Sound speeds, cracking and the stability of self-gravitating anisotropic compact objects

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Abstract
Using the concept of cracking we explore the influence that density fluctuations and local anisotropy have on the stability of local and non-local anisotropic matter configurations in general relativity. This concept, conceived to describe the behavior of a fluid distribution just after its departure from equilibrium, provides an alternative approach to consider the stability of self-gravitating compact objects. We show that potentially unstable regions within a configuration can be identified as a function of the difference of propagations of sound along tangential and radial directions. In fact, it is found that these regions could occur when, at a particular point within the distribution, the tangential speed of sound is greater than the radial one.

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1. Introduction
An increasing amount of theoretical evidence strongly suggests that a variety of very interesting physical phenomena may take place giving rise to local anisotropy, i.e., unequal radial and tangential stresses $P_r \neq P_\perp$ (see [1, 2] and references therein). In the Newtonian regime it was pointed out in the classic paper by J H Jeans [3], and in the context of general relativity it was first remarked by G Lemaître [4] that local anisotropy can relax the upper limits imposed on the maximum value of the surface gravitational potential. Since the pioneering work of R Bowers and E Liang [5] its influence in general relativity has been extensively studied.

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Any model for an anisotropic compact object is worthless if it is unstable against fluctuations of its physical variables, and different degrees of stability/instability will lead to different patterns of evolution in the collapse of self-gravitating objects. Therefore, as expected, the stability of anisotropic matter configurations in general relativity has been considered since the beginning of the efforts to understand the effects of tangential pressures on a self-gravitating matter configuration [5]. More recently, in 1976, W Hillebrandt and K O Steinmetz [6] considered the problem of stability of fully relativistic anisotropic neutron star models and showed (numerically) that there exists a stability criterion similar to the one obtained for isotropic models. Later, Chan, Herrera and Santos [7] studied the role played by the local anisotropy in the onset of dynamical instabilities. They found that small anisotropies might drastically change the evolution of the system. Recently, an analytical method to extend the traditional Chandrasekhar’s variational formalism [8] to anisotropic spheres [9] has been reported.

In 1992, L Herrera introduced the concept of cracking (or overturning) [10] which is a qualitatively different approach to identifying potentially unstable anisotropic matter configurations. The idea is that fluid elements, at both sides of the cracking point, are accelerated with respect to each other. It was conceived to describe the behavior of a fluid distribution just after its departure from equilibrium. Later, Herrera and collaborators [7] showed that even small deviations from local isotropy may lead to drastic changes in the evolution of the system as compared with the purely locally isotropic case. Moreover, they found that perturbations of density alone do not take the system out of equilibrium for anisotropic matter configurations. Only perturbations of both density and local anisotropy induce such departures [11, 12]. This concept refers only to the tendency of the configuration to split (or to compress) at a particular point within the distribution but not to collapse or to expand. The cracking, overturning, expansion or collapse has to be established from the integration of the full set of Einstein equations. Nevertheless, it should be clear that the occurrence of these phenomena could drastically alter the subsequent evolution of the system. If within a particular configuration no cracking (or overturning) is to appear, we could identify it as potentially stable (not absolutely stable), because other types of perturbations could lead to its expansions or collapse.

In the present paper we shall explore the influence that fluctuations of density and local anisotropy have on the possible cracking (or overturning) of local and non-local anisotropic matter configurations in general relativity. We show that, for particular dependent perturbations, potentially unstable regions within anisotropic matter configurations could occur when the tangential speed of sound, $\partial P_\perp / \partial \rho$, is greater than the radial, $\partial P_r / \partial \rho$. This can give a more clear physical insight when considering the stability of particular anisotropic configurations when independent perturbations occur.

This paper is organized as follows. Section 2 will describe our notation through a brief discussion of local anisotropy matter configurations. The concept of cracking for self-gravitating anisotropic matter configurations and its relation with the sound speeds is considered in section 3. The models and modeling strategy are presented in sections 4 and 5. Finally some results and conclusions are displayed in section 6.

2. Anisotropic matter configuration in general relativity

We shall consider a static spherically symmetric anisotropic distribution of matter, described by the Schwarzchild line element $ds^2 = e^\lambda(r) dt^2 - e^\nu(r) dr^2 - r^2(d\theta^2 + \sin \theta d\phi^2)$ and with an energy–momentum tensor represented by $T_{\mu}^\nu = \text{diag}[\rho, -P_r, -P_\perp, -P_\perp]$, where $\rho$ is
the energy density, $P_r$ the radial pressure and $P_\perp$ the tangential pressure. For these matter configurations, the general relativistic hydrostatic equilibrium equation can be written \cite{5} as

$$\frac{dP_r}{dr} + (\rho + P_r) \left( \frac{m + 4\pi r^3 P_r}{r(r - 2m)} \right) - \frac{2}{r} (P_\perp - P_r) = 0. \quad (1)$$

Obviously, in the isotropic case ($P_\perp = P_r$) it becomes the usual Tolman–Oppenheimer–Volkov (TOV) equation, which constrains the internal equilibrium structure of general relativistic, isotropic, static perfect fluid spheres and it is considered in standard textbooks of gravitation \cite{13, 14}.

It is clear that the last term in (1), $(P_\perp - P_r) \equiv \Delta$, represents a ‘force’ due to the local anisotropy. This ‘force’ is directed outwards when $P_\perp > P_r \iff \Delta > 0$ and inwards if $P_\perp < P_r \iff \Delta < 0$. Therefore we should have more massive configurations if $\Delta > 0$ and less massive ones if $\Delta < 0$. This becomes more evident when the most extreme situations—i.e. $P_\perp \neq 0$ and $P_r = 0$ or $P_\perp = 0$ and $P_r \neq 0$—are considered.

2.1. Ansätze for an anisotropic equation of state

If a density profile, $\rho = \rho(r)$, is given, it is possible to integrate (1) when the definition of mass and the two other equations of state,

$$m(r) = 4\pi \int_0^r \rho \hat{r}^2 \, d\hat{r}, \quad P_r = P_r(\rho) \quad \text{and} \quad P_\perp = P_\perp(P_r), \quad (2)$$

are provided. The first equation of state, $P_r = P_r(\rho)$, corresponds to the standard barotropic equation of state for time-independent systems. In order to close the system, we also have to provide a second equation of state relating radial and tangential pressures, $P_\perp = P_\perp(P_r)$. It has been shown \cite{15, 16} that there exists a unique global solution to (1) if $\rho$ is a continuous positive function, $P_\perp(r)$ is a continuous differentiable function, $P_r(r)$ is a solution to the equation with starting value $P_\perp(0) = P_r(0)$, and both pressures are positive at the center (i.e. $P_\perp(0) = P_r(0) \geq 0$), therefore in the whole interior of the body.

Much of the effort to disentangle the physics of very dense matter is reflected by the various ‘radial’ equations of state, $P_r = P_r(\rho)$, available (see \cite{17, 18} and references therein). In contrast, very little is known for the much less intuitive second equation of state $P_\perp = P_\perp(P_r)$. This is the reason why different ansätze are found to introduce anisotropy in matter configurations (see, for instance, \cite{1, 2, 5, 19–28}). The unknown physics in the ‘tangential’ equation of state is partially compensated by using heuristic criteria (geometric, simplicity or any other assumption relating radial and tangential pressures). Therefore, most of the exact solutions for the differential equation (1) found in the literature have been obtained from excessively simplifying heuristic assumptions and, in addition, some of the conditions becoming ‘physically acceptable fluids’ is not verified.

2.2. Acceptability conditions for anisotropic matter

The interior solution should satisfy some general physical requirements. Some of the ‘physical acceptability conditions’ for anisotropic matter have been stated elsewhere \cite{1, 2} as

(i) density $\rho$, radial pressure $P_r$ and tangential pressure $P_\perp$ should be positive everywhere inside the configuration;

(ii) gradients for density and radial pressure should be negative,

$$\frac{\partial \rho}{\partial r} \leq 0 \quad \text{and} \quad \frac{\partial P_r}{\partial r} \leq 0;$$
(iii) inside the static configuration the speed of sound should be less than the speed of light,
\[ \frac{\partial P_r}{\partial \rho} \leq 1 \quad \text{and} \quad \frac{\partial P_\perp}{\partial \rho} \leq 1; \]

(iv) in addition to the above intuitive physical requirements, the interior solution should satisfy [29] either
  
  - the strong energy condition \( \rho + P_r + 2P_\perp \geq 0, \rho + P_r \geq 0 \) and \( \rho + P_\perp \geq 0 \) or
  - the dominant energy condition \( \rho \geq P_r \) and \( \rho \geq P_\perp \).

(v) junction conditions [30] match the matter configuration to the exterior Schwarzschild solution. Because of the continuity of the first fundamental form, the definition of mass in (2), evaluated at the boundary, becomes the total mass, \( M = m(a) \), as measured by its external gravitational field. Moreover, the continuity of the second fundamental form forces the radial pressure to vanish at the boundary, \( r = a \), of the sphere \( P_r|_{r=a} = 0 \).

Note that as a consequence of the junction conditions, the radial pressure should vanish at the boundary, but not the tangential one. However, both should be equal at the center of the matter configuration. Also note that there is no restriction on the gradient for the tangential pressure.

These reasonable physical requirements validate the assumptions made for both equations of state and, in many cases, exclude possible mathematical solutions of the system (1) and (2). Delgaty and Lake [31], considering the isotropic case \( (P_r = P_\perp) \), found that from 127 published solutions only 16 satisfy the above conditions. In particular, it is worth mentioning that in order to have a causal theory of matter we have to demand that the sound speed be, at most, the speed of light. This important requirement, when obviated, violates the physical principles usually required for a matter physical theory [32].

3. Instability and cracking of anisotropic compact objects

As we have stressed above, in a series of papers Herrera and collaborators [10–12] elaborated the concept of cracking for self-gravitating isotropic and anisotropic matter configurations. It was introduced to describe the behavior of fluid distributions just after its departure from equilibrium when total non-vanishing radial forces of different signs appear within the system.

This section will describe the general framework of the cracking approach to identify potentially stable (and unstable) anisotropic matter configurations. We explicitly use some of the tacit assumptions for modeling cracking (or overturning) within these matter configurations, and, finally, we propose a more intuitive criterion based on the difference of sound speeds to estimate the relative magnitude for the density and anisotropy perturbations and to evaluate the stability of bounded distributions.

3.1. Cracking: the general framework

Herrera and collaborators state that there is cracking whenever the radial force is directed inwards in the inner part of the sphere and reverses its sign beyond some value of the radial coordinate; or, when the force is directed outwards in the inner part and changes sign in the outer part, we shall say that there is an overturning. These effects are related to the tidal accelerations of fluid elements [11, 33], defined by

\[ a^\alpha = \left[ -R_{\beta\gamma\mu}^\alpha u^\beta u^\gamma + h_\alpha^\mu \left( \frac{du^\beta}{ds} \right)_;\gamma - \frac{du^\alpha}{ds} \frac{dF^\gamma}{ds} \right] h^\gamma_\nu \delta x^\nu, \quad (3) \]
where $\delta x^\nu$ is a vector connecting the two neighboring particles, $h_{\nu}^\alpha$ denotes the projector onto the three-space orthogonal to the four-velocity $u^\alpha$ and $du^\alpha/ds \equiv u^\mu u^\nu_{\mu}$. Moreover, defining

$$R = \frac{dP_r}{dr} + (\rho + P_r) \left( \frac{m + 4\pi P_r r^3}{r(r - 2m)} \right) - \frac{2r}{r} \Delta,$$

it can be shown that (3) and (4), evaluated immediately after perturbation, lead to [11, 12]

$$R = -e^{\lambda/2} \rho \int_0^a \frac{e^{\nu/2} \bar{r}^2}{\Theta_1} ds,$$

where $\Theta_1$ represents the expansion. Again, here, $ds^2 = e^{\lambda(r)} dt^2 - e^{\nu(r)} dr^2 - r^2(d\theta^2 + \sin \theta d\phi^2)$, the static Schwarzchild line element, has been assumed (see [11] for details). Equation (4) is just the hydrostatic equilibrium equation (1) that vanishes for static (or slowly evolving) configurations. It can be appreciated from (5) that for cracking to occur at some value of $0 \leq r \leq a$, it is necessary that $d\Theta/ds$ vanishes somewhere within the configuration. The non-local nature of this effect is also clear, and small deviations from local isotropy may lead to drastic changes in the evolution of the system as compared with the purely locally isotropic case [12].

### 3.2. Cracking revisited

Following [12], we assume that the system having some pressure and density distributions satisfying $R = 0$ is perturbed from its hydrostatic equilibrium. Thus, fluctuations in density and anisotropy induce total radial forces ($R \neq 0$) which, depending on their spatial distribution, may lead to the cracking; i.e radial force directed inwards, ($R > 0$), or, overturning, directed outwards, ($R < 0$) of the source. Therefore, we will be looking for a change of the sign of $R$ beyond some value of the radial coordinate. We will exclusively consider perturbations on both density and local anisotropy, under which the system will be dynamically unstable. In other words, $\delta \rho$ and $\delta \Delta$ are going to be considered as independent perturbations; but fluctuations in mass and radial distribution pressure depend on density perturbations, i.e.,

$$\rho + \delta \rho \Rightarrow \left\{ \begin{array}{l} P_r(\rho + \delta \rho, r) \approx P_r(\rho, r) + \delta P_r \approx P_r(\rho, r) + \frac{\partial P_r}{\partial \rho} \delta \rho, \\
(\rho + \delta \rho) = 4\pi \int_0^r (\rho + \delta \rho) r^2 dr \approx m(\rho, r) + \frac{4\pi}{3} r^3 \delta \rho. \end{array} \right.$$  

Now, on expanding (4) we have, formally,

$$R \approx R_0(\rho, P_r, m, \Delta, r) + \frac{dR}{d\rho} \delta \rho + \frac{dR}{dP_r} \delta P_r + \frac{dR}{dm} \delta m + \frac{dR}{d\Delta} \delta \Delta,$$

and by using (6) it can be shown that

$$\tilde{R} = \delta \rho \left[ 2 \frac{\partial R}{\partial \rho} + \frac{4\pi}{3} r^3 \frac{\partial R}{\partial m} \right] - \frac{2\delta \Delta}{r} \delta \rho,$$

where

$$\frac{\partial R}{\partial \rho} = \frac{m + 4\pi P_r r^3}{r(r - 2m)} \geq 0 \quad \text{and} \quad \frac{\partial R}{\partial m} = \frac{(\rho + P_r)(1 + 8\pi P_r r^2)}{(r - 2m)^2} \geq 0. \quad (9)$$

It is immediately seen that, in order to have $\tilde{R} = 0$ and consequently a change in its sign,

- both anisotropy and density have to be perturbed;
- both anisotropy and density perturbations have to have the same sign, i.e., $\delta \Delta/\delta \rho > 0$.

In order words, potentially stable configurations should have $\delta \Delta/\delta \rho \leq 0$ everywhere because $\tilde{R}$ never changes its sign [34].
3.3. Cracking and sound speeds

When arbitrary and independent density and anisotropy perturbations are considered (as in all previous works concerning cracking [10–12, 34]), there are few physical criteria to establish the size (absolute and/or relative) of the perturbation, i.e., how small (or big) the perturbations should be. Different orders of magnitude (and relative size $\delta \Delta / \delta \rho$) of perturbations could produce a cracking but we could be describing an unphysical scenario. Additionally, all these previous works only consider constant perturbations. It is possible that variable perturbations could be more efficient inducing cracking within a particular matter configuration. Again, there is no criterion in establishing the functionality of the perturbation throughout the matter distribution.

We are going to consider a particular type of dependent perturbation whose relative order of magnitude could be bounded by the behavior of some physical variables and could be checked by physical intuition. Obviously, general perturbations should be independent because they emerge from non-related physical phenomena. But we are looking for some physical variables whose behavior could be checked in order to identify potential cracking.

It is easy to convince oneself that

$$\frac{\delta \Delta}{\delta \rho} \sim \frac{\delta (P_\perp - P_r)}{\delta \rho} \sim \frac{\delta P_\perp}{\delta \rho} - \frac{\delta P_r}{\delta \rho} \sim v_{s \perp}^2 - v_{s r}^2,$$

(10)

where $v_{s \perp}^2$ and $v_{s r}^2$ represent the radial and tangential sound speeds, respectively.

This will be the key concept we will use in revisiting Herrera’s approach to identify potentially unstable anisotropic matter configurations based on the concept of cracking. Now, by considering the sound speeds and evaluating (10), we could not only have a more precise idea of the relative order of magnitude of the perturbations ($\delta \Delta$ and $\delta \rho$) but also what regions are more likely to be potentially unstable within a matter configuration.

It is clear that because $0 \leq v_{s \perp}^2 \leq 1$ and $0 \leq v_{s r}^2 \leq 1$, we have $|v_{s \perp}^2 - v_{s r}^2| \leq 1$. Thus,

$$-1 \leq v_{s \perp}^2 - v_{s r}^2 \leq 1 \Rightarrow \begin{cases} -1 \leq v_{s \perp}^2 - v_{s r}^2 \leq 0 \hspace{1cm} \text{potentially stable,} \\ 0 < v_{s \perp}^2 - v_{s r}^2 \leq 1 \hspace{1cm} \text{potentially unstable.} \end{cases}$$

(11)

Therefore, we can now evaluate potentially unstable regions within anisotropic models based on the difference of the propagation of sound within the matter configuration. Those regions where $v_{s \perp}^2 > v_{s r}^2$ will be potentially unstable. On the other hand, if $v_{s \perp}^2 \leq v_{s r}^2$ everywhere within a matter distribution, no cracking will occur. It is worth mentioning that, concerning this criterion, one of the extreme matter configurations mentioned above ($P_\perp \neq 0$ and $P_r = 0$) is always potentially stable for cracking, and the other one ($P_\perp = 0$ and $P_r \neq 0$) becomes potentially unstable.

Moreover, for physically reasonable models, the magnitude of perturbations in anisotropy should always be smaller than those in density, i.e., $|v_{s \perp}^2 - v_{s r}^2| \leq 1 \Rightarrow |\delta \Delta| \leq |\delta \rho|$. When $\delta \Delta / \delta \rho > 0$, these perturbations lead to potentially unstable models.

The next section will be devoted to exploring the effectiveness of this criterion on the stability of bounded matter configurations, having different equations of state. Again, we recall that the concept of cracking refers only to the tendency of the configuration to split and its occurrence has to be established from the integration of the full set of Einstein equations. In addition, it is clear that there could also be some perturbations that do not induce cracking but could cause instabilities that lead the configuration to collapse or expand.
4. Perturbations and cracking for anisotropic configurations

In order to illustrate the workability of the above criterion (11), we shall work out several density profiles for models satisfying the physical acceptable conditions. Thus, in addition to the positivity of density and pressures profiles, their gradients and the fulfillment of the energy conditions (strong or dominant), we shall pay special and particular attention to the conditions bounding sound speeds (radial and tangential) within the matter configuration.

The idea will be to provide the density profile, then to obtain the radial pressure, \( P_r(r) \) from a ‘radial’ equation of state \( P_r = P_r(\rho(r)) \) and next to solve the tangential pressure \( P_\perp(r) \) from the anisotropic TOV (1). That is,

\[
\rho(r) \rightarrow P_r = P_r(\rho(r)) \rightarrow P_\perp = P_r + \frac{r}{2} \frac{dP_r}{dr} + \frac{1}{2} \left( \frac{m + 4\pi r^3 P_r}{r - 2m} \right).
\]

Then, the radial and tangential sound speeds are calculated and their difference \( |v^2_{sr} - v^2_{s\perp}| \) is evaluated. Next, by using (11) the potential stability or instability is established. This will be confirmed by a change in the sign of \( \tilde{R} \) described by (8) and (9).

To illustrate the above criterion (11) we shall analyze four cases concerning qualitatively different density profiles. We have selected two local (one singular and one non singular) and two non-local conformally flat anisotropic solutions. By local models we mean the standard way to express an equation of state where the energy density and radial pressure are related at a particular point within the configuration, i.e., \( P = P(\rho(r)) \). On the other hand, by non-local models we will understand those where the radial pressure \( P_r(r) \) is not only a function of the energy density, \( \rho(r) \), at that point but also its functional throughout the rest of the configuration. Any change in the radial pressure takes into account the effects of the variations of the energy density within the entire volume [27, 35, 36]. It has been shown that, in the static limit, this particular radial equation of state can be written as

\[
P_r(r) = \rho(r) - \frac{2}{r^2} \int_0^r \bar{r}^2 \rho(\bar{r}) d\bar{r}.
\]

It is clear that in equation (13) a collective behavior of the physical variables \( \rho(r) \) and \( P_r(r) \) is present.

4.1. Anisotropic Tolman VI model

This model was introduced by Cosenza, Herrera, Esculpi and Witten [20] starting from the singular Tolman VI density profile [37]. The original isotropic Tolman VI solution is not deprived of a physical meaning. It resembles a highly relativistic Fermi gas with the corresponding adiabatic exponent of 4/3. By using a heuristic method these authors determine other physical variables representing an anisotropic static matter configuration, i.e.,

\[
\rho = \frac{K}{r^2}, \quad \Rightarrow \quad P_r = \frac{3}{8\pi r^2} \left( \frac{1 - \sqrt{3}}{7 - 3\sqrt{3}} \right), \quad \Rightarrow \quad P_\perp = \frac{3}{224\pi r^2} \left( \frac{21 - 25\sqrt{3}}{7 - 3\sqrt{3}} \right)
\]

where the junction conditions force the adjustment of the parameter to \( K = 3/56\pi \), and the radius is given by \( a = 81/49 \).

Sound speeds can be determined from (14), and can be written as

\[
v^2_{sr} = \frac{7(7 + 3a - 9\sqrt{3})}{(7 - 3\sqrt{3})^2} \quad \text{and} \quad v^2_{s\perp} = \frac{3(49 + 25a - 70\sqrt{3})}{4(7 - 3\sqrt{3})^2}.
\]
By using (14) the particular expression for (8) can be obtained for this model as
\[
\tilde{R}_{\text{TolmanVI}} = \frac{2\delta \rho}{r} \left[ \frac{588 + 180\frac{\zeta}{a} - 672\sqrt{\frac{\zeta}{a}} \nu_{v, r}^2 + 539 + 195\frac{\zeta}{a} - 658\sqrt{\frac{\zeta}{a}}}{16(7 - 3\sqrt{\frac{\zeta}{a}})} - \frac{\delta \Delta}{\delta \rho} \right]. \tag{16}
\]

It is worth mentioning that this model does not fulfill all the acceptability conditions stated in 2.2. It is singular at the center, and near the boundary surface \((r/a \approx 0.706)\) the tangential pressure becomes negative. Despite this unphysical situation, this model is presented because, as it will later become clear in section 5, the difference in sound speed is constant through the whole configuration, which represents the above-mentioned constant perturbation relation considered in previous works [10–12, 34].

### 4.2. Non-local Stewart model 1

This model emerges from a density profile proposed by B W Stewart [21] to describe anisotropic conformally flat static bounded configurations, which was also, recently, considered for non-local anisotropic matter distributions [27]. Starting from this density profile we can find \(P_r(r)\) and \(P_\perp(r)\) as
\[
\rho = \frac{1}{8\pi r^2} \frac{(e^{2Kr} - 1)(e^{4Kr} + 8Kr e^{2Kr} - 1)}{(e^{2Kr} + 1)^3} \tag{17a}
\]
\[
P_r = \frac{1}{8\pi r^2} \frac{(1 - e^{2Kr})(e^{4Kr} - 8Kr e^{2Kr} - 1)}{(e^{2Kr} + 1)^3} \tag{17b}
\]
\[
P_\perp = \frac{2K^2 e^{4Kr}}{\pi [1 + e^{2Kr}]^4} \tag{17c}
\]

Again, the parameter \(K\) has to be obtained from the junction conditions, which means that \(K\) has to satisfy a transcendental equation,
\[
e^{4Ka} - 8Ka e^{2Ka} - 1 = 0 \quad \Rightarrow \quad K = \frac{1}{2a} \ln \left[ \frac{1 + \left(\frac{2M}{a}\right)^2}{1 - \left(\frac{2M}{a}\right)^2} \right]. \tag{18}
\]

From (17a), (17b) and (17c) the corresponding sound speeds can be found:
\[
v_{sr}^2 = \frac{8Kr e^{2Kr} [(e^{2Kr} - 1) + Kr [(e^{2Kr} - 2)^2 - 3]] - (e^{4Kr} - 1)^2}{(e^{4Kr} - 1)^2 + 8K^2r^2 e^{2Kr} [(e^{2Kr} - 2)^2 - 3]} \tag{19}
\]
and
\[
v_{s\perp}^2 = \frac{32K^3r^3 e^{4Kr} (e^{2Kr} - 1)(e^{2Kr} + 1)^{-1}}{(e^{4Kr} - 1)^2 + 8K^2r^2 e^{2Kr} [(e^{2Kr} - 2)^2 - 3]}. \tag{20}
\]

Now, equation (8) can also be obtained for this model as
\[
\tilde{R}_{\text{NLStewart1}} = \frac{2\delta \rho}{r} \left[ \frac{\left( e^{2Kr} + 1 \right)(5 + v_{sr}^2) + 4Kr (e^{2Kr} - 1)\left( e^{2Kr} - 1 \right)^{-1} K r}{6(e^{2Kr} + 1)^2(e^{2Kr} - 1)^{-1}} - \frac{\delta \Delta}{\delta \rho} \right]. \tag{21}
\]
4.3. Non-local Stewart model

This is a second density profile proposed by B W Stewart [21] which was recently proved to be non-local [36]:

\[
\rho = \frac{1}{8\pi r^2} \left[ 1 - \frac{\sin(2Kr)}{Kr} + \frac{\sin^2(Kr)}{K^2r^2} \right] \quad (22a)
\]

\[
P_r = -\frac{1}{8\pi r^2} \left[ 1 + \frac{\sin(2Kr)}{Kr} - \frac{3\sin^2(Kr)}{K^2r^2} \right] \quad (22b)
\]

\[
P_{\perp} = \frac{1}{8\pi r^2} \left[ 1 - \frac{\sin^2(Kr)}{K^2r^2} \right]. \quad (22c)
\]

As in the previous models, junction conditions \((M = m(a)\) and \(P_r(a) = 0\)) determine the coupling constant \(K\). In this case it also has to satisfy a transcendental equation:

\[
\frac{\sin Ka}{Ka} = \left( 1 - \frac{2M}{a} \right)^{1/2} \quad \text{and} \quad \cos Ka = \frac{1 - \frac{3M}{a}}{\sqrt{1 - \frac{2M}{a}}}. \quad (23)
\]

Thus,

\[
K = \sqrt{\frac{M}{a} \left( 4 - \frac{9M}{a} \right)} \quad (24)
\]

From (22a), (22b) and (22c) the corresponding sound speeds can be found:

\[
v_{sr}^2 = \frac{\sin^2(Kr)(3 - K^2r^2) - \frac{3}{2} \sin(2Kr)Kr}{[\cos(Kr)Kr - \sin(Kr)]^2} \quad \text{and} \quad v_{s\perp}^2 = \frac{Kr \left[ Kr + \frac{\sin(2Kr)}{2} \right] - \sin^2(Kr)}{[\cos(Kr)Kr - \sin(Kr)]^2}. \quad (25)
\]

Now, equation (8) can also be obtained for this model as

\[
\tilde{R}_{\text{NL Stewart}2} = \frac{2\delta\rho}{r} \left[ \frac{2\sin(2Kr)Kr - 9\sin^2(Kr)v_{sr}^2 + 1}{12\sin^2(Kr)[\sin(2Kr)Kr - 2\sin^2(Kr)]^{-1}} - \frac{\delta\Delta}{\delta\rho} \right]. \quad (26)
\]

4.4. Florides–Stewart–Gokhroo–Mehra model

This density profile is originally due to P S Florides [38], but also corresponds to different solutions considered by Stewart [21] and, more recently, by M K Gokhroo and A L Mehra [23]. The Florides–Stewart–Gokhroo–Mehra solution represents densities and pressures which, under particular circumstances [39], give rise to an equation of state similar to the Bethe–Börner–Sato Newtonian equation of state for nuclear matter [33, 40, 41]:

\[
\rho = \rho = \rho_c \left( 1 - \frac{Kr^2}{a^2} \right) \quad (27a)
\]

\[
P_r = \frac{\rho_c}{f} \left( 1 - \frac{2\mu r^2}{a^2} \left[ \frac{5 - \frac{3K^2r^2}{a^2}}{5 - 3K} \right] \right) \left( 1 - \frac{r^2}{a^2} \right)^n. \quad (27b)
\]
As can be appreciated from the parameters displayed in table 1, all the models considered have a radius \((a = 10 \text{ km})\) and total masses, \(M\), (in terms of solar mass \(M_\odot\)) that correspond to typical values for expected astrophysical compact objects. The boundary redshifts \(z_\odot\), surface and central densities \(\rho_s\) and \(\rho_c\) that emerge from our selection also fit the typical values for these objects.
Figure 1. Variations of the radial and tangential sound speeds for anisotropic configurations. Plates I, II, III and IV represent Tolman VI, NL Stewart 1, NL Stewart 2 and Gokhroo and Mehra, respectively. Curves a, b and c correspond to $v_{r}^2$, $v_{\perp}^2$ and $v_{\perp}^2 - v_{r}^2$, respectively.

Table 1. All parameters have been chosen to represent a possible compact object with $a = 10$ km. and the corresponding mass function satisfying the physical acceptability and energy conditions.

| Density profile | $M/a$ | $M(M_\odot)$ | $z_a$ | $\rho_a \times 10^{14}$ (g cm$^{-3}$) | $\rho_\odot \times 10^{15}$ (g cm$^{-3}$) |
|----------------|------|--------------|------|-----------------|-------------------|
| Tolman VI      | 0.21 | 1.42         | 0.31 | 2.30            | NA                |
| NL Stewart 1   | 0.32 | 2.15         | 0.65 | 6.80            | 1.91              |
| NL Stewart 2   | 0.39 | 2.68         | 1.19 | 8.49            | 2.14              |
| Gokhroo and Mehra | 0.26 | 1.76         | 0.44 | 0.00            | 2.09              |

Profiles for the radial, $v_{r}^2$, and tangential, $v_{\perp}^2$, sound speeds, as well as the difference, $v_{\perp}^2 - v_{r}^2$, are displayed in figure 1. The perturbation relation, $\delta \Delta / \delta \rho \equiv v_{\perp}^2 - v_{r}^2$, fulfills the physical restriction $-1 \leq \delta \Delta / \delta \rho \leq 1$ for all models considered. Note that $\delta \Delta / \delta \rho$, is constant within the matter distribution for the Tolman VI anisotropic model (plate I in figure 1). These types of constant perturbation relations were standard for modeling cracking in previous works [10–12, 34]. Because $\delta \Delta / \delta \rho < 0$, the sound speed stability criterion, (11), states that in the Tolman VI anisotropic model, no cracking will occur. Non-local Stewart models are sketched in plates II and III, respectively. For these two models we could implement variable perturbation relations, $\delta \Delta / \delta \rho$, through the matter configuration; because $-1 \leq \delta \Delta / \delta \rho \leq 0$, no cracking will occur in these models either. Finally, the most interesting scenario emerges from the Florides–Stewart–Gokhroo–Mehra model [23] with $j = 7, K = 1$ and $n = 2$, shown in plate IV. As is evident from this plate, the perturbation relation, $\delta \Delta / \delta \rho$, not only
has a variable profile, but it also changes its sign, alternating potentially stable and unstable regions within the distribution. In fact, this model presents two potentially unstable regions: $0 \lesssim \eta = r/a \lesssim 0.2570$ and $0.7565 \lesssim \eta = r/a \lesssim 1$ where $\delta \Delta / \delta \rho > 0$.

The profiles of $\tilde{R}$ for each model are plotted in figure 2 and the above stability assumptions can be contrasted with the change in sign for expressions (16), (21), (26) and (33). It is clear from the $\tilde{R}$-plots displayed in this figure that the models of Tolman VI, NL Stewart 1 and NL Stewart 2 models do not present any cracking point (plates I, II and III, respectively). On the other hand, the Florides–Stewart–Gokhroo–Mehra model displays a cracking point at $\eta \approx 0.17986$ within the first potentially unstable region, $0 \lesssim \eta = r/a \lesssim 0.2570$.

6. Results and conclusions

We have revisited the concept of cracking for self-gravitating anisotropic matter configurations introduced by L. Herrera and collaborators [10–12]. It has been shown that, for some particular dependent perturbations, the ratio for fluctuations in anisotropy to energy density, $\delta \Delta / \delta \rho$, can be interpreted in terms of the difference of sound speeds, i.e. $\delta \Delta / \delta \rho \sim v_{sr}^2 - v_{\perp}^2$, where $v_{sr}^2$ and $v_{\perp}^2$ represent the radial and tangential sound speeds, respectively. It is evident from (11) that regions where $v_{sr}^2 > v_{\perp}^2$ will be potentially unstable. On the other hand, if $v_{sr}^2 \leq v_{\perp}^2$ everywhere within a matter distribution, no cracking will occur and it could be considered as stable.

This reinterpretation could be useful for refining and making the concept of cracking more physically related to the potential instability due to the behavior of some physical variables.
within matter configurations. It is easy to determine each sound speed, their difference (10) and the sign of the difference. Thereafter, we could clearly identify from (8) which regions are more likely to be potentially unstable within a particular matter distribution. This can be appreciated from the Florides–Stewart–Gokhroo–Mehra model (figure 2 plate IV) which displays a cracking point at \( \eta \approx 0.17986 \) within the first potentially unstable region \( 0 \lesssim \eta = r/a \lesssim 0.2570 \).

Additionally, because each sound speed has to be less than the speed of light, it implies that their difference has the physical restriction \( \left| \frac{\delta}{\delta \rho} \right| \sim \left| v_s^2 - v_s^2 \right| \lesssim 1 \). This is very important in order to characterize a particular model as potentially unstable. It is possible to find cracking points within a configuration for an unphysical set of fluctuations in anisotropy and energy density, i.e. \( |\delta/\delta \rho| > 1 \), but the existence of these cracking points could not lead to physical potentially unstable models. Moreover, the physical restriction, \( |\delta/\delta \rho| \leq 1 \) also conditions the relative order of magnitude of the perturbations.

The ratio of perturbations, \( \delta \Delta / \delta \rho \), is now not necessarily constant. Models considered in previous works [10–12, 34] have constant fluctuations, because there were no other criteria to introduce in order to evaluate the change in the sign of \( \dot{R} \). Now, the possibility of introducing variable fluctuations based on the difference of sound speeds enriches the applicability of the cracking framework to evaluate instabilities within anisotropic matter configurations.

It is worth mentioning that, concerning this criterion, one of the extreme matter configurations mentioned above, \( P_\perp \neq 0 \) and \( P_r = 0 \), is always potentially stable, and the other, \( P_\perp = 0 \) and \( P_r \neq 0 \), could experiment a cracking (or overturning) scenario. The study of matter configurations with vanishing radial stresses traces back to G Lemaître [4] and, for non-static models, has been considered in [1]. Recently, this model has been studied [42, 43] concerning its relation with naked singularities, and conformally flat models have been considered in [24]. Extreme models with vanishing tangential stresses seem to be useful describing highly compact astrophysical objects having very large magnetic fields (\( B \gtrsim 10^{15} \text{ G} \)) [44].

As we have pointed out, any model for a static compact object is worthless if it is unstable against fluctuations of its physical variables. If a particular static model is unstable against these fluctuations it could follow different possible patterns in its subsequent evolution. It could collapse, expand, split or overturn. Perturbations play a crucial role not only in evaluating the stability of a particular static model, but also in identifying trends in the possible future evolution of the model. Their study should be considered from different points of view and formalisms. In this work we have considered only those perturbations, related through radial and tangential sound speeds, that lead to identifying potentially unstable regions. Independent perturbations (not related via any physical quantity) could also exist and could also lead to cracking (or overturning) points but, in this case, there is no criterion to quantify their order of magnitude. Other types of perturbations leading to expanding or collapsing evolutions could be considered in the standard Chandrasekhar’s variational formalism (see [8, 9] and references therein). Again, we stress the fact that the different possible evolution patterns for unstable configurations refer only to a tendency. Its occurrence has to be established from the integration of the full set of Einstein equations.

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