Local asymptotic properties for Cox-Ingersoll-Ross process with discrete observations

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Abstract
In this paper, we consider a one-dimensional Cox-Ingersoll-Ross (CIR) process whose drift coefficient depends on unknown parameters. Considering the process discretely observed at high frequency, we prove the local asymptotic normality property in the subcritical case, the local asymptotic quadraticity in the critical case, and the local asymptotic mixed normality property in the supercritical case. To obtain these results, we use the Malliavin calculus techniques developed recently for CIR process together with the estimation for positive and negative polynomial moments of the CIR process. In this study, we require the same conditions of high frequency and infinite horizon as in the case of ergodic diffusions with globally Lipschitz coefficients studied earlier in the literature. However, in the non-ergodic cases, additional assumptions on the decreasing rate are required due to the fact that the square root diffusion coefficient of the CIR process is not regular enough.

KEYWORDS
Cox-Ingersoll-Ross process, high-frequency observation, local asymptotic (mixed) normality property, local asymptotic quadraticity property, Malliavin calculus, parametric estimation, square root coefficient
1 INTRODUCTION

On a complete probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\), we consider a Cox-Ingersoll-Ross (CIR) process \(X^{a,b} = (X_t^{a,b})_{t \geq 0}\) in \(\mathbb{R}\) defined by

\[
X_t^{a,b} = x_0 + \int_0^t (a - bX_s^{a,b})ds + \int_0^t \sqrt{2\sigma X_s^{a,b}} dB_s,
\]

(1)

where \(X_0^{a,b} = x_0 > 0\) is a given initial condition and we assume that \(a \geq \sigma > 0\). Here, \(B = (B_t)_{t \geq 0}\) is a standard Brownian motion. The parameters \((a, b) \in \Theta \times \Sigma\) are unknown to be estimated, and \(\Theta\) and \(\Sigma\) are closed intervals of \(\mathbb{R}_+\) and \(\mathbb{R}\), respectively, where \(\mathbb{R}_+ = \mathbb{R} \setminus \{0\}\). Let \(\{\hat{\mathcal{F}}_t\}_{t \geq 0}\) denote the natural filtration generated by \(B\). We denote by \(\hat{\mathbb{P}}^{a,b}\) the probability measure induced by the CIR process \(X^{a,b}\) on the canonical space \((C(\mathbb{R}_+, \mathbb{R}), B(C(\mathbb{R}_+, \mathbb{R})))\) endowed with the natural filtration \(\{\hat{\mathcal{F}}_t\}_{t \geq 0}\). Here \(C(\mathbb{R}_+, \mathbb{R})\) denotes the set of \(\mathbb{R}\)-valued continuous functions defined on \(\mathbb{R}_+\), and \(B(C(\mathbb{R}_+, \mathbb{R}))\) is its Borel \(\sigma\)-algebra. We denote by \(\hat{\mathbb{E}}^{a,b}\) the expectation with respect to (w.r.t.) \(\hat{\mathbb{P}}^{a,b}\).

Let \(\rightarrow\), \(\rightarrow_L\), \(\rightarrow_{a.s.}\), and \(\Rightarrow\) denote the convergence in \(\hat{\mathbb{P}}^{a,b}\)-probability, in \(\hat{\mathbb{P}}^{a,b}\)-law, in \(\hat{\mathbb{P}}^{a,b}\)-almost surely, and equality in law, respectively. \(*\) denotes the transpose.

By applying Itô’s formula to (1), the unique strong solution of the Equation (1) which is nonnegative is given by

\[
X_t^{a,b} = x_0e^{-bt} + \int_0^t e^{-b(t-s)}ds + \int_0^t e^{-b(t-s)} \sqrt{2\sigma X_s^{a,b}} dB_s,
\]

(2)

for all \(t \geq 0\). Notice that condition \(a \geq \sigma > 0\) guarantees that the process \(X^{a,b}\) is always positive, that is, \(P(X_t^{a,b} > 0, \forall t \geq 0) = 1\).

Recall that the CIR process is extensively used in mathematical finance to model the evolution of short-term interest rates or to describe the dynamic of the stochastic volatility in the Heston model. The CIR process appears in the financial literature also as part of the class of affine processes, and a lot of interesting material can be found in this way, for example, in the work of Cuchiero, Filipović, Mayerhofer, and Teichmann (2011), Duffie, Filipović, and Schachermayer (2003), Kallsen (2006), Keller-Ressel and Mijatović (2012) and other authors.

A fundamental concept in asymptotic theory of statistics is the local asymptotic normality (LAN) property introduced by Le Cam (1960) and then extended by Jeganathan (1982) to the local asymptotic mixed normality (LAMN) property. The local asymptotic quadraticity (LAQ) property was introduced by, for example, Höpfner (2014), Le Cam and Lo Yang (1990). Initiated by Gobet (2001), the techniques of Malliavin calculus can be used to analyze the log-likelihood ratio of the discrete observation of continuous diffusion processes. Concretely, Gobet (2001), Gobet (2002) obtained the LAMN and LAN properties, respectively, for multidimensional elliptic diffusions and ergodic diffusions on the basis of discrete observations at high frequency. In the presence of jumps, several cases have been largely investigated, see, for example, Ar’i-Sahalia and Jacod (2007), Kawai (2013), Clément, Delattre, and Gloter (2014), Clément and Gloter (2015), Kohatsu-Higa, Nualart, and Tran (2014), Kohatsu-Higa, Nualart, and Tran (2017), and Tran (2017). However, all these results deal with the stochastic differential equations whose coefficients are continuously differentiable and satisfy a global Lipschitz condition. The case
where the coefficient functions of the model do not satisfy these standard assumptions, for instance, the square root diffusion function in the CIR model which is neither differentiable at 0 nor globally Lipschitz, still remains an open problem when the model is discretely observed at high frequency.

On the other hand, notice that most existing research works on statistics for CIR process and more generally for affine diffusions mainly focus on parameter estimation based on continuous observations. More precisely, Overbeck (1998) showed the asymptotic properties of maximum likelihood estimator (MLE) as well as the LAN, LAMN and LAQ properties for CIR process in the subcritical (ergodic), critical, and supercritical (nonergodic) submodels. Later, Ben Alaya and Kebaier (2012), Ben Alaya and Kebaier (2013) show various asymptotic properties of MLE associated to the partial and global drift parameters of the CIR process in both ergodic and nonergodic cases. Recently, Barczy and Pap (2016) have studied the asymptotic properties of MLE for Heston models. Later on, Benke and Pap (2017) have proved the LAN, LAMN, and LAQ properties for Heston models. More recently, Barczy, Ben Alaya, Kebaier, and Pap (2018), Barczy, Ben Alaya, Kebaier, and Pap (2019), have studied the asymptotic properties of MLE for jump-type Heston models and jump-type CIR process. Some results on parameter estimation based on discrete observations of CIR process can be found, for example, in Overbeck and Rydén (1997), Ben Alaya and Kebaier (2013). Overbeck and Rydén (1997) proved the LAN property for the ergodic CIR process based on discrete observations at low frequency and did one-step-improvement in the sense of Le Cam (1960).

However, as mentioned just above, it seems that the validity of the LAN, LAMN, and LAQ properties for CIR process in ergodic and nonergodic cases on the basis of discrete observations in a high frequency context has never been addressed in the literature. One difficulty comes from the fact that its diffusion coefficient is the square root function. Motivated by this fact, the main objective of this paper is to derive the LAN property in the subcritical case, the LAQ property in the critical case, and the LAMN property in the supercritical case for $X_{n}^{a,b}$ based on discrete observations.

Let us now recall the notion of the LAQ, LAMN, and LAN properties in our setting. Given the process $X_{t}^{a,b} = (X_{t}^{a,b})_{t \geq 0}$ and $n \geq 1$, we consider a discrete observation at equidistant and deterministic times $t_{k} = k\Delta_{n}$, $k \in \{0, \ldots, n\}$ of the process $X_{n}^{a,b}$, which is denoted by $X_{n,a,b} = (X_{t_{0}}^{a,b}, X_{t_{1}}^{a,b}, \ldots, X_{t_{n}}^{a,b})$, where $\Delta_{n} \leq 1$ for all $n \geq 1$. We suppose that the high-frequency and infinite horizon conditions satisfy: $\Delta_{n} \rightarrow 0$ and $n\Delta_{n} \rightarrow \infty$ as $n \rightarrow \infty$. We denote by $p_{n}^{a,b}$ and $p_{n}(\cdot; (a, b))$ the probability law and the density of the random vector $X_{n,a,b}$, respectively.

For fixed $(a_{0}, b_{0}) \in \Theta \times \Sigma$, we consider a discrete observation of the process $X_{n,a,b}^{a_{0},b_{0}}$ given by $X_{n,a,b}^{a_{0},b_{0}} = (X_{t_{0}}^{a_{0},b_{0}}, X_{t_{1}}^{a_{0},b_{0}}, \ldots, X_{t_{n}}^{a_{0},b_{0}})$. For $(u, v) \in \mathbb{R}^{2}$, we set $a_{n} := a_{0} + u\varphi_{1,n}(a_{0}, b_{0})$ and $b_{n} := b_{0} + v\varphi_{2,n}(a_{0}, b_{0})$, where we assume that $\varphi_{1,n}(a_{0}, b_{0})$ and $\varphi_{2,n}(a_{0}, b_{0})$ tend to zero as $n \rightarrow \infty$. Suppose that there exist a $\mathbb{R}^{2}$-valued random vector $U(a_{0}, b_{0})$ and a (random) matrix $I(a_{0}, b_{0})$ such that for all $\zeta = (u, v) \in \mathbb{R}^{2}$, as $n \rightarrow \infty$,

$$
\log \frac{dP_{n}^{a_{0},b_{0}}(X_{n,a,b}^{a_{0},b_{0}})}{dP_{n}^{a_{0},b_{0}}(X_{n,a_{0},b_{0}})} \xrightarrow{L^{(p_{n}^{a_{0},b_{0}})}} \zeta^{*}U(a_{0}, b_{0}) - \frac{1}{2}\zeta^{*}I(a_{0}, b_{0})\zeta.
$$

Then, we say that

(a) The LAQ property holds at $(a_{0}, b_{0})$ with rates of convergence $(\varphi_{1,n}^{-1}(a_{0}, b_{0}), \varphi_{2,n}^{-1}(a_{0}, b_{0}))$ and random matrix $I(a_{0}, b_{0})$ if further for all $\zeta = (u, v) \in \mathbb{R}^{2}$, we have
\[ E^{a_n,b_0} \left[ e^{\frac{1}{2} U(a_n,b_0) - \frac{1}{2} \varepsilon I(a_n,b_0) \varepsilon^T} \right] = 1. \]

(b) The LAMN property holds at \((a_0, b_0)\) with rates of convergence \((\varphi_{1,n}^{-1}(a_0, b_0), \varphi_{2,n}^{-1}(a_0, b_0))\) and asymptotic random Fisher information matrix \(I(a_0, b_0)\) if further we can write \(U(a_0, b_0) = I(a_0, b_0)^{\frac{1}{2}} \mathcal{N}(0, I_2)\), where \(\mathcal{N}(0, I_2)\) denotes a centered \(\mathbb{R}^2\)-valued Gaussian random vector with identity covariance matrix \(I_2\), which is independent of the conditional covariance matrix \(I(a_0, b_0)\).

In particular, when \(I(a_0, b_0)\) is deterministic, that is, \(U(a_0, b_0) = \mathcal{N}(0, I(a_0, b_0))\), we say that the LAN property holds at \((a_0, b_0)\) with rates of convergence \((\varphi_{1,n}^{-1}(a_0, b_0), \varphi_{2,n}^{-1}(a_0, b_0))\) and asymptotic Fisher information matrix \(I(a_0, b_0)\).

For the full power of the LAN, LAMN, or LAQ property, one should consider arbitrary bounded sequences \((u_n, v_n)_{n \geq 1}\) in \(\mathbb{R}^2\) when the local scale at \((a_0, b_0)\) is denoted by \((\varphi_n(a_0, b_0))_{n \geq 1} = (\varphi_{1,n}(a_0, b_0), \varphi_{2,n}(a_0, b_0))_{n \geq 1}\), that is, \(a_n := a_0 + u_n \varphi_{1,n}(a_0, b_0)\) and \(b_n := b_0 + v_n \varphi_{2,n}(a_0, b_0)\). For details, we refer the reader to section 7.1 of Höpfner (2014) or Le Cam and Lo Yang (1990).

As we will see, the rates of convergence for LAQ, LAMN, and LAN properties for CIR process depend strongly on the drift parameter \(b\). Indeed, the value of the drift parameter \(b > 0\), \(b = 0\) and \(b < 0\) determines, respectively, the subcritical, critical, and supercritical cases.

To show the convergence (3), one needs to derive an appropriate stochastic expansion of the log-likelihood ratio. For any \(t > s\), the law of \(X_{t|s}^{a,b}\) conditioned on \(X_{s|s}^{a,b} = x\) admits a positive explicit transition density \(p^{a,b}(t-s,x,y)\). From the explicit expression (5) and (6) below, the transition density \(p^{a,b}(t-s,x,y)\) is differentiable w.r.t. \(a\) and \(b\). Then using Markov property and the mean value theorem, the log-likelihood ratio can be decomposed as

\[
\begin{align*}
\log \frac{dP_{n.a_0,b_0}^{a_n,b_n}}{dP_{n.a_0,b_0}^{a_0,0}}(X_{n,a_0,b_0}^{n,a_0,b_0}) &= \log \frac{p_n(X^n_{a_0,b_0}; (a_n, b_n))}{p_n(X^n_{a_0,b_0}; (a_0, b_0))} \\
&= \log \frac{p_n(X^n_{a_0,b_0}; (a_n, b_n))}{p_n(X^n_{a_0,b_0}; (a_0, b_0))} + \log \frac{p_n(X^n_{a_0,b_0}; (a_n, b_n))}{p_n(X^n_{a_0,b_0}; (a_0, b_0))} \\
&= \sum_{k=0}^{n-1} \log \frac{p_{a_0,b_0}^{a_n,b_n}}{p_{a_0,b_0}^{a_0,b_0}}(\Delta_n, X_{tk}^{a_0,b_0}, X_{tk+1}^{a_0,b_0}) + \sum_{k=0}^{n-1} \log \frac{p_{a_0,b_0}^{a_n,b_n}}{p_{a_0,b_0}^{a_0,b_0}}(\Delta_n, X_{tk}^{a_0,b_0}, X_{tk+1}^{a_0,b_0}) \\
&= \sum_{k=0}^{n-1} u \varphi_{1,n}(a_0, b_0) \int_0^1 \frac{da}{p_{a(\varepsilon),b_0}(\Delta_n, X_{tk}^{a_0,b_0}, X_{tk+1}^{a_0,b_0})} (\Delta_n, X_{tk}^{a_0,b_0}, X_{tk+1}^{a_0,b_0}) \, d\varepsilon \\
&\quad + \sum_{k=0}^{n-1} v \varphi_{2,n}(a_0, b_0) \int_0^1 \frac{db}{p_{a_0,b(\varepsilon)}^{a_0,b_0}(\Delta_n, X_{tk}^{a_0,b_0}, X_{tk+1}^{a_0,b_0})} (\Delta_n, X_{tk}^{a_0,b_0}, X_{tk+1}^{a_0,b_0}) \, d\varepsilon, \quad (4)
\end{align*}
\]

where \(a_n = a_0 + u \varphi_{1,n}(a_0, b_0), b_n = b_0 + v \varphi_{2,n}(a_0, b_0), a(\varepsilon) := a_0 + \varepsilon u \varphi_{1,n}(a_0, b_0), b(\varepsilon) := b_0 + \varepsilon v \varphi_{2,n}(a_0, b_0)\).

Since we are dealing with the CIR process, one way to proceed could be to use an explicit expression for the transition density function which is characterized in terms of a noncentral chi-squared distribution (Cox, Ingersoll, & Ross, 1985). That is, for any \(t > 0\) and \(b \neq 0\),
\[ p^{a,b}(t, x, y) = \frac{-be^{bt}}{\sigma(1 - e^{bt})} \left( \frac{y}{xe^{-bt}} \right)^{\frac{\alpha}{2}} \exp \left\{ \frac{bxt + e^{bt}y}{\sigma(1 - e^{bt})} \right\} I_{\nu} \left( \frac{2\sqrt{xye^{bt}}}{\sigma(1 - e^{bt})} \right), \]

(5)

and \( b = 0 \),

\[ p^{a,0}(t, x, y) = \frac{1}{\sigma t} \left( \frac{y}{x} \right)^{\frac{\alpha}{2}} \exp \left\{ -\frac{x + y}{\sigma t} \right\} I_{\nu} \left( \frac{2\sqrt{xy}}{\sigma t} \right), \]

(6)

where \( \nu = \frac{a}{\sigma} - 1 \), and \( I_{\nu} \) is the modified Bessel function of the first kind of order \( \nu > 0 \) defined by

\[ I_{\nu}(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(\nu + n + 1)} \left( \frac{x}{2} \right)^{2n+\nu}, \]

for any \( x \in \mathbb{R} \). Here \( \Gamma(\cdot) \) is the Gamma function defined by \( \Gamma(z) = \int_{0}^{\infty} x^{z-1}e^{-x}dx \) for \( z \in \mathbb{R}_+ \). To use this approach, one needs to well understand the behavior of the logarithm derivatives of the transition density w.r.t. the parameters \( a \) and \( b \). However, it is not clear if using (5) and (6) would help especially since the parameter \( a \) appears in the index \( \nu \) of the modified Bessel function and we have at least to find appropriate estimates of several types of increments involving the parameter \( a \).

In this paper, for our purpose, we use an alternative strategy based on the Malliavin calculus approach initiated by Gobet (2001), Gobet (2002) in order to derive an explicit expression for the logarithm derivatives of the transition density in terms of a conditional expectation of a Skorohod integral. Let us mention here that Malliavin calculus for CIR process is established by Alòs and Ewald (2008) and then Malliavin calculus for constant elasticity of variance process is developed by Altmayer and Neuenkirch (2015). Furthermore, these articles give an explicit expression for the Malliavin derivative of the CIR process which will be useful for our purpose. Later on, the Malliavin differentiability of Heston stochastic volatility have recently used, for example, in Altmayer and Neuenkirch (2015), Davis and Johansson (2006), Kohatsu-Higa and Yasuda (2009), Petrou (2008).

The main results of this paper are the LAN property in the subcritical case, the LAQ property in the critical case, and the LAMN property in the supercritical case for CIR process. For this, let us formulate the following assumption on the ratio of the coefficients of Equation (1) we shall work with

\[ \left( A \right) \quad \frac{a}{\sigma} > \frac{11 + \sqrt{89}}{2}. \]

Under condition (A) and \( b_0 > 0 \), we obtain in Theorem 1 the LAN property for subcritical case with rates of convergence \((\varphi_{1,n}^{-1}(a_0, b_0), \varphi_{2,n}^{-1}(a_0, b_0)) = (\sqrt{n\Delta_n}, \sqrt{n\Delta_n})\). In the critical case, assuming condition (A), \( b_0 = 0 \) and that \( n\Delta^3_n \to 0 \) as \( n \to \infty \), we prove in Theorem 2 that the LAQ property holds with rates of convergence \((\varphi_{1,n}^{-1}(a_0, b_0), \varphi_{2,n}^{-1}(a_0, b_0)) = (\sqrt{\log(n\Delta_n)}, n\Delta_n)\). Finally, in the supercritical case when \( a \) is known and \( b \) is the only unknown parameter, assuming condition (A), \( b_0 < 0 \) and \( \Delta^3_n e^{-b_0 n\Delta_n} \to 0 \), we derive the validity of the LAMN property for the likelihood at \( b_0 \) with rate of convergence \( \varphi_{2,n}^{-1}(b_0) = e^{-b_0 \frac{\Delta_n}{2}} \).
When the LAMN property holds for the likelihood at \((a_0, b_0)\) with rates of convergence \((\varphi_{1,n}^{-1}(a_0, b_0), \varphi_{2,n}^{-1}(a_0, b_0))\) and asymptotic random Fisher information matrix \(I(a_0, b_0)\), convolution and minimax theorems can be applied. On the one hand, the convolution theorem (corollary 1 in Jeganathan (1982)) suggests the notion of asymptotically efficient estimators. That is, a sequence of estimators \((\hat{a}_n, \hat{b}_n)\) of the parameter \((a_0, b_0)\) is said to be asymptotically efficient at \((a_0, b_0)\) in the sense of Hájek–Le Cam convolution theorem if as \(n \to \infty\),

\[
\varphi_n^{-1}(a_0, b_0)((\hat{a}_n, \hat{b}_n) - (a_0, b_0))^* \mathcal{L}(\hat{a}_n^0, \hat{b}_n^0) \xrightarrow{\mathcal{L}} I(a_0, b_0)^{-\frac{1}{2}} \mathcal{N}(0, I_2),
\]

where \(\varphi_n^{-1}(a_0, b_0) := \text{diag}(\varphi_{1,n}^{-1}(a_0, b_0), \varphi_{2,n}^{-1}(a_0, b_0))\) is the diagonal matrix, \(I(a_0, b_0)\) and \(\mathcal{N}(0, I_2)\) are independent. On the other hand, as a consequence of the minimax theorem (proposition 2 in Jeganathan (1982)), the lower bound for the asymptotic variance of any estimators is given by \(I(a_0, b_0)^{-1}\). In the particular case when the asymptotic Fisher information matrix is deterministic, this property is called the LAN property. For details, we refer the reader to, for example, Jeganathan (1982), Le Cam and Lo Yang (1990). It is worth noticing that the estimators constructed from a discretization of the time-continuous MLE given in theorems 8 and 9 of Ben Alaya and Kebaier (2013) are asymptotically efficient in the subcritical and critical cases since their variance attains the lower bound for the asymptotic variance of estimators with the optimal rate of convergence (see Remark 2).

To obtain the aforementioned results, our Malliavin calculus approach allows to obtain an appropriate stochastic expansion of the log-likelihood ratio (see Proposition 1). To treat the main term in the asymptotic behavior of the expansion, in the subcritical case, we apply a central limit theorem for triangular arrays of random variables together with the ergodic property whereas in the critical and supercritical cases, the corresponding convergence results (9)–(11) and (12) on the CIR process are essentially used. The difficult part of the proof is to deal with the negligible terms of the expansion. In Gobet (2001), Gobet (2002), a change of transition density functions of the diffusion processes is performed in order to be able to use the upper and lower bounds of Gaussian type of the transition density functions. This allows to measure the deviation of the change of transition density functions when the parameters change. For the CIR process, the transition density estimates of Gaussian type may not exist since the diffusion coefficient and its derivative are not bounded. To overcome these difficulties, instead of changing transition density functions, change of measures is essentially used (see Lemma 9). Then a technical Lemma 10 is established in order to measure the deviation of the change of measures when the drift parameters change.

Furthermore, some \(L^p\)-norm estimation for positive and negative polynomial moments and exponential moment estimates of the CIR process taken from Ben Alaya and Kebaier (2013) and Bossy and Diop (2007) are needed in order to show the convergence of the negligible terms (see Lemmas 1 and 2 and Lemmas 5 and 6). For this, condition (A) above turns out to be crucial (see Remark 1). When using our strategy, we do not require some additional assumptions on the decreasing rate of \(\Delta_n\) such as \(n \Delta_p^p \to 0\) for some \(p > 1\) in the subcritical (ergodic) case. However, in the nonergodic cases, we require \(n \Delta_3^3 \to 0\) for the critical case and \(\Delta_2^2 e^{-b_0 n \Delta_n} \to 0\) for the supercritical case.

This paper is organized as follows. In Section 2, we state our main results in Theorem 1, 2, and 3 which correspond, respectively, to the subcritical, critical, and supercritical cases. Section 3 is devoted to the proof of the main results, which follows the aforementioned strategy. Using techniques of Malliavin calculus, Section 4 introduces technical results needed for the proof of
the main results, which are related to an explicit expression for the score functions in terms of a conditional expectation of a Skorohod integral, and a decomposition of the Skorohod integral. The proofs of these technical results are postponed to Appendix A1 in order to maintain the flow of the exposition. Section 5 focuses on studying the convergence of the remainder terms in the expansion (14) of the log-likelihood ratio. Finally, some useful results are presented in Appendix B.

2 MAIN RESULTS

In this section, we give a statement of our main results in this paper which is divided into three cases: subcritical, critical, and supercritical cases. Recall that our sampling scheme satisfies the high-frequency and infinite horizon conditions. That is, \( \Delta n \to 0 \) and \( n \Delta n \to \infty \) as \( n \to \infty \).

2.1 Subcritical case

Assume that \( b > 0 \). In this case, \( X^{a,b}_a \) is ergodic (Jin, Mandrekar, Rüdiger, & Trabelsi, 2013) and its unique stationary distribution which we denote by \( \pi_{a,b}(dx) \) is a Gamma law with shape parameter \( \frac{a}{\sigma} \) and scale parameter \( \frac{\sigma}{b} \) (Ben Alaya & Kebaier, 2013). That is,

\[
\pi_{a,b}(dx) = \left( \frac{b}{\sigma} \right)^{\frac{a}{\sigma}} \frac{1}{\Gamma\left(\frac{a}{\sigma}\right)} x^{\frac{a}{\sigma} - 1} e^{-\frac{b}{\sigma} x} dx.
\]

Furthermore, \( X^{a,b}_t \) converges in law as \( t \to \infty \) towards a random variable \( X^{a,b}_\infty \) whose distribution is given by \( \pi_{a,b}(dx) \) (propositions 3 and 4 of Ben Alaya and Kebaier (2012) or theorem 5.3 of Benke and Pap (2017)). Moreover, for any \( \pi_{a,b}(dx) \)-integrable function \( h \), \( X^{a,b} \) has the ergodic property in the sense that as \( t \to \infty \),

\[
\frac{1}{t} \int_0^t h(X^{a,b}_s) ds \to \mathbb{E}^{a,b}[h(X^{a,b}_\infty)] = \int_0^\infty h(x)\pi_{a,b}(dx), \quad \mathbb{P}^{a,b}\text{-a.s.}
\]

Now, for fixed \( (a_0, b_0) \in \Theta \times \Sigma_+ \) where \( \Sigma_+ \) is a closed interval of \( \mathbb{R}_+^2 \), we consider a discrete observation \( X^{n,a_0,b_0}_n = (X^{a_0,b_0}_{t_0}, X^{a_0,b_0}_{t_1}, \ldots, X^{a_0,b_0}_{t_n}) \) of the process \( X^{a_0,b_0} \). The first result of this paper is the LAN property.

**Theorem 1.** Assume condition (A) and \( b_0 > 0 \). Then, the LAN property holds for the likelihood at \( (a_0, b_0) \) with rate of convergence \( \sqrt{n \Delta_n}, \sqrt{n \Delta_n} \) and asymptotic Fisher information matrix \( I(a_0, b_0) \). That is, for all \( z = (u, v)^* \in \mathbb{R}^2 \), as \( n \to \infty \),

\[
\log \frac{dP^{n,a_0,b_0}_n}{dP^{a_0,b_0}_n}(X^{n,a_0,b_0}_n) \xrightarrow{L(\mathbb{P}^{a_0,b_0})} z^* \mathcal{N}(0, I(a_0, b_0)) - \frac{1}{2} z^* I(a_0, b_0) z,
\]

where \( a_n := a_0 + \frac{u}{\sqrt{n \Delta_n}}, b_n := b_0 + \frac{v}{\sqrt{n \Delta_n}}, \) and \( \mathcal{N}(0, I(a_0, b_0)) \) is a centered \( \mathbb{R}^2 \)-valued Gaussian vector with covariance matrix
2.2 Critical case

Assume that $b = 0$. In this case, from proposition 1 of Ben Alaya and Kebaier (2012), as $t \to \infty$,

$$
\left(\frac{X_{t}^{a,0}}{t}, \frac{1}{t^2} \int_{0}^{t} X_{s}^{a,0} ds\right) \Rightarrow \left(R_{1}^{a,0}, \int_{0}^{1} R_{s}^{a,0} ds\right).
$$

(9)

Here, $R_{t}^{a,0} = (R_{s}^{a,0})_{s \geq 0}$ is the CIR process starting from 0, solution to

$$
dR_{t}^{a,0} = adt + \sqrt{2\sigma R_{t}^{a,0}} dB_{t},
$$

(10)

where $R_{0}^{a,0} = 0$. Moreover, from proposition 2 of Ben Alaya and Kebaier (2012) when $a > \sigma$, as $t \to \infty$,

$$
\frac{1}{\log t} \int_{0}^{t} \frac{ds}{X_{s}^{a,0}} \overset{\mathcal{L}}{\to} \frac{1}{a - \sigma}.
$$

(11)

It is worth noticing that the formulae (9) can be obtained by a scaling argument. See Remark on p. 614 of Ben Alaya and Kebaier (2012).

For fixed $(a_{0}, 0) \in \Theta \times \Sigma$, consider a discrete observation $X_{n,a_{0},0} = (X_{t_{0}}^{a_{0},0}, X_{t_{1}}^{a_{0},0}, \ldots, X_{t_{n}}^{a_{0},0})$ of the process $X_{a_{0},0}$. The second result of this paper is the following LAQ property.

**Theorem 2.** Assume condition (A), $b_{0} = 0$ and that $n\Delta_{n}^{3} \to 0$ as $n \to \infty$. Then, the LAQ property holds for the likelihood at $(a_{0}, 0)$ with rates of convergence $(\sqrt{\log(n\Delta_{n})}, n\Delta_{n})$ and random matrix $I(a_{0}, 0)$. That is, for all $z = (u, v)^{\ast} \in \mathbb{R}^{2}$, as $n \to \infty$,

$$
\log \frac{dP_{n}^{a_{0},b_{n}}}{dP_{n}^{a_{0},0}}(X_{n,a_{0},0}) \overset{\mathcal{L}}{\to} Z^{\ast} U(a_{0}, 0) - \frac{1}{2} z^{\ast} I(a_{0}, 0) z,
$$

and

$$
\mathbb{E}_{a_{0},0}^{a_{0},0} \left[e^{z^{\ast} U(a_{0}, 0) - \frac{1}{2} z^{\ast} I(a_{0}, 0) z}\right] = 1,
$$

where $a_{n} := a_{0} + \frac{u}{\sqrt{\log(n\Delta_{n})}}$, $b_{n} := 0 + \frac{v}{n\Delta_{n}}$, and $U(a_{0}, 0)$ is a $\mathbb{R}^{2}$-valued random vector given by

$$
U(a_{0}, 0) := \left(\frac{1}{\sqrt{2\sigma(a_{0} - \sigma)}} G, \frac{a_{0} - R_{1}^{a_{0},0}}{2\sigma}\right)^{\ast},
$$

with covariance matrix
\[ I(a_0, 0) := \frac{1}{2\sigma} \begin{pmatrix} \frac{1}{a_0 - \sigma} & 0 \\ 0 & \int_0^1 R_s^{a_0, 0} ds \end{pmatrix}. \]

Here, \( G \) is a standard normal random variable independent of \((R_1^{a_0, 0}, \int_0^1 R_s^{a_0, 0} ds)\), where \((R_t^{a_0})_{t \geq 0}\) is the CIR process starting from 0 defined by (10).

We notice that Theorem 2 solves the problem left open in the last three lines in section 3.2 of Overbeck (1998) for time-continuous observations.

Next, we state the result in the case where there is only one unknown parameter.

**Corollary 1.** (i) Let \( b = 0 \) be known and let \( a \) be unknown parameter. Assume condition \((A)\). Then, the LAN property holds for the likelihood at \( a_0 \) with rate of convergence \( \sqrt{\log(n\Delta_n)} \) and asymptotic Fisher information \( I(a_0) = \frac{1}{2\sigma(a_0 - \sigma)} \). That is, for all \( u \in \mathbb{R} \), as \( n \to \infty \),

\[
\log \frac{dP_{n, a_0}^{a_0}}{dP_{n, a_0}}(X^{n,a_0}) \xrightarrow{\mathcal{L}(\theta^{a_0})} u \mathcal{N}(0, I(a_0)) - \frac{u^2}{2} I(a_0).
\]

(ii) Let \( a \) be known and let \( b = 0 \) be unknown parameter. Assume condition \((A)\) and that \( n\Delta_n^3 \to 0 \) as \( n \to \infty \). Then, the LAQ property holds for the likelihood at \( b_0 = 0 \) with rate of convergence \( n\Delta_n \) and random variable \( U(0) = \frac{a - R_1^{a_0}}{2\sigma} \) whose conditional variance is given by \( I(0) = \frac{1}{2\sigma} \int_0^1 R_s^{a_0, 0} ds \). That is, for all \( v \in \mathbb{R} \), as \( n \to \infty \),

\[
\log \frac{dP_{n, b_0}^{a_0}}{dP_{n, a_0}}(X^{n,a_0}) \xrightarrow{\mathcal{L}(\theta^{a_0})} vU(0) - \frac{v^2}{2} I(0),
\]

and

\[
\hat{\mathbb{E}}^{a_0} \left[ e^{vU(0) - \frac{v^2}{2} I(0)} \right] = 1.
\]

### 2.3 Supercritical case

Assume that \( b < 0 \). In this case, from proposition 3 of Ben Alaya and Kebaier (2012), as \( t \to \infty \),

\[
\left( e^{bt} X_t^{a,b}, e^{bt} \int_0^t X_s^{a,b} ds \right) \xrightarrow{\mathcal{L}(\theta^{a,b})} \left( R_t^{a_{-b} - \frac{1}{b}}, \frac{1}{b} R_t^{a_{-b}} \right).
\]

Here, \((R_t^{a})_{t \geq 0}\) is the CIR process starting from \( x_0 \), solution to

\[
dR_t^a = adt + \sqrt{2\sigma R_t^a} dB_t,
\]

where \( R_0^a = x_0 \). The first component in the right hand side in (12) specifies the limit variable called \( W \) in the last line of theorem 3 (i) of Overbeck (1998). This limit is also immediate from
the formula (8.10) on p. 236 of Ikeda and Watanabe (1989) combined with the formula (1.1) of Pinsky (1972).

Let \( a \) be known and let \( b < 0 \) be unknown parameter. For fixed \( b_0 \in \Sigma_- \) where \( \Sigma_- \) is a closed interval of \( \mathbb{R}^* = \mathbb{R}_- \setminus \{0\} \), we consider a discrete observation \( X^{n,a_0,b_0} = (X_{t_0}^{a_0,b_0}, X_{t_1}^{a_0,b_0}, \ldots, X_{t_n}^{a_0,b_0}) \) of the process \( X^{a_0,b_0} \) where \( a_0 \) is known. The last result of this paper is the following LAMN property in the case where \( b \) is the only unknown parameter.

**Theorem 3.** Assume condition (A), \( b_0 < 0 \) and that \( \Delta_n^2 e^{b_0 n \Delta_n} \to 0 \) as \( n \to \infty \). Then, the LAMN property holds for the likelihood at \( b_0 \) with rate of convergence \( e^{-b_0 n \Delta_n^2} \) and asymptotic random Fisher information \( I(b_0) := -\frac{1}{2} \sigma_{b_0}^2 R_t^a t \). That is, for all \( v \in \mathbb{R} \), as \( n \to \infty \),

\[
\log \frac{dP_{n}^{a_0,b_n}}{dP_{n}^{a_0,b_0}} \bigg|_{X^{n,a_0,b_0}} \xrightarrow{L} v \sqrt{I(b_0)} \mathcal{N}(0,1) - \frac{v^2}{2} I(b_0),
\]

where \( b_n := b_0 + \frac{v}{e^{b_0 n \Delta_n^2}} \), \( \mathcal{N}(0,1) \) is a centered standard Gaussian random variable which is independent of \( I(b_0) \). Here, \( (R_t^a)_{t \geq 0} \) is the CIR process starting from \( x_0 \) defined by (13).

We notice that Theorem 3 improves the result obtained by Overbeck (1989) in section 3.3 for time-continuous observations.

**Remark 1.** Let us mention that condition (A) on the ratio of the coefficients \( \frac{a}{\sigma} \) required in Theorem 1, 2, and 3 is similar to condition (10) in theorem 2.2 of Berkaoui, Bossy, and Diop (2008) which is used to prove the strong convergence of the symmetrized Euler scheme applied to CIR process. In fact, this technical condition comes from our techniques used in this paper and it is needed to treat Lemmas 11 and 14. It is worth noticing that the lower bound \( \frac{11 + \sqrt{89}}{2} \) appearing in condition (A) is fixed in an optimal way in the sense that the Hölder weights used to control the remainder terms are carefully chosen so that our final condition (A): \( \frac{a}{\sigma} \geq 10.22 \) is slightly different from the condition \( \frac{a}{\sigma} > 5 + 3\sqrt{2} \approx 9.24 \) needed for the representation of the score function in Proposition 1.

**Remark 2.** Combining our main results Theorem 1 and Theorem 2 together with theorems 8 and 9 in Ben Alaya and Kebaier (2013), the discrete MLE estimator given by (15) in Ben Alaya and Kebaier (2013) achieves the lower bound given by the inverse of the asymptotic Fisher information matrix \( I(a_0,b_0) \) and \( I(a_0,0) \) in the subcritical and critical cases, respectively. Note that the MLE estimator for \( a \) or \( (a,b) \) is not consistent in the supercritical case (see theorem 2 in Ben Alaya and Kebaier (2012) or theorem 7 in Ben Alaya and Kebaier (2013)). We can only estimate the parameter \( b \) for this supercritical case (see section 3.3 in Overbeck (1998)).

**Remark 3.** In Overbeck (1998), \( \sigma = 2 \) is considered. From sections 3.1, 3.2, and 3.3 of Overbeck (1998), the LAN and LAMN properties hold in the subcritical and supercritical cases for the continuous observations under condition \( \frac{a}{\sigma} > 1 \) and \( a > 0 \), respectively. Moreover, in the critical case, Overbeck (1998) studies the only parameter \( a \) or \( b \) to obtain the LAN or LAQ property, respectively, for the continuous observations under condition \( \frac{a}{\sigma} > 1 \). For the discrete observation setting, the LAN and LAMN properties for the subcritical and supercritical cases in Theorems 1 and 3 are established under condition (A) and have the same rate of convergence and asymptotic
3 | PROOF OF MAIN RESULTS

In this section, we give the proof of Theorems 1, 2, and 3. Let us first recall the following useful results taken from Ben Alaya and Kebaier (2013).

**Lemma 1** (proposition 3 of Ben Alaya and Kebaier (2013)).

(i) For all \( p < \frac{a}{\sigma} \) and \( b > 0 \), we have \( \sup_{t \geq 0} \hat{E}^{a,b}_t \left[ \frac{1}{(X^a_t)^p} \right] < \infty \).

(ii) For all \( p < \frac{a}{\sigma} \) and \( b = 0 \), we have

\[
\sup_{0 \leq t \leq 1} \hat{E}^{a,0}_t \left[ \frac{1}{(X^a_t)^p} \right] < \infty \quad \text{and} \quad \sup_{t \geq 1} \frac{\hat{E}^{a,0}_t \left[ \frac{1}{(X^a_t)^p} \right]}{t^p} < \infty.
\]

(iii) For all \( 0 < p < \frac{a}{\sigma} \) and \( b < 0 \), we have

\[
\sup_{0 \leq t \leq 1} \hat{E}^{a,b}_t \left[ \frac{1}{(X^a_t)^p} \right] < \infty \quad \text{and} \quad \sup_{t \geq 1} \frac{\hat{E}^{a,b}_t \left[ \frac{1}{(X^a_t)^p} \right]}{t^p} < \infty.
\]

Recall that (i) and (ii) of Lemma 1 are taken from proposition 3 of Ben Alaya and Kebaier (2013). The statement (iii) of Lemma 1 is due to the Comparison Theorem giving that \( X^a_t,0 \leq X^a_t,b \) a.s. for all \( t \geq 0 \) when \( b < 0 \) as their corresponding drift coefficients satisfy \( a - b x < a - b x \).

**Lemma 2** (propositions 4 and 5 of Ben Alaya and Kebaier (2013)). Let \( t > s \geq 0 \) satisfying that \( 0 < t - s < 1 \).

(i) Assume that \( b > 0 \). Then for any \( q \geq 1 \), there exists a constant \( C > 0 \) such that

\[
\hat{E}^{a,b}_t \left[ |X^{a,b}_t - X^{a,b}_s|^q \right] \leq C(t - s)^{\frac{q}{2}}.
\]

(ii) Assume that \( b = 0 \). Then there exists a constant \( C > 0 \) such that

\[
\hat{E}^{a,0}_t \left[ |X^{a,0}_t - X^{a,0}_s|^q \right] \leq \begin{cases} C(t - s)^{\frac{q}{2}} \sup_{u \in [s,t]} \hat{E}^{a,0}_u \left[ (X^{a,0}_u)^\frac{q}{2} \right], & \text{for any } q \geq 2, \\
C(at + x_0)^{\frac{q}{2}}(t - s)^{\frac{q}{2}}, & \text{for any } q \in (1, 2).
\end{cases}
\]

From (4), the log-likelihood ratio can be written as follows
\[
\log \frac{dP_{n,a,b_0}^\alpha}{dP_{n,a,b_0}^\alpha}(X_t^{n,a,b_0}) = \sum_{k=0}^{n-1} u\varphi_{1,n}(a_0, b_0) \int_0^1 \frac{\partial_{\alpha} P_{\alpha(\cdot),b_0}^a}{P_{\alpha(\cdot),b_0}^a}(\Delta_n, X_{t_k}^{a_0,b_0}, X_{t_{k+1}}^{a_0,b_0})d\alpha
\]
\[
+ \sum_{k=0}^{n-1} u\varphi_{2,n}(a_0, b_0) \int_0^1 \frac{\partial_{\alpha} P_{\alpha(\cdot),b_0}^a}{P_{\alpha(\cdot),b_0}^a}(\Delta_n, X_{t_k}^{a_0,b_0}, X_{t_{k+1}}^{a_0,b_0})d\alpha
\]
\[
= \sum_{k=0}^{n-1} (\xi_{k,n} + \eta_{k,n}) + \sum_{k=0}^{n-1} (R_{k,n} + H_{k,n}),
\]
(14)

where \( a(\cdot) = a_0 + \varepsilon u\varphi_{1,n}(a_0, b_0) \), \( b(\cdot) = b_0 + \varepsilon v\varphi_{2,n}(a_0, b_0) \), \( \xi_{k,n} \) and \( \eta_{k,n} \) are the main terms whereas \( R_{k,n} \), \( H_{k,n} \) are the remainder terms in the expansion. Two couples \((\xi_{k,n}, R_{k,n})\) and \((\eta_{k,n}, H_{k,n})\) will appear from the analysis of two corresponding score functions \( \frac{\partial_{\alpha} P_{\alpha(\cdot),b_0}^a}{P_{\alpha(\cdot),b_0}^a}(\Delta_n, X_{t_k}^{a_0,b_0}, X_{t_{k+1}}^{a_0,b_0}) \) and \( \frac{\partial_{\alpha} P_{\alpha(\cdot),b_0}^a}{P_{\alpha(\cdot),b_0}^a}(\Delta_n, X_{t_k}^{a_0,b_0}, X_{t_{k+1}}^{a_0,b_0}) \). The main terms are given by

\[
\xi_{k,n} = u\varphi_{1,n}(a_0, b_0) \frac{1}{2\sigma X_{t_k}^{a_0,b_0}} \left( \sqrt{2\sigma X_{t_k}^{a_0,b_0}(B_{t_{k+1}} - B_{t_k})} - \frac{1}{2} u\varphi_{1,n}(a_0, b_0)\Delta_n \right),
\]
\[
\eta_{k,n} = -\frac{\varepsilon v}{2\sigma} \varphi_{2,n}(a_0, b_0) \left( \sqrt{2\sigma X_{t_k}^{a_0,b_0}(B_{t_{k+1}} - B_{t_k})} - \Delta_n \left( u\varphi_{1,n}(a_0, b_0) - \frac{1}{2} v\varphi_{2,n}(a_0, b_0)X_{t_k}^{a_0,b_0} \right) \right),
\]
(15)

with in the subcritical case \((b_0 > 0)\) \((\varphi_{1,n}(a_0, b_0), \varphi_{2,n}(a_0, b_0)) = (\frac{1}{\sqrt{n\Delta_n}}, \frac{1}{\sqrt{n\Delta_n}})\), in the critical case \((b_0 = 0)\) \((\varphi_{1,n}(a_0, 0), \varphi_{2,n}(a_0, 0)) = (\frac{1}{\sqrt{\log(n\Delta_n)}}, \frac{1}{n\Delta_n})\), and in the supercritical case \((b_0 < 0)\) \((\varphi_{1,n}(a_0, b_0), \varphi_{2,n}(a_0, b_0)) = (0, e^{b_0\Delta_n} \frac{\Delta_n}{\varepsilon})\). The remainder terms \( R_{k,n}, H_{k,n} \) are implicitly defined by the decomposition (14) and they will be explicitly determined in the proof of Lemma 3. Then, our aim will be to study the asymptotic of the main terms \( \sum_{k=0}^{n-1} (\xi_{k,n} + \eta_{k,n}) \). Note that the convergence of this sum requires a weaker condition than (A), that is \( \frac{\xi}{\varepsilon} > 4 \). However, the convergence of the remainder terms \( \sum_{k=0}^{n-1} (R_{k,n} + H_{k,n}) \) given in the following lemma will require condition (A).

**Lemma 3.** Assume condition (A) and our setting \( \Delta_n \to 0 \) and \( n\Delta_n \to \infty \) as \( n \to \infty \). Then, for the subcritical case, as \( n \to \infty \),

\[
\sum_{k=0}^{n-1} (R_{k,n} + H_{k,n}) \to 0.
\]
(16)

Furthermore, if we assume the additional condition \( n\Delta_n^3 \to 0 \) for the critical case and \( \Delta_n^2 e^{-b_0 n\Delta_n} \to 0 \) for the supercritical case, then (16) remains valid for both cases.

The proof of this lemma will be given in Section 5.

From (14) and Lemma 3, to prove the main results, it suffices to show that as \( n \to \infty \),

\[
\sum_{k=0}^{n-1} (\xi_{k,n} + \eta_{k,n}) \to z^* U(a_0, b_0) - \frac{1}{2} z^* I(a_0, b_0) z.
\]

In order to control the convergence in the subcritical case, we will need the following classical discrete ergodic theorem which is analogous to lemma 3.1 in Gobet (2002) established in the case of ergodic diffusions with globally Lipschitz coefficients.
Lemma 4. Let $b_0 > 0$. Then, for all $p < \frac{1}{2} \left( \frac{a_0}{\sigma} - 1 \right)$, as $n \to \infty$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{(X_t^{a_0,b_0})^p} \to \int_0^\infty \frac{1}{x^p} \pi^{a_0,b_0}(dx).$$

Proof. First, using Itô’s formula, (i) of Lemma 1, and Burkholder–Davis–Gundy’s (BDG’s) inequality, for any $t_k \leq s \leq t_{k+1}$ we get

\[
\mathbb{E}^{a_0,b_0} \left[ \frac{1}{(X_s^{a_0,b_0})^p} - \frac{1}{(X_{t_k}^{a_0,b_0})^p} \right] \leq C(s-t_k) \left( \sup_{u \in [t_k,s]} \mathbb{E}^{a_0,b_0} \left[ \frac{1}{(X_u^{a_0,b_0})^p} \right] + \sup_{u \in [t_k,s]} \mathbb{E}^{a_0,b_0} \left[ \frac{1}{(X_u^{a_0,b_0})^{p+1}} \right] \right)
\]

\[
+ C \mathbb{E}^{a_0,b_0} \left[ \int_{t_k}^s \frac{dB_u}{(X_u^{a_0,b_0})^{p+rac{1}{2}}} \right]^{\frac{1}{2}} \leq C \Delta_n + C \left( s-t_k \right) \sup_{u \in [t_k,s]} \mathbb{E}^{a_0,b_0} \left[ \frac{1}{(X_u^{a_0,b_0})^{2p+1}} \right]^{\frac{1}{2}} \leq C \sqrt{\Delta_n},
\]

for some constant $C > 0$ which is independent of $t_k$ and $s$, where the moments above are bounded thanks to $p < \frac{1}{2} \left( \frac{a_0}{\sigma} - 1 \right)$. This shows that

\[
\mathbb{E}^{a_0,b_0} \left[ \frac{1}{n \Delta_n} \int_0^{n \Delta_n} ds \frac{1}{(X_s^{a_0,b_0})^p} - \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{(X_t^{a_0,b_0})^p} \right] \leq \frac{1}{n \Delta_n} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}^{a_0,b_0} \left[ \frac{1}{(X_s^{a_0,b_0})^p} - \frac{1}{(X_t^{a_0,b_0})^p} \right] ds \leq C \sqrt{\Delta_n},
\]

(17)

which tends to zero as $n \to \infty$. On the other hand, from (8), as $n \to \infty$,

\[
\frac{1}{n \Delta_n} \int_0^{n \Delta_n} ds \frac{1}{(X_s^{a_0,b_0})^p} \to \int_0^\infty \frac{1}{x^p} \pi^{a_0,b_0}(dx), \quad \mathbb{P}^{a_0,b_0} - a.s.
\]

(18)

where the limit above is finite. Thus, the result follows from (17) and (18). □
3.1 Proof of Theorem 1

Proof. From Lemma 3 with \((\varphi_1(a_0, b_0), \varphi_2(a_0, b_0)) = (\frac{1}{\sqrt{n}a_0}, \frac{1}{\sqrt{n}b_0})\), we need only to prove that as \(n \to \infty\),

\[
\sum_{k=0}^{n-1} (\xi_{k,n} + \eta_{k,n}) \xrightarrow{L(\hat{\psi}_{a_0,b_0})} \mathcal{Z}^* \mathcal{N}(0, I(a_0, b_0)) - \frac{1}{2} \mathcal{Z}I(a_0, b_0)z,
\]

where \(\xi_{k,n}\) and \(\eta_{k,n}\) are given by (15). To do so, we will use Theorem 4. At first, let us recall that in the subcritical case \((b > 0)\) the stationary distribution \(\pi_{a,b}(dx)\) (see (7)) satisfies

\[
\int_0^\infty x \pi_{a,b}(dx) = \frac{a}{b}, \quad \int_0^\infty \frac{1}{x} \pi_{a,b}(dx) = \frac{b}{a - \sigma}.
\]

Then, by Lemma 4, it is easy to check that as \(n \to \infty\),

\[
\sum_{k=0}^{n-1} \mathbb{E}^{a_0,b_0}[\xi_{k,n} | \hat{\tau}_{i_k}] = -\frac{u^2}{4\sigma} \sum_{k=0}^{n-1} \frac{1}{\mathcal{X}_{i_k}^{a_0,b_0}} \hat{\psi}_{a_0,b_0} - \frac{u^2}{4\sigma} \int_0^\infty \frac{1}{x} \pi_{a_0,b_0}(dx) = -\frac{u^2}{2} \frac{b_0}{2\sigma(a_0 - \sigma)},
\]

under condition \(\frac{a}{\sigma} > 3\), and

\[
\sum_{k=0}^{n-1} \mathbb{E}^{a_0,b_0}[\eta_{k,n} | \hat{\tau}_{i_k}] = \frac{uv}{2\sigma} - \frac{v^2}{4\sigma} \sum_{k=0}^{n-1} \mathcal{X}_{i_k}^{a_0,b_0} \hat{\psi}_{a_0,b_0} \rightarrow \frac{uv}{2\sigma} - \frac{v^2}{2\sigma b_0},
\]

which gives condition (i) of Theorem 4.

Next, we check condition (ii) of Theorem 4 similarly by using the same arguments. We get

\[
\sum_{k=0}^{n-1} (\hat{\psi}_{a_0,b_0}[\xi_{k,n} | \hat{\tau}_{i_k}])^2 = \frac{u^2}{2\sigma} \sum_{k=0}^{n-1} \frac{1}{\mathcal{X}_{i_k}^{a_0,b_0}} \hat{\psi}_{a_0,b_0} \rightarrow \frac{u^2}{2\sigma(a_0 - \sigma)};
\]

\[
\sum_{k=0}^{n-1} (\hat{\psi}_{a_0,b_0}[\eta_{k,n} | \hat{\tau}_{i_k}])^2 = \frac{v^2}{2\sigma} \sum_{k=0}^{n-1} \mathcal{X}_{i_k}^{a_0,b_0} \hat{\psi}_{a_0,b_0} \rightarrow \frac{v^2}{2\sigma b_0},
\]

\[
\sum_{k=0}^{n-1} (\hat{\psi}_{a_0,b_0}[\xi_{k,n} \eta_{k,n} | \hat{\tau}_{i_k}]) = -\frac{uv}{2\sigma},
\]

where the first convergence holds under condition \(\frac{a_0}{\sigma} > 3\). Finally, it remains to check condition (iii) of Theorem 4. Indeed, there exists \(c > 0\) such that

\[
\sum_{k=0}^{n-1} \mathbb{E}^{a_0,b_0}[\xi_{k,n}^4 | \hat{\tau}_{i_k}] \leq \frac{c}{n^2} \sum_{k=0}^{n-1} (\mathcal{X}_{i_k}^{a_0,b_0})^2 + \frac{c}{n^4} \sum_{k=0}^{n-1} (\mathcal{X}_{i_k}^{a_0,b_0})^4;
\]

\[
\sum_{k=0}^{n-1} \mathbb{E}^{a_0,b_0}[\eta_{k,n}^4 | \hat{\tau}_{i_k}] \leq \frac{c}{n^2} \sum_{k=0}^{n-1} (\mathcal{X}_{i_k}^{a_0,b_0})^2 + \frac{c}{n^3} \sum_{k=0}^{n-1} (\mathcal{X}_{i_k}^{a_0,b_0})^3;
\]
for some constant $c > 0$. Thus applying Lemma 1(i), under condition $\frac{a_0}{\sigma} > 4$ we have
\[ \mathbb{E}^{a_0,0}[\sum_{k=0}^{n-1} \mathbb{E}^{a_0,0}[\xi_k | \bar{I}_k]] \leq \frac{c}{n} \] and
\[ \mathbb{E}^{a_0,0}[\sum_{k=0}^{n-1} \mathbb{E}^{a_0,0}[\eta_k | \bar{I}_k]] \leq \frac{c}{n}, \] which tends to zero. Clearly, this condition is satisfied under (A). Thus, the result follows.

\[ \Box \]

### 3.2 Proof of Theorem 2

**Proof.** From (14), Lemma 3 with $(\varphi_{1,n}(a_0, 0), \varphi_{2,n}(a_0, 0)) = \left( \frac{1}{\sqrt{\log(n\Delta_n)}}, \frac{1}{n\Delta_n} \right)$, we need only to prove that as $n \to \infty$,

\[ \sum_{k=0}^{n-1} (\xi_{k,n} + \eta_{k,n}) \xrightarrow{L^2} z^* U(a_0, 0) - \frac{1}{2} z^* I(a_0, 0) z, \tag{19} \]

and

\[ \mathbb{E}^{a_0,0} \left[ e^{z^* U(a_0, 0) - \frac{1}{2} z^* I(a_0, 0) z} \right] = 1. \tag{20} \]

To do this, we rewrite

\[ \sum_{k=0}^{n-1} (\xi_{k,n} + \eta_{k,n}) = z^* U_n(a_0, 0) - \frac{1}{2} z^* I_n(a_0, 0) z + R_{n,1}^{a_0,0} + R_{n,2}^{a_0,0} + \frac{uv}{2\sigma \sqrt{\log(n\Delta_n)}}, \tag{21} \]

where $t_n = n\Delta_n$ and

\[ U_n(a_0, 0) = \frac{1}{\sqrt{2\sigma}} \left( \begin{array}{c} \frac{1}{\sqrt{\log t_n}} \int_0^{t_n} \frac{dB_s}{X_s^{a_0,0}} \\ -\frac{1}{t_n} \int_0^{t_n} \sqrt{X_s^{a_0,0}} dB_s \end{array} \right), \]

\[ I_n(a_0, 0) = \frac{1}{2\sigma} \left( \begin{array}{ccc} \frac{1}{\log t_n} \int_0^{t_n} \frac{ds}{X_s^{a_0,0}} & 0 \\ 0 & \frac{1}{t_n^2} \int_0^{t_n} X_s^{a_0,0} ds \end{array} \right), \]

\[ R_{n,1}^{a_0,0} = \frac{u^2}{4\sigma \log t_n} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left( \frac{1}{X_s^{a_0,0}} - \frac{1}{X_{t_k}^{a_0,0}} \right) ds + \frac{v^2}{4\sigma t_n^2} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (X_s^{a_0,0} - X_{t_k}^{a_0,0}) ds, \]

\[ R_{n,2}^{a_0,0} = -u \frac{1}{\sqrt{2\sigma \log t_n}} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left( \frac{1}{\sqrt{X_s^{a_0,0}}} - \frac{1}{\sqrt{X_{t_k}^{a_0,0}}} \right) dB_s \]
\[ + \frac{v}{\sqrt{2\sigma t_n}} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (\sqrt{X_s^{a_0,0}} - \sqrt{X_{t_k}^{a_0,0}} dB_s. \]

First, using Itô’s formula and Equation (1), we get that

\[ U_n(a_0, 0) = \frac{1}{\sqrt{2\sigma}} \left( \begin{array}{c} \frac{1}{\sqrt{2\sigma \log t_n}} \left( \log X_{t_n}^{a_0,0} - \log x_0 + (\sigma - a_0) \int_0^{t_n} \frac{ds}{X_s^{a_0,0}} \right) \\ -\frac{1}{\sqrt{2\sigma t_n}} (X_{t_n}^{a_0,0} - x_0 - a_0 t_n) \end{array} \right). \]
Moreover, it follows from the proof of theorem 6 in Ben Alaya and Kebaier (2013) and (11) that as \( n \to \infty \),

\[
\mathcal{L}\left( \hat{\beta}_{a_0}^{\beta_{a_0}} \right) \to \left( \sqrt{\frac{2\sigma}{a_0 - \sigma}}, \int_0^1 R_{a_0}^{\beta_{a_0}} ds, \frac{1}{a_0 - \sigma} \right),
\]

where \( G \) is a standard normal random variable independent of \( (R_{a_0}^{\beta_{a_0}}, \int_0^1 R_{a_0}^{\beta_{a_0}} ds) \). This implies that as \( n \to \infty \),

\[
(U_n(a_0, 0), I_n(a_0, 0)) \to (U(a_0, 0), I(a_0, 0)).
\]

Next, thanks to the proof of theorem 9 in Ben Alaya and Kebaier (2013), under condition \( \frac{a_0}{\sigma} > 2 \), we have that \( R_{a_0}^{\beta_{a_0}} \to 0 \) as \( n \to \infty \). Now, we treat \( R_{a_0}^{\beta_{a_0}} \). For this, we write \( R_{a_0}^{\beta_{a_0}} = \sum_{k=0}^{n-1} (\zeta_{k,n,1} + \zeta_{k,n,2}) \), where

\[
\zeta_{k,n,1} = -\frac{u}{\sqrt{2\sigma \sqrt{\log t_n}}} \int_{t_k}^{t_{k+1}} \left( \frac{1}{X_s^{\beta_{a_0}}} - \frac{1}{X_{t_k}^{\beta_{a_0}}} \right) dB_s,
\]

\[
\zeta_{k,n,2} = \frac{v}{\sqrt{2\sigma t_n}} \int_{t_k}^{t_{k+1}} \left( \sqrt{X_s^{\beta_{a_0}}} - \sqrt{X_{t_k}^{\beta_{a_0}}} \right) dB_s.
\]

We apply Lemma 18 to \( \zeta_{k,n,1} \) and \( \zeta_{k,n,2} \). As it is obvious that \( \sum_{k=0}^{n-1} \hat{E}^{\beta_{a_0}}[\zeta_{k,n,1}^{2} | \hat{F}_{t_k}] = 0 \), we only need to check that \( \sum_{k=0}^{n-1} \hat{E}^{\beta_{a_0}}[\zeta_{k,n,1}^{2} | \hat{F}_{t_k}] \to 0 \). To do so, we use Itô’s formula, Lemma 1(ii) and standard calculations to get

\[
\hat{E}^{\beta_{a_0}} \left[ \sum_{k=0}^{n-1} \hat{E}^{\beta_{a_0}}[\zeta_{k,n,1}^{2} | \hat{F}_{t_k}] \right] = \sum_{k=0}^{n-1} \hat{E}^{\beta_{a_0}}[\zeta_{k,n,1}^{2}]
\]

\[
= \frac{u^2}{2\sigma \log t_n} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \hat{E}^{\beta_{a_0}} \left[ \left( \int_{t_k}^{s} \frac{a_0}{2} + \frac{3\sigma}{4} \frac{du}{(X_u^{\beta_{a_0}})^3} \right) + \int_{t_k}^{s} \sqrt{\frac{\sigma}{2}} dB_u \right]^2 ds
\]

\[
\leq C \frac{u^2}{\log(n\Delta_n)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left( (s - t_k)^2 \sup_{u \in [t_k,s]} \hat{E}^{\beta_{a_0}} \left[ \frac{1}{(X_u^{\beta_{a_0}})^3} \right] + (s - t_k) \sup_{u \in [t_k,s]} \hat{E}^{\beta_{a_0}} \left[ \frac{1}{(X_u^{\beta_{a_0}})^2} \right] \right) ds
\]

\[
\leq C \frac{u^2 \Delta_n}{\log(n\Delta_n)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left( \sup_{u \in [t_k,s]} \hat{E}^{\beta_{a_0}} \left[ \frac{1}{(X_u^{\beta_{a_0}})^3} \right] + \sup_{u \in [t_k,s]} \hat{E}^{\beta_{a_0}} \left[ \frac{1}{(X_u^{\beta_{a_0}})^2} \right] \right) ds
\]

\[
\leq \frac{u^2 \Delta_n}{\log(n\Delta_n)} \left( C_1 + C_2 \left( \frac{1}{(n\Delta_n)^2} + \frac{1}{n\Delta_n} \right) \right),
\]
for some constants $C_1, C_2$, which tends to zero under $\frac{a_1}{\sigma} > 3$. Thus, $\sum_{k=0}^{n-1} \zeta_{k,n,1} \to 0$.

Similarly, we have $\sum_{k=0}^{n-1} \hat{E}^{a_{0,0}} \zeta_{k,n,2} | \hat{F}_{k,t} | = 0$. Next, applying Itô’s formula, Lemma 1(ii) and standard calculations, we get

$$
\hat{E}^{a_{0,0}} \left[ \sum_{k=0}^{n-1} \hat{E}^{a_{0,0}} \zeta_{k,n,2} | \hat{F}_{k,t} | \right] = \sum_{k=0}^{n-1} \hat{E}^{a_{0,0}} \zeta_{k,n,2} = \frac{\nu^2}{2 \sigma^2 t} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \hat{E}^{a_{0,0}} \left[ (\sqrt{X^0_{k,t}} - \sqrt{X^0_{k,t}})^2 \right] ds
$$

$$
= \frac{\nu^2}{2 \sigma^2 t} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left[ \left( \int_{t_k}^{s} \frac{(a_0 - \sigma)}{2} \frac{du}{\sqrt{X^0_{k,u}}} + \int_{t_k}^{s} \frac{\sigma}{2} dB_u \right)^2 \right] ds
$$

$$
\leq C \frac{\nu^2}{t} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left( (s - t_k)^2 \sup_{u \in [t_k, s]} \hat{E}^{a_{0,0}} \left[ \frac{1}{X^0_{k,u}} \right] + (s - t_k) \right) ds
$$

$$
\leq C \nu^2 \left( \frac{1}{(n \Delta_n)^2} \Delta^2_n (C_1 + C_2 \log(n \Delta_n)) + n \Delta^2_n \right),
$$

which tends to zero. Thus, $\sum_{k=0}^{n-1} \hat{E}^{a_{0,0}} \zeta_{k,n,2} \to 0$. Then, we have $\sum_{k=0}^{n-1} \zeta_{k,n,2} \to 0$. Consequently, under condition $\frac{a_1}{\sigma} > 3$, we have that $R^a_{n,0,0} \to 0$ as $n \to \infty$. Therefore, under $\frac{a_1}{\sigma} > 3$, $R^a_{n,1} + R^a_{n,2} \to 0$ as $n \to \infty$. This combined with (21) and (22) gives (19).

Finally, we treat (20). For this, let us consider the CIR process $R^{a,b} = (R^{a,b}_t)_{t \geq 0}$ starting from 0 defined by

$$
dR^{a,b}_t = (a - bR^{a,b}_t) dt + \sqrt{2 \sigma R^{a,b}_t} dB_t,
$$

where $R^{0,b}_0 = 0$, $(a, b) \in \mathbb{R}_+ \times \mathbb{R}$. We denote by $P^{a,b}_R$ the probability measure induced by the CIR process $R^{a,b}$ on the measurable space $(\mathbb{C}(\mathbb{R}_+, \mathbb{R}), B(\mathbb{C}(\mathbb{R}_+, \mathbb{R}))$. Let $T > 0$. For any $t > 0$, let $P^{a,b}_{R,t}$ be the restriction of $P^{a,b}_R$ on $\hat{F}_t$. As a consequence of lemma 3.1 in Barczy and Pap (2016), for any $a > \sigma, b \in \mathbb{R}$, the probability measures $P^{a,b}_{R,t}$ and $P^{a,b}_{R,t}$ are absolutely continuous with respect to each other and its Radon–Nikodym derivative is given by

$$
\frac{dP^{a,b}_{R,t}}{dP^{a,b}_{R,t}} ((R^a_0, s)_{s \in [0,t]}) = \left\{ \frac{b}{\sqrt{2 \sigma}} \int_0^t \sqrt{R^a_0} dB_s - \frac{b^2}{4 \sigma} \int_0^t R^a_0 ds \right\}.
$$

On the other hand, as a consequence of theorem 3.4 in chapter III of Jacod and Shiryaev (2003), the Radon–Nikodym derivative process $\frac{dP^{a,b}_{R,t}}{dP^{a,b}_{R,t}} ((R^a_0, s)_{s \in [0,t]})_{t \in [0,T]}$ is a martingale w.r.t. the filtration $(\hat{F}_t)_{t \in [0,T]}$.

Now, using the independence between $G$ and $(R^a_{1,0}, \int_0^1 R^a_{1,0} ds)$, we have

$$
\hat{E}^{a_{0,0}} \left[ e^{\sigma U(a_0) - \frac{1}{2} \sigma I(a_0) \zeta} \right] = E_1 E_2,
$$

where
Proof. (i) When \( b = 0 \) is known and \( a \) is unknown parameter, the result is obtained as a consequence of Theorem 2 by taking \( v = 0 \). This is the reason why the condition \( n \Delta_n^3 \rightarrow 0 \) as \( n \rightarrow \infty \) is removed.

(ii) When \( a \) is known and \( b = 0 \) is unknown parameter, the result is obtained as a consequence of Theorem 2 by taking \( u = 0 \).

3.4 Proof of Theorem 3

Proof. From (14), Lemma 3 with \((\varphi_{1,n}(a_0, b_0), \varphi_{2,n}(a_0, b_0)) = (0, e^{b_0^{\frac{b_0}{x}}})\), we need only to prove that as \( n \rightarrow \infty \),

\[
\sum_{k=0}^{n-1} (\xi_k, n + \eta_{k, n}) \xrightarrow{\mathcal{L}(\beta^{\alpha} b_0)} \nu \sqrt{I(b_0)} N(0, 1) - \frac{\nu^2}{2} I(b_0). \tag{24}
\]

To do this, we rewrite

\[
\sum_{k=0}^{n-1} (\xi_k, n + \eta_{k, n}) = \nu U_n(a_0, b_0) - \frac{\nu^2}{2} I_n(a_0, b_0) + R_{n,3}^{\alpha-b_0} + R_{n,4}^{\alpha-b_0}, \tag{25}
\]

where

\[
U_n(a_0, b_0) = \frac{-e^{b_0^{\frac{b_0}{x}}}}{\sqrt{2\sigma}} \int_0^{t_n} \sqrt{X_s^{a_0-b_0}} dB_s, \quad I_n(a_0, b_0) = \frac{1}{2\sigma} e^{b_0^{\frac{b_0}{x}}} \int_0^{t_n} X_s^{a_0-b_0} ds,
\]
\[ R_{n,3}^{\alpha_0, b_0} = \frac{v^2}{4\sigma e^{-b_0 n \Delta_n}} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (X_s^{\alpha_0, b_0} - X_t^{\alpha_0, b_0}) ds, \]
\[ R_{n,4}^{\alpha_0, b_0} = \frac{v}{\sqrt{2\sigma e^{-b_0 n \Delta_n}}} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (\sqrt{X_s^{\alpha_0, b_0}} - \sqrt{X_t^{\alpha_0, b_0}}) dB_s. \]

First, it follows from the proof of theorem 7.1 in Barczy and Pap (2016) that as \( n \to \infty \),
\[
\left( e^{b_0 s} \int_0^t \sqrt{X_s^{\alpha_0, b_0}} dB_s, \ e^{b_0 s} \int_0^t X_s^{\alpha_0, b_0} ds \right) \xrightarrow{\mathcal{L}(\bar{R}^{\alpha_0, b_0})} \left( \left( -\frac{1}{b_0} R_{\alpha_0, b_0} \right)^{\frac{1}{2}}, \mathcal{N}(0,1), -\frac{1}{b_0} R_{\alpha_0, b_0}^{\frac{1}{2}} \right),
\]
where \( \mathcal{N}(0,1) \) is a standard normal random variable independent of \( R_{\alpha_0, b_0}^{\frac{1}{2}} \). This implies that
\[
(U_n(a_0, b_0), I_n(a_0, b_0)) \xrightarrow{\mathcal{L}(\bar{R}^{\alpha_0, b_0})} (\sqrt{I(b_0)} \mathcal{N}(0,1), I(b_0)).
\]

Next, applying Itô’s formula, Lemma 1 (iii), (12) and standard calculations, we get
\[
\mathbb{E}^{\alpha_0, b_0} \left[ |R_{n,3}^{\alpha_0, b_0}| \right] \leq \frac{v^2}{4\sigma e^{-b_0 n \Delta_n}} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}^{\alpha_0, b_0} \left[ \left| \int_s^t (a_0 - b_0 X_u^{\alpha_0, b_0}) du + \int_s^t \sqrt{2\sigma X_u^{\alpha_0, b_0}} dB_u \right| \right] ds
\leq \frac{v^2}{4\sigma e^{-b_0 n \Delta_n}} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left( (s - t_k) + (s - t_k) \sup_{u \in [t_k, s]} \mathbb{E}^{\alpha_0, b_0} [X_u^{\alpha_0, b_0}] \right) ds \leq C \left( n \Delta_n^2 e^{b_0 n \Delta_n} + \Delta_n + \sqrt{n \Delta_n e^{b_0 n \Delta_n}} \right),
\]
which tends to zero. Thus, \( R_{n,3}^{\alpha_0, b_0} \xrightarrow{a.s.} 0 \). Finally, we treat \( R_{n,4}^{\alpha_0, b_0} \) by applying Lemma 18. First, we have \( \sum_{k=0}^{n-1} \mathbb{E}^{\alpha_0, b_0} [(R_{n,4}^{\alpha_0, b_0})^2 | \hat{F}_{t_k}] = 0 \). Next, applying Itô’s formula, Lemma 1(iii), (12) and standard calculations, we get
\[
\mathbb{E}^{\alpha_0, b_0} \left[ \sum_{k=0}^{n-1} \mathbb{E}^{\alpha_0, b_0} [(R_{n,4}^{\alpha_0, b_0})^2 | \hat{F}_{t_k}] \right] = \sum_{k=0}^{n-1} \mathbb{E}^{\alpha_0, b_0} [(R_{n,4}^{\alpha_0, b_0})^2]
= \frac{v^2}{2\sigma e^{-b_0 n \Delta_n}} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E}^{\alpha_0, b_0} \left[ \left| \int_s^t \left( \frac{a_0}{2} - \frac{\sigma}{4} \right) \frac{1}{\sqrt{X_u^{\alpha_0, b_0}}} du + \int_s^t \sqrt{\frac{\sigma}{2}} dB_u \right| \right] ds
\leq \frac{C v^2}{e^{-b_0 n \Delta_n}} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left( (s - t_k)^2 \sup_{u \in [t_k, s]} \mathbb{E}^{\alpha_0, b_0} [X_u^{\alpha_0, b_0}] + \sup_{u \in [t_k, s]} \mathbb{E}^{\alpha_0, b_0} [X_u^{\alpha_0, b_0}] \right) ds
\leq C e^{b_0 n \Delta_n} (\Delta_n^2 + \Delta_n^2 \log(n \Delta_n) + n \Delta_n^2) + C \Delta_n^2,
\]
which tends to zero under condition \( \frac{a_0}{\sigma} > 1 \). Thus, \( \sum_{k=0}^{n-1} \mathbb{E}^{\alpha_0, b_0} [(R_{n,4}^{\alpha_0, b_0})^2 | \hat{F}_{t_k}] \xrightarrow{a.s.} 0 \). Then, we have
\[
\sum_{k=0}^{n-1} R_{n,4}^{\alpha_0, b_0} \xrightarrow{a.s.} 0.
\]
Therefore, under \( \frac{a_0}{\sigma} > 1 \) as \( n \to \infty \).
Therefore, from (25) to (27), we conclude (24).

4 | TECHNICAL RESULTS

The aim of this section is to use techniques of Malliavin calculus in order to represent the remainder terms in Lemma 3 as a conditional expectation of Skorohod integral (see Proposition 1) that will be analyzed in Section 5. To be more precise, we first consider on a complete probability space \((\Omega, F, P)\) the flow process \(Y^{a,b}(s, x) = (Y^{a,b}_t(s, x), t \geq s), x \in \mathbb{R}^n\) on the time interval \([s, \infty)\) and with initial condition \(Y^{a,b}_s(s, x) = x\) satisfying

\[
Y^{a,b}_t(s, x) = x + \int_s^t (a - bY^{a,b}_u(s, x))du + \int_s^t \sqrt{2\sigma Y^{a,b}_u(s, x)}dW_u,
\]

for any \(t \geq s\), where \(W = (W_t)_{t \geq 0}\) is a standard Brownian motion. Let \(\{\tilde{F}_t\}_{t \geq 0}\) denote the natural filtration generated by \(W\). For all \(t \geq 0\) we denote \(Y^{a,b}_t \equiv Y^{a,b}_t(0, x_0)\) which is defined by

\[
Y^{a,b}_t = x_0 + \int_0^t (a - bY^{a,b}_s)ds + \int_0^t \sqrt{2\sigma Y^{a,b}_s}dW_s.
\]

Then, the Malliavin calculus on the Wiener space induced by \(W\) will be applied to obtain a new representation of score functions. We refer the reader to Nualart (2006) for a detailed exposition of the Malliavin calculus and the notations therein. We denote by \(D\) and \(\delta\) the Malliavin derivative and the Skorohod integral w.r.t. \(W\) on \([tk, tk+1]\). In what follows, let \(\mathbb{D}^{1,2}\) denote the Sobolev space of random variables that are differentiable w.r.t. \(W\) in the sense of Malliavin, and Dom \(\delta\) denote the domain of \(\delta\). The Malliavin calculus for CIR process is developed, for example, in Alòs and Ewald (2008) and Altmayer and Neuenkirch (2015).

For any \(k \in \{0, \ldots, n - 1\}\), the process \((Y^{a,b}_t(tk, x), t \in [tk, tk+1])\) is defined by

\[
Y^{a,b}_t(tk, x) = x + \int_{tk}^t (a - bY^{a,b}_u(tk, x))du + \int_{tk}^t \sqrt{2\sigma Y^{a,b}_u(tk, x)}dW_u.
\]

Recall that condition \(a \geq \sigma\) ensures that the CIR process \(Y^{a,b}\) remains almost surely strictly positive. Then, by theorem V.39 in Protter (2005), the process \((Y^{a,b}_t(tk, x), t \in [tk, tk+1])\) is differentiable w.r.t. \(x\) that we denote by \((\partial_x Y^{a,b}_t(tk, x), t \in [tk, tk+1])\). Furthermore, this process admits derivatives w.r.t. \(a\) and \(b\) that we denote by \((\partial_a Y^{a,b}_t(tk, x), t \in [tk, tk+1])\) and \((\partial_b Y^{a,b}_t(tk, x), t \in [tk, tk+1])\), respectively, since this problem is similar to the derivative w.r.t. the initial condition (see e.g. pp. 294-295 in Pagès (2015-2016)). Under condition \(a \geq \sigma\), these processes are solutions to the following equations

\[
\partial_x Y^{a,b}_t(tk, x) = 1 - b \int_{tk}^t \partial_x Y^{a,b}_u(tk, x)du + \int_{tk}^t \frac{\sqrt{\sigma}}{2Y^{a,b}_u(tk, x)} \partial_x Y^{a,b}_u(tk, x)dW_u.
\]
\[ \partial_u Y_{t}^{a,b}(t_k, x) = \int_{t_k}^t (1 - b \partial_u Y_{t}^{a,b}(t_k, x)) du + \int_{t_k}^t \frac{\sqrt{\sigma}}{\sqrt{2 Y_{u}^{a,b}(t_k, x)}} \partial_u Y_{u}^{a,b}(t_k, x) dW_u, \]

\[ \partial_b Y_{t}^{a,b}(t_k, x) = - \int_{t_k}^t \left( Y_{u}^{a,b}(t_k, x) + b \partial_b Y_{u}^{a,b}(t_k, x) \right) du + \int_{t_k}^t \frac{\sigma}{\sqrt{2 Y_{u}^{a,b}(t_k, x)}} dW_u. \]

Therefore, their explicit solutions are respectively given by

\[ \partial_{x} Y_{t}^{a,b}(t_k, x) = \exp \left\{ -b(t - t_k) - \frac{\sigma}{4} \int_{t_k}^t \frac{du}{Y_{u}^{a,b}(t_k, x)} + \frac{\sigma}{2} \int_{t_k}^t \frac{dW_u}{\sqrt{Y_{u}^{a,b}(t_k, x)}} \right\}, \]

\[ \partial_{a} Y_{t}^{a,b}(t_k, x) = \int_{t_k}^t \exp \left\{ -b(t - r) - \frac{\sigma}{4} \int_{t_k}^r \frac{du}{Y_{u}^{a,b}(t_k, x)} + \frac{\sigma}{2} \int_{t_k}^r \frac{dW_u}{\sqrt{Y_{u}^{a,b}(t_k, x)}} \right\} dr, \]

\[ \partial_{b} Y_{t}^{a,b}(t_k, x) = - \int_{t_k}^t Y_{r}^{a,b}(t_k, x) \exp \left\{ -b(t - r) - \frac{\sigma}{4} \int_{t_k}^r \frac{du}{Y_{u}^{a,b}(t_k, x)} + \frac{\sigma}{2} \int_{t_k}^r \frac{dW_u}{\sqrt{Y_{u}^{a,b}(t_k, x)}} \right\} dr. \]

Observe that from (34), (35), and (36), we can write

\[ \partial_{a} Y_{t}^{a,b}(t_k, x) = \int_{t_k}^t \partial_{x} Y_{t}^{a,b}(t_k, x) (\partial_{x} Y_{r}^{a,b}(t_k, x))^{-1} dr, \]

\[ \partial_{b} Y_{t}^{a,b}(t_k, x) = - \int_{t_k}^t Y_{r}^{a,b}(t_k, x) \partial_{x} Y_{t}^{a,b}(t_k, x) (\partial_{x} Y_{r}^{a,b}(t_k, x))^{-1} dr. \]

To analyze the score functions, some notations are introduced. Let \( \tilde{P}_{t}^{a,b} \) denote the probability measure induced by \( Y_{t}^{a,b} = (Y_{t}^{a,b})_{t \geq 0} \) on the canonical space \( (C(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R}))) \) endowed with the natural filtration \( \{ \tilde{F}_t \}_{t \geq 0} \), and \( \tilde{E}_{t}^{a,b} \) denote the expectation w.r.t. \( \tilde{P}_{t}^{a,b} \). For all \( k \in \{0, \ldots, n - 1\} \) and \( x \in \mathbb{R}^n_+ \), we denote by \( \tilde{P}_{t_k,x}^{a,b} \) and \( \tilde{E}_{t_k,x}^{a,b} \), respectively, the probability law of \( Y_{t_k}^{a,b} \) starting at \( x \) at time \( t_k \) and the expectation w.r.t. \( \tilde{P}_{t_k,x}^{a,b} \), that is, \( \tilde{P}_{t_k,x}^{a,b}(A) = \tilde{E}_{t_k,x}^{a,b}[Y_{t_k}^{a,b} = x] \) for all \( A \in \tilde{F} \) and \( \tilde{E}_{t_k,x}^{a,b}[V] = \tilde{E}_{t_k,x}^{a,b}[V|Y_{t_k}^{a,b} = x] \) for all \( \tilde{F} \)-measurable random variables \( V \). Thus, \( \tilde{E}_{t_k,x}^{a,b} \) is the expectation under the probability law of \( Y_{t}^{a,b} \) starting at \( x \) at time \( t_k \).

Next, we recall the following exponential moment estimate taken from Bossy and Diop (2007).

**Lemma 5** (Lemma 3.1 of Bossy and Diop (2007)). Assume that \( \frac{a}{\beta} > 2 \). For any \( \mu \leq \left( \frac{a}{\sigma} - 1 \right)^2 \frac{\sigma}{a} \), there exists a constant \( C > 0 \) such that for any \( k \in \{0, \ldots, n - 1\}, x \in \mathbb{R}^n_+ \) and \( t \in [t_k, t_{k+1}] \),

\[ \tilde{E}_{t_k,x}^{a,b} \left[ \exp \left\{ \mu \int_{t_k}^t \frac{du}{Y_{u}^{a,b}(t_k, x)} \right\} \right] \leq C \left( 1 + \frac{1}{x^{(\frac{a}{\beta} - 1)}(t_{k+1})} \right). \]

Moreover, the statement remains valid for \( Y_{t}^{a,b} \).
The proof for $b \geq 0$ is given in Lemma A.2 of Bossy and Diop (2007). The proof for $b < 0$ is based on the Comparison Theorem.

As mentioned before, our techniques are based on the Malliavin calculus, which will require some $L^p$-norm estimation for positive and negative polynomial moments of the flow process as well as the Malliavin derivative of the flow process and its inverse.

**Lemma 6.** Assume condition $a \geq \sigma$. For all $p \geq 1$, there exists $C > 0$ such that

$$
\tilde{E}_{t_k,x}[^{|Y_t^{a,b}(t_k,x)|^p}] \leq C(1 + x^p).
$$

For all $p \in [0, \frac{a}{\sigma} - 1)$, there exists $C > 0$ such that

$$
\tilde{E}_{t_k,x}[^{\frac{1}{|Y_t^{a,b}(t_k,x)|^p}}] \leq \frac{C}{x^p}.
$$

For all $p \geq -\frac{(\frac{a}{\sigma} - 1)^2}{2(\frac{a}{\sigma} - 1)^2}$, there exists $C > 0$ such that

$$
\tilde{E}_{t_k,x}[^{|\partial_x Y_t^{a,b}(t_k,x)|^p}] \leq C\left(1 + \frac{1}{x^{\frac{a}{\sigma} - 1}}\right),
$$

for any $k \in \{0, \ldots, n - 1\}$, $x \in \mathbb{R}^*_+$ and $t \in [t_k, t_{k+1}]$.

Now, let us recall that under condition $a \geq \sigma$, for any $t \in [t_k, t_{k+1}]$, the random variable $Y_t^{a,b}(t_k,x)$ belongs to $D^{1,2}$ (corollary 4.2 of Alòs and Ewald (2008)). From (30) and the chain rule of the Malliavin calculus, its corresponding Malliavin derivative satisfies the following equation

$$
D_s Y_t^{a,b}(t_k,x) = \sqrt{2 \sigma} Y_s^{a,b}(t_k,x) - b \int_s^t D_s Y_u^{a,b}(t_k,x)du + \int_s^t \frac{\sqrt{\sigma}}{2 Y_u^{a,b}(t_k,x)} D_s Y_u^{a,b}(t_k,x)dW_u,
$$

for $s \leq t$, and $D_s Y_t^{a,b}(t_k,x) = 0$ for $s > t$. Furthermore, using the same arguments as (2.59) in Nualart (2006), the Malliavin derivative $D_s Y_t^{a,b}(t_k,x)$ can be rewritten as

$$
D_s Y_t^{a,b}(t_k,x) = \sqrt{2 \sigma} Y_s^{a,b}(t_k,x) \partial_x Y_t^{a,b}(t_k,x) (\partial_x Y_s^{a,b}(t_k,x))^{-1} 1_{[t_k,t]}(s).
$$

On the other hand, by condition $a \geq \sigma$ and corollary 4.2 of Alòs and Ewald (2008), its explicit expression is given by

$$
D_s Y_t^{a,b}(t_k,x) = \sqrt{2 \sigma} Y_t^{a,b}(t_k,x) \exp \left\{ \int_s^t \left( -\frac{b}{2} - \left( \frac{a}{2} - \frac{\sigma}{4} \right) \frac{1}{Y_u^{a,b}(t_k,x)} \right) du \right\} 1_{[t_k,t]}(s).
$$

The Malliavin differentiability of the flow process and its inverse is guaranteed by the following lemma.

**Lemma 7.** Let $(a,b) \in \Theta \times \Sigma$, $k \in \{0, \ldots, n - 1\}$, $t \in [t_k, t_{k+1}]$, and $x \in \mathbb{R}^*_+$. 
(i) Assume condition \( \frac{a}{\sigma} > 4 \). Then, \( \partial_x Y_t^{a,b}(t_k, x) \in \mathbb{D}_{1,2} \). Furthermore,

\[
D_s(\partial_x Y_t^{a,b}(t_k, x)) = \partial_x Y_t^{a,b}(t_k, x) \left( \frac{\sqrt{\sigma}}{2} \frac{1}{\sqrt{Y_s^{a,b}(t_k, x)}} + \frac{\sigma}{4} \int_s^t \frac{1}{(Y_u^{a,b}(t_k, x))^2} D_s Y_u^{a,b}(t_k, x) du \right) \\
- \frac{1}{2} \frac{\sqrt{\sigma}}{2} \int_s^t \frac{1}{(Y_u^{a,b}(t_k, x))^2} D_s Y_u^{a,b}(t_k, x) dW_u \right) 1_{[t_k, t)}(s). \quad (45)
\]

(ii) Assume condition \( \frac{a}{\sigma} > \frac{9 + \sqrt{57}}{2} \). Then, \( (\partial_x Y_t^{a,b}(t_k, x))^{-1} \in \mathbb{D}_{1,2} \) and \( \frac{Y_t^{a,b}(x)}{\partial_x Y_t^{a,b}(t_k, x)} \in \mathbb{D}_{1,2} \). Furthermore,

\[
D_s \left( \frac{1}{\partial_x Y_t^{a,b}(t_k, x)} \right) = - \frac{1}{(\partial_x Y_t^{a,b}(t_k, x))^2} D_s(\partial_x Y_t^{a,b}(t_k, x)) 1_{[t_k, t)}(s), \quad (46)
\]

\[
D_s \left( \frac{Y_t^{a,b}(t_k, x)}{\partial_x Y_t^{a,b}(t_k, x)} \right) = \frac{1}{\partial_x Y_t^{a,b}(t_k, x)} D_s Y_t^{a,b}(t_k, x) + Y_t^{a,b}(t_k, x) D_s \left( \frac{1}{\partial_x Y_t^{a,b}(t_k, x)} \right). \quad (47)
\]

For any \( t > s \), the law of \( Y_t^{a,b} \) conditioned on \( Y_s^{a,b} = x \) possesses the transition density \( p^{a,b}(t - s, x, y) \). Following proposition 4.1 in Gobet (2001), the score functions are expressed in terms of a conditional expectation of a Skorohod integral.

**Proposition 1.** Assume condition \( \frac{a}{\sigma} > \frac{9 + \sqrt{57}}{2} \). Then, for all \( k \in \{0, \ldots, n - 1\} \), \( (a, b) \in \Theta \times \Sigma \), \( \beta \in \{a, b\} \) and \( x, y \in \mathbb{R}_+^* \),

\[
\frac{\partial_\beta p^{a,b}}{p^{a,b}}(\Delta_n, x, y) = \frac{1}{\Delta_n} E_{t_k, x}^a \left[ \delta(\partial_\beta Y_{t_{k+1}}^{a,b}(t_k, x) U^{a,b}(t_k, x)) | Y_{t_{k+1}}^{a,b} = y \right],
\]

where \( U^{a,b}(t_k, x) := (U_t^{a,b}(t_k, x), t \in [t_k, t_{k+1})] \) with \( U_t^{a,b}(t_k, x) := (D_t Y_{t_{k+1}}^{a,b}(t_k, x))^{-1} \). Furthermore, under condition \( \frac{a}{\sigma} > 5 + 3 \sqrt{2} \), the Skorohod integrals are decomposed as follows

\[
\delta(\partial_\alpha Y_{t_{k+1}}^{a,b}(t_k, x)) U^{a,b}(t_k, x)) = \frac{\Delta_n}{\sqrt{2} \sigma x} (W_{t_{k+1}} - W_{t_k}) + R_1^{a,b} + R_2^{a,b} + R_3^{a,b}, \quad (48)
\]

\[
\delta(\partial_\beta Y_{t_{k+1}}^{a,b}(t_k, x)) U^{a,b}(t_k, x))) = - \frac{\Delta_n}{\sqrt{2} \sigma} x (W_{t_{k+1}} - W_{t_k}) - x R_1^{a,b} + H_2^{a,b} + H_3^{a,b}, \quad (49)
\]

where

\[
R_1^{a,b} = R_1^{a,b}(t_k, x) = \Delta_n \int_{t_k}^{t_{k+1}} \left( \frac{\partial_x Y_s^{a,b}(t_k, x)}{\sqrt{2} \sigma Y_s^{a,b}(t_k, x)} - \frac{\partial_x Y_{t_k}^{a,b}(t_k, x)}{\sqrt{2} \sigma Y_{t_k}^{a,b}(t_k, x)} \right) dW_s,
\]

\[
R_2^{a,b} = R_2^{a,b}(t_k, x) = \int_{t_k}^{t_{k+1}} \left( \frac{1}{\partial_x Y_{t_k}^{a,b}(t_k, x)} - \frac{1}{\partial_x Y_{t_k}^{a,b}(t_k, x)} \right) dW_s \int_{t_k}^{t_{k+1}} \frac{\partial_x Y_{t_k}^{a,b}(t_k, x)}{\sqrt{2} \sigma Y_{t_k}^{a,b}(t_k, x)} dW_s,
\]

\[
R_3^{a,b} = R_3^{a,b}(t_k, x) = \Delta_n \int_{t_k}^{t_{k+1}} \left( \frac{\partial_x Y_{t_k}^{a,b}(t_k, x)}{\sqrt{2} \sigma Y_{t_k}^{a,b}(t_k, x)} - \frac{\partial_x Y_{t_k}^{a,b}(t_k, x)}{\sqrt{2} \sigma Y_{t_k}^{a,b}(t_k, x)} \right) dW_s.
\]
\[ R_3^{a,b} = R_3^{a,b}(t_k, x) = -\int_{t_k}^{t_{k+1}} \int_s^{t_{k+1}} D_3 \left( \frac{1}{\partial_x Y_r^{a,b}(t_k, x)} \right) dr \frac{\partial_x Y_s^{a,b}(t_k, x)}{2\sigma Y_s^{a,b}(t_k, x)} ds, \]

\[ H_2^{a,b} = H_2^{a,b}(t_k, x) = -\int_{t_k}^{t_{k+1}} \left( \frac{Y_r^{a,b}(t_k, x)}{\partial_x Y_r^{a,b}(t_k, x)} - \frac{Y_{t_k}^{a,b}(t_k, x)}{\partial_x Y_{t_k}^{a,b}(t_k, x)} \right) dr \int_{t_k}^{t_{k+1}} \frac{\partial_x Y_s^{a,b}(t_k, x)}{2\sigma Y_s^{a,b}(t_k, x)} dW_s, \]

\[ H_3^{a,b} = H_3^{a,b}(t_k, x) = \int_{t_k}^{t_{k+1}} \int_s^{t_{k+1}} D_3 \left( \frac{Y_r^{a,b}(t_k, x)}{\partial_x Y_r^{a,b}(t_k, x)} \right) dr \frac{\partial_x Y_s^{a,b}(t_k, x)}{2\sigma Y_s^{a,b}(t_k, x)} ds. \]

The following crucial estimates for the terms appearing in Proposition 1 will be needed.

**Lemma 8.** Let \((a, b) \in \Theta \times \Sigma, k \in \{0, \ldots, n - 1\}, \) and \(x \in \mathbb{R}_+^n.\)

(i) Under condition \(\frac{a}{\sigma} > 5 + 3\sqrt{2},\) we have

\[ E_{t_k, x}^{a,b}[R_1^{a,b} + R_2^{a,b} + R_3^{a,b}] = 0, \tag{50} \]

\[ E_{t_k, x}^{a,b}[-xR_1^{a,b} + H_2^{a,b} + H_3^{a,b}] = 0. \tag{51} \]

(ii) Let \(q \in \left[1, \frac{13 + \sqrt{69}}{20}\right]\) and assume condition (A). Then there exists a constant \(C > 0\) such that

\[ E_{t_k, x}^{a,b}[|R_1^{a,b} + R_2^{a,b} + R_3^{a,b}|^2] \leq C \Delta_n^4 q \left( \frac{1}{x^q} + \frac{1}{x^{\frac{q}{2}}} \right), \tag{52} \]

\[ E_{t_k, x}^{a,b}[|xR_1^{a,b} + H_2^{a,b} + H_3^{a,b}|^2] \leq C \Delta_n^4 (1 + x^q) \left( \frac{1}{x^q} + \frac{1}{x^{\frac{q}{2}}} \right). \tag{53} \]

5 PROOF OF LEMMA 3

**Proof.** To avoid confusion with the observed process \(X_1^{a,b}\) generated by the Brownian motion \(B,\) we introduce on the probability space \((\hat{\Omega}, \hat{P}, \hat{P})\) an independent copy \(Y_1 = (Y_1^{a,b})_{t \geq 0}\) of \(X_1^{a,b},\) which is associated to the Brownian motion \(W\) independent of \(B\) and which is defined by (29). Then \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) denotes the product filtered probability space of \((\hat{\Omega}, \hat{P}, \{\hat{P}_t\}_{t \geq 0}, \hat{P})\) and \((\hat{\Omega}, \hat{P}, \{\hat{P}_t\}_{t \geq 0}, \hat{P}),\) that is, \(\Omega = \hat{\Omega} \times \hat{\Omega}, \mathcal{F} = \hat{\mathcal{F}} \otimes \hat{\mathcal{F}}, P = \hat{P} \otimes \hat{P}, F_t = \hat{F}_t \otimes \hat{F}_t,\) where \(E, \hat{E}, \tilde{E}\) denote the expectation w.r.t. \(P, \hat{P}\) and \(\tilde{P}.

Using the decomposition (14) and Proposition 1, under condition \(\frac{a}{\sigma} > 5 + 3\sqrt{2},\) we obtain

\[ \sum_{k=0}^{n-1} (R_{k,n} + H_{k,n}) \]

\[ = \sum_{k=0}^{n-1} u_{1,n}(a_0, b_0) \int_0^1 \frac{\partial_d p_{a_d}(\xi, b_0)}{p_{a_d}(\xi, b_0)} (\Delta_n, X_{t_k}^{a_0, b_0}, X_{t_{k+1}}^{a_0, b_0}) d\xi \]

\[ + \sum_{k=0}^{n-1} u_{2,n}(a_0, b_0) \int_0^1 \frac{\partial_d p_{a_d}(\xi, b_0)}{p_{a_d}(\xi, b_0)} (\Delta_n, X_{t_k}^{a_0, b_0}, X_{t_{k+1}}^{a_0, b_0}) d\xi - \sum_{k=0}^{n-1} (\xi_{k,n} + \eta_{k,n}) \]
\[
= \sum_{k=0}^{n-1} u \varphi_1(n, b_0) \frac{\Delta_n}{\Delta_n} \int_0^1 E_{a(\varepsilon), b_0}^{(a, b_0)} \left[ \frac{\Delta_n}{2\sigma X^{a_0,b_0}_{t_k}} (W_{t_{k+1}} - W_{t_k}) + R^{a(\varepsilon), b_0}_{t_k} \right] Y^{a(\varepsilon), b_0}_{t_{k+1}} = X^{a_0,b_0}_{t_{k+1}} d\varepsilon
+ \sum_{k=0}^{n-1} v \varphi_2(n, b_0) \frac{\Delta_n}{\Delta_n} \int_0^1 E_{a(\varepsilon), b_0}^{(a, b_0)} \left[ \frac{\Delta_n}{2\sigma X^{a_0,b_0}_{t_k}} (W_{t_{k+1}} - W_{t_k}) \right] Y^{a_0,b_0}_{t_{k+1}} d\varepsilon - \sum_{k=0}^{n-1} \mathcal{F}_{k,n} + \eta_{k,n} \]

where

\[
R^{a(\varepsilon), b_0} = R_1^{a(\varepsilon), b_0} + R_2^{a(\varepsilon), b_0} + R_3^{a(\varepsilon), b_0}, \quad H^{a(\varepsilon), b_0} = -X^{a_0,b_0}_{t_k} R_1^{a_0,b_0} + H_2^{a_0,b_0} + H_3^{a_0,b_0}. \tag{54}
\]

Then, from Equation (29), we write

\[
W_{t_{k+1}} - W_{t_k} = \frac{1}{\sqrt{2\sigma Y^{a,b}_{t_k}}} \left( Y^{a,b}_{t_{k+1}} - Y^{a,b}_{t_k} - (a - bY^{a,b}_{t_k})\Delta_n + b \int_{t_k}^{t_{k+1}} (Y^{a,b}_s - Y^{a,b}_{t_k}) ds \right)
- \int_{t_k}^{t_{k+1}} \left( \sqrt{2\sigma Y^{a,b}_s} - \sqrt{2\sigma Y^{a,b}_{t_k}} \right) dW_s.
\]

This implies that

\[
\sum_{k=0}^{n-1} (R_{k,n} + H_{k,n})
= \sum_{k=0}^{n-1} u \varphi_1(n, b_0) \frac{\Delta_n}{\Delta_n} \int_0^1 E_{a(\varepsilon), b_0}^{(a, b_0)} \left[ \frac{\Delta_n}{2\sigma X^{a_0,b_0}_{t_k}} - \frac{1}{2\sigma Y^{a(\varepsilon), b_0}_{t_k}} \left( Y^{a(\varepsilon), b_0}_{t_{k+1}} - Y^{a(\varepsilon), b_0}_{t_k} \right) \right.
\times \left. \left( Y^{a(\varepsilon), b_0}_{t_{k+1}} - Y^{a(\varepsilon), b_0}_{t_k} - (a(\varepsilon) - b(\varepsilon)Y^{a(\varepsilon), b_0}_{t_k})\Delta_n + b \int_{t_k}^{t_{k+1}} (Y^{a,b}_s - Y^{a,b}_{t_k}) ds \right) \right]
- \int_{t_k}^{t_{k+1}} \left( \sqrt{2\sigma Y^{a,b}_s} - \sqrt{2\sigma Y^{a,b}_{t_k}} \right) dW_s + R^{a(\varepsilon), b_0}_{t_k} [Y^{a(\varepsilon), b_0}_{t_{k+1}} = X^{a_0,b_0}_{t_{k+1}}] d\varepsilon
+ \sum_{k=0}^{n-1} v \varphi_2(n, b_0) \frac{\Delta_n}{\Delta_n} \int_0^1 E_{a(\varepsilon), b_0}^{(a, b_0)} \left[ \frac{\Delta_n}{2\sigma X^{a_0,b_0}_{t_k}} - \frac{1}{2\sigma Y^{a(\varepsilon), b_0}_{t_k}} \left( Y^{a,a_0,b(\varepsilon)}_{t_{k+1}} - Y^{a,a_0,b(\varepsilon)}_{t_k} \right) \right.
- (a - b(\varepsilon)Y^{a,a_0,b(\varepsilon)}_{t_k})\Delta_n + b(\varepsilon) \int_{t_k}^{t_{k+1}} (Y^{a,b}_s - Y^{a,b}_{t_k}) ds \right]
- \int_{t_k}^{t_{k+1}} \left( \sqrt{2\sigma Y^{a,b}_s} - \sqrt{2\sigma Y^{a,a_0,b(\varepsilon)}_{t_k}} \right) dW_s + H^{a(\varepsilon), b(\varepsilon)}_{t_k} [Y^{a,a_0,b(\varepsilon)}_{t_{k+1}} = X^{a_0,b_0}_{t_{k+1}}] d\varepsilon - \sum_{k=0}^{n-1} \mathcal{F}_{k,n} + \eta_{k,n} \]
\[
\sum_{k=0}^{n-1} u \varphi_{1,n}(a_0, b_0) \int_0^1 \left\{ \frac{\Delta_n}{2 \sigma X_{t_k}^a b_0} (X_{t_{k+1}}^a b_0 - X_{t_k}^a b_0) - (a(\varepsilon) - b_0 X_{t_k}^a b_0) \Delta_n \right\} \, d\varepsilon + \sum_{k=0}^{n-1} \varphi_{2,n}(a_0, b_0) \Delta_n \\
\times \int_0^1 \left\{ -\frac{\Delta_n}{2 \sigma} (X_{t_{k+1}}^a b_0 - X_{t_k}^a b_0) - (a_n - b(\varepsilon) X_{t_k}^a b_0) \Delta_n \right. \\
+ \left. X_{t_k}^a b_0 \left[ R_4 a_n, b(\varepsilon) + H a_n, b(\varepsilon) | Y_{t_{k+1}}^{a_n, b(\varepsilon)} = X_{t_{k+1}}^a b_0 \right] \right\} \, d\varepsilon - \sum_{k=0}^{n-1} (\xi_{k,n} + \eta_{k,n}),
\]

where

\[
R_4^{a,b} = -\frac{\Delta_n}{2 \sigma X_{t_k}^a b_0 Y_{t_k}^a b} \int_{t_k}^{t_{k+1}} (Y_{s}^a b - Y_{t_k}^a b) \, ds,
\]

\[
R_5^{a,b} = \frac{\Delta_n}{2 \sigma X_{t_k}^a b_0 Y_{t_k}^a b} \int_{t_k}^{t_{k+1}} (\sqrt{Y_{s}^a b} - \sqrt{Y_{t_k}^a b}) \, dW_s.
\]

Next, using Equation (1), we write

\[
X_{t_{k+1}}^a b_0 - X_{t_k}^a b_0 = \sqrt{2 \sigma X_{t_k}^a b_0 (B_{t_{k+1}} - B_{t_k})} + (a_0 - b_0 X_{t_k}^a b_0) \Delta_n \\
- b_0 \int_{t_k}^{t_{k+1}} (X_{s}^a b - X_{t_k}^a b) \, ds + \int_{t_k}^{t_{k+1}} (\sqrt{2 \sigma X_{s}^a b_0} - \sqrt{2 \sigma X_{t_k}^a b_0}) \, dB_s.
\]

Consequently, using the definition of \( \xi_{k,n}, \eta_{k,n} \), we get

\[
\sum_{k=0}^{n-1} (R_{k,n} + H_{k,n})
= \sum_{k=0}^{n-1} \frac{u \varphi_{1,n}(a_0, b_0)}{\Delta_n} \int_0^1 \left\{ R_6^{a_0, b_0} + R_4^{a_0, b_0} + \bar{E}_{X_{t_k}^a b_0}^{a(\varepsilon), b_0} \right. \\
\left. - R_4^{a(\varepsilon), b_0} - R_5^{a(\varepsilon), b_0} \right\} \, d\varepsilon + \sum_{k=0}^{n-1} \varphi_{2,n}(a_0, b_0) \int_0^1 \left\{ -X_{t_k}^a b_0 R_6^{a_0, b_0} - X_{t_k}^a b_0 R_7^{a_0, b_0} \\
+ \bar{E}_{X_{t_k}^a b_0}^{a(\varepsilon), b_0} \left| X_{t_{k+1}}^{a_0, b_0} R_4^{a(\varepsilon), b_0} + X_{t_k}^a b_0 R_5^{a(\varepsilon), b_0} + H a_n, b(\varepsilon) | Y_{t_{k+1}}^{a_n, b(\varepsilon)} = X_{t_{k+1}}^a b_0 \right\} \right\} \, d\varepsilon,
\]

where

\[
R_6^{a_0, b_0} = -\frac{\Delta_n}{2 \sigma X_{t_k}^a b_0 b_0} \int_{t_k}^{t_{k+1}} (X_{s}^a b_0 - X_{t_k}^a b_0) \, ds,
\]

\[
R_7^{a_0, b_0} = \frac{\Delta_n}{2 \sigma X_{t_k}^a b_0 Y_{t_k}^a b} \int_{t_k}^{t_{k+1}} (\sqrt{X_{s}^a b_0} - \sqrt{X_{t_k}^a b_0}) \, dB_s.
\]
Remark that in the supercritical case with \( \varphi_{1,n}(a_0,b_0) = 0, \sum_{k=0}^{n-1}(R_{k,n} + H_{k,n}) \) contains only the second sum and \( a_n = a_0 \). This decomposition, together with Lemmas 11–16 below, finishes the proof of Lemma 3.

Now, it remains to prove Lemmas 11–16 below. To do so, we need the following preliminary part. Let \( \hat{P}_{t_k}^{a,b} \) and \( \hat{E}_{t_k}^{a,b} \) denote respectively the probability law of \( X_t^{a,b} \) starting at \( x \) at time \( t_k \) and the expectation w.r.t. \( \hat{P}_{t_k}^{a,b} \), that is, \( \hat{P}_{t_k}^{a,b}(A) = \hat{E}[1_A|X_{t_k}^{a,b} = x] \) for all \( A \in \hat{\mathcal{F}} \) and \( \hat{E}_{t_k}^{a,b}[V] = \hat{E}[V|X_{t_k}^{a,b} = x] \) for all \( \hat{\mathcal{F}} \) - measurable random variables \( V \). We introduce change of measures on \( I_k := [t_k, t_{k+1}] \). For all \( a,a_1 \in \mathbb{R}_+^*, b,b_1 \in \mathbb{R}, x \in \mathbb{R}_+^* \) and \( k \in \{0, \ldots, n-1\} \), by lemma 3.1 in Barczy and Pap (2016), for any \( a > \sigma \) the probability measures \( \hat{P}_{t_k}^{a,b} \) and \( \hat{P}_{t_k}^{a,b} \) are absolutely continuous w.r.t. each other and its Radon-Nikodym derivative is given by

\[
\frac{d\hat{P}_{t_k}^{a,b}}{d\hat{P}_{t_k}^{a,b}}((X_t^{a,b})_{t \in I_k}) = \exp \left\{ \int_{t_k}^{t_{k+1}} \frac{a_1 - a - (b_1 - b)X_s^{a,b}}{2 \sigma X_s^{a,b}} dX_s - \frac{1}{2} \int_{t_k}^{t_{k+1}} \frac{(a_1 - b_1 X_s^{a,b})^2 - (a - b X_s^{a,b})^2}{2 \sigma X_s^{a,b}} ds \right\}. \tag{55}
\]

To prove Lemma 3, we need the following useful lemma related to the change of measures. For this, let \( V \) denote a \( \hat{F}_{t_{k+1}} \) - measurable random variable which can be \( R_{1,b}^a, (R_{a,b}^a)^2, \hat{H}_{a,b}^a, (H_{a,b}^a)^2, R_{k,4}^a, (R_{k,4}^a)^2, R_{k,5}^a, (R_{k,5}^a)^2 \), and let \( \hat{V} \) be a \( \hat{F}_{t_{k+1}} \) - measurable random variable which can be \( R_{k,6}^a b^a \).

**Lemma 9.** Assume condition \( a > \sigma \). Then, for any \( k \in \{0, \ldots, n-1\} \) and \( x \in \mathbb{R}_+^* \),

\[
\hat{E}_{t_k}^{a,b}[\hat{E}_{t_k}^{b} [V|Y_{t_{k+1}}^{a,b} = x]] = \hat{E}_{t_k}^{a,b} [V] + \hat{E}_{t_k}^{a,b} \left[ \hat{E}_{t_k}^{a,b} [V|Y_{t_{k+1}}^{a,b}] \left( \frac{dP_{t_k}^{a,b}}{dP_{t_k}^{a,b}}((Y_{t_{k+1}}^{a,b})_{t \in I_k}) - 1 \right) \right]. \tag{56}
\]

Similarly, we have

\[
\hat{E}_{t_k}^{a,b}[\hat{V}] = \hat{E}_{t_k}^{a,b}[\hat{V}] + \hat{E}_{t_k}^{a,b} \left[ \hat{V} \left( \frac{dP_{t_k}^{a,b}}{dP_{t_k}^{a,b}}((X_{t_{k+1}}^{a,b})_{t \in I_k}) - 1 \right) \right]. \tag{57}
\]

**Proof.** For simplicity, we set \( g(x,y) := \hat{E}_{t_k}^{a,b}[V|Y_{t_{k+1}}^{a,b} = y] \) for all \( y \in \mathbb{R}_+^* \). Applying change of measures, we have that

\[
\hat{E}_{t_k}^{a,b}[\hat{E}_{t_k}^{a,b}[V|Y_{t_{k+1}}^{a,b} = X_{t_{k+1}}^{a,b}]] = \hat{E}_{t_k}^{a,b}[g(x, X_{t_{k+1}}^{a,b})] = \hat{E}_{t_k}^{a,b} \left[ g(x, X_{t_{k+1}}^{a,b}) \left( \frac{dP_{t_k}^{a,b}}{dP_{t_k}^{a,b}}((X_{t_{k+1}}^{a,b})_{t \in I_k}) - 1 \right) \right]
\]

\[
= \hat{E}_{t_k}^{a,b} \left[ g(x, Y_{t_{k+1}}^{a,b}) \left( \frac{dP_{t_k}^{a,b}}{dP_{t_k}^{a,b}}((Y_{t_{k+1}}^{a,b})_{t \in I_k}) - 1 \right) \right]
\]

\[
= \hat{E}_{t_k}^{a,b} \left[ \hat{E}_{t_k}^{a,b} \left[ V \left( \frac{dP_{t_k}^{a,b}}{dP_{t_k}^{a,b}}((X_{t_{k+1}}^{a,b})_{t \in I_k}) - 1 \right) \right] \right].
\]
where we have used the fact that $Y^{a,b}$ is the copy of $X^{a,b}$. Thus, (56) follows. The equality (57) is proven using the same arguments. ■

Next, the following lemma gives the estimate for the second term in Lemma 9.

**Lemma 10.** Let $(a, b) = (a(\epsilon), b_0)$ or $(a, b) = (a_n, b(\epsilon))$, $q_1$ be large enough, and $p, q > 1$ satisfy that $\frac{1}{p} + \frac{1}{q} = 1$ and $p < \frac{q}{\sigma} - 1$. Then, there exist constants $C_1, C_2 > 0$ such that for any $k \in \{0, \ldots, n-1\}$, $x \in \mathbb{R}_+^n$ and $n$ large enough,

$$
\left| \hat{E}_{t_k,x} \left[ \tilde{E}_{t_{k+1},x} [V] Y_{t_{k+1}}^{a,b} \right] \left( \frac{d\hat{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^{a,b}} (Y_t^{a,b})_{t \in I_k} - 1 \right) \right| 
\leq C_1 \sqrt{\Delta_n} (\tilde{E}_{t_{k+1},x}[V])^{1/2} e^{C_2(b_0-b)^{2}\Delta_n x} \left| \frac{1}{\sqrt{x}} + \sqrt{\Delta_n x} |a - a_0| + \sqrt{\Delta_n} |b - b_0| \right| 
\times \left( 1 + \frac{1}{x^{\sigma_1/2}} \right) + |b - b_0|(1 + \sqrt{x} + \sqrt{\Delta_n} |b - b_0|x) .
$$

(58)

Moreover, assume that $p < 2(\frac{\sigma}{\sigma} - 1)$. Then, there exist constants $C_1, C_2 > 0$ such that for any $k \in \{0, \ldots, n-1\}$ and $x \in \mathbb{R}_+^n$,

$$
\left| \hat{E}_{t_k,x} \left[ \tilde{V} \left( \frac{d\hat{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^{a,b}} (Y_t^{a,b})_{t \in I_k} - 1 \right) \right] \right| 
\leq C_1 \frac{\sqrt{\Delta_n}}{\sqrt{x}} \left| \int_a^{a_0} \left( \tilde{E}_{t_k,x}[V]^{1/2} \right) \int_b^{b_0} \left( \tilde{E}_{t_{k+1},x}[V]^{1/2} \right) \right| 
+ C_2 \sqrt{\Delta_n} (1 + \sqrt{x}) \left| \int_b^{b_0} \left( \tilde{E}_{t_{k+1},x}[V]^{1/2} \right) \right| .
$$

(59)

**Proof.** Using (55) for $Y$, we have the following decomposition

$$
\frac{d\hat{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^{a,b}} (Y_t^{a,b})_{t \in I_k} - 1 = \int_a^{a_0} \frac{\partial}{\partial \alpha} \left( \frac{d\hat{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^{a,b}} \right) (Y_t^{a,b})_{t \in I_k} d\alpha + \int_b^{b_0} \frac{\partial}{\partial \beta} \left( \frac{d\hat{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^{a,b}} \right) (Y_t^{a,b})_{t \in I_k} d\beta 
= \int_a^{a_0} \int_{t_k}^{t_{k+1}} \frac{1}{2\sigma Y_s^{a,b}} (dY_s^{a,b} - (\alpha - b_0 Y_s^{a,b}) ds) \frac{d\hat{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^{a,b}} (Y_t^{a,b})_{t \in I_k} d\alpha 
- \int_b^{b_0} \int_{t_k}^{t_{k+1}} \frac{1}{2\sigma} (dY_s^{a,b} - (\alpha - \beta Y_s^{a,b}) ds) \frac{d\hat{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^{a,b}} (Y_t^{a,b})_{t \in I_k} d\beta .
$$

(60)
Applying Equation (29), Hölder’s inequality with $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$, BDG’s and Jensen’s inequalities, (39) and (40) for $Y^{a,b}$, we get that

$$
\left| E_{t_k,x}^{a,b}\left[ E_{t_k,x}^b[V|Y_{t_{k+1}}^{a,b}] \left( \frac{d\tilde{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^a}((Y_{t_k}^{a,b})_{t\in I_{k}}) - 1 \right) \right]\right|
\leq \int_{a}^{b} E_{t_k,x}^{a,b}\left[ E_{t_k,x}^b[V|Y_{t_{k+1}}^{a,b}] \right] \left| \int_{t_k}^{t_{k+1}} \frac{1}{2\sigma Y_s^{a,b}} \left( \sqrt{2\sigma Y_s^{a,b}} dW_s + (a - \alpha - (b - b_0) Y_s^{a,b}) ds \right) \right|
\times \frac{d\tilde{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^a}(Y_t^{a,b})_{t\in I_{k}} \right| \right| \times \frac{1}{p_1} \right| \right|^{p_1} \frac{1}{q_1} \right|
\leq C_1 \left| \int_{a}^{b} \left( \frac{d\tilde{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^a}((Y_{t_k}^{a,b})_{t\in I_{k}}) \right)^{\frac{1}{q_1}} \sqrt{\Delta_n} \left( \frac{1}{\sqrt{x}} \right) \right|
+ \frac{\sqrt{\Delta_n}}{x} |a - a_0| + \sqrt{\Delta_n} |b - b_0| \right| \left( \frac{d\tilde{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^a}((Y_{t_k}^{a,b})_{t\in I_{k}}) \right)^{\frac{1}{q_1}} \left| \right| \left| \right|
+ C_2 \left| \int_{b}^{b_0} \left( \frac{d\tilde{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^a}((Y_{t_k}^{a,b})_{t\in I_{k}}) \right)^{\frac{1}{q_1}} \sqrt{\Delta_n} (1 + \sqrt{x} + \sqrt{\Delta_n} |b - b_0| x) \right|
\times \left( \frac{d\tilde{P}_{t_k,x}^{a,b}}{d\tilde{P}_{t_k,x}^a}((Y_{t_k}^{a,b})_{t\in I_{k}}) \right)^{\frac{1}{q_1}} \right| \right|
\left| \right|^{q_1} \right|

(61)

for some constants $C_1, C_2 > 0$, where $p_1$ is close to 1 and $p < \frac{a}{\sigma} - 1$. Then, using Cauchy–Schwarz inequality twice and Lemma 5, provided that we have an exponential martingale, for $n$ large enough we get
Next, using \((60)\) for some constants 
\(forsomeconstants\)

\[
\leq \left( \hat{E}_{t_k,x}^{a,b} \left[ \exp \left\{ p_{q_1}(2p_{q_1} - 1) \frac{1}{\sigma} \left( (a - a)^2 \int_{t_k}^{t_{k+1}} \frac{ds}{Y_{s}^{a,b}} + (b_0 - b)^2 \int_{t_k}^{t_{k+1}} Y_{s}^{a,b} \, ds \right) \right\} \right] \right)^{\frac{1}{2}}\]

\[
\leq C \left( 1 + \frac{1}{\chi^{\frac{1}{a-1}}} \right) \left( \hat{E}_{t_k,x}^{a,b} \left[ \exp \left\{ 2p_{q_1}(2p_{q_1} - 1) \frac{1}{\sigma}(b_0 - b)^2 \int_{t_k}^{t_{k+1}} Y_{s}^{a,b} \, ds \right\} \right] \right)^{\frac{1}{2}}.
\]

where we have used the fact that 
\(2p_{q_1}(2p_{q_1} - 1)\frac{1}{\sigma}(a - a)^2 \leq 2p_{q_1}(2p_{q_1} - 1)\frac{1}{\sigma}(a_0 - a)^2 \leq 2p_{q_1}(2p_{q_1} - 1)\frac{1}{\sigma}u^2\varphi_{2,n}(a_0, b_0) \leq \left( \frac{\varphi}{\sigma} \right)^2 \frac{c}{a} \) for \(n\) large enough since \(\varphi_{1,n}(a_0, b_0) \to 0\).

Next, using Jensen’s inequality and (1.9) of proposition 1.2.4 in Alfonsi (2015), for \(n\) large enough we get

\[
\leq \left( \hat{E}_{t_k,x}^{a,b} \left[ \exp \left\{ p_{q_1}(2p_{q_1} - 1)\frac{1}{\sigma}(b_0 - b)^2 \sum_{n=1}^{\infty} \left( \int_{t_k}^{t_{k+1}} Y_{s}^{a,b} \, ds \right) \right\} \right] \right)^{\frac{1}{2}}\]

\[
\leq C \left( 1 + \frac{1}{\chi^{\frac{1}{a-1}}} \right) \left( \hat{E}_{t_k,x}^{a,b} \left[ \exp \left\{ 2p_{q_1}(2p_{q_1} - 1)\frac{1}{\sigma}(b_0 - b)^2 \sum_{n=1}^{\infty} \left( \int_{t_k}^{t_{k+1}} Y_{s}^{a,b} \, ds \right) \right\} \right] \right)^{\frac{1}{2}}\]

\[
= C \left( 1 + \frac{1}{\chi^{\frac{1}{a-1}}} \right) \left( \hat{E}_{t_0,x}^{a,b} \left[ \exp \left\{ 2p_{q_1}(2p_{q_1} - 1)\frac{1}{\sigma}(b_0 - b)^2 \sum_{n=1}^{\infty} \left( \int_{t_k}^{t_{k+1}} Y_{s}^{a,b} \, ds \right) \right\} \right] \right)^{\frac{1}{2}}\]

\[
\leq C \left( 1 + \frac{1}{\chi^{\frac{1}{a-1}}} \right) e^{c(b_0 - b)^2 \sum_{n=1}^{\infty} \left( \int_{t_k}^{t_{k+1}} Y_{s}^{a,b} \, ds \right)},
\]

(62)

for some constants \(C, c > 0\). The same arguments can be used to check Novikov’s condition and deduce the validity of the exponential martingale property used above.

Proceeding as for the term 
\(\frac{dP_{t_k,x}^{a,b}}{dP_{t_0,x}^{a,b}}((Y_{t}^{a,b})_{t \in t_k})\) above, for \(n\) large enough we obtain

\[
\leq \left( \hat{E}_{t_k,x}^{a,b} \left[ \exp \left\{ 2p_{q_1}(2p_{q_1} - 1)\frac{1}{\sigma}(b_0 - b)^2 \sum_{n=1}^{\infty} \left( \int_{t_k}^{t_{k+1}} Y_{s}^{a,b} \, ds \right) \right\} \right] \right)^{\frac{1}{2}}\]

(63)

for some constants \(C, c > 0\). Thus, (58) follows from (61), (62), and (63).

Next, using (60) for \(X_{t}^{a,b}\), change of measures, Equation (1), Hölder’s, Jensen’s and BDG’s inequalities, (39) and (40) for \(X_{t}^{a,b}\), we get that

\[
\left| \hat{E}_{t_k,x}^{a,b} \left[ \hat{V} \left( \frac{dP_{t_k,x}^{a,b}}{dP_{t_0,x}^{a,b}}((X_{t}^{a,b})_{t \in t_k}) - 1 \right) \right] \right|
\]

\[
= \int_{a}^{\alpha_0} \hat{E}_{t_k,x}^{a,b} \left[ \hat{V} \int_{t_k}^{t_{k+1}} \frac{dB_s}{\sqrt{2\sigma X_{s}^{a,b}}} \right] \, da - \int_{b}^{b_0} \hat{E}_{t_k,x}^{a,b} \left[ \hat{V} \int_{t_k}^{t_{k+1}} \frac{\sqrt{X_{s}^{a,b}}}{\sqrt{2\sigma}} \, dB_s \right] \, db
\]
Now, we start showing condition (i) of Lemma 18. Using (56) of Lemma 9 with $V$ for some constants $C$ applied to the random variable $\hat{V}_k$. Proof.

**Lemma 11.** Assume condition (A). Then, for both subcritical and critical cases as $n \to \infty$, we have

$$
\sum_{k=0}^{n-1} u\varphi_{1,n}(a_0, b_0) \sqrt{\Delta_n} \int_0^1 \left( \tilde{E}_{\hat{a},b_0} \sum_{k=0}^{n-1} \left[ \tilde{E}_{\hat{a},b_0} \sum_{k=0}^{n-1} \right] \right)^{\frac{1}{2}} d\varepsilon \to 0,
$$

where $R^{a(\varepsilon),b_0}$ is given by (54).

Proof. It suffices to show that conditions (i) and (ii) of Lemma 18 satisfy under the measure $\tilde{P}^{a_0,b_0}$ applied to the random variable

$$
\zeta_{k,n} := \frac{u \varphi_{1,n}(a_0, b_0)}{\Delta_n} \int_0^1 \tilde{E}_{\hat{a},b_0} \sum_{k=0}^{n-1} \left[ \tilde{E}_{\hat{a},b_0} \sum_{k=0}^{n-1} \right] d\varepsilon.
$$

Now, we start showing condition (i) of Lemma 18. Using (56) of Lemma 9 with $V = R^{a(\varepsilon),b_0}$, the fact that $\tilde{E}_{\hat{a},b_0} \sum_{k=0}^{n-1} \left[ \tilde{E}_{\hat{a},b_0} \sum_{k=0}^{n-1} \right] = 0$ by (50), (58) of Lemma 10 with $q = p = 2$ and $q_1$ large enough, (52) of Lemma 8 with $q = 1$, and condition (A), we obtain for $n$ large enough,

$$
\sum_{k=0}^{n-1} \left| \frac{1}{\sqrt{\Delta_n}} \int_0^1 \left( \tilde{E}_{\hat{a},b_0} \sum_{k=0}^{n-1} \right)^{\frac{1}{2}} d\varepsilon \right| \leq C \sum_{k=0}^{n-1} \frac{1}{\sqrt{X^{a_0,b_0}}} \sum_{k=0}^{n-1} \left| \frac{1}{\sqrt{X^{a_0,b_0}}} \sum_{k=0}^{n-1} \right| \int_0^1 \left( \tilde{E}_{\hat{a},b_0} \sum_{k=0}^{n-1} \right)^{\frac{1}{2}} d\varepsilon 
$$

$$
\times \left| u \varphi_{1,n}(a_0, b_0) \right| \frac{1}{\sqrt{X^{a_0,b_0}}} \sum_{k=0}^{n-1} \left| \frac{1}{\sqrt{X^{a_0,b_0}}} \sum_{k=0}^{n-1} \right| \int_0^1 \left( \tilde{E}_{\hat{a},b_0} \sum_{k=0}^{n-1} \right)^{\frac{1}{2}} d\varepsilon 
$$

$$
\leq Cu^2 \Delta_n^{\frac{3}{2}} \left( \varphi_{1,n}(a_0, b_0) \right)^2 \sum_{k=0}^{n-1} \left| \frac{1}{\sqrt{X^{a_0,b_0}}} \sum_{k=0}^{n-1} \right| \int_0^1 \left( \tilde{E}_{\hat{a},b_0} \sum_{k=0}^{n-1} \right)^{\frac{1}{2}} d\varepsilon 
$$

$$
+ \frac{1}{\left( X^{a_0,b_0} \right)^{1+\frac{1}{2q_1} \frac{\varphi_{1,n}(a_0, b_0)}{\Delta_n}}} \left| u \varphi_{1,n}(a_0, b_0) \sqrt{\Delta_n} \right| \frac{1}{\left( X^{a_0,b_0} \right)^{1+\frac{1}{2q_1} \frac{\varphi_{1,n}(a_0, b_0)}{\Delta_n}}} d\varepsilon.
$$
for some constant $C > 0$. Now, we use the fact $a_0 - |u| \phi_{1,n}(a_0, b_0) \leq a(\varepsilon) = a_0 + \sqrt{a|u\phi_n(a_0, b_0) \leq a_0 + |u| \phi_{1,n}(a_0, b_0)$ and $a_0 - |u| \phi_{1,n}(a_0, b_0) \rightarrow a_0, a_0 + |u| \phi_{1,n}(a_0, b_0) \rightarrow a_0$. Then for any $\varepsilon > 0$ small enough and $n$ large enough, we have $a_0 - \varepsilon < a_0 - |u| \phi_{1,n}(a_0, b_0) < a_0 + \varepsilon$ and $a_0 - \varepsilon < a_0 + |u| \phi_{1,n}(a_0, b_0) < a_0 + \varepsilon$. This implies that

$$\frac{1}{(X_{t_k}^{a_0, b_0}) a(\varepsilon)} = \frac{1}{(X_{t_k}^{a_0, b_0}) a(\varepsilon)} + \frac{1}{(X_{t_k}^{a_0, b_0}) a_0 - \varepsilon} + \frac{1}{(X_{t_k}^{a_0, b_0}) a_0 + \varepsilon}. (64)$$

For subcritical case $b_0 > 0$ with $\phi_{1,n}(a_0, b_0) = \frac{1}{\sqrt{\log(n\Delta_n)}}$ (respectively, critical case $b_0 = 0$ with $\phi_{1,n}(a_0, 0) = \frac{1}{\sqrt{\log(n\Delta_n)}}$), taking the expectation in both sides and using (64), Lemma 1(i) (respectively, Lemma 1(ii)) and standard calculations, for $n$ large enough we get $E_{t_k}^{a_0, b_0} [\sum_{k=0}^{n-1} X_{t_k}^{a_0, b_0}] \leq C \sqrt{n\Delta_n}$, which tends to zero, since all the obtained powers are smaller than $\frac{\varepsilon^2}{|\varepsilon|}$ for $\varepsilon$ small enough and $q_1$ and $n$ large enough. This completes the proof of condition (i) of Lemma 18.

Similarly, we use Jensen’s inequality, (56) of Lemma 9 with $V = (R^2(\varepsilon, b_0))^2$, (58) of Lemma 10 with $q = q_0 = \frac{13 + \sqrt{99}}{20} > 1$ and $q_1$ large enough, (52) of Lemma 8 with $q \in \{1, q_0\}$, and condition (A), for $n$ large enough $\sum_{k=0}^{n-1} E_{t_k}^{a_0, b_0} [\delta_{t_k}^2 \hat{P}_{t_k}]$ is bounded by

$$\sum_{k=0}^{n-1} \frac{u^2(\phi_{1,n}(a_0, b_0))}{\Delta_n^2} \int_0^1 \frac{1}{(X_{t_k}^{a_0, b_0}) a(\varepsilon)} \left[ E_{t_k}^{a_0, b_0} \left[ (R(\varepsilon, b_0))^2 |Y_{t_k+1}^{a_0, b_0} = X_{t_k+1}^{a_0, b_0}\right] \right] d\varepsilon \leq \sum_{k=0}^{n-1} \frac{u^2(\phi_{1,n}(a_0, b_0))}{\Delta_n^2} \int_0^1 \left( E_{t_k}^{a_0, b_0} \left[ |(R(\varepsilon, b_0))^2| + C_1 \sqrt{\Delta_n} \left( \frac{1}{(X_{t_k}^{a_0, b_0}) a_0} \right) \right] \right) d\varepsilon \leq Cu^2(\phi_{1,n}(a_0, b_0))^2 \sum_{k=0}^{n-1} \int_0^1 \left( \frac{1}{(X_{t_k}^{a_0, b_0}) a_0} + \frac{1}{(X_{t_k}^{a_0, b_0}) a_0} \right) d\varepsilon + Cu^2 |u| \Delta_n(\phi_{1,n}(a_0, b_0))^3$$

for some constant $C > 0$, where $p = \frac{q_0}{q_0 - 1}$. It is worth noticing that when using (58) of Lemma 10 with $p, q = q_0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, the condition $p = \frac{q_0}{q_0 - 1} < \frac{a(\varepsilon)}{\sigma} - 1$ is valid under condition (A).

For subcritical case $b_0 > 0$ with $\phi_{1,n}(a_0, b_0) = \frac{1}{\sqrt{\log(n\Delta_n)}}$ (respectively, critical case $b_0 = 0$ with $\phi_{1,n}(a_0, 0) = \frac{1}{\sqrt{\log(n\Delta_n)}}$), taking the expectation in both sides and using (64), Lemma 1(i) (respectively, Lemma 1(ii)) and standard calculations, for $n$ large enough we get
\[ \hat{E}^{n} \left[ \sum_{k=0}^{n} \left| \frac{1}{\Delta} \right| \right] \leq C \Delta_n, \text{ which tends to zero, since all the obtained powers are smaller than } \frac{a_0}{\sigma} \text{ for } \epsilon \text{ small enough and } q_1 \text{ and } n \text{ large enough. This completes the proof.} \]

**Lemma 12.** Assume condition \( \frac{a_0}{\sigma} > 4. \) Then, for both subcritical and critical cases as \( n \to \infty, \) we have

\[
\frac{u \varphi_{1,n}(a_0, b_0)}{\Delta} \int_0^1 \left( R^a_{t_k} Y^a_{t_k} \right| Y^a_{t_k} = X^a_{t_k} \right) d\epsilon \to 0.
\]

**Proof.** Similarly as above, we apply Lemma 18 with

\[
\boxed{\begin{align*}
\tilde{\varsigma}_{k,n} & := \frac{u \varphi_{1,n}(a_0, b_0)}{\Delta} \int_0^1 \left( R^a_{t_k} Y^a_{t_k} \right| Y^a_{t_k} = X^a_{t_k} \right) d\epsilon.
\end{align*}}
\]

Using (56) of Lemma 9 with \( V = R^a_{t_k} Y^a_{t_k} \), \( \tilde{E}^{a_0, b_0}_{t_k} X^a_{t_k} Y^a_{t_k} = 0 \) and \( \tilde{E}^{a_0, b_0}_{t_k} X^a_{t_k} Y^a_{t_k} = 0 \), (58) of Lemma 10 with \( q = p = 2 \) and \( q_1 \) large enough, and condition \( \frac{a_0}{\sigma} > 3 \) we obtain for \( n \) large enough

\[
\left| \sum_{k=0}^{n-1} \tilde{E}^{a_0, b_0}_{t_k} \left[ \tilde{\varsigma}_{k,n} \right] \right| \leq C \left| \sum_{k=0}^{n-1} \frac{u \varphi_{1,n}(a_0, b_0)}{\Delta} \int_0^1 \tilde{E}^{a_0, b_0}_{t_k} Y^a_{t_k} \left| Y^a_{t_k} = X^a_{t_k} \right] d\epsilon \right| \leq C \frac{\left| \varphi_{1,n}(a_0, b_0) \right|^2 2^{n-1}}{\sqrt{\Delta_n}} \sum_{k=0}^{n-1} \int_0^1 \tilde{E}^{a_0, b_0}_{t_k} \left( X^a_{t_k} Y^a_{t_k} \right) d\epsilon \times \left( 1 + \left( \frac{1}{\tilde{X}^a_{t_k}} \frac{a_0}{\sigma} \right) \right)
\]

For \( p \geq 1, \) using BDG’s inequality, Itô’s formula, Hölder’s inequality, (39), (40), we get

\[
\hat{E}^{a_0, b_0}_{t_k} \left[ \left| Y^a_{t_k} \right|^{2p} \right] \leq C \frac{\Delta^{2p}_{n}}{\left( \tilde{X}^a_{t_k} \right)^{2p}} \int_{t_k}^{t_{k+1}} \hat{E}^{a_0, b_0}_{t_k} \left[ \sqrt{Y^a_{t_k} - Y^a_{t_k} \left| b \right|^p} \right] ds
\]

\[
\leq C \frac{\Delta^{3p-1}_{n}}{\left( \tilde{X}^a_{t_k} \right)^{2p}} \int_{t_k}^{t_{k+1}} \hat{E}^{a_0, b_0}_{t_k} \left[ \int_{t_k}^{s} \left( a - \frac{\sigma}{4} \right) \frac{1}{\sqrt{Y^a_{t_k}}} - b \frac{2}{2} \sqrt{Y^a_{t_k}} \right] ds + \int_{t_k}^{s} \sqrt{\frac{\sigma_d W_u}{2}} \right] \left[ \left| b \right|^p \right] ds
\]

\[
\leq C \frac{\Delta^{3p-1}_{n}}{\left( \tilde{X}^a_{t_k} \right)^{2p}} \int_{t_k}^{t_{k+1}} \left( s - t_k \right)^{2p-1} \left[ \int_{t_k}^{s} \hat{E}^{a_0, b_0}_{t_k} \left[ \frac{1}{\left| Y^a_{t_k} \right|^{p}} \right] ds + \left| b \right|^p \left[ \frac{1}{\left( \tilde{X}^a_{t_k} \right)^{2p}} \right] \right] \left( s - t_k \right)^p ds \leq C \Delta^{4p}_{n} \left( \frac{1}{\left( \tilde{X}^a_{t_k} \right)^{4p}} + \frac{\left| b \right|^p}{\left( \tilde{X}^a_{t_k} \right)^{4p}} + \frac{1}{\left( \tilde{X}^a_{t_k} \right)^{4p}} \right),
\]

(65)
provided that $p < \frac{q}{\sigma} - 1$. Then, applying (65) with $p = 1$, condition $\frac{a_0}{\sigma} > 3$, we get

$$
\left| \sum_{k=0}^{n-1} \hat{E}^{q_{\gamma}, b_0}_{k,x_k} | \hat{P}_{t_k} \right| \leq C u^2 (\varphi_{1,n}(a_0, b_0))^2 \Delta_n^3 \sum_{k=0}^{n-1} \int_0^1 \left( \frac{1}{(X_{t_k}^{a_0,b_0})^3} + \frac{|b_0|}{(X_{t_k}^{a_0,b_0})^2} + \frac{1}{X_{t_k}^{a_0,b_0}} \right) \sqrt{\Delta_n} d\epsilon.
$$

For subcritical case $b_0 > 0$ with $\varphi_{1,n}(a_0, b_0) = \frac{1}{\sqrt{n} \Delta_n}$ (respectively, critical case $b_0 = 0$ with $\varphi_{1,n}(a_0, 0) = \frac{1}{\sqrt{\log(n \Delta_n)}}$), taking the expectation in both sides and using (64), Lemma 1(i) (respectively, Lemma 1(ii)) and standard calculations, for $n$ large enough we get

$$
\hat{E}^{q_{\gamma}, b_0}_{k,x_k} \cdot \left| \sum_{k=0}^{n-1} \hat{E}^{q_{\gamma}, b_0}_{k,x_k} | \hat{P}_{t_k} \right| \leq C \Delta_n, \text{ which tends to zero, since all the obtained powers are smaller than } \frac{a_0}{\sigma} \text{ for } \epsilon \text{ small enough and } q_1 \text{ and } n \text{ large enough.}
$$

Similarly, we use Jensen’s inequality, (56) of Lemma 9 with $V = (R_0^{q_{\gamma}, b_0})^2$, (58) of Lemma 10 with $q = p = 2$ and $q_1$ large enough, (65) with $p \in \{1, 2\}$, and condition $\frac{a_0}{\sigma} > 3$ in the same way as above we obtain for $n$ large enough

$$
\sum_{k=0}^{n-1} \hat{E}^{q_{\gamma}, b_0}_{k,x_k} | \hat{P}_{t_k} |
\leq 2 u^2 (\varphi_{1,n}(a_0, b_0))^2 \sum_{k=0}^{n-1} \int_0^1 \left( \hat{E}^{q_{\gamma}, b_0}_{t_k,x_k} | (R_0^{q_{\gamma}})^2 \right) + \hat{E}^{q_{\gamma}, b_0}_{t_k,x_k} | (R_0^{q_{\gamma}})^2 | Y_{t_{k+1}}^{a_0,b_0} = X_{t_{k+1}}^{a_0,b_0} \right) \right) d\epsilon
\leq C u^2 \Delta_n (\varphi_{1,n}(a_0, b_0))^3
\times \sum_{k=0}^{n-1} \int_0^1 \left( \frac{1}{(X_{t_k}^{a_0,b_0})^3} + \frac{|b_0|}{X_{t_k}^{a_0,b_0}} + \frac{1}{(X_{t_k}^{a_0,b_0})^2} \right) \sqrt{\Delta_n} d\epsilon,
$$

where the estimate for $\hat{E}^{q_{\gamma}, b_0}_{k,x_k} | (R_0^{q_{\gamma}})^2$ is proceeded similarly as for (65).

For subcritical case $b_0 > 0$ (respectively, critical case $b_0 = 0$), taking the expectation in both sides and using (64), Lemma 1(i) (respectively Lemma 1(ii)) and standard calculations, under $\frac{a_0}{\sigma} > 4$ we get

$$
\hat{E}^{q_{\gamma}, b_0}_{k,x_k} \cdot \left| \sum_{k=0}^{n-1} \hat{E}^{q_{\gamma}, b_0}_{k,x_k} | \hat{P}_{t_k} \right| \leq C \Delta_n \text{ for } n \text{ large enough, which tends to zero, since all the obtained powers are smaller than } \frac{a_0}{\sigma} \text{ for } \epsilon \text{ small enough and } q_1 \text{ and } n \text{ large enough. This completes the proof.}$$
Lemma 13. Assume condition $\frac{\nu_0}{\sigma} > 3$. Then, for the subcritical case as $n \to \infty$, we have

$$
\sum_{k=0}^{n-1} u \varphi_{1,n}(a_0, b_0) \Delta_n \int_0^1 (R_6^{a_0, b_0} - \hat{E}^{a(\ell), b_0}_{k} [R_4^{a(\ell), b_0} | Y_{t_{k+1}}^{a(\ell), b_0} = X_{t_{k+1}}^{a_0, b_0}]) d\ell \to 0.
$$

For the critical case when $b_0 = 0$, we have $R_4^{a(\ell), b_0} = R_6^{a_0, b_0} = 0$.

Proof. Similarly as above, we apply Lemma 18 with

$$
\zeta_{k,n} := \frac{u \varphi_{1,n}(a_0, b_0)}{\Delta_n} \int_0^1 (R_6^{a_0, b_0} - \hat{E}^{a(\ell), b_0}_{k} [R_4^{a(\ell), b_0} | Y_{t_{k+1}}^{a(\ell), b_0} = X_{t_{k+1}}^{a_0, b_0}]) d\ell.
$$

Using (56) of Lemma 9 with $V = R_4^{a(\ell), b_0}$, (57) of Lemma 9 with $\hat{V} = R_6^{a_0, b_0}$, (58) and (59) of Lemma 10 with $q = p = 2$ and $q_1$ large enough, and condition $\frac{\nu_0}{\sigma} > 3$, for $n$ large enough $| \sum_{k=0}^{n-1} \hat{E}^{a_0, b_0}_{k} (\zeta_{k,n} | \hat{F}_k) |$ is bounded by

$$
C \frac{|u| \varphi_{1,n}(a_0, b_0)}{\Delta_n} \sum_{k=0}^{n-1} \int_0^1 \left( \frac{\sqrt{\Delta_n}}{X_{t_k}^{a_0, b_0}} \left[ \int_{a(\ell)}^{a_0} (R_4^{a(\ell), b_0} | Y_{t_k}^{a(\ell), b_0} |^2)^{\frac{1}{2}} d\alpha \right] \sqrt{\Delta_n} |u| \varphi_{1,n}(a_0, b_0) \right) \left( 1 + \frac{1}{(X_{t_k}^{a_0, b_0})^\frac{1}{2 - q_1}} \right) d\ell,
$$

where we have used $\hat{E}^{a(\ell), b_0}_{k} [R_6^{a(\ell), b_0} | Y_{t_{k+1}}^{a(\ell), b_0} = X_{t_{k+1}}^{a_0, b_0} ] = 0$ since $Y^{a(\ell), b_0}$ is the independent copy of $X^{a(\ell), b_0}$.

For $p \geq 1$, using Equation (29), BDG’s inequality, Hölder’s inequality and (39), we get

$$
\hat{E}^{a_0, b_0}_{k} \left[ \left( \int_{t_k}^{t_{k+1}} (Y_s^{a,b} - Y_{t_k}^{a,b}) ds \right)^{2p} \right] 
\leq \Delta_n^{2p-1} \int_{t_k}^{t_{k+1}} \hat{E}^{a_0, b_0}_{k} \left[ |Y_s^{a,b} - Y_{t_k}^{a,b}|^{2p} \right] ds
\leq C \Delta_n^{2p-1} \int_{t_k}^{t_{k+1}} \left( (s - t_k)^{2p-1} \left( \int_{t_k}^{s} (1 + |b| \hat{E}^{a_0, b_0}_{k} |Y_u^{a,b}|^{2p}) du + (s - t_k)^{p-1} \right) \times \int_{t_k}^{s} \hat{E}^{a_0, b_0}_{k} |Y_u^{a,b}|^p du \right) ds 
\leq C \Delta_n^{3p-1} (1 + |b| (X_{t_k}^{a_0, b_0})^{2p} + (X_{t_k}^{a_0, b_0})^p).
$$

Then, applying (66) with $p = 1$, we get

$$
\left| \sum_{k=0}^{n-1} \hat{E}^{a_0, b_0}_{k} \left[ \zeta_{k,n} | \hat{F}_k \right] \right| \leq C u^2 |b_0| (|\varphi_{1,n}(a_0, b_0)|^2 \Delta_n^{2} \sum_{k=0}^{n-1} \int_0^1 \left( \frac{1}{(X_{t_k}^{a_0, b_0})^\frac{1}{2}} + \frac{|b_0|}{\sqrt{X_{t_k}^{a_0, b_0}}} + \frac{1}{X_{t_k}^{a_0, b_0}} \right) \right).
$$
Using (56) of Lemma 9 with $p = 0$, from which we get that $\sum_{k=0}^{n-1} E_{\tilde{t}_k} \{ E_{\tilde{t}_k} \tilde{\zeta}_{k,n} | \tilde{\zeta}_{k,n} \} \leq C \Delta_n$, which tends to zero, since all the obtained powers are smaller than $\frac{p}{\sigma}$ for $\epsilon$ small enough and $q_1$ and $n$ large enough.

Next, using Jensen’s inequality, (56) of Lemma 9 with $V (R_{k,n}^{a,c})^2$, (68) of Lemma 10 with $q = p = 2$ and $q_1$ large enough, (66) with $p \in \{1, 2\}$, and condition $\frac{d_k}{\sigma} > 3$ in the same way as above we obtain for $n$ large enough, similarly,

$$\sum_{k=0}^{n-1} E_{\tilde{t}_k} \{ E_{\tilde{t}_k} \tilde{\zeta}_{k,n} | \tilde{\zeta}_{k,n} \} \leq C \Delta_n^2 (\varphi_{1,n}(a_0, b_0))^2 \sum_{k=0}^{n-1} \left( \frac{1}{X_{\tilde{t}_k}^{a_0,b_0}} + |b_0| + \frac{1}{X_{\tilde{t}_k}^{a_0,b_0}} \right)$$

$$+ C u^2 |b_0| \Delta_n^2 (\varphi_{1,n}(a_0, b_0))^2 \sum_{k=0}^{n-1} \left( \frac{1}{X_{\tilde{t}_k}^{a_0,b_0}} + |b_0| + \frac{1}{X_{\tilde{t}_k}^{a_0,b_0}} \right)$$

$$+ \frac{1}{(X_{\tilde{t}_k}^{a_0,b_0})^2 + \frac{1}{\sqrt{\Delta_n}} \frac{\sigma c}{\sigma c - 1}} + |u| \varphi_{1,n}(a_0, b_0) \frac{\sqrt{\Delta_n}}{(X_{\tilde{t}_k}^{a_0,b_0})^1 + \frac{1}{\sqrt{\Delta_n}} \frac{\sigma c}{\sigma c - 1}} \right) \right) d\epsilon.$$

where the estimate for $E_{\tilde{t}_k} \{ E_{\tilde{t}_k} \tilde{\zeta}_{k,n} | \tilde{\zeta}_{k,n} \}^2$ is proceeded similarly as for (66).

For subcritical case $b_0 > 0$, using (64) and Lemma 1(i), $E_{\tilde{t}_k} \{ E_{\tilde{t}_k} \tilde{\zeta}_{k,n} | \tilde{\zeta}_{k,n} \} \leq C \Delta_n^2$, which tends to zero, since all the obtained powers are smaller than $\frac{p}{\sigma}$ for $\epsilon$ small enough and $q_1$ and $n$ large enough. This completes the proof.

**Lemma 14.** Assume condition (A). Then, as $n \to \infty$, we have

$$\sum_{k=0}^{n-1} v \varphi_{2,n}(a_0, b_0) \Delta_n \int_{0}^{1} E_{\tilde{t}_k} \{ E_{\tilde{t}_k} \tilde{\zeta}_{k,n} | \tilde{\zeta}_{k,n} \} \langle H^{a_0,b_0}(\tilde{\epsilon}) | Y^{a_0,b_0}(\tilde{\epsilon}) \rangle = X_{\tilde{t}_{k+1}}^{a_0,b_0} \epsilon \to 0.$$

**Proof.** Similarly as above, we apply Lemma 18 with

$$\zeta_{k,n} := \frac{v \varphi_{2,n}(a_0, b_0)}{\Delta_n} \int_{0}^{1} E_{\tilde{t}_k} \{ E_{\tilde{t}_k} \tilde{\zeta}_{k,n} | \tilde{\zeta}_{k,n} \} \langle H^{a_0,b_0}(\tilde{\epsilon}) | Y^{a_0,b_0}(\tilde{\epsilon}) \rangle = X_{\tilde{t}_{k+1}}^{a_0,b_0} \epsilon.$$

Using (56) of Lemma 9 with $V = H^{a_0,b_0}$, $E_{\tilde{t}_k} \{ E_{\tilde{t}_k} \tilde{\zeta}_{k,n} | \tilde{\zeta}_{k,n} \} = 0$ by (51), (58) of Lemma 10 with $q = p = 2$ and $q_1$ large enough, (53) of Lemma 8 with $q = 1$, and condition (A), we obtain for $n$ large enough
\[
\sum_{k=0}^{n-1} E^{a_{t_k}, b_{t_k}} \left[ \zeta_{k,n} | P_{t_k} \right] \leq C \sum_{k=0}^{n-1} \frac{|v| \varphi_{2,n}(a_0, b_0)}{\Delta_n} \int_0^1 \sqrt{\Delta_n (E^{a_{t_k}, b_{t_k}} \left[ |H a_{t_k}, b_{t_k} |^2 \right])} \frac{1}{\Delta_n} \left| \sum_{r=0}^{n-1} |u| \varphi_{1,n}(a_0, b_0) \left( \sqrt{X_{t_k}^{a_{t_k}, b_{t_k}}} + \frac{\sqrt{\Delta_n}}{X_{t_k}^{a_{t_k}, b_{t_k}}} |u| \varphi_{1,n}(a_0, b_0) \right) + \sqrt{\Delta_n} |v| \varphi_{2,n}(a_0, b_0) \right| \right.
\]
\[
\left. \times \left( 1 + \frac{1}{(X_{t_k}^{a_{t_k}, b_{t_k}})^{\frac{1}{2}} \left( \frac{a_n}{\sigma} - 1 \right)} \right) \right) \leq C |v| \varphi_{2,n}(a_0, b_0) \Delta_n \sum_{k=0}^{n-1} e^{C; v^2 (\varphi_{2,n}(a_0, b_0))^2} \Delta_n X_{t_k}^{a_{t_k}, b_{t_k}}
\]
\[
\times (1 + X_{t_k}^{a_{t_k}, b_{t_k}}) \left( \sqrt{X_{t_k}^{a_{t_k}, b_{t_k}}} + \frac{1}{(X_{t_k}^{a_{t_k}, b_{t_k}})^{\frac{1}{2}} \left( \frac{a_n}{\sigma} - 1 \right)} \right) \right).
\]

Then, using Young’s inequality for products with \( \frac{1}{p_0} + \frac{1}{q_0} = 1 \) and \( \bar{q}_0 \) close to 1, for \( n \) large enough |\( \sum_{k=0}^{n-1} E^{a_{t_k}, b_{t_k}} \left[ \zeta_{k,n} | P_{t_k} \right] \)| is bounded by

\[
C |v| \varphi_{2,n}(a_0, b_0) \Delta_n \frac{1}{p_0} \left( |u| \varphi_{1,n}(a_0, b_0) + |v| \varphi_{2,n}(a_0, b_0) \right) \sum_{k=0}^{n-1} e^{C; v^2 (\varphi_{2,n}(a_0, b_0))^2} \Delta_n X_{t_k}^{a_{t_k}, b_{t_k}}
\]
\[
+ C |v| \varphi_{2,n}(a_0, b_0) \Delta_n \frac{1}{q_0} \sum_{k=0}^{n-1} \left( |u| \varphi_{1,n}(a_0, b_0) \left( X_{t_k}^{a_{t_k}, b_{t_k}} \right)^{\frac{q_0}{2}} \Delta_n \left( |u| \varphi_{1,n}(a_0, b_0) \right) + \left( \sqrt{\Delta_n} |v| \varphi_{2,n}(a_0, b_0) \right) \bar{q}_0 \left( X_{t_k}^{a_{t_k}, b_{t_k}} \right) \bar{q}_0 \right) + \left( 1 + \left( X_{t_k}^{a_{t_k}, b_{t_k}} \right)^{\frac{1}{2}} \bar{q}_0 \left( 1 + \frac{1}{(X_{t_k}^{a_{t_k}, b_{t_k}})^{\frac{1}{2}} \left( \frac{a_n}{\sigma} - 1 \right)} \right) \right) \right)
\]
\[
\times \left( 1 + \left( X_{t_k}^{a_{t_k}, b_{t_k}} \right)^{\frac{1}{2}} \bar{q}_0 \left( \sqrt{\Delta_n} |v| \varphi_{2,n}(a_0, b_0) \right) \bar{q}_0 \left( X_{t_k}^{a_{t_k}, b_{t_k}} \right) \bar{q}_0 \right) \left( \frac{1}{(X_{t_k}^{a_{t_k}, b_{t_k}})^{\frac{q_0}{2}}} + \frac{1}{(X_{t_k}^{a_{t_k}, b_{t_k}})^{\frac{q_0}{2}} \left( \frac{a_n}{\sigma} - 1 \right)} \right) \right).
\]

Now using (1.9) of proposition 1.2.4 in Alfonsi (2015), for \( n \) large enough, we get

\[
M_n := E^{a_{t_k}, b_{t_k}} \left[ e^{C; v^2 (\varphi_{2,n}(a_0, b_0))^2} \Delta_n X_{t_k}^{a_{t_k}, b_{t_k}} \right] \leq C,
\]

(67)
for some constant $C > 0$ for three cases. Indeed, we prove the convergence of $M_n$ toward 1 as $n \to \infty$. For this, for subcritical case $b_0 > 0$ (respectively, critical case $b_0 = 0$, respectively, supercritical case $b_0 < 0$), we take $\varphi_{2,n}(a_0, b_0) = \frac{1}{\sqrt{n\Delta_n}}$ (respectively $\varphi_{2,n}(a_0, 0) = \frac{1}{\sqrt{n\Delta_n}}$, respectively, $\varphi_{2,n}(a_0, b_0) = e^{\frac{b_0}{\sqrt{n\Delta_n}}}$) and use that $1 - e^{-b_0t_k} \leq b_0t_k, \frac{t_k}{n\Delta_n} \leq 1$ and $e^{-b_0t_k} \leq 1$ (respectively, $\frac{t_k}{n\Delta_n} \leq 1$, respectively, $e^{b_0(n\Delta_n-t_k)} \leq 1$ and $n\Delta_n \to \infty$).

Thus, for subcritical case $b_0 > 0$ with $(\varphi_{1,n}(a_0, b_0), \varphi_{2,n}(a_0, b_0)) = \left(\frac{1}{\sqrt{n\Delta_n}}, \frac{1}{\sqrt{n\Delta_n}}\right)$ (respectively, critical case $b_0 = 0$ with $(\varphi_{1,n}(a_0, b_0), \varphi_{2,n}(a_0, b_0)) = \left(\frac{1}{\sqrt{n\Delta_n}}, \frac{1}{\sqrt{n\Delta_n}}\right)$, respectively, supercritical case $b_0 < 0$ with $(\varphi_{1,n}(a_0, b_0), \varphi_{2,n}(a_0, b_0)) = (0, e^{\frac{b_0}{\sqrt{n\Delta_n}}})$, using (67), Lemma 1(i) (respectively, Lemma 1(ii), respectively, Lemma 1(iii), (12)) and standard calculations, we get $E_{a_0,b_0} [\sum_{k=0}^{n-1} E_{a_0,b_0} [\xi_{k,n} | \hat{P}_{t_k}]] \leq C \sqrt{n\Delta_n}$ (respectively, $E_{a_0,0} [\sum_{k=0}^{n-1} E_{a_0,0} [\xi_{k,n} | \hat{P}_{t_k}]] \leq C(n\Delta_n)^{\frac{3}{2}}$, respectively $E_{a_0,b_0} [\sum_{k=0}^{n-1} E_{a_0,b_0} [\xi_{k,n} | \hat{P}_{t_k}]] \leq C(e^{b_0n\Delta_n\Delta_n} + (\Delta_n^2 e^{-b_0n\Delta_n})^\frac{1}{2})$), which tends to zero, since all the obtained powers are smaller than $\frac{a_0}{\sigma}$ for $\bar{q}_0$ close to 1 and $q_1$ and $n$ large enough.

Next, using Jensen’s inequality, (56) of Lemma 9 with $V = (H_{a_n,b_n}(\cdot))^2$, (58) of Lemma 10 with $q = 0 = \frac{13+q}{20}$ and $q_1$ large enough, (53) of Lemma 8 with $q \in [1, q_0]$, condition (A), and Young’s inequality for products with $\frac{1}{p_0} + \frac{1}{q_0} = 1$ and $\bar{q}_0$ close to 1, the same arguments as above work here and then for $n$ large enough $\sum_{k=0}^{n-1} E_{a_0,b_0} [\xi_{k,n} | \hat{P}_{t_k}]$ is bounded by

\[\sum_{k=0}^{n-1} \frac{V^2(\varphi_{2,n}(a_0, b_0))^2}{\Delta_n} + \int_0^1 \left\{ e^{C_{\varphi_{2,n}(a_0, b_0)^2} \Delta_n X_{t_k}^{a_0,b_0}} \left[ \left| \varphi_{1,n}(a_0, b_0) \right| \right] + \sqrt{\Delta_n} \right\} \, d\varepsilon \]

\[\leq CV^2(\varphi_{2,n}(a_0, b_0))^2 \Delta_n \sum_{k=0}^{n-1} (1 + X_{t_k}^{a_0,b_0})^2 \left( \frac{1}{X_{t_k}^{a_0,b_0}} + \frac{1}{(X_{t_k}^{a_0,b_0})^\frac{a_0}{\sigma}} \right) + CV^2(\varphi_{2,n}(a_0, b_0))^2 \Delta_n^{\frac{3}{2}} \]

\[\times \left( \left| \varphi_{1,n}(a_0, b_0) \right| + \left| \varphi_{2,n}(a_0, b_0) \right| \right) \sum_{k=0}^{n-1} e^{C_{\varphi_{2,n}(a_0, b_0)^2} \Delta_n X_{t_k}^{a_0,b_0}} + CV^2(\varphi_{2,n}(a_0, b_0))^2 \]

\[\times \frac{\Delta_n}{\bar{q}_0} \sum_{k=0}^{n-1} \left( \left| \varphi_{1,n}(a_0, b_0) \right| \right) (X_{t_k}^{a_0,b_0})^\frac{3}{2} \bar{q}_0 + \left| \varphi_{2,n}(a_0, b_0) \right| (X_{t_k}^{a_0,b_0})^\frac{3}{2} \bar{q}_0 \]

\[+ \left( \frac{\Delta_n}{\bar{q}_0} \right)^{\frac{1}{2}} + \left( \frac{\Delta_n}{\bar{q}_0} \right)^{\frac{1}{2}} \left( \left| \varphi_{1,n}(a_0, b_0) \right| \right)^2 + \left| \varphi_{2,n}(a_0, b_0) \right| \]
and Young's inequality for products with $p > 1$ and Young's inequality for products with $1 < p < 2$, where $p = \frac{q_0}{q_0 - 1}$.

Thus, for subcritical case $b_0 > 0$ (respectively critical case $b_0 = 0$, respectively supercritical case $b_0 < 0$), using (67), Lemma 1(i) (respectively Lemma 1(ii), respectively Lemma 1(iii), (12)) and standard calculations, we get $E^{a_0, b_0} [\sum_{k=0}^{n-1} E^{a_0, b_0}_k \xi_{k,n} | \hat{F}_k ] \leq C \Delta_n$ (respectively, $E^{a_0, b_0}_k [\sum_{k=0}^{n-1} E^{a_0, b_0}_k \xi_{k,n} | \hat{F}_k ] \leq C \Delta_n$, respectively $E^{a_0, b_0}_k [\sum_{k=0}^{n-1} E^{a_0, b_0}_k \xi_{k,n} | \hat{F}_k ] \leq C(\Delta_n e^{-b_0 n \Delta_n} )^3$), which tends to zero, since all the obtained powers are smaller than $\frac{a_0}{\sigma}$ for $q_0$ close to 1 and $q_1$ and $n$ large enough. This completes the proof. 

\begin{lemma}
Assume condition $\frac{a_0}{\sigma} > 3$. Then, as $n \to \infty$, we have
\begin{equation}
\sum_{k=0}^{n-1} v \varphi_{2,n}(a_0, b_0) \Delta_n \int_0^1 (-X_{t_k}^{a_0, b_0} R_7 a_0 b_0 + E^{a_0, b(\epsilon)} X_{t_k}^{a_0, b(\epsilon)} \xi_{t_k}^{a_0, b(\epsilon)} | \tilde{Y}_{t_k}^{a_2, b(\epsilon)} = X_{t_k}^{a_2, b(\epsilon)}) d\epsilon' \to 0.
\end{equation}
\end{lemma}

\textbf{Proof.} Similarly as above, we apply Lemma 18 with
\begin{equation}
\xi_{k,n} := \frac{v \varphi_{2,n}(a_0, b_0)}{\Delta_n} \int_0^1 (-X_{t_k}^{a_0, b_0} R_7 a_0 b_0 + E^{a_0, b(\epsilon)} X_{t_k}^{a_0, b(\epsilon)} \xi_{t_k}^{a_0, b(\epsilon)} | \tilde{Y}_{t_k}^{a_2, b(\epsilon)} = X_{t_k}^{a_2, b(\epsilon)}) d\epsilon'.
\end{equation}

We use (56) of Lemma 9 with $V = X_{t_k}^{a_0, b_0} R_7 a_0 b_0$, the fact that $E^{a_0, b(\epsilon)} X_{t_k}^{a_0, b(\epsilon)} \xi_{t_k}^{a_0, b(\epsilon)} | \tilde{Y}_{t_k}^{a_2, b(\epsilon)} = 0$, (58) of Lemma 10 with $q = p = 2$ and $q_1$ large enough, (65) with $p = 1$, and Young’s inequality for products with $\frac{1}{b_0} + \frac{1}{q_0} = 1$ and $\tilde{q}_0$ close to 1, under $\frac{a_0}{\sigma} > 3$ for $n$ large enough $| \sum_{k=0}^{n-1} E^{a_0, b_0}_k \xi_{k,n} | \hat{F}_k |$ is bounded by
\begin{equation}
C |v \varphi_{2,n}(a_0, b_0) \Delta_n \sum_{k=0}^{n-1} \int_0^1 \sqrt{\Delta_n X_{t_k}^{a_0, b_0}} (E^{a_0, b(\epsilon)} X_{t_k}^{a_0, b(\epsilon)} | | R_7^{a_0, b(\epsilon)} |^2 )^{\frac{1}{2}} e^{C_2 \varphi_2(a_0, b_0)^2 \Delta_n X_{t_k}^{a_0, b_0}}
\times \left( |u| \varphi_{1,n}(a_0, b_0) \left( \frac{1}{\sqrt{X_{t_k}^{a_0, b_0}}} + \sqrt{\Delta_n} |v| \varphi_{2,n}(a_0, b_0) \right) \right)
\times \left( 1 + \frac{1}{(X_{t_k}^{a_0, b_0})^{\frac{2}{q_0}} (\frac{a_0}{\sigma} - 1)} \right) + |v| \varphi_{2,n}(a_0, b_0) (1 + \sqrt{X_{t_k}^{a_0, b_0}} + \sqrt{\Delta_n} |v| \varphi_{2,n}(a_0, b_0) X_{t_k}^{a_0, b_0})
\right) d\epsilon'
\end{equation}
\begin{equation}
\leq C |v \varphi_{2,n}(a_0, b_0) \Delta_n \frac{1}{P_0} \left( |u| \varphi_{1,n}(a_0, b_0) + |v| \varphi_{2,n}(a_0, b_0) \right) \sum_{k=0}^{n-1} e^{C_2 \varphi_{2}(a_0, b_0)^2 \Delta_n X_{t_k}^{a_0, b_0}}
\end{equation}
\begin{equation}
+ C |v \varphi_{2,n}(a_0, b_0) \Delta_n \frac{1}{q_0} \sum_{k=0}^{n-1} \left( |u| \varphi_{1,n}(a_0, b_0) \left( 1 + (|b_0| + |v| \varphi_{2,n}(a_0, b_0)) \right) \right)
\end{equation}
\[\times \left(\sqrt{\Delta_n} |v| \varphi_{2,n}(a_0, b_0)\right) \tilde{q}_0 \left(X_{t_k}^{a_0,b_0}\right) \frac{\sigma}{\tilde{q}_0} + \frac{1}{\left(X_{t_k}^{a_0,b_0}\right)^{(1+\frac{\sigma}{1-\tilde{q}_0})}} + \frac{\sqrt{\Delta_n} \tilde{q}_0}{\left(X_{t_k}^{a_0,b_0}\right)^{(\frac{2}{1-\tilde{q}_0})}} \right)

\times \left(|u| \varphi_{1,n}(a_0, b_0)\right) \tilde{q}_0 + |v| \varphi_{2,n}(a_0, b_0) \left(1 + \frac{1}{\left(X_{t_k}^{a_0,b_0}\right)^{\frac{\sigma}{\tilde{q}_0}}} + \left(|b_0| + |v| \varphi_{2,n}(a_0, b_0)\right) \tilde{q}_0 \right)

\times \left(X_{t_k}^{a_0,b_0}\right) \tilde{q}_0 + \left(|b_0| + |v| \varphi_{2,n}(a_0, b_0)\right) \tilde{q}_0 \left(\sqrt{\Delta_n} |v| \varphi_{2,n}(a_0, b_0)\right) \tilde{q}_0 \left(X_{t_k}^{a_0,b_0}\right)^{\frac{\sigma}{\tilde{q}_0}} \right).

Thus, for subcritical case \(b_0 > 0\) (respectively critical case \(b_0 = 0\), respectively supercritical case \(b_0 < 0\)), using (67), Lemma 1(i) (respectively Lemma 1(ii), respectively Lemma 1(iii), (12)) and standard calculations, we get

\[\tilde{E}^{a_0,b_0} \left[ \sum_{k=0}^{n-1} \tilde{E}^{a_0,b_0} [\tilde{F}_{t_k}] \right] \leq C \sqrt{\Delta_n} \text{ (respectively } \tilde{E}^{a_0,0} \left[ \sum_{k=0}^{n-1} \tilde{E}^{a_0,0} [\tilde{F}_{t_k}] \right] \leq C \sqrt{\Delta_n}, \text{ respectively } \tilde{E}^{a_0,b_0} \left[ \sum_{k=0}^{n-1} \tilde{E}^{a_0,b_0} [\tilde{F}_{t_k}] \right] \leq C \left(\Delta_n \tilde{E}^{b_0,n\pi} \right)^{\frac{1}{2}}),\]

which tends to zero, since all the obtained powers are smaller than \(\frac{a_0}{\tilde{q}_0}\) for \(\tilde{q}_0\) close to 1 and \(q_1\) and \(n\) large enough.

Next, using Jensen’s inequality, (56) of Lemma 9 with \(V = (X_{t_k}^{a_0,b_0} R_n^{a_0,b_0})^2\), (58) of Lemma 10 with \(q = p = 2\) and \(q_1\) large enough, (65) with \(p \in \{1, 2\}\), and Young’s inequality for products with \(\frac{1}{p_0} + \frac{1}{q_0} = 1\) and \(\tilde{q}_0\) close to 1, under \(\frac{a_0}{\tilde{q}_0} > 3\) for \(n\) large enough \(\sum_{k=0}^{n-1} \tilde{E}^{a_0,b_0} [\tilde{F}_{t_k}]\) is bounded by

\[\sum_{k=0}^{n-1} \left(\frac{1}{X_{t_k}^{a_0,b_0} + X_{t_k}^{a_0,b_0}} \right) + C \left(\frac{\sqrt{\Delta_n} \tilde{q}_0}{X_{t_k}^{a_0,b_0}} \right) \left(1 + \frac{1}{\left(X_{t_k}^{a_0,b_0}\right)^{\frac{\sigma}{\tilde{q}_0}}} \right) + \left(|b_0| + |v| \varphi_{2,n}(a_0, b_0)\right) \tilde{q}_0 \left(\sqrt{\Delta_n} |v| \varphi_{2,n}(a_0, b_0)\right) \tilde{q}_0 \left(X_{t_k}^{a_0,b_0}\right)^{\frac{\sigma}{\tilde{q}_0}} \right).

\[\leq C \varphi_{2,n}(a_0, b_0)^2 \Delta_{n} \sum_{k=0}^{n-1} \left(\frac{1}{X_{t_k}^{a_0,b_0}} + X_{t_k}^{a_0,b_0}\right) + C \left(\varphi_{2,n}(a_0, b_0)\right)^2 \Delta_{n} \frac{\sigma}{\tilde{q}_0} \left(|b_0| + |v| \varphi_{2,n}(a_0, b_0)\right) \tilde{q}_0 \left(\sqrt{\Delta_n} |v| \varphi_{2,n}(a_0, b_0)\right) \tilde{q}_0 \left(X_{t_k}^{a_0,b_0}\right)^{\frac{\sigma}{\tilde{q}_0}} \right).

\[\times \left(1 + \left(|b_0| + |v| \varphi_{2,n}(a_0, b_0)\right) \tilde{q}_0 \left(X_{t_k}^{a_0,b_0}\right)^{\frac{\sigma}{\tilde{q}_0}} \right) + \left(|b_0| + |v| \varphi_{2,n}(a_0, b_0)\right) \tilde{q}_0 \left(\sqrt{\Delta_n} |v| \varphi_{2,n}(a_0, b_0)\right) \tilde{q}_0 \left(X_{t_k}^{a_0,b_0}\right)^{\frac{\sigma}{\tilde{q}_0}} \right).

\[+ \varphi_{2,n}(a_0, b_0) \left(1 + \frac{1}{\left(X_{t_k}^{a_0,b_0}\right)^{\frac{\sigma}{\tilde{q}_0}}} \right) + \left(|b_0| + |v| \varphi_{2,n}(a_0, b_0)\right) \tilde{q}_0 \left(X_{t_k}^{a_0,b_0}\right)^{\frac{\sigma}{\tilde{q}_0}} \right).
Thus, for subcritical case $b_0 > 0$ (respectively critical case $b_0 = 0$, respectively supercritical case $b_0 < 0$), using (67), Lemma 1(i) (respectively Lemma 1(ii), respectively Lemma 1(iii), (12)) and standard calculations, we get $\hat{E}^{a_0,0} \left[ |\sum_{k=0}^{n-1} E^{a_0,0} \left[ |\tilde{x}_{k,n}^{2} | \tilde{F}_{k} \right] | \right] \leq C\Delta_n$ (respectively $\hat{E}^{a_0,0} \left[ |\sum_{k=0}^{n-1} E^{a_0,0} \left[ |\tilde{x}_{k,n}^{2} | \tilde{F}_{k} \right] | \right] \leq C\Delta_n$, respectively $\hat{E}^{a_0,0} \left[ |\sum_{k=0}^{n-1} E^{a_0,0} \left[ |\tilde{x}_{k,n}^{2} | \tilde{F}_{k} \right] | \right] \leq C(\Delta_n e^{-b_0 n\Delta_n})^\frac{1}{2}$), which tends to zero, since all the obtained powers are smaller than $\frac{a_0}{\sigma}$ for $q_0$ close to 1 and $q_1$ and $n$ large enough. This completes the proof.

**Lemma 16.** Assume condition $\frac{a_0}{\sigma} > 3$. Then, as $n \to \infty$, we have

$$\sum_{k=0}^{n-1} \frac{\nu_2, n(a_0, b_0)}{\Delta_n} \int_0^1 \left(-X_{t_k}^{a_0,b_0} R_0^{a_0,b_0} + \tilde{E}_n^{a_0,b_0} \left[ X_{t_k}^{a_0,b_0} R_0^{a_0,b_0} | Y_{t_{k+1}}^{a_0,b_0} = X_{t_{k+1}}^{a_0,b_0} \right] \right) d\nu \to 0.$$ 

**Proof.** We rewrite

$$\sum_{k=0}^{n-1} \frac{\nu_2, n(a_0, b_0)}{\Delta_n} \int_0^1 \left(-X_{t_k}^{a_0,b_0} R_0^{a_0,b_0} + \tilde{E}_n^{a_0,b_0} \left[ X_{t_k}^{a_0,b_0} R_0^{a_0,b_0} | Y_{t_{k+1}}^{a_0,b_0} = X_{t_{k+1}}^{a_0,b_0} \right] \right) d\nu = M_{k,n,1} + M_{k,n,2},$$

where

$$M_{k,n,1} = -\frac{v}{2\alpha} (\nu_2, n(a_0, b_0))^2 \int_0^1 \left( X_{t_k}^{a_0,b_0} - X_{t_k}^{a_0,b_0} \right) ds d\nu,$$

$$M_{k,n,2} = \frac{v}{2\alpha} \nu_2, n(a_0, b_0) \int_0^1 b(\nu) \left\{ \int_{t_k}^{t_{k+1}} \left( X_{s}^{a_0,b_0} - X_{s}^{a_0,b_0} \right) ds - \tilde{E}_n^{a_0,b(\nu)} \left[ \int_{t_k}^{t_{k+1}} \left( Y_{s}^{a_0,b(\nu)} - Y_{s}^{a_0,b(\nu)} \right) ds | Y_{t_{k+1}}^{a_0,b(\nu)} = X_{t_{k+1}}^{a_0,b(\nu)} \right] \right\} d\nu.$$
\[ \times e^C e^{(\varphi_{2,n}(a_0,b_0))^2 \Delta_n X_{tk}^{a_0,b_0}} \left\{ \left| u \right| \varphi_{1,n}(a_0,b_0) \left( \frac{1}{X_{tk}^{a_0,b_0}} + \frac{\sqrt{\Delta_n}}{X_{tk}^{a_0,b_0}} \left| u \right| \varphi_{1,n}(a_0,b_0) \right) + \sqrt{\Delta_n} \left| v \right| \varphi_{2,n}(a_0,b_0) \right\} \\
+ \sqrt{\Delta_n} \left| v \right| \varphi_{2,n}(a_0,b_0) \left( 1 + \frac{1}{(X_{tk}^{a_0,b_0})^{\frac{1}{4n} \left( \frac{2n}{n} - 1 \right)}} \right) + \left| v \right| \varphi_{2,n}(a_0,b_0) \left( 1 + \frac{\sqrt{\Delta_n}}{X_{tk}^{a_0,b_0}} \right) \right) \right) \right\} d\varepsilon \]

\[ \leq C_{\varphi_{2,n}(a_0,b_0)} \Delta_n^2 \sum_{k=0}^{n-1} \left\{ \left| u \right| \varphi_{1,n}(a_0,b_0) \right\} \left( \frac{1}{X_{tk}^{a_0,b_0}} + \sqrt{\Delta_n} \left| v \right| \varphi_{2,n}(a_0,b_0) \right) (1 + (X_{tk}^{a_0,b_0})^2) \]

where we use \( \hat{E}_{a_0,b_0}^{a_0,b_0} [V_{a_0,b_0}^{a_0,b_0}] = 0 \) since \( Y_{a_0,b}^{a_0,b} \) is the independent copy of \( X_{a_0,b}^{a_0,b} \).

Thus, for subcritical case \( b_0 > 0 \) (respectively critical case \( b_0 = 0 \), respectively supercritical case \( b_0 < 0 \)), using (67), Lemma 1(i) (respectively Lemma 1(ii), respectively Lemma 1(iii), (12)) and standard calculations, \( \hat{E}_{a_0,b_0}^{a_0,b_0} [\sum_{k=0}^{n-1} E_{a_0,b_0}^{a_0,b_0} [M_{k,n}, \hat{F}_{\hat{t}_k}]] \) \( \leq C \Delta_n \) (respectively \( \hat{E}_{a_0,b_0}^{a_0,b_0} [\sum_{k=0}^{n-1} E_{a_0,b_0}^{a_0,b_0} [M_{k,n,2}, \hat{F}_{\hat{t}_k}]] \) \( \leq C \sqrt{n \Delta_n^3} \), respectively \( \hat{E}_{a_0,b_0}^{a_0,b_0} [\sum_{k=0}^{n-1} E_{a_0,b_0}^{a_0,b_0} [M_{k,n,2}, \hat{F}_{\hat{t}_k}]] \) \( \leq C(\Delta_n^2 e^{-\beta_0 n \Delta_n}) \)), which tends to zero, since all the obtained powers are smaller than \( \frac{6}{\alpha} \) for \( \hat{q}_0 \) close to 1 and \( Q_1 \) and \( \hat{q}_0 \) large enough.

Next, using Jensen’s inequality, (56) of Lemma 9 with \( (V_{a_0,b}^{a_0,b})^2 \), (58) of Lemma 10 with \( q = p = 2 \) and \( q_1 \) large enough, (66) with \( p \in \{1, 2\} \), and the Young’s inequality for products with \( \frac{1}{p_0} + \frac{1}{\hat{q}_0} = 1 \) and \( \hat{q}_0 \) close to 1, under \( \frac{6}{\alpha} > 3 \) for \( n \) large enough \( \sum_{k=0}^{n-1} \hat{E}_{a_0,b_0}^{a_0,b_0} [M_{k,n,2}^2, \hat{F}_{\hat{t}_k}]] \) is bounded by

\[ C e^{(\varphi_{2,n}(a_0,b_0))^2 \Delta_n X_{tk}^{a_0,b_0}} \left\{ \left| u \right| \varphi_{1,n}(a_0,b_0) \left( \frac{1}{X_{tk}^{a_0,b_0}} + \frac{\sqrt{\Delta_n}}{X_{tk}^{a_0,b_0}} \right) \right\} + \sqrt{\Delta_n} \left| v \right| \varphi_{2,n}(a_0,b_0) \left( 1 + \frac{1}{(X_{tk}^{a_0,b_0})^{\frac{1}{4n} \left( \frac{2n}{n} - 1 \right)}} \right) + \left| v \right| \varphi_{2,n}(a_0,b_0) \left( 1 + \frac{\sqrt{\Delta_n}}{X_{tk}^{a_0,b_0}} \right) \right) \right\} d\varepsilon \]
\[
\begin{aligned}
&+ \frac{\sqrt{\Delta_n}}{X_{t_k}^{a_0,b_0}} |u| \varphi_{1,n}(a_0, b_0) + \sqrt{\Delta_n} |v| \varphi_{2,n}(a_0, b_0) \left(1 + \frac{1}{(X_{t_k}^{a_0,b_0})^{1 + \frac{1}{16q_1}\left(\frac{n}{\sigma} - 1\right)}}\right) \\
&+ |v| \varphi_{2,n}(a_0, b_0) \left(1 + \sqrt{X_{t_k}^{a_0,b_0}} + \sqrt{\Delta_n} |v| \varphi_{2,n}(a_0, b_0) X_{t_k}^{a_0,b_0}\right) \right) d\varepsilon \\
&\leq C\nu^2(\varphi_{2,n}(a_0, b_0))^2 \Delta_n^{3/2} \sum_{k=0}^{n-1} \left(1 + (X_{t_k}^{a_0,b_0})^2\right) \\
&+ C(\varphi_{2,n}(a_0, b_0))^2 \Delta_n^{1/2} \frac{1}{\tilde{P}_0} \left(|u| \varphi_{1,n}(a_0, b_0) + |v| \varphi_{2,n}(a_0, b_0)\right) \sum_{k=0}^{n-1} \nu^2(\varphi_{2,n}(a_0, b_0))^2 \Delta_n^{3/2} X_{t_k}^{a_0,b_0} \\
&+ C(\varphi_{2,n}(a_0, b_0))^2 \Delta_n^{1/2} \frac{1}{\tilde{q}_0} \sum_{k=0}^{n-1} \varphi_{1,n}(a_0, b_0) \left(1 + (|b_0| + |v| \varphi_{2,n}(a_0, b_0))\tilde{q}_0(X_{t_k}^{a_0,b_0})^{3/2} \tilde{q}_0 + \left(1 + (|b_0| + |v| \varphi_{2,n}(a_0, b_0))\tilde{q}_0(X_{t_k}^{a_0,b_0})^{3/2} \tilde{q}_0\right) \right) \right) \\

\end{aligned}
\]

Thus, for subcritical case \( b_0 > 0 \) (respectively critical case \( b_0 = 0 \), respectively super-critical case \( b_0 < 0 \), using (67), Lemma 1(i) (respectively Lemma 1(ii), respectively Lemma 1(iii), (12)) and standard calculations, \( \tilde{E}^{a_0,b_0}[\sum_{k=0}^{n-1} \tilde{E}^{a_0,b_0}[M_{k,n,2}^2 | \tilde{F}_k]] \leq C \Delta_n^2 \) (respectively \( \tilde{E}^{a_0,0}[\sum_{k=0}^{n-1} \tilde{E}^{a_0,0}[M_{k,n,2}^2 | \tilde{F}_k]] \leq C n \Delta_n^3 \) and respectively \( \tilde{E}^{a_0,b_0}[\sum_{k=0}^{n-1} \tilde{E}^{a_0,b_0}[M_{k,n,2}^2 | \tilde{F}_k]] \leq C \Delta_n^2 e^{-b_0 n \Delta_n} \), which tends to zero, since all the obtained powers are smaller than \( \frac{a_0}{\sigma} \) for \( \tilde{q}_0 \) close to 1 and \( q_1 \) and \( n \) large enough. This completes the proof. \[\blacksquare\]

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**APPENDIX A. PROOF OF TECHNICAL RESULTS**

**A.1 Proof of Lemma 6**

*Proof.* First, recall that the estimates (39) and (40) can be found in lemmas 2.1 and 3.1 of Bossy and Diop (2007) or lemma 3.1 of De Marco (2011). Next, we treat (41).
For all $a, a_1 \in \mathbb{R}_+^*$, $b, b_1 \in \mathbb{R}$, $x \in \mathbb{R}_+^*$ and $k \in \{0, \ldots, n-1\}$, by lemma 3.1 of Barczy and Pap (2016), for any $a > \sigma$ the probability measures $\tilde{P}_{t_k, x}^{a,b}$ and $\tilde{P}_{t_{k+1}, x}^{a,b}$ are absolutely continuous w.r.t. each other and its Radon–Nikodym derivative is given by

$$\frac{d\tilde{P}_{t_{k+1}, x}^{a_1, b_1}}{d\tilde{P}_{t_k, x}^{a,b}}((Y_s^{a,b}(t_k, x))_{s \in [t_k, t_{k+1}]})$$

$$= \exp \left\{ \int_{t_k}^{t_{k+1}} \frac{a_1 - a - (b_1 - b)Y_s^{a,b}(t_k, x)}{2\sigma Y_s^{a,b}(t_k, x)} dY_s^{a,b}(t_k, x) - \frac{1}{2} \int_{t_k}^{t_{k+1}} \frac{(a_1 - a - (b_1 - b)Y_s^{a,b}(t_k, x))^2 - (a - bY_s^{a,b}(t_k, x))^2}{2\sigma Y_s^{a,b}(t_k, x)} ds \right\}$$

$$= \exp \left\{ \int_{t_k}^{t_{k+1}} \frac{a_1 - a - (b_1 - b)Y_s^{a,b}(t_k, x)}{\sqrt{2\sigma} Y_s^{a,b}(t_k, x)} dW_s - \frac{1}{2} \int_{t_k}^{t_{k+1}} \frac{(a_1 - a - (b_1 - b)Y_s^{a,b}(t_k, x))^2}{2\sigma Y_s^{a,b}(t_k, x)} ds \right\}.$$

Using (34) and the change of measures, we have that for any $p \in \mathbb{R}$,

$$\tilde{E}_{t_k, x}^{a,b}[|\partial_x Y_t^{a,b}(t_k, x)|^p]$$

$$= e^{-bp(t-t_k)} \tilde{E}_{t_k, x}^{a,b} \left[ \exp \left\{ \frac{-p\sigma}{4} \int_{t_k}^{t} \frac{du}{Y_u^{a,b}(t_k, x)} + p \sqrt{\frac{\sigma}{2}} \int_{t_k}^{t} \frac{dW_u}{Y_u^{a,b}(t_k, x)} \right\} \right]$$

$$= e^{-bp(t-t_k)} \tilde{E}_{t_k, x}^{a,b} \left[ \exp \left\{ \frac{p(p-1)\sigma}{4} \int_{t_k}^{t} \frac{du}{Y_u^{a,b}(t_k, x)} \right\} \right]$$

$$\times \exp \left\{ p \sqrt{\frac{\sigma}{2}} \int_{t_k}^{t} \frac{dW_u}{Y_u^{a,b}(t_k, x)} - \frac{p^2\sigma}{4} \int_{t_k}^{t} \frac{du}{Y_u^{a,b}(t_k, x)} \right\}$$

$$= e^{-bp(t-t_k)} \tilde{E}_{t_k, x}^{a,b} \left[ \exp \left\{ \frac{p(p-1)\sigma}{4} \int_{t_k}^{t} \frac{du}{Y_u^{a,b}(t_k, x)} \right\} \right] \frac{d\tilde{P}_{t_k, x}^{a+\sigma p, b}}{d\tilde{P}_{t_k, x}^{a,b}}((Y_s^{a,b}(t_k, x))_{s \in [t_k, t_{k+1}]})$$

$$= e^{-bp(t-t_k)} \tilde{E}_{t_k, x}^{a+\sigma p, b} \left[ \exp \left\{ \frac{p(p-1)\sigma}{4} \int_{t_k}^{t} \frac{du}{Y_u^{a+\sigma p,b}(t_k, x)} \right\} \right]. \quad (A1)$$

Then, applying Lemma 5 to the probability measure $\tilde{P}_{t_k, x}^{a+\sigma p, b}$ with $\mu = \frac{p(p-1)\sigma}{4}$, we get that for any $p \geq -\frac{(\frac{\sigma}{2} - 1)^2}{2(\frac{\sigma}{2} - 1)}$ and $t \in [t_k, t_{k+1}]$,

$$\tilde{E}_{t_k, x}^{a+\sigma p, b} \left[ \exp \left\{ \frac{p(p-1)\sigma}{4} \int_{t_k}^{t} \frac{du}{Y_u^{a+\sigma p,b}(t_k, x)} \right\} \right] \leq C_p \left( 1 + \frac{1}{x^{\frac{1}{2}}} \right).$$

This, together with (A1), gives the desired estimate (41).
A.2 Proof of Lemma 7

Proof. (i) For $t_k \leq s \leq t \leq t_{k+1}$, using (44), we have

$$
\frac{1}{2} \sqrt{\frac{\sigma}{2}} \left( \frac{1}{Y_s^{a,b}(t_k,x)} \right)^{\frac{1}{2}} + \frac{\sigma}{4} \int_s^t \frac{1}{(Y_u^{a,b}(t_k,x))^2} D_s Y_u^{a,b}(t_k,x) du
$$

$$
- \frac{1}{2} \sqrt{\frac{\sigma}{2}} \int_s^t \frac{1}{(Y_u^{a,b}(t_k,x))^2} D_s Y_u^{a,b}(t_k,x) dW_u \right) 1_{[t_k,t]}(s) = \frac{\sigma}{4} \int_s^t \frac{1}{(Y_u^{a,b}(t_k,x))^2} D_s Y_u^{a,b}(t_k,x) du
$$

$$
- \frac{1}{2} \sqrt{\frac{\sigma}{2}} \int_s^t \frac{1}{(Y_u^{a,b}(t_k,x))^2} D_s Y_u^{a,b}(t_k,x) dW_u \right) 1_{[t_k,t]}(s).
$$

We will show that under condition $\frac{a}{\sigma} > 4$, this expression is contained in $L^2(\Omega \times [t_k, t_{k+1}])$. In fact, using the fact that the exponential terms can be bounded by a positive constant since $a > \sigma$, BDG’s and Hölder’s inequalities with $\frac{1}{p} + \frac{1}{q} = 1$, (40) and (41), we get that

$$
\frac{1}{2} \sqrt{\frac{\sigma}{2}} \left( \frac{1}{Y_s^{a,b}(t_k,x)} \right)^{\frac{1}{2}} + \frac{\sigma}{4} \int_s^t \frac{1}{(Y_u^{a,b}(t_k,x))^2} D_s Y_u^{a,b}(t_k,x) du
$$

$$
- \frac{1}{2} \sqrt{\frac{\sigma}{2}} \int_s^t \frac{1}{(Y_u^{a,b}(t_k,x))^2} D_s Y_u^{a,b}(t_k,x) dW_u \right) 1_{[t_k,t]}(s)^2
$$

$$
\leq C E_{t_k,x}^{a,b} \left[ \frac{\partial Y_t^{a,b}(t_k,x)}{Y_s^{a,b}(t_k,x)} \right]^{\frac{1}{2}} + C E_{t_k,x}^{a,b} \left[ \frac{\partial Y_t^{a,b}(t_k,x)}{Y_s^{a,b}(t_k,x)} \right]^{\frac{1}{2}}
$$

$$
+ C E_{t_k,x}^{a,b} \left[ (\partial X_t^{a,b}(t_k,x))^2 \int_s^t \frac{1}{(Y_u^{a,b}(t_k,x))^2} du \right]
$$

$$
\leq C \left( \frac{1}{Y_s^{a,b}(t_k,x)^p} \right)^{\frac{1}{p}} \left( E_{t_k,x}^{a,b} [\partial Y_t^{a,b}(t_k,x)]^{2q} \right)^{\frac{1}{2}}
$$

$$
+ C \left( \int_s^t \frac{du}{(Y_u^{a,b}(t_k,x))^2} \right)^{\frac{1}{2}} \left( E_{t_k,x}^{a,b} [\partial Y_t^{a,b}(t_k,x)]^{2q} \right)^{\frac{1}{2}}
$$

$$
+ C \left( \int_s^t \frac{du}{(Y_u^{a,b}(t_k,x))^2} \right)^{\frac{1}{2}} \left( E_{t_k,x}^{a,b} [\partial Y_t^{a,b}(t_k,x)]^{2q} \right)^{\frac{1}{2}}
$$

$$
\leq C \left( 1 + \frac{1}{X^{2q+1}} \right) + C \left( \Delta_n^{2p-1} \int_s^t \frac{1}{Y_u^{a,b}(t_k,x)} \left( \frac{1}{Y_u^{a,b}(t_k,x)} \right)^{3p} du \right)^{\frac{1}{2}} \left( 1 + \frac{1}{X^{2q+1}} \right)
$$
Proof. Let $p > 1$ and $p$ is close to 1. Indeed, the condition required here is $3p < \frac{a}{\sigma} - 1$ and $\frac{a}{\sigma} > 4$, we only need to choose $p \in (1, \frac{1}{3}(\frac{a}{\sigma} - 1))$. Thus, under $\frac{a}{\sigma} > 4$,

$$\sqrt{\frac{\sigma}{2}} E_{t_k, x} \left[ \int_{t_k}^{t_k+1} \left( \frac{1}{Y_{u}^{a,b}(t_k, x)} \right) du \right] \leq C \left( 1 + \frac{1}{x^3} \right) \sqrt{\frac{\sigma}{2}}.$$  

This combined with (41) with $p = 2$ guarantees that $\partial_x Y_t^{a,b}(t_k, x) \in D^{1,2}$ under $\frac{a}{\sigma} > 4$ thanks to Theorem 6, and furthermore, its Malliavin derivative is given by (45).

Next, to prove (ii) the following lemma will be useful.

**Proof.** Let $p > 1$. Assume condition $\frac{a}{\sigma} > \frac{7p+4+\sqrt{p(4p+16)}}{4}$ for (A2) and (A4), and condition $\frac{a}{\sigma} > p + 1 + \sqrt{p(p+1)}$ for (A3). Then for any $(a, b) \in \Theta \times \Sigma, k \in \{0, \ldots, n-1\}, t_k \leq s \leq t \leq t_{k+1}$, and $x \in \mathbb{R}^*_+$, there exists a constant $C > 0$ which does not depend on $x$ such that

$$\left| \frac{D_x Y_t^{a,b}(t_k, x)}{\partial_x Y_t^{a,b}(t_k, x)} \right|^p \leq C \left( 1 + \frac{1}{x^{\frac{p-1}{2}} \frac{p}{2}} \right) \left( 1 + \frac{1}{x^{\frac{p}{2}} \frac{p}{2}} \right), \quad (A2)$$

$$\left| D_y Y_t^{a,b}(t_k, x) \right|^p \leq C \left( 1 + \frac{1}{x^{\frac{p-1}{2}} \frac{p}{2}} \right) \left( 1 + x^2 \right), \quad (A3)$$

$$\left| Y_t^{a,b}(t_k, x) D_x \left( \frac{1}{\partial_x Y_t^{a,b}(t_k, x)} \right) \right|^p \leq C \left( 1 + \frac{1}{x^{\frac{p-1}{2}} \frac{p}{2}} \right) \left( 1 + x^2 \right), \quad (A4)$$

where $p_0 = \frac{21p+8+\sqrt{9p(4p+16)}}{8(3p+1)}$, $p_2 > 1$ and $p_2$ is close to 1.

(ii) In the same way we show that the expression $\frac{-1}{\partial_x Y_t^{a,b}(t_k, x)^{p}} D(\partial_x Y_t^{a,b}(t_k, x))$ is contained in $L^2(\tilde{\Omega} \times [t_k, t_{k+1}])$ under condition $\frac{a}{\sigma} > \frac{9\sqrt{57}}{2}$. Indeed, applying (A2) of Lemma 17 with $p = 2,$
condition $\frac{a}{\sigma} > \frac{9+\sqrt{57}}{2}$ ensures that

$$
E_{t_k,x}^{a,b} \left[ D_s \left( \partial_x Y_t^{a,b}(t_k, x) \right) \right]^2 \leq C \left( 1 + \frac{1}{x^{\frac{2}{3p_2} - 1}} \right) \left( \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^2} \right).
$$

where $p_0 = \frac{25+\sqrt{513}}{28}$. This implies that

$$
E_{t_k,x}^{a,b} \left[ \int_{t_k}^{t_{k+1}} \left| D_s \left( \partial_x Y_t^{a,b}(t_k, x) \right) \right|^2 \, ds \right] \leq C \Delta_n \left( 1 + \frac{1}{x^{\frac{2}{3p_2} - 1}} \right) \left( \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^2} \right) < +\infty.
$$

Moreover, applying Lemma 6 with $p = -2$, (41) is satisfied under condition $\frac{a}{\sigma} > 3 + \sqrt{6}$. This guarantees that $(\partial_x Y_t^{a,b}(t_k, x))^{-1} \in \mathbb{D}^{1,2}$ under condition $\frac{a}{\sigma} > \frac{9+\sqrt{57}}{2}$ thanks to Lemma 6 and, furthermore, its Malliavin derivative is given by (46). Finally, in the same way we show that under condition $\frac{a}{\sigma} > \frac{9+\sqrt{57}}{2}$,

$$
\frac{1}{\partial_x Y_t^{a,b}(t_k, x)} D Y_t^{a,b}(t_k, x) + Y_t^{a,b}(t_k, x) D \left( \frac{1}{\partial_x Y_t^{a,b}(t_k, x)} \right)
$$

is contained in $L^2(\tilde{\Omega} \times [t_k, t_{k+1}])$. Indeed, applying (A3) and (A4) of Lemma 17 with $p = 2$, condition $\frac{a}{\sigma} > \frac{9+\sqrt{57}}{2}$ ensures that

$$
E_{t_k,x}^{a,b} \left[ \frac{1}{\partial_x Y_t^{a,b}(t_k, x)} D_s Y_t^{a,b}(t_k, x) + Y_t^{a,b}(t_k, x) D_s \left( \frac{1}{\partial_x Y_t^{a,b}(t_k, x)} \right) \right]^2 \right] \leq 2E_{t_k,x}^{a,b} \left[ \frac{1}{\partial_x Y_t^{a,b}(t_k, x)} D_s Y_t^{a,b}(t_k, x) \right]^2 \right] + 2E_{t_k,x}^{a,b} \left[ Y_t^{a,b}(t_k, x) D_s \left( \frac{1}{\partial_x Y_t^{a,b}(t_k, x)} \right) \right]^2 \right] \leq C \left( 1 + \frac{1}{x^{\frac{2}{3p_2} - 1}} \right) (1 + x) + C \left( 1 + \frac{1}{x^{\frac{2}{3p_2} - 1}} \right) (1 + x^2) \left( \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^2} \right),
$$

where $p_0 = \frac{25+\sqrt{513}}{28}, p_2 > 1$ and $p_2$ is close to 1. This implies that under condition $\frac{a}{\sigma} > \frac{9+\sqrt{57}}{2}$,

$$
E_{t_k,x}^{a,b} \left[ \int_{t_k}^{t_{k+1}} \left| \frac{1}{\partial_x Y_t^{a,b}(t_k, x)} D_s Y_t^{a,b}(t_k, x) + Y_t^{a,b}(t_k, x) D_s \left( \frac{1}{\partial_x Y_t^{a,b}(t_k, x)} \right) \right|^2 \, ds \right] \leq C \Delta_n \left( 1 + \frac{1}{x^{\frac{2}{3p_2} - 1}} \right) (1 + x) + C \Delta_n \left( 1 + \frac{1}{x^{\frac{2}{3p_2} - 1}} \right) (1 + x^2) \left( \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^2} \right) < +\infty.
$$
On the other hand, using Hölder’s inequality with $\frac{1}{p_1} + \frac{1}{q_1} = 1$, we get

$$\mathbb{E}_{t_k,x}^a \left[ \frac{Y_t^{a,b}(t_k,x)}{\partial_x Y_t^{a,b}(t_k,x)} \right] \leq \left( \mathbb{E}_{t_k,x}^a \left[ \frac{1}{|\partial_x Y_t^{a,b}(t_k,x)|^{2p_1}} \right] \right)^{\frac{1}{p_1}} \left( \mathbb{E}_{t_k,x}^a \left[ |Y_t^{a,b}(t_k,x)|^{2q_1} \right] \right)^{\frac{1}{q_1}}$$

$$\leq C \left( 1 + \frac{1}{\sigma} \frac{\sqrt{a} - \sigma}{1} \right) (1 + x^2) < +\infty,$$

where $p_1 > 1$ and $p_1$ is close to 1. Indeed, the condition required here is $-2p_1 \geq -\left(\frac{\sigma - 1}{\sigma}\right)^2$ and $\frac{a}{\sigma} > 3 + \sqrt{6}$, thus we only need to choose $p_1 \in (1, \frac{(\sigma - 1)^2}{4(\sigma - 1)^2})$. Hence $\frac{Y_t^{a,b}(t_k,x)}{\partial_x Y_t^{a,b}(t_k,x)} \in L^2(\tilde{\Omega})$ under $\frac{a}{\sigma} > 3 + \sqrt{6}$. This guarantees thanks to Theorem 6 that $\frac{Y_t^{a,b}(t_k,x)}{\partial_x Y_t^{a,b}(t_k,x)} \in D^{1,2}$ under condition $\frac{a}{\sigma} > \frac{9 + \sqrt{57}}{2}$ and, furthermore, its Malliavin derivative is given by (47). Thus, the result follows.

### A.3 Proof of Lemma 17

**Proof.** First, we treat (A2). For $t_k \leq s \leq t \leq t_{k+1}$, from (45) and (44), under $\frac{a}{\sigma} > 4$ we have

$$\frac{1}{(\partial_x Y_t^{a,b}(t_k,x))^2} D_s(\partial_x Y_t^{a,b}(t_k,x))$$

$$= -\frac{1}{\partial_x Y_t^{a,b}(t_k,x)} \left\{ \sqrt{\frac{1}{2} Y_t^{a,b}(t_k,x)} \right\}$$

$$+ \sigma \int_s^t \frac{\sqrt{2\sigma}}{(Y_u^{a,b}(t_k,x))^{\frac{1}{2}}} \exp \left\{ \int_s^u \left( -\frac{b}{2} - \frac{(a - \sigma)}{4} Y_u^{a,b}(t_k,x) \right) d\xi \right\} dW_u.$$

Then, using the fact that the exponential terms can be bounded by a positive constant since $a > \sigma$, BDG’s and Hölder’s inequalities with $\frac{1}{p_i} + \frac{1}{q_i} = 1$, (40) and (41), we get that

$$\mathbb{E}_{t_k,x}^a \left[ \left| D_s(\partial_x Y_t^{a,b}(t_k,x)) \right|^{p} \right]$$

$$\leq C \mathbb{E}_{t_k,x}^a \left[ \frac{1}{\partial_x Y_t^{a,b}(t_k,x)} \frac{1}{Y_t^{a,b}(t_k,x)} \right]^{p}$$

$$+ C \mathbb{E}_{t_k,x}^a \left[ \frac{1}{\partial_x Y_t^{a,b}(t_k,x)} \int_s^t \frac{du}{(Y_u^{a,b}(t_k,x))^{\frac{1}{2}}} \right]^{p} + C \mathbb{E}_{t_k,x}^a \left[ \frac{1}{(\partial_x Y_t^{a,b}(t_k,x))^2} \int_s^t \frac{du}{(Y_u^{a,b}(t_k,x))^2} \right]^{\frac{p}{2}}.$$
The unique positive solution is given by
\[
\begin{align*}
\frac{1}{\partial x} Y^a, b(t_k, x) & \leq C \left( \frac{1}{|\partial_x Y^a, b(t_k, x)| |pp_4|} \right)^{\frac{1}{p_k}} \left( \frac{1}{|Y^a, b(t_k, x)| |pp_4|} \right)^{\frac{1}{p_k}} \\
& + C \left( \frac{1}{|\partial_x Y^a, b(t_k, x)| |pp_4|} \right)^{\frac{1}{p_k}} \left( \frac{1}{|Y^a, b(t_k, x)| |pp_4|} \right)^{\frac{1}{p_k}} \\
& + C \left( \frac{1}{|\partial_x Y^a, b(t_k, x)| |pp_4|} \right)^{\frac{1}{p_k}} \left( \frac{1}{|Y^a, b(t_k, x)| |pp_4|} \right)^{\frac{1}{p_k}} \\
& \leq C \left( 1 + \frac{1}{x^{\frac{1}{2} - \frac{\epsilon}{2}}} \right)^{\frac{1}{p_k}} + C \left( 1 + \frac{1}{x^{\frac{1}{2} - \frac{\epsilon}{2}}} \right) \left( \Delta_{q_k}^{p_4 - 1} \int_s^t \frac{1}{|Y^a, b(t_k, x)| |pp_4|} \right)^{\frac{1}{p_k}} \\
& + C \left( 1 + \frac{1}{x^{\frac{1}{2} - \frac{\epsilon}{2}}} \right) \left( \Delta_{q_k}^{p_4 - 1} \int_s^t \frac{1}{|Y^a, b(t_k, x)| |pp_4|} \right)^{\frac{1}{p_k}} \\
& \leq C \left( 1 + \frac{1}{x^{\frac{1}{2} - \frac{\epsilon}{2}}} \right)^{\frac{1}{p_k}} + C \Delta_{q_k}^{p_4} \left( 1 + \frac{1}{x^{\frac{1}{2} - \frac{\epsilon}{2}}} \right)^{\frac{1}{p_k}} + C \Delta_{q_k}^{p_4} \left( 1 + \frac{1}{x^{\frac{1}{2} - \frac{\epsilon}{2}}} \right)^{\frac{1}{p_k}} \\
& \leq C \left( 1 + \frac{1}{x^{\frac{1}{2} - \frac{\epsilon}{2}}} \right)^{\frac{1}{p_k}} + \frac{1}{x^{\frac{1}{2} - \frac{\epsilon}{2}}} + \frac{1}{x^{\frac{1}{2} - \frac{\epsilon}{2}}} \\
& \leq C \left( 1 + \frac{1}{x^{\frac{1}{2} - \frac{\epsilon}{2}}} \right)^{\frac{1}{p_k}} + \frac{1}{x^{\frac{1}{2} - \frac{\epsilon}{2}}} + \frac{1}{x^{\frac{1}{2} - \frac{\epsilon}{2}}}
\end{align*}
\]

Here, conditions are required as follows

\[ -pp_4 \geq \frac{(a - 1)^2}{2a - 1}, \quad \frac{3}{2} pq_4 < \frac{a}{\sigma} - 1. \]

This implies that

\[ \frac{a}{\sigma} \geq pp_4 + \sqrt{pp_4(pp_4 + 1)} + 1, \quad \frac{a}{\sigma} > \frac{3}{2} pq_4 + 1 = \frac{3}{2} pp_4 + 1. \]

Hence, the optimal choice for \( p_4 \) is solution to

\[ pp_4 + \sqrt{pp_4(pp_4 + 1)} = \frac{3}{2} pp_4 + 1. \]

The unique positive solution is given by \( p_4 = \frac{21p + 8 + \sqrt{9pp(49pp + 16)}}{8(3p + 1)} > 1. \) Thus, (A2) is valid under condition \( \frac{a}{\sigma} > \frac{7p + 4 + \sqrt{p(49p + 16)}}{8(3p + 1)}. \)

Next, we treat (A3). For \( t_k \leq s \leq t \leq t_{k+1} \), using (44), under condition \( a \geq \sigma \) we have

\[
\frac{1}{\partial x} Y^a, b(t_k, x) = \sqrt{2\sigma Y^a, b(t_k, x)} \exp \left\{ \int_s^t \left( -\frac{b}{2} \cdot a + \frac{\sigma}{2} \frac{1}{Y^a, b(t_k, x)} \right) du \right\}
\]
Then, using the fact that the exponential terms can be bounded by a positive constant since \( a > \sigma \), Hölder’s inequality with \( \frac{1}{p_2} + \frac{1}{q_2} = 1 \), (39) and (41), we get that

\[
\begin{align*}
\mathbb{E}_{t_k,x}^{a,b} \left[ \frac{1}{\partial_x Y_t^{a,b}(t_k,x)} \right]^{p_2} & \leq C \mathbb{E}_{t_k,x}^{a,b} \left[ \frac{1}{\partial_x Y_t^{a,b}(t_k,x)} \right]^{p_2} \\
& \leq C \left( 1 + \frac{1}{\sqrt{x}} \right) (1 + x^{\sigma/2}),
\end{align*}
\]

where \( p_2 > 1 \) and \( p_2 \) is close to 1. Indeed, the condition required here is \(-pp_2 \geq \frac{(\sigma - 1)^2}{2(\sigma - 2)}\) and \( a > p + 1 + \sqrt{p(p + 1)} \), thus we only need to choose \( p_2 \in (1, \frac{(\sigma - 1)^2}{2(\sigma - 2)} \). Hence, (A3) is valid under condition \( a > p + 1 + \sqrt{p(p + 1)} \). Finally, we treat (A4). For \( t_k \leq s \leq t \leq t_{k+1} \), using (46) and (A5), under \( \frac{a}{\sigma} > \frac{9 + \sqrt{37}}{2} \), we have

\[
Y_t^{a,b}(t_k,x) D_s \left( \frac{1}{\partial_x Y_t^{a,b}(t_k,x)} \right) = -\frac{Y_t^{a,b}(t_k,x)}{\partial_x Y_t^{a,b}(t_k,x)} \left\{ \sqrt{\frac{\sigma}{2}} \frac{1}{\sqrt{Y_s^{a,b}(t_k,x)}} \right. \\
+ \frac{\sigma}{4} \int_s^t \sqrt{2\sigma} \frac{1}{(Y_u^{a,b}(t_k,x))^2} \exp \left\{ \int_u^s \left( -\frac{b}{2} - \frac{a}{4} - \frac{\sigma}{4} \right) \frac{1}{Y_{\xi}^{a,b}(t_k,x)} \right\} du \right. \\
- \frac{\sigma}{2} \int_s^t \frac{1}{Y_u^{a,b}(t_k,x)} \exp \left\{ \int_u^s \left( -\frac{b}{2} - \frac{a}{4} - \frac{\sigma}{4} \right) \frac{1}{Y_{\xi}^{a,b}(t_k,x)} \right\} dW_u \right\}. \tag{A6}
\]

Then, using the fact that the exponential terms can be bounded by a positive constant since \( a > \sigma \), BDG’s and Hölder’s inequalities with \( \frac{1}{p_2} + \frac{1}{q_2} = 1 \) and \( \frac{1}{p_3} + \frac{1}{q_3} = 1 \), (39), (40) and (41), we get that

\[
\begin{align*}
\mathbb{E}_{t_k,x}^{a,b} \left[ Y_t^{a,b}(t_k,x) D_s \left( \frac{1}{\partial_x Y_t^{a,b}(t_k,x)} \right) \right]^{p_2} & \leq C \mathbb{E}_{t_k,x}^{a,b} \left[ Y_t^{a,b}(t_k,x) \frac{1}{\partial_x Y_t^{a,b}(t_k,x)} \sqrt{Y_s^{a,b}(t_k,x)} \right]^{p_2} \\
& + C \mathbb{E}_{t_k,x}^{a,b} \left[ \int_s^t \frac{1}{(Y_u^{a,b}(t_k,x))^2} \right]^{p_2} + C \mathbb{E}_{t_k,x}^{a,b} \left[ \int_s^t \frac{1}{(Y_u^{a,b}(t_k,x))^2} \right]^{p_2}.
\end{align*}
\]
Thus, the optimal choice for $p$ is $p = 3/2p_3$, where $p_3 > 1$ and $p_3$ is close to 1. Here, conditions are required as follows

$$-pp_3 p_3 \geq \frac{(a - 1)^2}{2(a - \frac{1}{2})}, \quad \frac{3}{2}pp_3 < \frac{a}{\sigma} - 1.$$

This implies that

$$pp_3 < \frac{(a - 1)^2}{2(a - \frac{1}{2})}, \quad \frac{3}{2}pq_3 < \frac{a}{\sigma} - 1.$$

Thus,

$$\frac{a}{\sigma} > pp_3 + \sqrt{pp_3(pp_3 + 1)} + 1, \quad \frac{a}{\sigma} > \frac{3}{2}pq_3 + 1 = \frac{3}{2} \frac{pp_3}{p_3 - 1} + 1.$$

Hence, the optimal choice for $p_3$ is solution to

$$pp_3 + \sqrt{pp_3(pp_3 + 1)} = \frac{3}{2} \frac{pp_3}{p_3 - 1}.$$
The unique positive solution is given by $p_3 = \frac{21p + 8 + \sqrt{9p(49p + 16)}}{8(p + 1)} > 1$. Thus, (A4) is valid under condition $\frac{a}{\sigma} > \frac{7p + 4 + \sqrt{49p(49p + 16)}}{4}$. 

A.4 Proof of Proposition 1

**Proof.** We are going to apply Theorem 5. First, we wish to show $\partial_a Y^{a,b}_{t_{k+1}}(t_k, x)U^{a,b}_{t_{k+1}}(t_k, x) \in \text{Dom} \ \delta$ under condition $\frac{a}{\sigma} > \frac{9 + \sqrt{57}}{2}$. For this, we write

$$\partial_a Y^{a,b}_{t_{k+1}}(t_k, x)U^{a,b}_{t_{k+1}}(t_k, x) = \frac{\partial_a Y^{a,b}_{t_{k+1}}(t_k, x)}{\sqrt{2\sigma Y^{a,b}_{t_{k+1}}(t_k, x)}} (\partial_a Y^{a,b}_{t_{k+1}}(t_k, x))^{-1} \partial_x Y^{a,b}_{t_{k+1}}(t_k, x) = Fu., \quad (A7)$$

where $F = \partial_a Y^{a,b}_{t_{k+1}}(t_k, x)(\partial_a Y^{a,b}_{t_{k+1}}(t_k, x))^{-1}$ and $u = \frac{1}{\sqrt{2\sigma Y^{a,b}_{t_{k+1}}(t_k, x)}} \partial_x Y^{a,b}_{t_{k+1}}(t_k, x)$. Here $u = (u_t, t \in [t_k, t_{k+1}])$ is an adapted process then it belongs to $\text{Dom} \ \delta$. By Theorem 5, it suffices to show that $F \in \mathbb{D}^{1,2}$ and $Fu \in L^2(\tilde{\Omega}; H) \cong L^2([t_k, t_{k+1}] \times \tilde{\Omega}, \mathbb{R})$, where $H = L^2([t_k, t_{k+1}], \mathbb{R})$.

From (37), we write

$$F = \frac{\partial_a Y^{a,b}_{t_{k+1}}(t_k, x)}{\partial_x Y^{a,b}_{t_{k+1}}(t_k, x)} = \int_{t_k}^{t_{k+1}} \frac{dr}{\partial_x Y^{a,b}_{t_r}(t_k, x)}. $$

By assertion (ii) of Lemma 7, under condition $\frac{a}{\sigma} > \frac{9 + \sqrt{57}}{2}$, $(\partial_a Y^{a,b}_{t_t}(t_k, x))^{-1} \in \mathbb{D}^{1,2}$ for any $t \in [t_k, t_{k+1}]$. Thus, it is straightforward that $F \in \mathbb{D}^{1,2}$ under condition $\frac{a}{\sigma} > \frac{9 + \sqrt{57}}{2}$. Furthermore, for $t_k \leq s \leq r \leq t_{k+1}$,

$$D_s F = \int_s^{t_{k+1}} D_s \left( \frac{1}{\partial_x Y^{a,b}_{t_r}(t_k, x)} \right) dr. $$

Next, we check $Fu \in L^2([t_k, t_{k+1}] \times \tilde{\Omega}, \mathbb{R})$. For this, using Hölder's inequality repeatedly with $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$, together with (40) and (41), we get that

$$\tilde{E}_{t_{k,x}}[|Fu|^2]$$

$$\leq \frac{1}{2\sigma} \left( \tilde{E}_{t_{k,x}} \left[ \left| \int_{t_k}^{t_{k+1}} \frac{dr}{\partial_x Y^{a,b}_{t_r}(t_k, x)} \sqrt{\frac{1}{\partial_x Y^{a,b}_{t_r}(t_k, x)}} \right|^{2p} \right]^\frac{1}{p} \right) \left( \tilde{E}_{t_{k,x}} \left[ | \partial_a Y^{a,b}_{t}(t_k, x)|^{2q} \right] \right)^\frac{1}{q}$$

$$\leq C \left( \tilde{E}_{t_{k,x}} \left[ \left| \int_{t_k}^{t_{k+1}} \frac{dr}{\partial_x Y^{a,b}_{t_r}(t_k, x)} \right|^{2pp_1} \right] \right)^\frac{1}{p_1} \left( \tilde{E}_{t_{k,x}} \left[ \frac{1}{|Y^{a,b}_{t_r}(t_k, x)|} \right]^{p_1} \right) \left( 1 + \frac{1}{x^{\frac{5}{2} - \frac{3\omega}{2}}} \right)^\frac{1}{q}.$$
\[ \leq C \left( \Delta_n^{2pp_1-1} \int_{t_k}^{t_{k+1}} \mathbb{E}_{t_k,x}^{a,b} \left[ \frac{1}{|\partial_x Y_r^{a,b}(t_k,x)|^{2pp_1}} \right] dr \right) \frac{1}{p_1} \frac{1}{x} \left( 1 + \frac{1}{x^\frac{p-1}{n+1}} \right) \]

\[ \leq C \Delta_n^2 \left( 1 + \frac{1}{x^\frac{p-1}{n+1}} \right) \frac{1}{x} \left( 1 + \frac{1}{x^\frac{p-1}{n+1}} \right) , \]

for some constant \( C > 0 \), where \( p > 1 \) and \( p \) is close to 1. This shows that

\[ \mathbb{E}_{t_k,x}^{a,b} \left[ \int_{t_k}^{t_{k+1}} (Fu)^2 dt \right] = \int_{t_k}^{t_{k+1}} \mathbb{E}_{t_k,x}^{a,b} (Fu)^2 dt < +\infty. \]

All conditions required here are as follows

\[-2pp_1 \geq -\frac{(\frac{a}{\sigma} - 1)^2}{2(\frac{a}{\sigma} - \frac{1}{2})}, \quad pq_1 < \frac{a}{\sigma} - 1.\]

That is,

\[-2p_1 > -\frac{(\frac{a}{\sigma} - 1)^2}{2(\frac{a}{\sigma} - \frac{1}{2})}, \quad q_1 < \frac{a}{\sigma} - 1.\]

This implies that

\[ \frac{a}{\sigma} > 2p_1 + \sqrt{2p_1(2p_1 + 1) + 1}, \quad \frac{a}{\sigma} > q_1 + 1 = \frac{p_1}{p_1 - 1} + 1. \]

Hence, the optimal choice for \( p_1 \) is solution to

\[ 2p_1 + \sqrt{2p_1(2p_1 + 1)} = \frac{p_1}{p_1 - 1}. \]

The unique positive solution is given by \( p_1 = \frac{9 + \sqrt{33}}{2} \). Thus, \( Fu \in L^2([t_k, t_{k+1}] \times \tilde{\Omega}, \mathbb{R}) \) under \( \frac{a}{\sigma} > \frac{9 + \sqrt{33}}{2} \). Hence, we have shown that \( \partial_a Y^{a,b}_{t_{k+1}}(t_k,x)U^{a,b}(t_k,x) \in \text{Dom } \delta \) under condition \( \frac{a}{\sigma} > \frac{9 + \sqrt{33}}{2} \).

Next, under condition \( \frac{a}{\sigma} > \frac{9 + \sqrt{57}}{2} \) we proceed as in the proof of proposition 3.1 of Kohatsu-Higa et al. (2017) with \( \beta = \tilde{a} \) (see pp. 441 and 442) to get the following representation of the score function

\[ \frac{\partial_a p^{a,b}}{p^{a,b}}(\Delta_n, x, y) = \frac{1}{\Delta_n} \mathbb{E}_{t_k,x}^{a,b} \left[ \delta(\partial_a Y^{a,b}_{t_{k+1}}(t_k,x)U^{a,b}(t_k,x)) \right] Y^{a,b}_{t_{k+1}} = y]. \]

Now, we wish to show (48). In fact, using condition \( \frac{a}{\sigma} > \frac{9 + \sqrt{57}}{2} \) and the fact that the Skorohod integral and the Itô integral of an adapted process coincide \( \delta(u) = \int_{t_k}^{t_{k+1}} \frac{\partial_y Y^{a,b}(t_k,x)}{\sqrt{2\sigma Y^{a,b}(t_k,x)}} dW_s \), we have
\[
F \delta(u) - < DF, u >_H = \int_{t_k}^{t_{k+1}} \frac{dr}{\partial_x Y_{a,b}^r(t_k,x)} \int_{t_k}^{t_{k+1}} \frac{\partial Y_{a,b}^r(t_k,x)}{\sqrt{2\sigma Y_{a,b}^r(t_k,x)}} dW_s - \int_{t_k}^{t_{k+1}} \int_s^{t_{k+1}} D_s \left( \frac{1}{\partial_x Y_{a,b}^r(t_k,x)} \right) dr \frac{\partial Y_{a,b}^r(t_k,x)}{\sqrt{2\sigma Y_{a,b}^r(t_k,x)}} ds. \tag{A8}
\]

We next add and subtract the term \( \frac{\partial Y_{a,b}^r(q_t,x)}{\sqrt{2\sigma Y_{a,b}^r(q_t,x)}} \) in the second integral, and the term \( \frac{1}{\partial_x Y_{a,b}^r(q_t,x)} \) in the first integral. This, together with \( Y_{a,b}^r(t_k,x) = x \), implies that

\[
F \delta(u) - < DF, u >_H = \frac{\Delta_n}{\sqrt{2\sigma}} (W_{t_{k+1}} - W_{t_k}) + R_1^{a,b} + R_2^{a,b} + R_3^{a,b}. \tag{A9}
\]

Then, applying (52) of Lemma 8 with \( q = 1 \), under condition \( \frac{\sigma}{\sigma} > 5 + 3\sqrt{2} \), we get

\[
E_{q,x}^a[(F \delta(u) - < DF, u >_H)^2] \leq \frac{\Delta_n^2}{\sigma x} [E_{q,x}^a((W_{t_{k+1}} - W_{t_k})^2) + 2E_{q,x}^a((R_1^{a,b} + R_2^{a,b} + R_3^{a,b})^2)] \leq \frac{\Delta_n^2}{\sigma x} + C\Delta_n^4 \left( \frac{1}{x} + \frac{1}{x^{\frac{\sigma}{\sigma - 1}}} \right) < +\infty.
\]

Thus, we have shown that \( F \delta(u) - < DF, u >_H \) is square integrable under condition \( \frac{\sigma}{\sigma} > 5 + 3\sqrt{2} \). Consequently, by Theorem 5, under condition \( \frac{\sigma}{\sigma} > 5 + 3\sqrt{2} \) we have

\(\delta(Fu) = F\delta(u) - < DF, u >_H\.\)

This, together with (A7) and (A9), gives (48) under condition \( \frac{\sigma}{\sigma} > 5 + 3\sqrt{2}\).

Finally, we write

\[
\partial_y Y_{a,b}^r(t_k,x) U_{a,b}^r(t_k,x) = \frac{\partial Y_{a,b}^r(t_k,x)}{\sqrt{2\sigma Y_{a,b}^r(t_k,x)}} (\partial Y_{a,b}^r(t_k,x))^{-1} \partial_y Y_{a,b}^r(t_k,x) = \overline{F} u, \tag{A10}
\]

where \( \overline{F} = \partial_y Y_{a,b}^r(t_k,x)(\partial Y_{a,b}^r(t_k,x))^{-1} \). From (38), we have

\[
\overline{F} = \frac{\partial Y_{a,b}^r(t_k,x)}{\partial_x Y_{a,b}^r(t_k,x)} = -\int_{t_k}^{t_{k+1}} \frac{Y_{r}^{a,b}(t_k,x)}{\partial_x Y_{a,b}^r(t_k,x)} dr.
\]

Then, by assertion (ii) of Lemma 7 and proceeding as above, we get that under condition \( \frac{\sigma}{\sigma} > \frac{9 + \sqrt{57}}{2} \), \( \partial_y Y_{a,b}^r(t_k,x) U_{a,b}^r(t_k,x) \in \text{Dom} \ \delta \) and

\[
\frac{\partial_y p_{a,b}^{r,a}}{p_{a,b}^{r,a}}(\Delta_n, x, y) = \frac{1}{\Delta_n} E_{q,x}^a [\delta(\partial_y Y_{a,b}^r(t_k,x) U_{a,b}^r(t_k,x)) \big| Y_{a,b}^r(x, t_k) = y].
\]
Furthermore, using the same arguments as for (A8), we can easily check that under condition 
\[
\frac{a}{\sigma} > \frac{9 + \sqrt{57}}{2},
\]
\[
\bar{F}\delta(u) - < D\bar{F}, u >_H = - \int_{t_k}^{t_{k+1}} \int_s^{t_{k+1}} \frac{Y_{r,s}^{a,b}(t_k,x)}{2\sigma Y_{r,s}^{a,b}(t_k,x)} \frac{d\sigma}{\sqrt{2\sigma Y_{r,s}^{a,b}(t_k,x)}} dr + \int_{t_k}^{t_{k+1}} \int_s^{t_{k+1}} D_s \left( \frac{Y_{r,s}^{a,b}(t_k,x)}{2\sigma Y_{r,s}^{a,b}(t_k,x)} \right) dr \frac{d\sigma}{\sqrt{2\sigma Y_{r,s}^{a,b}(t_k,x)}} ds.
\]

We next add and subtract the term \( \frac{Y_{r,s}^{a,b}(t_k,x)}{2\sigma Y_{r,s}^{a,b}(t_k,x)} \) in the first integral, and the term \( \frac{d\sigma}{\sqrt{2\sigma Y_{r,s}^{a,b}(t_k,x)}} \) in the second integral. This, together with \( Y_{t_k}^{a,b}(t_k,x) = x \), shows that
\[
\bar{F}\delta(u) - < D\bar{F}, u >_H = - \frac{\Delta_n}{\sqrt{2\sigma}} \sqrt{x(W_{t_{k+1}} - W_{t_k})} - x R_{11}^{a,b} + H_{22}^{a,b} + H_{33}^{a,b}.
\] (A11)

Then, applying (53) of Lemma 8 with \( q = 1 \), we get that \( \bar{F}\delta(u) - < D\bar{F}, u >_H \) is square integrable under condition \( \frac{a}{\sigma} > 5 + 3\sqrt{2} \). Consequently, from (B1) of Theorem 5, (A10) and (A11) we conclude (49) under condition \( \frac{a}{\sigma} > 5 + 3\sqrt{2} \). Thus, the result follows.

A.5 Proof of Lemma 8

Proof of (50) and (51). These facts follow easily from (48), (49), and properties of the moment of the Skorohod integral and the Brownian motion.

Proof of (52). Observe that
\[
\bar{E}_{t_k,x}^{a,b} \left[ \left| R_1^{a,b} + R_2^{a,b} + R_3^{a,b} \right|^{2q} \right] \leq 2^{2q-1} \left( \bar{E}_{t_k,x}^{a,b} \left[ R_1^{a,b} \right]^{2q} + \bar{E}_{t_k,x}^{a,b} \left[ R_2^{a,b} \right]^{2q} + \bar{E}_{t_k,x}^{a,b} \left[ R_3^{a,b} \right]^{2q} \right). \quad (A12)
\]

First, we treat the term \( R_1^{a,b} \). Using BDG’s inequality, we have that
\[
\bar{E}_{t_k,x}^{a,b} \left[ \left| R_1^{a,b} \right|^{2q} \right] \leq C\Delta_n^{2q} \bar{E}_{t_k,x}^{a,b} \left[ \int_{t_k}^{t_{k+1}} \left( \frac{\partial_x Y_{s}^{a,b}(t_k,x)}{\sqrt{2\sigma Y_{s}^{a,b}(t_k,x)}} - \frac{\partial_x Y_{t_k}^{a,b}(t_k,x)}{\sqrt{2\sigma Y_{t_k}^{a,b}(t_k,x)}} \right) ds \right]^{2q} 
\]
\[
\leq C\Delta_n^{2q} \Delta_n^{q-1} \int_{t_k}^{t_{k+1}} \frac{R_{11}^{a,b}}{s} ds,
\]
where
\[
R_{11}^{a,b} = \bar{E}_{t_k,x}^{a,b} \left[ \left( \frac{\partial_x Y_{s}^{a,b}(t_k,x)}{\sqrt{2\sigma Y_{s}^{a,b}(t_k,x)}} - \frac{\partial_x Y_{t_k}^{a,b}(t_k,x)}{\sqrt{2\sigma Y_{t_k}^{a,b}(t_k,x)}} \right)^2 \right]^{1/2q}.
\]
By Itô's formula, it can be checked that

\[
\frac{\partial_t Y^a_{t,s}(t_k, x)}{\sqrt{Y^a_{t,s}(t_k, x)}} - \frac{\partial_t Y^b_{t,k}(t_k, x)}{\sqrt{Y^a_{t,k}(t_k, x)}} = \int_{t_k}^{s} \frac{\partial_t Y^a_{t,u}(t_k, x)}{\sqrt{Y^a_{t,u}(t_k, x)}} \left( \frac{-a + \sigma}{(Y^a_{t,u}(t_k, x))^2} - \frac{b}{2 \sqrt{Y^a_{t,u}(t_k, x)}} \right) du,
\]

which, together with Hölder's inequality with \( \frac{1}{p_0} + \frac{1}{q_0} = 1 \), (40) and (41), implies that

\[
R_{11}^{a,b} \leq C \Delta_n^{2q-1} \int_{t_k}^{s} \left\{ \left( \frac{p_0}{2} \right)^{-1} \left( \frac{1}{|Y^a_{t,u}(t_k, x)|^{3q_0}} \right) \right\} du \leq C \Delta_n^{2q-1} \int_{t_k}^{s} \left\{ \left( 1 + \frac{1}{x^{q_0}} \right) \left( \frac{1}{x^{3q_0}} \right) \right\} du \leq C \Delta_n^{2q-1} \left\{ \left( 1 + \frac{1}{x^{q_0}} \right) \left( \frac{1}{x^{3q_0}} \right) \right\}.
\]

Here, \( q_0 \) should be chosen close to 1 in order that \( 3q_0 < \frac{a}{\sigma} - 1 \). Thus, by choosing \( q_0 \in (1, \frac{1}{3q} (\frac{a}{\sigma} - 1)) \), under condition \( \frac{a}{\sigma} > 3q + 1 \), we have

\[
\tilde{E}_{t_k,x}([R_{11}^{a,b} | 2q]) \leq C \Delta_n^{3q} \left\{ \left( 1 + \frac{1}{x^{q_0+q}} \right) \left( \frac{1}{x^{3q_0+q}} \right) \right\} \leq C \Delta_n^{3q} \left\{ \frac{1}{x^{q_0}} + \frac{1}{x^{q_0+4q}} \right\}, \tag{A13}
\]

where \( p_0 > 1 \) with \( q_0 = \frac{p_0}{p_0 - 1} \) close to 1. Then, when \( q \in [1, \frac{13 + \sqrt{89}}{20}] \) and \( \frac{a}{\sigma} > \frac{11 + \sqrt{89}}{2} \) it suffices to choose \( p_0 \) large enough to get \( \frac{a}{2p_0} + 4q < \frac{a}{\sigma} - 1 \).

Next, we treat the term \( R_{22}^{a,b} \). Using Hölder’s inequality with \( \frac{1}{p} + \frac{1}{q} = 1 \), we have

\[
\tilde{E}_{t_k,x}([R_{22}^{a,b} | 2q]) \leq (C^{2q})^{\frac{1}{q}} \left( R_{21}^{a,b} \right)^{\frac{1}{q}},
\]

where
\[ R_{21}^{a,b} = E_{b,x}^{a,b} \left[ \begin{array}{c} \int_{t_k}^{t_{k+1}} \left( \frac{1}{\partial_x Y_s^{a,b}(t_k, x)} - \frac{1}{\partial_x Y_s^{a,b}(t_k, x)} \right) ds \end{array} \right]^{2q_{\bar{p}}}, \]

\[ R_{22}^{a,b} = E_{b,x}^{a,b} \left[ \int_{t_k}^{t_{k+1}} \frac{\partial_x Y_s^{a,b}(t_k, x)}{\sqrt{2\sigma Y_s^{a,b}(t_k, x)}} dW_s \right]^{2q_{\bar{p}}}. \]

First, observe that \( R_{21}^{a,b} \leq C \Delta_n^{2q_{\bar{p}}-1} \int_{t_k}^{t_{k+1}} R_{212}^{a,b} ds \), where

\[ R_{212}^{a,b} = E_{b,x}^{a,b} \left[ \begin{array}{c} \frac{1}{\partial_x Y_s^{a,b}(t_k, x)} - \frac{1}{\partial_x Y_s^{a,b}(t_k, x)} \end{array} \right]^{2q_{\bar{p}}}. \]

From (31), Itô’s formula and \( \partial_x Y_{tk}^{a,b}(t_k, x) = 1 \), for any \( t \in [t_k, t_{k+1}] \), we have

\[
\frac{1}{\partial_x Y_s^{a,b}(t_k, x)} - \frac{1}{\partial_x Y_s^{a,b}(t_k, x)} = \int_{t_k}^{s} \left( \frac{b}{\partial_x Y_{u}^{a,b}(t_k, x)} + \frac{\sigma}{Y_{u}^{a,b}(t_k, x) \partial_x Y_{u}^{a,b}(t_k, x)} \right) du \\
- \int_{t_k}^{s} \sqrt{\sigma} \frac{1}{\sqrt{2Y_{u}^{a,b}(t_k, x) \partial_x Y_{u}^{a,b}(t_k, x)}} dW_u. \tag{A14}
\]

Therefore, \( R_{212}^{a,b} \leq C(R_{2121}^{a,b} + R_{2122}^{a,b} + R_{2123}^{a,b}) \), where

\[ R_{2121}^{a,b} = E_{b,x}^{a,b} \left[ \int_{t_k}^{s} \frac{du}{\partial_x Y_{u}^{a,b}(t_k, x)} \right]^{2q_{\bar{p}}}, \quad R_{2122}^{a,b} = E_{b,x}^{a,b} \left[ \int_{t_k}^{s} \frac{du}{Y_{u}^{a,b}(t_k, x) \partial_x Y_{u}^{a,b}(t_k, x)} \right]^{2q_{\bar{p}}}, \]

\[ R_{2123}^{a,b} = E_{b,x}^{a,b} \left[ \int_{t_k}^{s} \frac{1}{\sqrt{Y_{u}^{a,b}(t_k, x) \partial_x Y_{u}^{a,b}(t_k, x)}} dW_u \right]^{2q_{\bar{p}}}. \]

Using (41),

\[ R_{2121}^{a,b} \leq \Delta_n^{2q_{\bar{p}}-1} \int_{t_k}^{s} E_{b,x}^{a,b} \left[ \frac{1}{|\partial_x Y_{u}^{a,b}(t_k, x)|^{2q_{\bar{p}}}} \right] du \leq C \Delta_n^{2q_{\bar{p}}} \left( 1 + \frac{1}{\frac{q_{\bar{p}}}{\bar{q}} - q_{\bar{p}}} \right). \]

Next, using Hölder’s inequality with \( \frac{1}{p_1} + \frac{1}{q_1} = 1 \), (40) and (41),

\[ R_{2122}^{a,b} \leq \Delta_n^{2q_{\bar{p}}-1} \int_{t_k}^{s} E_{b,x}^{a,b} \left[ \frac{1}{Y_{u}^{a,b}(t_k, x) |\partial_x Y_{u}^{a,b}(t_k, x)|^{2q_{\bar{p}}}} \right] du \]

\[ \leq \Delta_n^{2q_{\bar{p}}-1} \int_{t_k}^{s} \left( E_{b,x}^{a,b} \left[ \frac{1}{Y_{u}^{a,b}(t_k, x)^{2p_1q_{\bar{p}}}} \right] \right)^{\frac{1}{p_1}} \left( E_{b,x}^{a,b} \left[ \frac{1}{|\partial_x Y_{u}^{a,b}(t_k, x)|^{2q_{\bar{p}}}} \right] \right)^{\frac{1}{q_1}} du. \]
Finally, using BDG’s and Hölder’s inequalities with \( \frac{1}{p_i} + \frac{1}{q_i} = 1 \), (40) and (41),

\[
R_{2123}^{a,b} \leq C \Delta_n^{q\overline{p}-1} \int_{t_k}^{s} \left( \frac{1}{|\sqrt{Y_{a,b}^{u}(t_k, x)}|^{2q_p, q_p}} \right) \frac{1}{p_i} \left( \frac{1}{|\partial_x Y_{a,b}^{u}(t_k, x)|^{q_i, q_i}} \right) \frac{1}{q_i} \, du
\]

Thus, we have shown that

\[
R_{21}^{a,b} \leq C \Delta_n^{q\overline{p}} \left( 1 + \frac{1}{x^{2q_p}} \right) + C \Delta_n^{q\overline{p}} \left( 1 + \frac{1}{x^{2q_p}} \right) + C \Delta_n^{q\overline{p}} \left( 1 + \frac{1}{x^{2q_p}} \right),
\]

which implies that

\[
R_{21}^{a,b} \leq C \Delta_n^{q\overline{p}} \left( 1 + \frac{1}{x^{2q_p}} \right) + C \Delta_n^{q\overline{p}} \left( 1 + \frac{1}{x^{2q_p}} \right) + C \Delta_n^{q\overline{p}} \left( 1 + \frac{1}{x^{2q_p}} \right).
\]

Next, using BDG’s and Hölder’s inequalities with \( \frac{1}{p_2} + \frac{1}{q_2} = 1 \), (40) and (41),

\[
R_{22}^{a,b} \leq C \Delta_n^{q\overline{q}-1} \left( \frac{1}{x^{2q_2}} \right) \int_{t_k}^{t_k+1} \left( \frac{\partial_x Y_{a,b}^{u}(t_k, x)^2}{Y_{a,b}^{u}(t_k, x)} \right) \, ds \leq C \Delta_n^{q\overline{q}-1} \left( \frac{1}{x^{2q_2}} \right) \int_{t_k}^{t_k+1} \left( \frac{\partial_x Y_{a,b}^{u}(t_k, x)}{\sqrt{Y_{a,b}^{u}(t_k, x)}} \right) \, ds
\]

Here, \( p_2 \) should be chosen close to 1 in order that \( p_2 q_2 < \sigma - 1 \). In order to apply (40) and (41) to estimate two terms above \( R_{21}^{a,b} \) and \( R_{22}^{a,b} \), all conditions required here are as follows

\[-2q_1 q_2 \sigma \leq \left( \frac{a - 1}{\sigma} \right)^2, \quad 2p_1 q_2 \sigma < \sigma - 1, \quad q_2 < \frac{a}{\sigma} - 1.\]
This implies that
\[
\frac{a}{\sigma} \geq 2q_1q\bar{p} + \sqrt{2q_1q\bar{p}(2q_1q\bar{p} + 1)} + 1, \quad \frac{a}{\sigma} > \frac{2q_1q\bar{p}}{q_1 - 1} + 1, \quad \frac{a}{\sigma} > \frac{q\bar{p}}{q\bar{p} - 1} + 1.
\]

Here, the optimal choice for \(q\) and \(q_1\) corresponds to choose them in a way which gives minimal restrictions on the ratio \(\frac{a}{\sigma}\). That is,
\[
2q_1q\bar{p} + \sqrt{2q_1q\bar{p}(2q_1q\bar{p} + 1)} = \frac{2q_1q\bar{p}}{q_1 - 1} = \frac{q\bar{p}}{q\bar{p} - 1}.
\]

Thus, the unique solution is given by \(\bar{p} = \frac{21q + 13\sqrt{q(49q + 8)}}{2(11q + 2 + \sqrt{q(49q + 8)})}\) and \(q_1 = \frac{11q + 2 + \sqrt{q(49q + 8)}}{2(6q + 1)}\), which implies that \(\frac{a}{\sigma} > \frac{7}{2}q + \frac{1}{2}\sqrt{q(49q + 8)} + 1\). Therefore, under \(\frac{a}{\sigma} > \frac{7}{2}q + \frac{1}{2}\sqrt{q(49q + 8)} + 1\), we have shown that
\[
\mathcal{E}_{t_k,x}[|R_2^{a,b}|2q] \leq C \Delta_t^{4q} \left(1 + \frac{1}{\lambda z} + q \right) \left(1 + \frac{1}{\lambda z - q}\right) \left(1 + \frac{1}{\lambda z + q}\right) \left(1 + \frac{q}{\lambda z}\right) \leq C \Delta_t^{4q} \left(1 + \frac{1}{\lambda z} + q \right)
\]

where \(\bar{p} = \frac{21q + 13\sqrt{q(49q + 8)}}{2(11q + 2 + \sqrt{q(49q + 8)})}\), \(\bar{q} = \frac{\bar{p}}{\bar{p} - 1} = \frac{21q + 13\sqrt{q(49q + 8)}}{2(6q + 1)}\), \(q_1 = \frac{11q + 2 + \sqrt{q(49q + 8)}}{2(6q + 1)}\), \(q_1 \bar{p} = \frac{21q + 13\sqrt{q(49q + 8)}}{4(6q + 1)}\), \(q_2 > 1\) with \(\frac{q_1}{q_2 - 1} = 1.39\) close to 1. When \(q \in [1, \frac{13 + \sqrt{89}}{20}]\) and \(\frac{a}{\sigma} > \frac{11 + \sqrt{89}}{2}\), it suffices to choose \(q_2\) large enough to get \((\frac{a}{\sigma} - 1)(\frac{1}{2\bar{p}} + \frac{1}{2q_2}) + 3q < \frac{a}{\sigma} - 1\) since \(\bar{p} > 1.04\).

Finally, we treat the term \(R_3^{a,b}\). Using Hölder’s inequality with \(\frac{1}{p_3} + \frac{1}{q_3} = 1\),
\[
\mathcal{E}_{t_k,x}[|R_3^{a,b}|2q] \leq \Delta_t^{2q - 1} \int_{t_k}^{t_{k+1}} \mathcal{E}_{t_k,x}^{a,b} \left[ \int_s^{t_{k+1}} D_s \left( \frac{1}{\sigma Y_r^{a,b}(t_k,x)} \right) dr \right] \left[ \frac{1}{\sqrt{2\sigma Y_r^{a,b}(t_k,x)}} \right] \left\| \frac{\partial_x Y_r^{a,b}(t_k,x)}{\sqrt{2\sigma Y_r^{a,b}(t_k,x)}} \right\|^{2q \frac{1}{p_3}} ds
\]

\[
\leq \Delta_t^{2q - 1} \int_{t_k}^{t_{k+1}} \left( \mathcal{E}_{t_k,x}^{a,b} \left[ \left\| \frac{\partial_x Y_r^{a,b}(t_k,x)}{\sqrt{2\sigma Y_r^{a,b}(t_k,x)}} \right\|^{2q \frac{1}{p_3}} \right] \right)^{\frac{1}{p_3}} ds
\]

\[
\leq \Delta_t^{2q - 1} \int_{t_k}^{t_{k+1}} \Delta_t^{2q_1 - 1} \int_{t_k}^{t_{k+1}} \mathcal{E}_{t_k,x}^{a,b} \left[ D_s \left( \frac{1}{\sigma Y_r^{a,b}(t_k,x)} \right) \right]^{2q \frac{1}{p_3}} ds
\]

\[
\leq \Delta_t^{2q - 1} \int_{t_k}^{t_{k+1}} \left( \mathcal{E}_{t_k,x}^{a,b} \left[ \left\| \frac{\partial_x Y_r^{a,b}(t_k,x)}{\sqrt{2\sigma Y_r^{a,b}(t_k,x)}} \right\|^{2q \frac{1}{p_3}} \right] \right)^{\frac{1}{p_3}} ds.
\]
By using (46) and the same computations as in the proof of (A2) with \( p = 2q_3 \), we get

\[
\tilde{E}_{t_k}^{a,b} \left[ D_5 \left( \frac{1}{\partial^2 Y_r^{a,b}(t_k,x)} \right) \right]^{2q_3} = \tilde{E}_{t_k}^{a,b} \left[ D_5 \left( \partial^2 Y_r^{a,b}(t_k,x) \right) \right]^{2q_3} \leq C \left( 1 + \frac{1}{\chi^{2q_3}} \right) \frac{1}{\chi^{q_3}},
\]

where \( p_4 \) and \( q_4 \) satisfying \( \frac{1}{p_4} + \frac{1}{q_4} = 1 \) are given in the proof of (A2). Next, using Hölder’s inequality with \( \frac{1}{p_4} + \frac{1}{q_5} = 1 \), \( (40) \) and \( (41) \),

\[
\tilde{E}_{t_k}^{a,b} \left[ \frac{\partial^2 Y_r^{a,b}(t_k,x)}{\sqrt{2\sigma Y_r^{a,b}(t_k,x)}} \right]^{2q_3} \leq C \left( \tilde{E}_{t_k}^{a,b} \left[ \partial^2 Y_r^{a,b}(t_k,x) \right]^{2q_3} \right)^{\frac{1}{q_5}} \left( \tilde{E}_{t_k}^{a,b} \left[ \frac{1}{Y_r^{a,b}(t_k,x)} \right]^{q_5} \right)^{\frac{1}{q_5}} \leq C \left( 1 + \frac{1}{\chi^{2q_3}} \right) \frac{1}{\chi^{q_3}}.
\]

Here, \( q_5 \) should be chosen close to 1 in order that \( q q_5 < \frac{a}{\sigma} - 1 \). In order to apply \( (40) \) and \( (41) \) to estimate the term \( R_3^{a,b} \), all conditions required here are as follows

\[
-2q_3 p_4 \geq \frac{\left( \frac{a}{\sigma} - 1 \right)^2}{2 \left( \frac{a}{\sigma} - \frac{1}{2} \right)}, \quad 3q_3 q_4 < \frac{a}{\sigma} - 1, \quad q q_5 < \frac{a}{\sigma} - 1.
\]

This implies that

\[
\frac{a}{\sigma} \geq 2q_3 p_4 + \sqrt{2q_3 p_4 (2q_3 p_4 + 1)} + 1, \quad \frac{q_3 p_4}{p_4 - 1} + 1, \quad \frac{a}{\sigma} > \frac{q p_3}{p_3 - 1} + 1.
\]

Here, the optimal choice for \( p_3 \) and \( p_4 \) corresponds to choose them in a way which gives minimal restrictions on the ratio \( \frac{a}{\sigma} \). That is,

\[
2q_3 p_4 + \sqrt{2q_3 p_4 (2q_3 p_4 + 1)} = \frac{3q_3 p_4}{p_4 - 1} = \frac{q p_3}{p_3 - 1}.
\]

Thus, the unique solution is given by \( p_3 = \frac{2\sqrt{1+8q}+4\sqrt{2q}}{2\sqrt{1+8q}+3\sqrt{2q}} \) and \( p_4 = 1 + \frac{3}{2} \sqrt{\frac{2q}{1+8q}} \), which implies that \( \frac{a}{\sigma} > 4q + 1 + \sqrt{2q(8q + 1)} \). Therefore, under \( \frac{a}{\sigma} > 4q + 1 + \sqrt{2q(8q + 1)} \), we obtain

\[
\tilde{E}_{t_k}^{a,b} \left[ R_3^{a,b} \right]^{2q} \leq C \frac{2q}{\Delta_n} \left( 1 + \frac{1}{\chi^{2q_3 q_4}} \right) \frac{1}{\chi^{q_3}} + \Delta_n \frac{4q}{\chi^{2q_3 q_4}} \frac{1}{\chi^{q_3}}.
\]
\[ + \Delta_n^{3q} \left( 1 + \frac{2q}{2q-p_3} \right) \frac{1}{\chi^{2q}} \leq C \Delta_n^{4q} \left( 1 + \frac{1}{\chi^{q}} \right) \left( 1 + \frac{1}{\chi^{q}} \right) \]

\[ \times \left( \frac{1}{\chi^{2q}} + \frac{1}{\chi^{4q}} + \frac{1}{\chi^{3q}} \right) \leq C \Delta_n^{4q} \left( \frac{1}{\chi^{2q}} + \frac{1}{\chi^{q}} \frac{2q}{2q-p_3} + \frac{1}{\chi^{4q}} + \frac{1}{\chi^{q}} \right), \quad (A16) \]

where \( p_3 = \frac{2\sqrt{1+8q}+4\sqrt{2q}}{2\sqrt{1+8q}+3\sqrt{2q}} \), \( q_3 = \frac{p_3}{p_3-1} = 4 + \sqrt{2(8+\frac{1}{q})} \), \( p_4 = 1 + \frac{3}{2} \sqrt{\frac{2q}{1+8q}} \), \( p_3p_4 = 1 + 2 \sqrt{\frac{2q}{1+8q}} \), \( p_5 > 1 \)

with \( \frac{p_3}{p_3-1} \) close to 1. Then, when \( q \in [1, \frac{13+\sqrt{89}}{20}] \) and \( \frac{1}{\sigma} > \frac{11+\sqrt{89}}{2} \), it suffices to choose \( p_5 \) large enough to get \( (\frac{2}{\sigma} - 1)(\frac{1}{2p_3p_4} + \frac{1}{2p_3q_5}) + 4q < \frac{1}{\sigma} - 1 \) since \( p_3p_4 > 1.89 \).

From (A12), (A13), (A15), and (A16), under condition \( \frac{1}{\sigma} > 4q + 1 + \sqrt{2q(8q+1)} \), we obtain

\[ \tilde{E}_{t_k,x}[|R_1^{a,b} + R_2^{a,b} + R_3^{a,b}|2q] \leq C \Delta_n^{4q} \left( \frac{1}{\chi^{2q}} + \frac{1}{\chi^{q}} \frac{2q}{2q-p_3} + \frac{1}{\chi^{4q}} + \frac{1}{\chi^{q}} \right). \]

Moreover, when \( q \in [1, \frac{13+\sqrt{89}}{20}] \) and condition (A) holds, we obtain

\[ \tilde{E}_{t_k,x}[|R_1^{a,b} + R_2^{a,b} + R_3^{a,b}|2q] \leq C \Delta_n^{4q} \left( \frac{1}{\chi^{2q}} + \frac{1}{\chi^{q}} \frac{2q}{2q-p_3} + \frac{1}{\chi^{4q}} + \frac{1}{\chi^{q}} \right). \]

Thus, we conclude the desired estimate (52).

Proof of (53). We proceed in the same way as in the proof of (52) where we use

\[ \frac{Y_{a,b}^{u}(t_k,x)}{\partial Y_{a,b}^{u}(t_k,x)} - \frac{Y_{a,b}^{u}(t_k,x)}{\partial Y_{a,b}^{u}(t_k,x)} = a \int_{t_k}^{s} \frac{dW_{u}}{\partial Y_{a,b}^{u}(t_k,x)} + \sqrt{\frac{\sigma}{2}} \int_{t_k}^{s} \frac{Y_{a,b}^{u}(t_k,x)}{\partial Y_{a,b}^{u}(t_k,x)} dW_{u} \]

instead of (A14).

Remark 4. When we use Cauchy–Schwarz’s inequality instead of Hölder’s inequality to estimate \( |R_1^{a,b} + R_2^{a,b} + R_3^{a,b}|2q \), the required condition will be \( \frac{2}{\sigma} > 8q + 1 + \sqrt{8q(8q+1)} \) which is actually bigger than \( 4q + 1 + \sqrt{2q(8q+1)} \).

APPENDIX B. USEFUL RESULTS

We recall a classical convergence result and a central limit theorem on triangular arrays of random variables. For each \( n \in \mathbb{N} \), let \((\xi_{k,n})_{k \geq 1}\) be a sequence of random variables defined on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\), and we assume that \( \xi_{k,n} \) are \( \mathcal{F}_{t_{k+1}} \)-measurable for all \( k \).

Lemma 18 (lemma 3.4 of Jacod (2012)). Assume that as \( n \to \infty \),

\( i \sum_{k=0}^{n-1} E[\xi_{k,n} | \mathcal{F}_{t_k}] \to 0 \), and \( ii \sum_{k=0}^{n-1} E[\xi_{k,n}^2 | \mathcal{F}_{t_k}] \to 0 \).
Then as \( n \to \infty \), \( \sum_{k=0}^{n-1} \xi_{k,n} \) \( p \to 0 \).

**Theorem 4** (Lemma 3.6 of Jacod (2012)). Assume that there exist real numbers \( M \) and \( V > 0 \) such that as \( n \to \infty \),

\[
\begin{align*}
\text{(i)} & \sum_{k=0}^{n-1} E[\xi_{k,n} | F_{t_k}] \overset{p}{\to} M, \\
\text{(ii)} & \sum_{k=0}^{n-1} (E[\xi_{k,n}^2 | F_{t_k}] - (E[\xi_{k,n} | F_{t_k}])^2) \overset{p}{\to} V, \quad \text{and} \\
\text{(iii)} & \sum_{k=0}^{n-1} E[\xi_{k,n}^4 | F_{t_k}] \overset{p}{\to} 0.
\end{align*}
\]

Then as \( n \to \infty \), \( \sum_{k=0}^{n-1} \xi_{k,n} \overset{L^p(f)}{\to} \mathcal{N} + M \), where \( \mathcal{N} \) is a centered Gaussian random variable with variance \( V \).

**Theorem 5** (Proposition 1.3.3 of Nualart (2006)). Let \( F \in \mathbb{D}^{1,2} \) and \( u \) be in the domain of \( \delta \) such that \( Fu \in L^2(\Omega; H) \). Then \( Fu \) belongs to the domain of \( \delta \) and the following equality is true

\[
\delta(Fu) = F\delta(u) - <DF, u>_H, \quad \text{for}\quad (B1)
\]

provided the right-hand side of \((B1)\) is square integrable.

**Theorem 6** (Lemma 2.1 of León, Navarro, and Nualart (2003)). Let \( \varphi \in C^1(\mathbb{R}) \) be a continuously differentiable function and let \( F \in \mathbb{D}^{1,2} \). Then \( \varphi(F) \in \mathbb{D}^{1,2} \) if and only if \( \varphi(F) \in L^2(\Omega) \) and \( \varphi'(F)DF \in L^2(\Omega \times [0, T]) \), and under these hypotheses

\[
D(\varphi(F)) = \varphi'(F)DF.
\]