THE KÄHLER-RICCI FLOW AND QUANTITATIVE BOUNDS FOR
DONALDSON-FUTAKI INVARIANTS OF OPTIMAL
DEGENERATIONS

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ABSTRACT. We establish a lower bound for the Donaldson-Futaki invariant of optimal
degenerations produced by the Kähler-Ricci flow in terms of the greatest Ricci lower
bound on arbitrary Fano manifolds. As an application, we can generalize the finiteness
of the Futaki invariants on Kähler-Ricci solitons obtained by Guo-Phong-Song-Sturm
to the space of all Fano manifolds. Also, we discuss the relation to Hisamoto’s in-
equality for the infimum of the $H$-functional.

1. Introduction

A central question in Kähler geometry is which Fano manifolds admit Kähler-Einstein
metrics. More precisely, Yau-Tian-Donaldson conjecture states that a Fano manifold
admits a Kähler-Einstein metric if and only if it is K-polystable, i.e. the Donaldson-
Futaki invariant $DF(\mathcal{X})$ is non-negative for all special degenerations $\mathcal{X}$ and equality
holds if and only if $\mathcal{X}$ is product. This conjecture was resolved by Chen-Donaldson-Sun
[CDS15a, CDS15b, CDS15c] and Tian [Tia15].

It seems that not so much things are known in unstable cases. Let $X$ be an $n$-
dimensional Fano manifold. According to [He16], we define the $H$-functional to be

$$H(\omega) := \int_X \rho e^\rho \omega^n, \quad \omega \in c_1(X),$$

where $\rho$ is the Ricci potential of $\omega$ defined by

$$\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} \rho, \quad \int_X e^\rho \omega^n = (-K_X)^n =: V.$$

By Jensen’s inequality we have $H(\omega) \geq 0$ with equality holds if and only if $\omega$ is Kähler-
Einstein. A concern in the unstable case is to construct a test configuration which
optimally destabilizes the Kähler or algebro-geometric structure of $X$. Geometric flows
are one of the most effective tool to attack this problem. Indeed, using the resolution
to the Hamilton-Tian conjecture [CW14], Dervan-Székelyhidi [DS16b] showed that the
destabilizer $\mathcal{X}_a$ produced by the Kähler-Ricci flow [CSW18] is optimal in the sense that

$$\inf_{\omega \in c_1(X)} H(\omega) = \sup_{\mathcal{X}} (-H(\mathcal{X})) = -H(\mathcal{X}_a),$$  \hspace{1cm} (1.1)
where the supremum of the RHS is taken over all special degenerations. The $H$-invariant
\( H(X_a) \) they introduced is computed on a (singular) Kähler-Ricci soliton, which arises
as the unique sequential polarized Gromov-Hausdorff limit along the flow (see Section
2 and Section 3 for precise details). However a problem is that it is hard to compute
\( H(X_a) \) and \( DF(X_a) \) directly since the soliton vector field is determined implicitly as
the (unique) critical point of a strictly convex function on the Lie algebra consisting of
holomorphic vector fields (cf. [BN14]).

So it has a meaning to establish the “quantitative” bounds for \( H(X_a) \) and \( DF(X_a) \),
that is, we want to get bounds by explicitly computable numbers. We consider the
greatest Ricci lower bound of \( X \) [Szé11] defined by
\[
R(X) := \sup \left\{ r \in [0, 1] \mid \exists \omega \in c_1(X) \text{ such that } \text{Ric}(\omega) > r\omega \right\}.
\]

It is known that the equality \( R(X) = 1 \) holds if and only if \( X \) is K-semistable (cf.
[BBJ15, Li17]), and the invariant \( R(X) \) is related to Berman-Fujitas’ \( \delta \)-invariant \( \delta(X) \)
by the formula \( R(X) = \min\{\delta(X), 1\} \) [BJ18, Fuj16]. Also, we have \( R(X) \leq S(X) \),
where \( S(X) \) is an algebro-geometric invariant defined in terms of uniform twisted K-
stability (see [Der16, Corollary 1.5]). A remarkable thing is that the invariant \( R(X) \) is
computed explicitly in many specific cases (e.g. [Li11, Del17]).

The main theorem in this paper is the following:

**Theorem 1.1.** For any \( n \)-dimensional Fano manifold \( X \), we have
\[
DF(X_a) \geq -\frac{1 - R(X)}{R(X)} nV,
\]
where the Donaldson-Futaki invariant \( DF(X_a) \) is computed on the \( \mathbb{Q} \)-Fano variety
admitting a (singular) Kähler-Ricci soliton \( (Y, \omega_Y, W_Y) \), which arises as the unique
sequential polarized Gromov-Hausdorff limit along the Kähler-Ricci flow starting from
any Kähler metric in \( c_1(X) \). Also, the algebraic invariant \( DF(X_a) \) coincides with the
integral invariant
\[
\text{Fut}(W_Y) := \int_Y |\nabla \rho_Y|^2 \omega_Y^n = \int_Y |W_Y|^2 \omega_Y^n,
\]
where \( \rho_Y \) denotes the Ricci potential of \( \omega_Y \).

It seems that Theorem 1.1 is new even in the case when \( Y \) is isomorphic to \( X \) (which
is equivalent to say that \( X \) admits a Kähler-Ricci soliton by [DS16b, Corollary 4.3]). To prove Theorem 1.1 we study the limit space \( (Y, \omega_Y, W_Y) \) by mean of the non-
Archimedean limits of energy functionals on the space of Kähler metrics developed in
[BJH17, BJH19].

Theorem 1.1 has some applications. First, by [Cam92, KMM92], we know that there exists
uniform constant \( C = C(n) > 0 \) (which is independent of \( X \)) such that \( V < C \).
Also from the uniform positive lower bound of the log canonical threshold [Bir16], there
exists a uniform constant \( \varepsilon = \varepsilon(n) > 0 \) (which is independent of \( X \)) such that \( R(X) > \varepsilon \)
(see [GPSS18, Corollary 2.1]). Combining with Theorem 1.1 we obtain the following:
Corollary 1.2. In the same setting as in Theorem 1.1, there exists a uniform constant \( F = F(n) > 0 \) such that for any \( n \)-dimensional Fano manifold \( X \), we have
\[
\operatorname{Fut}(W_Y) \geq -F.
\] (1.2)

In particular, when \( X \) admits a Kähler-Ricci soliton \((\omega_X, W_X)\), one can apply the argument in Section 3 to \( X \) directly, and get
\[
\operatorname{Fut}(W_X) \geq -F,
\]
which was conjectured in [PSS15, page 31], and thereafter solved by Guo-Phong-Song-Strum [GPSS18, Corollary 1.1] by showing the compactness of Kähler-Ricci solitons. So Corollary 1.2 can be regarded as a generalization of their result to the space of all Fano manifolds. Also we should remark that the property (1.2) does not hold for the space of all (singular) Kähler-Ricci solitons on \( Q \)-Fano varieties with at worst log terminal singularities as discussed in [PSS15, page 31].

Now we will explain another application. We have the following inequality (1.3) as a direct consequence from (1.1), Theorem 1.1 and \( \operatorname{DF}(X_a) \leq H(X_a) \) (cf. [DS16b, Lemma 2.5]):

Corollary 1.3. For any \( n \)-dimensional Fano manifold \( X \), we have
\[
\inf_{\omega \in c_1(X)} H(\omega) \leq \frac{1 - R(X)}{R(X)} nV.
\] (1.3)

Corollary 1.3 is inspired by Hisamoto’s inequality [His19, Proposition 5.4]: if \( R(X) > 1/4\pi \), we have
\[
\inf_{\omega \in c_1(X)} H(\omega) \leq (1 - R(X)) nV.
\] (1.4)

Hisamoto’s proof is quite different from ours. Indeed, he proved the inequality (1.4) by using the relation between \( \inf_{\omega} H(\omega) \) and the supremum of Perelman’s \( \mu \)-functional (based on [DS16b, Theorem 4.2]), and applying the log-Sobolev inequality. Although our inequality (1.3) is weaker than (1.4), an advantage is that it holds without any restrictions for \( R(X) \).

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2. Preliminaries

Fix a Kähler metric \( \hat{\omega} \in c_1(X) \). We start with the definitions and properties of several functionals on the space of Kähler potentials
\[
\mathcal{H} := \{ \phi \in C^\infty(X; \mathbb{R}) | \omega_\phi := \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \phi > 0 \}.
\]
The Mabuchi functional, or K-energy is defined by its variation
\[
\delta M(\phi) := -\int_X \delta \phi (S_\phi - n) \omega_\phi^n,
\]
where $S_\phi$ denotes the scalar curvature of $\omega_\phi$. The $I$ and $J$-functionals are defined by

$$I(\phi) := \int_X \phi(\widehat{\omega}^n - \omega_\phi^n),$$

$$\delta J(\phi) := \int_X \delta\phi(\widehat{\omega}^n - \omega_\phi^n).$$

For $\phi \in \mathcal{H}$ we define the Ricci potential $\rho_\phi$ by

$$\text{Ric}(\omega_\phi) - \omega_\phi = \sqrt{-1}\partial\bar{\partial}\rho_\phi, \quad \int_X e^{\rho_\phi}\omega_\phi^n = V,$$

and the $H$-functional by

$$H(\phi) := \int_X \rho_\phi e^{\rho_\phi}\omega_\phi^n.$$

Next we review the non-Archimedean limits of $M$, $I$ and $J$ in the sense of [BHJ17, BHJ19]. Here we will restrict our selves to special degenerations [Tia97]:

**Definition 2.1.** A special degeneration of $X$ is a flat normal $\mathbb{Q}$-Fano family $\pi : \mathcal{X} \to \mathbb{C}$, together with a holomorphic vector field $v$ on $\mathcal{X}$, which generates a $\mathbb{C}^*$-action on $\mathcal{X}$ covering the standard action on $\mathbb{C}$. In addition, the fiber $\mathcal{X}_\tau$ is required to be isomorphic to $X$ for one, and hence all $\tau \in \mathbb{C}^*$.

For a special degeneration $\mathcal{X}$, we consider the corresponding Hilbert and weight polynomials

$$N_k = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}),$$

$$w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}).$$

Then the Donaldson-Futaki invariant is defined by

$$DF(\mathcal{X}) := n!(2b_1 - nb_0).$$

Also we define

$$(I^{NA} - J^{NA})(\mathcal{X}) := n!(-b_0 + \lambda_{\text{max}} a_0).$$

where $\lambda_{\text{max}} \in \mathbb{Q}$ denotes the maximum weight of the $\mathbb{C}^*$-action on $X_0$ (indeed, the functional $I$, $J$ admits its own non-Archimedean limit $I^{NA}$, $J^{NA}$ respectively, but we will not use their definitions here). The functional $I^{NA} - J^{NA}$ is also called minimum norm of $\mathcal{X}$ (cf. [BHJ17 Remark 7.12], [Der16 Section 2.1]). Let us consider also the analytic aspects of these functionals. We choose a smooth Kähler metric $\omega_0 \in c_1(X_0)$, i.e. it is the restriction of a smooth metric under a projective embedding of $X_0$. We define its Ricci potential $\rho_0$ by

$$\text{Ric}(\omega_0) - \omega_0 = \sqrt{-1}\partial\bar{\partial}\rho_0, \quad \int_{X_0} e^{\rho_0}\omega_0^n = V.$$

So the function $\rho_0$ is continuous on $X_0$ and smooth on $X_{0,\text{reg}}$. Assume $\omega_0$ is $\text{Im}(v)$-invariant and let $\theta_0$ be a Hamiltonian for the induced holomorphic vector field $v$ on $X_0$. Then we have

$$DF(\mathcal{X}) = -\int_{X_0} \theta_0 e^{\rho_0}\omega_0^n + \int_{X_0} \theta_0 \omega_0^n,$$
\[(I^{NA} - J^{NA})(\mathcal{X}) = - \int_{\mathcal{X}_0} \theta_0 \omega_0^n + V \max_{\mathcal{X}_0} \theta_0.\]

Actually, to prove this, we take a resolution \(p : \tilde{\mathcal{X}}_0 \rightarrow \mathcal{X}_0\) and compute the algebraic Futaki invariant on \(\tilde{\mathcal{X}}_0\) with respect to the polarization \(p^*(-kK_{\mathcal{X}_0})\) for a large divisible \(k\), using the (equivariant) Riemann-Roch formula by a smooth background metric (cf. [CDS15c page 264]). In order to deal with the maximum weight \(\lambda_{\max}\), we should take \(\omega_0 = k^{-1}\omega_{FS}\) so that \(v\) has the Hamiltonian function

\[
\theta_0 = \frac{\sum_i \lambda_i |Z_i|^2}{\sum_i |Z_i|^2}
\]

for some suitable homogeneous coordinates \(\{Z_i\}\) (as in the proof of [DS16a, Lemma 12]). Since \(\mathcal{X}_0\) is not contained in any hyperplanes, we can take \(x_0 \in \mathcal{X}_0 \setminus \{Z_{\max} = 0\}\) and consider the gradient flow of \(\theta_0\) starting from \(x_0\) on \(\mathbb{P}^{N_k-1}\). Since \(\mathcal{X}_0\) is compact and invariant under the Hamiltonian action of \(v\), the flow converges to a limit point \(x_{\infty} \in \mathcal{X}_0\) where

\[
\max_{\mathcal{X}_0} \theta_0 \geq \theta_0(x_{\infty}) = \lambda_{\max} = \max_{\mathbb{P}^{N_k-1}} \theta_0.
\]

Then a consequence from [BHJ17, BHJ19] is the following:

**Theorem 2.2.** Let \(\mathcal{X}\) be a special degeneration of \(X\) and \(\{\phi_t\} \subset \mathcal{H}\) be an associated geodesic ray, then we have

\[
\lim_{t \to \infty} \frac{M(\phi_t)}{t} = DF(\mathcal{X}), \quad \lim_{t \to \infty} \frac{(I - J)(\phi_t)}{t} = (I^{NA} - J^{NA})(\mathcal{X}).
\]

On the other hand, the \(H\)-functional does not have such an non-Archimedean description for a certain energy functional. However according to [DS16b], we define the \(H\)-invariant \(H(\mathcal{X})\) to be

\[
H(\mathcal{X}) := 2n!b_1 - (n + 1)!b_0 + V \lim_{k \to \infty} \log \left( \frac{1}{N_k} \sum e^{\lambda_{k,i}} \right),
\]

where \(\lambda_{k,i}\) denotes the weight of \(\mathbb{C}^*\)-action on \(H^0(\mathcal{X}_0, -kK_{\mathcal{X}_0})\) generated by \(v\). As shown in [DS16b, Proposition 2.12], the \(H\)-invariant also has an analytic description

\[
H(\mathcal{X}) = - \int_{\mathcal{X}_0} \theta_0 e^{\phi_0} \omega_0^n + V \log \left( \frac{1}{V} \int_{\mathcal{X}_0} e^{\phi_0} \omega_0^n \right).
\]

By Jensen’s inequality, one can observe that \(DF(\mathcal{X}) \leq H(\mathcal{X})\) with equality if and only if \(\mathcal{X}\) is trivial (cf. [DS16b, Lemma 2.5]).

Finally, we remark on \(\mathbb{R}\)-degenerations, a generalization of the notion of a test configuration using the language of filtrations [DS16b]. Let \(Z\) be an arbitrary projective variety with ample line bundle \(L \rightarrow Z\). We define the graded coordinate ring

\[
R(Z, L) := \bigoplus_{k \geq 0} H^0(Z, kL).
\]

We write \(R_k = H^0(X, kL)\) for simplicity.
Definition 2.3. An \( \mathbb{R} \)-indexed filtration \( \{F^\lambda R_k\}_{\lambda \in \mathbb{R}} \) consists of the data satisfying for each \( k \)

- \( F \) is decreasing: \( F^\lambda R_k \subset F^{\lambda'} R_k \) if \( \lambda \geq \lambda' \).
- \( F \) is left-continuous: \( F^\lambda R_k = \bigcap_{\lambda' < \lambda} F^{\lambda'} R_k \).
- \( F^\lambda R_k = 0 \) for sufficiently large \( \lambda \) and \( F^\lambda R_k = R_k \) for sufficiently small \( \lambda \).
- \( F \) satisfies the multiplicative property:
  \[
  F^\lambda R_k \cdot R^{\lambda'} R_{k'} \subset F^{\lambda + \lambda'} R_{k+k'}
  \]
  for all \( \lambda, \lambda' \in \mathbb{R} \) and \( k, k' \geq 0 \).

The associated graded ring of the filtration is defined to be

\[
\text{gr} F^\lambda R(Z, L) := \bigoplus_{k \geq 0} \bigoplus_{i} F^{\lambda_{k,i}} R_k / F^{\lambda_{k,i+1}} R_k,
\]

where the \( \lambda_{k,i} \) are values of \( \lambda \) where the filtration of \( R_k \) is discontinuous.

Definition 2.4. An \( \mathbb{R} \)-degeneration for \( (Z, L) \) is a filtration of \( R(Z, mL) \) for some integer \( m > 0 \), whose associated graded ring is finitely generated.

For a given \( \mathbb{R} \)-degeneration, let \( \tilde{R} \) be the associated graded ring of the filtration and set \( Z_0 := \text{Proj} Z_0 \). Then the filtration gives a (possibly irrational) real one-parameter family which acts on \( F^{\lambda_{k,i}} R_k / F^{\lambda_{k,i+1}} R_k \) by multiplying the factor \( \tau^{\lambda_{k,i}} \). We may assume that we have an embedding of \( Z_0 \) to a projective space \( \mathbb{P}^{N-1} \) with \( N := \dim H^0(Z, L) \). Then the real one-parameter group is given by a projective automorphisms \( e^t \Lambda \) which preserves \( Z_0 \), where \( \Lambda := \text{diag}(\lambda_{1,1}, \ldots, \lambda_{1,N}) \) is a diagonal matrix. In addition, we also have an embedding \( Z \to \mathbb{P}^{N-1} \) so that \( \lim_{t \to \infty} e^{t \Lambda} \cdot Z = Z_0 \) in the Hilbert scheme.

By taking the closure of \( \{e^{\sqrt{-1}t \Lambda}\} \) in \( U(N) \), we obtain a real torus \( T \subset U(N) \) acting on \( Z_0 \). So the action of \( e^{t \Lambda} \) corresponds to a choice of \( \xi \in \mathfrak{t} := \text{Lie}(T) \). Then as discussed in [CSW18], we can take a sequence of \( \mathbb{C}^* \)-subgroup \( \nu_{\ell} \) in the complexified torus \( T^\mathbb{C} \) with \( \lim_{\ell \to 0} \nu_{\ell}(\tau) \cdot Z = Z_0 \) and \( \xi_{\ell} \to \xi \) as \( \ell \to \infty \), where \( \xi_{\ell} \) denotes the infinitesimal generator of \( \nu_{\ell} \). In this way, we can approximate any \( \mathbb{R} \)-degeneration by test configurations. Moreover, for \( \xi \in \mathfrak{t} \) and \( s \in \mathbb{C} \), we define the weight character by

\[
C(\xi, s) := \sum_{k \geq 0, \alpha \in \mathfrak{t}^*} e^{-sk} \alpha(\xi) \dim \tilde{R}_{k, \alpha},
\]

where \( \tilde{R}_{k, \alpha} \) denotes the components of the \( \mathbb{C}^* \times T^\mathbb{C} \)-action on \( \tilde{R} \) defined by the \( k \)-grading and \( T^\mathbb{C} \)-action on \( \tilde{R} \). Then \( C(\xi, s) \) has a Laurent series expansion as

\[
C(\xi, s) = \frac{b_0(n+1)!}{s^{n+2}} + \frac{b_1(n+2)!}{s^{n+1}} + O(s^{-n}),
\]

where the coefficients \( b_0, b_1 \) are smooth in \( \xi \), and coincide with \( b_0, b_1 \) which appear in the asymptotic expansion of the total weight \( w_k \) when \( \xi \) is rational. Also a transcendental term \( \lim_{k \to \infty} N_k^{-1} \sum_{i=1}^{N_k} e^{\lambda_{k,i}/k} \) in the \( H \)-invariant admits a continuous extension for \( \xi \in \mathfrak{t} \). So from the algebraic descriptions of each invariant, we can extend \( DF, I^{NA} - J^{NA} \) and \( H \) to be continuous under this approximation procedure (see [DS16b, Section 2.2] for more details).
3. Proof of Theorem 1.1

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. For any Kähler metric $\omega_0 \in c_1(X)$, we consider the normalized Kähler-Ricci flow starting from $\omega_0$:

$$\frac{d}{dt}\omega_t = -\text{Ric}(\omega_t) + \omega_t.$$

The evolution of scalar curvature along the flow (for instance, see [Ive93]) is given by

$$\frac{d}{dt}S_{\omega_t} = \Delta_{\omega_t} S_{\omega_t} + |\text{Ric}(\omega_t)|^2 - S_{\omega_t}.$$

The maximum principle shows that we have $S_{\omega_t} \geq \inf_X S_{\omega_0} \cdot e^{-t}$ for all positive time. We use the following, due to Chen-Wang [CW14] and Chen-Sun-Wang [CSW18]:

**Theorem 3.1.** Let $(X, \omega_t)$ be the solution of the Kähler-Ricci flow. Then the sequential polarized Gromov-Hausdorff limit of $(X, \omega_t)$ as $t \to \infty$ is a $\mathbb{Q}$-Fano variety $Y$, independent of choice of subsequences, which admits a (singular) Kähler-Ricci soliton $\omega_Y$ with soliton vector field $W_Y$. Moreover, this convergence is improved to be in $C^\infty$-Cheeger-Gromov topology away from the singular set $Y_{\text{sing}}$. Assume that $W_Y \neq 0$. Then there exists a “two-step” $\mathbb{R}$-degeneration from $X$ to $Y$, i.e. an $\mathbb{R}$-degeneration $X_a$ for $X$ with $\mathbb{Q}$-Fano central fiber $\bar{X}$, and an $\mathbb{R}$-degeneration $X_b$ for $\bar{X}$ with central fiber $Y$. The corresponding one-parameter subgroup of automorphisms on $X_b$ is induced by the soliton vector field $W_Y$ on $Y$.

Applying Theorem 3.1 to our flow $(X, \omega_t)$, we obtain $\mathbb{R}$-degenerations $X_a$ and $X_b$ as above. Without loss of generality, we assume that $X_a$ and $X_b$ are special degenerations (For general case, we simply approximate $\mathbb{R}$-degenerations by test configurations and use the continuity for DF, $I^{\mathbb{NA}} - J^{\mathbb{NA}}$ and $H$). As shown in the proof of [CSW18 Lemma 3.4, Proposition 3.5], the weight decompositions of $H^0(\bar{X}, -kK_{\bar{X}})$ and $H^0(Y, -kK_Y)$ are isomorphic for all sufficiently large and divisible $k$. Hence we have

$$H(X_a) = H(X_b), \quad DF(X_a) = DF(X_b), \quad (I^{\mathbb{NA}} - J^{\mathbb{NA}})(X_a) = (I^{\mathbb{NA}} - J^{\mathbb{NA}})(X_b)$$

by using algebraic descriptions for each invariant (where one can define the invariants $H(X_b), DF(X_b)$ and $(I^{\mathbb{NA}} - J^{\mathbb{NA}})(X_b)$ in an obvious way). Let $\rho_Y$ be the Ricci potential of $\omega_Y$ normalized by $\int_X e^{\rho_Y} \omega_Y^n = V$. Taking the trace we have

$$S_{\omega_Y} - n = \Delta_{\omega_Y} \rho_Y.$$

Also we compute

$$\sqrt{-1} \partial \bar{\partial} \left( \Delta_{\omega_Y} \rho_Y + \rho_Y + |\partial \rho_Y|_{\omega_Y}^2 \right) = 0$$

on $Y_{\text{reg}}$ by using the fact that the function $\rho_Y$ is the Hamiltonian of the holomorphic vector field $W_Y$ with respect to $\omega_Y$ (for instance, we can check it in the same way as the smooth case [TZ02 equation (1.11)] since the computation is local). Since $Y$ is normal, we also get

$$\Delta_{\omega_Y} \rho_Y + \rho_Y + |\partial \rho_Y|_{\omega_Y}^2 = c.$$

We know that $\rho_Y$ extends to a continuous function on $Y$. Also we know that $\nabla \rho_Y$ is bounded and the Minkowski codimension of $Y_{\text{sing}}$ strictly greater than 2. It follows
that integrating by parts works well since we can take an exhaustive \( K \subset Y_{\text{reg}} \) with the volume of \( Y \setminus K \) being small as well as we please (see the proof of [CSW18 Proposition 3.6] and [DS16b Lemma 3.5, Theorem 4.2]). Here we note that these regularity results do not imply that the metric \( \omega_Y \) is smooth in the sense of Section 2. Nevertheless we can detect the constant \( c \) as

\[
cV = \int_Y \rho_Y e^{\rho_Y} \omega_Y^n = -H(\mathcal{X}_b), \tag{3.1}
\]

and also

\[
\text{DF}(\mathcal{X}_b) = \int_Y \rho_Y \omega_Y^n + H(\mathcal{X}_b), \tag{3.2}
\]

where the last equality of (3.1) was shown in the proof of [DS16b Theorem 3.2]. To prove (3.2), we used also the fact that the invariant \( \text{DF}(\mathcal{X}_b) \) arises as the limit derivative of Ding functional along the flow generated by \( W_Y \) (cf. [BN14]). In particular, (3.1) and (3.2) imply that for any smooth metric \( \omega_0 \in c_1(Y) \) with Hamiltonian \( \theta_0 \) with respect to \( W_Y \) normalized by \( \int_Y e^{\theta_0} \omega_0^n = V \), we have

\[
\int_Y \rho_Y \omega_Y^n = \int_Y \theta_0 \omega_0^n.
\]

To deal with \( \max_Y \rho_Y \), we need an interpretation of Hamiltonians in terms of a lifted action on a line bundle and its holomorphic sections. Let \( J \) be a complex structure on \( Y_{\text{reg}} \), \( h_Y \) a continuous fiber metric on the \( \mathbb{Q} \)-line bundle \( -K_Y \) with curvature \( \omega_Y \) and \( \nabla \) the compatible connection on \( Y_{\text{reg}} \) arising as the sequential polarized Gromov-Hausdorff limit along the Kähler-Ricci flow \((X, \omega_t)\) in the sense of [DS14]. Also, there is a positive integer \( k \) (depending only on \((X, \omega_0)\)) such that we have an embedding of \( Y \hookrightarrow \mathbb{P}^{N-1} \) by \( L^2 \)-orthonormal sections \((s_1, \ldots, s_N)\) of \( H^0(Y, -kK_Y) \) with respect to \( h_Y \). We may let \( k = 1 \) for simplicity, and further assume that \( s_i \)'s are eigensections with weights \( \lambda_i \) with respect to the infinitesimal \( V_Y \)-action, where \( V_Y \) denotes the real part of \( W_Y \) (see also [CSW18 Section 3.2]). We know that the \( J V_Y \)-action on \( H^0(Y, -K_Y) \) has the following expression (cf. [Kob95]):

\[
R(s) := \frac{d}{dt} \exp(t J V_Y) \cdot s|_{t=0} = -\sqrt{-1} \rho_Y s - \nabla_{J V_Y \cdot s}, \quad s \in H^0(Y, -K_Y) \tag{3.3}
\]

on \( Y_{\text{reg}} \). Since \( \rho_Y \) and \( R(s) \) are continuous (here we used the fact that the \( J V_Y \)-action on holomorphic sections is compatible with the projective embedding of \( Y \)), we can extend the derivative in the vertical direction \( \sqrt{-1} \rho_Y s \) as well as horizontal direction \( -\nabla_{J V_Y \cdot s} \) as a continuous section of \( -K_Y \) over \( Y \). This together with the continuity of \( h_Y \) shows that the equality

\[
\sqrt{-1} \lambda_i |s_i|_{h_Y}^2 = \sqrt{-1} \rho_Y |s_i|_{h_Y}^2 - (\nabla_{J V_Y \cdot s_i, s_i})_{h_Y}
\]

holds on \( Y \) (where we note that the real part of \((\nabla_{J V_Y \cdot s_i, s_i})_{h_Y} \) actually vanishes since the function \( |s_i|_{h_Y}^2 \) is invariant under the \( J V_Y \)-action). Now we repeat the argument in Section 2 to find a point \( x \in Y \) such that:

- \( s_N(x) \neq 0 \) for the eigensection \( s_N \) corresponding to the maximum weight \( \lambda_N = \lambda_{\text{max}} \).
- \( s_i(x) = 0 \) whenever \( \lambda_i < \lambda_N \).
• $W_Y|_x = 0$ where we regard $W_Y$ as a holomorphic vector field on $\mathbb{P}^{N-1}$.

Moreover, since $x$ is a fixed point of $JV_Y$-action, the derivative in the horizontal direction vanishes at $x$ (in other words, $JV_Y$ acts on the fiber of $x$ with weight $\sqrt{-1}\rho_Y(x)$). So we have

$$((\nabla_{JV_Y}s_N))(x) = 0.$$  \hspace{1cm} (3.4)

We remark that if $x$ is a regular point of $Y$, we get (3.4) immediately from the third property $W_Y|_x = 0$. In general case, it seems to be difficult to prove the continuity of $\nabla_{JV_Y}s_N$ and (3.4) without the formula (3.3) since the connection $\nabla$ is intrinsic, defined only on $Y_{\text{reg}}$. Anyway, we have

$$\lambda_{\text{max}} = \rho_Y(x) \leq \max_Y \rho_Y,$$

which is enough to prove our statement.

By $S_{\omega_Y} \geq \inf_X S_{\omega_0} \cdot e^{-t}$ and $C^\infty$-Cheeger-Gromov convergence of the Kähler-Ricci flow away from $Y_{\text{sing}}$, we have $S_{\omega_Y} \geq 0$ on $Y_{\text{reg}}$. So we have

$$\rho_Y \leq S_{\omega_Y} + |\bar{\partial}\rho_Y|^2_{\omega_Y} + \rho_Y$$

$$= \Delta_{\omega_Y} \rho_Y + |\bar{\partial}\rho_Y|^2_{\omega_Y} + \rho_Y + n$$

$$= n - \frac{1}{V} H(\mathcal{X}_a)$$

on $Y_{\text{reg}}$. Since $\rho_Y$ is continuous, the above inequality actually holds on $Y$. Let $R(X)$ be the greatest lower bound of the Ricci curvature on $X$. From [Szé11] we know that for any $r \in (0, R(X))$ the twisted Mabuchi functional $M + (1-r)(I-J)$ is coercive. By taking the non-Archimedean limit we have

$$DF(\mathcal{X}_a) + (1-r)(I^{NA} - J^{NA})(\mathcal{X}_a) \geq 0.$$  

Thus

$$DF(\mathcal{X}_a) \geq (1-r)\left[ \int_Y \rho_Y \omega_Y^n - V \max_Y \rho_Y \right]$$

$$\geq (1-r)(DF(\mathcal{X}_a) - H(\mathcal{X}_a) + H(\mathcal{X}_a) - nV)$$

$$= (1-r)(DF(\mathcal{X}_a) - nV),$$

and hence

$$DF(\mathcal{X}_a) \geq -\frac{1-r}{r} nV.$$

By letting $r \nearrow R(X)$, we finish the proof. \qed
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