RUNGE APPROXIMATION AND STABILITY IMPROVEMENT FOR A PARTIAL DATA CALDERÓN PROBLEM FOR THE ACOUSTIC HELMHOLTZ EQUATION

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Abstract. In this article, we discuss quantitative Runge approximation properties for the acoustic Helmholtz equation and prove stability improvement results in the high frequency limit for an associated partial data inverse problem modelled on [3, 35]. The results rely on quantitative unique continuation estimates in suitable function spaces with explicit frequency dependence. We contrast the frequency dependence of interior Runge approximation results from non-convex and convex sets.

1. Introduction. In this article we study improvement of stability effects in Runge approximation originating from the interplay of geometry and an increasing frequency parameter for the acoustic Helmholtz equation. These effects had first been observed in [15] and have subsequently been the object of intensive study, both in the context of unique continuation [24, 23, 21, 51, 50] and with regards to their effects on inverse problems [10, 26, 22, 12, 13, 19, 20, 25, 28, 29, 5, 42, 6, 27]. Due to the notorious instability in many inverse problems, these improved stability estimates are of great significance, both from a theoretical and practical point of view [9]. We refer to Section 1.4 for an (incomplete) overview of the history and background of these type of results.

Building on the observation that Runge approximation properties are qualitatively and quantitatively dual to unique continuation [46] (see also [37, 53] and the references therein for analogous results in the control theory community), in this article we seek to study the effects of geometry and increasing frequency $k$ for acoustic Helmholtz equations

$$\left(\Delta + k^2 q(x) + V(x)\right)u(x) = 0 \tag{1}$$

on the associated Runge approximation properties under suitable conditions on the geometry and the potentials (see Section 1.1). For the special case of the (pure) Helmholtz equation ($q = 1$ and $V = 0$) quantitative Runge approximation had been deduced in [11, Lemma 2.1] in the context of approximation properties for

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dispersive equations. Relying on the duality between Runge approximation and unique continuation, we here prove quantitative unique continuation properties for the acoustic Helmholtz equation (1) in suitable, adapted function spaces, carefully tracking the parameter dependence.

As two of our main results, we deduce Runge approximation properties with exponential $k$-dependence without geometric assumptions (Theorems 1.1 and 1.2) and improved, polynomial behaviour under convexity conditions on the domain (Theorem 1.3).

In order to illustrate the importance, robustness and usefulness of these estimates, we consider the partial data inverse problem for the equation (1). Using the systematic duality strategy from [48], we prove improved stability results for this nonlinear inverse problem (Proposition 1). This generalizes the results from [35] to the case of acoustic Helmholtz equations.

1.1. Setting. In the following, we outline the precise geometric and functional assumptions under which our results are valid. Here, as a model system, we focus on generalizations of Helmholtz type equations with homogeneous Dirichlet conditions under the assumption that the increasing parameter is chosen with some distance to the spectrum. More precisely, for $n \geq 2$, $k \geq 1$ and $\Omega \subset \mathbb{R}^n$ a bounded, connected, open set with Lipschitz boundary we consider the acoustic Helmholtz equation (1) in $\Omega$, where

(i) $V \in L^\infty(\Omega)$ and zero is not a Dirichlet-eigenvalue of $\Delta + V$ in $\Omega$,
(ii) $q \in C^{0,1}(\Omega)$ and $q$ is strictly positive, that is $0 < \sigma^{-1} \leq q \leq \sigma$ for some $\sigma > 1$.

Let us comment on these conditions: The assumption that $V \in L^\infty(\Omega)$ ensures that the potential $V$ is subcritical in terms of scaling and that it is well in the regime in which unique continuation results are available. The second condition in (i) is a (technical) solvability condition. By domain perturbation arguments, this is generically satisfied [32].

The conditions formulated in (ii) on $q$ are two-fold: The sign condition and bounds on $q$ ensure that the acoustic equation (1) is of Helmholtz type; $q = 1$ corresponds to the Helmholtz equation with potential. The regularity condition $q \in C^{0,1}(\Omega)$ will be used in order to treat the term $k^2 q(x)$ as part of the principal symbol of the operator. In Sections 2 and 3 this will be a consequence of introducing an auxiliary new dimension in order to reduce the parameter dependent equation to a non-parameter dependent equation with Lipschitz continuous principal symbol. In Section 5 we will directly treat the parameter dependent term $k^2 q$ as part of the principal symbol of the Carleman estimate for which we again require some regularity on $q$. In order to do so, we will complement the condition (ii) with an additional radial monotonicity assumption (see (ii') in Section 1.3).

We consider (1) with homogeneous Dirichlet boundary conditions. In order to avoid solvability issues or a priori estimates without control on $k$, we make an additional technical assumption: We suppose that the real parameter $k \geq 1$ is chosen such that zero is not a Dirichlet eigenvalue of the operator in $\Omega$. More precisely, let $\Sigma_{V,q}$ denote the set of the inverse of eigenvalues of the operator $T := (-\Delta - V)^{-1} M_q$, where $M_q$ denotes the multiplication operator with $q$. We will then assume that

(a1) $\text{dist}(k^2, \Sigma_{V,q}) > ck^{2-n}$, for some $c \ll 1$.

We remark that generically this does not pose major restrictions, as it is always possible to find arbitrarily large values of $k$ such that the condition (a1) is fulfilled: Indeed, the operator $T$ is a classical pseudodifferential operator of order $-2$. We
denote the spectrum of $T^{-1}$ by $\Sigma_{V,q} = \{\lambda_n\}_{n \in \mathbb{N}}$. By Weyl’s law [14, 49]
\[ \#\{\lambda_n \leq E\} = CE^{-\frac{n}{2}} \text{ as } E \to \infty. \]
Thus, on average, the distance between consecutive eigenvalues is $C'E^{1-\frac{n}{2}}$ as $E \to \infty$. In this case, (a1) ensures that it is possible to find admissible frequencies in essentially all frequency ranges with $c \in (0, \frac{C'}{3})$.

We remark that also other boundary conditions would have been feasible. An alternative, natural condition would have been impedance conditions (including the potential $q$) which in the limit of growing domains would have approximated a Sommerfeld type radiation condition. This would also have had the advantage of avoiding the eigenvalue assumptions and discussions. Since we are working in finite domains, for simplicity, we here restrict our attention to the Dirichlet setting.

1.2. Runge approximation without convexity conditions. With the conditions stated above, we first address Runge approximation results without additional convexity assumptions on the domain. Our main results then provide quantitative Runge approximation results with a quantified dependence on the parameter $k$. Since these properties are dual to unique continuation properties for which exponential dependences on $k$ are unavoidable without additional geometric assumptions [8], the dependences on $k$ are expected to be exponential.

For the case of approximation in the domain in which a solution is prescribed we thus obtain the following result:

**Theorem 1.1.** Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open, bounded, connected Lipschitz domains such that $\Omega_1 \Subset \Omega_2$ and such that $\Omega_2 \setminus \Omega_1$ is connected. Let $\Gamma$ be a non-empty, open subset of $\partial \Omega_2$. Let $V$ and $q$ satisfy (i)-(ii) in $\Omega_2$. There exist constants $\mu > 1$, $s \geq n + 6$ and $C > 1$ depending on $n, \Omega_2, \Omega_1, \|V\|_{L^\infty(\Omega_2)}, \sigma$ and $\|q\|_{C^{0,1}(\Omega_2)}$ such that for any solution $v \in H^1(\Omega_1)$ of
\[ (\Delta + k^2 q + V)v = 0 \text{ in } \Omega_1, \]
with $k \geq 1$ satisfying (a1), and any $\epsilon > 0$, there exists a solution $u$ to
\[ (\Delta + k^2 q + V)u = 0 \text{ in } \Omega_2, \]
with $u|_{\partial \Omega_2} \in \bar{H}^{s/2}(\Gamma)$ such that
\[ \|u - v\|_{L^2(\Omega_1)} \leq \epsilon \|v\|_{H^1(\Omega_1)}, \quad \|u\|_{H^{s/2}(\partial \Omega_2)} \leq C \epsilon^{Ck^s} \|v\|_{L^2(\Omega_1)}. \]

In Section 3.1 we show that in the full data case and for $q = 1$, up to the precise values of $\mu$, $s$ and $C$, the bound in $\epsilon$ is optimal, see [46, Section 5] for the analogous result for the Laplacian without a large parameter.

If $\tilde{v}$ is a solution in a slightly larger domain than the one for which we seek to find a good approximation, the exponential dependence in $\epsilon$ changes to a polynomial dependence while the $k$-dependence remains exponential:

**Theorem 1.2.** Let $\Omega_1, \Omega_2, \Gamma, V$ and $q$ be as in Theorem 1.1. Further, let $\Omega_1$ be a bounded, Lipschitz domain such that $\Omega_1 \Subset \Omega_2$. There exist constants $\nu > 1$ and $C > 1$ depending on $n, \Omega_2, \Omega_1, \|V\|_{L^\infty(\Omega_2)}, \sigma$ and $\|q\|_{C^{0,1}(\Omega_2)}$ such that for any solution $\tilde{v} \in H^1(\Omega_1)$ of
\[ (\Delta + k^2 q + V)\tilde{v} = 0 \text{ in } \Omega_1, \]
with \( k \geq 1 \) satisfying (a1), there exists a solution \( u \) to
\[
(\Delta + k^2q + V)u = 0 \quad \text{in } \Omega
\]
with \( u|_{\partial \Omega_2} \in \tilde{H}^{1/2}(\Gamma) \) such that
\[
\|u - \tilde{v}\|_{L^2(\Omega_1)} \leq \epsilon\|\tilde{v}\|_{H^1(\Omega_1)}, \quad \|u\|_{H^{1/2}(\Gamma)} \leq Ce^{Ck^2} \epsilon^{-\nu}\|\tilde{v}\|_{L^2(\Omega_1)}.
\]

As an application of these results we prove a partial data uniqueness result for the Calderón problem with stability improvement in \( k \) under a priori assumptions on the potential in a neighbourhood of the boundary. This generalizes the results from [35] to acoustic equations. In particular, it thus combines ideas from [3, 47] with the observations from [15] (see also the references above and below). We further refer to [22] for similar results for different ranges of \( k \).

**Proposition 1.** Let \( n \geq 3 \), let \( \Omega \subset \mathbb{R}^n \) be a bounded, connected, smooth open set and let \( \Gamma \subset \partial \Omega \) be a nonempty open subset. Let \( \Omega' \subset \Omega \) be an open, Lipschitz subset such that \( \Omega \setminus \Omega' \) is connected. Let \( q_1, q_2, V_1, V_2 \) verify (i)-(ii) in \( \Omega \) and be such that
\[
\|q_j\|_{L^\infty(\Omega)} + \|V_j\|_{L^\infty(\Omega)} \leq B,
q_1 = q_2, \; V_1 = V_2 \quad \text{in } \Omega \setminus \Omega'.
\]

Then, there exists a constant \( C > 0 \) depending on \( n, \Omega, \Omega', \Gamma, \|q_j\|_{C^{0,1}(\Omega)}, \sigma \) and \( B \) such that for all \( k \geq 1 \) satisfying (a1) and \( \delta = \|\Lambda_{V_1,q_1}^k(k) - \Lambda_{V_2,q_2}^k(k)\|_{H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)} < 1 \), we have
\[
\|k^2(q_2 - q_1) + (V_2 - V_1)\|_{H^{-1}(\Omega)} \leq C \left( e^{Ck^{n+3}} \delta + \frac{1}{\left( k + |\log \delta|^{\frac{1}{\nu}} \right)^{\frac{n}{2}}} \right).
\]

**Remark 1.** We remark that if in Proposition 1 two measurements for different values of \( k \) are available, it is possible to provide stability both for \( q_j \) and \( V_j \) separately.

Earlier improvement of stability results had been obtained for the corresponding full data inverse problems in [42, 27]. While in the case of the Helmholtz equation \( (q = 1) \) with potential, the \( k \)-dependence of the Lipschitz contribution was proved to be polynomial instead of exponential, already in the full data acoustic case \( (q \neq 1) \) exponential \( k \)-dependences emerged.

### 1.3. Improvements of the Runge approximation results in convex geometries – the case of concentric balls.

Last but not least, in our final section, in line with the observations from [15] (and the literature building up on this, where the improved \( k \)-dependences in quantitative unique continuation are discussed in convex domain geometries), we deduce improved Runge approximation results in the interior in the presence of a large parameter in the specific convex domain geometry consisting of two concentric balls (see also [8, Section 3]). Here we impose both a regularity and a monotonicity condition on the potential:

(iii') \( q \in C^{0,1}(\Omega), \sigma^{-1} \leq q \leq \sigma \) for some \( \sigma > 1 \) and \( \nabla q \cdot x \geq 0 \) almost everywhere.

In this subsection, we do not argue by an extension of the problem to an elliptic equation in an additional dimension as in the previous subsection. Instead, we directly deduce a Carleman estimate for the operator at hand. Hence, a natural question arises with regards to the necessity of the regularity and monotonicity conditions for the potential. Here, due to the presence of the large parameter \( k^2 \),
in order to avoid exponential losses in $k$, we do not rely on perturbation arguments (which would only require $L^\infty$ regularity of the potential). Instead, as in other contexts in the literature (see, for instance, [4]), we exploit the regularity and monotonicity assumption which is imposed on the potential in order to interpret the potential as “part of the principal symbol” in the Carleman estimates (see the proof of Proposition 4). We also refer to [45] where this is used in the context of unique continuation in the presence of sublinear potentials and further highlight that these additional conditions on the potential $q$ are also commonly used in the context of the study of embedded eigenvalues [34] and quantitative unique continuation results [4]. Recently, in the article [39] for the two-dimensional Landis conjecture similar regularity conditions from [40] had been removed by Logunov, Malinnikova, Nadirashvili and Nazarov by using Beltrami equations techniques.

With this assumption we deduce improved (in $k$) dependences in the Runge approximation results in the interior for specific convex geometry determined by two concentric balls:

**Theorem 1.3.** Let $V$ and $q$ be as in (i)-(iii) in $\Omega_2 = B_2 \setminus \overline{B_{r_2}}$ and let $\Omega_1 = B_1 \setminus \overline{B_{r_2}}$ and $\tilde{\Omega}_1 = B_1 + \delta \setminus \overline{B_{r_2}}$, for some $\delta \in (0,1)$. There exist parameters $\nu > 1$, $s > 3$ and a constant $C > 1$ depending on $n$, $\|V\|_{L^\infty(\Omega_2)}$ and $\sigma$ such that for any solution $\tilde{v} \in H^1(\tilde{\Omega}_1)$ of

$$(\Delta + k^2 q + V)\tilde{v} = 0 \text{ in } \tilde{\Omega}_1,$$

$\tilde{v} = 0 \text{ on } \partial B_{r_2},$$

with $k \geq 1$ satisfying (a1), there exists a solution $u$ to

$$(\Delta + k^2 q + V)u = 0 \text{ in } \Omega_2,$$

$u = 0 \text{ on } \partial B_{r_2},$$

such that

$$\|u - \tilde{v}\|_{L^2(\Omega_1)} \leq \epsilon\|\tilde{v}\|_{H^1(\tilde{\Omega}_1)},$$

$$\|u\|_{H^{1/2}(\partial B_{r_2})} \leq Ck^s \epsilon^{-\nu}\|\tilde{v}\|_{L^2(\Omega_1)}.$$

These results will be derived by duality from improved unique continuation estimates for the dual equations. The choice of the specific geometry here should be viewed as a sample results. Based on the known quantitative unique continuation properties from, for instance, [15] and [9], it is expected that similar results remain valid for a larger class of convex domains, however, additional arguments are required for this (e.g., with respect to [15] additional shifts in the function spaces are necessary and with respect to [9] potentials have to be included in the estimates). Due to the associated technical difficulties, we postpone a further discussion of this to possible future work.

1.4. **Connection with the literature.** In order to put our results into a proper context, we recall some of the earlier literature on improved stability properties. Due to their ability to stabilize notoriously ill-posed inverse problems, the stabilization effects at high frequency which had first been established in [15] in the context of improved (interior) unique continuation properties were subsequently extended to improved unique continuation properties in various other geometric settings and other model equations [24, 23, 21, 51, 50, 10, 26, 22, 12, 13, 19, 20, 25, 28, 29, 5, 42, 6, 27]. The optimality of exponential $k$-dependences in unique continuation (in the form of three balls inequalities) was further established recently in [8] for the exact Helmholtz equation (which can be studied by investigating explicit behaviour
of Bessel functions). Earlier, in [31], the role of the geometry had already been highlighted for the closely connected wave equation, see also [33] for a systematic, microlocal argument for this.

Relying on these ideas further stability improvement results were also obtained for nonlinear inverse problems such as various variants of the Calderón problem. In this context, full data results were established in [27] for the Helmholtz equation with potential and in [42] for the acoustic equation. In recent work [35], this was extended to a partial data result for the Helmholtz equation with potential and impedance boundary conditions. Optimality of the improved stability estimates was discussed in a series of articles [17, 18, 16].

1.5. Outline of the remaining article. The remaining article is organised as follows: After briefly recalling some auxiliary results in Section 1.6, we turn to the quantitative unique continuation results without geometric assumptions in Section 2. In Section 3 a duality argument is used to transfer these into quantitative Runge approximation results. As an application we prove partial data stability for the Calderón problem for the acoustic equation with a priori information in a boundary layer in Section 4. Finally, in Section 5 we discuss improvements arising from convex geometries.

1.6. Notation and preliminaries. Before turning to the proofs of our main results we recall a number of auxiliary arguments and summarize our notation.

1.6.1. On spectral estimates. The following result contains a global estimate for the homogeneous Dirichlet problem depending on $\text{dist}(k^2, \Sigma_{V,q})$. It generalizes [7, Proposition 2] and together with the assumption (a1) allows us to invert the operator under consideration.

**Lemma 1.4.** Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $V$ and $q$ be as in (i)-(ii) in $\Omega$. Then there is a discrete set $\Sigma_{V,q} \subset \mathbb{R}$ such that for every $k^2 \notin \Sigma_{V,q}$, $k \geq 1$, there exists a unique solution $u \in H^1(\Omega)$ of

$$
(\Delta + k^2 q + V) u = f \text{ in } \Omega,
$$

$$
u = 0 \text{ on } \partial \Omega,
$$

where $f \in L^2(\Omega)$. In addition, there is a constant $C > 0$ depending on $\Omega$, $\sigma$ and $\|V\|_{L^\infty(\Omega)}$ such that

$$
\|u\|_{H^1(\Omega)} \leq C \left( 1 + \frac{k^3}{\text{dist}(k^2, \Sigma_{V,q})} \right) \|f\|_{L^2(\Omega)}.
$$

**Proof.** Recalling that zero is not a Dirichlet eigenvalue of $\Delta + V$ in $\Omega$, we consider the operator $T = (-\Delta - V)^{-1} M_q : H^1_0(\Omega) \to H^1_0(\Omega)$, where $M_q$ denotes the multiplication operator $M_q u = q u$. Then $T$ has eigenvalues $\alpha_n \in \mathbb{R}$ with $\alpha_n \to 0$ as $n \to \infty$. Let $\Sigma_{V,q} = \{\lambda_n = \alpha_n^{-1}\}_{n \in \mathbb{N}}$ and let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $L^2(\Omega)$ with $T e_n = \alpha_n e_n$.

Notice that (4) is equivalent to $(I - k^2 T) u = (-\Delta - V)^{-1} f =: h$, $u \in H^1_0(\Omega)$. If $k^2 \notin \Sigma_{V,q}$, by the Fredholm alternative, there is a unique solution $u$ to this problem. Moreover, we can write $u = \sum_{n \in \mathbb{N}} u_n e_n$, where $u_n = (u, e_n)_{L^2(\Omega)}$ is given by

$$
(1 - k^2 \alpha_n) u_n = h_n = (h, e_n)_{L^2(\Omega)} \quad \text{i.e.} \quad u_n = \frac{1}{1 - k^2 \alpha_n} h_n = \left( 1 + \frac{k^2}{\lambda_n - k^2} \right) h_n.
$$
Therefore,
\[ |u|_{L^2(\Omega)}^2 = \sum_{n \in \mathbb{N}} |u_n|^2 \leq \left( 1 + \frac{k^2}{\text{dist}(k^2, \Sigma_{V,q})} \right)^2 \sum_{n \in \mathbb{N}} |h_n|^2 = \left( 1 + \frac{k^2}{\text{dist}(k^2, \Sigma_{V,q})} \right)^2 \|h\|_{L^2(\Omega)}^2. \]
Taking into account that \( \|h\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \), we conclude
\[ \|u\|_{L^2(\Omega)}^2 \leq C \left( 1 + \frac{k^2}{\text{dist}(k^2, \Sigma_{V,q})} \right) \|f\|_{L^2(\Omega)}. \]
Finally, testing the equation with \( u \), we obtain \( \|\nabla u\|_{L^2(\Omega)} \leq C(1 + k)\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \), which in combination with the previous estimate yields the desired result. \( \square \)

1.6.2. Notation. For \( s \in \mathbb{R} \), the whole space Sobolev spaces are denoted by
\[ H^s(\mathbb{R}^n) := \{ f \in \mathcal{S}'(\mathbb{R}^n) : \|(1 + |\cdot|^2)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^n)} < \infty \}, \]
where
\[ \mathcal{F} f(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi} dx \]
denotes the Fourier transform.

Let \( \Omega \subset \mathbb{R}^n \) be an open set, then we define
\[ H^s(\Omega) := \{ f|_{\Omega} : f \in H^s(\mathbb{R}^n) \} \]
equipped with the quotient topology,
\[ \tilde{H}^s(\Omega) := \text{closure of } C^\infty_c(\Omega) \text{ in } H^s(\mathbb{R}^n). \]
For any \( s \in \mathbb{R} \) these spaces satisfy
\[ (H^s(\Omega))^* = H^{-s}(\Omega), \quad (\tilde{H}^s(\Omega))^* = H^{-s}(\Omega). \]
In addition, for \( \Gamma \subset \partial \Omega \), we set
\[ \tilde{H}^{s,2}(\Gamma) := \{ f \in H^{s,2}(\partial \Omega) : \text{supp } f \subset \Gamma \}, \]
which is a closed subspace of \( H^{s,2}(\partial \Omega) \) and its dual space may be identified with \( H^{-s,2}(\Gamma) \). We denote by \( \langle \cdot, \cdot \rangle_{L^2(\Omega)} \) the inner product in \( L^2(\Omega) \) and also use the abbreviation \( \langle \cdot, \cdot \rangle_{\partial \Omega} \) to denote \( \langle \cdot, \cdot \rangle_{L^2(\partial \Omega)} \).

Furthermore, for \( r > 0 \) and \( x_0 \in \mathbb{R}^n \), we denote the \( n \)-dimensional ball by \( B_r(x_0) \subset \mathbb{R}^n \) and we define the cylindrical \( (n + 1) \)-dimensional domain \( Q_r(x_0) := B_r(x_0) \times (-r, r) \subset \mathbb{R}^{n+1} \). In addition, given an open set \( \Omega \subset \mathbb{R}^n \), \( B_r^x(x_0) := B_r(x_0) \cap \Omega \).

2. Quantitative unique continuation. In this section we begin our analysis of the Runge approximation properties for the acoustic Helmholtz equation by proving quantitative unique continuation results without geometric assumptions on the underlying domains. Here we only assume the validity of the conditions (i)-(ii) (not necessarily the condition (a1)). The Runge approximation properties will be deduced as dual results in the next section. Since in this case exponential losses in \( k \) are expected to be unavoidable (see [8] for a proof of this in the closely related three balls inequalities), we do not prove these estimates by carefully tracking the \( k \)-dependence in the original equations but by embedding these equations into a family of elliptic equations without a large parameter but in an additional dimension. This is achieved by passing from \( u(x) \) to \( \tilde{u}(x,t) = e^{kt}u(x) \). We emphasize
that this is a well-known procedure (see for instance [36] and the references therein). The corresponding unique continuation properties follow from well-known results in the literature (e.g. [2]). The main novelty of this first part of our article – in which we do not pose geometric assumptions on our domains – are the quantitative (in \(k\)) Runge approximation results and the application of these to the stability of the partial data inverse problem which are deduced in the next sections.

Formulated for the original function \(u\) the unique continuation properties read as follows:

**Proposition 2.** Let \(\Omega\) be an open, bounded, connected Lipschitz domain and let \(\Gamma \subset \partial \Omega\) be a non-empty relatively open subset. Let \(V\) and \(q\) be as in (i)-(ii) in \(\Omega\). Let \(u \in H^1(\Omega)\) be a solution to

\[
\Delta u + k^2 qu + Vu = 0 \quad \text{in } \Omega,
\]

and let \(M, \eta\) be such that

\[
\|u\|_{H^1(\Omega)} \leq M, \\
\|u\|_{H^{1/2}(\Gamma)} + \|\partial_\nu u\|_{H^{-1/2}(\Gamma)} \leq \eta.
\]

Then there exist a parameter \(\mu \in (0, 1)\) and a constant \(C > 1\) depending on \(n, \Omega, \Gamma, \|V\|_{L^\infty(\Omega)}, \sigma\) and \(\|q\|_{C^{0,1}(\Omega)}\) such that

\[
\|u\|_{L^2(\Omega)} \leq Ck \left|\log \left( \frac{\eta}{M + \eta} \right) \right|^{-\mu} (M + \eta).
\]

In addition, if \(G\) is a bounded Lipschitz domain with \(G \Subset \Omega\), then there exist a parameter \(\nu \in (0, 1)\) and a constant \(C > 1\) (depending on \(n, \Omega, G, \Gamma, \|V\|_{L^\infty(\Omega)}, \sigma\) and \(\|q\|_{C^{0,1}(\Omega)}\)) such that

\[
\|u\|_{L^2(G)} \leq Ce^{Ck} \left( \frac{\eta}{M + \eta} \right)^\nu (M + \eta).
\]

As an auxiliary ingredient, the proof of Proposition 2 uses the following three-balls (boundary-bulk) inequalities derived from [2]:

**Lemma 2.1.** Under the same assumptions as in Proposition 2, there exist a parameter \(\alpha \in (0, 1)\) and a constant \(C > 1\) depending on \(\Omega, \|V\|_{L^\infty(\Omega)}, \sigma\) and \(\|q\|_{C^{0,1}(\Omega)}\) such that

\[
\|u\|_{L^2(B_r(x_0))} \leq Ce^{Ck} \|u\|_{L^2(B_{2r}(x_0))}^{1-\alpha} \|u\|_{L^2(B_{r/2}(x_0))}^\alpha,
\]

where \(x_0 \in \Omega\) and \(r > 0\) are such that \(B_{4r}(x_0) \subset \Omega\).

In addition, there exist a parameter \(\alpha_0 \in (0, 1)\) and a constant \(C > 1\) depending on \(\Omega, \Gamma; \|V\|_{L^\infty(\Omega)}, \sigma\) and \(\|q\|_{C^{0,1}(\Omega)}\) such that

\[
\|u\|_{L^2(B_r^+(x_0))} \leq Ce^{Ck} \left( \|u\|_{L^2(B_{2r}^+(x_0))} + \eta \right)^{1-\alpha_0} \eta^{\alpha_0},
\]

where \(x_0 \in \Gamma\) and \(r > 0\) are such that \(B_{4r}(x_0) \cap (\partial \Omega \setminus \Gamma) = \emptyset\) and \(B_r^+(x_0) = B_r(x_0) \cap \Omega\).

In order to invoke the quantitative uniqueness results for elliptic equations without a large parameter, we pass to equations in an additional dimension which is a well-known method in quantitative uniqueness for eigenfunctions [36, 38]. We remark that in the setting of Helmholtz equations where \(q = 1\), the \(k\)-dependence in this result is optimal as proved in [8].
Proof of Lemma 2.1. Let \( \tilde{\Omega} := \Omega \times (-d, d) \), with \( d = \text{diam}(\Omega) \). Let \( \tilde{u} \in H^1(\tilde{\Omega}) \) be a solution to
\[
(\Delta + q(x)\partial_t^2 + V(x))\tilde{u}(x, t) = 0 \quad \text{for} \quad (x, t) \in \tilde{\Omega},
\]
where \( \Delta \) denotes the Laplacian in \( x \). Recalling the assumption (ii), we observe that the operator \( \Delta + q(x)\partial_t^2 \) is elliptic with \( C^{0,1} \) coefficients. Hence, the results from [2] are applicable. Using the notation \( Q_r(x_0) = B_r(x_0) \times (-r, r) \), by [2, Theorem 1.10], there exist \( C > 1 \) and \( \alpha \in (0, 1) \) depending on \( \Omega, \|V\|_{L^\infty(\Omega)}, \sigma \) and \( \|q\|_{C^{0,1}(\Omega)} \) such that
\[
\|\tilde{u}\|_{L^2(Q_r(x_0))} \leq C\|\tilde{u}\|_{L^2(Q_{2r}(x_0))} \|\tilde{u}\|_{L^2(Q_{1/2}(x_0))},
\]
where \( B_{4r}(x_0) \subset \Omega \).

We now consider the particular solution \( \tilde{u}(x, t) = e^{kt}u(x) \), with \( u(x) \) satisfying (5). Then (8) follows from (10) together with the observation that
\[
2re^{-kd}\|u\|_{L^2(B_r(x_0))} \leq \|\tilde{u}\|_{L^2(Q_r(x_0))} \leq 2re^{kd}\|u\|_{L^2(B_r(x_0))}.
\]
Inserting this into (10) concludes the proof of (8).

In order to obtain (9), we observe that similarly, by [2, Theorem 1.7], there exist \( C > 1 \) and \( \alpha_0 \in (0, 1) \) depending on \( \Omega, \Gamma, \|V\|_{L^\infty(\Omega)}, \sigma \) and \( \|q\|_{C^{0,1}(\Omega)} \) such that
\[
\|\tilde{u}\|_{L^2(Q_r(x_0) \cap \tilde{\Omega})} \leq C\left(\|\tilde{u}\|_{L^2(Q_{2r}(x_0))} + \tilde{\eta}\right)^{1-\alpha_0}\tilde{\eta}^{\alpha_0}.
\]
Here \( \tilde{\Gamma} = \Gamma \times (-d, d) \) and
\[
\|u\|_{H^{1/2}(\tilde{\Gamma})} + \|\partial_t u\|_{H^{-1/2}(\tilde{\Gamma})} \leq \tilde{\eta}.
\]
The requirement \( \text{dist}(Q_r(x_0) \cap \tilde{\Omega}, \partial\tilde{\Omega} \setminus \tilde{\Gamma}) > 0 \) is satisfied since \( B_{4r}(x_0) \cap (\partial\tilde{\Omega} \setminus \tilde{\Gamma}) = \emptyset \).

Notice that on the lateral boundary \( \tilde{\Gamma} \) the normal derivative does not have any contribution in the \( t \) direction. Therefore, using the definition of the weak form of \( \partial_t \tilde{u} \) in terms of the bilinear form associated with (5), choosing \( \tilde{u}(x, t) = e^{kt}u(x) \) and \( \tilde{\eta} = Ce^{C\tilde{\eta}} \), the estimate (9) follows as above.

With Lemma 2.1 available, we next address the proof of Proposition 2.

Proof of Proposition 2. Let us define
\[
W_\epsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \epsilon\},
\]
\[
\Omega_\epsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \epsilon\},
\]
for \( \epsilon \in (0, \epsilon_0) \), for some \( \epsilon_0 < 1 \) such that \( \Omega_{\epsilon_0} \) is connected. We argue in three steps, estimating \( u \) separately on \( W_\epsilon \) and on \( \Omega_\epsilon \), and combining these bounds by means of a final optimization step (in \( \epsilon \)).

Step 1. Estimate on \( W_\epsilon \). By the Hölder and Sobolev inequalities we have
\[
\|u\|_{L^2(W_\epsilon)} \leq C\|u\|_{L^q(W_\epsilon)} \leq C\|u\|_{H^1(\Omega)} \leq C\|u\|_{L^2(\Omega)},
\]
with \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \) and the constant \( C > 0 \) depending on \( \Omega \).

Step 2. Estimate on \( \Omega_\epsilon \). We use Lemma 2.1 to propagate the smallness of \( \eta \) to \( \|u\|_{L^2(\Omega_\epsilon)} \). Firstly, we transport the information from the boundary to the interior of \( \Omega_\epsilon \). Let \( x_0 \in \Gamma \) and \( r_0 > 0 \) such that \( B_{4r_0}(x_0) \cap (\partial\Omega \setminus \Gamma) = \emptyset \). Then by (9) it holds
\[
\|u\|_{L^2(B_{r_0}(x_0))} \leq Ce^{C\tilde{\eta}(M + \eta)^{1-\alpha_0}\eta^{\alpha_0}}.
\]
Once we have reached the interior, we iterate (8) along a chain of balls which cover \( \Omega_\epsilon \) and such that \( B_{4r}(x) \subset \Omega \). This implies that it is necessary to iterate
By (14) we have

\begin{equation}
\|u\|_{L^2(\Omega_\epsilon)} \leq Ce^{\frac{C_k}{\log \epsilon}} (M + \eta)^{1-\alpha_0 \alpha N} \eta^{\alpha_0 \alpha N}.
\end{equation}

**Step 3. Optimization.** Combining (12) and (13), we obtain

\begin{equation}
\|u\|_{L^2(\Omega_\epsilon)} \leq C \left( \epsilon \right) + e^{C_k} \left( \frac{\eta}{M + \eta} \right)^{C_1} (M + \eta),
\end{equation}

where the constants depend on $\Omega, \Gamma, \|V\|_{L^\infty(\Omega)}, \sigma$ and $\|q\|_{C^{0,1}(\Omega)}$. Abbreviating

\begin{equation}
\tilde{\epsilon} := \epsilon^{C_2}, \quad \tilde{\eta} := \left( \frac{\eta}{\eta + M} \right)^{C_1}, \quad \gamma := \frac{1}{C_{2p}},
\end{equation}

we thus seek to optimize the expression

\[ F(\tilde{\epsilon}, \tilde{\eta}) := \tilde{\epsilon}^\gamma + e^{C_k \tilde{\eta}^\gamma} \]

by choosing $\tilde{\epsilon} = \tilde{\epsilon}(\tilde{\eta}) > 0$ appropriately. Setting $\epsilon := \frac{1}{(\log \tilde{\eta})^\beta} + \frac{C_k}{\log \tilde{\eta}} > 0$ for some $\beta \in (0, 1)$, we obtain

\[ |F(\tilde{\epsilon}, \tilde{\eta})| \leq \left( \frac{1}{(\log \tilde{\eta})^\beta} + \frac{k}{(\log \tilde{\eta})^\gamma} \right)^{\gamma} + e^{-|\log \tilde{\eta}|^{1-\beta}} \leq \frac{Ck^{\gamma}}{|\log \tilde{\eta}|^{\beta \gamma}} + \frac{1}{|\log \tilde{\eta}|^{1-\beta}}. \]

By (14) we have $\gamma < 1$ provided $p > 2$ in (12) is chosen big enough. This implies $k^{\gamma} < k$ for $k \geq 1$. Choosing $\beta = \frac{1}{1+\gamma}$ we infer (6) with $\mu = \beta \gamma < 1$. The bound (7) follows directly from Step 2 for a suitable choice of $\epsilon$ with $\nu = \alpha_0 \alpha N(\epsilon)$. \hfill \Box

### 3. Proof of the Runge approximation Theorems 1.1 and 1.2.

This section is devoted to the proofs of the quantitative Runge approximation results of Theorems 1.1 and 1.2. This relies on duality arguments and the quantitative unique continuation results from the previous section. In addition to the assumptions (i) and (ii), we will now always also assume the condition (a1) in $\Omega_2$ throughout the whole section in order to avoid solvability issues.

**Proposition 3.** Let $\Omega_1 \Subset \Omega_2$ and $\Gamma \subset \partial \Omega_2$ be as in Theorem 1.1. Let $V$ and $q$ satisfy the assumptions (i)-(ii) in $\Omega_2$. Let $u \in H^1(\Omega_2)$ be the unique solution to

\begin{equation}
\begin{aligned}
\Delta u + k^2 qu + Vu &= v^\sharp \Omega_1 \quad \text{in } \Omega_2, \\
u &= 0 \quad \text{on } \partial \Omega_2,
\end{aligned}
\end{equation}

with $v \in L^2(\Omega_1)$ and $k \geq 1$ satisfying the condition (a1). Then there exist a parameter $\nu_0 \in (0, 1)$ and a constant $C > 1$ depending on $n, \Omega_2, \Omega_1, \Gamma, \|V\|_{L^\infty(\Omega_2)}, \sigma$ and $\|q\|_{C^{0,1}(\Omega_2)}$ such that

\begin{equation}
\|u\|_{H^1(\Omega_2 \setminus \Omega_1)} \leq C k^{\nu_0 + 4} \left| \log \left( C \|\partial_{v} u\|_{H^{-1/2}(\Gamma)} \right) \right|^{-\nu_0} \|v\|_{L^2(\Omega_1)}.
\end{equation}

In addition, if $G$ is a bounded Lipschitz domain with $G \Subset \Omega_2 \setminus \Omega_1$, then there exist a parameter $\nu_0 \in (0, 1)$ and a constant $C > 1$ depending on $n, \Omega_2, \Omega_1, G, \Gamma, \|V\|_{L^\infty(\Omega_2)}$, $\sigma$ and $\|q\|_{C^{0,1}(\Omega_2)}$ such that

\begin{equation}
\|u\|_{H^1(G)} \leq C e^{C_k} \left( \frac{\|\partial_{v} u\|_{H^{-1/2}(\Gamma)}}{\|v\|_{L^2(\Omega_1)}} \right)^{\nu_0} \|v\|_{L^2(\Omega_1)}.
\end{equation}
Proof. We start by estimating \(\|u\|_{H^1(\Omega_0)}\) in terms of \(v\). By Lemma 1.4, there is a constant \(C > 1\) such that

\[
\|u\|_{H^1(\Omega_0)} \leq C \left( 1 + \frac{k^3}{\text{dist}(k^2, \Sigma_{V,q})} \right) \|v\|_{L^2(\Omega_1)} \leq C k^{n+1} \|v\|_{L^2(\Omega_1)},
\]

where for the last inequality we have used the assumption (a1).

Since \(u\) satisfies (1) in \(\Omega = \Omega_2 \setminus \overline{\Omega}_1\), which is connected, the results of Proposition 2 hold with

\[
\eta = \|\partial_r u\|_{H^{-\frac{1}{2}}(\Gamma)}, \quad M = C k^{n+1} \|v\|_{L^2(\Omega_1)}.
\]

In order to promote (6) and (7) to the gradient, we argue similarly as in Proposition 2. We consider the subsets \(W_\epsilon\) and \(\Omega_\epsilon\) defined in (11) with \(\Omega = \Omega_2 \setminus \overline{\Omega}_1\).

**Step 1’. Estimate on \(W_\epsilon\).** By the Hölder inequality

\[
\|\nabla u\|_{L^2(W_\epsilon)} \leq C k^{\frac{1}{p}} \|\nabla u\|_{L^p(\Omega_2)},
\]

where \(\frac{1}{p} + \frac{1}{q} = \frac{1}{2}\). By [41, Theorem 1] (together with [30, Theorem 0.5] for the admissibility of the Lipschitz domain) there exists \(q > 2\) such that

\[
\|\nabla u\|_{L^q(\Omega_0)} \leq C \|v\|_{L^q(\Omega_0)} - k^2 \sigma u_0 - V u_0 \|_{L^2(\Omega_2)} \leq C \left( \|v\|_{L^2(\Omega_1)} + (k^2 \sigma + \|V\|_{L^\infty(\Omega_2)}) \|u\|_{L^2(\Omega_2)} \right).
\]

In addition, from energy estimates for which we consider the weak version of the equation (15) applied with \(u\) itself as a test function, we obtain

\[
k\|u\|_{L^2(\Omega_2)} \leq C(\|u\|_{H^1(\Omega_2)} + \|v\|_{L^2(\Omega_1)}) \leq CM,
\]

with \(C\) depending on \(\sigma\) and \(\|V\|_{L^\infty(\Omega_2)}\). Therefore,

\[
\|\nabla u\|_{L^2(W_\epsilon)} \leq C k^{\frac{1}{p}} k M.
\]

**Step 2’. Estimate on \(\Omega_\epsilon\).** Let \(\chi\) be a smooth cut-off function supported in \(\Omega_{\epsilon/2}\) with \(\chi = 1\) in \(\Omega_\epsilon\) and \(\|\nabla \chi\| \leq c \epsilon^{-1}\). We then obtain the following Caccioppoli inequality by testing the equation with \(\chi^2 u\):

\[
\|\nabla u\|_{L^2(\Omega_\epsilon)} \leq C(\epsilon^{-1} + \|V\|^2_{L^\infty(\Omega_2)} + k \sigma) \|u\|_{L^2(\Omega_{\epsilon/2})} \leq C k \epsilon^{-1} \|u\|_{L^2(\Omega_{\epsilon/2})}.
\]

Inserting the estimate (7) with explicit \(\epsilon\) dependence coming from the Step 2 in the proof of Proposition 2, we infer

\[
\|\nabla u\|_{L^2(\Omega_\epsilon)} \leq C k \epsilon^{-1} e^{C k} \left( \frac{\eta}{M + \eta} \right) c_1^{c_2} (M + \eta).
\]

**Step 3’. Optimization.** Combining the previous two steps we obtain

\[
\|\nabla u\|_{L^2(\Omega_{\epsilon} \setminus \overline{\Omega}_1)} \leq C k \left( \epsilon^{\frac{1}{p}} + \epsilon^{-1} e^{C k} \left( \frac{\eta}{M + \eta} \right) c_1^{c_2} \right) (M + \eta).
\]

Optimizing in \(\epsilon\) as in the Step 3 in the proof of Proposition 2 yields

\[
\|\nabla u\|_{L^2(\Omega_{\epsilon} \setminus \overline{\Omega}_1)} \leq C k^2 \left| \log \left( \frac{\eta}{M + \eta} \right) \right|^{-\mu_0} (M + \eta)
\]

for a suitable \(\mu_0 \in (0, 1)\).
Introducing (18) and taking into account that by (19)
\[
\eta = \|\partial_\nu u\|_{H^{-12}(\Gamma)} \leq C(k^2\|u\|_{L^2(\Omega_2)} + \|\nabla u\|_{L^2(\Omega_2)} + \|v\|_{L^2(\Omega_1)}) \\
\leq CkM \leq Ck^{n+2}\|v\|_{L^2(\Omega_1)},
\]
we infer (16).

Estimate (17) follows from the Caccioppoli inequality in Step 2' for suitable choice of \(\epsilon\) together with the previous estimate for \(\eta\). \(\square\)

Using the results from Proposition 3 we now address the proof of Theorem 1.1:

Proof of Theorem 1.1. We seek to show that for any \(\alpha > 0\) there exists a solution \(u_\alpha\) to
\[
(\Delta + k^2q + V)u_\alpha = 0 \text{ in } \Omega_2
\]
with
\[
\|u_\alpha - v\|_{L^2(\Omega_1)} \leq C(\alpha, k)\|v\|_{H^1(\Omega_1)}, \quad \|u_\alpha\|_{H^{12}(\partial \Omega_2)} \leq \frac{1}{\alpha}\|v\|_{L^2(\Omega_2)}.
\]

Let \(X\) be the closure of \(\{u \in H^1(\Omega_1) \mid (\Delta + k^2q + V)u = 0 \text{ in } \Omega_1\}\) in \(L^2(\Omega_1)\). We then define
\[
A: \tilde{H}^{12}(\Gamma) \to X, \quad g \mapsto Ag = u|_{\Omega_1},
\]
where \(u \in H^1(\Omega_2)\) is the solution to (1) in \(\Omega_2\) satisfying the boundary condition \(u|_{\partial \Omega_2} = g \in \tilde{H}^{12}(\Gamma)\). We denote by \(A^*\) the Hilbert space adjoint of \(A\), which maps
\[
A^*: X \to \tilde{H}^{12}(\Gamma), \quad u \mapsto A^*u = R(\partial_\nu u|_{\Gamma}),
\]
where \(R\) is the Riesz isomorphism \(R: H^{-12}(\Gamma) \to \tilde{H}^{12}(\Gamma)\) and \(w \in H^1(\Omega_2)\) satisfies
\[
(\Delta + k^2q + V)w = \mathbb{1}_{\Omega_1}u \text{ in } \Omega_2, \quad w = 0 \text{ on } \partial \Omega_2.
\]
By [47, Lemma 4.1], \(A\) is a compact, injective operator with dense range in \(X\) and applying the spectral theorem to \(A^*A\) yields an orthonormal basis of eigenvectors \(\{\phi_j\}_{j=1}^\infty\) for \(\tilde{H}^{12}(\Gamma)\) and a sequence of positive, decreasing eigenvalues \(\{\mu_j\}_{j=1}^\infty\) with
\[
A^*\phi_j = \mu_j\phi_j.
\]
Then, setting \(\psi_j := \mu_j^{-1/2}A\phi_j\) yields an orthonormal basis \(\{\psi_j\}_{j=1}^\infty\) of \(X\). In particular, we have
\[
\|A^*\psi_j\|_{H^{12}(\partial \Omega_2)} \leq \mu_j^{1/2}. \quad (21)
\]
Returning to our setting, we notice that \(v \in X\), hence it admits a unique decomposition in the orthonormal basis \(v = \sum_{j=1}^\infty \beta_j \psi_j\). For \(\alpha > 0\), we define
\[
v_\alpha := \sum_{\alpha \geq \mu_j^{1/2}} \beta_j \psi_j
\]
and let \(w_\alpha\) be the solution of
\[
(\Delta + k^2q + V)w_\alpha = \mathbb{1}_{\Omega_1}v_\alpha \text{ in } \Omega_2, \quad w_\alpha = 0 \text{ on } \partial \Omega_2.
\]
Here the notation $\alpha \geq \mu_j^{1/2}$ is an abbreviation for the set $\{j \in \mathbb{N} : \alpha \geq \mu_j^{1/2}\}$. By (21), it holds

$$\|\partial_\nu w_\alpha\|_{H^{-1/2}(\Gamma)} = \|A^*v_\alpha\|_{H^{1/2}(\partial \Omega_2)} \leq \alpha \|v_\alpha\|_{L^2(\Omega_1)}.$$  

(22)

Now, we define $u_\alpha$ as the solution to (1) on $\Omega_2$ satisfying the boundary condition $u_\alpha = g_\alpha$ on $\partial \Omega_2$, with $g_\alpha = \sum_{\alpha \leq \mu_j^{1/2}} \beta_j \mu_j^{-1/2} \phi_j$. Note that at the boundary we have by the previous considerations

$$\|u_\alpha\|_{H^{1/2}(\partial \Omega_2)} = \|g_\alpha\|_{H^{1/2}(\partial \Omega_2)} = \left\|\sum_{\alpha \leq \mu_j^{1/2}} \beta_j \mu_j^{-1/2} \phi_j\right\|_{H^{1/2}(\partial \Omega_2)}$$

$$= \sum_{\alpha \leq \mu_j^{1/2}} \frac{\beta_j^2}{\mu_j} \leq \frac{1}{\alpha^2} \|v\|_{L^2(\Omega_1)}.$$

In addition, notice that

$$u_\alpha|_{\Omega_1} = Ag_\alpha = A \left(\sum_{\alpha \leq \mu_j^{1/2}} \beta_j \mu_j^{-1/2} \phi_j\right) = \sum_{\alpha \leq \mu_j^{1/2}} \beta_j \psi_j = v - v_\alpha.$$

Thus, it remains to obtain an explicit dependence on $\alpha$ and $k$ in

$$\|u_\alpha - v\|_{L^2(\Omega_1)} = \|v_\alpha\|_{L^2(\Omega_1)} \leq C(\alpha, k)\|v\|_{H^1(\Omega_1)}.$$

Orthogonality considerations show

$$\|v_\alpha\|_{L^2(\Omega_1)}^2 = \langle v, \partial_\nu w_\alpha \rangle_{\partial \Omega_1} - \langle \partial_\nu v, w_\alpha \rangle_{\partial \Omega_1}.$$

Using trace estimates for the solutions we find

$$\|v_\alpha\|_{L^2(\Omega_1)}^2 \leq C(1 + \|V\|_{L^\infty(\Omega_2)} + k^2\sigma)\|v\|_{H^1(\Omega_1)}\|w_\alpha\|_{H^1(\Omega_2 \setminus \tilde{\Omega}_1)}$$

(23)

with some constant $C > 0$ depending on $\Omega_1, \Omega_2 \setminus \Omega_1$. Using (16) to estimate the norm of $w_\alpha$ in (24) yields

$$\|v_\alpha\|_{L^2(\Omega_1)}^2 \leq Ck^{n+6} \left[\log\left(\frac{\|\partial_\nu w_\alpha\|_{H^{-1/2}(\Gamma)}}{\|v_\alpha\|_{L^2(\Omega_1)}}\right)\right]^{-\mu_0} \|v\|_{H^1(\Omega_1)}^{-\mu_0} \|w_\alpha\|_{H^1(\Omega_1)}.$$

Finally, dividing by $\|v_\alpha\|_{L^2(\Omega_1)}$, recalling (22) and using monotonicity, we arrive at

$$\|v_\alpha\|_{L^2(\Omega_1)} \leq Ck^{n+6} \log(C\alpha)|^{-\mu_0} \|v\|_{H^1(\Omega_1)}.$$}

We choose $\alpha < 1$ so that $Ck^{n+6}\log(C\alpha)|^{-\mu_0} = \epsilon$, i.e.

$$\frac{1}{\alpha} = Ce^{Ck^{n+6}\epsilon}$$

with $s = \frac{n+6}{\mu_0}$ and $\mu = \mu_0^{-1}$. This concludes the proof.

Relying on similar ideas, we also obtain the bounds from Theorem 1.2:

**Proof of Theorem 1.2.** We define $v_\alpha$ and $w_\alpha$ as in the proof of Theorem 1.1 with $v = \tilde{v}|_{\Omega_1}$. Equations (23) and (24) can be slightly modified to read

$$\|v_\alpha\|_{L^2(\Omega_1)}^2 = \langle \tilde{v}, \partial_\nu w_\alpha \rangle_{\partial \Omega_1} - \langle \partial_\nu \tilde{v}, w_\alpha \rangle_{\partial \Omega_1} \leq Ck^2\|\tilde{v}\|_{H^1(\tilde{\Omega}_1)}\|w_\alpha\|_{H^1(G)},$$

which completes the proof.

\[\square \]
where $G = \Omega'_2 \setminus \Omega_1 \subset \Omega_2 \setminus \Omega_1$ with $\Omega'_2 \subset \Omega_2$. Arguing as above and using the quantitative unique continuation result (17), we obtain
\[
\|u_\alpha - u\|_{L^2(\Omega_1)} = \|v_\alpha\|_{L^2(\Omega_1)} \leq C e^{Ck} \left( \frac{\|\partial_\nu u_\alpha\|_{H^{-1/2}(\partial \Omega_2)}}{\|u_\alpha\|_{L^2(\Omega_1)}} \right)^{\nu_0} \|v\|_{H^1(\Omega_1)} \\
\leq C e^{Ck} \alpha^{\nu_0} \|v\|_{H^1(\Omega_1)}.
\]
Choosing $\alpha$ so that $e^{Ck} \alpha^{\nu_0} = \epsilon$, i.e. $\frac{1}{\alpha} = C \left( \frac{Ck}{\epsilon} \right)^{\frac{1}{\nu_0}}$, the result follows with $\nu = \nu_0^{-1}$.

3.1. Optimality of the estimates in Theorem 1.2. In order to infer the optimality of the quantitative Runge approximation results in the parameter $\epsilon$, we consider the case $q = 1$ (i.e. the case of the Helmholtz equation). We remark that optimality results in $k$ for three balls inequalities were recently obtained in [8].

Lemma 3.1. Let $\Omega_2 = B_2, \Omega_1 = B_{2r}$ and $\Gamma = \partial B_1$. For fixed $k \geq 1$, there exists $N = N(k) \in \mathbb{N}$ and a sequence $(\nu_\ell)_{\ell \geq N}$ of solutions to $(\Delta + k^2)\nu_\ell = 0$ in $\Omega_1$ with $|v_\ell|_{H^1(\Omega_2)} = 1$ such that for any solution $u$ of $(\Delta + k^2)u = 0$ in $\Omega_2$ with $\|v_\ell - u\|_{L^2(\Omega_1)} \leq (2\lambda_\ell^2 + \lambda_\ell)\epsilon$ we have $\|u\|_{H^{1/2}(\Gamma)} \geq C k$.

Proof. Arguing by separation of variables, we obtain that any solution $u \in H^1(B_1)$ of $(\Delta + k^2)u = 0$ can be written with respect to the variables $r = |x|, \theta = \frac{x}{|x|} \in \mathbb{S}^{n-1}$ as
\[
u_\ell = u(r, \theta) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{N_\ell} c_{\ell m} R_\ell(k r) \psi_{\ell m}(\theta),
\]
where $\{\psi_{\ell m}\}_{\ell=0}^{\infty}$ is an orthonormal basis of $L^2(\mathbb{S}^{n-1})$ consisting of the spherical harmonics of degree $\ell$ and
\[
R_\ell(r) = r^{1-n/2} J_{\ell+n/2-1}(r),
\]
with $J_\alpha$ denoting the Bessel functions.

We consider $g_\ell(x) = R_\ell(k r) \psi_{\ell 1}(\theta)$ and define $v_\ell = \alpha_\ell g_\ell$ with $\alpha_\ell = \|g_\ell\|_{L^2(\Omega_1)}^{-1}$. Then, we may write $u = c v_\ell + w$, where $(w, g_\ell)_{L^2(B_1)} = 0$ and
\[
c \alpha_\ell = \|g_\ell\|_{L^2(B_1)}^{-2} (u, g_\ell)_{L^2(B_1)}.
\]
Therefore,
\[
u_\ell|_{\partial B_1} = c \alpha_\ell R_\ell(k) \psi_{\ell 1}(\theta) + \omega(\theta)
\]
with $(\omega, \psi_{\ell 1})_{L^2(\partial B_1)} = 0$ and $\omega(\theta) = w(x)|_{\partial B_1}$.

We are interested in estimating $\|u\|_{H^{1/2}(\Gamma)}$ from below. If we assume $\|u - v_\ell\|_{L^2(\Omega_1)} < \epsilon$, then $|c - 1| \alpha_\ell \|g_\ell\|_{L^2(\Omega_1)} < \epsilon$. Therefore,
\[
\|u\|_{H^{1/2}(\Gamma)} \geq (1 + \lambda_\ell^2)^{\frac{1}{2}} |c \alpha_\ell| R_\ell(k) \geq \frac{\lambda_\ell}{2} \left( |\alpha_\ell| - \epsilon \|g_\ell\|_{L^2(\Omega_1)}^{-1} \right) |R_\ell(k)| \geq \frac{\lambda_\ell}{2} \left( |\alpha_\ell| - \epsilon \|g_\ell\|_{L^2(\Omega_1)}^{-1} \right) |R_\ell(k)|,
\]
where $\lambda_\ell = \ell(\ell + n - 2)$.

Using [43, 10.22.27], we can estimate the $L^2(\Omega_1)$ norm of $g_\ell$ as follows:

$$
\|g_\ell\|_{L^2(\Omega_1)}^2 = \int_0^\infty R_\ell^2(kr)r^{n-1}dr = k^{-n} \int_0^\infty tJ_{\ell+\frac{n}{2}-1}(t)dt
$$

$$
= 2k^{-n} \sum_{m=0}^\infty \left( \ell + \frac{n}{2} + 2m \right) J_{\ell+\frac{n}{2}+2m}(\frac{k}{2}) \geq k^{-n}(2\ell + n)J_{\ell+\frac{n}{2}}^2(\frac{k}{2}).
$$

For the norm of the gradient, using integration by parts and the equation, we obtain

$$
\|\nabla g_\ell\|_{L^2(\Omega_1)}^2 = \int_{\partial B_{r^2}} g_\ell \partial_r g_\ell dH^{n-1}(\theta) - \int_{B_{r^2}} g_\ell \Delta g_\ell dx
$$

$$
= R_\ell \left( \frac{k}{2} \right) \partial_r R_\ell \left( \frac{k}{2} \right) + k^2 \|g_\ell\|_{L^2(\Omega_1)}^2.
$$

We next collect some properties of Bessel functions from [44] for $x \in (0, 1)$ and $\alpha > 0$:

\begin{align}
\text{(26) } & \quad 1 \leq \frac{J_\alpha(\alpha x)}{x^\alpha J_\alpha(\alpha)} \leq e^{\alpha(1-x)}, \quad 0 < \frac{1}{x} - \frac{J_\alpha'(\alpha x)}{J_\alpha(\alpha x)} < 1, \quad \frac{J_\alpha(\alpha x)}{J_{\alpha+1}(\alpha x)} < \frac{2\alpha + 2}{\alpha x}. \\
\end{align}

By the second estimate in (26) and the fact that $J_\alpha(\alpha x) > 0$ (e.g. [43, 10.14.2] together with the first estimate in (26) or [43, 10.14.7]), we have that $J_\alpha(\alpha x)$ is monotonously increasing for $x \in (0, 1)$. Moreover, we know [43, 10.19.1] that for $z \neq 0$ fixed

\begin{align}
\text{(27) } & \quad J_\alpha(z) \sim \frac{1}{\sqrt{2\pi}z} \left( \frac{e z}{2\alpha} \right)^\alpha, \quad \alpha \to \infty.
\end{align}

We assume from now on that $\ell + \frac{n}{2} - 1 > k$, so the previous estimates (26) can be applied. In particular, due to the monotonicity

$$
\|g_\ell\|_{L^2(\Omega_1)} = \left( \int_0^\infty R_\ell^2(kr)r^{n-1}dr \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2}} R_\ell \left( \frac{k}{2} \right),
$$

Inserting the previous estimates on $g_\ell$ into (25) yields

$$
\|u\|_{H^2(\Gamma)} \geq \ell^\frac{1}{2} \left( \frac{R_\ell(k)}{R_\ell(\frac{k}{2})} \right)^{\frac{1}{2}} \left( 1 + \frac{\alpha R_\ell(\frac{k}{2})}{R_\ell(k)} \right)^{\frac{1}{2}} - \epsilon \frac{R_\ell(k)k^\frac{n}{2}}{(2\ell + n)^\frac{1}{2}J_{\ell+\frac{1}{2}}(\frac{k}{2})},
$$

$$
= \ell^\frac{1}{2} \frac{J_\alpha(k)}{J_\alpha(\frac{k}{2})} \left( \frac{2^{1-\frac{\alpha}{2}}}{1 + k + \frac{\alpha R_\ell(\frac{k}{2})}{R_\ell(k)}} \frac{k}{2} - \epsilon \frac{J_\alpha(\frac{k}{2})}{(2\ell + n)^\frac{1}{2}J_{\alpha+1}(\frac{k}{2})} \right),
$$

where $\alpha = \ell + \frac{n}{2} - 1 > k$. Using the different estimates in (26), we deduce

$$
\frac{\partial_r R_\ell(\frac{k}{2})}{R_\ell(\frac{k}{2})} = \frac{J_\alpha'(\frac{k}{2})}{J_\alpha(\frac{k}{2})} - \frac{n - 2}{k} < \frac{2\alpha}{k} - \frac{n - 2}{k} = \frac{2\ell}{k},
$$

$$
\frac{k}{(2\ell + n)^\frac{1}{2}J_{\alpha+1}(\frac{k}{2})} \leq \frac{2\alpha + 2}{k^\frac{1}{2}} = 2(2\ell + n)^\frac{1}{2}.
$$

Therefore,

\begin{align}
\text{(28) } & \quad \|u\|_{H^2(\Gamma)} \geq 2\ell^\frac{1}{2} \frac{J_\alpha(k)}{J_\alpha(\frac{k}{2})} \left( \frac{2^{1-\frac{\alpha}{2}}}{1 + k + \frac{\alpha R_\ell(\frac{k}{2})}{R_\ell(k)}} \frac{k}{2} - \epsilon (2\ell + n)^\frac{1}{2} \right).
\end{align}
In order to finally obtain the optimality in $\epsilon$, we consider $\ell \gg \max\{k^2, n\}$ and $\epsilon = (2^{\frac{2}{1+4\ell}})^{-1}$. Then, by (27)
\[
\|u\|_{H^{1/2}(\Gamma)} \geq C 2^{-\frac{3}{2}} 2^\alpha \ell^\frac{1}{4} \left( \frac{1}{3\ell^2} - \frac{3}{16\ell^4} \right) > c_2 \ell.
\]

4. Stability for the Calderón problem for the Helmholtz equation with potential. As an application of the Runge approximation results from above, we present the proof of the stability estimate from Proposition 1 for a partial data Calderón problem with stability improvement for an increasing parameter $k$. For the Helmholtz setting with impedance boundary conditions this had earlier been deduced in [35].

More precisely, we assume the following set-up: We consider $n \geq 3$, $\Omega \subseteq \mathbb{R}^n$ a bounded connected open set with $C^\infty$ boundary and $V$ and $q$ as in (i)-(ii) and $k \geq 1$ satisfying (a1). Let $\Gamma$ be a non-empty open subset of $\partial \Omega$. We study the local Dirichlet-to-Neumann map 
\[
\Lambda_{V,q}^{\Gamma}(k) : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma),
\]
where $u \in H^1(\Omega)$ is the solution to
\[
(\Delta + k^2 q + V)u = 0 \text{ in } \Omega,
\]
\[
u \upharpoonright \Gamma = \partial \nu u \upharpoonright \Gamma.
\]

Theorem 1.2 allows us to obtain stability results for the inverse problem by using the strategy from [47, Proposition 6.1]. In particular, this reproves the result of [35, Theorem 1.2] in the case of Dirichlet boundary conditions and our spectral assumption (a1).

Proof of Proposition 1. We will use the short hand notation $b_j = k^2 q_j + V_j$ for $j = 1, 2$, for which $\|b_j\|_{L^2(\Omega)} \leq k^2 B$ and $b_1 = b_2$ in $\Omega \setminus \Omega'$.

We start with the construction of complex geometrical optics solutions $u_j$ solving $(\Delta + b_j)u_j = 0$ in $\Omega$ following [52]. We fix $\omega \in S^{n-1}$ and choose $\omega \perp$, $\tilde{\omega} \perp \in S^{n-1}$ such that
\[
\omega \cdot \omega \perp = \omega \cdot \tilde{\omega} \perp = \omega \perp \cdot \tilde{\omega} \perp = 0.
\]
We set for $\tau, r \in \mathbb{R}$ with $\tau \geq \frac{|r|^2}{2}$
\[
\xi_1 = \tau \omega \perp + i \left( \frac{r}{2} \omega + \sqrt{\tau^2 - \frac{r^2}{4}} \tilde{\omega} \perp \right), \quad \xi_2 = -\tau \omega \perp + i \left( \frac{r}{2} \omega - \sqrt{\tau^2 - \frac{r^2}{4}} \tilde{\omega} \perp \right).
\]
By [52], if $\tau \geq \max\{C_0 k^2 B, 1\}$, there are solutions $u_j$ for $j \in \{1, 2\}$ of the form
\[
u_j(x) = e^{\xi_j \cdot x}(1 + \psi_j(x)),
\]
where
\[
\|\psi_j\|_{L^2(\Omega)} \leq \frac{Ck^2 B}{\tau}, \quad \|\psi_j\|_{H^1(\Omega)} \leq Ck^2 B.
\]
This implies the following estimates for the solutions:
\[
\|u_j\|_{L^2(\Omega)} \leq C e^\tau, \quad \|u_j\|_{H^1(\Omega)} \leq C \tau e^\tau.
\]
Now we seek to approximate $u_j$ up to some order $\epsilon > 0$ which will be chosen later. We apply Theorem 1.2 with $\Omega_2 = \Omega$, $\Omega_1 = \Omega'$ and $\tilde{\Omega}_1 = \tilde{\Omega}'$, where the latter
is a slightly bigger domain containing $\Omega'$. This yields solutions $\tilde{u}_j$ to $(\Delta + b_j)\tilde{u}_j = 0$ in $\Omega$ with $\tilde{u}_j|_{\partial\Omega}$ supported in $\Gamma$ and

$$
\|u_j - \tilde{u}_j\|_{L^2(\Omega')} \leq \epsilon \|u_j\|_{H^1(\Omega')}, \quad \|\tilde{u}_j\|_{H^{1/2}(\partial\Omega)} \leq C\epsilon^{C_k}e^{-\nu\epsilon}\|u_j\|_{L^2(\Omega')}.
$$

In addition, since $b_1 = b_2$ in $\Omega \setminus \Omega'$ and using integration by parts, we obtain the following analog to Alessandrini’s identity [1]

$$
\int_{\Omega'} (b_2 - b_1)\tilde{u}_1\tilde{u}_2 \, dx = \int_{\Omega} (b_2 - b_1)\tilde{u}_1\tilde{u}_2 \, dx = \left( (\Lambda^1_{1,\eta_1}(k) - \Lambda^1_{2,\eta_2}(k))\tilde{u}_1, \tilde{u}_2 \right)_{L^2(\partial\Omega)}.
$$

Abbreviating $\delta := \|\Lambda^1_{1,\eta_1}(k) - \Lambda^1_{2,\eta_2}(k)\|_{H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)}$ and applying the previous estimates leads to

$$
\left| \int_{\Omega'} (b_2 - b_1)\tilde{u}_1\tilde{u}_2 \, dx \right| \leq \delta \|\tilde{u}_1\|_{H^{1/2}(\partial\Omega)} \|\tilde{u}_2\|_{H^{1/2}(\partial\Omega)} \leq C\delta e^{C_k}\epsilon^{-2\nu}\|u_1\|_{L^2(\Omega)}\|u_2\|_{L^2(\Omega)}.
$$

We extend $b_j$ by zero to $\mathbb{R}^n$. Now we seek to apply the previous steps to estimate

$$
|\mathcal{F}(b_2 - b_1)(r\omega)| = \left| \int_{\Omega'} (b_2 - b_1)e^{-ir\omega \cdot x} \, dx \right|
$$

for any $|r| \leq 2\tau$ and $\omega \in \mathbb{S}^{n-1}$. Notice that

$$
e^{-ir\omega \cdot x} = u_1u_2 - e^{-ir\omega \cdot x}(\psi_1 + \psi_2 + \psi_1\psi_2)
= -e^{-ir\omega \cdot x}(\psi_1 + \psi_2 + \psi_1\psi_2) + (u_1 - \tilde{u}_1)u_2 + (u_2 - \tilde{u}_2)\tilde{u}_1 + \tilde{u}_1\tilde{u}_2.
$$

Thus, using the Runge approximation bounds again and invoking the estimates for the functions $u_j$ and $\psi_j$, we obtain

$$
|\mathcal{F}(b_2 - b_1)(r\omega)| \leq CkB \left( \frac{k^2B}{\tau} + \epsilon \tau^2 \epsilon^{2\tau} \right) + C\delta e^{C_k}\epsilon^{-2\nu}\epsilon^{2\tau}.
$$

In order to estimate $\|b_2 - b_1\|_{H^{-1}(\Omega)}$, we notice that for any $\rho < 2\tau$

$$
\|b_2 - b_1\|^2_{H^{-1}(\Omega)} = \int_{\mathbb{R}^n} |\mathcal{F}(b_2 - b_1)(\zeta)|^2 (1 + |\zeta|^2)^{-1} \, d\zeta.
$$

$$
\leq \int_{|\zeta| < \rho} |\mathcal{F}(b_2 - b_1)(\zeta)|^2 (1 + |\zeta|^2)^{-1} \, d\zeta + \frac{1}{1 + \rho^2} \|b_1 - b_2\|^2_{L^2(\Omega)}
= C\rho^{n-2} \left( \frac{(k^2B)^4}{\tau^2} + (k^2B)^2\epsilon^{2\tau} + \delta^2 e^{C_k}\epsilon^{-2\nu}\epsilon^{2\tau} \right) + C\rho^{-2}(k^2B)^2.
$$

Choosing $\rho = \left( \frac{\tau}{k^2B} \right)^\frac{2}{n}$ and $\epsilon = \delta \frac{1}{\tau^{2\nu}}$ yields

$$
\|b_2 - b_1\|^2_{H^{-1}(\Omega)} \leq C \left( (k^2B)^{2+\frac{4}{n}} \tau^{-\frac{4}{n}} + e^{C_k}\epsilon e^{C_k}\delta \frac{1}{\tau^{2\nu}} \right).
$$

Now we assume $\delta < 1$, recall that $\tau \geq \max\{C_0(k^2B, 1\}$ and choose

$$
C\tau = C\delta^{k^2B} - \left( \frac{1}{1 + 2\nu} \right) \log \delta,
$$

which results in

$$
\|b_2 - b_1\|^2_{H^{-1}(\Omega)} \leq C \frac{1}{(k + k^{-(n+2)}|\log \delta|^{\frac{4}{n}} \delta e^{Ck^{n+3}\delta \frac{1}{\tau^{2\nu}}},
$$

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where the constant $C > 0$ now includes the $B$-dependence. Applying Young’s inequality we can estimate the last term as follows:

$$e^{Ck^{n+3}}\delta^{\frac{1}{n+2}} \leq C\left(e^{Ck^{n+3}}\delta^{2} + \frac{\delta^{\frac{2}{n+2}}}{k^{\frac{1}{n+2}}}\right).$$

Taking into account that $\delta^{\alpha}k^{-\frac{\alpha}{2}} \leq (k + \frac{\alpha}{2}|\log \delta|)^{-\frac{\alpha}{2}}$, then

$$\|b_2 - b_1\|_{H^{-1}(\Omega)} \leq C\left(\frac{1}{(k + k^{-(n+2)}|\log \delta|)^{\frac{1}{n+2}}} + e^{Ck^{n+3}}\delta\right).$$

In order to infer the desired result, we finally notice that

$$\frac{1}{k + k^{-(n+2)}|\log \delta|} \leq \frac{1}{k + |\log \delta|^{\frac{n+2}{n+3}}}.$$

Indeed, applying again Young’s inequality, we have

$$|\log \delta|^{\frac{1}{n+2}} \leq \frac{1}{\min\{p,q\}} \left((k^{\frac{n+2}{n+2}}|\log \delta|^{\frac{n+2}{n+2}})^{p} + k^{\frac{n+2}{n+2}}\right).$$

Choosing $p = n + 3$ and $q = \frac{n+3}{n+2}$, the previous claim follows. \(\square\)

5. Improved Carleman estimates and three balls inequalities in the presence of convexity – The case of concentric balls. Last but not least, in this section we show how in the presence of convexity of the domains the Runge approximation results can be improved by considering the specific geometric example of two concentric balls. This provides the Runge approximation counterpart to the improved stability estimates for unique continuation. Since we need quantitative unique continuation estimates in the natural trace spaces, we also provide the relevant Carleman estimates. In other functional settings similar results had been proved earlier in the literature, see for instance [15, 50]. In order to illustrate the effect, we consider the geometric setting of concentric balls. In view of the results from [15, 9, 8], it is expected that this extends to other convex geometries with additional technical effort (e.g. by shifting the Sobolev spaces as in [9]), which we postpone to a possible future study. We also refer to [31] for early observations on the role of convexity in the closely related stability analysis for the wave equation.

5.1. Improved unique continuation results. We seek to deduce improved unique continuation estimates in $k$. To this end, we first derive a Carleman estimate with improvements in $k$ for the model case of the acoustic equation without potential

$$(\Delta + k^{2}q)u = 0 \text{ in } \Omega.$$ 

Here we differ slightly from the argument by Isakov and use ideas from results on excluding embedded eigenvalues instead (see for instance [34]).

**Proposition 4.** Let $u : \mathbb{R}^{n} \rightarrow \mathbb{R}$ be compactly supported in $B_{2}\setminus B_{1} \subset \mathbb{R}^{n}\setminus\{0\}$ and solve

$$\begin{aligned}
(\Delta + k^{2}q)u = f + \sum_{j=1}^{n} \partial_{j}F^{j} \text{ in } \mathbb{R}^{n},
\end{aligned}$$

where $\Delta = \sum_{j=1}^{n} \partial_{j}^{2}$.
where \( q \) satisfies \((ii')\) in \( \mathbb{R}^n \) and \( f, F^j \in L^2(\mathbb{R}^n) \) with \( \text{supp} \, f, \text{supp} \, F^j \subset B_2 \setminus B_1 \). Let \( \phi(x) := \tau \log(|x|) \). Then, there exists \( \tau_0 > 0 \) such that for any \( \tau \geq \tau_0 \), there is a constant \( C > 0 \) depending on \( n \) and \( \sigma \) such that

\[
\tau \| e^{\phi} u \|_{L^1(\mathbb{R}^n)} + \| e^{\phi} |x| \nabla u \|_{L^2(\mathbb{R}^n)} + \tau^{1/2} k \| q^{1/2} |x| e^{\phi} u \|_{L^2(\mathbb{R}^n)} \leq C \left( \| e^{\phi} |x|^2 f \|_{L^2(\mathbb{R}^n)} + \max\{\tau, k\} \sum_{j=1}^n \| e^{\phi} |x| F^j \|_{L^2(\mathbb{R}^n)} \right). \tag{30} \]

**Remark 2.** As is common in stability improvement results, the main feature of the Carleman estimate from Proposition 4 is that the frequency is included in the right hand side (the main part of the operator) and that there is an improvement depending on \( k \) on the left hand side of the estimate. Such ideas also hold for more general operators (see, for instance, [34] or [23]).

**Proof.** We argue in three steps: First, we pass to conformal polar coordinates, then we invoke a splitting strategy in which we split the conjugated equation into an elliptic and a subelliptic contribution. For these we separately deduce the corresponding estimates. Finally, we combine these two estimates into the desired overall bound.

**Step 1. Coordinate transformation.** We pass to conformal polar coordinates, i.e. we set \( x = \psi(t, \theta) \) for

\[
\psi : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}^n \setminus \{0\}, \\
(t, \theta) \mapsto e^t \theta.
\]

For any \( \varphi \in L^1(\mathbb{R}^n, |x|^{-n} \, dx) \) we obtain with the area formula

\[
\int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} \varphi \circ \psi(t, \theta) d\mathcal{H}^{n-1}(\theta) \, dt = \int_{\mathbb{R}^n} \frac{1}{|x|^n} \varphi(x) \, dx,
\]

where \( \mathcal{H}^{n-1} \) denotes the \( n-1 \) dimensional Hausdorff measure on \( \mathbb{S}^{n-1} \). Thus, at least formally, \( d\mathcal{H}^{n-1}(\theta) \, dt = d\theta dt = |x|^{-n} \, dx \).

A standard calculation shows that in the new coordinates

\[
(|x|^2 \Delta u) \circ \psi = (\partial_t^2 + (n-2)\partial_t + \Delta_{\mathbb{S}^{n-1}}) (u \circ \psi).
\]

We can discard the first order term by conjugating the operator above with \( |x|^{-\frac{n-2}{2}} \), that is \( e^{-\frac{n-2}{2} t} \) in the new coordinates. Therefore (29) becomes the following equation for \( \hat{u}(t, \theta) = e^{\frac{n-2}{2} t} u \circ \psi(t, \theta) \):

\[
(\partial_t^2 + \Delta_{\mathbb{S}^{n-1}} + k^2 e^{2t} \hat{q} - c_n) \hat{u} = \hat{f} + \partial_t \hat{F}^t + \text{div}_{\mathbb{S}^{n-1}} \hat{F}^\theta, \tag{31}
\]

where

\[
\hat{q}(t, \theta) = q \circ \psi(t, \theta), \quad c_n = \left( \frac{n - 2}{2} \right)^2, \\
\hat{f}(t, \theta) = e^{2t} e^{\frac{n-2}{2} t} f \circ \psi(t, \theta) + \left( \frac{n}{2} - 1 \right) \hat{F}^t(t, \theta),
\]
\[
\tilde{F}^i(t, \theta) = e^{\frac{t}{h}} \left( \sum_{j=2}^{n} \frac{\theta_{j-1}^i}{\theta_{j-1}} F^j \circ \psi(t, \theta) \pm \left( 1 - \sum_{i=1}^{n-1} \frac{\theta_i^2}{\theta_i} \right)^{\frac{1}{2}} F^1 \circ \psi(t, \theta) \right),
\]
\[
\tilde{F}^{\theta_i}(t, \theta) = e^{\frac{t}{h}} F^{i+1} \circ \psi(t, \theta) - \theta_i \tilde{F}^i(t, \theta), \quad i \in \{1, \ldots, n-1\},
\]
\[
\tilde{F}^\theta(t, \theta) = \left( \tilde{F}^{\theta_1}(t, \theta), \ldots, \tilde{F}^{\theta_{n-1}}(t, \theta) \right),
\]
\[
\theta_i = \frac{x_{i+1}}{|x|}, \quad i \in \{1, \ldots, n-1\}.
\]

For ease of notation and later reference, we denote the operator on the left hand side of (31) by \( L \). In addition, for some given weight \( \Phi = \Phi(t) \), we denote by \( L_\Phi \) the conjugated operator given by

\[
L_\Phi = e^{\Phi} Le^{-\Phi} = \partial_t^2 + \Delta_{S^{n-1}} - 2\Phi \partial_t + k^2 e^{2t} \tilde{q} + \Phi'^2 - \Phi'' - c_n.
\]

**Step 2. Splitting strategy.** In order to deal with the divergence contributions, we use a splitting strategy and set \( u = u_1 + u_2 \), where \( \tilde{u}_1(t, \theta) := e^{\frac{2t}{h}} u_1 \circ \psi(t, \theta) \) is a weak solution to

\[
(L - D \max\{\tau^2, k^2 e^{2t} \tilde{q}\}) \tilde{u}_1 = \tilde{f} + \partial_t \tilde{F}^t + \text{div}_{S^{n-1}} \tilde{F}^\theta,
\]

for \( D > 0 \) large. A solution to this exists by Lax-Milgram. Indeed, this follows by considering the bilinear form

\[
B(h_1, h_2) = \int_{S^{n-1}} \int_{\mathbb{R}} \left( \partial_t h_1 \partial_t h_2 + \nabla_{S^{n-1}} h_1 \cdot \nabla_{S^{n-1}} h_2 + bh_1 h_2 \right) dt d\theta
\]

with \( b = D \max\{\tau^2, k^2 e^{2t} \tilde{q}\} - k^2 e^{2t} \tilde{q} + c_n > 0 \). An application of the Lax-Milgram theorem with this bilinear form then yields a solution \( \tilde{u}_1 \in H^1(\mathbb{R} \times S^{n-1}) \) and associated energy bounds in terms of the \( L^2 \) norms of \( \tilde{f}, \tilde{F}^t, \tilde{F}^\theta \).

The equation for the function \( \tilde{u}_2(t, \theta) = e^{\frac{2t}{h}} u_2 \circ \psi(t, \theta) \) is determined by considering the difference of \( \tilde{u} - \tilde{u}_1 \).

**Step 2a. Energy estimates for \( u_1 \).** We seek to complement the existence result for \( \tilde{u}_1 \) with exponentially weighted energy estimates. Since the support of \( \tilde{u}_1 \) is in general not bounded, we first consider the conjugated equation with a truncated weight. Energy estimates which are uniform in the truncation parameter and a limiting argument then allow us to pass to the desired weight. To this end, we consider a smooth weight \( \Phi_R \) for \( R \geq 2 \) such that \( \Phi_R(t) = (1 + \tau) t \) for \( t \leq R \) and \( \Phi_R(t) = (1 + \tau) \frac{3R}{2} \) for \( t \geq 2R \). In addition, \( \Phi_R' \leq 1 + \tau \) and \( \Phi_R'' \leq \frac{1+\tau}{R} \) in their corresponding supports. Let \( w_R = e^{\Phi_R} \tilde{u}_1 \), then it satisfies the equation

\[
(L_{\Phi_R} - D \max\{\tau^2, k^2 e^{2t} \tilde{q}\}) w_R = e^{\Phi_R} \left( \tilde{f} + \partial_t \tilde{F}^t + \text{div}_{S^{n-1}} \tilde{F}^\theta \right),
\]

where \( L_{\Phi_R} \) is given by (32). The weak version of the equation for \( w_R \) tested with \( w_R \) itself yields the following energy estimates

\[
\left\| \partial_t w_R \right\|_{L^2(\mathbb{R} \times S^{n-1})}^2 + \left\| \nabla_{S^{n-1}} w_R \right\|_{L^2(\mathbb{R} \times S^{n-1})}^2
\]
\[
+ \int_{S^{n-1}} \int_{\mathbb{R}} (D \max\{\tau^2, k^2 e^{2t} \tilde{q}\} - k^2 e^{2t} \tilde{q} + c_n) w_R^2 dt d\theta
\]
\[
+ \int_{S^{n-1}} \int_{\mathbb{R}} (\Phi_R' - \Phi_R'^2) w_R^2 dt d\theta + 2 \int_{S^{n-1}} \int_{\mathbb{R}} \Phi_R' w_R \partial_t w_R dt d\theta
\]
\[
= - \int_{S^{n-1}} \int_{\mathbb{R}} e^{\Phi_R} \left( \tilde{f} + \partial_t \tilde{F}^t + \text{div}_{S^{n-1}} \tilde{F}^\theta \right) w_R dt d\theta.
\]
Integrating by parts and applying Young’s inequality, together with the fact that supp \( \tilde{f} \), supp \( F^j \) \( \subset (0, \log 2) \times \mathbb{S}^{n-1} =: I \times \mathbb{S}^{n-1} \), we obtain the following estimates for \( \tau > 1 \)

\[
\int_{S^{n-1}} \int_{R} \left( \Phi''_R - \Phi''_R \right) w_R^2 dt \, d\theta \\
2 \int_{S^{n-1}} \int_{R} \Phi'_R w_R \partial_t w_R dt \, d\theta \\
\int_{S^{n-1}} \int_{R} e^{\Phi_R} \tilde{f} w_R dt \, d\theta
\leq 6 \tau^2 \| w_R \|_{L^2(R \times \mathbb{S}^{n-1})}^2,
\]

\[
\| w_R \|_{L^2(R \times \mathbb{S}^{n-1})}^2 + \frac{1}{4} \| \partial_t w_R \|_{L^2(R \times \mathbb{S}^{n-1})}^2,
\]

\[
\int_{S^{n-1}} \int_{R} e^{\Phi_R} \partial_t \tilde{F}^i w_R dt \, d\theta
\leq C \| (1 + r)^t \tilde{F}^i \|_{L^2(R \times \mathbb{S}^{n-1})}^2 + \frac{\tau^2}{4} \| w_R \|_{L^2(R \times \mathbb{S}^{n-1})}^2,
\]

\[
\int_{S^{n-1}} \int_{R} e^{\Phi_R} \nabla w_R dt \, d\theta
\leq \| (1 + r)^t \tilde{F}^i \|_{L^2(R \times \mathbb{S}^{n-1})}^2 + \frac{1}{4} \| \nabla w_R \|_{L^2(R \times \mathbb{S}^{n-1})}^2.
\]

Absorbing the terms with \( w_R \) and the non-positive terms for \( D \) sufficiently large, we obtain

\[
\tau \| w_R \|_{L^2(R \times \mathbb{S}^{n-1})} + \| \partial_t w_R \|_{L^2(R \times \mathbb{S}^{n-1})} + \| \tilde{F}^i \|_{L^2(R \times \mathbb{S}^{n-1})}
\leq C \left( \max \{ \tau, k \} \| (1 + r)^t f \|_{L^2(R \times \mathbb{S}^{n-1})} + \| (1 + r)^t \tilde{F}^i \|_{L^2(R \times \mathbb{S}^{n-1})} + \| (1 + r)^t \tilde{F}^j \|_{L^2(R \times \mathbb{S}^{n-1})} \right).
\]

Notice that the right hand side is finite and does not depend on \( R \), so taking \( R \to \infty \), we obtain similar estimates for \( w = e^{(1 + r)^t \tilde{u}_1} \). Multiplying the whole expression by \( \max \{ \tau, k \} \) and returning to the original coordinates we arrive at

\[
\tau^2 \| e^{\Phi} u_1 \|_{L^2(\mathbb{R}^n)} + \| \tilde{F}^i \|_{L^2(\mathbb{R}^n)} + \| q^{1/2} e^{\Phi} |x| u_1 \|_{L^2(\mathbb{R}^n)}
\leq C \left( \| e^{\Phi} |x|^2 f \|_{L^2(\mathbb{R}^n)} + \max \{ \tau, k \} \sum_{j=1}^{n} \| e^{\Phi} |x|^2 F^j \|_{L^2(\mathbb{R}^n)} \right).
\]

Arguing similarly for \( \tilde{\Phi}_R \) with \( \tilde{\Phi}_R = (2 + \tau)t \) if \( t \leq R \), we also deduce

\[
k^2 \| e^{\Phi} |x|^2 q^{1/2} u_1 \|_{L^2(\mathbb{R}^n)} \leq C \left( \| e^{\Phi} |x|^2 f \|_{L^2(\mathbb{R}^n)} + \max \{ \tau, k \} \sum_{j=1}^{n} \| e^{\Phi} |x|^2 F^j \|_{L^2(\mathbb{R}^n)} \right).
\]
Combining (33)-(34) and exploiting again the compact support of $f$ and $F^j$ and (ii’), we infer that

\[
\| D \max \{ \tau^2, k^2 |x|^2 q \} e^\phi u_1 \|_{L^2(\mathbb{R}^n)} \leq D \tau^2 \| e^\phi u_1 \|_{L^2(\mathbb{R}^n)} + D \tau^{1/2} k^2 \| |x|^2 q^{1/2} e^\phi u_1 \|_{L^2(\mathbb{R}^n)} \\
\leq C \left( \| e^\phi |x|^2 f \|_{L^2(\mathbb{R}^n)} + \max \{ \tau, k \} \sum_{j=1}^{n} \| e^\phi |x|^j F^j \|_{L^2(\mathbb{R}^n)} \right),
\]

where now $C$ also depends on $\sigma$.

**Step 2b. Carleman estimates for $u_2$.** We now consider the estimate for $u_2$. To this end, we note that $\tilde{u}_2$ solves the equation

\[
L \tilde{u}_2 = D \max \{ \tau^2, k^2 e^{2\tau} \} \tilde{u}_1 \quad \text{in } \mathbb{R} \times \mathbb{S}^{n-1}.
\]

We now carry out the conjugation with $e^\Phi$ for $\Phi(t) = (1 + \tau)t$ and split the operator $L_\Phi$ given in (32) into its symmetric and antisymmetric parts (with respect to the $L^2(\mathbb{R} \times \mathbb{S}^{n-1})$ scalar product)

\[
S_\Phi = \partial_t^2 + \Delta_{\mathbb{S}^{n-1}} + k^2 e^{2\tau} \tilde{q} + (1 + \tau)^2 - c_n, \quad A_\Phi = -2(1 + \tau) \partial_t.
\]

Let us set $v = e^\Phi \tilde{u}_2$. Expanding the right hand side of the last equality, we obtain

\[
\| L_\Phi v \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}^2 = \| S_\Phi v \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}^2 + \| A_\Phi v \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}^2 + ([S_\Phi, A_\Phi] v, v)_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}.
\]

In addition, by the definition of $L_\Phi$ and $v$,

\[
\| L_\Phi v \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} = \| e^\Phi L \tilde{u}_2 \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} = \| D \max \{ \tau^2, k^2 |x|^2 q \} e^\phi u_1 \|_{L^2(\mathbb{R}^n)}.
\]

We begin with a lower bound on the commutator. We calculate

\[
[S_\Phi, A_\Phi] v = [e^{2\tau} k^2 \tilde{q}, A_\Phi] v = 2(1 + \tau) \partial_t (k^2 e^{2\tau} \tilde{q}) v.
\]

As almost everywhere $\partial_t \tilde{q} = (\nabla q \cdot x) \circ \psi \geq 0$ and $\tilde{q} > 0$ by assumption, we thus find after returning to the standard coordinates

\[
([S_\Phi, A_\Phi] v, v)_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} \geq 4(1 + \tau) (e^{2\tau} k^2 \tilde{q} v, v)_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} = 4(1 + \tau) k^2 \int_{\mathbb{R}^n} |x|^2 e^{2\phi} u_2^2 \, dx.
\]

Therefore, we conclude

\[
4(1 + \tau) k^2 \int_{\mathbb{R}^n} |x|^2 e^{2\phi} u_2^2 \, dx \leq \| D \max \{ \tau^2, k^2 |x|^2 q \} e^\phi u_1 \|_{L^2(\mathbb{R}^n)}.
\]

Now we seek to estimate $\| v \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}$ in terms of

\[
\| A_\Phi v \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} = 2(1 + \tau) \| \partial_t v \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}.
\]

Using the compact support of $\tilde{u}$, we can apply the Poincaré inequality to the function $e^\Phi \tilde{u} (\cdot, \theta)$ for almost every $\theta \in \mathbb{S}^n$ as follows

\[
\| v \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} \leq \| e^\Phi \tilde{u} \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} + \| e^\Phi \tilde{u}_1 \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} \\
\leq C \left( \| \partial_t (e^\Phi \tilde{u}) \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} + \| e^\Phi \tilde{u}_1 \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} \right) \\
\leq C \left( \| \partial_t v \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} + \| \partial_t (e^\Phi \tilde{u}_1) \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} + \| e^\Phi \tilde{u}_1 \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} \right) \\
\leq C \left( \| \partial_t v \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} + \| e^\Phi \partial_t \tilde{u}_1 \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} + \tau \| e^\Phi \tilde{u}_1 \|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} \right),
\]
where $C$ depends on $n$ (and the support of $\tilde{u}$). Multiplying the whole inequality by $(1+\tau)$ we can write
\[
(1+\tau)\|e\|_{L^2(\mathbb{R}^n)} \leq C\left(\|A_{\Phi}e\|_{L^2(\mathbb{R}^n)} + \tau\|e\|_{L^2(\mathbb{R}^n)} + \tau^2\|e\|_{L^2(\mathbb{R}^n)}\right).
\]

Returning to Euclidean coordinates yields
\[
(1+\tau)\|e\|_{L^2(\mathbb{R}^n)} \leq C\left(\|D\max\{\tau^2, k^2|x|^2/q\}e\|_{L^2(\mathbb{R}^n)} + \tau^2\|e\|_{L^2(\mathbb{R}^n)}\right).
\]

Lastly, we deduce a gradient bound on $\tilde{u}_2$ using the symmetric part of the operator. Testing
\[
(S_k v, v)_{L^2(\mathbb{R}^n)} = -\|\nabla v\|_{L^2(\mathbb{R}^n)}^2 - \|\nabla v\|_{L^2(\mathbb{R}^n)}^2 + k^2(e^{2\tau}v, v)_{L^2(\mathbb{R}^n)} + ((1+\tau)^2 - c_n)\|v\|_{L^2(\mathbb{R}^n)}^2.
\]

Therefore,
\[
\|\nabla v\|_{L^2(\mathbb{R}^n)} + k^2(e^{2\tau}v, v)_{L^2(\mathbb{R}^n)} + (1+\tau)^2 - c_n\|v\|_{L^2(\mathbb{R}^n)}^2.
\]

Returning to the original coordinates and using (34)-(37) to estimate the right hand side yields
\[
\|\nabla v\|_{L^2(\mathbb{R}^n)} + k^2(e^{2\tau}v, v)_{L^2(\mathbb{R}^n)} + (1+\tau)^2\|v\|_{L^2(\mathbb{R}^n)}^2.
\]

Finally, combining (34),(37) and (38) with (33) and (35), we obtain for $\tau > 1$
\[
\tau\|e\|_{L^2(\mathbb{R}^n)} + \tau^2\|e\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}k\|q^{1/2}e\|_{L^2(\mathbb{R}^n)}
\]
\[
\leq C\left(\|e\|_{L^2(\mathbb{R}^n)} + \|e\|_{L^2(\mathbb{R}^n)} + \tau\|e\|_{L^2(\mathbb{R}^n)}\right).
\]

Step 3. Conclusion. The final estimate (30) follows by an application of the triangle inequality and the estimates (33) and (39) for $u_1$ and $u_2$, respectively. $\square$

Next, using the previous Carleman estimate, we deduce a quantitative unique continuation result which does not suffer from the losses in $k$.

**Theorem 5.1.** Let $V$ and $q$ be as in (i)-(ii) in $\Omega = B_2 \setminus B_1$. Let $u \in H^1(\Omega)$ be a solution to
\[
(\Delta + k^2 q + V)u = 0 \quad \text{in } \Omega,
\]
and let $M$, $\eta$ be such that
\[
\|u\|_{H^1(\Omega)} \leq M,
\]
\[
\|u\|_{H^1(\partial B_2)} + \|\partial u\|_{H^{-1/2}(\partial B_2)} \leq \eta.
\]
Assume further that $0 < k^2 \eta \leq M$. Then there exist a parameter $\mu \in (0,1)$ and a constant $C > 1$ depending on $n$, $\|V\|_{L^\infty(\Omega)}$ and $\sigma$ (but not on $k$) such that
\[
\|u\|_{L^2(\Omega)} \leq C \left(\log\left(\frac{k^2 \eta}{M}\right)\right)^{-\mu} M.
\]
In addition, if $G = B_2 \setminus B_{1+\delta}$ for some $\delta \in (0,1)$, then there exist a parameter $\nu \in (0,1)$ and a constant $C > 1$ (depending on $n, \delta, \|V\|_{L^\infty(\Omega)}, \sigma$ and $\|q\|_{C^{0,1}(\Omega)}$ but not on $k$) such that
\[
\|u\|_{L^2(G)} \leq C \left(k^3 \eta\right)^\nu M^{1-\nu}. \tag{42}
\]

We will prove Theorem 5.1 in several steps. First we prove a corresponding propagation of smallness result from the interior for divergence form equations. Combined with an extension argument this will then lead to the desired claim of Theorem 5.1.

**Proposition 5.** Let $V$ and $q$ be as in (i)-(ii') in $\Omega = B_2 \setminus B_1$. Let $u \in H^1(\Omega)$ with $\text{supp } u \subset B_2 \setminus B_1$ be a solution to
\[
(\Delta + k^2 q + V)u = f + \sum_{j=1}^n \partial_j F^j \quad \text{in } \Omega,
\]
where $f, F^j \in L^2(\Omega)$ with $\text{supp } f, \text{supp } F^j \subset B_2 \setminus B_1$. Let $M, \eta > 0$ be such that
\[
\|u\|_{H^1(\Omega)} \leq M, \quad \|f\|_{L^2(\Omega)} + \sum_{j=1}^n \|F^j\|_{L^2(\Omega)} \leq \eta.
\]
Assume further that $0 < k\eta \leq M$. Then there exist $\mu \in (0,1)$ and $C > 1$ (depending on $n, \|V\|_{L^\infty(\Omega)}$ and $\sigma$) such that
\[
\|u\|_{L^2(\Omega)} \leq C \left|\log \left(\frac{k\eta}{M}\right)\right|^{-\mu} M. \tag{43}
\]

In addition, if $G = B_2 \setminus \overline{B_{1+\delta}}$ for some $\delta \in (0,1)$, then there exist a parameter $\nu \in (0,1)$ and a constant $C > 1$ (depending on $n, \delta, \|V\|_{L^\infty(\Omega)}$ and $\sigma$) such that
\[
\|u\|_{L^2(G)} \leq C(k\eta)^\nu M^{1-\nu}. \tag{44}
\]

**Remark 3.** We emphasise that unlike in Proposition 4, in Proposition 5, we are not assuming that $u$ and the functions $f$ and $F^j$ vanish on the interior boundary $\partial B_1$.

**Proof of Proposition 5.** We use the Carleman inequality from Proposition 4 in combination with the Sobolev embedding theorem and an optimization argument. We argue in two steps, first proving (44) and then using this to prove (43).

**Step 1. Proof of (44).** We apply the Carleman estimate from Proposition 4 to the function $w := u\chi$, where $\chi$ is a smooth cut-off function which is equal to one on $B_2 \setminus B_{1+\delta/2}$, vanishes on $B_{1+\delta/4}$ and satisfies $|\nabla \chi| \leq C\delta^{-1}, |\Delta \chi| \leq C\delta^{-2}$. The function $w$ thus is compactly supported in $B_2 \setminus B_1$ and solves the equation
\[
(\Delta + k^2 q)w = -Vw + g + \sum_{j=1}^n \partial_j G^j \quad \text{in } B_2 \setminus B_1, \tag{45}
\]
where
\[
g = u\Delta \chi + 2\nabla u \cdot \nabla \chi + \chi f - \sum_{j=1}^n (\partial_j \chi)F^j, \quad G^j = \chi F^j.
\]
Hence, consider the function $\tilde{\nu}$ in $\Omega$. Proof of (43) then implies the desired result with (47).

Recalling that by assumption $k\eta$ in $\Omega$, $\partial_r \xi \geq 0$ in the bounded component of $\Omega \setminus \Omega'$ and $\partial_r \xi \leq 0$ otherwise. Now we consider the function $\tilde{q} = \xi q + (1 - \xi)(\sigma^{-1} 1_{B_{3/2}} + \sigma 1_{\mathbb{R}^n \setminus B_{3/2}})$, which coincides with $q$ in $\Omega'$. It is clear that $\tilde{q} \in C^{0,1}(\mathbb{R}^n)$ and $\sigma^{-1} \leq \tilde{q} \leq \sigma$ in $\mathbb{R}^n$. Finally, since $\nabla \tilde{q} \cdot x \geq 0$ in $\Omega$, $\partial_r \xi (q - \sigma^{-1}) \geq 0$ in $(\Omega \setminus \Omega') \cap B_{3/2}$ and $\partial_r \xi (q - \sigma) \geq 0$ in $(\Omega \setminus \Omega') \cap (\mathbb{R}^n \setminus B_{3/2})$ we deduce $\nabla \tilde{q} \cdot x \geq 0$ in $\mathbb{R}^n$.

Therefore, invoking Proposition 4, we obtain for $\tau > \tau_0$

$$\tau \|e^{\phi}w\|_{L^2(\mathbb{R}^n)} \leq C \left( \|e^{\phi}|x|^2Vw\|_{L^2(\mathbb{R}^n)} + \|e^{\phi}|x|^2g\|_{L^2(\mathbb{R}^n)} \right) + \max\{\tau, k\} \sum_{j=1}^n \|e^{\phi}|x|G_j\|_{L^2(\mathbb{R}^n)}.$$  

(46)

Considering $\tau \geq C\|V\|_{L^\infty(\Omega)}$, we can absorb the first term on the right hand side of (46) into the left hand side. Then, inserting the expressions for $w$, $g$, $G^j$ and $\phi$, we infer

$$\tau \|x|^\tau u\|_{L^2(B_2 \setminus B_{1+\delta/2})} \leq C \left( \delta^{-2} \|x|^{2+\tau} u\|_{L^2(B_1 + \delta/2 \setminus B_{1+\delta/4})} + \delta^{-1} \|x|^{2+\tau} \nabla u\|_{L^2(B_1 + \delta/2 \setminus B_{1+\delta/4})} + \|x|^{2+\tau} f\|_{L^2(\Omega)} \right) + \left( \delta^{-1} + \max\{\tau, k\} \right) \sum_{j=1}^n \|x|^{1+\tau} F^j\|_{L^2(\Omega)}.$$  

Hence,

$$\|u\|_{L^2(\mathbb{R}^n)} \leq C\delta^{-2} \left( \left( \frac{1 + \delta/2}{1 + \delta} \right)^{\tau+2} \|u\|_{H^1(B_{1+\delta/2} \setminus B_{1+\delta/4})} + 4\tau k \left( \|f\|_{L^2(\Omega)} + \sum_{j=1}^n \|F^j\|_{L^2(\Omega)} \right) \right) \leq C\delta^{-2} \left( \left( \frac{1 + \delta/2}{1 + \delta} \right)^{\tau} \|u\|_{H^1(\Omega)} + 4\tau k \left( \|f\|_{L^2(\Omega)} + \sum_{j=1}^n \|F^j\|_{L^2(\Omega)} \right) \right) \leq C\delta^{-2} \left( \left( \frac{1 + \delta/2}{1 + \delta} \right)^{\tau} M + 4\tau k\eta \right).$$

Recalling that by assumption $k\eta \leq M$ and optimizing the right hand side by choosing $\tau = \tau_1 + \tau_0 + C\|V\|_{L^\infty(\Omega)}$ for $\tau_1 > 0$ such that

$$\left( \frac{1 + \delta/2}{1 + \delta} \right)^{\tau_1} M \sim 4^{\tau_1} k\eta.$$  

(47)

This then implies the desired result with $\nu = 1 - \frac{\log 4}{\log 4 + \log \left( \frac{1 + \delta/2}{1 + \delta} \right)}$.

**Step 2. Proof of (43).** We argue by making (47) more explicit. If $\delta \leq \frac{1}{2}$ (which we can assume without loss of generality), then

$$\left( \frac{1 + \delta/2}{1 + \delta} \right)^{\tau} \leq \left( 1 - \frac{\delta}{3} \right)^{\tau}.$$
Hence, in the optimization argument we obtain
\[
\tau = \frac{1}{\log \left( \frac{4}{1 - \delta^3} \right)} \log \left( \frac{M}{k\eta} \right) + C ||V||_{L^\infty(\Omega)}.
\]

As a consequence,
\[
|u|_{L^2(B_2 \setminus B_1)} \leq C \delta^{-2} (k\eta)^{\alpha} M^{1-\alpha},
\]
with
\[\alpha = 1 - \frac{\log(2)}{\log(2) - \log(1 - \delta^3)} \geq c \delta \text{ and } C \text{ depending on } ||V||_{L^\infty(\Omega)}.\]

We combine this with an application of Hölder’s inequality and Sobolev embedding close to the boundary:
\[
|u|_{L^2(B_1 + \delta \setminus B_1)} \leq C \delta^{\frac{1}{n}} |u|_{L^{\frac{2n}{n-2}}(B_1 + \delta \setminus B_1)} \leq C \delta^{\frac{1}{n}} ||u||_{H^\frac{1}{2}(\Omega)} \leq C \delta^{\frac{1}{n}} M.
\]

The combination of the two estimates then yields
\[
|u|_{L^2(B_2 \setminus B_1)} \leq C \left( \delta^{\frac{1}{n}} + \delta^{-2} \left( \frac{k\eta}{M} \right)^{c\delta} \right) M.
\]

We now choose
\[\delta = c \log \left( \frac{M}{k\eta} \right)^{-\beta} \text{ for some } \beta \in (0, 1).\]

This implies the claim with
\[\mu = \frac{2\beta}{n} \text{ (and a corresponding constant } C > 0 \text{ which depends on } \beta).\]

With Proposition 5 at our disposal, we next address the proof of Theorem 5.1.

**Proof of Theorem 5.1.** We seek to reduce the problem with Cauchy data to the problem with a divergence form $H^{-1}$ right hand side. To this end, we argue by an extension argument. We note that by definition of the $H^{\frac{1}{2}}(\partial B_2)$ norm there exists a function $v \in H^1(B_3 \setminus B_2)$ such that
\[
|v|_{H^1(B_3 \setminus B_2)} \leq C |u|_{\partial B_2} ||\partial v||_{H^{\frac{1}{2}}(\partial B_2)}.
\]

Let now $\chi \in C^\infty(B_3 \setminus \overline{B_2})$ be a smooth cut-off function with $\chi|_{\partial B_2} = 1$ and $\chi|_{\partial B_3} = 0$. We then define
\[
\tilde{u} := \begin{cases} u \text{ in } B_2 \setminus B_1, \\ \chi u \text{ in } B_3 \setminus \overline{B_2}. \end{cases}
\]

This function then is an element of $H^1(B_3 \setminus B_1)$ with $\text{supp } \tilde{u} \subset B_3 \setminus B_1$. In addition, we claim that it is a weak solution to
\[
(\Delta + k^2 q + V)\tilde{u} = f + \sum_{j=1}^n \partial_j F^j \text{ in } B_3 \setminus B_1,
\]
where $f, F^j \in L^2(B_3 \setminus B_1)$ are functions supported in $B_3 \setminus B_1$ and satisfying the bounds
\[
|f|_{L^2(B_3 \setminus B_1)} + \sum_{j=1}^n ||F^j||_{L^2(B_3 \setminus B_1)} \leq C k^2 \left( ||u|_{\partial B_2} ||_{H^{1/2}(\partial B_2)} + ||\partial v u||_{H^{-1/2}(\partial B_2)} \right).
\]
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Indeed, by the weak formulation of (49), for \( \varphi \in C_\infty^\infty(B_3 \setminus B_1) \),

\[
\int_{B_3 \setminus B_1} (\nabla \tilde{u} \cdot \nabla \varphi + k^2 q \tilde{u} \varphi + V \tilde{u} \varphi) \, dx
\]

\[
= \int_{B_2 \setminus B_1} (\nabla u \cdot \nabla \varphi + k^2 q u \varphi + V u \varphi) \, dx
\]

\[
+ \int_{B_3 \setminus B_2} (\nabla (\chi v) \cdot \nabla \varphi + k^2 q \chi v \varphi + V \chi v \varphi) \, dx
\]

\[
= -\int_{\partial B_2} \partial_n u \varphi \, dx
\]

\[
+ \int_{B_3 \setminus B_2} (\nabla (\chi v) \cdot \nabla \varphi + k^2 q \chi v \varphi + V \chi v \varphi) \, dx.
\]

Note that the mapping

\[
\Psi : \varphi \mapsto \int_{\partial B_2} \partial_n u \varphi \, dx
\]

is bounded as an element in \( H^{-\frac{1}{2}}(\partial B_2) \) and also as an element in \( H^{-1}(B_2 \setminus B_1) \).
Indeed, for \( \varphi \in H^1(B_2 \setminus B_1) \) we have

\[
|\Psi(\varphi)| \leq \|\partial_n u\|_{H^{-\frac{1}{2}}(\partial B_2)} \|\varphi\|_{H^1(B_2 \setminus B_1)} \leq C \|\partial_n u\|_{H^{-\frac{1}{2}}(\partial B_2)} \|\varphi\|_{H^1(B_2 \setminus B_1)}.
\]

Therefore \( \Psi \in (H^1(B_2 \setminus B_1))^* \subset H^{-1}(B_2 \setminus B_1) \). Then, it admits a representation

\[
\Psi = g + \sum_{j=1}^n \partial_j G_j
\]

with \( g, G_j \in L^2(B_2 \setminus B_1) \) and

\[
\|g\|_{L^2(B_2 \setminus B_1)} + \sum_{j=1}^n \|G_j\|_{L^2(B_2 \setminus B_1)} = \|\Psi\|_{H^{-1}(B_2 \setminus B_1)} \leq C \|\partial_n u\|_{H^{-\frac{1}{2}}(\partial B_2)}.
\]

As a consequence, we obtain (49) with

\[
f = \begin{cases} -g & \text{in } B_2 \setminus B_1, \\ k^2 q \chi v + V \chi v & \text{in } B_3 \setminus B_2 \end{cases},
\]

\[
F_j = \begin{cases} -G_j & \text{in } B_2 \setminus B_1, \\ -\partial_j (\chi v) & \text{in } B_3 \setminus B_2 \end{cases},
\]

and (50) holds. The result of Proposition 5 (rescaled to \( B_3 \setminus B_1 \)) is therefore applicable with

\[
\tilde{\eta} = Ck^2 \eta \geq \|f\|_{L^2(B_3 \setminus B_1)} + \sum_{j=1}^n \|F_j\|_{L^2(B_3 \setminus B_1)},
\]

(51)

\[
\tilde{M} = C \mu \geq M + C \tilde{\eta} \geq \|\tilde{u}\|_{H^1(B_3 \setminus B_1)}.
\]

This yields the desired result. \( \square \)
5.2. Improved Runge approximation result. This section contains the proof of Theorem 1.3. We start by upgrading the interior quantitative estimate from Theorem 5.1 similarly as in Proposition 3.

**Proposition 6.** Let $V$ and $q$ be as in (i)-(ii') in $\Omega_2 = B_2 \setminus \overline{B_v}$ and let $\Omega_1 = B_1 \setminus \overline{B_v}$. Let $u \in H^1(\Omega_2)$ be the unique solution to
\[
\Delta u + k^2 qu + Vu = v \mathbb{1}_{\Omega_1} \quad \text{in } \Omega_2,
\]
\[
u = 0 \quad \text{on } \partial \Omega_2,
\]
with $v \in L^2(\Omega_1)$ and $k \geq 1$ satisfying (a1). Let $G = B_2 \setminus \overline{B_{1+\delta}}$ for some $\delta \in (0, 1)$. Then there exist parameters $\nu_0 \in (0, 1)$, $s_0 \in [3, n+1]$ and a constant $C > 1$ (depending on $n, \delta, \|V\|_{L^\infty(\Omega_2)}$ and $\sigma$) such that
\[
\|u\|_{H^1(G)} \leq C k^s \|\partial_v u\|_{H^{-1/2}(\partial G)} \|v\|_{L^2(\Omega_1)}^{1-\nu_0}.
\]

**Proof.** We start by estimating $\|u\|_{H^1(\Omega_2)}$ in terms of $\|v\|_{L^2(\Omega_1)}$ as in Proposition 3. By Lemma 1.4 and (a1), there is $C > 1$ such that
\[
\|u\|_{H^1(\Omega_2)} \leq C k^{n+1} \|v\|_{L^2(\Omega_1)}.
\]
Notice that then $u$ satisfies the assumptions in Theorem 5.1, so (44) holds with $M = C k^{n+1} \|v\|_{L^2(\Omega_1)}$, $\eta = \|\partial_v u\|_{H^{-1/2}(\partial G)}$. Let us now show that the bound (42) can be upgraded to an estimate for the $H^1$ norm. This is inherited from (44). Indeed, we argue as in Step 1 of the proof of Proposition 5, but now including into the left hand side of (46) the gradient term $\|e^\phi |x| \nabla u\|_{L^2(\mathbb{R}^n)}$ coming from Proposition 4. Therefore, if $k\tilde{\eta} \leq M$,
\[
\|\tilde{u}\|_{H^1(G)} \leq C \left( k\tilde{\eta} \right)^{\nu} M,
\]
where $\nu$ and $C$ depend in particular on $\delta$. Here $\tilde{u}$ is given by (48) and $M$ and $\tilde{\eta}$ are connected with $M$ and $\eta$ according to (51). Following the proof of Theorem 5.1, we then obtain
\[
\|u\|_{H^1(G)} \leq C \left( k^3 \eta \right)^{\nu_0} M,
\]
if $k^3 \eta \leq M$. Otherwise, if $k^3 \eta \geq M$, the estimate is immediate. Therefore the final bound (52) holds with $s_0 = 3\nu_0 + (n+1)(1-\nu_0)$. \qed

With Proposition 6 we deduce the proof of Theorem 1.3 similarly as in the analogous non-convex settings:

**Proof of Theorem 1.3.** The proof follows the proof of Theorem 1.2 in Section 3 with $\Gamma = \partial B_2$ in order to construct $u_\alpha, v_\alpha$ and $w_\alpha$. The difference appears at the time of estimating $\|u_\alpha\|_\alpha$. Applying the improved estimate (52) instead of (17), we obtain
\[
\|u_\alpha - u\|_{L^2(\Omega_1)} \leq C k^{\nu_0} \left( \frac{\|\partial_v u\|_{H^{-1/2}(\partial B_2)}}{\|v\|_{L^2(\Omega_1)}} \right)^{\nu_0} \|v\|_{H^1(\Omega_1)} \leq C k^{\nu_0} \alpha^{\nu_0} \|\tilde{v}\|_{H^1(\Omega_1)}.
\]
Choosing $\alpha$ such that $C k^{\nu_0} \alpha^{\nu_0} = \epsilon$, we finally deduce
\[
\|u_\alpha\|_{H^1(\Omega_1)} \leq \frac{1}{\alpha} \|\tilde{v}\|_{L^2(\Omega_1)} = C k^{\frac{\nu_0}{\alpha}} \epsilon^{-\frac{1}{\nu_0}} \|\tilde{v}\|_{L^2(\Omega_1)}.
\]
\qed
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