Resummation in (F)APT

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Abstract

We present new results on the summation of nonpower-series expansions in QCD Analytic Perturbation Theory (APT) in the one-loop approximation. We show how to generalize the approach suggested by one of us earlier to the cases of APT and Fractional APT (FAPT) with heavy-quark thresholds. Using this approach we analyze the Higgs boson decay $H^0 \rightarrow b\bar{b}$. We produce estimations of the higher-order corrections importance in (F)APT and present a very transparent interpretation of the resummation results in terms of $\Lambda_{\text{QCD}}$ shifts.

1 Introduction

This is a written version of the talk given at the Memorial Igor Solovtsov Seminar, held on January 17–18, 2008, at the Bogoliubov Laboratory of Theoretical Physics (BLTPh), JINR (Dubna). For this reason first we want to recollect our contacts with Igor Solovtsov in and outside the lab.

We met first time at BLTPh in the mid 90s when he conducted a series of seminars on the variational approach to quantum field theory, in particular to QCD. He was those times very enthusiastic about this approach which later on guided him to Analytic Perturbation Theory. Only after some time other people started to share his enthusiasm and understand what he knew at the very beginning. We met ones at the Institute for Theoretical Physics of Heidelberg University (presumably, in 1997) and participated in the seminar on Analytic Perturbation Theory he presented there — it was a kind of enjoiment, so clearly he presented the matter. We remember very well how he was preparing his doctorate thesis — calmly, with some kind of dignity, and remaining to be very open for discussions. Later on, we met also during some conferences. The one we remember more clearly, is the Research Workshop on Calculations for Modern and Future Colliders, held in Dubna in July 2000. The reason for this memory is especial: during the boat-trip along the Volga river it appeared that Igor was a very good singer — he had a good voice, an absolute pitch and a very nice song repertory. One of us, A.B., met Igor last time quite recently, during the...
XIV International Seminar “Nonlinear Phenomena in Complex Systems” (May 22–25, 2007, Minsk) and ones again enjoyed his nice voice. To resume, Igor was a talented physicist, a nice man, and a good friend. Let God bless his memory and Igor be remembered forever.

Now we continue with his ‘child’ — APT. Since the original works of Jones and Solovtsov [1, 2] and Shirkov and Solovtsov [3] appeared in 1995–97, the analytic approach to QCD perturbation theory has progressed considerably. The main object of this approach is the spectral density that provides the means to define the analytic running coupling in the Euclidean region using a dispersion relation. The same spectral density is used to define also the running coupling in Minkowski space, by employing the dispersion relation for the Adler function [4, 5]. The analytic approach was extended beyond the one-loop level [5, 6] and special analytic and numerical tools were elaborated [7–9]. APT appeared to be well-suited to deal with more than one (large) momentum scale [10, 11]. All these efforts resulted in a systematic approach, named Analytic Perturbation Theory (APT), recently reviewed in [12].

In order to treat hadronic amplitudes in the same manner one needs to use not only analytization of the running coupling, but also more complicated objects, like integrated couplings $\int_0^1 dx \int_0^1 dy \alpha_s(Q^2xy)f(x)f(y)$, as was firstly indicated in [13, 14]. The authors of this paper proposed the principle of analytization as a whole to treat such objects and new techniques was developed to produce needed analytic images. The same principle applies also to other hadronic amplitudes in the renormalization-group based QCD factorization approach, where complicated objects, like fractional powers of running coupling and its products with logarithms, $\ln(Q^2/\mu^2)$, naturally arise. A version of the “analytization” approach, which takes into account all terms that contribute to the spectral density, was developed in [15–17] and named Fractional Analytic Perturbation Theory (FAPT).

In this paper, we present new results on the resummation of non-power-series expansions in APT in the one-loop approximation. We show how to generalize the approach suggested by one of us earlier [18] to the cases of: (i) global (with taking into account heavy-quark thresholds) APT, and (ii) global FAPT. Using this approach we analyze the Higgs boson decay into a $\bar{b}b$-pair of quarks. We produce estimations of how important higher-order corrections can be and present very transparent interpretation of the resummation results in terms of $\Lambda_{\text{QCD}}$ shifts.

2 Fixed-$N_f$ and global APT

First, we need to explain our notation. In order to have a direct connection to our previous papers [15–17] and to simplify the main formulae in the sections,
Here we need also analogs of fixed-NL but rather of logarithms the same notation for coupling but with an argument placed in squares: $\rho_{n}(1a) - (1b)$ with the spectral densities $\bar{\rho}(1a)$ - (1b) with the spectral densities $\bar{\rho}$ notation [12]. These couplings also have integral representations of the type $\rho_{n}$ where, in order to make our formulae more compact, we denoted $m_{n}$.

In order to distinguish these two approaches, we introduce the upper index $\bar{\rho}$. Here we need also analogs of fixed-$N_{f}$ quantities with standard normalizations:

$$\bar{A}_{n}(Q^{2}) = \mathcal{A}_{E}[a^{n}] \equiv \int_{0}^{\infty} \frac{\rho_{n}(\sigma)}{\sigma + Q^{2}} d\sigma; \quad (1a)$$

$$\mathfrak{A}_{n}(s) = \mathcal{A}_{M}[\alpha^{n}] \equiv \int_{s}^{\infty} \frac{\rho_{n}(\sigma)}{\sigma} d\sigma \quad (1b)$$

with $\rho_{n}(\sigma) = \ln [a^{n}(-\sigma)]$ and $\beta_{f} = b_{0}(N_{f})/(4\pi) = (11 - 2N_{f}/3)/(4\pi)$ with $b_{0}(N_{f})$ being the first coefficient in the QCD $\beta$ function.

When we discuss a global version of (F)APT, where $Q^{2}$ (or $s$) varies in the whole domain $[0, \infty)$ and $N_{f}$ effectively becomes dependent on $Q^{2}$ (or $s$), we use the standard running coupling $\alpha_{s}(Q^{2})$, and analytic powers are defined for the corresponding powers of this coupling

$$\bar{A}_{n}^{\text{glob}}(Q^{2}) = \mathcal{A}_{E}[\alpha^{n}] \equiv \int_{0}^{\infty} \frac{\rho_{n}^{\text{glob}}(\sigma)}{\sigma + Q^{2}} d\sigma; \quad (1c)$$

$$\mathfrak{A}_{n}^{\text{glob}}(s) = \mathcal{A}_{M}[\alpha^{n}] \equiv \int_{s}^{\infty} \frac{\rho_{n}^{\text{glob}}(\sigma)}{\sigma} d\sigma \quad (1d)$$

In order to distinguish these two approaches, we introduce the upper index $\bar{\rho}$. Here we need also analogs of fixed-$N_{f}$ quantities with standard normalizations:

$$\bar{A}_{n}(Q^{2}; N_{f}) = \frac{\bar{A}_{n}(Q^{2})}{\beta_{f}^{n}} \quad ; \quad \mathfrak{A}_{n}(s; N_{f}) = \frac{\mathfrak{A}_{n}(s)}{\beta_{f}^{n}}, \quad (1e)$$

which correspond to analytic couplings $\bar{A}_{n}$ and $\mathfrak{A}_{n}$ in the Shirkov–Solovtsov notation [12]. These couplings also have integral representations of the type (1a)–(1b) with the spectral densities $\bar{\rho}_{n}(\sigma; N_{f}) = \rho_{n}(\sigma)/\beta_{f}^{n}$.

The spectral density $\rho_{n}^{\text{glob}}(\sigma)$ is simply related with the spectral density $\bar{\rho}_{n}(\sigma; N_{f})$ [3, 7, 19]:

$$\rho_{n}^{\text{glob}}(\sigma) = \theta [\sigma \leq m_{4}^{2}] \bar{\rho}_{n}(\sigma; 3) + \sum_{f=4}^{6} \theta [m_{f}^{2} < \sigma \leq m_{f+1}^{2}] \bar{\rho}_{n}(\sigma; N_{f}) , \quad (1f)$$

where, in order to make our formulae more compact, we denoted $m_{4} = m_{c}$, $m_{5} = m_{b}$, $m_{6} = m_{t}$, and $m_{7} = +\infty$.

We need also to speak about effective coupling as a function not of $Q^{2}$ or $s$, but rather of logarithms $L = \ln(Q^{2}/\Lambda^{2})$ or $L = \ln(s/\Lambda^{2})$. In these cases, we use the same notation for coupling but with an argument placed in squares: $a^{n}[L]$, $\bar{A}_{n}[L]$, and $\mathfrak{A}_{n}[L]$. Then, in the one-loop approximation we have

$$a[L] = \frac{1}{L}, \quad \rho_{1}[L] = \frac{1}{L^{2} + \pi^{2}}, \quad \bar{\rho}_{1}[L; N_{f}] = \frac{1}{\beta_{f}[L^{2} + \pi^{2}]} \quad . \quad (2)$$
2.1 Series summation in the one-loop APT with fixed \( N_f \)

Let us consider different sorts of perturbative series expansions of a typical physical quantity, like the Adler function, \( D[L] \), and write at the one-loop level [17],

\[
\begin{aligned}
\left\{ \begin{array}{c}
D[L] \\
D[L] \\
\mathcal{R}[L]
\end{array} \right\} &= d_0 + d_1 \sum_{n=1}^{\infty} \tilde{d}_n \left\{ \begin{array}{c}
a^n[L] \\
A_n[L] \\
\mathcal{A}_n[L]
\end{array} \right\},
\end{aligned}
\]

(3)

where \( L = \ln \left( \frac{Q^2}{\Lambda^2} \right) \) applies to the Euclidean \((D[L], D[L], a^n[L] and A_n[L])\) and \( L = \ln \left( \frac{s}{\Lambda^2} \right) \) to the Minkowski \((\mathcal{R}[L] and \mathcal{A}_n[L])\) regions, and \( \tilde{d}_n \equiv d_n/d_1 \).

It is useful to introduce a generating function \( P(t) \) for the series expansion whose specific form depends on a particular quantity in question:

\[
\tilde{d}_n = \int_0^\infty P(t) t^{n-1} dt \quad \text{with} \quad \int_0^\infty P(t) dt = 1.
\]

(4)

To shorten our formulae, we use a shorthand notation

\[
\langle \langle f(t) \rangle \rangle_P \equiv \int_0^\infty f(t) P(t) dt,
\]

(5)

where the brackets \( \langle \langle \ldots \rangle \rangle_P \) denote the average over \( t \) with the weight \( P(t) \).

The different couplings \( a, A_1, \) and \( \mathcal{A}_1 \) satisfy the same one-loop renormalization-group equation that can be rewritten as a one-loop recurrence relation [20–22]

\[
\left\{ \begin{array}{c}
a^{n+1}[L] \\
A_{n+1}[L] \\
\mathcal{A}_{n+1}[L]
\end{array} \right\} = \frac{1}{\Gamma(n+1)} \left( -\frac{d}{dL} \right)^n \left\{ \begin{array}{c}
a^n[L] \\
A_n[L] \\
\mathcal{A}_n[L]
\end{array} \right\}.
\]

(6)

Substituting Eqs. (6), (4) into the perturbative series expansion, Eq. (3), one obtains

\[
\begin{aligned}
\left\{ \begin{array}{c}
D[L] \\
D[L] \\
\mathcal{R}[L]
\end{array} \right\} &= d_0 + d_1 \sum_{n=0}^{\infty} \frac{\langle \langle (-t)^n \rangle \rangle_P}{n!} \frac{d^n}{dL^n} \left\{ \begin{array}{c}
a^n[L] \\
A_n[L] \\
\mathcal{A}_n[L]
\end{array} \right\} \\
&= d_0 + d_1 \left\{ \begin{array}{c}
\langle \langle a[L-t] \rangle \rangle_P \\
\langle \langle A_1[L-t] \rangle \rangle_P \\
\langle \langle \mathcal{A}_1[L-t] \rangle \rangle_P
\end{array} \right\},
\end{aligned}
\]

(7)

as it was shown by one of us (S.M.) in [18]. As long as we have not proved that the order of summation and integration can be interchanged, this representation has only a formal meaning. Note, however, that the integration over the Taylor expansion of the term \( a[L-t] \) in the integrand reproduces the initial series for any partial sum. It is not surprising that the integrand in the standard case (first
line in Eq. (7)) faces a pole singularity and is, therefore, ill-defined. On the other hand, for the last two cases the integral has a rigorous meaning, ensured by the finiteness of the couplings, i.e., $\mathcal{A}_1[t], \mathfrak{A}_1[t] \leq 1$. Since any coefficient $\tilde{d}_n$ is the moment of $P(t)$, this function should fall off faster than any power, e.g., like an exponential or faster. Therefore, the APT expressions on the RHS of Eq. (7) exist and are proportional to $\mathcal{A}_1[L - \bar{t}(L)]$ or $\mathfrak{A}_1[L - \bar{t}(L)]$, where $\bar{t}(L)$ can be treated for each $L$ as an average value of $t$ associated with this quantity.

Provided the generating function $P(t)$ is known, one can compute the average $\langle \langle \mathcal{A}_1[L - t] \rangle \rangle_P$ (or $\langle \langle \mathfrak{A}_1[L - t] \rangle \rangle_P$) in Eq. (7) explicitly and obtain all-order estimates for the expanded quantity under consideration. Unfortunately, only the first few $d_n$ coefficients are known for most relevant processes. However, the asymptotic tail calculated first by Lipatov et al. [23], see for review [24], allows us to construct a model of the generating function $P(t)$ with reasonable accuracy:

$$\tilde{d}_n \sim \Gamma(n + 1) n^\gamma c^n \left[ 1 + O \left( \frac{1}{n} \right) \right] \rightarrow P(t) \sim (t/c)^{\gamma+1} e^{-t/c}, \quad (8)$$

where $\gamma < 1$ and $c$ are numerical coefficients.

Now we illustrate this statement by the example of perturbative coefficients $\tilde{d}_n$ for the Adler function $D_S[L] = d_0 + d_1 \sum_{n=1}^{\infty} \tilde{d}_n \left( \frac{\alpha_s[L]}{\pi} \right)^n \quad (9)$

for scalar correlator, cf. [17, 25], see the first row in Table 1. To simulate the $n$-dependence of the coefficients $\tilde{d}_n$, in accordance with recipe (8), we use the model:

$$\tilde{d}_n^H = c^{n-1} \frac{\Gamma(n + 1) + \beta \Gamma(n)}{1 + \beta}, \quad (10a)$$

which is produced by the generating function

$$P_H(t) = \frac{(t/c) + \beta}{c(1 + \beta)} e^{-t/c}. \quad (10b)$$

In the second row of Table 1 we show the results obtained by fitting two known coefficients $\tilde{d}_2$ and $\tilde{d}_3$ using the model (10)$^1$. We see that it gives a very good prediction for $\tilde{d}_4^H = 662$, which appears to be quite close to the value 620 calculated by Chetyrkin et al. in [25] a year ago. If we use instead the fitting procedure with taking into account the fourth coefficient $\tilde{d}_4$, then the parameters $c$ and $\beta$ of our model (10) change slightly: $\{c = 2.5, \beta = -0.48\} \rightarrow \{c = 2.4, \beta = -0.52\}$. This shows the reasonability of our modelling.

$^1$Note that $\tilde{d}_1^H$ is automatically equal to unity.
| Model                                               | \( \tilde{d}_1 \) | \( \tilde{d}_2 \) | \( \tilde{d}_3 \) | \( \tilde{d}_4 \) | \( \tilde{d}_5 \) |
|----------------------------------------------------|------------------|------------------|------------------|------------------|------------------|
| pQCD results with \( N_f = 5 \)                   | 1                | 7.42             | 62.3             | 620              | —                |
| Model (10) with \( c = 2.5, \beta = -0.48 \)     | 1                | 7.42             | 62.3             | 662              | 8615             |
| Model (10) with \( c = 2.4, \beta = -0.52 \)     | 1                | 7.50             | 61.1             | 625              | 7826             |
| “NNA” prediction of [26]                           | 1                | 3.87             | 21.7             | 122              | 1200             |

Table 1: Coefficients \( \tilde{d}_n \) for Higgs boson decay series.

Predictions for \( \tilde{d}_n \) values in the so-called “Naive Non-Abelianization” (NNA) approximation [26] are presented in the fourth line of Table 1 and they appear to be significantly smaller than the exact values as well as our model estimates.

### 2.2 Series summation in the global APT

Consider the situation with resummation in the global APT when we take into account heavy-quark thresholds. We consider here the following series:

\[
\mathcal{R}_{\text{glob}}[L] = d_0 + d_1 \sum_{n=1}^{\infty} \tilde{d}_n \mathfrak{A}_n^{\text{glob}}[L]. \tag{11}
\]

Note that due to different normalizations of \( \mathfrak{A}_n[L] \) and \( \mathfrak{A}_n^{\text{glob}}[L] \) the coefficients \( \tilde{d}_n \) in Eqs. (3) and (11) are different. In Appendix A we consider in detail the case with only one threshold, corresponding to the transition \( N_f = 3 \to N_f = 4 \), and provide the result of summation in the Minkowski region in Eqs. (A6)–(A7).

With taking into account all other thresholds the final result reads

\[
\mathcal{R}_{\text{glob}}[L] = d_0 + d_1 \sum_{f=3}^{6} \theta(L_f \leq L < L_{f+1}) \langle \langle \mathfrak{A}_1 \left[ L + \lambda_f - \frac{t}{\beta_f}; f \right] \rangle \rangle_P
\]

\[
+ d_1 \sum_{f=3}^{5} \theta(L_f \leq L < L_{f+1}) \sum_{k=f+1}^{6} \langle \langle \Delta_k \mathfrak{A}_1[t] \rangle \rangle_P, \tag{12}
\]

where \( \lambda_f \equiv \ln(\Lambda_3^2/\Lambda_f^2) \) describes the shift of the logarithmic argument due to the change of the QCD scale parameter \( (\Lambda_3 \to \Lambda_f) \), \( L_f \equiv \ln(m_f^2/\Lambda_3^2) \), \( L_3 = -\infty \) and \( L_7 = +\infty \). The second term in Eq. (12) looks like a natural generalization of the fixed-\( N_f \) formula (7) with taking into account different QCD scales \( \Lambda_f \) for each fixed-\( N_f \) region, while the last term appears due to continuity of \( \mathcal{R}_{\text{glob}}[L] \) at heavy-quark thresholds. Using a toy model \( P_{\text{toy}}(t) = \delta(t - 2) \) we estimate a relative contribution of the \( \sum_{k=f+1}^{6} \langle \langle \Delta_k \mathfrak{A}_1[t] \rangle \rangle_P \)-term: it is of the order of 3%
for the $f = 4$ threshold, 0.5% for the $f = 5$ threshold, and 0.1% for the $f = 6$ threshold.

In the Euclidean region the result of summation, as shown in [27], is more complicated:

$$D_{\text{glob}}[L] = d_0 + d_1 \sum_{f=3}^{6} \left\langle \int_{L_f}^{L_{f+1}} \frac{\rho_1 [L_\sigma + \lambda_f; N_f]}{1 + e^{L_\sigma - t_\beta / \beta_f}} \right\rangle_P$$

$$+ d_1 \sum_{f=4}^{6} \left\langle \langle \Delta_f[L,t] \rangle \right\rangle_P , \quad (13)$$

where corrections $\left\langle \langle \Delta_f[L,t] \rangle \right\rangle_P$ to the naive expectation formula are defined as

$$\Delta_f[L,t] \equiv \int_0^1 \frac{\rho_1 [L_f + \lambda_f - t x / \beta_f; N_f]}{\beta_f \left[ 1 + e^{L_f - t x / \beta_f} \right]} dx$$

$$- \int_0^1 \frac{\rho_1 [L_f + \lambda_{f-1} - t x / \beta_{f-1}; N_{f-1}]}{\beta_{f-1} \left[ 1 + e^{L_f - t x / \beta_{f-1}} \right]} dx . \quad (14)$$

and now, in contrast to the Minkowski case, they explicitly depend on $L$. To estimate the effect of the thresholds in the Euclidean domain, we compare the numeric value of (13) for our toy model $P_{\text{toy}}(t) = \delta(t - 2)$ with the numeric value of the ‘naive’ summation formula

$$D_{\text{naive}}[L] = d_0 + d_1 \sum_{f=3}^{6} \int_{L_f}^{L_{f+1}} \left\langle \frac{\rho_1 [L_\sigma + \lambda_f; N_f]}{1 + e^{L_\sigma - t_\beta / \beta_f}} \right\rangle_{P_{\text{toy}}(t)} . \quad (15)$$

The relative values of the differences between (13) and (15) are of the order of 2.5% at $L \sim 5$ and are about 0.5% for $L \gtrsim 12$.

### 3 Fixed-$N_f$ and global FAPT

Sometimes perturbative series starts not with the unity term like $d_0$ in Eq. (3) but rather with $d_0 a^\nu[L]$ with power $\nu$ being a fractional number, see, for example, [17]. Then one needs to use FAPT in order to obtain an analytic image of the series. For this reason we need to generalize the APT summation method discussed in section 2 in order to apply it in FAPT case.

#### 3.1 Resummation of expansions in the fixed-$N_f$ FAPT

First, we consider the case of fixed $N_f$. We start with a power series in the standard perturbative QCD

$$D_{\nu}[L] = d_0 a^\nu[L] + d_1 \sum_{n=1}^{\infty} \tilde{d}_n a^{n+\nu}[L] , \quad (16)$$
with \( \nu > 0 \) that appears due to renormgroup summation. This type of series generates the following analytic image in the Minkowski region:

\[
R_\nu[L] = d_0 A_\nu[L] + d_1 \sum_{n=0}^{\infty} \tilde{d}_{n+1} A_{n+1,\nu}[L].
\]  

(17)

Due to the recurrence relation

\[
A_{n+\nu}[L] = \frac{\Gamma(\nu)}{\Gamma(n + \nu)} \left( -\frac{d}{dL} \right)^n A_\nu[L]
\]  

(18)

we have

\[
R_\nu[L] = d_0 A_\nu[L] + d_1 \langle\langle X(t; 1 + \nu)\rangle\rangle_P,
\]  

(19)

where

\[
X(t; 1 + \nu) \equiv \sum_{n=0}^{\infty} \left( -\hat{x}_t \right)^n \frac{\Gamma(1 + \nu)}{\Gamma(n + 1 + \nu)} A_{1+\nu}[L] \quad \text{with} \quad \hat{x}_t = t \frac{d}{dL}.
\]  

(20)

We have a nice representation for the ratio of \( \Gamma \)-functions (see formula 5.2.7.20 in [28])

\[
\frac{\Gamma(n + 1)\Gamma(\nu + 1)}{\Gamma(n + \nu + 1)} = \int_0^1 \left( 1 - u^{1/\nu} \right)^n du,
\]  

(21)

which gives us a possibility to process further with our series (20)

\[
\sum_{n=0}^{\infty} \left( -\hat{x}_t \right)^n \frac{\Gamma(1 + \nu)}{\Gamma(n + 1 + \nu)} = \int_0^1 \exp \left( \hat{x}_t \cdot u^{1/\nu} - \hat{x}_t \right) du.
\]

Now we recall that \( \hat{x}_t \) is the operator \( \hat{x}_t = t \cdot d/dL \). That means that the operator \( e^{z \hat{x}_t} \) when acting upon a function \( A[L] \) just shifts its argument: \( e^{z \hat{x}_t} A[L] = A[L + tz] \), and we have

\[
X(t; 1 + \nu) = \int_0^1 A_{1+\nu}[L + t \left( u^{1/\nu} - 1 \right)] du.
\]  

(22)

Substituting Eq. (22) in Eq. (19), we obtain as a result:

\[
R_\nu[L] = d_0 A_\nu[L] + d_1 \int_0^1 du \langle\langle A_{1+\nu}[L - t \left( 1 - u^{1/\nu} \right)] \rangle\rangle_P
\]

\[
= d_0 A_\nu[L] + d_1 \langle\langle A_{1+\nu}[L - t] \rangle\rangle_P,
\]  

(23)
where

\[ P_\nu(t) \equiv \int_0^1 P \left( \frac{t}{1-u^{1/\nu}} \right) \frac{du}{1-u^{1/\nu}} = \int_0^1 P \left( \frac{t}{1-x} \right) \Phi_\nu(x) \frac{dx}{1-x}; \]  

(24a)

\[ \Phi_\nu(x) \equiv \nu x^{\nu-1} \to \nu \to 0^+ \delta(x). \]

(24b)

Note here that \( \lim_{\nu \to 0^+} P_\nu(t) = P(t). \)

A similar formula can be obtained to sum up perturbative series in the Euclidean domain:

\[ D_\nu[L] = d_0 A_\nu[L] + d_1 \langle A_{1+\nu}[L-t] \rangle P_\nu, \]

(25)

We see that the main difference between FAPT and APT with respect to a series summation technique is that in FAPT one needs to modify the initial generating function \( P(t) \) into a new one, \( P_\nu(t) \). It is interesting to note that for our model

\[ \tilde{d}_n = c^{n-1} \frac{\Gamma(n+\delta)}{\Gamma(1+\delta)}, \quad P(t;\delta) = \left( \frac{t}{c} \right)^{\delta} e^{-t/c} \]

(26)

the integration for \( P_\nu(t) \) in Eq. (24a) can be carried out exactly to produce rather a complicated expression containing the regularized confluent hypergeometric function. For integer values of \( \delta = m \geq 0 \) formula is simplified to

\[ P_\nu(t; m) = \Gamma(\nu + 1) G_{1,2}^{2,0} \left( \frac{t}{c} \right| \nu \right)_0, m), \]

(27)

where \( G_{1,2}^{2,0}(z|... \) \) is the Meijer G-function defined as

\[ G_{1,2}^{2,0} \left( \frac{a}{b_1, b_2} \right) = \frac{1}{2 \pi i} \oint_c \frac{\Gamma(b_1 + s)\Gamma(b_2 + s)}{\Gamma(a + s)} \frac{ds}{z^s}, \]

(28)

where the contour \( C \) of integration is set up so that the poles of \( \Gamma(b_1 + s) \) and \( \Gamma(b_2 + s) \) lie in the same region (internal or external) with respect to the contour \( C \).

3.2 Resummation of expansions in the global FAPT

To generalize formulae obtained in the previous subsection to the case of the global FAPT, we follow the lines of subsection 2.2 and take into account the new recurrence relation (18) for couplings in FAPT. We start with the analytic image of perturbative series in the Minkowski region

\[ R_{\nu}^{\text{glob}}[L] = d_0 A_{\nu}^{\text{glob}}[L] + d_1 \sum_{n=0}^{\infty} \tilde{d}_{n+1} A_{\nu+1+\nu}^{\text{glob}}[L] \]

(29)
and proceeding carefully obtain the result

\[
\mathcal{R}^{\text{glob}}_{\nu}[L] = d_0 \mathfrak{A}^{\text{glob}}_{\nu}[L] + d_1 \sum_{f=3}^{6} \theta \left( L_f \leq L < L_{f+1} \right) \langle \langle \mathfrak{A}_{1+\nu} \left[ L + \lambda_f - t \beta_f ; L \right] \rangle \rangle \nu
\]

\[
+ d_1 \sum_{f=3}^{6} \theta \left( L_f \leq L < L_{f+1} \right) \sum_{k=f+1}^{6} \langle \langle \Delta_k \mathfrak{A}_{1+\nu} \left[ l \right] \rangle \rangle \nu. \tag{30}
\]

Formulae for the Euclidean region appear to be more complicated and we refer the interested reader to the review [27], where this case is analyzed and all formulas are derived.

4 Application to the Higgs boson decay

In order to apply the developed techniques to the estimation of real processes, we need first to model the generating function \( P(t) \) of the corresponding perturbative coefficients. As we know from the discussion in section 2, in the case of the Higgs boson decay \( H^0 \rightarrow \bar{b}b \) the model (10) is of reasonable accuracy.

First, we recall what is the object of interest in the case of the Higgs boson decay to a \( \bar{b}b \) pair. Here we have for the decay width

\[
\Gamma(H \rightarrow \bar{b}b) = \frac{G_F}{4\sqrt{2\pi}} M_H \tilde{R}_S(M_H^2), \tag{31}
\]

where \( \tilde{R}_S(M_H^2) \) is just \( m_b^{2}(M_H^2)R_S(M_H^2) \). In the one-loop FAPT this generates the following non-power expansion:

\[
\tilde{R}^{\text{FAPT}}_S[L] = 3 \hat{m}^{2}_{(1)} \left[ \mathfrak{A}^{\text{glob}}_{\nu_0}[L] + d_1 \sum_{n \geq 1} \frac{\hat{d}_n}{\pi^n} \mathfrak{A}^{\text{glob}}_{n+\nu_0}[L] \right], \tag{32}
\]

where \( \hat{m}^{2}_{(1)} \) is the renormalization-group invariant of the one-loop \( m_b^{2}(\mu^2) \) evolution

\[
m_b^{2}(Q^2) = \hat{m}^{2}_{(1)} \alpha_s^{\nu_0}(Q^2) \tag{33}
\]

with \( \nu_0 = 2\gamma_0/b_0(5) = 1.04 \) and \( \gamma_0 \) is the quark-mass anomalous dimension.

We take the model (10) and apply Eq. (30) to estimate how good is FAPT in approximating the whole sum \( \tilde{R}^{\text{FAPT}}_S[L] \) in the range \( L \in [11, 13.8] \) which corresponds to the range \( M_H \in [60, 170] \) GeV with \( \Lambda_{QCD}^{N_f=3} = 172 \) MeV and

\[\text{With the same notation } L_4 = -\infty \text{ and } L_7 = +\infty \text{ as in Eqs. (12) and (13).}\]

\[\text{Appearance of denominators } \pi^n \text{ in association with the coefficients } \hat{d}_n \text{ is a consequence of } d_n \text{ normalization, see Eq. (9).}\]
\[ A_1^\text{glob}(m_Z^2) = 0.120. \] In this range we have \( L_5 < L < L_6 \), so that Eq. (30) transforms into
\[
\frac{\tilde{R}_S^{\text{FAPT}}[L]}{3 \tilde{m}^2_{(1)}} = A_1^\text{glob}[L] + \frac{d_1}{\pi} \langle \tilde{A}_{1+\nu_0}[L+\lambda_5-t/\pi\beta_5;5] + \Delta_6 \tilde{A}_{1+\nu_0} \left( \frac{t}{\pi} \right) \rangle_{P_{\nu_0}} \tag{34}
\]
with \( P_{\nu_0}(t) \) defined via Eqs. (10) and (24a) with the parameters \( c = 2.4, \beta = -0.52, \) and \( \nu_0 = 1.04. \)

Now we analyze the accuracy of the truncated FAPT expressions
\[
\tilde{R}_S^{\text{FAPT}}[L; N] = 3 \tilde{m}^2_{(1)} \left[ A_0^\text{glob}[L] + d_1 \sum_{n=1}^{N} \frac{d_n}{\pi} A_n^{\text{glob}}[L] \right] \tag{35}
\]
and compare them with the total sum \( \tilde{R}_S^{\text{FAPT}}[L] \) in Eq. (34) using relative errors
\[
\Delta_N[L] = 1 - \frac{\tilde{R}_S^{\text{FAPT}}[L; N]}{\tilde{R}_S^{\text{FAPT}}[L]} \tag{36}
\]
In Fig. 1, we show these errors for \( N = 2, N = 3, \) and \( N = 4 \) in the analyzed range of \( L \in [11, 13.8] \). We see that already \( \tilde{R}_S^{\text{FAPT}}[L; 2] \) gives accuracy of the order of 2.5%, whereas \( \tilde{R}_S^{\text{FAPT}}[L; 3] \) of the order of 1%. That means that there is no need to calculate further corrections: at the level of accuracy of 1% it is quite enough to take into account only coefficients up to \( d_3. \) This conclusion is stable with respect to the variation of parameters of the model \( P_{\nu_0}(t). \) These estimates demonstrate also a good convergence of the considered series in FAPT.

It is also interesting to estimate how large is the effective shift \( \Delta L \) which models the exact result (34) by the simple formula
\[
\tilde{R}_S^{\text{FAPT}}[L] \approx 3 \tilde{m}^2_{(1)} \left[ A_0^\text{glob}[L] + \frac{d_1}{\pi(1+\nu_0)} A_1^{\text{glob}}[L-\Delta L] \right]. \tag{37}
\]
Numerical estimation shows that $\Delta L \approx 2.16$. We can transform this result into a finite shift of $\Lambda_{QCD}^{N_f=3} = 0.172$ MeV:

$$\Lambda_{QCD}^{N_f=3} \rightarrow \Lambda_{\text{eff}}^{N_f=3} = 2.94 \Lambda_{QCD}^{N_f=3} \approx 505 \text{ MeV}. \quad (38)$$

If one estimate this shift by using only the $d_2$-term, then the result is 10\% smaller: $\Lambda_{\text{eff}}^{N_f=3} \approx 450$ MeV.

## 5 Conclusions

In this paper, using recurrence relations of the one-loop (F)APT, see Eqs. (6) and (18), we construct all needed rules to sum up nonpower series expansions in the fixed-$N_f$ and global (F)APT. We show how these results can be used to estimate the rate of convergency of nonpower series expansion. In the case of the series appearing in the QCD description of the Higgs boson decay into a $\bar{b}b$-pair, this convergence is estimated to be moderately fast: in order to have 1\%-level of accuracy, it is sufficient to sum up the first four terms ($d_0$, $d_1$, $d_2$, and $d_3$). In this case, such summation is equivalent in the considered range of $L \in [11,13.8]$ to the constant shift of $\Lambda_{QCD}^{N_f=3}$ from 172 MeV to 505 MeV.

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## A Single quark threshold in the Minkowski APT

We consider here only one heavy-quark threshold corresponding to the transition $N_f = 3 \rightarrow N_f = 4$. In this case, the global one-loop spectral density $\rho_n^{\text{glob}}(s) = \rho_n^{\text{glob}}[L(s)]$ (with $L(s) = \ln (s/\Lambda_3^2)$) is expressed in terms of fixed-flavor spectral densities with 3 and 4 flavors, $\tilde{\rho}_n[L;3]$ and $\tilde{\rho}_n[L+\lambda_4;4]$, see Eq. (1f):

$$\rho_n^{\text{glob}}[L] = \theta (L < L_4) \tilde{\rho}_n [L;3] + \theta (L_4 \leq L) \tilde{\rho}_n [L+\lambda_4;4], \quad (A1)$$
with $\lambda_4 \equiv \ln (\Lambda_3^2 / \Lambda_4^2)$ describing a shift of the logarithmic argument due to the change of the QCD scale parameter ($\Lambda_3 \rightarrow \Lambda_4$), $L_4 \equiv \ln (m_4^2 / \Lambda_4^2)$, and $\tilde{\rho}_n [L; N_f]$ defined in Eq. (1). In the one-loop approximation we have a very useful property of the spectral densities $\rho_n [L]$ and $\tilde{\rho}_n [L; N_f]$:

$$\rho_{n+1}[L] = \frac{1}{n} \left( -\frac{d}{dL} \right) \rho_n[L] = \frac{1}{\Gamma(n+1)} \left( -\frac{d}{dL} \right)^n \rho_1[L]; \quad (A2a)$$

$$\tilde{\rho}_{n+1}[L; N_f] = \frac{1}{n \beta_f} \left( -\frac{d}{dL} \right) \tilde{\rho}_n[L; N_f]$$

$$= \frac{1}{\Gamma(n+1) \beta_f^{n+1}} \left( -\frac{d}{dL} \right)^n \rho_1[L], \quad (A2b)$$

which is valid for $n \geq 0$ (except only Eq. (A2b)) and allows us immediately to rewrite Eq. (1d) in a more explicit form, where all $n$-dependencies appear explicitly:

$$\mathcal{A}_{n+1}^{\text{glob}}[L] = \frac{\theta (L < L_4)}{\Gamma(n+1)} \left[ \frac{1}{\beta_3^n} \left( -\frac{d}{dL} \right)^n \bar{\mathcal{A}}_1[L; 3] - \left( -\frac{d}{dL_4} \right)^n \bar{\mathcal{A}}_1[L_4; 3] \right] + \frac{1}{\beta_4^n} \left( -\frac{d}{dL_4} \right)^n \bar{\mathcal{A}}_1[L_4 + \lambda_4; 4]$$

$$+ \frac{\theta (L \geq L_4)}{\Gamma(n+1) \beta_4^n} \left( -\frac{d}{dL} \right)^n \bar{\mathcal{A}}_1[L + \lambda_4; 4]. \quad (A3)$$

Thus the general structure of the $n$-dependence, which we have in Eq. (A3), is $x^n / \Gamma(n+1)$ with $x = [1/\beta_f](-d/dL)$ and we know how to sum up this structure, see Eq. (7):

$$\mathcal{R}^{\text{glob}}[L] \equiv d_0 + \sum_{n=0}^{\infty} d_{n+1} \mathcal{A}_{n+1}^{\text{glob}}[L] \equiv d_0 + d_1 \sum_i \theta_i[L] \mathcal{G}_{f;i}[L + \lambda_f]; \quad (A4)$$

$$\mathcal{G}_{f;i}[L] \sim \sum_{n=0}^{\infty} \frac{\left( \langle t^n \rangle \right)_P}{\beta_f^n \Gamma(n+1)} \left( -\frac{d}{dL} \right)^n \bar{\mathcal{A}}_1[L; N_f] = \langle \bar{\mathcal{A}}_1[L - \frac{t}{\beta_f}; N_f] \rangle_P. \quad (A5)$$

Collecting all types of different $\theta$-structures in Eq. (A3) and inserting them in Eq. (A5) we arrive at the answer

$$\mathcal{R}^{\text{glob}}[L] = d_0 + d_1 \theta (L < L_4) \langle \bar{\mathcal{A}}_1[L - \frac{t}{\beta_3}; 3] + \Delta_f \bar{\mathcal{A}}_1[t] \rangle_P$$

$$+ d_1 \theta (L \geq L_4) \langle \bar{\mathcal{A}}_1[L + \lambda_4 - \frac{t}{\beta_4}; 4] \rangle_P, \quad (A6)$$

where we denoted (with $\lambda_3 = 0$)

$$\Delta_f \bar{\mathcal{A}}_n[t] \equiv \bar{\mathcal{A}}_n[L_f + \lambda_f - \frac{t}{\beta_f}; N_f] - \bar{\mathcal{A}}_n[L_f + \lambda_{f-1} - \frac{t}{\beta_{f-1}}; N_{f-1}]. \quad (A7)$$
B Numerical details

We use the following heavy-quark masses: $m_c = 1.2$ GeV, $m_b = 4.3$ GeV, and $m_t = 175$ GeV that generate the following set of QCD scales and logarithms

$$\lambda_f = \ln \left[ \frac{\Lambda_f^2}{\Lambda_3^2} \right]$$

in the one-loop approximation:

$$\Lambda_3 = 172 \text{ MeV}, \quad \Lambda_4 = 147 \text{ MeV}, \quad \Lambda_5 = 111 \text{ MeV}, \quad \Lambda_6 = 55 \text{ MeV} ;$$

(B1)

$$\lambda_4 = 0.311, \quad \lambda_5 = 0.898, \quad \lambda_6 = 2.30, \quad \lambda_Z = \ln \frac{m_Z^2}{\Lambda_3^2} = 12.55 .$$

(B2)

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