ON AN ASYMPTOTIC CHARACTERISATION OF GRIFFITHS SEMIPOSITIVITY

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Abstract. We prove that a continuous Hermitian metric is Griffiths-semi positively curved if and only if an asymptotic extension property is met. This result answers a version of a question of Deng–Ning–Wang–Zhou in the affirmative.

1. Introduction

The positivity of the curvature $\Theta$ of the Chern connection of a Hermitian holomorphic vector bundle $(E,h)$ over a complex manifold $M$ plays an important role in algebraic geometry through extension problems. For a line bundle, there is only one notion of positivity, namely, the curvature being a Kähler form. For a vector bundle, there are several competing inequivalent notions, of which the most natural are Griffiths positivity ($\langle v, \sqrt{-1} \Theta v \rangle$ is a Kähler form) and Nakano positivity (The bilinear form defined by $\sqrt{-1} \Theta$ on $T^{1,0} M \otimes E$ is positive-definite). A famous conjecture of Griffiths [14] asks whether ample vector bundles ($O_E(1)$ over $\mathbb{P}(E)$) admits a positively-curved metric. Despite the conjecture being open, a considerable amount of work has been done to provide evidence for it [1, 4, 7, 9, 13, 16, 21, 22, 24, 28].

In the Kähler case, Demailly–Paun [8] proved a Nakai–Moizeshon–type criterion to characterise Kähler classes. Despite the assumptions and conclusions involving smooth objects, their proof used singular objects like positive currents and singular Kähler potentials. A similar phenomenon might play a role in the study of the Griffiths conjecture and hence it is fruitful to study singular Hermitian metrics on vector bundles. This topic has also been well-studied [5, 27, 3, 2, 17, 23, 26, 20] and seems to hold some surprises. For instance, even if a bundle is Griffiths-positively curved (in a certain sense), the curvature may not exist as a current (Theorem 1.3 in [27]).

In the quest for alternate characterisations of these notions of positivity, and defining similar notions for singular metrics, the following definition [11] involving the asymptotics of $L^2$-extension constants proved to be a useful measure of positivity [12, 10, 11, 18, 19].

Definition 1.1. Let $E$ be a holomorphic vector bundle over an $n$-dimensional complex manifold $X$. A singular Hermitian metric $h$ is said to satisfy the multiple coarse $L^2$-extension property if the following hold.

1. For every open subset $D \subset X$ and every holomorphic section $s : D \to E^*|_D$ that is not identically zero, the function $\ln \|s\|_{h^*}^2$ is upper-semicontinuous.
(2) Consider any cover of $M$ by relatively compact Stein trivialising coordinate neighbourhoods of the form $(\Omega'' \subset M, z, \{e_i\})$ such that there exists a subcover of Stein open subsets $\Omega' \supset \Omega \supset \Omega''$. Suppose there exist constants $C_m \forall m \geq 1$ satisfying the following conditions: $\lim_{m \to \infty} \frac{\ln C_m}{m} = 0$; and if $p \in \Omega'$ and $a \in E_p$ with $\|a\|_h < \infty$, for every integer $m \geq 1$, there exists a holomorphic extension $f_m : \Omega \to E$ (that is, $f_m(p) = a \otimes^m$) satisfying

$$\int_{\Omega'} \|f_m\|_{h \otimes^m} \left( \sqrt{-1} \partial \overline{\partial} |z|^2 \right)^n n! \leq C_m \|a\|_{h(p)}^{2m}.$$  

(1.1)

For the remainder of this paper, unless specified otherwise, an integral over a coordinate chart is understood to be an integral with the Euclidean volume form, similar to (1.1).

**Remark 1.2.** Actually, the definition given in [11] is slightly different from Definition 1.1 in that it did involve consider smaller open subsets. Nonetheless, we find this definition easier to work with. It is possible that our results can be generalised to the definition in [11].

In [11], it was proved that multiple coarse $L^2$-extension implies Griffiths semi-positivity. A question was raised as to whether it completely characterises Griffiths semipositivity. We answer that question in the affirmative:

**Theorem 1.3.** Let $h$ be a continuous Hermitian metric on a holomorphic vector bundle $E$ over a complex manifold $X$. The following are equivalent.

1. $(E, h)$ satisfies the multiple coarse $L^2$-extension property.
2. $(E, h)$ is Griffiths-semipositively curved.

**Remark 1.4.** Note that the constants $C_m$ are allowed to depend on the metric $h$ as per Definition 1.1. Indeed, in Theorem 1.3, $C_m$ depends on $\frac{\sup \det(h)}{\inf \det(h)}$.

It is interesting to know whether our result can be improved to general singular Hermitian metrics. Moreover, there are other positivity notions for vector bundles like MA-positivity for instance [25]. It might be fruitful to explore a similar extension/estimate-type characterisation for such notions as well.

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2. **Proof**

The assertion $1 \Rightarrow 2$ was proved earlier ([12, 10, 11]) in almost the form we stated. We recall a sketch of a proof for the sake of completeness.

**Proposition 2.1.** The multiple coarse extension property implies Griffiths semipositivity.


Proof. Griffiths semipositivity for singular metrics $h$ is clearly a local notion. Therefore, we restrict ourselves to a relatively compact Stein coordinate trivialising neighbourhood $\Omega' \subset \Omega \subset \Omega''$. We simply need to prove that for every holomorphic section $s : \Omega' \to E^*|_{\Omega}$, the continuous function $\phi : \Omega' \to \mathbb{R}$ defined by $\phi = \|s\|^2_h$, is plurisubharmonic (psh). It is enough to prove that $\ln \phi$ is psh.

At this juncture, we recall from the proof of Theorem 1.1 in [12] that to prove the plurisubharmonicity of a function $\psi : \Omega \to \mathbb{R}$, it is enough to show the following: For any $z_0 \in \Omega'$ such that $\psi(z_0) \neq -\infty$, and every $m \geq 1$, there exists a holomorphic function $g_m : \Omega' \to \mathbb{C}$ such that $g_m(z_0) = 1$ and

$$\int_{\Omega'} |g_m|^2 e^{-m\psi} \leq C_m e^{-m\psi(z_0)},$$

where $C_m$ are constants independent of $z_0, g_m$ such that $\lim_{m \to \infty} \frac{\ln C_m}{m} = 0$. Indeed, this theorem can be proved (akin to Demailly’s regularisation of psh functions [6]) by simply sandwiching the psh function $\psi_m = \frac{1}{m} \ln K_\psi$ (where $K_\psi$ is the Bergman kernel) for large $m$ between $\psi$ and $\sup_{\zeta \in B(z,e^{-m/\nu})} \psi(\zeta)$. The former is done using the assumed $L^2$-extension property and the latter using the sub-mean-value property of $|f|^2$. The key point is to observe that the constants $C_m$ should only be independent of $z_0, g_m$. They are allowed to depend on $\psi$.

We now need to verify the hypotheses of the aforementioned Theorem from [12] for $\psi = \phi$. Let $a \in E_{z_0}$ be such that $\langle a, s(z_0) \rangle = 1 = \langle a^{\otimes m}, s^{\otimes m}(z_0) \rangle$ and $\|a\|_{h^*} = \frac{1}{\|s(z_0)\|_h}$. Using the $L^2$-extension property, we can extend $a^{\otimes m}$ to a holomorphic section $f_m$ on $\Omega$ satisfying (1.1). Unfortunately, the obvious choice $g_m = \langle f_m, s^{\otimes m} \rangle$ does not satisfy the desired estimate. To remedy this issue, akin to [12], we apply the Ohsawa–Takegoshi extension theorem to produce a holomorphic function $g_m : \Omega \to \mathbb{C}$ such that $g_m(z_0) = 1$ and

$$\int_{\Omega'} |g_m|^2 e^{-\ln(f_m, s^{\otimes m})} \leq C,'$$

for some constant $C'$ depending only on $\Omega$. Applying the sub-mean-value property to the psh function $|g_m|^2 e^{-\ln(f_m, s^{\otimes m})}$, there exists another constant $C''$ (that depends on $\Omega'$) such that the following inequality is met on $\Omega'$:

\begin{equation}
|g_m|^2 e^{-\ln(f_m, s^{\otimes m})} \leq C''.
\end{equation}

Using (2.1),

$$\int_{\Omega'} |g_m|^2 e^{-m\phi} \leq C'' \int_{\Omega'} e^{\ln(f_m, s^{\otimes m}) - m\phi} = C'' \int_{\Omega'} \frac{|\langle f_m, s^{\otimes m} \rangle|}{\|s\|_{h^*}^m} \leq C' \int_{\Omega'} \|f_m\|_{h^{\otimes m}} \leq C' \left( \text{Vol}(\Omega') \int_{\Omega'} \|f_m\|_{h^{\otimes m}}^2 \right)^{1/2} \leq C' \left( \text{Vol}(\Omega') \right)^{1/2} \sqrt{C_m \|a\|_{h^*}^m} = C' \left( \text{Vol}(\Omega') \right)^{1/2} \sqrt{C_m e^{-m\phi(z_0)}},$$
Lemma 2.3. The subbundle $S^m E$ is orthogonal in the induced metric to $V$.

Proof. Indeed, suppose $x \in S^m E_q$, $y \in V_q$. Then $m!(x,y) = \sum_{g \in S_m} \langle g \cdot x, y \rangle = \sum_{g \in S_m} \langle x, g^{-1} \cdot y \rangle = \langle x, y_0 \rangle$ where $y_0 = \sum_{g \in S_m} g^{-1} \cdot y$ is in the fixed-point set $S^m E_q$. Hence, $y_0 = 0$ and so is $\langle x, y \rangle$. □

A key observation is that $a^{\otimes m} \in S^m E_p$. A result of Demailly–Skoda [9] shows that $S^m E \otimes \det(E)$ with the induced metric is Nakano positive. Endow $\Omega$ with the Euclidean metric. Since $E$ is trivial over $\Omega$, we pretend that $\det(E)$ is a trivial bundle. Thus, by the Ohsawa–Takegoshi theorem for vector bundles, there exists a universal constant $C$ (whose optimal value can be computed [15]) and an extension $f_{m,\epsilon} : \Omega \to S^m E \otimes \det(E)$ of $a^{\otimes m}$ such that

\begin{equation}
\int_{\Omega} \|f_{m,\epsilon}\|^2_{S^m h_{\epsilon}} \det(h) \leq C \|a^{\otimes m}\|^2_{S^m h_{\epsilon}(p)} \det(h_{\epsilon}).
\end{equation}

By Lemma 2.3, $\|b\|_{S^m h} = \|b\|_{h^{\otimes m}}$ if $b \in S^m E$. Rewriting (2.2),

$$
\int_{\Omega} \|f_{m,\epsilon}\|^2_{S^m h} e^{-m(r+1)\epsilon|z|^2} \det(h) \leq C \|a^{\otimes m}\|^2_{S^m h(p)} e^{-m(r+1)\epsilon|z|^2} \|a^{\otimes m}\|^2_{S^m h(p)} \det(h).
$$

The sections $u_{\epsilon,m} = f_{\epsilon,m} e^{-m(r+1)\epsilon|z|^2/2}$ are uniformly bounded in $L^2$ (independent of $\epsilon$). In fact,

$$
\bar{\partial} u_{\epsilon,m} = -u_{\epsilon,m} \frac{m(r+1)\epsilon z}{2} d\bar{z}.
$$

Hence, elliptic regularity shows that $\|u_{\epsilon,m}\|_{C^{1,\gamma}(\Omega')} \leq K_m$ where $0 < \gamma < 1$. The Arzela–Ascoli theorem implies that there is a convergent sequence $\epsilon_i \to 0$ such that $u_{\epsilon_i,m} \to u_m$ in $C^1$ where $\partial u_m = 0$. Thus $u_m$ is the desired holomorphic extension satisfying

$$
\int_{\Omega} \|u_m\|^2_{S^m h} \det(h) \leq C \|a^{\otimes m}\|^2_{S^m h(p)} \det(h)(p)
$$

$$
\Rightarrow \int_{\Omega'} \|u_m\|^2_{h^{\otimes m}} \leq \frac{C \sup_{\Omega'} \det(h)}{\inf_{\Omega'} \det(h)} \|a^{\otimes m}\|^2_{h^{\otimes m}(p)}.
$$

and the proof is complete. □
Clearly, the multiple coarse extension property is met. □

Remark 2.4. One might wonder as to why we used $e^{-\epsilon |z|^2}$ and took the limit as $\epsilon \to 0$. It would have been more expedient to simply choose a large enough $\epsilon$ so that $\tilde{h}_\epsilon$ is itself Nakano positive. However, such an act would lead to $C_m$ growing exponentially in $m$ and thus contradict the multiple coarse $L^2$-extension property.

Now we prove Proposition 2.2 for singular metrics. Akin to [19], we use an approximation argument to complete the proof. Indeed, Proposition 6.1 of [27] implies that the convolutions $h^{\nu}$ of $h$, increase to $h$ pointwise and are Griffiths-semipositively curved on a slightly small domain. Choose $\nu \leq \nu_0$ to be small enough so that the convolutions are defined on $\Omega$. Let $p \in \Omega'$ and $a \in E_p$. The proof of Proposition 2.2 shows the existence of extensions $f_{m,\nu}$ of $a$ on $\Omega''$ such that

$$\int_{\Omega_\delta} \|f_{m,\nu}\|^2_{h^{\otimes m}_{\nu}(p)} \leq C \sup_{\Omega_\delta} \det(h^{\nu}_\nu) \|a^{\otimes m}_{\nu}\|^2_{h^{\otimes m}_{\nu}(p)},$$

where $\Omega_\delta \subset \Omega'$ is a $\delta$-neighbourhood of $\Omega$ for some fixed small $\delta > 0$.

Since the $h^{\nu}$ uniformly converge to $h$ on $\Omega_\delta$, the extension constant is bounded above uniformly in $\nu$ (and $m$). Moreover, by the increasing property of $h^{\nu}$,

$$\int_{\Omega_\delta} \|f_{m,\nu}\|^2_{h^{\otimes m}_{\nu_{\nu_0}}} \leq \int_{\Omega_\delta} \|f_{m,\nu}\|^2_{h^{\otimes m}_{\nu_0}} \leq C^{\nu_0} \|a^{\otimes m}_{\nu_0}\|^2_{h^{\otimes m}_{\nu_0}(p)} \leq C^{\nu_0} \|a^{\otimes m}_{\nu_0}\|^2_{h^{\otimes m}_{\nu_0}(p)}.$$

Therefore, $f_{m,\nu}$ is uniformly bounded (independent of $\nu$) in $L^2(\Omega_\delta)$. The sub-mean value property shows that it is pointwise bounded in $\Omega_{\delta/2}$. Cauchy’s estimates show that $\|f_{m,\nu}\|_{\mathcal{C}^1(\Omega_{\delta/3})} \leq K_m$. Thus, by the Arzela–Ascoli theorem, there is a sequence $\nu_i \to 0$ such that $f_{m,\nu_i} \to v_m$ in $C^2(\Omega_{\delta/4})$. The limit $v_m$ is a holomorphic extension of $a^{\otimes m}$ over $\Omega$ and clearly satisfies

$$\int_{\Omega'} \|v_m\|^2_{h^{\otimes m}_{\nu_0}} \leq C^{\nu_0} \|a^{\otimes m}_{\nu_0}\|^2_{h^{\otimes m}_{\nu_0}(p)}.$$

Thus, the multiple coarse $L^2$-extension property is met and the proof of Theorem 1.3 is complete. □

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