DEFORMATIONS OF ASYMPTOTICALLY CYLINDRICAL
COASSOCIATIVE SUBMANIFOLDS WITH MOVING BOUNDARY

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ABSTRACT. In an earlier paper, [5], we proved that given an asymptotically cylindrical $G_2$-manifold $M$ with a Calabi–Yau boundary $X$, the moduli space of coassociative deformations of an asymptotically cylindrical coassociative 4-fold $C \subset M$ with a fixed special Lagrangian boundary $L \subset X$ is a smooth manifold of dimension $\dim V_+ + \dim \ker \Upsilon$, where $V_+$ is the positive subspace of the image of $H^2_{cs}(C, \mathbb{R})$ in $H^2(C, \mathbb{R})$. In order to prove this we used the powerful tools of Fredholm Theory for noncompact manifolds which was developed by Lockhart and McOwen [11, 12], and independently by Melrose [13, 14].

In this paper, we extend our result to the moving boundary case. Let $\Upsilon : H^2(L, \mathbb{R}) \to H^3_{cs}(C, \mathbb{R})$ be the natural projection, so that $\ker \Upsilon$ is a vector subspace of $H^2(L, \mathbb{R})$. Let $F$ be a small open neighbourhood of 0 in $\ker \Upsilon$ and $L_s$ denote the special Lagrangian submanifolds of $X$ near $L$ for some $s \in F$ and with phase $i$. Here we prove that the moduli space of coassociative deformations of an asymptotically cylindrical coassociative submanifold $C$ asymptotic to $L_s \times (R, \infty)$, $s \in F$, is a smooth manifold of dimension equal to $\dim V_+ + \dim \ker \Upsilon = \dim V_+ + b^2(L) - b^0(L) + b^3(C) - b^1(C) + b^4(C)$.

1. Introduction

Let $(M, \varphi, g_M)$ be a connected, complete, asymptotically cylindrical $G_2$-manifold with a $G_2$-structure $(\varphi, g_M)$ and asymptotic to $X \times (R, \infty)$, $R > 0$, with decay rate $\alpha < 0$, where $X$ is a Calabi–Yau 3-fold. An asymptotically cylindrical $G_2$-manifold $M$ is a noncompact Riemannian 7-manifold with zero Ricci curvature whose holonomy group $\text{Hol}(g_M)$ is a subgroup of the exceptional Lie group $G_2$. It is equipped with a covariant constant 3-form $\varphi$ and a 4-form $*\varphi$. There are two natural classes of noncompact calibrated submanifolds inside $M$ corresponding to $\varphi$ and $*\varphi$ which are called asymptotically cylindrical associative 3-folds and coassociative 4-folds, respectively.

The Floer homology programs for asymptotically cylindrical Calabi-Yau and $G_2$-manifolds lead to the construction of brand new Topological Quantum Field Theories. In order to construct consistent TQFT’s, the first step is to understand the deformations of asymptotically cylindrical calibrated submanifolds with some small decay rate inside Calabi-Yau and $G_2$-manifolds. A fundamental question is whether these noncompact submanifolds have smooth deformation spaces.

For this purpose, in [5] we studied the deformation space of an asymptotically cylindrical coassociative submanifold $C$ in an asymptotically cylindrical $G_2$-manifold $M$ with a Calabi-Yau boundary $X$. We assumed that the boundary $\partial C = L$ is a special Lagrangian submanifold of $X$. Using the analytic set-up which was developed for elliptic operators on
asymptotically cylindrical manifolds by Lockhart-McOwen and Melrose, [11], [12], [13], [14], we proved that for fixed boundary \( \partial C = L \) this moduli space is a smooth manifold.

In order to develop the Floer homology program for coassociative submanifolds the next step is to parametrize coassociative deformations with moving boundary. In this paper, we extend our previous result in [3] to the moving boundary case and show that if the special Lagrangian boundary is allowed to move, one still obtains a smooth moduli space. Moreover, we can determine the dimension of this space. These two technical results on fixed and moving boundary cases then parametrize all asymptotically cylindrical coassociative deformations. Remarkably, using this parametrization of deformations and basic algebraic topology we can verify one of the main claims of Leung in [10] to prove that the boundary map from the moduli space of coassociative cycles to the moduli space of special Lagrangian cycles is a Lagrangian immersion. With this result in hand, one can start defining the Floer homology of coassociative cycles with special Lagrangian boundary, which then leads to the construction of the TQFT of these cycles.

In this paper we prove the following theorem.

**Theorem 1.1.** Let \((M, \varphi, g)\) be an asymptotically cylindrical \(G_2\)-manifold asymptotic to \(X \times (R, \infty)\), \(R > 0\), with decay rate \(\alpha < 0\), where \(X\) is a Calabi–Yau 3-fold with metric \(g_X\). Let \(C\) be an asymptotically cylindrical coassociative 4-fold in \(M\) asymptotic to \(L \times (R', \infty)\) for \(R' > R\) with decay rate \(\beta\) for \(\alpha \leq \beta < 0\), where \(L\) is a special Lagrangian 3-fold in \(X\) with phase \(i\), and metric \(g_L = g_X|_L\).

Let also \(\Upsilon : H^2(L, R) \to H^2_{cs}(C, R)\) be the natural projection, so that \(\ker \Upsilon\) is a vector subspace of \(H^2(L, R)\) and let \(F\) be a small open neighbourhood of 0 in \(\ker \Upsilon\). Let also \(L_s\) be the special Lagrangian submanifolds of \(X\) near \(L\) for some \(s \in F\) and with phase \(i\). Then for any sufficiently small \(\gamma\) the moduli space \(\mathcal{M}_C^\gamma\) of asymptotically cylindrical coassociative submanifolds in \(M\) close to \(C\), and asymptotic to \(L_s \times (R', \infty)\) with decay rate \(\gamma\), is a smooth manifold of dimension \(\dim V_+ + \dim \ker \Upsilon = \dim V_+ + b_2(L) - b_0(L) + b_3(C) - b_1(C) + b_0(C)\), where \(V_+\) is the positive subspace of the image of \(H^2_{cs}(C, R)\) in \(H^2(C, R)\).

**Remark 1.2.** In Theorem 1.1 above, as in the fixed boundary case, [5], we require \(\gamma\) to satisfy \(\beta < \gamma < 0\), and \((0, \gamma^2, [\gamma, 0])\) to contain no eigenvalues of the Laplacian \(\Delta_L\) on functions on \(L\), and \([\gamma, 0]\) to contain no eigenvalues of the operator \(-\ast d\) on coexact 1-forms on \(L\). These hold provided \(\gamma < 0\) is small enough. This assumption is needed to guarantee that the linearized operator of the deformation map for asymptotically cylindrical coassociative submanifolds is Fredholm.

**Remark 1.3.** In our previous paper, [5], we proved that the dimension of the deformation space of asymptotically cylindrical coassociative submanifold \(C\) with fixed special Lagrangian boundary \(L\) is given as \(V_+\), where \(V_+\) is the positive subspace of the image of \(H^2_{cs}(C, R)\) in \(H^2(C, R)\). One should note that in Theorem 1.1 above, in the case when \(L\) is a special Lagrangian homology 3-sphere, \(b_3(L) = 0\) and hence \(L\) is rigid and there are no special Lagrangian deformations of \(L\) so it behaves like a fixed boundary. Then one can use basic algebraic topology to show that for this special case Theorem 1.1 gives us \(\dim V_+ + \dim \ker \Upsilon = \dim V_+ + b_2(L) - b_0(L) + b_3(C) - b_1(C) + b_0(C) = \dim V_+\) which is consistent with the result of our previous paper, [5], for fixed boundary.
Remark 1.4. Theorem 1.1 applies to examples of asymptotically cylindrical $G_2$-manifolds constructed by Kovalev, in [9]. These 7-manifolds are of the form $X \times S^1$ and are obtained by the direct product of asymptotically cylindrical 6-manifolds $X$ with holonomy $SU(3) \subset G_2$ and the circle $S^1$. It is still an open question whether there exist asymptotically cylindrical 7-manifolds with holonomy group $G_2$.

The outline of the paper is as follows: In §2 we begin with a general discussion of $G_2$ geometry, asymptotically cylindrical manifolds and basic tools used in elliptic theory such as weighted Sobolev spaces. In §3 we give an overview of deformations of an asymptotically cylindrical coassociative submanifold with fixed special Lagrangian boundary, followed by a sketch proof of our main theorem in [5], where we showed that the moduli space of such deformations is smooth and calculated its dimension. In §4 we introduce the analytic set-up for deformations with moving (free) boundary and prove Theorem 1.1. Finally, in §5 using Theorem 1.1 we verify one of the main claims of Leung in [10], which is necessary to prove that the boundary map from the moduli space of coassociative cycles into the moduli space of special Lagrangian cycles is a Lagrangian immersion.

2. $G_2$-manifolds and coassociative submanifolds

We now explain some background material on asymptotically cylindrical $G_2$-manifolds and their coassociative submanifolds. We will also review the Fredholm Theory for elliptic operators and weighted Sobolev spaces. A good reference for $G_2$ geometry is Harvey and Lawson [7]. The details of the elliptic theory on noncompact manifolds can be found in [11], [12], [13], [14].

2.1. $G_2$ Geometry. The imaginary octonions $\text{Im} \mathbb{O} = \mathbb{R}^7$ is equipped with the cross product $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ defined by $u \times v = \text{Im}(u \cdot \bar{v})$ where $\cdot$ is the octonionic multiplication. The exceptional Lie group $G_2$ can be defined as the linear automorphisms of $\text{Im} \mathbb{O}$ preserving this cross product operation. It is a compact, semisimple, and 14-dimensional subgroup of $SO(7)$.

Definition 2.1. A smooth 7-manifold $M$ has a $G_2$-structure if its tangent frame bundle reduces to a $G_2$ bundle.

It is known that if $M$ has a $G_2$-structure then there is a $G_2$-invariant 3-form $\varphi \in \Omega^3(M)$ which can be written in an orthonormal frame as

$$\varphi = e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^4 \wedge e^5 + e^1 \wedge e^6 \wedge e^7 + e^2 \wedge e^4 \wedge e^6 - e^2 \wedge e^5 \wedge e^7 - e^3 \wedge e^4 \wedge e^7 - e^3 \wedge e^5 \wedge e^6.$$

This $G_2$-invariant 3-form $\varphi$ gives an orientation $\mu \in \Omega^7(M)$ on $M$ and $\mu$ determines a metric $g = g_\varphi = \langle \cdot, \cdot \rangle$ on $M$ defined as

$$\langle u, v \rangle = |i_u(\varphi) \wedge i_v(\varphi) \wedge \varphi|/\mu,$$

where $i_v = v \cdot$ is the interior product with a vector $v$. So from now on we will refer to the term $(\varphi, g)$ as a $G_2$-structure and the term $(M, \varphi, g)$ as a manifold with $G_2$-structure.
Definition 2.2. \((M, \varphi, g)\) is a G2-manifold if \(\nabla \varphi = 0\), i.e. the G2-structure \(\varphi\) is torsion-free.

One can show that for a manifold with G2-structure \((M, \varphi, g)\), the following are equivalent:

(i) \(\text{Hol}(g) \subseteq G_2\),
(ii) \(\nabla \varphi = 0\) on \(M\), where \(\nabla\) is the Levi-Civita connection of \(g\),
(iii) \(d \varphi = 0\) and \(d^* \varphi = 0\) on \(M\).

Harvey and Lawson, \[7\], showed that there are minimal submanifolds of G2-manifolds calibrated by \(\varphi\) and \(\ast \varphi\).

Definition 2.3. Let \((M, \varphi, g)\) be a G2-manifold. A 4-dimensional submanifold \(C \subset M\) is called coassociative if \(\varphi|_C = 0\). A 3-dimensional submanifold \(Y \subset M\) is called associative if \(\varphi|_Y \equiv \text{vol}(Y)\); this condition is equivalent to \(\chi|_Y \equiv 0\), where \(\chi \in \Omega^3(M, TM)\) is the tangent bundle valued 3-form defined by the identity:

\[
\langle \chi(u, v, w), z \rangle = \ast \varphi(u, v, w, z)
\]

2.2. Asymptotically Cylindrical G2-Manifolds and Coassociative Submanifolds.

Next, we recall basic properties of asymptotically cylindrical G2-manifolds and the coassociative 4-folds. We also need these definitions and the analytic set-up in Section 4. More details on the subject can be found in \[5\].

Definition 2.4. A G2-manifold \((M_0, \varphi_0, g_0)\) is called cylindrical if \(M_0 = X \times \mathbb{R}\) and \((\varphi_0, g_0)\) is compatible with the product structure, that is,

\[
\varphi_0 = \text{Re} \Omega + \omega \wedge dt \quad \text{and} \quad g_0 = g_X + dt^2,
\]

where \(X\) is a connected, compact Calabi–Yau 3-fold with Kähler form \(\omega\), Riemannian metric \(g_X\) and holomorphic (3,0)-form \(\Omega\).

Definition 2.5. Let \(X\) be a Calabi–Yau 3-fold with metric \(g_X\). \((M, \varphi, g)\) is called an asymptotically cylindrical G2-manifold asymptotic to \(X \times (R, \infty)\), \(R > 0\), with decay rate \(\alpha < 0\) if there exists a cylindrical G2-manifold \((M_0, \varphi_0, g_0)\) with \(M_0 = X \times \mathbb{R}\), a compact subset \(K \subset M\), a real number \(R\), and a diffeomorphism \(\Psi : X \times (R, \infty) \to M \setminus K\) such that \(\Psi^*(\varphi) = \varphi_0 + \text{d} \xi\) for some smooth 2-form \(\xi\) on \(X \times (R, \infty)\) with \(\|\nabla^k \xi\| = O(e^{\alpha t})\) on \(X \times (R, \infty)\) for all \(k \geq 0\), where \(\nabla\) is the Levi-Civita connection of the cylindrical metric \(g_0\).

An asymptotically cylindrical G2-manifold \(M\) has one end modelled on \(X \times (R, \infty)\), and as \(t \to \infty\) in \((R, \infty)\) the G2-structure \((\varphi, g)\) on \(M\) converges to order \(O(e^{\alpha t})\) to the cylindrical G2-structure on \(X \times (R, \infty)\), with all of its derivatives. As in \[5\], we suppose \(M\) and \(X\) are connected, that is, we allow \(M\) to have only one end. We showed earlier that an asymptotically cylindrical G2-manifold can have at most one cylindrical end, or otherwise the holonomy reduces. \[16\].

We can also define calibrated submanifolds of noncompact G2-manifolds. In \[4\], we introduced definitions of cylindrical and asymptotically cylindrical coassociative submanifolds of a G2-manifold.
Definition 2.6. Let \((M_0, \varphi_0, g_0)\) be a cylindrical \(G_2\)-manifold. A 4-fold \(C_0\) is called cylindrical coassociative submanifold of \(M_0\) if \(C_0 = L \times \mathbb{R}\) for some compact special Lagrangian 3-fold \(L\) with phase \(i\) in the Calabi-Yau manifold \(X\).

Definition 2.7. Let \(M\) be an asymptotically cylindrical \(G_2\)-manifold asymptotic to \(X \times (R, \infty), R > 0\), with decay rate \(\alpha < 0\). Let also \(C\) be a connected, complete asymptotically cylindrical 4-fold in \(M\) asymptotic to \(L \times (R', \infty)\) for \(R' > R\) with decay rate \(\beta\) for \(\alpha \leq \beta < 0\) and let \(L\) be a compact special Lagrangian 3-fold in \(X\) with phase \(i\), and metric \(g_L = g_X|_L\).

Then \(C\) is called asymptotically cylindrical coassociative 4-fold if there exists a compact subset \(K' \subset C\), a normal vector field \(v\) on \(L \times (R', \infty)\) for some \(R' > R\), and a diffeomorphism \(\Phi : L \times (R', \infty) \to C \setminus K'\) such that the following diagram commutes:

\[
\begin{array}{ccc}
X \times (R', \infty) & \xrightarrow{\exp_p} & L \times (R', \infty) \\
\downarrow \scriptstyle{\subset} & & \downarrow \scriptstyle{\subset} \\
X \times (R, \infty) & \xrightarrow{\Psi} & (M \setminus K),
\end{array}
\]

Diagram (1) implies that \(C\) in \(M\) is asymptotic to the cylinder \(C_0\) in \(M_0 = X \times \mathbb{R}\) as \(t \to \infty\) in \(\mathbb{R}\), to order \(O(e^{\beta t})\). In other words, \(C\) can be written near infinity as the graph of a normal vector field \(v\) to \(C_0 = L \times \mathbb{R}\) in \(M_0 = X \times \mathbb{R}\), so that \(v\) and its derivatives are \(O(e^{\beta t})\). Here we require \(C\) but not \(L\) to be connected, so it is possible that \(C\) to have multiple ends.

2.3. Elliptic operators on noncompact manifolds. We will conclude this section with a brief review of the weighted Sobolev spaces and the results of Lockhart and McOwen, [11], [12], about Fredholm properties of elliptic operators on manifolds with cylindrical ends.

Let \(C\) and \(L\) be as in Definition 2.7 above. \(E_0\) is called a cylindrical vector bundle on \(L \times \mathbb{R}\) if it is invariant under translations in \(\mathbb{R}\). Let \(h_0\) be a smooth family of metrics on the fibres of \(E_0\) and \(\nabla_{E_0}\), a connection on \(E_0\) preserving \(h_0\), with \(h_0, \nabla_{E_0}\) invariant under translations in \(\mathbb{R}\). Let \(E\) be a vector bundle on \(C\) equipped with metrics \(h\) on the fibres, and a connection \(\nabla_E\) on \(E\) preserving \(h\). We say that \(E, h, \nabla_E\) are asymptotic to \(E_0, h_0, \nabla_{E_0}\) if there exists an identification \(\Phi_*(E) \cong E_0\) on \(L \times (R', \infty)\) such that \(\Phi_*(h) = h_0 + O(e^{\beta t})\) and \(\Phi_*(\nabla_E) = \nabla_{E_0} + O(e^{\beta t})\) as \(t \to \infty\). Then we call \(E, h, \nabla_E\) asymptotically cylindrical.

Definition 2.8. Let \(\rho : C \to \mathbb{R}\) be a smooth function satisfying \(\Phi^*(\rho) \equiv t\) on \(L \times (R', \infty)\).

For \(p \geq 1, k \geq 0\) and \(\gamma \in \mathbb{R}\) the weighted Sobolev space \(L^p_{k, \gamma}(E)\) is defined to be the set of sections \(s\) of \(E\) that are locally integrable and \(k\) times weakly differentiable and for which the norm

\[
\|s\|_{L^p_{k, \gamma}} = \left( \sum_{j=0}^{k} \int_C e^{-\gamma t} |\nabla_E^j s|^p dV \right)^{1/p}
\]

is finite.
Note that the weighted Sobolev space, $L^p_{k,\gamma}(E)$, is a Banach space.

Now suppose $E, F$ are two asymptotically cylindrical vector bundles on $C$, asymptotic to cylindrical vector bundles $E_0, F_0$ on $L \times \mathbb{R}$.

**Definition 2.9.** Let $A_0 : C^\infty(E_0) \to C^\infty(F_0)$ be a cylindrical elliptic operator of order $k$ invariant under translations in $\mathbb{R}$. Let also $A : C^\infty(E) \to C^\infty(F)$ be an elliptic operator of order $k$ on $C$. $A$ is asymptotic to $A_0$ if under the identifications $\Phi_*(E) \cong E_0$, $\Phi_*(F) \cong F_0$ on $L \times (R', \infty)$ we have $\Phi_*(A) = A_0 + O(e^{\beta t})$ as $t \to \infty$ for $\beta < 0$. Then $A$ is called an asymptotically cylindrical elliptic operator.

It is well known that if $A$ is an elliptic operator on a compact manifold then it should be Fredholm. But this is not the case for the noncompact manifolds; there are examples of elliptic operators which are not Fredholm, [11], [12] and it turns out that $A$ is Fredholm if and only if $\gamma$ does not lie in a discrete set $D_{A_0} \subset \mathbb{R}$ which can be defined as follows:

**Definition 2.10.** Let $A$ and $A_0$ be elliptic operators on $C$ and $L \times \mathbb{R}$, so that $E, F$ have the same fibre dimensions. Extend $A_0$ to the complexifications $A_0 : C^\infty(E_0 \otimes \mathbb{R} \mathbb{C}) \to C^\infty(F_0 \otimes \mathbb{R} \mathbb{C})$. Define $D_{A_0}$ to be the set of $\gamma \in \mathbb{R}$ such that for some $\delta \in \mathbb{R}$ there exists a nonzero section $s \in C^\infty(E_0 \otimes \mathbb{R} \mathbb{C})$ invariant under translations in $\mathbb{R}$ such that $A_0(e^{(\gamma + \delta) t} s) = 0$.

An important Fredholm property of elliptic operators on noncompact manifolds has been shown by Lockhart and McOwen in [12] Th. 1.1:

**Theorem 2.11.** Let $(C, g)$ be an asymptotically cylindrical Riemannian manifold asymptotic to $(L' \times \mathbb{R}, g_0)$, and $A : C^\infty(E) \to C^\infty(F)$ an asymptotically cylindrical elliptic operator on $C$ of order $k$ between asymptotically cylindrical vector bundles $E, F$ on $C$, asymptotic to the cylindrical elliptic operator $A_0 : C^\infty(E_0) \to C^\infty(F_0)$ on $L \times \mathbb{R}$. Let $D_{A_0}$ be defined as above.

Then $D_{A_0}$ is a discrete subset of $\mathbb{R}$, and for $p > 1$, $l \geq 0$ and $\gamma \in \mathbb{R}$, the extension $A^p_{k+l, \gamma} : L^p_{k+l, \gamma}(E) \to L^p_{k+l, \gamma}(F)$ is Fredholm if and only if $\gamma \notin D_{A_0}$.

Note that Theorem 2.11 plays an important role in our choice of weighted Sobolev spaces in Theorem 1.1.

### 3. Coassociative Deformations with Fixed Boundary

Using the set-up in Section 2 we proved the following theorem in [5], for asymptotically cylindrical coassociative deformations with fixed boundary.

**Theorem 3.1.** [5 Thm 1.1.] Let $(M, \varphi, g)$ be an asymptotically cylindrical $G_2$-manifold asymptotic to $X \times (R, \infty)$, $R > 0$, with decay rate $\alpha < 0$, where $X$ is a Calabi–Yau 3-fold with metric $g_X$. Let $C$ be an asymptotically cylindrical coassociative 4-fold in $M$ asymptotic to $L \times (R', \infty)$ for $R' > R$ with decay rate $\beta$ for $\alpha \leq \beta < 0$, where $L$ is a special Lagrangian 3-fold in $X$ with phase $i$, and metric $g_L = g_X|_L$.

Then for some small $\gamma$ the moduli space $M_{L}^{\gamma}$ of asymptotically cylindrical coassociative submanifolds in $M$ close to $C$, and asymptotic to $L \times (R', \infty)$ with decay rate $\gamma$, is a smooth
manifold of dimension \( \dim V_+ \), where \( V_+ \) is the positive subspace of the image of \( H^2_\omega(C, \mathbb{R}) \) in \( H^2(C, \mathbb{R}) \).

**Remark 3.2.** McLean proved the compact version of Theorem 3.1 in [15] and showed that the moduli space \( \mathcal{M}_C \) of coassociative 4-folds isotopic to a compact coassociative 4-fold \( C \) in \( M \) is a smooth manifold of dimension \( b^2_\omega(C) \). There he modelled \( \mathcal{M}_C \) on \( \tilde{P}^{-1}(0) \) for a nonlinear map \( \tilde{P} \) between Banach spaces, whose linearization \( d\tilde{P}(0, 0) \) at 0 was the Fredholm map between Sobolev spaces

\[
(3) \quad d_+ + d^* : L^p_{d+2}(\Lambda^{2}T^*C) \times L^p_{d+2}(\Lambda^{4}T^*C) \rightarrow L^p_{d+1}(\Lambda^{3}T^*C).
\]

McLean showed that \( d\tilde{P} \) is onto the image of \( \tilde{P} \) and used the Implicit Mapping Theorem for Banach spaces, [3, Thm 1.2.5] to conclude that \( \tilde{P}^{-1}(0) \) is smooth, finite-dimensional and locally isomorphic to \( \ker((d_+ + d^*)|_{L^p_{d+2}}) \).

**Sketch proof of Theorem 3.1.** Let \((M, \varphi, g) \) be an asymptotically cylindrical \( G_2 \)-manifold asymptotic to \( X \times (R, \infty) \), and \( C \) an asymptotically cylindrical coassociative 4-fold in \( M \) asymptotic to \( L \times (R', \infty) \).

Let \( \nu L \) be the normal bundle of \( L \) in \( X \) and \( \exp_L : \nu L \rightarrow X \) be the exponential map. For \( r > 0 \), \( B_r(\nu L) \) is the subbundle of \( \nu L \) with fibre at \( x \) the open ball about 0 in \( \nu L|_x \) with radius \( r \). Then for small \( \epsilon > 0 \), there is a tubular neighbourhood \( T_L \) of \( L \) in \( X \) such that \( \exp_L : B_{2\epsilon}(\nu L) \rightarrow T_L \) is a diffeomorphism. Also, \( \nu_L \times R \rightarrow L \times R \) is the normal bundle to \( L \times R \) in \( X \times R \) with exponential map \( \exp_L \times \id : \nu L \times R \rightarrow X \times R \). Then \( T_L \times \mathbb{R} \) is a tubular neighborhood of \( L \times \mathbb{R} \) in \( X \times \mathbb{R} \), and \( \exp_L \times \id : B_{2\epsilon}(\nu L) \times \mathbb{R} \rightarrow T_L \times \mathbb{R} \) is a diffeomorphism.

Let \( K, R, \Psi : X \times (R, \infty) \rightarrow M \setminus K \), and \( K', R' > R \), \( \Phi : L \times (R', \infty) \rightarrow C \setminus K' \), and \( \nu L \) the normal vector field \( \nu L \) on \( L \times (R', \infty) \) be as before so that Diagram (2) in Section 2.2 commutes. Then \( \nu L \) is a section of \( \nu L \times (R', \infty) \rightarrow L \times (R', \infty) \), decaying at rate \( O(e^{3t}) \) and by making \( K' \) and \( R' \) larger if necessary, we can suppose the graph of \( \nu L \) lies in \( B_{\epsilon_2}(\nu L) \times (R', \infty) \).

Let \( \pi : B_{\epsilon_2}(\nu L) \times (R', \infty) \rightarrow L \times (R', \infty) \) be the natural projection. Then we define a diffeomorphism

\[
(4) \quad \Xi : B_{\epsilon_2}(\nu L) \times (R', \infty) \rightarrow M \quad \text{by} \quad \Xi : w \mapsto \Psi[(\exp_L \times \id)(\nu L \times \pi(w) + w)].
\]

where \( w \) is a point in \( B_{\epsilon_2}(\nu L) \times (R', \infty) \), in the fibre over \( \pi(w) \in L \times (R', \infty) \). Under the identification of \( L \times (R', \infty) \) with the zero section in \( B_{\epsilon_2}(\nu L) \times (R', \infty) \), \( \Xi |_{L \times (R', \infty)} \equiv \Phi \). Using \( \Xi \) we can then define an isomorphism \( \xi \) between the vector bundles \( \nu L \times (R', \infty) \) and \( \Phi^\ast(\nu C) \) over \( L \times (R', \infty) \), where \( \nu C \) is the normal bundle of \( C \) in \( M \). This leads to the construction of another diffeomorphism \( \Theta : B_{\epsilon_3}(\nu C) \rightarrow T_C \) for appropriate choices of a small \( \epsilon' > 0 \), and a tubular neighborhood \( T_C \) of \( C \) in \( M \).

By choosing the local identification \( \Theta \) between \( \nu C \) and \( M \) near \( C \) that is compatible with the asymptotic identifications \( \Phi, \Psi \) of \( C, M \) with \( L \times \mathbb{R} \) and \( X \times \mathbb{R} \) we can then identify submanifolds \( \tilde{C} \) of \( M \) close to \( C \) with small sections of \( \nu C \). More importantly, the asymptotic convergence of \( \tilde{C} \) to \( C \), and so to \( L \times \mathbb{R} \), is reflected in the asymptotic convergence of sections of \( \nu C \) to 0.
We then define a map $Q : L^p_{t+2,\gamma}(B_c(\Lambda^2_+ T^*C)) \to \{3\text{-forms on } C\}$ by $Q(\zeta_+^2) = (\Theta \circ \zeta_+^2)^*(\varphi)$ for $p > 4$ and $l \geq 1$. That is, we regard the section $\zeta_+^2$ as a map $C \to B_c(\Lambda^2_+ T^*C)$, so $\Theta \circ \zeta_+^2$ is a map $C \to T_C \subset M$, and thus $(\Theta \circ \zeta_+^2)^*(\varphi)$ is a 3-form on $C$. Therefore if $\Gamma_\zeta^2$ is the graph of $\zeta_+^2$ in $B_c(\Lambda^2_+ T^*C)$ and $\tilde{C} = \Theta(\Gamma_\zeta^2)$ its image in $M$, then $\tilde{C}$ is coassociative if and only if $\varphi|_{\tilde{C}} \equiv 0$, which holds if and only if $Q(\zeta_+^2) = 0$. So basically $Q^{-1}(0)$ parametrizes coassociative 4-folds $\tilde{C}$ close to $C$. It turns out that $Q : L^p_{t+2,\gamma}(B_c(\Lambda^2_+ T^*C)) \to L^p_{t+1,\gamma}(\Lambda^3 T^*C)$ is a smooth map of Banach manifolds and the linearization of $Q$ at $0$ is $dQ(0) : \zeta_+^2 \to d\zeta_+^2$.

As in the proof of McLean’s Theorem, we then augment $Q$ by a space of 4-forms on $C$ to make it elliptic and define

$$P : L^p_{t+2,\gamma}(B_c(\Lambda^2_+ T^*C)) \times L^p_{t+2,\gamma}(\Lambda^4 T^*C) \to L^p_{t+1,\gamma}(\Lambda^3 T^*C)$$

by

$$P(\zeta_+^2, \zeta^4) = Q(\zeta_+^2) + d^* \zeta^4$$

for $p > 4$ and $l \geq 1$.

As mentioned before in Section 2.3, on a noncompact manifold, the ellipticity of a differential operator $A$ is not sufficient to ensure that $A$ is Fredholm. Using the analytical framework developed by Lockhart and McOwen in [11] and [12], we showed that in our case $dP$ is not Fredholm if and only if either $\gamma = 0$, or $\gamma^2$ is a positive eigenvalue of $\Delta = d^* d$ on functions on $L$, or $\gamma$ is an eigenvalue of $-d^* d$ on coexact 1-forms on $L$. Therefore we take $\gamma$ sufficiently small to guarantee that $dP$ is Fredholm.

We then show that $\text{Ker} ((d_+ + d^*)^l_{t+2,\gamma})$ is a vector space of smooth, closed, self-dual 2-forms and $\text{Coker} ((d_+^* + d)^l_{t+2,\gamma})$ is a vector space of smooth, closed and coclosed 3-forms. It turns out that the map $\text{Ker} ((d_+ + d^*)^l_{t+2,\gamma}) \to H^2(C, \mathbb{R})$, $\chi \mapsto \int \chi$ induces an isomorphism of $\text{Ker} ((d_+ + d^*)^l_{t+2,\gamma})$ with a maximal subspace $V_+$ of the subspace $V \subseteq H^2(C, \mathbb{R})$ on which the cup product $\cup : V \times V \to \mathbb{R}$ is positive definite. Hence

$$\dim \text{Ker} ((d_+ + d^*)^l_{t+2,\gamma}) = \dim V_+,$$

which is a topological invariant of $C, L$.

We finally show that $P$ maps $L^p_{t+2,\gamma}(B_c(\Lambda^2_+ T^*C)) \times L^p_{t+2,\gamma}(\Lambda^4 T^*C)$ to the image of $(d_+ + d^*)^l_{t+2,\gamma}$ and the image of $Q$ consists of exact 3-forms. The Implicit Mapping Theorem for Banach spaces then implies that $P^{-1}(0)$ is smooth, finite-dimensional and locally isomorphic to $\text{Ker} ((d_+ + d^*)^l_{t+2,\gamma})$. As $P^{-1}(0) = Q^{-1}(0) \times \{0\}$ we conclude that $Q^{-1}(0)$ is also smooth, finite-dimensional and locally isomorphic to $\text{Ker} ((d_+ + d^*)^l_{t+2,\gamma})$. Moreover, $Q^{-1}(0)$ is independent of $l$, and so consists of smooth solutions. This proves Theorem 3.1.

4. Coassociative Deformations with Moving Boundary and Proof of Theorem 1.1

We now prove Theorem 1.1 Let $(M, \varphi, g)$ be an asymptotically cylindrical $G_2$-manifold asymptotic to $X \times (R, \infty)$, $R > 0$, with decay rate $\alpha < 0$. Let $C$ be an asymptotically cylindrical coassociative 4-fold in $X$ asymptotic to $L \times (R', \infty)$ for $R' > R$ with decay rate $\beta$ for $\alpha \leq \beta < 0$. 

As in the proof of Theorem \([5.1]\) we suppose \(\gamma\) satisfies \(\beta < \gamma < 0\), and \((0, \gamma^2)\) contains no eigenvalues of the Laplacian \(\Delta_p\) on functions on \(L\), and \([\gamma, 0]\) contains no eigenvalues of the operator \(- * d\) on coexact 1-forms on \(L\). The reason for this assumption is that, as in the fixed boundary case, by Propositions 4.5, 4.6, we also have the same linearized operator given as

\[
(d_+ + d^*)_{l+2, \gamma} : F \times (L^p_{l+2, \gamma}(\Lambda^2 T^* C) \oplus L^p_{l+2, \gamma}(\Lambda^3 T^* C)) \rightarrow L^p_{l+1, \gamma}(\Lambda^3 T^* C)
\]

for \(p > 4\) and \(l \geq 1\) as in \([3]\).

The conditions on \(\gamma\) imply that \([\gamma, 0] \cap D_{(d_+ + d^*)_0} = \emptyset\) and hence \(\gamma \notin D_{(d_+ + d^*)_0}\), so that \((d_+ + d^*)_{l+2, \gamma}\) is Fredholm. Here \(D_{(d_+ + d^*)_0}\) is the discrete set derived from the linearized operator as in \([5]\).

Now, let \(K, R, \Psi : X \times (R, \infty) \rightarrow M \setminus K\), and \(K', R' > R\), \(\Phi : L \times (R', \infty) \rightarrow C \setminus K'\), and the normal vector field \(v\) on \(L \times (R', \infty)\) be as before so that Diagram \((1)\) in Section 2 commutes. Let \(\nu_C\) be the normal bundle of \(C\) in \(M\) and \(\nu_L\) be the normal bundle of \(L\) in \(X\). Then as in \([5]\), by choosing an appropriate local identification \(\Theta\) between \(\nu_C\) and \(M\) near \(C\) that is compatible with the asymptotic identifications \(\Phi, \Psi\) of \(C\), \(M\) with \(L \times \mathbb{R}\) and \(X \times \mathbb{R}\)

we can identify submanifolds \(\tilde{C}\) of \(M\) close to \(C\) with small sections of \(\nu_C\).

Let \(F \subset \mathbb{R}^d\) be an open subset of the space of special Lagrangian deformations of \(L\) in \((X, \omega, \Omega, g_{X})\). Let \(L_0 \in F\) be the starting point and \(L_s\) be nearby special Lagrangian submanifolds for some \(s \leq 0\) in \(F\). By McLean, \([15]\), the space of deformations \(\mathcal{M}_{L_0}\) of a special Lagrangian submanifold \(L_0\) can be identified with closed and coclosed 1-forms on \(L_0\) and so \(\mathcal{M}_{L_0}\) corresponds to \(H^1(L_0, \mathbb{R})\).

Moreover, one can naturally parametrize \(L_s\) by \(s \in H^2(L_0, \mathbb{R})\) and relate \(\mathcal{M}_{L_0}\) to \(H^2(L_0, \mathbb{R})\). Given a compact special Lagrangian \(L_0\), let \(U\) be a connected and simply connected open neighbourhood of \(L_0\) in \(\mathcal{M}_{L_0}\). One can construct natural local diffeomorphisms \(A : \mathcal{M}_{L_0} \rightarrow H^1(L_0, \mathbb{R})\) and \(B : \mathcal{M}_{L_0} \rightarrow H^2(L_0, \mathbb{R})\) as follows:

Let \(L_s \in U\) be a special Lagrangian submanifold with phase \(i\). Then there exists a smooth path \(\tilde{\gamma} : [0, 1] \rightarrow U\) with \(\tilde{\gamma}(0) = L_0\), and \(\tilde{\gamma}(1) = L_s\) which is unique up to isotopy. \(\tilde{\gamma}\) parametrizes a family of submanifolds of \(X\) diffeomorphic to \(L_0\), and can be lifted to a smooth map \(\Gamma : L_0 \times [0, 1] \rightarrow X\) with \(\Gamma(L_0 \times \{s\}) = \gamma(s)\). Now let \(\Gamma^*(\omega)\) be a 2-form on \(L_0 \times [0, 1]\). Then \(\Gamma^*(\omega)\big|_{L_0 \times \{s\}} \equiv 0\) for each \(s \in [0, 1]\) as each fiber \(\tilde{\gamma}(s)\) is Lagrangian. Hence \(\Gamma^*(\omega)\) can be written as \(\alpha_s \wedge ds\) where \(\alpha_s\) is a closed 1-form on \(L_0\) for each \(s \in [0, 1]\). Then we integrate \(\alpha_s\) with respect to \(s\) and take the cohomology class of the closed 1-form \(\int_0^1 \alpha_s ds\). So we can define \(A(L_s) = [\int_0^1 \alpha_s ds] \in H^1(L_0, \mathbb{R})\). Similarly, we can write \(\Gamma^*(\text{Im}(\Omega)) = \beta_s \wedge ds\) where \(\beta_s\) is a closed 2-form on \(L_0\) for each \(s \in [0, 1]\) and define \(B(L_s) = [\int_0^1 \beta_s ds] \in H^2(L_0, \mathbb{R})\). For more on the constructions of the \(H^2(L_0, \mathbb{R})\) coordinates on the moduli space of special Lagrangian submanifolds see \([6]\) and \([8]\).

We choose a smooth family of diffeomorphisms \(\vartheta_s : L_s \rightarrow L_0\) such that \(\vartheta_0 = \text{id}_{L_0}\) and \(L_s \cong L_0\) for small \(s\). Identify a tubular neighbourhood of \(L_0\) in \(X\) with a neighbourhood of
the zero-section in $T^*L_0$, then $L_s$ is the graph of $\zeta_s$, where $\zeta_s$ is a small closed and coclosed 2-form on $L_0$ such that $[\zeta_s] = s$ in $H^2(L_0, \mathbb{R})$.

Having got $\zeta_s$, for some small $s \in F$ we choose sections $\varrho_s$ of $\Lambda_+^2 T^*C$, that depend smoothly on $s$ where $\varrho_0 = 0$ and such that $\varrho_s$ is asymptotic to $\zeta_s = \zeta_s + dt \wedge *_{L_0}(\zeta_s)$ where $*_{L_0}$ is the star operator in $L_0$.

One way to do this is to take a smooth function $h : \mathbb{R} \to [0, 1]$ defined as $h(x) = 0$ for $x \leq R$ and $h(x) = 1$ for $x \geq R + 1$ and set $\varrho_s = \zeta_s \cdot h(x)$. Then $\varrho_s = \zeta_s$ in $(R + 1, \infty) \times X$ and $\varrho_s = 0$ in $K$, where $M = K \setminus (R, \infty) \times X$.

Given $\zeta_s$ as a section of $T^*L_0$, we can identify $T^*L_0$ with $\Lambda_+^2(L_0 \times \mathbb{R})$ and identify this with $\Lambda_+^2(C_0)$ near $\infty$. Then $\tilde{C}_s$ is the graph of $\varrho_s$, which is a 4-fold in $M$ asymptotic to $L_s \times \mathbb{R}$ but not necessarily coassociative. So we obtain a family of 4-folds $\tilde{C}_s$ such that $\tilde{C}_0 = C_0 = C$, which is a coassociative submanifold of $M$, and $\tilde{C}_s$ is asymptotic to $L_s \times \mathbb{R}$, where $L_s$ is special Lagrangian submanifold of $X$ as in Figure 1.

![Figure 1](image)

Note that when $\tilde{C}_s$ is asymptotic to $L_s \times (R, \infty)$ near $\infty$ then $\varphi|_{\tilde{C}_s} \equiv O(e^{s})$ near $\infty$ (see Proposition 4.1). On the other hand, since $L_s$ is a special Lagrangian, we can take $\varphi = 0$ in a small neighbourhood of the boundary $L_s$ in $L_s \times (R, \infty)$ and so $\varphi|_{\tilde{C}_s}$ is compactly supported and can take $[\varphi|_{\tilde{C}_s}] \in H^3_{ca}(\tilde{C}_s, \mathbb{R})$. Moreover, since $L_0$ and $L_s$ are isotopic (which follows from our choice of coordinates in $H^2(L_0, \mathbb{R})$) we can take $[\varphi|_{\tilde{C}_s}] \in H^3_{ca}(C_0, \mathbb{R})$ which is independent of the choice of $\tilde{C}_s$. And $[\varphi|_{\tilde{C}_s}]$ should be zero for there to exist a coassociative 4-fold $\tilde{C}_s$.

Now define $Q : F \times L_{p+2, q}^s(B_\varphi(\Lambda_+^2 T^*C)) \to \{3\text{-forms on } C\}$ by $Q(s, \zeta_s^+) = (\Theta \circ (\zeta_s^+ + \varrho_s))^*(\varphi)$. That is, we regard the sum of sections $\zeta_s^+ + \varrho_s$ as a map $C \to B_\varphi(\Lambda_+^2 T^*C)$, so $\Theta \circ (\zeta_s^+ + \varrho_s)$ is a map $C \to T_C \subset M$, and thus $(\Theta \circ (\zeta_s^+ + \varrho_s))^*(\varphi)$ is a 3-form on $C$. The point of this definition is that if $\Gamma(\zeta_s^+ + \varrho_s)$ is the graph of $\zeta_s^+ + \varrho_s$ in $B_\varphi(\Lambda_+^2 T^*C)$ and $\tilde{C}_s = \Theta(\Gamma(\zeta_s^+ + \varrho_s))$ its image in $M$, then $\tilde{C}_s$ is coassociative if and only if $\varphi|_{\tilde{C}_s} \equiv 0$, which
holds if and only if $Q(s, \xi^2) = 0$. So $Q^{-1}(0)$ parametrizes coassociative 4-folds $\tilde{C}_s$ close to $C$.

Note that to form $\Gamma(\zeta^2 + \varrho_s)$ we have chosen a diffeomorphism $U \subseteq \Lambda^2_+ T^* C_0 = \nu C_0 \to V \subseteq M$ where $U$ and $V$ are asymptotic to $U_0 \subset \Lambda^2_+ T^* (L_0 \times \mathbb{R})$ and $V_0 \subset X \times \mathbb{R}$, respectively. Also, we need $s$ to be sufficiently small in $F$ so that $L_s \times \mathbb{R}$ lies in $V_0$, in other words $L_s \times \mathbb{R}$ is the graph of $\zeta_s$ in $\Lambda^2_+ T^* (L_0 \times \mathbb{R})$.

Hence we consider $\zeta^2 + \varrho_s$ for $\zeta^2 \in L^p_{l+2,\gamma}(\Lambda^2_+ T^* C)$ and $\varrho_s \in C^\infty(\Lambda^2_+ T^* C)$ and define

$$Q(s, \xi^2) = \pi_*(\varphi|_{\Gamma(\zeta^2 + \varrho_s)}).$$

**Proposition 4.1.** $Q : F \times L^p_{l+2,\gamma}(B_\epsilon(\Lambda^2_+ T^* C)) \to L^p_{l+1,\gamma}(\Lambda^3 T^* C)$ is a smooth map of Banach manifolds. The linearization of $Q$ at 0 is $d Q(0,0) : (s, \xi^2) \mapsto d(\xi^2 + \varrho_s)$.

**Proof.** The functional form of $Q$ is

$$Q(s, \xi^2)|_x = H(s, x, \xi^2, \nabla \xi^2) \quad \text{for} \quad x \in C,$$

where $H$ is a smooth function of its arguments. Since $p > 4$ and $l \geq 1$ by Sobolev embedding theorem we have $L^p_{l+2,\gamma}(\Lambda^2_+ T^* C) \hookrightarrow C^1(\Lambda^2_+ T^* C)$. General arguments then show that locally $Q(s, \xi^2)$ is $L^p_{l+1}$.

From [5], we know that $Q(0, \xi^2)$ lies in $L^p_{l+1,\gamma}(\Lambda^3 T^* C)$. When $s \neq 0$, then $Q(s, 0) = \pi_*(\varphi|_{\Gamma(\xi^2)})$. By construction $\Gamma(\varrho_s)$ is asymptotic to $L_s \times \mathbb{R}$ which is coassociative in $X \times \mathbb{R}$ as $L_s$ is a special Lagrangian 3-fold. Identify $M \setminus K$ with $X \times \mathbb{R}$. Then $\varphi$ on $M \setminus K$ can be written as $\varphi = \omega \wedge dt + \Re(\Omega) + O(e^\alpha t)$ where $\alpha$ is the rate for $M$ converging to $X \times \mathbb{R}$. $\Gamma(\varrho_s)$ is the graph of $\varrho_s$ which is equal to $L_s \times (R + 1, \infty)$ in $X \times (R + 1, \infty)$. Since $L_s$ is a special Lagrangian submanifold with phase $i$, we get for $t > R + 1$, $\varphi|_{\Gamma(\varrho_s)} = (\omega \wedge dt + \Re(\Omega) + O(e^\alpha t))|_{L_s \times (R + 1, \infty)} = 0 + O(e^\alpha t)|_{L_s \times (R + 1, \infty)}$.

Therefore for $\Gamma(\varrho_s)$ the error term $\varphi|_{\Gamma(\varrho_s)}$ comes from the degree of the asymptotic decay. In particular, as $\alpha < \gamma$ we can assume $\varphi|_{\Gamma(\varrho_s)} \equiv O(e^\gamma t)$. We can easily arrange to choose $\varrho_s$ such that $\pi_*(\varphi|_{\Gamma(\varrho_s)}) \in L^p_{l+1,\gamma}(\Lambda^3 T^* C)$ and that $||\pi_*(\varphi|_{\Gamma(\varrho_s)})||_{L^p_{l+1,\gamma}} \leq c|s|$ for some constant $c$. This implies that $Q(s, \xi^2)$ lies in $L^p_{l+1,\gamma}(\Lambda^3 T^* C)$.

Finally, since we work locally, the linearization of $Q$ is again $d$ as before.

Next we show that the image of $Q$ consists of exact 3-forms in $L^p_{l,\gamma}(\Lambda^3 T^* C)$.

**Proposition 4.2.** $Q(F \times L^p_{l+2,\gamma}(B_\epsilon(\Lambda^2_+ T^* C))) \subseteq d(L^p_{l+1,\gamma}(\Lambda^2 T^* C)) \subseteq L^p_{l,\gamma}(\Lambda^3 T^* C)$.

**Proof.** This will be an adaptation of a similar proof in [5]. Consider the restriction of the 3-form $\varphi$ to the tubular neighborhood $T_C$ of $C$. As $\varphi$ is closed, and $T_C$ retracts onto $C$, and $\varphi|C \equiv 0$, we see that $\varphi|_{T_C}$ is exact. Thus we may write $\varphi|_{T_C} = d\theta$ for $\theta \in C^\infty(\Lambda^2 T^* T_C)$. Since $\varphi|C \equiv 0$ we may choose $\theta|C \equiv 0$. Also, as $\varphi$ is asymptotic to $O(e^\beta t)$ with all its derivatives to a translation-invariant 3-form $\varphi_0$ on $X \times \mathbb{R}$, we may take $\theta$ to be asymptotic to $O(e^\beta t)$ with all its derivatives to a translation-invariant 2-form on $T_L \times \mathbb{R}$.
the map \((s, \zeta^2_+ \mapsto (\Theta \circ (\zeta^2_+ + g_s))^\ast(\theta))\) maps \(F \times L^p_{l+2, \gamma}(B_r(\Lambda^2 T^\ast C)) \rightarrow L^p_{l+1, \gamma}(\Lambda^3 T^\ast C)\).

But

\[
Q(s, \zeta^2_+) = (\Theta \circ (\zeta^2_+ + g_s))^\ast(\varphi) = (\Theta \circ (\zeta^2_+ + g_s))^\ast(\theta) = d[(\Theta \circ (\zeta^2_+ + g_s))^\ast(\theta)],
\]

so we can conclude that \(Q(s, \zeta^2_+) \in d(L^p_{l+1, \gamma}(\Lambda^2 T^\ast C))\) for \(\zeta^2_+ \in L^p_{l+2, \gamma}(B_r(\Lambda^2 T^\ast C))\) and \(g_s \in C^\infty(\Lambda^2 T^\ast C)\).

As in [5], we now augment \(Q\) by a space of 4-forms on \(C\) to make it elliptic. That is, we define

\[
P : F \times L^p_{l+2, \gamma}(\Lambda^2 T^\ast C \oplus \Lambda^4 T^\ast C) \rightarrow L^p_{l+1, \gamma}(\Lambda^3 T^\ast C),
\]

by

\[
P(s, \zeta^2_+, \zeta^4) = \pi_2(\varphi|\zeta^2_+ + e_0) + d^\ast \zeta^4.
\]

Note that by the same discussion as in Proposition 4.1 we can also take that \(P(s, \zeta^2_+, \zeta^4)\) lies in \(L^p_{l+1, \gamma}(\Lambda^3 T^\ast C)\).

Proposition 4.1 implies that the linearization \(dP\) of \(P\) at 0 is the Fredholm operator \((d_+ + d^\ast)^p_{l+2, \gamma}\) of [6]. Define \(C\) to be the image of \((d_+ + d^\ast)^p_{l+2, \gamma}\). Then \(C\) is a Banach subspace of \(L^p_{l+1, \gamma}(\Lambda^3 T^\ast C)\), since \((d_+ + d^\ast)^p_{l+2, \gamma}\) is Fredholm. We show \(P\) maps into \(C\).

**Proposition 4.3.** \(P\) maps \(F \times L^p_{l+2, \gamma}(\Lambda^2 T^\ast C \oplus \Lambda^4 T^\ast C) \rightarrow C\).

**Proof.** Let \(s \in F\) and \((\zeta^2_+, \zeta^4) \in L^p_{l+2, \gamma}(B_r(\Lambda^2 T^\ast C)) \times L^p_{l+2, \gamma}(\Lambda^4 T^\ast C)\), so that \(P(s, \zeta^2_+, \zeta^4) = Q(s, \zeta^2_+) + d^\ast \zeta^4\) lies in \(L^p_{l+1, \gamma}(\Lambda^3 T^\ast C)\). We must show it lies in \(C\). Since \(\gamma \notin D_{(d_+ + d^\ast)}\), this holds if and only if

\[
\langle Q(s, \zeta^2_+) + d^\ast \zeta^4, \chi \rangle_{L^2} = 0 \quad \text{for all} \quad \chi \in \text{Ker}((d_+ + d)^q_{m+2, -\gamma}),
\]

where \(\frac{1}{p} + \frac{1}{q} = 1\) and \(m \geq 0\).

We proved earlier that \(\text{Ker}((d_+ + d)^q_{m+2, -\gamma})\) consists of closed and coclosed 3-forms \(\chi\). We also know that \(Q(s, \zeta^2_+) = d\lambda\) for \(\lambda \in \text{Ker}(\Lambda^2 T^\ast C)\) by Proposition 4.2. So

\[
\langle Q(s, \zeta^2_+) + d^\ast \zeta^4, \chi \rangle_{L^2} = \langle d\lambda, \chi \rangle_{L^2} + \langle d^\ast \zeta^4, \chi \rangle_{L^2} = \langle \lambda, d^\ast \chi \rangle_{L^2} + \langle \zeta^4, d\chi \rangle_{L^2} = 0
\]

for \(\chi \in \text{Ker}((d_+ + d)^q_{m+2, -\gamma})\), as \(\lambda\) is closed and coclosed, and the inner products and integration by parts are valid because of the matching of rates \(\gamma, -\gamma\) and \(L^p, L^q\) with \(\frac{1}{p} + \frac{1}{q} = 1\). So (7) holds, and \(P\) maps into \(C\). □

Proposition 4.3 implies that we can now apply Banach Space Implicit Mapping Theorem, and therefore conclude that \(P^{-1}(0)\) is smooth, finite-dimensional and locally isomorphic to \(\mathcal{A} = \text{Ker}((d_+ + d^\ast)^p_{l+2, \gamma}) \subset F \times L^p_{l+2, \gamma}(\Lambda^2 T^\ast C \oplus \Lambda^4 T^\ast C)\).

Note that our original map was \(Q\) and we needed the extra term from the space of 4-forms on \(C\) just to make \(Q\) elliptic. Therefore, we need to show that the following lemma holds.

**Lemma 4.4.** \(P^{-1}(0) = Q^{-1}(0) \times \{0\}\).
Proposition 4.5. Let \( (s, \zeta_2^+, \zeta_4^+) \in P^{-1}(0) \), so that \( Q(s, \zeta_2^+) + d^*\zeta_4^+ = 0 \). This should imply \( \zeta_4^+ = 0 \), so that \( Q(s, \zeta_2^+) = 0 \), and therefore \( P^{-1}(0) \subseteq Q^{-1}(0) \times \{0\} \). By Proposition 4.2 we have \( Q(s, \zeta_2^+) = d\lambda \) for \( \lambda \in L^p_{t+1, \gamma}(\Lambda^2 T^*C) \), so \( d\lambda = -d^*\zeta_4^+ \). Hence

\[
\|d^*\zeta_4^+\|^2_{L^2} = \langle d^*\zeta_4^+, d^*\zeta_4^+ \rangle_{L^2} = -\langle d^*\zeta_4^+, d\lambda \rangle_{L^2} = -\langle \zeta_4^+, d^2\lambda \rangle_{L^2} = 0,
\]

where the inner products and integration by parts are valid as \( L^p_{t+2, \gamma} \rightarrow L^2_2 \). Thus \( Q(s, \zeta_2^+) = d^*\zeta_4^+ = 0 \). But \( d^*\zeta_4^+ \equiv \nabla\zeta_4^+ \) as \( \zeta_4^+ \) is a 4-form, so \( \zeta_4^+ \) is constant. Since also \( \zeta_4^+ \rightarrow 0 \) near infinity in \( C \), we have \( \zeta_4^+ = 0 \).

By construction, it is straightforward to show that \( Q^{-1}(0) \times \{0\} \subseteq P^{-1}(0) \).

\[\square\]

Proposition 4.5. Let \( F \subset H^2(L, \mathbb{R}) \) be the subspace of special Lagrangian deformations of the boundary \( L \). Also let \( dP_{(0,0)}(s, \zeta_2^+, \zeta_4^+) \) represent the linearization of the deformation map \( P \) at 0 with moving boundary and \( dP_{(0,0)}^f(\zeta_2^+, \zeta_4^+) \) represent the linearization of the deformation map \( P^f \) at 0 with fixed boundary. Then

\[
\text{Index of } dP_{(0,0)} = \dim F + \text{index of } dP_{(0,0)}^f.
\]

Proof. At \( s = 0 \)

(9)

\[
P(0, \zeta_2^+, \zeta_4^+) = P^f(\zeta_2^+, \zeta_4^+) = \pi_*(\varphi|_{\Gamma(\zeta_2^+)}^\circ) + d^*\zeta_4^+
\]

Then at \( s = 0 \), the linearization at \( (0,0,0) \) is,

(10)

\[
dP_{(0,0,0)}(0, \zeta_2^+, \zeta_4^+) = dP_{(0,0)}^f(\zeta_2^+, \zeta_4^+)
\]

and since \( dP_{(0,0,0)} \) is linear we have

(11)

\[
dP_{(0,0,0)}(s, \zeta_2^+, \zeta_4^+) = dP_{(0,0,0)}(s, 0, 0) + dP_{(0,0,0)}(0, \zeta_2^+, \zeta_4^+)
\]

where \( dP_{(0,0,0)}(s, 0, 0) \) is finite dimensional, \( s \in T_0F = \mathbb{R}^d \).

Then

(12)

\[
\text{Index of } dP_{(0,0,0)} : F \times L^p_{t+2, \gamma}(\Lambda^2 T^*C \oplus \Lambda^4 T^*C) \rightarrow L^p_{t+1, \gamma}(\Lambda^3 T^*C)
\]

\[
= \text{index of } dP_{(0,0)}^f : F \times L^p_{t+2, \gamma}(\Lambda^2 T^*C \oplus \Lambda^4 T^*C) \rightarrow L^p_{t+1, \gamma}(\Lambda^3 T^*C)
\]

\[
= \dim F + \text{index of } dP_{(0,0)}^f : L^p_{t+2, \gamma}(\Lambda^2 T^*C \oplus \Lambda^4 T^*C) \rightarrow L^p_{t+1, \gamma}(\Lambda^3 T^*C).
\]

\[\square\]

In Proposition 4.6 we determine the set \( F \). First, we construct a linear map \( \Upsilon : H^2(L, \mathbb{R}) \rightarrow H^3_{cs}(C) \) and then show that \( F \) should be restricted to the kernel of \( \Upsilon \).
Proposition 4.6. There is a linear map $\Upsilon : H^2(L, \mathbb{R}) \to H^3_{cs}(C)$ given explicitly as $\Upsilon(\zeta) : [\zeta] \mapsto [d\zeta]$ and $F$ is an open subset of $\ker(\Upsilon)$.

Proof. $F$ is a subset of the special Lagrangian 3-fold deformations of $L$. So we could take $F \subset H^2(L, \mathbb{R})$. We can construct a linear map $\Upsilon : H^2(L, \mathbb{R}) \to H^3_{cs}(C)$ which comes from the standard long exact sequence in cohomology:

$$
\begin{array}{cccccccc}
0 & \longrightarrow & H^0(C) & \longrightarrow & H^0(L) & \longrightarrow & H^1_{cs}(C) & \longrightarrow & H^1(C) \\
& & & & \downarrow & & \searrow & & \\
& & & & H^2_{cs}(C) & \longrightarrow & H^2(C) & & \\
\end{array}
$$

(15)

where $H^k(C) = H^k(C, \mathbb{R})$ and $H^k(L) = H^k(L, \mathbb{R})$ are the de Rham cohomology groups, and $H^3_{cs}(C, \mathbb{R})$ is the compactly-supported de Rham cohomology group.

The potential problem here is that the image of $\Upsilon$ may not be zero in $H^3_{cs}(C)$. In [5], we studied the properties of the kernel and cokernel of the operator $d_\gamma + d^*$. The dimension of the cokernel is

$$\dim \ker((d_\gamma^* + d)d)^9_{m+2,-\gamma} = b^0(L) - b^0(C) + b^1(C).$$

The map $\ker((d_\gamma^* + d)d)^9_{m+2,-\gamma} \to H^3(C, \mathbb{R})$, $\chi \mapsto [\chi]$ is surjective, with kernel of dimension $b^0(L) - b^0(C) + b^1(C) - b^3(C) \geq 0$, which is the dimension of the kernel of $H^3_{cs}(C, \mathbb{R}) \to H^3(C, \mathbb{R})$. So we can think of $\ker((d_\gamma^* + d)d)^9_{m+2,-\gamma}$ as a space of closed and coclosed 3-forms filling out all of $H^3_{cs}(C, \mathbb{R})$ and $H^3(C, \mathbb{R})$. In other words cokernel has a piece looking like the kernel of $H^3_{cs}(C, \mathbb{R}) \to H^3(C, \mathbb{R})$. This is equivalent to the image of $\Upsilon : H^2(L, \mathbb{R}) \to H^3_{cs}(C)$ by exactness of the long exact sequence (15).

So the problem here is that if we allow the boundary to move arbitrarily then the image of $\Upsilon$ may not be zero in $H^3_{cs}(C)$. This means that, for $s \in H^2(L, \mathbb{R})$, the construction of asymptotically cylindrical coassociative 4-fold asymptotic to $L_f \times (R, \infty)$ is obstructed if and only if $\Upsilon(s) \neq 0$. Therefore the set $F$ is restricted to $\ker \Upsilon \subset H^2(L, \mathbb{R})$.

Next, we determine the dimension of the moduli space $\mathcal{M}_L^\gamma$.

Proposition 4.7. The dimension of $\mathcal{M}_L^\gamma$, the moduli space of coassociative deformations of an asymptotically cylindrical coassociative submanifold $C$ asymptotic to $L_f \times (R, \infty)$, $f \in F$, with decay rate $\gamma$ is

$$\dim(\mathcal{M}_L^\gamma) = \dim(V_+) + b^2(L) - b^0(L) + b^0(C) - b^1(C) + b^3(C).$$
Proof. Let \( b^k(C), b^k(L) \) and \( b^k_{cs}(C) \) be the corresponding Betti numbers as before. The actual dimension of the moduli space for free boundary is the sum of the dimension of the fixed boundary and the dimension of the kernel of \( \Upsilon \).

We know from Propositions 4.5, and 4.6 that

\[
\text{index of } d_P(0,0,0) = \dim F + \text{index of } d_P^f(0,0).
\]

In particular,

\[
\dim (\ker d_P(0,0,0)) = \dim (\ker \Upsilon) + \dim (V_+).
\]

Taking alternating sums of dimensions in (15), shows that the dimension of the kernel of \( \Upsilon \) is

\[
\dim (\ker \Upsilon) = b^2(L) - \text{Im}(\Upsilon) = b^2(L) - [b^0(L) - b^0(C) + b^1(C) - b^3(C)].
\]

Therefore the dimension of \( \mathcal{M}^C_{\gamma} \), the moduli space of coassociative deformations of an asymptotically cylindrical coassociative submanifold \( C \) asymptotic to \( L_s \times (R, \infty), s \in F \), with decay rate \( \gamma \) is

\[
\dim(\mathcal{M}^C_{\gamma}) = \dim(V_+) + \dim(\ker \Upsilon).
\]

\[
= \dim(V_+) + b^2(L) - b^0(L) + b^0(C) - b^1(C) + b^3(C).
\]

\( \square \)

And finally straightforward modifications of Propositions 4.6-4.8 in [5] provides us the standard elliptic regularity results and the bootstrapping argument to conclude that \( Q^{-1}(0) \) is a smooth, finite dimensional manifold, and so we complete the proof of Theorem 1.1. Here we will skip these details to avoid repetition, for more on regularity results see [5].

5. Applications: Topological Quantum Field Theory of Coassociative Cycles

In [10], it was mentioned that if one could show the analytical details of the deformation theory of asymptotically cylindrical coassociative submanifolds inside a \( G_2 \)-manifold then it would be possible to study global properties of these moduli spaces. In particular, one needs a result like Theorem 1.1 which shows that \( H^2(C, L) \) parametrizes the coassociative deformations of \( C \) with boundary \( L \). In this section using Theorem 1.1 and a well-known theory of (anti) self-dual connections (for more on the subject see [2], [3]) we will verify this claim.

Here are some basic definitions:

Let \( (M, \varphi, g) \) be an asymptotically cylindrical \( G_2 \)-manifold asymptotic to \( X \times (R, \infty) \), and \( C \) an asymptotically cylindrical coassociative 4-fold in \( M \) asymptotic to \( L \times (R', \infty) \). Let \( E \) be a fixed rank one vector bundle over \( C \). A connection \( D_E \) on \( C \) has finite energy if
\[ \int_C |F_E|^2 \, dV < \infty \]

where \( F_E \) is the curvature of the connection \( D_E \) and \( dV \) is the volume form with respect to the metric. Note that connections with finite energy play an important role in Yang-Mills Theory.

**Definition 5.1.** Let \((X, \omega, J, g_X, \Omega)\) be a Calabi–Yau 3-fold and \(L\) be a special Lagrangian submanifold of \(X\). Also let \((M, \varphi, g_M)\) be a \(G_2\)-manifold asymptotic to \(X \times (R, \infty)\). A pair \((C, D_E)\) is called a coassociative cycle if \(C\) is a coassociative submanifold of \(M\) asymptotic to \(L \times (R', \infty)\) and \(D_E\) is an anti-self-dual, unitary connection over \(C\) with finite energy.

**Definition 5.2.** Let \((X, \omega, J, g_X, \Omega)\) be a Calabi–Yau 3-fold and \(L\) be a special Lagrangian submanifold in \(X\). A pair \((L, D_{E'})\) is called a special Lagrangian cycle if \(D_{E'}\) is a unitary flat connection over \(L\) induced from \(D_E\).

Let \(\mathcal{M}^{slag}(X)\) denote the moduli space of special Lagrangian cycles in \(X\). In [8], Hitchin proved the following theorem:

**Theorem 5.3.** The tangent space to \(\mathcal{M}^{slag}(X)\) is naturally identified with the space \(H^2(L, \mathbb{R}) \times H^1(L, \text{ad}(E'))\). For line bundles over \(L\), the cup product \(\cup : H^2(L, \mathbb{R}) \times H^1(L, \mathbb{R}) \rightarrow \mathbb{R}\) induces a symplectic structure on \(\mathcal{M}^{slag}(X)\).

Now using Theorem [11] we verify the proof of the following theorem which was claimed by Leung [Claim 10, Sec.4] in [10].

**Theorem 5.4.** Let \(X\) be a Calabi–Yau 3-fold and \(L\) be a special Lagrangian submanifold in \(X\). Let \(M\) be a \(G_2\)-manifold asymptotic to \(X \times (R, \infty)\), and \(C\) a coassociative 4-fold in \(M\) asymptotic to \(L \times (R', \infty)\). Let also \(\mathcal{M}^{coas}(M)\) be the moduli space of coassociative cycles in \(M\) and \(\mathcal{M}^{slag}(X)\) be the moduli space of special Lagrangian cycles in \(X\). Then the boundary map

\[ b : \mathcal{M}^{coas}(M) \rightarrow \mathcal{M}^{slag}(X) \]

\[ b(C, D_E) = (L, D_{E'}) \]

is a Lagrangian immersion.

**Proof.** Let \(\delta : H^1(L) \rightarrow H^2_+(C, L)\), \(j^*: H^2_+(C, L) \rightarrow H^2_+(C)\), \(i^* : H^2_+(C) \rightarrow H^2(L)\) be the canonical maps. They give dual maps \(i_* : H_2(L) \rightarrow H_2^+(C)\), \(j_* : H_2^+(C) \rightarrow H_2^+(C, L)\) and \(\partial : H_2^+(C, L) \rightarrow H_1(C)\). Then we have the following long exact sequences:

\[
\begin{align*}
\ldots &\rightarrow H^1(L) \rightarrow H^2_+(C, L) \rightarrow H^2_+(C) \rightarrow H^2(L) \rightarrow H^3(C, L) \rightarrow \ldots \\
&\downarrow \cong \Downarrow \cong \downarrow \cong \Downarrow \cong \downarrow \cong \Downarrow \cong \\
\ldots &\rightarrow H_2(L) \rightarrow H_2^+(C) \rightarrow H_2^+(C, L) \rightarrow H_1(L) \rightarrow H_1(C) \rightarrow \ldots
\end{align*}
\]

Note that the classes coming from the boundary \(L\) should have zero self-intersection. So we can rewrite the long exact sequences above starting from 0, instead of \(H^1(L)\) and \(H_2(L)\).
From the sequences we get
\[ \ker \delta \cong \ker i_\ast \cong \im \partial \cong H_2^+(C, L)/\ker \partial = H_2^+(C, L)/\im(j_\ast) \]
which implies
\[ \dim (\ker \delta) = \dim (H_2^+(C, L)) - \dim(\im(j_\ast)) \tag{22} \]

Note that the only classes that survive in \( H_2^+(C, L) \) (i.e. do not go to zero) have self intersection 0. So one can identify \( \im(j_\ast) \) with \( H_2^+(C) \).

This implies that there is a short exact sequence
\[ 0 \rightarrow \im(j_\ast) \rightarrow H_2^+(C, L) \rightarrow \im \partial \rightarrow 0, \]
or equivalently
\[ 0 \rightarrow H_2^+(C) \rightarrow H_2^+(C, L) \rightarrow \ker \partial \rightarrow 0. \]

Then as a consequence of (22) and Theorem 1.1 we conclude that \( H_2^2(C, L) = V_+ \oplus \ker \delta \) parametrizes the deformations of \( C \) with moving boundary \( \partial C \). From our previous work in [5], it follows that \( H_2^2(C) \cong V_+ \) parametrizes the deformations of \( C \) with fixed boundary and McLean showed that \( H^2(L) \) gives the special Lagrangian deformations of \( L \).

Note that the linearization of the boundary value map \( \mathcal{M}^{\text{coas}}(M) \rightarrow \mathcal{M}^{\text{lag}}(X) \) in Theorem 5.4 is given by \( i^*: H_2^+(C) \rightarrow H^2(L) \) and for the connection part \( \beta: H^1(C) \rightarrow H^1(L) \). It is straightforward that \( \im(i^*)\oplus\im\beta \) is a subspace of \( H^2(L)\oplus H^1(L) \). By definition of cup product, the symplectic structure reduces to 0 on \( \im(i^*)\oplus\im\beta \) and by Poincaré Duality, \( \dim(\im(i^*)\oplus\im\beta) = \frac{1}{2} \dim(H^2(L)\oplus H^1(L)) \).

Thus we conclude that \( \im(i^*)\oplus\im\beta \) is a Lagrangian subspace of \( H^2(L)\oplus H^1(L) \) with the symplectic structure defined above and conclude the proof of Theorem 5.4.

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