REPRESENTABILITY OF MATROIDS BY $c$-ARRANGEMENTS IS UNDECIDABLE

BY

LUKAS KÜHNE∗

Einstein Institute of Mathematics, The Hebrew University of Jerusalem
Givat Ram, Jerusalem, 91904, Israel
and
Max Planck Institute for Mathematics in the Sciences
Inselstr. 22, 04103, Leipzig, Germany
e-mail: lukas.kuhne@mis.mpg.de

AND

GEVA YASHFE**

Einstein Institute of Mathematics, The Hebrew University of Jerusalem
Givat Ram, Jerusalem, 91904, Israel
e-mail: geva.yashfe@mail.huji.ac.il

ABSTRACT

For a natural number $c$, a $c$-arrangement is an arrangement of dimension $c$ subspaces satisfying the following condition: the sum of any subset of the subspaces has dimension a multiple of $c$. Matroids arising as normalized rank functions of $c$-arrangements are also known as multilinear matroids. We prove that it is algorithmically undecidable whether there exists a $c$ such that a given matroid has a $c$-arrangement representation, or equivalently whether the matroid is multilinear. It follows that certain problems on network coding and secret sharing schemes are also undecidable. In the proof, we encode group presentations in frame matroids of rank three which we call generalized Dowling geometries: the construction is inspired by Dowling geometries of finite groups and by the von Staudt construction. The idea is to construct a reduction from the uniform word problem for finite groups to multilinear representability of matroids. The $c$-arrangement condition gives rise to some difficulties and their resolution is the main part of the paper.

∗ L.K. was supported by a Minerva fellowship of the Max-Planck-Society, the Studienstiftung des deutschen Volkes and by ERC StG 716424 - CASe.

** G.Y. was supported by ISF grant 1050/16.

Received June 9, 2020 and in revised form May 6, 2021
1. Introduction

1.1. c-ARRANGEMENT REPRESENTATIONS. The main objects discussed in this article are matroids and their generalizations, polymatroids.

Definition 1.1: A polymatroid is a pair of a ground set $E$ together with a rank function $r : \mathcal{P}(E) \to \mathbb{R}_{\geq 0}$ which is

(i) monotone: $r(S) \leq r(T)$ for each $S \subseteq T \subseteq E$ and
(ii) submodular: $r(S) + r(T) \geq r(S \cup T) + r(S \cap T)$ for each $S, T \subseteq E$.

The pair $(E, r)$ is called a matroid if $r$ only takes integer values and additionally $r(S) \leq |S|$ holds for any subset $S \subseteq E$.

It is a classical problem to study matroid representations by vector configurations or equivalently hyperplane arrangements over some field. For an overview see [Oxl11, Chapter 6]. Goresky and MacPherson extended this notion by introducing $c$-arrangements in the context of stratified Morse theory [GM88]. For a fixed integer $c \geq 1$, these are arrangements of dimension $c$ subspaces of a vector space such that the dimension of each sum of these subspaces has dimension a multiple of $c$.

This condition ensures that the associated rank function, normalized by multiplication with $\frac{1}{c}$, is the rank function of a matroid. A matroid arising in this way is said to be representable as a $c$-arrangement.

This is a strict generalization of the usual notion of matroid representations, which are just 1-arrangements. For instance, Goresky and MacPherson showed that the non-Pappus matroid, which is not representable over any field, is representable as a 2-arrangement over $\mathbb{C}$.

Matroids arising as normalized rank functions of $c$-arrangement representations are also called multilinear matroids.

Fix a field $\mathbb{F}$. The following is the multilinear representability problem over $\mathbb{F}$:

Problem 1.2: Given a matroid $M$, does there exist a $c \geq 1$ such that $M$ is representable as a $c$-arrangement over $\mathbb{F}$?

This question was posed by Björner where he states “the question of $c$-representability of matroids is open, but probably hopeless.” [Bjö94, p. 333].

The main contribution of this article is a computability theoretic result for $c$-arrangement representations.

---

1 Goresky and Macpherson worked in the dual setting, where dimension is replaced by codimension and subspace sums are replaced by intersections.
Theorem 1.3: The multilinear representability problem is undecidable. This is true for any field $\mathbb{F}$. Moreover, the problem remains undecidable if the field remains unspecified, or is allowed to be taken from some given set.

In this sense, Björner was correct; the representability question is indeed “hopeless”.

Our proof works by constructing a reduction from the uniform word problem for finite groups, an undecidable problem in group theory, to the multilinear representability problem. We describe this problem in Section 3.4.

1.2. Related work. Multilinear matroids found applications to network coding capacity: in [ANLY00], Ahlswede et al. introduced a model for network information flow problems. When the coding functions are constrained to be linear, these problems are related to polymatroid representability. In [ESG10], El Rouayheb et al. constructed linear network capacity problems equivalent to multilinear matroid representability (cf. [DFZ07] for a related construction). In their language, Theorem 1.3 together with Proposition 18 in [ESG10] implies that the question whether an instance of the network coding problem has a linear vector coding solution is undecidable.

One can determine whether a given matroid is representable as a $1$-arrangement over an algebraically closed field via a Gröbner basis computation (see [Oxl11, p. 227]). In fact, the same method can be adapted to decide $c$-arrangement representability for a fixed $c \geq 1$. In his Ph.D. thesis, Mnëv proved a universality theorem for realization spaces of oriented matroids, see [Mnë88] for an exposition. Subsequently, Sturmfels observed that $1$-arrangement representability over the rational numbers $\mathbb{Q}$ is equivalent to Hilbert’s Tenth Problem for $\mathbb{Q}$, which asks whether a single multivariate polynomial equation over the integers has a solution in $\mathbb{Q}$ [Stu87]. It is not known whether this is decidable.

An important class of matroids are Dowling geometries which are defined from finite groups [Dow73]. In [BBEPT14], Beimel et al. characterized when a Dowling geometry is representable as a $c$-arrangement in terms of fixed point free representations of its underlying group. Our work is related to Dowling geometries and the above characterization: we generalize Dowling’s construction and construct matroids which encode finitely-presented groups. The issues we then have to deal with are directly related to the existence of fixed points in matrix representations of these groups.

Multilinear matroids are contained in two more general classes of matroids.
One of these is the class of matroids representable over a skew partial field, where a $c$-arrangement representation over a field $\mathbb{F}$ is equivalent to a representation over the skew partial field $(M_c(\mathbb{F}), GL_c(\mathbb{F}))$ [PvZ13]. An example of a matroid that is representable over a skew partial field (in fact a skew field) but is not multilinear is given in [KPY20].

The other class of matroids is the class of entropic matroids [Fuj78], or matroids representable by partitions [Mat99]. This is the class of matroids with rank function equal to a real multiple of the joint entropy function of a collection of discrete random variables; it has some applications to problems about communication (see also the references to work on network coding, above). It equals the class of secret sharing matroids [BD91], which describes the possible ideal perfect secret sharing schemes. Simonis and Ashikhmin observed that every multilinear matroid is a secret sharing matroid, and posed the still open problem whether every secret sharing matroid is multilinear [SA98].

### 1.3. Structure of the paper

This paper is organized as follows. In Section 2 we give a high-level outline of the proof. Section 3 gives definitions and basic properties which are used throughout the article. Generalized Dowling geometries and their relation to the uniform word problem are explained in Section 4. In Section 5 to Section 8 we develop several technical tools as explained in the outline. We put these tools together to prove Theorem 1.3 in Section 9.

### Acknowledgments

We would like to thank Karim Adiprasito for his mentorship and for introducing us to the topic of $c$-arrangements. We are grateful to Rudi Pendavingh for helpful conversations on von Staudt constructions. Our application of them is inspired by joint work with him in [KPY20]. We would also like to thank Eran Nevo, and other participants of the 2020-21 mathematical writing workshop at the Hebrew University, for helpful comments. Lastly, we thank the anonymous referee for carefully reading an earlier version of this article and for giving many suggestions which significantly improved the paper.

An extended abstract of this paper appeared as “Undecidability of $c$-Arrangement Matroid Representations” in a proceedings volume of Séminaire Lotharingien de Combinatoire, among extended abstracts from the 2020 FP-SAC conference [KY20].
2. Outline of the proof

We reduce the uniform word problem for finite groups (or UWPFG for short; see Section 3.4) to multilinear representability for matroids. This is done as follows. Given a group presentation $\langle S \mid R \rangle$ and a word $w$ in the generators, we construct a finite collection of matroids $\mathcal{M}$. This $\mathcal{M}$ is computable, and satisfies that the UWPFG instance corresponding to $\langle S \mid R \rangle$ and $w$ has a positive answer if and only if at least one $M \in \mathcal{M}$ is multilinear.

2.1. Weak $c$-arrangements. We begin with a simpler reduction, from the UWPFG to a problem about objects we call “weak $c$-arrangements”. These are described in Section 3 (see also the definitions of normalized rank functions). Essentially, a weak $c$-arrangement representation of a matroid $M$ in a vector space $V$ is an assignment of a subspace of $V$ to each element of the ground set of $M$. This assignment has to satisfy certain conditions on the dimensions of subspaces, which are weaker than the conditions for a $c$-arrangement.

The reduction is performed by constructing, from $\langle S \mid R \rangle$ and $w$, a matroid $N_{S,R}$ that we call a generalized Dowling geometry. This is a frame matroid of rank 3; such matroids are referred to as triangle matroids in the rest of the paper. It has a distinguished basis $\{b^{(1)}, b^{(2)}, b^{(3)}\}$ such that all other elements are on one of the lines $\{b^{(i)}, b^{(j)}\}$. Figure 1 shows a geometric depiction of a triangle matroid. In our construction, the points on each side of the triangle correspond with the elements of the generating set $S$ and their inverses. They are marked by an upper index to distinguish between different sides (both in Figure 1 and elsewhere). As in the construction of Dowling geometries, the circuits of the matroid describe multiplicative relations between the generators of the group.

The matroid $N_{S,R}$ can be thought of as encoding a group presentation equivalent to $\langle S \mid R \rangle$. Its representations as a weak $c$-arrangement correspond bijectively to $c$-dimensional representations of the corresponding group. Solving the given UWPFG instance is equivalent to deciding whether there is a weak $c$-arrangement representation of $N_{S,R}$ in which the linear transformation corresponding to $w$ (via the bijection to group representations) is not the identity. This is shown in Section 4.

Weak $c$-arrangements are usually not $c$-arrangements. The main part of the paper describes a construction to bridge this gap.
Let us briefly describe the issue we need to overcome. The bijection we construct between \( c \)-dimensional representations of \( G = \langle S \mid R \rangle \) and weak \( c \)-arrangement representations of the matroid \( N_{S,R} \) looks more or less like this:

From a representation \( \rho : G \rightarrow \text{GL}(V) \), we construct a large block matrix, with columns indexed by elements of \( N_{S,R} \). Each column is composed of three \( c \times c \) blocks. If \( s_1, s_2 \in S \) are two generators, \( N_{S,R} \) will contain elements \( b^{(1)}, b^{(2)}, s_1^{(1)}, s_2^{(1)} \), and the corresponding columns will be

\[
\begin{bmatrix}
I \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
I \\
0
\end{bmatrix},
\begin{bmatrix}
-I \\
\rho(s_1) \\
0
\end{bmatrix},
\begin{bmatrix}
-I \\
\rho(s_2) \\
0
\end{bmatrix},
\]

respectively. The corresponding subspaces in the weak \( c \)-representation are the column spans of these. For these blocks to be part of an honest \( c \)-arrangement representation, it is necessary that

\[
\text{rank} \begin{bmatrix}
-I & -I \\
\rho(s_1) & \rho(s_2) \\
0 & 0
\end{bmatrix} = 2c.
\]

Subtracting one block column from the other, we see that this requires that \( \rho(s_1) - \rho(s_2) \in \text{GL}(V) \) is invertible. This is a very restrictive condition,

Figure 1. A geometric depiction of a triangle matroid. Each side contains copies of the generators \( x, y, z, w \). The triples \( \{x^{(1)}, y^{(2)}, z^{(3)}\} \) and \( \{y^{(1)}, x^{(2)}, w^{(3)}\} \) form two circuits of the matroid.
and it cannot always be satisfied. It can happen that the rank of the above matrix is strictly less than $2c$. We depict this situation in Figure 2. In the figure, the blobs labeled $s_1^{(1)}, s_2^{(1)}$ depict the column spans of the corresponding block columns. Each such column span has dimension $c$. They intersect nontrivially, and thus the dimension of their sum (which is the rank of $\begin{bmatrix} -I & -I \\ \rho(s_1) & \rho(s_2) \end{bmatrix}$) is less than $2c$.

![Figure 2. Part of a weak $c$-arrangement representation of a triangle matroid. Each circle depicts a subspace of dimension $c$. The subspaces of $s_1^{(1)}, s_2^{(1)}$ are in the subspace spanned by $b^{(1)}$ and $b^{(2)}$.](image)

2.2. CONSTRUCTION OF $\mathcal{M}$. The correspondence between $c$-dimensional group representations and weak $c$-arrangements of corresponding matroids is not sufficient for the reason described above. We modify this correspondence to a more suitable one in several steps.

2.2.1. Inflation. Consider the example described above: in some given triangle matroid, there are elements $s_1^{(1)}, s_2^{(1)}$ on the line spanned by $b^{(1)}, b^{(2)}$. Suppose this matroid has a weak $c$-arrangement representation in which the subspaces corresponding to $s_1^{(1)}$ and $s_2^{(1)}$ are not equal, but intersect nontrivially. In this case, these subspaces are each of dimension $c$, but their sum has dimension strictly less than $2c$, and hence this weak $c$-arrangement representation is not an honest $c$-arrangement. We wish to fix this defect. The idea is to enlarge the subspaces in such a way that:

(a) the dimension of the subspace corresponding to each of $s_1^{(1)}, s_2^{(1)}$ becomes $3c$,

(b) the dimension of their intersection becomes $c$, and

(c) the sum of these two subspaces contains the subspace spanned by $b^{(1)}, b^{(2)}$.

---

2 The finite groups $G$ for there exists a representation $\rho$ (in characteristic coprime to $|G|$) for which $\rho(g_1) - \rho(g_2)$ is invertible for all distinct $g_1, g_2$ are precisely the Frobenius complements. Such representations are called fixed-point free [Fei67, Theorem 25.5].
We call this operation an algebraic inflation, or just inflation for short. Formally we do this in two steps, ensuring conditions (b) and (c) separately. This works as follows. Let $W_1$ and $W_2$ be the subspaces corresponding to $s_1^{(1)}$ and $s_2^{(1)}$, and let $B_1$ and $B_2$ be the subspaces corresponding to $b^{(1)}$ and $b^{(2)}$ respectively. Let $U_1$ and $U_2$ be $c$-dimensional subspaces of the ambient vector space such that:

- $\dim(U_1 \cap U_2) = c - \dim(W_1 \cap W_2)$, and
- $U_1 + U_2$ intersects the arrangement trivially.

In Figure 3a, $U_1$ and $U_2$ are the two subspaces above the line. We replace $W_1$ by $W_1 + U_1$ and $W_2$ by $W_2 + U_2$. This is the first step, which ensures that condition (b) holds. For the second step, let $U'$ be a generic $\dim(W_1 \cap W_2)$-dimensional subspace of $B_1 + B_2$. Let $U'_1$ and $U'_2$ be $c$-dimensional subspaces of the ambient vector space such that:

- $\dim(U'_1 \cap U'_2) = 0$, and
- $U'_1 + U'_2$ intersects $B_1 + B_2$ at $U'$.

We replace $W_1$ by $W_1 + U'_1$ and $W_2$ by $W_2 + U'_2$. This is the second step, after which condition (c) also holds. In Figure 3b, $U'$ is depicted by the small gray blob to the right of $b^{(1)}$, $W_1$ and $W_2$ are the large ovals, and $U'_1, U'_2$ are the circles above $W_1$ and $W_2$ respectively. A detailed algebraic description is given in Section 5.
Inflating takes us away from weak $c$-representations of matroids into a more general setting of subspace arrangements, because it is no longer true that the dimension of each subspace is $c$. However, the dimension of the sum of any subset of the four subspaces under consideration becomes a multiple of $c$. Hence this subspace arrangement is closer to being a $c$-arrangement in the sense most important to us.

Given a weak $c$-arrangement, we perform a sequence of inflation operations on various subsets of the subspaces which guarantees that the resulting subspace arrangement has the following property: the dimension of the sum of any subset of the subspaces is a multiple of $c$. We call such an arrangement $c$-admissible.

This sequence of operations is chosen in such a way that the resulting normalized rank function can be computed directly from the combinatorics of the original matroid, in a manner which does not depend on the specific $c$. This is done by applying a sequence of combinatorial operations to the matroid which mirror the sequence of algebraic inflations. We call these operations combinatorial inflations. They are defined in Section 6. Section 7 proves that appropriate sequences of combinatorial inflations compute the same rank function as the corresponding sequences of algebraic inflations, up to the normalization factor $\frac{1}{c}$.

2.2.2. From polymatroids to matroids again: separation and $c$-bases. Inflation operations resolve one issue but create another. They ensure that the sum of every subset of the subspaces has dimension a multiple of $c$. However, the normalized rank function of our inflated arrangements (or equivalently the polymatroid we compute by inflating the given matroid) depends entirely on the matroid, and not on a given weak $c$-arrangement representation at all. This means that the $c$-representability of the polymatroid we obtain by inflating $N_{S,R}$ tells us nothing about the corresponding group: we can always represent this polymatroid by applying inflations to the weak 1-representation corresponding to the trivial representation of the group.

We now want to modify the inflated polymatroid to achieve two goals. The first goal is to make sure that any $c$-representation of the new object corresponds to a weak $c$-arrangement representation of $N_{S,R}$ satisfying that the corresponding group representation $\rho$ has $\rho(w) \neq \rho(e)$ (here $\langle S \mid R \rangle$ and the word $w$ in the generators $S$ are the data of a UWPFG instance). On the level of subspace arrangements, this condition on $\rho$ is equivalent to asking that two subspaces
in the arrangement are distinct. The second goal is to obtain a matroid instead of a polymatroid. The ideas are described further in the beginnings of the subsections of Section 8.

3. Preliminaries

In this section we collect definitions and basic properties which will be used throughout the article. We also need some elementary matroid theory, but we will not cover it here. We recommend the beginning of Oxley’s book [Oxl11].

3.1. Subspace arrangements.

Definition 3.1: Let $V$ be a vector space over a field $\mathbb{F}$ and let $E$ be a finite set. A subspace arrangement $\mathcal{A}$ in $V$ is an indexed set $\{A_e\}_{e \in E}$ where each $A_e$ is a subspace of $V$. If $\mathcal{A} = \{A_e\}_{e \in E}$ is a subspace arrangement indexed by $E$ and $X \subseteq E$, we denote

$$A_X := \sum_{e \in X} A_e.$$ 

Further, if $c \geq 1$ is an integer:

(a) We call $\mathcal{A}$ $c$-homogeneous if $\dim A_e = c$ for all $e \in E$.

(b) We call $\mathcal{A}$ $c$-admissible if for any subset $X \subseteq \mathcal{A}$ the dimension of $A_X$ is a multiple of $c$.

(c) A $c$-arrangement is a subspace arrangement which is both $c$-homogeneous and $c$-admissible.

The notion of a $c$-admissible subspace arrangement is not standard, but will be useful in Section 8.

Remark 3.2: Note that $c$-arrangements are often defined as subspace arrangements such that all intersections are of codimension a multiple of $c$. We prefer to work in the dual framework, with subspaces of dimension $c$ and with sums instead of intersections. This duality is the same as that between matroid representations and hyperplane arrangements: it takes a subspace of a vector space $V$ to its annihilator in the dual $V^*$. Taking a sum of subspaces is dual to taking their intersection.
Definition 3.3: For a subspace arrangement $\mathcal{A} = \{A_e\}_{e \in E}$ over the field $\mathbb{F}$ we define two rank functions on the power set $\mathcal{P}(E)$. Fix a subset $X \subseteq E$.

(a) The usual rank function $r_{\mathcal{A}} : \mathcal{P}(E) \to \mathbb{N}$ is defined by
$$r_{\mathcal{A}}(X) := \dim(A_X).$$

(b) For any $c \geq 1$ we define the normalized rank function $r^c_{\mathcal{A}} : \mathcal{P}(E) \to \mathbb{Q}$ by setting
$$r^c_{\mathcal{A}}(X) := \frac{1}{c} \dim(A_X).$$

Remark 3.4: A $c$-homogeneous subspace arrangement $\mathcal{A}$ is a $c$-arrangement if and only if its normalized rank function $r^c_{\mathcal{A}}$ takes only integral values.

Now we can define when a matroid is representable by a $c$-arrangement. In our proofs we will additionally need a weaker notion of representations, due to the issues arising from subspaces having nontrivial intersections as discussed in Section 2.

Definition 3.5: Fix a matroid $M$ on the ground set $E$ with rank function $r$.

(a) The matroid $M = (E, r)$ is called multilinear of order $c$ over a field $\mathbb{F}$ if there exists a $c$-arrangement $\mathcal{A} = \{A_e\}_{e \in E}$ such that the normalized rank function of $\mathcal{A}$ equals $r$, i.e. $r(X) = r^c_{\mathcal{A}}(X)$ for all $X \subseteq E$. We say that the $c$-arrangement $\mathcal{A}$ represents the matroid $M$ in that case.

(b) To define a weaker representability notion, we fix a basis $B$ of the matroid $M$. We say that a $c$-homogeneous subspace arrangement $\mathcal{A} = \{A_e\}_{e \in E}$ is a weak $c$-arrangement representation of $M$ with respect to the basis $B$ (or a weak $c$-representation of $M$, for short) if $r(X) \geq r^c_{\mathcal{A}}(X)$ for all subsets $X \subseteq E$ and $r(Y) = r^c_{\mathcal{A}}(Y)$ for all subsets $Y \subseteq E$ with $|Y \setminus B| \leq 1$.

Our proofs often use the following dimension formula without explicitly mentioning it: If $U_1, U_2$ are finite dimensional subspaces of a vector space $V$, then
$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

3.2. Linear-algebraic calculations. The following three lemmas are collected here to avoid cluttering later sections. The first is a basic tool in Section 4. The other two will be used to bound the intersections of certain subspaces in Section 8.2.
Lemma 3.6: Let $\mathbb{F}$ be a field and $A, B, C \in M_k(\mathbb{F})$ any $k \times k$ matrices.

(a) The block matrix $\begin{bmatrix} -I_k & -I_k \\ I_k & B \\ 0 & 0 \end{bmatrix}$ has rank $k + \text{rk}(B - A)$.

(b) The block matrix $\begin{bmatrix} -I_k & 0 & C \\ A & -I_k & 0 \\ 0 & B & -I_k \end{bmatrix}$ has rank $2k + \text{rk}(BAC - I_k)$.

Proof. In the case of (a), we multiply the matrix from the right with the invertible block matrix $\begin{bmatrix} I_k & -I_k \\ 0 & I_k \end{bmatrix}$ which preserves its rank. Thus, we obtain the block matrix

$\begin{bmatrix} -I_k & 0 \\ A & B - A \\ 0 & 0 \end{bmatrix}$

which immediately implies the claim on its rank.

Analogously for the case (b), we multiply the matrix from the right with the invertible block matrix

$\begin{bmatrix} I_k & 0 & C \\ 0 & I_k & AC \\ 0 & 0 & I_k \end{bmatrix}$

which preserves its rank. Hence, we obtain the block matrix

$\begin{bmatrix} -I_k & 0 & 0 \\ A & -I_k & 0 \\ 0 & B & BAC - I_k \end{bmatrix}$

which finishes the proof.

Lemma 3.7: Let $\sigma \in S_k$ be a derangement (a permutation with no fixed points) and let $P_\sigma$ be the corresponding $k \times k$ permutation matrix over some field $\mathbb{F}$. Then $\text{rk}(P_\sigma - I_k) \geq \frac{k}{2}$.

Proof. Consider the graph $G_\sigma$ with vertices $\{1, \ldots, k\}$ having an edge $\{i, j\}$ if $\sigma(i) = j$. The matrix $P_\sigma - I_k$ is a representation matrix of the graphic matroid $M(G_\sigma)$ over $\mathbb{F}$ (see [Oxl11, Lemma 5.1.3]).

Suppose $\sigma$ has a decomposition into $r$ disjoint cycles. Then the graph $G_\sigma$ consists of $k$ edges arranged in $r$ disjoint cycles, or in other words, $M(G_\sigma)$ has $k$ elements and splits into a direct sum of $r$ circuits. Therefore we have

$\text{rk}(P_\sigma - I_k) = \text{rk}(M(G_\sigma)) = k - r$.

Since $\sigma$ is a derangement, each cycle has length at least two. The result now follows from the fact that $\sigma$ has at most $\frac{k}{2}$ cycles.
Corollary 3.8: Let $G$ be a finite group and let $\{P_g\}_{g \in G}$ be the permutation matrices of its regular representation (these are the permutation matrices of the action of $G$ on itself by left multiplication). Then for any distinct $g_1, g_2 \in G$:

$$\text{rk}(P_{g_1} - P_{g_2}) \geq \frac{|G|}{2}.$$ 

Proof. Note that $P_g$ is the permutation matrix of a derangement for any $g \in G$ other than $e$: otherwise, in the action of $G$ on itself by left multiplication, $g$ has a fixed point, say $h$, and $gh = h$; but this implies $g = e$.

For distinct $g_1, g_2 \in G$

$$\text{rk}(P_{g_1} - P_{g_2}) = \text{rk}((P_{g_1} - P_{g_2})P_{g_2}^{-1}),$$

since $P_{g_2}^{-1}$ is invertible. Thus Lemma 3.7 implies

$$\text{rk}(P_{g_1} - P_{g_2}) = \text{rk}(P_{g_1}g_{g_2}^{-1} - I_{|G|}) \geq \frac{|G|}{2}. \blacksquare$$

3.3. Group presentations. We collect basic definitions for group presentations and refer to the book by Lyndon and Schupp for details [LS77].

Definition 3.9: Let $\langle S \mid R \rangle$ be a finite presentation of a group. That is, $S$ is a finite list of symbols and $R$ is a finite set of words in $S$ and their inverses. Let $F_S$ be the free group on the elements of $S$, and let $N$ be the normal closure of the subgroup generated by $R$ in $F_S$. We define the group $G_{S,R} := F_S/N$. Two presentations $\langle S \mid R \rangle$ and $\langle S' \mid R' \rangle$ are equivalent if the groups $G_{S,R}$ and $G_{S',R'}$ are isomorphic.

To simplify the constructions below we will restrict ourselves to presentations with relations of length three. Using Tietze transformations, one can reduce to this situation from the general case. We state this in a lemma.

Lemma 3.10: For any finite presentation of a group $\langle S \mid R \rangle$, an equivalent finite presentation $\langle S' \mid R' \rangle$ can be found where $R'$ consists of relations of length three only.

Proof. Suppose the given presentation has a relation $s_{i_1}^{\epsilon_1}s_{i_2}^{\epsilon_2}\cdots s_{i_k}^{\epsilon_k} = e$ with $k > 3$, where $\epsilon_j = \pm 1$ for each $j$. Add a generator $\tilde{s}$ to $S$. Then replace the relation by the two relations

$$\tilde{s} = s_{i_k}^{\epsilon_k}s_{i_{k-1}}^{\epsilon_{k-1}},$$

$$s_{i_1}^{\epsilon_1}\cdots s_{i_{k-2}}^{\epsilon_{k-2}}\tilde{s} = e.$$
Substituting the first relation in the second one yields the original relation
\[ s_{i_1}^{c_1} s_{i_2}^{c_2} \cdots s_{i_k}^{c_k} = e. \]
Thus, we shortened the given relation, and the new presentation is equivalent to the given one. Continuing in this way eventually ensures that all relations left have length at most 3.

To avoid dealing with relations of length 1 or 2, add a generator \( t \) and the relation
\[ ttt^{-1} = e, \]
which implies \( t = e \). Now replace any relation of the form \( s_{i_1}^{c_1} = e \) with \( s_{i_1}^{c_1}tt = e \), and similarly replace any relation \( s_{i_1}^{c_1}s_{i_2}^{c_2} = e \) with \( s_{i_1}^{c_1}s_{i_2}^{c_2}t = e \).

### 3.4. The Uniform Word Problem for Finite Groups.

In the notation of Section 3.3, the uniform word problem for finite groups (UWPFG) is the following decision problem.

**Instance:** A finite presentation \( \langle S \mid R \rangle \) of a group and an element \( w \in G_{S,R} \).

**Problem:** Decide whether there exists a finite group \( G \) and a homomorphism \( \varphi : G_{S,R} \to G \) such that \( w \notin \ker(\varphi) \).

Our undecidability result relies on the following consequence of Slobodskoi’s work [Slo81].

**Theorem 3.11:** The uniform word problem for finite groups is undecidable.

Slobodskoi’s result is stronger: it shows that in fact the word problem for finite groups is undecidable for some specific \( \langle S \mid R \rangle \) (in the notation above, it is only the word \( w \) that is not fixed).

We will say the answer to an instance of the UWPFG is positive if there exist \( G \) and \( \varphi \) as above. In the rest of the paper, all instances of the UWPFG are assumed to satisfy that \( w \in S \) and that all relations in \( R \) have length 3.

A general instance of the problem can be transformed to one of this form as follows. First add \( w \) to \( S \), together with a relation of the form
\[ s_{i_1}^{c_1} \cdots s_{i_n}^{c_n} w^{-1} = e, \]
where \( s_{i_1}^{c_1} \cdots s_{i_n}^{c_n} \) is an expression for \( w \) in \( G_{S,R} \). Then apply the procedure of Lemma 3.10.
3.5. **Characteristic of the field.** Rado proved that a matroid which is linearly representable over a field $\mathbb{F}$ is also representable over a finite algebraic extension of the prime field of $\mathbb{F}$ (see [Rad57]). The proof uses Hilbert’s Nullstellensatz and directly generalizes to the situation of $c$-arrangement representations.

**Proposition 3.12:** Let $A$ be $c$-arrangement over a field of characteristic $p \in \mathbb{P} \cup \{0\}$ representing a matroid $M$. Then, $M$ has a $c'$-arrangement $A'$ representation over the prime field of characteristic $p$.

**Proof.** The discussion above shows that we can assume that $A$ is a $c$-arrangement in a vector space $V$ over a field $L$ which is a finite extension of degree $d$ over its prime field $\mathbb{F}$. Since $V$ is also a vector space over $\mathbb{F}$ and $\dim_{\mathbb{F}}(U) = d \dim_{L}(U)$ for any finite-dimensional subspace of $V$, the arrangement $A$ is naturally a $(c \cdot d)$-arrangement over the prime field $\mathbb{F}$. Their normalized rank functions agree. ■

When we work with $c$-arrangement representations we do not care about the specific $c$ involved. By the proposition, it is harmless to assume that our field is infinite. Thus the characteristic of the field is the only parameter important to us; but none of our constructions or proofs depend on the characteristic. We mention that the proofs also work for subspace representations over a specified set of characteristics.

3.6. **Genericity.** We will often choose generic subspaces of a given vector space. Essentially, a generic subspace is one which does not satisfy some given collection of Zariski-closed conditions.

Consider for example the following situation: we are given a vector space $V$ and a subspace $W$ of $V$. If $U$ is of dimension complementary to $W$ in $V$, we expect $W + U = V$; this is a generic condition on $U$, in the sense that it is satisfied on a dense Zariski-open subset of the subspaces of appropriate dimension.

The example is almost as general as we need here. When we say $U$ is chosen generically from some family $\mathcal{F}$ of subspaces of $V$, we mean the following: for each $W$ in some family $\mathcal{G}$ of subspaces of $V$,

$$\dim(U + W) = \max\{\dim(U' + W) \mid U' \in \mathcal{F}\}.$$
In this paper, the family $\mathcal{F}$ is always the family of $d$-dimensional subspaces of some fixed subspace of $V$. The family $\mathcal{G}$ is not specified explicitly, but there is always a suitable finite $\mathcal{G}$. Under these conditions, there is always a $U \in \mathcal{F}$ fulfilling the genericity condition as long as the field is infinite.

4. Generalized Dowling Geometries

In this section we reduce the uniform word problem for finite groups (UWPFG) to a problem on weak $c$-arrangements: given an instance of a UWPFG, we construct a matroid $M$ such that the given instance has a positive answer if and only if for some $c$, $M$ has a weak $c$-representation which satisfies an additional condition.

To do this we encode a finite group presentation in a matroid which we call a generalized Dowling geometry. These matroids all have a special form which simplifies many calculations, and which we describe first.

**Definition 4.1:** We call a matroid $M = (E, r)$ a **triangle matroid** if it is of rank three and there exists a basis

$$ B = \{b^{(1)}, b^{(2)}, b^{(3)}\} $$

such that all elements of $E$ are contained in the flats spanned by $\{b^{(1)}, b^{(2)}\}$, $\{b^{(1)}, b^{(3)}\}$ and $\{b^{(2)}, b^{(3)}\}$. We sometimes call such a basis a **distinguished basis** of $M$. To ease our notation we call

(a) the elements in $B$ the **vertices** of the triangle,

(b) the flats $\{b^{(1)}, b^{(2)}\}$, $\{b^{(1)}, b^{(3)}\}$ or $\{b^{(2)}, b^{(3)}\}$ the **sides** of the triangle, and

(c) the elements of $E$ **bottom, left, and right** elements if they are contained in the flats $\{b^{(1)}, b^{(2)}\}$, $\{b^{(1)}, b^{(3)}\}$, and $\{b^{(2)}, b^{(3)}\}$ respectively.

For a subset $S \subseteq E$ of the triangle matroid $M = (E, r)$, we define a subset $C_M(S)$ of $B$ as follows. If $S$ is either a vertex of the triangle, or of rank 2 and contained in a line of the triangle of $M$, or of rank 3, set $C_M(S)$ to be the unique subset of $B$ with closure equal to the closure of $S$. Otherwise, set $C_M(S) := \emptyset$.

Note that if $C_M(S) = \emptyset$, then there is no subset of $B$ with closure equal to the closure of $S$ in $M$. We will work with the notation $C_M(S)$ often, and usually arguments will be split into cases depending on whether or not $C_M(S) = \emptyset$. 
Remark 4.2: A triangle matroid is the same as a frame matroid of rank three as introduced by Zaslavsky [Zas94].

Figure 1 depicts a geometric representation of a triangle matroid. In the following we construct generalized Dowling geometries from group presentations.

Definition 4.3 (Generalized Dowling Geometry): Let $\langle S \mid R \rangle$ be a finite presentation of a group. By Lemma 3.10 we can assume that any relation in $R$ is of length three.

We construct a triangle matroid $N_{S,R}$ on the ground set $E_{S,R}$ with basis $B = \{b^{(1)}, b^{(2)}, b^{(3)}\}$ by describing its dependent flats $\mathcal{F}_{S,R}$ of rank 2, where we regard the indices cyclically modulo 3:

$$E_{S,R} := \{b^{(i)}, e^{(i)}, x^{(i)}, x^{-(i)} \mid 1 \leq i \leq 3 \text{ and } x \in S\},$$

$$\mathcal{F}_{S,R} := \left\{ \bigcup_{x \in S} \{x^{(i)}, x^{-(i)}\} \cup \{e^{(i)}, b^{(i)}, b^{(i+1)}\} \mid \text{for any fixed } 1 \leq i \leq 3 \right\}$$

$$\cup \{\{x^{(i)}, x^{-(j)}\}, e^{(k)}\} \mid \text{for } x \in S \text{ and pairwise different } 1 \leq i, j, k \leq 3 \}$$

$$\cup \{\{e^{(1)}, e^{(2)}, e^{(3)}\} \cup \{\{x^{(2)}, y^{(1)}, z^{(3)}\} \mid \text{for any } xyz \in R\}.\}$$

This defines a unique matroid of rank 3 with basis $B$, since $B$ is not contained in any of these subsets, and any two distinct such rank 2 flats intersect in at most one element (see [Oxl11, Proposition 1.5.6]).

To relate linear (group) representations of $\langle S \mid R \rangle$ to the matroid $N_{S,R}$, we investigate its weak $c$-representations with respect to the basis $B$ as defined in Definition 3.5 (b). To work with such representations, we will regard them as block matrices with $3 \times |E_{S,R}|$ blocks of size $c \times c$. The block columns are indexed by the elements of $E_{S,R}$, and the $c$ columns in each block column are a basis for the corresponding subspace in a weak representation.

Recall that the regular representation of a finite group $G$ with $n := |G|$ over a field $\mathbb{F}$ is the linear representation $\rho : G \mapsto GL_n(\mathbb{F})$ induced by the action of each element $g \in G$ on $G$ itself by left multiplication.

**Proposition 4.4:** Let $\langle S \mid R \rangle$ be a finite presentation of a group with relations of length three and let $G$ be a finite group with a homomorphism $\varphi : G_{S,R} \to G$. Set $n := |G|$ and fix any field $\mathbb{F}$. Let $\rho : G \to GL_n(\mathbb{F})$ be the regular representation of $G$. Then the $n$-homogeneous subspace arrangement $\mathcal{A}_{G,\varphi}$ over $\mathbb{F}$ given by the $3n \times |E_{S,R}|n$ block matrix $A_{G,\varphi}$ is a weak $n$-representation of the
matroid $N_{S,R}$ with respect to the basis $B$:

$$A_{G,\varphi} := \begin{bmatrix}
I_n & 0 & 0 & -I_n & \cdots & 0 & \cdots & \rho(\varphi(x)) & \cdots \\
0 & I_n & 0 & \rho(\varphi(x)) & \cdots & -I_n & \cdots & 0 & \cdots \\
0 & 0 & I_n & 0 & \cdots & \rho(\varphi(x)) & \cdots & -I_n & \cdots
\end{bmatrix}.$$

**Proof.** Each block column of $A_{G,\varphi}$ of size $3n \times n$ is indexed by an element of the matroid $N_{S,R}$. This defines a subspace arrangement $A = \{A_e\}_{e \in E_{S,R}}$ where the subspace $A_e$ is spanned by the column vectors of the block column indexed by $e$. So in particular we have

$$A_{x(1)} = \text{span}\begin{bmatrix}
-I_n \\
0 \\
\rho(\varphi(x))
\end{bmatrix}, \quad A_{x(2)} = \text{span}\begin{bmatrix}
0 \\
-I_n \\
\rho(\varphi(x))
\end{bmatrix} \quad \text{and} \quad A_{x(3)} = \text{span}\begin{bmatrix}
\rho(\varphi(x)) \\
0 \\
-I_n
\end{bmatrix},$$

for any $x \in S$ or $x^{-1} \in S$. The normalized rank $r_{AG,\varphi}$ of any subset of elements in $E_{S,R}$ containing only bottom, right or left elements is at most 2, since the submatrix of $A_{G,\varphi}$ consisting of columns indexed by such a subset has a vanishing block row.

Next consider any three elements $x^{(2)}, y^{(1)}, z^{(3)}$ of $E_{S,R}$ with $x, y, z$ elements of $S$, inverses of elements of $S$ or the identity element in $G_{S,R}$. By Lemma 3.6(b) we have

$$r^n_{A_{G,\varphi}}(\{x^{(2)}, y^{(1)}, z^{(3)}\}) = 2 + \frac{1}{n} \text{rk}(\rho(\varphi(x))\rho(\varphi(y))\rho(\varphi(z)) - I_n)$$

$$= 2 + \frac{1}{n} \text{rk}(\rho(\varphi(xyz - e))).$$

This implies that any circuit of $N_{S,R}$ of the form $\{x^{(2)}, y^{(1)}, z^{(3)}\}$ for a relation $xyz \in R$ corresponds to a subset of $A_{G,\varphi}$ of normalized rank 2 since we have $xyz = e$ in $G_{S,R}$ in this case. The same argument holds for circuits of the form $\{x^{(i)}, x^{-1}^{(j)}, e^{(k)}\}$ for any $x \in S$ and pairwise different indices $1 \leq i, j, k \leq 3$.

Clearly we have $r^n_{A_{G,\varphi}}(\{a\}) = 1$ for any element $a \in E_{S,R}$. Lastly, consider any subset of the form $\{b^{(i)}, x^{(j)}\}$ for indices $1 \leq i, j \leq 3$ and $x \in S$ or $x^{-1} \in S$. By symmetry we can without loss of generality assume $i = 1$ and $j = 1$. Hence,
this subset corresponds to the block matrix
\[
\begin{bmatrix}
I_n & -I_n \\
0 & \rho(\varphi(x)) \\
0 & 0
\end{bmatrix}
\]
which is of rank \(2n\) since the matrix \(\rho(\varphi(x))\) is by assumption invertible. Thus, we obtain \(r_{AG,\varphi}^n(\{b^{(1)}, x^{(1)}\}) = 2\). The cases of subsets containing two or three elements of the basis \(\{b^{(1)}, b^{(2)}, b^{(3)}\}\) and one additional element can be checked in the same way. This completes the proof that \(A_{G,\varphi}\) is a weak representation of the matroid \(N_{S,R}\) with respect to the basis \(\{b^{(1)}, b^{(2)}, b^{(3)}\}\).

Conversely, the next proposition shows how to obtain a group from a weak representation of \(N_{S,R}\).

**Proposition 4.5:** Again, let \(\langle S \mid R \rangle\) be a finite presentation of a group with relations of length three and consider the matroid \(N_{S,R}\). Fix some field \(\mathbb{F}\). Any weak \(c\)-representation \(A\) of \(N_{S,R}\) with respect to the basis \(\{b^{(1)}, b^{(2)}, b^{(3)}\}\) over \(\mathbb{F}\) yields a group \(G_A\) that is a finitely generated subgroup of \(GL_c(\mathbb{F})\) and a group homomorphism \(\varphi_A : G_{S,R} \to G_A\).

**Proof.** Let \(A = \{A_e\}_{e \in E_{S,R}}\) be a weak \(c\)-representation of the matroid \(N_{S,R}\) with respect to the basis \(\{b^{(1)}, b^{(2)}, b^{(3)}\}\) over the field \(\mathbb{F}\). The arrangements \(A\) yields a \(3c \times |E_{S,R}|c\) block matrix over \(\mathbb{F}\) where each \(3c \times c\) block column is indexed by an element of the matroid \(N_{S,R}\) and contains a basis of the corresponding subspace in \(A\).

We perform the following invertible operations which preserve the underlying combinatorial structure:

(a) After a change of coordinates of the ambient vector space \(\mathbb{F}^{3c}\), we can assume that the matroid basis \(\{b^{(1)}, b^{(2)}, b^{(3)}\}\) is represented by the block matrix
\[
\begin{bmatrix}
I_c & 0 & 0 \\
0 & I_c & 0 \\
0 & 0 & I_c
\end{bmatrix}
\]
This corresponds to multiplying the entire matrix by an invertible matrix of size \(3c \times 3c\) from the left.

(b) Consider \(x^{(1)}\) any bottom element of the matroid \(N_{S,R}\). Since
\[
r_{S,R}(\{b^{(1)}, x^{(1)}, b^{(2)}\}) = r_{S,R}(\{b^{(1)}, x^{(1)}\}) = r_{S,R}(\{x^{(1)}, b^{(2)}\}) = 2
\]
where $r_{S,R}$ is the rank function of the matroid $N_{S,R}$, the normalized rank $r^c_A$ of the corresponding sets of subspaces of $A$ must be 2 as well.

Hence, the block column of $x^{(1)}$ is of the form \[
\begin{bmatrix}
X'_1 \\ X''_1 \\
0
\end{bmatrix}
\] where $X', X''$ are invertible $c \times c$ matrices. Analogous arguments show that the block columns of any right and left elements of $N_{S,R}$ are of the form \[
\begin{bmatrix}
0 \\ X'_2 \\
X''_2
\end{bmatrix}
\] and \[
\begin{bmatrix}
X'_3 \\ 0 \\
X''_3
\end{bmatrix}
\] for suitable invertible $c \times c$ matrices $X'_1, X''_1, X'_2, X''_2, X'_3, X''_3$ respectively.

(c) By multiplying the block column $e^{(1)}$ by an invertible $c \times c$ matrix from the right and multiplying the second block row by an invertible $c \times c$ matrix from the left, we can assume that the block column of $e^{(1)}$ is \[
\begin{bmatrix}
-I_c \\ I_c \\
0
\end{bmatrix}.
\]

By applying the analogous operation to the block column $e^{(2)}$ and the third block row, we can assume that the block column of $e^{(2)}$ is \[
\begin{bmatrix}
0 \\ -I_c \\
I_c
\end{bmatrix}.
\]

Subsequently, we perform a multiplication from the right on the block column $e^{(3)}$ after which it is of the form \[
\begin{bmatrix}
E_3 \\ 0 \\
0
\end{bmatrix}
\] for some invertible $c \times c$ matrix $E_3$. Since $\{e^{(1)}, e^{(2)}, e^{(3)}\}$ is a circuit of $N_{S,R}$, Lemma 3.6 (b) implies that $E_3 - I_c = 0$ which implies $E_3 = I_c$.

(d) Lastly, by multiplying every block column again by a suitable invertible $c \times c$ matrix from the right we can assume that the block matrix defining $A$ is of the following form, where $T_{x^{(i)}}$ is an invertible matrix for each $x \in S$ and $1 \leq i \leq 3$:

\[
\begin{bmatrix}
I_c & 0 & 0 & -I_c & -I_c & \cdots & 0 & 0 & \cdots & I_c & T_{x^{(3)}} & \cdots \\
0 & I_c & 0 & I_c & T_{x^{(1)}} & \cdots & -I_c & -I_c & \cdots & 0 & 0 & \cdots \\
0 & 0 & I_c & 0 & 0 & \cdots & I_c & T_{x^{(2)}} & \cdots & -I_c & -I_c & \cdots
\end{bmatrix}
\]

For each relation $xyz \in R$ the elements $\{x^{(2)}, y^{(1)}, z^{(3)}\}$ form a circuit of the matroid $N_{S,R}$. Since the arrangement $A$ is a weak $c$-representation of $N_{S,R}$ the normalized rank of the corresponding subspaces is at most 2. This implies that the block matrix

\[
\begin{bmatrix}
0 & -I_c & T_{z^{(3)}} \\
-I_c & T_{y^{(1)}} & 0 \\
T_{x^{(2)}} & 0 & -I_c
\end{bmatrix}
\]

is of rank at most $2c$. Thus, Lemma 3.6 (b) implies

\[
T_{x^{(2)}}T_{y^{(1)}}T_{z^{(3)}} = I_c.
\]
Next let \( x \in S \) and consider the elements \( x^{(1)}, x^{(2)}, x^{(3)}, x^{-1^{(1)}}, x^{-1^{(2)}}, x^{-1^{(3)}} \): Applying Lemma 3.6 (b) to circuits of the form \( \{x^{(i)}, x^{-1^{(j)}}, e^{(k)}\} \) (where \( 1 \leq i, j, k \leq 3 \) are distinct) shows \( T_{x^{(i)}} = T_{x^{-1^{(j)}}} \) whenever \( i \neq j \). A second application of the same lemma implies \( T_{x^{(i)}} = T_{x^{(j)}} \) and \( T_{x^{-1^{(i)}}} = T_{x^{-1^{(j)}}} \) for all \( i \neq j \).

Now let \( G_A \) be the group generated by the matrices \( T_{x^{(1)}} \) for all \( x \in S \). This is a finitely generated subgroup of \( \text{GL}_c(\mathbb{F}) \). We define a group homomorphism \( \varphi_A : G_{S,R} \to G_A \) on the generating set \( S \) by setting \( \varphi_A(x) := T_{x^{(1)}} \) for each \( x \in S \). Equation (1) implies that this respects the relations \( R \) of \( G_{S,R} \) and thus defines a group homomorphism.

In the following theorem we establish the connection between the UWPFG and weak \( c \)-representations. Lemma 3.10 allows us to assume without loss of generality that the relations are of length three and the word \( w \) is an element of \( S \).

**Theorem 4.6**: Let \( \mathbb{F} \) be a field. Consider a UWPFG instance given by a finite presentation \( \langle S \mid R \rangle \) of a group and an element \( w \in S \). Then the answer to this instance is positive, i.e., there exists a finite group \( G \) with a homomorphism \( \varphi : G_{S,R} \to G \) such that \( \varphi(w) \neq e_G \), if and only if there exists a positive integer \( c \) and a weak \( c \)-representation \( \mathcal{A} = \{A_e\}_{e \in E_{S,R}} \) over \( \mathbb{F} \) of the matroid \( N_{S,R} \) with respect to the basis \( \{b^{(1)}, b^{(2)}, b^{(3)}\} \) such that

\[
(2) \quad r^c_\mathcal{A}(\{w^{(1)}, e^{(1)}\}) > 1.
\]

**Proof.** First, assume that there exists a finite group \( G \) with a homomorphism \( \varphi : G_{S,R} \to G \) such that \( \varphi(w) \neq e_G \). Set \( n := |G| \). Proposition 4.4 constructs a weak \( c \)-representation \( \mathcal{A}_{G,\varphi} = \{A_e\}_{e \in E_{S,R}} \) of \( N_{S,R} \) over \( \mathbb{F} \) from the regular representation \( \rho : G \to \text{GL}_n(\mathbb{F}) \). To compute the rank of \( \{w^{(1)}, e^{(1)}\} \) in this subspace arrangement, we can apply Lemma 3.6 (a):

\[
r^c_{\mathcal{A}_{G,\varphi}}(\{w^{(1)}, e^{(1)}\}) = \frac{1}{n} \text{rk} \left[ \begin{array}{cc} -I_n & -I_n \\ \rho(\varphi(w)) & \rho(\varphi(e)) \end{array} \right] \\
= 1 + \frac{1}{n} \text{rk}(\rho(\varphi(w)) - I_n).
\]

The assumption \( \varphi(w) \neq e_G \) implies \( \rho(\varphi(w)) \neq I_n \). Therefore Equation (2) holds.
Conversely, assume that there exists a weak \( c \)-representation \( \mathcal{A} = \{ A_e \}_{e \in E_{S,R}} \) over the field \( \mathbb{F} \) of the matroid \( N_{S,R} \) with respect to the basis \( \{ b^{(1)}, b^{(2)}, b^{(3)} \} \) such that Equation (2) holds. Proposition 4.5 shows that there exists a group \( G_{\mathcal{A}} \) that is a finitely generated subgroup of \( GL_c(\mathbb{F}) \) with a group homomorphism \( \varphi_{\mathcal{A}} : G_{S,R} \to G_{\mathcal{A}} \). By construction of \( G_{\mathcal{A}} \) we can compute using again Lemma 3.6(a)

\[
r^c_{\mathcal{A}}(\{ w^{(1)}, e^{(1)} \}) = \frac{1}{c} \rk \begin{bmatrix} -I_c & -I_c \\ \varphi_{\mathcal{A}}(w) - I_c \\ \varphi_{\mathcal{A}}(e) - I_c \end{bmatrix}
= 1 + \frac{1}{c} \rk(\varphi_{\mathcal{A}}(w) - I_c).
\]

Thus, Equation (2) implies \( \varphi_{\mathcal{A}}(w) \neq I_c \).

By Malcev’s theorem the group \( G_{\mathcal{A}} \) is residually finite\(^3\) since it is a finitely generated matrix group [Mal40]. Therefore, there exists a finite group \( H \) with a group homomorphism \( \varphi_H : G_{\mathcal{A}} \to H \) such that \( \varphi_H(\varphi_{\mathcal{A}}(w)) \neq e_H \), where \( e_H \) is the identity element in \( H \). Hence the given UWPFG instance has a positive answer. \( \blacksquare \)

5. Algebraic inflation

We develop an algebraic inflation procedure to produce a \( c \)-admissible arrangement from a weak \( c \)-representation of a triangle matroid, as outlined in Section 2. Inflation is an operation consisting of two steps. Both steps are instances of an elementary inflation procedure which we describe first.

5.1. Elementary inflation. Let \( \mathcal{U} = \{ U_e \}_{e \in E} \) be a subspace arrangement in a vector space \( V \), \( c \in \mathbb{N} \) and \( S \subseteq E \) a subset.

Intuitively, the idea is to pick a subspace \( W \) of \( V \), and then to extend each subspace \( U_e \) by a generic \( c \)-dimensional subspace of \( W \).

Formally, embed \( V \) together with the arrangement \( \mathcal{U} \) into a vector space \( \tilde{V} \). Given a subspace \( W \subseteq \tilde{V} \) of dimension at least \( c \), we construct a new subspace arrangement \( \tilde{\mathcal{U}} \) as follows:

(a) Choose \( |S| \)-many generic subspaces \( W_1, \ldots, W_{|S|} \) of \( W \), each of dimension \( c \).

---

\(^3\) A group \( G \) is **residually finite** if for any element \( e \neq g \in G \) there exists a group homomorphism \( \varphi : G \to H \) to a finite group \( H \) such that \( g \notin \ker \varphi \).
(b) Denote \( S = \{s_1, \ldots, s_{|S|}\} \). The new subspace arrangement \( \tilde{U} \) lives in \( \tilde{V} \) and consists of the subspaces \( \tilde{U}_{s_i} := U_{s_i} + W_i \) for \( i = 1, \ldots, |S| \) together with \( \tilde{U}_e := U_e \) for all \( e \in E \setminus S \).

**Remark 5.1:** We sometimes need the subspace \( W \) to intersect any sum of subspaces from \( U \) trivially. This is why it is necessary to work with a vector space \( \tilde{V} \) which is larger than \( V \). It is clear that it always suffices to take \( \tilde{V} \) of dimension \( \dim(V) + \dim(W) \).

Up to an automorphism of \( \tilde{V} \) fixing \( V \), a subspace \( W \subseteq \tilde{V} \) is determined by its dimension together with its intersection with \( V \). This will suffice for our uses of this construction, and we will give this data instead of explicitly constructing \( \tilde{V} \) and \( W \) in what follows.

**Definition 5.2:** The arrangement resulting from an application of the elementary inflation construction above to the arrangement \( U \), the subset \( S \subseteq E \), and a subspace \( W \) of dimension \( d \) with \( W' := W \cap V \) will be denoted by \( \mathcal{EI}_c(\mathcal{U}, S, d, W') \).

The subspace \( W \), when needed explicitly, will be denoted by \( W_c(\mathcal{U}, S, d, W') \).

The next lemma describes the effect of an elementary inflation on the rank function. To simplify the proof we make additional assumptions on \( d \) and \( W' \). These are satisfied in all our applications of the elementary inflation below.

**Lemma 5.3:** Let \( U = \{U_e\}_{e \in E} \) be a subspace arrangement in \( V \), \( S \subseteq E \) a subset, and \( W' \subseteq V \) a subspace. Fix \( c, d \in \mathbb{N} \) such that \( 0 \leq d - c(|S| - 1) \leq c \) and \( \dim(W') \leq d - c(|S| - 1) \).

Define \( U' := \mathcal{EI}_c(U, S, d, W') \) and denote \( U' = \{U'_e\}_{e \in E} \). Then for any \( T \subseteq E \)

\[
\dim(U'_T) - \dim(U_T) = \begin{cases} 
    c|S \cap T|, & S \not\subseteq T, \\
    d - \dim(U_T \cap W'), & S \subseteq T.
\end{cases}
\]

**Proof.** Denote the subspace used in the elementary inflation by

\[
W := W_c(U, S, d, W'),
\]

and denote \( W_{S \cap T} := \sum_{t \in T \cap S} W_t \) where \( W_t \) are the generic subspaces of \( W \) as defined in the construction of the elementary inflation. Then we compute

\[
\dim(U'_T) - \dim(U_T) = \dim(U_T + W_{S \cap T}) - \dim(U_T)
\]

\[
= \dim(W_{S \cap T}) - \dim(U_T \cap W_{S \cap T})
\]

\[
= \min\{d, c|S \cap T|\} - \dim(U_T \cap W_{S \cap T}),
\]

(3)
where the last equality holds since the subspaces $W_t$ are chosen generically in the subspace $W$ of dimension $d$. Now, we distinguish two cases:

**Case 1: $S \not\subseteq T$.** By the assumption on $W'$ we have
\[
\dim(U_T \cap W) \leq \dim(W') \leq d - c(|S| - 1).
\]
Thus, we have $\dim(U_T \cap W) \leq d - c(|S| - 1) \leq d - c|S \cap T|$. Hence, the genericity of $W_t$ implies $\dim(U_T \cap W_{S \cap T}) = 0$ and we obtain by Equation (3) $\dim(U'_T) - \dim(U_T) = c|S \cap T|$. 

**Case 2: $S \subseteq T$.** In this case, $W_{S \cap T} = W$ which implies that
\[
U_T \cap W_{S \cap T} = U_T \cap W'.
\]
Therefore, Equation (3) implies
\[
\dim(U'_T) - \dim(U_T) = d - \dim(U_T \cap W'). \quad \Box
\]

### 5.2. Extensions and Full Arrangements

The goal of the inflation construction is to extend a given weak $c$-representation $U = \{U_e\}_{e \in E}$ of a matroid $M$ with distinguished basis $B$ in such a way that after sufficiently many applications of the procedure, the resulting arrangement $U' = \{U'_e\}_{e \in E}$ has the following properties (precise formulations are in Section 7):

(a) The original weak representation $U$ can be reconstructed from $U'$ by “taking intersections with the basis”:
\[
U_e = U'_e \cap U'_B \quad \text{for all } e \in E,
\]
(b) the arrangement $U'$ is $c$-admissible, and
(c) the rank function $r_{U'}^c$ can be computed from $M$ by a procedure that does not depend on the weak representation $U$.

Showing that this can be arranged requires some bookkeeping. We first define a class of subspace arrangements containing those that arise from the iterated inflations we will apply.

**Definition 5.4:** Let $U = \{U_e\}_{e \in E}$ be a subspace arrangement in a vector space $V$ and let $M = (E, r)$ be a triangle matroid with a distinguished basis $B$. We call $U$ an extension of a weak $c$-representation of $M$ with respect to $B$, or for short an extension of $M$, if $\{U_e \cap U_B\}_{e \in E}$ is a weak $c$-representation of $M$ with respect to $B$ and we have for every $T \subseteq E$ and $D \subseteq B$:
\[
\dim(U_T \cap U_D) \leq c(r(T) + r(D) - r(T \cup D)).
\]
Similarly, if a weak $c$-representation $A$ of $M$ is given, an extension of $A$ is a subspace arrangement $U$ which is an extension of $M$ and which satisfies

$$A = \{ U_e \cap U_B \}_{e \in E}. $$

Observe that we can take $T = D$ in the equation above and obtain

$$\dim(U_D) = c \cdot r(D)$$

for any $D \subseteq B$.

Further, suppose $U = \{ U_e \}_{e \in E}$ extends a weak $c$-representation of $M$ in such a way that the goals above are achieved. Then for each $S \subseteq E$, the dimension of $U_S \cap U_B$ cannot depend on the original weak $c$-representation. By (4) we must have $\dim(U_S \cap U_B) \leq c \cdot r(S)$, and it is reasonable to demand an equality.

**Definition 5.5:** Let $M = (E, r)$ be a triangle matroid with a distinguished basis $B$ and $U = \{ U_e \}_{e \in E}$ an extension of a weak $c$-representation of $M$ with respect to $B$. For a subset $S \subseteq E$ we define the **defect** of $S$ to be

$$\text{def}_U(S) := c \cdot r(S) - \dim(U_S \cap U_B).$$

We call a subset $S \subseteq E$ **full with respect to the basis** $B$, or just **full** for short, if

$$\text{def}_U(S) = 0.$$  

The following lemmas are crucial to our inflation procedure. First we provide a bound for the defect of a subset.

**Lemma 5.6:** Let $M = (E, r)$ be a triangle matroid with distinguished basis $B$ and $U = \{ U_e \}_{e \in E}$ be an extension of a weak $c$-representation of $M$ with respect to $B$. Let $S \subseteq E \setminus B$ such that every $S' \subsetneq S$ is full. Then we have

$$0 \leq \text{def}_U(S) \leq c.$$  

**Proof.** The fact that $U$ is an extension of $M$ implies $\dim(U_S \cap U_B) \leq c \cdot r(S)$ which implies $0 \leq \text{def}_U(S)$. Now, choose a subset $S' \subsetneq S$ such that

$$r(S') = r(S) - 1.$$  

The assumption that $S'$ is full implies

$$c \cdot (r(S) - 1) = c \cdot r(S') = \dim(U_{S'} \cap U_B) \leq \dim(U_S \cap U_B).$$

Rearranging the terms yields $\text{def}_U(S) \leq c.$  

\[\square\]
Next we relate the defect with the subset of the basis $C_M(S)$. Recall that if $M = (E,r)$ is a triangle matroid with distinguished basis $B$, for each $S \subseteq E$ a set $C_M(S) \subseteq B$ is defined such that $r(S) = r(S \cup C_M(S)) = r(C_M(S))$ if such a subset exists, and $C_M(S) = \emptyset$ otherwise. Since the closures of distinct subsets of a basis are distinct, this defines $C_M(S)$ uniquely.

**Lemma 5.7:** Let $M = (E,r)$ be a triangle matroid with distinguished basis $B$ and $U = \{U_e\}_{e \in E}$ be an extension of a weak $c$-representation of $M$ with respect to $B$. Let $S \subseteq E \setminus B$. If $C_M(S) = \emptyset$ then $S$ is full in $U$.

**Proof.** Set $A := \{A_e := U_e \cap U_B\}_{e \in E}$ which is a weak $c$-arrangement by assumption. Consider what happens for each value of $r(S)$:

**Case 1:** $r(S) = 1$. In this case, $c \cdot r(S) = \dim(U_S \cap U_B)$ since $A$ is a weak $c$-arrangement. Hence $S$ is full in $U$.

**Case 2:** $r(S) = 2$. If $S$ is contained in a side of the triangle, say $C \subseteq B$, then $C = C_M(S) \neq \emptyset$ and the assumption of the lemma does not hold.

Suppose $S$ is not contained in a side of the triangle. In this case $C_M(S) = \emptyset$. Take $s_1, s_2 \in S$ which do not lie on the same side of the triangle and denote $\tilde{S} = \{s_1, s_2\}$. Let $C \subseteq B$ be the side of the triangle on which $s_1$ lies. Then $\dim A_C = 2c$. Since $A$ is a weak $c$-representation of $M$, the fact that $s_2$ does not lie on the side $C$ implies $\dim A_{C \cup \{s_2\}} = 3c$. Therefore

$$A_{C \cup \{s_2\}} = A_C \oplus A_{s_2}.$$  

Using the inclusion $A_{s_1} \subseteq A_C$, we obtain $\dim A_{\tilde{S}} = 2c$. Finally, since $U$ is an extension of $A$ and $U_{\tilde{S}} \subseteq U_S$, we have

$$2c \geq \dim(U_S \cap U_B) \geq \dim((U_{s_1} \cap U_B) + (U_{s_2} \cap U_B))$$

$$= \dim A_{\tilde{S}} = 2c,$$

which shows $S$ is full in $U$.

**Case 3:** $r(S) = 3$. In this case $C_M(S) = B \neq \emptyset$: the closure of $S$ in $M$ is equal to the closure of the entire basis $B$ since the matroid $M$ has rank three. Thus, the lemma holds in each case. $\blacksquare$

**Remark 5.8:** In the notation of the lemma, it follows that if $x, y \in E$ are not contained in any line of the basis $B$ then $\{x, y\}$ is a full subset of $U$, since in such a situation $r(\{x, y\}) = 2$. 

Lemma 5.9: Let \( M = (E, r) \) be a triangle matroid with distinguished basis \( B \) and \( \mathcal{U} = \{U_e\}_{e \in E} \) be an extension of a weak \( c \)-representation of \( M \) with respect to \( B \). Let \( S \subseteq E \) and let \( C_M(S) \subseteq B \) be the subset of the basis defined in Definition 4.1. If \( C_M(S) \neq \emptyset \) we have

\[
U_S \cap U_B \subseteq U_{C_M(S)}.
\]

In particular, \( U_S \cap U_B = U_S \cap U_{C_M(S)} \).

Proof. Since \( \mathcal{U} \) is an extension of \( M \) we have

\[
dim(U_{S \cup C_M(S)} \cap U_B) \leq c \cdot r(S \cup C_M(S)) = c \cdot r(C_M(S)) = dim(U_{C_M(S)}).
\]

The inequality implies \( U_{C_M(S)} = U_{S \cup C_M(S)} \cap U_B \) because \( U_{C_M(S)} \) is contained in the intersection: it is contained in \( U_{S \cup C_M(S)} \) and in \( U_B \) by definition. Therefore

\[
U_{C_M(S)} = U_{S \cup C_M(S)} \cap U_B \supseteq U_S \cap U_B
\]

as required. The claim \( U_S \cap U_B = U_S \cap U_{C_M(S)} \) follows trivially. \( \blacksquare \)

5.3. An inflation step. Let \( M = (E, r) \) be a triangle matroid and \( \mathcal{U} = \{U_e\}_{e \in E} \) be an extension of a weak \( c \)-representation of \( M \) with respect to \( B \). We now describe the inflation procedure. Given a subset \( S \) of \( E \setminus B \) such that each proper subset of \( S \) is full in \( \mathcal{U} \), this procedure yields a subspace arrangement \( \mathcal{I}(\mathcal{U}, S) \) which extends the same weak \( c \)-representation of \( M \), and in which \( S \) is also full. The procedure is split into two steps.

Step 1: We first perform an elementary inflation to inflate the subset \( S \). We call this step \textit{S-inflation}. The case in which \( S \) consists of two elements, both lying on the same side of the triangle of \( M \), is depicted in Figure 3a.

We elementarily inflate by setting

\[
\mathcal{U}^1 := \mathcal{E}\mathcal{I}_c(\mathcal{U}, S, c(|S| - 1) + \text{def}_\mathcal{U}(S), 0).
\]

At the end of this step, we have added a \( c \)-dimensional subspace to each of the subspaces in \( \{U_s\}_{s \in S} \). If \( m < |S| \), any \( m \) of these dimension-\( c \) subspaces spans a subspace of total dimension \( c \cdot m \). However, taken all together they span a subspace of dimension

\[
c(|S| - 1) + \text{def}_\mathcal{U}(S),
\]

which is in general less than \( c|S| \).
STEP 2: As second step we perform an elementary inflation, which we call $B$-inflation, to ensure that $U^1_S$ intersects $U^1_B$ in a subspace of the correct dimension. Again, the case of $S$ equal to two points lying on the same side of the triangle of $M$ is depicted in Figure 3b.

Consider the subset $C_M(S) \subseteq B$. If $C_M(S) = \emptyset$, we define $W' = 0$ (recall that Lemma 5.7 implies $\text{def}_U(S) = 0$ in this case). If $C_M(S) \neq \emptyset$, we take $W'$ to be a generic $\text{def}_U(S)$-dimensional subspace of $U^1_{C_M(S)}$.

Then we perform an elementary inflation by setting

$$U^2 := E\mathcal{I}_c(U^1, S, c|S|, W').$$

At the end of this step we have added disjoint $c$-dimensional subspaces to each subspace in $\{U_s\}_{s \in S}$ such that $S$ is a full subset in $U^2$. This will be proved in Corollary 5.11. We set

$$\mathcal{I}(U, S) := U^2.$$

The next result describes the difference of the rank functions after both inflation steps using the results of the previous two subsections.

**Theorem 5.10:** Let $U$ be an extension of a weak $c$-representation of a triangle matroid $M = (E, r)$ with respect to a distinguished basis $B$. Let $S \subseteq E \setminus B$ and assume that every subset $S' \subsetneq S$ is full. Let $U' = \mathcal{I}(U, S)$ be the inflation.

Then if $T \subseteq E$ is any subset disjoint from $S$ and $Z \subseteq S$, we have:

$$r^c_{U'}(T \cup Z) = \begin{cases} r^c_{U}(T \cup Z) + 2|Z|, & Z \subsetneq S, \\ r^c_{U}(T \cup S \cup C_M(S)) + 2|S| - 1, & Z = S. \end{cases}$$

**Proof.** Using the notation introduced in the definition of the inflation step, we set $U^1 := E\mathcal{I}_c(U, S, c(|S| - 1) + \text{def}_U(S), 0)$ and $U^2 := E\mathcal{I}_c(U^1, S, c|S|, W')$ which means

$$U^2 = U' = \mathcal{I}(U, S).$$

Lemma 5.6 yields

$$0 \leq \text{def}_U(S) \leq c.$$

This implies that both elementary inflations satisfy the assumptions of Lemma 5.3 which we will use repeatedly in the following proof.
Suppose first that $Z \subseteq S$. Lemma 5.3 yields
\[
\dim(U^2_Z) - \dim(U_Z) = (\dim(U^2_Z) - \dim(U^1_Z)) + (\dim(U^1_Z) - \dim(U_Z)) = 2c|Z|.
\]
Similarly, we obtain $\dim(U^2_T + U^2_Z) - \dim(U_T + U_Z) = 2c|Z|$. Thus, the fact $U^2_T = U_T$ implies $\dim(U^2_T \cap U^2_Z) = \dim(U_T \cap U_Z)$. Therefore, we can compute
\[
r_{U^2_T}(T \cup Z) = \dim(U^2_T) + \dim(U^2_Z) - \dim(U^2_T \cap U^2_Z) = \dim(U_T) + \dim(U_Z) + 2c|Z| - \dim(U_T \cap U_Z)
\]
so by definition $r_{U^2_T}(T \cup Z) = r_{U^2}(T \cup Z) + 2|Z|$ as required.

Now suppose $Z = S$. We first show the following claim:

**Claim 1:** $U^2_S \supseteq U^2_{CM(S)}$.

**Proof of Claim 1.** If $CM(S) = \emptyset$ this claim is trivial so assume $CM(S) \neq \emptyset$. Then we have by the construction of the elementary inflation $U^2_S \supseteq U_S + W'$ where $W'$ is the subspace of $U_{CM(S)}$ chosen in the elementary inflation. Therefore,
\[
U^2_S \cap U^2_{CM(S)} \supseteq (U_S + W') \cap U_{CM(S)} = (U_S \cap U_{CM(S)}) + W',
\]
where the sum distributes since $W' \subseteq U_{CM(S)}$. Consider the sum $(U_S \cap U_{CM(S)}) + W'$. By Lemma 5.9, $U_S \cap U_B = U_S \cap U_{CM(S)}$. Thus,
\[
\dim(U_S \cap U_{CM(S)}) = \dim(U_S \cap U_B) = c \cdot r(S) - \text{def}(U(S)) = \dim(U_{CM(S)}) - \text{def}(U(S)).
\]
Since $W'$ is a $\text{def}(U(S))$-dimensional generic subspace of $U_{CM(S)}$ we obtain
\[
\dim((U_S \cap U_{CM(S)}) + W') = c \cdot r(CM(S)).
\]
The summands are each contained in $U_{CM(S)}$ and the dimension of the sum is $\dim(U_{CM(S)})$. We obtain
\[
U_{CM(S)} = (U_S \cap U_{CM(S)}) + W'.
\]
Thus Equation (5) yields $U^2_S \supseteq U^2_{CM(S)}$ in $U^2$. ■
Using Claim 1, we may write

\[ r_{U^2}^c(T \cup Z) = r_{U^2}^c((T \cup C_M(S)) \cup (S \cup C_M(S))). \]

Therefore, we obtain

\[ r_{U^2}(T \cup Z) = \dim(U_T^2 + U_{C_M(S)}^2) + \dim(U_S^2 + U_{C_M(S)}^2) \]
\[ - \dim((U_T^2 + U_{C_M(S)}^2) \cap (U_S^2 + U_{C_M(S)}^2)). \]

Using Lemma 5.3 and the definition of the elementary inflation, we can compute

\[ \dim(U_S^2 + U_{C_M(S)}^2) = \dim(U_S^1 + U_{C_M(S)}^1) + c|S| - \text{def}_U(S) \]
\[ = \dim(U_S + U_{C_M(S)}) + c(|S| - 1) \]
\[ + \text{def}_U(S) + c|S| - \text{def}_U(S) \]
\[ = \dim(U_S + U_{C_M(S)}) + c(2|S| - 1). \]

By construction, we have \( U_T^2 + U_{C_M(S)}^2 = U_T + U_{C_M(S)}. \) Thus, we can compute using Equation (7) and its analogous form for \( U_T^2 + U_{C_M(S)}^2 + U_S^2 + U_{C_M(S)}^2: \)

\[ \dim((U_T^2 + U_{C_M(S)}^2) \cap (U_S^2 + U_{C_M(S)}^2)) \]
\[ = \dim(U_T^2 + U_{C_M(S)}^2) + \dim(U_S^2 + U_{C_M(S)}^2) \]
\[ - \dim(U_T^2 + U_{C_M(S)}^2 + U_S^2 + U_{C_M(S)}^2) \]
\[ = \dim(U_T + U_{C_M(S)}) + \dim(U_S + U_{C_M(S)}) + c(2|S| - 1) \]
\[ - \dim(U_T + U_{C_M(S)} + U_S + U_{C_M(S)}) - c(2|S| - 1) \]
\[ = \dim(U_T + U_{C_M(S)}) + \dim(U_S + U_{C_M(S)}) \]
\[ - \dim(U_T + U_S + U_{C_M(S)}). \]

This fact, combined with Equations (6) and (7) implies

\[ r_{U^2}(T \cup Z) = \dim(U_T + U_{C_M(S)}) + \dim(U_S + U_{C_M(S)}) + c(2|S| - 1) \]
\[ - (\dim(U_T + U_{C_M(S)}) + \dim(U_S + U_{C_M(S)}) \]
\[ - \dim(U_T + U_S + U_{C_M(S)})) \]
\[ = \dim(U_T + U_S + U_{C_M(S)}) + c(2|S| - 1). \]

This implies \( r_{U^2}^c(T \cup Z) = r_{U^c}(T \cup S \cup C_M(S)) + 2|S| - 1, \) as claimed. 

We apply the last theorem to prove that \( S \) is full in the inflation \( \mathcal{I}(U, S). \)
Corollary 5.11: Let $\mathcal{U}$ be an extension of a weak $c$-representation of a triangle matroid $M = (E, r)$ with respect to a distinguished basis $B$. Let $S \subseteq E \setminus B$ be a subset such that every $S' \subsetneq S$ is full in $\mathcal{U}$ and let $\mathcal{U}' = \mathcal{I}(\mathcal{U}, S)$ be the inflation. Then $\mathcal{U}'$ is an extension of $M$ and $S$ is full in $\mathcal{U}'$.

Proof. First note that $U'_{e} \cap U'_{B} = U_{e} \cap U_{B}$ for any $e \in E$ by construction of the inflation. Thus, since $\mathcal{U}$ is an extension of $M$, the subspace arrangement obtained by intersecting $\mathcal{U}$ with $U'_{B}$ is a weak $c$-representation of $M$.

Second, let $T \subseteq E$ and $D \subseteq B$. We need to show
\begin{equation}
\dim(U_{T}' \cap U_{D}') \leq c(r(T) + r(D) - r(T \cup D)).
\end{equation}

Suppose $S \not\subseteq T$. Then Theorem 5.10 combined with the fact $U_{D}' = U_{D}$ and the usual dimension formula implies $\dim(U_{T}' \cap U_{D}') = \dim(U_{T} \cap U_{D})$. Therefore, Equation (8) holds in this case due to the analogous statement for the extension $\mathcal{U}$.

Now suppose $S \subseteq T$. Using Theorem 5.10 together with the extension property on $\mathcal{U}$ yields
\begin{align*}
\dim(U_{T}' \cap U_{D}') &= \dim(U_{T \cup C_{M}(S)} \cap U_{D}) \\
&\leq c(r(T \cup C_{M}(S)) + r(D) - r(T \cup C_{M}(S) \cup D)) \\
&= c(r(T) + r(D) - r(T \cup D)),
\end{align*}
where we used the fact that $r(S) = r(S \cup C_{M}(S))$ by the definition of $C_{M}(S)$.

This shows $\mathcal{U}'$ extends $M$. It remains to prove that $S$ is full in $\mathcal{U}'$.

Using the same notation as in the construction of the inflation, Lemma 5.3 implies
\begin{equation*}
\dim(U_{S}') = \dim(U_{S}) + c(|S| - 1) + \text{def}_{\mathcal{U}}(S) + c|S| - \dim(W' \cap U_{S}).
\end{equation*}

Since $W'$ is a generic subspace of $U_{C_{M}(S)}$ of dimension $\text{def}_{\mathcal{U}}(S)$, which equals $\dim(U_{C_{M}(S)}) - \dim(U_{S} \cap U_{B})$, we obtain $\dim(W' \cap U_{S}) = 0$. Therefore, we compute using the usual dimension formula and Theorem 5.10 applied to the set $S \cup B$:
\begin{align*}
\dim(U_{S}' \cap U_{B}') &= \dim(U_{S}') + \dim(U_{B}') - \dim(U_{S}' + U_{B}') \\
&= \dim(U_{S}) + c(2|S| - 1) + \text{def}_{\mathcal{U}}(S) \\
&\quad + \dim(U_{B}) - \dim(U_{S} + U_{B}) - c(2|S| - 1) \\
&= \dim(U_{S} \cap U_{B}) + \text{def}_{\mathcal{U}}(S) = c \cdot r(S). \quad \blacksquare
6. Combinatorial inflation

This section describes a combinatorial inflation procedure for polymatroids which mirrors the algebraic one described in the previous section.

**Definition 6.1:** Let \( M = (E, r) \) be a rank three matroid with a distinguished basis \( B \). We call a polymatroid \( g \) defined on \( E \) an **extension** of \( M \) if for all \( C \subseteq B \) and \( S \subseteq E \) it satisfies

\[
(*) \quad g(C) + g(S) - g(S \cup C) = r(C) + r(S) - r(S \cup C).
\]

The condition in Equation (*) reflects the condition of a subspace arrangement extension given in Definition 5.4. It ensures that subspace arrangements representing \( g \) are weak \( c \)-representations of \( M \) when intersected with the subspace corresponding to \( B \) (this statement will be proved in Theorem 7.1). Note that for any \( C \subseteq B \) applying Equation (*) with \( S = C \) implies

\[ g(C) = r(C). \]

We define a combinatorial inflation operation on the family of all extension polymatroids \( g : \mathcal{P}(E) \rightarrow \mathbb{R}_{\geq 0} \) of a triangle matroid \( M = (E, r) \) with distinguished basis \( B \). This mirrors the algebraic inflation construction—compare Theorem 5.10. Further comparison of these constructions is carried out in the next section.

**Definition 6.2:** Given \( g : \mathcal{P}(E) \rightarrow \mathbb{R}_{\geq 0} \) which is an extension of a triangle matroid \( M = (E, r) \) with distinguished basis \( B \), together with a subset \( S \subseteq E \setminus B \), we define the inflated polymatroid \( g' \) as follows: let \( T \subseteq E \) be any subset disjoint from \( S \), and let \( Z \subseteq S \). Then we define

\[
g'(T \cup Z) := \begin{cases} 
g(T \cup Z) + 2|Z|, & Z \subsetneq S, 
g(T \cup S \cup C_M(S)) + 2|S| - 1, & Z = S. \end{cases}
\]

The rank function \( g' \) resulting from this construction, applied to \( g \) and the subset \( S \), will be denoted by \( I_{\text{comb}}(g, S) \).

**Proposition 6.3:** Let \( g \) be a polymatroid extending a matroid \( M = (E, r) \) with respect to the distinguished basis \( B \) and let \( S \subseteq E \setminus B \). Then \( g' = I_{\text{comb}}(g, S) \) also extends \( M \) with respect to \( B \).
Proof. Let $T \subseteq E$ be any subset disjoint from $S$. Since $g$ is a polymatroid extending $M$, we obtain by Equation (\*) that

$$g(S) + g(C_M(S)) - g(S \cup C_M(S)) = r(S) + r(C_M(S)) - r(S \cup C_M(S)).$$

Together with $g(C_M(S)) = r(C_M(S))$ and $r(S) = r(S \cup C_M(S))$, this equation implies $g(S) = g(S \cup C_M(S))$. The fact that $g$ is a polymatroid yields

$$g(T \cup S \cup C_M(S)) = g(T \cup S).$$

This shows that $g'$ satisfies

$$g'(T \cup Z) = \begin{cases} 
  g(T \cup Z) + 2|Z|, & Z \subsetneq S, \\
  g(T \cup Z) + 2|S| - 1, & Z = S.
\end{cases}$$

This is a polymatroid, since it is the sum of $g$ with the rank function of a polymatroid on $S$, namely the sum of a free rank function with a uniform rank function of rank $|S| - 1$.

It remains to prove that $g'$ also satisfies Equation (\*). Let $T' \subseteq E$ and $D \subseteq B$. The following equalities hold by definition of $g'$:

$$g'(T') = \begin{cases} 
  g(T') + 2|T' \cap S|, & S \not\subseteq T', \\
  g(T') + 2|T' \cap S| - 1, & S \subseteq T'.
\end{cases}$$

$$g'(T' \cup D) = \begin{cases} 
  g(T' \cup D) + 2|T' \cap S|, & S \not\subseteq T', \\
  g(T' \cup D) + 2|T' \cap S| - 1, & S \subseteq T'.
\end{cases}$$

Thus, it is always true that $g'(T') - g'(T' \cup D) = g(T') - g(T' \cup D)$. Using $g'(D) = g(D)$, we obtain in total

$$g'(D) + g'(T') - g'(T' \cup D) = g(D) + g(T') - g(T' \cup D).$$

We conclude $g'$ satisfies Equation (\*), since $g$ does. \(\blacksquare\)

7. Compatibility of algebraic and combinatorial inflation

This section proves two theorems which relate the algebraic and combinatorial inflation procedures introduced in the last two sections. We start by establishing the connection between weak $c$-representations and combinatorial polymatroid extensions.
THEOREM 7.1: Let $M = (E, r)$ be a matroid with a distinguished basis $B$ and let $g : \mathcal{P}(E) \to \mathbb{R}_{\geq 0}$ be a polymatroid extending $M$ (i.e. $g$ satisfies Equation (\ast) for all appropriate subsets). Suppose $U = \{U_e\}_{e \in E}$ represents $c \cdot g$, or in other words $r_U = c \cdot g$.

Denote $A_e = U_e \cap U_B$ for each $e \in E$. Then the arrangement $A = \{A_e\}_{e \in E}$ is a weak $c$-representation of $M$ with respect to $B$.

Remark 7.2: The proof is just an application of Equation (\ast) and basic linear algebra. The fact that it works is what justifies the definition of polymatroid extensions.

Proof. By definition, for any $C \subseteq B$ we have $g(C) = r(C)$ and $U_C = A_C$. Thus

$$r^c_A|_B = r|_B.$$

Let $e \in E \setminus B$ and $C \subseteq B$. Then, denoting $S := C \cup \{e\}$ we obtain

$$\begin{align*}
\dim A_S &= \dim A_C + \dim A_e - \dim (A_C \cap A_e) \\
&= \dim A_C + \dim (U_e \cap U_B) - \dim (U_C \cap (U_e \cap U_B)).
\end{align*}$$

(9)

Note that $U_C \cap U_e \cap U_B = U_C \cap U_e$ since $C \subseteq B$. Using Equation (9) together with the dimension formula and the identities $\dim U_T = c \cdot r^c_U(T) = c \cdot g(T)$ for any subset $T \subseteq E$, we obtain as required

$$\frac{1}{c} \dim A_S = r(C) + [g(B) + g(\{e\}) - g(B \cup \{e\})] - [g(C) + g(\{e\}) - g(C \cup \{e\})]$$

$$\overset{(*)}{=} r(C) + [r(B) + r(\{e\}) - r(B \cup \{e\})] - [r(C) + r(\{e\}) - r(C \cup \{e\})]$$

$$= r(C \cup \{e\}).$$

Suppose now that $S \subseteq E$ is a general subset. Then

$$r_A(S) = \dim(A_S) = \dim \left( \sum_{x \in S} (U_x \cap U_B) \right)$$

$$\leq \dim (U_S \cap U_B)$$

$$= \dim U_S + \dim U_B - \dim U_{S \cup B}$$

$$= c(g(S) + g(B) - g(S \cup B))$$

$$\overset{(*)}{=} c(r(S) + r(B) - r(S \cup B)) = c \cdot r(S).$$

Therefore, $A$ is a weak $c$-representation of $M$ as claimed. 

\[\square\]
Remark 7.3: In Equation (9) we needed the fact that \( e \) is a single element of the ground set: For a general \( S \subset E \) the subspace \( A_S = \sum_{e \in S} (U_e \cap U_B) \) may not equal \( U_S \cap U_B \).

Proposition 7.4: Let \( M = (E, r) \) be a triangle matroid with distinguished basis \( B \). Let \( D \subseteq B, S \subseteq E \). Any polymatroid \( g : \mathcal{P}(E) \to \mathbb{R}_{\geq 0} \) extending \( M \) satisfies

\[
g(D \cup S \cup C_M(S)) = g(S \cup B) - g(B) + r(D \cup S).
\]

Similarly, any subspace arrangement \( \mathcal{U} = \{U_e\}_{e \in E} \) extending a weak \( c \)-representation of \( M \) satisfies

\[
r^c_{\mathcal{U}}(D \cup S \cup C_M(S)) = r^c_{\mathcal{U}}(S \cup B) - r^c_{\mathcal{U}}(B) + r(D \cup S).
\]

Remark 7.5: Let \( r^c_{\mathcal{U}} \) be the rank function of an arrangement as above, and assume it has been obtained by a sequence of inflations of a weak \( c \)-representation of \( M \). Let \( r^c_{\mathcal{U}'} \) be the function obtained by inflating at an additional subset \( S \). Then \( r^c_{\mathcal{U}'}(D \cup S) \) can be expressed in terms of \( r^c_{\mathcal{U}}(D \cup S \cup C_M(S)) \) by Theorem 5.10.

The formula above thus expresses \( r^c_{\mathcal{U}'}(D \cup S) \) in terms of two simpler objects, namely \( r \) and the rank function

\[
S \mapsto r^c_{\mathcal{U}}(S \cup B) - r^c_{\mathcal{U}}(B)
\]

of the quotient by \( U_B \). We will see that this function does not depend on \( A \), but only on \( M \) and on the sequence of inflations we applied.

Proof of Proposition 7.4. By Equation (*) applied to \( D \cup S \cup C_M(S) \) and \( B \) we have

\[
g(D \cup S \cup C_M(S)) + g(B) - g(S \cup B) = r(D \cup S \cup C_M(S)) + r(B) - r(S \cup B).
\]

Since \( B \) is a basis for \( M \), we have \( r(B) = r(S \cup B) \) and the last two terms on the right cancel. Similarly, by definition of \( C_M(S) \) we have

\[
r(D \cup S \cup C_M(S)) = r(D \cup S).
\]

Substituting, we obtain

\[
g(D \cup S \cup C_M(S)) + g(B) - g(S \cup B) = r(D \cup S),
\]

and rearranging yields the required equation on \( g \).
For the claim on \( \mathcal{U} \) we will work with dimensions rather than the function \( r_{\mathcal{U}} \), since the latter provides no mechanism for considering intersections of subspaces. We have
\[
\dim(U_D \cup S \cup C_M(S)) + \dim(U_B)
= \dim(U_D \cup S \cup C_M(S) + U_B) + \dim(U_D \cup S \cup C_M(S) \cap U_B).
\]
Note that
\[
U_D \cup S \cup C_M(S) + U_B = (U_D + U_S + U_M(S)) + U_B = U_S + U_B,
\]
since \( D, C_M(S) \subseteq B \). Further, \( U_D \cup S \cup C_M(S) \cap U_B = (U_D \cup C_M(S) + U_S) \cap U_B \).
Since \( U_D \cup C_M(S) \subseteq U_B \), the intersection distributes over the sum. This yields
\[
(U_D \cup C_M(S) + U_S) \cap U_B = (U_D \cup C_M(S) \cap U_B) + (U_S \cap U_B).
\]
There are now two cases to consider:

**Case 1:** \( C_M(S) \neq \emptyset \). Since \( \mathcal{U} \) extends a weak \( c \)-representation of \( M \) Lemma 5.9 implies
\[
U_S \cap U_B \subseteq U_C(M(S)) \subseteq U_D \cup C_M(S),
\]
and \( U_D \cup C_M(S) = U_D \cup C_M(S) \cap U_B \). Thus
\[
(U_D \cup C_M(S) \cap U_B) + (U_S \cap U_B) = U_D \cup C_M(S).
\]
Further, \( \dim(U_D \cup C_M(S)) = c \cdot r(D \cup C_M(S)) = c \cdot r(D \cup S) \), where the first equality holds since \( \mathcal{U} \) extends a weak \( c \)-representation of \( M \).

**Case 2:** \( C_M(S) = \emptyset \). In this case, \( S \) is full in \( \mathcal{U} \) by Lemma 5.7. Thus, we obtain \( \dim(U_S \cap U_B) = c \cdot r(S) \). Since \( D \cup C_M(S) = D \subseteq B \), this implies:
\[
\dim((U_D \cup C_M(S) \cap U_B) + (U_S \cap U_B))
= \dim(U_D + (U_S \cap U_B))
= \dim(U_D) + \dim(U_S \cap U_B) - \dim(U_D \cap U_S \cap U_B)
= c \cdot (r(D) + r(S) - [r(D) + r(S) - r(D \cup S)])
= r(D \cup S).
\]
Thus in either case we obtain \( \dim((U_D \cup C_M(S) + U_S) \cap U_B) = c \cdot r(D \cup S) \), and on substituting this into the previous equation:
\[
\dim(U_D \cup S \cup C_M(S)) + \dim(U_B) = \dim(U_S \cup B) + c \cdot r(D \cup S).
\]
Rearranging and replacing the dimensions with ranks gives the claim. 

\[ \blacksquare \]
Construction 7.6: Let $M = (E, r)$ be a triangle matroid with distinguished basis $B$. We construct an extension $g(M)$ of $M$ as follows:

(a) Choose a linear ordering $S_0 = \emptyset, S_1, \ldots, S_n$ on $\mathcal{P}(E \setminus B)$ which refines the ordering given by inclusion. That is, if $S_i \subseteq S_j$ then $i \leq j$.

(b) We inductively define a sequence of polymatroids $g_0, \ldots, g_n$: set $g_0 := r$ and given $g_i$, define $g_{i+1} := I_{\text{comb}}(g_i, S_{i+1})$ for $i = 0, \ldots, n - 1$. Finally, set $g(M) := g_n$.

Proposition 6.3 implies that the extension $g(M)$ is indeed a polymatroid extension of $M$ with respect to $B$. In principle, the extension $g(M)$ depends on the chosen order on $\mathcal{P}(E \setminus B)$. We assume to be working with one fixed order from now on and therefore omit both this choice and the basis $B$ from the notation $g(M)$.

Theorem 7.7: Let $M = (E, r)$ be a triangle matroid with distinguished basis $B$. Let $c \geq 1$ be an integer. Then $M$ has a weak $c$-representation with respect to $B$ if and only if $c \cdot g(M)$ has a subspace arrangement representation $\mathcal{U}$, that is, $r^c_{\mathcal{U}} = g(M)$.

Further, given a weak $c$-representation $\mathcal{A}$ of $M$, the subspace arrangement $\mathcal{U}$ representing $c \cdot g(M)$ can be chosen to extend $\mathcal{A}$.

Proof. Let $S_0 = \emptyset, S_1, \ldots, S_n$ be the chosen order on $\mathcal{P}(E \setminus B)$ and

$g_0, \ldots, g_n = g(M)$

the sequence of inflations of Construction 7.6.

Theorem 7.1 implies that if $c \cdot g(M)$ is representable as the rank function of a subspace arrangement then $M$ is weakly $c$-representable.

For the other implication, suppose a weak $c$-representation $\mathcal{A} = \{A_e\}_{e \in E}$ of $M$ is given. We inductively produce a sequence of subspace arrangements $\mathcal{U}_0, \ldots, \mathcal{U}_n$ with $\mathcal{U}_i = \{U_{i,e}\}_{e \in E}$ as follows: set $\mathcal{U}_0 := \mathcal{A}$ and given $\mathcal{U}_i$, define

$\mathcal{U}_{i+1} := I(\mathcal{U}_i, S_{i+1})$

for $i = 0, \ldots, n - 1$. Lastly, set $\mathcal{U} := \mathcal{U}_n$.

Note we never inflate with respect to $S_0 = \emptyset$. Furthermore, the choice of order on $\mathcal{P}(E \setminus B)$ and Corollary 5.11 imply that in each step every proper subset of $S_i$ is full in $\mathcal{U}_{i-1}$. Thus the assumptions of Theorem 5.10 are satisfied in each inflation step.
We perform an induction to compare the combinatorial inflation operation $I_{\text{comb}}$ with the linear-algebraic inflation operation $I$, and prove that $U$ represents $c \cdot g(M)$ as desired. We do this using the following two claims.

**Claim 1**: For any $1 \leq i \leq n$ and any $T \subseteq E$ we have

$$c \cdot g_i(B \cup T) = r_{U_i}(B \cup T).$$

Note that the contractions of $c \cdot g_i$ and $r_{U_i}$ by $B$ are given by

$$T \mapsto c \cdot g_i(B \cup T) - c \cdot g_i(B) \quad \text{and} \quad T \mapsto r_{U_i}(B \cup T) - r_{U_i}(B)$$

respectively. Since $c \cdot g_i(B) = r_{U_i}(B)$, the claim shows that these contractions are equal.

**Claim 2**: If $D \subseteq B$ and $j \leq i$ then

$$c \cdot g_i(D \cup S_j) = r_{U_i}(D \cup S_j).$$

The theorem itself is directly implied by Claim 2 for the case $i = n$, since any subset of $E$ may be written as $A \cup S_j$ with $A \subseteq B$, $1 \leq j \leq n$.

**Proof of Claim 1.** For $i = 0$ and for any $T \subseteq E$,

$$g_i(T \cup B) = r_{U_i}^c(T \cup B) = r(B)$$

by definition.

Let $i > 0$. Applying Definition 6.2 we see that

$$g_i(T \cup B) = \begin{cases} g_{i-1}(T \cup B) + 2|T \cap S_i|, & S_i \not\subseteq T, \\ g_{i-1}(T \cup B \cup C_M(S_i)) + 2|T \cap S_i| - 1, & S_i \subseteq T. \end{cases}$$

It is clear that

$$g_i(T \cup B \cup C_M(S_i)) = g_i(T \cup B)$$

because $C_M(S_i) \subseteq B$. Subtracting $g_{i-1}(T \cup B)$ from both sides, we find

$$g_i(T \cup B) - g_{i-1}(T \cup B) = 2|T \cap S_i| - \begin{cases} 0, & S_i \not\subseteq T, \\ 1, & S_i \subseteq T. \end{cases}$$
Applying the same reasoning using Theorem 5.10 yields the same formula for \( r_{\mathcal{U}_i}(T \cup B) - r_{\mathcal{U}_{i-1}}(T \cup B) \). Therefore:

\[
g_i(T \cup B) - r(T \cup B) = g_i(T \cup B) - g_0(T \cup B) = \sum_{j=1}^{i} (g_j(T \cup B) - g_{j-1}(T \cup B)) = \sum_{j=1}^{i} \left( 2|T \cap S_{i+1}| - \begin{cases} 0, & S_i \not\subseteq T, \\ 1, & S_i \subseteq T, \end{cases} \right)
\]

\[
= \sum_{j=1}^{i} (r_{\mathcal{U}_j}^c(T \cup B) - r_{\mathcal{U}_{j-1}}^c(T \cup B)) = r_{\mathcal{U}_i}^c(T \cup B) - r_{\mathcal{U}_0}^c(T \cup B) = r_{\mathcal{U}_i}^c(T \cup B) - r(T \cup B).
\]

Hence we have the equality \( g_i(T \cup B) = r_{\mathcal{U}_i}^c(T \cup B) \).

**Proof of Claim 2.** We proceed again by induction on \( i \). The claim is trivially true for \( i = 0 \) by the definition of a weak \( c \)-arrangement: what it means is that \( r(D) = r_{\mathcal{A}}^c(D) \) for any \( D \subseteq B \).

Suppose the claim is true for some \( i < n \). Let us show it also holds for \( i + 1 \). For \( j \leq i \) and any \( D \subseteq B \) we set \( Z := S_{i+1} \cap S_j \) and \( T := S_j \setminus Z \) and obtain

\[
g_{i+1}(D \cup S_j) = g_{i+1}((D \cup T) \cup Z)
= g_i(D \cup T \cup Z) + 2|Z|
= g_i(D \cup S_j) + 2|Z|.
\]

The same computation, with \( r_{\mathcal{U}_{i+1}}^c \) and \( r_{\mathcal{U}_i}^c \) replacing \( g_{i+1} \) and \( g_i \), shows

\[
r_{\mathcal{U}_{i+1}}^c(D \cup S_j) = r_{\mathcal{U}_i}^c(D \cup S_j) + 2|Z|.
\]

By the induction hypothesis,

\[
c \cdot g_{i+1}(D \cup S_j) = c \cdot g_i(D \cup S_j) + 2c|Z| = r_{\mathcal{U}_i}(D \cup S_j) + 2c|Z| = r_{\mathcal{U}_{i+1}}(D \cup S_j).
\]

For \( j = i + 1 \), we apply Theorem 5.10 and Proposition 7.4:

\[
r_{\mathcal{U}_{i+1}}^c(D \cup S_{i+1}) = r_{\mathcal{U}_i}^c(D \cup C_M(S_{i+1}) \cup S_{i+1}) + 2|S_{i+1}| - 1 = r_{\mathcal{U}_i}^c(S_{i+1} \cup B) - r_{\mathcal{U}_i}^c(B) + r(D \cup S_{i+1}) + 2|S_{i+1}| - 1
= r_{\mathcal{U}_i}^c(S_{i+1} \cup B) - r(B) + r(D \cup S_{i+1}) + 2|S_{i+1}| - 1,
\]

where the last equality holds because \( \mathcal{U}_i \) extends a weak \( c \)-representation of \( M \), and thus \( r_{\mathcal{U}_i}^c(Z) = r(Z) \) for any \( Z \subseteq B \).
In exactly the same way, using Definition 6.2 in place of Theorem 5.10, we find:

\[ g_{i+1}(D \cup S_{i+1}) = g_i(S_{i+1} \cup B) - r(B) + r(D \cup S_{i+1}) + 2|S_{i+1}| - 1. \]

Hence

\[ r_{U_{i+1}}^{c}(D \cup S_{i+1}) - g_{i+1}(D \cup S_{i+1}) = r_{U_i}^{c}(S_{i+1} \cup B) - g_i(S_{i+1} \cup B), \]

and by Claim 1 the difference on the right side is 0. 

As noted above, this proves the theorem.

8. Bases of $c$-admissible arrangements

This section has two main purposes. The first is to translate questions about polymatroids and $c$-admissible arrangements to questions about matroids and $c$-arrangements. This is carried out in Section 8.1.

The second is more directly related to inflations and generalized Dowling geometries: Construction 7.6 gives a method by which to extend a weak $c$-representation $A$ of a triangle matroid into a $c$-admissible subspace arrangement $U$. This construction gives an arrangement of a combinatorial type that does not depend on $A$.

We want to apply group-theoretic undecidability results to this construction, where the weak arrangement $A$ is constructed from a group presentation as in Section 4. For this, we need to check whether some two subspaces $A_x, A_y$ of $A$ are different, and this needs to be encoded in the combinatorics of $U$’s rank function; but the rank function of $U$ contains no such information. It does not even know whether $A$ was constructed from a trivial representation of the group or a faithful one. Thus our second goal is to modify $U$ in such a way that the resulting rank function contains the required information. We call this construction “forcing an inequality”. It is carried out in Section 8.3; Section 8.2 sets up some necessary preliminaries.

8.1. Expansions and $c$-bases. We wish to translate problems about $c$-admissible subspace arrangements and polymatroids to problems about $c$-arrangements and matroids. This is entirely analogous to translating problems on subspace arrangement to problems on vector arrangements.
Given a subspace arrangement representing a polymatroid \(g\), one can construct a vector arrangement from it as follows: pick a basis for every subspace, and take the collection of all resulting basis vectors. If we keep track of which vector came from which subspace, the original subspace arrangement can be reconstructed.

This construction does not depend only on \(g\): the result depends on the chosen bases. However, if the ground field is large enough and the bases are chosen generically, we always obtain the same matroid. This matroid is called the free expansion \(\mathcal{F}(g)\).

In fact there are only finitely many possible vector arrangements obtained by picking bases for a subspace arrangement representing the polymatroid \(g\). The matroids arising in this way are called expansions of \(g\), and form a subset of the weak images of \(\mathcal{F}(g)\). For further details, see [Ngu86, Proposition 10.2.6].

In the discussion above, one can systematically replace subspace arrangements by \(c\)-admissible arrangements, and vector arrangements by \(c\)-arrangements. This section collects the relevant definitions and lemmas.

**Definition 8.1 (Arrangement \(c\)-bases and polymatroid expansions):**

(a) Let \(U = \{U_e\}_{e \in E}\) be a \(c\)-admissible subspace arrangement, and denote \(d_e = \frac{1}{c} \dim U_e\) for each \(e \in E\). A \(c\)-basis of \(U\) is a \(c\)-arrangement \(W = \{W_{e,i}\}_{e \in E, 1 \leq i \leq d_e}\) in the same ambient vector space as \(U\), satisfying that for each \(e \in E\):

\[
U_e = \sum_{i=1}^{d_e} W_{e,i}.
\]

In this situation, we denote \(W_e = \{W_{e,i}\}_{1 \leq i \leq d_e}\).

Note that if \(c = 1\), the subspaces \(W_{e,i}\) are lines, and, after identifying each \(W_{e,i}\) with an arbitrary nonzero point on it, we find that each \(W_e\) is a basis of the subspace \(U_e\). This is the sense in which these objects are \(c\)-bases.

(b) The **combinatorial type** of the \(c\)-basis \(W\) is the matroid given by \(r_W\).

(c) An **expansion** of a polymatroid \((E, g)\) is a matroid with rank function \(r\) on the ground set \(\{(e, i) \mid e \in E, 1 \leq i \leq g(e)\}\) satisfying the following property for any \(S \subseteq E\):

\[
r(\{(e, i) \mid e \in S, 1 \leq i \leq g(e)\}) = g(S).
\]
In particular, the combinatorial type of a $c$-basis of $U$ is an expansion of $r_U$. The converse is false, because expansions of representable rank functions are not always representable.

The next lemma clarifies the relation between $c$-bases and expansions.

**Lemma 8.2:** Let $g$ be a polymatroid on $E$ and let

$$
\{(e, i) \mid e \in E, 1 \leq i \leq g(e)\}, r
$$

be an expansion. If

$$
W = \{W_{e,i} \mid e \in E, 1 \leq i \leq g(e)\}
$$

is a $c$-arrangement representing $r$, then the arrangement $U = \{U_e \mid e \in E\}$ where $U_e = \sum_{i=1}^{g(e)} W_{e,i}$ is a subspace arrangement representing $c \cdot g$.

In other words, any $c$-arrangement representation of an expansion of $g$ is a $c$-basis of a corresponding subspace arrangement representing $c \cdot g$.

This is a direct consequence of the definitions.

**Definition 8.3:** Let $U = \{U_e \mid e \in E\}$ be a $c$-admissible subspace arrangement. Denote $d_e = \frac{1}{c} \dim(U_e)$ for each $e \in E$.

In each $U_e$, choose $d_e$ generic subspaces $W_{e,1}, \ldots, W_{e,d_e}$, each of dimension $c$. The arrangement $\{W_{e,i} \mid e \in E, 1 \leq i \leq d_e\}$ is called a **generic** $c$-basis of $U$.

It is a consequence of the following lemma that a generic $c$-basis is a $c$-arrangement.

**Lemma 8.4** (Splitting Lemma): Let $\{U_e \mid e \in E\}$ be a $c$-admissible subspace arrangement in a vector space $V$. For each $e \in E$ let $k_e$ be a nonnegative integer and let $W_{e,1}, \ldots, W_{e,k_e}$ be generic subspaces of $U_e$, each of dimension $c$. Then $\{W_{e,i} \mid e \in E, 1 \leq i \leq k_e\}$ is a $c$-admissible subspace arrangement.

**Proof.** We will prove by induction on $k := \sum_{e \in E} k_e$ that

$$
W := \{W_{e,i} \mid e \in E, 1 \leq i \leq k_e\} \cup \{U_e \mid e \in E\}
$$

is $c$-admissible. Since a subset of a $c$-admissible arrangement is also $c$-admissible, this proves the lemma.

For $k = 0$, the statement is just the assumption that $\{U_e \mid e \in E\}$ is $c$-admissible.
Suppose the statement is true for \( k - 1 \geq 0 \), and let \( W \) be any arrangement as above with \( \sum_{e \in E} k_e = k - 1 \). Let \( e \in E \) and let \( W \subseteq U_e \) be a generic subspace of dimension \( c \). We wish to show \( W \cup \{ W \} \) is also \( c \)-admissible. Given \( S \subseteq W \), denote \( W_S = \sum_{U \in S} U \). We want to show \( \dim(W_S + W) \in c \cdot \mathbb{N} \), and it suffices to prove that

\[
\dim(W_S \cap W) = \dim(W_S) + \dim(W) - \dim(W_S + W) \in c \cdot \mathbb{N}.
\]

Clearly if \( U_e \subseteq W_S \) we are done, since then also \( W \subseteq W_S \) and \( W_S \cap W = W \).

If \( U_e \not\subseteq W_S \), consider the intersection \( U_e \cap W_S \); it has dimension \( m \cdot c \) for some nonnegative integer \( m \), with \( mc < \dim(U_e) \). Since \( \dim(U_e) \in c \cdot \mathbb{N} \), we have \( \dim(U_e) \geq (m + 1)c \). By genericity of \( W \) in \( U_e \), the intersection \( W \cap (U_e \cap W_S) \) is trivial. Thus we have

\[
0 = W \cap (U_e \cap W_S) = (W \cap U_e) \cap W_S = W \cap W_S.
\]

Since \( S \subseteq W \) was arbitrary, the arrangement \( W \cup \{ W \} \) is \( c \)-admissible as required.

We now give a description of the matroid underlying a generic \( c \)-basis of a \( c \)-admissible arrangement. The construction of this matroid from the normalized rank function of the arrangement is the same as the construction of the free expansion of a polymatroid in [Ngu86, Proposition 10.2.7]. We omit the proof since we do not rely on this description in our subsequent proofs.

**Proposition 8.5:** Let \( U = \{ U_e \}_{e \in E} \) be a \( c \)-admissible subspace arrangement. For each \( e \in E \) let \( k_e \) be a nonnegative integer, and choose generic subspaces \( W_{e,1}, \ldots, W_{e,k_e} \) of \( U_e \), each of dimension \( c \). For each \( e \in E \) denote \( W_e = \{ W_{e,i} \}_{1 \leq i \leq k_e} \), and then define \( W = \bigcup_{e \in E} W_e \).

Let \( S \subseteq W \). Then

\[
\dim \left( \sum_{W \in S} W \right) < c |S|
\]

if and only if there is a subset \( F \subseteq E \) such that

\[
(10) \quad \dim \left( \sum_{f \in F} U_f \right) < c \sum_{f \in F} |S \cap W_f|.
\]

This gives a description of the independent sets in the matroid given by any generic \( c \)-basis of \( U \). Following [Ngu86], we call this matroid the free expansion of \( r^c_U \) and denote it by \( \mathcal{F}(r^c_U) \). Similarly, for a polymatroid \( g \) we will use the notation \( \mathcal{F}(g) \).
The following lemma clarifies the role of free expansions. Its corollary will be useful in the proof of the main theorem.

**Lemma 8.6:** Let \( g \) be a polymatroid on \( E \). Any expansion \( M \) of \( g \) is a weak image of the free expansion \( \mathcal{F}(g) \). That is, \( M \) and \( \mathcal{F}(g) \) are matroids on the same ground set, and any independent set in \( M \) is also independent in \( \mathcal{F}(g) \).

For the proof see [Ngu86, Proposition 10.2.6].

**Corollary 8.7:** A polymatroid \( g \) has only finitely many expansions, and the collection of these is computable from \( g \).

The proof is clear: a given matroid has finitely many weak images, and they are easy to list.

### 8.2. Well-separated Extensions

Let \( \mathcal{A} \) be a weak \( c \)-representation of a generalized Dowling geometry \( N_{S,R} \) as constructed in Section 4. As remarked at the beginning of this section, if \( \mathcal{U} \) is a \( c \)-admissible arrangement produced from \( \mathcal{A} \) by the sequence of iterated inflations described in Section 7, then \( r_c^U \) does not contain enough information to determine whether \( A_x \neq A_y \) for a pair of elements \( x, y \) on a line of \( N_{S,R} \). This kind of inequality is precisely what we need in order to apply Slobodskoi’s undecidability theorem, using Theorem 4.6.

Our strategy is to modify \( \mathcal{U} \) by adding a subspace which is contained in \( A_x \) but not in \( A_y \). The main difficulty in doing so is that the normalized rank function of the resulting arrangement should be computable in a manner independent of \( c \). Therefore we cannot just take a one-dimensional subspace \( \text{span}(v) \) which is contained in \( A_x \) but not in \( A_y \): the normalized rank function will then assign \( \text{span}(v) \) a value of \( \frac{1}{c} \). The goal is to show that \( \mathcal{A} \) and \( \mathcal{U} \) can be modified in such a way that there is a subspace \( W \) of dimension \( \frac{3}{2} \) which is contained in \( A_x \), and such that any \( U \in \mathcal{U} \) that intersects \( W \) nontrivially also contains \( A_x \). The separation condition defined below is sufficient to ensure that such a \( W \) can be found.

**Definition 8.8:** Let \( M = (E, r) \) be a triangle matroid with distinguished basis \( B \) and let \( \mathcal{A} = \{A_e\}_{e \in E} \) be a weak \( c \)-representation of \( M \). We call an extension \( \mathcal{U} = \{U_e\}_{e \in E} \) of \( \mathcal{A} \) **well-separated** with respect to a given \( x \in E \) if for any \( T \subseteq E \), either \( A_x \subseteq U_T \) or \( \dim(A_x \cap U_T) \leq \frac{1}{2} c \).
The next proposition shows that the full extensions of certain weak c-representations of the matroids $N_{S,R}$ are well-separated.

**Proposition 8.9:** Let $\langle S \mid R \rangle$ be a finite presentation and let $G$ be a finite group together with a homomorphism $\varphi : G_{S,R} \to G$. Set $n := |G|$ and let $A := A_{G,\varphi} = \{A_e\}_{e \in E_{S,R}}$ be the weak $n$-representation of the matroid $N_{S,R} = (E_{S,R},r)$ with respect to the distinguished basis $B = \{b^{(1)}, b^{(2)}, b^{(3)}\}$ constructed in Proposition 4.4.

Let $\mathcal{U} = \{U_e\}_{e \in E_{S,R}}$ be an extension of $A$ and assume that $U_T$ is full in $\mathcal{U}$ for any $T \subseteq E_{S,R}$. Then $\mathcal{U}$ is well-separated with respect to $x^{(1)} \in E_{S,R}$ for any $x \in S$.

**Proof.** Fix some $x \in S$ and $T \subseteq E_{S,R}$. We have to show $U_T \cap A_{x^{(1)}}$ is either equal to $A_{x^{(1)}}$ or has dimension at most $\frac{1}{2}n$. We split this into cases based on the value of $r(T)$, noting that $U_T \cap A_{x^{(1)}} = U_T \cap U_B \cap A_{x^{(1)}}$, since $A_{x^{(1)}} \subseteq U_B = A_B$.

**Case 1:** $r(T) = 1$. In this case $T = \{t\}$ for some $t \in E_{S,R}$, and $U_T \cap U_B = A_t$.

If $t$ is not a bottom element of $N_{S,R}$, then $\{t, x^{(1)}\}$ is full in $A$ by Remark 5.8. It has rank 2 in $N_{S,R}$, so $\dim(A_t + A_{x^{(1)}}) = 2n$, implying $A_t \cap A_{x^{(1)}} = 0$. If $t$ is a bottom element of $N_{S,R}$, then by Proposition 4.4,

$$A_{x^{(1)}} = \text{colspan} \left[ \begin{array}{c} -I_n \\ \rho(\varphi(x)) \end{array} \right], \quad A_t = \text{colspan} \left[ \begin{array}{c} -I_n \\ P_t \end{array} \right]$$

respectively, where $\rho$ is the regular representation of $G$ and $P_t$ is some matrix either of the form $\rho(\varphi(g))$ for some $g \in G$, or 0 if $t \in B$. Thus by Lemma 3.6(a),

$$\dim(A_{x^{(1)}} + A_t) = n + \text{rk}(\rho(\varphi(x)) - P_t) \geq \frac{3}{2}n$$

in either case: if $P_t = 0$ this is trivial, and otherwise it follows from Corollary 3.8 since $n = |G|$. This implies $A_{x^{(1)}} \cap A_t$ has dimension at most $\frac{1}{2}n$ as required.

**Case 2:** $r(T) = 2$. If $x^{(1)} \in T$ then $A_{x^{(1)}} \subseteq U_T \cap U_B$ holds trivially. If $T$ is contained in a side $\ell$ of the triangle matroid $N_{S,R}$ then $U_T \cap U_B$ either contains $A_{x^{(1)}}$ if $\ell$ is the bottom side or it intersects $A_{x^{(1)}}$ trivially if $\ell$ is the left or right side of $N_{S,R}$.

If $T$ is not contained in such a line, note that

$$U_T \cap U_B = U_T \cap A_B \supseteq A_T \cap A_B.$$
By Remark 5.8, $T$ is full in $\mathcal{A}$. Hence,

$$\dim(A_T \cap A_B) = \dim(A_T) = 2n.$$ 

Since $\dim(U_T \cap U_B) = 2n$, the containment implies equality:

$$U_T \cap U_B = A_T \cap A_B = A_T.$$ 

Now suppose $y^{(1)} \in T$ for some $y \in S$ with $x \neq y$. We allow $y^{(1)}$ to be $b^{(k)}$ for $k = 1, 2$ or $e^{(1)}$. Since $A_T$ intersects $A_{b^{(1)}, b^{(2)}}$ in $A_{y^{(1)}}$ we have $A_T \cap A_{x^{(1)}} = A_{y^{(1)}} \cap A_{x^{(1)}}$. Thus, we may reduce to the case $r(T) = 1$ by taking $T' := \{y^{(1)}\}$.

Lastly, assume $\{y^{(2)}, z^{(3)}\} \subseteq T$ for some $y, z \in S \cup \{e\}$. The intersection $A_T \cap A_{\{b^{(1)}, b^{(2)}\}}$ has dimension at most $n$, since the sum $A_T + A_{\{b^{(1)}, b^{(2)}\}}$ is the entire space $A_B$, and has dimension $3n$ (where $\dim(A_T) = 2n$).

The block columns representing $y^{(2)}, y^{(3)}$ in $\mathcal{A}$ are

$$\begin{bmatrix} 0 \\ -I_n \\ \rho(g_2) \\ 0 \\ -I_n \end{bmatrix}, \begin{bmatrix} \rho(g_3) \\ -I_n \\ 0 \\ \rho(g_2) \\ -I_n \end{bmatrix}$$

for some $g_2, g_3 \in G$ respectively where $\rho$ is the regular representation of $G$. By Lemma 3.6 (b) we see that the matrix

$$\begin{bmatrix} -I_n \\ \rho(g_2)^{-1} \rho(g_3)^{-1} \\ 0 \\ \rho(g_2)^{-1} \\ -I_n \end{bmatrix}$$

has rank $2n$, where the first block column is in $A_{\{b^{(1)}, b^{(2)}\}}$. Thus, the intersection above is given by the block column span of the first block column. By the proof of the case $r(T) = 1$ and the fact that $\rho(g_2)^{-1} \rho(g_3)^{-1} = \rho((g_3 g_2)^{-1})$, we see that

$$A_{x^{(1)}} \cap \text{colspan} \begin{bmatrix} -I_n \\ \rho(g_2)^{-1} \rho(g_3)^{-1} \\ 0 \end{bmatrix} \leq \frac{1}{2} n,$$

since we can assume $\varphi(x) \neq (g_2 g_3)^{-1}$ (otherwise we would be in the previous case). This implies the claim.

**Case 3:** $r(T) = 3$. In this case $U_T \cap U_B$ has dimension $3n$ since $T$ is full in $\mathcal{U}$. Using the fact $\dim(U_B) = 3n$ we obtain $U_T \supseteq U_B \supseteq A_{x^{(1)}}$ as desired. ■
8.3. Forcing an Inequality. We put the ideas from the two previous subsections together, and construct certain c-bases of well-separated arrangements.

**Definition 8.10:** The double of a subspace arrangement \( \mathcal{U} = \{U_e\}_{e \in E} \) in a vector space \( V \) is the arrangement \( \mathcal{W} = \{W_e\}_{e \in E} \) in \( V \oplus V \), where

\[
W_e := U_e \oplus U_e.
\]

Note that in this situation, \( r^2_W = r^c_U \) and \( r^c_W = 2r^c_U \). Hence if \( r^c_U \) represents some polymatroid so does \( r^2_W \).

If \( \mathcal{U} \) represents some polymatroid \( c \cdot g \), or extends a weak \( c \)-representation \( \mathcal{A} \) of some matroid \( M \), it will be convenient to think of \( \mathcal{W} \) as an arrangement representing \( 2c \cdot g \) or an extension of the weak \( 2c \)-representation \( \mathcal{A} \oplus \mathcal{A} \) of \( M \).

The point is that the tools of the previous section require that we work with subspaces of dimension \( \frac{c'}{2} \). Doubling a \( c \)-arrangement, and considering it as a \( c' = 2c \)-representation of the same polymatroid, makes \( \frac{c'}{2} = c \) the “basic unit of measurement” (while also guaranteeing that it is an integer, i.e. that \( c' \) is even).

**Notation:** We use the following notation in the rest of this section:

Let \( M = (E, r) \) be a triangle matroid with respect to the distinguished basis \( B = \{b^{(1)}, b^{(2)}, b^{(3)}\} \), and let \( \mathcal{W} = \{W_e\}_{e \in E} \) be a 2c-admissible extension of the weak 2c-representation \( \mathcal{A} = \{A_e\}_{e \in E} \) of \( M \). Let \( d_e = \frac{1}{c} \dim W_e \) for each \( e \in E \). Suppose \( x \in E \) is in the line \( \ell \) of \( M \) spanned by \( \{b^{(1)}, b^{(2)}\} \), i.e. the bottom line of \( M \).

**Definition 8.11:** Let

\[
\mathcal{W}^B := \{W_{e,i}\} \quad \text{for} \quad 1 \leq i \leq d_e
\]

be a \( c \)-basis for \( \mathcal{W} \). Similarly, let \( g \) be a polymatroid extension of \( M \) and let

\[
N = (\{(e, i)\} \quad \text{for} \quad 1 \leq i \leq 2g(e) \}, r_{\exp})
\]

be an expansion (in the sense of Definition 8.1(c)) of the doubled polymatroid \( 2g \). Let \( y \in E \) be an element on the bottom line of \( M \).

(a) The \( c \)-basis \( \mathcal{W}^B \) separates \( x \) from \( y \) if

\[
W_{x,1} \subseteq W_{b^{(1)}} + W_{b^{(2)}} \quad \text{and} \quad W_{x,1} \cap W_y = 0.
\]
(b) The expansion $N$ separates $x$ from $y$ if
\[ r_{\exp}(\{(x, 1)\} \cup \{(b^{(i)}, j)\}_{1 \leq i, j \leq 2}) = 4 \]
and
\[ r_{\exp}(\{(x, 1)\} \cup \{(y, i)\}_{1 \leq i \leq 2g(y)}) = 2g(y) + 1. \]

The following proposition shows that a $c$-basis of $\mathcal{W}$ separates $x$ from $y$ if and only if the corresponding expansion of $r_{\mathcal{W}}$ does. It also describes the main consequence of the existence of a $c$-basis separating $x$ from $y$: in the weak $c$-representation $A$ corresponding to $\mathcal{W}$, the subspaces corresponding to $x$ and to $y$ are distinct.

**Proposition 8.12:**

(a) If the arrangement $\mathcal{W}$ has a $c$-basis $\mathcal{W}^B$ which separates $x$ from $y$, then $A_x \neq A_y$.

(b) A $c$-basis $\mathcal{W}^B$ separates $x$ from $y$ if and only if the expansion $r_{\mathcal{W}}^c$ of $r_{\mathcal{W}}$ separates $x$ from $y$, or equivalently, the following two conditions hold:

\[ r_{\mathcal{W}}^c(\{(x, 1)\} \cup \{(b^{(i)}, j)\}_{1 \leq i, j \leq 2}) = 4, \]

\[ r_{\mathcal{W}}^c(\{(x, 1)\} \cup \{(y, i)\}_{1 \leq i \leq d_y}) = d_y + 1. \]

**Proof.**

(a) Suppose the $c$-basis $\mathcal{W}^B$ of $\mathcal{W}$ separates $x$ and $y$. Then
\[ W_{x,1} \subseteq W_x \cap W_B = A_x \]
is not contained in $W_y \supseteq A_y$. Therefore $A_x$ is also not contained in $A_y$, and $A_x \neq A_y$.

(b) The condition $W_{x,1} \subseteq W_{b^{(1)}} + W_{b^{(2)}}$ is equivalent to the equation
\[ W_{b^{(1)}} + W_{b^{(2)}} = W_{x,1} + W_{b^{(1)}} + W_{b^{(2)}}, \]
and this occurs if and only if both sides have the same dimension, which is $4c$ by construction.

The condition $W_{x,1} \cap W_y = 0$ is equivalent to the equation
\[ \dim(W_{x,1} + W_y) = \dim(W_{x,1}) + \dim(W_y), \]
and by construction the right-hand side equals $cd_y + c$.
Construction 8.13: Suppose \( W \) is well-separated with respect to some \( x \in E \). We construct the following \( c \)-basis, which as we will show separates \( x \) from any \( y \in \ell \) such that \( A_x \neq A_y \):

(a) For each \( z \in E \setminus \{x\} \), let \( \{W_{z,1}, \ldots, W_{z,d_z}\} \) be generic \( c \)-dimensional subspaces of \( W_z \).

(b) Let \( W_{x,1} \) be a generic \( c \)-dimensional subspace of \( W_x \cap W_{\{b^{(1)}, b^{(2)}\}} = A_x \).

(c) Let \( \{W_{x,2}, \ldots, W_{x,d_x}\} \) be generic \( c \)-dimensional subspaces of \( W_x \).

The collection of all these subspaces will be denoted \( W^{B,x} \).

Lemma 8.14: \( W^{B,x} \) is a \( c \)-basis of \( W \).

Proof. We need to show \( W^{B,x} \) is a \( c \)-arrangement and that \( \sum_{i=1}^{d_z} W_{e,i} = W_e \) for each \( e \in E \). The second part is clear from genericity. Let us prove \( W^{B,x} \) is a \( c \)-arrangement.

By construction, the subspaces \( W_{z,i}^{B,x} \) (for any \( (z, i) \neq (x, 1) \)) are generic subspaces of the subspaces \( W_z \), each of dimension \( c \). Also, \( \{W_z\}_{z \in E} \cup \{W_{x,1}\} \) is a \( c \)-admissible subspace arrangement, essentially since \( W \) is well-separated: If \( T \subseteq E \) is any subset, then \( W_T \cap A_x \) is either of dimension at most \( c \) or \( A_x \subseteq W_T \). In the first case, \( W_T \) intersects \( A_x \) in dimension at most \( c \), and since \( W_{x,1} \) is generic of dimension \( c \) in \( A_x \) it satisfies \( W_{x,1} \cap W_T = 0 \). In the second case, \( A_x \subseteq W_T \) so also \( W_{x,1} \subseteq W_T \), and \( W_{x,1} + W_T = W_T \) again has dimension a multiple of \( c \). Hence \( W \cup \{W_{x,1}\} \) is \( c \)-admissible and the claim follows from the Splitting Lemma 8.4: \( W^{B,x} \) is a subspace arrangement comprised of some generic \( c \)-dimensional subspaces of the \( \{W_z\}_{z \in E} \) and of \( W_{x,1} \) (note that any \( c \)-dimensional subspace of \( W_{x,1} \) is equal to \( W_{x,1} \)).

Proposition 8.15: In the notation of Construction 8.13, \( W^{B,x} \) separates \( x \) from \( y \) for any \( y \in \ell \) such that \( A_x \neq A_y \).

Proof. The equality

\[
\rho_{W^{B,x}}(\{(x,1)\} \cup \{(b^{(i)}, j)\}_{1 \leq i, j \leq 2}) = 4
\]

is clear from the construction as \( W_{x,1} \) is contained in the \( 4c \)-dimensional subspace

\[
W_{b^{(1)}} + W_{b^{(2)}} = \sum_{i,j=1}^{2} W_{b^{(i)},j}.
\]
Suppose \( A_x \neq A_y \). Then \( A_x \not\subseteq W_y \), as otherwise

\[
A_y = W_y \cap W_B = (W_y + A_x) \cap W_B \supseteq A_x
\]
gives a contradiction. Hence \( A_x \cap W_y \) has dimension at most \( c \) by well-separatedness. As \( W_{x,1} \subseteq A_x \) is generic of dimension \( c \), it has trivial intersection with \( W_y \). This proves that

\[
r_{W_{B,x}}^c \left( \{(x, 1)\} \cup \{(y, i)\}_{1 \leq i \leq d_y} \right) = \frac{1}{c} \dim(W_{x,1} + W_y) \\
= \frac{1}{c} \left( \dim(W_{x,1}) + \dim W_y - \dim(W_{x,1} \cap W_y) \right)
\]
equals \( \frac{1}{c}(\dim(W_{x,1}) + \dim(W_y)) = 1 + d_y \). Therefore by Proposition 8.12, the \( c \)-basis \( W_{B,x} \) separates \( x \) from \( y \).

9. Proof of Theorem 1.3

In this section we connect our previous results to prove Theorem 1.3. As explained in Section 2, we reduce the uniform word problem for finite groups to the multilinear representability problem.

For the rest of this section fix an instance of the UWPFG as in Section 3.4: a finite presentation \( \langle S \mid R \rangle \) of a group together with an element \( w \in S \).

**Construction 9.1:** Let \( N_{S,R} = (E,r) \) be the matroid constructed in Definition 4.3 and let \( B = \{b^{(1)}, b^{(2)}, b^{(3)}\} \) be its distinguished basis.

Let \( g(N_{S,R}) \) be the polymatroid extension defined in Construction 7.6. Among the expansions of \( 2g(N_{S,R}) \), let \( \mathcal{M}_{S,R} \) be the set of all those which separate \( w^{(1)} \) from \( e^{(1)} \).

Note that the set \( \mathcal{M}_{S,R} \) is finite and computable from \( \langle S \mid R \rangle \) and \( w \) by Corollary 8.7. Thus the construction computes a finite set of matroids from the given UWPFG instance.

We work with the notation of the construction. Our goal is to prove that at least one of the matroids in \( \mathcal{M}_{S,R} \) is representable as a \( c \)-arrangement for some \( c \geq 1 \) if and only if the answer to the given UWPFG instance is positive.

**Proposition 9.2:** Suppose there exists a weak \( c \)-representation \( \mathcal{A} = \{A_f\}_{f \in E} \) of \( M \) over a field \( \mathbb{F} \) such that \( A_{w^{(1)}} \neq A_{e^{(1)}} \). Then at least one matroid in \( \mathcal{M}_{S,R} \) is representable as a \( c \)-arrangement over \( \mathbb{F} \).
Proof. By Proposition 4.5, there is a finitely generated matrix group $G_A$ and a homomorphism $\varphi : G_{S,R} \to G_A$ corresponding to the weak $c$-representation $A$. By construction, the elements $w^{(1)}, e^{(1)} \in E \setminus B$ correspond to the two elements $w, e$ of $G_{S,R}$, and since $A_{w^{(1)}} \neq A_{e^{(1)}}$ their images in $G_A$ are distinct. Using Malcev’s theorem, $G_A$ has a finite quotient $G'_A$ in which the images of $w, e$ are distinct. Replace $G_A$ by $G'_A$, and the homomorphism $\varphi$ by its composition with the quotient map.

Define $A' = A_{G_A; \varphi} = \{A'_f\}_{f \in E}$ as in Proposition 4.4, and note that also that $A'_{w^{(1)}} \neq A'_{e^{(1)}}$. Let $U$ be the extension of $A'$ constructed in Theorem 7.7. The arrangement $U$ is well-separated with respect to $w^{(1)}$ by Proposition 8.9.

By Lemma 8.14, Construction 8.13 produces a $c$-basis for the double $W$ of $U$. This is a $c$-arrangement representing an expansion of $2g(N_{S,R})$. By Proposition 8.15, it separates $w^{(1)}$ from $e^{(1)}$. Thus its normalized rank function is in $\mathcal{M}_{S,R}$.

**Proposition 9.3:** If any $M \in \mathcal{M}_{S,R}$ has a $c$-arrangement representation over a field $F$ then $N_{S,R}$ has a weak $c$-representation $A = \{A_f\}_{f \in E}$ over $F$ with $A_{w^{(1)}} \neq A_{e^{(1)}}$.

**Proof.** By Lemma 8.2, a representation of any $M \in \mathcal{M}_{S,R}$ as a $c$-arrangement gives a representation of the polymatroid $2c \cdot g(N_{S,R})$. This yields a weak $2c$-representation $A = \{A_f\}_{f \in E}$ of $N_{S,R}$ by the intersection procedure of Theorem 7.1. Since $M$ is an expansion of $2g(N_{S,R})$ separating $w^{(1)}$ from $e^{(1)}$, Proposition 8.12 shows $A_{w^{(1)}} \neq A_{e^{(1)}}$.

**Theorem 9.4:** Let $F$ be a field. The given UWPFG instance has a positive answer if and only if at least one matroid $M \in \mathcal{M}_{S,R}$ is multilinear over $F$.

**Remark 9.5:** We mention the field $F$ explicitly in order to emphasize that our results are entirely independent of it. Observe that the theorem implies that if the given UWPFG instance has a positive answer, then for every field $F$ there is some $F$-multilinear $M \in \mathcal{M}_{S,R}$; and conversely, if at least one $M \in \mathcal{M}_{S,R}$ is multilinear over at least one field, then the given UWPFG instance has a positive answer.

**Proof.** By Theorem 4.6, $N_{S,R}$ has a weak $c$-representation $A = \{A_f\}_{f \in E}$ over $F$ with $A_{w^{(1)}} \neq A_{e^{(1)}}$ if and only if the given UWPFG instance has a positive answer. Thus, Propositions 9.2 and 9.3 yield the two directions of the claimed equivalence.
The construction and the theorem essentially prove Theorem 1.3.

**Proof of Theorem 1.3.** A decision algorithm for the multilinear representability problem can be used to solve the uniform word problem for finite groups: apply Construction 9.1 to compute a finite set of matroids $\mathcal{M}$ from a given instance of UWPFG. By the previous theorem, the answer to the UWPFG instance is positive if and only if at least one matroid in $\mathcal{M}$ is multilinearly representable.

The UWPFG is undecidable by [Slo81], and therefore the multilinear representability problem is undecidable as well. Remark 9.5 explains why this remains true if the field is unspecified, or is allowed to be taken from some given set. ■

**References**

[ANLY00] R. Ahlswede, N. Cai, S.-Y. R. Li and R. W. Yeung, Network information flow, Institute of Electrical and Electronics Engineers. Transactions on Information Theory 46 (2000), 1204–1216.

[BBEPT14] A. Beimel, A. Ben-Efraim, C. Padró and I. Tyomkin, Multi-linear secret-sharing schemes, in Theory of Cryptography, Lecture Notes in Computer Science, Vol. 8349, Springer, Heidelberg, 2014, pp. 394–418.

[BD91] E. F. Brickell and D. M. Davenport, On the classification of ideal secret sharing schemes, Journal of Cryptology 4 (1991), 123–134.

[Bjö94] A. Björner, Subspace arrangements, in First European Congress of Mathematics, Vol. I (Paris, 1992), Progress in Mathematics, Vol. 119, Birkhäuser, Basel, 1994, pp. 321–370.

[DFZ07] R. Dougherty, C. Freiling and K. Zeger, Networks, matroids, and non-Shannon information inequalities, Institute of Electrical and Electronics Engineers. Transactions on Information Theory 53 (2007), 1949–1969.

[Dow73] T. A. Dowling, A class of geometric lattices based on finite groups, Journal of Combinatorial Theory. Series B 14 (1973), 61–86.

[ESG10] S. El Rouayheb, A. Sprintson and C. Georgiades, On the index coding problem and its relation to network coding and matroid theory, Institute of Electrical and Electronics Engineers. Transactions on Information Theory 56 (2010), 3187–3195.

[Fei67] W. Feit, Characters of Finite Groups, W. A. Benjamin, New York–Amsterdam, 1967.

[Fuj78] S. Fujishige, Polymatroidal dependence structure of a set of random variables, Information and Control 39 (1978), 55–72.

[GM88] M. Goresky and R. MacPherson, Stratified Morse theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 14, Springer, Berlin, 1988.

[KPY20] L. Kühne, R. Pendavingh and G. Yashfe, Von staudt constructions for skew-linear and multilinear matroids, https://arxiv.org/abs/2012.07361.
[KY20] L. Kühne and G. Yashfe, Undecidability of c-arrangement matroid representations, Séminaire Lotharingien de Combinatoire 84B (2020), Article no. 87.

[LS77] R. C. Lyndon and P. E. Schupp, Combinatorial Group Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 89 Springer, Berlin–New York, 1977.

[Mal40] A. Malcev, On isomorphic matrix representations of infinite groups, Matematicheskiĭ Sbornik 8 (1940), 405–422.

[Mat99] F. Matúš, Matroid representations by partitions, Discrete Mathematics 203 (1999), 169–194.

[Mnë88] N. E. Mnëv, The universality theorems on the classification problem of configuration varieties and convex polytopes varieties, in Topology and Geometry—Rohlin Seminar, Lecture Notes in Mathematics, Vol. 1346, Springer, Berlin, 1988, pp. 527–543.

[Ngu86] H. Q. Nguyen, Semimodular Functions, in Theory of Matroids, Encyclopedia of Mathematics and its Applications, Vol. 26, Cambridge University Press, Cambridge, 1986, pp. 272–297.

[Oxl11] J. Oxley, Matroid Theory, Oxford Graduate Texts in Mathematics, Vol. 21, Oxford University Press, Oxford, 2011.

[PvZ13] R. A. Pendavingh and S. H. van Zwam, Skew partial fields, multilinear representations of matroids, and a matrix tree theorem, Advances in Applied Mathematics 50 (2013), 201–227.

[Rad57] R. Rado, Note on independence functions, Proceedings of the London Mathematical Society 7 (1957), 300–320.

[SA98] J. Simonis and A. Ashikhmin, Almost affine codes, Designs, Codes and Cryptography 14 (1998), 179–197.

[Slo81] A. M. Slobodskoi, Unsolvability of the universal theory of finite groups, Algebra and Logic 20 (1981), 139–156.

[Stu87] B. Sturmfels, On the decidability of Diophantine problems in combinatorial geometry, Bulletin of the American Mathematical Society 17 (1987), 121–124.

[Zas94] T. Zaslavsky, Frame matroids and biased graphs, European Journal of Combinatorics 15 (1994), 303–307.