Quantum coherence of spatial photonic qudits: experimental measurement and path-marker analysis

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Abstract
Discussions about quantum interference, indistinguishability, and superposition between quantum states go back to the beginning of quantum mechanics. However, the theoretical problem, concerning quantitative measures for quantum coherence, was only recently solved by Baumgratz et al. (2014 Phys. Rev. Lett. 113 140401). Since then, many works have explored one of the possible coherence measures: the $l_1$ norm, which has not yet been obtained experimentally for spatial photonic states in high dimensions. In this article, we study the measurement of spatial qudits quantum coherence. We treat the optical measure-operators as path-markers and show in which condition an interference pattern can be used for measure $l_1$ norm. We analyze the validity of a $l_1$ norm measurement method, proposed by Paul and Qureshi (2017 Phys. Rev. A 95 042110), implementing it experimentally in qutrit states, and propose an alternative method for qudits states in which one of their assumptions does not apply.

Keywords: quantum coherence, spatial qudits, optical interference, path-markers

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum coherence is a feature of quantum systems related to an inability to distinguish between two states of a quantum superposition, with two or more states [1]. Historically, it has been associated with the indistinguishability of quantum trajectories in interferometers. Primarily motivated by questions concerning particle-wave dual behavior, interferometric dualities have been a subject of many works since the beginning of quantum mechanics [2, 3].

In the 90s it was well established that for two-dimensional states, or interferometers with two trajectories, the visibility of the observed interference pattern quantifies the indistinguishability between the two states in superposition, that is, between the two trajectories in the interferometer [4]. Englert was the first to show, quantitatively, that the observation of interference fringes depend not only on the state studied but also on the states of the detector used for the measurements [5]. Despite quantum coherence is a state property of a quantum system, the observed quantum coherence also depends on the measurement apparatus, and is less than or equal to that belonging to the superposition state.

Over the years, many authors tried to generalize the Englert quantifiers for higher dimension quantum states, but without success [6–8]. Only in 2014 Baumgratz, Cramer and Plenio obtained adequate measures for quantum coherence through resource theory [9, 10]. Since then, quantum coherence particulars have been widely studied, as well as its usefulness and applicability in quantum information developments. In [33], for example, quantum coherence of a two-photon state was studied in a coupled cavity system. In [30], the authors showed the relation between entanglement and quantum coherence, providing a new measure that is entangled based. In [31, 32], it is discussed how to design universal quantum gates in a decoherence free subspace and how to extend the coherence time in superconducting circuits.

Although the theoretical problem has been solved, experimental measurement of qudits remains to be done. One of the quantum coherence measures, presented in [9], was demonstrated experimentally in [11] for photonic states in...
polarization variables. Concerning transverse spatial photonic states, a theoretical method for measuring spatial qudits quantum coherence was proposed in [12]. In this article, we analyze this method validity and implement it experimentally in two specific qutrit states, for which the method proved to be valid. We also present a theoretical and general proposal, which is valid for any qudit state, including the states for which the method of Paul and Qureshi is not applicable.

The article is organized as follows: in section 2 we discuss how Englert analysis is present in transverse spatial qudits, and in section 3 we analyze the link between quantum coherence and the interference pattern in detail. In section 4 we present, for qutrit states, experimental results obtained based on the proposal of Paul and Qureshi. In sequence, in section 5, we discuss an alternative way to measure qudits quantum coherence for cases in which this method is not applicable. Finally, we conclude in section 6. In appendix A, we show that we can regard the system passing through the measurement apparatus, as an interaction between the photonic path state and a path-marker and what are the requirements for the measurement apparatus such that the interference pattern is observed and quantum coherence is measured.

2. Near- and far-field operations as path-marker in a context of spatial states

Quantum coherence is a property of a quantum state system. Its measurement depends on the observation of the interference pattern between the vector states components (paths) of the system. However, the interference pattern observation requires a path-marker being in a superposition state such that the detection system is unable to distinguish the state component in which the system is (or which path of interferometer the particle followed), as pointed out by Englert [5].

In this article, we deal with photons in a state superposition of transverse path states. The state system is prepared after a photon crosses multiple slits, and in such a way that the state components are defined by the possible photon paths through the apertures of the multiple slit set (figure 1(a)). The quantum coherence measurement is done by measuring interference patterns using a lens placed in the path of the photon after it crosses the multiple slits towards the detection plane (figure 1(c)). As we discuss in appendix A, the ratio between the distance multiple slit set-lens plane and the distance lens plane-detector plane is essential (figures 1(b) and (c)).

By calculations explained in detail in appendix A, we demonstrate that the propagation of the spatial photonic state through free space towards a lens and then crossing the lens and propagating towards the detection plane plays a role as a path marker. So, we show how Englert’s discussion [5] is present in the context of photonic spatial states (prepared with slits) and interference patterns for quantum coherence measurements.

Considering a setup similar to that in figures 1(b) and (c) (multiple slits–lens–detector), the probability distribution of photons linear momentum, by a multiple slit set. Photons pass through multiple slits positioned transversely with respect to the direction of photons propagation. Only photons with linear momentum within a specific range of values pass through some aperture, and |l⟩ is the state of a photon crossing a specific aperture l. (b) A lens with a focal length f, is placed at distance 2f from both the multiple slits and the detector plane, allowing the measurement of the photonic spatial probability distribution at the image plane. (c) A lens with a focal length f is placed at distance f from both, multiple slits and the detector plane, allowing the measurement of the photonic spatial probability distribution at the Fourier plane, i.e. the multiple-slit interference pattern.
detecting a photon has oscillation functions which the coefficients are proportional to the off-diagonal elements of the density matrix. We can see this density matrix as the reduced one from a system state in a larger Hilbert space, after we trace out \( \{ E^+(x,\z_d) | l \} \{ m | E^-(x,\z_d) \} \).

It is notable the similarity between our analyses and the Englert treatment (equations (A6) and (A1), and between the equations (A5) and (A2)). It is also remarkable the fact that if \( \{ m | E^+(x,\z_d) E^+(x,\z_d) | l \} = 0 \), we do not observe any coherence. In this case, we do not have access to the off-diagonal elements of the joint state from the interaction state - path marker (equation (A5)). On the other hand, if the elements of \( \{ (m | E^+(x,\z_d) E^+(x,\z_d) | l \} \) have the same amplitudes and different phases, we can observe an oscillation controlled only by the off-diagonal elements of the initial system state (equation (1)).

We interpret the spatial photonic state as interacting with a path-marker before the detection plane at \( \z_d \), similarly to the Englert’s treatment. We can identify each path-marker possible state, as the state of a photon which is annihilated at the detection plane after it propagates from the aperture \( f \) and passes through an optical arrangement. Therefore, the optical setup configuration between the multiple slits plane, at \( \z_d \), and the detection plane, at \( \z_d \), determines the possible path-marker states.

In configurations where the lens equation is obeyed, the probability distribution of detecting one photon in the plane at \( \z_d \) reproduces the multiple slit spatial profile. A lens projects, on the detection plane, the multiple slit image, whose magnification depends on the setup configuration.

The measurement of quantum coherence of a qudit state is viable only in the condition that the possible path-marker states are maximally indistinguishable. Concerning path states prepared with multiple slits, this condition occurs when the optical Fourier transform of the slits plane coincides with the detection plane. We can obtain this condition in a configuration in which the distance between the detector and the lens is equal to its focal length. In this case, the probability distribution of detecting one photon is an interference pattern, with the spatial oscillation depending on the state quantum coherence. In our experiment, we use the configuration of figure 1(c), in which the multiple slits and the detector are equidistant from the lens by the focal length.

Equation (3) describes mathematically an interference pattern at \( \z_d \) with contributions of both, diagonal (\( \rho_m \)) and off-diagonal (\( \rho_{lm}, l \neq m \)) coefficients of \( \rho \). However, quantum coherence, which we intend to determine, involves only the absolute values of the off-diagonal coefficients (equation (4)). Therefore, we can infer quantum coherence from an interference pattern, when the possible path-marker states are non-orthogonal and maximally indistinguishable. The path-marker states, in photonic spatial interference patterns, are produced by the optical configuration that produces the interference pattern, as demonstrated here.

The possibility to infer quantum coherence from an interference pattern, where the possible path-marker states are non-orthogonal and maximally indistinguishable, accompanies the question: how to determine the value of the quantum coherence from the detected spatial interference pattern, i.e. from the photonic spatial probability distribution at the Fourier plane? We aim to answer this question in the next section.

### 3. Quantum coherence and photonic interference pattern

A simple method to prepare spatial photonic states consists of discretizing the transverse profile of an ensemble of photons, identically generated, using multiple slits [14, 16–18]. Figure 1(a) shows a multiple slits array which selects photons within specific intervals of linear momentum. We can describe the state immediately after the apertures by the density operator

\[
\rho = \sum_{l=-\lambda}^{\lambda} \sum_{m=-\lambda}^{\lambda} \rho_{lm} | l \rangle \langle m |,
\]

where \( \lambda = \frac{D-1}{2} \), \( D \) is the dimension of the state given by the number of slits, \( \{ \rho_{lm} \} \) is the set of density matrix coefficients and \( \{ | j \rangle \} \) is the chosen basis with \( | j \rangle \) indicating here the state of a photon transmitted through the aperture \( j \). We use multiple slits set with thin apertures with \( 2\lambda \) width, so we can consider only the detection of photons that crossed the multiple slits set with \( D \) apertures. After photons are transmitted through slits, their path states form a discrete state set. In this scenario, the probability of detecting one photon on detection plane at the longitudinal position \( \z_d \), and along the transverse \( x \)-direction is [19–21, 27]

\[
P(x,\z_d) = Tr(\Gamma \rho),
\]

where \( \Gamma = E^-(x,\z_d) E^+(x,\z_d) \) is a positive operator describing the measurement and \( E^+(x,\z_d) \) is the electrical field operator proportional to the annihilation operator \( a(x) \) that indicates the destruction of a photon at the transverse position \( x \) in the plane at \( \z_d \), which is the plane where the detector is scanned along the \( x \)-direction. The density operator \( \rho \) is written in terms of \( | j \rangle \) defined at the multiple-slit plane and \( E^+(x,\z_d) \) is constructed from the multiple slits to the detection plane and therefore, includes the optical configuration that guides photons from the multiple slits to the detector. Figures 1(b) and (c) show two possible optical configurations: detection of the photons at the image plane and the Fourier plane, respectively.

The probability distribution of detecting a photon at \( x \) transverse position, when scanning detector in the Focal plane [15, 27], is

\[
P(x,\z_d) \propto \sin^2 \left( \frac{kax}{f} \right) \left[ \sum_m \rho_{mm} \right.
\]

\[
+ \sum_{l:\neq m, l< m} 2 | \rho_{lm} | \cos (\gamma (l-m) x - \varphi_{lm}) \right],
\]

where \( \sum_m \rho_{mm} = 1 \), \( \varphi_{lm} \) is the phase of \( \rho_{lm} \), \( k \) is the wavenumber of the photons, \( f \) is the lens focal length, and \( \gamma \)...
is the phase that photons acquire propagating from the slits to the plane at \( z_0 \) [13].

Quantum coherence defined in [9] for the state represented in equation (1), in a normalized form [23], is

\[
C = \frac{1}{D-1} \sum_{l,m}^{\lambda} |\rho_{lm}| .
\]  

(4)

In order to extract the quantum coherence \( C \) from equation (3), we define an oscillation function \( \Theta(x,z_d) \), namely

\[
\Theta(x,z_d) = \frac{P(x,z_d) - P_{\text{diag}}(x,z_d)}{P_{\text{diag}}(x,z_d)}
\]

\[
= \sum_{l,m} \cos (\gamma (l-m) x - \varphi_{lm}) |\rho_{lm}| ,
\]  

(5)

where \( P_{\text{diag}}(x,z_d) \) is the probability to detect one photon from a state \( P_{\text{diag}} \), on the far-field plane (equation (3)). The state \( \rho_{\text{diag}} \) is an auxiliary state, supposed to be previously characterized as incoherent and having the same diagonal elements of \( \rho \). We consider that both \( \rho \) and \( P_{\text{diag}} \) are normalized in equation (5).

From now on, we will analyze qutrits photon state in path variables, as an example, but one could extend our discussions for higher dimensions without difficulties. For qutrit systems, we can write a general and normalized state by

\[
\rho = \begin{pmatrix}
-1 & 0 & 1 \\
0 & \rho_{01} & \rho_{10} \\
1 & \rho_{10}^* & \rho_{11}
\end{pmatrix}
\]

(6)

and its oscillation function as

\[
\Theta(x,z_d) = 2 |\rho_{01}| \cos (\gamma x + \varphi_{10}) + 2 |\rho_{10}| \cos (\gamma x + \varphi_{01})
\]

\[
+ 2 |\rho_{11}| \cos (2\gamma x + \varphi_{11}) .
\]  

(7)

Figure 2 shows the qutrit oscillation function (equation (7)) for different states, that is, qutrit states with different coefficients \( \rho_{lm} \) and phases \( \varphi_{lm} \). In all the graphs, the top lines (dashed blue lines) correspond to the respective quantum coherence in a non-normalized form, namely the value of 2 \((|\rho_{01}| + |\rho_{10}| + |\rho_{11}|)\). The bottom lines (dotted orange lines) correspond to the negative of these values.

For states with null-phases in figures 2(a) and (b) (or phases equal to \( \pi \) in figures 2(c) and (d)), we can determine quantum coherence by the maximum (or the minimum) value achieved by the oscillation curve. We can write

\[
C = \begin{cases}
\frac{\text{max}_{\varphi_{lm}}(x_{\text{max}}, z_d)}{D-1} & \text{when } \{\varphi_{lm} = 0\} ,
\frac{\text{max}_{\varphi_{lm}}(x_{\text{min}}, z_d)}{D-1} & \text{when } \{\varphi_{lm} = \pi\} .
\end{cases}
\]  

(8)

This is possible because, in the case of phases of the density matrix elements equal to 0 (or \( \pi \)), each cosine term in equation (7) reaches a maximum (or minimum) value at the same point \( x_{\text{max}} \) (\( x_{\text{min}} \)). It makes the overall maximum (or overall minimum) equal to the sum of the absolute values of the off-diagonal \( \rho \) elements.

The determination of \( l_1 \) norm using the central maximum of the interference pattern in equation (3), and the diffraction pattern from \( P_{\text{diag}} \), was theoretically proposed in [12]. The authors proposed the use of the equivalent to a single point of the oscillation curve in equation (5). The authors treated the case where the phases of all density matrix elements are null. However, figure 2 shows that if \( \rho \) elements have phases different from zero (or \( \pi \)), or even if it is completely unknown, we cannot determine its quantum coherence by using the plot of its oscillation function, neither employing the contrast of the interference patterns.

In the next section we describe an experiment for \( l_1 \) norm quantum coherence determination, considering qutrit states for which equation (8) is valid. In section 5, we present an alternative for the cases for which equation (8) is not applicable.

4 Experiment with qutrits

We determined the quantum coherence of two different spatial qutrit states. Each of them is part of a bipartite \( 2 \times 3 \) state prepared with multiple slits and twin photons generated by spontaneous parametric down conversion (SPDC) crossing the slits [24]. The twin photons are generated in a collinear regime by
single photon qutrit, in each coincidence registered between the two detectors. This operation is equivalent to the partial trace of the density matrices obtained marginaly from the 2 × 3 states (prepared as in [24]) proposed in [25]. The states in equation (9) have null phases, it is possible to determine their quantum coherence using the functions in equation (10). Their symmetries also enable us to determine the usual contrast of a typical two-dimensional spatial interference pattern [26]. We use the contrast to compare the results obtained with different methods for measuring quantum coherence. The results comparison enables us to evaluate the reliability of the method that we present. Therefore, we study qutrit states with simple density matrices and simple interference patterns, aiming to verify if the method works well and if it could be applied to any qudit state with photonic spatial correlations are prepared as shown in [24]. The photon in the qubit state passes trough a lens (L1) that projects the slits image on a detection plane at $z_D$. The photon in the qubit state passes trough a lens (L1) that projects the optical Fourier transform on a detection plane at $z_D$. 100 μm single slit is coupled to the bucket detector $D_1$ such that the spatial fringes can be resolved. The qubit detector ($D_2$) is a bucket detector which collects all transmitted photons such that the counts of qubit photons work as a trigger to the counts of the qutrit photons. The qubit detector ($D_2$) records the photons transverse distribution by scanning $D_1$ along the $x$-direction.

The black squares in figures 5(a) and (c), are the probability distributions $P(x,z_D)$ for the states in equation (9), measured on the far-field plane. The fits of the theoretical expressions to the experimental results are shown in the full lines superimposing the experimental data (black curves). The absolute values of the off-diagonal elements of $\rho_1$ and $\rho_2$ are the free parameters.

To simulate the presence of an auxiliary incoherent state proposed in [12], we obtain $P_{\text{diag}}(x,z_D)$ by measuring the spatial photon counts distribution coming from each aperture individually and adding them. We registered all counts the bucket detector and the qutrit fringes resolving detector. In other words, it warrants that we are working with a quantum source of light. Besides that, with the partial trace operation on the qubit system, we manage to prepare qutrit states with the symmetries shown in the density operators, in equation (9). A lens $L_1$ projects the optical Fourier transform of the triple slit plane on the plane at $z_D$, where we scan an avalanche photodiode ($D_1$) along the $x$-direction. 810 nm interference filters, with a bandwidth of 10 nm, are used to select the photon pairs. The results are compared with the theoretical predictions (full lines). The fits of the theoretical expressions to the experimental results are shown in black curves. The absolute values of the off-diagonal elements of $\rho_1$ and $\rho_2$ are the free parameters.
in coincidence with the detector $D_2$, and add the diffraction patterns of each aperture. It reproduces the result that would be achieved for a diagonal state without the requirement of a second light source, or setup preparation. Although it is necessary to realize extra diffraction measurements, concerning the number of optical elements in the setup, the experimental complexity is smaller. The experimental procedure that we used for measuring the diffraction patterns of the individual slits is shown in figure 4(a), and the normalized spatial photons distribution measured in coincidence with $D_2$, resulting from the diffraction sums, are the red dots presented in figures 5(a) and (c), and the fits of the theoretical curves to the experimental diffraction sums are superposed to them (full red line).

By obtaining $\Theta (x, z_\alpha)$ using the experimental $P(x, z_D)$ and $P_{diag} (x, z_D)$, as shown in the first line of equation (5), we obtain the experimental oscillations shown in figures 5(b) and (d). The full line (blue) superposed to the experimental data are the theoretical oscillations curves used to fit the experimental data and $A_1$ and $A_2$ (equation (10)) are the free parameters. At the beginning and end of the diffraction oscillations, one can observe relative fluctuations of the counts, which induce bigger fluctuation at the ends of the oscillation curves. These fluctuations are statistical and are more pronounced in the beginning and the end due to the low counts in the extreme positions of the detector. We highlight that since all data were used in the analyses, in the fits and the average calculation, these fluctuations did not interfere in the achievement of results in accordance with theoretical prediction.

Table 1 presents the values of quantum coherence that we obtained from the theoretical curve fits of the interference patterns (equation (3)) and the oscillation curves (equation (9)). The fits were obtained by a data analyzes software. It obtains the parameters of the interference curve by comparing the general expression, introduced in it, and the experimental data. The experimental data are the coincidence counts and their experimental errors are calculated as the square root of the counts. The software calculates the best adjustable parameters corresponding to the statistical uncertainties. To obtain the quantum coherence, we sum the parameters given by the fit (absolute values of the off diagonal density matrix elements) and propagate the errors according to error propagation rules. We observe a good agreement between the values that the theoretical curve fitting provided for the sum of the modulus of the off-diagonal elements of $\rho$ in equation (3) and equation (10) considering the states shown in equation (9). The sum of these terms are equal to the quantum coherence of the states. In general, fitting the experimental data with a mathematical function having adjustable parameters demands knowledge about the theoretical expression of the patterns to be measured. Because of that, we need a different analysis when ‘all phases of the density matrix elements is equal to zero’ is the only state information at hand. Table 2 presents the values of the quantum coherence that we obtained by calculating the usual interference pattern visibility, the average of the oscillation curve maximum points and by using the expression proposed in [12]. In these calculations, we considered the uncertainty of each data as the square root of the data itself (as in a Poissonian distribution), and propagate them according to error propagation rules. We observe an agreement between the
obtained experimental values and point that the experimental error is larger in the method suggested in [12].

We can measure quantum coherence from the usual interference pattern visibility for all states which produce interference patterns with only one oscillation frequency, independently of the dimension. For the cases where null coefficients phases is the only information available about the state, the average of the peaks of the oscillation curve can also be employed, which can be verified by the agreement between the values in table 2.

5. An alternative for a non-null phases

Now we present a general method that one can use to determine the quantum coherence of spatial qudits prepared with multiple slits with oscillation curves as the ones shown in figures 2(e) and (f). This method is valid for any spatial qudit state, including that for which the method of [12] is not applicable. It also enables us to obtain the absolute value of each off-diagonal element of the density matrix of any spatial qudit state.

In equation (3), each absolute value of the upper off-diagonal elements of \( \rho \) \((\rho_{ij}, i < j)\) controls the amplitudes of one of the cosine terms. We relate each cosine term to an impossibility to distinguish photons coming from one of the two apertures of a possible slit pair belonging to a multiple-slit. We can say that the interference in equation (3) results from the sum of interference patterns generated from pairs of apertures, or in other words, equation (3) is a combination of different double-slit interference patterns. Therefore, by thinking in quantum coherence, in equation (4), as the sum of 'coherences' of the state pairs belonging to the state basis, it is intuitive to determine the total coherence by summing these 'partial coherences'.

Over again, we take qutrit states as a practical example. Considering the state in equation (6), if we use a procedure to prevent photons from passing through aperture ‘0’1’, for example, the state after the multiple slits and available for detection at \( z_d \) is

\[
\rho_{12} = \frac{1}{\rho_{11} + \rho_{00}} \begin{pmatrix} \rho_{11} & e^{i\varphi_{10}} e^{-i\varphi_{12}} & 0 \\ e^{-i\varphi_{10}} e^{i\varphi_{12}} & \rho_{00} & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The \( \rho_{12} \) probability distribution for detecting photons in the far-field plane is

\[
P_{10}(x,z_d) \propto \sin^2 \left( \frac{k_{ax}}{\lambda} \right) \left[ 1 + V_{10} \cos \left( \gamma x - \varphi_{10} \right) \right],
\]

where \( V_{10} = 2 |\rho_{10}| / (\rho_{11} + \rho_{00}) \).

Equation (11) is equivalent to a density operator expected for a two-dimensional spatial state, whose quantum coherence is determined by the visibility \( V_{10} \), i.e. the oscillation contrast presented in equation (12) [1]. Therefore, we can determine the quantum coherence \( \mathcal{C} \) of the state in equation (6) by summing the visibilities of interference patterns of all the possible aperture pairs that constitute the multiple-slit. For a three-dimensional state, as in equation (6), we have three aperture pairs and

\[
\mathcal{C} = \frac{1}{2} \frac{V_{10} (\rho_{11} + \rho_{00} + \rho_{10}) + V_{11} (\rho_{11} + \rho_{01} + \rho_{10}) + V_{01} (\rho_{00} + \rho_{11})}{\rho_{11} + \rho_{00} + \rho_{10}},
\]

where the diagonal coefficients \( \rho_{11}, \rho_{00} \) and \( \rho_{10} \) can be determined, by performing a triple slit image measurement in coincidence with detector \( D_2 \), as explained above. \( V_{ij} \) is the interference pattern visibility resulted from the interference between the apertures \( i \) and \( j \) recorded by detector \( D_j \) in coincidence with detector \( D_2 \).

For a qudit state with dimension \( D \), it is necessary to measure \( N \) interference patterns, being

\[
N = \frac{D!}{2(D - 2)!}.
\]

This method provides us an accurate result for states for which equation (8) [12] is not useful to determine qudits quantum coherence. For states with density matrix off-diagonal elements with null phases, or for those that produce interference patterns with only one oscillation frequency, both the methods are applicable and give us the same result. In this case, the method presented in [12] requests fewer measures for dimensions \( D \geq 4 \). The results that one can obtain with both methods coincide for states with null phases or with a unique frequency interference pattern. For the other states, our alternative method is the only one useful for the experimental determination of quantum coherence of qudits i.e. when the null phases assumption proposed in [12] is not valid.

Although \( N \) increases quickly with the dimension of the state, it does not require the presence of any additional equipment to correct the coefficients phases, such as a spatial light modulator (SLM) for example. From the measurements of pair-to-pair interference patterns, we can also determine the absolute value of each off-diagonal coefficient of \( \rho \). Besides that, the displacement between two of these interference patterns gives us the relative phase between two matrix elements. The displacement between the maximum points of \( P_{10}(x,z_d) \) and \( P_{10}(x,z_d) \), for example, allow us to obtain \( |\varphi_{10} - \varphi_{10}| \).
Concerning the examples that we explored, equation (13) also allows us to evaluate the maximum expected quantum coherence, for the prepared qutrit quantum states. States whose density matrices have the symmetries of equation (9) do not have quantum coherence equal to one. The oscillation amplitudes measured experimentally reflect the quantum coherence of our prepared states. Notice that for the state \( \rho_I, V_{-1,1} = 0 \) ideally, so that quantum coherence for this state is

\[
C_I = \frac{1}{2} V_{-10} (\rho_{-1-1} + \rho_{00}) + V_{01} (\rho_{0+1} + \rho_{11}).
\]

By state normalization \( \rho_{-1-1} + \rho_{00} + \rho_{11} = 1 \), such that \( C_I = V_{-10} (\rho_{-1-1} + \rho_{00}) + V_{01} (\rho_{0+1} + \rho_{11}) \). Even with maximal coherence between the modes \([-1]\) and \([0]\), and between \([1]\) and \([0]\), i.e. even for \( V_{-10} = V_{01} = 1 \) the theoretical quantum coherence \( C_I \) is smaller than 1, because \( (\rho_{-1-1} + \rho_{00}) < 1 \) and \( (\rho_{0+1} + \rho_{11}) < 1 \). As discussed in [5, 10], some complementary equations show that the distinguishability between the possible states of a system decreases the state quantum coherence. In other words, for any dimension, the quantum coherence upper bound, of a state with some symmetry, is reached when all the density matrix diagonal elements are equal, as we can conclude from equation (3) of [29]. Considering a state with the symmetry of \( \rho_I \) and with all diagonal elements equal to 1/3, we have \( C_I = \frac{V_{-10} + V_{01}}{3} \). Therefore, the quantum coherence upper bound, in this case, is \( 2/3 \approx 0.67 \), and the experimental values shown in the tables I and II for \( \rho_I \) is slightly smaller than the upper bound. Similarly, for the state \( \rho_{II}, V_{10} = V_{01} = 0 \) and \( C_{II} = V_{11} (\rho_{-1-1} + \rho_{00}) \). If all diagonal elements are equal to 1/3, \( C_{II} = \frac{V_{11}}{3} \) and the upper bound for quantum coherence is \( 1/3 \approx 0.33 \). We observe that the measured quantum coherence of the state \( \rho_{II} \) is near the upper bound within the experimental error range. This happens because there is an influence of matrix density elements that are not absolutely null in the experimental prepared state. Therefore, the results presented in tables I and II are in accordance with the expected upper bound for the quantum coherence of the states that we have prepared.

In this article, we focused on studying the quantum coherence measurement of spatial photonic states prepared with multiple slits, so that the method presented in this section is general because it is valid for qudits of any dimension prepared with multiple slits. However, it could be used for photonic qudits in other degrees of freedom. For example, for photonic qutrits prepared in angular momentum modes, a spatial light modulator (SLM) can be programmed to filter out (or block) one of the angular momentum modes in order to make possible the measurement of the analogous remaining angular momenta mode pair interference [28].

6. Conclusion

In this article, we studied the experimental quantum coherence measurement of spatial qudits. A photon path qutrit state, having a quantum coherence associated with it, can be prepared with photons transmitted through multiple slits with \( D \) apertures. We determined the spatial qutrit quantum coherence experimentally, employing a method based on the theoretical proposal made in [12]. We discussed the method validity and presented an alternative that is always valid for qudits states, without assumptions about the phases of the density matrix elements. We also analyzed the multiple-slit experiment from the perspective of an interferometer containing path-markers. We showed that quantum coherence measurement must be realized, in an optical configuration, such that the interference patterns are detected with path-markers states maximally indistinguishable.

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Appendix A. Detailed calculation for near- and far-field measurement analyses

Consider the density operator describing a general state of a quantum system, shown above in equation (1), and its quantum coherence, defined in equation (4) [9, 23]. In the general treatment of quantum coherence measurement, an ensemble of particles, initially in the state described in equation (1), interacts with a path-marker inside the setup. Each possible path-marker state marks a particle in one specific state of the system base state. The resulting joint state after the interaction is [5–7, 23]

\[
\rho_T = \sum_{l,m=-\lambda}^{\lambda} \rho_{lm} |l\rangle \langle m| \otimes |d_l\rangle \langle d_m|,
\]

where \( \{|d_l\}\) are the possible path-marker states. By tracing out \( \rho_T \) over the path-marker states, we obtain the available state to the particles register, since we detect them without measuring the path-marker state directly. Thus, the probability distribution of detecting a particle at the interferometer output ports involves the matrix elements of the reduced state operator

\[
\rho_2 = \sum_{l,m=-\lambda}^{\lambda} \rho_{lm} |l\rangle \langle m| |d_m\rangle \langle d_l|,
\]

which depends on the internal products between the possible path-marker states.

As a consequence, the observation of quantum coherence depends on the path-marker states and can be equal to or less than the associated with the initial state \( \rho \). If the states belonging to \( \{|d_l\}\) are entirely distinguishable, that is, orthogonal,
we observe a null quantum coherence even if the initial state has coherence. In this case, we do not have access to the off-diagonal elements of \( \rho \). Therefore, experimental determination of quantum coherence requires path-markers, or measurement apparatus, which the states do not preclude the observation of the modulus of off-diagonal elements of \( \rho \) or the direct quantum coherence value.

As mentioned above, concerning the spatial photonic states, a common method to prepare them consists of selecting photons within specific intervals of linear momentum \([14, 16–18]\) (figure 1(a)). Figures 1(b) and (c) show two possible optical configurations of detecting the photonic spatial probability distribution after the multiple slits, at the longitudinal position \( z_2 \) and along the transverse \( x \)-direction. In this scenario, the probability of detecting one photon on a detection plane is given by equation (2). For the state preparations, we use a multiple-slits set with thin apertures with \( 2a \) width separated by \( d \), so we can consider only the detection of photons that crossed the multiple-slits set passing through some aperture \( l \). Each slit \( l \) defines a state vector of the base \( \{ |l \rangle \} \) and \( |l \rangle \) represents a photon that crossed the slit \( l \). Therefore, the operator \( \hat{a}(q) \) acts only on photons in some mode described by \( |l \rangle \), which can be written in terms of the Fock state \( |1q \rangle \) in the transverse momentum variable \( q \): \( |l \rangle = \sqrt{\frac{1}{2\pi}} \int dq e^{iqld \text{sinc}(qa)} |1q \rangle \) [16, 20]. By rewriting \( \Gamma = E^{(-)}(x,z_2)E^{(+)}(x,z_2) \) in the \( \{ |l \rangle \} \) base, we obtain \([15, 16, 19, 21]\)

\[
\Gamma = \lambda \sum_{m=-\lambda}^{\lambda} \sum_{l=-\lambda}^{\lambda} \left( \langle m | E^{(-)}(x,z_2)E^{(+)}(x,z_2) | l \rangle \right) |m \rangle |l \rangle. \tag{A3}
\]

We can write the probability shown in equation (2) in the form

\[
P = \sum_{l,m=-\lambda}^{\lambda} \langle m | E^{(-)}(x,z_2)E^{(+)}(x,z_2) | l \rangle \rho_{lm}, \tag{A4}
\]

which is a sum of different terms. Notice that it is possible to write a state operator \( \rho_{lm} \) in such a way that its matrix elements are the different terms in the sum of equation (A4)

\[
\rho_{lm} = \sum_{l,m=-\lambda}^{\lambda} \rho_{lm} |l \rangle \langle m | E^{(-)}(x,z_2)E^{(+)}(x,z_2) | l \rangle. \tag{A5}
\]

We can also see \( \rho_{lm} \) as the reduced density operator obtained from the joint state

\[
\rho_T = \sum_{l,m=-\lambda}^{\lambda} \rho_{lm} |l \rangle \otimes E^{(+)}(x,z_2) | l \rangle \otimes | m | E^{(-)}(x,z_2), \tag{A6}
\]

after we apply the trace operation. We can observe a notable similarity between equations (A6) and (A1), and equations (A5) and (A2), what points to the validity of Englert’s discussion [5] in context of transverse spatial qudits.

For a measurement setup composed by apertures-lens-detector, as in figures 1(b) and (c) \([34]\)

\[
E^{(+)}(x,z_d) | l \rangle \propto e^{iax^2} \int \int e^{iqa(x,z_d)\text{sinc}(qa)} dq \times
\]

\[
e^{ia(l-x)^2}(\frac{l-x}{l+x})^{i} \frac{1}{2} \leq l \leq x \geq 0 \leq x \leq 0 \leq l \leq l \leq l, \tag{A7}
\]

where the functions are integrated in all domain; \( |0 \rangle \) indicates no photons in any mode of the set \( \{ |l \rangle \} \) and is resulting from the operation of the annihilation operator \( \hat{a}(l) \) over \( |l \rangle \); \( d \) and \( a \) are, respectively, the separation between two adjacent identical apertures and their width; \( f \) is the lens focal length, \( x_1 \) and \( x_2 \) are, respectively, the transverse positions in the apertures and lens planes; \( \alpha_{l(2)} = 2k/d_{l(2)} \); \( d_1 \) and \( d_2 \) are the distances apertures-lens and lens-detector, respectively.

If (1/d_1 + 1/d_2 – 1/f) = 0 the lens projects the aper- tures image on the detector plane. In the special case where \((z_2 - z_0) / 2 = (z_d - z_0) / 2 \) (figure 1(b))

\[
\sum_{l=-\lambda}^{\lambda} E^{(+)}(x,z_2) | l \rangle \propto \sum_{l=-\lambda}^{\lambda} \prod_{m=-\lambda}^{\lambda} \left( \frac{x+ld}{2a} \right) |0 \rangle, \tag{A8}
\]

where \( \prod_{m=-\lambda}^{\lambda} \left( \frac{x+ld}{2a} \right) \) is the function that describes the rectangular aperture centered in \( x = ld \), with width \( 2a \) \([14]\). The probability distribution to detect a photon at \( x \) transverse position (figure 1(b)) is

\[
P(x,z_D) \propto \sum_{l,m=-\lambda}^{\lambda} \prod_{m=-\lambda}^{\lambda} \left( \frac{x+ld}{2a} \right) \prod_{m=-\lambda}^{\lambda} \left( \frac{x+md}{2a} \right) \rho_{lm}. \tag{A9}
\]

By construction, rectangular functions in equation (A8) are orthogonal to each other within the range \( 2a \) and consequently, \( P(x,z_2) \neq 0 \) only if \( l = m \). In this case, we can identify which aperture each counted photon comes from and determine the initial state diagonal elements \( \{ \rho_{ll} \} \). On the other hand, if \( d_2 = f \), the lens projects the apertures optical Fourier transform on the detector plane. In the special case where \((z_2 - z_0) = (z_d - z_0) = f \) (figure 1(c)) \([15, 27]\),

\[
\sum_{l} E^{(+)}(x,z_d) | l \rangle \propto \sum_{l} e^{iax^2} |0 \rangle, \tag{A10}
\]

setting the elements of \( \{ E^{(+)}(x,z_d) | l \} \) in a non-orthogonal form. In this case, it is not possible to distinguish the photons paths and \( P(x,z_D) \) is given by equation (3).

Note after proof

After we submitted this article, we became aware of a similar proposition of Tabish Qureshi to measure coherence using interference from pairs of slits in a multi-slit experiment for arbitrary phases in the density matrix elements \([35]\).
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