Maximum principle for stochastic optimal control problem of forward–backward stochastic difference systems

Shaolin Jia and Haodong Liu

Abstract

In this paper, we study the maximum principle for stochastic optimal control problems of forward–backward stochastic difference systems (FBSΔSs). Two types of FBSΔSs are investigated. The first one is described by a partially coupled forward–backward stochastic difference equation (FBSΔE) and the second one is described by a fully coupled FBSΔE. By adopting an appropriate representation of the product rule and an appropriate formulation of the adjoint process, we deduce the adjoint difference equation. Finally, the maximum principle for this optimal control problem with the control domain being convex is established.

1. Introduction

The maximum principle is one of the principal approaches in solving the optimal control problems. A lot of work has been done on the maximum principle for forward stochastic system. See, for example, Bensoussan (1982); Bismut (1978); Kushner (1972); Peng (1990). Peng also first studied one kind of forward–backward stochastic control system (FBSCS) in Peng (1993) and obtained the maximum principle for this kind of control system with control domain being convex. The FBSCSs have wide applications in many fields. As the stochastic differential recursive utility, which is a generalisation of a standard additive utility, can be regarded as a solution of a backward stochastic differential equation (BSDE). The recursive utility optimisation problem can be described by an optimisation problem for an FBSCS (see Schroder and Skiadas (1999)). Besides, in the dynamic principal-agent problem with unobservable states and actions, the principal’s problem can be formulated as a partial information optimal control problem of an FBSCS (see Williams (2009)). We refer to Chala and Hafayed (2020); Dokuchaev and Zhou (1999); Hu et al. (2018, 2020); Huang and Shi (2012); Huang et al. (2009); Khalilout and Chala (2020); Moon (2020); Xu (1995); Yong (2010); Zhang et al. (2018) for other works on maximum principle for optimisation problems of FBSCSs.

In this paper, we will discuss the maximum principle for optimal control of discrete time systems described by forward–backward stochastic difference equations (FBSΔEs). To the best of our knowledge, there are few results on such optimisation control problems. In fact, the discrete time control systems are of great value in practice. For example, the digital control can be formulated as discrete time control problems, where the sampled data is obtained at discrete instants of time. Besides, the forward–backward stochastic difference system (FBSΔS) can be used for modelling in financial markets. For example, the solution to the backward stochastic differential equation (BSΔE) can be used to construct time-consistent nonlinear expectations (see Cohen & Elliott, 2010, 2011) and be used for pricing in the financial markets (see Bielecki et al., 2015). However, the formulation of BSΔE is quite different from its continuous time counterpart. Many works are devoted to the study of BSΔEs (see, e.g. Allan & Cohen, 2016; An et al., 2013; Bielecki et al., 2015; Cheridito & Stadje, 2013; Cohen & Elliott, 2010, 2011; Stadje, 2010). Based on the driving process, there are mainly two types of formulations of BSΔEs. One is driving by a finite state process which takes values from the basis vectors (as in Allan & Cohen, 2016; Cohen & Elliott, 2010) and the other is driving by a martingale with independent increments (as in Bielecki et al., 2015; Cheridito & Stadje, 2013). For the latter case, the solution of the BSΔE is a triple of processes which is due to the discrete time version of the Kunita–Watanabe decomposition. In this paper, we adopt the second type of formulation to investigate the optimisation problems for FBSΔSs.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)\) be a probability space, and \(W_t\) be a martingale process with independent increments. Define the difference operator \(\Delta\) as \(\Delta V_t = V_{t+1} - V_t\). Here we consider two types of controlled FBSΔSs.

Problem 1.1: [partially coupled system] The controlled system is

\[
\begin{align*}
\Delta X_t &= b(t, X_t, u_t) + \sum_{i=1}^d \sigma_i(t, X_t, u_t) \Delta W^i_t, \\
X_0 &= x_0, \\
\Delta Y_t &= -f(t + 1, X_{t+1}, Y_{t+1}, Z_{t+1}, u_{t+1}) \\
&\quad + Z_t \Delta W_t + \Delta N_t, \\
Y_T &= y_T.
\end{align*}
\]
and the cost functional is

$$J(u(\cdot)) = \mathbb{E}\left[\sum_{t=0}^{T-1} I(t, X_t, Y_t, Z_t, u_t) + h(X_T)\right].$$  \hspace{1cm} (2)$$

**Problem 1.2:** [fully coupled system] The controlled system is

$$\begin{align*}
\Delta X_t &= b(t, X_t, Y_t, Z_t, u_t) + \sum_{i=1}^{d} \sigma_i(t, X_t, Y_t, Z_t, u_t) \Delta W_t^i, \\
X_0 &= x_0, \\
\Delta Y_t &= -f(t+1, X_{t+1}, Y_{t+1}, Z_{t+1}, u_{t+1}) + Z_t \Delta W_t + \Delta N_t, \\
Y_T &= y_T,
\end{align*}$$

and the cost functional is

$$J(u(\cdot)) = \mathbb{E}\left[\sum_{t=0}^{T-1} I(t, X_t, Y_t, Z_t, u_t) + h(X_T)\right].$$  \hspace{1cm} (4)$$

Let \(\{U_t\}_{t\in\{0,1,\ldots,T-1\}}\) be a sequence of nonempty convex subset of \(\mathbb{R}^d\). We denote the set of admissible controls \(U\) by \(U = \{u(\cdot) \in \mathcal{M}^2(0, T - 1; \mathbb{R}^d) \mid u(t) \in U_t\}\). It can be seen that in Problem 1.1, \(b\) and \(\sigma\) do not contain the solution \((Y, Z)\) of the backward equation. This kind of FBS\(\Delta E\) is called the partially coupled FBS\(\Delta E\). Meanwhile, the problem in Problem 1.2 is called the fully coupled FBS\(\Delta E\).

The optimal control problem is to find the optimal control \(u \in U\), such that the optimal control and the corresponding state trajectory can minimise the cost functional \(J(u(\cdot))\). In this paper, we assume the control domain is convex. By making the perturbation of the optimal control at a fixed time point, we obtain the maximum principle for Problems 1.1 and 1.2.

To build the maximum principle, the key step is to find the adjoint process which can be applied to deduce the variational inequality. In Lin and Zhang (2015), the authors studied the maximum principle for a discrete time stochastic optimal control problem in which the state equation is only governed by a forward stochastic difference equation. By applying the Riesz representation theorem, they explicitly obtained the adjoint process and establish the maximum principle. But to solve our problems, we need to construct the adjoint difference equations since generally the adjoint process cannot be obtained explicitly for our case. However, the techniques which are adopted to formulate the adjoint process in the continuous time framework as in Peng (1990, 1993) are not applicable in our discrete time framework. In this paper, we first choose a suitable representation of the product rule

$$\Delta(X_t, Y_t) = \langle \Delta X_t, Y_t \rangle + \langle X_t, \Delta Y_t \rangle + \langle \Delta X_t, \Delta Y_t \rangle,$$

the form of which is

$$\Delta(X_t, Y_t) = \langle X_{t+1}, \Delta Y_t \rangle + \langle \Delta X_t, Y_t \rangle.  \hspace{1cm} (5)$$

Second, we formulate the BS\(\Delta E\) as in (6). In other words, the generator \(f\) of the BS\(\Delta E\) (6) depends on time \(t + 1\). It is worth pointing out that this kind of formulation is just the formulation of the adjoint equations for stochastic optimal control problems (see Lin and Zhang (2015) for the classical case). Based on these two techniques, we can deduce the adjoint difference equations. The readers may refer to Remark 3.6 for more details.

The remainder of this paper is organised as follows. In Section 2, two types of the controlled FBS\(\Delta E\)s are formulated. We deduce the maximum principle for the partially coupled controlled FBS\(\Delta E\)s in Section 3. Finally, we establish the maximum principle for the fully coupled controlled FBS\(\Delta E\) in Section 4.

**2. Preliminaries and model formulation**

Let \(T\) be a deterministic terminal time, and let \(\bar{T} := \{0, 1, \ldots, T\}\). Consider a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})\), with \(\mathcal{F}_0 = \{\emptyset, \Omega\}\) and \(\mathcal{F} = \mathcal{F}_T\). Here we define the difference operator \(\Delta\) as \(\Delta U_t = U_{t+1} - U_t\). Let \(W\) be a fixed \(\mathbb{R}^d\)-valued square integrable martingale process with independent increments, i.e. \(\mathbb{E}[\Delta W_t|\mathcal{F}_t] = \mathbb{E}[\Delta W_t] = 0\) for any \(t \in \{0, \ldots, T - 1\}\). Also we suppose that \(\mathbb{E}[\Delta W_t|\Delta W_s] = I_{ts}\) for any \(t \in \{0, \ldots, T - 1\}\). Here \((\cdot)^+\) denotes vector transposition. We assume that \(\mathcal{F}_t\) is the completion of the \(\sigma\)-algebra generated by the process \(W\) up to time \(t\).

Denote by \(L^2(\mathcal{F}_t; \mathbb{R}^n)\) the set of all \(\mathcal{F}_t\)-measurable square integrable random variables \(X_t\) taking values in \(\mathbb{R}^n\) and by \(\mathcal{M}^2(0, t; \mathbb{R}^n)\) the set of all \(\mathcal{F}_t\)-adapted square integrable processes \(X_t\) taking values in \(\mathbb{R}^n\). Moreover, we define \(e_t = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n\) and mention that an inequality on a vector quantity is to hold componentwise.

Consider the following backward stochastic difference equation (BS\(\Delta E\)):

$$\begin{align*}
\Delta Y_t &= -f(t+1, Y_{t+1}, Z_{t+1}) + Z_t \Delta W_t + \Delta N_t, \\
Y_T &= \eta,
\end{align*}$$

where \(\eta \in L^2(\mathcal{F}_T; \mathbb{R}^n), f : \Omega \times \{1, 2, \ldots, T\} \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^n\).

**Assumption 2.1:** (i) The function \(f(t, y, z)\) is uniformly Lipschitz continuous and independent of \(z\) at \(t = T\), i.e. there exists constants \(c_1, c_2 > 0\), such that for any \(t \in \{1, 2, \ldots, T - 1\}\), \(y_1, y_2 \in \mathbb{R}^n, z_1, z_2 \in \mathbb{R}^{n \times d}\),

\[
\begin{align*}
|f(T, y_1, z_1) - f(T, y_2, z_2)| &\leq c_1 |y_1 - y_2|, \\
|f(t, y_1, z_1) - f(t, y_2, z_2)| &\leq c_1 |y_1 - y_2| \\
+ c_2 \|z_1 - z_2\|, &\quad P - a.s.
\end{align*}
\]

(ii) \(f(t, 0, 0) \in L^2(\mathcal{F}_t; \mathbb{R}^n)\) for any \(t \in \{1, 2, \ldots, T\}\).

**Remark 2.2:** The BS\(\Delta E\) (6) is analogous to the continuous time BSDE driven by a general martingale (cf. El Karoui and Huang (1997)), and the solution is a triple of processes.

**Definition 2.3:** A solution to BS\(\Delta E\) (6) is a triple of processes \((Y, Z, N) \in \mathcal{M}^2(0, T; \mathbb{R}^n) \times \mathcal{M}^2(0, T - 1; \mathbb{R}^{n \times d}) \times \mathcal{M}^2(0, T; \mathbb{R}^n)\) which satisfies equality (6) for all \(t \in \{0, 1, \ldots, T - 1\}\), and \(N\) is a martingale process strongly orthogonal to \(W\).

The following theorem provides the existence and uniqueness result of BS\(\Delta E\) (6). The proof can be seen in Ji and Liu (2019).
**Theorem 2.4:** Suppose that Assumption 2.1 holds. Then for any terminal condition \( \eta \in L^2(F_T; \mathbb{R}^n) \), the BS\(\Delta\)E (6) has a unique adapted solution \((Y, Z, N)\).

Now we consider the control systems (1)–(2) and (3)–(4). Let the coefficients in system (1)–(2) be such that:

\[
\begin{align*}
\phi(t, x, u) & : \Omega \times [0, 1, \ldots, T - 1] \times \mathbb{R}^d \times \mathbb{R}^r \to \mathbb{R}^d, \\
\psi(t, x, u) & : \Omega \times [0, 1, \ldots, T - 1] \times \mathbb{R}^d \times \mathbb{R}^r \to \mathbb{R}, \\
\mu(t, x, u) & : \Omega \times [0, 1, \ldots, T - 1] \times \mathbb{R}^d \times \mathbb{R}^r \to \mathbb{R}. 
\end{align*}
\]

And the coefficients in system (3)–(4) be such that:

\[
\begin{align*}
\phi(t, x, u, z) & : \Omega \times [0, 1, \ldots, T - 1] \times \mathbb{R}^d \times \mathbb{R}^r \to \mathbb{R}^d, \\
\psi(t, x, u, z) & : \Omega \times [0, 1, \ldots, T - 1] \times \mathbb{R}^d \times \mathbb{R}^r \to \mathbb{R}, \\
\mu(t, x, u, z) & : \Omega \times [0, 1, \ldots, T - 1] \times \mathbb{R}^d \times \mathbb{R}^r \to \mathbb{R}. 
\end{align*}
\]

**Remark 2.5:** The cost functional in Peng (1993) consists of three parts: the running cost functional, the terminal cost functional of \(X_T\), the initial cost functional of \(Y_0\). In our formulation, if we take \(l(\omega, x, y_0, z_0, u_0) = y(\omega, Y_0)\), then the cost functional (4) for our discrete time framework can be reduced to the cost functional in Peng (1993) formally.

For system (1)–(2), we assume that:

**Assumption 2.6:** For \( \phi = b, \sigma, f, l, h \), we assume that

1. \( \phi \) is an adapted map, i.e. for any \((x, y, z, u) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^r \), \( \phi(\cdot, t, 0, 0, 0) \in L^2(F_t) \). Moreover, \( \phi(\cdot, t, 0, 0, 0) \in L^2(F_t) \).
2. \( \forall t \in [0, 1, \ldots, T] \), \( \phi(t, \cdot, \cdot, \cdot, \cdot) \) is continuously differentiable with respect to \( x, y, z, u \), and \( \dot{x}, \dot{y}, \dot{z}, \dot{u} \) are uniformly bounded by a positive constant \( P - a.s. \). Also, for \( t = T, f_T = 0, i.e. f \) is independent of \( z \) at time \( T \). Here we use \( z_i \) to represent the \( i \)th column of the matrix \( z \).

Assume that \( G \) is an \( n \times m \) full-rank matrix and let

\[
\lambda = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A(t, \lambda; u) = \begin{pmatrix} -G^2 \phi \\ Gb \\ G\sigma \end{pmatrix} (t, \lambda; u).
\]

For control system (3)–(4), we additionally assume that:

**Assumption 2.7:** For any \( u \in \mathcal{U} \), the coefficients in (3) satisfy the following monotone conditions, i.e. when \( t \in [1, \ldots, T - 1] \),

\[
\begin{align*}
&\langle A(t, \lambda_1; u) - A(t, \lambda_2; u), \lambda_1 - \lambda_2 \rangle \\
&\leq -\beta_1 |G(x_1 - x_2)|^2 - \beta_2 |G^2(y_1 - y_2)|^2 \\
&- \beta_2 \|G^2(z_1 - z_2)\|^2, \quad P - a.s., \quad \forall \lambda_1, \lambda_2 \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n \times d}; \\
\end{align*}
\]

when \( t = T \),

\[
\begin{align*}
&\| -Gf(T, x_1, y_1, z_1, u) + G^2f(T, x_2, y_1, z_1, u), x_1 - x_2 \| \\
&\leq -\beta_1 |G(x_1 - x_2)|^2, \quad P - a.s.; \\
\end{align*}
\]

when \( t = 0 \),

\[
\begin{align*}
&\| Gb(0, \lambda_1; u) Gb(0, \lambda_2; u), y_1 - y_2 \| + \| G\sigma(0, \lambda_1; u) \\
&- G\sigma(0, \lambda_2; u), z_1 - z_2 \| \\
&\leq -\beta_2 |G^2(y_1 - y_2)|^2 - \beta_2 \|G^2(z_1 - z_2)\|^2, \quad P - a.s., \\
\end{align*}
\]

where \( \beta_1 \) and \( \beta_2 \) are given nonnegative constants with \( \beta_1 + \beta_2 \geq 0 \). Moreover we have \( \beta_1 > 0 \) (resp. \( \beta_2 > 0 \)) when \( n > m \) (resp. \( m > n \)).

Besides, in the following, we formally denote \( b(T, x, y, z, u) = 0, \sigma(T, x, y, z, u) = 0, l(T, x, y, z, u) = 0, f(0, x, y, z, u) = 0 \).

### 3. Maximum principle for the partially coupled FBS\(\Delta\)E system

For any \( u \in \mathcal{U} \), it is obvious that there exists a unique solution \( \{X_t\}_{t=0}^T \in \mathcal{M}^2(0, T; \mathbb{R}^m) \) to the forward stochastic difference equation in the system (1). Then, by Theorem 2.4, the backward equation in the system (1) has a unique solution \((Y, Z, N)\) where

\[
Y = \{Y_t\}_{t=0}^T, Z = \{Z_t\}_{t=0}^T, \text{ and } N = \{N_t\}_{t=0}^T.
\]

Suppose that \( \bar{u} = \{\bar{u}_t\}_{t=0}^T \) is an optimal control of problem (1)–(2) and \((\bar{X}, \bar{Y}, \bar{Z})\) is the corresponding optimal trajectory. For a fixed time \( 0 \leq s \leq T \), choose any \( \Delta v \in L^2(F_s; \mathbb{R}^r) \) such that \( \bar{u}_s + \Delta v \) takes values in \( U_s \). For any \( \varepsilon \in [0, 1] \), construct the perturbed admissible control

\[
\bar{u}_t + \delta_t (\bar{u}_t + \varepsilon \Delta v) = \bar{u}_t + \delta_t \varepsilon \Delta v, \quad (7)
\]

where \( \delta_t = 1 \) for \( t = s \), \( \delta_t = 0 \) for \( t \neq s \) and \( t \in [0, 1, \ldots, T] \). Since \( U_t \) is a convex set, \( \{\bar{u}_t + \delta_t \varepsilon \Delta v\}_{t=0}^T \in \mathcal{U} \) is an admissible control. Let \((X^\varepsilon, Y^\varepsilon, Z^\varepsilon, N^\varepsilon)\) be the solution of (1) corresponding to the control \( u^\varepsilon \).

Set

\[
\bar{\phi}(t) = \phi(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t), \quad \phi^\varepsilon(t) = \phi(t, X^\varepsilon_t, Y^\varepsilon_t, Z^\varepsilon_t, \bar{u}_t),
\]

\[
\bar{\phi}^\varepsilon(t) = \phi(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t), \quad \phi^\mu(t) = \phi^\mu(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t),
\]

where \( \phi = b, \sigma, f, l, h \) and \( \mu = x, y, z \) and \( u \).

Then, we have the following estimates.
Lemma 3.1: Under Assumption 2.6, we have
\[
\sup_{0 \leq t \leq T} \mathbb{E} \left| X_t^e - \bar{X}_t \right|^2 \leq C \varepsilon^2 \mathbb{E} |\Delta v|^2. 
\] (9)

Proof: In the following, the positive constant C may change from line to line.
When \( t = 0, \ldots, s, X_t^e = \bar{X}_t \).
When \( t = s + 1 \),
\[
X_{s+1}^e - \bar{X}_{s+1} = \bar{b}'(s) - \bar{b}(s) + \sum_{i=1}^{d} \left[ \sigma_i^e(s) - \sigma_i(s) \right] \Delta W_i^j.
\]
Then,
\[
\mathbb{E} \left| X_{s+1}^e - \bar{X}_{s+1} \right|^2 \leq (d+1) \mathbb{E} \left[ \left| \bar{b}'(s) - \bar{b}(s) \right|^2 + \sum_{i=1}^{d} \left[ \sigma_i^e(s) - \sigma_i(s) \right]^2 \Delta W_i^j \right].
\]
By the boundness of \( b_{ul} \), we have
\[
\mathbb{E} \left[ \left| \bar{b}'(s) - \bar{b}(s) \right|^2 \right] \leq C \varepsilon \mathbb{E} \left[ |\Delta v|^2 \right].
\]
By the boundness of \( \sigma_{iu} \), we have
\[
\mathbb{E} \left[ \left( \sigma_i^e(s) - \sigma_i(s) \right)^2 \right] \leq C \varepsilon^2 \mathbb{E} \left[ |\Delta v|^2 \right],
\]
which leads to
\[
\mathbb{E} \left| X_{s+1}^e - \bar{X}_{s+1} \right|^2 \leq C \varepsilon^2 \mathbb{E} \left[ |\Delta v|^2 \right].
\]
When \( t = s+2, \ldots, T \),
\[
\mathbb{E} \left| X_t^e - \bar{X}_t \right|^2 \leq (d+1) \mathbb{E} \left[ b \left| t - 1, X_{t-1}^e, \bar{u}_{t-1} \right| \right] \left| \bar{b}(t - 1, \bar{X}_{t-1}, \bar{u}_{t-1}) \right|^2 + \sum_{i=1}^{d} \left[ \sigma_i(t - 1, X_{t-1}^e, \bar{u}_{t-1}) \right] \left| \Delta W_i^j \right|^2.
\]
Due to the boundness of \( b_{x}, \sigma_{ix} \), we obtain \( \mathbb{E} \left| X_t^e - \bar{X}_t \right|^2 \leq C \varepsilon \mathbb{E} \left[ |\Delta v|^2 \right] \). Thus by induction we prove the result. \( \blacksquare \)

Let \( \xi = \{ \xi_t \}_{t=0}^{T} \) be the solution to the following difference equation,
\[
\begin{align*}
\Delta \xi_t = & b_x(t) \xi_t + \delta_x b_u(t) \varepsilon |\Delta v| \\
+ & \sum_{i=1}^{d} \left[ \sigma_{ix}(t) \xi_t + \delta_{t \epsilon} \sigma_{iu}(t) \Delta V^j \right] \Delta W_i^j, \\
\xi_0 = & 0.
\end{align*}
\] (10)

It is easy to check that
\[
\sup_{0 \leq t \leq T} \mathbb{E} |\xi_t|^2 \leq C \varepsilon^2 \mathbb{E} |\Delta v|^2,
\] (11)
and we have the following result:

Lemma 3.2: Under Assumption 2.6, we have
\[
\sup_{0 \leq t \leq T} \mathbb{E} \left| X_t^e - \bar{X}_t - \xi_t \right|^2 = o(\varepsilon^2).
\]

Proof: When \( t = 0, \ldots, s, X_t^e = \bar{X}_t \) and \( \xi_t = 0 \) which lead to)
\[
X_{s+1}^e - \bar{X}_{s+1} - \xi_{s+1} = \left[ \tilde{b}_u(s) - b_u(s) \right] \varepsilon |\Delta v| \\
+ \sum_{i=1}^{d} \left[ \tilde{\sigma}_{iu}(s) - \sigma_{iu}(s) \right] \varepsilon |\Delta v| \Delta W_i^j,
\]
where
\[
\tilde{b}_u(s) = \int_{0}^{s} b_u(s, \bar{X}_s, \bar{u}_s + \lambda (u_s' - \bar{u}_s)) \, d\lambda, \\
\tilde{\sigma}_{iu}(s) = \int_{0}^{s} \sigma_{iu}(s, \bar{X}_s, \bar{u}_s + \lambda (u_s' - \bar{u}_s)) \, d\lambda.
\]
Then
\[
\mathbb{E} \left| X_{s+1}^e - \bar{X}_{s+1} - \xi_{s+1} \right|^2 \\
\leq (d+1) \mathbb{E} \left[ \left( \tilde{b}_u(s) - b_u(s) \right) \varepsilon |\Delta v| \right]^2 \\
+ \sum_{i=1}^{d} \left[ \tilde{\sigma}_{iu}(s) - \sigma_{iu}(s) \right] \varepsilon |\Delta v| \Delta W_i^j \\
\leq C \varepsilon \mathbb{E} \left[ \left| \tilde{b}_u(s) - b_u(s) \right|^2 \Delta V^j \right] \\
+ \sum_{i=1}^{d} \left\| \tilde{\sigma}_{iu}(s) - \sigma_{iu}(s) \right\|^2 \left| \Delta v \right|^2 \varepsilon^2.
\]
Since \( \| \tilde{b}_u(s) - b_u(s) \| \to 0 \) and \( \| \tilde{\sigma}_{iu}(s) - \sigma_{iu}(s) \| \to 0 \) as \( \varepsilon \to 0 \), we have
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathbb{E} \left| X_{s+1}^e - \bar{X}_{s+1} - \xi_{s+1} \right|^2 = 0.
\]
When \( t = s+2, \ldots, T \),
\[
X_t^e - \bar{X}_t - \xi_t = \tilde{b}_x(t - 1) \left( X_{t-1}^e - \bar{X}_{t-1} - \xi_{t-1} \right) \\
+ \left[ \tilde{b}_x(t - 1) - b_x(t - 1) \right] \xi_{t-1} \\
+ \sum_{i=1}^{d} \left[ \tilde{\sigma}_{ix}(t - 1) \left( X_{t-1}^e - \bar{X}_{t-1} - \xi_{t-1} \right) \\
+ \left[ \tilde{\sigma}_{ix}(t - 1) - \sigma_{ix}(t - 1) \right] \xi_{t-1} \right] \Delta W_i^j,
\]
where
\[
\tilde{b}_x(t) = \int_{0}^{1} b_x(s, \bar{X}_1 + \lambda (X_s^e - \bar{X}_1), \bar{u}_s) \, d\lambda,
\]
\[\tilde{\sigma}_{ix}(t) = \int_0^1 \sigma_{ix}(t, \tilde{X}_t + \lambda (X^e_{t+1} - \tilde{X}_t), \tilde{u}_t) \, d\lambda.\]

Then
\[
\mathbb{E}|X^e_{t+1} - \tilde{X}_t - \xi_t|^2 \\
\leq CE \left( \left\| \tilde{b}_x(t-1) \right\|^2 |X^e_{t-1} - \tilde{X}_{t-1} - \xi_{t-1}|^2 \\
+ \sum_{i=1}^d \left\| \tilde{\sigma}_{ix}(t-1) \right\|^2 |X^e_{t-1} - \tilde{X}_{t-1} - \xi_{t-1}|^2 \\
+ \sum_{i=1}^d \left\| \tilde{\sigma}_{ix}(t-1) - \sigma_{ix}(t-1) \right\|^2 |\xi_{t-1}|^2 \right).
\]

\[
\tilde{b}_x(t-1) - b_x(t-1) \to 0 \text{ and } \tilde{\sigma}_{ix}(t-1) - \sigma_{ix}(t-1) \to 0 \text{ as } \varepsilon \to 0. \text{ Since } \tilde{b}_x(t-1) \text{ and } \tilde{\sigma}_{ix}(t-1) \text{ are bounded, by the estimation (11), we have}
\]

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathbb{E}|X^e_{t+1} - \tilde{X}_t - \xi_t|^2 = 0.
\]

This completes the proof.

**Lemma 3.3:** Under Assumption 2.6, we have
\[
\sup_{0 \leq t \leq T} \mathbb{E}|Y^e_t - \bar{Y}_t|^2 \leq C \varepsilon^2 \mathbb{E}|\Delta v|^2 .
\]

\[\sup_{0 \leq t \leq T} \mathbb{E} |Z^e_t - \bar{Z}_t|^2 \leq C \varepsilon^2 \mathbb{E}|\Delta v|^2 .
\]

**Proof:** It is obvious that \(Y^e_T - \bar{Y}_T = 0\) at time \(T\).

When \(t = s, \ldots , T - 1\) (if \(s = T\), skip this part), we have
\[
\mathbb{E}[f(t + 1, X^e_{t+1}, Y^e_{t+1}, Z^e_{t+1}, \bar{u}_{t+1}) - f(t + 1, \tilde{X}_{t+1}, \tilde{Y}_{t+1}, \tilde{Z}_{t+1}, \tilde{u}_{t+1})]^2 \\
\leq CE \left[ |X^e_{t+1} - \tilde{X}_{t+1}|^2 + |Y^e_{t+1} - \tilde{Y}_{t+1}|^2 \\
+ |Z^e_{t+1} - \tilde{Z}_{t+1}|^2 \right] \\
\leq CE \left[ |Y^e_{t+1} - \tilde{Y}_{t+1}|^2 + |Z^e_{t+1} - \tilde{Z}_{t+1}|^2 \right] \\
+ C_1 \varepsilon^2 \mathbb{E}|\Delta v|^2.
\]

It yields that
\[
\mathbb{E}|Y^e_t - \tilde{Y}_t|^2 \leq CE \left[ |Y^e_{t+1} - \tilde{Y}_{t+1}|^2 + |Z^e_{t+1} - \tilde{Z}_{t+1}|^2 \right] \\
+ C_2 \varepsilon^2 \mathbb{E}|\Delta v|^2.
\]

Similarly, we have
\[
\mathbb{E}|Z^e_t - \tilde{Z}_t|^2 \leq CE \left[ |Y^e_{t+1} - \tilde{Y}_{t+1}|^2 + |Z^e_{t+1} - \tilde{Z}_{t+1}|^2 \right] \\
+ C_2 \varepsilon^2 \mathbb{E}|\Delta v|^2.
\]

When \(t = s - 1\), by similar analysis,
\[
\mathbb{E}|Y^e_{t+1} - \tilde{Y}_{t+1}|^2 \leq CE \left[ |Y^e_{t+1} - \tilde{Y}_{t+1}|^2 + |Z^e_{t+1} - \tilde{Z}_{t+1}|^2 \right] \\
+ C_2 \varepsilon^2 \mathbb{E}|\Delta v|^2 ,
\]

\[
\mathbb{E}|Z^e_{t+1} - \tilde{Z}_{t+1}|^2 \leq CE \left[ |Y^e_{t+1} - \tilde{Y}_{t+1}|^2 + |Z^e_{t+1} - \tilde{Z}_{t+1}|^2 \right] \\
+ C_2 \varepsilon^2 \mathbb{E}|\Delta v|^2.
\]

(If \(s = T\), we have \(\mathbb{E}|Y^e_{T+1} - \tilde{Y}_T|^2 \leq C \varepsilon^2 \mathbb{E}|\Delta v|^2, \mathbb{E}|Z^e_{T+1} - \tilde{Z}_T|^2 \leq C \varepsilon^2 \mathbb{E}|\Delta v|^2.)

When \(t = 0, \ldots , s - 2\), we have
\[
\mathbb{E}|Y^e_{t+1} - \tilde{Y}_{t+1}|^2 \leq CE \left[ |Y^e_{t+1} - \tilde{Y}_{t+1}|^2 + |Z^e_{t+1} - \tilde{Z}_{t+1}|^2 \right] ,
\]

\[
\mathbb{E}|Z^e_{t+1} - \tilde{Z}_{t+1}|^2 \leq CE \left[ |Y^e_{t+1} - \tilde{Y}_{t+1}|^2 + |Z^e_{t+1} - \tilde{Z}_{t+1}|^2 \right] .
\]

Thus, there exists \(C > 0\), such that for any \(t \in \{0, 1, \ldots , T\}, \)
\[
\mathbb{E}|Y^e_t - \tilde{Y}_t|^2 \leq C \varepsilon^2 \mathbb{E}|\Delta v|^2,
\]

\[
\mathbb{E}|Z^e_t - \tilde{Z}_t|^2 \leq C \varepsilon^2 \mathbb{E}|\Delta v|^2.
\]

This completes the proof.

Let \((\eta, \xi, V)\) be the solution to the following BS\AE,
\[
\Delta \eta_t = -f_x(t + 1) \xi_{t+1} \\
- \sum_{i=1}^d f_{x_i}(t + 1) \xi_{t+1} + \xi_{t+1} \xi_t W_t + \Delta V_t,
\]

\[
\eta_T = 0.
\]

Notice that \(f_s(T) = f_s(T, \tilde{X}_T, \tilde{Y}_T, \tilde{u}_T)\) since \(f\) is independent of \(Z\), also as \(f_s(T, \tilde{u}_T)\).

It is easy to check that
\[
\sup_{0 \leq t \leq T} \mathbb{E} |\eta_t|^2 \leq C \varepsilon^2 \mathbb{E}|\Delta v|^2,
\]

\[
\sup_{0 \leq t \leq T} \mathbb{E} |\xi_t|^2 \leq C \varepsilon^2 \mathbb{E}|\Delta v|^2,
\]

and we have the following result:

**Lemma 3.4:** Under Assumption 2.6, we have
\[
\sup_{0 \leq t \leq T} \mathbb{E} |Y^e_t - \tilde{Y}_t - \eta_t|^2 = o(\varepsilon^2),
\]

\[
\sup_{0 \leq t \leq T} \mathbb{E} |Z^e_t - \tilde{Z}_t - \xi_t|^2 = o(\varepsilon^2).
\]

**Proof:** When \(t = T\), \(Y^e_T - \tilde{Y}_T - \eta_T = 0\).

When \(t \in \{0, 1, \ldots , T - 1\}\), we have
\[
Y^e_t - \tilde{Y}_t - \eta_t
\]
\[\begin{align*}
&= \mathbb{E} \left[ Y_{t+1}^\mu + f^\mu (t + 1) - \bar{Y}_{t+1} \right] \\
&- \bar{f}_t (t + 1) - \eta_{t+1} + f_k (t + 1) \xi_{t+1} \\
&- \delta_{(t+1)} f_u (t + 1) \varepsilon \Delta v | F_t \right] \\
&= \mathbb{E} \left[ Y_{t+1}^\mu - \bar{Y}_{t+1} - \eta_{t+1} + \bar{f}_t (t + 1) \right] \\
&\times \left( X_{t+1}^\mu - \bar{X}_{t+1} + \bar{f}_t (t + 1) (Y_{t+1}^\mu - \bar{Y}_{t+1}) + \delta_{(t+1)} \bar{f}_u (t + 1) \varepsilon \Delta v \right) \\
&\sum_{i=1}^d \bar{f}_z (t + 1) (Z_{t+1}^\mu - \bar{Z}_{t+1} + \xi_{t+1} \eta_{t+1}) \\
&- \delta_{(t+1)} f_u (t + 1) \varepsilon \Delta v | F_t \right],
\end{align*}\]

where

\[\bar{f}_\mu (t) = \int_0^1 f_\mu (t, \bar{X}_t + \lambda (X_t^\mu - \bar{X}_t), \bar{Y}_t + \lambda (Y_t^\mu - \bar{Y}_t), \bar{Z}_t + \lambda (Z_t^\mu - \bar{Z}_t), \bar{u}_t + \lambda (u_t^\mu - \bar{u}_t) \right. \]

for \( \mu = x, y, z \) and \( u \). Then,

\[\mathbb{E} \left[ Y_{t+1}^\mu - \bar{Y}_{t+1} - \eta_{t+1} \right]^2 \]

\[\leq \mathbb{E} \left[ \left( Y_{t+1}^\mu - \bar{Y}_{t+1} - \eta_{t+1} \right)^2 + \bar{f}_x (t + 1) \left( X_{t+1}^\mu - \bar{X}_{t+1} - \xi_{t+1} \right)^2 \right] \\
+ \left[ \left| \bar{f}_x (t + 1) - f_x (t + 1) \xi_{t+1} \right|^2 + \left| \bar{f}_y (t + 1) \left( Y_{t+1}^\mu - \bar{Y}_{t+1} - \eta_{t+1} \right)^2 + \left[ \left| \bar{f}_y (t + 1) - f_y (t + 1) \right|^2 + \left| \bar{f}_z (t + 1) \eta_{t+1} \right|^2 \right] \\
+ \delta_{(t+1)} \left[ \left| \bar{f}_u (t + 1) - f_u (t + 1) \right|^2 \right] \varepsilon \Delta v | F_t \right] \]

and

\[\mathbb{E} \left[ Z_{t+1}^\mu - \bar{Z}_{t+1} - \xi_{t+1} \right]^2 \]

\[\leq \mathbb{E} \left[ \left( Y_{t+1}^\mu - \bar{Y}_{t+1} - \eta_{t+1} \right)^2 + \bar{f}_x (t + 1) \left( X_{t+1}^\mu - \bar{X}_{t+1} - \xi_{t+1} \right)^2 \right] \\
+ \left| \bar{f}_x (t + 1) - f_x (t + 1) \xi_{t+1} \right|^2 + \left| \bar{f}_y (t + 1) \left( Y_{t+1}^\mu - \bar{Y}_{t+1} - \eta_{t+1} \right)^2 \right] + \left| \bar{f}_z (t + 1) \eta_{t+1} \right|^2 \]

\[+ \delta_{(t+1)} \left| \bar{f}_u (t + 1) - f_u (t + 1) \right|^2 \varepsilon \Delta v | F_t \right] \]

Notice that \( \bar{f}_x (t) - f_x (t) \to 0, \bar{f}_y (t) - f_y (t) \to 0, \bar{f}_z (t) - f_z (t) \to 0, \bar{f}_u (t) - f_u (t) \to 0 \) as \( \varepsilon \to 0 \). We obtain that

\[\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E} \left| Y_{t+1}^\mu - \bar{Y}_{t+1} - \eta_{t+1} \right|^2 = 0, \]

\[\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E} \left| Z_{t+1}^\mu - \bar{Z}_{t+1} - \xi_{t+1} \right|^2 = 0. \]

This completes the proof.

By Lemmas 3.2 and 3.4, we have

\[J (u^\varepsilon (\cdot)) - J (\bar{u} (\cdot)) = \mathbb{E} \sum_{i=0}^{T-1} \left[ \left| l_{\xi_i} (t) + l_j (t) \right| + \left| l_{\eta_i} (t) \right| \right] + \mathbb{E} \int_{t=0} \left[ \varepsilon \Delta v \right] + \mathbb{E} \left[ h_k (X_T), \xi_T \right] + o (\varepsilon) \]

Introducing the following adjoint equation:

\[\Delta p_t = -b_x^* (t + 1) p_{t+1} - \sum_{i=1}^d \sigma_{i+t}^* (t + 1) q_{i+t} \eta_{t+1} \]

\[+ f_x^* (t + 1) k_{t+1} + l_k (t + 1) + q_k \Delta W_t + \Delta Q_t, \]

\[\Delta k_t = f_y^* (t) k_t + l_j (t) + \sum_{i=1}^d \left[ f_{z_i}^* (t) k_t + l_{z_i} (t) \right] \Delta W_t^i, \]

\[p_T = -h_k (X_T), \]

\[k_0 = 0, \]

\[(14)\]

where \( W \) and \( Q \) are square integrable martingale processes and \( Q \) is strongly orthogonal to \( W \).

Obviously the forward equation in (14) admits a unique solution \( k \in M^2 (0, T; \mathbb{R}^m) \). Then, based on the solution \( k \), according to Theorem 2.4, the backward equation in (14) has a unique solution \( (p, q, Q) \in M^2 (0, T; \mathbb{R}^m) \times M^2 (0, T - 1; \mathbb{R}^{m \times d}) \times M^2 (0, T; \mathbb{R}^m) \). So FBSDE has a unique solution \( (p, q, Q, k) \).

We obtain the following maximum principle for the optimal control problem (1)-(2).

Define the Hamiltonian function

\[H (\omega, t, u, x, y, z, p, q, k) = b^* (\omega, t, x, u) p + \sum_{i=1}^d \sigma_i^* (\omega, t, x, u) q_i \]

\[+ f^* (\omega, t, x, y, z, u) k - l (\omega, t, x, y, z, u). \]
Theorem 3.5: Suppose that Assumption 2.6 holds. Let \( u \) be an optimal control of the problem (1)-(2), \((\bar{X}, \bar{Y}, \bar{Z})\) be the corresponding optimal trajectory and \((p, q, k)\) be the solution to the adjoint equation (14). Then for any \( t \in \{0, 1, \ldots, T\} \), for any \( v \in U_t \), we have
\[
\langle H_u (t, \bar{u}_t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, p_t, q_t, k_t), v - \bar{u}_t \rangle \leq 0, \quad P - a.s. \quad (15)
\]

\textbf{Proof:} For \( t \in \{0, 1, \ldots, T-1\} \), we have
\[
\Delta \langle \xi_t, p_t \rangle = \langle \xi_{t+1}, \Delta p_t \rangle + \langle \Delta \xi_t, p_t \rangle
\]
\[
= \langle \xi_{t+1}, -b_x (t+1) p_{t+1} \rangle
\]
\[
- \sum_{i=1}^{d} \sigma^a_i (t+1) q_{i,t+1} b_i + f^a_i (t+1) k_{i,t+1} + l_i (t+1)
\]
\[
+ \sum_{j=1}^{d} \left[ \sigma_{xj} (t) \xi_j + \delta_{v} \xi_j \right] \sum_{i=1}^{d} \left[ f_{zi}^a (t) k_i + l_z_i (t) \right] \Delta W^i_{t}
\]
\[
+ \left[ b_x (t) \xi_t + \delta_{v} b_u (t) \right] \epsilon \Delta v, p_t \Phi_t + \Phi_t,
\]

(16)

where
\[
\Phi_t = \langle \xi_t + b_x (t) \xi_t + \delta_{v} b_u (t) \epsilon \Delta v, q_t, \Delta W_t \rangle
\]
\[
+ \sum_{j=1}^{d} \left[ \sigma_{xj} (t) \xi_j + \delta_{v} \xi_j \right], \sum_{i=1}^{d} \left[ f_{zi}^a (t) k_i + l_z_i (t) \right] \Delta W^i_{t}, \Delta Q_t
\]
\[
+ \langle \xi_t + b_x (t) \xi_t + \delta_{v} b_u (t) \epsilon \Delta v, \Delta Q_t \rangle
\]
\[
+ \sum_{j=1}^{d} \left[ \sigma_{xj} (t) \xi_j + \delta_{v} \xi_j \right] \sum_{i=1}^{d} \left[ f_{zi}^a (t) k_i + l_z_i (t) \right] \Delta W^i_{t}, \Delta Q_t
\]

It is obvious that \( \mathbb{E}[\Phi_t] = 0 \). We have
\[
\mathbb{E} \left[ \sum_{j=1}^{d} \sigma_{xj} (t) \xi_j \Delta W^j_{t}, q_t, \Delta W_t \right]
\]
\[
= \mathbb{E} \left[ \sum_{j=1}^{d} \sigma_{xj} (t) \xi_j \Delta W^j_{t}, \sum_{j=1}^{d} q_j \xi_j, \Delta W_t \right]
\]
\[
= \mathbb{E} \left[ \sum_{j=1}^{d} \left[ \xi_j, \sum_{j=1}^{d} \sigma_{xj} (t) q_j \xi_j, \epsilon \mathbb{E}[\Delta W^j_{t}, \Delta W_t] \right] \right]
\]
\[
= \mathbb{E} \left[ \sum_{j=1}^{d} \left[ \xi_j, \sigma_{xj} (t) q_j \xi_j \right] \right]
\]

and
\[
\mathbb{E} \left[ \sum_{j=1}^{d} \delta_{v} \xi_j \epsilon \sum_{i=1}^{d} f_{zi}^a (t) k_i + l_z_i (t) \right]
\]
\[
= \mathbb{E} \left[ \delta_{v} \epsilon \sum_{i=1}^{d} \left[ \sigma_{zi} (t) \Delta v, q_i \xi_i \right] \right]
\]

Similarly, it can be shown that for \( t \in \{0, 1, \ldots, T-1\} \), we have
\[
\Delta \langle \eta_t, k_t \rangle = \langle -f_x (t+1) \xi_{t+1} - f_y (t+1) \eta_{t+1} \rangle
\]
\[
- \sum_{i=1}^{d} \left[ f_{zi}^a (t+1) k_i + l_z_i (t) \right] \Delta W^i_{t}
\]
\[
+ \left[ \eta_t, f^a_w (t) k_t + l_y (t) \right] + \psi_t,
\]

where
\[
\psi_t = \langle \xi_t \Delta W_t, k_t + f^a_w (t), k_t + l_y (t) \rangle
\]
\[
+ \left[ \eta_t, \sum_{i=1}^{d} \left[ f_{zi}^a (t) k_i + l_z_i (t) \right] \Delta W^i_{t}
\]
\[
+ \left[ k_t + f^a_w (t) k_t + l_y (t) \right] \Delta V_t
\]
\[
+ \left( \sum_{i=1}^{d} \left[ f_{zi}^a (t) k_i + l_z_i (t) \right] \Delta W^i_{t}, \Delta V_t \right).
\]

It is easy to check that
\[
\mathbb{E} \left[ \xi_t \Delta W_t, \sum_{i=1}^{d} f_{zi}^a (t) k_i + l_z_i (t) \right] = \mathbb{E} \left[ \sum_{i=1}^{d} \left[ f_{zi} (t) \xi_t e_i, k_i \right] \right],
\]
\[
\mathbb{E} \left[ \xi_t \Delta W_t, \sum_{i=1}^{d} l_z_i (t) \right] = \mathbb{E} \left[ \sum_{i=1}^{d} \left[ l_z_i (t), \xi_t e_i \right] \right].
\]

Then we have
\[
\mathbb{E} \left[ \Delta \left( \langle \xi_t, p_t \rangle + \langle \eta_t, k_t \rangle \right) \right]
\]
\[
= \mathbb{E} \left[ \left[ -b_x (t+1) \xi_{t+1} + p_{t+1} \right] + \left[ b_x (t) \xi_t \right] \right]
\]
\[
- \sum_{i=1}^{d} \left[ \xi_{t+1}, \sigma^a_i (t+1) q_{i,t+1} \right] + \sum_{i=1}^{d} \left[ \xi_i, \sigma^a_{zi} (t) \right] \sum_{i=1}^{d} \left( f_{zi} (t+1) \eta_{t+1} + k_{i,t+1} \right)
\]
\[
- \left[ f_y (t+1) \eta_{t+1} + k_{i,t+1} \right] + \left[ f_y (t) \eta_t + k_t \right]
\]
\[
- \sum_{i=1}^{d} \left[ f_{zi} (t+1) \xi_{t+1} e_i, k_{i,t+1} \right]
\]
\[
+ \sum_{i=1}^{d} \left[ f_{zi} (t) \xi_t e_i, k_t \right] + \left( l_x (t+1), \xi_{t+1} e_i \right) + \left( \eta_t, l_y (t) \right)
\]
\[
+ \sum_{i=1}^{d} \left[ l_z_i (t), \xi_t e_i \right] - \epsilon \left( \delta_{(t+1)} f_{zi} (t+1) \Delta v, k_{i,t+1} \right)
\]
\[
+ \epsilon \left( \delta_{v} b_u (t) \Delta v, p_t \right) + \delta_{v} \epsilon \sum_{i=1}^{d} \left[ \sigma_{zi} (t) \Delta v, q_i \xi_i \right].
\]

Therefore,
\[
- \mathbb{E} \left[ h_x (\bar{X}_t, \bar{X}_t^T) \right]
\]
\[\begin{align*}
&= \mathbb{E} \left[ (\xi_T, p_T) + \langle \eta_T, k_T \rangle - \langle \xi_0, p_0 \rangle - \langle \eta_0, k_0 \rangle \right] \\
&= \sum_{t=0}^{T-1} \mathbb{E} \left[ \langle \xi_t, p_t \rangle + \langle \eta_t, k_t \rangle \right] \\
&= \mathbb{E} \left[ \langle b_x(0) \xi_0, p_0 \rangle + \sum_{i=1}^{d} \langle \xi_0, \sigma_{ix}^a(0) q_0 e_i \rangle \\
&+ \langle f_x(0) \eta_0, k_0 \rangle + \sum_{i=1}^{d} \langle f_x(0) \xi_0 e_i, k_0 \rangle \right] \\
&+ \sum_{t=0}^{T-1} \mathbb{E} \left[ \langle l_x(t), \xi_t \rangle + \langle l_y(t), \eta_t \rangle + \sum_{i=1}^{d} \langle l_z(t), \xi_t e_i \rangle \right] \\
&+ \sum_{t=0}^{T-1} \delta_{t,0} \mathbb{E} \left[ \langle b_y^a(t) p_t, \Delta v \rangle + \sum_{i=1}^{d} \langle \sigma_{yi}^a(t) q_i e_i, \Delta v \rangle \right] \\
&+ \sum_{i=1}^{d} \langle \sigma_{yi}^a(t) q_i e_i, \Delta v \rangle - \langle f_y^a(t) k_t, \Delta v \rangle \right].
\end{align*}\]

Thus it is easy to obtain Equation (15) since \( s \) is taking arbitrarily. This completes the proof.

**Remark 3.6:** In the introduction, we point out that we need a reasonable representation of the product rule and the BSAE. By virtue of these techniques, the term \( \Delta \langle \xi_t, p_t \rangle \) in (16) is represented as \( -b_x(t+1) \xi_{t+1}, p_{t+1} \rangle + \langle b_x(t) \xi_t, p_t \rangle + \cdots \). By summing and rearranging these terms in (18), we obtain the dual relation (19).

When \( f \equiv 0 \), our control system (1)–(2) degenerates to the classical discrete control system which only contains a forward stochastic difference equation as in Lin and Zhang (2015). For this special case, the adjoint equation becomes

\[\begin{align*}
\Delta p_t &= -b_x^a(t+1) p_{t+1} \\
&- \sum_{i=1}^{d} \sigma_{xi}^a(t+1) q_i e_i + l_x(t+1) + q_1 \Delta W_t + \Delta Q_t, \\
p_T &= -h_\xi (\hat{\mathcal{X}}_T),
\end{align*}\]

and the Hamiltonian function becomes

\[H(\omega, t, x, u, p, q) = b^a(\omega, t, x, u) p + \sum_{i=1}^{d} \sigma_{xi}^a(\omega, t, x, u) q_i e_i - l(\omega, t, x, u).\]

The adjoint equation has the following explicit solution:

\[\begin{align*}
p_{T-1} &= -\mathbb{E} \left[ h_\xi (\hat{\mathcal{X}}_{T-1}) | \mathcal{F}_{T-1} \right], \\
q_{T-1} &= -\mathbb{E} \left[ h_\xi (\hat{\mathcal{X}}_{T-1}) (\Delta W_{T-1})^* | \mathcal{F}_{T-1} \right], \\
p_t &= \mathbb{E} \left[ \left[ I + b_x^a(t+1) \right] p_{t+1} - l_x(t+1) + \sum_{i=1}^{d} \sigma_{xi}^a(t+1) q_i e_i | \mathcal{F}_t \right], \\
q_t &= \mathbb{E} \left[ \left[ I + b_x^a(t+1) \right] p_{t+1} - l_x(t+1) + \sum_{i=1}^{d} \sigma_{xi}^a(t+1) q_i e_i (\Delta W_{t})^* | \mathcal{F}_t \right].
\end{align*}\]

which coincides with the results in Lin and Zhang (2015).

**4. Maximum principle for the fully coupled FBS\AE system**

In this section, we suppose \( W \) to be one-dimensional driving process. Let \( \bar{u} = (\bar{u}_t)_{t \geq 0} \) be the optimal control for the control problem (3)–(4) and \( (\bar{X}, \bar{Y}, \bar{Z}) \) be the corresponding optimal trajectory. Note that the existence and uniqueness of \((\bar{X}, \bar{Y}, \bar{Z})\) is guaranteed by the results in Ji and Liu (2019). The perturbed control \( u^\varepsilon \) is the same as (7) and we denote by \((X^\varepsilon, Y^\varepsilon, Z^\varepsilon)\) the corresponding trajectory.

Let

\[\begin{align*}
\bar{X}_t &= X^\varepsilon_t - \bar{X}_t, \\
\bar{Y}_t &= Y^\varepsilon_t - \bar{Y}_t, \\
\bar{Z}_t &= Z^\varepsilon_t - \bar{Z}_t, \\
\bar{N}_t &= N^\varepsilon_t - \bar{N}_t.
\end{align*}\]

Using the similar notations (8) in Section 3, we have

\[\begin{align*}
\Delta \bar{X}_t &= b^\varepsilon(t) - \bar{F}(t) + (\sigma^\varepsilon(t) - \bar{\sigma}(t)) \Delta W_t, \\
\Delta \bar{Y}_t &= -f^\varepsilon(t+1) + \tilde{J}(t+1) + \bar{Z}_t \Delta W_t + \Delta \bar{N}_t, \\
\bar{X}_0 &= 0, \\
\bar{Y}_T &= 0.
\end{align*}\]

**Lemma 4.1:** Under Assumptions 2.6 and 2.7, we have

\[\mathbb{E} \left( \sum_{t=0}^{T} |\bar{X}_t|^2 + \sum_{t=0}^{T} |\bar{Y}_t|^2 + \sum_{t=0}^{T} |\bar{Z}_t|^2 \right) \leq C \varepsilon^2 \mathbb{E} |\Delta v|^2.\]

**Proof:** By (21),

\[0 = \mathbb{E} \langle \mathcal{GX}_T, \hat{\bar{Y}}_T \rangle - \mathbb{E} \langle \mathcal{GX}_0, \hat{\bar{Y}}_0 \rangle = \sum_{t=0}^{T} \Delta \langle \mathcal{GX}_t, \hat{\bar{Y}}_t \rangle \]

\[= \mathbb{E} \sum_{t=0}^{T} \left\langle \left[ \mathcal{GX}_t, -f^\varepsilon(t) + \tilde{J}(t) \right] + \left\langle \hat{\bar{Y}}_t, G\sigma^\varepsilon(t) - G\bar{\sigma}(t) \right\rangle \right\rangle \]

\[= \mathbb{E} \sum_{t=1}^{T} \langle A(t, \bar{X}_t; u_t^\varepsilon) - A(t, \bar{X}_t; u_t), \hat{\bar{X}}_t \rangle \\
&+ \mathbb{E} \langle \mathcal{GX}_T, -f^\varepsilon(T) + \tilde{J}(T) \rangle + \langle \hat{\bar{Y}}_0, G\sigma^\varepsilon(0) - G\bar{\sigma}(0) \rangle.
\]
+ \left[ \mathcal{Z}_0, G\sigma^e (0) - G\bar{\sigma}^e (0) \right] \\
+ \mathbb{E} \sum_{t=0}^{T} \left[ \left( \mathcal{G}\mathcal{X}_t, -\bar{f}^e (t) + \bar{f} (t) \right) \mathcal{Y}_t, \mathcal{G}\sigma^e (t) - \mathcal{G}\bar{\sigma} (t) \right] \\
+ \mathbb{E} \sum_{t=0}^{T-1} \left[ \left( \mathcal{G}\mathcal{X}_t, -\bar{f}^e (s) + \bar{f} (s) \right) \mathcal{Y}_s, \mathcal{G}\sigma^e (s) - \mathcal{G}\bar{\sigma} (s) \right] .

By the monotone condition, we obtain

\[ \mathbb{E} \left[ \sum_{t=0}^{T} |\mathcal{G}\mathcal{X}_t|^2 + \beta_2 \sum_{t=0}^{T} |\mathcal{G}\sigma^e (t)|^2 + \beta_2 \sum_{t=0}^{T-1} |\mathcal{G}\mathcal{G}\mathcal{Z}_t|^2 \right] \leq \mathbb{E} \left[ \sum_{t=0}^{T} |\mathcal{G}\mathcal{X}_t|^2 + |\mathcal{G}\sigma^e (s)|^2 - \mathcal{G}\bar{\sigma} (s) \right] + \frac{K_1}{2} \mathbb{E} |\Delta \mathcal{V}|^2 . \quad (23) \]

where the constant \( K_1 \) depends on the constants \( \alpha, \beta_1, \beta_2 \) and \( G \).

On the other hand, by applying the induction method to the BS\( \Delta \mathcal{E} \) part in Equation (21), we can obtain

\[ \mathbb{E} \left[ \sum_{t=0}^{T} |\mathcal{Y}_t|^2 + \sum_{t=0}^{T-1} |\mathcal{Z}_t|^2 \right] \leq K_2 \mathbb{E} \left[ \sum_{t=0}^{T} |\mathcal{G}\mathcal{Z}_t|^2 + \varepsilon^2 |\Delta \mathcal{V}|^2 \right] , \quad (24) \]

where the constant \( K_2 \) depends on the constants \( \alpha, \beta_1, G \) and \( T \).

Besides, by applying the induction method to the S\( \Delta \mathcal{E} \) part in Equation (21), we can obtain

\[ \mathbb{E} \left[ \sum_{t=0}^{T} |\mathcal{X}_t|^2 \right] \leq K_3 \mathbb{E} \left[ \sum_{t=0}^{T} |\mathcal{G}\mathcal{Z}_t|^2 + \sum_{t=0}^{T-1} |\mathcal{Z}_t|^2 + \varepsilon^2 |\Delta \mathcal{V}|^2 \right] , \quad (25) \]

where the constant \( K_3 \) depends on the constants \( \alpha, \beta_2, G \) and \( T \).

In the case \( \beta_1 > 0 \) (resp. \( \beta_2 > 0 \)), combining Equations (23), (24) (resp. Equations (23), (25)), we always have

\[ \mathbb{E} \left[ \sum_{t=0}^{T} |\mathcal{X}_t|^2 + \sum_{t=0}^{T} |\mathcal{Y}_t|^2 + \sum_{t=0}^{T-1} |\mathcal{Z}_t|^2 \right] \leq C \mathbb{E} |\Delta \mathcal{V}|^2 , \]

where the constant \( C \) depends only on \( \alpha, \beta_1, \beta_2, G \) and \( T \). This completes the proof. \( \blacksquare \)

Next we introduce the following variational equation:

\[ \Delta \xi_t = b_x (t) \xi_t + b_y (t) \eta_t + b_z (t) \zeta_t + \delta_{01} \mathcal{B}_u (t) \varepsilon \Delta \mathcal{V} \]
\[ + (\sigma_x (t) \xi_t + \sigma_y (t) \eta_t + \sigma_z (t) \zeta_t + \delta_{12} \mathcal{A}_u (t) \varepsilon \Delta \mathcal{V}) \Delta \mathcal{W}_t, \]
\[ - \delta_{(t+1)} \mathcal{B}_u (t+1) \varepsilon \Delta \mathcal{V} + \xi_t \Delta \mathcal{W}_t + \Delta \mathcal{V}_t, \]
\[ \xi_0 = 0, \]
\[ \eta_T = 0. \quad (26) \]

By Assumptions 2.6 and 2.7, when \( t \in \{1, \ldots, T - 1\} \),

\[ \begin{pmatrix} b_x (t) \\ b_y (t) \\ b_z (t) \\ \sigma_x (t) \\ \sigma_y (t) \\ \sigma_z (t) \end{pmatrix} \leq - \begin{pmatrix} \beta_1 I_m \\ 0 \\ 0 \\ 0 \\ \beta_2 I_n \\ \beta_3 I_n \end{pmatrix}, \quad P - a.s.; \quad (27) \]

when \( t = 0 \),

\[ \begin{pmatrix} b_x (0) \\ b_y (0) \end{pmatrix} \leq - \begin{pmatrix} \beta_2 I_n \\ 0 \end{pmatrix}, \quad P - a.s.; \quad (28) \]

when \( t = T \),

\[ -f_x (T) \leq -\beta_1 I_m, \quad P - a.s. \quad (29) \]

Thus, the coefficients of (26) satisfy the monotone condition and there exists a unique solution \((\xi, \eta, \zeta, \mathcal{V})\) to (26). Similar to the proof of Lemma 4.1, we have

\[ \mathbb{E} \left[ \sum_{t=0}^{T} |\xi_t|^2 + \sum_{t=0}^{T} |\eta_t|^2 + \sum_{t=0}^{T-1} |\zeta_t|^2 \right] \leq C \varepsilon^2 \mathbb{E} |\Delta \mathcal{V}|^2 . \quad (30) \]

Define

\[ \varphi_\mu (t) = \int_{0}^{t} \varphi_\mu (s) (t, x_t, \lambda (X_t^e - x_t), \bar{Y}_t, \bar{Y}_t, (Y_t^e - \bar{Y}_t), \bar{Z}_t + \lambda (Z_t^e - \bar{Z}_t) + \mathcal{B}_u (t) \varepsilon \Delta \mathcal{V}) d\lambda, \]

where \( \varphi = b, \sigma, f, s, h \) and \( \mu = x, y, z \) and \( u \).

**Lemma 4.2:** Under Assumptions 2.6 and 2.7, we have

\[ \mathbb{E} \left[ \sum_{t=0}^{T} |\mathcal{X}_t - x_t|^2 + \sum_{t=0}^{T} |\mathcal{Y}_t - \eta_t|^2 + \sum_{t=0}^{T-1} |\mathcal{Z}_t - \zeta_t|^2 \right] = o (\varepsilon^2) . \]

**Proof:** Note that

\[ \varphi^e (t) - \varphi (t) = \varphi_x (t) (X_t^e - \bar{X}_t) + \varphi_y (t) (Y_t^e - \bar{Y}_t) + \varphi_z (t) (Z_t^e - \bar{Z}_t) + \delta_{12} \varphi_\mu (t) \varepsilon \Delta \mathcal{V}. \]

Set

\[ \mathcal{X}_t = \bar{X}_t - x_t, \quad \mathcal{Y}_t = \bar{Y}_t - \eta_t, \]

where the constant \( C \) depends only on \( \alpha, \beta_1, \beta_2, G \) and \( T \). This completes the proof. \( \blacksquare \)
Then,
\[
\begin{cases}
\Delta \tilde{X}_t = b_x(t) \tilde{X}_t + b_y(t) \tilde{Y}_t + b_z(t) \tilde{Z}_t + \Lambda_1(t) \\
+ [\sigma_x(t) \tilde{X}_t + \sigma_y(t) \tilde{Y}_t + \sigma_z(t) \tilde{Z}_t + \Lambda_2(t)] \Delta W_t, \\
\Delta \tilde{Y}_t = -f_x(t+1) \tilde{X}_{t+1} - f_y(t+1) \tilde{Y}_{t+1} \\
- f_z(t+1) \tilde{Z}_{t+1} - \Lambda_3(t+1) + \tilde{Z}_t \Delta W_t + \Delta \tilde{N}_t, \\
\tilde{X}_0 = 0, \\
\tilde{Y}_T = 0,
\end{cases}
\]

where
\[
\begin{align*}
\Lambda_1(t) &= (\tilde{b}_x(t) - b_x(t)) \tilde{X}_t + (\tilde{b}_y(t) - b_y(t)) \tilde{Y}_t \\
&\quad + (\tilde{b}_z(t) - b_z(t)) \tilde{Z}_t + \delta_{t0} (\tilde{b}_u(t) - b_u(t)) \varepsilon \Delta v, \\
\Lambda_2(t) &= (\tilde{\sigma}_x(t) - \sigma_x(t)) \tilde{X}_t + (\tilde{\sigma}_y(t) - \sigma_y(t)) \tilde{Y}_t \\
&\quad + (\tilde{\sigma}_z(t) - \sigma_z(t)) \tilde{Z}_t + \delta_{t0} (\tilde{\sigma}_u(t) - \sigma_u(t)) \varepsilon \Delta v, \\
\Lambda_3(t) &= - (\tilde{f}_x(t) - f_x(t)) \tilde{X}_t - (\tilde{f}_y(t) - f_y(t)) \tilde{Y}_t \\
&\quad - (\tilde{f}_z(t) - f_z(t)) \tilde{Z}_t + \delta_{t0} (\tilde{f}_u(t) - f_u(t)) \varepsilon \Delta v.
\end{align*}
\]

Similar to the proof of Lemma 4.1, we have
\[
\begin{align*}
E \left[ \sum_{t=0}^{T} |\tilde{X}_t|^2 + \sum_{t=0}^{T} |\tilde{Y}_t|^2 + \sum_{t=0}^{T-1} |\tilde{Z}_t|^2 \right] \\
\leq C E \left[ \sum_{t=0}^{T-1} |\Lambda_1(t)|^2 + \sum_{t=0}^{T-1} |\Lambda_2(t)|^2 + \sum_{t=1}^{T} |\Lambda_3(t)|^2 \right].
\end{align*}
\]

Note that
\[
E \sum_{t=0}^{T-1} |\Lambda_1(t)|^2 \\
\leq E \sum_{t=0}^{T-1} |\tilde{b}_x(t) - b_x(t)|^2 |\tilde{X}_t|^2 \\
+ E \sum_{t=0}^{T-1} |\tilde{b}_y(t) - b_y(t)|^2 |\tilde{Y}_t|^2 \\
+ E \sum_{t=0}^{T-1} |\tilde{b}_z(t) - b_z(t)|^2 |\tilde{Z}_t|^2 \\
+ \varepsilon^2 E |\tilde{b}_u(s) - b_u(s)|^2 |\Delta v|^2.
\]

When \( \varepsilon \to 0 \), \( ||\tilde{b}_\mu(t) - b_\mu(t)|| \to 0 \) for \( \mu = x, y, z \) and \( u \). Then, by Lemma 4.1,
\[
E \sum_{t=0}^{T-1} |\Lambda_1(t)|^2 = o (\varepsilon^2).
\]

Similar results hold for the other terms in (32). Finally, we have
\[
E \left[ \sum_{t=0}^{T} |\tilde{X}_t|^2 + \sum_{t=0}^{T} |\tilde{Y}_t|^2 + \sum_{t=0}^{T-1} |\tilde{Z}_t|^2 \right] = o (\varepsilon^2).
\]

This completes the proof.

By Lemma 4.2, we obtain
\[
J (u^\varepsilon (\cdot)) - J (\bar{u} (\cdot)) = E \sum_{i=0}^{T-1} \left[ (l_x (t), \xi_t) + (l_y (t), \eta_t) + (l_z (t), \zeta_t) \\
+ \delta_t (l_u(s), \varepsilon \Delta v) + E [h_x (\tilde{X}_T), \xi_T] + o (\varepsilon) \right].
\]

Introduce the following adjoint equation:
\[
\begin{align*}
\Delta p_t &= -b_x^* (t+1) p_{t+1} - \sigma_x^* (t+1) q_{t+1} \\
+ f_x^* (t+1) k_{t+1} + l_x (t+1) + q_t \Delta W_t + \Delta Q_t, \\
\Delta k_t &= f_x^* (t) k_t - b_x^* (t) p_t - \sigma_x^* (t) q_t + l_t (t) \\
+ f_x^* (t) k_t - b_x^* (t) p_t - \sigma_x^* (t) q_t + l_t (t) \Delta W_t, \\
p_T &= -h_x (\tilde{X}_T), \\
k_0 &= 0.
\end{align*}
\]

Define the Hamiltonian function as follows:
\[
H (\omega, t, x, y, z, p, q, k) \\
= b^* (\omega, t, x, y, z, u) p + \sum_{i=1}^{d} \sigma_i^* (\omega, t, x, y, z, u) q e_i \\
- f^* (\omega, t, x, y, z, u) k - l (\omega, t, x, y, z, u).
\]

**Theorem 4.3:** Suppose that Assumptions 2.6 and 2.7 hold. Let \( \bar{u} \) be an optimal control for (3)-(4), \( (\tilde{X}, \tilde{Y}, \tilde{Z}) \) be the corresponding optimal trajectory and \( (p, q, k) \) be the solution to the adjoint Equation (33). Then, for any \( t \in \{0, 1, \ldots, T\} \) and any \( v \in U_t \), we have
\[
\{H_u (t, u, \tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t, p_t, q_t, k_t), v - \tilde{u}_t \} \leq 0, \ P - a.s.
\]

**Proof:** From the expression of \( \xi_t, p_t \) for \( t \in \{0, 1, \ldots, T-1\} \), we have
\[
\Delta \xi_t, p_t \approx \langle \xi_{t+1} + \Delta p_t \rangle + \langle \Delta \xi_t, p_t \rangle \\
= \langle \xi_{t+1} - b_x^* (t+1) p_{t+1} - \sigma_x^* (t+1) q_{t+1} \\
+ f_x^* (t+1) k_{t+1} + l_x (t+1) + q_t \Delta W_t + \Delta Q_t \rangle \\
+ \langle \sigma_x^* (t) \xi_t + \sigma_y^* (t) \eta_t + \sigma_z^* (t) \zeta_t \\
+ \delta_t e \sigma_u (t) \Delta v \rangle \Delta W_t, q_t \Delta W_t \\
+ \langle b_x (t) \tilde{X}_t + b_y (t) \tilde{Y}_t + b_z (t) \tilde{Z}_t \\
+ \delta_t b_u (t) \varepsilon \Delta v, p_t \rangle + \Phi_t,
\]

where
\[
\Phi_t = \langle \xi_t + b_x (t) \xi_t + b_y (t) \eta_t + b_z (t) \zeta_t \\
+ \delta_t b_u (t) \varepsilon \Delta v, q_t \Delta W_t \rangle \\
+ \langle \sigma_x^* (t) \xi_t + \sigma_y^* (t) \eta_t + \sigma_z^* (t) \zeta_t \\
+ \delta_t e \sigma_u (t) \Delta v \rangle \Delta W_t, p_t \rangle \\
+ \langle \xi_t + b_x (t) \xi_t + b_y (t) \eta_t + b_z (t) \zeta_t \\
+ \delta_t b_u (t) \varepsilon \Delta v, q_t \Delta W_t \rangle + \Phi_t.
\]
\[
\Delta \langle \eta_t, k_t \rangle = \langle \Delta \eta_t, k_{t+1} \rangle + \langle \eta_t, \Delta k_t \rangle \\
= (-f_x(t + 1) \xi_{t+1} - f_y(t + 1) \eta_{t+1}) \\
- f_x(t + 1) \xi_{t+1} - \delta_{t+1} \sigma_u(t) (t+1) \Delta v, Q_{t+1}) \\
+ \langle \xi_t, t \rangle \Delta W_t, k_t + b_x^*(t) p_t \\
- \sigma_x^*(t) q_t + l_x(t) \rangle \\
+ \langle \eta_t, f_y^*(t) k_t - b_y^*(t) p_t \rangle \\
- \sigma_y^*(t) q_t + l_y(t) \rangle + \Psi_t,
\]

where

\[
\Psi_t = \left\langle \xi_t \Delta W_t, k_t + f_x^*(t) k_t - b_x^*(t) p_t - \sigma_x^*(t) q_t + l_x(t) \right\rangle \\
+ \langle \eta_t, f_y^*(t) k_t - b_y^*(t) p_t - \sigma_y^*(t) q_t + l_y(t) \rangle \Delta W_t \\
+ \left\langle k_t + f_x^*(t) k_t - b_x^*(t) p_t - \sigma_x^*(t) q_t + l_x(t) , \Delta V_t \right\rangle \\
+ \left\langle f_y^*(t) k_t - b_y^*(t) p_t - \sigma_y^*(t) q_t + l_y(t) \right\rangle \Delta W_t, \Delta V_t.
\]

Furthermore,

\[
E \left[ \langle \sigma_x(t) \xi_t + \sigma_y(t) \eta_t + \sigma_z(t) \xi_t + \delta_{t+1} \sigma_u(t) (t+1) \Delta v, Q_{t+1} \rangle \Delta W_t, k_t + b_x^*(t) p_t - \sigma_x^*(t) q_t + l_x(t) \right]\rangle \\
= E \left[ \langle \sigma_x(t) \xi_t + \sigma_y(t) \eta_t + \sigma_z(t) \xi_t + \delta_{t+1} \sigma_u(t) (t+1) \Delta v, q_t \Delta W_t \rangle \right]\rangle \\
= E \left[ \sum_{i=1}^d \langle \xi_i, \sigma_{x_i}^*(t) q_i \rangle \right]\rangle
\]

and

\[
E \left\langle \xi_t \Delta W_t, f_x^*(t) k_t - b_x^*(t) p_t - \sigma_x^*(t) q_t + l_x(t) \right\rangle \Delta W_t \\
= E \left\langle f_x^*(t) k_t - b_x^*(t) p_t \right\rangle \Delta W_t \\
- \sigma_x^*(t) q_t + l_x(t) \right\rangle E \left[ \Delta W_t^2 | F_t \right]\rangle \\
= E \left[ \sum_{i=1}^d \xi_i c_i \Delta W_t, f_x^*(t) k_t \Delta W_t \right]\rangle \\
= E \left[ \sum_{i=1}^d \left\langle f_x^*(t) \xi_i c_i, k_t \right\rangle \right]\rangle.
\]

Then, we obtain

\[
E \left[ \Delta \left( \langle \xi_t, p_t \rangle + \langle \eta_t, k_t \rangle \right) \right] \\
= E \left[ -b_x(t + 1) \xi_{t+1} + \langle b_x(t) \xi_t, p_t \rangle \\
- \langle \xi_{t+1}, \sigma_x^*(t + 1) q_{t+1} \rangle + \langle \xi_t, \sigma_x^*(t) q_t \rangle \right] \\
- \left\langle f_x(t + 1) \eta_{t+1}, k_t \right\rangle \\
- \left\langle f_x(t + 1) \eta_{t+1}, k_t \right\rangle \\
+ \left\langle l_x(t + 1) \xi_{t+1}, \eta_t \right\rangle + \left\langle l_x(t + 1) \xi_{t+1}, \eta_t \right\rangle \\
+ \delta_{t+1} \sigma_u(t) (t+1) \Delta v, q_t \right]\rangle \\
- E \left[ \delta_{t+1} \sigma_u(t) (t+1) \Delta v, q_t \right]\rangle.
\]

Therefore,

\[
- E \left[ \langle b_x(t) \xi_t, \xi_t \rangle \right] \\
= E \left[ \langle \xi_t, p_t \rangle + \langle \eta_t, k_t \rangle - \langle \xi_0, p_0 \rangle - \langle \eta_0, k_0 \rangle \right] \\
= \sum_{t=0}^{T-1} E \left\langle \left( \xi_t, p_t \right) + \langle \eta_t, k_t \rangle \right\rangle \\
= E \left[ \langle b_x(t) \xi_0, p_0 \rangle + \sum_{i=1}^d \langle \xi_i, \sigma_{x_i}^*(0) q_0 \rangle \\
+ \langle f_x(t) 0) \eta_0, k_0 \rangle + \langle f_x(t) 0) \eta_0, k_0 \rangle \right] \\
+ \sum_{t=0}^{T-1} E \left[ \langle l_x(t) + \eta_t \rangle + \langle l_x(t) + \eta_t \rangle \right] \\
+ \sum_{t=0}^{T} \delta_{t+1} E \left[ \langle b_u(t) p_t, \Delta v \rangle + \langle \sigma_u^*(t) q_t, \Delta v \rangle \\
- \langle f_u(t) k_t, \Delta v \rangle \right].
\]

Notice that \( \xi_0 = 0, k_0 = 0 \). So

\[
E \sum_{t=0}^{T-1} \left[ \langle l_x(t) + \eta_t \rangle + \langle l_x(t) + \eta_t \rangle \right] + E \left[ \sigma_x^*(0) q_0 \right] \\
n = -E \left[ \langle b_u(s) p_s + \sigma_u^*(s) q_s - f_u^*(s) k_s, \Delta v \rangle \right].
\]

Since \( \lim_{s \to 0} \frac{1}{s} [f(u^*(\cdot)) - f(u(\cdot))] \geq 0 \), we obtain

\[
E \left[ \langle b_u^*(s) p_s + \sigma_u^*(s) q_s - f_u^*(s) k_s - l_u(s), \Delta v \rangle \right] \leq 0.
\]

Then, (34) holds due to that \( s \) is taking arbitrarily. This completes the proof. 

Theorem 4.3 gives a necessary condition of the optimal control problem (3)–(4). Next, we discuss the sufficient conditions for optimality by adding additional convexity conditions.

**Theorem 4.4:** Let Assumptions 2.6 and 2.7 hold. Let \( \bar{u} \) be an admissible control for (3)–(4) with \( \bar{X}, \bar{Y}, \bar{Z} \) be the corresponding trajectory and \( (p, q, k) \) be the solution to the adjoint Equation (33). Suppose that \( h(\cdot) \) is convex, \( H(\omega, t, \cdot, \cdot, \cdot, \cdot, p(t), q(t), k(t)) \) is concave for all \( t \in [0, 1, \ldots, T] \) almost surely, and the maximum condition (34) holds. Then \( \bar{u} \) is the optimal control of the problem (3)–(4).

**Proof:** For another admissible control \( v \), denote \( \bar{X}, \bar{Y}, \bar{Z} \) as the corresponding trajectory. By virtue of the same technique above, we can get

\[
E \left[ \Delta \left( \langle p_t, X_t - \bar{X}_t \rangle \right) \right]
\]
\begin{align*}
  &\Delta E = \Delta X_t = AX_t + BY_t + C\bar{Z}_t + Du_t \\
  &+ (\overline{AX}_t + \overline{BY}_t + \overline{CZ}_t + \overline{Du}_t) \Delta W_t, \\
  &\Delta Y_t = \Delta X_{t+1} + \overline{BY}_{t+1} + \overline{CZ}_{t+1} \\
  &+ \overline{Du}_{t+1} + Z_t \Delta W_t + \Delta N_t, \\
  &X_0 = x_0, \\
  &Y_T = y_T,
\end{align*}

and the cost functional is
\[ J(u(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \sum_{t=0}^{T-1} \left[ QX_t^2 + \overline{QY}_t^2 + \overline{QZ}_t^2 + Ru_t^2 \right] + \overline{QX}_T^2 \right]. \]

Here all the coefficients are deterministic real numbers, \{\Delta W_t\} are independent identically distributed with probability \( P(\Delta W_t = 1) = P(\Delta W_t = -1) = \frac{1}{2} \).

In this case, the adjoint equation becomes
\begin{align*}
  \Delta \rho_t &= -Ap_{t+1} - \overline{A}q_{t+1} - \overline{\Delta}k_{t+1} \\
  + QX_{t+1} + q_t \Delta W_t + \Delta N_t, \\
  \Delta k_t &= -\overline{B}k_t - Bp_t - \overline{B}q_t + \overline{Q}Y_t \\
  + [\overline{C}k_t - Cp_t - \overline{C}q_t + \overline{Q}Z_t] \Delta W_t, \\
  p_T &= -\overline{Q}X_T, \\
  k_0 &= 0.
\end{align*}

And the maximum condition is
\[ u_t = R^{-1}(Dp_t + \overline{D}q_t + \overline{D}k_t), \quad p \text{ a.s.} \quad (35) \]

Furthermore, we can provide a numerical method for this problem. Giving an arbitrary control law \{u_t^0\}, the linear state equation and adjoint equation can be solved (see Ji & Liu, 2019). Then we can update the control law as \{u_t^1\} according to Equation (35). Repeat the procedure and we can get the numerical solution for the optimal control \{\hat{u}_t\}. For example, let
\begin{align*}
  A &= 0.2, & B &= -1, & C &= -0.5, & D &= 0.5, \\
  \overline{A} &= 0.1, & \overline{B} &= -0.5, & \overline{C} &= -1, & \overline{D} &= 0.5, \\
  \Delta &= 0.5, & \overline{\Delta} &= -0.2, & \overline{C} &= -0.1, & \overline{\Delta} &= 0.5, \\
  Q &= 1, & R &= 1, & \overline{Q} &= 0, & \overline{Q} &= 1, \\
  \overline{Q} &= 1, & x_0 &= 1, & y_T &= 0, & T &= 9.
\end{align*}

The values of the cost functional at each iteration step are illustrated in Figure 1.

Figure 1 shows that as the iterate times increase, the values of the cost functional decline. And the numerical optimal value \( J(\overline{u}) = 2.1149 \).

Besides, the stochastic trajectories of \((X, Y, Z, u)\) are shown in Figure 2.

5. Conclusion

In this paper, we study the maximum principle for stochastic optimal control problems of forward–backward stochastic difference systems. Through the appropriate formulation of the backward difference equations both in the state equations and in the adjoint equations, we establish the maximum
principle for this optimal control problem. The main
theorems are Theorem 3.5 for the partially coupled FBSΔE and
Theorem 4.3 for the fully coupled FBSΔE. Furthermore some
sufficient conditions are given and an example is provided to
support the results.

Throughout the article, we need the assumption that the
control domain is convex. Usually spike variation method
is adopted to handle the nonconvex control domain case.
However, this method does not work in the discrete time fram-
work. More effort is needed to relax the convex assumption
for the control domain in the discrete-time stochastic con-
trol problems. Besides, according to the maximum principle,
obtaining the feedback representation of the optimal control in
the linear-quadratic FBSΔE control problem is practically useful
in application. We refer to future work that will address these
issues.
Disclosure statement

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