COISOTROPIC BRANES IN TORIC CALABI-YAU 3-FOLDS

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Abstract. We study disk-instantons ending on coisotropic branes preserved by real torus action in toric Calabi-Yau 3-folds. In particular, we find fermion zero modes on disk multi-covers ending on a coisotropic brane in local $\mathbb{P}^1$ geometry with normal bundle $\mathcal{O}(-a) \oplus \mathcal{O}(a-2)$. It turns out that, independent of $a$, disk multi-cover formula is the same as for disks ending on a Lagrangian brane in resolved conifold. We further construct an example of a coisotropic brane in Calabi-Yau 3-fold used in geometric engineering of 4d $\mathcal{N} = 2$ $SU(N)$ gauge theory, where this brane provides a surface defect.
1. Introduction

Since the work [1], it is known that boundary conditions in topological A-model [2] include not only Lagrangian branes but also coisotropic branes. The latter are supported on higher than middle dimensional submanifolds of the symplectic target manifold and are characterized by nontrivial curvature 2-form of the connection of the complex line bundle on the worldvolume. Generalization to higher rank coisotropic branes was given in [3]. The existence of coisotropic branes has important implications for Homological Mirror Symmetry [4],[5].

Computation of world-sheet instantons ending on Lagrangian branes in A-model is a well-developed subject. In [6],[7], the proof of mirror symmetry [8],[9] was used to provide counting of holomorphic maps from genus zero Riemann surface with boundary to holomorphic disks in toric 3-folds ending on Lagrangian branes. In particular, the formula for disk multi-covers, predicted in [10], was derived. This formula was confirmed in [11],[12],[13] by the virtual localization method. Later all genus world-sheet instantons ending on Lagrangian branes in toric Calabi-Yau 3-folds were summed up [14],[15]. This story was generalized for degenerate torus action [16].

The subject of world-sheet instantons ending on coisotropic branes is much less explored. So far explicit examples of coisotropic branes and topological strings ending on them were discussed in the special case of space-filling coisotropic brane [17],[18],[19].

In this note we study disks ending on coisotropic branes preserved by real torus action in toric Calabi-Yau 3-folds. In particular, we find fermion zero modes on disk multi-covers ending on a coisotropic brane in local $\mathbb{P}^1$ geometry with normal bundle $\mathcal{O}(-a) \oplus \mathcal{O}(a-2)$. Surprisingly, the weights of these fermion zero modes under the torus action are exactly the same as for disk multi-covers ending on a toric Lagrangian brane in resolved conifold computed in [11]. These weights completely determine the counting of holomorphic disk multi-covers, allowing to write the multiple-cover formula for disks ending on a coisotropic brane.

We further construct a coisotropic brane $Y$ in toric Calabi-Yau 3-fold $\mathcal{X}_{SU(N)}$ which is used in geometric engineering [20] of $\mathcal{N} = 2, 4d$ $SU(N)$ gauge theory in IIA string theory. By considering $D6$ brane supported on $R^{1,1} \times Y$, one gets a surface defect in 4d theory with (2,2) supersymmetric gauge theory in $R^{1,1}$. We write the multiple-cover formula for holomorphic disks in $\mathcal{X}_{SU(N)}$ ending on $Y$. This provides the leading contribution to the superpotential for the chiral field $y$ in 2d (2,2) gauge theory.

This note is organized as follows. In Section 2 we review basic facts about disk instantons in topological A-model. In Section 3 we give a warm-up example of a coisotropic brane in $\mathbb{C}^3$. Section 4 provides construction of a coisotropic brane in local $\mathbb{P}^1$ geometry with normal bundle $\mathcal{O}(-a) \oplus \mathcal{O}(a-2)$ and study of fermion zero modes on disk multi-covers ending on
this brane. In Section 5 we construct coisotropic brane in $X_{SU(N)}$ and discuss its application as providing a surface defect in 4d gauge theory.

2. Disk instantons and A-branes: Review

Let us consider topological A-model [2] with Kähler target space $X$ and let $Y \subset X$ be a brane in this model. Let $L$ be a complex line bundle on $Y$, $A$ a connection on $L$, and $F$ the curvature 2-form of $A$. Let $\beta \in H_2(X,Y)$, $\beta \neq 0$, and let $\mathcal{M}_{g,h}(X;Y,L,A;\beta)$ be a stable map compactification of the moduli space of holomorphic maps $f : (\Sigma, \partial \Sigma) \rightarrow (X,Y)$ from a genus $g \geq 0$ Riemann surface $\Sigma$ with $h \geq 1$ boundary components with boundary conditions specified by the triple $(Y,L,A)$, and $f_*[\Sigma] = \beta$. From a physical point of view, the data $(X,Y,L,A)$ determines a boundary topological A-model [1]. The space $\mathcal{M}_{g,h}(X,Y,L,A;\beta)$ should be thought of as a stable compactification of the moduli space of instantons in this A-model coupled to topological gravity.

The key in the path-integral approach to counting open world-sheet instantons ending on a brane is to find fermion zero modes on a world-sheet with appropriate boundary conditions specified by boundary matrix $R$. Let $G$ denote the restriction of the Kähler metric on $X$ to $Y$. Note that the restriction $T_X|_Y$ of the complexified tangent bundle $T_X$ admits a direct sum decomposition (of $C^\infty$-bundles)

\[(2.1)\quad T_X|_Y \simeq N_Y \oplus T_Y\]

where $N_Y$ is the normal bundle to $Y$ in $X$, and $T_Y$ is the tangent bundle to $Y$. The boundary matrix $R : T_X|_Y \rightarrow T_X|_Y$ is the linear map defined by the following block matrix [20]

\[(2.2)\quad R = \begin{bmatrix} -1_{N_Y} & 0 \\ 0 & (G - F)^{-1}(G + F) \end{bmatrix}\]

with respect to the decomposition (2.1).

In topological A-model on $\Sigma$ there are fermions

$\chi \in \Gamma\left(f^*\left(T_X^{1,0}\right)\right)$  \quad  $\bar{\chi} \in \Gamma\left(f^*\left(T_X^{0,1}\right)\right)$  \quad  $\psi \in \Gamma\left(\Omega^1_\Sigma \otimes f^*\left(T_X^{1,0}\right)\right)$  \quad  $\bar{\psi} \in \Gamma\left(\Omega^1_\Sigma \otimes f^*\left(T_X^{0,1}\right)\right)$.

They arise from fermions $\Psi_\pm \in \Gamma\left(S^\pm_\Sigma \otimes TX\right)$ in untwisted $\sigma$-model as [2]

\[(2.3)\quad \begin{array}{c|c}
\text{A-model} & \sigma-model \\
\hline
\chi & \Psi_\pm \partial z_i \\
\bar{\chi} & \Psi^-_i \partial \bar{z}_i \\
\psi & \Psi_\pm^i \partial z_i \\
\bar{\psi} & \Psi^\pm_\bar{i} \partial \bar{z}_i
\end{array}\]

where $z_i, \bar{z}_i$ are local complex coordinates on $X$.

On $\partial \Sigma$ fermions satisfy boundary conditions

\[(2.4)\quad \bar{\chi}|_{\partial \Sigma} = \bar{R}_+(\chi|_{\partial \Sigma}) \quad \bar{\psi}|_{\partial \Sigma} = R_+(\psi|_{\partial \Sigma}).\]

where $R_+$ and $\bar{R}_+$ are linear maps $R_+, \bar{R}_+ : T_X^{1,0}|_Y \rightarrow T_X^{0,1}|_Y$ determined by $R$. Equations (2.4) follow from boundary condition $\Psi_+ = R(\Psi_-)$ on $\partial \Sigma$ in untwisted $\sigma$-model using relations (2.3).

Recall that in path-integral approach one uses the coupling $\int_\Sigma R_{mik\bar{j}} \chi^m \bar{\chi}^k \psi^\bar{j}$ in the action of A-model to saturate fermion zero modes. (For closed $\Sigma$ this was done in [21].) In this way one gets integral of the Euler class of certain vector bundle over the moduli space.

\footnote{Recall that path-integral of A-model localizes to holomorphic maps.}
For general \((X,Y)\) it is difficult to evaluate this integral. However, in many cases of interest for physics one uses global symmetry to localize this integral to a sum.

In this note we initiate the study of open string instantons for coisotropic branes preserved by the canonical (real) torus action in toric target spaces. Therefore it will be assumed in the following that \(X\) is a toric Calabi-Yau threefold and that 5-dimensional coisotropic cycle \(Y\) is preserved by the canonical \(U(1)^3\)-action on \(X\). Moreover, \(L\) is equipped with an equivariant structure so that the two-form \(F\) is invariant under the torus action. When a rigorous construction of a moduli space equipped with a torus equivariant perfect tangent-obstruction theory is available, explicit computations rely on the virtual localization theorem \([22]\). In certain cases of physical interest, these steps can be formally carried out even in the absence of a rigorous construction of a virtually smooth moduli space. Such an approach has been implemented for stable maps with Lagrangian boundary conditions in many examples \([11, 12, 13, 23, 24, 25]\), the results being consistent with mirror symmetry and large \(N\) duality.

Let \(z_i, 1 \leq i \leq 3\), be affine toric coordinates on a toric coordinate patch \(U \subset X, U \simeq \mathbb{C}^3\). Suppose there is a holomorphic disk \(D \subset U\) defined by the equations

\[
|z_1| \leq 1 \quad z_2 = z_3 = 0
\]

so that \(\partial D \subset Y\). As mentioned above, throughout this section it will be assumed that the cycle \(Y\) and curvature 2-form \(F\) are invariant under the canonical \(U(1)^3\)-action on \(X\). In fact it suffices to work with a single one-parameter subgroup \(T \simeq U(1) \subset U(1)^3\). Then there is a natural induced action \(T \times M_{g,h}(X, Y, \beta) \to M_{g,h}(X, Y, \beta)\) on the moduli space of open string instantons. This allows to evaluate the integral of the Euler class over the moduli space as a sum of local contributions from torus fixed points.

Let \(\Delta\) be the disk \(|t| \leq 1\) in the complex \(t\)-plane and consider a degree \(d \geq 1\) map \(f: \Delta \to D\) of the form

\[
f(t) = (e^t, 0, 0).
\]

Note that [(\(\Delta, f\))] is the unique torus invariant degree \(d\) multi-cover of \(D\) of type \((g, h) = (0, 1)\) up to isomorphism. In particular it determines an isolated fixed point for the torus action on the moduli space of degree \(d\) instantons of the topological \(A\)-model defined by the data \((Y, L, A)\). Let \(\mathcal{T}_{D, R_+}(\mathcal{T}_{D, R_+})\) be the sheaf of germs of holomorphic sections of the bundle \(T_{X, D}^{1, 0}\) with boundary conditions \((2.4)\). Let \(\mathcal{T}_\Delta\) be the sheaf of germs of holomorphic sections of the tangent bundle \(T_{\Delta}^{1, 0}\) with natural real boundary conditions along \(\partial \Delta\).

The local contribution of the isolated fixed point [(\(\Delta, f\))] to the virtual localization formula is

\[
\frac{1}{d} e_T \left( H^1(\Delta, f^*\mathcal{T}_{D, R_+}) \right) e_T \left( H^0(\Delta, \mathcal{T}_\Delta) \right) \frac{1}{d} e_T \left( H^0(\Delta, \mathcal{T}_\Delta) \right)
\]

where \(e_T\) denotes the equivariant Euler class. Recall that for a vector space \(V\) with \(T\)-action, \(e_T(V)\) is the product of \(T\)-weights of basic vectors. One may think of vector spaces \(H^1(\Delta, f^*\mathcal{T}_{D, R_+})\) and \(H^0(\Delta, f^*\mathcal{T}_{D, R_+})\) as \(\psi\) and \(\chi\) zero modes respectively. The factor \(e_T(H^0(\Delta, \mathcal{T}_\Delta))\) takes into account the \(Aut\) group acting on \(\Delta\).

Once the local contribution of disk multi-covers are given, the local contribution of a generic torus fixed point in \(\overline{M}_{g,h}(X, Y, \beta)\) is a standard exercise in virtual localization.

In \([11, 12, 13]\) the formula \((2.5)\) was used for the case of Lagrangian branes\(^2\). We use \((2.5)\) in Section 4 to compute contribution of disk multi-covers ending on a coisotropic brane in the total space of a holomorphic rank two bundle of the form \(\mathcal{O}(-a) \oplus \mathcal{O}(-b)\) over \(\mathbb{P}^1\), where \(a + b = 2\).

\(^2\)with \(R_+, \tilde{R}_+\) which follow from \(R = \begin{pmatrix} -1_{NY} & 0 \\ 1_{TY} & 1_{TY} \end{pmatrix} \)
3. A COISOTROPIC A-BRANE IN $X = \mathbb{C}^3$

Our warm-up example is a toric coisotropic $A$-brane in $X = \mathbb{C}^3$. As above, let $z_i$, $i = 1, 2, 3$ be linear coordinates on $X$. The cycle $Y \subset X$ is given by $|z_1| = 1$, therefore $Y \simeq S^1 \times \mathbb{C}^2$. The line bundle $L$ is trivial, and we set

$$A = \frac{1}{2} (z_2 dz_3 + \bar{z}_2 d\bar{z}_3),$$

which implies

$$F = \frac{1}{2} (dz_2 \wedge dz_3 + d\bar{z}_2 \wedge d\bar{z}_3).$$

Note that the restriction of the standard symplectic Kähler form to $Y$ is

$$\omega|_Y = \frac{i}{2} \sum_{j=2}^3 dz_j \wedge d\bar{z}_j,$$

Note that $Y$ is a coisotropic cycle in $X$ since it is real codimension 1. In order to check the remaining conditions it is more convenient to use the real coordinates

$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = x_2 + iy_2, \quad z_3 = x_3 + iy_3$$

on the open subset $z_1 \neq 0$, which covers $Y$. In terms of these coordinates

$$F = dx_2 \wedge dx_3 - dy_2 \wedge dy_3$$

$$\omega|_Y = dx_2 \wedge dy_2 + dx_3 \wedge dy_3.$$ 

The symplectic complement $T^\perp_Y$ of the tangent space $T_Y \subset T_X$ is spanned by $\partial_{\theta_i}$. Obviously, there is a natural direct sum decomposition

$$T_Y \simeq T^\perp_Y \oplus F_Y$$

where $F_Y \subset T_Y$ is spanned by $\partial_{x_i}, \partial_{y_i}, i = 2, 3$. Moreover, $\iota_{\partial_{\theta_1}} F = 0$, therefore $F$ determines a section $F$ of $\Lambda^2 F^*_Y$. Similary $\iota_{\partial_{\theta_1}} \omega = 0$, hence $\omega$ determines a section $\sigma$ of $\Lambda^2 F^*_Y$. Let $J : F_Y \to F_Y$ be the linear map determined by

$$\mathcal{F}(h_1, h_2) = \sigma(h_1, J(h_2))$$

for any $h_1, h_2 \in F_Y$. Then a straightforward computation yields

$$J(\partial_{x_2}) = -\partial_{y_3}, \quad J(\partial_{y_2}) = -\partial_{x_3}, \quad J(\partial_{x_3}) = \partial_{y_2}, \quad J(\partial_{y_3}) = \partial_{x_2}.$$ 

This implies that $J^2 = -1$, therefore $J$ is an almost complex structure on $F_Y$.

Finally, note that $Y$ is preserved by the $T$-action on $X$

$$e^{i\varphi} \times (z_j) \to (e^{i\varphi} z_j), \quad j = 1, 2, 3,$$

for any weights $w_j$. In order for $F$ to be invariant under the $T$-action, the weights must satisfy $w_2 = -w_3$.

In order to compute the boundary matrix $R$ note that the Kähler metric $G$ is given by

$$G = dr_1 \otimes dr_1 + r_1^2 d\theta_1 \otimes d\theta_1 + \sum_{i=2}^3 (dx_i \otimes dx_i + dy_i \otimes dy_i)$$

in the above real coordinate chart. Therefore there is a natural $G$-orthogonal decomposition

$$T_X|_Y \simeq N_Y \oplus T^\perp_Y \oplus F_Y$$
where \( N_Y \) is spanned by \( \partial_r \) and \( T_Y^\perp \) is the symplectic complement of \( T_Y \subset T_X|_Y \), spanned by \( \partial_{\theta_1} \). Then \( R \) has the following block form with respect to the decomposition (3.8)

\[
R = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & S
\end{bmatrix}
\]

(3.9)

where \( S : F_Y \rightarrow F_Y \) is the linear map determined by the condition

\[
(G + F)(h_1, h_2) = (G - F)(h_1, S(h_2))
\]

(3.10)

for any \( h_1, h_2 \in F_Y \). Using equations (3.9), (3.11), a straightforward computation yields

\[
S(\partial_{x_2}) = -\partial_{x_3}, \quad S(\partial_{y_2}) = \partial_{y_3}, \quad S(\partial_{x_3}) = \partial_{x_2}, \quad S(\partial_{y_3}) = -\partial_{y_2}.
\]

(3.11)

In terms of holomorphic coordinate vector fields equations (3.9), (3.11) imply

\[
R(\partial_{z_1}) = -e^{-2i\theta_1} \partial_{\overline{z}_1}, \quad R(\partial_{z_2}) = -\partial_{\overline{z}_3}, \quad R(\partial_{z_3}) = \partial_{\overline{z}_2}.
\]

(3.12)

The remaining nontrivial \( R \)-matrix elements follow from (3.12) by conjugation.

4. Coisotropic A-branes in local \( \mathbb{P}^1 \) geometry

In this section we present a more elaborate construction of a coisotropic A-brane in the total space of a holomorphic rank two bundle of the form \( O(-a) \oplus O(-b) \) over \( \mathbb{P}^1 \), where \( a + b = 2 \). We find fermion zero modes on holomorphic disk multi-covers ending on it and write a formula counting these disks.

4.1. Construction. The total space \( X \) of the rank two bundle \( O(-a) \oplus O(-b) \) on \( \mathbb{P}^1 \), \( a, b \in \mathbb{Z}_{\geq 0} \) admits a symplectic quotient construction \( \mathbb{C}^4/U(1) \). In terms of complex linear coordinates \( (X_1, X_2, U, V) \) the symplectic form on \( \mathbb{C}^4 \) is

\[
\tilde{\omega} = \frac{i}{2} (dX_1 \wedge d\overline{X}_1 + dX_2 \wedge d\overline{X}_2 + dU \wedge d\overline{U} + dV \wedge d\overline{V})
\]

(4.1)

and the symplectic \( U(1) \) actions is given by

\[
e^{i\alpha} \times (X_1, X_2, U, V) \rightarrow \left(e^{i\alpha} X_1, e^{i\alpha} X_2, e^{i\alpha} U, e^{i\alpha} V\right).
\]

(4.2)

The moment map of the symplectic \( U(1) \)-action is

\[
\mu(X_1, X_2, U, V) = |X_1|^2 + |X_2|^2 - a|U|^2 - b|V|^2.
\]

(4.3)

For any \( \zeta \in \mathbb{R}_{>0} \), the level set \( \mathcal{Z}_\zeta = \mu^{-1}(\zeta) \) is a smooth manifold and the restriction of the \( U(1) \) action to \( \mathcal{Z}_\zeta \) is free. The quotient \( X = \mathcal{Z}_\zeta/U(1) \) is a smooth manifold equipped with a symplectic Kähler form \( \omega \). Moreover there is a canonical principal \( U(1) \)-bundle structure \( q : \mathcal{Z}_\zeta \rightarrow X \) so that

\[
\tilde{\omega}|_{\mathcal{Z}_\zeta} = q^* \omega.
\]

(4.4)

By construction \( X \) is isomorphic to the total space of the rank two bundle \( O(-a) \oplus O(-b) \) on \( \mathbb{P}^1 \), and the homogeneous toric coordinates \( X_1, X_2 \) are naturally identified with homogeneous coordinates on \( \mathbb{P}^1 \). Let \( \pi : X \rightarrow \mathbb{P}^1 \) denote the projection map.

Let \( M \subset \mathcal{Z}_\zeta \) be the codimension one cycle determined by the equation

\[
|X_2|^2 = |X_1|^2 + c \quad c \geq 0
\]

(4.5)

Obviously, \( M \) is preserved by the symplectic \( U(1) \)-action. Since the later is free as observed above, the quotient \( Y = M/U(1) \) is a codimension one cycle in \( X \). Therefore \( M \) is a coisotropic submanifold of \( X \). In fact equation (4.5) determines a circle \( S^1 \subset \mathbb{P}^1 \), and \( Y = \pi^{-1}(S^1) \). Therefore, again \( Y \simeq S^1 \times \mathbb{C}^2 \) as in Section 3. In the following we will determine a global
vector field $\eta$ on $Y$ which generates the symplectic complement $T^\perp \subset T_Y$ at any point. This vector field enters the definition of coisotropic brane \cite{1} as

$$L_\eta F = 0, \quad L_\eta (\omega|_Y) = 0.$$  

First we define a suitable coordinate chart on $Z_\zeta$ covering $M$ and write down some explicit formulas for future reference. Given the moment map equation \eqref{4.3}, condition \eqref{4.5} implies that $X_1, X_2$ cannot vanish on $M$ as long as $a, b \in \mathbb{Z}_{\geq 0}$ and we fix parameter $\zeta$ such that $\zeta > c$. Therefore $M$ is contained in the open subset $\mathcal{U} \subset Z_\zeta$ defined by $X_1 \neq 0$, $X_2 \neq 0$. The requirement of non-negativity of $a, b$ can be relaxed when local $\mathbb{P}^1$ is part of more general geometry and there are other reasons that $X_1, X_2$ cannot vanish on $M$. We discuss examples with $a > 0$ but $b < 0$ in Section 5.

The following real coordinate functions are well defined on $\mathcal{U}$.

\begin{equation}
X_j = r_j e^{i\theta_j}, \quad j = 1, 2, \quad U = U_1 + iU_2 \quad V = V_1 + iV_2.
\end{equation}

In certain formulas it will be more convenient to use mixed coordinates of the form $(r_i, \theta_i, U, \overline{U}, V, \overline{V})$, $i = 1, 2$. Then the Kähler form $\tilde{\omega}$ and the moment map equation read

\begin{equation}
\tilde{\omega}|_{\mathcal{U}} = r_1 dr_1 \wedge d\theta_1 + r_2 dr_2 \wedge d\theta_2 + \frac{i}{2} (dU \wedge d\overline{U} + dV \wedge d\overline{V}).
\end{equation}

\begin{equation}
r_1^2 + r_2^2 - a|U|^2 - b|V|^2 = \zeta.
\end{equation}

The restriction $T_{Z_\zeta}|_{\mathcal{U}}$ of the tangent space to $Z_\zeta$ is isomorphic to the kernel of the differential 1-form

$$\sum_{i=1}^{2} (r_idr_i - aU_idU_i - bV_idV_i)$$

on $T_{\mathbb{C}^4}|_{\mathcal{U}}$. An elementary computation shows that $T_{Z_\zeta}|_{\mathcal{U}}$ is generated by the vector fields

$$\left( \frac{\partial_{r_2} - \partial_{r_1}}{r_2} \right) \partial_{\theta_i} \quad \xi_i = \partial_{U_i} + \frac{aU_i}{2} \left( \frac{\partial_{r_1}}{r_1} + \frac{\partial_{r_2}}{r_2} \right) \quad \eta_i = \partial_{V_i} + \frac{bV_i}{2} \left( \frac{\partial_{r_1}}{r_1} + \frac{\partial_{r_2}}{r_2} \right)$$

with $i = 1, 2$. Note that these are well defined since $r_1 \neq 0$, $r_2 \neq 0$ in $\mathcal{U}$. Since the defining equation of $M \subset \mathcal{U}$ is $r_2^2 = r_1^2 + c$, it follows that there is a direct sum decomposition

\begin{equation}
T_{Z_\zeta}|_M \simeq N_M \oplus T_M
\end{equation}

where the normal bundle $N_M$ is spanned by $(\partial_{r_2}/r_2 - \partial_{r_1}/r_1)|_M$ and $T_M$ is spanned by $(\partial_{\theta_i}|_M, \xi_i|_M, \eta_i|_M), i = 1, 2$.

Note also that equation \eqref{4.8} and the defining equation of $M$ yield

\begin{equation}
\tilde{\omega}|_M = \frac{i}{2} \left( dU \wedge d\overline{U} + dV \wedge d\overline{V} \right) + \frac{1}{4} \left( aU d\overline{U} + a\overline{U} dU + bV d\overline{V} + b\overline{V} dV \right) \wedge d\theta_+\!
\end{equation}

where $\theta_+ = \theta_1 + \theta_2$. So that

\begin{equation}
L_{\partial_{\theta_+}} (\tilde{\omega}|_M) = 0.
\end{equation}

The restriction of the symplectic $U(1)$-action to the neighborhood $\mathcal{U}$ has the form

\begin{equation}
e^{i\alpha} \times (r_1, r_2, \theta_1, \theta_2, U, \overline{U}, V, \overline{V}) \longrightarrow \left( r_1, r_2, \theta_1 + \alpha, \theta_2 + \alpha, e^{-ia\alpha} U, e^{ia\alpha} \overline{U}, e^{-ib\alpha} V, e^{ib\alpha} \overline{V} \right).
\end{equation}

The canonical vector field $\xi$ determined by the $U(1)$-action on $\mathcal{U}$ is

\begin{equation}
\xi = \partial_{\theta_1} + \partial_{\theta_2} + ia \left( \overline{U} \partial_U - U \partial_{\overline{U}} \right) + ib \left( \overline{V} \partial_V - V \partial_{\overline{V}} \right).
\end{equation}

Note that

\begin{equation}
t_\xi (\tilde{\omega}|_\mathcal{U}) = 0.
\end{equation}
For future reference note that there is a canonical exact sequence

\[(4.15) \quad 0 \to T_q \to T_{Z\zeta \mid q} \to q^* T_X \to 0\]

where \(T_q\) is the vertical tangent bundle of the \(p\)-torus on \(Z_\zeta \to X\). At each point \(z \in Z_\zeta\) the map \(p_z : T_{Z\zeta \mid z} \to (q^* T_X)_z\) is the differential of \(q\) at \(z\). By construction, the restriction \(T_q \mid_M\) is spanned by the vector field \(\xi\). Since \(M\) is preserved by the \(U(1)\)-action \((4.12)\), it follows that \(T_q \mid_M \subset T_M\). Moreover, as \(p\) maps \(T_M \subset T_{Z\zeta \mid M}\) to \(q^* T_Y\), there is an exact sequence

\[(4.16) \quad 0 \to T_q \mid_M \to T_M \to p^* q^* T_Y \to 0.\]

Equation \((4.4)\) implies that

\[(4.17) \quad \bar{\omega}_0 \mid_{Z\zeta}(h_1, h_2) = (q^* \omega)(p(h_1), p(h_2))\]

for any two vectors \(h_1, h_2 \in T_{Z\zeta}\). This implies in turn

\[(4.18) \quad (q^* T_Y)^\perp = p(T_M^\perp)\]

where \((q^* T_Y)^\perp \subset q^* T_X \mid_M\) is the symplectic complement with respect to \(q^* \omega\) and \(T_M^\perp \subset T_{Z\zeta \mid M}\) is the complement with respect to \(\bar{\omega}_0\).

Since \(M \subset U\) is defined by \(r^2 = r_1^2 + c\), equations \((4.7), (4.14)\) imply that \(T_{Z\zeta}^\perp\) is spanned by \(\xi \mid_M\) and \((\partial_{\theta_2} - \partial_{\theta_1})\mid_M\). Therefore \((q^* T_Y)^\perp\) is spanned by \(p(\partial_{\theta_2} - \partial_{\theta_1})\mid_M\). The direct sum decomposition \((4.9)\) implies that \((\partial_{\theta_2} - \partial_{\theta_1})\mid_M\) belongs to \(T_M\), hence \(p(\partial_{\theta_2} - \partial_{\theta_1})\mid_M\) belongs to \(q^* (T_Y)\). Moreover \((\partial_{\theta_2} - \partial_{\theta_1})\mid_M\) is invariant under the \(U(1)\)-action \((4.12)\), that is the Lie derivative with respect to \(\xi \mid_M\) vanishes,

\[L_{\xi \mid_M}(\partial_{\theta_2} - \partial_{\theta_1})\mid_M = 0.\]

Therefore \(p(\partial_{\theta_2} - \partial_{\theta_1})\mid_M\) descends to a nonvanishing vector field \(\eta\) on \(Y\) which generates \(T_Y^\perp \subset T_Y\) at any point.

Next we have to construct the line bundle \(L\) with \(U(1)\)-connection \(A\) on \(Y\). We will take \(L\) to be the trivial complex line bundle on \(Y\). In order to construct the \(U(1)\)-connection on \(L\), one can construct in principle a \(U(1)\) connection on the trivial line bundle on \(M\) which descends to a connection on \(L\) satisfying the coisotropic \(A\)-brane conditions. Since \(L\) is trivial it suffices to construct a closed real 2-form \(\tilde{F}\) on \(M\) which descends to a closed real 2-form \(F\) on \(Y\) so that \(F\) satisfies the required conditions. Since there are no closed 2-cycles on \(Y\), one can then find a globally defined connection 1-form \(A\) on \(Y\) so that \(F = dA\).

Invariance under \(U(1)\) generated by \(\xi \mid_M\) suggests the following ansatz for \(\tilde{F}\)

\[(4.19) \quad \tilde{F} = \frac{1}{2} e^{i(\theta_1 + \theta_2)} \left( E_{(-a)} \wedge E_{(-b)} + i \left( \lambda_1 U E_{(-b)} + \lambda_2 V E_{(-a)} \right) \wedge (d\theta_1 - d\theta_2) \right) + c.c.\]

where

\[E_{(-a)} = DU + iAU d\theta_1, \quad E_{(-b)} = DV + iBV d\theta_2\]

transform under the \(U(1)\) in \((4.12)\) as

\[E_{(-a)} \mapsto e^{-ia\alpha} E_{(-a)}, \quad E_{(-b)} \mapsto e^{-ib\alpha} E_{(-b)}\]

Now we impose

\[(4.20) \quad d\tilde{F} = 0, \quad L_{\partial_{\theta_2}} \tilde{F} = 0, \quad \theta_- = \theta_2 - \theta_1\]

which fixes a linear combination of the two parameters in our ansatz:

\[(4.21) \quad \lambda_1 - \lambda_2 = a - 1\]

\[\text{Recall the definition of Lie derivative } \mathcal{L}_v = \iota_v d + d \iota_v.\]
Therefore in order to prove that $\tilde{F}$ is by construction preserved by the torus action (4.12), equations (4.20) imply that there exists a closed real 2-form $F$ on $Y$ so that $\tilde{F} = q^*F$ and
\begin{equation}
\mathcal{L}_\eta F = 0
\end{equation}
on $Y$. Recall that $\eta$ generates the symplectic complement $T^\perp_Y \subset T_Y$, as shown below equation (4.18).

Moreover, $\mathcal{L}_\eta(\omega|_Y) = 0$ as a consequence of (4.11). Therefore $F, \omega|_Y$ determine global sections $\mathcal{F}, \sigma$ of $\Lambda^2 F^*_Y$, where $F_Y = T_Y/T^\perp_Y$. Let $J : F_Y \to F_Y$ be the linear map determined by
\begin{equation}
\mathcal{F}(h_1, h_2) = \sigma(h_1, J(h_2))
\end{equation}
for any $h_1, h_2 \in F_Y$. In order to complete the construction, one has to check that $J^2 = -1$, hence $J$ defines an almost complex structure on $F_Y$.

Let $F_M = T_M/T^\perp_M$, where $T^\perp_M \subset T_M$ is the complement of $T_M$ in $T_Z\subset M$ with respect to $\tilde{\omega}|_Z$. Since $T^\perp_M$ is spanned by $\xi|_M$ and $\partial_{\theta, i}|_M$, equations (4.11), (4.20), (4.22) imply that $\tilde{F}, \tilde{\omega}|_M$ determine global sections $\tilde{\mathcal{F}}, \tilde{\sigma}$ of $\Lambda^2 F^*_M$. Let $\tilde{J} : F_M \to F_M$ be the linear map determined by
\begin{equation}
\tilde{\mathcal{F}}(h_1, h_2) = \tilde{\sigma}(h_1, \tilde{J}(h_2))
\end{equation}
for any $h_1, h_2 \in F_M$.

Now note that the exact sequence (4.16) and equation (4.18) imply that there exists a commutative diagram of the form
\begin{equation}
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & & \\
0 & T_q & T_q & 0 \\
\downarrow & & & \\
0 & T^\perp_M & T_M & F_M \to 0 \\
\downarrow & & & \\
0 & q^*T^\perp_Y & q^*T_Y & q^*F_Y \to 0 \\
\downarrow & & & \\
0 & 0 & 0 & 0
\end{array}
\end{equation}
where all columns and rows are exact. In particular the pushforward map $p : T_M \to T_Y$ yields a natural isomorphism $F_M \simeq q^*F_Y$, which will also be denoted by $p$ in the following. Moreover, by construction
\begin{equation}
p \circ \tilde{J} = (q^*J) \circ p.
\end{equation}
Therefore in order to prove that $J^2 = -1$ it suffices to prove that $\tilde{J}^2 = -1$. To this end, note that there is a direct sum decomposition
\begin{equation}
T_M \simeq T^\perp_M \oplus F_M
\end{equation}
where $F_M$ is identified with the linear sub-bundle of $T_M$ spanned by the vector fields $(\xi_i|_M, \eta_i|_M)$, $i = 1, 2$. Recall that
\begin{equation}
\xi_i = \partial_{U_i} + \frac{aU_i}{2} \left( \frac{\partial r_1}{r_1} + \frac{\partial r_2}{r_2} \right) \quad \eta_i = \partial_{V_i} + \frac{bV_i}{2} \left( \frac{\partial r_1}{r_1} + \frac{\partial r_2}{r_2} \right)
\end{equation}
for $i = 1, 2$. Equations (4.5), (4.10) imply that
\begin{align*}
\tilde{F}(\xi_i, \xi_j) &= \tilde{F}(\partial_{U_i}, \partial_{U_j}) \\
\tilde{F}(\xi_i, \eta_j) &= \tilde{F}(\partial_{V_i}, \partial_{V_j}) \\
\tilde{F}(\eta_i, \eta_j) &= \tilde{F}(\partial_{U_i}, \partial_{V_j}) \\
\tilde{\omega}(\xi_i, \xi_j) &= \tilde{\omega}(\partial_{U_i}, \partial_{U_j}) \\
\tilde{\omega}(\eta_i, \eta_j) &= \tilde{\omega}(\partial_{U_i}, \partial_{V_j}) \\
\tilde{\omega}(\xi_i, \eta_j) &= \tilde{\omega}(\partial_{U_i}, \partial_{V_j})
\end{align*}
for all $i, j = 1, 2$. Then a straightforward computation yields
\begin{equation}
\begin{split}
\tilde{J}(\xi_1) = -\sin \theta_{ab} \eta_1 - \cos \theta_{ab} \eta_2 \\
\tilde{J}(\xi_2) = -\cos \theta_{ab} \eta_1 + \sin \theta_{ab} \eta_2 \\
\tilde{J}(\eta_1) = \sin \theta_{ab} \xi_1 + \cos \theta_{ab} \xi_2 \\
\tilde{J}(\eta_2) = \cos \theta_{ab} \xi_1 - \sin \theta_{ab} \xi_2.
\end{split}
\end{equation}
where $\theta_{ab} = a \theta_1 + b \theta_2$. Equations (4.24) easily imply $\tilde{J}^2 = -1$, concluding the construction.

### 4.2. Boundary matrix and disk multi-covers.
As explained in Section 2, evaluation of disk amplitudes via localization requires an explicit expression for the boundary matrix $R$ in terms of local holomorphic coordinates on $X$. The standard affine toric coordinate patches on $X$ are
\begin{equation}
\begin{align*}
U_1 &: X_2 \neq 0 \\
& z_1 = \frac{X_1}{X_2} \\
& u_1 = X_2^q U \\
& v_1 = X_2^q V \\
U_2 &: X_1 \neq 0 \\
& z_2 = \frac{X_2}{X_1} \\
& u_2 = X_1^q U \\
& v_2 = X_1^q V.
\end{align*}
\end{equation}
Note that $Y \subset U_1 \cap U_2 = q(U)$, where $U \subset Z_\zeta$ is the open subset $X_1 \neq 0, X_2 \neq 0$ introduced in the previous subsection.

To simplify the computation of $R$, we may set parameter $c = 0$ in the defining equation for $M$. The weights of fermion zero modes under the torus $\mathbf{T}$, and hence the contribution of the isolated fixed point to the virtual localization formula (2.5), do not depend on $c$.

There are two torus invariant holomorphic disks in $X$ with boundary on $Y$
\begin{equation}
\begin{align*}
D_1 & : |z_1| \leq 1, \quad u_1 = v_1 = 0 \\
D_2 & : |z_2| \leq 1, \quad u_2 = v_2 = 0.
\end{align*}
\end{equation}
It suffices to do the explicit computations only for $D_1$, since $D_2$ is entirely analogous. The boundary of $D_1$ is contained in both holomorphic coordinate charts $U_1, U_2$. Then a straightforward computation yields
\begin{equation}
q_*(X_1 \partial X_2 |_M) = \partial z_2 \quad q_*(X_1^{-a} \partial U |_M) = \partial u_2 \quad q_*(X_1^{-b} \partial V |_M) = \partial v_2.
\end{equation}
Let us define real coordinate functions on $U_1 \cap U_2$ by
\begin{equation}
z_2 = \rho_2 e^{i \phi_2} \quad u_2 = x_2 + i y_2 \quad v_2 = x_3 + i y_3.
\end{equation}
In terms of the real coordinates (4.26),
\begin{equation}
\begin{align*}
q_*(\partial_{\rho_1})|_{\partial D_1} &= -\frac{1}{r} \partial_{\rho_1}|_{\partial D_1} \\
q_*(\partial_{\rho_2})|_{\partial D_1} &= \frac{1}{r} \partial_{\rho_2}|_{\partial D_1} \\
q_*(\partial_{\theta_1})|_{\partial D_1} &= -\partial_{\phi_2}|_{\partial D_1} \\
q_*(\partial_{\theta_2})|_{\partial D_1} &= \partial_{\phi_2}|_{\partial D_1} \\
q_*\left(\frac{1}{r \alpha}(\cos(\alpha \theta_1) \partial U_1 - \sin(\alpha \theta_1) \partial U_2)\right)|_{\partial D_1} &= \partial x_2|_{\partial D_1} \\
q_*\left(\frac{1}{r \alpha}(\sin(\alpha \theta_1) \partial U_1 + \cos(\alpha \theta_1) \partial U_2)\right)|_{\partial D_1} &= \partial y_2|_{\partial D_1} \\
q_*\left(\frac{1}{r \beta}(\cos(\beta \theta_1) \partial V_1 - \sin(\beta \theta_1) \partial V_2)\right)|_{\partial D_1} &= \partial x_3|_{\partial D_1} \\
q_*\left(\frac{1}{r \beta}(\sin(\beta \theta_1) \partial V_1 + \cos(\beta \theta_1) \partial V_2)\right)|_{\partial D_1} &= \partial y_3|_{\partial D_1}
\end{align*}
\end{equation}
where $r = r_1 = r_2$ on $\partial D_1$ satisfies the moment map equation $2r^2 = \zeta$. Note that the restriction $N_Y|_{\partial D_1}$ of the normal bundle to $Y$ in $X$ to the boundary of $D_1$ is generated by $\partial_{\rho_2}|_{\partial D_1}$. The restriction $T_Y|_{\partial D_1}$ is generated by $\partial_{\rho_2}|_{\partial D_1}, \partial_{\theta_1}|_{\partial D_1}, \partial_{\phi_1}|_{\partial D_1}, i = 2, 3$. The restriction of the symplectic complement $T^\perp_Y|_{\partial D_1}$ is generated by $\eta|_{\partial D_1} = \partial_{\phi_2}|_{\partial D_1}$. Therefore the restriction of the tangent space $T_X|_{\partial D_1}$ admits the direct sum decomposition
\begin{equation}
T_X|_{\partial D_1} \simeq N_Y|_{\partial D_1} \oplus T^\perp_Y|_{\partial D_1} \oplus F_Y|_{\partial D_1}
\end{equation}
where $F_Y|_{\partial D_1}$ is generated by $\partial_{x_i}, \partial_{y_i}, i = 2, 3$.

Rewriting equation (4.19) in real coordinates yields

(4.29) \[ \tilde{F} = \cos(\theta_1 + \theta_2) \left( (dU_1 - aU_2 d\theta_1) \wedge (dV_1 - bV_2 d\theta_2) - (dU_2 + aU_1 d\theta_1) \wedge (dV_2 + bV_1 d\theta_2) \right) 
- \sin(\theta_1 + \theta_2) \left( (dU_1 - aU_2 d\theta_1) \wedge (dV_2 + bV_1 d\theta_2) + (dU_2 + aU_1 d\theta_1) \wedge (dV_1 - bV_2 d\theta_2) \right) 
- \lambda_1 \cos(\theta_1 + \theta_2) \left( U_1 (dV_2 + bV_1 d\theta_2) + U_2 (dV_1 - bV_2 d\theta_2) \right) \wedge (d\theta_1 - d\theta_2) 
- \lambda_1 \sin(\theta_1 + \theta_2) \left( U_1 (dV_1 - bV_2 d\theta_2) - U_2 (dV_2 + bV_1 d\theta_2) \right) \wedge (d\theta_1 - d\theta_2) 
- \lambda_2 \cos(\theta_1 + \theta_2) \left( V_1 (dU_2 + aU_1 d\theta_1) + V_2 (dU_1 - aU_2 d\theta_1) \right) \wedge (d\theta_1 - d\theta_2) 
- \lambda_2 \sin(\theta_1 + \theta_2) \left( V_1 (dU_1 - aU_2 d\theta_1) - V_2 (dU_2 + aU_1 d\theta_1) \right) \wedge (d\theta_1 - d\theta_2) 
\]

where $\lambda_2 \in \mathbb{R}$ and $\lambda_1 = a - 1 + \lambda_2$. Then, using the identity

\[ \tilde{F}(h_1, h_2) = F(q_s h_1, q_s h_2) \]

for any two tangent vectors $h_1, h_2 \in T_M$, one obtains the following

\[ F(\partial_{x_2}, \partial_{y_2})|_{\partial D_1} = 0 \quad F(\partial_{x_3}, \partial_{y_3})|_{\partial D_1} = 0 \]

(4.30) \[ F(\partial_{x_2}, \partial_{x_3})|_{\partial D_1} = \frac{1}{r^2} \cos(\phi_2) \quad F(\partial_{y_2}, \partial_{y_3})|_{\partial D_1} = -\frac{1}{r^2} \cos(\phi_2) \]

\[ F(\partial_{x_2}, \partial_{y_3})|_{\partial D_1} = -\frac{1}{r^2} \sin(\phi_2) \quad F(\partial_{x_3}, \partial_{y_2})|_{\partial D_1} = \frac{1}{r^2} \sin(\phi_2) \]

\[ F(\partial_{\theta_2}, \partial_{x_1})|_{\partial D_1} = F(\partial_{\phi_2}, \partial_{y_1})|_{\partial D_1} = 0 \]

for $i = 2, 3$.

In order to compute the boundary matrix $R$, we also have to evaluate the quotient Kähler metric $G$ on the coordinate vector fields $\partial_{\theta_s}, \partial_{x_i}, \partial_{y_i}, i = 2, 3$, restricted to $\partial D_1$. The local expression of the Kähler metric $\tilde{G}|_{\mathcal{U}}$ is

(4.31) \[ \tilde{G}|_{\mathcal{U}} = \sum_{i=1}^{2} (dr_i \otimes dr_i + r_i^2 d\theta_i \otimes d\theta_i)|_{\mathcal{U}} + \sum_{i=1}^{2} (dU_i \otimes dU_i + dV_i \otimes dV_i)|_{\mathcal{U}}. \]

The inverse image $q^{-1}(\partial D_1) \subset \mathcal{U}$ is determined by the equations

\[ r_1 = r_2 \quad U = V = 0. \]

Therefore

\[ \xi|_{q^{-1}(\partial D_1)} = (\partial_{\theta_1} + \partial_{\theta_2})|_{q^{-1}(\partial D_1)} = 2\partial_{\theta_s}|_{q^{-1}(\partial D_1)} \]

Then the orthogonal complement of $\xi|_{q^{-1}(\partial D_1)}$ in $T_M|_{q^{-1}(\partial D_1)}$ with respect to the metric $\tilde{G}|_{\mathcal{M}}$ is generated by

\[ \partial_{\theta_-}|_{q^{-1}(\partial D_1)}, \quad \partial_{U_i}|_{q^{-1}(\partial D_1)}, \quad \partial_{V_i}|_{q^{-1}(\partial D_1)}, \quad i = 1, 2, \]

where $\theta_- = \theta_2 - \theta_1$ and $\partial_{\theta_-} = \frac{1}{2} (\partial_{\theta_2} - \partial_{\theta_1})$. Moreover, note that the induced metric on the orthogonal complement of $\xi$ is invariant under the $U(1)$ action. Therefore, using relations (4.27) one obtains

(4.32) \[ G(\partial_{x_2}, \partial_{x_2})|_{\partial D_1} = G(\partial_{y_2}, \partial_{y_2})|_{\partial D_1} = \frac{1}{r^2} \quad G(\partial_{x_3}, \partial_{x_3})|_{\partial D_1} = G(\partial_{y_3}, \partial_{y_3})|_{\partial D_1} = \frac{1}{r^2} \]

\[ G(\partial_{\theta_2}, \partial_{\phi_2})|_{\partial D_1} = r^2, \]

all other matrix elements of $G$ being trivial.
Equations (4.30), (4.32) imply that the linear map $R|_{\partial D_1} : T_X|_{\partial D_1} \to T_X|_{\partial D_1}$ has the following block form with respect to the direct sum (4.28)

\[
R|_{\partial D_1} = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & S
\end{bmatrix}
\]

(4.33)

where $S : F_Y|_{\partial D_1} \to F_Y|_{\partial D_1}$ is determined by the following condition

\[(G + F)(h_1, h_2)|_{\partial D_1} = (G - F)(h_1, S(h_2))|_{\partial D_1}\]

for any $h_1, h_2 \in F_Y|_{\partial D_1}$. Then a straightforward computation yields

\[
\begin{align*}
S(\partial_{x_2}) &= \tau^2 - 2a (\cos(\phi_2) \partial_{x_3} + \sin(\phi_2) \partial_{y_3}) \\
S(\partial_{y_2}) &= \tau^2 - 2a (\sin(\phi_2) \partial_{x_3} + \cos(\phi_2) \partial_{y_3}) \\
S(\partial_{x_3}) &= \tau^2 - 2b (\cos(\phi_2) \partial_{x_2} - \sin(\phi_2) \partial_{y_2}) \\
S(\partial_{y_3}) &= \tau^2 - 2b (\sin(\phi_2) \partial_{x_2} - \cos(\phi_2) \partial_{y_2})
\end{align*}
\]

where recall that $\tau^2 = \frac{\xi}{\xi}$ on $\partial D_1$. In terms of holomorphic coordinate vector fields we obtain

\[
R(\partial z_2)|_{\partial D_1} = -e^{-2i\phi_2} \partial z_2|_{\partial D_1}, \quad R(\partial u_2)|_{\partial D_1} = -e^{i\phi_2} \tau^2 - 2a \partial u_2|_{\partial D_1}, \quad R(\partial v_2)|_{\partial D_1} = e^{i\phi_2} \tau^2 - 2b \partial v_2|_{\partial D_1}
\]

and complex-conjugated equations.

Let $f : \Delta \to X$ be a degree $d \geq 1$ torus invariant holomorphic map with coisotropic boundary conditions along $Y$ factors through the embedding $D_1 \subset X$.

Let $\mathcal{V}_1, \mathcal{V}_2 \subset \Delta$ be an open cover of $\Delta$ so that $\mathcal{V}_1 = \Delta \setminus \partial \Delta$, and $\mathcal{V}_2 = \Delta \setminus \{0\}$. Let $t_1, t_2$ be affine coordinates on $\mathcal{V}_1, \mathcal{V}_2$ so that the map $f$ is locally given by

\[
z_1 = t_1^d, \quad z_2 = t_2^d.
\]

(4.35)

Note that $t_1$ is centered at the origin and $t_1 t_2 = 1$ on the overlap $\mathcal{V}_1 \cap \mathcal{V}_2$.

As reviewed in Section 2, $R$ determines boundary conditions for fermions

\[
\Psi_+ = R(\Psi_-)
\]

where by $| \cdot |_{\partial \Delta}$ and in the coordinate chart $U_2$

\[
\Psi_\pm = \Psi_\pm^z \partial_{z_2} + \Psi_\pm^u \partial u_2 + \Psi_\pm^v \partial v_2 + \Psi_\pm^z \partial_{\bar{z}_2} + \Psi_\pm^u \partial \bar{u}_2 + \Psi_\pm^v \partial \bar{v}_2.
\]

In components, (4.36) implies the following boundary conditions

\[
\begin{align*}
\Psi_+^z &= -e^{2i\phi_2} \Psi_\pm^z, & \Psi_+^z &= -e^{-2i\phi_2} \Psi_\pm^z, & \Psi_-^z &= e^{-i\phi_2} \tau^2 - 2a \Psi_\pm^z, \\
\Psi_+^z &= e^{i\phi_2} \tau^2 - 2b \Psi_\pm^z, & \Psi_-^z &= e^{-i\phi_2} \tau^2 - 2a \Psi_\pm^z, & \Psi_-^z &= e^{i\phi_2} \tau^2 - 2b \Psi_\pm^z
\end{align*}
\]

where $r^2 = \frac{\xi}{\xi}$ and, as follows from (4.33), $e^{i\phi_2} = \frac{t_2}{t_1^d}$. Now we use relation (2.3) between $\Psi_\pm$ and fermions in A-model where in the present case $i = z_2, u_2, v_2$, and we find boundary conditions $\chi$

\[
\begin{align*}
\chi^z &= -e^{2i\phi_2} \chi^z, & \chi^z &= e^{-2i\phi_2} \tau^2 - 2b \chi^z, & \chi^v &= -e^{-i\phi_2} \tau^2 - 2b \chi^v
\end{align*}
\]

and for $\psi$

\[
\begin{align*}
\psi^z &= -e^{2i\phi_2} \psi^z, & \psi^z &= -e^{-2i\phi_2} \tau^2 - 2b \psi^z, & \psi^v &= e^{-i\phi_2} \tau^2 - 2b \psi^v
\end{align*}
\]

\[4\text{In untwisted }\sigma\text{-model reality condition on fermions } (\Psi^i_\pm)^* = \epsilon_{\alpha \beta} J^i_\beta \Psi^\alpha_\beta \text{ uses almost complex structure } J \text{ on the target space. In topological A-model one can choose any } J. \text{ We take } J \text{ such that fermionic reality conditions are } (\chi^z)^* = \chi^u, (\chi^z)^* = -\chi^v, (\chi^z)^* = \chi^z, (\chi^u)^* = \psi^z, (\psi^z)^* = -\psi^v, (\psi^z)^* = \psi^z.\]
In Appendix A.1 we found \( \chi \) zero modes

\[
\chi^{(zm)} = \left( \sum_{m=0}^{2d} \alpha_m \ell^m \right) \partial_{z_2}, \quad \overline{\alpha}_m = -\alpha_{2d-m}.
\]

If \( T \) acts on \( X \) as

\[
X_1 \mapsto e^{i\varphi} X_1, \quad X_2 \mapsto X_2, \quad U \mapsto e^{i(n+1)\varphi} U, \quad V \mapsto e^{-i(n+1)\varphi} V
\]

then we find the weights of \( \chi^{(zm)} \):

\[
\left\{ \frac{1}{d}, \ldots, \frac{d-1}{d}, 1 \right\} \quad \& \quad 0_R
\]

where \( 0_R \) corresponds to the real mode with \( m = d \) in (4.39) and weights in the bracket correspond to complex modes for \( m = 0, \ldots, d-1 \).

In Appendix A.2 we found \( \psi \) zero modes (which exist only if \( d > 1 \))

\[
\psi^{(zm)} = \sum_{k=1}^{d-1} \left( b_k t_2^{k-ad} \partial_{u_2} + c_k t_2^{k-bd} \partial_{v_2} \right) \quad b_k = -r^2 - 2b e^{-d-k} \quad d > 1.
\]

The weights of \( \psi^{(zm)} \) under (4.40) are

\[
\left\{ n + \frac{1}{d}, \ldots, n + \frac{d-1}{d} \right\}.
\]

In (4.41) and (4.43) we recognize the weights of fermion zero modes for disk multi-covers ending on a toric Lagrangian brane in resolved conifold [11]. Then the localization formula (2.5) gives the contribution of the multi-covers of \( D_1 \)

\[
W = \sum_{d=1}^{\infty} \frac{N_d(n)}{d^2} e^{-dy} e^{-dt/2} \quad N_1 = 1 \quad N_d(n) = \prod_{j=1}^{d-1} (j + nd) (d-1)! \in \mathbb{Z} \quad \text{for} \quad d > 1.
\]

In (4.44) \( y = c + i\mathcal{A} \) where \( c \) is a parameter in \( |X_2|^2 = |X_1|^2 + c \) and \( \mathcal{A} \) is a Wilson line around \( \partial D_1 \) in \( Y \), and \( t \) is complexified Kähler modulus of the local geometry. Note that (4.44) is independent of \( a \) specifying the normal bundle \( \mathcal{O}(-a) \oplus \mathcal{O}(a-2) \) but depends on \( n \) which enters in the choice of \( T \)-action (4.40).

So far we discussed multi-covers of \( D_1 \). One can do an analogous computation for \( D_2 \), the second torus invariant holomorphic disk in \( X \) with boundary on \( Y \). One finds fermion zero modes

\[
\chi^{(zm)} = \left( \sum_{m=0}^{2d} \alpha_m \ell^m \right) \partial_{z_1}, \quad \overline{\alpha}_m = -\alpha_{2d-m}.
\]

\[
\psi^{(zm)} = \sum_{k=1}^{d-1} \left( b_k t_1^{k-ad} \partial_{u_1} + c_k t_1^{k-bd} \partial_{v_1} \right) \quad b_k = -r^2 - 2b e^{-d-k} \cdot
\]

and the contribution of the multi-covers of \( D_2 \) is

\[
W = \sum_{d=1}^{\infty} \frac{N_d(\tilde{n})}{d^2} e^{-dy} e^{-dt/2}
\]

where \( \tilde{n} = n + 1 - b \).
5. Surface defect in geometrically engineered SU(N) gauge theory

Here we give an example of coisotropic brane Y in toric Calabi-Yau 3-fold $X_{SU(N)}$ used in IIA string theory for geometric engineering [26] of 4d $\mathcal{N} = 2$ SU(N) gauge theory. By considering D6 brane supported on $R^{1,1} \times Y$, one gets a surface defect in 4d theory with (2,2) supersymmetric gauge theory on $R^{1,1}$. Similar to [27,10,28], counting disk multi-covers in $X_{SU(N)}$ ending on $Y$ gives contribution to the superpotential for the chiral field $y$ supported on the surface defect.

Let us first consider SU(2) gauge theory arising from $X_{SU(2)}$ defined as a toric manifold

$$
\begin{array}{c|cccc}
\mathbb{C}^*_1 & X_1 & X_2 & X_3 & X_4 & W \\
\hline
\mathbb{C}^*_1 & 1 & 1 & 1 & 0 & -3 \\
\mathbb{C}^*_2 & 0 & 0 & 1 & 1 & -2 \\
\end{array}
$$

The construction starts from defining $M$ as real codimension 3 submanifold in $\mathbb{C}^5$

$$|X_1|^2 + |X_2|^2 + |X_3|^2 - 3|W|^2 = \zeta_1, \quad |X_3|^2 + |X_4|^2 - 2|W|^2 = \zeta_2, \quad |X_2|^2 = |X_1|^2 + c. $$

To ensure that $X_1, X_2 \neq 0$ on $M$ and hence $\theta_1$ and $\theta_2$ are well-defined, we must choose $\zeta_1$ and $\zeta_2$ such that $\zeta_1 - \zeta_2 - c > 0$. $Y$ is defined as a quotient of $M$ by the two symplectic $U(1)$ actions

$$
\begin{align*}
\xi_1 : \{X_1, X_2, X_3, X_4, W\} & \mapsto \{e^{i\alpha}X_1, e^{i\alpha}X_2, e^{i\alpha}X_3, X_4, e^{-3i\alpha}W\} \\
\xi_2 : \{X_1, X_2, X_3, X_4, W\} & \mapsto \{X_1, X_2, e^{i\beta}X_3, e^{i\beta}X_4, e^{-2i\beta}W\}
\end{align*}
$$

The following closed 2-form is invariant under both $\xi_1$ and $\xi_2$ actions

$$\tilde{F} = e^{i(\theta_1 + \theta_2)} \left( (X_3dX_4 - X_4dX_3) \wedge dW - 2WdX_3 \wedge dX_4 - iX_4(X_3dW + 2WdX_3) \wedge d\theta_1 \\
+ iW(X_3dX_4 - X_4dX_3) \wedge d\theta_2 - WX_3X_4d\theta_1 \wedge d\theta_2 \right) + c.c
$$

and gives rise to the closed 2-form $F$ on $Y$. Let us clarify that $\tilde{F}$ provides the appropriate brane flux for $Y$ to be coisotropic brane in $X_{SU(2)}$.

In the patch $X_3 \neq 0$ we may define $U = \frac{X_3}{X_4}$ and $V = WX_3^2$ so that $X_1, X_2, U, V$ describe local $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In this patch $\tilde{F}$ coincides with (4.19) for $a = b = 1$ and $\lambda_1 = \lambda_2 = 0$ which we used before to define coisotropic brane in resolved conifold.

Meanwhile, in the patch $X_4 \neq 0$ we define

$$
\tilde{U} = WX_4^2, \quad \tilde{V} = \frac{X_3}{X_4}
$$

so that $X_1, X_2, \tilde{U}, \tilde{V}$ describe local $\mathbb{P}^1$ geometry with normal bundle $\mathcal{O}(-3) \oplus \mathcal{O}(1)$. In this patch $\tilde{F}$ coincides with (4.19) for $a = 3, b = -1$ and $\lambda_1 = 1, \lambda_2 = -1$. In this way in both patches $Y$ is coisotropic brane in the corresponding local geometry.

Let us choose torus $T$ acting on $X_{SU(2)}$ as

$$
T : X_1 \mapsto e^{i\varphi}X_1, \quad X_2 \mapsto X_2, \quad X_3 \mapsto X_3, \quad X_4 \mapsto e^{in\varphi}X_4, \quad W \mapsto e^{-i(n+1)\varphi}W.
$$

Contribution to the chiral superpotential on the surface defect from disk multi-covers in both local $\mathbb{P}^1$ geometries is

$$
W = \sum_{d=1}^{\infty} \left( N_d(n)e^{-dt_1/2} + N_d(n-1)e^{-dt_2/2} \right) e^{dy} + \sum_{d=1}^{\infty} \left( N_d(n)e^{-dt_1/2} + N_d(n+1)e^{-dt_2/2} \right) e^{-dy}
$$

where $t_1, t_2$ are complexified Kähler moduli.
As in [6], there are other contributions to $W$ from holomorphic maps with reducible domain $\Sigma$ that contains $k$ copies of $\mathbb{P}^1$ and d-multi-covers of the disk. These are suppressed at large $\text{Re}(t)$ as $e^{-(k+d/2)t}$ with appropriate $t$. Assuming the standard Kähler function for $y$, the stability of the surface defect can be argued using (5.2).

The story is easy to generalize. Let us consider toric manifold $X_{SU(N)}$ which is typically used in geometric engineering of $\mathcal{N} = 2, 4d$ $SU(N)$ gauge theory with $N \geq 3$.

\begin{equation}
\begin{array}{cccccccc}
\mathbb{C}_*^{(1)} & W_1 & W_2 & W_3 & \cdots & W_{N-1} & W_N & W_{N+1} & X_1 & X_2 \\
\mathbb{C}_*^{(2)} & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbb{C}_*^{(N-1)} & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
\mathbb{C}_*^{(N)} & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\end{equation}

In this case is real codimension $N + 1$ submanifold of $\mathbb{C}^{N+3}$

$$|W_1|^2 - 2|W_2|^2 + |W_3|^2 = \zeta_1, \quad |W_2|^2 - 2|W_3|^2 + |W_4|^2 = \zeta_2, \quad \ldots,$$

$$|W_{N-1}|^2 - 2|W_N|^2 + |W_{N+1}|^2 = \zeta_{N-1}, \quad |X_1|^2 + |X_2|^2 - |W_1|^2 - |W_2|^2 = \zeta_N, \quad |X_2|^2 = |X_1|^2 + c$$

where we choose $\zeta_N > c$ so that $X_1, X_2 \neq 0$ on $M$. $Y$ is the quotient of $M$ by symplectic $U(1)^N$ action.

Let $X_a = r_a e^{i\theta_a}$ $a = 1, 2$. The following closed 2-form on $M$

$$\tilde{F} = e^{i(\theta_1 + \theta_2)} \left( \prod_{j=3}^{N+1} W_j (dW_1 + iW_1d\theta_1) \wedge (dW_2 + iW_2d\theta_2) + \sum_{k=3}^{N+1} \left( \prod_{j=3, j \neq k}^{N+1} W_j \right) \left( (k-1)W_2(dW_1 + iW_1d\theta_1) + (k-2)W_1(dW_2 + iW_2d\theta_2) \right) \wedge dW_k \right)$$

is invariant under symplectic $U(1)^N$ and descends to the closed 2-form $F$ on $Y$. Similarly to the previously considered $X_{SU(2)}$, $\tilde{F}$ provides the appropriate brane flux for $Y$ to be coisotropic brane in $X_{SU(N)}$. Namely, there are $N$ local $\mathbb{P}^1$ geometries in $X_{SU(N)}$. In each of these geometries, $\tilde{F}$ coincides with (4.19) for the appropriate $a, b$ and $\lambda_1, \lambda_2$.

For example, in the patch $W_j \neq 0 j = 3, \ldots, N + 1$ we define

$$U = \frac{W_1}{\prod_{j=3}^{N+1} (W_j)^{j-2}}, \quad V = W_2 \prod_{j=3}^{N+1} (W_j)^{j-1}$$

so that $X_1, X_2, U, V$ describe local $\mathbb{P}^1$ geometry with normal bundle $(-1, -1)$. In this patch $\tilde{F}$ coincides with (4.19) for $a = b = 1$ and $\lambda_1 = \lambda_2 = 0$ which we used before to define coisotropic brane in resolved conifold.

Let us choose $T$ acting on $X_{SU(N)}$ as

$$T : X_1 \mapsto e^{i\varphi}X_1, \quad X_2 \mapsto X_2, \quad X_3 \mapsto X_3, \quad \ldots \quad X_N \mapsto X_N, \quad X_{N+1} \mapsto X_{N+1}, \quad W_1 \mapsto e^{i\varphi}W_1, \quad W_2 \mapsto e^{-i(n+1)\varphi}W_2.$$
Disk multi-covers ending on coisotropic brane $Y$ in each of $N$ local $\mathbb{P}^1$ geometries contribute to chiral superpotential on the surface defect

$$W = \sum_{k=0}^{N-1} \sum_{d=1}^{\infty} \left( \frac{N_d(n-k)e^{-dtk/2}}{d^2} e^{dy} + \frac{N_d(n+k)e^{-dtk/2}}{d^2} e^{-dy} \right)$$

where $t_k$ for $k = 0, \ldots, N - 1$ are complexified Kähler moduli.

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**Appendix A. Computation of fermion zero modes**

Let $X$ be a local $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-a) \oplus \mathcal{O}(-b)$ with $a + b = 2$. Let $f : \Delta \to X$ be a degree $d \geq 1$ torus invariant holomorphic map with coisotropic boundary conditions along $Y$ which factors through the embedding $D_1 \subset X$. This appendix consists of Čech cochain computations of cohomology spaces $H^0(\Delta, f^*\mathcal{T}_{D_1,R^+_\Delta})$ and $H^1(\Delta, f^*\mathcal{T}_{D_1,R^+_\Delta})$ which appear in virtual localization formula (2.5).

Let $\mathcal{V}_1, \mathcal{V}_2 \subset \Delta$ be an open cover of $\Delta$ so that $\mathcal{V}_1 = \Delta \setminus \partial \Delta$, and $\mathcal{V}_2 = \Delta \setminus \{0\}$. Let $t_1, t_2$ be affine coordinates on $\mathcal{V}_1, \mathcal{V}_2$ so that the map $f$ is locally given by

$$z_1 = t_1^d \quad z_2 = t_2^d.$$ 

Note that $t_1$ is centered at the origin and $t_1 t_2 = 1$ on the overlap $\mathcal{V}_1 \cap \mathcal{V}_2$.

A.1. $\chi$ zero modes. Let us first determine $\chi$ zero modes. For this we compute $H^0(\Delta, f^*\mathcal{T}_{D_1,R^+_\Delta})$.

The local sections are of the form

$$s_1 = \left( \sum_{n=0}^{\infty} \alpha_n t_1^n \right) \partial_{z_1} + \left( \sum_{n=0}^{\infty} \beta_n t_1^n \right) \partial_{u_1} + \left( \sum_{n=0}^{\infty} \gamma_n t_1^n \right) \partial_{v_1}$$

$$s_2 = \left( \sum_{n=-\infty}^{\infty} \alpha_n t_2^n \right) \partial_{z_2} + \left( \sum_{n=-\infty}^{\infty} \beta_n t_2^n \right) \partial_{u_2} + \left( \sum_{n=-\infty}^{\infty} \gamma_n t_2^n \right) \partial_{v_2}$$

Equation (4.37) yields the following boundary conditions for $s_2$

$$\alpha'_n + \alpha'_{2d-n} = 0 \quad \beta'_n = v^{2-2b-2d-n}$$

Moreover

$$\partial_{z_1} = -z_2^2 \partial_{z_2} \quad \partial_{u_1} = z_2^{-a} \partial_{u_2} \quad \partial_{v_1} = z_2^{-b} \partial_{v_2}$$

on the overlap $\mathcal{V}_1 \cap \mathcal{V}_2$. Therefore

$$\delta(s_1, s_2) = - \left( \sum_{n=0}^{\infty} \alpha_n t_2^{2d-n} \right) \partial_{z_2} + \left( \sum_{n=0}^{\infty} \beta_n t_2^{-ad-n} \right) \partial_{u_2} + \left( \sum_{n=0}^{\infty} \gamma_n t_2^{-bd-n} \right) \partial_{v_2}$$

$$- \left( \sum_{m=-\infty}^{\infty} \alpha'_m t_2^m \right) \partial_{z_2} - \left( \sum_{m=-\infty}^{\infty} \beta'_m t_2^m \right) \partial_{u_2} - \left( \sum_{m=-\infty}^{\infty} \gamma'_m t_2^m \right) \partial_{v_2}$$

and we find zero modes of $\chi$ as $\text{Ker} \delta$

$$\chi^{zm} = \left( \sum_{m=0}^{2d} \alpha'_m t_2^m \right) \partial_{z_2}, \quad \overline{\alpha'_m} = -\alpha'_{2d-m}.$$
A.2. ψ zero modes. Let us now determine ψ zero modes. For this we compute $H^1(\Delta, f^* T D_1, R_+)$.

The local sections are of the form

$$s_1 = \left( \sum_{n=0}^{\infty} \alpha_n t_1^n \right) \partial z_1 + \left( \sum_{n=0}^{\infty} \beta_n t_1^n \right) \partial u_1 + \left( \sum_{n=0}^{\infty} \gamma_n t_1^n \right) \partial v_1$$

$$s_2 = \left( \sum_{n=-\infty}^{\infty} \alpha'_n t_2^n \right) \partial z_2 + \left( \sum_{n=-\infty}^{\infty} \beta'_n t_2^n \right) \partial u_2 + \left( \sum_{n=-\infty}^{\infty} \gamma'_n t_2^n \right) \partial v_2$$

Equation (4.38) yields the following boundary conditions for $s_2$

$$\alpha'_n + \alpha'_{d-n} = 0 \quad \beta'_n = -r^{2-2b} \gamma'_{d-n}$$

As before we use that

$$\partial z_1 = -z_2^a \partial z_2 \quad \partial u_1 = z_2^{-a} \partial u_2 \quad \partial v_1 = z_2^{-b} \partial v_2$$

on the overlap $\mathcal{V}_1 \cap \mathcal{V}_2$. Therefore

$$\text{Im} \delta = -\left( \sum_{n=0}^{\infty} \alpha'_n \gamma'_{2d-n} \right) \partial z_2 + \left( \sum_{n=0}^{\infty} \beta'_n \gamma'_{2d-n} \right) \partial u_2 + \left( \sum_{n=0}^{\infty} \gamma'_n \gamma'_{2d-n} \right) \partial v_2$$

$$- \left( \sum_{m=-\infty}^{\infty} \alpha'_m \gamma'_{m2} \right) \partial z_2 - \left( \sum_{m=-\infty}^{\infty} \beta'_m \gamma'_{m2} \right) \partial u_2 + \left( \sum_{m=-\infty}^{\infty} \gamma'_m \gamma'_{m2} \right) \partial v_2$$

Zero modes of $\psi$ are 1-chains which cannot be written as $\text{Im} \delta$

$$\psi(z^m) = \sum_{k=1}^{d-1} \left( b_k t_2^{k-2a} \partial u_2 + c_k t_2^{k-2b} \partial v_2 \right) \quad b_k = -r^{2-2b} c_{d-k}.$$
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