Social power evolution in influence networks with stubborn individuals

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Abstract—This paper studies the evolution of social power in influence networks with stubborn individuals. Based on the Friedkin-Johnsen opinion dynamics and the reflected appraisal mechanism, two models are proposed over issue sequences and over a single issue, respectively. These models generalize the original DeGroot-Friedkin (DF) model by including stubbornness. To the best of our knowledge, this paper is the first attempt to investigate the social power evolution of stubborn individuals basing on the reflected appraisal mechanism. Properties of equilibria and convergence are provided. We show that the models have same equilibrium social power and convergence property, where the equilibrium social power depends only upon interpersonal influence and individuals’ stubbornness. Roughly speaking, more stubborn individual has more equilibrium social power. Moreover, unlike the DF model without stubbornness, we prove that for the models with stubbornness, autocracy can never be achieved, while democracy can be achieved under any network topology.

Keywords Opinion dynamics, influence networks, social power, reflected appraisal, dynamical systems, mathematical sociology

I. INTRODUCTION

Problem description and motivation: This paper investigates the evolution of social power in influence networks with stubborn individuals. Two models are formulated over issue sequences and over a single issue, respectively. The first model incorporates the Friedkin-Johnsen (FJ) opinion dynamics and the reflected appraisal mechanism to characterize the process of opinion change on each issue and evolution of social power over issue sequences, respectively. The second model is a variation of the first model, in which the processes of opinion dynamics and reflected appraisal take place on a single issue. In the DeGroot-Friedkin (DF) model, the process of opinion dynamics is described by the DeGroot model, where individuals are completely open to interpersonal influence. However, it has been shown by empirical evidence that the FJ model is more realistic and predictive in modelling opinion changes. This paper extends the original DF model by including stubbornness. Rigorous analysis and numerical experiments are provided for equilibria and convergence properties. We aim to uncover the difference between the evolution of social power in groups with and without stubbornness.

Literature review: The investigation of social networks has attracted much attention from applied mathematics, sociology, control theory and economics, etc., over the last several decades. Classic dynamic models of interest concern on how individuals exchange and integrate opinions on a certain issue [25], [26], including the DeGroot model [7], [14], [6], the Abelson model [1], the FJ model [11] and the Hegselmann-Krause model [15], [20], to name but a few. In this literature, the FJ model, which generalizes the DeGroot model by introducing stubbornness, is particularly of interest, due to its predictive ability in human-subject experiments [9], [10], [12], [11]. Further investigations of the FJ model include [13], [27], [24] and the references therein.

Recently, the evolution of social power, namely, the amount of influence or relative control of individuals during opinion discussion, has drawn considerable interest. The study of social power dynamics was initiated by Friedkin [8] with a mathematization of the psychological mechanism of reflected appraisal. A rigorous mathematical model and dynamical system analysis was provided by Jia et al. [18], known as the DF model, which integrates, respectively, the DeGroot model and the reflected appraisal to describe the opinion dynamics on each issue and the social power evolution over issue sequences. Empirical evidence in support of the reflected
appraisal mechanism was provided in [10]. Several extensions and variations of the DF model has been presented since its introduction. Jia et al. [17] extended it to the case that the relative interaction matrix is reducible. A single timescale DF model was proposed and investigated in [16], where reflected appraisal and opinion dynamics take place on a single issue. A modified DF model was proposed in [28], where social power is updated before opinion consensus. A novel stability analysis method for nonlinear Markov chains formulated on the DF model was provided in [2]. Chen et al. [5] extended the DF model to the scenario where the relative interaction matrix is switching and stochastic. In [29], Ye et al. extended the DF model to the setting that the relative interaction matrix is switching in a finite set, an approach based on nonlinear contraction analysis [19] was employed to address the convergence properties.

Contributions: This paper extends the DF model by including stubbornness. First, we propose two models on social power evolution of stubborn individuals over issue sequences and over a single issue, respectively. These models cover two prevalent scenarios in the practice. That is, for specially designed groups, it is feasible or necessary to appraise each member’s performance or importance after discussion on each issue; for loosely assembled or spontaneously arisen groups, discussion on issues may be persistent and reflected appraisal may take place after each opinion update.

Second, we study the properties of equilibria. We prove that for the two models, equilibrium social power is equivalent. Based on the equivalence, we derive the condition for uniqueness of equilibrium social power under general topology, and provide lower and upper bounds for the equilibrium social power. Moreover, we analyze the relationship between equilibrium social power and stubbornness, interpersonal influence, respectively. A sufficient and necessary condition for the existence of democratic equilibrium social power is also provided. For the case that the influence network is star topology, we analyze the uniqueness of the equilibrium social power in the settings that the center node is fully stubborn and partially stubborn, respectively. In the former case, we prove that the center node occupies the largest equilibrium social power, while the ordering of the equilibrium social power of partially stubborn individuals is consistent with the ordering of their stubbornness. In the later case, we show that individuals’ social power at equilibrium increases as their stubbornness or influence weights accorded by center node increase.

Third, we establish the convergence properties. For the model over issue sequences, we prove that all its trajectories globally converge to the unique equilibrium exponentially fast. The convergence properties under star topologies with fully stubborn and partially stubborn center node are also provided, respectively. For the model over a single issue, we prove that individuals’ social power globally exponentially converges to the unique equilibrium if their stubborn levels are higher than 1/2. Moreover, in the case that the relative interaction matrix is doubly-stochastic and individuals are uniformly stubborn, we prove that individuals’ social power globally exponentially converges to the democratic social power structure. Finally, based on the simulation results and the Chernoff bound, we provide a conjecture for the uniqueness and global attractivity of the equilibrium.

Our investigation reveal some findings which are of sociological interest. First, the equivalence of equilibrium social power implies that the reflected appraisal mechanism is robust with respect to variations in the time scales at which opinions and social power evolve. Second, individuals will forget their initial social power exponentially fast, and the equilibrium social power only depends on interpersonal influence and stubbornness. Third, the social power of stubborn individual can never be 0, which means that stubbornness leads to social power. Moreover, for individuals embedded in symmetric influence networks or accorded same influence weights by partially stubborn individuals, more stubbornness leads to more social power. Finally, in groups consisting of stubborn individuals, autocratic social power never emerges, while democratic social power can be achieved regardless of the network topology. From this perspective, stubbornness enables groups to prevent the emergence of autocracy and to achieve democracy.

Lastly, compared with our preliminary conference paper [21], this article contains several results and updates not found in [21]. First, we propose a new model on the social power evolution of stubborn individuals over a single issue, and analyze properties of equilibria and convergence. Second, for the model over issue sequences, we derive a milder condition for the uniqueness of equilibrium social power, and provide convergence analysis, which is not addressed in [21] except the case that the influence network is doubly-stochastic and individuals are homogeneous. Third, we discuss the properties of the equilibrium social power and its relationship with the influence network and individuals’ stubbornness.

Paper organization: In Section II, we propose the DF model with stubbornness over issue sequences and over a single issue, respectively. In Section III, properties of the equilibrium social power is analyzed. In Section IV, we establish the convergence properties. Simulations and a conjecture are provided in Section V. Section VI concludes the paper and all proofs are in Appendices.

Notations: Let 1n and In denote the n × 1 all-ones vector and the n × n identity matrix, respectively. e1 denotes the 1-th standard basis vector with proper dimension. Given δ ∈ Rn, diag(δ) denotes a diagonal
ensures that the FJ opinion dynamics holds. Let

\[ \Theta = \text{diag}(\theta) \]

have at least one stubborn individual, and \( \theta \) denotes individual \( i \)'s susceptibility to interpersonal influence, i.e., \( 1 - \theta_i \) represents its stubbornness to initial opinion. Assume that during the discussion of issue \( s \), the self-appraisal of individual \( i \), denoted by \( x_i(s) \in \{0,1\} \), is static, and each individual forms its opinion according to the FJ model

\[ y_i(s, k + 1) = \theta_i \sum_{j=1}^{n} W_{ij}(s)y_j(s, k) + (1 - \theta_i)y_i(s, 0). \]

Assume that \( W_{ij}(s) = x_i(s) \), and \( W_{ij}(s) = (1 - x_i(s))C_{ij} \), i.e., individuals' self-weights are equal to their self-appraisals. Let \( y(s, k) \) and \( \theta \) denote the vectors of individuals' opinions and susceptibilities, we have

\[ y(s, k + 1) = \Theta W(x(s))y(s, k) + (I_n - \Theta)y(s, 0), \]

where \( \Theta = \text{diag}(\theta) \), \( x(s) \in \Delta_n \), and \( W(x(s)) = \text{diag}(x(s)) + (I_n - \text{diag}(x(s)))C \).

**Assumption 1** Suppose that every sink SCC of \( G(C) \) has at least one stubborn individual, and \( \theta_i < 1 \) if \( x(0) = e_i \).

Assumption 1 ensures that the FJ opinion dynamics converges on each issue. By Lemma III.1 in [21], \( \Theta W(x(s)) \) is strictly row-substochastic for any \( s \geq 0 \) under Assumption 1. Hence, on each issue \( s \), there holds

\[ y(s, \infty) = V(x(s))y(s, 0), \]

where \( V(x(s)) = (I_n - \Theta W(x(s)))^{-1}(I_n - \Theta) \) is row-stochastic.

Equation (2) implies that each individual’s opinion converges to a convex combination of all individuals’ initial opinions. In other words, \( V_{ij}(s) \) is the influence of individual \( j \)'s initial opinion to individual \( i \)'s final opinion on issue \( s \). Consequently, \( (1/n)\sum_{i=1}^{n} V_{ij}(s) \), which represents individual \( j \)'s relative control on other individuals’ final opinions, is individual \( j \)'s social power exerted on issue \( s \), as defined in [4]. According to the reflected appraisal mechanism [8], individuals’ self-appraisals on each issue are set equal to their social power they exerted over prior issue. That is,

\[ x(s + 1) = V(x(s))^{T}\frac{1_n}{n}. \]  

Since \( V(x(s)) \) is row-stochastic, equation (3) ensures that \( x(s + 1) \in \Delta_n \).

**Definition 1** (The DeGroot-Friedkin model with stubborn individuals over issue sequences) Consider an influence network with \( n \geq 2 \) individuals discussing a sequence of issues \( s = 0, 1, 2, \ldots \) in an influence network formulated by weighted digraph \( G(C) \), where \( C \) is the row-stochastic, zero-diagonal relative interaction matrix. Let \( y_i(s, k) \in \mathbb{R} \) denote the opinion of individual \( i \) on issue \( s \) at time \( k \). \( \theta_i \in [0, 1] \) denotes individual \( i \)'s susceptibility to interpersonal influence, i.e., \( 1 - \theta_i \) represents its stubbornness to initial opinion. Assume that during the discussion of issue \( s \), the self-appraisal of individual \( i \), denoted by \( x_i(s) \in \{0,1\} \), is static, and each individual forms its opinion according to the FJ model

\[ y_i(s, k + 1) = \theta_i \sum_{j=1}^{n} W_{ij}(s)y_j(s, k) + (1 - \theta_i)y_i(s, 0). \]

Assume that \( W_{ij}(s) = x_i(s) \), and \( W_{ij}(s) = (1 - x_i(s))C_{ij} \), i.e., individuals’ self-weights are equal to their self-appraisals. Let \( y(s, k) \) and \( \theta \) denote the vectors of individuals’ opinions and susceptibilities, we have

\[ y(s, k + 1) = \Theta W(x(s))y(s, k) + (I_n - \Theta)y(s, 0), \]

where \( \Theta = \text{diag}(\theta) \), \( x(s) \in \Delta_n \), and \( W(x(s)) = \text{diag}(x(s)) + (I_n - \text{diag}(x(s)))C \).

Define \( F : \Delta_n \rightarrow \Delta_n \) as

\[ F(x) = (I_n - \Theta)(I_n - W(x)^{T}\Theta)^{-1}\frac{1_n}{n}. \]

Then, system (4) can be written as \( x(s + 1) = F(x(s)) \).

System (4) generalized the original DF model to the case that individuals are anchored to their initial opinions during the discussion of each issue. Empirical evidence supporting this generalization is provided in [11], [10] and [9], which substantiate that the presence of stubbornness is prevalent in human-subject experiments, and the model including stubbornness is more predictive. Note that if \( \Theta = I_n \), then system (4) is the original DF model. Whereas, at the presence of stubbornness, individuals’ final opinions on each issue depend not only on the relative influence network, but also on their stubbornness, and generally can not achieve consensus [27]. This is different from the original DF model, in which individuals’ social power can be captured by the dominant left eigenvector of \( C \) under the assumption that all sink SCCs of \( G(C) \) are aperiodic.

According to Definition 1, for any \( s > 0 \) and \( x(0) \in \Delta_n \), if \( \theta_i = 1 \), then \( x_i(s) \equiv 0 \); if \( \theta_i = 0 \) for all \( i \),
then \(x(s) \equiv 1_n/n\). For simplicity, we have the following assumption.

**Assumption 2** Suppose that \(\theta_i < 1\) for any \(i \in \{1, \ldots, n\}\), and there exists at least one individual \(j\) with \(\theta_j > 0\).

Note that Assumption 2 implies Assumption 1.

**Remark 1** In model (4) individual’s relative control over the prior discussion is appraised by computing \((I_n - W(x(s))^T \Theta)^{-1}\) and by averaging the columns of \(V(x(s))\); both steps are unrealistic for an individual to perform in a large group because of information and computational requirements. Here we propose a simple distributed dynamical process by which individuals can perceive their social power by using the local interpersonal influence information. Assume that each individual knows the group size \(n\), the susceptibilities of individuals who accord interpersonal influence to it and the accorded influence weights. At each issue \(s\) and time \(k\), let \(p_i(s, k)\) denote the perceived social power of individual \(i\), \(W(s) = \text{diag}(x(s)) + (I - \text{diag}(x(s)))C\) denote the influence matrix. Then, individual \(i\) perceives its social power during the discussion of issue \(s\) according to

\[
p_i(s, k + 1) = (1 - \theta_i) \sum_{j=1}^{n} \frac{\theta_j W_{ij}(s)p_j(s, k)}{1 - \theta_j} + \frac{1 - \theta_i}{n}.
\]

That is,

\[
p(s, k + 1) = \tilde{W}(s)p(s, k) + (I_n - \Theta)\frac{1_n}{n},
\]

where \(\tilde{W}(s) = (I_n - \Theta)W(s)^T \Theta(I_n - \Theta)^{-1}\), whose spectral radius is strictly less than 1 under Assumption 2. Hence, \(p(s, \infty) = (I_n - \Theta)/(I_n - W(s)^T \Theta)^{-1}1_n/n = x(s + 1)\) for any \(p(s, 0) \in \mathbb{R}^n\).

**B. The DF model with stubborn individuals over a single issue**

We now propose a variation of model (4), in which the processes of reflected appraisal and opinion dynamics take place on the same timescale. Consider \(n \geq 2\) individuals discussing a single issue on timescale \(k = 0, 1, 2, \ldots\) according to the FJ model

\[
y(k + 1) = \Theta W(x(k))y(k) + (I_n - \Theta)y(0),
\]

where \(W(x(k)) = \text{diag}(x(k)) + (I_n - \text{diag}(x(k)))C\), \(x(k)\) is the individuals’ social power, \(y(k)\) is the opinion vector, \(\Theta\) is the diagonal matrix describing individuals’ susceptibilities to interpersonal influence, and \(C\) is the row-stochastic and zero-diagonal relative interaction matrix. By equation (6), we have

\[
y(k + 1) = V(k + 1)y(0),
\]

where \(V(k + 1)\) is row-stochastic, and satisfies \(V(k + 1) = \Theta W(x(k))V(k) + I_n - \Theta\) with \(V(0) = I_n\).

Similarly, in equation (7), the \(i\)-th column of \(V(k + 1)\) is the relative control of individual \(i\)’s initial opinion onto all others’ opinions at time \(k\). Based on the reflected appraisal mechanism, we suppose that each individual’s self-appraisal at time \(k + 1\) equals its social power at time \(k\), that is, \(x(k + 1) = V(k + 1)^T 1_n/n\).

**Definition 2** (The DeGroot-Friedkin model with stubborn individuals over a single issue) Consider an influence network with \(n \geq 2\) individuals discussing a single issue over timescale \(k = 0, 1, 2, \ldots\). Let \(C\) and \(\Theta = \text{diag}(\theta_1, \theta_2, \ldots, \theta_n)\) be the row-stochastic, zero-diagonal relative interaction matrix and the diagonal matrix representing individuals’ susceptibilities, respectively. Then, the DeGroot-Friedkin model with stubborn individuals over a single issue is

\[
\begin{align*}
V(k + 1) &= \Theta W(x(k))V(k) + I_n - \Theta, \\
x(k + 1) &= \frac{V(k + 1)^T 1_n}{n},
\end{align*}
\]

with \(W(x(k)) = \text{diag}(x(k)) + (I_n - \text{diag}(x(k)))C\) and \(V(0) = I_n\).

**Remark 2** In the formulation of reflected appraisal mechanism [8], both individual’s self-weights for current opinions and stubbornness are postulated as the reflected appraisals of its social power. In this paper, we focus on the case that individual’s self-weights for its current opinions equal to its manifested social power.

Let \(\Gamma_n = \{W \in \mathbb{R}^{n \times n} \mid W \geq 0, W1_n = 1_n\}\) denote the set of \(n \times n\) row-stochastic real matrices. Define \(G : \Gamma_n \times \Delta_n \to \Gamma_n \times \Delta_n\) by \(G(V, x) = (G_V(V, x), G_x(V, x))\) with \(G_V(V, x) = \Theta W(x)V + I_n - \Theta\) and \(G_x(V, x) = G_V(V, x)^T 1_n/n\). Then, system (8) can be expressed by

\[
\begin{align*}
V(k + 1) &= G_V(V(k), x(k)), \\
x(k + 1) &= G_x(V(k), x(k)).
\end{align*}
\]

**III. Equilibrium Analysis**

This section studies the properties of the equilibria of models (4) and (8).

**A. Equivalence of equilibrium social power**

Since \(F(x)\) and \(G(V, x)\) are both continuous functions from, respectively, \(\Delta_n\) and \(\Gamma_n \times \Delta_n\) to themselves, where \(\Delta_n\) and \(\Gamma_n \times \Delta_n\) are convex and compact subsets of Banach space. Then, following the Schauder fixed point theorem [3], i.e., every continuous function from a convex compact subset of a Banach space to itself has a fixed point, systems (4) and (8) have at least one equilibrium, respectively.
Lemma 1 (Equivalence of equilibrium social power) Suppose that Assumption 1 holds, system (4) and (8) have the same relative interaction matrix $C$ and susceptibility matrix $\Theta$. Then, $x^*$ is an equilibrium of system (4) if and only if for $V^*=(I_n-\Theta W(x^*))^{-1}(I_n-\Theta)\in \Gamma_n$, $(V^*, x^*)$ is an equilibrium of system (8).

Lemma 1 implies that the reflected appraisal mechanism is robust with respect to variations in the time scales at which opinions and social power evolve. Moreover, since non-stubborn individual has 0 equilibrium social power in system (4), it also have 0 equilibrium social power in system (8). Thus, we assume that Assumption 2 holds for model (8) in the sequel.

B. Properties of equilibrium social power with general topology

Since systems (4) and (8) have same equilibrium social power, we focus on equilibria of system (4). In what follows, let $\theta_{min} = \min_j \theta_j$, $\theta_{ave} = \sum_{j=1}^{n} \theta_j/n$, and $\theta_{max} = \max_j \theta_j$. Moreover, let $\mathcal{V}_l$ and $\mathcal{V}_p$ denote the sets of individuals who are fully stubborn ($\theta_i = 0$) and partially stubborn ($\theta_i > 0$), respectively. Without loss of generality, assume $\mathcal{V}_l = \{1, \ldots, r\}$ and $\mathcal{V}_p = \{r + 1, \ldots, n\}$ with $r < n$.

Lemma 2 (Properties of $F(x)$) For the map $F : \Delta_n \rightarrow \Delta_n$ defined by $F(x) = (I_n - \Theta)(I_n - W(x)\Theta)^{-1}1_n/n$ with $W(x) = \text{diag}(x) + (I_n - \text{diag}(x))C$, the following statements hold true:

(i) $F$ is differentiable on int $\Delta_n$ and continuous on $\Delta_n$;

(ii) the Jacobian of $F$ is $\frac{\partial F}{\partial x} = (I_n - \Theta)(I_n - W(x)^T\Theta)^{-1}(I_n - C^T\Theta(I_n - \Theta) - \text{diag}(F(x)))$;

(iii) for any $x \in \Delta_n$, $(1 - \theta_i)/n \leq F_i(x) \leq (1 + \zeta)/n$, where $\zeta = n\theta_{ave} - \theta_{min}$.

Theorem 1 (Equilibrium social power with general topology) Consider systems (4) and (8) with $n \geq 2$ and $x(0) \in \Delta_n$. Suppose that Assumption 2 holds, and $C$ is row-stochastic and zero-diagonal. Then, we have that:

(i) there exists at least one equilibrium of systems (4) and (8), which satisfies

a) $x^* \in \text{int } \Delta_n$;

b) $x^*_i \geq 1/n$ for $i \in \mathcal{V}_l$, and $x^*_i = 1/n$ if and only if $C_{ki} = 0$ for any $j \in \mathcal{V}_p$;

c) $x^*_i > (1 - \theta_i)/n$ for $i \in \mathcal{V}_p$, and $x^*_i < 1/n$ if $C_{ji} = 0$ for any $j \in \mathcal{V}_p$;

d) $\max_i x^*_i < 1/n + \theta_{ave}$;

(ii) the equilibrium social power $x^*$ is unique if $\theta_{max} < \frac{n}{n + 2(1 + \zeta)}$, with $\zeta = n\theta_{ave} - \theta_{min}$.

Remark 3 In Theorem 1 we prove that if $\theta_{max} < \frac{n}{n + 2(1 + \zeta)}$, then $F(x)$ is contractive on $\Delta_n$, which also implies that the equilibrium social power only depends upon $C$ and $\Theta$. Since $\zeta < n - 1$, then we have $\frac{n}{n + 2(1 + \zeta)} > 1/3$, which implies that $\theta_{max} < \frac{n}{n + 2(1 + \zeta)}$ is a milder restriction compared with that proposed in [21]. Moreover, note that $\frac{n}{n + 2(1 + \zeta)} = \frac{1}{1 + 2\theta_{ave} + \frac{2}{n}(1 - \theta_{min})}$, that is, $\theta_{max} < \frac{n}{n + 2(1 + \zeta)}$ is a restriction on the distribution of individuals’ stubbornness. For clarification, now consider a special case. Suppose that $r \geq 1$. Then, we have that $\zeta < n - r$. Thus, it follows that $\frac{n}{n + 2(1 + \zeta)} > \frac{1}{1 + 2/n}$ as $r/n$ approaches 1. That is, $\theta_{max}$ can be arbitrarily close to 1 in a large group where the majority is fully stubborn.

Note that the relative interaction matrix $C$ is just required row-stochastic and zero-diagonal in Theorem 1, which means that the autocratic social power (i.e., there is exactly one individual has social power 1, and all others’ are 0) can never emerge in systems (4) and (8), even though the initial social power is autocratic or $\mathcal{G}(C)$ is star topology. This is a key difference between models (4), (8) and the original DF model, in which the autocratic social power can be achieved under both irreducible and reducible influence networks [18], [17].

Corollary 1 (Properties of equilibrium social power) Consider systems (4) and (8) with $n \geq 2$ and $x(0) \in \Delta_n$. Suppose that Assumption 2 holds, and $C$ is row-stochastic and zero-diagonal. Then the equilibrium social power of systems (4) and (8), i.e., $x^*$, satisfies:

(i) for any $i \in \mathcal{V}_l$ and $j \in \mathcal{V}_p$, if $C_{ki} = C_{kj}$ holds for any $k \in \mathcal{V}_p \setminus \{j\}$, then $x^*_i > x^*_j$;

(ii) for any $i, j \in \mathcal{V}_p$, suppose that $C_{ki} = C_{kj}$ holds for any $k \in \mathcal{V}_p \setminus \{i, j\}$ and $C_{ij} = C_{ji}$. Then $x^*_i < x^*_j$ holds if and only if $\theta_i > \theta_j$;

(iii) suppose that $C$ is symmetric. Then for any $i, j$, if $\theta_i > \theta_j$, then $x^*_i < x^*_j$.

Corollary 1 shows that if two individuals are accorded same influence weights by partially stubborn individuals, or the relative interaction matrix is symmetric, then the more stubborn individual has more equilibrium social power. In the DF model without stubbornness, the democratic social power structure, i.e., $x^* = 1_n/n$, is achieved only if the network is irreducible and doubly-stochastic. Next, we show that for systems (4) and (8), the democracy can be achieved even if the network is neither doubly-stochastic nor irreducible.

Corollary 2 (Existence of democratic equilibrium social power) Consider system (4) and (8) with $n \geq 2$
and \( x(0) \in \Delta_n \). Suppose that Assumption 2 holds, and \( C \) is row-stochastic and zero-diagonal. Then, \( \mathbf{1}_n/n \) is an equilibrium of systems (4) and (8) if and only if \( \Theta(I_n - \Theta)^{-1}\mathbf{1}_n \) is a left eigenvector of \( C \) corresponding to eigenvalue 1.

The proof of Corollary 2 can be readily obtained by substituting \( x \) and \( F(x) \) for \( x^* = \mathbf{1}_n/n \) in equation (4).

C. Properties of equilibrium social power with star topology

First, we consider the scenario where the center node of \( G(C) \) belongs to \( V_i \).

**Theorem 2** (Equilibrium social power under star topology with fully stubborn center node) Consider system (4) and (8) with \( n \geq 2 \) and \( x(0) \in \Delta_n \). Suppose that Assumption 2 holds, and \( C \) is row-stochastic and zero-diagonal with \( G(C) \) being a star topology with center node \( l \) satisfying \( \theta_l = 0 \). Then, the equilibrium social power of systems (4) and (8) is unique, and satisfies:

(i) \( x^* \in \text{int } \Delta_n \);
(ii) \( x^*_i = 1/n \) for \( i \in V_i \setminus \{l\} \);
(iii) \( x^*_i = \frac{n - \sqrt{n^2 - 4n\theta_l(1 - \theta_i)}}{2n\theta_l} \) when \( x^*_i \) decreases with respect to \( \theta_i \) for \( i \in V_p \);
(iv) \( x^*_i = \frac{1}{n} + \frac{1}{n} \sum_{j=r+1}^{n} \frac{\theta_j(1 - x^*_j)}{1 - \theta_j x^*_j} \) for \( i \in V_p \).

Theorem 2 shows that for systems (4) and (8) under star topology with fully stubborn center node, the center node has the largest equilibrium social power, which is strictly larger than \( 1/n \). And other fully stubborn individuals’ equilibrium social power is \( 1/n \), while all partially stubborn individuals’ equilibrium social power is strictly less than \( 1/n \). Moreover, the ordering of equilibrium social power of partially stubborn individuals is consistent with the ordering of their stubbornness. Now, we consider the scenario where the center node of \( G(C) \) belongs to \( V_p \).

**Theorem 3** (Equilibrium social power under star topology with partially stubborn center node) Consider systems (4) and (8) with \( n \geq 2 \) and \( x(0) \in \Delta_n \). Suppose that Assumption 2 holds, and \( C \) is row-stochastic and zero-diagonal with \( G(C) \) being a star topology with center node \( l \) satisfying \( 1 > \theta_l > 0 \). Then,

(i) the equilibrium social power of systems (4) and (8) has the following properties:

(a) \( x^* \in \text{int } \Delta_n \);
(b) for \( i \in V_i \) if \( C_{ii} = 0 \), then \( x^*_i = 1/n \); otherwise, \( x^*_i > 1/n \);
(c) for \( i \in V_p \setminus \{l\} \) if \( C_{ii} = 0 \), then \( x^*_i \) is unique,

\[ x^*_i = \frac{n - \sqrt{n^2 - 4n\theta_l(1 - \theta_i)}}{2n\theta_l} \] when \( x^*_i \) decreases with respect to \( \theta_i \).

(ii) Moreover, if there holds \( C_{ii} = 0 \) for all \( i \in V_p \setminus \{l\} \), then the equilibrium social power of systems (4) and (8) is unique, and satisfies:

(a) \( x^*_i = \frac{n - \sqrt{n^2 - 4n\theta_l(1 - \theta_i)}}{2n\theta_l} \) and decreases with respect to \( \theta_i \) for \( i \in V_p \setminus \{l\} \);
(b) \( x^*_i = \frac{n - \sqrt{n^2 - 4n\theta_l(1 - \theta_i)}}{2n\theta_l} \);\]
(c) \( x^*_i = \frac{1}{n} + \frac{(\xi^* - x^*_i)}{n} C_{ii} \) for \( i \in V_i \).

where \( \xi^* = n - r - n \sum_{j=1}^{n} x^*_j \).

Theorem 3 shows that all individuals have positive equilibrium social power, while the partially stubborn center does not necessarily have the largest equilibrium social power. The following examples show that under the same star topology with partially stubborn center node, both fully stubborn individual and partially stubborn individual (whether if it is center node or not) can obtain the largest equilibrium social power, which depends upon individuals’ stubbornness.

**Numerical examples on star topology with partially stubborn center node:** Consider system (4) with \( n = 3 \). Suppose that \( C = [0, 0.2, 0.8; 1, 0, 0; 0, 0, 0] \), i.e., individual 1 is the center node. Then, under different settings of \( \Theta \), we obtain the trajectories of \( x(s) \), shown in Fig. (1). It is observed that in Fig. (1), the center node 1 occupies the largest equilibrium social power when \( \theta = (0.1, 0, 0.6)^T \), while the fully stubborn node and partially stubborn node which are not center node can also obtain largest equilibrium social power under the same influence network but different settings of \( \theta \).

**Corollary 3** (Ordering of equilibrium social power under star topology with partially stubborn center node) Consider systems (4) and (8) with \( n \geq 2 \) and \( x(0) \in \Delta_n \). Suppose that Assumption 2 holds, and \( C \) is row-stochastic and zero-diagonal with \( G(C) \) being a star topology with center node \( l \) satisfying \( 1 > \theta_l > 0 \). Then, the equilibrium social power of systems (4) and (8) satisfies:
(i) for any \(i, j \in \mathcal{V}_l\), if \(C_{li} > C_{lj}\), then \(x_i^* > x_j^*\);
(ii) for any \(i \in \mathcal{V}_l\) and \(j \in \mathcal{V}_p \setminus \{l\}\), if \(C_{li} = C_{lj}\), then \(x_i^* > x_j^*\);
(iii) for any \(i, j \in \mathcal{V}_p \setminus \{l\}\) with \(C_{li} = C_{lj}\), \(x_i^* > x_j^*\) if and only if \(\theta_i < \theta_j\);
(iv) for any \(i, j \in \mathcal{V}_p \setminus \{l\}\) with \(\theta_i = \theta_j\), \(x_i^* > x_j^*\) if and only if \(C_{li} > C_{lj}\).

IV. CONVERGENCE ANALYSIS

This section studies the convergence of systems (4) and (8).

A. Convergence of the DF model with stubborn individuals over issue sequences

**Theorem 4 (Convergence with general topology)** Consider system (4) with \(n \geq 2\) and \(x(0) \in \Delta_n\). Suppose that Assumption 2 holds, and \(C\) is row-stochastic and zero-diagonal. Let \(\zeta = n\theta_{\max} - \theta_{\min}\). If \(\theta_{\max} < \frac{n}{n + 2(1 + \zeta)}\), then all trajectories of system (4) converge to the unique equilibrium social power \(x^*\) characterized in Theorem 1 exponentially fast.

In the proof of Theorem 1, we show that if \(\theta_{\max} < \frac{n}{n + 2(1 + \zeta)}\), then all trajectories of system (4) converge to the unique equilibrium social power \(x^*\) characterized in Theorem 1 exponentially fast.

**Corollary 4 (Convergence under star topology with fully stubborn center node)** Consider system (4) with \(n \geq 2\) and \(x(0) \in \Delta_n\). Suppose that Assumption 2 holds, and \(C\) is row-stochastic and zero-diagonal with \(G(C)\) being a star topology with center node \(l\) satisfying \(\theta_l = 0\). Then, all trajectories of system (4) exponentially converge to the unique equilibrium social power \(x^*\) characterized in Theorem 2.

Next we consider the case that the center node is partially stubborn.

**Theorem 5 (Convergence property under star topology with partially stubborn center node)** Consider system (4) with \(n \geq 2\) and \(x(0) \in \Delta_n\). Suppose that Assumption 2 holds, and \(C\) is row-stochastic and zero-diagonal with \(G(C)\) being a star topology with center node \(l\) satisfying \(1 > \theta_l > 0\). Then,

(i) for \(i \in \mathcal{V}_p \setminus \{l\}\), if \(C_{li} = 0\), then \(x_i(s)\) exponentially converges to \(x_i^* = \frac{n - \sqrt{n^2 - 4n\theta_l(1 - \theta_l)}}{2n\theta_i}\);
(ii) moreover, if there holds \(C_{li} = 0\) for all \(i \in \mathcal{V}_p \setminus \{l\}\) and \(\sum_{j \in \mathcal{V}_p \setminus \{l\}} \theta_j \leq \frac{4n}{5} + 1\), then all trajectories of system (4) exponentially converge to the equilibrium social power \(x^*\) characterized in statement (ii) of Theorem 3.

B. Convergence of the DF model with stubborn individuals over a single issue

First, we consider doubly-stochastic influence network with uniformly stubborn individuals.

**Lemma 3 (Convergence with doubly-stochastic topology and uniform stubbornness)** Consider system (8) with \(n \geq 2\) and \(x(0) \in \Delta_n\). Suppose that \(\theta_l = \theta\) for all \(l \in \{1, 2, \ldots, n\}\), \(C\) is doubly-stochastic and zero-diagonal. Then all trajectories of system (8) exponentially converge to the democratic equilibrium \(1^n/n\).

Since \(\theta_l = \theta\) and \(C\) is doubly-stochastic, we have \(x(k + 1) = \theta x(k) + (1 - \theta) 1^n/n\). Note that \(\theta \in (0, 1)\), thus \(x(k) \to 1^n/n\). It is clear that in system (8), if \(V(k)\) converges, then \(x(k)\) converges. Let \(V \in \Gamma_n\) be a row-stochastic matrix, and \(V_l\) denote the \(l\)-th column of \(V\). Then \(\chi = [V_1^T V_2^T \ldots V_n^T]^T\) denote the vector by vectorizing \(V\), then \(\chi \in \mathcal{A} = \{x \mid x \in \mathbb{R}^n, x \geq 0, \sum_{i=0}^{n-1} x_{ni+t} = 1\}\). Let \(\nu = [(1 - \theta_1)e_1^T (1 - \theta_2)e_1^T \ldots (1 - \theta_n)e_n^T]^T \in \mathbb{R}^n\). Define \(\hat{G} : \mathcal{A} \to \mathcal{A}\) by

\[
\hat{G}(x) = I_n \otimes \Theta W(x)x + \nu,
\]
where \(x \in \mathcal{A}\), \(W(x) = \text{diag}(\omega) + (I_n - \text{diag}(\omega))C\) with \(\omega \in \mathbb{R}^n\) and \(\omega_l = V_l^T 1^n/n\). Now, we present our convergence result for system (8) with general topology.

**Theorem 6 (Convergence with general topology)** Consider system (8) with \(n \geq 2\) and \(x(0) \in \Delta_n\). Suppose that Assumption 2 holds, and \(C\) is row-stochastic and zero-diagonal. If \(\theta_{\max} < 1/2\), then all trajectories of system (8) converge to the unique equilibrium social power \(x^*\) characterized in Theorem 1 exponentially fast.

Note that even though systems (4) and (8) have the same equilibrium social power, their trajectories may be different. In the proof of Theorem 6, we show that system (8) is contractive if \(\theta_{\max} < 1/2\). However, this condition is not necessary. In next section, we will propose a conjecture on the contractivity of systems (4) and (8).

V. SIMULATIONS AND CONJECTURE

As we shown, the equilibrium social power of systems (4) and (8) is unique if \(\theta_{\max} < \frac{n}{n + 2(1 + \zeta)}\). However, for the general case, the uniqueness of equilibrium social power of systems (4) and (8) is equivalent to that the quadratic equations \((I_n - C^T\Theta)(I_n - \Theta)^{-1}x - (I_n - C^T\Theta)(I_n - \Theta)^{-1}\text{diag}(x)x = 1^n/n\) has exactly one solution in \(\text{int} \Delta_n\), which is difficult to prove due to the entanglement of \(C\) and \(\Theta\). In this subsection, we shall estimate the probability that systems (4) and (8) converge to unique equilibrium social power for any initial social power and matrix pair \((C, \Theta)\).
**Monte Carlo validation:** Since systems (4) and (8) have the same equilibrium social power, here we just focus on system (4). For given matrix pair \((C, \Theta)\), where \(C\) is row-stochastic and zero-diagonal, \(\Theta\) satisfies Assumption 2, we randomly pick \(\hat{x}(0)\) and compute the final social power \(\hat{x}^*\) by running system (4). Let \(x \in \Delta_n\) be a random variable representing the initial social power, and \(x^*_n\) denote the corresponding final social power of system (4). Then, define \(pr(C, \Theta) = Pr\{J(x) = 0\}\) as the probability that system (4) converges to \(\hat{x}^*\) with initial social power \(x\), where \(J : \Delta_n \to \mathbb{R} = x^*_n - \hat{x}^*\) is a measurable performance function. Now, we can estimate \(pr(C, \Theta)\) as follows. First, we generate \(N\) independent identically distributed random samples of the initial social power \(x^1, x^2, \ldots, x^N\), where \(N\) is a positive integer. Second, define an indicator function \(I_{J,C,\Theta} : \Delta_n \to \{0, 1\}\) by \(I_{J,C,\Theta}(x) = 1\) if \(J(x) = 0\), and 0 otherwise. Finally, we compute the empirical probability as

\[
\hat{pr}(C, \Theta) = \frac{1}{N} \sum_{i=1}^{N} I_{J,C,\Theta}(x^i).
\]

Then, for any accuracy \(\epsilon \in (0, 1)\) and confidence level \(1 - \eta \in (0, 1)\), by the Chernoff bound we have that

\[
Pr\{ | \hat{pr}(C, \Theta) - pr(C, \Theta) | < \epsilon \} \geq 1 - 2 \exp(-2\epsilon^2 N).
\]

If there holds \(N \geq \log(2/\eta)/(2\epsilon^2)\), then we have \(1 - 2 \exp(-2\epsilon^2 N) > 1 - \eta\). That is, the probability that \(| \hat{pr}(C, \Theta) - pr(C, \Theta) | < \epsilon \) is greater than \(1 - \eta\). In [22], the authors computed that for \(\epsilon = \eta = 0.01\), the Chernoff bound is satisfied by \(N = 27000\). That is to say, for given \((C, \Theta)\), if system (4) converges to \(x^*_n\) for all 27000 samples of initial social power, we can say that for the given \(C\) and \(\Theta\), with confidence level 99\%, there is at least 99\% probability that system (4) converges to unique equilibrium social power for any initial social power.

Similarly, consider random variable \((C, \Theta)\) where \(C\) is row-stochastic and zero-diagonal, \(\theta_i \in [0, 1)\) for \(i \in \{1, \ldots, n\}\). If there holds that for each of 27000 samples of matrix pairs, system (4) converges to same equilibrium social power for all 27000 samples of initial social power, then we can say that for any \(C\) and \(\Theta\) satisfying Assumption 2, with confidence level 99\%, there is at least 99\% probability that system (4) converges to unique equilibrium social power for any initial social power.

**Numerical examples on uniqueness and convergence:** Based on above discussion, we run 27000\(^2\) experiments for systems (4) and (8) with randomly generated initial social power \(x^i, i \in \{1, \ldots, 27000\}\) for each randomly generated matrix pair \((C^j, \Theta^j), j \in \{1, \ldots, 27000\}\). Figure 2 depicts the trajectories of 6 nodes for 100 initial social power with 3 matrix pairs. The experiments show that for systems (4) and (8) with each of the 27000 samples of \((C, \Theta)\), the trajectories beginning at all 27000 samples of initial social power converge to the same equilibrium social power. Therefore, our experiments establish the following conjecture.

**Conjecture 1** Consider system (4) and (8) with \(n \geq 2\) and \(x(0) \in \Delta_n\). Suppose that Assumption 2 holds, and \(C\) is row-stochastic and zero-diagonal. Then, all trajectories of systems (4) and (8) exponentially converge to an unique equilibrium social power, which only depends upon \(C\) and \(\Theta\).

**VI. CONCLUSIONS**

This paper has investigated the evolution of social power of stubborn individuals. Two models are proposed to characterize the social power evolution over issue sequences and over a single issue, respectively. Analytical and numerical results are provided. We prove that the model over a single issue has the same equilibrium social power with the model over issue sequences. Based on this equivalence, uniqueness and properties of the equilibrium social power are analyzed under different settings of the influence network topology. Then, we establish convergence of the equilibrium.

Our investigations reveal several features for social power evolution of stubborn individuals. First, the re-
flected appraisal mechanism is robust with respect to variations in the time scales at which opinions and social power evolve. Second, individuals will exponentially forget their initial social power, and the equilibrium social power only depends upon the relative interaction matrix and their stubbornness. Third, individuals will have positive equilibrium social power if they are stubborn, and more stubbornness leads to more social power. Finally, for an influence network in which all individuals are stubborn, the autocratic social power structure never emerges, while the democratic social power can be achieved with any network topologies. Future works will focus on the co-evolution of individuals’ stubbornness with their social power.

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APPENDIX A

PROOF OF LEMMA 1

Regarding necessity, suppose that $x^* \in \Delta_n$ is an equilibrium of system (4). Then, by equation (4) we have $x^* = (I_n - \Theta)(I_n - W(x^*)^T\Theta)^{-1}1_n/n$. Let
\[ V^* = (I_n - \Theta W(x^*))^{-1}(I_n - \Theta), \text{ then } V^* \in \Gamma \text{ and } x^* = (V^*)^T 1_n/n. \]

Since
\[ \Theta W(x^*) V^* + I_n - \Theta \]
\[ = (\Theta W(x^*) + I_n - \Theta W(x^*)) (I_n - \Theta W(x^*))^{-1} (I_n - \Theta) \]
\[ = V^*, \]
then \((V^*, x^*)\) is an equilibrium of system (8).

Regarding sufficiency, suppose that \((V^*, x^*)\) is an equilibrium of system (8). By equation (8) we have
\[ V^* = \Theta W(x^*) V^* + I_n - \Theta \text{ and } x^* = (V^*)^T 1_n/n, \]
which implies that \((V^*, x^*)\) is an equilibrium of system (4).

**APPENDIX B**

**PROOF OF LEMMA 2**

Denote \( \Delta(x) = I_n - W(x)^T \Theta \). Regarding (i), note \((I_n - W(x)^T \Theta)^{-1} = \Delta^*(x)/\det(\Delta(x)) \), where \( \Delta^*(x) \) is the adjoint matrix of \( \Delta(x) \). Then,
\[ F_i(x) = \frac{(1 - \theta_i)}{n \times \det(\Delta(x))} \sum_{k=1}^{n} \Lambda^*_i x_k(x), \]
where \( \det(\Delta(x)) \) and \( \Lambda^*_i x_k(x) \) are both analytic functions of \( x \). Since \( \det(\Delta(x)) \neq 0 \), \( F_i(x) \) is differentiable on \( \Delta_n \) and continuous on \( \Delta_n \) for any \( i \). That is, \( F(x) \) is differentiable on \( \Delta_n \) and continuous on \( \Delta_n \).

Regarding (ii), since \( \theta_i < 1 \), \((I_n - \Theta)^{-1} \) exists. By (5) we obtain \((I_n - W(x)^T \Theta)(I_n - \Theta)^{-1} F(x) = 1/n \).

Then, taking the derivatives of both sides, we get
\[ (I_n - W(x)^T \Theta)(I_n - \Theta)^{-1} \frac{\partial F_i(x)}{\partial x} = (I_n - C \Theta)^T \Theta(I_n - \Theta)^{-1} \text{diag}(F(x)). \]

Hence,
\[ \frac{\partial F}{\partial x} = (I_n - \Theta)^{-1} (I_n - W(x)^T \Theta)^{-1} (I_n - C \Theta)^T \Theta(I_n - \Theta)^{-1} \text{diag}(F(x)). \]

Regarding (iii), since \( \Lambda^-(x) (I_n - W(x)^T \Theta) = I_n \), we have that
\[ \Lambda^i_{ii}^{-1} x_i (1 - \theta_i x_i) - \theta_i (1 - x_i) \sum_{k=1}^{n} \Lambda^i_{ik} x_i C_{ik} = 1. \]

For any \( i \in \{1, \ldots, n\} \), since \( 1 - \theta_i x_i > 0 \), there holds
\[ \Lambda^i_{ii}^{-1} x_i = \frac{1 + \theta_i (1 - x_i) \sum_{k=1}^{n} \Lambda^i_{ik} x_i C_{ik}}{1 - \theta_i x_i} > 1 \]

By \( F(x) = (I_n - \Theta) \Lambda^-(x) 1_n/n \), we have that
\[ F_i(x) \geq \frac{1}{n} (1 - \theta_i) \Lambda^i_{ii}^{-1} x_i \geq \frac{1 - \theta_i}{n} > 0. \]

On the other hand, by \( \sum_{k \neq i}^{n} F_k(x) = 1 \) there holds
\[ F_i(x) = 1 - \sum_{k \neq i}^{n} F_k(x) \leq 1 - \sum_{k \neq i}^{n} \frac{1 - \theta_k}{n} x_i = 1 - \frac{1}{n} \sum_{k \neq i}^{n} \theta_k. \]

Since \( \sum_{k \neq i}^{n} \theta_k \leq n \theta_{\min} - \theta_{\min} = \zeta \) for any \( i \in \{1, \ldots, n\} \), we obtain \( F_i(x) \geq (1 + \zeta)/n \).

**APPENDIX C**

**PROOF OF THEOREM 1**

Since the equilibrium social power of systems (4) and (8) is equivalent, we just need to show that (i) and (ii) hold for system (4). Regarding (i), by Lemma 2 we have that \( F_i(x) \in (0, 1) \), i.e., \( x^* \in \Delta_n \). According to the definitions of \( \mathcal{V}_f \) and \( \mathcal{V}_p \), we write
\[ \Theta = \begin{bmatrix} 0_r & r \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} C_f & C_p \end{bmatrix}. \]

Let \( x^*_r \in \mathbb{R}^r \) and \( x^*_p \in \mathbb{R}^{n-r} \) denote the equilibrium social power vectors of fully stubborn and partially stubborn individuals, respectively. Then, by equation (4),
\[
\begin{cases}
   x^*_r = \frac{1}{n} + C_p^T \Theta_p (I_n - \Theta^* - \text{diag}(x^*_p)) x^*_p,
   \\
   (I_n - C_p^T \Theta_p) (I_n - \Theta^* - \text{diag}(x^*_p)) x^*_p = \frac{1}{n} - (I_n - C_p^T \Theta_p) (I_n - \Theta^* - \text{diag}(x^*_p)) x^*_p,
\end{cases}
\]

Since \( x^* \in \Delta_n \), we have \( \text{diag}(x^*_p) x^*_p > 0 \) and \( (I_n - \text{diag}(x^*_p)) x^*_p > 0 \), which imply that \( x^*_r \geq 1/n \) and \( x^*_p \geq (1 - \theta_j)/n + \theta_j (x^*_p)^2 \) for any \( i \in V_f, j \in V_p \).

Moreover, for \( i \in V_f \), if \( C_{ji} = 0 \) for all \( j \in V_p \), we have that \( x^*_r = 1/n \); otherwise, \( x^*_r > 1/n \). For \( i \in V_p \), if \( C_{ji} = 0 \) for all \( j \in V_p \), we have that \( x^*_r = (1 - \theta_i)/n + \theta_i (x^*_p)^2 \). Since \( x^*_r < 1/n \) and \( \theta_i > 0 \), there holds \( x^*_r < \frac{n - \sqrt{n^2 - 4n\theta_i (1 - \theta_i)}}{2n\theta_i} < 1/n \). Finally, \( x^*_p = 1/n + \theta_{\text{ave}} \) follows from that \( x^* \in \Delta_n \).

Regarding (ii), first, we show that \( F(x) \) is contractive on \( \Delta_n \) if \( \theta_{\text{max}} < \frac{n}{n + 1} \). Since \( \|(I_n - \Theta)(I_n - W(x)^T \Theta)^{-1}\|_1 = 1 \) and \( \|(I_n - C)^T\|_1 = 2 \), we have
\[
\|\frac{\partial F}{\partial x}\|_1 \leq 2\|\Theta (I_n - \Theta)^{-1}\|_1 \|\text{diag}(F(x))\|_1 \frac{2\theta_{\text{max}}}{1 - \theta_{\text{max}}} \max_{i} F_i(x).
\]

Since \( \max_{i} F_i(x) \leq (1 + \zeta)/n \) and \( \theta_{\text{max}} < \frac{n}{n + 1} \), we have
\[
\frac{\|\partial F\|}{\|\partial x\|_1} \leq \frac{2\theta_{\text{max}}}{n(1 - \theta_{\text{max}})} (1 + \zeta) < 1.
\]

Denote \( \kappa = \frac{2\theta_{\text{max}} (1 + \zeta)}{n(1 - \theta_{\text{max}})} \). Now, following the mean value inequality (Theorem 3.2.3, [23]), we have that for any \( y, z \in \Delta_n \), there holds
\[
\|F(y) - F(z)\|_1 \leq \sup_{0 \leq t \leq 1} \left\| \frac{\partial F}{\partial x} \right\|_{x=tz+(1-t)y} \|y - z\|_1 \leq \kappa \|y - z\|_1 < \|y - z\|_1,
\]
i.e., \( F(x) \) is contractive on \( \Delta_n \). Moreover, for any \( y \in \Delta_n \setminus \Delta_n \) and \( z \in \Delta_n \), since \( \Delta_n \) is compact,
there exists a Cauchy sequence \( \{y_k\}_{k=0}^\infty \) which satisfies \( y_k \in \text{int} \Delta_n \) and \( \lim_{k \to \infty} y_k = y \). Therefore,

\[
\begin{align*}
\|F(y) - F(z)\|_1 &= \|F(\lim_{k \to \infty} y_k) - F(z)\|_1 \\
&\leq \lim_{k \to \infty} \sup_{0 \leq t \leq 1} \left\| \frac{\partial F}{\partial x} \right\|_{x=x+(ty-y)} \|y_k - z\|_1 \\
& \leq \kappa \lim_{k \to \infty} \|y_k - z\|_1 = \kappa \|y - z\|_1 < \|y - z\|_1.
\end{align*}
\]

Similarly, for any \( y, z \in \Delta_n \setminus \text{int} \Delta_n \), there holds \( \|F(y) - F(z)\|_1 \leq \kappa \|y - z\|_1 \). That is, for any \( y, z \in \Delta_n \), there holds \( \|F(y) - F(z)\|_1 < \|y - z\|_1 \). Thus, \( F(x) \) is contractive on \( \Delta_n \). Then the uniqueness of \( x^* \) follows from the Banach fixed point theorem.

APPENDIX D
PROOF OF COROLLARY 1

Regarding (i), for any \( i \in \mathcal{V}_i \) and \( j \in \mathcal{V}_p \), equation (9) implies that

\[
x_i^* - x_j^* = \frac{\theta_i}{n} - \theta_j(x_j^*)^2 + C_{ji} \frac{\theta_i}{1 - \theta_j} x_j^*(1 - x_j^*) + \theta_j \sum_{k \in \mathcal{V}_i \setminus \{i\}} \frac{\theta_k}{1 - \theta_k} x_k^*(1 - x_k^*).
\]

Note that \( \sum_{k \in \mathcal{V}_i \setminus \{i\}} C_{kj} \frac{\theta_k}{1 - \theta_k} x_k^*(1 - x_k^*) = \frac{1 - \theta_j x_j^*}{1 - \theta_j} - \frac{1}{n} \). Then,

\[
x_i^* - x_j^* = \frac{\theta_j(C_{ij}(1 - x_j^*)}{1 - \theta_j} x_j^* + x_j^*(1 - \theta_j x_j^* - x_j^*)) > 0,
\]

Therefore, \( x_i^* > x_j^* \).

Regarding (ii), for any \( i, j \in \mathcal{V}_p \), by equation (9),

\[
x_i^* - x_j^* = \theta_i(x_i^*)^2 - \theta_j(x_j^*)^2 + \frac{\theta_j - \theta_i}{n} + C_{ji} \frac{\theta_j}{1 - \theta_i} x_i^*(1 - x_i^*) - C_{ij} \frac{\theta_j}{1 - \theta_i} x_j^*(1 - x_j^*) + \sum_{k \in \mathcal{V}_p \setminus \{i, j\}} C_{ki} \frac{\theta_k(\theta_1 - \theta_i)}{1 - \theta_k} x_k^*(1 - x_k^*).
\]

Note that \( \theta_i(x_i^*)^2 - \theta_j(x_j^*)^2 = (\theta_i - \theta_j)(x_j^*)^2 + C_{ji} \frac{\theta_i}{1 - \theta_j} x_i^*(1 - x_i^*) = \frac{1 - \theta_i}{1 - \theta_j} x_i^* - \frac{1}{n} - \theta_j C_{ij} x_j^*(1 - \theta_j x_j^*) \), and

\[
\theta_j C_{ij} x_j^*(1 - x_j^*) \frac{1 - \theta_i}{1 - \theta_j} - \theta_i C_{ij} x_i^*(1 - x_i^*) \frac{1 - \theta_j}{1 - \theta_i} = C_{ij}(\theta(x_i^* - x_j^*) - \frac{1}{1 - \theta_i}) + (\theta_j x_j^*) \frac{1 - \theta_i}{1 - \theta_j} (\theta_j - \theta_i).
\]

Then, we obtain

\[
(x_i^* - x_j^*) \left( 1 - \theta_i(x_i^*)^2 - \theta_j(x_j^*)^2 + \frac{\theta_j - \theta_i}{n} + C_{ji} \frac{\theta_j}{1 - \theta_i} x_i^*(1 - x_i^*) - C_{ij} \frac{\theta_j}{1 - \theta_i} x_j^*(1 - x_j^*) + \sum_{k \in \mathcal{V}_p \setminus \{i, j\}} C_{ki} \frac{\theta_k(\theta_1 - \theta_i)}{1 - \theta_k} x_k^*(1 - x_k^*) \right) = \left( \theta_j - \theta_i \right) x_j^* \left( 1 - \theta_i \right) + C_{ij} \left( x_j^* \left( 1 - \theta_i \right) \right) \left( 1 - x_j^* \right) + \left( \theta_j - \theta_i \right) \left( x_j^* \right) \left( 1 - \theta_j \right) x_j^* \left( 1 - x_j^* \right),
\]

where \( \theta_i > \theta_j \) and \( 1 - \theta_i x_i^* + \theta_j C_{ij}(1 - x_j^* - x_j^*) \) indicate that the right hand side is negative. Moreover, since \( x_i^* + x_j^* < 1 \), then \( 1 - \theta_i x_i^* + x_j^* + \theta_j C_{ij}(1 - x_j^* - x_j^*) > 0 \), which implies that \( x_i^* < x_j^* \).

Regarding (iii), for any \( i, j \),

\[
x_i^* - x_j^* = (\theta_i - \theta_j)(x_j^*)^2 - \theta_j(x_j^*)^2 + \frac{\theta_j}{1 - \theta_i} x_i^*(1 - x_i^*) + C_{ji} \frac{\theta_j}{1 - \theta_i} x_j^*(1 - x_j^*) + C_{ij} \frac{\theta_j}{1 - \theta_i} x_i^*(1 - x_i^*) = \left( \theta_j - \theta_i \right) x_j^* \left( 1 - \theta_i \right) + C_{ij} \left( x_j^* \left( 1 - \theta_i \right) \right) \left( 1 - x_j^* \right) + \left( \theta_j - \theta_i \right) \left( x_j^* \right) \left( 1 - \theta_j \right) x_j^* \left( 1 - x_j^* \right).
\]

Moreover, we have

\[
(1 - \theta_j) \left( \sum_{k \in \mathcal{V}_i \setminus \{j\}} C_{ki} \frac{\theta_k x_k^*(1 - x_k^*)}{1 - \theta_k} \right) + (1 - \theta_i) \left( \sum_{k \in \mathcal{V}_j \setminus \{i\}} C_{kj} \frac{\theta_k x_k^*(1 - x_k^*)}{1 - \theta_k} \right) + (1 - \theta_j) \left( \sum_{k \in \mathcal{V}_i \setminus \{i\}} C_{ki} \frac{\theta_k x_k^*(1 - x_k^*)}{1 - \theta_k} \right) = \left( \theta_j - \theta_i \right) \left( \sum_{k \in \mathcal{V}_i \setminus \{j\}} C_{ki} \frac{\theta_k x_k^*(1 - x_k^*)}{1 - \theta_k} \right) + (1 - \theta_j) \left( \sum_{k \in \mathcal{V}_i \setminus \{j\}} C_{ki} \frac{\theta_k x_k^*(1 - x_k^*)}{1 - \theta_k} \right)
\]

and

\[
(1 - \theta_j) \left( \sum_{k \in \mathcal{V}_i \setminus \{j\}} C_{ki} \frac{\theta_k x_k^*(1 - x_k^*)}{1 - \theta_k} \right) = \left( \theta_j - \theta_i \right) \left( \sum_{k \in \mathcal{V}_j \setminus \{i\}} C_{kj} \frac{\theta_k x_k^*(1 - x_k^*)}{1 - \theta_k} \right) + (1 - \theta_j) \left( \sum_{k \in \mathcal{V}_i \setminus \{j\}} C_{ki} \frac{\theta_k x_k^*(1 - x_k^*)}{1 - \theta_k} \right).
\]

Then it follows that

\[
(x_i^* - x_j^*) C_{ij} \theta_j(1 - x_j^* - x_j^*) \frac{1 - \theta_i}{1 - \theta_j} = (\theta_j - \theta_i) \left( \sum_{k \in \mathcal{V}_i \setminus \{j\}} C_{ki} \frac{\theta_k x_k^*(1 - x_k^*)}{1 - \theta_k} \right) + (1 - \theta_j) \left( \sum_{k \in \mathcal{V}_i \setminus \{j\}} C_{ki} \frac{\theta_k x_k^*(1 - x_k^*)}{1 - \theta_k} \right) + \left( \theta_j - \theta_i \right) \left( x_j^* \right) \left( 1 - \theta_i \right) x_j^* \left( 1 - x_j^* \right),
\]

Since \( 1 - \theta_i x_i^* x_j^* > 0 \), we obtain that \( x_i^* < x_j^* \) holds if and only if \( \theta_i > \theta_j \).

APPENDIX E
PROOF OF THEOREM 2

According to Lemma 1, we just need to show the statements hold for system (4). Without loss of generality, let node 1 be the center node. Then, \( \Theta \) and \( C \) can be written as

\[
\Theta = \begin{bmatrix} 0_r \times r & 0 \\ 0 & \Theta_p \end{bmatrix}, \quad C = \begin{bmatrix} C_f & C_{fp} \\ C_{pf} & 0 \end{bmatrix}.
\]
where $\Theta_p = \text{diag}(\theta_{r+1}, \theta_{r+2}, \ldots, \theta_n)$, and $C_{pf} = I_{n-r}e^T_r$ with $e_1$ being a $r$-dimensional vector whose first element is 1 and others are 0. Let $V(x) = (I_n - \Theta W(x))^{-1}(I_n - \Theta)$, we have

$$V(x) = (I_n - \Theta \text{diag}(x) - \Theta C + \Theta \text{diag}(x) C)^{-1}(I_n - \Theta)$$

$$= \left[ -\Theta_p (I_{n-r} - \text{diag}(x)) C_{pf} I_{n-r} - \Theta_p \text{diag}(x) \right]^{-1} I_r$$

$$\times \left[ I_r 0 \begin{array}{c} 1 \end{array} \right] = \left[ \begin{array}{c} I_r 0 \begin{array}{c} 1 \end{array} \end{array} \right] V_p(x) V_p(x)^T,$$

where $x_p = (x_{r+1}, x_{r+2}, \ldots, x_n)^T$, $V_p(x) = (I_n - \Theta_p \text{diag}(x))^{-1} \Theta_p (I_{n-r} - \text{diag}(x)) C_{pf}$, and $V_p(x) = (I_n - \Theta_p \text{diag}(x))^{-1}(I_n - \Theta_p)$. Therefore,

$$F(x) = \frac{1}{n} \left[ 1 + e_1^T n_{r} \text{diag}(x) - \text{diag}(x) \right] 1_n,$$

where $V' = (I_{n-r} - \Theta_p \text{diag}(x))^{-1}$. Note that $V = \begin{pmatrix} I_r & 0 \\ 0 & V_p(x) \end{pmatrix}$, we obtain that $F_i(x) = \frac{1}{n} 1_n$ for $i \in \mathcal{V}_1 \setminus \{1\}$; 2) $F_i(x) = \frac{1}{n} 1_n$ for $i \in \mathcal{V}_p$.

By Theorem 1, we have $x_i^* \in \text{int} \Delta_n$. Regarding (ii), since $F_i(x) = 1/n$ for $i \in \mathcal{V}_1 \setminus \{1\}$, then $x_i^* = 1/n$ for any $i \in \mathcal{V}_1 \setminus \{1\}$. Regarding (iii), for $i \in \mathcal{V}_p$, we have that $F_i(x) = F_i(x_i) = \frac{1}{n} 1_n$. Since $F_j(x) \geq 1/n$ for any $j \in \mathcal{V}_1$, $F_i(x) \in [0, 1 - r/n]$. Next, we show that $F_i(x_i)$ is contractive on $[0, 1 - r/n]$. For any $x', x'' \in [0, 1 - r/n]$, we have

$$| F_i(x') - F_i(x'') | = \frac{\theta_i (1 - \theta_i) | x' - x'' |}{n(1 - \theta_i x_i)(1 - \theta_i x_i')},$$

in which $\frac{\theta_i (1 - \theta_i)}{n(1 - \theta_i (1-r/n)^2)} < 1$ follows from the fact that $\left( \frac{n-(n-r)}{2} \right) \theta_i^2 - 2(n-r)^2 \theta_i + n > 0$ for any $0 < \theta_i < 1$. Therefore, $F_i(x_i)$ is contractive on $[0, 1 - r/n]$ for any $i \in \mathcal{V}_p$. By the Banach fixed point theorem, $x_i(s)$ globally converges to unique equilibrium for any $x(0) \in \Delta_n$. In conclusion, for any $i \neq l$, $x_i(s)$ converges to unique $x^*_i$. Moreover, for any $i \in \mathcal{V}_p$, by the proof of Theorem 1, we obtain $x_i^* = \frac{n - \sqrt{n^2 - 4n\theta_i(1-\theta_i)}}{2n\theta_i}$. Suppose that $x_i^*$ is non-decreasing with respect to $\theta_i$, then, taking the derivative of $x_i^*$ with respect to $\theta_i$, we obtain

$$\frac{2n\theta_i - 4n\theta_i^2}{\sqrt{n^2 - 4n\theta_i(1-\theta_i)}} - n + \sqrt{n^2 - 4n\theta_i(1-\theta_i)} \geq 0,$$

which indicates that $n - 2\theta_i - \sqrt{n^2 - 4n\theta_i(1-\theta_i)} \geq 0$. Since $n - 2\theta_i > 0$, we have $(n - 2\theta_i)^2 \geq n^2 - 4n\theta_i(1-\theta_i)$, i.e., $1 \geq n$, which is a contradiction. Thus, $x_i^*$ is decreasing with respect to $\theta_i$. Regarding (iv), since $s_i = 1/n + \frac{\theta_i (1 - \theta_i)}{n(1 - \theta_i x_i)}$, then $x_i(s)$ globally converges to $1/n + \frac{n \theta_i (1 - \theta_i)}{n(1 - \theta_i x_i)} > 1/n$.

**APPENDIX F**

**PROOF OF THEOREM 3**

Let $\beta_i = \theta_i (1 - x_i)$ and $\gamma_i = 1 - \theta_i x_i$. Without loss of generality, let node $r + 1$ be the center node of $\mathcal{G}(C)$, i.e., $l = r + 1$. Similarly, $C$ can be written as

$$C = \begin{pmatrix} 0 & C_{pf} \\ C_{pf}^T & C_{p} \end{pmatrix},$$

where $C_{pf} = e_1 (C_{11}, C_{12}, \ldots, C_{1r})$, $C_{p} = (I_n - e_1 e_1^T + \epsilon_0, C_{l+1}, \ldots, C_{ln})$ with $e_1$ being a $(n - r)$-dimensional vector whose first element is 1 and others are 0. Then,

$$V(x) = \begin{pmatrix} I_{r \times r} & 0 \\ \hat{V}^{-1} \Theta_p (I_{n-r} - \text{diag}(x)) C_{pf} \\ \hat{V}^{-1}(I_n - \Theta_p) \end{pmatrix},$$

where $x_p = (x_{r+1}, x_{r+2}, \ldots, x_n)^T$, $\hat{V} = I_n - \Theta_p \text{diag}(x) - \Theta_p (I_{n-r} - \text{diag}(x)) C_{pf}$, and $\hat{V}(x) = (I_n - \Theta_p \text{diag}(x))^{-1}(I_n - \Theta_p)$. Therefore,

$$F(x) = \frac{1}{n} \left[ 1 + C_{pf}^T \Theta_p (I_{n-r} - \text{diag}(x)) (\hat{V}^{-1})^T 1_n - \Theta \hat{V}(x) \right].$$

By column transformations, we obtain the first column of $\hat{V}^{-1}$ is $\frac{\partial}{\partial x_i}$, and the i-th column of $\hat{V}^{-1}$ is

$$\frac{\partial}{\partial x_i} \frac{\gamma_{l-r-1} \beta_i}{\alpha_{l-r+1}}$$

for $2 \leq i \leq n - r$, where $\alpha = (\beta_1, \beta_2, \ldots, \beta_{n-r+1})^T$, $\beta = (\beta_1, \beta_2, \ldots, \beta_{n-r+1})^T$, $\gamma_i = \gamma_i - \beta_i \sum_{j \neq i} \gamma_j$. Thus, we have that $F_i(x) = \frac{\xi_i (1 - \theta_i)}{\alpha_i}$ for any $i \in \mathcal{V}_1$, and $F_i(x) = \frac{\xi_i (1 - \theta_i)}{\alpha_i}$ for any $i \in \mathcal{V}_p \setminus \{l\}$, where $\xi = 1 + \sum_{j \neq i} \gamma_j$. Similarly, we only need to show that the equilibrium social power of system (4) satisfies all statements. Denote $\beta_i^*, \gamma_i^*$, $\alpha^*$ and $\xi^*$ as $\beta_i$, $\gamma_i$, $\alpha$ and $\xi$ corresponding to $x_i^*$, respectively. Regarding (i), a) by Theorem 1, we have $x_i^* \in \text{int} \Delta_n$. b) For $i \in \mathcal{V}_1$, we have that $x_i^* = \frac{1 + \xi_i \beta_i^*}{\alpha_i}$, then, if $C_{li} = 0$, $x_i^* = \frac{1}{n}$. Otherwise, $x_i^* > \frac{1}{n}$ follows from that $\alpha^*$, $\xi^*$ and $\beta_i^*$ are all positive. c) For $i \in \mathcal{V}_p \setminus \{l\}$, if $C_{li} = 0$, we have
that \( x_i^* = \frac{1 - \theta_i}{n(1 - \theta_i x_i^* - \Delta)} \). Then, by the proof of Theorem 2 we have that \( x_i^* = \frac{n - \sqrt{n^2 - 4n\theta_i(1 - \theta_i)}}{2n\theta_i} \) and is decreasing with respect to \( \theta_i \).

Regarding (ii), since \( C_{li} = 0 \) for any \( i \in V_P \setminus \{l\} \), we have that \( \alpha(s) = \gamma_i(s) \) and \( \alpha^* = \gamma_i^* \). A) For \( i \in V_P \setminus \{l\} \), since \( C_{li} = 0 \), by Theorem 2 \( x_i(s) \) globally converges to \( x_i^* = \frac{\xi^*(1 - \theta_i)}{n\gamma_i^*} \).

b) Note that \( x_i^* = \frac{\xi^*(1 - \theta_i)}{n\gamma_i^*} \), then \( x_i^* = \frac{n - \sqrt{n^2 - 4n\theta_i(1 - \theta_i)\xi^*}}{2n\theta_i} \) since \( x_i^* < 1 \).

Moreover, for \( i \in V_P \setminus \{l\} \), since \( C_{li} = 0 \), we have that \( n x_i^* = \frac{\xi^*}{n\gamma_i^*} \), which implies that \( \gamma_i^* = 1 - 1 - \theta_i = 1 - nx_i^* \). Therefore, \( \xi^* = 1 + \sum_{j \in V_N \setminus \{l\}} \frac{\beta_j^*}{\gamma_j^*} = 1 + \sum_{j \in V_N \setminus \{l\}} (1 - nx_j^*) = n - r - n \sum_{j \in V_N \setminus \{l\}} x_j^* \).

Then, the uniqueness of \( x_i^* \) follows from the uniqueness of \( \xi^* \).

c) For \( i \in V_l \), we have \( x_i^* = \frac{1}{n} + \frac{\xi^*}{n\gamma_i^*} \), which implies that \( x_i^* = \frac{1}{n} + \frac{\xi^*}{n\gamma_i^*} \).

Finally, the uniqueness of \( x_i^* \) follows from the fact that \( \xi^* \) and \( x_i^* \) are both unique.

**APPENDIX G**

**PROOF OF COROLLARY 3**

First, we show that the equilibrium social power of system (4) satisfies all statements. Regarding (i), for any \( i, j \in V_l \), since \( \alpha^* \), \( \xi^* \) and \( \beta_j^* \) are all positive, we have that \( x_i^* = \frac{1}{n} + \frac{\xi^*}{n\gamma_i^*} \).

Regarding (ii), for any \( i \in V_l \) and \( j \in V_P \setminus \{l\} \) with \( C_{li} = C_{lj} \),

\[
x_i^* - x_j^* = \frac{1}{n}(1 - \theta_j) + \frac{\xi^* \beta_j^* C_{lj}}{n\alpha^*} (1 - \frac{1 - \theta_j}{\gamma_j^*}) = \frac{1}{n}(1 + \frac{\xi^* \beta_j^* C_{lj}}{n\alpha^*} (1 - \theta_j)),
\]

Since \( 1 + \frac{\xi^* \beta_j^* C_{lj}}{n\alpha^*} > 0 \) and \( 1 - \frac{\theta_j}{\gamma_j^*} > 0 \), we obtain that \( x_i^* - x_j^* > 0 \).

Regarding (iii), for any \( i, j \in V_P \setminus \{l\} \) with \( C_{li} = C_{lj} \),

\[
x_i^* - x_j^* = \frac{1}{n}(1 + \xi^* \beta_j^* C_{lj}) \left( \frac{1 - \theta_i}{\gamma_i^*} - \frac{1 - \theta_j}{\gamma_j^*} \right) = \frac{1 - \theta_i}{n} (1 + \xi^* \beta_j^* C_{lj}) \left( \frac{1 - \theta_i}{\gamma_i^*} - \frac{1 - \theta_j}{\gamma_j^*} \right)
\]

where the last equation follows from that \( C_{li} = C_{lj} \) and \( \frac{\theta_i - \theta_j}{\gamma_i^* - \gamma_j^*} = \frac{1}{\gamma_i^*} (\theta_i - \theta_j) + \frac{1}{\gamma_j^*} (\theta_i - \theta_j) \). Note that

\[
1 - \frac{\theta_i}{\gamma_i^*} (1 + \xi^* \beta_j^* C_{lj}) = \frac{n}{\gamma_i^*} \gamma_j^* \left( 1 - \frac{\theta_i}{\gamma_i^*} \right) - x_i^* \left( 1 - \theta_i \right) - \theta_j.
\]

That is,

\[
x_i^* - x_j^* = \frac{n}{\gamma_i^*} \gamma_j^* \left( 1 - \frac{\theta_i}{\gamma_i^*} \right) - x_i^* \left( 1 - \theta_i \right) - \theta_j.
\]

Therefore,

\[
(x_i^* - x_j^*)(1 - \theta_j (x_i^* + x_j^*)) = (x_i^* (1 - \frac{\theta_i}{\gamma_i^*}) (1 - \theta_j)).
\]

Since \( x_j^* \in \Delta_n \) and \( \theta_j < 1 \), we have that \( 1 - \theta_j (x_j^* + x_j^*) > 0 \) and \( \gamma_j^*/(1 - \theta_j) > 1 > x_j^* \), which means that \( x_i^* > x_j^* \).

Regarding (iv), for any \( i, j \in V_P \setminus \{l\} \) with \( \theta_i = \theta_j \), we obtain that

\[
x_i^* - x_j^* = \frac{1}{n} - \theta_i \left( \frac{1}{\gamma_i^*} - \frac{1}{\gamma_j^*} \right) + \frac{\xi^* \beta_j^* C_{lj}}{n\alpha^*} \left( \frac{C_{li} - C_{lj}}{\gamma_j^*} \right)
\]

Thus, it follows that

\[
(x_i^* - x_j^*)(1 - \theta_j (x_i^* + x_j^*)) = (1 - \theta_i \xi^* \beta_j^* C_{lj} (C_{li} - C_{lj}))
\]

with \( \frac{1}{n} \xi^* \beta_j^* > 0 \), which means that \( x_i^* > x_j^* \) if and only if \( C_{li} > C_{lj} \).

**APPENDIX H**

**PROOF OF THEOREM 5**

The proof of (i) follows from Theorem 2. Regarding (ii), for \( i \in V_P \setminus \{l\} \), Theorem 2 implies that \( x_i(s) \) converges to \( x_i^* \) for any \( x(0) \in \Delta_n \). Since \( \xi(s) = 1 + \sum_{j \in V_N \setminus \{l\}} (1 - \theta_j) \) only depends on \( x(s) \), \( \xi(s) \) converges to \( \xi^* \) for any \( x(0) \in \Delta_n \). For \( i \in V_l \), note that \( x_i(s + 1) = \frac{1}{n} + \frac{\xi(s) \beta_j} {n\gamma_j} C_{lj} \), which depends on \( \xi(s) \) and \( x_i(s) \).

Therefore, \( x_i(s) \) converges if \( x_i(s) \) converges for any \( x(0) \in \Delta_n \). Since \( F_i(x) \geq 1/n \) for all \( i \), \( F_i(x) \geq 1/(1 - \theta_i) \) for \( i \in V_P \), we have \( x(s) \in \{ x \in \Delta_n | (1 - \theta_i)/n \leq x_i \leq a_i, i \in V_P \} \) for any \( s \geq 1 \), where \( a_i = \frac{n-r}{n} - \sum_{j \in V_N \setminus \{l\}} \frac{1 - \theta_j}{n} \).

Next, we show that \( F_i(x') = \frac{\xi'}{n} \xi'' \) is contractive on \( \{ x \in \Delta_n | (1 - \theta_i)/n \leq x_i \leq a_i, i \in V_P \} \). Consider \( x', x'' \in \{ x \in \Delta_n | (1 - \theta_i)/n \leq x_i \leq a_i, i \in V_P \} \), we have that

\[
|F_i(x') - F_i(x'')| = \frac{1}{n} \left| \frac{\xi'}{\gamma_i} - \frac{\xi''}{\gamma_i} \right| 
\leq \frac{1}{n} \left| \frac{\xi'}{\gamma_i} - \frac{\xi''}{\gamma_i} \right| + \frac{1}{n} \left| \frac{\xi'}{\gamma_i} - \frac{\xi''}{\gamma_i} \right|.
\]
On one hand,
\[ |\xi' - \xi''| = \left| \sum_{j \in V_p \setminus \{l\}} \left( \frac{1 - \theta_j}{\gamma_j} - \frac{1 - \theta_j}{\gamma_j} \right) \right| \]
\[ \leq \sum_{j \in V_p \setminus \{l\}} \left( 1 - \theta_j \right) \frac{1 - \gamma_j}{\gamma_j} \cdot \frac{1}{n} \cdot \frac{1}{(1 - \theta_j(1 - r/n))^2} . \]

Because \( a_j < 1 - r/n \) for \( j \in V_p \setminus \{l\} \), by Theorem 1, we have that for any \( j \in V_p \setminus \{l\} \),
\[ |\frac{1}{\gamma_j} - \frac{1}{\gamma''_j}| = \frac{\theta_j}{\gamma_j \gamma''_j} |x'_j - x''_j| \leq \frac{\theta_j}{(1 - \theta_j(1 - r/n))^2} |x'_j - x''_j| . \]

Therefore, there holds
\[ |\xi' - \xi''| \leq \sum_{j \in V_p \setminus \{l\}} \frac{\theta_j(1 - \theta_j)}{(1 - \theta_j(1 - r/n))^2} |x'_j - x''_j| . \]

On the other hand, for the center node,
\[ |\frac{1}{\gamma_l} - \frac{1}{\gamma''_l}| = \frac{\theta_l}{\gamma_l \gamma''_l} |x'_l - x''_l| \leq \frac{\theta_l}{(1 - \theta_l(1 - r/n))^2} |x'_l - x''_l| . \]

Thus,
\[ |F_l(x') - F_l(x'')| \leq \xi''_l \frac{\theta_l(1 - \theta_l)}{n(1 - \theta_l a_l)^2} |x'_l - x''_l| \]
\[ + \frac{1 - \theta_l}{n \gamma''_l} \max_{j \in V_p \setminus \{l\}} \frac{\theta_j(1 - \theta_j)}{(1 - \theta_j(1 - r/n))^2} \sum_{j \in V_p \setminus \{l\}} |x'_j - x''_j| . \]

Denote \( \lambda \) by
\[ \max \left\{ \frac{1 - \theta_l}{n \gamma''_l} \max_{j \in V_p \setminus \{l\}} \frac{\theta_j(1 - \theta_j)}{(1 - \theta_j(1 - r/n))^2}, \frac{\xi''_l \theta_l(1 - \theta_l)}{n(1 - \theta_l a_l)^2} \right\} . \]

Then, we have
\[ |F_l(x') - F_l(x'')| \leq \lambda n \sum_{j \in V_p \setminus \{l\}} |x'_j - x''_j| \leq \lambda \|x' - x''\|_1 . \]

By the proof of Theorem 1, we have that
\[ \frac{\theta_j(1 - \theta_j)}{(1 - \theta_j(1 - r/n))^2} < n \text{ for any } j \in V_p \setminus \{l\}, \]

i.e., \( \max_{j \in V_p \setminus \{l\}} \frac{\theta_j(1 - \theta_j)}{(1 - \theta_j(1 - r/n))^2} < n \).

Moreover, since \( \frac{1 - \theta_l}{n \gamma''_l} \max_{j \in V_p \setminus \{l\}} \frac{\theta_j(1 - \theta_j)}{(1 - \theta_j(1 - r/n))^2} < 1 \), we have that
\[ \frac{\xi''_l \theta_l(1 - \theta_l)}{n(1 - \theta_l a_l)^2} < 1 \text{ for any } j \in V_p \setminus \{l\} . \]

We prove that \( \xi''_l \theta_l(1 - \theta_l) < \frac{4n}{5} - 1 \). Note that \( \frac{\xi''_l \theta_l(1 - \theta_l)}{n(1 - \theta_l a_l)^2} < 1 \)

means \( (na_l^2 + \xi'') \theta_l^2 - 2na_l + \xi'') \theta_l + n > 0 \). That is,
\[ (2na_l + \xi'')^2 - 4n(\xi''^2 + \xi'') < 0 \text{, which is equivalent to } \xi' < 4n(1 - a_l) = 4(n - 1) - 4 \sum_{j \in V_p \setminus \{l\}} \theta_j . \]

Since \( \xi'' < 1 + \sum_{j \in V_p \setminus \{l\}} \theta_j \), and \( \sum_{j \in V_p \setminus \{l\}} \theta_j \leq \frac{4n}{5} - 1 \), we have that \( \xi'' < 4(n - 1) - 4 \sum_{j \in V_p \setminus \{l\}} \theta_j . \)

In conclusion, for any \( x', x'' \in \{x \in \Delta_n | (1 - \theta_j)/n \leq x_i \leq a_j, i \in V_p \} \), we have that \( |F_l(x') - F_l(x'')| \leq \lambda \|x' - x''\|_1 \), which means \( F_l(x) \) is contractive on \( \{x \in \Delta_n | (1 - \theta_j)/n \leq x_i \leq a_j, i \in V_p \} \).

By the Banach fixed point theorem, we have \( x_l(s) \) converges for any \( x(0) \in \Delta_n \), which implies that \( x(s) \) globally converges to \( x^* \) for any \( x(0) \in \Delta_n \) exponentially fast.

APPENDIX I

PROOF OF THEOREM 6

Since \( \hat{G}(x) \) is an analytic function of \( x \in \mathbb{A} \), it is differentiable on \( \text{int} \, \mathbb{A} \) and continuous on \( \mathbb{A} \). Let \( B = \Theta(I_n - C)V \) and \( B_i \) be the \( i \)-th row of \( B \). Then,
\[ I_n \otimes \Theta W(x) = \begin{bmatrix} \Theta CV_1 \\ \Theta CV_2 \\ \vdots \\ \Theta CV_n \end{bmatrix} + \begin{bmatrix} \text{diag}(\omega)\Theta(I_n - C)V_1 \\ \text{diag}(\omega)\Theta(I_n - C)V_2 \\ \vdots \\ \text{diag}(\omega)\Theta(I_n - C)V_n \end{bmatrix} . \]

Furthermore, since \( \frac{\partial}{\partial V_i} \text{diag}(\omega)\Theta(I_n - C)V_j = \text{diag}(\omega)\Theta(I_n - C)V_j \), we obtain
\[ \frac{\partial}{\partial V_i} \text{diag}(\omega)\Theta(I_n - C)V_j = \frac{B_{ij}}{n} e_i e_j^T \]

for any \( j \neq i \), where \( B_{ij} \) is the \( ij \)-th entry of \( B = \Theta(I_n - C)V \). Hence,
\[ \frac{\partial}{\partial V_i} (I_n \otimes \Theta W(x)) x = \begin{bmatrix} 0_{n \times n} \\ 0_{n \times n} \\ \vdots \\ \Theta C \end{bmatrix} + \begin{bmatrix} B_{11} e_1 e_1^T / n \\ B_{22} e_2 e_2^T / n \\ \vdots \\ B_{nn} e_n e_n^T / n \end{bmatrix} . \]

where \( B_{ii} = \text{diag}(\omega)\Theta(I_n - C) \). Consequently, \( \frac{\partial G}{\partial x} = I_n \otimes \Theta W(x) + H/n \), where \( H = \begin{bmatrix} B_{11} \otimes (e_1 e_1^T/n) \\ B_{22} \otimes (e_2 e_2^T/n) \\ \vdots \\ B_{nn} \otimes (e_n e_n^T/n) \end{bmatrix} \).

Note that
\[ \left\| \frac{\partial \hat{G}}{\partial x} \right\|_\infty \leq \|\Theta W(x)\|_\infty + \frac{\|H\|_\infty}{n} = \theta_{\text{max}} + \max_{i,j} |B_{ij}| . \]

Since \( 0 \leq V_{ij} \leq 1 \) and \( 0 \leq \sum_{k=1}^{n} C_{ik} V_{kj} \leq 1 \), we have \( |V_{ij} - \sum_{k=1}^{n} C_{ik} V_{kj}| \leq V_{ij} \leq 1 \) if \( V_{ij} \geq \sum_{k=1}^{n} C_{ik} V_{kj} \), and \( |V_{ij} - \sum_{k=1}^{n} C_{ik} V_{kj}| \leq \sum_{k=1}^{n} C_{ik} V_{kj} \) if \( V_{ij} \leq \sum_{k=1}^{n} C_{ik} V_{kj} \). Thus, it follows that \( |B_{ij}| = \theta_{\text{max}} + \max_{i,j} |V_{ij}| - \sum_{k=1}^{n} C_{ik} V_{kj} \leq \theta_{\text{max}} . \) Therefore, we obtain
that $\partial \hat{G}/\partial x \leq 2\theta_{\text{max}} < 1$. Similar with the proof of Theorem 1, we obtain that $\hat{G}(x)$ is contractive on $\mathcal{A}$. Then, exponential convergence of system (8) follows from the Banach fixed point theorem.