Isomonodromic problem for $K_{2}^{2}$ analogue of the Painlevé equations

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Abstract. A fourth–order analogue for the Painlevé equations that is self–similar reduction of the modified Sawada–Kotera and Kaup–Kupershmidt equation is considered. Zero curvature representation of Lax pair for the modified Sawada–Kotera equation is presented. Linear problem for a fourth–order analogue for the Painlevé equations is obtained from Lax pair for the modified Sawada–Kotera equation using self-similar variables.

1. Introduction
Nonlinear ordinary differential equations play significant role in the description of different physical phenomena as exact mathematical models or arise as symmetry reductions of nonlinear partial differential equations [1, 2, 3]. Properties of the second–order differential equations of the form
\[ w_{zz} = R(z, w, w_{z}) \]  \hspace{1cm} (1)
have been studied by Painlevé [4] and Gambier [5]. They proved that there are fifty canonical equations of the form (1) with general solutions without critical movable points. Since that ordinary differential equations having solutions without critical movable points are called equations of the $P$–type. Thus all second-order differential equations of the $P$–type can be transformed to the integrating equations or to the one of the six non–autonomous equations which also called the Painlevé equations. It was proved that these equations have no movable critical points and determines new special functions which called Painlevé transcendents.

It is well known that equations of the $P$-type are closely connected with solvable by inverse scattering transform nonlinear evolution equations. According to Painlevé conjecture all exact reductions of the nonlinear partial differential equations solvable by inverse scattering transform are equations of the $P$-type. So investigation of nonlinear ordinary differential equations of $P$-type which can determine new special functions is important issue in modern mathematical physics.

Generalizations of the Painlevé equations were obtained by Kudryashov ($K$ and $P$ hierarchies) [6], Flashka and Newell ($FN$–hierarchy) [7], Jimbo and Miwa ($JM$–hierarchy) [8]. The following generalizations of the Painlevé give us higher order equations of $P$-type which can determine new higher transcendents. $FN$ and $JM$ hierarchies of the Painlevé equations were founded as the compatibility of the linear problem given by $2 \times 2$ matrix systems of the form
\[ \frac{\partial_{\mu}}{\partial \mu} \Psi = F \Psi \]
\[ \frac{\partial_{\tau}}{\partial \tau} \Psi = G \Psi \]  \hspace{1cm} (2)
So the existence of linear isomonodromic problem for $FN$ and $JM$ hierarchies is completely closed problem. A different situation arises with the $K$–hierarchies which were obtained as generalization of fourth–order equations which pass Painlevé test and define new special functions [6]. Lately in works [9] the linear problem representation was obtained for $P$ and $K_1$ hierarchies as the $2 \times 2$ matrix systems. Exact form of the linear problem representation which also called isomonodromic problem for $K_2$ hierarchy is still open problem. To get more closer to solution of this problem we are going to find the linear problem representation for the fourth—order member of the $K_2$–hierarchy which can be written in the following form

$$w_{zzzz} + 5w_z w_{zz} - 5w^2 w_{zz} - 5ww_z^2 + w^5 - zw - \beta = 0. \quad (3)$$

It is well known that equation (3) can be obtained taking into account self–similar variables

$$v = \frac{1}{(5t)^{\frac{1}{5}}} w(z), \quad z = \frac{x}{(5t)^{\frac{1}{5}}}, \quad (4)$$

from the following nonlinear partial differential equation

$$v_t - v_{xxxxx} + 5v_x^2 + 5v_x v_{xxx} + 5v_x^3 + 20vv_x v_{xx} + 5v^2 v_{xxx} - 5v^4 v_x = 0 \quad (5)$$

Equation (5) is the modified Kaup–Kupershmidt and Sawada–Kotera equations. It is known that solutions of the Kaup–Kupershmidt and Sawada–Kotera equations can be found via the solutions of equation (5) by means of the Miura transformations. These equations play significant role in the description of non–linear waves in a liquid with gas bubbles and in other application [2, 3].

The Lax pair representation for equation (5) was obtained in work [10]. The authors of this work show that equation (5) can be obtained as the compatibility of the following linear system

$$L \psi = \lambda \psi, \quad \psi_t = B \psi \quad (6)$$

where $L$ and $B$ are differential operators given by

$$L = \partial_x^2 - 3v \partial_x^2 + 2(v^2 - v_x) \partial_x$$
$$B = -\partial_x + 45v \partial_x^2 + 75(v^2 - v_x) \partial_x^2 + (60v_{xxx} - 165vv_x + 45v^3) \partial_x^2$$
$$+ 20(v_{xxxx} - 45v_x^2 - 70vv_x v_{xx} + 70v^2 v_x - 5v^4) \partial_x$$

The connection between equation (3) and equation (5) gives us opportunity to find isomonodromic problem for equation (3) using self–similar variables (4) for the Lax pair (6)–(7).

2. Main results
Let us rewrite the Lax pair (6)–(7) in zero curvature representation form

$$L_1 \Psi = 0, \quad L_2 \Psi = 0$$
$$L_1 = \partial_x - A, \quad L_2 = \partial_t - M$$
$$[L_1, L_2] = [\partial_x - A, \partial_t - M] = A_t - M_x + [A, M] = 0 \quad (8)$$

where $\Psi$ is $N$-vector, $A$ and $M$ are $N \times N$ matrices, $N$ is order of the Lax pair. The order of the Lax pair (6)–(7) is equal 3. Let us choose the following 3-vector

$$\Psi = \begin{pmatrix} \psi_{xx} \\ \psi_x \\ \psi \end{pmatrix} \quad (9)$$
Obtaining all derivatives from the second order by \( x \) and all mixed derivatives of \( \psi \) from equations (6)-(7) we find the form of matrices \( A \) and \( B \) in the following form

\[
A = \begin{pmatrix}
3v & -2(v^2 - v_x) & \lambda \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad M = \begin{pmatrix}
m_{11} & m_{12} & m_{13} \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{pmatrix}
\]  

(10)

where

\[
m_{11} = 3v^5 + 8v^3v_x - 7v^2v_{xx} - 12v^2\lambda + vv_x^2 - 2vv_{xxx} - 5v_xv_{xx} - 6v_x\lambda + v_{xxxx}
\]

\[
m_{12} = -2v^6 - 6v^4v_x - 6v^3v_{xx} + 18v^3\lambda - 22v^2v_x^2 - 6v^2v_{xxx} - 50vv_xv_{xx} + 12v\lambda v_x
\]

\[- 10v_x^3 + 2vv_{4x} - 10v_xv_{xxx} - 10v_{xx}^2 + 3v_{xxxx}\lambda - 9\lambda^2 + 2v_{5x}\]

\[
m_{13} = \lambda m_{21}, \quad m_{21} = v^4 + 4v^2v_x - vv_{xx} - 9v\lambda + 3v_x^2 - v_{xxx}
\]

\[
m_{22} = -8v^3v_x - 8v^2v_{xx} + 15v^2\lambda - 16vv_x^2 + 2vv_{xxx} - 10v_xv_{xx} + 3v_x\lambda + 2v_{xxxx}
\]

\[
m_{23} = -9\lambda^2, \quad m_{31} = 6w_v - 3v_{xx} - 9\lambda, \quad m_{32} = v^4 - 14v^2v_x + 2vv_{xx} + 18v\lambda - 3v_x^2 + 2v_{xxx}
\]

\[
m_{33} = -3\lambda(v^2 - v_x).
\]

Obtained zero curvature representation can be rewritten as the following one–form

\[
d\Psi = A(v, x, t)\Psi dx + M(v, x, t)\Psi dt.
\]  

(11)

The goal is to find one-form representation corresponding to (3) using transformation (4)

\[
v = \frac{1}{(5t)^\frac{4}{3}}g(z), \quad z = \frac{x}{(5t)^\frac{2}{3}}.
\]  

(12)

Using transformation (4) we obtain that

\[
dx = (5t)^\frac{1}{3}dz + \frac{z}{(5t)^\frac{2}{3}}dt.
\]  

(13)

So equation (11) can be rewritten in the following form

\[
d\Psi = \dot{A}(g, z, t)dz + \dot{B}(g, z, t)dt
\]

\[
\dot{A}(g, z, t) = (5t)^\frac{1}{3} A \left( \frac{g(z)}{(5t)^\frac{2}{3}}, z(5t)^\frac{1}{3}, t \right)
\]

\[
\dot{B}(g, z, t) = \frac{z}{(5t)^\frac{2}{3}} A \left( \frac{g(z)}{(5t)^\frac{2}{3}}, z(5t)^\frac{1}{3}, t \right) + B \left( \frac{g(z)}{(5t)^\frac{2}{3}}, z(5t)^\frac{1}{3}, t \right)
\]  

(14)

Choosing new variable \( \mu \) in the following way

\[
\mu = \left( \frac{4}{5} \right)^\frac{1}{3}, \quad \frac{d\mu}{dt} = \frac{dt}{t}
\]  

(15)

we get rid of variable \( t \) and obtain the following form of matrices

\[
d\Psi = N(g, z, \mu)dz + F(g, z, \mu)d\mu
\]  

(16)
where

\[ N = \begin{pmatrix} 3g & \frac{2g^2 - g_z}{\mu} & \mu \lambda \\ \mu & 0 & 0 \\ 0 & \mu & 0 \end{pmatrix}, \] (17)

and

\[
5^{-\frac{1}{2}} F = 45\lambda \mu^4 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ 1 & 0 & 0 \end{pmatrix} - 45\lambda g \mu^3 \begin{pmatrix} 0 & 0 & \lambda \end{pmatrix} + \\
+ 15\lambda \mu^2 \begin{pmatrix} 2(2g^2 - g_z) & 0 & 0 \\ 0 & -5g^2 + g_z & 0 \\ 0 & 0 & g^2 + g_z \end{pmatrix} + 15\mu \begin{pmatrix} 0 & \lambda m_1(g) & 0 \\ 0 & 0 & 0 \\ -m_1(g) + 6(gg_z - g^2) & 0 & 0 \end{pmatrix} - \\
- \begin{pmatrix} m_{0,0}(g) & 0 & \lambda m_{0,0}(g) \\ 0 & m_{0,0}(g) + m_{0,1}(g) & 0 \end{pmatrix} + \frac{1}{\mu} \begin{pmatrix} m_{-1,0}(g) & 0 & 0 \\ 0 & 0 & 3m_{-1,1}(g) \end{pmatrix} + \\
+ \frac{1}{\mu^2} \begin{pmatrix} 0 \ m_{-2}(g) \ 0 \\ 0 \ 0 \ 0 \\ 0 \ 0 \ 0 \end{pmatrix}
\]

\[ m_1(g) = g_{zz} - 4g_{g_z} + 6g, \quad m_{0,0}(g) = 5g_{zzz} + 5g^4 - 20g^2 g_z - 5gg_{zz} + 15g^2 - z \]
\[ m_{0,1}(g) = -15g_{zzzz} + 90g^2 g_z + 15gg_{zz} - 30g^2 \]
\[ m_{-1,0}(g) = 5g_{zzzz} + 15g^5 - 40g^3 g_z - 35g_{g_z} g^2 + 5gg_{zz} + 25g_z z g_z + 10g_{zzzz} g - 32g \]
\[ m_{-1,1}(g) = 5g_{zzzz} + g^5 - 20g^2 g_z + 15g^2 - 5gg_z - z \]
\[ m_{-2}(g) = 5g_{zzzz} + 5g^6 - 15g^4 g_z + 55g^2 g^2_z + 15g^3 g_{zz} - 25g^3 - \\
- 125gg_{zzz} - 15g^2 g_{zzz} - g^2 z + 25g_z z g_z + 25g_{zz} g_z - 5gg_{zzzzz}. \] (18)

The isomonodromic deformations equations of system (16)-(18) reduce to the following equation

\[ g_{zzzz} - 5g_z g_z - 5gg_z^2 - 5g^2 g_z z + g^3 - \frac{1}{5}z g + \alpha = 0, \] (19)

which can be transformed to (3) by the following transformation

\[ w(z') = -5^{\frac{1}{5}} g(z), \quad z' = 5^{-\frac{1}{4}} z, \quad \beta = 5\alpha. \] (20)

Using scaling deformations (20) for the following linear system

\[
\begin{align*}
\Psi_z &= N \Psi \\
\Psi_\mu &= F \Psi
\end{align*}
\]

we obtain isomonodromic problem representation for $K_2^2$ equation.
3. Conclusion
In this work we have considered a fourth order analogue of the Painlevé equations (3). We have shown that equation (3) can be obtained from (5) using self–similar variables. Using Lax pair representation (6)-(7) for equation (5) we have constructed linear system which isomonodromic deformations equation gives equation (19) which can be transformed to equation (3). Obtained linear representation gives us chance to look for asymptotic solutions of (3) using isomonodromic deformation technic and find corresponding Riemann–Hilbert problem.

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