On the constants in some inequalities for the Sobolev norms and pointwise product.

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Abstract

We consider the Sobolev norms of the pointwise product of two functions, and estimate from above and below the constants appearing in two related inequalities.

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1 Introduction.

For $d \in \mathbb{N}_0 := \mathbb{N} \setminus \{0\}$ and $n \in [0, +\infty)$, let us consider the Sobolev (or Bessel potential) space $H^n(\mathbb{R}^d, \mathbb{C})$, with the standard norm $\| \cdot \|_n$; this is defined setting

$$\|f\|_n := \|\sqrt{1-\Delta}^n f\|_{L^2} = \left(\int_{\mathbb{R}^d} dk \, (1+|k|^2)^n |(\mathcal{F}f)(k)|^2 \right)^{\frac{1}{2}},$$  

(1.1)

where $\Delta$ is the Laplacian and $\mathcal{F}$ is the Fourier transform, normalised so that

$$\mathcal{F}f(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx \, e^{-ik\cdot x} f(x).$$  

(1.2)

(For $n$ integer, $\|f\|_n$ can also be expressed in terms of the partial derivatives of $f$ of all orders $\leq n$, see Eq. (3.5)). In the sequel, we consider besides $n$ another real number $a$. The following statement is known in the literature:

1.1 Proposition. i) Let $n, a$ be such that $0 \leq n \leq d/2 < a$. Then, for each $f \in H^a(\mathbb{R}^d, \mathbb{C})$, $g \in H^n(\mathbb{R}^d, \mathbb{C})$ it is $fg \in H^n(\mathbb{R}^d, \mathbb{C})$; also, there is a constant $K_{n,a,d}$ such that

$$\|fg\|_n \leq K_{n,a,d} \|f\|_a \|g\|_n$$  

(1.3)

for all $f \in H^a(\mathbb{R}^d, \mathbb{C})$, $g \in H^n(\mathbb{R}^d, \mathbb{C})$.

ii) Let $n > d/2$. Then, for each $f, g \in H^n(\mathbb{R}^d, \mathbb{C})$ it is $fg \in H^n(\mathbb{R}^d, \mathbb{C})$; furthermore, for each $a$ such that $n \geq a > d/2$ there is a constant $K_{n,a,d}$ such that

$$\|fg\|_n \leq K_{n,a,d} \text{Max} \left(\|f\|_a \|g\|_n, \|f\|_n \|g\|_a\right)$$  

(1.4)

for all $f, g \in H^n(\mathbb{R}^d, \mathbb{C})$.

⋄

Eq.(1.4) makes sense because $H^n \subset H^a$ for $n \geq a$. For $n$ integer, (1.4) follows from the so-called ”Moser calculus inequality” [14] [7] [21]. Both for integer and noninteger $n$ (and $a$), Eqs (1.3) and (1.4) can be obtained specializing to the classical Sobolev spaces a more general result on the Triebel-Lizorkin spaces: see, e.g., [15]. The inequalities in Prop.1.1 have interesting applications to nonlinear PDE’s; in particular, they can be employed to derive ”tame” estimates (in the sense of the Nash-Moser theory) in the case of polynomial nonlinearities [3] [12] (in these applications, one generally asks $n$ to be an arbitrary integer, and $a$ the smallest integer $> d/2$, i.e., $a = [d/2] + 1$).

From here to the end of the paper, we intend that $K_{n,a,d}$ is the sharp (i.e., the minimum) constant satisfying for all $f, g$ the inequality (1.3) if $0 \leq n \leq d/2 < a$, or the inequality (1.4) if $n \geq a > d/2$. 

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Neither in the quoted references, nor in any other of our knowledge, these constants are estimated; this is a nontrivial task, because essentially one must evaluate the quantity \( \| \sqrt{1 - \Delta}^n (fg) \|_{L^2}^2 = \int_{\mathbb{R}^d} dk \ (1 + |k|^2)^n \ |\mathcal{F}(fg)(k)\|^2 \), quartic in the pair \((f, g)\). A direct variational approach seems to be very difficult, even for \(n\) and \(a\) integers (to say the least: the formal stationarity condition of the functional to be maximised for finding \(K_{n,a,d}\) yields a system of cubic PDE’s of order \(2 \max(n, a)\) for the pair \((f, g)\); one can hope to treat it for special values of \(n, a\) at most).

On the other hand explicit estimates on \(K_{n,a,d}\), even non optimal, holding for all values of \(n, a\) (and \(d\)) would be useful for the strictly quantitative aspects of the previously mentioned applications; the aim of the present article is just to give some results of this sort. The paper is organized as follows.

In the next Section we describe the results to be derived in the subsequent ones; these are, essentially, the upper bounds on \(K_{n,a,d}\) coming from a general argument, and the lower bounds obtained by substituting convenient classes of trial functions in Eq.s (1.3) (1.4). These bounds allow to estimate \(K_{n,a,d}\) from above and below, for general values of \(n, a, d\). The explicit numerical values of the bounds are reported for a number of cases with low \(n, a\) and \(d\); for large \(n\), our estimates have the form \(\text{const.}2^n/(n+a)^{a/2+d/4} \lesssim K_{n,a,d} \lesssim \text{const.}2^n\).

In Sect.3 our normalizations and notational conventions on Sobolev spaces and their norms are fixed; also, we report an estimate [13] on the constants in the classical imbedding inequalities of Sobolev into \(L^r\) spaces, to be employed later on.

In Sect.4 we define some auxiliary nonlinear operators \(f \mapsto D_n(f)\) on Sobolev spaces, that will be a basic tool to prove our upper bounds on the constants \(K_{n,a,d}\); as a first step towards this goal, we infer a non conventional ”Leibnitz” inequality for \(D_n(fg)\), where \(f, g\) are two functions.

In Sect.5 we derive some estimates on pointwise products of the form \(D_l(f)D_m(g)\); in the subsequent Sect.6, these estimates are employed to obtain a new proof of Prop.1.1, and to to derive our upper bounds on \(K_{n,a,d}\).

In the final Sect.7, 8 and 9 we derive the already mentioned, different kinds of lower bounds on \(K_{n,a,d}\), corresponding to different trial functions.

2 Description of the results.

Many of our bounds will be given in terms of the usual \(\Gamma\) function, and of the function \(E\) defined by

\[
E(s) := s^s \quad \text{for } s \in (0, +\infty) \quad , \quad E(0) := 1 . \tag{2.1}
\]

We will employ the coefficients

\[
S_{a,d} := \frac{1}{(4\pi)^{d/4}} \sqrt{\frac{\Gamma(a - d/2)}{\Gamma(a)}} , \tag{2.2}
\]
\[ E_{\ell,a,d} := \left( \frac{E \left( \frac{\ell}{2a} \right) E \left( \frac{1}{2} - \frac{\ell}{2a} \right)}{E \left( \frac{1}{2} + \frac{\ell}{2a} \right) E \left( 1 - \frac{\ell}{2a} \right)} \right)^{d/2} \]  

(\( a > d/2, \ 0 \leq \ell \leq a \)); as explained in the sequel, these can be interpreted in terms of the imbedding inequalities of Sobolev into \( L^r \) spaces. One finds by elementary means that the function \( \ell \in [0, a] \mapsto E_{\ell,a,d} \) attains its maximum at \( \ell = 0, \ a \) and its minimum at \( \ell = a/2 \); these are, respectively,

\[ E_{0,a,d} = E_{a,a,d} = 1, \quad E_{a/2,a,d} = (16/27)^{d/4}. \]  

(2.4)

Our estimates will rely on some combinations of the above constants with the binomial coefficients; to deal with Sobolev spaces of noninteger order, certain generalised binomial coefficients will be necessary. For each \( n \in [0, +\infty) \), we will put

\[ n_+ := \text{Min} \{ m \in \mathbb{N} \mid m \geq n \} ; \]  

(2.5)

(clearly, \( n_+ \) is an integer approximation of \( n \) from above, not to be confused with the standard integer part \( \lfloor n \rfloor := \text{maximum integer \leq n} \)).

We will employ the "lattice" (with initial point 0 and final point \( n \))

\[ \Lambda(n) := \{ j \frac{n}{n_+} \mid j = 0, \ldots, n_+ \} \]  

(2.6)

and the coefficients

\[ \binom{n}{\ell}_+ := \binom{n_+}{j} \quad \text{for} \ \ell = j \frac{n}{n_+} \in \Lambda(n). \]  

(2.7)

In the above formula, ( ) denotes the usual binomial coefficients; also, it is understood that \( n_+/n := 1 \) if \( n = 0 \). For any \( n \in \mathbb{N} \), it is \( n_+ = n, \ \Lambda(n) = \{0, 1, \ldots, n\} \), and \( \binom{n}{\ell}_+ \) are the usual binomial coefficients \( \binom{n}{\ell} \). In general, we have

\[ \sum_{\ell \in \Lambda(n)} \binom{n}{\ell}_+ = 2^{n_+}. \]  

(2.8)

**General upper bounds on** \( K_{n,a,d} \). These will be obtained by combining the Sobolev imbedding inequalities of \( H^n \) into \( L^r \) spaces (Sect.3) with a "Leibnitz" inequality for the \( H^n \) norms of products (Sect.4); the final result will be the following.
2.1 Proposition. Let \( d \in \mathbb{N}_0 \) and either \( 0 \leq n \leq d/2 < a \) or \( n \geq a > d/2 \); then the sharp constant in the inequality (1.3) or (1.4) admits the upper bound

\[
K_{n,a,d} \leq S_{a,d} \sum_{\ell \in \Lambda(n)} \left( \frac{n}{\ell} \right) E_{\ell,a,d},
\]

where the coefficients in the r.h.s. are defined as follows. In the case \( 0 \leq n \leq d/2 < a \), we put

\[
E_{n,\ell,a,d} := E_{\ell,a,d},
\]

while for \( n \geq a > d/2 \) we put

\[
E_{n,\ell,a,d} := \begin{cases} 
E_{\ell,a,d} & \text{if } 0 \leq \ell < a/2 , \\
(16/27)^{d/4} & \text{if } a/2 \leq \ell \leq n - a/2 , \\
E_{n-\ell,a,d} & \text{if } n - a/2 < \ell \leq n .
\end{cases}
\]

Eqs (2.9-2.11) imply weaker bounds: for all \( n, a \), they give

\[
K_{n,a,d} \leq S_{a,d}^{2^{n^+}}
\]

and, for \( n \geq a > d/2 \), they imply

\[
K_{n,a,d} \leq (16/27)^{d/4} S_{a,d} u_{n,a,d} 2^{n^+} ,
\]

\[
u_{n,a,d} := 1 + \left( \frac{27}{16} \right)^{d/4} - 1 \left( \frac{n^+}{n^+ - a_n + 1} \right), \quad a_n := \left( \frac{n^+}{n} \cdot \frac{a}{2} \right)^+ .
\]

(Note that, by construction, \( \lim_{n \to +\infty} u_{n,a,d} = 1 \) for fixed \( a \) and \( d \).)

General method to derive lower bounds on \( K_{n,a,d} \). Of course, Eqs (1.3) (1.4) imply

\[
K_{n,a,d} \geq \frac{\|fg\|_n}{\|f\|_a \|g\|_n}
\]

for \( 0 \leq n \leq d/2 < a \) and any nonzero \( f \in H^a(\mathbb{R}^d, \mathbb{C}) \), \( g \in H^a(\mathbb{R}^d, \mathbb{C}) \), and

\[
K_{n,a,d} \geq \frac{\|fg\|_n}{\max(\|f\|_a \|g\|_n, \|f\|_n \|g\|_a)}
\]

for \( n \geq a > d/2 \) and any nonzero \( f, g \in H^a(\mathbb{R}^d, \mathbb{C}) \).

All the results presented hereafter will be derived inserting convenient trial functions \( f, g \) into Eqs (2.15) (2.16). In particular, we will use the functions (or some rescaled variants)

\[
f_{n,d} := F^{-1} \left( \frac{1}{(1 + |k|^2)^n} \right) \quad (n > 0) ,
\]

where \( F \) is the Fourier transform.
where \( \mathcal{F} \) is again the Fourier transform, and \( |k| \) denotes the map \((k_1, \ldots, k_d) \in \mathbb{R}^d \mapsto \sqrt{k_1^2 + \cdots + k_d^2}\). From the above definition, it is clear that \( f_{n,d} \in H^n(\mathbb{R}^d, \mathbb{C}) \) if \( n > d/2 \). By a known formula for radially symmetric Fourier transforms [3], we have

\[
 f_{n,d} = \frac{|x|^{n-d/2}}{2^{n-1}\Gamma(n)} K_{n-d/2}(|x|),
\]

(2.18)

where \( x \) is the function \( x \mapsto \sqrt{x_1^2 + \cdots + x_d^2} \) and \( K(\cdot) \) are the modified Bessel functions of the third kind, or Macdonald functions, see, e.g., [19].

Another family of trial functions, useful for our purposes, is made of the functions

\[
 x = (x_1, \ldots, x_d) \mapsto f_{p, \sigma, d}(x) := e^{ipx_1} e^{-\left(\sigma/2\right)|x|^2} \quad (p, \sigma \in (0, +\infty))
\]

(2.19)

(with \( |x| := \sqrt{x_1^2 + \cdots + x_d^2} \)); here, the Fourier character \( e^{ipx_1} \) is regularised at infinity by the rapidly vanishing, Gaussian factor \( e^{-\left(\sigma/2\right)|x|^2} \). We shall mainly deal with these functions in the limit of small \( \sigma \) and large \( p \).

Let us present three kinds of lower bounds obtained from the above trial functions; for fixed \( d \) and \( a \), these are interesting for \( n \) very low, \( n \) close to \( a \) and \( n \) large, respectively.

**Ground level** lower bounds on \( K_{n,a,d} \). First of all one can show that, in any case, \( K_{n,a,d} \) is bounded from below by a constant independent of \( n \). Either in Eq.(2.15) or in (2.16), we insert the trial functions \( f := f_{a,d} \) (see Eq.s (2.17) (2.18)) and \( g := \) a smooth approximant of the Dirac \( \delta \) distribution. This yields the following estimate.

**2.2 Proposition.** Let \( d \in \mathbb{N}_0 \) and either \( 0 \leq n \leq d/2 < a \) or \( n \geq a > d/2 \); then

\[
 K_{n,a,d} \geq S_{a,d}.
\]

(2.20)

\( \diamond \)

The constant \( S_{a,d} \) is the same appearing in the upper bounds (2.9). In particular, for \( n = 0 \) the upper and lower bounds (2.3) (2.20) coincide, allowing to individuate the sharp constant; in other terms, we have

**2.3 Corollary.** For \( d \in \mathbb{N}_0 \), \( a > d/2 \) and \( n = 0 \) the sharp constant in the inequality (1.3) is

\[
 K_{0,a,d} = S_{a,d}.
\]

(2.21)

\( \diamond \)
For \( n \) low, the upper and lower bounds \((2.9)-(2.20)\) are not far, thus confining the sharp constant to a fairly small interval; for example, if \( n = 1 \leq d/2 \) this interval is described by the inequalities

\[
S_{a,d} \leq K_{1,a,d} \leq S_{a,d} (1 + E_{1,a,d}) .
\] (2.22)

“Bessel” lower bounds on \( K_{n,a,d} \). We assume \( n \geq a > d/2 \), and insert into Eq.(2.16) the functions \( f := g := f_{\lambda,n,d} \), where

\[
f_{\lambda,n,d} := \frac{1}{\lambda^d} F^{-1} \left( \frac{1}{(1 + |k|^2/\lambda^2)^n} \right) \quad (\lambda > 0) ;
\] (2.23)

these come from rescaling by a factor \( \lambda \) of the function \( f_{n,d} \) in Eq.(2.17), i.e., \( f_{\lambda,n,d}(x) = f_{n,d}(\lambda x) \). We write down Eq.(2.16) for these functions and maximise w.r.t. \( \lambda \); the conclusions stemming from this analysis can be summarised as follows.

2.4 Proposition. For \( d \in \mathbb{N}_0, n \geq a > d/2 \) the sharp constant in Eq.(1.4) is such that

\[
K_{n,a,d} \geq \sup_{\lambda > 0} \frac{\| f_{\lambda,n,d}^2 \|_n}{\| f_{\lambda,n,d} \|_a \| f_{\lambda,n,d} \|_n} .
\] (2.24)

The norms in the r.h.s. of Eq.(2.24) can be expressed in terms of hypergeometric functions (or one-dimensional integrals of them).

The explicit expressions of the above norms will be given in Sect.8. Experimentally, the lower bounds \((2.24)\) are not too far from the upper bounds \((2.9)\) when \( n \) and \( a \) are fairly close to \( d/2 \).

Numerical examples corresponding to the upper bounds \((2.9)\) and to the “ground level” and “Bessel” lower bounds. The numerical values reported hereafter have been obtained from the (analytical) estimates mentioned previously by means of the MATHEMATICA package. We consider the cases \( d = 1, 2, 3, a = [d/2] + 1 \) and \( n \) integer, \( 0 \leq n \leq a \).

Let \( d = 1, a = 1 \); for \( n = 0 \) we apply Eq.(2.21), whereas for \( n = 1 \) we use Eq.s \((2.24)-(2.9)\); this yields the estimates

\[
0.71 < K_{0,1,1} = 1/\sqrt{2} < 0.72 ,
\] (2.25)

\[
0.84 < K_{1,1,1} < 1.42 .
\] (2.26)

Let \( d = 2, a = 2 \); using Eq.(2.21) for \( n = 0 \), Eq.(2.22) for \( n = 1 \) and Eq.s \((2.24)-(2.9)\) for \( n = 2 \) we get, respectively,

\[
0.27 < K_{0,2,2} = 1/(2\sqrt{\pi}) < 0.28 ,
\] (2.27)
Finally, let $d = 3, a = 2$; using again Eq.\((2.21)\) for $n = 0$, Eq.\((2.22)\) for $n = 1$ and Eq.s \((2.24)\) \((2.29)\) for $n = 2$ we get, respectively,

\[
0.19 < K_{0,2,3} = 1/(2\sqrt{2\pi}) < 0.20 ,
\]

\[
0.19 < K_{1,2,3} < 0.34 ,
\]

\[
0.24 < K_{2,2,3} < 0.67 .
\]

"Fourier" lower bounds on $K_{n,a,d}$. We insert into Eq.\((2.15)\) or \((2.16)\) the trial functions $f := g := f_{p,\sigma,d}$, where $f_{p,\sigma,d}(x) := e^{ipx} e^{-(\sigma/2)|x|^2}$ as in Eq. \((2.19)\). As anticipated, the Gaussian factor $e^{-(\sigma/2)|x|^2}$ is used only to regularise at infinity the Fourier character $e^{ipx}$; it will be ultimately taken as close as possible to unity, setting $\sigma$ close to zero. The norms of the functions \((2.19)\) are evaluated in Sect.\(9\); in the same Section, we will subsequently choose $p = \text{const.}\sqrt{n + a}, \sigma = \text{const.}/\sqrt{n + a}$. The estimate arising in this way is interesting for $n$ great, but can be nominally written for small $n$ too; it can be expressed as follows.

2.5 Proposition. Let $d \in \mathbb{N}_0$, and either $1/2 \leq n \leq d/2 < a$ or $n \geq a > d/2$; then, the sharp constant in Eq.\((1.3)\) or \((1.4)\) admits the lower bounds

\[
K_{n,a,d} \geq R_{a,d} v_{n,a,d} \frac{2^n}{(n + a)^{a/2 + d/4}} ,
\]

\[
R_{a,d} := \frac{e^{-a/2}}{(2\pi)^d/4} \sqrt{E(d/2)E(a - d/2)} ,
\]

\[
v_{n,a,d} := \left(1 - \frac{d}{2(n + a)} + \frac{d^2 an}{4(n + a)^4}\right)^{d/4} \frac{1}{e} \frac{(4a - d)dn + 2da^2}{8(n + a)^2 - 4dn} - \frac{da^2}{4(n + a)^2 - 2da} .
\]

(Note that $\lim_{n \to +\infty} v_{n,a,d} = 1$ for fixed $a$ and $d$).

For $n \geq a > d/2$, Eq.\((2.33)\) implies the weaker bound

\[
K_{n,a,d} \geq R_{a,d} \left(1 - \frac{d}{2a}\right)^{d/4} \frac{2^n}{(n + a)^{a/2 + d/4}} .
\]

\[
\diamondsuit
\]

The $n \to +\infty$ limit for the constants $K_{n,a,d}$. From Prop.s \(2.1\) and \(2.5\) we see that $K_{n,a,d}$ has upper and lower bounds behaving essentially as $2^n$ and $2^n/n^{a/2 + d/4}$, respectively. These yield upper and lower bounds for the ratio $(1/n) \log_2 K_{n,a,d}$, both converging to 1 for $n \to \infty$. So, Prop.s \(2.1\) and \(2.5\) imply
2.6 Corollary. For fixed \(a\) and \(d\), it is

\[
\lim_{n \to +\infty} \frac{\log_2 K_{n,a,d}}{n} = 1.
\]

(2.37)

\[\Box\]

3 Some facts on Sobolev spaces.

Notations. We stick to the previous paper [13]. As usually: \(d \in \mathbb{N}_0 = \mathbb{N} \setminus \{0\}\) is a fixed space dimension; the running variable in \(\mathbb{R}^d\) is \(x = (x_1, \ldots, x_d)\), and \(k = (k_1, \ldots, k_d)\) when using the Fourier transform; we write \(|x|\) for the function \((x_1, \ldots, x_d) \mapsto \sqrt{x_1^2 + \ldots + x_d^2}\); and intend \(|k|\) similarly. We denote with

\[
\mathcal{F}, \mathcal{F}^{-1} : S'(\mathbb{R}^d, \mathbb{C}) \to S'(\mathbb{R}^d, \mathbb{C})
\]

(3.1)

the Fourier transform of tempered distributions and its inverse, choosing normalizations so that (1.2) holds (whenever the integral makes sense, say, for \(f\) in \(L^1(\mathbb{R}^d, \mathbb{C})\)). The restriction of \(\mathcal{F}\) to \(L^2(\mathbb{R}^d, \mathbb{C})\), with the standard inner product and the associated norm \(\| \cdot \|_{L^2}\), is a Hilbertian isomorphism.

Consider a real number \(n \geq 0\), and introduce the operator

\[
S'(\mathbb{R}^d, \mathbb{C}) \to S'(\mathbb{R}^d, \mathbb{C}), \quad f \mapsto \sqrt{1 - \Delta}^n f := \mathcal{F}^{-1} \left( \sqrt{1 + |k|^2}^n \mathcal{F} f \right)
\]

(3.2)

(if \(n\) is an even integer, this is a power of 1 minus the distributional Laplacian \(\Delta\), in the elementary sense). The \(n\)-th order Sobolev (or Bessel potential) space of \(L^2\) type, denoted with \(H^n(\mathbb{R}^d, \mathbb{C})\), and its norm \(\| \cdot \|_n\) are defined as follows:

\[
H^n(\mathbb{R}^d, \mathbb{C}) := \left\{ f \in S'(\mathbb{R}^d, \mathbb{C}) \mid \sqrt{1 - \Delta}^n f \in L^2(\mathbb{R}^d, \mathbb{C}) \right\} = \left\{ f \in S'(\mathbb{R}^d, \mathbb{C}) \mid \sqrt{1 + |k|^2}^n \mathcal{F} f \in L^2(\mathbb{R}^d, \mathbb{C}) \right\}
\]

(3.3)

\[
\| f \|_n := \| \sqrt{1 - \Delta}^n f \|_{L^2} = \| \sqrt{1 + |k|^2}^n \mathcal{F} f \|_{L^2}.
\]

(3.4)

Of course, if \(n \leq n'\), it is \(H^{n'}(\mathbb{R}^d, \mathbb{C}) \subset H^n(\mathbb{R}^d, \mathbb{C})\) and \(\| \cdot \|_n \leq \| \cdot \|_{n'}\); also, \(H^0 = L^2\).

The connections between \(H^n\) spaces and Bessel functions are known after [3]. For completeness, let us write down the expression of \(\| \cdot \|_n\) in terms of distributional derivatives, when \(n\) is integer; in this case, we have

\[
\| f \|_n = \sqrt{\sum_{m=0}^{n} \binom{n}{m} \sum_{\alpha \in \mathbb{N}_d, |\alpha| = m} \frac{m!}{\alpha!} \| \partial^\alpha f \|_{L^2}^2},
\]

(3.5)
where, for each \( \alpha = (\alpha_1, \ldots, \alpha_d) \), it is intended that \( \partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} \), \( |\alpha| := \alpha_1 + \ldots + \alpha_d \), \( \alpha! := \alpha_1! \cdots \alpha_d! \). Eq.\( (3.3) \) can be derived with appropriate manipulations from Eq. \( (3.4) \); the sums over \( m \) and \( \alpha \) come essentially from the expansion of \( (1 - \Delta)^n \) or \( (1 + |k|^2)^n \).

**Imbedding and interpolation inequalities.** For real \( r \geq 2 \) or \( r = \infty \), we consider the space \( L^r(\mathbb{R}^d, \mathbb{C}) \) and its norm \( \| \cdot \|_{L^r} \). For each real \( n \geq 0 \), \( \| \cdot \|_n \) is again the Sobolev norm \( (3.4) \); \( E(\cdot) \) is the function of Eq.(2.1).

The imbedding inequalities of \( H^n \) into \( L^r \) spaces are well known; in this paper, we will use the following estimate on the imbedding constants \( [13] \).

### Proposition (Imbedding inequality).

Let \( d \in \mathbb{N}_0 \), and either

\[
\begin{align*}
    n &= 0, \quad r = 2, \quad \text{or} \quad 0 < n < d/2, \quad 2 \leq r < \frac{d}{d/2 - n}, \quad \text{or} \\
    n &= d/2, \quad 2 \leq r < \infty, \quad \text{or} \quad n > d/2, \quad 2 \leq r \leq \infty.
\end{align*}
\]

Then \( H^n(\mathbb{R}^d, \mathbb{C}) \subset L^r(\mathbb{R}^d, \mathbb{C}) \), and for each \( f \in H^n(\mathbb{R}^d, \mathbb{C}) \) it is

\[
\|f\|_{L^r} \leq S_{r,n,d} \|f\|_n,
\]

\[
S_{r,n,d} := \frac{1}{(4\pi)^{d/4 - d/(2r)}} \left( \frac{\Gamma\left(\frac{n}{1-2/r} - \frac{d}{2}\right)}{\Gamma\left(\frac{n}{1-2/r}\right)} \right)^{1/2-1/r} \left( \frac{E(1/r)}{E(1-1/r)} \right)^{d/2} \quad (r \neq 2, \infty),
\]

\[
S_{2,n,d} := 1, \quad S_{\infty,n,d} := \frac{1}{(4\pi)^{d/4}} \left( \frac{\Gamma(n-d/2)}{\Gamma(n)} \right)^{1/2}.
\]

\( S_{r,n,d} \) is the sharp \( (i.e., \text{minimum}) \) imbedding constant for \( n \geq 0, \ r = 2, \) and \( n > d/2, \ r = \infty; \) concerning the latter case, the equality \( \|f\|_{L^\infty} = S_{\infty,n,d}\|f\|_n \) holds if \( f = f_{n,d} = \text{the function in Eq.s (2.17) (2.18)}. \)

**Remarks.**

i) The constant denoted with \( S_{a,d} \) in Eq.(2.2) is just \( S_{\infty,a,d} \) with the notations of the above Prop.3.1.

ii) The estimates of Prop.3.1 on the imbedding constants come from application of the Hausdorff-Young and Hölder inequalities to the Fourier transform, as shown in \([13]\); they were previously written, for particular cases, in a number of works mentioned therein.

iii) For \( 0 < n < d/2 \), the imbedding inequality of \( H^n(\mathbb{R}^d, \mathbb{C}) \) into \( L^r(\mathbb{R}^d, \mathbb{C}) \) still holds in the limit case \( r = d/(d/2 - n) \), not covered by the previous Proposition.

An analysis of the reliability of the estimates \((3.8) \ (3.9)\) was made in \([13]\); here, the statement that \( S_{r,n,d} \) is the sharp imbedding constant for \( r = 2 \) and \( n > d/2, \ r = \infty \).
was completed showing that $S_{r,n,d}$ is generally very close to the (unknown) sharp constant for $n > d/2$ and arbitrary $r$ in $\mathbb{R}$.

For $0 < n \leq d/2$, one could derive different estimates on the imbedding constants using the Hardy-Littlewood-Sobolev inequality, with the method indicated in [11] and the expression determined in [8] for the sharp Hardy-Littlewood-Sobolev constant. (The imbedding constants derived in this way become, in the particular cases $r = d/(d/2 - n)$, $n = 1$ and 2, the constants determined in papers [4] [10] prior to [8], and in [18], respectively; these are sharp for the inequalities considered therein, strictly related to the above indicated cases of (3.7)). Nevertheless, all the numerical experiments we performed using the alternative, Hardy-Littlewood-Sobolev estimates on the imbedding constants yielded no essential improvement for the main purpose of the present paper, i.e., estimating the constants $K_{n,a,d}$ in the product inequalities (1.3) (1.4). For this reason, whenever the imbedding constants will be needed in the sequel, we will always stick to the result of Prop.3.1.

The second inequality we need in the sequel is the following, known in the literature.

3.2 Proposition (Multiplicative interpolation inequality). Let $(b,b'),(i,i'),(c,c') \in [0, \infty)^2$; assume $(i,i')$ to lie on the segment of extremes $(b,b')$ and $(c,c')$, i.e.,

$\{(i,i') = (1-t)(b,b') + t(c,c') \text{ with } t \in [0,1]\}$

Then, for each $f \in H^{Max(b,c)}(\mathbb{R}^d, C)$, $g \in H^{Max(b',c')}(\mathbb{R}^d, C)$ it is

$$\|f\|_i \|g\|_{i'} \leq Max\left(\|f\|_b \|g\|_{b'}, \|f\|_c \|g\|_{c'}\right) \tag{3.10}$$

Proof. We have the inequalities (à la Browder-Ehrling-Gagliardo-Nirenberg [4])

$$\|f\|_i \leq \|f\|_b^{1-t} \|f\|_c^t \tag{3.11}$$

(understanding that $0^t := 0$; these can be derived from the Fourier representation (3.4) of the Sobolev norms and Hölder’s inequality). The thesis (3.10) follows multiplying, and recalling the elementary inequality $v^{1-t}w^t \leq \text{Max}(v,w)$ for $v, w \in [0, +\infty)$ and $t \in [0,1]$. (Similar arguments were employed in [14], and in [3] for the Sup-norms).  

4 Operators $D_n$. "Leibnitz" inequality.

As anticipated, these maps will be basic in our strategy to estimate the pointwise product of two functions.
4.1 Definition. For each \( n \in [0, +\infty) \), we put

\[
\mathcal{D}_n : H^n(\mathbb{R}^d, \mathbb{C}) \rightarrow L^2(\mathbb{R}^d, \mathbb{C}) \, , \quad f \mapsto \mathcal{D}_n(f) := \mathcal{F}^{-1} \left( \sqrt{1 + |k|^2}^n |\mathcal{F}f| \right) .
\] (4.1)

The notation \(|\mathcal{F}f|\) stands for the function \( \mathbb{R}^d \rightarrow [0, +\infty), \ k \mapsto |(\mathcal{F}f)(k)| \); due to the presence of the modulus \(||\), the map \( \mathcal{D}_n \) is homogeneous but nonlinear, and differs (on its domain) from the linear operator \( \sqrt{1 - \Delta}^n \) of Eq.(3.2). However, comparing with the Fourier representations of Sobolev spaces and their norms in Eqs (3.3) (3.4), we see that the definition of \( \mathcal{D}_n \) is well posed, and that

\[
\|f\|_n = \|\mathcal{D}_n(f)\|_{L^2} \quad \text{for } n \in [0, +\infty), \ f \in H^n(\mathbb{R}^d, \mathbb{C}) ;
\]

(4.2)

\[
\mathcal{D}_\ell \left( H^{\ell+m}(\mathbb{R}^d, \mathbb{C}) \right) \subset H^m(\mathbb{R}^d, \mathbb{C}), \quad \|\mathcal{D}_\ell(f)\|_m = \|f\|_{\ell+m} ,
\]

(4.3)

\[
\mathcal{D}_m(\mathcal{D}_\ell(f)) = \mathcal{D}_{\ell+m}(f) \quad \text{for } \ell, m \in [0, +\infty), \ f \in H^{\ell+m}(\mathbb{R}^d, \mathbb{C}) .
\]

Let us consider the action of \( \mathcal{D}_n \) on the pointwise product of two functions \( f \) and \( g \), and obtain an inequality concerning \( \|\mathcal{D}_n(fg)\|_{L^2} = \|fg\|_n \); this is stated in the following Proposition, to be proved after two Lemmas.

4.2 Proposition ("Leibnitz" inequality). Let \( n \in [0, +\infty), \) and \( f, g \in H^n(\mathbb{R}^d, \mathbb{C}) \) be such that \( \mathcal{D}_\ell(f)\mathcal{D}_{n-\ell}(g) \in L^2(\mathbb{R}^d, \mathbb{C}) \) for each \( \ell \in \Lambda(n) \). Then \( fg \in H^n(\mathbb{R}^d, \mathbb{C}) \), and

\[
\|fg\|_n \leq \sum_{\ell \in \Lambda(n)} \binom{n}{\ell} \|\mathcal{D}_\ell(f)\mathcal{D}_{n-\ell}(g)\|_{L^2} .
\]

(4.4)

Remarks. i) Recall that \( \Lambda(n) \), \( \binom{n}{\ell} \) are defined by Eqs (2.6) (2.7).

ii) For \( n \) integer, one could express \( \|fg\|_n \) via Eq.(3.5), and the partial derivatives \( \partial^n(fg) \) appearing therein in terms of products \( \partial^\lambda f \partial^\mu g \) (\( \lambda, \mu \) multiindices), with the usual Leibnitz rule. Apart from working also for noninteger \( n \), the estimate (4.4) is more efficient for evaluating the constants \( K_{n,a,d} \) in Eqs (1.3) (1.4).

Here are the two Lemmas to be employed for proving Prop.4.2.

4.3 Lemma. For all real \( a, b > 0, n \geq 0 \) it is

\[
(a + b)^n \leq \sum_{\ell \in \Lambda(n)} \binom{n}{\ell} a^\ell b^{n-\ell} .
\]

(4.5)
Proof. It is obvious if \( n = 0 \). Let \( n > 0 \); being \( 0 < n/n_+ \leq 1 \), we have
\[
(a + b)^n = \left((a + b)^{n/n_+}\right)^{n_+} \leq \left(a^{n/n_+} + b^{n/n_+}\right)^{n_+};
\]
the thesis follows expanding the last expression with the standard binomial formula for integer exponent \( n_+ \). \( \diamond \)

4.4 Lemma. For all \( \xi, \eta \in \mathbb{R}^d \), it is
\[
\sqrt{1 + |\xi + \eta|^2} < \sqrt{1 + |\xi|^2} + \sqrt{1 + |\eta|^2}.
\]
Proof. It is: \( 1 + |\xi + \eta|^2 \leq 1 + |\xi|^2 + |\eta|^2 + 2 |\xi| |\eta| \leq (1 + |\xi|^2) + (1 + |\eta|^2) + 2\sqrt{1 + |\xi|^2} \sqrt{1 + |\eta|^2} \).

The last tool we need to prove Prop. 4.2 is the convolution product. Let us write \( F * G \) for the convolution of two distributions \( F, G \in \mathcal{D}'(\mathbb{R}^d, \mathbb{C}) \), whenever this exists as an element of \( \mathcal{D}'(\mathbb{R}^d, \mathbb{C}) \) (see, e.g., [17]). We have
\[
(F * G)(k) = \int_{\mathbb{R}^d} dh \, F(k - h)G(h)
\]
if \( F, G \in L^1_{loc}(\mathbb{R}^d, \mathbb{C}) \) and the integral in the r.h.s. exists, defining an \( L^1_{loc} \) function of \( k \); these conditions are satisfied, in particular, if \( F, G \in L^2(\mathbb{R}^d, \mathbb{C}) \), which is the case considered in the forthcoming proof. With the chosen normalizations for \( F \) and \( * \), we have \( F(fg) = (2\pi)^{-d/2} (F f) * (F g) \) for sufficiently regular tempered distributions \( f \) and \( g \), e.g., for \( f, g \in L^2(\mathbb{R}^d, \mathbb{C}) \).

Proof of Prop. 4.2. We put for brevity \( F := Ff \), \( G := Fg \). By the relations between \( F \), the \( L^2 \) norm and the convolution, we see that the thesis (4.4) is proved if we show that
\[
\sqrt{1 + |k|^2}^n |F * G| \leq \int_{\mathbb{R}^d} dh \, \sqrt{1 + |k|^2}^n |F(k - h)| |G(h)|
\]
(4.8)
(4.9)

(note that the convolutions in the r.h.s are \( L^2 \) by our assumptions). In order to derive Eq. (4.8) we observe that, for \( k \in \mathbb{R}^d \),
\[
\sqrt{1 + |k|^2}^n |F * G|(k) \leq \int_{\mathbb{R}^d} dh \, \sqrt{1 + |k|^2}^n |F(k - h)| |G(h)| \leq \int_{\mathbb{R}^d} dh \, \left(\sqrt{1 + |k - h|^2} + \sqrt{1 + |h|^2}\right)^n |F(k - h)| |G(h)| \leq \sum_{\ell \in \Lambda(n)} \binom{n}{\ell} \int_{\mathbb{R}^d} dh \, \sqrt{1 + |k - h|^2}^\ell |F(k - h)| \sqrt{1 + |h|^2}^{n-\ell} |G(h)| ;
\]
in the last two steps, we have employed Lemmas 4.4 and 4.3. Summing up, we have the pointwise inequality
\[ \sqrt{1 + |k|^2} |F \ast G| \leq \sum_{\ell \in \Lambda(n)} \left( \frac{n}{\ell} \right) \left( \sqrt{1 + |k|^2} \right) \left( \sqrt{1 + |k|^2}^{-n+\ell} |G| \right). \] (4.9)

The functions in the r.h.s. are $L^2$, so the same happens for the l.h.s.; taking the $L^2$ norms of both sides, we get the desired inequality in (4.8). \[ \diamond \]

5 Inequalities for products $\mathcal{D}_\ell(f) \mathcal{D}_m(g)$.

Our approach is similar to the usual argument for proving the Moser calculus inequality \[ ]^{14} \[ ]^{17} \[ ]^{21}, with the following differences: the $\mathcal{D}_\ell$ operators replace systematically the partial derivatives appearing in the cited works, and all the constants are estimated. In the sequel, $d \in \mathbb{N}_0$ is an arbitrary space dimension.

5.1 Lemma. Let $a, \ell, m$ be real numbers such that $a > d/2$, $0 \leq \ell \leq a$, $m \geq 0$. For each $f \in H^a(\mathbb{R}^d, C)$, $g \in H^{\ell+m}(\mathbb{R}^d, C)$ it is $\mathcal{D}_\ell(f) \mathcal{D}_m(g) \in L^2(\mathbb{R}^d, C)$, and
\[ \|\mathcal{D}_\ell(f) \mathcal{D}_m(g)\|_{L^2} \leq E_{\ell, a, d} S_{a, d} \|f\|_a \|g\|_{\ell+m}, \] (5.1)
with $S_{a, d}$, $E_{\ell, a, d}$ defined as in Eq.s (2.2) (2.3).

Proof. We put
\[ p := \frac{2a}{\ell}, \quad q := \frac{2a}{a-\ell} \]
(intending $1/0 := \infty$); by construction $p, q \geq 2$ and $1/p + 1/q = 1/2$. By the Hölder and the imbedding inequality (Prop.5.1), the functions we consider are in the spaces indicated below, and
\[ \|\mathcal{D}_\ell(f) \mathcal{D}_m(g)\|_{L^2} \leq \|\mathcal{D}_\ell(f)\|_{L^p} \|\mathcal{D}_m(g)\|_{L^q} \leq \]
\[ \leq \left( S_{p, a-\ell, d} \|\mathcal{D}_\ell(f)\|_{a-\ell} \right) \left( S_{q, \ell, d} \|\mathcal{D}_m(g)\|_{\ell} \right) = S_{p, a-\ell, d} S_{q, \ell, d} \|f\|_a \|g\|_{\ell+m} \cdot \] On the other hand, using Eq.s (3.8)(3.9) with the above definitions of $p$ and $q$, and comparing with Eq.s (2.2) (2.3), one checks the equality $S_{p, a-\ell, d} S_{q, \ell, d} = E_{\ell, a, d} S_{a, d}$, yielding the thesis. \[ \diamond \]

5.2 Lemma. Let $a, \ell, m$ be real numbers such that $a > d/2$, $\ell, m \geq a/2$. For each $f, g \in H^{\ell+m}(\mathbb{R}^d, C)$ it is $\mathcal{D}_\ell(f) \mathcal{D}_m(g) \in L^2(\mathbb{R}^d, C)$ and
\[ \|\mathcal{D}_\ell(f) \mathcal{D}_m(g)\|_{L^2} \leq \left( \frac{16}{27} \right)^{d/4} S_{a, d} \max(\|f\|_a \|g\|_{\ell+m}, \|f\|_{\ell+m} \|g\|_a), \] (5.2)
where $S_{a, d}$ is again as in Eq.(2.3).
Proof. We use the relation $1/2 + 1/4 = 1/2$ with the Hölder, the imbedding and the multiplicative interpolation inequality (Prop. 3.2). These ensure that the functions in consideration are in the spaces indicated below, and give the estimates

$$
\|D_\ell(f)D_m(g)\|_{L^4} \leq \|D_\ell(f)\|_{L^4}\|D_m(g)\|_{L^4} \leq \left(S_{4a/2,d}\right)^2 \|D_\ell(f)\|_{a/2}\|D_m(g)\|_{a/2} = \\
= \left(S_{4a/2,d}\right)^2 \|f\|_{\ell+a/2}\|g\|_{m+a/2} \leq \left(S_{4a/2,d}\right)^2 \operatorname{Max}(\|f\|_a, \|g\|_{\ell+m}, \|f\|_{\ell+m}, \|g\|_a).
$$

On the other hand, comparing the definitions (3.8) and (2.2) (2.3) one checks that

$$
\left(S_{4a/2,d}\right)^2 = (16/27)^{d/4} S_{a,d} \text{ (this also equals } E_{a/2,a,d} S_{a,d} \text{). The proof is concluded.}
$$

6 Proof of Prop. 2.1: upper bounds on $K_{n,a,d}$.

Our approach will rely on the Leibnitz inequality (Prop. 4.2) and Lemmas 5.1, 5.2; by the way, the argument employed to derive the upper bounds of Prop. 2.1 on the constants $K_{n,a,d}$ will also give a non conventional proof of Prop. 1.4. We divide the proof in some steps.

Proof of Eqs. (2.9) (2.10), case $0 \leq n \leq d/2 < a$. Prop. 4.2 reduces the problem to analysing the products $D_\ell(f)D_{n-\ell}(g) \in L^2(\mathbb{R}^d, \mathbb{C})$ for $\ell \in \Lambda(n) \subset [0,n]$. To estimate them, we use Lemma 5.1 (with $m = n - \ell$); this suffices to get the thesis.

Proof of Eqs. (2.9) (2.10), case $n \geq a > d/2$. Again, we must analyse the products $D_\ell(f)D_{n-\ell}(g)$ for $\ell \in \Lambda(n) \subset [0,n]$.

If $\ell < a/2$, we use Lemma 5.1 (with $m = n - \ell$). If $a/2 \leq \ell \leq n - a/2$, we use Lemma 5.2 (with $m = n - \ell$). Finally, if $n - a/2 < \ell$, we employ Lemma 5.1 with the pairs $(\ell, m)$, $(f, g)$ replaced by $(n - \ell, \ell)$, $(g, f)$. This yields the thesis.

Proof of the weaker bounds (2.12) (2.13). Eq. (2.12) follows trivially from (2.9) (2.10) and the fact that $E_{n,\ell,a,d} \leq 1$ for all $\ell$. Let us prove Eq. (2.13), assuming $n \geq a > d/2$. In this case, from $E_{\ell,a,d} \leq 1$ we infer

$$
K_{n,a,d} \leq S_{a,d} \left( \sum_{\ell < a/2} \binom{n}{\ell} + (16/27)^{d/4} \sum_{a/2 \leq \ell \leq n - a/2} \binom{n}{\ell} + \sum_{\ell > n - a/2} \binom{n}{\ell} \right);
$$

in all sums, it is intended that $\ell$ takes values in the lattice $\Lambda(n)$. The first and the last sum are equal to $\sum_{j=0}^{a_n-1} \binom{n_+}{j}$, with $a_n$ as in Eq. (2.14); this implies

$$
K_{n,a,d} \leq S_{a,d} \left( (16/27)^{d/4} 2^{n_n} + 2 \sum_{j=0}^{a_n-1} \binom{n_+}{j} \left( 1 - (16/27)^{d/4} \right) \right). \quad (6.1)
$$
To go, on we need the elementary inequality \( \binom{m}{j} \leq \left( \frac{m}{m-k} \right) \binom{k}{j} \), holding for \( 0 \leq j \leq k \leq m \) integers. Applying it with \( m = n_+, k = a_n - 1 \) and summing over \( j \) we obtain from (6.1) the thesis (2.13).

7 "Ground level" lower bounds on \( K_{n,a,d} \).

The aim of this Section is to prove Prop.2.2. The major step consists in proving the following

7.1 Lemma. Let \( d \in \mathbb{N}_0 \) and either \( 0 \leq n \leq d/2 < a \) or \( n \geq a > d/2 \); then, the sharp constant in Eq.(1.3) or (1.4) is such that

\[
K_{n,a,d} \geq \frac{|f(0)|}{\|f\|_a} (7.1)
\]

for each nonzero \( f \in H^a(\mathbb{R}^d, \mathbb{C}) \).

Remark. Evaluation of \( f \) at zero makes sense because \( H^a(\mathbb{R}^d, \mathbb{C}) \subset C(\mathbb{R}^d, \mathbb{C}) \).

Proof. The idea is very simple; let us introduce it heuristically, say for \( n \leq d/2 < a \).

Let \( f \in H^a(\mathbb{R}^d, \mathbb{C}) \), and put \( g := \delta \) (the Dirac distribution). Formally, we have \( fg = f(0)\delta \), \( \|fg\|_n = |f(0)| \|\delta\|_n \), \( \|g\|_n = \|\delta\|_n \); inserting these trial functions into Eq.(2.13), and simplifying \( \|\delta\|_n \) as if it were well defined, we get (7.1). All the rest of the proof is simply a rigorization of this idea, also working for \( n \geq a > d/2 \).

First of all, we note that it suffices to prove the thesis (7.1) for \( f = F - 1 \), \( F \in L^\infty(\mathbb{R}^d, \mathbb{C}), F \neq 0 \), \( \text{Supp } F \) bounded

\[
F = \mathcal{F}^{-1} F, \quad F \in L^\infty(\mathbb{R}^d, \mathbb{C}), \quad F \neq 0, \quad \text{Supp } F \text{ bounded} \quad (7.2)
\]

(where \( \text{Supp} \) is the essential support; functions of the above kind are dense in all Sobolev spaces, and the evaluation map \( f \mapsto f(0) \) is continuous in the \( \| \cdot \|_a \) norm). So, let us assume (7.2), and consider a one parameter family of functions

\[
g_\epsilon := \mathcal{F}^{-1} (G_\epsilon) \quad (\epsilon > 0); \quad G_\epsilon(k) := G(\epsilon k) \quad \text{for } k \in \mathbb{R}^d; \quad (7.3)
\]

\[G \in C^1(\mathbb{R}^d, \mathbb{C}), \quad G \neq 0, \quad \text{Supp } G \text{ bounded} \]

The function \( g_\epsilon \) belongs to \( H^n \) for any \( n \geq 0 \); we will ultimately consider the limit \( \epsilon \to 0 \), under which \( G_\epsilon \) behaves like a constant, and \( g_\epsilon \) like a constant \( \times \delta \). We have

\[
\|g_\epsilon\|_n^2 = \int_{\mathbb{R}^d} dk (1 + |k|^2)^n |G(\epsilon k)|^2 = \frac{1}{\epsilon^{2n+d}} \int_{\mathbb{R}^d} dk (\epsilon^2 + |k|^2)^n |G(k)|^2 ,
\]

whence

\[
\|g_\epsilon\|_n \sim \frac{1}{\epsilon^{n+d/2}} \sqrt{\int_{\mathbb{R}^d} |k|^{2n} |G(k)|^2} = \frac{1}{\epsilon^{n+d/2}} \| |k|^n G \|_{L^2} \quad \text{for each } n \geq 0. \quad (7.4)
\]
Due to this asymptotics, for \( n > a \) the product \( \|f\|_a \|g\|_n \) clearly dominates \( \|f\|_n \|g\|_a \) when \( \epsilon \) is small; so, we have

\[
\text{Max}(\|f\|_a \|g\|_n, \|f\|_n \|g\|_a) \sim \frac{1}{\epsilon^{n+d/2}} \|f\|_a \|\mathbf{k}^n G\|_{L^2} \quad \text{for } n \geq a .
\] (7.5)

Let us pass to evaluate \( \|fg_\epsilon\|_n \), for any \( n \geq 0 \). Reexpressing the pointwise product via Fourier transform and convolution, we get

\[
\|fg_\epsilon\|_n^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dk (1+|k|^2)^n |(F \ast G_\epsilon)(k)|^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dp \int_{\mathbb{R}^d} dq \overline{F(p)}F(q) \int_{\mathbb{R}^d} dk (1+|k|^2)^n G(ek-\epsilon p)G(ek-\epsilon q) = \frac{1}{(2\pi)^d \epsilon^{2n+d}} \int_{\mathbb{R}^d} dp \int_{\mathbb{R}^d} dq \overline{F(p)}F(q) \int_{\mathbb{R}^d} dk (\epsilon^2+|k|^2)^n G(k-\epsilon p)G(k-\epsilon q) .
\] (7.6)

In the last two equalities, we have explicitated the convolution and rescaled by \( \epsilon \) the integration variable. With our assumptions on \( G \), it is not difficult to prove that

\[
\text{last integral in Eq.}(7.6) \rightarrow \int_{\mathbb{R}^d} dp \int_{\mathbb{R}^d} dq \overline{F(p)}F(q) \int_{\mathbb{R}^d} dk \epsilon^2 |(\mathbf{k}^n G)(k)|^2 = \left| \int_{\mathbb{R}^d} dp \ F(p) \right|^2 \|\mathbf{k}^n G\|_{L^2}^2 = (2\pi)^d |f(0)|^2 \|\mathbf{k}^n G\|_{L^2}^2 .
\]

Inserting this into Eq.(7.6), we finally get

\[
\|fg_\epsilon\|_n \sim \frac{1}{\epsilon^{n+d/2}} |f(0)| \|\mathbf{k}^n G\|_{L^2} \quad \text{for all } n \geq 0 .
\] (7.7)

Let \( 0 \leq n \leq d/2 < a \). We write Eq.(2.13) with \( f \) and \( g = g_\epsilon \) as above; sending \( \epsilon \) to zero, and using the asymptotics (7.7), (7.4), we get the thesis (7.1). For \( n \geq a > d/2 \), the thesis (7.1) follows from Eq.(2.16) and from the asymptotics (7.7) (7.3). \( \diamond \)

Now, we are ready to give the

**Proof of Prop.2.2** \( K_{n,a,d} \geq S_{a,d} \). Due to the previous Lemma, it suffices to find a nonzero function \( f \in H^a(\mathbb{R}^d, \mathbb{C}) \) such that

\[
\frac{|f(0)|}{\|f\|_a} = S_{a,d} .
\] (7.8)
As observed in [13], this equality holds for \( f := f_{a,d} = \text{the function in Eq.s (2.17) (2.18)}, \) with \( n = a. \)

**Applications.** As previously noted, Prop.2.2 combined with the upper bound (2.9) implies Corollary 2.3, i.e., the equality \( K_{0,a,d} = S_{a,d}. \) Also, the combination of these results confines \( K_{n,a,d} \) to a fairly small interval when \( n \) is low (see, e.g., Eq.(2.22)). For \( d = 1, 2, 3, a = \lfloor d/2 \rfloor + 1 \) and \( n < a \) integer, we find

\[
K_{0,1,1} = \frac{1}{\sqrt{2}}, \quad (7.9)
\]

\[
K_{0,2,2} = \frac{1}{2\sqrt{\pi}}, \quad \frac{1}{2\sqrt{\pi}} \leq K_{1,2,2} \leq \frac{1}{2\sqrt{\pi}} \left( 1 + \frac{4\sqrt{3}}{9} \right), \quad (7.10)
\]

\[
K_{0,2,3} = \frac{1}{2\sqrt{2\pi}}, \quad \frac{1}{2\sqrt{2\pi}} \leq K_{1,2,3} \leq \frac{1}{2\sqrt{2\pi}} \left( 1 + \frac{8}{39/4} \right). \quad (7.11)
\]

These results correspond to the numerical bounds already written in Sect.2. For completeness, let us also give the functions \( f = f_{a,d} \) for the above values of \( a \) and \( d. \) From the general representation (2.18) in terms of the Macdonald functions \( K(\_\_) \) (and from the equality \( \rho^{1/2} K_{1/2}(\rho) = \frac{\sqrt{\pi/2}}{e^{-\rho}} \)) we get

\[
f_{1,1} = \sqrt{\frac{\pi}{2}} e^{-|x|}, \quad f_{2,2} = \frac{1}{2} |x| K_1(|x|), \quad f_{2,3} = \sqrt{\frac{\pi}{8}} e^{-|x|}. \quad (7.12)
\]

8° ”Bessel” lower bounds on \( K_{n,a,d}. \)

These bounds are expressed by Prop.2.4, and rest on the functions \( f_{\lambda,n,d} \) in Eq.(2.23). In this section we will compute the norms of \( f_{\lambda,n,d} \) and of its square \( f_{\lambda,n,d}^2; \) after this, in a number of cases we will explicitate the bound they give on \( K_{n,a,d} \) maximising w.r.t. \( \lambda. \) Our results will be expressed in terms of the Beta function \( B(z,w) = \Gamma(z)\Gamma(w)/\Gamma(z+w) \) and of the Gauss hypergeometric function \( F = \, _2F_1. \)

**8.1 Lemma.** For \( d \in \mathbb{N}_0, n > d/2 \) and \( \lambda > 0 \) it is \( f_{\lambda,n,d} \in H^n(\mathbb{R}^d, C), \) and

\[
\|f_{\lambda,n,d}\|^2_n = \frac{2 \pi^{d/2}}{\Gamma(d/2)\lambda^d} \int_0^{+\infty} ds \, s^{d-1}(1 + \lambda^2 s^2)^n (1 + s^2)^{-2n} = \quad (8.1)
\]

\[
= \frac{\pi^{d/2}}{\Gamma(d/2)\lambda^d} \left[ B(2n - d/2, d/2) \, F(d/2, -n, 1 + d/2 - 2n; \lambda^2) + \right.
\]

\[
\left. + \lambda^{4n-d} \, B(n - d/2, d/2 - 2n) \, F(2n, n - d/2, 1 - d/2 + 2n; \lambda^2) \right].
\]

In particular, for \( n \) integer it is

\[
\|f_{\lambda,n,d}\|^2_n = \frac{\pi^{d/2}}{\Gamma(d/2)\lambda^d} \sum_{\ell=0}^{n} \binom{n}{\ell} B(\ell + d/2, 2n - d/2 - \ell) \lambda^{2\ell}. \quad (8.2)
\]
Proof. We have
\[ \|f_{\lambda,n,d}\|_n^2 = \int_{\mathbb{R}^d} dk \, (1 + |k|^2)^n |F f_{\lambda,n,d}|^2 = \frac{1}{\lambda^{2d}} \int_{\mathbb{R}^d} dk \frac{(1 + |k|^2)^n}{(1 + |k|^2/\lambda)^{2n}} . \] (8.3)

On the other hand, on radially symmetric functions depending only on \( \rho := |k| \), it is \( \int_{\mathbb{R}^d} dk = 2\pi^{d/2}/\Gamma(d/2) \int_0^{+\infty} d\rho \, \rho^{d-1} \); from here, expressing the second integral of Eq. (8.3) in terms of the rescaled radial variable \( s = |k|/\lambda \) we get the first equality (8.1). The second equality (8.1) comes from a known expression of the above integral over \( s \) in terms of hypergeometric functions (in the singular cases \( 2n - d/2 - 1 \in \mathbb{N} \), the first hypergeometric in (8.1) must be appropriately intended, as a limit from nonsingular values). For \( n \) integer, the integral over \( s \) can be computed expanding the binomial \( (1 + \lambda^2 s^2)^n \), and integrating term by term; when this is done, Eq. (8.2) follows recalling that \( \int_0^{+\infty} ds \, s^\alpha/(1 + s^2)^\gamma = (1/2) B(\alpha/2 + 1/2, \gamma - \alpha/2 - 1/2) \).

8.2 Lemma. For \( d \in \mathbb{N}_0 \), \( n \geq a > d/2 \) and \( \lambda > 0 \) it is \( f_{\lambda,n,d} \in H^a(\mathbb{R}^d, C) \), and
\[ \|f_{\lambda,n,d}\|_a^2 = \frac{2 \pi^{d/2}}{\Gamma(d/2)\lambda^d} \int_0^{+\infty} ds \, s^{d-1} \frac{(1 + \lambda^2 s^2)^a}{(1 + s^2)^{2n}} = \frac{\pi^{d/2}}{\Gamma(d/2)\lambda^d} \left[ B(2n - d/2, d/2) \, F(d/2, -a, 1 + d/2 - 2n; \lambda^2) + \lambda^{4n-d} \, B(2n - a - d/2, d/2 - 2n) \, F(2n, 2n - a - d/2, 1 - d/2 + 2n; \lambda^2) \right]. \] (8.4)

In particular, for \( a \) integer it is
\[ \|f_{\lambda,n,d}\|_a^2 = \frac{\pi^{d/2}}{\Gamma(d/2)\lambda^d} \sum_{\ell=0}^{a} \binom{a}{\ell} B(\ell + d/2, 2n - d/2 - \ell) \, \lambda^{2\ell} . \] (8.5)

Proof. Proceed as in the proof of Lemma 8.1.

8.3 Lemma. For \( d \in \mathbb{N}_0 \), \( n > d/2 \) and \( \lambda > 0 \) it is \( f^2_{\lambda,n,d} \in H^n(\mathbb{R}^d, C) \), and
\[ \|f^2_{\lambda,n,d}\|_n^2 = \frac{2 \pi^{d/2}}{\Gamma(d/2)\lambda^d} \frac{\Gamma^2(2n - d/2)}{\Gamma^2(2n)} \times \int_0^{+\infty} ds \, s^{d-1} (1 + 4\lambda^2 s^2)^n F(2n - d/2, n, n + 1/2; -s^2)^2 . \] (8.6)

\[ \diamond \]
Remark. If one is able to express the hypergeometric in the above equation in terms of elementary functions, for some integer \( n = n_0 \), one can derive an expression via elementary functions for all integers \( n \geq n_0 \), by repeated application of differentiation operations (see, e.g., [1]). For \( d \) odd and \( n \) integer it is

\[
F(2n - d/2, n, n + 1/2; -s^2) = \frac{a \text{ polynomial of order } (n - d/2 - 1/2) \text{ in } s^2}{(1 + s^2)^{2n-d/2-1/2}};
\]

inserting this into Eq.(8.6), we can reduce the integral therein to a linear combination of elementary integrals of the form \( \int_0^{+\infty} ds s^\alpha/(1 + s^2)^\gamma \).

\[\diamond\]

Proof of Lemma 8.3. It is \( f^2_{\lambda,n,d} \in H^n(\mathbb{R}^d, \mathbb{C}) \) by Prop.1.1 and the fact that \( f_{\lambda,n,d} \in H^n(\mathbb{R}^d, \mathbb{C}) \). To compute the norm of this function, we start from its Fourier transform. For all (sufficiently regular) radially symmetric functions

\[
f : \mathbb{R}^d \to \mathbb{C}, \quad f(x) = \varphi(r), \quad r := |x|
\]
the Fourier transform \( \mathcal{F}f \) is also radially symmetric and given by [3]

\[
(\mathcal{F}f)(k) = \frac{1}{\rho^{d/2-1}} \int_0^{+\infty} dr r^{d/2} J_{d/2-1}(\rho r) k^2_{n-d/2}(\rho r), \quad \rho := |k|, \quad (8.8)
\]
where \( J_{\cdot} \) are the Bessel functions of the first kind.

In particular, for the radial function \( f = f^2_{\lambda,n,d} \) we obtain from Eqs (2.18) (2.23)

\[
(\mathcal{F} f^2_{\lambda,n,d})(k) = \frac{1}{\rho^{d/2-1}} \int_0^{+\infty} dr r^{d/2} J_{d/2-1}(\rho r) (\lambda r)^{2n-d} 2^{2n-2} \Gamma(2n) K^2_{n-d/2}(\lambda r) =
\]

\[
= \frac{2^{2-2n}}{\Gamma(2n) \lambda^{d/2+1} \rho^{d/2-1}} \int_0^{+\infty} dr r^{2n-d/2} J_{d/2-1}(\rho r) K^2_{n-d/2}(r) =
\]

\[
= \frac{2^{2-2n}}{\Gamma(2n) \lambda^{d/2+1} \rho^{d/2-1}} \frac{\sqrt{\pi}}{2^{d/2}} \frac{\Gamma(n) \Gamma(2n-d/2)}{2 \Gamma(n+1/2)} F(2n-d/2, n+1/2; -1; \frac{1}{4 \lambda^2}) =
\]

\[
= \frac{1}{2^{d/2} \lambda^d} \frac{\Gamma(2n-d/2)}{\Gamma(2n)} F(2n-d/2, n+1/2; -1; \frac{1}{4 \lambda^2}) \quad (8.9)
\]

with \( \rho := |k| \). (In the last three steps: we have rescaled \( r \) into \( r/\lambda \); we have used a known relation between integrals of Bessel functions and \( F \); we have employed the identity \( \Gamma(n) \Gamma(n+1/2) = 2^{1-2n} \sqrt{\pi} \Gamma(2n) \)). Putting (8.9) into the Fourier representation (3.4) of \( \|f^2_{\lambda,n,d}\|_n \), and introducing the scaled radial variable \( s := (1/2)|k|/\lambda = (1/2)(\rho/\lambda) \), we readily obtain the thesis (8.6).

\[\diamond\]

At last, we have explicit expressions for the norms of \( f_{\lambda,n,d} \) and its square, to be inserted into Eq.(2.24); after doing this and maximising with respect to \( \lambda \) we obtain the "Bessel" lower bound on \( K_{n,a,d} \). We will compute this in three cases.
Case $d = 1$, $n = a = 1$. Eq. (7.12) (rescaled by $\lambda$) and Lemma 8.2 give

$$ f_{\lambda,1,1} = \sqrt{\frac{\pi}{2}} e^{-\lambda|x|}, \quad \|f_{\lambda,1,1}\|_1 = \sqrt{\frac{\pi}{2}} \sqrt{\lambda + \frac{1}{\lambda}}. \quad (8.10) $$

To compute the norm of $f_{\lambda,1,1}^2$ we do not even need Lemma 8.3 because $f_{\lambda,1,1}^2 = \frac{\sqrt{\pi}}{2} f_{2\lambda,1,1}$, which implies

$$ \|f_{\lambda,1,1}^2\|_1 = \sqrt{\frac{\pi}{2}} \sqrt{2\lambda + \frac{1}{2\lambda}}. \quad (8.11) $$

The Bessel lower bound for the present case is

$$ K_{1,1,1} \geq \sup_{\lambda > 0} K_{1,1,1}(\lambda), \quad K_{1,1,1}(\lambda) := \frac{\|f_{\lambda,1,1}^2\|_1}{\|f_{\lambda,1,1}\|_1^2}. \quad (8.12) $$

The function $K_{1,1,1}(\lambda)$ attains its absolute maximum at $\lambda = \sqrt{9 + \sqrt{97}}/(2\sqrt{2}) \simeq 1.53$, which yields the lower bound reported in Eq. (2.26):

$$ K_{1,1,1} \geq K_{1,1,1} \left( \frac{\sqrt{9 + \sqrt{97}}}{2\sqrt{2}} \right) > 0.84. \quad (8.13) $$

Case $d = 2$, $n = a = 2$. Eq. (7.12) and Lemma 8.2 give

$$ f_{\lambda,2,2} = \frac{\lambda}{2} |x| K_1(\lambda|x|), \quad \|f_{\lambda,2,2}\|_2 = \sqrt{\frac{\pi}{3}} \sqrt{\lambda^2 + 1 + \frac{1}{\lambda^2}}. \quad (8.14) $$

Concerning the square $f_{\lambda,2,2}^2$, by Lemma 8.3 we have

$$ \|f_{\lambda,2,2}^2\|_2 = \sqrt{\frac{2\pi}{3\lambda}} \sqrt{\int_0^{+\infty} ds \, s(1 + 4\lambda^2s^2)^2 F(3,2,5/2;-s^2) } \quad (8.15) $$

the corresponding hypergeometric is such that

$$ F(3,2,5/2;-s^2) = \frac{3(2s^2 - 1)}{16s^2(1 + s^2)^2} + \frac{3(1 + 4s^2)}{16s^3(1 + s^2)^{5/2}} \operatorname{ArcSinh}(s), \quad (8.16) $$

and the integral in Eq. (8.15) can be computed numerically. The Bessel lower bound is, in the present case,

$$ K_{2,2,2} \geq \sup_{\lambda > 0} K_{2,2,2}(\lambda), \quad K_{2,2,2}(\lambda) := \frac{\|f_{\lambda,2,2}^2\|_2}{\|f_{\lambda,2,2}\|_2^2}. \quad (8.17) $$
A numerical study of the function $K_{2,2,2}(\cdot)$ shows that it attains its absolute maximum at a point close to $\lambda = 1.35$. In agreement with Eq. (2.29), we have

$$K_{2,2,2} \geq K_{2,2,2}(1.35) > 0.36. \quad (8.18)$$

**Case** $d = 3$, $n = a = 2$. Eq. (7.12) and Lemma 8.2 give

$$f_{\lambda,2,3} = \sqrt{\frac{\pi}{8}} e^{-\lambda|x|}, \quad \|f_{\lambda,2,3}\|_2 = \frac{\pi}{\sqrt{8}} \sqrt{5\lambda + \frac{2}{\lambda} + \frac{1}{\lambda^3}}. \quad (8.19)$$

Also, being $f_{\lambda,2,3}^2 = \sqrt{\frac{\pi}{8}} f_{2\lambda,2,3}$, we have

$$\|f_{\lambda,2,3}^2\|_2 = \frac{\pi^{3/2}}{8} \sqrt{10\lambda + \frac{1}{\lambda} + \frac{1}{8\lambda^3}}. \quad (8.20)$$

The Bessel lower bound is

$$K_{2,2,3} \geq \text{Sup}_{\lambda > 0} K_{2,2,3}(\lambda), \quad K_{2,2,3}(\lambda) := \frac{\|f_{\lambda,2,3}^2\|_2}{\|f_{\lambda,2,3}\|_2^2}. \quad (8.21)$$

The function $K_{1,1,1}(\cdot)$ attains its absolute maximum at a point close to $\lambda = 1.31$, and in agreement with Eq. (2.32) we get

$$K_{2,2,3} \geq K_{2,2,3}(1.31) > 0.24. \quad (8.22)$$

### 9 “Fourier” lower bounds on $K_{n,a,d}$.

As anticipated, these are based on the trial functions $f_{p,\sigma,d}(x) := e^{ipx_1} e^{-(\sigma/2)|x|^2}$ (with $p, \sigma > 0$; see Eq. (2.13)). Let us apply Eqs (2.13) (2.16) with $f = g = f_{p,\sigma,d}$, taking into account that $(f_{p,\sigma,d})^2 = f_{2p,2\sigma,d}$; in this way we obtain

$$K_{n,a,d} \geq \frac{\|f_{2p,2\sigma,d}\|_n}{\|f_{p,\sigma,d}\|_n \|f_{p,\sigma,d}\|_a} \quad \text{for each } p, \sigma, \quad (9.1)$$

both for $0 \leq n \leq d/2 < a$ and for $n \geq a > d/2$. We wish to infer from here the lower bounds of Prop. 2.5; this result will follow from a number of Lemmas. First of all, we will evaluate the norms of the trial functions and give upper and lower bounds for them, both interesting for $p$ large and $\sigma/p^2$ small. Then, insert these bounds in (9.1) and get lower bounds for $K_{n,a,d}$, depending on $p, \sigma$. Finally, we will choose $p$ and $\sigma$ suitably, and obtain the lower bounds on $K_{n,a,d}$ of Prop. 2.5.
9.1 Lemma. For each \( p, \sigma > 0 \) and \( n \geq 0 \), the Fourier transform and the \( n \)-th norm of \( f_{p,\sigma,d} \) are given by

\[
( \mathcal{F} f_{p,\sigma,d} ) (k) = \frac{1}{\sigma^{d/2}} e^{-((k_1-p)^2+k_2^2+\ldots+k_d^2)/(2\sigma)} ,
\]

(9.2)

\[
\| f_{p,\sigma,d} \|_n^2 = \frac{1}{\sigma^d} \int_{\mathbb{R}^d} dk \left( 1 + (k_1 + p)^2 + k_2^2 + \ldots + k_d^2 \right)^n e^{-|k|^2/\sigma} .
\]

(9.3)

Proof. An elementary computation relying on \( \int_{-\infty}^{+\infty} d\xi \ e^{h\xi e^{-\alpha\xi^2}} = \sqrt{\pi/\alpha} e^{h^2/(4\alpha)} \) (for \( h \in \mathbb{C}, \alpha > 0 \)). \( \diamond \)

9.2 Lemma. For each \( p, \sigma > 0 \) and \( n \geq 1/2 \), it is

\[
\| f_{p,\sigma,d} \|_n^2 \geq \pi^{d/2} \frac{p^{2n}}{\sigma^{d/2}} .
\]

(9.4)

Proof. Eq. (9.3) implies

\[
\| f_{p,\sigma,d} \|_n^2 \geq \frac{1}{\sigma^d} \int_{\mathbb{R}} dk_1 \ |k_1 + p|^{2n} e^{-|k_1|^2/\sigma} \int_{\mathbb{R}^{d-1}} dk_2 \ldots dk_d \ e^{-k_2^2/\sigma} \ldots e^{-k_d^2/\sigma} =
\]

\[
= \frac{\pi^{d/2-1/2}}{\sigma^{d/2+1/2}} \int_{\mathbb{R}} dk_1 \ |k_1 + p|^{2n} e^{-|k_1|^2/\sigma} (9.5)
\]

(of course, the intermediate integral over \( \mathbb{R}^{d-1} \) is intended to be 1 if \( d = 1 \)). On the other hand,

\[
\int_{\mathbb{R}} dk_1 \ |k_1 + p|^{2n} e^{-|k_1|^2/\sigma} \geq \int_{-p}^{+\infty} dk_1 (k_1 + p)^{2n} e^{-|k_1|^2/\sigma} =
\]

\[
= p^{2n} \sqrt{\sigma} \int_{-p/\sqrt{\sigma}}^{+\infty} dt \ (1+\sqrt{\sigma}/p)^{2n} e^{-t^2} \geq p^{2n} \sqrt{\sigma} \left( \int_{-p/\sqrt{\sigma}}^{+\infty} dt \ e^{-t^2} + 2n \sqrt{\sigma} \int_{-p/\sqrt{\sigma}}^{+\infty} dt \ te^{-t^2} \right)
\]

(in the last two steps: the variable change \( t = k_1/\sqrt{\sigma} \) has been performed, and the Bernoulli inequality \( (1+u)^m \geq 1 + mu \) for \( m \geq 1, u > -1 \) has been employed with \( u = (\sqrt{\sigma}/p) t \) and \( m = 2n \)). Computing the above two integrals, we get

\[
\int_{\mathbb{R}} dk_1 \ |k_1 + p|^{2n} e^{-|k_1|^2/\sigma} \geq p^{2n} \sqrt{\pi\sigma} U_n(\sqrt{\sigma}/p) , \ U_n(\xi) := \frac{1 + \text{Erf}(1/\xi)}{2} + \frac{n\xi}{\sqrt{\pi}} e^{-1/\xi^2} .
\]

(9.6)

Here, Erf denotes as usually the error function. An elementary analysis shows that, for each \( n \geq 1/2 \), the function \( U_n(\cdot) \) is monotonically increasing on the domain \( (0, +\infty) \); on the other hand \( U_n(\xi) \to 1 \) for \( \xi \to 0^+ \). So, \( U_n(\xi) > 1 \) for all \( \xi > 0 \); inserting this into Eq. (9.6), and the result into (9.5) we get the thesis. \( \diamond \)
9.3 Lemma. Let $p, \sigma > 0$ and $n \geq 0$ be such that $n\sigma / p^2 < 1$. Then

$$
\|f_{p,\sigma,d}\|_n^2 \leq \pi^{d/2} \frac{1}{(1 - n\sigma / p^2)^{d/2}} \frac{n^2\sigma}{p^2} \frac{n}{p^2} \frac{p^{2n}}{\sigma^{d/2}} .
$$

(9.7)

Proof. The elementary inequality $1 + u \leq e^u$ for $u \in \mathbb{R}$ implies

$$(v + w)^n \leq v^n e^{nw/v} \quad \text{for } v > 0, v + w > 0, n \geq 0 .
$$

(9.8)

Applying (9.8) with $v := p^2, w := 1 + 2k_1p + |k|^2$, and inserting the outcome into Eq.(9.3) we obtain

$$
\|f_{p,\sigma,d}\|_n \leq \frac{p^{2n}}{\sigma^d} e^{n/p^2} \int_{-\infty}^{+\infty} dk_1 e^{-(1/\sigma - n/p^2)k_1^2 + 2nk_1/p} \Pi_{i=2}^d \int_{-\infty}^{+\infty} dk_i e^{-(1/\sigma - n/p^2)k_i^2}
$$

(intending the last product to be 1 if $d = 1$). Our assumptions on $n, p, \sigma$ ensure all the above Gaussian integrals to converge, and their computation yields the thesis (9.7). \hfill \Box

9.4 Lemma. Let either $1/2 \leq n \leq d/2 < a$ or $n \geq a > d/2$ and $p, \sigma > 0$ be such that $\text{Max}(n, a) \sigma / p^2 < 1$. Then

$$
K_{n,a,d} \geq \frac{1}{(2\pi)^{d/4}} \frac{(1 - n\sigma / p^2)^{d/4} (1 - a\sigma / p^2)^{d/4}}{e^{1 - n\sigma / p^2} 2p^2 + 1 - a\sigma / p^2 2p^2 + \frac{n + a}{2p^2} \sigma^{d/4} 2^n} .
$$

(9.9)

Proof. We apply Eq.(9.1), using the following estimates: Eq.(9.4) for $\|f_{2p,2\sigma,d}\|_n$, Eq.(9.7) for $\|f_{p,\sigma,d}\|_n$ and Eq.(9.7) (with $n$ replaced by $a$) for $\|f_{p,\sigma,d}\|_a$. The term $2^n$ in Eq.(9.4) appears because $(2p)^n = 2^n p^n$. \hfill \Box

Now, we use the freedom we have for the choice of $p, \sigma$ in the previous Lemma; of course, we would like to maximise the r.h.s. of Eq.(9.9), or at least to go close to the maximum. The choice we will present is the result of a careful inspection of Eq.(9.9), and approximates well the maximum for large $n$; it leads directly to the

Proof of Prop.2.5. We apply Lemma 9.4 with

$$
p := \frac{\sqrt{n + a}}{\sqrt{\lambda}} , \quad \sigma := \frac{\mu/\lambda}{n + a} , \quad \lambda > 0 , \quad 0 < \mu < a .
$$

(9.10)

In this way, after a tedious computation we get

$$
K_{n,a,d} \geq \frac{1}{(2\pi)^{d/4}} \phi_{a/2-d/4}(\lambda) \phi_{d/4}(\mu) v_{\mu,n,a,d} \frac{2^n}{(n + a)^{2d/4}} ,
$$

(9.11)
\[ \phi_\alpha(\xi) := \xi^\alpha e^{-\xi/2} \quad \text{for } \alpha, \xi > 0, \]

\[ v_{\mu,n,a,d} := \left(1 - \frac{\mu}{n + a} + \frac{\mu^2 an}{(n + a)^4}\right)^{d/4} \frac{(2a - \mu)\mu n + \mu a^2}{e^{2(n + a)^2 - 2\mu n}} - \frac{\mu a^2}{2(n + a)^2 - 2\mu a}. \]

By construction, it is \( \lim_{n \to +\infty} v_{\mu,n,a,d} = 1 \) for each \( \mu \), so this factor becomes irrelevant for large \( n \). Now, we choose \( \lambda, \mu \) so as to maximise the factors \( \phi_{\lambda/2-d/4}(\lambda)\phi_{d/4}(\mu) \).

The maximising values are \( \lambda = a - d/2, \mu = d/2 \); inserting them into Eq. (9.11), we finally get the lower bound (2.33) for \( K_{n,a,d} \) (the factor \( v_{n,a,d} \) appearing in (2.33) is just the present coefficient \( v_{\mu,n,a,d} \) with \( \mu = d/2 \)).

To conclude, we must derive the weaker bound (2.36) for \( n \geq a > d/2 \); this follows readily from the expression (2.35) of \( v_{n,a,d} \) and from the inequalities

\[ 1 - \frac{d}{2(n + a)} + \frac{d^2 an}{4(n + a)^4} \geq 1 - \frac{d}{2a}, \quad \frac{(4a - d)dn + 2da^2}{8(n + a)^2 - 4dn} - \frac{da^2}{4(n + a)^2 - 2ad} \geq 0. \]

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