A spectral method for approximate solving a second-order linear differential equation

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Abstract. We suggest an approximate method for the linear differential equation \( x''(t) + 2\alpha x'(t) + Ax(t) = f(t) \), where \( A \) is an unbounded self-adjoint operator, \( \alpha \) is a given scalar and the operator \( A - \alpha^2 1 \) is positive semidefinite. A priori estimates of the approximation error are obtained. The results of numerical experiments for the hyperbolic equation are presented. The suggested approach can be used in the remodeling problems for complicated objects and systems if the initial mathematical model is such an equation.

1. Introduction
We consider a linear differential equation of the second order
\[
x''(t) + 2\alpha x'(t) + Ax(t) = f(t)
\] (1)
with an unbounded self-adjoint operator coefficient \( A \). We assume that the operator \( Q = A - \alpha^2 1 \) is positive semidefinite. In this case, Equation (1) is a hyperbolic equation in the abstract form [1]. The exact solution of such an equation can be written in terms of cosine and sine operator functions [2] generated by the operator \( Q \).

In this article we suggest a new approximate method for solving Equation (1). It is based on the approximation of sine operator function by some rational function of the operator \( Q \). If the free term \( f \) in Equation (1) has the form \( f(t) = bv(t) \), where \( b \) is a given vector and \( v \) is a scalar function, we suggest a special version of our method. It allows one to calculate the solution at a given point \( t \) without finding the solution at the previous points. This, in particular, distinguishes our method from the difference schemes [3, 4], where the calculations are carried out sequentially from the initial to the final point. For both versions of the method we obtain (Theorems 2 and 3) a priori estimates of the approximation error. The proof technique uses the spectral theorem for self-adjoint operators and a functional calculus theorem.

A similar approach is applied for solving an abstract parabolic equation in [5], and some close ideas for solving second-order differential equations with matrix coefficients are discussed in [6].

In Section 2, we refine the statement of the problem, describe the method and prove theorems on the estimation of the approximation error.

In Section 3, we present the results of numerical experiments.
2. Method

Let $H$ be a Hilbert space (real or complex), and let $A: D(A) \subset H \to H$ be a linear unbounded operator. By $1$ we denote the identity operator.

We consider a second order linear differential equation with zero initial conditions

$$x''(t) + 2\alpha x'(t) + Ax(t) = f(t),$$
$$x(0) = 0,$$
$$x'(0) = 0.$$  \tag{2}

We assume that the function $f: [0, +\infty) \to H$ is continuous, $\alpha \geq 0$, the operator $A$ is self-adjoint, and the operator $Q = A - \alpha^2 1$ is positive semidefinite.

We recall that the spectrum $\sigma(Q)$ of the positive semidefinite operator $Q$ is contained in $[0, +\infty)$. Since the operator $Q$ is self-adjoint, there exists a unique resolution of identity $E$ and one can define a functional calculus $g \mapsto g(Q)$ which maps every Borel measurable function $g$ to the densely defined closed operator

$$g(Q) = \int_0^{+\infty} g(\xi) \, dE(\xi).$$

If the function $g$ is bounded on $\sigma(Q)$, then the operator $g(Q)$ is bounded, and the inequality

$$\|g(Q)\| \leq \sup_{\xi \in \sigma(Q)} |g(\xi)|$$  \tag{3}

holds.

The impulse response of problem (2) is defined as the operator function

$$U(t) = s_t(Q) = \int_0^{+\infty} s_t(\xi) \, dE(\xi),$$

where

$$s_t(\xi) = e^{-\alpha t} \sin\sqrt{\xi} t \sqrt{\xi}.$$  

The function

$$x(t) = \int_0^t U(t-\tau)f(\tau) \, d\tau$$  \tag{4}

is called a (generalized) solution of problem (2).

Sometimes we discuss the version of problem (2) with the free term $f(t) = bv(t)$, where $b \in H$ is a given vector and $v: [0, +\infty) \to \mathbb{R}$ is a given continuous function, i.e. we solve the problem

$$x''(t) + 2\alpha x'(t) + Ax(t) = bv(t),$$
$$x(0) = 0,$$
$$x'(0) = 0.$$  \tag{5}

We define the auxiliary function

$$\theta_t(\xi) = \int_0^t s_{t-\tau}(\xi)v(\tau) \, d\tau = \int_0^t s_t(\xi)v(t-\tau) \, d\tau$$ \tag{6}

or, in more detail,

$$\theta_t(\xi) = \frac{1}{\sqrt{\xi}} \int_0^t e^{-\alpha(t-\tau)} \sin(\sqrt{\xi}(t-\tau))v(\tau) \, d\tau = \frac{1}{\sqrt{\xi}} \int_0^t e^{-\alpha(t-\tau)} \sin(\sqrt{\xi}\tau)v(t-\tau) \, d\tau.$$
We note that for any fixed \( t > 0 \) the functions \( s_t \) and \( \theta_t \) are bounded on \( \sigma(Q) \), therefore, the operators \( s_t(Q) \) and \( \theta_t(Q) \) are bounded.

**Theorem 1.** The (generalized) solution of problem (5) can be represented in the form

\[
x(t) = \theta_t(Q)b.
\]

**Proof.** From formula (4) for \( f(t) = bu(t) \) we have

\[
x(t) = \int_0^t U(t - \tau)bu(\tau) d\tau = \int_0^t \int_0^{+\infty} s_{t-\tau}(\xi) dE(\xi) bu(\tau) d\tau = \\
= \int_0^t \int_0^{+\infty} s_{t-\tau}(\xi) v(\tau) dE(\xi) d\tau b = \int_0^{+\infty} \int_0^t s_{t-\tau}(\xi) v(\tau) d\tau dE(\xi) b = \\
= \int_0^{+\infty} \theta_t(\xi) dE(\xi) b = \theta_t(Q)b.
\]

The proof of the theorem is complete.

Let us construct an approximate impulse response of problem (2). We fix \( t > 0 \) and approximate on \( \sigma(Q) \subset [0, +\infty) \) the function \( \xi \mapsto s_t(\xi) \) by a rational function

\[
r_t(\xi) = p_0 + \sum_{k=1}^{K_1} \frac{p_k}{\xi - q_k} + \sum_{k=1}^{K_2} \frac{\alpha_k + \beta_k \xi}{\xi^2 + \mu_k \xi + \nu_k}
\]  
(7)

where the roots of the denominators lie outside \( \sigma(Q) \).

For the approximate impulse response of problem (2) we take the operator function

\[
\tilde{U}(t) = r_t(Q).
\]  
(8)

To calculate a rational function of the operator \( Q \), it is convenient to use the formula [5]

\[
r_t(Q) = p_0 \mathbf{1} + \sum_{k=1}^{K_1} p_k(Q - q_k \mathbf{1})^{-1} + \sum_{k=1}^{K_2} (\alpha_k \mathbf{1} + \beta_k Q)(Q^2 + \mu_k Q + \nu_k \mathbf{1})^{-1}.
\]

We define the approximate solution of problem (2) by the formula

\[
\tilde{x}(t) = \int_0^t \tilde{U}(t - \tau)f(\tau) d\tau = \int_0^t r_{t-\tau}(Q)f(\tau) d\tau.
\]  
(9)

In the case of the problem (5), we can immediately construct an approximate solution at a given point \( t > 0 \). To do this, we approximate on \( \sigma(Q) \subset [0, +\infty) \) the function \( \xi \mapsto \theta_t(\xi) \) by rational function (7), and for approximate solution of problem (5) we take the function

\[
\tilde{x}(t) = r_t(Q)b = p_0b + \sum_{k=1}^{K_1} p_k(Q - q_k \mathbf{1})^{-1}b + \sum_{k=1}^{K_2} (\alpha_k \mathbf{1} + \beta_k Q)(Q^2 + \mu_k Q + \nu_k \mathbf{1})^{-1}b.
\]  
(10)

We note that to find vectors of the form \( w = (Q - q_k \mathbf{1})^{-1}b \) from formula (10) it is sufficient [5] to solve (exactly or approximately) equations

\[
Qw - q_k w = b.
\]
Similarly, to calculate vectors of the form \( w = (Q^2 + \mu_k Q + \nu_k \mathbf{1})^{-1} b \) it is sufficient to solve equations

\[
Q^2 w + \mu_k Q w + \nu_k w = b.
\]

To estimate the absolute errors of approximate impulse response (8) and approximate solution (9) of problem (2) we present the following theorem.

**Theorem 2.** Let \( \varepsilon: [0, +\infty) \to [0, +\infty) \) be a given function, such that for each \( t > 0 \) the estimate

\[
|r_t(\xi) - s_t(\xi)| \leq \varepsilon(t), \quad \xi \in [0, +\infty)
\]

holds. Then the approximate impulse response (8) of problem (2) satisfies the estimate

\[
\|\tilde{U}(t) - U(t)\| \leq \varepsilon(t),
\]

and the approximate solution (9) of problem (2) satisfies the estimate

\[
\|\tilde{x}(t) - x(t)\| \leq \sup_{\tau \in [0,t]} \varepsilon(\tau) \int_{0}^{t} \|f(\tau)\| \, d\tau.
\]

**Proof.** Consider the function \( g_t(\xi) = r_t(\xi) - s_t(\xi) \). Obviously, the function \( g_t \) is bounded on \( \sigma(Q) \subset [0, +\infty) \). Using estimates (3) and (11) we obtain

\[
\|\tilde{U}(t) - U(t)\| = \|r_t(Q) - s_t(Q)\| = \|g_t(Q)\| \leq \sup_{\xi \in \sigma(Q)} |g_t(\xi)| \leq \varepsilon(t).
\]

This implies

\[
\|\tilde{x}(t) - x(t)\| = \left\| \int_{0}^{t} (\tilde{U}(t - \tau) - U(t - \tau)) f(\tau) \, d\tau \right\| \leq \\
\leq \int_{0}^{t} \left\| \tilde{U}(t - \tau) - U(t - \tau) \right\| f(\tau) \, d\tau \leq \int_{0}^{t} \varepsilon(t - \tau) \|f(\tau)\| \, d\tau \leq \\
\leq \sup_{\tau \in [0,t]} \varepsilon(\tau) \int_{0}^{t} \|f(\tau)\| \, d\tau.
\]

The proof of the theorem is complete.

Now we estimate the absolute error of approximate solution (10) of problem (5).

**Theorem 3.** We fix \( t > 0 \). Let the rational function \( \xi \mapsto r_t(\xi) \) approximates the function \( \xi \mapsto \theta_t(\xi) \) on \([0, +\infty)\) with the absolute error \( \varepsilon(t) \), i.e.

\[
|r_t(\xi) - \theta_t(\xi)| \leq \varepsilon(t), \quad \xi \in [0, +\infty).
\]

Then approximate solution (10) of problem (5) satisfies the estimate

\[
\|\tilde{x}(t) - x(t)\| \leq \varepsilon(t) \|b\|.
\]

**Proof.** Consider the function \( g_t(\xi) = r_t(\xi) - \theta_t(\xi) \). Obviously, the function \( g_t \) is bounded on \( \sigma(Q) \subset [0, +\infty) \). Using estimates (3) and (12) we obtain

\[
\|\tilde{x}(t) - x(t)\| = \|g_t(Q) b\| \leq \|g_t(Q)\| \|b\| \leq \sup_{\xi \in \sigma(Q)} |g_t(\xi)| \|b\| \leq \varepsilon(t) \|b\|.
\]

The proof of the theorem is complete.
3. Results and discussion
The main theoretical results are given above in Theorems 2 and 3, where the estimates of the approximation error are constructed. In this Section we discuss the results of the numerical experiments for the suggested method.

3.1. Input function
Let the input function \( v \) has the form

\[
v(t) = \begin{cases} 
0, & t < 0; \\
\frac{t}{r}, & 0 \leq t < r; \\
1, & r \leq t < w + r; \\
-(t - (w + 2r))/r, & w + r \leq t < w + 2r; \\
0, & t \geq w + 2r.
\end{cases}
\] (13)

The graph of this function in the case \( r = 0.1 \) and \( w = 1 \) is shown in Figure 1.

![Figure 1. The graph of the input function \( v \)](image)

**Theorem 4.** In the case of the input function (13), the function (6) admits the representation

\[
\theta(t) = e^{-\alpha t} \left\{ \begin{array}{ll}
0, & t < 0; \\
\theta_1(t), & 0 \leq t < r; \\
\theta_2(t), & r \leq t < w + r; \\
\theta_3(t), & w + r \leq t < w + 2r; \\
\theta_4(t), & t \geq w + 2r.
\end{array} \right.
\]

where

\[
\begin{align*}
\theta_1(t) &= e^{\alpha t} \sqrt{\xi} (-2\alpha + \alpha^2 t + \xi t) + (\alpha^2 - \xi) \sin \left(\sqrt{\xi} t\right) + 2\alpha \sqrt{\xi} \cos \left(\sqrt{\xi} t\right), \\
\theta_2(t) &= (\alpha^2 + \xi) \sqrt{\xi} e^{\alpha t} + (\xi - \alpha^2) e^{\alpha r} \sin \left(\sqrt{\xi} (t - r)\right) - 2\alpha \sqrt{\xi} e^{\alpha r} \cos \left(\sqrt{\xi} (t - r)\right) + \\
&\quad + (\alpha^2 - \xi) \sin \left(\sqrt{\xi} t\right) + 2\alpha \sqrt{\xi} \cos \left(\sqrt{\xi} t\right), \\
\theta_3(t) &= 2(\alpha^2 + \xi) \sqrt{\xi} e^{\alpha t} + (\xi - \alpha^2) e^{\alpha r} \sin \left(\sqrt{\xi} (t - r)\right) - 2\alpha \sqrt{\xi} e^{\alpha r} \cos \left(\sqrt{\xi} (t - r)\right) + \\
&\quad + (\alpha^2 - \xi) e^{\alpha (r + w)} \sin \left(\sqrt{\xi} (r - t + w)\right) - 2\alpha \sqrt{\xi} e^{\alpha (r + w)} \cos \left(\sqrt{\xi} (r - t + w)\right) - \\
&\quad - (\alpha^2 + \xi) \sqrt{\xi} e^{\alpha t} + (\alpha^2 - \xi) \sin \left(\sqrt{\xi} t\right) + 2\alpha \sqrt{\xi} e^{\alpha t} + \\
&\quad + 2\alpha \sqrt{\xi} \cos \left(\sqrt{\xi} t\right) + (\alpha^2 + \xi) \sqrt{\xi} we^{\alpha t}.
\end{align*}
\]
\begin{align*}
\theta_1^t(\xi) &= (\xi - \alpha^2)e^{\alpha t} \sin \left(\sqrt{\xi}(t-r)\right) - 2\alpha \sqrt{\xi} e^{\alpha t} \cos \left(\sqrt{\xi}(t-r)\right) + \\
&\quad + (\alpha^2 - \xi)e^{\alpha(2r+w)} \sin \left(\sqrt{\xi}(-2r+t-w)\right) + (\alpha^2 - \xi)e^{\alpha(r+w)} \sin \left(\sqrt{\xi}(r-t+w)\right) + \\
&\quad + 2\alpha \sqrt{\xi} e^{\alpha(2r+w)} \cos \left(\sqrt{\xi}(-2r+t-w)\right) - 2\alpha \sqrt{\xi} e^{\alpha(r+w)} \cos \left(\sqrt{\xi}(r-t+w)\right) + \\
&\quad + (\alpha^2 - \xi) \sin \left(\sqrt{\xi}t\right) + 2\alpha \sqrt{\xi} \cos \left(\sqrt{\xi}t\right).
\end{align*}

In particular, for \( r = 0 \) we have

\[ \theta_1(\xi) = \frac{e^{-at}}{\sqrt{\xi}(\alpha^2 + \xi)} \begin{cases} 
0, & t < 0; \\
\sqrt{\xi} e^{at} - \alpha \sin \left(\sqrt{\xi}t\right) - \sqrt{\xi} \cos \left(\sqrt{\xi}t\right), & 0 \leq t < w; \\
-\alpha \sin \left(\sqrt{\xi}t\right) - \sqrt{\xi} \cos \left(\sqrt{\xi}t\right) + e^{aw} \alpha \sin \left(\sqrt{\xi}(t-w)\right) + \sqrt{\xi} \cos \left(\sqrt{\xi}(t-w)\right), & t \geq w.
\end{cases} \]

**Proof.** The proofs of these formulas reduce to direct calculations.

### 3.2. Example 1

We consider the problem of solving the hyperbolic equation

\[ \frac{\partial^2}{\partial t^2} u(t, s) + \frac{\partial}{\partial t} u(t, s) - \frac{\partial^2}{\partial s^2} u(t, s) + u(t, s) = b(s)v(t), \quad t \geq 0, \quad s \in [0, 1], \quad (14) \]

with the boundary conditions of the third kind

\[ u(t, 0) - \frac{\partial u}{\partial s}(t, 0) = 0, \quad u(t, 1) + \frac{\partial u}{\partial s}(t, 1) = 0, \quad (15) \]

and with the initial conditions

\[ u(0, s) = 0, \quad \frac{\partial u}{\partial t}(0, s) = 0. \quad (16) \]

Here \( b \in L_2[0, 1] \) is a real function, and \( v: [0, +\infty) \to \mathbb{R} \) is a continuous function.

In the Hilbert space \( H = L_2[0, 1] \) we consider the operator \( A = 1 - \frac{d^2}{dx^2} \) with the domain

\[ D(A) = \{ z \in W_2^2[0, 1] : z(0) - z'(0) = 0, \; z(1) + z'(1) = 0 \} . \]

We set \( \alpha = \frac{1}{2} \) and \( x(t)(s) = u(t, s) \). In this notation, Problem (14)–(16) has the form (5). It is easy to verify that the operator \( Q = A - \alpha^2 1 = \frac{3}{4} 1 - \frac{d^2}{dx^2} \) is self-adjoint and positive semidefinite; therefore, we can use formula (10) to construct an approximate solution. Obviously, \( D(Q) = D(A) \).

The vectors \( w = (Q - q_k 1)^{-1} b \) from formula (10) are given by the explicit formula

\[ (Q - q_k 1)^{-1} b(s) = C_1(q_k, s) \int_0^s e^{\frac{3}{2}\sqrt{\xi - 4q_k}} b(\xi) \, d\xi + C_2(q_k, s) \int_0^s e^{-\frac{3}{2}\sqrt{\xi - 4q_k}} b(\xi) \, d\xi + C_3(q_k, s) \int_1^s e^{\frac{3}{2}\sqrt{\xi - 4q_k}} b(\xi) \, d\xi + C_4(q_k, s) \int_1^s e^{-\frac{3}{2}\sqrt{\xi - 4q_k}} b(\xi) \, d\xi, \quad (17) \]
where

\[
C_1(q_k, s) = \frac{1}{C_0(q_k)} \left( e^{-\frac{1+2}{2} \sqrt{3-4q_k}} (1 + 4q_k) - e^{\frac{1+2}{2} \sqrt{3-4q_k}} (\sqrt{3-4q_k} + 2)^2 \right),
\]

\[
C_2(q_k, s) = \frac{1}{C_0(q_k)} \left( -e^{-\frac{1+2}{2} \sqrt{3-4q_k}} (\sqrt{3-4q_k} - 2)^2 + e^{\frac{1+2}{2} \sqrt{3-4q_k}} (1 + 4q_k) \right),
\]

\[
C_3(q_k, s) = \frac{1}{C_0(q_k)} \left( e^{-\frac{1+2}{2} \sqrt{3-4q_k}} (1 + 4q_k) - e^{\frac{1+2}{2} \sqrt{3-4q_k}} (\sqrt{3-4q_k} - 2)^2 \right),
\]

\[
C_4(q_k, s) = \frac{1}{C_0(q_k)} \left( e^{-\frac{1+2}{2} \sqrt{3-4q_k}} (1 + 4q_k) - e^{\frac{1+2}{2} \sqrt{3-4q_k}} (1 + 4q_k) \right),
\]

\[
C_0(q_k) = \sqrt{3-4q_k} (\sqrt{3-4q_k} - 2)^2 e^{-\frac{1}{2} \sqrt{3-4q_k}} + e^{\frac{1}{2} \sqrt{3-4q_k}} \sqrt{3-4q_k}(2 + \sqrt{3-4q_k})^2.
\]

We note that to obtain formula (17) it suffices [5] to solve the boundary value problem

\[-w'' + \left( \frac{3}{4} - q_k \right) w = b,\]

\[w(0) - w'(0) = 0,\]

\[w(1) + w'(1) = 0.\]

In particular, for \(b(s) = 1, s \in [0, 1]\), formula (17) has the form

\[(Q - q_k 1)^{-1} b(s) = \frac{8 - 8e^{\frac{1}{2} \sqrt{3-4q_k}} - 8e^{\frac{1}{2} \sqrt{3-4q_k}} + 4e^{\frac{1}{2} \sqrt{3-4q_k}} (\sqrt{3-4q_k} + 2) - 4\sqrt{3-4q_k}}{(e^{\frac{1}{2} \sqrt{3-4q_k}} (\sqrt{3-4q_k} + 2) - \sqrt{3-4q_k} + 2) (3 - 4q_k)}.
\]

Also there exists an explicit formula for the vectors \(w = (\alpha_k 1 + \beta_k Q)(Q^2 + \mu_k Q + \nu_k 1)^{-1} b\) from formula (10), but it is very cumbersome, so we do not write it here. We note only that it follows from the formula for the \(Q\) of the solution of the boundary value problem

\[w^{IV}(s) - \left( \frac{3}{2} + \mu_k \right) w''(s) + \left( \frac{9}{16} + \frac{3}{4} \mu_k + \nu_k \right) w(s) = b,\]

\[w(0) - w'(0) = 0,\]

\[w(1) + w'(1) = 0,\]

\[w''(0) - w'''(0) = 0,\]

\[w''(1) + w'''(1) = 0.\]

To construct an approximate solution \(\tilde{x}\) as a rational function \(\xi \mapsto r_t(\xi)\) we use the Padé approximation [10] of degree (3,9) for the function \(\xi \mapsto \theta_t(\xi)\) at zero. To compute \(\tilde{x}(t)\), we expand \(r_t(\xi)\) into the sum of the elementary fractions.

In Figure 2 we present the graph of the approximate solution \(\tilde{x}\) of Problem (14)–(16) for \(t \in [0, 10]\), corresponding to the input function (13) with \(r = 0, w = 1\) and \(b(s) = 1, s \in [0, 1]\). In this case we have

\[\max_{t \in [0, 10]} \max_{\xi \in [0, +\infty]} |\theta_t - r_t(\xi)| \leq \varepsilon = 0.0508.
\]

From this estimate, by virtue of Theorem 3, we obtain the estimate for the error of the approximate solution

\[\|\tilde{x}(t) - x(t)\| \leq \varepsilon \|b\| = 0.0508.
\]
3.3. Example 2

We consider the problem of solving the hyperbolic equation

$$
\frac{\partial^2}{\partial t^2} u(t, s) + \frac{\partial}{\partial t} u(t, s) - \frac{\partial^2}{\partial s^2} u(t, s) + u(t, s) = b(s)v(t), \quad t \geq 0, \quad s \in [0, 1],
$$

(18)

with the boundary conditions of the first kind

$$
u(t, 0) = 0, \quad u(t, 1) = 0,
$$

(19)

and with the initial conditions

$$
u(0, s) = 0, \quad \frac{\partial u}{\partial t}(0, s) = 0.
$$

(20)

Here $b \in L_2[0, 1]$ is a real function, and $v: [0, +\infty) \to \mathbb{R}$ is a continuous function.

Similar to Example 1 we set $\alpha = \frac{1}{2}$, $x(t)(s) = u(t, s)$ and consider the operators $A = 1 - \frac{d^2}{ds^2}$ and $Q = A - \alpha^2 1 = \frac{3}{4} 1 - \frac{d^2}{ds^2}$ with the domain

$$
D(Q) = D(A) = \{ z \in W_2^2[0, 1] : z(0) = 0, \quad z(1) = 0 \}.
$$

In this notation, problem (18)–(20) has the form (5).

To calculate the vectors $w = (Q - q_k 1)^{-1} b$ from formula (10) it suffices to solve the boundary value problem

$$
-w'' + \left(\frac{3}{4} - q_k\right) w = b,
$$

$$
w(0) = 0,
$$

$$
w(1) = 0,
$$
and to obtain an explicit formula for the vectors \( w = (\alpha_k + \beta_k Q)(Q^2 + \mu_k Q + \nu_k 1)^{-1}b \) it suffices to solve the boundary value problem

\[
\begin{align*}
    w^{IV}(s) - \left( \frac{3}{2} + \mu_k \right) w''(s) + \left( \frac{9}{16} + \frac{3}{4} \mu_k + \nu_k \right) w(s) &= b, \\
    w(0) &= 0, \\
    w(1) &= 0, \\
    w''(0) &= 0, \\
    w''(1) &= 0.
\end{align*}
\]

**Figure 3.** The graph of the approximate solution \( \tilde{x}(t)(s) = \tilde{u}(t, s) \) of Problem (18)–(20)

In Figure 3 we present the graph of the approximate solution \( \tilde{x} \) of Problem (18)–(20) for \( t \in [0, 10] \), corresponding to the input function (13) with \( r = 0, \ w = 1 \) and \( b(s) = 1, \ s \in [0, 1] \).

Since the input function is the same as in Example 1, we can use the same functions as scalar rational functions \( r_t \) which approximate \( \theta_t \) on \([0, +\infty)\). Thus, due to Theorem 3, we have the same estimates of the approximate solution. However, the domains of the operator \( Q \) from Examples 1 and 2 do not coincide. Therefore, the results of substituting the operator \( Q \) into the function \( r_t \) are not equal and the corresponding solutions are different.

4. Conclusion
The suggested method allows one to construct an approximate solution by means of the rational approximation of the function \( s_t \) or \( \theta_t \). If one can write an explicit formula for the resolvent of operator \( Q \) or for the vectors \( w = (Q - q_k 1)^{-1}b \) the constructed solution is numerical-analytical. If the analytic form of the solution of the equation \( Qw - q_k w = b \) is unknown, one can solve it approximately and apply the suggested approach. The numerical experiments show the effectiveness of the suggested method. To improve the accuracy of the approximation one
can increase the degree of the Padé approximation or use the rational functions of the best approximation.

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