A three-way decision method with tolerance dominance relations in decision information systems

Wenjie Wang1 · Jianming Zhan1 · Weiping Ding2 · Shuping Wan3

Published online: 25 November 2022 © The Author(s), under exclusive licence to Springer Nature B.V. 2022

Abstract
In recent years, various medical diagnosis problems have been addressed from the perspective of multi-attribute decision making. Among them, the three-way decision theory can provide a novel scheme to solve medical diagnosis issues under the framework of multi-attribute decision making via considering transforming relationships between loss functions and decision matrices. In this paper, we primarily explore a three-way decision method with tolerance dominance relations in ordered decision information systems. In existing three-way decision models, all objects can be divided into two states, we utilize decision attributes to obtain the set of two states in ordered decision information systems. Then, in order to improve the accuracy of patient classifications, the paper simultaneously considers the influence of loss and gain functions for each object, and uses loss and gain functions to obtain net profit functions as new measurement functions. Meanwhile, a class of three-way decisions in terms of multi-attribute decision making rules based on a tolerance dominance relation is established. In light of the proposed three-way decision method, we further construct a multi-attribute decision making method by using tolerance dominance relations and the constructed method is applied to a medical diagnosis issue of Lymphography. Finally, a comparison analysis and an experimental evaluation are performed to illustrate the feasibility and effectiveness of the presented methodology.

Keywords Three-way decision · Multi-attribute decision making · Ordered decision information system · Tolerance dominance relation

Jianming Zhan
zhanjianming@hotmail.com

Wenjie Wang
wangwenjie9706@163.com

Weiping Ding
dwp9988@163.com

Shuping Wan
shupingwan@163.com

1 School of Mathematics and Statistics, Hubei Minzu University, Enshi 445000, Hubei, China
2 School of Information Science and Technology, Nantong University, Nantong 226019, China
3 School of Information Technology, Jiangxi University of Finance and Economics, Nanchang 330013, China
1 Introduction

With the development of social economy, medical technologies are stepping into the big data era. Globally, according to the latest figures from the World Health Organization WHO (https://www.who.int/about/iarc/zh/), there are more than 14 million new cases of cancers each year, and that number is expected to mount to more than 21 million by 2030. Cancers kill approximately 8.8 million people each year, accounting for nearly one in six cancer deaths worldwide. Moreover, many cancer cases are diagnosed too late, and most cancer cases are diagnosed and detected at an advanced stage with little chance or hope of a cure. In addition, one of the hallmarks for cancers is the rapid formation of abnormal cells that grow beyond their normal range, invade nearby parts of the body and spread to other organs. Thus, with the development of medical technologies, the diagnosis of cancers is a particularly vital issue. Radiography is a commonly used X-ray examination method, which can be widely used in various fields, such as COVID-19 Mardani et al. (2020), focal liver lesions Chen et al. (2017), etc. For instance, Lymphography is used to determine if a patient has lymphoma disease, where Lymphography is a simple, safe and reliable diagnostic method for the detection of lymph node lesions. The changes in the internal structure of the lymph nodes can be observed from diverse aspects, thus realizing the diagnosis of cancer diseases. However, existing multi-attribute decision making (MADM for short) methods can only obtain the optimal target and cannot classify the target. In response to this challenge, Yao (2010) pioneered the three-way decision (TWD for short) theory, which can effectively reduce decision risks. In addition to immediately deciding what is and is not acceptable, decision makers also tend to defer decisions that cannot be immediately made. At present, the combination of TWD with MADM is becoming a popular issue in the research of decision theories. The corresponding TWD with tolerance dominance relations (TWD-TDR for short) models in an ordered decision information system is discussed via utilizing a tolerance dominance relation instead of an equivalence relation, and the model is especially applied to Lymphography in the current paper.

During the past decades, a variety of MADM methods have been extensively used in medical diagnosis Wang et al. (2019, 2021), image processing Li et al. (2020), resource allocations Liang et al. (2019b), economic managements Mendonca et al. (2020), staff recruitment Chen and Hong (2014), investment decisions Jia and Liu (2019), person-job fit Zhang et al. (2020), fire explosions Ren et al. (2017) and many other real-life applications. The determination of attribute weights is a key issue when modeling MADM problems. Pena et al. (2020) presented a taxonomy of existing methods and critically discussed explicit weighting methods. In detail, MADM can be regarded as an evaluation of schemes according to ordered decision information systems, then an evaluation matrix can be obtained. In earlier years, some classic MADM schemes have been successively put forward, such as the EDAS (Evaluation based on Distance from Average Solution) method Ghorabaee et al. (2015), the TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) method Hwang and Yoon (1981), the WAA (Weighted Average Algorithm) operator method Harsanyi (1955), etc. At present, MADM has been widely developed and promoted. Xie et al. (2016) constructed a ranking method based on weighted standardized cardinalities and interval-valued fuzzy sets for two types of inclusion indicators. For the sake of handling the interaction between attributes in the context of interval type-2 fuzzy sets, Liang et al. (2019a) constructed a non-additive integration of heterogeneous multi-attribute behavior TWD by using 2-additive Choquet integral. Liu and Liu Liu and Liu (2021) used dual generalized weighted Bonferroni mean operators and dual generalized
weighted Bonferroni geometric mean operators of 2-dimensional uncertain linguistic variables for developing a new MADM method. However, the equivalence relation in MADM methods is usually too strict, and the studies of some binary relations have recently become an interesting research hotspot.

In the ranking and comparison problems of MADM, Greco et al. (1999) established new rough approximations from the perspective of preference relations. Afterwards, a series of rough set models have been presented in light of dominance relations Greco et al. (2002a, b), which have enriched the research content of rough sets. In order to deal with mult-criteria classification problems, Greco et al. (2002a) considered the preference order of attribute domains and decision class sets at the same time, and further proposed a rough set method based on dominance relations. Then, the rough set approximation in light of dominance relations was applied to many information systems, such as dynamic information systems Li and Li (2015) and Li et al. (2013), incomplete information systems Du and Hu (2016), multi-scale information systems Huang et al. (2019), group information systems Chakhar et al. (2016) and interval-valued decision systems Yang et al. (2020). Meanwhile, plenty of scholars have explored novel practical applications by virtue of the above theoretical models, such as supplier selections Huang et al. (2019), speed control Augeri et al. (2015), salary managements Liu and Liang (2017), and so forth. When dealing with real-valued MADM problems, the rough set method based on dominance relations is too strict. Thus, many scholars have promoted dominance relations. For instance, Li and Xu (2014) constructed a new probabilistic rough set model in light of dominance relations by transforming the improbability measure. Yang et al. (2015) explored the notion of \(\alpha\)-dominance relations in the context of interval-valued information systems. Cocloho (2015) defined the concept of a linear limit state function and presented an evaluation method for the probability dominance. Du and Hu (2017) proposed two attribute reduction methods from the perspective of heuristic strategies and discernibility matrices. Huang et al. (2020) put forward an incremental approximation updating method in light of matrices in compound ordered information systems. With the development of dominance relations in the decision making field, there are several new studies in 2021. For instance, Yang et al. (2021) explored large-scale fuzzy decision making issues by virtue of dominance-based neighborhood rough sets as well as preference relations. Szlapczynski and Szlapczynska (2021) presented an approach for addressing multi-objective meta-heuristics via fusing decision makers’ preferences. In addition, in order to solve the problem that the classic dominance relation is too strict in sequential information systems, which may lead to the failure of ranking methods, Wang et al. (2021) studied TWD-based MADM (TWMADM for short) problems based on probabilistic dominance relations. Since the probabilistic dominance relation considered the pros and cons between the evaluation values of the objects under each attribute, it cannot reflect the pros and cons of evaluation values of objects under each attribute. Thus, Chen et al. (2014) introduced the concept of tolerance dominance relations and established a ranking method based on tolerance dominance relations. Generalizations on Chen et al. (2014) are temporarily scarce, and existing ranking methods based on tolerance dominance relations cannot solve MADM and TWD problems. Given the advantages of tolerance dominance relations, this paper will study TWMADM problems based on a tolerance dominance relation.

Uncertainties are ubiquitous in real life and it is significant to explore valid methods for addressing various uncertainties. In light of the above study motivation, Ma et al. (2021) synthesized three uncertainty theories, i.e., the soft set theory, the fuzzy set theory and the rough set theory, and reviewed data mining methods based on two types of mixed soft models: soft rough fuzzy sets and soft fuzzy rough sets. The rough set theory is a new
mathematical method for dealing with imprecise, uncertain and incomplete data. Nowadays, the research on rough sets involves feature selections Wang et al. (2020), attribute reductions Wang et al. (2020), data reductions Tan et al. (2020), MADM Zhan and Alcantud (2019); Zhan and Sun (2020) and TWD Yao (2020). TWD plays a leading role in decision making problems, where it can reduce decision risks more effectively than traditional two-way counterparts. In the following, we will review a number of main concepts of TWD. First, Yao (2010) established the TWD theory, and the intact process of TWD lies in separating a whole part into three stages to acquire an optimal result. The concept of decision-theoretic rough sets (DTRSs for short) can be semantically explained, hence TWD is largely investigated via the framework of rough sets. In recent years, we can see that TWD has been widely used and researched in a wider range of fields, such as artificial intelligence Zhang et al. (2020), medical diagnosis Li et al. (2020), supplier selections Liu et al. (2020), and so forth. Moreover, considering the significance of semantic interpretations for TWD, plenty of scholars have studied DTRSs in the early development phase of TWD Yao (2010, 2020). Recently, TWD models have been extended from different perspectives. Wang et al. (2020) constructed a regret-based TWD in the context of interval type-2 fuzzy information systems. Liu and Liang (2017) presented a TWD model in an ordered information system. Ye et al. (2020) put forward a new method in light of TWD in fuzzy information systems. Liang et al. (2017) used the analytic hierarchy process to express the loss functions, and proposed a method to determine tripartite decisions via using DTRSs. Shen et al. (2020) defined a cross-level blocking distribution matrix and proposed a hierarchical classification model in light of TWD. In order to avoid serious risks caused by a far-fetched classification of uncertain instances, Yue et al. (2020) explored a novel shadowed set for constructing the shadowed neighborhood of uncertain data classifications, and constructed a three-way classification method via using step functions. Recently, Yao (2019) systematically studied a series of TWD models from the perspective of conflict analysis. Sun et al. (2020) proposed a new updated conflict analysis model by virtue of probabilistic rough sets and TWD. Lang et al. (2020) investigated a three-way group conflict analysis in the context of pythagorean fuzzy information systems. Moreover, a set theory model of TWD was proposed by Yao (2021). More interestingly, studies combining TWD with medical diagnostic problems have received particular attention from lots of scholars. For instance, Wang et al. (2021) studied a TWD method for the diagnosis of infectious diseases in a hesitant fuzzy environment. Huang and Zhan (2021) proposed a TWD method for hepatitis diagnosis based on the regret theory under multi-scale information systems.

In this paper, we primarily explore a TWD method with tolerance dominance relations in ordered decision information systems. According to previous literature analysis, we present several key contributions and innovations of the paper below:

- In many extended versions of TWD, the loss functions of all objects are presented by viable semantic explanations Zhan et al. (2021). However, in real life, there are not only certain losses but also certain benefits for each object. For the sake of cutting down the decision risks of only considering losses or costs, this paper plans to introduce the loss and gain of each object to get the net gain, which is more in line with practical scenarios.
- In classic TWD models, a set of states are usually subjectively given by decision makers Li et al. (2020). For the sake of cutting down the influence of subjective decision making and supporting experts to make reasonable decisions, an objective expression for obtaining the set of states will be explored in this paper. Hence, the decision attribute is used to construct a set of states in this paper.
In extended TWD models, the dominance relations Greco et al. (2002a) and equivalence relations Yao (2010) are too strict to deal with information systems with real values, and corresponding application range is restricted. Thus, we propose a more general approach for settling a wide range of decision making issues. Then, a novel TWD model based on a tolerance dominance relation is established, which is an extension of existing TWD models.

Many existing MADM methods Ghorabaee et al. (2015), Harsanyi (1955) and Hwang and Yoon (1981) only consider information tables and fail to consider the losses and benefits of each object when making decisions. Consequently, we mean to research a more efficient approach via combining the losses with benefits of all objects. By taking the merits of TWD, the current paper plans to introduce a novel MADM approach via the notion of TWD, which is conducive to enriching both the theoretical aspect of TWD and the application range of MADM studies.

The rest of the paper is organized as follows: Sect. 2 briefly revisits some fundamental concepts of ordered decision information systems, tolerance dominance relations and TWD. In Sect. 3, a novel TWD model based on a tolerance dominance relation is constructed in ordered decision information systems. In Sect. 4, we establish a novel MADM approach and utilize it to medical cases in the UCI (University of California, Irvine) database. In Sect. 5, a comparative study is made by comparing the presented approach and existing counterparts, which shows that the presented TWD model is effective. In Sect. 6, the stability of the proposed approach is demonstrated by several experimental analysis. At last, Sect. 7 sums up the presented approach and illustrates potential future study options. The framework of this article is shown in Fig. 1.

2 Related works

In this section, we recall the knowledge of ordered decision information systems, tolerance dominance relations and TWD based on DTRSs.

2.1 Ordered decision information systems

In order to deal with multi-criteria classification problems, Greco et al. (2002a) simultaneously considered the preference order of conditional attributes and decision attributes, and further proposed a dominance-based rough set in an ordered decision information system. Now, we present the concept of ordered decision information systems in Definition 2.1.

**Definition 2.1** Greco et al. (2002a) An ordered decision information system is given as a four-tuple \( S = (U, C \cup D, V, f) \), where \( U = \{z_1, z_2, \ldots, z_n\} \) is a non-empty finite set of \( n \) objects, \( C = \{c_1, c_2, \ldots, c_m\} \) is a non-empty finite set of \( m \) condition attributes, \( D = \{d_1, d_2, \ldots, d_q\} \) is a finite set that consists of \( q \) decision attributes, \( V = \bigcup_{c_j \in C} V_{c_j} \), here \( V_{c_j} \) is the domain of values for \( c_j \), and \( f = \{f_j : U \rightarrow V_{c_j}\} \) is a mapping set, such that \( f(z_i, c_j) = c_j(z_i) = z_{ij} \in V_{c_j} \) for any \( z_i \in U \). If \( l > s \), then \( d_l > d_s \). An ordered decision information system is given in Table 1.
Then, in order to obtain the upper and lower approximations of rough sets in ordered decision information systems, Greco et al. (2002a) put forward the concept of upward and downward unions by considering the order relation of decision attributes. Now, we give the concepts of upward and downward unions in Definition 2.2.

Table 1 An ordered decision information system $S = (U, C \cup D)$

| $U/C \cup D$ | $c_1$ | $c_2$ | $\cdots$ | $c_m$ | $D$ |
|-------------|-------|-------|-----------|-------|-----|
| $z_1$       | $z_{11}$ | $z_{12}$ | $\cdots$ | $z_{1m}$ | $d_1$ |
| $z_2$       | $z_{21}$ | $z_{22}$ | $\cdots$ | $z_{2m}$ | $d_2$ |
| $\cdots$    | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $z_n$       | $z_{n1}$ | $z_{n2}$ | $\cdots$ | $z_{nm}$ | $d_q$ |
Definition 2.2 Greco et al. (2002a) Assume that \( U \) is divided into \( q \) classes \( CL = \{ Cl_t, t \in T \} \) by the decision attributes \( D \), where \( T = \{ 1, \cdots, q \} \). Each \( z \in U \) belongs to one and only one class \( Cl_t \). The classes from \( CL \) are preference-ordered in light of ascending order of class indices, i.e., for all \( r, s \in T \) such that \( r > s \), the objects from \( Cl_r \) are preferred to the objects from \( Cl_s \). For \( t \in T \), the upward and downward unions are represented by \( Cl_t^\supseteq \) and \( Cl_t^\subseteq \), respectively.

\[
Cl_t^\supseteq = \bigcup_{s \geq t} Cl_s \quad \text{and} \quad Cl_t^\subseteq = \bigcup_{s \leq t} Cl_s.
\]

Remark 2.1 \( z \in Cl_t^\supseteq \) represents that \( z \) belongs at least to \( Cl_t \), whereas \( z \in Cl_t^\subseteq \) represents that \( z \) belongs at most to \( Cl_t \). Obviously, the following statements hold.

1. \( Cl_1^\supseteq = Cl_1^\subseteq = U, Cl_q^\supseteq = Cl_q^\subseteq = Cl_1 \).
2. \( Cl_t^\supseteq = U - Cl_{t+1}^\subseteq, t = 2, 3, \cdots, q \).
3. \( Cl_t^\subseteq = U - Cl_{t-1}^\supseteq, t = 1, 2, \cdots, q - 1 \).

Considering the order relation of evaluation values in ordered decision information systems, Greco et al. (2002a) put forward the concept of \( Q \)-dominating and \( Q \)-dominated sets. In what follows, we give the concept of \( Q \)-dominating and \( Q \)-dominated sets in Definition 2.3.

Definition 2.3 Greco et al. (2002a) For \( z_1, z_2 \in U \) and \( Q \subseteq C \), if all \( p \in Q, z_1 \succeq_p z_2 \) is true, we use \( z_1D_Qz_2 \) to indicate that \( z_1 \) dominates \( z_2 \) with respect to \( Q \). \( Q \)-dominating and \( Q \)-dominated sets are represented below:

\[
D_Q^+(z_1) = \{ z_2 \in U : z_2D_Qz_1 \},
\]

\[
D_Q^-(z_1) = \{ z_2 \in U : z_1D_Qz_2 \}.
\]

In order to deal with multi-criteria problems, Greco et al. (2002a) combined the relation of upward and downward unions with \( Q \)-dominating and \( Q \)-dominated sets to give a rough set model based on dominance relations. In the following, we give a rough set model based on the dominance relation in Definition 2.4.

Definition 2.4 Greco et al. (2002a) For any \( Y \subseteq U \), the \( Q \)-lower and the \( Q \)-upper approximations of \( Cl_t^\supseteq \) are represented below:

\[
\underline{Q}(Cl_t^\supseteq) = \{ z \in U | D_Q^+(z) \subseteq Cl_t^\supseteq \},
\]

\[
\overline{Q}(Cl_t^\supseteq) = \{ z \in U | D_Q^-(z) \cap Cl_t^\supseteq \neq \emptyset \}.
\]

In a similar way, the \( Q \)-lower and the \( Q \)-upper approximations of \( Cl_t^\subseteq \) are represented below:

\[
\underline{Q}(Cl_t^\subseteq) = \{ z \in U | D_Q^-(z) \subseteq Cl_t^\subseteq \},
\]

\[
\overline{Q}(Cl_t^\subseteq) = \{ z \in U | D_Q^+(z) \cap Cl_t^\subseteq \neq \emptyset \}.
\]
2.2 Tolerance dominance relations

In 2014, Chen et al. (2014) introduced the concept of tolerance dominance relations and established a ranking method of objects in an ordered decision information system. In what follows, we first revisit the knowledge of the dominated rate for the values of the objects in Definition 2.5.

**Definition 2.5** Chen et al. (2014) Let $S = (U, C \cup D, V, f)$ be an ordered decision information system. Then, for any $z_i, z_j \in U$ and $c \in C$, we have

$$e_c(z_i, z_j) = \begin{cases} \frac{f(z_j, c) - f(z_i, c)}{f(z_j, c)} & f(z_i, c) < f(z_j, c), \\ 0 & \text{otherwise,} \end{cases}$$

we call $e_c(z_i, z_j)$ the dominated rate of the values of the object $z_i$ relative to $z_j$ under the attribute $c$. $e_c(z_i, z_j) = 0$ means that the value of $z_i$ is at least not worse than the value of $z_j$ under the attribute $c$. Otherwise, $e_c(z_i, z_j) = 1$ means that the value of $z_i$ must be worse than the value of $z_j$ under the attribute $c$.

In order to aggregate the dominated rate of the object $z_i$ relative to the object $z_j$ under each attribute, we review the concept of a dominated matrix in Definition 2.6.

**Definition 2.6** Chen et al. (2014) Let $S = (U, C \cup D, V, f)$ be an ordered decision information system, for any $z_i, z_j \in U$ and $c \in C$, we have

$$E_C(z_i, z_j) = \frac{\sum_{c \in C} e_c(z_i, z_j)}{|C|},$$

where $|C|$ denotes the number of attribute sets. All $E = [E_C(z_i, z_j)]_{m \times m}$ of $E_C(z_i, z_j)$ is titled a dominated matrix of $S$ in terms of $c$.

Recently, Chen et al. (2014) gave the tolerance dominance relation according to the relation between the dominance matrix and the parameter $\sigma$. In the following, we review the concept of tolerance dominance relations below.

**Definition 2.7** Chen et al. (2014) Let $S = (U, C \cup D, V, f)$ be an ordered decision information system. For any $z_i, z_j \in U$ and $c \in C$, a tolerance dominance relation of $S$ in terms of $c$ is given as:

$$R_c^{\leq \sigma} = \{(z_i, z_j) : (z_i, z_j) \in U \times U | E_C(z_i, z_j) \leq \sigma, 0 \leq \sigma \leq 1\},$$

where $\sigma$ is titled a tolerance rate, $R_c^{\leq \sigma}$ is titled a $\sigma$-tolerance dominance relation.

Based on the concept of tolerance dominance relations, the object $z_j$ that satisfies the tolerance dominance relation with $z_i$ can be obtained. Therefore, Chen et al. (2014) proposed the concept of tolerance dominance classes. In what follows, we review the concept of tolerance dominance classes of $z_i$ in Definition 2.8.
**Definition 2.8** Chen et al. (2014) Let $S = (U, C \cup D, V, f)$ be an ordered decision information system. For $\forall c \in C$, given the tolerance rate $\sigma$, $[z_i]_C^{\mathcal{E}_\sigma} = \{z_j \in U | (z_j, z_i) \in \mathcal{R}_C^{\mathcal{E}_\sigma}\}$, then $[z_i]_C^{\mathcal{E}_\sigma}$ is the $\sigma$-tolerance dominance class of $z_i$ in terms of $C$.

### 2.3 TWD with DTRSs

Let $\Omega = \{\zeta, \neg \zeta\}$ denote a set of states, $H = \{h_P, h_B, h_N\}$ represent a set of actions, $h_P$, $h_B$, and $h_N$ denote the three domains in classifying an object $z_i$, i.e., $z_i \in \text{POS}(\zeta)$, $z_i \in \text{BND}(\zeta)$ and $z_i \in \text{NEG}(\zeta)$, respectively. Then, when $z_i \in \zeta$, let $\lambda_{PP}$, $\lambda_{BP}$ and $\lambda_{NP}$ represent the losses of taking the actions $h_P$, $h_B$ and $h_N$. Similarly, when $z_i \in \neg \zeta$, let $\lambda_{PN}$, $\lambda_{BN}$ and $\lambda_{NN}$ represent the losses of taking the actions $h_P$, $h_B$ and $h_B$. For any object $h_i \in Y$, $R(h_{\ast}|[z_i])(\ast = P; B; N)$ shows the expected losses of taking the actions $h_{\ast}$. In light of the Bayesian decision theory, we obtain the action with the least expected loss as the optimal case. Therefore, we can get three minimum decision rules. Since the conditional probabilities that satisfy $P(\zeta|[h_i]) + P(\neg \zeta|[h_i]) = 1$, the corresponding rules can be obtained:

- **(P)** If $P(\zeta|[z_i]) \geq \alpha$ and $P(\zeta|[z_i]) \geq \gamma$, decide $z_i \in \text{POS}(\zeta)$.
- **(B)** If $P(\zeta|[z_i]) \leq \alpha$ and $P(\zeta|[z_i]) \geq \beta$, decide $z_i \in \text{BND}(\zeta)$.
- **(N)** If $P(\zeta|[z_i]) \leq \beta$ and $P(\zeta|[z_i]) \leq \gamma$, decide $z_i \in \text{NEG}(\zeta)$.

where $\alpha$, $\beta$, and $\gamma$ are calculated as follows:

$$\alpha = \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})},$$

$$\gamma = \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{PN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{PP})},$$

$$\beta = \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})}.$$

### 3 TWD in light of a tolerance dominance relation

In what follows, we get a set of two states based on decision attributes. On this basis, we consider the combination of loss functions with gain functions and design a new TWD model in light of a tolerance dominance relation.

**Definition 3.1** Give an ordered decision information system $S = (U, C \cup D, V, f)$, where $U = \{z_1, z_2, \cdots, z_n\}$ and $D = \{d_1, d_2, \cdots, d_q\}$. Let $l, s \in \{1, 2, \cdots, q\}$. If $l > s$, then $d_l > d_s$. It is assumed that the larger the values of $l$, the closer the state of the object is to the good state case. For instance, if we consider a diagnosis as a good state set, the severity of the disease is divided into three cases: normal, more severe, very severe which can be denoted by 1, 2, 3 respectively, we can know that 3 is closer to the diagnosis. Then the set $\Omega = \{\zeta, \neg \zeta\}$ is expressed as:

![Image](https://example.com/springer-image)
For Definition 3.1, we give the following explanation: If $l > s$, then $d_l > d_s$. Also when the values of $l$ is larger, the state of the object is closer to the good state. In classic TWD theories, $\mu_1$ represents the good state and $\neg \mu_1$ represents the bad state. Therefore, for fairness we divide the values of $d_l$ into two parts according to the average of the values of $d_l$. By combining with the majority principle, the corresponding object can be considered as good when the values of the decision attribute is better than the other half of the values of the decision attribute. Conversely, when the values of the decision attribute is inferior to the other half of the values of the decision attribute, the corresponding object can be considered as bad. We use an example of an enterprise project investment to present this concept.

**Example 3.1** An enterprise decides to select suitable investment projects to use its idle funds. The decision makers choose eight alternatives based on five indicators. The evaluation values are listed in Table 2.

From the data in Table 2, we can get that $\zeta = \{z_1, z_5, z_7\}$ and $\neg \zeta = \{z_2, z_3, z_4, z_6, z_8\}$ based on Definition 3.1.

**Remark 3.1** Suppose that $S = (U, C \cup D)$ is an ordered decision information system. Then $\Omega = \{\zeta, \neg \zeta\}$ meets the two conditions as: (1) $\zeta \cup \neg \zeta = U$ and (2) $\zeta \cap \neg \zeta = \emptyset$.

Based on the above analysis, the set of two states is obtained by Definition 3.1. Then, we calculate the conditional probability by using $\Pr(\zeta | [z_i]_{C}^{\geq s}) = \frac{[\zeta \cap [z_i]_{C}^{\geq s}]}{[z_i]_{C}^{\geq s}}$. Unlike classic DTRSs described in Sect. 2.2, we consider that all objects own gain functions and loss functions. Additionally, all objects may own diverse views as losses and gains. For instance, given the risks included in conducting a diagnostic decision in Lymphography, patients with more severe cases suffer less loss and more gain than those with less severe cases. In this paper, let $z_{ij} \in [z_{\min}, z_{\max}]$, based on the result in Jia and Liu (2019), the relative loss functions $L_j(z_i)$ and gain functions $G_j(z_i)$ according to the evaluation values of attributes are expressed as:

| $U/C \cup D$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $D$ |
|----------------|-------|-------|-------|-------|-----|
| $z_1$          | 0.8   | 0.6   | 0.7   | 0.8   | 4   |
| $z_2$          | 0.9   | 0.5   | 0.5   | 0.7   | 2   |
| $z_3$          | 0.3   | 0.6   | 0.4   | 0.4   | 1   |
| $z_4$          | 0.5   | 0.8   | 0.8   | 0.7   | 2   |
| $z_5$          | 0.7   | 0.4   | 0.4   | 0.5   | 3   |
| $z_6$          | 0.4   | 0.2   | 0.3   | 0.7   | 1   |
| $z_7$          | 0.9   | 0.5   | 0.9   | 0.8   | 3   |
| $z_8$          | 0.6   | 0.2   | 0.2   | 0.3   | 1   |
According to the semantic interpretation of the net profit functions $Q_{\star}(z_i)$ in real life, we can get the following conditions: (a) $Q_{PP}^i \leq Q_{BP}^i < Q_{NP}^i$ and (b) $Q_{NN}^i \leq Q_{BN}^i < Q_{PN}^i$. 

---

**Remark 3.2** According to the semantic interpretation of the net profit functions $Q_{\star}(z_i)$ in real life, we can get the following conditions: (a) $Q_{PP}^i \leq Q_{BP}^i < Q_{NP}^i$ and (b) $Q_{NN}^i \leq Q_{BN}^i < Q_{PN}^i$. 

---

**Table 3** The net profit function table for all objects

| $z_1$ | $Q_{PP}^1$ | $Q_{BP}^1$ | $Q_{NP}^1$ | $Q_{PN}^1$ | $Q_{NN}^1$ | $Q_{NN}^1$ |
| $z_2$ | $Q_{PP}^2$ | $Q_{BP}^1$ | $Q_{NP}^1$ | $Q_{PN}^1$ | $Q_{NN}^1$ | $Q_{NN}^1$ |
| ...  | ...        | ...        | ...        | ...        | ...        | ...        |
| $z_n$ | $Q_{PP}^n$ | $Q_{BP}^n$ | $Q_{NP}^n$ | $Q_{PN}^n$ | $Q_{NN}^n$ | $Q_{NN}^n$ |

\[
L_{\star}(z_i) = \left( \begin{array}{c} L_{PP}^j(z_i) \\ L_{BP}^j(z_i) \\ L_{NP}^j(z_i) \end{array} \right) = \left( \begin{array}{c} 0 \\ \xi_j(z_{ij} - z_{\min}^j) \\ z_{ij} - z_{\min}^j \end{array} \right) \quad (14)
\]

and

\[
G_{\star}(z_i) = \left( \begin{array}{c} G_{PP}^j(z_i) \\ G_{BP}^j(z_i) \\ G_{NP}^j(z_i) \end{array} \right) = \left( \begin{array}{c} (z_{ij} - z_{\min}^j) \\ (1 - \xi_j)(z_{ij} - z_{\min}^j) \\ 0 \end{array} \right) \quad (15)
\]

where $\xi_j \in [0, 0.5]$ denotes the risk avoidance coefficient.

For the sake of better explaining the loss function and the gain function along with their applications, we consider the relation between the loss functions and the gain functions. By virtue of the relation between the gain functions and the loss functions, the net profit functions are shown below:

\[
Q_{\star}(z_i) = \left( \begin{array}{c} Q_{PP}^j(z_i) \\ Q_{BP}^j(z_i) \\ Q_{NP}^j(z_i) \end{array} \right) = \left( \begin{array}{c} G_{PP}^j(z_i) - L_{PP}^j(z_i) \\ G_{BP}^j(z_i) - L_{BP}^j(z_i) \\ G_{NP}^j(z_i) - L_{NP}^j(z_i) \end{array} \right)
\quad (16)
\]

Since each attribute has a different weight value, we express the net profit functions of all attributes in more detail shown as follows:

\[
Q_{\star}(z_i) = \left( \begin{array}{c} Q_{PP}(z_i) \\ Q_{BP}(z_i) \\ Q_{NP}(z_i) \\ Q_{NN}(z_i) \end{array} \right) = \left( \begin{array}{c} \sum_{j=1}^{m} w_j(z_{ij} - z_{\min}^j) \\ \sum_{j=1}^{m} w_j(z_{ij} - z_{\min}^j) \end{array} \right) = \left( \begin{array}{c} \sum_{j=1}^{m} w_j(1 - 2\xi_j)(z_{ij} - z_{\min}^j) \\ \sum_{j=1}^{m} w_j(1 - 2\xi_j)(z_{ij} - z_{\min}^j) \end{array} \right) \quad (17)
\]

The net profit functions for each object is shown in Table 3.
For any object $z_i \in U$, $Q(h_\star|[z_i])$($\star = P; B; N$) shows the expected net profit of taking the actions $h_\star$, which are calculated as follows:

$$Q(h_P|[z_i]) = Q_{PP}Pr(\zeta|[z_i]) + Q_{PN}Pr(\gamma|[z_i]). \quad (18)$$

$$Q(h_B|[z_i]) = Q_{BP}Pr(\zeta|[z_i]) + Q_{BN}Pr(\gamma|[z_i]). \quad (19)$$

$$Q(h_N|[z_i]) = Q_{NP}Pr(\zeta|[z_i]) + Q_{NN}Pr(\gamma|[z_i]). \quad (20)$$

Based on the principle of net profit maximization, we can obtain the decision rules (P1)-(N1). Furthermore, the future $POS(\zeta)$, $BND(\zeta)$ and $NEG(\zeta)$ can be abbreviated as $(P)$, $(B)$ and $(N)$ respectively in this paper.

(P1) Decide $z_i \in POS(\zeta)$, if $Q(h_P|[z_i]) \geq Q(h_B|[z_i])$, and $Q(h_P|[z_i]) \geq Q(h_N|[z_i])$.

(B1) Decide $z_i \in BND(\zeta)$, if $Q(h_B|[z_i]) \geq Q(h_P|[z_i])$, and $Q(h_B|[z_i]) \geq Q(h_N|[z_i])$.

(N1) Decide $z_i \in NEG(\zeta)$, if $Q(h_N|[z_i]) \geq Q(h_P|[z_i])$, and $Q(h_N|[z_i]) \geq Q(h_B|[z_i])$.

By virtue of the statement of Remark 3.2, the decision rules (P1)-(N1) can be re-expressed below:

(P1’) Decide $z_i \in POS(\zeta)$, if $Pr(\zeta|z_i) \geq \alpha_i$ and $Pr(\zeta|z_i) \geq \gamma_i$,

(B1’) Decide $z_i \in BND(\zeta)$, if $Pr(\zeta|z_i) \leq \alpha_i$ and $Pr(\zeta|z_i) \geq \beta_i$,

(N1’) Decide $z_i \in NEG(\zeta)$, if $Pr(\zeta|z_i) \leq \beta_i$ and $Pr(\zeta|z_i) \leq \gamma_i$,

where the values of $\alpha_i$, $\beta_i$ and $\gamma_i$ are computed by

$$\alpha_i = \frac{Q^i_{PN} - Q^i_{BN}}{(Q^i_{PN} - Q^i_{BN}) + (Q^i_{BP} - Q^i_{PP})} \sum^m_j w_j (1 - \xi_j) (\xi^{j}_{\max} - z_{ij})$$

$$= \sum^m_j w_j (1 - \xi_j) (\xi^{j}_{\max} - z_{ij}) + \sum^m_j w_j \xi_j (\xi^{j}_{\min} - z_{ij}), \quad (21)$$

$$\beta_i = \frac{Q^i_{BN} - Q^i_{NN}}{(Q^i_{BN} - Q^i_{NN}) + (Q^i_{NP} - Q^i_{BP})} \sum^m_j w_j \xi_j (\xi^{j}_{\max} - z_{ij})$$

$$= \sum^m_j w_j \xi_j (\xi^{j}_{\min} - z_{ij}) + \sum^m_j w_j (1 - \xi_j) (\xi^{j}_{\max} - z_{ij}), \quad (22)$$
We can obtain two cases of thresholds: (1) When $z_{ij} = z_{\min}^j$, $\alpha_i = \beta_i = \gamma_i = 1$, it means that there is only one decision result of the rejection action; (2) When $z_{ij} = z_{\max}^j$, $\alpha_i = \beta_i = \gamma_i = 0$, it means that there is only one decision result of the acceptance action.

**Theorem 3.1** Consider $z_{ij} \in [z_{\min}^j, z_{\max}^j]$, the following three conclusions hold.

1. $\alpha_i$ is monotonically decreasing in $\xi$.
2. $\beta_i$ is monotonously increasing in $\xi$.
3. $\gamma_i$ is uncorrelated in $\xi$.

**Proof** Let $Z_A = z_{ij} - z_{\max}^j$, $Z_B = z_{\max}^j - z_{ij}$ and $Z_C = z_{ij} - z_{\min}^j$. Then

1. If $\alpha_i(\xi)$ is regarded as the function of $\xi$, the derivation of $\alpha_i(\xi)$ is shown as follows:

$$
\alpha_i'(\xi) = \frac{Z_A((1 - \xi)Z_B + \xi Z_C) - (1 - \xi)Z_B(Z_C - Z_B)}{((1 - \xi)Z_B + \xi Z_C)^2},
$$

where $((1 - \xi)Z_B + \xi Z_C)^2 > 0$ and

$$
Z_A((1 - \xi)Z_B + \xi Z_C) - (1 - \xi)Z_B(Z_C - Z_B) = Z_A((1 - \xi)Z_B + \xi Z_C) + (1 - \xi)Z_A + (1 - \xi)Z_C < 0.
$$

Therefore, $\alpha_i'(\xi) < 0$ and $\alpha_i$ is monotonically decreasing in the $\xi$.

2. In a similar way, one gets that

$$
\beta_i'(\xi) = \frac{Z_B(\xi Z_B + (1 - \xi)Z_C) - \xi Z_B(Z_B - Z_C)}{((\xi Z_B + (1 - \xi)Z_C)^2},
$$

where $((\xi Z_B + (1 - \xi)Z_C)^2 > 0$ and

$$
Z_B(\xi Z_B + (1 - \xi)Z_C) - \xi Z_B(Z_B - Z_C) = Z_B(\xi Z_B + (1 - \xi)Z_C) - \xi Z_B + \xi Z_C > 0.
$$

Therefore, $\beta_i'(\xi) > 0$ and $\beta_i$ is monotonically increasing in the $\xi$.

3. According to the expression form of $\gamma_i$, we can know that it is uncorrelated in $\xi$. 

\[\sum_{j} w_j(z_{\max}^j - z_{ij}) \]
Next, we analyze the effective result of the risk aversion coefficient.

From Table 4, when $\xi = 0$, we can get $\beta_i = 0$, $\alpha_i = 1$, this means that with only one uncertain decision, TWD can be reduced to original rough set models; when $0 < \xi < 0.5$, the thresholds satisfy the condition of TWD; When $\xi = 0.5$, we can get $\beta_i = \gamma_i = \alpha_i$ that means TWD becomes two-way decisions, and the boundary region is empty.

## 4 A novel TWD based MADM method in light of tolerance dominance relations

In what follows, we intend to build a novel TWD-MADM based on the above indicators that can be used for medical diagnostics, capacity assessment, and so forth.

### 4.1 Descriptions of the established MADM method

Let $U = \{z_1, z_2, \cdots, z_n\}$ be a non-empty finite set and $C \cup D = \{c_1, c_2, \cdots, c_m\} \cup \{d_1, d_2, \cdots, d_q\}$ be a non-empty finite set of properties. Generally speaking, the condition attribute set includes cost attributes and benefit attributes. Let $z_{ij}$ be an evaluation function related to the attribute $c_j$, each object can be evaluated by $z_{ij}$. Besides, a weight vector $W = (w_1, w_2, \cdots, w_m)^T$ is given to denote the relative significance of attributes with $w_j \in [0, 1]$ and $\sum_j w_j = 1$. In some particular cases, the attribute weights are unknown. In what follows, we give an ordered decision information system to succinctly express an MADM problem and calculate the normalized ordered decision information system by the following formulas:

$$t_{ij} = \frac{z_{ij} - c_{j}^{\min}}{c_{j}^{\max} - c_{j}^{\min}} \quad \text{if } c_j \text{ is a benefit attribute,} \quad (24)$$

$$t_{ij} = \frac{c_{j}^{\max} - z_{ij}}{c_{j}^{\max} - c_{j}^{\min}} \quad \text{if } c_j \text{ is a cost attribute.} \quad (25)$$

When dealing with MADM problems, different types of attributes may lead to different degrees of influences on decision outcomes. Therefore, it is necessary to consider that each attribute has its own attribute weight values. Among many calculation methods of attribute weight values, we choose the method of maximum deviation weights. In the method of the maximum deviation weight, the smaller the difference between the attribute values of all objects under that attribute, the less the influence of that attribute on the decision making of the object. Conversely, the greater the importance of the object. From the perspective of object ranking, the greater the deviation of the attribute values, the greater weight value.
should be given. By the above analysis, the method of the maximum deviation weight is considered to be a better method to obtain the objective weight values, and this method is described as follows.

**Definition 4.1** Wang (1998) Let \( S = (U, C \cup D) \) be an ordered decision information system, the formula to calculate the maximum deviation weight \( w_j \) for the \( j \)-th attribute is shown as:

\[
w_j = \frac{\sum_{i=1}^{n} \sum_{k=1}^{n} |z_{ij} - z_{kj}|}{\sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{n} |z_{ij} - z_{kj}|},
\]

(26)

where \( |z_{ij} - z_{kj}| \) expresses for the distance between two evaluative values and \( z_{ij} \) for \( i, k \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, m\} \).

Hence, the maximum dispersion weight of all attributes can be computed in light of Definition 4.1.

For an MADM problem, the protocol sequencing can help physicians select the most severe patients or some of the more severe patients. According to the TWMADM model, all candidates are divided into one of three decision regions: \( POS(\zeta) \), \( BND(\zeta) \) or \( NEG(\zeta) \). Furthermore, the semantic interpretation of the rules in these three cases can be expressed as the expected net benefits associated with them. That is, for each object \( z_i \), its relevant expected net profit is calculated according to the decision rules \( (P_1^\prime)-(N_1^\prime) \) as follows:

\[
N(z_i) = \begin{cases} 
Q(h_{P}\mid z_i) & \text{if } z_i \in POS(\zeta), \\
Q(h_{B}\mid z_i) & \text{if } z_i \in BND(\zeta), \\
Q(h_{N}\mid z_i) & \text{if } z_i \in NEG(\zeta).
\end{cases}
\]

(27)

Finally, we rank all objects according to the expected net profit \( N(z_i) \) and the rule of \( (P) > (B) > (N) \).

### 4.2 A decision process of the TWMADM method

**Input:** An ordered decision information system \( S = (U, C \cup D, V, f) \).

**Output:** The ranking results for all objects.

**Step 1:** The ordered decision information system on \([0,1]\) is acquired by the normalization rule of Eqs. (24–25).

**Step 2:** The tolerance dominance classes \( [z_i]_{C}^{\xi, \eta} \) are obtained by Definition 2.8 and the state set \( \Omega = \{\xi, \neg \zeta\} \) by Definition 3.1.

**Step 3:** Calculate the conditional probability \( Pr(\zeta \mid [z_i]_{C}^{\xi, \eta}) \) for \( z_i \in U \).

**Step 4:** Obtain the loss (gain) functions of each object, the thresholds \( \alpha_i, \beta_i, \gamma_i \) are obtained by Eqs. (21–23).

**Step 5:** The region \( (P, B, N) \) of each object is determined by the decision rules \( (P_1^\prime)-(N_1^\prime) \) in the new TWD model.

**Step 6:** Based on Eqs. (18–20), calculate the expected net profit \( Q(h_\star \mid [z_i]) (\star = P, B, N) \).
Step 7: Rank all objects via the expected net profit \( N(z_i) \) and \((P) > (B) > (N)\) by Eq. (27).

Remark 4.1 In a TWMADM problem, suppose that \( n \) denotes the number of elements in the object set and \( m \) denotes the number of elements in the attribute set. In what follows, a detailed complexity analysis of Algorithm 1 will be listed below. In Step 1, the ordered decision information system on \([0,1]\) is acquired by the normalization rule of Eqs. (24–25), and the complexity degree is \( O(m) \). In Step 2, the tolerance dominance classes for all objects are obtained via Definition 2.8 and \( \Omega = \{\zeta, \neg\zeta\} \) via Definition 3.1, hence the complexity degree is \( O(n^2m) \). Step 3 calculates the conditional probability \( \Pr\left[\zeta | (z_i)^{[\xi]}\right] \) for \( z_i \in U \), and the complexity is \( O(m) \). Step 4 calculates the gain functions and the loss functions of each object and the thresholds \( \alpha_i, \gamma_i, \beta_i \) are obtained by Eqs. (21–23), and the complexity degree is \( O(n) \). Step 5 presents the regions \((P, B, N)\) of each object through the new TWD model, and the complexity degree is \( O(n) \). Step 6 calculates the expected net profit \( Q(h_\star) ([z_i]) (\star = P, B, N) \) by Eqs. (18–20), and the complexity is \( O(n) \). Step 7 ranks all objects via the relevant expected net profit \( N(z_i) \) and \((P) > (B) > (N)\) by Eq. (27), and the complexity degree is \( O(n) \). By the above analysis, the overall complexity degree of the constructed algorithm is \( O(n^2m) \).

To sum up, we build a TWMADM method in the ordered decision information system. The constructed model’s core points are to acquire the state sets by the order relation of decision relation, calculate the loss and gain functions by attribute evaluation values and obtain the value of the thresholds, respectively. Furthermore, we can acquire corresponding decision rules and determine decision results for each object. On the basis of the above-mentioned eight steps, we summarize the algorithm process of the built TWMADM model in Algorithm 1.

Algorithm 1: The algorithm step of the TWMADM model with the tolerance dominance relation.

| Step | Description |
|------|-------------|
| 1.   | Given the parameters \( \mu = 0.01 \), \( v = 0.02 \), \( d = 1 \), \( \delta = 0.1 \), \( \gamma = 0 \), \( \sigma = 0 \), \( \mu \in [0, 1) \), \( v \in [0, 1) \), \( d \in [0, 1) \), \( \delta \in [0, 1) \), \( \gamma \in [0, 1) \), \( \sigma \in [0, 1) \). |
| 2.   | Obtain a normalized ordered decision information system. \( \text{by Eqs. (24)-(25)} \). |
| 3.   | \( i = 1 \) to \( n \) do |
| 4.   | Calculate the tolerance dominance classes \( \xi \) for each object \( z_i \). \( \text{by Definition 2.8} \). |
| 5.   | Obtain the regions \((P, B, N)\) for each \( z_i \). \( \text{by Definition 3.1} \). |
| 6.   | Calculate the conditional probability \( \Pr\left[\zeta | (z_i)^{[\xi]}\right] \) for each \( z_i \). \( \text{by Eqs. (21)-(23)} \). |
| 7.   | Obtain the expected net profit \( Q(h_\star) ([z_i]) \). \( \star = P, B, N \). \( \text{by Eqs. (18)-(20)} \). |
| 8.   | Rank all objects via the relevant expected net profit \( N(z_i) \) and \((P) > (B) > (N)\). \( \text{by Eq. (27)} \). |
| 9.   | Return the decision rules \((\xi)\) and the ranking results \((\xi)\). |

 Springer
4.3 An illustrative example

We illustrate the presented method with an MADM example in the context of medical diagnosis from the UCI repository Kononenko and Cestnik (1988). Then, we suppose a hospital plans to use Lymphography to screen some patients for lymphoma. The example contains 148 objects \( z_i \), 18 conditional attributes \( c_j \), 1 decision attribute \( d_i \), and 4 possible final diagnostic classes Clark and Niblett (1987), where \( d = \{ d_1, d_2, d_3, d_4 \} \) stands for normal find, metastases, malign lymph and fibrosis, respectively. The conditional attributes refer to the data of disease conditions, which are obviously benefit attributes. The dataset is obtained from the Institute of Oncology of the University Medical Center in Ljubljana, Yugoslavia. The decision making process of Lymphography is shown as: these patients first undergo Lymphography. Then, doctors use the dataset to determine if a patient has lymphoma. Finally, the patient receives further treatment if true. In addition, the weight vector obtained according to Definition 4.1 is

\[
W = \{0.0518, 0.0882, 0.0517, 0.0161, 0.0657, 0.0893, 0.0225, 0.0746, 0.0105, 0.0526, 0.0509, 0.0561, 0.0474, 0.0617, 0.0617, 0.0719, 0.0799, 0.0591, 0.0500\}.
\]

According to the constructed TWMADM model, the ranking and classification result of Lymphography domain problem can be obtained, as shown in Fig. 2 and Table 5. In the TWMADM model, the risk aversion coefficient \( \xi = \{0.35, 0.4, 0.4, 0.4, 0.35, 0.4, 0.4, 0.4, 0.35, 0.4, 0.4, 0.4, 0.35, 0.4, 0.4, 0.35, 0.3, 0.3\} \) and the tolerance rate \( \sigma = 0.1 \).

It is clear from the ranking results in Fig. 2 and according to the ranking results obtained by processing the evaluation values obtained by lymphography with the proposed method shows that out of 148 patients, the 108th patient is most likely to be suffering from lymphoma. In Table 5, it is possible to visualize the division of all objects into three regions, in which both 72 patients are classified as \( POS(\xi) \), 46 patients are classified as \( BND(\xi) \), and
In this section, several classic decision making methods are selected for comparative analysis of the diagnosis of tumor diseases with decision problems, including the TOPSIS.
method Hwang and Yoon (1981), the EDAS method Ghorabaee et al. (2015) and the WAA operator method Harsanyi (1955), where the weight value is consistent with the weight value in this paper. We can know that the optimal results obtained by different MADM methods are consistent, i.e., the object $z_{108}$. This is consistent with the results of the presented method, which shows the validity and practicability of the presented method. The sequencing results of 148 objects obtained by several existing methods are shown in Fig. 3.

The principle of the three methods compared in Fig. 3 is introduced as follows: The TOPSIS method Hwang and Yoon (1981) is to use the distance between the evaluation value of the object and the positive ideal solution and the negative ideal solution to calculate the relative approximation of the object to the positive ideal solution and rank all the objects. The EDAS method Ghorabaee et al. (2015) uses the positive and negative distance between the evaluation value of the object and the average solution to evaluate the object. The WAA operator method Harsanyi (1955) is to rank the objects by weighting the evaluation values of the objects and then aggregate them. MADM refers to the decision making problem of selecting the optimal object and obtaining the complete order of the objects under the condition of considering multiple attributes. It is not difficult to find that Fig. 3, which evaluates the ranking results of proposed method with those of several different classic MADM methods, shows that the overall trend of the ranking results is similar. In addition, the complete ranking obtained by these four methods has its optimal object as $z_{108}$. That is, in the process of diagnosing whether a patient has lymphoma, the physician is more
likely to have lymphoma in patient $z_{108}$ compared to other patients. The proposed method is combined with TWD and divides all patients into three regions. Due to the different principles of the above methods, the ranking results may be different. When the most objects are consistent and other rankings are similar, it can be shown that the proposed method has certain effectiveness. Compared to the other three methods, the proposed method takes into account delayed decision making and reduces decision risks, which includes the TOPSIS method Hwang and Yoon (1981), the EDAS method Ghorabaee et al. (2015) and the WAA operator method Harsanyi (1955). Thus it shows that proposed method is not only effective in solving the decision problem, but also superior to other methods.

Afterwards, in order to analyze the similarity between the rankings obtained by the above compared methods from a qualitative point of view, Fig. 4 shows the Spearman rank correlation coefficient (SRCC for short) values obtained by the ranking results of four methods. When $n$ is the number of objects, $x_i$ and $y_i$ represent the ranking position of object $z_i$ in different methods. The SRCC represents the correlation between variables and its formula Gauthier (2001) is as follows:

$$SRCC = 1 - \frac{6 \sum_{i=1}^{n} (x_i - y_i)^2}{(n^3 - n)}.$$  

From Fig. 4, the SRCC value of the rankings obtained by the proposed model and the WAA operator method has the lowest value that is 0.7906, and the SRCC value of the rankings obtained by the proposed model and the TOPSIS method Hwang and Yoon (1981) has a greater value 0.8664. The SRCC values of the rankings obtained by the proposed model and the EDAS method Ghorabaee et al. (2015) are 0.8298. According to the Spearman rank correlation threshold table, when the sample size is 100 and the significance level of the two-sided test is 0.01, the SRCC value obtained by comparing the two samples is greater than 0.257, indicating that the two samples are highly correlated. The SRCC values of the proposed method and these methods are all greater than 0.7906, indicating that the ranking performance of these methods is highly consistent. Therefore, this further demonstrates the efficiency and rationality of the decision result derived by the proposed TWMADM model.
5.2 Comparison results with existing TWD methods

In this section, several existing TWD methods are selected for comparative analysis of the diagnosis of tumor diseases with decision making problems, including Jia and Liu’s method (2019), Zhan et al.’s method (2021)\(^1\) and Wang et al.’s method (2021)\(^2\), where the loss functions are consistent with those in this paper. We can know that the optimal results acquired by different TWD methods are consistent, i.e., the object \(z_{108}\). This is consistent with the results obtained by the presented method, which shows the effectiveness and practicability of the presented method. The sequencing results of 148 objects obtained by several existing methods are shown in Fig. 5.

The principle of the three methods compared in Fig. 5 is introduced as follows: Jia and Liu’s method (2019) uses the relative loss function to obtain the thresholds, and further uses the threshold and subjective conditional probability to rank and classify objects. Zhan et al.’s method (2021) uses the fuzzy state set to obtain the objective conditional probability based on the outranking relations, and then combines the subjective loss function to obtain the expected loss of each object, and then obtains the ranking

\(^1\) The parameters in the ELECTRE-I method \(\sigma = 0.6\) and \(p = 0.2\). Choose a pessimistic strategy.

\(^2\) The confidence level \(\sigma = 0.7\).
and classification results of the objects. Wang et al.’s method (2021) gives a way to obtain the objective state set, and then uses the objective conditional probability and subjective loss function based on the probabilistic dominance relation to obtain the expected loss of each object, and finally obtains the ranking and classification results of the objects. TWMADM refers to the decision making problem for selecting the optimal object, obtaining the complete ranking and the classification results of the objects under the condition of considering multiple attributes. It is worth noting that Fig. 4 shows the evaluation of the ranking results of different TWMADM methods, and it can be seen that the optimal object in the complete ranking obtained by Jia and Liu’s method (2019), Zhan et al.’s method (2021), and Wang et al.’s method (2021) with the proposed method and the three methods in Sect. 5.1 is $z_{108}$. Due to the different principles of the above methods, the ranking results may vary. When the proposed method is compared with existing methods, if most objects are consistent and other rankings are similar, it can indicate that the proposed method has certain effectiveness. However, in terms of the ranking analysis, the complete ranking results obtained by proposed method with Jia and Liu’s method (2019) and Zhan et al.’s method (2021) are all higher than 0.7474, indicating that the ranking performance of these methods is highly consistent. On the other hand, we find that the SRCCs between the proposed method and Jia and Liu’s method (2019), Zhan et al.’s method (2021) are all higher than these five decision methods and SRCCs between Wang et al.’s method (2021), showing

![Fig. 6 The ranking of all objects with different TWD methods](image)

Further, in order to analyze the similarity between the rankings obtained by the above compared methods from a qualitative point of view, Fig. 6 shows the SRCC values obtained by the ranking results of four methods.

According to the Spearman rank correlation threshold table, when the sample size is 100 and the significance level of the two-sided test is 0.01, the SRCC value obtained by comparing the two samples is greater than 0.257, indicating that the two samples are highly correlated. As can be seen from Fig. 6, the proposed method and two decision making methods, including Jia and Liu’s method (2019) and Zhan et al.’s method (2021) are all higher than 0.7474, indicating that the ranking performance of these methods is highly consistent. On the other hand, we find that the SRCCs between the proposed method and Jia and Liu’s method (2019), Zhan et al.’s method (2021) are all higher than these five decision methods and SRCCs between Wang et al.’s method (2021), showing
that proposed method outperforms Wang et al.’s method (2021). Therefore, this further demonstrates the efficiency and superiority of the decision result derived by the proposed model.

In addition, objects can be divided into three classes in TWD by different methods. Figure 7 shows the classification results of all patients with different methods.

In light of Fig. 7, only the proposed method can divide all objects into three regions relatively uniformly. Zhan et al.’s method (2021) and Wang et al.’s method (2021) can only divide almost all objects into bounding domains. When all objects are almost divided into the boundary domain, it is equivalent to need to classify the objects again, so the meaning of three classifications is lost. There are almost no objects in the negative domain of Jia and Liu’s method (2019), but the classification method of this method is obtained by using the threshold obtained by the relative loss function and the conditional probability given subjectively. Therefore, the distribution of the number of classified objects obtained by the method in this paper is more uniform and more in line with actual situations.

**Remark 5.1** The purpose of the proposed method is to solve the problem of ranking, classifying and selecting optimal objects under multiple attributes. The MADM problem is to study the acquisition of optimal objects and complete ranking under multiple attributes, so Figs. 3 and 5 show the ranking results of the proposed method and the six methods. Due to the different principles of the above methods, it may lead to differences in ranking results. When the principles of the methods are different, if the ranking positions of the optimal object and most objects are consistent, it can be shown that the proposed method has certain practical value. It can be seen that the optimal objects are consistent and the ranking trend is the same. As can be seen from Fig. 7, only the proposed method can divide all objects into three regions more uniformly. Overall, the proposed method has certain practicality and effectiveness.

**5.3 Discussions with the related methods**

In the above content, the feasibility of the TWD method constructed in this paper can be analyzed through the above different decision making methods. By comparing with the

![Fig. 7 The classification of all patients with different methods](image-url)
above several methods, the method constructed in this paper can divide objects into positive regions, boundary regions and negative regions. Some existing methods can only rank objects and choose the optimal. In addition, the method constructed in this paper not only considers the loss functions of each object, but also considers the gain functions. For instance, in the Lymphography problem, if the patient with the disease is diagnosed, there will be a relative loss, and at the same time, there will be a relative gain from early diagnosis and treatment, so we consider the net profit functions. Next, we show the differences among the above seven methods in Table 6.

According to the application of the above medical cases and the comparative analysis with other counterparts, we can sum up the following interesting findings from Table 6.

On the one side, the major diverse aspects between the presented method and existing counterparts are listed below:

(1) It is inevitable that some risks and losses exist in various decision making scenarios. This method considers the loss functions and the gain functions, whereas the traditional MADM matrix method only considers the MADM matrix, without considering the cost and the gain of decisions. For instance, assuming that the patient is unworthy of selections, this will bring relatively more losses and relatively less gains to the patient; On the contrary, assuming that the patient is a worthwhile choice, the patient will have relatively few losses and relatively more gains. Under the background of choice problems, the method utilizes the relative costs and benefits of the candidates to rank respective objects. However, some other MADM methods [the TOPSIS method Hwang and Yoon (1981), the EDAS method Ghorabaee et al. (2015), the WAA operator method Harsanyi (1955), Jia and Liu’s method (2019), Zhan et al.’s method (2021) and Wang et al.’s method (2021)] fail to incorporate costs and gain of decisions, therefore these methods cannot rank efficiently.

(2) By using the TWD model, the method divides the optional object into three regions, and then decides to accept the decision, defers the decision and rejects the decision. Some existing methods only prioritize the objects and choose the optimal choice. In some traditional MADM methods, the selection is divided into two regions, and then the acceptance and rejection decisions are made. Thus, this method reduces decision risks.

(3) This paper builds a TWD model based on the tolerance dominance relation and compares it with the probability dominance relation Wang et al. (2021). In this paper, the

| Different methods               | Set of objective states | Loss/Gain functions | Conditional probability | Ranking | Classification |
|--------------------------------|-------------------------|---------------------|-------------------------|---------|---------------|
| Proposed method                | ✓                       | ✓                   | ✓                       | ✓       | ✓             |
| TOPSIS method Hwang and Yoon (1981) | ×                      | ×                   | ✓                       | ×       | ×             |
| EDAS method Ghorabaee et al. (2015) | ×                      | ×                   | ×                       | ✓       | ×             |
| WAA operator method Harsanyi (1955) | ×                      | ×                   | ×                       | ✓       | ×             |
| Jia and Liu’s method (2019)     | ×                       | ✓                   | ✓                       | ✓       | ✓             |
| Zhan et al.’s method (2021)     | ×                       | ✓                   | ✓                       | ✓       | ✓             |
| Wang et al.’s method (2021)     | ×                       | ✓                   | ✓                       | ✓       | ✓             |
degree of superiority is considered in the tolerance dominance relation, whereas in the probabilistic dominance relation, there is only 0 or 1.

(4) Compared with the other six MADM methods, the proposed method considers decision attributes. This paper presents a method to obtain the set of objective states according to decision attributes. It can effectively avoid decision risks brought by the set of subjective states.

On the other side, based on what has been discussed above, the advantages of the new method are shown below:

(1) In the newly proposed method, we consider decision makers’ evaluations for alternatives, thus our model is built on the basis of an ordered information system. Hence we can get a set of two states based on the decision attribute.
(2) In actual decision making problems, the certain risks and losses will always exist when making decisions. In addition, there is a corresponding benefit when more beneficial decisions are made. Compared with Zhan et al.’s method (2021) and Wang et al.’s method (2021), the loss functions are given by the decision makers according to practical experiences, which may lead to the strong subjectivity. In order to decrease the decision risk and increase the profit brought by the decision, the method considers six loss functions and six gain functions of each object, as well as the net profit functions obtained by the integration of the loss functions and the gain functions, which is more consistent with actual situations and semantic interpretations.
(3) In order to better satisfy the demands of experts, we can adjust the credibility of $\sigma$-tolerance dominance class in the model of this method in light of experts’ diverse decision contexts and decision making preferences. At this point, it shows the flexibility of the approach.
(4) A comparative analysis is performed by comparing the presented method with the existing MADM counterparts, and the presented TWMADM enriches the research content of MADM, such as the edge boundary, the net profit functions, etc. Therefore, it can aid experts to make valid decisions.

In addition to the above differences and advantages, the proposed method has certain limitations, which are shown as follows: A direct limitation of this method is that the proposed method can only be used for MADM problems where there is a partial order relation between the evaluation values of objects. In addition, when there is no decision attribute in the original information table, the decision attributes of the objects need to be obtained according to the known information before making a decision.

6 Experimental evaluations

In this section, according to the instance of medical diagnosis in Sect. 4.3, the validity and stability of the TWD method presented in this paper are proved via three experiments.

6.1 Weight evaluations

There are $m$ objects $z_1, z_2, \cdots, z_m$ and $n$ indicators $c_1, c_2, \cdots, c_m$, where $z_{ij}$ is the index value of $z_i$ under $c_j$, $w_j$ is the weight of $c_j$ and $w_j \in [0, 1]$. Then $U = (z_{ij})_{m \times n}$ and $w_j$ can express
the assessment value in an initial decision matrix. In what follows, we define the notion of entropy weights.

**Definition 6.1** Hwang and Yoon (1981) Assume that $S = (U, C \cup D)$ is an ordered decision information system, the entropy weight value $(e_j)$ for all $j$th criteria is given below:

$$e_j = -\frac{1}{\ln n} \sum_{i=1}^{n} p_{ij} \ln p_{ij},$$

(29)

where $e_j > 0$ and $p_{ij} = \frac{z_{ij}}{\sum z_{ij}}$ for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$. For the case $z_{ij} = 0$, $p_{ij} \ln p_{ij} = 0$. Then, the entropy weight $w_j$ for the $j$th criterion can be further given below:

$$w_j = \frac{1 - e_j}{\sum_{j=1}^{m} (1 - e_j)}, j \in \{1, 2, \ldots, m\}.$$  

(30)

In what follows, in order to exhibit the validity of the newly constructed MADM method, the entropy weight in Definition 6.1 is applied instead of the weight value of the Lymphography mentioned in this paper, the entropy weights are shown as follows:

$W = \{0.0441, 0.0538, 0.0456, 0.0165, 0.0539, 0.0567, 0.0225, 0.0402, 0.0307, 0.0580, 0.0295, 0.0447, 0.0392, 0.0907, 0.0625, 0.0445, 0.0298, 0.2371\}.$

In order to analyze the stability of the ranking results of the proposed method under different weight values, Fig. 8 will show the ranking of all objects with different weights for different methods.

By Fig. 8, we observe that the optimal result for proposed method, the TOPSIS method Hwang and Yoon (1981), the EDAS method Ghorabaee et al. (2015), the WAA operator...
method Harsanyi (1955) is still the object $z_{108}$. Meanwhile, we find that the trend of the ranking results obtained by four different methods is similar under two different weights of attributes. Therefore, the proposed method is also efficient from the viewpoint of the ranking result under different attribute weights. It has been demonstrated for the different weights considered in this example that the proposed method is stable.

6.2 Parameter evaluations

In the presented method, the parameter $\sigma$ plays a decisive role in determining the conditional probability. Meanwhile, the parameter $\xi$ has an important effect on the classification of regions. Thus, it is necessary to explore the influence of parameter values on decision results.

6.2.1 Parameter evaluations based on the parameter $\sigma$

First, the range of $\sigma$ is determined. According to Definition 2.7, we know that $\sigma \in [0, 1]$. Furthermore, the parameter is a confidence level that indicates the degree to which one object is inferior to another one under the attribute subset. Further, when we consider $\sigma = 1$, it means that the value difference is infinite, that is, it is easy to meet the condition of superiority. When $\sigma = 0$, the requirement cannot accept any error, that is, it is difficult to meet the superior condition. Hence, the condition $\sigma > 0.5$ is very easy to satisfy, which makes the decision less rigorous, so we can determine that $\sigma \in (0, 0.5]$. Based on the example given in Sect. 4.3, we set the step size to 0.1$^3$, five different values are selected for the analysis. The result is shown in Fig. 9.

---

$^3$ The risk aversion coefficient $\xi = \{0.35, 0.4, 0.4, 0.4, 0.35, 0.4, 0.4, 0.4, 0.35, 0.4, 0.4, 0.4, 0.3, 0.3, 0.3, 0.3\}$. 

---

Fig. 9 The ranking of all objects with different values for the parameter $\sigma$
We can intuitively find from Fig. 9 that when the value of $\sigma$ changes, the overall trend of the ranking results is similar and consistent. Thus, the method presented in this paper is stable.

In addition, when $\sigma \in (0, 0.5]$ and set the step size to 0.1, Fig. 10 can intuitively describe the change of classification results when the value of $\sigma$ changes.

As can be seen from Fig. 10, when the parameters gradually increase, the number of objects in $BND(\zeta)$ and $NEG(\zeta)$ increases, whereas the number of objects in $POS(\zeta)$ decreases. Therefore, we know that the higher the fault tolerance, the smaller the number of objects in the accepted domain.

**6.2.2 Parameter evaluations based on the parameter $\xi$**

The decision model constructed in this paper considers the combination of relative loss functions with relative gain functions to get the net profit functions, and the net profit functions play an important place in the TWD decision process for classification. What is more worth analyzing is that the risk aversion factor may affect the net profit function. Figure 11 visually shows the ground segmentation obtained when $\xi \in (0, 0.5)$ and the step size is 0.05$^4$.

When we fix the tolerance rate $\sigma = 0.1$, we can observe from Fig. 11 that the ranking results obtained from the change in the value of $\xi$ for all objects are close. And all the optimal objects are obtained as $\varepsilon_{108}$. Therefore, it can be shown that proposed method is stable.

In addition, we select datasets from UCI database to classify and analyze the presented method. The description of the experimental datasets are shown in Table 7.

The classification results of several datasets are shown in Fig. 12.

Figure 12 shows the classification results of four datasets, namely, Lymphography, Connectionist Bench, Balance Scale and Wdbc when the value of the risk aversion coefficient

$^4$ The tolerance rate $\sigma = 0.1$. 

 Springer
ξ changes under the condition of σ = 0.1. The value of σ is fixed to 0.1 in order to keep it consistent with the value of question in Sect. 4.3. It is noted that the numbers of objects in the boundary domain decrease gradually with the increase of the parameter ξ, which is consistent with the theoretical knowledge in Sect. 3.

### 6.2.3 Parameter evaluations based on the combination between parameters σ and ξ

Based on the above analysis, it is also worth studying that the values of the parameters σ and ξ change simultaneously. Therefore, for σ ∈ (0, 0.5] and ξ ∈ (0, 0.5), set the step size
to 0.1 and 0.05, respectively, Fig. 13 can intuitively show the influence of simultaneous changes of the parameters on the number of classifications.

As can be seen from Fig. 13, when the parameters $\sigma$ and $\xi$ gradually increase at the same time, the number of objects in $\text{POS}(\xi)$ and $\text{NEG}(\xi)$ also increases gradually, whereas the number of objects in $\text{BND}(\xi)$ decreases gradually. Hence, we can acquire that the larger the value of the parameter is, the more domains the object in the certain classification will be obtained. At the same time, it can help decision makers make choices.

### 6.3 Error rate evaluations

According to the four datasets in Table 7, the decision result may vary with the change of parameters. Hence, it is necessary to use the decision attributes of the four datasets to assess the performance of the presented model. We apply the error rate to evaluate the performance of our proposed TWD based on the tolerance dominance relation. The error rate is given below Liang et al. (2019b):

![Graphs showing the classification of all objects for the four datasets](image-url)
Fig. 13  The classification of all objects for the parameters $\sigma$ and $\xi$

Fig. 14  The changes in error rate results for all objects in the four datasets
where $U$ denotes the total numbers of objects. $n_{\xi \rightarrow \text{NEG}(\zeta)}$ denotes the numbers of objects classified into the negative region $\text{NEG}(\zeta)$ belonging to $\xi$. $n_{\xi \rightarrow \text{POS}(\zeta)}$ denotes the numbers of objects classified into the positive region $\text{POS}(\zeta)$ belonging to $\xi$. 

$$\text{Error rate} = \frac{n_{\xi \rightarrow \text{NEG}(\zeta)} + n_{\xi \rightarrow \text{POS}(\zeta)}}{|U|},$$ (31)

where $U$ denotes the total numbers of objects.
of objects classified into the positive region \( \text{POS}(\zeta) \) belonging to \( \neg\zeta \). Obviously, the smaller the error rate, the better the performance of the proposed TWD model based on the tolerance dominance relation. Figures 14, 15 and 16 show the error rates for each object in the four datasets.

From Fig. 14, when \( \sigma = 0.1 \), the error rates of the four datasets vary with the increases of values of \( \xi \), where the abscissa represents \( \xi \in (0, 0.5) \) and step size of 0.05, and the ordinate represents the error rates with the ranges \((0, 0.3)\). From Fig. 15, when \( \xi = 0.2 \), the error rates of the four datasets change with the increases of values of the parameter \( \sigma \), where the abscissa represents the parameter \( \sigma \in (0, 0.5) \) and step size of 0.1, and the ordinate represents the error rates with the ranges \((0, 0.08)\).

As can be seen from Fig. 16, when \( \sigma \) and \( \xi \) simultaneously change, the error rates of the four datasets change. As can be seen from the whole, the error rates belong to \((0, 0.4)\) for the four datasets. It can be seen from Lymphography, Connectionist Bench, Balance Scale and Wdbc that under different \( \sigma \) values, with the increase of risk aversion coefficient, as the number of objects in the boundary domain increases, the error rates of the datasets decrease. In realistic decision making procedures, decision makers can enlarge the boundary area and reduce the error rates by changing the parameter values. This reflects the effectiveness of the presented method.

7 Conclusions and future research directions

In summary, by combining the MADM method, the TWD theory with decision makers’ preferences, we present a novel method to address MADM by tolerance dominance relations. By using the presented model, decision makers can not only acquire the ordering of objects, but also the classification of objects. The feasibility, validity and stability of the method are verified by the sample of Lymphography in the UCI database. The main contributions of the current work can be listed below:

- For the sake of obtaining a more applicable model, we have used the dominance relations to measure the relation among objects to obtain the dominance class, and then replace the equivalence class in the classic TWD model. When dealing with real-valued MADM problems, the tolerance dominance relation can avoid the limitation that the condition of the equivalence relation is too strict. We have introduced a novel TWD model in light of the tolerance dominance relation.
- In order to avoid the decision risk brought by different subjective state sets from existing methods, this paper proposes a method to obtain the target state set by considering the decision attributes in the TWD model. Furthermore, an objective conditional probability based on the tolerance dominance relation can be obtained.
- According to the idea of Zhan et al. (2021), we have considered not only the more reasonable semantic interpretation of the six loss functions, but also the six gain functions. In addition, we have considered the net benefit functions obtained by the combination of six loss functions and six gain functions, and the proposed TWD model has six relative net benefit functions for each object.
- In addition to the above key contributions, the application of TWD models in multivariate decision problems in this paper is also very valuable. Therefore, we establish a decision making method that combines TWD model with decision makers’ preference.
We use the data from UCI database to verify that the presented method can effectively address MADM by experimental analysis, error rate analysis and comparative analysis.

Furthermore, there are a series of questions in future studies: (1) The process of combining loss functions with gain functions to get net benefit functions is very meaningful to study in the future. (2) The presented TWD model can be expanded to diverse information systems. For instance, fuzzy coverings in information systems, hesitation fuzzy information systems, interval-valued information systems, feature selection problems and categorical data, etc. (3) The linkage between TWD and MADM is further worth studying.

Acknowledgements The authors would like to express their sincere thanks to the editors and anonymous reviewers for their most valuable comments and suggestions in improving this paper. The work of the first and second authors was supported by grants from the NNSFC (61866011; 11961025; 12161036). This work of the third author was supported in part by the NNSFC (61976120), in part by the Natural Science Foundation of Jiangsu Province under Grant BK20191445.

Author contributions Wenjie Wang: Investigation, Conceptualization, Methodology, Writing—original draft. Jianming Zhan: Methodology, Investigation, Writing—original draft. Weiping Ding: Writing—Reviewing and Editing. Shuping Wan: Writing—Reviewing and Editing.

Declarations

Conflicts of interest The authors declared that they have no conflicts of interest to this work.

References

Augeri MG, Cozzo P, Greco S (2015) Dominance-based rough set approach: an application case study for setting speed limits for vehicles in speed controlled zones. Knowl-Based Syst 89:288–300
Chakhar S, Ishizaka A, Labib A, Saad I (2016) Dominance-based rough set approach for group decisions. Eur J Oper Res 251:206–224
Chen SM, Hong JA (2014) Fuzzy multiple attributes group decision-making based on ranking interval type-2 fuzzy sets and the TOPSIS method. IEEE Trans Syst.Man Cybern Syst 44(12):1665–1673
Chen WC, Lv YJ, Weng SZ (2014) Sorting method and its application based on tolerance dominance relation. J Comput Appl 34(8):2170–2174 (in Chinese)
Chen YF, Yue XD, Fujita H, Fu SY (2017) Three way decision support for diagnosis on focal liver lesions. Knowl-Based Syst 127:85–99
Clark P, Niblett T (1987) Induction in noisy domains. In: Eur. Mach. Learn. Conference. EWSL, pp 11–30
Coelho RF (2015) Probabilistic dominance in multiobjective reliability-based optimization: theory and implementation. IEEE Trans Evol Comput 19(2):214–224
Du WS, Hu BQ (2016) Dominance-based rough set approach to incomplete ordered information systems. Inf Sci 346:106–129
Du WS, Hu BQ (2017) Dominance-based rough fuzzy set approach and its application to rule induction. Eur J Oper Res 261:690–703
Gauthier DT (2001) Detecting trends using Spearman’s rank correlation coefficient. Environ Forensic 2(4):359–362
Ghorabaee MK, Zavadskas EK, Olfat L, Turskis Z (2015) Multi-criteria inventory classification using a new method of evaluation based on distance from average solution (EDAS). Informatica 26(3):435–451
Greco S, Matarazzo B, Slowinski R (1999) Rough approximation of a preference relation by dominance relations. Eur J Oper Res 117:63–83
Greco S, Matarazzo B, Slowinski R (2002a) Rough approximation by dominance relations, Inform. J Intell Syst 17:153–171
Greco S, Matarazzo B, Slowinski R (2002b) Rough sets theory for multicriteria decision analysis. Eur J Oper Res 129:1–47
Harsanyi JC (1955) Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. J Polit Econ 63:309–321
A three-way decision method with tolerance dominance relations…

Huang XF, Zhan JM (2021) TWD-R: a three-way decision approach based on regret theory in multi-scale decision information systems. Inf Sci 581:711–739

Huang B, Li HX, Feng GF, Zhou XZ (2019) Dominance-based rough sets in multi-scale intuitionistic fuzzy decision tables. Appl Math Comput 348:487–512

Huang QQ, Li TR, Huang YY, Yang X, Fujita H (2020) Dynamic dominance rough set approach for processing composite ordered data. Knowl-Based Syst 187:104829

Hwang CL, Yoon K (1981) Multiple attribute decision making methods and applications. Springer, Berlin Heidelberg

Jia F, Liu PD (2019) A novel three-way decision model under multiple-criteria environment. Inf Sci 471:29–51

Kononenko I, Cestnik B (1988) UCI repository of machine learning databases. http://archive.ics.uci.edu/ml/datasets/Lymphography

Lang GM, Miao DQ, Fujita H (2020) Three-way group conflict analysis based on pythagorean fuzzy set theory. IEEE Trans Fuzzy Syst 28(3):447–461

Li SY, Li TR (2015) Incremental update of approximations in dominance-based rough sets approach under the variation of attribute values. Inf Sci 294:348–361

Li WT, Xu WH (2014) Probabilistic rough set model based on dominance relation. Rough Sets Knowl Technol 78:856–863

Li SY, Li TR, Liu D (2013) Dynamic maintenance of approximations in dominance-based rough set approach under the variation of the object set. Int J Intell Syst 28(8):729–751

Li ZW, Zhang PF, Xie NX, Zhang GQ, Wen CF (2020) A novel three-way decision method in a hybrid information system with images and its application in medical diagnosis. Eng Appl Artif Intell 92:103651

Liang DC, Pedrycz W, Liu D (2017) Determining three-way decisions with decision-theoretic rough sets using a relative value approach. IEEE Trans Syst Man Cybern Syst 47(8):1785–1799

Liang DC, Wang MW, Xu ZS (2019a) Heterogeneous multi-attribute nonadditivity fusion for behavioral three-way decisions in interval type-2 fuzzy environment. Inf Sci 496:242–263

Liang DC, Wang MW, Xu ZS, Chen X (2019b) Risk interval-valued three-way decisions model with regret theory and its application to project resource allocation. J Oper Res Soc 72(1):180–199

Liu D, Liang DC (2017) Three-way decisions in ordered decision system. Knowl-Based Syst 137:182–195

Liu PD, Liu WQ (2021) Dual generalized Bonferroni mean operators based on 2-dimensional uncertain linguistic information and their applications in multi-attribute decision making. Artif Intell Rev 54:491–517

Liu PD, Wang YM, Jia F, Fujita H (2020) A multiple attribute decision making three-way model for intuitionistic fuzzy numbers. Int J Approx Reason 119:177–203

Ma XL, Zhan JM, Ali MI, Mehmood N (2021) A survey of decision making methods based on two classes of hybrid soft set models. Artif Intell Rev 49:511–529

Mardani A, Saraji MK, Mishra AR, Rani P (2020) A novel extended approach under hesitant fuzzy sets to design a framework for assessing the key challenges of digital health interventions adoption during the COVID-19 outbreak. Appl Soft Comput 96:106613

Mendonca GHM, Ferreira FGDC, Cardoso RTN, Martins FVC (2020) Multi-attribute decision making applied to financial portfolio optimization problem. Expert Syst Appl 158:113527

Pena J, Napoles G, Salgueiro Y (2020) Explicit methods for attribute weighting in multi-attribute decision-making: a review study. Artif Intell Rev 53:3127–3152

Ren PJ, Xu ZS, HaoZN (2017) Hesitant fuzzy thermodynamic method for emergency decision making based on prospect theory. IEEE Trans Cybern 47(9):2531–2543

Shen W, Wei ZH, Li QW, Zhang HY, Miao DQ (2020) Three-way decisions based blocking reduction models in hierarchical classification. Inf Sci 523:63–76

Sun BZ, Chen XT, Zhang LY, Ma WM (2020) Three-way decision making approach to conflict analysis and resolution using probabilistic rough set over two universes. Inf Sci 507:809–822

Szlapczynski R, Szlapczynska J (2021) W-dominance: tradeoff-inspired dominance relation for preference-based evolutionary multi-objective optimization. Swarm Evol Comput 63:100866

Tan AH, Shi SW, Wu WZ, Li JJ, Pedrycz W (2020) Granularity and entropy of intuitionistic fuzzy information and their applications. IEEE Trans Cybern. https://doi.org/10.1109/TCYB.2020.2973379

Wang YM (1998) Using the method of maximizing deviations to make decision for multi-indices. Syst Eng Electron 7:24–31 (in Chinese)

Wang P, Zhang PF, Li ZW (2019) A three-way decision method based on Gaussian kernel in a hybrid information system with images: an application in medical diagnosis. Appl Soft Comput 77:734–749

Wang CZ, Huang Y, Shao MW, Hu QH, Chen DG (2020a) Feature selection based on neighborhood self-information. IEEE Trans Cybern 50(9):4031–4042
Wang CZ, Wang Y, Shao MW, Qiao YH, Chen DG (2020b) Fuzzy rough attribute reduction for categorical data. IEEE Trans Fuzzy Syst 28(5):818–830
Wang TX, Li HX, Qian YH, Huang B, Zhou XZ (2020) A regret-based three-way decision model under interval type-2 fuzzy environment. IEEE Trans Fuzzy Syst. https://doi.org/10.1109/TFUZZ.2020.3033448
Wang WJ, Zhan JM, Zhang C (2021) Three-way decisions based multi-attribute decision making with probabilistic dominance relations. Inf Sci 599:75–96
Wang JJ, Ma XL, Xu ZS, Zhan JM (2021) Three-way multi-attribute decision making under hesitant fuzzy environments. Inf Sci 552:328–351
World Health Organization. https://www.who.int/about/iarc/zh/
Xie B, Li LJ, Mi JS (2016) A novel approach for ranking in interval-valued information systems. J Intell Fuzzy Syst 30:523–534
Yang XB, Qi Y, Yu DJ, Yu HL, Yang JY (2015) $\alpha$-Dominance relation and rough sets in interval-valued information systems. Inf Sci 294:334–347
Yang DD, Deng TQ, Fujita H (2020) Partial-overall dominance three-way decision models in interval-valued decision systems. Int J Approx Reason 126:308–325
Yang SY, Zhang HY, Baets BD, Jah M, Shi G (2021) Quantitative dominance-based neighborhood rough sets via fuzzy preference relations. IEEE Trans Fuzzy Syst 29(3):515–529
Yao YY (2010) Three-way decisions with probabilistic rough sets. Inf Sci 180:341–353
Yao YY (2019) Three-way conflict analysis: reformulations and extensions of the Pawlak model. Knowl-Based Syst 180:26–37
Yao YY (2020) Three-way granular computing, rough sets, and formal concept analysis. Int J Approx Reason 116:106–125
Yao YY (2021) Set-theoretic models of three-way decision. Granul Comput 6:133–148
Ye J, Zhan JM, Xu ZS (2020) A novel decision-making approach based on three-way decisions in fuzzy information systems. Inf Sci 541:362–390
Yue XD, Zhou J, Yao YY, Miao DQ (2020) Shadowed neighborhoods based on fuzzy rough transformation for three-way classification. IEEE Trans Fuzzy Syst 28(5):978–991
Zhan JM, Alcantud JCR (2019) A novel type of soft rough covering and its application to multicriteria group decision making. Artif Intell Rev 52:2381–2410
Zhan JM, Sun BZ (2020) Covering-based intuitionistic fuzzy rough sets and applications in multi-attribute decision-making. Artif Intell Rev 53:671–701
Zhan JM, Jiang HB, Yao YY (2021) Three-way multi-attribute decision-making based on outranking relations. IEEE Trans Fuzzy Syst 29:2844–2858
Zhang C, Li DY, Liang JY (2020) Multi-granularity three-way decisions with adjustable hesitant fuzzy linguistic multigranulation decision-theoretic rough sets over two universes. Inf Sci 507:665–683
Zhang QH, Zhao F, Yang J, Wang GY (2020) Three-way decisions of rough vague sets from the perspective of fuzziness. Inf Sci 523:111–132

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.