Ghost-free infinite derivative quantum field theory

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ABSTRACT: In this paper we will study Lorentz-invariant, infinite derivative quantum field theories, where infinite derivatives give rise to non-local interactions, which manifests at the energy scale $M_s$, beyond the Standard Model. We will study a specific class, where there are no new dynamical degrees of freedom other than the original ones of the corresponding local theory. We will show in a systematic way that the Green functions are modified by a non-local extra term that is responsible for acausal effects, which are confined in the region of non-locality, i.e. $M_s^{-1}$. Indeed, the standard time-ordered structure of the causal Feynman propagator is not preserved; moreover, the retarded Green function turns out to be non-vanishing for space-like separations. Physically, formulating such theories in the non-local region with Minkowski signature is not sensible, but they have a beautiful Euclidean interpretation. We will show how such non-local construction ameliorates ultraviolet/short-distance singularities suffered typically in the local quantum field theory, in particular the Euclidean 2-point correlation function is singularity-free on the lightcone surface. We will show that non-locality and acausality are inherently off-shell in nature, and as such classical physics do not make sense inside the non-local region, or for momenta higher than $M_s$, but only quantum amplitudes are physically meaningful, so that all the perturbative quantum corrections have to be consistently taken into account. Moreover, the analytic continuation of the Wightman function, i.e. the Schwinger function, does not coincide with the Euclidean propagator mainly due to the fact that it solves the homogeneous field equations, and is not at all affected by the non-locality. As a consequence, the local commutativity and the reflection positivity conditions still hold.
1 Introduction and beyond 2 derivatives

In nature, a simple 2 derivative field theory captures all aspects of local interactions - both in a classical and in a quantum sense. However neither locally nor globally, nature forbids going beyond 2 derivative kinetic terms. In this sense, there is no prohibition in constructing higher derivative Lorentz-invariant (and diffeomorphism invariant in the context of curved spacetime) kinetic terms. Higher derivative kinetic terms may harbor certain kind of classical and quantum instabilities depending on the nature of the sign of the kinetic terms; for example, Ostr’ogradsky instability [1] can arise, due to the fact that the Hamiltonian density is unbounded from below. This classical instability can also be seen at a quantum level, in the Lagrangian formalism, especially when there are extra propagating degrees of freedom, which comes with a negative residue in the propagator— an indication of a ghost-like degree of freedom. Typically, such instabilities are considered to be safe in low energy effective field theories, at energy scales much below the cut-off, but the ghost problem becomes important at high energies, towards the ultraviolet (UV) scales. There is one particular avenue, where higher derivatives play a very significant role - which is massless gravitational interaction.

It has been known for a while that the quadratic curvature theory of gravity is renormalizable in 4 dimensions [2] \(^1\), but contains a massive spin-2 Weyl ghost as a dynamical degree of freedom. Indeed, being a 4-dimensional higher derivative theory of gravity, it improves the short distance behavior of gravitational interaction, but not sufficiently strong enough to resolve some of the thorny issues of gravity - such as classical singularity problems in cosmology and blackhole solutions; these singular solutions still persist. Recently, it has been noticed that theories with kinetic terms made of derivatives of infinite order are better equipped to handle the issue of ghost. In fact, this classic observation was made in the context of gravity and gauge theory first [3–6].

In particular, in Ref.[7] it was explicitly shown that the most general quadratic curvature gravitational action (parity-invariant and torsion-free), with infinite covariant derivatives can make the gravitational sector free from the Weyl ghost and, moreover, the infinite derivative action is free from classical singularities, such as blackhole type [7–16] \(^2\) and cosmological type [19–24]. The modified graviton propagator around the Minkowski background in 4 dimensions is given by [7]

\[
\Pi(-k^2) = \frac{1}{a(-k^2)} \Pi_{GR}(-k^2),
\]

where \(\Pi_{GR}(-k^2) = \mathcal{P}^2/k^2 - \mathcal{P}^0/2k^2\) is the graviton propagator in Einstein’s general relativity (GR) expressed in terms of the spin-projection operators along the spin-2 and spin-0 components, respectively \(^3\). The presence of infinite covariant derivatives are captured by

\(^1\)Quadratic curvature action contains terms like \(R^2\), \(R_{\mu\nu}R^{\mu\nu}\), \(C^\nu_{\mu\rho\sigma}C^{\mu\rho\sigma}\), where \(\mu, \nu = 0, 1, 2, 3\), \(C\) stands for the Weyl tensor. In 4 dimensions one can further reduce the action with the help of Gauss-Bonnet identity.

\(^2\)Previously, arguments were provided regarding non-singular solutions in Refs. [17, 18].

\(^3\)See Refs. [7, 25, 26] for a pedagogical review on the spin-projection formalism and its application to the computation of the graviton propagator.
\( a(-k^2) \), which can contain in principle \textit{infinitely} many poles. This means \textit{infinitely} many new degrees of freedom, other than the massless graviton propagating in 4 dimensions. The key observation here is to \textit{avoid} the presence of the extra degrees of freedom, and keep \textit{solely} the original transverse and traceless graviton as the only dynamical degree of freedom.

Thus in order to avoid \textit{extra} poles in the propagator the form of \( a(-k^2) \) is constrained by [5–7, 19]:

\[
a(-k^2) = e^{\gamma(k^2/M_s^2)}, \quad a(-k^2) \to 1 \text{ if } k/M_s \to 0,
\]

where \( \gamma(k^2) \) is an \textit{entire function} which ensures that there are no extra poles in the complex plane and \( M_s \) is the new scale of physics \(^4\). The absence of \textit{ghosts} can be understood by the fact that there are no new dynamical degrees of freedom left in the propagator\(^5\).

Indeed, at low energies \( k \ll M_s \), the quadratic curvature graviton propagator in Eq. (1.1) reduces to that of the Einstein-Hilbert propagator of GR in 4 dimensions, as expected; while at high energies, \( k \gg M_s \), the graviton propagator is exponentially suppressed. Gravitational interaction is derivative in nature, therefore the vertex operator gets modified by an exponential enhancement. The interplay between the graviton propagator and the vertex operator leads to the non-locality in the momentum space. Indeed, the structure of non-locality is hidden in the form factor \( a(-k^2) \), as shown in Ref. [7].

Besides having very interesting applications in resolving singularities in blackhole physics and in cosmology, at a quantum level, it is believed that the introduction of such form-factors can make the gravitational theory UV-finite, beyond 1-loop, as discussed into details in Refs.\([4, 6, 29, 30]\). In this respect, non-local interactions, indeed, ameliorate the UV aspects of gravity at short distances and small time scales\(^6\).

The appearance of non-locality in string theory is very well known, the infinite derivative operators appear in the string field theory (SFT) \([31, 32]\), where they are known as \( \alpha' \) corrections. In STF vertices arise of the following form \([33]\)

\[
V \sim e^{c \alpha' \Box}
\]

where \( c \sim \mathcal{O}(1) \) is a dimensionless constant that can change depending on whether one considers either open or closed string, and \( \alpha' \) is the so called universal Regge slope, and \( \Box = \eta_{\mu\nu} \partial^\mu \partial^\nu \) is the d’Alambertain operator in flat spacetime, where \( \eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1) \).

Note that \( \alpha' = 1/M_s^2 \) is a dimensionful coupling, where \( M_s \) is denoted to be the string tension. At the phenomenological level, there have been attempts to construct a model

\(^4\)It is worth mentioning that such a scale of non-locality \( M_s \) has been constrained in different field theories. For instance, in the case of gravitational interaction one has the lower bound \( M_s > 0.004\text{eV} \) coming from torsion balance experiments as pointed out in Ref.\([27]\).

\(^5\)See also Ref.\([28]\) where the ghost problem has been discussed in the non-perturbative scenario of asymptotic safety.

\(^6\) Note that the arguments are mostly based on power counting renormalizability. In \([4]\) 1-loop counter terms have been presented which are local, but the veracity of the results have not been checked elsewhere.

In Ref.\([30]\) attempts have been made to replicate the results of \([4]\), in infinite derivative scalar field theory which has all the properties of gravitational interaction around a constant curvature background. It was shown explicitly in Ref.\([30]\) that up to 2-loops there are no divergences, provided propagator and vertices are properly dressed.
with infinite derivative Higgs and fermion sector, which indeed ameliorates the UV aspects of the Abelian Higgs [34, 35].

Motivated by the success of the infinite derivative gravity, and the success of open SFT, it is worthwhile to investigate some quantum aspects of infinite derivative field theories in more detail \(^7\). We wish to study some properties of Lorentz-invariant infinite derivative quantum field theories with exponential analytic form-factors made of derivatives of infinite order. We will treat the simplest case of a scalar field. The paper is organized as follows.

- In Section 2, we will introduce the action for a real scalar field and analyze into details the structure of the propagator, and emphasize that non-locality is important only when the interactions are switched on. We will see how to perform calculations with operators involving derivatives of infinite orders.

- In Section 3, we will show that non-locality leads to a violation of causality in a space-time region whose size is given by the scale of non-locality \(l_s = 1/M_s\); in particular, we will see that the retarded Green function becomes acausal, while local commutativity is still preserved.

- In Section 4, we will show that because of the violation of causality the concepts of space and time are not well-defined in the region of non-locality, where the usual notion of the Minkowski spacetime does not apply; therefore we are physically forced to describe the region of non-locality with tools of Euclidean field theory, in which there is no concept of real time. We will show that in Euclidean space the infinite derivative theory acquires a beautiful interpretation. Indeed, we will compute the Euclidean 2-point correlation function and show that it is non-singular on the light-cone surface, unlike in local QFT, due to the regularizing property of non-local interactions.

- In Section 5, we will discuss the smearing out on a region of size \(1/M_s\) in the vertex. In particular, non-locality is inherently off-shell, and as such only quantum scattering amplitudes are physically meaningful inside the non-local region, or for momenta \(k^2 > M_s^2\), so that all perturbative quantum corrections have to be consistently taken into account. For instance, once the propagator is dressed all scattering amplitudes, s- t- and u-channels, are well-defined and exponentially suppressed in the UV regime.

\(^7\)In general, non-locality can be thought at least in two different ways: (i) as discretization of the space-time; (ii) or purely related to the interaction in systems defined in a continuum space-time. In the case (i) there would be a minimal length-scale given by the size of the unit-cell in such a discrete background, and it is often identified with the Planck length, \(\ell_p \sim 1/M_p\), where \(M_p \sim 10^{19}\) GeV, or the string scale below the Planck scale in 4 dimensions. As for (ii), the non-locality does not affect the kinematics at the level of free-theory, but it becomes relevant only when dynamics is considered. In the free-theory such a non-locality would not play any role, but it would become relevant as soon as the interaction is switched on. In this regard, we will be investigating the latter scenario, where we will consider a continuum space-time and introduce non-locality through form-factors into either the kinetic operator or the interaction vertex. First attempts along (ii) trace back in the fifties, when people were still facing the problem of ultraviolet (UV) divergences in quantum field theory and renormalization was still not very well understood, thus an alternative possibility to deal with divergences was the introduction of non-local interactions with the aim to regularize the theory and make it finite in the UV. These developments also encouraged a deeper understanding of field theories from an axiomatic point of view [36, 37].
• In Section 6, we will show that the analytic continuation of the Wightman function, i.e. the Schwinger function, does not coincide with the Euclidean propagator in infinite derivative quantum field theory, unlike in the local case, and discuss its consequences.

• In Section 7, we will present summary and conclusions.

2 Infinite derivative action

We now wish to introduce a Lorentz-invariant infinite derivative field theory for a real scalar field $\phi(x)$ by an action:

$$ S = \frac{1}{2} \int d^4x d^4y \phi(x) K(x-y) \phi(y) - \int d^4x V(\phi(x)), \quad (2.1) $$

where the operator $K(x-y)$ in the kinetic term makes explicit the dependence on the field variables at finite distances $x-y$, signaling the presence of a non-local nature; the second contribution to the action is a standard local potential term. We can rewrite the kinetic term as follow

$$ S_K = \frac{1}{2} \int d^4x d^4y \phi(x) K(x-y) \phi(y) $$

$$ = \frac{1}{2} \int d^4x d^4y \phi(x) \int \frac{d^4k}{(2\pi)^4} F(-k^2) e^{i k \cdot (x-y)} \phi(y) $$

$$ = \frac{1}{2} \int d^4x \phi(x) \int \frac{d^4k}{(2\pi)^4} e^{i k \cdot (x-y)} \phi(y) $$

$$ = \frac{1}{2} \int d^4x \phi(x) F(\Box) \phi(x), \quad (2.2) $$

where $F(-k^2)$ is the Fourier transform of $K(x-y)$, and we have used the integral representation of the Dirac delta, $\int \frac{d^4k}{(2\pi)^4} e^{i k \cdot (x-y)} = \delta^{(4)}(x-y)$. From Eq.(2.2) one can notice that the operator $K(x-y)$ has the following general form [38]:

$$ K(x-y) = F(\Box) \delta^{(4)}(x-y). \quad (2.3) $$

Note that the action in Eqs.(2.1)-(2.2) is manifestly Lorentz invariant, thus it is possible to define a divergenceless stress-energy momentum tensor [39]. Note that $\Box$ is dimensionful, and strictly speaking we should write $\Box/M_s^2$. For brevity, we will suppress $M_s$ in the definition of the form factors from now on.

Note that inspite of this, the action without the potential has no non-locality, as we will show below. The homogeneous solution obeys the local equations of motion, i.e., the plane wave solution of the local field theory, see our discussion in Section 2.2.

2.1 Choice of kinetic form factor

So far we have not required any property for the form factor $F(\Box)^8$, other than being Lorentz invariant; however it has to satisfy special conditions in order to define a consistent quantum field theory, in particular absence of ghosts at the tree level. We will restrict the

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*In the following we will not refer to the operator $K(x-y)$ anymore, but we will speak in terms of $F(\Box)$. 

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class of operators by demanding $F(\Box)$ to be an entire analytic function. We can now apply the Weierstrass factorization-theorem for entire functions, so that we can write:

$$F(\Box) = e^{-f(\Box)} \prod_{i=1}^{N} (\Box - m_i^2), \quad (2.5)$$

where $f(\Box)$ is also an entire function, $N$ can be either finite or infinite and it is related to the number of zeros of the entire function $F(\Box)$. From a physical point of view, $2N$ counts the number of poles in the propagator that is defined as the inverse of the kinetic operator in Eq.(2.5). The exponential function does not introduce any extra degrees of freedom and it is suggestive of a cut-off factor that could improve the UV-behavior of loop-integrals in perturbation theory, moreover it contains all information about the infinite-order derivatives:

$$e^{-f(\Box)} = \sum_{n=0}^{\infty} \frac{f_n}{n!} \Box^n, \quad (2.6)$$

where $f_n := \partial^{(n)} e^{-f(\Box)}/\partial \Box^n \big|_{\Box=0}$. By inverting the kinetic operator in Eq. (2.5), we obtain the propagator that in momentum space reads

$$\Pi(k) = e^{f(-k^2)} \prod_{i=1}^{N} \frac{-i}{k^2 + m_i^2}. \quad (2.7)$$

One can immediately notice that if $N > 1$ ghosts appear. Indeed, we can decompose the propagator in Eq.(2.7) as

$$e^{f(-k^2)} \prod_{i=1}^{N} \frac{1}{k^2 + m_i^2} = e^{f(-k^2)} \sum_{i=1}^{N} \frac{c_i}{k^2 + m_i^2}, \quad (2.8)$$

where the coefficients $c_i$ contain the sign of the residues of the propagator at each pole; then by multiplying with $e^{-f(-k^2)} k^2$, and taking the limit $k^2 \to \infty$, we obtain

$$0 = \sum_{i=1}^{N} c_i, \quad (2.9)$$

which means that at least one of the coefficients $c_i$ must be negative in order to satisfy the equality in Eq.(2.9), namely, at least one of the degrees of freedom must be ghost like.

Let us remind that an entire function is a complex-valued function that is holomorphic at all finite points in the whole complex plane. It is worthwhile to mention that in literature there are also examples of field theory where the operator is a non-analytic function. For instance in causal-set theory \cite{40, 41}, the Klein-Gordon operator for a massive scalar field is modified as follows

$$F(\Box + m^2) = \Box + m^2 - \frac{3\ell_p^2}{2\pi^2} (\Box + m^2)^2 \left[3\gamma - 2 + \ln \left(\frac{3\ell_p^2 (\Box + m^2)^2}{2\pi}\right)\right] + \cdots, \quad (2.4)$$

where $\gamma$ is the Euler-Mascheroni constant and $\ell_p$ is the appropriate length scale; note also the presence of branch cuts once analyticity is given up.

We adopt the convention in which the propagator in the Minkowski signature is defined as the inverse of the kinetic term times the imaginary number $'i'$. 
this paper, instead, we will focus on the case $N = 1$, so that tree-level unitarity will be preserved and no ghosts whatsoever will be present in the physical spectrum of the theory.

Let us now fix the function $f(\Box)$ in the exponential. As we have already mentioned, it has to be an entire function, moreover it has to recover the local Klein-Gordon operator, i.e. 2-derivatives differential operator, in the IR regime, $\Box/M_s^2 \to 0$. In this paper we will mainly consider polynomial functions of $\Box$, in particular we will study the simplest operator

$$
11 f(\Box) = (-\Box + m^2)^n M_s^{2n} \implies F(\Box) = e^{-\frac{1}{2} f(\Box)} (\Box - m^2),
$$

where $n$ is a positive integer and we have explicitly reinstated $M_s$. In the infinite derivative gravitational action, the form of $f(\Box)$ remains very similar, except $m = 0$ [7].

### 2.2 Field redefinition and non-local interaction

The infinite derivative field theory introduced in Eqs.(2.1) and (2.2) shows a modification in the kinetic term. However, note that we can also define an infinite derivative field theory where the kinetic operator corresponds to the usual local Klein-Gordon operator by making the following field re-definition:

$$
\tilde{\phi}(x) = e^{-\frac{1}{2} f(\Box)} \phi(x)
= \int d^4 y F(x - y) \phi(y),
$$

where $F(x - y) := e^{-\frac{1}{2} f(\Box)} \delta^{(4)}(x - y)$; the quantity $F(x - y)$ is the kernel of the differential operator $e^{-\frac{1}{2} f(\Box)}$. By inserting such a field redefinition into the action in Eq.(2.1), we obtain an equivalent action that we can still name by $S$ [12]:

$$
S = \frac{1}{2} \int d^4 x \tilde{\phi}(x) (\Box - m^2) \tilde{\phi}(x) - \int d^4 x V \left( e^{\frac{1}{2} f(\Box)} \tilde{\phi}(x) \right).
$$

From Eq.(2.12) it is evident that now the form factor $e^{\frac{1}{2} f(\Box)}$ appears in the interaction term and that non-locality only plays a crucial role when the interaction is switched on as the free-part is just the standard local Klein-Gordon kinetic term. Such a feature of non-locality is relevant only at the level of interaction, this will become more clear below, when we will discuss homogeneous (without interaction-source), and inhomogeneous (with interaction-source) field equations.

### 2.3 Homogeneous field equations: Wightman function

We can now determine the field equation for a free massive scalar field by varying the kinetic action in Eq.(2.2) in the case of $N = 1$ degree of freedom, see section 2.1, and we obtain

$$
F(\Box) \phi(x) = 0 \iff e^{-f(\Box)} (\Box - m^2) \phi(x) = 0,
$$

See also Ref.[6, 27, 29] for other possible choices of entire functions that improve the UV-behavior.

However, the real fields of the theory are $\phi$, and not $\tilde{\phi}$. In fact, the presence of infinite derivatives in the kinetic operators is crucial to make the Euclidean Green functions singularity-free on the lightcone surface (see Section 4.1). Furthermore, in gravity and in gauge theories infinite derivatives can appear in both kinetic operators and interaction vertices.
that is a homogeneous differential equation of infinite order. One of the first question one needs to ask is how to formulate the Cauchy problem corresponding to Eq.(2.13) or, in other words, whether we really need to assign an infinite number of initial conditions in order to find a solution; if this is the case we would lose physical predictability as we would need an infinite amount of information to uniquely specify a physical configuration. Fortunately, as pointed out in Ref.[42], what really fixes the number of independent solutions is the pole structure of the inverse operator $F^{-1}(\square)$. For instance, as for Eq.(2.13) we have two poles solely given by the Klein-Gordon operator $\square - m^2$, which implies that the number of initial conditions and independent solutions is also two.

In particular, note that the equality $(\square - m^2)\phi(x) = 0$ also solves Eq.(2.13), namely the two independent solutions of Eq.(2.13) are given by the same two independent solutions of the standard local Klein-Gordon equation:\footnote{See also Ref.[43] for an alternative method to count the number of initial conditions and degrees of freedom with infinite order differential equations.}

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} (a^{\dagger}_{\vec{k}}e^{ik\cdot x} + a^*_{\vec{k}}e^{-ik\cdot x}), \quad (2.14)$$

where $k \cdot x = -\omega_k x_0 + \vec{k} \cdot \vec{x}$, with $\omega_k = \sqrt{k^2 + m^2}$. The coefficients $a_{\vec{k}}$ and $a^*_{\vec{k}}$ are fixed by the initial conditions and once a quantization procedure is applied they become the usual creation and annihilation operators satisfying the following commutation relations:

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}'), \quad [a^\dagger_{\vec{k}}, a_{\vec{k}'}] = 0 = [a_{\vec{k}}, a^*_{\vec{k}'}]. \quad (2.15)$$

Furthermore, let us remind that the \textit{Wightman function} is defined as a solution of the homogeneous differential equation Eq.(2.13), thus from the above considerations it follows that it is not affected by the infinite derivative modification.

Indeed, in a local field theory the Wightman function is found by solving the homogeneous Klein-Gordon equation, and reads\footnote{The normalization factor $\frac{1}{(2\pi)^3\sqrt{2\omega_k}}$ in the field-decomposition Eq.(2.14) is consistent with the following conventions for the creation operator $a^\dagger_{\vec{k}}(0) = \frac{1}{\sqrt{2\omega_k}}|\vec{k}\rangle$, for the states-product $|\vec{k}\rangle|\vec{k}'\rangle = 2\omega_k(2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}')$ and for the identity in the Fock space $\mathbb{I} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{2\omega_k} |\vec{k}\rangle\langle\vec{k}|$. With such conventions, the canonical commutation relation for free-fields reads $[\phi(x), \pi(y)]_{x^0 = y^0} = \delta^{(3)}(\vec{x} - \vec{y})$, where $\pi(y)$ is the conjugate momentum to $\phi(y)$.}

$$W_L(x - y) = \int \frac{d^4k}{(2\pi)^4} \theta(k^0)\delta^{(4)}(k^2 + m^2)e^{ik\cdot(x-y)}. \quad (2.16)$$

The corresponding infinite derivative Wightman function would be defined by acting on Eq.(2.16) with the operator $e^{f(\square)}$. However, because of the Lorentz-invariance of the operator $e^{f(\square)}$, with $f(\square)$ being an entire analytic function, Eq.(2.16) will only depend on $k^2$ in momentum space. Therefore, given the on-shell nature of $W_L(x - y)$ through the presence
of $\delta(4)(k^2 + m^2)$, one has\footnote{Note that Wightman function for the free-theory can get modified in field theories with non-analytic form factors, see Refs. [40, 41], in our scenario this is not the case.}

$$W(x - y) = e^{f(\square)}W_L(x - y)$$
$$= \int \frac{d^4k}{(2\pi)^3} e^{f(-k^2)\theta(k^0)\delta(4)(k^2 + m^2)e^{ik(x - y)}}$$
$$= e^{f(m^2)} \int \frac{d^4k}{(2\pi)^3} \theta(k^0)\delta(4)(k^2 + m^2)e^{ik(x - y)}.$$  \hspace{1cm} (2.17)

The exponential operator only modifies the local Wightman function by an overall constant factor $e^{f(m^2)}$ that can be appropriately normalized to 1:

$$e^{f(m^2)} = 1.$$  \hspace{1cm} (2.18)

For instance, in the case of exponential of polynomials, as in Eq.(2.10), one has

$$e^{-(-k^2 - m^2)n/M^2n} = 1,$$

once we go on-shell, $k^2 = -m^2$. Thus, infinite derivatives do not modify the Wightman function.

It is now also clear that the commutation relations between the two free-fields evaluated at two different space-time points will not change:

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = W(x - y) - W(y - x) = W_L(x - y) - W_L(y - x).$$  \hspace{1cm} (2.19)

Let us remind that for a massive scalar field, one has:

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = W_L(x - y) - W_L(y - x)$$
$$= \int \frac{d^4k}{(2\pi)^3} \theta(k^0)\delta(4)(k^2 + m^2) \left( e^{ik(x - y)} - e^{-ik(x - y)} \right)$$
$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left( e^{ik(x - y)} - e^{-ik(x - y)} \right)$$
$$= -\frac{i}{2\pi^2 r} \int_0^\infty d|k| |k| \sin(\sqrt{k^2 + m^2}t) \sin(|k|r) \sqrt{k^2 + m^2}$$
$$\equiv i\Delta(t, r)$$  \hspace{1cm} (2.20)

where we have defined $t = x^0 - y^0$ and $\vec{r} = \vec{x} - \vec{y}$; in order to go from the second to the third line in the above equation we have used the equality $\theta(k^0)\delta(k^2 + m^2) = \delta(k^0 - \omega_k)/2\omega_k$; while to go from the third to the fourth we have moved to polar coordinates, with $k \equiv |k|$; $\Delta(t, r)$ is called Pauli-Jordan function. The integral in the fourth line of Eq.(2.20) can be calculated, and in the massive case this is given by [44]:

$$\Delta(t, r) = -\frac{1}{2\pi} \varepsilon(t)\delta(\rho) + \frac{m}{4\pi \sqrt{\rho}} \theta(\rho)\varepsilon(\rho) J_1(m\sqrt{\rho}),$$  \hspace{1cm} (2.21)
where \( \rho := t^2 - r^2 \), \( \varepsilon(t) = \theta(t) - \theta(-t) \), and \( J_1 \) is the Bessel function of the first kind. It is clear that \( \Delta(t, r) \) has support only within the past and future lightcones, indeed it vanishes for space-like separations \( (\rho < 0) \).

Instead, when \( m = 0 \), i.e. in the case of a massless scalar field, one has

\[
\Delta(t, r)|_{m=0} = \frac{1}{4\pi r} [\delta(t + r) - \delta(t - r)],
\]

which has support only on the lightcone surface. By defining the lightcone coordinates \( u = t - r \) and \( v = t + r \), the massless fields are parametrized by \( u = 0 = v \), as indicated by the Dirac deltas in Eq. (2.22), so it follows that the commutation relations in Eqs. (2.21), (2.22) define the lightcone structure of the theory, which is not modified by infinite derivatives.

**Hence, we have shown that the Wightman function, defined as a homogeneous solution, remains the same for both local and infinite derivative theories.**

### 2.4 Inhomogeneous field equations: propagator

From the previous considerations it is very clear that non-locality in infinite derivative theories is not relevant at the level of free-theory, but it will play a crucial role when interactions are included. In fact, in presence of the potential term the field equation is

\[ \mathcal{L} \Phi(x) = \mathcal{L}_V \Phi(x) \]

and in this case the general solution cannot be simply found by solving the local Klein-Gordon equation, but the exponential operator \( e^{-f(\Box)} \) will play a crucial role. Hence, solutions of the inhomogeneous field equation will feel the non-local modification. The simplest example of inhomogeneous equation is the one with a delta source

\[ e^{f(\Box)}(\Box - \mathbf{m}^2)\phi(x) = \frac{\partial V(\phi)}{\partial \phi(x)}, \]

and in this case the general solution cannot be simply found by solving the local Klein-Gordon equation, but the exponential operator \( e^{-f(\Box)} \) will play a crucial role. Hence, solutions of the inhomogeneous field equation will feel the non-local modification. The simplest example of inhomogeneous equation is the one with a delta source \( \delta^{(4)}(x - y) = \delta(x^0 - y^0)\delta^{(3)}(\mathbf{x} - \mathbf{y}) \), whose solution corresponds to the propagator of the theory. In Minkowski signature, the propagator \( \Pi(x - y) \) satisfies the following differential equation:

\[ e^{-f(\Box)}(\Box - \mathbf{m}^2)\Pi(x - y) = i\delta^{(4)}(x - y), \]

\[ \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \frac{i}{2\pi^2 r} \int \frac{d^3k}{(2\pi)^3} e^{-\frac{\mathbf{k}^2}{M^2}} \sin(|\mathbf{k}| t) \sin(|\mathbf{k}| r) \]

\[ = \frac{i M_\pi}{8\pi^{3/2}} \left[ e^{-\frac{1}{4} M^2 (r^2 + t^2)} - e^{-\frac{1}{4} M^2 (r - t)^2} \right]. \]

It is evident from Eq. (2.23) that the massless commutator is different from zero either inside and outside the lightcone on a region of size \( \sim 1/M_\pi \) around the lightcone surface \( u = 0 = v \).
whose solution can be expressed as

\[
\Pi(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{-ie^{f(-k^2)}}{k^2 + m^2 - i\epsilon} e^{ik(x-y)},
\]

(2.26)

where

\[
\Pi(k) = -\frac{ie^{f(-k^2)}}{k^2 + m^2 - i\epsilon},
\]

(2.27)
is the Fourier transform of the propagator in Minkowski signature.

We now wish to explicitly show that the propagator in Eq.(2.26) can not be identified with the time-ordered product of two fields,

\[
\Pi(x - y) \neq \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle.
\]

(2.28)

As we have already seen for the Wightman function, the quantity \(\Pi(x - y)\) can be expressed in terms of the local one, \(\Pi_L(x - y)\), by acting on the latter with the operator \(e^{f(\square_x)}\):

\[
\Pi(x - y) = e^{f(\square_x)}\int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + m^2 - i\epsilon} e^{ik(x-y)}
\]

\[
= e^{f(\square_x)}\Pi_L(x - y)
\]

\[
= e^{f(\square_x)}\left[ \theta(x^0 - y^0)W_L(x - y) + \theta(y^0 - x^0)W_L(y - x) \right],
\]

(2.29)

where we have used the fact that the local propagator \(\Pi_L(x - y)\) corresponds to the time-ordered product between two fields \(\phi(x)\) and \(\phi(y)\). Because of the time-derivative component of the d’Alambertian in the exponential function \(f(\square_x)\), it is clear that the propagator cannot maintain the same causal structure of the Feynman propagator of the standard local field theory.

We now want to find the explicit form of the propagator in the coordinate-space, and in order to do so we need to understand how to deal with the differential operators of infinite order. By using the identity

\[
\Box^n_x = (-\partial^2_{x^0} + \nabla^2_x)^n = \sum_{p=0}^{n} \binom{n}{p} \left( -\partial^2_{x^0} \right)^{(p)} \left( \nabla^2_x \right)^{(n-p)},
\]

(2.30)

and the generalized Leibniz product-rule,

\[
\partial^{(2p)}_{x^0} \left[ g(x^0)h(x^0) \right] = \sum_{q=0}^{2p} \binom{2p}{q} \partial^{(q)}_{x^0} g(x^0) \partial^{(2p-q)}_{x^0} h(x^0),
\]

(2.31)

\textsuperscript{18}The identity in Eq.(2.30) holds in flat spacetime as \([\partial^2_{x^0}, \nabla^2_x] = 0\). In curved spacetime one has to deal with covariant derivatives and \(\Box = g_{\mu\nu} \nabla^\mu \nabla^\nu\), so that the simple decomposition in Eq.(2.30) is not possible.
we can manipulate the expression in the last line of Eq.(2.29), as follows:

\[ e^{f(\Box_x)} \left[ \theta(x^0 - y^0) W_L(x - y) \right] = \sum_{n=0}^{\infty} \frac{f_n}{n!} \sum_{p=0}^{n} \left( \frac{n}{p} \right) (-\partial_{x^0}^2)^{\langle p \rangle} \left[ \theta(x^0 - y^0) (\nabla_x^2)^{\langle n-p \rangle} W_L(x - y) \right] \]

\[ = \sum_{n=0}^{\infty} \frac{f_n}{n!} \sum_{p=0}^{n} \left( \frac{n}{p} \right) (-1)^p \sum_{q=1}^{2p} \left( \frac{2p}{q} \right) \partial_{x^0}^{q-1} \delta(x^0 - y^0) \times \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot (x-y)}}{2\omega_k} (-\vec{k}^2)^{n-p} (-i\omega_k)^{2p-q} \]

\[ = \theta(x^0 - y^0)e^{f(\Box_x)} W(x - y) + \sum_{q=1}^{\infty} \sum_{p=0}^{q-1} \left( \frac{2p}{q} \right) i^q \theta(2p - q) \partial_{x^0}^{q-1} \delta(x^0 - y^0) \times \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot (x-y)}}{2\omega_k} (-\vec{k}^2)^{n-p} \omega_k^{2p-q}, \]

(2.32)

where in the last equality we have introduced the step-function \( \theta(2p - q) \), so that we can extend the summation over \( q \) up to infinity. Let us now compute:

\[ \frac{1}{q!} \frac{\partial^q e^{f(-\vec{k}^2)}}{\partial k^{\langle q \rangle}} = \frac{1}{q!} \sum_{n=0}^{\infty} \frac{f_n}{n!} \sum_{p=0}^{n} \left( \frac{n}{p} \right) \frac{\partial^q (k^0)^2 - \vec{k}^2)^n}{\partial k^{\langle q \rangle}} \]

\[ = \frac{1}{q!} \sum_{n=0}^{\infty} \frac{f_n}{n!} \sum_{p=0}^{n} \left( \frac{n}{p} \right) 2p(2p - 1) \cdots (2p - q + 1) \theta(2p - q) (k^0)^{2p-q} (-\vec{k}^2)^{n-p} \]

\[ = \sum_{n=0}^{\infty} \frac{f_n}{n!} \sum_{p=0}^{n} \left( \frac{n}{p} \right) \frac{(2p)}{q} \theta(2p - q) (k^0)^{2p-q} (-\vec{k}^2)^{n-k}. \]

(2.33)

Note, that the result in Eq.(2.33) allows us to rewrite Eq.(2.32) as follows:

\[ e^{f(\Box_x)} \left[ \theta(x^0 - y^0) W_L(x - y) \right] = \theta(x^0 - y^0)W(x - y) + i \sum_{q=1}^{\infty} \frac{q-1}{q!} \partial_{x^0}^{q-1} \delta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot (x-y)}}{2\omega_k} \frac{\partial^q e^{f(-\vec{k}^2)}}{\partial k^{\langle q \rangle}} \bigg|_{k^0=\omega_k}. \]

(2.34)

Following the same steps for the second term in Eq.(2.29), one has

\[ e^{f(\Box_x)} \left[ \theta(y^0 - x^0) W_L(y - x) \right] = \theta(y^0 - x^0)W(y - x) - i \sum_{q=1}^{\infty} \frac{q-1}{q!} \partial_{x^0}^{q-1} \delta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot (y-x)}}{2\omega_k} \frac{\partial^q e^{f(-\vec{k}^2)}}{\partial k^{\langle q \rangle}} \bigg|_{k^0=\omega_k}. \]

(2.35)

We can now substitute Eqs.(2.34),(2.35) into Eq.(2.29), and obtain a very interesting ex-
pression for the propagator:  

\[ \Pi(x - y) = \theta(x^0 - y^0)W(x - y) + \theta(y^0 - x^0)W(y - x) \]

\[ + i \sum_{q=1}^{\infty} \frac{i^{q-1}}{q!} \partial^{(q-1)}_{x^0} \delta(x^0 - y^0)[W^{(q)}(x - y) - W^{(q)}(y - x)], \]  

(2.36)

where we have defined

\[ W^{(q)}(x-y) := \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot (x-y)}}{2\omega_k^2} \frac{\partial\theta(k^0) e^{f(-k^2)}}{\partial k^0} |_{k^0 = \omega_k}. \]  

(2.37)

From Eq. (2.36) it is very clear that the propagator is not just a time ordered product, but it also has an extra term that breaks the causal structure of the local Feynman propagator: this is a first example of causality violation induced by non-local interactions, as already been shown in Ref. [38].

Note that in the standard local quantum field theory, the time-ordered product corresponds to the Feynman causal propagator that is constructed such that particles with positive-energy travels forward in time, while particles with negative energy (anti-particles) travel backwards in time. Such a structure is not preserved in infinite derivative field theory and causality is violated within \( 1/M_s \). For energies below \( M_s \), the form factor reduces to \( e^{-f(\Box)} \rightarrow 1 \), and hence reaches the local field theory limit in the IR. In Section 3, we will quantify the violation of causality in more detail.

We can also define causal and non-causal (or acausal) parts of the propagator in Eq. (2.36), as follows:

\[ \Pi_c(x - y) = \theta(x^0 - y^0)W(x - y) + \theta(y^0 - x^0)W(y - x) \]  

(2.38)

and

\[ \Pi_{nc}(x - y) = i \sum_{q=1}^{\infty} \frac{i^{q-1}}{q!} \partial^{(q-1)}_{x^0} \delta(x^0 - y^0)[W^{(q)}(x - y) - W^{(q)}(y - x)], \]  

(2.39)

so that the propagator in Eq. (2.29) can be rewritten as

\[ \Pi(x - y) = \Pi_c(x - y) + \Pi_{nc}(x - y). \]  

(2.40)

Since for free-fields \( W(x - y) = W_L(x - y) \), one has \( \Pi_c(x - y) = \Pi_L(x - y) \).

Hence, we have shown that the propagator, which is an inhomogeneous solution, differs for local and infinite derivative theories. In particular, for infinite derivative theories the propagator is not time-ordered.

\[ \text{Eq. (2.36) is in agreement with the result obtained in Ref. [38], where the author has followed a different procedure.} \]

\[ \text{Note that in the non-causal term} \ \Pi_{nc}(x - y) \ \text{there is an infinite number of contact terms that cannot be absorbed through counterterms, thus they will still be there once the theory is renormalized [38].} \]

\[ \text{The concept of propagator assumes physical meaning only when we consider propagation between two interaction vertices; thus, such a causality violation does not appear at the level of free-theory within infinite derivative theory, but only when the interaction is switched on.} \]
2.5 Källén-Lehmann representation

In local quantum field theory, important properties of interacting fields can be described in terms of the so-called Källén-Lehmann representation [46] of several quantities, like Wightman function, commutator and propagator, which can be expressed in terms of vacuum expectation values. In particular, the corresponding representation for the propagator strongly relies on its time-ordered structure.

We have seen that in infinite derivative field theory the propagator is not simply given by a time-ordered product, but it is also made of an acausal contribution which breaks the causal structure of the Feynman propagator; see Eq.(2.40). For this reason the usual Källen-Lehmann representation cannot be defined for the propagator; this remains an open question that still need to be solved.

However, the Wightman function being the solution of the homogeneous field equation, as pointed in the Subsection 2.3, is not affected by the exponential form factor and in the case of interacting fields we can still use the Källen-Lehmann representation. Indeed, the usual steps that bring to the introduction of a spectral density \( \rho(s) \) in local field theory can be followed, and the interacting Wightman function can be expressed as follows

\[
W(x - y) = \int_{0}^{\infty} ds \rho(s)W^{(0)}(x - y; s),
\]

where \( W^{(0)}(x - y; s) \) is the free Wightman function with mass squared \( s \) and \( \rho(s) \geq 0 \). The same can be done for the interacting commutator, or equivalently for the interacting Pauli-Jordan function, which can be expressed as

\[
\Delta(x - y) = \int_{0}^{\infty} ds \rho(s)\Delta^{(0)}(x - y; s).
\]

Thus, it is clear that the Wightman function and the commutator are unaffected by infinite order derivatives also at the level of interacting theory.

3 Causality

In this section, after briefly reviewing the concept of causality which can be stated through different definitions, we will explicitly show that the presence of non-local interactions violate causality in a region whose size is given by \( l_s \sim 1/M_s \) in coordinate space, and for momenta \( k^2 > M_s^2 \) in momentum space.

3.1 A brief reminder

Let us consider a real scalar field \( \phi(x^0, \vec{x}) \) that evolves by means a differential operator \( F(\Box) \) in presence of a source \( j(x^0, x) \), so that it satisfies the following differential equation:

\[
F(\Box)\phi(x^0, \vec{x}) = -j(x^0, \vec{x}).
\]
A formal solution to Eq. (3.1) is given by
\[
\phi(x^0, \vec{x}) = \phi_o(x^0, \vec{x}) + i \int dy^0 d^3y G(x^0 - y^0, \vec{x} - \vec{y}) j(y^0, \vec{y}),
\]
where \(\phi_o(x^0, \vec{x})\) is the solution of the homogeneous equation, and \(G(x^0 - y^0, \vec{x} - \vec{y})\) is the Green function of the differential operator \(F(\Box)\), defined by
\[
F(\Box)G(x^0 - y^0, \vec{x} - \vec{y}) = i \delta(x^0 - y^0) \delta^{(3)}(\vec{x} - \vec{y}).
\]

A system whose evolution is governed by Eq. (3.1) is said to be causal if the corresponding Green function \(G(x^0, \vec{x})\) can be chosen, such that
\[
G(x^0, \vec{x}) = 0, \quad \text{if } x^0 < 0.
\]

The statement in Eq. (3.4) means that a physical system cannot respond to an interaction-source before the source was turned on.\(^{22}\)

The previous definition of causal response holds for both relativistic and non-relativistic systems. A stronger version of the condition in Eq. (3.4) is given by the concept of sub-luminality \([47]\), which is a property that has to be satisfied by any relativistic system. A physical system is said to be sub-luminal, if the Green function \(G(x^0, \vec{x})\) is causal and also vanishes outside the light cone, i.e.
\[
G(x^0, \vec{x}) = 0, \quad \text{if } x^0 < |\vec{x}|.
\]

In this paper, by causality we will also refer to the concept of sub-luminality, namely a causal system will be characterized by a vanishing Green function for space-like separations. Such a Green function is often indicated with a subscript \("R"\) due to its retarded behavior, and we will use the symbol \(G_{L,R}\) in the case of the standard local field theory.

Another definition of the concept of causality is given through the commutator of two fields evaluated in two different space-time points. From a physical-measurement point of view, to preserve causality, we would require that the commutator of the two observables has to vanish outside the lightcone, i.e. for space-like separations. For a real scalar field, such a property can be formulated in the following way\(^{23}\):
\[
[\phi(x), \phi(y)] = 0, \quad \text{if } (x - y)^2 > 0.
\]

When two observables commute, it means that they can be measured simultaneously, i.e. namely one measurement cannot influence the other. If the condition in Eq. (3.6) is violated, there would be correlations between the two measurements performed at two different spacetime points with space-like separation, implying transmission of information\(^{22}\)\(^{23}\).

---

\(^{22}\)Note that such a definition of causality in terms of the Green function, not only holds for classical fields, but also for the expectation value of quantum fields in presence of a source, \(\langle \phi(x) \rangle\).

\(^{23}\)Let us remind that in the mostly positive metric signature \((x - y)^2 > 0\) stands for space-like separation and \((x - y)^2 < 0\) for time-like separation. In the mostly negative convention we would have had the opposite situation.
at a speed faster than light, thus violating causality. The property in Eq.(3.6) is called local commutativity, or sometime microcausality.

Note that the two conditions of causality given in terms of the Green function, see Eq. (3.5), and local commutativity, see Eq. (3.6), are closely related in local field theory. Let us consider a Hamiltonian interaction between a real scalar field $\phi(x)$ and a source \( j(x) \), \( H_{\text{int}} = \int d^3x j(x) \phi(x) \). Consider an initial configuration with a vacuum state at a time \( y^0 = -\infty \) and then switch on the source at a later time. The expectation value of $\phi$ at a spacetime point \((x^0, \vec{x})\), with \( x^0 > y^0 \), can be calculated in the interaction picture, and it is given by [47]

\[
\langle \phi(x) \rangle_j = \langle 0 | e^{i \int_{-\infty}^{x^0} dy^0 d^3y \phi(y) \phi(x)} e^{-i \int_{-\infty}^{x^0} dy^0 d^3y \phi(y) \phi(0)} | 0 \rangle \\
= \langle \phi(x) \rangle_{j=0} + \int_{-\infty}^{x^0} dy^0 d^3y j(y) i \langle 0 | [\phi(y), \phi(x)] | 0 \rangle + \cdots \\
= \langle \phi(x) \rangle_{j=0} - \int_{-\infty}^{x^0} d^3y j(y) i \theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle + \cdots
\]  

(3.7)

where the dots stand for higher order contributions in the interaction-source term. By comparing Eq.(3.2) with Eq.(3.7) we can identify \( \phi_o(x) = \langle \phi(x) \rangle_{j=0} \), and also note that in local field theory, the expectation value of the commutator between the two fields is related to the retarded Green function through the following relation:

\[
G_{L,R}(x - y) = -\theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle .
\]

(3.8)

Hence, if the commutator vanishes for space-like separations, the interaction-source can only generate non-zero modes inside its future lightcone, and thus the definition of causality given in terms of the Green function is consistent with the local commutativity condition. For completeness, we can also write the analog of Eq.(3.8) for the advanced Green function:

\[
G_{L,A}(x - y) = \theta(y^0 - x^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle .
\]

(3.9)

### 3.2 Acausal Green functions in infinite derivative field theory

We have already given an example of causality violation in Section 2, where we have shown that the propagator is not simply given by a time-ordered product, but it has an extra non-causal term which becomes relevant inside the non-local region. We now want to show that the presence of non-local interactions leads inevitably to a violation of causality inside the region \( \sim 1/M_\nu \). In particular, we wish to show explicitly that the non-local retarded Green function, \( e^{iL}[G_{L,R}(x^0, \vec{x})] \), is not vanishing outside the light-cone. We will simply indicate the non-local retarded Green function with the symbol \( G_R \), meaning that it is a non-local quantity, while in presence of the subscript ”L” we would refer to local quantities.

Let us remind that in local quantum field theory the retarded Green function is defined in terms of its Fourier transform as

\[
-G_{L,R}(x - y) = \int_{C_R} \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{(k^0 + i\epsilon)^2 - \vec{k}^2 - m^2}.
\]

(3.10)
where the integration contour $C_R$ is given by the real axis where both the poles: $\pm \omega_{k \cdot k} = \pm \sqrt{k^2 + m^2}$ are avoided from above with two semi-circles. By evaluating the integral in Eq.(3.10) in the massless case, one obtains the retarded Green function in coordinate space:

$$-iG_{L,R}(x - y) = \frac{1}{4\pi r} \delta(t-r) \left[ \frac{1}{\pi} \theta(t) \delta(\rho) \right],$$

(3.11)

where $t = x^0 - y^0$, $\vec{r} = \vec{x} - \vec{y}$ and $\rho = t^2 - \vec{r}^2$. From Eq.(3.11) it is obvious that the retarded propagator is vanishing outside the light-cone, i.e. in the region $t < r$.

We now want to treat the case of infinite derivative field theory and explicitly see that the retarded Green function shows an acausal behavior due to non-local interactions. By following the steps in Eqs.(2.34) and (2.35) together with Eq.(3.8), one can write the non-local retarded Green function as follows

$$G_R(x - y) = e^{\int(\Box)}G_{L,R}(x - y) = -\theta(x^0 - y^0) \langle \{\phi(x), \phi(y)\} | 0 \rangle - \Pi_{nc}(x - y),$$

(3.12)

Note the presence of the acausal (non-causal) term $\Pi_{nc}$ introduced in Eq.(2.39). In particular, we will consider form-factors with polynomial exponents as in Eq.(2.10), and for this specific choice we will see which is the form of $\Pi_{nc}$.

First of all, note that such form factors are divergent at infinity along some directions in the complex plane $k^0$: for example, it can happen that they diverge at $-\infty$ and $+\infty$ along the real axis making it impossible to compute the integral in Eq.(3.11) in Minkowski signature. These kind of divergences make also impossible to define the usual Wick-rotation; this is one of the mathematical reason why in infinite derivative field theory one has to define all amplitudes in the Euclidean space, and in the end of the calculation go back to Minkowski signature by analytic continuation. Below we will give a more detailed discussion about this last observation.

However, note that the in the case of the exponential choice in Eq.(2.10) with even $n$ the non-local form-factor does not diverge along the real axis at infinity, and we can still compute the principal value of the integral in Minkowski signature. Therefore, in this subsection we will consider the following form factors $^{24}$:

$$e^{-f(\Box)} = e^{(\frac{\Box}{M_s^2})^{2n}},$$

(3.13)

and we will work in the massless case for simplicity, $m = 0$. The aim is to compute the following integral:

$$-iG_R(x - y) = \int_{C_R} d^4 k \ e^{-\left(\frac{k^2}{M_s^2}\right)^{2n}} \left(\frac{1}{2\pi^3} \right) \ e^{ik \cdot (x-y)}/(k^0 + i\epsilon)^2 - k^2).$$

(3.14)

$^{24}$Form factors with odd power of $\Box$ can be computed in the region $\sim 1/M_s$ once we go to the Euclidean signature, where one has a very interesting scenario in which all the Euclidean Green functions turn out to be non-singular on the lightcone surface, for any power $n$. See Section 4.1, where we will consider the case for $n = 1$. However, as we will emphasize in Section 4, because of the presence of acausal effects inside the non-local region, all Green functions, with any power $n$, can be physically interpreted only in Euclidean signature for $|x - y| \leq 1/M_s$. 




The integral in Eq.(3.14) can be split into its principal value plus the contribution coming from the two semi-circles that avoid the two poles from above:

\[-iG_R = I_{PV} + I_{2C},\]

(3.15)

where \(I_{2C}\) can be calculated by using the residue theorem, and one can easily show that it is equal to

\[I_{2C} = \frac{1}{8\pi^2} [\delta(t-r) - \delta(t+r)]\]

(3.16)

As for the principal value, one has

\[I_{PV} = \frac{1}{16i\pi^3} \int \frac{1}{r} kdk \text{ P.V.} \int \frac{e^{-\frac{(k^2 + 1)^2}{4M^2_s}}}{k^2 - k^2} \left(e^{i(\omega^2 - k^2)t} - e^{-i(\omega^2 + k^2)t}\right),\]

(3.17)

where \(k \equiv |\vec{k}|\) and \(\omega^2 \equiv k\), as we are working in the massless case. Note that all information about non-locality is contained in the principal value \(I_{PV}\), while \(I_{2C}\) is just a local contribution as it is evaluated at the residues, i.e. on-shell.

After some manipulations, one can show that the principal value in Eq.(3.17) can be recast in the following form:

\[I_{PV} = \frac{1}{4\pi^3} \frac{1}{\rho} \int \frac{\partial}{\partial \rho} \left\{ \varepsilon(\rho) \int_0^\infty \frac{d\zeta e^{-\frac{\zeta^4 n}{M^2_s \rho^2}} \left[K_0(\zeta) + \frac{\pi}{2} Y_0(\zeta)\right]}{\zeta^{4n}} \right\},\]

(3.18)

where \(\varepsilon(\rho)\) is equal to +1 if \(\rho > 0\) (time-like separation), while it is −1 if \(\rho < 0\) (space-like separation); \(Y_0\) and \(K_0\) are Bessel functions of the second kind and the modified Bessel function, respectively.

We can now find an explicit form for the integral in Eq.(3.18), for example we can consider the power \(2n = 2\). In such a case the integral can be computed and expressed in terms of the Meijer-G functions [49], so that the acausal Green function in Eq.(3.14),(3.15) reads

\[-iG_R = \frac{1}{4\pi} \varepsilon(\rho)\delta(\rho) + \frac{1}{2\pi^3} \varepsilon(\rho) \left\{ G_{2,5}^{4,1} \left( \begin{array}{c} 0, 0, 0, 1, 1, 2 \\ 0, 1, 1, 1, 1, 1 \end{array} \right) \frac{M^4_s \rho^2}{256} \right\} + 2\pi^2 G_{3,6}^{4,1} \left( \begin{array}{c} 0, 0, 0, 1, 1, 1, 1, 1 \\ 0, 0, 1, 1, 1, 1, 1, 1 \end{array} \right) \frac{M^4_s \rho^2}{256}.\]

(3.19)

From Eq.(3.19) it follows that the Green function \(G_R\) is not vanishing for space-like separation (\(\rho < 0\)). In Fig. 1 we have plotted such a Green function for \(\rho < 0\) so that it is very clear that it assumes values different from zero, but for large value of \(\rho\), i.e. for \(M^2_s \rho \to \infty\), \(G_R \to 0\), as expected. Thus, the violation of causality is restricted to the spacetime region of size approximatively given by \(~1/M_s\). Such an acausal behavior also implies that the

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25See Appendix 7.1 for all the details of the calculation.
Figure 1. In this plot we have shown the behavior of the retarded Green function as a function of the space-like distance $|\rho|^{1/2} = |x - y|$, $\rho < 0$, for several values of the power in the exponent: $2n = 2$ (continuous thick blue line), $2n = 4$ (dashed orange line), $2n = 8$ (dotted red line) and $2n = 10$ (continuous thin green line). The first two cases can be computed analytically and expressed in terms of the Meijer-G functions (see Eq.(3.19) for the case $2n = 2$), while the last two cases have been obtained numerically. We have set $M_s = 1$ as we are only interested in the qualitative behavior of the functions. It is evident that for small distances non-locality is relevant and we have an acausal behavior, but as soon as $|\rho|^{1/2}$ increases non-locality becomes less important and the Green function tends to a zero value recovering the local result, as expected. The oscillation-effects induced by non-locality increase with the power $2n$.

The evolution of interacting fields will depend acausally upon the initial data (see Appendix 7.2).

In the limit $M_s^2 \rho \rightarrow \infty$ the integral in Eq. (3.18) reduces to:

$$\lim_{M_s^2 \rho \rightarrow \infty} I_{PV} = \frac{1}{8\pi\rho} \left[ \delta(t + \rho) + \delta(t - \rho) \right], \quad (3.20)$$

so that the sum of the two contributions $I_{2C} + I_{PV}$ would recover the local result in Eq.(3.11).

It is worth mentioning that for other values of $2n$ the integral in Eq.(3.18) also shows an acausal behavior, for example we have checked that in the case $2n = 4$ the integral can be still expressed as a combination of Meijer-G functions; for larger values of $2n$ one can proceed numerically. In Fig. 1 we have also shown the behavior of the acausal Green function for $2n = 4, 8, 10$. Moreover, the same procedure that we have used above can be used to compute the advanced Green function, and it will lead to an opposite situation in which $G_A$ will be non-vanishing for time-like separations.

---

26In Appendix 7.2 we will see that such a causality violation also manifests if we move the form factor from the kinetic operator to the interaction term.
3.2.1 Region of non-locality

We have seen that in a Lorentz-invariant quantum field theory, non-local interactions can yield causality violation. In the case of only time - or space-dependence the non-local region is: \( t < 1/M_s \) or \( r < 1/M_s \), respectively. However, in \((3+1)\)-dimensions acausality is confined in a spacetime region defined by the following inequalities:

\[
-\frac{1}{M_s^2} < (x - y)^2 < \frac{1}{M_s^2}.
\]

By looking at the double inequalities in Eq.(3.21), one could be brought to think that causality violation extends on macroscopic scales in the direction of the lightcone surface, i.e. for large values of both \( t \) and \( r \). Indeed, by looking at the structure of the Green function, being Lorentz invariant it will only depend on \( \rho = -t^2 + r^2 \), and will be non-zero for \( r, t \to \infty \), with \( t \lesssim r \). However, as we will now explain, acausal effects only emerge in the region \( r, t < 1/M_s \) when studying the evolution of a field in terms of non-local Green functions.

Let us consider a field \( \phi(x) \) evolving in presence of an interaction-source \( j(x) \), so that its dynamics will be governed by the non-local Green function \( G_R(x - y) \) through the following integral equation:

\[
\phi(x) = \phi_o(x) + i \int d^4yG_R(x - y)j(y),
\]

where \( \phi_o \) resolves the homogeneous field equation. The non-local form factor can be moved on the source under the integral sign, so that the integral in Eq. (3.22) can be written as

\[
\int_{B(x^0, \vec{x})} d^4yG_{R,L}(x - y)e^{i(\rho^0)}j(y),
\]

where now the integration region \( B(x^0, \vec{x}) := \{ (y^0, \vec{y}) : |\vec{x} - \vec{y}| \leq x^0 - y^0 \} \) has support inside the lightcone as \( G_{R,L}(x - y) = 0 \) for \( |\vec{x} - \vec{y}| > x^0 - y^0 \), thus no acausal effect extends on macroscopic scales \((t, r > 1/M_s)\) along the direction of the lightcone surface.

However, the presence of non-local interaction plays a crucial role when studying the initial value problem for the field evolution in Eq.(3.22). Indeed, as discussed in Ref.[38], acausal effects are such that the existence of solutions to the initial value problem can be established, but the uniqueness is lost, which is due to the fact that to obtain a solution on a time interval \([t_0, t_1]\) one has to specify not only initial data for past delays but also for future delays, and the latter would be of the order of the scale of non-locality, \( t_1 + 1/M_s \).

From a physical point of view, such an acausal time delay cannot be measured because every measurement process would average over time-scales longer than \( 1/M_s \).

Hence, all acausal effects in infinite derivative field theory only emerge at microscopic scales, namely inside the non-local region defined by \( t, r < 1/M_s \).

3.3 Local commutativity

We have shown that the presence of non-local interaction implies an acausal behavior of the Green functions. In local field theory the retarded and advanced Green functions
are proportional to the fields commutator, see Eqs.(3.8) and (4.5); however, in infinite derivative theories the Green functions are made of a non-causal extra term $\Pi_{nc}$ other than the piece containing the expectation value of the commutator, see Eq.(3.12). We have learnt that it is responsible for the violation of causality within $1/M_s$, but we have not investigated yet whether also the fields commutator can introduce additional acausal effects through a violation of local commutativity condition.

In Section 2.3, we have shown that the commutator for free-fields is not modified by infinite derivatives, maintaining the same structure of the local theory. We can also show that local commutativity is still preserved even when non-local interactions are switched on, due to the fact that we can still define a Källén-Lehmann representation for the Wightman function, see Eqs.(2.41) and (2.42). Indeed, by looking at the spectral representation in Eq.(2.42), one can notice that for the space-like separation the free Pauli-Jordan function vanishes as

$$\Delta^{(0)}(x-y; s) = \Delta_L^{(0)}(x-y; s) = 0,$$

implying that for interacting fields, one also has

$$\langle 0| [\phi(x), \phi(y)] |0 \rangle = i \Delta(x-y) = \int_0^\infty ds \rho(s) i \Delta^{(0)}(x-y; s) = 0 \quad \text{if} \quad (x-y)^2 > 0. \quad (3.24)$$

In principle, by looking at Eq.(3.12), there can be two terms related to the causality violation that emerges from the Green functions. However, we have shown that local commutativity is not violated by infinite order derivatives, indeed the commutator between two fields is unaffected, meaning that all acausal effects are taken into account by the term $\Pi_{nc}$ coming from the breaking of the time-ordered structure of the propagator.

We wish to point out that such a scenario is due to the special choice of non-local form factors that we have made in Subsection 2.1. In fact, if Lorentz invariance is given up one would violate local commutativity but we could still preserve the time ordered structure: for example, it happens with the choice $f(\nabla^2) = \nabla^2/M_s^2$ as shown in the footnote 17. One can have cases in which the Wightman function is modified but both local commutativity and time-ordered structure are still preserved, as for example in causal-set theory [40, 41], where non-analytic form factors were considered, such that the Wightman shows the presence of branch cuts. Moreover, the choice in terms of the exponential of an entire function for the form factors is fundamental in order to keep unchanged the pole-structure of the Green functions, thus to avoid the introduction of new degrees of freedom in the physical particle spectrum.

**Hence, the choice of the Lorentz invariant analytic form factors in terms of exponential of entire functions are very special while preserving local commutativity: the commutation relations do not alter from local quantum field theory to infinite derivative quantum field theory.**

4 Euclidean prescription

From a physical point of view the presence of acausal effects means that there is no concept of Minkowski spacetime in the non-local region, where there is no concept of past and future;
non-locality is such that we cannot define the usual concepts of space and time. We cannot define clock and rulers to make any kind of measurements inside $1/M_s$. For this reason, we believe that defining physical quantities in Minkowski signature in such a region would not make sense from a physical point of view, but the appropriate way to proceed would be to define Euclidean amplitudes and Euclidean correlators. Indeed, in Euclidean space we do not have any concept of real time, all Euclidean distances are space-like by definition.

Such a physical argument also has a mathematical counterpart. As we have already briefly mentioned in the previous subsection, in infinite derivative field theory the form factors introduce some ambiguities when performing calculations of integrals in momentum space. For example, the exponential form factors with polynomial exponents introduced in Eq. (2.10) can always appear in loop-integral and amplitudes in the form $e^{-(k^2/M_s)n}$ where $n$ is a positive integer. For example, for the calculation of either propagator or any other Green functions, one has to deal with integrals of the following type:

$$I(x) = \int_{-\infty}^{\infty} \frac{d\vec{k}^0}{\omega_k^2} e^{-\frac{(-\vec{k}^2 + \vec{x}^2)}{M_s^2}} e^{ik^0x^0 + i\vec{k} \cdot \vec{x}}.$$  \hspace{1cm} (4.1)

It is easy to understand that the presence of the form factor gives divergent contributions along certain directions in the complex plane $k^0$; for instance, we can consider as examples $n = 1$ and $n = 2$.

• In the case $n = 1$ one has:

$$e^{\frac{k^2 - \vec{k}^2}{M_s^2}} e^{-i\vec{k} \cdot \vec{x} + i\vec{k} \cdot \vec{x}} \sim e^{\frac{\text{Re}(k^0)}{M_s^2}} e^{\frac{\text{Im}(k^0)}{M_s^2}} e^{\frac{\text{Im}(k^0)}{M_s^2}},$$  \hspace{1cm} (4.2)

which diverges at infinity along the directions belonging to the region $|\text{Re}(k^0)| > |\text{Im}(k^0)|$, while it converges to zero along the directions such that $|\text{Re}(k^0)| < |\text{Im}(k^0)|$.

• In the case $n = 2$, the relevant contribution at infinity is given by:

$$e^{\frac{(-\vec{k}^2 + \vec{x}^2)^2}{M_s^2}} e^{-i\vec{k} \cdot \vec{x} + i\vec{k} \cdot \vec{x}} \sim e^{\frac{\text{Re}^2(k^0)}{M_s^2}} e^{\frac{\text{Im}^2(k^0)}{M_s^2}} e^{6 \frac{\text{Im}^2(k^0) \text{Re}^2(k^0)}{M_s^4}},$$  \hspace{1cm} (4.3)

that only diverges at infinity along the directions $\text{Im}(k^0) = \pm \text{Re}(k^0)$, while in the rest of the complex plane it goes to zero at infinity.

Note that such divergences make it almost always impossible to calculate integrals in Minkowski signature, for example the usual Feynman contour prescription does not work anymore, because the contribution coming from the semi-circle in either the lower or the upper half of the complex plane receive an infinite contribution at infinity. It implies that the usual Wick-rotation cannot be defined. Furthermore, in Minkowski signature the optical theorem is not satisfied for amplitudes and unitarity seems to be lost [50]; however one can show that by working in Euclidean space and then analytically continuing the external momenta to Minkowski, the theory turns out to be unitary [50–54]. An important property of such exponential form-factors is that they always go to zero along the imaginary
axis directions, \( \text{Im}(k^0) \rightarrow \pm \infty \), so that amplitudes in Euclidean signature are well-defined and can be legitimately computed.

The recipe to follow is to formulate the infinite derivative quantum field theories in Euclidean signature, where we can define and compute all the amplitudes, and only after doing the calculations we can analytically continue the physical quantities to their real values. Physically, we cannot probe the region of non-locality, in fact making a measurement inside the region \( \sim 1/M_s \) would make no sense. However, we can still falsify field theories for example through scattering processes, indeed, the amplitudes for any process with non-local interactions can be well-defined, see for example Ref.[34]. The region of non-locality can be imagined as an off-shell region in which interactions take place, and we can measure effects by making, as usual, on-shell detections at in - and out-states.

In the next section, we will show how non-locality, through infinite derivatives, can resolve the lightcone singularity from which the local quantum field theory suffers.

4.1 Euclidean 2-point correlation function

In local quantum field theory one has to deal with infinities which need to be regularized in order to give physical meaning to the theory. There are at least three kind of divergences that one can encounter:

1. UV divergences \( (k \rightarrow \infty) \);
2. IR divergences \( (k \rightarrow 0) \);
3. lightcone singularities \( (|x - y| \rightarrow 0) \).

In principle, one can cure IR and UV divergences but, even after the renormalization procedure has been applied, lightcone singularities still remain uncured in both Minkowski and Euclidean signature.

In this section we wish to compute the 2-point correlation function in infinite derivative quantum field theory; as an example we will consider \( \phi^4 \)-theory. In particular, we want to analyze its behavior on the light-cone surface, and see whether non-local interactions can regularize the divergence at \( (x - y) \rightarrow 0 \) from which the local theory suffers. For simplicity, we will focus on the form-factor \( e^{\frac{k^4}{4M_s^2}} \). As we have strongly stressed in Subsection 4, we will be formulating our theory in the Euclidean space; thus let us consider the following Euclidean generating functional:

\[
\mathcal{Z}[J] = \int \mathcal{D}\phi e^{-S_E[\phi]} + \int d^4x J \phi,
\]

where \( J(x) \) is the source-term and the Euclidean action is given by:

\[
S_E[\phi] = \int d^4x \left( -\frac{1}{2} \phi(x)e^{-(\not\!x-m^2)/M_s^2}(\not\!x - m^2)\phi(x) + \frac{\lambda}{4!}\phi^4(x) \right). \tag{4.5}
\]

The functional in Eq.(4.5) can be rewritten in the following way:

\[
\mathcal{Z}[J] = e^{-\frac{\lambda}{4!} \int d^4x \left[ \frac{\phi^4(x)}{M_s^4} \right]} \mathcal{Z}_0[J], \tag{4.6}
\]
where $Z_0[J]$ is the free generating functional:

$$Z_0[J] = \int D\phi e^{\frac{1}{2} \int d^4x \phi e^{-\frac{\Box}{2M^2}} \phi + \int d^4x J\phi + \int d^4y J(x)\Pi(x-y)J(y)},$$

(4.7) and $\Pi(x - y)$ is the propagator in the Euclidean signature:

$$\Pi(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-\frac{k^2 + m^2}{M^2}} e^{ik\cdot(x-y)}}{k^2 + m^2},$$

(4.8)

where, now, $k \equiv (k_4, \vec{k})$ stands for the Euclidean momentum, with $k^4 = -ik^0$, and $x \equiv (x^4, \vec{x})$ for the Euclidean coordinate, with $x^4 = ix^0$.

We are interested in computing the 2-point correlation function that is defined as

$$G(x - y) := \left. \frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)} \right|_{J=0}.$$

(4.9)

By expanding the exponential in Eq.(4.6), we can compute perturbatively the correlator $G(x - y)$; for instance up to the first order in $\lambda$, we obtain:

$$G(x - y) = \Pi(x - y) - \frac{\lambda}{2} \Pi(0) \int d^4z \Pi(x - z)\Pi(z - y) + O(\lambda^2),$$

(4.10)

where at zeroth order we have the free-propagator, while at the first order a tadpole contribution.

Let us start by analyzing the zeroth order of the perturbative expansion in Eq.(4.10), i.e. the Euclidean propagator introduced in Eq.(4.8), in both massless and massive case\textsuperscript{27}.

- In the massless case the integral in Eq.(4.8) can be easily calculated as follows:

$$\Pi(x - y)|_{m=0} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-\frac{k^2}{M^2}} e^{ik\cdot(x-y)}}{k^2} = \int_0^\infty dt \int \frac{d^4k}{(2\pi)^4} \frac{e^{-\frac{k^2}{M^2}} e^{-\frac{t}{M^2} + ik\cdot(x-y)}}{k^2 + \frac{t}{M^2}}$$

$$= \frac{M^4}{16\pi^2} \int_0^\infty dt \frac{e^{-\frac{M^2(x-y)^2}{4(t + M^2)}}}{(1 + M^2t)^2} = \frac{1}{4\pi^2(x - y)^2} \left(1 - e^{-\frac{M^2(x-y)^2}{4}}\right),$$

(4.11)

where we have made use of the Schwinger parametrization,

$$\frac{1}{k^2} = \int_0^\infty dt e^{-tk^2}.$$

(4.12)

\textsuperscript{27}Both these examples comprise the case of odd power of $\Box$ that could not be computed in Minkowski signature, as discussed in section 3.2, and discussion surrounding Eq.(3.13).
First of all, note that in the limit \(M^2(x - y)^2 \to \infty\), we recover the local massless propagator:

\[
\Pi_L(x - y)|_{m=0} = \frac{1}{4\pi^2(x - y)^2}.
\]

(4.13)

More importantly, note that in the limit in which non-locality becomes relevant, i.e. \(M^2_s(x - y)^2 \to 0\), unlike the local case (Eq.(4.13)) the massless propagator in Eq.(4.11) does not diverge but it tends to a finite constant value:

\[
\Pi(0)|_{m=0} = \frac{M_s^2}{16\pi^2}.
\]

(4.14)

The result in Eq.(4.14) is extremely important for what concerns the UV behavior of the theory. The quantity \(\Pi(0)|_{m=0}\) appears as a coefficient of the perturbative series in Eq.(4.10), and in local field theory the renormalization problem arises because of the presence of divergent coefficients. Thus, we have seen a first concrete example of how non-local interaction can improve the UV behavior of the theory. In particular, \(\phi^4\)-theory with non-local interaction becomes finite as discussed in Ref.[30, 34].

- As for the massive propagator, by using again the Schwinger parametrization for \(1/(k^2 + m^2)\), we can write

\[
\Pi(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-(k^2+m^2)/M_s^2}e^{ik(x-y)}}{k^2 + m^2}
\]

\[
= \int_0^\infty dt e^{-tm^2} e^{-\frac{m^2}{M_s^2}} \int \frac{d^4k}{(2\pi)^4} e^{-k^2(t+1/M_s^2)+ik(x-y)}
\]

\[
= \frac{M_s^2}{16\pi^2} \int_1^\infty \frac{dt}{t^2} e^{-\frac{tM_s^2}{41}} e^{-\frac{M_s^2(x-y)^2}{4t}}
\]

\[
= \frac{m^2}{16\pi^2} \int_{\frac{m^2}{M_s^2}}^{\infty} ds e^{-s} e^{-\frac{m^2(x-y)^2}{4s}}.
\]

(4.15)

where from the second to the third line we have made the change of variable \(1 + M_s^2 t \to t\), while from the third to the fourth \(t \to s = tm^2/M_s^2\). Although the integral in Eq.(4.15) cannot be solved in terms of elementary functions as in the massless case (Eq.(4.11)), it can be expressed in terms of the so called cylindrical incomplete function of Sonine-Schlaefli:

\[
\Pi(x - y) = -\frac{1}{4\pi} \frac{m}{|x - y|} S_1\left(-\frac{m^2}{M_s^2}, -\infty; im|x - y|\right),
\]

(4.16)

where the Sonine-Schlaefli function is defined as [57]

\[
S_\nu(-p, -q; iz) := \frac{e^{-iz\nu}}{2\pi i} \left(\frac{z}{2}\right)^\nu \int_p^q dt t^{-\nu-1} e^{-t-\frac{z^2}{4t}}.
\]
We can study the limit \((x - y) \to 0\), and note that the massive propagator is not singular on the light-cone surface. Indeed, in such a case the integral in Eq.(4.15) gives

\[
\Pi(0) = \frac{M^2 e^{-\frac{m^2}{M^2}}}{16\pi^2} \left[ 1 + e^{\frac{m^2}{M^2}} \text{Ei} \left( -\frac{m^2}{M^2} \right) \right],
\]

where

\[
\text{Ei}(x) := - \int_{-x}^{\infty} dt \frac{e^{-t}}{t}
\]

is the so-called exponential-integral function.

4.1.1 First-order correction \(\mathcal{O}(\lambda)\)

So far we have learnt that at the zeroth order in the perturbative expansion in Eq.(4.10) the 2-point correlation function is regular on the light-cone surface unlike the local case where singularities are present. We now want to study the first order correction (tadpole) in Eq.(4.10) and see whether such a regularization property is maintained.

- In the massive case, one can check numerically that the first order correction is non-singular on the lightcone surface.
- In the massless case, the first order correction to the 2-point correlator is singular on the lightcone surface, as we will now show with an explicit calculation. However, this can be made non-singular by dressing the propagator.

At the first order in perturbation theory, the 2-point function for the massless case is given by

\[
G^{(1)}(x - y) \big|_{m=0} = -\frac{\lambda}{2} \Pi(0) \big|_{m=0} \int d^4 z \, \Pi(x - z) \big|_{m=0} \Pi(z - y) \big|_{m=0}
\]

\[
= -\frac{\lambda}{2} \Pi(0) \big|_{m=0} \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-\frac{2k^2}{M^2}} e^{ik \cdot (x-y)}}{k^4} \int_0^\infty d\frac{k}{4\pi^2} \frac{J_1(k|x-y|)}{k^2},
\]

where we have moved to polar coordinates in 4-dimensions: \(d^4 k = k^3 \sin\theta \sin^2\alpha dkd\theta d\alpha d\varphi\).

First of all, note that the integral in Eq.(4.18) has an IR divergence, as we can see more explicitly by introducing an IR cut-off \(L\):

\[
G^{(1)}(x - y) \big|_{m=0} = -\frac{\lambda}{2} \Pi(0) \big|_{m=0} \left\{ -\frac{1}{2\pi^2} \frac{1}{M^2} \frac{1}{(x-y)^2} \left( 1 - e^{-\frac{M^2(x-y)^2}{8}} \right) \right. \\
+ \frac{1}{16\pi^2} \text{Ei} \left( -\frac{M^2(x-y)^2}{8} \right) \left. + \lim_{L \to \infty} \frac{1}{4\pi^2} \frac{L}{|x-y|} \left( 1 - e^{-\frac{|x-y|}{4L}} \right) - \frac{1}{16\pi^2} \text{Ei} \left( -\frac{|x-y|}{4L} \right) \right\}.
\]

It is obvious that in the limit \(L \to \infty\), the IR divergence comes from

\[
\lim_{L \to \infty} \text{Ei} \left( -\frac{|x-y|}{4L} \right) = \infty.
\]
Note that the IR divergence comes together with the lightcone singularity in the massless case - the correlator at the first order in $\lambda$ in Eq. (4.19) also diverges for $|x - y| \to 0$.\(^{28}\)

It is now very natural to ask ourself the following question - whether non-local interaction is sufficient to regularize the Euclidean 2-point correlation function on the lightcone surface and avoid unphysical singularities?

We have seen that the first order correction seems to suggest that non-locality is not sufficient to regularize the lightcone singularity of the 2-point correlation function in the massless case. However, as we will show below, if we consider the full correlator, namely taking into account all quantum perturbative corrections through the so dressed propagator, then we will see that the physical 2-point function becomes regular on the lightcone surface.

### 4.1.2 Dressed 2-point correlation function

Let us consider the Fourier transform $G(k)$ of the correlator in Eq. (4.10), in the more general case of massive scalar field and then we will also specialize to the massless case. It is well known that once one takes into account all perturbative corrections to the 2-point correlation function in momentum space one obtains the so called dressed-propagator that can be expressed in terms of the self-energy $\Sigma(k)$. Thus, we have [34]:

\[
G(k) = \Pi(k) \left( 1 - \Sigma(k)\Pi(k) + \Sigma(k)\Pi(k)\Sigma(k)\Pi(k) - \cdots \right)
= \Pi(k) \sum_{n=0}^{\infty} (-1)^n [\Sigma(k)\Pi(k)]^n
= \frac{\Pi(k)}{1 + \Sigma(k)\Pi(k)} e^{-\frac{k^2 + m^2}{M_s^2}}
= \frac{k^2 + m^2 + \Sigma(k) e^{-\frac{k^2 + m^2}{M_s^2}}}{k^2 + m^2 + \Sigma(k) e^{-\frac{k^2 + m^2}{M_s^2}}},
\]

The self-energy at 1-loop is independent on the external momenta, and reads:

\[
\Sigma = \lambda \int \frac{d^4p}{(2\pi)^4} \frac{e^{-\frac{p^2 + m^2}{M_s^2}}}{p^2 + m^2}, \tag{4.22}
\]

The integral in Eq. (4.22) turns out to be finite in both massless and massive cases; in fact it has the same expression of the Euclidean propagator evaluated at the origin in Eqs. (4.14) and (4.17) for massless and massive cases, respectively.

We are interested in the coordinate-space dressed-correlator, so we need to consider

\(^{28}\)Such a lightcone singularity will also appear for any power of the d’Alambertian $\Box^n$, with any $n$; indeed, it is purely related to the infrared divergence that one has in the massless case. It so happens that infrared divergence and lightcone singularity are mixed.
Figure 2. In this plot we have shown the behavior of the full 2-point correlation function, where all perturbative corrections are taken into account, that we obtained by solving numerically the integral in Eq.(4.23). The blue-line represent the massless case, while the orange line represents the massive case with $m = 1$. We have set $M_s = 1$ for simplicity, as we are only interested in the qualitative behavior around the origin. We can notice that for $|x - y| \to 0$ the correlators tend to a finite value that of course differ for massless and massive cases. The important result is that the 2-point correlation function in infinite derivative quantum field theory is singularity-free unlike the one in local-field theory that instead diverges on the light-cone surface.

the following Fourier transform:

$$
\mathcal{G}(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{\frac{k^2 + m^2}{M_s^2}} e^{ik(x-y)}}{k^2 + m^2 + \Sigma(k)e^{\frac{k^2 + m^2}{M_s^2}}}
$$

$$
= \int \frac{d^4k}{(2\pi)^4} \frac{e^{\frac{k^2 + m^2}{M_s^2}} (k^2 + m^2) + \Sigma(k)}{e^{\frac{k^2 + m^2}{M_s^2}}} e^{ik(x-y)}
$$

$$
= \frac{1}{4\pi^2|x - y|} \int_0^{\infty} \frac{k^2 J_1(k|x - y|)dk}{e^{\frac{k^2 + m^2}{M_s^2}} (k^2 + m^2) + \Sigma(k)}.
$$

The integral in Eq.(4.23) cannot be solved analytically, but we can calculate it numerically, and we note that the full 2-point correlation function is singularity-free on the light-cone surface for both massive and massless cases. In Fig.2, we have shown the numerical solutions of the integral in Eq.(4.23) for both massive and massless cases.

5 Scattering amplitudes

In this section we will clarify the amplitudes in infinite derivative quantum field theory and how can they be well-defined and physically meaningful. First of all, note that some apparent ambiguities may appear when working with amplitudes with the exponentials of the kind $e^{-(k^2/M_s^2)^n}$:
1. $e^{-(k^2/M_s^2)^{2n}}$: tree-level amplitudes with even power of the exponent would be always exponentially suppressed for both time-like and space-like momentum exchange;

2. $e^{-(k^2/M_s^2)^{2n+1}}$: tree-level amplitudes with odd power of the exponent would be exponentially suppressed only for space-like momentum exchange, but they blow up for time-like exchange, i.e. for $k^2 < 0$.

As a consequence, for both even and odd powers the tree-level scattering amplitudes turn out to be exponentially suppressed in the case of $t$- and $u$-channels, while the $s$-channel amplitude is exponentially suppressed only for even powers, but it blows up for odd powers as in this case the momentum exchange is time-like, giving a positive exponent in the exponential which causes the divergence for high energies. Such a divergence behavior appears when the amplitude is made of an internal propagator that connects two cubic vertices (e.g. $\phi^3$-theory).

However, this apparent unphysical divergence only manifest at the level of the bare propagator. In fact, once all quantum corrections are taken into account through the dressed propagator, all the scattering amplitudes become exponentially suppressed, thus physically well-defined; see also for discussions in Ref. [56], where the tree-level scattering amplitudes were computed, and also the dressed vertices and the dressed propagator. The importance of using the dressed propagator, instead of the bare one, also arises when studying the renormalizability of non-local quantum field theories with infinitely many derivatives. As shown in Ref. [30], the procedure of dressing the propagators ameliorates the UV aspects of the theory, making all loop-integrals finite.

Note that in the region of non-locality $\leq 1/M_s$, or in momentum space, for momenta $k^2 \geq M_s^2$, we cannot define any classical concept of spacetime point, but vertices are smeared out such that the external legs and internal lines do not join in one point but they overlap in a region of size $1/M_s$. The crucial role is played by the acausal term $\Pi_{nc}$ defined in Eq.(2.39), which implies causality violation in the vertices. In momentum space $\Pi_{nc}$ reads [38]:

$$
\Pi_{nc}(k) = \int d^4x \Pi_{nc}(x)e^{-ik\cdot x} = i \sum_{q=1}^{\infty} \frac{1}{q!} \frac{1}{2\omega_k} \left[ \frac{\partial(q) e^{f(-k^2)}}{\partial k^0(q)} \right]_{k^0=\omega_k}^{q-1} - \frac{\partial(q) e^{f(-k^2)}}{\partial k^0(q)} \right]_{k^0=-\omega_k}^{q-1} (k^0 + \omega_k)^q \right],
$$

and it is evident that it has no poles and has no absorptive components, namely it is not made of on-shell intermediate states, but it is purely off-shell; indeed, it can be seen as a non-local vertex: $\Pi_{nc}(k) = iV(k)$. Thus, in momentum space the non-local propagator in Eq.(2.40) reads:

$$
\Pi(k) = \Pi_c(k) + iV(k),
$$

where $\Pi_c(k)$ is causal and $V(k)$ tends to zero for $k^2 \ll M_s^2$, but would be relevant for $k^2 \geq M_s^2$. Note that the non-local acausal part of the propagator, $iV(k)$, is made of infinite derivatives and this is the main cause of the smearing of the vertices, which are
Figure 3. We have shown a pictorial illustration for local (left side) and non-local (right side) vertices. We can notice that above the scale of non-locality, $k^2 \geq M^2_s$, non-local interactions are such that the vertices are smeared out on a region of size $1/M_s$. Thus, from Eqs.(5.1,5.2) it is now more clear that in infinite derivative field theories non-locality and acausality manifest as off-shell effects, so that for momenta $k^2 \geq M^2_s$ one has to consider any amplitudes as quantum and consistently take into account all perturbative quantum corrections. In the standard local quantum field theory all internal lines of a Feynman diagram are seen as off-shell, while in infinite derivative quantum field theory the degree of ”off-shellness” increases as also the vertices become non-local. In particular, there is no energy and momentum conservation in one single point, as legs and internal lines overlap on a smeared region of size $1/M_s$, or in terms of momentum, $M_s$. See Fig. 3 for an illustration of local and non-local vertices.

In this respect, bare amplitudes, which can be seen as classical amplitudes, do not make sense within the non-local regime, where we cannot define any classical concept of space-time point, but instead vertices are smeared out. Indeed, what makes sense is the quantum scattering, and therefore the correct procedure will be always to consider the dressed vertices and dressed propagators irrespective of the cases of even powers $e^{-(k^2/M^2_s)^{2n}}$, or odd powers $e^{-(k^2/M^2_s)^{2n+1}}$, for any kind of amplitudes. We will compute these amplitudes for $\phi^3$ interaction.

5.1 $s$- and $t$-channels

In this subsection we will show that once all perturbative corrections are consistently taken into account for any channels and any power $n$ of the d’Alambertian, all scattering amplitudes are well-defined and have the same asymptotic behavior in the UV regime. As it is worth emphasizing that infinitely many derivatives can smear out point like source. In fact, by acting with infinite derivatives on a delta Dirac distribution, which has a point-like support, we obtain a non-point support [15]. For example, in the case of an exponential we obtain a Gaussian smearing:

$$e^{\alpha\partial^2} \delta(x) = \frac{1}{\sqrt{2\alpha}} e^{-\frac{x^2}{4\alpha}},$$  

where we have used the Fourier transform of the Dirac delta distribution.
we have already mentioned above, some ambiguities can arise when considering odd powers, $e^{-(k^2/M_s^2)^{2n+1}}$, in the case of s-channel, where the momentum exchange is time-like, $k^2 < 0$, giving a divergence at high energies, $k^2 \gg M_s^2$.

We now want to explicitly show that by correctly dressing the propagator no such ambiguity would arise. For simplicity, we will consider the case of $\lambda \phi^3$-theory with a non-local kinetic operator, and work in the massless case. A generic tree-level scattering amplitude will be given by:

$$M_n \sim \lambda^2 e^{-(k^2/M_s^2)^n} \frac{k^2}{k^2 + \Sigma_n(k)}.$$  \hspace{1cm} (5.4)

Once we dress the propagator, we will obtain (see Eq.(4.21))

$$M_n \sim \lambda^2 \frac{e^{-(k^2/M_s^2)^n}}{k^2 + \Sigma_n(k)} e^{-(k^2/M_s^2)^n},$$  \hspace{1cm} (5.5)

where the self-energy $\Sigma_n(k)$ for $\lambda \phi^3$-theory, for example at 1-loop, reads

$$\Sigma_n(k) = \lambda^2 \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-(p^2/M_s^2)^n}}{p^2 (p - k)^2} e^{-(p^2/M_s^2)^n},$$  \hspace{1cm} (5.6)

which for even power of $n$ turns out to be always exponentially suppressed for any value of $k^2$, while for odd power of $n$ can blow up for time-like momenta, $k^2 < 0$. Note that, generally, integrals of the type in Eq.(5.6) can blow up for some values of the integration variable $p$; however, by working in Euclidean signature, $p^0 = ip^4$ and $k^0 = i k^4$, the integrals can be computed, and after the computation is made the momentum $k$ can be analytically continued back to Minkowski signature, $k^4 = -ik^0$; see for example for discussions Refs.\[31, 34, 51]. Below we will make an explicit example for the case $n = 1$.

Let us analyze $t$-channel and $s$-channel, for both even and odd powers.

5.1.1 Even powers $\Box^{2n}$

Note that in the case of even powers, $e^{-(k^2/M_s^2)^{2n}}$, the high energy behavior of the scattering amplitudes is the same for both bare and dressed propagator. Indeed, in the UV regime, $k^2/M_s \to \infty$ we have the following asymptotic behavior for the dressed amplitude in Eq.(5.5) with even powers:

$$M_{2n} \sim e^{-(k^2/M_s^2)^{2n}},$$  \hspace{1cm} (5.7)

as $\Sigma_{2n}(k)e^{-(k^2/M_s^2)^{2n}} \to 0$ for $k^2/M_s^2 \to \infty$. Moreover, the asymptotic behavior is also the same for both $t$- and $s$-channels, as the even power, $2n$, does not distinguish between space-like and time-like momentum exchange: $(\pm k^2)^{2n} = (k^2)^{2n}$.

Thus, we have shown that in both cases of bare and dressed propagator, and $s$- and $t$-channels, the asymptotic behaviors of the scattering amplitudes are the same for even powers ($2n$) of the d’Alambertian.

5.1.2 Odd powers $\Box^{2n+1}$

We now want to address the case of odd powers, $e^{-(k^2/M_s^2)^{2n+1}}$. Let us distinguish the cases of $t$-channel and $s$-channel.
• **t-channel scattering**: In the case of t-channel scattering, the momentum exchange is space-like, $k^2 > 0$, and the asymptotic behavior for high energies turns out to be the same for both bare and dressed propagator$^{30}$, as it also happens for the case of even power. Indeed, for space-like momentum exchange, for $k^2/M_s^2 \to \infty$, we have:

$$\mathcal{M}_{2n+1} \sim e^{-(k^2/M_s^2)^{2n+1}},$$

as $\Sigma_{2n+1}(k)e^{-(k^2/M_s^2)^{2n+1}} \to 0$.

• **s-channel scattering**: In the case of s-channel scattering, the momentum exchange is time-like, $k^2 < 0$, and it is clear that for high energy the propagator blows up. However, as we have already emphasized, non-locality is inherently off-shell, and as such quantum effects are not negligible and what is physically meaningful is the dressed propagator in the region of non-locality, $k^2 > M_s^2$.

By dressing the propagator, it so happens that for high energy the amplitude has the same asymptotic behavior as for the t-channel, with the same exponential suppression, as one would expect for consistency. We will show this by making an explicit calculation for the simplest case $n = 1$. In this case the s-channel amplitude with dressed propagator is given by:

$$\mathcal{M}_1 \sim \lambda^2 \frac{e^{-k^2/M_s^2}}{k^2 + \Sigma_1(k)e^{-k^2/M_s^2}}. \quad (5.9)$$

Note that the self-energy $\Sigma_1(k)$ at 1-loop can be explicitly computed by performing the integration in Euclidean space and then analytically continuing back to Minkowski the momentum $k$; indeed by using the Schwinger parameterization we obtain:

$$\Sigma_1(k) = \frac{\lambda^2}{16\pi^2} \int_0^\infty dt_1 \int_0^\infty dt_2 \frac{e^{-(k^2/2M_s^2)}(t_1 + t_2)}{(t_1 + t_2)^2}$$

$$= \frac{\lambda^2}{16\pi^2} \left[ \frac{2M_s^4}{k^2} \left( e^{-k^2/2M_s^2} - e^{-k^2/M_s^2} \right) + \text{Ei} \left( -\frac{k^2}{2M_s^2} \right) - \text{Ei} \left( -\frac{k^2}{M_s^2} \right) \right], \quad (5.10)$$

which in the high energy regime goes to zero for space-like momentum exchange, while diverges for time-like exchange. In particular, the asymptotic behavior of the self-energy for $k^2/M_s^2 \to \infty$ is given by:

$$\Sigma_1(k) \sim e^{-k^2/M_s^2}. \quad (5.11)$$

$^{30}$We have understood that the region of non-locality is quantum in nature and has to be studied by consistently taking into account all perturbative quantum correction by dressing vertices and propagators. To be consistent we should do the same also when computing the gravitational force between two masses exchanging a graviton, in infinite derivate gravity [7, 19], namely a purely classical computation may not be enough to describe the gravitational interaction in the non-local region. However, let us remind that to compute the gravitational force one has to consider a space-like graviton exchange, i.e., a t-channel scattering, but in such a case, as we have shown above, the asymptotic behavior in the UV regime is the same for both bare and dressed propagator. It means that a classical computation of the gravitational force (or potential) is still acceptable, as the dressing procedure will not modify neither the UV behavior nor the IR.
It is now clear that the interplay between the divergences of the self-energy and the bare propagator, for time-like exchange, is such that the fully dressed $s$-channel scattering amplitude turns out to be exponentially suppressed in the UV regime, showing the same behavior as in the case of the $t$-channel. Indeed, the asymptotic behavior of the $s$-channel amplitude in Eq. (5.9) is given by:

$$M_1 \sim \lambda^2 \frac{e^{-k^2/M_s^2}}{k^2} + \lambda^2 e^{-k^2/M_s^2} \cdot e^{-k^2/M_s^2} \sim e^{k^2/M_s^2},$$  \hspace{1cm} (5.12)

which is exponentially suppressed in the UV regime, $k^2/M_s \to \infty$, with $k^2 < 0$.

Therefore, we have shown that all scattering amplitudes, for any kind of channel,\textsuperscript{31} and for any power of $n$ of the d’Alambertian, are physically well-defined and have exactly the same UV behavior, once the propagator is consistently dressed.

Furthermore, in the case of 1-loop or multi-loops amplitudes, we still have finite results, for example, quartic interaction with propagators that are exponentially suppressed was considered in Ref. [34], where scattering amplitudes ($\phi\phi \rightarrow \phi\phi$) and cross sections were computed. One can also consider loop-amplitudes for decay of unstable particles ($\phi \rightarrow \psi\psi$), but in this case, although there is no UV divergence, the amplitudes blow up for some values of the integration variables. However, such divergent integrals can be regularized with appropriate prescriptions and are still physically meaningful; see for example Refs.[51, 55].

5.2 Shifting the scale of non-locality

In momentum space the interaction vertices become non-local, they are not point-like as in the local quantum field theory, but they are smeared out for momenta $k^2 > M_s^2$. Note that the scale $M_s$ itself can be modified in such a way that the non-local region can be made larger and the smearing effect can increase, if more interaction vertices are introduced. For instance, in the simplest case $f(\Box) = \Box/M_s^2$, for a large number $N$ of interaction vertices, the scale of non-locality can be modified to

$$M_{\text{eff}} \sim \frac{M_s}{\sqrt{N}}; \hspace{1cm} (5.13)$$

see Ref. [12] for discussion. We can also understand it by looking, for example, at tree-level $s$-channel scattering amplitudes for cubic interaction vertices, where for $N$-external legs there are $(N - 2)$-vertices and $(N - 3)$-internal propagators, which have to be consistently dressed as explained in the previous subsection. For instance, by working in the frame of the mass-centre, a 4-point scattering amplitude, $1 + 2 \rightarrow 3 + 4$, in the UV goes like $M \sim e^{-E_{\text{CM}}^2/M_s^2}$, where $E_{\text{CM}}$ is the mass-centre energy; an $N$-point scattering amplitude, $1 + 2 \rightarrow 3 + 4 \cdots + N$, is roughly given by\textsuperscript{32}

$$M \sim e^{-(N-3)E_{\text{CM}}^2/M_s^2},$$  \hspace{1cm} (5.14)

\textsuperscript{31}Note that the case of $u$-channel is similar to the $t$-channel where the momentum exchange is space-like.

\textsuperscript{32}Detailed calculations to show that the scattering amplitude becomes more exponentially suppressed for increasing number of external legs, $M_s \rightarrow M_s/\sqrt{N}$, will be presented elsewhere [58].
and for $N \gg 1$ we can write $\mathcal{M} \sim e^{-E_{CM}^2/(M_s/\sqrt{N})^2} = e^{-E_{CM}^2/M_{\text{eff}}^2}$, which is strongly suppressed as $E_{CM} > 0$. Thus, the scale of non-locality moves towards the IR regime as the number of interacting particles $N$ increases, the smearing effect becomes stronger so that the non-local region increases in size, and the amplitudes become more exponentially suppressed.

6 Distinction between Schwinger function and Euclidean propagator

We have seen that the non-local interaction in infinite derivative field theories discriminates between the Wightman function and the propagator, the former is not affected, while the latter is. The main reason is that non-locality is only relevant when interaction are switched on, and since the Wightman function satisfies the homogeneous field equation, we would not expect any effect, while the propagator feels non-locality, as it solves the inhomogeneous field equation with a delta-source.

Furthermore, we have noticed that for the Wightman function the standard Källén-Lehmann representation can be still defined, while this is not true in the case of the propagator, as the usual time-ordered structure is broken by the presence of infinite contact terms which induce violation of causality. In this section we want to show that in the infinite derivative field theory there is a net distinction between these two quantities, in particular between their analytic continuation, i.e. the Schwinger function and the Euclidean propagator.

Note that the Schwinger function is defined as the analytic continuation of the Wightman function in the Euclidean space:

$$S(x, y) \equiv S(x^4, \vec{x}^4; y^4, \vec{y}^4) = W(ix^4, \vec{x}; iy^4, \vec{y}),$$  \hspace{1cm} (6.1)

where $x^4, y^4 \in \mathbb{R}$ are Euclidean time-coordinates.

6.1 Local quantum field theory

In the standard local quantum field theory it is well known that in the Euclidean space the Schwinger function and the Euclidean propagator coincide and both satisfy the following differential equation:

$$(\Box_x - m^2)G_L(x - y) = -\delta^{(4)}(x - y),$$ \hspace{1cm} (6.2)

where $G_L(x - y)$ is the local Euclidean correlator that can represent both the Schwinger function and the Euclidean propagator. Such a matching between these two quantities is due to the fact that in the Minkowski signature the Wightman function and the propagator are equal for space-like separations. Since in the Euclidean signature all the distances are space-like by definition, we would expect that in the Euclidean space the Schwinger function and the Euclidean propagator would coincide in the local case. We can explicitly see it in both massless and massive cases.

- In the massless case the local Wightman function and the local propagator are given by

$$W_L(x - y)|_{m=0} = \frac{-1}{4\pi^2(\rho - i\epsilon(x^0 - y^0))}, \quad \Pi_L(x - y)|_{m=0} = \frac{-1}{4\pi^2(\rho + i\epsilon)},$$ \hspace{1cm} (6.3)
where $\rho = (x^0 - y^0)^2 - (\vec{x} - \vec{y})^2$. It is very clear that going to the Euclidean signature the two quantities coincide and satisfy the same differential equation in Eq. (6.2) with $m = 0$:

$$S_L(x - y)|_{m=0} = \Pi_L(x - y)|_{m=0} = \frac{1}{4\pi^2 \rho_E},$$

where $\rho_E := (x^4 - y^4)^2 + (\vec{x} - \vec{y})^2 > 0$.

- In the massive case the local Wightman function and the local propagator are given by [44] $^{33}$

\[
W_L(x - y) = -\frac{i}{4\pi} \epsilon(x^0 - y^0) \delta(\rho) + \frac{im}{8\pi \sqrt{\rho}} \theta(\rho) [\epsilon(x^0 - y^0) J_1(m \sqrt{\rho}) - iY_1(m \sqrt{\rho})] \\
+ \theta(-\rho) \frac{m}{4\pi^2 \sqrt{-\rho}} K_1(m \sqrt{-\rho}),
\]

\[
\Pi_L(x - y) = -\frac{i}{4\pi} \delta(\rho) + \frac{im}{8\pi \sqrt{\rho}} \theta(\rho) [J_1(m \sqrt{\rho}) - iY_1(m \sqrt{\rho})] \\
+ \theta(-\rho) \frac{m}{4\pi^2 \sqrt{-\rho}} K_1(m \sqrt{-\rho}),
\]

(6.5)

where $J_1$ and $Y_1$ are the Bessel functions of the first and the second kind, respectively, and $K_1$ is a modified Bessel function. One can easily notice that for the space-like distances, ($\rho < 0$), the quantities $W_L(x - y)$ and $\Pi_L(x - y)$ in Eq.(6.5) coincide, indeed when analytically continued to the Euclidean space both the Schwinger function and the Euclidean propagator satisfy Eq.(6.2), and are given by

\[
S_L(x - y) = \Pi_L(x - y) = \frac{m}{4\pi^2 \sqrt{\rho_E}} K_1(m \sqrt{\rho_E}).
\]

(6.6)

Thus, we have explicitly seen that in the local quantum field theory, the Schwinger function and the Euclidean propagator do coincide and satisfy the same differential equation, Eq.(6.2).

### 6.2 Nonlocal, infinite derivative quantum field theory

We now want to show that in the infinite derivative quantum field theory, with non-local interaction, such a close relation between the Schwinger function and the Euclidean propagator does not exist; we can understand this by considering the massless case.

As we have already mentioned several times, the Wightman function is not affected by the presence of infinite derivatives, and of course the same will also hold for its analytic continuation, indeed the Schwinger function reads

\[
S(x - y)|_{m=0} = \frac{1}{4\pi^2 \rho_E},
\]

and satisfies the Euclidean Klein-Gordon equation in Eq.(6.2), with $m = 0$.

Instead, the Euclidean massless propagator is given by (see Eq.(4.11))

\[
\Pi(x - y)|_{m=0} = \frac{1}{4\pi^2 \rho_E} \left( 1 - e^{-\frac{m^2 \rho_E}{4}} \right),
\]

(6.8)

$^{33}$Note that in the massive case we are not explicitly including the “$i\epsilon$” to make the formula less heavy.
and satisfies the following Euclidean differential equation:
\[ e^{-\Box x/M^2} \Box x \Pi(x-y) = -\delta^{(4)}(x-y). \] (6.9)

By looking at the Eqs. (6.7) and (6.8) it is clear that the Schwinger function and the Euclidean propagator do not coincide in the presence of non-local interactions, and therefore cannot be treated at the same footing, in fact they become two completely different quantities when infinite order derivatives are introduced. It so happens that in the local field theory the Schwinger function and the Euclidean propagator end up coinciding, but generally they can be two different quantities as we have shown in the case of infinite derivative quantum field theory.

### 6.3 Reflection positivity

Furthermore, it is worth mentioning that, since the Wightman function is related to the expectation values in the Hilbert space, through the reconstruction theorem the Schwinger function will be related, too. In fact, one can show that in order to have measurable expectation values and non-negative norms in the Hilbert space, the condition of reflection positivity has to hold [59]:
\[ \int d^4x d^4y \Theta(g(x)) S(x,y) g(y) \geq 0, \] (6.10)
for any continuous function \( g(x) \) with compact support in \( \mathbb{R}^+ \times \mathbb{R}^3 \), with \( \mathbb{R}^+ \) meaning the upper half \( x^4 > 0 \); the function \( \Theta \) is defined by \( \Theta(g(x^4, \vec{x})) = g^*(-x^4, \vec{x}) \).

Since the Schwinger function is not affected by non-local interaction, the condition of reflection positivity will still hold in the infinite derivative quantum field theories with non-local interaction, and no states with negative norms will appear, thus unitarity is still preserved at the level of states in the Hilbert space.

Moreover, one can show that the principle of reflection positivity is a necessary and sufficient condition for the Schwinger function (or Wightman function) to possess a Källén-Lehmann representation [60], and this is consistent with our results in Subsection 2.2, where we have shown that in the infinite derivative field theory the Wightman function can be expressed in terms of the Källén-Lehmann representation.

### 7 Summary and conclusions

In this paper we have studied quantum aspects of infinite derivative scalar field theory. We have shown that the action can be made non-local by introducing Lorentz invariant analytic form factor either in the kinetic operator or in the interaction vertex. We have shown that in order to not introduce any ghost-like degree of freedom, we require the form factors ought to be exponential of entire function; in particular, we have considered exponentials of polynomials of the d’Alambertian \( \Box \), see Eq.(2.10). Since we have kept the Lorentz invariance and chosen the form factors to be exponential of entire functions, it follows that non-locality only plays a crucial role when the interaction is switched on, otherwise the theory is local.
• We have investigated the main consequences induced by the presence of non-local interactions. In particular, we have explicitly shown that the non-local propagator is not simply defined in terms of a time-ordered product, unlike the local theory, but it is made of an acausal contribution. Moreover, the retarded Green function assumes an acausal behavior, indeed it is non-vanishing for space-like separations. However, acausal effects are only confined to \( t, r < \frac{1}{M_s} \); while outside this region, i.e. in the IR regime, physics is causal, namely causality is preserved on macroscopic scales, \( t, r > \frac{1}{M_s} \). Moreover, local commutativity is preserved both for the free- and interacting-theory.

• In the non-local region, we cannot define any concept of space and time due to the presence of acausal effects. Such a statement is also mathematically justified by the fact that amplitudes are ill-defined with the Minkowski signature due to the presence of the exponential form factors, which can diverge along some direction in the complex plane, making it impossible to define the Wick-rotation. For this reasons, the recipe is to define the theory in the Euclidean space, where all the amplitudes can be well-defined, and after having performed the computations, we can analytically continue back the external momenta to the Minkowski signature.

• We have also analyzed singularity properties in the case of non-local interactions; as an example we have considered \( \phi^4 \)-theory. First of all, we have noticed that the theory turns out to be UV finite, unlike the local case, where there are divergent integrals that need to be renormalized. Moreover, we have studied the structure of the Euclidean 2-point correlation function, and shown that it is non-singular on the lightcone surface, while in the local field theory lightcone singularities remain even after the renormalization is performed.

• We have emphasized that scattering amplitudes can be well-defined in infinite derivative quantum field theories, and that for momenta \( k^2 \geq M_s^2 \) classical physics do not make sense as the vertices are smeared out in momentum space; indeed, non-locality and acausality manifest as off-shell phenomena, which means that all amplitudes have to be seen as quantum for momenta \( k^2 \geq M_s^2 \), and all perturbative quantum corrections have to be taken into account by dressing propagators and vertices. In this way all scattering amplitudes, for any channel, and for both odd and even powers of the d’Alambertian, turn out to be exponentially suppressed in the UV regime.

• In the end, we have made some remarks on the distinction between the Schwinger function and the Euclidean propagator. Once the principle of locality is given up by means the introduction of infinite derivatives, these two quantities become completely different objects. While the Schwinger (or Wightman) function maintain the same properties of the local theory, the propagator is drastically modified by the presence of non-local interaction. For instance, the usual Källén-Lehmann representation cannot be defined due to the fact that the time-ordered structure is not maintained. While for the Wightman function the standard Källén-Lehmann representation can be still
defined, and this implies that the local commutativity condition in Eq.(3.6) still holds for both free and interacting fields also in infinite derivative theory.

Although, infinite derivative quantum field theory shows many interesting features, there are still some open questions that need to be possibly answered. For instance, as we have already mentioned, the usual Källén-Lehmann representation for the non-local propagator is not possible; see Ref.[6] for some attempts aimed to generalize such a representation for the propagator to the case of infinite derivative interactions. Furthermore, systematic methods to proof the unitarity and macrocausality conditions at the level of the S-matrix have not been developed yet. In the local quantum field theory, it is well known that the unitarity can be proven by using the largest time equation [61]. Such an approach strongly relies on two crucial hypothesis: (i) the propagator has a time-ordered structure, (ii) vertices are local. It is clear that when the principle of locality is given up at the level of interaction, and infinite derivative are introduced, the largest time equation cannot be consistently applied to check the unitarity, as the propagator is not simply a time ordered product and the interaction vertices become non-local.

Moreover, in local quantum field theory the causality condition is often formulated in terms of the S-matrix through the Bogoliubov causality condition [44]. We have learnt that once non-locality is introduced, acausal effects emerge only at the microscopic scales, in the region $\sim 1/M_s$, and that physics remains causal outside this region, i.e. in the IR, so that a condition of macrocausality still holds. However, a systematic method that generalizes the Bogoliubov condition to the case of non-local interactions is also still lacking; see Ref.[38] for some progress along these directions.

The presence of non-local interaction seems to be very important to avoid singularities of several types, thanks to its beautiful regularizing nature. We believe that infinite derivative field theory is very important to construct a consistent theory of quantum gravity, in particular it could be the right root to follow in order to overcome ambiguities and paradoxes when dealing with blackhole physics; see for example Ref.[12, 15]. For these reasons, we strongly believe that infinite derivative field theories deserve further and deeper investigations.

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8 Appendix

8.1 Principal value computation for acausal Green function

We now want to show the computation that leads to the expression in Eq.(3.19) for the acausal Green function in infinite derivative field theory. In particular, we want to compute

\[ t, r \leq 1/M_s \]

but physics is still causal on macroscopic scales \( t, r \gg 1/M_s \).

34Macrocausality is a generalization of the concept of causality in which one can have the presence of acausal effects at microscopic scales (\( t, r \leq 1/M_s \)), but physics is still causal on macroscopic scales (\( t, r \gg 1/M_s \)).
the principal-value integral in Eq. (3.17) that we recall for clarity:

$$I_{PV} = \frac{1}{16\pi^3} \frac{1}{r} \int_{-\infty}^{\infty} k dk \text{ P.V.} \int_{-\infty}^{\infty} dt_0^2 \left( \frac{-\frac{k_0^2 + k^2}{M^2}}{k_0^2 - k^2} \right)^{2n} \left( e^{i(kr - k_0^2t)} - e^{-i(kr + k_0^2t)} \right), \quad (8.1)$$

where, let us remind that \( k \equiv |\vec{k}| \) and \( \omega_k = k \), as we are working with the massless case.

Since we are interested in the modification of the local retarded Green function we will consider the case \( t > 0 \)\(^{35}\). To compute \( I_{PV} \) we need to consider several cases corresponding to different regions of the planes \( t-r \) and \( k^0-k \). As for the plane \( t-r \) we have to distinguish \(^{36}\):

1. \( t > 0, \ t^2 > r^2 \iff (x - y)^2 < 0 \) (time-like separation):
   
   \[ t = \rho^{1/2}\cosh^2\alpha, \ r = \rho^{1/2}\sinh^2\alpha, \ t^2 - r^2 = \rho > 0; \]

2. \( t > 0, \ t^2 < r^2 \iff (x - y)^2 > 0 \) (space-like separation):
   
   \[ t = \rho^{1/2}\cosh^2\alpha, \ r = \rho^{1/2}\sinh^2\alpha, \ t^2 - r^2 = \rho < 0. \]

Instead, as for the plane \( k^0-k \) we will split the double integral in Eq. (8.1) in the following two regions:

(i) \( k_0^2 > k^2 \):

\[ k = R\sinh\beta, \ k^0 = R\cosh\beta, \ R^2 = k_0^2 - k^2 > 0, \ -\infty < \beta, R < \infty; \]

(ii) \( k_0^2 < k^2 \):

\[ k = R\cosh\beta, \ k^0 = R\sinh\beta, \ -R^2 = k_0^2 - k^2 < 0, \ -\infty < \beta, R < \infty. \]

By moving to the new integration variables \( R, \beta \) we get a Jacobian factor \(|R|\) so that the integral in Eq.(8.1) in the case 1. (\( \rho > 0 \)) reads

$$I_{PV} = \frac{i}{16\pi^3} \int_{-\infty}^{\infty} dR \beta e^{-R^4n/M^4n} \left| \frac{R}{R} \right| \left\{ \sinh\beta \left[ e^{-iR^{1/2}\cosh(\beta - \alpha)} - e^{-iR^{1/2}\cosh(\beta + \alpha)} \right] \right. $$

$$ \left. - \cosh\beta \left[ e^{iR^{1/2}\sinh(\beta - \alpha)} - e^{iR^{1/2}\sinh(\beta + \alpha)} \right] \right\} $$

$$ = \frac{i}{16\pi^3} \int_{-\infty}^{\infty} dR \beta e^{-R^4n/M^4n} \left| \frac{R}{R} \right| \left\{ \sinh\beta \left[ -e^{-iR^{1/2}\cosh(\beta + \alpha)} - e^{-iR^{1/2}\cosh(\beta - \alpha)} \right] \right. $$

$$ \left. - \cosh\beta \left[ e^{-iR^{1/2}\sinh(\beta + \alpha)} + e^{iR^{1/2}\sinh(\beta - \alpha)} \right] \right\} $$

$$ = -\frac{1}{8\pi^3} \int_{-\infty}^{\infty} dR \beta e^{-R^4n/M^4n} \left| \frac{R}{R} \right| \left\{ \sinh\beta \sin \left( R^{1/2}\cosh(\beta + \alpha) \right) \right. $$

$$ \left. - \cosh\beta \sin \left( R^{1/2}\sinh(\beta + \alpha) \right) \right\}. \quad (8.2) $$

---

\(^{35}\)If we considered the case \( t < 0 \) we would study the modification of the advanced Green function.

\(^{36}\)In Ref.[37] the authors consider the same calculation for the case \( 2n = 2 \).
Defining the new integration variable \( \theta = \beta + \alpha \), the integral in Eq.(8.2) becomes:

\[
I_{PV} = \frac{1}{8\pi^3\rho^{1/2}} \int_{-\infty}^{\infty} dR d\theta e^{-R^{4n}/M^{4n}_2} \left| R \right| \left\{ \cosh \theta \sin \left( R\rho^{1/2} \cos \theta \right) - \sinh \theta \sin \left( R\rho^{1/2} \sin \theta \right) \right\}
\]

\[
= \frac{1}{2\pi^3\rho^{1/2}} \int_{0}^{\infty} dR d\theta e^{-R^{4n}/M^{4n}_2} \left\{ \cosh \theta \sin \left( R\rho^{1/2} \cos \theta \right) - \sinh \theta \sin \left( R\rho^{1/2} \sin \theta \right) \right\}
\]

\[
= \frac{1}{\pi^3} \frac{\partial}{\partial \rho} \int_{0}^{\infty} dR e^{-R^{4n}/M^{4n}_2} \int_{0}^{\infty} d\theta \left\{ \cos \left( R\rho^{1/2} \cos \theta \right) - \cos \left( R\rho^{1/2} \sin \theta \right) \right\}.
\]

(8.3)

The integrals in \( \theta \)-variable can be expressed in terms of Bessel functions [48]:

\[
\int_{0}^{\infty} d\theta \cos \left( R\rho^{1/2} \sin \theta \right) = K_0(R\rho^{1/2}), \quad \int_{0}^{\infty} d\theta \cos \left( R\rho^{1/2} \cos \theta \right) = -\frac{\pi}{2} Y_0(R\rho^{1/2});
\]

(8.4)

thus the principal-value integral in Eq.(8.3) becomes

\[
I_{PV} = \frac{1}{\pi^3} \frac{\partial}{\partial \rho} \int_{0}^{\infty} dR e^{-R^{4n}/M^{4n}_2} \left[ K_0(R\rho^{1/2}) + \frac{\pi}{2} Y_0(R\rho^{1/2}) \right].
\]

(8.5)

One can also introduce the dimensionless variable \( \zeta = R\rho^{1/2} \) so that the integral in Eq.(8.5) can be rewritten as

\[
I_{PV} = \frac{1}{\pi^3} \frac{\partial}{\partial \rho} \int_{0}^{\infty} \frac{d\zeta}{\zeta} e^{-\frac{\zeta^4}{M^4_2\rho^{2n}}} \left[ K_0(\zeta) + \frac{\pi}{2} Y_0(\zeta) \right].
\]

(8.6)

The last result holds for the case 1. when \( \rho > 0 \), but we can also take into account the case 2., when \( \rho < 0 \), by considering the following expression:

\[
I_{PV} = \frac{1}{\pi^3} \frac{\partial}{\partial \rho} \left\{ \varepsilon(\rho) \int_{0}^{\infty} \frac{d\zeta}{\zeta} e^{-\frac{\zeta^4}{M^4_2\rho^{2n}}} \left[ K_0(\zeta) + \frac{\pi}{2} Y_0(\zeta) \right] \right\},
\]

(8.7)

where the function \( \varepsilon(\rho) \) is equal to +1 if \( \rho > 0 \) (time-like separation), while it is −1 if \( \rho < 0 \) (space-like separation). The result in Eq.(8.7) correspond to the integral in Eq.(3.18).

The integral in Eq.(8.7) can be computed analytically for \( 2n = 2 \) and can be expressed in terms of the Meijer-G functions [49]; indeed for space-like separation (\( \rho < 0 \)) one has

\[
I_{PV} = \frac{1}{2\pi^3} \frac{\partial}{\partial \rho} \left\{ \int_{0}^{\infty} \frac{d\zeta}{\zeta} e^{-\frac{\zeta^4}{M^4_2\rho^{2n}}} \left[ K_0(\zeta) + \frac{\pi}{2} Y_0(\zeta) \right] \right\}
\]

\[
= \frac{2}{\pi^3 M^4_2 \rho^{2n}} \left\{ \int_{0}^{\infty} d\zeta e^{-\frac{\zeta^4}{M^4_2\rho^{2n}}} \zeta^3 \left[ K_0(\zeta) + \frac{\pi}{2} Y_0(\zeta) \right] \right\}
\]

\[
= \frac{1}{2\pi^3} \frac{1}{\rho} \left\{ G_{2,5}^{4,1} \left( 0, 0, 0, \frac{1}{2}, \frac{1}{2} \right) \frac{M^4_2 \rho^2}{256} + 2\pi^2 G_{3,6}^{4,1} \left( 0, 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4} \right) \frac{M^4_2 \rho^2}{256} \right\},
\]

(8.8)

which explains the expression in Eq.(3.19) for the acausal retarded Green function \( G_R \).
8.2 Acausality for interacting fields

In this appendix we wish to show that non-local interaction implies the presence of acausality in the evolution of the fields; in particular we will see that the fields can depend acausal upon the initial data.

Let us consider the Lagrangian for a real scalar field $\phi(x)$ with a quartic interaction \[37:]
\[
\mathcal{L} = \frac{1}{2} \phi(x)(\Box - m^2)\phi(x) - \frac{\lambda}{4!} \left( e^{\frac{i}{2}f(\Box)} \phi(x) \right)^4, \quad (8.9)
\]
with corresponding field equations given by
\[
(-\Box + m^2)\phi(x) = -\frac{\lambda}{3!} e^{\frac{i}{2}f(\Box)} \left( e^{\frac{i}{2}f(\Box)} \phi(x) \right)^3, \quad (8.10)
\]
where $\lambda$ is a dimensionless coupling constant. The field equation in Eq. (8.10) can be solved perturbatively by continuous iterations; the zeroth and first order are given by:
\[
(-\Box + m^2)\phi^{(0)}(x) = 0,
\]
\[
(-\Box + m^2)\phi^{(1)}(x) = -\frac{\lambda}{3!} e^{\frac{i}{2}f(\Box)} \left( e^{\frac{i}{2}f(\Box)} \phi^{(0)}(x) \right)^3, \quad (8.11)
\]
where the zeroth order is nothing but the homogeneous Klein-Gordon equation, whose local solutions are given by the free-field decomposition in Eq. (2.14), that we rewrite for clarity:
\[
\phi^{(0)}(x^0, \vec{x}) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{2\omega_k}} \left( a_k e^{-i\omega_k x^0 + i\vec{k} \cdot \vec{x}} + a_k^* e^{i\omega_k x^0 - i\vec{k} \cdot \vec{x}} \right). \quad (8.12)
\]
Note that the Fourier-transform of the field $\phi^{(0)}(x^0, \vec{x})$ with respect to the spatial coordinate $\vec{x}$, $\tilde{\phi}^{(0)}(x^0, \vec{k})$, can be expressed in terms of the initial field configuration, $\tilde{\phi}^{(0)}(x^0, \vec{k})$ and $\tilde{\phi}^{(0)}(x^0, \vec{k})$, as follows \[38:]
\[
\tilde{\phi}^{(0)}(x^0, \vec{k}) = \tilde{\phi}^{(0)}(x^0, \vec{k}) \cos(\omega_k x^0) + \tilde{\phi}^{(0)}(x^0, \vec{k}) \frac{\sin(\omega_k x^0)}{\omega_k}, \quad (8.13)
\]
so that the free-field in Eq. (8.12) can be rewritten as
\[
\phi^{(0)}(x^0, \vec{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}} \left( \tilde{\phi}^{(0)}(x^0, \vec{k}) \cos(\omega_k x^0) + \tilde{\phi}^{(0)}(x^0, \vec{k}) \frac{\sin(\omega_k x^0)}{\omega_k} \right). \quad (8.14)
\]

Let us now compute the variation of the free-field with respect to an initial field-configuration $\phi^{(0)}(y)$, with $y \equiv (0, \vec{y})$, such that the distance between $x$ and $y$ is space-like, $(x - y)^2 > 0$ (or, equivalently, $|\vec{x} - \vec{y}| > x^0$):
\[
\frac{\delta \phi^{(0)}(x^0, \vec{x})}{\delta \phi^{(0)}(0, \vec{y})} = \int \frac{d^d k}{(2\pi)^d} e^{-i\vec{k} \cdot (\vec{x} - \vec{y})} \cos(\omega_k x^0) = -\hat{\Delta}(x^0, \vec{x} - \vec{y}), \quad (8.15)
\]
\[37\text{We could have considered any kind of interaction, but as an example we have chosen }\phi^4. \text{ Moreover, we are working in the case in which the kinetic term is the standard Klein-Gordon operator and the interaction term is modified by the introduction of a form factor; of course the results would be the same if we considered non-local kinetic operator and local interaction vertices.}
\[38\text{Note that the symbol }$\cdot$\text{ means derivative with respect to }x^0.$
\[
\frac{\delta \phi^{(0)}(x^0, \vec{x})}{\delta \phi^{(0)}(0, \vec{y})} = \int \frac{d^3 k}{(2\pi)^3} e^{-i k (\vec{x} - \vec{y})} \frac{\sin(\omega_k x^0)}{\omega_k}
\]
where we have used
\[
\tilde{\phi}^{(0)}(0, \vec{k}) = \int d^3 x' e^{-i \vec{k} \cdot \vec{x'}} \phi^{(0)}(0, \vec{x'}),
\]
and
\[
\frac{\delta \phi^{(0)}(0, \vec{x}')}{\delta \phi^{(0)}(0, \vec{y})} = \delta^{(3)}(\vec{x}' - \vec{y}).
\]
Note that we have obtained the Pauli-Jordan function introduced in the Subsection 2.2 as a result of the functional differentiation. As we have already emphasized, \(\Delta(x^0, \vec{x} - \vec{y})\) has only support inside the lightcone, and the same holds for its time-derivative; thus for space-like separations they vanish and causality is preserved at the level of free-theory.

Let us now consider the first order in perturbation, i.e. the differential equation in the second line of Eq.(8.12). The solution \(\phi^{(1)}(x)\) is given by the sum of the homogeneous and the particular solutions, which we can indicate by \(\phi_o^{(1)}(x)\) and \(\phi_p^{(1)}(x)\), respectively:
\[
\phi^{(1)}(x) = \phi_o^{(1)}(x) + \phi_p^{(1)}(x),
\]
where the particular solution has the physical meaning of interacting field. The homogeneous solution satisfies the same equation of \(\phi^{(0)}(x)\), while the particular solution can be formally expressed in terms of the retarded Green function as in Eq.(3.2):
\[
\phi_p^{(1)}(x^0, \vec{x}) = \frac{i \lambda}{3!} \int dx^0 d^3 x' G_{L,R}(x^0 - x^0, \vec{x} - \vec{x'}) e^{\frac{i}{2} f(\square_v)} \left( e^{\frac{i}{2} \frac{f(\square_v)}{\lambda}} \phi^{(0)}(x^0, \vec{x'}) \right)^3 = \frac{i \lambda}{3!} \int dx^0 d^3 x' e^{\frac{i}{2} f(\square_v)} [G_{L,R}(x^0 - x^0, \vec{x} - \vec{x'})] \left( \phi^{(0)}(x^0, \vec{x'}) \right)^3,
\]
where we have used the fact that \(e^{\frac{i}{2} f(\square_v)} \phi^{(0)}(x) = \phi^{(0)}(x)\), at the zeroth order we have a free-field field propagation - satisfying the homogeneous Klein-Gordon equation, and we have also made use of the kernel representation of the exponential differential operator:
\[
e^{\frac{i}{2} f(\square_v)} g(x) = \int d^4 y e^{\frac{i}{2} f(\square_v)} \delta^{(4)}(x - y) g(y).
\]

All information about the presence of non-local interactions is contained in the particular solution; thus let us now calculate, as done for the zeroth order in Eqs.(8.15) and (8.16), the variation of the field \(\phi_p^{(1)}(x^0, \vec{x})\) with respect an initial field configuration \(\phi^{(0)}(0, \vec{y})\) and \(\phi^{(0)}(0, \vec{y})\):
\[
\frac{\delta \phi_p^{(1)}(x^0, \vec{x})}{\delta \phi^{(0)}(0, \vec{y})} = -\frac{i \lambda}{2} \int dx^0 d^3 x' e^{\frac{i}{2} f(\square_v)} [G_{L,R}(x^0 - x^0, \vec{x} - \vec{x'})] \Delta(x^0, \vec{x'} - \vec{y}) \left( \phi^{(0)}(x^0, \vec{x'}) \right)^2,
\]
\[(8.19)\]

\[39\]A similar computation was also done, for example, in Ref. [33] in the case of scalar field with cubic interaction, \([e^{\lambda/M^2} \phi(x)]^3\), that can represent the interaction vertex for a dilaton field in string field theory.
\[
\frac{\delta \phi^{(1)}_p(x^0, \vec{x})}{\delta \phi^{(0)}(0, \vec{y})} = -\frac{i\lambda}{2} \int dx'^0 d^3x' e^{i \mathcal{L}(\mathcal{L})} [G_{L,R}(x^0 - x'^0, \vec{x} - \vec{x}') \Delta(x'^0, \vec{x} - \vec{y}) \left( \phi^{(0)}(x'^0, \vec{x}') \right)^2].
\]

The action of the differential operator on the local Green function in the integrals in Eqs. (8.19) and (8.20) makes the interacting field \( \phi^{(1)}_p \) depending acausally upon the initial data: in fact, the integrals in Eq. (8.19) and (8.20) are not vanishing for space-like separations \( |\vec{x} - \vec{y}| > x^0 \) due to the non-zero contribution coming from the integration-region \( x'^0 < |\vec{x}'| \) as the function \( e^{i \mathcal{L}(\mathcal{L})} [G_{L,R}(x'^0, \vec{x}')] \) exhibits an acausal behavior, i.e. it is non-vanishing for the space-like separations \( x'^0 < |\vec{x}'| \), as shown in section 3 for the case \( f(\mathcal{L}) = ( -\mathcal{L} + m^2)^n / M^2 n \). Indeed, more explicitly one has the following situations.

- In the local case, \( f(\mathcal{L}) = 0 \), the integrals in Eqs. (8.19), (8.20) get a vanishing contribution from the space-like region \( x^0 - x'^0 < |\vec{x} - \vec{x}'| \) as the retarded Green function \( G_{L,R} \) is causal, thus a non-vanishing contribution has to come from the region

\[
x^0 - x'^0 \geq |\vec{x} - \vec{x}'|.
\]

Moreover, since the initial time condition is \( y^0 = 0 \) and we are looking at the future evolution by means the retarded Green function, the following inequality has to hold:

\[
x^0 > x'^0 > 0.
\]

We can now ask if the field \( \phi^{(1)}_p(x) \) depends acausally upon the initial spacetime configuration \( (0, \vec{y}) \). One can easily show that if \( x^0 < |\vec{x} - \vec{y}| \), by using also the inequalities in Eqs. (8.21), (8.22), one obtains the following inequality:

\[
x'^0 < |\vec{x}' - \vec{y}|,
\]

which implies that the Pauli-Jordan function and its derivative in Eqs. (8.19) and (8.20), respectively, vanish. Thus, in local field theory the field evolution turns out to be causal.

- In the case of non-local interactions, \( f(\mathcal{L}) \neq 0 \), the Green function \( G_R \) shows an acausal behavior, i.e. it is non-vanishing for space-like separations, thus we cannot use the inequality in Eq. (8.21) as done above for the local case. It follows that for space-like separation \( x^0 < |\vec{x} - \vec{y}| \) the functional derivatives in Eqs. (8.19), (8.20) do not vanish and the field can depend acausally on the initial data.

The acausal behavior is confined to a region of size \( \sim 1/M_s \), as it would be more explicit once a specific choice for the form factor is made.

In conclusion, the retarded Green function assumes an acausal behavior, such that it is not vanishing outside the light-cone, thus making \( \phi^{(1)}_p(x) \) depending acausally upon the initial conditions. It is also clear that such a causality violation will be also present at higher order in perturbation theory.
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