Static contact of belt and pulleys with account for shear and gravity

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Abstract. The fitting of a looped belt on two pulleys of equal radii is considered. The belt is subject to the gravity force. A geometrically nonlinear model with account for tension and transverse shear is applied for modeling the belt. The pulleys are considered rigid bodies, and the belt-pulley contact is assumed frictionless. The problem has an axis of symmetry, therefore the boundary value problem is formulated and solved for a half of the belt. The considered part consists of three segments, two free span segments and a contact segment between them. The introduction of a dimensionless material coordinate at all segments leads to a system of ordinary differential equations. The nonlinear boundary value problem for this system and boundary conditions is solved numerically with the finite difference method. As a result, the belt shape including the rotation angle, the forces, moments and contact pressure are determined. The contact pressure increases near the end point of contact areas, however no concentrated contact forces occur. The influence of gravity force on the contact pressure is discussed.

1. Introduction
The first systematic study of belt drives (and elastic creep in it) was reported by Reynolds in [1], where he used the string model. Until recently, one-dimensional models of elastic strings were widely used, cf. [2, 3, 4]. The exact solution for nonlinear steady state equations of extensible string is obtained in [3] by assuming zones of perfect and sliding friction contact between the belt and the pulley.

A spatial description of the belt contour motion is suggested in [2]. An idealized point friction model allows extending this result to the transient dynamics, see [4].

However it turned out that the model of extensible string describes just a part of important effects in belt mechanics. Friction forces transmit power between the belt and the pulleys. They are applied on the belt from one side and result not only in tangent forces, but also in distributed moments. The model of extensible string without bending stiffness cannot resist the moment loading. Therefore, we use the rod model (i.e., the one with at least bending stiffness).

The calculation of belt-pulley interaction can be found in works [3, 5, 6, 7, 8, 9, 10]. The study of interaction should be made with account for the transverse shear, because the importance of shear in contact problems of the rod theory is well-known, cf. [11, 12, 13, 14]. A contrary point of view is presented in [15]. The introduction of shear deformation causes the absence of lumped
contact forces and promotes better understanding of the contact force distribution. The shear is also required to describe the effect of elastic microslip, see, e. g. [16, 17] for general friction modeling. Particularly the shear is of importance in friction belt drive operation [18, 19, 20].

However, shear is known to be taken into account only together with tension (compression) if the nonlinear theory of rods is applied. The goal of this work is to model the belt as a rod with bending and shear stiffness in the static problem where the belt is set on the pulleys. The system of equations of the corresponding rod theory can be found in [21, 22, 23] (without direct application to belt drive mechanics, though).

The penalty formulation is in common use in numerical modeling of contact problems [24] and includes the choice of penalty coefficient or its distribution (see e. g. [25] for application to contact between rods). This approach is applied for belt drives in [10]. In the present paper we obtain the contact pressure and the stress-strain state in a different manner by describing the form of the belt lying on the rigid pulleys (the constraint formulation or the Hertz—Signorini—Moreau conditions for frictionless contact according to [24]). The similar approach of rod segmentation in contact problem can be found e. g. in [26, 27, 28], however without transverse shear.

We assume that the contact is full in a classical sense, i. e. it is continuous in an interval of the belt which length is to be determined. This assumption will be verified by obtaining the positive contact pressure. In the present study we use computer mathematics to solve difficult boundary value problems (BVP) for the systems of ordinary differential equations (ODE). The application of standard methods is possible due to introduction of transformed material coordinate.

The presented study is an extension of the work [29, 30]. Here we introduce significant complexity by including the gravity force neglected there. We intend to use these results to verify the developed finite element models [31, 32, 33].

2. Equations of nonlinear rod theory

Figure 1 displays a general calculation scheme of a rod with transverse shear and tension in the xy plane.

We consider the deformation of a belt as an elastic rod with tension and shear [21, 22, 23] subject to static contact with rigid pulley circles without friction. The position vector in the deformed configuration \( r(s) \) is a function of material coordinate \( s \), and \( \partial(\cdots)/\partial s = (\cdots)' \).

The angular orientation of each particle of the rod is given by the orthonormal unit vectors \( e_1, e_2 \). The angle \( \varphi \) denotes the angle between unit vector \( e_1 \) and Cartesian axis \( x \), the latter having the unit vector \( i \).

For plane deformations the system of equations of nonlinear theory of rods is as follows [21, 22, 23]:

\[
Q' + q = 0, \quad M' + k \cdot r' \times Q + m = 0, \quad \varphi' - \varphi'_0 = AM, \quad r' = P \cdot r'_0 + B \cdot Q = e_1 (1 + B_1 Q_1) + e_2 B_2 Q_2. \tag{1}
\]

Here we denote: \( Q \) is the internal force vector, \( M \) is the bending moment, and \( q, m \) stand for the external distributed forces and moments, respectively; \( k \) is the unit vector of Cartesian axis \( z \) perpendicular to the plane of drawing. The index zero indicates the values in initial state, and \( P = e_i e_{i0} \) is the rotation tensor. Elastic characteristics of the rod in plane are determined by three scalar compliances: the bending compliance \( A \), the tension compliance \( B_1 \) and the shear compliance \( B_2 \). The unit vectors \( e_1, e_2 \) are directed along the principal axes of the compliance tensor \( B \).

Algorithms and results for the contact of belt and pulleys without shear are presented in [5, 6]. The peculiarity of the considered problem is due to the fact that the function \( r(s) = R(\sigma) \) is unknown in the contact segment because the arc coordinate \( \sigma \) of the belt in its actual configuration (solid line in figure 2) is not yet determined.
We use the following geometrical relations:

\[ e_1 = i \cos \varphi + j \sin \varphi, \quad e_2 = -i \sin \varphi + j \cos \varphi. \]  

(2)

We will also use the expression for \( r' \) from (1):

\[ r'(s)ds = \dot{R}(\sigma)d\sigma; \quad |\dot{R}| = 1; \quad \sigma' = |r'| = D = \sqrt{(1 + B_1 Q_1)^2 + (B_2 Q_2)^2}, \quad \dot{s} = D^{-1}. \]  

(3)

The last equations determine the relation of material coordinate \( s \) with arc coordinate \( \sigma \) (a dot indicates the derivative with respect to \( \sigma \)).

3. Contact segment

Let us consider the full contact with the pulley. We write the expression for the tangent unit vector and integrate it

\[ R = i \sin \kappa \sigma + j \cos \kappa \sigma; \quad r' = D \dot{R}; \quad R = \kappa^{-1} [i (1 - \cos \kappa \sigma) + j \sin \kappa \sigma], \]  

(4)

where \( \kappa \) is the curvature of the curved belt in its actual configuration, which in the contact segment coincides with the circular pulley contour, and \( R(0) = 0 \) is assumed. We note that

\[ r' \cdot e_1 = 1 + B_1 Q_1 = D \sin (\varphi + \kappa \sigma), \quad r' \cdot e_2 = B_2 Q_2 = D \cos (\varphi + \kappa \sigma). \]  

(5)

Hence we may write the force component \( Q_2 \) as a function of coordinate \( \sigma \), angle \( \varphi \) and another force component \( Q_1 \):

\[ Q_2 (\sigma, \varphi, Q_1) = B_2^{-1} (1 + B_1 Q_1) \cot \alpha; \quad \alpha \equiv \varphi + \kappa \sigma. \]  

(6)

Then the relation between the derivatives (3) may be reduced to the form:

\[ D = \frac{1 + B_1 Q_1}{\sin \alpha}. \]  

(7)
As a result we have four unknown functions in the contact segment which are \( \sigma, \varphi, M, Q_1 \). Let us assume no tangential contact force between the belt and the pulley. Therefore the contact pressure \( p \) and the tangent vector \( \mathbf{r}' \) are orthogonal:

\[
(Q' - \rho g \mathbf{j}) \cdot \mathbf{r}' = 0; \quad \mathbf{e}'_1 = \varphi' \mathbf{e}_2, \quad \mathbf{e}'_2 = -\varphi' \mathbf{e}_1;
\]

\[ (Q'_1 - \varphi' Q_2 - \rho g \sin \varphi) (1 + B_1 Q_1) + (Q'_2 + \varphi' Q_1 - \rho g \cos \varphi) B_2 Q_2 = 0. \tag{8} \]

Here we may use (6) to simplify the equation (8) and write the derivatives of \( \sigma, \varphi, Q_2 \):

\[
Q'_2 = \frac{\partial Q_2}{\partial Q'_1} Q'_1 + \frac{\partial Q_2}{\partial \sigma} D + \frac{\partial Q_2}{\partial \varphi} \varphi' = \frac{B_1}{B_2} Q'_1 \cot \alpha - \frac{1 + B_1 Q_1}{B_2 \sin^2 \alpha} (\varphi' + \kappa \sigma'). \tag{9} \]

The combination of (8) and (9) gives the differential equation for \( Q_1 \):

\[
Q'_1 = - \left[ \left( \varphi'_0 + AM \right) \left( \frac{1 + B_1 Q_1}{B_2} \left( \frac{2}{1 - \cos 2\alpha} + 1 \right) - Q_1 \right) \cot \alpha + \frac{8 \kappa (1 + B_1 Q_1)^2 \cos \alpha}{B_2 (2 \cos 4\alpha - 4 \cos 2\alpha + 3)} + \rho g \left( \cos \varphi \cot \alpha + \sin \varphi \right) \right] \left( \frac{B_1}{B_2 (1 + \cos 2\alpha)} + 1 \right). \tag{10} \]

Let us substitute \( \mathbf{r}' \) into the balance of moments (1) (with distributed moment \( m = 0 \) in the current frictionless problem) and obtain

\[
M' = ((B_2 - B_1) Q_1 - 1) Q_2. \tag{11} \]

Finally we determine the contact pressure \( p \):

\[
Dp = \mathbf{p} \cdot \mathbf{k} \times \mathbf{r}' = \mathbf{k} \cdot (Q' - \rho g \mathbf{j}) \times \mathbf{r}',
\]

\[
p = \frac{B_2 - B_1}{B_2} Q'_1 \cos \alpha + \left( \frac{1}{B_2} + \frac{B_1 - B_2}{B_2} Q_1 \right) \varphi' \sin \alpha + \frac{\kappa (1 + B_1 Q_1)^2}{B_2 \sin^2 \alpha} + \rho g \sin \kappa \sigma. \tag{12} \]

The contact pressure must be non-negative. Unlike the case without shear in the considered problem there must be no concentrated contact reactions. The similar case of the classical contact between an initially straight nonlinear shearable rod and a rigid straight obstacle is discussed e. g. in [13].

4. Free segment

The scheme of fitting the belt on the pulleys is shown in figure 2. In the initial undeformed state the belt is the circle of radius \( R \). Hence the initial angle and its derivative are \( \varphi_0 = \pi - s/R, \varphi'_0 = -1/R \). The pulleys have curvature \( \kappa \). The center distance \( H \) (i. e. the distance between the pulley centers) is prescribed. Because of the problem symmetry it is sufficient to consider only the left half \( 0 \leq s \leq \pi R \). In the segment \( s_1 \leq s \leq s_2 \) the full contact is assumed, however the coordinates \( s_1 \) and \( s_2 \) are yet unknown. The formulation for this segment is presented above.

Let us turn to the free segments (lower \( 0 \leq s \leq s_1 \) and upper \( s_2 \leq s \leq \pi R \) belt spans). Firstly we consider the lower span. The balance of forces and boundary conditions (as follows from the symmetry of the problem)

\[
Q' - \rho g \mathbf{j} = 0; \quad Q'_x = 0, \quad Q'_y - \rho g = 0; \quad Q(0) = -P_1 \mathbf{i} \tag{13} \]

allow obtaining the explicit relations for the force components:

\[ Q_x = -P_1 = \text{const}, \quad Q_y = \rho gs. \] (14)

Here \( Q(0) = -P_1i \) is yet unknown horizontal force between the belt left and right parts.

Then we substitute these components into the balance of moments (11), which also holds in the free segments, and hence we need the relations

\[
\begin{align*}
Q_1 &= Q_x \cos \varphi + Q_y \sin \varphi = -P_1 \cos \varphi + \rho gs_1 \sin \varphi, \\
Q_2 &= -Q_x \sin \varphi + Q_y \cos \varphi = P_1 \sin \varphi + \rho gs_1 \cos \varphi.
\end{align*}
\] (15)

The differential equations for the angle and coordinates \((r = xi + yj)\) derived from (1) are

\[
\begin{align*}
x' &= \cos \varphi - P_1 \left( B_1 + \frac{B_2 - B_1}{2} (1 - \cos 2\varphi) \right) + \rho gs \frac{B_1 - B_2}{2} \sin 2\varphi, \\
y' &= \sin \varphi - P_1 \frac{B_1 - B_2}{2} \sin 2\varphi + \rho gs \left( B_1 + \frac{B_2 - B_1}{2} (1 + \cos 2\varphi) \right).
\end{align*}
\] (16)

As a result in the free segment we have four unknown functions: \(M, \varphi, x, y\) and an unknown constant \(P_1\). The similar equations may be written for the second (upper) free span. The balance of forces and the boundary conditions (from the problem symmetry) allow deriving the force components

\[ Q_x = P_2 = \text{const}, \quad Q_y = \rho g (s - \pi R). \] (17)

We write the balance of moments in the form (11) where we have the force components

\[
\begin{align*}
Q_1 &= P_2 \cos \varphi + \rho g (s - \pi R) \sin \varphi, \\
Q_2 &= -P_2 \sin \varphi + \rho g (s - \pi R) \cos \varphi.
\end{align*}
\] (18)

The remaining differential equations read

\[
\begin{align*}
x' &= \cos \varphi + P_2 \left( B_1 + \frac{B_2 - B_1}{2} (1 - \cos 2\varphi) \right) + \rho g(s - \pi R) \frac{B_1 - B_2}{2} \sin 2\varphi, \\
y' &= \sin \varphi + P_2 \frac{B_1 - B_2}{2} \sin 2\varphi + \rho g(s - \pi R) \left( B_1 + \frac{B_2 - B_1}{2} (1 + \cos 2\varphi) \right).
\end{align*}
\] (19)

5. **Formulation of a boundary value problem with fixed boundaries**

The introduction of the new nondimensional coordinate \(0 \leq \xi \leq 1\) (see figure 3) allows rewriting the equations in the three segments into one general BVP with fixed boundaries (see for the similar technique in general [34] and in quasi-static problem [35]). We need the relations between the derivatives:

\[
\begin{align*}
0 \leq s \leq s_1: & \quad s = s_1 \xi \Rightarrow \frac{d}{d\xi} = s_1 \frac{d}{ds}; \\
s_1 \leq s \leq s_2: & \quad s = s_1 - \xi (s_1 - s_2) \Rightarrow \frac{d}{d\xi} = - (s_1 - s_2) \frac{d}{ds}; \\
s_2 \leq s \leq \pi R: & \quad s = \pi R - \xi (\pi R - s_2) \Rightarrow \frac{d}{d\xi} = - (\pi R - s_2) \frac{d}{ds}.
\end{align*}
\] (20)

We combine the equations at all three segments into a single system and distinguish the values by the indices \((\cdots)^{(1)}, (\cdots)^{(2)}, (\cdots)^{(3)}\), where \((\cdots)^{(1)}\) indicates the lower free segment,
\[ Z(s) = \begin{pmatrix} x^{(1)} \\ y^{(1)} \\ \sigma^{(2)} \\ Q_1^{(2)} \\ M^{(2)} \\ \varphi^{(2)} \\ M^{(3)} \\ x^{(3)} \\ y^{(3)} \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ \xi = 1 \\ \xi = 0 \\ \xi = 1 \\ \xi = 0 \\ \xi = 0 \\ \xi = 0 \end{pmatrix} \]

Figure 3. Segmentation of belt and coordinate transformation.

\[
\begin{align*}
\xi &= 0 : \quad \varphi^{(1)} = \pi, \\
\xi &= 1 : \quad x^{(3)} = \kappa^{-1} \left(1 - \cos \kappa \sigma^{(2)}\right), \quad y^{(3)} = \kappa^{-1} \sin \kappa \sigma^{(2)}, \\
&\quad \varphi^{(3)} = \varphi^{(2)}, \\
&\quad P_2 \cos \varphi^{(3)} + \rho g (s_2 - \pi R) \sin \varphi^{(3)} = Q_1^{(2)}, \\
&\quad M^{(3)} = M^{(2)}.
\end{align*}
\]

Here we use the prescribed orientation and position \((x, y, \varphi, Q_1, Q_2, M)\) at the boundary points of contact zones \(s_1\) and \(s_2\) (no lumped forces and moments are applied there).

At the point with material coordinate \(s_1\) we get the different values of nondimensional coordinate \(\xi\) in the first and second segment. We may write the boundary conditions in the following way:

\[
\begin{align*}
\xi = 1 & : \quad \varphi^{(1)} = \pi, \\
\varphi^{(1)} &\bigg|_{\xi = 1} = \varphi^{(2)} \bigg|_{\xi = 0}, \\
&\quad P_1 \cos \varphi^{(1)} + \rho g s_1 \sin \varphi^{(1)} \bigg|_{\xi = 1} = Q_1^{(2)} \bigg|_{\xi = 0}, \\
&\quad M^{(1)} \bigg|_{\xi = 1} = M^{(2)} \bigg|_{\xi = 0}, \\
&\quad P_1 \sin \varphi^{(1)} + \rho g s_1 \cos \varphi^{(1)} \bigg|_{\xi = 1} = B_2^{-1} \left(1 + B_1 Q_1^{(2)} \right) \cot \left(\varphi^{(1)} \bigg|_{\xi = 1} + \kappa \sigma^{(2)} \bigg|_{\xi = 0}\right).
\end{align*}
\]

We note that to obtain the arc coordinate \(\sigma(s)\) in the free segments and the whole belt length after the deformation we need to solve two ordinary differential equations (3) with the values of \(\sigma\) at \(s_1\) and \(s_2\) as initial conditions.
6. Numerical results
The formulated nonlinear BVP (21), (22), (23) may be solved by the shooting method. However we prefer the finite difference method because of its efficiency. We implement it in Wolfram Mathematica with the scheme described in [29, 30]. The formulated problem may be also solved by the collocation method with the solver of Matlab package Chebfun [36].

The parameters of the benchmark example are: \( H = 2 \) m, \( \kappa = 1/0.15 \) m\(^{-1}\), \( L = 1.05(H + \pi/\kappa) \), \( R = L/\pi \), \( E = 10^9 \) Pa, \( \nu = 0.5, \mu = E/(2(1 + \nu)) \), \( \rho = 0.48 \) kg/m \((\rho = 0.16 \) kg/m\), \( A = 1/0.07 \) N\(^{-1}\)m\(^{-2}\), \( B_1 = 1/2000 \) N\(^{-1}\), \( F = 1/(EB_1) \), \( B_2 = 6/(5\mu F) \), \( g = 9.81 \) m/s\(^2\). We draw the belt form in figure 4 and pressure distribution in figure 5.

![Figure 4. Belt and pulley. Solid line is the belt current configuration, \( \rho = 0.48 \) kg/m; dot-dashed line for \( \rho = 0.16 \) kg/m, thin circle is the pulley.](image)

![Figure 5. Pressure distribution vs material coordinate \( s \). Solid line for \( \rho = 0.48 \) kg/m, dashed line for \( \rho = 0.16 \) kg/m.](image)

The pressure distribution is symmetric in the case without gravity, and it tends to concentrate at one (upper) side with gravity included. In figure 6, 7 we show the forces \( P_1, P_2 \) acting at the middle of the spans and the contact angles \( \beta_1, \beta_2 \) (central angles of contact zones on pulleys, see figure 2). For \( y^{(1)}|_{\xi=1} < 0 \) we have \( \beta_1 > 0 \), and for \( y^{(3)}|_{\xi=1} > 0 \) we have \( \beta_2 > 0 \). The lower span has lower tension force, than the upper one. The gravity force may significantly influence the angle of contact \( \beta_2 + \beta_1 \) and, therefore, the friction behaviour of the belt drive.

7. Conclusion
The main results of the present study are listed below:

- a system of ODE for the drive belt as a plane nonlinear elastic rod with account for tension and transverse shear is derived;
- the considered part of the belt is divided into three segments, the contact segment and two free segments;
- the equations for the free and contact segments which have qualitatively different formulation are combined in a single system of ODE;
- the formulated BVP is suitable for numerical solving by the shooting, finite difference or collocation method;
- the effect of gravity force acting on the belt is clarified.

The generalization to the cases with pulleys of unequal radii and an inclination of the gravity force relative to \( y \)-axis is possible. These results will be used in further work including the
study of belt-pulley interaction with friction and the dynamical modeling, both stationary and transient.

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