Orthocenters of triangles in the n-dimensional space

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Abstract

In this paper we present a way to define a set of orthocenters for a triangle in the n-dimensional space $\mathbb{R}^n$ and we will see some analogies of these orthocenters with the classic orthocenter of a triangle in the Euclidean plane.

1 Introduction

In the Euclidean plane, the orthocenter $H$ of a triangle $\triangle ABC$ is defined as the point where the altitudes of the triangle converge, i.e., the point at which lines perpendicular to the sides of the triangle passing through the opposite vertex to such sides converge. If $O$ and $G$ are the circumcenter and the centroid of the triangle respectively, the classical Euler’s theorem asserts that $O$, $G$ and $H$ are collinear and $OG = 2GH$.

Another property of the orthocenter of a triangle is the following: the orthocenter is where concur the circles of radius equal to the circumscribed passing through two vertices of the triangle, i.e if the circumcircle is reflected with respect to the midpoints of the sides of the triangle, then the three circles obtained concur in the orthocenter of the triangle. Since this definition of orthocenter not have to initially do with the notion of orthogonality we speak in this case of $C$-orthocenter. Moreover, by the definition of $C$-orthocenter of a triangle, it is the circumcenter of the triangle whose vertices are the symmetric of the circumcenter with respect to the midpoint of the sides.

If the triangle $\triangle ABC$ is not a right triangle, then the triangles $\triangle HBC$, $\triangle AHC$ and $\triangle ABH$ have the points $A$, $B$, $C$ as orthocenters, respectively, i.e., triangles with three vertices in the set $\{A, B, C, H\}$ has as orthocenter the remaining point. A set of four points satisfying the above property is called an orthocentric system. Basic references to the orthocentric system in Minkowski planes are in [Martini and Spirova (2007)].

When we review the properties related to the orthocenter such as; Euler line, Feuerbach circumference, $C$-orthocenter, orthocentric system, we realize that their validity essentially depend on the relationship between vertices and the circumcenter of the triangle, i.e, equidistance. In this paper we will use this idea to define an ‘orthocenter’ associated with each point that is equidistant from the vertices of a triangle in the n-dimensional space and we will see some properties similar to those of the orthocenter in the Euclidean plane.

2 Notation and Preliminaries

$\mathbb{R}^n$ denote the classical $n$-dimensional Euclidean space, its elements as vector space or affine space will call points and denoted them with capital letters. if $A$ and $B$ are two points, then $\overrightarrow{AB}$ and $AB$ denote the vector and the standard segment with ends $A$ and $B$ respectively, i.e, $\overrightarrow{AB} = B - A$ and $AB = \|B - A\|$. 
A triangle \( \triangle A_0A_1A_2 \) is determined by three non collinear points \( A_0, A_1 \) and \( A_2 \) in the space \( R^n \), the points \( A_i \) are called vertices of the triangle, the segment denoted by \( a_i \) whose endpoints are the vertices other than \( A_i \) is called side of the triangle and is said to \( A_i \) is his opposite vertex. Denote by \( O, C, r \) and \( G \) the circumcenter, the circumcircle, the circumradius and the centroid of the triangle \( \triangle A_0A_1A_2 \) , respectively, i.e \( O \) it is the only point on plane determined by \( A_0, A_1, A_2 \) equidistant from them, \( C \) is the circumference on afore mentioned plane passing through \( A_0, A_1, A_2 \), \( r = O A_0 = O A_1 = O A_2 \) and \( G = \frac{A_0+ A_1+ A_2}{3} \), and \( M_i \) the centroid (midpoint) of side \( a_i \). We also recall the medial or Feuerbach triangle \( \triangle M_0M_1M_2 \) of the triangle \( \triangle A_0A_1A_2 \), and denote its circumcenter by \( Q_O \). Note that \( Q_O = \frac{1}{2} (A_0 + A_1 + A_2 - O) \).

If \( P \) is a point of \( R^n \) and \( \lambda \) is a scalar, the homothetic with center \( P \) and ratio \( \lambda \), is the application \( H_{P,\lambda} : R^n \to R^n \) defined by

\[
H_{P,\lambda}(X) = (1 - \lambda) P + \lambda X,
\]

for all \( X \) in \( R^n \). \( H_{P,-1} \) we will symbolize by \( S_P \) which is called the point reflection with respect to \( P \).

The following list contains some of the properties satisfied by the orthocenter (see Figure 1).

For the triangle \( \triangle A_0A_1A_2 \), the orthocenter \( H \) is expressed as a function of the circumcenter \( O \) and the vertices of the triangle by the formula \( H = A_0 + A_1 + A_2 - 2O \) and it is not difficult to see that \( H \) is the circumcenter of the triangle \( \triangle B_0B_1B_2 \), where \( B_i \) is the symmetric of \( O \) with respect to \( M_i \) for \( i = 0,1,2 \), i.e \( B_i = A_j + A_k - O \). The points \( B_0, B_1 \) and \( B_2 \) are the circumcenters of the triangles \( \triangle H A_1A_2 \), \( \triangle A_0H A_2 \) and \( \triangle A_0A_1H \) respectively, the circumscribed circles of these triangles are denoted by \( C_0, C_1 \) and \( C_2 \), and all of them have radius \( r \). The triangles \( \triangle A_0A_1A_2 \) and \( \triangle B_0B_1B_2 \) are symmetrical and in [Pacheco and Rosas (2014)] the triangle \( \triangle B_0B_1B_2 \) is called the antitriangle of the triangle \( \triangle A_0A_1A_2 \) associated with \( O \). The center of symmetry between both triangles is the point \( Q_O \).

![Figure 1: Orthocenter properties](image)

1. The points \( O, G \) and \( H \) are collinear, with \( G \) in between, and \( 2OG = GH \) (Euler property).
2. If \( N_0, N_1 \) and \( N_2 \) are the midpoints of the sides of the triangle \( \triangle B_0B_1B_2 \), the circumference of center \( Q_O \) and radio \( r/2 \) (Feuerbach circumference) passes through the points \( M_0, M_1, M_2, N_0, N_1 \).
Furthermore, the following assertions hold:

and $N_2$. It also passes through the midpoints of the segments that joint $H$ with the points of the circumcircle of $\triangle A_0A_1A_2$. and the midpoints of the segments that joint $O$ with the points of the circumcircle of $\triangle B_0B_1B_2$.

3. the points $O$, $Q_O$, $G$, and $H$, form a harmonic range, being satisfied $\frac{OQ_O}{OQ} = \frac{QH}{HQ} = 2$.

4. The following sets are orthocentrics systems $\{A_0, A_1, A_2, H\}$, $\{B_0, B_1, B_2, O\}$, $\{M_0, M_1, M_2, O\}$, $\{N_0, N_1, N_2, H\}$ and $\{G_0, G_1, G_2, G\}$, where $G_0$, $G_1$ and $G_2$ are the centroids of the triangles $\triangle H A_1 A_2$, $\triangle H A_0 A_2$ and $\triangle A_0 A_1 H$ respectively.

5. If $\{A_0, A_1, A_2, A_3\}$ is an orthocentric system, then $\overrightarrow{A_i A_j} \perp \overrightarrow{A_k A_l}$ for $\{i, j, k, l\} = \{0, 1, 2, 3\}$.

3 Results

Given three non collinear points $A_0$, $A_1$, $A_2$ in the Euclidean space, there is only one point that is equidistant from them, which is precisely the circumcenter of the triangle $\triangle A_0A_1A_2$. However, if the points $A_0$, $A_1$, $A_2$ are in an $n$-dimensional space, with $r$, then the set of equidistant points from $A_0$, $A_1$ and $A_2$ form an $(n-2)$-dimensional affine subspace, which we denote by $\mathcal{C}(\triangle A_0A_1A_2)$. Each of these points is the center of an $n$-dimensional sphere passing through the points $A_0$, $A_1$ and $A_2$. The following theorem allows us to introduce the notion of an “orthocenter” associated with each point in $\mathcal{C}(\triangle A_0A_1A_2)$, and provides a generalization of the notion of C-orthocenter in the plane.

**Theorem 1.** Let $\triangle A_0A_1A_2$ be a triangle in $\mathbb{R}^n$, $G$ its centroid and $H$ its orthocenter. If $P \in \mathcal{C}(\triangle A_0A_1A_2)$ and $S$ is the sphere of center $P$ passing through the points $A_0$, $A_1$ and $A_2$, and $r$ his radius, then the spheres $S_0$, $S_1$, and $S_2$ that are symmetrical to $S$ with respect to the midpoints $M_0$, $M_1$ and $M_2$ of the sides of the triangle $\triangle A_0A_1A_2$ concur in the points $H$ and $H_P = A_0 + A_1 + A_2 - 2P$. Furthermore, the following assertions hold:

1. If $B_0$, $B_1$ and $B_2$ are the centers of $S_0$, $S_1$ and $S_2$ respectively, then the triangles $\triangle A_0A_1A_2$ and $\triangle B_0B_1B_2$ are symmetrical and the center of symmetry is the point $Q_P = \frac{1}{2}(A_0 + A_1 + A_2 - P)$.

2. The points $P$, $G$ and $H_P$ are collinear with $G$ between $P$ and $H_P$, and with $2PG = GH_P$. (Euler property)

3. If $N_0$, $N_1$ and $N_2$ are the midpoints of the sides of the triangle $\triangle B_0B_1B_2$, the sphere $S_H$ of Center $O$ and radius $r/2$ passes through the points $M_0$, $M_1$, $M_2$, $N_0$, $N_1$ and $N_2$. It also passes through the midpoints of the segments that joint $H_P$ with the points of $S$, and the midpoints of the segments that joint $O$ with the points of the sphere $S_H$ of center $H_P$ and radius $r$ (Feuerbach sphere).

4. The points $P$, $Q_P$, $G$, and $H_P$, form a harmonic range, being satisfied $\frac{PG}{OQ_P} = \frac{PH_P}{HQ_P} = 2$.

**Proof.** Since the circumferences $C_0$, $C_1$ and $C_2$ are included in $S_0$, $S_1$ and $S_2$ respectively, then the point $H$ is in the spheres $S_0$, $S_1$ and $S_2$. In order to see $H_P = A_0 + A_1 + A_2 - 2P$ is in $H_P = A_0 + A_1 + A_2 - 2P$, it is enough to take a look at $H_PB_i = r$, for $i = 0, 1, 2$, where $B_i$ is the center of $H_PB_i = r$. Note that $B_i = A_j + A_k - P$, for $\{i, j, k\} = \{0, 1, 2\}$. From which.

$$H_PB_i = \| (A_j + A_k - P) - (A_0 + A_1 + A_2 - 2P) \| = \| P - A_i \| = r$$

1. Note that $A_i + B_i = A_i + A_j + A_k - P$, where $\{i, j, k\} = \{0, 1, 2\}$. Therefore, the midpoint of $A_iB_i$ is $Q_P = \frac{1}{2}(A_0 + A_1 + A_2 - P)$, for $i = 0, 1, 2$.

2. Since $2(G - P) = \frac{2}{3}(A_0 + A_1 + A_2 - 3P) = \frac{2}{3}(H_P - P) = H_P - G$, it follow that $P$, $G$ and $H_P$ are collinear and $2PG = GH_P$. 

3
3. By 1. we know that $\mathcal{A}_{Q_P} (\triangle A_0A_1A_2) = \triangle B_0B_1B_2$, from where $\mathcal{A}_{Q_P} (M_i = N_i)$, for $i = 0, 1, 2$, For the first part only remains to show that $M_iQ_P = r/2$, for $i = 0, 1, 2$. Indeed,

$$M_iQ_P = \left\| \frac{1}{2} (A_0 + A_1 + A_2 - P) - \frac{1}{2} (A_J + A_K) \right\| = \frac{1}{2} \| A_i - P \| = r/2,$$

for $i = 0, 1, 2$.

For the second part note that $\mathcal{H}_{H_P, r/2}(S) = S_M$ $\wedge$ $\mathcal{H}_{P, r/2}(S_H) = S_M$ which implies the assertion.

4. Since 2 holds, $PG = \frac{1}{3} PH_P$. On the other hand

$$GQ_P = \left\| \left(\frac{1}{2} (A_0 + A_1 + A_2 - P) - \frac{1}{3} (A_0 + A_1 + A_2) \right) \right\| = \frac{1}{6} \|(A_0 + A_1 + A_2 - 3P)\| = \frac{1}{6} PH_P$$

and

$$Q_PH_P = \left\| \left( (A_0 + A_1 + A_2 - 2P) - \frac{1}{2} (A_0 + A_1 + A_2 - P) \right) \right\| = \frac{1}{2} \|(A_0 + A_1 + A_2 - 3P)\| = \frac{1}{2} PH_P,$$

Finally, the assertion of the statement follows from the above relations.

We call the point $H_P$ the orthocenter of the triangle $\triangle A_0A_1A_2$ associated to $P$ and the set of all these orthocenters is denote by $\mathcal{H}(\triangle A_0A_1A_2)$. The above theorem says that the Euler property is satisfied,
We know that $W_3 = W_1 = W_2 = 0$. Furthermore, the orthocenter of the triangle $\triangle H_P A_i A_j$ associated to $B_k$ is the point $A_k$, where $\{i, j, k\} = \{0, 1, 2\}$. Thus, the notion of orthocentric system can be generalized to an $n$-dimensional space, and we say that a set of four points $\{A_0, A_1, A_2, A_3\}$ is an orthocentric system, if there is a point $I$ such that the properties about orthocentric systems in the plane previously listed are also valid in this context. In fact, the following lemma is used for this purpose.

**Lemma 2.** The homothetic image of a $C$-orthocentric system is a $C$-orthocentric system.

**Proof.** Let $\{A_0, A_1, A_2, A_3\}$ be a $C$-orthocentric system, then there exists $P_4 \in \mathcal{C}(\triangle A_0 A_1 A_2)$ such that $A_3 = A_0 + A_1 + A_2 - 2P_4$. We will see that the properties about orthocentric systems in the plane previously listed are also valid in this context. In fact, the following lemma is used for this purpose.

Let $B_i = \mathcal{H}_{C, \lambda}(A_i)$, for $i = 0, 1, 2, 3$, and $R = \mathcal{H}_{C, \lambda}(P)$. Clearly $R \in \mathcal{C}(\triangle B_0 B_1 B_2)$ and

$$B_0 + B_1 + B_2 - 2R = (1 - \lambda)C + \lambda A_0 + ((1 - \lambda)C + \lambda A_1) + ((1 - \lambda)C + \lambda A_2) - 2((1 - \lambda)C + \lambda P) = (1 - \lambda)C + \lambda (A_0 + A_1 + A_2 - 2P) = (1 - \lambda)C + \lambda A_3 = A_0,$$

which completes the proof.

**Theorem 3.** Let $\triangle A_0 A_1 A_2$ be a triangle in $R^n$, $G$ its centroid, $P \in \mathcal{C}(\triangle A_0 A_1 A_2)$ and $H_P$ the orthocenter associated with $P$. Then the sets of points $\{A_0, A_1, A_2, H_P\}$, $\{B_0, B_1, B_2, P\}$, $\{M_0, M_1, M_2, O\}$, $\{N_0, N_1, N_2, H_P\}$ and $\{G_0, G_1, G_2, G\}$ are orthocentric systems, where $G_0$, $G_1$ and $G_2$ are the centroids of the triangles $\triangle H_P A_1 A_2$, $\triangle A_0 H_P A_2$ and $\triangle A_0 A_1 H_P$ respectively.

**Proof.** We know that $M_i = \mathcal{H}_{G, -\frac{1}{2}}(A_i)$, for $i = 0, 1, 2, 3$, and $\mathcal{H}_{G, -\frac{1}{2}}(H_P) = \frac{3}{2}G - \frac{1}{2}H_P = P$, from which $\{M_0, M_1, M_2, P\} = \mathcal{H}_{G, -\frac{1}{2}}(\{A_0, A_1, A_2, H_P\})$

If $Q_P = \frac{1}{3}(A_0 + A_1 + A_2 - P)$, then $\mathcal{I}_{Q_P}(P) = H_P$. Thus, $\mathcal{I}_{Q_P}(\{A_0, A_1, A_2, H_P\}) = \{H_0, H_1, H_2, P\}$ and $\mathcal{I}_{Q_P}(\{M_0, M_1, M_2, P\}) = \{N_0, N_1, N_2, H_P\}$.

Finally, $G_i = \frac{A_i + 2A_1 + 2A_2 - 2P}{3}$, from which

$$\mathcal{H}_{Q_P, -\frac{1}{3}}(A_i) = \frac{4}{3}Q_P - \frac{1}{3}A_i = \frac{2}{3}(A_0 + A_1 + A_2 - P) - \frac{1}{3}A_i = G_i$$

and

$$\mathcal{H}_{Q_P, -\frac{1}{3}}(H_P) = \frac{4}{3}Q_P - \frac{1}{3}H_P = \frac{2}{3}(A_0 + A_1 + A_2 - P) - \frac{1}{3}(A_0 + A_1 + A_2 - 2P) = G.$$
Thus, $\mathcal{H}_{Q_p, -1/3} (\{A_0, A_1, A_2, H_P\}) = \{G_0, G_1, G_2, G\}$.

**Theorem 4.** If $\{A_0, A_1, A_2, A_3\}$ is a orthocentric system, then $\overrightarrow{A_iA_j} \perp \overrightarrow{A_kA_l}$ for $\{i, j, k, l\} = \{0, 1, 2, 3\}$.

**Proof.** Since orthogonality in $\mathbb{R}^n$ is equivalent to isosceles orthogonality, we just need to see that $\|\overrightarrow{A_iA_j} - \overrightarrow{A_kA_l}\| = \|\overrightarrow{A_iA_j} + \overrightarrow{A_kA_l}\|$.

Indeed, consider the case; $i = 0$, $j = 1$, $k = 2$, $l = 3$. Let $P \in C(\triangle A_0A_1A_2)$, be such that $A_3 = A_0 + A_1 + A_2 - 2P$ and $r$ be the radius of the sphere with center $P$ passing through $A_0$, $A_1$ and $A_2$. Then

$$\|\overrightarrow{A_0A_1} - \overrightarrow{A_2A_3}\| = \|(A_1 - A_0) - (A_3 - A_2)\| = \|2(P - A_0)\| = 2r$$

and

$$\|\overrightarrow{A_0A_1} + \overrightarrow{A_2A_3}\| = \|(A_1 - A_0) + (A_3 - A_2)\| = \|2(A_1 - P)\| = 2r.$$

The other cases are shown analogously.

The above theorem tells us also that if $\{A_0, A_1, A_2, A_3\}$ is an orthocentric system and $A_3$ is not on the plane determined $A_0$, $A_1$ and $A_2$. Then the tetrahedron $A_0A_1A_2A_3$ is an orthocentric tetrahedron, i.e, the altitudes of this tetrahedron concur.

It is also important to note that, in our proof of the previous theorems we do not use of the orthogonality properties in $\mathbb{R}^n$ hence the results presented here are still valid if we take any norm in $\mathbb{R}^n$, i.e., in Minkowsky spaces in general.

**References**

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