A new methodology in the study of acoustic fields in the almost stratified ocean

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Abstract. In this work, we present the mathematical expressions for the study of the behaviour of the acoustic field in almost stratified ocean. These expressions are suitable for numerical simulation and are derived from the spectral parameter power series method applied in problems of underwater acoustics. Analytical treatment for the obtention of the dispersion equation and normed eigenfunctions of the problem is given.

1. Introduction
Sound waves propagate over long distances in the ocean making them appropriate for applications in acoustics communications. Sound propagation in the ocean is mathematically described by the wave equation, whose parameters and boundary conditions describe the ocean environment. The main physical characteristic of the ocean is the stratification where density, sound speed and other parameters are a function of the depth [7], [8]. This stratification has considerable effect on the contribution of the acoustic field in the ocean [11]. As summarized in Figure 1, there exist a variety of different available techniques for solving numerically the wave equation for estimations of the sound propagation in the sea. In Figure 1, abbreviations correspond to: Finite Element (FE), Finite Difference (FD), Normal Modes (NM), Parabolic Equation (PE), Fast Field Program (FFP), Ray Tracing (RT) [12]. Models for solutions of the wave equation that permit the ocean environment to vary with depth are termed range independent. A model that permits horizontal variations in the ocean environment is termed range dependent. Solutions based on spectral techniques (FFP) and normal modes (NM) are applied making the assumption that the environment is range independent. Both techniques can be extended and applied to solve range dependent problems. To deal directly with range varying environments, the Ray, PE and FD/FE solutions are computed. For almost stratified media, combination of the mode decomposition with respect to the eigenfunctions of the transversal Sturm-Liouville operator depending on the horizontal parameter and the ray method is considered in this work.

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2. Statement of the problem
This brief document is devoted to a mathematical analysis of the acoustic wave propagation from a point source in the almost stratified model of the real ocean. The almost stratification of the ocean means that density $\rho_0$, and sound speed $c_0$, of the water layer depend on the small parameter $\varepsilon > 0$, that is

$\rho_0 = \rho_0(\varepsilon x, z), \ c_0 = c_0(\varepsilon x, z)$. 

The interface $\Omega$ between the water layer and the bottom is slowly varying in the horizontal direction,  

$\Omega = \{(x, z) \in \mathbb{R}^2 : z = \phi(x), x \in \mathbb{R}^2\}$

In [15], a methodology for combination of the mode decomposition with respect to the eigenfunctions of the transversal Sturm-Liouville operator depending on the horizontal is proposed. The mathematical model of the acoustic wave propagation in the ocean is given by the following partial differential equations (see [7], [8], [11], [12])

$$
\left(\Delta_x + \rho(z) \frac{\partial}{\partial z} \rho^{-1}(z) \frac{\partial}{\partial z} + k_0^2(z)\right) u(x, z) = -\delta(x, z - z_0), \quad x \in \mathbb{R}^2, z \in (0, H)
$$

$$
\left(\Delta_x + \frac{\partial^2}{\partial z^2} + k_1^2\right) u(x, z) = 0, z \in (H, +\infty).
$$

(1)

where $\Delta_x$ is the two-dimensional Laplacian operator, $x$ the horizontal variable, and $z$ the vertical variable. $u(x, z)$ is the amplitude of the harmonic vibrations of the acoustic pressure. Finally, $\delta$ gives the spatial position of the point source. The acoustic energy is provided by a point source depending on time ($t$) and frequency ($\omega$), $w(t, x, z) = e^{-i\omega t} u(x, z)$. $k_0$ and $k_1$ are the wave numbers of the problem.

$k(z) = \begin{cases} k_0 = \omega / c_0(z), & z \in [0, H] \\
 k_1 = \omega / c_1, & z \in [H, +\infty) \end{cases}$

The density of the stratified ocean is denoted by

$\rho(z) = \begin{cases} \rho_0(z), & z \in [0, H] \\
 \rho_1, & z \in [H, +\infty) \end{cases}$

And the sound speed

$c(z) = \begin{cases} c_0(z), & z \in [0, H] \\
 c_1, & z \in [H, +\infty) \end{cases}$

It must follow that $c_1 > c_0(z)$, and $\rho_1 > \rho_0(z)$. $c_1$ and $\rho_1$ are the sound speed and density in the half-space respectively (the fluid bottom), meanwhile $c_0(z)$ and $\rho_0(z)$ are the sound speed and density in...
the waveguide. \( H \) is the depth of the waveguide and \( r \) is the separation distance (horizontal variable) between source and receiver in the waveguide.

\[
\text{Figure 2. Source and receiver in an ocean waveguide.}
\]

The boundary condition

\[
u(x, 0) = 0, \ x \in R^2,
\]

simulates the interface between the ocean surface and the atmosphere. And

\[
u(x, H - 0) = u(x, H + 0),
\]

is the continuity of the acoustic pressure between the waveguide and the half-space. The continuity of the normal component of the fluid particles velocity on \( \Omega \) is accepted as in [13], [14]

\[
\frac{1}{\rho(H)} \frac{\partial u(x, H+0)}{\partial z} = \frac{1}{\rho_1} \frac{\partial u(x, H-0)}{\partial z}
\]

Equation (1) is connected with the spectral Sturm-Liouville problem. We apply the Fourier Transform to (1)-(4), and we obtain

\[
\left( \rho(z) \frac{\partial}{\partial z} \rho^{-1} \frac{\partial}{\partial z} + k_0^2(z) - k_1^2 \right) \varphi(z) = \mu^2 \varphi(z), z \in [0, H].
\]

\[
\varphi(0) = 0, [\varphi]_{z=H} = 0, [\rho^{-1}(z)u']_{z=H} = 0.
\]

The solution of the problem (5) is seek on the form

\[
u_{e}(x, z) = \sum_j a_j(x)\varphi_j(x, z)e^{i\sigma_j(x)} + O(\varepsilon^2).
\]

Where \( \varphi_j(x, z) \) are eigenfunctions of the transversal Sturm-Liouville operator \( L(x) \) depending of the point \( x \in R^2 \) as a parameter, \( s_j(x) \) is the solution of the two-dimensional eikonal equation

\[
(\nabla s_j(x))^2 = \mu_j^2(x), x \in R^2,
\]

where \( \mu_j^2 \) is an eigenvalue of the operator \( L(x) \). The coefficient \( a_j \) satisfies the transport equation

\[
2\nabla s_j(x) \cdot \nabla x a_j(x) + \nabla^2 s_j(x) a_j(x) = 0
\]

The two-dimensional ray method is applied for the solution of equations (7) and (8). The numerical simulation, of the problem of wave propagation in the almost stratified ocean, needs a quick and accurate solution of the transversal Sturm-Liouville problem depending on the parameter, which varies on long distances. For this aim, we propose the spectral parameter power series method, which presents solutions of the second order ordinary differential equation

\[
\rho(z) \frac{d}{dz} \rho^{-1}(z) \frac{d u(z, \lambda)}{dz} + q(z) = \lambda u(z, \lambda), z \in [0, H],
\]
in the form of series

\[ u(z, \lambda) = \sum_{j=1}^{\infty} a_j(z) \lambda^j. \]  

(10)

Where \( a_j(z) \) are defined by the recurrent formulas containing only integration [1] and \( \lambda \) are the values of the discrete spectrum of the problem.

3. Dispersion equation for the almost stratified ocean

For the study of the behavior of acoustic fields in almost stratified ocean, it is needed eigenvalues and eigenfunctions of the vertical Sturm-Liouville spectral problem depending on the horizontal parameter \( x \in R^2 \). For this aim, we will apply the spectral parameter power series method for a rapid and exact numerical solution of the Sturm-Liouville problem. This method was applied in the problem of underwater acoustics of the stratified ocean [13]. We seek a solution \( \varphi \) of the spectral problem of the form

\[ \varphi(z, \mu) = \begin{cases} f(z, \mu), & 0 < z < H, \\ C_1(\mu)e^{\mu(z-H)} + C_2(\mu)e^{-\mu(z-H)}, & z \geq H. \end{cases} \]  

(11)

\( f \) is a solution of the Cauchy problem

\[ f''(z, \mu) + k_0^2 f(z, \mu) = \mu^2 f(z, \mu), \quad z \in [0, H] \]  

\[ f(0, \mu) = 0, \quad f'(0, \mu) = 1. \]  

(12)

Solution of the Cauchy problem (12) is sought in the form

\[ f(z, \mu) = \sum_{k=0}^{\infty} a_k(z) \mu^{2k}. \]

With coefficients \( a_k \) defined by some recursive formulas. Applying boundary conditions (2), (3), and (4) and the theory of spectral parameter power series method (for details see [1], [13], [14]), the following dispersion equation is obtained

\[ D(\mu) = \mu \frac{\rho_0}{\rho_1} f(H, \mu) + f'_H(H, \mu) = 0, \quad 0 \leq \mu \leq \sqrt{k_1^2 - k_0^2} \]  

(13)

\( D(\mu) = 0 \) is the dispersion equation for the definition of the eigenvalues \( \mu_j \in (k_1, k_{max}) \) of the Sturm-Liouville problem (detailed procedure for calculation of these expression is found in [1]). The normalized eigenfunctions are given as

\[ \varphi_j(z) = \frac{1}{M_j} \begin{cases} f(z, \mu_j), & 0 < z < H, \\ C_2(\mu_j)e^{-\mu_j(z-H)}, & z \geq H, \end{cases}, \]  

(14)

where

\[ \mu_j^2 = \frac{1}{2\mu_0 \rho_1} + \frac{1}{\rho_0} \left| f(z, \mu_j) \right|^2 dz. \]  

(15)

For applications in underwater acoustics, expression (13) will be applied for calculation of approximate values \( \mu_j^2 \) of the eigenvalues for the spectral problem defined in (5). This expression is suitable for numerical treatment in software as Wolfram Mathematica or Matlab. Expressions (14) and (15) are recursive formulas that contributes to calculate the losses of the acoustic field. The behavior of the acoustic field is an important parameter for some applications and studies in shallow water [6], [9], [11], [12]. It is expected that the contribution of this work will give and analytic methodology suitable for numerical and more efficient calculation of the contribution of the acoustic field in almost stratified environments of the ocean.
4. Conclusions
In forthcoming works we will apply expressions (13), (14), and (15) to study the contribution of the acoustic field in almost stratified ocean. This study would have applications in the study of climate change with accurate data, ocean dynamics. Special interest is pay attention on bottom interaction and inversion, signal processing in shallow water, and applications in marine biology.

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