The Physical Role of Gravitational and Gauge Degrees of Freedom in General Relativity - II: Dirac versus Bergmann Observables and the Objectivity of Space-Time

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Abstract

This is the second of a couple of papers in which we aim to show the peculiar capability of the Hamiltonian ADM formulation of metric gravity to grasp a series of conceptual and technical problems that appear to have not been directly discussed so far. In this paper we also propose new viewpoints about issues that, being deeply rooted into the foundational level of Einstein theory, seem particularly worth of clarification in connection with the alternative programs of string theory and loop quantum gravity. The achievements of the present work include:

1) the analysis of the so-called Hole phenomenology in strict connection with the Hamiltonian treatment of the initial value problem. The work is carried through in metric gravity for the class of Christoudoulou-Klainermann space-times, in which the temporal evolution is ruled by the weak ADM energy. It is crucial to our analysis the re-interpretation of active diffeomorphisms as passive and metric-dependent dynamical symmetries of Einstein’s equations, a re-interpretation which enables to disclose their (nearly unknown) connection to gauge transformations on-shell; this is expounded in the first paper (gr-qc/0403081).

2) the utilization of the Bergmann-Komar intrinsic pseudo-coordinates, defined as suitable functionals of the Weyl curvature scalars, as tools for a peculiar gauge-fixing to the super-hamiltonian and super-momentum constraints;

3) the consequent construction of a physical atlas of 4-coordinate systems for the 4-dimensional mathematical manifold, in terms of the highly non-local degrees of freedom of the gravitational field (its four independent Dirac observables). Such construction embodies the physical individuation of the points of space-time as point-events, both in absence and presence of matter, and associates a non-commutative structure to each gauge fixing or 4-dimensional coordinate system.

4) a clarification of the multiple definition given by Peter Bergmann of the concept of (Bergmann) observable in general relativity. This clarification leads to the proposal of a main conjecture asserting the existence of i) special Dirac’s observables which are also Bergmann’s observables, ii) gauge variables that are coordinate independent (namely they behave like the tetradic scalar fields of the Newman-Penrose formalism). A by-product of this achievements is the falsification of a recently advanced argument asserting the absence of (any kind of) change in the observable quantities of general relativity.

5) a proposal showing how the physical individuation of point-events could in principle be implemented as an experimental setup and protocol leading to a standard of space-time more or less like atomic clocks define standards of time.

In the end, against the well-known Einstein’s assertion according to which general covariance takes away from space and time the last remnant of physical objectivity, we conclude that point-events maintain a peculiar sort of objectivity. Also, besides being operationally essential for building measuring apparatuses for the gravitational field, the role of matter in the non-vacuum gravitational case is also that of participating directly in the individuation process, being involved in the determination of the Dirac observables. Finally, some hints following from our approach for the quantum gravity programme are suggested.

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I. INTRODUCTION.

In a first paper[1] (hereafter referred to as I), we have shown how the capabilities of the ADM Hamiltonian approach to metric gravity enables us to get new insights into a series of technical problems concerning the physical status of the gauge variables and the Dirac observables (DO), as well as the dynamical nature of the simultaneity and gravito-magnetism conventions. We have shown in particular that i) before solving Einstein-Hamilton equations different conventions within the same space-time universe simply correspond to different gauge choices; ii) each solution of Einstein-Hamilton equations dynamically selects a preferred convention. In the present paper we exploit the technical achievements obtained in I to get new insights into issues deeply rooted into the foundational level of the theory that we deem still worth of clarification. The superiority of the Hamiltonian treatment is essentially due to the fact that it allows to work off shell, i.e., without immediate restriction to the solution of Einstein’s equations. On the other hand, such kind of analysis could hardly be dealt with in a satisfactory way within the standard Lagrangian approach because of the non-hyperbolic nature of Einstein’s equations. It is not by chance that the modern treatment of the initial value problem within the Lagrangian configurational approach [2, 3] must in fact mimic the Hamiltonian methods.

The adoption of the Hamiltonian viewpoint entails that the range of our analysis and conclusions be confined to a particular class of models of general relativity, namely those which are compatible with a 3+1 splitting of space-time. In particular, we shall work with globally-hyperbolic, non-compact, topologically trivial, asymptotically flat at spatial infinity (and with suitable boundary conditions there) space-times, which are of the type classified by Christodoulou and Klainermann[4].

The main specific issues we want to scrutinize are: a) the long-standing issue of the objectivity of point-events of space-time. Although this question is nowadays mainly of interest to philosophers of science and appears to have been nicely bypassed in the standard physical literature, we intend to show that it maintains interesting technical aspects which could even be relevant for the forefront physics, namely string theory and loop quantum gravity. b) The concept of observable in general relativity, in particular the relation between the notion of Dirac observable (DO) and that of Bergmann observable (BO), two notions that do not simply overlap. Actually, we shall show that the relation between these two concepts contains the seeds of interesting developments concerning not only the concept of observable itself but also a possible invariant notion of generalized inertial effects in general relativity and thereby new insights into the equivalence principle.

The paper should be read in sequence after I which contains various technical premises for the present analysis. Previous partial accounts of the material of this paper can be found in Refs. [5, 6, 7].

General relativity is commonly thought to imply that space-time points have no intrinsic physical meaning due to the general covariance of Einstein’s equations. This feature is implicitly described in standard modern textbooks by the statement that solutions to the Einstein’s equations related by (active) diffeomorphisms have physically identical properties, so that only the equivalence class of such solutions represents a space-time geometry. Such kind of equivalence, which also embodies the modern understanding of Einstein’s historical Hole Argument, has been named as Leibniz equivalence in the philosophical literature by Earman and Norton[8]. In this paper we will not examine any philosophical aspect of
the issue¹, although our analysis is inspired by the belief that Leibniz equivalence is not and cannot be the last word about the intrinsic physical properties of space-time, well beyond the needs of the empirical grounding of the theory. Specifically, our contribution should be inscribed in the list of the various attempts made in the literature to gain an intrinsic dynamical characterization of space-time points in terms of the gravitational field itself, besides and beyond the trivial mathematical individuation furnished to them by the coordinates. We refer in particular to old hints offered by Synge, and to the attempts successively sketched by Komar, Bergmann and Stachel. Actually, we claim that we have pursued this line of thought till its natural end.

The original Hole Argument is naturally spelled out within the configurational Lagrangian framework of Einstein’s theory. It is essential to realize from the beginning that - by its very formulation - the Hole Argument is inextricably entangled with the initial value problem although, strangely enough, it has never been explicitly discussed in that context in a systematic way. Possibly the reason is that most authors have implicitly adopted the Lagrangian approach, where the Cauchy problem is intractable because of the non-hyperbolic nature of Einstein’s equations. The proper way to deal with such problem is indeed the ADM Hamiltonian framework with its realm of DO and gauge variables. But then the real difficulty is just the connection between such different frameworks, particularly from the point of view of symmetries. The clarification of this issue in I started from a rediscovery of a nearly forgotten paper by Bergmann and Komar [10] which enabled us to enlighten this correspondence of symmetries and, in particular, that existing between active diffeomorphisms of the configurational approach and gauge transformations of the Hamiltonian viewpoint.

At first sight it could seem that in facing the original Hole Argument Einstein simply equated general covariance with the unavoidable arbitrariness of the choice of coordinates, a fact that, in modern language, can be translated into invariance under passive diffeomorphisms. The so-called point-coincidence argument (a terminology introduced by Stachel in 1980), satisfied Einstein doubts at the end of 1915 but offered mainly a pragmatic solution of the issue and was based on a very idealized model of physical measurement where all possible observations reduce to the intersections of the world-lines of observers, measuring instruments, and measured physical objects. Furthermore, this solution left unexplored some important aspects of the role played by the metric tensor in the Hole Argument as well as of the related underlying full mathematical structure of the theory.

That the Hole Argument was in fact a subtler issue that Einstein seemingly thought in 1915 [11] and that it consisted in much more than mere arbitrariness in the choice of the coordinates², has been revealed by a seminal talk given by John Stachel in 1980 [13], which gave new life to the original Hole Argument.

The Hole Argument, in its modern version, runs as follows. Consider a general-relativistic space-time, as specified by the four-dimensional mathematical manifold $\mathcal{M}^4$ and by a metric tensor field $^4g$ which represents at the same time the chrono-geometrical and causal structure of space-time and the potential for the gravitational field. The metric $^4g$ is a solution of the generally-covariant Einstein equations. If any non-gravitational physical fields are present,

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¹ A philosophical critique following from the technical results of the present couple of papers can be found in Ref.[9]

² In fact, however, Einstein’s argument was not so naive, see Norton [12].
they are represented by tensor fields that are also dynamical fields, and that appear as sources in the Einstein equations.

Assume now that $M^4$ contains a Hole $\mathcal{H}$: that is, an open region where all the non-gravitational fields are zero. On $M^4$ we can prescribe an active diffeomorphism (see I, Section II) $D_A$ that re-maps the points inside $\mathcal{H}$, but blends smoothly into the identity map outside $\mathcal{H}$ and on the boundary. Now, just because Einstein's equations are generally covariant so that they can be written down as geometrical relations, if $^4g$ is one of their solutions, so is the drag-along field $^4g' = D_A \cdot ^4g$. By construction, for any point $p \in \mathcal{H}$ we have (geometrically) $^4g'(D_A \cdot p) = ^4g(p)$, but of course $^4g'(p) \neq ^4g(p)$ (also geometrically). Now, what is the correct interpretation of the new field $^4g'$? Clearly, the transformation entails an active redistribution of the metric over the points of the manifold, so the crucial question is whether, to what extent, and how the points of the manifold are primarily individuated.

In the mathematical literature about topological spaces, it is always implicitly assumed that the entities of the set can be distinguished and considered separately (provided the Hausdorff conditions are satisfied), otherwise one could not even talk about point mappings or homeomorphisms. It is well known, however, that the points of a homogeneous space cannot have any intrinsic individuality. There is only one way to individuate points at the mathematical level that we are considering: namely by coordinatization, a procedure that transfers the individuality of 4-tuples of real numbers to the elements of the topological set. Precisely, we introduce by convention a standard coordinate system for the primary individuation of the points (like the choice of standards in metrology). Then, we can get as many different names, for what we consider the same primary individuation, as the coordinate charts containing the point in the chosen atlas of the manifold. We can say, therefore, that all the relevant transformations operated on the manifold $M^4$ (including active diffeomorphisms which map points to points), even if viewed in purely geometrical terms, must be realizable in terms of (possibly generalized) coordinate transformations.

Let us go back to the effect of this primary mathematical individuation of manifold points. If we now think of the points of $\mathcal{H}$ as also physically individuated spatio-temporal events even before the metric is defined, then $^4g$ and $^4g'$ must be regarded as physically distinct solutions of the Einstein equations (after all, as already noted, $^4g'(p) \neq ^4g(p)$ at the same point $p$). This, however, is a devastating conclusion for the causality, or better, determinateness of the theory, because it implies that, even after we completely specify a physical solution for the gravitational and non-gravitational fields outside the Hole - for example, on a Cauchy surface for the initial value problem, assuming for the sake of argument that this intuitive and qualitative wording is mathematically correct, see Section III - we are still unable to predict uniquely the physical solution within the Hole. Clearly, if general relativity has to make any sense as a physical theory, there must be a way out of this foundational quandary, independently of any philosophical consideration.

In the modern understanding, the most widely embraced escape from the (mathematical) strictures of the Hole Argument (which is essentially an update to current mathematical terms of the pragmatic solution adopted by Einstein), is to deny that diffeomorphically related

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$^3$ As Hermann Weyl [14] puts it: "There is no distinguishing objective property by which one could tell apart one point from all others in a homogeneous space: at this level, fixation of a point is possible only by a demonstrative act as indicated by terms like this and there."
mathematical solutions represent physically distinct solutions. With this assumption (i.e., the mathematical basis of Leibnitz equivalence), an entire equivalence class of diffeomorphically related mathematical solutions represents only one physical solution. This statement, is implicitly taken as obvious in the contemporary specialized literature (see, e.g. Ref. [15]).

It is seen at this point that the conceptual content of general covariance is far more deeper than the simple invariance under arbitrary changes of coordinates. Actually (see Stachel [16, 17]) asserting that \(4g\) and \(D_A \cdot 4g\) represent one and the same gravitational field entails that the mathematical individuation of the points of the differentiable manifold by their coordinates has no physical content until a metric tensor is specified. In particular, coordinates lose any physical significance whatsoever [11]. Furthermore, if \(4g\) and \(D_A \cdot 4g\) must represent the same gravitational field, they cannot be physically distinguishable in any way. So when we act on \(4g\) with an active diffeomorphisms to create the drag-along field \(D_A \cdot 4g\), no element of physical significance can be left behind: in particular, nothing that could identify a point \(p\) of the manifold as the same point of space-time for both \(4g\) and \(D_A \cdot 4g\). Instead, when \(p\) is mapped onto \(p' = D_A \cdot p\), it brings over its identity, as specified by \(4g'(p') = 4g(p)\). 

This conclusion led Stachel to the conviction that space-time points must be physically individuated before space-time itself acquires a physical bearing, and that the metric itself plays the privileged role of individuating field: a necessarily unique role in the case of space-time without matter. More precisely, Stachel claimed that this individuating role should be implemented by four invariant functionals of the metric, already considered by Bergmann and Komar [18] (see Section II). However, he did not follow up on his suggestion. As a matter of fact, as we shall see, the question is not straightforward.

There are many reasons why one should revisit the Hole Argument nowadays, quite apart from any conceptual interest.

First of all, the crucial point of the Hole issue is that the mathematical representation of space-time provided by general relativity under the condition of general covariance evidently contains superfluous structure hidden behind Leibniz equivalence and that this structure must be isolated. At the level of general covariance, only the equivalence class is physically real so that, on this understanding, general covariance is invariably an unbroken symmetry and the physical world is to be described in a diffeomorphically invariant way. Of course, the price to be paid is that the values of all fields at manifold points as specified by the coordinates, are not physically real. On the other hand, this isolation appears to be required de facto both by any explicit solution of Einstein’s equation, which requires specification of the arbitrariness of coordinates, and by the empirical foundation of the theory: after all any effective kind of measurement requires in fact a definite physical individuation of space-

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4 A further important point made by Stachel is that simply because a theory has generally covariant equations, it does not follow that the points of the underlying manifold must lack any kind of physical individuation. Indeed, what really matters is that there can be no non-dynamical individuating field that is specified independently of the dynamical fields, and in particular independently of the metric. If this was the case, a relative drag-along of the metric with respect to the (supposedly) individuating field would be physically significant and would generate an inescapable Hole problem. Thus, the absence of any non-dynamical individuating field, as well as of any dynamical individuating field independent of the metric, is the crucial feature of the purely gravitational solutions of general relativity as well as of the very concept of general covariance.
time points in terms of physically meaningful coordinates. Summarizing, it is evident that breaking general covariance is a pre-condition for the isolation of the superfluous structure hidden within Leibniz equivalence, namely the generalized inertial effects analyzed in I.

Secondly, the program of the physical individuation of space-time points must be completed because, as it will appear evident in Section II, the mere recourse to the four functional invariants of the metric alluded to by Stachel cannot do, by itself, the job of physically individuating space-time points. In the context of the Hamiltonian formalism, we find the tools for completing Stachel’s suggestion and exploiting the old proposal advanced by Bergmann and Komar for an intrinsic labeling of space-time points by means of the eigenvalues of the Weyl tensor. Precisely, Bergman and Komar, in a series of papers [18, 19, 20] introduced suitable invariant scalar functionals of the metric and its first derivatives as invariant pseudo-coordinates. We shall show that such proposal can be utilized in constructing a peculiar gauge-fixing to the super-hamiltonian and super-momentum constraints in the canonical reduction of general relativity. This gauge-fixing makes the invariant pseudo-coordinates into effective individuating fields by forcing them to be numerically identical with ordinary coordinates: in this way the individuating fields turn the mathematical points of space-time into physical point-events. Eventually, we discover that what really individuates space-time points physically are the very degrees of freedom of the gravitational field. As a consequence, we advance the claim that - physically - Einstein’s vacuum space-time is literally identified with the autonomous physical degrees of freedom of the gravitational field, while the specific functional form of the invariant pseudo-coordinates matches these latter into the manifold’s points. The introduction of matter has the effect of modifying the Riemann and Weyl tensors, namely the curvature of the 4-dimensional substratum, and to allow measuring the gravitational field in a geometric way for instance through effects like the geodesic deviation equation. It is important to emphasize, however, that the addition of matter does not modify the construction leading to the individuation of point-events, rather it makes it conceptually more appealing.

Finally, our procedure of individuation transfers, as it were, the noncommutative Poisson-Dirac structure of the DO onto the individuated point-events. The physical implications of this circumstance might deserve some attention in view of the quantization of general relativity. Some hints for the quantum gravity programme will be offered in the final Section of the paper (Concluding Remarks).

A Section of the paper is devoted to our second main topic: the clarification of the multiple and rather ambiguous concept of Bergmann’s observable (BO) [22]. Bergmann’s definition has various facets, namely a configurational side having to do with invariance under passive diffeomorphisms, an Hamiltonian side having to do with Dirac’s concept of observable, and the property of predictability which is entangled with both sides. According to Bergmann, (his) observables are passive diffeomorphisms invariant quantities (PDIQ) ”which can be predicted uniquely from initial data”, or ”quantities that are invariant under a coordinate transformation that leaves the initial data unchanged”. Bergmann says in addition that they are further required to be gauge invariant, a statement that could only be interpreted as implying that Bergmann’s observables are simultaneously DO. Yet, he offers no explicit demonstration of the compatibility of this bundle of statements.

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5 Actually, the first suggestion of specifying space-time points absolutely in terms of curvature invariants is due to Synge [21].
Actually, once fully clarified, the concept of predictability implies in its turn that, in order Bergmann’s multiple definition be consistent, only four of such observables can exist for the vacuum gravitational field, and can be nothing else than tensorial Lagrangian counterparts of the Hamiltonian DO. We formalize this result and related consequences into a main conjecture, which essentially amounts to claiming the internal consistency of Bergmann’s multiple definition. Our conjecture asserts: i) the existence of special Dirac’s observables which are also Bergmann’s observables, as well as ii) the existence of gauge variables that are coordinate independent.

As anticipated in I, the hoped for validity of the main conjecture would add new emphasis to the physical meaning of the separation between DO, as related to tidal-like effects, on the one hand, and gauge variables, as related to generalized inertial effects, on the other. Actually, in spite of the physical relevance of distinction as it stands, its weakness is that the separation of the two autonomous degrees of freedom of the gravitational field from the gauge variables is, as yet, a coordinate (i.e. gauge) - dependent concept. The known examples of pairs of conjugate DO are neither coordinate-independent (they are not PDIQ) nor tensors. Bergmann asserts that the only known method (at the time) to build BO is based on the existence of Bergmann-Komar invariant pseudo-coordinates. The results of this method, however, are of difficult interpretation, so that, in spite of the importance of this alternative non-Hamiltonian definition of observables, no explicit determination of them has been proposed so far. A possible starting point to attack the problem of the connection of DO with BO seems to be a Hamiltonian re-formulation of the Newman-Penrose formalism [23] (it contains only PDIQ) employing Hamiltonian null-tetrads carried by the time-like observers of the congruence orthogonal to the admissible space-like hyper-surfaces. This suggests the technical conjecture that special Darboux bases for canonical gravity should exist in which generalized inertial effects (related to gauge variables fixings) are described by PDIQ while the autonomous degrees of freedom (DO) are also BO. Therefore, the validity of the conjecture would render our distinction above an invariant statement, providing a remarkable contribution to the old-standing debate about the equivalence principle. Note in addition that, since Newman-Penrose PDIQ are tetradic quantities, the validity of the conjecture would also eliminate the existing difference between the observables for the gravitational field and the observables for matter, built usually by means of the tetrads associated to some time-like observer. Furthermore, this would also provide a starting point for defining a metrology in general relativity in a generally covariant way6, replacing the empirical metrology [24] used till now. It would also enable to replace the test matter of the axiomatic approach to measurement theory by dynamical matter (see Appendix A).

Incidentally, our results about the definition of Bergmann observable help in showing that various recent claims [25] about the absence of any kind of change in general relativity as such are not mathematically justified and that - with reference to the class of models we are considering - the so-called issue of time raises no particular difficulty: even more, such models provide an explicit counterexample to the frozen time argument. The role of the generator of real time evolution in such space-times is played in fact by the so-called weak ADM energy, while the super-hamiltonian constraint has nothing to do with temporal change and is only the generator of gauge transformations connecting different admissible 3+1 splittings of space-time. We argue, therefore, that in these space-times there is neither

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6 Recall that this is the main conceptual difference from the non-dynamical metrology of special relativity
a frozen reduced phase space nor a possible Wheeler-De Witt interpretation based on some
local concept of time as in compact space-times. In conclusion, we claim that our gauge-
invariant approach to general relativity is perfectly adequate to accommodate real temporal
change, so that all the consequent developments based on it are immune to criticisms like
those referred to above.

A final step of our analysis consists in suggesting how the physical individuation of space-
time points, introduced at the conceptual level, could in principle be implemented with a
well-defined empirical procedure, an experimental set-up and protocol for positioning and
orientation. This suggestion is outlined in correspondence with the abstract treatment of
the empirical foundation of general relativity as exposed in the classical paper of Ehlers,
Pirani and Schild [26]. The conjunction of the Hamiltonian treatment of the initial value
problem, with the correlated physical individuation of space-time points, and the practice
of general-relativistic measurement, on the backdrop of the axiomatic foundation closes, as
it were, the coordinative circuit of general relativity.

Section IIA is devoted to the issue of the individuation of the mathematical points of \( M^4 \)
as physical point-events by means of a peculiar gauge-fixing to Bergmann-Komar intrinsic
pseudo-coordinates. In Section IIB, we sketch the implementation of the physical individuation
in terms of well-defined experimental procedures which realize the axiomatic structure
of general relativity proposed by Ehlers, Pirani and Schild. The analysis of the concept of
BO and the criticism of the frozen time and universal no-change arguments are the content
of Section III where our main conjecture is advanced concerning the relations between DO
and BO. The Conclusion contains some general comments about the gauge nature of gen-
eral relativity and some hints in view of the quantum gravity programme while Appendix
A reviews the Ehlers, Pirani and Schild axiomatic approach.
II. PHYSICAL INDIVIDUATION OF SPACE-TIME POINTS BY MEANS OF GAUGE FIXINGS TO BERGMANN-KOMAR INTRINSIC COORDINATES.

Let us now exploit the results of I, Sections II and III, to the effect of clarifying the issue of the physical individuality of space-time point-events in general relativity and its implications for the theory of measurements with test objects.

A. The Physical Individuation of Space-Time Points.

Let us begin by recalling again that the ADM formulation assumes the existence of a mathematical 4-manifold, the space-time $M^4$, admitting 3+1 splittings with space-like leaves $\Sigma_\tau \approx R^3$. All fields (also matter fields when present) depend on $\Sigma_\tau$-adapted coordinates $(\tau, \vec{\sigma})$ for $M^4$.

We must insist again that a crucial component of the individuation issue is the inextricable entanglement of the Hole Argument with the initial value problem, which has been dealt with at length in I. Now, however, we have at our disposal the right framework for dealing with the initial value problem, so our main task should be to put all things together. Finally, another fundamental tool at our disposal is the clarification we gained in Section II of I concerning the relation between active diffeomorphisms in their passive view and the dynamical gauge symmetries of Einstein’s equations in the Hamiltonian approach.

We are then ready to move forward by conjoining Stachel’s suggestion with the proposal advanced by Bergmann and Komar [18] that, in the absence of matter fields, the values of four invariant scalar fields built from the contractions of the Weyl tensor (actually its eigenvalues) can be used to build intrinsic pseudo-coordinates.

The four invariant scalar eigenvalues $\Lambda^{(k)}_W(\tau, \vec{\sigma})$, $k = 1, ..., 4$, of the Weyl tensor, written in Petrov compressed notations, are

$$
\Lambda^{(1)}_W = Tr(4C^4g^4C^4g), \\
\Lambda^{(2)}_W = Tr(4C^4g^4C^4\epsilon), \\
\Lambda^{(3)}_W = Tr(4C^4g^4C^4g^4C^4g), \\
\Lambda^{(4)}_W = Tr(4C^4g^4C^4g^4C^4\epsilon),
$$

where $4C$ is the Weyl tensor, $4g$ the metric, and $4\epsilon$ the Levi-Civita totally anti-symmetric tensor.

Bergmann and Komar then propose that we build a set of (off-shell) intrinsic coordinates for the point-events of space-time as four suitable functions of the $\Lambda^{(k)}_W$’s,

$$
\vec{s}^A(\sigma) = F^A[\Lambda^{(k)}_W(4g(\sigma), \partial^4g(\sigma))], (\vec{A} = 1, 2, ..., 4). \tag{2.2}
$$

Indeed, under the hypothesis of no space-time symmetries, we would be tempted (like Stachel) to use the $F^A[\Lambda^{(k)}_W]$ as individuating fields to label the points of space-time, at least locally.

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7 As shown in Ref.[27] in general space-times with matter there are 14 algebraically independent curvature scalars for $M^4$.

8 Our attempt to use intrinsic coordinates to provide a physical individuation of point-events would prima
Of course, since they are invariant functionals, the \( F^A \Lambda^{(k)}_W \)'s are quantities invariant under passive diffeomorphisms (PDIQ), therefore, as such, they do not define a coordinate chart for the atlas of the mathematical Riemannian 4-manifold \( M^4 \) in the usual sense (hence the name of pseudo-coordinates and the superior bar we used in \( F^A \)). Moreover, the tetradic 4-metric which can be built by means of the intrinsic pseudo-coordinates (see the next Section) is a formal object invariant under passive diffeomorphisms that does not satisfy Einstein’s equations (but possibly much more complex derived equations). Therefore, the action of active diffeomorphisms on the tetradic metric is not directly connected to the Hole argument. All this leads to the conclusion that the proposal advanced by Bergmann [22] (”we might then identify a world point (location-plus-instant-in-time) by the values assumed by (the four intrinsic pseudo-coordinates)” to the effect of individuating point-events in terms of intrinsic pseudo-coordinates is not - as it stands - physically viable in a tractable way. This is not the final verdict, however, and we must find a dynamical bridge between the intrinsic pseudo-coordinates and the ordinary 4-coordinate systems which provide the primary identification of the points of the mathematical manifold.

Our procedure starts when we recall that, within the Hamiltonian approach, Bergmann and Komar [18] proved the fundamental result that the Weyl eigenvalues \( \Lambda^{(A)}_W \), once re-expressed as functionals of the ADM canonical variables, do not depend on the lapse and shift functions but only on the 3-metric and its conjugate canonical momentum, \( \Lambda^{(k)}_W [^4g(\tau, \bar{\sigma}), \partial^4g(\tau, \bar{\sigma})] = \bar{\Lambda}^{(k)}_W [^3g(\tau, \bar{\sigma}), ^3\Pi(\tau, \bar{\sigma})] \). This result entails that the intrinsic pseudo-coordinates \( \bar{\sigma}^A \) are natural quantities to be used to build four gauge fixing constraints for the canonical reduction procedure depending only on a single hyper-surface \( \Sigma_\tau \) and not on how these surfaces are packed in the foliation.

Taking into account the results of Section III of I, we know that, in a completely fixed gauge, both the four intrinsic pseudo-coordinates and the ten tetradic components of the metric field (see Eq.(3.2) of the next Section) become gauge dependent functions of the four DO of that gauge. For the Weyl scalars in particular we can write:

\[ F^A [^4\Lambda^{(k)}_W [^4g(\sigma), \partial^4g(\sigma)]] \] become degenerate. This objection was originally raised by Norton [28] as a critique to manifold-plus-further-structure (MPFS) substantivalism (according to which the points of the manifold, conjoined with additional local structure such as the metric field, can be considered physically real; see for instance [29]). Several responses are possible. First, although most of the known exact solutions of the Einstein equations admit one or more symmetries, these mathematical models are very idealized and simplified; in a realistic situation (for instance, even with two masses) space-time is filled with the excitations of the gravitational degrees of freedom, and admits no symmetries at all. A case study is furnished by the non-symmetric and non-singular space-times of Christodoulou-Klainermann [4]. Second, the parameters of the symmetry transformations can be used as supplementary individuating fields, since, as noticed by Komar [20] and Stachel [17] they also depend upon metric field, through its isometries. To this move it has been objected [30] that these parameters are purely mathematical artifacts, but a simple rejoinder is that the symmetric models too are mathematical artifacts. Third, and most important, in our analysis of the physical individuation of points we are arguing a question of principle, and therefore we must consider generic solutions of the Einstein equations rather than the null-measure set of solutions with symmetries.

Problems might arise if we try to extend the labels to the entire space-time: for instance, the coordinates might turn out to be multi-valued.
\[
\Lambda_W^{(k)}(\tau, \vec{\sigma})|_G = \tilde{\Lambda}_W^{(k)}[\sigma^g(\tau, \vec{\sigma}), \Pi^G(\tau, \vec{\sigma})]|_G = \Lambda_G^{(k)}[\tilde{r}^G_m(\tau, \vec{\sigma}), \pi_m^G(\tau, \vec{\sigma})].
\]
where \(|_G\) denotes the specific gauge. Conversely, by the inverse function theorem, in each gauge, the DO of that gauge can be expressed as functions of the 4 eigenvalues restricted to that gauge: \(\Lambda_W^{(k)}(\tau, \vec{\sigma})|_G\).

Our program is implemented in the following way: after having selected a completely arbitrary mathematical radar-type (see Ref.[31]) coordinate system \(\sigma^A \equiv [\tau, \sigma^a]\) adapted to the \(\Sigma_\tau\) surfaces, we choose as physical individuating fields four suitable functions \(F^A[\Lambda_W^{(k)}(\tau, \vec{\sigma})]\), and express them as functionals \(\tilde{F}^A\) of the ADM variables

\[
F^A[\Lambda_W^{(k)}(\tau, \vec{\sigma})] = F^A[\tilde{\Lambda}_W^{(k)}[\sigma^g(\tau, \vec{\sigma}), \Pi^G(\tau, \vec{\sigma})]] = \tilde{F}^A[\sigma^g(\tau, \vec{\sigma}), \Pi^G(\tau, \vec{\sigma})].
\]

The space-time points, mathematically individuated by the quadruples of real numbers \(\sigma^A\), become now physically individuated point-events through the imposition of the following gauge fixings to the four secondary constraints

\[
\tilde{x}^A(\tau, \vec{\sigma}) \overset{df}{=} \sigma^A - \sigma^A(\tau, \vec{\sigma}) = \sigma^A - \tilde{F}^A[\sigma^g(\tau, \vec{\sigma}), \Pi^G(\tau, \vec{\sigma})] \approx 0.
\]

Then, following the standard procedure, we end with a completely fixed Hamiltonian gauge, say \(G\). This will be a correct gauge fixing provided the functions \(F^A[\Lambda_W^{(k)}(\tau, \vec{\sigma})]\) are chosen so that the \(\tilde{x}^A(\tau, \vec{\sigma})\)'s satisfy the orbit conditions

\[
\det |\{\tilde{x}^A(\tau, \vec{\sigma}), \tilde{H}^{B}(\tau, \vec{\sigma})\}_1| \neq 0,
\]
where \(\tilde{H}^{B}(\tau, \vec{\sigma}) = (\tilde{\mathcal{H}}(\tau, \vec{\sigma}); \tilde{\mathcal{H}}^x(\tau, \vec{\sigma})) \approx 0\) are the super-hamiltonian and super-momentum constraints of Eqs.(3.2) of I. These conditions enforce the Lorentz signature on Eq.(2.5), namely the requirement that \(F^x\) be a time variable, and imply that the \(F^A\)'s cannot be DO.

The above gauge fixings allow in turn the determination of the four Hamiltonian gauge variables \(\xi^i(\tau, \vec{\sigma}), \pi_\sigma(\tau, \vec{\sigma})\) of Eqs.(3.7) of I. Then, their time constancy induces the further gauge fixings \(\tilde{\psi}^A(\tau, \vec{\sigma}) \approx 0\) for the determination of the remaining gauge variables, i.e., the lapse and shift functions in terms of the DO in that gauge as

\[
\dot{\tilde{x}}^A(\tau, \vec{\sigma}) = \frac{\partial \tilde{x}^A(\tau, \vec{\sigma})}{\partial \tau} + \{\tilde{\sigma}^A(\tau, \vec{\sigma}), \tilde{H}_D\} = \delta^A r + \int d^3\sigma_1 \left[ n(\tau, \vec{\sigma}_1) \{\sigma^A(\tau, \vec{\sigma}), \mathcal{H}(\tau, \vec{\sigma}_1)\} + n_r(\tau, \vec{\sigma}_1) \{\sigma^A(\tau, \vec{\sigma}), \mathcal{H}^x(\tau, \vec{\sigma}_1)\} \right] = \tilde{\psi}^A(\tau, \vec{\sigma}) \approx 0.
\]
Finally, \(\tilde{\psi}^A(\tau, \vec{\sigma}) \approx 0\) determines the Dirac multipliers \(\lambda^A(\tau, \vec{\sigma})\).

In conclusion, the gauge fixings (2.5) (which break general covariance) constitute the crucial bridge that transforms the intrinsic pseudo-coordinates into true physical individuating coordinates.

As a matter of fact, after going to Dirac brackets, we enforce the point-events individuation in the form of the identity

12
\[ \sigma^A \equiv \bar{\sigma}^A = F^A_G[r^G_a(\tau, \bar{\sigma}), \pi^G_a(\tau, \bar{\sigma})] = F^A_G[\Lambda^{(k)}_W(\tau, \bar{\sigma})]_G. \] (2.8)

In this physical 4-coordinate grid, the 4-metric, as well as other fundamental physical entities, like e.g. the space-time interval \( ds^2 \) with its associated causal structure, and the lapse and shift functions, depend entirely on the DO in that gauge. The same is true, in particular, for the solutions of the eikonal equation [4] \( 4 g^{AB}(\sigma^D) \frac{\partial U(\sigma^D)}{\partial \sigma^A} \frac{\partial U(\sigma^D)}{\partial \sigma^B} = 0 \), which define generalized wave fronts and, therefore, through the envelope of the null surfaces \( U(\sigma^D) = \text{const} \) at a point, the light cone at that point.

Let us stress that, according to the results of I, only on the solutions of Einstein’s equations the completely fixed gauge \( G \) is equivalent to the fixation of a definite 4-coordinate system \( \sigma^A_G \). Our gauge fixing (2.5) ensures that on-shell we get \( \sigma^A = \sigma^A_G \). In this way we get a physical 4-coordinate grid on the mathematical 4-manifold \( M^4 \) dynamically determined by tensors over \( M^4 \) with a rule which is invariant under \( {}_p\text{Diff}M^4 \) but such that the functional form of the map \( \sigma^A \mapsto \text{physical 4 - coordinates} \) depends on the complete chosen gauge \( G \): we see that what is usually called the local point of view [32] (see later on) is justified a posteriori in every completely fixed gauge.

Summarizing, the effect of the whole procedure is that the values of the DO, whose dependence on space (and on parameter time) is indexed by the chosen radar coordinates \( (\tau, \bar{\sigma}) \), reproduces precisely such \( (\tau, \bar{\sigma}) \) as the Bergmann-Komar intrinsic coordinates in the chosen gauge \( G \). In this way we get mathematical points have become physical individuated point-events by means of the highly non-local structure of the DO. If we read the identity (2.8) as \( \sigma^A \equiv f^A_G(r^G_a, \pi^G_a) \), we see that each coordinate system \( \sigma^A \) is determined on-shell by the values of the 4 canonical degrees of freedom of the gravitational field in that gauge. This is tantamount to claiming that the physical role and content of the gravitational field in absence of matter is just the very identification of the points of Einstein space-times into physical point-events by means of its four independent phase space degrees of freedom. The existence of physical point-events in general relativity appears here as a synonym of the existence of the DO, i.e. of the true physical degrees of freedom of the gravitational field.

As said in the Introduction, the addition of matter does not change this conclusion, because we can continue to use the gauge fixing (2.5). However, matter changes the Weyl tensor through Einstein’s equations and contributes to the separation of gauge variables from DO in the quasi-Shanmugadhasan canonical transformation through the presence of its own DO. In this case we have DO both for the gravitational field and for the matter fields, which satisfy coupled Hamilton equations. Therefore, since the gravitational DO will still provide the individuating fields for point-events according to our procedure, matter will come to influence - on-shell only - the very physical individuation of points.

We have seen that, once the orbit conditions are satisfied, the Bergmann-Komar intrinsic pseudo-coordinates \( F^A_G[\Lambda^{(k)}_W g(\tau, \bar{\sigma}), \Pi(\tau, \bar{\sigma})]_G \) become just the individuating fields Stachel was looking for. Indeed, by construction, the intrinsic pseudo-coordinates are both invariant under \( {}_p\text{Diff}M^4 \) and also numerically invariant under the drag along induced by active diffeomorphisms (in the notations of the Introduction we have \( [\phi^*F^A]\equiv [F^A](\phi^{-1}, p) \), a fact that is also essential for maintaining a connection to the Hole Argument.
A better understanding of our point of view can be achieved by exploiting Bergmann-Komar’s group of passive transformations $Q$ discussed in Section II of I. We can argue in the following way. Given a 4-coordinate system $\sigma^A$, the passive view of each active diffeomorphism $\phi$ defines a new 4-coordinate system $\sigma^A_\phi$ (drag-along coordinates produced by a generalized Bergmann-Komar transformation (2.4) of I). This means that there will be two functions $F^A$ and $F^A_\phi$ realizing these two coordinates systems through the gauge fixings

$$
\sigma^A - F^A[\Lambda^{(k)}(\sigma)] \approx 0,
\sigma^A_\phi - F^A_\phi[\Lambda^{(k)}(\sigma_\phi)] \approx 0,
$$

(2.9)

It is explicitly seen in this way that the functional freedom in the choice of the four functions $F^A$ allows to cover all those coordinates charts $\sigma^A$ in the atlas of the mathematical space-time $M^4$ which are adapted to any allowed 3+1 splitting. By using gauge fixing constraints more general than those in Eq.(2.5) (like the standard gauge fixings used in ADM metric gravity) we can reach all the 4-coordinates systems of $M^4$. Here, however, we wanted to restrict to the class of gauge fixings (2.5) for the sake of clarifying the interpretational issues.

Let us conclude by noting that the gauge fixings (2.5), (2.7) induce a coordinate-dependent non-commutative Poisson bracket structure upon the physical point-events of space-time by means of the associated Dirac brackets implying Eqs.(2.8). More exactly, on-shell, each coordinate system gets a well defined non-commutative structure determined by the associated functions $\tilde{F}^A_G(r^G_a(\tau, \vec{\sigma}), \pi^G_{1a}(\tau, \vec{\sigma}))$, for which we have

$$\{\tilde{F}^A_G(r^G_a(\tau, \vec{\sigma}), \pi^G_{1a}(\tau, \vec{\sigma})), \tilde{F}^B_G(r^G_a(\tau, \vec{\sigma}_1), \pi^G_{1a}(\tau, \vec{\sigma}_1))\}^* \neq 0.$$ 

The meaning of this structure in view of quantization is worth investigating (see the Concluding Remarks).

B. Implementing the Physical Individuation of Point-Events with Well-Defined Empirical Procedures: a Realization of the Axiomatic Structure of Ehlers, Pirani and Schild.

The problem of the individuation of space-time points as point-events cannot be methodologically separated from the problem of defining a theory of measurement consistent with general covariance. This means that we should not employ the absolute chrono-geometric structures of special relativity, like it happens in all the formulations on a given background (gravitational waves as a spin two field over Minkowski space-time, string theory,...). Moreover matter (either test or dynamical) is now an essential ingredient for defining the experimental setup.

At present we do not have such a theory, but only preliminary attempts and an empirical metrology [24], in which the standard unit of time is a coordinate time and not a proper time. As already said, a global non-inertial space-time laboratory with its standards corresponds to a description realized by a completely fixed Hamiltonian gauge viz., being on-shell, in an atlas of uniquely determined 4-coordinate systems.

We shall take into account the following pieces of knowledge.

A) Ehlers, Pirani and Schild [26] developed an axiomatic framework for the foundations of general relativity and measurements (reviewed in Appendix A). These authors exploit
the notions of test objects as idealizations to the effect of approximating the conformal, projective, affine and metric structures of Lorentzian manifolds; such structures are then used to define ideal geodesic clocks [33]. The axiomatic structure refers to basic objects such as test light rays and freely falling test particles. The first ones are used in principle to reveal the conformal structure of space-time, the second ones the projective structure. Under an axiom of compatibility which is well corroborated by experiment (see Ref.[34]), it can be shown that these two independent classes of observations determine completely the structure of space-time. Let us remark that one should extend this axiomatic theory to tetrad gravity (space-times with frames) in order to include objects like test gyroscopes needed to detect gravito-magnetic effects.\textsuperscript{10}

B) De Witt [36] introduced a procedure for measuring the gravitational field based on a reference fluid (a stiff elastic medium) equipped with material clocks. This phenomenological test-fluid is then exploited to bring in Bergmann-Komar invariant pseudo-coordinates $\zeta^A$, $A = 1, \ldots, 4$, as a method for coordinatizing the space-time where to do measurements and also for grounding space-time geometry operationally, at least in the weak field regime. De Witt essentially proposes to simulate a mesh of local clocks and rods. Even if De Witt considers the measurement of a weak quantum gravitational field smeared over such a region, his procedure could even be adopted classically. In this perspective, our approach furnishes the ingredients of the Hamiltonian description of the gravitational field, which were lacking at the time De Witt developed his preferred covariant approach.

C) Antennas and interferometers are the tools used to detect gravitational waves on the Earth. The mechanical prototype of these measurements are test springs with end masses feeling the gravitational field as the tidal effect described by the geodesic deviation equation [33, 37]. Usually, however, one works on the Minkowski background in the limit of weak field and non-relativistic velocities. See Ref.[38] for the extension of this method to a regime of weak field but with relativistic velocities in the framework of a background-independent Hamiltonian linearization of tetrad gravity.

Lacking solutions to Einstein’s equations with matter corresponding to simple systems to be used as idealizations for a measuring apparatuses described by matter DO (hopefully also BO), a generally covariant theory of measurement as yet does not exist. We hope, however, that some of the clarifications achieved in this paper of the existing ambiguities about observables will help in developing such a theory.

In the meanwhile we want to sketch here a scheme for implementing - at least in principle - the physical individuation of points as an experimental setup and protocol for positioning and orientation. Our construction should be viewed in parallel to the axiomatic treatment of Ehlers, Pirani and Schild. We could reproduce the logical scheme of this axiomatic approach in the following way.

a) A radar-gauge system of coordinates can be defined in a finite four-dimensional volume by means of a network of artificial spacecrafts similar to the Global Position System (GPS) [39]. Let us consider a family of spacecrafts, whose navigation is controlled from the Earth by means of the standard GPS. Note that the GPS receivers are able to determine their

\textsuperscript{10} Stachel [35], stresses the dynamical (not axiomatic) aspect of the general relativistic space-times structures associated to the behavior of ideal measuring rods (geometry) and clocks (chronometry) and free test particles (inertial structures)
actual position and velocity because the GPS system is based on the advanced knowledge of the gravitational field of the Earth and of the satellites’ trajectories, which in turn allows the coordinate synchronization of the satellite clocks. During the navigation the spacecrafts are test objects. Since the geometry of space-time and the motion of the spacecrafts are not known in advance in our case, we must think of the receivers as obtaining four, so to speak, conventional coordinates by operating a full-ranging protocol involving bi-directional communication to four super-GPS that broadcast the time of their standard a-synchronized clocks (see the discussion given in Ref.[5] and Refs.[40] for other proposals in the same perspective). This first step parallels the axiomatic construction of the conformal structure of space-time.

Once the spacecrafts have arrived in regions with non weak fields, like near the Sun or Jupiter, they become the (non test but with world-lines assumed known from GPS space navigation) elements of an experimental setup and protocol for the determination of a local 4-coordinate system and of the associated 4-metric.

Each spacecraft, endowed with an atomic clock and a system of gyrosopes, may be thought as a time-like observer (the spacecraft world-line assumed known) with a tetrad (the time-like vector is the spacecraft 4-velocity (assumed known) and the spatial triad is built with gyrosopes) and one of them is chosen as the origin of the radar-4-coordinates we want to define. This means that the natural framework should be tetrad gravity instead of metric gravity.

b) At this point we have to synchronize the atomic clocks by means of radar signals [41]. As shown in I, in an Einstein space-time there is a dynamical determination of the simultaneity convention. However, since - again - the geometry of space-time is not known in advance in our case, we could only lay down the lines of an approximation procedure starting from an arbitrary simultaneity convention like in special relativity. As shown in Section VI of Ref.[31], the spacecraft A chosen as origin (and using the proper time τ along the assumed known world-line) sends radar signals to the other spacecrafts, where they are reflected back to A. For each radar signal sent to a spacecraft B, the spacecraft A records four data: the emission time τ₀, the emission angles θ₀, φ₀ and the absorption time τ_f. Given four admissible (see Ref.[31]) functions \( E(τ₀, θ₀, φ₀, τ_f) \), \( G(τ₀, θ₀, φ₀, τ_f) \) the point \( P_B \) of the world-line of the spacecraft B, where the signal is reflected, is given radar coordinates \( \tau | Q = \tau (R) (P_B) \). This allows establishing a radar-gauge system of 4-coordinates (more exactly a coordinate grid) lacking any direct metric content

\[
σ^A_{(R)} = (\tau_{(R)}; σ^r_{(R)}), \tag{2.10}
\]

in a finite region, with \( \tau_{(R)} = const \) defining the radar simultaneity surfaces of this convention. By varying the functions \( E, G \) we change the simultaneity convention among the admissible ones 11.

Note that by replacing test radar signals (conformal structure) with test particles (projective structure) in the measurements, we would define a different 4-coordinate system.

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11 Einstein’s simultaneity convention corresponds to \( E = \frac{1}{2} \) and to space-like hyper-planes as simultaneity surfaces.
Then the navigation system provides determination of the 4-velocities (time-like tetrads) of the satellites and the $4g_{(R)r}(\sigma^A_{(R)})$ component of the 4-metric in these coordinates.

In the framework of metric gravity the spacecrafts make repeated measurements of the motion of four test particles. In this way they test also the projective structure in a region of space-time with a vacuum gravitational field. By the motion of gyroscopes they measure the shift components $4g_{(R)r}(\sigma^A_{(R)})$ of the 4-metric and end up (in principle) with the determination of all the components of the four-metric with respect to the radar-gauge coordinate system:

$$4g_{(R)AB}(\tau_{(R)}, \sigma^r_{(R)}).$$  \hspace{1cm} (2.11)

The tetrad gravity alternative, employing test gyroscopes and light signals (i.e. only the conformal structure), is the following. By means of exchanges (one-way signals) of polarized light it should be possible to determine how the spatial triads of the satellites are rotated with respect to the triad of the satellite chosen as origin (see also Ref.[42]). Once we have the light it should be possible to determine how the spatial triads of the satellites are rotated.

1) To check whether Einstein’s equation in radar-gauge coordinates are satisfied. If not, this means that the chosen simultaneity $\tau_{(R)} = const.$ is not the dynamical simultaneity of the Einstein solution describing the solar system. By changing the functions $E, \tilde{G}$, we can put up an approximation procedure converging towards the dynamical simultaneity.

2) If $(\tau_R, \bar{\sigma}_R)$ are the radar coordinates corresponding to the dynamical synchronization of clocks, we can get a numerical determination of the intrinsic coordinate functions $\sigma^A_R$ defining the radar gauge by the gauge fixings $\sigma^A_R - \bar{\sigma}^A_R(\sigma_R) \approx 0$. Since we know the eigenvalues of the Weyl tensor in the radar gauge, it is possible to solve in principle for the functions $F^A$ that reproduce the radar-gauge coordinates as radar-gauge intrinsic coordinates

$$\sigma^A_{(R)} = F^{-1}[\bar{\Lambda}_{W}^{(k)}g(\tau, \bar{\sigma}), 3\Pi(\tau, \bar{\sigma})],$$  \hspace{1cm} (2.12)

consistently with the gauge-fixing that enforces just this particular system of coordinates:

$$\chi^A(\tau, \bar{\sigma}) = \sigma^A - \bar{\sigma}^A(\tau, \bar{\sigma}) = \sigma^A - F^A[\bar{\Lambda}_{W}^{(k)}g(\tau, \bar{\sigma}), 3\Pi(\tau, \bar{\sigma})] \approx 0. $$  \hspace{1cm} (2.13)

Finally, the intrinsic coordinates are reconstructed as functions of the DO of the radar gauge, at each point-event of space-time, as the identity.
\[ \sigma^A \equiv \bar{\sigma}^\bar{A} = F_G^\bar{A}[\nu^{(R)}_\bar{a}(\tau, \sigma), \pi^{(R)}_\bar{a}(\tau, \sigma)], \quad (2.14) \]

This procedure of principle would close the *coordinative circuit* of general relativity, linking individuation to operational procedures [5].
III. BERGMANN OBSERVABLES AS TENSORIAL DIRAC OBSERVABLES AND THE ISSUE OF THE OBJECTIVITY OF CHANGE.

This Section is devoted to some crucial aspects of the definition of observable in general relativity. While, for instance in astrophysics, matter observables are usually defined as tetradic quantities evaluated with respect to the tetrads of a time-like observer so that they are obviously invariant under $\rho \text{Diff} M^4$ (PDIQ), the definition of the notion of observable for the gravitational field without matter faces a dilemma. Two fundamental definitions of observable have been proposed in the literature.

1) The off-shell and on-shell Hamiltonian non-local Dirac observables (DO) which, by construction, satisfy hyperbolic Hamilton equations of motion and are, therefore, deterministically predictable. In general, as already said, they are neither tensorial quantities nor invariant under $\rho \text{Diff} M^4$ (PDIQ).

2) The configurational Bergmann observables (BO) [22]: they are quantities defined on $M^4$ which not only are independent of the choice of the coordinates, [i.e. they are either scalars or invariants under $\rho \text{Diff} M^4$ (PDIQ)], but are also "uniquely predictable from the initial data". An equivalent, but according to Bergmann more useful, definition of a (PIDQ) BO, is "a quantity that is invariant under a coordinate transformation that leaves the initial data unchanged".

Let us note, first of all, that PDIQ’s are not in general DO, because they may also depend on the eight gauge variables $n, n_r, \xi^a, \pi_\phi$. Thus most, if not all, of the curvature scalars are gauge dependent quantities at least at the kinematic off-shell level. For example, each 3-metric in the conformal gauge orbit has a different 3-Riemann tensor and different 3-curvature scalars. Since 4-tensors and 4-curvature scalars depend: i) on the lapse and shift functions (and their gradients); ii) on $\pi_\phi$, both implicitly and explicitly through the solution of the Lichnerowicz equation (and this affects the 3-curvature scalars), most of these objects are in general gauge dependent variables from the Hamiltonian point of view. The simplest relevant off-shell scalars with respect to $\rho \text{Diff} M^4$, which exhibit such gauge dependence, are the bilinears $R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$, $R_{\mu\nu\rho\sigma} \epsilon^{\rho\sigma\alpha\beta}$ $R_{\alpha\beta\rho\sigma}$ and the four eigenvalues of the Weyl

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12 For other approaches to the observables of general relativity see Refs.[43]: the perennials introduced in this Reference are essentially our DO. See Ref.[44] for the difficulties in observing perennials experimentally at the classical and quantum levels as well as for their quantization. See Ref.[45] about the non-existence of observables built as spatial integrals of local functions of Cauchy data and their first derivatives, in the case of vacuum gravitational field in a closed universe. Also, Rovelli’s evolving constants of motion and partial observables [46] are related with DO; however, the holonomy loops used in loop quantum gravity [47] are PDIQ but not DO. On the other hand, even recently Ashtekar [48] noted that The issue of diffeomorphism invariant observables and practical methods of computing their properties is one among the relevant challenges.

13 In Ref.[19] Bergmann defines: i) a scalar as a local field variable which retains its numerical value at the same world point under coordinate transformations (passive diffeomorphisms), $\varphi'(x') = \varphi(x)$; ii) an invariant I as a functional of the given fields which has been constructed so that if we substitute the coordinate transforms of the field variables into the argument of I instead of the originally given field variables, then the numerical value of I remains unchanged.
tensor exploited in Section V. What said here does hold, in particular, for the line element $ds^2$ and, therefore, for the causal structure of space-time.

On the other hand, BO are those special PDIQ which are also predictable. Yet, the crucial question is now "what does it precisely mean to be predictable within the configurational framework?". Bergmann, gave in fact a third definition of BO or, better, a third part of the original definition, as "a dynamical variable that (from the Hamiltonian point of view) has vanishing Poisson brackets with all the constraints", i.e., essentially, is also a DO. This means that Bergmann thought, though only implicitly and without proof, that predictability implied that a BO must also be projectable to phase space to a special subset of DO that are also PDIQ.

The unresolved multiplicity of Bergmann’s definitions leads to an entangled net of problems. First of all, as shown at length in Ref.[3], in order to tackle the Cauchy problem at the configuration level 14 one has firstly to disentangle the Lagrangian constraints from Einstein’s equations, then to take into account the Bianchi identities, and finally to write down a system of hyperbolic equations. As a matter of fact one has to mimic the Hamiltonian approach, but with the additional burden of lacking an algorithm for selecting those predictable configurational field variables whose Hamiltonian counterparts are just the DO. The only thing one might do is to adopt an inverse Legendre transformation, to be performed after the Shanmugadhasan canonical transformation characterizing a possible complete set of DO. Yet, this just corresponds to the inverse of Bergmann's statement that the BO are projectable to special (PDIQ) DO. In conclusion, configurational predictability must be equivalent to the statement of off-shell Hamiltonian gauge invariance. The moral is that the complexity of the issue should warn against any naive utilization of geometric intuitions in dealing with the initial value problem of general relativity within the configurational approach.

This Hamiltonian predictability of BO entails in turn that only four functionally independent BO can exist for the vacuum gravitational field, since the latter has only two pairs of conjugate independent degrees of freedom. Let us see now why Bergmann’s multiple definition of BO raises additional subtle problems.

Bergmann himself proposed a constructive procedure for the BO. This is essentially based on his re-interpretation of Einstein’s coincidence argument in terms of the individuation of space-time points as point-events by using intrinsic pseudo-coordinates. In his - already quoted - words [22]: "we might then identify a world point (location-plus-instant-in-time) by the values assumed by (the four intrinsic pseudo-coordinates) and ask for the value, there and then, of a fifth field". As an instantiation of this procedure, Bergmann refers to Komar’s [20] pseudo-tensorial transformation of the 4-metric tensor to the intrinsic pseudo-coordinate system [$\sigma^A = \sigma^A(\bar{\sigma}^A)$ is the inversion of Eqs.(2.2)]

$$^{4}\bar{g}_{AB}(\bar{\sigma}^C) = \frac{\partial \sigma^C}{\partial \bar{\sigma}^A} \frac{\partial \sigma^D}{\partial \bar{\sigma}^B} ^4 g_{CD}(\sigma). \tag{3.1}$$

The $^{4}\bar{g}_{AB}$ represent ten invariant scalar (PDIQ or tetradic) components of the metric; of course, they are not all independent since must satisfy eight functional restrictions following

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14 In the theory of systems of partial differential equations this is done in a passive way in a given coordinate system and then extended to all coordinate systems.
from Einstein’s equations.

Now, Bergmann claims that the ten components $\bar{g}_{AB}(\bar{\sigma}^C)$ are a complete, but non-minimal, set of BO. This claim, however, cannot be true. As already pointed out, since BO are predictable they must in fact be equivalent to (PDIQ) DO so that, for the vacuum gravitational field, exactly four functions at most, out of the ten components $\bar{g}_{AB}(\bar{\sigma}^C)$, can be simultaneously BO and DO, while the remaining components must be non-predictable PDIQ, counterparts of ordinary Hamiltonian gauge variables.

On the other hand, as shown in Section II, the four independent degrees of freedom of the pure vacuum gravitational field, even for Bergmann, have allegedly already been exploited for the individuation of point-events. Besides, as Bergmann explicitly asserts in his purely passive interpretation, the PDIQ $\bar{g}_{AB}(\bar{\sigma}^C)$ identify on-shell a 4-geometry, i.e. an equivalence class in $\mathcal{G}_{\text{geom}} = \mathcal{G}_{\text{Riem}}/\text{DiffM}_4$. Furthermore, as shown in Section II of I, Eq.(2.8), the identification of the same 4-geometry starting from active diffeomorphisms can be done by using their passive re-formulation (the group $Q$). Finally, let us remark that Bergmann’s intention to first exploit the intrinsic pseudo-coordinates and then “ask for the value, there and then, of a fifth field” makes sense only if such “fifth field” is a matter field. Asking the question for purely gravitational quantities like $\bar{g}_{AB}(\bar{\sigma}^C)$ would be at least tautological since, as we have seen, only four of them can be independent and have already been exploited. If the individuation procedure is intended to be effective, it would make little sense to assert that point-events have such and such values in terms of point-events.

But now, Bergmann’s incorrect claim is relevant also to another interesting quandary. Indeed, Bergmann’s main configurational notion of observable and its implications are accepted as they stand in a paper of John Earmann, Ref.[25]. In particular Earmann notes that the intrinsic coordinates $[\bar{\sigma}^C]$ can be used to support Bergmann’s observables and says “one can speak of the event of the metric - components - $\bar{g}_{AB}(\bar{\sigma}^C)$ - having - such - and - such - values - in - the - coordinate - system - $\{\bar{\sigma}^C\}$ - at - the - location - where - the - $\bar{\sigma}^C$ - take - on - values - such - and - so” and (aptly) calls such an item a Komar event, adding moreover that “the fact that a given Komar event occurs (or fails to occur) is an observable matter in Bergmann’s sense, albeit in an abstract sense because how the occurrence of a Komar event is to be observed/measured is an unresolved issue”. Earmann’s principal aim, however, is to exploit Bergmann’s definition of BO to show that ”it implies that there is no physical change, i.e., no change in the observable quantities, at least not for those quantities that are constructible in the most straightforward way from the materials at hand”. Although we are not committed here to object to what Earmann calls ”modern Mc-Taggart argument” about change, we are obliged to take issue against Earmann’s radical universal no-change argument because, if sound, it would contradict the substance of Bergmann’s definition of predictability and would jeopardize the relation between BO and DO which is fundamental to our program.

In order to scrutinize this point, let us resume, for the sake of clarity, the essential basic ingredients of the present discussion.

One: the equations of motion derived from Einstein-Hilbert Action and those derived from ADM Action have exactly the same physical content: the ADM Lagrangian leads, through the Legendre transformation, to Hamilton equations equivalent to Einstein’s equations.

Two: Hamiltonian predictability must, therefore, be equivalent to Lagrangian predictability: specification of the latter, however, is an awkward task.
Three: the only functionally independent Hamiltonian predictable quantities for the vacuum gravitational field, are four DO.

Four: by inverse Legendre transformation, every DO has a Lagrangian predictable counterpart.

Then, a priori, one among the following three possibilities might be true: i) all the existing BO must also be DO; this means however that only four functionally independent BO can exist; ii) some of the existing BO are also DO while other are not; iii) no one of the existing BO is also a DO. Possibilities ii) and iii) entail that Bergmann’s multiple definition (that including the third part) of BO is inconsistent, so that no BO satisfying such multiple definition would exist. Yet, the third part of Bergmann definition is essential for the overall meaning of it since no Lagrangian definition of predictability independent of its Hamiltonian counterpart can exist because of Two. Thus cases ii) and iii) imply inconsistency of the very concept of Bergmann’s observability. Of course, it could be that even i) is false since, after all, Bergmann did not prove the self-consistency of his multiple definition: but this would mean that no Lagrangian predictable quantity could exist which simultaneously be a PDIQ. Here, we are assuming that Bergmann’s multiple definition is consistent and that i) is true. We will formalize this assumption into a definite constructive conjecture later on in this Section.

Let us take up again the discussion about the reality of change. As already noted, the discussion in terms of BO in the language of Komar events (or coincidences) must be restricted to the properties of matter fields because, consistently with the multiple Bergmann’s definition, only four of the BO can be purely gravitational in nature. And, if these latter have already been exploited for the individuation procedure, it would again make little sense to ask whether point-events do or do not change. Therefore, let us consider Earman’s argument by examining his interpretation of predictability and the consequent implications for a BO, say $B(p), p \in M^4$, which, besides depending on the 4-metric and its derivatives up to some finite order, also depends on matter variables, and is of course a PDIQ. In order to simplify the argument, Earman concentrates on the special case of the vacuum solutions to the Einstein’s field equations, asserting however that the argument easily generalizes to non-vacuum solutions. Since we have already excluded the case of vacuum solutions, let us take for granted that this generalization is sound. Earman argues essentially in the following way: 1) There are existence and uniqueness proofs for the initial value problem of Einstein’s equations, which show that for appropriate initial data associated to a three manifold $\Sigma_0 \subset M^4$, there is a unique up to diffeomorphism (obviously to be intended active) maximal development for which $\Sigma_0$ is a Cauchy surface; 2) By definition, a BO is a PDIQ whose value $B(p)$ at some point $p$ in the future of $\Sigma_0$ is predictable from initial data on $\Sigma_0$. If $D_A : M^4 \rightarrow M^4$ is an active diffeomorphism that leaves $\Sigma_0$ and its past fixed, the point $p$ will be sent to the point $p' = D_A \cdot p$. Then, the general covariance of Einstein’s equations, conjoined with predictability, is interpreted to imply $B(p) = B(D_A \cdot p)$. This result, together with the definition $B'(p) = B(D_A^{-1} \cdot p)$ of the drag along of $B$ under the

\[ \text{Note that Earman deliberately deviates here from the purely passive viewpoint of Bergmann (and of the standard Cauchy problem for partial differential equations) by resorting to active diffeomorphisms in place of the coordinate transformations that leave the initial data unchanged or, possibly, in place of their extension in terms of the passive re-interpretation of active diffeomorphisms (Q group).} \]
active diffeomorphism $D_A$, entails $D_A^* B = B$ for a BO. In conclusion, since $\Sigma_o$ is arbitrary, a matter BO should be constant everywhere in $M^4$.

It is clear that, within our class of space-times, this conclusion cannot hold true for any matter dependent BO that is projectable to the DO of the gravitational field *cum* matter, if only for the fact that such BO are in fact ruled by the weak ADM energy which generates real temporal change (see Section IIIID, Eq.(3.8), of I). The crucial point in Earman’s argument is the assertion that predictability implies $B(p) = B(\phi \cdot p)$. But this does *not* correspond to the property of *off-shell gauge invariance* spelled above as the main qualification of predictable quantities, except of course for the trivial case of quantities everywhere constant. As clarified in Sections II and III of I, the relations between active diffeomorphisms and dictable quantities, except of course for the trivial case of quantities everywhere constant. Precisely, because of the properties of the group $Q$ of Bergmann and Komar, we have to distinguish between the active diffeomorphisms in $Q$ that *do belong* to $Q_{can}$ and those that *do not belong* to $Q_{can}$. Actually recall that:

i) The intersection $Q_{can} \cap p\text{Diff} M^4$ identifies the space-time passive diffeomorphisms which, respecting the 3+1 splitting of space-time, are projectable to $G_{4,p}$ in phase space;

ii) The remaining elements of $Q_{can}$ are the projectable subset of active diffeomorphisms in their passive view. The union $Q_{can} \cup p\text{Diff} M^4$ exhausts the Hamiltonian view of Leibniz equivalence.

iii) The elements of $Q$ which do not belong to $Q_{can}$ are not projectable to phase space at all and have, therefore, nothing to do with Lagrangian predictability. In particular the non-projectable active diffeomorphisms (passively reinterpreted) do not correspond to Hamiltonian gauge transformations acting within a given universe, solution of Einstein’s equations. Actually many of them are maps on the space of Cauchy data (i.e. maps among different universes) and consequently are unrelated to Leibniz equivalence.

In conclusion, for most active diffeomorphisms\textsuperscript{16}, the conclusion $B(p) = B(D_A \cdot p)$ cannot hold true. This erroneous conclusion seems to be just an instantiation of how misleading may be any loose geometrical and non-algorithmic interpretation of $\Sigma_o$ as a Cauchy surface within the Lagrangian configuration approach to the initial value problem of general relativity.

Having settled this important point, let us come back to tetradic fields. Besides the tetradic components (3.1) of the 4-metric, we have to take into account the extrinsic curvature tensor $3K^{AB} (\sigma) = \frac{\partial \sigma^A}{\partial \tau} \frac{\partial \sigma^B}{\partial x^r} 3K^{\mu\nu} (x)$. In the coordinates $\sigma^A$ adapted to $\Sigma_r$, it has the components $3K^{rr} (\sigma) = 3K^{\tau r} (\sigma) = 0$ and $3K^{rs} (\sigma)$ and we can rewrite it as

$$3K^{AB} (\sigma^C) = \frac{\partial \sigma^A}{\partial \sigma^C} \frac{\partial \sigma^B}{\partial \sigma^r} 3K^{\lambda \beta} (\sigma) = \frac{\partial \sigma^A}{\partial \sigma^C} \frac{\partial \sigma^B}{\partial \sigma^r} 3K^{rs} (\sigma).$$

(3.2)

In this way we get 10 additional scalar (tetradic) quantities (only six of which are independent due to the vanishing of the lapse and shift momenta) replacing $3K^{rs} (\sigma)$ and, therefore, the ADM momenta $3\Pi^{rs} (\sigma) = e_k [\sqrt{g} (3K^{rs} - 3g^{rs} 3K)] (\sigma)$.

In each intrinsic coordinate system $\bar{\sigma}^A = F^A [\Lambda^k_W (\sigma)]$, we have consequently the 20 scalar (tetradic) components $3\bar{g}_{AB} (\bar{\sigma}^C)$ and $3\bar{K}^{AB} (\bar{\sigma}^C)$ of Eqs.(3.1), (3.2), only 16 of which are functionally independent. However, four of them are scalar intrinsic constraints $H^A (\bar{\sigma}^C) = \ldots\ldots$

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\textsuperscript{16} In particular, the special $D_A$’s considered by Earman.
\[ \frac{\partial \sigma^A}{\partial \sigma^a} \mathcal{H}^A(\sigma) \approx 0 \] replacing the super-hamiltonian and super-momentum constraints \( \mathcal{H}^a(\sigma) = \left( \mathcal{H}(\sigma); \mathcal{H}^r(\sigma) \right) \approx 0 \).

The various aspects of the discussion given above strongly suggest that, in order to give consistency to Bergmann's unresolved multiple definition of BO and, in particular, to his (strictly speaking unproven) claim [22] about the existence of DO that are simultaneously (PDIQ) BO, the following conjecture should be true:

**A Main Conjecture:** "The Darboux basis whose 16 ADM variables consist of the 8 Hamiltonian gauge variables \( n, n_r, \xi, \pi_\phi \), the 3 Abelianized constraints \( \bar{\pi}^\mathbf{H} \approx 0 \), the conformal factor \( \phi \) (to be determined by the super-hamiltonian constraint) and the (non-tensorial) DO \( r_\bar{a}, \pi_\bar{a} \), appearing in the quasi-Shanmugadhasan canonical basis (3.7) of I can be replaced by a Darboux basis whose 16 variables are all PDIQ (or tetradic variables), such that four of them are simultaneous DO and BO, eight vanish because of the first class constraints, and the other 8 are coordinate-independent gauge variables."

If this conjecture is sound, it would be possible to construct an intrinsic Darboux basis of the Shanmugadhasan type (Eq.(3.7) of I). Then a suitable transformation performed off-shell before adding the gauge fixings \( \sigma^A - \bar{\sigma}^A(\sigma) \approx 0 \), should exist bringing from the non-tensorial Darboux basis (3.7) of I to this new intrinsic basis. Since the final result would be a representation of the gauge variables as coordinate-independent (PIDQ) gauge variables and of the DO as Dirac-and-Bergmann observables, the freedom of the above transformation reduces to the possibility of mixing the PDIQ gauge variables among themselves and of making canonical transformations in the subspace of the Dirac-Bergmann observables.

More precisely, we would have a family of quasi-Shanmugadhasan canonical bases in which all the variables are PDIQ and include 7 PDIQ first class constraints (not the one corresponding to the super-hamiltonian constraint) that play the role of momenta. It would be interesting, in particular, to check the form of the constraint replacing the standard super-hamiltonian constraint. By re-expressing the 4 Weyl eigenvalues in terms of anyone of these PDIQ canonical bases, we could still define a Hamiltonian gauge, namely an on-shell 4-coordinate system and then derive the associated individuation of point-events by means of gauge-fixings of the type (2.5). **Note that this would break general covariance even if the canonical basis is PDIQ!** The only difference with respect to the standard Hamiltonian bases would be that, instead of being non-tensorial quantities, both \( r^G_a, \pi^G_a \) and \( \bar{F}^A_G \) in Eq.(2.8) would be PDIQ.

As anticipated in the Introduction, further strong support to the conjecture comes from Newman-Penrose formalism [23] where the basic tetradic fields are the 20 Weyl and Ricci scalars which are PDIQ by construction. While the vanishing of the Ricci scalars is equivalent to Einstein’s equations (and therefore to a scalar form of the super-hamiltonian and super-momentum constraints), the 10 Weyl scalars plus 10 scalars describing the ADM momenta (restricted by the four primary constraints) should lead to the construction of a Darboux basis spanned only by PDIQ restricted by eight PDIQ first class constraints. Again, a quasi-Shanmugadhasan transformation should produce the Darboux basis of the

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\[ ^{17} \text{Note that Bergmann’s constructive method based on tetradic 4-metric is not by itself conclusive in this respect!} \]
conjecture. The problem of the phase space re-formulation of Newman-Penrose formalism is now under investigation.

A final important logical component of the issue of the objectivity of change is the particular question of temporal change. This aspect of the issue is not usually tackled as a sub-case of Earman’s no-universal-change argument discussed above in terms of BO, so it should be answered separately. We shall confine our remarks to the objections raised by Belot and Earmann [49] and Earman [25] (see also Refs.[49, 50, 51, 52] for the so called problem of time in general). According to these authors, the reduced phase space of general relativity is a frozen space without evolution. Belot and Earman draw far reaching conclusions about the absence of real (temporal) change in general relativity from the circumstance that, in spatially compact models of general relativity, the Hamiltonian temporal evolution boils down to a mere gauge transformation and is, therefore, physically meaningless. We want to stress, however, that this result does not apply to all families of Einstein space-times. In particular, there exist space-times like the Christodoulou-Klainermann space-times [4] we are using in this paper that constitute a counterexample to the frozen time argument. They are defined by suitable boundary conditions, are globally hyperbolic, non-compact, and asymptotically flat at spatial infinity as shown in Section III of I. The existence of such meaningful counterexamples entails, of course, that we are not allowed to draw negative conclusions in general about the issue of temporal change in general relativity.

We can conclude that in these space-times there is neither a frozen reduced phase space nor a Wheeler-DeWitt interpretation based on some local concept of time like in compact space-times. Therefore, our models of general relativity are perfectly adequate to accommodate objective temporal change.

Let us remark that the definitions given in this Section of the notion of observable in general relativity are in correspondence with two different points of view, existing in the physical literature, that are clearly spelled out in Ref.[53] and related references, namely:

i) The non-local point of view of Dirac [54], according to which causality implies that only gauge-invariant quantities, i.e., DO, can be measured. As we have shown, this point of view is consistent with general covariance. For instance, \(4 R(\tau, \vec{\sigma})\) is a scalar under diffeomorphisms, and therefore a BO, but it is not a DO - at least the kinematic level - and therefore, according to Dirac, not an observable quantity. Even if \(4 R(\tau, \vec{\sigma}) = 0\) in absence of matter, the other curvature scalars do not vanish in force of Einstein’s equations and, lacking known solutions without Killing vectors, it is not clear their connection with the DO. The 4-metric tensor \(4 g_{\mu \nu}\) itself as well as the line element \(ds^2\) are not DO so a completely fixed gauge is needed to get a definite functional form for them in terms of the DO in that gauge. This means that all standard procedures for defining measures of length and time [55, 56, 57] and the very definition of angle and distance properties of the material bodies forming the reference system, are gauge dependent. Then they are determined only after a complete gauge fixing and after the restriction to the solutions of Einstein’s equations has been made\(^{18}\). Likewise, only after a complete gauge fixing the procedure for measuring the Riemann tensor with

\(^{18}\) Note that in standard textbooks these procedures are always defined without any reference to Einstein’s equations.
$n \geq 5$ test particles described in Ref.[57] (see also Ref.[21]) becomes completely meaningful, just as it happens for the electro-magnetic vector potential in the radiation gauge.

Note finally that, after the introduction of matter, even the measuring apparatuses should be described by the gauge invariant matter DO associated with the given gauge.

ii) The local point of view, according to which the space-time manifold $M^4$ is a kind of postulated (often without any explicit statement) background manifold of physically determinate events, like it happens in special relativity with its absolute chrono-geometric structure. Space-time points are assumed physically distinguishable, because any measurement is performed in the frame of a given reference system interpreted as a physical laboratory. In this view the gauge freedom of generally covariant theories is reduced to mere passive coordinate transformations. See for instance Ref. [58] for a refusal of the concept of DO in general relativity based on the local point of view. This point of view, however, discount the Hole Argument completely and must renounce to a deterministic evolution, so that it is ruled out by our results.

Rovelli ([53]) accepts the non-local point of view and proposes to introduce some special kind of matter for defining a material reference system (not to be confused with a coordinate system) to localize points in $M^4$. The aim is to recover the local point of view in some approximate way \(^1\) since the analysis of classical experiments shows that both approaches lead to the same conclusions in the weak field regime. This approach relies therefore upon matter to solve the problem of the individuation of space-time points as point-events, at the expense of loosing determinism. The emphasis on the fundamental role of matter for the individuation issue is present also in Refs.[36, 52, 59], where material clocks and reference fluids are exploited as test matter.

As we have shown, however, the problem of the individuation can be solved before and without the introduction of matter. The presence of matter has the only effect of modifying the individuation and, of course, is fundamental in trying to establish a general-relativistic theory of measurement.

\(^1\) The main approximations are: 1) neglect, in Einstein equations, the energy-momentum tensor of the matter forming the material reference system (it’s similar to what happens for test particles); 2) neglect, in the system of dynamical equations, the entire set of equations determining the motion of the matter of the reference system (this introduces some indeterminism in the evolution of the entire system).
IV. CONCLUDING REMARKS.

The aim of this paper and the previous one (I) was to show that the Hamiltonian approach to general relativity in the ADM formulations has the capability to get new insights into both deep foundational issues and technical problems of the theory, including its experimental forefront.

In paper I we have clarified the correspondence between the active diffeomorphisms operating in the configurational manifold $M^4$, on the one hand, and the on-shell gauge transformations of the ADM canonical approach to general relativity, on the other. Understanding such correspondence is fundamental for fully disclosing the connection of the Hole phenomenology, at the Lagrangian level, with the correct formulation of the initial value problem of the theory and its gauge invariance, at the Hamiltonian level. The upshot is the discovery that, as concerns both the Hole Argument and the issue of predictability, only the active diffeomorphisms of $M^4$, which are also elements of $Q_{\text{can}}$ (i.e. only the projectable maps of Ref.[10]), play an effective role to define a correct mathematical setting of the initial value problem at the Lagrangian level.

Secondly, we have identified a class of solutions of Einstein’s equations (of the type of the Christodoulou-Klainermann space-times [7]), which are particularly interesting for both our main program and a unified description of gravity and particle physics as well as the analysis of the gravitational phenomenology in the solar system. Such class allows in particular: i) exploiting the 3+1 splitting of space-time required by the ADM Hamiltonian approach to general relativity; ii) an effective time evolution ruled by the so-called weak ADM energy: they provide thereby a counterexample to the frozen time argument and are free of any Wheeler-De Witt interpretation; iii) a possible accommodation of the standard model of elementary particles; iv) the vanishing of super-translations and consequent definability of the total angular momentum; v) the definition of asymptotic idealized structures playing the role of the fixed stars of the empirical astronomy. Finally, by means of suitable restrictions upon the admissible simultaneity hyper-surfaces, they become Lichnerowicz manifolds [60] and allow thereby for the existence of generalized Fourier transforms and the definition of positive and negative asymptotic frequencies. The last option paves the way for a quantization program.

Within this background, we have shown that, unlike the case of standard special relativity, the admissible notions of distant simultaneity in canonical metric gravity turn out to be dynamically determined on-shell, while off-shell different conventions within the same universe are merely different gauge-related options like in special relativity [31]. This gives new insight into the old - and outdated - debate about the so-called conventionality of distant simultaneity in special relativity, showing the trading between conventionality and gauge freedom. On this backdrop, we have furthermore recognized the distinct physical role played by the DO, as embodying tidal-like dynamical effects, on the one hand, and that played by off shell gauge variables as connected to generalized inertial effects, on the other.

The main results of the present paper are:

1) A definite procedure for the physical individuation of the mathematical points of the would-be space-time manifold $M^4$ into physical point-events, through a gauge-fixing identifying the mathematical 4-coordinates with the intrinsic pseudo-coordinates of Komar and Bergmann (defined as suitable functionals of the Weyl scalars). This has led to the conclusion
that each of the point-events of space-time is endowed with its own physical individuation as
the value, as it were, at that point, of the four canonical coordinates or DO (just four!), which
describe the dynamical degrees of freedom of the gravitational field. Since such degrees of
freedom are non-local functionals of the 3-metric and 3-curvature\(^20\), they are unsolvably en-
tangled with the whole metrical texture of the simultaneity surfaces in a way that is strongly
both gauge-dependent and highly non-local with respect to the background mathematical
coordinatization. Still, once they are calculated, they appear as local fields in terms of the
background mathematical coordinatization, a fact that makes the identity Eq.(2.8) possible
and shows, in a sense, a Machian flavor within a non-Machian environment. We can also say,
on the other hand, that any coordinatization of the manifold can be seen as embodying the
physical individuation of points, because it can be implemented as the Komar-Bergmann
intrinsic pseudo-coordinates, after a suitable choice of the functions of the Weyl scalars and
of the gauge-fixing. Moreover, as stressed in Section III, only on-shell matter will come to influence the very physical individuation of points. We claim that our results bring the
Snyge-Bergmann-Komar-Stachel program of the physical individuation of space-time points
to its natural end.

2) It should be clear by now that the Hole Argument has little to do with an alleged
indeterminism of general relativity as a dynamical theory. For, in our analysis of the initial-
value problem within the Hamiltonian framework, we have shown that on shell a complete
gauge-fixing (which could in theory concern the whole space-time) is equivalent, among other things, to the choice of an atlas of coordinate charts on the space-time manifold, and
in particular within the Hole. At the same time, we have shown that a peculiar subset
of the active diffeomorphisms of the manifold can be interpreted as passive Hamiltonian
gauge transformations. Actually, only this subset, realizing the essential content of Leibnitz
equivalence, plays an effective role in connection to the Hole Argument.

3) An outline of the implementation (in principle) of the physical individuation of point-
events as an experimental setup and protocol for positioning and orientation, which closes,
as it were, the practical coordinative circuit of general relativity.

4) A clarification of Bergmann’s multiple ambiguous definition of observable in general
relativity. This has led to formulate our main conjecture concerning the unification of
Bergmann’s and Dirac’s concepts of observable, as well as to restate the issue of change
and, in particular and independently, of temporal change, within the Hamiltonian approach
to Einstein equations. When concretely carried out, this program would provide even ex-
licitly evidence for the invariant objectivity of point-events. Furthermore, the existence of
simultaneous Bergmann-Dirac observables and PDIQ gauge variables would lead to a de-
scription of tidal-like and inertial-like effects in a coordinate independent way, while the
Dirac-Bergmann observables only would remain as the only quantities subjected to a causal

\(^{20}\) Admittedly, at least at the classical level, we don’t know of any detailed analysis of the relationship
between the notion of non-local observable (the predictable degrees of freedom of a gauge system), on one
hand, and the notion of a quantity which has to be operationally measurable by means of local apparatuses,
on the other. Note that this is true even for the simple case of the electro-magnetic field where the Dirac
observables are defined by the transverse vector potential and the transverse electric field. Knowledge of
such fields at a definite mathematical time involves data on the whole Cauchy surface at that time. Even
more complex is the situation in the case of Yang-Mills theories [61]
evolution. If the conjecture about the existence of simultaneous DO-BO observables is sound, it would open the possibility of a new type of coordinate-independent canonical quantization of the gravitational field. Only the DO should be quantized in this approach, while the gauge variables, i.e., the *appearances* of inertial effects, should be treated as c-number fields (a prototype of this quantization procedure is under investigation [62] in the case of special relativistic and non-relativistic quantum mechanics in non-inertial frames). This would permit to preserve causality (the space-like character of the simultaneity Cauchy 3-surfaces), the property of having only the 3-metric quantized (with implications similar to loop quantum gravity for the quantization of spatial quantities), and to avoid any talk of *quantization of the 4-geometry* (see more below), a talk we believe to be deeply misleading (in this connection see Ref. [63])

We want to conclude our discussion with some general remarks about the foundation of general relativity and some venture-some suggestions concerning quantum gravity.

A) First of all, our program is substantially grounded upon the *gauge nature* of general relativity. Such property of the theory, however, is far from being a simple matter and we believe that it is not usually spelled out in a sufficiently explicit and clear fashion. The crucial point is that general relativity *is not* a standard gauge theory like, e.g., electromagnetism or Yang-Mills theories in some relevant respects. The relevant fact is that, while from the point of view of the constrained Hamiltonian formalism general relativity is a gauge theory like any other, it is radically different from the physical point of view. For, in addition to creating the distinction between what is observable\(^{21}\) and what is not, the gauge freedom of general relativity is unavoidably entangled with the definition-constitution of the very *stage*, space–time, where the *play* of physics is enacted. In other words, the gauge mechanism has the double role of making the dynamics unique (as in all gauge theories), and of fixing the spatio-temporal reference background. It is only after a complete gauge-fixing is specified (i.e. after the individuation of a *well defined* physical laboratory network that we have called a *global non-inertial space-time laboratory*), and after going on shell, that even the mathematical manifold \(M^4\) gets a *physical individuation* and becomes the spatio-temporal carrier of well defined physical *tidal-like* and *generalized inertial* effects.

In gauge theories such as electromagnetism, we can rely from the beginning on empirically validated, gauge-invariant dynamical equations for the *local* fields. This is not the case for general relativity: in order to get dynamical equations for the basic field in a *local* form, we must pay the price of Einstein’s general covariance which, by ruling out any background structure at the outset, weakens the objectivity that the spatiotemporal description could have had *a priori*.

The isolation of the superfluous structure hidden behind the Leibniz equivalence (namely the gauge variables describing inertial effects) renders even more glaring the ontological diversity and prominence of the gravitational field with respect to all other fields, as well as the difficulty of reconciling the deep nature of the gravitational field with the standard wisdom of theories based on background space-time like effective quantum field theory and string theory. Any procedure of linearizing and quantizing these latter unavoidably leads to looking at gravity as to a spin-2 theory in which the graviton stands on the same ontological level of other quanta: in the standard approach, photons, gluons and gravitons all live on the

\(^{21}\) In the Dirac or Bergmann sense.
stage on equal footing. From the point of view we gained in this paper, however, quantum DO, i.e. non-perturbative gravitons, do in fact constitute the stage for the causal play of photons, gluons as well as of other matter actors like electrons and quarks. More precisely, if our main conjecture is sound, the non-perturbative graviton would be represented by a pair of scalar fields.

B) Let us close this survey with some hints that our results tend to suggest for the quantum gravity programme. As well-known this programme is documented nowadays by two inequivalent quantization methods: i) the perturbative background-dependent either string or effective QFT formulations, on a Fock space containing elementary particles; ii) the non-perturbative background-independent loop quantum gravity approach, based on the non-Fock polymer Hilbert space. In this connection, see Ref.[64] for an attempt to define a coarse-grained structure as a bridge between standard coherent states in Fock space and some shadow states of the discrete quantum geometry associated to a polymer Hilbert space. As well-known, this approach still fails to accommodate elementary particles.

Now, the individuation procedure we have proposed transfers, as it were, the non-commutative Poisson-Dirac structure of the Dirac observables onto the individuated point-events even if, of course, the coordinates on the l.h.s. of the identity Eq.(2.8) are c-numbers quantities. Of course, no direct physical meaning can be attributed to this circumstance at the classical level. One could guess, however, that such feature might deserve some attention in view of quantization, for instance by maintaining that the identity (Eq.(2.8)) could still play some role at the quantum level. We will assume here that the main conjecture is verified so that all the quantities we consider are manifestly covariant. On the other hand, this is a logical necessity in order to get a coordinate-independent quantization procedure.

Let us first lay down some qualitative premises concerning the status of Minkowski space-time in relativistic quantum field theory (call it micro space-time, see Ref.[65]). Such status is quite peculiar. From the chrono-geometric point of view, the micro space-time is a universal, classical, non-dynamical space-time, just Minkowski’s space-time of the special theory of relativity, utilized without any scale limitation from below. However, it is introduced into the theory through the group-theoretical requirement of relativistic invariance of the statistical results of measurements with respect to the choice of macroscopic reference frames. The micro space-time is therefore anchored to the macroscopic medium-seized objects that asymptotically define the experimental conditions in the laboratory. It is, in fact, in this asymptotic sense that a physical meaning is attributed to the classical spatiotemporal coordinates upon which the quantum fields’ operators depend as parameters. Thus, the spatiotemporal properties of the micro Minkowski manifold, including its basic causal structure, are, as it were, projected on it from outside.

Summarizing, the role of Minkowski’s micro space-time seems to be essentially that of an instrumental external translator of the symbolic structure of quantum theory into the causal language of the macroscopic irreversible traces that constitute the experimental findings within macro space-time. The conceptual status of this external translator fits then very well with that of epistemic precondition for the formulation of relativistic quantum field theory.

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22 One should not forget that the Minkowski structure of the micro-space-time has been probed down to the scale of $10^{-18}$ m., yet only from the point of view of scattering experiments, involving a limited number of real particles.
in the sense of Bohr, independently of one’s attitude towards the interpretation of quantum theory of measurement. Thus, barring macroscopic Schrödinger Cat states of the would-be quantum space-time, any conceivable formulation of a quantum theory of gravity would have to respect, at the operational level, the epistemic priority of a classical spatiotemporal continuum. Talking about the quantum structure of space-time needs overcoming a serious conceptual difficulty concerning the localization of the gravitational field: indeed, what does it even mean to talk about the values of the gravitational field at a point, to the effect of points individuation, if the field itself is subject to quantum fluctuations? One needs in principle some sort of reference structure in order to give physical operational meaning to the spatiotemporal language, one way or the other. It is likely, therefore, that in order to attribute some meaning to the individuality of points that lend themselves to the basic structure of standard quantum theory, one should split, as it were, the individuation of point-events from the true quantum properties, i.e., from the fluctuations of the gravitational field and the micro-causal structure.

Now, it seems that our canonical analysis of the individuation issue, tends to prefigure a new approach to quantization having in view a Fock space formulation. Accordingly, unlike loop quantum gravity, this approach could even lead to a background-independent incorporation of the standard model of elementary particles (provided the Cauchy surfaces admit Fourier transforms). Two options present themselves for a quantization program respecting relativistic causality 23:

1) The procedure for the individuation outlined in Section II suggests to quantize the DO=BO of each Hamiltonian gauge, as well as all the matter DO, and to use the weak ADM energy of that gauge as Hamiltonian for the functional Schrödinger equation (of course there might be ordering problems). This quantization would yield as many Hilbert spaces as 4-coordinate systems, which would likely be grouped in unitary equivalence classes (we leave aside asking what could be the meaning of inequivalent classes, if any). In each Hilbert space the DO=BO quantum operators would be distribution-valued quantum fields on a mathematical micro space-time parametrized by the 4-coordinates $\tau, \vec{\sigma}$ associated to the chosen gauge. Strictly speaking, due to the non-commutativity of the operators $\hat{F}^A$ associated to the classical gauge-fixing (2.5) $\sigma^A - F^A \approx 0$ defining that gauge, there would be no space-time manifold of point-events to be mathematically identified by one coordinate chart over the micro-space-time: only a gauge-dependent non-commutative structure which is likely to lack any underlying topological space structure. However, for each Hilbert space, a coarse-grained space-time of point-events might be associated to each solution of the functional Schrödinger equation, through the expectation values of the operators $\hat{F}^A$:

$$\Sigma^A(\tau, \vec{\sigma}) = \langle \Psi | \hat{F}^A_G[R^a(\tau, \vec{\sigma}), \Pi_a(\tau, \vec{\sigma})] | \Psi \rangle, \quad a = 1, 2;$$

(4.1)

where $R^a(\tau, \vec{\sigma})$ and $\Pi_a(\tau, \vec{\sigma})$ are scalar Dirac operators.

Let us note that, by means of Eq.(4.1), the non-locality of the classical individuation of point-events would directly get imported at the basis of the ordinary quantum non-locality.

Also, one could evaluate in principle the expectation values of the operators corresponding to the lapse and shift functions of that gauge. Since we are considering a quantization of

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23 Recall that a 3+1 splitting of the mathematical space-time, including the notions of space-like, light-like, and time-like directions, is presupposed from the beginning.
the 3-geometry (like in loop quantum gravity), evaluating the expectation values of the quantum 3-metric and the quantum lapse and shift functions could permit to reconstruct a coarse-grained foliation with coarse-grained WSW hyper-surfaces\textsuperscript{24}.

2) In order to avoid inequivalent Hilbert spaces, we could quantize \textit{before} adding any gauge-fixing (i.e. independently of the choice of the 4-coordinates and the individuation of point-events), using e.g., the following rule of quantization, which respects relativistic causality: in a given canonical basis of the conjecture, quantize the two pairs of DO=BO observables and the matter DO, but leave the 8 gauge variables $\zeta^a(\tau, \vec{\sigma})$, $a = 1, ..., 8$, as \textit{c-number classical fields} (\textit{c-number generalized times}). Like in Schrödinger's theory with time-dependent Hamiltonian, where $i \frac{\partial}{\partial t}$ is equated to the action of the Hamiltonian operator, the momenta conjugate to the gauge variables would be represented by the functional derivatives $i \delta / \delta \zeta^a(\tau, \vec{\sigma})$. Assuming that, in the chosen canonical basis of our \textit{main conjecture}, 7 among the eight constraints be gauge momenta, we would get 7 Schrödinger equations $i \delta / \delta \zeta^a(\tau, \vec{\sigma}) \Psi(R^a|\tau; \zeta^a) = 0$ from them. Let $H(\text{new}) \approx 0$ be the super-Hamiltonian constraint and $E_{ADM}(\text{new})$ the weak ADM energy, in the new basis. Both would become operators $\hat{H}$ or $\hat{E}_{ADM}(\hat{r}_a, \hat{\pi}_a, \zeta^a, i \delta / \delta \zeta^a)$. If an ordering existed such that the 8 quantum constraints $\hat{\phi}_a$ and $\hat{E}_{ADM}$ satisfied a closed algebra $[\hat{\phi}_a, \hat{\phi}_b] = \hat{C}_{\alpha\beta\gamma} \hat{\phi}_a$, and $[\hat{E}_{ADM}, \hat{\phi}_a] = \hat{B}_{\alpha\beta} \hat{\phi}_b$ (with the quantum structure functions tending to the classical ones for $\hbar \to 0$), we might quantize by imposing the following 9 coupled integrable functional Schrödinger equations

\[
i \frac{\delta}{\delta \zeta^a(\tau, \vec{\sigma})} \Psi(R^a|\tau; \zeta^a) = 0, \quad \alpha = 1, ..., 7, \quad \Rightarrow \quad \Psi = \Psi(R^a|\tau; \zeta^8),
\]

\[
\hat{H}(R^a, i \frac{\delta}{\delta R_a}, \zeta^a, i \frac{\delta}{\delta \zeta^a}) \Psi(R^a|\tau; \zeta^a) = 0, \quad \bar{a} = 1, 2,
\]

\[
i \frac{\partial}{\partial \tau} \Psi(R^a|\tau; \zeta^a) = \hat{E}_{ADM}(R^a, i \frac{\delta}{\delta R_a}, \zeta^a, i \frac{\delta}{\delta \zeta^a}) \Psi(R^a|\tau; \zeta^a),
\]

(4.2)

with the associated usual Schroedinger scalar product $\langle \Psi | \Psi \rangle$ being independent of $\tau$ and $\zeta^a$'s because of Eq.(4.2). This is similar to what happens in the quantization of the two-body problem in relativistic mechanics [62, 67, 68].

If the previously described quasi-Shanmugadhasan canonical basis exists, the wave functional would depend on 8 functional field parameters $\zeta^a(\tau, \vec{\sigma})$, besides the mathematical time $\tau$ (actually only on $\zeta^8$). Each \textit{curve} in this parameter space would be associated to a Hamiltonian gauge in the following sense: for each solution $\Psi$ of the previous equations, the classical gauge-fixings $\sigma^A - F^A_G \approx 0$ implying $\zeta^a = \zeta^{(G)\alpha}(R^a, \Pi_a)$, would correspond to expectation values $\langle \Psi | \zeta^{(G)\alpha}(\tau, \vec{\sigma}) | \Psi \rangle = \zeta^{(G)\alpha}(\tau, \vec{\sigma})$ defining the \textit{curve} in the parameter space. Again, we would have a \textit{mathematical micro space-time} and a \textit{coarse-grained space-time of "point-events"}. At this point, by going to \textit{coherent states}, we could try to recover classical gravitational fields\textsuperscript{25}. The 3-geometry (volumes, areas, lengths) would be quantized, perhaps in a way coherent with the results of loop quantum gravity.

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\textsuperscript{24} This foliation is called [66] the Wigner-Sen-Witten (WSW) foliation due to its properties at spatial infinity (see footnote 14 of I).

\textsuperscript{25} At the classical level, we have the ADM Poincaré group at spatial infinity on the asymptotic Minkowski
It is important to stress that, according to both of our suggestions, only the DO would be quantized. The upshot is that fluctuations in the gravitational field (better, in the DO) would entail fluctuations of the point texture that lends itself to the basic space-time scheme of standard relativistic quantum field theory: such fluctuating texture, however, could be recovered as a coarse-grained structure. This would induce fluctuations in the coarse-grained metric relations, and thereby in the causal structure, both of which would tend to disappear in a semi-classical approximation. Such a situation should be conceptually tolerable, and even philosophically appealing, as compared with the impossibility of defining a causal structure within all of the attempts grounded upon quantization of the full 4-geometry.

Besides, in space-times with matter, our procedure entails quantizing the tidal effects and action-at-a-distance potentials between matter elements but not the inertial aspects of the gravitational field. As shown before, the latter are connected with the gauge variables whose variations reproduce all the possible viewpoints of local accelerated time-like observers. Thus, quantizing the gauge variables would be tantamount to quantizing the metric and the passive observers and their reference frames associated to the congruences studied in Section IV of I. Of course, such observers have nothing to do with the dynamical observers, which should be realized in terms of the DO of matter.

Finally, concerning different ways of looking at inertial forces, consider for the sake of completeness the few known attempts of extending non-relativistic quantum mechanics from global inertial frames to global non-inertial ones [69] by means of time-dependent unitary transformations \( U(t) \). The resulting quantum potentials \( V(t) = i\dot{U}(t)U^{-1}(t) \) for the fictitious forces in the new Hamiltonian \( \tilde{H} = U(t)HU^{-1}(t) + V(t) \) for the transformed Schrödinger equation, as seen by an accelerated observer (passive view), are often re-interpreted as action-at-a-distance Newtonian gravitational potentials in an inertial frame (active view). This fact, implying in general a change in the emission spectra of atoms, is justified by invoking an extrapolation of the non-relativistic limit of the weak equivalence principle (universality of free fall or identity of inertial and gravitational masses) to quantum mechanics. Our Hamiltonian distinction among tidal, inertial and action-at-a-distance effects supports Synge's criticism [21] b) of Einstein's statements about the equivalence of uniform gravitational fields and uniform accelerated frames. Genuine physical uniform gravitational fields do not exist over finite regions and must be replaced by tidal and action-at-a-distance effects: these, however, are clearly not equivalent to uniform acceleration effects. From our point of view, the latter are generated as inertial effects whose appearance depends upon the gauge variables. Consequently, the non-relativistic limit of our quantization procedure should be consistent with the previous passive view in which

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26 Note that as it happens with the time-dependent Foldy-Wouthuysen transformation [70], the operator \( \tilde{H} \) describing the non-inertial time evolution is no more the energy operator.

27 Nor is their definition a unambiguous task in general [71].
atom spectra are not modified by pure inertial effects, and should match the formulation of standard non-relativistic quantum mechanics of Ref.[62].
APPENDIX A: AXIOMATIC FOUNDATIONS AND THEORY OF MEASUREMENT IN GENERAL RELATIVITY.

In this Appendix we review the Ehlers-Pirani-Schild axiomatic approach [26] to the theory of measurement in general relativity, based on idealized test matter.

After a critique of the Synge’s chronometrical axiomatic approach [21], Ehlers, Pirani and Schild [26], reject clocks as basic tools for setting up the space-time geometry and propose to use light rays and freely falling particles. The full space-time geometry can then be synthesized from a few local assumptions about light propagation and free fall.

a) The propagation of light determines at each point of space-time the infinitesimal null cone and thus establishes its conformal structure $\mathcal{C}$. In this way one introduces the notions of being space-like, time-like and null and one can single out as null geodesics the null curves contained in a null hyper-surface (the light rays).

b) The motions of freely falling particles determine a family of preferred $\mathcal{C}$-time-like curves. By assuming that this family satisfies a generalized law of inertia (existence of local inertial frames in free fall, equality of inertial and passive gravitational mass), it follows that free fall defines a projective structure $\mathcal{P}$ in space-time such that the world lines of freely falling particles are the $\mathcal{C}$-time-like geodesics of $\mathcal{P}$.

c) Since, experimentally, an ordinary particle (positive rest mass), though slower than light, can be made to chase a photon arbitrarily close, the conformal and projective structures of space-time are compatible, in the sense that every $\mathcal{C}$-null geodesic is also a $\mathcal{P}$-geodesic. This makes $M^4$ a Weyl space $(M^4, \mathcal{C}, \mathcal{P})$. A Weyl space possesses a unique affine structure $\mathcal{A}$ such that $\mathcal{A}$-geodesics coincide with $\mathcal{P}$-geodesics and $\mathcal{C}$-nullity of vectors is preserved under $\mathcal{A}$-parallel displacement. In conclusion, light propagation and free fall define a Weyl structure $(M^4, \mathcal{C}, \mathcal{A})$ on space-time (this is equivalent to an affine connection due to the presence of both the projective and the conformal structure).

d) In a Weyl space-time, one can define an arc length (unique up to linear transformations) along any non-null curve. Applying such definition to the time-like world line of a particle

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28 Synge accepts as basic primitive concepts particles and (standard) clocks. Then he introduces the 4-metric as the fundamental structure, postulating that whenever $x, x + dx$ are two nearby events contained in the world line or history of a clock, then the separation associated with $(x, x + dx)$ equals the time interval as measured by that (and by other suitably scaled) clock. These axioms are good for the deduction of the subsequent theory, but are not a good constructive set of axioms for relativistic space-times geometries. The Riemannian line element cannot be derived by clocks alone without the use of light signals. The chronometrical determination of the 4-metric components does not compellingly determine the behavior of freely falling particles and light rays and Synge has to add a further axiom (the geodesic hypothesis). On the basis of this axiom it is then possible (Marzke [41], Kundt-Hoffmann [72]) to construct clocks by means of freely falling particles and light rays (i.e. to give a physical interpretation of the 4-metric in terms of time). Therefore the chronometrical axioms appear either as redundant or, if the term clock is interpreted as atomic clock, as a link between macroscopic gravitation theory and atomic physics: these authors claim for the equality of gravitational and atomic time. It should be better to test this equality experimentally (in radar tracking of planetary orbits atomic time has been used only as an ordering parameter, whose relation to gravitational time was to be determined from the observations) or to derive it eventually from a theory that embraces both gravitational and atomic phenomena, rather than to postulate it as an axiom.
P (not necessarily freely falling), we obtain a \textit{proper time} (= arc length) $t$ on $P$, provided two events on $P$ have been selected as \textit{zero point} and \textit{unit point of time}. The (idealized) Kundt-Hoffmann experiment [72] designed to measure proper time along a time-like world line in Riemannian space-time by means of light signals and freely falling particles can be used without modifications to measure the proper time $t$ in a Weyl space-time.

e) In absence of a \textit{second clock effect} a Weyl space $(M^4, \mathcal{C}, \mathcal{A})$ becomes a Riemannian space, in the sense that there exists a Riemannian 4-metric $\mathcal{M}$ compatible with $\mathcal{C}$ (i.e. having the same null-cones) and having $\mathcal{A}$ as its metric connection. The Riemannian metric is necessarily unique up to a constant positive factor. Since $\mathcal{A}$ determines a \textit{curvature tensor} $R$, the use of the equation of geodesic deviation shows that $(M^4, \mathcal{C}, \mathcal{A})$ is Riemannian if and only if the proper times $t, t'$ of two arbitrary, infinitesimally close, freely falling particles $P, P'$ are \textit{linearly} related (to first order in the distance) by \textit{Einstein simultaneity} (see Ref.[26]).

In Newtonian space-time the role of $\mathcal{C}$ is played by the \textit{absolute time}. It is also easy to add a physically meaningful axiom that singles out the space-time of special relativity, either by requiring homogeneity and isotropy of $M^4$ with respect to $(\mathcal{C}, \mathcal{A})$, or by postulating vanishing relative accelerations between arbitrary, neighboring, freely falling particles.

Now, Perlick [34] states that experimental data on standard atomic clocks confirm the absence of the \textit{second clock effect}, so that our actual space-time is not Weyl but pseudo-Riemannian and it is possible to introduce a notion of \textit{ideal rigid rod}.

Let us note that the previous axiomatic approach should be enlarged to cover tetrad gravity, because of the need of test gyroscopes to define the triads of the tetrads of time-like observers. Then the axiomatic would include the possibility of measuring gravito-magnetism and would have to face the question of whether or not the free fall of macroscopic test gyroscopes is geodesic.

An associated theory of the measurement of time-like and space-like intervals has been developed by Martzke-Wheeler [33, 41], using Schild geodesic clock (if it is a standard clock, Perlick’s definition of rigid rod can be used): \textit{the axiomatics is replaced by the empirical notion of a fiducial interval as standard}. Pauri and Vallisneri [73] have further developed the Martzke-Wheeler approach, showing that, given the \textit{whole} world-line of an accelerated time-like observer, it is possible to build an associated space-time foliation with simultaneity space-like non-overlapping 3-surfaces. However, since, like in the local construction of Fermi coordinate systems, the 3-surfaces are orthogonal to the observer world-line, its validity is limited to a neighborhood of the observer, determined by the acceleration radii. See the discussion in Subsection IIB and Section VI of Ref.[31], for the construction of good foliations with simultaneity 3-surfaces not orthogonal to the observer world-line.

As already said, material (test) reference fluids were introduced by various authors [36, 53, 59] for simulating the axioms.

\footnote{The first clock effect is essentially the twin paradox effect. On the other hand, if the time unit cannot be fixed for all standard clocks simultaneously in a consistent way, Perlick [34] speaks of a \textit{second clock effect}.}
[1] L.Lusanna and M.Pauri, *The Physical Role of Gravitational and Gauge Degrees of Freedom in General Relativity - I: Dynamical Synchronization and Generalized Inertial Effects* (gr-qc/0403081).

[2] J.Stachel, *The Cauchy Problem in General Relativity - The Early Years*, in *Historical Studies in General Relativity*, Einstein Studies, Vol. 3, eds. J.Eisenstaedt and A.J.Kox (Birkhäuser, Boston, 1992), pp.407-418.

[3] H.Friedrich and A.Rendall, *The Cauchy Problem for Einstein Equations*, in *Einstein’s Field Equations and their Physical Interpretation*, ed. B.G.Schmidt (Springer, Berlin, 2000) (gr-qc/0002074).

A.Rendall, *Local and Global Existence Theorems for the Einstein Equations*, Online journal Living Reviews in Relativity 1, n. 4 (1998) and 3, n. 1 (2000) (gr-qc/0001008).

[4] D.Christodoulou and S.Klainerman, *The Global Nonlinear Stability of the Minkowski Space* (Princeton, Princeton, 1993).

[5] M.Pauri and M.Vallisneri, *Ephemeral Point-Events: is there a Last Remnant of Physical Objectivity?*, essay for the 70th birthday of R.Torretti, Dialogos 79, 263 (2002) (gr-qc/0203014).

[6] L.Lusanna, *Space-Time, General Covariance, Dirac-Bergmann Observables and Non-Inertial Frames*, talk at the 25th Johns Hopkins Workshop 2001: A Relativistic Space-Time Odyssey, Firenze September 3-5, 2001 (gr-qc/0205039).

L.Lusanna, *The Chrono-Geometrical Structure of Special and General Relativity: towards a Background-Independent Description of the Gravitational Field and Elementary Particles*, invited paper for the book *Progress in General Relativity and Quantum Cosmology Research* (Nova Science) (gr-qc/0404122).

[7] L.Lusanna and M.Pauri, *General Covariance and the Objectivity of Space-Time Point-Events: The Physical Role of Gravitational and Gauge Degrees of Freedom in General Relativity* (gr-qc/0301040).

[8] J.Earman and J.Norton, *What Price Space-Time Substantivalism? The Hole Story*, British Journal for the Philosophy of Science, 38, 515 (1987).

[9] M.Dorato and M.Pauri, *Holism and Structuralism in Classical and Quantum General Relativity*, Pittsburgh-Archive, ID code 1606, February 10, 2004.

[10] P.G.Bergmann and A.Komar, *The Coordinate Group Symmetries of General Relativity*, Int.J Theor.Phys. 5, 15 (1972).

[11] A.Einstein, *Die Grundlage der allgemein Relativitätstheorie*, Annalen der Physik 49, 769 (1916); translated by W.Perrett and G.B.Jeffrey, *The Foundations of the General Theory of Relativity*, in *The Principle of Relativity* (Dover, New York, 1952), pp.117-118.

[12] J.Norton, *Einstein, the Hole Argument and the Reality of Space*, in *Measurement, Realism and Objectivity*, ed. J.Forge (Reidel, Dordrecht, 1987).

J.Norton, *General Covariance and the Foundations of General Relativity: Eight Decades of Dispute*, Rep.Prog.Phys. 56, 791 (1993).

[13] J.Stachel, *Einstein’s Search for General Covariance, 1912-1915*, paper read at the Ninth International Conference on General Relativity and Gravitation, Jena 1980; published in *Einstein and the History of General Relativity*, Einstein Studies, Vol.1, eds. D.Howard and J.Stachel (Birkhäuser, Boston, 1985), pp.63-100.

[14] H.Weyl, *Groups, Klein’s Erlangen Program. Quantities*, ch.I, sec.4 of *The Classical Groups*,
their Invariants and Representations, 2nd ed., (Princeton University, Princeton, 1946), pp.13-23.

[15] R.M.Wald, General Relativity (University of Chicago, Chicago, 1984), pp.438-439.

[16] J.Stachel, How Einstein Discovered General Relativity: A Historical Tale with Some Contemporary Morals, in Proc. GR11 General Relativity and Gravitation, ed. M.A.H. MacCallum (Cambridge University Press, Cambridge, 1987) p.200.

[17] J.Stachel, The Meaning of General Covariance, in Philosophical Problems of the Internal and External Worlds, Essays in the Philosophy of A.Grünaum, eds. J.Earman, A.I.Janis, G.J.Massey and N.Rescher (Pittsburgh Univ. Press, Pittsburgh, 1993).

[18] P.G.Bergmann and A.Komar, Poisson Brackets between Locally Defined Observables in General Relativity, Phys.Rev.Letters 4, 432 (1960).

[19] P.G.Bergmann, The General Theory of Relativity, in Handbuch der Physik, Vol. IV, Principles of Electrodynamics and Relativity, ed. S.Flugg (Springer, Berlin, 1962) pp.247-272.

[20] A.Komar, Construction of a Complete Set of Independent Observables in the General Theory of Relativity, Phys.Rev. 111, 1182 (1958).

[21] J.L.Synge, a) Relativity: the Special Theory (North Holland, Amsterdam, 1956); b) Relativity: the General Theory (North Holland, Amsterdam, 1960).

[22] P.G.Bergmann, Observables in General Relativity, Rev.Mod.Phys. 33, 510 (1961).

[23] J.Stewart, Advanced General Relativity (Cambridge Univ. Press, Cambridge, 1993).

[24] M.H.Soffel, Relativity in Astrometry, Celestial Mechanics and Geodesy (Springer, Berlin, 1989).

[25] J.Earman, Thoroughly Modern McTaggart or what McTaggart would have said if He had read the General Theory of Relativity, Philosophers’ Imprint 2, No.3 August 2002 (http://www.philosophersimprint.org/002003/).

[26] J.Ehlers, F.A.E.Pirani and A.Schild, The Geometry of Free-Fall and Light Propagation in General Relativity, Papers in Honor of J.L.Synge, ed. L.O’Raifeartaigh (Oxford Univ.Press, London, 1972).

[27] J.Géhéniau and R.Debever, Les quatorze invariants de courbure de l’espace Riemannien a’ quatre dimensions in Jubilee of Relativity Theory, eds. A.Mercier and M.Kervaire, Bern 1955, Helvetica Physica Acta Supplementum IV (Birkhäuser, Basel, 1956).

E.Zakhary and C.B.G.McIntosh, A Complete Set of Riemann Invariants, Gen.Rel.Grav. 29, 539 (1997).

[28] J.Norton, The Hole Argument, PSA 1988, Vol. 2, pp.56-64.

[29] T.Maudlin, The Essence of Space-Time, PSA 1988, Vol.2, pp.82-91.

[30] S.Saunders, Indiscernibles, General Covariance and Other Symmetries (2001), www.philsociety-archive.pitt.edu/documents/disk0/00/04/016.

[31] D.Alba and L.Lusanna, Simultaneity, Radar 4-Coordinates and the 3+1 Point of View about Accelerated Observers in Special Relativity (gr-qc/0311058).

[32] A.Einstein, letter of January 3rd 1916 in Albert Einstein and Michele Besso Correspondence 1903-1955, ed. P.Speziali (Hermann, Paris, 1972).

Relativity and the Problem of Space in Relativity: the Special and General Theory (Crown, New York, 1961).

M.Jammer, Concepts of Space (Harvard Univ.Press, Cambridge, 1954).

[33] C.W.Misner, K.S.Thorne and J.A.Wheeler, Gravitation (Freeman, New York, 1973).

[34] V.Perlick, Characterization of Standard Clocks by means of Light Rays and Freely Falling Particles, Gen.Rel.Grav. 19, 1059 (1987).
Characterization of Standard Clocks in General Relativity, in Semantic Aspects of Space-Time Theories, eds. U.Majer and H.J.Schmidt (BI-Wissenschaftsverlag, Mannheim, 1994).

[35] J.Stachel, A Brief History of Space-Time, contribution at the 25th Johns Hopkins Workshop 2001: A Relativistic Spacetime Odyssey, eds. I.Ciufolini, D.Dominici and L.Lusanna (World Scientific, Singapore, 2003).

[36] B.S.De Witt, The Quantization of Geometry, in Gravitation, ed. L.Witten (Wiley, New York, 1962).

[37] B.F.Schutz, A First Course in General Relativity (Cambridge University Press, Cambridge, 1989).

[38] J.Agresti, R.DePietri, L.Lusanna and L.Martucci, Hamiltonian Linearization of the Rest-Frame Instant Form of Tetrad Gravity in a Completely Fixed 3-Orthogonal Gauge: a Radiation Gauge for Background-Independent Gravitational Waves in a Post-Minkowskian Einstein Space-Time, to appear in Gen.Rel.Grav. (gr-qc/0302084).

[39] N.Ashby and J.J.Spilker, Introduction to Relativistic Effects on the Global Positioning System, in Global Positioning System: Theory and Applications, Vol.1, eds. B.W.Parkinson and J.J.Spilker (American Institute of Aeronautics and Astronautics, 1995).

[40] C.Rovelli, GPS Observables in General Relativity, e-print 2001 (gr-qc/0110003).

[41] R.F.Marzke and J.A.Wheeler, Gravitation as Geometry- I: the Geometry of the Space-Time and the Geometrodynamical Standard Meter, in Gravitation and Relativity, eds. H.Y.Chiu and W.F.Hoffman (Benjamin, New York, 1964).

[42] R.A.Coleman and H.Korte’, A Semantic Analysis of Model and Symmetry Diffeomorphisms in Modern Space-Time Theories in Semantic Aspects of Space-Time Theories, eds. U.Majer and H.J.Schmidt (BI Wissenschaftsverlag, Mannheim, 1994).

[43] P.Hájíček, Group Quantization of Parametrized Systems. I. Time Labels, J.Math.Phys. 36, 4612 (1995).

[44] K.Kuchar, Canonical Quantum Gravity in General Relativity and Gravitation Int.Conf. GR13, Cordoba (Argentina) 1992, eds. R.J.Gleiser, C.N.Kozameh and O.M.Moreschi (IOP, Bristol, 1993).

[45] C.G.Torre, Gravitational Observables and Local Symmetries, Phys.Rev. D48, R2373 (1993).

[46] C.Rovelli, Quantum Evolving Constants of Motion: Reply to Comment on 'Time in Quantum Gravity: an Hypothesis', Phys.Rev. D44, 1339 (1991).

[47] C.Rovelli, Partial Observables, Phys.Rev. D65, 124013 (2002)(gr-qc/0110035).

[48] C.Rovelli, Loop Quantum Gravity, Living Rev.Rel. 1, 1 (1998) (gr-qc/9710008).

[49] A.Ash, Quantum Geometry and Gravity: Recent Advances, (gr-qc/0112038) Dec.2001.

[50] G.Belot and J.Earman, From Metaphysics to Physics, in From Physics to Philosophy, eds
J.Butterfield and C.Pagonis (Cambridge University Press, Cambridge, 1999) p.166.  

Pre-Socratic Quantum Gravity, in Physics Meets Philosophy at the Planck Scale. Contemporary Theories in Quantum Gravity, ed. C.Callender (Cambridge University Press, Cambridge, 2001).

[50] C.J.Isham, Canonical Quantum Gravity and the Problem of Time, in Integrable Systems, Quantum Groups and Quantum Field Theories, eds.L.A.Ibort and M.A.Rodriguez, Salamanca 1993 (Kluwer, London, 1993).  
Conceptual and Geometrical Problems in Quantum Gravity, in Recent Aspects of Quantum Fields, Schladming 1991, eds. H.Mitter and H.Gausterer (Springer, Berlin, 1991).  
Prima Facie Questions in Quantum Gravity and Canonical Quantum Gravity and the Question of Time, in Canonical Gravity: From Classical to Quantum, eds. J.Ehlers and H.Friedrich (Springer, Berlin, 1994).

[51] J.Butterfield and C.J.Isham, Space-Time and the Philosophical Challenge of Quantum Gravity, Imperial College preprint TP/98-99/45 (gr-qc/9903072).  
On the Emergence of Time in Quantum Gravity, Imperial College preprint TP/98-99/23 (gr-qc/9901024).

[52] K.Kuchar, Time and Interpretations of Quantum Gravity, in Proc.4th Canadian Conf. on General Relativity and Relativistic Astrophysics, eds. G.Kunstatter, D.Vincent and J.Williams (World Scientific, Singapore, 1992).

[53] C.Rovelli, What is Observable in Classical and Quantum Gravity?, Class. Quantum Grav. 8, 297; Quantum Reference Systems, 8, 317 (1991).

[54] P.A.M.Dirac, Lectures on Quantum Mechanics, Belfer Graduate School of Science, Monographs Series (Yeshiva University, New York, N.Y., 1964).

[55] L.Landau and E.Lifschitz, The Classical Theory of Fields (Addison-Wesley, Cambridge, 1951).

[56] N.Straumann, General Relativity and Relativistic Astrophysics (Springer, Berlin, 1984).

[57] I.Ciufolini and J.A.Wheeler, Gravitation and Inertia (Princeton Univ.Press, Princeton, 1995).

[58] S.S.Feng and C.G. Huang, Can Dirac Observability Apply to Gravitational Systems?, Int.J.Theor.Phys. 36, 1179 (1997).

[59] J.D.Brown and K.Kuchar, Dust as a Standard of Space and Time in Canonical Quantum Gravity, Phys.Rev. D51, 5600 (1995).

[60] A.Lichnerowicz, Propagateurs, Commutateurs et Anticommutateurs en Relativité Generale, in Les Houches 1963, Relativity, Groups and Topology, eds. C.DeWitt and B.DeWitt (Gordon and Breach, New York, 1964).  
C.Moreno, On the Spaces of Positive and Negative Frequency Solutions of the Klein-Gordon Equation in Curved Space-Times, Rep.Math.Phys. 17, 333 (1980).

[61] I.Lusanna, Classical Yang-Mills Theory with Fermions, I) General Properties of a System with Constraints, Int.J.Mod.Phys. A10, 3531 (1995); II) Dirac’s Observables, Int.J.Mod.Phys. A10, 3675 (1995).

[62] D.Alba and I.Lusanna, Multi-Temporal Quantization for Relativistic and Non-Relativistic Particles in Non-Inertial Frames in Absence of Gravity, in preparation.

[63] S.Weinstein, Naive quantum gravity, in Physics Meets Philosophy at the Planck Scale, C.Callender and N.Huggett eds. (Cambridge University Press, Cambridge 2001), pp.90-100.

[64] A.Ashtekar, S.Fairhurst and J.L.Willis, Quantum Gravity, Shadow States and Quantum Mechanica, preprint 2002 (gr-qc/0207106).

[65] M.Pauri, Leibniz, Kant, and the Quantum: A Provocative Point of View about Observation, Space-Time, and the Mind-Body Issue, in The Reality of the Unobservable - Observability, Un-
observability and Their Impact on the Issue of Scientific Realism, E.Agazzi and M.Pauri eds., Boston Studies in the Philosophy of Science n.215 (Kluwer Academic Publishers, Dordrecht, 2000), pp.257-282.

[66] L.Lusanna, The Rest-Frame Instant Form of Metric Gravity, Gen.Rel.Grav. 33, 1579 (2001) (gr-qc/0101048).

[67] L.Lusanna, Towards a Unified Description of the Four Interactions in Terms of Dirac-Bergmann Observables, invited contribution to the book Quantum Field Theory: a 20th Century Profile, of the Indian National Science Academy, ed.A.N.Mitra, forewords by F.J.Dyson (Hindustan Book Agency, New Delhi, 2000) (hep-th/9907081).

Tetrad Gravity and Dirac’s Observables, talk given at the Conf. Constraint Dynamics and Quantum Gravity 99, Villasimius 1999 (gr-qc/9912091).

The Rest-Frame Instant Form of Dynamics and Dirac’s Observables, talk given at the Int.Workshop Physical Variables in Gauge Theories, Dubna 1999.

Classical Observables of Gauge Theories from the Multi-Temporal Approach, Contemp. Math. 132, 531 (1992).

[68] L.Lusanna, Solving Gauss’ Laws and Searching Dirac Observables for the Four Interactions, talk at the Second Conf. on Constrained Dynamics and Quantum Gravity, S.Margherita Ligure 1996, eds. V.De Alfaro, J.E.Nelson, G.Bandelloni, A.Blasi, M.Cavaglià and A.T.Filippov, Nucl.Phys. (Proc.Suppl.) B57, 13 (1997) (hep-th/9702114).

Unified Description and Canonical Reduction to Dirac’s Observables of the Four Interactions, talk at the Int.Workshop New non Perturbative Methods and Quantization on the Light Cone, Les Houches School 1997, eds. P.Grangé, H.C.Pauli, A.Neveu, S.Pinsky and A.Werner (Springer, Berlin, 1998) (hep-th/9705154).

The Pseudo-Classical Relativistic Quark Model in the Rest-Frame Wigner-Covariant Gauge, talk at the Euroconference QCD97, ed. S.Narison, Montpellier 1997, Nucl.Phys. (Proc. Suppl.) B64, 306 (1998).

[69] E. Schmutzer and J. Plebanski, Fortschritte der Physik, 25, 37 (1978).

D. M. Greenberger and A. W. Overhauser, Rev. of Mod. Phys. 51, 43 (1979).

K. Kuchař, Phys. Rev. D22, 1285 (1980).

W. H. Klink, Ann. of Phys. 260, 27 (1998).

H. Rauch and S.A. Werner, Neutron Interferometry: Lessons in Experimental Quantum Mechanics (Clarendon Press, Oxford, 2000).

[70] M.M.Nieto, Hamiltonian Expectation Values for Time-Dependent Foldy-Wouthuysen Transformations: Implications for Electrodynamics and Resolution of the External Field πN Ambiguity, Phys.Rev.Lett. 38, 1042 (1977).

T. Goldman, Gauge Invariance, Time-Dependent Foldy-Wouthuysen Transformations and the Pauli Hamiltonian, Phys. Rev. D15, 1063 (1977).

H.W.Fearing, G.I.Poulis and S.Scherer, Effective Hamiltonians with Relativistic Corrections: 1) The Foldy-Wouthuysen Transformation versus the Direct Pauli Reduction, Nucl.Phys. A570, 657 (1994) (nucl-th/9302014).

[71] M.Pauri and M.Vallisneri, Classical Roots of the Unruh and Hawking Effects, Found.Phys. 29, 1499 (1999) (gr-qc/9903052).

[72] W.Kundt and B.Hoffmann, Determination of Gravitational Standard Time, in Recent Development in General Relativity a book dedicated to Leopold Infeld’s 60th birthday, p. 303 (Polish Scientific Publishers, Warsaw, 1962).

[73] M.Pauri and M.Vallisneri, Marzke-Wheeler Coordinates for Accelerated Observers in Special
Relativity, Found. Phys. Lett. 13, 401 (2000) (gr-qc/0006095).