Predicting Choice with Set-Dependent Aggregation

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Abstract

Providing users with alternatives to choose from is an essential component in many online platforms, making the accurate prediction of choice vital to their success. A renewed interest in learning choice models has led to significant progress in modeling power, but most current methods are either limited in the types of choice behavior they capture, cannot be applied to large-scale data, or both.

Here we propose a learning framework for predicting choice that is accurate, versatile, theoretically grounded, and scales well. Our key modeling point is that to account for how humans choose, predictive models must capture certain set-related invariances. Building on recent results in economics, we derive a class of models that can express any behavioral choice pattern, enjoy favorable sample complexity guarantees, and can be efficiently trained end-to-end. Experiments on three large choice datasets demonstrate the utility of our approach.

1 Introduction

One of the most prevalent activities of online users is choosing. In almost any online platform, users constantly face choices: what to purchase, who to follow, where to dine, what to watch, and even simply where to click. As the prominence of online services becomes ever more reliant on such choices, the accurate prediction of choice is quickly becoming vital to their success. The availability of large-scale choice data has spurred hopes of feasible individual-level prediction, and many recent works have been devoted to the modeling and prediction of choice [3, 31, 15, 14, 23, 36, 24, 6, 29].

In a typical choice scenario, a user is presented with a set of items \(s = \{x_1, \ldots, x_k\}, x_i \in \mathcal{X}\), called the choice set. Of these, the user chooses an item \(y \in s\). In economics this is known as the problem of discrete choice [16]. We let the collection of choice sets \(\mathcal{X}\) include all sets of at most \(n\) items, \(\mathcal{X} = \bigcup_{k \leq n} \mathcal{X}_k\), and assume choice sets and choices are drawn i.i.d. from an unknown joint distribution \(D\). Given a set of \(m\) examples \(S = \{(s_i, y_i)\}_{i=1}^m\) sampled from \(D\), our goal is to learn a choice predictor \(\hat{y} = h(s)\) that generalizes well to unseen sets, i.e., has low expected error w.r.t. \(D\).

A natural way to predict choice is to learn a score function \(f(x)\) over items. If \(f\) learns to score chosen items higher than their alternatives, then the argmax rule \(h_f(s) = \arg\max_{x \in s} P_f(x|s)\) where \(P_f(x|s) \propto e^{f(x)}\) will lead to good predictions. This approach may seem appealing, but is in fact constrained by an undesired artifact known as the Independence of Irrelevant Alternatives (IIA) [16]:

**Definition 1.** For any \(s \in \mathcal{X}\) and \(a, b \in s\), IIA holds if \(P(a|\{a, b\})/P(b|\{a, b\}) = P(a|s)/P(b|s)\).

IIA states that the likelihood of choosing \(a\) over \(b\) should not depend on what other alternatives are available. This means that for any choice set \(s\), if \(y \in s\) is predicted for \(s\), then \(y\) will also be predicted out of any subset of items \(s' \subset s\) in which \(y\) is included. IIA is a rigid constraint that imposes a fundamental limitation on what choice behavior can be expressed, and cannot be mitigated by simply increasing score function complexity (e.g., adding layers or clever non-linearities). From a practical point of view, this is discouraging, as there is ample empirical evidence of regular and consistent violations of IIA in real choice data (e.g., [33]). This has led to a surge of interest in machine learning models that go beyond IIA [25, 27, 3, 31, 28, 32, 6, 34, 30].

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The naïve way to avoid IIA is to directly model all possible subsets, but this is likely to make learning intractable [22, 34]. A common approach for resolving this difficulty is to impose structure, typically in the form of a probabilistic choice model encoding certain inter-item dependencies. While allowing for violations, this approach presents several practical limitations: First, explicitly modeling the dependency structure restricts how IIA can be violated. Second, as these methods focus mostly on asymptotic properties (e.g., consistency) rather than on finite-sample expected accuracy, they may not perform well in practice. Finally, surprisingly few of these methods apply to large-scale online choice data, where the number of instances can be prohibitively large, choice sets rarely appear more than once, and items can be complex structured objects whose number is virtually unbounded.

To complement most current works, and motivated by the growing need for accuracy and scalability, here we propose a framework for choice prediction that is purely discriminative, and hence directly optimizes for accuracy. Our framework is based on the idea of set-dependent aggregation, a principled approach in economics for modeling IIA violations [1], that to the best of our knowledge has not been considered from a machine learning perspective. The challenge in designing a useful discriminative choice framework that supports violations is twofold. On the one hand, it must be general enough to allow for IIA violations without explicitly modeling how they can occur. On the other hand, it must be specific enough to efficiently capture actual behavioral violations emerging from the data. In this paper, we show that set-dependent aggregation achieves both. Specifically, our framework makes the following contributions:

- **Efficient and scalable discriminative training.** Our approach is geared towards predictive performance: it directly optimizes for accuracy, can be efficiently trained end-to-end, and scales well to realistically large and complex prediction scenarios.

- **Rigorous error bounds.** We bound both approximation and estimation error of aggregation. Our results show how the complexity of the hypothesis class controls the type and number of accountable violations, uncovering a data-driven mechanism for targeting important violations.

- **Behavioral inductive bias.** Aggregation can provably express any form of violation [1], but may require an intractable number of parameters to do so. To control for overfitting, we infuse our model with inductive bias taken from behavioral decision theory, balancing flexibility and specificity without imposing any explicit structure.

- **Thorough empirical evaluation.** We conduct experiments on three large choice datasets—flight itineraries, hotel reservations, and news recommendations—demonstrating the utility of our approach. Our analysis reveals how accuracy is driven by the flexibility in modeling violations, which resonates well with our theoretical results.

Overall, our work presents a practical and theoretically-grounded approach for predicting choice.

**Paper organization.** We begin with a review of the related literature (Sec. 1.1). We then present our model of set-dependent aggregation in Sec. 2. This is followed in Sec. 3 by our main theoretical results, bounding the approximation error (Sec. 3.1) and estimation error (Sec. 3.2) of aggregators. Finally, we present our experimental results in Sec. 4, and give concluding remarks in Sec. 5.

### 1.1 Related material

IIA begins with Luce’s Axiom of Choice [16], which for a certain noise distribution, induces the popular Multinomial Logit model (MNL) [20, 39]. Two common extensions—Nested MNL and Mixed MNL—relax IIA by grouping items via a tree structure [19] or modeling a mixture population [21], respectively. These, however, pose restrictive assumptions on the nature of violations, require elaborate hand-coded auxiliary inputs, and are in many cases intractable. Although recent progress has alleviated some difficulties [25, 3, 6], applying these models in practice remains difficult. Recent works have proposed other probabilistic models that deviate from IIA, by modeling pairwise utilities [32], $k$th-order interactions [31, 34], and general subset relations [4]. These, however, do not optimize for predictive accuracy, and rarely apply to complex choice settings with many items and sparse choice sets. Others suggest discriminative solutions, but these include models that tend to be either over-specific or over-general [23, 30] and provide no guarantees as to what violations can be captured.

Our work draws on the literature of utility aggregation (or “multi-self” models), studied extensively in economics [11, 7, 18, 8, 1], psychology [41, 35, 44], and marketing [13]. While these typically focus
We now describe the learning framework and propose a concrete architecture for SDA. Aggregation is a popular approach to enriching the hypothesis class, but it is easy to show that aggregation is not set-dependent. At the core of our approach is the idea of weights \( \phi \) that are a function of \( s \) when removing an item can change the scoring direction (b). When \( \phi \) is a function of \( s \), removing an item can change the spatial position of items (c). Both of these allow set-dependent embeddings \( v, F \) to have different maximizing items, and hence different predictions.

Set-dependent weights: Let \( s \in X \). Starting with \( g(x;v) = \langle v, F(x) \rangle \), consider replacing each scalar weight \( v_i \) with a set function \( w_i : X \rightarrow \mathbb{R} \). This gives the following aggregation model:

\[
\text{SDW:} \quad g(x,s;w) = \langle w(s), F(x) \rangle, \quad w(s) = (w_1(s), \ldots, w_{\ell}(s))
\]

Denote \( \hat{y} = \arg\max_{x \in s} \langle w(s), F(x) \rangle \), and let \( s' = s \setminus \{x\} \) for some \( x \neq \hat{y} \). Recall that IIA is violated when removing \( x \) from \( s \) changes the prediction. Here, because removing \( x \) changes the input to each \( w_i \), the contribution of each \( f_i \) to \( g \) changes from \( w_i(s) \) to \( w_i(s') \). Geometrically, the scoring direction changes, and therefore the choice prediction can change as well (Fig. 1b).

Set-dependent embeddings: Now, consider instead replacing each item score function \( f_i \in F \) with an item-centric set function \( \varphi_i : X \times X \rightarrow \mathbb{R} \). The aggregation model is now:

\[
\text{SDE:} \quad g(x,s;v,\varphi) = \langle v, \varphi(x,s) \rangle, \quad \varphi(x,s) = (\varphi_1(x,s), \ldots, \varphi_{\ell}(x,s))
\]

The vector \( \varphi(x,s) \) is a set-dependent embedding of \( x \) (the dependence on \( F \) is implicit). The predicted item is that whose embedding is closest to \( v \)'s. Here, when the set changes (i.e., by removing an item), items shift in the embedded space, and the prediction can change (Fig. 1c).

Our final and most general set-dependent aggregation model combines both mechanisms:

\[
\text{SDA:} \quad g(x,s;w,\varphi) = \langle w(s), \varphi(x,s) \rangle
\]

We now describe the learning framework and propose a concrete architecture for SDA.
2.1 Learning

Our goal is to learn an aggregator \( g \) that minimizes the expected risk \( \varepsilon(g) = \mathbb{E}_D[\Delta(y, g(s))] \) where \( \Delta \) is the 0/1 loss. We approximate this by minimizing the empirical risk:

\[
\min_{g \in \mathcal{G}} \sum_{i=1}^{m} \Delta(y_i, g(s_i)) + \lambda \mathcal{R}(g), \quad \Delta(y, g(s)) = 1_{\{y(s) \neq y\}}
\]

(4)

where \( \mathcal{G} \) is an aggregator class, and \( \mathcal{R} \) is a regularization term with coefficient \( \lambda \). Note that the decision rule \( h_y(s) = \arg\max_{x \in s} g(x, s) \) is now set-dependent. Because aggregation is simply a template for combining score functions, if \( \mathcal{F}, w, \) and \( \varphi \) are differentiable, \( g \) is differentiable as well, and can therefore be efficiently optimized end-to-end using standard gradient methods. In Sec. 3 we bound the expected risk of aggregator classes \( \mathcal{G} \).

2.2 Inductive bias

To allow for efficient learning, we implement \( w(s) \) and \( \varphi(x, s) \) using key principles from behavioral choice theory, suggesting that the choices are based on a relative, a-symmetric, and context-dependent perception of utility. In accordance, we instantiate:

\[
w(s) = w(F(s)), \quad \varphi(x, s) = \mu(F(x) - r(F(s)))
\]

(5)

where \( F(s) = \{F(x)\}_{x \in s} \). Here, \( F \) models item utilities, \( r \) is a set-specific reference point for comparing utilities [42], \( \mu \) is a loss-averse evaluation mechanism [10, 43], and \( w \) integrates multiple evaluations in a context-dependent manner [40, 44]. This construction generalizes many models from the fields of economics, psychology, and marketing:

Claim 1. The aggregation model in Eq. (5) generalizes choice models in [40, 20, 19, 12, 11, 13, 26].

See Appendix A for details. In practice, we implement \( F, w \) and \( r \) using appropriate neural networks, and \( \mu \) using an a-symmetric s-shaped nonlinearity [17].

3 Theoretical Analysis

In this section we bound the expected risk of aggregators. Our results demonstrate the power of set-dependent aggregation: expressive enough to allow for arbitrary violations, and concise enough to be efficiently learned. The risk of any \( g \in \mathcal{G} \) breaks down into two fundamental error types [37]:

\[
\varepsilon(g) = \varepsilon^*(\mathcal{G}) + \varepsilon(g) - \varepsilon^*(\mathcal{G}) = \min_{g' \in \mathcal{G}} \varepsilon_D(g')
\]

(6)

Approximation error is a property of the class \( \mathcal{G} \), measuring its suitability for examples drawn from \( D \). Estimation error is specific to \( g \), measuring its performance relative to \( g' \). Our results show that the tradeoff in both errors is controlled by two factors: the complexity of the utility function class \( \mathcal{F} \) and the aggregation dimension \( \ell \).

3.1 Approximation Error

Our first result shows how aggregation combines predictors that cannot express violations into a model that can account for any form of violation. Our main theorem reveals how this is achieved: aggregation uses score functions to ‘isolate’ regions of violating sets, and within each region, operate independently. Technically, this is shown by decomposing the approximation error of \( \mathcal{G} \) over these regions, where the error in each region depends on the error of \( \mathcal{F} \).

We begin with some definitions. Recall that IIA mandates that if \( y \) is chosen from \( s \), then it should also be chosen from any \( s' \subseteq s \) with \( y \in s' \). This motivates a definition of IIA violation due to [1]:

Definition 2 (IIA Violation, [1]). A set \( s \) violates IIA if there exists \( s' \subset s \) with \( c(s) \neq c(s') \in s \).

Denote the set of all violating sets by \( V \) and non-violating sets by \( U = \mathbb{X} \setminus V \). Let the predicate \( s \ni x \subseteq s' \) be true if \( f \) scores items in \( s \) higher than items in \( s' \), i.e., \( f(x) > f(x') \forall x \in s, x' \in s' \).

\(^1\) The definition in [1] additionally requires \( s \) to be of maximal size, which they need for a counting argument.
We are now ready to state our main result (see Appendix B for proof and further details).

**Theorem 1.** The approximation error \( \varepsilon(G) \) decomposes over:

1. non-violating sets with risk at most \( \varepsilon^*(F|U) \)
2. separable violating regions with risk at most \( \varepsilon^*(F|\Omega_f) \) per region

and the set of separable regions is optimal under budget constraint \( \ell \). Specifically, we have:

\[
\varepsilon^*(G) \leq \sum_{R \in \Omega \setminus \Omega_f} p_R \varepsilon^*(F|R) + p_{\Omega \setminus \Omega_f}
\]

for the optimal \( F \in F^\ell \), where \( \Omega_f = \bigcup_{f \in F} \Omega_f \), \( \ell' \leq (\ell - 1)/5 \), and \( p_A = P(s \in A) \).

Theorem 1 hints at how aggregation can account for violations: it partitions \( \mathbb{X} \) into regions, and applies \( F \) to each region independently. In this process, \( F \) plays a corresponding dual role of both determining separable regions \( \Omega_f \) and predicting within these regions. This demonstrates how the type of accountable violations is controlled by the expressivity of \( F \), and how the number of violations (via the number of regions) is controlled by \( \ell \). In principal, aggregation can account for any violation:

**Corollary 1.** If \( y = c(s) \) for some choice function \( c \) and for all \( s \in \mathbb{X} \), then with sufficiently large \( \ell \) and sufficiently expressive \( F \), the approximation error vanishes, i.e., \( \varepsilon^*(G) = 0 \).

Our proof takes the main building blocks of [1]—objects called “triple bases”—and carefully weaves them within a statistical framework. As in [1], results apply to classes satisfying a natural property of scale-invariance; these include all but one model from Claim 1. The main lemma in the proof requires that \( G \) is defined over an auxiliary neural-network class of score functions that is slightly richer than \( F \), and shows how “implementing” triple bases using simple neural circuits enables a decomposition of the error over violating regions.

The general statement of Theorem 1 is that the more complex \( F \) and the larger \( \ell \), the better the approximation error. Our next results quantifies how this trades off with estimation error.

### 3.2 Estimation Error

Our next results establish the learnability of aggregation. We give sample complexity bounds on the empirical risk \( \varepsilon_S(g) = \mathbb{E}_S|\Delta(y, g(s))| \) showing how learning both weight and embedding mechanisms (Sec. 2) depends on \( F, \ell \), and the aggregation components in \( g \). Proofs are based on Rademacher bounds and are given in Appendix C. To simplify the analysis, we focus on set operations and on a linear base class:

\[
F^\rho_{lin} = \{ F(x) = x^\top \Theta \mid \Theta \in \mathbb{R}^{d \times \ell}; \|\Theta\|_\rho \leq 1 \}
\]

where \( \|\cdot\|_\rho \) is the induced \( \rho \)-norm. This suffices to cover the models from Claim 1.

For any function \( g \), we denote its Lipschitz constant by \( \lambda_g \) when it is scalar-valued and by \( \lambda'_g \) when it is vector valued and with respect to the \( \rho \)-norm. We also use \( X_\rho = \max_{x \in S} \|x\|_\rho \).

The first bound applies to the set-dependent weight mechanism (Eq. (1)).

**Theorem 2.** Let \( G \) be a class of aggregators over \( F^\infty_{lin} \) of the form \( g(x, s) = (w(s), F(x)) \). Then for all \( D \) and any \( \delta \in [0, 1] \), it holds that

\[
\varepsilon(G) \leq \varepsilon_S(G) + 4X^2 \lambda^\rho_w \sqrt{\frac{2 \log 2d}{m}} + O \left( \sqrt{\frac{\log(1/\delta)}{m}} \right)
\]

with probability of at least \( 1 - \delta \).

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2 Here we assume that the approximation error does not include the irreducible error.

3 All except for one of the three models proposed in [13].
The second bound applies to the set-dependent embedding mechanism (Eq. (2)), where for concreteness we set $\varphi$ according to the inductive bias model in Eq. (5).

**Theorem 3.** Let $G$ be a class of aggregators over $\mathcal{F}_{\text{lin}}$ of the form $g(x, s) = \langle v, \varphi(x, s) \rangle$ where $\varphi(x, s) = \langle w, \mu(F(x) - r(s)) \rangle$ as in Eq. (5). Then for all $D$ and any $\delta \in [0, 1]$, it holds that

$$\varepsilon(G) \leq \varepsilon_S(G) + 4W_1X_\infty\lambda \mu(1 + \lambda^d)\sqrt{\frac{2 \log 2d}{m}} + O\left(\sqrt{\frac{\log(1/\delta)}{m}}\right)$$

with probability of at least $1 - \delta$, where $W_1 = \|w\|_1$.

Both results show what governs learnability: the dimension $\ell$ plays an implicit role via the Lipschitz constants of the set operations, and $\mathcal{F}$ affects the bound via the norm of $w$ (also depending on $\ell$).

### 4 Experiments

We now present our experimental evaluation on click prediction tasks on online platforms.

**Datasets.** We evaluate our method on three large datasets: flight itineraries from Amadeus\(^4\), hotel reservations from Expedia\(^5\), and news recommendations from Outbrain\(^6\). Each dataset includes sets of alternatives and the corresponding user choices. We focus on examples where users clicked on exactly one item. Features describe items (e.g., price, quality, or category) and context (e.g., query, date and time). Unfortunately, for reasons of privacy, very little user information is available. Appendix D.1 includes further details.

**Baselines.** We compare against three baseline categories: methods that satisfy IIA, methods that target specific violations of IIA, and methods that do not satisfy IIA but also do not directly model violations. This lets us explore whether capturing IIA violations is useful, and if so, what form of deviation is preferable. We focus on approaches that can be applied to large-scale data as the above.

For IIA methods, we use Multinomial Logit (MNL) \([20, 39]\), SVMRank \([9]\), and RankNet \([5]\). These methods predict based on item-centric score functions, but differ in the way they are optimized. For non-IIA methods, we use a discrete Mixed MNL model \([39]\), AdaRank \([45]\), and Deep Sets \([46]\). These differ in how they consider item dependencies: Mixed MNL models user populations, ListNet captures set dependencies via the loss function, and Deep Sets attempts to universally approximate general set functions. We also compare to simple baselines based on price or quality, and to random.

**Aggregation models.** For our approach, we evaluate the performance of three models, each corresponding to one of the aggregation mechanisms from Sec. 2.

- **Set-dependent weights:** $g(x, s) = \langle w(s), F(x) \rangle$ [SDW, Eq. (1)]
- **Set-dependent embeddings:** $g(x, s) = \langle v, \varphi(x, s) \rangle$ [SDE, Eq. (2)]
- **Combined inductive bias model:** $g(x, s) = \langle w(s), \varphi(x, s) \rangle$ [SDA, Eq. (3)]

The set function $w$ is implemented by a small permutation-invariant neural network \([46]\) having 2 hidden layers of size 16 with tanh activations and use mean pooling. In the combined model, we use a slight generalization of Eq. (3), letting $\varphi(x, s) = \mu(\langle F(x), r(s) \rangle)$ where $F$ and $r$ are now

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\(^4\)See [23]
\(^5\)www.kaggle.com/c/expedia-personalized-sort
\(^6\)www.kaggle.com/c/outbrain-click-prediction
When comparing predictive top-1 accuracy, non-IIA methods tend to outperform IIA methods by a small margin. See Appendix D.3 for further details.

To highlight the contribution of aggregation, for all models we use a simple linear base class $\mathcal{F}$, allowing a clean differential comparison to IIA methods such as MNL (linear score functions, no set operations) and to non-IIA methods such as DeepSets (neural set functions, no linear components). Our main results use $\ell = 24$, which strikes a good balance between performance and runtime (in general, accuracy increases with $\ell$, but most of the gain is already reached at $\ell = 4$; see Fig. 3 left).

**Setup.** Results are based on averaging 10 random 50:25:25 train-validation-test splits. The methods we consider are varied in their expressive power and computational requirements. Hence, for a fair comparison (and to allow reasonable run times), each trial includes 10,000 randomly sampled examples. Performance is measured using top-1 accuracy, top-5 accuracy, and mean reciprocal rank (MRR). For all methods we tuned regularization, dropout, and learning rate (when applicable) using Bayesian optimization. Default parameters were used for other hyper-parameters. For optimization we used Adam with step-wise exponential decay. See Appendix D.3 for further details.

4.1 Results

Our main results are presented in Table 1, which compares the different methods for choice sets up to 10, 10, 12 items for each dataset (the full results for a range of maximal number of items can be found in Appendix D.5). As can be seen, SDA outperforms other baselines in all settings.

When comparing predictive top-1 accuracy, non-IIA methods tend to outperform IIA methods by a small margin. Discrete MNL, which deviates from IIA by targeting certain inter-item dependencies, is observed as the most competitive baseline. DeepSets, which is our most general non-IIA baseline, shows mixed results, and relatively high standard errors. Deepsets portrays the invariances required for expressing violations with significantly more parameters than SDA (Appendix D.2), but lacks the inductive bias that SDA holds, thus demonstrating the importance of the latter.

For measures that extend beyond the first item (such as top-5 and MRR), IIA ranking methods perform well, likely because their objective consider relations between items. Nonetheless, they are still outperformed by SDA, even though its objective considers only the top item. We conjecture that training SDA with a ranking loss will increase performance further.

4.2 Analysis

**Accuracy and violation capacity.** Figure 3 (left) presents the accuracy of an aggregator for increasing $\ell$. Results show that accuracy steadily increases, although roughly 90% of the gain is accuracy is achieved by $\ell = 4$. Theorem 1 suggests that increasing $\ell$ helps by covering additional regions of
violating sets. To empirically quantify this, we measure violation capacity:

$$\kappa_S(g) = \frac{1}{|S|} \sum_{(s,y) \in S} \frac{1}{|s| - 1} \sum_{y \neq x \in s} 1 \{ g(x, s - x) \neq y \}$$  \hspace{1cm} (8)

where $s - x = s \setminus \{x\}$. Violation capacity measures how frequently the prediction changes when single items are removed from the choice set, and is a first-order approximation of the violation frequency captured by a model. Figure 3 (left) reveals a tight correlation between accuracy and $\kappa$.

**Violation budget allocation.** The decomposition in Theorem 1 shows how aggregation allocates one score function to cover all non-violating sets $U$, and uses the remaining $\ell - 1$ score functions for targeting and handling violating regions. Empirically, this suggests that $\kappa$ should vary across choice sets: high $\kappa$ for targeted violating regions, low $\kappa$ for $U$ and all other regions. Figure 3 (right) compares the violation capacity of SDA MNL, and Deep Sets. For each method, the diagram shows average $\kappa$ values partitioned according to intersecting correctness regions (i.e., examples that only SDA was correct on, examples that SDA and MNL were correct on but not Deep Sets, etc.). The results highlight how SDA efficiently allocates its violation budget, executing very little violation capacity on examples that MNL is correct on, and allocating the minimally necessary amount to the rest. In contrast, MNL has no violation capacity (as it cannot express violations), and Deep Sets over-utilizes its unconstrained capacity to violate.

**Comparing aggregators** Claim 1 states that SDA generalizes many aggregation models from the literature, and by preserving their structural form (i.e., Eq. (5)), introduces useful inductive bias. Figure 3 (center) compares the performance of SDA to the models from Claim 1 on Amadeus. As can be seen, SDA clearly outperforms other models. Many of these models were designed for mathematical or behavioral tractability, having simple components with light parameterization. These results demonstrate the benefit of replacing these with highly flexible parametric neural components.

## 5 Conclusions

In this paper we proposed a method for accurately predicting choice. Human choice follows complex and intricate patterns, and capturing it requires models that on the one hand are sufficiently expressive, but on the other are specific enough to be learned efficiently. Our goal in this paper was to show that aggregation strikes a good balance between expressivity and specificity, in theory and in practice.

Our work is motivated by the growing need for methods that are accurate and scalable. Aggregation is a good candidate for two main reasons. First, aggregation accounts for general violations without imposing any structural assumptions. This is important since accurately predicting choice is likely impossible within the confines of IIA or of targeted forms of violation. Second, it provides a simple template into which differential components can be cast. Aggregators can therefore capitalize on the success of discriminative learning and neural architectures to efficiently optimize predictive accuracy.

There are two main avenues in which our work can extend. First, our theoretical results explore the connection between generalization and violation of IIA. We conjecture that this connection runs...
deeper, and that a properly defined “violation complexity” can be useful in giving exact characterizations of learnability. Second, our work focused on a rudimentary choice task: choosing an item from a set. There are, however, many other important choice tasks, such as sequential or subset selection, posing interesting modeling challenges which we leave for future work.

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Appendix A  Inductive Bias

As stated in Claim 1, SDA with inductive bias Eq. (5) generalizes many models that have been proposed in the discrete choice literature. Below, we summarize the specific choices of $w$, $r$ and $\mu$ in each paper.

| Type    | Extends       | $w$          | $r$          | $\mu$          |
|---------|---------------|--------------|--------------|----------------|
| -       | MNL ([20])    | one          | zero         | identity       |
| SDW     | [40] (max-min)$^\rho$ | zero         | $\log \sum$ | log            |
|         | [19] linear    |              |              |                |
|         | [11] softmax   |              | zero         | identity       |
|         | [26] linear    |              | w. average   | kinked lin.    |
| SDE     | [12] sum       | min          | log          |                |
|         | [13] (LAM)     |              | (max+min)/2  | kinked lin.    |
|         | [13] (CCM)     |              | min          | power($\rho$)  |
| SDA     | [13] (NCCM)   | max-min      | min          | norm. pow($\rho$) |
| (ours)  | set-nn        | set-nn       |              | kinked tanh    |

Table 2: Discrete choice models as set-aggregation models.

As discussed in Sec. 4, we chose our specification of SDA based on set neural networks and a 'kinked' tanh function. In Appendix D.4, we describe each part of our model in further detail.

Appendix B  Approximation Error

We begin with some useful definitions and lemmas due to [1], rephrased to align with our setup. We then briefly highlight the differences in setting and results between our paper and [1]. Finally, we give the proof of Theorem 1.

B.1 Definitions and lemmas from [1]

As in many works in economics, [1] use the concept of item utility functions $u$, mapping each item to a scalar representing its utility. From our point of view, a score function $f$ correspond to a utility function if it is fully parameterize, i.e., has one parameter for every item that represents its utility. We will think of $u$ either as functions $u(x)$ or vectors $u_x$ in appropriate context. We use $u$ to define a collection of utilities $u_i$ in the same way that $F$ represents a collections of score functions $f_i$, and accordingly define $g_u$ as an aggregator of utilities $u$.

It will be useful to notationally differentiate the aggregation mechanism $M$, which is the functional form defining how individual score functions are combined, from aggregator classes $G$, which include the actual aggregators (i.e., functions), whose form is given by $M$.

Definition 4 (Triple basis, [1]). Let $M$ be an aggregation mechanism and $k \in \mathbb{N}$, then $u = \{u_i\}_{i=1}^k$ with $u_i \in \mathbb{R}^3$ is a triple basis for $T = (x_1, x_2, x_3)$ under $M$ if:

1. $g_u(x_1, \{x_1, x_2\}) > g_u(x_2, \{x_1, x_2\})$, and
2. $g_u(x, s') = g_u(x', s') \forall s' \neq \{x_1, x_2\}, x, x' \in s'$.

where $g_u$ denotes an aggregation with item values given by $u$.

When used in an aggregator, triple bases serve two purposes: they determine the choice from $\{x_1, x_2\}$, but make sure this does not effect choices in other choice sets. Triple bases can be extended to handle set-triples $(x, s, t)$, $x \in \mathcal{X}$, $s, t \subseteq \mathcal{X}$ with $x \in s$, where $x$ plays the role of $x_1$, subsets $s' \subseteq s$ the role of $x_2$, and subsets $t' \subseteq t$ the role of $x_3$: the triple bases will choose $x$ from $s'$ (when $x \in s'$) and will be indifferent otherwise. The following excerpt from the main proof of [1] (stated here as a lemma) formalizes this notion. As we now consider multiple items, it will be useful to think of utilities $u$ as
mappings from items to scalar utilities, \( u : \mathcal{X} \rightarrow \mathbb{R} \), and we will use both representations of utilities (as vectors and as functions) interchangeably.

**Lemma 1** (Triple basis for sets, [1]). Let \( \mathbf{u} = \{u_i\}_{i=1}^k \) with utilities \( u_i : \mathcal{X} \rightarrow \mathbb{R} \), and consider some set-triple \((x, s', t)\) with \( x \in \mathcal{X}, s' \subseteq \mathcal{X} \) and \( x \in s' \). If each \( u_i(z) \) is the same for all \( z \in s' \setminus \{x\} \) and is also the same for all \( z \in t \), then if \( \mathbf{u} \) is a triple-basis for some \((x_1, x_2, x_3)\), it is also a set triple basis for \((x, s', t)\), in the sense that:

1. \( x \) is chosen from any \( s' \subseteq s \cup \{x\} \) with \( x \in s' \), and
2. \( \mathbf{u} \) is indifferent on all other choice sets, i.e., those without \( x \) or that include items from \( t \).

Triple bases serve as the main building block in [1]. The surprising finding in [1] is that once a triple basis is known for an arbitrary triplet \((x_1, x_2, x_3)\), using Lemma 1, it can be applied to any violating set \( s \) by setting \( x = y \) and \( t = \mathcal{X} \setminus s \). Hence, the existence of a triple basis is a property of the aggregation mechanism, and not specific to certain items or choice sets (and in our case, to score function classes or the distribution). The authors of [1] provide explicit constructions of triple bases for several aggregation (a.k.a. ‘multi-self’) models from the literature, as well as a general recipe that applies to a large class of aggregation mechanisms that satisfy, in addition to certain natural axioms (e.g., Neutrality and Consistency), the property of scale invariance:

**Definition 5** (Scale invariance, [1]). An aggregator \( g_u \) is scale invariant if there exists an invertible and odd function \( \Phi \) such that for every \( \alpha > 0 \), \( g_{u,\alpha}(x, s) = \Phi(\alpha)g_u(x, s) \) for all \( x \in \mathcal{X}, s \in \mathcal{X} \).

Hence, for any aggregation mechanism that satisfies Def. 5 there exists some triple basis \( \mathbf{u} \). Furthermore, the authors show that \( \mathbf{u} \) includes at most \( k = 5 \) utilities. Scale invariance simply means that the ranking over items (and hence prediction) induced by the aggregator do not depend on the scale of their internal score functions, and hence, scaling these does not change predictions. Accordingly, and following [1], our results herein apply to such mechanisms, for which we assume the existence of a corresponding triple basis.

The authors of [1] show that all of their results follow through when exact triple bases are replaced with approximate triple basis, where the equalities hold only up to some precision \( \epsilon \). This will also be the case in our proof.

**B.2 Comparing our setting and results to those of [1]**

Before proceeding with our proof, we describe the differences between our result and those of [1], thus highlighting some of the challenges encountered while proving our result.

The setting of [1] differs from ours in three crucial aspects. First, they focus on a realizable setting: they assume the existence of a choice function \( c : \mathcal{X} \rightarrow \mathcal{X} \) designating choices \( y = c(s) \in s \) for all \( s \in \mathcal{X} \), and aim to recover it from the class of all choice functions. We, on the other hand, focus on the agnostic setting, where labels (i.e., choices) \( y \) are not necessarily generated from a function within the class we consider (and in fact, can be sampled from an unknown conditional distribution \( D_{Y|S} \)).

Second, [1] provide worst case results for the successful reconstruction of \( c \) from any collection of labeled examples. In contrast, we focus on a statistical setting where a sample set of alternatives and choices are drawn from some unknown distribution. We are interested in minimizing the expected loss, i.e., the probability of correctly predicting choice from choice sets drawn from the same distribution.

Finally, and perhaps most importantly, [1] assume that all items can be given arbitrary scores (which they refer to as ‘utilities’). In other words, score functions are fully parameterized and can assign any value to any item. This of course means that the number of parameters required for each score function (of which there can be many in the context of aggregation) is equal to the number of items, which in practice can be rather large, and in principle can be unbounded. In contrast, we focus on the parametric settings common in machine learning, where items are described by features (e.g., vectors), and score functions are parametric functions of those features. Critically, without the assumption of full parameterization, the results of [1] break down, as they require the ability to assign arbitrary values to each item.

\(^7\)While \( u \) is a mapping of items, for notational clarity we write it as a function of (observed) features.
B.3 Proof of Theorem 1

Let $M$ be a scale-invariant aggregation mechanism as in [1], and let $u$ be a corresponding triple basis of size $k$, which we assume fixed throughout the proof. As noted in Sec. 3.1, proof requires that the class of aggregators $G$ be defined over a class of score functions that is slightly more expressive $F$, denoted $F_+$. Specifically, $F_+$ includes combinations of pairs of score functions from $F$, given by $F_+ = \{a(b(x), b'(x)) : a \in N, b, b' \in F\}$ where $N$ is a class of small neural networks (2 inputs, 2 layers with 2 units each, sigmoidal activations) whose precise definition will be given in the proof. Note that by construction we will have $F \subset F_+$.

To reduce notational clutter we will use $\varepsilon^*_D(\cdot) = \varepsilon^*(\cdot)$ for the minimal expected error and $\varepsilon^*_A(\cdot) = \varepsilon^*(\cdot | A)$ for the minimal expected error conditioned on the event $s \in A$.

We can now restate a slightly tighter variant of Theorem 1 in finer detail:

**Theorem 1.** Let $M$ be a scale-invariant aggregation mechanism as in [1] with a triple basis $u$ of size $k$. Let $G = G^{(\ell)}_F$ be a corresponding class of aggregators of dimension $\ell$ defined over the class of item score functions $F_+$. Then:

$$\varepsilon^*_D(G) \leq p u \varepsilon^*_D(F_+) + \min_{B \in F^{\ell'}} \sum_{b \in B} p_{B_b} \varepsilon^*_D(F) + p_{\Omega_B} \varepsilon^*_D(F)$$

(9)

where $\Omega_B = \cup_{b \in B} \Omega_b$, $\ell' = (\ell - 1)/k$, and $p_A = P(s \in A)$.

Before giving the proof, we state a practical corollary of Theorem 1.

**Corollary 2.** Let $N^{N,K}$ be a class of neural networks with $N \geq 2$ fully connected layers with $K \geq 2$ units each and with sigmoidal activations (i.e., multilayer perceptrons). Then if $G$ is defined over score functions $N^{N,K}$, the bound in Eq. 9 holds for $F = N^{N-2,K/2}$.

The corollary shows how for fully-connected neural networks, the error on each violating region is bounded by the error of slightly less expressive neural networks. The result holds since $F_+ \subseteq N^{N,K}$.

**Proof.** We begin with two useful definitions.

**Definition 6** (Implementation). Let $F \in (F_+)_k$, then if $F(x') = u(x')$ for all items $x'$ appearing in a (set-)triple $T$, we say that $F$ implements $T$. Similarly, if $F(x') \approx u(x')$ for all $x'$, we say that $F$ approximates-implies $T$.

**Definition 7** (Isolation). Let $s \in X$, then if $F$ (approximately) implements the set-triple $(z, s, \Gamma(s))$ for some $z \in s$ and for $\Gamma(s) = X \setminus s$, we say that $F$ (approximately) isolates $s$.

Implementation simply states that the values of items in the triple-basis under $F$ are (approximately) those under $u$. Isolation considers the implementation of triple bases for which the choice set is $s$ (but the actual choice $z$ does not matter). As we will see, aggregation will target certain choice sets by “isolating” them from others, letting some of the aggregated score functions effect $s$, but guaranteeing that others are indifferent to it. Note that implementation and isolation, as well as separation (Def. 3), are inherited properties, i.e., that if they hold for $s$, they also hold for any $s' \subset s$.

For our next lemma, we will make explicit the class $N$. Each $a \in N$ is a neural networks taking as input vectors of size two and outputting a single scalar. The networks have two fully-connected hidden layers, each with two units, and sigmoidal activations. The final layer is a 2-to-1 linear layer. We parameterize units using $r(\cdot; \alpha, \beta) = \langle \alpha, \cdot \rangle + \beta$ with $\alpha \in \mathbb{R}^2$, $\beta \in \mathbb{R}$, and use $\theta \in \Theta$ to denote all of the network’s parameters. A network $a \in N$ with parameters $\theta$ is denoted $a_\theta$. We assume w.l.o.g. that sigmoidal activations $\sigma$ are scaled to $[0,1]$.

Recall that each function $f \in F_+$ is composed of a pair $b,b' \in F$ whose outputs are combined via some $a \in N$, given by $f_\alpha(x; b,b') = a_\alpha(b(x), b'(x))$. We further denote:

$$F_\theta(x; b,b') = (f_{\theta_1}(x; b,b'), \ldots, f_{\theta_k}(x; b,b'))$$

The next lemma shows how functions in $F_+$ can implement triple bases when separation holds.

**Lemma 2** (Neural implementation of triple bases). Let $T = (x_1, x_2, x_3)$, and let $b, b' \in F$ be such that $b$ separates $x_1, x_2$ from $x_3$ and $b'$ separates $x_1, x_2$ from $x_3$, i.e., $\{x_1, x_2\} \succ_3 x_3$ and $x_1 \succ_2 x_2$. Then, there exists $\theta \in \Theta^k$ for which $F_\theta(x; b,b')$ (approximately) implements $T$. 

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Proof. We give a general recipe for constructing \( \theta \). Assume w.l.o.g. that \( \sigma \) maps to \([0, 1]\). The idea is to choose \( \alpha, \beta \) such that \( \sigma \) will give items on the r.h.s. and l.h.s. of the separation operator \( \succeq \) values that are arbitrarily close to 0 and 1, respectively. We now construct each unit of the the neural network (see diagram in Fig. 4).

- Since \( \{x_1, x_3\} \succ x_2 \), there exist \( \alpha, \beta \) such that \( r((b(x),b'(x)); \alpha, \beta) \approx 1 \) for \( x = x_1 \) and 0 for \( x = x_3 \). This is because \( \sigma \) is a sigmoidal, and hence \( \alpha \) and \( \beta \) simply shift and scale the sigmoid so that the higher-valued \( x_1 \) and \( x_2 \) are “pushed” towards 1 and the lower-valued \( x_3 \) towards 0. We denote this unit by \( r_1 \).

- Since \( x_1 \succ \gamma x_2 \), there exist \( \alpha, \beta \) such that \( r((b(x),b'(x)); \alpha, \beta) \approx 1 \) for \( x = x_1 \) and 0 for \( x = x_2 \). We denote this unit by \( r_2 \). Note that the values for \( x = x_3 \) are arbitrary but bounded in \([0, 1]\).

- There exist \( \alpha, \beta \) such that \( r((r_1(x), r_2(x)); \alpha, \beta) \approx 1 \) for \( x = x_1 \) and 0 for \( x = x_2, x_3 \). This is because \( r_1 \) and \( r_2 \) contribute 1 to \( x_1 \), while \( x_2 \) and \( x_3 \) never get 1 from \( r_1 \) and \( r_2 \). We denote this unit by \( r_3 \).

- Because \( \sigma \) is sigmoidal, there exist \( \alpha = (\epsilon, 0) \) with small enough \( \epsilon \) such that with \( \beta = 0, r \) approximates the identity function on the first input.\(^8\) We denote this unit by \( r_4 \).

Note that measuring the outputs of \( r_3 \) and \( r_4 \) when plugging \( x_1, x_2, x_3 \) into \( b, b' \) gives:

\[
\begin{align*}
    r_3 &\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\
    r_4 &\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}
\end{align*}
\]

(10)

which together with the vector \( e = (1, 1, 1) \), form an (approximate) linear basis for \( \mathbb{R}^3 \). This means that any utility \( u_i \in u \) when applied to \( x_1, x_2, x_3 \) can be expressed as a linear combination of \( r_3, r_4 \), and \( e \). Specifically, there exist \( \alpha, \beta \) such that the linear layer \( (r_3, r_4) \) gives \( \langle \alpha, (r_3(x), r_4(x)) \rangle + \beta = u_i(x) \) for all \( x \in \{x_1, x_2, x_3\} \). Altogether, we get that there exist \( \theta_i \) with \( f_{\theta_i}(x; b, b') = u_i \) for all \( i \), giving \( \theta = \{\theta_i\}^{k}_{i=1} \) as required.

Corollary 3. Lemma 2 applies to set-triples \( (x, s, t) \) when \( s \succ r_\gamma t \) and \( x \succ \gamma s \).

Proof. According to Lemma 1, it suffices to ensure that scores for all \( z \in s \setminus \{x\} \) and for all \( z' \in t \) are the same. Since the construction in Lemma 2 applies to arbitrary separable triples, the above can be achieved by choosing parameters (i.e., scaling and shifting) such that the properties of each \( r_i \) hold for all \( x \in s, t \).

Lemma 2 give a “template” for generating set-triple bases from a pair of score functions \( b, b' \), where \( b \) determines the choice set \( s \) (by separating \( s \) from \( t \)), and \( b' \) determines the choice \( x \in s \) (by separating \( x \) from \( s \)). This notion is formalized in the next result relating separation and isolation.

Lemma 3 (Separation entails isolation). If \( b \) separates \( s \), then for any \( b' \in \mathcal{F} \), we have that \( F_\theta(x; b, b') \) isolates \( s \). Furthermore, the prediction is determined by \( b' \), i.e.,

\[
h_{g_F}(s) = \arg\max_{x \in s} b'(x)
\]

\(^8\) Alternatively, \( N \) can be defined with only one unit in the second layer.
Proof. Let $z = \arg\max_{x \in S} b'(x)$, then instantiate Lemma 2 for $b, b'$ on $T = (z, s, \Gamma(s))$. 

Note that $\theta$ is determined by the aggregation mechanism $M$ (this is because $u$ is determined by $M$). Hence, once $\theta$ is fixed, the triple-basis template is fixed, and learning can focus on targeting choice sets (by optimizing $b$) and predicting within those choice sets (by optimizing $b'$). In the remainder of the we will use $\theta$ to denote the template parameters corresponding to $u$ (note that Lemma 2 implies that such $\theta$ exists for any triple basis). Because the approximation in Eq. (10) can be made arbitrarily small (i.e., by scaling the appropriate $\alpha$-s), the approximate-indifference of the implemented triple-basis can also be arbitrarily small. Thus, $\theta$ can be made to give $\epsilon$-approximate indifference for any necessary $\epsilon$.

We can now define the following class:

$$F^b_+ = \{F_\theta(x; b, b') : b' \in F\}$$

which includes all score functions that implement set-triple bases that effect choice sets separated by $b$ (while predictions can vary across functions according to $b'$).

**Corollary 4.** If $b$ separates $s$, then $s$ is isolated by all $F \in F^b_+$.

Isolation will be our main building block for showing error decomposition. For $b \in F$, we denote:

$$\Lambda_b = \{s \in V : s \text{ is isolated by all } F \in F^b_+\}$$

Corollary 4 implies that $\Omega_b \subseteq \Lambda_b$.

Note that although each $F_\theta(x; b, b')$ includes only two base score functions $b, b' \in F$, it is in fact composed of $k$ score functions $F_1, \ldots, F_k \in F^b_+$, each determined by one of $\theta_1, \ldots, \theta_k \in \theta$, or in other words, $F^b_+ \subseteq (F^b_+)^k$. We denote by $G^b_+ \subseteq (F^b_+)^k$. The following lemma shows how isolation helps in decomposing the error of $G$. By “isolating” a separable region of $X$, a budget of $k$ (out of $\ell$) score functions can be allocated to that region, and aggregation will ensure that these score functions will only effect predictions of choice sets within the region. This is the main component in the decomposition bound of the error of $G$.

**Lemma 4** (Decomposition by isolation). For any $R \subseteq X$ with $\Lambda_b \subseteq R$ and any $n \geq k$, it holds that:

$$\varepsilon^*_R(G^b_{F^b_+}) \leq p_{\Lambda_b} \varepsilon^*_R(F) + p_{R \setminus \Lambda_b} \varepsilon^*_R(F \setminus \Lambda_b)$$  \hspace{1cm} (11) 

**Proof.** Let $G' = G \times G^{(n-k)}$, i.e., aggregators whose first $k$ score functions give some $F \in F^b_+$.

Since $G' \subseteq G$, the minimal error cannot decrease:

$$\varepsilon^*_R(G^{(n)}_{F^b_+}) \leq \varepsilon^*_R(G')$$  \hspace{1cm} (12) 

Next, fix some constant $c$, and denote by $\tilde{G} \subseteq G^{(n-k)}$ the aggregators whose absolute values are at most $c$, and by $\bar{G} \subseteq G^{(n-k)}$ the aggregators whose absolute values are at least $c^9$ Denoting $G'' = \tilde{G} \times \bar{G}$, because $\tilde{G} \subseteq G^{(n-k)}$ and $\bar{G} \subseteq G^{(n-k)}$, we have:

$$\varepsilon^*_R(G') \leq \varepsilon^*_R(G'')$$  \hspace{1cm} (13) 

For any $g \in G''$, since $g$ is an aggregator, we can write $g = \bar{g} + \tilde{g}$ where $g \in \bar{G}$ and $\tilde{g} \in \tilde{G}$. We now consider how different $s \in R$ are treated by $g, \tilde{g}$, and $g$. If $s \in \Lambda_b$, then because $g(x, s) > \bar{g}(x', s)$ for all $x, x' \in s$, $g$ ‘dominates’ $\tilde{g}$ in the sense that only $g$ effects predictions, i.e., $h_g(s) = h_g$. On the other hand, if $s \in R \setminus \Lambda_b$, then because $\tilde{g} \subseteq G^{(n-k)}$, it is approximately-indifferent on $s$ (to see this, note that due to the inheritance of isolation, there is no $t \supseteq s$ with $t \in \Lambda_b$, and so $s$ falls in the second class of sets from Lemma 1). And since this approximate-indifference can be made arbitrarily small

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$^9$Here we make the technical assumption that $g(x, s) \neq 0$ for all $x, s$. 

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(by adjusting $\theta$ as described above), $g$ does not effect predictions, i.e., $h_g(s) = h_{\bar{g}}$. Overall, for any $g \in \mathcal{G}'$, we have:

$$h_g(s) = h_g(s) \mathbf{1}_{\{s \in \Lambda_b\}} + h_{\bar{g}}(s) \mathbf{1}_{\{s \in \Lambda \setminus \Lambda_b\}}$$

This allows us to decompose the error:

$$\varepsilon^*_{\mathcal{F}}(G') \leq p_{\Lambda_b} \varepsilon^*_{\Lambda_b}(\mathcal{F}) + p_{\Lambda \setminus \Lambda_b} \varepsilon^*_{\Lambda \setminus \Lambda_b}(\bar{G})$$

and because $M$ is scale-invariant (Definition 5), scaling does not change predictions (and thus does not change the expressivity of the classes), allowing us to remove the constraints on $c$:

$$\varepsilon^*_{\Lambda_b}(\mathcal{G}) = \varepsilon^*_{\Lambda_b}(\mathcal{F}) \quad \varepsilon^*_{\Lambda \setminus \Lambda_b}(\bar{G}) = \varepsilon^*_{\Lambda \setminus \Lambda_b}(\mathcal{G})$$

Finally, because the predictions of any $F(x; b, b') \in \mathcal{F}_b(b)$ are determined by $b'$ (Lemma 3), we can replace $\mathcal{G}_{\mathcal{F}_b(b)}$ in Eq. (15) with $\mathcal{F}$, and combining Eqs. (12)-(15) concludes the proof.

We now tie all lemmas together to derive Eq. (9). Recall that $\ell = k \ell' + 1$. For any $B = \{b_1, \ldots, b_{\ell'}\}$ with $\cap_{b \in B} \Lambda_b = \emptyset$, we can iteratively applying Lemma 4 to isolate separable regions until our budget is depleted:

$$\varepsilon^*_{\mathcal{F}}(\mathcal{G}_{\mathcal{F}_+}^{(\ell)}) \leq p_{\Lambda_{b_1}} \varepsilon^*_{\Lambda_{b_1}}(\mathcal{F}) + p_{\Lambda \setminus \Lambda_{b_1}} \varepsilon^*_{\Lambda \setminus \Lambda_{b_1}}(\mathcal{G}_{\mathcal{F}_+}^{(\ell - k)})$$

$$\leq p_{\Lambda_{b_2}} \varepsilon^*_{\Lambda_{b_2}}(\mathcal{F}) + p_{\Lambda \setminus \Lambda_{b_2}} \varepsilon^*_{\Lambda \setminus \Lambda_{b_2}}(\mathcal{F}) + p_{\Lambda \setminus (\Lambda_{b_1} \cup \Lambda_{b_2})} \varepsilon^*_{\Lambda \setminus (\Lambda_{b_1} \cup \Lambda_{b_2})}(\mathcal{G}_{\mathcal{F}_+}^{(\ell - 2k)})$$

$$\leq \ldots$$

$$\leq \sum_{b \in B} p_{\Lambda_b} \varepsilon^*_{\Lambda_b}(\mathcal{F}) + p_{\Lambda \setminus \Lambda_B} \varepsilon^*_{\Lambda \setminus \Lambda_B}(\mathcal{G}_{\mathcal{F}_+})$$

where $\Lambda_B = \cup_{b \in B} \Lambda_b$, and the last inequality is due to $\mathcal{F}_+ \subseteq \mathcal{G}_{\mathcal{F}_+}^{(1)}$ (note that also $\mathcal{F} \subseteq \mathcal{F}_+$, giving the version of the Theorem in the paper).

Because the above holds for any $B$, we can plug in the optimal $B$. We now have:

$$\varepsilon^*_{\mathcal{F}}(\mathcal{G}_{\mathcal{F}_+}^{(\ell)}) \leq \min_{B \in \mathcal{F}_+} \sum_{b \in B} p_{\Lambda_b} \varepsilon^*_{\Lambda_b}(\mathcal{F}) + p_{\Lambda \setminus \Lambda_B} \varepsilon^*_{\Lambda \setminus \Lambda_B}(\mathcal{F}_+)$$

$$\leq \min_{B \in \mathcal{F}_+} \sum_{b \in B} p_{\Lambda_b} \varepsilon^*_{\Lambda_b}(\mathcal{F}) + p_{\Lambda \setminus \Lambda_B} + p_{\Omega} \varepsilon^*_{\Omega}(\mathcal{F}_+)$$

$$\leq \min_{B \in \mathcal{F}_+} \sum_{b \in B} p_{\Lambda_b} \varepsilon^*_{\Lambda_b}(\mathcal{F}) + p_{\Lambda \setminus \Lambda_B} + p_{\Omega} \varepsilon^*_{\Omega}(\mathcal{F}_+)$$

The first inequality holds because isolation is an inherited property, and hence the argmin $B^*$ necessarily has disjoint isolated regions $\{\Lambda_b\}_{b \in B}$. The second inequality holds because $\Lambda_B$ only includes sets in $\mathcal{V}$, and $\varepsilon^*_{\mathcal{V} \setminus \Lambda_B}(\cdot) \leq 1$. The third inequality holds since by Lemma 3, separation entails isolations, and so $\Omega_b \subseteq \Lambda_b$, meaning that sets $s \in \Lambda_B \setminus \Omega_b$ are assumed to receive the worst-case error of one (i.e., are “moved” from the error term $\varepsilon^*_{\Lambda_b}$ to the maximal-error term $p_{\Lambda \setminus \Lambda_B}$), concluding the proof.

**Appendix C  Estimation Error**

We begin with the case $g(x, s) = \langle w(s), F(x) \rangle$ with:

$$F(x) = (f_1(x), \ldots, f_k(x)) \quad w(s) = (w_1(s), \ldots, w_k(s))$$

where $w_1(s) = w(f_1(s)), f_i(s) = (f_i(x_1), \ldots, f_i(x_{\ell_i}))$, and each $f_i(x) = \langle \theta_i, x \rangle$. 

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We can write \( F(x) = x^T \Theta \) where \( \Theta_i, = \theta_i \) are rows, and denote columns by \( \bar{\theta}_j = \Theta_j \). Note that:

\[
g(x, s) = \sum_{i=1}^{\ell} w_i(s)\langle \theta_i, x \rangle = \sum_{k=1}^{d} \sum_{i=1}^{\ell} \Theta_{ik} w_i(s)x_k = \sum_{k=1}^{d} \langle \bar{\theta}_k, w(s) \cdot x_k \rangle \tag{I}
\]

We now bound the Rademacher complexity of each component.

\[
R_f \leq X_\infty X_\infty \sqrt{\frac{2 \log 2d}{m}} \tag{[37], Lemma 26.11}
\]
\[
R_w \leq \lambda_w^{(\rho)} R_f \tag{[37], Lemma 26.9}
\]
\[
R_{(I)} \leq \max_x \| x \|_\infty R_w \tag{[37], Lemma 26.6}
\]
\[
R_{(II)} \leq 2 \max_\theta \| \bar{\theta} \|_1 \cdot R_{(I)} = 2 \| \Theta \|_1 R_{(I)} \tag{[38], Sec. 4}
\]

Assuming \( \| \Theta \|_1 \leq 1 \), combining the above gives:

\[
R_g \leq 2 X_\infty^2 \lambda_w^{(\rho)} \sqrt{\frac{2 \log 2d}{m}}
\]

Using standard Rademacher-based generalization bounds (i.e., [2]) concludes the proof:

\[
\varepsilon(G) \leq \varepsilon_S(G) + 2 R_g + O \left( \sqrt{\frac{\log(1/\delta)}{m}} \right)
\]

which gives the desired result.

We now analyze the case \( g(x, s) = \langle v, \mu(F(x) - r(s)) \rangle \), where \( v \in \mathbb{R}^\ell, r(s) = (r_1(s), \ldots, r_\ell(s)) \), and \( r_i(s) = r_i(f(s)) \). The Rademacher complexity of each component is:

\[
R_{\mu} \leq \lambda_{\mu}(R_f + R_r) \tag{[37], Lemma 26.6}
\]
\[
R_r \leq \lambda_r^{(\rho)} \tag{[37], Lemma 26.6}
\]
\[
R_g \leq 2W_1 R_{\mu} \tag{[37], Lemma 26.7}
\]

and \( R_f \) is as before. Together, this gives:

\[
R_g \leq 2W_1 R_{\mu} \leq 2W_1 \lambda_{\mu}(1 + \lambda_r^{(\rho)}) X_\infty \sqrt{\frac{2 \log 2d}{m}}
\]

which concludes our proof.

### Appendix D Experiments

#### D.1 Datasets

For Expedia and Outbrain, the preprocessed data used to train the models can be found here\(^{10}\). For Amadeus, the data is not publicly available. Please contact the authors of [23] for the data. Note that we did not conduct any feature engineering on Amadeus dataset. Here are the details of the features and the preprocessing steps of each dataset.

1. **Amadeus**: Each item is a recommended flight itinerary, and user clicks one. Key features include flight origin/destination, price and number of transfers. User features are excluded from the original dataset due to privacy concerns. We did not conduct any feature engineering, and used the dataset from [23].

2. **Expedia**: Each item is a recommended hotel, and user clicks one. Key features include hotel price, rating, length of stay, booking window, user’s average past ratings and visitor’s location. We applied the following standard preprocessing steps for different variable types:

\(^{10}\)https://drive.google.com/file/d/1G0Rmsa9M1D5NKbryY4mTmZCX9i6MuY2F/view
• continuous: depending on the distribution, log or square root transform to make the distribution look Gaussian.
• ordinal: one-hot encode.
• datetime: get week and month to capture seasonality of hotel pricing.
• categorical: one-hot encode. For those with too many categories, group unpopular ones as "others".
• new features: we created one new feature, popularity score, based on a popular blog post on this dataset\textsuperscript{11}.

3. Outbrain: Each item is a news article, and user clicks one. When users see an article, they also see these recommended articles at the bottom of the page. Key features include article category, advertiser ID, and geo-location of the views. For preprocessing steps, we followed one of the leading solutions in the Outbrain click prediction Kaggle competition\textsuperscript{12}.

Table 3 includes further details.

| Dataset  | m   | |X|   | max(n) | avg(n) | d  |
|----------|-----|-----|------|--------|--------|----|
| Amadeus  | 34K | 1.0M| 50   | 32.1   | 17     |
| Expedia  | 199K| 129K| 38   | 25     | 8      |
| Outbrain | 16.8M| 478K| 12   | 5.2    | 10     |

D.2 Baselines

As shown in Claim 1, SDA generalizes many known variants of MNL. Our implementation of SDA is capable of running these specific instances of MNL variants. MNL as well as all the models in Figure 3 (center) are implemented within our framework.

• MNL: our implementation.
• SVMRank: used an open source code on GitHub\textsuperscript{13}, with minor modifications.
• RankNet: used learning2rank library open sourced on GitHub\textsuperscript{14}.
• MixedMNL: our implementation.
• AdaRank: used an open source code on GitHub\textsuperscript{15}.
• DeepSets: used source code provided by the authors\textsuperscript{16}.

For neural network based models SDA, RankNet, DeepSets, the number of parameters are $816+784d$, $525312+1024d$, $196864+256d$, respectively, where $d$ is the number of features in a dataset. For a reasonable range of $d$, the number of SDA parameters is significantly lower than that of other models. This further illustrates how SDA reduces model complexity by incorporating inductive bias in clever ways.

D.3 Setup

Implementation

All code was implemented in Python, using Tensorflow\textsuperscript{17}. The source code can be found here\textsuperscript{18}.

Evaluation Metrics Definition

\textsuperscript{11}https://ajourneyintodatascience.quora.com/Learning-to-Rank-Personalize-Expedia-Hotel-Seraches-ICDM-2013-Feature-Engineering
\textsuperscript{12}Up to Step 5 of https://github.com/alexeygrigorev/outbrain-click-prediction-kaggle
\textsuperscript{13}https://gist.github.com/coreylynch/4150976/1a2983d3a896f4caba33e11a06a5c48962e6060c0
\textsuperscript{14}https://github.com/shiba24/learning2rank
\textsuperscript{15}https://github.com/rueycheng/AdaRank
\textsuperscript{16}https://github.com/manzilzaheer/DeepSets
\textsuperscript{17}https://www.tensorflow.org/
\textsuperscript{18}https://drive.google.com/file/d/1KZVbqfVR6QNp3y8e8ptzVdbi_GQH4/view
• top-1 accuracy: the conventional accuracy. 1 if the model choice prediction is the same as the chosen item, 0 otherwise.
• top-5 accuracy: 1 if the model’s 5 highest probability choice predictions include the chosen item, 0 otherwise.
• mean reciprocal rank (MRR): a rank-based measure commonly used in the information retrieval literature. Let rank\(_i\) to indicate the rank of the chosen item in our prediction (by probability). Then, MRR = \(\frac{1}{m} \sum_{i=1}^{m} \frac{1}{\text{rank}_i}\). Because we want the rank of the model prediction to be higher (and thus the reciprocal rank to be lower), the lower MRR the better.

**Hyperparameters**

For all methods, we tuned regularization, dropout, and learning rate (when applicable) using Bayesian optimization using an open source library Optuna\(^{19}\). We tuned the hyperparameters on the validation set for 100 trials of Bayesian optimization. The range of hyper-parameters considered is as follows:

- learning rate: log uniformly sample from 1E-05 ~ 1E-03
- weight decay: log uniformly sample from 1E-10 ~ 1E-03
- dropout keep probability: uniformly sample from 0.5 ~ 1.0

For exponential decay, we used decay rate of 0.95 with decay step of 10 for all models. For batch size, we used 128 for all models. Finally, we applied early stopping to all models based on the validation accuracy with an early stop window of 25 epochs.

**Computing Infrastructure**

We optimized the hyperparameters and trained our model on Slurm Worklord Manager\(^{20}\). All the training was done on CPUs, and the CPU core type we used are AMD "Abu Dhabi" and Intel "Broadwell".

**D.4 Ablation Study**

**Details on Model Specification**

Recall that SDA is of the form:

\[ g(x, s; w, \varphi) = \langle w(s), \varphi(x, s) \rangle \]

We further introduced the general form of inductive bias:

\[ w(s) = w(F(s)), \quad \varphi(x, s) = \mu(F(x) - r(F(s))) \]

In this section, we first elaborate on the different components of \( g \) to supplement Sec. 2, then justify our choices of \( w, r \) and \( \mu \) described in Sec. 4.

For a single dimension \( i \in [\ell] \) in the embedded space, items are first evaluated using some \( f_i(x) \in \mathcal{F} \). Then, a "reference" valuation is constructed via a set function \( r_i(s) \). Next, \( r_i(s) \) is subtracted from \( f_i(x) \) to "standardize" the scores with respect to this reference. As noted in 4, we have found it useful to generalize this by letting \( f_i \) and \( r_i \) be vector valued, and taking their inner product instead. The item-reference relation is then fed into an a-symmetric non-linearity \( \mu(\cdot) \). Finally, all \( \ell \) valuations are aggregated using set-dependent weights \( w(s) \).

Eq. (5) injects inductive bias inspired by the following key principles in behavioral choice theory:

- **P1: Asymmetry.** Losses and gains are perceived in an a-symmetric, non-linear manner. This is the hallmark of Kahneman and Tversky’s Prospect Theory [10, 43].
- **P2: Relativity.** Valuations are relative, and are considered with respect to a mutual (and possibly hypothetical) referral item (e.g., [42]).
- **P3: Integrability.** Subjective value is multi-dimensional, and context determines how valuations are integrated into a single choice (e.g., [40, 44]).

\(^{19}\)https://optuna.org/

\(^{20}\)https://slurm.schedmd.com/documentation.html
Proposition 1. \( g(x, s) \) in Eq. (5) satisfies P1, P2, and P3.

To see this, note that for each dimension \( i \), item scores \( f_i(x) \) are compared to a mutual set-dependent reference point \( r_i(s) \) via \( d \), satisfying P2. Using a pointwise a-symmetric non-linearity for \( \mu \) gives P1. Finally, in accordance with P3, valuations are aggregated in \( g \) via \( w(s) \).

We now describe our specific choice of components in Sec. 4. We model \( w(s) \) and \( r(s) \) as set neural networks [46]. These are general neural networks whose architecture guarantees permutation invariance, meaning that permuting \( s \) will not change the output. For the scalar function \( \mu \), we have found it useful to use a kinked tanh function:

\[
\mu(z) = c \cdot \text{tanh}(z) \cdot \mathbb{1}_{\{z < 0\}} + \text{tanh}(z) \cdot \mathbb{1}_{\{z \geq 0\}}, \quad c > 1
\]

which is inspired by the s-shaped utility functions used in prospect theory [17].

Since all four elements of \( g \) are differentiable, \( g \) can be optimized over any appropriate loss function (i.e., cross entropy) using standard gradient methods. The learned parameters include the weights in \( w \) and in \( r \), and \( c \) for \( \mu \). We have noted in Sec. 2 that Eq. (5) encompasses several models from the multi-self literature. Since these are usually designed for mathematical tractability, they do not always include explicit functional forms, and use simple non-parametric set operations for \( w \) and \( r \).

The predictive advantage of our parametrized model is demonstrated empirically in Sec. 4.

Ablation Study

We now investigate the contribution of each component of SDA in an ablation study.

In 2, we motivated our model choice from the behavioral decision theory perspective. To motivate our design decisions also from a machine learning point of view, we conducted an ablation study. In particular, we decomposed SDA into \( F, \ell, w, \varphi \) (which can consist of \( r, \mu \)), removed each component, and analyzed the performance. The full set of ablation models is detailed in Table 4 and the experimental results are presented in Table 5. The experiment setup is exactly the same as Sec. 4.

| Table 4: Specification of all ablated models |
|-----------------|---|---|---|---|---|
| F               | \( \ell \) | \( w \) | \( \varphi \) | \( r \) | \( \mu \) |
| SDA             | linear 24 | Set NN | \((F(x), r(s))\) | Set NN | kinked tanh |
| SDA with \( g = \tanh \) | linear 24 | Set NN | \((F(x), r(s))\) | Set NN | tanh |
| SDA with no \( g \) | linear 24 | Set NN | \((F(x), r(s))\) | Set NN | - |
| SDA with \( w \) vector variable | linear 24 | vector variable | \((F(x), r(s))\) | Set NN | kinked tanh |
| SDE             | linear 24 | vector variable | Set NN | - | kinked tanh |
| SDW             | linear 24 | Set NN | F(x) | - | - |
| MNL with \( w \) setnn | linear 1 | Set NN | F(x) | - | - |
| MNL [20]        | linear 1 | Scalar vector | F(x) | - | - |
|                  | Amadeus | Expedia | Outbrain |
|------------------|---------|---------|----------|
|                  | Top-1   | Top-5   | MRR      |
|                  | 10      | 20      | 30       | 40   | 50   |
|                  | 10      | 20      | 30       | 40   | 50   |
|                  | MRR     | Top-1   | Top-5   | MRR  |
|                  | 12      | 12      | 12       |      |
| SDA              | 31.49   | 26.81   | 21.96    | 18.36 |
|                  | ±0.2    | ±0.1    | ±0.1     | ±0.2 |
| SDA with \( g = \tanh \) | 31.19   | 26.26   | 21.81    | 18.10 |
|                  | ±0.1    | ±0.2    | ±0.2     | ±0.2 |
| SDA with \( g \neq \tanh \) | 30.90   | 26.52   | 21.88    | 18.20 |
|                  | ±0.2    | ±0.2    | ±0.2     | ±0.2 |
| SDA with vector variable | 25.05   | 22.43   | 18.24    | 17.63 |
|                  | ±0.2    | ±0.2    | ±0.2     | ±0.2 |
| SDE              | 31.47   | 26.86   | 21.85    | 18.14 |
|                  | ±0.2    | ±0.2    | ±0.2     | ±0.2 |
| SDW              | 31.27   | 26.80   | 21.88    | 18.22 |
|                  | ±0.2    | ±0.2    | ±0.2     | ±0.2 |
| MNL with \( w \) setnn | 30.04   | 25.52   | 20.82    | 16.64 |
|                  | ±0.2    | ±0.2    | ±0.2     | ±0.2 |
| MNL [20]         | 30.06   | 25.59   | 20.61    | 16.65 |
|                  | ±0.2    | ±0.2    | ±0.2     | ±0.2 |

Table 5: Ablation Experiment Result.
D.5 Full Result

We ran our experiments for the following number of maximum items:

1. Amadeus: 10, 20, 30, 40, 50
2. Expedia: 10, 20, 30, 40
3. Outbrain: 12

The result is shown in Table 6.
|        | Top-1 |     | 10  | 20  | 30  | 40  | 50  | Top-5 |     | 10  | 20  | 30  | 40  | 50  | MRR |     | 10  | 20  | 30  | 40  | 50  |
|--------|-------|-----|-----|-----|-----|-----|-----|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|        |       |     |     |     |     |     |     |      |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| **SDA** | 45.42 ± 0.5 | 33.48 ± 0.3 | 29.26 ± 0.0 | 26.57 ± 0.3 | 23.23 ± 0.2 | 9.37 ± 0.0 | 80.40 ± 0.3 | 73.77 ± 0.4 | 69.64 ± 0.0 | 62.35 ± 0.1 | 2.31 ± 0.3 | 3.50 ± 0.0 | 4.33 ± 0.2 | 4.93 ± 0.4 | 6.37 ± 0.0 |
| **SDE** | 39.62 ± 0.4 | 32.26 ± 0.4 | 27.62 ± 0.3 | 26.73 ± 0.2 | 20.62 ± 0.5 | 9.170 ± 0.3 | 79.75 ± 0.3 | 72.18 ± 0.5 | 68.82 ± 0.2 | 58.89 ± 0.3 | 2.52 ± 0.0 | 3.58 ± 0.0 | 4.48 ± 0.1 | 4.91 ± 0.0 | 6.90 ± 0.1 |
| **SDW** | 39.98 ± 0.4 | 32.03 ± 0.3 | 27.85 ± 0.3 | 26.29 ± 0.3 | 20.15 ± 0.2 | 9.189 ± 0.3 | 79.57 ± 0.3 | 71.82 ± 0.3 | 68.62 ± 0.2 | 58.55 ± 0.2 | 2.50 ± 0.0 | 3.59 ± 0.0 | 4.51 ± 0.0 | 5.04 ± 0.0 | 6.97 ± 0.0 |
| **MNL [20]** | 38.42 ± 0.5 | 27.93 ± 0.3 | 23.54 ± 0.3 | 22.31 ± 0.1 | 18.39 ± 0.2 | 9.102 ± 0.3 | 76.51 ± 0.3 | 68.36 ± 0.4 | 65.10 ± 0.4 | 56.20 ± 0.3 | 2.57 ± 0.0 | 3.92 ± 0.0 | 4.94 ± 0.0 | 5.60 ± 0.1 | 7.55 ± 0.1 |
| **SVMRank [9]** | 40.27 ± 0.4 | 28.17 ± 0.3 | 23.99 ± 0.3 | 23.02 ± 0.2 | 18.64 ± 0.2 | 9.94 ± 0.3 | 76.82 ± 0.3 | 68.56 ± 0.4 | 66.52 ± 0.2 | 57.50 ± 0.2 | 2.49 ± 0.0 | 3.87 ± 0.0 | 4.85 ± 0.0 | 5.35 ± 0.0 | 7.10 ± 0.1 |
| **RankNet [5]** | 37.44 ± 0.7 | 26.77 ± 0.6 | 23.81 ± 0.3 | 20.29 ± 0.7 | 16.99 ± 0.5 | 8.467 ± 1.7 | 66.06 ± 1.9 | 61.59 ± 0.9 | 49.45 ± 1.8 | 44.98 ± 0.6 | 3.02 ± 0.1 | 4.98 ± 0.2 | 5.96 ± 0.1 | 8.35 ± 0.3 | 11.07 ± 0.1 |
| **Mixed MNL [39]** | 37.96 ± 0.3 | 27.00 ± 0.3 | 22.98 ± 0.3 | 21.68 ± 0.2 | 17.67 ± 0.3 | 90.40 ± 0.3 | 74.80 ± 0.3 | 65.87 ± 0.4 | 62.50 ± 0.3 | 52.87 ± 0.4 | 2.62 ± 0.0 | 4.09 ± 0.0 | 5.26 ± 0.0 | 6.00 ± 0.1 | 8.39 ± 0.1 |
| **AdaRank [45]** | 37.27 ± 0.4 | 25.79 ± 0.3 | 18.79 ± 0.7 | 15.79 ± 0.5 | 11.89 ± 0.2 | 72.34 ± 0.3 | 58.28 ± 0.3 | 51.64 ± 0.6 | 47.85 ± 1.4 | 39.08 ± 0.2 | 4.03 ± 0.0 | 5.75 ± 0.0 | 7.14 ± 0.1 | 8.05 ± 0.3 | 11.55 ± 0.1 |
| **Deep Sets [46]** | 40.36 ± 0.5 | 31.02 ± 0.3 | 26.66 ± 0.4 | 25.48 ± 0.3 | 20.55 ± 0.4 | 91.92 ± 0.3 | 79.31 ± 0.2 | 71.18 ± 0.4 | 68.76 ± 0.4 | 59.88 ± 0.4 | 2.48 ± 0.0 | 3.64 ± 0.0 | 4.58 ± 0.0 | 5.01 ± 0.0 | 6.75 ± 0.0 |
| Price/Quality | 36.44 ± 0.3 | 25.44 ± 0.2 | 22.40 ± 0.2 | 20.26 ± 0.2 | 16.11 ± 0.1 | 87.23 ± 0.2 | 67.77 ± 0.2 | 58.80 ± 0.2 | 54.44 ± 0.2 | 45.43 ± 0.3 | 2.79 ± 0.0 | 4.86 ± 0.0 | 6.32 ± 0.0 | 7.27 ± 0.0 | 10.90 ± 0.1 |
| Random | 25.15 ± 0.5 | 14.78 ± 0.2 | 11.58 ± 0.2 | 9.91 ± 0.2 | 6.24 ± 0.2 | 32.87 ± 0.0 | 42.60 ± 0.3 | 34.54 ± 0.2 | 14.20 ± 0.2 | 11.04 ± 0.2 | 6.49 ± 0.0 | 8.98 ± 0.2 | 12.03 ± 0.0 | 18.11 ± 0.0 | 23.55 ± 0.1 |

Table 6: Full experimental results.