SCHWARZSCHILD MANIFOLD AND
NON-REGULAR COORDINATE TRANSFORMATIONS
(A CRITICO-HISTORICAL NOTE)

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ABSTRACT. A careful analysis of the maximally extended metrics of Schwarzschild manifold shows that the original Schwarzschild’s solution (1916) and Brillouin’s solution (1923) are the only ones that are adequate from the physical standpoint. Contrary to the other maximally extended metrics, they represent faithfully the gravity field created by the mass-point.

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1. – According to the current literature, the maximally extended metrics of Schwarzschild manifold, which is created by a gravitating point-mass \( m \), are in primis the following: i) the Eddington-Finkelstein \([1]\) metric, ii) the Lemaître \([2]\) and Robertson \([3]\) metrics, iii) the Kruskal-Szekeres \([4]\) metric.

We anticipate their main defects: \( \alpha \) the “soft” singularity at \( r = 2m \) of the standard metric (see eq. (1) of sect. 2) is “hidden” in the differentials of the new coordinates with respect to the standard ones; \( \beta \) an impairing of the permanent gravitational field of mass-point (a consequence of \( \alpha \)); \( \gamma \) a time-dependent \( ds^2 \) (with the only exception of metric \([1]\), which is stationary, non-reversible) for a static problem.

The older maximally extended metrics by Schwarzschild \([5]\) and by Brillouin \([6]\) are generally ignored: a fact that will be clarified by the future history of physics. We hope.

2. – The Eddington-Finkelstein metric \([1]\) is obtained from the standard (Hilbert-Droste-Weyl) metric

\[
(1) \quad ds^2 = \left( \frac{r}{r - \alpha} \right) dr^2 + r^2 d\omega^2 - \left( \frac{r - \alpha}{r} \right) dt^2
\]

– where: \( \alpha \equiv 2m; \ c = G = 1; \ d\omega^2 \equiv d\theta^2 + \sin^2 \theta d\varphi^2 \) – with the following transformation of time coordinate \( t \):

\[
(2) \quad t = t' \pm \alpha \ln |r - \alpha| \ , \quad \Rightarrow
\]
we see that both eqs. \((2)\) and \((2')\) contain the “soft” singularity at \(r = \alpha\) of eq. \((1)\), that Finkelstein intended to remove from the \(ds^2\). We have:

\[
ds^2 = (\frac{r + \alpha}{r}) dr^2 + r^2 d\omega^2 - (\frac{r - \alpha}{r}) dt'^2 \mp \frac{2\alpha}{r} dr dt' = \\
= dr^2 + r^2 d\omega^2 - dt'^2 \mp \frac{\alpha}{r} (dr + dt')^2 \tag{3}
\]

the behaviour of the radial light-rays is particularly interesting; from \(0 = ds^2 = d\theta = d\varphi\), we get two pairs of values for \(\frac{dr}{dt'}\):

\[
(4) \quad \frac{r - \alpha}{r + \alpha} ; \quad -1 ; \\
(4') \quad 1 ; \quad -\frac{r - \alpha}{r + \alpha} ;
\]

light velocity depends on the considered direction (positive or negative) of coordinate \(r\). Remark that eqs. \((3)\), \((4)\), \((4')\) hold for \(r > 0\). At \(r = 0\) we have a “hard” singularity (Kretschmann scalar = \(\infty\)), the same of standard \(ds^2\) of eq. \((1)\). Eqs. \((4)\) and \((4')\) tell us that light-rays do not “feel” the Hilbertian gravitational repulsion \([7]\); the non-reversibility of the Eddington-Finkelstein field plays here an important role. And we can also affirm that the radial geodesics of test-particle too are not subjected to a Hilbertian repulsion. Transformation \((2) - (2')\) has sensibly impaired the action of the gravitational field of mass \(m\).

The “hard” singularity at \(r = 0\) of eq. \((1)\) can be removed with a simple change of the radial coordinate \(r\); for instance:

\[
(5) \quad r \rightarrow (r^3 + \alpha^3)^{1/3} , \\
\]

or

\[
(5') \quad r \rightarrow r + \alpha ;
\]

with these substitutions, the form \((1)\) yields, respectively, the original Schwarzschild form \([5]\) or the Brillouin form \([6]\). Quite analogously, \((5)\), or \((5')\) remove the singularity at \(r = 0\) of eq. \((3)\). Physics remains unaltered! Indeed, the true physics of standard and Schwarzschild-Brillouin forms of \(ds^2\) concerns, respectively, the spatial regions \(r > \alpha\) and \(r > 0\); the distorted physics of eq. \((3)\), and of its transformed with substitution \((5)\) or \((5')\), concerns, respectively, the spatial regions \(r > 0\) and \(r \geq 0\).

2bis. – Let us consider the pair of expressions \((1)\) and perform the integration of \((dr/dt') = [(r - \alpha)/(r + \alpha)]\), and of \((dr/dt') = -1\). We have
\[ t' = r + 2\alpha \ln |r - \alpha| + \text{const} \]  

\[ t' = -r + \text{const} \]

According to (6.1), the light-rays which start from an \( r > \alpha \) move away from the origin \( r = 0 \), while those which start from an \( r < \alpha \) go towards \( r = 0 \), and reach it in a finite time. According to (6.2), all rays move towards \( r = 0 \), and reach it in a finite time. The pair of expressions (6.1) gives results that can be obtained from eqs. (6.1) and (6.2) with the substitution \( t' \rightarrow -t' \).

Remark that, in particular, the equation \( \frac{dr}{dt'} = -\left[\frac{r - \alpha}{r + \alpha}\right] \) tells us that the light-rays which start from an \( r > \alpha \) will reach \( r = \alpha \) in an infinite time interval. Analogously, if in eq. (3) we make, e.g., the substitution (5.2), the integration of equation \( \frac{dr}{dt'} = -\left[\frac{r}{r + 2\alpha}\right] \) gives, for a generic \( \bar{r} \):

\[ \Delta t' = -\int_{\bar{r}}^{0} \frac{r + 2\alpha}{r} \, dr = +\infty \]

For the radial light-rays of standard \( ds^2 \) (eq. (11)), the integration of \( \frac{dr}{dt'} = \pm \left[\frac{r - \alpha}{r + \alpha}\right] \) gives:

\[ t = \pm \left[ r + \alpha \ln \left| \frac{r - \alpha}{\alpha} \right| \right] + \text{const} , \quad (r > \alpha) \]

the surface \( r = \alpha \) is reached in an infinite time interval (an instance of Hilbertian repulsion).

3. – The Lemaître time-dependent metric \( ds^2 \) is obtained from eq. (11) by means of the following coordinate transformations:

\[ r = \left[ \frac{3}{2} \alpha^{1/2} (\tau - \chi) \right]^{2/3} \]

\[ t = \tau + 2(\alpha r)^{1/2} + \alpha \ln \left| \frac{r^{1/2} - \alpha^{1/2}}{r^{1/2} + \alpha^{1/2}} \right| \]

from which:

\[ dt = d\tau + \frac{r}{r - \alpha} \alpha^{1/2} r^{-1/2} \, dr \]

We have:

\[ ds^2 = \frac{\alpha}{r} \, d\chi^2 + r^2 \, d\omega^2 - d\tau^2 \]

The speed of the radial (\( d\omega = 0 \)) light-rays is:
\[ \frac{d\chi}{d\tau} = \pm \left( \frac{r}{\alpha} \right)^{1/2}; \]

thus: \( d\chi/d\tau = 0 \) for \( r = 0 \); \( d\chi/d\tau = \pm 1 \) for \( r = \alpha \); \( d\chi/d\tau = \pm \infty \) for \( r = \infty \). As it is clear, there is a Hilbertian repulsion along the whole trajectory (as it happens for eq. (1), cfr. eq. (7)). And the test-particles moving along radial geodesics will “feel” the Hilbertian repulsion in some portions of their trajectories (as it happens for eq. (1)).

The “hard” singularity at \( r = 0 \) of eq. (9) can be removed by one \( (ad \ libitum) \) of the substitutions (5), (5'), exactly as in the case of Eddington-Finkelstein form of \( ds^2 \).

Robertson metric [3] can be obtained from Lemaître metric with the transformation

\[ \chi = -\frac{2}{3} \chi^{3/2} \alpha^{-1/2}; \]

clearly, this metric has the same general properties of Lemaître’s one.

4. – The interval \( ds^2 \) of Kruskal-Szekeres metric is [4]:

\[ ds^2 = 4 \frac{\alpha^3}{r} \exp \left( -\frac{r}{\alpha} \right) (du^2 - dv^2) + r^2 d\omega^2; \]

where \( r \) is a function of the space-like coordinate \( u, (-\infty < u < +\infty) \), and of the time-like coordinate \( v, (-\infty < v < +\infty) \); more precisely:

\[ \left( \frac{r}{\alpha} - 1 \right) \exp \left( \frac{r}{\alpha} \right) = u^2 - v^2; \text{ and} \]

\[ t = 2\alpha \arctanh \left( \frac{v}{u} \right). \]

This metric is invariant under the substitutions \( u \to -u \) and \( v \to -v \). Each point of metric (11) has a twofold representation in metric (12): an odd-looking \textit{embarras de richesse}.

Seemingly, the singularity at \( r = \alpha \) of eq. (1) does not appear in eq. (12). Now, the differentials \( du, dv \) of the functions \( u(r,t), v(r,t) \) are singular at \( r = \alpha \) ! (See sect. A3 of the Appendix).

Metric (12) suffers from various defects, for instance: \( i) \) it is \( v \)-dependent, \( i.e. \) dependent on a time-like coordinate, \( ii) \) the radial \( (d\omega = 0) \) light-rays do not “feel” the gravity: \( ds^2 = 0 \) gives \( du = \pm dv \), the light-cones are “open” as in \textit{special} relativity: an apparent worth, a physical fault, a revenge of the “soft” singularity at \( r = \alpha \) of eq. (11), which has been swept away into a commonly unobserved corner (the differentials \( du, dv \)).

4bis. – The “hard” singularity at \( r = 0 \) of metric (12) can be removed with a suitable substitution of the radial coordinate, \( e.g. \) with (5) or (5'). The
new \( r = 0 \) represents the previous \( r = \alpha \); the interior region \( r < \alpha \) loses any meaning and dies away from existence: a trivial consequence of the fact that the choice of the radial coordinate is quite free, and allows a shifting of the standard \( r \), which eliminates the “hard” singularity at \( r = 0 \), in spite of the infinite value of its Kretschmann scalar. (Physics does not always coincide with geometry).

If, for instance, we perform the shifting \( r \to r + \alpha \), write for clarity’s sake \( r = \bar{r} + \alpha \), and call \( U, V \) the new space-like and time-like coordinates, eq. (12) becomes:

\[
\begin{align*}
\text{(12')} \\
\text{d} s^2 &= -\frac{4\alpha^3}{\varrho + \alpha} \exp\left(-\frac{\varrho + \alpha}{\alpha}\right) (\text{d}V^2 - \text{d}U^2) + (\varrho + \alpha)^2 \text{d}\omega^2 ; \quad (0 \leq \varrho < \infty) .
\end{align*}
\]

We have:

\[
\begin{align*}
\{ & U(\varrho, t) = \left(\frac{\varrho}{\alpha}\right)^{1/2} \exp\left(\frac{\varrho + \alpha}{2\alpha}\right) \cosh\left(\frac{t}{2\alpha}\right) , \\
& V(\varrho, t) = \left(\frac{\varrho}{\alpha}\right)^{1/2} \exp\left(\frac{\varrho + \alpha}{2\alpha}\right) \sinh\left(\frac{t}{2\alpha}\right) ;
\end{align*}
\]

from which:

\[
\begin{align*}
\text{(14')} & \quad \left(\frac{\varrho}{\alpha}\right) \exp\left(\frac{\varrho + \alpha}{\alpha}\right) = U^2 - V^2 . \quad \\
\text{(14''')} & \quad t = 2\alpha \text{arctanh}\left(\frac{V}{U}\right) .
\end{align*}
\]

Apart from the substitutions \( U \to -U \) and \( V \to -V \), we have here a unique form (14) for the functions \( U(\varrho, t), V(\varrho, t) \). On the contrary, in the Kruskal-Szekeres metric (12) there are four different pairs of coordinates: two for \( r > \alpha \), and two for \( r < \alpha \): a real patchwork. (See sect.A3 of the Appendix).

4ter. – A constant radial coordinate, \( r = \text{const} \), is represented in a Cartesian plane \( (u, v) \) – or \( (U, V) \) – by an equilateral hyperbola \( u^2 - v^2 = \text{const} \) – or \( U^2 - V^2 = \text{const} \). If we choose as new coordinates, say \( (u', v') \) – or \( (U', V') \), the asymptotes of these hyperbolae, their equations become \( u'v' = \text{const} \) – or \( U'V' = \text{const} \), but we have lost the difference between space-like and time-like coordinates: a not negligible disadvantage, from a physical standpoint. The null lines of radial \( (\text{d}\omega = 0) \) light-rays are represented by equations \( u' = \text{const}, v' = \text{const} \) – or \( U' = \text{const}, V' = \text{const} \).

Some authors take the equations \( u' = \text{const}, v' = \text{const} \) as a starting point for a direct derivation of Kruskal-Szekeres metric, avoiding any reference to eq. (1). Thus, they postulate that the radial light-rays are not subjected to the gravitational field of mass \( m \). This is an ad hoc assumption: ad hoc,
because its aim is the exclusion from the metric of the “soft” singularity at \( r = \alpha \) (finite value of Kretschmann scalar). Now, this singularity, which could be qualified as non-physical because the mass-point is in \( r = 0 \), is not a mere spurious hindrance: as it was first demonstrated by Schwarzschild [5] in the original construction of the homonymous manifold, it plays actually a fundamental role. Indeed Schwarzschild \( ds^2 \) is:

\[
(15) \quad ds^2 = \left( \frac{R}{R - \alpha} \right) dR^2 + R^2 d\omega^2 - \left( \frac{R - \alpha}{R} \right) dt^2 ,
\]

where: \( R \equiv (r^3 + \alpha^3)^{1/3} \); \( 0 < r < \infty \). The point-mass \( m \) is situated in \( r = 0 \), see the correspondence with Newton theory; the singularity at \( r = 0 \) of eq. (15) coincides with the singularity at \( r = \alpha \) of eq. (1).

When we look for a solution of Einstein equations \( R_{jk} = 0 \), \( (j, k = 1, 2, 3, 4) \), a solution with singularities, we mean –, we must be very careful about the choice of the reference system. Indeed, a system which appears simple and reasonable from a geometrical standpoint, can originate some misleading properties, as for instance a weakening of the permanent gravitational fields.

Kruskal-Szekeres metric [4] “does not make physical sense”, as Bonnor wrote in the article quoted in [4], and a similar negative judgement was expressed by this author on the Novikov metric (1963) of Schwarzschild manifold, which “throws some light on the Kruskal diagram \([u,v]\), without removing all its obscurities.”

Conclusion – We have evidenced the shortcomings of the metrics by Eddington-Finkelstein, Lemaître and Robertson, Kruskal-Szekeres. Schwarzschild’s \( ds^2 \) [5] and Brillouin’s \( ds^2 \) [6] give maximally extended metrics which describe perfectly the physical reality, and make clear that standard \( ds^2 \) (eq. (1)) holds only for \( r > \alpha \). A fact confirmed by Hilbertian gravitational repulsion [7].

APPENDIX

“Bildräume” (Weyl) and representative spaces (Synge)

A1. – The notion of Bildraum (picture space) has been introduced in GR by Weyl (see, e.g., [8]). Synge spoke of a “representative space”, theorized its use and applied it in his study on “The gravitational field of a particle” [2]. In a particular and important problem, Fock employed a “conformal space” [9]. Eddington utilized the concept in a subtle and indirect way when he wrote, at the beginning of his treatment of Schwarzschild manifold [10]: “In a flat space-time the interval, referred to spherical polar coordinates and time, is \( ds^2 = -dt^2 - r^2 d\vartheta^2 - r^2 \sin^2 \vartheta d\varphi^2 + dr^2 \). – If we consider what modifications of this can be made without destroying the spherical symmetry in space,
the symmetry as regards past and future time, or the static condition, the most general possible form appears to be \( \text{d} s^2 = -U(r)\text{d}r^2 - V(r)\text{d}r\text{d}\vartheta + r^2 \sin^2 \vartheta \text{d}\varphi^2 + W(r)\text{d}t^2 \)." Boyer and Lindquist, in their study of Kerr’s metric \cite{11}, introduced a “Euclidean 3-space with Cartesian coordinates”.

The great majority of the authors utilize implicitly representative spaces, and often without a clear distinction between features of the considered space-time of GR and features of its picture space.

A2. – In sect.2 of paper \cite{2} Synge wrote: “Once we have decided on the idealized experiments which we shall use, we have thereby set up a system of coordinates \( x^r \) in a space-time. […]. The next step is to make a geometrical representation of space-time. […]. We think then of a space \( V_4 \) of four dimensions – a representative space. […]. The representative space is a map of space-time, and like every map it is a mixture of the intrinsic properties of the thing mapped and certain conventionalities introduced for our human convenience in understanding and interpreting. […]. Modifications for convenience may be introduce later, but let us start with the idea that our representative space \( V_4 \) is a Euclidean space of four dimensions.” In sect.3 of \cite{2} we read: “Let \((u, v)\) be two variables ranging from \(-\infty\) to \(+\infty\). They will be taken, for purposes of representation, as rectangular Cartesians in a Euclidean plane \( U_2 \).” And in sect.4 of \cite{2}: “The rest of that section [i.e., of sect.3] was devoted to the definition of certain functions of \((u^2 - v^2)\). Among these functions was \( r \) […]. The plane \( U_2 \) forms half of our representation. The other half is provided through a family of concentric spheres on which the variables \( \vartheta \) and \( \varphi \) are respectively colatitude and azimuth referred to a common pole \( \vartheta = 0 \) and to a common base plane \( \varphi = 0 \). […]. – To the assigned pair of values \((u, v)\) there corresponds a point \( P \) in the plane \( U_2 \) and also […]. a value of \( r \) in the range \( 0 \leq r < +\infty \). Hence there corresponds a sphere \( S_2 \) of radius \( r \) in the above mentioned concentric family. Assigned values of \((\vartheta, \varphi)\) fix a point \( Q \) on \( S_2 \). […]. We shall define our representative space \( V_4 \) by saying that a point of \( V_4 \) is a point-pair \((P, Q)\). […]. So far nothing of space-time. […]. Hypothesis A: All events in space-time containing a single gravitating particle may be put in one to one correspondence with the points of the representative space \( V_4 \) described above.

As regards the line-element of space-time, let us set down for consideration the form \( \Phi = \text{d}u^2 - \text{d}v^2 + (v\text{d}u - u\text{d}v)^2 F + r^2 (\text{d}\vartheta^2 + \sin^2 \vartheta \text{d}\varphi^2), \) \( F \) and \( r \) being functions of \((u^2 - v^2)\) as defined in Section 3, these functions involving a positive constant \( a \equiv 2m \).”

The above \( \Phi, \) i.e. \( \text{d}s^2 \), is the clou of a complex investigation, which inspired Kruskal \cite{4} and Szekeres \cite{4}, who succeeded in giving a simplified and more manageable version of Synge’s results.

A3. – Back to Kruskal-Szekeres metric. The representative space is identical to Synge’s one: a Euclidean plane \( U_2 \), referred to Cartesian orthogonal axes \((u, v)\), and a set \( S_2 \) of concentric spheres on which a colatitude \( \vartheta \) and an
azimuth $\varphi$ are defined. Metric (12) is referred to four different pairs of coordinates; accordingly, representative plane $U_2$ is divided into four regions I, II, III, IV.

\[
\begin{cases}
  u_I = \left( \frac{r}{\alpha} - 1 \right)^{1/2} \exp \left( \frac{r}{2\alpha} \right) \cosh \left( \frac{t}{2\alpha} \right) ; \\
  v_I = \left( \frac{r}{\alpha} - 1 \right)^{1/2} \exp \left( \frac{r}{2\alpha} \right) \sinh \left( \frac{t}{2\alpha} \right),
\end{cases}
\]

for $r > \alpha$, and

\[
\begin{cases}
  u_{II} = \left( 1 - \frac{r}{\alpha} \right)^{1/2} \exp \left( \frac{r}{2\alpha} \right) \sinh \left( \frac{t}{2\alpha} \right) ; \\
  v_{II} = \left( 1 - \frac{r}{\alpha} \right)^{1/2} \exp \left( \frac{r}{2\alpha} \right) \cosh \left( \frac{t}{2\alpha} \right),
\end{cases}
\]

for $r < \alpha$. From (A1)–(A2) we have, in particular:

\[
\begin{cases}
  \left( \frac{r}{\alpha} - 1 \right) \exp \left( \frac{r}{\alpha} \right) = \left\{ \begin{array}{c}
  u_I^2 - v_I^2 ; \\
  u_{II}^2 - v_{II}^2.
\end{array} \right.
\end{cases}
\]

The Cartesian plane $(u, v)$ is divided into four quadrants I, II, III, IV by the null lines $u = v$, $u = -v$, that are null lines of light-rays. Right-hand quadrant I and upper quadrant II cover the entire Schwarzschild space-time; left-hand quadrant III is a pendant of I: $u_{III} = -u_I$, $v_{III} = -v_I$; lower quadrant IV is a pendant of II: $u_{IV} = -u_{II}$, $v_{IV} = -v_{II}$. Formula (A3) holds also for quadrants III and IV. (Formula (13) of sect. 4 holds too for all quadrants).

The above patchwork is not only redundant, because quadrants I and II are sufficient to describe both the exterior and the interior regions of $r = \alpha$, but has also this surprising property: if we substitute in metric (12) any whatever of the four coordinate pairs $(u_I, v_I)$, $(u_{II}, v_{II})$, $(u_{III}, v_{III})$, $(u_{IV}, v_{IV})$, we obtain always the standard $ds^2$ of eq. (1), without any distinction between the exterior and the interior region of surface $r = \alpha$.

Kruskal-Szekeres metric is a good example of the heuristic and interpretative value of a convenient Bildraum. However, Synge’s representative space and Kruskal-Szekeres metric do not give a faithful description of physical reality, owing to the defects that we have pointed out in sect. 4.

A4. – In sect. A1 we have written that, in his construction of Schwarzschild manifold, Eddington utilized the concept of Bildraum in a subtle and indirect way. Indeed, for the spherical symmetry in space-time of GR he drew inspiration from the Minkowskian $ds^2$ expressed with spherical polar coordinate, and wrote $ds^2 = -U(r)dt^2 - V(r)(r^2d\theta^2 + r^2\sin^2 \theta d\varphi^2) + W(r)dr^2$. We emphasize that in GR the notion of spherical symmetry is not a well
defined and understood concept (Synge [2]). Thus, Eddington (as – more or less implicitly – all the Fathers of Relativity) took advantage of the fact that, on the contrary, spherical symmetry can be perfectly mastered in SR. Then, he wrote: \( r^2 V(r) \rightarrow r^2 \); and we can say that the free choice of the radial coordinate allowed him to exploit Synge’s family \( S_2 \) of concentric spheres. Finally, at p.94 of [10], our Author pointed out that the general solution in spherical polar coordinates of Schwarzschild problem can be obtained by substituting the \( r \) of standard form of solution (eq. (1)) with any regular function \( f(r) \). A result that can be recovered by solving equations \( R_{jk} = 0 \) for \( g_{rr} = -U(r) \); \( g_{\vartheta \vartheta} = -V(r) r^2 \); \( g_{\varphi \varphi} = -V(r) r^2 \sin^2 \vartheta \); \( g_{tt} = W(r) \) – see, e.g., the Appendix of Abrams [12].

The instance of Schwarzschild manifold is emblematic: as a matter of fact, in all problems of GR the pseudo-Riemannian manifold is not known \textit{a priori}, it is obtained by solving the concerned Einsteinian equations. Consequently, the starting point of the investigation is always the (implicit or explicit) consideration of a \textit{Bildraum}, that we choose taking heed of the general properties (e.g., spherical symmetry) of our problem.

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Schwarzschild’s original solution – a value that is generally overlooked by the current literature.

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