NONEXISTENCE OF EXCEPTIONAL IMPRIMITIVE Q-POLYNOMIAL ASSOCIATION SCHEMES WITH SIX CLASSES

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Abstract. Suzuki (1998) showed that an imprimitive Q-polynomial association scheme with first multiplicity at least three is either Q-bipartite, Q-antipodal, or with four or six classes. The exceptional case with four classes has recently been ruled out by Cerzo and Suzuki (2009). In this paper, we show the nonexistence of the last case with six classes. Hence Suzuki’s theorem now exactly mirrors its well-known counterpart for imprimitive distance-regular graphs.

1. Introduction

Distance-regular graphs have been extensively studied. They form the class of metric (or P-polynomial) association schemes, and play a central role in Algebraic Combinatorics. Cometric (or Q-polynomial) association schemes are the “dual version” of distance-regular graphs. Besides their importance in the theory of association schemes, cometric schemes are of particular interest because of their connections, e.g., to combinatorial/spherical designs, (Euclidean) lattices and also mutually unbiased bases in quantum information theory. For recent activity on cometric schemes and related topics, see [2, 7, 6] and the references therein.

Major theorems concerning distance-regular graphs often have their counterparts for cometric schemes, but the proofs can be very different in terms of both techniques and difficulties; compare, e.g., the proofs of the long-standing Bannai–Ito Conjecture and its dual [11] [8] [5]. It is a well-known elementary fact [3, p. 315] that an imprimitive distance-regular graph with valency $k > 2$ is bipartite or antipodal. On the cometric side, Suzuki [9] did show in 1998 that an imprimitive cometric scheme with first multiplicity $m > 2$ and more than six classes is Q-bipartite or Q-antipodal, but there remained two cases of open parameter sets with four and six classes, respectively. The exceptional case with four classes has recently been ruled out by Cerzo and Suzuki [5]. In this paper, we finally show that the other case with six classes does not occur (Theorem 3). Hence Suzuki’s theorem now exactly mirrors the result for imprimitive distance-regular graphs.

The contents of the paper are as follows. §§2, 3 review basic terminology, notation and facts concerning cometric schemes and their imprimitivity. §§4, 5 are concerned with the exceptional imprimitive cometric scheme with six classes. §4 deals with the description of its eigenmatrix, and §5 with the calculation of a structure constant. The nonexistence of the scheme is established in §6.

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1In [3, p. 237], Bannai and Ito conjectured that there are only finitely many distance-regular graphs with any given valency $k > 2$. 
2. Association schemes

In this section and the next, we recall necessary definitions and results. The reader is referred to [3, 4, 7] for more background material.

Let \( X \) be a finite set and \( \mathcal{R} = \{ R_0, R_1, \ldots, R_d \} \) a set of symmetric binary relations on \( X \). For each \( i \), let \( A_i \) be the adjacency matrix of the graph \((X, R_i)\). The pair \((X, \mathcal{R})\) is a symmetric association scheme with \( d \) classes if

- (AS1) \( A_0 = I \), the identity matrix;
- (AS2) \( \sum_{i=0}^{d} A_i = J \), the all ones matrix;
- (AS3) \( A_i A_j \) is a linear combination of \( A_0, A_1, \ldots, A_d \) for \( 0 \leq i, j \leq d \).

By (AS1) and (AS3), the vector space \( A \) spanned by the \( A_i \) is an algebra; this is the Bose–Mesner algebra of \((X, \mathcal{R})\). Note that \( A \) is semisimple as it is closed under conjugate transposition, so that it has a basis consisting of the primitive idempotents; \( E_0 = \frac{1}{|X|} J, E_1, \ldots, E_d \), i.e., \( E_i E_j = \delta_{ij} E_i, \sum_{i=0}^{d} E_i = I \). By (AS2), \( A \) is closed under entrywise multiplication, denoted \( \circ \). The \( A_i \) are the primitive idempotents of \( A \) with respect to \( \circ \), i.e., \( A_i \circ A_j = \delta_{ij} A_i, \sum_{i=0}^{d} A_i = J \).

We define \( p_{ij}^h \), \( q_{ij}^h \) \((0 \leq i, j, h \leq d)\) by the equations

\[
A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h, \quad E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^{d} q_{ij}^h E_h.
\]

The \( p_{ij}^h \) are nonnegative integers. On the other hand, since each \( E_i \circ E_j \) (being a principal submatrix of \( E_i \otimes E_j \)) is positive semidefinite, it follows that the \( q_{ij}^h \) are real and nonnegative. The eigenmatrices \( P, Q \) are defined by

\[
A_i = \sum_{j=0}^{d} P_{ij} E_j, \quad E_i = \frac{1}{|X|} \sum_{j=0}^{d} Q_{ij} A_j.
\]

Note that \( P_{0i}, P_{1i}, \ldots, P_{di} \) give the eigenvalues of \( A_i \). Let \( k_i = P_{0i}, \quad m_i = Q_{0i} \).

It follows that \( k_i \) is the valency of \((X, R_i)\) and \( m_i = \text{rank}(E_i) \). The \( m_i \) are called the multiplicities of \((X, \mathcal{R})\).

We say that \((X, \mathcal{R})\) is primitive if the graphs \((X, R_i)\) \((1 \leq i \leq d)\) are connected, and imprimitive otherwise. Let \( I_r \) (resp. \( J_r \)) denote the \( r \times r \) identity (resp. all ones) matrix. Then

**Lemma 1.** The following are equivalent:

(i) \((X, \mathcal{R})\) is imprimitive.

(ii) There are \( \mathcal{I}, \mathcal{J} \subseteq \{0, 1, \ldots, d\} \) such that \( \frac{1}{r} \sum_{i \in \mathcal{I}} A_i = \sum_{i \in \mathcal{J}} E_i = \frac{1}{s} J_r \otimes I_s \) for some integers \( r, s \geq 2 \) and an ordering of \( X \).

Moreover, if (i), (ii) hold then \( r = \sum_{i \in \mathcal{I}} k_i \) and \( s = \sum_{i \in \mathcal{J}} m_i \).

See, e.g., [2] §2.9. Suppose now that \((X, \mathcal{R})\) is imprimitive, so that \( \bigcup_{i \in \mathcal{I}} R_i \) is an equivalence relation on \( X \). Then there is a natural structure of a symmetric association scheme \((\tilde{X}, \tilde{\mathcal{R}})\) on the set \( \tilde{X} \) of all equivalence classes, called a quotient scheme of \((X, \mathcal{R})\). Let \( E = \frac{1}{|X|} J_r \otimes I_s \) be as in (ii) above. The Bose–Mesner algebra \( \tilde{A} \) of \((\tilde{X}, \tilde{\mathcal{R}})\) is canonically isomorphic to the “Hecke algebra” \( E A E \) (which is also

\[\text{Observ}e\text{ that }\frac{1}{|X|} J \text{ is an idempotent in } A \text{ with rank one, hence must be primitive.}\]
an ideal of $A$); to be more precise, the primitive idempotents $\{\hat{E}_i\}_{i \in J}$ of $\hat{A}$ are indexed by $J$ and satisfy $E_i = \frac{1}{2} J_i \otimes \hat{E}_i$ ($i \in J$), and for every $i$ ($0 \leq i \leq d$) we have $EA_i = (\sum_{j \in J} p_{ij}) \cdot \frac{1}{2} J_r \otimes \hat{A}$ for some adjacency matrix $\hat{A}$ of $(\hat{X}, \hat{R})$. 

3. IMPRIMITIVE COMETRIC SCHEMES

We say that $(X, R)$ is cometric (or $Q$-polynomial) with respect to the ordering $\{E_i\}_{i=0}^d$ if for each $i$ ($0 \leq i \leq d$) there is a polynomial $v_i^\ast$ with degree $i$ such that $Q_{ji} = v_i^\ast(Q_{j1})$ ($0 \leq j \leq d$). Such an ordering is called a $Q$-polynomial ordering. Note that $(X, R)$ is cometric with respect to the above ordering if and only if for all $i, j, h$ ($0 \leq i, j, h \leq d$) we have $q_{ij}^h = 0$ if $i + j < h$ and $q_{ij}^h \neq 0$ if $i + j = h$. Suppose now that $(X, R)$ is cometric and set

$$m = m_1, \quad a_1^* = q_{11}^1, \quad b_1^* = q_{1,j+1}^1, \quad c_1^* = q_{1,j-1}^1.$$ 

It follows that

$$a_1^* + b_1^* + c_1^* = m, \quad b_0^* = m, \quad a_0^* = b_2^* = c_0^* = 0, \quad c_1^* = 1.$$ 

In fact, the $q_{ij}^h$ are entirely determined by the Krein array $v_i^\ast(X, R) = \{b_0^*, b_1^*, \ldots, b_{d-1}^*; c_1^*, c_2^*, \ldots, c_d^*\}$. The $v_i^\ast$ are also determined recursively:

$$v_0^1(x) = 1, \quad v_i^1(x) = x, \quad xv_i^1(x) = c_{i+1}^*v_{i+1}^1(x) + a_i^*v_i^1(x) + b_{i-1}^*v_{i-1}^1(x).$$

Moreover, if we formally define $v_{d+1}^* = \sum_{i=0}^d (x - Q_{i1})$. Then we find

$$c_1^*c_2^* \cdots c_d^*v_{d+1}^*(x) = \prod_{i=0}^d (x - Q_{i1}).$$

It is well known [3, p. 315] that an imprimitive distance-regular graph with valency $k > 2$ is bipartite or antipodal. On the cometric side, Suzuki proved

**Theorem 2.** If $(X, R)$ is an imprimitive cometric scheme with $Q$-polynomial ordering $\{E_i\}_{i=0}^d$ and $m > 2$, then at least one of the following holds:

(i) $(X, R)$ is $Q$-bipartite: $a_i^* = 0$ for $0 \leq i \leq d$.

(ii) $(X, R)$ is $Q$-antipodal: $b_i^* = c_{d-i}^*$ for $0 \leq i \leq d$, except possibly $i = \lfloor \frac{d}{2} \rfloor$.

(iii) $d = 4$ and $v^\ast(X, R) = \{m, m - 1, 1, b_2^*, 1, c_2^*, m - b_3^*, 1\}$.

(iv) $d = 6$ and $v^\ast(X, R) = \{m, m - 1, 1, b_3^*, b_4^*, 1; 1, c_3^*, m - b_5^*, 1, c_5^*, m\}$, where $a_2^* = a_4^* + a_5^*$. 

It should be remarked that, with the notation of Lemma 1, the cases (i)–(iv) above correspond to $J = \{0, 2, 4, \ldots \}$, $J = \{0, d\}$, $J = \{0, 3\}$ and $J = \{0, 3, 6\}$, respectively. Case (iii) in the theorem has recently been ruled out by Cerzo and Suzuki [5] based on the integrality conditions of the $P_{ij}$. In this paper, we prove

**Theorem 3.** Case (iv) in Theorem 2 does not occur. In particular, an imprimitive cometric scheme with $m > 2$ is $Q$-bipartite or $Q$-antipodal.
4. Discussions: $Q$

For the rest of this paper, we shall always assume that we are in case (iv) of Theorem 2 so that $(X, \mathcal{R})$ is a conetric scheme with six classes and Krein array

\[ i^*(X, \mathcal{R}) = \{m, m - 1, 1, b_3^*, b_4^*, 1; 1, c_2^*, m - b_3^*, 1, c_5^*, m\}, \quad a_2^* = a_4^* + a_5^*. \]

Note that

\[ a_2^* = m - 1 - c_2^*, \quad a_4^* = m - 1 - b_4^*, \quad a_5^* = m - 1 - c_5^*, \quad a_1^* = a_3^* = a_6^* = 0. \]

This section is devoted to the description of $Q$. First, we routinely obtain

\[ c_2^*v_2^*(x) = x^2 - m, \]
\[ c_2^*c_3^*v_3^*(x) = x^3 - a_2^*x^2 - (m + c_2^*(m - 1))x + ma_2^*; \]
\[ v_4^*(x) = xv_3^*(x) - v_2^*(x), \]
\[ c_5^*v_5^*(x) = (x - a_4^*)v_4^*(x) - b_5^*v_3^*(x), \]
\[ mc_5^*v_5^*(x) = (x^2 - a_2^*x - c_2^*(m - 1))v_4^*(x) - b_5^*(x - a_5^*)v_3^*(x). \]

We now describe the $Q_{i1}$ $(0 \leq i \leq 6)$.

**Lemma 4.** Concerning (2), it follows that

\[ mc_5^*c_3^*v_7^*(x) = (x^3 - a_2^*x^2 - (m + c_2^*(m - 1))x + ma_2^* - a_5^*c_3^*) \times (x^2 + c_2^*x - m)(x - m)(x + 1). \]

**Proof.** By definition,

\[ v_7^*(x) = xv_6^*(x) - v_5^*(x). \]

Eliminating $v_6^*$, $v_5^*$ we find

\[ mc_5^*v_7^*(x) = (x^3 - a_2^*x^2 - (m + c_2^*(m - 1))x + ma_2^*)v_4^*(x) \]
\[ - b_5^*(x^2 - a_5^*x - m)v_3^*(x) \]
\[ = (c_5^*c_3^*v_7^*(x) - ma_4^*)v_4^*(x) - b_5^*c_5^*v_3^*(x)v_4^*(x) + a_5^*b_5^*xv_3^*(x); \]

the replacement $xv_3^*(x) = v_4^*(x) + v_2^*(x)$ in the last term implies:

\[ = (c_5^*v_5^*(x) - a_5^*)(c_3^*v_4^*(x) - b_5^*v_2^*(x)) \]
\[ = (c_5^*v_5^*(x) - a_5^*)(c_3^*xv_3^*(x) - mv_2^*(x)). \]

Now the result follows from a straightforward calculation. \( \square \)

Let $x_i = Q_{i1}$ $(0 \leq i \leq 6)$\footnote{It is customary to use $\theta_i^*$ instead of $x_i$, but we decided to avoid introducing too many asterisks.} We may assume $x_0 = m$, $x_6 = -1$, and $x_1, x_2, x_3$ $(x_1 > x_2 > x_3)$ are the roots of

\[ x^3 - a_2^*x^2 - (m + c_2^*(m - 1))x + ma_2^* - a_5^*c_3^* = 0, \]

and $x_4, x_5$ $(x_4 > x_5)$ are the roots of

\[ x^2 + c_2^*x - m = 0. \]

In particular, $x_1, x_2, x_3$ satisfy

\[ x_1 + x_2 + x_3 = a_2^*, \]
\[ x_1x_2 + x_1x_3 + x_2x_3 = -m - c_2^*(m - 1). \]
The value of the left-hand side in (3) at $x = m - 1$ equals $-mc^2_2 - a^*_3c^3_5 < 0$, from which it follows that

$$m - 1 < x_1 < m.$$  

The other five columns of $Q$ are determined by the recursion (1):

$$Q = \begin{bmatrix} 1 & m & m(m-1) & m_3 & m_6 \\ 1 & x_1 & \frac{m^2}{c_2^2} & a^*_1 & m_{m-1} \\ 1 & x_2 & \frac{m^2}{c_2^2} & a^*_2 & b^*_2 \\ 1 & x_3 & \frac{m^2}{c_2^2} & a^*_3 & b^*_2 \\ 1 & x_4 & -x_1 & -\frac{m}{c_3} & b^*_2b^*_3 \\ 1 & x_5 & -x_2 & -\frac{m}{c_3} & b^*_2b^*_3 \\ 1 & -1 & -\frac{m-1}{c_2} & m_3 & m_6 \end{bmatrix}$$

where $m_3 = \frac{m(m-1)}{c_2^2c_5^2}$, $m_6 = \frac{(m-1)b^*_5b^*_3}{c_2^2c_5^2}$. In particular:

$$|X| = \sum_{i=0}^{6} m_i = (m+1)(1+m_3+m_6).$$

It should be remarked that the validity of (7) can be checked by the formula $B^*Q^T = Q^T\text{diag}(x_0, x_1, \ldots, x_6)$, where $B^*_1$ is the tridiagonal matrix defined by $(B^*_1)_{ij} = q^*_1_i$ (cf. [3] p. 91).

5. Discussions: $p^1_{16} + 1$

The section is devoted to the computation of $p^1_{16} + 1$. It should be remarked that there is a formula which expresses the $p^1_{16}$ as rational functions of the $Q_{ij}$ (see [3] p. 65). However, the required calculation turns out to be extremely involved, so that we provide a more conceptual (computer-free) approach looking at the quotient scheme $(\tilde{X}, \tilde{K})$.

We have $\mathcal{J} = \{0, 3, 6\}$ with the notation of Lemma 1 so that $(\tilde{X}, \tilde{K})$ has two classes. Since $E_0 + E_3 + E_6 = \frac{1}{m+1}(A_0 + A_6)$ by (7) and (8), we find $I = \{0, 6\}$ and $r = k_6 + 1 = m + 1$.

Note also that $E_0, E_3, E_6$ are linear combinations of $A_0 + A_6, A_1 + A_2 + A_3, A_4 + A_5$. Hence we may write

$$A_1 + A_2 + A_3 = J_r \otimes \tilde{A}, \quad A_4 + A_5 = J_r \otimes \tilde{A},$$

where $\tilde{A}, \tilde{A}'$ are the nontrivial adjacency matrices of $(\tilde{X}, \tilde{K})$. Let $k$ be the valency of the graph corresponding to $\tilde{A}$. Then

**Lemma 5.** The following hold:

(i) $k_1 + k_2 + k_3 = k(m+1)$.

(ii) $k_1x_1 + k_2x_2 + k_3x_3 = 0$.

(iii) $k_1x^2_1 + k_2x^2_2 + k_3x^2_3 = km(m+1)$.

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4We did, however, double-check the formula (9) below using a software package MAGMA (http://magma.maths.usyd.edu.au/magma/).
Proof. (i) is the (constant) row sum of $A_1 + A_2 + A_3 = J_r \otimes \tilde{A}$. For (ii), observe
$$0 = (PQ)_{01} = (k_1 x_1 + k_2 x_2 + k_3 x_3) + (k_4 x_4 + k_5 x_5),$$
$$0 = c_1^2 c_3^4 (PQ)_{05} = -b_4^2 c_5^4 (k_1 x_1 + k_2 x_2 + k_3 x_3) + b_5^2 c_2^3 (k_4 x_4 + k_5 x_5).$$
Since $b_5^2 c_3^5 + b_4^2 c_5^4 > 0$ we obtain (ii), as well as $k_4 x_4 + k_5 x_5 = 0$; the latter, together with $(PQ)_{02} = 0$ and (i) above, implies (iii).

Solving (i)–(iii) above for $k_1, k_2, k_3$ we routinely obtain, in particular,
$$k_1 = \frac{k(m + 1)(x_2 x_3 + m)}{(x_1 - x_2)(x_1 - x_3)}.$$
Note that $A_1$ has constant row sum $k_1 = k(p_{16}^1 + 1)$, from which it follows that
$$p_{16}^1 + 1 = \frac{(m + 1)(x_2 x_3 + m)}{(x_1 - x_2)(x_1 - x_3)}.$$  \[(9)\]
Moreover, using (4), (5) we obtain
$$x_2 x_3 + m = x_1^2 - a_2^2 x_1 - c_2^3 (m - 1),$$
$$0 < (x_1 - x_2)(x_1 - x_3) = 3 x_1^2 - 2 a_2^2 x_1 - m - c_2^3 (m - 1).$$  \[(10)\]  \[(11)\]
6. Proof of Theorem 3

We are now ready to prove Theorem 3.

Suppose first $p_{16}^1 = 0$. Then (9)–(11) would imply
$$0 = (m + 1)(x_1^2 - a_2^2 x_1 - c_2^3 (m - 1)) - (3 x_1^2 - 2 a_2^2 x_1 - m - c_2^3 (m - 1))$$
$$= (x_1 - m)((m - 2) x_1 - 1 + c_2^3 (m - 1)).$$  \[\text{However, since}\]
$$\frac{1 - c_2^3 (m - 1)}{m - 2} < \frac{1}{m - 2} < m - 1,$$
this contradicts (9). Hence $p_{16}^1 \geq 1$.

Fix a scalar $\alpha$ satisfying
$$\frac{m(m + 1)}{m^2 + 1} < \alpha < \min \left\{ \frac{m + 1}{3}, 2 \right\}.$$  \[\text{Since}\]
$$p_{16}^1 + 1 \geq 2 > \alpha,$$  \text{by (9)–(11) we find}
$$(m + 1)(x_1^2 - a_2^2 x_1 - c_2^3 (m - 1)) \geq \alpha (3 x_1^2 - 2 a_2^2 x_1 - m - c_2^3 (m - 1)),$$
or equivalently,
$$\alpha (m + 1 - 3 \alpha) x_1^2 - (m + 1 - 2 \alpha) a_2^2 x_1 - (m + 1 - \alpha) c_2^3 (m - 1) + ma \geq 0.$$  \[\text{On the other hand, since}\]
$$(m + 1 - 3 \alpha) x_1^2 - (m + 1 - 2 \alpha) a_2^2 x_1 - (m + 1 - \alpha) c_2^3 (m - 1) + ma \geq 0.$$  \[\text{or equivalently,}\]
$$\alpha a_2^2 (m - 1) \leq \alpha a_2^2 x_1 \leq m(m + 1) - 2 a_2^2 (m - 1) - 2 ma,$$
from which it follows that
$$0 < \alpha a_2^2 (m - 1) \leq m(m + 1) - \alpha(m^2 + 1).$$
However, since the right-hand side above is negative, this is absurd. The proof is complete.

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