Zeta Strings

Branko Dragovich

Institute of Physics
Pregrevica 118, P.O. Box 57, 11001 Belgrade, Serbia

Abstract

We introduce nonlinear scalar field models for open and open-closed strings with spacetime derivatives encoded in the operator valued Riemann zeta function. The corresponding two Lagrangians are derived in an adelic approach starting from the exact Lagrangians for effective fields of $p$-adic tachyon strings. As a result tachyons are absent in these models. These new strings we propose to call zeta strings. Some basic classical properties of the zeta strings are obtained and presented in this paper.

1 Introduction

There are just twenty years from the publication of the first paper on the $p$-adic string [1]. So far $p$-adic structures have been observed not only in string theory but also in many other models of modern mathematical physics (for a review of the early days developments, see e.g. [2], [3]).

One of the greatest achievements in $p$-adic string theory is an effective field description of scalar open and closed $p$-adic strings [4], [5]. The corresponding Lagrangians are very simple and exact. They describe not only four-point scattering amplitudes but also all higher ones at the tree-level.

This $p$-adic string theory has been significantly pushed forward when was shown [6] that it describes tachyon condensation and brane descent relations...
simpler than by ordinary bosonic strings. After that success, many aspects of p-adic string dynamics have been investigated and compared with dynamics of ordinary strings (see, e.g. [7] and references therein). Noncommutative deformation of p-adic string world-sheet with a constant B-field was investigated in [8] (on p-adic noncommutativity see also [9]). A systematic mathematical study of spatially homogeneous solutions of the nonlinear equation of motion was performed in [10]. Some possible cosmological implications of p-adic string theory have been also investigated [11]. It was recently proposed [12] that p-adic string theories provide lattice discretization to the world-sheet of ordinary strings. As a result of these developments many nontrivial features of ordinary string theory have been reproduced from the p-adic effective action.

Adelic approach to the string scattering amplitudes is a very useful way to connect p-adic and ordinary counterparts (see [2] as a review). Moreover, it eliminates unwanted prime number parameter p contained in p-adic amplitudes and also cures the problem of p-adic causality violation. Adelic generalization of quantum mechanics was also successfully formulated, and was found a connection between adelic vacuum state of the harmonic oscillator and the Riemann zeta function [13]. Recently, an interesting approach toward foundation of a field theory and cosmology based on the Riemann zeta function was proposed in [14]. An adelic approach with the Riemann zeta function is one of the motivations for this paper.

The present paper is a result of the attempt to integrate all p-adic effective field actions into one bosonic field theory. In the next two sections we explore the cases of open and open-closed bosonic strings.

2 Open scalar zeta string

The exact tree-level Lagrangian for effective scalar field \( \phi \) which describes open p-adic string tachyon is

\[
\mathcal{L}_p = \frac{1}{g^2} \frac{p^2}{p - 1} \left[ -\frac{1}{2} \phi p^{-\frac{D}{2}} \phi + \frac{1}{p + 1} \phi^{p+1} \right],
\]

where \( p \) is any prime number, \( \Box = -\partial_t^2 + \nabla^2 \) is the \( D \)-dimensional d’Alembertian and we adopt metric with signature \((- + \ldots +)\). An infinite number of space-
time derivatives follows from the expansion

\[ p^{-\Box} = \exp \left( -\frac{1}{2} \ln p \Box \right) = \sum_{k \geq 0} \left( -\frac{\ln p}{2} \right)^k \frac{1}{k!} \Box^k. \]

The equation of motion for (1) is

\[ p^{-\Box} \varphi = \varphi^p, \quad (2) \]

and its properties have been studied by many authors (see e.g. \[10\] and references therein).

It is worth noting that prime number \( p \) in (1) and (2) can be replaced by natural number \( n \geq 2 \) and such expressions also make sense. Moreover, when \( p = 1 + \varepsilon \to 1 \) there is the limit of (1) which is related to the ordinary bosonic string in the boundary string field theory \([15]\).

Now we want to introduce a model which incorporates all the above \( p \)-adic string Lagrangians in a restricted adelic way. To this end, let us take the sum of the above Lagrangians \( \mathcal{L}_n \) (1) in the form

\[
L = \sum_{n \geq 1} C_n \mathcal{L}_n = \sum_{n \geq 1} \frac{n-1}{n^2} \mathcal{L}_n
= \frac{1}{g^2} \left[ -\frac{1}{2} \phi \sum_{n \geq 1} n^{-\frac{\Box}{2}} \phi + \sum_{n \geq 1} \frac{1}{n+1} \phi^{n+1} \right], \quad (3)
\]

where coefficients \( C_n = \frac{n-1}{n^2} \) are inverses of those within \( \mathcal{L}_n \). Note that this choice is formally equivalent to the following one: \( \frac{1}{g^2} \frac{n^2}{n-1} \to \frac{1}{g^2} \) and \( C_n = 1 \), but it seems to be less natural. Thus we retain the string coupling constant \( g \). To emphasize that Lagrangian (3) describes a new field, which takes into account all \( p \)-adic fields, we introduced notation \( \phi \) instead of \( \varphi \). The term \( C_1 \mathcal{L}_1 = 0 \), because of its two equal parts of opposite sign, but these parts give contribution to kinetic and potential terms of the total Lagrangian \( L \).

According to the famous Euler product formula one can write

\[
\sum_{n \geq 1} n^{-\frac{\Box}{2}} = \prod_p \frac{1}{1 - p^{-\frac{\Box}{2}}}. \]

Recall that the Riemann zeta function is defined as
\[
\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \tag{4}
\]

Employing usual expansion for the logarithmic function and definition (4) we can rewrite (3) in the form

\[
L = -\frac{1}{g^2} \left[ \frac{1}{2} \phi \zeta \left( \frac{\Box}{2} \right) \phi + \phi + \ln(1 - \phi) \right], \tag{5}
\]

where \(|\phi| < 1\).

\[\zeta \left( \frac{\Box}{2} \right)\] acts as pseudodifferential operator in the following way (see also [14]):

\[
\zeta \left( \frac{\Box}{2} \right) \phi(x) = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta \left( -\frac{k^2}{2} \right) \tilde{\phi}(k) \, dk, \quad -k^2 = k_0^2 - \vec{k}^2 > 2 + \varepsilon, \tag{6}
\]

where \(\tilde{\phi}(k) = \int e^{-ikx} \phi(x) \, dx\) is the Fourier transform of \(\phi(x)\). The region of integration in (6) is \(-k^2 = k_0^2 - \vec{k}^2 > 2 + \varepsilon\), where \(\varepsilon\) is an arbitrary small positive number, and it follows from the definition of zeta function (4). Here and in the sequel, it is understood that this zeta function depends also on \(\varepsilon\).

The usual tachyon with mass \(m^2 = -k^2 = -2\) is absent in this theory and the energy of this new string is bounded from below by \(k_0^2 = 2\) in the string mass scale.

Dynamics of this field \(\phi\) is encoded in the (pseudo)differential form of the Riemann zeta function. When the d’Alembertian is an argument of the Riemann zeta function we shall call such string a zeta string. Consequently, the above \(\phi\) is an open scalar zeta string.

The equation of motion for the zeta string \(\phi\) is

\[
\zeta \left( \frac{\Box}{2} \right) \phi = \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2 + \varepsilon} e^{ixk} \zeta \left( -\frac{k^2}{2} \right) \tilde{\phi}(k) \, dk = \frac{\phi}{1 - \phi}, \tag{7}
\]

which has an evident solution \(\phi = 0\).

The above zeta string potential is given by

\[
V(\phi) = \frac{1}{g^2} [\phi + \ln(1 - \phi)] = -\frac{1}{g^2} \sum_{n \geq 2} \frac{\phi^n}{n}, \tag{8}
\]
where $V(\phi) \leq 0$ for $-1 < \phi < 1$: it increases from $V(\phi \to -1) = -\frac{0.33}{g^2}$ to the maximum $V(\phi = 0) = 0$ and then $V(\phi)$ decreases so that $V(\phi) \to -\infty$ as $\phi \to +1$.

For the case of time dependent spatially homogeneous solutions one has to consider the equation of motion

$$
\zeta \left( \frac{-\partial_t^2}{2} \right) \phi(t) = \frac{1}{(2\pi)} \int_{|k_0|>\sqrt{2+\varepsilon}} e^{-ik_0 t} \zeta \left( \frac{k_0^2}{2} \right) \tilde{\phi}(k_0) \, dk_0 = \frac{\phi(t)}{1-\phi(t)}. \tag{9}
$$

In the weak field approximation ($|\phi(t)| \ll 1$) the above expression $\phi/(1 - \phi) \approx \phi$ and (9) becomes a linear equation which can be written in the form

$$
\int_\mathbb{R} e^{-ik_0 t} \left[ \zeta \left( \frac{k_0^2}{2} \right) \theta(|k_0| - \sqrt{2} - \varepsilon) - 1 \right] \tilde{\phi}(k_0) \, dk_0 = 0, \tag{10}
$$

where $\theta$ is the Heaviside function. Since $\zeta \left( \frac{k_0^2}{2} \right) > 1$ when $|k_0| > \sqrt{2}$ the equation (10) has solution only for $\tilde{\phi}(k_0) = 0$. This also means the absence of mass.

### 3 Open and closed scalar zeta strings

The exact Lagrangian for the coupled open and closed $p$-adic tachyons is presented in [2] and it reads

$$
L' = \frac{1}{g^2} \frac{p^2}{p - 1} \left[ - \frac{1}{2} \varphi \psi^{p-1} \varphi + \frac{1}{p + 1} \psi^{p+1} (\varphi^{p+1} - 1) \right] + \frac{1}{h^2} \frac{p^4}{p^2 - 1} \left[ - \frac{1}{2} \psi \psi^{p-1} \psi + \frac{1}{p^2 + 1} \psi^{p+1} \right] = \frac{1}{g^2} \mathcal{M}_p(\varphi, \psi) + \frac{1}{h^2} \mathcal{N}_p(\psi), \tag{11}
$$

where $g$ and $h$ are the open and closed string coupling constants ($h \sim g^2$), respectively. Tachyon condensation with this model was analyzed in [16]. According to the above adopted approach for construction of the zeta string Lagrangian we start with
\[ L' = \frac{1}{g^2} \sum_{n \geq 1} \frac{n-1}{n^2} \mathcal{M}_n(\phi, \theta) + \frac{1}{\hbar^2} \sum_{n \geq 1} \frac{n^2 - 1}{n^4} \mathcal{N}_n(\theta) \]
\[ = \frac{1}{g^2} \sum_{n \geq 1} \left[ -\frac{1}{2} \phi n^{1+\phi} \phi + \frac{1}{n+1} \theta^{n(n-1)2}(\phi^{n+1} - 1) \right] \]
\[ + \frac{1}{\hbar^2} \sum_{n \geq 1} \left[ -\frac{1}{2} \theta n^{1+\theta} \theta + \frac{1}{n^2+1} \theta^{n^2+1} \right], \tag{12} \]

where infinite power series are convergent for \(|\phi| < 1\) and \(|\theta| < 1\).

Using again Riemann’s zeta function definition \((4)\) one obtains Lagrangian
\[ L' = \frac{1}{g^2} \left[ -\frac{1}{2} \phi \zeta\left(\frac{\Box}{2}\right) \phi + \sum_{n \geq 1} \frac{1}{n+1} \theta^{n(n-1)2}(\phi^{n+1} - 1) \right] \]
\[ + \frac{1}{\hbar^2} \left[ -\frac{1}{2} \theta \zeta\left(\frac{\Box}{4}\right) \theta + \sum_{n \geq 1} \frac{1}{n^2+1} \theta^{n^2+1} \right], \tag{13} \]

for the coupled zeta strings \(\phi\) and \(\theta\), which are open and closed, respectively.

The equations of motion are
\[ \zeta\left(\frac{\Box}{2}\right) \phi = \frac{1}{(2\pi)^D} \int e^{ikx} \left( -\frac{k^2}{2} \right) \tilde{\phi}(k) \, dk = \sum_{n \geq 1} \theta^{n(n-1)2} \phi^n, \tag{14} \]
\[ \zeta\left(\frac{\Box}{4}\right) \theta = \frac{1}{(2\pi)^D} \int e^{ikx} \left( -\frac{k^2}{4} \right) \tilde{\theta}(k) \, dk \]
\[ = \sum_{n \geq 1} \left[ \theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{n(n-1)2}(\phi^{n+1} - 1) \right], \tag{15} \]

and one can easily see trivial solution \(\phi = \theta = 0\). The weak field approximation obtains taking only terms with \(n = 1\) and related equations of motion are:
\[ \zeta\left(\frac{\Box}{2}\right) \phi = \phi, \quad \zeta\left(\frac{\Box}{4}\right) \theta = \theta \tag{16} \]
whose spatially homogeneous solutions have only trivial solutions, i.e. \(\phi(t) = \theta(t) = 0\) and there is no mass.
The corresponding potential is

\[
V(\phi, \theta) = \sum_{n \geq 1} \left[ \frac{1}{g^2} \frac{1}{n + 1} \theta^{n(n-1)/2} (1 - \phi^{n+1}) - \frac{1}{h^2} \frac{1}{n^2 + 1} \theta^{n^2+1} \right],
\]

for which \(V(\phi, \theta = 0) = 0\) but its behavior in the region \(-1 < \phi < 1, -1 < \theta < 1\) is more complex than potential (8) and needs an analysis in detail.

4 Concluding remarks

We derived effective field Lagrangians for description of open and open-closed bosonic strings, which contain all p-adic Lagrangians. As a result one obtains that an infinite number of spacetime derivatives and related nonlocality are governed by the Riemann zeta function. Potentials are nonlinear. The tachyon is absent in this theory, although it is contained in the constitutive p-adic Lagrangians. In this model there is no mass. Energy is bounded from below. p-Adic ingredients can be easily restored from the whole Lagrangian using just an inverse procedure for its construction.

This paper contains only some classical field properties of the introduced zeta strings. There are still many classical aspects which need to be investigated. One of them is a systematic study of the equations of motion. In the quantum sector it is very desirable to derive and explore scattering amplitudes.

In this short paper we have restricted our attention to the case when fields satisfy \(|\phi| < 1\) and \(|\theta| < 1\), and the Riemann zeta function \(\zeta(s)\) is defined for \(\Re s > 1\). Analytic continuation of the potentials and the zeta function should provide more complete insights.

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References

[1] I.V. Volovich, p-Adic space-time and string theory, Theor. Math. Phys. 71 (1987) 340; see also p-Adic string, Class. Quantum Grav. 4 (1987) L83 - L87.

[2] l. Brekke and P.G.O. Freund, p-Adic numbers in physics, Phys. Rep. 233 (1993) 1.

[3] V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, p-Adic analysis and mathematical physics, World Scientific, Singapore, 1994.

[4] L. Brekke, P.G.O. Freund, M. Olson and E. Witten, Non-archimedean string dynamics, Nucl. Phys. B 302 (1988) 365.

[5] P.H. Frampton and Y. Okada, Effective scalar field theory of p-adic string, Phys. Rev. D 37 (1988) 3077; The p-adic string N-point function, Phys. Rev. Lett. 60 (1988) 484.

[6] D. Ghoshal and A. Sen, Tachyon condensation and brane descent relations in p-adic string theory, Nucl. Phys. B 584 (2000) 300, hep-th/0003278

[7] J.A. Minahan, Mode interactions of the tachyon condensate in p-adic string theory, JHEP 0103 (2001) 028, hep-th/0102071; A. Sen, Time evolution in open string theory, JHEP 0210 (2002) 003, hep-th/0207105; N. Moeller and B. Zwiebach, Dynamics with infinitely many time derivatives and rolling tachyons, JHEP 0210 (2002) 034, hep-th/0207107; H. Yang, Stress tensors in p-adic string theory and truncated OSFT, JHEP 0211 (2002) 007; I.Ya. Aref’eva, L.V. Joukovskaya and A.S. Koshelev, Time evolution in superstring field theory on non BPS-brane. I. Rolling tachyon and energy-momentum conservation, JHEP 0309 (2003) 012, hep-th/0301137

[8] D. Ghoshal and T. Kawano, Towards p-adic string in constant B-field, Nucl. Phys. B 710 (2005) 577, hep-th/0409311; P. Grange, Deformation of p-adic string amplitudes in a magnetic field, Phys. Lett. B 616 (2005) 135, hep-th/0409305

[9] B. Dragovich and I.V. Volovich, p-Adic strings and noncommutativity, in 'Noncommutative Structures in Mathematics and Physics', eds. S.
[10] V.S. Vladimirov and Ya.I. Volovich, *On the nonlinear dynamical equation in the p-adic string theory*, Theor. Math. Phys. 138 (2004) 297, math-ph/0306018.

[11] I.Ya. Aref’eva, *Nonlocal string tachyon as a model for cosmological dark energy*, AIP Conf. Proc. 826 (2006) 301, astro-ph/0410443; N. Barnaby, T. Biswas and J.M. Cline, *p-Adic inflation*, hep-th/0612230.

[12] D. Ghoshal, *p-Adic string theories provide lattice discretization to the ordinary string worldsheet*, Phys. Rev. Lett. 97 (2006) 151601.

[13] B. Dragovich, *Adelic model of harmonic oscillator*, Theor. Math. Phys. 101 (1994) 1404; *Adelic harmonic oscillator*, Int. J. Mod. Phys. A 10 (1995) 2349, hep-th/0404160.

[14] I.Ya Aref’eva and I.V. Volovich, *Quantization of the Riemann zeta-function and cosmology*, hep-th/0701284.

[15] A. Gerasimov and S. Shatashvili, *On exact tachyon potential in open string field theory*, JHEP 10 (2000) 034, hep-th/0009103.

[16] N. Moeller and M. Schnabl, *Tachyon condensation in open-closed p-adic string theory*, hep-th/0304213.