Non-Gaussian sampling effects on the CMB power spectrum estimation

Charlotte Cheung and Jo˜ ao Magueijo
Theoretical Physics, The Blackett Laboratory, Imperial College, Prince Consort Road, London, SW7 2BZ, U.K.

We introduce a non-Gaussian model of structure formation, and show how it produces surprising sampling effects on the estimation of the CMB power spectrum. Estimates of the average power spectrum based on small sky patches produce qualitatively different results from patch to patch, and vary also with the patch size. For instance a system of secondary Doppler peaks is observed in small enough patches, but at different positions in different patches. However, no secondary Doppler peaks are observed in large sky patches. These effects question usual methods of power spectrum estimation, and techniques like bootstrapping. The non-Gaussianity of this model fails to be detected by standard tests.

PACS Numbers : 98.80.Cq, 95.35+d

Considerable effort is currently being put into the estimation of the power spectrum \( C_\ell \) of CMB temperature fluctuations \([1,2]\). The \( C_\ell \) are expected to provide a wealth of information on theories of the early Universe, discriminating between paradigms \([3,11]\) and allowing highly accurate measurements of cosmological parameters \([3,4]\). The assumption that the temperature fluctuations constitute a Gaussian random field plays a central role in nearly all data-analysis methods (see however \([3]\)). This assumption has been checked in some data sets (eg. \([8]\)), although one can argue that the tests applied may not be conclusive \([10]\). One may certainly devise subtle non-Gaussian theories which pass the traditional Gaussianity tests.

Although non-Gaussianity cannot show up at the level of the average power spectrum \( C_\ell \), it may affect its estimation from a single sky. Larger \( C_\ell \) cosmic variance error bars, and correlations between all-sky \( C_\ell \) are familiar non-Gaussian effects known to be present in some models \([9]\). These effects would certainly affect comparison between theoretical and experimental \( C_\ell \). Correlations between \( C_\ell \) would allow more structure to be present in each observed \( C_\ell \) spectrum than in the average \( C_\ell \). Also one would have to revise the allocation of error bars to experimental \( C_\ell \) estimates, since these errorbars normally include cosmic variance.

More interesting still is the possibility that non-Gaussianity might affect in a similar way sampling statistics, that is, cosmic variance in sky patches and the coordination of estimates derived from different sky patches. It could for instance happen that estimates of the average power spectrum based on small sky patches produce different results from patch to patch. New results again could be obtained by changing the size of the patch.

To give a motivated example, take a defect theory known not to display secondary Doppler peaks \([12]\). It could happen that power spectrum estimates using small patches of the sky revealed a structure of secondary Doppler peaks, but that different patches saw peaks at different positions. The \( C_\ell \) average could then display no secondary Doppler peaks, and it is this average that would be sampled by a large enough patch of the sky. We will see that this non-Gaussian effect may be achieved with a coherent \([13]\) non-Gaussian source. For large sky patches such a source mimics the incoherent phenomena studied in \([12]\). There is therefore a loophole in the argument connecting absence of secondary peaks and incoherence, if one allows for non-Gaussianity.

As we shall see, it is possible to construct theories which display this effect while still passing the traditional Gaussian tests. This is somewhat disturbing, as for signals coming from such theories the usual methods for estimating the power spectrum would be grossly wrong, and in any case miss the theory’s rich structure of spectra. Techniques like bootstrapping would not be appropriate.

Here we present a simple model with all the properties mentioned above. We regard it as a valid model of structure formation deserving attention in its own right. However we reserve for a longer publication \([14]\) a presentation of the details of the model, and concentrate here on its unusual sampling effects. The idea is that structure is due to seeds in the early Universe, which act as a separate inhomogeneous component. All we know about them is that their formation and evolution satisfies energy conservation and that their power spectrum respects the causality constraints \([3]\). In this sense these seeds are like the seeds in the isocurvature models of \([16]\). They represent an ad-hoc construction led merely by the guiding principles of causality and energy conservation.

However we postulate some dynamics. We require these seeds to be explosions, as in the mimic models discussed in \([17,18]\). They differ from mimic explosions only in that they do not explode at the Big Bang, but at a given blast time \( \eta_b \). When they explode they start seeding structure. Before the explosion they merely lurk about imparting no gravitational action on the surrounding matter. As in \([17,18]\) we require these seeds to be scale-invariant.

A final but central ingredient of the model is that the blast time \( \eta_b \) is random, with a probability distribution \( P(\eta_b) \), and is uncorrelated beyond a given distance \( \xi \). This is the only qualitative novelty of our model, but this feature will be responsible for all the weird non-Gaussian behaviour to be derived. In summary structure is seeded by bombs which have always been around, but which re-
main dormant for part of their lives, then explode, not at a pre-set time, but whenever it pleases them. Do not confuse this scenario with the explosive scenarios proposed in [24]. Our explosions satisfy energy conservation separately, and only interact with the rest of the Universe gravitationally.

More concretely we define our bombs by means of their stresses. If $\Theta_{\mu\nu}$ is their stress energy tensor, then we have $\Theta_{ij} = p^s \delta_{ij} + \langle \partial_i \partial_j - \frac{1}{3} \delta_{ij} \partial^2 \rangle \Theta^s$. The scalars $p^s$ and $\Theta^s$ may then be Fourier transformed. We choose $\Theta^s = 0$ and $p^s = 0$ for $\eta < \eta_b$, but

$$p^s = \frac{1}{\eta^{1/2}} \frac{\sin Ak(\eta - \eta_b)}{Ak(\eta - \eta_b)}$$

for $\eta > \eta_b$. For definiteness we take $A = 1$. Switching on stresses rather than energy ensures that we do not violate energy conservation. It is also reminiscent of the switching on of topological defects during a phase transition [10]. The other components of the stress energy tensor are determined by the energy conservation equations.

The above specifies each individual explosion. If we set up a Poisson process of such explosions (with fixed $\eta_b$) then the resulting field power spectrum equals the density of explosions times the square of the Fourier transform of each individual explosion [14]. If furthermore the density of explosions is very large we have a Gaussian random field within each correlated region. We assume that the source is coherent [13] within each correlated region. The calculation of the $C_\ell$ conditional to a given $\eta_b$ can then be made in the usual way [17, 18].

We finally choose a distribution of blast times lifted from radioactive decay:

$$P(\eta_b) = \frac{\exp(-\eta_b/\tau_b)}{\tau_b}$$

where $\tau_b$ is the average blast time. We assume that the spatial correlation between the blast times $\eta_b$ dies off at a scale $\xi$ which when projected on the last scattering surface obeys $\xi \ll 4\pi$ (in radians).

The CMB anisotropies produced by such a theory exhibit a rich structure of power spectra. For patches of sky with a size $L$ smaller than $\xi$ the system is best described by noting that the temperature probability distribution conditional to a given $\eta_b$ is Gaussian. The observed power spectrum $\bar{C}_\ell$ conditional to $\eta_b$ is the usual $\chi^2$ distribution. We define the conditional average power spectrum as:

$$C_\ell(\eta_b) = \int d\hat{C}_\ell \bar{C}_\ell P(\hat{C}_\ell|\eta_b)$$

The sample variance is the usual

$$\sigma^2(\hat{C}_\ell) = \frac{2}{N_\ell} C_\ell^2(\eta_b)$$

where $N_\ell = (2\ell + 1)f_{\text{sky}}$ is the total number of modes contributing to the estimate, with $f_{\text{sky}}$ the fraction of sky covered by the patch. Conditional to a given $\eta_b$ there are no correlations between the $\bar{C}_\ell$ (other than the ones mentioned in [21], which we assume have been bypassed). This is the simplest description for patches with $L < \xi$: “a spectrum of spectra $C_\ell(\eta_b)$, and the distribution of the parameter $\eta_b$. In other words, the statistical ensemble is best described in terms of a set of Gaussian subensembles, with a power spectrum indexed by a variable $\eta_b$. From the point of view of the overall ensemble $\eta_b$ is itself a random variable, with a distribution $P(\eta_b)$. In Fig. [1] we show the various power spectra $C_\ell(\eta_b)$ which any estimate derived from small patches would produce.

Applying the traditional description, in terms of an average $C_\ell$ plus a sample variance error bar $\sigma^2(C_\ell)$, is a bad idea for patches with $L < \xi$. Although the temperature distribution conditional to $\eta_b$ is Gaussian, the marginal distribution is not. Non-Gaussian effects on the power spectrum therefore emerge. The marginal average power spectrum is

$$C_\ell = \langle \hat{C}_\ell \rangle = \int d\hat{C}_\ell d\eta_b \bar{C}_\ell P(\hat{C}_\ell, \eta_b)$$

Using $P(\hat{C}_\ell, \eta_b) = P(\hat{C}_\ell|\eta_b)P(\eta_b)$ leads to

$$C_\ell = \int d\eta_b P(\eta_b) C_\ell(\eta_b)$$

The sample variance is now

$$\sigma^2(\hat{C}_\ell) = \left(1 + \frac{2}{N_\ell}\right) \int d\eta_b C_\ell^2(\eta_b) P(\eta_b) - C_\ell^2$$

always larger than the Gaussian sample variance $2C_\ell^2/N_\ell$ by virtue of the Schwarz inequality. If the distribution $P(\eta_b)$ is reasonably peaked and $C_\ell(\eta_b)C_\ell^2(\eta_b) \ll C_\ell^2(\eta_b)$ we get the “error propagation formula”: ...
at the fact that the average power spectrum
Clearly the larger sample variance error bars merely hint
between
Larger sample variance error bars and correlations be-
ter off without them, which in this case means abandon-
comparison between theory and experiment. One is bet-
et to the sky with
the same as above with
C
it, not the
C
of the
the sky with
same as above with
N

There are also correlations between ˆ
C
ℓ

Even if
N
ℓ
= ∞, there would be a residual, purely para-
metric, cosmic variance:

There are also correlations between ˆ
C
ℓ
:

Larger sample variance error bars and correlations be-
tween
C
(“cosmic covariance” [11]) no doubt complicate
comparison between theory and experiment. One is bet-
ter off without them, which in this case means abandon-
ing the
C
ℓ
description, and taking up instead
C
ℓ
(ηb) and
P
(ηb).

In Fig. 2 we plotted the average power spectrum
C
ℓ
. Clearly the larger sample variance error bars merely hint
at the fact that the average power spectrum
C
ℓ
fails to
describe what would be observed in each small patch of
the sky with
L < ξ.
What is seen in any patch is one
of the
C
ℓ
(ηb) spectra with Gaussian fluctuations about
it, not the
C
ℓ
description, and taking up instead
C
ℓ
(ηb) and
P
(ηb).

If we choose a patch with
L ≫ ξ then the situation is
different. Power spectrum estimates based on the whole
patch amount to an average of the estimates that would
be supplied by each correlated region with size
ξ. Let us
index these regions by
i
and let there be
N = (L/ξ)2
of them, so that
C
ℓ
i
= ∑
C
ℓ
i/N. The statistics of
C
ℓ
i
are the
same as above with
N
ℓ
= N
N
ℓ
, and so we have

and

We see that the excess sample variance, and the corre-
lations between ˆ
C
ℓ
get smaller as the number of uncor-
related regions in the patch
N increases. As
N → ∞
the residual parametric cosmic variance mentioned above
also disappears. Essentially what happened is that for
large patches the average power spectrum
C
ℓ
has become
representative of the observed
C
ℓ.
Observed
C
ℓ
for large
patches do scatter around a curve as in Fig. 3.
We illustrate this phenomenon further in Fig. 3. We
chose
ξ = 5°, generated maps [14], and found the measured
power spectrum in a square patch 4° across, and in
a patch 20° across. A maximum likelihood estimate of
the average power spectrum based on small patches would
reveal a system of secondary peaks, at different positions in
different patches. The same exercise in a large patch would
suggest absence of secondary peaks.

We chose
ξ = 5°, generated maps [14], and found the measured
power spectrum in a square patch 4° across, and in
a patch 20°. Secondary Doppler peaks are observed
in each realization for small sky patches. However the
peak’s positions vary from patch to patch, or from real-
ization to realization. Hence the average over realiz-
ations does not exhibit secondary Doppler peaks. Similarly es-
timates based on large patches do not show secondary
Doppler peaks. This effect was found by Turok in cosmic
string simulations [22].
The theory we have just proposed is somewhat distressing. The non-Gaussianity of its maps fails to be detected by the traditional tests, such as skewness and kurtosis, density of peaks above a given height, genus number and the 3-point correlation function. We show how this is the case with the skewness \( \gamma_3 \) and the kurtosis \( \gamma_4 \). In Fig. 4 we plot histograms of \( \gamma_3 \) and \( \gamma_4 \) as estimated from 25600 pixels for 10000 realizations. Plots with dashes correspond to our theory, with lines to a Gaussian theory with the same average power spectrum. As far as skewness is concerned the two theories are essentially the same. Our theory has a slightly positive kurtosis, but not beyond what could be explained from cosmic variance for a Gaussian theory. Even this latter effect can be eliminated by changing the distribution \( P(\eta_b) \) so as to ensure that the most likely power spectra \( C_\ell(\eta_b) \) have the same \( \int d\ell^2 C_\ell \). This amounts to reducing the probability of patches with \( \eta_b \approx 0 \).

Despite its apparent Gaussianity, the theory proposed shows a sampling effect on the power spectrum that invalidates traditional estimation methods. In particular bootstrapping becomes very misleading. One should rather aim at independent power spectrum estimates for all possible combinations of patch location and size. The outcome would be \( C_\ell(\eta_b) \), \( P(\eta_b) \), and \( \xi \). A standard estimation of the power spectrum would be highly misrepresentative of the actual structure of spectra of the theory.

There is an irritating inconsistency between power spectrum estimates produced by different experiments, using different patches of the sky, or patches with different sizes [23]. Although these may be due to imperfect foreground subtraction, or calibration errors, it is curious to point out that such inconsistencies might not be inconsistencies at all. They may be telling us about the unusual power spectrum structure non-Gaussianity may induce.

Finally our theory shows that maybe we should be more careful when ruling out defects on the grounds of unflattering average power spectra \( C_\ell \) [24-26]. We know very little about non-Gaussianity in these models, but nothing tells us that comparison between theoretical and experimental \( C_\ell \) should be made in the same way as for Gaussian theories.

ACKNOWLEDGEMENTS: We would like to thank Kim Baskerville and Pedro Ferreira for many helpful comments. We have used a modified version of the Boltzmann code of Seljak and Zaldarriaga [27] in our calculations. C.C was supported by PPARC and J.M. by the Royal Society.

---

[1] K. M. Gorski, Proceedings of the XXXlst Recontres de Moriond.
[2] L. Knox, J. Bond, A. Jaffe, astro-ph/9702110.
[3] M. Tegmark, A. Hamilton, astro-ph/9702019.
[4] P. Steinhardt, Cosmology at the Crossroads to appear in the Proceedings of the Snowmass Workshop on Particle Astrophysics and Cosmology, E. Kolb and R.Peccei, eds. (1995) astro-ph/9502024.
[5] T.W.B. Kibble, J. Phys., A9 1387-1398 (1976), A. Vilenkin and P. Shellard, Cosmic Strings and other Topological Defects. Cambridge University Press, Cambridge (1994).
[6] J. Magueijo and M. Hobson, to be published in Phys.Rev.D, astro-ph/9610107.
[7] J. Bond, G. Efstathiou, and M. Tegmark, astro-ph/9702104.
[8] M.Hobson, MEM.
[9] A. Kogut et al, astro-ph/9601062.
[10] P. Ferreira and J. Magueijo, Phys.Rev. D 55 3358 (1997).
[11] J. Magueijo, Phys. Rev. D 52 4361 (1995).
[12] A. Albrecht et al, Phys.Rev.Lett 76 2617 (1996).
[13] J.Magueijo et al,Phys.Rev D 54 3727 (1996).
[14] C. Cheung and J. Magueijo, in preparation.
[15] J.Traschen, Phys. Rev. D, 31 283-289 (1985); J.Traschen, Phys. Rev. D, 29 1563-1574 (1984).
[16] P. J. E. Peebles, Astrophys. J. Lett. 315 L-73 (1987);
[17] N. Turok, *Phys. Rev. Lett.* **77**, 4138 (1996).
[18] W. Hu, D. Spergel, and M. White, astro-ph/9605193.
[19] J. Magueijo, *Phys. Rev. D* **46**, 1368 (1992).
[20] J. P. Ostriker and L. L. Cowie, *Ap. J* **243**, L127 (1981).
[21] M. Hobson and J. Magueijo, *MNRAS* **283**, 1133 (1996).
[22] N. Turok, astro-ph/9606087.
[23] E. Gaztanaga, P. Fosalba, E. Elizalde, astro-ph/9705116.
[24] U. Seljak, U. Pen, N. Turok, astro-ph/9704165.
[25] B. Allen et al, astro-ph/9704160.
[26] A. Albrecht, R. Battye, J. Robinson, astro-ph/9707129.
[27] U. Seljak and M. Zaldarriaga, astro-ph/9603033.