The algorithmic details of polynomials application in the problems of heat and mass transfer control on the hypersonic aircraft permeable surfaces

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Abstract. The hypersonic aircraft permeable surfaces heat and mass transfer effective control mathematical modeling problems are considered. The analysis of the control (the blowing) constructive and gasdynamical restrictions is carried out for the porous and perforated surfaces. The functions classes allowing realize the controls taking into account the arising types of restrictions are suggested. Estimates of the computational complexity of the W. G. Horner scheme application in the case of using the C. Hermite interpolation polynomial are given.

1. Introduction

To solve the hypersonic aircraft permeable surfaces heat and mass transfer effective control (with the use of blowing into boundary layer) mathematical modeling problems it is necessary to take into account some constructive and gasodynamical restrictions \[1, 2\].

To simulate the controls tending to discontinuous the use of glued functions \[3, 4\] is suggested. The right choice of polynomials arithmetics algorithms is very important from the point of view of computational complexity for the multinode meshes and the high degree polynomials \[5, 6, 7\] realizing the glued functions elements.

2. Special blowing laws for permeable porous surfaces

In the case of porous area for some problems with special blowing laws taking into account the constructive restrictions in the stagnation point neighborhood \[1\]

\[m(0) = \begin{cases} m^* & \text{for } x \in [0; x^*]; \\ m_{opt}(x) & \text{for } x \in [x^*; 1], \end{cases}\]

where the blowing system power \[8\] satisfies the condition \(N(m) \leq N_c\) for \(N_c = N(m_{IV})\), the following variants are considered:

the law 1: \(m(0) = m_1, \quad m_1 \leq m_{\text{analyt}}(0)\);

the law 2: \(m(0) = 0\);

the law 3: according to \([9, 10]\): \(m^* = 0\),

and displayed on the left half of fig. 1.
Taking into account the necessity to guarantee the determined “level” of weak blowing let’s introduce an additional gazdynamical restriction into laws 1-3:

\[ m(x) \leq m^{\text{lim}}_{\text{weak}}, \]  

where

\[ m^c_{\text{IV}} \leq m^{\text{lim}}_{\text{weak}}, \]

and obtain laws 4-6 displayed on the right half of fig. 1.

![Figure 1. The special blowing laws](image)

The numerical solutions obtained for strengthened conditions:

\[ m^* \to 0 \quad \text{and/or} \quad x^* \to 1 \quad \text{and/or} \quad m^{\text{lim}}_{\text{weak}} \to m^c_{\text{IV}} \]

tend to discontinuous solutions, which can not be realized physically for porous blowing.

### 3. Special blowing laws for permeable perforated surfaces

Let’s consider some problems with special blowing laws simulating blowing through holes.

The blowing law variant (“decreasing triangle”) similar to “simple” law 1 from [8] and realized for surface perforated with the ratio \( d/h = 1/5 \) (here \( d \) is the hole diameter and \( h \) is the distance between holes) is displayed on fig. 2 as an example. The ratio 1/5 is recommended in [11].

The necessity to simulate the controls corresponding to blowing through holes [2] comes as a variant to the use of rectangular “impulses” (the “jets”):

\[ m(x) = \begin{cases} 
0 & \text{for} \quad x \in [x_0; x_1); \\
m_k & \text{for} \quad x \in (x_{2k-1}; x_{2k}); \\
0 & \text{for} \quad x \in (x_{2k}; x_{2k+1}), 
\end{cases} \]  

having for \( m_k > 0 \) discontinuous character in the joint points \( x_{2k-1}, x_{2k} \).

### 4. The boundary layer on hypersonic aircraft permeable surfaces effective control problems solution classes

Let’s describe the functions [12] similar to the glued functions introduced in [3, 4].
Let the segmentation be fixed for $X = [0; 1]$:

$$x_0 = 0 < x_1 < \ldots < x_{n-1} < x_n = 1.$$  \hfill (4)

Each segment $[x_{j-1}; x_j]$, $j = 1, \ldots, n$ is associated with an “element” $m_j(x)$ of function $m(x)$:

$$m(x) = m_j(x) \quad \text{for} \quad x \in [x_{j-1}; x_j].$$  \hfill (5)

The following opportunities were added to the computation experiment algorithm scheme for the fixed integer number $\nu \geq 0$.

4.1. Restrictions on segments

Let for the continuity intervals $[x_{j-1}; x_j]$ of the elements $m_j$ (here $j = 1, \ldots, n$) and for $k = 0, \ldots, \nu$ the continuous functions $b_{j,k}(x)$ and $t_{j,k}(x)$ be preset given as:

$$b_{j,k}(x) \leq t_{j,k}(x) \quad \text{for} \quad x \in [x_{j-1}; x_j].$$  \hfill (6)

Let’s the elements $m_j(x)$ of unknown glued function satisfy the conditions

$$m_j^{(k)}(x) \in I_{j,k}(x) \quad \text{for} \quad x \in [x_{j-1}; x_j],$$  \hfill (7)

where $I_{j,k}(x) = [b_{j,k}(x); t_{j,k}(x)]$.

4.2. Restrictions for jumps

For the joint points $x_j$ of the segments $[x_{j-1}; x_j]$ and $[x_j; x_{j+1}]$ (here $j = 1, \ldots, n - 1$) and for $k = 0, \ldots, \nu$ let’s define the jumps

$$\Delta m_{j,k} = m_{j+1}^{(k)}(x_j) - m_j^{(k)}(x_j) = m^{(k)}(x_j + 0) - m^{(k)}(x_j - 0).$$  \hfill (8)

Let the restrictions $\Delta m_{j,k}$ be given as follows:

$$\Delta m_{j,k} \leq \Delta m_{j,k}.$$

(9)
Let’s the jumps $\Delta m_{j,k}$ of unknown glued function satisfy the conditions

$$\underline{\Delta m_{j,k}} \leq \Delta m_{j,k} \leq \Delta m_{j,k}.$$  \hfill (10)

This allows choose one of the following alternatives.

“The jump of fixed size”:  \hfill (11)

$$\Delta m_{j,k} = \Delta m_{j,k} \neq 0;$$

“the necessity of “+ jump””:  \hfill (12)

$$0 < \Delta m_{j,k} < \Delta m_{j,k};$$

“the necessity of “– jump””:  \hfill (13)

$$\Delta m_{j,k} < \Delta m_{j,k} < 0;$$

“the jump prohibition”:  \hfill (14)

$$\Delta m_{j,k} = 0 = \Delta m_{j,k}.$$  \hfill (15)

For  \hfill (16)

$$0 < \Delta m_{j,k}$$

and/or for

$$\Delta m_{j,k} < 0$$

the jump is possible, but not obligatory.

4.3. Notes

1) In the case, when $m_j(x)$ are polynomials, the glued function is a generalization of spline.

2) It is preferable for the functions $b_{j,k}(x)$ and $t_{j,k}(x)$ to be simple enough, for example, they may be the polynomials of not high degree.

3) The values of

$$b_{j-1,k}(x_j - 0), \quad b_{j,k}(x_j + 0), \quad t_{j-1,k}(x_j - 0), \quad t_{j,k}(x_j + 0)$$

must satisfy the conditions (6) only.

4) The interval conditions (7) and the joint conditions (10) may require mutual coordination.

5) If on a certain proper subinterval $[x_+; x^*] \subset [x_{j-1}; x_j]$ for a certain $k^*, \ 0 \leq k^* \leq \nu$ the restrictions

$$b_{j,k^*}(x) = t_{j,k^*}(x) \quad \text{for} \quad x \in [x_+; x^*],$$  \hfill (17)

coincide; besides, the values $x_+$, $x^*$ are exact, i.e. for them such value $\delta_{\text{isol}} > 0$ exists that for

$$x \in [x_+ - \delta_{\text{isol}}; x^* + \delta_{\text{isol}}] \cap [x_{j-1}; x_j], \quad x \notin [x_+; x^*] \text{ it follows that } b_{j,k^*}(x) < t_{j,k^*}(x),$$

then such points $x_+$, $x^*$ must be included in the nodes system (4).

The procedure for including nodes of this type in the system (4) should be executed until such subintervals finish, or the size $(x^*-x_+)$ of each remaining subinterval becomes less than the preset given threshold value $\delta_{\text{min}} > 0$.

For $k^* = 0$ the unknown function $m(x)$ may be obtained on $[x_+; x^*]$ with the use of (17) and (7). For $k^* > 0$ the function $m(x)$ may be obtained with the accuracy upon $k^*$ integration constants.

6) If in a certain interior point $x^*$ of interval $[x_{j-1}; x_j]$ for a certain $k^*, \ 0 \leq k^* \leq \nu$ the restrictions

$$b_{j,k^*}(x^*) = t_{j,k^*}(x^*),$$  \hfill (18)

coincide; besides, $x^*$ is an isolated point, i.e. for it such value $\delta_{\text{isol}} > 0$ exists that for

$$x \in [x^* - \delta_{\text{isol}}; x^* + \delta_{\text{isol}}] \cap [x_{j-1}; x_j], \quad x \notin x^* \text{ it follows that } b_{j,k^*}(x) < t_{j,k^*}(x),$$

then such point $x^*$ must be included in the nodes system (4).
The procedure for including nodes of this type in the system (4) should be executed until such nodes finish, or the including of any of such remaining nodes into system (4) gives the interval which length is shorter than the preset given threshold value $\delta_{\min} > 0$.

The unknown function $m(x)$ must satisfy the point condition (18) and (7). The jump prohibtion (14) (with the order $k^*$) must be prescribed for the point $x^*$, which was included into the system of nodes.

5. The polynomials application algorithmic details

Among functions being representable on computer one of the simpliest variant allowing satisfy the conditions (7) is the case of the polynomials. Taking into account the necessity of the conditions (7) control on the intervals ends

$$m_j^{(k)}(x_{j-1} + 0) \in [b_{j,k}(x_{j-1} + 0); t_{j,k}(x_{j-1} + 0)],$$

and the fulfillment of conditions (10), it is convenient to use the boundaries $x_{j-1}$, $x_j$ of segments (4) as the points, where the polynomials parameters are given. Because the numbers of conditions are $(\nu + 1)$ both for the left and the right ends of the segment $[x_{j-1}; x_j]$, then a special case of C. Hermite interpolation polynomial (with two nodes) having degree $2\mu - 1 \geq 2\nu + 1$ may be used as polynomial.

5.1. Complete construction of interpolation polynomial by means of divided differences

Let for $a < b$ and $\mu \geq 1$ on the segment $[a; b]$ the values

$$\left( f^{(p-1)}(a) \right)_{p=1,\ldots,\mu}, \quad \left( f^{(p-1)}(b) \right)_{p=1,\ldots,\mu}$$

of derivatives in the points $a$ and $b$ having the order from 0 to $\mu - 1$ be given as the control $f(x)$ initial approximation. To obtain the polynomial let’s make a table of a special form. Let’s place the values

$$f_{p,0} = f(a,\ldots,a) = \frac{f^{(p-1)}(a)}{(p-1)!}, \quad f_{0,q} = f(b,\ldots,b) = \frac{f^{(q-1)}(b)}{(q-1)!} \text{ for } p, q = 1, \ldots, \mu$$

in its left and upper headings. Let’s denote $\Delta = b - a$. Then, from left to right and from top to bottom let’s compute the divided differences

$$f_{p,q} = f(a,\ldots,a,b,\ldots,b) = \frac{f_{p-1,q} - f_{p,q-1}}{\Delta} \text{ for } p, q = 1, \ldots, \mu.$$  

In table 1 and further the arrow to the right means the construction “by $a$, then by $b$”, and the arrow to the left means the construction “by $b$, then by $a$”. The C. Hermite interpolation polynomial constructed for the nodes $a$ and $b$ can be written in the following forms:

$$P_{\mu,\mu}(x; a, b) = L_\mu^\rightarrow(x; a, b) + H_\mu^\rightarrow(x; a, b) =$$

$$= L_\mu^\leftarrow(x; a, b) + H_\mu^\leftarrow(x; a, b).$$

Hereinafter $L$ is the lower part of $P$ and $H$ is the higher part of $P$:

$$L_\mu^\rightarrow(x; a, b) = \sum_{k=0}^{\mu-1} \lambda_k^\rightarrow \cdot (x - a)^k, \quad H_\mu^\rightarrow(x; a, b) = (x - a)^\mu \cdot \sum_{k=0}^{\mu-1} \eta_k^\rightarrow \cdot (x - b)^k,$$
\begin{table}
\centering
\begin{tabular}{cccc}
\hline
 & \(\lambda_0^\rightarrow = f_{0,1} = f(b)\) & \(f_{0,2} = f'(b)\) & \(\ldots\) & \(f_{0,\mu} = \frac{f^{(\mu-1)}(b)}{\mu!}\) \\
\hline
\(\lambda_0^\rightarrow = f_{1,0} = f(a)\) & \(f_{1,1}\) & \(f_{1,2}\) & \(\ldots\) & \(\eta_0^\rightarrow = f_{1,\mu}\) \\
\hline
\(f_{2,0} = f'(a)\) & \(f_{2,1}\) & \(f_{2,2} = \frac{f_{1,2} - f_{2,1}}{\Delta}\) & \(\ldots\) & \(f_{2,\mu}\) \\
\hline
\(\vdots\) & \(\vdots\) & \(\vdots\) & \(\ldots\) & \(\vdots\) \\
\hline
\(f_{\mu,0} = \frac{f^{(\mu-1)}(a)}{(\mu-1)!}\) & \(\eta_{\mu,0}^\rightarrow = f_{\mu,1}\) & \(f_{\mu,2}\) & \(\ldots\) & \(f_{\mu,\mu}\) \\
\hline
\end{tabular}
\caption{}
\end{table}

\[L_{\mu}^\leftarrow (x; a, b) = \sum_{k=0}^{\mu-1} \lambda_k^\rightarrow \cdot (x-b)^k, \quad H_{\mu}^\leftarrow (x; a, b) = (x-b)^\mu \cdot \sum_{k=0}^{\mu-1} \eta_k^\rightarrow \cdot (x-a)^k, \quad (27)\]

\[\lambda_k^\rightarrow = f_{k+1,0}, \quad \eta_k^\rightarrow = f_{\mu,k+1}, \quad \lambda_k^\leftarrow = f_{0,k+1}, \quad \eta_k^\leftarrow = f_{k+1,\mu}, \quad k = 0, \ldots, \mu - 1. \quad (28)\]

To compute the divided differences (23) it is necessary to accomplish \(\mu^2\) divisions (further on for the complexity estimates the operations with floating-point numbers are considered everywhere) and \(2\mu\) divisions (22) by factorials. It takes \(2\mu\) memory cells to compute one of the forms (24) or (25) and requires \((4\mu - 1)\) cells for two forms.

5.2. Interpolation polynomials construction by means of basis polynomials

The interpolation polynomials may be constructed in a different way: let’s prepare the “basis” polynomials beforehand [13], i.e. for the preset given values of \(a, b, \mu\) let’s construct the system

\[F = (F_{p,0}, F_{0,p})_{p=1,\ldots,\mu}\]

of such polynomials, that

\[F^{(k-1)}_{j,0}(x) = \begin{cases} 
\frac{1}{k!} \cdot \delta_j^k \text{ for } x = a; \\
0 \text{ for } x = b,
\end{cases} \quad F^{(k-1)}_{0,j}(x) = \begin{cases} 
0 \text{ for } x = a; \\
\frac{1}{k!} \cdot \delta_j^k \text{ for } x = b,
\end{cases} \quad (29)\]

for \(j, k = 1, \ldots, \mu\), where the L. Kronecker’s delta is \(\delta_j^k = \begin{cases} 
0 \text{ for } j \neq k; \\
1 \text{ for } j = k.
\end{cases}\) In this case

\[P_{\mu,\mu}(x; a, b) = \sum_{k=0}^{\mu-1} F_{k,0}(x; a, b) \cdot f^{(k-1)}(a) + \sum_{k=0}^{\mu-1} F_{0,k}(x; a, b) \cdot f^{(k-1)}(b). \quad (30)\]

In particular for \(\mu = 2:\)

| \(F_{1,0}\) | 0 | 0 | \(F_{2,0}\) | 0 | 0 |
|-------|---|---|-------|---|---|
| 1     | -1/\Delta & +1/\Delta^2 |
| 0     | -1/\Delta^2 & +2/\Delta^3 | 1 | -1/\Delta & +1/\Delta^2 |
\[ F_{1,0}(x; a, b) = \left[ 1 + 0 \cdot (x - a)^1 \right] + (x - a)^2 \cdot \left[ \frac{-1}{\Delta^2} + \frac{+2}{\Delta^3} \cdot (x - b)^1 \right] = (31) \]
\[ = \left[ 0 + 0 \cdot (x - b)^1 \right] + (x - b)^2 \cdot \left[ \frac{+1}{\Delta^2} + \frac{+2}{\Delta^3} \cdot (x - a)^1 \right] = (32) \]
\[ = 1 + 0 \cdot (x - a)^1 + \frac{-3}{\Delta^2} (x - a)^2 + \frac{+2}{\Delta^3} \cdot (x - a)^3 = (33) \]
\[ = 0 + 0 \cdot (x - b)^1 + \frac{+3}{\Delta^2} (x - b)^2 + \frac{+2}{\Delta^3} \cdot (x - b)^3, (34) \]
\[ F_{2,0}(x; a, b) = \left[ 0 + 1 \cdot (x - a)^1 \right] + (x - a)^2 \cdot \left[ \frac{-1}{\Delta^2} + \frac{+1}{\Delta^3} \cdot (x - b)^1 \right] = (35) \]
\[ = \left[ 0 + 0 \cdot (x - b)^1 \right] + (x - b)^2 \cdot \left[ 0 + \frac{+1}{\Delta^3} \cdot (x - a)^1 \right] = (36) \]
\[ = 0 + 1 \cdot (x - a)^1 + \frac{-2}{\Delta} \cdot (x - a)^2 + \frac{+1}{\Delta^2} \cdot (x - a)^3 = (37) \]
\[ = 0 + 0 \cdot (x - b)^1 + \frac{+1}{\Delta^2} \cdot (x - b)^2 + \frac{+1}{\Delta^3} \cdot (x - b)^3. (38) \]

The formulas for \( F_{0,1}(x; a, b) \) and \( F_{0,2}(x; a, b) \) may be constructed by analogy. The first two forms: (31), (35) and (32), (36) are two-point, constructed with the use of the tables of divided differences. The next two forms: (33), (37) and (34), (38) are one-point, obtained by the substitution \( (x - a) = (x - b) + \Delta \).

Generally, table 2 of divided differences for \( F_{1,0} \) is a set of coefficients at monomials \( a^p b^q \) in expansion \( -\left( \frac{a - b}{\Delta} \right)^{p+q} \). It is the union of two fragments of the B. Pascal triangle with signs changing from column to column and with degrees of \( \Delta \) increasing in denominators.

| \( \lambda_0 \) | \( \lambda_0^{-} = F^{(0)}_{1,0}(b) = 0 \) | \( F^{(1)}_{1,0}(b) = 0 \) | \( \ldots \) | \( F^{(\mu-1)}_{1,0}(b) = 0 \) |
|------------------|----------------|-----------------|-----------------|----------------|
| \( F^{(0)}_{1,0}(a) = 1 \) | \( \frac{-1}{\Delta^1} \) | \( \frac{+1}{\Delta^{1+2-1}} \) | \( \ldots \) | \( \frac{(-1)^{\mu}}{\Delta^\mu} \cdot \left( \frac{\mu - 1}{\mu - 1} \right) \) |
| \( F^{(1)}_{1,0}(a) = 0 \) | \( \frac{-1}{\Delta^{2+1-1}} \) | \( \frac{(-1)^2}{\Delta^{2+2-1}} \cdot \left( \frac{2}{1} \right) \) | \( \ldots \) | \( \frac{(-1)^{\mu}}{\Delta^{2+\mu-1}} \cdot \left( \frac{2\mu - 2}{\mu - 1} \right) \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( F^{(\mu-1)}_{1,0}(a) = 0 \) | \( \eta_0^{-} = \frac{-1}{\Delta^\mu} \left( \frac{\mu - 1}{0} \right) \) | \( \frac{+1}{\Delta^{\mu+2-1}} \cdot \left( \frac{\mu}{1} \right) \) | \( \ldots \) | \( \frac{(-1)^{\mu}}{\Delta^{2\mu-1}} \cdot \left( \frac{2\mu - 2}{\mu - 1} \right) \) |

The tables for \( F_{p,0} \) may be obtained from \( F_{1,0} \) by means of shift down for \( (p - 1) \) lines (the top lines becoming empty must be filled in by zeros) and by means of the result division by \( (p - 1)! \).

The table for \( F_{0,1} \) is a copy of the one for \( F_{1,0} \): only a sign replacement and the headers correction are necessary. The tables for \( F_{0,p} \) may be obtained from \( F_{0,1} \) by means of shift to the right (with the use filling in of columns by zeros) and division by \( (p - 1)! \).
For $p = 1, \ldots, \mu$ the form “by $a$, then by $b$” can be written as

$$F_{p,0} (x; a, b) = \frac{(x - a)^{p-1}}{(p-1)!} + \frac{(x - a)^{\mu}}{(p-1)!} \cdot \left[ \sum_{k=0}^{\mu-1} \frac{(-1)^{k+1}}{\Delta^{\mu+k-p+1}} \cdot \left( \frac{\mu + k - p}{k} \right) \cdot (x - b)^k \right],$$

(39)

$$F_{0,p} (x; a, b) = \frac{(x - a)^{\mu}}{(p-1)!} \cdot \left[ \sum_{k=p}^{\mu-1} \frac{(-1)^{k+p-1}}{\Delta^{\mu+k-p+1}} \cdot \left( \frac{\mu + k - p}{k - p + 1} \right) \cdot (x - b)^k \right].$$

(40)

It is reasonably to compute only once (by means of integer additions) and to save $(\mu + 1) \cdot \mu/2$ binomial coefficients.

To compute $P_{\mu,\mu}$ in (30) it is necessary to add up $2\mu$ basis polynomials of the same form multiplied by the corresponding $2\mu$ preset given coefficients. Formally it requires to execute $4\mu^2$ multiplications. However, in two-point form (“by $a$, then by $b$”) for the lower part of $F_{p,0} (x; a, b)$ only one coefficient is not equal to zero; for $F_{0,p} (x; a, b)$ the lower part entirely is equal to zero, and in the high part the coefficients are equal to zero at the items $(x - a)^{\mu} \cdot (x - b)^k$ for $k < p - 1$. In fact, it requires to accomplish $(\mu^2 + \mu ) \cdot 3/2$ multiplications and to prepare $\mu$ negative degrees of $\Delta$. The first one-point form (“by $a$”) almost always has $\mu$ non-zero coefficients in the higher parts $F_{k,0} (x; a, b)$ because of decomposition $(x - a)^k = \sum_{p=0}^{k} (x - b)^p \cdot \Delta^{k-p} \cdot \left( \frac{k}{p} \right)$.

Unlike of subsection 5.1 the method of basis polynomials is better because (when the basis polynomials and $P_{\mu,\mu} (x; a, b)$ are already constructed) it allows replace $k$ values in the set (21) (for example, for variation) by $O(2k\mu)$ operations. But if all $2\mu$ parameters vary, then the subsection 5.1 method is preferable.

5.3. On $P_{\mu,\mu}$ computation and reconstructions

**Statement 1.** For $c \in [a; b]$ the computation of value $P_{\mu,\mu} (c; a, b)$ requires $2\mu$ multiplications.

**Statement 2.** The presentation $P_{\mu,\mu} (x; a, b) = T_{2\mu} (x; a)$ (one-point – in the point $a$), where

$$T_{2\mu} (x; a) = \sum_{k=0}^{2\mu-1} t_k (a) \cdot (x - a)^k, \quad t_k (a) = \frac{T_{2\mu}^{(k)} (a; a)}{k!}, \quad k = 0, \ldots, 2\mu - 1$$

(41)

may be obtained from (24) by $O(\mu)$ multiplications/divisions.

Really, the coefficients $(\lambda_k)_{k=0, \ldots, \mu - 1}$ coincide with $t_k (a)$ for $k = 0, \ldots, \mu - 1$. Then, for $(\eta_k)_{k=0, \ldots, \mu - 1}$ it is necessary to find the coefficients $(A_{\mu+k})_{k=0, \ldots, \mu - 1}$ in the multiplier

$$\sum_{k=0}^{\mu-1} \eta_k \cdot (x - b)^k = \sum_{k=0}^{\mu-1} A_{\mu+k} \cdot (x - a)^k$$

(42)

of the polynomial $P_{\mu,\mu} (x; a, b)$ higher part $H_{\mu} (x; a, b)$. It requires $O(\mu)$ multiplications/divisions. At last, $t_k (a) = A_k$ for $k = \mu, \ldots, 2\mu - 1$.

**Statement 3.** If the coefficients of $T_{2\mu} (x; a)$ are known, then for $c \in (a; b]$ the presentation (one-point – in the point $c$) $P_{\mu,\mu} (x; a, b) = T_{2\mu} (x; c)$, where

$$T_{2\mu} (x; c) = \sum_{k=0}^{2\mu-1} t_k (c) \cdot (x - c)^k, \quad t_k (c) = \frac{T_{2\mu}^{(k)} (c; c)}{k!}, \quad k = 0, \ldots, 2\mu - 1$$

(43)

may be obtained by $O(\mu)$ multiplications/divisions.
Really, using the coefficients \((t_k(a))_{k=0, \ldots, 2m-1}\) we can find the coefficients \((t_k(c))_{k=0, \ldots, 2m-1}\) in the presentation
\[
\sum_{k=0}^{2\mu-1} t_k(a) \cdot (x - a)^k = \sum_{k=0}^{2\mu-1} t_k(c) \cdot (x - c)^k
\]
(44)
of the polynomial \(P_{\mu,\mu}(x; a, b)\), that requires \(O(2\mu)\) multiplications/divisions.

**Statement 4.** If the coefficients of \(P_{\mu,\mu}(x; a, b)\) are known, then for \([\tilde{a}; \tilde{b}] \subset [a; b]\) the presentation (two-point “by \(\tilde{a}\), then by \(\tilde{b}\)”) \[
P_{\mu,\mu}(x; \tilde{a}, \tilde{b}) = L_{\mu}^{\rightarrow}(x; \tilde{a}, \tilde{b}) + H_{\mu}^{\rightarrow}(x; \tilde{a}, \tilde{b}),
\]
(45) \[
L_{\mu}^{\rightarrow}(x; \tilde{a}, \tilde{b}) = \sum_{k=0}^{\mu-1} \tilde{\lambda}_{k}^{\rightarrow} \cdot (x - \tilde{a})^k,
\]
(46)
may be obtained by \(O(3\mu)\) multiplications/divisions.

Really, after \(O(2\mu)\) multiplications/divisions, according to Statements 2 or 3 we can find
\[
T_{2\mu}(x; \tilde{a}) = \sum_{k=0}^{2\mu-1} t_k(\tilde{a}) \cdot (x - \tilde{a})^k
\]
(47)
and with the use of the found \(\tilde{\lambda}_{k}^{\rightarrow} = t_k(\tilde{a})\) for \(k = 0, \ldots, \mu - 1\) we can determinate \(L_{\mu}^{\rightarrow}(x; \tilde{a}, \tilde{b})\).

The pass to \(H_{\mu}^{\rightarrow}\) requires additionally \(O(\mu)\) multiplications/divisions:
\[
(x - \tilde{a})^\mu \cdot \sum_{k=0}^{\mu-1} \tilde{\eta}_{k}^{\rightarrow} \cdot (x - \tilde{a})^k = (x - \tilde{a})^\mu \cdot \sum_{k=0}^{\mu-1} \tilde{\eta}_{k}^{\rightarrow} \cdot (x - \tilde{b})^k = H_{\mu}^{\rightarrow}(x; \tilde{a}, \tilde{b}).
\]
(48)

### 5.4. Notes

1. In Statement 1 the variant of the W. G. Horner scheme [14] is applied.
2. In Statements 2-4 the M. Shaw and J. F. Traub variant [14] of the W. G. Horner scheme is applied. This variant has the complexity \(O(\mu)\) multiplications/divisions. Besides, this variant contains double cycle. So it has the complexity \(O(\mu^2)\) additions/subtractions.
3. The pass from \(P_{\mu,\mu}(x; a, b)\) to \(P_{\mu,\mu}(x; \tilde{a}, \tilde{b})\) is applied for example when it is necessary to localize on subintervals [6] initial approximations being preset given on continuity intervals. In particular, it applies for addition of points in (4), when the condition (17) or (18) is satisfied. Further, the individual polynomials will be used on the subintervals (for the controls variation for the functionals [8, 15, 16] optimization).
4. From the presented estimates it follows that the use of explicit expressions as the algorithmic base for computation of derivatives \(P_{\mu,\mu}^{(k)}(c; a, b)\) for \(k = 0, \ldots, \mu - 1\) at \(c \in [a; b]\)
\[
P_{\mu,\mu}^{(k)}(c; a, b) = \sum_{p=0}^{\mu-1} f_{p+1,0} \cdot \frac{p! \cdot (c - a)^{p-k}}{(p-k)!} + \sum_{q=0}^{k} \sum_{s=0}^{\mu-1} \sum_{q=k+1}^{k} f_{\mu,q+1} \cdot \frac{k! \cdot \mu! \cdot q! \cdot (c - a)^{\mu-(k-s)} \cdot (c - b)^{q-s}}{s! \cdot (k-s)! \cdot (\mu - (k-s))! \cdot (q-s)!}
\]
(49)
at high \(\mu\) is worse than the use of \(T_{2\mu}(x; c)\), where \(P_{\mu,\mu}^{(k)}(c; a, b) = k! \cdot t_k(c)\).
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References

[1] Bilchenko G G and Bilchenko N G 2015 Comparative analysis of some special blowing laws application in the problems of heat and mass transfer optimal control on the permeable porous surfaces of hypersonic aircraft “Actual Problems of Applied Mathematics, Informatics and Mechanics” Proc. Int. Scientific and Technical Conf. (Russia, Voronezh: “Research Publishers”) pp 137-139 (in Russian)

[2] Bilchenko G G and Bilchenko N G 2015 Comparative analysis of some special blowing laws application in the problems of heat and mass transfer optimal control on the permeable perforated surfaces of hypersonic aircraft “Actual Problems of Applied Mathematics, Informatics and Mechanics” Proc. Int. Scientific and Technical Conf. (Russia, Voronezh: “Research Publishers”) pp 134-136 (in Russian)

[3] Bilchenko G G 2011 On valency of Cisotti’s generalized integrals “Modern Problems of Applied Mathematics, Control Theory and Mathematical Modeling” Proc. IV Int. Scientific Conf. (Russia, Voronezh: VSU Publishing Center) pp 31-33 (in Russian)

[4] Bilchenko G G 2011 Starlike modification of strictly close-to-convex Cisotti’s integral “Actual Problems of Applied Mathematics, Informatics and Mechanics” Proc. Int. Scientific Conf. (Russia, Voronezh: VSU Publishing Center) pp 69-71 (in Russian)

[5] Bilchenko G G and Bilchenko N G 2016 On the possibility of control restoration in some inverse problems of heat and mass transfer 11th Int. Conf. “Mesh methods for boundary-value problems and applications” IOP Conf. Series: Mater. Sci. Eng. 158:1 012020 pp 1-6 (Russia, Kazan: IOP Publishing) DOI: 10.1088/1757-899X/158/1/012020 URL: http://iopscience.iop.org/article/10.1088/1757-899X/158/1/012020/pdf

[6] Bilchenko G G and Bilchenko N G 2016 On computational experiments in some inverse problems of heat and mass transfer 11th Int. Conf. “Mesh methods for boundary-value problems and Applications” IOP Conf. Series: Mater. Sci. Eng. 158:1 012021 pp 1-5 (Russia, Kazan: IOP Publishing) DOI: 10.1088/1757-899X/158/1/012021 URL: http://iopscience.iop.org/article/10.1088/1757-899X/158/1/012021/pdf

[7] Bilchenko G G and Bilchenko N G 2017 Inverse problems of heat and mass transfer on hypersonic aircrafts permeable surfaces III. On the statement of two dimensional problems and the ranges of allowed values “heat-friction” Proc. Voronezh State Univ. Ser. System Analysis and Inform. Technologies 1 18-25 (in Russian) URL: http://www.vestnik.vsu.ru/pdf/analiz/2017/01/2017-01-03.pdf

[8] Bilchenko N G 2015 Permeable surfaces hypersonic aircraft optimal heat protection mathematical modeling Proc. 2015 Int. Conf. “Stability and Control Processes” SCP-2015 (in Memory of V I Zubov) ed L A Petrosyan and A P Zhabko (Russia, Saint-Petersburg: IEEE) pp 310-313 DOI: 10.1109/SCP.2015.7342145 URL: http://ieeexplore.ieee.org/document/7342145/

[9] Polyakov A F and Reviznikov D L 1999 Singularities of thermal protection of the front edge under conditions of combination porous penetration and convection-conduction cooling High Temperature 37:6 895-898

[10] Varaksin A Yu, Polyakov A F, Reviznikov D L, Strat’ev V K and Tre’t’yakov A F 2002 Investigation of flow in the vicinity of porous material on a cylindrical body subjected to flow High Temperature 40:6 843-849 URL: https://doi.org/10.1023/A:1021473116156

[11] Polezhaev Yu V and Yurevich F B 1976 Thermal Protection (Russia, Moscow: “Energiya”) 392 (in Russian)

[12] Bilchenko G G and Bilchenko N G 2016 The classes of solutions of the problems of boundary layer optimal control on the permeable surfaces of hypersonic aircrafts Proc. Int. Conf. “Voronezh Winter Math. School of S G Krein” ed V A Kostin (Russia, Voronezh: “Nauchnaya kniga” Publishers) pp 92-86 (in Russian)

[13] Samarskiy A A and Gulin A V 1989 Numerical Methods (Russia, Moscow: “Nauka”) 432 (in Russian) ISBN 5-502-013996-3

[14] Knuth D E 1997 The Art of Computer Programming, vol 2. Seminumerical Algorithms. – 3rd ed. (Reading, Massachusetts: Addison-Wesley) xiv+762 ISBN 0-201-89684-2

[15] Bilchenko G G and Bilchenko N G 2016 Construction of range of extreme values of functionals Diff. Equations and Control Processes 2:2 “Herzen Readings 2016. Some Actual Probl. of Modern Math. and Math. Educ.” ed V F Zaytsev, V D Budaev and A V Flegontov (Russia, Saint-Petersburg) pp 56-61 (in Russian) URL: http://www.math.spb.ru/diffjournal/pdf/herzen2016.pdf

[16] Bilchenko G G and Bilchenko N G 2016 Functional extreme values range Proc. 2016 Int. Conf. “Stability and Oscillations of Nonlinear Control Systems” (Pyatnitskiy’s Conf.) ed V N Tkhai (Russia, Moscow: IEEE) pp 1-2 DOI: 10.1109/STAB.2016.7541166 URL: http://ieeexplore.ieee.org/document/7541166/