Twists of Albanese variety of cyclic multiple planes with arbitrary large rank over function fields

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Abstract

In [16], we proved a structure theorem on the Mordell-Weil group of abelian varieties over function fields that arise as twists of abelian varieties by cyclic covers of quasi projective varieties in terms of Prym varieties associated to covers. In this paper, we are going to provide an explicit way to construct abelian varieties with arbitrary large rank over function fields. This will be done by applying the above mentioned theorem to the twists of Albanese variety of cyclic multiple plane.

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1 Introduction

Let $A$ be an abelian variety defined over an arbitrary global field $k$. Denote by $A(k)$ the set of $k$-rational points on $A$. It is proved that $A(k)$ is a finitely generated abelian group [6, 9]. Thus, we have $A(k) \cong A(k)_{\text{tors}} \oplus \mathbb{Z}^r$ where $A(k)_{\text{tors}}$ is a finite group called the torsion group of $A/k$; and $r$ is a positive number called the Mordell-Weil rank or simply the rank of $A/k$ and denoted by $\text{rk}(A(k))$. We note that study on the ranks of abelian varieties is more difficult than the torsion subgroups. In particular, finding abelian varieties of any dimension and arbitrary large rank over global fields is one of the challenging problems in modern number theory.

The aim of this paper is to provide an explicit method of the construction of abelian varieties with arbitrary large rank over the function fields of certain defined quotient varieties. The main tools of this work is the twisting theory.

Given the integers $n, r \geq 2$, we assume that $k$ is a global field of characteristic $\geq 0$, not dividing $n$, that contains an $n$th root of unity $\zeta_n$. We denote by $\mathbb{P}^\ell$ the projective space of dimension $\ell$ over $\bar{k}$ an algebraically closed field containing $k$. Given any polynomial $f(x, y) \in k[x, y]$, we denote by $X_n$ a non-singular model of the hypersurface $X_n$ defined by the affine equation $w^n = f(x, y)$. For any integer $m \geq 1$, we define $U_m$ to be the fibered product of $m$ copies of $X_n$ and $V_m$ be the quotient of $U_m$ by a certain cyclic subgroup of $\text{Aut}(U_m)$. See Section 5 for more details. We define $\tilde{X}_n$ to be the twist of $X_n$ by the cyclic extension $\mathcal{L}|\mathcal{K}$, where $\mathcal{K} = k(V_m)$ and $\mathcal{L} = k(U_m)$ are the function fields of $U_m$ and $V_m$ respectively. We denote
by $d_n$ the dimension of $\text{Alb}(X_n)$ the Albanese variety of $X_n$. We also let $\text{Alb}(X_n)[n](k)$ be the group of $k$-rational $n$-division points on $\text{Alb}(X_n)$. The main results of the paper are as follows.

**Theorem 1.1.** Notation being as above, we assume that there exists a $k$-rational point on $X_n$ and hence on $X_n$. Then, as an isomorphism of abelian groups, we have:

$$
\text{Alb}(\tilde{X}_n)(K) \cong (\text{End}_k(\text{Alb}(X_n)))^m \oplus \text{Alb}(X_n)[n](k),
$$

and hence, $\text{rk}(\text{Alb}(\tilde{X}_n)(K)) \geq m \cdot \text{rk}(\text{End}_k(\text{Alb}(X_n))).$

For some small values of $n$, say $3 \leq n \leq 12$ but $n \neq 7, 9, 11$, we have the following result.

**Theorem 1.2.** Keeping the notations and hypothesis of as above theorem, we assume that $k \subset \mathbb{C}$ is a field containing $\mathbb{Q}(\zeta_n)$. Then, we have

$$
\text{rk}(\text{Alb}(\tilde{X}_n)(K)) \geq m \cdot c_n,
$$

where $q_n$ is the irregularity of $X_n$ and $c_n = 2q_n$ if $n = 3, 4, 6$; and $c_n = 2d_n$ if $n = 5, 8, 10, 12$. In particular, for the polynomial

$$
f(x, y) = x^3 - 3xy(y^3 - 8) + 2(y^6 - 20y^3 - 8),
$$

the image of $X_6$ in $\text{Alb}(X_6)$ is a surface and $\text{rk}(\text{Alb}(\tilde{X}_6)(K)) \geq 2mq_6$.

**Remark 1.3.** We notice that Theorem 1.1 can be generalized for the twists of Albanese variety of an $n$-cover of the projective space $\mathbb{P}^\ell$ with $\ell \geq 3$. Indeed, given any polynomial $f(u_1, \cdots, u_\ell)$ with coefficients in $k$, if $X_n$ denotes a non-singular model of the hypersurface defined by affine equation $w^n = f(u_1, \cdots, u_\ell)$ and $\tilde{X}_n, U_m, V_m, K$ and $L$ are defined as above, then the assertions of Theorem holds for $\tilde{X}_n$ by adapting its proof.

The present paper has been organized as follows. Section 2 is devoted to recall the basics on the Albanese and Prym varieties. In Section 3 we review briefly the theory of $G$-sets and twists of varieties by Galois extensions. We provides some fundamental results on the multiple planes $X_n$ in Section 4. Finally, in Section 5 we give the proofs of main results.

## 2 Albanese and Prym varieties

In this section, we let $\mathcal{X}$ be a smooth (quasi-) projective variety over a field $k$ and $\text{Pic}(\mathcal{X})$ denotes its reduced Picard variety.

**Definition 2.1.** The **Albanese variety** of $\mathcal{X}$ is an abelian variety $\text{Alb}(\mathcal{X})$ together with a morphism $\alpha: \mathcal{X} \to \text{Alb}(\mathcal{X})$ satisfying the universal property: For any morphism $\alpha': \mathcal{X} \to \mathcal{A}$ such that $\mathcal{A}$ is an abelian variety, then there exist a unique homomorphism up to translation $\alpha'': \text{Alb}(\mathcal{X}) \to \mathcal{A}$ such that the diagram below commutes:

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\alpha} & \text{Alb}(\mathcal{X}) \\
\downarrow \alpha' & \cong & \downarrow \alpha'' \\
\mathcal{A} & \xrightarrow{id} & \mathcal{A}
\end{array}
$$

(2.1)

The morphism $\alpha$ is called the **Albanese morphism** and $\alpha(\mathcal{X})$ is known as the **Albanese image** of $\mathcal{X}$. Moreover, the dimension of $\alpha(\mathcal{X})$ is called the **Albanese dimension** of $\mathcal{X}$ and is denoted by $\text{Albdim}(\mathcal{X})$. 

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It can be shown that if \( \text{Alb}(\mathcal{X}) \) exists then it is unique up to unique isomorphism and it is equal to \( \text{Pic}^v(\mathcal{X}) \) the dual of the Picard variety of \( \mathcal{X} \). For example, when \( \mathcal{X} \) is a smooth projective curve then Albanese variety is nothing but \( J(\mathcal{X}) \) the Jacobian variety of \( \mathcal{X} \). Moreover, the albanese variety \( \text{Alb}(\mathcal{X}) \) is generated by \( \alpha(\mathcal{X}) \), the albanese image of \( \mathcal{X} \) by \( \alpha \). In other words, there is no abelian subvariety of \( \text{Alb}(\mathcal{X}) \) containing \( \alpha(\mathcal{X}) \). In particular, \( \alpha(\mathcal{X}) \) is not reduced to a point if \( \text{Alb}(\mathcal{X}) \) is not point.

We note that \( \text{Alb}(\mathcal{X}') \) is a functor from the category of smooth (quasi-) projective varieties to the category of abelian varieties. In other words, for any morphism \( \pi : \mathcal{X}' \to \mathcal{X} \) of smooth projective varieties, there exists a unique morphism \( \text{Alb}(\pi) : \text{Alb}(\mathcal{X}') \to \text{Alb}(\mathcal{X}) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{\alpha'} & \mathcal{X} \\
\downarrow{\pi} & & \downarrow{\alpha} \\
\text{Alb}(\mathcal{X}') & \xrightarrow{\text{Alb}(\pi)} & \text{Alb}(\mathcal{X})
\end{array}
\]

(2.2)

It is remarkable that if the morphism \( \pi \) is surjective, then the morphism \( \text{Alb}(\pi) \) is surjective too. This suggest the following definition of Prym variety, which is introduced by Mumford in [14] for double covers of curves over complex field and extensively studied by Beauville in [1]. It is considered for double covers of smooth surfaces by Khashin in [7]. In [5], it is generalized for double covers of quasi-projective varieties by F. Hazam. In [16], we extended the notion of Prym variety to the case of cyclic \( n \)-cover \( \pi : \mathcal{X}' \to \mathcal{X} \) of quasi-projective varieties, for an integer \( n \geq 2 \), as follows.

**Definition 2.2.** The Prym variety of the cyclic \( n \)-cover \( \pi : \mathcal{X}' \to \mathcal{X} \) is defined by the quotient abelian variety

\[
\text{Prym}_{\mathcal{X}'/\mathcal{X}} := \frac{\text{Alb}(\mathcal{X}')}{\text{Im}(id + \tilde{\gamma} + \cdots + \tilde{\gamma}^{n-1})},
\]

where \( \tilde{\gamma} \) is the automorphism of \( \text{Alb}(\mathcal{X}') \) induced by an order \( n \) automorphism \( \gamma \) of \( \mathcal{X}' \).

In the case that the irreducible quasi-projective varieties \( \mathcal{X}, \mathcal{X}' \) and the \( n \)-cover \( \pi \) are defined over \( k \), there is a \( k \)-isogeny of abelian varieties,

\[
\text{Prym}_{\mathcal{X}'/\mathcal{X}} \simeq_k \ker(id + \tilde{\gamma} + \cdots + \tilde{\gamma}^{n-1} : \text{Alb}(\mathcal{X}') \to \text{Alb}(\mathcal{X}'))^0,
\]

where \( (\ast)^0 \) means the connected component of its origin. We refer the reader to Lemma 2.2 in [16] for its proof.

It is useful to construct a new \( n \)-cover using the given ones. To this end, we let \( \pi_i : \mathcal{X}'_i \to \mathcal{X}_i \) \( (i = 1, 2) \) be \( n \)-covers of irreducible quasi-projective varieties, \( \gamma_i \in \text{Aut}(\mathcal{X}_i) \) be an automorphism of order \( n \geq 2 \) for \( i = 1, 2 \), all defined over \( k \). Moreover, we assume that there exist \( k \)-rational points \( x'_i \in \mathcal{X}'_i(k) \) for \( i = 1, 2 \). Then, we have a \( k \)-rational isogeny of abelian varieties,

\[
\text{Prym}_{\mathcal{X}'_1 \times \mathcal{X}'_2/\mathcal{Y}} \simeq_k \text{Prym}_{\mathcal{X}'_1/\mathcal{X}_1} \times \text{Prym}_{\mathcal{X}'_2/\mathcal{X}_2},
\]

where \( \mathcal{Y} = \mathcal{X}'_1 \times \mathcal{X}'_2/G \) is the intermediate cover and \( G \) is the cyclic group generated by \( \gamma = (\gamma_1, \gamma_2) \in \text{Aut}(\mathcal{X}_1 \times \mathcal{X}_2) \). For the proof, one can see Proposition 2.3 in [16].
3 G-sets and twists

In this section, we briefly recall two equivalent definition of twist an algebraic variety and its basic properties. To see more on the subject, we refer the reader to [3].

Let \( K \) be a field and \( L|K \) a finite extension with Galois group \( G = Gal(L|K) \). A \( G \)-set is a discrete topological space \( \mathcal{X} \) such that the left action of \( G \) on \( \mathcal{X} \) is continuous. For every \( x \in \mathcal{X} \) and \( u \in G \), we denote by \( ^u x \) the left action of \( u \) on \( x \). A \( G \)-group is a \( G \)-set \( \mathcal{A} \) equipped with a group structure invariant under action of \( G \), i.e., \( ^u(x \cdot y) = ^u x \cdot ^u y \) for \( x, y \in \mathcal{A} \) and \( u \in G \). Any continuous application \( a : u \mapsto a_u \) of \( G \) to a \( G \)-set \( \mathcal{A} \) is called a cochain of \( G \) with values in \( \mathcal{A} \). A cochain \( a = (a_u) \) is called a 1-cocycle of \( G \) with values in \( \mathcal{A} \) if \( a_{uv} = a_u \cdot ^u a_v \) for \( u, v \in G \). For any 1-cocycle \( a = (a_u) \), one has \( a_{id} = 1 \) and \( a_u \cdot ^u a_v = 1 \), where \( u \in G \) and \( 1 \in \mathcal{A} \) denotes the identity element. The set of 1-cocycles of \( G \) with values in a \( G \)-set \( \mathcal{A} \), is denoted by \( Z^1(G, \mathcal{A}) \). We say that a \( G \)-group \( \mathcal{A} \) acts on the \( G \)-set \( \mathcal{X} \) from left, in a compatible way with action of \( G \), if there is an application \( (a, x) \rightarrow a \cdot x \) of \( \mathcal{A} \times \mathcal{X} \) to \( \mathcal{X} \) satisfying the following conditions:

(i) \( ^u (a \cdot x) = ^u a \cdot ^u x \) \( (a \in \mathcal{A}, x \in \mathcal{X}, u \in G) \)

(ii) \( a \cdot (b \cdot x) = (a \cdot b) \cdot x \), and \( 1 \cdot x = x \), \( (a, b \in \mathcal{A}, x \in \mathcal{X}) \).

Let \( \mathcal{A} \) be a \( G \)-group, \( (a_u) \in Z^1(G, \mathcal{A}) \) be a 1-cocycle of \( \mathcal{A} \), and \( \mathcal{X} \) a \( G \)-set that is compatible with the group action of \( G \). For any \( u \in G \) and \( x \in \mathcal{X} \), define \( ^u x := a_u \cdot ^u x \). The \( G \)-set with this action of \( G \) is denoted by \( \mathcal{X}_a \) and is called the twist of \( \mathcal{X} \) obtained by the 1-cocycle \( a \).

In what follows, we will give other definition of twists in terms of schemes and its relation with the above one. Let \( \mathcal{X} \) be a quasi-projective scheme defined over \( K \), \( Aut(\mathcal{X}) \) be the automorphism scheme of \( \mathcal{X} \) and \( a = (a_u) \in Z^1(G, Aut(\mathcal{X})) \) be a 1-cocycle. Then, \( Alb(a) := (Alb(a_u)) \) satisfies the 1-cocycle condition, i.e., \( Alb(a) \in \mathcal{Z}^1(G, Aut(Alb(\mathcal{X}))) \). Indeed, the equality \( a_{uv} = a_u \circ ^u a_v \) implies \( Alb(a_{uv}) = Alb(a_u) \circ ^u Alb(a_v) \), since the construction of Albanese variety is compatible with base change. Here, we have a proposition that provide second definition for twist. To see its proof, we cite the reader to the propositions 2.6 and 2.7 in [3].

**Proposition 3.1.** Keeping above notations, there exist a unique quasi-projective \( K \)-scheme \( \mathcal{Y} \) and a unique \( L \)-isomorphism \( g : \mathcal{X} \otimes K \mathcal{L} \rightarrow \mathcal{Y} \otimes K \mathcal{L} \) such that \( ^u g = g \circ a_u \) holds for any \( u \in G \). The map \( g \) induces an isomorphism of the twisted \( G \)-set \( \mathcal{X}(\mathcal{L})_a \) onto the \( G \)-set \( \mathcal{Y}(\mathcal{L}) \).

The scheme \( \mathcal{Y} \) in above theorem is denoted by \( \mathcal{X}_a \) and is called the twist of \( \mathcal{X} \) by the extension \( \mathcal{L}|K \), or equivalently by the 1-cocycle \( a = (a_u) \).

The following proposition shows the relation between Albanese variety and the twists.

**Proposition 3.2.** Under above conditions, the twist \( Alb(\mathcal{X})_{Alb(a)} \) of \( Alb(\mathcal{X}) \) by the 1-cocycle \( Alb(a) \) is \( K \)-isomorphic to \( Alb(\mathcal{X}_a) \).

**Proof.** If \( g : \mathcal{X} \otimes K \mathcal{L} \rightarrow \mathcal{X}_a \otimes K \mathcal{L} \) denotes the isomorphism such that \( ^u g = g \circ a_u \), then the induced isomorphism of Albanese varieties

\[
Alb(g) : Alb(\mathcal{X}) \otimes K \mathcal{L} \rightarrow Alb(\mathcal{X}_a) \otimes K \mathcal{L}
\]

satisfies \( ^u Alb(g) = Alb(g) \circ ^u Alb(a_u) \) for any \( u \in G = Gal(L|K) \), by functoriality of the twist. Therefore, the uniqueness of the twist implies that \( Alb(\mathcal{X})_{Alb(a)} \) is \( K \)-isomorphic to \( Alb(\mathcal{X}_a) \).
In our recent paper [16], we proved a structural theorem on the set of rational points of twists of abelian varieties by cyclic covers over function field of quasi-projective varieties. In order to make this work self-contained and for convenience of the reader, we recall here the main theorem of [16] that has an essential rule in the proof of Theorem 1.1.

Given an integer \( n \geq 2 \), let \( \pi : \mathcal{X} \to \mathcal{X}' \) be a cyclic \( n \)-cover of irreducible quasi-projective varieties, both as well as \( \pi \) defined over a field \( k \). Denote by \( K \) and \( L \) the function fields of \( \mathcal{X} \) and \( \mathcal{X}' \) respectively. Assume that \( A \) is an abelian variety with an automorphism \( \sigma \) of order \( n \), and let \( A[n](k) \) be the group of \( k \)-rational \( n \)-division points on \( A \). Define \( A_a \) to be the twist of \( A \) by the cyclic extension \( L|K \), or equivalently, by the 1-cocycle \( a = (a_u) \in Z^{1}(G, \text{Aut}(A)) \), where \( a_{id} = id \) and \( a_{\gamma^j} = \sigma^j \) for \( j = 0, \cdots, n-1 \). To see more on the twist theory, we refer the reader to Section 3. The following theorem describes the structure of the Mordell-Weil group \( K \)-rational points on the twist \( A_a \) that extends the main result of [5, 23]. See Theorem 1.1 in [16] for its proof.

**Theorem 3.3.** Notation being as above, we assume that there exist a \( k \)-rational point on \( \mathcal{X}' \). Then, as an isomorphism of abelian groups, we have:

\[
A_a(K) \cong \text{Hom}_k(\text{Prym}_{\mathcal{X}'/\mathcal{X}}, A) \oplus A[n](k).
\]

Moreover, if \( \text{Prym}_{\mathcal{X}'/\mathcal{X}} \) is \( k \)-isogenous with \( A^m \times B \) for some integer \( m > 0 \) and \( B \) is an abelian varieties defined over \( k \) such that \( \dim(B) = 0 \) or \( \dim(B) > \dim(A) \) and none of irreducible components of \( B \) is \( k \)-isogenous to \( A \), then \( \text{rk}(A_a(K)) \geq m \cdot \text{rk}(\text{End}_k(A)) \).

4 The cyclic multiple plane \( \mathcal{X}_n \)

Given any integer \( n \geq 2 \) and an irreducible polynomial \( f(x, y) \in k[x, y] \), we suppose that \( F(u_0, u_1, u_2) \) is a homogeneous polynomial such that \( f(x, y) = F(1, x, y) \). Denote by \( \mathcal{C} \) the projective plane curve defined by \( F = 0 \) and let \( \mathcal{C} \) be the affine curve \( f = 0 \). Let \( L_{\infty} \) be the line at infinity, say \( u_0 = 0 \). Define \( \mathcal{X}_n \) to be the normalization of the hypersurface \( X_n : w^n = f(x, y) \). Then, it can be expressed by \( u_3^a = u_0^{n_0}F(u_0, u_1, u_2) \), in the weighted projective space \( \mathbb{P}^2_{(1,1,1, e)} \), where \( n_0 = n - \deg(f) \). The map \( X_0 \to \mathbb{P}^2 \), defined by dropping the last coordinate, ramifies over \( \mathcal{C} \) if \( n_0 \) and over \( \mathcal{C} \cup L_{\infty} \) for \( n_0 \neq 0 \). Let \( \phi : \mathcal{X}_n \to \bar{X}_n \) be the desingularization and \( \pi : \mathcal{X}_n \to \mathbb{P}^2 \) be the composition map.

Define \( q_n \) to be the irregularity of \( \mathcal{X}_n \), i.e., \( q_n = \dim H^0(\mathcal{X}_n, \Omega^1_{\mathcal{X}_n}) \), where \( \Omega^1_{\mathcal{X}_n} \) denotes the vector space of differential 1-forms on \( \mathcal{X}_n \). If we suppose that \( f(x, y) \) has a decomposition \( f = f_1^{m_1} \cdots f_d^{m_d} \) in to irreducible factors such that \( \gcd(x, m_1, \cdots, m_d) = 1 \), then there exists a trivial bound for \( q_n \) given by,

\[
2q_n \leq \begin{cases} 
(n - 1)(\sum_{i=1}^d r_i - 2), & \text{if } s|r \\
(n - 1)(\sum_{i=1}^d r_i - 1), & \text{otherwise,}
\end{cases}
\]

where \( r_i = \deg(f_i) \) for \( i = 1, \cdots, d \).

In order to give examples of multiple plane with positive irregularity, let us to consider the following situation. Denote by \( D_\ell \) an \( \ell \)-cyclic covering of the projective line \( \mathbb{P}^1 \) defined by the equation \( v_2^\ell = \prod_{i=1}^l (b_i v_0 - a_i v_1)^{s_i} \) with \( \ell \sum s_i \). Let \( \varphi : \mathbb{P}^2 \to \mathbb{P}^1 \) be a rational map given by \( \varphi(u_0 : u_1 : u_2) = (F_1(u_0, u_1, u_2) : F_2(u_0, u_1, u_2)) \), where \( F_1 \) and \( F_2 \) are both homogeneous
polynomials of degree $s$. If $n|s \cdot \sum_{i=1}^{t} s_i$ and $\ell|n$, then the cyclic multiple plane $X_n$ associated to the hypersurface

$$X_n : w^n = f(x, y) = \prod_{i=1}^{t} (b_i F_1(1, x, y) - a_i F_2(1, x, y))^{s_i},$$

factor through $D_{\ell}$. In this case, it is said that $X_n$ factors through a pencil. One can see that if $D_{\ell}$ is a curve of positive genus, then $X_n$ has positive irregularity.

Now, assume that $k \subset \mathbb{C}$ is a field that contains an $n$-th root of unity denoted by $\zeta_n$. Consider the $n$-cyclic covering $\pi : \mathcal{X}_n \to \mathbb{P}^2$ and let $L$ and $K$ be the function fields of $\mathcal{X}_n$ and $\mathbb{P}^2$, respectively. Denote by $G = Gal(L|K)$ the Galois group of the extension $L|K$, which is a cyclic group generated by $\gamma \in \text{Aut}(\mathcal{X}_n)$. Let us denote by $\gamma$ and $\gamma^*$ the induced map on $\text{Alb}(\mathcal{X}_n)$ and $\Omega^n_1$, respectively. The Galois group $Gal(L|K)$ acts on $\mathcal{X}_n$ and naturally on $\text{Alb}(\mathcal{X}_n)$ the Albanese variety of $\mathcal{X}_n$. The following proposition gives a description of the structure of $\text{Alb}(\mathcal{X}_n)$. To see its proof, we refer the reader to [15].

**Proposition 4.1.** Suppose that $k$ contains an $n$-th root of unity $\zeta_n \neq \pm 1$ and $q_n > 0$.

(i) If there exist two linearly independent $w, w' \in H^0(\mathcal{X}_n, \Omega^n_1)$ such that $\gamma^* w = \zeta_n w$ and $\gamma^* w' = \zeta_n^{-1} w'$, then the Albanese image $\alpha(\mathcal{X}_n)$ is a curve. In this case, $X_n$ factors through a pencil;

(ii) If $\gamma^* w = \zeta_n w$ for all $w \in H^0(\mathcal{X}_n, \Omega^n_1)$, then $\zeta_n \in \{ \pm i, \pm \rho, \pm \rho^2 \}$ with $\rho = e^{2\pi i/3}$. Furthermore, $\text{Alb}(\mathcal{X}_n) \cong \mathcal{E}_{\zeta_n}^{a_n}$, where $\mathcal{E}_{\zeta_n}$ is the elliptic curve associated to the lattice $\Lambda_{\zeta_n} = \mathbb{Z} \oplus \zeta_n \mathbb{Z}$.

To the next result, we denote by $\mathcal{E}_i$ and $\mathcal{E}_\rho$ the elliptic curves associated to the lattices $\Lambda_\rho = \mathbb{Z} \oplus \rho \mathbb{Z}$ and $\Lambda_i = \mathbb{Z} \oplus i \mathbb{Z}$, where $\rho = \zeta_3$ and $i = \zeta_4$. We also let $C_1$ and $C_2$ be the genus 2 affine curves $y^2 = x^5 + 1$ and $y^2 = x^5 + x$, respectively, and denote by $J(C_1)$ and $J(C_2)$ their Jacobians varieties. The following proposition describes $\text{Alb}(\mathcal{X}_n)$ for some small values of $n$.

For its proof, we cite to Section 5 of [1] and its references.

**Proposition 4.2.** Keeping the above notations and the hypothesis of Proposition 4.1 in mind, for $3 \leq n \leq 12$ but $n \neq 7, 9$ and $11$, we have:

(i) $\mathcal{X}_2$ factors through a pencil;

(ii) For $n = 3, 6$, if $\alpha(\mathcal{X}_n)$ is a surface, then $\text{Alb}(\mathcal{X}_n) \cong \mathcal{E}_\rho^{a_n}$;

(iii) For $n = 4$, either $\mathcal{X}_4$ factors through a pencil, or $\text{Alb}(\mathcal{X}_4) \cong \mathcal{E}_i^{q_4}$;

For $n = 5, 8, 10$, and $12$, $d_n \dim(\text{Alb}(\mathcal{X}))$ is an even integer and

(iv) $\text{Alb}(\mathcal{X}_n) \cong J(C_1)^{d_n/2}$ for $n = 5, 10$;

(iv) $\text{Alb}(\mathcal{X}_8) \cong J(C_2)^{n_1 \times \mathcal{E}_i^{2n_2}}$ with $n_1 + n_2 = d_8/2$;

(v) $\text{Alb}(\mathcal{X}_{12}) \cong \mathcal{E}_i^{2n_1} \times \mathcal{E}_\rho^{2n_2}$ with $n_1 + n_2 = d_{12}/2$.

The Albanese dimension of $C : f = 0$ is defined as $\text{Albdim}(C) = \max \dim_{n \in \mathbb{N}} \alpha(\mathcal{X}_n)$. It is a well known fact that $\text{Albdim}(C) = 0, 1$, or 2. In [8], Kulikov provides a sufficient condition to have $\text{Albdim}(C) = 2$ without giving some concrete examples. In Theorem 0.2 of [18], Tokunga demonstrated an explicit irreducible affine plane curve enjoying the condition as follows.
Proposition 4.3. Let \( C \) be the sextic plane curve defined by the polynomial
\[
F(u_0, u_1, u_2) = u_0^3 - 3u_1u_2(u_0^3 - 8) + 2(u_2^3 + 20u_0^3u_2^3 - 8u_0^6) = 0.
\]
Then the affine curve \( C : f(x, y) = 0 \) is irreducible and its normalization is a rational curve. The Albanese image \( \alpha(X_6) \) is a surface and hence \( \text{Albdim}(C) = 2 \), where \( X_6 \) is the multiple plane associate to hypersurface \( X_6 : w^6 = f(x, y) \).

5 Proofs of the main results

It is clear that the hypersurface \( X_n \) defined by \( w^n = f(x, y) \) admits an order \( n \) automorphism \( \tau : (x, y, w) \mapsto (x, y, \zeta \cdot w) \) that induces an automorphism on its non-singular model \( X_n \), denoted by \( \overline{\tau} \). For any integer \( m \geq 1 \) and each \( 1 \leq i \leq m \), we define \( U_m := X_n^{(i)} \times \cdots \times X_n^{(m)} \) where \( X_n^{(i)} \) is a copy of \( X_n \) given by affine equation \( w_i^n = f(x_i, y_i) \), and denote by \( \tau_i \) the corresponding automorphism. Then \( \gamma = (\tau_1, \cdots, \tau_m) \) is an order \( n \) automorphism of \( U_m \) that induces naturally an order \( s \) automorphism \( \tilde{\gamma} = (\tilde{\tau}_1, \cdots, \tilde{\tau}_m) \) of the fibered product \( U_m = X_n^{(1)} \times \cdots \times X_n^{(m)} \), where \( X_n^{(i)} \) is a nonsingular model of \( X_n^{(i)} \) for each \( 1 \leq i \leq m \). Note that \( U_m \) can be seen as a non-singular model of \( U_m \). Denote by \( L \) and \( \mathcal{L} \) the function field of \( U_m \) and \( U_m \), respectively. We let \( \text{Aut}(*) \) denotes the automorphism group of its origin and \( G = \langle \gamma \rangle \) and \( G = \langle \tilde{\gamma} \rangle \) be the cyclic subgroup of \( \text{Aut}(U_m) \) and \( \text{Aut}(U_m) \) generated by \( \gamma \) and \( \tilde{\gamma} \), respectively. We define \( V_m \) and \( V_m \) be the quotient of \( U_m \) and \( U_m \) by \( G \) and \( G \), resp., and denote by \( K \) and \( \mathcal{K} \) the function fields of \( V_m \) and \( V_m \), respectively. Then, both of the extensions \( L|K \) and \( \mathcal{L}|\mathcal{K} \) are finite cyclic of order \( n \). Indeed, we have \( L \subset k(x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_m, w_1, \cdots, w_m) \), where \( x_i \)'s and \( y_i \)'s are independent transcendental variables and \( w_i \) is related by the equations,
\[
w_i^n - f(x_i, y_i) = 0 \quad (i = 1, \cdots, m). \tag{5.1}
\]
Then, by definition, the field \( K \) is the \( G \)-invariant elements of \( L \), i.e.,
\[
K = L^G \subset k(x_1, \cdots, x_m, y_1, \cdots, y_m, w_1^{n-1}w_2, \cdots, w_1^{n-1}w_m-1).
\]
Since \( (w_1^{n-1}w_i+1)^n = f(x_1, y_1)^n f(x_i+1, y_i+1) \) for \( 1 \leq i \leq m-1 \), so by defining \( z_i := w_i^{n-1}w_i+1 \) the variety \( V_m \) can be expressed by the equations
\[
z_i^n = f(x_1, y_1)^n f(x_i+1, y_i+1) \quad (i = 1, \cdots, m-1). \tag{5.2}
\]
Thus, the extension \( L|K \) is a cyclic of degree \( n \) determined by \( w_1^n = f(x_1, y_1) \), i.e.,
\[
L = K(w_1) \subset k(x_1, \cdots, x_m, y_1, \cdots, y_m, z_1, \cdots, z_{m-1})(y_1).
\]
We note that the cyclic field extension \( \mathcal{L}|\mathcal{K} \) can be determined by considering the homogenization of the equations \( f(x_1, y_1)z^n = f(x, y) \). Furthermore, the variety \( V_m \) can be expressed by the homogenization of the equations \( f(x_1, y_1)z^n = f(x, y) \). Let \( X_n \) and \( \overline{X}_n \) be the twist of \( X_n \) and \( X_n \) by the extensions \( L|K \) and \( \mathcal{L}|\mathcal{K} \), respectively. Then, one can check that the twist \( X_n \) is given by the affine equation,
\[
f(x_1, y_1)z^n = f(x, y), \tag{5.3}
\]
Proposition 2.3 in [16], we get an 

\[ \tilde{X}' \]

\[ \tilde{X}' \] contains \( m \) \( K \)-rational points:

\[ P_i := (x_i, y_i, 1) \]

\[ P_{i+1} := \left( \frac{x_i}{f(x_1, y_1)}, \frac{y_i}{f(x_1, y_1)} \right) \]

for \( 1 \leq i \leq m - 1 \). \hspace{1cm} (5.4)

Let us denote by \( P_i \) the point corresponding to \( P_i \) on \( \tilde{X}' \) for \( i = 1, \ldots, m \). Now, applying Lemma 2.2 in [16] to the \( n \)-cover \( \pi : \tilde{X}' \rightarrow \mathbb{P}^2 \), we obtain that

\[ \text{Prym}_{\tilde{X}' / \mathbb{P}^2} = \frac{\text{Alb}(\tilde{X}'^{(i)})}{\text{Im}(id + i + \cdots + \tilde{r}^{n-1})} \sim_k \ker(id + i + \cdots + \tilde{r}^{n-1})^{\circ}. \]

Since \( 0 = id - \tilde{r}^m = (id - \tilde{i})(id + i + \cdots + \tilde{r}^{n-1}) \) and \( id \neq \tilde{i} \), we have

\[ 0 = id + i + \cdots + \tilde{r}^{n-1} \in \text{End}(\text{Alb}(\tilde{X}'^{(i)})) = \text{End}(\text{Alb}(X_n)), \]

which implies that \( \text{Prym}_{\tilde{X}' / \mathbb{P}^2} = \text{Alb}(\tilde{X}'^{(i)}) = \text{Alb}(X_n) \) for each \( i = 1, \ldots, m \). Then, using Proposition 2.3 in [16], we get an \( k \)-isogeny of abelian varieties,

\[ \text{Prym}_{\tilde{X}' / \mathbb{P}^2} = \text{Alb}(X_n)^m. \]

Now, let us consider the 1-cocycle \( a = (a_u) \in Z^1(\tilde{G}, \text{Aut}(\text{Alb}(X_n))) \) defined by \( a_{id} = id \) and \( a_{\tilde{r}^j} = \tilde{r}^j \) where \( \tilde{r}^j \in \tilde{G} \) and \( \tilde{r}^j : \text{Alb}(X_n) \rightarrow \text{Alb}(X_n) \) is the automorphism induced by \( \tilde{r} : X_n \rightarrow X_n \). Denote by \( \text{Alb}(X_n)_a \) the twist of \( \text{Alb}(X_n) \) with the 1-cocycle \( a = (a_u) \). Then, using Proposition 3.2 we conclude that \( \text{Alb}(X_n)_a = \text{Alb}(\tilde{X}_a) \). Therefore, by Theorem 1.1 for \( X' = U_m, X = V_m, \) and \( A = \text{Alb}(X_n) \), we have:

\[ \text{Alb}(\tilde{X}_a)(K) \cong \text{Hom}_k(\text{Prym}_{U_m / V_m}, \text{Alb}(X_n)) \oplus \text{Alb}(X_n)[n](k) \]

\[ \cong \text{Hom}_k(\text{Alb}(X_n)^n, \text{Alb}(X_n)) \oplus \text{Alb}(X_n)[n](k) \]

\[ \cong (\text{End}_k(\text{Alb}(X_n))^m) \oplus \text{Alb}(X_n)[n](k). \]

We denote by \( \tilde{Q}_i \) the image of \( \tilde{P}_i \) by map \( \tilde{X}_a \rightarrow \text{Alb}(\tilde{X}_a) \) for \( i = 1, \ldots, m \). Then, by tracing back the above isomorphisms, one can see that the points \( \tilde{Q}_1, \cdots, \tilde{Q}_m \) form a subset of independent generators of for the Mordell-Weil group \( \text{Alb}(\tilde{X}_a)(K) \). Hence, as \( \mathbb{Z} \)-modules, we have

\[ \text{rk}(\text{Alb}(\tilde{X}_a)(K)) \geq m \cdot \text{rk}(\text{End}_k(\text{Alb}(X_n))). \]

In order to prove Theorem 1.2 let us to investigate the ring of endomorphisms of \( \text{Alb}(\tilde{X}_a) \) for \( 3 \leq n \leq 12 \) and \( n \neq 7, 9, 11 \) case by case. We assume that \( k \) contains \( \mathbb{Q}(\zeta_n) \) and there exists a \( k \)-rational point on \( X_n \) and hence on \( X_n \). By Theorem 1.1 we have

\[ \text{Alb}(\tilde{X}_a)(K) \cong (\text{End}_k(\text{Alb}(X_n)))^m \oplus \text{Alb}(X_n)[n](k). \]

Since \( \text{Alb}(X_n)[n](k) \) is a trivial or finite group, so it does not distribute on the rank of \( \text{Alb}(\tilde{X}_a)(K) \) as a \( \mathbb{Z} \)-module. The endomorphism groups \( \text{End}_k(\mathcal{E}_1) \) and \( \text{End}(\mathcal{E}_p) \) contain the rings \( \mathbb{Z}[i] \) and \( \mathbb{Z}[\rho] \), respectively, that both are rank 2 \( \mathbb{Z} \)-modules. The Jacobian variety \( J(C_1) \)
is a simple abelian variety. Its endomorphism ring $\text{End}_k(J(C_1))$ contains $\mathbb{Z}[\zeta_5]$ that has rank 4 as $\mathbb{Z}$-module. The Jacobian variety $J(C_2)$ splits as the product of the following elliptic curves,

$$E_1 : y^2 = x^3 + x^2 - 3x + 1, \quad E_2 : y^2 = x^3 - x^2 - 3x + 1.$$  

We notice that $\text{End}_k(E_1)$ and $\text{End}_k(E_2)$ contain the rings $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{-3}]$, respectively, that both are rank 2 $\mathbb{Z}$-modules. We refer the reader to LMFDB [13] for more details on the above facts. Now, by considering Proposition 4.2 we obtain that

$$\text{rk}(\text{Alb}(\tilde{X}_n)(K)) \geq m \cdot c_n,$$

where $q_n$ is the irregularity of $X_n$ and $c_n = 2q_n$ if $n = 3, 4, 6$; and $c_n = 2d_n$ if $n = 5, 8, 10, 12$ as desired. Finally, the proof of last assertion is a direct consequence of Proposition 4.3 and Theorem 1.2.

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