DYNAMICS OF A DIFFUSIVE LESLIE-GOWER
PREDATOR-PREY MODEL IN SPATIALLY HETEROGENEOUS
ENVIRONMENT

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(Communicated by Yuan Lou)

Abstract. In this paper, we are concerned with a diffusive Leslie-Gower predator-prey model in heterogeneous environment. The global existence and boundedness of solutions are shown. By analyzing the sign of the principal eigenvalue corresponding to each semi-trivial solution, we obtain the linear stability and global stability of semi-trivial solutions. The existence of positive steady state solution bifurcating from semi-trivial solutions is obtained by using local bifurcation theory. The stability analysis of the positive steady state solution is investigated in detail. In addition, we explore the asymptotic profiles of the steady state solution for small and large diffusion rates.

1. Introduction. In ecological systems, the interaction of predator population and prey population has abundant dynamical feature. Many biological models have been put forward since the pioneering works of Lotka [25] and Volterra [34]. Moreover, more realistic models are proposed according to laboratory experiments and observations. For example, the Leslie-Gower predator-prey model was proposed in [20, 21, 29], which takes the form

\[
\begin{aligned}
\frac{du}{dt} &= u(a - bu) - muv, \\
\frac{dv}{dt} &= v\left(d - \frac{c}{u}\right),
\end{aligned}
\]

where \(u(t)\) and \(v(t)\) represent the densities of prey and predators at time \(t\), respectively; \(u/c\) represents the environmental carrying capacity of predators. Thus, the amount of predators is only affected by its favorite food \(u\). When the quantity of the favorite food \(u\) is insufficient, the predator has to switch over to other population but its growth will be limited. To take into account the effect of this factor,
Aziz-Alaoui and Okiya [3] have proposed the following Leslie-Gower predator-prey system with saturated functional responses

\[
\begin{align*}
\frac{du}{dt} &= u \left( a_1 - b_1 u - \frac{c_1 v}{u + k_1} \right), \\
\frac{dv}{dt} &= v \left( a_2 - \frac{c_2 v}{u + k_2} \right),
\end{align*}
\]  

where \( a_1 \) and \( a_2 \) stand for the growth rates per capita of prey \( u \) and predator \( v \), respectively; \( b_1 \) measures the strength of intraspecific competition among individuals of species \( u \), which is related to the carrying capacity of the prey; \( c_1 \) and \( c_2 \) represent the maximum value of the per capita reduction rate of \( u \) due to \( v \) and the maximum growth per capita of \( v \) due to predation of \( u \), respectively; \( k_1 \) and \( k_2 \) measure the extent to which environment provides protection to prey \( u \) and predator \( v \), respectively.

In view of the inhibitory effect of high concentrations on microbial growth, Andrews [2] suggested a Monod-Haldane response function \( p(u) = \frac{mu}{k_1 + k_2 u + u^2} \). In experiments on the uptake of phenol by pure culture of Pseudomonas later, Sokol and Howell [33] proposed a simplified Monod-Haldane type function of the form \( p(u) = \frac{mu}{k_1 + u^2} \). In the interactions between predator and prey, we can obtain the following Leslie-Gower predator-prey model by using Monod-Haldane type functional response function

\[
\begin{align*}
\frac{du}{dt} &= u \left( a_1 - b_1 u - \frac{c_1 v}{u + k_1} \right), \\
\frac{dv}{dt} &= v \left( a_2 - \frac{c_2 v}{u + k_2} \right).
\end{align*}
\]  

Considering the effects of dispersal and environmental heterogeneity on the dynamic behavior of populations, our concern is the following Leslie-Gower prey-predator system

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + u \left( a(x) - u - \frac{v}{1 + u^2} \right) \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial v}{\partial t} &= d_2 \Delta v + v \left( b(x) - \frac{v}{1 + u^2} \right) \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, \infty), \\
u(x, 0) &= \varphi(x) \geq 0, \quad v(x, 0) = \psi(x) \geq 0, \quad \text{in } \Omega,
\end{align*}
\]  

where \( \Delta \) denote the Laplacian operator on \( \mathbb{R}^N \), \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) \((N \geq 1)\) with smooth boundary \( \partial\Omega \), \( n \) is the outward unit normal vector on \( \partial\Omega \). The homogeneous Neumann boundary condition means that the two species have zero flux across the boundary \( \partial\Omega \). Throughout this paper, we always assume that \( a(x) \) and \( b(x) \in C^r(\Omega) \) with \( r \in (0, 1) \) are nonconstant, and that \( a(x) > 0 \) and \( b(x) > 0 \) in \( \Omega \). For system (4) with \( \Omega = (0, \pi) \), \( d_1 = 1 \), and constant-valued functions \( a(x) \) and \( b(x) \), Li et al. [24] investigated the Hopf bifurcation and steady-state bifurcation by taking \( d_2 \) as the bifurcation parameter and described both the global structure of the steady-state bifurcation from simple eigenvalues and the local structure of the steady-state bifurcation from double eigenvalues by using space decomposition and the implicit function theorem.
Many researchers have paid more attention to the effect of interspecies interaction on the dynamic behaviors of reaction-diffusion population models in which it is assumed that the environment is spatially homogeneous, that is, all the coefficients are constant [12, 13, 14, 15, 23, 28, 30, 37, 38]. It has been observed in many scientific experiments that spatial heterogeneity has a profound effect on ecosystems. Therefore, it is very important and significant to study the heterogeneous effects of the environments on the dynamics of biological population models. In recent two decades, more and more researchers have paid attention to the effects of the spatial heterogeneity on the dynamics of biological population models and have put forward many interesting mathematical problems; See, for example, [7, 8, 9, 10, 16, 17, 18, 19, 27, 35, 36]. In particular, some predator-prey models with degeneracy have been investigated in [7, 10], in which some coefficients of the prey population vanish to zero in some areas of life. Du and Shi [9] proposed a predator-prey model with a protected area and assumed that only the bait population is free to enter or leave the protected area. In [16, 17, 18, 19], the researchers considered the following diffusive Lotka-Volterra competition system with spatially heterogeneous resources

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= d_1 \Delta u + u(m_1(x) - u - cv), & (x, t) \in \Omega \times \mathbb{R}^+,
\frac{\partial v}{\partial t} &= d_2 \Delta v + v(m_2(x) - bu - v), & (x, t) \in \Omega \times \mathbb{R}^+,
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & (x, t) \in \partial \Omega \times \mathbb{R}^+,
u(x, 0) = \varphi(x) > 0, \quad v(x, 0) = \psi(x) \geq 0, & x \in \Omega.
\end{aligned}
\]

In a weak competition case, Lam and Ni [19] showed that system (5) with \(m_1(x) = m_2(x)\) has an unique coexistence steady state solution which is globally asymptotically stable. In the case where \(b = c = 1\) (i.e., the two species have the same competition) and \(\int_{\Omega} m_1(x) dx = \int_{\Omega} m_2(x) dx\) (i.e., the total resources are the same), He and Ni [16] analyzed the effect of the diffusion coefficients of system (5) on the stability of the steady state solution, and obtained the global asymptotic stability when the diffusion coefficients \(d_1\) and \(d_2\) are sufficiently large or sufficiently small, respectively. In particular, in the case where \(m_2(x)\) is a constant, the species \(u\) with heterogeneous resource allocation has more obvious competitive advantage. He and Ni [17] analyzed the effect of diffusion coefficients on the stability of semi-trivial steady-states and coexistence steady state solutions of system (5). He and Ni [18] discussed the existence and uniqueness of coexistence steady state solutions and the global asymptotic stability of the semi-trivial steady state solution of system (5). Recently, Lou and Wang [27], Wang and Zhang [35] discussed the following reaction-diffusion predator-prey system with heterogeneous resource allocation

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= d_1 \Delta u + u\left(m(x) - u - \frac{cv}{1 + u}\right), & (x, t) \in \Omega \times \mathbb{R}^+,
\frac{\partial v}{\partial t} &= d_2 \Delta v + v\left(lu \frac{1}{1 + u} - bv\right), & (x, t) \in \Omega \times \mathbb{R}^+,
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & (x, t) \in \partial \Omega \times \mathbb{R}^+,
u(x, 0) = \varphi(x) > 0, \quad v(x, 0) = \psi(x) \geq 0, & x \in \Omega.
\end{aligned}
\]

More precisely, Lou and Wang [27] focused on the effect of the variation of diffusion coefficient on the stability of the semi-trivial steady-state solution of system (6),
while Wang and Zhang \[35\] investigated the local bifurcation of semi-trivial steady state solutions of system (6).

Note that the functional response function in system (5) is linear, and that only the prey population in system (6) has spatial resource heterogeneity. Our system (4) has not only a non-linear functional response function, but also has the spatial resource heterogeneity for both species. Because the functional response function is non-linear, there is some difficulty in the analysis of the existence and asymptotic behavior of positive steady-state solutions. Moreover, the structure of positive steady-state solutions will change essentially due to the heterogeneity of spatial resources. Therefore, it is very meaningful to investigate the influence of spatial resource heterogeneity on the dynamic behavior. Motivated by \[16, 19, 35\], in this paper we shall pay more attention to the existence, stability, and asymptotic behavior of positive steady-state solutions of system (4).

The organization of the remaining part of the paper is as follows. In section 2 some concepts are introduced for later use. In section 3 we obtain the global existence and boundedness of solutions to (4) by applying comparison methods. In section 4 we derive the stability of semi trivial solutions by analyzing the sign of the principal eigenvalue. Section 5 is devoted to the existence and stability of positive steady state solution which bifurcates from semi-trivial steady state of system (4). Finally, in section 6 we first employ the fixed point index theory in a cone to investigate the existence of positive steady-state solutions of system (4) and then investigate the limiting behaviours of positive steady state solution as the dispersal rates tend to 0 or \(\infty\).

Throughout the paper, we denote
\[
\tilde{f} = \min_{\Omega} f(x), \quad \hat{f} = \max_{\Omega} f(x) \quad \text{and} \quad \bar{f} = \frac{1}{|\Omega|} \int_{\Omega} f(x) \, dx
\]
for any given continuous function \(f\) on \(\bar{\Omega}\).

2. Preliminaries. In this section, we will present several lemmas which shall be used in the subsequence analysis. First, we consider the following steady-state problem for the logistic equation with linear diffusion
\[
\begin{cases}
\Delta u + u(m(x) - u) = 0 & x \in \Omega, \\
\frac{\partial u}{\partial n} = 0 & x \in \partial \Omega,
\end{cases}
\]
where \(m(x) \in C^r(\bar{\Omega})\) with \(r \in (0,1)\) is nonconstant such that \(m > 0\) in a set of positive measure in \(\Omega\). It is well known that (see, for example, \[4\]) system (7) has a unique positive solution, denoted by \(\theta_{d,m}\), if and only if \(\mu_1(d,m) < 0\), and that \(\theta_{d,m} \in C^2(\Omega)\), where \(\mu_1(d,m) < 0\) is given later (see Definition 2.2). Dividing both sides of the first equation of (7) by \(\theta_{d,m}\) and integrating over \(\Omega\), we obtain that
\[
\int_{\Omega} (m(x) - \theta_{d,m}) \, dx = -d \int_{\Omega} \frac{\nabla \theta_{d,m}}{\theta_{d,m}^2} \, dx < 0,
\]
which indicates that \(\int_{\Omega} m(x) \, dx < \int_{\Omega} \theta_{d,m} \, dx\).

Next, we will present several results on properties of \(\theta_{d,m}\), the unique positive solution of (7), which shall be used in the subsequent analysis.

**Lemma 2.1** (\[16\]). Suppose that \(m(x) \in C^r(\bar{\Omega})\) is nonconstant and \(\int_{\Omega} m(x) \, dx \geq 0\). Then we have the following results.
(i): $d \to \theta_{d,m}$ is continuous from $\mathbb{R}^+\to W^{2,p}(\Omega) \cap C^2(\bar{\Omega})$. Moreover, 
\[
\lim_{d \to 0} \theta_{d,m} = \max\{m(x), 0\} \quad \text{and} \quad \lim_{d \to \infty} \theta_{d,m} = \bar{m} \quad \text{uniformly on } \Omega.
\]

(ii): For any $d > 0$, we have $\hat{\theta}_{d,m} < \hat{m}$ and $\hat{\theta}_{d,m} > \hat{m}$. In particular, 
\[
\|\theta_{d,m}\|_{L^\infty(\Omega)} < \|m\|_{L^\infty(\Omega)}.
\]

To characterize the principal eigenvalue of system (7), we need to introduce the following eigenvalue problem with indefinite weight
\[
\begin{cases}
\Delta \varphi + \lambda h(x)\varphi = 0 & x \in \Omega, \\
\frac{\partial \varphi}{\partial n} = 0 & x \in \partial \Omega,
\end{cases}
\] (8)

where $h \neq$ constant, could change sign in $\Omega$. We say that $\lambda_1(h)$ is a principal eigenvalue if (8) with $\lambda = \lambda_1(h)$ has a positive solution (Notice that 0 is always a principal eigenvalue). Regarding the property of the principal eigenvalue $\lambda_1(h)$ of (8), detailed information can be found in [16]. Next, we collect some facts concerning the eigenvalue problem (8).

**Definition 2.2.** Given a positive constant $d$ and a function $h \in L^\infty(\Omega)$, denote by $\mu_k(d,h)$ the $k$-th eigenvalue (counting multiplicities) of the following eigenvalue problem:
\[
\begin{cases}
d\Delta \psi + h(x)\psi + \mu \psi = 0 & x \in \Omega, \\
\frac{\partial \psi}{\partial n} = 0 & x \in \partial \Omega,
\end{cases}
\] (9)

In particular, we call $\mu_1(d,h)$ the principal eigenvalue of (8), which has the following variational characterization
\[
\mu_1(d,h) = \inf_{\psi \in H_1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (d|\nabla \psi|^2 - h(x)\psi^2)dx}{\int_{\Omega} \psi^2dx}. \tag{10}
\]

In the following, we give some important properties of $\mu_1(d,h)$ in connection with $\lambda_1(h)$ and refer to [4] for a proof.

**Proposition 1 ([4]).** The principal eigenvalue $\mu_1(d,h)$ of (9) depends smoothly on $d > 0$ and continuously on $h \in L^\infty(\Omega)$. Moreover, it has the following properties:

(i): If $\int_{\Omega} h \geq 0$ and $h \neq 0$, then $\mu_1(d,h) < 0$ for all $d > 0$.

(ii): If $\int_{\Omega} h < 0$ and $h$ changes sign in $\Omega$, then
\[
\begin{cases}
\mu_1(d,h) < 0, & d < \frac{1}{\lambda_1(h)}; \\
\mu_1(d,h) = 0, & d = \frac{1}{\lambda_1(h)}; \\
\mu_1(d,h) > 0, & d > \frac{1}{\lambda_1(h)}.
\end{cases}
\]

(iii): $\mu_1(d,h)$ is strictly increasing and concave in $d > 0$. Furthermore,
\[
\lim_{d \to 0} \mu_1(d,h) = \min_{\Omega}(-h), \quad \lim_{d \to \infty} \mu_1(d,h) = -\bar{h},
\]
where $\bar{h}$ is the average of $h$.

(iv): If $h_1(x) \leq h_2(x)$ in $\Omega$, then $\mu_1(d,h_1) \geq \mu_1(d,h_2)$ with equality holds if and only if $h_1 = h_2$ a.e. in $\Omega$. Assume in addition to that $h$ is nonconstant, then $\mu_1(d_1,h) < \mu_1(d_2,h)$ if $d_1 < d_2$. In particular, $\mu_1(d,h) > 0$ if $h \leq (\neq) 0$. 
3. Global existence and boundedness. In the section, we investigate the global existence and boundedness of the solution to (4). In what follows, we will state the relevant results.

**Theorem 3.1.** If \( \varphi \not\equiv 0 \) and \( \psi \not\equiv 0 \), but \( \varphi(x) \geq 0 \) and \( \psi(x) \geq 0 \) for all \( x \in \overline{\Omega} \), then (4) has a unique solution \((u(x, t), v(x, t))\), which is bounded and \((u(x, t), v(x, t)) \in \mathbb{R}_+^2 \) for \( t > 0 \) and \( x \in \Omega \).

**Proof.** First, the local existence of the solution to (4) follows from standard theory. Denote by \( T \) the maximal existence time of solution. Since \( u(x, t) \) satisfies

\[
\begin{cases}
\frac{\partial u}{\partial t} \leq d_1 \Delta u + u(a(x) - u) & \text{in } \Omega \times (0, \infty), \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \times (0, \infty), \\
u(x, 0) = \varphi(x) \geq 0 & \text{in } \Omega.
\end{cases}
\]

Then from comparison principle of the parabolic equations, it is easy to verify that

\[ u(x, t) \leq \theta_{d_1, a} + \max_{\Omega} \varphi(x) \quad \text{for all } (x, t) \in \overline{\Omega} \times [0, T_{\text{max}}). \]

Moreover, it is easy to see that \( v(x, t) \) satisfies

\[
\begin{cases}
\frac{\partial v}{\partial t} \leq d_2 \Delta v + v \left( b(x) - \frac{v}{1 + K^2} \right) & \text{in } \Omega \times (0, \infty), \\
\frac{\partial v}{\partial n} = 0 & \text{on } \partial \Omega \times (0, \infty), \\
v(x, 0) = \psi(x) \geq 0 & \text{in } \Omega,
\end{cases}
\]

where \( K = \hat{\theta}_{d_2, b} + \hat{\varphi} \). Again using the comparison principle of the parabolic equations, we obtain that

\[ v(x, t) \leq (1 + K^2)\theta_{d_2, b} + \max_{\Omega} \psi(x) \quad \text{for all } (x, t) \in \overline{\Omega} \times [0, T_{\text{max}}). \]

Hence, it follows from [1] that \( T_{\text{max}} = \infty \). Moreover, we conclude that \( u(x, t) > 0 \) and \( v(x, t) > 0 \) due to the maximum principle and Hopf boundary lemma for parabolic equations. Therefore, the solution of system (4) exists globally and is bounded. The proof is completed. \( \square \)

4. Stability analysis of semi-trivial solutions. It is easy to see that system (4) has a trivial steady state \((0, 0)\) and two semi-trivial steady states \((\theta_{d_1, a}, 0)\) and \((0, \theta_{d_2, b})\). Here, \( \theta_{d_1, a} \) (respectively, \( \theta_{d_2, b} \)) is the unique positive solution of (7) with \( d = d_1 \) and \( m(x) = a(x) \) (respectively, \( d = d_2 \) and \( m(x) = b(x) \)). If a steady state \((u, v)\) of (4) satisfying \( u \geq 0 \) and \( v \geq 0 \) is neither a trivial nor a semi-trivial steady state, then by the maximum principle, we have \( u > 0 \) and \( v > 0 \) on \( \Omega \). In this case, we call \((u, v)\) a co-existing steady state. The purpose of this section is to analyze the local stability of semi-trivial steady states of (4) and to obtain the following results.

**Theorem 4.1.**

(i): The semi-trivial steady state \((\theta_{d_1, a}, 0)\) of system (4) is unstable.

(ii): The semi-trivial steady state \((0, \theta_{d_2, b})\) of system (4) is locally asymptotically stable if \( \mu_1(d_1, a(x) - \theta_{d_2, b}) > 0 \), while it is unstable if \( \mu_1(d_1, a(x) - \theta_{d_2, b}) < 0 \). More precisely,
(a): if $\hat{\theta}_{d_2,b} > \hat{a}$, then the steady state $(0, \theta_{d_2,b})$ is locally asymptotically stable.
(b): if $\hat{\theta}_{d_2,b} < \hat{a}$, then the steady state $(0, \theta_{d_2,b})$ is unstable.
(c): Suppose that $\hat{\theta}_{d_2,b} > \hat{a}$ and $a(x) - \theta_{d_2,b}$ changes sign in $\Omega$, then there exists $d_1^* > d_2^*$ such that for every $d_1 < d_1^*$, the steady state $(0, \theta_{d_2,b})$ is unstable, while for every $d_1 > d_1^*$, the steady state $(0, \theta_{d_2,b})$ is locally asymptotically stable.

**Proof.** We first prove conclusion (i). From the linearization principle, the stability of $(\theta_{d_1,a}, 0)$ can be determined by studying the following eigenvalue problem

$$
\begin{align*}
&d_1 \Delta \Phi + (a(x) - 2\theta_{d_1,a}) \Phi - \frac{\theta_{d_1,a}}{1 + \theta_{d_1,a}^2} \Psi + \lambda \Phi = 0, & x \in \Omega, \\
&d_2 \Delta \Psi + b(x) \Psi + \lambda \Psi = 0, & x \in \Omega, \\
&\frac{\partial \Phi(x)}{\partial n} = \frac{\partial \Psi(x)}{\partial n} = 0, & x \in \partial \Omega.
\end{align*}
\tag{11}
$$

Let $\lambda$ be an eigenvalue of (11) with an associated eigenfunction $(\Phi, \Psi)$. If $\Psi \neq 0$, then $\lambda$ belongs to the spectrum of the self-adjoint operator $-d_1 \Delta - b(x)$ (with zero Neumann boundary condition). Therefore, $\lambda$ must be real and satisfy $\lambda \geq \mu_1(d_2, b(x))$. Alternatively, if $\Psi \equiv 0$, then $\Phi \neq 0$, and $\lambda$ belongs to the spectrum of $-d_1 \Delta - a(x) + 2\theta_{d_1,a}$ (with zero Neumann boundary condition), which must again be real and satisfy $\lambda \geq \mu_1(d_1, a(x) - 2\theta_{d_1,a})$. This implies

$$
\lambda \geq \min \{\mu_1(d_2, b(x)), \mu_1(d_1, a(x) - 2\theta_{d_1,a})\}.
$$

Moreover, in view of proposition 1 (iv), we have

$$
\mu_1(d_2, b(x)) < \mu_1(d_2, b(x) - \theta_{d_2,b}) = 0
$$

and

$$
\mu_1(d_1, a(x) - 2\theta_{d_1,a}) > \mu_1(d_1, a(x) - \theta_{d_1,a}) = 0.
$$

Therefore, we obtain $\lambda \geq \mu_1(d_2, b(x))$. Let $\psi$ be the first eigenfunction corresponding to $\mu_1(d_2, b(x))$, then $\mu_1(d_2, b(x))$ is an eigenvalue of (11) with eigenfunction

$$
(\Phi, \Psi) = \left( L^{-1} \frac{\theta_{d_1,a} \psi}{1 + \theta_{d_1,a}^2}, \psi \right).
$$

where $L = d_1 \Delta + (a(x) - 2\theta_{d_1,a}) + \mu_1(d_2, b(x))$. Note that

$$
\mu_1(d_1, a(x) - 2\theta_{d_1,a}) + \mu_1(d_2, b(x)) = \mu_1(d_1, a(x) - 2\theta_{d_1,a}) - \mu_1(d_2, b(x)) > 0,
$$

then every eigenvalue of $L$ is positive and hence $L$ is invertible. Thus, $\min \lambda = \mu_1(d_2, b(x)) < 0$. This implies that the principal eigenvalue of (11) exists and is negative. It follows from [32] that the semi-trivial steady state $(\theta_{d_1,a}, 0)$ is unstable and hence the proof of conclusion (i) is completed.

Next, we discuss the stability of $(0, \theta_{d_2,b})$, which is determined by the following eigenvalue problem

$$
\begin{align*}
&d_1 \Delta \Phi + (a(x) - \theta_{d_2,b}) \Phi + \lambda \Phi = 0, & x \in \Omega, \\
&d_2 \Delta \Psi + (b(x) - 2\theta_{d_2,b}) \Psi + \lambda \Psi = 0, & x \in \Omega, \\
&\frac{\partial \Phi(x)}{\partial n} = \frac{\partial \Psi(x)}{\partial n} = 0, & x \in \partial \Omega.
\end{align*}
\tag{12}
$$

Let $\lambda$ be an eigenvalue of (12) with an associated eigenfunction $(\Phi, \Psi)$. From the first equation of system (12), it is easy to see that $\lambda$ belongs to the spectrum
of the self-adjoint operator $-d_1 \Delta - a(x) + \theta_{d_2,b}$ (with zero Neumann boundary condition). Hence, $\lambda$ must be real and satisfy $\lambda \geq \mu_1(d_1, a(x) - \theta_{d_2,b})$. Similarly, it follows from the second equation of system (12) that $\lambda$ must be real and satisfy $\lambda \geq \mu_1(d_2, b(x) - 2\theta_{d_2,b})$. Therefore, we have

$$\lambda \geq \min\{\mu_1(d_1, a(x) - \theta_{d_2,b}), \mu_1(d_2, b(x) - 2\theta_{d_2,b})\}.$$ 

Let $\varphi$ be the first eigenfunction corresponding to $\mu_1(d_1, a(x) - \theta_{d_2,b})$, then we have $\mu_1(d_1, a(x) - \theta_{d_2,b})$ is an eigenvalue of (12) with an associated eigenfunction $(\Phi, \Psi)=(\varphi, 0)$. Similarly, let $\psi$ be the first eigenfunction associated with $\mu_1(d_2, b(x) - 2\theta_{d_2,b})$, then $\mu_1(d_2, b(x) - 2\theta_{d_2,b})$ is an eigenvalue of (12) with an associated eigenfunction $(\Phi, \Psi)=(0, \psi)$. Hence,

$$\min \lambda = \min\{\mu_1(d_1, a(x) - \theta_{d_2,b}), \mu_1(d_2, b(x) - 2\theta_{d_2,b})\}.$$ 

On the other hand, it follows from Proposition 1 (iv) that

$$\mu_1(d_2, b(x) - 2\theta_{d_2,b}) > \mu_1(d_2, b(x) - \theta_{d_2,b}) = 0.$$ 

Based on the above discussion, we see that the principal eigenvalue of (12) exists and has the same sign as that of the first eigenvalue $\mu_1(d_1, a(x) - \theta_{d_2,b})$. By [32], the semi-trivial steady state $(0, \theta_{d_2,b})$ is locally asymptotically stable if $\mu_1(d_1, a(x) - \theta_{d_2,b}) > 0$, while the semi-trivial steady state $(0, \theta_{d_2,b})$ is unstable if $\mu_1(d_1, a(x) - \theta_{d_2,b}) < 0$. Moreover, Proposition 1 says that $\mu_1(d_1, a(x) - \theta_{d_2,b})$ is strictly increasing with respect to $d_1$, and satisfies

$$\lim_{d_1 \to 0} \mu_1(d_1, a(x) - \theta_{d_2,b}) = \min(\theta_{d_2,b} - a(x)) \geq \bar{\theta}_{d_2,b} - \bar{a}$$

and

$$\lim_{d_1 \to \infty} \mu_1(d_1, a(x) - \theta_{d_2,b}) = \tilde{\theta}_{d_2,b} - \bar{a}.$$ 

Thus, if $\tilde{\theta}_{d_2,b} < \bar{a}$, then $\mu_1(d_1, a(x) - \theta_{d_2,b}) < 0$ for all $d_1 > 0$. If $\tilde{\theta}_{d_2,b} > \bar{a}$, then $\mu_1(d_1, a(x) - \theta_{d_2,b}) > 0$ for all $d_1 > 0$. Next, we discuss the case where $\bar{\theta}_{d_2,b} > \bar{a}$. In this case, if $a(x) - \theta_{d_2,b}$ changes sign in $\Omega$, then it follows from Proposition 1 that there exists $d_1^* = 1/\lambda_1(a(x) - \theta_{d_2,b})$ such that $\mu_1(d_1, a(x) - \theta_{d_2,b}) > 0$ when $d_1 > d_1^*$ and that $\mu_1(d_1, a(x) - \theta_{d_2,b}) < 0$ when $d_1 < d_1^*$. This completes the proof of Theorem 4.1 (ii).}

\textbf{Remark 1.} Using a similar argument, we know that the linear stability of the trivial solution $(0,0)$ of system (4) is determined by $\min\{\mu_1(d_1, a(x)), \mu_1(d_2, b(x))\}$. In fact, the trivial solution $(0,0)$ is unstable for all $d_1 > 0$ and $d_2 > 0$.

Next, we consider the global stability of $(0, \theta_{d_2,b})$.

\textbf{Theorem 4.2.} The semi-trivial $(0, \theta_{d_2,b})$ is globally asymptotically stable if

$$\mu_1\left(d_1, a - \frac{\theta_{d_2,b}}{1 + \bar{\theta}_{d_2,b}}\right) > 0. \quad (13)$$

More precisely, the steady state $(0, \theta_{d_2,b})$ is globally asymptotically stable if either

$$\frac{\theta_{d_2,b}}{1 + \bar{a}^2} \geq \bar{a} \quad \text{or} \quad \text{the following condition (H) is satisfied},$$

\textbf{(H):} $\frac{\theta_{d_2,b}}{1 + \bar{a}^2} > \bar{a}$ and $a(x) - \frac{\theta_{d_2,b}}{1 + \bar{a}^2}$ changes sign in $\Omega$ and $d_1$ is large enough.
Proof. In virtue of Proposition 1 and (13), we have
\[ \mu_1(d_1, a - \theta_{d_2,b}) \geq \mu_1 \left( d_1, a - \frac{\theta_{d_2,b}}{1 + \theta^2_{d_1,a}} \right) > 0, \]
which together with Theorem 4.1 implies that \((0, \theta_{d_2,b})\) is locally asymptotically stable. Moreover, according to Proposition 1 (iv) and Lemma 2.1, we have
\[ \mu_1 \left( d_1, a - \frac{\theta_{d_2,b}}{1 + \theta^2_{d_1,a}} \right) \geq \mu_1 \left( d_1, a - \frac{\theta_{d_2,b}}{1 + a^2} \right). \]

In view of Proposition 1 (iii), we know that \(\mu_1 \left( d_1, a - \frac{\theta_{d_2,b}}{1 + a^2} \right)\) is strictly increasing with respect to \(d_1\) and satisfies
\[ \lim_{d_1 \to 0} \mu_1 \left( d_1, a(x) - \frac{\theta_{d_2,b}}{1 + a^2} \right) = \min \left( \frac{\theta_{d_2,b}}{1 + a^2} - a(x) \right) \geq \frac{\theta_{d_2,b}}{1 + a^2} - \hat{a}. \]

Thus, if \(\frac{\theta_{d_2,b}}{1 + a^2} \geq \hat{a}\) then \(\mu_1 \left( d_1, a(x) - \frac{\theta_{d_2,b}}{1 + a^2} \right) \geq 0\) for all \(d_1 > 0\). In addition, if \(\frac{\theta_{d_2,b}}{1 + a^2} > \hat{a}\) and \(a(x) - \frac{\theta_{d_2,b}}{1 + a^2}\) changes sign in \(\Omega\), then it follows from Proposition 1 (ii) that there exists \(\tilde{d}_1 = 1/\lambda_1 \left( a(x) - \frac{\theta_{d_2,b}}{1 + a^2} \right) \) such that \(\mu_1 \left( d_1, a(x) - \frac{\theta_{d_2,b}}{1 + a^2} \right) \geq 0\) when \(d_1 \geq \tilde{d}_1\).

In what follows, we shall prove the global attractivity of \((0, \theta_{d_2,b})\). From the first equation of system (4) and \(v \geq 0\), we obtain that
\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &\leq d_1 \Delta u(x,t) + u(x,t)(a(x) - u(x,t)) \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial u(x,t)}{\partial n} &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
u(x,0) &= \varphi(x) \geq 0 \quad \text{in } \Omega.
\end{align*}
\]

Then by the comparison principle of parabolic equations, it is easy to verify that
\[ \limsup_{t \to \infty} u(x,t) \leq \theta_{d_1,a} \text{ uniformly for } x \in \Omega. \]

More precisely, for any given \(\varepsilon > 0\), there exists \(t_1 > 0\) such that
\[ u(x,t) \leq \theta_{d_1,a} + \varepsilon, \quad \forall x \in \Omega, \quad t \geq t_1. \tag{14} \]

Obviously, \(v(x,t)\) satisfies
\[
\begin{align*}
\frac{\partial v(x,t)}{\partial t} &\geq d_2 \Delta v(x,t) + v(x,t)(b(x) - v(x,t)) \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial v(x,t)}{\partial n} &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
v(x,0) &= \psi(x) \geq 0 \quad \text{in } \Omega.
\end{align*}
\]

Similarly, we have
\[ \liminf_{t \to \infty} v(x,t) \geq \theta_{d_2,b} \text{ uniformly for } x \in \Omega. \tag{15} \]

Then there exists \(t_2\) such that
\[ v(x,t) \geq \theta_{d_2,b} - \varepsilon, \quad \forall x \in \Omega, \quad t \geq t_2. \tag{16} \]
Therefore, in light of (14) and (16), there exists \( t \geq t_3 = \max\{t_1, t_2\} \) such that \( u(x, t) \) satisfies

\[
\begin{cases}
\frac{\partial u(x, t)}{\partial t} \leq d_1 \Delta u(x, t) + u(x, t) \left[ a(x) - u(x, t) - \frac{\theta d_2 - \varepsilon}{1 + (\theta d_1 a + \varepsilon)^2} \right] & \text{in } \Omega \times [t_3, \infty), \\
\frac{\partial u(x, t)}{\partial n} = 0 & \text{on } \partial \Omega \times [t_3, \infty), \\
u(x, t_3) > 0 & \text{in } \Omega.
\end{cases}
\]

In view of (13), using the comparison principle, we have

\[
\lim_{t \to \infty} u(x, t) = 0 \text{ uniformly for } x \in \bar{\Omega}.
\]

Thus, there exists \( t > t_4 \) such that

\[
0 \leq u(x, t) \leq \varepsilon \text{ for all } x \in \bar{\Omega}, \quad t \geq t_4.
\]

It then follows that \( v(x, t) \) satisfies

\[
\begin{cases}
\frac{\partial v(x, t)}{\partial t} \leq d_2 \Delta v(x, t) + v(x, t) \left( b(x) - \frac{v(x, t)}{1 + \varepsilon^2} \right) & \text{in } \Omega \times [t_4, \infty), \\
\frac{\partial v(x, t)}{\partial n} = 0 & \text{on } \partial \Omega \times [t_4, \infty), \\
v(x, t_4) > 0 & \text{in } \Omega.
\end{cases}
\]

By the standard comparison theorem again, we can get

\[
\lim_{t \to \infty} \sup v(x, t) \leq \theta d_2, b(1 + \varepsilon^2) \text{ uniformly for } x \in \bar{\Omega}.
\]

Hence, combining with (15) and the arbitrariness of \( \varepsilon \), we obtain

\[
\lim_{t \to \infty} v(x, t) = \theta d_2, b \text{ uniformly for } x \in \bar{\Omega}.
\]

Therefore, the solution \((u, v)\) of system (4) such that \( \lim_{t \to \infty}(u(x, t), v(x, t)) = (0, \theta d_2, b) \) uniformly for \( x \in \bar{\Omega} \). The proof is completed.

5. **Local bifurcation of steady states.** In this section, we shall apply the local bifurcation theory [5, 31] to analyze the bifurcation phenomena of semi-trivial steady states of (4) by regarding dispersal rates of the prey and predator as bifurcation parameters. Note that positive steady states of (4) satisfy the following equation:

\[
\begin{cases}
d_1 \Delta u + u \left( a(x) - u - \frac{v}{1 + u^2} \right) = 0 & \text{in } \Omega, \\
d_2 \Delta v + v \left( b(x) - \frac{v}{1 + u^2} \right) = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(17)

To find solutions of (17), we define the operator \( F(d_1, u, v): \mathbb{R}^+ \times X \to Y \) by

\[
F(d_1, u, v) = \begin{pmatrix}
  d_1 \Delta u + u \left( a(x) - u - \frac{v}{1 + u^2} \right) \\
  d_2 \Delta v + v \left( b(x) - \frac{v}{1 + u^2} \right)
\end{pmatrix},
\]

where
It is easy to see that in order to apply the bifurcation theory of Crandall and Rabinowitz [5], it suffices to obviously, it follows from dividing both sides by the straightforward computations, it is easy to see that the derivatives $D_{d_1} F(d_1, u, v)$, $F(u,v)(d_1, u, v)$ and $D_{d_1} F(u,v)(d_1, u, v)$ exist and are continuous in a neighborhood of $(d_1, 0, \theta_{d_2,b})$.

**Lemma 5.1.** If $\bar{a} < \theta_{d_2,b}$ and $a(x) - \theta_{d_2,b}$ changes sign in $\Omega$, then a branch of positive steady-state solutions of (4) bifurcates from the semi-trivial solution curve $(d_1, 0, \theta_{d_2,b})$ if and only if $d_1 = d_1^* \triangleq 1/\lambda_1(a(x) - \theta_{d_2,b})$. More precisely, there exists a small positive number $\delta$ such that all positive steady-state solutions of (4) near $(d_1^*, 0, \theta_{d_2,b}) \in \mathbb{R}^+ \times X$ can be parameterized as

$$
\{(d_1, u(s), v(s)) = (d_1(s), s \varphi^* + s^2 u_1(s), \theta_{d_2,b} + s^2 v_1(s)) \in \mathbb{R} \times X : 0 < s < \delta\}, \quad (18)
$$

where $\varphi^*$ is defined in (19) and $d_1(0) = d_1^*$ and $(u_1(s), v_1(s))$ is a family of bounded smooth functions and lies in the complement of the kernel of $F_{(u,v)}(d_1^*, 0, \theta_{d_2,b})$ in $X$. Moreover, the bifurcation direction of the solution $(d_1^*, 0, \theta_{d_2,b})$ can be characterized by $d_1'(0) < 0$.

**Proof.** Obviously, it follows from the assumption of Lemma 5.1 and Proposition 1(ii) that $\mu_1(d_1^*, a(x) - \theta_{d_2,b}) = 0$. Since $\mu_1(d_1, b(x) - 2\theta_{d_2,b}) > 0$, then $\mu_1(d_1^*, a(x) - \theta_{d_2,b}) = 0$ is the principal eigenvalue of (12). Therefore, there exists $\varphi^* > 0$ such that

$$
\begin{align*}
\begin{cases}
d_1^* \Delta \varphi^* + (a(x) - \theta_{d_2,b}) \varphi^* = 0 & \text{in } \Omega, \\
\frac{\partial \varphi^*}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\end{align*}
$$

(19)

It is easy to see that $F(u,v)(d_1^*, 0, \theta_{d_2,b})$ is a self-adjoint operator with the kernel $\mathcal{N}(F(u,v)(d_1^*, 0, \theta_{d_2,b}))$ spanned by $(\varphi^*, 0)$, and $\dim \mathcal{N}(F(u,v)(d_1^*, 0, \theta_{d_2,b})) = 1$. Hence the Fredholm alternative theorem tells us that $\text{codim} \mathcal{R}(F(u,v)(d_1^*, 0, \theta_{d_2,b})) = 1$. In order to apply the bifurcation theory of Crandall and Rabinowitz [5], it suffices to check the following transversality condition:

$$
D_{d_1} F_{(u,v)}(d_1^*, 0, \theta_{d_2,b}) \begin{pmatrix} \varphi^* \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta \varphi^* \\ 0 \end{pmatrix} \notin \mathcal{R}(F_{(u,v)}(d_1^*, 0, \theta_{d_2,b})).
$$

Obviously, it follows from $\int_{\Omega} |\nabla \varphi|^2 \neq 0$ that the equation $d_1^* \Delta \varphi^* + (a(x) - \theta_{d_2,b}) \varphi^* = \Delta \varphi^*$ has no positive solutions. Substituting (18) into the first equation of (17) and dividing both sides by $s$, we have

$$
\frac{d_1 - d_1^*}{s} \Delta \varphi^* + d_1 \Delta u_1(s) + u_1(s)(a(x) - \theta_{d_2,b}) - \varphi^* = s(2 \varphi^* u_1(s) + \varphi^* v_1(s)) + o(s).
$$

(20)

Multiplying (20) by $\varphi^*$ and applying integration by parts and the boundary condition of $\varphi^*$, we have

$$
d_1'(0) \int_{\Omega} |\nabla \varphi^*|^2 = - \int_{\Omega} (\varphi^*)^2
$$

with $s \to 0$. Clearly, we obtain $d_1'(0) < 0$. Therefore, the proof is completed. \qed
In what follows, we will study the linear stability of \((u(s), v(s))\) which bifurcates from semi-trivial steady state \((0, \theta_{d_\delta, b})\). Firstly, we need to prove the following Lemma.

**Lemma 5.2.** Let \((u(s), v(s))\) be the positive steady-state solution given in Lemma 5.1, then as \(s \to 0\), \((u(s), v(s)) \to (0, \theta_{d_\delta, b})\), \(u(s)/\|u(s)\|_{L^\infty} \to \varphi^*\) and \(\varphi \to \varphi^*\) in \(C^1(\Omega)\), where \(\varphi\) is the corresponding eigenfunction of the principal eigenvalue of system (12) such that \(\|\varphi\|_{L^\infty} = 1\).

**Proof.** In view of (18), we conclude that \((u(s), v(s)) \to (0, \theta_{d_\delta, b})\) as \(s \to 0\). Denote by \(\tilde{u} = u(s)/\|u(s)\|_{L^\infty}\) the \(L^\infty\) normalization of \(u(s)\). Then \(\tilde{u}\) satisfies

\[
\begin{aligned}
&d_1(s)\Delta \tilde{u} + \tilde{u} \left(a(x) - u(s) - \frac{v(s)}{1 + u^2(s)}\right) = 0 & \text{in } \Omega, \\
&\frac{\partial \tilde{u}}{\partial n} = 0 & \text{on } \partial \Omega.
\end{aligned}
\]

(21)

Note that \(\|\tilde{u}\|_{L^\infty} = 1\), then it follows from the elliptic regularity theory and Sobolev embedding theorem that \(\tilde{u} \to \varphi^*\) in \(C^1(\Omega)\) as \(s \to 0\). Using a similar argument, we obtain \(\mu_1(d_1, a(x) - \theta_{d_\delta, b}) \to 0\) and \(\varphi \to \varphi^*\) in \(C^1(\Omega)\) as \(s \to 0\). \(\Box\)

**Lemma 5.3.** Under the assumption of Lemma 5.1, for every small \(s > 0\), the bifurcating solution \((d_1, u(s), v(s)) = (d_1(s), s\varphi^* + s^2u_1(s), -\theta_{d_\delta, b} + + s^2v_1(s))\) is locally asymptotically stable.

**Proof.** In order to investigate the stability of bifurcating solution \((u(s), v(s))\) of system (17) for small \(s\), we consider the following linear eigenvalue problem

\[
\begin{aligned}
&d_1\Delta \Phi + \left(a(x) - 2u(s) - \frac{v(s)(1 - u^2(s))}{1 + u^2(s)}\right) \Phi - \frac{u(s)}{1 + u^2(s)} \Psi + \eta \Phi = 0, & x \in \Omega, \\
&d_2\Delta \Psi + \left(b(x) - \frac{2v(s)}{1 + u^2(s)}\right) \Psi + \frac{2u(s)v^2(s)}{(1 + u^2(s))^2} \Phi + \eta \Psi = 0, & x \in \Omega, \\
&\frac{\partial \Phi(x)}{\partial n} = \frac{\partial \Psi(x)}{\partial n} = 0, & x \in \partial \Omega.
\end{aligned}
\]

(22)

Let us introduce two operators \(\Gamma_s\) and \(\Gamma_0\): \(X \to Y\) as follows:

\[
\begin{aligned}
\Gamma_s \left(\Phi, \Psi\right) &= \left(d_1(s)\Delta \Phi + \left(a(x) - 2u(s) - \frac{v(s)(1 - u^2(s))}{1 + u^2(s)}\right) \Phi - \frac{u(s)}{1 + u^2(s)} \Psi\right) \\
&\quad \quad + d_2\Delta \Psi + \left(b(x) - \frac{2v(s)}{1 + u^2(s)}\right) \Psi + \frac{2u(s)v^2(s)}{(1 + u^2(s))^2} \Phi \\
\Gamma_0 \left(\varphi, \psi\right) &= \left(d_1^*\Delta \varphi + \left(a(x) - \theta_{d_\delta, b}\right) \varphi\right) \\
&\quad \quad + d_2\Delta \psi + \left(b(x) - 2\theta_{d_\delta, b}\right) \psi.
\end{aligned}
\]

By virtue of Lemma 5.2, we have \((u(s), v(s)) \to (0, \theta_{d_\delta, b})\) in \(C^1(\Omega)\) as \(s \to 0\). Hence \(\Gamma_s \to \Gamma_0\) uniformly in operator norm as \(s \to 0\). Moreover, it is easy to see that the kernel of \(\Gamma_0\) is spanned by \((\varphi^*, 0)\), and zero is a simple eigenvalue of \(\Gamma_0\). Then it follows that there exists a unique \(K\)-simple eigenvalue \(\sigma_1 = \sigma_1(s)\) of \(\Gamma_s\) such that \(\sigma_1 \to 0\) as \(s \to 0\). Obviously, \(\sigma_1 = -\eta\).

Let \((\Phi, \Psi)\) be an eigenfunction associated with the eigenvalue \(\sigma_1\) of (22). First, if \(\Phi \neq 0\), we assume that \(\|\Phi\|_{L^\infty(\Omega)} = 1\) after scaling and \(\Phi\) is positive somewhere in \(\Omega\). Using a similar argument as before, we conclude that \((\phi^*, 0)\) in \(C^1(\Omega)\) as \(s \to 0\) since \((u(s), v(s)) \to (0, \theta_{d_\delta, b})\) and \(\sigma_1 \to 0\) as \(s \to 0\). Multiplying the first
Under the assumption of Lemma 5.1, there exists some small equation of (22) by $u(s)$ and the first equation of (17) with $(u, v) = (u(s), v(s))$ by $\Phi$. Under the boundary conditions of $\Phi$ and $u(s)$, applying integration by parts, we have

$$
\sigma_1 \int_{\Omega} \Phi u(s) = - \int_{\Omega} u^2(s) \Phi + \int_{\Omega} \frac{2u^3(s)v(s)}{(1 + u^2(s))^2} \Phi - \int_{\Omega} \frac{u^2(s)}{1 + u^2(s)} \Psi.
$$

Dividing the above equation by $\|u(s)\|_{L^\infty(\Omega)}^2$ and applying Lemma 5.2 and $\Psi \to 0$ in $C^1(\bar{\Omega})$ as $s \to 0$, we obtain

$$
\lim_{s \to 0} \frac{\|u(s)\|_{L^\infty(\Omega)}}{\|u(s)\|_{L^\infty(\Omega)}} = - \frac{\int_{\Omega} (\varphi^*)^2}{\int_{\Omega} (\varphi^*)^2}.
$$

Note that $\varphi^* > 0$, then we have $\sigma_1 < 0$ for small $s$.

Next, suppose that $\Phi \equiv 0$ in $\bar{\Omega}$. Then $\Psi \not\equiv 0$ and

$$
d_2 \Delta \Psi + \left(b(x) - \frac{2v(s)}{1 + u^2(s)}\right) \Psi = \sigma_1 \Psi \quad \text{in } \Omega, \quad \frac{\partial \Psi}{\partial n} = 0 \quad \text{on } \partial \Omega.
$$

In view of Proposition 1, we have $\mu_1(d_2, b(x) - 2\theta_{d_2, b}) > 0$, which is the principal eigenvalue of $-d_2 \Delta - (b(x) - 2\theta_{d_2, b})$ with homogeneous Neumann boundary condition. Combining with the fact that $(u(s), v(s)) \to (0, \theta_{d_2, b})$ as $s \to 0$, we conclude that $\sigma_1 < 0$ for small $s$. Therefore, all eigenvalues of (22) have positive real parts, that is, $(u(s), v(s))$ is locally asymptotically stable. Hence, we complete the proof.

In view of the above lemmas, we have the following conclusion.

**Theorem 5.4.** Under the assumption of Lemma 5.1, there exists some small $\delta > 0$ such that a branch of steady state solutions $(u(s), v(s))$ of (4) bifurcates from $(0, \theta_{d_2, b})$ at $d_1 = d_1^* \triangleq 1/\lambda_1(a(x) - \theta_{d_2, b})$, which can be parameterized by $d_1 \in (d_1^* - \delta, d_1^*)$ and these solutions are locally asymptotically stable for $d_1 \in (d_1^* - \delta, d_1^*)$.

6. **Asymptotic profiles of steady states.** In this section, our interest is to investigate the asymptotic behavior of (17). To this end, we first obtain some a priori estimates for positive solutions and investigate the existence of positive steady state solutions of (17).

**Lemma 6.1.** Let $(u, v)$ be any positive solution of (17), then

$$
\|u\|_{L^\infty(\Omega)} \leq \hat{a}, \quad 0 < \hat{b} \leq \|v\|_{L^\infty(\Omega)} \leq (1 + \hat{a}^2)\hat{b}.
$$

**Proof.** Let $w = u - \hat{a}$. It follows from $d_1 \Delta u + u(a(x) - u) \geq 0$ that $w$ satisfies

$$
\begin{aligned}
&d_1 \Delta w + w(a(x) - u - \hat{a}) \geq 0 & x & \in \Omega, \\
&\frac{\partial w}{\partial n} = 0 & x & \in \partial \Omega.
\end{aligned}
$$

Obviously, $a(x) - u - \hat{a} < 0$. Let $x_1 \in \Omega$ be a maximum point of $w$, i.e., $w(x_1) = \max_{x \in \Omega} w(x)$. Applying the maximum principle [11, 26], we have $w(x_1) \leq 0$. Hence $w = u - \hat{a} \leq 0$ in $\Omega$. Therefore, we obtain $\|u\|_{L^\infty(\Omega)} \leq \hat{a}$. Similarly, $\theta = v - \hat{b}(1 + \hat{a}^2)$ satisfies

$$
\begin{aligned}
&d_2 \Delta \theta + \theta \left(b(x) - v \frac{\hat{b}(1 + \hat{a}^2)}{1 + u^2(x)}\right) \geq 0 & x & \in \Omega, \\
&\frac{\partial \theta}{\partial n} = 0 & x & \in \partial \Omega.
\end{aligned}
$$
It follows from \( \|u\|_{L^\infty(\Omega)} \leq \hat{a} \) that
\[
b(x) - \frac{v}{1 + u^2(x)} - \frac{(1 + \hat{a}^2)b}{1 + u^2(x)} \leq 0.
\]
Using a similar argument, we have \( \hat{v} = v - \hat{b}(1 + \hat{a}^2) \leq 0 \) in \( \Omega \). Thus, \( v \leq \hat{b}(1 + \hat{a}^2) \).
Define \( v(x_1) = \min_{\Omega} v(x) \), then by using the maximum principle [11, 26] for the second equation of system (17), we obtain
\[
b(x_1) - \frac{v(x_1)}{1 + u^2(x_1)} \leq 0.
\]
Obviously,
\[
0 < \hat{b} \leq b(x_1) \leq \frac{v(x_1)}{1 + u^2(x_1)} \leq v(x_1).
\]
Therefore, the proof is completed. \( \square \)

Next, we will establish the existence result of positive solutions of (17) by the Leray-Schauder degree theory. Now, we recall the fixed point index theory in a cone [6, 22]. Let \( E \) be a real Banach space and \( W \subset E \) be a closed convex set. For all \( \beta \geq 0 \), if \( \beta W \subset W \) and \( W - W = E \), then \( W \) is called a total wedge, moreover, when \( W \cap (-W) = 0 \), a wedge is said to be a cone. For any point \( y \in W \), define \( W_y = \{ x \in E : y + \gamma x \in W \text{ for some } \gamma > 0 \} \) and \( S_y = \{ x \in W_y : -x \in W_y \} \).

Let \( T \) be a compact linear operator on \( E \) which satisfies \( T(W_y) \subset W_y \), then we say that \( T \) has property \( \alpha \) on \( W_y \) if there exist \( t \in (0, 1) \) and \( w \in W_y \setminus S_y \) such that \( w - tTw \in S_y \). Let \( U \) be an open subset of \( W \). We define \( \text{index}_W(T, U, W) = \text{deg}_W(I - T, U, 0) \). Then the fixed point index of compact operator \( T \) at \( y \) in \( W \) is defined by \( \text{index}_W(T, y) = \text{index}_W(T, U(y), W) \), where \( U(y) \) is a small open neighborhood of \( y \) in \( W \). The following result is standard in the theory of fixed point index.

**Proposition 2** ([6, 22]). Let \( L = T'(y) \) be the Fréchet derivative of \( T \) at \( y \). Suppose that \( I - L \) is invertible on \( W_y \).

(i): If \( L \) has property \( \alpha \) on \( W_y \), then \( \text{index}_W(T, y) = 0 \).

(ii): If \( L \) does not have property \( \alpha \) on \( W_y \), then \( \text{index}_W(T, y) = (-1)^\sigma \), where \( \sigma \) is the sum of algebraic multiplicities of the eigenvalues of \( L \) which are greater than 1.

For a linear operator \( L \), we denote the spectral radius of \( L \) by \( r(L) \). Then we give the relationship between \( \mu_1(d, h) \) and \( r(L) \) as follows (see [22]).

**Proposition 3.** Suppose that \( h(x) \in C(\Omega) \) and \( M \) is a positive constant such that \( M + h(x) > 0 \) on \( \Omega \). Then we have

(i): If \( \mu_1(d, h) < 0 \), then \( r[(M - d\Delta)^{-1}(M + h(x))] > 1 \);

(ii): If \( \mu_1(d, h) > 0 \), then \( r[(M - d\Delta)^{-1}(M + h(x))] < 1 \);

(iii): If \( \mu_1(d, h) = 0 \), then \( r[(M - d\Delta)^{-1}(M + h(x))] = 1 \).

For convenience, we introduce some notations: \( E = X \times X \), where \( X = \{ u \in C^1(\Omega) : \frac{\partial u}{\partial n} = 0 \} \); \( W = K \times K \), where \( K = \{ u \in X : u \geq 0 \} \); \( D = \{ (u, v) \in W : u < 1 + \hat{a}, v < 1 + (1 + \hat{a}^2)b \} \); \( D' = (\text{int}(D)) \cap W \). Therefore, it is easy to see that \( \overline{W}_{(0,0)} = K \times K \), \( S_{(0,0)} = \{ (0,0) \} \), \( \overline{W}_{(\theta_1, a, 0)} = X \times K \), \( S_{(\theta_1, a, 0)} = X \times \{ 0 \} \), \( \overline{W}_{(0, \theta_2, b)} = K \times X \), \( S_{(0, \theta_2, b)} = \{ 0 \} \times X \), \( \overline{W}_{(\theta_2, a, \theta_2, b)} = S_{(\theta_2, a, \theta_2, b)} = X \times X \).
Define an operator $\mathcal{A}_t$: $\overline{D} \to E$ by
$$
\mathcal{A}_t(u, v) = \left[ (M - d_1 \Delta)^{-1} u \left( a(x) - u - \frac{tv}{1+u^2} \right) + Mu, \right.
(M - d_2 \Delta)^{-1} v \left( b(x) - \frac{v}{1+tu^2} \right) + Mv \right],
$$
where $t \in [0, 1]$ and $M$ is a sufficiently large number such that
$$
u \left( a(x) - u - \frac{tv}{1+u^2} \right) + Mu \geq 0, \quad v \left( b(x) - \frac{v}{1+tu^2} \right) + Mv \geq 0
$$
for all $(u, v) \in \overline{D}$. Then it follows from the standard elliptic regularity theory that $\mathcal{A}_t$ is a completely continuous operator for all $t \in [0, 1]$. Therefore, it suffices to prove that $\mathcal{A}_t$ has a nontrivial fixed point in $D$ in order to show the existence of positive solutions of (17). For each $t \in [0, 1]$, a fixed point of $\mathcal{A}_t$ is a solution of the following problem
$$
\begin{aligned}
d_1 \Delta u + u \left( a(x) - u - \frac{tv}{1+u^2} \right) &= 0 \quad \text{in } \Omega, \\
d_2 \Delta v + v \left( b(x) - \frac{v}{1+tu^2} \right) &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega.
\end{aligned}
$$
Moreover, it follows from Lemma 6.1 that all fixed points of $\mathcal{A}_t$ are in $D'$. Hence, the degree $\deg_W(I - \mathcal{A}_t, D, 0)$ is well defined for all $t \in [0, 1]$. By the homotopy invariance of degree, we have $\deg_W(I - \mathcal{A}_1, D, 0) = \deg_W(I - \mathcal{A}_0, D, 0)$. For simplicity, we denote $\mathcal{A} := \mathcal{A}_1$. When $t = 0$, it is clear that system (23) has a trivial solution $(0, 0)$, semi-trivial solutions $(\theta_{d_1,a}, 0)$ and $(0, \theta_{d_2,b})$, and a positive solution $(\theta_{d_1,a}, \theta_{d_2,b})$. Then by the additivity property of fixed point index, we obtain that
$$
\begin{aligned}
\deg_W(I - \mathcal{A}_0, D, 0) &= \text{index}_W(\mathcal{A}_0, (0, 0)) + \text{index}_W(\mathcal{A}_0, (\theta_{d_1,a}, 0)) \\
&\quad + \text{index}_W(\mathcal{A}_0, (0, \theta_{d_2,b})) + \text{index}_W(\mathcal{A}_0, (\theta_{d_1,a}, \theta_{d_2,b})).
\end{aligned}
$$
Furthermore, we have the following lemma.

**Lemma 6.2**. The following conclusions hold true

(i): $\deg_W(I - \mathcal{A}, D, 0) = 1$.

(ii): $\text{index}_W(\mathcal{A}, (0, 0)) = \text{index}_W(\mathcal{A}, (\theta_{d_1,a}, 0)) = 0$.

(iii): $\text{index}_W(\mathcal{A}, (0, \theta_{d_2,b})) = 0$ if $\mu_1(d_1, a - \theta_{d_2,b}) < 0$;
$\text{index}_W(\mathcal{A}, (0, \theta_{d_2,b})) = 1$ if $\mu_1(d_1, a - \theta_{d_2,b}) > 0$.

**Proof.** (i) By using the similar arguments to those parts (ii)-(iii), we can calculate the following indexes
$$
\begin{aligned}
\text{index}_W(\mathcal{A}_0, (0, 0)) &= \text{index}_W(\mathcal{A}_0, (\theta_{d_1,a}, 0)) = \text{index}_W(\mathcal{A}_0, (0, \theta_{d_2,b})) = 0, \\
\text{index}_W(\mathcal{A}_0, (\theta_{d_1,a}, \theta_{d_2,b})) &= 1.
\end{aligned}
$$
Consequently, $\deg_W(I - \mathcal{A}, D, 0) = 1$. The proof (i) is completed.

(ii) Denote $L = \mathcal{A}(u,v)(0,0)$, then
$$
L = \begin{bmatrix}
(M - d_1 \Delta)^{-1}(a(x) + M) & 0 \\
0 & (M - d_2 \Delta)^{-1}(b(x) + M)
\end{bmatrix}.
$$
Obviously, $I - L$ is invertible on $W(0,0)$. Moreover, it follows from Proposition 3 that $r_0 = r((M - d_1 \Delta)^{-1}(a(x) + M)) > 1$ and $r_0$ is the principal eigenvalue of the operator $(M - d_1 \Delta)^{-1}(a(x) + M)$ with a corresponding eigenfunction $\phi(x) > 0$ in
\( \Omega \). Set \( t_0 = 1/r_0 \in (0,1) \). Then \( (I - t_0L)(\phi,0) = (0,0) \in S_{(0,0)} \). This shows that \( L \)
has property \( \alpha \), and thus \( \text{index}_W(\mathcal{A},(0,0)) = 0 \) by Proposition 2 (i).

Similarly, let \( L = \mathcal{A}_{(u,v)}(\theta_{d_1,a},0) \), then

\[
L = \begin{bmatrix}
(M - d_1\Delta)^{-1}(a(x) - 2\theta_{d_1,a} + M) & -\frac{\theta_{d_1,a}}{1 + \theta_{d_1,a}^2}M
\end{bmatrix}.
\]

Assume that \( L(\xi_1,\xi_2) = (\xi_1,\xi_2) \) for some \( (\xi_1,\xi_2) \in W_{\mathcal{A},(0,0)} \), that is, \( (\xi_1,\xi_2) \)
satisfies

\[
\begin{align*}
d_1\Delta\xi_1 + (a(x) - 2\theta_{d_1,a})\xi_1 - \frac{\theta_{d_1,a}}{1 + \theta_{d_1,a}^2}\xi_2 &= 0, & x \in \Omega, \\
d_2\Delta\xi_2 + b(x)\xi_2 &= 0, & x \in \Omega, \\
\frac{\partial\xi_1(x)}{\partial n} &= \frac{\partial\xi_2(x)}{\partial n} = 0, & x \in \partial\Omega.
\end{align*}
\]

Since \( \mu_1(d_2,b(x)) < 0 \), then \( \xi_2 = 0 \). Moreover, \( \xi_1 = 0 \) by \( \mu_1(d_1,a(x) - 2\theta_{d_1,a}) > 0 \).
Obviously, \( I - L \) is invertible on \( W_{\mathcal{A},(0,0)} \). Therefore, it follows from Proposition
3 that \( r_1 = r((M - d_2\Delta)^{-1}(b(x) + M)) > 1 \) and \( r_1 \) is the principal eigenvalue of the
operator \( (M - d_2\Delta)^{-1}(b(x) + M) \) with a corresponding eigenfunction \( \phi(x) > 0 \) in \( \Omega \).
Set \( t_1 = 1/r_1 \in (0,1) \). Then \( (I - t_1L)(0,0) = (0,0) \in S_{(\theta_{d_1,a},0)} \), which implies that \( L \)
has property \( \alpha \). Therefore \( \text{index}_W(\mathcal{A},(\theta_{d_1,a},0)) = 0 \) by Proposition 2 (i).

(iii) Let \( L = \mathcal{A}_{(u,v)}(\theta_{d_2,b},0) \), then

\[
L = \begin{bmatrix}
(M - d_1\Delta)^{-1}(a(x) - \theta_{d_2,b} + M) & 0 \\
0 & (M - d_2\Delta)^{-1}(b(x) - 2\theta_{d_2,b} + M)
\end{bmatrix}.
\]

Assume that \( L(\xi_1,\xi_2) = (\xi_1,\xi_2) \) for some \( (\xi_1,\xi_2) \in W_{(0,\theta_{d_2,b})} \), that is, \( (\xi_1,\xi_2) \)
satisfies

\[
\begin{align*}
d_1\Delta\xi_1 + (a(x) - \theta_{d_2,b})\xi_1 &= 0, & x \in \Omega, \\
d_2\Delta\xi_2 + b(x)\xi_2 - 2\theta_{d_2,b}\xi_2 &= 0, & x \in \Omega, \\
\frac{\partial\xi_1(x)}{\partial n} &= \frac{\partial\xi_2(x)}{\partial n} = 0, & x \in \partial\Omega.
\end{align*}
\]

Since \( \mu_1(d_2,b(x) - 2\theta_{d_2,b}) > 0 \), then \( (\xi_1,\xi_2) = (0,0) \) if \( \mu_1(d_1,a(x) - \theta_{d_2,b}) \neq 0 \).
Obviously, \( I - L \) is invertible on \( W_{(0,\theta_{d_2,b})} \).

We claim that \( L \) has property \( \alpha \) on \( W_{(0,\theta_{d_2,b})} \) if \( \mu_1(d_1,a(x) - \theta_{d_2,b}) < 0 \). By
Proposition 3, we have \( r_2 = r((M - d_1\Delta)^{-1}(a(x) - \theta_{d_2,b} + M)) > 1 \) and \( r_2 \) is the
principal eigenvalue of the operator \( (M - d_1\Delta)^{-1}(a(x) - \theta_{d_2,b} + M) \) with a
Corresponding eigenfunction \( \phi(x) > 0 \) in \( \Omega \). Set \( t = 1/r_2 \in (0,1) \). Then \( (I - tL)(\phi,0) = (0,0) \in S_{(0,\theta_{d_2,b})} \). This implies that \( L \) has property \( \alpha \), and hence \( \text{index}_W(\mathcal{A},((0,\theta_{d_2,b})) = 0 \) by Proposition 2 (i).

In the case where \( \mu_1(d_1,a(x) - \theta_{d_2,b}) > 0 \), we show that \( L \) has no property
\( \alpha \) on \( W_{(0,\theta_{d_2,b})} \). We argue by contradiction. Suppose that \( L \) has property \( \alpha \) on
\( W_{(0,\theta_{d_2,b})} \). Then there exist \( t \in (0,1) \) and \( (\phi_1,\phi_2) \in W_{(0,\theta_{d_2,b})} \) such that
\( (I - tL)(\phi_1,\phi_2) \in W_{(0,\theta_{d_2,b})} \). Denote \( B = (M - d_1\Delta)^{-1}(a(x) - \theta_{d_2,b} + M) \), thus we obtain
\[
B\phi_1 = 1/t\phi_1.
\]
In light of \( \phi_1 \in \mathcal{K} \setminus \{0\} \), we have \( 1/t > 1 \) is an eigenvalue of the operator of \( B \).
On the other hand, if \( \mu_1(d_1,a(x) - \theta_{d_2,b}) > 0 \), it follows from Proposition 3 that
$r(B) < 1$. This is a contraction. Therefore, $L$ has no property $\alpha$ on $\overline{W}_0(\theta_{d_2,b})$ and it follows from Proposition 2 (ii) that

$$\text{index}_W(A_\ast, (0, \theta_{d_2,b})) = (-1)^{\sigma},$$

where $\sigma$ is the sum of the multiplicities of all real eigenvalues of $L$ which are greater than 1. Suppose that $L$ has an eigenvalue $\mu > 1$. Therefore there exists $(\zeta_1, \zeta_2) \neq (0,0)$ satisfies

$$
\begin{align*}
&d_1 \Delta \zeta_1 + \frac{1}{\mu}(a(x) - \theta_{d_2,b} + M - \mu M)\zeta_1 = 0, \quad x \in \Omega, \\
&d_2 \Delta \zeta_2 + \frac{1}{\mu}(b(x) - 2\theta_{d_2,b} + M - \mu M)\zeta_2 = 0, \quad x \in \Omega, \\
&\frac{\partial \zeta_1(x)}{\partial n} = \frac{\partial \zeta_2(x)}{\partial n} = 0, \quad x \in \partial \Omega.
\end{align*}
$$

In virtue of proposition 1 (iv), we have $\mu_1(d_2, \frac{1}{\mu}(b(x) - 2\theta_{d_2,b} + M - \mu M)) > \mu_1(d_2, b(x) - 2\theta_{d_2,b}) > 0$ and $\mu_1(d_1, \frac{1}{\mu}(a(x) - \theta_{d_2,b} + M - \mu M)) > \mu_1(d_1, a(x) - \theta_{d_2,b}) > 0$. Therefore $(\zeta_1, \zeta_2) = (0,0)$, which is a contradiction with $(\zeta_1, \zeta_2) \neq (0,0)$. This contraction implies that $L$ has no eigenvalues being greater than 1. Thus we obtain $\text{index}_W(A_\ast, (0, \theta_{d_2,b})) = 1$. The proof (iii) is completed.

Based on the above lemma, we gives the following theorem for the existence of positive solutions of (17).

**Theorem 6.3.** If $\mu_1(d_1, a - \theta_{d_2,b}) < 0$, then system (17) has at least one positive steady state solution.

**Proof.** From Lemma 6.2, we have

$$\text{deg}_W(I - A, D, 0) - \text{index}_W(A_\ast, (0,0)) - \text{index}_W(A_\ast, (\theta_{d_1,a}, 0)) - \text{index}_W(A_\ast, (0, \theta_{d_2,b})) = 1$$

if $\mu_1(d_1, a - \theta_{d_2,b}) < 0$. Therefore, system (17) has at least one positive steady state solution if $\mu_1(d_1, a - \theta_{d_2,b}) < 0$. This completes the proof.

Theorem 6.3 implies system (17) has positive steady state solutions when the trivial solution $(0,0)$, the semi-trivial solutions $(\theta_{d_1,a}, 0)$ and $(0, \theta_{d_2,b})$ are unstable. In the case where $\mu_1(d_1, a - \theta_{d_2,b}) = 0$, we obtain the existence result of positive steady state solutions in Theorem 5.4 if $\bar{a} < \theta_{d_2,b}$ and $a(x) - \theta_{d_2,b}$ changes sign in $\Omega$ and $d_1 = 1/\lambda(a(x) - \theta_{d_2,b})$. In addition, Lemma 2.1 implies that $\lim_{d_2 \to 0} \theta_{d_2,b} = b$ and $\lim_{d_2 \to \infty} \theta_{d_2,b} = b$. Proposition 1 means that $\lim_{d_1 \to 0} \mu_1(d_1, h) = \min_{\Omega}(-h)$ and $\lim_{d_1 \to \infty} \mu_1(d_1, h) = -h$. Hence, combining with Theorems 6.3 and 5.4, the existence of positive steady state solutions of (17) can be established when $d_1 \to \infty$ or $d_2 \to 0$ as follows.

**Theorem 6.4.** (i): If $d_1 \to \infty$, then system (17) has at least one positive steady state solution provided $\bar{a} > \theta_{d_2,b}$.

(ii): If $d_1 \to \infty$ and $d_2 \to \infty$, then system (17) has at least one positive steady state solution provided $\bar{a} > b$

(iii): If $d_2 \to 0$, then system (17) has at least one positive steady state solution provided either $\bar{a} \geq b$ and $a(x) \neq b(x)$ in $\Omega$ or $\bar{a} < b$ and $a(x) - b(x)$ changes sign in $\Omega$ and $d_1 \leq 1/\lambda(a-b)$.

(iv): If $d_1 \to 0$ and $d_2 \to 0$, then system (17) has at least one positive steady state solution provided $a(x) > b(x)$. 

6.1. **Asymptotic of solutions as** $d_1 \to \infty$. In what follows, we study the asymptotic behavior of positive solutions of (17) as $d_1 \to \infty$. In virtue of the asymptotic analysis, we can obtain the following results.

**Theorem 6.5.** (i): If $\bar{a} > \bar{b}$ and $d_j \to \infty$ ($j = 1, 2$), then $(u, v) \to (u_\infty, v_\infty)$ in $C^1(\bar{\Omega})$, where

$$u_\infty = \bar{a} - \bar{b}, \quad v_\infty = \bar{b}(1 + (\bar{a} - \bar{b})^2)).$$

(ii): If $d_1 \to \infty$ and $d_2$ is fixed, then $(u, v) \to (u_\infty, v_\infty)$ in $C^1(\bar{\Omega})$ for $\bar{a} \gt \bar{\theta} \in d_2, b$, where

$$u_\infty = \bar{a} - \bar{\theta} \in d_2, b, \quad v_\infty = \bar{\theta} \in d_2, b(1 + (\bar{a} - \bar{\theta} \in d_2, b)^2)).$$

**Proof.** We first prove part (i). Obviously, system (17) can be rewritten as

$$\begin{cases}
\Delta u + \frac{1}{d_1}(a(x) - u - \frac{v}{1 + u^2}) u = 0 & \text{in } \Omega, \\
\Delta v + \frac{1}{d_2}(b(x) - \frac{v}{1 + u^2}) v = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}$$

(24)

It follows from Lemma 6.1 and a standard compactness argument that there exists a subsequence of $d_j (j = 1, 2)$, denoted by $d_{jn}$ satisfying $d_{jn} \to \infty$, such that a corresponding positive solution $(u_n, v_n)$ of (17) with $d_j = d_{jn}$ satisfies $u_n \to u_\infty$ and $v_n \to v_\infty$ in $C^1(\bar{\Omega})$ as $n \to \infty$, where $u_\infty \geq 0$ and $v_\infty > 0$ in $\bar{\Omega}$ due to Lemma 6.1. Moreover, $u_\infty$ and $v_\infty$ satisfy the following equations

$$\begin{cases}
\Delta u_\infty = \Delta v_\infty = 0 & \text{in } \Omega, \\
\frac{\partial u_\infty}{\partial n} = \frac{\partial v_\infty}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}$$

Clearly, $u_\infty$ and $v_\infty$ are nonnegative constants. Define

$$\tilde{u}_n = \frac{u_n}{\|u_n\|_{L^\infty}}.$$ 

then $(\tilde{u}_n, v_n)$ satisfies

$$\begin{cases}
d_1 \Delta \tilde{u}_n + \left(a(x) - u_n - \frac{v_n}{1 + u_n^2}\right) \tilde{u}_n = 0 & \text{in } \Omega, \\
d_2 \Delta v_n + \left(b(x) - \frac{v_n}{1 + u_n^2}\right) v_n = 0 & \text{in } \Omega, \\
\frac{\partial \tilde{u}_n}{\partial n} = \frac{\partial v_n}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}$$

(24)

Hence, integrating the second equation of (24) yields

$$\int_\Omega \left(b(x) - \frac{v_n}{1 + u_n^2}\right) v_n dx = 0.$$ 

Since $v_\infty$ is a positive constant, letting $n \to \infty$ in the above equation leads to

$$\bar{b} |\Omega| = \int_\Omega b(x) dx = \int_\Omega \frac{v_\infty}{1 + u_\infty^2} dx.$$ 

(25)
According to Lemma 6.1 and the Soblev embedding theorem, we see that \( \tilde{u}_n \) is uniformly bounded in \( C^1(\Omega) \) for all \( n \geq 1 \). Passing to a subsequence if necessary, we obtain \( \tilde{u}_n \to \tilde{u}_\infty \) in \( C^1(\Omega) \), where \( \tilde{u}_\infty \) satisfies
\[
\begin{cases}
\Delta \tilde{u}_\infty = 0 & x \in \Omega, \\
\frac{\partial \tilde{u}_\infty}{\partial n} = 0 & x \in \partial \Omega.
\end{cases}
\]

By the maximum principle, \( \tilde{u}_\infty \) is a constant. Since \( \| \tilde{u}_n \|_{L^\infty} = 1 \), we have \( \tilde{u}_\infty = 1 \). Similarly, integrating the first equation of (24), we obtain
\[
\int_\Omega \left( a(x) - u_n - \frac{v_n}{1 + u_n^2} \right) \tilde{u}_n \, dx = 0. \tag{26}
\]

Letting \( n \to \infty \), it follows from \( \tilde{u}_\infty = 1 \) that
\[
\bar{a}|\Omega| = \int_\Omega a(x) \, dx = \int_\Omega \left( u_\infty + \frac{v_\infty}{1 + u_\infty^2} \right) \, dx. \tag{27}
\]

Therefore, together with (25), we obtain
\[
u_\infty = \bar{a} - \bar{b} \quad \text{and} \quad v_\infty = \bar{b} (1 + (\bar{a} - \bar{b})^2).
\]

Next, we prove part (ii). It follows from the proof process of part (i) that \( u_n \to u_\infty \) as \( d_1 \to \infty \) and \( u_\infty \) is a nonnegative constant. Furthermore, (27) holds. Letting \( n \to \infty \), it follows from (24) that
\[
\begin{cases}
d_2 \Delta v_\infty = \left( b(x) - \frac{v_\infty}{1 + u_\infty^2} \right) v_\infty & x \in \Omega, \\
\frac{\partial v_\infty}{\partial n} = 0 & x \in \partial \Omega.
\end{cases} \tag{28}
\]

Define \( w_\infty = v_\infty / (1 + u_\infty^2) \). Since \( u_\infty \) is a constant, then system (28) can be rewritten as
\[
\begin{cases}
d_2 \Delta w_\infty = (b(x) - w_\infty) w_\infty & x \in \Omega, \\
\frac{\partial w_\infty}{\partial n} = 0 & x \in \partial \Omega.
\end{cases} \tag{29}
\]

Note that system (29) has a unique positive solution \( \theta_{d_2,b} \), then we obtain \( v_\infty = (1 + u_\infty^2) \theta_{d_2,b} \). Therefore, combining with (27), we can assert that \( u_\infty = \bar{a} - \theta_{d_2,b} \).

\section*{6.2. Asymptotic of solutions as \( d_2 \to 0 \)}

Now, we discuss the asymptotic behavior of positive solutions of (17) when \( d_2 \to 0 \), and have the following conclusions.

\begin{theorem}
(i): Suppose that \( \bar{a} > \bar{b} \). If \( d_1 \to \infty \) and \( d_2 \to 0 \), then
\[
(u, v) \to (\bar{a} - \bar{b}, b(x)(1 + (\bar{a} - \bar{b})^2)) \quad \text{in} \quad C^1(\Omega).
\]

(ii): If \( d_1 \) is fixed and \( d_2 \to 0 \), then \( (u, v) \to (\theta_{d_1,\bar{a} - \bar{b}}, b(x)(1 + \theta_{d_1,\bar{a} - \bar{b}}^2)) \) when either \( \bar{a} \geq \bar{b} \) and \( a(x) \notin b(x) \) in \( \Omega \) or \( (a, b) \in \Sigma_1 \), \( (u, v) \to (0, b(x)) \) when \( (a, b) \in \Sigma_2 \) in \( C^1(\Omega) \), where
\[
\Sigma_1 = \left\{ (a(x), b(x))a(x) - b(x) \text{ changes sign in } \Omega \text{ and } \bar{a} < \bar{b} \text{ and } d_1 < \frac{1}{\lambda_1(\bar{a} - \bar{b})} \right\},
\]
\[
\Sigma_2 = \left\{ (a(x), b(x))a(x) - b(x) \text{ changes sign in } \Omega \text{ and } \bar{a} < \bar{b} \text{ and } d_1 = \frac{1}{\lambda_1(\bar{a} - \bar{b})} \right\}.
\]
\end{theorem}
(iii): Suppose that $a(x) > b(x)$. If $d_j \to 0$ $(j = 1, 2)$, then

$$(u, v) \to (a(x) - b(x), 1 + (a(x) - b(x))^2) b(x)) \quad \text{in} \quad C^1(\bar{\Omega}).$$

Proof. We first prove (i). Lemma 6.1 and the usual compactness argument imply that there exist a subsequence of $d_2$, still denote by $d_{2n}$ satisfying $d_{2n} \to 0$ as $n \to \infty$, and a corresponding positive solution $(u_n, v_n)$ of (17) with $d_2 = d_{2n}$, such that

$$u_n \to u_* \quad \text{and} \quad v_n \to v_* \quad \text{uniformly on} \quad \bar{\Omega}, \quad \text{as} \; n \to \infty,$$

where both $u_*$ and $v_* \in C^1(\bar{\Omega})$, $u_* \geq 0$ and $v_* > 0$. Obviously, $(u_n, v_n)$ satisfies (17) with $d_1 = d_{1n}$, that is,

$$\begin{cases}
d_{1n} \Delta u_n + \left( a(x) - u_n - \frac{v_n}{1 + u_n^2} \right) u_n = 0, & x \in \Omega, \\
d_{2n} \Delta v_n + \left( b(x) - \frac{v_n}{1 + u_n^2} \right) v_n = 0, & x \in \Omega, \\
\partial u_n / \partial n = v_n / \partial n = 0, & x \in \partial \Omega.
\end{cases} \quad (30)$$

Letting $n \to \infty$ in the second equation of system (30), we have

$$b(x) - \frac{v_*}{1 + u_*^2} = 0. \quad (31)$$

From the proof of Theorem 6.5, it is easy to see that $u_n \to u_* = u_{\infty}$ as $d_{1n} \to \infty$ and $u_*$ is a nonnegative constant. Hence, (26) is satisfied. Letting $n \to \infty$ in (26), we have

$$\int_{\Omega} \left( a(x) - u_* - \frac{v_*}{1 + u_*^2} \right) dx = 0,$$

which, together with (31), implies that $u_* = \bar{a} - \bar{b}$ and $v_* = b(x)(1 + u_*^2)$. This completes the proof of conclusion (i).

Next, we prove conclusion (ii). It follows from the above discussion that $u_n \to u_*$ as $d_2 \to 0$, where $u_*$ is the unique positive solution of the following equations

$$\begin{cases}
d_1 \Delta u_* = u_* \left( a(x) - u_* - \frac{v_*}{1 + u_*^2} \right), & x \in \Omega, \\
\partial u_* / \partial n = 0, & x \in \partial \Omega.
\end{cases} \quad (32)$$

Substituting (31) into (32), we have

$$\begin{cases}
d_1 \Delta u_* = u_*(a(x) - b(x) - u_*), & x \in \Omega, \\
\partial u_* / \partial n = 0, & x \in \partial \Omega.
\end{cases} \quad (33)$$

Obviously, system (33) has a unique positive solution $\theta_{d_1, a-b}$ if and only if $\mu_1(d_1, a-b) < 0$. Applying Proposition 1 (i), we know that $\mu_1(d_1, a-b) < 0$ for all $d_1 > 0$ when $\bar{a} > \bar{b}$. If $\bar{a} < \bar{b}$ and $a(x) - b(x)$ changes sign in $\Omega$, then by Proposition 1 (ii), we get

$$\mu_1(d_1, a-b) < 0 \quad \text{for all} \; d_1 < \frac{1}{\lambda_1(a-b)}$$

and

$$\mu_1(d_1, a-b) = 0 \quad \text{for all} \; d_1 = \frac{1}{\lambda_1(a-b)}.$$
Hence, system (33) has a unique positive solution $\theta_{d_1,a-b}$ if either $\bar{a} \geq \bar{b}$ and $a(x) \neq b(x)$ in $\Omega$ or $(a, b) \in \Sigma_1$. This implies that $u^\ast = \theta_{d_1,a-b}$. In light of Theorem 5.4, assume that $(a, b) \in \Sigma_2$ so that $\mu_1(d_1, a - b) = 0$ and thus (17) admits positive solutions for all small $d_2$. Obviously, if $(a, b) \in \Sigma_2$, then $u^\ast = 0$. By (31), we obtain $u_\ast = b(x)(1 + u_\ast^2)$. Therefore, we complete the proof of conclusion (ii).

Finally, we consider the case where $d_j \to 0$. It follows from the above arguments that (31) and (32) are satisfied as $d_2 \to 0$. From system (32) with $d_1 \to 0$, we obtain

$$u_\ast(a(x) - b(x) - u_\ast) = 0. \quad (34)$$

Combining with (34) yields either $u_\ast = 0$ or $u_\ast = a(x) - b(x)$. Suppose that $u_\ast \neq a(x) - b(x)$, then by (31), we deduce that

$$a(x) - u_\ast - \frac{v_n}{1 + u_n} \neq 0 \quad \text{in} \quad \Omega$$

when $n$ is sufficiently large. Together with $u_n > 0$ in $\Omega$, we obtain

$$\int_\Omega \left( a(x) - u_\ast - \frac{v_n}{1 + u_n} \right) u_n dx \neq 0 \quad (35)$$

for sufficiently large $n$. On the other hand, integrating the first equation of system (30) over $\Omega$ yields

$$\int_\Omega \left( a(x) - u_\ast - \frac{v_n}{1 + u_n} \right) u_n dx = 0,$$

which contradicts (35). Therefore, $u_\ast = a(x) - b(x)$. Moreover, we have $v_\ast = b(x)(1 + u_\ast^2)$ as the result of (31).

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Received June 2019; revised November 2019.

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