Epistemic Learning Programs
A Calculus for Describing Epistemic Action Models

Mohammad Ardeshir and Rasoul Ramezanian
Department of Mathematical Sciences,
Sharif University of Technology,
P. O. Box 11365-9415, Tehran, Iran
mardeshir@sharif.edu, ramezanian@sharif.edu

Abstract
Dynamic Epistemic Logic makes it possible to model and reason about information change in multi-agent systems. Information change is mathematically modeled through epistemic action Kripke models introduced by Baltag et al. Also, van Ditmarsch interprets the information change as a relation between epistemic states and sets of epistemic states and to describe it formally, he considers a special constructor $L_B$ called learning operator. Inspired by this, it seems natural to us that the basic source of information change in a multi-agent system should be learning an announcement by some agents together, privately, concurrently or even wrongly. Hence moving along this path, we introduce the notion of a learning program and prove that all finite $K45$ action models can be described by our learning programs.

1 Introduction

A computable function over strings of a finite alphabet is a function that can be computed by a Turing machine. A Turing machine takes a string as input, performs a sequence of elementary changes on the string and if it halts, it provides another string as output of the function. In recursion theory all Turing computable function can be obtained via some initial functions: zero, successor, and projections through applying some basic operations such as composition, primitive recursion and least search. In this paper, our goal is to develop a similar methodology for a class of epistemic functions. Following the same terminology, an epistemic function is a function that takes the epistemic state of a multi-agent system as input and yields a new epistemic state as output. The notion of epistemic function is the focus of Dynamic Epistemic Logic, Baltag et al. [4] [5], and is formalized in action models. These functions act on Kripke models via an update operator and produce an update Kripke model. In this paper, we concentrate on those epistemic functions which can be coded as $K45$ action models. $K45$ models are those models which accessibly relations transitive and Euclidian. We claim that there are possible information changes which are not possible to encode them by $KD45$ or $S5$ action models. Consequently $K45$ action models are more powerful to describe epistemic functions than $KD45$ and $S5$ action models. It is why we consider $K45$ action models instead of $S5$ or $KD45$ models.

So far no one has looked at it from a computational aspect to answer the following question

what are the initial functions and the basic operations by which all $K45$ epistemic functions can be obtained?
The basic source of information change in a multi-agent system is learning an announcement by some agents together, privately, concurrently or even wrongly. So the basic operators should be different kinds of learning. Van Ditmarsch et al. introduced a learning constructor in [7, 8]. We define our own learning operator which is different from van Ditmarsch’s.

As our initial functions, we take the test of any facts \( \phi \), that is \( ?\phi \). For the basic operations, we take the following different kinds of learning: 1- alternative learning, 2- concurrent learning, 3- wrong learning, and finally, 4- recursive learning. Following the footsteps of recursion theory, we prove that all epistemic functions can be obtained through the test of facts by applying the above four basic operations.

Epistemic logic, started with Hintikka’s groundwork [14], models and reasons about the knowledge of agents in a group [12]. In Epistemic logic, the notions of knowledge and belief are modeled in terms of the possible worlds (states). An agent knows or believes a fact if it is true in all the worlds that the agent considers possible as alternatives for the actual world.

As information is transmitted, knowledge and belief of agents in a multi-agent system may change. The simplest cause for an information change is to announce some truth in public. Plaza in 1989 [15], introduced a logic to formalize these changes and called it the public announcement logic. In public announcement logic, a fact is publicly announced in a multi-agent system and all agents together update their knowledge and belief. However, more complex actions than such as private or dishonest announcements may occur, this is whereby different agents may have different views on some action [4, 5, 7], or the announcement may not be truthful.

Our work may be considered as a bridge between two paradigms in dynamic epistemic logic, namely Baltag et al. style action model and van Ditmarsch et al. epistemic actions.

The paper is organized as follows. In section 2, we recall the definition of action models from [4], and we explain what it means that a group of agents learns an action model. We discuss \( K45 \) Kripke models which only have transitive and Euclidian properties. We also introduce a new notion called the applicable formulas, and restrict the definition of satisfaction relation only to applicable formulas.

In section 3, we introduce the initial functions and basic operations as the building blocks of the finite epistemic functions.

In section 4, we add the new operator of recursion in constructing the finite epistemic functions.

Finally, in section 5, we compare our work with other related works, and further works may be done.

2 Backgrounds

In this section, we introduce two significant notion: 1- learning an action model, 2- \( K45 \) Kripke models and applicable formulas.

2.1 Action models

We start by recalling the definition of an action model from Baltag [4]. A pointed action model \((N, s_0)\) is a tuple \(N = \langle S, (\neg\alpha)_{\alpha \in A}, pre\rangle\), where \(S\) is a set of events, \(A\) is a set of
agents, $\rightarrow_a$ is an accessibility relation for agent $a$ on events in $S$, and $\text{pre}$ is a function that assigns to each event, a formula of an appropriate epistemic language, as a precondition for that event. $s_0 \in S$ is called the actual event.

Announcements of facts in multi-agent systems give rises to information changes, where different agents have different access to the resource of the announcement, and also different views about the access of other agents to the resource.

For instance, consider the pointed action model $(N_1, s)$ in Figure 1, where $S = \{s, t\}$, $s \rightarrow_a t$, $t \rightarrow_a t$, $\text{pre}(s) = \varphi$, $\text{pre}(t) = \psi$.

![Figure 1](image1)

Here $(N_1, s)$ encodes the following information change

“$\varphi$ is announced whereas agent $a$ (wrongly) learns $\psi$."

One may assume that $\varphi = \text{green}$ and $\psi = \text{blue}$, are two different colors. Then the action model $(N_1, s)$ says that a green ball is shown to $a$, whereas agent $a$ thinks that she sees a blue ball.

Consider the pointed model $(N_2, s)$ in Figure 2, where $S = \{s, t\}$, $s \rightarrow_b s$, $s \rightarrow_a t$, $t \rightarrow_a t$, $\text{pre}(s) = \varphi$ and $\text{pre}(t) = \psi$.

![Figure 2](image2)

This action model encodes the following information change: “a green ball is shown to agents $a$ and $b$, agent $b$ sees a green ball and is aware that agent $a$ has a color-blindness and sees a blue ball. Agent $a$ just sees a blue ball and has no idea about what $b$ sees”.

**Remark 2.1** The word “having no idea” used above is vague, and needs to be clarified. In the action model $N_1$, agent $b$ is not present in the state $t$. So agent $a$ has no idea about the information of agent $b$. There could be lots of possibilities about the color-blindness of $b$ at state $t$, but the action model says nothing about it. We will later introduce the notion of applicable formulas (see Definition 2.7) to formally model this case.

For another example, consider the pointed model $(N_3, s)$ in Figure 3, where $S = \{s, t\}$, $s \rightarrow_b s$, $t \rightarrow_b s s \rightarrow_a t$, $t \rightarrow_a t$, $\text{pre}(s) = \varphi$ and $\text{pre}(t) = \psi$. 

![Figure 3](image3)
The action model \((N_3, s)\) encodes the information change that “a green ball is shown to agents \(a\) and \(b\), agent \(b\) sees a green ball and is aware that agent \(a\) has a color-blindness and sees a blue ball. Agent \(a\) sees a blue ball and wrongly thinks that agent \(b\) has a color-blindness and sees a green ball. Moreover, both agents are aware about each other’s thoughts”.

According to the above discussion, a pointed action model \((N, s)\), where \(N = (S, (\neg_a )_{a \in A}, pre)\), encodes the information change that

\[ \text{the fact } pre(s) \text{ is announced whereas each agent (relevant to its accessibility relation in the action model) acquires information of what may have been announced and what other agents may have heard.} \]

### 2.2 Learning an Action Model

We should also clarify what it means that a group of agents learns an *action model*. Suppose \((N, s)\) is a pointed action model, and \(B\) is a group of agents. The case that

“group \(B\) learns \((N, s)\),”

is a new action model which encodes the following information change

“the fact \(pre(s)\) is announced and group \(B\) learns the fact \(pre(s)\) and the information that other agents (excluding \(B\)) acquires due to learning \((N, s)\)”.

For example, recall the pointed action model \((N_1, s)\) in Figure 1, where it encodes the following information change

“\(\varphi\) is announced whereas agent \(a\) (wrongly) learns \(\psi\)."

An agent \(b\) learns \((N_1, s)\) means

“\(\varphi\) is announced and agent \(b\) learns that \(\varphi\) is announced and the fact that agent \(a\) (wrongly) learns \(\psi\). Agent \(a\) still wrongly learns \(\psi\) and has no idea about what \(b\) learns.”

The case that agent \(b\) learns \((N_1, s)\), denoted by \(L_b((N_1, s))\), is encoded by the pointed action model \((N_2, s)\) in Figure 2.

Four different kinds of learning, in a multi-agent system, may be distinguished:

1. **Alternative Learning**: a group of agents, \(B\), together learns that among a set of actions \((M_1, s_1), (M_2, s_2), ..., (M_k, s_k)\), one of them is the actual action.
Example 2.2 Let \((M_1, s_1)\) be the pointed action model \((N_1, s_1)\) in Figure 1, and \((M_2, s_2)\) be another action model, in which \(M_2 = \langle S = \{s_2\}, \emptyset, \text{pre}(s_2) = \chi \rangle\). Then the pointed action model of \(L_b((M_1, s_1), (M_2, s_2))\) is

\[\text{Figure 4}\]

See Definition 3.2 for details.

2. Concurrent Learning: two disjoint groups of agents \(B_1\) and \(B_2\) learn concurrently but not together; group \(B_1\) learns \((N_1, s_1)\) and group \(B_2\) learns \((N_2, s_2)\) concurrently but not together.

Example 2.3 Let \((M_1, s_1)\) be the pointed action model \((N_1, s_1)\) in Figure 1, and \((M_2, s_2)\) be another pointed action model, in which \(M_2 = \langle S = \{s_2\}, (s_2 \rightarrow b, s_2), \text{pre}(s_2) = \varphi \rangle\). Then the pointed action model of \((M_1, s_1) \cap (M_2, s_2)\) would be

\[\text{Figure 5}\]

See Definition 3.2 for details.

3. Wrong Learning: whereas a fact \(\psi\) is announced, a group of agents, \(B\), wrongly learns something else.

Example 2.4 \(\varphi\) is announced and agent \(a\) learns \(\psi\). See the pointed action model \((N_1, s_1)\) in Figure 1.

4. Recursive Learning: a group of agents, \(B_1\), learns what another group of agents, \(B_2\), learns, and group \(B_2\) learns what group \(B_1\) learns.

Example 2.5 Consider the pointed action model \((N_3, s)\) in Figure 3.
The action model \((N_3, s)\) means
\(\varphi\) is announced and agent \(b\) learns
\[\{ \varphi \text{ and the case that agent } a \text{ (wrongly) learns} \}
\[\{ \psi \text{ and about what } b \text{ (wrongly) learns} \}\].

2.3 Epistemic Logic

In this section we briefly go through the syntax and semantics of epistemic logic. The syntax of epistemic logic is as usual, but the semantics is a little bit different from the standard one.

**Definition 2.6** Let \(P\) be a non-empty set of propositional variables, and \(A\) be a set of agents. The language \(L(A, P)\) is the smallest superset of \(P\) such that
\[
\text{if } \varphi, \psi \in L(A, P) \text{ then } \neg \varphi, (\varphi \land \psi), K_i \varphi \in L(A, P),
\]
for \(i \in A\).

For \(i \in A\), \(K_i \varphi\) has to be read as ‘agent \(i\) believes (knows) \(\varphi\)”. For a group of agents \(B \subseteq A\), \(K_B \varphi\) means that \(K_i \varphi\), for all \(i \in B\).

Epistemic logic models the notions of knowledge and belief in terms of the notion of possible worlds in Kripke semantics.

**Definition 2.7** A Kripke model \(M\) is a tuple \(M = \langle S, (\rightarrow_i)_{i \in A}, V \rangle\), where \(S\) is a non-empty set of worlds (states) \(s \in S\), \(V\) is a function from \(P\) to \(2^S\), and each \(\rightarrow_i\) is a binary accessibility relation between worlds. We define the group present at the state \((M, s)\) as follows:
\[
gr((M, s)) = \{ i \in A \mid (\exists t \in S) \ s \rightarrow_i t \}.
\]

For an epistemic state \((M, s)\), the set of applicable formulas at the state \((M, s)\), denoted by \(\Phi_{(M, s)} \subseteq L(A, P)\), is the smallest subset satisfying the following conditions
1. \(P \subseteq \Phi_{(M, s)}\),
2. if \(\varphi, \psi \in \Phi_{(M, s)}\) then \(\varphi \land \psi \in \Phi_{(M, s)}\) and \(\neg \varphi \in \Phi_{(M, s)}\),
3. $K_i \varphi \in \Phi_{(M,s)}$ if and only if $i \in \text{gr}((M,s))$ and for all $t$ such that $s \rightarrow_i t$, $\varphi \in \Phi_{(M,t)}$.

Intuitively, the applicable formulas of an epistemic state, are those formulas that can sensibly be assigned a truth value. For example, consider the Kripke model in Figure 6.

![Figure 6](image)

As agent $b$ is not present in the world $t$, formulas like $K_b \chi$ are not applicable in the world $t$ (it is not possible to talk about the truth of $K_b \chi$ in world $t$, where agent $b$ is not present in this world). Also the formula $K_a (K_b p \lor K_b \neg p)$ is not applicable at the world $s$. In the next definition, we restrict the definition of truth to applicable formulas.

**Definition 2.8** In order to determine whether an applicable formula $\varphi \in \Phi_{(M,s)}$ is true in the epistemic state $(M,s)$, denoted by $(M,s) \models \varphi$, we look at the structure of $\varphi$:

- $(M,s) \models p$ iff $s \in V(p)$
- $(M,s) \models (\varphi \land \psi)$ iff $(M,s) \models \varphi$ and $(M,s) \models \psi$
- $(M,s) \models \neg \varphi$ iff not $(M,s) \models \varphi$ $(M,s) \not\models \varphi$
- $(M,s) \models K_i \varphi$ iff for all $t$ such that $s \rightarrow_i t$, $(M,t) \models \varphi$

Note that the satisfaction relation is just defined for applicable formulas.

The standard epistemic logic $S5$ consists of axioms $A1 - A5$ and the derivation rules $R1$ and $R2$ given below

- $R1$: $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$
- $R2$: $\vdash \varphi \Rightarrow K_i \varphi$, for all $i \in A$

- $A1$: Axioms of propositional logic
- $A2$: $(K_i \varphi \land K_i(\varphi \to \psi)) \to K_i \psi$
- $A3$: $K_i \varphi \to \varphi$
- $A4$: $K_i \varphi \to K_i K_i \varphi$
- $A5$: $\neg K_i \varphi \to K_i \neg K_i \varphi$

If instead of $A3$, we assume the weaker axiom $D$ (given below), the logic of belief $KD45$ will be specified.

- $D$: $\neg(K_i \varphi \land K_i \neg \varphi)$

**Definition 2.9** Let $M = (S, (\rightarrow_a)_{a \in A}, V)$ be a Kripke model. For each $a \in A$, we say that the relation $\rightarrow_a$ is

1) reflexive if and only if for all $s \in S$, $s \rightarrow_a s$;

2) serial if and only if for all $s \in S$, there exists $t \in S$ such that $s \rightarrow_a t$;

3) transitive if and only if for all $s,t,u \in S$, if $s \rightarrow_a t$ and $t \rightarrow_a u$ then $s \rightarrow_a u$;

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4) Euclidean if and only if for all three states \( s, t, u \in S \) if \( s \rightarrow_a t \) and \( s \rightarrow_a u \) then \( t \rightarrow_a u \).

We say a relation \( \rightarrow_a \) is

- **S5** whenever it is reflexive, transitive and Euclidean,
- **KD45** whenever it is serial, transitive and Euclidean, and
- **K45** whenever it is transitive and Euclidean.

A Kripke model \( M \) is called an \( S5 \) model if and only if for each agent \( a, \rightarrow_a \) is \( S5 \). It is obvious that every \( S5 \) model is a model of the standard epistemic logic. A Kripke model \( M \) is called an \( KD45 \) model if and only if for each agent \( a, \rightarrow_a \) is \( KD45 \).

We also say a Kripke model \( M \) to be a \( K45 \) model if and only if for each agent \( a, \rightarrow_a \) is \( K45 \).

We recall definition of bisimulation of actions from [9]. Consider two action models \( N = \langle S, (\rightarrow_a)_{a \in A}, \text{pre} \rangle \) and \( N' = \langle S', (\rightarrow'_a)_{a \in A}, \text{pre}' \rangle \). The pointed action model \( (N, s) \) is bisimilar to \( (N', s') \), denoted by \( (N, s) \simeq (N', s') \), whenever there is a relation \( \mathcal{R} \subseteq S \times S' \) satisfying the following conditions, for each agent \( a \in A \):

**Initial.** \( \mathcal{R}(s, s') \).

**Forth.** If \( \mathcal{R}(t, t') \) and \( t \rightarrow_a v \), then there is a \( v' \in S' \) such that \( \mathcal{R}(v, v') \) and \( t' \rightarrow'_a v' \).

**Back.** If \( \mathcal{R}(t, t') \) and \( t' \rightarrow'_a v' \), then there is a \( v \in S \) such that \( \mathcal{R}(v, v') \) and \( t \rightarrow_a v \).

**Pre.** If \( \mathcal{R}(t, t') \), then \( \text{pre}(t) \) is equivalent to \( \text{pre}(t') \) in \( KD45 \) belief logic.

We define another notion of equivalence on action models and call it **agent-bisimulation**.

**Definition 2.11** Let \( a \in A \) be an arbitrary agent. Two pointed action models \( (N, s) \) and \( (N', s') \) are \( a \)-bisimilar whenever

- **Forth.** If \( s \rightarrow_a t \), then there is a \( t' \in S' \) such that \( s' \rightarrow'_a t' \) and \( (N, t) \simeq (N', t') \).

- **Back.** If \( s' \rightarrow'_a t' \), then there is a \( t \in S \) such that \( s \rightarrow_a t \) and \( (N, t) \simeq (N', t') \).
The execution of a pointed action model \((N, t) \in FAct\) on an epistemic state \((M, s) \in Mod\) is a new epistemic state \((M \ast N, (s, t))\), where \(M \ast N = \langle S, \rightarrow_i \rangle_{i \in A}, V \rangle\) and

\[
S = \{(s_1, t_1) \mid s_1 \in N, t_1 \in M, (M, s_1) \models \text{pre}(t_1)\},
\]

\((s_1, t_1) \rightarrow_i (s_2, t_2)\) iff \(s_1 \rightarrow_i s_2\) and \(t_1 \rightarrow_i t_2\),

\((s_1, t_1) \in V(p)\) iff \(s_1 \in V(p)\).

**Proposition 2.12** Suppose \((N, t) \in FAct\) and \((M, s) \in Mod\). Then \((M \ast N, (s, t)) \in Mod\), i.e., it is a \(K45\) model.

**Proof.** It is straightforward. \(\dashv\)

### 2.4 Why \(K45\) Models and Applicable Formulas?

\(K45\) Kripke models are more general than \(KD45\) models, as they necessarily do not have the serial property. The reason that we consider serial property for \(KD45\) model is that, we want the agent’s belief to be consistent. For \(K45\) Kripke models, we consider the definition of satisfaction relation just for *applicable* formulas. In this way, the agent’s beliefs are consistent at each state. Moreover,

regarding *applicable* formulas, the class of \(K45\) models is a (sound and complete) semantics for logic of belief \(KD45\).

Assume that a formula \(\varphi\) is derivable from the logic of belief, i.e., \(KD45 \vdash \varphi\). Then for every \(K45\) pointed model \((M, s)\), if \(\varphi\) is applicable at this state, then \((M, s) \models \varphi\). Also, note that every \(KD45\) model is also a \(K45\) model. So for any formula \(\varphi\), if for all \(K45\) pointed model \((M, s)\) which \(\varphi\) is applicable at the state, we have \((M, s) \models \varphi\) then \(KD45 \vdash \varphi\).

Since \(K45\) models are more general than \(KD45\) models, we can encode more epistemic functions in \(K45\) models, as we do not have to determine the cases where an agent does not have any idea about the belief of another agent. For example, consider the following information change:

a green ball is shown to agents \(a\) and \(b\), agent \(b\) sees a green ball and is aware that agent \(a\) has a color-blindness and sees a blue ball. Agent \(a\) just sees a blue ball and has no idea about what \(b\) sees.

In the above information change, agent \(a\) has no idea about what \(b\) sees. Agent \(a\) says I do not have any idea, there could be lots of possibilities and I do not know even how many possibilities exist, may be an infinite number of them. It would be possible that agent \(b\) sees a cube instead of a ball, or even an elephant, and etc. It would be possible that agent \(b\) sees the ball in a color which is unknown for me.

In \(KD45\) models, we are forced to encode all possibilities, but if it happens that some possibilities are unknown, then we don’t know what to do. However such information change can be encoded by the \(K45\) action model in figure 2.

Therefore, considering applicable formulas, the class of \(K45\) models is still a semantics for logic of belief (similar to the class of \(KD45\) models) and moreover, they are enough general than \(KD45\) models for describing information changes formally.
3 Basic Learning Programs

As mentioned in the introduction, our goal is to introduce some initial functions and basic operators as the building blocks of all finite epistemic functions (FAcnt). In this section, we introduce the first class of epistemic learning programs, called basic learning programs. The reason these are called basic is that they do not include any recursion in their structure.

**Definition 3.1** Let $\Phi$ be a set of epistemic formulas over a set of atomic formulas $\mathcal{P}$ and a set of agents $\mathcal{A}$. The set of basic learning programs $\text{BLP}(\Phi)$ is defined as follows:

i. **Test.** for all $\varphi \in \Phi$, $\varphi$ is a basic learning program, and we define $\text{group}(\varphi) = \emptyset$, and $\text{pre}(\varphi) = \varphi$,

ii. **Alternative Learning.** for all $n \in \mathbb{N}$, and $B \subseteq A$, if $\alpha_1, \alpha_2, ..., \alpha_n$ are basic learning programs, then $L_B(\alpha_1, \alpha_2, ..., \alpha_n)$ is a basic learning program, and we define $\text{group}(L_B(\alpha_1, \alpha_2, ..., \alpha_n)) = B \cup \text{group}(\alpha_1)$ and $\text{pre}(L_B(\alpha_1, \alpha_2, ..., \alpha_n)) = \text{pre}(\alpha_1)$,

iii. **Concurrent Learning.** if $\alpha_1, \alpha_2$ are basic learning programs such that $\text{pre}(\alpha_1) = \text{pre}(\alpha_2)$ and $\text{group}(\alpha_1) \cap \text{group}(\alpha_2) = \emptyset$, then $\alpha_1 \cap \alpha_2$ is a basic learning program. We define $\text{group}(\alpha_1 \cap \alpha_2) = \text{group}(\alpha_1) \cup \text{group}(\alpha_2)$, and $\text{pre}(\alpha_1 \cap \alpha_2) = \text{pre}(\alpha_1)$,

iv. **Wrong Learning.** if $\alpha_1$ is a basic learning program and $\psi$ is an epistemic formula, then $\psi|_B^{\alpha_1}$ is a basic learning program whenever $B \subseteq \text{group}(\alpha_1)$. We define $\text{group}(\psi|_B^{\alpha_1}) = B$ and $\text{pre}(\psi|_B^{\alpha_1}) = \psi$.

To each basic learning program, we associate a pointed action model as follows.

**Definition 3.2** Semantics of BLP.

1. for all $\varphi \in \Phi$, the pointed action model of the program $\varphi$, is $(N_{\varphi}, s_{\varphi})$, where

   $$N_{\varphi} = \langle \{s_{\varphi}\}, (\sim_a)_{a \in \mathcal{A}}, \text{pre} \rangle,$$

   in which for all $a \in \mathcal{A}$, $\sim_a = \emptyset$ and $\text{pre}(s_{\varphi}) = \varphi$.

2. Suppose $\alpha_1, \alpha_2, ..., \alpha_k$ are basic learning programs and their associated action models are $(N_1, s_1), (N_2, s_2), ..., (N_k, s_k)$, where $N_l = \langle S_l, (\sim^l_a)_{a \in \mathcal{A}}, \text{pre}_l \rangle$, for $1 \leq l \leq k$, and $\cap_{1 \leq i \leq k} S_k = \emptyset$. Then the associated action model to the basic learning program $L_B(\alpha_1, \alpha_2, ..., \alpha_n)$ is $(N, s)$, where $N = \langle S, (\sim_a)_{a \in \mathcal{A}}, \text{pre} \rangle$ and

   - $S = \{(s_1, 1), (s_2, 1), ..., (s_k, 1)\} \cup S_1 \cup S_2 \cup ... \cup S_k$,
   - for all $b \in B$, for all $1 \leq i, j \leq k$, if $s_i$ is $b$-bisimilar to $s_j$ then $(s_i, 1) \rightarrow_b (s_j, 1)$,
   - for all $1 \leq j \leq k$, for all $a \in \text{gr}(N_j, s_j) - B$, for all $t \in S_j$, if $s_j \rightarrow^*_a t$ then $(s_j, 1) \rightarrow_a t$,
   - for all $a \in \mathcal{A}$, for all $1 \leq i \leq k$, for all $v, t \in S_i$, if $v \rightarrow^*_a t$ then $v \rightarrow_a t$,
   - $s = (s_1, 1)$,
   - $\text{pre}((s_l, 1)) = \text{pre}(s_l)$ for all $1 \leq l \leq k$. 

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The associated action model of $L_B(\alpha_1, \alpha_2, \ldots, \alpha_n)$ means: $\text{pre}(\alpha_1)$ is announced whereas agents in $B$ learn that either $\alpha_1$ or $\alpha_2$ or ... or $\alpha_k$ has been executed, and what other agents learn in each alternative case.

3. Suppose $\alpha_1, \alpha_2$ are basic learning programs and their associated action models are $(N_1, s_1)$ and $(N_2, s_2)$ respectively, where $N_i = (S_i, (\neg a_{a \in A_i}, \text{pre}_i))$ for $l = 1$ or $l = 2$, $A_1 \cap A_2 = \emptyset$ and $S_1 \cap S_2 = \emptyset$. Then the pointed action model of the basic learning program $\alpha_1 \cap \alpha_2$ is $(N, s)$ with $N = (S, (\neg a_{a \in A}, \text{pre}))$, where

- $S = \{s\} \cup S_1 \cup S_2$, for some $s \notin S_1 \cup S_2$,
- for all $1 \leq i \leq 2$, for all $a \in A_i$, if $v \neg\neg a^i_t$ then $v \rightarrow a^i_t$,
- for all $1 \leq j \leq 2$, for all $a \in A_j$, if $s_j \neg\neg a^j_t$ then $s \rightarrow a^j_t$,
- $\text{pre}(s) = \text{pre}(s_1)$.

The associated action model of $\alpha_1 \cap \alpha_2$ means: $\text{pre}(\alpha_1)$ is announced whereas agents in group($\alpha_1$) learn according to execution of $\alpha_1$, agents in group($\alpha_2$) learn according to execution of $\alpha_2$.

4. Suppose $\alpha$ is a basic learning program and its associated action model is $(N_1, s_1)$, where $N_1 = (S, (\neg a_{a \in A}, \text{pre}_1))$. Then the associated action model of $\psi|_{B\alpha}$ is $(N, s)$ with $N = (S, (\neg a_{a \in A}, \text{pre}))$, where

- $S = \{s\} \cup S_1$, for some $s \notin S_1$,
- for all $a \in A$, if $v \neg\neg a^1_t$ then $v \rightarrow a^1_t$,
- for all $b \in B$, if $s_1 \neg\neg b^1_t$ then $s \neg\neg b^1_t$,
- $\text{pre}(s) = \psi$.

The associated action model of $\psi|_{B\alpha}$ means: $\psi$ is announced whereas agents in group $B$ wrongly learn according to execution of $\alpha_1$.

The action model of a program $\alpha$ is denoted by $(N_\alpha, s_\alpha)$. An epistemic action $(N, t) \in \text{FA}(\Phi)$ is called basic learning action whenever there is a basic learning programs $\alpha \in \text{BLP}(\Phi)$ such that $(N_\alpha, s_\alpha)$ is bisimilar to $(N, t)$.

**Example 3.3** The action model associated with the basic learning program

$$L_b(\varphi|_aL_a?\psi).$$

is bisimilar to the action model illustrated in Figure 2.

The action model illustrated in Figure 4 is a basic learning action. It is bisimilar to the action model associated with the basic learning program

$$L_b(\varphi|_aL_a?\psi, ?\chi).$$

Also, the action model associated with the basic learning program

$$\varphi|_aL_a?\psi \land \varphi|_bL_b?\varphi.$$

is bisimilar with the action model illustrated in Figure 5.
Definition 3.4 Let \( N = \langle S, (\neg a)_{a \in A}, \text{pre} \rangle \) be an \( K45 \) action model.

- An \( S5 \) submodel of \( N \) is an \( S5 \) action model \( M = \langle S', (\neg a')_{a \in B}, \text{pre}' \rangle \), where \( S' \subseteq S \), \( \text{pre}' = \text{pre}|_{S'} \), \( B \subseteq A \) and for all \( a \in B \), \( \neg a' = \neg a \mid_{S'} \). An \( S5 \) submodel is called connected whenever all two different states of the model are reachable from each other. It is called closed whenever for all \( s, t \in S \), if \( s \in S' \) and for some \( a \in B \), \( s \xrightarrow{a} t \), then \( t \in S' \).

- Let \( M' = \langle S', (\neg a')_{a \in B}, \text{pre}' \rangle \) and \( M'' = \langle S'', (\neg a'')_{a \in C}, \text{pre}'' \rangle \) be two closed connected \( S5 \) action submodels of \( N \). We write \( M' \leq M'' \) whenever \( S' \subseteq S'' \) and \( B \subseteq C \). One may easily verify that \( \leq \) is a partial order relation.

- Assume \( M^i = \langle S^i, (\neg a^i)_{a \in B^i}, \text{pre}^i \rangle \), \( 1 \leq i \leq k \) are all different maximal closed connected \( S5 \) submodel of \( N \) with respect to the partial order relation \( \leq \). We construct the action model \( T(N) = \langle S', (\neg a)_{a \in A}, \text{pre}' \rangle \) as follows:
  
  - \( S' = \bigcup_{1 \leq i \leq k} (S^i \times \{ i \}) \)
  - \( (s, i) \xrightarrow{a^i} (t, j) \) if and only if either \( i = j \) and \( s \xrightarrow{a} t \), or \( i \neq j \), \( a \notin B^j \), \( a \in B^i \), and \( s \xrightarrow{a} t \) (in the action model \( N \))
  - \( \text{pre}'((s, i)) = \text{pre}^i(s) \)

- The projection of a state \( (s, i) \) in the action model \( T(N) \), denoted by \( \Pi((s, i)) \), is defined to be the state \( s \) in the action model \( N \).

- For the action model \( N \), we define the directed graph of \( N \), denoted by \( G(N) \) as follows:
  
  - Each node \( G(N) \) is a maximal closed connected \( S5 \) submodel \( M^i \) of \( N \).
  - Between two different nodes \( M^i, M^j \), there exists a directed edge \( M^i \xrightarrow{k} M^j \) if and only if there exists an accessibility relation in the action model \( T(N) \), \( (s, i) \xrightarrow{k} (t, j) \), for some \( s \in S_i, a \in A \), and \( t \in S_j \).

It is obvious that an \( S5 \) submodel of a \( K45 \) action model may be connected but not closed and vice versa.

Proposition 3.5 Let \( N = \langle S, (\neg a)_{a \in A}, \text{pre} \rangle \) be a \( K45 \) action model. Then \( (N, s_0) \) is bisimilar to \( (T(N), w_0) \) for all \( s_0 \) and \( w_0 \) for which the state \( s_0 \) is the projection of \( w_0 \).

Proof. See the Appendix.\( \dashv \)

Theorem 3.6 If a pointed \( K45 \) action model \( (N, s) \) is a basic learning action, then its graph \( G(N) \) is a tree.

Proof. See the Appendix.\( \dashv \)

Example 3.7 Consider the action model \( (N_3, s) \) shown in Figure 3. There is no basic learning program \( \alpha \) such that its action model is \( (N_3, s) \). That is because \( G(N_3) \) is not a tree.
We showed that the graph of the basic learning actions are trees. The converse is also true, that is, for each $K45$ pointed action model $(N,s)$, if its graph is a tree then it is associated to a basic learning program up to bisimilarity.

**Proposition 3.8** All $S5$ pointed action models are basic learning actions.

**Proof.** See the Appendix. \( \dashv \)

**Theorem 3.9** If the graph of a finite $K45$ pointed model $(N,s)$ is a finite tree, then $(N,s)$ is a basic learning action.

**Proof.** See the Appendix. \( \dashv \)

### 3.1 Comparing two Learning Operators

In this part, we aim to compare our proposed learning operator with the learning operator introduced in [8]. We begin with the same example “Lecture or Amsterdam” discussed in [8].

Anne and Bert are in a bar, sitting at a table. A messenger comes in and delivers a letter addressed to Anne. The letter contains either an invitation for a night out in Amsterdam or an obligation to give a lecture instead. Anne and Bert commonly know that these are the only alternatives.

Consider the following information change scenario:

- (spy-seeing). Bert says good bye to Anne and leaves the bar. During his leaving, he secretly spies from the window of the bar that whether Anne reads the letter, Anne does not get aware that Bert spies on her, and wrongly thinks that she is alone (Bert is not present) while she reads the letter.

Suppose that $p$ stands for “Anna is invited for a night out in Amsterdam”, and also assume that in fact $p$ is true.

It is not possible to model the above information change scenario in concurrent dynamic epistemic logic [8]. But we can model above scenario by the following basic learning programs.

$$L_b(L_a?p, L_a?\neg p).$$

The pointed action model associated to the above learning program is $(N,s)$, where $S = \{s,t,v,u\}$, $s \rightarrow_a v$, $v \rightarrow_a v$, $t \rightarrow_a u$, $u \rightarrow_a u$, and $s \rightarrow_b s$, $s \rightarrow_b t$, $t \rightarrow_b s$, $t \rightarrow_b t$, and $pre(s) = pre(v) = p$, $pre(t) = pre(u) = \neg p$. 
A candidate to describe the action “spy-seeing” in formalization presented in [8] could be \( L_b(!L_a?p \cup L_a?¬p) \). But the term \( L_b(!L_a?p \cup L_a?¬p) \) is not a well-formed action in concurrent dynamic epistemic logic since it does not satisfy definition 6 of [8]. In this definition if \( L_B\alpha \) is a well-formed action then \( gr(\alpha) \subseteq B \). Whereas, \( gr(L_a?p \cup L_a?¬p) = \{a\} \), and \( \{a\} \nsubseteq \{b\} \).

For another example, we discuss the following information change scenario:

- **(spy-reading).** Bert says good bye to Anne and leaves the bar. using a hidden camera, Bert spies on Anne when she reads the letter, and Bert gets aware of the contents of the letter using the camera. Anne wrongly thinks that she is alone, and there is no spy on her.

Again, it is not possible to model the above information change scenario in concurrent dynamic epistemic logic [8]. But we can model above scenario by the following basic learning programs \( L_b(L_a?p) \).

The pointed action model associated to \( L_b(L_a?p) \)
2- \textbf{(read)}. Both Bert and Anne are present in the bar, sitting on a table, and Bert is seeing that Anne reads the letter: $L_{ab}(L_a ? p, L_a ? \neg p)$.

![Diagram of pointed action model associated to $L_{ab}(L_a ? p, L_a ? \neg p)$]

The pointed action model associated to $L_{ab}(L_a ? p, L_a ? \neg p)$

3- \textbf{(mayread)} Bert orders a drink at the bar so that Anne may have read the letter (and actually Anne reads the letter)

$$L_{ab}(\alpha, \beta, \gamma),$$

where

- $\alpha := L_a ? p \land p \mid b L_b ? \top$ (Anne reads the letter, and in fact $p$ is true),
- $\beta := L_a ? \neg p \land \neg p \mid b L_b ? \top$ (Anne reads the letter, and in fact $\neg p$ is true),
- $\gamma := L_a ? \top \land L_b ? \top$ (Anne does not read the letter)

- The pointed action model associated to $\alpha$, denoted by $(N_1, s_1)$, is $S = \{s_1, t_1, v_1\}$, $s_1 \rightarrow_a t_1$, $s_1 \rightarrow_b v_1$, $t_1 \rightarrow_a t_1$, $v_1 \rightarrow_b v_1$, and $\text{pre}(s_1) = \text{pre}(t_1) = p$, $\text{pre}(v_1) = \top$.
- The pointed action model associated to $\beta$, denoted by $(N_2, s_2)$, is $S = \{s_2, t_2, v_2\}$, $s_2 \rightarrow_a t_2$, $s_2 \rightarrow_b v_2$, $t_2 \rightarrow_a t_2$, $v_2 \rightarrow_b v_2$, and $\text{pre}(s_2) = \text{pre}(t_2) = \neg p$, $\text{pre}(v_2) = \top$.
- The pointed action model associated to $\gamma$, denoted by $(N_3, s_3)$, is $S = \{s_3, t_3, v_3\}$, $s_3 \rightarrow_a t_3$, $s_3 \rightarrow_b v_3$, $t_3 \rightarrow_a t_3$, $v_3 \rightarrow_b v_3$, and $\text{pre}(s_3) = \text{pre}(t_3) = \text{pre}(v_3) = \top$.

One may easily check that for every $i, j \in \{1, 2, 3\}$, $(N_i, s_i)$ is $b$-bisimilar to $(N_j, s_j)$. Also, for every $i, j \in \{1, 2, 3\}$, $(N_i, s_i)$ is $a$-bisimilar to $(N_j, s_j)$ if and only if $i = j$.

Therefore, the action model associated to $L_{ab}(\alpha, \beta, \gamma)$ is $(N, 1)$, where $S = \{1, 2, 3\}$, $1 \rightarrow_a 1$, $2 \rightarrow_a 2$, $3 \rightarrow_a 3$, and $1 \rightarrow_b 2$, $2 \rightarrow_b 3$, $3 \rightarrow_b 1$, and $\text{pre}(1) = p$, $\text{pre}(2) = \neg p$, and $\text{pre}(3) = \top$.

4- \textbf{(bothmayread)}. Bert orders a drink at the bar while Anne goes to the bathroom. Both may have read the letter (and actually both of them have read).

$$L_{ab}(L_a ? p \land L_b ? p, L_a ? \neg p \land L_b ? \neg p, L_a ? \top \land L_b ? \top, L_a ? p \land p \mid b L_b ? \top, L_a \neg p \land \neg p \mid b L_b ? \top, L_b ? p \land p \mid a L_a ? \top, L_b \neg p \land \neg p \mid a L_a ? \top).$$

Four above information change scenarios are also modelled in concurrent dynamic epistemic logic in example 7, of [S]. Also, in Figure 1 of [S], page 4, Epistemic states resulted from the execution of actions (for these scenarios) described in concurrent dynamic epistemic logic (example 7, of [S]) is shown. It easy to verify that

Epistemic states resulted from the execution of pointed action models associated to basic learning programs for these scenarios (introduced above) are exactly the same Epistemic states resulted from the execution actions described in concurrent dynamic epistemic logic [S] (and shown in Figure 1 of the same reference, in page 4).
4 Learning by Recursion

By learning by recursion, we mean the cases where agents learn about each other’s learning, i.e., an agent \( a \) learns something about learning of another agent \( b \) and agent \( b \) also learns about learning of agent \( a \). In this way, a recursive learning occurs. In this section, we introduce recursive learning actions to model this type of learning.

In the following definition, \( \text{undf} \), indicates that the term is undefined.

**Definition 4.1** Let \( \Phi \) be a set of epistemic formulas over a set of atomic formulas \( P \) and a set of agents \( A \), and \( \text{Var} = \{ X, Y, Z, ... \} \) be a set of variables. The set of open terms \( \text{OpenT}(\Phi) \) is defined as follows:

1. for all \( X \in \text{Var} \), \( X \) is an open term, we let \( \text{pre}(X) = \text{undf} \) and \( \text{group}(X) = \text{undf} \),
2. for all \( \varphi \in \Phi \), \( ?\varphi \) is an open term, \( \text{group}(?\varphi) = \emptyset \), and \( \text{pre}(?\varphi) = \varphi \),
3. for all \( n \in \mathbb{N} \), and \( B \subseteq A \), if \( \alpha_1, \alpha_2, ..., \alpha_n \) are open terms, then \( L_B(\alpha_1, \alpha_2, ..., \alpha_n) \) is an open term, and we let
   \[
   \text{group}(L_B(\alpha_1, \alpha_2, ..., \alpha_n)) = B \cup \text{group}(\alpha_1), \text{and} \quad \text{pre}(L_B(\alpha_1, \alpha_2, ..., \alpha_n)) = \text{pre}(\alpha_1)
   \]
   Note that the left sides are defined if \( \text{group}(\alpha_1) \) and \( \text{pre}(\alpha_1) \) are defined.
4. \( \alpha_1 \cap \alpha_2 \) is an open term whenever \( \alpha_1 \) and \( \alpha_2 \) are open terms. If both \( \text{group}(\alpha_1) \), \( \text{group}(\alpha_2) \) are defined and \( \text{group}(\alpha_1) \cap \text{group}(\alpha_2) = \emptyset \), and if both \( \text{pre}(\alpha_1), \text{pre}(\alpha_2) \) are defined and \( \text{pre}(\alpha_1) = \text{pre}(\alpha_2) \), then we let \( \text{group}(\alpha_1 \cap \alpha_2) = \text{group}(\alpha_1) \cup \text{group}(\alpha_2) \) and \( \text{pre}(\alpha_1 \cap \alpha_2) = \text{pre}(\alpha_1) \),
5. \( \psi|_B \alpha_1 \) is an open term whenever \( \alpha_1 \) is an open term. If \( \text{group}(\alpha_1) \) is defined and \( B \subseteq \text{group}(\alpha_1) \). We then define \( \text{group}(\psi|_B \alpha_1) = B \) and \( \text{pre}(\psi|_B \alpha_1) = \psi \).

**Definition 4.2** Assume \( \alpha(X_1, X_2, ..., X_k) \) is an open term with \( k \) variables, the tuple

\[
(\beta_1, \beta_2, ..., \beta_k) \in \text{OpenT}(\Phi)
\]

is a suitable substitution for \( \alpha(X_1, X_2, ..., X_k) \), whenever \( \alpha(\beta_1, \beta_2, ..., \beta_k) \in \text{OpenT}(\Phi) \).

Now we define open action models, in order to be associated to open terms. For each variable \( X \), let \( (N_X, s_X) \) be a variable pointed action model, where \( N_X = (S_X, R_a^X, \text{pre}^X) \) and the set \( S_X \), the relations \( R_a^X \), and the function \( \text{pre}^X \) are variables. The open action model of an open term \( \alpha(X_1, X_2, ..., X_k) \) is simply constructed using variable pointed action models \( (N_{X_1}, s_{X_1}), (N_{X_2}, s_{X_2}), ..., (N_{X_k}, s_{X_k}) \) and Definition 3.2. Substituting an action model \( (N, s) \) in an open term \( \alpha(X) \) is obtained by considering \( s \) instead of \( s_X \).

**Example 4.3** Suppose \( A = \{ a, b \} \), and consider the open term \( L_b L_a(X) \). The open action model \( (N_{L_b L_a(X)}, s_{L_b L_a(X)}) \) is constructed as follows. We consider three states \( s_0, s_1 \) and \( s_X \), where \( \text{pre}(s_0) = \text{pre}(s_1) = \text{pre}(s_X) \). The actual state is \( s_0 \), and the accessibility relations are \( s_0 \rightarrow_b s_0, s_0 \rightarrow_a s_1, s_1 \rightarrow_a s_1, \) and \( s_1 \rightarrow_b t \) for all \( t \in b[s_X] \), where \( b[s_X] \) is the set of all states in the hypothetical model \( N_X \) in which \( s_X \) has \( b \)-accessibility to them. The open action model of the term \( L_b L_a(X) \) is illustrated in Figure 8.
1. Let \((M, t_0)\) be the action model of the term \(L_{ab}(\varphi)\), and \(S = \{t_0\}\), \(pre(t_0) = \varphi\), \(t_0 \rightarrow_a t_0\) and \(t_0 \rightarrow_b t_0\).

Substitution of \((M, t_0)\) for \((N_X, s_X)\) means to substitute \(t_0\) for \(s_X\). Therefore, \(pre(s_X) = pre(t_0) = \varphi\) and \(b[s_X] = b[t_0] = \{t_0\}\). Substituting \((M, t_0)\) in \((N_{LbL_a(X)}, s_{LbL_a(X)})\) yields to the following action model

which is the action model of \(L_aL_bL_{ab}(\varphi)\).

**Example 4.4** Suppose \(A = \{a, b\}\), and consider the open term \(L_b(\varphi|_a L_a(\psi|_b L_b(X)))\). The open action model \((N_{L_b(\varphi|_a L_a(\psi|_b L_b(X)))}, s_{L_b(\varphi|_a L_a(\psi|_b L_b(X)))})\) is shown in Figure 11.
\[
(N_{L_b}\sigma(N_X,s_X)), s_{L_b}\sigma(N_X,s_X)\) is obtained by adding the new state \((s_X,1)\) as the actual state and adding new accessibility relations \((s_X,1) \rightarrow_b (s_X,1)\), and \((s_X,1) \rightarrow_a t\), for all \(t \in a[s_X]\). We have \(pre((s_X,1)) = pre(s_X)\), and \(s_{L_b}(\sigma(N_X,s_X)) = (s_X,1)\).

\((N\psi|b L_a(X), s\psi|b L_a(X))\) is obtained by adding the new state \(s\psi|b L_a(X)\) to \(N_{L_b}(\sigma(N_X,s_X))\), and adding new accessibility relations \(s\psi|b L_a(X) \rightarrow_b t\), for all \(t \in b[s_{L_b}(\sigma(N_X,s_X))]\). Moreover, \(pre(s\psi|b L_a(X)) = \psi\).

Continuing the above scenario and using Definition 3.2, the open action model in Figure 11 is constructed.

To obtain the open action model of \(L_b(\varphi|a L_a(\psi|b L_a(\varphi|a L_a(\psi|b L_a(X))))))\), one may simply consider two copies of the open action model \((N_{L_b}(\sigma|a L_a(\psi|b L_a(X)))\)) and replace the actual state \(s_{L_b}(\sigma|a L_a(\psi|b L_a(X)))\) of one of them for \(s_X\) in another one, and obtain the model shown in Figure 12. By substituting \(s_{L_b}(\sigma|a L_a(\psi|b L_a(X)))\) for \(s_X\), the set \(a[s_X] = \{s\psi|b L_a(X)\}\).
Fixed point of $L_b(\varphi|_a L_a(\psi|_b L_b(X)))$. To construct an action model as a fixed point of the term $L_b(\varphi|_a L_a(\psi|_b L_b(X)))$, we consider the open action model $(N_{L_b}(\varphi|_a L_a(\psi|_b L_b(X))))$ and replace its actual state $s_{L_b}(\varphi|_a L_a(\psi|_b L_b(X)))$ for $s_X$ (see Figure 13, the symbol $s_\mu X$ will be defined in the next section).

It is easy to verify that the pointed action model in Figure 13 is bisimilar to the pointed action model $(N_3, s)$ explained in Example 2.5 as a recursive learning action.
4.1 Recursive Learning Programs

One may check that the graph of the action model shown in Figure 3 is not a tree and thus by Theorem 3.6, it is not a basic learning action. So we cannot describe it in terms of alternative learning, \( L_B(\cdot, \cdot, \cdot, \cdot) \), concurrent learning \( \cap \), and wrong learning, \( |_B \), operators. We add a new operator \( \mu \) to the language for recursive learning and show that the action model shown in Figure 3 is a recursive learning action. To do this, we need to slightly modify the Definition of open terms 4.1 as follows.

**Definition 4.5** Let \( \Phi \) be a set of epistemic formulas over a set of atomic formulas \( P \) and a set of agents \( A \), and \( \text{Var} = \{X, Y, Z, \ldots\} \) be a set of variables. The set of open terms, \( \text{OT}(\Phi) \), is defined as follows:

1. For all \( X \in \text{Var} \), \( X \) is an open term, we let \( \text{pre}(X) = \text{undf} \) and \( \text{group}(X) = \text{undf} \).
2. For all \( \varphi \in L(\Phi) \), \( \varphi \) is an open term, \( \text{group}(\varphi) = \emptyset \), and \( \text{pre}(\varphi) = \varphi \).
3. For all \( n \in \mathbb{N} \), and \( B \subseteq A \), if \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are open terms then \( L_B(\alpha_1, \alpha_2, \ldots, \alpha_n) \) is an open term, and we let

\[
\text{group}(L_B(\alpha_1, \alpha_2, \ldots, \alpha_n)) = B \cup \text{group}(\alpha_1) \quad \text{and,} \\
\text{pre}(L_B(\alpha_1, \alpha_2, \ldots, \alpha_n)) = \text{pre}(\alpha_1)
\]

Note that the left sides are defined if \( \text{group}(\alpha_1) \) and \( \text{pre}(\alpha_1) \) are defined.

4. \( \alpha_1 \cap \alpha_2 \) is an open term whenever \( \alpha_1 \) and \( \alpha_2 \) are open terms. If both \( \text{group}(\alpha_1) \), \( \text{group}(\alpha_2) \) are defined and \( \text{group}(\alpha_1) \cap \text{group}(\alpha_2) = \emptyset \), and both \( \text{pre}(\alpha_1) \), \( \text{pre}(\alpha_2) \) are defined and \( \text{pre}(\alpha_1) = \text{pre}(\alpha_2) \). Then we let \( \text{group}(\alpha_1 \cap \alpha_2) = \text{group}(\alpha_1) \cup \text{group}(\alpha_2) \), and \( \text{pre}(\alpha_1 \cap \alpha_2) = \text{pre}(\alpha_1) \).

5. \( \psi|_B \alpha_1 \) is an open term whenever \( \alpha_1 \) is an open term. If \( \text{group}(\alpha_1) \) is defined and \( B \subseteq \text{group}(\alpha_1) \), We define \( \text{group}(\psi|_B \alpha_1) = B \) and \( \text{pre}(\psi|_B \alpha_1) = \psi \).

6. If \( \alpha(X_1, X_2, \ldots, X_k) \) is an open term, and both

\[
\text{group}(\alpha(X_1, X_2, \ldots, X_k)) \quad \text{and,} \\
\text{pre}(\alpha(X_1, X_2, \ldots, X_k))
\]

are defined, then \( \mu X_1.\alpha(X_1, X_2, \ldots, X_k) \) is an open term with \( k - 1 \) free variables; the variable \( X_1 \) is bound under \( \mu X_1 \). We define

\[
\text{group}(\mu X.\alpha(X_1, X_2, \ldots, X_k)) = \text{group}(\alpha(X_1, X_2, \ldots, X_k)) \quad \text{and,} \\
\text{pre}(\mu X.\alpha(X_1, X_2, \ldots, X_k)) = \text{pre}(\alpha(X_1, X_2, \ldots, X_k)).
\]

A term \( \alpha \in \text{OT}(\Phi) \) is closed if it has no unbounded variable.

**Definition 4.6** Let \( \Phi \) be a set of epistemic formulas. The set of recursive learning programs, \( \text{RLP}(\Phi) \), is the set of all closed terms in \( \text{OT}(\Phi) \).
**Definition 4.7 (Semantics of \(\mu X.\alpha(X)\)).** The associated pointed action model to \(\mu X.\alpha(X)\) is obtained by replacing the actual state \(s_\alpha(X)\) of the action model \(N_\alpha(X)\) for the state \(s_X\).

The associated action model of \(\mu X.\alpha(X)\) is actually a **fixed point** of the open action model of \(\alpha(X)\), see Example 4.4.

**Example 4.8** The semantics of the recursive learning program

\[ \alpha := \mu X_1. L_c(\chi|a\mu X_2. L_a(\varphi|b L_b((\psi|a X_2) \cap (\psi|c X_1)))) \]

is constructed in the following way. The model of the open term

\[ L_a(\varphi|b L_b((\psi|a X_2) \cap (\psi|c X_1))) \]

is shown in Figure 14.

![Figure 14](image)

The model of the open term

\[ \mu X_2. L_a(\varphi|b L_b((\psi|a X_2) \cap (\psi|c X_1))) \]

is shown in Figure 15.

![Figure 15](image)
The action model of
\[
L_c(\chi|_a \mu X_2. L_\alpha(\varphi|_b L_b((\psi|_a X_2) \cap (\psi|_c X_1))))
\]
is illustrated in Figure 16.

![Figure 16](image)

Finally the action model of the recursive learning program \(\alpha\) is shown in Figure 17.

![Figure 17](image)

As two other examples, one may check that the action model associated to the recursive learning program
\[
\mu X. L_\alpha(\varphi|_a \mu Y. L_b(\psi|_a X \cap \psi|_c L_c(\theta|_b Y))
\]
is the pointed action model illustrated in Figure 18,
Figure 18

and the action model associated to the recursive learning program

\[ \mu X.L_a(\varphi|L_b(\varphi|X), \psi|L_b(\varphi|X)) \]

is bisimilar to the pointed action model \((N,s)\) illustrated in Figure 19.

Figure 19

4.2 Two Main Theorems

In this subsection, we present two main theorems of the paper. In the first one, Theorem 4.12, we show that every finite \(K45\) action model is associated to some recursive learning program and conversely, for every learning program, there is a finite \(K45\) action model associated with it. Then we introduce a hierarchy over learning programs with respect to the number of recursive learning operators. In our second main Theorem 4.15, it is shown that the hierarchy of the learning programs is strict, i.e., it is not possible to describe all \(K45\) action models by a determined finite number of recursion in learning. That means that the hierarchy does not collapse.

4.2.1 Representing Epistemic Action Models

Definition 4.9 Let \(N = \langle S, (\neg_a)_{a \in A}, pre \rangle\) be a \(K45\) action model. For each agent \(a \in A\), an \(a\)-component of \(N\) is \(M_a = \langle S'_a, (\neg_a')_{a \in \{a\}}, pre' \rangle\), where \(M_a\) is an \(S5\) closed connected submodel of \(N\).

Assume for all \(a \in A\), \(M_a^i = \langle S_a^i, (\neg_a^i)_{a \in \{a\}}, pre_a^i \rangle, 1 \leq i \leq k_a\) are all different \(a\)-component of \(N\). We construct the action model \(T'(N) = \langle S', (\neg_a')_{a \in A}, pre' \rangle\) as follows:

- \(S' = \bigcup_{a \in A} \bigcup_{1 \leq i \leq k_a} (S_a^i \times \{(i, a)\})\),
- \((s, (i, a)) \rightarrow_b^i (t, (j, b))\) if and only if either \(i = j\), \(a = b\) and \(s \rightarrow_a^i t\), or \(i \neq j\), \(b \neq a\), and \(s \rightarrow_b t\) (in the action model \(N\)).
pre′((s, (i, a))) = pre′i(s).

The projection of an state (s, (i, a)) in the action model \( T'(N) \), denoted by \( \Pi((s, (i, a))) \), is defined to be the state s in the action model N.

**Example 4.10** Consider the action models \( N_1 \) and \( N_2 \) in Figure 20. Their action models \( T'(N_1) \) and \( T'(N_2) \) are illustrated in Figure 20 as well.

![Diagram](image)

**Figure 20**

**Proposition 4.11** Let \( N = \langle S, (\rightarrow a)_{a \in A}, \text{pre} \rangle \) be an K45 action model. Then \( (N, s_0) \) is bisimilar to \( (T'(N), w_0) \) for all \( s_0 \) and \( w_0 \), in which the state \( s_0 \) is the projection of \( w_0 \).

**Proof.** The proof is similar to the proof of Proposition 3.5. ⊣

**Theorem 4.12** All finite epistemic actions are recursive learning programs, i.e.,

\[ \text{FAct}(\Phi) = \text{RLP}(\Phi). \]

**Proof.** See the Appendix. ⊣

**Example 4.13** It is shown in Figure 21, how to construct a program for the pointed action model \( (N_1, s) \) through the instruction argued in proof of the Theorem 4.12.
4.2.2 A Hierarchy of Learning

Definition 4.14 For each \( k \in \mathbb{N} \), we define the class \( kRLP \), of all finite \( K45 \) pointed action models which can be described by a recursive learning program with at most \( k \) times of dependent use of the recursive operator \( \mu \).

The term ‘dependent’ in the above definition is crucial. In a program, two operators \( \mu_X \) and \( \mu_Y \) are called to be dependent if it is not possible to use one variable for both operators and achieve the same action model.

For example, in the program,
\[
(\chi|_a \mu X. L_a(\varphi|_b L_b(\psi|_a X))) \cap (\chi|_b \mu Y. L_b(\psi|_a L_a(\varphi|_b Y)),
\]
the operators \( \mu X \) and \( \mu Y \) are independent. Note that the program
\[
(\chi|_a \mu Z. L_a(\varphi|_b L_b(\psi|_a Z))) \cap (\chi|_a \mu Z. L_b(\psi|_a L_a(\varphi|_b Z)),
\]
describes the same action model. In contrast with the above example, in the following program,
\[
\mu X. L_a(\varphi|_b \mu Y. L_b(\psi|_a X \cap \psi|_c L_c(\theta|_b Y))),
\]
the operators \( \mu X \) and \( \mu Y \) are dependent.

It is easy to observe that the class \( 0RLP \) is the class of all basic learning programs, \( BLP \). We also wish to name \( 1RLP \) as the class of primitive recursive learning programs and denote it also by \( PRLP \).

To prove the next theorem, we need to clarify some notions in graph theory. Let \( G = (N,E) \) be a directed graph. A simple loop \( L \) in the graph \( G \) is a sequence of nodes \( \langle s_0, s_1, s_2, ..., s_n \rangle \), such that for all \( i < n \), \( (s_i, s_{i+1}) \in E \), \( (s_n, s_0) \in E \), and for all \( 0 \leq i, j \leq n \), if \( i \neq j \) then \( s_i \neq s_j \). We call \( s_0 \) the start-point of the simple loop \( L \). Let \( L = \langle s_0, s_1, s_2, ..., s_n \rangle \) and \( L' = \langle s_0', s_1', s_2', ..., s_m' \rangle \) be two simple loops. We say \( L' \) is connected to \( L \) by its start-point, if there exists \( 0 \leq t \leq n \), \( s_t \neq s_0 \) and \( s_t = s_0' \), and for all \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \) if \( i \neq t \) and \( j \neq 0 \), then \( s_i \neq s_j \). A \( k \)-nested loop is a sequence \( \langle L_1, L_2, ..., L_k \rangle \) of simple loops, such that for each \( i > 1 \), \( L_i \) is connected to \( L_{i-1} \) by its start-point.
Theorem 4.15 We have the following hierarchy of recursive learning programs

$$\text{BLP} \subsetneq \text{PRLP} \subsetneq \text{2RLP} \subsetneq \ldots \subsetneq \text{kRLP} \subsetneq \ldots$$

Moreover, it does not collapse to RLP, i.e., for all \( k \),

$$\text{kRLP} \neq \text{RLP},$$

Proof. See the Appendix. \( \Box \)

5 Concluding Remarks and Further Work

5.1 Related Works

We may compare epistemic learning programs with other approaches, like concurrent dynamic epistemic logic \[8\] and epistemic programs \[5\].

In concurrent dynamic epistemic logic \[8\], an epistemic action is interpreted as a relation between \( S5 \) epistemic states and sets of \( S5 \) epistemic states. There are two main differences between the interpretation of epistemic action in concurrent dynamic epistemic logic and epistemic learning programs.

1. An epistemic action in concurrent dynamic epistemic logic is a relation between epistemic states whereas in epistemic learning programs, it is a function from epistemic states to epistemic states.

2. Concurrent dynamic epistemic logic is just about \( S5 \) models whereas epistemic learning programs also considers \( K45 \) models.

Another difference is in the interpretation of the notion of learning. Our learning operator is an operator on action models, and \( L_B(\alpha_1, \alpha_2, \ldots, \alpha_k) \) is a new action model, expressing the condition that agents in \( B \) learn that an action among \( \alpha_1, \alpha_2, \ldots, \alpha_k \) has occurred, whereas the action \( \alpha_1 \) has actually occurred. For example, \( L_b(\varphi, \psi) \) is an action model which says:

\[ \varphi \text{ is announced and agent } b \text{ is suspicious whether he learns } \varphi \text{ or learns } \psi. \]

One may compare the above learning program with the action \( L_b(!\varphi \cup ?\psi) \) in dynamic epistemic logic, and observe that for all \( S5 \) epistemic state \((M, s)\), we have that \((M, s) \times (N_{L_b(\varphi, \psi)}, t_{L_b(\varphi, \psi)})\) is bisimilar to \((M, s)[[L_b(\varphi \cup ?\psi)]]\). However, note that the learning program \( L_b(L_b(\varphi, ?\psi))) \) is equal to the learning program \( L_b(\varphi) \), whereas, in dynamic epistemic logic, the action \( L_b(L_b(\varphi \cup ?\psi)) \) is equal to \( L_b(\varphi \cup ?\psi) \). Thus there is a difference between the notion of learning we consider for learning programs and the notion of learning considered in dynamic epistemic logic.

Despite of the above arguments, it seems possible to translate a class of action terms, say \( \alpha \), in concurrent dynamic epistemic logic to a recursive learning program \( \text{tr}(\alpha) \), such that for all \( S5 \) epistemic state \((M, s)\), we have that \((M, s) \times (N_{\text{tr}(\alpha)}, t_{\text{tr}(\alpha)})\) is bisimilar to \((M, s)[[\alpha]]\).
Another way to represent information change is via the notion of epistemic program introduced in [5]. Whereby the notion of action signature is introduced and by adding this notion to the propositional dynamic logic PDL [13, 3], a logical language is obtained to represent information change. However in this setting, no learning operator is considered, and the information change is represented through action signature, alternative, sequential, and iteration compositions. We focus on different kinds of learning; as the primitive notion of information change is learning something by agents.

Another work related to ours is [10] where the epistemic programs are discussed by adding a parallel composition operator to non-deterministic sum and sequential composition.

We showed that all finite K45 action models can be described by recursive learning programs. It is also announced in [6] that every S5 action model can be described as a concurrent epistemic action.

We showed that K45 models are models of the belief KD45 logic for applicable formulas. In this way, to preserve the belief consistency of an agent, the agent is absent in the states that conflicts his beliefs. A similar work has been done in [16], which assumes that a rational agent rejects those incoming information which dispute his beliefs.

By introducing K45 models and actions, we may think of a theory of multi-agent belief revision. A related work is [2], which generalize AGM [1], to a multi-agent belief revision theory.

Our work presents a method to construct K45 action models through some basic constructors. Also in [11], it is introduced operators to compose epistemic models in order to construct large models by small components representing agents’ partial observational information.

5.2 Further work

5.2.1 A functional Semantics

As a semantics of epistemic learning programs, we associated a pointed action model to every basic learning program. We may propose a functional semantics for the basic learning programs, in the manner that each program is associated to a partial function from epistemic states Mod to Mod. In this semantics, the meaning of learning operator is different form the meaning we discussed in the introduction. Here, learning in epistemic states (M, s) deals with two things, a set U of states of M, which includes the actual state s, and a set of agents B ⊆ A (where A refers to the set of all agents). Learning with B ⊆ A and U ⊆ S in the epistemic states (M, s) means that:

agents in B become aware that the actual state is among the states in U, and
other agents in A – B believe that nothing has occurred.

Let Φ be a set of epistemic formulas over a set of atomic formulas P and a set of agents A. To each α ∈ BLP(Φ), we associate a pair (f_α, U_α), where f_α : Mod → Mod is a partial function, and for each epistemic state (M, s), U_α((M, s)) is a subset of S', where f_α((M, s)) = (M', s') with M' = (S', (−1)i∈A, V').

For a recursive learning program μX.α(X), the associated partial function should satisfy the fixed point equation, i.e., f_{μX.α(X)} = f_{μX.α(X)} ◦ f_{μX.α(X)} = f_{μX.α(X)} ◦ f_{μX.α(X)}. As our forthcoming work,
we aim to study this functional semantics. It seems to us that functional semantics and recursive learning take us beyond the action models, that is, by functional semantics, epistemic learning programs can encode information changes which cannot be encode by action models.

5.2.2 A Logic for RLP

We need to provide a proof system for RLP as it is done for other approaches, like concurrent dynamic epistemic logic [8], and action models [4, 5].

5.2.3 Notions of Learning

In Introduction, we put forward two meanings for 1. pointed action models (see 2.1) and 2. learning of an action model (see 2.2). We supposed that an action model describes what is announced and what agents perceive based on their accesses to the resource of announcement. We also assumed that the learning of an action by a set of agents is to learn about the way information change. So our meanings of action models and learning refer to the occurrence of information change.

We may propose two other meanings for pointed action models and learning of an action model, which refer to disability in information change. In this way, an action model describes the disability of agents in hearing or accessing the resource of announcement. For example, the new meaning of the pointed action model \((N_1, s)\) in Figure 1,

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Figure 1
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is

“in the case of announcement of \(\varphi\) agent \(a\) hears \(\psi\).”

Note that the above meaning does not speak about what occurs in information change, but it just describes a disability of agent \(a\). Suppose \(\varphi = \text{green}\) and \(\psi = \text{blue}\). The new meaning of the pointed action model \((N_1, s)\) is that agent \(a\) has a color-blindness and if a green ball is shown to her then she thinks that she sees a blue ball. Similarly, the meaning of learning an action changes. The new meaning is learning about disability not about occurrence. The learning of an action by a set of agents is to learn about the disabilities that the agents have. Figure 22 shows two pointed action models where both refer to \(L_a((N_1, s))\) (agent \(a\) learns the pointed action model \((N_1, s)\)), but one considers the occurrence meaning and the other considers the disability meaning.
In the occurrence meaning, agent $a$ learns that $\varphi$ is announced. In the disability meaning, agent $a$ becomes aware of her color-blindness, and after this learning, if she sees a blue ball, she is suspicious whether it is green or blue.

Acknowledgement. The authors would like to thank Hans van Ditmarsch for his careful reading of our manuscript, and his very helpful comments and suggestions. We also would like to thank Mehrnoosh Sadrzadeh for her very helpful comments and suggestions to improve the readability of the manuscript.

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Lemma 6.1 Let $M = (S, (\rightarrow_a)_{a \in A}, V)$ be a Kripke model such that for each $a \in A$, $\rightarrow_a$ is Euclidean. Then for all $s \in S$, if there is a state $v \in S$ such that $v \rightarrow_a s$ then there exists $t \in S$ such that $s \rightarrow_a t$.

Proof. Let $S$ be a set of states and $\rightarrow \subseteq S \times S$ be an Euclidean relation. Suppose $s, v \in S$ are arbitrary and $v \rightarrow s$. By Euclidean property, we derive $s \rightarrow s$, and we are done. \(\square\)

Proof. Consider $T(N) = (S', (\rightarrow'_a)_{a \in A}, \text{pre}')$. Define $R \subseteq S \times S'$ as follows. For all $s \in S$ and $w \in S'$, $sRw$ if and only if $\Pi(w) = s$.

We show that $R$ is a bisimulation relation. Suppose all different the maximal closed connected $S5$ submodels of $N$ are $M^1, M^2, \ldots$ and $M^k$. Assume $sRw$. Then $w = (s, i)$, for some $1 \leq i \leq k$.

- Forth. Let $s \rightarrow_a t$. Either $a \in B^i$ or $a \notin B^i$. In the first case, since $M^i$ is closed, we have $t \in S^i$ and thus $(s, i) \rightarrow'_a (t, i)$, and since $tR(t, i)$, we are done. In the second case, since $N$ is a K45 model, and $s \rightarrow_a t$, we have $t \rightarrow_a t$, by Lemma 6.1 Therefore there exists a maximal closed connected $S5$ submodel of $N$, say $M^j$, such that $t \in S^j$ and $a \in B^j$. By Definition 3.4 $(s, i) \rightarrow'_a (t, j)$, and since $tR(t, j)$ we are done.

- Back. Suppose $(s, i) \rightarrow'_a (t, j)$. Then by Definition 3.4, $s \rightarrow_a t$, and we are done.

- Pre. It is straightforward.
The proof is by induction on the structure of basic learning programs.

First of all, note that for each epistemic formula \( \varphi \), the graph of the action model \((N_\varphi, s_\varphi)\) is a tree.

Let \( \alpha \) be a basic learning program and its graph \( G(N_\alpha) \) be a tree. We show that the graph \( G(N_\psi|_{BA}) \) is a tree, for any arbitrary formula \( \psi \) and \( B \subseteq \text{group}(\alpha) \). The maximal closed connected \( S5 \) submodels of the action model \( N_\psi|_{BA} \) are all the maximal closed connected \( S5 \) submodels of \( N_\alpha \), say \( M^1, M^2, \ldots, M^K \), and the maximal closed connected \( S5 \) submodel containing the state \( s_\psi|_{BA} \), which is \( M^0 = \langle \{s_\psi|_{BA}\}, (\neg a)^{0}_{a \in B^0}, \text{pre}^0(s_\psi|_{BA}) = \psi \rangle \). One may check that

1. for all \( i, j \geq 1 \), the edges between two nodes \( M^i, M^j \) in the graph \( G(N_\psi|_{BA}) \) are the same edges in the graph \( G(N_\alpha) \),

2. for all \( i \geq 1 \), there is no directed edge from \( M^i \) to \( M^0 \), as group of \( M^0 \) is empty,

3. for all \( i \geq 1 \), \( M^0 \rightarrow M^i \) if and only if there exists \( t \in S^i, a \in B^i \cap B \), such that \( s_\alpha \rightarrow a \ t \) in the model \( N_\alpha \).

Hence, if the graph \( G(N_\alpha) \) has no loop, the graph of \( G(N_\psi|_{BA}) \) would have no loop as well.

Let \( \alpha_1 \) and \( \alpha_2 \) be two basic learning programs such that \( \text{group}(\alpha_1) \cap \text{group}(\alpha_2) = \emptyset \), and \( \text{pre}(\alpha_1) = \text{pre}(\alpha_2) \) and \( G(N_{\alpha_1}) \) and \( G(N_{\alpha_2}) \) are trees. Then the graph \( G(N_{\alpha_1 \cap \alpha_2}) \) is a tree for the following reasons. The maximal closed connected \( S5 \) submodels of \( N_{\alpha_1 \cap \alpha_2} \) consist of all the maximal closed connected \( S5 \) submodels of \( N_{\alpha_1} \), and all the maximal closed connected \( S5 \) submodels of \( N_{\alpha_2} \), and the maximal closed connected \( S5 \) submodel containing the state \( s_{\alpha_1 \cap \alpha_2} \), which is \( M^0 = \langle \{s_{\alpha_1 \cap \alpha_2}\}, (\neg a)^{0}_{a \in B^0}, \text{pre}^0(s_{\alpha_1 \cap \alpha_2}) = \text{pre}(\alpha_1) \rangle \). One may check that, since \( \text{group}(\alpha_1) \cap \text{group}(\alpha_2) = \emptyset \), there is no edge between the nodes of the subtrees \( G(N_{\alpha_1}) \) and \( G(N_{\alpha_2}) \). So if the graphs \( G(N_{\alpha_1}) \) and \( G(N_{\alpha_2}) \) have no loop, the graph of \( G(N_{\alpha_1 \cap \alpha_2}) \) would have no loop as well.

Assume \( \alpha_1, \alpha_2, \ldots, \alpha_m \) are basic learning programs such that their graphs have no loop. The maximal closed connected \( S5 \) submodels of \( N_{L_B(\alpha_1, \alpha_2, \ldots, \alpha_m)} \) are the followings:

1. the \( S5 \) model \( M_0 \) consists of \( m \) states \( (s_{\alpha_1}, 1), (s_{\alpha_2}, 1), \ldots, (s_{\alpha_m}, 1) \), with accessibility relations produced by agent-bisimilarity of group \( B \) (see Definition \ref{def:s5_bisimilarity}).

2. all the maximal closed connected \( S5 \) submodels of \( N_{\alpha_1}, N_{\alpha_2}, \ldots, N_{\alpha_m} \).

The node \( M_0 \) is the root of the graph \( G(N_{L_B(\alpha_1, \alpha_2, \ldots, \alpha_m)}) \), and all the graphs \( G(N_{\alpha_1}), G(N_{\alpha_2}), \ldots, G(N_{\alpha_m}) \) are disjoint subgraphs of \( G(N_{L_B(\alpha_1, \alpha_2, \ldots, \alpha_m)}) \), such that the root may be connected to them. Hence if \( G(N_{\alpha_1}), G(N_{\alpha_2}), \ldots, G(N_{\alpha_m}) \) are trees, then \( G(N_{L_B(\alpha_1, \alpha_2, \ldots, \alpha_m)}) \) is a tree. \( \Box \)

**Proof** \( \ref{lem:basic_learning_programs} \) Suppose \((N, s_0)\) is an \( S5 \) pointed action model. Let \( N = \langle S, (\neg a)_{a \in A}, \text{pre} \rangle \), where \( S = \{s_0, s_1, \ldots, s_k\} \), and \( A = \{a_1, a_2, \ldots, a_m\} \). As the model is \( S5 \), all accessibility relations are equivalence relations. For each agent \( a_i \), let \( P_i = \{D_i^1, D_i^2, \ldots, D_i^n\} \) be the equivalence classes of the relation \( \rightarrow_{a_i} \), which partitions the set \( S \). Consider \( n_i \) epistemic formulas \( \psi_{i,1}, \psi_{i,2}, \ldots, \psi_{i,n_i} \), where none of them are \( KD45 \) equivalent to each other. For each \( s_j \), \( 0 \leq j \leq k \), consider the basic learning program \( \alpha_j \).
\[ \alpha_j = \beta_{j,1} \cap \beta_{j,2}, ..., \cap \beta_{j,m}, \]

where each \( \beta_{j,l} \) is a basic learning program defined as follows

\[ \beta_{j,l} = \text{pre}(s_j)|_{a_l} L_{a_l}(\omega_{t,h}), \]

where \( s_j \in D^h \).

The action model associated to the basic learning program \( L_A(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_k) \) is \((N,s_0)\). Since for each agent \( a \), the action models of two programs \( \alpha_j \) and \( \alpha_i \) are \( a \)-bisimilar if and only if \( s_j \) and \( s_i \) are in the same equivalence classes induced by the relation \( \rightarrow_a \). \( \dashv \)

**Proof 3.9** By induction on the height of the tree. Let \( h_N \) be the height of the tree \( G(N) \). If \( h_N = 1 \), the action model \( N \) is an \( S5 \) model and by the Proposition 3.8 \((N,s)\) is a basic learning action. Suppose that for all \( h < k \), if the graph of a \( K45 \) pointed model \((M,t)\) is a tree with height \( h \), then \((M,t)\) is a basic learning action. Assume \( h_N = k \). Let \( M^0, M^1, ..., M^{k-1} \) are all maximal connected closed submodels of \( N \), and \( s \) is a state of \( M^0 \). We consider \( M^0 \) as the root of the tree, and delete all nodes which are not reachable from \( M^0 \). That is because we want to state a program for the pointed action model \((N,s)\), and by deleting those maximal connected closed submodels which are not reachable from \( s \), we do not loose anything up to bisimilarity. Let \( M^0 = (s^0, (\pre_0)_{a \in B^0}, \text{pre}_0) \), where \( S^0 = \{s^0_1, s^0_2, ..., s^0_n\} \), \( B^0 = \{a^0_1, a^0_2, ..., a^0_m\} \) and \( s^0 = s \). As \( M^0 \) is an \( S5 \) action model, by using Proposition 3.8 there is a basic learning program \( L_{B^0}(\alpha_1, \alpha_2, ..., \alpha_n) \), such that its action model is \((M^0, s^0)\), and each \( \alpha_i \) corresponds to a state \( s^0_i \) (see the proof of Proposition 3.8).

For each \( s^0_i \), let \( \{b^i_1, b^i_2, ..., b^i_{l_i}\} = \text{group}_{T(N)}(s^0_i) - B^0 \), where \( \text{group}_{T(N)}(s^0_i) \) is the group of agents of the state \( s^0_i \) in the model \( T(N) \). For each \( 1 \leq i \leq l_i \), if \( s^0_i \rightarrow_{b^i_j} t_{i,j} \), for some state \( t_{i,j} \) of the model \( T(N) \), then as \( M_0 \) is the root and is not accessible from \( t_j \), the graph of the pointed action model \((T(N), t_{i,j})\) is a tree with height less than \( k \), and by the induction hypothesis, there is a basic leaning program \( \beta_{i,j} \) such that its action model is \((T(N), t_{i,j})\). For each \( s^0_i \), let

\[ \gamma_i = \alpha_i \cap (\text{pre}(s^0_i)|_{b^i_1} \beta_{i,1}) \cap (\text{pre}(s^0_i)|_{b^i_2} \beta_{i,2}) \cap ... \cap (\text{pre}(s^0_i)|_{b^i_{l_i}} \beta_{i,l_i}). \]

The program \( L_{B^0}(\gamma_1, \gamma_2, ..., \gamma_n) \) is the desired program, that is, its associated action model is \((N,s)\). \( \dashv \)

**Proof 4.12** Let \((N,s_0)\) be a \( K45 \) pointed action model. Consider the action model \( T'(N) \). One may check that the followings hold true. Suppose \( M^a_1 \) and \( M^b_2 \) are two different components in \( T'(N) \), then

1. if \( a = b \), as both components are closed \( S5 \) models, there is no accessibility relation for agent \( a \) between the two components,

2. if for some state \( s \in M^a_1 \) and \( t \in M^b_2 \), we have \( s \rightarrow_b t \), then for all \( v \in M^b_2 \), we have \( s \rightarrow_b v \), by transitivity and connectedness of \( M^b_2 \).

Let \( n_0, n_1, n_2, ..., n_k \) be all different components of the model \( N \). Also suppose \( n_0 \) is a component in which the actual state \( s_0 \) appears. The model \( T'(N) \) is a directed labeled
graph in which the nodes are $n_0, n_1, n_2, ..., n_k$, and the edges are agents in $A$. To each node $n_i$, we correspond a variable $X_i$.

If the graph is a tree, then we are done and then we can construct a basic learning program describing $(N, s_0)$. If the graph is not a tree, we unwind it to an infinite tree with the root $n_0$.

![Image of unwinding and cutting process]

Figure 23

In the unwound infinite tree, there could be infinite nodes with the same name, say $n_i$. For all nodes $w$ of the unwound tree, if $w$ is a node with name $n_i$ (for some $i$) and exactly one of its parents has the same name $n_i$, then we cut the subtree rooted from $w$ and change the name of $w$ from $n_i$ to variable $X_i$. In this way, a finite tree $T''(N)$ is obtained.

Now we are ready to construct the desired program. We start from down to the top of the finite tree $T''(N)$.

1. First note that each leaf of the tree is either a variable or an $a$-component. If it is an $a$-component, then we associate to that leaf, the program

$$L_a(\text{pre}(v_1), \text{pre}(v_2), \ldots, \text{pre}(v_m),$$

where $v_1, v_2, \ldots, v_m$ are all the states of the component. We note that as the $a$-component is a connected $S5$ action model, it is associated to the program

$$L_a(\text{pre}(v_1), \text{pre}(v_2), \ldots, \text{pre}(v_m)).$$

2. Suppose that $n_j$ is the name of a node $w$ in $T''(N)$, which either all of its children are corresponded to a variable or a program. Two cases are possible:

- Case 1. Among the children of $w$ there is no node corresponding to the variable with the same index $j$, that is $X_j$.

For this case, suppose $n_j$ refers to a $b$-component with the states $v_1, v_2, \ldots, v_m$. For each state $v_l$, and each agent $a \in A$, if there is a directed edge with label $a$ starting from the state $v_l$ to a children of $w$, say $u$, in the tree $T''(N)$, consider

$$\text{pre}(v_l)|_a P_{a,l}$$

where $P_{a,l}$ is a program or variable corresponding to the node $u$. Then we associate to the node $w$, the program

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\[ L_b(\bigcap_{a \in A} \text{pre}(v_1)|_a P_{a,1}, \bigcap_{a \in A} \text{pre}(v_2)|_a P_{a,2}, \ldots, \bigcap_{a \in A} \text{pre}(v_m)|_a P_{a,m}) \]

- Case 2. Among the children of \( n_j \) there are some nodes corresponding to the variable with the same index \( j \), that is \( X_j \). For this case, we do exactly the same as we did in the first case, thus obtaining the program

\[ L_b(\bigcap_{a \in A} \text{pre}(v_1)|_a P_{a,1}, \bigcap_{a \in A} \text{pre}(v_2)|_a P_{a,2}, \ldots, \bigcap_{a \in A} \text{pre}(v_m)|_a P_{a,m}). \]

Then we associate the following program to the node

\[ \mu X_j. L_b(\bigcap_{a \in A} \text{pre}(v_1)|_a P_{a,1}, \bigcap_{a \in A} \text{pre}(v_2)|_a P_{a,2}, \ldots, \bigcap_{a \in A} \text{pre}(v_m)|_a P_{a,m}) \]

The program corresponding to the root of the tree \( T''(N) \) is a recursive learning program which describes pointed action model \((N, s_0)\). \( \dashv \)

**Proof.** In Theorem 3.6 it is proved that the graph of the action model of a basic learning program is a tree. So none of the operations: alternative learning, concurrent learning, wrong learning, produces any loops in the graph of a learning program. It is easily seen by Definition 4.7 that, the only operation that makes loops in the semantics of a learning program is the recursive learning operator. Therefore, if in a learning program, there exist \( k \) times of dependent use of the recursive operator \( \mu \), then there exists at most a \( k \)-nested loop in its graph. That is, for each \( k \in \mathbb{N} \), the graph of an action model associated to a program in kRLP has at most \( k \)-nested loops.

For each \( k > 0 \), we introduce a learning program \( \alpha^k \), such that its associated action model belongs to kRLP but not (\( k-1 \))RLP.

- \( k = 1 \). Let \( \alpha^1 = \mu X. L_b(\varphi|_a L_a(\psi|_b L_b(X))) \), where \( \varphi \) is not logically equivalent to \( \psi \).

The associated action model of the learning program \( \alpha^1 \) is the action model \((N_3, s)\) in Figure 3. Since \( \varphi \) and \( \psi \) are not logically equivalent, the two states \((N_3, s)\) and \((N_3, t)\) (see Figure 3) are not bisimilar. If there exists a program \( \beta \) without any recursive operator that its associated action model is bisimilar to \((N_3, s)\), then the action model \( N_3 \) would be bisimilar to a finite action model \( M' \in \text{FA}ct \), such that its graph is a tree. Suppose \( R \) is a bisimilarity relation between \( N_3 \) and \( M' \), and \( sRs' \). Because of bisimilarity, since \( s \xrightarrow{a} t \), there exists an state \( t' \) in model \( M' \), such that \( tRt' \) and \( s' \xrightarrow{a} t' \). Again, since \( t \xrightarrow{b} s \), there exists an state \( s'' \) in \( M' \), such that \( sRs'' \) and \( t' \xrightarrow{b} s'' \). The model \( M' \) is a tree, so we have \( s'' \neq s' \), and as \( s \) and \( t \) are not bisimilar, we have \( s'' \neq t' \). Again, by bisimilarity, there exists an state \( t'' \) in \( M' \), such that \( s'' \xrightarrow{a} t'' \) and \( tRt'' \). The new state \( t'' \) is different from other states of \( M' \), since \( M' \) has no loop. In this way, \( M' \) is an infinite model, and we derive a contradiction.

- \( k = 2 \). The above argument can be done for \( k = 2 \), by considering the associated action model of the program \( \alpha^2 = \mu X. L_a(\varphi|_b \mu Y. L_b(\psi|_c L_c(\theta|_d Y))) \) (see Figure 17), where none of the formulas \( \varphi, \psi \) and \( \theta \) are logically equivalent. If there is a program \( \beta \) with at most one use of recursive operation, then the action model in Figure 17 (which has a 2-nested loop) would be bisimilar to an action model \( M' \), that its graph has just one loop. This can easily be shown, since none of the states of the action model in Figure 18 are bisimilar to each other, so the action model \( M' \) cannot be finite.
So for any arbitrary $k$, we can construct an action model having one $k$-nested loop, where none of its nodes are bisimilar to each other. Then this action model is in $k$RLP but not $(k - 1)$RLP. \(\dashv\)