AUSLANDER-REITEN AND HUNEKE-WIEGAND
CONJECTURES OVER QUASI-FIBER PRODUCT RINGS

T. H. FREITAS, V. H. JORGE PÉREZ, R. WIEGAND, AND S. WIEGAND

Abstract. In this paper we explore consequences of the vanishing of Ext for finitely generated modules over a quasi-fiber product ring $R$; that is, $R$ is a local ring such that $R/(x)$ is a non-trivial fiber product ring, for some regular sequence $x$ of $R$. Equivalently, the maximal ideal of $R/(x)$ decomposes as a direct sum of two nonzero ideals. Gorenstein quasi-fiber product rings are AB-rings and are Ext-bounded. We show in Theorem 3.31 that quasi-fiber product rings satisfy a sharpened form of the Auslander-Reiten Conjecture. We also make some observations related to the Huneke-Wiegand conjecture for quasi-fiber product rings.

This article is dedicated to the memory of Nicholas Baeth

1. Introduction

This article is motivated by the celebrated Auslander-Reiten Conjecture (ARC) and the Huneke-Wiegand Conjecture for integral domains (HWC$_d$); see [8, p. 70], [31], and [32, pp. 473–474]:

Definition 1.1. Let $R$ be a commutative Noetherian local ring.

(ARC) **Auslander-Reiten Conjecture.** If $M$ is a finitely generated $R$-module such that $\text{Ext}_R^i(M, M \oplus R) = 0$, for all $i \geq 1$, then $M$ is free.

(HWC$_d$) **Huneke-Wiegand Conjecture (for domains).** If $R$ is a Gorenstein local domain and $M$ is a finitely generated torsion-free $R$-module $M$ such that $M \otimes R M^*$ is reflexive, then $M$ is free.

Here $M^*$ denotes the algebraic dual of $M$, namely, $\text{Hom}_R(M, R)$. Recall that an $R$-module $M$ is *torsion-free* provided every non-zerodivisor of $R$ is a non-zerodivisor on $M$.

Several positive cases for (ARC) are known; see, for instance, work of Huneke, Leuschke, Goto, Takahashi, Nasseh, Sather-Wagstaff, Christensen, Holm, Avramov, and Iyengar in [31], [27], [16], [40], [19] and [11]. Huneke
and R. Wiegand [32] established (HWC$_d$) over hypersurfaces (see Remark 4.2), but (HWC$_d$) is still open for Gorenstein domains, even if $M$ is assumed to be an ideal of the ring; see the article of Huneke, R. Wiegand, and Iyengar [30] or Celikbas [17].

At the other extreme, we know of no counterexample to the following general form of the conjecture:

\[(G\text{-}HWC_d) \quad \text{Huneke-Wiegand Conjecture (generalized, domain).} \]

Let $R$ be a local domain, and let $M$ be an $R$-module. If $M \otimes_R M^*$ is maximal Cohen-Macaulay (henceforth abbreviated “MCM”), then $M$ is free.

Any attempt to solve (G-HWC$_d$) is likely to involve knowing what properties (weaker than being free) one can conclude about $M$ from the assumption that $M \otimes_R M^*$ is MCM. For example, we might conjecture that $M$ is forced to be torsion-free:

\[(G\text{-}HWC_{tf}) \quad \text{Let } R \text{ be a local domain, and let } M \text{ be an } R\text{-module.} \]

If $M \otimes_R M^*$ is maximal Cohen-Macaulay, then $M$ is torsion-free.

For convenience we usually assume $M$ is torsion-free for our discussion here, as in (HWC$_d$). (Assuming that $M$ is torsion-free in (HWC$_d$) avoids the trivial case where $M$ is torsion, and hence $M^* = 0 = M \otimes_R M^*$.)

Some partial results concerning these conjectures appear in articles by Huneke, Iyengar, and Wiegand [30]; Celikbas [14], [17]; Goto, Takahashi, Taniguchi and Truong [28]; and Garcia-Sanchez and Leamer [26]. By a result from Celikbas and R. Wiegand [18], reproduced here as Proposition 2.8, the truth of (HWC$_d$) in the one-dimensional case would imply the general case. (On the other hand, the truth of a Huneke-Wiegand conjecture for Gorenstein quasi-fiber product rings—the focus of this article—does not seem to reduce to the one-dimensional case; see Remark 2.9). By Proposition 2.8, the truth of (HWC$_d$) also would imply (ARC) for Gorenstein domains of arbitrary dimension. It is not known whether or not (ARC) implies (HWC$_d$).

Concerning (ARC), there are several situations in which the vanishing of $\text{Ext}^i_R(M, M \oplus R)$ for a specific finite set of values of $i$ is enough to deduce that $M$ is free, or, perhaps, that $M$ has finite projective dimension; see for example, results of Huneke and Leuschke [31, Main Theorem]; Araya [1, Corollary 10]; and Goto and Takahashi [27, Theorem 1.5]. Our Proposition 3.30 and Theorems 3.25 and 3.31 are results along these lines. In addition, we consider a general version of (ARC) that involves two modules:

**Question 1.2.** For $M$ and $N$ finitely generated $R$-modules, can one find integers $s$ and $t$, with $1 \leq s \leq t$, such that the vanishing of $\text{Ext}^i_R(M, N \oplus R)$, for all $i$ with $s \leq i \leq t$, ensures that $M$ or $N$ has finite projective dimension?

The main body of this paper is an investigation of Question 1.2 and of these conjectures over a quasi-fiber product ring. (See Setting 2.1 for conventions and definitions.) Nasseh and Takahashi [42] introduced the notion
of a “local ring with quasi-decomposable maximal ideal” as an extension of
the notion of fiber product ring; we call it a “quasi-fiber product ring” here.

The class of quasi-fiber product rings includes, for instance, every regular
local ring of dimension \( d \geq 2 \) and every non-hypersurface Cohen-Macaulay
ring with minimal multiplicity and with infinite residue field, as well as every
two-dimensional non-Gorenstein normal domain with a rational singularity
([12, Examples 4.7 and 4.8]) and, of course, every fiber product ring. Re-
cently the study of fiber product rings has become an active research topic,
as is evident in articles by Nasseh, Sather-Wagstaff, Takahashi and Vande-
Bogart [40, 41, 42, 43, 46], and the current authors [25].

We briefly describe the contents of the paper. Section 2 gives the main
definitions and basic facts for the rest of this work. Section 3 concerns
the vanishing of \( \operatorname{Ext} \) over a quasi-fiber product ring. In Notation and Re-
marks [6] we define and discuss “AB rings” and “Ext-bounded rings”, in-
troduced in Huneke’s and Jorgensen’s article [29]; we prove, in Theorem 3.3
and Proposition 3.7 that a Gorenstein quasi-fiber product ring has both of
these properties. In Corollary 3.8 we verify (ARC) for quasi-fiber product
rings. Theorems 3.13 and 3.20 and their corollaries give implications of the
vanishing of finitely many \( \operatorname{Ext}^i_R(M, N) \) over a quasi-fiber product ring \( R \)
under an additional assumption of Tor-rigidity for \( N \).

The remaining theorems of Section 3 concern implications of the vanishing
of finitely many \( \operatorname{Ext}^i_R(M, M) \) over quasi-fiber product rings without the
additional assumption of Tor-rigidity. We show in Theorem 3.31 that quasi-
fiber product rings satisfy a sharper version of (ARC): For \( M \) a finitely
generated module over a quasi-fiber product ring, there is a positive integer
\( b \) such that, if \( \operatorname{Ext}^i_R(M, M \oplus R) = 0 \), for every \( i \) with \( 1 \leq i \leq b \), then \( M \)
is free. Moreover Corollary 3.32 states that, if \( M \) is a finitely generated
module over a fiber product ring and \( \operatorname{Ext}^i_R(M, M \oplus R) = 0 \), for every \( i \) such
that \( 1 \leq i \leq 6 \), then \( M \) is free. This improves Nasseh and Sather-Wagstaff’s
result that fiber product rings satisfy (ARC) [40].

In Section 4 we apply the results of Section 3 to obtain some positive re-
sults related to (HWC\(_d\)) and we consider a more general condition involving
two modules.

2. Setup and Background

This section gives basic definitions and properties that are used in later
sections.

Setting 2.1. Throughout this paper, \((R, \mathfrak{m}, k)\), or simply \((R, \mathfrak{m})\), denotes
a local ring with maximal ideal \( \mathfrak{m} \) and residue field \( k \). Local rings are
always assumed to be commutative and Noetherian, and modules are always
assumed to be finitely generated.

(i) \((R, \mathfrak{m}, k)\) is the fiber product ring \( S \times_k T \) of two local rings \( S \) and \( T \),
with the same residue field \( k \), if \( R \) is the subring of \( S \times T \) consisting
of pairs \((s, t)\) such that \( s \in S \), \( t \in T \) and \( \pi_S(s) = \pi_T(t) \), where \( \pi_S \)
and \( \pi_T \) denote reduction modulo the maximal ideals \( m_S \) and \( m_T \). We always assume fiber product rings are non-trivial; that is, neither \( S \) nor \( T \) is equal to \( k \).

(i) \((R, m, k)\) has decomposable maximal ideal if \( m = I \oplus J \), where \( I \) and \( J \) are nonzero ideals of \( R \).

(ii) \((R, m, k)\) is a quasi-fiber product ring if there exists an \( R \)-sequence \( \underline{x} := x_1, \ldots, x_n \), of length \( n \geq 0 \), such that \( R/(\underline{x}) \) is a non-trivial fiber product ring.

(ii') \((R, m, k)\) has quasi-decomposable maximal ideal if there exists an \( R \)-sequence \( \underline{x} := x_1, \ldots, x_n \), of length \( n \geq 0 \), such that \( m/(\underline{x}) \) is decomposable.

Ogoma [44, Lemma 3.1] observed the following:

Fact 2.2. A local ring \((R, m, k)\) has decomposable maximal ideal if and only if \( R \) can be realized as a non-trivial fiber product. In fact, if \( m = I \oplus J \), the map \( R \to S \times_k T \), given by \( r \mapsto (r + I, r + J) \) is an isomorphism, where \( S = R/I \) and \( T = R/J \). For the converse, if \( R = S \times_k T \), then \( m = m_S \oplus m_T \), where \( m_S \) and \( m_T \) are the maximal ideals of \( S \) and \( T \), respectively.

Similarly, items (ii) and (ii') are equivalent. We often say that \((R, m, k)\) is a quasi-fiber product ring with respect to the regular sequence \( \underline{x} \), or \( m \) is quasi-decomposable with respect to \( \underline{x} \). The case \( n = 0 \) is the case of a fiber product ring, equivalently, a local ring with decomposable maximal ideal. In this article all fiber product rings and quasi-fiber product rings are assumed to be non-trivial.

Examples of quasi-fiber rings abound. For example, every regular local ring of dimension \( d \geq 2 \) is a quasi-fiber ring. (If \( x_1, \ldots, x_d \) generate the maximal ideal, then \( R/(x_1x_2, x_3, x_4, \ldots, x_d) \) has decomposable maximal ideal.) Many interesting examples of quasi-fiber product rings can be found in the paper [42, §4] by Nasseh and Takahashi.

Fact 2.3. Let \((R, m, k)\) be a fiber product ring and let \( M \) be a finitely generated \( R \)-module. Then:

1. depth \( R \leq 1 \) ([37] or [25, Remark 1.9]).
2. If pd\( R M < \infty \), then pd\( R M \leq 1 \).

For (2), use (1) and Remark 2.4, the Auslander-Buchsbaum Formula:

Remark 2.4. Auslander-Buchsbaum Formula [38, A.5, Theorem, p. 310]

Let \( M \) be a nonzero module of finite projective dimension (pd) over a local ring \( R \). Then depth \( M + \text{pd}_R M = \text{depth} R \). Thus pd\( R M \leq \text{depth} R \).

Definitions and Remarks 2.5. Recall that a finitely generated module \( M \) over a local ring \( R \) has rank provided there is an integer \( r \) such that \( M_P \) is \( R \)-free of rank \( r \) for every \( P \in \text{Ass}(R) \). Equivalently, \( M \otimes_R K \) is free as a \( K \)-module, where \( K \) is the total quotient ring of \( R \), namely \( K = \{\text{non-zerodivisors of } R\}^{-1}R \). If \( R \) is an integral domain, \( M \) always has rank. It is probably better, for moving about from one ring to another, not
to assume that $R$ is a domain, but to invoke the weaker hypothesis that $M$ have rank. (Example \[3,18\] shows why some such hypothesis is needed.)

A hypersurface ring is a local ring $(R, \mathfrak{m})$ whose $\mathfrak{m}$-adic completion $\hat{R}$ has the form $\hat{R} = S/fS$, where $(S, \mathfrak{m}_S)$ is a complete regular local ring and $f \in \mathfrak{m}_S$. More generally a local ring $R$ with maximal ideal $\mathfrak{m}$ is a complete intersection if the $\mathfrak{m}$-adic completion $\hat{R}$ has the form $S/(f)$, where $f$ is a regular sequence and $S$ is a complete regular local ring. (By Cohen’s Structure Theorem, the ring $S$ is a ring of formal power series over a field or over a discrete valuation ring.)

An $R$-module is torsion-free provided the natural map $M \to M \otimes_R K$ is injective. Equivalently, every non-zerodivisor in $R$ is a non-zerodivisor on $M$. This leads to the following version of (HWC$_d$):

**Definition 2.6.** Let $R$ be a local ring (not necessarily an integral domain).

(HWC) **Huneke-Wiegand Conjecture.** Assume $R$ is Gorenstein, and $M$ is a torsion-free $R$-module with rank. If $M \otimes_R M^*$ is maximal Cohen-Macaulay, then $M$ is free.

Following \[16\], we consider conditions, labeled (AR) and (HW) here, on a local ring $(R, \mathfrak{m})$:

**Definition 2.7.** Let $R$ be a local ring.

(AR) **Artin-Reiten Condition.** For every finitely generated torsion-free $R$-module $M$,

$$\Ext^i_R(M, M \oplus R) = 0 \text{ for every } i \geq 1 \implies M \text{ is free.}$$

(HW) **Huneke-Wiegand Condition.** For every finitely generated torsion-free module $M$ with rank,

$$M \otimes_R M^* \text{ is MCM } \implies M \text{ is free.}$$

Thus (ARC) says that every local ring $(R, \mathfrak{m})$ satisfies (AR), and (HWC) says that every local Gorenstein ring satisfies (HW).

Proposition \[2.8\] gives the connection between the Huneke-Wiegand Conjecture and the commutative version of the Auslander-Reiten Conjecture. The proof of Proposition \[2.8\] in \[18\] is more explicit than in \[1\] and \[16\].

**Proposition 2.8.** \[15\] Proposition 8.6. Let $R$ be a local Gorenstein ring. Consider the following statements regarding the conditions of Definition \[2.7\]:

(i) $R$ satisfies (HW).

(ii) $R_p$ satisfies (HW), for every prime ideal $p$ with height $p \leq 1$.

(iii) $R$ satisfies (AR).

(iv) $R_p$ satisfies (AR), for every prime ideal $p$ with height $p \leq 1$.

Then (ii) $\implies$ (i) $\implies$ (iii) $\iff$ (iv).

The main ideas in the proof of (ii) $\implies$ (i) are
(a) A module $M$ is free $\iff$ the natural map $M \otimes_R M^* \to \text{Hom}_R(M, M)$, taking $x \otimes f$ to the homomorphism $y \mapsto (f(x))y$, for $x, y \in M$ and $f \in M^*$, is an isomorphism; and

(b) A map from a reflexive module to a torsion-free module is an isomorphism if and only if it is an isomorphism at each height-one prime ideal.

Both (a) and (b) were used by Auslander in his proof of [5, Proposition 3.3]. The implication (iv) $\implies$ (iii) is due to Araya [1]. Two obvious questions: Does (i) $\implies$ (ii)? Does (iii) $\implies$ (iv)?

Remark 2.9. By Proposition 2.8, the condition (HW) being satisfied for one-dimensional local Gorenstein rings would ensure that it holds for every local Gorenstein ring. Attempts to prove (HW) for Gorenstein quasi-fiber rings, however, do not immediately reduce to the one-dimensional case, since localizations of two-dimensional quasi-fiber product rings at height-one primes are not necessarily quasi-fiber rings. For example, a two-dimensional regular local ring is a quasi-fiber ring, but its localizations at height-one primes are discrete valuation rings, which are not quasi-fiber rings.

Fact 2.10. Let $(R, m)$ be a fiber product ring, say, $R = S \times_k T$. The following statements are equivalent:

(i) $R$ is Gorenstein.

(ii) $R$ is a 1-dimensional hypersurface, as in Remark 2.5

(iii) $S$ and $T$ are discrete valuation rings.

The implications (i) $\iff$ (ii) $\implies$ (iii) constitute the Main Theorem of [33], while the implication (iii) $\implies$ (i) is Part (3) of [24, Proposition 2.2].

Fact 2.11. Let $R$ be a Gorenstein quasi-fiber product ring of dimension 1. Then $R$ is a Gorenstein fiber product ring.

To see this, let $x = x_1, \ldots, x_n$ be an $R$-sequence such that $R/(x)$ is a fiber product ring. By Fact 2.10, the dimension of $R/(x)$ is 1. Thus $n = 0$ and $R$ is a Gorenstein fiber product ring.

Combining Fact 2.10 with Fact 2.3, we have:

Remark 2.12. Let $R$ be a quasi-fiber product ring, and let $n$ be the length of a regular sequence $x$ such that $R/(x)$ is a fiber product ring. Then $n$ is equal to either depth $R - 1$ or depth $R$. Moreover, if $R$ is Gorenstein, then $n = \text{depth } R - 1$.

Notation 2.13. For an $R$-module $M$, let $\Omega^i_R M$ denote the $i$th syzygy of $M$ with respect to a minimal $R$-free resolution. We often write $\Omega_R M$ for $\Omega^1_R M$.

Lemma 2.14. [42, Lemma 5.1] Let $R$ be a local ring and $M$ an $R$-module. Let $x = x_1, \ldots, x_n$ be an $R$-sequence. Then $x$ is a regular sequence on $\Omega^n_R M$. 

Definitions and Remarks 2.15. (i) The Auslander transpose, written as $D^1 M$ or $D^1_1 M$, of a finitely generated module $M$ over a local ring $R$ is defined to be the cokernel of the map $F^*_0 \to F^*_1$, where $F_1 \to F_0 \to M \to 0$ is a minimal resolution of $M$, with the $F_i$ free $R$-modules. Thus one has the exact sequence

$$0 \to M^* \to F^*_0 \to F^*_1 \to D^1 M \to 0. \tag{2.15.0}$$

(ii) More generally, for $n \geq 1$ and a minimal free resolution $F$ of $M$ over $R$, $F : \cdots F_n \to \cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$ define the $n$th Auslander transpose $D^n M$ by $D^n M := \text{coker}(F^*_n \to F^*_n-1)$.

(iii) In [35, p.4462], Jorgensen uses the notation “$D^0 M$” to mean the same as our “$D^1 M$”, and “$D^n$” to mean the same as our “$D^{n+1}$”. Note that, for every $i$ with $0 \leq i \leq n$, $D_n M = D_{n-i} \Omega^i_R M$ [35, p. 4462].

After one adjusts the notation as in Definitions and Remarks 2.15(iii) above, [35] Proposition 3.1(1) states:

Proposition 2.16. Let $R$ be a commutative Noetherian ring, $M$ be a finitely generated $R$-module, and $n \geq 1$. If $\text{Ext}^i_R(M, M \oplus R) = 0$, for every $i$ with $1 \leq i \leq n$, then:

1. $\text{Tor}^R_i(D_{n+1} M, M) = 0$, for every $i$ with $1 \leq i \leq n$.
2. The following sequence is exact

$$0 \to \text{Tor}^R_{n+2}(D_{n+1} M, M) \to \text{Hom}_R(M, R) \otimes_R M$$

$$\to \text{Hom}_R(M, M) \to \text{Tor}^R_{n+1}(D_{n+1} M, M) \to 0,$$

where the middle homomorphism $(\text{Hom}_R(M, R) \otimes_R M \to \text{Hom}_R(M, M))$ is the natural one.

Facts 2.17. Here are some well-known facts concerning reflexive, maximal Cohen-Macaulay (MCM), and torsion-free $R$-modules. Let $R$ be a local ring and $M$ a non-zero $R$-module.

(i) If $R$ is Gorenstein and $M$ is MCM, then $M$ is reflexive, and the dual module $M^*$ is also MCM. (These follow from the fact that $R$ is its own canonical module. See [13] Theorems 3.3.7 and 3.3.10(d)).

(ii) If $R$ is 1-dimensional and Cohen-Macaulay, then $M$ is MCM if and only if $M$ is torsion-free.

(iii) If $R$ is Cohen-Macaulay and $n \geq \dim R$, then $\Omega^n_R M$ is MCM, by the Depth Lemma [38, Lemma A.4].

(iv) Suppose $R$ is Cohen-Macaulay and $\dim R \leq 2$. If $M$ is reflexive, then $M$ is MCM, since, by Equation (2.15.0), $M = M^{**}$ is the second syzygy of $D(M^*)$.

Since we have not found a proof of (ii) in the literature, we include one here: Since $R$ is CM, there is a non-zerodivisor $f \in m$. If $M$ is torsion-free, then $f$ is a non-zerodivisor on $M$, and hence $\text{depth } M \geq 1$, that is, $M$ is
MCM. Conversely, suppose $M$ is MCM, and let $r$ be a zero-divisor on $M$. Then $r \in p$ for some $p \in \text{Ass } M$. Now $p \neq m$ since $M$ is MCM, and hence $p$ is a minimal prime ideal of $R$. Therefore $r$ is a zero-divisor of $R$; this shows that $M$ is torsion-free.

**Remark 2.18.** The truth of (HWC), the local ring version of the Huneke-Wiegand Conjecture in Definition 2.6, would imply the truth of (HWC$_d$), the integral domain version in Definition 1.1. Assume (HWC). By Proposition 2.8 (HWC$_d$) reduces to the one-dimensional case. Fact 2.17 parts (i) and (iv), implies that the non-zero reflexive modules over a one-dimensional Gorenstein local ring are exactly the MCM modules. Thus a Gorenstein local domain satisfying (HW) of Definition 2.7 also satisfies (HW$_d$), the domain form of the condition:

(HW$_d$) If $M$ is a finitely generated torsion-free $R$-module such that $M \otimes_R M^*$ is reflexive, then $M$ is a free module.

The original conjecture (HWC$_d$) in Definition 1.1 is that every Gorenstein local domain satisfies (HW$_d$).

3. Vanishing of Ext and the Auslander-Reiten Conjecture

The results in this section can be compared with those in Section 6 of [42]. We begin by recalling the Auslander Condition (AC) and the Uniform Auslander Condition (UAC) on a ring $R$:

(AC) For each $R$-module $M$, there is a non-negative integer $b = b_M$ such that, for every $R$-module $N$, one has

\[
\text{Ext}^i_R(M, N) = 0 \quad \forall i \gg 0 \quad \implies \quad \text{Ext}^i_R(M, N) = 0, \quad \forall i \geq b. \quad (3.0.0)
\]

(UAC) There is an integer $b \geq 0$ such that (3.0.0) holds for every pair $M, N$ of $R$-modules.

**Notation and Remarks 3.1.** (i) A number $b$ with the property required in (UAC) is called a uniform Auslander bound.

(ii) The smallest number $b$ with this property is called the Ext-index of $R$ [29].

(iii) To our knowledge, it is unknown whether every local ring satisfying (AC) actually satisfies the stronger condition (UAC).

(iv) A Gorenstein local ring satisfying (UAC) is called an AB ring [29]. For a local AB ring, the Ext-index is known to be equal to $\dim R$ [29, Proposition 3.1].

(v) Modules $M$ and $N$ over an AB ring $R$ satisfy the following symmetry [29, Theorem 4.1]:

\[
\text{Ext}^i_R(M, N) = 0, \quad \forall i \gg 0 \iff \text{Ext}^i_R(N, M) = 0, \quad \forall i \gg 0. \quad (3.1.0)
\]

(vi) Auslander conjectured [3, p. 795] that finite-dimensional modules over a finite-dimensional $k$-algebra satisfy (AC). This conjecture was disproved by Jorgensen and Šega [36]. Their counterexample is a finite-dimen-
sional commutative Gorenstein $k$-algebra, where $k$ can be taken to be any field that is not algebraic over a finite field.

**Lemma 3.2.** Let $R$ be a quasi-fiber product ring with respect to an $R$-sequence of length $n$. If $M$ and $N$ are $R$-modules such that $\text{pd}_R M < \infty$ or $\text{id}_R N < \infty$, then $\text{Ext}^i_R(M, N) = 0$, for every $i > n + 1$.

**Proof.** By Remark 2.12, $\text{depth}_R R < \infty$ implies $\text{depth}_R R \leq n + 1$. The Auslander-Buchsbaum Formula (Remark 2.4) and Bass Formula [13, Theorem 3.1.17]) show that $\text{pd}_R M \leq \text{depth}_R R \leq n + 1$ or $\text{id}_R N = \text{depth}_R R \leq n + 1$. In either case, $\text{Ext}^i_R(M, N) = 0$, for every $i$ with $i > n + 1$. □

**Theorem 3.3.** If $R$ is a quasi-fiber product ring with respect to an $R$-sequence of length $n$, then the Ext-index of $R$ is at most $n + 2$. In particular, every quasi-fiber product ring satisfies (UAC). If, further, $R$ is Gorenstein, then $R$ is an AB ring.

**Proof.** Assume that $\text{Ext}^i_R(M, N) = 0$ for all $i \gg 0$. Then [42, Corollary 6.8] asserts that $\text{pd}_R M < \infty$ or $\text{id}_R N < \infty$. By Lemma 3.2, $\text{Ext}^i_R(M, N) = 0$, for every $i$ with $i > n + 1$. Therefore $R$ satisfies (UAC).

The last statement is clear by definition; an AB ring is a Gorenstein local ring satisfying (UAC). □

**Definition 3.4.** Let $M$ and $N$ be $R$-modules, and let $g$ and $m$ be positive integers. We say that $\text{Ext}_R(M, N)$ has a *gap of length* $g$ with *lower bound* $m$ if

$$\text{Ext}^i_R(M, N) \neq 0,$$

for $i = m - 1$ and for $i = m + g$; and

$$\text{Ext}^i_R(M, N) = 0,$$

whenever $m \leq i \leq m + g - 1$.

Then $(m, g)$ is called a *gap pair* for $\text{Ext}_R(M, N)$. Set

$$\text{Ext} \text{- gap}_R(M, N) := \sup \{g \mid \text{Ext}(M, N) \text{ has a gap of length } g\};$$

and

$$\text{Ext} \text{- gap}(R) := \sup \{\text{Ext} \text{- gap}_R(M, N) \mid M \text{ and } N \text{ are } R\text{-modules}\}.$$

**Example 3.5.** In Example 3.18 where $k$ is a field, $R$ is the fiber product ring $R = k[[x,y]]/(xy)$, and $M = R/(y)$, we show $\text{Ext}^i_R(M, M) = 0$ for every odd $i > 0$ and $\text{Ext}^i_R(M, M) = 0$, for every even $i > 0$. Thus every odd positive integer $m$ is a lower bound for a gap of length 1; the set of gap pairs for $\text{Ext}_R(M, M)$ is $\{(m, 1) \mid m \text{ is an odd integer} \}$.

**Remarks 3.6.** Let $R$ be a quasi-fiber product ring, let $M$ and $N$ be finitely generated $R$-modules, and let $n$ be the length of an $R$-sequence $x$ such that $R/(x)$ is a fiber product ring. With this setting, Nasseh and Takahashi give these interesting and useful results related to “Tor-gaps”, Question 1.2 and Ext-gaps in [42]:

(1) [42, Corollary 6.5] If there exists an integer $t$ with $t \geq \max\{5, n + 1\}$ such that $\text{Tor}_t^R(M, N) = 0$, for every $i$ with $t + n \leq i \leq t + n + \text{depth} R$, then $\text{pd}_R M < \infty$ or $\text{pd}_R N < \infty$.

(2) [42, Corollary 6.6] Assume $R$ is a $d$-dimensional Cohen-Macaulay ring, and set $s := d - \text{depth} M$. If there exists an integer $t$ such that $t \geq 5$ and
Let \( \Ext^i_R(M, N) = 0 \), for every \( i \) with \( t + s \leq i \leq t + s + d \), then \( \pd_R M < \infty \) or \( \id_R N < \infty \).

(2') (Restating (2)) If \( R \) is a \( d \)-dimensional Cohen-Macaulay ring such that both \( \pd_R M \) and \( \id_R N \) are infinite, if \( s := d - \depth M \), and if \( t \geq 5 \), then there exists an integer \( j \) with \( \Ext^j_R(M, N) \neq 0 \) and \( t + s \leq j \leq t + s + d \).

In \cite{29}, a local ring is called Ext-bounded if its Ext-gap is finite. It seems to be unknown whether every AB ring is Ext-bounded \cite{29} §6, Question 4.

Proposition \( \ref{p3.7} \) shows that quasi-fiber product rings would not be a good place to look for a counterexample. If \( \pd_R M < \infty \) or \( \id_R N < \infty \), the gap is at most the dimension. For convenience we give bounds on the size of the gaps associated with various values of the lower bound \( m \) if \( \pd_R M \) and \( \id_R N \) are both infinite. That is, we give restrictions on the possible gap pairs having various values for \( m \).

**Proposition 3.7.** Let \( R \) be a Cohen-Macaulay quasi-fiber product ring of dimension \( d \), let \( m \) and \( g \) be positive integers, and let \( M \) and \( N \) be finitely generated \( R \)-modules such that \( \Ext_R(M, N) \) has a gap of length \( g \) with lower bound \( m \). Set \( s := d - \depth M \). Then:

1. \( g \leq 2d + 4 \). Thus the Ext-gap of \( R \) is at most \( 2d + 4 \), and hence \( R \) is Ext-bounded.
2. If \( \pd_R M \) or \( \id_R N \) is finite, then \( g \leq d \) and \( \Ext-gap_R(M, N) \leq d \).
3. If \( m \geq s + 5 \), then \( g \leq d \).
4. If \( s \leq m \leq s + 4 \), then \( g \leq d + 5 - (m - s) \). (Thus, for example, \( m = s + 4 \implies g \leq d + 1 \), and \( m = s \implies g \leq d + 5 \).)
5. If \( m \geq 5 \), then \( g \leq s + d \), so \( g \leq 2d \).
6. If \( 0 < m \leq 4 \), then \( g \leq s + d + (5 - m) \). so \( g \leq 2d + (5 - m) \); e.g. \( m = 1 \implies g \leq 2d + 4 \).
7. (i) If \( s = 0 \) (that is, \( \dim R = \depth M \)) and \( m \geq 5 \), then \( g \leq d \).
   (ii) If \( s = 0 \) and \( 0 < m < 5 \), then \( g \leq d + (5 - m) \).
8. If \( d = 0 \), then \( m < 5 \).

**Proof.** By Definition \( \ref{d3.4} \) we have, for every \( i \) with \( m \leq i \leq m + g - 1 \),

\[
\Ext^i_R(M, N) = 0 \quad \text{and} \quad \Ext^{m+g}_R(M, N) \neq 0. \quad \tag{3.7a}
\]

Let \( n \) be the length of a regular sequence \( \underline{x} \) such that \( R/(\underline{x}) \) is a fiber product ring. Remark \( \ref{r2.12} \) and the fact that \( R \) is Cohen-Macaulay yield

\[
d = \depth R \in \{n, n + 1\}. \quad \tag{3.7b}
\]

Part (1) follows from parts (2), (5) and (6).

For part (2), Lemma \( \ref{l3.2} \) implies that \( \Ext^i_R(M, N) = 0 \) for every \( i > n + 1 \), and hence \( m + g \leq n + 1 \); that is, \( g \leq n - (m - 1) \leq n \). Since \( n \leq d \) by \( \ref{p3.7b} \), we have \( g \leq d \). This proves item (2).

For the remainder of the proof, assume \( \pd_R M = \infty \) and \( \id_R N = \infty \). Then, by Remark \( \ref{r3.6} \) (2'), for every \( t \geq 5 \), there is an integer \( j \) satisfying

\[
t + s \leq j \leq t + s + d \quad \text{and} \quad \Ext^j_R(M, N) \neq 0. \quad \tag{3.7c}
\]
To prove (3), assume that \( m \geq 5 + s \). Let \( t := m - s \geq 5 \). By (3.7.c), there exists \( j \) with
\[
m = t + s \leq j \leq t + s + d = m + d \quad \text{and} \quad \text{Ext}_R^j(M, N) \neq 0.
\]
By (3.7.a), \( g - 1 < d \). Thus \( g \leq d \).

To prove (4), assume that \( m = a + s \) for \( 0 \leq a \leq 4 \). Let \( t := 5 \). By (3.7.c), there exists \( j \) with
\[
a + s = m < 5 + s \leq j \leq 5 + s + d = (5 - a) + (a + s) + d = m + (5 - a) + d
\]
and \( \text{Ext}_R^j(M, N) \neq 0 \). By (3.7.a), \( g - 1 < (5 - a) + d \). Thus \( g \leq d + 5 - a \).

For (5), let \( t := m \). By (3.7.c), there exists a positive integer \( j \) with
\[
m + s \leq j \leq m + s + d \quad \text{and} \quad \text{Ext}_R^j(M, N) \neq 0.
\]
Definition 3.4 implies that \( m + g - 1 < m + s + d \). Now \( g \leq s + d \leq 2d \), since \( 0 \leq s \leq d \).

For (6), let \( t := 5 \) and \( a := 5 - m \). By (3.7.c), there exists \( j \) with
\[
a + m + s = 5 + s \leq j \leq 5 + s + d = a + m + s + d \quad \text{and} \quad \text{Ext}_R^j(M, N) \neq 0.
\]
By Definition 3.4, \( j > m + g - 1 \). Therefore \( m + g - 1 < a + m + s + d \), and hence \( g \leq a + s + d \leq a + 2d \).

Observe that \( g \leq 2d + 4 \) by parts (2), (5) and (6). Thus (1) holds.

For (7)(i), use (5), \( g \leq s + d = d \), and, for (7)(ii), use (6),
\[
g \leq (5 - m) + s + d = (5 - m) + d.
\]

For (8), if \( d = 0 \), then also \( s = 0 \). If \( m \geq 5 \), then, by (3.7.a), there exists \( j \) with
\[
m + s \leq j \leq m + s + d \quad \text{and} \quad \text{Ext}_R^j(M, N) \neq 0.
\]
That is, \( m \leq j \leq m \implies j = m \implies \text{Ext}_R^m(M, N) \neq 0 \). This contradicts (3.7.a). Thus \( m < 5 \). \( \square \)

The Auslander-Reiten Conjecture (ARC) is the case \( n = 1 \) of the Generalized Auslander-Reiten Conjecture (GARC):

(GARC) If \( \text{Ext}_R^i(M, M \oplus R) = 0 \) for all \( i \geq n \), then \( \text{pd}_R M < n \).

Corollary 3.8. Every quasi-fiber product ring satisfies (GARC), and so also (ARC).

Proof. Divers [23] introduced the notion of “finitistic extension degree” \( \text{fed}(A) \), for a left Noetherian ring \( A \):
\[
\text{fed}(A) := \sup\{\sup\{i \mid \text{Ext}_A^i(M, M) \neq 0\}\},
\]
where the outside sup is over all finitely generated left \( A \)-modules \( M \). Divers proved [23, Corollary 2.12] that (GARC) holds for \( A \) if \( \text{fed}(A) < \infty \). By Theorem 3.3, every quasi-fiber product ring \( R \) satisfies (UAC), which certainly implies \( \text{fed}(R) \) is finite. Thus (GARC) holds for quasi-fiber product rings. \( \square \)
It also follows from Nasseh and Sather-Wagstaff’s theorem [10, Theorem 4.5] that every quasi-fiber product ring $R$ satisfies (ARC) – the case $n = 1$ of (GARC). They show that fiber product rings (i.e., with decomposable maximal ideal) satisfy (ARC). By Celikbas’ theorem [16, Theorem 4.5 (1)], the (AR) property lifts modulo a regular sequence.

**Discussion 3.9. Auslander sequence and Tor-rigidity.** We introduce an important tool concerning the Auslander transpose (see Definition 2.15).

(i) From [6, Theorem 2.8 (b)] or [35, (1.1.1)], we have, for each $i \geq 0$, the following exact sequence:

\[ \text{Tor}_2^R(D \Omega_R^i M, N) \to \text{Ext}_R^i(M, R) \otimes_R N \to \text{Ext}_R^i(M, N) \to \text{Tor}_1^R(D \Omega_R^i M, N) \to 0. \]

The sequence (3.9.1) is known as the Auslander sequence.

(ii) An $R$-module $M$ is said to be Tor-rigid provided that the following holds for every $R$-module $N$ and every $n \geq 1$ (see [5]):

\[ \text{Tor}_n^R(M, N) = 0 \Rightarrow \text{Tor}_{n+1}^R(M, N) = 0. \]

(iii) Let $M$ be an $R$-module and let $h : M \to M^{**}$ be the canonical map. Recall that $M$ is torsionless provided $h$ is injective, and reflexive if $h$ is bijective. By mapping a free module onto $M^*$ and dualizing, we see that $M^{**}$ embeds in a free module. It follows that torsionless modules are torsion-free. From [13, Exercise 1.4.21] we obtain an exact sequence

\[ 0 \to \text{Ext}_R^1(D M, R) \to M \xrightarrow{h} M^{**} \to \text{Ext}_R^2(D M, R) \to 0. \]

Thus $M$ is torsionless if and only if $\text{Ext}_R^1(D M, R) = 0$, and reflexive if and only if $\text{Ext}_R^1(D M, R)$ and $\text{Ext}_R^2(D M, R)$ are both zero.

We use Auslander’s Theorem 3.10 and Lemma 3.11 due to Jothilingam and Duraivel:

**Theorem 3.10.** [5, Theorem 1.2] Let $R$ be a local ring, and let $M$ and $N$ be nonzero $R$-modules such that $\text{pd} M < \infty$. Let $q$ be the largest integer such that $\text{Tor}_q^R(M, N) \neq 0$. If $\text{depth}(\text{Tor}_q^R(M, N)) \leq 1$ or $q = 0$, then $\text{depth} N = \text{depth}(\text{Tor}_q^R(M, N)) + \text{pd} M - q$.

**Lemma 3.11.** [34, Lemma, p. 2763] Let $R$ be a local ring, and let $M$ and $N$ be $R$-modules such that $N \neq 0$ and

\[ \text{Ext}_R^i(M, N) = 0, \text{ for } i = 1, \ldots, \max\{1, \text{depth}_R N - 2\}. \]

Then $\text{depth} N \leq \text{depth}(\text{Hom}_R(M, N))$.

Our next result, Theorem 3.13, can be compared to the following result, due to Jothilingam and Duraivel:

**Theorem 3.12.** [34, Theorem 1]: Let $M$ be a module over a regular local ring $R$. If

\[ \text{Ext}_R^i(M, N) = 0, \text{ for } i = 1, \ldots, \max\{1, \text{depth}_R N - 2\}, \]
for some nonzero $R$-module $N$, then $M^*$ is free.

**Theorem 3.13.** Let $R$ be a quasi-fiber product ring. Let $M$ and $N$ be nonzero $R$-modules. If $N$ is Tor-rigid and

$$\text{Ext}_R^i(M, N) = 0 \text{ for } i = 1, \ldots, \max\{1, \text{depth}_R N - 2\},$$

then $M^*$ is free or $\text{pd}_R N < \infty$.

**Proof.** Since $\text{Ext}_R^1(M, N) = 0$, the Auslander sequence (3.9.1) shows that $\text{Tor}_R^1(D \Omega_R^1 M, N) = 0$.

The Tor-rigidity of $N$ implies that $\text{Tor}_R^i(D \Omega_R^1 M, N) = 0$ for all $i \geq 1$. (3.13.1)

The Auslander sequence (3.9.1) implies that $\text{Ext}_R^1(M, R) \otimes_R N = 0$. Since $N$ is nonzero, Nakayama's lemma implies that $\text{Ext}_R^1(M, R) = 0$.

Using the minimal free resolution

$$F_2 \to F_1 \to F_0 \to M \to 0$$

to compute $\text{Ext}_R(M, R)$, we have

$$0 = \text{Ext}_R^1(M, R) = \frac{\ker(F_1^* \to F_2^*)}{\text{im}(F_0^* \to F_1^*)}.$$ 

Therefore

$$D M = \text{coker}(F_0^* \to F_1^*) = \frac{F_1^*}{\text{im}(F_0^* \to F_1^*)} = \frac{F_1^*}{\ker(F_1^* \to F_2^*)},$$

and hence the map $F_1^* \to F_2^*$ factors as $F_1^* \to D M \to F_2^*$. This shows that $D M \cong \text{im}(F_1^* \to F_2^*)$. Since $D \Omega_R^1 M = \text{coker}(F_1^* \to F_2^*)$, we get a short exact sequence

$$0 \to D M \to F_2^* \to D \Omega_R^1 M \to 0.$$ (3.13.2)

Together (3.13.1) and (3.13.2) imply that

$$\text{Tor}_R^j(D M, N) = 0 \text{ for all } j \geq 1.$$ (3.13.3)

Using (3.13.3) and (2.15.0), we see that

$$\text{Tor}_R^j(M^*, N) = 0 \text{ for all } j \geq 1.$$ (3.13.4)

By (3.13.3) and Remark 3.6(1),

$$\text{pd}_R M^* < \infty \text{ or } \text{pd}_R N < \infty.$$

If $\text{pd}_R N < \infty$, we are done.

To complete the proof of Theorem 3.13, assume $\text{pd}_R M^* < \infty$. We show that $M^*$ is free. By (3.13.3) and the Auslander sequence (3.9.1) in the case $i = 0$, we have the isomorphism

$$M^* \otimes_R N = \text{Hom}_R(M, R) \otimes_R N \cong \text{Hom}_R(M, N).$$ (3.13.5)

Since $\text{pd}_R M^* < \infty$, Theorem 3.10 with $q = 0$ implies

$$\text{depth}_R N = \text{depth}_R(M^* \otimes_R N) + \text{pd}_R M^*.$$ (3.13.6)
By (3.13.5) and Lemma 3.11,
\[ \text{depth}_R(M^* \otimes R N) = \text{depth}_R(\text{Hom}_R(M, N)) \geq \text{depth}_R N. \]  \hspace{1cm} (3.13.7)

Putting together (3.13.7) and (3.13.6) yields that \( \text{pd}_R M^* = 0 \); that is, \( M^* \) is free, as desired.

Corollary 3.14. Let \( R \) be a quasi-fiber product ring with \( \dim R \leq 3 \). Let \( M \) and \( N \) be nonzero \( R \)-modules such that \( N \) is Tor-rigid, \( \text{Ext}^1_R(M, N) = 0 \), and \( \text{pd}_R N = \infty \). Then \( M^* \) is free.

Proof. Since \( \text{depth}_R N \leq \text{depth} R \leq 3 \), Theorem 3.13 applies. \( \square \)

The next result provides an answer for Question 1.2.

Corollary 3.15. Let \( R \) be a quasi-fiber product ring. Let \( M \) and \( N \) be nonzero \( R \)-modules, with \( \text{pd}_R N = \infty \). Suppose that
(i) \( N \) is Tor-rigid.
(ii) \( \text{Ext}^i_R(M, N) = 0 \), for all \( i \) with \( 1 \leq i \leq \max\{1, \text{depth}_R N - 2\} \).
(iii) \( \text{Ext}^j_R(M, R) = 0 \), for every \( j \) with \( 1 \leq j \leq \text{depth} R \).

Then \( M \) is free.

Proof. By Theorem 3.13, \( M^* \) is free. By [22, Lemma 3.3], \( M^* \) free and \( \text{Ext}^j_R(M, R) = 0 \), for all \( j = 1, \ldots, \text{depth} R \), together imply that \( M \) is free. \( \square \)

Corollary 3.16. Let \( R \) be a Cohen-Macaulay fiber product ring. Let \( M \) and \( N \) be nonzero \( R \)-modules. If \( N \) is Tor-rigid and \( \text{Ext}^i_R(M, N) = 0 \), then \( M \) is free or \( \text{pd}_R N \leq 1 \).

Proof. If \( \text{pd}_R N < \infty \), then \( \text{pd}_R N \leq 1 \), by Fact 2.3(2). Thus we assume \( \text{pd}_R N = \infty \). By Fact 2.3(1), \( \text{depth} R \leq 1 \), and so \( \dim R \leq 1 \). Since \( \text{Ext}^1_R(M, N) = 0 \), and \( N \) is Tor-rigid, the Auslander sequence (3.9.1) implies
\[ \text{Tor}^i_R(D\Omega^{-1}_R M, N) = 0, \] for all \( i \geq 1 \).

Again from the Auslander sequence we deduce \( \text{Ext}^1_R(M, R) \otimes R N = 0 \). Since \( N \) is nonzero, Nakayama’s Lemma implies \( \text{Ext}^1_R(M, R) = 0 \). Now apply Corollary 3.15. \( \square \)

Corollary 3.17. Let \( R \) be a Cohen-Macaulay fiber product ring. Let \( M \) and \( N \) be nonzero \( R \)-modules. If \( N \) is Tor-rigid and \( \text{Ext}^i_R(M, N) = 0 \) for some \( i > 1 \), then \( \text{pd}_R M \leq 1 \) or \( \text{pd}_R N \leq 1 \).

Proof. Assume that \( \text{pd}_R N > 1 \). Since \( \text{Ext}^i_R(\Omega^{-1}_R M, N) \cong \text{Ext}^i_R(M, N) = 0 \), Corollary 3.16 implies that \( \Omega^{-1}_R M \) is free. Therefore \( \text{pd}_R M < \infty \), and now Fact 2.3 implies that \( \text{pd}_R M \leq 1 \). \( \square \)

Corollary 3.17 is somewhat relevant to a question raised by Jorgensen [35, Question 2.7]: If \( \text{Ext}^i_R(M, M) = 0 \), over a complete intersection \( R \), must \( M \) have projective dimension at most \( n - 1 \)?
Example 3.18. [35, Remark 2.6], [9, Example 4.3] Let $k$ be a field, and let $R = k[[x, y]]/(xy)$. Let $M = R/(y) \cong R$. It is easy to compute $\text{Ext}_R(M, M)$ and $\text{Tor}_R(M, M)$ using the periodic free resolution

$$
\cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} 0
$$

of $M$. Then $\text{Ext}_R^1(M, M) = 0$, for every odd $i > 0$, and $\text{Ext}_R^i(M, M) \cong k$, for every even $i > 0$. Therefore $\text{pd}_R M = \infty$. Also, $\text{Tor}_R^i(M, M)$ alternates between 0 and $k$, but with $k$ at the odd indices. Thus $M$ is not Tor-rigid. This example justifies the rigidity hypothesis in Corollary 3.17.

In addition, this example shows why one should insist that $R$ be a domain (or at least that $M$ have rank) in (HWC$_d$). One checks that $M^* \cong M$, and hence $M \otimes_R M^* \cong (R/(y)) \otimes_R (R/(y)) \cong R/(y) \cong M$, which is torsion-free, hence reflexive (as $R$ is one-dimensional and Gorenstein — see Facts 2.17). But of course $M$ is not free.

In the proofs of the next theorems we use the change-of-rings isomorphisms (i) and (iii) of Lemma 2 in Chapter 18 of Matsumura’s book [39]. We record those formulas here, adjusting the notation to suit our situation:

Remark 3.19. Let $R$ be a local ring, and let $N$ and $Z$ be $R$-modules. Let $\underline{x} = x_1, \ldots, x_n$ be an $R$-sequence that is also an $N$-sequence, and assume that $(\underline{x})Z = 0$. Put $\overline{R} = R/(\underline{x})$ and $\overline{N} = N/(\underline{x})N$. The following formulas hold for every non-negative integer $i$:

(a) $\text{Ext}_R^{i+n}(Z, N) \cong \text{Ext}_{\overline{R}}^i(Z, \overline{N})$.

(b) $\text{Tor}_R^i(Z, N) \cong \text{Tor}_{\overline{R}}^i(Z, \overline{N})$.

Theorem 3.20. Let $R$ be a quasi-fiber product ring with respect to the regular sequence $\underline{x} = x_1, \ldots, x_n$. Let $M$ and $N$ be nonzero $R$-modules. If $N$ is a Tor-rigid maximal Cohen-Macaulay module and, for some $t > n$,

$$
\text{Ext}_R^i(M, N) = 0 \text{ whenever } t \leq i \leq t + n,
$$

then $\text{pd}_R M < \infty$ or $N$ is free.

Proof. First observe that $R$ is Cohen-Macaulay, since it admits a Tor-rigid maximal Cohen-Macaulay module ([5, Theorem 4.3] or [48, Corollary 4.7]).

If $n = 0$, $R$ is a fiber product ring. In this case, $\text{pd}_R N \leq 1$, by Corollaries 3.16 and 3.17. Since $N$ is maximal Cohen-Macaulay, $N$ is actually free, by the Auslander Buchsbaum Formula (Remark 2.4).

Assume from now on that $n \geq 1$. Putting $X := \Omega^n_R(M)$, noting that $t - n \geq 1$, and shifting merrily, we obtain

$$
\text{Ext}_R^i(X, N) = 0 \text{ whenever } t - n \leq i \leq t. \tag{3.20.0}
$$

Put $Y_j = X/(x_1, \ldots, x_j)X$, for $0 \leq j \leq n$. Now $\underline{x}$ is $X$-regular by Lemma 2.13 and hence we have short exact sequences

$$
0 \rightarrow Y_{j-1} \xrightarrow{x_j} Y_j \rightarrow Y_j \rightarrow 0, \quad 1 \leq j \leq n. \tag{3.20.1}
$$
We want to deduce, from the long exact sequence of Ext, that
\[ \text{Ext}^t_R(Y_n, N) = 0. \] (3.20.2)
To do this, let \( T \) be the isosceles right-triangular region
\[ T := \{ (i, j) \mid 0 \leq j \leq i + n - t \leq n \}, \]
with vertices \((t - n, 0), (t, 0)\) and \((t, n)\) in the \((i, j)\) plane, and let
\[ S = \{ (i, j) \in T \mid \text{Ext}^i_R(Y_j, N) = 0 \}. \]
We claim that \( S = T \). The bottom leg of \( T \), namely, \( \{ (i, 0) \mid t - n \leq i \leq t \} \), is contained in \( S \), by (3.20.0). Therefore, to prove the claim it suffices to show that
\( (i, j - 1) \in S, (i + 1, j) \in S, \) and \( j \leq n \Rightarrow (i + 1, j) \in S. \) (3.20.3)
But (3.20.3) follows immediately from the following snippet of the long exact sequence of Ext stemming from the \( j \)th short exact sequence (3.20.1):
\[ \text{Ext}^i_R(Y_{j-1}, N) \rightarrow \text{Ext}^{i+1}_R(Y_j, N) \rightarrow \text{Ext}^{i+1}_R(Y_{j-1}, N) \rightarrow \]
This verifies the claim that \( S = T \). In particular, \((t, n)\) belongs to \( S \), and (3.20.2) is verified.

Since \( N \) is maximal Cohen-Macaulay, the \( R \)-regular sequence \( \mathfrak{x} \) is also \( N \)-regular, and so Remark 3.19 (a) shows that
\[ \text{Ext}^{t-n}_R(X/\mathfrak{x}X, N/\mathfrak{x}N) = 0. \] (3.1)
Since \( N \) is Tor-rigid, we see from Remark 3.19 (b) that \( N/\mathfrak{x}N \) is a Tor-rigid \( R/\mathfrak{x} \)-module. Now \( R \) is a Cohen-Macaulay fiber product ring, so Corollaries 3.17 and 3.16 yield \( \text{pd}_R X/\mathfrak{x}X \leq 1 \) or \( \text{pd}_R N/\mathfrak{x}N \leq 1 \). By [13, Lemma 1.3.5], \( \text{pd}_R X \leq 1 \) or \( \text{pd}_R N \leq 1 \).

If \( \text{pd}_R X \leq 1 \), then \( \text{pd}_R M < \infty \), and we are done. Assume \( \text{pd}_R N \leq 1 \). The Auslander-Buchsbaum Formula (Remark 2.4) and the fact that \( N \) is maximal Cohen-Macaulay now imply \( \text{pd}_R N = 0 \), and so \( N \) is free. \( \square \)

If \( R \) is assumed to be Gorenstein, we can get by without the assumption that \( N \) is maximal Cohen-Macaulay. For this we need Remark 3.21.

**Remark 3.21.** [29, Formula (1.2), p. 164] Let \( R \) be a local Gorenstein ring, let \( M \) be a finitely generated maximal Cohen-Macaulay \( R \)-module, and let \( N \) be a finitely generated \( R \)-module. Then \( \text{Ext}^i_R(M, R) = 0 \), for every \( i \geq 1 \). It follows that \( \text{Ext}^i_R(M, N) = \text{Ext}^{i+j}_R(M, \Omega^j_R N), \) for every \( i \geq 1 \) and \( j \geq 0 \).

**Corollary 3.22.** Let \( R \) be a \( d \)-dimensional Gorenstein quasi-fiber product ring with respect to the regular sequence \( \mathfrak{x} = x_1, \ldots, x_n \). Let \( M \) and \( N \) be nonzero \( R \)-modules. If \( N \) is Tor-rigid, and there is an integer \( t > n \) such that
\[ \text{Ext}^i_R(M, N) = 0 \] whenever \( t \leq i \leq t + n \),
then \( \text{pd}_R M < \infty \) or \( \text{pd}_R N < \infty \).
Proof. By Remark 2.12, \( n = d - 1 \). By Fact 2.17(iii), \( X := \Omega^n_R(M) \) is Cohen-Macaulay. We have
\[
\text{Ext}^i_R(X, N) = 0 \quad \text{whenever} \quad t - d \leq i \leq t + n - d.
\]
If \( Y := \Omega^n_R(N) \), the hypothesis that \( R \) is Gorenstein and the fact that \( X \) is maximal Cohen-Macaulay imply, by Remark 3.21, that
\[
\text{Ext}^i_R(X, Y) = 0 \quad \text{whenever} \quad t \leq i \leq t + n.
\]
For \( R \)-modules \( U \) and \( V \), the isomorphisms \( \text{Tor}^R_m(U, \Omega^1_R(V)) \cong \text{Tor}^R_{m+1}(U, V) \), for all \( m \geq 1 \), imply that a syzygy of a Tor-rigid module is Tor-rigid. Thus \( Y \) is Tor-rigid and maximal Cohen-Macaulay. It follows from Theorem 3.20 that at least one of \( X \) and \( Y \) has finite projective dimension, and, of course, the same must hold for \( M \) and \( N \). □

Recall that an \( R \)-module \( N \) is said to be a rigid-test module provided the following holds for all \( R \)-modules \( Z \) (see [48, Definition 2.3]):
\[
\text{Tor}^R_n(Z, N) = 0 \quad \text{for some} \quad n \geq 1 \implies \text{Tor}^R_{n+1}(Z, N) = 0 \quad \text{and} \quad \text{pd}_R Z < \infty.
\]

Corollary 3.23. Let \( R \) be a Gorenstein quasi-fiber product ring with respect to the sequence \( \underline{x} = x_1, \ldots, x_n \). Let \( M \) and \( N \) be nonzero \( R \)-modules. Assume \( N \) is a rigid-test module and, for some \( t > n \),
\[
\text{Ext}^i_R(M, N) = 0 \quad \text{whenever} \quad t \leq i \leq t + n.
\]
Then \( \text{pd}_R M < \infty \) or \( R \) is regular.

Proof. Corollary 3.22 implies \( \text{pd}_R M < \infty \) or \( \text{pd}_R N < \infty \), since \( N \) is Tor-rigid. If \( \text{pd}_R M < \infty \), we are done.

Suppose \( \text{pd}_R N < \infty \). By [48, Theorem 1.1], a local ring having a rigid test module of finite projective (or injective) dimension must be regular. Therefore the desired conclusion follows. □

Perhaps it is worth stating the case \( n = 0 \) of Corollary 3.23:

Corollary 3.24. Let \( R \) be a Gorenstein fiber product ring. Let \( M \) and \( N \) be nonzero \( R \)-modules such that \( N \) is a rigid-test module. If \( \text{Ext}^i_R(M, N) = 0 \) for some \( i > 1 \), then \( \text{pd}_R M \leq 1 \). □

Proof. By Corollary 3.23, one has \( \text{pd}_R M > \infty \) or \( R \) is regular. Since \( R \) is a not a domain, it cannot be regular. Therefore \( \text{pd}_R M < \infty \), and now Fact 2.3 shows that \( \text{pd}_R M \leq 1 \). □

Some additional results. For the rest of this section, we investigate the vanishing of Ext without the assumption of Tor-rigidity.

Recall that an \( R \)-module \( M \) satisfies Serre’s condition \((S_n)\) if
\[
\text{depth}_{R_p} M_p \geq \min\{n, \text{dim} R_p\},
\]
for all \( p \in \text{Supp}_R(M) \).

(Warning: Some sources define \((S_n)\) by the inequality
\[
\text{depth}_{R_p} M_p \geq \min\{n, \text{dim} M_p\}.
\]

(*)
The two definitions are not equivalent. For example, using the definition with the inequality (\(\ast\)), every module of finite length satisfies (\(S_n\)) for every \(n\), whereas, by the definition using inequality (\(\ast\)), a nonzero module of finite length over a ring of positive dimension does not even satisfy (\(S_1\)).

For Gorenstein quasi-fiber product rings, we obtain the following result.

**Theorem 3.25.** Let \(R\) be a \(d\)-dimensional Gorenstein quasi-fiber product ring with respect to the regular sequence \(x = x_1, \ldots, x_n\). Suppose that

\[
\operatorname{Ext}^i_R(M, M) = 0 \quad \text{for} \quad 2 \leq i \leq d + 1.
\]

(i) If \(M\) is a maximal Cohen-Macaulay \(R\)-module, then \(M\) is free.

(ii) If \(M\) satisfies \((S_k)\) for some \(k \geq n\), then \(\operatorname{pd}_R M \leq 1\).

**Proof.** (i) Since \(M\) is maximal Cohen-Macaulay over a Gorenstein ring, \(M\) is a \(d\)th syzygy, by \([33, \text{Corollary A.15}]\). Then Lemma \(2.14\) implies that \(x\) is an \(M\)-sequence. As in the proof of Theorem \(3.20\) apply the long exact sequence of Ext to the short exact sequences

\[
0 \to M/(x_1, \ldots, x_{j-1})M \xrightarrow{x_j} M/(x_1, \ldots, x_{j-1})M \to M/(x_1, \ldots, x_j)M \to 0,
\]

for \(1 \leq j \leq n\). We deduce that

\[
\operatorname{Ext}^i_R(M/x M, M) = 0 \quad \text{for} \quad 2 + n \leq i \leq d + 1.
\]

Now \(3.19\) (a) shows that

\[
\operatorname{Ext}^i_{R/(x)}(M/x M, M/x M) = 0 \quad \text{for} \quad 2 \leq i \leq 1 + d - n. \tag{3.25.0}
\]

Since \(R/(x)\) is a Gorenstein fiber product ring, Fact \(2.10\) yields that \(R/(x)\) is a \(1\)-dimensional hypersurface. Hence \(n = d - 1\) by Remark \(2.12\) and hence \(3.25.0\) just says that \(\operatorname{Ext}^2_{R/(x)}(M/x M, M/x M) = 0\). Now \([33, \text{Proposition 2.5}]\) yields \(\operatorname{pd}_{R/(x)} M/x M \leq 1\).

Therefore \(\operatorname{pd}_R M \leq 1\). Since \(M\) is maximal Cohen-Macaulay, \(M\) is free by the Auslander-Buchsbaum Formula.

(ii) Since \(M\) satisfies \((S_k)\) for some \(k \geq n\) and \(R\) is Gorenstein, \(M\) is \(k\)th syzygy by \([33, \text{Corollary A.15}]\), and Lemma \(2.14\) says that \(x\) is also an \(M\)-sequence. The proof is now similar to the proof of part (i). \(\square\)

The next result is a consequence of Theorem \(3.25\) since Gorenstein fiber product rings are \(1\)-dimensional (see Fact \(2.10\)).

**Corollary 3.26.** Let \(R\) be a Gorenstein fiber product ring. Let \(M\) be a maximal Cohen-Macaulay \(R\)-module. If \(\operatorname{Ext}^2_R(M, M) = 0\), then \(M\) is free.

Since a Gorenstein fiber product ring is a hypersurface (Fact \(2.10\)), and hence a complete intersection, Corollary \(3.20\) agrees with \([33, \text{Proposition 2.5}]\).

Remarks \(3.27\) contains a summary of related terminology and results from Avramov’s and Buchweitz’ article \([9]\); Theorem \(3.28\) and Corollary \(3.29\) follow from these remarks.
Remarks 3.27. [9] Let $R$ be a a Noetherian ring.

(1) For $R$ local, a quasi-deformation (of codimension $c$) is a diagram of
local homomorphisms $R \longrightarrow R' \leftarrow Q$, the first being faithfully flat
and the second surjective with kernel generated by a $Q$-regular sequence
(of length $c$). If $(R, m)$ is a complete intersection (Remark 2.5), then there
exists a quasi-deformation $R \longrightarrow \hat{R} \leftarrow Q$.

(2) Let $M \neq 0$ be a finitely generated module over $R$. If $R$ is local,
the complete intersection dimension over $R$ is defined by

$$\text{CI-dim}_R M := \inf \{ \text{pd}(M \otimes_R R') - \text{pd}_R R', \text{such that} \quad R \longrightarrow R' \leftarrow Q \text{ is a quasi-deformation} \}.$$ 

For $R$ not local, the complete intersection dimension over $R$ is defined by

$$\text{CI-dim}_R M := \sup \{ \text{CI-dim}_R m \mid m \in \text{Max}(R) \}; \quad \text{CI-dim}_R 0 = 0.$$ 

(3) [9, 4.1.4.1.5] Let $R$ be a local ring, let $M$ be a finitely generated
$R$-module with $\text{CI-dim}_R M < \infty$, and let $R \longrightarrow R' \leftarrow Q$ be a quasi-deformation of codimension $c$ such that the module $M' = M \otimes_R R'$ has
finite projective dimension over $Q$. By [10, (1.4)], the Auslander-Buchsbaum
Formula extends to: $\text{CI-dim}_R M = \text{depth } R - \text{depth}_R M \leq \text{dim } R$.
Thus $\text{CI-dim}_R M < \infty$.

(4) [9, 5.1] If $R$ is a local complete intersection and $M$ is a finitely generated
$R$-module, then $\text{pd}_Q(M \otimes_R \hat{R}) < \infty$, for any quasi-deformation

$$R \longrightarrow \hat{R} \leftarrow Q,$$

with $Q$ a regular ring.

(5) [9, Theorem 4.2] Let $M$ be a finitely generated module of finite CI-
dimension over $R$. Then $M$ has finite projective dimension if and only if $\text{Ext}^{2i}_R(M, M) = 0$ for some $i \geq 1$.

Remarks 3.27 yield Theorem 3.28, due to Avramov and Buchweitz.

Theorem 3.28. [9, Theorem 4.2, (5.1), p. 24] If $R$ is a local complete inter-
section and $M$ is a finitely generated module such that $\text{Ext}^{2i}_R(M, M) = 0$
for some $i \geq 1$, then $M$ has finite projective dimension.

Proof. Let $R \longrightarrow \hat{R} \leftarrow Q$ be a quasi-deformation with $Q$ a complete
regular local ring. By (4) of Remarks 3.27 $\text{pd}_Q(M \otimes_R \hat{R}) < \infty$. By (3) of
Remarks 3.27 $\text{CI-dim}_R M < \infty$. Now (5) implies $\text{pd}(M) < \infty$. \hfill \Box

By Fact 2.3 we have a generalization of Corollary 3.26.

Corollary 3.29. If $R$ is a fiber product complete intersection ring and $M$
is a finitely generated module such that $\text{Ext}^{2i}_R(M, M) = 0$ for some $i \geq 1,$
ring, then $\text{pd}_R M \leq 1$.

Using Avramov’s and Buchweitz’ Theorem 3.28 and arguments similar to
those in the proof of Theorem 3.20 we deduce the following:

Proposition 3.30. Let $R$ be a Gorenstein quasi-fiber product ring with re-
spect to the sequence $x = x_1, \ldots, x_n$. Let $M$ be an $R$-module. If for some
even number \( t > n \)
\[
\text{Ext}_R^{i}(M, M) = 0 \quad \text{whenever} \quad t \leq i \leq t + n,
\]
then \( \text{pd}_R M < \infty \).

**Proof.** Put \( X = \Omega_R^n M \). By Lemma 2.14, \( x \) is \( X \)-regular. Then
\[
\text{Ext}_R^i(X, M) = 0 \quad \text{for} \quad t - n \leq i \leq t.
\]
With \( Y_n = X/x \) and \( N = X \), we obtain Equation (3.20.2), exactly as in the proof of Theorem 3.20. Thus
\[
\text{Ext}_{R/x}^{t-n}(X/x, X/x) = 0.
\]
Now Remark 3.19 shows that \( \text{Ext}_R^t(X, X) = 0 \). Avramov’s and Buchweitz’s Theorem 3.28 implies \( \text{pd}_R X < \infty \), since \( t \) is an even positive integer. Since \( X \) is a syzygy of \( M \), it follows that \( \text{pd}_R M < \infty \).

The next result provides a bound \( b \) such that if \( \text{Ext}_R^i(M, M \oplus R) = 0 \), for every \( i \) with \( 1 \leq i \leq b \), then \( M \) is free. This gives a positive answer to Question 1.2 in case \( N = M \oplus R \) and improves Corollary 3.8.

**Theorem 3.31.** Let \( R \) be a quasi-fiber product ring with respect to the sequence \( x = x_1, \ldots, x_n \). Let \( M \) be a finitely generated \( R \)-module. If \( t \geq \max\{5, n+1\} \) and
\[
\text{Ext}_R^t(M, M \oplus R) = 0, \quad \text{for every} \quad i \quad \text{with} \quad 1 \leq i \leq t + \text{depth} R + n,
\]
then \( M \) is free. Thus
\[
\begin{align*}
\text{n > 4, } & \text{Ext}_R^t(M, M \oplus R) = 0, \quad \text{for} \quad 1 \leq i \leq 3n + 2 \implies M \text{ is free.} \\
\text{n \leq 4, } & \text{Ext}_R^t(M, M \oplus R) = 0, \quad \text{for} \quad 1 \leq i \leq 5 + 2n + 1 \implies M \text{ is free.} \\
\text{n \leq 4, } & \text{Ext}_R^t(M, M \oplus R) = 0, \quad \text{for} \quad 1 \leq i \leq 14 \implies M \text{ is free.}
\end{align*}
\]

**Proof.** Set \( \alpha := t + \text{depth} R + n \). By Proposition 2.16(1),
\[
\text{Tor}_t^R(D_{\alpha+1} M, M) = 0, \quad \text{for every} \quad i \quad \text{with} \quad 1 \leq i \leq \alpha. \quad (3.31.1)
\]
Since \( R \) is a quasi-fiber product ring, Equation (3.31.1) and Remark 3.6(1) imply \( \text{pd}_R M < \infty \) or \( \text{pd}_R(D_{\alpha+1} M) < \infty \). By the Auslander-Buchsbaum Formula, \( \text{pd}_R M \leq \text{depth} R \) or \( \text{pd}_R(D_{\alpha+1} M) \leq \text{depth} R \). Thus
\[
\text{Tor}_i^R(D_{\alpha+1} M, M) = 0, \quad \text{for every} \quad i \quad \text{with} \quad i > \text{depth} R. \quad (3.31.2)
\]
Now Proposition 2.16(2) provides the exact sequence
\[
\text{Hom}_R(M, R) \otimes_R M \rightarrow \text{Hom}_R(M, M) \rightarrow \text{Tor}_R^{\alpha+1}(D_{\alpha+1} M, M) \rightarrow 0,
\]
where the homomorphism \( \text{Hom}_R(M, R) \otimes_R M \rightarrow \text{Hom}_R(M, M) \) is the natural one.
Since \( \alpha + 1 > \text{depth} R \), \( \text{Tor}_R^{\alpha+1}(D_{\alpha+1} M, M) = 0 \), by (3.31.2). Thus the previous exact sequence yields a surjective map
\[
\text{Hom}_R(M, R) \otimes_R M \rightarrow \text{Hom}_R(M, M).
\]
The freeness of $M$ follows by Proposition 2.8(a), ([7, A.1]).

The “Thus” statement follows by setting $t = n + 1$ for the first line and setting $t = 5$ in the second line, and using that depth $R \leq n + 1$ by Remark 2.12.

A natural consequence of Theorem 3.31 of particular interest is the case that $R$ is a fiber product ring; that is, $n = 0$. Corollary 3.32 improves Nasseh and Sather-Wagstaff’s result [40, Theorem 4.5 (b)] that (ARC) holds for fiber product rings.

Corollary 3.32. Let $R$ be a fiber product ring and let $M$ be an $R$-module. If $\text{Ext}_R^i(M, M \oplus R) = 0$, for $1 \leq i \leq 6$, then $M$ is free.

4. The Huneke-Wiegand Conjecture

Conjecture (HWC) is related to another condition on local rings we call (HW$^2$), also considered by Huneke and R. Wiegand in [32].

Definition 4.1. Let $(R, m)$ be a local ring; define (HW$^2$) on $R$ as follows:

(HW$^2$) For every pair $M$ and $N$ of finitely generated $R$-modules such that $M$ or $N$ has rank, if $M \otimes_R N$ is a maximal Cohen-Macaulay module, then $M$ or $N$ is free.

Theorem 4.2, one of the main results of [32], yields that every hypersurface satisfies (HW$^2$).

Theorem 4.2. [32, Theorem 3.1] Let $(R, m, k)$ be an abstract hypersurface, and let $M$ and $N$ be finitely generated $R$-modules such that $M$ or $N$ has rank. If $M \otimes_R N$ is a maximal Cohen-Macaulay module, then both $M$ and $N$ are maximal Cohen-Macaulay modules and one of $M$ or $N$ is free.

They also give Example 4.3 below to show that even complete intersection rings may not satisfy (HW$^2$).

Example 4.3. [32, Example 4.3]. Let $R := k[[T^4, T^5, T^6]]$, $I := (T^4, T^5)$, and $J := (T^4, T^6)$. Then $R$ is a complete intersection and $I \otimes_R J$ is torsion-free, and so Cohen-Macaulay by Facts 2.17(ii), yet neither $I$ nor $J$ is free. For the proof that $I \otimes_R J$ is torsion-free, see [32].

Proposition 4.4. Let $R$ be a Gorenstein fiber product ring and let $M$ be a torsion-free $R$-module with rank. If $M \otimes_R M^*$ is torsion-free, then $M$ is free.

Proof. Since $R$ is a Gorenstein fiber product ring, $R$ is a 1-dimensional hypersurface (Fact 2.10). By [32, Theorem 3.7], either $M$ or $M^*$ is free. If $M^*$ is free, then so is $M^{**}$. Using (ii), and then (i), of Facts 2.17 we see that $M$ is reflexive, and hence is free in either case.

Corollary 4.5. If $R$ is a Gorenstein fiber product ring, then $R$ satisfies (HW). That is, if $M$ is a torsion-free $R$-module with rank and $M \otimes_R M^*$ is MCM, then $M$ is free.
Corollary 4.6. If $R$ is a one-dimensional Gorenstein quasi-fiber product ring, then $R$ satisfies (HW).

To see Corollary 4.6, apply Fact 2.11.

Next we give an example showing why for higher dimensional rings simply assuming in (HWC) (see Definition 2.6) that $M \otimes_R M^*$ is torsion-free (rather than reflexive) over a Gorenstein domain $R$ is not enough to conclude that $M$ is free.

Example 4.7. Let $(R, \mathfrak{m}) = k[[x, y]]$, a Gorenstein local domain with maximal ideal $\mathfrak{m} = Rx + Ry$. Then $\mathfrak{m}^{-1} = \{ \alpha \in K \mid \alpha \mathfrak{m} \subseteq R \}$, where $K$ is the quotient field of $R$.

We claim that $\mathfrak{m}^{-1} = R$. Of course $\mathfrak{m}^{-1}$ is naturally isomorphic to $\mathfrak{m}^*$. Since $\mathfrak{m}^{-1} \supseteq R$, we prove the reverse inclusion. Let $\alpha \in \mathfrak{m}^{-1}$, and write $\alpha = \frac{a}{b}$ in lowest terms (recall that $R$ is a UFD). Suppose, by way of contradiction, that $b$ is not a unit of $R$, and let $p$ be a prime divisor of $b$. Since $\frac{a}{b}x \in R$ and $\frac{a}{b}y \in R$, we have $ax \in Rb$ and $ay \in Rb$. Therefore $p | ax$ and $p | ay$. The assumption about lowest terms means that $p \nmid a$, and hence $p | x$ and $p | y$. That’s impossible, and the claim is proved. Now $\mathfrak{m} \otimes_R \mathfrak{m}^* \cong \mathfrak{m} \otimes_R R \cong \mathfrak{m}$, which is torsion-free. But, of course, $\mathfrak{m}$ is not free.

Conjecture 4.8. Let $(R, \mathfrak{m})$ be a Gorenstein local ring and let $x \in \mathfrak{m}$ be a non-zerodivisor. If $R/(x)$ satisfies (HW), then $R$ satisfies (HW).

Conjecture 4.8 is related to Conjecture 4.9.

Conjecture 4.9. Gorenstein quasi-fiber product rings satisfy (HW).

In particular, we are interested in the case when $R$ has dimension two:

Question 4.10. If $M$ is a finitely generated torsion-free module with rank over a two-dimensional Gorenstein quasi-fiber product ring $R$ and $M \otimes_R M^*$ is maximal Cohen-Macaulay, must $M$ be free?

We have some results related to the conjectures with additional hypotheses. First we make a remark and prove a lemma.

Remark 4.11. If $R$ is a 0-dimensional local ring, then $R$ satisfies (HW). This follows trivially since every $R$-module with rank is free.

One stumbling block to proving Conjectures 4.8 and 4.9 and answering Question 4.10 is that “torsion-free with rank” for $M$ as an $R$-module is not necessarily inherited by $M/IM$ as an $R/I$-module for an ideal $I$ of $R$, even if $I$ is a principal ideal generated by a non-zerodivisor on both $R$ and $M$.

Example 4.12. Let $k$ be a field, let $R = k[[x, y]]$, and let $M$ be the maximal ideal $Rx + Ry$. Then $M$ is a torsion-free $R$-module. However $M/xM$ is not torsion-free as a module over $\overline{R} = R/(x)$, since $x \in M \setminus xM$ implies $\overline{x} \neq 0$ in $\overline{M}$, but $yx \in xM$, and so $\overline{y} \cdot \overline{x} = 0$. (Note that $\overline{y}$ is a non-zerodivisor of $\overline{R} = k[[y]]$.)
Also, for \( N = R/(y) \) and \( f = xy \), the module \( N \) has rank (because \( R \) is a domain), but \( \overline{N} := N/fN \) does not have rank as an \( R/(f) \)-module. To see this: The ring \( S := R/(f) \) has two associated primes, namely \( P := S\mathfrak{p} \) and \( Q := S\mathfrak{q} \). Now \( yN = 0 \implies \overline{y} \cdot \overline{N} = 0 \implies \overline{N}P = 0 \), since \( \overline{y} \) is a unit of \( SP \). To see that \( \overline{N}Q \neq 0 \), consider the element \( \overline{t} \in \overline{N} \), that is, the coset \( 1 + (y) + xyN \). If \( \overline{t} = 0 \) in \( \overline{N}Q \), then, for some \( t \in S \setminus S\overline{y} \), we would have that \( t \cdot \overline{t} \) is the coset \( 0 + (y) + xyN \). Now \( t \in S \setminus S\overline{y} \implies t \) is a coset of form \( g_1(x) + yg_2(y) + xyR \), where \( 0 \neq g_1(x) \in k[[x]] \) and \( g_2(y) \in k[[y]] \). Then
\[
N/xyR, \ xyN \Rightarrow \overline{xyN} = \overline{g_1(x)} + (y) + xyN \neq 0,
\]
a contradiction. Thus \( \overline{N}Q \neq 0 \).

On the other hand, we do have Lemma 4.13.

**Lemma 4.13.** Let \( N \) be a finitely generated module over a local ring \((R, \mathfrak{m})\). Let \( \underline{x} = \{x_1, \ldots, x_n\} \) be a regular sequence of \( R \) such that \( \underline{x} \) is a regular sequence on \( N \) and \( N/(\underline{x})N \) is free. Then \( N \) is free.

**Proof.** First consider the case \( n = 1 \), that is, \( \underline{x} = \{x\} \), where \( x \in \mathfrak{m} \) and \( x \) is a non-zero divisor on \( R \) and \( N \). By [13, Lemma 4.9], \( \operatorname{pd} N = \operatorname{pd}(N/xN) \).

Since \( N/xN \) is free, \( \operatorname{pd} N = \operatorname{pd}(N/xN) = 0 \), and so \( N \) is free.

For \( n > 1 \), use induction and the equation
\[
N/(x_1, \ldots, x_n)N = \frac{N/(x_1, \ldots, x_{n-1})N}{x_n(N/(x_1, \ldots, x_{n-1})N)}. \;
\]

By applying Lemma 4.13 we have Proposition 4.14.

**Proposition 4.14.** Let \( R \) be a Gorenstein quasi-fiber product ring, and let \((\underline{x})\) be a regular sequence such that \( R/(\underline{x}) \) is a fiber product ring. Let \( M \) be a finitely generated \( R \)-module such that
\begin{enumerate}
  \item \( \underline{x} \) is a regular sequence on \( M \),
  \item \( M/(\underline{x})M \) is torsion-free and has rank as an \( (R/(\underline{x})) \)-module,
  \item \( (M/(\underline{x})M) \otimes_R (R/(\underline{x})) \operatorname{Hom}_{R/(\underline{x})}(M/(\underline{x})M, R/(\underline{x})) \) is torsion-free as an \( (R/(\underline{x})) \)-module.
\end{enumerate}

Then \( M \) is free.

**Proof.** By Proposition 4.4 \( M/(\underline{x})M \) is free as an \( (R/(\underline{x})) \)-module. Now Lemma 4.13 implies that \( M \) is free.

Proposition 4.14 in the case dim \( R = 2 \) yields a partial affirmative answer to Question 4.10 in Corollary 4.15. We need to replace the conditions on \( M \) in 4.10 by conditions on \( M/xM \), where \( x \) is a regular element of \( R \) such that \( R/(x) \) is a fiber product ring and \( x \) is regular on \( M \).

**Corollary 4.15.** Let \((R, \mathfrak{m})\) be a two-dimensional Gorenstein quasi-fiber product ring, let \( x \) be a non-zero divisor of \( R \) such that \( x \in \mathfrak{m} \) and \( R/(x) \) is a fiber product ring, and let \( M \) be a finitely generated \( R \)-module such that \( x \) is regular on \( M \). If \( M/xM \) is a torsion-free \( (R/(x)) \)-module with rank,
and \((M/xM) \otimes_{R/(x)} \text{Hom}_{R/(x)}(M/xM, R/(x))\) is maximal Cohen-Macaulay as an \((R/(x))\)-module, then \(M\) is free.

**Proof.** By (ii) of Fact 2.17 and Proposition 4.4, one has \(M/xM\) is free as an \((R/(x))\)-module; now Lemma 4.13 says that \(M\) is free over \(R\). 

**Remark 4.16.** Another stumbling block: It is easily seen that \(M \otimes_R M^*\) MCM implies that tensoring the terms with \(R/(x)\) preserves MCM, so

\[
(M/(x)M) \otimes_{R/(x)} (M^*/(x)M^*)
\]

is MCM as an \((R/(x))\)-module (equivalently torsion-free as an \((R/(x))\)-module, by (ii) of Facts 2.17). But it does not necessarily follow that

\[
(M/(x)M) \otimes_{R/(x)} \text{Hom}_{R/(x)}(M/(x)M, R/(x))
\]

is MCM as an \((R/(x))\)-module. If \(M^*/(x)M^* = \text{Hom}_{R/(x)}(M/(x)M, R/(x))\), then Expression (4.16.1) would be MCM.

Lemma 4.17 gives a condition that implies \(\overline{M}^* = (\overline{M})^*\), for certain \(R\)-modules \(M\) and \(x \in R\), where \(I := xR\), \(\overline{M}^* := M^*/IM^*\), \(\overline{M} := M/IM\) and

\[
(\overline{M})^* = (M/IM)^* := \text{Hom}_{R/I}(M/IM, R/I).
\]

**Lemma 4.17.** Let \((R, m)\) be a local ring and \(M\) and \(N\) be \(R\)-modules. Let \(x \in m\) be a NZD in \(R\) and also a NZD on \(N\). For any \(R\)-module \(V\), denote \(V/xV\) by \(\overline{V}\). The natural map \(\text{Hom}_R(M, N) \to \text{Hom}_{R/x}(\overline{M}, \overline{N})\) induces an injective homomorphism \(\overline{\text{Hom}}_R(M, N) \to \text{Hom}_{R/x}(\overline{M}, \overline{N})\). If, in addition, \(\text{Ext}^1_R(M, N) = 0\), then the injective homomorphism is in fact an isomorphism.

**Proof.** Let \(\pi_M : M \to \overline{M}\) and \(\pi_N : N \to \overline{N}\) be the natural homomorphisms. Given an \(R\)-homomorphism \(f : M \to N\), there is a unique \(\overline{R}\)-homomorphism \(\overline{f} : \overline{M} \to \overline{N}\) making the following diagram commute:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{\pi_M} & & \downarrow{\pi_N} \\
\overline{M} & \xrightarrow{\overline{f}} & \overline{N}.
\end{array}
\]

The kernel of the resulting homomorphism \(\text{Hom}_R(M, N) \to \text{Hom}_{R/x}(\overline{M}, \overline{N})\) taking \(f\) to \(\overline{f}\) is \(K := \{f \in \text{Hom}_R(M, N) \mid f(M) \subseteq xN\}\). We claim that \(K = x \text{Hom}_R(M, N)\). Obviously \(x \text{Hom}_R(M, N) \subseteq K\). For the reverse inclusion, suppose \(f \in K\). Then, for each \(m \in M\), we have \(f(m) = xn\) for some \(n \in M\). Moreover, the element \(n\) is unique, since \(x\) is a NZD on \(N\). The correspondence \(g : M \to N\) taking \(m\) to \(n\) is easily seen to be an \(R\)-homomorphism. Since \(f = xg\), this proves the claim and provides a natural injection \(\overline{\text{Hom}}_R(M, N) \to \text{Hom}_{R/x}(\overline{M}, \overline{N})\).
For the last statement, just apply \( \text{Hom}_R(M, -) \) to the short exact sequence
\[
0 \to N \xrightarrow{\varphi} N \to N/xN \to 0.
\]
(See [13, Proposition 3.3.3] for the details.) □

Corollary 4.18 now follows from Proposition 4.14.

**Corollary 4.18.** Let \( R \) be a Gorenstein quasi-fiber product ring, and let \((\bar{x})\) be a regular sequence such that \( R/(\bar{x}) \) is a fiber product ring. Let \( M \) be a finitely generated \( R \)-module such that
1. \( \bar{x} \) is a regular sequence on \( M \),
2. \( M/(\bar{x})M \) is torsion-free and has rank as an \( (R/(\bar{x})) \)-module,
3. \( M \otimes M^* \) is MCM as an \( R \)-module.
4. \( \text{Ext}^1_R(M, R) = 0 \).

Then \( M \) is free.

**Proof.** The new condition (4) implies \( M^* = M^* \) by Lemma 4.17 and so by Remark 4.16 and Proposition 4.14 the corollary holds. □

Using Theorem 4.2 and Lemma 4.13, we obtain the next corollary, which may be useful for Question 4.10.

**Corollary 4.19.** Let \((R, \mathfrak{m})\) be a local ring, let \( M \) and \( N \) be finitely generated \( R \)-modules and let \( \bar{x} \in \mathfrak{m} \) be a regular sequence on \( M \) and \( N \) such that
1. \( R/(\bar{x}) \) is a hypersurface,
2. \( M/(\bar{x})M \) has rank as an \( (R/(\bar{x})) \)-module, and
3. \( M \otimes_R N \) is MCM as an \( R \)-module.

Then \( M \) or \( N \) is free.

**Proof.** By Remark 4.2, \( R/(\bar{x}) \) satisfies (HW\(^2\)). Now \( M \otimes_R N \) a MCM \( R \)-module implies
\[
(M/(\bar{x})M) \otimes_R (N/(\bar{x})N) = (M \otimes_R N) \otimes_R (R/(\bar{x}))
\]
is MCM as an \( (R/(\bar{x})) \)-module. Therefore \( M/(\bar{x})M \) or \( N/(\bar{x})N \) is free as an \( (R/(\bar{x})) \)-module. By Lemma 4.13 \( M \) or \( N \) is free. □

**References**

[1] T. Araya, *The Auslander-Reiten conjecture for Gorenstein rings*, Proc. Amer. Math. Soc. 137 (2009), no.6, 1941–1944.
[2] T. Araya, O. Celikbas, A. Sadeghi, R. Takahashi, *On the vanishing of self extensions over Cohen–Macaulay local rings*, Proc. Amer. Math. Soc. 146 (2018), 4563–4570.
[3] M. Auslander, *Selected works of Maurice Auslander*, I. Reiten, S. O. Smalø, and Ø. Solberg, Editors, Amer. Math. Soc., Collected Works, vol. 10 (1999).
[4] M. Auslander, *Coherent Functors*, in: Proc. Conf. Categorical Algebra, La Jolla, CA, 1965, Springer, New York, 1966, pp. 189-231.
[5] M. Auslander, *Modules over unramified regular local rings*, Illinois J. Math., 5, (1961), 631–647.
[6] M. Auslander, M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc., 94, American Mathematical Society, Providence, R.I, 1969.
[7] M. Auslander, O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc., 97 (1960), 1-24.
[8] M. Auslander and I. Reiten, *On a generalized version of the Nakayama Conjecture*, Proc. Amer. Math. Soc. 32 (1975), 69–74.
[9] L. L. Avramov and R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. 142 (2000), 285–318.
[10] L. L. Avramov, V. N. Gasharov, I.V. Peeva, *Complete intersection dimension*, Publ. Math. I.H.E.S. 86 (1997), 67-114.
[11] L. L. Avramov, S. B. Iyengar, S. Nasseh and S. Sather-Wagstaff, *Persistence of homology over commutative noetherian rings*, (2020), https://arxiv.org/pdf/2005.10808.pdf.
[12] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Zeitschr. 82 (1963), 8-28.
[13] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Stud. Adv. Math. 39, Cambridge Univ. Press, Cambridge, 1993.
[14] O. Celikbas, *Vanishing of Tor over complete intersections*, J. Commut. Algebra 3 (2011), 169–206.
[15] O. Celikbas and H. Dao, *Necessary conditions for the depth formula over Cohen-Macaulay local rings*, J. Pure Appl. Algebra 218 (2014), no. 3, 522–530.
[16] O. Celikbas and R. Takahashi, *Auslander-Reiten conjecture and Auslander-Reiten duality*, J. Algebra 382 (2013), 100–114.
[17] O. Celikbas, S. Goto, R. Takahashi, and N. Taniguchi, *On the ideal case of a conjecture of Huneke and Wiegand*, Proc. Edinburgh Math. Soc. 62 (2019), 847–859.
[18] O. Celikbas and R. Wiegand, *Vanishing of Tor, and why we care about it*, J. Pure Appl. Algebra 219 (2015), no. 3, 429-448.
[19] L.W. Christensen and H. Holm, *Algebras that satisfy Auslander’s condition on the vanishing of cohomology*, Math. Z. 265 (1) (2010), 21–40.
[20] L. W. Christensen and H. Holm, *Vanishing of cohomology over Cohen-Macaulay rings*, Manuscripta Math. 139 (2012), 535–544.
[21] L.W. Christensen, J. Striuli, O. Veliche, *Growth in the minimal injective resolution of a local ring*, J. Lond. Math. Soc. (2) 81 (1) (2010), 24–44.
[22] H. Dao, M. Eghbali, and J. Lyle, *Hom and Ext, revisited*, J. Algebra 571 (2021), 75–93.
[23] K. Diveris, *Finitistic extension degree*, Algebr. Represent. Theory 17 (2014), no. 2, 495–506.
[24] N. Endo, S. Goto, and R. Isobe, *Almost Gorenstein rings arising from fiber products*, Canad. Math. Bull. 64(2) (2021), 383–400.
[25] T. H. Freitas, V. H. Jorge Pérez, R. Wiegand, and S. Wiegand, *Vanishing of Tor over fiber products*, Proc. Amer. Math. Soc. 149, (2021) 1817–1825.
[26] P. A. García-Sánchez, M. J. Leamer, *Huneke-Wiegand conjecture for complete intersection numerical semigroup rings*, J. Algebra 391 (2013), 114–124.
[27] S. Goto, R. Takahashi, *On the Auslander-Reiten conjecture for Cohen-Macaulay local rings*, Proc. Amer. Math. Soc. 145 (2017), no. 8, 3289–3296.
[28] S. Goto, R. Takahashi, N. Taniguchi, H. Le Truong, *Huneke-Wiegand conjecture and change of rings*, J. Algebra 422 (2015), 33–52.
[29] C. Huneke, D. A. Jorgensen, *Symmetry in the vanishing of Ext over Gorenstein rings*, Math. Scand. 93 (2003), 161–184.
[30] C. Huneke, S. B. Iyengar, and R. Wiegand, *Rigid ideals in Gorenstein rings of dimension one*, Acta Math. Vietnam. 44 (2019), no. 1, 31–49.
[31] C. Huneke and G. J. Leuschke, *On a conjecture of Auslander and Reiten*, J. Algebra 275(2) (2004) 781–790.
[32] C. Huneke, R. Wiegand, Tensor products of modules and the rigidity of Tor, Math. Ann. 229 (1994), 449–476.
[33] C. Huneke, R. Wiegand, Correction to “Tensor products of modules and the rigidity of Tor, Math. Ann., 299 (1994), 449–476, Math. Ann. 338 (2007), no. 2, 291–293.
[34] P. Jothilingam and T. Duraivel, Test Modules for Projectivity of Duals, Comm. Alg. 38(8) (2010), 2762–2767.
[35] D. A. Jorgensen, Finite projective dimension and the vanishing of Ext(R(M, M), Comm. Algebra 36 (2008), 4461–4471.
[36] D. A. Jorgensen and L. Şega, Nonvanishing cohomology and classes of Gorenstein rings, Adv. Math. 188 (2004), 470–490.
[37] J. Lescot, La série de Bass d’un produit fibré d’anneaux locaux, C. R. Acad. Sci. Paris 293 (1981), 569–571.
[38] G. L. Leuschke and R. Wiegand, Cohen-Macaulay Representations. Mathematical Surveys and Monographs 181, American Mathematical Society 2012.
[39] H. Matsumura, Commutative ring theory, Translated from the Japanese by M. Reid, Second edition, Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press, Cambridge, 1989.
[40] S. Nasseh, S. Sather-Wagstaff, Vanishing of Ext and Tor over fiber products, Proc. Amer. Math. Soc. 145 (2017), no. 11, 4661–4674.
[41] S. Nasseh, S. Sather-Wagstaff, R. Takahashi, and K. VandeBogert, Applications and homological properties of local rings with decomposable maximal ideals, J. Pure Appl. Algebra 223 (2019), no. 3, 1272–1287.
[42] S. Nasseh and R. Takahashi, Local rings with quasi-decomposable maximal ideal, Math. Proc. Camb Phil. Soc. 168 (2020), 305–322.
[43] S. Nasseh, R. Takahashi, and K. VandeBogert, On Gorenstein fiber products and applications, (2017), https://arxiv.org/pdf/1701.08689.pdf.
[44] T. Ogoma, Existence of dualizing complexes, J. Math. Kyoto Univ. 24 (1984), 27–48.
[45] J. Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979.
[46] R. Takahashi, Direct summands of syzygy modules of the residue class field, Nagoya Math. J. 189 (2008), 1–25.
[47] Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, London Math. Soc. Lecture Note Ser. 146, Cambridge Univ. Press, Cambridge, 1990.
[48] M. R. Zargar, O. Celikbas, M. Gheibi, and A. Sadeghi, Homological dimensions of rigid modules, Kyoto J. Math. 58 (2018), 639–669.

Universidade Tecnológica Federal do Paraná, 85053-525, Guarapuava-PR, Brazil.
Email address: freitas.thf@gmail.com

Universidade de São Paulo - ICMC, Caixa Postal 668, 13560-970, São Carlos-SP, Brazil.
Email address: vhjperez@icmc.usp.br

University of Nebraska-Lincoln
Email address: rwiegand@unl.edu

University of Nebraska-Lincoln