A CONSTRUCTIVE PROOF OF THE DVORETZKY–ROGERS THEOREM
IN $\ell_p$ SPACES

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Abstract. The Dvoretzky–Rogers Theorem asserts that in every infinite-dimensional Banach space $X$ there exists an unconditionally convergent series $\sum x^{(j)}$ such that $\sum \|x^{(j)}\|^{2 - \varepsilon} = \infty$ for all $\varepsilon > 0$. Their proof is non-constructive and, according to the literature, the case $X = \ell_1$ is critical. A constructive proof when $X = \ell_1$ in the particular case $\varepsilon \leq 1$ was obtained by MacPhail in 1947. However, to the best of the authors’ knowledge there is no explicit construction of a series satisfying the whole statement of the Dvoretzky–Rogers theorem for $X = \ell_1$ (and even for $\ell_p$ with $p \leq 2$; the case $p > 2$ is trivial). In this note we provide an example of such series for $1 \leq p \leq 2$. Our approach rests in a suitable handling of certain special matrices that date back to the works of Toeplitz.

1. Introduction

A series in a Banach space is said to be unconditionally convergent if all rearrangements of that series converge (to the same vector). An old result due to Dirichlet (1829) asserts that a series $\sum x^{(j)}$ of real or complex scalars is unconditionally convergent precisely when it is absolutely convergent, i.e., when $\sum \|x^{(j)}\|$ converges. The same characterization holds for finite-dimensional Banach spaces. In the infinite-dimensional framework it seems that unconditionally convergent series were first investigated by Orlicz about 100 years after Dirichlet. The subject attracted the attention of Banach, Mazur and others, who proposed the following problem (see [8, Problem 122] and also [10] for further details): in an arbitrary infinite-dimensional Banach space, does there exist an unconditionally convergent series that does not converge absolutely?

In most of the classical Banach spaces, examples of unconditionally convergent series that do not converge absolutely were easily obtained. For instance, the sequence $(j^{-1}e_j)_{j=1}^\infty$ is unconditionally summable in $\ell_2$ but it fails to be absolutely summable. The same example works for all $\ell_p$ spaces for any $p > 1$, but it is useless for $\ell_1$. In 1947, MacPhail ([6]) proved that the answer was also positive in $\ell_1$. MacPhail’s approach has shown that $\ell_1$ is essentially the critical case for all infinite-dimensional Banach spaces and, according to [3, pages 2 and 20], inspired Dvoretzky and Rogers in 1950 ([4]) to transplant in a highly nontrivial fashion the construction of MacPhail to any infinite-dimensional Banach space and obtain a definitive solution to the problem. Some alternative proofs of MacPhail’s result and of the Dvoretzky–Rogers Theorem appeared later, but they were not constructive (see, for instance, [5, page 145] and [1, 11] and the references therein). We also refer the interested reader to [2] for a non-constructive approach in $\ell_1$ using the Kahane–Salem–Zygmund inequality (see also [7] for details on the Kahane–Salem–Zygmund inequality).

The striking result of Dvoretzky–Rogers attracted the attention of Grothendieck who referred to the Dvoretzky–Rogers Theorem as the unique decisive result in the finer metric theory of general Banach spaces known at that time (see [3, page 20]). Additionally, the result of Dvoretzky and Rogers answers much more than what is asked in the original problem of Banach’s school. In more precise terms, if $X$ is an infinite-dimensional Banach space, the Dvoretzky–Rogers Theorem assures the existence of an unconditionally convergent series $\sum x^{(j)}$ in $X$ such

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that
\begin{equation}
\sum_{j=1}^{\infty} \|x^{(j)}\|^{2-\varepsilon} = \infty
\end{equation}
for all \(\varepsilon > 0\). However, the proof of the Dvoretzky–Rogers Theorem do not offer an explicit construction of such sequence. In this note we provide an elementary and constructive proof of such a sequence in \(\ell_p\), for all \(p \in [1, 2]\) (the case \(p > 2\) is obvious and for this reason we do not consider it); unlike MacPhail’s approach which is valid for \(\varepsilon \leq 1\), our construction completely satisfies the above statement, i.e., we provide an explicit construction of an unconditionally convergent series \(\sum x^{(j)}\) in \(\ell_p\) satisfying (1.1) for all \(\varepsilon > 0\).

Our main result is:

**Theorem 1.1.** Let \(p \in [1, 2]\) and \(\alpha > 1\) a positive integer. For all positive integers \(j\), let
\[
A_j := \left[ 1/2 + \sqrt{1/4 + \log_\alpha ((j + 1)/2)}, 1/2 + \sqrt{1/4 + \log_\alpha j}\right] \cap \mathbb{N}.
\]
The sequence \((x^{(j)})_{j=1}^{\infty}\) defined by
\[
x^{(j)} = \alpha^{(1-k_j)(k_j(2p+2p))} \cdot \sum_{s=1}^{\alpha^{k_j(k_j-1)}} \left( \exp \left( j\alpha^{k_j(1-k_j)} + \alpha^{k_j(1-k_j)} - 1 \right) 2\pi s i \right) \cdot e^{\alpha^{k_j(k_j-1)} + s - 1}
\]
if \(A_j \neq \emptyset\), where \(k_j\) is the unique element of \(A_j\), and \(x^{(j)} = 0\) otherwise, is unconditionally summable in \(\ell_p\) and
\[
\sum_{j=1}^{\infty} \|x^{(j)}\|^{2-\varepsilon} = \infty
\]
for all \(\varepsilon > 0\).

The statement may seem somewhat complicated at first glance, but the construction is fairly simple. We shall first observe that the interval
\[
\left[ 1/2 + \sqrt{1/4 + \log_\alpha ((j + 1)/2)}, 1/2 + \sqrt{1/4 + \log_\alpha j}\right]
\]
has size smaller than 1 and thus \(A_j\) is, in fact, either empty or formed by a single integer. A straightforward computation shows that when \(j \in \{\alpha^{k(k-1)}, \alpha^{k(k-1)} + 1, \ldots, 2\alpha^{k(k-1)} - 1\}\) for a certain positive integer \(k\), we have \(A_j = \{k\}\); otherwise \(A_j\) is empty.

2. THE PROOF

The core of the proof rests in the following matrices that date back to the works of Toeplitz [12]. For each positive integer \(n\), consider the \(n \times n\) matrix \((a^{(n)}_{rs})\) with \(a^{(n)}_{rs} = \exp(2\pi irs/n)\); it is obvious that \(|a^{(n)}_{rs}| = 1\) and one can also check that
\begin{equation}
\sum_{s=1}^{n} a^{(n)}_{rs} a^{(n)}_{is} = n \delta_{rt},
\end{equation}
where \(\delta_{rt}\) denotes the Kronecker delta. In order to prove (2.1) note that
\[
\sum_{s=1}^{n} a^{(n)}_{rs} a^{(n)}_{is} = \sum_{s=1}^{n} \exp \left( 2\pi irs/n \right) \cdot \exp \left( -2\pi it/n \right) = \sum_{s=1}^{n} \exp \left( 2\pi i (r - t)s/n \right).
\]
If \(r = t\), it is obvious that \(\sum_{s=1}^{n} a^{(n)}_{rs} a^{(n)}_{is} = n\). On the other hand, if \(r \neq t\), we have
\[
\sum_{s=1}^{n} \exp \left( 2\pi i (r - t)s/n \right) = \frac{\exp \left( 2\pi i (r - t)s/n \right) - \exp \left( 2\pi i (r - t)(n+1)/n \right)}{1 - \exp \left( 2\pi i (r - t)/n \right)}.
\]
and, recalling that \( \exp(\pi i) = -1 \), a straightforward computation gives us
\[
\sum_{s=1}^{n} a_{rs}^{(n)} \overline{a_{rs}^{(n)}} = 0.
\]

Let \( p^* \) be the conjugate of \( p \), i.e.,
\[
\frac{1}{p} + \frac{1}{p^*} = 1
\]
and let \( \ell_p^n \) denote \( \mathbb{C}^n \) with the \( \ell_p \)-norm. For unit vectors \( y^{(1)} \in \ell_p^n \) and \( y^{(2)} \in \ell_{\infty}^n \), we have
\[
(2.2) \quad \left| \sum_{i_1, i_2 = 1}^{n} a_{i_1 i_2}^{(n)} y_{i_1}^{(1)} y_{i_2}^{(2)} \right| \leq n^{\frac{1}{2} + \frac{1}{p^*}}.
\]
The proof of (2.2) can be found in [9, pages 30 and 31], but we repeat the argument for the sake of completeness. By the H"older inequality, we have
\[
\left| \sum_{i_1, i_2 = 1}^{n} a_{i_1 i_2}^{(n)} y_{i_1}^{(1)} y_{i_2}^{(2)} \right| \leq \left( \sum_{i_2 = 1}^{n} |y_{i_2}^{(2)}|^2 \right)^{\frac{1}{2}} \left( \sum_{i_2 = 1}^{n} \left| \sum_{i_1 = 1}^{n} a_{i_1 i_2}^{(n)} y_{i_1}^{(1)} \right|^2 \right)^{\frac{1}{2}}
\]
\[
\leq n^{\frac{1}{2}} \left( \sum_{i_2 = 1}^{n} \left| \sum_{i_1 = 1}^{n} a_{i_1 i_2}^{(n)} y_{i_1}^{(1)} \right|^{2} \right)^{\frac{1}{2}}.
\]
Since
\[
\left( \sum_{i_2 = 1}^{n} \left| \sum_{i_1 = 1}^{n} a_{i_1 i_2}^{(n)} y_{i_1}^{(1)} \right|^2 \right)^{\frac{1}{2}} = \left( \sum_{i_2 = 1}^{n} \sum_{i_1 = 1}^{n} y_{i_1}^{(1)} y_{i_1}^{(1)} a_{i_1 i_2}^{(n)} a_{i_1 i_2}^{(n)} \right)^{\frac{1}{2}} = \left( \sum_{i_2 = 1}^{n} y_{i_1}^{(1)} y_{i_1}^{(1)} a_{i_1 i_2}^{(n)} a_{i_1 i_2}^{(n)} \right)^{\frac{1}{2}},
\]
by (2.1) and by the H"older inequality, we have
\[
\left| \sum_{i_1, i_2 = 1}^{n} a_{i_1 i_2}^{(n)} y_{i_1}^{(1)} y_{i_2}^{(2)} \right| \leq n^{\frac{1}{2}} \left( \sum_{i_1 = 1}^{n} y_{i_1}^{(1)} y_{i_1}^{(1)} n \delta_{i_1 i_1} \right)^{\frac{1}{2}}
\]
\[
= n \left( \sum_{i_1 = 1}^{n} |y_{i_1}^{(1)}|^2 \right)^{\frac{1}{2}}
\]
\[
\leq n \left( \sum_{i_1 = 1}^{n} \right)^{\frac{1}{2} - \frac{1}{p}} \left( \sum_{i_1 = 1}^{n} |y_{i_1}^{(1)}|^{p} \right)^{\frac{1}{p}}
\]
\[
= n^{\frac{1}{2} + \frac{1}{p^*}}.
\]
Thus, by (2.2), we obtain
\[
(2.3) \quad \sup_{\|\varphi\|_{p^*} \leq 1} \sum_{r=1}^{n} \sum_{s=1}^{n} \varphi_s a_{rs}^{(n)} = \sup_{\|\varphi\|_{p^*} \leq 1} \sup_{\|\psi\|_{\infty} \leq 1} \sum_{r,s=1}^{n} \psi_r \varphi_s a_{rs}^{(n)} \leq n^{\frac{1}{2} + \frac{1}{p^*}},
\]
where \( \varphi = (\varphi_1, \ldots, \varphi_n) \) and \( \psi = (\psi_1, \ldots, \psi_n) \).
Let \( j_k := \alpha^{k(k-1)} \) for all positive integers \( k \). Define the sequence \((x^{(j)})_{j=1}^{\infty}\) as follows: if \( j \in \{j_k, \ldots, 2j_k - 1\} \), for a certain positive integer \( k \), consider
\[
x^{(j)} = j_k^{-\left(\frac{1}{2} + \frac{1}{p} + \frac{1}{q}\right)} \left( \sum_{s=1}^{j_k} a_{rs}^{(j_k)} e_{j_k+s-1} \right),
\]
where \( r = j - j_k + 1 \); otherwise take \( x^{(j)} = 0 \). For all \( \varphi \in (\ell_p)^* \) it is plain that
\[
(2.4) \quad \sup_{\|\varphi\|_{\ell_p} \leq 1} \sum_{j=1}^{\infty} \|\varphi (x^{(j)})\| \leq \sup_{\|\varphi\|_{\ell_p} \leq 1} \left( \sum_{j=j_k}^{2j_k-1} \sum_{m=1}^{2j_k-1} \|\varphi (x^{(j)})\| \right).
\]
By the definition of \( x^{(j)} := (x^{(m)})_{m=1}^{\infty} \), we obtain
\[
(2.5) \quad \sum_{j=j_k}^{2j_k-1} \|\varphi (x^{(j)})\| = \sup_{\|\varphi\|_{\ell_p} \leq 1} \sum_{j=j_k}^{2j_k-1} \sum_{m=1}^{2j_k-1} \varphi m a_m [j] a_{j-1}[m-1]
= \sup_{\|\varphi\|_{\ell_p} \leq 1} \left( \sum_{j=j_k}^{2j_k-1} \sum_{m=1}^{2j_k-1} \varphi m a_{[j-1][m-1]} \right)
\]
for all positive integers \( k \). Note that we can assign new indices to the sums \( \sum_{j=j_k}^{2j_k-1} \) and \( \sum_{m=j_k}^{2j_k-1} \) as follows:
\[
\sum_{j=j_k}^{2j_k-1} j_k^{-\left(\frac{1}{2} + \frac{1}{p} + \frac{1}{q}\right)} \sum_{m=j_k}^{2j_k-1} \varphi m a_{[j-1][m-1]} = \sum_{r=1}^{j_k} j_k^{-\left(\frac{1}{2} + \frac{1}{p} + \frac{1}{q}\right)} \sum_{s=1}^{j_k} \varphi (j_k+s-1) a_{j_k+s-1}
\]
Hence, for \( n \geq 2 \), by (2.3) and (2.5), we have
\[
(2.6) \quad \sum_{k=n}^{\infty} \left( \sup_{\|\varphi\|_{\ell_p} \leq 1} \sum_{j=j_k}^{2j_k-1} \|\varphi (x^{(j)})\| \right) = \sum_{k=n}^{\infty} \left( \sup_{\|\varphi\|_{\ell_p} \leq 1} \sum_{j=j_k}^{2j_k-1} j_k^{-\left(\frac{1}{2} + \frac{1}{p} + \frac{1}{q}\right)} \sum_{m=j_k}^{2j_k-1} \varphi m a_{[j-1][m-1]} \right)
= \sum_{k=n}^{\infty} \left( \sup_{\|\varphi\|_{\ell_p} \leq 1} \sum_{r=1}^{j_k} j_k^{-\left(\frac{1}{2} + \frac{1}{p} + \frac{1}{q}\right)} \sum_{s=1}^{j_k} \varphi (j_k+s-1) a_{j_k+s-1} \right)
\]
\[
\leq \sum_{k=n}^{\infty} \left( j_k^{-\left(\frac{1}{2} + \frac{1}{p} + \frac{1}{q}\right)} \frac{1}{j_k^{\frac{1}{p}}} \right)
= \sum_{k=n}^{\infty} \alpha^{1-k} < \infty.
\]
A well-known necessary and sufficient condition for a series \( \sum_{n=1}^{\infty} y_n \) to be unconditionally summable is that, given \( \delta > 0 \), there is \( n_{\delta} \in \mathbb{N} \) such that
\[
\left\| \sum_{n \in M} y_n \right\| < \delta
\]
whenever \( M \) is a finite subset of \( \mathbb{N} \) with \( \min M > n_{\delta} \).

By (2.4) and (2.6) we have
\[
\lim_{n \to \infty} \sup_{\|\varphi\|_{\ell_p} \leq 1} \sum_{j=n}^{\infty} \|\varphi (x^{(j)})\| = 0.
\]
Hence, given $\delta > 0$, there is $n_\delta \in \mathbb{N}$ such that $n \geq n_\delta$ implies
\[
\sup_{\|\varphi\|_{\ell^p} \leq 1} \sum_{j=n}^{\infty} |\varphi(x^{(j)})| < \delta.
\]
Then, for all finite sets $M \subset \mathbb{N}$ such that $M \subset \{n_\delta, n_\delta + 1, \ldots\}$, it follows that
\[
\left\| \sum_{j \in M} x^{(j)} \right\| \leq \sup_{\|\varphi\|_{\ell^p} \leq 1} \sum_{j \in M} |\varphi(x^{(j)})| \leq \sup_{\|\varphi\|_{\ell^p} \leq 1} \sum_{j=n}^{\infty} |\varphi(x^{(j)})| < \delta.
\]
Therefore, $(x^{(j)})_{j=1}^\infty$ is unconditionally summable.

On the other hand, when $j \in \{j_k, \ldots, 2j_k - 1\}$, for a certain positive integer $k$, we have
\[
\|x^{(j)}\| = \left\| j_k \cdot \left( j_k^{-\left(\frac{1}{r} + \frac{1}{p} + \frac{1}{q}\right)} \right) \right\|^\frac{1}{p} = j_k^{-\left(\frac{1}{r} + \frac{1}{p} + \frac{1}{q}\right)};
\]
otherwise $\|x^{(j)}\| = 0$. Then, we have
\[
\sum_{j=1}^{\infty} \|x^{(j)}\|^r = \sum_{k=1}^{\infty} j_k \cdot \left( j_k^{-\left(\frac{1}{r} + \frac{1}{p} + \frac{1}{q}\right)} \right) = \sum_{k=1}^{\infty} j_k^{-\frac{1}{r} - \frac{1}{p} - \frac{1}{q}} = \sum_{k=1}^{\infty} \alpha^{k(k-1)(1-\frac{1}{r} - \frac{1}{p} - \frac{1}{q})},
\]
that diverges for all $r < 2$, because $\lim_{k \to \infty} \alpha^{k(k-1)(1-\frac{1}{r} - \frac{1}{p} - \frac{1}{q})} \neq 0$ whenever $r < 2$.

Note that, considering, for all positive integers $j$, \[A_j := \left[1/2 + \sqrt{1/4 + \log_{\alpha}((j+1)/2)}, 1/2 + \sqrt{1/4 + \log_{\alpha}j}\right] \cap \mathbb{N},\] each $x^{(j)}$ is precisely
\[
x^{(j)} = \alpha^{\frac{(1-k_j)(k_j(2p+2)+2p)}{2p}} \cdot \alpha^{k_j(k_j-1)} \sum_{s=1}^{\infty} \left( \exp \left( j\alpha^{k_j(1-k_j)} + \alpha^{k_j(1-k_j)} - 1 \right) 2\pi si \right) \cdot e^{\alpha^{k_j(k_j-1)} + s-1})
\]
when $A_j \neq \emptyset$, where $k_j$ is the unique element of $A_j$, and $x^{(j)} = 0$ otherwise. This concludes the proof.

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