SEMI-CLASSICAL STATES
FOR NON-SELF-ADJOINT
SCHRÖDINGER OPERATORS

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Abstract

We prove that the spectrum of certain non-self-adjoint Schrödinger operators is unstable in the semi-classical limit $h \to 0$. Similar results hold for a fixed operator in the high energy limit. The method involves the construction of approximate semi-classical modes of the operator by the JWKB method for energies far from the spectrum.

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1 Introduction

It is well known that the complex spectrum of many non-self-adjoint differential operators is highly unstable under small perturbations [2, 3, 8, 9]; this has been investigated in detail for the Rayleigh equation in hydrodynamics in [3, Ch 4]. One way of exploring this fact is by defining the pseudospectrum of such an operator $H$ by

$$\text{Spec}_\varepsilon(H) := \text{Spec}(H) \cup \{z : \|R(z)\| > \varepsilon^{-1}\}$$

where $\varepsilon > 0$ and $R(z)$ is the resolvent of $H$. It is known that $\text{Spec}_\varepsilon(H)$ contains the $\varepsilon$-neighbourhood of the spectrum, and that it is contained in the $\varepsilon$-neighbourhood of the numerical range of $H$. Theorem 1 states that for Schrödinger operators with complex potentials $\text{Spec}_\varepsilon(H)$ expands to fill a region $U$ of the complex plane.
much larger than the spectrum in the semi-classical limit \( h \to 0 \). More precisely we obtain an explicit lower bound on \( \| R(z) \| \) which increases rapidly as \( h \to 0 \) for all \( z \) in the region \( U \) defined by

\[
U := \{ z = \eta^2 + V(a) : \eta \in \mathbb{R} \setminus \{0\} \text{ and } \text{Im}(V'(a)) \neq 0 \}.
\]

For \( h = 1 \) we deduce that for large enough \( z \) within a suitable region the resolvent norms of Schrödinger operators with complex potentials become very large even though \( z \) may be far from the spectrum. This had apparently not been noticed in other spectral investigations of these operators [1, 4, 5, 7], apart from [3], whose results are greatly extended and improved in Theorem 2 below.

Our results have a positive aspect. Our proofs use a JWKB analysis to construct a continuous family of approximate eigenstates, which we call semi-classical modes, for the operators in question. These modes have complex energies far from the spectrum, but could be used to investigate the time evolution of fairly general initial states by expanding these in terms of the modes.

## 2 The estimates

For reasons which will become clear in the next section, we consider operators somewhat more general than those described in the last section. We assume that \( H \) acts in \( L^2(\mathbb{R}) \) according to the formula

\[
Hf(x) := -h^2 \frac{d^2 f}{dx^2} + V_h(x)f(x)
\]

Where \( V_h \) are smooth potentials for all small enough \( h > 0 \) which depend continuously on \( h \), together with their derivatives of all orders. In the applications in the next section \( V_h \) has an expansion involving fractional powers of \( h \), but this is invisible since we treat \( V_h \) as it stands, and only need an asymptotic expansion involving integer powers of \( h \); this simplification is essential to our solution of the problem. Technically we assume that \( H \) is some closed extension of the operator initially defined on \( C_\infty^\infty(\mathbb{R}) \). We assume throughout the section that

\[
z := \eta^2 + V_h(a)
\]

where \( \eta \in \mathbb{R} \setminus \{0\} \) and \( \text{Im}(V'_h(a)) \neq 0 \). The \( h \)-dependence of \( z \) as defined above may be eliminated by a uniformity argument spelled out in the next section, but we do not focus on this issue here. Our goal is to prove the upper bound

\[
\| H\tilde{f} - z\tilde{f} \| / \| \tilde{f} \| = O(h^n)
\]
as \( h \to 0 \), for all \( n > 0 \), where \( \tilde{f} \in C^\infty_c(\mathbb{R}) \) depends upon \( h \) and \( n \). This immediately implies that \( \| (H - zI)^{-1} \| \) diverges as \( h \to 0 \) faster than any negative power of \( h \). Although the above equation may be interpreted as stating that such \( z \) are approximate eigenvalues for small enough \( h > 0 \), it does not follow that they are close to true eigenvalues, and indeed the examples studied in \([2, 3, 8, 9]\) show that there is a strong distinction between the spectrum and pseudospectrum.

**Theorem 1** There exists \( \delta > 0 \) and for each \( n > 0 \) a positive constant \( c_n \) and an \( h \)-dependent function \( \tilde{f} \in C^\infty_c(\mathbb{R}) \) such that if \( 0 < h < \delta \) then

\[
\| H \tilde{f} - z\tilde{f} \| / \| \tilde{f} \| \leq c_n h^n.
\]

**Proof** The proof is a direct construction. We put \( \tilde{f}(a + s) := \xi(s)f(s) \) for all \( s \in \mathbb{R} \) where \( \xi \in C^\infty_c(\mathbb{R}) \) satisfies \( \xi(s) = 1 \) if \( |s| < \delta/2 \) and \( \xi(s) = 0 \) if \( |s| > \delta \), and \( \delta > 0 \) is determined below. We take \( f \) to be the smooth but not square integrable JWKB function \( f := \exp(-\psi) \) where

\[
\psi(s) := \sum_{m=-1}^{n} h^m \psi_m(s)
\]

and \( \psi_m \) are defined below. A direct computation shows that

\[
H f - z f = \left( \sum_{m=0}^{2n+2} h^m \phi_m \right) f
\]

where \( \phi_m \) are given by the formulae

\[
\begin{align*}
\phi_0 & := - (\psi'_{-1})^2 + V_h - z \\
\phi_1 & := \psi''_{-1} - 2\psi'_{-1}\psi'_0 \\
\phi_2 & := \psi''_0 - 2\psi'_{-1}\psi'_1 - (\psi'_0)^2 \\
\phi_3 & := \psi''_1 - 2\psi'_{-1}\psi'_2 - 2\psi'_0\psi'_1 \\
& \quad \vdots \\
\phi_{2n+2} & := -(\psi'_n)^2.
\end{align*}
\]

By setting \( \phi_m = 0 \) for \( 0 \leq m \leq n + 1 \), we obtain a series of equations which enable us to determine all \( \psi_m \), provided \( \delta > 0 \) is small enough.

The key equation is the complex eikonal equation

\[
\psi'_{-1}(s)^2 = V_h(a + s) - V_h(a) - \eta^2
\]
whose solution is

\[
\psi_{-1}(s) := \int_0^s \left( V_h(a + t) - V_h(a) - \eta^2 \right)^{1/2} dt
= \int_0^s i\eta \left( 1 - \frac{V_h(a + t) - V_h(a)}{\eta^2} \right)^{1/2} dt
= i\eta s - \frac{iV'_h(a)s^2}{4\eta} + O(s^3).
\]

We have assumed that \(\text{Im}(V'_0(a)) \neq 0\) and this implies that \(\text{Im}(V'_h(a)) \neq 0\) for all small enough \(h > 0\). We choose \(\eta\) to be of the same sign as \(\text{Im}(V'_h(a))\) so that for a suitable constant \(\gamma > 0\) we have

\[
\gamma s^2 \leq \text{Re}(\psi_{-1}(s)) \leq 3\gamma s^2
\]

for all small enough \(s\) and \(h\). We also assume that \(s\) and \(h\) are small enough for \(\rho := (2\psi'_{-1})^{-1}\) to satisfy a bound of the form \(|\rho(s)| \leq \beta\).

We now force \(\phi_m = 0\) for \(0 \leq m \leq n + 1\) by putting

\[
\begin{align*}
\psi'_0 &= \rho\psi'_{-1} \\
\psi'_1 &= \rho(\psi''_0 - (\psi'_0)^2) \\
\psi'_2 &= \rho(\psi''_1 - 2\psi'_0\psi'_1) \\
\text{etc.}
\end{align*}
\]

We determine the functions uniquely by also imposing \(\psi_m(0) = 0\) for all \(0 \leq m \leq n\). Each of the functions is bounded provided \(s\) and \(h\) are small enough, and the same is true of the remaining functions \(\phi_m\). Specifically we assume that for some \(\delta > 0\) and constants \(c_m, c'_m\) we have

\[
|\psi_m(s)| \leq c_m
\]

for \(0 \leq m \leq n\), and

\[
|\phi_m(s)| \leq c'_m
\]

for \(n + 2 \leq m \leq 2n + 2\), provided \(|s| \leq \delta\) and \(0 < h < \delta^2\).

In the following calculations \(a_i\) denote various positive constants, independent of \(h\) and \(s\). We have

\[
\|f\|_2^2 \geq \int_{-\delta/2}^{\delta/2} |f(s)|^2 \, ds \\
\geq \int_{-\delta/2}^{\delta/2} e^{-3\gamma s^2 h^{-1} - a_1} \, ds
\]
\[
= \int^{\delta h^{-1/2}/2}_{-\delta h^{-1/2}/2} e^{-3\gamma t^2-a_1 h^{1/2}} \, dt
\geq \int^{1/2}_{-1/2} e^{-3\gamma t^2-a_1 h^{1/2}} \, dt
= a_2 h^{1/2}.
\]

We also have

\[
\|H \tilde{f} - z \tilde{f}\|_2 = \| - h^2 f \xi'' - 2 h^2 f' \xi' + \xi (H f - z f)\|_2 \\
\leq h^2 \|f \xi''\|_2 + 2 h^2 \|f' \xi'\|_2 + \sum_{m=n+2}^{2n+2} h^m \|\xi \phi_m f\|_2
\]

and need to estimate each of the norms. Since \( \xi' \) has support in \( \{s : \delta/2 \leq |s| \leq \delta\} \), we have

\[
\|\xi'' f\|_2^2 \leq a_3 \int_{\delta/2 \leq |s| \leq \delta} e^{-\gamma s^2 h^{-1} + a_4} \, ds \\
\leq a_5 e^{-\gamma \delta^2/4h}.
\]

In other words \( \|\xi'' f\|_2 \) decreases at an exponential rate as \( h \to 0 \). A similar argument applies to \( \|\xi' f'\|_2 \). Since \( \phi_m \) is bounded on \( \{s : |s| \leq \delta\} \), uniformly for \( |h| \leq \delta^2 \), we see that

\[
\|\xi \phi_m f\|_2^2 \leq a_6 \int_{-\delta}^{\delta} |f(s)|^2 \, ds \\
\leq \int_{-\delta}^{\delta} e^{-\gamma s^2 h^{-1} + a_7} \, ds \\
\leq a_8 h^{1/2}
\]

by an argument similar to that used for \( \tilde{f} \) above. Putting the various inequalities together we obtain the statement of the theorem.

### 3 High Energy Spectrum

By a change of scale our theorem can be applied to prove the instability of the high energy spectrum of a non-self-adjoint Schrödinger operator with complex potential. The results in this section extend those of [3] both by providing greater insight into the mechanism involved and by obtaining much stronger estimates. Adopting quantum mechanical notation we assume that \( h = 1 \) and that the operator \( H \) acting in \( L^2(\mathbb{R}) \) is given by

\[
H := P^2 + \sum_{m=1}^{n} c_m Q^m
\]
where \( n \) is even and the constant \( c_n \) has positive real and imaginary parts.

**Theorem 2** If \( z \in \mathbb{C} \) satisfies \( 0 < \arg(z) < \arg(c_n) \) and \( \sigma > 0 \) then

\[
\| (H - \sigma z I)^{-1} \|
\]
diverges to infinity faster than any power of \( \sigma \) as \( \sigma \to +\infty \).

**Proof** If \( u > 0 \) then the operator \( H \) is unitarily equivalent to the operator

\[
H_1 := u^{-2} P^2 + \sum_{m=1}^{n} c_m u^m Q^m
\]

Putting \( u := \sigma^{1/n} \) we obtain

\[
\| (H - \sigma z I)^{-1} \| = \sigma^{-1} \| (H_2 - z I)^{-1} \|
\]

where

\[
H_2 := \sigma^{-1} H_1 = u^{-2-n} P^2 + \sum_{m=1}^{n} c_m u^{m-n} Q^m
\]

Putting \( h := u^{-(n+2)/2} \) we have

\[
H_2 := h^2 P^2 + \sum_{m=1}^{n} c_m h^{2(n-m)/(n+2)} Q^m = h^2 P^2 + V_h(Q).
\]

This is precisely the form of operator to which Theorem 1 applies. We have \( V'_0(a) = c_n a^n \) so \( \Im(c_n) > 0 \) implies that the conditions of Theorem 1 are satisfied for any \( z \) in the sector

\[
U := \{ z : 0 < \arg(z) < \arg(c_n) \}.
\]

This completes the proof, except for a technical point which we now address. In Theorem 1 we assumed that \( z = \eta^2 + V_h(a) \) where \( \Im(V'_0(a)) \neq 0 \), so \( z \) is apparently dependent on \( h \), with the limit \( z_0 := \eta^2 + V_0(a) \) as \( h \to 0 \). We rectify this problem by fixing \( z \in U \) and making \( a \) and \( \eta \) depend upon \( h \) in such a way that

\[
z = \eta_h^2 + V_h(a_h)
\]

where \( \eta_h \to \eta \) and \( a_h \to a \) as \( h \to 0 \). We now have to check that all the estimates of Section 2 are locally uniform with respect to \( \eta \) and \( a \), so that the result we claim does indeed follow.

The method of this paper can be extended to treat certain rotationally invariant problems in higher space dimensions. The condition \(-2 \leq p(1) \) in the next theorem is included because it is relevant to the existence of a closed extension of the operator, by virtue of an application of the theory of sectorial forms.
Theorem 3 Let the operator $H$ acting in $L^2(\mathbb{R}^N)$ be some closed extension of the operator given by

$$Hf(x) := -\Delta f(x) + \sum_{m=1}^{n} c_m |x|^{p(m)} f(x)$$

for all $f$ in the initial domain $C_\infty^\infty(\mathbb{R}^N\setminus\{0\})$, where $c_n$ has positive real and imaginary parts, $p(n) > 0$ and

$$-2 \leq p(1) < p(2) < \ldots < p(n).$$

If $z \in \mathbb{C}$ satisfies $0 < \arg(z) < \arg(c_n)$ and $\sigma > 0$ then

$$\|(H - \sigma z I)^{-1}\|$$

diverges to infinity faster than any power of $\sigma$ as $\sigma \to +\infty$.

Proof The difference from Theorem 2 is that after restricting to the usual angular momentum sectors the operators act in $L^2(0, \infty)$ and include angular momentum terms in the potential. However, it may be seen that the analysis of Theorem 2 can be extended to operators of the form

$$H := P^2 + \sum_{m=1}^{n} c_m Q^{p(m)}$$

so the incorporation of the angular momentum terms causes no difficulties.

Note Since the supports of the test functions used in the proof of the theorem are compact and move to infinity, Theorem 3 remains valid if we add a non-central potential to $H$, provided that potential decreases at infinity faster than any negative power of $|x|$. Weaker versions of Theorems 2 and 3 hold if one adjoins a potential which decreases more slowly at infinity.

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