EXISTENCE AND REGULARITY OF MAXIMAL METRICS FOR THE FIRST LAPLACE EIGENVALUE ON SURFACES

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Abstract. We investigate in this paper the existence of a metric which maximizes the first eigenvalue of the Laplacian on Riemannian surfaces. We first prove that, in a given conformal class, there always exists such a maximizing metric which is smooth except at a finite set of conical singularities. This result is similar to the beautiful result concerning Steklov eigenvalues recently obtained by Fraser and Schoen (Sharp eigenvalue bounds and minimal surfaces in the ball, 2013). Then we get existence results among all metrics on surfaces of a given genus, leading to the existence of minimal isometric immersions of smooth compact Riemannian manifold $(M,g)$ of dimension 2 into some $k$-sphere by first eigenfunctions. At last, we also answer a conjecture of Friedlander and Nadirashvili (Int Math Res Not 17:939–952, 1999) which asserts that the supremum of the first eigenvalue of the Laplacian on a conformal class can be taken as close as we want of its value on the sphere on any orientable surface.

1 Introduction

Let $(\Sigma, g)$ be a smooth compact Riemannian surface without boundary. The eigenvalues of the Laplacian $\Delta_g = -\text{div}_g(\nabla)$ form a discrete sequence

$$0 = \lambda_0 < \lambda_1 (\Sigma, g) \leq \lambda_2 (\Sigma, g) \leq \cdots$$

Getting bounds on these eigenvalues depending on the metric or the topology of $\Sigma$ has been the subject of intensive studies in the past decades. In this paper, we shall focus on the first eigenvalue $\lambda_1$. One can for instance consider the first conformal eigenvalue of $(\Sigma, g)$ defined by

$$\Lambda_1 (\Sigma, [g]) = \sup_{\tilde{g} \in [g]} \lambda_1 (\tilde{g}) \text{Vol}_{\tilde{g}} (\Sigma).$$

(1.1)

If one looks at the infimum of the first eigenvalue in a given conformal class, it is always 0. Now one can also study invariants which depend only on the topology of the surface. For orientable surfaces, one can define for any genus $\gamma \geq 0$

$$\Lambda_1 (\gamma) = \sup_g \lambda_1 (g) \text{Vol}_g (\Sigma) = \sup_{[g]} \Lambda_1 (\Sigma, [g])$$

(1.2)
where $\Sigma$ is a compact orientable surface of genus $\gamma$. One can also look at

$$\inf_{[g]} \Lambda_1 (\Sigma, [g]).$$

Natural questions about these quantities are to get explicit values or explicit bounds on it, and whether or not the supremum (or infimum) in their definition is achieved by some metric and, if yes, how regular these extremal metrics are. Yang and Yau [YY80] (see also [LY82]) obtained an upper-bound for the first eigenvalue of the Laplacian on a surface, depending only on the genus $\gamma$ of the surface. In case of orientable surfaces, this reads as

$$\Lambda_1 (\gamma) \leq 8\pi \left[ \frac{\gamma + 3}{2} \right]. \quad (1.3)$$

Colbois and El Soufi [CE03] gave an explicit lower bound of $\Lambda_1 (\Sigma, [g])$ on any closed Riemannian surface and proved that

$$\Lambda_1 (\Sigma, [g]) \geq \Lambda_1 (S^2, [\text{can}])$$

and by the work of Hersch [Her70], we know that $\Lambda_1 (S^2, [\text{can}]) = 8\pi$. A lower bound for $\Lambda_1 (\gamma)$ can be obtained from [BM01] and [BBD88] (see [FS13a]):

$$\Lambda_1 (\gamma) \geq \frac{3\pi}{4} (\gamma - 1). \quad (1.4)$$

Exact values of these quantities were obtained for small genus and for specific conformal classes. Let us mention the sphere (Hersch [Her70]), the projective plane (Li and Yau [LY82]), the torus (Girouard [Gir09] and Nadirashvili [Nad96]), the Klein bottle (El Soufi et al. [EGJ06] and Jakobson et al. [JNP06]), the genus 2 surfaces (Jakobson et al. [JLNNP05], Karpukhin [Kar13]).

Concerning $\Lambda_1 (\Sigma, [g])$, we prove the following theorem:

**Theorem 1.** Let $(\Sigma, g)$ be a compact Riemannian surface without boundary. Then

$$\Lambda_1 (\Sigma, [g]) > \Lambda_1 (S^2, [\text{can}]) = 8\pi$$

if $\Sigma$ is not diffeomorphic to $S^2$. Moreover, there is an extremal metric $\tilde{g} \in [g]$, smooth except maybe at a finite number of points corresponding to conical singularities, such that $\Lambda_1 (\Sigma, [g]) = \lambda_1 (\tilde{g}) \ Vol_{\tilde{g}} (\Sigma)$.

This theorem contains a rigidity result which states that the sphere is characterized by having the minimal first conformal eigenvalue. It also contains an existence result of “smooth” maximal metrics. Note that, on the sphere, we know since the work of Hersch [Her70] that maximal metrics exist and are all smooth since they consist in all metrics isometric to the standard one. As observed in [Kok11], conical singularities naturally appear for extremal metrics. Indeed, the conformal factor relating $\tilde{g}$ to $g$ is $|\nabla \Phi|^2_{\tilde{g}}$ where $\Phi$ is some smooth harmonic map from $M$ into some