Algebraic representations and constructible sheaves

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Abstract. I survey what is known about simple modules for reductive algebraic groups. The emphasis is on characteristic $p > 0$ and Lusztig’s character formula. I explain ideas connecting representations and constructible sheaves (Finkelberg–Mirković conjecture) in the spirit of the Kazhdan–Lusztig conjecture. I also discuss a conjecture with S. Riche (a theorem for $GL_n$) which should eventually make computations more feasible.

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Mathematics Subject Classification (2010): 17B10, 20G15, 14F05

Introduction

Let $G$ denote an algebraic group over an algebraically closed field $\mathbb{k}$. A representation of $G$ is a $\mathbb{k}$-vector space $V$ and a homomorphism $G \to GL(V)$ of algebraic groups. In this article we discuss various approaches to the representation theory of reductive algebraic groups (like $GL_n, Sp_{2n}, \ldots, E_8$) via constructible sheaves.

Studying the representation theory of $G$ can be thought of as “harmonic analysis in algebraic geometry”. Over fields of characteristic zero the theory is well understood and extremely useful. It parallels the theory of compact Lie groups.

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Much research over the last five decades has focused on the case of characteristic \( p > 0 \). Here the theory is highly developed, however several fundamental questions remain unsolved.

The deepest result in the field (at least on the level of characters) is Lusztig’s formula. It gives character formulas for certain simple modules, from which the characters of all simple modules can be deduced\(^1\). If we fix the root system of our group and let \( p \) vary, then we know that Lusztig’s character formula holds if \( p \) is very large. However only in very few cases (e.g. \( SL_2 \), \( SL_3 \), \( SL_4 \), \( Sp_4 \), \( G_2 \)) do we know precisely when it holds! We also don’t understand well what happens when it fails.

Lusztig’s character formula was motivated by the Kazhdan–Lusztig conjecture, which gives the characters of simple highest weight representations of complex semi-simple Lie algebras. The Kazhdan–Lusztig conjecture was first proved by establishing a bridge to constructible sheaves on the flag variety. Once one has traversed such a bridge, deep theorems concerning constructible sheaves (e.g. the decomposition theorem, the Weil conjectures, \ldots) can be used to deduce the Kazhdan–Lusztig conjecture\(^2\).

By analogy with the Kazhdan–Lusztig conjecture one would like to build a bridge between representations of \( G \) and constructible sheaves. The goal being to better understand Lusztig’s character formula (amongst other things). Building such a bridge turns out to be much harder in this setting. The most satisfactory such statements are the geometric Satake equivalence and the Finkelberg–Mirković conjecture\(^3\). Both results purport an equivalence between the representation theory of \( G \) and a category of perverse sheaves on the affine Grassmannian \( \mathcal{G} \) associated to the (complex) Langlands dual group. Under both such equivalences the base field of the representation theory corresponds to the coefficients of the perverse sheaves. The space \( \mathcal{G} \), however, is fixed.

The Finkelberg–Mirković conjecture is easily seen to imply Lusztig’s character formula for large \( p \). It also gives a character formula for all \( p \) in terms of the Euler characteristic of the stalks of intersection cohomology complexes with \( k \)-coefficients. In this way, deciding for which \( p \) Lusztig’s character formula holds becomes a question about controlling torsion in certain local integral intersection cohomology groups. Roughly speaking, it was by producing many unexpected torsion classes that the author was recently able to show that Lusztig’s character formula cannot hold with the hoped-for bounds.

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1. If our characteristic \( p \) is not too small. Such subtleties will be ignored in the introduction.
2. In the words of Bernstein [Ber]: “The amazing feature of the proof is that it does not try to solve the problem but just keeps translating it in languages of different areas of mathematics (further and further away from the original problem) until it runs into Deligne’s method of weight filtrations which is capable to solve it.”
3. The reader is warned that the Finkelberg–Mirković conjecture is still a conjecture. However it is very useful as a guiding principle. Furthermore, recent work of Achar, Mautner, Riche and Rider seems to bring us close to a proof.