Tensor-network study of quantum phase transition on Sierpiński fractal

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The transverse-field Ising model on the Sierpiński fractal, which is characterized by the fractal dimension $\log_3 3 \approx 1.585$, is studied by a tensor-network method, the Higher-Order Tensor Renormalization Group. We analyze the ground-state energy and the spontaneous magnetization in the thermodynamic limit. The system exhibits the second-order phase transition at the critical magnetic field $h_c = 1.865$. The critical exponents $\beta \approx 0.198$ and $\delta \approx 8.7$ are obtained. Complementary to the tensor-network method, we make use of the real-space renormalization group and improved mean-field approximations for comparison.

PACS numbers:

I. INTRODUCTION

The classification of quantum phase transitions remains one of the major interests in the condensed matter physics. Although there are groups of exactly solvable models in physics, a vast majority of the physical systems calls for different approaches, in particular, for numerical calculations. Some of them are straightforwardly applicable, such as the Monte Carlo (MC) simulations, whereas the other ones, including renormalization group techniques, require development of novel algorithms.

This work is oriented for classification of the quantum phase transition on a fractal lattice, which is the infinite-size Sierpiński fractal (triangle or gasket), whose Hausdorff fractal dimension is $\log_3 3 \approx 1.585$. It is known that the classical Ising model exhibits no phase transition on the Sierpiński fractal [1,2]. Substantially less is known about its quantum counterpart. A couple of recent works investigated quantum spin models on fractals by means of real-space renormalization-group methods and by classical MC simulations [3,6]. In order to bring more light into the quantum fractal system, we consider a different methodology. It should be noted that the tensor-network viewpoint is efficient if we consider a recursive structure of the fractal lattices [5]. In particular, we generalized the Higher-Order Tensor Renormalization Group (HOTRG) method [9] to adapt the method to the Sierpiński fractal.

We focus on the quantum Ising model on the Sierpiński fractal, and analyze the ground-state energy per site $E_0$ and the spontaneous magnetization with respect to the magnetic field. We first determine the critical transverse magnetic field $h_c$, and then we estimate the magnetic critical exponents $\beta$ and $\delta$ from the calculated order parameter, in order to classify the quantum phase transition. We discuss the numerical stability in the HOTRG, as it deserves an attention due to the correct initialization of the tensors, where the entanglement entropy is calculated to clarify the appropriate initialization of the tensors. We provide two additional complementary calculations by improved mean-field approximations and by a real-space renormalization-group technique, which we modify to determine the phase transition field $h_c$ only (not $\beta$ or $\delta$).

II. MODEL AND CONVENTIONAL APPROXIMATIONS

Figure 1 shows the structure of the Sierpiński fractal that we consider in this article. The lattice is recursively constructed by connecting three units, as shown in (a)-(d). In that manner, a bond is shared by the two units. The black dots represent the lattice sites, and the full lines represent the nearest-neighboring connections. For

FIG. 1: Structure of the Sierpiński fractal. (a) The smallest unit consists of a site shown by the black circle, from which the three bonds labeled by $i$, $j$, and $k$ emerge. (b)-(d) If connecting the three units, one can iteratively expand the size of the unit to form the fractal lattice. The vertical dotted lines correspond to the imaginary-time evolution, which is considered in Sec. III.
the moment, we omit the vertical (dotted) lines and disregard the tensor notations shown in the figure. The Hamiltonian of the transverse-field Ising model on the lattice has the form

\[ H = -J \sum_{\langle a,b \rangle} \sigma^x_a \sigma^x_b - h_x \sum_a \sigma^x_a - h_z \sum_a \sigma^z_a, \]  

where \( \sigma^x_a \) and \( \sigma^z_a \) represent the Pauli spin operators acting on the lattice site \( a \). The uniform magnetic fields \( h_x \) and \( h_z \), respectively, are applied to the transverse (\( x \)) and longitudinal (\( z \)) directions. The ferromagnetic Ising interaction \( J > 0 \) is present between the nearest-neighboring spin pairs \( \sigma^x_a \) and \( \sigma^x_b \). The interacting pairs are denoted by the symbol \( \langle a,b \rangle \) positioned on bonds in the fractal lattice. Throughout this article we focus on the ground-state of this system and its quantum phase transition with respect to the transverse field \( h_x \). Hereafter we assume \( J = 1 \). The parameter \( h_z \) is set to zero unless its value is specified (when calculating the critical exponent \( \delta \)).

Before we start explaining the details of the HOTRG method, we briefly introduce the two conventional approximation schemes. The first one is the mean-field approximation, which offers a rough insight into the ground-state of this system and its quantum phase transition with respect to the transverse field \( h_x \). Hereafter we assume \( J = 1 \). The parameter \( h_z \) is set to zero unless its value is specified (when calculating the critical exponent \( \delta \)).

We focus on the numerical analysis of the quantum fractal system by means of the HOTRG method [9], which has yielded a high numerical accuracy for two- and three-dimensional classical Ising model. The method has also been applied to one- and two-dimensional quantum Ising model through the quantum-classical correspondence, which is a discrete imaginary-time path-integral representation. The imaginary-time evolution expressed by the density operator \( \rho = e^{-\tau H} \) is essential, as it behaves as the projection to the ground-state in the large \( \tau \) limit.

Let us consider the Hamiltonian in Eq. (1) and divide the imaginary-time span \( \tau \) into \( \Delta \tau \) intervals \( \Delta \tau = \tau/\ell \). We express \( \rho \) in the form of product

\[ \rho = (e^{-\Delta \tau H})^\ell = [e^{-\Delta \tau (H_{zz} + H_x + H_z)}]^{\ell} \]  

among imaginary-time intervals, where \( H_{zz}, H_x, \) and \( H_z \), respectively, correspond to the first, second, and third term in the r.h.s. of Eq. (1). Although \( H_x \) does not commute with \( H_{zz} \) or \( H_z \), a good approximation of \( \rho \) can be obtained by means of the Trotter-Suzuki decomposition

\[ \rho \approx \left[ e^{-\Delta \tau (H_{zz} + H_z)} e^{-\Delta \tau H_x} \right]^\ell, \]
provided that $\Delta \tau$ is sufficiently small. Each imaginary-time interval, which corresponds to $e^{-\Delta \tau \mathcal{H}}$, plays the role of the transfer matrix.

Applying a duality transformation, which introduces new two-state variables in the middle of the connected bonds, we can express the transfer matrix in terms of the tensor network. Figure 1 shows the structure of the transfer matrix for the elementary unit, in (a) and for the extended ones, in (b)-(d). This time, we regard the black dots as local tensors, which have three legs in the spatial direction shown by the lines, and two legs in the imaginary-time directions shown by the vertical dotted lines. Each local tensor is given by

$$T_{ijk,st}^{(0)} = \sum_{\sigma} W_{\sigma i} W_{\sigma j} W_{\sigma k} P_{\sigma s} P_{\sigma t} G_{\sigma},$$

where the matrix

$$W = \begin{pmatrix} \sqrt{\cosh(J\Delta \tau)} & \sqrt{\sinh(J\Delta \tau)} \\ \sqrt{\cosh(J\Delta \tau)} & -\sqrt{\sinh(J\Delta \tau)} \end{pmatrix}$$

originates from the Ising interaction in $\mathcal{H}_{zz}$ between the neighboring spins. The other matrix

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp(h_z \Delta \tau/2) & \exp(-h_z \Delta \tau/2) \\ \exp(h_z \Delta \tau/2) & -\exp(-h_z \Delta \tau/2) \end{pmatrix}$$

corresponds to the spin-flipping effect by the transverse field $h_z$ in $\mathcal{H}_{zz}$, and the column vector

$$G = \begin{pmatrix} \exp(h_z \Delta \tau) \\ \exp(-h_z \Delta \tau) \end{pmatrix}$$

represents the effect of external field $h_z$ along the $z$-direction in $\mathcal{H}_{zz}$. All the indices $i, j, k, s, t$ of $T_{ijk,st}^{(0)}$ thus carry two degrees of the freedom.

The transfer matrices $T_{ijk,st}^{(1)}$, $T_{ijk,st}^{(2)}$, and $T_{ijk,st}^{(3)}$ for the extended units, respectively, shown in Fig. 1(b), (c), and (d), can be obtained by contracting horizontal legs in a recursive manner. Actually, we do not directly treat these extended transfer matrices. Our aim is to obtain the local thermodynamic quantities

$$\langle O \rangle = \frac{\text{Tr} \left( O \rho \right)}{\text{Tr} \left( \rho \right)} = \frac{\text{Tr} \left( O e^{-\tau \mathcal{H}} \right)}{\text{Tr} \left( e^{-\tau \mathcal{H}} \right)},$$

where $O$ represents a local operator, for a sufficiently wide system when $\tau$ is large enough. For this purpose, we do not have to construct $\rho$ in a faithful manner, but only need to consider a series of finite-size clusters represented as a stack of the extended transfer matrices, i.e.,

$$\left[ T_{ijk,st}^{(1)} \right]^2, \left[ T_{ijk,st}^{(2)} \right]^4, \left[ T_{ijk,st}^{(3)} \right]^8, \cdots, \left[ T_{ijk,st}^{(\ell)} \right]^{2^\ell}, \cdots$$

The HOTRG method is appropriate for this purpose. We create the stack of the transfer matrices in a renormalized form, through the recursive contraction processes,

$$A_{ijm',ab}^{(l)} = \sum_{jnm',a} T_{ijm,n}^{(l)} T_{km',a'b'}^{(l)} U_{s's',a} U_{tt',b'},$$

$$B_{ijk,st}^{(l)} = \sum_{aum',ab'} A_{ijm',ab}^{(l)} U_{s's',a} U_{tt',b'},$$

$$T_{ijk,st}^{(l+1)} = \sum_{umn',aan'} B_{mn'o',ut}^{(l)} B_{mn'o',ut}^{(l)} U_{m'm',a} U_{n'n',b},$$

which are also depicted by diagrams in Fig. 2. The projectors $U$, $U'$, and $U''$, which are also called as isometries, are quasi unitary rectangular matrices of the size $D^2 \times D$, with $D$ being the degree of the freedom for a tensor index. These matrices are obtained from the higher-order singular value decomposition (SVD), whenever two tensors are combined and consequently reshaped into a matrix form [9]. We keep the states that correspond to $D$ largest singular values. Hence, the larger the $D$, the better the approximation is reached [8, 12].

In this manner the HOTRG method, applied to the discrete path-integral representation of the quantum fractal system, enables us to build up a sufficiently large finite-size system. Note that during the recursive extension of the system, we can obtain thermodynamic functions such as the ground-state energy $E_0$ per site, as it is has been done for transverse-field Ising model on the square lattice. Further details on the calculation of $E_0^{(\text{HOTRG})}$ can be found in Refs. [5, 9]. One-point functions, such as magnetization $\langle \sigma^z \rangle$, can also be calculated by introducing an impurity tensor, as discussed in Ref. [8]. The expansion procedure is stopped after all of the thermodynamic functions (normalized per site) completely converge.
Remarks on initialization

It is known that the Trotter-Suzuki decomposition in Eq. (6) introduces an error of the order of $(\Delta \tau)^2$. Thus it is more suitable to keep $\Delta \tau$ relatively small. We choose $\Delta \tau$ of the order $10^{-2}$ in most of the numerical calculations. When $\Delta \tau$ is too small, however, there is a conspicuous anisotropy between the space and imaginary-time directions. Thus, a naive application of the iterative processes in Eqs. (12a)-(12c) causes a numerical instability when $\Delta \tau$ is small. In order to avoid this problem, we redefine $T^{(0)}_{ijk,st}$ by stacking a number of the local weights $T^{(0)}_{ijk,st}$ in Eq. (7) vertically, right before we start the iterative processes in Eqs. (12a)-(12c).

We introduce an integer variable $n = 0, 1, 2, \ldots$, which enumerates the number of the initial stacking process. At $n = 0$, we set $T^{(0,0)} = T^{(0)}$ as has been defined in Eq. (7). The local tensor is vertically stacked by means of the contraction process

$$T^{(0,n+1)}_{ijk,st} = \sum_{abcd} T^{(0,n)}_{abc,stu} T^{(0,n)}_{def,at} T^{(n)}_{ad,i} U^{(n)}_{be,j} U^{(n)}_{cf,k},$$

(13)

where the geometry in this contraction is identical to Eq. (12). This contraction is performed along the following processes. We first stack two tensors vertically and take the configuration sum

$$M^{(n)}_{ia,jkbsv} = \sum_{t} T^{(0,n)}_{ijk,stu} T^{(0,n)}_{abc,stu},$$

(14)

for the vertical index $t$. We perform the singular value decomposition (SVD) that factorizes $M^{(n)}$ into three matrices,

$$M^{(n)}_{ia,jkbsv} = \sum_{\xi=1} \xi^{(n)}_{ia,\xi} \xi^{(n)}_{jkbsv},$$

(15)

in order to obtain the unitary projector $U^{(n)}$. Finally, we apply $U^{(n)}$ to $M^{(n)}$, as shown in Eq. (13). The D largest singular values $\omega^{(n)}_{\xi}$ are kept in this renormalization-group transformation. After repeating the stacking processes in Eq. (13) $n$ times, $2^n$ numbers of the original $T^{(0)}_{ijk,stu}$ are contracted by means of a tree-tensor network, which corresponds to $U^{(n)}$.

It is possible to define the entanglement entropy (out of the singular values $\omega^{(n)}_{\xi}$ in Eq. (15))

$$\varepsilon^{(n)}_{\text{vertical}} = -\sum_{\xi} \frac{[\omega^{(n)}_{\xi}]^2}{\Omega} \ln \left( \frac{[\omega^{(n)}_{\xi}]^2}{\Omega} \right),$$

(16)

where $\Omega = \sum_{\xi} [\omega^{(n)}_{\xi}]^2$ normalizes the probability. In a similar manner, we can define another type of the entanglement entropy $\varepsilon^{(n)}_{\text{planar}}$ with respect to the horizontal contraction between two $T^{(0,n)}_{ijk,stu}$. The calculation process proceeds in the following

$$\tilde{M}^{(n)}_{su,ikabtv} = \sum_{j} T^{(0,n)}_{ijk,stu} T^{(0,n)}_{abj,uv} = \sum_{\xi} \tilde{U}^{(n)}_{su,\xi} \tilde{V}^{(n)}_{\xi,ikabtv},$$

$$\varepsilon^{(n)}_{\text{planar}} = -\sum_{\xi} \frac{[\omega^{(n)}_{\xi}]^2}{\Omega} \ln \left( \frac{[\omega^{(n)}_{\xi}]^2}{\Omega} \right).$$

(17)

These two entanglement entropies have different behavior. Figure 3 shows $\varepsilon^{(n)}_{\text{vertical}}$ and $\varepsilon^{(n)}_{\text{planar}}$ with respect to $n$ at zero transverse field $h_x = 0$, at the critical field $h_x = h_c$ (triangles), and $h_x = 3$ (squares).

FIG. 3: (Color online) The entanglement entropy $\varepsilon^{(n)}_{\text{vertical}}$ and $\varepsilon^{(n)}_{\text{planar}}$, with respect to $n$ at $h_x = 0$ (circles), at $h_x = h_c$ (triangles), and $h_x = 3$ (squares).

IV. NUMERICAL RESULTS

The mean-field approximation (MFA) offers a rough insight into phase transitions. As we have introduced in Sec. II, we consider series of the three approximations, MFA1, MFA3, and MFA4, respectively, where the interactions inside the (extended) units shown in Fig. 1 (a), (b), and (c) are treated exactly. We have also introduced the RSRG method, which can capture critical behavior of the model, provided that a sufficiently large number of the block-spin states $D$ is kept. however, the improvement with respect to the increasing $D$ is rather slow. In contrast, the numerical precision of the HOTRG method significantly improves if $D$ increases. We have confirmed
that $D = 8$ is large enough to obtain well-converged results for the fractal lattice. We present the numerical results up to $D = 20$.

We first compare the three types of mean-field approximations with the HOTRG method when $D = 8$. The ground-state energy per site $E_0$ with respect to the transverse field $h_x$ is shown in Fig. 4. The ground state energy obtained by the HOTRG method is always the lower than those obtained by the mean-field approximations. This can be better visible by comparing the HOTRG ground-state energy $E_0$ around the phase transition $h = h_c$ with the MFA ones on the top inset. The bottom inset shows the two spontaneous magnetizations by calculating the parallel $\langle \sigma^z \rangle$ and the transversal $\langle \sigma^x \rangle$ local spin operators we have defined in Eq. (11). Since the fractal structure does not form a homogeneous lattice, the local spin operator $\langle \sigma^z \rangle$ is calculated by averaging three independent impurity operators $[14]$ located at $T^{(1)}$, cf Fig. 1(b). From the data obtained by the HOTRG method, the critical field is determined $h_c = 1.865$.

The RSRG method provides relatively accurate $E_0$ when $D = 24$ block-spin states are kept. Figure 5 shows $E_0$ obtained by RSRG ($D = 24$) compared with HOTRG ($D = 8$). The inset shows the energy difference in $E_0$, within the range $0 \leq h_x \leq 3$, which is not conspicuous. Figure 6 shows the longitudinal and transverse magnetizations, respectively, $\langle \sigma^z \rangle$ and $\langle \sigma^x \rangle$. The difference between both of the methods is better visible below the criticality. The RSRG method gives the critical field $h_c = 1.864$. As we have stated above, the HOTRG method gives the critical field $h_c = 1.865$. (These values of the critical fields in both methods have been calculated from $\langle \sigma^z \rangle$.) The magnetization $\langle \sigma^x \rangle$, is calculated by making use of the Hellman-Feynman theorem [15], i.e.

\[
\langle \sigma^x \rangle = -\frac{dE_0}{dh_x}.
\]

It also exhibits a singularity at the critical magnetic field $h_x = h_c$. (We still consider the case $h_z = 0$).

Finally, we increase $D$ in HOTRG to the value of $D = 20$, which is still computationally feasible in order to determine the critical field $h_c$ and the critical exponents $\beta$ and $\delta$ more precisely. The critical exponent $\beta$ is associated with the critical behavior of the spontaneous magnetization $\langle \sigma^z \rangle \propto (h_c - h_x)^{\beta}$. Using the least-square fitting on the magnetization profile of $\langle \sigma^z \rangle$, we have been able to estimate the critical magnetic field $h_c = 1.865$ precisely and the exponent $\beta = 0.20$. For confirmation, we plot $\langle \sigma^z \rangle^{1/\beta}$ with the obtained exponent in Fig. 7 where the linear behavior of $\langle \sigma^z \rangle^{1/\beta}$ below $h_c$ is evident. The other exponent $\delta$ comes from the scaling $\langle \sigma^z \rangle \propto h_x^{1/\delta}$.
TABLE I: Comparison of $h_c$, $\beta$, and $\delta$ for the transverse-field Ising model on a chain ($d_H = 1$), Sierpiński fractal lattice ($d_H = \log_2 3$) [3], and the square lattice ($d_H = 2$) [9][10].

| $d_H$   | $h_c$ | $\beta$ | $\delta$ | method used          |
|---------|-------|---------|----------|----------------------|
| log₂ 2 = 1 | 1     | 0.125   | 15       | exact solution        |
| log₂ $\approx 1.585$ | 1.865 | 0.20    | 8.7      | HOTRG, MC             |
| log₂ 4 = 2 | 3.0439| 0.3295  | 4.8      | HOTRG, MC             |

if measured at the critical field $h_z = h_c$. It is obvious from the inset, that the linearity of $\langle \sigma^z \rangle^\delta$ with respect to $h_z$ yields the second critical exponent $\delta = 8.7$.

V. SUMMARY

The transverse-field Ising model on the Sierpiński fractal was studied by the three methods: (1) the mean-field approximation, (2) the RSRG method, which can be easily adapted for the fractal structure, and (3) the HOTRG method, which had reproduced very reliable results for the transverse-field Ising model on the square lattice [9]. The numerical algorithm in the original HOTRG method has been generalized in order to design a tensor network with the fractal structure. We provided the entanglement-entropy analysis for improving the numerical accuracy in the HOTRG method at the pre-initial stage.

We have confirmed the existence of the second-order phase transition in the quantum Ising model on the Sierpiński fractal, whose Hausdorff dimension $d_H \approx 1.585$. The critical magnetic field is $h_c = 1.865$, and the two critical magnetic exponents $\beta = 0.20$ and $\delta = 8.7$ are obtained. Our results are in a good agreement with MC simulations, where $h_c = 1.865(2)$ and $\beta = 0.19(2)$ [3]. Table I summarizes the transition point $h_c$ and the exponents $\beta$ and $\delta$ for one- and two-dimensional lattices and compares them with exact and numerical methods.

Knowledge of the quantum phase transitions on such non-typical lattices can be helpful in initializing neural networks. To add, recent studies on neural networks [17, 18] has also motivated the current study. There is an open problem to be solved: It concerns of the classification of the quantum phase transition on a fractal with $d_H = \log_4 12$, as we have recently studied for the classical system [8]. This particular fractal can be easily generalized to the present tensor-network structure. Moreover, we can also vary the fractal dimensions and generate a set of the Hausdorff dimensions $1 < d_H < 2$. The purpose of performing these calculations is to confirm validity of the hyper-scaling relations since the fractional dimensionality has not been fully investigated yet.

ACKNOWLEDGMENTS

This work was supported by the projects EXSES APVV-16-0186 and VEGA Grant No. 2/0130/15. T. N. and A. G. acknowledge the support of Grant-in-Aid for Scientific Research. J. G. and T. N. were supported by JSPS KAKENHI Grant Number 17K05578 and P17750.

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