Harmonic superspace: new directions

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Abstract

We sketch recent applications of the harmonic superspace approach for off-shell formulations of $(4, 4)$, $2D$ sigma models with torsion and for constructing super KdV hierarchies associated with ”small” and ”large” $N = 4$ superconformal algebras.

1. Introduction. Harmonic superspace (HSS) has been proposed in 1984 by our Dubna group headed by late Viktor Isaakovich Ogievetsky \[1\] as an efficient tool to treat theories with extended SUSY. This concept allowed to solve the long-standing problem of constructing off-shell superfield formulations of all $N = 2$, $4D$ supersymmetric theories: $N = 2$ matter (sigma models), $N = 2$ super Yang-Mills and supergravity theories [1 - 4] as well as of $N = 3$ super Yang-Mills theory [5]. Later on, the same method was applied to purely bosonic problems to achieve a new formulation of the Ward construction for self-dual Yang-Mills fields \[6\] and to find unconstrained potentials for hyper-Kähler and quaternionic geometries [7, 8].

The essence of the harmonic (super)space approach consists in extending the original (super)manifold by some extra variables which parametrize the automorphism group of the given (super)manifold. In the $N = 2$, $4D$ case it was the $SU(2)$ automorphism group acting on the Grassmann coordinates, the relevant additional variables being isospinor $SU(2)$ harmonics. The basic advantage of considering such extended manifolds is the possibility to single out in them a submanifold of lower dimension, the so-called “analytic subspace”. In most examples the unconstrained functions on this subspace, analytic (super)fields, turn out to be the fundamental entities of the given theory.
Since its invention, the HSS approach has been advanced and developed along several directions. One of new trends was applications to covariant quantization of superparticles and superstrings, as well as to constructing variants of the twistor-like formulation of these theories (see, e.g., refs. [9 - 13]). Another line was the further exploration of the relationships between complex target geometries and types of extended worldsheet supersymmetries in 2D sigma models. Using the SU(2) harmonic (super)space language the most general action for sigma models with heterotic (0, 4) worldsheet supersymmetry has been found and the relevant bosonic target geometry has been revealed and studied [14, 15] (in general, such sigma models possess a non-trivial torsion). Closely related development was the recent application of the same harmonic formalism for giving the (0, 4) superspace version of Witten’s sigma model construction for ADHM instantons [16]. To set up general off-shell actions of torsionful (4, 4) supersymmetric sigma models, a new type of HSS, SU(2) × SU(2) one, has been recently proposed [17, 18]. One of the aims of this report is to give a brief account of the SU(2) × SU(2) harmonic approach and its applications in sigma models.

One more interesting and perspective domain where the HSS techniques proved to be helpful is supersymmetric integrable models. Recently, the HSS methods were successfully used to find general solutions to self-dual super Yang-Mills and supergravity equations [19], as well as to construct invariant actions for these systems [20]. This method also plays a central role in interpreting N = 3 super Yang-Mills theory as an integrable 4D theory [21] (the Lorentz sl(2, C) harmonics are relevant in this case). An example of Lorentz invariant 2D integrable system with (4, 4) supersymmetry, the (4, 4) SU(2) Liouville-WZNW theory [22, 23], admits a nice description in SU(2) × SU(2) HSS [17]. Quite recently, the one-dimensional version of SU(2) HSS was used for constructing and studying an N = 4 superextension of the text-book example of integrable equation, the KdV one, based on the ”small” N = 4 superconformal algebra as the second hamiltonian structure [24, 25]. The survey of the harmonic superspace formulation of this algebra and associated N = 4 KdV hierarchy is another subject of the present contribution. We will also briefly comment on the ”large” N = 4 SCA (with SO(4) × U(1) affine subalgebra) and the associate KdV hierarchy. In this case, the adequate formalism proves to be a version of the SU(2) × SU(2) HSS one.

2. SU(2)×SU(2) harmonic superspace. The SU(2) × SU(2) HSS is an extension of the standard real (4, 4) 2D superspace by two independent sets of harmonic variables \( u^{\pm i} \) and \( v^{\pm a} \) (\( u^{1 i} u_i^{-1} = u^{1 a} u_a^{-1} = 1 \)) associated with the automorphism groups SU(2)\(_L\) and SU(2)\(_R\) of the left and right sectors of (4, 4) supersymmetry [17] (see also [26]). The corresponding analytic subspace is spanned by the following set
of coordinates
\[(\zeta, u, v) = (x^+, x^-, \theta^{1,0}_1, \theta^{0,1}_2, u^\pm_1, v^\pm_ a),\] (1)
where we omitted the light-cone indices of odd coordinates (the first and second \(\theta_s\) in (1) carry, respectively, the indices + and −). The superscript “\(n, m\)” stands for two independent strictly preserved harmonic \(U(1)\) charges, left (\(n\)) and right (\(m\)) ones. The additional doublet indices, \(i\) and \(a\), refer to two extra automorphism groups \(SU(2)'_L\) and \(SU(2)'_R\). Together with \(SU(2)_L\) and \(SU(2)_R\) they form the full \((4,4)\) supersymmetry automorphism group \(SO(4)_L \times SO(4)_R\). We point out that the \(SU(2) \times SU(2)\) harmonic superspace admits a manifest realization of the whole automorphism group of \((4,4)\) \(2D\) supersymmetry. This is one of the reasons why a more general type of \((4,4)\) sigma models, those with torsion on the bosonic manifold, can be described within its framework. This type of sigma models is interesting mainly because they can provide non-trivial backgrounds for superstrings (see, e.g., ref. \([27]\)).

More precisely, such a description becomes possible largely due to the fact that the \(SU(2) \times SU(2)\) HSS provides a natural framework for off-shell formulation of the twisted \((4,4)\) multiplet. Until now, the latter was the basic object used for constructing sigma models of this type (actually, a subclass of them with mutually commuting left and right quaternionic structures \([23, 29]\)). It is described by a real analytic \(SU(2) \times SU(2)\) harmonic superfield \(q^{1,1}(\zeta, u, v)\) subjected to the harmonic constraints \([17]\)

\[D^{2.0}q^{1,1} = D^{0.2}q^{1,1} = 0.\] (2)
Here
\[D^{2.0} = \partial^{2.0} + i \theta^{1.0}_1 \theta^{1.0}_2 \partial_{++}, \quad D^{0.2} = \partial^{0.2} + i \theta^{0.1}_2 \theta^{0.1}_2 \partial_{--},\] (3)
\[(\partial^{2.0} = u^i_1 \frac{\partial}{\partial u^{-1}_i}, \quad \partial^{0.2} = v^a_1 \frac{\partial}{\partial v^{-1}_a})\]
are the left and right mutually commuting analyticity-preserving harmonic derivatives. These constraints leave in \(q^{1,1}\) 8 + 8 independent components that is just the irreducible off-shell component content of \((4,4)\) twisted multiplet.

The general off-shell action of \(n\) superfields \(q^{1,1}_M\) \((M = 1, 2, \ldots n)\) is given by the following integral over the analytic superspace \([17]\)

\[S_q = \int \mu^{-2,-2} h^{2,2}(q^{1,1}_M, u^\pm_1, v^\pm_1),\] (4)
\(\mu^{-2,-2}\) being the relevant integration measure. The analytic superfield lagrangian \(h^{2,2}\) is an arbitrary function of its arguments (the only restriction on its dependence on the harmonics \(u\) and \(v\) is the consistency with the external \(U(1)\) charges 2, 2).
Let us shortly characterize the relevant target bosonic geometry. 4n physical bosons $q^{aM}_0(x)$ appear as the first component in the $u,v$ harmonic expansion of $q^{1,1M} = q^{aM}_0(x)u^1_i v^1_a + \ldots$. The component physical bosons part of the action (4) contains both the metric and torsion terms. Both the target metric and torsion are expressed in terms of the single symmetric $n \times n$ matrix function

$$G_{MN}(q_0) = \int dudv \ g_{MN}(q^{1,1}_0, u, v), \quad g_{MN}(q^{1,1}_0, u, v) = \frac{\partial^2 h^{2,2}}{\partial q^{1,1M} \partial q^{1,1N}} \bigg|_{\theta=0}, \quad (5)$$

where $q^{1,1}_0 \equiv q^{1,1}|_{\theta=0}$. By its definition, this "metric" is a solution of the constraints

$$(a) \quad \partial_{[M} \partial_{N]} G_{N} = 0, \quad (b) \quad \partial_{M} \partial_{N} G_{L} = 0. \quad (6)$$

For four-dimensional targets there remains only one component in $G_{MN}$, so the relevant target metric is reduced to a conformal factor. The first condition in the above set is obeyed identically, while the second one becomes the Laplace’s equation. This agrees with general conditions on the bosonic target in torsionful (4,4) sigma models with four-dimensional targets [28, 30].

As a non-trivial example of the $q^{1,1}$ action with four-dimensional bosonic manifold we quote the action of (4,4) extension of the $SU(2) \times U(1)$ WZNW sigma model

$$S_{wznw} = \frac{1}{\kappa^2} \int \mu^{-2,-2} \left( q^{1,1} \right)^2 \left( \frac{\ln(1 + X)}{X^2} - \frac{1}{(1 + X)X} \right). \quad (7)$$

Here

$$q^{1,1} = q^{1,1} - c^{1,1}, \quad X = c^{-1,-1} q^{1,1}, \quad c^{\pm 1, \pm 1} = c^a \ u^\pm_i v^\pm_a, \quad c^a c_i^a = 2. \quad (8)$$

Despite the presence of an extra quartet constant $c^a$ in the analytic superfield lagrangian, the action (7) actually does not depend on $c^a$ because it is invariant under arbitrary rigid rescalings and $SU(2) \times SU(2)$ rotations of this constant. Its physical bosons part eventually turns out to be expressed through the single function $G(q_0)$. Up to the overall coupling constant, it reads [17],

$$G(q_0) \propto \int dudv \frac{1 - X}{(1 + X)^3} = 2(q^{a}_0 q_0^a)^{-1}. \quad (9)$$

The component action, in the appropriate parametrization, coincides with the standard (4,4), $SU(2) \times U(1)$ WZNW action.

It is worth mentioning that the action (9) uniquely follows from requiring invariance under one of two different (though isomorphic) $N = 4$ $SU(2)$ superconformal groups which one may realize in the $SU(2) \times SU(2)$ analytic HSS [12, 17] (they close
on the “large” \( N = 4 \) \( SO(4) \times U(1) \) superconformal group \([33, 23]\). Also notice that there exist a few equivalent forms of the superfield lagrangian in (7) which differ from each other by full harmonic derivatives. As an example we give two such forms of the \( SU(2) \times U(1) \) WZNW action

\[
S_{\text{wznw}} = -\frac{1}{\kappa^2} \int \mu^{-2,-2} \left( \hat{q}^{1,1} \right)^2 c^{-1,1} c_{1,-1} = -\frac{1}{\kappa^2} \int \mu^{-2,-2} c^{1,1} \hat{q}^{1,1} \frac{1}{1 + X} .
\]

Let us discuss massive deformations of the action (4). Surprisingly, and this is a crucial difference of the considered (4, 4) case from, say, (2, 2) sigma models with torsion, the only massive term of \( q^{1,1 M} \) consistent with analyticity and off-shell (4, 4) supersymmetry (not modified by central charges) is the following one \([32, 17]\)

\[
S_{m} = m \int \mu^{-2,-2} \theta^{-1,0} \bar{\theta}^{0,1} \frac{1}{2} C_{\frac{M}{\bar{A}}}^{M} q^{1,1 M} ; \quad [m] = cm^{-1} .
\]

Here \( C_{\frac{M}{\bar{A}}}^{M} \) are arbitrary constants (subject to the appropriate reality conditions).

After adding the term (11) to the action (4), passing to components and eliminating auxiliary fields, the effective addition to the (4, 4) sigma model component action is given by

\[
S_{\text{pot}} = \frac{m^2}{2} \int d^2 z \ G_{M N} (q_0) \ (C_{\frac{M}{\bar{A}}}^{M} C_{\frac{A}{\bar{M}}}^{A} N) .
\]

Here \( G_{M N} (q_0) \) is the inverse of \( G_{M N} (q_0) \) defined in eq. (5). Thus we see that the potential term in the case in question is uniquely determined by the form of the bosonic target metric. In particular, in the case of (4, 4) \( SU(2) \times U(1) \) WZNW model one gets the Liouville potential term for \( u(x) \propto \ln \{ \text{tr} \ q_0^a \} \), so the massive deformation of this model is nothing but the (4, 4) super Liouville theory \([22]\). It would be interesting to inquire whether (4, 4) extensions of other integrable 2D theories can be obtained as massive deformations of some appropriate (4, 4) sigma models with torsion.

3. A dual formulation of the twisted multiplet and its generalization.

The above \( SU(2) \times SU(2) \) HSS description of (4, 4) twisted multiplet suggests a new off-shell formulation of the latter via unconstrained analytic superfields. After implementing the constraints (2) in the action with superfield lagrange multipliers and adding this term to (4) we arrive at the following new action \([17]\)

\[
S_{q,\omega} = \int \mu^{-2,-2} \left\{ q^{1,1 M} ( D^{2,0} \omega^{-1,1 M} + D^{0,2} \omega^{1,-1 M} ) + \hbar^{2,2} (q^{1,1}, u, v) \right\} .
\]

In (13) all the involved superfields are unconstrained analytic, so from the beginning the action (13) contains an infinite number of auxiliary fields coming from the double harmonic expansions with respect to the harmonics \( u^{\pm 1,1}, v^{\pm 1,1} \). Varying with
respect to the superfields $\omega^{1,-1}_M, \omega^{-1,1}_M$ takes one back to the action (1) and constraints (2). On the other hand, varying with respect to $q^{1,1}_M$ yields an algebraic equation for eliminating this superfield. This enables one to get a new dual off-shell representation of the twisted multiplet action through unconstrained analytic superfields $\omega^{-1,1}_M, \omega^{1,-1}_M$.

The crucial feature of the action (13) (and its $\omega$ representation) is the abelian gauge invariance

$$\delta \omega^{1,-1}_M = D^{2,0}_0 \sigma^{-1,-1}_1, \delta \omega^{-1,1}_M = -D^{0,2}_0 \sigma^{-1,-1}_1,$$

where $\sigma^{-1,-1}_1$ are unconstrained analytic superfield parameters. This gauge freedom ensures the on-shell equivalence of the $q, \omega$ or $\omega$ formulations of the twisted multiplet action to its original $q$ formulation (1). Namely, it neutralizes superfluous physical dimension component fields in the superfields $\omega^{1,-1}_M$ and $\omega^{-1,1}_M$ and thus equalizes the number of propagating fields in both formulations. It holds already at the free level, with $h^{2,2}$ quadratic in $q^{1,1}_M$. So it is natural to expect that any reasonable generalization of the action (13) respects this symmetry or a generalization of it.

It is well known that with making use of $(4,4)$ twisted multiplets one may construct invariant off-shell actions only for those torsionful $(4,4)$ sigma models for which left and right triplets of covariantly constant complex structures on the bosonic target mutually commute [28, 29]. This is true of course for both the actions (1) and (13). However, it turns out that the second one is a good starting point for constructing more general actions. They admit no inverse duality transformation to the twisted multiplets actions and yield an off-shell description of sigma models with non-commuting left and right complex structures.

In refs. [18] we started from the most general analytic superspace action of the triple of superfields $q^{1,1}_M, \omega^{1,-1}_M, \omega^{-1,1}_M$. Exploiting the freedom of target space reparametrizations together with the constraints which stem from the important self-consistency condition

$$[D^{2,0}_0, D^{0,2}_0] = 0,$$

we reduced the action to the form

$$S_{q,\omega} = \int \mu^{-2,-2} \left\{ q^{1,1}_M D^{0,2}_0 \omega^{1,-1}_M + q^{1,1}_M D^{2,0}_0 \omega^{-1,1}_M + \omega^{1,-1}_M h^{1,3}_M + \omega^{-1,1}_M h^{3,1}_M + \omega^{1,-1}_M \omega^{-1,1}_M h^{2,2}[M,N] + h^{2,2} \right\}.$$

Here, the involved potentials depend only on $q^{1,1}_M$ and target harmonics and still satisfy some additional constraints following from eq. (15). Most important of them
is as follows
\[ h^{2,2} [N,T] \frac{\partial h^{2,2} [M,L]}{\partial q^{1,1} T} + h^{2,2} [L,T] \frac{\partial h^{2,2} [N,M]}{\partial q^{1,1} T} + h^{2,2} [M,T] \frac{\partial h^{2,2} [L,N]}{\partial q^{1,1} T} = 0. \] (17)

It ensures the action to be invariant under some non-abelian and nonlinear gauge transformations which generalize (14). Their role is to maintain the correct number of physical fields (4n bosons and 8n fermions). They affect not only the \( \omega \) superfields, but \( q^{1,1} M \) as well
\[ \delta q^{1,1} M = \sigma^{-1,-1} N h^{2,2} [N,M]. \] (18)

In general, these gauge transformations close with a field-dependent Lie bracket parameter:
\[ \delta_{br} q^{1,1} M = \sigma_{br}^{-1,-1} N h^{2,2} [N,M], \quad \sigma_{br}^{-1,-1} N = -\sigma_{2}^{-1,-1} L \sigma_{2}^{-1,-1} T \frac{\partial h^{2,2} [L,T]}{\partial q^{1,1} N}. \] (19)

We see that eq. (17) guarantees the nonlinear closure of the algebra of gauge transformations (18), and so it is a group condition similar to the Jacobi identity. Note that the non-abelian character of these transformations is directly related to the presence of the new non-vanishing potential \( h^{2,2} [N,M] \).

It is a matter of straightforward computation to demonstrate that in the latter case the left and right complex structures on the bosonic target do not commute. We checked this property explicitly \[18\] for a particular class of the above models corresponding to the ansatz
\[ h^{1,3} N = h^{3,1} N = 0, \quad h^{2,2} [N,M] = b^{1,1} f^{NML} q^{1,1} L, \quad b^{1,1} = b^{ia} u_1^{1,1} v_1^{1,1}, \quad b^{ia} = \text{const}. \] (20)

Here, the totally antisymmetric real constants \( f^{NML} \) are structure constants of some \( n \)-dimensional semi-simple Lie algebra.

Thus it is the presence of the potential \( h^{2,2} [N,M] \) that leads to the above gauge symmetry and simultaneously to the non-commutativity of the left and right complex structures \[18\]. So, only for \( n \geq 2 \) a new class of torsionful \((4,4)\) sigma models emerges. The action with non-vanishing \( h^{2,2} [N,M] \) does not admit any duality transformation to the pure \( q^{1,1} M \) form and involves an infinite number of auxiliary fields.

Important problems ahead are to find out possible stringy applications of this new class of off-shell \((4,4)\) sigma models and to construct more general \((4,4)\) sigma model actions by incorporating other types of twisted \((4,4)\) multiplets within the \( SU(2) \times SU(2) \) HSS (or its further extensions). It would be also interesting to couple these sigma models to \((4,4)\) supergravity.
4. \textbf{N=4 super KdV hierarchies.} A powerful way to construct generalized KdV hierarchies is to associate them to the proper conformal algebras and superalgebras as the second hamiltonian structure \[33\].

The second hamiltonian structure for \( N=4 \) superextensions of KdV is provided by \( N=4 \) SCA’s. There exist two \( N=4 \) SCA’s which are different in their affine Kac-Moody subalgebras: the minimal one with the \( SU(2) \) affine subalgebra \[34\] and a more extensive SCA with the subalgebra \( SO(4) \times U(1) \) \[31\]. Both of them and the associated super KdV hierarchies admit a natural formulation in the framework of \( N=4, 1D \) HSS.

The \( N=4, SU(2) \) SCA is represented by the analytic harmonic spin 1 supercurrent \( V^{++}(\zeta) \), \( \zeta = (z, \theta^+i, u^\pm) \), subjected to the harmonic constraint

\[
D^{++}V^{++} = 0 \, , \quad D^{++} = \partial^{++} + i\theta^+\theta^+_i\partial_i \tag{21}
\]

(the notation is basically the same as in the previous sections). With this constraint, the superfield \( V^{++} \) displays the irreducible current contents of \( N=4, SU(2) \) SCA: the spin 1 affine current \( v^{(ik)} \), the spin 3/2 fermionic current \( \xi^{ik} \) and the spin 2 conformal stress-tensor \( T \). It is easy to write superfield Poisson brackets for \( V^{++} \) which reproduce \( N=4, SU(2) \) SCA for these currents \[24\]

\[
\{ V^{++}(1), V^{++}(2) \} = D^{(++|++)} \Delta(1-2)
\]

\[
D^{(++|++)} = (D^+_1)^2(D^+_2)^2 \left( \left[ \frac{u^+_1 u^-_2}{u^+_1 u^-_2} \right] - \frac{1}{2} D^-_2 \right) V^{++}(2) - \frac{k}{4} \partial_2 \tag{22}
\]

Here \( \Delta(1-2) = \delta(x_1 - x_2) (\theta^1 - \theta^2)^4 \) is the ordinary 1D \( N=4 \) superspace delta function and

\[
(D^+_i)^2 = D^+_i D^+_i .
\]

Note that the harmonic singularity in the r.h.s. of (22) is fake: it is cancelled after decomposing the harmonics \( u^+_2 \) over \( u^+_1 \) with making use of the completeness relation \( u^+_1 u^-_k - u^-_k u^+_1 = \epsilon_{ik} \).

Now it is straightforward to derive the relevant evolution equation, the \( N=4, SU(2) \) super KdV equation \[24\]

\[
\partial_t V^{++} = \left\{ H_3, V^{++} \right\} , \tag{23}
\]

\[
H_3 = \int [dZ] V^{++}(D^{--})^2 V^{++} - i \int [d\zeta^{-2}] e^{-4(u)} (V^{++})^3 . \tag{24}
\]

Here, \([dZ]\) and \([d\zeta^{-2}]\) are appropriate integration measures over the full 1D HSS and its analytic subspace, \( D^{--} \) is the second harmonic derivative (it does not preserve the
analyticity) and $c^{-4} = c^{ijkl} u_i^- u_j^- u_k^- u_l^- \text{ is a } SU(2) \text{ breaking tensor (in the explicit form, this equation is given in } [24, 25]). \text{ The } N = 4 \text{ super KdV is integrable (generates the whole hierarchy and is bi-hamiltonian) under the following restrictions on } c^{ijkl} [24, 25].$

\begin{align}
(a) \quad c^{ijkl} &= \frac{1}{3} \left( a^{(ij)} a^{(kl)} + a^{(ik)} a^{(jl)} + a^{(il)} a^{(jk)} \right); \quad (b) \quad a^{(ij)} a^{(ij)} \propto -\frac{1}{k}, \quad (25)
\end{align}

$k$ being the level of $SU(2)$ Kac-Moody subalgebra. There exist two non-equivalent reductions of this system to $N = 2$ super KdV, yielding the $a = 4$ and $a = -2$ integrable cases $[35]$ of the latter. An interesting unsolved problem is to construct the Lax representation for this $N = 4$ super KdV.

The $N = 4, SO(4) \times U(1)$ SCA is also tightly related to 1D harmonic analyticity, this time to the $SU(2) \times SU(2)$ one $[36]$. Its basic object is a spin $1/2$ fermionic supercurrent $J^{1,1}$. It lives on the three-theta analytic subspace of the $SU(2) \times SU(2), 1D$ HSS

\begin{equation}
D^{1,1} J^{1,1} = 0 \iff J^{1,1} = J^{1,1}(\xi_3), \quad \xi_3 = (z, \theta^{1,1}, \theta^{1,-1}, \theta^{-1,1}, u_i^{\pm 1}, v_k^{\pm 1}), \quad (26)
\end{equation}

(with $D^{\pm 1,\pm 1} = D^{ik} u_i^{\pm 1} v_k^{\pm 1}$ in the central basis) and obeys the harmonic constraints

\begin{equation}
D^{2,0} J^{1,1} = D^{0,2} J^{1,1} = 0. \quad (27)
\end{equation}

Its irreducible field contents include the spin $1/2$ current $j^{ij}$, the affine $SO(4) \times U(1)$ spin 1 currents $v^{(ik)}, v^{(ij)}, v$, the spin $3/2$ currents $\xi^{ik}$ and the stress-tensor $T$. This is just the set forming $N = 4, SO(4) \times U(1)$ SCA. It is easy to establish the proper Poisson brackets between $J^{1,1}$’s and to write the related superfield evolution equation

\begin{equation}
\partial_t J^{1,1} = \left\{ H'_3, J^{1,1} \right\}, \quad (28)
\end{equation}

$H'_3$ being the most general dimension 3 $N = 4$ supersymmetric hamiltonian composed of $J^{1,1}$ and its derivatives. This equation is a kind of the “master” one, as all the previously known super KdV equations are expected to follow from it via proper reductions. For instance, the supercurrents of two $N = 4, SU(2)$ SCA’s present in $N = 4, SO(4) \times U(1)$ SCA are defined as

\begin{equation}
V^{2,0} \equiv D^{1,-1} J^{1,1}, \quad V^{0,2} \equiv D^{-1,1} J^{1,1}. \quad (29)
\end{equation}

Besides the analyticity condition $[26]$, they meet extra analyticitics

\begin{equation}
D^{1,-1} V^{2,0} = 0, \quad D^{-1,1} V^{0,2} = 0, \quad (30)
\end{equation}
which imply them to live on two different two-theta analytic subspaces of the $SU(2) \times SU(2)$, 1D HSS

\[ V^{2,0} = V^{2,0}(\xi_2), \quad V^{2,0} = V^{2,0}(\xi'_2), \]

\[ \xi_2 = (z', \theta^{1,1}, \theta^{1,-1}, u^\pm_1, v^\pm_1), \quad \xi'_2 = (z'', \theta^{1,1}, \theta^{-1,1}, u''^\pm_1, v''^\pm_1). \quad (31) \]

Both supercurrents satisfy the bi-harmonic shortness condition like (27). In the limits $V^{2,0} = 0$ or $V^{0,2} = 0$ the remaining supercurrent is expected to satisfy the $N = 4, SU(2)$ KdV equation as a consequence of the $SO(4) \times U(1)$ one (28). The analysis of self-consistency of these reductions as well as the issue of integrability of (28) (the existence of an infinite set of conserved quantities, Lax representation, ...) are now under study.

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