Dual Path Integral
a non-perturbative approach to strong coupling

Vitaly Vanchurin
Department of Physics, University of Minnesota, Duluth, Minnesota, 55812
Duluth Institute for Advanced Study, Duluth, Minnesota, 55804
E-mail: vvanchur@d.umn.edu

Abstract. We develop a non-perturbative method for calculating partition functions of strongly coupled quantum mechanical systems with interactions between subsystems described by a path integral of a dual system. The dual path integral is derived starting from non-interacting subsystems at zeroth order and then by introducing couplings of increasing complexity at each order of an iterative procedure. These orders of interactions play the role of a dual time and the full quantum partition function is expressed as a transition amplitude in the dual system. More precisely, it is expressed as a path integral from a deformation-operators dependent initial state at zero time/order to the inverse-temperature dependent final state at later time/order. We provide three examples of strongly coupled systems with first-order, second-order and higher-order interactions and discuss a possible emergence of space-time, quantum field theories and general relativity in context of the dual path integral.
1 Introduction

There are two main puzzles in theoretical physics which so far have no satisfactory resolutions. The first one is how to calculate observables in strongly coupled theories and the second one is how to derive general relativity from quantum mechanics, i.e. the problem of quantum gravity. While the former puzzle is purely mathematical, as it is usually well defined although the analytical calculations can be difficult to carry out, the latter puzzle is more physical than mathematical, as it is not even clear what is the right problem to solve. Of course, it would be nice if the two problems were actually related and in fact there are evidences suggesting that it might be the case. For example, in context of the AdS/CFT correspondence [1, 2], general relativity in the bulk emerges from a quantum theory on the boundary and at the same time observables of a strongly coupled conformal field theory can be obtained by performing perturbative calculations on the anti-de Sitter background. Unfortunately, the space-time we live in is not anti-de Sitter and so the AdS/CFT conjecture is at most an educated guess of what one might expect to see in a true theory of quantum gravity.

In this paper our main motivation is to study the emergence of space-time and general relativity from strongly coupled theories beyond the AdS/CFT duality, but the initial task will be a lot more modest. What we want first is to understand how a fully interacting quantum mechanical partition function can be obtained from the partition functions of non-interacting subsystems. In Ref. [3] we constructed one such transformation described by a second order differential equation for the “child” partition function with initial conditions specified by a “parent” partition function. In certain cases the parent partition function can describe non-interacting subsystems (see Sec. 3), but the framework was not rich enough to handle more complex systems including many strongly coupled systems. In this paper we will extend the analysis to a lot more general transformations (and thus more general interactions) described by a path integral with initial state specified by non-interacting partition functions (See Sec. 5). By construction the analysis is non-perturbative with couplings of different orders of complexity introduced at different orders of an iterative procedure.
At a very minimum, the dual description can be viewed as a non-perturbative method for calculating partition functions of strongly coupled quantum mechanical systems such as spin chain models (see Sec. 7), but more ambitiously it may also tell us something new about the quantum origin of the space-time and gravity. In Ref. [4] we argued that the anti-commutativity of quantum operators may be responsible for the emergence of space-time and in this paper we see that essentially the same phenomena occurs for more general quantum systems (see Sec. 3). Of course, it is premature to claim that we have derived the fully non-perturbative equations of general relativity (see, however, some promising recent ideas in context of emergent gravity [5–7]), but some progress in this directions can be made. In particular, we will argue that the fully interacting quantum partition function can often be represented by a dual path integral of a local quantum filed theory (See Sec. 6) where the perturbative methods are of value. With this respect the proposed framework can be considered as a generalization of the AdS/CFT duality mapping to more general quantum systems. Even more generally, the dual path integral may differ from the Feynman path integral and then some significant deviations from the quantum field theory framework are expected.

The paper is organized as follows. In the next section we construct a quantum system with a particular type of interactions which can be modeled with an additional (or first order) system. In Sec. 3 we consider a simple system with first-order interactions which can be described through a transformation of an extended partition function in an emergent space-time. In Sec. 4 we obtain an analytically continued expression of the the partition function which is used in Sec. 5 to derive the dual path integral representation of a fully interacting partition function. In Sec. 6 we argue that the local quantum field theory may emerge in certain limits of the dual path integral, but some significant deviations from the Feynman path integral are also expected. In Sec. 7 we demonstrate how the dual description of a spin chain model can be used to study strongly coupled systems. In Sec. 8 we summarize and discuss the main results of the paper, i.e. the dual path integral and emergent phenomena.

2 First-order interactions

Consider a quantum system described by a Hamiltonian operator

$$\hat{H}_0 \equiv \sum_{k=1}^{K} J_k^0 \hat{H}_0^k \tag{2.1}$$

where $J_k^0$’s are the real coefficients which parametrize deformations of the corresponding Hermitian operators $\hat{H}_0^k$’s. At finite temperature the system is described by the quantum partition function

$$Z[-\beta, J] = tr \left[ \exp \left( -\beta \sum_{k=1}^{K} J_k^0 \hat{H}_0^k \right) \right] \tag{2.2}$$

where $\beta$ is the inverse temperature. To simplify calculations in Sec. 4 we shall analytically continue the parameter $\beta$ to complex plane, but it will be assumed throughout the paper that we are dealing with a quantum system in thermal equilibrium. The time will eventually emerge in the dual descriptions first as a relativistic coordinate in Sec. 3 and later as a parameter in a path integral of Sec. 5 despite of the fact that the original quantum system is in a time-invariant thermal state.
In general the partition function (2.2) is difficult to calculate, but the analysis is greatly simplified if the operators \(\hat{H}_k^0\) describe Hamiltonian operators of non-interacting subsystems. This happens when the entire Hilbert space can be decomposed into a tensor product of Hilbert spaces
\[
\mathcal{H}_0 = \mathcal{H}_1^0 \otimes \mathcal{H}_2^0 \otimes \cdots \otimes \mathcal{H}_K^0
\]
with each of the operators \(\hat{H}_k^0\) acting non-trivially on the corresponding factor \(\mathcal{H}_k^0\) only. Then the full partition function can be expanded into a product,
\[
Z_0[-\beta, J] = \text{tr}_0 \left[ \exp \left( -\beta \sum_{k=1}^{K} J_k^0 \hat{H}_k^0 \right) \right]
= \prod_{k=1}^{K} \text{tr}_0^k \left[ \exp \left( -\beta J_k^0 \hat{H}_k^0 \right) \right]
= \prod_{k=1}^{K} Z_k^k[-\beta J_k], \tag{2.4}
\]
where \(\text{tr}_k^k[\_]\) is the trace over \(k\)'th subsystem and \(Z_k^k[-\beta J_k]\) is the partition function of the \(k\)'th subsystem.

Perhaps, more realistically, the Hamiltonian operator should contain interactions between subsystems. For example, a tensor product operator \(\hat{H}_j^0 \otimes \hat{H}_k^0\) acts non-trivially on two Hilbert space factors \(\mathcal{H}_j^0\) and \(\mathcal{H}_k^0\), but an inclusion of such operators would spoil the factorization of the partition function and could lead to the problem of strong coupling. In this paper we will take a somewhat different approach and to describe interactions between zeroth-order subsystems \(\mathcal{H}_k^0\)'s we will employ a collection of additional (or the first-order) operators \(\hat{H}_k^1, \ldots, \hat{H}_k^K\) which act non-trivially on an additional (or the first-order) system \(\mathcal{H}_1\) only. Then the full Hilbert space is a tensor product of the zeroth-order and first-order Hilbert spaces, i.e.
\[
\mathcal{H}_1 \otimes \mathcal{H}_0 = \mathcal{H}_1 \otimes \mathcal{H}_1^1 \otimes \mathcal{H}_2^0 \otimes \cdots \otimes \mathcal{H}_K^0.
\]
and an interacting Hamiltonian can be defined as
\[
\hat{H} = \sum_{k=1}^{K} \hat{H}_k^1 \otimes \hat{H}_k^0 J_k^0.
\]
This is certainly not the most general operator which can be defined in the Hilbert space (2.5), but it is general enough to enable us to study certain strongly coupled systems whose partition function can be expressed through a dual path integral.

Different operators \(\hat{H}_k^1 \otimes \hat{H}_k^0\) in (2.6) act non-trivially on the same factor of the Hilbert space, namely \(\mathcal{H}_1\), and as a result the interacting partition function
\[
Z[-\beta, J] = \text{tr} \left[ \exp \left( -\beta \sum_{k=1}^{K} \hat{H}_k^1 \otimes \hat{H}_k^0 J_k^0 \right) \right]
\]
(2.7)
does not factorize as was the case for non-interacting partition function (2.4). To derive a useful expression of the interacting partition function it will be convenient to first define the first-order partition function
\[
Z_1[-\beta, p_1, \ldots, p_K] = \text{tr}_1 \left[ \exp \left( -\beta \sum_{k=1}^{K} \hat{H}_k^1 p_k \right) \right]\tag{2.8}
\]
where \( tr_1[ ] \) is a trace over only the first-order system \( \mathcal{H}_1 \). The key observation is that for the system described by (2.6) the interacting partition function (2.7) can be expressed as

\[
\mathcal{Z}[-\beta, J] = tr_1 \left[ \exp \left( -\beta \sum_{k=1}^{K} \hat{H}_k \frac{\partial}{\partial x_0^k} \right) \prod_{k=1}^{K} \mathcal{Z}_0^k[x_0^k, J_k] \right]_{x_0^1 = \ldots = x_0^K = 0} \tag{2.9}
\]

or in terms of the first-order partition function (2.8) as

\[
\mathcal{Z}[-\beta, J] = Z_1 \left[ -\beta, \frac{\partial}{\partial x_0^1}, \ldots, \frac{\partial}{\partial x_0^K} \right] \prod_{k=1}^{K} \mathcal{Z}_0^k[x_0^k, J_k] \bigg|_{x_0^1 = \ldots = x_0^K = 0}. \tag{2.10}
\]

Note that such a representation of the partition function with operator \( Z_1 \bigg[ -\beta, \frac{\partial}{\partial x_0^1}, \ldots, \frac{\partial}{\partial x_0^K} \bigg] \) acting on a non-interacting partition function \( \prod_{k=1}^{K} \mathcal{Z}_0^k[x_0^k, J_k] \) was only possible because the zeroth-order subsystems \( \mathcal{H}_0^k \)'s were not coupled to each other directly, but through interactions in the first-order system as is evident from the Hamiltonian expression (2.6).

3 Extended partition function

In order to better understand the transformation (2.10) from a non-interacting partition function \( \prod_{k=1}^{K} \mathcal{Z}_0^k[x_0^k, J_k] \) to an interacting partition function partition function \( \mathcal{Z}[-\beta, J_1, \ldots, J_K] \), consider a first-order system described by anti-commuting first-order operators

\[
\{ \hat{H}_1^k, \hat{H}_1^k \} = 2 \delta_{jk} \hat{I}_1, \tag{3.1}
\]

where \( \hat{I}_1 \) is the identity operator in \( \mathcal{H}_1 \). The corresponding first-order partition function can be calculated by separating odd and even terms in the Taylor series expansion, i.e.

\[
\mathcal{Z}_1[-\beta, p_1, \ldots, p_K] = tr_1 \left( \exp \left( -\beta \sum_{k=1}^{K} \hat{H}_1^k p_k \right) \right) \tag{3.2}
\]

\[
= \sum_{n=0}^{\infty} tr_1 \left[ \frac{(-\beta)^n}{n!} \left( \sum_{k=1}^{K} \hat{H}_1^k p_k \right)^n \right] = \sum_{m=0}^{\infty} tr_1 \left( \frac{(-\beta)^{2m}}{(2m)!} \left( \sum_{k=1}^{K} \hat{H}_1^k p_k \right)^{2m} \right) + \sum_{m=0}^{\infty} tr_1 \left( \frac{(-\beta)^{2m+1}}{(2m+1)!} \left( \sum_{k=1}^{K} \hat{H}_1^k p_k \right)^{2m+1} \right).
\]

From the anti-commutation relation (3.1) we get

\[
\left( \sum_{k=1}^{K} \hat{H}_1^k p_k \right)^2 = \sum_{k=1}^{K} \hat{I}_1 p_k^2 \tag{3.3}
\]

which can be substituted into (3.2),

\[
\mathcal{Z}_1[-\beta, p_1, \ldots, p_K] = \sum_{m=0}^{\infty} tr_1 \left( \frac{(-\beta)^{2m}}{(2m)!} \left( \sum_{k=1}^{K} \hat{I}_1 p_k^2 \right)^{m} \right) + \sum_{m=0}^{\infty} tr_1 \left( \frac{(-\beta)^{2m+1}}{(2m+1)!} \left( \sum_{k=1}^{K} \hat{I}_1 p_k^2 \right)^{m} \left( \sum_{k=1}^{K} \hat{H}_1^k p_k \right) \right). \tag{3.4}
\]
If we also assume that the anti-commuting operators are traceless

\[ tr_1 \left( \hat{H}_1 \right) = 0 \]  

(3.5)

then the second term in (3.4) vanishes and the partition function is greatly simplified

\[
\mathcal{Z}_1[-\beta, p_1, ..., p_K] = tr_1 \left( \hat{I}_1 \right) \sum_{m=0}^{\infty} \frac{(-\beta)^{2m}}{(2m)!} \left( \sum_{k=1}^{K} p_k^2 \right)^m
\]

\[
= \mathcal{N}_1 \cosh \left( \beta \sum_{k=1}^{K} p_k^2 \right)
\]

(3.6)

where \( \mathcal{N}_1 \) is the trace of the identity operator \( \hat{I}_1 \) in the first-order system \( \mathcal{H}_1 \) (see Ref. [4] for details).

By combining (3.6) and (2.10) we obtain an equation for a fully interacting partition function

\[
\mathcal{Z}[-\beta, J] = \mathcal{N}_1 \cosh \left( \beta \sum_{k=1}^{K} \left( \frac{\partial}{\partial x_0^k} \right)^2 \right) \prod_{k=1}^{K} \mathcal{Z}_0^k[x^k, J_k]
\]

(3.7)

which can be calculated by following the analysis of Ref. [4]. Let us define an extended partition function

\[
z[-\beta, x^1, ..., x^K, J_1, ..., J_K] \equiv tr \left[ \exp \left( -\beta \sum_{k=1}^{K} \hat{H}_1^k \frac{\partial}{\partial x_k} \right) \right] \prod_{k=1}^{K} \mathcal{Z}_0^k[x^k, J_k]
\]

\[
= \mathcal{N}_1 \cosh \left( \beta \sum_{k=1}^{K} \left( \frac{\partial}{\partial x_0^k} \right)^2 \right) \prod_{k=1}^{K} \mathcal{Z}_0^k[x^k, J_k]
\]

(3.8)

that contains information about both interacting and non-interacting partition functions. For example, at \( \beta = 0 \) the function reduces to a product of non-interacting partition functions of the zeroth-order subsystems, i.e.

\[
z[0, x^1, ..., x^K, J_1, ..., J_K] = \mathcal{N}_1 \prod_{k=1}^{K} \mathcal{Z}_0^k[x^k, J_k]
\]

(3.9)

and at \( x^1 = ... = x^K = 0 \) the function reduces to the fully interacting partition function, i.e.

\[
z[-\beta, 0, ..., 0, J_1, ..., J_K] = \mathcal{Z}[-\beta, J_1, ..., J_K].
\]

(3.10)

Then one might wonder if it may be possible to derive a differential equation whose solution would be the entire extended partition function (including (3.10)) starting from initial conditions at \( \beta = 0 \), i.e. from (3.9).

The answer to the question is affirmative and the corresponding equation can be found by differentiating (twice) the extended partition function (3.8) with respect to the inverse.
temperature, i.e.

$$\frac{\partial^2}{\partial \beta^2} z[-\beta, x, J] = \left( \sum_{k=1}^{K} \left( \frac{\partial}{\partial x_k} \right)^2 \right) N_1 \cosh \left( \beta \sum_{k=1}^{K} \left( \frac{\partial}{\partial x_k} \right)^2 \right) \prod_{k=1}^{K} Z_0^k[x^k, J_k]$$

$$= \left( \sum_{k=1}^{K} \left( \frac{\partial}{\partial x_k} \right)^2 \right) z[-\beta, x, J]. \quad (3.11)$$

Then it is obvious that the extended partition function must be a solution of a relativistic wave equation

$$\Box z[\beta, x, J] = \left( \frac{\partial^2}{\partial \beta^2} - \sum_{k=1}^{K} \left( \frac{\partial}{\partial x_k} \right)^2 \right) z[\beta, x, J] = 0, \quad (3.12)$$

but since (3.12) is a second order differential equation in $\beta$, the correct initial conditions must be specified at $\beta = 0$ for both “position” $z[0, x, J]$ and “velocity” $\frac{\partial}{\partial \beta} z[0, x, J]$. These conditions can be deduced directly from the definition (3.8), i.e.

$$z[0, x, J] = N_1 \prod_{k=1}^{K} Z_0^k[x^k, J_k] \quad (3.13)$$

$$\frac{\partial}{\partial \beta} z[0, x, J] = 0. \quad (3.14)$$

Evidently, equation (3.12) describes a transformation from the initial state $z[0, x, J]$ specified by a non-interacting partition function to final state $z[\beta, x, J]$ which describes the fully interacting partition function $z[\beta, 0, J] = Z[-\beta, J]$ at inverse temperature $\beta$.

In the remainder of the paper we shall generalize this transformation to a lot more general couplings between subsystems, but before we proceed let us emphasize that the transformation is described by an equation which is relativistic (3.12). Recall that the original quantum system was in a time-invariant thermal state and it is only in the dual description the relativistic dynamics of the extended partition function emerged from a particular coupling between subsystems.

4 Analytic continuation

So far, we have analyzed some very specific first-order interactions between zeroth-order subsystems described by a quantum Hamiltonian (2.6). This allowed us to derive the corresponding interacting partition function (2.10) and to study a possible emergence of space-time (3.12). The next step is to generalize the construction to allow more general interactions between subsystems and then (if we are lucky) to study a possible emergence of general relativity from strong coupling. In the following section we will describe one such generalization by including the higher-order interactions, but it turns out that such interactions are a lot more transparent when expressed in terms of the analytically continued functions.

Consider the quantum partition function (2.7) analytically continued to complex plane, \footnote{The parameter $ix$ is purely imaginary, but it is assumed that at the end of the calculations it is to be analytically continued back to the negative real line, i.e. $-\beta$.}

$$Z[ix, J] = \text{tr} \left[ \exp \left( ix \sum_{k=1}^{K} \hat{H}^k \otimes \hat{H}_0^k J_k \right) \right]. \quad (4.1)$$
This function was shown to be given by the equation (2.10) which can also be written using the Dirac delta functions,

\[
Z[ix, J] = \int \left( \prod_{k=1}^{K} dx_0^k \delta(x_0^k) \right) Z_1 \left[ ix, \frac{\partial}{\partial x_0^1}, ..., \frac{\partial}{\partial x_0^K} \right] \prod_{k=1}^{K} Z_0^k[x_0^k, J_k]
= \int \left( \prod_{k=1}^{K} dx_0^k \delta(x_0^k) \right) tr_1 \left[ \exp \left( x \sum_{k=1}^{K} \hat{H}_1^k \frac{\partial}{\partial x_0^k} \right) \right] \prod_{k=1}^{K} Z_0^k[x_0^k, J_k]
= \int \left( \prod_{k=1}^{K} \frac{dx_0^k dp_1^k}{2\pi} e^{-ix_0^k p_1^k} \right) tr_1 \left[ \exp \left( x \sum_{k=1}^{K} \hat{p}_k \hat{H}_1^k \right) \right] \prod_{k=1}^{K} Z_0^k[x_0^k, J_k].
\]

Upon integration by parts and neglecting the (vanishing) boundary terms we get

\[
Z[ix, J] = \int \prod_{k=1}^{K} \frac{dx_0^k dp_1^k}{2\pi} \prod_{k=1}^{K} Z_0^k[x_0^k, J_k] tr_1 \left[ \exp \left( -x \sum_{k=1}^{K} \hat{H}_1^k \frac{\partial}{\partial x_0^k} \right) \right] e^{-i \sum_{k=1}^{K} x_0^k p_1^k}
= \int \prod_{k=1}^{K} \frac{dx_0^k dp_1^k}{2\pi} \prod_{k=1}^{K} Z_0^k[x_0^k, J_k] tr_1 \left[ \exp \left( ix \sum_{k=1}^{K} \hat{p}_k \hat{H}_1^k \right) \right] e^{-i \sum_{k=1}^{K} x_0^k p_1^k}
= \int \prod_{k=1}^{K} \frac{dx_0^k dp_1^k}{2\pi} \prod_{k=1}^{K} Z_0^k[x_0^k, J_k] e^{-i \sum_{k=1}^{K} x_0^k p_1^k} Z_1[ix, p_1^1, ..., p_1^K].
\]

Just like before (2.10) the new equation (4.3) describes a transformation from a zeroth-order partition function of non-interacting subsystems \(\prod_{k=1}^{K} Z_0^k[x_0^k, J_k]\) (in the Hilbert space \(\mathcal{H}_0\)) to a fully interacting partition function \(Z[ix, J]\) (in the Hilbert space \(\mathcal{H}_1 \otimes \mathcal{H}_0\)) with interactions described by the first-order partition function \(Z_1[ix, p_1^1, ..., p_1^K]\). For example, the analytically continued partition function (3.7) can now be written as

\[
Z[ix, J] = \int \prod_{k=1}^{K} \frac{dx_0^k dp_1^k}{2\pi} \prod_{k=1}^{K} Z_0^k[x_0^k, J_k] e^{-i \sum_{k=1}^{K} x_0^k p_1^k} N_1 \cos \left( x \sum_{k=1}^{K} \left( p_1^k \right)^2 \right).
\]

The key idea is, whenever possible, to expand \(Z_1[ix, p_1^1, ..., p_1^K]\) into a product of the first-order non-interacting partition functions \(Z_1^j[ix, p_1^1, ..., p_1^K]\) similarly to how the zeroth-order partition function was expanded into a product of the zeroth-order non-interacting partition functions (2.4). Then the combined first-order partition function is

\[
Z_1[ix, p_1^1, ..., p_1^K] = tr \left[ \exp \left( ix \sum_{j=1}^{K} \sum_{k=1}^{K} \hat{H}_1^j p_k \right) \right]
= \prod_{j=1}^{K} tr_1^j \left[ \exp \left( ix \sum_{k=1}^{K} \hat{H}_1^j p_k \right) \right]
= \prod_{j=1}^{K} Z_1^j[ix, p_1^1, ..., p_1^K].
\]
single factor $\mathcal{H}^k_0$ for all $k$. To simplify calculations, but without losing the generality, we assume that the number of factors in the first-order Hilbert space is once again $K$ and the full Hilbert space is

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_0 = \left( \bigotimes_{j=1}^{K} \mathcal{H}_1^j \right) \left( \bigotimes_{k=1}^{K} \mathcal{H}_0^k \right).$$

(4.6)

The factorization of the first-order partition function (4.5) can be substituted into (4.3) to obtain

$$Z[ix,J] = \text{tr}_1 \left[ \exp \left( ix \sum_{k=1}^{K} \left( \sum_{j=1}^{K} \hat{H}^j_1 \otimes \hat{H}^k_0 J_{jk}^0 \right) \right) \right]$$

$$= \int \Pi_{k=1}^{K} \frac{dx_k^0 dp_k^1}{2\pi} \Pi_{k=1}^{K} \mathcal{Z}_0^k[ix_0^k, J_k] e^{-i\sum_{k=1}^{K} x_k^0 p_k^1} \Pi_{j=1}^{K} e^{-i\sum_{k=1}^{K} x_k^1 p_k^2} \mathcal{Z}_1^j[ix_1^k, p^1] \mathcal{Z}_2^k[ix,p^2].$$

(4.7)

This is the most general partition function with interactions between zeroth-order subsystems $\mathcal{H}^k_0$'s described by the first-order subsystems $\mathcal{H}^k_1$'s which are not yet interacting.

5 Dual path integral

To derive a path integral with higher-order interactions all that we have to do is to iterate the procedure developed above. For example, the second-order interactions between the first-order subsystems $\mathcal{H}^k_1$'s can be introduced by employing a second-order system $\mathcal{H}_2$ which (for starters) can be assumed to factor into a tensor product of non-interacting second-order subsystems $\mathcal{H}^k_2$'s. Then the full Hilbert space consists of a tensor product of the zeroth-, first- and second-order systems, i.e.

$$\mathcal{H} = \mathcal{H}_2 \otimes \mathcal{H}_1 \otimes \mathcal{H}_0 = \left( \bigotimes_{l=1}^{K} \mathcal{H}_2^l \right) \left( \bigotimes_{j=1}^{K} \mathcal{H}_1^j \right) \left( \bigotimes_{k=1}^{K} \mathcal{H}_0^k \right).$$

(5.1)

By analogy with the first-order operators we define the second-order operators $\hat{H}^{ij}_2$ which act non-trivially on only a single factor $\mathcal{H}^l_2$ of the Hilbert space. Then the full partition function can be obtained trivially from (4.7) by adding the second order of interactions, i.e.

$$Z[ix,J] = \text{tr} \left[ \exp \left( ix \sum_{l=1}^{K} \sum_{j=1}^{K} \sum_{k=1}^{K} \hat{H}^{ij}_2 \otimes \hat{H}^j_1 \otimes \hat{H}^k_0 J_{jk}^0 \right) \right]$$

$$= \int \Pi_{k=1}^{K} \frac{dx_k^0 dp_k^1 dx_k^1 dp_k^2}{(2\pi)^2} \Pi_{k=1}^{K} \mathcal{Z}_0^k[ix_0^k, J_k] e^{-i\sum_{k=1}^{K} x_k^0 p_k^1} \Pi_{j=1}^{K} \mathcal{Z}_1^j[ix_1^k, p^1] e^{-i\sum_{k=1}^{K} x_k^1 p_k^2} \Pi_{k=1}^{K} \mathcal{Z}_2^k[ix,p^2].$$

(5.2)

This is now the partition function with up to the second order of interactions complexity, but we can keep going and introduce third-order interactions, fourth-order interactions etc.

To describe a fully interacting partition function (with up to some finite order $N$ of interactions) we define a tensor product Hilbert space

$$\mathcal{H} = \bigotimes_{n=0}^{N} \mathcal{H}_n = \bigotimes_{n=0}^{N} \bigotimes_{k=1}^{K} \mathcal{H}^k_n,$$
Consequently, the operators $\hat{H}^j_{kn}$ are assumed to act non-trivially on only respective Hilbert space factors $H^j_n$. From this point on it will be useful to borrow some terminology from the textbook quantum mechanics, e.g. propagator, wave-functions, Hamiltonian, path integral, etc., although as we shall see the analogy is not exact. To avoid confusions and also to distinguish the approximate notions of the dual description from the standard terminology we shall be adding the word “dual” wherever appropriate, e.g. dual propagator, dual wave-functions, dual Hamiltonian, dual path integral, etc.

For example, it will be convenient to define a “dual Hamiltonian density” as

$$H^k_n(x, p_1, ..., p_K) \equiv xp_k + i \log \left( Z^k_n[ix, p_1, ..., p_K] \right).$$  \hspace{1cm} (5.4)

where the $n$'th order partition functions of non-interacting subsystems are given by

$$Z^k_n[ix, p_1, ..., p_K] = tr_n \left[ \exp \left( ix \sum_{j=1}^{K} \sum_{k=1}^{K} \hat{H}^j_{nk} p_k \right) \right]$$  \hspace{1cm} (5.5)

and $tr_n[\cdot]$ is a trace over $H^k_n$ factor. Then the combined partition function of all $n$'th order subsystems can be written as

$$Z_n[ix, p_1, ..., p_K] = tr_n \left[ \exp \left( ix \sum_{j=1}^{K} \sum_{k=1}^{K} \hat{H}^j_{nk} p_k \right) \right]$$  \hspace{1cm} (5.6)

where $tr_n[\cdot]$ is a trace over $H_n = \bigotimes_{k=1}^{K} H^k_n$. At this point it is not clear why (5.4) should be called the dual Hamiltonian density, but as we shall see shortly the choice is very well motivated.

To obtain a “dual path integral” expression for the fully interacting partition function we iterate the procedure which had led to (5.2). The final result is

$$Z[ix, J] \equiv \int d^K x_0 d^K x_N \Psi_{out}[x_N] K[x_N; x_0] \Psi_{in}[x_0]$$  \hspace{1cm} (5.7)

where the “dual propagator” is defined as

$$K[x_N; x_0] = \int \prod_{k=1}^{K} dp_k^N dx_1 dx_k^1 ... dx_{N-1} dx_n^k \frac{e^{-i \sum_{n=1}^{N} \sum_{k=1}^{K} (H^j_{nk}(x_n^p, p_n^1, ..., p_n^K) - (x_n^k - x_{n-1}^k)p_n^k)}}{(2\pi)^N}$$  \hspace{1cm} (5.8)

and the initial and final “dual wave-functions” as

$$\Psi_{in}[x_0^1, ..., x_0^K] \equiv \prod_{k=1}^{K} Z^k_0(x_0^k, J_k)$$  \hspace{1cm} (5.9)

$$\Psi_{out}[x_N^1, ..., x_N^K] \equiv \prod_{k=1}^{K} \delta(x - x_N^k).$$  \hspace{1cm} (5.10)
Figure 1. Interactions network of a dual path integral.

See Fig. 1 for an illustration of interactions in a generic dual path integral with boxes representing separate factors of the Hilbert space $H^k_n$, solid lines representing tensor products of operator from the respective factors and dashed lines connected to circles representing variables of the full partition function $Z[ix, J]$, i.e. $x$ and $J_1, ..., J_K$. Note that the dual path integral (5.7) can be interpreted as a transition amplitude of a dual system from the deformation-operators dependent initial state (5.9) to inverse-temperature dependent final state (5.10). This in a sharp contrast to the standard path integral representation of thermal partition functions where the inverse temperature plays the role of the size of extra dimension (see for example Refs. [8–10]).

6 Quantum field theories

The dual path integral expression (5.7) is exact, but it involves calculating a transition amplitude from initial state (5.9) to final state (5.10) in a system described by a “dual time”-dependent (or $n$-dependent) and “dual space”-dependent (or $k$-dependent) Hamiltonian density $H^k_n(x^k, p_1, ..., p_K)$. This can be considered as a generalization of the Feynman path integral for local quantum field theories where the Hamiltonian density is usually assumed to be the same everywhere in space and time. However, for more general geometries with dynamical gravitational degrees of freedom the Feynman path integral is not adequate and with this respect the dual path integral (5.7) may give us the desired definition of quantum gravity.

To demonstrate how a local quantum field theory can emerge from a dual path integral, consider a collection of $n$th order “local” operators $\hat{H}^j_n$ which are non-zero only for $j \in \{k-$
$\{\hat{H}_n^{k+1,k},\hat{H}_n^{k-1,k}\} = 2\delta_{ij}\hat{H}_n^{kk} = 2\delta_{ij}\hat{I}_n^k$ \hfill (6.1)

and a tracelessness condition

$$tr\left(\hat{H}_n^{jk}\right) = \delta_{jk} tr\left(\hat{H}_n^{kk}\right) = \delta_{jk} N_n^j.$$ \hfill (6.2)

then the $n$’th order partition function is a straightforward generalization of Eq. (3.6),

$$Z_n^k(ix^k,p_1,...,p_K) = \mathcal{N}_n^k e^{ix^k p_k^2 \cos\left(x^k \sqrt{p_{k-1}^2 + p_{k+1}^2}\right)}.$$ \hfill (6.3)

The corresponding dual Hamiltonian density (5.4) is purely imaginary

$$H_n^k(x^k,p_1,...,p_K) = i \log\left(\mathcal{N}_n^k \cos\left(x^k \sqrt{p_{k-1}^2 + p_{k+1}^2}\right)\right)$$ \hfill (6.4)

and as such cannot be interpreted as the standard Hamiltonian density of a local quantum field theory. However, recall that the variable $ix$ of the fully interacting quantum partition function must be analytically continued to a real line $-\beta$ (see Sec. 4). Then if we are only interested in a zero temperature limit, i.e. $\beta \to \infty$, the dominant contribution would come from only a negative frequency mode, i.e.

$$\cos\left(x^k \sqrt{p_{k-1}^2 + p_{k+1}^2}\right) = \frac{e^{ix^k \sqrt{p_{k-1}^2 + p_{k+1}^2}} + e^{-ix^k \sqrt{p_{k-1}^2 + p_{k+1}^2}}}{2} \to \frac{e^{-ix^k \sqrt{p_{k-1}^2 + p_{k+1}^2}}}{2}.$$ \hfill (6.5)
Therefore, for the purpose of describing ground states (i.e. $\beta \to \infty$) the local dual Hamiltonian density can be approximated as

$$H^k_n(x^k, p_1, \ldots, p_K) \approx i \log \left( \frac{N^k_n}{2} \exp \left( -ix^k \sqrt{p_{k-1}^2 + p_{k+1}^2} \right) \right) = x^k \sqrt{p_{k-1}^2 + p_{k+1}^2} + \text{const}$$

(6.6)

where the irrelevant constant does not produce any observable effects and can be dropped. As a result the dual Hamiltonian is the desired sum of the local dual Hamiltonian density terms

$$H_n[x^1, \ldots, x^K, p_1, \ldots, p_K] = \sum_{k=1}^K H^k_n(x^k, p_1, \ldots, p_K) = \sum_{k=1}^K x^k \sqrt{p_{k-1}^2 + p_{k+1}^2}$$

(6.7)

as is the case for local quantum field theories. This puts the corresponding dual path integral in the same form as the Feynman path integral for a local quantum field theory on a lattice,

$$Z[ix, J] \equiv \int \prod_{k=1}^K dx_k^0 dp_k^1 \ldots dp_N^0 dx_k^N \frac{N^N}{(2\pi)^N} \Psi_{\text{out}}[x_N] e^{-i \sum_{n=1}^N \sum_{k=1}^K (x^k \sqrt{p_{k-1}^2 + p_{k+1}^2} - (x^n_k - x^{n-1}_k)p^n_k)} \Psi_{\text{in}}[x_0]$$

(6.8)

On the other hand, if we would have kept both modes in (6.5), which would be appropriate for example at finite temperatures, then the dual Hamiltonian would not be real and the standard quantum mechanical interpretation would be lost. This is the limit where we expect to see some non-trivial deviations from the Feynman path integral and where the effects of quantum gravity should become important.

### 7 Spin chain model

In the previous sections we stated a possibility of using the dual path integral (5.7) to model or, more precisely, to define quantum gravity. This resonates well with the AdS/CFT proposal, but for the time being it also remains highly speculative. On a more practical level, it would be important to explore in greater details the strongly coupled quantum systems which can be solved using the dual description. In particular, are there any spin chain models whose partition functions can be expressed using the dual path integral representation developed in this paper?

To construct a semi-simple example of such a model it is sufficient to consider a dual path integral with only two orders of interactions, i.e. $N = 2$. See Fig. 3 for an illustration of interactions in the corresponding path integral with boxes representing factors of the Hilbert space $\mathcal{H}_k^k$ and solid lines representing tensor products of non-vanishing operator from the respective factors. As is evident from the figure, the only non-vanishing operators are $\hat{H}_1^{2k,2k}, \hat{H}_1^{2k,2k-1}, \hat{H}_2^{2k,2k}, \hat{H}_2^{2k,2k-2}$ which we assume to satisfy the following anti-commutation relations

$$\{ \hat{H}_1^{2k,2k}, \hat{H}_1^{2k,2k-1} \} = 0$$
$$\{ \hat{H}_2^{2k,2k}, \hat{H}_2^{2k,2k-2} \} = 0$$

(7.1)
where $K$ is even and all indices are periodic modulo $K$. These operators can be expressed in terms of spin operators

\begin{align}
\hat{H}^{2k,2k-1}_1 &= \hat{\sigma}^x_{2k-1} \\
\hat{H}^{2k,2k}_1 &= \hat{\sigma}^y_{2k-1} \\
\hat{H}^{2k,2k-2}_2 &= \hat{\sigma}^x_{2k} \\
\hat{H}^{2k,2k}_2 &= \hat{\sigma}^y_{2k}
\end{align}

(7.2) – (7.5)

where $\hat{\sigma}^x_k$ and $\hat{\sigma}^y_k$ are the anti-commuting Pauli spin operators on lattice sites $k = 1, ..., K$. If we (for simplicity) also assume that

\[\hat{H}^0_0 = \hat{I}_0\]

(7.6)

then the corresponding quantum system can be written as a 1D spin chain model with Hamiltonian

\[\hat{H} = \sum_{k=1}^{K/2} (J_{2k-1}\hat{\sigma}^x_{2k-1} + J_{2k}\hat{\sigma}^y_{2k-1}) (\hat{\sigma}^x_{2k-2} + \hat{\sigma}^y_{2k})\]

(7.7)

As a spin chain model the system of Hamiltonian (7.7) is strongly coupled (for generic $J_k$’s), but as a dual path integral it can be easily solved. All that we have to do is to plug in the known partition functions of the zeroth-, first- and second-order subsystems

\begin{align}
Z^0_0[i\pi, p] &= \exp(i\pi p) \\
Z^{2k}_1[i\pi, p_1, p_2] &= \cos(x\sqrt{(p_1)^2 + (p_2)^2}) \\
Z^{2k}_2[i\pi, p_1, p_2] &= \cos(x\sqrt{(p_1)^2 + (p_2)^2})
\end{align}

(7.8) – (7.10)
$$Z[ix, J] = \int \prod_{k=1}^{K/2} \frac{dx_k dp_k}{(2\pi)^{3/2}} \prod_{k=1}^{K/2} dp_{2k} e^{i \sum_{k=1}^{K/2} x_k (J_k - p_k)} \prod_{k=1}^{K/2} Z_{1k}^{2k}[ix^{2k}, p^{2k}]$$

$$Z[-\beta, J] = \int \prod_{k=1}^{K/2} \frac{dx_k dp_k}{2\pi} \prod_{k=1}^{K/2} \cos \left( x_k \sqrt{J_{2k}^2 + J_{2k-1}^2} \right) e^{-i \sum_{k=1}^{K/2} x_k p_k} \prod_{k=1}^{K/2} \cosh \left( \beta \sqrt{p_k^2 + p_{k-1}^2} \right).$$

It is now straightforward to analytically continue $ix$ to $-\beta$ to obtain an exact expression for the fully interacting quantum partition function of a spin model (7.7),

$$Z[-\beta, J] = \int \prod_{k=1}^{K/2} \frac{dx_k dp_k}{2\pi} \prod_{k=1}^{K/2} \cos \left( x_k \sqrt{J_{2k}^2 + J_{2k-1}^2} \right) e^{-i \sum_{k=1}^{K/2} x_k p_k} \prod_{k=1}^{K/2} \cosh \left( \beta \sqrt{p_k^2 + p_{k-1}^2} \right).$$

This shows that the dual description can be used to study the strongly coupled systems including certain spin chain models.

8 Discussion

In this paper we simultaneously achieved two parallel results: derived a dual path integral representation of some strongly coupled systems (5.7) and then argued that the dual description may be responsible for the emergence of space-time, quantum field theories and gravity. While the first result is purely mathematical and should be viewed as a non-perturbative method for calculating partition functions, the second result is an attempt to study how the essential ingredients of any successful theory of gravity may emerge from a dual path integral representation of the partition function for strongly coupled systems. With this respect our approach is similar to the AdS/CFT with the main difference that we do not a priori assume a specific symmetry of the interactions (e.g. conformal) nor a specific geometry of the dual space-time (e.g. anti-de Sitter) and rely completely on the interactions to determine the geometry of the dual space-time. Our approach is also similar to the more recent ideas of describing space-time using entanglement of subsystems [11–15] or using quantum circuits [16–18], but our derivation of the dual description is certainly very different.

To demonstrate how the non-perturbative method works in practice, we considered three examples of quantum mechanical systems with up to the first-order (Sec. 3), up to the second-order (Sec. 7) and up to an arbitrary high-order (Sec. 6) of interactions complexity and calculated the respective dual expressions for the fully interacting partition functions (4.4), (7.12) and (6.8). In the latter example we showed that the zero temperature limit of the dual path integral can be approximated as a path integral of a 1 + 1D quantum field theory with a non-canonical kinetic term and with the orders of interactions playing the role of a dual time. In this limit the dual system is a legitimate quantum field theory on a lattice, but for more general systems we expect the dual path integral to differ from the Feynman path integral. This is where our analysis diverges significantly from the AdS/CFT proposal in which one is supposedly dealing with legitimate quantum theories on both sides of the duality. Of course, in a true theory of quantum gravity the Feynman path integral may not be adequate and with this respect the dual path integral could give us the desired non-perturbative definition of quantum gravity.
To study the emergent phenomena we first followed the analysis of Ref. [4] and defined an extended partition function (Sec. 3) which solves a relativistic wave equation and simultaneously describes both non-interacting and interacting partition functions up to the first order in interactions complexity. Although the original system was in a time-invariant thermal state (Sec. 2), in the dual description the relativistic dynamics emerges from interactions between subsystems. With this respect not only time, but also space-time can emerge as was first noted in Ref. [4]. Unfortunately, such systems are not rich enough to describe many strongly coupled systems and therefore not suitable for describing the more complex emergent phenomena such as quantum field theories, curved space-time or gravity. To study the more general emergent phenomena we proposed to use the dual path integral representation of the partition functions with arbitrary high orders of interactions. The higher-order interactions are expected in a generic strongly coupled system and in Sec. 6 we defined one such system using the interactions network of Fig. 2. As was already mentioned the corresponding dual path integral can be approximated as the Feynman path integral of a 1+1D quantum field theory on a flat background, but a generalization of Fig. 2 to more general curved backgrounds does not seem out of reach. Also note that it is not too difficult to come up with examples of different, but equivalent, dual path integral representations of the same quantum system and so at some level the emergence of local symmetries is expected. What is, however, less obvious is how to derive fully non-perturbative equations of general relativity, but that is work in progress [19].

Acknowledgments. The work was supported in part by Foundational Questions Institute (FQXi).
References

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Int. J. Theor. Phys. 38, 1113 (1999)
[2] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998)
[3] V. Vanchurin, “Differential equation for partition functions and a duality pseudo-forest,” arXiv:1910.11268 [hep-th].
[4] V. Vanchurin, “A quantum-classical duality and emergent space-time,” arXiv:1903.06083 [hep-th].
[5] T. Jacobson, “Thermodynamics of space-time: The Einstein equation of state,” Phys. Rev. Lett. 75, 1260 (1995)
[6] E. P. Verlinde, “On the Origin of Gravity and the Laws of Newton,” JHEP 1104, 029 (2011)
[7] V. Vanchurin, “Covariant Information Theory and Emergent Gravity,” Int. J. Mod. Phys. A 33, no. 34, 1845019 (2018)
[8] D.A. Kirznits and A.D. Linde, “Macroscopic Consequences of the Weinberg Model,” Phys. Lett. B42 (1972) 471
[9] Weinberg, S. (1974), “Gauge and Global Symmetries at High Temperature,” Phys. Rev. D. 9 (12): 3357-3378.
[10] L. Dolan, and R. Jackiw (1974), “Symmetry behavior at finite temperature,” Phys. Rev. D. 9 (12): 3320-3341.
[11] S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” Phys. Rev. Lett. 96, 181602 (2006)
[12] M. Van Raamsdonk, “Building up spacetime with quantum entanglement,” Gen. Rel. Grav. 42, 2323 (2010)
[13] B. Swingle, “Entanglement Renormalization and Holography,” Phys. Rev. D 86, 065007 (2012)
[14] C. Cao, S. M. Carroll and S. Michalakis, “Space from Hilbert Space: Recovering Geometry from Bulk Entanglement,” Phys. Rev. D 95, 024031 (2017)
[15] V. Vanchurin, “Information Graph Flow: a geometric approximation of quantum and statistical systems,” Found. Phys. 48, no. 6, 636 (2018)
[16] A. Almheiri, X. Dong and D. Harlow, “Bulk Locality and Quantum Error Correction in AdS/CFT,” JHEP 1504, 163 (2015)
[17] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle and Y. Zhao, “Complexity Equals Action,” arXiv:1509.07876 [hep-th]
[18] V. Vanchurin, “Dual Field Theories of Quantum Computation,” JHEP 1606, 001 (2016)
[19] V. Vanchurin, “Quantum-Gravity Duality”, in progress