An Even Simpler Proof of the Right-Hand Rule

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Suppose you are stranded at great altitude on the positive z-axis of a standard (“right-handed”) Cartesian coordinate system (see Figure 1), looking vertiginously down on the vast, flat xy-plane (imagine being the first person to see Kansas from Mt. Everest). As hypothermia and hypoxia set in, you ask yourself, “How did I end up here? And is there a proof of the right-hand rule that requires nothing more difficult than high school trigonometry?”

Your first question is unanswerable, but before we demonstrate that the answer to your second question is “yes,” recall that the cross product of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is denoted $\mathbf{a} \times \mathbf{b}$ and defined analytically as the vector $\langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$. It is easy to prove directly from this definition that $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$ and, if either $\mathbf{a}$ or $\mathbf{b}$ is the zero vector or $\mathbf{a}$ is a multiple of $\mathbf{b}$, then $\mathbf{a} \times \mathbf{b}$ is the zero vector. Otherwise, there are two vectors of the correct length but of opposite directions that are orthogonal to $\mathbf{a}$ and $\mathbf{b}$. The right-hand rule states that to determine the direction of $\mathbf{a} \times \mathbf{b}$, place your right hand...
so that the little-finger edge of your hand is parallel to \( \mathbf{a} \), curl your fingers toward \( \mathbf{b} \), and move your thumb away from your other fingers. Then your thumb points in the direction of \( \mathbf{a} \times \mathbf{b} \).

Several proofs of the right-hand rule have been published, e.g., [1, p. 192], [2], [3, ch. 7], [4, p. 616], of which Gao [2] is the simplest. Here is an even simpler proof.

Let \( \mathbf{a}_p = (a_1, a_2, 0) \) and \( \mathbf{b}_p = (b_1, b_2, 0) \) be the projections of \( \mathbf{a} \) and \( \mathbf{b} \) onto the \( xy \)-plane, respectively. First, assume that \( \mathbf{a}_p \) is not a multiple of \( \mathbf{b}_p \), i.e., that the plane that contains \( \mathbf{a} \) and \( \mathbf{b} \) is not perpendicular to the \( xy \)-plane. From the perspective of the positive \( z \)-axis, we will determine if \( \mathbf{a} \) is to the right or left of \( \mathbf{b} \) by the following procedure: First, look at the smaller of the two angles that the vectors form with each other. Then, without crossing your arms, move your arms so that one arm is parallel to \( \mathbf{a} \) and move your other arm so that it is parallel to \( \mathbf{b} \). If your right arm is parallel to \( \mathbf{a} \), then \( \mathbf{a} \) is to the right of \( \mathbf{b} \) and, if your left arm is parallel to \( \mathbf{a} \), then \( \mathbf{a} \) is to the left of \( \mathbf{b} \).

Now, the right-hand rule says that from the perspective of the positive \( z \)-axis, if \( \mathbf{a} \) is to the right of \( \mathbf{b} \), then the \( z \)-component of \( \mathbf{a} \times \mathbf{b} \) will be positive since your thumb will point up and, if \( \mathbf{a} \) is to the left of \( \mathbf{b} \), then the \( z \)-component of \( \mathbf{a} \times \mathbf{b} \) will be negative since your thumb will point down. Note that the left-right orientation of \( \mathbf{a}_p \) and \( \mathbf{b}_p \) is the same as the left-right orientation of \( \mathbf{a} \) and \( \mathbf{b} \), i.e., if \( \mathbf{a} \) is to the right of \( \mathbf{b} \), then \( \mathbf{a}_p \) is to the right of \( \mathbf{b}_p \), and if \( \mathbf{a} \) is to the left of \( \mathbf{b} \), then \( \mathbf{a}_p \) is to the left of \( \mathbf{b}_p \).

Let \( \alpha \) and \( \beta \) be the measures of the angles that \( \mathbf{a}_p \) and \( \mathbf{b}_p \) form with the positive \( x \)-axis, respectively, where \( 0 \leq \alpha, \beta < 2\pi \) and, as usual, positive angle measurements correspond to counterclockwise rotations. Note that \( \mathbf{a} \times \mathbf{b} \) and \( \mathbf{a}_p \times \mathbf{b}_p \) have the same \( z \)-component, namely

\[
a_1 b_2 - a_2 b_1 = \left( |\mathbf{a}_p| \cos \alpha \right) \left( |\mathbf{b}_p| \sin \beta \right) - \left( |\mathbf{a}_p| \sin \alpha \right) \left( |\mathbf{b}_p| \cos \beta \right)
= |\mathbf{a}_p| |\mathbf{b}_p| \left( \sin \beta \cos \alpha - \sin \alpha \cos \beta \right)
= |\mathbf{a}_p| |\mathbf{b}_p| \sin (\beta - \alpha).
\]

Suppose \( \mathbf{a} \) is to the right of \( \mathbf{b} \). Then \( \mathbf{a}_p \) is to the right of \( \mathbf{b}_p \), so \( 0 < \beta - \alpha < \pi \) (see Figure 2(a)) or \( -2\pi < \beta - \alpha < -\pi \) (see Figure 2(b)). In both cases, \( \sin (\beta - \alpha) > 0 \), so the \( z \)-component of \( \mathbf{a} \times \mathbf{b} \) is positive, as stated by the right-hand rule. Similarly, if \( \mathbf{a} \) is to the left of \( \mathbf{b} \) then \( \mathbf{a}_p \) is to the left of \( \mathbf{b}_p \), so \( -\pi < \beta - \alpha < 0 \) or \( \pi < \beta - \alpha < 2\pi \),

![Figure 2](image_url)

**Figure 2.** (a) The view from the positive \( z \)-axis, where \( \mathbf{a}_p \) is to the right of \( \mathbf{b}_p \) and \( 0 < \beta - \alpha < \pi \). (b) As in (a) with \( -2\pi < \beta - \alpha < -\pi \).
so \(\sin(\beta - \alpha) < 0\), thus the \(z\)-component is negative. This concludes the proof of the right-hand rule when \(a_p\) is not a multiple of \(b_p\).

If \(a_p\) is a multiple of \(b_p\), the proof is almost identical, except that we redefine \(a_p\) and \(b_p\) by letting \(a_p = \langle 0, a_2, a_3 \rangle\) and \(b_p = \langle 0, b_2, b_3 \rangle\) be the projections of \(a\) and \(b\) onto the \(yz\)-plane, respectively. Let \(\alpha\) and \(\beta\) be the angles that \(a_p\) and \(b_p\) make with the positive \(y\)-axis, respectively, where again \(0 \leq \alpha, \beta < 2\pi\), but now positive angle measurements correspond to counterclockwise rotations as seen from a vantage point on the positive \(x\)-axis. Then the right-hand rules says that, from the perspective of the positive \(x\)-axis, the \(x\)-component of \(a \times b\) is positive if and only if \(a\) is to the right of \(b\). Again, the left-right orientation of \(a_p\) and \(b_p\) is the same as the left-right orientation of \(a\) and \(b\). Now, \(a \times b\) and \(a_p \times b_p\) have the same \(x\)-component, and the proof proceeds as above.

**Summary.** We present a trigonometric proof of the right-hand rule.

**References**

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