Isometric Embeddings via Heat Kernel

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Abstract

We combine the heat kernel embedding and Günther’s implicit function theorem to obtain isometric embeddings of compact Riemannian manifolds with vanishing Einstein tensor into Euclidean spaces. As the heat flow time $t \to 0_+$, the second fundamental form of the embedded images has certain normal form, and the mean curvature vectors converge to (large) constant length. These embeddings are canonical in the sense that they are constructed by the eigenfunctions of the Laplacian and intrinsic perturbations.

1 Introduction

Given a $n$-dimensional Riemannian manifold $(M, g)$, one seeks for the embeddings $u : M \to \mathbb{R}^q$ for some $q$ such that the induced metric is $g$, i.e. $u^* g_0 = g$, where $g_0$ is the standard Euclidean metric in $\mathbb{R}^q$. This is called the isometric embedding problem and has long history. In 1873, Schlaefli conjectured that every $n$-dimensional smooth Riemannian manifold always admits a smooth local isometric embedding in $\mathbb{R}^{s_n}$, where $s_n := n(n+1)/2$. When the metric $g$ is analytic, existence of local isometric embeddings was established by Janet (1926, [J]), Cartan (1927, [C]) and Burstin (1931, [Bu]) with $q = s_n$ using the Cauchy-Kowalewski theorem. For global isometric embeddings, Nash (1954, [N1]) and Kuiper (1955, [K]) proved the existence of isometric embeddings of class $C^1$ with $n < q \leq 2n + 1$, provided that the metric $g$ is continuous. For $s > 3$ and $s = \infty$, in a celebrated paper [N2] in 1956, Nash proved the existence of isometric embeddings of class $C^s$ for $g \in C^s$, with $q = 3s_n + 4n$ in the compact case, and $q = (n + 1)(3s_n + 4n)$ in the noncompact case. Gromov (1986, [Gr]) gave $q = s_n + 2n + 3$ as the best value for compact $M$ with smooth metric.

Nash’s proof used the so-called hard implicit function theorems or Nash-Moser technique, which involves smoothing operators in the Newton iteration to preserve the differentiability of approximate solutions of the isometric problem. Günther (1989, [G1]) significantly simplified Nash’s proof by inventing a new iteration scheme for the isometric embedding problem, such that there is no loss of differentiability in the iteration so the usual contracting mapping theorem is enough. A good exposition of his method is in [G2].
Nash and Günther's isometric embedding is very flexible. It can start from any short embedding as the approximate solution, i.e. any embedding $u: M \to \mathbb{R}^q$ such that the induced metric is less or equal to $g$, to produce an isometric embedding. On the other hand, such great flexibility of the initial embeddings usually makes the resulted isometric embeddings non-canonical. In their methods it is needed to apply the implicit function theorem on the local coordinate charts of the manifolds, and in each iteration step to perturb the mapping $u_k: M \to \mathbb{R}^q$ to a free mapping, i.e. the $n$ first derivative vectors $\partial_i u_k(x)$ and the $n(n + 1)/2$ second derivative vectors $\partial_i \partial_j u_k(x)$ are linearly independent at every $x$ on $M$ (see Definition 9), and to estimate the right inverse of the matrix spanned by these derivative vectors.

In 1994, Bérard, Besson and Gallot [BBG] constructed the "asymptotically isometric embedding" of compact Riemannian manifolds $M$ into the space $l^2$ of real-valued, square summable series, using the normalized heat kernel embedding for $t > 0$:

$$\psi_t: x \to \sqrt{2} \left(4\pi t\right)^{n/4} \cdot \left\{ e^{-\lambda_j t/2} \phi_j(x) \right\}_{j \geq 1},$$

where $\lambda_j$ is the $j$-th eigenvalue of the Laplacian $\Delta_g$ of $(M,g)$ and $\{\phi_j\}_{j \geq 0}$ is the $L^2$ orthonormal eigenbasis of $\Delta_g$. The advantage is that the embeddings $\psi_t: M \to l^2$ are canonical, in the sense that they are constructed by the eigenfunctions of the Laplacian of $(M,g)$. If the heat flow time $t \to 0^+$ then the embedding $\psi_t$ becomes more and more close to an isometric one. However for any given $t > 0$, $\psi_t$ usually is not an isometric embedding (with an error of order $O(t)$), and the target space $l^2$ is infinite dimensional.

We are able to combine the almost isometric embeddings $\psi_t$ in [BBG] and Günther's implicit function theorem in [G2] to produce isometric embeddings of $M$ into $\mathbb{R}^q$ for some $q$ depending on $t$. More precisely we have

**Theorem 1 (Isometric embedding)** Let $(M,g)$ be a $n$-dimensional compact Riemannian manifold with a smooth metric $g$ satisfying $\text{Ric}_g = \frac{1}{2}S_g \cdot g$, i.e. $M$ is Ricci flat if $n \neq 2$, or $M$ is a Riemann surface of constant curvature. Given any $\rho > 0$ and $0 < \alpha < \frac{1}{2}$, there exists $C$ and $t_0 > 0$ depending on $(g,\rho,\alpha)$, such that for the integer $q = q(t) \sim Ct^{-\frac{n}{2}}\rho$ and $0 < t \leq t_0$, the truncated heat kernel embedding

$$\psi_t: M \to \mathbb{R}^q \subset l^2$$

can be perturbed to an isometric embedding $i_t: M \to \mathbb{R}^q$, with the perturbation of $\psi_t$ of order $O \left(t^{\frac{n}{2} - \frac{n}{2}}\right)$ in the $C^{2,\alpha}$-norm.

The interest on these embeddings is that they are canonical, in the sense that they are close to the canonical embeddings $\psi_t: M \to l^2$ in [BBG], and our perturbation only uses the intrinsic information of $(M,g)$. More concretely, we choose the smoothing operator in Günther’s iteration to be associated with $g$ (in [G1] it could be associated to any auxiliary Riemannian metric $\tilde{g}$). The iteration attempts to adjust $\psi_t$ to the nearest isometric embedding in each
step with a unique minimal movement. Our method has the following advantages. The heat kernel embedding $\psi_t : M \to \mathbb{R}^q(t)$ is automatically a free mapping for small $t$ (Theorem [15]). Furthermore, the row vectors $\{\partial_i \psi_t(x)\}_{i=1}^n$ and $\{\partial_i \partial_j \psi_t(x)\}_{1 \leq i \leq j \leq n}$ span a matrix $P(\psi_t)$ with an explicit right inverse bound (Corollary [29]) on the whole $M$. There is no need of sophisticated perturbation arguments in local charts to achieve this right inverse bound as in Nash and Günther’s methods. Our method also yields the higher derivative estimates of $\psi_t$ (Proposition [15]).

If we compose our isometric embeddings into $\mathbb{R}^q$ with the inverse stereographic projection $\mathbb{R}^q \to S^q$, we obtain conformal embeddings of $M$ into the unit sphere $S^q$. These embeddings may relate the conformal geometry of $M$ to the conformal group actions (generalized Möbius transforms) on its embedded images in $S^q$, following the lines in [LY]. Especially when $n = 2$, every Riemann surface $M$ admits a constant curvature metric in its conformal class, so our theorem gives canonical conformal embeddings $M \to S^q$. It will be interesting to see the relation to the conformal volume defined in [LY].

There are limitations of our method though. First, we need the condition that $(M^n, g)$ satisfies

$$\text{Ric}_g - \frac{1}{2} S_g \cdot g = 0,$$

i.e. the Einstein tensor $(-\text{Ric}_g - \frac{1}{2} S_g \cdot g)$ vanishes on $M$, where $\text{Ric}_g$ is the Ricci curvature and $S_g$ is the scalar curvature. This condition is used in the last step of our implicit function theorem to make the error to isometric embedding of $O(t^2)$ small, because from [BBG] Theorem 5

$$\psi_t^* g_0 = g + \frac{t}{3} \left( \frac{1}{2} S_g \cdot g - \text{Ric}_g \right) + O(t^2),$$

where $g_0$ is the standard metric in $t^2$, and the convergence is in $C^l$ sense for any $l \geq 0$ (see [BeGaM] p. 213). Taking trace in (1) we have

$$S_g - \frac{n}{2} S_g = 0 \Rightarrow n = 2 \text{ or } S_g = 0.$$  

So our manifolds $M$ restrict to compact Ricci flat manifolds when $\dim M \neq 2$, or 2-dimensional surfaces of constant curvature (For example, all compact Calabi-Yau manifolds and $G_2$ manifolds satisfy our condition. For isometric embeddings of 2-dimensional surfaces in $\mathbb{R}^3$, see the excellent book [HH]). It will be interesting to see if condition (1) can be removed. Second, the dimension $q$ of the target $\mathbb{R}^q$ is of order

$$q(t) \sim Ct^{-\frac{n}{2}-\rho}$$

for any given $\rho > 0$ and for $0 < t \leq t_0$. Our bound of $q$ may be not as good as Nash-Günther’s bounds. On the other hand, the $t_0$ is determined by the quadratic remainder terms in (2), which might be made explicit by refining [BBG] using higher order expansion of heat kernel in terms of curvature terms.
(see Proposition 17). So it seems that the curvature and diameter of \((M,g)\) gives an intrinsic minimal embedding dimension \(q(t_0)\), with the meaning that the first \(q(t_0)\) eigenfunctions should be enough to recover all metric information of \((M,g)\), via the implicit function theorem on \(\psi_{t_0}: M \to \mathbb{R}^{q(t_0)}\).

In our proof, a crucial step is the following \emph{uniform linear independence property} of the \((\text{higher})\)-derivative vectors of \(\psi_t\), which is of independent interest (Theorem 10 with a different scaling factor before \(\psi_t\), and \textit{without} the vanishing Einstein tensor assumption):

**Theorem 2** (Uniform linear independence) Let \(M\) be a compact Riemannian manifold with smooth metric. For any \(x\) on \(M\), let \(\{V_i\}_{1 \leq i \leq n}\) be an orthonormal basis in its neighborhood. Then for the \(n\) first derivatives vectors \(\nabla_i \psi_t \ (1 \leq i \leq n)\) and the \(n(n+1)/2\) second derivatives vectors \(\nabla_i \nabla_j \psi_t \ (1 \leq i \leq j \leq n)\) of \(\psi_t\) with respect to \(\{V_i\}_{1 \leq i \leq n}\), as \(t \to 0_+\) we have

\[
\frac{\langle \nabla_i \nabla_j \psi_t, \nabla_k \psi_t \rangle}{|\nabla_i \nabla_j \psi_t| |\nabla_k \psi_t|} \to \delta_{ij}, \quad \text{and} \quad \frac{\langle \nabla_i \nabla_j \psi_t, \nabla_k \psi_t \rangle}{|\nabla_i \nabla_j \psi_t| |\nabla_k \psi_t|} \to 0,
\]

and for \(i \neq j\) or \(k \neq l\),

\[
\frac{\langle \nabla_i \nabla_j \psi_t, \nabla_k \nabla_l \psi_t \rangle}{|\nabla_i \nabla_j \psi_t| |\nabla_k \nabla_l \psi_t|} \to 0,
\]

except when \(\{i, j\} = \{k, l\}\) as sets. Furthermore, for \(i \neq j\),

\[
\frac{\langle \nabla_i \nabla_j \psi_t, \nabla_j \nabla_j \psi_t \rangle}{|\nabla_i \nabla_j \psi_t| |\nabla_j \nabla_j \psi_t|} \to \frac{1}{3}.
\]

The convergence is uniform for all \(x\) on \(M\).

Theorem 2 when combined with Proposition 12 and Theorem 1 has the following consequence on the \textit{second fundamental form} and \textit{mean curvature} of the heat kernel embedding image \(\psi_t(M) \subset l^2\) and the isometric embedding image \(\psi_t(M) \subset \mathbb{R}^{q(t)}\) (the proof is in Corollary 22 and Remark 25):

**Corollary 3** (Second fundamental form) Let \(M\) be a compact Riemannian manifold with smooth metric. For any \(x\) on \(M\), let \((x_1, \cdots, x_n)\) be a local coordinates near \(x\) such that the coordinate vectors \((\frac{\partial}{\partial x_i})_{i=1}^n\) are orthonormal at \(x\). We write the second fundamental form \(A(x,t) = \sum_{1 \leq i,j \leq n} h_{ij}(x,t) \, dx^i dx^j\) of the submanifold \(\psi_t(M) \subset l^2\) as

\[
A(x,t) = \frac{1}{\sqrt{2t}} \left( \sum_{i=1}^n \sqrt{3} a_{ii} (x,t) \, (dx^i)^2 + \sum_{1 \leq j < k \leq n} 2a_{jk} (x,t) \, dx^j dx^k \right),
\]

where \(a_{jk}(x,t)\) \((1 \leq j \leq k \leq n)\) are vectors in \(l^2\). Then as \(t \to 0_+\), for any two sets \(\{i, j\}\) and \(\{k, l\}\) \(\subset \{1, 2, \cdots, n\}\) (it is allowed that \(i = j\) or \(k = l\)),

\[
\langle a_{ij}, a_{ij} \rangle \to 1, \text{ for any } i, j,
\]

\[
\langle a_{ii}, a_{jj} \rangle \to \frac{1}{3}, \text{ if } i \neq j,
\]

\[
\langle a_{ij}, a_{kl} \rangle \to 0, \text{ if } \{i, j\} \neq \{k, l\} \text{ and } \{i, k\} \neq \{j, l\}.
\]
and the mean curvature vector $H(x,t)$, after scaled by a factor $\sqrt{t}$, converges to constant length:

$$\sqrt{t} |H(x,t)| \to \sqrt{\frac{n+2}{2n}}.$$

The above statements also hold true for the isometric embedding image $i_t(M) \subset \mathbb{R}^{q(t)}$ in Theorem 4.

Our construction of the isometric embedding is based on the heat kernel of the Laplace-Beltrami operator of $(M,g)$. It seems possible to consider other operators canonically associated with $(M,g)$ for isometric embedding. For example, [Wu] recently studied the connection Laplacian associated with the Levi-Civita connection of $(M,g)$, and established results similar to [BBG] for the embeddings of compact Riemannian manifolds into $l^2$ and pre-compactness. There are also vast literatures of heat kernels when $M$ is equipped with bundles.

The organization of the paper is follows: In Section 2 we review the heat kernel embedding $\psi_t : M \to l^2$ in [BBC]. In Section 3 we truncate the embedding to $\mathbb{R}^q \subset l^2$ and estimate the remainder. In Section 4 we recall the matrix $E(u)$ appeared in the linearization of the isometric embedding problem, and review Günther’s iteration scheme and implicit function theorem. In Section 5 we give higher derivative estimates of $\psi_t$ using the off-diagonal heat kernel expansion method, and establish the crucial uniform linear independence property of the matrix $E(\psi_t)$. Consequences of the second fundamental form, mean curvature and center of mass of the embedded image $\psi_t(M)$ are derived. In Section 6 we establish the uniform quadratic estimate of the nonlinear operator $Q(u)$ in the isometric embedding problem for all $\mathbb{R}^q$. In Section 7 we apply Günther’s implicit function theorem to obtain isometric embeddings. In Section 8 we illustrate our method by explicit calculations on $M = S^1$ and $S^1 \times S^1$ cases. In the Appendix we make the constant in Günther’s implicit function theorem explicit, and discuss the minimal embedding dimension of our method.

Convention: In this paper, unless otherwise remarked, the constant $C$ only depends on $(M,g)$, its dimension $n$, and $k, \alpha$ in the $C^{k,\alpha}$-Hölder norm, but not on $t$ and $q$ of $\mathbb{R}^q$. In a sequence of inequalities, the constant $C$ in successive appearances can be assumed to increase.

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2 The heat kernel embedding into $l^2$

Let $l^2$ be the Hilbert space of real series $\{a_i\}_{i \geq 1}$ such that $\sum_{i=1}^{\infty} a_i^2 < \infty$. 

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**Definition 4** Let \((M, g)\) be a \(n\)-dimensional compact Riemannian manifold with smooth metric \(g\), and \(\{\phi_j(x)\}_{j \geq 0} \subset C^\infty (M)\) be a \(L^2\)-orthonormal basis of real eigenfunctions of the Laplacian of \(M\), that is, such that for eigenvalues \(0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots\), \(\Delta g \phi_j = \lambda_j \phi_j\) for \(\forall j\). We define the family of maps

\[
\Phi_t : M \to l^2 \quad \{e^{-\lambda_j t/2} \phi_j(x)\}_{j \geq 1},
\]

for \(t > 0\) and call \(\Phi_t\) the heat kernel embedding.

In [BBG], Bérard, Besson and Gallots proved the following

**Theorem 5** (BBG Theorem 5) Let \(\psi_t : M \to l^2\) be the normalized heat kernel embedding

\[
\psi_t : x \to \sqrt{2} (4\pi)^{n/4} t^{n/2} \cdot \{e^{-\lambda_j t/2} \phi_j(x)\}_{j \geq 1},
\]

then as \(t \to 0_+\),

\[
\psi_t^* g_0 = g - \frac{t}{3} \left( \frac{1}{2} S_g \cdot g - Ric_g \right) + O(t^2),
\]

where \(g_0\) is the standard metric in \(l^2\).

### 3 Truncation for the embedding into \(\mathbb{R}^q\)

To get the embedding into \(\mathbb{R}^q\), we truncate \(l^2\) to \(\mathbb{R}^q \subset l^2\) by taking the first \(q\) components. Fix a small constant \(\rho > 0\). We prove

**Theorem 6** For \(q = q(t) \sim Ct^{-(\frac{n}{2} + \rho)}\), the truncated heat kernel embedding

\[
\psi_t^{q(t)} : (M, g) \to \mathbb{R}^{q(t)}
\]

still satisfies the asymptote

\[
\left(\psi_t^{q(t)}\right)^* g_{st} = g + \frac{t}{3} \left( \frac{1}{2} S_g \cdot g - Ric_g \right) + O(t^2).
\]

**Proof.** Choose an orthonormal basis \(\{V_k\}_{1 \leq k \leq n}\) near \(x \in M\). Then

\[
\psi_t^* g_{st}(x) (V_i, V_j) = \langle \nabla_i \psi_t(x), \nabla_j \psi_t(x) \rangle = \Sigma_{s \geq 1} e^{-\lambda_s t} \nabla_i \phi_s(x) \nabla_j \phi_s(x).
\]

To estimate the truncation error

\[
\Sigma_{s \geq q(t)+1} e^{-\lambda_s t} \nabla_i \phi_s(x) \nabla_j \phi_s(x),
\]

we recall that

\[
|\phi_s|_{C^0(M)} \leq C \lambda_s^{n-1}.
\]
in [H], [S] and [Gr]. Then by the inequality that
\[ |\nabla \phi_s|_{C^0(M)} \leq C \lambda_s^{n+1} |\phi_s|_{L^2(M)} \]
in [X] (Lemma 2.7), [YY] (Theorem 1.8.5) we have the following asymptotic gradient estimate of eigenfunctions
\[ |\nabla \phi_s|_{C^0(M)} \leq C \lambda_s^{n+1}. \quad (5) \]
Combining the Weyl’s asymptotic formula (p.9 in [Ch]) for eigenvalues on compact manifolds \( M \) that
\[ \lambda_n \sim \frac{4\pi^2}{(\omega_n \text{Vol}(M))^{\frac{2}{n}}} s^\frac{2}{n} := As^\frac{2}{n} \]
as \( s \to \infty \), where \( \omega_n \) is the volume of the unit disk in \( \mathbb{R}^n \), we have
\[
\begin{align*}
|\sum_{s \geq q(t)} e^{-\lambda_s t} \nabla_i \phi_s (x) \nabla_j \phi_s (x)| & \leq C \sum_{s \geq q(t)} (s^\frac{2}{n})^{n+1} e^{-As^\frac{2}{n} t} t \leq C \int_{q(t)}^{\infty} s^{\frac{2(n+1)}{n}} e^{-As^\frac{2}{n} t} ds \\
& = C \frac{n}{2} \int_{q(t)}^{\infty} s^{-\frac{2}{n}} e^{-A\sigma t} d\sigma (\sigma = s^\frac{2}{n}) \\
& \leq Ct^{-\left(\frac{2}{n}+1\right)} \int_{Aq(t)^{\frac{2}{n}}}^{\infty} \mu^{-\frac{2}{n}} e^{-\mu} d\mu (\mu = A\sigma t) \\
& \leq Ct^{-\left(\frac{2}{n}+1\right)} \int_{A\sigma t^{-\frac{2}{n}}}^{\infty} \mu^{-\frac{2}{n}} e^{-\mu} d\mu (\because q(t) \sim t^{-\left(\frac{n}{2}+\rho\right)}) \\
& \leq Ct^{-\left(\frac{2}{n}+1\right)} \int_{A\sigma t^{-\frac{2}{n}}}^{\infty} e^{-\frac{2}{n} \sigma} d\mu \\
& \leq Ct^2,
\end{align*}
\]
where in the last inequality we have used that
\[ t^{-\eta} e^{-\frac{1}{2} t^{-\frac{2}{n}}} = o(t^2) \quad (6) \]
for any real number \( \eta > 0 \) as \( t \to 0_+ \). Therefore
\[ \left(\psi_t^{g(t)}\right)^* g_0 = (\psi_t)^* g_0 + O(t^2) = g + \frac{t}{3} \left( \frac{1}{2} S_g \cdot g - Ric_g \right) + O(t^2) \]
in C⁰ norm. Similarly we can prove the truncation error (3) is of order \( o(t^2) \) in C⁰-norm for any \( k \), if we use the higher derivative estimates of \( \phi_j \) in the following Lemma[S] and notice that (5) holds for any \( \eta > 0 \).
Remark 7 As can be seen from the above, the condition \( q(t) \sim t^{-\left(\frac{n}{2} + \rho\right)} \) is to make the lower limit \( A(q(t))^t \) of the integral \( \int_{A(t)}^\infty t \mu^{\frac{n}{2}} e^{-\mu} d\mu \) to go to \( \infty \), while the precise control of \( |\nabla \phi_s|_{C^0(\Omega)} \) by \( C(\lambda_s)^\eta \) with power order \( \eta = \frac{n+1}{2} \) is not important, i.e. for any \( \eta > 0 \) we only need \( q(t) \sim t^{-\left(\frac{n}{2} + \rho\right)} \) in order to have

\[
|\Sigma_{s \geq t} e^{-\lambda_s t} \nabla_i \phi_s(x) \nabla_j \phi_s(x)| \leq C t^2.
\]

For the same reason, the truncation order \( q(t) \sim t^{-\left(\frac{n}{2} + \rho\right)} \) is also valid for higher derivatives of \( \psi_t \), which are involved in the truncation of the \( n \times n \) matrix \( P(\psi_t) \) to \( \frac{n(n+3)}{2} \times q \) one in Section 7.

With (4) and (5) it is easy to get the higher order derivative estimate of \( \phi_j \) by the elliptic estimate as in the following lemma. The \( \|\phi_j\|_{C^k(M)} \) estimate (with no Schauder \( \alpha \)-derivative) already appeared in [X].

Lemma 8 For any integer \( k \geq 0 \) and \( 0 < \alpha < 1 \), we have

\[
\|\phi_j\|_{C^k,\alpha(M)} \leq C \lambda_j^{\frac{n+k+\alpha}{2}}, \tag{7}
\]

\[
\|\nabla (k) \phi_j\|_{C^0(M)} \leq C \lambda_j^{\frac{n+k+1}{2}}. \tag{8}
\]

Proof. We prove it by induction on \( k \). For \( k = 0, 1 \), (8) are consequences of (4) and (3). For any \( 0 < \alpha < 1 \), by (4) and (3) we have

\[
\|\phi_j\|_{C^0(M)} \leq C \lambda_j^{\frac{n+\alpha}{2}}, \tag{9}
\]

by an interpolation argument similar to Proposition 10.

For \( k \geq 3 \), suppose the inequalities (7) and (8) are true up to \( k - 1 \). Then by the elliptic estimate, we have

\[
\|\phi_j\|_{C^k,\alpha(M)} \leq C \left( \|\Delta \phi_j\|_{C^{k-2,\alpha}(M)} + \|\phi_j\|_{C^0(M)} \right)
\]

\[
= C \left( \lambda_j \|\phi_j\|_{C^{k-2,\alpha}(M)} + \|\phi_j\|_{C^0(M)} \right)
\]

\[
\leq C \left( \lambda_j \cdot \lambda_j^{\frac{n+k+\alpha}{2}} + \lambda_j^{\frac{n+k+1}{2}} \right)
\]

\[
\leq C \lambda_j^{\frac{n+k+\alpha}{2}}.
\]

Therefore

\[
\|\nabla (k) \phi_j\|_{C^0(M)} \leq \|\phi_j\|_{C^k,\alpha(M)} \leq C \lambda_j^{\frac{n+k+\alpha}{2}} \leq C \lambda_j^{\frac{n+k+1}{2}}.
\]

The Lemma follows. Q.E.D.
4 Günther’s iteration for isometric embedding

4.1 Free mappings and Günther’s implicit function theorem

Definition 9 (Free mapping) A $C^2$ map $u : M \to \mathbb{R}^q$ is called a free mapping if the $\frac{n(n+3)}{2}$ vectors $\{ \partial_i u(x), \partial_j \partial_k u(x) \}_{1 \leq i,j,k \leq n}$ in $\mathbb{R}^q$ are linearly independent at any $x \in M$, where $\partial_i$ is the derivative with respect to a coordinate $\{ x_i \}_{i=1}^n$ of $M$ near $x$. Note that this property is independent on choice of coordinates.

Given a free mapping $u : M \to \mathbb{R}^q$, a vector field $h = (h_i(x))_{1 \leq i \leq n} \in C^{s,\alpha}(M, TM)$ and a symmetric 2-tensor field $f = (f_{jk}(x))_{1 \leq j,k \leq n} \in C^{s,\alpha}(M, S_+^{(2)}M)$ with $s \geq 2$ and $0 < \alpha < 1$ (where $S_+^{(2)}M$ is the bundle of symmetric 2-tensors on $M$), we consider the following system of linear equations with unknown $v(x) \in C^{s,\alpha}(M, \mathbb{R}^q)$ that

\[
(\nabla_i u)^T \cdot v = h_i, \quad (\nabla_j \nabla_k u)^T \cdot v = f_{jk},
\]

for $1 \leq i \leq n$ and $1 \leq j \leq k \leq n$, where $\cdot$ is matrix multiplication and $^T$ is the transpose, i.e.

\[
\begin{bmatrix}
(\nabla_i u(x))^T & (\nabla_j \nabla_k u(x))^T \\
\end{bmatrix}_{1 \leq i,j,k \leq n} \cdot v(x) = \begin{bmatrix}
h_i(x) \\
f_{jk}(x) \\
\end{bmatrix}_{1 \leq i,j,k \leq n},
\]

where $\begin{bmatrix}
(\nabla_i u(x))^T & (\nabla_j \nabla_k u(x))^T \\
\end{bmatrix}_{1 \leq i,j,k \leq n}$ (note $j \leq k$) is a $\frac{n(n+3)}{2} \times q$ matrix (where $\frac{n(n+3)}{2}$ is fixed, but $q$ can be large). We require that $v$ has minimal Euclidean norm in $\mathbb{R}^q$ so it is unique. We denote the matrix

\[
P(u) = \begin{bmatrix}
(\nabla_i u(x))^T & (\nabla_j \nabla_k u(x))^T \\
\end{bmatrix}_{1 \leq i,j,k \leq n}
\]

and the solution

\[
v = E(u)(h, f) = E(u) \cdot \begin{bmatrix}
h_i \\
f_{jk} \\
\end{bmatrix}_{1 \leq i,j,k \leq n}.
\]

In [G2]

\[
E(u) : C^{s,\alpha}(M, TM) \times C^{s,\alpha}(M, S_+^{(2)}M) \to C^{s,\alpha}(M, \mathbb{R}^q)
\]

is defined globally over $M$. We define the Schauder norm of $E(u) = [e_{ij}(x)]$ as

\[
\|E(u)\|_{C^{s,\alpha}(M)} = \Sigma_{j=1}^{n(n+3)} \|E_j\|_{C^{s,\alpha}(M)},
\]

where $E_j$ is the $j$-th column vector of $E(u)$, regarded as a $\mathbb{R}^q$-valued function on $M$. 

9
Remark 10 The matrix \( P(u)(x) \) is the coordinate expression of the 2-jet \((du, d^2u)\) of the map \( u: M \to \mathbb{R}^q \) at \( x \). There is a coordinate free description of \( P(u) \) as a section of the 2-jet bundle \( J^2(M \times \mathbb{R}^q) \), for example see [Hir] or [EM]. Note that \( J^2(M \times \mathbb{R}^q) \to J^1(M \times \mathbb{R}^q) \) is an affine bundle, so we need to do coordinate transform of the system (11) in an affine way to define \( E(u) \) globally on \( M \).

Theorem 11 (G) Let \( u: (M^n, g) \to \mathbb{R}^q \) be a \( C^\infty \) embedding, and \( u \) is a free mapping. For \( f \in C^{s,\alpha}\left(M,S^2(M)\right) \) with \( s \geq 2 \) or \( s = \infty \) and \( 0 < \alpha < 1 \), there is a positive number \( \theta \) (independent on \( u, s \) and \( f \)) with the following property: If

\[
\|E(u)\|_{C^{2, \alpha}(M)} \|E(u)(0, f)\|_{C^{2, \alpha}(M)} \leq \theta,
\]

then there exists a \( v \in C^{s,\alpha}(M, \mathbb{R}^q) \) such that

\[
d(u + v) \cdot d(u + v) = du \cdot du + f
\]
on \( M \).

In Günther’s implicit function theorem, the \( \| \cdot \|_{C^{2, \alpha}(M)} \) norm for the \( q \times \frac{n(n+3)}{2} \) matrix \( E(u) \) is the summation of the \( \| \cdot \|_{C^{2, \alpha}(M)} \) norm of all column vectors, each regarded as a \( \mathbb{R}^q \)-valued function (See Section 6.1). As shown in Section 6 and Section 7 Günther’s implicit function theorem has a uniform quadratic estimate in \( \mathbb{R}^q \) for all \( q \), i.e. the \( \theta \) above is uniform for all \( \mathbb{R}^q \). Fixing any \( x \in M \), this norm can be related to the operator norms of the matrices \( E(u)(x) \), \( \nabla_i E(u)(x) \), \( \nabla_i \nabla_k E(u)(x) \) and \( [\nabla_i \nabla_k E(u)]_{\alpha,M}(x) : \mathbb{R}^{\frac{n(n+3)}{2}} \to \mathbb{R}^q \). Due to the special “uniform linear independence” property of \( E(u) \) for \( u = \psi_i \) that we will prove in Section 5 asymptotes of these operator norms can be estimated.

4.2 Günther’s iteration scheme

We recall Günther’s iteration scheme. The general form of the iteration is given in [G1] and is independent on the choice of coordinates, thus is globally defined on \( M \). To make the idea stand out we assume \( M \) has a global coordinate \( (x^i)_{i=1}^n \) as in [G2]. Fixing a constant \( \Lambda_0 \) that is not an eigenvalue of the Laplacian \( \Delta \) on \((M,g)\), then \( \Delta - \Lambda_0 \) is an invertible elliptic operator on \( M \), with

\[
(\Delta - \Lambda_0)^{-1} : C^{\alpha}(M,R) \to C^{2,\alpha}(M,R)
\]
a bounded linear map for any \( \alpha \in (0,1) \) by the elliptic regularity. Let its operator norm be

\[
\sigma(\Lambda_0, \alpha, M) := \left\| (\Delta - \Lambda_0)^{-1} \right\| .
\]

(\( \Delta - \Lambda_0 \))\(^{-1} \) is called the smoothing operator. (For general \( M \) with no global coordinates, we need to use the Laplacians for vector fields and symmetric 2-tensors on \( M \), as in [G1] equations (12)^-(21)).
For a free mapping \( u \in C^{2,\alpha}(M, \mathbb{R}^q) \) and a symmetric 2-tensor \( f \in C^{2,\alpha}(M, S_s^{(2)} M) \), to solve the equation
\[
d(u+v) \cdot d(u+v) = du \cdot du + f
\]
of \( v \) (where \( \cdot \) is the standard inner product in \( \mathbb{R}^q \)), Günther \[G2\] introduced the following iteration scheme \( \Upsilon_u : C^{2,\alpha}(M, \mathbb{R}^q) \to C^{2,\alpha}(M, \mathbb{R}^q) \)
\[
\Upsilon_u(v) = E(u)f + Q(u)(v, v), \tag{15}
\]
where \( Q(u)(v, v) \) is a quadratic functional in \( v \) that
\[
Q_i(u, v) \cdot u_i = Q_i(v, v), \quad Q(u)(v, v) \cdot u_{ij} = Q_{ij}(v, v)
\]
where
\[
Q_i(v, v) = (\Delta - \Lambda_0)^{-1}(\partial_i v \cdot (\Delta - \Lambda_0)v),
\]
\[
Q_{ij}(v, v) = (\Delta - \Lambda_0)^{-1}(\Sigma_k \partial_{ik} v \cdot \partial_{jk} v - \partial_{ij} v \cdot (\Delta - \Lambda_0)v), \tag{16}
\]
and \( \cdot \) is the inner product in \( \mathbb{R}^q \). In other words
\[
Q(u)(v, v) = E(u)([Q_i(v, v)], [Q_{ij}(v, v)])
\]
where the notation \([a_{ij}]\) means a matrix (or vector) with entries \(a_{ij}\). We want to solve the fixed points of
\[
\Upsilon_u(v) = E(u)f + E(u)([Q_i(v, v)], [Q_{ij}(v, v)]). \tag{17}
\]
\section{Uniform linear independence property of \( P(\psi_t) \)}
Recall that the (un-normalized) heat kernel embedding \( \Phi_t : (M, g) \to l^2 \) in \[BBG\] is
\[
\Phi_t : x \in M \to \left(e^{-\frac{\lambda q}{t}} \nabla_1 \phi_1(x), e^{-\frac{\lambda q}{t}} \nabla_2 \phi_2(x), \ldots, e^{-\frac{\lambda q}{t}} \nabla_n \phi_n(x), \ldots\right) \in l^2.
\]
For any \( x \in M \) we take an orthonormal basis \( \{V_i\}_{1 \leq i \leq n} \) in its neighborhood. Following our notation \( P(u) \) for a smooth map \( u : \bar{M} \to \mathbb{R}^q \), we consider the following matrix \( P(\Phi_t) \) (with \( q = \infty \)) consisting of the \( n \) first derivatives of \( \Phi_t \) and the \( n(n+1)/2 \) second derivatives of \( \Phi_t \) with respect to \( \{V_i\}_{1 \leq i \leq n} \):
\[
P(\Phi_t) = \\
\begin{bmatrix}
e^{-\frac{\lambda q}{t}} \nabla_1 \phi_1 & e^{-\frac{\lambda q}{t}} \nabla_1 \phi_2 & \cdots & e^{-\frac{\lambda q}{t}} \nabla_1 \phi_q & \cdots \\
e^{-\frac{\lambda q}{t}} \nabla_2 \phi_1 & e^{-\frac{\lambda q}{t}} \nabla_2 \phi_2 & \cdots & e^{-\frac{\lambda q}{t}} \nabla_2 \phi_q & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
e^{-\frac{\lambda q}{t}} \nabla_n \phi_1 & e^{-\frac{\lambda q}{t}} \nabla_n \phi_2 & \cdots & e^{-\frac{\lambda q}{t}} \nabla_n \phi_q & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
e^{-\frac{\lambda q}{t}} \nabla_j \nabla_k \phi_1 & e^{-\frac{\lambda q}{t}} \nabla_j \nabla_k \phi_2 & \cdots & e^{-\frac{\lambda q}{t}} \nabla_j \nabla_k \phi_q & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
e^{-\frac{\lambda q}{t}} \nabla_n \nabla_1 \phi_1 & e^{-\frac{\lambda q}{t}} \nabla_n \nabla_2 \phi_2 & \cdots & e^{-\frac{\lambda q}{t}} \nabla_n \nabla_n \phi_q & \cdots \\
\end{bmatrix}
\]
Let $\langle , \rangle$ be the standard inner product in $l^2$ and $|\cdot|$ be the standard norm in $l^2$. We will prove $P(\Phi_t)(x)$ has full rank for any $x$ on $M$, so $\Phi_t$ is a free mapping. For this we will compute the inner products of the row vectors of $P(\Phi_t)$ in the following subsection.

### 5.1 Derivative estimates of the heat kernel embedding map $\psi_t$

**Proposition 12** For any $x$ on $M$, let $\{V_i\}_{i=1}^n$ be an orthonormal basis near $x$. Then as $t \to 0_+$, we have

\[
\langle \nabla_i \Phi_t, \nabla_j \Phi_t \rangle (x) \to \frac{1}{(4\pi t)^{n/2}}
\]

\[
\langle \nabla_j \nabla_i \Phi_t, \nabla_m \nabla_k \Phi_t \rangle (x) \to \left( \frac{1}{2t} \right)^2 \frac{1}{(4\pi t)^{n/2}} (\delta_{ij} \delta_{km} + \delta_{im} \delta_{jk} + \delta_{ik} \delta_{jm}),
\]

\[
|\langle \nabla_j \nabla_i \Phi_t, \nabla_k \Phi_t \rangle (x) | = t^{-n/2-1} \cdot O(1).
\]

The convergence is uniform for all $x$ on $M$. For a general frame field $\{V_i\}_{i=1}^n$ near $x$, we only need to replace $\delta_{ij}$ by $g_{ij} := g(V_i, V_j)$ in the above expressions.

In the proof of the above Proposition, the following Lemma in [BBC] will be used.

**Lemma 13** Let $r = r(x, y)$ be the shortest distance between $x$ and $y$ on $M$. For $x$ and $y$ which are close enough to each other, $r : M \times M \to \mathbb{R}$ is smooth. Using the orthonormal basis $\{V_i\}_{1 \leq i \leq n}$ near $x$, we have

\[
\nabla_{V_i} r^2 (x, y) |_{x=y} = \nabla_{V_i} r^2 (x, y) |_{x=y} = 0,
\]

\[
\nabla_{V_i} \nabla_{V_i} r^2 (x, y) |_{x=y} = -\nabla_{V_i} \nabla_{V_i} r^2 (x, y) |_{x=y} = 2 \delta_{ij}.
\]

**Proof.** (of Proposition 12). Let $\{V_i\}_{i=1}^n$ be an orthonormal basis near $x$, and $H(t, x, y)$ be the heat kernel of $(M, g)$,

\[
H(t, x, y) = \sum_{s=1}^{\infty} e^{-\lambda_s t} \phi_s (x) \phi_s (y).
\]

Let $\nabla_{V_i}$ be the partial derivative of $H(t, x, y)$ with respect to $V_i$ in the $x$ variables, similarly for $\nabla_{V_i}$. We use the off-diagonal expansion of the heat kernel method. It is easy to check that

\[
\langle \Phi_t, \Phi_t \rangle (x) = H(t, x, y) |_{x=y},
\]

\[
\langle \nabla_i \Phi_t, \nabla_j \Phi_t \rangle (x) = \nabla_{V_i} \nabla_{V_j} H(t, x, y) |_{x=y},
\]

\[
\langle \nabla_{ij} \Phi_t, \nabla_k \Phi_t \rangle (x) = \nabla_{V_{ij}} \nabla_{V_k} H(t, x, y) |_{x=y},
\]

\[
\langle \nabla_{ij} \Phi_t, \nabla_{km} \Phi_t \rangle (x) = \nabla_{V_{ij}} \nabla_{V_m} \nabla_{V_k} H(t, x, y) |_{x=y}.
\]
On the other hand, as in [BBG],

\[ H (t, x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} U (t, x, y), \]

where \( r = r (x, y) \) is the distance function for points \( x \) and \( y \) on \( M \), and

\[ U (t, x, y) = u_0 (x, y) + tu_1 (x, y) + \cdots + t^p u_p (x, y) + O (t^{p+1}), \tag{20} \]

where

\[ u_0 (x, y) = [\theta (x, y)]^{-1/2} \quad \text{(for } x \text{ and } y \text{ close enough)}, \]

and

\[ \theta (x, y) = \frac{\text{volume density at } y \text{ read in the normal coordinate around } x }{r^{n-1}} \quad \text{(21)} \]

with \( r = r (x, y) \) ([BeGaM], p. 208). Especially

\[ \theta (x, x) = 1 = u_0 (x, x). \tag{22} \]

From this we immediately see from (19) as \( t \to 0_+ \),

\[ \langle \Phi_t, \Phi_t \rangle (x) \to \frac{1}{(4\pi t)^{n/2}}. \]

For simplicity we let \( \nabla_{V_i}^x = \partial_i \) and \( \nabla_{V_k}^y = \partial_k \) from now on (the notation \( V^y_k \) means that the derivative is taken with respect to \( V_k \) for the \( y \) variable). We have

\[ \partial_i H (t, x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} \left[ -\frac{\partial_i (r^2)}{4t} U + \partial_i U \right] \tag{23} \]

and

\[ \partial_j \partial_i H (t, x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} \left[ -\frac{\partial_j (r^2)}{4t} \left( -\frac{\partial_i (r^2)}{4t} U + \partial_i U \right) \right. \]

\[ + \left. \left( -\frac{\partial_j \partial_i (r^2)}{4t} U - \frac{\partial_i (r^2)}{4t} \partial_j U + \partial_j \partial_i U \right) \right]. \tag{24} \]

Especially when \( x = y \), by Lemma [13] we have

\[ \partial_j \partial_i H (t, x, y) \big|_{x=y} = \frac{1}{(4\pi t)^{n/2}} \left[ -\frac{\delta_{ij}}{2t} + \text{lower order terms} \right], \]

and similarly for \( \partial_j \partial_i H (t, x, y) \) we have

\[ \partial_j \partial_i H (t, x, y) \big|_{x=y} = \frac{1}{(4\pi t)^{n/2}} \left[ \frac{\delta_{ij}}{2t} + \text{lower order terms} \right]. \tag{25} \]
(The sign before $\delta_{ij}$ changes because $\partial_j \partial_i (r^2) |_{x=y} = -\partial_j \partial_i (r^2) |_{x=y}$).

Differentiating (24) by $\nabla^2_{V_1}$ we have

$$
\frac{1}{(4\pi t)^{n/2}} e^{-\frac{x^2}{4t}} \cdot A(t, x, y),
$$

where in (27) we denote the complicated expression inside $\{}$ by $A(t, x, y)$ for convenience of later computation. Using Lemma 13 and letting $x = y$ we have

$$
\frac{1}{(4\pi t)^{n/2}} \left\{ -\frac{1}{4t} \left[ \partial_k \partial_j (r^2) \partial_i U + \partial_j \partial_i (r^2) \partial_k U + \partial_k \partial_j (r^2) \partial_i U + \partial_k \partial_j (r^2) U \right] \left|_{x=y} \right. 
\right. 
+ \text{lower order terms} 
$$

$$
= \frac{1}{2t} \left( \frac{1}{4\pi t} \right)^{n/2} \left[ -\delta_{ij} \partial_k U + \delta_{ij} \partial_k U - \delta_{ik} \partial_j U + \frac{\partial_k \partial_j (r^2)}{2} U \right] \left|_{x=y} \right. 
+ \text{lower order terms} 
$$

(28)

For fixed $x$ and $0 \leq s \leq 1$ letting $x(s) = \exp_s (sV_1 (x))$, then by [BeGaM]

$$
\theta (x(s), 0) = 1 - \text{Ric} (\dot{x} (s), \dot{x} (s)) \frac{s^2}{3!} + O \left( |s|^3 \right).
$$

Noticing that

$$
u_0 (x, y) = [\theta (x, y)]^{-1/2},
$$

we have

$$
\partial_i U (x, y) |_{x=y} = \nabla^2_{V_1} u_0 (x, y) + \text{lower order terms}
$$

$$
= \frac{d}{ds} \left|_{s=0} \right. \left[ \theta (x(s), 0) \right]^{-1/2} + \text{lower order terms}
$$

$$
= 0 + \text{lower order terms}
$$
Therefore by (28) we have
\[
\langle \nabla_i \nabla_j \Phi_t, \nabla_k \Phi_t \rangle \rightarrow \frac{1}{2t} \frac{1}{(4\pi t)^{n/2}} \left[ -\delta_{kj} \partial_t U + \delta_{ij} \partial_k U - \delta_{ik} \partial_j U - \frac{\partial_r \partial_j \partial_i (r^2) U}{2} \right] |_{x=y} (29)
\]
\[
= t^{-n/2-1} \cdot O(1)
\]
(30)
as \( t \to 0_+ \), where in the last row we have used that \( \partial_r \partial_j \partial_i (r^2) |_{x=y} \), \( \partial_j u_0 (x, y) |_{x=y} \)
and \( \partial_k u_0 (x, y) |_{x=y} \) are bounded on \( M \).

Finally we compute \( \partial_m \partial_k \partial_j \partial_i H (t, x, y) \). Differentiating (26) by \( \nabla^y_u \) we have
\[
\partial_m \partial_k \partial_j \partial_i H (t, x, y)
\]
\[
= \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} \cdot \left\{ -\frac{\partial_m (r^2)}{4t} A(t, x, y) + \frac{\partial_m \partial_k (r^2)}{4t} \left[ -\frac{\partial_j (r^2)}{4t} U + \partial_i U \right] + \left( -\frac{\partial_j \partial_i (r^2)}{4t} U - \frac{\partial_j (r^2)}{4t} \partial_i U + \partial_j \partial_i U \right] \right\}
\]
\[
- \frac{\partial_m \partial_k \partial_j \partial_i (r^2)}{4t} \left[ -\frac{\partial_i (r^2)}{4t} U + \partial_j U \right] - \frac{\partial_m \partial_k \partial_j \partial_i (r^2)}{4t} \left[ -\frac{\partial_j \partial_i (r^2)}{4t} U - \frac{\partial_j (r^2)}{4t} \partial_i U + \partial_j \partial_i U \right] \]
\[
- \frac{\partial_m \partial_k \partial_j \partial_i (r^2)}{4t} U - \frac{\partial_m \partial_k \partial_j \partial_i (r^2)}{4t} \partial_k U - \frac{\partial_m \partial_k \partial_j \partial_i (r^2)}{4t} \partial_j U - \frac{\partial_m \partial_k \partial_j \partial_i (r^2)}{4t} \partial_j U + \partial_m \partial_k \partial_j \partial_i U \}
\]
(31)
Using Lemma [13] and letting \( x = y \) we have
\[
\partial_m \partial_k \partial_j \partial_i H (t, x, y) |_{x=y}
\]
\[
= \left( \frac{1}{4t} \right)^2 \frac{1}{(4\pi t)^{n/2}} \left[ \partial_m \partial_k (r^2) \partial_j \partial_i (r^2) + \partial_m \partial_j (r^2) \partial_m \partial_i (r^2) + \partial_m \partial_j (r^2) \partial_r \partial_i (r^2) \right] U |_{x=y}
\]
+ lower order terms
\[
= \frac{1}{4t^2} \frac{1}{(4\pi t)^{n/2}} (\delta_{ij} \delta_{km} + \delta_{im} \delta_{jk} + \delta_{ik} \delta_{jm}) + \text{lower order terms},
\]
(32)
where in the last equation we have used that \( u_0 (x, x) = 1 \) (see (22)). Therefore
\[
\langle \nabla_j \nabla_i \Phi_t, \nabla_m \nabla_k \Phi_t \rangle \rightarrow \left( \frac{1}{2t} \right)^2 \frac{1}{(4\pi t)^{n/2}} (\delta_{ij} \delta_{km} + \delta_{im} \delta_{jk} + \delta_{ik} \delta_{jm}) + \text{lower order terms}.
\]
To obtain the results with respect to a general frame field \( \{V_i\}_{i=1}^n \), we only need to use

\[
\nabla_{V_i} V_j r^2 (x, y) |_{x=y} = 2g (V_i, V_j)
\]

in the above argument, instead of (18).

**Remark 14** In the expression (31) about \( \partial \bar{m} \partial \bar{k} \partial j \partial i H (t, x, y) \), the term \( \partial \bar{m} \partial k \partial j \partial i (r^2) U \) involves 4-th derivative of \( r^2 \) hence the curvature terms on \( M \), but it is of order \( t^{-1} \) and is not the leading term in (31). The leading terms are \( \left( \frac{1}{4\pi} \right)^2 \partial \bar{m} \partial k (r^2) \partial j \partial i (r^2) U \) etc., which are of order \( t^{-2} \).

The following Proposition generalizes Proposition 12 to higher order derivative estimates of \( \Phi_t \). It also gives a conceptual, shorter proof of Proposition 12 (modulo the precise coefficients), in the sense that it predicts all possible highest order terms in \( \langle \nabla^i \Phi_t, \nabla^j \Phi_t \rangle, \langle \nabla^j \nabla^i \Phi_t, \nabla^k \Phi_t \rangle \) and \( \langle \nabla^j \nabla^i \Phi_t, \nabla^m \nabla^k \Phi_t \rangle \) as \( t \to 0_+ \). We note that the higher derivative estimates of the truncated \( H (0, x, y) \) up to the \( k \)-th eigenfunctions already appeared in [X].

**Proposition 15** Let \( \vec{\alpha} = (x^{\beta}, y^{\gamma}) \) be the multi-indices in \( x \) and \( y \) variables in \( M \times M \), \( D^{\vec{\alpha}} \) be the corresponding multi-derivative operator, and \( |\vec{\alpha}| \) be the total multiplicity of the derivative \( D^{\vec{\alpha}} \). Then as \( t \to 0_+ \), we have

\[
|D^{\vec{\alpha}} H (t, x, y) |_{x=y}| \leq Ct^{-\frac{n}{2} - \left[ \frac{|\vec{\alpha}|}{2} \right]},
\]

and for \( \vec{\alpha} \) only involving \( x \),

\[
|D^{\vec{\alpha}} \Phi_t (x) |^2 \leq Ct^{-\frac{n}{2} - |\vec{\alpha}|},
\]

where \([b]\) is the largest integer less or equal to a given real number \( b \).

**Proof.** We write

\[
H (t, x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} U (t, x, y),
\]

and

\[
D^{\vec{\alpha}} H (t, x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} P_{\vec{\alpha}} (t, x, y),
\]

where \( P_{\vec{\alpha}} (t, x, y) \) is a polynomial in

\[
D^{\vec{\alpha}} r^2 (x, y) \quad \text{and} \quad D^{\vec{\alpha}} U (t, x, y)
\]

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for multi-indices $\vec{\mu}_j$ and $\vec{\eta}_k$. For example, when $\vec{\alpha} = \partial_x$,

$$P_{\vec{\alpha}} (t, x, y) = -\frac{1}{4} \partial_t \left( \frac{r^2 (x, y)}{t} \right) U (t, x, y) + \partial_t U (t, x, y).$$

We have

1. Each summand of $P_{\vec{\alpha}} (x, t)$ is of the form

$$\left( \Pi_{j=1}^s D_{\vec{\mu}_j} \left( \frac{r^2 (x, y)}{t} \right) \right) |_{x=y} \cdot D_{\vec{\eta}} U (t, x, x)$$

with

$$\sum_{s=1}^s |\vec{\mu}_j| + |\vec{\eta}| = |\vec{\alpha}|. \tag{33}$$

2. As $t \to 0_+$, the nonzero terms involving the highest power of $\frac{1}{t}$ must have $\vec{\mu}_j$s with $|\vec{\mu}_j| = 2$ as many as possible, and may have one $\vec{\mu}_j$ with $|\vec{\mu}_j| = 3$ if $|\vec{\alpha}|$ is odd. This is because

$$\partial_i \left( \frac{r^2 (x, y)}{t} \right) |_{x=y} = 0,$$

$$\partial_{ij} \left( \frac{r^2 (x, y)}{t} \right) |_{x=y} = 2 \partial_{ij} \left( \frac{r^2 (x, y)}{t} \right) |_{x=y},$$

$$\left| D_{\vec{\mu}_j} \left( \frac{r^2 (x, y)}{t} \right) \right|_{x=y} \leq C \cdot \frac{1}{t}, \tag{34}$$

and if there are more than one $\vec{\mu}_j$ with $|\vec{\mu}_j| > 3$, the summand

$$\left( \Pi_{j=1}^s D_{\vec{\mu}_j} \left( \frac{r^2 (x, y)}{t} \right) \right) |_{x=y} \cdot D_{\vec{\eta}} U$$

loses the potential to have the maximal number of factors $\frac{1}{t}$ from the equations in (34) and the total degree condition (33).

Therefore we have

$$\left| D_{\vec{\alpha}} H (t, x, y) \right|_{x=y} \leq C \cdot \frac{1}{(4\pi t)^{n/2}} \cdot t^{-\left\lfloor \frac{|\vec{\alpha}|}{2} \right\rfloor}.$$

and

$$\left| D_{\vec{\alpha}} \psi_t (x) \right|^2 = \left| D_{\vec{\alpha}} D_y H (t, x, y) \right|_{x=y} \leq C \cdot \frac{1}{(4\pi t)^{n/2}} \cdot t^{-\left\lfloor \frac{2|\vec{\alpha}|}{2} \right\rfloor} = C \cdot \frac{1}{(4\pi t)^{n/2}} \cdot t^{-|\vec{\alpha}|}.$$

Later we will need the H"older derivative estimate of $D_{\vec{\alpha}} \psi_t (x)$. This can be obtained by interpolation between integer derivative estimates in the above Proposition. We have
Proposition 16 As \( t \to 0^+ \), for any \( \alpha \in (0,1) \), the Hölder derivative \( \left[ D^{\overline{\alpha}} \Phi_t (x) \right]_{\alpha;M} \) satisfies

\[
D^{\overline{\alpha}} \Phi_t (x) \leq Ct^{- \frac{n}{4} - \frac{1}{k+1}} \left\| x \right\|_{C^{k,\alpha}(M)} \leq Ct^{- \frac{n}{4} - \frac{k+1}{k}},
\]

and so

\[
\left\| \Phi_t (x) \right\|_{C^{k,\alpha}(M)} \leq Ct^{- \frac{n}{4} - \frac{k+1}{k}} + \frac{1}{2}.
\]

Proof. Let \( \left\| \overline{\alpha} \right\| = k \). We compute the Hölder \( \alpha \)-derivative of \( D^{\overline{\alpha}} \Phi_t (x) \). (We remind readers that \( \overline{\alpha} \) is a multiple index for the derivative operator \( D^{\overline{\alpha}} \), and \( \alpha \in (0,1) \) is a number; they are not the same). We consider two cases:

For \( x,y \in M \) with \( \text{dist}(x,y) \leq \frac{1}{2} \), we have

\[
\left| \frac{D^{\overline{\alpha}} \Phi_t (x) - D^{\overline{\alpha}} \Phi_t (y)}{\left( \text{dist}(x,y) \right)^{\alpha}} \right| \leq \frac{2}{\left( \text{dist}(x,y) \right)^{\alpha}} \left\| D^{\overline{\alpha}} \Phi_t (y) \right\|_{C^{\alpha}(M)} \leq Ct^{- \frac{n}{4} - \frac{k+1}{k}} \cdot t^{- \frac{\alpha}{2}} = Ct^{- \frac{n}{4} - \frac{k+1}{k}}.
\]

For \( x,y \in M \) with \( \text{dist}(x,y) \geq \frac{1}{2} \), we have

\[
\left| \frac{D^{\overline{\alpha}} \Phi_t (x) - D^{\overline{\alpha}} \Phi_t (y)}{\left( \text{dist}(x,y) \right)^{\alpha}} \right| \leq \frac{2}{\left( \text{dist}(x,y) \right)^{\alpha}} \left\| D^{\overline{\alpha}} \Phi_t (y) \right\|_{C^{\alpha}(M)} \leq Ct^{- \frac{n}{4} - \frac{k+1}{k}} \cdot t^{- \frac{\alpha}{2}} = Ct^{- \frac{n}{4} - \frac{k+1}{k}}.
\]

Combining the two cases we have

\[
D^{\overline{\alpha}} \Phi_t (x) \leq Ct^{- \frac{n}{4} - \frac{1}{k+1}} \left\| x \right\|_{C^{k,\alpha}(M)} \leq Ct^{- \frac{n}{4} - \frac{k+1}{k}} + \frac{1}{2},
\]

The Proposition follows. (We note similar estimates were obtained in Theorem 3 (iii) of [BBG] by different method, where \( D^{\overline{\alpha}} \) are powers of the Laplacian).

For future application of our implicit function theorem, we refine the asymptote (2) to isometric embedding in [BBG] to the quadratic terms:

Proposition 17 Same notations as in Proposition 16. Then as \( t \to 0^+ \), we have

\[
(\psi^*_t g_0)(x) = g(x) + t^3 \left( \frac{1}{2} S_g \cdot g - \text{Ric}_g \right)
\]

\[
+ t^2 \left[ u_2 (x,x) + \sum_{i,j=1}^n 2 \partial_j \partial_i u_1 (x,y) \big|_{x=y} \right] dx^i dx^j + O \left( t^3 \right).
\]
Proof. From [24] we have
\[
\partial_j \partial_i H(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} \left[ \frac{\partial_j (r^2)}{4t} \left( -\frac{\partial_i (r^2)}{4t} U + \partial_i U \right) + \left( \frac{\partial_j \partial_i (r^2)}{4t} \right) (u_0 + tu_1 + t^2 u_2 + O(t^3)) \right].
\]

Letting \( x = y \) and \( i = j \) and using Lemma [13] we have
\[
(\psi_0^* g_0) (V_i V_i) (x) = 2 (4\pi)^{\frac{n}{2}} t^{\frac{n+2}{2}} \left( \partial_i \partial_j H(t,x,y) \right)_{x=y} = 2 (4\pi)^{\frac{n}{2}} t^{\frac{n+2}{2}} \left[ \frac{1}{2t} \left( 1 + \frac{r S_{ij}}{6} + t^2 u_2 + O(t^2) \right) \right] + \left( -\frac{1}{6} \text{Ric}_g (V_i V_i) + t \partial_i \partial_j u_1 (x,y) \right)_{x=y} + O(t) \right)
\]
\[
= \left[ g - \frac{t}{3} \left( \Delta \text{Ric}_g - \frac{1}{2} S_g \cdot g \right) \right] (V_i V_i) + t^2 \left( u_2 (x,x) + 2 \partial_i \partial_j u_1 (x,y) \right)_{x=y} + O(t^3).
\]

Since \((\psi_0^* g_0) (V,W)\) is bilinear in \( V \) and \( W \), the Proposition follows. \[\blacksquare\]

Remark 18 Using the higher order expansion of \( H(t,x,y) \) in terms of curvature terms, it seems possible to make the quadratic terms in the above proposition explicit. On p. 245 of [BeGaM] there is an explicit
\[
u_2 (x,x) = \frac{1}{180} |R_g (x)|^2 - \frac{1}{180} |\text{Ric}_g (x)|^2 + \frac{1}{72} |S_g (x)|^2 + \delta (n) \Delta S_g (x),
\]
where \( R_g \) is the Riemannian curvature tensor, \( \delta (n) \) is a constant, \( |R_g (x)|^2 = \sum_{i \leq j \leq 1 \leq n} R_g (x) (V_i, V_j, V_k, V_l) \) for orthonormal basis \( \{ V_i \}_{1 \leq i \leq n} \) at \( x \), similarly for \( |\text{Ric}_g (x)|^2 \). It remains to compute \( \partial_i \partial_j u_1 (x,y) \) \( \left. \right|_{x=y} \). If we know these quadratic terms, we can estimate the upper bound of \( t_0 \) in our implicit theorem, hence give the minimal dimension \( q (t_0) \sim C (\rho) t_0^{\frac{2}{5} - \rho} \) for the isometric embedding \( M \rightarrow \mathbb{R}^{q (t_0)} \) (The constant \( C (\rho) \) depends on \( \rho \). We may need to vary \( \rho \) to make \( q (t_0) \) as small as possible). For future application, we denote
\[
G := \sup_{x \in M} \left| u_2 (x,x) + \sum_{i,j=1}^n 2 \partial_i \partial_j u_1 (x,y) \right|_{x=y}.
\]

5.2 Uniform linear independence property of \( P (\psi_t) \)

Now we will prove the matrix \( P (\Phi_t) (x) \) has full rank for all \( x \) on \( M \) for small \( t \). Actually we will prove much stronger results as the following
Theorem 19 (Uniform linear independence) The row vectors of the above matrix are uniform linearly independent. More precisely, as \( t \to 0^+ \),

\[
\langle \nabla_i \Phi_t, \nabla_j \Phi_t \rangle_{\nabla_i \Phi_t \mid \nabla_j \Phi_t} \to \delta_{ij}, \quad \text{and} \quad \langle \nabla_i \nabla_j \Phi_t, \nabla_k \Phi_t \rangle_{\nabla_i \nabla_j \Phi_t \mid \nabla_k \Phi_t} \to 0,
\]

and for \( i \neq j \) or \( k \neq l \),

\[
\langle \nabla_i \nabla_j \Phi_t, \nabla_k \nabla_l \Phi_t \rangle_{\nabla_i \nabla_j \Phi_t \mid \nabla_k \nabla_l \Phi_t} \to 0,
\]

except when \( (i, j) = (k, l) \). Furthermore, for \( i \neq j \),

\[
\langle \nabla_i \nabla_j \Phi_t, \nabla_i \nabla_j \Phi_t \rangle_{\nabla_i \nabla_j \Phi_t \mid \nabla_i \nabla_j \Phi_t} \to \frac{1}{3},
\]

The above convergence is uniform for all \( x \) on \( M \). Furthermore, if we truncate \( \Phi_t : M \to \mathbb{R}^q \subset l^2 \) for \( q = q(t) \sim t^{-\frac{n}{2} - \rho} \) the above results are still true. The results also holds if we replace \( \Phi_t \) by \( \psi_t \).

Proof. The remainder of the truncation for the matrix \( P(\Phi_t) \) from \( \frac{n(n+3)}{2} \times \infty \) to \( \frac{n(n+3)}{2} \times q \) can be estimated by Lemma 8, using the asymptotic estimate of the derivatives of \( \Phi_t \) up to order 2, with the argument similar to the proof in Theorem 6. The truncation order of \( q = q(t) \sim t^{-\frac{n}{2} - \rho} \) is explained in Remark 7. So in the following we will only work with \( q = \infty \). We will use Proposition 12, which gives the precise orders of \( \langle \nabla_i \Phi_t, \nabla_j \Phi_t \rangle \), \( \langle \nabla_i \nabla_j \Phi_t, \nabla_k \Phi_t \rangle \) and \( \langle \nabla_j \nabla_i \Phi_t, \nabla_m \nabla_k \Phi_t \rangle \) as \( t \to 0^+ \).

By (25) we have

\[
|\nabla_i \Phi_t|^2 \to \frac{1}{2t} \frac{1}{(4\pi t)^{n/2}} \tag{37}
\]

and

\[
\langle \nabla_i \Phi_t, \nabla_j \Phi_t \rangle_{\nabla_i \Phi_t \mid \nabla_j \Phi_t} \to \delta_{ij}. \tag{38}
\]

From (32) we have

\[
\langle \nabla_j \nabla_i \Phi_t, \nabla_m \nabla_k \Phi_t \rangle \to \left( \frac{1}{2t} \right)^2 \frac{1}{(4\pi t)^{n/2}} (\delta_{ij} \delta_{km} + \delta_{im} \delta_{jk} + \delta_{ik} \delta_{jm}) + \text{lower order terms.}
\]

Especially for \( i \neq j \)

\[
\langle \nabla_i \nabla_j \Phi_t, \nabla_i \nabla_j \Phi_t \rangle \to \left( \frac{1}{2t} \right)^2 \frac{1}{(4\pi t)^{n/2}} \cdot 3 + \text{lower order terms}
\]

\[
\langle \nabla_j \nabla_i \Phi_t, \nabla_j \nabla_i \Phi_t \rangle \to \left( \frac{1}{2t} \right)^2 \frac{1}{(4\pi t)^{n/2}} \cdot 1 + \text{lower order terms} \tag{39}
\]

\[
\langle \nabla_i \nabla_j \Phi_t, \nabla_k \nabla_i \Phi_t \rangle \to \left( \frac{1}{2t} \right)^2 \frac{1}{(4\pi t)^{n/2}} \cdot 1 + \text{lower order terms}
\]

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and
\[
\langle \nabla_j \nabla_i \Phi_t, \nabla_m \nabla_k \Phi_t \rangle \to \left( \frac{1}{2\pi} \right)^2 \frac{1}{(4\pi t)^{n/2}} \cdot 0 + \text{lower order terms}
\]
if \( i \neq j \) or \( k \neq m \) and \( (i, j) \neq (k, m) \). So we conclude that
\[
\frac{\langle \nabla_i \nabla_j \Phi_t, \nabla_k \nabla_l \Phi_t \rangle}{|\nabla_i \nabla_j \Phi_t| \cdot |\nabla_k \nabla_l \Phi_t|} \to \begin{cases} 0, & \text{if } (i, j) \neq (k, l) \text{ and } (i, k) \neq (j, l), \\ 1/3, & \text{if } i = j \text{ and } k = l, \text{ but } i \neq k. \end{cases}
\]
(40)

Using (39) we have
\[
|\nabla_j \nabla_i \Phi_t|^2 \to C t^{-\frac{n}{2} - 2},
\]
so combining (37) and (30) we have
\[
|\langle \nabla_i \nabla_j \Phi_t, \nabla_k \Phi_t \rangle| \leq C t^{-\frac{n}{2} - 1} \left( t^{-n/2 - 2} \cdot t^{-n/2 - 1} \right)^{1/2} = C t^{1/2} \to 0. \quad (41)
\]
The linear independence of the row vectors follows from (38), (40), (41) and by taking \( \alpha_i = \nabla_i \nabla_i \Phi_t(x) \) for \( i = 1, \ldots, n \) in the following linear algebra lemma. ■

**Lemma 20** For \( n \) vectors \( \alpha_1, \ldots, \alpha_n \) in a real linear space \( V \) equipped with an inner product \( \langle \cdot, \cdot \rangle \), if there is a constant \( \sigma \in (0, 1) \) such that
\[
\frac{\langle \alpha_i, \alpha_j \rangle}{|\alpha_i| |\alpha_j|} = \sigma, \quad \text{for all } i \neq j.
\]
then \( \{\alpha_i\}_{1 \leq i \leq n} \) are linearly independent.

**Proof.** We will find an explicit invertible linear transform \( L_0 : V \to V \) to change the angle \( \cos^{-1} \sigma \) between \( \alpha_i \) and \( \alpha_j \) to \( \frac{\pi}{2} \). With out loss of generality we can assume all \( \alpha_i \) are unit vectors. Let
\[
\alpha_0 = \frac{\alpha_1 + \cdots + \alpha_n}{n},
\]

\[
c = c(\sigma, n) = \sqrt{\frac{1 + (n - 1)\sigma}{1 - \sigma}},
\]

\[
\tilde{\alpha}_i = \alpha_0 + c(\alpha_i - \alpha_0) := L_0 \alpha_i, \text{ for } i = 1, \ldots, n,
\]
(43)
then \( \tilde{\alpha}_i \) is nonzero by checking the orthogonal relation \( \langle \alpha_0, \alpha_i - \alpha_0 \rangle = 0 \). We also have
\[
\langle \tilde{\alpha}_i, \tilde{\alpha}_j \rangle = \frac{|\alpha_0|^2 + c^2 (\alpha_i - \alpha_0) (\alpha_j - \alpha_0)}{n} = \frac{(n - 1)\sigma + 1 + (n - 1)\sigma}{1 - \sigma} \left( \sigma - \frac{(n - 1)\sigma + 1}{n} \right)\]
\[
= 0 \text{ for } i \neq j,
\]
so \( \{\tilde{\alpha}_i\}_{1 \leq i \leq n} \) is an orthogonal set. Since \( \{\tilde{\alpha}_i\}_{1 \leq i \leq n} \) is obtained from linear combinations of \( \{\alpha_i\}_{1 \leq i \leq n} \), \( \{\alpha_i\}_{1 \leq i \leq n} \) must be linearly independent. ■
Remark 21 Using the explicit transform $L_0 : \alpha_i \to \tilde{\alpha}_i$ in (43) with the coefficient $c = c(1/3, n)$ in (42), we can make all row vectors of the matrix $P(\Phi_t)$ orthogonal as $t \to 0_+$. Then it is easy to construct the right inverse matrix of $P(\Phi_t)$ by transposing. It is straightforward to compute $|\tilde{\alpha}_i| = \sqrt{1 + (n - 1)\sigma}$ for all $i$, so the operator norm $\|L_0\| = \sqrt{1 + (n - 1)\sigma}$ only depends on $\sigma$ and $n$.

5.3 Geometry of the embedded image $\psi_t(M)$ in $l^2$

In this section we study the geometry of the embedded image $\psi_t(M)$ in $l^2$. We first combine Theorem 19 and Proposition 12 to give the following consequence on the second fundamental form and mean curvature of the embedded image $\psi_t(M) \subset l^2$:

**Corollary 22** For any $x \in M$, let $(x_1, \cdots, x_n)$ be a local coordinates near $x$ such that the coordinate vectors $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ are orthonormal at $x$. The second fundamental form $A(x, t) = \sum_{1 \leq i \leq j \leq n} h_{ij}(x, t) \, dx^i \, dx^j$ of the submanifold $\psi_t(M) \subset l^2$ can be written as

$$A(x, t) = \frac{1}{\sqrt{2t}} \left( \sum_{i=1}^n \sqrt{3} a_{ii}(x, t) \, (dx^i)^2 + \sum_{1 \leq j < k \leq n} 2 a_{jk}(x, t) \, dx^j \, dx^k \right),$$

where $a_{jk}(x, t) (1 \leq j \leq k \leq n)$ are vectors in $l^2$. Then as $t \to 0_+$,

1. For any two sets $\{i, j\}$ and $\{k, l\} \subset \{1, 2, \cdots, n\}$ (it is allowed that $i = j$ or $k = l$),
   \begin{align*}
   \langle a_{ij}, a_{ij} \rangle &\to 1, \\
   \langle a_{ij}, a_{kl} \rangle &\to 0, \text{ if } \{i, j\} \neq \{k, l\} \text{ and } \{i, k\} \neq \{j, l\}, \\
   \langle a_{ii}, a_{jj} \rangle &\to \frac{1}{3}, \text{ if } i \neq j.
   \end{align*}

2. The mean curvature vector $H(x, t) = \frac{1}{n} \sum_{i=1}^n h_{ii}(x, t)$, after scaled by a factor $\sqrt{t}$, converges to constant length:

$$\sqrt{t} |H(x, t)| \to \sqrt{\frac{n + 2}{2n}}. \quad (47)$$

*The convergence is uniform for all $x$ on $M$.*

**Proof.** We first refine our estimates in Proposition 12 to show

$$|\langle \nabla_i \nabla_j \Phi_t, \nabla_k \Phi_t \rangle(x)| = O \left( t^{-n/2} \right)$$

instead of $|\langle \nabla_i \nabla_j \Phi_t, \nabla_k \Phi_t \rangle(x)| = O \left( t^{-n/2-1} \right)$. This is due to

$$\nabla_y^y \nabla_z^z \nabla_x^x \nabla_y^2 (x, y) |_{x=y} = 0$$

(49)
from the Taylor expansion of the metric (e.g. Proposition 3.1, p. 41)

\[ g_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{iklj} x^k x^l + \frac{1}{6} R_{iklj,s} x^k x^l x^s + \frac{1}{20} R_{iklj, st} x^k x^l x^s x^t + O(r^5), \]

where \( r \) is the distance to the base point \( x_0 \), and

\[ \partial_i U|_{x=y} = \partial_i u_0 (x, y) |_{x=y} + t \partial_i u_1 (x, y) |_{x=y} + O \left( t^2 \right) = O (t), \quad (50) \]

where we have used that

\[ u_0 (x, 0) = \left[ \theta (x (s), 0) \right]^{-\frac{1}{2}} = \left[ 1 - \text{Ric} (\dot{x} (s), \dot{x} (s)) \frac{s^2}{3!} + O \left( |s|^3 \right) \right]^{-\frac{1}{2}} = 1 + \frac{1}{12} \text{Ric} (\dot{x} (s), \dot{x} (s)) s^2 + O \left( |s|^3 \right), \]

thus \( \partial_i u_0 (x, y) |_{x=y} = 0 \). Plugging the refined estimates \((49)\) and \((50)\) into \((29)\), we obtain \((48)\).

Therefore for the normalized heat kernel embedding

\[ \psi_t = \sqrt{2} (4\pi t)^{n/4} t^{-n/2} \cdot \Phi_t, \]

its first derivative and second derivative vectors become orthogonal as \( t \to 0^+ \) by \((48)\):

\[ \langle \nabla_i \nabla_j \psi_t, \nabla_k \psi_t \rangle (x) \to 2 \left( \frac{1}{4\pi t} \right)^{n/2} t \frac{n+2}{2} \cdot t^{-n/2} \cdot O (1) = C t \cdot O (1) \to 0. \]

So as \( t \to 0^+ \), the second fundamental form at \( \psi_t (x) \) on \( \psi_t (M) \subset \mathbb{R}^2 \) is approximated by the second order terms in the Taylor expansion of \( \psi_t : M \to \mathbb{R}^2 \) near \( x \) on \( M \), i.e.

\[ \lim_{t \to 0^+} [ A (x, t) - (\Sigma \leq i, j \leq n \nabla_i \nabla_j \psi_t (x, t) \, dx^i dx^j) ] = 0. \]

From the asymptotes

\[ \langle \nabla_j \nabla_i \Phi_t, \nabla_m \nabla_k \Phi_t \rangle (x) \to 2 \left( \frac{1}{4\pi t} \right)^{n/2} \frac{1}{2t} \left( \frac{1}{4\pi t} \right)^{n/2} \left( \frac{1}{2t} \right)^{2} (\delta_{ij} \delta_{km} + \delta_{im} \delta_{jk} + \delta_{ik} \delta_{jm}) \]

in Proposition \([12]\) we have

\[ \langle \nabla_j \nabla_i \psi_t, \nabla_m \nabla_k \psi_t \rangle (x) \to 2 \left( \frac{1}{4\pi t} \right)^{n/2} t \frac{n+2}{2} \cdot \left( \frac{1}{2t} \right)^{2} \frac{1}{2t} \left( \frac{1}{4\pi t} \right)^{n/2} \left( \frac{1}{2t} \right)^{2} (\delta_{ij} \delta_{km} + \delta_{im} \delta_{jk} + \delta_{ik} \delta_{jm}) \]

\[ = \frac{1}{2t} (\delta_{ij} \delta_{km} + \delta_{im} \delta_{jk} + \delta_{ik} \delta_{jm}). \quad (51) \]
Especially as \( t \to 0^+ \),
\[
\begin{align*}
|\frac{1}{\sqrt{2t}} \cdot \sqrt{3}a_{ii} (x, t)| &= |\nabla_i \nabla_i \psi_t| \to \frac{1}{\sqrt{2t}} \cdot \sqrt{3}, \\
|\frac{1}{\sqrt{2t}} \cdot a_{jk} (x, t)| &= |\nabla_j \nabla_k \psi_t| \to \frac{1}{\sqrt{2t}} \cdot 1 \ (j \neq k),
\end{align*}
\]
so (44) follows. (45) and (46) also follow from (51). For the mean curvature, we have
\[
H (x, t) = \frac{1}{n} \sum_{i=1}^{n} h_{ii} (x, t) = \frac{1}{n} \sqrt{3} \left( \sum_{i=1}^{n} a_{ii} (x, t) \right).
\]
Using \( |a_{ii}| \to 1 \) and \( \langle a_{ii}, a_{jj} \rangle \to \frac{1}{3} \) as \( t \to 0^+ \), we have
\[
|H (x, t)|^2 \to \frac{1}{n^2} \frac{3}{2t} \left( n \cdot 1 + n (n - 1) \cdot \frac{1}{3} \right) = \frac{1}{2t} \frac{n + 2}{n},
\]
so (47) follows. \( \blacksquare \)

**Remark 23** In the above corollary,
1. It is unknown if \( \lim_{t \to 0^+} a_{jk} (x, t) \) exists, but \( \lim_{t \to 0^+} \langle a_{ij}, a_{kl} \rangle \) exists. There exists isometry \( I (x, t) : l^2 \to l^2 \), such that
\[
a_{jk} := \lim_{t \to 0^+} I (x, t) \cdot a_{jk} (x, t) \ (1 \leq j \leq k \leq n)
\]
exists, and \( \{a_{jk}\}_{1 \leq j \leq k \leq n} \) is a fixed basis in \( \mathbb{R}^{\frac{n(n+1)}{2}} \subset l^2 \) satisfying the inner product relations in item 1 of the above corollary;
2. The length of the mean curvature of \( \psi_t (M) \subset l^2 \) converges to constant on \( M \), but the length is large (of order \( t^{-\frac{1}{2}} \)) as \( t \to 0^+ \) by (47). Intuitively, this is because the embedding \( \psi_t \) uses more and more high frequency eigenfunctions in its \( l^2 \) norm as \( t \to 0^+ \), making the image \( \psi_t (M) \) evenly bumpy at all \( x \) on \( M \).
3. If we want to construct constant mean curvature submanifolds in \( \mathbb{R}^q \), perhaps we can consider
\[
t^{-\frac{1}{4}} \psi_t : M \to l^2,
\]
because as \( t \to 0^+ \), the mean curvature vectors of the submanifold \( t^{-\frac{1}{4}} \psi_t (M) \subset l^2 \) converge to constant length \( \sqrt{\frac{2n+2}{2n}} \). Then we can truncate \( l^2 \) to some \( \mathbb{R}^q \) and use the implicit function theorem.

We know from Proposition 12 that
\[
|\psi_t (x)| = \left| \sqrt{2} (4\pi)^{n/4} l^{\frac{n+2}{4}} \cdot \Phi_t (x) \right| \to \sqrt{2t}
\]
as \( t \to 0^+ \). So the image \( \frac{\psi_t (x)}{\sqrt{2t}} \) is almost on a unit sphere as \( t \to 0^+ \). In the following proposition we estimate the center of mass of \( \psi_t (M) \subset l^2 \). The motivation is for balanced conformal embedding \( M \to S^q \), which was related to the conformal volume and first eigenvalue estimate of \( \Delta_g \) in [LY].
Proposition 24 Let $m = \int_M \psi_t \, d\text{vol}_{\psi_t^* g_0} / \int_M d\text{vol}_{\psi_t^* g_0} \in L^2$ be the center of mass of $\psi_t(M) \subset L^2$, then

$$|m| \leq C t^{\frac{7}{2}}.$$

If $M$ is of constant scalar curvature, then

$$|m| \leq C t^{\frac{5}{2}}.$$

Proof. From (2) we have

$$d\text{vol}_{\psi_t^* g_0} = \det \left( [\psi_t^* g_0]_{ij} \right)^{1/2} \, dx^1 \cdot \ldots \cdot dx^n$$

$$= \det \left( g - \frac{t}{3} \left( \frac{1}{2} S_g \cdot g - \text{Ric}_g \right) + O \left( t^2 \right) \right)_{ij} \, dx^1 \cdot \ldots \cdot dx^n$$

$$= \left( 1 - \frac{1}{2} \text{tr} \left( \frac{t}{3} \left( \frac{1}{2} S_g \cdot g - \text{Ric}_g \right) + O \left( t^2 \right) \right) \right) \det \left( g_{ij} \right)^{1/2} \, dx^1 \cdot \ldots \cdot dx^n$$

$$= \left( 1 - \frac{n - 2}{12} t S_g + O \left( t^2 \right) \right) d\text{vol}_g.$$

Since each $\phi_j (x) (j \geq 1)$ is $L^2$ orthogonal to 1, we have $\int_M \psi_t \, d\text{vol}_g = 0$. Combining $|\psi_t(x)| \to \sqrt{2t}$ as $t \to 0^+$, we have

$$\int_M \psi_t \, d\text{vol}_{\psi_t^* g_0} = \int_M \psi_t \left( 1 - \frac{n - 2}{12} t S_g + O \left( t^2 \right) \right) \, d\text{vol}_g$$

$$= 0 - \frac{2 - n}{12} \int_M \psi_t S_g \, d\text{vol}_g + \int_M \psi_t \cdot O \left( t^2 \right) \, d\text{vol}_g$$

$$= \begin{cases} 2 - n & O \left( \sqrt{t} \right) + O \left( \sqrt{t} \cdot O \left( t^2 \right) \right), \\ 0 + O \left( \sqrt{t} \cdot O \left( t^2 \right) \right), \text{ if } S_g \text{ is constant.} \\ O \left( t^{\frac{7}{2}} \right), \\ O \left( t^{\frac{5}{2}} \right), \text{ if } S_g \text{ is constant.} \end{cases}$$

We also have

$$\int_M d\text{vol}_{\psi_t^* g_0} = \int_M \left( 1 - \frac{n - 2}{12} t S_g + O \left( t^2 \right) \right) \, d\text{vol}_g$$

$$= \text{Vol}(M) + O \left( t \right).$$

Since $m = \int_M \psi_t \, d\text{vol}_{\psi_t^* g_0} / \int_M \, d\text{vol}_{\psi_t^* g_0}$, the proposition follows. ■

Remark 25 (On the isometric embedding image $i_t(M)$) In Section 7 we will prove Theorem 11 that the isometric embedding $i_t : M \to \mathbb{R}^{q(t)}$ can be obtained by $C^{2, \alpha}$-perturbation of $\psi_t : M \to \mathbb{R}^{q(t)}$ of order $O \left( t^{\frac{7}{2}} \right)$, with $q(t) \sim Ct^{-\frac{2}{3}} - \rho$. Since the second fundamental form and mean curvature of any embedding $f : M \to \mathbb{R}^q$ are determined by up to the second order derivatives of $f$,
the statements in Corollary 22 also hold true for the isometric embedding image \(i_t(M) \subset \mathbb{R}^{q(t)}\). For the center of mass \(\tilde{m}\) of \(i_t(M)\), if we replace \(\psi_t\) by \(i_t\) in (52) and notice \(\|\psi_t - i_t\|_{C^{2,\alpha}(M,\mathbb{R}^{q(t)})} \leq Ct^{\frac{1}{2}-\frac{\alpha}{2}}\), we see \(|\tilde{m}| \leq Ct^{\frac{1}{2}}\). The estimate is slightly weaker than the unperturbed image \(\psi_t(M)\), which has \(|m| \leq Ct^{\frac{1}{2}}\).

5.4 Operator norm estimate of \(E(\psi_t)\)
We start with the following elementary linear algebra lemma.

**Lemma 26** Let \(A\) be a \(m \times n\) matrix. Regarding \(A\) as a linear map from \(\mathbb{R}^n\) to \(\mathbb{R}^m\), the operator norm \(\|A\|\) of \(A\), defined as
\[
\|A\| = \sup_{v \in \mathbb{R}^n, \|v\|=1} \frac{|Av|}{|v|},
\]
is less or equal to the length of its longest column vector. If the column vectors are orthogonal to each other, then \(\|A\|\) is equal to the length of the longest column vector.

From now we consider the normalized heat kernel embedding \(\psi_t = \sqrt{2} (4\pi)^{n/4} t^{\frac{n+2}{2}} \Phi_t\),
\[
\psi_t := x \rightarrow \sqrt{2} (4\pi)^{n/4} t^{\frac{n+2}{2}} \left\{ e^{-\lambda_j t/2} \phi_j (x) \right\}_{j \geq 1}.
\]
Then from Proposition 10 (noticing difference between \(\psi_t\) and \(\Phi_t\) by a factor \(t^{\frac{n+2}{2}}\)), we have
\[
\left[ D^2 \psi_t (x) \right]_{\alpha;\beta;M} \leq Ct^{\frac{1}{2} - \frac{\alpha + \beta}{2}},
\]
\[
\|\psi_t (x)\|_{C^{k,\alpha}(M)} \leq Ct^{\frac{1}{2} - k \alpha / 2}.
\] (53)

**Corollary 27** The matrix \(P(\psi_t) (x)\) has a right inverse \(E(\psi_t) (x)\) with uniform operator norm bound \(C\) for all \(q \sim Ct^{-\frac{\alpha}{2} - \rho}\) and all \(x \in M\) as \(t \to 0_+\).

**Proof.** By the uniform linear independence property of \(P(\psi_t)\), its right inverse
\[
E(\psi_t) = LP(\psi_t)^T L^T
\] (54)
where \(P(\psi_t)\) is the normalization of \(P(\psi_t)\) by dividing every row vector by its length square, i.e. the row vectors are
\[
\frac{\nabla_i \psi_t}{|\nabla_i \psi_t|^2}, \frac{\nabla_j \nabla_k \psi_t}{|\nabla_j \nabla_k \psi_t|^2}, \text{ for } 1 \leq i, j, k \leq n,
\]“\(^T\)” is the transpose, and \(L\) is a \(\frac{n(n+3)}{2} \times \frac{n(n+3)}{2}\) block diagonal matrix with a \(n \times n\) block \(L_0\) adjusting \(n\) vectors (corresponding to \(\{\nabla_i \nabla_j \psi_t (x)\}_{i=1}^n\) in \(V\) in (43) with mutual angles \(\cos^{-1} \left( \frac{1}{4} \right)\) to \(\frac{\pi}{2}\), and with all other diagonal elements 1.
The column vectors of $\tilde{P}(\psi_t)^T L^T$ are orthogonal, so by Lemma 26 the operator norm of $P(\psi_t)^T L^T$ is controlled by the length of the longest row vector of $\tilde{P}(\psi_t)(x)$, which must be some $\frac{\nabla_j \psi_i(x)}{|\nabla_i \psi_i|}$ (for $\frac{\nabla_j \nabla_k \psi_i(x)}{|\nabla_i \psi_i|}$ are shorter) as $t \to 0_+$. By Proposition 15, From (57) (and $\psi_t = \sqrt{2}(4\pi)^{n/4} t^{-\frac{3}{2}} \Phi_t$) we see

$$\|E(\psi_t)(x)\| \leq \|L\| \cdot \left| \frac{\nabla_i \psi_i(x)}{|\nabla_i \psi_i|} \right| \cdot \|L^T\| \leq C \|L\|^2$$

as $t \to 0_+$.  

**Proposition 28** For any $x \in M$, the operator norms of linear maps $E(\psi_t)(x)$, $\nabla_i E(\psi_t)(x)$, $\nabla_i \nabla_k E(\psi_t)(x)$ and $[\nabla_i \nabla_k E(\psi_t)](x)$ are of order $Ct^{-\frac{1}{2}}, Ct^{-1}$ and $Ct^{-1-\frac{q}{2}}$ respectively for all $q \sim Ct^{-\frac{3}{2}-\rho}$ and $1 \leq i, j, k \leq n$ as $t \to 0_+$.

**Proof.** We have $E(\psi_t) = \tilde{P}(\psi_t)^T L^T$. Therefore

$$\nabla_i E(\psi_t) = L \nabla_i \tilde{P}(\psi_t)^T L^T,$$

$$\nabla_j \nabla_k E(\psi_t) = L \nabla_j \nabla_k \tilde{P}(\psi_t)^T L^T,$$

$$[\nabla_j \nabla_k E(\psi_t)](x) = L \left[ \nabla_j \nabla_k \tilde{P}(\psi_t)^T \right]_{\alpha, M} L^T,$$

where in the last inequality we have used that as $t \to 0_+$, the lengths

$$|\nabla_i \psi_i(x)| \to 1, |\nabla_j \nabla_k \psi_i(x)| \to Ct^{-1/2}$$

uniformly on $M$ in the $C^3$ norm, so as $t \to 0_+$ we can regard them to be independent on $x \in M$, treating them like constant factors when we take the $C^{2,\alpha}$ (Schauder) derivatives with respect to $x$ on the normalized row vectors $\nabla_i \psi_i$ and $\nabla_i \nabla_j \psi_j$.

By Proposition 15 we have the upper bound estimates of the length of the $\frac{n(n+3)}{2}$-row vectors of each $\tilde{P}(\psi_t)(x)$, $\nabla_i \tilde{P}(\psi_t)(x)$, $\nabla_i \nabla_k \tilde{P}(\psi_t)(x)$ and $[\nabla_i \nabla_k \tilde{P}(\psi_t)]_{\alpha, M}(x)$, which in term control the operator norms of $E(\psi_t)(x)$, $\nabla_i E(\psi_t)(x)$, $\nabla_i \nabla_k E(\psi_t)(x)$ and $[\nabla_i \nabla_k E(\psi_t)]_{\alpha, M}(x)$ respectively by Lemma 26. So the Proposition follows from equations (55).

**Corollary 29** For $q \sim Ct^{-\frac{3}{2}-\rho}$, the operator norm $\|E(\psi_t)\|_E$ of $E(\psi_t) : C^{2,\alpha}(M, TM) \times C^{2,\alpha}(M, S^1(M)) \rightarrow C^{2,\alpha}(M, \mathbb{R}^q)$ is of order $t^{-1-\frac{q}{2}}$, so is the Schauder norm $\|E(\psi_t)\|_{C^{2,\alpha}(M)}$, i.e.

$$\|E(\psi_t)\|_E \|E(\psi_t)\|_{C^{2,\alpha}(M)} \leq Ct^{-1-\frac{q}{2}}$$

for a constant $C_E > 0$. 

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Proof. By Proposition 15, we have the upper bound of the lengths of 
\[\nabla_k \nabla_i \nabla_j \psi_t(x), \nabla_i \nabla_k \nabla_i \nabla_j \psi_t(x) \text{ and } [\nabla_i \nabla_k \nabla_i \nabla_j \psi_t(x)]_\alpha\n\]
for \(1 \leq i, j, k, l \leq n\). Thus by Lemma 29 we can control the operator norms of \(E(u)(x), \nabla_i E(\psi_t)(x), \nabla_j \nabla_k E(\psi_t)(x)\) and \([\nabla_j \nabla_k E(\psi_t)]_{\alpha,M}(x) : \mathbb{R}^g \to \mathbb{R}^{n(n+1)/2}\) respectively. We have for any \(x \in M\),
\[
\|E(u)(x)\| \leq C,
\|\nabla_i E(\psi_t)(x)\| \leq Ct^{-\frac{1}{2}},
\|\nabla_j \nabla_k E(\psi_t)(x)\| \leq Ct^{-1},
\|[\nabla_j \nabla_k E(\psi_t)]_{\alpha,M}(x)\| \leq Ct^{-1-\frac{\alpha}{2}}.
\]
(56)

Therefore for any \(\varphi \in C^{2,\alpha}(M,TM) \times C^{2,\alpha}(M,S_2^{(2)}M)\), by the Leibniz rule and (61) we have
\[
[\nabla_j \nabla_k (E(u)(x) \varphi(x))]_{\alpha,M} \leq Ct^{-1-\frac{\alpha}{2}} \|\varphi\|_{C^{2,\alpha}(M,\mathbb{R}^g)}
\]
Taking sup for \(x \in M\) in the above inequalities, we have
\[
\|E(u) \varphi\|_{C^{2,\alpha}(M,\mathbb{R}^g)} \leq Ct^{-1-\frac{\alpha}{2}} \|\varphi\|_{C^{2,\alpha}(M,\mathbb{R}^g)},
\]
so the operator norm of \(E(\psi_t) : C^{2,\alpha}(M,TM) \times C^{2,\alpha}(M,S_2^{(2)}M) \to C^{2,\alpha}(M,\mathbb{R}^g)\) is of order \(Ct^{-1-\frac{\alpha}{2}}\). Note this operator norm agrees with the \(C^{2,\alpha}\)-Schauder norm of \(E(u)\) by (56).

Definition 30 (The constant \(C_E\)): Due to the importance of the operator norm of \(E(\psi_t)\), we denote the constants \(C\) appeared in the coefficients of the above estimates of \(\|E(\psi_t)(x)\|, \|E(\psi_t)\|\) and \(\|E(u)\|_{C^{2,\alpha}(M,\mathbb{R}^g)}\) by \(C_E\), where “\(E\)” indicates \(E(\psi_t)\).

In previous discussions, in the construction of the matrix \(P(\psi_t)\) we have used an orthonormal basis \(\{V_i\}_{1 \leq i \leq n}\) in the neighborhood of \(x \in M\), such that
\[
\langle V_i, V_j \rangle = \delta_{ij}.
\]
It is convenient to allow any smooth frame field \(\{V_i\}_{1 \leq i \leq n}\), dropping the orthonormal condition. For any smooth frame field \(\{V_i\}_{1 \leq i \leq n}\), we define the matrix \(P(\psi_t)\) and \(E(\psi_t)\) as before. The estimates on \(E(\psi_t)\) still carry through as the following Proposition, so they are independent on the choice of coordinates on \(M\). This is useful since the \(\nabla V_j \nabla V_i \psi_t\) involved in \(E(\psi_t)\) is not tensorial under basis change.

Proposition 31 Let \(U\) be a compact, connected region in \(M\) with a smooth frame field \(\{V_i\}_{1 \leq i \leq n}\) and let \(P(\psi_t)\) be the matrix constructed from \(\{V_i\}_{1 \leq i \leq n}\) as in (11). Then the row vectors of \(P(\psi_t)\) are still linearly independent, and \(E(\psi_t)\) has the operator bounds as in Proposition 28 and Corollary 29.
Proof. For any \( x_0 \in U \), we take an orthonormal basis \( \{ W_i \}_{1 \leq i \leq n} \) nearby \( x_0 \) such that \( (W_i, W_j)(x) = \delta_{ij} \). There exists a smooth family of matrices \( \Theta(x) \in SL(T_x M, g_x) \) for \( x \) near \( x_0 \) such that
\[
V_i = \Theta(x) W_i, \text{ for } 1 \leq i \leq n.
\]
We have
\[
\nabla_{V_i} \psi_t = \Theta(x) \nabla_{W_i} \psi_t,
\]
\[
\nabla_{V_i} \nabla_{V_j} \psi_t = \Theta(x) (\nabla_{W_i} \nabla_{W_j} \psi_t) \Theta(x)^T + (\Theta(x) \nabla_{W_i} \Theta(x)) \cdot \nabla_{W_j} \psi_t
\]
(57)
Since \( \Theta(x) \) is smooth on the compact region \( U \), we have
\[
\| \Theta(x) \nabla_{W_i} \Theta(x) \|_{C^5(U)} \leq C_5
\]
for a constant \( C_5 \) only depending on \( (U, g, \{ V_i \})_{1 \leq i \leq n} \) (we use \( C^5(U) \) norm because we need \( C^{4, \alpha} \) estimate of \( \psi_t \) to obtain the operator norm bound of \( E(\psi_t) : C^{2, \alpha}(M, \mathbb{R}^q) \to C^{2, \alpha}(M, \mathbb{R}^q) \)). Let \( L(\{ W_i \}, \{ V_j \}) \) be the matrix acting on \( P(\psi_t; \{ W_i \}) \) by adding \( (\Theta(x) \nabla_{W_j} \Theta(x)) \) times of its \( i \)-th row \( \nabla_{W_j} \psi_t \) to the \( j \)-th row corresponding to \( \nabla_{W_j} \psi_t \) for \( 1 \leq i, j \leq n \). Then
\[
L(\{ W_i \}, \{ V_j \}) P(\psi_t; \{ W_i \}) = P(\psi_t; \{ V_j \})
\]
(58)
and \( L(\{ W_i \}, \{ V_j \}) \) and \( L(\{ W_i \}, \{ V_j \})^{-1} \) are uniformly bounded in their operator norms. Since we have proved that \( P(\psi_t; \{ W_i \}) \) has the uniform linear independence property, so is \( P(\psi_t; \{ V_j \}) \) by (58).

From the Leibniz rule of differentiation we get the relations between the higher derivatives of \( \psi_t \) with respect to basis \( \{ W_i \}_{1 \leq i \leq n} \) and \( \{ V_i \}_{1 \leq i \leq n} \) respectively, similar to the second order term relations in (57). Then by the smoothness of \( \Theta(x) \) on \( U \) we see Proposition 13 still holds. From that estimate we can derive the operator bounds of \( E(\psi_t) \) in Proposition 28 and Corollary 29 as before. ■

6 Quadratic estimate of \( Q(u) \)

6.1 \( C^{k, \alpha} \) norms for \( \mathbb{R}^q \)-valued functions

We first define the \( C^{k, \alpha} \) norms for \( \mathbb{R}^q \)-valued functions for any integer \( k \geq 0 \) and \( \alpha \in (0, 1) \). Since our \( q = q(t) \sim Ct^{-\frac{3}{2} - \rho} \to \infty \) as \( t \to 0^+ \), several equivalent \( C^{k, \alpha} \) norms for any fixed \( q \) will diverge from each other as \( q \to \infty \). To get the uniform quadratic estimate for all \( q \), we will carefully choose the definition of the \( C^{k, \alpha} \) norm.

Definition 32 Let \( f : M \to \mathbb{R}^q \) be a \( \mathbb{R}^q \)-valued function, \( f = (f_1, \cdots, f_q) \), where each \( f_j : M \to \mathbb{R} \). We let \( | | \) be the standard Euclidean norm in \( \mathbb{R}^q \), \( \nabla \) be
Lemma 33

Let \( \beta \geq 0 \) be an integer, and let

\[
\|f\|_{C^0(M, \mathbb{R}^q)} = \sup_{x \in M} |f(x)| = \sup_{x \in M} \left( \sum_{j=1}^q f_j^2(x) \right)^{1/2},
\]

\[
\left\| \nabla^\beta f \right\|_{C^0(M, \mathbb{R}^q)} = \sup_{x \in M} \left( \sum_{j=1}^q \left| \nabla^\beta f_j(x) \right|^2 \right)^{1/2},
\]

\[
\|f\|_{C^k(M, \mathbb{R}^q)} = \sum_{0 \leq \beta \leq k} \left\| \nabla^\beta f \right\|_{C^0(M, \mathbb{R}^q)},
\]

\[
[f]_{\alpha, M; \mathbb{R}^q} = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{\text{dist}(x, y)\alpha},
\]

\[
\|f\|_{C^{k, \alpha}(M, \mathbb{R}^q)} = \|f\|_{C^k(M, \mathbb{R}^q)} + \left[ \nabla^k f \right]_{\alpha, M; \mathbb{R}^q}. \tag{59}
\]

Note that our \( \|f\|_{C^0(M, \mathbb{R}^q)} \) is different from the usual \( C^0 \)-norm of \( f \), which is

\[
\sum_{j=1}^q \|f_j\|_{C^0(M)}.
\]

We clearly have

\[
\frac{1}{q} \sum_{j=1}^q \|f_j\|_{C^0(M)} \leq \|f\|_{C^0(M, \mathbb{R}^q)} \leq \sum_{j=1}^q \|f_j\|_{C^0(M)}
\]

so these two norms are equivalent for any fixed \( q \), but to get the uniform quadratic estimate estimate for \( Q(u, u) \) for all \( q \), it is important to use our definitions. The following type inequality appeared in Günther’s [G1]. We adapt it to \( \mathbb{R}^q \)-valued functions and further observe that the constant \( C(k, \alpha, M) \) is uniform for all \( q \) of \( \mathbb{R}^q \).

**Lemma 33** Let \( \cdot \) be the standard inner product in \( \mathbb{R}^q \). For \( f \) and \( g \) in \( C^{k, \alpha}(M, \mathbb{R}^q) \), we have

\[
\|f \cdot g\|_{C^{k, \alpha}(M)} \leq C(k, \alpha, M) \|f\|_{C^{k, \alpha}(M, \mathbb{R}^q)} \|g\|_{C^{k, \alpha}(M, \mathbb{R}^q)}, \tag{60}
\]

where the constant \( C(k, \alpha, M) \) is uniform for all \( q \).

**Proof.** Let \( |\cdot| \) be the standard Euclidean norm in \( \mathbb{R}^q \), and \( \cdot \) be the standard inner product. Let \( C^{k, \alpha}(M) \) be the usual Hölder norm for scalar functions and \( C^{k, \alpha}(M, \mathbb{R}^q) \) be the Hölder norm for \( \mathbb{R}^q \)-valued functions defined above. We have from Cauchy-Schwartz inequality that

\[
|f(x) \cdot g(x)| \leq |f(x)||g(x)|,
\]

therefore by taking sup for \( x \) on \( M \) we get

\[
\|f \cdot g\|_{C^0(M)} \leq \|f\|_{C^0(M, \mathbb{R}^q)} \|g\|_{C^0(M, \mathbb{R}^q)}. \tag{61}
\]

Since for each \( 1 \leq i \leq n \),

\[
(\nabla_i)^k (f \cdot g) = \sum_{s=1}^k C_k^s (\nabla_i)^s f(x) \cdot (\nabla_i)^{k-s} g(x).
\]
Similar identities holds for mixed derivatives $D^\beta$ of degree $k$ by the Leibniz rule, with the combinatorial constants (similar to $C^k_1$) bounded by $n^k$. Applying (61) to the above identity and taking sup for $x$ in $M$ we get
\[
\| f \cdot g \|_{C^k(M)} \leq n^k \| f \|_{C^k(M,\mathbb{R}^q)} \| g \|_{C^k(M,\mathbb{R}^q)}.
\] (62)

Last we notice that
\[
\frac{|f(x) \cdot g(y) - f(y) \cdot g(y)|}{\text{dist} (x,y)} \leq \frac{|f(x) \cdot g(y) - f(x) \cdot g(y)|}{\text{dist} (x,y)} \leq \frac{|f(x)| \cdot |g(y) - g(y)|}{\text{dist} (x,y)} + \frac{|f(x) - f(y)| \cdot |g(y)|}{\text{dist} (x,y)}.
\]
(by Cauchy-Schwartz), so taking sup for $x \neq y$ in $M$ we get
\[
[f \cdot g]_{\alpha,M,\mathbb{R}^q} \leq \| f \|_{C^k(M,\mathbb{R}^q)} \| g \|_{C^k(M,\mathbb{R}^q)} + \| g \|_{C^k(M,\mathbb{R}^q)} \| f \|_{C^k(M,\mathbb{R}^q)}.
\] (63)

Similarly for $D^\beta$ of degree $k$, where $\beta$ is the multi-indices, using the Leibniz rule to $D^\beta (f \cdot g)$ and applying (63) to each summand, we have
\[
\left[D^\beta (f \cdot g)\right]_{\alpha,M,\mathbb{R}^q} \leq n^k \left( \| f \|_{C^k(M,\mathbb{R}^q)} \| g \|_{C^{k,\alpha}(M,\mathbb{R}^q)} + \| g \|_{C^k(M,\mathbb{R}^q)} \| f \|_{C^{k,\alpha}(M,\mathbb{R}^q)} \right).
\]

Putting (61), (62) and (63) together, we have
\[
\| f \cdot g \|_{C^{k,\alpha}(M)} \leq C(k, \alpha, M) \| f \|_{C^{k,\alpha}(M,\mathbb{R}^q)} \| g \|_{C^{k,\alpha}(M,\mathbb{R}^q)},
\]
where we can take the constant $C(k, \alpha, M) = n^k$, independent on $q$ of $\mathbb{R}^q$.  

### 6.2 Uniform quadratic estimate of $Q(u)$

For any given map $u \in C^{2,\alpha}(M,\mathbb{R}^q)$, the quadratic estimate of $Q(u)$ was established in \[4\] Lemma 4. In this section we show the constant in the quadratic estimate is uniform for all $\mathbb{R}^q$. This is essentially due to Lemma \[3\] where the constant $C(k, \alpha, M)$ is uniform for all $q$.

**Proposition 34** For any $v \in C^{2,\alpha}(M,\mathbb{R}^q)$, we have
\[
\| Q_i (v, v) \|_{C^{2,\alpha}(M,\mathbb{R}^q)} \leq C(A_0, \alpha, M) \| v \|_{C^{2,\alpha}(M,\mathbb{R}^q)}^2,
\]
\[
\| Q_{ij} (v, v) \|_{C^{2,\alpha}(M,\mathbb{R}^q)} \leq C(A_0, \alpha, M) \| v \|_{C^{2,\alpha}(M,\mathbb{R}^q)}^2,
\]
\[
\| Q (\psi_i) (v, v) \|_{C^{2,\alpha}(M,\mathbb{R}^q)} \leq C_E C(A_0, \alpha, M) t^{-1 - \frac{1}{2}} \| v \|_{C^{2,\alpha}(M,\mathbb{R}^q)}^2,
\]
where the constant $C(A_0, \alpha, M) = C(A_0, \alpha, M)$ is uniform for all $q$ (The constants $\sigma(A_0, \alpha, M)$, $C(2, \alpha, M)$ and $C_E$ are in \[14\], Lemma \[3\] and Definition \[2\] respectively).
Proof. We prove the case when $M$ has a global coordinate first. For brevity we write $C^{k,\alpha}(M,\mathbb{R}^q)$ as $C^{k,\alpha}(M)$. Recall that

$$Q_i(v, v) = (\Delta - \Lambda_0)^{-1}(\partial_i v \cdot (\Delta - \Lambda_0) v),$$

$$Q_{ij}(v, v) = (\Delta - \Lambda_0)^{-1}(\Sigma_{k=1}^n \partial_{ik} v \cdot \partial_{jk} v - \partial_{ij} v \cdot (\Delta - \Lambda_0) v),$$

where $\cdot$ is the standard inner product in $\mathbb{R}^q$. We prove that

$$\|Q_{ij}(v, v)\|_{C^{2,\alpha}(M)} \leq \sigma (\Lambda_0, \alpha, M) C (2, \alpha, M) \|v\|^2_{C^{2,\alpha}(M)},$$

and the first inequality is similar. By the elliptic estimate for $\Delta - \Lambda_0$ on $M$ and (60) we have

$$\|Q_{ij}(v, v)\|_{C^{2,\alpha}(M)} \leq \sigma (\Lambda_0, \alpha, M) \|\Sigma_{k=1}^n \partial_{ik} v \cdot \partial_{jk} v - \partial_{ij} v \cdot (\Delta - \Lambda_0) v\|_{C^{\alpha}(M)}$$

$$\leq \sigma (\Lambda_0, \alpha, M) (\Sigma_{k=1}^n \|\partial_{ik} v\|_{C^{\alpha}(M)} + \|\partial_{ij} v\|_{C^{\alpha}(M)} ) (\|\Delta - \Lambda_0\|_{C^{\alpha}(M)} )$$

$$\leq \sigma (\Lambda_0, \alpha, M) C (2, \alpha, M) \|v\|^2_{C^{2,\alpha}(M)}.$$

Finally, since

$$Q(u)(v, v) = E(u) ([Q_i(u)(v, v)], [Q_{ij}(u)(v, v)])$$

and the operator norms of $E(v) : C^{2,\alpha}(M, \mathbb{R}^q) \to C^{2,\alpha}(M, \mathbb{R}^q)$ is of order $C E t^{-1 - \frac{\alpha}{2}}$ by Corollary [29] we have

$$\|Q(v, v)\|_{C^{2,\alpha}(M)} \leq C E \sigma (\Lambda_0, \alpha, M) C (2, \alpha, M) t^{-1 - \frac{\alpha}{2}} \|v\|^2_{C^{2,\alpha}(M)}$$

for some constant $C$ uniform for all $q$.

For the $Q(v)$ defined for general manifolds (without global coordinate) in [G1], we still have the above uniform quadratic estimate for all $\mathbb{R}^q$. We only need to replace the elliptic estimates of $(\Delta - \Lambda_0)^{-1}$ in the above by that of the smoothing operators $(\Delta_{11} - \epsilon)^{-1}$ and $(\Delta_{22} - \epsilon)^{-1}$ in [G1], and apply Lemma [33] in the same way in the above estimates.
7 The implicit function theorem: isometric embedding

In previous sections we have considered the $\frac{n(n+3)}{2} \times \infty$ matrix $P(\psi_t)$ and its right inverse $E(\psi_t)$. If we truncate $l^2$ to $\mathbb{R}^q(t)$ with $q(t) \sim Ct^{-\frac{3}{2}-\rho}$, and consider $\psi_t : M \to \mathbb{R}^q(t)$, then $E(\psi_t)$ is a $q(t) \times \frac{n(n+3)}{2}$ matrix. By Lemma 8 and Remark 7, $E(\psi_t)$ still has the operator bounds as in Proposition 28 and Corollary 29, because from our construction (54),

$$E(\psi_t) = \hat{L} \bar{P}(\psi_t) ^T \bar{L} ^T$$

is essentially the transpose of $P(\psi_t)$ consisting of the row vectors $\partial_i \psi_t$ and $\partial_i \partial_j \psi_t$ ($1 \leq i \leq j \leq n$).

Now we are ready to give the proof of Theorem 1. We divide it into two propositions: isometric immersion and one-to-one map.

**Proposition 35 (Isometric immersion)** Under the conditions of Theorem 1, there exists $C$ and $t_0 > 0$ depending on $(g, \rho, \alpha)$, such that for the integer $q = q(t) \sim Ct^{-\frac{3}{2}-\rho}$ and $0 < t \leq t_0$, the truncated heat kernel embedding

$\psi_t : M \to \mathbb{R}^q \subset l^2$

can be perturbed to an isometric embedding $i_t : M \to \mathbb{R}^q$, with the perturbation of $\psi_t$ of order $O \left( t^{\frac{3}{2} - \frac{\alpha}{2}} \right)$ in the $C^{2,\alpha}$-norm.

**Proof.** Given the truncated heat kernel embedding $u = \psi_t^q : M \to \mathbb{R}^q$ with $q = q(t) \sim t^{\frac{3}{2} - \rho}$, and the error $h = \left( \psi_t^q \right)^{*} g_0 - g$ to the isometric embedding, we consider the nonlinear functional

$$F : C^{2,\alpha}(M, \mathbb{R}^q) \to C^{2,\alpha}(M, \mathbb{R}^q),$$

$$F(v) = v - E(\psi_t)(0, h) + E(\psi_t) \left( \left[ Q_i (u) (v, v) \right], \left[ Q_{jk} (u) (v, v) \right] \right). \quad (65)$$

We stress that this iteration is *coordinate free* and is defined on the whole $M$, as it is the coordinate expression of the iteration of tensors (see equations (12)–(21) in [G1]). We want to find the zeros of $F$. By the general implicit function theorem (e.g. Proposition A.3.4. in [MS]), the operator norm estimate in Corollary 29, and the uniform quadratic estimates in Proposition 34, it is enough to verify that

$$\| E(\psi_t) \|_{C^{2,\alpha}(M)} \| E(\psi_t)(0, h) \|_{C^{2,\alpha}(M, \mathbb{R}^q)} \to 0$$

as $t \to 0_+$. By Corollary 29 we have

$$\| E(\psi_t) \|_{C^{2,\alpha}(M)} \leq C_E t^{-\frac{3}{2} - \frac{\alpha}{2}}.$$

If we have

$$\text{Ric}_g - \frac{1}{2} S_g \cdot g = 0,$$

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then by Theorem 6
\[ h = (\psi_t)^* g_0 - g = -\frac{t}{3} \left( \frac{1}{2} S_g \cdot g - \text{Ric}_g \right) + O(t^2) = O(t^2) \quad (66) \]
in $C^3$ norm, i.e. for small $t$,
\[ \|h\|_{C^{2,\alpha}(M,S^2 g)} \leq G t^2 \]
from our definition of $G$ in (69). By our construction in (54), $E(\psi_t) = L \widehat{P}_{\psi_t}^{T} L^T$, and
\[ |\nabla_i \nabla_k \psi_t(x)|^2 \to \frac{3}{2} t^{-1} \text{ in } C^3 \text{ norm by Proposition 12} \]
\[ \|\nabla_i \nabla_k \psi_t\|_{C^{2,\alpha}(M,R^n)} \leq C_E t^{-\frac{3}{2}-\alpha} \text{ by (53)}, \]
we have
\[ \|E(\psi_t)(0,h)\|_{C^{2,\alpha}(M,R^n)} = \left\| \frac{L}{L^T h} \left[ \frac{\nabla_i \nabla_k \psi_t}{|\nabla_i \nabla_k \psi_t|^2} \right]_{1 \leq i \leq k \leq n} \right\|_{C^{2,\alpha}(M,R^n)} \]
\[ \leq C_E \|L\| \cdot \left( \frac{t^{-\frac{3}{2}-\alpha}}{2 t^{-1}} \right) \cdot \|L^T\| \cdot G t^2 \]
\[ = \frac{2}{3} C_E \|L\|^2 G t^{\frac{3}{2}-\alpha}. \]
where $[\cdot]_{1 \leq i \leq k \leq n}$ is the notation for a matrix. Hence
\[ \|E(\psi_t)\|_{C^{2,\alpha}(M)} \|E(\psi_t)(0,h)\|_{C^{2,\alpha}(M,R^n)} \]
\[ \leq \frac{2}{3} C_E^2 \|L\|^2 G t^{-1-\frac{3}{2}} \cdot t^{\frac{3}{2}-\alpha} \]
\[ = \frac{2}{3} C_E^2 \|L\|^2 G t^{\frac{3}{2}-\alpha} \to 0 \quad (67) \]
as $t \to 0_+$ for $0 < \alpha < \frac{1}{2}$. By Günther’s implicit function theorem we obtain the smooth map $i_t : M \to R^q$ such that
\[ i_t^* g_0 = g. \]
From this we immediately see $i_t$ is an isometric immersion. From the implicit function theorem we also see the needed perturbation from $\psi_t^{q(0)}$ to $i_t$ is of order $O(t^{\frac{3}{2}-\alpha})$ in the $C^{2,\alpha}$-norm (For readers interested in more details, see the Appendix). □

**Remark 36** (67) is the place that we need the condition $\text{Ric}_g = \frac{1}{2} S_g \cdot g$ to conclude $\|h\|_{C^{2,\alpha}(M,S^2 g)}$ is of order $O(t^2)$. Otherwise its order is $O(t)$ and
\[\|E(\psi_t)\|_{C^{2,\alpha}(M)} \|E(\psi_t)(0,h)\|_{C^{2,\alpha}(M,\mathbb{R}^q)} \sim Ct^{-\frac{d}{2} - \alpha} \to 0.\]
If we can make the quadratic terms in \[66\] explicit, then we can give the estimate of the smallness of \(t\) in the above implicit function theorem. See \[35\] for partial results in this direction.

To show the map \(i_t : M \to \mathbb{R}^q(t)\) is one-to-one for small enough \(t > 0\), we prove the following

**Proposition 37 (One-to-one map)** Let \((M, g)\) be a compact Riemannian manifold with smooth metric \(g\). Then there exists \(\delta_0 > 0\), such that for \(0 < t < \delta_0\), and \(q(t) \sim Ct^{-\frac{d}{2} - \rho}\), the truncated heat kernel mapping \(\psi^{q(t)}_t : M \to \mathbb{R}^q(t)\) can distinguish any two points on the manifold, i.e. for any \(x \neq y\) on \(M\), \(\psi^{q(t)}_t(x) \neq \psi^{q(t)}_t(y)\). The same is true for the isometric immersion \(i_t : M \to \mathbb{R}^q(t)\).

**Proof.** If there is no such \(\delta_0\), then there is a sequence of \(t_k \to 0\), and \(x_k \neq y_k\) on \(M\), such that

\[\psi^{q(t_k)}_{t_k}(x_k) = \psi^{q(t_k)}_{t_k}(y_k) .\] (68)

Since \(M\) is compact, taking a subsequence we can assume \(\lim_{k \to \infty} x_k = x_\infty\) and \(\lim_{k \to \infty} y_k = y_\infty\) for some \(x_\infty, y_\infty\) on \(M\). Suppose \(x_\infty \neq y_\infty\), \(\text{dist}(x_\infty, y_\infty) = 0 > 0,\) and \(\gamma_0 : [0, l_0] \to M\) is the minimal geodesic joining \(x_\infty\) and \(y_\infty\). Since \(\psi_t : M \to I^2\) is an embedding for any \(t > 0\) (\([35]\)), \(\psi_t(x_\infty) \neq \psi_t(y_\infty)\).

For any path \(\chi_t(s)\) \((a \leq s \leq b)\) in \(\psi_t(M)\) joining \(\psi_t(x_\infty)\) to \(\psi_t(y_\infty)\), \(\gamma_t(s) := (\psi_t^{-1} \circ \chi_t)(s)\) is a path in \(M\) joining \(x_\infty\) to \(y_\infty\). By \([2]\) there exists \(\delta_0 > 0\), such that for \(0 < t \leq \delta_0\), \(\psi_t^* g_0 \geq \frac{1}{2} g\) on \(M\), so

\[
\text{Length}_{(\psi_t(M), g_0 |_{\psi_t(M)})}(\chi_t)
= \text{Length}_{(M, \psi_t^* g_0)}(\gamma_t)
= \int_a^b \sqrt{\psi_t^* g_0 \left( \frac{\partial \gamma_t}{\partial s}, \frac{\partial \gamma_t}{\partial s} \right)} ds
\geq \frac{1}{2} \int_a^b \sqrt{g \left( \frac{\partial \gamma_t}{\partial s}, \frac{\partial \gamma_t}{\partial s} \right)} ds
= \frac{1}{2} \text{Length}_{(M, g)}(\gamma_t)
\geq \frac{1}{2} \text{Length}_{(M, g)}(\gamma_0)
= \frac{1}{2} l_0 > 0,
\]

and so

\[
\text{dist}_{(\psi_t(M), g_0 |_{\psi_t(M)})}(\psi_t(x_\infty), \psi_t(y_\infty)) \geq \frac{1}{2} l_0 > 0.\] (69)

On the other hand, we have

\[
\lim_{k \to \infty} \text{dist}_{(\psi_{t_k}(M), g_0 |_{\psi_{t_k}(M)})}(\psi_{t_k}(x_\infty), \psi_{t_k}(y_\infty))
= \lim_{k \to \infty} \text{dist}_{(\psi_{t_k}(M), g_0 |_{\psi_{t_k}(M)})}(\psi_{t_k}(x_k), \psi_{t_k}(y_k))
\text{ (since} \nabla_i \psi_i | \to 1 \text{ in Prop} [12] \text{ as } t \to 0_+, \psi_t \text{ is equicontinuous})
\]
Taking a subsequence again, we may assume the geodesics joining \(x\) and \(y\) have some limiting direction, i.e. there exists some unit vector \(V\) such that for \(k\to\infty\),
\[
\exp_{p}^{-1}(x_{k}) - \exp_{p}^{-1}(y_{k}) \to V \in T_{p}M.
\]
By Proposition \ref{proposition:72} as \(t \to 0_{+}\), \(|\nabla_{V_{i}}\psi_{t}(x)| \to 1\) uniformly on \(M\) for any orthonormal basis \(\{V_{i}\}_{i=1}^{n}\) at \(x\). So there exists \(\delta_{0} > 0\), such that for \(0 < t \leq \delta_{0}\), \(|\nabla_{V_{i}}\psi_{t}(x)| \geq \frac{2}{3}\) for all \(x\) on \(M\) and \(1 \leq i \leq n\). By the computations in Theorem \ref{theorem:6}, this passes to the truncated heat kernel embedding \(\psi_{t}^{(t)} : M \to \mathbb{R}^{q(t)}\) for \(q(t) \sim Ct^{-\frac{n}{2} - \frac{1}{d}}\) as well, more precisely
\[
|\nabla_{V_{i}}\psi_{t}^{(t)}(x)| \geq \frac{1}{2} \quad \text{for} \quad 0 < t \leq \delta_{0}, \ 1 \leq i \leq n \quad \text{and all} \quad x \in M. \tag{71}
\]
This contradicts with \(\ref{corollary:36}\). So \(\psi_{t}^{(t)}\) is one-to-one, and is an immersion by \(\ref{corollary:36}\).

Hence \(\psi_{t}^{(t)}\) is an embedding.

The proof of the one-to-one property of \(i_{t} : M \to \mathbb{R}^{q(t)}\) is almost identical to that of \(\psi_{t}\). The reason is that we have \(\|i_{t} - \psi_{t}\|_{C^{2,\alpha}(M)} \leq C_{\varepsilon}t^{-\frac{n}{2} - \frac{1}{d}}\) from Proposition \ref{proposition:6}, so the key estimates \(\ref{corollary:36}\), \(\ref{corollary:36}\), \(\ref{corollary:6}\), \(\ref{corollary:71}\) and \(\ref{corollary:72}\) in the argument of \(\psi_{t}\) still hold for \(i_{t}\).

We have an interesting

**Corollary 38** Let \((M,g)\) be a compact Riemannian manifold with smooth metric \(g\). Then there exists an integer \(N_{0} > 0\) depending on \(g\), such that the first \(N_{0}\) eigenfunctions \(\{\phi_{j}\}_{j=1}^{N_{0}}\) can distinguish any two points on \(M\), i.e. for any \(x \neq y\) on \(M\), there exists some \(j_{0} \in \{1, 2, \cdots N_{0}\}\), such that \(\phi_{j_{0}}(x) \neq \phi_{j_{0}}(y)\).

**Proof.** Take \(N_{0} = q(t_{0})\) in the above Lemma, and note
\[
\psi_{t}^{N_{0}}(x) = \psi_{t}^{N_{0}}(y) \iff \phi_{j}(x) = \phi_{j}(y) \quad \text{for} \quad 1 \leq j \leq N_{0}.
\]
\[\square\]
8 Examples

We can write down the matrix $P ( u )$ and $E ( u )$ explicitly in the $M = S^1$ case, and compute the inner product of the row vectors of the matrix $P ( u )$ in $M = T^n$ case for $n = 1, 2$ directly. The general $T^n$ ($n \geq 3$) case is still computable but this elementary method becomes increasingly tedious. We do these examples to illustrate the proofs of our main theorems in the simplest situations.

8.1 $M = S^1$ case

We can explicitly compute $E ( u )$ in $M = S^1 \simeq [0, 2\pi]$ case, where $n = 1$. For eigenvalue $\lambda_{2k-1} = \lambda_{2k} = k^2$, the $L^2$ orthonormal eigenfunctions are pairs

$$\phi_{2k-1} (x) = \frac{1}{\sqrt{\pi}} \cos kx \text{ and } \phi_{2k} (x) = \frac{1}{\sqrt{\pi}} \sin kx.$$  

The heat kernel embedding $u : S^1 \to \mathbb{R}^{2q}$ is

$$u (x) = \sqrt{\frac{2}{\pi}} (4\pi)^{n/4} t^{\frac{n+2}{4}} \left( e^{-\frac{2\pi}{t}} \cos x, e^{-\frac{2\pi}{t}} \sin x, \cdots, e^{-\frac{2\pi}{t}} \cos qx, e^{-\frac{2\pi}{t}} \sin qx \right),$$

so the system (10) becomes

$$\sqrt{\frac{2}{\pi}} (4\pi)^{n/4} t^{\frac{n+2}{4}} : \Sigma_{k=1}^q e^{-k^2 t} (-k \sin k \cdot v_{2k-1} + k \cos k \cdot v_{2k}) = h_1 (x)$$

$$\sqrt{\frac{2}{\pi}} (4\pi)^{n/4} t^{\frac{n+2}{4}} : \Sigma_{k=1}^q e^{-k^2 t} (-k^2 \cos k \cdot v_{2k-1} - k^2 \sin k \cdot v_{2k}) = f_{11} (x).$$

The $2 \times 2q$ matrix of the above linear system is

$$\sqrt{\frac{2}{\pi}} (4\pi)^{n/4} t^{\frac{n+2}{4}} : \left[ \cdots \begin{array}{ccc} -ke^{-\frac{2\pi}{t}} \sin kx & ke^{-\frac{2\pi}{t}} \cos kx \cdots \\ \cdots & -k^2 e^{-\frac{2\pi}{t}} \cos kx & -k^2 e^{-\frac{2\pi}{t}} \sin kx & \cdots \end{array} \right]_{1 \leq k \leq q}.$$  

Note that the two row vectors $R_1$ and $R_2$ are orthogonal, where

$$R_1 (x) = \sqrt{\frac{2}{\pi}} (4\pi)^{n/4} t^{\frac{n+2}{4}} \left\{ \left( -e^{-\frac{2\pi}{t}} k \sin kx, e^{-\frac{2\pi}{t}} k \cos kx \right) \right\}_{1 \leq k \leq q},$$

$$R_2 (x) = \sqrt{\frac{2}{\pi}} (4\pi)^{n/4} t^{\frac{n+2}{4}} \left\{ \left( -e^{-\frac{2\pi}{t}} k^2 \cos kx, -e^{-\frac{2\pi}{t}} k^2 \sin kx \right) \right\}_{1 \leq k \leq q},$$

since for all $k$,

$$(-k e^{-\frac{2\pi}{t}} \sin kx) \cdot (-k^2 e^{-\frac{2\pi}{t}} \cos kx) + (ke^{-\frac{2\pi}{t}} \cos kx) \cdot (-k^2 e^{-\frac{2\pi}{t}} \sin kx) = 0.$$  

So we are solving

$$R_1 \cdot v = h_1 (x)$$

$$R_2 \cdot v = f_{11} (x).$$
Therefore the solution \( v \) with minimal Euclidean norm is

\[
v(x) = \frac{h_1(x)}{|R_1(x)|^2} R_1(x) + \frac{f_{11}(x)}{|R_2(x)|^2} R_2(x).
\]

**Remark 39** For general linear equation system whose coefficient row vectors \( R_1, \cdots, R_{n(n+3)} \) are not orthogonal, to solve the \( v \) with minimal Euclidean norm is not so easy. But we can use Gram-Schmidt procedure (row operations to the linear system of equations) to get an equivalent system with orthonormal row vectors \( \tilde{R}_1, \cdots, \tilde{R}_{n(n+3)}\):

\[
\tilde{R}_i \cdot v(x) = \tilde{h}_i(x),
\]

\[
\tilde{R}_{jk} \cdot v(x) = \tilde{f}_{jk}(x).
\]

Then the solution \( v \) with minimal Euclidean norm is

\[
v(x) = \sum_i \tilde{h}_i(x) \tilde{R}_i(x) + \sum_{j,k} \tilde{f}_{jk}(x) \tilde{R}_{jk}(x).
\]

This, in principle, can be solved with explicit formula. The method should be the QR decomposition in matrix analysis.

Hence

\[
E(u) = \left[ \frac{R_1(x)}{|R_1(x)|^2} \bigg| \frac{R_2(x)}{|R_2(x)|^2} \right]
\]

and

\[
E(u)(0, f_{11}) = \frac{f_{11}(x)}{|R_2(x)|^2} R_2(x).
\]

**Lemma 40** We have

\[
\lim_{t \to 0^+} t^{m+1} \sum_{k=1}^{\infty} k^m e^{-k^2 t} = \int_0^\infty \mu^m e^{-\mu^2} d\mu,
\]

\[
\sum_{k=1}^{\infty} k^m e^{-k^2 t} \leq K t^{-\frac{m+1}{2}},
\]

where the constant \( K = \int_0^\infty \mu^m e^{-\mu^2} d\mu \).

**Proof.** We notice that

\[
t^{m+1} \sum_{k=1}^{\infty} k^m e^{-k^2 t} = \sum_{k=1}^{\infty} \left(k \sqrt{t} \right)^m e^{-(k \sqrt{t})^2} \cdot \sqrt{t}
\]

is a Riemann sum for \( \int_0^\infty \mu^m e^{-\mu^2} d\mu \) with the partition \( \sqrt{t} \mathbb{Z} \) on \( \mathbb{R} \). When \( t \to 0^+ \) the diameter of subintervals of this partition goes to 0, so

\[
\lim_{t \to 0^+} t^{m+1} \sum_{k=1}^{\infty} k^m e^{-k^2 t} = \int_0^\infty \mu^m e^{-\mu^2} d\mu.
\]
The second inequality follows immediately since $\int_0^\infty \mu^m e^{-\mu^2} d\mu$ is convergent.

For $q = q(t) \sim Ct^{-\frac{7}{4}}$ large, using

$$\int_0^\infty \mu^2 e^{-\mu^2} d\mu = \frac{\sqrt{\pi}}{4}, \text{ and } \int_0^\infty \mu^4 e^{-\mu^2} d\mu = \frac{3\sqrt{\pi}}{8},$$

we have

$$|R_1(x)|^2 = \frac{2}{\pi} (4\pi)^{n/4} t^{n+2} \sum_{k=1}^n k^2 e^{-k^2 t} \rightarrow \frac{2}{\pi} (4\pi)^{1/2} t^{3/2} \cdot t^{-\frac{7}{4}} \int_0^\infty \mu^2 e^{-\mu^2} d\mu = 1,$$

$$|R_2(x)|^2 = \frac{2}{\pi} (4\pi)^{n/4} t^{n+2} \sum_{k=1}^n k^4 e^{-k^2 t} \rightarrow \frac{2}{\pi} (4\pi)^{1/2} t^{3/2} \cdot t^{-\frac{7}{4}} \int_0^\infty \mu^4 e^{-\mu^2} d\mu = \frac{3}{2} t^{-1}.$$

These agree with $\psi_* g_0 \rightarrow g$ in (2) and the mean curvature length

$$|H(x,t)| \rightarrow \sqrt{\frac{3}{2} t^{-\frac{7}{4}}} = \sqrt{\frac{1 + 2}{2} \cdot t^{-\frac{7}{4}}}$$

in Corollary 3 respectively.

Thus for $q = q(t)$ large, in $C^3$ convergence we have

$$E(u) \rightarrow \left[ \frac{R_1(x)}{3} \frac{R_2(x)}{R_2(x)} \right],$$

and

$$E(u)(0,f_{11}) \rightarrow \frac{2t}{3} R_2(x) f_{11}(x). \quad (73)$$

We have the $C^2$ norm (according to our Definition 32 for vector-valued functions)

$$\|R_1(x)\|_{C^2(M)} = \sqrt{\frac{2}{\pi} (4\pi)^{n/4} t^{n+2} \left[ \sum_{k=1}^n k^2 e^{-k^2 t} k^6 \right]^{1/2}} \rightarrow Ct^{\frac{3}{2}} \cdot t^{-\frac{7}{4}} = Ct^{-1},$$

$$\|R_2(x)\|_{C^2(M)} = \sqrt{\frac{2}{\pi} (4\pi)^{n/4} t^{n+2} \left[ \sum_{k=1}^n k^4 e^{-k^2 t} k^8 \right]^{1/2}} \rightarrow Ct^{\frac{3}{2}} \cdot t^{-\frac{7}{4}} = Ct^{-\frac{3}{2}}.$$

So

$$\|E(u)\|_{C^2(M)} \rightarrow \sqrt{\frac{2}{\pi} (4\pi)^{n/4} t^{n+2} \left[ \sum_{k=1}^n k^4 e^{-k^2 t} k^6 + t^2 \cdot \sum_{k=1}^n e^{-k^2 t} k^8 \right]^{1/2}} \rightarrow Ct^{\frac{3}{2}} \left[ t^{-\frac{7}{4}} + t^2 t^{-\frac{7}{4}} \right]^{1/2} = Ct^{-1}.$$

Notice that for $S^1$, the curvature tensor $R \equiv 0$, so $\text{Ric}_g - \frac{1}{2} g \cdot S_g \equiv 0$. By [BBG] Theorem 5 we have

$$f_{11} = \psi_* g_0 - g = O(t^2)$$
as \( t \to 0_+ \) (it may be of higher vanishing order \( O(t^p) \) for some \( p > 2 \), but here we only use \( O(t^2) \) to illustrate our method). So by (73) we have

\[
\|E(u)(0, f_{11})\|_{C^2(M)} \to C \|t^3 R_2(x)\|_{C^2(M)} = Ct^3 \cdot \|R_2(x)\|_{C^2(M)} = Ct^3 \cdot t^{-\frac{2}{3}} = Ct^{\frac{7}{3}}.
\]

Then

\[
\|E(u)\|_{C^2(M)} \to \|E(u)(0, f_{11})\|_{C^2(M)} \to Ct^{-\frac{1}{2}} \cdot Ct^{\frac{7}{3}} = Ct^{\frac{5}{6}},
\]

and similarly (using Proposition 14 to estimate the Schauder norm)

\[
\|E(u)\|_{C^2,\alpha(M)} \to \|E(u)(0, f_{11})\|_{C^2,\alpha(M)} \to Ct^{-\frac{1}{2}} \cdot Ct^{\frac{7}{3}} = Ct^{\frac{5}{6} - \alpha} \to 0
\]

for \( 0 < \alpha < \frac{1}{2} \). We see the estimates of the orders are exactly the same as obtained by the off-diagonal expansion of heat kernel method.

By Günther’s implicit function theorem we obtain the isometric embeddings of \( S^1 \) into \( \mathbb{R}^{q(t)} \).

### 8.2 \( M = S^1 \times S^1 \) case

Here \( \dim M = n = 2, M = S^1 \times S^1 = [0, 2\pi] \times [0, 2\pi] \). For eigenvalue \( \lambda_{4k+1} = \lambda_{4k+2} = \lambda_{4k+3} = \lambda_{4k+4} = k^2 + l^2 \) for integers \( k \) and \( l \), the \( L^2 \) orthonormal eigenfunctions are

\[
\phi_{4k+1}(x, y) = \frac{1}{\pi} \cos kx \cos ly, \quad \phi_{4k+2}(x) = \frac{1}{\pi} \sin kx \cos ly,
\]

\[
\phi_{4k+3}(x, y) = \frac{1}{\pi} \cos kx \sin ly, \quad \phi_{4k+4}(x) = \frac{1}{\pi} \sin kx \sin ly.
\]

The heat kernel embedding \( u : S^1 \times S^1 \rightarrow \mathbb{R}^{4q} \) is

\[
u (x, y) = ct^{\frac{n+2}{4}} \left\{ \cdots, e^{-\frac{k^2 + l^2}{2} t} \cos kx \cos ly, \cos kx \sin ly, \sin kx \sin ly \right\}_{k,l \geq 1},
\]

where \( c = \frac{\sqrt{2}}{\pi} (4\pi)^{n/4} \). We have

\[
\partial_x \partial_x u = ct^{\frac{n+2}{4}} \left\{ \cdots, -k^2 \cdot e^{-\frac{k^2 + l^2}{2} t} \cos kx \cos ly, \cos kx \sin ly, \sin kx \sin ly \right\}_{k,l \geq 1},
\]

\[
\partial_y \partial_y u = ct^{\frac{n+2}{4}} \left\{ \cdots, -l^2 \cdot e^{-\frac{k^2 + l^2}{2} t} \cos kx \cos ly, \cos kx \sin ly, \sin kx \sin ly \right\}_{k,l \geq 1},
\]

\[
\partial_x \partial_y u = ct^{\frac{n+2}{4}} \left\{ \cdots, kl \cdot e^{-\frac{k^2 + l^2}{2} t} \sin kx \sin ly, -\cos kx \sin ly, -\sin kx \sin ly, \cos kx \cos ly \right\}_{k,l \geq 1}.
\]
Then

\[
|\partial_x \partial_z u|^2 = c^2 t^2 \sum_{k,l \geq 1} k^4 e^{-(k^2 + l^2) t} \sim c^2 t^2 \cdot \int_0^\infty \int_0^\infty x^4 e^{-(x^2 + y^2) t} dxdy
\]

\[
= c^2 t^2 \cdot \int_0^{\frac{\pi}{2}} \int_0^\infty r^4 \cos^4 \theta e^{-r^2 t} r dr d\theta
\]

\[
= c^2 t^2 \cdot \int_0^{\frac{\pi}{2}} \int_0^\infty r^4 \cos^4 \theta d\theta \cdot \int_0^\infty r^5 e^{-r^2 t} dr
\]

\[
= \frac{3\pi}{16} c^2 t^{-1} \int_0^\infty \mu^5 e^{-\mu^2} d\mu,
\]

\[
|\partial_y \partial_y u|^2 = c^2 t^2 \sum_{k,l \geq 1} k^2 l^2 e^{-(k^2 + l^2) t} \sim c^2 t^2 \cdot \frac{1}{2} \int_0^\infty \int_0^\infty x^2 y^2 e^{-(x^2 + y^2) t} dxdy
\]

\[
= c^2 t^2 \cdot \int_0^{\frac{\pi}{2}} \int_0^\infty r^4 \sin^2 \theta \cos^2 \theta e^{-r^2 t} r dr d\theta
\]

\[
= c^2 t^2 \cdot \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^2 \theta d\theta \cdot \int_0^\infty r^5 e^{-r^2 t} dr
\]

\[
= \frac{\pi}{16} c^2 t^{-1} \int_0^\infty \mu^5 e^{-\mu^2} d\mu,
\]

where we have used that

\[
\int_0^{\frac{\pi}{2}} \sin^4 \theta d\theta = \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta = \frac{3\pi}{16}, \text{ and } \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^2 \theta d\theta = \frac{\pi}{16}.
\]

Similar to Lemma [40] the above ”\sim” is actually ”=” as \( t \to 0^+ \) because we have used the lattice

\[
\sqrt{t}Z^2 = \left\{ (\sqrt{t}k, \sqrt{t}l) \mid k, l \in \mathbb{Z} \right\}
\]

in the Riemann sum to approximate the integral \( \int_0^\infty \int_0^\infty x^4 e^{-(x^2 + y^2) t} dxdy \) etc. As \( t \to 0^+ \) the lattice has periods going to zero, so the Riemann sum has the limit equal to the integral.

We also have

\[
(\partial_x \partial_z u, \partial_y \partial_y u) = c^2 t^2 \sum_{k,l \geq 1} \left[ k^2 l^2 e^{-(k^2 + l^2) t} \right] = |\partial_x \partial_y u|^2 \sim \frac{\pi}{16} c^2 t^{-1} \int_0^\infty \mu^5 e^{-\mu^2} d\mu,
\]

\[
(\partial_x \partial_z u, \partial_x \partial_x u) = c^2 t^2 \sum_{k,l \geq 1} \left[ -k^3 l (\cos kx \cos ly \cdot \sin kx \sin ly - \sin kx \cos ly \cdot \cos kx \sin ly - \cos kx \sin ly \cdot \sin kx \cos ly + \sin kx \sin ly \cdot \cos kx \cos ly) \right]
\]

\[
= 0.
\]
Therefore as $t \to 0_+$,
\[
\frac{||\langle \partial_x \partial_y u, \partial_x \partial_y u \rangle||}{|\partial_x \partial_y u| |\partial_x \partial_y u|} \to \frac{1}{3} = \frac{\int_0^\pi \sin^2 \theta \cos^2 \theta d\theta}{\int_0^\pi \sin^4 \theta d\theta},
\]
\[
\frac{||\langle \partial_x \partial_y u, \partial_x \partial_y u \rangle||}{|\partial_x \partial_y u| |\partial_x \partial_y u|} \to 0 = \frac{||\langle \partial_y \partial_x u, \partial_x \partial_y u \rangle||}{|\partial_y \partial_x u| |\partial_x \partial_y u|}.
\]

The results agree with Theorem 19.

9 Appendix

If we apply the following Proposition A.3.4. of [MS] (an abstract implicit function theorem) to the nonlinear function $F : C^{2,\alpha} (M, \mathbb{R}^q) \to C^{2,\alpha} (M, \mathbb{R}^q)$ in Section 7 we recover Ginther’s implicit function theorem (Theorem 11), and obtain the following more information: First, the constant $\theta$ in his Theorem can be made explicit in (75); Second, the needed perturbation of $\psi_t$ can be shown of order $O \left( \frac{t^{2 - \frac{1}{2}}}{\tau} \right)$ in the $C^{2,\alpha}$ norm.

Proposition 41 (Proposition A.3.4. of [MS]) Let $X, Y$ be Banach spaces and $U$ be an open set in $X$. The map $F : X \to Y$ is continuous differentiable. For $x_0 \in U$, $D := dF(x_0) : X \to Y$ is surjective and has a bounded linear right inverse $Q : Y \to X$, with $\|Q\| \leq c$. Suppose that there exists $\delta > 0$ such that $x \in B_\delta(x_0) \subset U$

\[ x \in B_\delta(x_0) \subset U \implies \|dF(x) - D\| \leq \frac{1}{2c}. \]

Suppose $\|F(x_0)\| \leq \frac{\phi}{4c}$, then there exists a unique $x \in B_\delta(x_0)$ such that

\[ F(x) = 0, \quad x - x_0 \in \text{Image} Q, \quad \|x - x_0\| \leq 2c \|F(x_0)\|. \]

Applying the Proposition to our case, we have (following their notations)

\[ X = C^{2,\alpha} (M, \mathbb{R}^q), \quad Y = C^{2,\alpha} (M, \mathbb{R}^q), \quad F : X \to Y, \]

\[ F(v) = v - E(\psi_t)(0, h) + E(\psi_t)([Q_t(u)(v, v)], [Q_{2t}(u)(v, v)]), \]

\[ x_0 = 0, \quad x \text{ solution, } F(x) = 0, \]

\[ c = \left\| (dF(0))^{-1} \right\| = \left\| (id)^{-1} \right\| = 1, \]

\[ \|F(0)\| = \|E(\psi_t)(0, h)\|_{C^{2,\alpha}(M, \mathbb{R}^q)} \leq \frac{2}{3} CE \|L\|^2 Gt^{2 - \frac{1}{\tau}}, \]

\[ \|dF(v) - dF(0)\| \leq \|E(\psi_t)\|_{C^{2,\alpha}(M)} \cdot \sigma (\Lambda_0, \alpha, M) C(2, \alpha, M) \cdot \|v\|_{C^{2,\alpha}(M, \mathbb{R}^q)}, \]

where the last inequality is from [51]. Since the $\delta$ should satisfy

\[ \|v - 0\| \leq \delta \implies \|dF(v) - dF(0)\| \leq \frac{1}{2c}. \]
we can take
\[
\delta = \frac{1}{2 \| E(u) \|_{C^{2,\alpha}(M)} : \sigma (\Lambda_0, \alpha, M) C(2, \alpha, M) \frac{t^{1+\frac{\alpha}{2}}}{2 C E \sigma (\Lambda_0, \alpha, M) C(2, \alpha, M)}}.
\]
The condition
\[
\| F(0) \| \leq \frac{\delta}{4c} \tag{74}
\]
is translated to
\[
\| E(\psi_t)(0,h) \|_{C^{2,\alpha}(M,R^k)} \leq \frac{1}{4} \cdot \frac{1}{2 \| E(\psi_t) \|_{C^{2,\alpha}(M)} : \sigma (\Lambda_0, \alpha, M) C(2, \alpha, M)}.
\]

or
\[
\| E(\psi_t)(0,h) \|_{C^{2,\alpha}(M,R^k)} \| E(u) \|_{C^{2,\alpha}(M)} \leq (8 \sigma (\Lambda_0, \alpha, M) C(2, \alpha, M))^{-1} := \theta. \tag{75}
\]
So the constant \( \theta \) is essentially determined by \( \sigma (\Lambda_0, \alpha, M) = \| (\Delta - \Lambda_0)^{-1} \| \) of the smoothing operator (Note \( C(2, \alpha, M) = n^2 \)). In terms of \( t \) the condition \( \| x \| \leq \frac{2}{3} C E \| L \| \frac{G t \frac{1+\alpha}{2}}{2} \) is
\[
\frac{2}{3} C E \| L \| ^{2} G t \frac{1+\alpha}{2} \leq \frac{t^{1+\frac{\alpha}{2}}}{2 C E \sigma (\Lambda_0, \alpha, M) C(2, \alpha, M)},
\]
i.e. we need
\[
t \leq \left[ \frac{3}{4 C E \sigma (\Lambda_0, \alpha, M) G \cdot C(2, \alpha, M) \| L \|^2} \right] ^{\frac{2}{2-\alpha}} := t_0. \tag{76}
\]

So we see the smallness of \( t \) in our implicit function theorem depends on (i) the constant \( C_E \) in the operator bound for \( E(\psi_t) \), (ii) the constant \( \sigma (\Lambda_0, \alpha, M) \) for the smoothing operator \( (\Delta - \Lambda_0)^{-1} \), and (iii) the quantity \( G \) from the quadratic expansion of the heat kernel of \( (M, g) \) (the constants \( C(2, \alpha, M) = n^2 \) and \( \| L \|^2 \) are independent on the geometry of \( (M, g) \)). If we know \( t_0 \), we can obtain the estimate of the minimal embedding dimension \( q(t) \sim C t_0^{\frac{\alpha}{2} - \rho} \).

From the above Proposition the solution \( x \) satisfies \( \| x \| \leq 2c \| F(0) \| \), i.e. the perturbation of \( \psi_t \) is of order
\[
\| x \|_{C^{2,\alpha}(M,R^k)} \leq \frac{2}{3} C E \| L \| ^{2} G t \frac{1+\alpha}{2} = O \left( t^{\frac{1}{2} - \frac{\alpha}{2}} \right).
\]

References

[BBG] P. Bérard, G. Besson, S. Gallot, Embedding Riemannian Manifolds by Their Heat kernel, Geom. and Func. Analysis, vol. 4, No. 4, (1994).
[BeGaM] M. Berger, P. Gauduchon, E. Mazet, *Le spectre d’une variété riemannienne*, Springer Lecture Notes in Math. 194, 1971.

[Bu] Burstin, C, *Ein Beitrag zum Problem der Einbettung der Riemannschen Räume in euklidischen Räumen.*, Mat. Sb. USSR (1931), 38, 74-85.

[C] Cartan, E., *Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien*, Ann. Soc. Polon. Math. (1927), 6, 1-7.

[Ch] I. Chavel, *Eigenvalues in Riemannian Geometry*, Vol 115 of Pure and Applied Mathematics, Academic Press, 1984.

[EM] Y. Eliashberg, N. Mishachev, *Introduction to the h-Principle* (Graduate Studies in Mathematics), Vol 48, Amer Mathematical Society, 2002.

[G1] M. Günther, *On the perturbation problem associated to isometric embeddings of Riemannian manifolds*, Ann. Global Anal. Geom. Vol. 7, No. 1 (1989), 69-77.

[G2] M. Günther, *Isometric embeddings of Riemannian manifolds*, Proceedings of the International Congress of Mathematicians, Kyoto, Japan, 1990.

[Gr] M.L. Gromov, *Partial differential relations*, (1986), Springer, New York Berlin Heidelberg.

[Gri] D. Grieser, *Uniform bounds for eigenfunctions of the Laplacian on manifolds with boundary*. Comm. P.D.E., 27 (7-8), 1283-1299 (2002).

[H] L. Hörmander, *The spectral function of an elliptic operator*, Acta Math. 88 (1968), 341-370.

[HH] Q. Han, J.X. Hong, *Isometric Embedding of Riemannian Manifolds in Euclidean Spaces*, (2006), American Mathematical Society.

[Hir] Hirsch, M., *Differential Topology*, GTM 33, Springer-Verlag, 1976.

[J] Janet, M., *Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien*, Ann. Soc. Polon. Math. (1926), 5, 38-43.

[K] Kuiper, N.H. *On $C^1$-isometric imbeddings I*, Nederl. Akad. Wetensch. Proc. (1955), Ser. A. 58: 545–556.

[LY] P. Li, S.T. Yau, *A New Conformal Invariant and Its Applications to the Willmore Conjecture and the First Eigenvalue of Compact Surfaces*, Invent. math. 69, 269-291, (1982).

[MS] D. McDuff, D. Salamon, *J-holomorphic Curves and Symplectic Topology*, Colloquium Publications, vol 52, AMS, Providence RI, 2004.
[N1] Nash, John (1954), $C^1$-isometric imbeddings, Annals of Mathematics 60 (3): 383–396.

[N2] Nash, John (1956), The imbedding problem for Riemannian manifolds, Annals of Mathematics 63 (1): 20–63.

[S] C. D. Sogge, Concerning the $L^p$ norm of spectral clusters for second-order elliptic operators on compact manifolds, J. Funct. Anal. 77 (1988), no. 1, 123–134.

[T] Takashi Sakai. Riemannian geometry, Vol 149 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1996.

[Wu] H.T. Wu, Embedding Riemannian Manifolds by the Heat Kernel of the Connection Laplacian, preprint, [arXiv:1305.4232]

[X] B. Xu, Derivatives of the Spectral Function and Sobolev Norms of Eigenfunctions on a Closed Riemannian Manifold, Annals of Global Analysis and Geometry, Vol. 26, no.3, Oct. (2004), 231-252 (22).

[YV] Yu Safarov, D. Vassiliev, The asymptotic distribution of eigenvalues of partial differential operators, Translations of Mathematical Monographs Series , #155, (1996), American Mathematical Society.

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