GLOBAL EXISTENCE FOR THE VLASOV-POISSON SYSTEM IN BOUNDED DOMAINS

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Abstract. In this paper we prove global existence for solutions of the Vlasov-Poisson system in convex bounded domains with specular boundary conditions and with a prescribed outward electrical field at the boundary.

1. Introduction

In this paper we study global solutions for Vlasov-Poisson system in a convex bounded domain \( \Omega \) with specular reflection on the boundary:

\[
\begin{align*}
 f_t + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f &= 0, & x \in \Omega \subset \mathbb{R}^3, & v \in \mathbb{R}^3, & t > 0 \\
 \Delta \phi &= \int_{\mathbb{R}^3} f \, dv, & x \in \Omega, & t > 0 \\
 \frac{\partial \phi}{\partial n_x} (t, x) &= h(x), & x \in \partial \Omega, & t > 0 \\
 f(0, x, v) &= f_0(x, v), & x \in \Omega, & v \in \mathbb{R}^3 \\
 f(t, x, v) &= f(t, x, v^*), & x \in \Omega, & v \in \mathbb{R}^3, & t > 0
\end{align*}
\]

where \( \Omega \) is a convex bounded domain with \( C^5 \) boundary, \( n_x \) denotes the outer normal to \( \partial \Omega \) and

\[
f_0(x, v) \geq 0
\]

Here \( f(t, x, v) \) denotes the distribution density of electrons, \( \phi(t, x) \) is the electric potential. The function \( h \) in (1.3) will be assumed to be positive and satisfy the following compatibility condition:

\[
\int_{\Omega} f_0(x, v) \, dx \, dv = \int_{\partial \Omega} h(x) \, dS_x
\]

We will also assume that \( f_0 \) is compactly supported in \( \bar{\Omega} \times \mathbb{R}^3 \). In (1.5) and in the rest of the paper we use the following notation. Given \( (x, v) \) in \( \partial \Omega \times \mathbb{R}^3 \) we define:

\[
v^* \equiv v - 2n_x \cdot v
\]

where from now on \( n_x \) is the outer normal vector to \( \partial \Omega \) at the point \( x \).

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The density of particles at a given point \( x \) is given by:

\[
\rho(t, x) \equiv \int_{\mathbb{R}^3} f(t, x, v) \, dv
\]

In the case of \( \Omega = \mathbb{R}^3 \), the solutions of the system \((1.1)-(1.4)\) are globally defined in time for general initial data, as it was proved in [13] as well as in [11] using different methods. However, in the case of domains with boundaries the mathematical theory of well-posedness for the solutions of the Vlasov-Poisson system is not so complete as in the case of the whole space. It was proved in [6] that classical solutions for the problem \((1.1)-(1.5)\) may not exist in general without the nonnegativity assumption \((1.6)\) if \( \Omega \) is the half-space \( \mathbb{R}^3_+ \). On the other hand, it was also proved in [6] that even with the assumption \((1.6)\) the derivatives of the solutions of \((1.1)-(1.5)\) cannot be uniformly bounded near the boundary of \( \Omega \) due to the fact that a Lipschitz estimate for the characteristics in terms of the initial data is not possible.

One of the main technical difficulties that must be considered in order to solve \((1.1)-(1.5)\), even for short times, is a careful study of the evolution of the characteristic curves associated to \((1.1)\) that remain close during their evolution to the so-called singular set, that is defined as follows:

\[
\Gamma = \{ (x, v) \in \Omega \times \mathbb{R}^3 : x \in \partial \Omega, \ v \in T_x \partial \Omega \}
\]

where \( T_x \partial \Omega \subset \mathbb{R}^3 \) is the tangent plane to \( \partial \Omega \) at the point \( x \).

Notice that the projection of such characteristic curves in the domain \( \Omega \) bounces repeatedly at the boundary \( \partial \Omega \).

For the case of a half-space \( \Omega = \mathbb{R}^3_+ \) the global existence result was shown in [7] if the initial data \( f_0 \) is assumed to be constant in a neighbourhood of the singular set. The method there is to adapt the high-moment technique in [11]. Recently, we have developed in [10] a new proof of global existence modifying Pfaffelmoser’s idea (cf. [13]).

In [9] the case of a general convex bounded domain was considered and solutions of the linear approximate problem of \((1.1)-(1.5)\) were constructed under the assumption that the initial data \( f_0 \) is constant near the singular set. The absorbing condition was assumed for the distribution density \( f \) at the boundary to obtain global existence of solution to the full VP system. Global existence results in \( \Omega = B_R(0) \) were also obtained in the paper, under the same assumption, for a class of radially symmetric data that rule out possible singular behaviors at the origin.

Actually, several of the technical difficulties that arise in the study of the characteristics near the singular set had been already addressed in [7] in the particular case \( \Omega = \mathbb{R}^3_+ \). On the other hand the results in [9] provide techniques for the problem \((1.1)-(1.5)\) in more general domains. However, the assumption of \( f_0 \) being constant near the singular set imposes some restrictions on the initial data, but its main consequence is to make it possible to
ignore the evolution of the characteristic curves that are close to the singular set.

The main contribution of this paper is to show how to adapt Pfaffelmoser’s ideas and to introduce geometric methods to the problem of general bounded convex domains with curvatures in order to prove global existence in time for the solutions of (1.1)-(1.5). It turns out that the effect of the geometry of the domains modifies in a stronger way than that of the electric field the dynamics of the characteristics. One of the key ideas consists in approximating the dynamics of the characteristic curves that are close to the singular set by means of the dynamics of a hamiltonian system whose trajectories are constrained to the boundary $\partial \Omega$. This approach allows us to include easily in the estimates the effects of the curvature of the domain. We will then be able to adapt the ideas in [6], [9], [13] to prove global existence.

The plan of the paper is the following. In Section 2 we introduce a coordinate system that makes it easier to study the trajectories near the singular set and that we will use in the rest of the paper. In this Section we also introduce a suitable flatness condition for the initial data $f_0$ near the singular set that will make it possible to obtain solvability for the initial value problem for the VP system. In Section 3 we describe an iterative procedure that defines a sequence of functions $\{f_n\}$ whose limit as $n \to \infty$ yields the desired global solution of the problem. In Section 4 we obtain suitable estimates for the solutions of the so-called "linear problem" that is the system (1.1), (1.4), (1.5) with prescribed $\phi$. In Section 5 we prove that the sequence $\{f_n\}$ converges to a limit function that is defined as long as a suitable functional $Q(t)$ is bounded. Section 6 contains some standard energy estimates for the solutions. Section 7 provides a proof of the boundedness of the functional $Q(t)$ adapting the ideas of Pfaffelmoser for this problem, in order to deal with the geometrical complexity of the domain. This concludes the proof of the Theorem.

2. Preliminary notation and statement of the main result.

2.1. A more convenient coordinate systems near the singular set.

By assumption $\partial \Omega$ is a $C^5$ surface and we will parametrize it locally using a set of coordinates $(\mu_1, \mu_2)$. Let us denote as $x_\parallel (\mu_1, \mu_2)$ the point of $\partial \Omega$ characterized by the values of the parameters $(\mu_1, \mu_2)$. We will denote as $n(\mu_1, \mu_2)$ the outer normal to $\partial \Omega$ at the point $x_\parallel (\mu_1, \mu_2)$.

The Implicit Function Theorem shows that for $\delta > 0$ sufficiently small we can parametrize uniquely the set of points $\partial \Omega + B_\delta (0) \subset \mathbb{R}^3$ by means of the unique values $(\mu_1, \mu_2, x_\perp)$ solving the equation:

\begin{equation}
(2.1) \quad x = x_\parallel (\mu_1, \mu_2) - x_\perp n(\mu_1, \mu_2)
\end{equation}

Given $x \in \partial \Omega + B_\delta (0)$ we will represent any vector $v \in \mathbb{R}^3$ as:

\begin{equation}
(2.2) \quad v = v_\parallel (\mu_1, \mu_2) - v_\perp n(\mu_1, \mu_2)
\end{equation}
where \( v_{\|}(\mu_1, \mu_2) \in T_{x_{\|}(\mu_1, \mu_2)}(\partial \Omega), v_{\perp} \in \mathbb{R} \) and \((\mu_1, \mu_2)\) are as in (2.1). Moreover, we will represent \( v_{\|} = v_{\|}(\mu_1, \mu_2) \) using the two coordinates \((w_1, w_2)\) defined by means of:

\[
(2.3) \quad v_{\|} = w_1 u_1 + w_2 u_2
\]

where \(\{u_1, u_2\}\) are the basis of \(T_{x_{\|}(\mu_1, \mu_2)}(\partial \Omega)\) given by:

\[
(2.4) \quad u_i = \frac{\partial x_{\|}(\mu_1, \mu_2)}{\partial \mu_i}, \quad i = 1, 2.
\]

The system of coordinates \((\mu_1, \mu_2, x_{\perp}, w_1, w_2, v_{\perp})\) will provide a more convenient representation of the set of points in the phase space \(\Omega \times \mathbb{R}^3\) that are close to the singular set \(\Gamma\) defined in (1.10). The form that the original equation (1.1) takes in this new set of coordinates is given in the following Lemma:

**Lemma 1.** The equation (1.1) can be rewritten for \((x, v) \in [\partial \Omega + B_\delta(0)] \times \mathbb{R}^3\), and using the set of coordinates \((\mu_1, \mu_2, x_{\perp}, w_1, w_2, v_{\perp})\) in the form:

\[
(2.5) \quad \frac{\partial f}{\partial t} + \sum_{i=1}^{2} w_i \frac{\partial f}{\partial \mu_i} + v_{\perp} \frac{\partial f}{\partial x_{\perp}} + \sum_{i=1}^{2} \sigma_i \frac{\partial f}{\partial w_i} + F \frac{\partial f}{\partial v_{\perp}} = 0
\]

where:

\[
(2.6) \quad \sigma_i \equiv E_i - \frac{v_{\perp} w_i k_i}{1 + k_i x_{\perp}} - \sum_{j, \ell=1}^{2} \Gamma_{i,j,\ell} w_j w_\ell, \quad F \equiv E_{\perp} + \sum_{j=1}^{2} \frac{w_j^2 b_j}{1 + k_j x_{\perp}}
\]

where \(k_j\) are the principal curvatures, \(b_j\) are the coefficients \(e\) and \(g\) from the second fundamental form according to the notation in [15] and \(\Gamma_{i,j,\ell}\) are the Christoffel symbols of the surface \(\partial \Omega\). The vector \(E = \nabla_{x\phi}\) has been written in the form

\[
(2.7) \quad E = E_1 u_1 + E_2 u_2 - E_{\perp} n(\mu_1, \mu_2)
\]

where \(u_1, u_2\) are as in (2.4).

**Proof.** The proof of this result is just a standard lengthy change of variables that makes use of the classical Gauss-Weingarten equations (cf. [15], page 124).

**Remark 1.** The choice of coordinates (2.7) implies that the coefficient from the second fundamental form that is denoted as \(f\) in [15] is identically zero.

**Remark 2.** Notice that since the domain \(\Omega\) is convex, and due to (1.6) we have \(F < 0\).
2.2. Compatibility conditions for the initial data. In order to obtain classical solutions of \((1.1)-(1.5)\) we need to impose some compatibility conditions on the initial data \(f_0(x,v)\) at the reflection points of \(\partial \Omega \times \mathbb{R}^3\) (cf. \([6],[9]\)). These conditions are the following:

\begin{equation}
(2.8) \quad f_0(x,v) = f_0(x,v^*)
\end{equation}

\begin{equation}
(2.9) \quad v^+ \left[ \nabla_x^+ f_0(x,v^*) + \nabla_x^+ f_0(x,v) \right] + 2E_x(0,x) \nabla_v^+ f_0(x,v) = 0
\end{equation}

where \(E_x(0,x)\) is the decomposition of the field \(E(0,x)\) given by \((2.7)\) and \(\nabla_x^+, \nabla_v^+\) are the normal components to \(\partial \Omega\) of the gradients \(\nabla_x, \nabla_v\) respectively.

2.3. Flatness condition. The usual way of dealing with the impossibility of obtaining smooth solutions for general initial data \(f_0\) near the singular set consists is assuming that \(f_0\) is constant near such a set (cf. \([7]\) as well as \([9]\)). More precisely we will assume that \(f_0 \in C^{1,\mu}\) satisfies the following flatness condition near the singular set \(\Gamma\):

\begin{equation}
(2.10) \quad f_0(x,v) = \text{constant}, \quad \text{dist}((x,v), \Gamma) \leq \delta_0
\end{equation}

for some \(\delta > 0\) small.

We need to introduce some functional spaces for technical reasons. We define for \(\mu \in (0,1)\):

\[\|f\|_{C^{1,\mu}(\bar{\Omega} \times \mathbb{R}^3)} = \sup_{(x,v),(x',v') \in \bar{\Omega} \times \mathbb{R}^3} \left( \frac{|\nabla f(x,v) - \nabla f(x',v')|}{|x-x'|^\mu + |v-v'|^\mu} \right) + \|f\|_{L^\infty(\bar{\Omega} \times \mathbb{R}^3)} \quad \nabla = (\nabla_x, \nabla_v)\]

\[C^{1,\mu}_0(\bar{\Omega} \times \mathbb{R}^3) = \{ f \in C^{1,\mu}(\bar{\Omega} \times \mathbb{R}^3) : f \text{ compactly supported}, \|f\|_{C^{1,\mu}(\bar{\Omega} \times \mathbb{R}^3)} < \infty \}\]

\[\|f\|_{C^{1,\mu}_{t,x}(\Omega) \times \mathbb{R}^3}) = \sup_{x,x' \in \Omega, t,t' \in [0,T]} \frac{|\nabla_x f(t,x) - \nabla_x f(t',x')|}{|x-x'|^\mu} + \|f\|_{C([0,T] \times \Omega)} + \|f_t\|_{C([0,T] \times \Omega)}\]

\[\|f\|_{C^{1,\mu}_{t,x,v}(\Omega) \times \mathbb{R}^3}) = \sup_{x,x' \in \Omega, t,t' \in [0,T]} \frac{|\nabla_x f(t,x,v) - \nabla_x f(t',x',v)| + |\nabla_v f(t,x,v) - \nabla_v f(t',x',v')|}{|x-x'|^\mu + |v-v'|^\mu} + \|f\|_{C([0,T] \times \Omega \times \mathbb{R}^3)} + \|f_t\|_{C([0,T] \times \Omega \times \mathbb{R}^3)}\]

We define the spaces \(C([0,T] \times \hat{\Omega})\), \(C([0,T] \times \hat{\Omega} \times \mathbb{R}^3)\) as the spaces of continuous functions bounded in the uniform norm.

**Remark 3.** Theorem [7] that is the main result of this paper can be proved under a more general condition than \((2.10)\), namely under the a vanishing
condition similar to the one used in (6) in the half-line case. More precisely, Theorem 1 is valid under the assumption:

(2.11)\[ |f_0(x,v)| + \frac{1}{x_\perp + v_\perp^2} \left| \frac{\partial f_0}{\partial x}(x,v) \right| + \frac{1}{x_\perp + v_\perp^2} \left| \frac{\partial f_0}{\partial v}(x,v) \right| \leq C \left( x_\perp + v_\perp^2 \right)^\theta, \theta > 1 \]

We will explain at the relevant points the modifications that should be needed in the proof. However, we have focused mostly in the details of the proof under the more stringent assumption (2.10) for simplicity.

2.4. The main result: Global existence Theorem. The main result of this paper is the following:

**Theorem 1.** Let \( f_0 \in C^{1,\mu}_0(\Omega \times \mathbb{R}^3), f_0 \geq 0 \) for some \( 0 < \mu < 1 \) and satisfy (2.10) and suppose that \( h \in C^{2,\mu}(\partial \Omega) \) satisfies (1.7) and \( h(x_2, x_3) > 0 \). Then there exists a unique solution \( f \in C^{1,\lambda}(\Omega \times \mathbb{R}^3), \phi \in C^{1,3,\lambda}(\partial \Omega) \) with compact support in \( x \) and \( v \).

**Remark 4.** Theorem 1 could be derived using similar arguments for the case that the function \( h \) depends on time and that \( \frac{\partial h}{\partial t} \) is smooth enough. The arguments would require minor changes but we will just give the details of the argument in the case that \( \frac{\partial h}{\partial t} = 0 \) for simplicity.

3. Iterative procedure

The usual procedure of proving the existence of solutions for Vlasov-Poisson models consists in obtaining such a solution as the limit of a sequence of functions \( f^n \) that are defined by means of an iterative procedure. More precisely, we define:

\[
(3.1) \quad f^0(t,x,v) = f_0(x,v), \quad t \geq 0, \quad x \in \Omega, \quad v \in \mathbb{R}^3
\]

\[
(3.2) \quad f^n_t + v \cdot \nabla_x f^n + \nabla_x \phi^{n-1} \cdot \nabla_v f^n = 0, \quad x \in \Omega \subset \mathbb{R}^3, \quad v \in \mathbb{R}^3, \quad t > 0
\]

\[
(3.3) \quad \Delta \phi^{n-1} = \rho^{n-1}(x) \equiv \int_{\mathbb{R}^3} f^{n-1} dv, \quad x \in \Omega, \quad t > 0
\]

\[
(3.4) \quad \frac{\partial \phi^{n-1}}{\partial n} = h, \quad x \in \partial \Omega, \quad t > 0
\]

\[
(3.5) \quad f^n(0,x,v) = f_0(x,v), \quad x \in \Omega, \quad v \in \mathbb{R}^3
\]

\[
(3.6) \quad f^n(t,x,v) = f^n(t,x,v^*) \quad x \in \partial \Omega, \quad v \in \mathbb{R}^3, \quad t > 0
\]

for \( n = 1, 2, \ldots \). We assume that \( f_0, h \) satisfy (1.6), (1.7) as well as (2.10).

We will also use the notation:

\[
(3.7) \quad E^n = \nabla \phi^n
\]
Our goal is to show that the sequence $f^n$ converges as $n \to \infty$ for all $0 \leq t < \infty$. To this end we need to show as a first step that this sequence is globally defined in time for each $n \geq 0$.

4. Linear problem.

In order to show that the sequence $\{f^n\}$ is well defined we first study the well-posedness of the problem (1.1), (1.4), (1.5) under the assumption that the field $E = \nabla \phi$ is given and satisfies suitable smoothness conditions. Closely related results have been obtained in [6] for the half-line case where geometric complications are not present.

For further reference we define the evolution of the characteristic curves associated to (1.1), (1.5). More precisely, given the field $E = \nabla x \phi$ we define for each $(x, v) \in \Omega \times \mathbb{R}^3$ the generalized characteristic curve $(X(t; t, x, v), V(t; t, x, v))$ by the following differential equations:

\begin{align}
\frac{dX}{ds} &= V \\
\frac{dV}{ds} &= E = \nabla x \phi \\
X(t; t, x, v) &= x, \quad V(t; t, x, v) = v
\end{align}

as long as $X \in \Omega$. We extend this definition to arbitrarily long times assuming that at the times $s = s^*$ when $X_n(s^*; t, x, v) \in \partial\Omega$ the velocity $V$ bounces elastically at the boundary, i.e.:

\begin{align}
V((s^*)^\dagger; t, x, v) &\equiv \lim_{s \to s^*, s > s^*} V(s; t, x, v) = (V((s^*)^-; t, x, v))^* \equiv \left( \lim_{s \to s^*, s < s^*} V(s; t, x, v) \right)^*
\end{align}

where $(\cdot)^*$ is as in (1.8).

**Theorem 2.** Assume that $E \in C^{0;\mu}_{t; x}([0, T] \times \Omega)$ for some $\mu \in (0, 1)$, and $E \cdot n = h > 0$ at $\partial\Omega$. Suppose that $f_0 \in C^{1,\mu}_{0; x} (\Omega \times \mathbb{R}^3)$, $f_0 \geq 0$ for some $\mu > 0$. Then there exists a unique $f \in C^{1,1,\lambda}_{t; (x,v)} ([0, T] \times \Omega \times \mathbb{R}^3)$, solution to the linear Vlasov-Poisson system (1.1), (1.4), (1.5), for some $0 < \lambda < \mu$. Moreover the function $f$ satisfies:

\begin{align}
f(t, x, v) &\geq 0 \\
\int f(t, x, v) \, dx dv &= \int f_0(x, v) \, dx dv \quad t \in [0, T]
\end{align}

We will introduce a new coordinate system that will be convenient to study the dynamics of the characteristic curves for bouncing trajectories. Suppose that $(\mu_1, \mu_2, x_\perp, w_1, w_2, v_\perp)$ are as in (2.1)-(2.3). We then define two new coordinates $(\alpha(t, \mu_1, \mu_2, x_\perp, w_1, w_2, v_\perp), \beta(t, \mu_1, \mu_2, x_\perp, w_1, w_2, v_\perp))$ as
follows:

\[
\alpha(t, \mu_1, \mu_2, x_\perp, w_1, w_2, v_\perp) = \frac{v_\perp^2}{2} - F(t, \mu_1, \mu_2, 0, w_1, w_2) x_\perp,
\]

(4.7)

\[
\beta(t, \mu_1, \mu_2, x_\perp, w_1, w_2, v_\perp) - 2\pi H(t, \mu_1, \mu_2, x_\perp, w_1, w_2, v_\perp) = \pi \left( 1 - \frac{v_\perp}{\sqrt{2\alpha}} \right) .
\]

(4.8)

where the function \( H \) will increase by one at each bounce and \( F \) is as in (2.6). Since \( v_\perp \) changes from \(-\sqrt{2\alpha}\) to \(\sqrt{2\alpha}\) in each bounce, it follows that \( \beta \) is continuous along characteristics. Notice that \( \beta \) is just a coordinate that indicates the specific point in the surface \( \{ \alpha = \text{constant} \} \) where the trajectory lies. It does not have the specific meaning of an angle, although we have normalized their variation by \(2\pi\) between bounces. Its functional form has been chosen only for convenience. In all the following we will write for simplicity \( F(t, 0) \) instead of \( F(t, \mu_1, \mu_2, 0, w_1, w_2) \) and we will drop the dependence of \( H \) on the variables \( \mu_1, \mu_2, x_\perp, w_1, w_2, v_\perp \) if there is no risk of confusion. We can then write:

\[
x_\perp = -\frac{\alpha}{F(t, 0)} \left[ 1 - \left( 1 - \frac{\beta - 2\pi H(t)}{\pi} \right)^2 \right] ,
\]

\[
v_\perp = \sqrt{2\alpha} \left( 1 - \frac{\beta - 2\pi H(t)}{\pi} \right) .
\]

Making the change of variables \((t, \mu_1, \mu_2, x_\perp, w_1, w_2, v_\perp) \rightarrow (t, \mu_1, \mu_2, \alpha, w_1, w_2, \beta)\) we transform the system (2.5) as follows:

\[
\begin{align*}
\frac{\partial f}{\partial t} + \sum_{i=1}^{2} \frac{w_i}{1+k_i x_\perp} \frac{\partial f}{\partial \mu_i} + \sum_{i=1}^{2} \sigma_i \frac{\partial f}{\partial w_i} \\
+ \left[ v_\perp (F(t, x_\perp) - F(t, 0)) - x_\perp \left\{ \sum_{i=1}^{2} \left( \frac{w_i}{1+k_i x_\perp} \frac{\partial F(t, 0)}{\partial \mu_i} + \sigma_i \frac{\partial F(t, 0)}{\partial w_i} \right) \right\} \frac{\partial f}{\partial \alpha} \\
+ \left[ -\frac{\pi v_\perp^2 F(t, 0)}{(2\alpha)^{3/2}} \frac{2\pi F(t, 0) (F(t, x_\perp) x_\perp)}{(2\alpha)^{3/2}} - \frac{\pi x_\perp v_\perp}{(2\alpha)^{3/2}} \left\{ \sum_{i=1}^{2} \left( \frac{w_i}{1+k_i x_\perp} \frac{\partial F(t, 0)}{\partial \mu_i} + \sigma_i \frac{\partial F(t, 0)}{\partial w_i} \right) \right\} \frac{\partial f}{\partial \beta} \right] \right) = 0,
\end{align*}
\]

(4.9)
Proof. The inequality $F(t, \mu_1, \mu_2, 0, w_1, w_2) \leq -\varepsilon_0 < 0$ is a consequence of the convexity of $\Omega$ which implies that the term $\sum_{j=1}^{2} \frac{w_{j}^{2} b_j}{1 + k_j x_{\perp}}$ in (2.6) is negative near $\partial \Omega$. On the other hand the continuity of the function $h$ in (1.3) implies that $E_{\perp} = -h \leq -\varepsilon_0 < 0$ whence the result follows. The derivation of the equation (4.9) follows from (4.7)-(4.8). □

Remark 5. We can rewrite (4.9) as:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{2} \frac{w_i}{1 + k_i x_{\perp}} \frac{\partial f}{\partial \mu_i} + \sum_{i=1}^{2} \sigma_i \frac{\partial f}{\partial w_i} + \left[ \frac{v_{\perp} (F(t, x_{\perp}) - F(t, 0)) - x_{\perp}}{2 (2\alpha)^{3/2}} - \frac{\pi F(t, 0)}{\sqrt{2\alpha}} + 2\pi F(t, 0) \left[ F(t, x_{\perp}) - F(t, 0) \right] x_{\perp} - \frac{\pi x_{\perp} v_{\perp}}{(2\alpha)^{3/2}} \right] \frac{\partial f}{\partial \alpha}$$

$$= 0,$$

(4.10)

Remark 6. Notice that the dynamics of the tangential part to $\partial \Omega$ of the characteristics at the singular set is given by the equations:

$$\frac{d\mu_i}{dt} = w_i, \quad \frac{dw_i}{dt} = \sigma_i$$

Remark 7. The compatibility conditions (2.8), (2.9) imply that the original data $f_0$ written in the variables $(t, \mu_1, \mu_2, \alpha, w_1, w_2, \beta)$ is a $C^{1,\mu}$ function. Moreover, the estimate (2.10) implies that $f_0$ is constant for $0 \leq \alpha \leq C\delta_0$ for some $C > 0$ fixed.

4.1. Velocity Lemma. The following result has been proved in [6, 9] in a slightly different manner.

Lemma 3. Given $\delta > 0$ fixed we define $\Gamma_\delta \equiv \left[ \partial \Omega + B_{\delta}(0) \right] \cap \Omega \times \mathbb{R}^{3}$. Suppose that $E$ satisfies the regularity assumptions in Theorem 2. Then, the characteristic equations (4.1)-(4.3) can be solved in the interval $t \in [0, T]$ for any $(x, v) \in \Omega \times \mathbb{R}^{3}$. Moreover, there exist positive constants $C_1, C_2$ depending only on $T, f_0, \|\nabla E\|_{L^\infty([0, T], C^{1/2}(\Omega))}$ such that, for any $(x, v) \in \Gamma_\delta$, the following estimate holds:

$$C_1 \left( v_{\perp}^{2}(0) + x_{\perp}(0) \right) \leq \left( v_{\perp}^{2}(t) + x_{\perp}(t) \right) \leq C_2 \left( v_{\perp}^{2}(0) + x_{\perp}(0) \right), \quad 0 \leq t \leq T$$

(4.11)

Proof. Using the estimate $F \leq -\varepsilon_0 < 0$ in Lemma 2 as well as (4.7), it follows that $\alpha$ is equivalent to $v_{\perp}^{2} + x_{\perp}$. Due to the boundedness of $\Omega$ it is enough to prove this result for small values of $v_{\perp}^{2}(0) + x_{\perp}(0)$, i.e. for
points that are close to the singular set. Using (4.10) we obtain along the characteristic curves:

\[
\frac{d\alpha}{dt} = v_\perp (F(t, x_\perp) - F(t, 0)) - x_\perp \left\{ \sum_{i=1}^{2} \left( \frac{w_i}{1 + k_i x_\perp} \frac{\partial F(t, 0)}{\partial \mu_i} + \sigma_i \frac{\partial F(t, 0)}{\partial w_i} \right) \right\}
\]

Notice, however that keeping this term would not change the essence of the argument if \(\frac{\partial h}{\partial t}\) is smooth as indicated in Remark 4.

Using our regularity assumptions on \(E\) we obtain the estimate:

\[
\left| \frac{d\alpha}{dt} \right| \leq C \| \nabla E \|_{C^{1/2}(\Omega)} |v_\perp| (x_\perp)^{1/2} + C x_\perp
\]

where \(C\) depends only on \(h\) and the geometric properties of \(\partial \Omega\).

Since \(v_\perp^2 + x_\perp \leq K\alpha\) for some positive constant \(K\) depending on \(h, \partial \Omega\). It then follows that:

(4.12) \[
\left| \frac{d\alpha}{dt} \right| \leq C\alpha
\]

Therefore:

\[
C_1 \alpha(0) \leq \alpha(t) \leq C_2 \alpha(0), \quad 0 \leq t \leq T
\]

for some \(C_1, C_2\) depending on \(f_0, \| \nabla E \|_{C^{1/2}(\Omega)}\), whence (4.11) follows. \(\square\)

4.2. Well-posedness of the linear problem.

Proof of Theorem 2: The proof of this theorem just follows by integrating the equation along the characteristics, combined with the reflecting boundary condition. The existence and uniqueness of solutions to (1.1)-(1.5) are not immediate due to the bounces of the characteristics at the boundary \(\partial \Omega\), but the well-posedness of this evolution has been obtained in [9]. We will define the sequence of functions \(f_n\) by means of the evolution of the characteristics associated to (4.1)-(4.4). Then:

(4.13) \[
f(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v))
\]

In order to show that this procedure defines the function \(f\) globally in time we need to show that for each \((x, v) \in \Omega \times \mathbb{R}^3\) the curves defined by means of (4.1)-(4.4) intersect the boundary \(\partial \Omega \times \mathbb{R}^3\) at most a finite number of times, and in particular they never intersect with the singular set. This fact is a consequence of the Lemma 3, since for any trajectory starting at \(\Omega \times \mathbb{R}^3\) at time \(s = t\) we have \(\alpha > 0\) at time \(s = t\), and therefore \(\alpha\) remains bounded below and above during the evolution of the trajectory in the interval \(s \in [0, t]\). Moreover, the characteristics starting in the region \(\{\alpha \leq C \delta_0\}\) where \(f_0\) is constant, remain in a set of the form \(\{\alpha \leq C_1(T) \delta_0\}\) for all \(0 \leq s \leq T\), and the characteristics starting in the region \(\{\alpha > C \delta_0\}\) where \(f_0\) is not necessarily constant remain in a set of the form \(\{\alpha > C_2(T) \delta_0\}\) for \(0 \leq s \leq T\). In the first set \(f = \text{constant}\). In the second one we have that \(\frac{d\alpha}{dt}\) is bounded by \(\frac{C}{\sqrt{\delta_0}}\), where the bound on the number of bounces is...
uniformly bounded by $\frac{C}{\sqrt{\delta}}$. Finally we notice that, since $E \in C^{1,\mu}_x$, classical regularity estimates for the solutions of ODEs show that the functions $X(s; t, x, v), \ V(s; t, x, v)$ are $C^{1,\mu}$ with respect to the variables $(x, v)$ as long as there is no bounces. Moreover, if a trajectory intersects $\partial \Omega \times \mathbb{R}^3$ our regularity assumptions on $\partial \Omega$ imply also $C^{1,\mu}$ regularity with respect to $(x, v)$ for the values of the coordinates $X$ where the function $X(s; t, x, v)$ intersects $\partial \Omega$, as well as for the time $s = s(t, x, v)$ when such intersection takes place (cf. [7], [9]). Therefore, for trajectories bouncing a finite number of times, the functions $X(s; t, x, v), \ V(s; t, x, v)$ can be written as the composition of a finite number of $C^{1,\mu}$ functions with respect to the variables $(x, v)$. This proves that the function $f$ defined by means of (4.13) is H"{o}lder with respect to $(x, v)$. The uniqueness of the solution is due to the fact that the solution is uniquely determined by the evolution of the characteristic curves. This concludes the proof of the result. □

Remark 8. The proof of this Theorem is where there would be a relevant difference if the assumption (2.11) had been used instead of (2.10). Indeed, if (2.11) had been assumed, the number of bounces would not be uniformly bounded for $0 \leq t \leq T$. However, it is possible to argue as in [6] to show that Theorem 3 holds replacing (2.10) by (2.11). In our setting the way of proving this would be to rewrite the characteristic equations associated to (4.10) as:

\[
\begin{align*}
\frac{d\mu_i}{dt} &= h_1 (\mu_i, w_i, \alpha, \beta \sqrt{\alpha}) \\
\frac{dw_i}{dt} &= h_2 (t, \mu_i, w_i, \alpha, \beta \sqrt{\alpha}) \\
\frac{d\alpha}{dt} &= v_\perp (F^n(t, x_\perp) - F^n(t, 0)) - x_\perp h_3 (\mu_i, w_i, \alpha, \beta \sqrt{\alpha}) \\
\frac{d\beta}{dt} &= -\frac{\pi F^n(t, 0)}{\sqrt{2\alpha}} + \frac{2\pi F^n(t, 0) (F^n(t, x_\perp) - F^n(t, 0)) x_\perp}{(2\alpha)^{\frac{3}{2}}} + h_4 (\mu_i, w_i, \alpha, \beta \sqrt{\alpha})
\end{align*}
\]

where the function $h_k$ are smooth in their arguments. The Velocity Lemma implies that, for a given trajectory the order of magnitude of $\alpha$ does not change in a time interval $0 \leq t \leq T$. Suppose that we consider trajectories with $\alpha$ of order $R \leq 1$. Using the change of variables $\alpha = R\tilde{\alpha}, \beta = \frac{1}{\sqrt{R}}\tilde{\beta}$ we would transform (4.14) in a similar system of equations for the variables $\tilde{\zeta} = (\mu_i, w_i, \tilde{\alpha}, \tilde{\beta})$ with the nonlinearities and their derivatives bounded. Classical regularity theory for this system would imply that, for $\tilde{\alpha}$ of order one $\frac{\partial \tilde{\zeta}}{\partial \tilde{\alpha}_0}$ with $\tilde{\zeta}_0 = (\mu_i, w_i, 0, 0)\tilde{\beta}_0$ would be bounded, as well as the H"{o}lder norms evaluated at the points with $\tilde{\alpha}$ of order one. Returning to the original variables $\alpha, \beta$ it would then follow that the worse derivative would be $\frac{\partial \beta}{\partial \alpha_0}$ that would be bounded as $\frac{1}{\sqrt{R}}$. In general, the rescaling for each
factor $\alpha$ would include a term of order $\sqrt{R}$. Due to the decay of $f_0$, $\nabla f_0$ near the singular set it would then follows that $\nabla f$ would be bounded like a power law near the singular set. Estimates for the Hölder norms of $\nabla f$ would then be obtained using the holderianity of $f_0$ at distances of order one from the singular set, and using the decay of $\nabla f$ near the singular set, as well as the fact that $\frac{|\nabla f(x_1,v_1)-\nabla f(x_2,v_2)|}{|(x_1,v_1)-(x_2,v_2)|} \leq C \frac{\alpha^3}{\alpha}$ where $\alpha$ is the largest one of the corresponding value associated to $(x_1,v_1)$ or $(x_2,v_2)$.

5. On the convergence of the sequence $\{f_n\}$.

5.1. Representation formula for the solutions of the Poisson equation with Neumann boundary conditions. The following result is standard. We just include it here for further references:

**Proposition 1.** Given a bounded domain $\Omega$ in $\mathbb{R}^3$ with a smooth boundary $\partial \Omega$, there exists a Green’s function $G(x,y)$ for the Laplacian operator with Neuman boundary conditions:

\begin{align}
\Delta \phi &= \rho(x), \; x \in \Omega \\
\frac{\partial \phi}{\partial n} &= h, \; x \in \partial \Omega
\end{align}

with the compatibility condition

\begin{equation}
\int_\Omega \rho(x) \, dx = \int_{\partial \Omega} h(x) \, dS_x
\end{equation}

is given by the following representation formula:

\begin{equation}
\phi(x) = \int_\Omega G(x,y) \rho(y) \, dy - \int_{\partial \Omega} G(x,y) h(y) \, dS_y
\end{equation}

Any other solution of (5.1)-(5.3) is given by (5.4) up to an additive constant. The function $G(x,y)$ satisfies the following estimates:

\begin{align}
|G(x,y)| &\leq \frac{C}{|x-y|}, \quad |\nabla_x G(x,y)| \leq \frac{C}{|x-y|^2} \\
|\nabla_x^2 G(x,y)| &\leq \frac{C}{|x-y|^3}, \quad x, y \in \Omega
\end{align}

where $C$ depends only on the domain $\Omega$.

**Proof.** This result is well known. See for instance [3]. \hfill \Box

5.2. The iterative sequence $\{f_n\}$ is globally defined in time. We will need the following auxiliary Lemma that states that given $f$ bounded in the space $C_{t,x,v}^{1;1,\lambda} \left([0,T] \times \Omega \times \mathbb{R}^3\right)$, the corresponding field $E$ defined by $E = \nabla \phi$, with $\phi$ satisfying (1.2), (1.3) satisfies the regularity estimates required in the Theorems 2.
Given a function \( g : \Omega \to \mathbb{R} \), we will denote as \([\cdot]_{0, \lambda; x}\) the seminorm:

\[
[g]_{0, \lambda; x} = \sup_{x, y \in \Omega} \frac{|g(x) - g(y)|}{|x - y|^\lambda}
\]

We define:

\[(5.6) \quad Q(t) \equiv \sup \{ |v| \mid (x, v) \in \text{supp } f(s), \quad 0 \leq s \leq t \}.
\]

We then have the following result

**Lemma 4.** Suppose that \( f \in C^{1,1, \lambda}_{t, (x, v)}([0, T] \times \Omega \times \mathbb{R}^3) \). Then, the following estimates hold:

\[(5.7) \quad |\rho(t, x)| = \left| \int f(t, x, v) \, dv \right| \leq C(T) \|f\|_{C^{1,1, \lambda}_{t, (x, v)}([0, T] \times \Omega \times \mathbb{R}^3)}, \quad x \in \Omega \times [0, T]
\]

\[(5.8) \quad |E(t, x)| \leq C(T) \left( \|f\|_{C^{1,1, \lambda}_{t, (x, v)}([0, T] \times \Omega \times \mathbb{R}^3)} + 1 \right), \quad x \in \Omega \times [0, T]
\]

\[(5.9) \quad |F(t, x)| \leq C(T) \left( \|f\|_{C^{1,1, \lambda}_{t, (x, v)}([0, T] \times \Omega \times \mathbb{R}^3)} + 1 \right), \quad x \in \partial \Omega \times [0, T]
\]

\[(5.10) \quad |\nabla \rho(t, x)| \leq \int |\partial_x f(t, x, v)| \, dv \leq C(T) \|f\|_{C^{1,1, \lambda}_{t, (x, v)}([0, T] \times \Omega \times \mathbb{R}^3)}, \quad x \in \Omega \times [0, T]
\]

\[(5.11) \quad |\nabla E(t, x)| + |\nabla (E(t, \cdot))_{0, \lambda; x} + [\nabla^2 E(t, \cdot)]_{0, \lambda; x} \leq C(T) \|f\|_{C^{1,1, \lambda}_{t, (x, v)}([0, T] \times \Omega \times \mathbb{R}^3)}, \quad x \in \Omega \times [0, T]
\]

\[(5.12) \quad |\rho_t(t, x)| \leq C(T) \|f\|_{C^{1,1, \lambda}_{t, (x, v)}([0, T] \times \Omega \times \mathbb{R}^3)}, \quad x \in \Omega \times [0, T]
\]

\[(5.13) \quad |E_t(t, x)| \leq C(T) \|f\|_{C^{1,1, \lambda}_{t, (x, v)}([0, T] \times \Omega \times \mathbb{R}^3)}, \quad x \in \Omega \times [0, T]
\]

where \( C(T) > 0 \) depends on \( Q(T) \) and \( T \).

**Proof.** The estimate (5.7) is a consequence of the boundedness of the support of \( f \) as well as the boundedness of the \( L^\infty \) norm of \( f \). The inequality (5.8) follows using a standard regularity theory for the Poisson equation (cf. [4], [12]). The estimate (5.9) is a consequence of the definition of \( F \) in (2.6) and our regularity assumptions on \( \partial \Omega \) and \( h \). The inequality (5.10) is just a consequence of the regularity properties of \( f \) and the boundedness of its support. Then (5.11) is a consequence of classical regularity theory for the Poisson equation. Similarly we can deduce (5.12) and (5.13). Thus the proof is complete. \( \square \)
Proposition 2. Let \( \mu, \lambda \in (0, 1) \), satisfying \( \mu > \lambda \). Let \( f_0 \in C_{0}^{1, \mu}(\Omega \times \mathbb{R}^{3}) \), \( f_0 \geq 0 \) satisfy \((2.7)\). Suppose \( h \in C_{1}^{1, \mu}(\partial \Omega) \), \( h > 0 \). Then, the sequence of functions \( f^n \) is globally defined for each \( x \in \Omega, v \in \mathbb{R}^{3} \) and \( 0 \leq t < \infty \). Moreover we have \( f^n \in C_{t(x,v)}^{1,1, \lambda}([0,T] \times \Omega \times \mathbb{R}^{3}) \) for any \( T > 0 \) and \( \parallel f^n \parallel_{\infty} = \parallel f_0 \parallel_{\infty} \), \( \int \rho_n(x,t) \, dx = \int f_0 (t,x,v) \, dxdv \).

Proof. We argue by induction. If \( n = 1 \) we use the fact that \( |\nabla \phi_0| \) is bounded to obtain that \( f^1 \) is supported in the region where \( |v| \leq C (1 + t) \). Then, \( \rho_1 \) is bounded by \( C (1 + t)^3 \). Moreover, due to the regularity of \( h \), we can apply Theorem 2 to show that \( f^1 \) is bounded in \( C_{t(x,v)}^{1,1, \lambda}([0,T] \times \Omega \times \mathbb{R}^{3}) \) for any \( T > 0 \). Applying then Lemma 4 it follows that the norm \( \parallel E^n \parallel_{C_{t(x,v)}^{1,1, \lambda}([0,T] \times \Omega)} \) is bounded. We can then apply Theorem 2 to prove that \( f^2 \) is well defined in \( C_{1}^{1, \lambda} \) for \( 0 \leq t < \infty \). Moreover, the support of \( f^2 \) would be contained in the region \( |v| \leq Ch_2(t) \) where \( h_2(t) \) is a continuous increasing function in \( t \) and using again Theorem 2 it follows that \( f^2 \) is bounded in \( C_{t(x,v)}^{1,1, \lambda}([0,T] \times \Omega \times \mathbb{R}^{3}) \) for any \( T > 0 \). Iterating the argument we obtain that the sequence \( f^n \) is defined as indicated. Finally, using the fact that \( f^n \) just propagates along the characteristics we obtain the conservation of \( \parallel f^n \parallel_{\infty} \). The conservation of the total mass just follows integrating the equation (1.1) with respect to the variables \( x,v \) whence the proposition follows. \( \square \)

5.3. The sequence \( \{ f^n \} \) converges to a solution of the VP system if the sequence \( \{ Q^n \} \) is bounded. We define the following measure for the maximal velocities reached for the distribution \( f^n \):

\[
Q^n(t) \equiv \sup \{ |v| \mid (x,v) \in \text{supp} \, f^n(s), \; 0 \leq s \leq t \}.
\]

Proposition 3. Under the assumptions of Theorem 1, suppose that \( Q^n(t) \leq K \) for \( n \geq n_0 \), \( 0 \leq t \leq T \). Then, \( f^n \to f \) in \( C_{t(x,v)}^{1,1, \lambda}([0,T] \times \Omega \times \mathbb{R}^{3}) \) as \( n \to \infty \) with \( 0 < \lambda < \mu \), \( 0 < \nu < 1 \) and where \( f \in C_{t(x,v)}^{1,1, \lambda}([0,T] \times \Omega \times \mathbb{R}^{3}) \) is a solution of (1.1)–(1.5).

In the Proof of this Proposition we will use some auxiliary Lemmas. The first one similar to the one in Theorem 6.2, p 309 of [1].

Lemma 5. Suppose the assumptions on Theorem 1 are satisfied and that \( Q^n(t) \leq K \) for \( n \geq n_0 \), \( 0 \leq t \leq T \). Then:

\[
|E^n(x,t)| \leq C(t)
\]

\[
|E^n(x,t)|_{C_{\gamma}(\Omega)} \leq C(T), \; \text{for any } \gamma \in (0,1)
\]

for \( n \geq n_0+1, \; 0 \leq t \leq T \), where \( C(T) \) depends only on \( K, \parallel f_0 \parallel_{L^\infty(\Omega \times \mathbb{R}^{3})} \), \( T \).
proof. Estimate \([5.15]\) follows from classical regularity theory for the Poisson equation as well as the fact that the density \(\rho^n\) can be estimated in \(L^\infty\) in terms of only \(Q^n(t)\) and the initial data.

Indeed, the uniform boundedness of \(Q^n(t)\) implies that we can estimate \(\rho^n\) in the interval \(0 \leq t \leq T\) uniformly in \(n\). Therefore, regularity theory for the Poisson equation implies that \(E^n\) is bounded in \(W^{1,p}(\Omega)\) for any \(1 < p < \infty\) uniformly in \(n\). Classical embedding results then imply that \(E^n\) is bounded in \(C^\gamma(\Omega)\) for any \(\gamma \in (0,1)\) and the result follows. \(\square\)

We now prove the following basic Lemma that shows that the boundedness of \(Q^n\) implies the boundedness of \(f\) in the form:

\[
\|f^n\|_{C^{1,1,\lambda}_{t;\Omega}([0,T] \times \Omega \times \mathbb{R}^3)} \leq C(T), \quad 0 \leq t \leq T
\]

for \(\lambda < \mu, n \geq n_0 + 1\), where \(C(T)\) depends only on \(K\) and \(T\).

Proof. Indeed, the estimates \([5.15], [5.16]\) imply that \(E^n\) is uniformly bounded on the Hölder spaces \(C^\gamma\) for \(0 < \gamma < 1\). Notice that choosing \(\gamma > \frac{1}{2}\) we can derive the estimates in the velocity Lemma (cf. [3]) uniformly in \(n\). In particular this implies that \(\alpha \geq C\delta_0\) uniformly in \(n\) for \(t \in [0,T]\).

We can write the characteristic equations for \([4.10]\) as:

\[
\begin{align*}
\frac{d\mu_i}{dt} &= f_1(\mu_i, w_i, \alpha, \beta) \\
\frac{dw_i}{dt} &= f_2(t, \mu_i, w_i, \alpha, \beta) \\
\frac{d\alpha}{dt} &= v_\perp (F^n(t, x_\perp) - F^n(t, 0)) - x_\perp f_3(\mu_i, w_i, \alpha, \beta) \\
\frac{d\beta}{dt} &= -\frac{\pi F^n(t, 0)}{\sqrt{2\alpha}} + \frac{2\pi F^n(t, 0) (F^n(t, x_\perp) - F^n(t, 0)) x_\perp}{(2\alpha)^{\frac{3}{2}}} + f_4(\mu_i, w_i, \alpha, \beta)
\end{align*}
\]

where the functions \(f_1, f_3, f_4\) depend only on the regularity properties of \(\partial\Omega\) and the boundary value \(h\), and therefore are smooth. The function \(f_2\) depends also on the field \(E\).

We can now take the Hölder derivative of \([5.18]\) with respect to the initial data \(\zeta_0 = (\mu_{i,0}, w_{i,0}, \alpha_0, \beta_0)\). Let us write also \(\zeta = (\mu_i, w_i, \alpha, \beta)\) and:

\[
[g]_{\lambda;\zeta_0} = \sup_{|(x_0, v_0) - (x_0', v_0')| \leq 1} \frac{|g(x_0, v_0) - g(x_0', v_0')|}{|\zeta_0 - \zeta_0'|^{\lambda}}
\]

Since, \(\alpha\) is uniformly bounded below and using also the uniform boundedness of \(Q^n\) it follows that for any \(\lambda < \mu\) \([\zeta]_{\lambda;\zeta_0}\) satisfies an inequality of the form:

\[
\left| \frac{d}{dt} ([\zeta]_{\lambda;\zeta_0}) \right| \leq C[\zeta]_{\lambda;\zeta_0}
\]
Therefore \( \| \zeta \|_{\lambda, \Omega} \leq C(T) \) where \( C(T) \) depends only on \( K \) and \( T \). Using the formula \( f_0(x_0, v_0) = f^n(t, x, v) \), we obtain a uniform estimate for \( f^n(t, \cdot, \cdot) \) in \( C^\lambda_{(x, v)}(\Omega \times \mathbb{R}^3) \) for any \( 0 < \lambda < \mu \). It then follows that \( \rho^n \) is uniformly bounded in \( C^\lambda_{\Omega}(\Omega) \) and using classical regularity theory for the Poisson equation it then follows that \( E^n(t, \cdot) \) is uniformly bounded in \( C^\lambda_{(x, v)}(\Omega) \).

Using the formula \( f_0(x_0, v_0) = f^n(t, x, v) \), we obtain a uniform estimate for \( f^n(t, \cdot, \cdot) \) in \( C_{\lambda, \Omega}(\Omega \times \mathbb{R}^3) \) for any \( 0 < \lambda < \mu \). It then follows that \( \rho^n \) is uniformly bounded in \( C_{\lambda, x, \Omega}(\Omega) \) and using classical regularity theory for the Poisson equation it then follows that \( E^n(t, \cdot) \) is uniformly bounded in \( C^\lambda_{(x, v)}(\Omega \times \mathbb{R}^3) \).

Using again the lower bound for \( \alpha \) we can now apply standard regularity results for ODEs to (5.18) to obtain that

\[
\frac{\partial \zeta}{\partial \zeta_0} \lambda, \zeta_0 \text{ is uniformly bounded for } 0 \leq t \leq T.
\]

Therefore \( f^n \) is uniformly bounded in \( C^\lambda_{(x, v)}(\Omega \times \mathbb{R}^3) \).

We finally conclude the Proof of Proposition 3.

**Proof of Proposition 3.** We recall that the sequence \( f^n \) has been defined by the iteration (3.1)-(3.7).

We claim that the sequence is a Cauchy sequence in \( L^1([0, T] \times \Omega \times \mathbb{R}^3) \).

To prove the claim, let \( f^{n+1} \) and \( f^n \) be consecutive elements of the sequence \( \{f^n\} \):

\[
\begin{align*}
f^{{n+1}t} + v \cdot \nabla_x f^{n+1} + \nabla_x \phi^n \cdot \nabla_v f^{n+1} &= 0, \\
f^n + v \cdot \nabla_x f^n + \nabla_x \phi^{n-1} \cdot \nabla_v f^n &= 0.
\end{align*}
\]

Subtracting (5.21) from (5.20) yields

\[
(f^{n+1} - f^n) + v \cdot \nabla_x (f^{n+1} - f^n) + \nabla_x \phi^{n} \cdot \nabla_v (f^{n+1} - f^n) = (\nabla_x \phi^{n-1} - \nabla_x \phi^{n}) \cdot \nabla_v f^n.
\]

**Remark 9.** The result in Lemma 6 can be proved if (2.10) is replaced by (2.11) using the rescaling argument in Remark 8.

The following simplectic property is a standard consequence of the fact that the evolution of the characteristics curves is hamiltonian.

**Lemma 7.** Moreover, let us denote as \( (X(s; t, x, v), V(s; t, x, v)) \) the solution of the characteristic equations (4.1)-(4.3). For any \( s \in [0, T] \) the transformation

\[
(x, t) \rightarrow (X(s; t, x, v), V(s; t, x, v))
\]

is simplectic. In particular:

\[
dX(s; t, x, v) dV(s; t, x, v) = dxdv
\]

We finally conclude the Proof of Proposition 3.
where:

\[ \text{where:} \]

\[ \frac{dX}{ds} = V, \quad \frac{dV}{ds} = \nabla_x \phi^n (s, X(s)) \]

By integrating (5.22) along the trajectory \((X(t), V(t))\) with \(X(t) = x\) and \(V(t) = v\) we get:

\begin{align*}
(f^n + f^n) (x, v, t) &= (f^n |_{t=0} - f^n |_{t=0}) (X(0), V(0)) \\
&+ \int_0^t (\nabla_x \phi^{n-1} - \nabla_x \phi^n) (s, X(s)) \cdot \nabla_v f^n (s, X(s), V(s)) \, ds \\
&= \int_0^t (\nabla_x \phi^{n-1} - \nabla_x \phi^n) (s, X(s)) \cdot \nabla_v f^n (s, X(s), V(s)) \, ds
\end{align*}

where:

\[ \frac{dX}{ds} = V, \quad \frac{dV}{ds} = \nabla_x \phi^n (s, X(s)) \]

Applying the representation formula (5.4) and the estimates (5.5) to compute the difference \((\nabla_x \phi^{n-1} - \nabla_x \phi^n) (s, x)\) we obtain:

\[ |(\nabla_x \phi^{n-1} - \nabla_x \phi^n) (s, x)| \leq C \int \frac{\rho^n (y) - \rho^{n-1} (y)}{|x - y|^2} \, dy \]

Integrating (5.23) over the phase space \((x, v)\) we obtain, applying Fubini’s theorem:

\[ \| f^n + f^n (t) \|_{L^1 (\Omega \times \mathbb{R}^3)} \]

\[ \leq \int_0^t \int_{\Omega \times \mathbb{R}^3} |(\nabla_x \phi^{n-1} - \nabla_x \phi^n) (s, X(s))| \| \nabla_v f^n (s, X(s), V(s)) \|  \, dv \, dx \, ds \\
\]

\[ \leq C \int_0^t \int_{\Omega} K^n (y, s) |\rho^n (s, y) - \rho^{n-1} (s, y)| \, dy \, ds, \]

where:

\[ K^n (y, s) \equiv \int_{\Omega \times \mathbb{R}^3} \frac{1}{|X(s) - y|^2} |\nabla_v f^n (s, X(s), V(s))| \, dX(s) \, dV(s) \]

\[ = \int_{\Omega \times \mathbb{R}^3} \frac{1}{|x - y|^2} |\nabla_v f^n (s, x, v)| \, dx \, dv. \]

In the last identity we used the Liouville principle [5, 19]. We now estimate \(K^n (x, t)\) as follows:

\[ K^n (x, t) = \int_{|y-x| \leq r} \frac{\| \nabla_v f^n (t, y, \cdot) \|_{L_x^1}}{|x - y|^2} \, dy + \int_{|y-x| \geq r} \frac{\| \nabla_v f^n (t, y, \cdot) \|_{L_x^1}}{|x - y|^2} \, dy \]

\[ \leq C r \| \nabla_v f^n (t, y) \|_{L^\infty (L_x^1)} + \frac{\| \nabla_v f^n (t, \cdot) \|_{L_x^1 (L_v^1)}}{r^2}. \]

We choose \(r\) to optimize the right hand side of the above inequality, namely:

\[ r \| \nabla_v f^n (t) \|_{L^\infty (L_x^1)} = \frac{\| \nabla_v f^n (t) \|_{L_x^1 (L_v^1)}}{r^2}. \]
where from now on we use by shortedness the notation \( L^\infty_x (L^1_v) = L^\infty (\Omega, L^1 (\mathbb{R}^3)) \), \( L^1_x (L^1_v) = L^1 (\Omega \times \mathbb{R}^3) \). Thus we have

\[
K^n (t, x) \leq C \| \nabla_v f^n (t) \|^2_{L^\infty_x (L^1_v)} \| \nabla_v f^n (t) \|^{1/3}_{L^1_x (L^1_v)}.
\]

Since \( f^n \) is bounded in \( W^{1, \infty} \) and the supports are bounded uniformly in \( n \) (due to the global bound on \( Q^n (t) \), for \( 0 \leq t \leq T \)) it is easy to see that \( \| \nabla_v f^n (t) \|_{L^1_v (L^1_v)} \) and \( \| \nabla_v f^n (t) \|_{L^\infty (L^1_v)} \) are uniformly bounded in \( n \) and hence \( K^n (t, x) \) is uniformly bounded.

Thus we obtain

\[
(5.24)
\]

\[
\| f^{n+1} (t) - f^n (t) \|_{L^1 (\Omega \times \mathbb{R}^3)} \leq C (T) \int_0^t \| f^n (s) - f^{n-1} (s) \|_{L^1 (\Omega \times \mathbb{R}^3)} ds
\]

where \( C = C (T) \) depends only on \( T, Q (T) \), and the initial data. Notice that \( (5.24) \) implies by iteration that:

\[
(5.25)
\]

\[
\| f^{n+1} (t) - f^n (t) \|_{L^1 (\Omega \times \mathbb{R}^3)} \leq C_1 \theta^n
\]

for some \( \theta < 1 \), and \( 0 \leq t \leq \varepsilon_0 \), if \( \varepsilon_0 \) is sufficiently small depending only on \( T \). Then \( (5.24) \) implies:

\[
\| f^{n+1} (t) - f^n (t) \|_{L^1 (\Omega \times \mathbb{R}^3)} \leq C (T) C_1 \theta^n + C (T) \int_{\varepsilon_0}^t \| f^n (s) - f^{n-1} (s) \|_{L^1 (\Omega \times \mathbb{R}^3)} ds
\]

Therefore we obtain that \( (5.25) \) is valid for \( t \in [\varepsilon_0, 2\varepsilon_0] \) changing \( C_1 \) if needed. Iterating the argument, it then follows that \( \{ f^n \} \) is a Cauchy sequence in the space \( L^\infty (\Omega \times \mathbb{R}^3) \).

Once we know that \( f^n \) is a Cauchy sequence in \( L^1 \left( [0, T] \times \Omega \times \mathbb{R}^3 \right) \), we can show that the sequence is Cauchy in \( C^{\nu;1,\lambda} \left( [0, T] \times \Omega \times \mathbb{R}^3 \right) \) for any \( 0 < \lambda < \mu \), \( 0 < \nu < 1 \) arguing by interpolation. Indeed, using \( (5.17) \) with \( \lambda \) replaced by \( \lambda \) satisfying \( \lambda < \lambda < \mu \) and interpolating between \( L^1 \left( [0, T] \times \Omega \times \mathbb{R}^3 \right) \) and \( W^{1, \infty} \left( [0, T] \times \Omega \times \mathbb{R}^3 \right) \) we obtain that \( f^n \) is Cauchy in \( W^{1,p} \left( [0, T] \times \Omega \times \mathbb{R}^3 \right) \) for any \( p > 1 \). Using Sobolev’s embeddings we can obtain that \( f^n \) is a Cauchy sequence in \( C^{\tilde{\nu};1,\lambda} \left( [0, T] \times \Omega \times \mathbb{R}^3 \right) \) for any \( 0 < \tilde{\nu} < \nu \). Interpolating then in Schauder spaces between \( C^{1,\lambda} \) and \( C^{\tilde{\nu}} \) we obtain the desired convergence. In order to check that \( f \in C^{1;1,\lambda} \left( [0, T] \times \Omega \times \mathbb{R}^3 \right) \), we use \( (5.21) \) to obtain:

\[
f^n (t) = f_0 - \int_0^t \left[ v \cdot \nabla_x f^n (s) + \nabla_x \phi^{n-1} (s) \cdot \nabla_v f^n (s) \right] ds
\]

Passing to the limit in this equation as \( n \to \infty \) it follows that:

\[
f (t) = f_0 - \int_0^t \left[ v \cdot \nabla_x f (s) + \nabla_x \phi (s) \cdot \nabla_v f (s) \right] ds
\]

and this implies the desired differentiability for \( f \), and the Proposition follows. \( \square \)
5.4. Convergence of the sequence \( Q^n \) to \( Q \).

**Proposition 4.** Let \( Q^n, \ Q \) be as in (5.14), (5.6) respectively. Suppose that \( \max \{ \sup_{n \geq n_0} Q^n (t), Q (t) \} \leq K \) for \( 0 \leq t \leq T \). Let us assume also that \( f^n \to f \) in \( C^{\nu,1,\lambda}_{t;[x,v]} ([0,T] \times \Omega \times \mathbb{R}^3) \) for any \( 0 < \lambda < \mu, \ 0 < \nu < 1 \). Then \( \lim_{n \to \infty} Q^n (t) = Q (t) \) uniformly on \([0,T]\).

**Proof.** The characteristics starting in \( \alpha (0) \geq C \delta_0 \) remain during their evolution in the set \( \{ \alpha (t) \geq C \delta_0 \} \) due to Lemma 3. Therefore, these characteristics remain separated from the singular set and we can estimate their difference as \( n \to \infty \) as it was made in [10]. Indeed, for these characteristics the number of bounces is uniformly bounded in \( n \) in the interval \( 0 \leq t \leq T \). Moreover, the times where these bounces take place for the functions \( f^n \) converge to the corresponding times for the bouncing times for the characteristics associated to \( f \) and using the fact that \( E^n \to E \) as \( n \to \infty \) we obtain also that the corresponding characteristic curves associated to \( f^n \) converge to the ones associated to \( f \) between bounces, and due to the boundedness of the number of such bounces, it follows that the functions \( V^n (s,t;x,v) \) converge uniformly to \( V (s,t;x,v) \) as \( n \to \infty \). Since \( Q^n (t) \) is the maximum value of \( |V^n| \) associated to characteristics \( (X^n (s,t;x,v),V^n (s,t;x,v)) \) which at \( s=0 \) lie in the support of \( f_0 \), it follows that \( Q^n (t) \to Q (t) \), and the result follows. \( \square \)

5.5. Prolongability of uniform estimates for the functions \( f^n \).

**Proposition 5.** Suppose that for some \( T \geq 0 \) there exist \( K > 0 \) and \( n_0 \geq 0 \) such that for any \( n \geq n_0 \) and \( 0 \leq t \leq T \) we have \( Q^n (t) \leq K \). Then, there exists \( \varepsilon_0 = \varepsilon_0 (K, \| f_0 \|_\infty) > 0 \) such that for \( 0 \leq t \leq T + \varepsilon_0 \) and \( n \geq n_0 \) the following estimate holds:

\[
Q^n (t) \leq 2K
\]

**Proof.** Notice that:

\[
(5.26) \quad |\rho^n| = \left| \int f^n dv \right| \leq \| f_0 \|_\infty (Q^n (t))^3
\]

Using the representation formula (5.4) for the solutions of the Poisson equation in \( \Omega \) with Neumann boundary conditions in Proposition 1 we obtain:

\[
|\nabla \phi^n| \leq C \left[ \| f_0 \|_\infty (Q^n (t))^2 + \int \rho_n dx + \| h \|_{C^{1,\mu}} \right].
\]

On the other hand, using (4.2) and (5.14) we obtain the following inequality for \( t \geq T \):

\[
Q^{n+1} (t) \leq Q^n (T) + C \| f_0 \|_\infty \int_T^t (Q^n (s))^3 ds + C (t-T)
\]
where $C > 0$ is just a numerical constant independent of $n$ and $Q^n$. Using the assumption on $Q^n$ we obtain:

$$Q^{n+1}(t) \leq K + C \|f_0\|_{\infty} \int_T^t (Q^n(s))^3 \, ds + C (t - T) \quad , \quad t \geq T$$

Defining $R^n(t) = \max \{Q^\ell(t) : n_0 \leq \ell \leq n\}$, we obtain:

$$R^{n+1}(t) \leq K + C \|f_0\|_{\infty} \int_T^t (R^{n+1}(s))^3 \, ds + C (t - T) \quad , \quad t \geq T$$

Let us select $\varepsilon_0 = \frac{K}{C [8 \|f_0\|_{\infty} K^3 + 1]}$. It then follows, using a Gronwall type of argument that:

$$Q^n(t) \leq 2K \quad , \quad n \geq n_0 \quad , \quad 0 \leq t \leq T + \varepsilon_0 .$$

\[ \square \]

6. Energy estimate and consequences.

The following energy estimates are standard for the Vlasov-Poisson system (cf. [5]). We state them here for further reference.

**Proposition 6.** Suppose that $f$ is a solution of (1.1)-(1.6) on the time interval $t \in [0, T]$. Then:

(6.1) \[ \frac{d}{dt} \left( \int_{\Omega \times \mathbb{R}^3} v^2 f \, dx dv + \int_{\Omega} (E)^2 \, dx \right) = 0 \]

*Proof.* It is similar to the proof of the analogous result in the whole space (see [5], page 120). The only difference is that we need to take into account the contribution of some boundary terms. More precisely, using (1.1) we obtain, after some integration by parts:

(6.2) \[ \frac{d}{dt} \left( \int_{\Omega \times \mathbb{R}^3} v^2 f \, dx dv \right) = 2 \int_{\Omega \times \mathbb{R}^3} \nabla_x \phi \cdot v f \, dx dv - 2 \int_{\partial \Omega \times \mathbb{R}^3} v^2 v \cdot n_x f dS_x dv \]

(6.3) \[ \frac{d}{dt} \left( \int_{\Omega \times \mathbb{R}^3} E^2 \, dx \right) = -2 \int_{\Omega \times \mathbb{R}^3} \nabla_x \phi \cdot v f \, dx dv + 2 \int_{\partial \Omega} \phi \left( \frac{\partial \phi}{\partial n_x} \right)_t dS_x + 2 \int_{\partial \Omega \times \mathbb{R}^3} \phi f v \cdot n_x dS_x dv \]

The boundary term in (6.2) vanishes, since $\int_{\partial \Omega \times \mathbb{R}^3} v^2 v \cdot n_x f dS_x dv = 0$ for each $x \in \partial \Omega$, due to the specular boundary condition (1.6). On the other hand, the two last terms in (6.3) vanish due to the fact that $\frac{\partial \phi}{\partial n_x} = h$ is independent of $t$ and $\int_{\mathbb{R}^3} f v \cdot n_x dx = 0$ for each $x \in \partial \Omega$. Adding then (6.2), (6.3) we obtain (6.1). \[ \square \]
Proposition 7. Suppose that $f$ is a solution of (1.1)-(1.5) defined in $0 \leq t \leq T$ with $f(0, x, v) = f_0(x, v)$, where $f_0$ is as in (1.6). There exists $C$ depending only on $T$ and on the regularity norms assumed for $f_0$ in Theorem 1 such that:

\begin{align}
\sup_{0 \leq t \leq T} \int_{\Omega} v^2 f(x, t) \, dv \, dx \leq C \\
\sup_{0 \leq t \leq T} \left[ \|\rho(t, \cdot)\|_{L^2(\Omega)} \right] \leq C \\
\|f(t)\|_{L^p(\Omega \times \mathbb{R}^3)} = \|f_0\|_{L^p(\Omega \times \mathbb{R}^3)}, \text{ for all } 1 \leq p \leq \infty.
\end{align}

Proof. This result is just a consequence of (6.1) and its proof is standard in kinetic theory. See for instance [5]. Finally (6.6) follows multiplying (1.1) by $(f)^{n-1}$, and integrating by parts using the specular boundary conditions.

\section{Pfaffelmoser’s argument: Global bound for $Q(t)$}

In this section we show that the function $Q(t)$ can be bounded in any time interval $0 \leq t \leq T$ and therefore that the corresponding solutions of (1.1)-(1.5) can be extended to arbitrarily long intervals. The estimate of $Q(t)$ will be derived using the ideas of Pfaffelmoser (cf. [13]) in the case of bounded domains with purely reflected boundary conditions at $\partial\Omega$. The main content of the result is a uniform estimate for $Q(t)$ as long as $f$ is defined.

Arguing as in the derivation of (5.26) we obtain the following estimate:

\begin{align}
\|\rho\|_{L^\infty} \leq \|f\|_{L^\infty} Q(t)^3
\end{align}

where $\rho$ is in (1.2).

The main result of this paper, that we will prove using Pfaffelmoser method, is as follows:

\textbf{Theorem 3.} Let $f_0 \in C^{1,\mu}(\Omega \times \mathbb{R}^3)$ with $0 < \mu < 1$. Suppose that $f \in C^{1,1,\lambda}_{t, (x,v)} ([0, T] \times \Omega \times \mathbb{R}^3)$ is a solution of (1.1)-(1.5) with $\lambda \in (0, 1)$, $0 < T < \infty$. There exists $\sigma(T) < \infty$ depending only on $T$, $Q(0)$ and $\|f_0\|_{C^{1,\mu}(\Omega \times \mathbb{R}^3)}$ such that:

\begin{align}
Q(t) \leq \sigma(T), \quad 0 \leq t \leq T.
\end{align}

\subsection{Bounds for $Q(t)$}

Suppose that $(\hat{X}(s), \hat{V}(s))$ is any fixed characteristic curve such that:

\begin{align}
(\hat{X}(0), \hat{V}(0)) \in \text{supp } f_0.
\end{align}

The basic idea in the Pfaffelmoser’s method consists in deriving estimates for $E = \nabla \phi$. Using the estimates in Proposition 1 it follows that:
\begin{equation}
(7.3) \int_{t-\Delta}^{t} ds \left| E \left( s, \hat{X} (s) \right) \right| \leq C \int_{t-\Delta}^{t} ds \int_{\Omega \times \mathbb{R}^3} \frac{f (s, y, w)}{|y - \hat{X} (s)|^2} dy dw + C \| h \|_{\infty} \Delta
\end{equation}

\begin{equation}
= C \int_{t-\Delta}^{t} ds \int_{\Omega \times \mathbb{R}^3} \frac{f (t, x, v)}{|X (s) - \hat{X} (s)|^2} dx dv + C \| h \|_{\infty} \Delta
\end{equation}

where we are using the following change of variables:

\[(X (s), V (s)) = (y, w) \rightarrow (x, v) = (X (t), V (t))\]

We also note the measure preserving property \((dy dw = dx dv)\) that is due to the fact that the evolution of the characteristics is hamiltonian away from the boundary and that the measure \(dx dv\) is also preserved by reflection on the boundary.

Pfaffelmoser’s method is based on the idea of splitting the region of integration in (7.3) into three different sets that are usually termed \(the \ good, \ the \ bad, \ and \ the \ ugly\). Fix \(Q > 0\), that will be precised later. In the rest of the argument we will use two numbers \(\Delta\) and \(P\) defined by

\begin{equation}
(7.4) P \equiv Q^{3/4 - \delta}, \quad \delta > 0 \; \text{small}
\end{equation}

\begin{equation}
(7.5) \Delta \equiv \frac{c_0 P}{Q^2}
\end{equation}

where \(c_0\) is small, but fixed number (independent on \(Q\)).

Given \(Q, \Delta, P\) we define \(the \ good, \ the \ bad, \ and \ the \ ugly\) respectively by

\[G \equiv \left\{ (s, y, w) \in [t - \Delta, t] \times \Omega \times \mathbb{R}^3 : |w| \leq P, \ |w - \hat{V} (t)| \leq P, \ |w - \hat{V}^+ (t)| \leq P \right\}\]

\[B \equiv \left\{ (s, y, w) \in [t - \Delta, t] \times \Omega \times \mathbb{R}^3 : |y - X (s)| \leq \varepsilon_0, \ |w| \geq P, \ |w - \hat{V} (t)| \geq \varepsilon_0 \right\}\]

\[U \equiv \left\{ (s, y, w) \in [t - \Delta, t] \times \Omega \times \mathbb{R}^3 : |y - X (s)| \geq \varepsilon_0, \ |w| \geq P, \ |w - \hat{V} (t)| \geq \varepsilon_0 \right\}\]

where

\[\varepsilon_0 \equiv \frac{R}{|v|^2} \frac{1}{|v - \hat{V} (t)|}\]

if the characteristic curve \(\hat{X} (s)\) does not intersect \(\partial \Omega\) on the interval \(s \in [t - \Delta, t]\), and otherwise

\[\varepsilon_0 \equiv \frac{R}{|v|^2} \left( \frac{1}{|v - \hat{V} (t)|} + \frac{1}{|v - \hat{V}^+ (t)|} \right)\].
Here
\[ \hat{V}^+(t) = \hat{V}^*(s_0), \]
where
\[ s_0 = \sup \left\{ s \in [t - \Delta, t] : \hat{X}(s) \in \partial \Omega \right\}. \]

The change due to the field can be estimated as:

**Lemma 8.** Under the assumptions in Theorem 3 we have the following estimate:

\[ \int_{t-\Delta}^{t} |E(X(s), s)| \, ds \leq C \Delta \left[ \left( Q(t) \right)^{4/3} + 1 \right], \Delta \leq t \leq T \]

**Proof.** It is just an adaptation of a similar estimate in [5], page 122. The last constant term in (7.6) is a consequence of the Neumann boundary condition, i.e., \( E = h \) at the boundary (cf. (1.3)). \( \square \)

We now estimate the change of \( \hat{V}(s) \) due to geometry of the domain and a basic geometric property that relates the change of \( \hat{V}(s) \) with the curvature will be investigated: The idea is as follows:

(i) The total length that the characteristic moves, including reflections is bounded as \( C Q \Delta \).

(ii) The change of the normal vector in a distance of order \( \ell_i \) is bounded by \( C \ell_i \).

(iii) Therefore, the change of the angle of the vectors \( \hat{V}(s), \hat{V}^*(s) \) with respect to the previous reflection is bounded by \( C Q \ell_i + \int_{t_i}^{t_{i+1}} |E| \, ds \). (The change is the difference of outcoming vector with respect to the next outcoming one, and incoming vectors with respect to the incoming ones).

(iv) The total change of these vectors is then bounded by \( C Q^2 \Delta + \int_{t-\Delta}^{t} |E| \, ds \leq C \left( Q^2 + Q^{4/3} \right) \Delta \leq C Q^2 \Delta \), where the constant \( C \) depends only on the maximum of the curvature.

We now give its explicit formulation and the rigorous proof. To this end we first recall basic ingredients such as Formulas of Frenet in differential geometry (see [15] p. 18, p. 94 for reference) that we will combine with Lemma 2 in this paper.

**Lemma 9.** Let \( T \) and \( N \) be the unit tangential and normal vector on \( \partial \Omega \) respectively. Let \( \kappa \) be the curvature along a curve and \( \kappa_N \) be the normal curvature in the direction \( dx \) of the line of curvature. Then we have

\[ dN + \kappa_N dx = 0, \quad dT = \kappa N ds, \]

where \( s \) is the arc length.

**Proof.** See [15] p. 18, p. 94. \( \square \)
In view of Lemma 2 the equations describing the evolution of the characteristics in a geometrical form are:

\[
\frac{d\mu_i}{dt} = \frac{w_i}{1 + k_i x_\perp} \tag{7.7}
\]

\[
\frac{dw_i}{dt} = E_i - \frac{v_\perp w_i k_i}{1 + k_i x_\perp} - \sum_{j,\ell=1}^2 \frac{\Gamma_{i,j,\ell} w_j w_\ell}{1 + k_j x_\perp} \tag{7.8}
\]

\[
\frac{dx_\perp}{dt} = v_\perp \tag{7.9}
\]

\[
\frac{dv_\perp}{dt} = F = E_\perp + \sum_{j=1}^2 \frac{w_j^2 b_j}{1 + k_j x_\perp} \tag{7.10}
\]

Note that these equations work only near the boundary \(\partial\Omega\), but inside the domain the maximum displacement of the characteristics is \(Q\Delta\) that can be made small.

**Lemma 10.** Let \((X(s), V(s))\) and \((\hat{X}(s), \hat{V}(s))\) be two characteristics and let \(Q\) be as in (5.6). Then we have

\[
\min \left\{ |\hat{V}(s) - V(t)|, |\hat{V}(s) - \hat{V}^+(t)| \right\} \leq C \Delta Q^2(t) , \ s \in [t - \Delta, t] \tag{7.11}
\]

\[
\min \left\{ |V(s) - V(t)|, |V(s) - V^+(t)| \right\} \leq C \Delta Q^2(t) , \ s \in [t - \Delta, t] \tag{7.12}
\]

*Proof.* Estimates equally apply to \(V\) and \(\hat{V}\) and so we only give a proof for \(V\). Unlike the whole space case, we have to take care of the possible sign change of \(v_\perp\) at the bounces, i.e. if it becomes zero coming from the region \(v_\perp < 0\). Using the equations (7.8) and (7.10), it follows that

\[
\left| \frac{d(|v_\perp|)}{ds} \right| + \left| \frac{dw_i}{dt} \right| \leq C \Delta Q^2 , \ s \in [t - \Delta, t] \tag{7.11}
\]

and this implies

\[
|w_i(s) - w_i(t)| \leq C \Delta Q^2(t - s) , \ (7.12)
\]

(Notice that the equation above is valid even if we cross the bounces, and that \(|v_\perp|\) is differentiable). If the number of jumps of the normal velocities
is even, then by (7.11)-(7.12) and by Lemma 9 we have
\[ |V(s) - V(t)| = \left| \left( \sum_{i=1}^{2} w_i u_i + v_N \right)(s) - \left( \sum_{i=1}^{2} w_i u_i + v_N \right)(t) \right| \]
\[ \leq \left| \sum_{i=1}^{2} (w_i(s) - w_i(t)) u_i(t) + (v_{\perp}(s) - v_{\perp}(t)) N(t) \right| \]
\[ + \sum_{i=1}^{2} |w_i(s)| |u_i(t) - u_i(s)| + |v_{\perp}(s)||N(t) - N(s)| \]
\[ \leq CQ^2 (t - s). \]

In a similar manner, we deduce that if the number of jumps of the normal velocities is odd, then
\[ |V(s) - V^+(t)| \leq CQ^2 (t - s). \]

Thus we conclude the assertion of the lemma. In general, we can split the time interval \([s, t]\) into several time sub-intervals in such a way that particles are governed completely by the equation (2.5) near the boundary or by the usual Vlasov equation (1.1) away from the boundary on each sub-interval. Then the estimates above combined with those in the whole space and noticing that the effect \(Q(t)^{4/3}\) of the electric field alone (Lemma 8) is negligible to that \(Q(t)^2\) of the geometry yield the lemma. \(\square\)

**Lemma 11.** In the sets \(B\) and \(U\), we have
\[ \frac{|w|}{2} \leq |v| \leq 2 |w|, \]
\[ \frac{|w - \hat{V}(t)|}{2} + \frac{|w - \hat{V}^+(t)|}{2} \leq |v - \hat{V}(t)| + |v - \hat{V}^+(t)| \leq 2 |w - \hat{V}(t)| + 2 |w - \hat{V}^+(t)|. \]

**Proof.** In the sets \(B\) and \(U\), we have \(|w| \geq P, |w - \hat{V}(t)| \geq P, |w - \hat{V}^+(t)| \geq P\). By Lemma 10 we have either \(|w - V(t)| \leq CQ^2\) or \(|w - V^+(t)| \leq CQ^2\). If \(|w - V(t)| \leq CQ^2\), we have
\[ |w - \hat{V}(t)| - |V(t) - w| \leq |v - \hat{V}(t)| \leq |w - \hat{V}(t)| + |V(t) - w|, \]
\[ |w - \hat{V}^+(t)| - |V(t) - w| \leq |v - \hat{V}^+(t)| \leq |w - \hat{V}^+(t)| + |V(t) - w| \]
Since, for small \(c_0\),
\[ CQ^2 \Delta \leq \frac{P}{4} \leq \frac{|w - \hat{V}(t)|}{4}, \]
\[ CQ^2 \Delta \leq \frac{P}{4} \leq \frac{|w - \hat{V}^+(t)|}{4},. \]
we have
\[ \frac{|w - \hat{V}(t)|}{2} \leq \left| v - \hat{V}(t) \right| \leq 2 \left| w - \hat{V}(t) \right|, \]
\[ \frac{|w - \hat{V}^+(t)|}{2} \leq \left| v - \hat{V}^+(t) \right| \leq 2 \left| w - \hat{V}^+(t) \right|. \]

In the other case, we similarly obtain
\[ \frac{|w - \hat{V}(t)|}{2} \leq \left| v - \hat{V}(t) \right| \leq 2 \left| w - \hat{V}(t) \right|, \]
\[ \frac{|w - \hat{V}^+(t)|}{2} \leq \left| v - \hat{V}^+(t) \right| \leq 2 \left| w - \hat{V}^+(t) \right|. \]

Therefore, we deduce our lemma. \( \Box \)

We denote \( X_{\parallel} = \mu_1 u_1 + \mu_2 u_2, \) \( X_{\perp} = x_{\perp} \) and \( V_{\parallel} = w_1 u_1 + w_2 u_2, \) \( V_{\perp} = v_{\perp} \).

**Lemma 12.** If a trajectory \((X, V)\) has more than one bounce in the interval \([t - \Delta, t]\), then we have, for all \( s \in [t - \Delta, t]\),
\[ |V_{\perp}(s)| \leq CQ^2(t) \Delta. \]

**Proof.** If a trajectory \((X, V)\) has more than one bounce, then we have \( V_{\perp}(\hat{s}) = 0 \), for some \( \hat{s} \in [t - \Delta, t] \). Since
\[ (7.13) \quad \left| \frac{d[V_{\perp}]}{ds} \right| \leq CQ^2(t), \]
the lemma follows. \( \Box \)

**Lemma 13.** Let \((X(s), V(s))\) and \((\hat{X}(s), \hat{V}(s))\) be trajectories over \([t - \Delta, t]\).

Suppose that
\[ \left| X_{\perp}(s_0) - \hat{X}_{\perp}(s_0) \right| = \min_{s \in [t - \Delta, t]} \left| X_{\perp}(s) - \hat{X}_{\perp}(s) \right|, \quad s_0 \in (t - \Delta, t). \]

Then either both \( X_{\perp}(s_0) > 0, \hat{X}_{\perp}(s_0) > 0 \) or both \( X_{\perp}(s_0) = \hat{X}_{\perp}(s_0) = 0 \).

**Proof.** We prove the lemma by contradiction. Suppose \( X_{\perp}(s_0) > 0 \) and \( \hat{X}_{\perp}(s_0) = 0 \). The case \( \hat{X}_{\perp}(s_0) > 0 \) and \( X_{\perp}(s_0) = 0 \) can be studied in a symmetric way. Notice that the function \( \lambda(s) = \left| X_{\perp}(s) - \hat{X}_{\perp}(s) \right|^2 \) is differentiable in the interval \((t - \Delta, t)\) except at a finite set of points. Therefore, at the point \( s = s_0 \) where the minimum of \( \lambda \) is achieved we have:
\[ \frac{d\lambda}{ds}(s_0) = \frac{d}{ds} \left| X_{\perp}(s_0) - \hat{X}_{\perp}(s_0) \right|^2 \leq 0 \]
where from now on \( f(s_0) = \lim_{s_0 < s \to s_0} f(s), \) \( f(s_0 +) = \lim_{s_0 < s < s_0} f(s) \). We then have:
\[ X_{\perp}(s_0) \left( V_{\perp}(s_0) - \hat{V}_{\perp}(s_0) \right) \leq 0, \]
which implies
\begin{equation}
V_\perp (s_0-) \leq \hat{V}_\perp (s_0-) < 0.
\end{equation}

The fact that $\hat{V}_\perp (s_0-) \neq 0$ is a consequence of Lemma 3.

Notice that, since $X_\perp (s_0) = 0$ there is a reflection of $V_\perp$ at $s = s_0$. On the other hand, since $X_\perp (s_0) > 0$, $V_\perp$ is continuous at $s = s_0$. Therefore:
\[ V_\perp (s_0+) = V_\perp (s_0-) , \quad \hat{V}_\perp (s_0+) = -\hat{V}_\perp (s_0-) \]

Thus we have
\[ |X_\perp (s) - \hat{X}_\perp (s)| = X_\perp (s) - \hat{X}_\perp (s) = X_\perp (s_0) + \left( V_\perp (s_0-) + \hat{V}_\perp (s_0-) \right) (s - s_0) + o(s - s_0) \]
as $s \to s_0$, $s > s_0$. Due to (7.14), we have $|X_\perp (s) - \hat{X}_\perp (s)| < X_\perp (s_0) = |X_\perp (s_0) - \hat{X}_\perp (s_0)|$ for $s - s_0 > 0$ sufficiently small, but this contradicts the fact that $|X_\perp (s) - \hat{X}_\perp (s)|$ reaches its minimum at $s = s_0$. Therefore the result follows.

We show the following crucial separation property.

**Lemma 14.** *(Separation property)* In the ugly set $U$, there exist $s_0, s_1 \in [t - \Delta, t]$ such that the following separation holds:
\begin{equation}
|X(s) - \hat{X}(s)| \geq C \left( \varepsilon_0 + \min \left\{ |v - \hat{V}(t)|, |v - \hat{V}^+(t)|, |s - s_0|, |s - s_1| \right\} \right) , \quad s \in [t - \Delta, t]
\end{equation}
where $C$ is a universal constant depending only on the curvature of $\partial \Omega$.

**Proof.** We separate into two cases:

**Case 1:** Both trajectories $(X(s), V(s)), (\hat{X}(s), \hat{V}(s))$ have at most one bounce in the time interval $[t - \Delta, t]$. Let $t - \Delta \leq t_1 < t_2 \leq t$ be possible two bouncing times with $t_1, t_2$ corresponding to $(X(s), V(s)), (\hat{X}(s), \hat{V}(s))$ respectively. We split the time interval $[t - \Delta, t]$ into a maximum of three sub-intervals, namely $[t - \Delta, t_1] \cup [t_1, t_2] \cup [t_2, t]$. In the most general case, some of these intervals could be empty. Let us describe the argument in the most general case, since for a smaller number of reflections the argument required is just a minor simplification of it. Pick $s_0$ and $s_1$ such that
\[ \min_{s \in [t - \Delta, t_1] \cup [t_2, t]} \left| \left( X - \hat{X} \right)(s) \right| = \left| \left( X - \hat{X} \right)(s_0) \right| , \]
\[ \min_{s \in [t_1, t_2]} \left| \left( X - \hat{X} \right)(s) \right| = \left| \left( X - \hat{X} \right)(s_1) \right| . \]

In the interval $[t_1, t_2]$ there is no bounces along the trajectories. Then, we can argue exactly as in the case without boundaries (cf. [3], pages 128-129) to show that:
\[ |X(s) - \hat{X}(s)| \geq C |v - \hat{V}^+(t)| |s - s_1| , \quad \text{for } s \in [t_1, t_2] . \]
On the other hand, the portion of the trajectories \( X(s), \hat{X}(s) \) for \( s \in [t_2, t] \) might be reflected with respect to the boundary \( \partial \Omega \). Suppose for the moment that the boundary \( \partial \Omega \) is flat as in the half space case. The trajectories over \([t_2, t]\) obtained by reflections together with the original trajectories \( X(s), \hat{X}(s) \) for \( s \in [t - \Delta, t_1] \) yield portion of new trajectories without bounces, and satisfying an equation of the form:

\[
\frac{dX}{ds} = V, \quad \frac{dV}{ds} = \tilde{E},
\]

where \( \int_{t-\Delta}^{t} \left| \tilde{E}(X(s), s) \right| ds \leq C\Delta \left( (Q(t))^{4/3} + 1 \right) \). We can argue then exactly as in the case of the whole space (cf. [5]), to estimate the difference between the trajectories, and since the reflection with respect to the plane \( x_1 = 0 \) is an isometry, we finally obtain:

\[
(7.16) \quad \left| X(s) - \hat{X}(s) \right| \geq C \left| V(t) - \hat{V}(t) \right| |s - s_0|, \quad \text{for } s \in [t - \Delta, t_1] \cup [t_2, t].
\]

Now if the boundary \( \partial \Omega \) is not flat then the change of the normal vectors between two reflections can be bounded by \( C\Delta Q \) and the corresponding change of the vectors \( V \) and \( \hat{V} \) can be bounded by \( C\Delta Q^2 \), which is smaller than \( c_0P \) for sufficiently small \( c_0 \). Therefore the change of these vectors is small compared to \( \left| V(t) - \hat{V}(t) \right| \) in the ugly set and the inequality \((7.16)\) above is valid for the case of domain with curvature.

**Case 2:** At least one of the trajectories has more than one bounce in \([t - \Delta, t]\). Let \( \hat{X}, \hat{V} \) have more than one bounce in \([t - \Delta, t]\).

We first consider the case

\[
(7.17) \quad |w_i(s) - \hat{w}_i(s)| \geq \frac{1}{2} \left| V(s) - \hat{V}(s) \right|, \quad \text{for all } s \in [t - \Delta, t],
\]

i.e., the tangential part of \( V - \hat{V} \) dominates along the trajectory in the whole interval \([t - \Delta, t]\). Let

\[
\min_{s \in [t - \Delta, t]} |\mu_i(s) - \hat{\mu}_i(s)| = |\mu_i(s_0) - \hat{\mu}_i(s_0)|.
\]

Note that the tangential part of the trajectory, \( X_\parallel \) is \( C^1 \). Consider the equation \((7.7)\) and \((7.8)\) for the tangential components of the position and the velocity:

\[
\frac{dw_i}{dt} = E_i - \frac{v_i w_i k_i}{1 + k_i x_\perp} - \sum_{j,\ell=1}^{2} \Gamma_{i,\ell} w_j w_\ell \frac{1}{1 + k_j x_\perp} = \mathcal{O} \left( Q^{4/3} \right) + \mathcal{O} \left( Q^2 \right),
\]

\[
\frac{d\mu_i}{ds} = \frac{w_i}{1 + k_i x_\perp} = w_i(s_0) + \mathcal{O} \left( \Delta Q^2 \right) + O \left( x_\perp |w_i| \right) = w_i(s_0) + \mathcal{O} \left( \Delta Q^2 \right),
\]
where the second term comes from the change on the velocity \( w_i \) due to the field and to the geometry and constants depend only on the geometry of the domain. We also used the fact that

\[ |x_\perp| |w_i| = O(\Delta Q^2). \]

Then, integrating the tangential part \( \mu_i - \hat{\mu}_i \) of the difference of \( (X,V) \) and \( (\hat{X}, \hat{V}) \), we get:

\[
(\mu_i - \hat{\mu}_i)(s) = (\mu_i - \hat{\mu}_i)(s_0) + (w_i(s_0) - \hat{w}_i(s_0)) (s - s_0) + O(\Delta Q^2) (s - s_0)
\]

By Lemma 10 and (7.17), we have

\[
|w_i(s_0) - \hat{w}_i(s_0)| \geq \frac{1}{2} |V(s_0) - \hat{V}(s_0)| \geq \frac{1}{2} \min \left\{ |v - \hat{V}(t)|, |v - \hat{V}^+(t)| \right\} + O(Q^2(t)\Delta).
\]

Since \( |(\mu_i - \hat{\mu}_i)(s)|^2 \) is a \( C^1 \) function it follows that at the point \( s_0 \in [t - \Delta, t] \) where \( |(\mu_i - \hat{\mu}_i)(s)|^2 \) attains the minimum we have

\[
(s - s_0) \frac{d}{ds} \left( |(\mu_i - \hat{\mu}_i)(s)|^2 \right) = [\mu_i(s_0) - \hat{\mu}_i(s_0)] \cdot [w_i(s_0) - \hat{w}_i(s_0)] (s - s_0) \geq 0.
\]

Note that the inequality above takes into account the possibility that the minimum of \( |(\mu_i - \hat{\mu}_i)(s)|^2 \) can be achieved at the end points \( s_0 = t - \Delta, t \). Since

\[
O(Q^2(t)\Delta) = c_0 P,
\]

we deduce

\[
|(\mu_i - \hat{\mu}_i)(s)| \geq \frac{1}{4} \min \left\{ |v - \hat{V}(t)|, |v - \hat{V}^+(t)| \right\} (s - s_0),
\]

for sufficiently small \( c_0 \). Using the definition of the ugly set we have \( |(\mu_i - \hat{\mu}_i)(s)| \geq \varepsilon_0 \). Thus (7.15) follows.

We now consider the complementary case, i.e., there is \( \bar{s} \in [t - \Delta, t] \) such that

\[
\left| V_\perp(\bar{s}) - \hat{V}_\perp(\bar{s}) \right| \geq \frac{1}{2} |V(\bar{s}) - \hat{V}(\bar{s})|.
\]

By (7.13), we have

\[
\left| |V_\perp(s)| - |\hat{V}_\perp(s)| \right| \geq \left| |V_\perp(\bar{s})| - |\hat{V}_\perp(\bar{s})| \right| - CQ^2(t)\Delta.
\]

On the other hand, note that

\[
\left| |V_\perp(s)| - |\hat{V}_\perp(s)| \right| = \left| V_\perp(s) - \text{sgn}(V_\perp(s)) \hat{V}_\perp(s) \right|.
\]
Using Lemma 12 it follows that
\[ |V_\perp (s) - sgn(V_\perp (s)) \hat{V}_\perp (s)| \]
\[ = |V_\perp (s) - \hat{V}_\perp (s) + \hat{V}_\perp (s) - sgn(V_\perp (s)) \hat{V}_\perp (s)| \]
\[ \leq |V_\perp (s) - \hat{V}_\perp (s)| + 2 |\hat{V}_\perp (s)| \]
\[ \leq |V_\perp (s) - \hat{V}_\perp (s)| + CQ^2 (t) \Delta. \]

Similarly, we get
\[ |V_\perp (\bar{s}) - sgn(V_\perp (\bar{s})) \hat{V}_\perp (\bar{s})| \]
\[ = |V_\perp (\bar{s}) - \hat{V}_\perp (\bar{s}) + \hat{V}_\perp (\bar{s}) - sgn(V_\perp (\bar{s})) \hat{V}_\perp (\bar{s})| \]
\[ \geq |V_\perp (\bar{s}) - \hat{V}_\perp (\bar{s})| - 2 |\hat{V}_\perp (\bar{s})| \]
\[ \geq |V_\perp (\bar{s}) - \hat{V}_\perp (\bar{s})| - CQ^2 (t) \Delta. \]

Thus, we obtain, for all \( s \in [t - \Delta, t] \),
\[ (7.18) \quad |V_\perp (s) - \hat{V}_\perp (s)| \geq |V_\perp (\bar{s}) - \hat{V}_\perp (\bar{s})| - CQ^2 (t) \Delta \]
\[ \geq \frac{1}{2} |V (\bar{s}) - \hat{V} (\bar{s})| - CQ^2 (t) \Delta \]
\[ \geq \frac{1}{2} \min\{ |v - \hat{V} (t)|, |v - \hat{V}^+ (t)| \} - CQ^2 (t) \Delta \]
\[ \geq \frac{1}{2} \min\{ |v - \hat{V} (t)|, |v - \hat{V}^+ (t)| \} - c_0 P \]
\[ \geq \frac{1}{4} \min\{ |v - \hat{V} (t)|, |v - \hat{V}^+ (t)| \} \geq \frac{P}{4}. \]

Using Lemma 12 it follows, choosing \( c_0 \) in (7.5) sufficiently small, that:
\[ (7.19) \quad |V_\perp (s)| \geq \frac{P}{8} \]

for all \( s \in [t - \Delta, t] \). Taking into account (7.6) it follows that \( V_\perp (s) \) changes sign, by reflection, at most once in the interval \( s \in [t - \Delta, t] \) if \( c_0 \) is sufficiently small. Combining Lemma 12, (7.18), and (7.19) it follows that \( V_\perp (s) - \hat{V}_\perp (s) \) changes sign at most once for \( s \in [t - \Delta, t] \). Indeed, \( \hat{V}_\perp (s) \) is small compared to \( |V_\perp (s)| \geq \frac{P}{8} \) in the interval \( [t - \Delta, t] \), and \( V_\perp (s) \) changes sign only once at most. Suppose that \( V_\perp (s) \) changes sign at \( s = s_0 \). Since \( X_\perp (s) \geq 0 \), it follows that \( \text{sgn}(V_\perp (s)) = \text{sgn} (V_\perp (s) - \hat{V}_\perp (s)) = \text{sgn}(s - s_0) \), for \( s \in [t - \Delta, t] \). We have
\[ X_\perp (s) - \hat{X}_\perp (s) = \int_{s_0}^{s} [V_\perp (\tau) - \hat{V}_\perp (\tau)] d\tau. \]
Then, using (7.18), we have:
\[ |X_\perp (s) - \hat{X}_\perp (s)| \geq \frac{1}{4} \min \{ |V (t) - \hat{V} (t)|, |V (t) - \hat{V}^+ (t)| \} |s - s_0| . \]
Since in the ugly set \( U \),
\[ |X_\perp (s) - \hat{X}_\perp (s)| \geq \varepsilon_0 , \]
we obtain (7.15). The proof is complete.

**Proof of Theorem 3.** The key point in the proof is to estimate the right-hand side of (7.3). Let us assume without loss of generality that \( Q \geq 1 \), since otherwise the leading contribution would be \( C \Delta \) in (7.3). In order to make this estimate we separate the contributions of the sets \( G, B \) and \( U \):
\[ \int_{t - \Delta}^{t} ds \int_{\Omega \times IR^3} f (y, w, s) dw dy dw = \int_{G} \ldots ds dy dw + \int_{B} \ldots ds dy dw + \int_{U} \ldots ds dy dw \]
In order to estimate the contribution of the good set we define:
\[ \rho_G (y, s) \equiv \int_{G} f (y, w, s) dw \]
Standard estimates yield
\[ \| \rho_G \|_\infty \leq CP^3 \]
where from now on \( C \) depends on \( \| f_0 \|_\infty \), but not on \( P, \varepsilon_0, R, Q, \Delta \).
Arguing as in the derivation of (7.6) we obtain:
\[ (7.20) \int_{G} \ldots ds dy dw \leq \Delta P^{4/3} \]
In order to estimate the contribution of the bad set, notice that Lemma 11 implies
\[ \varepsilon_0 \leq \frac{8 R}{|w|^2} \left( \frac{1}{|w - \hat{V} (t)|} + \frac{1}{|w - \hat{V}^+ (t)|} \right) . \]
Then
\[ \int_{B} f (y, w, s) dw dy dw \leq C \int_{t - \Delta}^{t} ds \int_{|w| \leq Q} \varepsilon_0 dw = \]
\[ (7.21) C \int_{t - \Delta}^{t} ds \int_{|w| \leq Q} \frac{R}{|w|^2} \left( \frac{1}{|w - \hat{V} (t)|} + \frac{1}{|w - \hat{V}^+ (t)|} \right) dw \leq CR \Delta \log (Q) \]
where by assumption \( |\hat{V} (t)| \geq 1 \), since otherwise the corresponding characteristic does not have effect in the variation of \( Q \).
Lastly we estimate the integral over the ugly set:
The aim is to estimate
\[
\int_{U} f(y, w, s) \frac{1}{|y - \hat{X}(s)|^2} ds dy dw = \int_{U} f(x, v, t) \frac{1}{|X(s) - \hat{X}(s)|^2} ds dx dv
\]
Using (7.15) we can estimate this integral as
\[
C \int_{U} f(x, v, t) \left(\varepsilon_0 + \min \left\{|v - \hat{V}(t)| |s - s_0|, |v - \hat{V}^+(t)| |s - s_1|\right\}\right)^2 ds dx dv
\]
The integration with respect to time \(s\) can be estimated as
\[
\int_{t-\Delta}^{t} ds \left(\varepsilon_0 + \min \left\{|v - \hat{V}(t)| |s - s_0|, |v - \hat{V}^+(t)| |s - s_1|\right\}\right)^2 \leq C \frac{\varepsilon_0}{\varepsilon_0} \left(\frac{1}{|v - \hat{V}(t)|} + \frac{1}{|v - \hat{V}^+(t)|}\right) \leq C \frac{1}{\varepsilon_0} \left(\frac{1}{|v - \hat{V}(t)|} + \frac{1}{|v - \hat{V}^+(t)|}\right) = C \frac{v^2}{R}
\]
Then
\[
(7.22) \quad \int_{U} [... \, ds dy dw \leq C \frac{v^2}{R} \int v^2 f(x, v, t) dv dx \leq C \frac{\Delta}{R \Delta}
\]
Combining the estimates (7.20), (7.21), and (7.22) for the Good, Bad, and Ugly set, we obtain
\[
\int_{t-\Delta}^{t} \left|E(s, \hat{X}(s))\right| ds \leq C \Delta \left(P^{4/3} + R \log(Q) + \frac{1}{R \Delta}\right) = C \Delta \left(P^{4/3} + R \log(Q) + \frac{Q^{4/3}}{RP}\right)
\]
Choosing \(R = Q^{1-\delta}\), \(P = Q^{3/4-\delta}\) we obtain
\[
\int_{t-\Delta}^{t} \left|E(s, \hat{X}(s))\right| ds \leq C \Delta Q^\beta
\]
where \(\beta < 1\).
Therefore
\[
\frac{Q(t) - Q(t - \Delta)}{\Delta} \leq C (Q(t))^\beta,
\]
and a standard iteration yields \(Q(t)\) bounded in any interval \(0 \leq t \leq T\). Thus this completes the proof.
\[\square\]
Proof of Theorem 1. In order to prove the existence of a solution \( f \) of (1.1)-(1.6) globally defined in \( t \) we will show that the sequence of functions \( f^n \) defined by (3.1)-(3.7) converges as \( n \to \infty \) to a solution of (3.2)-(3.6) for arbitrary values of \( t \). To this end, it suffices to show that the functions \( Q^n (t) \) are uniformly bounded in each compact set of \( t \in [0, \infty) \). The desired limit property would then follow from Proposition 3.

To this end, we define
\[
L(t) = \sup_n Q^n (t).
\]
We have that \( L(t) \) is increasing in \( t \). We denote as \( T_{\text{max}} \) the time where
\[
\lim_{t \to T_{\text{max}}} L(t) = \infty.
\]
Our goal is to show that \( T_{\text{max}} = \infty \). Let us assume that \( T_{\text{max}} < \infty \). We define \( \epsilon_0 = \epsilon_0 (2\sigma (T_{\text{max}}), \| f_0 \|_{L^\infty}) \), where the function \( \epsilon_0 (\cdot) \) is as in Proposition 5 and the function \( \sigma (T) \) is as in Theorem 3. Notice that the functions \( Q^n (t) \) are uniformly bounded for \( t \in [0, T_{\text{max}} - \epsilon_0^2] \) by definition of \( T_{\text{max}} \).

Therefore, Proposition 5 implies that \( f^n \to f \) in \( C^{\nu,1,\lambda}_{t,(x,v)} \) for \( 0 \leq t \leq T_{\text{max}} - \frac{\epsilon_0^2}{2} \). In particular, \( \lim_{n \to \infty} Q^n (t) = Q (t) \leq \sigma (T_{\text{max}}) \), for \( t = T_{\text{max}} - \frac{\epsilon_0^2}{2} \). Therefore \( Q^n (t) \leq 2\sigma (T_{\text{max}}) \) for \( n \geq n_0 \) with \( n_0 \) large. Then, Proposition 5 implies that the sequence \( Q^n (t) \) is uniformly bounded for \( t \in [0, T_{\text{max}} + \frac{\epsilon_0^2}{2}] \), whence \( L(t) \) is bounded as \( t \to T_{\text{max}} \). This contradicts the definition of \( T_{\text{max}} \) and concludes the proof of the existence of a solution of (1.1)-(1.6) in \( C^{1,1,\lambda}_{t,(x,v)} \) for some \( 0 < \lambda < \mu \) and for \( 0 \leq t \leq T \) as asserted in Theorem 1. In order to prove uniqueness we argue as in the proof of (5.24) to obtain, that two \( C^{1,1,\lambda}_{t,(x,v)} \) solutions of (1.1)-(1.6) with the same initial and boundary satisfy
\[
\| f_1 (t) - f_2 (t) \|_{L^1} \leq C (T) \int_0^t \| f_1 (s) - f_2 (s) \|_{L^1} ds,
\]
Therefore \( f_1 = f_2 \). This completes the proof of the Theorem. \( \Box \)

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