The representation category of the Woronowicz quantum group $S_\mu U(d)$ as a braided tensor $C^*$–category

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Abstract
An abstract characterization of the representation category of the Woronowicz twisted $SU(d)$ group is given, generalizing analogous results known in the classical case.

1 Introduction

In [3] Doplicher and Roberts proved a duality theory for compact groups which allows one to recognize the representation category of a group $G$ among all tensor $C^*$–categories as those endowed with a symmetry of the permutation group, and for which each object has a conjugate.

This paper is part of the program of generalizing the Doplicher–Roberts duality theory to Woronowicz compact quantum groups. As Woronowicz showed in [10], the representation categories in question are precisely the tensor $C^*$–subcategories of Hilbert space categories for which each object has a conjugate in the sense of [7]. Therefore, at this level of generality, we lose any symmetry in the category.

However, motivated by Quantum Field Theory in low dimensions, and with the final aim of a possible application to the physical situation, we are interested in studying tensor $C^*$–categories which have a braided symmetry: a representation of the braid group $B_n$ in the intertwiner spaces $(\rho^n, \rho^n)$ between powers of each object $\rho$ of the category which preserves the tensor structures.

Here, for example, we look at the Woronowicz quantum group $S_\mu U(d)$. Its representation category contains, among its intertwiners, a representation of the braid group: the Jimbo–Woronowicz representation [6], [10]. This is of a very particular form, as the basic intertwiner $g_q$ is selfadjoint with only two eigenvalues: $\mu^2$ and $-1$. This means that the braided symmetry in fact factors through a representation of the Hecke algebra $H_\infty(\mu^2)$.

We show that actually this Hecke symmetry satisfies more: its basic intertwiner $g_q$ can be suitably normalized so that the resulting braided symmetry $\varepsilon$
makes the representation category of \( S_\mu U(d) \) into a \textit{braided tensor category}, in the sense that
\[
\varepsilon(g_1 \ldots g_m) T \otimes 1_H = 1_H \otimes T \varepsilon(g_1 \ldots g_n)
\]
whenever \( T \in (H^{\otimes n}, H^{\otimes m}) \) is an intertwiner in the representation category of \( S_\mu U(d) \) between tensor powers of the defining representation of \( S_\mu U(d) \) on the \( d \)-dimensional Hilbert space \( H \) (Cor. 5.2).

Our interest in braiding is motivated by the fact that the tensor \( C^* \)-categories arising from low dimensional QFT are braided, albeit with a \textit{unitary} braiding.

The aim of this paper is to characterize the representation category of \( S_\mu U(d) \) among all braided tensor \( C^* \)-categories with conjugates (Theorem 6.2).

In section 2 we review the notion of Hecke algebra and the properties we shall need. In particular, we introduce special cases of the Young symmetrizers and antisymmetrizers which have appeared in the literature concerning representations of Hecke algebras (see, e.g., [1]).

In section 3 we study Hecke symmetries in tensor \( C^* \)-categories and we generalize results previously known for permutation symmetries due to Doplicher and Roberts [2]. The main result of this section is the generalization of the theorem about the restriction of the ‘statistics parameter’ values to the values \( \lambda_d := q^d \frac{q^{-1} - 1}{q^{-1} - 1}, \) with \( d \in \mathbb{Z} \). Here we consider both the case where \( q \) is real and the case where \( q \) is a root of unity.

We show that each of these allowed values determines uniquely the kernel of the Hecke algebra representations given by the symmetry: If \( q > 0 \) and \( d \) is negative, then the kernel is the same as the kernel of the Jimbo–Woronowicz representation on \( H \otimes H \) with \( H \) a \( d \)-dimensional Hilbert space. Symmetries where this situation occurs will be then called of dimension \( d \).

In the root of unity case we find out that the given Hecke algebra representation with parameter \( \lambda_d \) has the same kernel as the Wenzl’s representation \( \pi^{(d,m)} \) (see Theorem 3.3 for a precise statement).

Later on we specialize to the case where \( q > 0 \). In section 5 we introduce the basic intertwiner, namely the quantum determinant, \( S \) of \( S_\mu U(d) \), with \( \mu = \sqrt{q} \), and we compute the conjugate representation of the defining representation (Theorem 5.5). For future reference, we perform the computations in a slight more generality, considering also quantum determinants of proper subspaces of \( H \). In doing so we discover that in the case where \( q \neq 1 \) there exist more left inverses of \( H \) which are faithful on the image of the Jimbo–Woronowicz representation than in the classical case (Cor. 5.2). We realize also that these smaller quantum determinants do not satisfy the afore mentioned braid relation.

In section 6 we prove the main result which characterizes the representation category of \( S_\mu U(d) \) among tensor \( C^* \)-categories by means of its Hecke symmetry and the conjugate representation of the fundamental representation.
2 Preliminaries on the Hecke algebra

For any integer \( n \) let \( B_n \) denote the braid group with generators \( g_i, i = 1, \ldots, n-1 \) and relations
\[
g_i g_j = g_j g_i \quad i, j : |i-j| > 1 \quad (2.1)
\]
\[
g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad (2.2)
\]
For any complex number \( q \), let \( H_n(q) \) denote the Hecke algebra of order \( n \), namely the quotient of the complex group algebra \( \mathbb{C}[B_n] \) by the relations
\[
g_i^2 = (q-1)g_i + q, i = 1, \ldots, n-1. \quad (2.3)
\]

We also consider the inductive limit \( H_\infty(q) \) of \( (H_n(q), \iota_{n,m}) \), where, for \( m > n \), \( \iota_{n,m} \) takes the element \( g_i \in H_n(q) \) to the element \( g_i \in H_m(q) \). Therefore the map \( g_i \to g_{i+1} \) extends uniquely to a monomorphism \( \sigma \) of \( H_\infty(q) \) into itself.

Let us introduce, for \( q \neq -1 \), the elements \( e_i = \frac{1 + q}{q+1} \). Then (2.3) is equivalent to the fact that \( e_i \) is an idempotent: \( e_i^2 = e_i \). Therefore \( g_i \) is a linear combination of two mutually orthogonal idempotents:
\[
g_i = qe_i - (1 - e_i).
\]

We define, for \( q \neq 0 \), special elements \( A_k, k \in \mathbb{Z} - \{0\} \), of \( H_\infty(q) \):
\[
A_1 = A_{-1} = 1,
\]
\[
A_{n+1} = \sigma(A_n) + g_1 \sigma(A_n) + \cdots + g_n \ldots g_1 \sigma(A_n),
\]
\[
A_{-n-1} = \sigma(A_{-n}) - q^{-1} g_1 \sigma(A_n) + \cdots + (-q^{-1})^n g_n \ldots g_1 \sigma(A_{-n}).
\]

Notice that both \( A_2 = (1 + q)e_1 \) and \( A_{-2} = (1 + q^{-1})(1 - e_1) \). These elements are special cases of symmetrization and antisymmetrization operators \( \mathbb{H} \). Notice also that \( A_3 \) is a remarkable element of \( H_n(q) \). Indeed, \( A_3 = 1 + g_1 + g_2 + g_1 g_2 + g_2 g_1 + g_2 g_1 g_2 \), and it is known that the quotient of \( H_n(q) \) by the ideal generated by \( A_3 \) is the Temperley-Lieb algebra \( TL_n((2 + q + q^{-1})^{-1}) \) (see, e.g., \( \mathbb{H} \)).

We next show that for \( q \) generic, the \( A_k \)'s are nonzero scalar multiples of idempotents. We shall need the quantum factorials: for \( n \in \mathbb{N} \),
\[
1!_q := 1
\]
\[
n!_q := (1 + q + \cdots + q^{n-1})(n-1)!_q.
\]

2.1 Lemma For \( q \neq 0, -1 \) and for all \( n \geq 2 \) and \( j = 1, \ldots, n-1 \), and \( k = 0, \ldots, n-1 \),
a) \( g_j A_n = A_n g_j = q A_n \),
b) \( g_j A_{-n} = A_{-n} g_j = -A_{-n} \).
c) \( \sigma^k(A_{n-k})A_n = (n-k)!qA_n \), \( \sigma^k(A_{-(n-k)})A_{-n} = (n-k)!_{1/q}A_{-n} \). In particular, \( A_n^2 = n!qA_n \), \( A_{-n}^2 = n!_{1/q}A_{-n} \).

d) If \( q^n = 1 \), \( A_n \) and \( A_{-n} \) are central nilpotent elements of \( H_n(q) \), and therefore they belong to the radical of that algebra.

e) If \( q^p \neq 1 \) for \( p = 2, \ldots, n \), \( E_n := \frac{1}{n!q}A_n \) and \( E_{-n} := \frac{1}{n!_{1/q}}A_{-n} \) are minimal and central idempotents of \( H_n(q) \).

**Proof** Notice that \( h_i := -q^{-1}g_i \) satisfy the presentation relations for \( H_\infty(q^{-1}) \). Since the \( A_{-n} \) corresponds to the \( A_n \) in the algebra \( H_\infty(q^{-1}) \) generated by the \( h_i \)'s, b) and the second relation in c) will follow from a) and the first relation in c), respectively. a) We prove that \( h_iA_n = qA_n \) by induction on \( n \). For \( n = 2 \), \( A_2 = 1 + g_1 = (q + 1)e_1 \) and \( g_1A_2 = g_1 + (g_1)^2 = qg_1 + q = qA_2 \). On the other hand for \( i \geq 1 \),

\[
g_1g_{i+1}g_i \cdots g_1 = g_{i+1}g_i \cdots g_1g_2g_1 = g_{i+1} \cdots g_2g_1g_2.
\]

So, assuming that the claim holds for \( n \),

\[
g_1g_{i+1} \cdots g_1\sigma(A_n) = g_{i+1} \cdots g_1\sigma(g_1A_n) = qg_{i+1} \cdots g_1\sigma(A_n),
\]

thus

\[
g_1A_{n+1} = g_1\sigma(A_n) + (g_1)^2\sigma(A_n) + q \sum_{i=2}^{n} g_i \cdots g_1\sigma(A_n) = qA_{n+1},
\]

and the claim follows for \( n + 1 \) and \( j = 1 \). Similar computations show that the claim holds also for \( j = 2, \ldots, n \). We now show that \( A_n g_j = qA_n \) as well, for \( j = 1, \ldots, n - 1 \). First consider, for \( n \in \mathbb{N} \), the element \( B_n \) defined by: \( B_1 = 1 \),

\[
B_{n+1} = \sigma(B_n) + \sigma(B_n)g_1 + \cdots + \sigma(B_n)g_1 \cdots g_n.
\]

Similar arguments show that \( B_n g_j = qB_n \) for \( j = 1, \ldots, n - 1 \). We claim that \( B_n = A_n \) for all \( n \). It suffices to assume \( n \geq 1 \). In fact, \( A_1 = B_1 = 1 \). Assume inductively that \( A_k = B_k \) for \( k = 1, \ldots, n \). Then computing \( A_{n+1} \) by means of \( A_n = B_n \) and \( B_{n-1} \) gives

\[
A_{n+1} = \sigma(A_n) + \sum_{i=1}^{n} g_i \cdots g_1\sigma(A_n) =
\]

\[
\sigma(A_n) + \sum_{i=1}^{n} g_i \cdots g_1\sigma(B_n) =
\]

\[
\sigma(A_n) + \sum_{i=1}^{n} g_i \cdots g_1\sigma^2(B_{n-1}) + \sum_{i=1}^{n} \sum_{h=1}^{n-1} g_i \cdots g_1\sigma^2(B_{n-1})\sigma(g_1 \cdots g_h) =
\]
\[ \sigma(A_n) + \sum_{i=2}^{n} \sigma(g_{i-1} \ldots g_1 \sigma(B_{n-1})g_1 + \sigma^2(B_{n-1})g_1 + \sum_{i=2}^{n-1} \sum_{h=1}^{i-1} \sigma(g_{i-1} \ldots g_1 \sigma(B_{n-1})g_1 \ldots g_{h+1} + \sum_{h=1}^{n-1} \sigma^2(B_{n-1})g_1 \ldots g_{h+1} = \sigma(B_n) + \sigma(B_n)g_1 + \sum_{h=1}^{n-1} \sigma(B_n)g_1 \ldots g_{h+1} = B_{n+1}. \]

c) We fix \( n \). \( A_2 = 1 + g_1, \sigma^{-1}(A_2)A_{n+1} = (1+q)A_{n+1} \) by a). Taking into account the definition of the \( A_j \)'s, one obtains iteratively that \( \sigma^k(A_{n+1-k})A_{n+1} = (n + 1 - k)!qA_{n+1} \) for \( k = n, \ldots, 0 \). d) By c), \( A_n^2 = A_{-n}^2 = 0 \), so \( A_n \) and \( A_{-n} \) are nilpotent in \( H_\infty(q) \). On the other hand, they are central in \( H_n(q) \), therefore they are properly nilpotent elements of \( H_n(q) \). e) It is clear that \( E_n \) and \( E_{-n} \) are central idempotents of \( H_n(q) \). a) and b) show that for all \( X \in H_n(q) \), \( E_nX \) is a scalar multiple of \( E_n \), so \( E_n \) is a minimal idempotent.

We are (eventually) interested in representing Hecke algebras in tensor \( C^* \)-categories. Therefore we look for \( * \)-involutions on \( H_\infty(q) \). If \( q \) is real, a natural involution making \( H_\infty(q) \) a \( * \)-algebra is that one for which \( g_i \)'s become selfadjoint elements, while, if \( |q| = 1 \), one would like to have \( g_i^* = g_i^{-1} \). In both cases the \( e_i \) become selfadjoint idempotents.

It is known that there is indeed a unique \( * \)-involution on \( H_\infty(q) \) making the \( e_i \)'s selfadjoint idempotents if and only if \( q \) is real or \( |q| = 1 \). We shall refer to this involution as the standard involution.

**2.2 Corollary** Assume that \( q \neq 0, q \neq -1 \). For \( q \) real or \( |q| = 1 \), let us endow \( H_\infty(q) \) with its standard involution.

a) If \( q^n = 1, A_n^*A_n = 0 \). In particular, \( A_n \) lies in the kernel of any Hilbert space \( * \)-representation of \( H_n(q) \).

b) If \( q^p \neq 1 \) for \( p = 2, \ldots, n \), the idempotents \( E_n \) and \( E_{-n} \) defined in the previous lemma are selfadjoint.

**Proof** Since the \( e_i \) are selfadjoint, a straightforward computation shows that \( g_j^* = \frac{q+1}{q+1}g_j + \frac{q}{q+1} \). Thus by a) in the previous lemma, \( g_j^*A_n = qA_n \) for \( j = 1, \ldots, n - 1 \). It follows that \( A_n^*A_n = (1 + q + \cdots + (q)^{n-1})\sigma(A_n^*A_n) \). If \( q^n = 1, A_n^*A_n = 0 \), and this shows a). b) If instead \( q^p \neq 1, p = 2, \ldots, n \), \( A_n^*A_n = n!qA_n = |n!q|^2E_n \), which shows that \( E_n \) is selfadjoint. Replacing the \( g_j \)'s by the \( -q^{-1}g_j \)'s and \( q \) by \( q^{-1} \), amounts to showing that \( E_{-n} \) is selfadjoint as well.

For \( q \neq 0, q \neq -1, q^p \neq 1, p = 2, \ldots, n, H_n(q) \) is a semisimple algebra \( \mathbb{F} \): as for the the symmetric group on \( n \) symbols, its irreducibles \( \{ \pi_\lambda \} \) are labeled by Young diagrams \( \lambda = [\lambda_1, \ldots, \lambda_k] \) with \( n \) squares. For the explicit formula of the representation \( \pi_\lambda \) we shall apply formula (2.3) in \( \mathbb{F} \) to our idempotents.
2 PRELIMINARIES ON THE HECKE ALGEBRA

e_i's. Notice, however, that we adopt a different convention than in \[9\]: our idempotents e_i correspond to Wenzl's 1−ei's.

Let us assume q real, q ≠ −1, or |q| = 1, and let us endow \(H_\infty(q)\) with its standard *−involution. A *−representation of \(H_n(q)\), with \(n\) possibly infinite, on a Hilbert space is called trivial if it is the direct sum of the one dimensional representations: \(\pi_0(e_i) = 0, \pi_1(e_i) = 1\) for all i. \(H_n(q)\) admits a non trivial Hilbert space *−representation for \(n\) arbitrarily large only if \(q \geq 0\) or \(q = e^{2πi}\), for some \(m \in \mathbb{Z}\) with \(|m| = 3, 4, \ldots\). For \(q > 0\) the above representations \(\pi_i\) are *−representations.

For \(q = e^{2πi}\), with \(m \in \mathbb{Z}\), \(|m| \geq 4\), and for any positive integer \(k \leq m − 1\), Wenzl defines a semisimple, irreducible *−representation \(\pi_\lambda^{(k,m)}\) of \(H_n(q)\) associated with every \((k,m)\)-diagram \(\lambda = [\lambda_1, \ldots, \lambda_k]\) (a Young diagram with \(n\) squares, at most \(k\) rows and such that \(\sum \lambda_j \leq m − k\)). Let \(\Lambda_n^{(m)}\) denote the collection of all Young \((k,m)\)-diagrams with \(n\) squares, for some \(k \leq m − 1\).

The semisimple representation \(\pi_n^{(m)} = \bigoplus_{\lambda \in \Lambda_n^{(m)}} \pi_\lambda\) of \(H_n(q)\) is, in general, not faithful.

2.3 Lemma Assume that \(q \neq 0, −1, q^p \neq 1, p = 2, \ldots, k\). For any positive integer \(k\), \(E_k\) (resp. \(E_{−k}\)) is the minimal central idempotent of \(H_k(q)\) corresponding to the Young diagram \([1^k]\) (resp. \([k]\)).

Proof By Lemma 2.1 e) \(E_k\) is a minimal central idempotent of \(H_k(q)\). It defines a one dimensional representation \(\chi\) of \(H_k(q)\) by \(X E_k = \chi(X) E_k\) and such that \(\chi(q_i) = q\), or, equivalently, \(\chi(e_i) = 1, i \leq k − 1\). We are left to show that \(\pi_{[1^k]}(e_i) = 1\), and this follows from formula (2.3) in \[3\].

For generic values of \(q\), \(H_n(q)\) is a semisimple algebra. Its irreducibles are labeled by Young diagrams with \(n\) squares. We have the problem of determining the Young diagrams corresponding to those irreducibles whose central supports sum up to the central support of the ideal of \(H_n(q)\) generated by \(E_k\), for \(k \leq n\). Notice that, for \(k = 3\), the quotient by that ideal is precisely the Temperley–Lieb algebra \(TL_n(q + q^{−1} + 2)^{−1}\), which is known to correspond, in the above sense, to the set of diagrams with at most 2 rows, see \[?\].

2.4 Proposition If \(q \neq 0, q \neq −1, q^p \neq 1, p = 2, \ldots, n\), for any positive integer \(k \leq n\), the ideal of \(H_n(q)\) generated by the idempotent \(E_k\) (resp. \(E_{−k}\)) corresponds to the set of Young diagrams with at least \(k\) rows (resp. columns).

Proof Let \(p_\lambda\) denote a minimal central idempotent of \(H_n(q)\) corresponding to the Young diagram \(\lambda\). By the previous lemma \(E_k\) is the minimal central idempotent of \(H_n(q)\) corresponding to the Young diagram \([1^k]\). In order to decide whether or not \(p_\lambda E_k \neq 0\) it is necessary and sufficient that \(\lambda > [1^k]\) (see formula (2.6) in \[3\]) i.e. that \(\lambda\) contain at least \(k\) rows. One similarly shows the remaining part.
3 Hecke symmetries in tensor $C^*$-categories

Let us introduce the braid category $\mathbb{B}$ with set of objects the non negative integers and arrows
\[(n, m) = 0, \quad \text{if } n \neq m,\]
\[(0, 0) = (1, 1) = \mathbb{C},\]
\[(n, n) = \mathbb{C}[\mathbb{B}_n], \quad n \geq 2,\]
the complex group algebra of the braid group $\mathbb{B}_n$ on $n-1$ generators: $g_1, \ldots, g_{n-1}$.

with their natural structure of complex vector spaces, and composition of arrows
arising from the algebra structure of $(n, n)$. $\mathbb{B}$ becomes a strict tensor category if we define the tensor product between objects as $n \otimes m = n + m$ and between arrows as
\[S \otimes T = S\sigma^n(T), \quad S \in (n, n), T \in (m, m),\]
where $\sigma$ denotes, as before, the monomorphism $\mathbb{B}_m \to \mathbb{B}_{m+1}$ taking $g_i$ to $g_{i+1}$ for $i = 1, \ldots, m - 1$.

Let us consider a tensor category $\mathcal{T}$ with objects the tensor powers of a single object $\rho$, with tensor identity object given by $\iota$.

A braided symmetry for $\rho$ will be given by a tensor functor
\[\varepsilon : \mathbb{B} \to \mathcal{T}\]
such that $\varepsilon(0) = \iota$ and $\varepsilon(1) = \rho$.

More explicitly, we shall need group representations $\varepsilon_n : \mathbb{B}_n \to (\rho^n, \rho^n)$ such that, for $b \in \mathbb{B}_n$,
\[\varepsilon_{n+1}(b) = \varepsilon_n(b) \otimes 1_\rho,\]
\[\varepsilon_{n+1}(\sigma(b)) = 1_\rho \otimes \varepsilon_n(b).\]

It is convenient to select the subcategory $\mathcal{T}^\varepsilon$ of $\mathcal{T}$ with the same objects as $\mathcal{T}$ and arrows
\[(\rho^n, \rho^m)^\varepsilon = \{ T \in (\rho^n, \rho^m) : \varepsilon(g_1 \ldots g_m)T \otimes 1_\rho = 1_\rho \otimes T\varepsilon(g_1 \ldots g_n) \}. \quad (3.1)\]

$\mathcal{T}$ is called a braided tensor category if $\mathcal{T}^\varepsilon = \mathcal{T}$.

In a similar way, one can define the Hecke category $H(q)$ with objects the non negative integers and arrows $(n, m) = 0$ for $n \neq m$ and $(n, n) = H_n(q)$.

A braided symmetry $\varepsilon : \mathbb{B} \to \mathcal{T}$ will be called a Hecke symmetry if it factors through a functor $\varepsilon : H(q) \to \mathcal{T}$ from the Hecke category.

If $\mathcal{T}$ equals $\mathbb{B}$ (resp. $H(q)$), the identity functor does define a braided symmetry (resp. a Hecke symmetry) for the object 1 making it into a braided tensor category. The only nontrivial relation that we need to check is (3.1), namely that
\[g_1 \ldots g_n b = \sigma(b)g_1 \ldots g_n, \quad b \in \mathbb{B}_n,\]
which follow from the presentation relations (2.1)-(2.2) of the braid group.
Let us assume that $H(q)$ admits a $^*$-involution making it into a tensor $^*$-category. In this case, a $^*$-preserving Hecke symmetry into another tensor $^*$-category $\mathcal{T}$ will be called a $^*$-symmetry.

As an example, if $q$ is real or if $|q| = 1$, $H(q)$ becomes naturally a tensor $^*$-category with the involution inherited from the standard involution of $H_n(q)$.

Let $\Phi$ be a left inverse of $\rho$: a set $\{\Phi_n, n \in \mathbb{N}\}$ of $(i, i)$-linear mappings

$$\Phi_n : (\rho^n, \rho^n) \rightarrow (\rho^{n-1}, \rho^{n-1})$$

(with $\rho^0 = i$) preserving right tensoring by $1_\rho$ and satisfying for $S \in (\rho^n, \rho^n)$, $T \in (\rho^{n-1}, \rho^{n-1})$,

$$\Phi_n(S_1 \rho \otimes T) = \Phi_n(ST),$$

$$\Phi_n(1_\rho \otimes TS) = T\Phi_n(S).$$

If $\mathcal{T}$ is a tensor $C^*$-category, $\Phi$ will be called positive if each $\Phi_n$ is positive. The bimodule property then implies that $\Phi_n$ is completely positive.

**3.1 Lemma** Let $\varepsilon$ be a braided symmetry for an object $\rho$ of a tensor $C^*$-category $\mathcal{T}$, and let $\Phi$ be a positive left inverse of $\rho$ such that $\Phi(\varepsilon(g_1))$ is invertible. If $T \in (\rho^n, \rho^n)^*$ satisfies $\Phi_n(T^*T) = 0$ then $T = 0$.

**Proof** Set $\varepsilon_n := \varepsilon(g_1 \ldots g_n)$. Since $\varepsilon_n T \otimes 1_\rho = 1_\rho \otimes T \varepsilon_n$, $T^* \otimes 1_\rho \varepsilon_m \varepsilon_n T \otimes 1_\rho = \varepsilon_n 1_\rho \otimes T^*T \varepsilon_n$. Assume that $\Phi_n(T^*T) = 0$. Then the left hand side of the above equation is in the kernel of $\Phi_{n+1}$ by positivity. Writing $\varepsilon_n = \varepsilon(g_1) \otimes 1_{\rho^n-1} 1_\rho \otimes \varepsilon_{n-1}$ shows that $\varepsilon(g_1)^* \otimes 1_{\rho^n-1} 1_\rho \otimes T^*T \varepsilon(g_1) \otimes 1_{\rho^n-1}$ lies in the kernel of $\Phi_{n+1}$. Complete positivity implies $\Phi_{n+1}(A^*A) \geq \Phi_{n+1}(A)^*\Phi_{n+1}(A)$, which, in turn, implies $0 = \Phi_{n+1}(1_\rho \otimes T \varepsilon(g_1) \otimes 1_{\rho^n-1}) = T\Phi_2(\varepsilon(g_1)) \otimes 1_{\rho^n-1}$, and the conclusion follows.

For a given Hecke $^*$-symmetry $\varepsilon : H(q) \rightarrow \mathcal{T}_\rho$ we have the problem of determining the kernels of the corresponding $^*$-homomorphisms $H_n(q) \rightarrow (\rho^n, \rho^n)$. If $\rho$ has a positive left inverse $\Phi$ such that $\Phi(\varepsilon(g_1))$ is an invertible element in $(i, i)$, we show that this element determines the kernels of $\varepsilon$ uniquely.

**3.2 Corollary** Let $H(q)$ be endowed with a $^*$-involution making it into a tensor $^*$-category, and let $\varepsilon : H(q) \rightarrow \mathcal{T}_\rho$ be a Hecke $^*$-symmetry into a tensor $C^*$-category. If there is a positive left inverse $\Phi$ for $\rho$ such that $\Phi(\varepsilon(g_1))$ is an invertible element in $(i, i)$ then, for all $n \in \mathbb{N}$, the kernel of the $^*$-representation $\varepsilon : H_n(q) \rightarrow (\rho^n, \rho^n)$ depends only on $\Phi_2(\varepsilon(g_1))$.

**Proof** Let us consider the quotient map $\pi_q : \mathbb{C}[\mathbb{B}_n] \rightarrow H_n(q)$. Notice that $\pi_1 : \mathbb{C}[\mathbb{B}_n] \rightarrow \mathbb{C}[\mathbb{F}_n]$ associates to a braid on $n$ threads a corresponding permutation of $(1, \ldots, n)$. There is a natural section $s_q$ of $\pi_q$ (i.e. a linear map $s_q : H_n(q) \rightarrow \mathbb{C}[\mathbb{B}_n]$ such that $\pi_q \circ s_q$ is the identity map) constructed in the following way. First we consider the subset $B_n$ of $\mathbb{B}_n$ defined by: $B_1 = \{1\}$,

$$B_{n+1} = \sigma(B_n) \cup g_1\sigma(B_n) \cup \cdots \cup g_n \cdots g_1\sigma(B_n).$$

Set $H_n = \pi_q(B_n)$, for all $n$. $H_n$ is a linear basis of $H_n(q)$, therefore the inverse map $s_q$ of the restriction of $\pi_q$ to $\pi_q : B_n \rightarrow H_n$ extends uniquely to the desired
linear section $s_q$. If $h \in H_n \subset H_n(q)$ is of the form $h = \sigma(h')$, with $h' \in H_{n-1}$ then $\Phi(\varepsilon(h)) = \varepsilon(h')$. If instead $h = g_1 \ldots g_{2\ell} \sigma(h'')$ with $h'' \in H_{n-1}$, then $\Phi(\varepsilon(h)) = \lambda \varepsilon(h')$, where $h' = g_1 \ldots g_{2\ell} h''$. In both cases, $h'$ is the result of the image under $\pi_n \circ s_1$ of that permutation of $F_{n-1}$ obtained from $\pi_1 s_q(h)$ by deleting 1 from its cycle in its decomposition into disjoint cycles, and then writing $n-1$ in place of $n$. For all $n$, if $h \in H_n$, $\Phi_2 \circ \cdots \circ \Phi_n(\varepsilon(h))$ is a power of $\lambda := \Phi(\varepsilon(g_1))$, therefore we get a $(i,i)$-valued, positive linear map $\omega_\lambda$ on $H_\infty(q)$, which, by the above, can be computed explicitly: $\omega_\lambda(h)$ is the product of factors of the form $\lambda^{k-1}$ for each cycle on length $k$ in the decomposition of $\pi_1 s_q(h)$. Now notice that $\varepsilon(H_n(q)) \subset (\rho^n, \rho^n)^e$, so, if $\lambda$ is invertible, by the previous lemma, an element $h \in H_n(q)$ is in the kernel of $\varepsilon$ if and only if $\Phi_n(\varepsilon(h^* h)) = 0$. On the other hand, $\Phi_n(\varepsilon(H_n(q)))$ is contained in the algebra generated by $\varepsilon(H_{n-1}(q))$ and $(i,i)$, which is in turn contained in $(\rho^n-1, \rho^n-1)^e$, therefore $\Phi_n(\varepsilon(h^* h)) = 0$ is equivalent to $\Phi_{n-1}(\Phi_n(\varepsilon(h^* h))) = 0$. Iterating the left inverse $n$ times we get that $\varepsilon(h) = 0$ if and only if $\omega_\lambda(h^* h) = 0$.

Recall that if $g_i$ generate $H_\infty(q)$, $h_i = -\frac{a_i}{q}$ generate $H_\infty(\frac{a_i}{q})$. We thus have an invertible tensor $\ast$-functor $\phi : H(\frac{a_i}{q}) \to H(q)$ such that $\phi(h_i) = -\frac{a_i}{q}$. If $\varepsilon$ is a $\ast$-symmetry of $H(q)$ in $\mathcal{T}$, for $q \geq 1$ or $q = e^{2\pi i/m}$, $m = 3, 4, \ldots$, then $\varepsilon \phi$ is a $\ast$-symmetry of $H(\frac{a_i}{q})$ in the same category, and any such $\ast$-symmetry arises in this way. Therefore we can restrict the values of $q$ to $[1, +\infty) \cup \{e^{2\pi i/m}, m = 3, 4, \ldots\}$.

3.3 Theorem Assume that $q \in [1, +\infty) \cup \{e^{2\pi i/m}, m = 3, 4, \ldots\}$. Let us endow $H(q)$ with its standard $\ast$-involution: $e_i^* = e_i$, for all $i$. Let $\varepsilon : (H(q), \ast) \to \mathcal{T}$ be a Hecke $\ast$-symmetry for an object $\rho$ of a tensor $C^\ast$-category, and let $\Phi$ be a unital, positive left inverse of $\rho$ for which $\Phi(\varepsilon(g_i)) = : \lambda$ is a complex number. Set, for $d \in \mathbb{Z}$, $d \neq 0$, $\lambda_d := \frac{\varepsilon(q-1)}{d-1}$. Then $\lambda$ satisfies one of the following conditions:

a) if $q \geq 1$, then either $0 \leq \lambda \leq q-1$ or $\lambda = \lambda_d$ for some $d \in \mathbb{Z}$, $d \neq 0$. If $\lambda \in [0, q-1]$, all the maps $\varepsilon : H_n(q) \to (\rho^n, \rho^n)$ are monomorphisms, whereas if $\lambda = \lambda_d$, the kernel of the homomorphism $H_n(q) \to (\rho^n, \rho^n)$ is the ideal generated by $E_{-d} - \frac{d}{d-1}$.

b) if $q = e^{2\pi i/m}$ then $\lambda = \lambda_d$ for some $d \in \mathbb{Z}$, $|d| \leq m-1$, $d \neq 0$. The kernel of $\varepsilon : H_n(q) \to (\rho^n, \rho^n)$ coincides, for $d$ negative, with the kernel of Wenzl’s representation $\pi_n^{(-d,m)}$, and, for $d$ positive with the kernel of $\pi_n^{(d,m)} \circ \alpha$, with $\alpha$ the automorphism of $H_n(q)$ defined by $\alpha(g_i) = q-1-g_i$.

Proof In the proof of corollary 2.2 we have seen that $A_{n+1}^* A_{n+1} = (n+1)! q A_{n+1}$ and $A_{-n-1}^* A_{-(n+1)} = (n+1)! q A_{-(n+1)}$, so

$$
\Phi(\varepsilon(A_{n+1}^* A_{n+1})) = (n+1)! q \Phi(\varepsilon(A_{n+1})) = (n+1)! q (A_n + \lambda A_n + \lambda g_1 A_n + \cdots + \lambda g_{n-1} \cdots g_1 A_n) = (n+1)! q [1 + \lambda (1 + q + \cdots + q^{n-1})] \varepsilon(A_n) =
$$
\[1 + \hat{q} + \cdots + \hat{q}^n\] \[1 + \lambda + q + \cdots + q^{n-1}\] \varepsilon(A_n^*A_n).

Similarly,
\[\Phi(\varepsilon(A_{-(n+1)}^*A_{-(n+1)})) =\]
\[1 + (\hat{q})^{-1} + \cdots + (\hat{q})^{-n}\] \[1 - q^{-1}\lambda + q^{-1} + \cdots + (q^{-1})^{n-1}\] \varepsilon(A_{-n}^*A_{-n}).

Assume \(q \geq 1\). If \(\Phi(\varepsilon(A_k^*A_k)) \neq 0\) for all integers \(k\), positivity of \(\Phi\) implies \(\lambda \in [0, q - 1]\). If, on the contrary, \(d\) is the integer with smallest absolute value for which \(\Phi(\varepsilon(A_k^*A_k)) = 0\), then necessarily \(|d| \geq 2\). If \(d\) is negative, \(\lambda = \lambda_{-d-1}\), whereas if \(d\) is positive, \(\lambda = \lambda_{-d+1}\). In the case where \(q = e^{2\pi i/m}\), \(A_{m}^*A_{m} = 0 = A_{-m}^*A_{-m}\). Defining \(d\) still as above, now implies \(|d| \leq m\) and \(d\) still needs to assume the stated values.

Let \(\omega_{\lambda}\) be defined as in the proof of the previous corollary. For \(q \geq 1\) and \(\lambda \in [0, q - 1]\), the previous computations show that \(\omega_{\lambda}(A_k^*A_k) > 0\) for all \(k\), so \(\varepsilon(A_k^*A_k) \neq 0\), and this shows that \(\varepsilon\) is faithful. It is also evident that if \(\lambda = \lambda_{d}\) then \(E_{-d-\varepsilon}h\), and therefore the ideal it generates in \(H_n(q)\), lies in the kernel of \(\varepsilon\). Taking into account the previous corollary, the proof of a) will be complete if we produce for each \(d \in \mathbb{Z}\), \(d \neq 0\), a model \(^*\)–symmetry \(\varepsilon\) of \(H_\infty(q)\), with left inverse \(\Phi\) for which \(\Phi(\varepsilon(g_j)) = \lambda_{-d}\) and having the kernel as stated. This is the content of the next section. In case b), let us assume that \(\Phi(\varepsilon(g_j)) = \lambda_{-d}\) for an appropriate positive \(d\). We have seen that an element \(h \in H_\infty(q)\) is annihilated by \(\varepsilon\) if and only if \(\omega_{\lambda_{-d}}(h^*_h) = 0\). Now \(\omega_{\lambda_{d}}\) is a positive Markov trace on \(H_\infty(q)\) factoring through the \(C^*\)–representation \(\varepsilon\). This condition says that \(h\) lies in the kernel of the GNS representation associated to this trace. By Theorem 3.6 in [11], the latter is equivalent to \(\pi_n^{(d,m)}\).

## 4 The Model Hecke \(^*\)–symmetry

Notice that if \(g_i, i = 1, 2, \ldots\), are generators of \(H_\infty(q)\), then the elements \(l_i := q - 1 - g_i, i = 1, 2, \ldots\) still satisfy the presentation relations of \(H_\infty(q)\). In other words, we have an invertible tensor endofunctor \(\alpha\) of \(H(q)\) which associates \(l_i\) to \(g_i\).

With every Hecke symmetry \(g_i \to \varepsilon(g_i)\) of \(H(q)\) in a tensor category we can associate another Hecke symmetry of \(H(q)\) in the same category by
\[\varepsilon' = \varepsilon \circ \alpha.\]

We shall refer to \(\varepsilon'\) as the dual symmetry.

If we endow \(H(q)\) with its standard involution for appropriate values of \(q\), \(\alpha\) becomes a \(^*\)–functor. Obviously, if \(\varepsilon\) is a \(^*\)–symmetry with respect to the standard involution, \(\varepsilon'\) is a \(^*\)–symmetry, as well.

Let \(H\) be a finite dimensional vector space of dimension \(d\). We consider the representation of the Hecke algebra \(H_\infty(q)\) on \(H^\otimes n\) discovered by Jimbo [6] and Woronowicz [10]. Let \(g_i\) denote the operator on \(H \otimes H\):
\[g_i \psi_i \otimes \psi_j = -\mu \psi_j \otimes \psi_i, \quad i < j,\]
$g_q \psi_i \otimes \psi_i = -\psi_i \otimes \psi_i$

$g_q \psi_i \otimes \psi_j = (q - 1) \psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i, \quad i > j,$

with $\psi_1, \ldots, \psi_d$ a basis of $H$ and $\mu$ a fixed complex square root of $q$. One can check that, on the vector space $H^\otimes n$, the operators $g_1 = g_q \otimes 1_{H^\otimes n-2}$, $g_2 = 1_H \otimes g_q \otimes 1_{H^\otimes n-3}$, ..., $g_{n-1} = 1_{H^\otimes n-2} \otimes g_q$ do satisfy all the relations (2.1)–(2.3), and therefore we obtain in this way a model Hecke symmetry $\varepsilon : H(q) \to \mathcal{L}_H$ with $\mathcal{L}_H$ the tensor category of vector spaces with objects the tensor powers of $H$.

One can compute the images of the idempotents $e_i = \frac{1 + g_q}{q+1}$ by taking the translates of $e_q := \frac{1+g_q}{q+1}$,

$e_q \psi_i \otimes \psi_j = \frac{1}{q+1} (\psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i), \quad i < j$

$e_q \psi_i \otimes \psi_i = 0$

$e_q \psi_i \otimes \psi_j = \frac{1}{q+1} (q \psi_i \otimes \psi_j - \mu \psi_j \otimes \psi_i), \quad i > j.$

Let us consider first the special case where $d = 2$. It is then known, and easy to check, that the corresponding representation of $H_n(q)$ is not faithful for $n > 2$: it factors through a representation of the Temperley–Lieb algebra $\mathcal{TL}_n((2+q+q^{-1})^{-1})$, which is precisely the quotient of the Hecke algebra $H_n(q)$ by the ideal generated by the idempotent $E_3$. This fact generalizes to higher dimensions.

In general, we want to identify the kernels of the homomorphisms $\varepsilon : H_n(q) \to (H^\otimes n, H^\otimes n)$.

For values of $q$ for which the algebras $H_n(q)$ are semisimple, those kernels are determined by specifying the quasiequivalence class of the above representation of $H_n(q)$ on $H^\otimes n$.

4.1 Proposition If $q \neq 0$, $q \neq -1$, and $q^p \neq 1$ for $p = 2, \ldots, n$, the quasiequivalence class of $\varepsilon : H_n(q) \to B(H^\otimes n)$, with $d \leq n - 1$, corresponds to the set of Young diagrams with $n$ squares and at most $d$ rows. Alternatively, it coincides with the ideal of $H_n(q)$ generated by $E_{d+1}$. If $d \geq n$, $\varepsilon$ is faithful.

Proof The first assertion is shown in subsection 2.1 of [3]. Notice that our model representation of $H_n(q)$ defined by $g_q$ is equivalent to the representation defined by the operator $1 - q - R$, where $R$ is defined in section 2 of [3]. Also, our irreducible representation $\pi_\lambda$ associated to the Young diagram $\lambda$ corresponds to their $V_{\lambda'}$, where $\lambda'$ is the diagram obtained from $\lambda$ exchanging rows with columns.
In particular, if $d \geq n$, any Young diagram with $n$ boxes gives rise to a subrepresentation of $\varepsilon$, so $\varepsilon$ is faithful.

Therefore the kernel of $H_n(q) \to (H^\otimes n, H^\otimes n)$ corresponds to the Young diagrams with $\geq d + 1$ rows. The conclusion follows from Prop 2.4.

It follows that the dimension of $H$ is uniquely determined by the quasi-equivalence class of the model symmetry $\varepsilon$.

An easy inductive argument shows that, for $k \in \mathbb{N}$, $k \neq 0$, $\alpha(A_k) = q^{k(k-1)/2}A_k$, (see, e.g., [8]). Therefore, if we have a Hecke symmetry of $H(q)$ in a tensor category with kernel the ideal generated by $E_{d+1}$, passing to the dual symmetry will give a Hecke symmetry with kernel the ideal generated by $E_{-d-1}$.

In analogy with the case of a permutation symmetry treated in [2], we introduce the concept of the integral dimension of a Hecke symmetry.

**4.2 Definition** Assume that $q \neq 0, q \neq -1, q^n \neq 1$ for all $n \geq 2$. Let $d$ be any integer with $|d| \geq 2$. We shall say that a Hecke symmetry $\varepsilon : H(q) \to \mathcal{T}_p$ has dimension $d$ if, for all $n$, the kernel of the homomorphism $H_n(q) \to (\rho^n, \rho^n)$ is the ideal generated by $E_{d+|d|}$.

In particular, for generic values of $q$, a 2-dimensional Hecke symmetry is just a symmetry factoring over the Temperley–Lieb algebras $\mathcal{T}_n((2 + q + q^{-1})^{-1})$ and faithful on them.

Assume that $H$ is a Hilbert space and that $\psi_1, \ldots, \psi_d$ is an orthonormal basis. The Hilbert space adjoint of $e_q$ is given by

$$e_q^* \psi_i \otimes \psi_j = \frac{1}{q + 1} (\psi_i \otimes \psi_j - \bar{\mu} \psi_j \otimes \psi_i), \quad i < j$$

$$e_q^* \psi_i \otimes \psi_i = 0$$

$$e_q^* \psi_i \otimes \psi_j = \frac{1}{q + 1} (\bar{\mu} \psi_i \otimes \psi_j - \bar{\mu} \psi_j \otimes \psi_i), \quad i > j.$$  

These computations show the following fact.

**4.3 Proposition** The idempotent $e_q$ is selfadjoint if and only if $q$ is a positive real number. Therefore the Hecke symmetry $\varepsilon$ is a $^*$-symmetry with respect to the standard involution of $H(q)$ if and only if $q > 0$.

We next show by simple computations that in the case where $q$ is not a positive real, the Hilbert space $H$ can not accomodate interesting $^*$-symmetries. We introduce the symmetry $\varepsilon_* : H(\mathcal{T}) \to \mathcal{L}_H$, which takes the basic generator of $H(\mathcal{T})$ to $g_q^*$, the Hilbert space adjoint of $g_q$.

The following simple proposition shows that if $|q| = 1$, the smallest tensor $^*$-subcategory of $\mathcal{L}_H$ containing the operators $g_q$ is permutation symmetric!
4.4 Proposition If $|q| = 1$, the smallest $^*$-subalgebra of $(H \otimes H, H \otimes H)$ containing $1$ and $g_q$ also contains the permutation operator $\theta$ which exchanges the order of factors in $H \otimes H$.

Proof We can assume $q \neq 1$. An computation shows that if $|q| = 1$,

$$(g_q^{-1})^\ast \psi_i \otimes \psi_j = -\mu \psi_j \otimes \psi_i + (q - 1) \psi_i \otimes \psi_j, \quad i < j,$$

$$(g_q^{-1})^\ast \psi_i \otimes \psi_i = -\psi_i \otimes \psi_i$$

and

$$(g_q^{-1})^\ast \psi_i \otimes \psi_j = -\mu \psi_j \otimes \psi_i, \quad i > j$$

therefore $(g_q^{-1})^\ast - g_q = (q - 1)P$, where

$$P \psi_i \otimes \psi_j = \psi_i \otimes \psi_j, \quad i < j,$$

$$P \psi_i \otimes \psi_i = 0,$$

$$P \psi_i \otimes \psi_j = -\psi_i \otimes \psi_j, \quad i > j.$$ 

On the other hand,

$$1/2 [(g_q^{-1})^\ast g_q + g_q (g_q^{-1})^\ast] - q = (1 - q)T$$

with

$$T \psi_i \otimes \psi_j = \mu \psi_j \otimes \psi_i, \quad i \neq j,$$

$$T \psi_i \otimes \psi_i = \psi_i \otimes \psi_i.$$ 

Now the equality $T = \mu \theta + (1 - \mu)(1 - P^2)$ completes the proof.

5 Special Intertwiners and Conjugates

In this section we specialize to the case where $q > 0$. Let $\mu$ denote the positive square root of $q$. Consider the element of $H^{\otimes d}$ defined by

$$S = \sum_{p \in \mathbb{P}_d} (-\mu)^{i(p)} \psi_{p(1)} \otimes \cdots \otimes \psi_{p(d)}.$$ 

This is the fundamental intertwiner of the representation category of the Woronowicz quantum group $S_\mu U(d)$. Here $i(p)$ is number of inversed pairs on $p$, i.e. the cardinality of the set $\{(i, j) : i < j, p(i) > p(j)\}$. Note that $S$ is not normalized for $d > 1$. In fact,

$$\|S\|^2 = \sum_{p \in \mathbb{P}_d} q^{i(p)}$$

and this is known to coincide with the quantum $d$ factorial defined at the beginning of section 2. Indeed, the claim is true for $d = 1$. For $d > 1$, let us represent $\mathbb{P}_d$ as the disjoint union of left cosets $\mathbb{P}_d = \mathbb{P}_d' - 1 \cup \cup_{h=1}^{d-1} (hh + 1) \ldots (12) \mathbb{P}_d'$,
with $\mathbb{P}'_{d-1} := \{ p \in \mathbb{P} : p(1) = 1 \}$, which is a copy of $\mathbb{P}_{d-1}$ contained in $\mathbb{P}_d$.

Since $i((hh+1) \ldots (12)q) = h + i(q)$ for all $q \in \mathbb{P}'_{d-1}$,

$$
\sum_{p \in \mathbb{P}_d} q^{i(p)} = (1 + q + \cdots + q^{d-1}) \sum_{p \in \mathbb{P}_{d-1}} q^{i(p)}.
$$

More generally, choose $n \leq d$ increasing indices $i_1 < i_2 < \cdots < i_n$ among $\{1, \ldots, d\}$ and consider the $n$–dimensional subspace $H_{i_1, \ldots, i_n}$ generated by $\psi_{i_1}, \ldots, \psi_{i_n}$ and the corresponding symmetric tensor:

$$
S_{i_1, \ldots, i_n} := \sum_{p \in \mathbb{P}_n} (-\mu)^{i(p)} \psi_{i_1(p)} \otimes \cdots \otimes \psi_{i_n(p)}.
$$

5.1 Lemma Let $\varepsilon$ be the model Hecke $^*$–symmetry defined in Sect. 4, and $\varepsilon' = \varepsilon \circ \alpha$ its dual symmetry.

a) For $i = 1, \ldots, n-1$,

$$
\varepsilon(g_i)S_{i_1, \ldots, i_n} = qS_{i_1, \ldots, i_n},
\varepsilon'(g_i)S_{i_1, \ldots, i_n} = -S_{i_1, \ldots, i_n},
$$

b) $\varepsilon(A_n)$ has support on the linear span of vectors $\psi_{i_1} \otimes \cdots \otimes \psi_{i_n}$ with $i_h \neq i_k$ for $h \neq k$. In particular, $\varepsilon(A_n) = 0$ for $n > d$. Furthermore, for $n \leq d$, and for $i_1 < i_2 < \cdots < i_n$,

$$
\varepsilon(A_n)\psi_{i_{p-1}(1)} \otimes \cdots \otimes \psi_{i_{p-1}(n)} = (-\mu)^{i(p)}S_{i_1, \ldots, i_n}.
$$

c) \[
\sum_{i_1 < \cdots < i_n} S_{i_1, \ldots, i_n} S_{i_1, \ldots, i_n}^* = \varepsilon(A_n).
\]

In particular,

$$
SS^* = \varepsilon(A_d).
$$

d) For $n \leq d$, $i_1 < i_2 < \cdots < i_n$,

$$
\varepsilon(g_1 \ldots g_n)S_{i_1, \ldots, i_n} \otimes 1_{H_{i_1, \ldots, i_n}} = -(-\mu)^{n-1}1_{H_{i_1, \ldots, i_n}} \otimes S_{i_1, \ldots, i_n},
\varepsilon'(g_1 \ldots g_n)S_{i_1, \ldots, i_n} \otimes 1_{H_{i_1, \ldots, i_n}} = \mu^{n+1}1_{H_{i_1, \ldots, i_n}} \otimes S_{i_1, \ldots, i_n}.
$$

In particular,

$$
\varepsilon(g_1 \ldots g_d)(S \otimes 1_H) = -(-\mu)^{d-1}1_H \otimes S,
\varepsilon'(g_1 \ldots g_d)(S \otimes 1_H) = \mu^{d+1}1_H \otimes S.
$$
e) Set: \( H^{(0)} = \text{span}\{\psi_r, r > i_n\} \), \( H^{(n)} = \text{span}\{\psi_r, r < i_1\} \) if those sets are not empty, and, for \( h = 1, \ldots, n - 1 \), \( H^{(h)} = \text{span}\{\psi_r, i_{n-h} < r < i_{n-h+1}\} \), if \( |i_{n-h+1} - i_{n-h}| \geq 2 \). Then

\[
S_{i_1, \ldots, i_n}^* \otimes 1 H \otimes 1 \otimes g_q S_{i_1, \ldots, i_n} \otimes 1 H = -(n-1)! q 1 H_{i_1, \ldots, i_n} + \\
(n-1)! q \sum_h (q^h - 1) 1 H^{(h)}
\]

where the sum is taken over all \( h = 0, \ldots, n \) for which \( H^h \) makes sense.

**Proof** We omit the proof of a), as it is straightforward. b) Consider, for \( n \in \mathbb{N} \) a vector in \( H^{\otimes n} \) of the form \( \psi = \psi_{i_1} \otimes \cdots \otimes \psi_{i_n} \). One can easily check by induction on \( n \) that, if \( i_1 < i_2 < \cdots < i_n \),

\[
\epsilon(s(p))\psi = (-\mu)^{i(p)}\psi_{i_{p-1}(1)} \otimes \cdots \otimes \psi_{i_{p-1}(n)},
\]

with \( s : \mathbb{P}_n \rightarrow H_n(q) \) the section already considered in the proof of Corollary 3.2. Combining this observation with the equation \( \epsilon(A_n)\epsilon(s(p)) = q^{i(p)} \epsilon(A_n) \), which follows from a) in Lemma 2.1, shows that \( \epsilon(A_n)\psi_{i_{p-1}(1)} \otimes \cdots \otimes \psi_{i_{p-1}(n)} = (-\mu)^{i(p)} \epsilon(A_n)\psi \). Also, \( \epsilon(A_n)\psi = \sum_{p \in \mathbb{P}_n} \epsilon(s(p))\psi = \sum_{p \in \mathbb{P}_n} (-\mu)^{i(p)} \psi_{i_{p-1}(1)} \otimes \cdots \otimes \psi_{i_{p-1}(n)} = S_{i_1, \ldots, i_n} \) as \( i(p) = i(p^{-1}) \). If, more generally, \( i_1 \leq i_2 \leq \cdots \leq i_n \), and, for instance, \( i_1 = i_2 \) then applying both sides of \( \epsilon(A_n g_1) = q \epsilon(A_n) \) to \( \psi \) yields \( \epsilon(A_n)\psi = 0 \), as \( q \neq -1 \). On the other hand, as above, the relation \( \epsilon(A_n)\epsilon(s(p)) = q^{i(p)} \epsilon(A_n) \), due to Lemma 2.1 a), with \( s \) the section defined as in the proof of Corollary 3.2, applied to \( \psi \) shows that \( \epsilon(A_n)\psi_{i_{p-1}(1)} \otimes \cdots \otimes \psi_{i_{p-1}(n)} \) is a scalar multiple of \( \epsilon(A_n)\psi \), and the latter vanishes if at least two indices repeat.

c) Notice that if two ordered \( n \)-tuples \( i_1 < i_2 < \cdots < i_n, j_1 < \cdots < j_n \) are different, the corresponding \( S \)'s are orthogonal. Also, \( \sum_{j_1 < \cdots < j_n} S_{j_1, \ldots, j_n} S_{j_1, \ldots, j_n}^* \psi_{i_{p(1)}} \otimes \cdots \otimes \psi_{i_{p(n)}} = (-\mu)^{i(p)} S_{i_1, \ldots, i_n} \)

\[
\epsilon(A_n)\psi_{i_{p(1)}} \otimes \cdots \otimes \psi_{i_{p(n)}},
\]

d) Since \( S_{i_1, \ldots, i_n} = \epsilon(A_n)\psi_{i_1} \otimes \cdots \otimes \psi_{i_n} \),

\[
\epsilon(g_1 \cdots g_n) S_{i_1, \ldots, i_n} \otimes \psi_n = \epsilon(g_1 \cdots g_n A_n) \psi_{i_1} \otimes \cdots \otimes \psi_{i_n} \otimes \psi_n = 
\]

\[
\epsilon(\sigma(A_n) g_1 \cdots g_n) \psi_{i_1} \otimes \cdots \otimes \psi_{i_n} \otimes \psi_n = -\epsilon(\sigma(A_n)) \psi_{i_1} \otimes \cdots \otimes \psi_{i_n} \otimes \psi_n = -(\mu)^{n-1} \psi_{i_1} \otimes \psi_{i_1} \otimes \cdots \otimes \psi_{i_n} = \]

while, for \( i_n > r \) and \( r \in \{i_1, \ldots, i_n\} \),

\[
\epsilon(g_1 \cdots g_n) S_{i_1, \ldots, i_n} \otimes \psi_r = -\mu \epsilon(\sigma(A_n)) \sum_{i_1 < \cdots < i_n} \psi_{i_1} \otimes \cdots \otimes \psi_{i_{n-1}} \otimes \psi_r \otimes \psi_n + \\
(q-1) \epsilon(\sigma(A_n)) \epsilon(g_1 \cdots g_{n-1}) \psi_{i_1} \otimes \cdots \otimes \psi_{i_{n-1}} \otimes \psi_r \otimes \psi_n.
\]
For any permutation \( \sigma \) that reverses the order of the ordered integers will take to the desired result. One can proceed in the following way. Let \( \psi_r \) be the last addendum in the above sum vanishes as the index \( r \) appears twice in \( \psi_{i_1} \otimes \cdots \otimes \psi_{i_{n-1}} \otimes \psi_r \). Iterating this procedure gives the desired result.

For the analogous relation relative to the dual symmetry, similar computations will take to the desired result. One can proceed in the following way. Let \( p_n \) be that permutation that reverses the order of the ordered integers \( i_1 < \cdots < i_n \), and write

\[
S_{i_1, \ldots, i_n} = (-\mu)^{i(p_n)} \varepsilon(A_n) \psi_{i_n} \otimes \cdots \psi_{i_1}.
\]

Then

\[
\varepsilon'(g_1 \cdots g_n) S_{i_1, \ldots, i_n} \otimes \psi_r = (-\mu)^{-i(p_n)} \varepsilon(\sigma(A_n)) \varepsilon'(g_1 \cdots g_n) \psi_{i_n} \otimes \cdots \psi_{i_1} \otimes \psi_r.
\]

Now for \( r = i_1 \), as before, we easily get the result, and for \( r > i_1 \) we just need to know that for \( i \neq j \), the new coefficient \( \psi_i \otimes \psi_j, \varepsilon'(g_1) \psi_j \otimes \psi_i \) is \( \mu \).

We give another proof in the case \( n = d \). By (c),

\[
dl_{q} \varepsilon'(g_1 \cdots g_d) S \otimes 1_H = \varepsilon'(g_1 \cdots g_d) \varepsilon(A_d) S \otimes 1_H = \varepsilon(\sigma(A_d)) \varepsilon'(g_1 \cdots g_d) S \otimes 1_H = 1_H \otimes S(1_H \otimes S^* \varepsilon'(g_1 \cdots g_d) S \otimes 1_H).
\]

Now by the previous part

\[
1_H \otimes S^* \varepsilon'(g_1 \cdots g_d) S \otimes 1_H = (-1)^d \frac{1}{\mu^{d-1}} S^* \otimes 1_H \varepsilon(g_d \cdots g_1) \varepsilon'(g_1 \cdots g_d) S \otimes 1_H = \mu^{d+1} dl_{q}
\]

as \( \varepsilon(g_i) \varepsilon'(g_i) = -q \).

e) We apply the left hand side to all basis vectors \( \psi_r \). We start from the case where \( r \) is of the form \( i,j \).

\[
S^*_{i_1, \ldots, i_n} \otimes 1_H (- \sum_{p(n)=j} (-\mu)^{i(p)} \psi_{i_{p(1)}} \otimes \cdots \otimes \psi_{i_{p(n-1)}} \otimes \psi_{i_j} +
\]

\[
- \mu \sum_{p(n) \neq j} (-\mu)^{i(p)} \psi_{i_{p(1)}} \otimes \cdots \otimes \psi_{i_{p(n-1)}} \otimes \psi_{i_j} \otimes \psi_{i_{p(n)}} +
\]

\[
(q-1) \sum_{p(n)>j} (-\mu)^{i(p)} \psi_{i_{p(1)}} \otimes \cdots \otimes \psi_{i_{p(n-1)}} \otimes \psi_{i_j}.
\]

For any permutation \( p \) for which \( p(n) \neq j \), \( j \) appears twice in \( (p(1), p(2), \ldots, p(n-1), j) \), therefore the second sum, when multiplied on the left by \( S^*_{i_1, \ldots, i_n} \otimes 1_H \), vanishes, and the computation equals

\[
- \sum_{p(n)=j} (-\mu)^{i(p)} S^*_{i_1, \ldots, i_n} \otimes 1_H \psi_{i_{p(1)}} \otimes \cdots \otimes \psi_{i_{p(n-1)}} \otimes \psi_{i_j} +
\]
\[
(q - 1) \sum_{k=j+1}^{n} \sum_{p(n)=k} (-\mu)^i(p) S_{i_1,\ldots,i_n}^* \otimes 1_H \psi_{p(1)} \otimes \cdots \otimes \psi_{p(n)} \otimes \psi_{i_j}.
\]

Let us write each permutation for which \( p(n) = k \) in the form \( p = (kk - 1)(k - 1k - 2)\ldots(21)p''(123\ldots n - 1n) \) with \( p''(1) = 1 \). We claim that \( i(p) = i(p'') + n - k \). In fact, for any permutation \( q \) for which \( q(1) = h \) the permutation \( (h - 1h)q \) has one less inversed pair than \( q \), so \( i((h - 1h)q) = i(q) - 1 \), which shows that \( i(p'') = i(p(1\ldots n)^{-1}) - (k - 1) \). Now if we want to compare the inversed pairs of \( p \) and \( p(1\ldots n)^{-1} \) we see that the two sets of inversed pairs have a common intersection with cardinality, say \( m \), but for \( p \) we need to count how many inversed pairs we have in the set \{ \( (p(1),\ldots,(p(n - 1),k) \)\}, which are \( n - k \), whereas for \( p(1\ldots n)^{-1} \) we need the cardinality of the subset of inversed pairs in \{ \( (k,p(1)),\ldots,(k,p(n - 1)) \)\}, which is \( k - 1 \). Therefore \( i(p) = m + n - k \) and \( i(p(1\ldots n)^{-1}) = m + k - 1 \), which implies \( i(p(1\ldots n)^{-1}) = i(p) + 2k - n - 1 \) and the claim is proved. Thus

\[
\sum_{p(n)=k} (-\mu)^i(p) S_{i_1,\ldots,i_n}^* \otimes 1_H \psi_{p(1)} \otimes \cdots \otimes \psi_{p(n)} \otimes \psi_{i_j} = \sum_{p'' \in \mathbb{P}_{n-1}} q^{i(p'')} + n - k \psi_{i_j},
\]

and the conclusion follows from simple computations in this case. Assume now that \( r \) is none of the \( \psi_{i_j} \)'s, and, to fix ideas, assume that \( i_{n-h} < r < i_{n-h+1} \). Then similar computations yield:

\[
S_{i_1,\ldots,i_n}^* \otimes 1_H 1_{H^{\otimes(n-1)}} \otimes g_q S_{i_1,\ldots,i_n} \otimes 1_H (\psi_r) =
\]

\[
S_{i_1,\ldots,i_n}^* \otimes 1_H (-\mu \sum_p (-\mu)^i(p) \psi_{p(1)} \otimes \cdots \otimes \psi_{p(n-1)} \otimes \psi_r \otimes \psi_{p(n)}) +
\]

\[
(q - 1) \sum_{\psi_{p(n)} \geq r} (-\mu)^i(p) \psi_{p(1)} \otimes \cdots \otimes \psi_{p(n)} \psi_r \ = (q - 1) \sum_{\psi_{p(n)} \geq r} q^{i(p)} \psi_r.
\]

Notice that the condition \( i_{p(n)} > r \) amounts to \( p(n) = k \) with \( k = n - h + 1,\ldots,n \). Therefore, as before, we write each such permutation in the form \( p = (kk - 1)\ldots(21)p''(12\ldots n) \) with \( p''(1) = 1 \) and \( i(p) = i(p'') + n - k \). We thus conclude that the above term equals

\[
(q - 1) \sum_{k=n-h+1}^{n} \sum_{p''} q^{i(p'')} + n - k \psi_r = (q - 1)(n - 1)!q^n \sum_{k=n-h+1}^{n} q^k \psi_r =
\]

\[
(q^n - 1)(n - 1)!q \psi_r.
\]

In the case where \( r < i_1 \) the computations are simpler, as \( i_{p(n)} > r \) for all \( p \in \mathbb{P}_n \), so the result is \( (q - 1)n!q \psi_r = (q^n - 1)(n - 1)!q \psi_r \). Finally, if \( r > i_n \) then \( S_{i_1,\ldots,i_n}^* \otimes 1_H 1_{H^{\otimes(n-1)}} \otimes g_q S_{i_1,\ldots,i_n} \otimes 1_H \psi_r = 0 \).

The previous lemma has various consequences. We start with the following one.
5.2 Corollary The representation category of \( S_\mu U(d) \) is a braided tensor category.

Proof Let \( \omega \) be a complex number such that \( \omega^d = -(\mu)^{d-1} \), and consider the representations \( \tilde{\varepsilon}_n : B_n \to (H^{\otimes n}, H^{\otimes n}) \) given by \( \tilde{\varepsilon}_n(g_1) = \frac{\varepsilon_i \Phi}{\omega} \). This is obviously a braided symmetry in the category \( \mathcal{L}_H \) (but not a Hecke symmetry anymore) satisfying relation (3.1) at least for \( T = S \) and \( T \in \varepsilon(H_n(q)) \). Since these arrows generate the representation category of \( S_\mu U(d) \) just as a \( C^* \)-category, we deduce that equation (3.1) holds on that whole category.

We will see, as a consequence of Lemmas 5.1 and 5.4 together, that for \( n < d \) the elements \( S_{i_1,...,i_n} \) do not belong to the maximal braided subcategory \( \mathcal{L}_H \).

Another consequence of Lemma 5.1 is that part e) allows one to construct left inverses of \( H \).

5.3 Corollary Let \( q > 0 \), and let, as before, \( \mathcal{L}_H \) be the tensor \( * \)-category of Hilbert spaces with objects tensor powers of the \( d \)-dimensional Hilbert space \( H \). Then, for \( i_1 < \cdots < i_n \), the map

\[
\Phi^{i_1,...,i_n} : (H^{\otimes r}, H^{\otimes r}) \to (H^{\otimes(r-1)}, H^{\otimes(r-1)}),
\]

\[
\Phi_{r}^{i_1,...,i_n}(T) := \frac{1}{n! q} S_{i_1,...,i_n}^{*} \otimes 1_{H^{\otimes(r-1)}} 1_{H^{\otimes(n-1)}} \otimes TS_{i_1,...,i_n} \otimes 1_{H^{\otimes(r-1)}}
\]

is a positive left inverse of \( H \). If \( n = d \), \( \Phi_{2}^{1,...,d}(g_{q_i}) = \lambda_{-d}1_{H} \). If \( n < d \), but \( i_n = d \) and \( q \neq 1 \), then \( \Phi_{2}^{1,...,i_n}(g_{q}) \) is an invertible diagonal operator of \( (H, H) \). In both cases these left inverses are faithful on the range algebras of the Jimbo-Woronowicz Hecke symmetry \( \varepsilon : H_n(q) \to (H^{\otimes n}, H^{\otimes n}) \).

Proof Clearly \( \Phi_{r}^{i_1,...,i_n} \) is a positive unital linear map with range contained in the space of linear maps on \( H^{\otimes(r-1)} \). For \( S \in (H^{\otimes r}, H^{\otimes r}) \) and \( T \in (H^{\otimes(r-1)}, H^{\otimes(r-1)}) \),

\[
\Phi_{r}^{i_1,...,i_n}(S 1_{H} \otimes T) = \Phi_{r}^{i_1,...,i_n}(S) T,
\]

so it is a left inverse of \( H \). The Jimbo-Woronowicz Hecke \( * \)-symmetry is a braided symmetry for \( \mathcal{L}_H \) for which the range of \( H_r(q) \) is a \( * \)-subalgebra of \( (H^{\otimes r}, H^{\otimes r})^r \), therefore whenever \( \Phi_{2}^{1,...,i_n}(g_{q}) \) is invertible, \( \Phi_{r}^{i_1,...,i_n} \) is faithful on that range, by Lemma 3.1.

Remark Let \( q > 0 \), and let \( H(q) \) be endowed with its standard involution. We have seen in section 3 that if we have a Hecke \( * \)-symmetry \( \varepsilon : H(q) \to \mathcal{H} \) in a tensor \( C^* \)-category with a left inverse \( \Phi \) for \( \rho \) such that \( \Phi(\varepsilon(g_1)) \) is a nonzero scalar, then the kernel of the \( * \)-homomorphisms \( \varepsilon : H_n(q) \to (\rho^n, \rho^n) \) are determined by that scalar. Therefore if \( \Phi(\varepsilon(g_1)) = \lambda_{-d} \), \( \varepsilon \) has dimension \( d \).

The following computations will serve to define a certain tensor \( C^* \)-category with conjugates in the sense of [7].
5.4 Lemma Assume \( q > 0 \). For \( n \leq d \), and indices \( i_1 < \cdots < i_n \) in \( 1, \ldots, d \) consider the \( n \)-dimensional subspace \( H_{i_1, \ldots, i_n} \) of \( H \) generated by \( \psi_{i_1}, \ldots, \psi_{i_n} \). Then
\[
S^*_{i_1, \ldots, i_n} \otimes 1_H \circ 1_H \otimes S_{i_1, \ldots, i_n} = (n-1)!q(-\mu)^{n-1}1_{H_{i_1, \ldots, i_n}},
\] (5.1)
\[
S^*_{i_1, \ldots, i_n} \otimes 1_{H \otimes(n-1)} \circ 1_{H \otimes(n-1)} \otimes S_{i_1, \ldots, i_n} = (-\mu)^{n-1}\varepsilon(A_{n-1})1_{H_{i_1, \ldots, i_n} \otimes(n-1)}.
\] (5.2)

In particular,
\[
S^* \otimes 1_H \circ 1_H \otimes S = (d-1)!q(-\mu)^{d-1}1_H,
\] (5.3)
\[
S^* \otimes 1_{H \otimes(d-1)} \circ 1_{H \otimes(d-1)} \otimes S = (-\mu)^{d-1}\varepsilon(A_{d-1}).
\] (5.4)

Proof Set
\[
\hat{\psi}_j = \sum_{p(1)=j} (-\mu)^{i(p)}\psi_{i_p(2)} \otimes \cdots \otimes \psi_{i_p(n)}
\]
and
\[
\tilde{\psi}_h = \sum_{p(d)=h} (-\mu)^{i(p)}\psi_{i_p(1)} \otimes \cdots \otimes \psi_{i_p(n-1)}
\]
and write
\[
S = \sum_j \hat{\psi}_j \otimes \tilde{\psi}_j = \sum_h \tilde{\psi}_h \otimes \hat{\psi}_h.
\]
The left hand side of (5.1) is the operator \( T \) on \( H \) with initial and final support in \( H_{i_1, \ldots, i_n} \) such that \( (\psi_{i_h}, T\psi_{i_j}) = (\hat{\psi}_j, \tilde{\psi}_h) \). Now, for \( j \neq h \), \( (\hat{\psi}_j, \tilde{\psi}_h) = 0 \), while
\[
(\hat{\psi}_j, \tilde{\psi}_j) = \sum_{p(1)=p'(n)=j} (-\mu)^{i(p)}(-\mu)^{i(p')}\psi_{i_p(2)} \otimes \psi_{i_p(1)} \cdots \psi_{i_p(n)} \psi_{i_{p'}(1)} \cdots \psi_{i_{p'}(n-1)}.
\]
Thus for a fixed \( p \) for which \( p(1) = j \) we need to choose \( p' \) so that \( p'(1) = p(2), \ldots, p'(n-1) = p(n), p'(n) = j \). Since \( i(p') = i(p) + n + 1 - 2j \),
\[
(\hat{\psi}_j, \tilde{\psi}_j) = \sum_{p(1)=j} (-\mu)^{i(p)}(-\mu)^{i(p)+n+1-2j}.
\]
Writing each such \( p \) in the form \( p = (j - 1) \cdots (23)(12)p'' \) with \( p''(1) = 1 \), we see that \( i(p) = i(p'') + j - 1 \), so \( i(p') = i(p'') + n - j \) and
\[
(\hat{\psi}_j, \tilde{\psi}_j) = (n-1)!q(-\mu)^{n-1}.
\]
Taking into account the elements \( \hat{\psi}_j \) and \( \tilde{\psi}_h \), the left hand side of (5.2) is
\[
\sum_j \theta_{\hat{\psi}_j, \tilde{\psi}_j}, \text{ with } \theta_{\eta, \xi}(\zeta) = \langle \eta, \zeta \rangle \xi. \text{ Defining, for each permutation } p \text{ for which } p(n) = j, \text{ the permutation } p' \text{ such that } p'(1) = j, p'(2) = p(1), \ldots, p'(n) = p(n-1) \text{ shows that } i(p) = i(p')+n+1-2j, \text{ so } \tilde{\psi}_j = (-\mu)^{n+1-2j}\tilde{\psi}_j, \text{ and we deduce that the left hand side of (5.2) coincides with } \sum_j (-\mu)^{n+1-2j}\theta_{\hat{\psi}_j, \tilde{\psi}_j}. \text{ Notice that } (\hat{\psi}_j, \tilde{\psi}_j) = (n-1)!q^{d-1}. \text{ Let } (j) \text{ denote the ordered } n-1 \text{-tuple } 1, \ldots, n \text{ with the index } j \text{ suppressed. Then the relation with the elements } S_{(j)} \text{ defined}.
in lemma (5.1), is \(\hat{\psi}_j = (-\mu)^{j-1}S_{(j)}\), therefore, in conclusion, \(\sum_j \theta\hat{\psi}_j, \hat{\psi}_j = (-\mu)^{n-1}S_{(j)}\). We give a proof in the case where \((i_1, \ldots, i_n) \neq (1, \ldots, n). Then parts a) and e) allow us to compute

\[
S^*_{i_1 \cdots i_n} \otimes 1_H \varepsilon(g_1 \cdots g_n)S_{i_1 \cdots i_n} \otimes 1_H = q^{n-1}S^*_{i_1 \cdots i_n} \otimes 1_H 1_H \otimes 1_H \otimes \cdots \otimes 1_H
\]

which must be a sum of scalar multiples of orthogonal projections where, for some \(h > 0\), a nonzero multiple of \(1_H \otimes 1_H\) does appear. On the other hand by Lemma 5.4 \(S^*_{i_1 \cdots i_n} \otimes 1_H 1_H \otimes S_{i_1 \cdots i_n}\) is just a scalar multiple of \(1_{H_{i_1 \cdots i_n}}\). In the case where \((i_1, \ldots, i_n) = (1, \ldots, n)\), arguing as in the proof of d) 5.1 shows that

\[
\varepsilon(g_1 \cdots g_n)S_{1, \ldots, n} \otimes 1_H = (-\mu)^n 1_{H^{(0)}} \otimes S_{1, \ldots, n}
\]

which together with

\[
\varepsilon(g_1 \cdots g_n)S_{1, \ldots, n} \otimes 1_{H_{i_1 \cdots i_n}} = (-\mu)^{n-1}1_{H_{i_1 \cdots i_n}} \otimes S_{1, \ldots, n}
\]

and the fact that \(1_H = 1_{H_{i_1 \cdots i_n}} + 1_{H^{(0)}}\), shows the claim.

Recall that we defined the idempotents \(E_1 = 1, E_n = \frac{1}{n_q}A_n\) of the Hecke algebra \(H_\infty(q)\).

5.5 Theorem Assume \(q > 0\). Set, for \(n \leq d\) and indices \(i_1 < \cdots < i_n\) in \(1, \ldots, d\), \(\overline{H}_{i_1 \cdots i_n} := \varepsilon(E_{n-1})H_{i_1 \cdots i_n} \otimes 1_{H^{(n-1)}}\). Then

\[
R_{i_1 \cdots i_n} := \frac{1}{\mu^{n-1/2}(n-1)!q}S_{i_1 \cdots i_n} \in \overline{H}_{i_1 \cdots i_n} \otimes H_{i_1 \cdots i_n}
\]

and

\[
\overline{R}_{i_1 \cdots i_n} := (-1)^{n-1}R_{i_1 \cdots i_n} = \varepsilon(n-1, 1)R_{i_1 \cdots i_n} \in H_{i_1 \cdots i_n} \otimes \overline{H}_{i_1 \cdots i_n}
\]

satisfy the conjugate equations:

\[
\overline{R^*}_{i_1 \cdots i_n} \otimes 1_{H_{i_1 \cdots i_n}} \otimes 1_{H_{i_1 \cdots i_n}} \otimes R_{i_1 \cdots i_n} = 1_{H_{i_1 \cdots i_n}},
\]

\[
R^*_{i_1 \cdots i_n} \otimes 1_{\overline{H}_{i_1 \cdots i_n}} \otimes \overline{R}_{i_1 \cdots i_n} = 1_{\overline{H}_{i_1 \cdots i_n}},
\]

so \(\overline{H}_{i_1 \cdots i_n}\) is a conjugate object for \(H_{i_1 \cdots i_n}\) in the smallest tensor \(^\ast\)-subcategory of \(\mathcal{L}_H\) containing \(S_{i_1 \cdots i_n}\).

Proof We give a proof in the case \(n = d\). In the general case, it suffices to replace \(H\) by the subspace \(H_{i_1 \cdots i_n}\). We keep the notation of the proof of the two previous lemmas. By b) in Lemma 5.1, the range of \(\varepsilon(A_{d-1})\) is the linear span of \(\{S_{(1)}, \ldots, S_{(d)}\}\). Since each \(S_{(j)}\) is a multiple of both \(\hat{\psi}_j\) and \(\hat{\psi}_j\),
Now, if \( H \) is defined by
\[
\psi = R \psi
\]
In order to simplify the following notation we shall prove (5.4). Set
\[
R_k = \frac{1}{\mu^{k(d-k)/2} \sqrt{k!q} \sqrt{(d-k)!q}} S_i
\]
\[
\tilde{R}_k = (-1)^{k(d-k)} R_k = \varepsilon'(d-k, k) R_k.
\]

**Proof** In order to simplify the following notation we shall prove (5.5) with \( d-k \) in place of \( k \). Set
\[
\tilde{\psi}_{i_1, \ldots, i_k} = \sum_{p(d-k)+1 = i_1, \ldots, p(d) = i_k} (-\mu)^{i(p)} \psi_{i_1, \ldots, i_k}. 
\]
Then
\[
S = \sum_{i_1 < \cdots < i_k} \sum_{p \in P_k} \tilde{\psi}_{i_1, \ldots, i_k, i_{p(1)}, \ldots, i_{p(k)}} \psi_{i_1, \ldots, i_k}.
\]
Now, if \( i_1 < \cdots < i_k \),
\[
\tilde{\psi}_{i_1, \ldots, i_k} = (-\mu)^{i(p)} \tilde{\psi}_{i_1, \ldots, i_k},
\]
so
\[
S = \sum_{i_1 < \cdots < i_k} \tilde{\psi}_{i_1, \ldots, i_k} \otimes S_{i_1, \ldots, i_k},
\]
as \( i(p) = i(p^{-1}) \). Similarly, \( S = \sum_{i_1 < \cdots < i_k} \tilde{\psi}_{i_1, \ldots, i_k} \otimes \overline{\psi}_{i_1, \ldots, i_k} \) with
\[
\tilde{\psi}_{i_1, \ldots, i_k} = \sum_{p(1)=i_1, \ldots, p(k)=i_k} (-\mu)^{i(p)} \psi_{i_1, \ldots, i_k} \otimes \cdots \otimes \psi_{p(d)}.
\]
Thus the left hand side of (5.5), with, recall, \( k \) and \( d-k \) replaced, equals
\[
\sum_{i_1 < \cdots < i_k} (S_{i_1, \ldots, i_k}, S_{i_1, \ldots, i_k}) \theta_{\tilde{\psi}_{i_1, \ldots, i_k}, \overline{\psi}_{i_1, \ldots, i_k}},
\]
where \( (S_{i_1, \ldots, i_k}, S_{i_1, \ldots, i_k}) = k!q \). Now \( \sum_{i_1 < \cdots < i_k} \theta_{\tilde{\psi}_{i_1, \ldots, i_k}, \overline{\psi}_{i_1, \ldots, i_k}} \) annihilates all simple tensors \( \psi_{i_1} \otimes \cdots \otimes \psi_{j_{d-k}} \) with \( j_h = j_k \) for some \( h \neq k \), and applied on a simple tensor of the form \( \psi_{\!p(1)} \otimes \cdots \otimes \psi_{\!p(d-k)} \), with \( j_1 < \cdots < j_{d-k} \) and
p \in \mathbb{P}_{d-k}, \text{ yields } (-\mu)^{i(p)+N} \tilde{\psi}_{i_1, \ldots, i_k}, \text{ with } (i_1, \ldots, i_k) \text{ obtained from } (1, \ldots, d) \text{ removing the indices } j_1, \ldots, j_{d-k}, \text{ and with } N \text{ the number of pairs } (i_h, j_k) \text{ for which } i_h > j_k. \text{ On the other hand it is not difficult to check that }

\tilde{\psi}_{i_1, \ldots, i_k} = S_{j_1, \ldots, j_{d-k}} (-\mu)^{k(d-k)-N},

so the image of the left hand side of (5.5) on that simple tensor is

\begin{align*}
(-\mu)^{i(p)} k! q (-\mu)^{k(d-k)} S_{j_1, \ldots, j_{d-k}},
\end{align*}

the same as the image under \((-\mu)^{k(d-k)} k! q \varepsilon(A_{d-k})\).

Remark Choosing \(k = 2\) shows that \(\varepsilon(E_2)\), and therefore the Hecke symmetry \(\varepsilon\), is determined uniquely by the intertwiner \(S\) in the tensor \(C^*\)-category \(L_H\).

Let us consider the smallest tensor \(\ast\)-subcategory of \(L_H\) with subobjects, containing \(S\). This is the representation category \(\text{Rep}(S_{\mu}U(d))\) of the Woronowicz compact quantum group \(S_{\mu}U(d)\). It is a tensor \(C^*\)-category of Hilbert spaces with conjugates and with a Hecke symmetry whose dimension equals the dimension of \(H\).

6 An abstract characterization of \(\text{Rep}(S_{\mu}U(d))\)

In the previous section we have shown the following relations between the fundamental intertwiner \(S\) of the representation category of the Woronowicz quantum group \(S_{\mu}U(d)\) and the model Woronowicz–Jimbo \(q\)-Hecke \(\ast\)-symmetry, with \(q = \mu^2\).

\begin{align*}
S^* S &= d! q, \quad (6.1) \\
S^* \otimes 1_H \circ 1_H \otimes S &= (d-1)! q (-\mu)^{d-1}, \quad (6.2) \\
S S^* &= \varepsilon(A_d), \quad (6.3) \\
\varepsilon(g_1 \ldots g_d) S \otimes 1_H &= -(-\mu)^{d-1} 1_H \otimes S. \quad (6.4)
\end{align*}

The second and third relations above serve to define conjugates in the tensor \(C^*\)-category \(\text{Rep}(S_{\mu}U(d))\), while the last one asserts that that category is a braided tensor category with respect to a renormalization of the Woronowicz-Jimbo Hecke symmetry.

Let \(\mathcal{T}\) denote a tensor \(C^*\)-category with tensor unit \(\iota\) such that \((\iota, \iota) = \mathbb{C}\) and objects \(\{\iota, \rho, \rho^2, \ldots\}\), the tensor powers of a fixed object \(\rho\). We shall also assume the existence of a Hecke \(\ast\)-symmetry \(\varepsilon : H_n(q) \rightarrow (\rho^n, \rho^n)\) for some \(q > 0\).

In this framework we introduce the notion of \textit{special object} which generalizes the corresponding notion in permutation symmetric tensor \(C^*\)-categories given by Doplicher and Roberts in \(\mathbb{K}\).

6.1 Definition The object \(\rho\) of \(\mathcal{T}\) is called \textit{special with dimension} \(d\) if there is an intertwiner \(R \in (\iota, \rho^d)\) such that properties (6.1)–(6.4) hold for \(R\) and \(\rho\) in
place of $S$ and $H$ respectively and $\mu$ the positive square root of $q$. $R$ will be referred to as a special intertwiner.

The aim of this section is to show that the above properties are sufficient to construct an embedding of tensor $C^*$-categories from $\text{Rep}(S_\mu U(d))$ to $\mathcal{T}$.

6.2 Theorem Let $\rho$ be a special object of $\mathcal{T}$ with dimension $d \geq 2$. Then there is a unique faithful tensor $^*$-functor $\text{Rep}(S_\mu U(d)) \to \mathcal{T}$ taking the fundamental representation of $\epsilon_1 = e_1 \cdots e_d$ to $\rho$ and $S$ to a special intertwiner $R$.

Proof Notice that for $q = 1$, this theorem reduces to Theorems 4.1 and 4.4 in [2]. Here we argue in the same way. We first show that our assumptions imply that $\epsilon$ has dimension $d$. By Theorem 3.3 we need to show that $\rho$ has a positive left inverse $\Phi$ such that $\Phi(\epsilon_1) = \lambda$.

$$\Phi(T) := \frac{1}{d!} \epsilon R^* \otimes 1_\rho \circ 1_{\rho^{-1}} \otimes T \circ R \otimes 1_\rho$$

is a positive, unital left inverse of $\rho$ such that, since by (6.1), (6.3) and Lemma 2.1 a), $\epsilon(g)R = qR$, $j = 1, \ldots, d - 1$,

$$\Phi(\epsilon(g)) = \frac{1}{d!} R^* \otimes 1_\rho(\epsilon(g_1 \cdots g_d))^{-1} \epsilon(g_1 \cdots g_d)R \otimes 1_\rho =$$

$$= \frac{1}{q^{d-1}d!} R^* \otimes 1_\rho(\epsilon(g_1 \cdots g_d))R \otimes 1_\rho = -\frac{1}{(-\mu)^{d-1}d!} R^* \otimes 1_\rho 1_\rho \otimes R =$$

$$= \frac{1}{1 + q + \cdots + q^{d-1}} = \lambda_{d-1}.$$

Therefore the Hecke symmetry $H(q) \to \mathcal{T}$ factors over the model symmetry, and this defines a faithful tensor $^*$-functor from homogeneous subcategory (the subcategory with the same objects and with intertwiners with the same domain and range) of $\text{Rep}(S_\mu U(d))$ to $\mathcal{T}$. Now any intertwiner space $(H^{\otimes r}, H^{\otimes s})$ in $\text{Rep}(S_\mu U(d))$ is nonzero if and only if $s - r$ is an integer multiple of $d$, and, for $s = r + kd$, $k > 0$, one can write $T \in (H^{\otimes r}, H^{\otimes s})$ uniquely in the form $T = T_0 \otimes S \otimes \cdots \otimes S$ with $T_0 \in (H^{\otimes r}, H^{\otimes s})$ satisfying $T_0 = T_0 \circ SS^* \otimes \cdots \otimes SS^*$. One can thus uniquely extend that functor from the homogeneous subcategory to a faithful tensor $^*$-functor on the whole of $\text{Rep}(S_\mu U(d))$, taking $S$ to $R$.

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