Space-Time Compactification Induced By Lightlike Branes

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Abstract

The aim of the present paper is two-fold. First we describe the Lagrangian dynamics of a recently proposed new class of lightlike p-branes and their interactions with bulk space-time gravity and electromagnetism in a self-consistent manner. Next, we discuss the role of lightlike branes as natural candidates for wormhole “throats” and exemplify the latter by presenting an explicit construction of a new type of asymmetric wormhole solution where the lightlike brane connects a “right” universe with Reissner-Nordström geometry to a “left” Bertotti-Robinson universe with two compactified space dimensions.

Keywords: traversable wormholes; non-Nambu-Goto lightlike branes; dynamical brane tension; black hole’s horizon “straddling”

1 Introduction

Lightlike branes (LL-branes for short) play an increasingly significant role in general relativity and modern non-perturbative string theory. Mathematically they represent singular null hypersurfaces in Riemannian space-time which provide dynamical description of various physically important cosmological and astrophysical phenomena such as:

(i) Impulsive lightlike signals arising in cataclysmic astrophysical events (supernovae, neutron star collisions) [1];
(ii) Dynamics of horizons in black hole physics – the so called “membrane paradigm” [2];
(iii) The thin-wall approach to domain walls coupled to gravity [3, 4, 5].

More recently, the relevance of LL-branes in the context of non-perturbative string theory has also been recognized, specifically, as the so called H-branes describing quantum horizons (black hole and cosmological) [6], as Penrose limits of baryonic D-branes [7], etc (see also Refs.[8]).

A characteristic feature of the formalism for LL-branes in the pioneering papers [3, 4, 5] in the context of gravity and cosmology is that they have been exclusively treated in a “phenomenological” manner, i.e., without specifying an underlying Lagrangian dynamics from which they may originate. As a partial exception, in a more recent paper [9] brane actions in terms of their pertinent extrinsic geometry have been proposed which generically describe non-lightlike branes, whereas the lightlike branes are treated as a limiting case.

On the other hand, in the last few years we have proposed in a series of papers [10, 11, 12, 13] a new class of concise manifestly reparametrization invariant world-volume Lagrangian actions, providing a derivation from first principles of the LL-brane dynamics. The following characteristic features of the new LL-branes drastically distinguish them from ordinary Nambu-Goto branes:

(a) They describe intrinsically lightlike modes, whereas Nambu-Goto branes describe massive ones.
(b) The tension of the LL-brane arises as an additional dynamical degree of freedom, whereas Nambu-Goto brane tension is a given ad hoc constant. The latter characteristic feature significantly distinguishes our LL-brane models from the previously proposed tensionless p-branes (for a review, see Ref.[14]). The latter rather resemble p-dimensional continuous distributions of independent massless point-particles without cohesion among the latter.
(c) Consistency of LL-brane dynamics in a spherically or axially symmetric gravitational background of codimension one requires the presence of an event horizon which is automatically occupied by the LL-brane (“horizon straddling” according to the terminology of Ref.[4]).
(d) When the LL-brane moves as a test brane in spherically or axially symmetric gravitational backgrounds its dynamical tension exhibits exponential “inflation/deflation” time behavior [11] – an effect
similar to the “mass inflation” effect around black hole horizons [15].

An intriguing novel application of $LL$-branes as natural self-consistent gravitational sources for wormhole space-times has been developed in a series of recent papers [12, 13, 16, 17].

Before proceeding let us recall that the concept of “wormhole space-time” was born in the classic work of Einstein and Rosen [18], where they considered matching along the horizon of two identical copies of the exterior Schwarzschild space-time region (subsequently called Einstein-Rosen “bridge”). Another cornerstone in wormhole physics is the seminal work of Morris and Thorne [19], who studied for the first time traversable Lorentzian wormholes.

In what follows, when discussing wormholes we will have in mind the physically important class of “thin-shell” traversable Lorentzian wormholes first introduced by Visser [20, 21]. For a comprehensive review of wormhole space-times, see Refs. [21, 22].

In our earlier work [12, 13, 16, 17] we have constructed various types of wormhole solutions in self-consistent systems of bulk gravity and bulk gauge fields (Maxwell and Kalb-Ramond) coupled to $LL$-branes where the latter provide the appropriate stress energy tensors, electric currents and dynamically generated space-varying cosmological constant terms consistently derived from well-defined world-volume $LL$-brane Lagrangian actions.

The original Einstein-Rosen “bridge” manifold [18] appears as a particular case of the construction of spherically symmetric wormholes produced by $LL$-branes as gravitational sources occupying the wormhole throats (Refs. [16, 13]). Thus, we are lead to the important conclusion that consistency of Einstein equations of motion yielding the original Einstein-Rosen “bridge” as well-defined solution necessarily requires the presence of $LL$-brane energy-momentum tensor as a source on the right hand side.

More complicated examples of spherically and axially symmetric wormholes with Reissner-Nordström and rotating cylindrical geometry, respectively, have been explicitly constructed via $LL$-branes in Refs. [12, 13]. Namely, two copies of the exterior space-time region of a Reissner-Nordström or rotating cylindrical black hole, respectively, are matched via $LL$-brane along what used to be the outer horizon of the respective full black hole space-time manifold. In this way one obtains a wormhole solution which combines the features of the Einstein-Rosen “bridge” on the one hand (with wormhole throat at horizon), and the features of Misner-Wheeler wormholes [23], i.e., exhibiting the so called “charge without charge” phenomenon.

Recently the results of Refs. [12, 13] have been extended to the case of asymmetric wormholes, describing two “universes” with different spherically symmetric geometries of black hole type connected via a “throat” materialized by the pertinent gravitational source – an electrically charged $LL$-brane, sitting on their common horizon. As a result of the well-defined world-volume $LL$-brane dynamics coupled self-consistently to gravity and bulk space-time gauge fields, it creates a “left universe” comprising the exterior Schwarzschild-de-Sitter space-time region beyond the Schwarzschild horizon and where the cosmological constant is dynamically generated, and a “right universe” comprising the exterior Reissner-Nordström region beyond the outer Reissner-Nordström horizon with dynamically generated Coulomb field-strength. Both “universes” are glued together by the $LL$-brane occupying their common horizon. Similarly, the $LL$-brane can dynamically generate a non-zero cosmological constant in the “right universe”, in which case it connects a purely Schwarzschild “left universe” with a Reissner-Nordström-de-Sitter “right universe”.

In the present paper we will further broaden the application of $LL$-branes in the context of wormhole physics by constructing a new type of wormhole solution to Einstein-Maxwell equations describing a “right universe”, which comprises the exterior Reissner-Nordström space-time region beyond the outer Reissner-Nordström horizon, connected through a “throat” materialized by a $LL$-brane with a “left universe” being a Bertotti-Robinson spacetime with two compactified spatial dimensions [24] (see also [25]).

Let us note that previously the junction of a compactified space-time (of Bertotti-Robinson type) to an uncompactified space-time through a wormhole has been studied in a different setting using time-like matter on the junction hypersurface [26]. Also, in a different context a string-like (flux tube) object with similar features to Bertotti-Robinson solution has been constructed [27] which interpolates between uncompactified space-time regions.

2 World-Volume Formulation of Lightlike Brane Dynamics

There exist two equivalent dual to each other manifestly reparametrization invariant world-volume Lagrangian formulations of $LL$-branes [10, 11, 12, 13, 16, 28]. First, let us consider the Polyakov-type for-
mulation where the \( LL\)-brane world-volume action is given as:

\[
S_{LL} = \int d^{p+1} \sigma \Phi \left[ -\frac{1}{2} \gamma^{ab} g_{ab} + L(F^2) \right].
\] (1)

Here the following notions and notations are used:
(a) \( \Phi \) is alternative non-Riemannian integration measure density (volume form) on the \( p \)-brane world-volume manifold:

\[
\Phi = \frac{1}{(p+1)!} \varepsilon^{a_1 \ldots a_{p+1}} H_{a_1 \ldots a_{p+1}}(B),
\] (2)

\[
H_{a_1 \ldots a_{p+1}}(B) = (p+1) \partial_{[a_1} B_{a_2 \ldots a_{p+1}]},
\] (3)

instead of the usual \( \sqrt{-\gamma} \). Here \( \varepsilon^{a_1 \ldots a_{p+1}} \) is the alternating symbol \( (a_1^a \ldots p^a = 1) \), \( \gamma_{ab} \) \( (a,b = 0,1, \ldots , p) \) indicates the intrinsic Riemannian metric on the world-volume, and \( \gamma = \det |\gamma_{ab}| \). \( H_{a_1 \ldots a_{p+1}}(B) \) denotes the field-strength of an auxiliary world-volume antisymmetric tensor gauge field \( B_{a_1 \ldots a_p} \) of rank \( p \). As a special case one can build \( H_{a_1 \ldots a_{p+1}} \) in terms of \( p+1 \) auxiliary world-volume scalar fields \( \{ \varphi^I \} \).

\[ H_{a_1 \ldots a_{p+1}} = \varepsilon_{\mu_1 \ldots \nu_{p+1}} \partial_{\mu_1} \varphi^1 \ldots \partial_{\nu_p} \varphi^p \] (4)

Note that \( \gamma_{ab} \) is independent of the auxiliary world-volume fields \( B_{a_1 \ldots a_p} \) or \( \varphi^I \). The alternative non-Riemannian volume form (2) has been first introduced in the context of modified standard (non-lightlike) string and \( p \)-brane models in Refs.[29].

(b) \( X^\mu(\sigma) \) are the \( p \)-brane embedding coordinates in the bulk \( D \)-dimensional space-time with bulk Riemannian metric \( G_{\mu\nu}(X) \) with \( \mu, \nu = 0,1, \ldots , D-1 \);

\[ (\sigma) \equiv (\sigma^0 = \tau, \sigma^i) \] with \( i = 1, \ldots , p \); \( \partial_\mu = \frac{\partial}{\partial \sigma^\mu} \).

(c) \( g_{ab} \) is the induced metric on world-volume:

\[ g_{ab} \equiv \partial_\mu X^a \partial_\nu X^b G_{\mu\nu}(X), \] (5)

which becomes singular on-shell (manifestation of the lightlike nature, cf. second Eq.(10) below).

(d) \( L(F^2) \) is the Lagrangian density of another auxiliary \((p-1)\)-rank antisymmetric tensor gauge field \( A_{a_1 \ldots a_{p-1}} \) on the world-volume with \( p \)-rank field-strength and its dual:

\[ F_{a_1 \ldots a_p} = p \partial_{[a_1} A_{a_2 \ldots a_p]}, \quad F^{*a} = \frac{1}{p!} \sqrt{-\gamma} F_{a_1 \ldots a_p}. \] (6)

\( L(F^2) \) is arbitrary function of \( F^2 \) with the short-hand notation:

\[
F^2 \equiv F_{a_1 \ldots a_p} F_{b_1 \ldots b_p} \gamma^{a_1 b_1} \ldots \gamma^{a_p b_p}.
\]

Rewriting the action (1) in the following equivalent form:

\[
S = - \int d^{p+1} \sigma \sqrt{-\gamma} \left[ \frac{1}{2} \gamma^{ab} g_{ab} + L(F^2) \right],
\]

\[
\chi \equiv \Phi \sqrt{-\gamma}
\] (7)

with \( \Phi \) the same as in (2), we find that the composite field \( \chi \) plays the role of a dynamical (variable) brane tension

1

Let us now consider the equations of motion corresponding to (1) w.r.t. \( B_{a_1 \ldots a_p} \):

\[ \partial_a \left[ \frac{1}{2} \gamma^{cd} g_{cd} - L(F^2) \right] = 0 \rightarrow \frac{1}{2} \gamma^{cd} g_{cd} - L(F^2) = M, \] (8)

where \( M \) is an arbitrary integration constant. The equations of motion w.r.t. \( \gamma^{ab} \) read:

\[
\frac{1}{2} g_{ab} - F^2 L'(F^2) \left( \gamma_{ab} - \frac{F_a F_b}{F^* + F^2} \right) = 0, \] (9)

where \( F^* \) is the dual field strength (6). Eqs.(9) can be viewed as \( p \)-brane analogues of the string Virasoro constraints.

Taking the trace in (9) and comparing with (8) implies the following crucial relation for the Lagrangian function \( L(F^2) \): \( L(F^2) - pF^2 L'(F^2) + M = 0 \), which determines \( F^2 \) on-shell as certain function of the integration constant \( M \) (8), i.e. \( F^2 = F^2(M) \) = const. Here and below \( L'(F^2) \) denotes derivative of \( L(F^2) \) w.r.t. the argument \( F^2 \).

The next and most profound consequence of Eqs.(9) is that the induced metric (5) on the world-volume of the \( p \)-brane model (1) is singular on-shell (as opposed to the induced metric in the case of ordinary Nambu-Goto branes):

\[ g_{ab} F^{*a} b \equiv \partial_a X^b G_{\mu\nu} \left( \partial_b X^\nu F^{*a} b \right) = 0. \] (10)

Eq.(10) is the manifestation of the lightlike nature of the \( p \)-brane model (1) (or (7)), namely, the tangent vector to the world-volume \( F^* a \partial_a X^\nu \) is lightlike w.r.t. metric of the embedding space-time.

Further, the equations of motion w.r.t. world-volume gauge field \( A_{a_1 \ldots a_{p-1}} \) (with \( \chi \) as defined in (7) read:

\[
\partial_a \left( F^* a \right) \chi = 0. \] (11)

Finally, the \( X^\mu \) equations of motion produced by the (1) read:

\[
\partial_a \left( \chi \sqrt{-\gamma} \gamma^{ab} \partial_b X^\mu \right) + \chi \sqrt{-\gamma} \gamma^{ab} \partial_a X^\nu \partial_b X^\lambda \Gamma_{\nu\lambda}^a = 0
\] (12)

where \( \Gamma_{\nu\lambda}^a = \frac{1}{2} G^{\mu\nu} \left( \partial_\mu G_{\kappa\lambda} + \partial_\lambda G_{\kappa\mu} - \partial_\kappa G_{\mu\lambda} \right) \) is the Christoffel connection for the external metric.

1 The notion of dynamical brane tension has previously appeared in different contexts in Refs.[30].
Eq.(11) allows us to introduce the dual “gauge” potential \( u \) (dual w.r.t. world-volume gauge field \( A_{a_1...a_{p-1}}(6) \)):

\[
F^*_a = c_p \frac{1}{\chi} \partial_a u \quad \text{,} \quad c_p = \text{const} \ .
\] (13)

Relation (13) enables us to rewrite Eq.(9) (the light-like constraint) in terms of the dual potential \( u \) in the form:

\[
\gamma_{ab} = \frac{1}{2a_0} g_{ab} - \frac{(2a_0)^{p-2}}{\chi^2} \partial_a u \partial_b u
\]

\[a_0 \equiv F^2 L' (F^2) \bigg|_{F^2 = F^2 (M)} = \text{const} \ .
\] (14)

\((L/(F^2)\) denotes derivative of \((L/(F^2)\) w.r.t. the argument \( F^2 \). From (13) we obtain the relation:

\[
\chi^2 = -(2a_0)^{p-2} \gamma_{ab} \partial_a u \partial_b u \ ,
\] (15)

and the Bianchi identity \( \nabla_a F^{*a} = 0 \) becomes:

\[
\partial_a \left( \frac{1}{\chi} \sqrt{-\gamma} \gamma^{ab} \partial_b u \right) = 0 \ .
\] (16)

It is straightforward to prove that the system of equations (12), (16) and (15) for \((X^\mu, u, \chi)\), which are equivalent to the equations of motion (8)–(11),(12) resulting from the original Polyakov-type \( LL\)-brane action (1), can be equivalently derived from the following \( dual \) Nambu-Goto-type world-volume action:

\[
S_{NG} = -\int d^{p+1} \sigma T \sqrt{\det g_{ab} - \epsilon \frac{1}{T^2} \partial_a u \partial_b u} \ ,
\] (17)

with \( \epsilon = \pm 1 \). Here again \( g_{ab} \) indicates the induced metric on the world-volume (5) and \( T \) is dynamical variable tension simply proportional to \( \chi \) \( (\chi^2 = (2a_0)^{p-1} T^2 \)with \( a_0 \) as in (14)). The choice of the sign in (17) does not have physical effect because of the non-dynamical nature of the \( u \)-field.

Henceforth we will stick to the Polyakov-type formulation of world-volume \( LL\)-brane dynamics since within this framework one can add in a natural way \([10, 11, 12]\) couplings of the \( LL\)-brane to bulk space-time Maxwell \( A_\mu \) and Kalb-Ramond \( A_{\mu_1...\mu_D} \) gauge fields (in the case of codimension one \( LL\)-branes, i.e., for \( D = (p + 1) + 1 \):

\[
\tilde{S}_{LL} = S_{LL} - q \int d^{p+1} \sigma \varepsilon^{a_1...a_p} b_p F_{b_1...b_p} \partial_a X^\mu A_\mu
\]

\[-\frac{\beta}{(p+1)!} \int d^{p+1} \sigma \varepsilon^{a_1...a_{p+1}} \partial_a X^{\mu_1}...\partial_{a_{p+1}} X^{\mu_{p+1}}
\]

\[\times A_{\mu_1...\mu_{p+1}}
\] (18)

with \( S_{LL} \) as in (1). The \( LL\)-brane constraint equations (8)–(9) are not affected by the bulk space-time gauge field couplings whereas Eqs.(11)–(12) acquire the form:

\[
\partial_{\mu} \left( F^*_\mu \right) + \frac{q}{4} \partial_\mu X^\nu \partial_\nu X^\nu F_{\mu \nu} = 0 \ ;
\] (19)

\[
\partial_\mu \left( \chi \sqrt{-\gamma} \gamma^{ab} \partial_b X^\mu \right) + \chi \sqrt{-\gamma} \gamma^{ab} \partial_\mu X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu \lambda}
\]

\[-q \varepsilon^{a_1...a_{p+1}} b_p F_{b_1...b_p} \partial_a X^\nu F_{\lambda \nu \lambda}
\]

\[\epsilon \frac{1}{(p+1)!} \varepsilon^{a_1...a_{p+1}} \partial_a X^{\mu_1}...\partial_{a_{p+1}} X^{\mu_{p+1}}
\]

\[\sqrt{-F_{\nu \sigma \lambda}} G^{\mu \nu} G^{\lambda \mu} = 0 \ .
\] (20)

Here \( \chi \) is the dynamical brane tension as in (7), \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and

\[
F_{\mu_1...\mu_D} = D \partial_{[\mu_1} A_{\mu_2...\mu_D]} = F_{\nu \sigma \lambda} G^{\nu \sigma} G^{\lambda \mu}
\] (21)

are the field-strengths of the electromagnetic \( A_\mu \) and Kalb-Ramond \( A_{\mu_1...\mu_D} \) gauge potentials \([31]\).

### 3 Lightlike Brane Dynamics in Various Types of Gravitational Backgrounds

World-volume reparametrization invariance allows us to introduce the standard synchronous gauge-fixing conditions:

\[
\gamma^{0i} = 0 \quad (i = 1, ..., p) \ , \quad \gamma^{00} = -1 \ .
\] (22)

Also, we will use a natural ansatz for the “electric” part of the auxiliary world-volume gauge field-strength (6):

\[
F^{0i} = 0 \quad (i = 1, ..., p) \ , \quad \text{i.e.,} \quad F_0u_1...u_{p-1} = 0 \ ,
\] (23)

meaning that we choose the lightlike direction in Eq.(10) to coincide with the brane proper-time direction on the world-volume \( F^{0a} \partial_a \sim \partial_t \). The Bianchi identity \( \nabla_a F^{*a} = 0 \) together with (22)–(23) and the definition for the dual field-strength in (6) imply:

\[
\partial_t \gamma^{(p)} = 0 \quad \text{where} \quad \gamma^{(p)} \equiv \det \| \gamma_{ij} \| \ .
\] (24)

Taking into account (22)–(23), Eqs.(9) acquire the following gauge-fixed form (recall definition of the induced metric \( g_{ab} \) (5)):

\[
g_{00} \equiv X^\mu G_{\mu \nu} X^\nu = 0 \ , \quad g_{0i} = 0 \ , \quad g_{ij} - 2a_0 \gamma_{ij} = 0 \ ,
\] (25)

where \( a_0 \) is the same constant as in (14).
3.1 Spherically Symmetric Backgrounds

Here we will be interested in static spherically symmetric solutions of Einstein-Maxwell equations (see Eqs.(35)–(36) below). We will consider the following generic form of static spherically symmetric metric:

\[ ds^2 = -A(\eta)dt^2 + \frac{d\eta^2}{A(\eta)} + C(\eta)h_{ij}(\tilde{\theta})d\tilde{\theta}^i d\tilde{\theta}^j \ , \quad (26) \]

or, in Eddington-Finkelstein coordinates \([32]\) \((dt = dv - \frac{d\eta}{A(\eta)})\):

\[ ds^2 = -A(\eta)dv^2 + 2dv d\eta + C(\eta)h_{ij}(\tilde{\theta})d\tilde{\theta}^i d\tilde{\theta}^j \ . \quad (27) \]

Here \(h_{ij}\) indicates the standard metric on the sphere \(S^p\). The radial-like coordinate \(\eta\) will vary in general from \(-\infty\) to \(+\infty\).

We will consider the simplest ansatz for the \(LL\)-brane embedding coordinates:

\[ X^0 = v = \tau \ , \quad X^i = \eta = \eta(\tau) \]
\[ X^i = \tilde{\theta}^i = \sigma^i \quad (i = 1, \ldots, p) \ . \quad (28) \]

Now, the \(LL\)-brane equations (25) together with (24) yield:

\[ -A(\eta) + 2 \eta = 0 \ , \quad \partial_\tau C = \frac{\eta}{C} \partial_\eta C \bigg|_{\eta=\eta(\tau)} = 0 \ . \quad (29) \]

First, we will consider the case of \(C(\eta)\) as non-trivial function of \(\eta\) (i.e., proper spherically symmetric space-time). In this case Eqs.(29) imply:

\[ \eta = 0 \rightarrow \eta(\tau) = \eta_0 = \text{const} \ , \quad A(\eta_0) = 0 \ . \quad (30) \]

Eq.(30) tells us that consistency of \(LL\)-brane dynamics in a proper spherically symmetric gravitational background of codimension one requires the latter to possess a horizon (at some \(\eta = \eta_0\)), which is automatically occupied by the \(LL\)-brane (“horizon straddling” according to the terminology of Ref.[4]). Similar property – “horizon straddling”, has been found also for \(LL\)-branes moving in rotating axially symmetric (Kerr or Kerr-Newman) and rotating cylindrically symmetric black hole backgrounds [12, 13].

With the embedding ansatz (28) and assuming the bulk Maxwell field to be purely electric static one \(\mathcal{F}_\eta = \mathcal{F}_{v\eta} \neq 0\), the rest being zero; this is the relevant case to be discussed in what follows), Eq.(19) yields the simple relation: \(\partial_\tau \chi = 0\ , \ i.e. \ \chi = \chi(\tau)\). Further, the only non-trivial contribution of the second order (w.r.t. world-volume proper time derivative) \(X^\mu\)-equations of motion (20) arises for \(\mu = v\), where the latter takes the form of an evolution equation for the dynamical tension \(\chi(\tau)\). In the case of absence of couplings to bulk space-time gauge fields, the latter yields exponential “inflation”/“deflation” at large times for the dynamical \(LL\)-brane tension:

\[ \chi(\tau) = \chi_0 \exp \left\{ -\tau \left( \frac{1}{2} \psi_0 + \psi \right) \right\} \ , \quad (31) \]

\(\chi_0 = \text{const}\). Similarly to the “horizon straddling” property, exponential “inflation”/“deflation” for the \(LL\)-brane tension has also been found in the case of test \(LL\)-brane motion in rotating axially symmetric and rotating cylindrically symmetric black hole backgrounds (for details we refer to Refs.[11, 12, 13]). This phenomenon is an analog of the “mass inflation” effect around black hole horizons [15].

3.2 Product-Type Gravitational Backgrounds: Bertotti-Robinson Space-Time

Consider now the case \(C(\eta) = \text{const} \) in (27), i.e., the corresponding space-time manifold is of product type \(\Sigma_2 \times S^p\). A physically relevant example is the Bertotti-Robinson [24, 25] space-time in \(D = 4\) (i.e., \(p = 2\)) with (non-extremal) metric (cf.[25]):

\[ ds^2 = r_0^2 \left[ -\eta^2 dt^2 + \frac{d\eta^2}{\eta^2} + d\theta^2 + \sin^2 \theta d\varphi^2 \right] \ , \quad (32) \]

or in Eddington-Finkelstein (EF) form \((dt = \frac{1}{r_0^2}dv - \frac{d\eta}{\eta})\):

\[ ds^2 = -\frac{\eta^2}{r_0^2} dv^2 + 2dv d\eta + r_0^2 \left[ d\theta^2 + \sin^2 \theta d\varphi^2 \right] \ . \quad (33) \]

At \(\eta = 0\) the Bertotti-Robinson metric (32) (or (33)) possesses a horizon. Further, we will consider the case of Bertotti-Robinson universe with constant electric field \(\mathcal{F}_{v\eta} = \pm \frac{1}{2r_0 \sqrt{\eta}}\). In the present case the second Eq.(29) is trivially satisfied whereas the first one yields: \(\eta(\tau) = \eta(0) \left( 1 - \tau \frac{\eta(0)}{2r_0^2} \right)^{-1}\). In particular, if the \(LL\)-brane is initially (at \(\tau = 0\)) located on the Bertotti-Robinson horizon \(\eta = 0\), it will stay there permanently.

4 Self-Consistent Wormhole Solutions Produced By Lightlike Branes

4.1 Lagrangian Formulation of Bulk Gravity-Matter System Coupled to Lightlike Brane

Let us now consider self-consistent bulk Einstein-Maxwell-Kalb-Ramond system coupled to a charged
codimension-one lightlike p-brane (i.e., \(D = (p + 1) + 1\)). It is described by the following action:

\[
S = \int d^Dx \sqrt{-G} \left[ \frac{R(G)}{16\pi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{D^2} F_{\mu_1...\mu_D} F^{\mu_1...\mu_D} \right] \equiv \tilde{S}_{LL} .
\]

Here \(F_{\mu\nu}\) and \(F_{\mu_1...\mu_D}\) are the Maxwell and Kalb-Ramond field-strengths \((21)\). The last term on the r.h.s. of \((34)\) indicates the reparametrization invariant world-volume action \((18)\) of the LL-brane coupled to the bulk space-time gauge fields.

The pertinent Einstein-Maxwell-Kalb-Ramond equations of motion derived from the action \((34)\) read:

\[
R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = 8\pi \left( T_{\mu\nu}^{(EM)} + T_{\mu\nu}^{(KR)} + T_{\mu\nu}^{(brane)} \right) ,
\]

\[
\partial_\nu \left( \sqrt{-G} F^{\mu\nu} \right) + \frac{1}{2} G_{\mu\nu} \partial_\sigma \delta(D) \left( x - X(\sigma) \right) \cdot \varepsilon^{a_1...a_p} \partial_{a_1} X^\mu \, ,
\]

\[
\varepsilon^{\mu_1...\mu_{p+1}} \partial_\nu F - \beta \int d^{p+1}\sigma \delta(D) \left( x - X(\sigma) \right) \cdot \varepsilon^{a_1...a_p} \partial_{a_1} X^\mu , \quad \partial_{\sigma} X^{\mu_{p+1}} = 0 ,
\]

where in the last equation we have used the last relation \((21)\). The explicit form of the energy-momentum tensors read:

\[
T_{\mu\nu}^{(EM)} = F_{\mu\alpha} F_{\nu\lambda} G^{\alpha\lambda} - G_{\mu\nu} \frac{1}{4} F_{\rho\sigma} F_{\sigma\rho} G^{\rho\sigma} G^{\alpha\lambda} , \quad T_{\mu\nu}^{(KR)} = \frac{1}{(D - 1)!} \left[ F_{\mu\lambda_1...\lambda_{D-1}} F_{\nu}^{\lambda_1...\lambda_{D-1}} \right] - \frac{1}{2} G^{\mu\nu} F^{2} , \quad T_{\mu\nu}^{(brane)} = -G_{\mu\kappa} G_{\nu\lambda} \int d^{p+1}\sigma \frac{\delta(D) \left( x - X(\sigma) \right)}{\sqrt{-G}} \cdot \varepsilon^{a_1...a_p} \partial_{a_1} X^\kappa \partial_{a_p} X^\lambda ,
\]

where the brane stress-energy tensor is straightforwardly derived from the world-volume action \((1)\) (or, equivalently, \((7)\); recall \(\chi \equiv \frac{\Phi}{\sqrt{-g}}\) is the variable brane tension).

Using again the embedding ansatz \((28)\) together with \((30)\) as well as \((22)\)–\((25)\), the Kalb-Ramond equations of motion \((37)\) reduce to:

\[
\partial_\eta F + \beta \delta(\eta - \eta_0) = 0
\]

implying

\[
F = F_{(+)\theta(\eta - \eta_0) + F_{(-)}\theta(\eta_0 - \eta) , \quad F_{(\pm)} = \text{const} , \quad F_{(-)} - F_{(+) = \beta} \]
terms in the respective bulk space-time Einstein-Maxwell Eqs.(35)–(36) (now \( r_0 = 0 \)). The discontinuity problem is resolved following the approach in Ref.[3] (see also the regularization approach in Ref.[33], Appendix A) by taking mean values of the “force” terms across the discontinuity at \( \eta = 0 \). Furthermore, we will require \( \chi = \text{const} \) (independent of the LL-brane proper time \( \tau \)) for consistency with the matching relations (46). Thus, in the case of the LL-brane embedding (28) the \( X^\mu \)-equation (20) for \( \mu = v \) with \( D = 4, \, p = 2, \) no Kalb-Ramond coupling, i.e., \( F = 0 \), and using the gauge-fixing (22), becomes:

\[
\chi \left( \frac{1}{4} \left[ \partial_\eta A \big|_{+0} + \partial_\eta A \big|_{-0} \right] \right) + a_0 \left( \partial_\eta \ln C \big|_{+0} + \partial_\eta \ln C \big|_{-0} \right) - q \sqrt{2} a_0 \left[ F_{vq} \big|_{+0} + F_{vq} \big|_{-0} \right] = 0 \tag{47}
\]

In the present wormhole solution we will take “left” Bertotti-Robinson “universe” with:

\[
A(\eta) = \frac{\eta^2}{r_0^2} \, , \, C(\eta) = r_0^2 \, , \, F_{vq} = \pm \frac{1}{2 \sqrt{\pi} r_0} \tag{48}
\]

for \( \eta < 0 \), and “right” Reissner-Nordström “universe” with:

\[
A(\eta) \equiv A_{RN}(r_0 + \eta) = 1 - \frac{2m}{r_0 + \eta} + \frac{Q^2}{(r_0 + \eta)^2} \, ,
\]

\[
C(\eta) = (r_0 + \eta)^2 \, , \, F_{vq} \equiv F_{vr} \big|_{RN} = \frac{Q}{\sqrt{4\pi} (r_0 + \eta)} \tag{49}
\]

for \( \eta > 0 \), and

\[
A(0) \equiv A_{RN}(r_0) = 0 \, , \, \partial_\eta A \big|_{+0} \equiv \partial_\eta A_{RN} \big|_{r=r_0} > 0 \tag{50}
\]

where \( F_{vq} \)'s are the respective Maxwell field-strengths and where \( Q = r_0 \left[ \sqrt{\frac{2\pi}{a_0}} q_0 \pm 1 \right] \) is determined from the discontinuity of \( F_{vq} \) in Maxwell equations (36) across the charged LL-brane. Here we have used the standard coordinate notations for the Reissner-Nordström metric coefficients and Coulomb field strength:

\[
A_{RN}(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \, , \, F_{vr} \big|_{RN} = \frac{Q}{\sqrt{4\pi} r^2} \, . \tag{51}
\]

Since obviously both Bertotti-Robinson (48) and Reissner-Nordström (49) metrics do satisfy the “vacuum” Einstein-Maxwell equations (Eqs.(35)–(36) without the LL-brane stress-energy tensor) it remains to check the matching of both metrics at the “throat” \( \eta = 0 \) (the location of the LL-brane) according to Eqs.(46)–(47). In this case the latter equations give:

\[
\partial_r A_{RN} \big|_{r=r_0} = -16\pi \chi \, , \, \partial_r \ln r^2 \big|_{r=r_0} = -\frac{4\pi}{a_0} \chi \tag{52}
\]

\[
\chi \left[ \frac{1}{4} \partial_r A_{RN} \big|_{r=r_0} + a_0 \partial_\tau \ln r^2 \big|_{r=r_0} \right]
- 2q^2 \mp \frac{q}{r_0} \sqrt{\frac{2a_0}{\pi}} = 0 \tag{53}
\]

From (52)–(53) we get:

\[
r_0 = \frac{a_0}{2\pi |\chi|} \, , \, m = \frac{a_0}{2\pi |\chi|} (1 - 4a_0) \, , \tag{54}
\]

implying that the dynamical LL-brane tension \( \chi \) must be negative, thus identifying the LL-brane as “exotic matter” [19, 21]. Further we obtain a quadratic equation for \( |\chi| \):

\[
\chi^2 + \frac{q^2}{4\pi} \pm \frac{q}{2\sqrt{2\pi} a_0} |\chi| = 0 \, , \tag{55}
\]

which dictates that we have to choose the sign of \( q \) to be opposite to the sign in the expression for the Bertotti-Robinson constant electric field (last Eq.(48)). There are two positive solutions for \( |\chi| \):

\[
|\chi| = \frac{|q|}{4\sqrt{2\pi} a_0} (1 \pm \sqrt{1 - 8a_0}) \quad \text{for} \quad a_0 < 1/8 \, . \tag{56}
\]

Using (54) and (56) the expression for \( Q^2 \) reads:

\[
Q^2 = \frac{a_0^3}{4\pi^2 \chi^2} (1 - 8a_0) = \frac{8a_0^3}{\pi q^2} \left( \frac{1}{1 + \sqrt{1 - 8a_0}} \right)^2 \tag{57}
\]

Thus, we have constructed a solution to Einstein-Maxwell equations (35)–(36) in \( D = 4 \) describing a wormhole space-time manifold consisting of a “left” Bertotti-Robinson universe with two compactified space dimensions and a “right” Reissner-Nordström universe connected by a “throat” materialized by a LL-brane. The “throat” is a common horizon for both universes where for the “right” universe it is the external Reissner-Nordström horizon. All wormhole parameters, including the dynamical LL-brane tension, are determined in terms of the surface charge density \( q \) of the LL-brane (cf. Eq.(18)) and the integration constant \( a_0 \) (14) characterizing LL-brane dynamics in a bulk gravitational field.

5 Conclusions. Travel to Compactland Through a Wormhole

In this work we have explored the use of (codimension-one) LL-branes for construction of new
asymmetric wormhole solutions of Einstein-Maxwell equations. We have put strong emphasize on the crucial properties of the dynamics of LL-branes interacting with gravity and bulk space-time gauge fields:

(i) “Horizon straddling” – automatic position of the LL-brane on (one of) the horizon(s) of the bulk space-time geometry;

(ii) Intrinsic nature of the LL-brane tension as an additional dynamical degree of freedom unlike the case of standard Nambu-Goto p-branes;

(iii) The LL-brane stress-energy tensor is systematically derived from the underlying LL-brane Lagrangian action and provides the appropriate source term on the r.h.s. of Einstein equations to enable the existence of consistent non-trivial wormhole solutions;

(iv) Electrically charged LL-branes naturally produce asymmetric wormholes with the LL-brane itself materializing the wormhole “throat” and uniquely determining the pertinent wormhole parameters.

Finally, let us point out that the above asymmetric wormhole connecting Reissner-Nordström universe with a Bertotti-Robinson universe through a lightlike hypersurface occupied by a LL-brane is traversable w.r.t. the proper time of a traveling observer. The latter property is similar to the proper time traversability of other symmetric and asymmetric wormholes with LL-brane sitting on the “throat” [12, 13, 16, 17]. Indeed, let us consider test particle (“traveling observer”) dynamics in the asymmetric wormhole background given by (48)–(49), which is described by the action:

\[ S_{\text{particle}} = \frac{1}{2} \int d\lambda \left[ \frac{1}{e} \mathcal{E} \right] \mathcal{G}_{\mu \nu} - e m_0^2 \]  

Using energy \( \mathcal{E} \) and orbital momentum \( J \) conservation and introducing the proper world-line time \( s \) \( (\frac{ds}{d\lambda} = em_0) \), the “mass-shell” equation (the equation w.r.t. the “einbein” \( e \) produced by the action (58)) yields:

\[ \left( \frac{d\eta}{ds} \right)^2 + V_{\text{eff}}(\eta) = \frac{\mathcal{E}^2}{m_0^2}, \quad V_{\text{eff}}(\eta) \equiv A(\eta) \left( 1 + \frac{J^2}{m_0^2 C(\eta)} \right) \]  

with \( A(\eta), C(\eta) \) – the same metric coefficients as in (48)–(50).

For generic values of the parameters the effective potential in the Bertotti-Robinson universe (48) (i.e., for \( \eta < 0 \)) has harmonic-oscillator-type form. Therefore, a traveling observer starting in the Reissner-Nordström universe (49) (i.e., at some \( \eta > 0 \)) and moving “radially” along the \( \eta \)-direction towards the horizon, will cross the wormhole “throat” \( (\eta = 0) \) within finite interval of his/her proper time, then will continue into the Bertotti-Robinson universe subject to harmonic-oscillator deceleration force, will reverse back at the turning point and finally will cross the “throat” back into the Reissner-Nordström universe.

Let us stress that, as in the case of the previously constructed symmetric and asymmetric wormholes via LL-branes sitting on their “throats” [12, 13, 16, 17], the present Reissner-Nordström-to-Bertotti-Robinson wormhole is not traversable w.r.t. the “laboratory” time of a static observer in either universe.

**Acknowledgments**

E.N. and S.P. are supported by Bulgarian NSF grant DO 02-257. E.G. thanks the astrophysics and cosmology group at PUCV, Chile, for hospitality. Also, all of us acknowledge support of our collaboration through the exchange agreement between the Ben-Gurion University of the Negev and the Bulgarian Academy of Sciences.

**References**

[1] C. Barrabés and P. Hogan, “Singular Null-Hypersurfaces in General Relativity” (World Scientific, Singapore, 2004).

[2] K. Thorne, R. Price and D. Macdonald (Eds.), “Black Holes: The Membrane Paradigm” (Yale Univ. Press, New Haven, CT, 1986).

[3] W. Israel, Nuovo Cim. B44, 1 (1966); erratum, Nuovo Cim. B48, 463 (1967).

[4] C. Barrabés and W. Israel, Phys. Rev. D43, 1129 (1991).

[5] T. Dray and G. ’t Hooft, Class. Quantum Grav. 3, 825 (1986).

[6] I. Kogan and N. Reis, Int. J. Mod. Phys. A16, 4567 (2001) [hep-th/0107163].

[7] D. Mateos and S. Ng, JHEP 0208, 005 (2002) [hep-th/0205291].

[8] J. Harvey, P. Kraus and F. Larsen, Phys. Rev. D63, 026002 (2001) [hep-th/0008064]; D. Mateos, T. Mateos and P.K. Townsend, JHEP 0312, 017 [hep-th/0309114]; A. Bredthauer, U. Lindström, J. Persson and L. Wulff, JHEP 0402, 051 [hep-th/0401159].

[9] C. Barrabés and W. Israel, Phys. Rev. D71, 064008 (2005) [gr-qc/0502108].
[10] E. Guendelman, A. Kaganovich, E. Nissimov and S. Pacheva, Phys. Rev. D72, 0806011 (2005) [hep-th/0507193]; Fortschr. der Physik 55, 579 (2007) [hep-th/0612091]; in “Fourth Internat. School on Modern Math. Physics”, eds. B. Dragovich and B. Sazdovich (Belgrade Inst. Phys. Press, Belgrade, 2007), p. 215 [hep-th/0703114]; in “Lie Theory and Its Applications in Physics”, eds. V. Dobrev and H. Doebner (Heron Press, Sofia, 2008), p. 79 [arxiv:0711.1841[hep-th]].

[11] E. Guendelman, A. Kaganovich, E. Nissimov and S. Pacheva, Centr. Europ. Journ. Phys. 7, 668 (2009) [arxiv:0711.2877[hep-th]]; in “Fifth Summer School in Modern Mathematical Physics”, eds. B. Dragovich and Z. Rakic (Belgrade Inst. Phys. Press, Belgrade, 2009) [arxiv:0810.5008[hep-th]].

[12] E. Guendelman, A. Kaganovich, E. Nissimov and S. Pacheva, Phys. Lett. 673B, 457 (2009) [arxiv:0904.3198[hep-th]].

[13] E. Guendelman, A. Kaganovich, E. Nissimov and S. Pacheva, Asymmetric Wormholes via Electrically Charged Lightlike Branes, arxiv:0911.0940[hep-th], to appear in “Lie Theory and Its Applications in Physics VIII”, eds. H.-D. Doebner and V. Dobrev, Heron Press, Sofia (2010).

[14] A. Einstein and N. Rosen, Phys. Rev. 43, 73 (1935).

[15] M. Morris and K. Thorne, Am. J. Phys. 56 (1988) 395; M. Morris, K. Thorne and U. Yurtsever, Phys. Rev. Lett. 61, 1446 (1988).

[16] E. Guendelman, A. Kaganovich, E. Nissimov and S. Pacheva, Phys. Lett. 681B, 457 (2009) [arxiv:0904.3198[hep-th]].

[17] E. Guendelman, A. Kaganovich, E. Nissimov and S. Pacheva, Asymmetric Wormholes via Electrically Charged Lightlike Branes, arxiv:0911.0940[hep-th], to appear in “Lie Theory and Its Applications in Physics VIII”, eds. H.-D. Doebner and V. Dobrev, Heron Press, Sofia (2010).

[18] A. Einstein and N. Rosen, Phys. Rev. 43, 73 (1935).

[19] M. Morris and K. Thorne, Am. J. Phys. 56 (1988) 395; M. Morris, K. Thorne and U. Yurtsever, Phys. Rev. Lett. 61, 1446 (1988).

[20] M. Visser, Phys. Rev. D39, 3182 (1989); Nucl. Phys. B328, 203 (1989).

[21] M. Visser, “Lorentzian Wormholes. From Einstein to Hawking” (Springer, Berlin, 1996).

[22] J. Lemos, F. Lobo and S. de Oliveira, Phys. Rev. D68, 064004 (2003) [gr-qc/0302049]; S. Sushkov, Phys. Rev. D71, 043520 (2005) [gr-qc/0502084]; F. Lobo, Exotic Solutions in General Relativity: Traversable Wormholes and “Warp Drive” Spacetimes , arxiv:0710.4474[gr-qc].

[23] C. Misner and J. Wheeler, Ann. of Phys. 2, 525-603 (1957).

[24] B. Bertotti, Phys. Rev. D116, 1331 (1959); I. Robinson, Bull. Acad. Pol., 7 (1959) 351.

[25] A.S. Lapedes, Phys. Rev. D17, 2556 (1978).

[26] E. Guendelman, Gen. Rel. Grav. 23 (1991) 1415.

[27] V. Dzhunushaliev, Gen. Rel. Grav. bf 35 (2003) 1481 [gr-qc/0301046].

[28] E. Guendelman, A. Kaganovich, E. Nissimov and S. Pacheva, Int. J. Mod. Phys. A25, 1571 (2010) [arxiv:0908.4195[hep-th]].

[29] E. Guendelman, Class. Quantum Grav. 17, 3673 (2000); Phys. Rev. D63, 046006 (2001).

[30] P. Townsend, Phys. Lett. 277B, 285 (1992); E. Bergshoeff, L. London and P. Townsend, Class. Quantum Grav. 9, 2545 (1992) [hep-th/9206026]; J. de Azcarraga, J. Izquierdo and P. Townsend, Phys. Rev. D45, R3321 (1992).

[31] A. Aurilia, H. Nicolai and P.K. Townsend, Nucl. Phys. B176, 509 (1980); A. Aurilia, Y. Takahashi and P.K. Townsend, Phys. Lett. 95B, 265 (1980).

[32] A. Eddington, Nature, 113, 192 (1924); D. Finkelstein, Phys. Rev. D110, 965 (1958).

[33] S. Blau, E. Guendelman and A. Guth, Phys. Rev. D35, 1747 (1987).