Anholonomic Frames and Thermodynamic Geometry of 3D Black Holes

Sergiu I. Vacaru*, Panyiotis Stavrinos † and Denis Gontsa ‡

* Physics Department, CSU Fresno, Fresno, CA 93740-8031, USA,
Centro Multidisciplinar de Astrofisica - CENTRA, Departamento de Fisica,
Instituto Superior Tecnico, Av. Rovisco Pais 1, Lisboa, 1049-001, Portugal,
† Department of Mathematics, University of Athens,
15784 Panepistimiopolis, Athens, Greece
and
‡ Department of Physics, St. Petersbourg State University
P. O. Box 122, Petergoff, St. Petersbourg, 198904, Russia

May 31, 2002

Abstract

We study new classes three dimensional black hole solutions of Einstein equations written in two holonomic and one anholonomic variables with respect to anholonomic frames. Thermodynamic properties of such (2 + 1)–black holes with generic local anisotropy (having elliptic horizons) are studied by applying geometric methods. The corresponding thermodynamic systems are three dimensional with entropy \( S \) being a hypersurface function on mass \( M \), anisotropy angle \( \theta \) and eccentricity of elliptic deformations \( \varepsilon \). Two–dimensional curved thermodynamic geometries for locally anistropic deformed black holes are constructed after integration on anisotropic parameter \( \theta \). Two approaches, the first one based on two–dimensional hypersurface parametric geometry and the second one developed in a Ruppeiner–Mrugala–Janyszek fashion, are analyzed. The thermodynamic curvatures are computed and the critical points of curvature vanishing are defined.

*e–mail: vacaru@fisica.ist.utl.pt, sergiu_vacaru@yahoo.com
†e–mail: pstavrin@cc.uoa.gr
‡e–mail: d_gontsa@yahoo.com
1 Introduction

This is the second paper in a series in which we examine black holes for spacetimes with generic local anisotropy. Such spacetimes are usual pseudo–Riemannian spaces for which an anholonomic frame structure By using moving anholonomic frames one can construct solutions of Einstein equations with deformed spherical symmetries (for instance, black holes with elliptic horizons (in three dimensions, 3D), black tora and another type configurations) which are locally anisotropic \([23, 27]\).

In the first paper [26] (hereafter referred to as Paper I) we analyzed the low–dimensional locally anisotropic gravity (we shall use terms like locally anisotropic gravity, locally anisotropic spacetime, locally anisotropic geometry, locally anisotropic black holes and so on) and constructed new classes of locally anisotropic \((2 + 1)\)-dimensional black hole solutions. We emphasize that in this work the splitting \((2 + 1)\) points not to a space–time decomposition, but to a spacetime distribution in two isotropic and one anisotropic coordinate.

In particular, it was shown following [24] how black holes can recast in a new fashion in generalized Kaluza–Klein spaces and emphasized that such type solutions can be considered in the framework of usual Einstein gravity on anholonomic manifolds. We discussed the physical properties of \((2 + 1)\)-dimensional black holes with locally anisotropic matter, induced by a rotating null fluid and by an inhomogeneous and non–static collapsing null fluid, and examined the vacuum polarization of locally anisotropic spacetime by non–rotating black holes with ellipsoidal horizon and by rotating locally anisotropic black holes with time oscillating and ellipsoidal horizons. It was concluded that a general approach to the locally anisotropic black holes should be based on a kind of nonequilibrium thermodynamics of such objects imbedded into locally anisotropic spacetime background. Nevertheless, we proved that for the simplest type of locally anisotropic black holes theirs thermodynamics could be defined in the neighborhoods of some equilibrium states when the horizons are deformed but constant with respect to a frame base locally adapted to a nonlinear connection structure which model a locally anisotropic configuration.

In this paper we will specialize to the geometric thermodynamics of, for simplicity non–rotating, locally anisotropic black holes with elliptical horizons. We follow the notations and results from the Paper I which are reestablished in a manner compatible in the locally isotropic thermodynamic [1] and spacetime [1] limits with the Banados–Teitelboim–Zanelli (BTZ) black hole. This new approach (to black hole physics) is possible for locally anisotropic spacetimes and is based on classical results [10, 15, 16, 22].

Since the seminal works of Bekenstein [4], Bardeen, Carter and Hawking [2] and Hawking [12], black holes were shown to have properties very similar to those of ordinary thermodynamics. One was treated the surface gravity on the event horizon as the temperature of the black hole and proved that a quarter of the event horizon area corresponds to the entropy of black holes. At present time it is widely believed that a black hole is a thermodynamic system (in spite of the fact that one have been developed a number of realizations of thermodynamics involving radiation) and the problem of statistical interpretation of the black hole entropy is one of the most fascinating subjects of modern investigations in gravitational and string theories.

In parallel to the ‘thermodynamilazation’ of black hole physics one have developed
a new approach to the classical thermodynamics based of Riemannian geometry and its generalizations (a review on this subject is contained in Ref. [20]). Here is to be emphasized that geometrical methods have always played an important role in thermodynamics (see, for instance, a work by Blaschke [3] from 1923). Buchdahl used in 1966 a Euclidean metric in thermodynamics [8] and then Weinhold considered a sort of Riemannian metric [28]. It is considered that the Weinhold’s metric has not physical interpretation in the context of purely equilibrium thermodynamics [19, 20] and Ruppeiner introduced a new metric (related via the temperature $T$ as the conformal factor with the Weinhold’s metric).

The thermodynamical geometry was generalized in various directions, for instance, by Janyszek and Mrugala [13, 14, 18] even to discussions of applications of Finsler geometry in thermodynamic fluctuation theory and for nonequilibrium thermodynamics [22].

Our goal will be to provide a characterization of thermodynamics of $(2+1)$–dimensional locally anisotropic black holes with elliptical (constant in time) horizon obtained in [25, 26]. From one point of view we shall consider the thermodynamic space of such objects (locally anisotropic black holes in local equilibrium with locally anisotropic space-time ether) to depend on parameter of anisotropy, the angle $\theta$, and on deformation parameter, the eccentricity $\varepsilon$. From another point, after we shall integrate the formulas on $\theta$, the thermodynamic geometry will be considered in a usual two–dimensional Ruppeiner–Mrugala–Janyszek fashion. The main result of this work are the computation of thermodynamic curvatures and the proof that constant in time elliptic locally anisotropic black holes have critical points of vanishing of curvatures (under both approaches to two–dimensional thermodynamic geometry) for some values of eccentricity, i.e. for under corresponding deformations of locally anisotropic spacetimes.

The paper is organized as follows. In Sec. 2, we briefly review the geometry pseudo–Riemannian spaces provided with anholonomic frame and associated nonlinear connection structure and present the $(2+1)$–dimensional constant in time elliptic black hole solution. In Sec. 3, we state the thermodynamics of nearly equilibrium stationary locally anisotropic black holes and establish the basic thermodynamic law and relations. In Sec. 4 we develop two approaches to the thermodynamic geometry of locally anisotropic black holes, compute thermodynamic curvatures and the equations for critical points of vanishing of curvatures for some values of eccentricity. In Sec. 5, we draw a discussion and conclusions.

2 Locally Anisotropic Spacetimes and Black Holes

In this section we outline for further applications the basic results on $(2+1)$–dimensional locally anisotropic spacetimes and locally anisotropic black hole solutions [25, 26].

2.1 Anholonomic frames and nonlinear connections in $(2+1)$–dimensional spacetimes

A $(2+1)$–dimensional locally anisotropic spacetime is defined as a 3D pseudo–Riemannian space provided with a structure of anholonomic frame with two holonomic coordinates $x^i, i = 1, 2$ and one anholonomic coordinate $y$, for which $u = (x, y) = \{u^\alpha = (x^i, y)\}$, the
Greek indices run values $\alpha = 1, 2, 3$, when $u^3 = y$. We shall use also underlined indices, for instance $\overline{\alpha}$, in order to emphasize that some tensors are given with respect to a local coordinate base $\partial_{\overline{\alpha}} = \partial/\partial u^{\overline{\alpha}}$.

An anholonomic frame structure of triads (dreibein) is given by a set of three independent basis fields

$$e_{\alpha}(u) = e_{\alpha}^{\alpha}(u)\partial_{\overline{\alpha}}$$

which satisfy the relations

$$e_{\alpha}e_{\beta} - e_{\beta}e_{\alpha} = w_{\gamma}^{\alpha\beta}e_{\gamma},$$

where $w_{\gamma}^{\alpha\beta} = w_{\gamma}^{\alpha\beta}(u)$ are called anholonomy coefficients.

We investigate anholonomic structures with mixed holonomic and anholonomic triads when

$$e_{\alpha}^{\alpha}(u) = \{e_{\overline{j}}^{\overline{i}} = \delta_{\overline{i}}^{\overline{j}}, e_{3}^{\overline{3}} = N_{3}^{\overline{3}}(u) = w_{i}(u), e_{3}^{\overline{3}} = 1\}.$$ 

In this case we have to apply ‘elongated’ by N–coefficients operators instead of usual local coordinate basis $\partial_{\alpha} = \partial/\partial u^{\alpha}$ and $d^{\alpha} = du^{\alpha}$, (for simplicity we shall omit underling of indices if this does not result in ambiguities):

$$\delta_{\alpha} = (\delta_{i}, \partial_{(y)}) = \delta^{i}$$

$$(\partial x^{i} = \partial x^{i} - w_{i}(x, y) \partial y, \partial_{(y)} = \partial y)$$

and their duals

$$\delta^{\beta} = (d^{i}, \delta^{(y)}) = \delta y$$

$$=(d^{i} = dx^{i}, \delta^{(y)} = \delta y = dy + w_{k}(x, y) dx^{k}).$$

The coefficients $N = \{N_{3}^{\overline{3}}(x, y) = w_{i}(x, y)\}$, are associated to a nonlinear connection (in brief, N–connection, see [3]) structure which on pseudo-Riemannian spaces defines a locally anisotropic, or equivalently, mixed holonomic–anholonomic structure. The geometry of N–connection was investigated for vector bundles and generalized Finsler geometry [17] and for superspaces and locally anisotropic (super)gravity and string theory [24] with applications in general relativity, extra dimension gravity and formulation of locally anisotropic kinetics and thermodynamics [17, 23, 26, 27]. In this paper (following the Paper I) we restrict our considerations to the simplest case with one anholonomic (anisotropic) coordinate when the N–connection is associated to a subclass of anholonomic triads (1), and/or (2), defining some locally anisotropic frames (in brief, anholonomic basis, anholonomic frames).

With respect to a fixed structure of locally anisotropic bases and their tensor products we can construct distinguished, by N–connection, tensor algebras and various geometric objects (in brief, one writes d–tensors, d–metrics, d–connections and so on).

A symmetrical locally anisotropic metric, or d–metric, could be written with respect to an anholonomic basis (2) as

$$\delta s^{2} = g_{\alpha\beta}(u^{\tau}) \delta u^{\alpha} \delta u^{\beta}$$

$$= g_{ij}(x^{k}, y) dx^{i}dx^{j} + h(x^{k}, y)(\delta y)^{2}.$$
We note that the anisotropic coordinate $y$ could be both type time–like ($y = t$, or space–like coordinate, for instance, $y = r$, radial coordinate, or $y = \theta$, angular coordinate).

### 2.2 Non–rotating black holes with ellipsoidal horizon

Let us consider a 3D locally anisotropic spacetime provided with local space coordinates $x^1 = r$, $x^2 = \theta$ when as the anisotropic direction is chosen the time like coordinate, $y = t$. We proved (see the Paper I) that a d–metric of type (3),

$$
\delta s^2 = \Omega^2 (r, \theta) \left[ a(r)dr^2 + b(r, \theta) d\theta^2 + h(r, \theta)\delta t^2 \right],
$$

where

$$
\delta t = dt + w_1 (r, \theta) dr + w_2 (r, \theta) d\theta, \\
w_1 = \partial_r \ln |\ln \Omega|, w_2 = \partial_\theta \ln |\ln \Omega|,
$$

for $\Omega^2 = \pm h(r, \theta)$, satisfies the system of vacuum locally anisotropic gravitational equations with cosmological constant $\Lambda_{[0]}$,

$$
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R - \Lambda_{[0]} g_{\alpha\beta} = 0
$$

if

$$
a (r) = 4 r^2 |\Lambda_0|, b(r, \theta) = \frac{4}{|\Lambda_0|} \Lambda^2 (\theta) \left[ r^2_+ (\theta) - r^2 \right]^2
$$

and

$$
h (r, \theta) = - \frac{4}{|\Lambda_0| r^2} \Lambda^3 (\theta) \left[ r^2_+ (\theta) - r^2 \right]^3.
$$

The functions $a(r), b(r, \theta)$ and $h(x^i, y)$ and the coefficients of nonlinear connection $w_i (r, \theta, t)$ (for this class of solutions being arbitrary prescribed functions) were defined as to have compatibility with the locally isotropic limit.

We construct a black hole like solution with elliptical horizon $r^2 = r^2_+ (\theta)$, on which the function (5) vanishes if we chose

$$
r^2_+ (\theta) = \frac{p^2}{[1 + \varepsilon \cos \theta]^2},
$$

where $p$ is the ellipse parameter and $\varepsilon$ is the eccentricity. We have to identify

$$
p^2 = r^2_{+[0]} = -M_0 / \Lambda_0,
$$

where $r_{+[0]}, M_0$ and $\Lambda_0$ are respectively the horizon radius, mass parameter and the cosmological constant of the non–rotating BTZ solution [1] if we wont to have a connection with locally isotropic limit with $\varepsilon \to 0$. In the simplest case we can consider that the elliptic horizon (3) is modeled by an anisotropic mass

$$
M (\theta, \varepsilon) = \frac{M_0}{2 \pi (1 + \varepsilon \cos (\theta - \theta_0))^2} = \frac{r^2_+}{2 \pi}
$$

(7)
and constant effective cosmological constant, $\Lambda(\theta) \simeq \Lambda_0$. The coefficient $2\pi$ was introduced in order to have the limit

$$\lim_{\varepsilon \to 0} 2 \int_{0}^{\pi} M(\theta, \varepsilon) d\theta = M_0.$$  \hspace{1cm} (8)

Throughout this paper, the units $c = \hbar = k_B = 1$ will be used, but we shall consider that for an locally anisotropically renormalized gravitational constant $8G_{(gr)}^{(a)} \neq 1$, see [26].

3 On the Thermodynamics of Elliptical Black Holes

In this paper we will be interested in thermodynamics of locally anisotropic black holes defined by a d–metric (1).

The Hawking temperature $T(\theta, \varepsilon)$ of a locally anisotropic black hole is anisotropic and is computed by using the anisotropic mass (7):

$$T(\theta, \varepsilon) = \frac{M(\theta, \varepsilon)}{2\pi r_+ (\theta, \varepsilon)} = \frac{r_+ (\theta, \varepsilon)}{4\pi^2} > 0. \hspace{1cm} (9)$$

The two parametric analog of the Bekenstein–Hawking entropy is to be defined as

$$S(\theta, \varepsilon) = 4\pi r_+ = \sqrt{32\pi^3 M(\theta, \varepsilon)} \hspace{1cm} (10)$$

The introduced thermodynamic quantities obey the first law of thermodynamics (under the supposition that the system is in local equilibrium under the variation of parameters $(\theta, \varepsilon)$)

$$\Delta M(\theta, \varepsilon) = T(\theta, \varepsilon) \Delta S, \hspace{1cm} (11)$$

where the variation of entropy is

$$\Delta S = 4\pi \Delta r_+ = 4\pi \frac{1}{\sqrt{M(\theta, \varepsilon)}} \left( \frac{\partial M}{\partial \theta} \Delta \theta + \frac{\partial M}{\partial \varepsilon} \Delta \varepsilon \right).$$

According to the formula $C = (\partial m/\partial T)$ we can compute the heat capacity

$$C = 2\pi r_+ (\theta, \varepsilon) = 2\pi \sqrt{M(\theta, \varepsilon)}.$$ 

Because of $C > 0$ always holds the temperature is increasing with the mass.

The formulas (7)–(11) can be integrated on angular variable $\theta$ in order to obtain some thermodynamic relations for black holes with elliptic horizon depending only on deformation parameter, the eccentricity $\varepsilon$.

For a elliptically deformed black hole with the outer horizon $r_+$ given by formula (10) the depending on eccentricity [26] Bekenstein–Hawking entropy is computed as

$$S^{(a)}(\varepsilon) = \frac{L_+}{4G_{(gr)}^{(a)}},$$
were
\[ L_+(\varepsilon) = 4 \int_0^{\pi/2} r_+(\theta, \varepsilon) \, d\theta \]
is the length of ellipse’s perimeter and \( G^{(a)}_{(gr)} \) is the three dimensional gravitational coupling constant in locally anisotropic media (the index \((a)\) points to locally anisotropic renormalizations), and has the value
\[ S^{(a)}(\varepsilon) = \frac{2p}{G^{(a)}_{gr} \sqrt{1 - \varepsilon^2}} \arctg \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}. \] (12)

If the eccentricity vanishes, \( \varepsilon = 0 \), we obtain the locally isotropic formula with \( p \) being the radius of the horizon circumference, but the constant \( G^{(a)}_{(gr)} \) could be locally anisotropic renormalized.

The total mass of a locally anisotropic black hole of eccentricity \( \varepsilon \) is found by integrating (7) on angle \( \theta \):
\[ M(\varepsilon) = M_0 \left(1 - \varepsilon^2\right)^{3/2} \] (13)
which satisfies the condition (8).

The integrated on angular variable \( \theta \) temperature \( T(\varepsilon) \) is to be defined by using \( T(\theta, \varepsilon) \) from (9),
\[ T(\varepsilon) = 4 \int_0^{\pi/2} T(\theta, \varepsilon) \, d\theta = \frac{2\sqrt{M_0}}{\pi^2 \sqrt{1 - \varepsilon^2}} \arctg \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}. \] (14)

Formulas (12)–(14) describe the thermodynamics of \( \varepsilon \)-deformed black holes.

Finally, in this section, we note that a black hole with elliptic horizon is to be considered as a thermodynamic subsystem placed into the anisotropic ether bath of spacetime. To the locally anisotropic ether one associates a continuous locally anisotropic medium assumed to be in local equilibrium. The locally anisotropic black hole subsystem is considered as a subsystem described by thermodynamic variables which are continuous field on variables \((\theta, \varepsilon)\), or in the simplest case when one have integrated on \( \theta \), on \( \varepsilon \). It will be our first task to establish some parametric thermodynamic relations between the mass \( m(\theta, \varepsilon) \) (equivalently, the internal locally anisotropic black hole energy), temperature \( T(\theta, \varepsilon) \) and entropy \( S(\theta, \varepsilon) \).

4 Thermodynamic Metrics and Curvatures of Anisotropic Black Holes

We emphasize in this paper two approaches to the thermodynamic geometry of nearly equilibrium locally anisotropic black holes based on their thermodynamics. The first one is to consider the thermodynamic space as depending locally on two parameters \( \theta \) and \( \varepsilon \) and to compute the corresponding metric and curvature following standard formulas from
curved bidimensional hypersurface Riemannian geometry. The second possibility is to take as basic the Ruppeiner metric in the thermodynamic space with coordinates \((M, \varepsilon)\), in a manner proposed in Ref. [7] with that difference that as the extensive coordinate is taken the black hole eccentricity \(\varepsilon\) (instead of the usual angular momentum \(J\) for isotropic \((2 + 1)\)-black holes). Of course, in this case we shall background our thermodynamic geometric constructions starting from the relations (12)–(14).

4.1 The thermodynamic parametric geometry

Let us consider the thermodynamic parametric geometry of the elliptic \((2 + 1)\)-dimensional black hole based on its thermodynamics given by formulas (7)–(11).

Rewriting equations (11), we have

\[
\Delta S = \beta(\theta, \varepsilon) \Delta M(\theta, \varepsilon),
\]

where \(\beta(\theta, \varepsilon) = 1/T(\theta, \varepsilon)\) is the inverse to temperature (9). This case is quite different from that from [4, 5] where there are considered, respectively, BTZ and dilaton black holes (by introducing Ruppeiner and Weinhold thermodynamic metrics). Our thermodynamic space is defined by a hypersurface given by parametric dependencies of mass and entropy. Having chosen as basic the relative entropy function,

\[
\varsigma = \frac{S(\theta, \varepsilon)}{4\pi\sqrt{M_0}} = \frac{1}{1 + \varepsilon \cos \theta},
\]

in the vicinity of a point \(P = (0, 0)\), when, for simplicity, \(\theta_0 = 0\), our hypersurface is given locally by conditions

\[
\varsigma = \varsigma(\theta, \varepsilon) \text{ and } \text{grad}_{P}\varsigma = 0.
\]

For the components of bidimensional metric on the hypersurface we have

\[
\begin{align*}
g_{11} &= 1 + \left(\frac{\partial \varsigma}{\partial \theta}\right)^2, \\
g_{12} &= \left(\frac{\partial \varsigma}{\partial \theta}\right)\left(\frac{\partial \varsigma}{\partial \varepsilon}\right), \\
g_{22} &= 1 + \left(\frac{\partial \varsigma}{\partial \varepsilon}\right)^2,
\end{align*}
\]

The nonvanishing component of curvature tensor in the vicinity of the point \(P = (0, 0)\) is

\[
R_{1212} = \frac{\partial^2 \varsigma}{\partial \theta^2} \frac{\partial^2 \varsigma}{\partial \varepsilon^2} - \left(\frac{\partial^2 \varsigma}{\partial \varepsilon \partial \theta}\right)^2
\]

and the curvature scalar is

\[
R = 2R_{1212}. \quad (15)
\]

By straightforward calculations we can find the condition of vanishing of the curvature (15) when

\[
\varepsilon = -1 \pm \frac{(2 - \cos^2 \theta)}{\cos \theta (3 - \cos^2 \theta)}.
\]
So, the parametric space is separated in subregions with elliptic eccentricities $0 < \varepsilon_{\pm} < 0$ and $\theta$ satisfying conditions (16).

Ruppeiner suggested that the curvature of thermodynamic space is a measure of the smallest volume where classical thermodynamic theory based on the assumption of a uniform environment could conceivably work and that near the critical point it is expected this volume to be proportional to the scalar curvature $[20]$. There were also proposed geometric equations relating the thermodynamic curvature via inverse relations to free energy. Our definition of thermodynamic metric and curvature in parametric spaces differs from that of Ruppeiner or Weinhold and it is obvious that relations of type (16) (stating the conditions of vanishing of curvature) could be related with some conditions for stability of thermodynamic space under variations of eccentricity $\varepsilon$ and anisotropy angle $\theta$. This interpretation is very similar to that proposed by Janyszek and Mrugala $[13]$ and supports the viewpoint that the first law of thermodynamics makes a statement about the first derivatives of the entropy, the second law is for the second derivatives and the curvature is a statement about the third derivatives. This treatment holds good also for the parametric thermodynamic spaces for locally anisotropic black holes.

4.2 Thermodynamic Metrics and Eccentricity of Black Hole

A variant of thermodynamic geometry of locally anisotropic black holes could be grounded on integrated on anisotropy angle $\theta$ formulas (12)–(14). The Ruppeiner metric of elliptic black holes in coordinates $(M, \varepsilon)$ is

$$ds^2 = -\left(\frac{\partial^2 S}{\partial M^2}\right)_\varepsilon dM^2 - \left(\frac{\partial^2 S}{\partial \varepsilon^2}\right)_M d\varepsilon^2.$$  \hspace{1cm} (17)

For our further analysis we shall use dimensionless values $\mu = M(\varepsilon)/M_0$ and $\zeta = S^{(a)} G_{gr}^{(a)}/2p$ and consider instead of (17) the thermodynamic diagonal metrics $g_{ij}(a_1, a_2) = g_{ij}(\mu, \varepsilon)$ with components

$$g_{11} = -\frac{\partial^2 \zeta}{\partial \mu^2} = -\zeta_{,11} \quad \text{and} \quad g_{22} = -\frac{\partial^2 \zeta}{\partial \varepsilon^2} = -\zeta_{,22},$$  \hspace{1cm} (18)

where by commas we have denoted partial derivatives.

The expressions (12) and (13) are correspondingly rewritten as

$$\zeta = \frac{1}{\sqrt{1 - \varepsilon^2}} \arctan \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}$$

and

$$\mu = (1 - \varepsilon^2)^{-3/2}.$$

By straightforward calculations we obtain

$$\zeta_{,11} = -\frac{1}{9} (1 - \varepsilon^2)^{5/2} \arctan \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}} + \frac{1}{9\varepsilon} (1 - \varepsilon^2)^3 + \frac{1}{18\varepsilon^2} (1 - \varepsilon^2)^4$$
and
\[
\zeta_{22} = \frac{1 + 2\varepsilon^2}{(1 - \varepsilon^2)^{5/2}}\arctg\sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}} - \frac{3\varepsilon}{(1 - \varepsilon^2)^2}.
\]

The thermodynamic curvature of metrics of type (18) can be written in terms of second and third derivatives \[13\] by using third and second order determinants:
\[
R = \frac{1}{2} \begin{vmatrix}
-\zeta_{11} & 0 & -\zeta_{22} \\
-\zeta_{111} & -\zeta_{112} & 0 \\
-\zeta_{112} & 0 & -\zeta_{222}
\end{vmatrix} \times \begin{vmatrix}
-\zeta_{11} & 0 \\
0 & -\zeta_{22}
\end{vmatrix}^{-2}
= -\frac{1}{2} \left(\frac{1}{\zeta_{11}}\right)_2 \times \left(\frac{\zeta_{11}}{\zeta_{22}}\right)_2.
\]

The conditions of vanishing of thermodynamic curvature (19) are as follows
\[
\zeta_{112}(\varepsilon_1) = 0 \quad \text{or} \quad \left(\frac{\zeta_{11}}{\zeta_{22}}\right)_2(\varepsilon_2) = 0
\]
for some values of eccentricity, \(\varepsilon = \varepsilon_1\) or \(\varepsilon = \varepsilon_2\), satisfying conditions \(0 < \varepsilon_1 < 1\) and \(0 < \varepsilon_2 < 1\). For small deformations of black holes, i.e. for small values of eccentricity, we can approximate \(\varepsilon_1 \approx 1/\sqrt{5.5}\) and \(\varepsilon_2 \approx 1/(18\lambda)\), where \(\lambda\) is a constant for which \(\zeta_{11} = \lambda\zeta_{22}\) and the condition \(0 < \varepsilon_1 < 1\) is satisfied. We omit general formulas for curvature (19) and conditions (20), when the critical points \(\varepsilon_1\) and/or \(\varepsilon_2\) must be defined from nonlinear equations containing \(\arctg\sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}\) and powers of \((1 - \varepsilon^2)\) and \(\varepsilon\).

5 Discussion and Conclusions

In closing, we would like to discuss the meaning of geometric thermodynamics following from locally anisotropic black holes.

(1) Nonequilibrium thermodynamics of locally anisotropic black holes in locally anisotropic spacetimes. In this paper and in the Paper I [20] we concluded that the thermodynamics in locally anisotropic spacetimes has a generic nonequilibrium character and could be developed in a geometric fashion following the approach proposed by S. Sieniutycz, P. Salamon and R. S. Berry [22, 21]. This is a new branch of black hole thermodynamics which should be based on locally anisotropic nonequilibrium thermodynamics and kinetics [27].

(2) Locally Anisotropic Black holes thermodynamics in vicinity of equilibrium points. The usual thermodynamical approach in the Bekenstein–Hawking manner is valid for anisotropic black holes for a subclass of such physical systems when the hypothesis of local equilibrium is physically motivated and corresponding renormalizations, by locally anisotropic spacetime parameters, of thermodynamical values are defined.

(3) The geometric thermodynamics of locally anisotropic black holes with constant in time elliptic horizon was formulated following two approaches: for a parametric thermodynamic space depending on anisotropy angle \(\theta\) and eccentricity \(\varepsilon\) and in a standard Ruppeiner–Mrugala–Janyszek fashion, after integration on anisotropy \(\theta\) but maintaining locally anisotropic spacetime deformations on \(\varepsilon\).
(4) The thermodynamic curvatures of locally anisotropic black holes were shown to have critical values of eccentricity when the scalar curvature vanishes. Such type of thermodynamical systems are rather unusual and a corresponding statistical model is not that for ordinary systems composed by classical or quantum like gases.

(5) Thermodynamic systems with constraints require a new geometric structure in addition to the thermodynamical metrics which is that of nonlinear connection. We note this object must be introduced both in spacetime geometry and in thermodynamic geometry if generic anisotropies and constrained field and/or thermodynamic behavior are analyzed.

Acknowledgements

The authors are grateful to D. Singleton, Radu Miron and M. Anastasiei for help and collaboration. The S. V. work is supported both by a 2000–2001 California State University Legislative Award and a NATO/Portugal fellowship grant at the Instituto Superior Tecnico, Lisboa.

References

[1] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849
M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D 48 (1993) 1506.

[2] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. 31, 161 (1973).

[3] W. Barthel, J. Reine Angew. Math. 212, 120 (1963).

[4] J. D. Bekenstein, Phys. Rev. D 7, 949, 2333 (1973).

[5] W. Blaschke, Vorlesungen uber Differential Geometrie, Vol. II (Springer, Berlin, 1923); reprinted in 1967 by Chelsea Publishing company, New York.

[6] H. A. Buchdahl, The Concepts of Classical Thermodynamics (Cambridge University, New York, 1966).

[7] Rong–Gen Cai and Jin–Ho Cho, Phys. Rev. D60 (1999) 067502.

[8] E. Cartan, Les Espaces de Finsler (Herman, Paris, 1934).

[9] S. Ferrara, G. W. Gibbons and R. Kallosh, Nucl. Phys. B500, 75 (1997).

[10] P. Glansdorff and I. Progogine, Thermodynamic Theory of Structure, Stability and Fluctuations (Wiley–Interscience, London, 1971).

[11] S. W. Hawking and C. F. R. Ellis, The Large Scale Structure of Spacetime (Cambridge University Press, 1973).
[12] S. Hawking, Nature **248**, 30 (1974); Commun. Math. Phys. **43**, 199 (1975).

[13] H. Janyszek and R. Mrugala, Phys. Rev. A **39**, 6515 (1989); J. Phys. A **23**, 467 (1990).

[14] H. Janyszek and R. Mrugala, in *Advances in Thermodynamics, vol 3*, eds S. Sieniutycz and P. Salamon (Taylor & Francis, London, 1990).

[15] Yu. L. Klimontovichi, *Statistical Physics* (Nauka, Moscow, 1982) [in Russian].

[16] H. J. Kreuzer, *Nonequilibrium Thermodynamics and its Statistical Foundations* (Clarendon Press, Oxford, 1981).

[17] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications* (Kluwer Academic Publishers, Dordrecht, Boston, London, 1994).

[18] R. Mrugala, J. D. Nulton, J. C. Schon, and P. Salamon, Phys. Rev. A **41**, 3156 (1990).

[19] G. Ruppeiner, Phys. Rev. A **20**, 1608 (1979).

[20] G. Ruppeiner, Rev. Mod. Phys. **67** (1995) 605; **68** (1996) 313 (E).

[21] P. Salamon and R. Berry, Phys. Rev. Lett. **51**, 1127 (1983).

[22] S. Sieniutycz and R. R. Berry, Phys. Rev. A **43**, 2807 (1991); A **40**, 348 (1989).

[23] J. L. Synge, *Relativity: General Theory* (North–Holland, 1966).

[24] S. Vacaru, Ann. Phys. (NY) **256**, 39 (1997); Nucl. Phys. **B434**, 590 (1997); Phys. Lett. B **498**, 74 (2001).

[25] S. Vacaru, gr–qc/ 0001020; JHEP **0104**, 009 (2001) Ann. Phys. (NY) **290**, 83 (2001); S. Vacaru, et al., Phys. Lett. B **519**, 249 (2001); S. Vacaru and F. C. Popa, Class. Quant. Gravity. **18**, 4921 (2001); S. Vacaru and O. Tintareanu–Mircea, Nucl. Phys. B626 (2002) 239; S. Vacaru and D. Singleton, J. Math. Phys. **43**, 2486 (2002); S. Vacaru and D. Singleton., Class. Quant. Gravity. **19**, 2793 (2002).

[26] S. Vacaru, P. Stavrinou and E. Gaburov, Anholonomic Triads and New Classes of (2+1)–Dimensional Black Hole Solutions, gr–qc/ 0106068.

[27] S. Vacaru, Ann. Phys. (NY) **290**, 83 (2001); Ann. Phys. (Leipzig) **11**, 5 (2000).

[28] F. Weinhold, J. Chem. Phys. **63**, 2479, 2484, 2488, 2496 (1975); **65**, 559 (1976).