The Path Integral for a Particle in Curved Spaces and Weyl Anomalies

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The computation of anomalies in quantum field theory may be carried out by evaluating path integral Jacobians, as first shown by Fujikawa. The evaluation of these Jacobians can be cast in the form of a quantum mechanical problem, whose solution has a path integral representation. For the case of Weyl anomalies, also called trace anomalies, one is immediately led to study the path integral for a particle moving in curved spaces. We analyze the latter in a manifestly covariant way and by making use of ghost fields. The introduction of the ghost fields allows us to represent the path integral measure in a form suitable for performing the perturbative expansion. We employ our method to compute the Hamiltonian associated with the evolution kernel given by the path integral with fixed boundary conditions, and use this result to evaluate the trace needed in field theoretic computation of Weyl anomalies in two dimensions.

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1. Introduction

Anomalies arise in quantum field theory whenever the symmetries of a classical system cannot be all preserved by the quantization procedure \([1]\). In a path integral context, this happens when the path integral measure is not invariant under certain symmetries. When deriving Ward identities for these particular symmetries by performing a change of path integration variables, a Jacobian appears. This Jacobian is identified as the anomaly \([2]\). For change of variables given by infinitesimal symmetry transformations, the Jacobian differs from unity by a trace which may be evaluated in several ways. One of the most elegant ways, which goes back to Schwinger \([3]\), consists in representing the operator we need to compute the trace of by a quantum mechanical operator acting in a fictitious (non-physical) Hilbert space. Using then the reformulation of quantum mechanics due to Feynman \([4]\), the trace computation is recast as a path integral with specific boundary conditions and evaluated. Such a program was beautifully carried out in the computation of chiral anomalies \([5]\), where supersymmetric quantum mechanics was used. The topological characterization of chiral anomalies is responsible for the relative easiness in computing the corresponding path integral. In fact, because of the topological character, any approximation may be used to evaluate such a path integral and a semiclassical approximation gives the full result. However, the method is general and one may try to use it to compute other types of anomalies. In the present article, we look at the case of Weyl anomalies, also called trace anomalies \([6]\). Weyl transformations are defined by arbitrary rescalings of the metric tensor. A field theory invariant under such rescalings (and corresponding rescalings of the dynamical fields) has necessarily an energy-momentum tensor with a vanishing trace. An anomaly in the Weyl symmetry is then seen as a non-vanishing trace in the energy-momentum tensor arising quantum mechanically (hence the alternative name of trace anomalies). In the following, we will consider the special case of a scalar field coupled to gravity in a Weyl invariant way. After having identified the trace giving the Weyl anomaly, we look for a quantum mechanical representation of such a trace. The relevant quantum mechanical system turns out to be that of a particle moving in a curved space. The path integral for such a system has been a favorite topic of study over the years \([7]\),\([8]\). However, we try to do better of what we could find in the literature and develop a method for computing perturbatively the path integral for the particle in a manifestly geometrical way (i.e. using Riemann normal coordinates) and making use of ghost fields. These ghost fields are essential to exponentiate a certain path integral Jacobian, thereby
leaving a translational invariant measure suitable for performing a perturbative expansion. The Weyl anomaly does not seem to have a topological interpretation and this may be the reason why a semiclassical approximation is not enough to get the complete result. Our findings are that the Weyl anomaly for the Klein Gordon field in $2k$ dimensions (Weyl anomalies are present only in even dimensions) is obtained by an $k + 1$ loop computation in the corresponding quantum mechanical system.

The paper is structured as follows. In section 2 we describe some generalities on Weyl transformations and identify the “Fujikawa” trace which gives the anomaly for the Klein-Gordon field in arbitrary dimensions. In section 3 we discuss the quantum mechanics of a particle in a curved space. We define its path integral in a manifestly covariant manner and by making use of ghost fields. In section 4 we evaluate our reformulation of the path integral in the two loop approximation and obtain the Hamiltonian associated with the evolution kernel. This gives enough information for obtaining the Weyl anomaly in two dimensions. Eventually, we state our conclusions in section 5, where we comment on the Weyl anomaly in higher dimensions as well as on the connection of our computation with the Schwinger-DeWitt method for evaluating the Klein-Gordon propagator in curved spaces.

2. Generalities on Weyl transformations

Let’s consider a classical field theory in a curved space defined by an action functional $S[\phi, g_{\mu\nu}]$, where the dynamical fields are collectively denoted by $\phi$ and where $g_{\mu\nu}$ is the metric tensor of the manifold, considered as a background. A Weyl transformation is defined as an arbitrary rescaling of the metric

$$g'_{\mu\nu}(x) = \Omega(x)g_{\mu\nu}(x) \quad (2.1)$$

The theory defined by the action $S[\phi, g_{\mu\nu}]$ is Weyl invariant if one can define transformation rules on the dynamical fields $\phi$ which, together with (2.1), leave the action unchanged. In those cases where Weyl invariance can be achieved, the energy-momentum tensor, defined as $T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$, is necessarily traceless. This follows by considering an infinitesimal Weyl transformation, which being a symmetry leaves the action unchanged

$$\delta S[\phi, g_{\mu\nu}] = \int d^nx \left( \frac{\delta S}{\delta g_{\mu\nu}(x)} \delta g_{\mu\nu}(x) + \frac{\delta S}{\delta \phi(x)} \delta \phi(x) \right) = 0 \quad (2.2)$$
Infinitesimally \( \delta g_{\mu\nu}(x) = \sigma(x) g_{\mu\nu}(x) \) with \( \sigma(x) \) an arbitrary local function and it follows from (2.2) that, on the shell of the \( \phi \) equations of motion \( (\frac{\delta S}{\delta \phi} = 0) \), the trace of the energy momentum vanishes

\[
T \equiv g^{\mu\nu} T_{\mu\nu} = 0 \tag{2.3}
\]

Strictly speaking, this is not a true symmetry as long as the metric is not dynamical. In fact, changing the metric is like changing coupling constants, which are defining parameters of the theory. Truly dynamical symmetries are obtained by considering the conformal group. This is defined as the subgroup of coordinate transformations which change the metric only up to a scale factor \( f(x) \)

\[
g'_{\mu\nu}(x') = g_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^{\sigma}}{\partial x'^\nu} = f(x) g_{\mu\nu}(x) \tag{2.4}
\]

Then, by fine tuning \( \Omega(x) = (f(x))^{-1} \) in (2.1) to compensate for the change in (2.4), one is able to construct transformation rules which leave \( g_{\mu\nu} \) inert, while only the dynamical fields \( \phi \) get transformed. In the case of flat space-time of signature \((q,p)\), one obtains the usual conformal group, locally isomorphic to \( SO(p + 1, q + 1) \)†.

A typical example of a Weyl invariant system is given by the massless Klein Gordon field with action

\[
S_{KG} = \int d^n x \sqrt{g} \frac{1}{2} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \xi R \phi^2) \tag{2.5}
\]

where \( n \) is the space-time dimensions, \( \xi = \frac{n-2}{4(n-1)} \) and \( R \) is the curvature scalar. Our conventions for the curvature tensor and the various related quantities are encoded in the following relations: \([\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\mu\sigma} V^\sigma, \ R_{\mu\nu} = R_{\mu\sigma\sigma\nu} \) and \( R = g^{\mu\nu} R_{\mu\nu} \), where \( V^\rho \) is an arbitrary vector and \( \nabla_\mu \) is the usual covariant derivative. The Weyl transformation rules which leave the action (2.5) invariant are defined as follows

\[
g'_{\mu\nu} = \Omega g_{\mu\nu} \\
\phi' = \Omega^{\frac{2-n}{4}} \phi \tag{2.6}
\]

and infinitesimally they read as

\[
\delta g_{\mu\nu} = \sigma g_{\mu\nu} \\
\delta \phi = \frac{1}{4} (2 - n) \sigma \phi \tag{2.7}
\]

† Except in one and two dimensions, where the conformal group is infinite dimensional.
where $\sigma = \ln \Omega$ is taken to be small. The coupling to the scalar curvature $R$ is necessary to insure invariance under local Weyl transformations. For $\xi \neq \frac{n-2}{4(n-1)}$ only a few rigid Weyl symmetries survive, and precisely those for which $\sigma$ is harmonic (i.e. satisfies $\nabla^2 \sigma = 0$). An example of this case is given by the transformation with $\sigma$ constant, which implies a less restrictive condition then (2.3), namely the vanishing of the trace of the energy-momentum tensor integrated over space-time.

To analyze the quantum theory, we consider the Euclidean path integral

$$Z[g_{\mu\nu}] = \int \mathcal{D}\phi \exp\left[-\frac{1}{\hbar} S_{KG}\right]$$

(2.8)

The Ward identities which express the Weyl invariance of the classical theory can be derived by performing a dummy change of path integration variables $\phi \rightarrow \phi' = \phi + \delta \phi$ in (2.8) with $\delta \phi$ given in (2.7), and using the invariance of $S_{KG}$. One obtains

$$\int d^n x \sqrt{g} \langle T \rangle \sigma = -2\hbar \operatorname{Tr}\left(\frac{\partial \delta \phi}{\partial \phi}\right)$$

(2.9)

where $\langle \cdots \rangle$ denotes the vacuum expectation value computed with (2.8). The trace in the right hand side of (2.9), if non-vanishing, gives the anomaly. However, such a trace is still a formal expression because the operator $t(x,y) \equiv \frac{\partial \delta \phi(x)}{\partial \phi(y)}$ is infinite dimensional and one must discuss the necessary regularization to define its trace. This is achieved by using a negative definite operator $\mathcal{R}$ and defining the regularized trace by

$$\operatorname{Tr}(t) = \lim_{m \to \infty} \operatorname{Tr}(t e^{-\frac{\mathcal{R}}{m^2}})$$

(2.10)

The effect of $e^{-\frac{\mathcal{R}}{m^2}}$ is to exponentially damp the contribution of the higher frequency modes, making the trace convergent. Choices of $\mathcal{R}$ are not arbitrary if one wants to obtain a consistent anomaly [10]. A general method [11] to identify such a regulator is to appeal to a Pauli-Villars regularization, which guarantees the consistency of the anomaly. For further details on this procedure, we refer directly to [11]. For our purposes, we use the Fujikawa variables $\tilde{\phi} \equiv g^{1/4} \phi$ to derive the Ward identity in (2.9), so that one is assured that there will be no gravitational anomalies [2], and we obtain the following expression for the Weyl anomaly

$$\int d^n x \sqrt{g} \langle T \rangle \sigma = -\hbar \operatorname{Tr}(\sigma) = -\hbar \lim_{m \to \infty} \operatorname{Tr}\left(\sigma e^{-\frac{\nabla^2 + \xi R}{m^2}}\right)$$

(2.11)
where the regulator, obtained according to refs. [11], is simply given by the kinetic operator \( R = \nabla^2 + \xi R \).

As anticipated in the introduction, we would like to compute the anomaly by representing the trace on the right hand side of (2.11) as a trace in the Hilbert space of a quantum mechanical system. Therefore, we should try to identify a system with action \( S[q] \) which has the operator \( H = -(\nabla^2 + \xi R) \) as quantum Hamiltonian, so that we can use path integral methods to compute the trace. The latter would then read as follows

\[
\text{Tr} \left( \sigma e^{-\beta H} \right) = \int_{PBC} (\mathcal{D}q) \sigma \exp(-S[q])
\]

where the euclidean time \( \beta \) is identified with \( m^{-2} \) and \( PBC \) refers to the periodic boundary condition \( q(0) = q(\beta) \) [12]. The correct system which does the job is that of a particle moving in a curved space. To see this, express the Laplacian \( \nabla^2 \) by using the momentum operator \( \hat{p}_\mu = -ig^{-\frac{1}{4}} \frac{\partial}{\partial q^\mu} g^{\frac{1}{4}} \)

\[
-\nabla^2 = g^{-\frac{1}{4}} \hat{p}_\mu \sqrt{g} g^{\mu\nu} \hat{p}_\nu g^{-\frac{1}{4}}
\]

where \( g = \det g_{\mu\nu}(q) \). Note that \( g_{\mu\nu}(q) \) is a function of the coordinates \( q^\mu \) (for notational convenience, we sometimes do not indicate such a functional dependence). In the classical limit, when the \( p \)'s commute with the \( q \)'s, it corresponds to the classical Hamiltonian

\[
H_{cl} = g^{\mu\nu} \hat{p}_\mu \hat{p}_\nu
\]

which can be obtained from the configuration space Lagrangian

\[
L = g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu
\]

where \( \dot{q}^\mu = \frac{d}{dt} q^\mu \). This Lagrangian describes a particle with coordinates \( q^\mu \) moving in a manifold with metric \( g_{\mu\nu}(q) \). We dedicate the next section to the analysis of the path integral quantization of such a system.

3. The covariant path integral

In this section, we want to consider the quantization of the particle in curved spaces using a manifestly covariant path integral. The object of interest is the transition amplitude
\[ \langle q^\mu_f, t_f | q^\mu_i, t_i \rangle \] for the particle starting at point \( q^\mu_i \) at time \( t_i \) and reaching point \( q^\mu_f \) at time \( t_f \). This is given by the covariant path integral†

\[
\langle q^\mu_f, t_f | q^\mu_i, t_i \rangle = \int_{q(t_i) = q_i}^{q(t_f) = q_f} (\tilde{D}q) \exp \left[ -\frac{1}{\hbar} S[q] \right]
\]

\[
S[q] = \int_{t_i}^{t_f} dt \left( \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu + v_\mu \dot{q}^\mu + s \right) \tag{3.1}
\]

\[
(\tilde{D}q) = \prod_{t_i < t < t_f} \sqrt{g(q(t))} d^n q(t)
\]

where we have included in the action a vector potential \( v_\mu \) and a scalar potential \( s \) to be as general as possible. This path integral is manifestly covariant since it is built from manifestly covariant objects. One can give a derivation of it by starting from the corresponding phase space path integral and showing that the integration over the momenta explicitly produces the \( \sqrt{g} \) factors in the measure of (3.1) (see e.g. [13]). However, the path integral in (3.1) is so natural that one may simply take it as a definition of the quantum theory. As it stands, it is still a formal expression and one must tell how to compute it, i.e. how to discretize it. It is known that different prescriptions for discretizing the path integral correspond to different orderings of the \( p \)'s and \( q \)'s in the quantum Hamiltonian. We will compute the path integral by making a mode expansion of the histories \( q^\mu(t) \), and eventually recognize the quantum Hamiltonian by deriving the Schrödinger equation†† satisfied by the transition amplitude \( \langle q^\mu_f, t_f | q^\mu_i, t_i \rangle \). We would like to compute the path integral perturbatively, in a loop expansion. Unfortunately, the form in (3.1) is not directly suitable for such an expansion since the measure \( (\tilde{D}q) \) is not translational invariant and the usual methods for deriving the propagators are not applicable. To overcome this difficulty, we introduce two anticommuting ghost fields, \( b \) and \( c \), and exponentiate the unwanted piece sitting in the measure. We get

\[
\langle q^\mu_f, t_f | q^\mu_i, t_i \rangle = \int_{q(t_i) = q_i}^{q(t_f) = q_f} (Dq)(Db)(Dc) \exp \left[ -\frac{1}{\hbar} (S[q] + S_{gh}[b, c, q]) \right] \tag{3.2}
\]

where now the measures are translational invariant, i.e.

\[
(Dq) = \prod_{t_i < t < t_f} d^n q(t), \quad (Db) = \prod_{t_i < t < t_f} db(t), \quad (Dc) = \prod_{t_i < t < t_f} dc(t) \tag{3.3}
\]

† After Wick rotation to Euclidean time.

†† More properly we should call it the diffusion equation, since we use the Euclidean path integral.
and the ghost action is given by

$$S_{gh}[b, c, q] = \int_{t_i}^{t_f} dt \ b \sqrt{g(q)} c$$  \hspace{1cm} (3.4)$$

Such a reformulation of the path integral is still manifestly covariant, provided one transforms $b$ and $c$ as weight $\frac{1}{2}$ densities under change of coordinates. Explicitly

$$q^\mu \rightarrow q^{\mu'} = q^{\mu'}(q')$$

$$b \rightarrow b' = b \left( \frac{\partial q'}{\partial q} \right)^{\frac{1}{2}}$$

$$c \rightarrow c' = c \left( \frac{\partial q'}{\partial q} \right)^{\frac{1}{2}}$$  \hspace{1cm} (3.5)$$

Under these rules the ghost action $S_{gh}[b, c, q]$ as well as the full measure $(Dq)(Db)(Dc)$ are invariant objects.

4. Two loop expansion and the quantum Hamiltonian

We come now to the task of evaluating in a perturbative expansion the covariant path integral defined in the previous section. We will set $\hbar = 1$ in the following. The easiest way to proceed is to make use of the general covariance of the path integral by choosing Riemann normal coordinates (see e.g. [14]). We will use Riemann normal coordinates $z^\mu$ centered at the final point $q_f^\mu$. Then, by definition, the Riemann normal coordinates of a neighbouring point $q^\mu$ are the components of the tangent vector at $q_f^\mu$ of the geodesic joining $q_f^\mu$ to $q^\mu$ in unit time. In particular $q_f^\mu$ has Riemann normal coordinates $z_f^\mu = 0$. Thus we see that the Riemann normal coordinates have a clear geometrical meaning, namely they are tangent vectors belonging to the tangent space of the manifold at the point $q_f^\mu$. This property will be useful for recognizing the various geometrical quantities in the perturbative expansion of the transition amplitude (3.2). Let’s denote the Riemann normal coordinates of the initial point $q_i^\mu$ in (3.2) by $x^\mu$ and set $\beta = t_f - t_i$. Then the geodesic

$$z_{cl}^\mu = -x^\mu \tau; \hspace{1cm} \tau = t - t_f \beta$$  \hspace{1cm} (4.1)$$

is the classical path for vanishing potentials $v_\mu$ and $s$. Before proceeding further, let us switch to the dimensionless variable $\tau = \frac{t - t_f}{\beta}$, so that the action reads as follows (with $\dot{q}^\mu = \frac{d}{d\tau} q^\mu$)

$$S = \frac{1}{\beta} \int_{-1}^{0} d\tau \left( \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^{\nu} + \beta v_\mu \dot{q}^\mu + \beta^2 s \right)$$  \hspace{1cm} (4.2)$$
and remark that to obtain the Schrödinger equation we need terms up to $\beta^2$ in the transition amplitude (3.2) (see our derivation in eq. (4.21)). This approximation corresponds to a two loop expansion with $\beta$ the loop counting parameter.

In order to efficiently carry out the Riemann normal coordinates expansion of the action in (4.2), it is useful to consider the field $q^\mu(\tau, \lambda)$, defined for each $\tau$ to be the geodesic connecting $q^\mu_f$ at $\lambda = 0$ to $q^\mu(\tau)$ at $\lambda = 1$. Then by construction $q^\mu(\tau, \lambda)$ satisfies

$$\frac{D}{d\lambda} \frac{dq^\mu}{d\lambda} \equiv \frac{d^2 q^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dq^\nu}{d\lambda} \frac{dq^\rho}{d\lambda} = 0 \quad (4.3)$$

and the Riemann normal coordinates of $q^\mu(\tau)$ are given by

$$z^\mu = \frac{dq^\mu}{d\lambda} \bigg|_{\lambda=0} \quad (4.4)$$

The expansion of $S$ is carried out by noticing that

$$S[q(\tau)] = S[q(\tau, \lambda = 1)] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n S[q]}{d\lambda^n} \bigg|_{\lambda=0} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{D^n S[q]}{d\lambda^n} \bigg|_{\lambda=0} \quad (4.5)$$

so that one can proceed using covariant derivatives as well as the following identities

$$\frac{Dz^\mu}{d\lambda} = \frac{Dg_{\mu\nu}}{d\lambda} = 0; \quad \frac{Dq^\mu}{d\lambda} = \frac{Dz^\mu}{d\tau}; \quad \left[ \frac{D}{d\lambda}, \frac{D}{d\tau} \right] V^\mu = z^\nu q^\sigma R^\mu_{\nu\sigma} \rho V^\rho \quad (4.6)$$

where in the last equation $V^\mu$ is an arbitrary vector and $z^\mu = \frac{d}{d\lambda} q^\mu$ for any $\lambda$.

We obtain the following Riemann normal coordinates expansion of the action in (4.2)

$$S = \frac{1}{\beta} \int_{-1}^{0} d\tau \left[ \frac{1}{2} g_{\mu\nu}(0) \dot{z}^\mu \dot{z}^\nu + \frac{1}{6} R_{\mu\nu\rho\sigma}(0) z^\mu \dot{z}^\nu \dot{z}^\rho z^\sigma + \beta \left( v^\mu(0) \dot{z}^\mu + \nabla_\mu v^\mu(0) z^\mu \dot{z}^\nu \right) + \beta^2 s(0) + \cdots \right] \quad (4.7)$$

where we kept only terms contributing to two loops. Next, we expand a general path $z^\mu$ around the classical solution as

$$z^\mu = z^\mu_{cl} + y^\mu \quad (4.8)$$

$$y^\mu = \sum_{n=1}^{\infty} y^\mu_n \sin(n\pi \tau)$$
with \( z_{cl}^\mu \) given in (4.1) and where \( y^\mu \) is the quantum variable satisfying the boundary conditions \( y^\mu(0) = y^\mu(-1) = 0 \). Inserting this expansion in the quadratic leading piece of (4.7), we obtain

\[
S^{(2)} \equiv \frac{1}{\beta} \int_{-1}^{0} d\tau \frac{1}{2} g_{\mu\nu}(0) \dot{z}^\mu \dot{z}^\nu = \frac{1}{2\beta} g_{\mu\nu}(0) x^\mu x^\nu + \frac{\pi^2}{4\beta} g_{\mu\nu}(0) \sum_{n=1}^{\infty} n^2 y_n^\mu y_n^\nu \tag{4.9}
\]

This leads to the propagators for the Fourier coefficients of the quantum field \( y^\mu \)

\[
\langle y_m^\mu y_n^\nu \rangle = \frac{2\beta}{\pi^2 n^2} \delta_{m,n} g^{\mu\nu}(0) \tag{4.10}
\]

as well as to the main normalization of the measure

\[
(Dz) = A \prod_{m=1}^{\infty} \sqrt{g(0)} \prod_{\mu=1}^{n} \left( \frac{\pi m^2}{4\beta} \right)^{\frac{1}{2}} dy_m^\mu \tag{4.11}
\]

with \( A \) an yet unfixed normalization factor (we have required here that apart from the normalization factor \( A \), the gaussian integrals over each Fourier mode give unity; the full normalization will be derived later on, in the derivation of the Schrödinger equation in eq. (4.21)). Using the above propagator and treating the remaining part of (4.7) as a perturbation, we get a first contribution to the transition amplitude

\[
\Delta_1 \langle 0, 0 | x^\mu, -\beta \rangle = A \exp \left( -\frac{1}{2\beta} g_{\mu\nu}(0) x^\mu x^\nu \right) \left( 1 - \left( \frac{1}{18} R_{\mu\nu}(0) x^\mu x^\nu + \frac{\beta}{36} R(0) \right) M \right.
\]

\[
- \frac{5}{72} R_{\mu\nu}(0) x^\mu x^\nu + \frac{\beta}{36} R(0) + v_\mu(0) x^\mu + \frac{1}{2} v_\mu(0) v_\nu(0) x^\mu x^\nu \right)
\]

\[
+ \frac{1}{2} \nabla_\mu v_\nu(0) x^\mu x^\nu - \beta s(o) \right)
\]  

(4.12)

where a divergent piece has been regulated by truncating the mode expansion at the \( M^{th} \) mode, while in the converging part the sum \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \) has been used. In evaluating (4.12) we have kept only terms up to order \( \beta \) or terms which potentially can generate pieces of order \( \beta \) when an integration over the initial point \( x^\mu \) is performed. The highest loop graph contributing to (4.12) is a two loop one.

Next, we look at the ghost piece. The expansion of the metric in Riemann normal coordinates is read off from (4.7)

\[
g_{\mu\nu}(z) = g_{\mu\nu}(0) + \frac{1}{3} R_{(\mu\nu)\rho} z^\rho z^\sigma \tag{4.13}
\]

We use time translational invariance to set \( t_f = 0 \).
where \((\ldots)\) denotes symmetrization in the enclosed indices. Inserting this in the ghost action we get
\[
S_{gh} = \int_{t_i}^{t_f} dt \ b \sqrt{\frac{g(q)}{g(0)}} c = \beta \int_{-1}^{0} d\tau \ b(1 + \frac{1}{6} R_{\mu\nu}(0) z^\mu z^\nu + \ldots) c
\] (4.14)

Note that, for convenience, we left a \(z^\mu\)-independent factor \(\sqrt{g(0)}\) in the measure (4.11). At this point we have to discuss in more details the boundary conditions on the ghost fields \(b, c\).

Since we need to reproduce the Jacobian in the measure (3.1), which excludes the end points \(t_f\) and \(t_i\), we should not integrate on the boundary values of \(b\) and \(c\). Therefore we require the following boundary condition for the ghost fields:
\[
b(t_f) = b(t_i) = c(t_f) = c(t_f) = 0,
\]
which is the only boundary condition consistent with the algebraic character of the ghost action (i.e., it is the only boundary condition consistent with the classical ghost field equations). Taking this into account, we use the mode decomposition
\[
b(\tau) = \sum_{n=1}^{\infty} b_n \sin(n\pi \tau); \quad c(\tau) = \sum_{n=1}^{\infty} c_n \sin(n\pi \tau)
\] (4.15)

and derive from the quadratic piece of the ghost action the ghost propagator
\[
\langle b_n c_m \rangle = -\frac{2}{\beta} \delta_{n,m}
\] (4.16)

Together with the correct normalization of the ghost measure
\[
(\mathcal{D}b)(\mathcal{D}c) = \prod_{m=1}^{\infty} \frac{2\hbar}{\beta} db_md_cm
\] (4.17)

Then, the interacting term in (4.14) gives the remaining perturbative contribution to the transition amplitude
\[
\Delta_2<0,0|x^\mu, -\beta> = A \exp\left(-\frac{1}{2\beta} g_{\mu\nu}(0)x^\mu x^\nu\right) \left(1 + \left(\frac{1}{18} R_{\mu\nu}(0)x^\mu x^\nu + \frac{\beta}{36} R(0)\right) M - \frac{1}{72} R_{\mu\nu}(0)x^\mu x^\nu + \frac{\beta}{72} R(0)\right)
\] (4.18)

Summing (4.12) with (4.18), we see that the potentially divergent terms cancel and we are left with a finite expression for the transition amplitude in the two loop approximation
\[
<0,0|x^\mu, -\beta> = A \exp\left(-\frac{1}{2\beta} g_{\mu\nu}(0)x^\mu x^\nu\right) \left(1 - \frac{1}{12} R_{\mu\nu}(0)x^\mu x^\nu + \frac{\beta}{24} R(0) + v_\mu(0)x^\mu + \frac{1}{2} v_\mu(0) v_\nu(0)x^\mu x^\nu + \frac{1}{2} \nabla_\mu v_\nu(0)x^\mu x^\nu - \beta s(o)\right)
\] (4.19)

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It is worth to note that we have performed the full path integral even in the limit $\beta \to 0$. Approximating the transition amplitude by considering only the contribution coming from the extremal classical trajectory is incorrect when the space is curved.

We derive now the Schrödinger equation to recognize the quantum Hamiltonian associated with the transition amplitude (4.19) and fix the coefficient $A$ in it. The transition amplitude is used to evolve the wave function $\Psi(q^\mu, t)$ as follows

$$\Psi(q_f^\mu, t_f) = \int d^n q_i \sqrt{g(q_i)} \langle q_f^\mu, t_f | q_i^\mu, t_i \rangle \Psi(q_i^\mu, t_i)$$ (4.20)

Using (4.20) in Riemann normal coordinates and Taylor expanding the wave function as well as the measure on the right hand side, we get

$$\Psi(0, 0) = \int d^n x \sqrt{g(0)} (1 + \frac{1}{6} R_{\mu\nu}(0) x^\mu x^\nu + \cdots) \langle 0, 0 | x^\mu, -\beta \rangle$$ (4.21)

and inserting (4.19) in (4.21), we obtain as a consistency requirement

$$A = (2\pi \beta)^{-\frac{3}{2}}$$ (4.22)

as well as the Euclidean Schrödinger equation (diffusion equation)

$$-\dot{\psi} = H\psi$$

$$H = -\frac{1}{2} \nabla^2 - v^\mu \partial_\mu - \frac{1}{2} v_\mu v^\mu - \frac{1}{2} (\nabla_\mu v^\mu) - \frac{1}{8} R + s$$ (4.23)

This is the correct quantum Hamiltonian associated with the path integrals (3.1) and (3.2). It can be rewritten in the following manifestly gauge invariant way

$$H = -\frac{1}{2} (\nabla^\mu + v^\mu)(\partial_\mu + v_\mu) - \frac{1}{8} R + s$$ (4.24)

Now, as promised, we turn to the computation of the Weyl anomaly in two dimensions. We proceed in full generality and make use of (4.19) and (4.23) to derive a lemma, already presented in [15], for computing general algebraic anomalies. Namely, we want to compute

$$I = \lim_{m \to \infty} Tr(\sigma e^{i R \frac{m}{2}})$$

$$R = \nabla^2 + V^\mu \partial_\mu + S$$ (4.25)
which corresponds to

$$I = \lim_{\beta \to 0} \int_{PBC} (Dq) \exp(-S[q])$$

(4.26)

where $PBC$ stands for the periodic boundary conditions $q^\mu(0) = q^\mu(-\beta)$, and where, according to (4.23), we have to use the potentials

$$s = -\frac{1}{2} S + \frac{1}{8} R + \frac{1}{2} v_\mu v^\mu + \frac{1}{2} \nabla_\mu v^\mu$$

(4.27)

$$v_\mu = \frac{1}{2} V_\mu$$

to recover $\mathcal{R}$ as $-2$ times the quantum Hamiltonian. Actually, in the limit $\beta \to 0$ there are also divergent pieces in (4.26) which should be discarded. This corresponds to the fact that in computing the effective action in quantum field theory, one renormalizes away all the divergent pieces. In particular, anomalies are always finite. Thus, one should interpret the “$\lim_{\beta \to 0}$” symbol as meaning “pick the $\beta$ independent part of”. With these cautionary words for interpreting our formulae, we get

$$I = \lim_{\beta \to 0} \int d^2 q \sqrt{g(q)} \sigma(q) \langle q^\mu, 0 | q^\mu, -\beta \rangle$$

$$= \int \frac{d^2 q}{4\pi} \sqrt{g(q)} \sigma(q) \left( S - \frac{1}{2} \nabla_\mu V^\mu - \frac{1}{4} V_\mu V^\mu - \frac{1}{6} R \right)$$

(4.28)

In particular, for vanishing $S$ and $V^\mu$, we get the well-known result for the Weyl anomaly in (2.11), specialized to the case of a scalar field in two dimensions

$$\langle T \rangle = \frac{\hbar}{24\pi} R$$

(4.29)

5. Conclusions

We have shown that Weyl anomalies can be computed by using a quantum mechanical representation of the “Fujikawa” Jacobian. From the normalization (4.22) and the formula (2.11), we can see that the Weyl anomaly in $n$ dimensions is given by the $\beta^{\frac{n}{2}}$ perturbative contribution to the quantum mechanical path integral, indicating that in odd dimensions the Weyl anomaly always vanish, since fractional powers of $\beta$ never arise in the quantum mechanical loop expansion. In even dimensions $n = 2k$, the terms contributing to the anomaly contain connected graphs up to $k + 1$ loops. We have explicitly computed the Weyl anomaly for a scalar field in two dimension. This result has been achieved by using a novel reformulation of the path integral for a particle in curved spaces which makes use of ghost
fields. The advantage of using ghost fields comes from the fact that one can obtain translational invariant path integral measures, suitable for generating the perturbative expansion. Our approach has been rather pragmatical in that we used the quantum mechanics as a mathematical tool to compute an infinitesimal path integral Jacobian, as pioneered in the computation of chiral anomalies [5], where supersymmetric quantum mechanics was used. From this perspective, one does not care about the nature of the particular quantum mechanical system used to represent the required trace. Typically, one considers the quantum mechanical Hilbert space to be of a fictitious, non-physical nature [9]. However, at a closer inspection, one can reinterpret our computation as being the evaluation of the Weyl anomaly using the first quantized description of a scalar field coupled to gravity. The quantum mechanical Hilbert space is then not at all fictitious, but it is directly related to the Hilbert space of the first quantized Klein-Gordon field. This interpretation can in fact be extended to the Schwinger-De Witt method for calculating the Klein-Gordon propagator in curved spaces [9]. In that method, the Klein-Gordon propagator is obtained by integrating over the proper time a kernel which satisfies the heat equation, and which, in fact, is given by our equation (4.19). The proper time can be identified as the integral over the moduli space of the one dimensional metric, which enters the action of the first quantized scalar field. Such an action would reduce to equation (4.2) upon gauge fixing and by ignoring the corresponding ghost fields, which however do not couple to the target-space geometry and which should not be confused with the ghosts introduced in section 3.

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