Dynamic Interventions With Limited Knowledge in Network Games

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Abstract—This article studies the problem of intervention design for steering the actions of noncooperative players in quadratic network games to the social optimum. The players choose their actions with the aim of maximizing their individual payoff functions, while a central regulator uses interventions to modify their marginal returns and maximize the social welfare function. This work builds on the key observation that the solution to the steering problem depends on the knowledge of the regulator on the players’ parameters and the underlying network. We therefore consider different scenarios based on limited knowledge and propose suitable static, dynamic, and adaptive intervention protocols. We formally prove convergence to the social optimum under the proposed mechanisms. We demonstrate our theoretical findings on a case study of Cournot competition with differentiated goods.

Index Terms—Economic networks, game theory, networked control systems.

I. INTRODUCTION

NETWORK games have emerged as a powerful tool for studying the scenarios where the well-being of individuals depends on their own decisions as well as the actions of their neighbors in an interaction network. These games have a broad spectrum of applications such as studying crime networks [1] pricing in social networks [2], [3]; public good provision [4]; firm competition [5]; and telecommunication [6]. We refer to [7] for a systematic analysis of the outcome of network games via the use of variational inequalities. In economics, the problem of influencing the outcome of network games by interventions has been of great interest, and this has led to various works typically studying the effects of the network topology on optimal policies, see, e.g., [1], [8], [9], [10].

Generally speaking, noncooperative games involve players who are self-interested/selfish and pursue their own well-being. Such selfish behavior of the players entails degradation of performance in comparison to the scenarios where the players would cooperate to maximize the social welfare. The deterioration in performance has led to the definitions of two performance metrics termed the price of anarchy [11] and the price of stability [12], and their quantification is extensively studied in different applications such as resource allocation [13], congestion games [14], [15], and supply chains [16].

An active line of research concerns improving the performance of noncooperative games and realigning the preferences of the players with the social optimum through interventions. To this end, a central regulator provides incentives to coordinate the players and alter their strategies toward the social optimum. The main challenge, however, is that optimal incentives depend on private information of the players, generally unknown to the regulator [17]. The celebrated Vickrey–Clarke–Groves (VCG) mechanism [18] is adopted in different disciplines, and especially in economics, to address this problem. In this setup, the mechanism generates a payment rule with the aim of incentivizing the players to announce their private information to the regulator. This information is then used to reach to the social optimum. See [19] for more details on the topic.

Another methodology for enhancing the performance and achieving the social optimum in noncooperative games is to exploit control-theoretic tools. In this case, the players do not report their private information, but their actions are observed over time by the regulator. The problem is then regarded as a feedback control problem where the desired outcome is the social optimum, and the control effort is implemented through interventions [20]. Devising suitable control laws is straightforward when the regulator has perfect information on the game and the payoffs of the players, whereas it becomes much more intricate when some of the players’ private information and/or network level parameters are unknown. To overcome this lack of information, dynamical protocols are proposed in [20], [21], and [22].

In [20], a dynamic pricing mechanism is proposed that addresses the problem for players with separable utility functions, that is, the players’ utility functions only depend on their own actions, not those of others. When the utility functions are nonseparable, Alpcan et al. [21] utilized side information to steer the players toward social optimum. In this case, the pricing mechanism incorporates the utility functions evaluated at the Nash equilibrium and their gradient information. In the context of congestion control, the mechanism presented in [22] ensures convergence, provided that the network manager possesses information on the aggregate flow on each link and the delay cost experienced by the users. The aforementioned mechanisms, however, are not generally applicable to network games, where players’ payoff functions are nonseparable and the information available to the regulator is limited.
It is worth mentioning that a large number of studies have also explored employing pricing to improve efficiency in different contexts, see, e.g., communication networks [23], [24], [25]. These studies, however, are limited to specific setups, which are not directly applicable to network games.

In this work, we address the problem of steering the actions of noncooperative players in quadratic network games to the solution of the social welfare maximization problem. We consider selfish players who maximize their individual payoff functions by following pseudogradient dynamics. The regulator, on the other hand, is aimed at nudging the players toward the social optimum, and to do this, he or she modifies the marginal returns of the players through interventions. Essential to our results is the observation that the choice of interventions structurally depends on the information available to the regulator. Therefore, we differentiate among notable cases concerning the knowledge available to the regulator: Full game information, the network structure, or an estimate of social optimum. Unavailability of such information gives rise to a fourth scenario, where an adaptive protocol is proposed to achieve the social optimum. We provide analytical convergence guarantees for all the proposed protocols, and accompany our findings with a numerical case study of Cournot competition.

The challenges of the current work are the coupling among players’ actions, the limited information available to the regulator, and the presence of constraints on both actions and interventions. The limited information requires adapting the type of the control policy accordingly (open-loop, static, dynamic, adaptive), whereas the presence of constraints requires using tools from variational inequalities and projected dynamical systems.

The structure of the article is as follows. Notations and preliminaries are provided in Section II. Section III discusses the network game model and characterizes the optimization problem faced by the regulator. Section IV includes the intervention protocols and presents their convergence guarantees to the social optimum. The case study is provided in Section V, and concluding remarks and future research directions are stated in Section VI. Existence of a unique social optimum and boundedness analysis of the adaptive mechanism are presented in the Appendix.

II. NOTATIONS AND PRELIMINARIES

This section introduces notational conventions and provides a few basic notions on convex analysis.

A. Notations

The set of real and nonnegative real numbers is denoted by \( \mathbb{R} \) and \( \mathbb{R}_{\geq 0} \), respectively. We denote the standard Euclidean norm by \( \| \cdot \| \). The symbol \( 0 \) denotes a vector/matrix of all zeros. For given vectors \( x_1, \ldots, x_m \in \mathbb{R}^n \), we use the notation \( \text{col}(x_i) := [x_i^1, \ldots, x_i^m]^\top \). We use \( P \succ 0 (\prec 0) \) to denote that \( P = P^\top \in \mathbb{R}^{n \times n} \) is positive definite (negative definite). Given a matrix \( P = P^\top \in \mathbb{R}^{n \times n} \), we denote its Frobenius norm by \( \| P \|_F = \sqrt{\text{Tr}(P^\top P)} \), where \( \text{Tr}(\cdot) \) is the trace operator. Moreover, the notation \( \lambda_i(P) \) with \( i \in \{ 1, \ldots, n \} \) denotes the eigenvalues of \( P \), and \( \lambda_{\min}(P) \) and \( \lambda_{\max}(P) \) are the minimum and the maximum eigenvalues of \( P \), respectively. The weighted Euclidean norm of a vector \( x \in \mathbb{R}^n \) is given by \( \| x \|_r = \sqrt{x^\top P x} \) where \( P \succ 0 \). A function \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is hypomonotone, monotone, and strongly monotone, respectively, if it satisfies \( (x - y)^\top (F(x) - F(y)) \geq \mu \| x - y \|^2 \) for all \( x, y \in \mathbb{R}^n \), with \( \mu \leq 0, \mu = 0, \) and \( \mu > 0 \), respectively. For a piecewise continuous function \( x : [0, \infty) \rightarrow \mathbb{R}^n \), we define the \( \mathcal{L}_2 \) and \( \mathcal{L}_2 \) norms as \( \| x \|_{\infty} := \sup_{t \geq 0} \| x(t) \|_2 \) and \( \| x \|_2 := (\int_0^\infty \| x(t) \|_2^2 \, dt)^{\frac{1}{2}} \), respectively. Moreover, we say \( x \in \mathcal{L}_2 \) when \( \| x \|_{\infty} \) is finite, and \( x \in \mathcal{L}_2 \) when \( \| x \|_2 \) is finite.

B. Convex Analysis

Consider a nonempty, closed, and convex set \( \mathcal{X} \subseteq \mathbb{R}^n \). We denote the projection of a point \( z \in \mathbb{R}^n \) on to the set \( \mathcal{X} \) by \( \text{proj}_\mathcal{X}(z) := \arg \min_{x \in \mathcal{X}} \| y - z \|_2 \). Given a point \( x \in \mathcal{X} \), the set \( \mathcal{N}_\mathcal{X}(x) := \{ y \in \mathbb{R}^n \mid y^\top (s - x) \leq 0, \forall s \in \mathcal{X} \} \) is the normal cone to \( \mathcal{X} \) at \( x \), and the tangent cone is denoted by \( \mathcal{T}_\mathcal{X}(x) := \text{cl}(\cup_{y \in \mathcal{X}} \mathcal{N}_\mathcal{X}(y - x)) \), where \( \text{cl}(\cdot) \) is the closure. Given a point \( z \in \mathbb{R}^n \), we denote its projection on to \( \mathcal{T}_\mathcal{X}(x) \) by \( \Pi_{\mathcal{T}_\mathcal{X}(x)}(z) \). It also follows from Moreau’s decomposition theorem [26, Th. 3.2.5] that \( z = \text{proj}_{\mathcal{N}_\mathcal{X}(x)}(z) + \text{proj}_{\mathcal{T}_\mathcal{X}(x)}(z) \). Given the set \( \mathcal{X} \) and a map \( F : \mathcal{X} \rightarrow \mathbb{R}^n \), the variational inequality problem VI(\( \mathcal{X}, F \)) consists of finding a point \( x \in \mathcal{X} \) such that \((x - \bar{x})^\top F(x) \geq 0 \) for all \( x \in \mathcal{X} \). We write SOL(\( \mathcal{X}, F \)) to denote the set of solutions to VI(\( \mathcal{X}, F \)).

III. PROBLEM FORMULATION

We consider a game with the population of \( \mathcal{I} := \{ 1, \ldots, n \} \) players/agents that interact repeatedly with a central regulator as well as with each other according to an underlying interaction network. We denote the adjacency matrix of this network by \( P \in \mathbb{R}^{n \times n} \), where \( P_{ij} \in [0, 1] \) denotes the influence of player \( j \)’s strategy/action on the utility function of player \( i \). We assume that the network has no self-loop; thus, \( P_{ii} = 0 \) for all \( i \in \mathcal{I} \), and the set of neighbors of player \( i \) is denoted by \( \mathcal{N}_i = \{ j \in \mathcal{I} \mid P_{ij} > 0 \} \). Each player \( i \in \mathcal{I} \) is associated with a payoff function \( U_i(x_i, z_i(x), u_i) \) that depends on his or her own action \( x_i \in \mathcal{X}_i \subseteq \mathbb{R} \), the aggregate of his or her neighbors’ actions

\[ z_i(x) := \sum_{j \in \mathcal{N}_i} P_{ij} x_j \]

(1)

with \( x = \text{col}(x_i) \), and a scalar intervention \( u_i \), which will be determined by the central regulator. We restrict our attention to linear quadratic payoff functions of the form

\[ U_i(x_i, z_i(x), u_i) = W_i(x_i, z_i(x)) + x_i u_i \]

(2)

with

\[ W_i(x_i, z_i(x)) := -\frac{1}{2} x_i^2 + x_i (a z_i(x) + b_i) \]

(3)

where \( a \in \mathbb{R} \setminus \{ 0 \} \) captures the impact of neighbors aggregate actions \( z_i(x) \), and \( b_i \in \mathbb{R} \) is the stand-alone marginal return. The intrinsic payoff function \( W_i \) is used in the literature to model peer effects in social and economic processes, see, e.g., [1], [27], and [28]. The term \( x_i u_i \) is included to capture the intervention of the
central regulator in modifying the stand-alone marginal return $b_i$ to $b_i + u_i$ [8], [10].

In our setup, the interventions $u = \text{col}(u_i)$ take values from a set $U \subseteq \mathbb{R}^n$. The action and intervention constraint sets satisfy the following assumption.

**Assumption III.1:** The constraint sets $X_i \subseteq \mathbb{R}$ and $U \subseteq \mathbb{R}^n$ are nonempty, closed, and convex. Moreover, the set $U$ contains the origin.

**Remark III.2:** We note that while the constraints on the action set are local, namely, $x_i \in X_i$, the interventions constraint set $U$ allows both local, e.g., $U = \mathbb{R}^n$, and coupled constraints, e.g., $U = \{u \in \mathbb{R}^n \mid \|u\| \leq c\}$, for some $c > 0$. Another notable example is given by $U = \{u \in \mathbb{R}^n \mid u_i \in \mathbb{R} \ \forall i \in I \text{ and } u_i = 0 \ \forall i \in I \setminus \mathcal{I}\}$, which can accommodate the case where the regulator applies the intervention to a subset of players only.

**Problem overview:** The players are noncooperative and merely interested in maximizing their individual payoff functions by choosing their actions. The resulting self-interested behavior may deviate or be in contrast with what is desired for the group as a whole. Motivated by this, the central regulator coordinates the players by applying suitable interventions with the aim of steering the players to a more desirable group behavior. The latter target behavior in this article is taken as the social optimum defined as the maximizer of the intrinsic payoff functions, namely, $\max_{y \in X} W_i(y)$. In the next two sections, we discuss the dynamic model capturing the strategies of the players, and characterize the optimization problem faced by the regulator.

### A. Players’ Strategy

Each player aiming at maximizing his or her individual payoff function given the aggregated actions of his or her neighbors and the current value of the intervention signal. To capture this, we consider that the action of each player $i \in \mathcal{I}$ evolves over time according to the following pseudogradient dynamics$^1$:

$$\dot{x}_i(t) = \Pi_{X_i} \left( -\nabla_{x_i} F(x_i(t)) + \alpha \sum_{j \in I} P_{ij} x_j(t) + b_i + u_i(t) \right),$$

where $u_i(t)$ is the intervention designed by the regulator. Noting the definition of $z_i(x)$ given by (1) and the fact that $P_{ii} = 0$, we can rewrite dynamics above as

$$\dot{x}_i(t) = \Pi_{X_i} \left( -\nabla_{x_i} F(x_i(t)) + \alpha \sum_{j \in I} P_{ij} x_j(t) + b_i + u_i(t) \right).$$

Note that in the case of no intervention, i.e., $u_i(t) \equiv 0$, the equilibrium of (4) coincides with the Nash equilibrium of the game, namely, the action profile $x^\text{NE}$ satisfies

$$x^\text{NE} = \max_{y \in X_i} W_i(y, z_i(x^\text{NE})) \quad \forall i \in \mathcal{I},$$

where $W_i$ is given by (3). The Nash equilibrium $x^\text{NE}$ can also be expressed as a solution of the variational inequality $\text{VI}(\mathcal{X}, F)$,

$$\text{VI}(\mathcal{X}, F) = \max_{x \in \mathcal{X}} \sum_{i \in \mathcal{I}} W_i(x, z_i(x)),$$

where $\mathcal{X} = \prod_{i \in \mathcal{I}} X_i$ and $F(x) := (I - aP)x - b$. That is

$$x^\text{NE} \in \text{SOL}(\mathcal{X}, F).$$

Next, we look at the problem from the regulator’s side.

### B. Regulator’s Objective

The central regulator aims to implement suitable interventions to coordinate the players and maximize the total payoff. More precisely, he or she aims at designing the intervention signal $\text{col}(u_i(t))$ such that the actions of the players converge to a social optimum $x_{opt}$, defined as a solution of the social welfare maximization problem

$$x_{opt} \in \text{arg max}_{y \in X} \sum_{i \in \mathcal{I}} W_i(y_i, z_i(y))$$

where $y = \text{col}(y_i)$ and $W_i$ is given by (3). Any social optimum $x_{opt}$ is also a solution to the following variational inequality problem [34, Prop. 2.1.2]:

$$x_{opt} \in \text{SOL}(\mathcal{X}, H)$$

where $H(x) = (I - a(P + P^T)x - b$. Note that $-H$ is the gradient of the social welfare function $\sum_{i \in \mathcal{I}} W_i(x_i, z_i(x))$.

Observe that $x_{opt}$ differs from the Nash equilibrium in (5). The regulator, therefore, aims to design intervention mechanisms that solve the following problem.

**Problem formulation:** Design intervention mechanisms $u \in \mathcal{U}$ that asymptotically steer the action profile $x$ of the players in (4) to the social optimum $x_{opt}$ given by (7).

### IV. INTERVENTION PROTOCOLS

Before proceeding with the intervention protocols, we discuss the existence of a unique social optimum and comment on the feasibility of the formulated problem.

**Lemma IV.1:** Let Assumption III.1 hold. Then, the social welfare maximization problem (6) has a unique solution if

$$\max_{i \in \mathcal{I}} \lambda_i(P + P^T) < 1.$$ 

**Proof:** See the Appendix.

Note that for each player, the intrinsic payoff function $W_i$ is strongly concave with respect to her action $x_i$. The condition (8), therefore, implies that the coupling between the payoff functions, induced by $aP$, is not too strong as to disturb the strong concavity property, thereby ensuring the existence of a unique maximizer. We also note that the sufficient condition (8) is in general necessary if one looks at arbitrary constraint set $X_i$ satisfying Assumption III.1. A notable example is given by $X_i = \mathbb{R}$ for all $i \in \mathcal{I}$; see [35, Lem. II.1].

Motivated by Lemma IV.1, we impose the following standing assumption throughout the article.

**Assumption IV.2:** The adjacency matrix $P \in \mathbb{R}^{n \times n}$ and the parameter $a \in \mathbb{R}$ satisfy $\max_{i \in \mathcal{I}} a \lambda_i(P + P^T) < 1$. 

\[ \text{Existence of a Nash equilibrium follows from analogous arguments to the proof of [32, Cor. 4.2], and the relation in (5) is satisfied using [35, Prop. 1.4.2].} \]

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Remark IV.3: The matrix $P + P^\top$ is symmetric with the diagonal elements equal to zero. This implies that the matrix $P + P^\top$ has only real eigenvalues and their sum is zero. Hence

$$\lambda_{\min}(P + P^\top) < 0 < \lambda_{\max}(P + P^\top).$$

It follows from the above inequalities that Assumption IV.2 is satisfied if and only if either (i) $a > 0$ and $a \lambda_{\max}(P + P^\top) < 1$ or (ii) $a < 0$ and $a \lambda_{\min}(P + P^\top) < 1$.

As a consequence of Assumption IV.2, the social welfare function on the right-hand side of (6) is strongly concave and thus admits a unique maximizer that is also the solution of (7), namely

$$x_{\text{opt}} = \text{SOL}(X, H)$$

with $H(x) = (I - a(P + P^\top))x - b$.

Having established the uniqueness of the social optimum $x_{\text{opt}}$, we shift our attention to the feasibility of the problem formulated at the end of the previous section.

Noting (4), we recall that the action profile evolves according to the following projected pseudogradient dynamics:

$$\dot{x}(t) = \Pi_X(x(t), -F(x(t)) + u(t))$$

where $u \in \mathcal{U}$ and

$$F(x) = (I - aP)x - b.$$  \hfill (11)

The dynamics (10) at steady-state reads as $0 = \Pi_X(\bar{x}, -F(\bar{x}) + \bar{u})$ for constant action-intervention pairs $(\bar{x}, \bar{u}) \in \mathcal{X} \times \mathcal{U}$. We thus deduce from Moreau’s decomposition theorem that $0 = -F(\bar{x}) + \bar{u} - \text{proj}_{\mathcal{N}_X}(\bar{x})(-F(\bar{x}) + \bar{u})$ or, equivalently, $-F(\bar{x}) + \bar{u} \in \mathcal{N}_X(\bar{x})$. By [36, Ex. 6.13], the pair $(\bar{x}, \bar{u})$ satisfies the latter inclusion only if $\bar{x}$ belongs to the set below

$$S := \{\bar{x} \in \mathcal{X} | \exists \bar{u} \in \mathcal{U} \text{ such that } \bar{x} = \text{SOL}(\mathcal{X}, F - \bar{u})\}.$$  \hfill (12)

The set $S$ contains all assignable equilibria (action profile) of (10), which necessitates the following assumption on $x_{\text{opt}}$.

Assumption IV.4: The social optimum $x_{\text{opt}}$ given by (9) belongs to the set $S$ in (12).

In general, the above condition depends on the network game parameters. This dependence can be explicitly observed in the special case of $\mathcal{X} = \mathbb{R}^n$, where one can show that the Assumption IV.4 reduces to the condition

$$aP^\top \left(I - a \left(P + P^\top\right)\right)^{-1} b \in \mathcal{U}.$$  

This imposes the budget to be sufficiently big, determined by the network game parameters. For the general case, a sufficient condition for Assumption IV.4 is given by $aP^\top x_{\text{opt}} \in \mathcal{U}$.

The Role of Limited Knowledge: In what follows, we provide several intervention protocols that are able to steer the action profile toward the social welfare $x_{\text{opt}}$. Key to our results is the observation that the suitable intervention depends on the knowledge of the regulator on the underlying game parameters. We emphasize that Assumptions III.1, IV.2, and IV.4 are assumed to hold throughout this section.

3Note that in (12), the variational inequality problem VI($\mathcal{X}, F - \bar{u}$) has a unique solution since $F$ is strongly monotone (see (14) and [33, Th. 2.3.3]).

A. Static Open-Loop Intervention

The first case that we consider is where the regulator has full access to the game information, i.e., $(aP, b)$ and $\mathcal{X}_i$’s. The regulator, therefore, can use this knowledge to compute $x_{\text{opt}}$ and its corresponding intervention $u_{\text{opt}} \in \mathcal{U}$, with $\text{SOL}(\mathcal{X}, F - u_{\text{opt}}) = x_{\text{opt}}$. Note that such $u_{\text{opt}}$ exists by Assumption IV.4. The regulator can then implement the protocol $u(t) \equiv u_{\text{opt}}$ to steer the action profile to $x_{\text{opt}}$. This is formalized in the following proposition.

Proposition IV.5: Consider the pseudogradient dynamics (10). Let $u_{\text{opt}} \in \mathcal{U}$ be such that $\text{SOL}(\mathcal{X}, F - u_{\text{opt}}) = x_{\text{opt}}$. Then, for any initial condition $x(0) \in \mathcal{X}$, the static open-loop intervention $u(t) \equiv u_{\text{opt}}$ steers the action profile $x(t)$ to the social optimum $x_{\text{opt}}$. Moreover, $u_{\text{opt}}$ satisfies

$$u_{\text{opt}} = (I - aP)x_{\text{opt}} - b + v$$

for some $v \in \mathcal{N}_X(x_{\text{opt}})$.

Proof: We first use the relation $x_{\text{opt}} = \text{SOL}(\mathcal{X}, F - u_{\text{opt}})$ to show that $u_{\text{opt}}$ admits the form (13). To see this, note that

$$(y - x_{\text{opt}})^\top (F(x_{\text{opt}}) - u_{\text{opt}}) \geq 0 \quad \forall y \in \mathcal{X}.$$  

This implies that $v := -F(x_{\text{opt}}) + u_{\text{opt}} \in \mathcal{N}_X(x_{\text{opt}})$. The latter yields $u_{\text{opt}} = F(x_{\text{opt}}) + v$, which together with (11) establishes (13).

Next we prove that the dynamics (10) under the input (13) has a unique solution $x(t)$ that converges to the social optimum. In this regard, we rewrite the overall dynamics as follows:

$$\dot{x} = \Pi_X(x, -T(x))$$

where $T(x) := F(x) - F(x_{\text{opt}}) - v$. We note that for the mapping $T$, the following holds:

$$(x - y)^\top (F(x) - F(y)) = \|x - y\|^2 \left(1 - \frac{1}{2}a(P + P^\top)\right)$$

$$\geq \frac{1}{2}\|x - y\|^2 \quad \forall x, y \in \mathbb{R}^n$$

where we have used Assumption IV.2 to obtain the inequality. This means that $F$ is strongly monotone, and in turn, the mapping $T$ is also strongly monotone. In addition, the set $X$ is closed and convex. It then follows from [37, Th. 1] that, for any initial condition $x(0) \in \mathcal{X}$, the above dynamics has a unique solution $x(t)$ for all $t \geq 0$.

Next consider the Lyapunov candidate $V(x) = \frac{1}{2}\|\bar{x}\|^2$ with $\bar{x} = x - x_{\text{opt}}$. The time-derivative of the evolution of $V$ along the solution of the system satisfies

$$\nabla V(x)^\top \Pi_X(x, -T(x)) = -\bar{x}^\top T(x)$$

$$-\bar{x}^\top \text{proj}_{\mathcal{N}_X}(x)(-T(x))$$

where we used Moreau’s decomposition theorem. Note that $-\bar{x}^\top \text{proj}_{\mathcal{N}_X}(x)(-T(x)) \leq 0$ as $x, x_{\text{opt}} \in \mathcal{X}$. It then follows from the definition of $T$ that:

$$\nabla V(x)^\top \Pi_X(x, -T(x)) \leq -\bar{x}^\top (F(x) - F(x_{\text{opt}})) + \bar{x}^\top v.$$
Recalling that $v \in N_X(x_{opt})$, we have $\bar{x}^Tv \leq 0$ and, in turn, we obtain

$$\nabla V(x)^T \Pi_X(x, -T(x)) \leq -\frac{1}{2}\|\bar{x}\|^2$$

where we have used (14). The above inequality implies that $V$ decreases monotonically along the solution of the closed-loop dynamics and the action profile $x(t)$ converges to $x_{opt}$. \hfill $\square$

To compute the optimal value $u_{opt}$, the regulator can begin by solving the convex optimization problem (6) to obtain the social optimum $x_{opt}$. Then, she should compute $N_X(x_{opt})$, where $X$ is the Cartesian product of closed intervals $X_i$ in $\mathbb{R}$ (Assumption III.1). This can be accomplished using [36, Ex. 6.10]. Finally, the regulator can solve the convex feasibility problem of finding a vector $v \in N_X(x_{opt})$ that satisfies $(I-aP)x_{opt} - b + v \in U$, and subsequently obtain $u_{opt}$ as in (13).

B. Static Feedback Intervention

We next consider the case where the regulator has only access to $aP$, but neither $b$ nor $\mathcal{X}_i$’s. This means that the regulator has complete knowledge about the network topology and the impact of the actions of the players on each other. Leveraging this information, we show that under a weak coupling condition, the regulator can steer the players to the social optimum by employing a static state feedback protocol.

**Proposition IV.6:** Consider the pseudogradient dynamics (10). Assume that $aP^Tx_{opt} \in U$ and

$$\|aP\| < \frac{1}{2},$$

(15)

Then, for any initial condition $x(0) \in \mathcal{X}$, the static feedback intervention

$$u(t) = \text{proj}_{\mathcal{X}}(aP^Tx(t))$$

(16)

steers the action profile $x(t)$ to the social optimum $x_{opt}$. \hfill $\square$

**Proof:** The closed-loop dynamics of (10) and (16) is

$$\dot{x} = \Pi_X(x, -T(x))$$

(17)

with $T(x) = F(x) - \text{proj}_{\mathcal{X}}(aP^Tx)$. For the map $T$, we have

$$(x - y)^T(T(x) - T(y)) = \|x - y\|^2_{(I-aP^T)} - (x - y)^T(\text{proj}_{\mathcal{X}}(aP^Tx)) - (x - y)^T(\text{proj}_{\mathcal{X}}(aP^Ty))$$

(18)

for all $x, y \in \mathbb{R}^n$. Note that the projection operator is nonexpansive [34, Prop. 2.1.3], we thus deduce that the second term on the right-hand side of the relation satisfies

$$(x - y)^T(\text{proj}_{\mathcal{X}}(aP^Tx) - \text{proj}_{\mathcal{X}}(aP^Ty)) \leq \|aP\|\|x-y\|^2.$$ 

As a result, it follows from (18) that:

$$(x - y)^T(T(x) - T(y)) \geq (1-2\|aP\|\|x-y\|^2) \quad \forall x, y \in \mathbb{R}^n$$

(19)

This means that $T$ is hypomonotone, hence the dynamics (17) has a unique solution $x(t)$ for all $t \geq 0$ [37, Th. 1].

Let $\bar{x} := x - x_{opt}$, and consider the Lyapunov candidate $V(\bar{x}) = \frac{1}{2}\|\bar{x}\|^2$. The time-derivative of the evolution of $V$ along the solution of (17) satisfies

$$\nabla V(\bar{x})^T \Pi_X(x, -T(x)) = -\bar{x}^T \Pi_X(x, -T(x))$$

$$-\bar{x}^T \text{proj}_{\mathcal{X}}(aP^Tx_{opt})$$

where we have used Moreau’s decomposition theorem. Recall that $-\bar{x}^T \text{proj}_{\mathcal{X}}(aP^Tx_{opt}) \leq 0$ since $x, x_{opt} \in \mathcal{X}$. We therefore have

$$\nabla V(\bar{x})^T \Pi_X(x, -T(x)) \leq -\bar{x}^T T(x).$$

(20)

Note from (9) that $(y - x_{opt})^TH(x_{opt}) \geq 0$ for all $y \in \mathcal{X}$. Adding the left-hand side of this inequality evaluated at $y = x$ to the right-hand side of (20) yields

$$\nabla V(\bar{x})^T \Pi_X(x, -T(x)) \leq -\|\bar{x}\|^2_{(I-a(P+P^T))} + \bar{x}^T (\text{proj}_{\mathcal{X}}(aP^Tx) - aP^Tx_{opt})$$

(21)

where the definitions of $T$ and $H$ are used. The relation $aP^Tx_{opt} \in U$ implies that $aP^Tx_{opt} = \text{proj}_{\mathcal{X}}(aP^Tx_{opt})$. This together with (18) evaluated at $y = x_{opt}$ means that the right-hand side of (21) is equal to $-\bar{x}^T(T(x) - T(x_{opt}))$. We therefore deduce from (19) that

$$\nabla V(\bar{x})^T \Pi_X(x, -T(x)) \leq -(1 - 2\|aP\|)\|\bar{x}\|^2.$$ 

It then follows from (15) that $V$ decreases monotonically along the solution of the closed-loop dynamics and the action profile $x(t)$ converges to $x_{opt}$. \hfill $\square$

Based on Proposition IV.6, the static feedback intervention (16) steers the actions of the players to the social optimum under the condition (15). Interestingly, this condition can be dropped in the case where the constraint set $U$ is sufficiently “large,” namely, if $aP^Tx \in U$ for all $x \in \mathcal{X}$; a trivial example is given by $U = \mathbb{R}^n$. The following corollary summarizes this argument.

**Corollary IV.7:** Consider the pseudogradient dynamics (10), and assume for all $\bar{x} \in \mathcal{X}$, we have $aP^T \bar{x} \in U$. Then, for any initial condition $x(0) \in \mathcal{X}$, the static feedback intervention

$$u(t) = aP^Tx(t)$$

(22)

steers the action profile $x(t)$ to the social optimum $x_{opt}$. \hfill $\square$

**Proof:** Note that the state feedback intervention (16) is equivalent to (22) as $aP^Tx \in U$, that is $\text{proj}_{\mathcal{X}}(aP^Tx) = aP^Tx$. We therefore deduce from the proof of Proposition IV.6 that the closed-loop system has a unique solution $x(t)$ for all $t \geq 0$. Moreover, given the Lyapunov candidate $V(\bar{x}) = \frac{1}{2}\|\bar{x}\|^2$ with $\bar{x} = x - x_{opt}$, its time-derivative along $x(t)$ satisfies (21). Next we use $\text{proj}_{\mathcal{X}}(aP^Tx) = aP^Tx$ and rewrite (21) as follows:

$$\nabla V(\bar{x})^T \Pi_X(x, -T(x)) \leq -\|\bar{x}\|^2_{(I-a(P+P^T))}.$$ 

We conclude from $I - a(P + P^T) \succ 0$ (cf. Assumption IV.2) that $V$ decreases monotonically along the solution of the closed-loop dynamics and $x(t)$ converges to $x_{opt}$. \hfill $\square$

**Remark IV.8:** It is worth mentioning that modifying the stand-alone marginal returns in (2) by setting $u_t = a \sum_{j \in I} f_{ji}x_j$,
transforms the network game into a "potential game" with the potential function being the social welfare, namely, $\sum_{i\in\mathcal{I}} W_i(x_i, z_i(x))$. In fact, bearing in mind that $P_{ji}$ reflects the influence of player $i$ on player $j$, the aforementioned modification balances the game such that the mutual effects between any pair of players become identical. The protocol (22) provides a dynamic counterpart of this marginal returns modification.

C. Dynamic Intervention With Estimated Social Optimum

Next we consider the scenario where the regulator is not aware of the game information $(aP, b)$ and $\mathcal{X}_i$’s, but instead has a reliable estimate of the social optimum $x_{\text{opt}}$, which we denote by $x_s$. The choice $x_s = x_{\text{opt}}$ is a special case. Given the estimate $x_s$, the regulator can resort to an integral control-based intervention to steer the action profile to this point. As before, we require that the target action profile, in this case $x_s$, belongs to the set of assignable equilibria given in (12), that is, $x_s \in \mathcal{S}$.

We present such an intervention and its convergence guarantees in the following proposition.

**Proposition IV.9:** Consider the pseudogradient dynamics (10). Let $x_s \in \mathcal{S}$ and consider the dynamic intervention

$$\dot{u}(t) = \Pi_U (u(t), x_s - x(t)) \tag{23}$$

Then, for any initial condition $x(0) \in \mathcal{X}$, the above intervention protocol steers the action profile $x(t)$ to the point $x_s$.

**Proof:** By using (10) and (23), the dynamics of the overall closed-loop system is given by

$$\dot{\xi} = \Pi_\Lambda (\xi, -T(\xi)) \tag{24}$$

where $\xi = \text{col}(x, u)$, $\Lambda = \mathcal{X} \times \mathcal{U}$, and

$$T(\xi) = \begin{bmatrix} F(x) - u \\ x - x_s \end{bmatrix}.$$ 

We deduce from strong monotonicity of $F$ [see (14)] that the above mapping is monotone, and the set $\Lambda$ is closed and convex. We then obtain from [37, Th. 1] that, for any initial condition $x(0) \in \Lambda$, the dynamics (24) admits a unique solution $\xi(t)$ for all $t \geq 0$.

It follows from $x_s \in \mathcal{S}$ that there exists a $u_s \in \mathcal{U}$ such that:

$$(y - x_s)^T (F(x_s) - u_s) \geq 0 \forall y \in \mathcal{X}. \tag{25}$$

Next we use the inequality above and prove that $(x(t), u(t))$ converges to $(x_s, u_s)$. To this end, consider the Lyapunov candidate $V(\xi) = \frac{1}{2}||\tilde{x}||^2 + \frac{1}{2}||\tilde{u}||^2$ with $\tilde{x} = x - x_s$ and $\tilde{u} = u - u_s$. The time-derivative of the evolution of $V$ along the solution of (24) satisfies

$$\nabla V(\xi)^T \Pi_\Lambda (\xi, -T(\xi)) = \tilde{x}^T (F(x) + u) - \tilde{u}^T \tilde{x}$$

where we have used the Moreau’s decomposition theorem. Since $x, x_s \in \mathcal{X}$ and $u, u_s \in \mathcal{U}$, we have $-\tilde{x}^T \text{proj}_{\mathcal{X}_i}(F(x) + u) \leq 0$ and $-\tilde{u}^T \text{proj}_{\mathcal{X}_i}(\tilde{x}) \leq 0$. We then obtain that

$$\nabla V(\xi)^T \Pi_\Lambda (\xi, -T(\xi)) \leq \tilde{x}^T (F(x) + u) - \tilde{u}^T \tilde{x}.$$ 

Now we add the left-hand side of (25) evaluated at $y = x$ to the right-hand side of the foregoing inequality to get

$$\nabla V(\xi)^T \Pi_\Lambda (\xi, -T(\xi)) \leq -\frac{1}{2}||\tilde{x}||^2$$

where the equality follows from the definition of $F$ given by (11). Note that $I - \frac{1}{2}a(P + P^T)$ is a consequence of Assumption IV.2. Hence, we deduce that

$$\nabla V(\xi)^T \Pi_\Lambda (\xi, -T(\xi)) \leq -\frac{1}{2}||\tilde{x}||^2.$$ 

Let $\xi_0 \in \Lambda$, and $\xi(t)$ be a solution starting from the initial condition $\xi(0) = \xi_0$. Moreover, let $\Theta := V(\xi_0)$ and define the set $\Omega := \{\xi \in \Lambda | \ V(\xi) \leq \Theta\}$. Note that $\xi(0) \in \Omega$, and $\Omega$ is compact since $V(\xi) \to \infty$ as $||\xi|| \to \infty$. It also follows from the inequality above that the solution $\xi(t)$ remains in $\Omega$. We then use the invariance principle for discontinuous systems [39, Prop. 2.1] to conclude that the solution of the closed-loop system converges to the largest invariant set contained in $\{\xi \in \Omega | \ \nabla V(\xi)^T \Pi_\Lambda (\xi, -T(\xi)) = 0\}$. This together with the inequality above imply that $\xi(t)$ also converges to the largest invariant set in $\{\xi \in \Omega | \ |x_s| = 0\}$. We therefore conclude that for any initial condition $x(0) \in \Lambda$, the action profile $x(t)$ converges to $x_s$, and this completes the proof.

D. Adaptive Intervention With Known Stand-Alone Marginal Returns

Recall that in case the regulator knows $aP$ or the social optimum $x_{\text{opt}}$, she can steer the players to the social optimum by implementing the previously discussed interventions. Here, we shift our focus to the case where both $aP$ and $x_{\text{opt}}$ are unknown to the regulator, and she merely has knowledge about the individual stand-alone marginal returns of the players $b_i$. It turns out that such limited knowledge substantially complicates the problem faced by the regulator. To partially tame this complexity, we restrict our attention in this section to the case of unconstrained actions and interventions, i.e., $\mathcal{X}_i = \mathbb{R}$ and $\mathcal{U} = \mathbb{R}^n$, and undirected networks, i.e., $P = P^T$. As a result, the pseudogradient dynamics (10) simplifies to the following:

$$\dot{x}(t) = (-I + aP)x(t) + b + u(t).$$

A natural approach to tackle this problem is to resort to adaptive control techniques, which potentially allow to compensate for lack of complete knowledge on the system dynamics. However, there are certain obstacles that hinder an application of standard adaptive control schemes. First, a control design based on the regulation error $x(t) - x_{\text{opt}}$ is not feasible since $x_{\text{opt}}$ is unknown.

A second attempt would be to try to estimate $x_{\text{opt}}$ by using a reference model such as $\dot{x}_m(t) = (-I + 2aP)x_m(t) + b$. However, while $x_m(t)$ converges to $x_{\text{opt}}$ (see Corollary IV.7 with
$P = P^T$), the reference model is not implementable as the network matrix $aP$ is unknown.

To overcome these challenges, we propose the adaptive feedback intervention protocol

$$u(t) = K(t)x(t)$$  \hfill (27)

with an adaptive gain matrix $K(t)$ determined by the following extended nonlinear dynamics:

$$\dot{z}(t) = -z(t) + K(t)x(t) + b + u(t)$$  \hfill (28a)
$$\dot{w}(t) = -w(t) + e(t)x^\top(t)x(t)$$  \hfill (28b)
$$\dot{K}(t) = e(t)x^\top(t)$$  \hfill (28c)

where

$$e(t) := x(t) - z(t) - w(t).$$

Note that the intervention only uses information on $b$, and no knowledge on $aP$ or $x_{opt}$ is required. The first dynamics (28a) aims to replicate the pseudogradient dynamics (26) and generate $z(t)$ such that it tracks the action profile $x(t)$. The second dynamics (28b) is included for technical reasons and is needed to guarantee boundedness of all solutions. The last dynamics (28c) is chosen such that sign-indefinite terms in the time-derivative of the Lyapunov function are canceled out. As a result, all solutions of the closed-loop system are bounded as stated in the following lemma.

**Lemma IV.10:** Consider the pseudogradient dynamics (26) and let $P = P^T$. Then, under the adaptive feedback intervention given by (27) and (28), all solutions of the closed-loop system are bounded.

**Proof:** See the Appendix.

The next result establishes convergence to the social optimum $x_{opt}$.

**Theorem IV.11:** Let $P = P^T$ and consider the pseudogradient dynamics (26) interconnected with the adaptive feedback intervention given by (27) and (28). Then, the action profile $x(t)$ converges to the social optimum $x_{opt}$.

**Proof:** Let $\xi := (x, e, \Psi)$ with $\Psi = K - aP$. Then, bearing in mind (26), (27), and (28), $\xi$ admits the following dynamics:

$$\dot{x} = (-I + 2aP)x + b + \Psi x$$  \hfill (29a)
$$\dot{e} = -e - \Psi x - ex^\top x$$  \hfill (29b)
$$\dot{\Psi} = ex^\top.$$  \hfill (29c)

We proceed by following similar arguments as in the proof of the LaSalle’s invariance principle [40, Th 4.4], but the proof is tailored for a single (yet arbitrary) trajectory. Let $\xi_0 := (x_0, e_0, \Psi_0)$ with some $x_0, e_0 \in \mathbb{R}^n$ and $\Psi_0 \in \mathbb{R}^{n \times n}$, and $\xi(t)$ be a solution from the initial condition $\xi(0) = \xi_0$. It follows from Lemma IV.10 that this solution is bounded. Thus, there exists a compact set $D$ such that $\xi(t) \in D$ for all $t \geq 0$. It also follows from [40, Lemma 4.1] that the positive limit set $\Omega$ of $\xi(t)$ is nonempty, compact, and invariant. Moreover, $\xi(t)$ approaches $\Omega$ as $t$ tends to infinity.

We now consider the function

$$V(\xi) := \frac{1}{2}||x||^2 + \frac{1}{2}||\Psi||^2_F$$

where we recall that $||\Psi||_F$ is the Frobenius norm. The derivative of $V$ along the solutions of (29) is

$$\dot{V} = -||e||^2 - e^\top \Psi x - ||e||^2||x||^2 + \text{Tr}(\Psi^\top ex^\top)$$
$$= -||e||^2 - ||e||^2||x||^2$$  \hfill (30)

where the last equality is obtained using $e^\top Px = \text{Tr}(\Psi^\top ex^\top)$. Therefore, we have $V \geq 0$ and $\dot{V} \leq 0$, which implies that $V(\xi(t))$ has a limit $V_\infty \geq 0$ as $t \to \infty$. Pick any point $\xi' \in \Omega$, then there is a sequence $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$, such that $\xi(t_n) \to \xi'$ as $n \to \infty$. We obtain from continuity of $V$ that $V(\xi') = \lim_{n \to \infty} V(\xi(t_n)) = V_\infty$. Therefore, since $\xi'$ is chosen arbitrary, we deduce that $V(\xi) = V_\infty$ for all $\xi \in \Omega$, which means that on the invariant set $\Omega$, the function $V$ is constant. Moreover, we have $V(\xi(t)) = 0$ for all $\xi(t) \in \Omega$. Let $E := \{\xi \in D \mid V(\xi) = 0\}$, then we have $\Omega \subset E$. Now let $M$ be the largest invariant set inside $E$, subsequently we have the following relations:

$$\Omega \subset M \subset E \subset D.$$

Noting that $\xi(t)$ approaches $\Omega$ as $t \to \infty$, we obtain that $\xi(t)$ approaches $M$ as $t \to \infty$.

The last step is to find the set $M$. Note from the definition of $E$ and (30) that $E = \{\xi \in D \mid e = 0\}$. Thus, on the invariant set $M$, the dynamics of (29) reads as

$$\dot{x} = (-I + 2aP)x + b$$
$$0 = -\Psi x$$
$$\dot{\Psi} = 0.$$

Noting that $-I + 2aP$ is Hurwitz as a consequence of Assumption IV.2, the largest invariant set in $E$ is given by

$$M = \{\xi \in D \mid x = x_{opt}, e = 0, \Psi x_{opt} = 0\}.$$

Consequently, we conclude that $x(t)$ converges to $x_{opt}$ as desired.

**V. ILLUSTRATIVE EXAMPLES**

We consider a Cournot competition, where a set of $I = \{1, \ldots, 10\}$ firms produce differentiated goods [27]. For each firm $i$, we denote the amount of good by $x_i \in X_i$, and its corresponding price is obtained from the inverse demand function $p_i(x) = \alpha_i - \frac{1}{2}(x_i + 2\beta \sum_{j \neq i} P_{ij} x_j)$. In this equation, $\alpha_i > 0$ is the maximum price that consumers would pay for the good, $\beta P_{ij} > 0$ is the degree of product substitutability, where $P_{ij} \in (0, 1]$ if the product of firm $j$ is a substitute for firm $i$ and $P_{ij} = 0$ otherwise. The payoff function of firm $i$, therefore, can be written in the form of (2) and (3) as

\[ P_i(x) = [\alpha_i - \frac{1}{2}(x_i + 2\beta \sum_{j \neq i} P_{ij} x_j)] x_i. \]
Actions of the players and their distance to social optimum

follows:
\[
U_i(x_i, x, u_i) = x_i p_i(x) - x_i d_i + x_i u_i
\]
\[
= -\frac{1}{2} x_i^2 + x_i \left( a \sum_{j \neq i} P_{ij} x_j + b_i \right) + x_i u_i
\]
where \(a = -\beta\) and \(b_i = \alpha_i - d_i\) with \(d_i > 0\) being the marginal cost and \(u_i\) reflects taxes or subsidies provided by the regulator.

Next we present the simulation results under our interventions and illustrate convergence of the players’ actions to the social optimum.

A. Open-Loop, Static Feedback, and Dynamic Interventions

Here, we consider a competition where \(x_i \in X_i = \mathbb{R}_{>0}\) and \(\beta = 0.2\), and the products of the firms are substitutable according to the weighted directed graph depicted in Fig. 1. In this graph, the weight of each link from firm \(j\) to firm \(i\) denotes the weight \(P_{ij}\), and the number next to each node \(i\) indicates its stand-alone marginal return, namely, \(\alpha_i - d_i\).

The social optimum of this game is
\[
x_{\text{opt}} = \text{col}(2.19, 0.01, 0.99, 0.49, 1.34, 3.4, 0, 0, 0.99, 0.04).
\]
In this setup, the firms take their actions using the pseudogradients dynamics (10) with arbitrary initialization. On the other hand, the regulator incentivizes the firms to the social optimum by applying bounded taxes \(u \in \mathcal{U}\), where \(\mathcal{U} = [-2, 0]^\mathcal{N}\). Next we use intervention mechanisms to obtain suitable taxes.

Open-loop intervention: Having full information of the game, the regulator can find the social optimum given above as well as \(N_X(x)\). Noting that \(X = \mathbb{R}_{>0}\), the latter can be expressed as
\[
N_X(x) = \prod_{i \in I} N_{X_i}(x_i), \quad \text{where} \quad [36, \text{Ex. 6.10}]
\]
\[
N_{X_i}(x_i) = \begin{cases} 
0, & \text{if } 0 < x_i \\
\mathbb{R}_{\leq 0}, & \text{if } x_i = 0 \\
0, & \text{otherwise}.
\end{cases}
\]
Therefore, the regulator can obtain \(N_X(x_{\text{opt}})\) and find a vector \(v \in N_X(x_{\text{opt}})\) that results in \((I - aP)x_{\text{opt}} - b + v \in \mathcal{U}\). It then follows from (13) that \(u_{\text{opt}} = (I - aP)x_{\text{opt}} - b + v\). Carrying out these calculations results in \(u_{\text{opt}} = \text{col}(0.001, 0.96, 0.003, 0.09, 0.08, 0.11, 0.01, 0.29, 0)\). Fig. 2 shows the simulation results obtained by applying the intervention \(u(t) = u_{\text{opt}}\). The figure includes two plots: The upper plot depicts the evolution of the players’ action profile over time, with each line representing a different player’s action. The lower plot shows the convergence error of the action profile to the social optimum \(x_{\text{opt}}\). The obtained results demonstrate the effectiveness of the intervention and convergence of the players’ actions to the social optimum.

Static Feedback Intervention: Under the assumption that the regulator knows \(aP\), she can implement (16). Given the set \(\mathcal{U} = [-2, 0]\), we can explicitly express this intervention as follows:
\[
u_i(t) = \begin{cases}
z_i(t), & \text{if } -2 \leq z_i(t) \leq 0 \\
-2, & \text{if } z_i(t) < -2 \\
0, & \text{if } z_i(t) > 0
\end{cases}
\]
with \(z(t) := aP^T x(t)\). Fig. 3 presents the simulation results of the intervention implementation, showing the convergence of the players’ actions to the social optimum. The actions of the players, consequently, converge to the social optimum as desired.

Dynamic intervention: We assume that the regulator only knows the value of the social optimum, she can then implement (23) with \(x_s = x_{\text{opt}}\). In this mechanism, the projection operator

Fig. 1. Directed network illustrating asymmetrical product substitutability.

Fig. 2. Actions of the players and their distance to social optimum under static open-loop intervention.

Fig. 3. Actions of the players and their distance to social optimum under static feedback intervention.

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Actions of the players and their distance to social optimum is compact as a result of strong concavity of the social optimum $x(t) \in X$. The inequality $x(t) + p(t) \leq 0$ and $\in X$ implies that the set $X$ is closed and convex. The argument that maximizes the social welfare $W(x)$ is given by $\arg \max_{x \in X} W(x)$. We divide the proof into three parts, that include 1) proving that certain signals of the overall closed-loop system are bounded; 2) upper bounding all the closed-loop state variables by a common signal, denoted by $\ell(t)$; and 3) showing that $\ell(t)$ and thus all the state-variables are bounded.

### Appendix

#### Proofs of the Technical Lemmas

**Proof of Lemma IV.1:** The inequality (8) is equivalent to the matrix inequality $-I + a(P + P^T) < 0$ and thus to strong concavity of $x : \sum_{i \in I} W_i(x_i, z_i(x))$. The latter map admits at most one maximizer over the closed convex set $X$ [34, Prop. 2.1.1]. To show the existence of such unique maximizer, pick a point $p \in X$ and define the following set:

$$\mathcal{Y} := \left\{ y \in X \mid \sum_{i \in I} W_i(p_i, z_i(p)) \leq \sum_{i \in I} W_i(y_i, z_i(y)) \right\}.$$  

The set $\mathcal{Y}$ is compact as a result of strong concavity of the social welfare function, and the maximization problem (6) is equivalent to $x_{opt} \in \arg \max_{y \in \mathcal{Y}} \sum_{i \in I} W_i(y_i, z_i(y))$.

The existence of $x_{opt}$ then follows from Weierstrass’ theorem [34, Prop. A.8], and this concludes the proof. □

**Proof of Lemma IV.10:** We divide the proof into three parts, that include 1) proving that certain signals of the overall closed-loop system are bounded; and 2) upper bounding all the closed-loop state variables by a common signal, denoted by $\ell(t)$; and 3) showing that $\ell(t)$ and thus all the state-variables are bounded.
Step 1 (\(L_\infty\) and \(L_2\) analysis): We start our proof by analyzing evolution of \((e, \Psi)\), where \(\Psi := K - aP\). It follows from (26), (27), and (28) that:

\[
\begin{align*}
\dot{e} &= -e - \Psi x - ex^T x \\
\dot{\Psi} &= ex^T.
\end{align*}
\]

(31a) (31b)

Consider the Lyapunov candidate

\[
V(e, \Psi) := \frac{1}{2} \|e\|^2 + \frac{1}{2} \|\Psi\|^2_F
\]

where \(\|\Psi\|^2_F\) is the Frobenius norm. The derivative of \(V\) along the solutions of (31) is

\[
\begin{align*}
\dot{V} &= -\|e\|^2 - e^T \dot{\Psi} x - \|e\|^2 \|x\|^2 + \text{Tr}(\dot{\Psi}^T ex^T) \\
&= -\|e\|^2 - \|e\|^2 \|x\|^2 + \text{Tr}(\dot{\Psi}^T ex^T)
\end{align*}
\]

(33)

where the last equality is obtained using \(e^T \dot{\Psi} x = \text{Tr}(\dot{\Psi}^T ex^T)\). Therefore, we have \(\dot{V} \geq 0\) and \(V \leq 0\), which results in

\[
V_\infty := \lim_{t \to \infty} V(e(t), \Psi(t)) \leq V(0, \Psi(0)).
\]

(34)

Thus, we obtain \(e, \Psi \in L_\infty\). We proceed to show that the closed-loop signals \(\dot{\Psi}, e, \|e\|\) belong to \(L_2\), for any \(x\). Note from (33) that

\[
\dot{V} \leq -\|e\|^2.
\]

We integrate both sides of the inequality above and use (34) to get

\[
\int_{0}^{\infty} \|e(\tau)\|^2 d\tau \leq V(0, \Psi(0)) - V_\infty < \infty.
\]

Consequently, we have \(e \in L_2\). Moreover, we deduce from an analogous analysis for \(e\|x\|\) in (33) that \(e\|x\| \in L_2\). Now we rewrite the dynamics of \(\dot{\Psi}\) given in (31b) as follows:

\[
\dot{\Psi} = e(1 + \|x\|) \frac{x^T}{1 + \|x\|}.
\]

(35)

Note that for any \(x\), we have \(x/(1 + \|x\|) \in L_\infty\), and \(e(1 + \|x\|) \in L_2\) since \(e, \|x\| \in L_2\). Thus, we derive from (35) that \(\Psi \in L_2\). We record below our findings in Step 1 of the proof for a later use:

1) \(\dot{\Psi}, e \in L_\infty\);
2) \(\dot{\Psi}, e, \|x\| \in L_2\).

Step 2 (Determining a Common Upper Bound): Consider a solution \((x(t), z(t), w(t), K(t))\) of the closed-loop system, made of (26), (27), and (28), starting at an arbitrary initial condition. Note that \(K(t)\) is bounded as \(\Psi(t) \in L_\infty\). Next, we find a common upper bound for the closed-loop signals \((x(t), z(t), w(t))\) using the properties established in the previous step. This will allow us to prove boundedness of all the closed-loop signals in Step 3.

We proceed the analysis by introducing the following normalizing signal:

\[
\ell(t) := \sqrt{1 + \|x(t)\|^2_{2\delta}}.
\]

(36)

where \(\|x(t)\|_{2\delta}\) is the exponentially weighted \(L_2\) norm of \(x\) defined as

\[
\|x(t)\|_{2\delta} := \left(\int_{0}^{\tau} \exp(-\delta(t - \tau)) x^T(\tau)x(\tau) d\tau\right)^{\frac{1}{2}}
\]

for a given \(\delta \geq 0\). Next we show that the closed-loop signals \((x, z, w)\) can be bounded from above by an affine function of \(\ell\). Noting \(u = Kx\) and \(\Psi = K - aP\), we rewrite (26) as

\[
\dot{x} = -(I + 2aP)x + b + \Psi x.
\]

(37)

Note that \(-(I + 2aP)\) is Hurwitz as a consequence of Assumption IV.2, thus there exist constants \(k_0, \alpha_0 > 0\) that satisfy

\[
\|\exp \left(-(I + 2aP)(t - \tau)\right)\| \leq k_0 \exp(-\alpha_0(t - \tau))
\]

(38)

for all \(\tau \in [0, t]\). It then follows from (37), the established property \(\Psi \in L_\infty\), and [41, Lem. 3.3.3(ii)] that for any given \(\delta \in [0, 2\alpha_0]\), there exist constants \(c_0, c_1 > 0\) such that \(\|x\| \leq c_0 + c_1 \|x\|_{2\delta}\). Similarly, we obtain from (28a) that for any \(\delta \in [0, 2]\), we have \(\|z\| \leq c_2 + c_3 \|x\|_{2\delta}\) for some \(c_2, c_3 > 0\). Regarding \(w\), we employ \(e \in L_\infty\), together with the definition of \(e\) and the upper bounds on \(\|x\|\) and \(\|z\|\) to deduce that for any \(\delta \in [0, 2\min(1, \alpha_0)]\), there are \(c_4, c_5 > 0\) such that \(\|w\| \leq c_4 + c_5 \|x\|_{2\delta}\). Therefore, we can use \(\|x\|_{2\delta}\) and bound from above \(x, z, w\). Note from the definition of \(\ell\) that \(\|x\|_{2\delta} \leq \ell\). Consequently, for any \(\delta \in [0, 2\min(1, \alpha_0)]\), the following holds:

\[
\|x\| \leq c_0 + c_1 \ell, \quad \|z\| \leq c_2 + c_3 \ell, \quad \|w\| \leq c_4 + c_5 \ell.
\]

(39)

We see from the above relations that the signal \(\ell\) provides a common upper bound for all the closed-loop signals, thus these signals are bounded provided that \(\ell\) is bounded.

Step 3 (Boundedness Analysis): Here, we address boundedness analysis of \(\ell\) using the fact that some of the signals belong to \(L_2\). We perform the analysis in two steps. First, we find an explicit upper bound of \(\ell\), which includes the \(L_2\) signals. We then use Bellman–Gronwall Lemma to find an explicit upper bound of \(\ell\) and conclude its boundedness.

In this part of the proof, we ease the notation by using \(c > 0\) to denote all positive constants whose actual values do not affect stability of the system. In other words, the forthcoming analysis is oblivious to the exact value of \(c\), and \(e\) is used merely for simplicity of the presentation. We remark that such notational convention is used in the classical textbook [42].

Bearing in mind the definition of \(\ell\) given by (36), we use (37) and [41, Lem. 3.3.3(ii)] to infer that for any \(\delta \in [0, 2\alpha_0]\), we have

\[
\|x\|_{2\delta} \leq c + c\|\Psi x\|_{2\delta}
\]

(38)

where \(\alpha_0\) is defined in (38). Thus, we obtain from (36) that

\[
\ell^2 \leq c + c\|\Psi x\|_{2\delta}.
\]

(40)

It follows from the inequality above that the signal \(\ell\) is bounded from above by the norm of \(\Psi x\). As a result, we proceed by analyzing \(\dot{\Psi} x\), and for that, we introduce the following dynamics:

\[
\begin{align*}
\dot{p} &= -\beta p + \dot{\Psi} x + \dot{x}, \\
\dot{q} &= -\beta q + \dot{\Psi} x,
\end{align*}
\]

(41)
where $\beta > 0$. It is then straightforward to verify that $\Psi x = p + q$. Moreover, note from (31a) that $\Psi x = -\dot{e} - e - ex^T x$. This, together with the dynamics of $q$, allows us to write $q = q_1 + q_2 - \beta e$ where

\[
\begin{align*}
q_1 &= -\beta q_1 - \beta (1 - \beta)e, && q_1(0) = \beta e(0) \\
q_2 &= -\beta q_2 - \beta ex^T x, && q_2(0) = 0.
\end{align*}
\]  

(42)

As a result, we have obtained $\Psi x = p + q_1 + q_2 - \beta e$. We next use this relation to find an upper bound of $\|\Psi x\|_{L_2}$, which explicitly depends on $\beta$ and includes the $L_2$ signals. Consequently, this provides us an upper bound for $\ell$. Toward this end, we consider the introduced dynamics (41) and (42) and obtain from [41, Lem. 3.3.3(ii)] that for any $\delta \in [0, \delta_1)$, where $\delta_1 \in (0, 2\beta)$ is arbitrary, the following relations hold:

\[
\begin{align*}
\|p\|_{L_2} &\leq c + h(\beta) \|\Psi x + \Psi \dot{x}\|_{L_2} \\
\|q_1\|_{L_2} &\leq c + h(\beta) (1 - \beta) \|e\|_{L_2} \\
\|q_2\|_{L_2} &\leq c (h(\beta) \|e\|_{L_2}) \\
\text{where} \quad h(\beta) &:= \frac{1}{(\sqrt{\delta_1 - \delta})(2\beta - \delta_1)}.
\end{align*}
\]

The upper bound of $\|q_2\|_{L_2}$ does not have an additive constant term $c$ as we have $q_2(0) = 0$. Let $\beta > 1$ and $\delta_1 = 1$, we then deduce that for any given $\delta \in [0, 1)$, the succeeding inequalities are satisfied for all $\beta > 1$:

\[
\begin{align*}
\|p\|_{L_2} &\leq c + c \beta^{-\frac{1}{2}} (\|\Psi x\|_{L_2} + \|\Psi \dot{x}\|_{L_2}) \\
\|q_1\|_{L_2} &\leq c + c \beta^{\frac{1}{2}} \|e\|_{L_2} \\
\|q_2\|_{L_2} &\leq c \beta^{\frac{1}{2}} \|e\|_{L_2}
\end{align*}
\]

where we have used the triangular inequality to get the first relation. Note that $c$ does not depend on $\beta$, this property will be useful later in establishing boundedness of $\ell$. We now employ the above inequalities in the relation $\Psi x = p + q_1 + q_2 + \beta e$ to obtain

\[
\begin{align*}
\|\Psi x\|_{L_2} &\leq c + c \beta^{\frac{1}{2}} (\|\dot{\psi} x\|_{L_2} + \|\psi \dot{x}\|_{L_2}) \\
&+ c \beta^{\frac{3}{2}} \|e\|_{L_2} + c \beta^{\frac{1}{2}} \|e\|_{L_2} + \beta \|e\|_{L_2}.
\end{align*}
\]  

(43)

Next, we further bound the right-hand side of the inequality above by the $L_2$ signals $e\|x\|$, $\Psi$ and the normalizing signal $\ell$. For the term $\beta^{\frac{3}{2}} \|\dot{\psi} x\|_{L_2}$, we use the definition of the exponentially weighted $L_2$ norm to get

\[
\begin{align*}
\beta^{-1} \|\psi \dot{x}\|_{L_2}^2 &\leq \beta^{-1} \int_0^t \exp (-\delta (t - \tau)) \|\Psi (\tau)\|^2 \|x(\tau)\|^2 d\tau \\
&\leq \beta^{-1} \int_0^t \exp (-\delta (t - \tau)) \|\Psi (\tau)\|^2 (e + e\ell(\tau))^2 d\tau \\
&\leq c + c \beta^{-1} \|\Psi \ell\|_{L_2}^2.
\end{align*}
\]  

(44)

where we used (39) to find the second inequality, and the last inequality follows from $\Psi \in L_2$ and $\beta > 1$. Similarly, we analyze the term $\beta^{\frac{1}{2}} \|\dot{\psi} x\|_{L_2}$ on the right-hand side of (43). First, note from the pseudogradient dynamics (37), the upper bound on $\|x\|$ in (39), and $\Psi \in L_\infty$ that $\|\dot{x}\| \leq c + c \ell$. Therefore, we obtain

\[
\begin{align*}
\beta^{-1} \|\dot{x}\|_{L_2}^2 &\leq \beta^{-1} \int_0^t \exp (-\delta (t - \tau)) \|\Psi (\tau)\|^2 \|\dot{x}(\tau)\|^2 d\tau \\
&\leq \beta^{-1} \int_0^t \exp (-\delta (t - \tau)) \|\Psi (\tau)\|^2 (c + c\ell(\tau))^2 d\tau \\
&\leq c + c \beta^{-1} \|\ell\|_{L_2}^2.
\end{align*}
\]  

(45)

where the last relation follows from $\Psi \in L_\infty$ and $\beta > 1$. We now consider the term $\beta^{\frac{1}{2}} \|\dot{x}\|_{L_2}$.

(46)

Finally, we have $\|\dot{x}\|_{L_2} \leq c$ since $e \in L_2$. Having found the relations (44)–(46), and $\|\ell\|_{L_2} \leq c$, we conclude from (43) and (40) that for any $\delta \in [0, \min\{1, 2\alpha_0\})$, the following implication holds for all $\beta > 1$:

\[
\ell^2 \leq c \beta^3 + c \beta^{-1} \left( \|\Psi \|_{L_2}^2 + \|e\|_{L_2}^2 + c \beta \|e\|_{L_2} \|x\| \|\ell\|_{L_2}^2. \right.
\]

The above inequality provides an implicit upper bound of $\ell$, which includes the $L_2$ signals $\Psi, e\|x\|$. Next, we obtain an explicit upper bound of $\ell$ to conclude its boundedness. For that, we use the definition of the exponentially weighted $L_2$ norm to deduce that

\[
\ell^2(t) \leq c \beta^3 + c \int_0^t \exp (-\delta (t - \tau)) k(\tau) \ell^2(\tau) d\tau \quad \forall t \geq 0
\]

where

\[
k(\tau) := \beta^{-1} \|\Psi(\tau)\|^2 + \beta^{-1} + \|e(\tau)\|^2 \|x(\tau)\|^2.
\]  

(47)

It then follows from Bellman–Gronwall Lemma [41, Lem. 3.3.9] that:

\[
\ell^2(t) \leq c \beta^3 \Phi(t, 0) + c \beta^3 \delta \int_0^t \Phi(t, \tau) d\tau \quad \forall t \geq 0
\]

(48)

where

\[
\Phi(t, \tau) := \exp \left(-\delta (t - \tau) + c \int_\tau^t k(s) ds \right),
\]

Since $\Psi, e\|x\| \in L_2$, we obtain from (47) that

\[
c \int_\tau^t k(s) ds \leq c(\beta^{-1} + \beta) + c \beta^{-1}(t - \tau)
\]

and this implies that

\[
\Phi(t, \tau) \leq c(\beta^{-1} + \beta) \exp \left(-\delta (t - \tau) \right).
\]

Now select $\delta > 0$ in the interval $[0, \min\{1, 2\alpha_0\})$. Also note that $\beta > 1$ can be selected independent of $\delta$ and $c$ is oblivious of $\beta$. Thus, we choose $\beta$ sufficiently large such that $\delta - c \beta^{-1} > 0$. It then follows from the inequality above and (48) that $\ell \in L_\infty$. Therefore, bearing (39) in mind, we conclude that all signals of the closed-loop system are uniformly bounded. \hfill $\square$

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