DISPERSIONLESS BIGRADED TODA HIERARCHY AND ITS ADDITIONAL SYMMETRY

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Abstract. In this paper, we firstly give the definition of dispersionless bigraded Toda hierarchy (dBTH) and introduce some Sato theory on dBTH. Then we define Orlov-Schulman’s $M_L, M_R$ operator and give the additional Block symmetry of dBTH. Meanwhile we give tau function of dBTH and some some related dispersionless bilinear equations.

Mathematics Subject Classifications(2000). 37K05, 37K10, 37K20.
Keywords: dispersionless bigraded Toda hierarchy, additional symmetry, dispersionless Hirota bilinear identity.

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1. Introduction

The Toda lattice equation is a nonlinear evolutionary differential-difference equation introduced by Toda [1] describing an infinite system of masses on a line that interact through an exponential force. This equation is completely integrable, i.e. admits infinite conserved quantities and has important applications in many different fields such as classical and quantum field...
theory, in particular in the theory of Gromov-Witten invariants [2]. Considering its application to 2D topological field theory [3, 4], one extended the interpolated Toda lattice hierarchy into the so-called extended Toda hierarchy [5, 6]. In the paper [7], it generalized the Toda lattice hierarchy (TH) and extended the Toda lattice hierarchy by considering $N + M$ dependent variables and used them to provide a Lax pair definition of the extended bigraded Toda hierarchy (EBTH). This hierarchy later lead to a series of results [8, 9, 10, 11]. In fact that model has been proposed in [12] because the dispersionless EBTH can be obtained from the dispersionless KP hierarchy and the dispersionless Toda (dToda) hierarchy describes the genus zero-limit of the Landau-Ginzburg formulation of two-dimensional string theory [13, 14, 15].

Additional symmetries have been analyzed in the explicit form of the additional flows of KP hierarchy given by Orlov and Shulman [16]. This kind of additional flows include dynamic variables explicitly. That additional symmetries form a centerless $W_{1+\infty}$ algebra which is closely related to matrix model [3, 17] because of the Virasoro constraint and string equation. About Toda hierarchy, there was parallel results [15] which was used to give string equations and Riemann-Hilbert problem of dispersionless Toda hierarchy [18]. Because of the close reduction relation between Toda hierarchy and the bigraded Toda hierarchy (BTH), it motivated us to consider the additional symmetry of the BTH. In another paper [10], we give a novel Block type additional symmetry of the BTH. This is the first time to find the direct relation between integrable hierarchy and the Block type algebra. The representation theory of the Block type infinite algebra has been studied intensively in references [19]-[26].

Dispersionless integrable systems have been found very important in the study of all kinds of nonlinear phenomenon in various fields of physics and mathematics, particularly in the application in topological field theory [15] and matrix model theory [27, 28]. In particular, the dispersionless integrable systems have many typical properties as usual integrable systems such as Lax pair, infinite conservation law, symmetry and the Hirota bilinear equations (HBEs). There is a dispersionless limit to get the dBTH from the BTH. However, after taking this limit of the BTH, whether the Block type Lie algebraic structure can be preserved is still an interesting question. So the high relevance of the TH and BTH in mathematical physics motivates us to focus on the HBEs and the additional symmetry of the dBTH in this paper.

The paper is organized as follows. In Section 2 the definition of dBTH and corresponding dispersionless version of Sato theory are introduced. In Section 3, we define Orlov-Schulman’s $\mathcal{M}_L$, $\mathcal{M}_R$ function. The Block type additional symmetry of dBTH will be given in Section 4. In Section 5, we give the quasi-classical limit of BTH to get dBTH. In Section 6, we give the dispersionless Hirota bilinear identity of dBTH which provide a very sound mathematical background in its possible applications. Section 7 is devoted to conclusions and discussions.

2. The dBTH

Introduce firstly the lax operator of dispersionless bigraded Toda hierarchy (dBTH) as following

\begin{equation}
\mathcal{L} = k^N + u_{N-1}k^{N-1} + \cdots + u_M k^{-M}
\end{equation}
(\(N, M \geq 1\) are two fixed positive integers). The variables \(u_j\) are functions of the real variable \(x\). The Lax operator \(L\) can be written in two different dressed ways
\[
L = e^{adϕ}(k^N) = e^{adφ}e^{adϕ}(k^{-M}),
\]
which in fact gives the constraint (string equation) \([29]\) of the two-dimensional dispersionless Toda hierarchy. The two dressing functions have the following form
\[
ϕ_L = w_1 k^{-1} + w_2 k^{-2} + \ldots, \quad (2.2)
\]
\[
ϕ_R = \tilde{w}_1 k + \tilde{w}_2 k^2 + \ldots, \quad (2.3)
\]
We can get the following relation
\[
u_{N-1} = -N \frac{∂w_1}{∂x}, \quad u_{-M}(x) = ϕ(x + M). \quad (2.4)
\]
The pair is unique up to adding some Laurent series about variable \(k\) with coefficients which do not depend on \(x\).

**Definition 2.1.** The dispersionless bigraded Toda hierarchy (dBTH) consists of the system of flows given in the Lax pair formalism by
\[
\frac{∂L}{∂t_{γ,n}} = \{A_{γ,n}, L\} := k(\frac{∂A_{γ,n}}{∂k} \frac{∂L}{∂x} - \frac{∂A_{γ,n}}{∂x} \frac{∂L}{∂k}), \quad (2.5)
\]
for \(γ = N, N-1, N-2, \ldots, -M + 1\) and \(n \geq 0\). The operators \(A_{γ,n}\) are defined by
\[
A_{γ,n} = (L^{n+1-\frac{γ}{2}})_+ \quad \text{for} \quad γ = N, N-1, \ldots, 1 \quad (2.6a)
\]
\[
A_{γ,n} = -(L^{n+1+\frac{γ}{2}})_- \quad \text{for} \quad γ = 0, \ldots, -M + 1. \quad (2.6b)
\]

Particularly for \(N = 1 = M\) this hierarchy coincides with the dispersionless Toda hierarchy.

To see the dBTH clearly, we will introduce two examples as following, i.e. (1,2)-dBTH and (2,2)-dBTH.

**2.1. Example as the (1,2)-dBTH.** The Lax operator is
\[
L = k + u_0 + u_{-1} k^{-1} + u_{-2} k^{-2}. \quad (2.7)
\]
Then there will be one fraction power of \(L\), denoted as \(L^{\frac{1}{2}}\) as following form
\[
L^{\frac{1}{2}} = b_{-1} k^{-1} + b_0 + b_1 k + b_2 k^2 + \ldots. \quad (2.8)
\]
We can get some relations of \(\{b_j; j \geq -1\}\) with \(\{u_i; -M \leq i \leq N-1\}\) as following
\[
b_{-1}^2 = u_{-2}. \quad (2.9)
\]
Then by Lax equation, we get the \(t_{1,0}\) flow of (1,2)-BTH which is equivalent to \(t_{0,0}\) flow as following
\[
∂_{1,0}L = \{k + u_0, L\} \quad (2.10)
\]
which corresponds to
\begin{equation}
\begin{cases}
\partial_{1,0} u_0 &= \frac{\partial u_{-1}}{\partial x} \\
\partial_{1,0} u_{-1} &= \frac{\partial u_{-2}}{\partial x} + u_{-1} \frac{\partial u_0}{\partial x} \\
\partial_{1,0} u_{-2} &= 2 \frac{\partial u_0}{\partial x} u_{-2}
\end{cases}
\tag{2.11}
\end{equation}

By Lax equation, we get the \( t_{-1,0} \) flow of (1,2)-BTH
\begin{equation}
\partial_{-1,0} {\mathcal L} = \{-u_{-2}^{-1} k, \mathcal L\}
\end{equation}
which correspond to
\begin{equation}
\begin{cases}
\partial_{-1,0} u_0 &= \frac{\partial u_{-2}}{\partial x} \\
\partial_{-1,0} u_{-1} &= u_{-2} \frac{\partial u_0}{\partial x} \\
\partial_{-1,0} u_{-2} &= \frac{\partial u_{-1}}{\partial x} u_{-2} - \frac{\partial u_{-2}}{\partial x} u_{-1} \\
0 &= \frac{\partial u_{-2}}{\partial x} u_{-2} - 2 \frac{\partial u_{-2}}{\partial x} u_{-2}
\end{cases}
\tag{2.13}
\end{equation}

2.2. Example as the (2,2)-dBTH. The Lax operator is
\begin{equation}
\mathcal L = k^2 + u_1 k + u_0 + u_{-1} k^{-1} + u_{-2} k^{-2}
\end{equation}
Then there will be two different fraction power of \( \mathcal L \), denoted as \( \mathcal L_{N}^{\frac{1}{2}} \) and \( \mathcal L_{M}^{\frac{1}{2}} \) respectively as following form
\begin{equation}
\mathcal L_{N}^{\frac{1}{2}} = k + a_0 + a_{-1} k^{-1} + a_{-2} k^{-2} + \ldots,
\end{equation}
\begin{equation}
\mathcal L_{M}^{\frac{1}{2}} = a_{-1}^0 k^{-1} + a_{-1}^1 k + a_{-2}^1 k k^2 + \ldots.
\end{equation}
We can get some relations of \( \{a_i; i \leq 0\}, \{a_j^i; j \geq -1\} \) with \( \{u_i; -M \leq i \leq N - 1\} \) as following
\begin{equation}
u_1 = 2a_0, \quad a_{-1}^2 = u_{-2}.
\end{equation}
Then by Lax equation, we get the \( t_{2,0} \) flow of (2,2)-BTH
\begin{equation}
\partial_{2,0} {\mathcal L} = \{k + \frac{1}{2} u_1, \mathcal L\}
\end{equation}
which corresponds to
\begin{equation}
\begin{cases}
\partial_{2,0} u_1 &= \frac{\partial u_0}{\partial x} - \frac{1}{2} \frac{\partial u_1}{\partial x} \\
\partial_{2,0} u_0 &= \frac{\partial u_{-1}}{\partial x} \\
\partial_{2,0} u_{-1} &= \frac{\partial u_{-2}}{\partial x} + \frac{1}{2} \frac{\partial u_{-1}}{\partial x} \\
\partial_{2,0} u_{-2} &= \frac{\partial u_{-2}}{\partial x}
\end{cases}
\tag{2.19}
\end{equation}
Similarly by Lax equation, we get the \( t_{1,0} \) flow of (2,2)-BTH which is equivalent to \( t_{0,0} \) flow as following
\begin{equation}
\partial_{1,0} {\mathcal L} = \{k^2 + u_1 k + u_0, \mathcal L\}
\end{equation}
which corresponds to

\[
\begin{align*}
\partial_{1,0} u_1 &= 2 \frac{\partial u_{-1}}{\partial x} \\
\partial_{1,0} u_0 &= 2 \frac{\partial u_{-2}}{\partial x} + 2 u_1 \frac{\partial u_{-1}}{\partial x} - u_1 \frac{\partial u_0}{\partial x} \\
\partial_{1,0} u_{-1} &= u_1 \frac{\partial u_{-2}}{\partial x} + 2 u_{-2} \frac{\partial u_{-1}}{\partial x} + u_{-1} \frac{\partial u_0}{\partial x} \\
\partial_{1,0} u_{-2} &= 2 \frac{\partial u_0}{\partial x} u_{-2} - 2 \frac{\partial u_{-2}}{\partial x} u_{-1}.
\end{align*}
\]

(2.21)

By Lax equation, we get the \(t_{-1,0}\) flow of (2,2)-BTH

\[
\partial_{-1,0} \mathcal{L} = \{ -u_{-2} \frac{\partial}{\partial x}, \mathcal{L} \}
\]

(2.22)

which corresponds to

\[
\begin{align*}
\partial_{-1,0} u_1 &= 2 \frac{\partial u_{-1}}{\partial x} \\
\partial_{-1,0} u_0 &= \frac{\partial u_{-2}}{\partial x} u_1 \\
\partial_{-1,0} u_{-1} &= u_{-2} \frac{\partial u_{-1}}{\partial x} - \frac{\partial u_{-2}}{\partial x} u_{-1} \\
\partial_{-1,0} u_{-2} &= \frac{\partial u_{-2}}{\partial x} u_{-1} - 2 \frac{\partial u_{-1}}{\partial x} u_{-2}.
\end{align*}
\]

(2.23)

For the convenience to lead to the Sato equation, we will define the following functions:

\[
B_{\gamma,n} := \begin{cases} 
\mathcal{L}^{n+1-\gamma} & \gamma = N \ldots 1 \\
\mathcal{L}^{n+1+\frac{\gamma}{N}} & \gamma = 0 \ldots - M + 1.
\end{cases}
\]

(2.24)

Proposition 2.2. The following identities hold true

\[
(\mathcal{L}^\frac{\gamma}{M})_{t_{\gamma,p}} = \{ - (B_{\gamma,p})_-, \mathcal{L}^\frac{\gamma}{M} \}
\]

(2.25)

\[
(\mathcal{L}^\frac{1}{M})_{t_{\gamma,p}} = \{ (B_{\gamma,p})_+, \mathcal{L}^\frac{1}{M} \}.
\]

(2.26)

The proposition above can lead to the following proposition.

Proposition 2.3. If \(\mathcal{L}\) satisfies the Lax equations then we have the following Zakharov-Shabat equations

\[
(A_{\alpha,m})_{t_{\beta,n}} - (A_{\beta,n})_{t_{\alpha,m}} + \{ A_{\alpha,m}, A_{\beta,n} \} = 0
\]

(2.27)

for \(-M + 1 \leq \alpha, \beta \leq N\), \(m, n \geq 0\).

Using the Zakharov-Shabat eqs.\((2.27)\) the flows of eqs.\((2.5)\) can be proved to commute pairwise.

Lemma 2.4. The following Zakharov-Shabat identities hold

\[
\partial_{\beta,n} (B_{\alpha,m})_- - \partial_{\alpha,m} (B_{\beta,n})_- - \{ (B_{\alpha,m})_-, (B_{\beta,n})_- \} = 0
\]

(2.28)

\[
- \partial_{\beta,n} (B_{\alpha,m})_+ + \partial_{\alpha,m} (B_{\beta,n})_+ - \{ (B_{\alpha,m})_+, (B_{\beta,n})_+ \} = 0
\]

(2.29)
here, \(-M + 1 \leq \alpha, \beta \leq N\), \(m, n \geq 0\).

Then following proposition will appear.

**Proposition 2.5.** There exists \(\phi = \phi(t, x)\) which is characterized by

\[
(2.30) \quad d\phi = \sum_{\gamma = -M+1}^{N} \sum_{n=1}^{\infty} \text{Res}(B_{\gamma,n}d\log k)dt_{\gamma,n} + \frac{1}{M} \log u_{-M} dx,
\]

where “d” means total differentiation in \((t, x)\), and \(d\log k = dk/k\). Furthermore \(\phi\) satisfies the well-known dispersionless (long-wave) limit of the two-dimensional Toda field equation

\[
(2.31) \quad \partial_{t_{-M+1,0}} \partial_{t_{N,0}} \phi + \partial_{x} \exp(\partial_{x} \phi) = 0.
\]

**Proof.** The equation \(2.30\) is a compact form of the following system

\[
(2.32) \quad \partial_{t_{\alpha,m}} \phi = (B_{\alpha,n})_0,
\]

\[
(2.33) \quad \partial_{x} \phi = \frac{1}{M} \log u_{-M}.
\]

Taking the projection of \(2.29\) to \(k^0\) term will lead to following identity

\[-\partial_{\beta,n}(B_{\alpha,m})_0 + \partial_{\alpha,m}(B_{\beta,n})_0 - \{(B_{\alpha,m})_+, (B_{\beta,n})_+\}_0 = 0\]

Because \(\{(B_{\alpha,m})_+, (B_{\beta,n})_+\}_0 = 0\), we get

\[\partial_{\alpha,m}(B_{\beta,n})_0 = \partial_{\beta,n}(B_{\alpha,m})_0\]

i.e. the \(t_{\alpha,m}\) flow and \(t_{\beta,n}\) flow of \(2.32\) are compatible. Now we can see that the solution \(\phi\) of \(2.30\) exists. From \(2.35\), we consider the \(k^M\) part.

\[
(2.34) \quad \frac{\partial u_{-M}}{\partial t_{\alpha,m}} = [k(\frac{\partial (B_{\alpha,m})_+}{\partial k} \frac{\partial L}{\partial x} - \frac{\partial (B_{\alpha,m})_+}{\partial x} \frac{\partial L}{\partial k})]|_{k^M}
\]

\[
(2.35) \quad = M \frac{\partial (B_{\alpha,m})_0}{\partial x} u_{-M},
\]

which is

\[
(2.36) \quad \frac{\partial \log u_{-M}}{\partial t_{\alpha,m}} = M \frac{\partial (B_{\alpha,m})_0}{\partial x},
\]

i.e. eq. \(2.32\) and eq. \(2.33\) are compatible. Considering a special case of \(2.27\)

\[
(2.37) \quad (A_{N,0})_{t_{-M+1,0}} - (A_{-M+1,0})_{t_{N,0}} + \{A_{N,0}, A_{-M+1,0}\} = 0
\]

whose \(k^0\) part is

\[
(2.38) \quad \partial_{t_{-M+1,0}}(A_{N,0})_0 - \partial_{t_{N,0}}(A_{-M+1,0})_0 + \{A_{N,0}, A_{-M+1,0}\}_0 = 0.
\]

So we can get

\[
(2.39) \quad \partial_{t_{-M+1,0}}(B_{N,0})_0 + \{(L^+)_+, -(L^+)_-\}_0 = 0,
\]
which implies

\[ \partial_{t-M+1,0} \partial_{N,0} \phi + \partial_x (u_{-M}) \frac{\phi}{x} = 0, \]

i.e.

\[ \partial_{t-M+1,0} \partial_{N,0} \phi + \partial_x \exp(\partial_x \phi) = 0. \]

This is the end of proof. \( \square \)

Eq. (2.41) is just the dispersionless limit of the generalized two-dimensional Toda field equation.

**Proposition 2.6.** \( L \) is the Lax function of the dispersionless BTH if and only if there exists two Laurent series \( \phi_L, \phi_R \) (dressing function) which satisfies the equations

\[ \nabla_{t,\gamma,n} \phi_L = -(B_{\gamma,n})_-, \quad \nabla_{t,\gamma,n} \phi_R = (B_{\gamma,n})_+, \]

where \(-M + 1 \leq \gamma \leq N, n \geq 0. \) \( \phi_L \) and \( \phi_R \) have the following form

\[ \phi_L = w_1 k^{-1} + w_2 k^{-2} + \ldots, \]

\[ \phi_R = \tilde{w}_1 k + \tilde{w}_2 k^2 + \ldots, \]

where

\[ \text{ad}\phi(\psi) = \{\phi, \psi\}, \quad \nabla_{t,\gamma,n} \phi = \sum_{m=0}^{\infty} \frac{1}{(m+1)!} (\text{ad}\psi)^m \left( \frac{\partial \phi}{\partial t_{\gamma,n}} \right). \]

Such Laurent series \( \phi_L, \phi_R \) are unique up to transformation \( \phi_L \mapsto H(\phi_L, \psi_L), \phi_R \mapsto H(\phi_R, \psi_R), \)

with a constant Laurent series \( \psi_L = \sum_{n=1}^{\infty} \psi_{Ln} k^{-n} \) (\( \psi_{Ln} \): constant), \( \psi_R = \sum_{n=1}^{\infty} \psi_{Rn} k^n \) (\( \psi_{Rn} \): constant) respectively, where \( H(X, Y) \) is the Hausdorff series defined by

\[ \exp(\text{ad}H(\phi, \psi)) = \exp(\text{ad}\phi) \exp(\text{ad}\psi). \]

**Proof.** The proof is standard and similar as proof in [18]. So we omit it here. \( \square \)

With the above preparation, in the next section we will consider the Block type additional symmetry of the dBTH.

### 3. Orlov-Schulman’s \( \mathcal{M}_L, \mathcal{M}_R \) Functions

To introduce the additional symmetry of the dBTH, We firstly define the following Orlov-Schulman’s \( \mathcal{M}_L \) functions as

\[ \mathcal{M}_L = e^{\text{ad} \phi_L} (\Gamma_L) = e^{\text{ad} \phi_L} e^{\text{ad} t_L(k)} \left( \frac{x}{N} k^{-N} \right), \]

where

\[ \Gamma_L = \frac{x}{N} k^{-N} + \sum_{n \geq 0} \sum_{\alpha=1}^{N} (n + 1 - \alpha - 1) \left( 1 - \frac{N}{\alpha} \right) k^{N(n-\frac{N}{\alpha})} t_{\alpha,n}, \]
\[ t_L(k) = \sum_{n \geq 0} \sum_{\alpha=1}^{N} k^{(n+1-\frac{\alpha-1}{N})} t_{\alpha,n}. \] (3.47)

\( M_L \) can be written in another form as following

\[ M_L = \frac{x}{N} L^{-1} + \sum_{m=0}^{\infty} \sum_{\gamma=1}^{N} v_{\gamma,m}(t, x) L^{-(m+2+\frac{1}{M})} + \sum_{n \geq 0} \sum_{\alpha=1}^{N} (n + 1 - \frac{\alpha-1}{N}) L^{n-\frac{\alpha-1}{N}} t_{\alpha,n}, \] (3.48)

where \( L^{-1}_L := e^{ad\xi}(k^{-N}) \). Similarly we define the following Orlov-Schulman’s \( M_R \) functions as following

\[ M_R = e^{ad\xi_R}(\Gamma_R) = e^{ad\xi} e^{ad\varphi_R}(\Gamma_R) = e^{ad\xi} e^{ad\varphi_R}(k^{-1}) \left( -\frac{x}{M} k^M \right), \] (3.49)

where

\[ \Gamma_R = -\frac{x}{M} k^M - \sum_{n \geq 0} \sum_{\beta=-M+1}^{0} (n + 1 + \frac{\beta}{M}) k^{-M(n+\frac{\beta}{M})} t_{\beta,n}, \] (3.50)

\[ t_R(k^{-1}) = \sum_{n \geq 0} \sum_{\beta=-M+1}^{0} k^{-M(n+1+\frac{\beta}{M})} t_{\beta,n}. \] (3.51)

\( M_R \) can be written as following form

\[ M_R = -\frac{x}{M} L^{-1}_R + \sum_{m=0}^{\infty} \sum_{\gamma=-m+1}^{0} \tilde{v}_{\gamma,m}(t, x) L^{-(m+2+\frac{1}{M})} - \sum_{n \geq 0} \sum_{\beta=-M+1}^{0} (n + 1 + \frac{\beta}{M}) L^{n+\frac{\beta}{M}} t_{\beta,n}, \] (3.52)

where \( L^{-1}_R := e^{ad\xi_R}(k^M) \).

A direct calculation shows that the Orlov-Schulman’s functions satisfies the following theorem.

**Theorem 3.1.** The following identities hold

\[ \{\mathcal{L}, M_L\} = 1, \{\mathcal{L}, M_R\} = 1, \] (3.53)

\[ \partial_{\gamma,n} M_L = \{A_{\gamma,n}, M_L\}, \quad \partial_{\gamma,n} M_R = \{A_{\gamma,n}, M_R\}, \] (3.54)

\[ \frac{\partial M_L^k}{\partial t_{\gamma,n}} = \{A_{\gamma,n}, M_L^k\}, \quad \frac{\partial M_R^k}{\partial t_{\gamma,n}} = \{A_{\gamma,n}, M_R^k\}, \] (3.55)

where \( 1 - M \leq \gamma \leq N \).

**Proof.** Before the proof, we firstly set \( 1 \leq \alpha \leq N, -M+1 \leq \beta \leq 0 \). Then the following calculation will lead to one part of the first equation of eq. (3.54)

\[ \partial_{\alpha,n} M_L = \partial_{\alpha,n} e^{ad\xi}(\Gamma_L) = e^{ad\xi_L} \partial_{\alpha,n}(\Gamma_L) + \{\nabla_{\alpha,n} \varphi_L, M_L\}. \]
\[
\begin{align*}
&= e^{ad\phi}L\left(\sum_{n\geq0}^N(n + 1 - \frac{\alpha - 1}{N})k^{N(n - \frac{\alpha - 1}{N})}\right) + \{-\mathcal{B}_{\alpha,n}^-,\mathcal{M}_L\} \\
&= (n + 1 - \frac{\alpha - 1}{N})\mathcal{L}^{n-\frac{\alpha - 1}{N}} + \{-\mathcal{B}_{\alpha,n}^-,\mathcal{M}_L\} \\
&= \{\mathcal{L}^{n+1-\frac{\alpha - 1}{N}},\mathcal{M}_L\} + \{-\mathcal{B}_{\alpha,n}^-,\mathcal{M}_L\} \\
&= \{\mathcal{B}_{\alpha,n},\mathcal{M}_L\} + \{-\mathcal{B}_{\alpha,n}^-,\mathcal{M}_L\} \\
&= \{(\mathcal{B}_{\alpha,n})^+\mathcal{M}_L\}
\end{align*}
\]

Similarly \(t_{\beta,n}\) flow of \(\mathcal{M}_L\) is as following
\[
\begin{align*}
\partial_{t_{\beta,n}}\mathcal{M}_L &= \partial_{t_{\beta,n}}e^{ad\phi}L(\Gamma_L) \\
&= e^{ad\phi}\partial_{t_{\beta,n}}(\Gamma_L) + \{\nabla_{t_{\beta,n}}\phi_L,\mathcal{M}_L\} \\
&= \{-\mathcal{B}_{\beta,n}^-\mathcal{M}_L\},
\end{align*}
\]

which imply the other part of the first equation of eq.(3.54).

In the same way, by calculation we can prove results
\[
\begin{align*}
\partial_{t_{\alpha,n}}\mathcal{M}_R &= \{(\mathcal{B}_{\alpha,n})^+\mathcal{M}_R\} \quad (3.56)
\end{align*}
\]
and
\[
\begin{align*}
\partial_{t_{\beta,n}}\mathcal{M}_R &= \{-\mathcal{B}_{\beta,m}^-\mathcal{M}_R\} \quad (3.57)
\end{align*}
\]

Till now we have finished the proof of eq.(3.54). By eq.(3.54) and eq.(2.5), we can prove eq.(3.55) easily.

□

We can formulate the following 2-form
\[
\omega = \frac{dk}{k} \wedge dx + \sum_{\alpha=-M+1}^N\sum_{n\geq0} dA_{\alpha,n} \wedge dt_{\alpha,n} \quad (3.58)
\]

which satisfies
\[
\begin{align*}
\omega &= 0, \quad (3.59)
\end{align*}
\]

and the following proposition.

**Proposition 3.2.** The identity
\[
\omega \wedge \omega = 0 \quad (3.60)
\]
is equivalent to the Zakharov-Shabat equations.

**Proof.** Eq.(3.58) can have the following detailed representation
\[
\omega = \frac{dk}{k} \wedge dx + \sum_{\alpha,\beta=-M+1}^N\sum_{m,n\geq0} \frac{\partial A_{\alpha,n}}{\partial t_{\beta,m}} dt_{\beta,m} \wedge dt_{\alpha,n} + \sum_{\alpha=-M+1}^N\sum_{n\geq0} \left(\frac{\partial A_{\alpha,n}}{\partial k} dk + \frac{\partial A_{\alpha,n}}{\partial x} dx\right) \wedge dt_{\alpha,n}.
\]
We can construct wedge product $\omega \wedge \omega$ as following

$$\omega \wedge \omega = \left( \frac{dk}{k} \wedge dx + \sum_{\alpha, \beta = -M+1}^{N} \sum_{n \geq 0} \frac{\partial A_{\alpha,n}}{\partial \beta,m} dt_{\beta,m} \wedge dt_{\alpha,n} + \sum_{\alpha = -M}^{N} \sum_{n \geq 0} \left( \frac{\partial A_{\alpha,n}}{\partial k} dk + \frac{\partial A_{\alpha,n}}{\partial x} dx \right) \wedge dt_{\alpha,n} \right) \wedge \left( \frac{dk}{k} \wedge dx + \sum_{\alpha, \beta = -M+1}^{N} \sum_{n \geq 0} \frac{\partial A_{\alpha,n}}{\partial \beta,m} dt_{\beta,m} \wedge dt_{\alpha,n} + \sum_{\alpha = -M}^{N} \sum_{n \geq 0} \left( \frac{\partial A_{\alpha,n}}{\partial k} dk + \frac{\partial A_{\alpha,n}}{\partial x} dx \right) \wedge dt_{\alpha,n} \right)$$

$$= \frac{2}{k} \sum_{\alpha, \beta = -M+1}^{N} \sum_{n \geq 0} \left( \frac{\partial A_{\alpha,n}}{\partial k} \frac{\partial A_{\beta,m}}{\partial t_{\alpha,n}} - \frac{\partial A_{\beta,m}}{\partial k} \frac{\partial A_{\alpha,n}}{\partial t_{\alpha,n}} \right) dk \wedge dx \wedge dt_{\beta,m} \wedge dt_{\alpha,n}$$

$$+ 2 \sum_{\alpha, \beta, \gamma = -M+1}^{N} \sum_{n \geq 0} \left[ \left( \frac{\partial A_{\alpha,n}}{\partial \beta,m} \frac{\partial A_{\beta,m}}{\partial t_{\alpha,n}} - \frac{\partial A_{\beta,m}}{\partial \beta,m} \frac{\partial A_{\alpha,n}}{\partial t_{\alpha,n}} \right) \frac{\partial A_{\gamma,l}}{\partial k} dk \wedge dt_{\beta,m} \wedge dt_{\alpha,n} \wedge dt_{\gamma,l} \right.$$

$$+ \left( \frac{\partial A_{\gamma,l}}{\partial t_{\alpha,n}} - \frac{\partial A_{\alpha,n}}{\partial t_{\alpha,n}} \right) \frac{\partial A_{\beta,m}}{\partial k} dk \wedge dt_{\beta,m} \wedge dt_{\alpha,n} \wedge dt_{\gamma,l}\right.$$

$$+ 2 \sum_{\alpha, \beta, \gamma = -M+1}^{N} \sum_{n \geq 0} \left[ \left( \frac{\partial A_{\alpha,n}}{\partial \beta,m} \frac{\partial A_{\beta,m}}{\partial t_{\alpha,n}} - \frac{\partial A_{\beta,m}}{\partial \beta,m} \frac{\partial A_{\alpha,n}}{\partial t_{\alpha,n}} \right) \frac{\partial A_{\gamma,l}}{\partial x} \right.$$

$$+ \left( \frac{\partial A_{\gamma,l}}{\partial t_{\alpha,n}} - \frac{\partial A_{\alpha,n}}{\partial t_{\alpha,n}} \right) \frac{\partial A_{\beta,m}}{\partial x} \right) dx \wedge dt_{\beta,m} \wedge dt_{\alpha,n} \wedge dt_{\gamma,l}.$$ 

So $\omega \wedge \omega = 0$ is equivalent to $\frac{\partial A_{\alpha,n}}{\partial \beta,m} \frac{\partial A_{\beta,m}}{\partial t_{\alpha,n}} + k \left( \frac{\partial A_{\alpha,n}}{\partial k} \frac{\partial A_{\beta,m}}{\partial x} - \frac{\partial A_{\beta,m}}{\partial k} \frac{\partial A_{\alpha,n}}{\partial x} \right) = 0$, i.e. $\frac{\partial A_{\alpha,n}}{\partial \beta,m} \frac{\partial A_{\beta,m}}{\partial t_{\alpha,n}} + \{A_{\alpha,n}, A_{\beta,m}\} = 0$ which is just eq. (2.27). □

Similarly we get the following corollary.

**Proposition 3.3.** The $dBTH$ is equivalent to the following exterior differential equations

(3.61) $dL \wedge dM_L = dL \wedge dM_R = \omega$

**Proof.** By expanding the left side of identity

(3.62) $dL \wedge dM_L = \omega,$

the following result can be got.

$$dL \wedge dM_L = \left( \frac{\partial L}{\partial k} dk + \frac{\partial L}{\partial x} dx + \sum_{\alpha = -M+1}^{N} \sum_{n \geq 0} \frac{\partial L}{\partial t_{\alpha,n}} dt_{\alpha,n} \right) \wedge$$
\[
\left( \frac{\partial M_L}{\partial k} dk + \frac{\partial M_L}{\partial x} dx + \sum_{\alpha = -M+1}^{N} \sum_{n \geq 0} \frac{\partial M_L}{\partial t_{\alpha,n}} dt_{\alpha,n} \right)
\]
\[
= \left( \frac{dk}{k} \wedge dx + \sum_{\alpha, \beta = -M+1}^{N} \sum_{m,n \geq 0} \frac{\partial A_{\alpha,n}}{\partial t_{\beta,m}} dt_{\beta,m} \wedge dt_{\alpha,n} + \sum_{\alpha = -M+1}^{N} \sum_{n \geq 0} \frac{\partial A_{\alpha,n}}{\partial k} dk \right)
\]

By comparing the coefficients of \(dk \wedge dx\), we get the canonical relation \(\{L, M_L\} = 1\). By comparing the coefficients of \(dk \wedge dt_{\alpha,n}, dx \wedge dt_{\alpha,n}, dt_{\beta,m} \wedge dt_{\alpha,n}\), we get the canonical relation

\[
\frac{\partial L}{\partial k} \frac{\partial M_L}{\partial t_{\alpha,n}} - \frac{\partial M_L}{\partial k} \frac{\partial L}{\partial t_{\alpha,n}} = \frac{\partial A_{\alpha,n}}{\partial k}
\]

\[
\frac{\partial L}{\partial x} \frac{\partial M_L}{\partial t_{\alpha,n}} - \frac{\partial M_L}{\partial x} \frac{\partial L}{\partial t_{\alpha,n}} = \frac{\partial A_{\alpha,n}}{\partial x}
\]

\[
\frac{\partial L}{\partial t_{\beta,m}} \frac{\partial M_L}{\partial t_{\alpha,n}} - \frac{\partial M_L}{\partial t_{\beta,m}} \frac{\partial L}{\partial t_{\alpha,n}} = \frac{\partial A_{\alpha,n}}{\partial t_{\beta,m}} - \frac{\partial A_{\beta,m}}{\partial t_{\alpha,n}}
\]

eq. (3.63) and eq. (3.64) imply eq. (2.5), eq. (3.54). Eq. (2.5), eq. (3.54) can lead to eq. (3.65) as following.

\[
\frac{\partial L}{\partial t_{\beta,m}} \frac{\partial M_L}{\partial t_{\alpha,n}} - \frac{\partial M_L}{\partial t_{\beta,m}} \frac{\partial L}{\partial t_{\alpha,n}} = \frac{A_{\alpha,n}}{\partial t_{\beta,m}} - \frac{A_{\beta,m}}{\partial t_{\alpha,n}}
\]

In the same way, we can get all the equations for \(M_R\) from the second identity in eq. (3.61). □

Using equation (3.48) and (3.52), taking derivatives of them will lead to following lemma.
Lemma 3.4. Following formula will hold
\[\frac{\partial v_{\gamma,n}(t,x)}{\partial t_{\gamma_2,m}} = \text{Res} B_{\gamma_1,n} d_k A_{\gamma_2,m},\]
where \(-M + 1 \leq \gamma, \gamma_1, \gamma_2 \leq N.\)

Proof. Firstly we take derivatives of \(M_L\) by \(t_{\alpha,n}\) and get following calculation,
\[\frac{\partial M_L}{\partial t_{\alpha,n}} = \frac{\partial M_L}{\partial L} \frac{\partial L}{\partial t_{\alpha,n}} + \sum_{n=0}^{N} \sum_{\gamma=1}^{N} \frac{\partial v_{\gamma,m}(t,x)}{\partial t_{\alpha,n}} L^{-(m+1+\frac{1-\gamma}{N})} + \sum_{n=0}^{N} \sum_{\alpha=1}^{N} (n+1+\frac{1-\gamma}{N}) B_{\alpha,n-1} \]
where \(v_{1,0} = \frac{x}{N},\) which leads to
\[\frac{\partial v_{\gamma,m}(t,x)}{\partial t_{\alpha,n}} = \text{Res} L^{m+1-\frac{1-\gamma}{N}} (\frac{\partial M_L}{\partial t_{\alpha,n}} d_k \mathcal{L} - \frac{\partial M_L}{\partial L} \frac{\partial L}{\partial t_{\alpha,n}} d_k \mathcal{L} - \sum_{n=0}^{N} \sum_{\alpha=1}^{N} (n+1+\frac{1-\gamma}{N}) B_{\alpha,n-1} d_k \mathcal{L}) \]
which leads to
\[\frac{\partial v_{\gamma,m}(t,x)}{\partial x} = \text{Res} L^{m+1-\frac{1-\gamma}{N}} (\frac{\partial M_L}{\partial k} d_k \mathcal{L} - \frac{\partial M_L}{\partial L} \frac{\partial L}{\partial k} d_k \mathcal{L}) \]
Now we take derivatives of \(M_L\) by \(x\) and get following calculation,
\[\frac{\partial M_L}{\partial x} = \frac{\partial M_L}{\partial L} \frac{\partial L}{\partial x} + \sum_{n=0}^{N} \sum_{\gamma=1}^{N} \frac{\partial v_{\gamma,m}(t,x)}{\partial x} L^{-(m+1+\frac{1-\gamma}{N})},\]
which leads to
\[\frac{\partial v_{\gamma,m}(t,x)}{\partial x} = \text{Res} L^{m+1-\frac{1-\gamma}{N}} (\frac{\partial M_L}{\partial x} d_k \mathcal{L} - \frac{\partial M_L}{\partial L} \frac{\partial L}{\partial x} d_k \mathcal{L}) \]
The other cases can be proven in similar ways. □

By the Lemma above, it is time to introduce $S$ functions which is included in the following proposition.

**Proposition 3.5.** There exist functions $S_L$ and $S_R$ which satisfy

$$
(3.71) \quad dS_L = M_L d\mathcal{L} + \log k dx + \sum_{\alpha = -M+1}^{N} \sum_{n \geq 0} A_{\alpha,n} \wedge dt_{\alpha,n},
$$

$$
(3.72) \quad dS_R = M_R d\mathcal{L} + \log k dx + \sum_{\beta = -M+1}^{N} \sum_{n \geq 0} A_{\beta,n} \wedge dt_{\beta,n},
$$

where $S_L, S_R$ have the following Laurent expansion.

$$
(3.73) \quad S_L = \frac{x}{N} \log \mathcal{L} - \sum_{m=0}^{\infty} \sum_{\gamma=1}^{N} v_{\gamma,m}(t,x) \left( \frac{\mathcal{L}^{-(m+1+\frac{1-\gamma}{N})}}{m+1+\frac{1-\gamma}{N}} \right) + \sum_{n \geq 0} \sum_{\alpha=1}^{N} B_{\alpha,n} t_{\alpha,n},
$$

$$
(3.74) \quad S_R = -\frac{x}{M} \log \mathcal{L} - \sum_{m=0}^{\infty} \sum_{\gamma=-M+1}^{0} v_{\gamma,m}(t,x) \left( \frac{\mathcal{L}^{-(m+1+\frac{\gamma}{M})}}{m+1+\frac{\gamma}{M}} \right) - \sum_{n \geq 0} \sum_{\beta=-M+1}^{0} B_{\beta,n} t_{\beta,n}.
$$

**Proof.** We can prove the right hand sides of eq.(3.71) and eq.(3.72) are closed according to eq.(3.61). That implies the existence of $S_L, S_R$.

By Lemma 3.4 we can prove the form of $S_L$ is correct using following computation,

$$
\frac{\partial S_L}{\partial t_{\alpha,n}}|_{t_{\beta,m}(\beta \neq \alpha, \text{ or } n \neq m),x}
= -\sum_{m=0}^{\infty} \sum_{\gamma=1}^{N} \partial v_{\gamma,m}(t,x) \left( \frac{\mathcal{L}^{-(m+1+\frac{1-\gamma}{N})}}{m+1+\frac{1-\gamma}{N}} \right) + B_{\alpha,n}
= -\sum_{m=0}^{\infty} \sum_{\gamma=1}^{N} (\text{Res } B_{\gamma,m} d_k A_{\alpha,n}) \left( \frac{\mathcal{L}^{-(m+1+\frac{1-\gamma}{N})}}{m+1+\frac{1-\gamma}{N}} \right) + B_{\alpha,n}
= \sum_{m=0}^{\infty} \sum_{\gamma=1}^{N} (\text{Res } A_{\alpha,n} d_k B_{\gamma,m}) \left( \frac{\mathcal{L}^{-(m+1+\frac{1-\gamma}{N})}}{m+1+\frac{1-\gamma}{N}} \right) + B_{\alpha,n}
= \sum_{m=0}^{\infty} \sum_{\gamma=1}^{N} (\text{Res } A_{\alpha,n} \mathcal{L}^{m+\frac{1-\gamma}{N}} d_k \mathcal{L}) \left( \frac{\mathcal{L}^{-(m+1+\frac{1-\gamma}{N})}}{m+1+\frac{1-\gamma}{N}} \right) + B_{\alpha,n}
= -(B_{\alpha,n})_- + B_{\alpha,n}
$$
Now, we will define the tau function of dBTH in following proposition.

**Proposition 3.6.** There exists a function \( \tau_d(t, x) \) which satisfies

\[
\frac{d}{dt} \log \tau_d(t, x) = \sum_{n \geq 0} \sum_{\alpha=1}^{N} v_{\alpha,n} dt_{\alpha,n} + \sum_{n \geq 0} \sum_{\beta=-M+1}^{0} \bar{v}_{\beta,n} dt_{\beta,n} + \phi dx
\]  

(3.75)

**Proof.** To prove the existence of tau function, we need to prove the compatibility of all the time flows which can be shown in following calculation,

\[
\frac{\partial v_{\alpha_1,n_1}}{\partial t_{\alpha_2,n_2}} = \text{Res } B_{\alpha_1,n_1} d_k(B_{\alpha_2,n_2}) + \text{Res}(B_{\alpha_1,n_1}) - d_k(B_{\alpha_2,n_2}) + \text{Res}(B_{\alpha_1,n_1}) - d_k B_{\alpha_2,n_2} = \frac{\partial v_{\alpha_2,n_2}}{\partial t_{\alpha_1,n_1}},
\]

\[
\frac{\partial v_{\alpha,n}}{\partial t_{\beta,m}} = - \text{Res } B_{\alpha,n} d_k(B_{\beta,m}) - \text{Res}(B_{\alpha,n}) + d_k(B_{\beta,m}) - \text{Res}(B_{\alpha,n}) + d_k B_{\beta,m} = \frac{\partial \bar{v}_{\beta,m}}{\partial t_{\alpha,n}},
\]

\[
\frac{\partial v_{\gamma,n}}{\partial x} = \text{Res } B_{\gamma,n} d \log k = (B_{\gamma,n})_0 = \frac{\partial \phi}{\partial t_{\gamma,n}}.
\]

The other cases for commutativity can be proved in similar ways. So the 1-form of the right side of eq. (3.75) is closed. Therefore, there exist a tau function which satisfies eq. (3.75). □

---

**4. Additional Symmetries of dBTH**

We are now in a position to define the additional flows, and then to prove that they are symmetries, which are called additional symmetries of the dBTH. We introduce additional independent variables \( t^*_{m,l} \), and define the action of the additional flows on the wave operator as

\[
\nabla_{t^*_{m,l}\phi_L} \phi_L = - \left( (\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l \right)_- ,
\]

(4.76)
\[ \nabla_{t^*_m,l} \varphi_R \varphi_R = ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^+ . \]

**Proposition 4.1.** The additional flows act on \( \mathcal{L} \) and \( \mathcal{M}_L, \mathcal{M}_R \) as

\[ \frac{\partial \mathcal{L}}{\partial t^*_m,l} = \left\{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^+, \mathcal{L} \right\}, \]

\[ \frac{\partial \mathcal{M}_L}{\partial t^*_m,l} = -\left\{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^-, \mathcal{M}_L \right\}, \]

\[ \frac{\partial \mathcal{M}_R}{\partial t^*_m,l} = \left\{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^+, \mathcal{M}_R \right\}. \]

**Proof** By performing the derivative on \( \mathcal{L} \) and using eq. (4.78), we get

\[ \partial_{t^*_m,l} \mathcal{L} = \partial_{t^*_m,l} \left( e^{\text{ad} \varphi_L(k^N)} \right) \]

\[ = \{ \nabla_{t^*_m,l} \varphi_L, e^{\text{ad} \varphi_L(k^N)} \} \]

\[ = -\left\{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^-, \mathcal{L} \right\}. \]

For the action on \( \mathcal{M}_L \) and \( \mathcal{M}_R \) given in eq. (4.79), there exists similar derivation as \( \partial_{t^*_m,l} \mathcal{L} \), i.e.

\[ \partial_{t^*_m,l} \mathcal{M}_L = \partial_{t^*_m,l} \left( e^{\text{ad} \varphi_L(\Gamma_L)} \right) \]

\[ = \{ \nabla_{t^*_m,l} \varphi_L, e^{\text{ad} \varphi_L(\Gamma_L)} \} \]

\[ = -\left\{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^-, \mathcal{M}_L \right\}. \]

Here the fact that \( \Gamma_L \) does not depend on the additional flows variables \( t^*_m,l \) has been used.

Other identities can also be got in the similar way. \( \square \)

From that, we can prove the following corollary:

**Corollary 4.2.** The following several equations hold

\[ \frac{\partial \mathcal{L}^n}{\partial t^*_{m,l}} = \{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^+, \mathcal{L}^n \}, \quad \frac{\partial \mathcal{L}^n}{\partial t^*_{m,l}} = -\{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^-, \mathcal{L}^n \} \]

\[ \frac{\partial B_{\alpha,n}}{\partial t^*_{m,l}} = -\{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^-, B_{\alpha,n} \}, \quad \frac{\partial B_{\beta,n}}{\partial t^*_{m,l}} = \{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^+, B_{\beta,n} \} \]

\[ \frac{\partial \mathcal{M}^n_L}{\partial t^*_{m,l}} = -\{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^-, \mathcal{M}^n_L \}, \quad \frac{\partial \mathcal{M}^n_R}{\partial t^*_{m,l}} = \{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^+, \mathcal{M}^n_R \} \]

\[ \frac{\partial \mathcal{M}^n_L \mathcal{L}^k}{\partial t^*_{m,l}} = -\{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^-, \mathcal{M}^n_L \mathcal{L}^k \}, \]

\[ \frac{\partial \mathcal{M}^n_R \mathcal{L}^k}{\partial t^*_{m,l}} = \{ ((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}_l)^+, \mathcal{M}^n_R \mathcal{L}^k \}. \]
Proof We present here only the proof of the first equation. The others can be proved in a similar way. The derivative of $L^n$ with respect to $t^*_m,l$ leads to

$$\frac{\partial L^n}{\partial t^*_m,l} = \frac{\partial L}{\partial t^*_m,l} L^{n-1} + L \frac{\partial L}{\partial t^*_m,l} L^{n-2} + \cdots + L^{n-2} \frac{\partial L}{\partial t^*_m,l} L + L^{n-1} \frac{\partial L}{\partial t^*_m,l} = \sum_{k=1}^n L^{k-1} \frac{\partial L}{\partial t^*_m,l} L^{n-k},$$

and then taking $\frac{\partial L}{\partial t^*_m,l} = \{(M_L - M_R)^m L^l\}_+, L\} \text{ into the above formula. After that, we get}$

$$\frac{\partial L^n}{\partial t^*_m,l} = \sum_{k=1}^n L^{k-1} \{(M_L - M_R)^m L^l\}_+, L\} L^{n-k} = \{(M_L - M_R)^m L^l\}_+ L^n.$$

Proposition 4.3. The additional flows $\frac{\partial}{\partial t^*_m,l}$ commute with the dispersionless bigraded Toda hierarchy flows $\frac{\partial}{\partial \varphi_{c,n}}$, i.e.

$$(4.86) \quad [\partial^*_{m,l}, \partial_{c,n}] L = 0,$$

where $-M + 1 \leq c \leq N$. Here $\partial^*_{m,l} = \frac{\partial}{\partial t^*_m,l}, \partial_{c,n} = \frac{\partial}{\partial \varphi_{c,n}}$.

Proof. According to the definition and using the action of the additional flows on $L$, we get

$$[\partial^*_{m,l}, \partial_{c,n}] L = \partial^*_{m,l} \partial_{c,n} e^{ad\varphi_L} (k^N) - \partial_{c,n} \partial^*_{m,l} e^{ad\varphi_L} (k^N)$$

$$= \partial^*_{m,l} \{\nabla_{c,n}, \varphi_L, e^{ad\varphi_L} (k^N)\} - \partial_{c,n} \{\nabla^*_{m,l}, \varphi_L, e^{ad\varphi_L} (k^N)\}$$

$$= \partial^*_{m,l} \{-\mathcal{B}_{c,n}, L\} - \partial_{c,n} \{(M_L - M_R)^m L^l\}_+, \mathcal{L}\}$$

$$= \{-((M_L - M_R)^m L^l)_+, \mathcal{B}_{c,n}\}, \mathcal{L}\} + \{-\mathcal{B}_{c,n}, \{(M_L - M_R)^m L^l\}_+, \mathcal{L}\} - \{(M_L - M_R)^m L^l\}_+, \{-\mathcal{B}_{c,n}, \mathcal{L}\}\}$$

$$= \{-((M_L - M_R)^m L^l)_+, \mathcal{L}\} + \{-\mathcal{B}_{c,n}, 1\} + \{(M_L - M_R)^m L^l\}_+, \mathcal{B}_{c,n}\} + \{\mathcal{L}, \{(M_L - M_R)^m L^l\}_+, \mathcal{B}_{c,n}\}$$

$$= 0.$$
Proposition 4.4. Additional flows $\partial^*_{m, l}$ (m ≥ 0, l ≥ 0) form the following Block type Lie algebra

\[(4.87) \quad [\partial^*_{m, l}, \partial^*_{n, k}] \mathcal{L} = (km - nl)\partial^*_{m+n-1, k+l-1} \mathcal{L}.\]

where m, n ≥ 0; l, k ≥ 0.

Proof. By using Proposition [4.1] we get

\[\begin{align*}
[\partial^*_{m, l}, \partial^*_{n, k}] \mathcal{L} &= \partial^*_{m, l} (\partial^*_{n, k} \mathcal{L}) - \partial^*_{n, k} (\partial^*_{m, l} \mathcal{L}) \\
&= -\partial^*_{m, l} \{((\mathcal{M}_L - \mathcal{M}_R)^n \mathcal{L}^k)_-, \mathcal{L}\} + \partial^*_{n, k} \{((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}^l)_-, \mathcal{L}\} \\
&= -\{\partial^*_{m, l} (\mathcal{M}_L - \mathcal{M}_R)^n \mathcal{L}^k)_-, \mathcal{L}\} - \{((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}^l)_-, \partial^*_{n, k} \mathcal{L}\} \\
&\quad + \{\partial^*_{n, k} (\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}^l)_-, \mathcal{L}\} + \{((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}^l)_-, \partial^*_{n, k} \mathcal{L}\},
\end{align*}\]

which further leads to the following calculation

\[\begin{align*}
[\partial^*_{m, l}, \partial^*_{n, k}] \mathcal{L} &= - \left\{ \sum_{p=0}^{n-1} (\mathcal{M}_L - \mathcal{M}_R)^p (\partial^*_{m, l} (\mathcal{M}_L - \mathcal{M}_R))(\mathcal{M}_L - \mathcal{M}_R)^{n-p-1} \mathcal{L}^k + (\mathcal{M}_L - \mathcal{M}_R)^n (\partial^*_{m, l} \mathcal{L}^k)_-, \mathcal{L} \right\} \\
&\quad - \{((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}^l)_-, \partial^*_{n, k} \mathcal{L}\} \\
&\quad + \left\{ \sum_{p=0}^{m-1} (\mathcal{M}_L - \mathcal{M}_R)^p (\partial^*_{n, k} (\mathcal{M}_L - \mathcal{M}_R))(\mathcal{M}_L - \mathcal{M}_R)^{m-p-1} \mathcal{L}^l + (\mathcal{M}_L - \mathcal{M}_R)^m (\partial^*_{n, k} \mathcal{L}^l)_-, \mathcal{L} \right\} \\
&\quad + \{((\mathcal{M}_L - \mathcal{M}_R)^m \mathcal{L}^l)_-, \partial^*_{n, k} \mathcal{L}\} \\
&= \{((nl - km) \mathcal{M}_L - \mathcal{M}_R)^{m+n-1} \mathcal{L}^{k+l-1})_-, \mathcal{L} \} \\
&= (km - nl)\partial^*_{m+n-1, k+l-1} \mathcal{L}.
\end{align*}\]

From Proposition [4.4] it can be seen that the additional symmetry has a nice Block type Lie algebraic structure whose structure theory and representation theory have recently received much attention. The difference of this Block Lie algebra from the one in [10] is the representation space here is functional space and the one in [10] is space of operators. Similarly as [10], the action of this kind of Lie algebra on tau function space is still hard to handle with.

5. Quasi-classical limit of the BTH

To consider the quasi-classical limit of the BTH, it is convenient to introduce the order and the principal symbol of functions of difference operators. The order and the principal symbol are defined for the difference operators as follows [18].

Define the order of operator $a_{n,m}(t,x)e^{n\partial_x}$ as following

\[ord \left( \sum a_{n,m}(t,x)e^{n\partial_x} \right) = \max \{n \mid a_{n,m}(t,x) \neq 0 \}.\]
The principal symbol of a difference operator $A = \sum a_{n,m} e^{m \partial_x}$ is defined as

$$\sigma^\epsilon(A) = \epsilon^{-\text{ord}(A)} \sum_{n=\text{ord}(A)} \sum_{m} a_{n,m} k^m.$$

To see the limit, we need to recall the Lax operator of the BTH given by the Laurent polynomial of $\Lambda$ [7]

$$L := \Lambda^N + u_{N-1} \Lambda^{N-1} + \cdots + u_0 + \cdots + u_{-M} \Lambda^{-M}.$$  

The $L$ can be written in two different ways by dressing the shift operator

$$L = \mathcal{P}_L \Lambda^N \mathcal{P}_L^{-1} = \mathcal{P}_R \Lambda^{-M} \mathcal{P}_R^{-1},$$

where the dressing operators have the form,

$$\mathcal{P}_L = 1 + w_1 \Lambda^{-1} + w_2 \Lambda^{-2} + \ldots,$$

$$\mathcal{P}_R = \tilde{w}_0 + \tilde{w}_1 \Lambda + \tilde{w}_2 \Lambda^2 + \ldots.$$

Eq. (5.2) are quite important because it gives the reduction condition from the two-dimensional Toda lattice hierarchy. The pair is unique up to multiplying $\mathcal{P}_L$ and $\mathcal{P}_R$ from the right by operators in the form $1 + a_1 \Lambda^{-1} + a_2 \Lambda^{-2} + \ldots$ respectively with coefficients independent of $x$. Given any difference operator $A = \sum k A_k \Lambda^k$, the positive and negative projections are defined by $A_+ = \sum_{k \geq 0} A_k \Lambda^k$ and $A_- = \sum_{k < 0} A_k \Lambda^k$.

To write out explicitly the Lax equations of BTH, fractional powers $L^{\frac{1}{N}}$ and $L^{\frac{1}{M}}$ are defined by

$$L^{\frac{1}{N}} = \Lambda + \sum_{k \leq 0} a_k \Lambda^k, \quad L^{\frac{1}{M}} = \sum_{k \geq -1} b_k \Lambda^k,$$

with the relations

$$(L^{\frac{1}{N}})^N = (L^{\frac{1}{M}})^M = L.$$

Acting on free function, these two fraction powers can be seen as two different locally expansions around zero and infinity respectively. It was stressed that $L^{\frac{1}{N}}$ and $L^{\frac{1}{M}}$ are two different operators even if $N = M(N,M \geq 2)$ in [7] due to two different dressing operators. They can also be expressed as following

$$L^{\frac{1}{N}} = \mathcal{P}_L \Lambda \mathcal{P}_L^{-1}, \quad L^{\frac{1}{M}} = \mathcal{P}_R \Lambda^{-1} \mathcal{P}_R^{-1}.$$

Definition 5.1. The bigraded Toda hierarchy consists of the system of flows given in the Lax pair formalism by

$$\frac{\partial L}{\partial t_{\alpha,n}} = [A_{\alpha,n}, L]$$

for $\alpha = N, N-1, N-2, \ldots, -M + 1$ and $n \geq 0$. The operators $A_{\alpha,n}$ are defined by

$$A_{\alpha,n} = (L^{n+1-\frac{\alpha}{N}})_+ \quad \text{for} \quad \alpha = N, N-1, \ldots, 1,$$

$$A_{\alpha,n} = -(L^{n+1+\frac{\alpha}{M}})_- \quad \text{for} \quad \alpha = 0, \ldots, -M + 1.$$
The coefficients $u_i$ of $L$ are set to be regular to $\epsilon$, i.e. $u_i(\epsilon, t, x) = u_0^i(t, x) + O(\epsilon)$.

**Proposition 5.2.** If $L$ satisfies the Lax equations then we have the following Zakharov-Shabat equations

$$
\epsilon (A_{\alpha,m})_{t_{\beta,n}} - \epsilon (A_{\beta,n})_{t_{\alpha,m}} + [A_{\alpha,m}, A_{\beta,n}] = 0
$$

for $-M + 1 \leq \alpha, \beta \leq N$, $m, n \geq 0$.

Using the Zakharov-Shabat eqs. (2.27) we can prove that the flows of eqs. (5.5) can commute pairwise.

**Lemma 5.3.**

$$
\epsilon \partial_{\beta,n} (B_{\alpha,m})_+ - \epsilon \partial_{\alpha,m} (B_{\beta,n})_+ - [(B_{\alpha,m})_-, (B_{\beta,n})_-] = 0
$$

$$
\epsilon \partial_{\beta,n} (B_{\alpha,m})_- + \epsilon \partial_{\alpha,m} (B_{\beta,n})_- - [(B_{\alpha,m})_+, (B_{\beta,n})_+] = 0
$$

here, $-M + 1 \leq \alpha, \beta \leq N$, $m, n \geq 0$.

where

$$
B_{\gamma,n} := \begin{cases} 
L^{n+1-\frac{\gamma}{N}} & \gamma = N \ldots 1 \\
L^{n+1+\frac{\gamma}{M}} & \gamma = 0 \ldots -M + 1.
\end{cases}
$$

**Theorem 5.4.** $L$ is a solution to the BTH if and only if there is a pair of dressing operators $\mathcal{P}_L$ and $\mathcal{P}_R$, which satisfies the following Sato equations:

$$
\epsilon \partial_{\gamma,n} \mathcal{P}_L = -(B_{\gamma,n})_- \mathcal{P}_L,
$$

$$
\epsilon \partial_{\gamma,n} \mathcal{P}_R = (B_{\gamma,n})_+ \mathcal{P}_R,
$$

where, $-M + 1 \leq \gamma \leq N$, $n \geq 0$.

In paper [10], we defined the Orlov-Schulman’s $M_L$, $M_R$ operators as following

$$
M_L = \mathcal{P}_L \Gamma_L \mathcal{P}_L^{-1}, \quad M_R = \mathcal{P}_R \Gamma_R \mathcal{P}_R^{-1},
$$

where

$$
\Gamma_L = \frac{x}{N\epsilon} \Lambda^{-N} + \sum_{n \geq 0} \sum_{\alpha = 1}^{N} (n + 1 - \frac{\alpha - 1}{N}) \epsilon^{-1} \Lambda^{N(n-\frac{\alpha-1}{N})} t_{\alpha,n},
$$

$$
\Gamma_R = -\frac{x}{M\epsilon} \Lambda^{M} - \sum_{n \geq 0} \sum_{\beta = -M+1}^{0} (n + 1 + \frac{\beta}{M}) \epsilon^{-1} \Lambda^{-M(n+\frac{\beta}{M})} t_{\beta,n}.
$$

Therefore $M_L$ and $M_R$ can be written in another form as following

$$
M_L = \frac{x}{N} L^{-1} + \sum_{m=0}^{\infty} \sum_{\gamma=1}^{N} u_{\gamma,m}(t, x) L^{-(m+2+\frac{1}{N})} + \sum_{n \geq 0} \sum_{\alpha=1}^{N} (n + 1 - \frac{\alpha - 1}{N}) L^{n-\frac{\alpha-1}{N}} t_{\alpha,n},
$$
\(M_R = -\frac{x}{M}L_R^{-1} + \sum_{m=0}^{\infty} \sum_{\gamma=-m+1}^{0} \bar{v}_{\gamma,m}(t,x)L^{-(m+2+\frac{2}{M})} - \sum_{n\geq 0} \sum_{\beta=-M+1}^{0} (n + 1 + \frac{\beta}{M})L^{n+\frac{2}{M}}t_{\beta,n},\)

where \(L_R^{-1} := P_L \Lambda^{-N}P_L^{-1}, \quad L_R^{-1} := P_R \Lambda^M P_R^{-1}.\)

To consider the quasi-classical limit, we set
\[
\begin{align*}
\mathcal{P}_L &= \exp(\epsilon^{-1}X_L(\epsilon, t, x)), \\
\mathcal{P}_R &= \exp(\phi(\epsilon, t, x)) \exp(\epsilon^{-1}X_R(\epsilon, t, x)), \\
\text{ord}(X_L(\epsilon, t, x)) &= \text{ord}(X_R(\epsilon, t, x)) = 0, \\
\sigma^\epsilon(X_L) &= \varphi_L, \quad \sigma^\epsilon(X_R) = \varphi_R,
\end{align*}
\]

where \(\text{ord}(\phi(\epsilon, t, x)) \leq 0\) and
\[
\begin{align*}
\partial_{\alpha,n} \phi &= (B_{\alpha,n})_0, \\
\frac{1}{\epsilon} (\phi(\epsilon, t, x) - \phi(\epsilon, t, x - \epsilon)) &= \frac{1}{M} \log u - M,
\end{align*}
\]

where \((\ )_0\) is projection to operators not containing \(\Lambda\), i.e. \(\Lambda^0\) term.

Now, we do the following change \([\ , \ ] \rightarrow \{ \ , \ }, \epsilon^{-1}X_L(\epsilon, t, x) \rightarrow \varphi_L, \epsilon^{-1}X_R(\epsilon, t, x) \rightarrow \varphi_R, \phi(\epsilon, t, x) \rightarrow \phi(t, x), \Lambda \rightarrow k.\) Then we have the following two propositions with quasi-classical limit.

**Proposition 5.5.** Symbols \(\sigma^\epsilon(X_L), \sigma^\epsilon(X_R), \sigma^\epsilon(\phi)\) give dressing functions \(\varphi_L, \varphi_R\) and potential function \(\phi\) of the dispersionless Lax function \(L = \sigma^\epsilon(L)\) respectively. Conversely if \(\varphi_L, \varphi_R\) and \(\phi_0\) are dressing functions and potential function of dispersionless BTH respectively, then there exist a solution \(L\) of the dBTH and dressing operators \(\mathcal{P}_L = \exp(\epsilon^{-1}X_L(\epsilon, t, x))\) and \(\mathcal{P}_R = \exp(\phi) \exp(\epsilon^{-1}X_R(\epsilon, t, x))\) such that \(\sigma^\epsilon(L) = L, \sigma^\epsilon(X_L) = \varphi_L, \sigma^\epsilon(X_R) = \varphi_R, \sigma^\epsilon(\phi) = \phi_0.\)

**Proposition 5.6.** \(\text{ord}^\epsilon(\epsilon M_L) = \text{ord}^\epsilon(\epsilon M_R) = 0\) and \(\mathcal{M}_L = \sigma^\epsilon(M_L)\) and \(\mathcal{M}_R = \sigma^\epsilon(M_R)\) are the Orlov functions of the dispersionless BTH whose Lax function is \(\mathcal{L} = \sigma^\epsilon(L)\).

To see the limit clearly, we firstly introduce spectral \(z\) and two functions \(w_L(t, x, z)\) and \(w_R(t, x, z)\) which have forms
\[
\begin{align*}
w_L(t, x, z) &= \mathcal{P}_L(x, \Lambda)e^{\xi_L(t, x, z)}, \\
w_R(t, x, z) &= \mathcal{P}_R(x, \Lambda)e^{\xi_R(t, x, z)},
\end{align*}
\]

where
\[
\begin{align*}
\xi_L(t, x, z) &= \sum_{n\geq 0} \sum_{\alpha=1}^{N} \epsilon^{-1}z^{n+1-\frac{\alpha}{N}}t_{\alpha,n} + \frac{x}{N\epsilon} \log z, \\
\xi_R(t, x, z) &= -\sum_{n\geq 0} \sum_{\beta=-M+1}^{0} \epsilon^{-1}z^{n+1+\frac{\beta}{M}}t_{\beta,n} - \frac{x}{M\epsilon} \log z.
\end{align*}
\]
We call these two functions \( w_L(t, x, z) \) and \( w_R(t, x, z) \) wave functions. These two wave functions are a little different from ones in [10]. Similarly as [10], the following proposition holds.

**Proposition 5.7.** The functions \( w_L(t, x, z) \) and \( w_R(t, x, z) \) satisfy the following linear equations

\[
\begin{cases}
Lw_L(t, x, z) = zw_L(t, x, z), \\
M_Lw_L(t, x, z) = \partial_z w_L(t, x, z), \\
\varepsilon\partial_{\gamma,n} w_L(t, x, z) = A_{\gamma,n} w_L(t, x, z),
\end{cases}
\]

(5.29)

\[
\begin{cases}
Lw_R(t, x, z) = zw_R(t, x, z), \\
M_Rw_R(t, x, z) = \partial_z w_R(t, x, z), \\
\varepsilon\partial_{\gamma,n} w_R(t, x, z) = A_{\gamma,n} w_R(t, x, z),
\end{cases}
\]

(5.30)

where, \( -M + 1 \leq \gamma \leq N, n \geq 0 \).

**Proof.** The proof is similar as paper [10]. \( \square \)

Using Proposition 5.7, \( w_L(\epsilon, t, z) \) and \( w_R(\epsilon, t, z) \) can be written using functions \( S_L(\epsilon, t, x, z) \) and \( S_R(\epsilon, t, x, z) \) in the following proposition.

**Proposition 5.8.** The Baker function \( w_L(\epsilon, t, x, z) \) and \( w_R(\epsilon, t, x, z) \) take a WKB asymptotic form as \( \epsilon \to 0 \):

\[
w_L(\epsilon, t, x, z) = \exp \left( \epsilon^{-1} S_L(t, x, z) + O(\epsilon^0) \right),
\]

(5.31)

\[
w_R(\epsilon, t, x, z) = \exp \left( \epsilon^{-1} S_R(t, x, z) + O(\epsilon^0) \right),
\]

(5.32)

where

\[
S_L(t, x, z) = \sum_{\alpha \geq 0}^{N} \sum_{\beta = 1}^{N} z^{(n+1-\alpha-1)\beta \alpha, n} \sum_{m = 0}^{\infty} \sum_{\gamma = 1}^{N} v_{\gamma,m}(t, x) z^{-(m+1+\frac{\gamma}{N})} m + 1 + \frac{1}{\gamma},
\]

(5.33)

\[
S_R(t, x, z) = -\sum_{n \geq 0}^{\infty} \sum_{\beta = -M+1}^{0} z^{n+1+\frac{\beta}{M}} t_{\beta,n} - \sum_{m = 0}^{\infty} \sum_{\gamma = -M+1}^{0} v_{\gamma,m}(t, x) z^{-(m+1+\frac{\gamma}{M})} m + 1 + \frac{1}{\gamma}.
\]

(5.34)

Then we get following proposition similar as [18]

**Proposition 5.9.** The spectral \( z \) of Lax function has following representations

\[ z = e^{N\partial_z S_L(x, t, z)} + u_{-N-1} e^{(N-1)\partial_z S_L(x, t, z)} + \cdots + u_{-M} e^{-M\partial_z S_L(x, t, z)} = \sigma^e(L)|_{k = \partial_z S_L(x, t, z)}, \]

\[ z = e^{N\partial_z S_R(x, t, z)} + u_{-N-1} e^{(N-1)\partial_z S_R(x, t, z)} + \cdots + u_{-M} e^{-M\partial_z S_L(x, t, z)} = \sigma^e(L)|_{k = \partial_z S_R(x, t, z)}, \]

and derivatives of \( S \) function have formulas

\[
dS_L(x, t, z) = \mathcal{M}_L(z) dz + \frac{\partial S_L}{\partial x} dx + \sum_{n \geq 0}^{N} \sum_{\gamma = -M+1}^{0} \mathcal{A}_{\gamma,n}(e^{\theta_z S_L}) dt_{\gamma,n},
\]

(5.35)
\begin{align}
\frac{dS_R(x, t, z)}{dz} &= M_R(z)dz^{-1} + \frac{\partial S_R}{\partial x} dx + \sum_{n \geq 0}^{N} \sum_{\gamma = -M+1} \mathcal{A}_{\gamma,n}(e^{\partial_z S_R}) dt_{\gamma,n}.
\end{align}

**Proof.** By eqs. (5.29), we can do the following computation
\begin{align}
zw_L(t, x, z) &= Lw_L(t, x, z) = (\Lambda^N + u_{N-1}\Lambda^{N-1} + \cdots + u_{-M}\Lambda^{-M}) \exp \left( \epsilon^{-1} S_L(t, x, z) + O(\epsilon^0) \right).
\end{align}
After that, we can pursue the WKB analysis, regarding $S_L(t, x, z)$ as the phase function which further leads to
\begin{align}
z &= e^{N\partial_z S_L(x,t,z)} + u_{N-1}e^{(N-1)\partial_z S_L(x,t,z)} + \cdots + u_{-M}e^{-M\partial_z S_L(x,t,z)} = \sigma^\epsilon(L)|_{k=\partial_z S_L(x,t,z)}.
\end{align}
By eqs. (5.29), we can also get
\begin{align}
M_Lw_L(t, x, z) &= \partial_z w_L(t, x, z) = \partial_z \exp \left( \epsilon^{-1} S_L(x, t, z) + O(\epsilon^0) \right)
= \left( \epsilon^{-1} \partial_z S_L(x, t, z) + O(\epsilon^0) \right)e^{\epsilon^{-1} S_L(x,t,z)+o(\epsilon)}
= \epsilon^{-1}(\sum_{n \geq 0}^{N}(n+1 - \frac{\alpha - 1}{N})z^{(n+\frac{\alpha-1}{N})}t_{\alpha,n} + \frac{x}{Nz} - \sum_{m=0}^{\infty} \sum_{\gamma=1}^{N} v_{\gamma,m}(t, x)z^{-(m+2+\frac{1}{z})})w_L(t, x, z)
+ O(\epsilon^0)w_L(t, x, z),
\end{align}
which implies $M_Lw_L(t, x, z) = \frac{M_L}{\epsilon}w_L(t, x, z) = \epsilon^{-1}\sigma^\epsilon(M_L)|_{L=\epsilon}w_L(t, x, z)$. Also by the equations above, we can get $\partial_z S_L(x,t,z) = M_L = \sigma^\epsilon(M_L)|_{L=\epsilon}$.

Because
\begin{align}
\epsilon \partial_{\gamma,n} w_L(t, x, z) &= \epsilon \partial_{\gamma,n} \exp \left( \epsilon^{-1} S_L(x, t, z) + O(\epsilon^0) \right)
= \left( \partial_{\gamma,n} S_L(x, t, z) + O(\epsilon^0) \right) \exp \left( \epsilon^{-1} S_L(x, t, z) + O(\epsilon^0) \right)
\end{align}
and
\begin{align}
\epsilon \partial_{\gamma,n} w_L(t, x, z) &= A_{\gamma,n}(\Lambda)w_L(t, x, z)
= A_{\gamma,n}(\Lambda) \exp \left( \epsilon^{-1} S_L(x, t, z) + O(\epsilon^0) \right)
= \left( A_{\gamma,n}(e^{\partial_z S_L}) + O(\epsilon^0) \right) \exp \left( \epsilon^{-1} S_L(x, t, z) + O(\epsilon^0) \right),
\end{align}
we can get
\begin{align}
\partial_{\gamma,n} S_L(x, t, z) &= A_{\gamma,n}(e^{\partial_z S_L}).
\end{align}
This implies
\begin{align}
\partial_{\gamma,n}(\partial_z S_L(x, t, z)) = \frac{\partial A_{\gamma,n}(e^{\partial_z S_L})}{\partial x}.
\end{align}
So we can consider $\partial_z S_L(x, t, z)$ which will be denoted as $\log k_1$ later as conserved density.

By eqs. (5.30), we can do the following computation
\begin{align}
zw_R(t, x, z) &= Lw_R(t, x, z) = (\Lambda^N + u_{N-1}\Lambda^{N-1} + \cdots + u_{-M}\Lambda^{-M}) \exp \left( \epsilon^{-1} S_R(x, t, z) + O(\epsilon^0) \right).
\end{align}
We can also pursue the WKB analysis, also regarding $S_R(t, x, z)$ as the phase function which leads to
\[ z = e^{N\partial_z S_R(t, x, z)} + u_{N-1}e^{(N-1)\partial_z S_R(t, x, z)} + \cdots + u_{-M}e^{-M\partial_z S_R(t, x, z)} = \sigma^f (L)|_{k = \partial_z S_R(t, x, z)}. \]

Similar result can also be got as following
\[ M_R w_R (t, x, z) = \frac{M_R}{\epsilon} w_R(t, x, z) = \epsilon^{-1}\sigma^f (M_R)|_{L = z} w_R(t, x, z). \]

Also by the equations above, we can get $\partial_z S_R(t, x, z) = M_R = \sigma^f (M_R)|_{L = z}$. Direct calculation can lead to
\[ \epsilon \partial_{\gamma, n} w_R (t, x, z) = (\partial_{\gamma, n} S_R(t, x, z) + O(\epsilon^0)) \exp (\epsilon^{-1} S_R(t, x, z) + O(\epsilon^0)). \]

By formula eqs. (5.30), following identities hold
\[ \epsilon \partial_{\gamma, n} w_R (t, x, z) = A_{\gamma, n} (\Lambda) w_R(t, x, z), \]
\[ = A_{\gamma, n} (\Lambda) \exp (\epsilon^{-1} S_R(t, x, z) + O(\epsilon^0)) \]
\[ = A_{\gamma, n} (e^{\partial_z S_R}) + O(\epsilon^0)) \exp (\epsilon^{-1} S_R(t, x, z) + O(\epsilon^0)). \]

So we can get
\[ \partial_{\gamma, n} S_R(t, x, z) = A_{\gamma, n} (e^{\partial_z S_R}). \]

So the following derivatives will be correct
\[ dS_L (x, t, z) = M_L (z) dz + \frac{\partial S_L}{\partial x} dx + \sum_{n \geq 0} \sum_{\gamma = -M+1}^N A_{\gamma, n} (e^{\partial_z S_L}) dt_{\gamma, n}, \]
\[ dS_R (x, t, z) = M_R (z) dz^{-1} + \frac{\partial S_R}{\partial x} dx + \sum_{n \geq 0} \sum_{\gamma = -M+1}^N A_{\gamma, n} (e^{\partial_z S_R}) dt_{\gamma, n}. \]

Then it it time to consider the Legendre-type transformation as following.

**Proposition 5.10.** The Legendre-type transformation $(t, x, z) \rightarrow (t, x, k_i), i = 1, 2$ is defined by
\[ e^{\partial_z S_L (t, x, z)} = k_1, \quad e^{\partial_z S_R (t, x, z)} = k_2, \]
the spectral parameters $z$ turns into the $L$-function of the dispersionless BTH which has two expressions by $k_1$ and $k_2$ respectively
\[ z = L (t, x, k_1) = k_1^N + u_{N-1}k_1^{N-1} + \cdots + u_{-M}k_1^{-M}, \]
\[ z = L (t, x, k_2) = k_2^N + u_{N-1}k_2^{N-1} + \cdots + u_{-M}k_2^{-M}, \]
and $S_L, S_R$ become the corresponding $S$-functions.

After these, we will start to consider the free energy of the dBTH in the next section.
6. Tau Function and Free Energy of dBTH

As paper \[10\], the relation of tau function for the BTH with wave functions is as following

\[
\begin{align*}
\tau(t,x,z) &= \mathcal{P}_L(x,L) \exp(\sum_{n \geq 0} \sum_{\alpha=1}^{N} \epsilon^{-1} z^{(n+1-\frac{n-1}{N})} t_{\alpha,n} + \frac{x}{N \epsilon} \log z) \\
&= \mathcal{P}_L(x,L) \exp(\frac{1}{\epsilon} (t_L(z^{\frac{1}{N}}) + \frac{x}{N} \log z)) \\
&= \frac{\tau(x - \epsilon/2, t - [z^{\frac{1}{N}}]^N)}{\tau(x - \epsilon/2, t)} \exp(\frac{1}{\epsilon} (t_L(z^{\frac{1}{N}}) + \frac{x}{N} \log z)),
\end{align*}
\]

\[
\begin{align*}
\tau(t,x,z) &= \mathcal{P}_R(x,L) \exp(-\sum_{n \geq 0} \sum_{\beta=-M+1}^{0} \epsilon^{-1} z^{n+1+\frac{\beta}{M}} t_{\beta,n} - \frac{x}{M \epsilon} \log z) \\
&= \mathcal{P}_R(x,L) \exp(\frac{1}{\epsilon} (-t_R(z^{\frac{1}{M}}) - \frac{x}{M} \log z)) \\
&= \frac{\tau(x + \epsilon/2, t + [z^{\frac{1}{M}}]^M)}{\tau(x - \epsilon/2, t)} \exp(-\frac{1}{\epsilon} t_R(z^{\frac{1}{M}}) - \frac{x}{M} \log z),
\end{align*}
\]

where

\[
\begin{align*}
[z^{-\frac{1}{N}}]_n^{\alpha} &= \begin{cases} \frac{e^{-\frac{n+1-\frac{\alpha}{N}}{N(n+1-\frac{\alpha}{N})}}}{N(n+1-\frac{\alpha}{N})}, & \alpha = N, N-1, \ldots, 1, \\ 0, & \alpha = 0, -1 \cdots - M+1, \end{cases} \\
[z^{\frac{1}{M}}]_n^{\alpha} &= \begin{cases} 0, & \alpha = N, N-1, \ldots, 1, \\ \frac{e^{-\frac{n+1+\frac{\beta}{M}}{M(n+1+\frac{\beta}{M})}}}{M(n+1+\frac{\beta}{M})}, & \alpha = 0, -1, \cdots - M+1, \end{cases}
\end{align*}
\]

Define free energy \(F(t,x)\) as following

\[
\log \tau(\epsilon, t, x) = \epsilon^{-2} F(t, x) + O(\epsilon^{-1}) \quad (\epsilon \to 0).
\]

Taking the logarithm of identities above and comparing them with the form eq.\[5.31\], one can find that \(\log \tau\) should behave as

\[
\begin{align*}
\epsilon^{-1}(\sum_{n \geq 0} \sum_{\alpha=1}^{N} z^{(n+1-\frac{\alpha}{N})} t_{\alpha,n} + \frac{x}{N} \log z - \sum_{m=0}^{\infty} \sum_{\gamma=1}^{N} v_{\gamma,m}(t, x) z^{-(m+1+\frac{1}{N})} m + 1 + \frac{1}{N} O(\epsilon^0) \\
&= \sum_{n \geq 0} \sum_{\alpha=1}^{N} \frac{z^{(n+1-\frac{\alpha}{N})}}{\epsilon} t_{\alpha,n} + \frac{x}{N \epsilon} \log z + \log \tau(\epsilon, t - [z^{\frac{1}{N}}]^N, x) - \log \tau(\epsilon, t, x),
\end{align*}
\]

which further leads to following relation

\[
v_{\alpha,m}(t, x) = \partial_{\alpha,m} F, \quad 1 \leq \alpha \leq N, m \geq 0.
\]
Similarly we can also get
\begin{equation}
\bar{v}_{\beta,m}(t,x) = \partial_{\bar{\beta},m}F, \quad -M + 1 \leq \beta \leq 0, m \geq 0.
\end{equation}

Considering Lemma 3.4, the second derivatives of free energy will have formula in following lemma.

**Lemma 6.1.** The derivatives of free energy have the following formula
\begin{equation}
F_{\alpha,n;\beta,m} = \text{Res}_{\lambda} B_{\alpha,n}(B_{\beta,m})+, \quad -M + 1 \leq \alpha, \beta \leq N, m, n \geq 0,
\end{equation}
where $F_{\alpha,n;\beta,m} := \partial_{\alpha,n, \partial_{\beta,m}F}$.

**Proof.** Bringing eq. (6.4) and eq. (6.5) into Lemma 3.4 will lead to this lemma. □

Eqs. (5.33), (5.34) and eqs. (5.39), (5.43) can lead to following proposition.

**Proposition 6.2.** The following identities hold
\begin{align}
A_{\gamma,n}(t,x,k_1) &= z^{(n+1-\frac{\gamma}{N})} - \sum_{m=0}^{\infty} \sum_{\alpha=1}^{N} F_{\alpha,m;\gamma,n}(t,x) z^{-\left(m+1+\frac{\alpha}{\beta}\right)} m + 1 + \frac{1-\alpha}{N}, \\
A_{\gamma,n}(t,x,k_2) &= -\sum_{m=0}^{\infty} \sum_{\alpha=1}^{N} F_{\alpha,m;\gamma,n}(t,x) z^{-\left(m+1+\frac{\alpha}{\beta}\right)} m + 1 + \frac{1-\alpha}{N}, \\
A_{\gamma,n}(t,x,k_2) &= -z^{(n+1-\frac{\gamma}{M})} + \sum_{m=0}^{\infty} \sum_{\beta=-M+1}^{0} F_{\beta,m;\gamma,n}(t,x) z^{-\left(m+1+\frac{\beta}{\gamma}\right)} m + 1 + \frac{1-\alpha}{N},
\end{align}
where $-M + 1 \leq \gamma \leq N, n \in \mathbb{Z}_+$.

\begin{equation}
e^{\partial_{S_L}(t,x,z)} = k_1, \quad e^{\partial_{S_R}(t,x,z)} = k_2, \quad z = \mathcal{L}(t,x,k_1) = \mathcal{L}(t,x,k_2).
\end{equation}

After the definition of free energy of the dBTH, the dispersionless Hirota bilinear identities about tau function will be derived in the next section.

**7. Dispersionless Hirota bilinear identities**

In our paper [9], we have got the following Hirota bilinear identity of the BTH in one corollary
\begin{align}
\text{Res}_{\lambda} \left\{ \lambda^{m-1}\tau(x, t - [\lambda^{-1}]^N) \times \tau(x - (m-1)i, t' + [\lambda^{-1}]^N) e^{\xi_L(t-t')} \right\} \\
= \text{Res}_{\lambda} \left\{ \lambda^{m-1}\tau(x + \epsilon, t + [\lambda]^M) \times \tau(x - me, t' - [\lambda]^M) e^{\xi_R(t-t')} \right\},
\end{align}

where $\mathcal{L}(t,x,k_1) = \mathcal{L}(t,x,k_2)$.
where
\begin{equation}
[\lambda^{-1}]_{a,n}^N := \begin{cases} 
\frac{e^{\lambda^{-N(n+1-\frac{\alpha-1}{N})}}}{N(n+1-\frac{\alpha-1}{N})}, & \alpha = N, N - 1, \ldots, 1, \\
0, & \alpha = 0, -1 \cdots M + 1,
\end{cases}
\end{equation}

\begin{equation}
[\lambda]^M_{\alpha,n} := \begin{cases} 
0, & \alpha = N, N - 1, \ldots, 1, \\
\frac{e^{\lambda^{M(n+1+\beta)}_{\alpha,n}}}{M(n+1+\frac{\beta}{M})}, & \alpha = 0, -1, \cdots M + 1,
\end{cases}
\end{equation}

\begin{equation}
\xi_L(t - t') = \sum_{n \geq 0} \sum_{a = 1}^{N} \lambda^{N(n+1-\frac{\alpha-1}{N})}(t_{a,n} - t'_{a,n}),
\end{equation}

\begin{equation}
\xi_R(t - t') = -\sum_{n \geq 0} \sum_{\beta = -M+1}^{0} \lambda^{-M(n+1+\frac{\beta}{M})}(t_{\beta,n} - t'_{\beta,n}).
\end{equation}

In order to understand the properties of the dBTH more, it is useful to get the fay-like identities from the ones of BTH.

We can choose different values for \(m, t, t'\) which lead to the following dispersionless Fay-like identities:

I. \(m = 0, t' = t - [\lambda^{-1}]^N - [\lambda^{-1}]^N\). In this case the Hirota bilinear identity (7.1) will lead to

\[
\begin{align*}
\text{Res}_\lambda \left\{ \tau(x, t - [\lambda^{-1}]^N) \times \tau(x + \epsilon, t' + [\lambda^{-1}]^N) \frac{1}{(1 - \lambda\lambda^{-1})(1 - \lambda\lambda^{-1})} \frac{1}{\lambda} \right\} & = \text{Res}_\lambda \left\{ \tau(x + \epsilon, t + [\lambda]^M) \times \tau(x, t' - [\lambda]^M) \frac{1}{\lambda} \right\}.
\end{align*}
\]

Using
\[
(1 - \lambda^{-1}\lambda)^{-1}(1 - \lambda_2^{-1}\lambda)^{-1} = \frac{\lambda_1^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}}(1 - \lambda_1^{-1}\lambda)^{-1} - \frac{\lambda_2^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}}(1 - \lambda_2^{-1}\lambda)^{-1},
\]

we get
\[
\frac{\lambda_1^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}} \tau(x, t - [\lambda_1^{-1}]^N) \tau(x + \epsilon, t' + [\lambda_1^{-1}]^N) - \frac{\lambda_2^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}} \tau(x, t - [\lambda_2^{-1}]^N) \tau(x + \epsilon, t' + [\lambda_2^{-1}]^N)
\]
\[
= \tau(x + \epsilon, t)\tau(x, t').
\]

It further leads to
\[
\frac{\lambda_1^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}} \tau(x, t - [\lambda_1^{-1}]^N) \tau(x + \epsilon, t - [\lambda_2^{-1}]^N) - \frac{\lambda_2^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}} \tau(x, t - [\lambda_2^{-1}]^N) \tau(x + \epsilon, t - [\lambda_1^{-1}]^N)
\]
\[
= \tau(x + \epsilon, t)\tau(x, t - [\lambda_1^{-1}]^N - [\lambda_2^{-1}]^N).
\]

(7.4)
Dividing both sides of eq. (7.4) by $\tau(x, t - [\lambda_1^{-1}]^N)\tau(x + \epsilon, t - [\lambda_2^{-1}]^N)$ will lead to the following equation

$$\frac{\lambda_1^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}} \frac{\lambda_2^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}} \frac{\tau(x, t - [\lambda_2^{-1}]^N)\tau(x + \epsilon, t - [\lambda_1^{-1}]^N)}{\tau(x, t - [\lambda_1^{-1}]^N)\tau(x + \epsilon, t - [\lambda_2^{-1}]^N)} = \frac{\tau(x + \epsilon, t)\tau(x, t - [\lambda_1^{-1}]^N)\tau(x + \epsilon, t - [\lambda_1^{-1}]^N)}{\tau(x, t - [\lambda_1^{-1}]^N)\tau(x + \epsilon, t - [\lambda_2^{-1}]^N)}.$$  

(7.5)

Finally we can get the following bilinear equations on free energy,

$$\lambda_1^{-1} \exp \left( - \sum_{n \geq 0} \sum_{\alpha = 1}^N \frac{\lambda_2^{-N(n+1-\frac{\alpha-1}{N})}}{N(n + 1 - \frac{\alpha-1}{N})} \partial^2 F \right) - \lambda_2^{-1} \exp \left( - \sum_{n \geq 0} \sum_{\alpha = 1}^N \frac{\lambda_1^{-N(n+1-\frac{\alpha-1}{N})}}{N(n + 1 - \frac{\alpha-1}{N})} \partial^2 F \right) = (\lambda_1^{-1} - \lambda_2^{-1}) \exp \left( \sum_{m,n \geq 0, \alpha,\alpha' = 1}^N \frac{\lambda_1^{-N(n+1-\frac{\alpha-1}{N})}}{N^2(n + 1 - \frac{\alpha-1}{N})(m + 1 - \frac{\alpha'-1}{N})} \partial^2 F \right).$$

II. $m = 0, t' = t + [\lambda_1]^M + [\lambda_2]^M$. In this case the Hirota bilinear identity (7.1) will lead to

$$\text{Res}_\lambda \left\{ \tau(x, t - [\lambda^{-1}]^N) \tau(x + \epsilon, t' + [\lambda^{-1}]^N) \lambda^{-1} \right\} = \text{Res}_\lambda \left\{ \tau(x + \epsilon, t + [\lambda]^M) \tau(x, t' - [\lambda]^M) \lambda^{-1} \frac{1}{(1 - \lambda^{-1}\lambda_1)(1 - \lambda^{-1}\lambda_2)} \right\}.$$

Using

$$(1 - \lambda^{-1}\lambda_1)^{-1}(1 - \lambda_2\lambda^{-1})^{-1} = \frac{\lambda_1}{\lambda_1 - \lambda_2} (1 - \lambda_1\lambda^{-1})^{-1} - \frac{\lambda_2}{\lambda_1 - \lambda_2} (1 - \lambda_2\lambda^{-1})^{-1},$$

we get

$$\tau(x, t)\tau(x + \epsilon, t') = \frac{\lambda_1}{\lambda_1 - \lambda_2} \tau(x + \epsilon, t + [\lambda_1]^M)\tau(x, t' - [\lambda_1]^M) - \frac{\lambda_2}{\lambda_1 - \lambda_2} \tau(x + \epsilon, t + [\lambda_2]^M)\tau(x, t' - [\lambda_2]^M).$$

It further leads to

$$\tau(x, t)\tau(x + \epsilon, t + [\lambda_1]^M + [\lambda_2]^M) = \frac{\lambda_1}{\lambda_1 - \lambda_2} \tau(x + \epsilon, t + [\lambda_1]^M)\tau(x, t + [\lambda_2]^M) - \frac{\lambda_2}{\lambda_1 - \lambda_2} \tau(x + \epsilon, t + [\lambda_2]^M)\tau(x, t + [\lambda_1]^M).$$

Finally we can get the following bilinear equations on free energy,

$$\exp \left( \sum_{m,n \geq 0, \beta,\beta' = -M+1}^0 \frac{\lambda_1^{M(n+1+\frac{\beta}{M})}}{M^2(n + 1 + \frac{\beta}{M})(m + 1 + \frac{\beta'}{M})} \partial^2 F \right) = \frac{\lambda_1}{\lambda_1 - \lambda_2} \exp \left( - \sum_{n \geq 0}^0 \frac{\lambda_2^{M(n+1+\frac{\beta}{M})}}{M(n + 1 + \frac{\beta}{M})} \partial^2 F \right).$$
III. \(m = 1, t' = t - [\lambda_1^{-1}]^N + [\lambda_2]^M\). In this case the Hirota bilinear identity (7.1) will lead to

\[
\text{Res}_\lambda \left\{ \tau(x, t - [\lambda_1^{-1}]^N) \times \tau(x, t' + [\lambda_1^{-1}]^N) \frac{1}{1 - \lambda \lambda_1^{-1}} \right\}
\]

\[
= \text{Res}_\lambda \left\{ \tau(x + \epsilon, t + [\lambda]^M) \times \tau(x - \epsilon, t' - [\lambda]^M) \frac{1}{1 - \lambda \lambda_2} \right\},
\]

which is equivalent to

\[
\lambda_1 (\tau(x, t - [\lambda_1^{-1}]^N) \tau(x, t' + [\lambda_1^{-1}]^N) - \tau(x, t) \tau(x, t')) = \lambda_2 \tau(x + \epsilon, t + [\lambda]^M) \tau(x - \epsilon, t' - [\lambda]^M).
\]

It further implies

\[
\lambda_1 (\tau(x, t - [\lambda_1^{-1}]^N) \tau(x, t + [\lambda_2]^M) - \tau(x, t) \tau(x, t - [\lambda_1^{-1}]^N + [\lambda_2]^M))
\]

\[(7.8)\]

\[
= \lambda_2 \tau(x + \epsilon, t + [\lambda]^M) \tau(x - \epsilon, t - [\lambda_1^{-1}]^N).
\]

Similarly we can also get the following Hirota bilinear equation on free energy,

\[
1 - \exp \left( \sum_{m, n \geq 0} \sum_{\beta = -M+1}^{0} \sum_{\alpha = 1}^{N} \lambda_1^{-N(n+1-\frac{\alpha-1}{N})} \lambda_2^{M(m+1+\frac{\beta}{M})} \frac{\partial^2 F}{\partial t_{\alpha, n} \partial t_{\beta, m}} \right)
\]

\[
= \lambda_1^{-1} \lambda_2 \exp \left( \frac{\partial^2 F}{\partial x^2} + \sum_{n \geq 0} \sum_{\alpha = 1}^{N} \frac{\lambda_1^{-N(n+1-\frac{\alpha-1}{N})}}{N(n+1-\frac{\alpha-1}{N})} \frac{\partial^2 F}{\partial t_{\alpha, n} \partial x} \right)
\]

\[+ \sum_{n \geq 0} \sum_{\beta = -M+1}^{0} \lambda_2^{M(m+1+\frac{\beta}{M})} \frac{\partial^2 F}{\partial t_{\beta, n} \partial x} \right).
\]

(7.9)

The properties of the dBTH mentioned above provide a very sound mathematical background in its possible applications in deriving solutions and some combinatorics meaning in matrix model.

8. Conclusions and Discussions

We define Orlov-Schulman’s \(M_L, M_R\) function of the dBTH and give the additional symmetry of the dBTH. We find this kind of Block type Lie algebraic structure is still preserved by the dBTH. Also we give the quasi-classical limit relation of the BTH and the dBTH and some Hirota bilinear equations of the dBTH. We hope the additional symmetry and dispersionless Hirota bilinear identity of the dBTH can be used more in other fields of mathematical physics, particularly topological fields theory and string theory.

The main difference of Block symmetry and HBEs of the dBTH from the ones of the BTH is that the representation space here is a directly a functional space and the HBEs of the dBTH here are in form of free energy using WKB analysis.
Acknowledgments This work is supported by NSF of Zhejiang Province under Grant No. LY12A01007, NSF of China under Grant No.10971109 and K.C.Wong Magna Fund in Ningbo University. Jingsong He is also supported by Program for NCET under Grant No. NCET-08-0515 and NSF of Ningbo under Grant No.2011A610179. We also thank Professor Yishen Li(USTC, China) for long-term encouragements and supports. Chuanzhong Li would like to thank Professor Yuji Kodama in Department of Mathematics at Ohio State University for his useful discussions.
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