutron stars, that involve the first four non-zero multipole moments or relations between them as parameters.

Acknowledgments
We would like to thank Kostas Kokkotas for the hospitality at the University of Tübingen and Nikos Stergioulas for providing us access to his RNS numerical code. This work has been supported by the I.K.Y. (IKYDA 2010). G.P. would also like to acknowledge DAAD scholarship number A/12/71258.

References
[1] Geroch R 1970 J. Math. Phys. 11 1955-1961; Geroch R 1970 J. Math. Phys. 11 2580-2588
[2] Hansen, R O 1974 J. Math. Phys. 15 46-52
[3] Simon W and Beig R 1983 J. Math. Phys. 24 1163-1171
[4] Thorne K S 1980 Rev. Mod. Phys. 52 299-399
[5] Quevedo H 1990 Fortschr. Phys. 38 733-840
[6] Fodor G, Hoenselaers C and Perjés Z 1989 J. Math. Phys. 30 2252-2257
[7] Ryan F 1995 Phys. Rev. D 52 5707-5718
[8] van der Klis M 2006 Compact Stellar X-Ray Sources ed Lewis W H G and van der Klis M (Cambridge Univ. Press) p 39
[9] Stella L, Vietri M 1998 ApJ 492 L59
[10] Pappas G 2012 MNRAS 422 2581
[11] Lannarckers W G, Poisson E 1999 ApJ 512 282-287
[12] Sotiriou T P, Pappas G 2005 J. Phys.: Conf. Series 8 23
[13] Pachón L A, Rueda J A, Sanabria-Gómez J D 2006 Phys. Rev. D 73 104038
[14] Pappas G 2009 J. Phys.: Conf. Series 189 012028
[15] Pachón L A, Rueda J A, Valenzuela-Toledo C A 2012 ApJ 756 82
[16] Pappas G, Apostolatos T A Preprint arXiv:1209.6148 [gr-qc]
[17] Hartle J B, Thorne K S 1968 ApJ 153 807
[18] Boshkayev K, Quevedo H, Ruffini R 2012 Physical Review D 86 064043
[19] Pappas G, Apostolatos T A 2012 Phys. Rev. Lett. 108 231104
[20] Manko V S, Miille E W, Sanabria-Gómez J D 2000 Phys. Rev. D 61 081501
[21] Manko V S, Sanabria-Gómez J D and Manko O V 2000 Phys. Rev. D 62 044048
[22] Stergioulas N and Friedman J L 1995 ApJ 444 306; Stergioulas N, rns, (November, 1997), [public domain code]: cited on 19 November 1997, http://www.gravity.phys.uwm.edu/rns
[23] Butterworth E M and Ipser J R 1976 ApJ 204 200-223
[24] Komatsu H, Eriuchi Y and Hechisu I 1989 MNRAS 237 355-379
[25] Cook G B, Shapiro S L and Teukolsky S A 1994 ApJ 424 823-845
[26] Berti E and Stergioulas N 2004 MNRAS 350 1416
[27] Manko V S, Martin J, Ruiz J E 1995 J. Math. Phys. 36 3063