Degree distribution of an inhomogeneous random intersection graph

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Abstract

We show the asymptotic degree distribution of the typical vertex of a sparse inhomogeneous random intersection graph.

key words: degree distribution, random graph, random intersection graph, power law

2000 Mathematics Subject Classifications: 05C80, 05C07, 05C82

1 Introduction

Let $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ be independent non-negative random variables such that each $X_i$ has the probability distribution $P_1$ and each $Y_j$ has the probability distribution $P_2$. Given realized values $X = \{X_i\}_{i=1}^m$ and $Y = \{Y_j\}_{j=1}^n$, we define the random bipartite graph $H_{X,Y}$ with the bipartition $V = \{v_1, \ldots, v_n\}$, $W = \{w_1, \ldots, w_m\}$, where edges $\{w_i, v_j\}$ are inserted with probabilities $p_{ij} = \min\{1, X_i Y_j (nm)^{-1/2}\}$ independently for each $\{i, j\} \in [m] \times [n]$. The inhomogeneous random intersection graph $G(P_1, P_2, n, m)$ defines the adjacency relation on the vertex set $V$: vertices $v', v'' \in V$ are declared adjacent (denoted $v' \sim v''$) whenever $v'$ and $v''$ have a common neighbour in $H_{X,Y}$.

The degree distribution of the typical vertex of the random graph $G(P_1, P_2, n, m)$ has been first considered by Shang [20]. The proof of the main result of [20] contains gaps and the result is incorrect in the regime where $m/n \to \beta \in (0, +\infty)$ as $m, n \to +\infty$. We remark that this regime is of particular importance, because it leads to inhomogeneous graphs with the clustering property: the clustering coefficient $P(v_1 \sim v_2 | v_1 \sim v_3, v_2 \sim v_3)$ is bounded away from zero provided that $EX_1^3 < \infty$ and $EY_1^2 < \infty$, see [8]. The aim of the present paper is to show the asymptotic degree distribution in the case where $m/n \to \beta$ for some $\beta \in (0, +\infty)$.

We consider a sequence of graphs $\{G_n = G(P_1, P_2, n, m)\}$, where $m = m_n \to +\infty$ as $n \to \infty$, and where $P_1, P_2$ do not depend on $n$. We denote $a_i = EX_i^3$, $b_i = EY_i^2$. By $d(v_j) = d_{G_n}(v_j)$ we denote the degree of a vertex $v_j$ in $G_n$ (the number of vertices adjacent to $v_j$ in $G_n$). We remark that for every $n$ the random variables $d_{G_n}(v_1), \ldots, d_{G_n}(v_n)$ are identically distributed.

In Theorem 1 below we show the asymptotic distribution of $d(v_1)$.

Theorem 1. Let $m, n \to \infty$. 

1
(i) Assume that \( m/n \to 0. \) Suppose that that \( \mathbb{E}X_1 < \infty. \) Then \( P(d(v_1) = 0) = 1 - o(1). \)

(ii) Assume that \( m/n \to \beta \) for some \( \beta \in (0, +\infty). \) Suppose that \( \mathbb{E}X_i^2 < \infty \) and \( \mathbb{E}X_1 < \infty. \) Then \( d(v_1) \) converges in distribution to the random variable

\[
d_\ast = \sum_{j=1}^{\Lambda_1} \tau_j,
\]

where \( \tau_1, \tau_2, \ldots \) are independent and identically distributed random variables independent of the random variable \( \Lambda_1. \) They are distributed as follows. For \( r = 0, 1, 2, \ldots, \) we have

\[
P(\tau_1 = r) = \frac{r + 1}{\mathbb{E}\Lambda_2} P(\Lambda_2 = r + 1) \quad \text{and} \quad P(\Lambda_i = r) = \mathbb{E} e^{-\lambda_i} \frac{\lambda_i^r}{r!}, \quad i = 1, 2.
\]

Here \( \lambda_1 = Y_1 a_1 \beta^{1/2} \) and \( \lambda_2 = X_1 b_1 \beta^{-1/2}. \)

(iii) Assume that \( m/n \to +\infty. \) Suppose that \( \mathbb{E}X_i^2 < \infty \) and \( \mathbb{E}Y_1 < \infty. \) Then \( d(v_1) \) converges in distribution to a random variable \( \Lambda_3 \) having the probability distribution

\[
P(\Lambda_3 = r) = \mathbb{E} e^{-\lambda_3} \frac{\lambda_3^r}{r!}, \quad r = 0, 1, \ldots.
\]

Here \( \lambda_3 = Y_1 a_2 b_1. \)

Remark 1. The probability distributions \( P_{\Lambda_1} \) of \( \Lambda_i, \; i = 1, 2, 3, \) are Poisson mixtures. One way to sample from the distribution \( P_{\Lambda_i} \) is to generate random variable \( \lambda_i \) and then, given \( \lambda_i, \) to generate Poisson random variable with the parameter \( \lambda_i. \) The realized value of the Poisson random variable has the distribution \( P_{\Lambda_i}. \)

Remark 2. The asymptotic distributions \( \{1\} \) and \( \{3\} \) admit heavy tails. In the case (ii) we obtain a power law asymptotic degree distribution \( \{1\} \) provided that at least one of \( P_1 \) and \( P_2 \) has a power law and \( P_1(0), P_2(0) < 1. \) In the case (iii) we obtain a power law asymptotic degree distribution \( \{3\} \) provided that \( P_2 \) has a power law.

Remark 3. Since the second moment \( a_2 \) does not show up in \( \{1\}, \{2\} \) we expect that in the case (ii) the second moment condition \( \mathbb{E}X_1^2 < \infty \) is redundant and could perhaps be replaced by the weaker first moment condition \( \mathbb{E}X_1 < \infty. \)

Random intersection graphs have attracted considerable attention in the recent literature, see, e.g., \( \{1\}, \{2\}, \{3\}, \{5\}, \{6\}, \{7\}, \) \{11\}, \{12\}, \{14\}, \{15\}, \{18\}. Starting with the paper by Karoński et al \{16\}, see also \{21\}, where the case of degenerate distributions \( P_1 = P_{1n}, P_2 = P_{2n} \) depending on \( n \) was considered (i.e., \( P_{1n}(c_n) = P_{2n}(c_n) = 1, \) for some \( c_n > 0), \) several more complex random intersection graph models were later introduced by Godehardt and Jaworski \{13\}, Spirakis et al. \{17\}, Shang \{20\}. The asymptotic degree distribution for various random intersection graph models was shown in \( \{4\}, \{5\}, \{6\}, \{7\}, \{11\}, \{14\}, \{15\}, \{19\}, \{22\}.

2 Proofs

Before the proof we introduce some notation and give two auxiliary lemmas.

The event that the edge \( \{w_i, v_j\} \) is present in \( H = H_{X,Y} \) is denoted \( w_i \to v_j. \) We denote

\[
\mathbb{1}_{ij} = \mathbb{1}_{\{w_i \to v_j\}}, \quad \mathbb{1}_i = \mathbb{1}_{1i}, \quad u_i = \sum_{2 \leq j \leq n} \mathbb{1}_{ij}, \quad L = \sum_{i=1}^{m} u_i \mathbb{1}_i.
\]
We remark that \( u_i \) counts all neighbours of \( w_i \) in \( H \) belonging to the set \( V \setminus \{v_1\} \). Denote
\[
\hat{a}_k = m^{-1} \sum_{1 \leq i \leq m} X_i^k, \quad \hat{b}_k = n^{-1} \sum_{2 \leq j \leq n} Y_j^k, 
\]
\[
\lambda_{ij} = \frac{X_i Y_j}{\sqrt{nm}}, \quad S_{XY} = \sum_{i=1}^{m} \sum_{j=2}^{n} \frac{X_i}{\sqrt{nm}} \lambda_{ij} \min\{1, \lambda_{ij}\},
\]
and introduce the event \( A_1 = \{\lambda_{i1} < 1, 1 \leq i \leq m\} \). By \( P_1 \) and \( E_1 \) we denote the conditional probability and conditional expectation given \( Y_1 \). By \( \hat{P} \) and \( \hat{E} \) we denote the conditional probability and conditional expectation given \( X, Y \). By \( d_{TV}(\zeta, \xi) \) we denote the total variation distance between the probability distributions of random variables \( \zeta \) and \( \xi \). In the case where \( \zeta, \xi \) and \( X, Y \) are defined on the same probability space we denote by \( d_{TV}(\zeta, \xi) \) the total variation distance between the conditional distributions of \( \zeta \) and \( \xi \) given \( X, Y \).

In the proof below we use the following simple fact about the convergence of a sequence of random variables \( \{\nu_n\} \):
\[
\nu_n = o_P(1), \quad E \sup_n \nu_n < \infty \quad \Rightarrow \quad E\nu_n = o(1). \tag{5}
\]

We remark that the condition \( E \sup_n \nu_n < \infty \) (which assumes implicitly that all \( \nu_n \) are defined on the same probability space) can be replaced by the more restrictive condition that there exists a constant \( c > 0 \) such that \( |\nu_n| \leq c \), for all \( n \). In the latter case we do not need all \( \nu_n \) to be defined on the same probability space. In particular, given a sequence of bivariate random vectors \( \{\eta_n, \theta_n\} \) such that, for every \( n \) and \( m = m_n \) random variables \( \eta_n, \theta_n \) and \( \{X_i\}_{i=1}^m, \{Y_j\}_{j=1}^n \) are defined on the same probability space, we have
\[
\hat{d}_{TV}(\eta_n, \theta_n) = o_P(1) \quad \Rightarrow \quad d_{TV}(\eta_n, \theta_n) \leq E\hat{d}_{TV}(\eta_n, \theta_n) = o(1).
\]

**Lemma 1.** Assume that \( E\nu_i^2 < \infty \) and \( E\nu_1 < \infty \). We have as \( n, m \to +\infty \)
\[
P(d(v_1) \neq L) = o(1), \tag{6}
\]
\[
P(A_1) = 1 - o(1), \tag{7}
\]
\[
S_{XY} = o_P(1), \quad E S_{XY} = o(1). \tag{8}
\]

**Proof of Lemma 1.** We observe that the event \( A = \{d(v_1) \neq L\} \) occurs in the case where for some \( 2 \leq j \leq n \) and some distinct \( i_1, i_2 \in [m] \) the event \( A_{i_1,i_2,j} = \{w_{i_1} \to v_1, w_{i_1} \to v_j, w_{i_2} \to v_1, w_{i_2} \to v_j\} \) occurs. From the union bound and the inequality \( P(A_{i_1,i_2,j}) \leq m^{-2} n^{-2} X_{i_1}^2 X_{i_2}^2 Y_1^2 Y_j^2 \) we obtain
\[
P(A) = \hat{P} \left( \bigcup_{(i_1, i_2) \in [m]} \bigcup_{2 \leq j \leq n} A_{i_1,i_2,j} \right) \leq n^{-1} \hat{b}_2 Y_1^2 Q_X. \tag{9}
\]

Here \( Q_X = m^{-2} \sum_{i_1,i_2 \in [m]} X_{i_1}^2 X_{i_2}^2 \). We note that \( Q_X \) and \( Y_1^2 \) are stochastically bounded and \( n^{-1} \hat{b}_2 = o_P(1) \) as \( n \to +\infty \). Therefore, \( P(A) = o_P(1) \). Now \( (3) \) implies \( (6) \).

**Proof of (7).** Let \( A_1 \) denote the complement event to \( A_1 \). By the union bound and Markov’s inequality
\[
P_1(A_1) \leq \sum_{i \in [m]} P_1(\lambda_{i1} \geq 1) \leq (nm)^{-1} Y_1^2 \sum_{i \in [m]} E\nu_i^2 = n^{-1} a_2 Y_1^2.
\]
Hence we obtain $P_1(\mathcal{A}_1) = o_P(1)$. Now (5) implies $P(\mathcal{A}_1) = EP_1(\mathcal{A}_1) = o(1)$.

Proof of (8). Since the first bound of (8) follows from the second one, we only prove the latter. Denote $X_1 = \max\{X_1, 1\}$ and $Y_1 = \max\{Y_1, 1\}$. We observe that $EX_1^2 < \infty$, $EY_1 < \infty$ implies

$$\lim_{t \to +\infty} E\hat{X}_1^2[X_1 > t] = 0, \quad \lim_{t \to +\infty} E\hat{Y}_1^2[Y_1 > t] = 0.$$ 

Hence one can find a strictly increasing function $\varphi : [1, +\infty) \to [0, +\infty)$ with $\lim_{t \to +\infty} \varphi(t) = +\infty$ such that

$$E\hat{X}_1^2\varphi(\hat{X}_1) < \infty, \quad E\hat{Y}_1\varphi(\hat{Y}_1) < \infty.$$ 

We remark that (11) and (12) imply the third and the first inequality of (13), respectively.

In addition, we can choose $\varphi$ satisfying

$$\varphi(t) < t \quad \text{and} \quad \varphi(st) \leq \varphi(t)\varphi(s), \quad \forall s, t \geq 1.$$ 

For this purpose we take a sufficiently slowly growing concave function $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi(0) = 0$ and define $\varphi(t) = e^{\psi(\ln(t))}$. We note that the second inequality of (11) follows from the concavity property of $\psi$. We remark, that (11) implies

$$t/(st) = s^{-1} \leq 1/\varphi(s) \leq \varphi(t)/\varphi(st), \quad s, t \geq 1.$$ 

Let $S_{XY}$ be defined as in (4) above, but with $\lambda_{ij}$ replaced by $\hat{\lambda}_{ij} = \hat{X}_i\hat{Y}_j/\sqrt{mn}$. We note, that $S_{XY} \leq \hat{S}_{XY}$. Furthermore, from the inequalities

$$\min\{1, \hat{\lambda}_{ij}\} \leq \min\left\{1, \frac{\varphi(\hat{X}_i\hat{Y}_j)}{\varphi(\sqrt{mn})}\right\} \leq \frac{\varphi(\hat{X}_i\hat{Y}_j)}{\varphi(\sqrt{mn})} \leq \frac{\varphi(\hat{X}_i)\varphi(\hat{Y}_j)}{\varphi(\sqrt{mn})}$$

we obtain $S_{XY} \leq \hat{S}_{XY} \leq S^*_{XY}/\varphi(\sqrt{mn})$, where

$$S^*_{XY} = (mn)^{-1} \left(\sum_{1 \leq i \leq m} \hat{X}_i^2\varphi(\hat{X}_i)\right) \left(\sum_{2 \leq j \leq n} \hat{Y}_j\varphi(\hat{Y}_j)\right).$$

We remark that (11) and (12) and imply the third and the first inequality of (13), respectively. Finally, the bound $ES_{XY} = o(1)$ follows from the inequality $S_{XY} \leq S^*_{XY}/\varphi(\sqrt{mn})$ and the fact that $ES^*_{XY}$ remains bounded as $n, m \to +\infty$, see (10). \hfill \Box

In the proof of Theorem 1 we use the following inequality refered to as LeCam’s lemma, see e.g., [23].

Lemma 2. Let $S = \mathbb{I}_1 + \mathbb{I}_2 + \cdots + \mathbb{I}_n$ be the sum of independent random indicators with probabilities $P(\mathbb{I}_i = 1) = p_i$. Let $\Lambda$ be Poisson random variable with mean $p_1 + \cdots + p_n$. The total variation distance between the distributions $P_S$ of $P_\Lambda$ of $S$ and $\Lambda$

$$\sup_{A \subseteq \{0, 1, 2, \ldots\}} |P(S \in A) - P(\Lambda \in A)| = \frac{1}{2} \sum_{k \geq 0} |P(S = k) - P(\Lambda = k)| \leq \sum_{i} p_i^2.$$  

(14)

Proof of Theorem 1. The case (i). We have $P(d(v_1) > \varepsilon) \leq P(L > \varepsilon)$, for each $\varepsilon > 0$. We prove that $P(L > \varepsilon) = o(1)$, for any $\varepsilon > 0$. In view of the identity $P(L > \varepsilon) = EP_1(L > \varepsilon)$ and (5) it suffices to show that $P_1(L > \varepsilon) = o_P(1)$. For this purpose we write, by the union bound and Markov’s inequality,

$$P_1(L > \varepsilon) \leq \sum_{1 \leq k \leq m} P_1(\mathbb{I}_k = 1) \leq E_1 \sum_{1 \leq k \leq m} \lambda_{kl} = \sqrt{m/n}Y_1EX_1 = o_P(1).$$
In cases (ii) and (iii) we apply (6). In view of (6) the random variables \( d(v_1) \) and \( L \) have the same asymptotic distribution (if any). Hence, it suffices to show the convergence in distribution of \( L \).

**The case (ii).** Here we prove that \( L \) converges in distribution to (1). We first approximate \( L \) by the random variable \( L_3 = \sum_{k=1}^{m} \eta_k \xi_{3k} \). Then we show that \( L_3 \) converges in distribution to (1). Here \( \eta_1, \ldots, \eta_m, \xi_{31}, \ldots, \xi_{3m} \) are conditionally independent (given \( X,Y \)) Poisson random variables with \( E \eta_k = \lambda_{k1} \) and \( E \xi_{3k} = X_k(n/m)^{1/2}b_1 \). We assume, in addition, that given \( X,Y \), the sequences \( \{\eta_k\}_{k=1}^{m} \) and \( \{\xi_{3k}\}_{k=1}^{m} \) are conditionally independent.

Given \( X,Y \), we generate independent Poisson random variables \( \xi_{11}, \ldots, \xi_{1m}, \Delta_{11}, \ldots, \Delta_{1m} \), with the conditional mean values

\[
\tilde{E} \xi_{1k} = \sum_{2 \leq j \leq n} p_{kj}, \quad \tilde{E} \Delta_{1k} = \sum_{2 \leq j \leq n} (\lambda_{kj} - p_{kj}), \quad 1 \leq k \leq m.
\]

We assume that, given \( X,Y \), these Poisson random variables are conditionally independent of the sequence \( \{\eta_k\}_{k=1}^{m} \). We suppose, in addition, that \( \{\eta_k\}_{k=1}^{m} \) is conditionally independent (given \( X,Y_1 \)) of the set of edges of \( H \) that are not incident to \( v_1 \). We define \( \xi_{2k} = \xi_{1k} + \Delta_{1k} \) and observe that \( \xi_{2k} \) has the same distribution as \( \xi_{1k} \) and \( \Delta_{1k} \). Hence, it suffices to show the convergence in distribution to (1). Here the first inequality follows from (7).

In order to show that \( L_2 \) and \( L_3 \) have the same asymptotic probability distribution (if any) we prove that

\[
d_{TV}(L_2, L_3) = o(1), \quad d_{TV}(L_0, L_1) = o(1), \quad d_{TV}(L_0, L_1) = o(1), \quad d_{TV}(L_0, L_1) = o(1).
\]

Here \( \tilde{L}_2 \) and \( \tilde{L}_3 \) are marginals of the random vector \( (\tilde{L}_2, \tilde{L}_3) \) constructed below which has the property that \( \tilde{L}_2 \) has the same distribution as \( L_2 \) and \( \tilde{L}_3 \) has the same distribution as \( L_3 \).

Let us prove the first bound of (15). In view of (6) it suffices to show that \( d_{TV}(L_0, L) = o_P(1) \). In order to prove the latter bound we apply the inequality

\[
d_{TV}(L_0, L) \|_{A_1} \leq n^{-1}Y_1^2\hat{a}_2 \]

shown below. We remark that (17) implies

\[
d_{TV}(L_0, L) \leq d_{TV}(L_0, L) \|_{A_1} \leq n^{-1}Y_1^2\hat{a}_2 + \|_{A_1} = o_P(1).
\]

Here \( n^{-1}Y_1^2\hat{a}_2 = o_P(1) \), because \( Y_1^2\hat{a}_2 \) is stochastically bounded. Furthermore, the bound \( \|_{A_1} = o_P(1) \) follows from (7).

It remains to prove (17). We denote \( L_k' = \sum_{i=1}^{k} \xi_i u_i + \sum_{i=k+1}^{m} \eta_i u_i \) and write, by the triangle inequality,

\[
d_{TV}(L_0, L) \leq \sum_{k=1}^{m} d_{TV}(L_k', L_k').
\]

Then we estimate \( d_{TV}(L_k', L_k') \leq d_{TV}(\eta_k, \xi_k) \leq (nm)^{-1}Y_1^2X_k^2 \). Here the first inequality follows from the properties of the total variation distance. The second inequality follows from Lemma 2 and the fact that on the event \( A_1 \) we have \( p_{k1} = \lambda_{k1} \).
Let us prove the second bound of (15). In view of (5) it suffices to show that \( \tilde{d}_{TV}(L_0, L_1) = O(1) \). We denote \( L_k^* = \sum_{i=1}^k \eta_i u_i + \sum_{i=k+1}^m \eta_i \xi_i \) and write, by the triangle inequality,

\[
\tilde{d}_{TV}(L_0, L_1) \leq \sum_{k=1}^m \tilde{d}_{TV}(L_{k-1}, L_k^*).
\]

Here

\[
\tilde{d}_{TV}(L_{k-1}, L_k^*) \leq \tilde{d}_{TV}(\eta_k u_k, \eta_k \xi_{1k}) \leq \tilde{P}(\eta_k \neq 0) \tilde{d}_{TV}(u_k, \xi_{1k}).
\]

Now, invoking the inequalities \( \tilde{P}(\eta_k \neq 0) = 1 - e^{-\lambda_{k1}} \leq \lambda_{k1} \) and \( \tilde{d}_{TV}(u_k, \xi_{1k}) \leq \sum_{j=2}^n \tilde{p}_{k, j}^2 \), we obtain from (18), (19) and (8) that

\[
\tilde{d}_{TV}(L_0, L_1) \leq \sum_{k=1}^m \lambda_{k1} \sum_{j=2}^n \tilde{p}_{k, j}^2 \leq Y_1 S_{XY} = o(1).
\]

Let us prove the first bound of (16). We observe that

\[
|L_2 - L_1| = L_2 - L_1 = \sum_{1 \leq k \leq m} \eta_k \Delta_{1k}
\]

and

\[
\tilde{E} \sum_{1 \leq k \leq m} \eta_k \Delta_{1k} = \sum_{1 \leq k \leq m} \lambda_{k1} \sum_{2 \leq j \leq n} (\lambda_{k, j} - 1) I_{\{\lambda_{k, j} > 1\}} \leq Y_1 S_{XY}.
\]

We obtain \( E|L_2 - L_1| \leq EY_1 ES_{XY} = o(1) \), see (8).

Let us prove the second bound of (16). Given \( X, Y \), generate independent Poisson random variables \( \xi_{31}, \ldots, \xi_{3m}, \Delta_{21}, \ldots, \Delta_{2m}, \Delta_{31}, \ldots, \Delta_{3m} \) which are conditionally independent of the sequence \( \{\eta_k\}_{k=1}^m \) and have the conditional mean values

\[
\tilde{E}\xi_{3k} = X_k(n/m)^{1/2} b, \quad \tilde{E}\Delta_{2k} = X_k(n/m)^{1/2} \delta_2, \quad \tilde{E}\Delta_{3k} = X_k(n/m)^{1/2} \delta_3.
\]

Here \( b = \min\{b_1, b_2\}, \delta_2 = \dot{b}_1 - b, \delta_3 = b_1 - b \). We note that \( \delta_2, \delta_3 \geq 0 \) and observe that the random vector

\[
(L_2', L_3'), \quad L_2' = \sum_{1 \leq k \leq m} \eta_k (\xi_{3k} + \Delta_{2k}), \quad L_3' = \sum_{1 \leq k \leq m} \eta_k (\xi_{3k} + \Delta_{3k})
\]

has the marginal distributions of \( (L_2, L_3) \). In addition, we have

\[
\tilde{E}|L_2' - L_3'| = (\delta_2 + \delta_3) Y_1 \dot{a}_2 = |\dot{b}_1 - b_1| Y_1 \dot{a}_2 = O_P(1).
\]

In the last step we used the fact that \( Y_1 \dot{a}_2 = O_P(1) \) and \( \dot{b}_1 - b_1 = o_P(1) \), by the law of large numbers. Finally, we show that (20) implies the bound \( |L_2' - L_3'| = o_P(1) \). Denoting, for short, \( H = |L_2' - L_3'| \) and \( h = E\{H \geq \varepsilon\} \), we write, for \( \varepsilon \in (0, 1) \),

\[
P(H \geq \varepsilon) = Eh = Eh\{H \geq \varepsilon^2\} + Eh\{H < \varepsilon^2\}.
\]

Using the simple inequality \( h \leq 1 \) and the inequality, \( h \leq \varepsilon^{-1} E H \), which follows from Markov’s inequality, we obtain

\[
Eh\{E \geq \varepsilon^2\} \leq Eh\{E \geq \varepsilon^2\} = P(EH \geq \varepsilon^2) = o(1),
\]

\[
Eh\{E < \varepsilon^2\} \leq E(\varepsilon^{-1} EH) \geq o(1)
\]

6
Invoking these inequalities in (21) we obtain \( P(H \geq \varepsilon) < \varepsilon + o(1) \). Hence \( H = o_P(1) \).

Now we prove that \( L_3 \) converges in distribution to \( \hat{L} \). Introduce the random variable \( \bar{L} = \sum_{1 \leq k \leq m} \eta_k \hat{\xi}_k \), where, given \( X, Y \), the random variables \( \xi_1, \ldots, \xi_m \) are conditionally independent of \( \{\eta_k\}_{k=1}^m \) and have the conditional mean values \( \bar{E}\hat{\xi}_k = X_k \beta^{-1/2} \). Proceeding as in the proof of the second bound of (16) above, we construct a random vector \((\bar{L}_3, \bar{L})\) with the same marginals as \((L_3, \bar{L})\) and such that

\[
\mathbb{E}|\bar{L}_3 - \bar{L}| \leq |1 - (m/n)1/2\beta^{-1/2}|Y_1 b_1 \tilde{a}_2 = o_P(1).
\] (22)

In the last step we used the fact that \( Y_1 \tilde{a}_2 = O_P(1) \) and \( m/n \to \beta \). Now, (22) implies \( L_3 - \bar{L}' = o_P(1) \). We conclude that \( L_3 \) and \( \bar{L} \) have the same asymptotic distribution (if any).

Next we prove that \( \bar{L} \) converges in distribution to \( \hat{L} \). For this purpose we show that \( \mathbb{E} e^{itL} \to \mathbb{E} e^{it\bar{L}} \), for each \( t \in (-\infty, +\infty) \). Denote \( \Delta(t) = e^{itL} - e^{it\bar{L}} \). We shall show below that, for any real \( t \) and any realized value \( Y_1 \) there exists a positive constant \( c = c(t, Y_1) \) such that for every \( 0 < \varepsilon < 0.5 \) we have

\[
\limsup_{n,m \to +\infty} |\mathbb{E}(\Delta(t)|Y_1)| < c\varepsilon.
\] (23)

Clearly, (23) implies \( \mathbb{E}(\Delta(t)|Y_1) = o(1) \). This fact together with the simple inequality \(|\Delta(t)| \leq 2\) yields \( \mathbb{E}\Delta(t) = o(1) \), by Lebesgue’s dominated convergence theorem. Observing that \( \mathbb{E}\Delta(t) = \mathbb{E} e^{itL} - \mathbb{E} e^{it\bar{L}} \), we conclude that \( \mathbb{E} e^{itL} \to \mathbb{E} e^{it\bar{L}} \).

We fix \( 0 < \varepsilon < 0.5 \) and prove (23). Before the proof we introduce some notation. Denote

\[
f_{\tau}(t) = \mathbb{E} e^{it\tau_1}, \quad \bar{f}_{\tau}(t) = \sum_{r \geq 0} e^{itr} \bar{p}_r, \quad \bar{p}_r = \bar{\lambda}^{-1} \sum_{1 \leq k \leq m} \lambda_k 1_{\{\bar{\xi}_k = r\}}, \quad \bar{\lambda} = \sum_{k=1}^m \lambda_k, \\
\delta = (\bar{f}_{\tau}(t) - 1)\bar{\lambda} - (f_{\tau}(t) - 1)\lambda_1, \quad f(t) = \mathbb{E} e^{it\bar{L}}, \quad \bar{f}(t) = \mathbb{E} e^{it\bar{L}}.
\]

Here \( \mathbb{E} \) denotes the conditional expectation given \( X, Y \) and \( \bar{\xi}_1, \ldots, \bar{\xi}_m \). Introduce the event \( \mathcal{D} = \{ |\bar{a}_1 - a_1| < \varepsilon \min\{1, a_1\} \} \) and let \( \overline{\mathcal{D}} \) denote the complement event. Furthermore, select the number \( T > 1/\varepsilon \) such that \( P(\tau_1 \geq T) < \varepsilon \). By \( c_1, c_2, \ldots \) we denote positive numbers which do not depend on \( n, m \).

We observe that, given \( Y_1 \), the conditional distribution of \( \bar{a}_1 \) is the compound Poisson distribution with the characteristic function \( f(t) = e^{\lambda_1 f_{\tau_1}(t-1)} \). Similarly, given \( X, Y \) and \( \bar{\xi}_1, \ldots, \bar{\xi}_m \), the conditional distribution of \( \bar{L} \) is the compound Poisson distribution with the characteristic function \( \bar{f}(t) = e^{\bar{\lambda} f(t-1)} \). In the proof of (23) we exploit the convergence \( \bar{\lambda} \to \lambda_1 \) and \( \bar{f}_{\tau}(t) \to f_{\tau}(t) \).

In what follows we assume that \( m, n \) are so large that \( \beta \leq 2m/n \leq 4\beta \).

Let us prove (23). We write

\[
\mathbb{E}_1 \Delta(t) = I_1 + I_2, \quad I_1 = \mathbb{E}_1 \Delta(t) 1_{\mathcal{D}}, \quad I_2 = \mathbb{E}_1 \Delta(t) 1_{\overline{\mathcal{D}}},
\]

Here \( |I_2| \leq 2\mathbb{P}(\overline{\mathcal{D}}) = 2\mathbb{P}(\overline{\mathcal{D}}) = o(1) \), by the law of large numbers. Next we estimate \( I_1 \). Combining the identity \( \mathbb{E}_1 \Delta(t) = \mathbb{E}_1 f(t)(e^\delta - 1) \) with the inequalities \(|f(t)| \leq 1 \) and \(|e^s - 1| \leq |s| e^{|s|} \), we obtain

\[
|I_1| \leq \mathbb{E}_1 |\delta| e^{\delta} 1_{\mathcal{D}} \leq c_1 \mathbb{E}_1 |\delta| 1_{\mathcal{D}}.
\] (24)

Here we estimated \( e^{\delta} \leq e^{8\lambda_1} =: c_1 \) using the inequalities

\[
|\delta| \leq 2\bar{\lambda} + 2\lambda_1, \quad \bar{\lambda} = Y_1 (m/n)^{1/2} \tilde{a}_1 \leq 3\lambda_1.
\]
We remark that the last inequality holds for \( m/n \leq 2\beta \) provided that the event \( \mathcal{D} \) occurs. Finally, we show that \( E_1[|\delta|] \leq (c_2 + \lambda_1 c_3 + \lambda_1 c_4)\varepsilon + o(1) \). We first write

\[
\delta = (\bar{f}_r(t) - 1)(\tilde{\lambda} - \lambda_1) + (\bar{f}_r(t) - f_r(t))\lambda_1,
\]

and estimate \(|\delta| \leq 2|\tilde{\lambda} - \lambda_1| + \lambda_1|\bar{f}_r(t) - f_r(t)|\). From the inequalities \(|\tilde{a}_1 - a_1| < \varepsilon \) and \( m/n \leq 2\beta \) we obtain

\[
|\tilde{\lambda} - \lambda_1| \leq Y_1|\tilde{a}_1 - a_1|(m/n)^{1/2} + Y_1a_1|(m/n)^{1/2} - \beta^{1/2}| \leq 2Y_1\beta^{1/2}\varepsilon + o(1).
\]

Hence \( E_1[|\tilde{\lambda} - \lambda_1|] \leq c_2\varepsilon + o(1) \), where \( c_2 = 2Y_1\beta^{1/2} \). We secondly show that

\[
E_1[|\bar{f}_r(t) - f_r(t)|] \leq (c_3 + c_4)\varepsilon + o(1).
\]

To this aim we split

\[
\bar{f}_r(t) - f_r(t) = \sum_{r \geq 0} e^{itr}(\bar{p}_r - p_r) = R_1 - R_2 + R_3,
\]

and estimate separately the terms

\[
R_1 = \sum_{r \geq T} e^{itr}\bar{p}_r, \quad R_2 = \sum_{r \geq T} e^{itr}p_r, \quad R_3 = \sum_{0 \leq r < T} e^{itr}(\bar{p}_r - p_r).
\]

Here we denote \( p_r = P(\tau_1 = r) \). The upper bound for \( R_2 \) follows by the choice of \( T \)

\[
|R_2| \leq \sum_{r \geq T} p_r = P(\tau_1 \geq T) < \varepsilon.
\]

Next, combining the identity \( p_r = (\tilde{a}_1)^{-1}\sum_{1 \leq k \leq m} X_k\mathbb{I}_{\{\tilde{\xi}_k = r\}} \) with the inequalities

\[
|p_r| \leq (\tilde{a}_1)^{-1}\sum_{r \geq T} \sum_{1 \leq k \leq m} X_k\mathbb{I}_{\{\tilde{\xi}_k \geq T\}} = (\tilde{a}_1)^{-1}\sum_{1 \leq k \leq m} X_k\mathbb{I}_{\{\tilde{\xi}_k \geq T\}}.
\]

and \( \tilde{a}_1^{-1} \mathbb{I}_D \leq 2a_1^{-1} \), we estimate

\[
E_1[|R_1|] \leq 2a_1^{-1}E_1(X_1\mathbb{I}_{\{\tilde{\xi} \geq T\}}) \leq 2a_1^{-1}T^{-1}E_1(X_1\tilde{\xi}) = 2a_1^{-1}a_2b_1\beta^{-1/2}\varepsilon.
\]

Hence \( E_1[|R_1|] \leq c_4\varepsilon \), where \( c_4 = 2a_1^{-1}a_2b_1\beta^{-1/2} \).

Now we estimate \( R_3 \). We denote \( p'_r = (\tilde{a}_1/a_1)\bar{p}_r \) and observe that the inequality \(|\tilde{a}_1 - a_1| \leq \varepsilon a_1 \) implies \(|\tilde{a}_1a_1^{-1} - 1| \leq \varepsilon \) and

\[
|\sum_{0 \leq r \leq T} e^{itr}(\bar{p}_r - p'_r)| \leq \varepsilon \sum_{0 \leq r \leq T} \bar{p}_r \leq \varepsilon.
\]

In the last inequality we use the fact that the probabilities \( \{\bar{p}_r\}_{r \geq 0} \) sum up to 1. It follows now that

\[
|R_3| \leq \varepsilon + \sum_{0 \leq r \leq T} |p'_r - p_r|.
\]

Furthermore, observing that \( E_1p'_r = a^{-1}E_1X_k\mathbb{I}_{\{\tilde{\xi}_k = r\}} = p_r \), for \( 1 \leq k \leq m \), we obtain

\[
E_1|p'_r - p_r|^2 = m^{-1}E_1|a^{-1}X_k\mathbb{I}_{\{\tilde{\xi}_k = r\}} - p_r|^2 \leq m^{-1}a_1^{-2}E_1X^2_1.
\]
Hence, \( E_1[p'_r - p_r] = O(m^{-1/2}) \). We conclude that
\[
E_1|R_3|1_D \leq \varepsilon + O(\sqrt{|T|m^{-1/2}}) = \varepsilon + o(1).
\]

**The case (iii).** We start with introducing some notation. Denote \( m/n = \beta_n \). Given \( \varepsilon \in (0,1) \) introduce random variables
\[
\gamma = \sum_{1 \leq k \leq m} I_k \lambda_k \gamma_k, \quad \gamma_k = X_k \beta_n^{-1/2} b_1, \quad Y_k = \mathds{1}_{\{X_k \beta_n^{-1/2} b_1 \leq \varepsilon\}}.
\]

Given \( X,Y \), let \( \tilde{I}_1, \ldots, \tilde{I}_m \) be conditionally independent Bernoulli random variables with success probabilities
\[
\tilde{P}(\tilde{I}_k = 1) = 1 - \tilde{P}(\tilde{I}_k = 0) = \gamma_k.
\]
We assume that, given \( X,Y \), the sequences \( \{\tilde{I}_k\}_{k=1}^m \), \( \{\tilde{I}_k\}_{k=1}^m \) and \( \{\xi_k\}_{k=1}^m \) are conditionally independent. Introduce random variables
\[
L_4 = \sum_{1 \leq k \leq m} \tilde{I}_k \xi_{3k}, \quad L_5 = \sum_{1 \leq k \leq m} \tilde{I}_k \tilde{I}_k \tilde{I}_k \xi_{3k}, \quad L_6 = \sum_{1 \leq k \leq m} \tilde{I}_k \tilde{I}_k \tilde{I}_k.
\]

Furthermore, we define the random variable \( L_7 \) as follows. We first generate \( X,Y \). Then, given \( X,Y \), we generate a Poisson random variable with the conditional mean value \( \gamma \). The realized value of the Poisson random variable is denoted \( L_7 \). Thus, we have \( P(L_7 = r) = E e^{-\gamma \gamma_r / r!} \), for \( r = 0,1, \ldots \).

We note that \( L \) and \( L_3 \) have the same asymptotic distribution (if any), by (15), (16). Now we prove that \( L_3 \) converges in distribution to \( \Lambda_3 \). For this purpose we show that for any \( \varepsilon \in (0,1) \)
\[
\begin{align*}
d_{\text{TV}}(L_3, L_4) &= o(1), \quad E(L_4 - L_5) = o(1), \quad (26) \\
d_{\text{TV}}(L_5, L_6) &\leq a_2 b_1^2 \varepsilon, \quad d_{\text{TV}}(L_6, L_7) = o(1), \quad (27) \\
E e^{it L_7} - E e^{it \Lambda_3} &= o(1). \quad (28)
\end{align*}
\]
Let us prove (26), (27), (28). The first bound of (26) is obtained in the same way as the first bound of (15). To show the second bound of (26) we invoke the inequality
\[
E(L_4 - L_5) = \sum_{1 \leq k \leq m} (1 - \tilde{I}_k) \tilde{I}_k \tilde{I}_k \tilde{I}_k \xi_{3k} \leq Y_1 b_1 m^{-1} \sum_{1 \leq k \leq m} (1 - \tilde{I}_k) X_k^2
\]
and obtain
\[
E(L_4 - L_5) = E \bar{E}(L_4 - L_5) \leq b_1^2 E X_1^2 \mathds{1}_{\{X_1 \geq \varepsilon b_1^{-1} \beta_n^{1/2}\}} = o(1).
\]
We note that the right hand side tends to zero since \( \beta_n \to +\infty \).

Let us prove the first inequality of (27). Proceeding as in (18), (19) and using the identity \( \tilde{I}_k = \tilde{I}_k \tilde{I}_k \tilde{I}_k \) we write
\[
\begin{align*}
\tilde{d}_{\text{TV}}(L_5, L_6) &\leq \sum_{1 \leq k \leq m} \tilde{I}_k \tilde{I}_k \tilde{I}_k \hat{P}(\tilde{I}_k \neq 0) \tilde{d}_{\text{TV}}(\xi_{3k}, \tilde{I}_k).
\end{align*}
\]
Next, we estimate \( \tilde{I}_k \tilde{I}_k \hat{d}_{\text{TV}}(\xi_{3k}, \tilde{I}_k) \leq \gamma_k^2 \), by LeCam’s inequality (14), and invoke the inequality \( \hat{P}(\tilde{I}_k \neq 0) \leq \lambda_k \). We obtain
\[
\begin{align*}
\tilde{d}_{\text{TV}}(L_5, L_6) &\leq \sum_{1 \leq k \leq m} \tilde{I}_k \tilde{I}_k \tilde{I}_k \gamma_k \leq \varepsilon \sum_{1 \leq k \leq m} \tilde{I}_k \tilde{I}_k \tilde{I}_k \gamma_k \leq \varepsilon Y_1 b_1 a_2.
\end{align*}
\]
Here we estimated $\gamma^2 \leq \varepsilon \gamma k$. Now the inequalities $d_{\mathcal{TV}}(L_5, L_6) \leq \mathbf{E}d_{\mathcal{TV}}(L_5, L_6) \leq a_2 b_1^2 \varepsilon$ imply the first relation of (27).

Let us prove the second relation of (27). In view of (5) it suffices to show that $\tilde{d}_{\mathcal{TV}}(L_6, L_7) = o_P(1)$. For this purpose we write

$$d_{\mathcal{TV}}(L_6, L_7) \leq \mathbb{I}_{\mathcal{A}_1} \tilde{d}_{\mathcal{TV}}(L_6, L_7) + \mathbb{I}_{\mathcal{A}_1},$$

where $\mathbb{I}_{\mathcal{A}_1} = o_P(1)$, see (7), and estimate using LeCam’s inequality (14)

$$
\mathbb{I}_{\mathcal{A}_1} \tilde{d}_{\mathcal{TV}}(L_6, L_7) \leq \mathbb{I}_{\mathcal{A}_1} \sum_{1 \leq k \leq m} \mathbf{P}^2(\mathbb{I}_k \tilde{I}_k = 1) \mathbb{I}_k \leq b_1^2 Y_1^2 m^{-2} \hat{a}_4 = o_P(1).
$$

Here we used the fact that $EX_1^2 < \infty$ implies $m^{-2} \hat{a}_4 = o_P(1)$.

Finally, we show (28). We write $\mathbf{E}e^{itL_7} = e^{\gamma(e^{it-1})}$ and observe that

$$Y_1 b_1 a_2 - \gamma = o_P(1).$$

Furthermore, since for any real $t$ the function $z \rightarrow e^{\gamma(e^{it-1})}$ is bounded and uniformly continuous for $z \geq 0$, we conclude that (29) implies the convergence

$$
\mathbf{E}e^{itL_7} = \mathbf{E}e^{\gamma(e^{it-1})} \rightarrow \mathbf{E}e^{Y_1 b_1 a_2 (e^{it-1})} = \mathbf{E}e^{it\Lambda_3}.
$$

It remains to prove (29). We write $Y_1 b_1 a_2 - \gamma = Y_1 b_1 (a_2 - \hat{a}_2) + Y_1 b_1 \hat{a}_2 - \gamma$ and note that $a_2 - \hat{a}_2 = o_P(1)$, by the law of large numbers, and

$$0 \leq \mathbf{E}(Y_1 b_1 \hat{a}_2 - \gamma) = b_1^2 \mathbf{E}X_1^2 \mathbb{I}_{\{X_1 \geq b_1^{-1} \beta_n^{1/2}\}} = o(1).$$

\[\square\]

Acknowledgement. Research of M. Bloznelis was supported by the Research Council of Lithuania grant MIP-053/2011.

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