A numerical analysis of planar central and balanced configurations in the \((n + 1)\)-body problem with a small mass

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Abstract

Two numerical algorithms for analyzing planar central and balanced configurations in the \((n + 1)\)-body problem with a small mass are presented. The first one relies on a direct solution method of the \((n + 1)\)-body problem by using a stochastic optimization approach, while the second one relies on an analytic-continuation method, which involves the solutions of the \(n\)-body and the restricted \((n + 1)\)-body problem, and the application of a local search procedure to compute the final \((n + 1)\)-body configuration in the neighborhood of the configuration obtained at the first two steps. Some exemplary central and balanced configurations in the cases \(n = 4, 5, 6\) are shown.

1 Introduction

The \(n\)-body problem is the problem of predicting motions of a group of celestial objects interacting with each other gravitationally. A central configuration is an initial configuration such that if the particles were all released with zero velocity, they would all collapse toward the center of mass at the same time. The central configurations are important in \(n\)-body problems because they are
(i) bifurcation points for the topological classification of the coplanar $n$-body problem, and (ii) the starting points for finding some new classes of periodic solutions.

A central configuration in the $(n+1)$-body problem is a configuration which is the limit of central configurations in the full $(n+1)$-body problem as the mass of the $(n+1)$-th particle tends to zero while the remaining particles approach definite positive values (Hagihara 1970). Thus central configuration in the $(n+1)$-body problem must be continuuable to positive masses, and this is achieved by a proper non-degeneracy condition. The restricted three-body problem (both circular and elliptical) was explored extensively during the 19th and 20th century. The problem has undergone extensive treatment both in terms of analytical and numerical tools. Inspired by the restricted three-body problem, the study of the four-body problem was also simplified by firstly considering one of the bodies to be of negligible mass and position of the other three bodies according to different configurations such as equilateral triangular configuration or bicircular formation. In this context, it should be mentioned that Arenstorf (1982) obtained the number of central configurations in the four-body problem by starting with one zero mass and then analytically continue it into positive masses. By also using the method of analytic continuation, Xia (1991) found the exact numbers of central configurations for some open sets of $n$ positive masses for any choice of $n$. Starting with two zero masses, the corresponding central configurations, especially the ones in which two zero masses are at the same point, were obtained. Under certain conditions, these central configurations were analytically continued into a full $n$-body problem with all masses positive.

In this work we analyze planar central and balanced configurations in the $(n+1)$-body problem from a numerical point of view. The case considered here consists in the computation of central and balanced configurations when the masses $m_1, \ldots, m_n, m_{n+1}$ are given, and $m_{n+1}$ is finite but very small as compared to $m_i$, $i = 1, \ldots, n$. For this purpose, we will use (i) a stochastic optimization approach (Doicu et al. 2020) for solving directly the $(n+1)$-body problem, and (ii) a new algorithm relying on an analytic-continuation method.

The paper is organized as follows. A succinct mathematical description of central and balanced configurations in the $n$-body problem, as well as an overview of the stochastic optimization algorithm are provided in Section 2. In Section 3, the analytic-continuation method for computing central and balanced configurations in the $(n+1)$-body problem with a small mass is discussed, and the underlying algorithm is described. Numerical results are given in Section 4, and some conclusions are summarized in Section 5.
2 Planar central and balanced configurations in the $n$-body problem

Consider $n$ point masses $m_1, \ldots, m_n > 0$ with positions $q_1, \ldots, q_n$, where $q_i = (x_i, y_i)^T \in \mathbb{R}^2$. Define the mass and configuration vectors $m = (m_1, \ldots, m_n)^T \in \mathbb{R}^{n+1}$ and $q = (q_1^T, \ldots, q_n^T)^T \in \mathbb{R}^{2n}$, respectively, and let

$$\Delta = \{ q = (q_1^T, \ldots, q_n^T)^T \in \mathbb{R}^{2n} \mid q_i = q_j \text{ for some } i \neq j \},$$

be the subspace of $\mathbb{R}^{2n}$ consisting of collisions.

Let

$$U_n(m, q) = \sum_{1 \leq i < j \leq n} m_i m_j \frac{||q_j - q_i||^3}{||q_j - q_i||},$$

(2.1)

be the Newtonian force function for the configuration $q \in \mathbb{R}^{2n} \setminus \Delta$, where $||\cdot||$ is the Euclidean norm in $\mathbb{R}^2$.

The gradient of $U_n$ with respect to the coordinates of $q_i$ is

$$\nabla_i U_n(m, q) = \left( \frac{\partial U_n}{\partial x_i}(m, q), \frac{\partial U_n}{\partial y_i}(m, q) \right) = \sum_{j=1, j \neq i}^n \frac{m_i m_j}{||q_j - q_i||^4} (q_j - q_i),$$

(2.2)

the center of mass of the system of point masses, and $S \in \mathbb{R}^{2 \times 2}$ a positive definite symmetric matrix.

**Definition 1.** A configuration $q = (q_1^T, \ldots, q_n^T)^T \in \mathbb{R}^{2n} \setminus \Delta$ is said to form a balanced configuration (in short BC($S$)) if there exists a $\lambda \in \mathbb{R} \setminus \{0\}$ such that the equations

$$\nabla_i U_n(m, q) + m_i \lambda S(q_i - c(m, q)) = 0,$$

(2.4)

are satisfied for all $i = 1, \ldots, n$. A configuration $q = (q_1^T, \ldots, q_n^T)^T \in \mathbb{R}^{2n} \setminus \Delta$ is said to form a central configuration (in short CC) if there exists a $\lambda \in \mathbb{R} \setminus \{0\}$ for which Eqs. (2.4) are satisfied with $S = \sigma I_{2 \times 2}$ for some $\sigma \in \mathbb{R}_+$.

Obviously, central configurations are special cases of balanced configurations with e.g. $S = I_{2 \times 2}$.

Consider the diagonal action of $O(2)$ on $\mathbb{R}^{2n}$, defined by

$$O(2) \times \mathbb{R}^{2n} \setminus \Delta \to \mathbb{R}^{2n} \setminus \Delta$$

$$O(q) \mapsto Oq,$$

(2.5)

where

$$Oq = \begin{pmatrix} Oq_1 \\ \vdots \\ Oq_n \end{pmatrix},$$

We note the following result, which is a direct consequence of Definition 1.
Lemma 2. Let \( q \in \mathbb{R}^{2n} \setminus \Delta \) be a BC(S) and \( O \in O(2) \) an orthogonal matrix. Then \( Oq \) is a BC(OSO\(^T\)).

Two direct consequences of this lemma are the following results:

1. If \( q \in \mathbb{R}^{2n} \setminus \Delta \) forms a central configuration and \( O \in O(2) \) is an orthogonal matrix, then \( Oq \) is also a central configuration with the same \( \lambda \).

2. The positive definite \( 2 \times 2 \) matrix \( S \) can be assumed to be diagonal, i.e.,

\[
S = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{pmatrix}
\] (2.6)

with \( \sigma_x, \sigma_y > 0 \) (indeed, if \( q \) is a BC(S) and \( O \in O(2) \) is an orthogonal matrix such that \( OSO^T = \text{diag}(\sigma_x, \sigma_y) \), then \( Oq \) is a BC(OSO\(^T\)) = BC(diag(\sigma_x, \sigma_y))).

In view of these results, we assume in the following that the matrix \( S \) is diagonal, i.e., \( S \) is as in Eq. (2.6) with \( \sigma_x, \sigma_y > 0 \).

For \( \xi, \eta \in \mathbb{R}^2 \), we define their inner product with respect to the positive definite diagonal matrix \( S \) by

\[
\langle \xi, \eta \rangle_S := \xi^T S \eta, \quad ||\xi||_S^2 = \xi^T S \xi,
\] (2.7)

and accordingly, the \( S \)-weighted moment of inertia by

\[
I_S(m, q) = \sum_{j=1}^{n} m_j ||q_j - c||_S^2.
\] (2.8)

In this context, assuming that the configuration \( q \) forms a BC(S), taking the inner product of Eq. (2.4) with \( q_i - c \), and summing up over all \( i = 1, \ldots, n \), we find

\[
\lambda = \frac{U(m, q)}{I_S(m, q)} > 0.
\] (2.9)

Thus, in the definition of balanced configurations, the parameter \( \lambda \) cannot be chosen arbitrary; it depends on \( q \) and \( S \).

In the next step, we consider the \( S \)-normalized configuration space defined by

\[
N(m, S) = \{ q \in \mathbb{R}^{2n} \setminus \Delta \mid c(m, q) = 0, I_S(m, q) = 1 \} \subset \mathbb{R}^{2n}.
\]

Starting from a BC with respect to \( S \), it is possible to normalize this configuration so that the new configuration is a BC with respect to \( S \) in \( N(m, S) \).

Actually, by the change of variable \( \tilde{q}_i = \sqrt{1/I_S(m, q)}(q_i - c(m, q)) \) it can be shown that \( \tilde{q} = (\tilde{q}_1^T, \ldots, \tilde{q}_n^T)^T \) is a BC(S) with the parameter \( \tilde{\lambda} = U_n(m, \tilde{q}) > 0 \), and has the center of mass \( \tilde{c}(\tilde{m}, \tilde{q}) = 0 \) and the \( S \)-weighted moment of inertia \( I_S(m, \tilde{q}) = 1 \); thus, \( \tilde{q} \in N(m, S) \).

According to Moeckel (2014a), for a positive definite symmetric \( 2 \times 2 \) matrix \( S \), a configuration \( q \) is a BC(S) if and only if its corresponding normalized
configuration \( \tilde{q} \in \mathcal{N}(m, S) \) is a critical point of \( \tilde{U}_n = U_n|_{\mathcal{N}(S)} : \mathcal{N}(S) \to \mathbb{R} \). The nullity at a critical point is defined as \( \text{null}(\tilde{q}) := \dim(\ker(H(\tilde{q}))) \), where \( H(\tilde{q}) \) is the Hessian quadratic form of \( \tilde{U}_n \) on \( T_{\tilde{q}}\mathcal{N}(m, S) \). Note that the Hessian \( H(\tilde{q}) \) of \( \tilde{U}_n : \mathcal{N}(m, S) \to \mathbb{R} \) at a critical point \( \tilde{q} \in \text{Crit}(\tilde{U}_n) \) is given by \( H(\tilde{q})v = v^T H(\tilde{q})v \), where
\[
H(\tilde{q}) = D^2U_n(\tilde{q}) + U_n(m, \tilde{q})\hat{S}M
\] (2.10)
and
\[
\hat{S} = \begin{pmatrix} S & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & S \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & \cdots & 0 \\
\cdots & \ddots & \cdots \\
0 & \cdots & m_n \end{pmatrix}.
\]

In the case of central configurations, the normalized configuration space \( \mathcal{N}(m, S) \) and the Newtonian force function \( \tilde{U}_n \) are invariant under the diagonal \( O(2) \)-action. Hence, \( \tilde{U}_n \) descends to a function \( \tilde{U}_n : \mathcal{N}(m, S)/O(2) \to \mathbb{R} \). For a central configuration \( \tilde{q} \in \mathcal{N}(m, S) \), \( \text{null}(\tilde{q}) \geq 1 \) and the equivalence class \( [\tilde{q}] \in \mathcal{N}(m, S)/O(2) \) is a critical point of \( \tilde{U}_n \). Consequently, the Hessian \( \tilde{H}(\tilde{q}) \) of \( \tilde{U}_n \) at \([\tilde{q}]\) is obtained by descending \( H(\tilde{q}) \) to the space \( T_{[\tilde{q}]} (\mathcal{N}(m, S)/O(2)) \).

In Moeckel (2014a) and Doicu et al. (2020), the non-degeneracy of a critical point is defined as follows.

**Definition 3.** Let \( q \) be a BC(S) with \( S = \text{diag}(\sigma_x, \sigma_y) \). Then

Case 1: for \( \sigma_x = \sigma_y \), the configuration \( q \) is called non-degenerate if the Hessian \( \tilde{H}(\tilde{q}) \) is non-degenerate, while

Case 2: for \( \sigma_x \neq \sigma_y \), the configuration \( q \) is called non-degenerate if the Hessian \( H(\tilde{q}) \) is non-degenerate,

where \( \tilde{q} \) represents the corresponding normalized configuration of \( q \).

**Remark 4.** A more convenient way to define non-degenerateness of central and balanced configurations was suggested in Moczurad and Zgliczynski (2019) and Moeckel (2014a). In the case \( \sigma_x = \sigma_y \) a normalized central configuration \( \tilde{q} \) is called non-degenerate if the matrix \( H(\tilde{q}) = D^2U_n(\tilde{q}) + \lambda SM \), where \( \lambda = U_n(m, \tilde{q}) \), is of rank \( 2n - 1 \), while in the case \( \sigma_x \neq \sigma_y \), a normalized configuration \( \tilde{q} \) is called non-degenerate if the matrix \( H(\tilde{q}) = D^2U_n(\tilde{q}) + \lambda SM \) is of full rank \( 2n \). It is easy to check that both definitions are equivalent.

In summary, from Eqs. (2.2) and (2.4) in conjunction with \( \tilde{\lambda} = U_n(m, \tilde{q}) \), \( \bar{c}(m, \tilde{q}) = 0 \), and \( I_{\tilde{S}}(m, \tilde{q}) = 1 \), we infer that the position vectors of a balanced configuration BC(S) satisfies the relative equilibrium equations
\[
I^{(n)}_{\tilde{S}}(m, \tilde{q}) := \sum_{j=1, j \neq i}^n \frac{m_j}{||\tilde{q}_j - \tilde{q}_i||^3} (\tilde{q}_j - \tilde{q}_i) + U_n(m, \tilde{q})\tilde{S}\tilde{q}_i = 0, \quad (2.11)
\]
Algorithm 1 The main steps of the stochastic optimization method.

- Initialize the set of distinct solutions $Q = \emptyset$.
- Generate a set $S = \{s_k\}_{k=1}^{N_s}$ of $N_s$ sample points in the box $B$.

For $k = 1, \ldots, N_s$ do
- Initialize the number of distinct solutions at step $k$, $N_{sol}(k) = |Q|$.
- $s = s_k$.
  - If $s$ is a start point then
    - Start a local search $q = L(s)$.
    - If $q \notin Q$, insert $q$ in the set of distinct solutions $Q$ and update $N_{sol}(k) \leftarrow N_{sol}(k) + 1$.
  - End if
- If $N_{sol}(k)$ does not change within a prescribed number of iteration steps $k^*$ exit

End for

for all $i = 1, \ldots, n$. In the following we will deal only with normalized configurations and renounce on the tilde character "~".

A stochastic optimization algorithm for analyzing planar central and balanced configurations in the $n$-body problem, was designed in Doicu et al. (2020). This numerical approach is a modified version of the Minfinder method of Tsoulos and Lagaris (2006) and is devoted to the solution of the generic system of nonlinear equations

$$ f(q) = 0, \quad (2.12) $$

where $q \in \mathbb{R}^N$ is assumed to lie in the box $B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N$, $f(q) = (f_1(q), f_2(q), \ldots, f_M(q))^T$, $M \geq N$, and $f_i : B \to \mathbb{R}$ are continuous functions. Actually, the solution of the system of equations (2.12) is equivalent to the solution of an optimization problem consisting in the computation of all local minima of the objective function $F : B \subset \mathbb{R}^N \to \mathbb{R}$ given by

$$ F(q) = \frac{1}{2} ||f(q)||^2. $$

The stochastic optimization approach is illustrated in Algorithm 1, where $L$ is a deterministic local optimization method, while $L(s)$ is the point where the local search procedure $L$ terminates when started at point $s$.

The following key elements of the algorithm can be emphasized.

1. Generation of sampling points. A sampling method should create data that accurately represent the underlying function and preserve the statistical characteristics of the complete dataset. The following sampling methods are implemented: (i) pseudo-random number generators (Marsaglia and Tsang 2000; Matsumoto and Nishimura 1998), (ii) chaotic method (Dong et al. 2012; Gao and Wang 2007; Gao and Liu 2012), (iii) low discrepancy method including Halton, Sobol, Niederreiter, Hammersley, and
Faure sequences, (iv) Latin hypercube (McKay et al. 1979), (v) quasi-oppositional differential evolution (Rahnamayan et al. 2006; Rahnamayan et al. 2008), and (vi) centroidal Voronoi tessellation (Du et al. 2010).

2. Selection of a starting point for the local search. A point is considered to be a start point if it is not too close to some already located minimum or another sample, whereby the closeness with a local minimum or some other sample is guided through the so-called typical distance (Tsoulos and Lagaris 2006).

3. Local optimization method. Several optimization software packages for nonlinear least squares and general function minimization are implemented. These include (i) the BFGS algorithm of Byrd et al. (1995), (ii) the TOLMIN algorithm of Powell (1989), (iii) the DQED algorithm due to Hanson and Krogh (1992), and (iv) the optimization algorithms implemented in the Portable, Outstanding, Reliable and Tested (PORT) library. In the latter case, a trust-region method in conjunction with a Gauss-Newton and a Quasi-Newton model are used to compute the trial step (Dennis Jr. et al. 1981a; Dennis Jr. et al. 1981b). These deterministic optimization algorithms can be used in conjunction with several stochastic solvers, as for example: (i) evolutionary strategy, (ii) genetic algorithms, and (iii) simulated annealing.

4. Stopping rule. The algorithm must stop when all minima have been collected with certainty. As default, (i) Bayesian stopping rules (Zieliński 1981; Boender and Kan 1987; Boender and Romeijn 1995), and (ii) the double-box stopping rule proposed by Lagaris and Tsoulos (2008) are implemented. However, because the Bayesian and the double-box stopping rule are not very efficient for this type of applications (in order to capture a large number of solutions, either the tolerances of the stopping rules should be very small or the number of local searches should be extremely large) we adopted an additional termination criterion: if the number of solutions does not change within a prescribed number of iteration steps \( k^* \), the algorithm stops.

To analyze planar central and balanced configurations for the \( n \)-body problem, the stochastic optimization algorithm is used with \( N = 2n \) and \( M = 2n \), and is adapted as follows.

1. According to Eq. \((2.11)\) and for \( S = \text{diag}(\sigma_x, \sigma_y) \), the functions that determine the objective function \( F(q) \) are

\[
\begin{align*}
\overline{f}^{(n)}_{2i-1}(m, q) &= \sum_{j=1}^{n} m_j \frac{x_j - x_i}{\|q_j - q_i\|^3} + U_n(m, q)\sigma_x x_i, \\
\overline{f}^{(n)}_{2i}(m, q) &= \sum_{j=1}^{n} m_j \frac{y_j - y_i}{\|q_j - q_i\|^3} + U_n(m, q)\sigma_y y_i.
\end{align*}
\] (2.13) (2.14)
for $i = 1, \ldots, n$.

2. For $q_i = (x_i, y_i)^T$ and in view of the normalization condition for the moment of inertia $I_S(m, q) = 1$, i.e., $\sum_{i=1}^n m_i q_i^T S q_i = 1$, the following simple bounds on the variables:

$$-l_{xi} \leq x_i \leq l_{xi}, \quad -l_{yi} \leq y_i \leq l_{yi}$$

with

$$l_{xi} = \frac{1}{\sqrt{m_i \sigma_x}}, \quad l_{yi} = \frac{1}{\sqrt{m_i \sigma_y}},$$

are imposed.

3. To specify the set of distinct solutions $Q$, we take into account that for a central configuration, if $q$ is a solution, then any (i) permuted solution $Pq$, (ii) rotated solution of angle $\alpha$, $R_\alpha q$, and (iii) conjugated solutions $C_x q$ and $C_y q$ are also solutions. Here, $P$ and $R_\alpha$ are the permutation and the rotation operator of angle $\alpha$, respectively, while $C_x$ and $C_y$ stand for the reflection operators with respect to the $x$- and $y$-axis, respectively. For a balanced configuration, if $q$ is a solution, then (i) any permuted solution, (ii) a solution rotated by $\alpha = \pi$, and (iii) any conjugated solutions are also solutions.

4. The decision that a solution $q_i$, computed by means of a local optimization method will be included in the set of (distinct) solutions $Q = \{q_i\}_{i=1}^{N_{sol}}$ is taken according to the following rule: if (i) the objective function at $q_i$ is smaller than a prescribed tolerance and (ii) the ordered set of mutual distances $\{R_{ij}\}$ corresponding to $q_i$ does not coincides with the ordered set of mutual distances $\{R'_{ij}\}$ corresponding to any $q' \in Q$, then $q_i$ is inserted in the set of solutions $Q$.

5. In the post-processing stage, several solution tests have been incorporated. These are related to the fulfillment of the normalization condition for the moment of inertia, the center-of-mass equation, the Albouy-Chenciner equations (Albouy and Chenciner 1998), the Morse equality, and the uniqueness of the solutions. Note that the solution uniqueness is checked by means of an approach based on the Krawczyk operator method (Lee and Santoprete 2009; Moczurad and Zgliczynski 2019, 2020).

3 Planar central and balanced configurations in the $(n + 1)$-body problem with a small mass

Let $(q_1^T, \ldots, q_n^T, q_{n+1}^T)^T$ be a normalized BC(S) for the $(n + 1)$-body problem with masses $(m_1, \ldots, m_n, m_{n+1})^T$, that is, the configuration satisfies the relative
equilibrium equations

\[ f_i^{(n+1)}(m, m_{n+1}, q, q_{n+1}) := \sum_{j=1, j \neq i}^{n+1} \frac{m_j}{||q_j - q_i||^3}(q_j - q_i) + U_{n+1}(m, m_{n+1}, q, q_{n+1})S q_i = 0, \] (3.1)

for all \( i = 1, \ldots, n+1 \), where \( m = (m_1, \ldots, m_n)^T \), \( q = (q_1^T, \ldots, q_n^T)^T \), and

\[ U_{n+1}(m, m_{n+1}, q, q_{n+1}) = U_n(m, q) + \sum_{i=1}^{n} \frac{m_i m_{n+1}}{||q_{n+1} - q_i||}. \] (3.2)

In particular, for \( i = 1, \ldots, n \), the system of equations (2.11) reads as

\[ f_i^{(n+1)}(m, m_{n+1}, q, q_{n+1}) := \frac{m_{n+1}}{||q_{n+1} - q_i||^3}(q_{n+1} - q_i) + \sum_{j=1, j \neq i}^{n} \frac{m_j}{||q_j - q_i||^3}(q_j - q_i) + U_{n+1}(m, m_{n+1}, q, q_{n+1})S q_i = 0, \] (3.3)

while for \( i = n+1 \), we have

\[ f_{n+1}^{(n+1)}(m, m_{n+1}, q, q_{n+1}) := \sum_{j=1}^{n} \frac{m_j}{||q_j - q_{n+1}||^3}(q_j - q_{n+1}) + U_{n+1}(m, m_{n+1}, q, q_{n+1})S q_{n+1} = 0. \] (3.4)

The problem that we intend to solve consists in the computation of the balanced and central configurations when the masses \( m_1, \ldots, m_n, m_{n+1} \) are specified and \( m_{n+1} \ll m_i \) for all \( i = 1, \ldots, n \).

A first option is to solve the \((n+1)\)-body problem directly by using the above stochastic optimization algorithm. However in this case, the standard algorithm should be slightly changed. The reason for this change is that the bounds \( l_{x,n+1} \) and \( l_{y,n+1} \) computed as in Eq. (2.16) are very large (because \( m_{n+1} \) is very small), and so, a large number of sampling points is required to find as many solutions as possible. Instead, the choice \( l_{x,n+1} = 2 \max_{i=1,\ldots,n} \{l_{xi}\} \) and \( l_{y,n+1} = 2 \max_{i=1,\ldots,n} \{l_{yi}\} \) leads to a substantial reduction of the computational time.

A second option is to design a numerical algorithm relying on an analytic-continuation result. Let \( q = (q_1^T, \ldots, q_n^T)^T \) be a normalized BC(S) with masses \( m = (m_1, \ldots, m_n)^T \), i.e., the configuration \( q \) satisfies the relative equilibrium equations

\[ f_i^{(n)}(m, q) = 0, \] (3.5)

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for all $i = 1, \ldots, n$. Consider the function

$$V_{Sn}(m, q, q_{n+1}) = \sum_{j=1}^{n} \frac{m_j}{||q_j - q_{n+1}||} + \frac{1}{2} U_n(m, q)q_{n+1}^T S q_{n+1}. \quad (3.6)$$

**Definition 5.** A critical point $q_{n+1}$ of $V_{S_n}(m, q, \cdot)$, i.e.

$$\nabla_{q_{n+1}} V_{S_n}(m, q, q_{n+1}) = \sum_{j=1}^{n} \frac{m_j}{||q_j - q_{n+1}||^3} (q_j - q_{n+1}) + U_n(m, q)S q_{n+1} = 0, \quad (3.7)$$

is called a BC(S) of the restricted $(n + 1)$-body problem. A BC(S) of the restricted $(n+1)$-body problem $q_{n+1}$ is called non-degenerate if $D_{q_{n+1}}^2 V_{S_n}(m, q, q_{n+1})$ is of full rank $2$.

From Eqs. (3.2)–(3.4), it is readily seen that $U_{n+1}(m, 0, q, q_{n+1}) = U_n(m, q)$, and consequently, if $q$ is a normalized BC(S) with masses $m$, and $q_{n+1}$ is a BC(S) of the restricted $(n + 1)$-body problem, that

$$f_{i}^{(n+1)}(m, 0, q, q_{n+1}) = f_i^{(n)}(m, q) = 0, \quad i = 1, \ldots, n, \quad (3.8)$$

$$f_{n+1}^{(n+1)}(m, 0, q, q_{n+1}) = \nabla_{q_{n+1}} V_{S_n}(m, q, q_{n+1}) = 0. \quad (3.9)$$

The next result is a simplified statement of Proposition 1 given in Xia (1991) adapted to the BC case.

**Proposition 6.** Let $q_0 = (q_0^T, \ldots, q_{0n}^T)^T$ be a non-degenerate normalized BC(S) for the $n$-body problem with masses $m = (m_1, \ldots, m_n)^T$ and $q_{0,n+1}$ a non-degenerate BC(S) for the restricted $(n + 1)$-body problem. Then the configuration $(q_0^T, q_{0n+1}^T)^T = (q_0^T, \ldots, q^{T}_{0n}, q_{0,n+1}^T)^T$ for the masses $(m^T, 0)^T = (m_1, \ldots, m_n, 0)^T$ can be analytically continued to a normalized BC(S) for the $(n + 1)$-body problem for any mass $m_{n+1}$ in an open neighborhood of $0$.

**Proof.** By hypothesis, we have that $f_i^{(n)}(m, q_0) = 0$ for all $i = 1, \ldots, n$, and $\nabla_{q_{n+1}} V_{S_n}(m, q_0, q_{0,n+1}) = 0$. The aim is to show that for $(m^T, m_{n+1})^T$ with $m_{n+1} \in U_+(0)$, where $U_+(0) \subset \mathbb{R}_+$ is an open neighborhood of $0$, there exist $q$ and $q_{n+1}$ in the neighborhoods of $q_0$ and $q_{0,n+1}$, respectively, such that $f_i^{(n+1)}(m, m_{n+1}, q, q_{n+1}) = 0$ for all $i = 1, \ldots, n + 1$. For

$$f_i^{(n+1)}(m, m_{n+1}, q, q_{n+1}) := \sum_{j=1}^{n+1} \frac{m_j}{||q_j - q_i||^3} (q_j - q_i) + \lambda S q_i = 0,$$

with $\lambda = U_{n+1}(m, 0, q, q_{n+1}) = U_n(m, q)$ fixed, consider the function

$$f(m, q, q_{n+1}) : \mathbb{R}_+ \times \mathbb{R}^{2n} \times \mathbb{R}^2 \to \mathbb{R}^{2n+2}.$$  

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defined by

\[ f(m_{n+1}, q, q_{n+1}) = \begin{pmatrix} f^{(n+1)}_1(m, m_{n+1}, q, q_{n+1}) \\ \vdots \\ f^{(n+1)}_{n+1}(m, m_{n+1}, q, q_{n+1}) \end{pmatrix}. \tag{3.10} \]

From Eqs. (3.8)–(3.9), we see that \( f^{(n+1)}_i(m, 0, q_0, q_{0,n+1}) = 0, \) \( i = 1, \ldots, n, \) and \( f^{(n+1)}_{n+1}(m, 0, q_0, q_{0,n+1}) = 0; \) hence, \( f(0, q_0, q_{0,n+1}) = 0. \) Moreover, for \( \sigma_x \neq \sigma_y, \) it can be checked using the non-degeneracy assumption together with Remark 4 that the Jacobian \( D_{(q, q_{n+1})}f(0, q_0, q_{0,n+1}) \)

\[ D_{(q, q_{n+1})}f(0, q_0, q_{0,n+1}) = \begin{pmatrix} Dqf^{(n)}_1(m, q_0) & 0 \\ \vdots & \vdots \\ Dqf^{(n)}_{n+1}(m, q_0) & 0 \end{pmatrix} (D^2_{q_{n+1}}V_{S_n})(m, q_0, q_{0,n+1}) \]

is non-singular, i.e., \( \text{rank}(D_{(q, q_{n+1})}f(0, q_0, q_{0,n+1})) = 2n + 2. \) In this regard, according to the implicit function theorem, there exist an open neighborhood of 0, \( U_+(0) \subset \mathbb{R}_+, \) and a function

\[ g : U_+(0) \to \mathbb{R}^{2n} \times \mathbb{R}^2 \]

such that \( (q_0, q_{0,n+1}) = g(0) \) and \( f(m_{n+1}, g(m_{n+1})) = 0 \) for all \( m_{n+1} \in U_+(0). \)

Thus, \( f^{(n+1)}_i(m, m_{n+1}, g(m_{n+1})) = 0 \) for all \( m_{n+1} \in U_+(0) \) and \( i = 1, \ldots, n + 1. \)

In the case of central configurations \( (\sigma_x = \sigma_y), \) a direct application of the implicit function theorem has to involve an additional reduction procedure, since the rotational direction is the kernel of \( D_{(q, q_{n+1})}f(0, q_0, q_{0,n+1}). \) Instead, here we apply an equivariant version of the implicit function theorem as in (Bettiol et al. (2014)). To prove the Proposition, we first consider the \((n+1)\)th equation of the \((n+1)\)-body problem, i.e.,

\[ f^{(n+1)}_{n+1}(m, m_{n+1}, q, q_{n+1}) = \sum_{j=1}^{n} \frac{m_j}{|q_j - q_{n+1}|^3} (q_j - q_{n+1}) + \lambda S q_{n+1}. \]

Since the Jacobian of \( f^{(n+1)}_{n+1} \) with respect to \( q_{n+1} \) is assumed to be non-degenerate, the implicit function theorem provides a unique real analytic map \( g_0 : U_+(0) \times U(q_0) \to \mathbb{R}^2, \) where \( U_+(0) \subset \mathbb{R}_+ \) and \( U(q_0) \subset \mathbb{R}^{2n} \) are neighborhoods of 0 and \( q_0, \) respectively, such that \( g_0(0, q_0) = q_0, n+1 \) and

\[ f^{(n+1)}_{n+1}(m, m_{n+1}, q, g_0(q)) = 0, \tag{3.11} \]
or equivalently,

\[ \nabla_{q_{n+1}} U_{n+1}(m, m_{n+1}, q_0, q_0(m_{n+1}, q)) + \frac{\lambda}{2} \nabla_{q_{n+1}} I_{n+1}(m, m_{n+1}, q_0, g_0(m_{n+1}, q)) = 0, \]

(3.12)

where

\[ I_{n+1}(m, m_{n+1}, q, q_{n+1}) = \sum_{i=1}^{n} m_i q_i^T q_i + m_{n+1} q_{n+1}^T q_{n+1}. \]

Moreover, using the slice theorem it is possible to show that the map \( g_0 \) extends to an equivariant map with respect to the \( SO(2) \) diagonal action defined on \( U_+(0) \times \hat{U}(q_0) \), where \( \hat{U}(q_0) \) is a \( SO(2) \)-invariant neighborhood of the orbit \( SO(2)q_0 \), i.e., for all \( O \in SO(2) \) and all \((m_{n+1}, q) \in U_+(0) \times \hat{U}(q_0)\), we have \( g_0(m_{n+1}, Oq) = O g_0(m_{n+1}, q) \). Next, for the first \( n \) equations of the \((n+1)\)-body problem, we set

\[ f(m_{n+1}, q) = \begin{pmatrix} f_1^{(n+1)}(m, m_{n+1}, q, g_0(m_{n+1}, q)) \\ \vdots \\ f_n^{(n+1)}(m, m_{n+1}, q, g_0(m_{n+1}, q)) \end{pmatrix}. \]

(3.13)

\( f(m_{n+1}, q) \) has an \( SO(2) \) symmetry, and therefore, the kernel of the Jacobian of \( f(m_{n+1}, q) \) with respect to \( q \) is the rotational direction. Because the implicit function theorem cannot be applied directly, we use an equivariant version of the implicit function theorem due Bettiol et al. (2014). For doing this, we consider the function

\[ f : U_+(0) \times \hat{U}(q_0) \to \mathbb{R}, \]

defined by

\[ f(m_{n+1}, q) := U_{n+1}(m, m_{n+1}, q, g_0(m_{n+1}, q)) + \frac{\lambda}{2} I_{n+1}(m, m_{n+1}, q, g_0(m_{n+1}, q)). \]

By Eq. (3.12) and the definition of \( I_{n+1} \), we have

\[ \nabla_{q} f(m_{n+1}, q) \\
= \nabla_{q} U_{n+1}(m, m_{n+1}, q, g_0(m_{n+1}, q)) + \frac{\lambda}{2} \nabla_{q} I_{n+1}(m, m_{n+1}, q, g_0(m_{n+1}, q)) \\
+ \left( \frac{\partial g_0}{\partial q} \right)^T (\nabla_{q_{n+1}} U_{n+1}(m, m_{n+1}, q, g_0(m_{n+1}, q)) \\
+ \frac{\lambda}{2} \nabla_{q_{n+1}} I_{n+1}(m, m_{n+1}, q, g_0(m_{n+1}, q))) \\
= \nabla_{q} U_{n+1}(m, m_{n+1}, q, g_0(m_{n+1}, q)) + \frac{\lambda}{2} \nabla_{q} I_{n+1}(m, m_{n+1}, q, g_0(m_{n+1}, q)). \]

(3.14)
\[ D^2_q f(m_{n+1}, q) = D^2_q U_{n+1}(m, m_{n+1}, q, g_0(m_{n+1}, q)) \]
\[ + \frac{\lambda}{2} D^2_q I_{n+1}(m, m_{n+1}, q, g_0(m_{n+1}, q)) \]
\[ + m_{n+1} G(q), \] \hspace{1cm} (3.15)

where \( G(q) \) is a \( 2n \times 2n \) matrix valued analytic function. Inspecting Eqs. \((3.13)\) and \((3.14)\), we find \( \nabla_q f(m_{n+1}, q) = f(m_{n+1}, q) \), yielding \( D^2_q f(m_{n+1}, q) = D^2_q I_{n+1}(m_{n+1}, q) \). The non-degeneracy assumption in conjunction with Eq. \((3.15)\) implies that the requirement of the implicit function theorem in \((\text{Bettiol et al.} (2014))\), that is,

\[
\ker (D^2_q f(0, q_0)) = T_{q_0} (SO(2) \cdot q_0)
\]
is fulfilled. As a result, there exists a real analytic map \( g : U_+(0) \to \mathbb{R}^{2n} \) such that \( g(0) = q_0 \) and \( (m_{n+1}, g(m_{n+1}), g_0(m_{n+1}, g(m_{n+1}))) \in U_+ \times \mathbb{R}^{2n} \times \mathbb{R}^2 \) is a central configuration for all \( m_{n+1} \in U_+(0) \). \( \square \)

From a computational point of view, Proposition \( \Box \) shows that a solution of the \((n+1)\)-body problem with given masses \((m^T, m_{n+1})^T = (m_1, \ldots, m_n, m_{n+1})^T\) and \( m_{n+1} \ll m_i \) for all \( i = 1, \ldots, n \), exists in a neighborhood of a configuration formed by the solutions of the \(n\)-body problem and the restricted \((n+1)\)-body problem. In other words, for given masses \((m^T, m_{n+1})^T = (m_1, \ldots, m_n, m_{n+1})^T\) with \( m_{n+1} \ll m_i \) for all \( i = 1, \ldots, n \), \((q_0^T, q_{0,n+1}^T) = (q_0^T, \ldots, q_{0,m}^T, q_{0,n+1}^T)\) can be regarded as an “approximate” BC(\(S\)) for the \((n+1)\)-body problem, while an “exact” BC(\(S\)) can be obtained by solving the \((n+1)\)-body problem \((3.3)-(3.4)\) in a neighborhood of \((q_0^T, q_{0,n+1}^T) = (q_0^T, \ldots, q_{0,m}^T, q_{0,n+1}^T)\) as a result of the analytic-continuation result. It is fully based on the stochastic optimization algorithm, which is used in Step 1 to compute all solutions of the \(n\)-body problem, and in Step 2 to compute all solutions of the restricted \((n+1)\)-body problem. In the second case, corresponding to \( N = M = 2 \), we

1. impose the simple bounds on the variables \((2.15)\), where
\[
l_{x,n+1} = 2 \max_{i=1,\ldots,n} \{l_{xi}\}, \quad l_{y,n+1} = 2 \max_{i=1,\ldots,n} \{l_{yi}\},
\]
and \(l_{xi}\) and \(l_{yi}\) are given by Eq. \((2.16)\),

2. adopt as sampling method, the pseudo-random number generator method, and

3. assume for simplicity, that \(q_1\) and \(q_2\) belong to the set of distinct solutions \(Q\) if \(q_1 \neq q_2\).

Thus, in principle, we do not exclude from the final set of solutions, the rotated and reflected solutions. Finally, in Step 3, only the local search procedure
Algorithm 2 Analytic-continuation algorithm for computing balanced and central configurations for the \((n+1)\)-body problem with a small mass.

**Step 1.** Compute all solutions \(q_0^{(k)} = (q_0^{(k)T}, \ldots, q_n^{(k)T})^T \in \mathbb{R}^{2n}, \ k = 1, \ldots, N_{sol}(n)\) of the \(n\)-body problem (2.11).

**Step 2.** For each \(k\)th solution \(q_0^{(k)}\), compute all solutions \(q_0^{(k,l)} \in \mathbb{R}^2, \ l = 1, \ldots, N_{sol}(k, n)\) of the restricted \((n+1)\)-body problem (3.7).

**Step 3.** For each initial guess

\[
q_0^{(k,l)} = (q_0^{(k)T}, q_{0_{n+1}}^{(k,l)T})^T = (q_0^{(k)T}, \ldots, q_n^{(k)T}, q_{0_{n+1}}^{(k,l)T})^T \in \mathbb{R}^{2n+2},
\]

solve the \((n+1)\)-body problem (3.3)-(3.4) for

\[
q^{(k,l)} = (q_1^{(k)T}, q_{n+1}^{(k,l)T})^T = (q_1^{(k)T}, \ldots, q_n^{(k)T}, q_{n+1}^{(k,l)T})^T
\]

in a neighborhood of \(q_0^{(k,l)}\), e.g., in a box

\[
B_\delta(q_0^{(k,l)}) = \{q^{(k,l)} \mid |x_i^{(k,l)} - x_{0i}^{(k,l)}| \leq \delta|x_{0i}^{(k,l)}|, \ |y_i^{(k,l)} - y_{0i}^{(k,l)}| \leq \delta|y_{0i}^{(k,l)}|, \ i = 1, \ldots, n+1\},
\]

where \(q_i^{(k,l)} = (x_i^{(k,l)}, y_i^{(k,l)})^T, \ q_0^{(k,l)} = (x_{0i}^{(k,l)}, y_{0i}^{(k,l)})^T\), and \(\delta\) is sufficiently small.
\( q^{(k,l)} = \mathcal{L}(q_0^{(k,l)}) \) is applied, and a solution \( q^{(k,l)} \) is accepted if the objective function is below a prescribed tolerance.

The main peculiarities of the algorithm are that (i) the initial guess \( q_0^{(k,l)} = (q_0^{(k)})^T, q_{0n+1}^{(k,l)}T \) corresponds to the case \( m_{n+1} = 0 \) (hence, it does not depend on the mass \( m_{n+1} \)), and (ii) \( B_\delta(q_0^{(k,l)}) \) is a small box around \( q_0^{(k,l)} \). As a result,

1. the number of solutions of the \((n+1)\)-body problem

\[
N_{\text{sol}}(n + 1) = \sum_{k=1}^{N_{\text{sol}}(n)} N_{\text{sol}}(k,n)
\]

is independent on \( m_{n+1} \),

2. the set of solution \( \{q^{(k,l)}\} \) corresponds to a sufficiently small value of \( m_{n+1} \),

3. the solution \( q^{(k,l)} = (q^{(k)})^T, q_{n+1}^{(k,l)}T \) \( \in B_\delta(q_0^{(k,l)}) \) is close to the initial guess \( q_0^{(k,l)} = (q_0^{(k)})^T, q_{0n+1}^{(k,l)}T \), whereby \( q^{(k)} \) is near the solution of the \( n \)-body problem \( q_0^{(k)} \), and \( q_{n+1}^{(k,l)} \) is near the solution of the restricted \((n+1)\)-body problem \( q_{0n+1}^{(k,l)} \), and

4. \( ||q^{(k,l)} - q_0^{(k,l)}|| \to 0 \) as \( m_{n+1} \to 0 \);

Actually, the solution \( q^{(k,l)} \) belongs to the box \( B_\delta(q_0^{(k,l)}) \), and there is no guarantee that for example, when \( m_{n+1} \) is above an upper bound \( m_{n+1} \), there are no other solutions outside the domain \( \cup_{k,l} B_\delta(q_0^{(k,l)}) \). However, the analytic-continuation method is more efficient than the direct method. The reason is that two optimization problems of lower dimensions are solved consecutively (the dimensions are \( M = 2n, N = 2n \) in Step 1 and \( M = 2, N = 2 \) in Step 2), and therefore, the number of sampling points required to capture a large number of solutions is smaller than in the case of the direct method.

4 Numerical simulations

The goal of our numerical analysis is twofold. First, to illustrate some central and balanced configurations for the \((n+1)\)-body problem with a small mass, and second, to analyze the accuracy and efficiency of the analytic-continuation algorithm, and in particular, to provide a numerical verification of the analytic-continuation method, according to which, a solution of the \((n+1)\)-body problem exists in a neighborhood of a configuration formed by the solutions of the \( n \)-body problem and the restricted \((n+1)\)-body problem.

In our simulations we choose \( m_i = m = 0.1 \), for \( i = 1, \ldots, n \), and \( m_{n+1} = \varepsilon m \), where the mass parameter \( \varepsilon \) takes the values \( 10^{-8}, 10^{-9}, \text{and} 10^{-10} \). For central configurations we set \( \sigma_x = \sigma_y = 1.0 \), while for balanced configurations we use \( \sigma_x = 1.0 \) and \( \sigma_y = 0.3 \). The parameter \( \delta \) specifying the dimension of the
box $B_3(q_0^{(k,l)})$ in Step 3 of Algorithm 2 is $5 \times 10^{-2}$. For the $n$-body problem, the number of sample points is $N_s = 10^3$, the number of iteration steps within the number of solutions does not change is $k^* = 200$, and the sampling method is a Faure sequence. For the restricted $(n+1)$-body problem, the number of sample points is $N_s = 9 \times 10^6$, the number of iteration steps within the number of solutions does not change is $k^* = 3000$, and the sampling method is a chaotic method. Note that this large number of sample points is required in order to capture as many solutions as possible.

The deviation of the solution $q^{(k,l)}$ of the $(n+1)$-body problem computed by the analytic-continuation method with respect to the initial guess $q_0^{(k,l)}$, where $k = 1, \ldots, N_{sol}(n)$ and $l = 1, \ldots, N_{sol}(k,n)$, is characterized through the RMS of the absolute error in Cartesian coordinates

$$
\Delta q_{okl} = \sqrt{\frac{1}{n+1} \sum_{i=1}^{n+1} ||q_i^{(k,l)} - q_0^{(k,l)}||^2},
$$

and the average RMS

$$
\Delta q_0 = \sqrt{\frac{1}{N_{sol}(n)} \sum_{k=1}^{N_{sol}(n)} \left( \frac{1}{N_{sol}(k,n)} \sum_{l=1}^{N_{sol}(k,n)} \Delta q_{okl}^2 \right)}.
$$

The deviation of the solution $\hat{q}^{(m)}$ of the $(n+1)$-body problem computed by the direct method with respect to an analytic-continuation solution $q^{(k,l)}$, where $m = 1, \ldots, \hat{N}_{sol}(n+1)$ and $\hat{N}_{sol}(n+1)$ is the number of solutions, is quantified as follows. For each $\hat{q}^{(m)}$, we determine the corresponding solution $q^{(k_0,l_0)}$ computed by the analytic-continuation method as

$$(k_0, l_0) = \arg \min_{(k,l)} \sum_{1 \leq i < j \leq n} |R_{ij}^{(m)} - R_{ij}^{(k,l)}|^2,$$

and accordingly, calculate the RMS of the absolute error in radial distances

$$
\Delta R_m = \sqrt{\frac{1}{N} \sum_{1 \leq i < j \leq n} |\hat{R}_{ij}^{(m)} - R_{ij}^{(k_0,l_0)}|^2},
$$

and the average RMS

$$
\Delta R = \sqrt{\frac{1}{\hat{N}_{sol}(n+1)} \sum_{m=1}^{\hat{N}_{sol}(n+1)} \Delta R_m^2},
$$

where in general, the $R_{ij}$ are the mutual distances of the configuration $q$, and $N = n(n+1)/2$.

The results of our numerical analysis are available at website: https://github.com/AlexandruDoicu/Central-and-Balanced-Configurations-with-a-small-mass. The simulations were performed on a computer Intel Core x86_64 CPU 2.70GHz. The output files contain the following data.
1. Results related to the analytic-continuation method:

(a) the number of solutions of the $n$-body problem $N_{\text{sol}}(n)$, and for each $k$th solution of the $n$-body problem, the number of solutions of the restricted $(n + 1)$-body problem $N_{\text{sol}}(k, n)$;

(b) the (total) number of solutions of the $(n+1)$-body problem $N_{\text{sol}}(n+1)$ and the number of distinct solutions excluding symmetries $N_{\text{sol}}^0(n + 1)$;

(c) for each configuration $q^{(k,l)}$ of the $(n + 1)$-body problem, where $k = 1, \ldots, N_{\text{sol}}(n)$ and $l = 1, \ldots, N_{\text{sol}}(k, n)$: (i) the Cartesian coordinates of the point masses, (ii) the residual of the relative equilibrium equations, and (iii) the RMS of the absolute error in Cartesian coordinates with respect to the initial guess $\Delta q_{0kl}$;

(d) the residual of the normalization condition for the moment of inertia and the Cartesian coordinates of the center of mass.

2. Results related to the direct method:

(a) the number of solutions of the $(n + 1)$-body problem $\hat{N}_{\text{sol}}(n + 1)$;

(b) for each configuration $\hat{q}^{(m)}$ of the $(n + 1)$-body problem, where $m = 1, \ldots, \hat{N}_{\text{sol}}(n + 1)$: (i) the Cartesian coordinates of the point masses, (ii) the residual of the relative equilibrium equations, and (iii) the RMS of the absolute error in radial distances with respect to the analytic-continuation solution $\Delta R_m$;

3. the average RMS of the absolute errors in Cartesian coordinates and mutual distances, and the computational times.

4.1 Central configurations

The central configurations computed by the analytic-continuation method in the cases $n = 4, 5,$ and $6$ are illustrated in Figs. 4.1, 4.2, and 4.3 respectively. The following features are apparent.

1. If a configuration $q_0^{(k)}$ of the $n$-body problem has an axis of symmetry (reflection axis), the corresponding configuration $q^{(k,l)}$ of the $(n + 1)$-body problem inherits this symmetry.

2. The solutions $q^{(k,l)}$ of the $(n + 1)$-body problem plotted in Figs. 4.1, 4.3 include discrete rotated and reflected solutions. The distinct central configurations without these symmetries are illustrated in Figs. 4.4, 4.6. Note that these configurations are almost identical with the configurations delivered by the direct method.

In Table 1, we provide the numbers of central configurations, the RMS errors, and the computational times for the analytic-continuation and the direct method. The following conclusions can be drawn.
Figure 4.1: Central configurations $\mathbf{q}^{(k,l)} = (\mathbf{q}^{(k)} T, \mathbf{q}^{(k,l+1)} T)$ in the case $n = 4$.

The solutions $\mathbf{q}^{(k)}$, corresponding to the solutions $\mathbf{q}^{(k)}_0$ of the $n$-body problem, are marked with filled circles and are shown in each of the four plots. For each $\mathbf{q}^{(k)}$, the solutions $\mathbf{q}^{(k,l)}_{n+1}$, corresponding to the restricted $(n+1)$-body problem, are marked with open circles. The number of configurations for the $n$-body problem is $N_{\text{sol}}(n) = 4$, while the number of configurations for the $(n+1)$-body problem is $N_{\text{sol}}(n+1) = 38$.

1. For the mass parameter $\varepsilon = 10^{-8}, 10^{-9}, 10^{-10}$, the numbers of solutions do not change, and the number of distinct solutions (excluding discrete rotated and reflected solutions) computed by the analytic-continuation method coincides with the number of solutions computed by the direct method, i.e., $N_{\text{sol}}^0(n+1) = \tilde{N}_{\text{sol}}(n+1)$. This is an indication that the analytic-continuation algorithm presumably delivers all solutions of the $(n+1)$-body problem.

2. The RMS error $\Delta q_0$ decreases with $\varepsilon$; this result suggests that $||\mathbf{q}^{(k,l)} - \mathbf{q}^{(k,l)}_0|| \to 0$ as $m_{n+1} \to 0$.

3. The RMS error $\Delta R$ is small; hence, the solutions corresponding to the analytic-continuation and the direct method are very close.

4. The analytic-continuation method is on average 7 times faster than the direct method.

4.2 Balanced configurations

The balanced configurations computed by the analytic-continuation method in the cases $n = 4, 5,$ and $6$ are illustrated in Figs. 4.7, 4.8, and 4.9 respectively, while the numbers of balanced configurations, the RMS errors, and the computational times for the analytic-continuation and the direct method are given in Table 2. As in the case of central configurations, we are led to the following conclusions.
Figure 4.2: The same as in Fig. 4.1 but for $n = 5$. The number of configurations for the $n$-body problem is $N_{\text{sol}}(n) = 5$, while the corresponding number of configurations for the $(n + 1)$-body problem is $N_{\text{sol}}(n + 1) = 60$.

Figure 4.3: The same as in Fig. 4.1 but for $n = 6$. The number of configurations for the $n$-body problem is $N_{\text{sol}}(n) = 9$, while the number of configurations for the $(n + 1)$-body problem is $N_{\text{sol}}(n + 1) = 131$. 

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Figure 4.4: Distinct central configurations (excluding discrete rotated and reflected solutions) in the case $n = 4$. The number of configurations for the $n$-body problem is $N_{\text{sol}}(n) = 4$, while the number of distinct configurations for the $(n + 1)$-body problem is $N_{\text{sol}}^0(n + 1) = 17$. Note that these configurations also correspond to the direct method.

Figure 4.5: The same as in Fig. 4.4 but for $n = 5$. The number of configurations for the $n$-body problem is $N_{\text{sol}}(n) = 5$, while the number of distinct configurations for the $(n + 1)$-body problem is $N_{\text{sol}}^0(n + 1) = 27$. 

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Figure 4.6: The same as in Fig. 4.4 but for $n = 6$. The number of configurations for the $n$-body problem is $N_{\text{sol}}(n) = 9$, while the number of distinct configurations for the $(n+1)$-body problem is $N_{\text{sol}}^0(n+1) = 55$.

Table 1: Numbers of central configurations, the RMS errors, and the computational times corresponding to the analytic-continuation method and the direct method. Here, $n$ is the number of point masses, $\varepsilon$ the mass parameter, $N_{\text{sol}}(n)$ and $N_{\text{sol}}(n+1)$ the number of central configurations computed by the analytic-continuation method for the $n$- and $(n+1)$-body problems, respectively, $N_{\text{sol}}^0(n+1)$ the number of distinct solutions (excluding discrete rotated and reflected solutions), $\tilde{N}_{\text{sol}}(n+1)$ the number of central configurations computed by the direct method for the $(n+1)$-body problem, $\Delta q_0$ the average RMS of the absolute errors in Cartesian coordinates, and $\Delta R$ the average RMS of the absolute errors in mutual distances. The RMS value $x.yz(e)$ should be understood as $x.yz \times 10^e$, and the computational time is given in minutes:seconds.

| $n$ | $\varepsilon$ | Analytic-continuation method | Direct method |
|-----|---------------|-----------------------------|--------------|
|     |               | $N_{\text{sol}}(n)$ | $N_{\text{sol}}(n+1)$ | $N_{\text{sol}}^0(n+1)$ | RMS $\Delta q_0$ | Time | $\tilde{N}_{\text{sol}}(n+1)$ | RMS $\Delta R$ | Time |
| 4   | $10^{-8}$     | 4                          | 38            | 17            | 3.03(-8) | 0:30 | 17                          | 6.85(-8) | 8:13 |
|     | $10^{-9}$     |                            |               |               | 3.04(-9) | 0:22 |                            | 5.86(-9) | 6:14 |
|     | $10^{-10}$    |                            |               |               | 4.44(-10) | 0:20 |                            | 5.79(-10) | 5:32 |
| 5   | $10^{-8}$     |                            |               |               | 1.09(-8) | 0:54 |                            | 3.21(-9) | 8:20 |
|     | $10^{-9}$     | 5                          | 60            | 27            | 1.10(-9) | 0:53 |                            | 3.41(-10) | 8:31 |
|     | $10^{-10}$    |                            |               |               | 2.40(-10) | 0:53 |                            | 1.03(-10) | 10:18 |
| 6   | $10^{-8}$     |                            |               |               | 4.64(-8) | 2:34 |                            | 1.19(-9) | 17:05 |
|     | $10^{-9}$     | 9                          | 131           | 55            | 4.64(-9) | 2:57 |                            | 1.43(-10) | 15:42 |
|     | $10^{-10}$    |                            |               |               | 4.69(-10) | 2:52 |                            | 2.81(-10) | 14:58 |
The solutions $q^{(k)}$, corresponding to the solutions $q_0^{(k)}$ of the $n$-body problem, are marked with filled circles, and for each $q^{(k)}$, the solutions $q^{(k,l)}_{n+1}$ corresponding to the restricted $(n+1)$-body problem, are marked with open circles. The number of configurations for the $n$-body problem is $N_{\text{sol}}(n) = 7$, while the number of configurations for the $(n+1)$-body problem is $N_{\text{sol}}(n+1) = 79$.

1. If a configuration $q_0^{(k)}$ of the $n$-body problem has an axis of symmetry with respect to the $x$- or $y$-axis, or is symmetric with respect to the origin of the coordinate system, the same happens with the corresponding configuration $q^{(k,l)}$ of the $(n+1)$-body. Parenthetically we note that the number of solutions is significantly higher than that of central configurations.

2. For the mass parameter $\varepsilon = 10^{-8}, 10^{-9}, 10^{-10}$, (i) the number of distinct solutions (excluding reflected solutions) computed by the analytic-continuation method coincides with the number of solutions computed by the direct method, i.e., $N_{\text{sol}}(n+1) = N_{\text{sol}}(n+1)$, (ii) the RMS error $\Delta q_0$ decreases with $\varepsilon$, (iii) the RMS error $\Delta R$ is small, and (iv) the analytic-continuation method is much more efficient than the direct method.
Figure 4.8: The same as in Fig. 4.7 but for \( n = 5 \). The number of configurations for the \( n \)-body problem is \( N_{sol}(n) = 12 \), while the number of configurations for the \((n+1)\)-body problem is \( N_{sol}(n + 1) = 170 \).
Figure 4.9: The same as in Fig. 4.7 but for \( n = 6 \). The number of configurations for the \( n \)-body problem is \( N_{\text{sol}}(n) = 22 \), while the number of configurations for the \((n + 1)\)-body problem is \( N_{\text{sol}}(n + 1) = 366 \).
5 Conclusions

Planar central and balanced configurations in the \((n + 1)\)-body problem with a small mass have been analyzed from a numerical point of view. For this purpose, two algorithms have been designed. The first one relies on a direct solution method of the \((n + 1)\)-body problem by using a stochastic optimization approach, while the second one is based on an analytic-continuation method. The analytic-continuation algorithm involves three computational steps. These include the solutions of the \(n\)-body and the restricted \((n + 1)\)-body problem, and the application of a local search procedure to compute the final \((n + 1)\)-body configuration in the neighborhood of the configuration obtained at the first two steps. Our numerical experiments have showed that

1. if a configuration of the \(n\)-body problem has an axis of symmetry, the corresponding configuration of the \((n + 1)\)-body problem inherits this symmetry,
Table 2: The same as in Table 1 but for balanced configurations. Here, the 
number of distinct solutions computed by the analytic-continuation method 
$N_0(n+1)$ excludes reflected solutions.

| $n$ | $\varepsilon$ | Analytic-continuation method | Direct method |
|-----|---------------|------------------------------|---------------|
|     |               | $N_{sol}(n)$ | $N_{sol}(n+1)$ | RMS $\Delta q_0$ | Time | $N_{sol}(n+1)$ | RMS $\Delta R$ | Time |
|     |               | $N_{sol}(n+1)$ | $N_{sol}(n+1)$ | RMS $\Delta q_0$ | Time | $N_{sol}(n+1)$ | RMS $\Delta R$ | Time |
| $10^{-2}$ | 4 | 7 | 79 | 42 | 1.84(-8) | 0.31 | 2.09(-9) | 0.31 | 42 | 1.80(-9) | 5:47 |
| $10^{-10}$ | 5 | 12 | 170 | 96 | 5.81(-10) | 0.30 | 1.29(-9) | 1.10 | 96 | 8.89(-10) | 9:10 |
| $10^{-5}$ | 6 | 22 | 366 | 210 | 1.04(-8) | 3.91 | 1.07(-9) | 2.19 | 210 | 1.44(-9) | 18:13 |
| $10^{-10}$ |               | 1.52(-10) | 2.28 |               | 5.67(-10) | 18:50 |

2. for sufficiently small values of the mass $m_{n+1}$, both algorithms deliver 
almost identical configurations,

3. the algorithm based on the analytic-continuation method is on average 7 
times faster than the algorithm based on the direct method.

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