Spacetime torsion changes the polarization but not the speed of gravitational waves

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(Dated: March 13, 2019)

Abstract

Gravitational waves, and multi-messenger astronomy, in particular, have opened up a new window into the Universe. For instance, the recently established experimental fact that electromagnetic and gravitational waves travel at the same speed in vacuum wiped out whole families of alternative gravitational theories. In this paper, we show in a model-independent way that the presence of spacetime torsion affects the polarization, but not the speed, of gravitational waves as compared with a torsionless spacetime. These results are the consequence of a careful examination of the Weitzenböck–Lichnerowicz wave operator, the Weitzenböck identity, their generalizations for spaces with torsion, and the study of the eikonal limit on Riemann–Cartan geometries. The analysis is general enough to include waves for other fields and to discover when torsion affects their polarization.

PACS numbers: 04.50.+h

Keywords: Nonvanishing Torsion, Gravitational Waves, Riemann–Cartan Geometry, Eikonal Approximation.

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I. INTRODUCTION

On August 17, 2017, the detection of a binary neutron star merger heralded a new era of multi-messenger astronomy. On one hand, the LIGO/Virgo collaboration announced the detection of the gravitational wave (GW) signal GW170817 [1], while on the other hand, Fermi and other observatories detected its electromagnetic counterpart across the spectrum [2, 3]. Remarkably, the almost simultaneous observations of both signals set a bound on the GW speed in the sense that the difference between the speed of photons and gravitons must be less than one part in $10^5$. This observation starkly reduces the number of viable alternative theories of gravity, refuting many in the Horndeski family, among others [4–10].

This event dramatically shows that comparing the propagation of different kind of waves on a curved spacetime can be a useful tool to put whole families of theories to the test. The purpose of this article is to develop the tools to do the same with torsion and to test
whether a specific region of spacetime has torsion or not.

The central idea is to study the propagation of different kinds of waves on a background with nonvanishing torsion. This task is nontrivial since, as we will prove in the current article, torsion does not change the dispersion relation, and therefore it does not change the speed of propagation of waves. Therefore, massless waves in the “geometric optics” limit propagate on null geodesics at the speed of light regardless of torsion. However, in some cases, torsion affects the propagation of polarization and “ray density.” In carrying out our analysis, we do not assume any particular theory or Lagrangian but focus exclusively on the wave equation. Then we compare how different kinds of fields propagate on a fixed background with curvature and torsion in the eikonal limit.

The wave equation for different $p$-form fields leads to different behaviors. For instance, a classical electromagnetic wave will be oblivious to torsion (it is “torsion transparent”). In contrast, gravitational waves should have different behavior regarding their polarization.

A spacetime geometry with nonvanishing torsion may have different origins. In the standard Einstein–Cartan–Sciamma–Kibble (ECSK) theory [11–13], torsion stems from a non-vanishing spin tensor, but the effect would be feeble unless we have a region with a very high fermion density (see Ref. [14, Sec. 8.4] and Ref. [15]). Recently, we considered another possible source of torsion in $d = 4$ [16], where we studied Horndeski’s theory in the context of Riemann–Cartan (RC) geometry. In that work, we found that, in general, non-minimal couplings between geometry and scalar fields as well as second derivatives of scalars in the Lagrangian are generic sources of propagating torsion, agreeing with other previous works [17, 18].

Regardless of its origins, detecting torsion appears extremely difficult. Among the Standard Model particles, fermions should interact with torsion [14, 19]. However, the effect is so weak that it is hard to imagine any foreseeable particle physics experiment which might detect it (see, e.g., Ref. [14, Sec. 8.4]). That is why it becomes important to analyze whether gravitational waves could provide an alternative way to test or even rule out the presence of torsion.

This paper is organized as follows. In order to work with the wave operator efficiently on spaces with curvature and torsion, we define some operators and study their properties in Sec. II. After that, in Sec. III we review the well-known Weitzenböck–Lichnerowicz identity relating the de Rham and Beltrami wave operators in the standard torsionless case, but using
the operators introduced in Sec. II. In this section, we revisit some well-known examples (electromagnetic waves and the wave equation for the Riemann curvature) in order to allow the reader to get familiar with the notation. The wave operator that arises from the Einstein equations in the case of nonvanishing torsion \[16, 20\] inspired us to introduce in Sec. IV a generalized wave operator compatible with background geometries with torsion. This wave operator only differs from the standard case for fields with free Lorentz indices on spaces with torsion. In Sec. V, we study the eikonal limit for this operator. We show that the dispersion relation is not affected by torsion, but a nonvanishing torsion changes the propagation of polarization and the conservation of the density of rays in the geometric optics limit.

In Sec. VI we offer our conclusions and a discussion of how these results may apply to the case of gravitational waves, and how they could be used to detect torsion.

II. NOTATION AND SOME BASIC OPERATORS

Let \( M^{(d)} \) be a \( d \)-dimensional manifold endowed with a metric \( g = g_{\mu\nu} dx^\mu \otimes dx^\nu \), and let \( e^a = e^a_\mu dx^\mu \) be its associated basis of orthonormal one-forms, often called the vielbein, defined implicitly by

\[
g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu, \tag{1}\]

where \( \eta_{ab} = \text{diag} (-1, \ldots, -1, +1, \ldots, +1) \), with a signature of \( \eta_+ \) negative directions and \( d-\eta_- \) positive directions. Similarly, we define an orthonormal basis of vectors as \( \vec{e}_a = e_a^\mu \partial_\mu \), satisfying

\[
g^{\mu\nu} = \eta^{ab} e_a^\mu e_b^\nu. \tag{2}\]

We use Greek indices to denote coordinate basis (“spacetime”) components and lowercase Latin indices to denote orthonormal basis (“tangent space”) components.

Given a metric, we can map \( p \)-forms into \( (d-p) \)-forms using the Hodge dual \( * : \Omega^p (M^{(d)}) \to \Omega^{d-p} (M^{(d)}) \), acting on a \( p \)-form \( \alpha \) as

\[
*\alpha = \frac{\sqrt{|g|}}{p! (d-p)!} \alpha^{\mu_1 \cdots \mu_p} e_{\mu_1} \cdots e_{\mu_p} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{d-p}}. \tag{3}\]

In terms of the Hodge and the vielbein forms, we can define the operators \( I^{a_{1} \cdots a_{q}} \) as the map from \( p \)-forms to \( (p-q) \)-forms,

\[
I^{a_{1} \cdots a_{q}} : \Omega^p (M^{(d)}) \to \Omega^{p-q} (M^{(d)}), \tag{4}\]
given by
\[ I_{a_1 \cdots a_q} = (-1)^{(d-p)(p-q)+n-} \ast (e^{a_1} \wedge \cdots \wedge e^{a_q} \wedge \ast). \]  

These operators have many properties and obey an interesting algebra,
\[ I_{a_1 \cdots a_m} I_{b_1 \cdots b_n} = I_{b_1 \cdots b_n a_1 \cdots a_m}. \]

The most useful case is \( q = 1 \),
\[ I^a = (-1)^{d(p-1)+n-} (e^a \wedge \ast), \]
because it satisfies the Leibniz identity,
\[ I_a (\alpha^{(p)} \wedge \beta^{(q)}) = I_a \alpha^{(p)} \wedge \beta^{(q)} + (-1)^p \alpha^{(p)} \wedge I_a \beta^{(q)}. \]

We define the covariant derivative in both bases using the affine connection \( \Gamma^\lambda_{\mu \nu} \) and the spin connection one-form \( \omega^{ab} \). Given a vector \( \vec{V} = V^\mu \partial_\mu = V^a \vec{e}_a \), we have
\[ \nabla \vec{V} = \left( \partial_\mu V^\lambda + \Gamma^\lambda_{\mu \nu} V^\nu \right) dx^\mu \otimes \partial_\lambda, \]
and
\[ D\vec{V} = (dV^a + \omega^a_b V^b) \otimes \vec{e}_a. \]

Demanding that \( \omega^{ab} \) and \( \Gamma^\lambda_{\mu \nu} \) codify the same parallel transport for \( \vec{V} \) implies \( \nabla \vec{V} = D\vec{V} \). This leads us to the “vielbein postulate,”
\[ \partial_\mu e^a_\nu + \omega^a_{b \mu} e^b_\nu - \Gamma^\lambda_{\mu \nu} e^a_\lambda = 0. \]

The vielbein postulate implies the equivalence of using \( D \) or \( \nabla \) on any tensorial zero-form. The same is not true for general \( p \)-forms. On \( p \)-forms, both derivatives have a different action. The covariant derivative \( D \) includes the antisymmetrization wedge operator and maps \( p \)-forms into \( (p+1) \)-forms, \( D : \Omega^p (M^{(d)}) \to \Omega^{p+1} (M^{(d)}) \). However, even more importantly, the operator \( D \) does not perform parallel transport on spacetime indices. For instance, when writing the torsion two-form \( T^a \) as
\[ T^a = De^a = de^a + \omega^a_{b \mu} e^b_\nu, \]
it is important to notice that we are not performing any parallel transport on the “hidden” spacetime index of \( e^a = e^a_\mu dx^\mu \).
In contrast, the $\nabla$-operator performs parallel transport on every index, regardless of kind. This difference between both operators is simple but frequently misunderstood. Once the vielbein postulate is imposed, the choice of basis is irrelevant. This is not about $\nabla$ being a “spacetime” derivative and D a “tangent space” derivative. The difference between $\nabla$ and D is just that the former applies parallel transport on everything that is acts on while the second one is a bit more selective.

For instance, that is why we have

$$D^2 \vec{V} = R^a{}_b V^b \otimes \vec{e}_a,$$

but

$$\nabla \wedge \nabla \vec{V} = \frac{1}{2} \left[ R^\rho{\sigma \mu \nu} V^\sigma - T^\lambda{\mu \nu} \nabla_\lambda V^\rho \right] dx^\mu \wedge dx^\nu \otimes \partial_\rho,$$

where the curvature two-form components are given by

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b,$$

$$= e^a{\rho}e_b{\sigma} \frac{1}{2} R^\rho{\sigma \mu \nu} dx^\mu \wedge dx^\nu,$$

with

$$R^\rho{\sigma \mu \nu} = \partial_\mu \Gamma^\rho{\nu \sigma} - \partial_\nu \Gamma^\rho{\mu \sigma} + \Gamma^\rho{\mu \lambda} \Gamma^\lambda{\nu \sigma} - \Gamma^\rho{\nu \lambda} \Gamma^\lambda{\mu \sigma}, \quad (15)$$

$$T^\lambda{\mu \nu} = \Gamma^\lambda{\mu \nu} - \Gamma^\lambda{\nu \mu}. \quad (16)$$

The difference between D and $\nabla \wedge$ depends on torsion. Indeed, it is possible to show that

$$D = \nabla \wedge + T^c \wedge I_c. \quad (17)$$

It proves useful to define a third kind of covariant derivative besides D and $\nabla$ in order to explore the wave operator in spaces with nonvanishing torsion. The new derivative, denoted by $\mathcal{D}_a$, maps $p$-forms to $p$-forms, $\mathcal{D}_a : \Omega^p (M^{(d)}) \rightarrow \Omega^p (M^{(d)})$, and is defined explicitly by the anticommutator

$$\mathcal{D}^a = I^a D + DI^a. \quad (18)$$

The $\mathcal{D}^a$-operator has many properties, but the most important of them is that it satisfies the Leibniz rule (without any sign correction),

$$\mathcal{D}^a (\alpha^{(p)} \wedge \beta^{(p)}) = \mathcal{D}^a \alpha^{(p)} \wedge \beta^{(p)} + \alpha^{(p)} \wedge \mathcal{D}^a \beta^{(p)}, \quad (19)$$
and that it satisfies a commutation relation with the Hodge dual given by

\[[\mathcal{D}_a, \ast] = I_a T^b \wedge I_b \ast.\]  \hspace{1cm} (20)

It is possible to prove that the covariant derivative \(\nabla_a = e_a^\mu \nabla_\mu\) is related to \(\mathcal{D}_a\) (and therefore, to \(\mathcal{D}\)) through the expression

\[
\nabla_a = \mathcal{D}_a - I_a T^b \wedge I_b, \\
= I_a \mathcal{D} + \mathcal{D} I_a - I_a T^b \wedge I_b.
\] \hspace{1cm} (21)

From eqs \((17)\) and \((21)\), and comparing eq. \((13)\) with \((14)\), it is clear that these derivatives are equivalent when torsion vanishes.

We use a circle above a quantity to denote its torsionless version. For instance, the spin connection \(\omega^{ab}\) splits as \(\omega^{ab} = \hat{\omega}^{ab} + \kappa^{ab}\), where the torsionless piece \(\hat{\omega}^{ab}\) satisfies

\[
\hat{D} e^a = d e^a + \hat{\omega}^a_b \wedge e^b = 0
\] \hspace{1cm} (22)

and the contorsion one-form \(\kappa^{ab}\) is related to torsion through \(T^a = \kappa^a_b \wedge e^b\). In the same way, the Lorentz curvature splits as

\[
R^a_{\ b} = \hat{R}^a_{\ b} + \hat{D} \kappa^a_{\ b} + \kappa^a_{\ c} \wedge \kappa^c_{\ b},
\] \hspace{1cm} (23)

where \(\hat{R}^{ab}\) is the Riemann curvature two-form.

In the torsionless case, the covariant derivatives are related as

\[
\hat{D} = \hat{\nabla} \wedge, \\
\hat{\nabla}_a = \hat{D}_a = I_a \hat{\mathcal{D}} + \hat{D} I_a.
\] \hspace{1cm} (24) (25)

In Secs. III A and III B we present the standard Weitzenböck identity for the wave operator using the operators we just introduced. In Sec. IV we study a generalized Weitzenböck identity for spaces with torsion, and we analyze how it affects the eikonal approximation and polarization for an arbitrary \(p\)-form wave in Sec. V.

Since the goal of this paper is to study the propagation of waves on spaces with torsion, in Secs. III and IV we use the term “wave operator,” but the results presented here are equally valid for other Laplacians regardless of the metric signature.
III. THE WAVE OPERATOR AND THE CANONICAL WEITZENBÖCK IDENTITY

There are at least two wave operators, $\Box : \Omega^p(M(d)) \to \Omega^p(M(d))$, that are relevant in physics. One of them is the Laplace–de Rham operator

$$
\Box_{dR} = d^\dagger d + d d^\dagger,
$$

(26)
defined in terms of the standard exterior derivative, $d : \Omega^p(M(d)) \to \Omega^{p+1}(M(d))$, and the coderivative operator, $d^\dagger : \Omega^p(M(d)) \to \Omega^{p-1}(M(d))$, which is given by

$$
d^\dagger = -(-1)^{d(p+1)+\eta-} * d *.
$$

(27)

The other important wave operator is the standard Laplace–Beltrami operator, $\Box_B = -\nabla^\lambda \nabla_{\lambda}$. Both operators are related via the Weitzenböck identity, a relation of the form

$$
\Box_{dR} = \Box_B + \text{(Riemann curvature terms)}.
$$

(28)

These “Riemann curvature terms” change depending on the degree of the form that the wave operator acts on. The identity is well-known but cumbersome (see, e.g., Ref. [21, Ch. V, Sec. B.4] and Ref. [22, Ch. 6.3]).

The purpose of Secs. III A and III B is not to develop new mathematics or physics, but instead to show how to use the operators introduced in Sec. II to study this problem in a way that is simpler than usual. For instance, much of the content and examples of Secs. III A and III B has already been thoroughly studied in Ref. [23], with a different notation and procedures as here. In Secs. IV and V we use these operators and procedures to develop a new physical approach to the propagation of waves in spaces with nonvanishing torsion.

A. Case of scalar $p$-forms

A scalar $p$-form is a form $\Phi = \frac{1}{p!} \Phi_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ with only spacetime indices and no other indices. For instance, the electromagnetic potential one-form $A = A_\mu dx^\mu$ and the field strength two-form $F = dA$ are scalar forms. The vielbein $e^a$, the torsion $T^a$, and the curvature $R^{ab}$ are nonscalar forms.
The Weitzenböck formula can be understood easily by observing that the coderivative on scalar $p$-forms can be rewritten as

\[ d^\dagger \Phi = -I^a \bar{D}_a \Phi. \]  

(29)

It is important to notice that this last identity just includes the torsionless piece of the geometry, regardless of whether torsion vanishes or not on the manifold. The left hand side only has information on the metric and not on the contorsion. Therefore, the right hand side can not include such information either.

Since the scalar form $p$-form $\Phi$ lacks any index to perform parallel transport with the $\hat{D}$ operator, we can also write $d\Phi = \hat{D}\Phi = \bar{D}\Phi$. This means that for de Rham’s definition,

\[ \square_{\text{dR}} \Phi = d^\dagger d\Phi + dd^\dagger \Phi, \]  

(30)

we can write

\[ dd^\dagger \Phi = -\hat{D}I^a \bar{D}_a \Phi, \]  

(31)

\[ d^\dagger d\Phi = -I^a \bar{D}I_a \Phi. \]  

(32)

However, let us observe that

\[ d^\dagger d\Phi = -I^a \bar{D}I_a \hat{D}\Phi, \]

\[ = -I^a \hat{D} \left( \hat{D}_a - \bar{D}_a \right) \Phi, \]

\[ = - \left( \hat{D}^a - \bar{D}^a \right) \hat{D}_a \Phi + I^a \bar{D}I_a \Phi, \]

\[ = -\bar{D}^a \hat{D}_a \Phi + \bar{D}I^a \left( I_a \hat{D}\Phi + \bar{D}_a \Phi \right) + I^a \bar{D}I_a \Phi. \]

Since $I^a I_a = 0$, $-dd^\dagger \Phi = \hat{D}I^a \bar{D}_a \Phi$, and $-\bar{D}^a \hat{D}_a = -\hat{\nabla}^\lambda \hat{\nabla}_\lambda = \square_B$, we have that

\[ \square_{\text{dR}} \Phi = \square_B \Phi + I_a \hat{D}^2 I^a \Phi, \]  

(33)

and therefore

\[ \square_{\text{dR}} \Phi = \square_B \Phi + I_a \left( \hat{R}^a_{\ b} \wedge I^b \Phi \right) \]

\[ = \square_B \Phi + I_a \hat{R}^a_{\ b} \wedge I^b \Phi + \hat{R}^a_{\ b} \wedge I_a I^b \Phi. \]

From here we arrive at the standard Weitzenböck identity written in terms of the operators defined in eqs. (5) and (7):

\[ \square_{\text{dR}} \Phi = \square_B \Phi + I_a \hat{R}^a_{\ b} \wedge I^b \Phi - \hat{R}^{ab} \wedge I_{ab} \Phi. \]  

(34)
The best known examples of this relation are given by classical electromagnetism in curved space. For instance, the Maxwell equations in vacuum can be written as

\[ \mathbf{d}^\dagger \mathbf{F} = \mathbf{d}^\dagger \mathbf{d} \mathbf{A} = 0. \]  \hspace{1cm} (35)

Choosing the Lorenz gauge, \( \mathbf{d}^\dagger \mathbf{A} = 0 \), we have

\[ \Box_{\text{dR}} \mathbf{A} = \mathbf{d}^\dagger \mathbf{d} \mathbf{A} + \mathbf{d} \mathbf{d}^\dagger \mathbf{A} = 0, \]  \hspace{1cm} (36)

and therefore

\[ \Box_B \mathbf{A} + \mathbf{I}_a \hat{\mathbf{R}}^a_b \wedge \mathbf{I}^b \mathbf{A} = 0. \]  \hspace{1cm} (37)

In terms of coordinate components, this last equation reads

\[ \left( -\hat{\nabla}_\lambda \hat{\nabla}^\lambda A_\nu + \hat{\mathbf{R}}^\lambda_{\mu\nu} A^\mu \right) dx^\nu = 0, \]  \hspace{1cm} (38)

where \( \hat{\mathbf{R}}_{\mu\nu} = \hat{\mathbf{R}}^\lambda_{\mu\lambda\nu} \) is the canonical torsionless Ricci tensor.

For the field strength, from \( \mathbf{d}^\dagger \mathbf{F} = 0 \) and \( \mathbf{d} \mathbf{F} = 0 \) we have \( \Box_{\text{dR}} \mathbf{F} = \mathbf{d}^\dagger \mathbf{d} \mathbf{F} + \mathbf{d} \mathbf{d}^\dagger \mathbf{F} = 0 \), and therefore

\[ \Box_B \mathbf{F} + \mathbf{I}_a \hat{\mathbf{R}}^a_b \wedge \mathbf{I}^b \mathbf{F} - \hat{\mathbf{R}}^{ab}_b \mathbf{I}_{ab} \mathbf{F} = 0. \]  \hspace{1cm} (39)

In terms of coordinate components, this last equation leads to the canonical relation

\[ \frac{1}{2} \left[ -\hat{\nabla}_\lambda \hat{\nabla}^\lambda F_{\mu\nu} + \hat{\mathbf{R}}_{\lambda\mu} F_{^\lambda\nu} - \hat{\mathbf{R}}_{\lambda\nu} F_{^\lambda\mu} - \hat{\mathbf{R}}_{\rho\sigma}^\rho F_{\mu\nu} - \hat{\mathbf{R}}_{\lambda\mu}^\rho F_{\rho\sigma} \right] dx^\mu \wedge dx^\nu = 0. \]  \hspace{1cm} (40)

An important point here is that all derivatives and curvatures in these expressions are torsionless, regardless of whether the background space has torsion or not. The de Rham operator \( \Box_{\text{dR}} = \mathbf{d}^\dagger \mathbf{d} + \mathbf{d} \mathbf{d}^\dagger \) only includes information on the metric through the Hodge dual, and it doesn’t need the spin connection to be defined. Therefore, such information cannot appear in the curvature terms at the right hand side of the Weitzenböck identity.

**B. Case of \( p \)-forms with free indices on torsionless geometries**

When considering \( p \)-forms \( \Psi^A = \frac{1}{p!} \Psi^A_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \) in some matrix representation of the local \( \text{SO}(d) \) group algebra,

\[ \delta \Psi^A = \frac{1}{2} \lambda^{ab} [J_{ab}]^A_B \Psi^B, \]  \hspace{1cm} (41)
we find that the de Rham operator \( \Box_{dR} = d^\dagger d + dd^\dagger \) is not covariant under these transformations. This means that operators such as \( \Box_{dR} \Psi^A = (d^\dagger d + dd^\dagger) \Psi^A \) are ill-defined.

On spaces without torsion, this problem has a simple solution. In analogy with the de Rham definition, we can define a covariant coderivative, \( \tilde{D}^\dagger : \Omega^p(M^{(d)}) \rightarrow \Omega^{p-1}(M^{(d)}) \), as

\[
\tilde{D}^\dagger = -(-1)^{(d-1)p} \ast D^*,
\]

and in terms of it we define the Lichnerowicz–de Rham operator \([24, 25]\) as the covariant generalization of \( \Box_{dR} \),

\[
\Box_L = \tilde{D}^\dagger \tilde{D} + \tilde{D} \tilde{D}^\dagger.
\]

Again, we have that the identity

\[
\tilde{D}^\dagger \Psi^A = -I^a \tilde{D}_a \Psi^A
\]

is satisfied. From this, we find that the result from the last section still holds true,

\[
\Box_L \Psi^A = \Box_B \Psi^A + I_c \tilde{D}^2 I^c \Psi^A,
\]

\[
= \Box_B \Psi^A + I_c \left( \tilde{R}^c_e \wedge I^e \Psi^A + \frac{1}{2} \tilde{R}^{ab} [J^K_{ab}]^A_B \wedge I^c \Psi^B \right),
\]

and therefore

\[
\Box_L \Psi^A = \Box_B \Psi^A + I_a \tilde{R}^a_{cb} \wedge I^b \Psi^A - \tilde{R}^{ab} \wedge I_{ab} \Psi^A + \frac{1}{2} I_c \tilde{R}^{ab} [J^K_{ab}]^A_B \wedge I^c \Psi^B.
\]

Perhaps the best known example of this case is provided by applying \( \Box_L \) to the Riemann curvature itself (see exercise 15.2 of Ref. \([26]\)). Solving such an example in the standard tensorial language is a long algebraic nightmare, but with the operators introduced here it is a trivial exercise. In fact, a vacuum solution of Einstein equations satisfies \( \tilde{D}^\dagger \tilde{R}^{ab} = 0 \), and using the Bianchi identity \( \tilde{D} \tilde{R}^{ab} = 0 \), we find in this case that \( \left[ \tilde{D}^\dagger \tilde{D} + \tilde{D} \tilde{D}^\dagger \right] \tilde{R}^{ab} = 0 \). After a couple of lines of algebra, eq. \((45)\) lead us to

\[
\Box_B \tilde{R}^{ab} + I_c \tilde{R}^{ce} \wedge I_e \tilde{R}^{ab} - \tilde{R}^{ce} I_{ce} \tilde{R}^{ab} - 2 I_c \tilde{R}^a_c \wedge I^c \tilde{R}^{be} = 0.
\]

Writing this in the coordinate basis we find

\[
\frac{1}{2} \left[ -\nabla^\lambda \nabla_\mu \tilde{R}^{\alpha\beta}_{\mu\nu} + \tilde{R}^\lambda_\mu \tilde{R}^{\alpha\beta}_{\lambda\nu} - \tilde{R}^\lambda_\nu \tilde{R}^{\alpha\beta}_{\lambda\mu} - \tilde{R}^{\rho\sigma}_{\mu\nu} \tilde{R}^{\alpha\beta}_{\rho\sigma} + \\
-2 \left( \tilde{R}^{\alpha}_{\rho\mu} \tilde{R}^{\beta\rho\sigma}_{\nu} - \tilde{R}^{\alpha}_{\rho\nu} \tilde{R}^{\beta\rho\sigma}_{\mu} \right) \right] dx^\mu \wedge dx^\nu = 0.
\]
On a vacuum solution of the Einstein equations, $\ddot{R}^\lambda_\nu = 0$, we recover the standard expression

$$-\nabla^\lambda \nabla_\lambda \dot{R}^{\alpha^3} - \dot{R}^{\rho^3} \nabla^\rho \dot{R}^{\alpha^3} - 2 \left( \dot{R}^{\alpha^3} \nabla^\rho \dot{R}^{\beta^3} - \dot{R}^{\alpha^3} \nabla^\rho \dot{R}^{\beta^3} \right) = 0. \quad (48)$$

Of course, the Lichnerowicz–de Rham operator $\Box_L = \dot{D}^\dagger \dot{D} + \ddot{D}^\dagger$ reduces to the standard de Rham operator $\Box_{dR} = d^\dagger d + dd^\dagger$ when applied to a scalar $p$-form. The difference between both is subtle but important. The de Rham operator $\Box_{dR} = d^\dagger d + dd^\dagger$ does not care whether the background geometry has torsion or not. In any case, the wave operator on a scalar $p$-form $\Phi$ is sensitive only to the torsionless piece of the geometry. The same is not true in the case of a non-scalar $p$-form $\Psi^A$. In this case we had to impose the torsionless condition in order to define the Lichnerowicz–de Rham operator $\Box_L = \dot{D}^\dagger \dot{D} + \ddot{D}^\dagger$.

As we shall see in the next section, it is nontrivial to extend these results to spaces with nonvanishing torsion. In fact, there are two nonequivalent ways of extending the Lichnerowicz–de Rham operator in these cases.

IV. THE WAVE OPERATOR AND TORSION

It is not trivial to generalize the wave operator $\Box_L$ for non-scalar forms $\Psi^A$ in the case of nonvanishing torsion. The problem lies in the fact that there is no single, clear way of generalizing the coderivative operator $d^\dagger = -(-1)^{d(p+1)+\eta-} * d*$ in the torsional case. The most straightforward way of doing it may seem to be with $D^\dagger : \Omega^p (M^{(d)}) \to \Omega^{p-1} (M^{(d)})$, given by

$$D^\dagger = -(-1)^{d(p+1)+\eta-} * D*. \quad (49)$$

However, the proofs of the last section strongly suggest the alternative definition, $D^\dagger : \Omega^p (M^{(d)}) \to \Omega^{p-1} (M^{(d)})$, given by

$$D^\dagger = -I^a D_a \quad (50)$$

as the “right” generalization of $d^\dagger$ in this case.

The problem does not appear for scalar $p$-forms and vanishing torsion, because in this case $\dot{D}^\dagger = D^\dagger$. But both coderivatives differ in the torsional case, as the following equation shows:

$$D^\dagger \Psi^A = D^\dagger \Psi^A + I_a \kappa^a_b I^b \Psi^A + \kappa^{ab} \wedge I_{ab} \Psi^A, \quad (51)$$

where $\kappa^{ab}$ are the components of the contorsion one-form, $T^a = \kappa^a_b \wedge e^b$. 

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Both coderivatives seem “reasonable,” but an important distinction should be done. First, only $D^\dagger$ lead us to a “clean” version of the Weitzenböck identity. When using $D^\ddagger$ it seems unavoidable to end up with a “messier” version of the Weitzenböck identity involving derivatives of torsion and $\Psi^A$. The second point is that when applying it to the gravitational wave case, the wave operator given by the dynamics seem to be naturally constructed from $D^\dagger$ instead of from $D^\ddagger$ [10].

In fact, when defining a generalized de Rham operator, $\Box_{\text{GdR}} : \Omega^p (M^{(d)}) \to \Omega^p (M^{(d)})$, as

$$\Box_{\text{GdR}} = D^\dagger D + DD^\dagger$$

we get a generalized Weitzenböck identity with the same form as before,

$$\Box_{\text{GdR}} \Psi^A = \Box_{\text{GB}} \Psi^A + I_a R^a_{\ b} \wedge I^b \Psi^A - R^{ab} \wedge I_{ab} \Psi^A + \frac{1}{2} I_c R^{ab} [J_{ab}]^A_B \wedge I^c \Psi^B,$$

but with the Lorentz curvature $R^a_{\ b} = \hat{R}^a_{\ b} + \hat{D} \kappa^a_{\ b} + \kappa^a_{\ c} \wedge \kappa^c_{\ b}$, which encodes metric and affine degrees of freedom, instead of only the purely metric Riemann curvature two-form, $\hat{R}^a_{\ b}$.

Here it is important to notice that the generalized Beltrami operator, $\Box_{\text{GB}} : \Omega^p (M^{(d)}) \to \Omega^p (M^{(d)})$, is given by

$$\Box_{\text{GB}} = -D^a D_a,$$

and therefore constructed in terms of $D_a : \Omega^p (M^{(d)}) \to \Omega^p (M^{(d)})$,

$$D_a = I_a D + DI_a$$

$$= \nabla_a + I_a T^b \wedge I_b,$$

instead of $\nabla_a$, as the “usual” Beltrami operator $\Box_B = -\nabla^a \nabla_a$.

A. Generalized Weitzenböck identity for an arbitrary Lie (super)algebra

Using the above introduced definitions it is straightforward to write down a “gauge Weitzenböck identity” for any Lie (super)algebra including a Lorentz subalgebra. Let us consider a generic algebra-valued gauge connection one-form and a matrix representation of
the Lie algebra generators \( [T_A]^\alpha_{\beta} \),

\[
A^A [T_A]^\alpha_{\beta} = \frac{1}{2} \omega^{ab} [J_{ab}]^\alpha_{\beta} + \text{(other terms)},
\]

such that the algebra-valued gauge curvature two-form can be written as

\[
F^C [T_C]^\alpha_{\beta} = \left( dA^C + \frac{1}{2} C_{AB}^C A^A \wedge A^B \right) [T_C]^\alpha_{\beta}
= \frac{1}{2} R^{ab} [J_{ab}]^\alpha_{\beta} + \text{(other curvature terms)},
\]

where \( C_{AB}^C \) are the structure constants of the Lie (super)algebra.

Let us call now

\[
D\Psi^\alpha = d\Psi^\alpha + A^A \wedge [T_A]^\alpha_{\beta} \Psi^\beta
\]

and let us again make the same definitions but in terms of the full gauge derivative, i.e.,

\[
D^\dagger = -I^a D I_a, \quad D_a = I_a D + D I_a, \quad \Box_{\text{GB}} = -D^a D_a, \quad \text{etc.}
\]

In this case, it is trivial to get the “gauge-generalized” Weitzenböck identity as

\[
\Box_{\text{GdR}} \Psi^\alpha = \Box_{\text{GB}} \Psi^\alpha + I_a R^a_b \wedge I^b \Psi^\alpha - R^{ab} \wedge I_a \Psi^\alpha + \frac{1}{2} I_c F^A [T_A]^\alpha_{\beta} \wedge I^\beta \Psi^\beta
\]

for a general gauge theory. The only difference is the appearance of the full gauge curvature in the last term.

It is important to emphasize that all the general definitions and the generalized Weitzenböck identity we are proposing reduce to the standard case when considering scalar forms or vanishing torsion.

In the next section we will use the generalized Weitzenböck identity and the properties of \( D_a = I_a D + D I_a \) to study the propagation of a \( p \)-form wave \( \Box_{\text{GdR}} \Psi^A = 0 \) on a space with curvature and torsion.

V. THE EIKONAL LIMIT IN SPACES WITH CURVATURE AND TORSION

In this section we study the behavior of a nonscalar \( p \)-form \( \Psi^A \) satisfying the generalized wave equation \( \Box_{\text{GdR}} \Psi^A = 0 \).

In order to accomplish this, we use the well known two-parameter eikonal approximation, or WKB approximation in the context of quantum mechanics (see, e.g., Ref. [26, Ch. 22.5] and Ref. [27, Ch. 1.5.1]). When the eikonal approximation conditions are not fulfilled, it
is possible to find anomalous dispersion relations even in standard general relativity, e.g., electromagnetic waves not traveling on null geodesics [28, 29]. Since the purpose of the current article is to study the effects of torsion on the propagation of waves, and not to study generic anomalous dispersion relations for other geometrical reasons, we will assume that the eikonal conditions hold throughout.

Let us consider a background geometry described by slowly-varying curvature $R^{ab}$ and torsion $T^a$, changing on characteristic lengths given by $L_R$ and $L_T$. On this geometry we have a $p$-form field $\Psi^A$ satisfying the generalized wave equation

$$(D^\dagger D + DD^\dagger) \Psi^A = 0.$$  

This $p$-form field $\Psi^A$ can be written as

$$\Psi^A = e^{i\theta} \psi^A,$$

where $\theta$ is a rapidly-changing real phase with a characteristic length $\lambda$, and $\psi^A$ is a complex valued $p$-form. The complex valued $p$-form $\psi^A$ changes slowly over scales given by a length parameter $L_\psi$. Defining the parameter

$$\varepsilon = \frac{\lambda}{\min (L_\psi, L_R, L_T)},$$

the eikonal limit corresponds to the $\varepsilon \ll 1$ case. We can expand the amplitude as

$$\psi^A = \sum_{n=0}^{\infty} \psi^A_{(n)},$$

where $\psi^A_{(n)}$ is a term of order $\varepsilon^n$. In this way we can split the amplitude as a dominant, $\lambda$-independent $p$-form amplitude $\psi^A_{(0)}$ piece plus small “geometric optics” deviations due to the finite wavelength.

Let us call $k = d\theta = k_a e^a = k_\mu dx^\mu$ the “wave one-form.” Using eq. (58) and the generalized Weitzenböck identity [56], it is possible to show that

$$(D^\dagger D + DD^\dagger) \Psi^A = e^{i\theta} \left[ k^2 \psi^A - 2i \left( k^a D_a \psi^A - \frac{1}{2} \psi^A D^\dagger k \right) + (D^\dagger D + DD^\dagger) \psi^A \right],$$

where $k^2 = k^a k_a$. 

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Using the eikonal approximation to split orders in \( \varepsilon \) we get

\[
k^2 \psi^A_{(0)} = 0, \tag{62}
\]

\[
k^2 \psi^A_{(1)} - 2i \left( k^a \mathcal{D}_a \psi^A_{(0)} - \frac{1}{2} \psi^A_{(0)} \mathcal{D}^a k \right) = 0, \tag{63}
\]

\[
\sum_{n=0}^{\infty} \left[ k^2 \psi^A_{(n+2)} - 2i \left( k^a \mathcal{D}_a \psi^A_{(n+1)} - \frac{1}{2} \psi^A_{(n+1)} \mathcal{D}^a k \right) + \left( \mathcal{D}^a \mathcal{D} + \mathcal{D}^a \right) \psi^A_{(n)} \right] = 0. \tag{64}
\]

From these we read the dispersion relation,

\[
k^2 = 0, \tag{65}
\]

the propagation of the leading term of the complex-valued amplitude \( p \)-form \( \psi^A_{(0)} \),

\[
k_a \mathcal{D}^a \psi^A_{(0)} - \frac{1}{2} \psi^A_{(0)} \mathcal{D}^a k = 0, \tag{66}
\]

and the higher-order deviations from “geometric optics,”

\[
\sum_{n=0}^{\infty} \left[ \left( \mathcal{D}^a \mathcal{D} + \mathcal{D}^a \right) \psi^A_{(n)} - 2i \left( k^a \mathcal{D}_a \psi^A_{(n+1)} - \frac{1}{2} \psi^A_{(n+1)} \mathcal{D}^a k \right) \right] = 0. \tag{67}
\]

It is remarkable that torsion does not affect the dispersion relation. The “wave one-form” is null. From this and the fact that \( dk = d^2 \theta = 0 \), it is straightforward to prove that regardless of whether the background geometry has torsion or not, one always has

\[
k^\mu \nabla_\mu k^\nu = 0, \tag{68}
\]

i.e., \( k = k_\mu dx^\mu \) moves along null geodesics (and not along null self-parallels). The dispersion relation does not care at all whether the background geometry has torsion or not. The same is not true regarding polarization.

We shall write the complex-valued amplitude \( p \)-form \( \psi^A_{(0)} \) as the product of a real scalar amplitude \( \varphi \) and a complex-valued polarization \( p \)-form \( P^A \),

\[
\psi^A_{(0)} = \varphi P^A, \tag{69}
\]

normalized as

\[
(-1)^{n-} \left( \bar{P}_A \wedge *P^A \right) = 1, \tag{70}
\]

whence

\[
\varphi^2 = (-1)^{n-} \left( \bar{\psi}^A_{(0)} \wedge *\psi^A_{(0)} \right). \tag{71}
\]
Here we have to observe that in the standard scalar $p$-form/torsionless geometry case it is possible to prove that, in the eikonal limit,

$$k^A \nabla_\lambda P^A = 0$$  \hspace{1cm} (72)

(i.e., polarization is parallel transported along geodesics), and that the one-form current $J = \varphi^2 k$ (the “ray density” or “number of photons” in the electromagnetic case) is conserved,

$$d^\dagger J = 0.$$  \hspace{1cm} (73)

Neither of these are true in the torsional case.

Let us show how the conservation of $J = \varphi^2 k$ and the propagation of $P^A$ get affected by the presence of torsion. In order to do this, let us observe that from eq. (71) we have that

$$k^a D_a \varphi^2 = (-1)^{n-} k^a D_\alpha (\bar{\psi}_A(0) \wedge *\psi^A(0)), \hspace{1cm} (74)$$

and using the property (20) we get from here that

$$k^a D_a \varphi^2 = (-1)^{n-} k^a D_\alpha (\bar{\psi}_A(0) \wedge *\psi^A(0)) - \varphi^2 T_{abc} \Pi^{ab} k_c, \hspace{1cm} (75)$$

where $T_{abc}$ are the orthonormal components of the torsion two-form, $T_a = \frac{1}{2} T_{abc} e^b \wedge e^c$, and $\Pi^{ab}$ given by

$$\Pi^{ab} = \eta^{ab} - \frac{1}{2} (-1)^{n-} \left[ P^A \wedge *P^A + \bar{P}_A \wedge *\bar{P}_A \right]. \hspace{1cm} (76)$$

Using eq. (66) for the propagation of the amplitude, eq. (75) takes the form

$$k^a \mathcal{D}_a \varphi^2 = -D^a k_a (-1)^{n-} [\bar{\psi}_A(0) \wedge *\psi^A(0)] - \varphi^2 T_{abc} \Pi^{ab} k_c, \hspace{1cm} (77)$$

and from eq. (71) we get

$$\mathcal{D}_a J^a = -T_{abc} \Pi^{ab} J^c \hspace{1cm} (78)$$

with $J^a = \varphi^2 k^a$. It is possible to write eq. (78) in terms of the coderivative as

$$d^\dagger J = \left( \eta^{ab} + \Pi^{ab} \right) J^c. \hspace{1cm} (79)$$

From this, it is clear that in the case of nonscalar forms on a background with nonvanishing torsion, the conservation of the “$\psi$-ray density” $J$ is broken, $d^\dagger J \neq 0$, and depending on the sign of the right hand side of eq. (79), torsion can either reinforce or damp the propagation of the wave.
This result also implies that the polarization is not going to be parallel-propagated unless torsion vanishes. In fact, replacing eq. (69) in eq. (66) we get

\[ k^a \mathcal{D}_a P^A + \frac{1}{2\varphi^2} \mathcal{D}_a J^a P^A = 0. \] (80)

Using eq. (78) it is clear that in this case

\[ k^a \mathcal{D}_a P^A = \frac{1}{2} T_{abc} \Pi^{ab} k^c P^A, \] (81)

and finally, using eq. (17),

\[ k^a \nabla_a P^A = \frac{1}{2} T_{abc} \Pi^{ab} k^c P^A - k^a I_a T^b \wedge I_b P^A. \] (82)

Therefore, the polarization form is not parallel-propagated on self-parallel, and neither on geodesics in this case.

We have arrived at a complex picture for a \( p \)-form \( \Psi^A \) wave \((D^\dagger D + DD^\dagger) \Psi^A = 0\) on a space with torsion. The dispersion relation is the standard one: the wave will propagate at the speed of light on null geodesics regardless of torsion. However, its geometric optics limit will have some anomalous features: the associated “ray density” one-form is not conserved \( d^\dagger (\varphi^2 k) \neq 0 \), and even further, polarization is not be propagated through null geodesics, \( \nabla_{\vec{k}} P^A \neq 0 \). Both the scalar amplitude \( \varphi \) and the polarization \( p \)-form \( P^A \) are interacting with the background torsion while the wave is propagating on null geodesics.

VI. CONCLUSIONS: GRAVITATIONAL WAVES AND TORSION

In this work, we have studied the wave operator and the Weitzenböck identity for spaces endowed with torsion, and we have seen how this can affect the “geometrical optics” limit for the field under consideration. The main conclusion is that, regardless of the type of wave, the presence of torsion does not change the dispersion relation, but it does change the propagation of polarization for fields carrying free Lorentz indices. This result is also valid for gravitational waves, and therefore the detection of GW170817 and its optical counterpart does not rule out torsion. However, comparing the propagation of polarization for different kinds of waves we could in principle test whether a specific region of spacetime has torsion or not.
Gravitational waves carry a free Lorentz index and therefore could be used to test torsion. Linear and second-order perturbations on a geometry with torsion have been considered in Ref. [20], and gravitational waves in this case have been briefly considered in Ref. [16].

In short, gravitational waves can be studied in this case as perturbations on a background vielbein $e^a$ and spin connection $\omega^{ab}$, parametrized as

$$
e^a \mapsto \bar{e}^a = e^a + \frac{1}{2} H^a,$$

$$
\omega^{ab} \mapsto \bar{\omega}^{ab} = \omega^{ab} + U^{ab}(H) + V^{ab},
$$

where the vielbein perturbation $H^a = H^a_\lambda d\lambda$ in eq. (83) is related to the standard perturbation on the metric $\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ by [20]

$$H^a_\mu = e^a_\alpha \left[ h^{\alpha}_\mu - \frac{1}{4} h^{\alpha}_\beta h^{\beta}_\mu + \frac{1}{8} h^{\alpha}_\beta h^{\beta}_\gamma h^{\gamma}_\mu - \cdots \right].$$

The spin connection perturbation can be split in eq. (84) as a term $U^{ab}$ carrying the $H^a$-dependence, and $V^{ab}$, corresponding to a “torsionon” [20] or “roton” [30] perturbation.

When working with gravitational waves on a background with curvature and torsion, $H^a$ satisfies a wave equation given by the operator $\square_{GdR} = D^4D + DD^4$ plus some extra couplings with torsion and curvature. Working in a simplified case where $(D^4D + DD^4) H^a = 0$, we can use the eq. (81) on the polarization $P^a = P^a_\lambda d\lambda$ of the gravitational wave $H^a$ to find that while being transported along the null geodesic it interacts with the background torsion via

$$k^c \nabla_c P_m = \frac{1}{2} k^c \left[ T_{abc} \Pi^{ab} P_m - (T_{mac} - T_{mac} + T_{cma}) P^a - (T_{acp} - T_{cap} + T_{pca}) e^p I_m P^a \right].$$

The details of this interaction will strongly depend on the solution being considered and deserve a separate treatment, which will be reported elsewhere. However, regardless of these details, it seems plausible that an arrange of gravitational wave detectors separated by a long distance on the path of a gravitational wave could measure changes on the polarization of the wave along the path. A parallel-transported polarization $k^c \nabla_c P_m = 0$ along this path would indicate the absence of torsion; a different result would signal to the contrary.

In some aspects, the effect could be thought of as a gravitational version of the optical rotation (Faraday effect) for electromagnetic waves. However, the analogy is broken because of the absence of birefringence; the refraction index is always $n = 1$ regardless of the polarization.
ACKNOWLEDGMENTS

We are grateful to Yuri Bonder, Fabrizio Canfora, Oscar Castillo-Felisola, Fabrizio Cordonier-Tello, Cristóbal Corral, Nicolás González, Perla Medina, Daniela Narbona, Julio Oliva, Francisca Ramírez, Patricio Salgado, Sebastián Salgado, Jorge Zanelli, and Alfonso Zerwekh for many enlightening conversations. JB acknowledges financial support from CONICYT grant 21160784. JB also thanks the Institute of Mathematics of the Czech Academy of Sciences, where part of this work was carried out, for their warm hospitality. FI acknowledges financial support from the Chilean government through Fondecyt grants 1150719 and 1180681. OV acknowledges VRIIP UNAP for financial support through Project VRIIP0258-18.

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