Quantum 3D Tensionless String in Light-cone Gauge

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1. **Introduction** ~Critical dimension~

Quantizations of a bosonic (or supur-) string give us the critical dimension, 26 (or 10).

These dimensions are required by the anomaly-free conditions for the Lorentz symmetry.

Examples

- $[J^{-I}, J^{-K}] = 0$ in Light-cone quantization.
- $Q_{BRST}^2 = 0$ in BRST quantization.
- $\vdots$
Recently, Mezincescu and Townsend found that a 3D tensile string in light-cone gauge has no Lorentz anomaly.

Because there is one transverse direction in the light-cone coordinate, a dangerous commutator of Lorentz generators vanishes trivially.

\[ [J^{-I}, J^{-K}] \equiv 0 \quad (I = K = 2) \]
1. Introduction  ~”Anyon” from string~

[L. Mezincescu and P. K. Townsend, ‘10]

“Anyons” exist in the spectrum of a 3D tensile string.

In the representation of 3D Lorentz group (some multiple cover of $SL(2, R)$), massive states can have non half-integer spin. [B. Binegar, ‘81]

The appearance of anyons which can’t be found by covariant quantizations like BRST method is interesting.

- only boson and fermion
- preserves spin-statistics relation
1. Introduction  ~This talk~

The light-cone quantization gives the different result from the BRST quantization.

This fact is not understood completely.

- Is the light-cone method inadequate as quantization?
- Is the BRST method imperfect?
- Is anyon the origin of difference?

⇒ It is important to find other examples. Like anyons in a 3D tensile string, we may find some interesting result.

This talk:

We considered a 3D tensionless bosonic closed string in light-cone gauge.
Outline

1. Introduction
2. Tensionless String ←next
3. 3D Tensionless Closed String in Light-cone Gauge
4. Summary
2. Tensionless String

In sec. 2, “What is a tensionless string?” and “What kind of properties are known?” will be explained.

Terminologies:

- \( T \rightarrow 0 \) “tensionless limit”
- \( T = 0 \) “tensionless”
- \( T \neq 0 \) “tensile”

Comment:

We consider properties of “\( T = 0 \)” string, not behaviors of a tensile string in “\( T \rightarrow 0 \)”. 
2. Formulations of a massless point-particle

- **Massive point particle**

\[
S = -m \int d\tau \sqrt{-\dot{x}^2} = \frac{1}{2} \int d\tau \left[ e \dot{x}^2 - m^2 e^{-1} \right] = \int d\tau \left[ \dot{x} \cdot p - \frac{1}{2} e^{-1} (p^2 + m^2) \right]
\]

- **Massless point particle** \((m^2 = 0)\)

Action: \(S = \frac{1}{2} \int d\tau e \dot{x}^2 = \int d\tau \left[ \dot{x} \cdot p - \frac{1}{2} e^{-1} p^2 \right]
\]

We consider a tensionless string as the analogy of massless point particle.
2. Formulation of a tensionless string

- **Tensile string**

\[ S = -T \int d^2 \xi \sqrt{-\gamma} \]

\[ = \frac{1}{2} \int d^2 \xi \left[ \phi \gamma - T^2 \phi^{-1} \right] \]

\[ \phi = \frac{T}{\sqrt{-\gamma}} \]

\[ = \int d^2 \xi \left[ \dot{X} \cdot P - \frac{1}{2} V (P^2 + T^2 X'^2) - U X' \cdot P \right] \]

\[ \dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}, X'^\mu = \frac{\partial X^\mu}{\partial \sigma}, \gamma = \dot{X}^2 X'^2 - (\dot{X} \cdot X')^2 \text{ and } V^{-1} = \phi X'^2 \]

- **Tensionless string** \((T = 0)\)

\[ S = \frac{1}{2} \int d^2 \xi \phi \gamma \]

\[ = \int d^2 \xi \left[ \dot{X} \cdot P - \frac{1}{2} V P^2 - U X' \cdot P \right] \]

Hamiltonian Formalism with constraints

Our Starting Action!
2. Properties of a tensionless string : I
~Absence of critical dimensions~

◆ BRST quantization [F. Lizzi, B. Rai, G. Sparano & A. Srivastava, ’86]

Constraints:
\[
\begin{align*}
\Phi_V &= P(\sigma)^2 \\
\Phi_U &= \frac{1}{2} (P \cdot X' + X' \cdot P)(\sigma)
\end{align*}
\]

BRST charge: \( \eta^V \) and \( \eta^U \) are ghosts for \( \Phi_V \) and \( \Phi_U \).
\[
Q = \oint d\sigma \left[ \Phi_V \eta^V + \Phi_U \eta^U + \overline{\eta^U} \eta'^U \eta^U + \overline{\eta^V} (\eta'^U \eta^V + \eta'^V \eta^U) \right]
\]

Hermitian order ver. : \( Q_{HO} = \frac{1}{2} (Q + Q^\dagger) \Rightarrow Q_{HO}^2 = 0 \)

◆ Light-cone quantization
\[
[J^{-I}, J^{-K}] \propto \oint d\sigma X' \cdot P = 0
\]

Constraint (corresponding to Level matching in a tensile string)
2. Another formulation ~Conformal string~

Tensionless string: \( S = \frac{1}{2} \int d^2 \xi \phi \left[ \ddot{X}^2 \dot{X}'^2 - (\dot{X} \cdot \dot{X}')^2 \right] \)

We consider next (D+2)-dimensional space to introduce the (classically) equivalent action of this.

Action: \( S = \int d^2 \xi \text{det} \left( \partial_\alpha Y^{\hat{\mu}} \partial_\beta Y^{\hat{\nu}} G_{\hat{\mu} \hat{\nu}} \right) \)

with the light-cone condition, \( G_{\hat{\mu} \hat{\nu}} Y^{\hat{\mu}} Y^{\hat{\nu}} = 0 \)

(D+2)-coordinates: \( Y^{\hat{\mu}} = (\varphi X^\mu, \varphi, \chi), \varphi = \phi^{\frac{1}{4}}, \hat{\mu} = 0, \ldots, D + 1 \)

(D+2)-metric: \( G_{\hat{\mu} \hat{\nu}} = \begin{pmatrix} \eta_{\mu\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \)

Isometry of this (D+2)-dim. space

= D-dim. conformal symmetry
2. Properties of a tensionless string: II

~Anomaly for spacetime conformal symmetry~

We expect a quantum tensionless string to have the "spacetime"-conformal symmetry, like the Lorentz symmetry for a tensile string.

◆ BRST quantization

[H. Gustafsson, U. Lindstrom, P. Saltsidis, B. Sundborg & R. Unge, ’95]

Nilpotency of BRST charge for conformal string ⇒ D=2

◆ Light-cone quantization

- \( D = 2 \) case ⇒ O.K. ← Trivial!
- \( D \geq 3 \) case [J. Isberg, U. Lindstrom, B. Sundborg & G. Theodridis, ’94]

The commutators, \([J^{-I}, K^K] = i \eta^{IK} K^-\), have an anomaly in the traceless part of superscripts, I and K.
2. **D=3 case**?

In D=3, \([J^{-1}, K^K]\) \((I = K = 2)\) has no traceless part. The difference of the trace part is absorbed in the redefinition of \(K^-\).

Like Lorentz anomaly for 3D tensile string, We expect that 3D tensionless string has no anomaly in spacetime conformal symmetry.

But other commutators still remain to be checked. New dangerous commutator exists.

This check is one of topics in this talk!
3. Quantum 3D Tensionless Closed String in Light-cone Gauge

- The First Half
  Checking the absence of anomalies in commutators

- The Second Half
  Investigation of the spectrum
3. Light-cone coordinate

We consider a 3D tensionless closed string.

Action: \[ S = \int d\tau \frac{d\sigma}{2\pi} \left[ \dot{X} \cdot P - \frac{1}{2} V P^2 - U X' \cdot P \right] \]

Symmetry:
\[
\begin{align*}
\delta X^\mu &= \alpha P^\mu + \beta X'^\mu \\
\delta P_\mu &= (\beta P_\mu)' \\
\delta V &= \dot{\alpha} + U' \alpha - U \alpha' + V' \beta - V \beta' \\
\delta U &= \dot{\beta} + U' \beta - U \beta'
\end{align*}
\]

\[ \alpha = \alpha(\tau, \sigma), \beta = \beta(\tau, \sigma) : \text{parameters} \]

Light-cone coordinate
\[
\begin{align*}
X^\pm &= \frac{1}{\sqrt{2}} (X^1 \pm X^0), \quad X = X^2 \\
P^\pm &= \frac{1}{2} (P_1 \pm P_0) = P^\mp, \quad P = P_2
\end{align*}
\]
3. Lagrangian in the light-cone gauge

Gauge fixing:
\[
\begin{align*}
X^+(\tau, \sigma) &= \tau \Rightarrow \alpha = 0 \\
P_-(\tau, \sigma) &= p_-(\tau) \neq 0 \Rightarrow \beta = \beta_0(\tau)
\end{align*}
\]

For a given \( F(\tau, \sigma) \), we decompose it into \( f \) and \( \bar{F} \):
\[
f(\tau) \equiv \oint \frac{d\sigma}{2\pi} F(\tau, \sigma), \quad \bar{F}(\tau, \sigma) \equiv F(\tau, \sigma) - f(\tau) \quad [F = X^-, X, P_+, P, U]
\]

Lagrangian:
\[
L = \dot{x}p + x^- p_- + p_+ + \oint \frac{d\sigma}{2\pi} \dot{X} \bar{P} - u \oint \frac{d\sigma}{2\pi} X' \bar{P} - \oint \frac{d\sigma}{2\pi} \bar{U} \bar{X}' P + p_- \oint \frac{d\sigma}{2\pi} \left[ \bar{U}' \bar{X}^- - V \left( P_+ + \frac{1}{2p_-} P^2 \right) \right]
\]

Solving constraints:
\[
\begin{align*}
\delta \bar{X}^- &\Rightarrow \bar{U} = 0 \\
\delta \bar{U} &\Rightarrow p_-(X^-)' = -\bar{X}' P \\
\delta V &\Rightarrow P_+ = -\frac{1}{2p_-} P^2
\end{align*}
\]
3. Lagrangian in the light-cone gauge

Gauge fixing:
\[
\begin{align*}
X^+(\tau, \sigma) &= \tau \Rightarrow \alpha = 0 \\
\left[ P_-(\tau, \sigma) = p_-(\tau) \right. &\left. \neq 0 \Rightarrow \beta = \beta_0(\tau) \right.
\end{align*}
\]

For a given \( F(\tau, \sigma) \), we decompose it into \( f \) and \( \overline{F} \):
\[
f(\tau) \equiv \oint \frac{d\sigma}{2\pi} F(\tau, \sigma), \quad \overline{F}(\tau, \sigma) \equiv F(\tau, \sigma) - f(\tau) \quad [F = X^-, X, P_+, P, U]
\]

Lagrangian:
\[
L = \dot{x}p + x^- p_- + \oint \frac{d\sigma}{2\pi} \dot{X} \overline{P} - H - u \oint \frac{d\sigma}{2\pi} \overline{X}' \overline{P}
\]
\[
H \equiv -p_+ = \frac{1}{2p_-} (p^2 + \mathcal{M}^2), \quad \mathcal{M}^2 \equiv 2p_+p_- - p^2 = \oint \frac{d\sigma}{2\pi} p^2
\]

Residual symmetry by \( \beta_0 \Rightarrow u = 0 \)

Only one constraint is left:
\[
\oint \frac{d\sigma}{2\pi} \overline{X}' \overline{P} = 0
\]
3. Fourier expansion & E.o.M

We got the Lagrangian with 1 unsolved constraint.

Lagrangian: \( L = \dot{x}p + x^- p_- + \oint \frac{d\sigma}{2\pi} \dot{X} \overline{P} - H \)

\[
H \equiv \frac{1}{2p_-} (p^2 + \mathcal{M}^2), \quad \mathcal{M}^2 \equiv \oint \frac{d\sigma}{2\pi} p^2
\]

Constraint: \( \oint \frac{d\sigma}{2\pi} \overline{X'} \overline{P} = 0 \quad \leftarrow \text{unsolved} \)

Relations: \( p_- (X^-)' = -\overline{X'}P \)

\[
P_+ = -\frac{1}{2p_-} P^2
\]

Fourier expansion:

\[
X = \sum_{n=-\infty}^{\infty} X_n e^{in\sigma}, X_0 = x, (X_n)^* = X_{-n}
\]

\[
P = \sum_{n=-\infty}^{\infty} P_n e^{in\sigma}, P_0 = p, (P_n)^* = P_{-n}
\]
3. Fourier expansion & E.o.M

We got the Lagrangian with 1 unsolved constraint.

Lagrangian: \( L = \dot{x}p + x^{-}p_{-} + \sum_{n \neq 0} \dot{X}_{n}P_{-n} - H \)

\[
H \equiv \frac{1}{2p_{-}} (p^{2} + \mathcal{M}^{2}), \quad \mathcal{M}^{2} \equiv 2 \sum_{n > 0} P_{n}P_{-n}
\]

Constraint: \( M_{0} = 0 \) ← unsolved

Relations: \( X^{-} = x^{-} - \frac{1}{p_{-}} \sum_{n \neq 0} \frac{i}{n} M_{n} e^{i n \sigma} \)

\[
P_{+} = -\frac{1}{2p_{-}} \sum_{n \neq 0} L_{n} e^{i n \sigma}
\]

\[
M_{n} \equiv -i \sum_{m} mX_{m}P_{n-m}, \quad L_{n} = \sum_{m} P_{m}P_{n-m}
\]

Equations of motion: \( \dot{p} = p_{-} = P_{n} = 0 \)

\[
\dot{x} = \frac{p}{p_{-}}, \quad \dot{x}^{-} = -\frac{H}{p_{-}}, \quad \dot{X}_{n} = \frac{P_{n}}{p_{-}}
\]

Linear moving with uniform acceleration
3. Quantization & Vacuum

- Commutation relations: $\left[ x^-, p_- \right] = i$, $\left[ X(\sigma), P(\sigma') \right] = 2\pi i \delta(\sigma - \sigma')$
  
  $\left[ x, p \right] = i$, $\left[ X_n, P_m \right] = i \delta_{n+m,0}$

- Vacuum & Ordering:
  
  The principle to determine the vacuum for a tensionless string is not known, but expect the smooth connection with the tensile string vacuum.
3. Quantization & Vacuum

- Commutation relations: ⟷normal quantization

\[ [x^-, p_-] = i, \quad [X(\sigma), P(\sigma')] = 2\pi i \delta(\sigma - \sigma') \]

\[ [x, p] = i, \quad [X_n, P_m] = i \delta_{n+m,0} \]

- Vacuum & Ordering:

Tensile string case \( T \to 0 \)

\[ \alpha_n = -i \sqrt{\frac{T}{2}} nX_n + \frac{1}{\sqrt{2T}} P_n \quad \rightarrow \quad \frac{1}{\sqrt{2T}} P_n \]

\[ \tilde{\alpha}_n = -i \sqrt{\frac{T}{2}} nX_{-n} + \frac{1}{\sqrt{2T}} P_{-n} \quad \rightarrow \quad \frac{1}{\sqrt{2T}} P_{-n} \]

Ground state \( |0\rangle \): \( \alpha_n |0\rangle = \tilde{\alpha}_n |0\rangle = 0 \) for all positive \( n \)

Ordering: All positive modes are to the right of negative modes.
3. Quantization & Vacuum

- Commutation relations: \( [x^-, p_-] = i, [X(\sigma), P(\sigma')] = 2\pi i \delta(\sigma - \sigma') \)
  \( [x, p] = i, [X_n, P_m] = i \delta_{n+m,0} \)

- Vacuum & Ordering:
  Tensionless string case

\[ \alpha_n \rightarrow \frac{1}{\sqrt{2T}} P_n \]
\[ \tilde{\alpha}_n \rightarrow \frac{1}{\sqrt{2T}} P_{-n} \]

Ground state \( |0\rangle: P_n |0\rangle = 0 \) for all \( n \neq 0 \)

Ordering: All P-modes are to the right of X-modes. We call this the Reference ordering, R-ordering.
3. Expected symmetry

|                  | $T \neq 0$ string                                      | $T = 0$ string                                      |
|------------------|-------------------------------------------------------|----------------------------------------------------|
| **Mass**         | $\mathcal{M}^2 = T \sum_{n>0} [\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n]$ | $\mathcal{M}^2 = 2 \sum_{n>0} P_{-n}P_n$          |
| **Ground state** | $\alpha_n |0\rangle = \tilde{\alpha}_n |0\rangle = 0$ for $n > 0$ | $P_n |0\rangle_R = 0$ for $n \neq 0$ |
| **Constraint**   | $\sum_{n>0} \alpha_{-n}\alpha_n = \sum_{n>0} \tilde{\alpha}_{-n}\tilde{\alpha}_n$ Level-matching condition | $M_0 \equiv -i \sum_n nX_nP_{-n} = 0$              |
| **Symmetry**     | Lorentz symmetry $\Rightarrow$ D=26                  | Lorentz symmetry $\Rightarrow \forall D$          |
| **Critical dim.**|                                        | Conformal symmetry $\Rightarrow$ D=?               |

Because a classical tensile string has the Lorentz symmetry, we require that a quantum tensile string has the Lorentz symmetry and get the critical dimension.

A classical tensionless string has the **spacetime conformal symmetry**.

So we expect **it** for a quantum tensionless string.
3. Generators (R-ordering) → Hermitian vers.

Translation: \( \mathcal{P}_\mu = \oint \frac{d\sigma}{2\pi} P_\mu(\sigma) \)
\[ \mathcal{P}^+_R = p_-, \mathcal{P}_R = p, \mathcal{P}^-_R = -H \quad \mathcal{P}^2 = \mathcal{M}^2 \text{ time-independent} \]

Lorentz: \( J^\mu = \epsilon^{\mu\nu\rho} \oint \frac{d\sigma}{2\pi} X_\nu P_\rho \)
\[ J^+_R = \tau p - x p_-, J_R = x^- p_-, J^-_R = -x^- p - xH + \frac{\Lambda}{p_-} \]
\[ \Lambda = \mathcal{P} \cdot J = p_- \oint \frac{d\sigma}{2\pi} [\mathcal{X} \mathcal{P}^+ - \bar{\mathcal{X}}^+ \mathcal{P}] = \sum_{n \neq 0} \left( -\frac{1}{2} X_n L_n - \frac{i}{n} M_n P_{-n} \right) \]

Dilatation: \( \mathcal{D}_R = \oint \frac{d\sigma}{2\pi} X^\mu P_\mu = x^- p_0 - \tau H + \sum_n X_n P_{-n} \)

Special: \( \mathcal{K}^\mu = \oint \frac{d\sigma}{2\pi} \left[ X^\mu (X \cdot P) - \frac{1}{2} (X \cdot X) P^\mu \right] \)
\[ \mathcal{K}^+_R = -\frac{1}{2} \sum_n X_n X_{-n} p_- + \tau \sum_n X_n P_{-n} - \tau^2 H \]
\[ \mathcal{K}_R = xx^- p_- + x^- \sum_{n \neq 0} \frac{i}{n} X_n M_{-n} + \frac{1}{2} \sum_{n,m} X_n X_m P_{-n-m} + \tau \mathcal{D} \]
\[ \mathcal{K}^-_R = x^- x^- p_- + x^- \sum_n X_n P_{-n} + \frac{1}{4p_-} \sum_{n,m} X_n X_m L_{-n-m} \]
\[ -\frac{i}{p_-} \sum_n \sum_{m \neq 0} \left( \frac{n}{m^2} + \frac{1}{m} \right) X_n M_m P_{-n-m} \]
3. 3D conformal group in light-cone gauge

Poincaré group

\[
\begin{align*}
[J^\pm, P^\mp] &= \pm iP, \\
[J, P^\pm] &= \pm iP^\pm, \\
[J^\pm, P] &= \mp iP^\pm, \\
[J, J^\pm] &= \pm iP^\pm, \\
[J^+, J^-] &= iJ,
\end{align*}
\]

other commutators vanish.

\([J^{-1}, J^{-K}]\) is a dangerous commutator in \(D>3\),
but its counterpart in \(D=3\) is trivially zero, \([J^-, J^-]=0\).

Conformal group

\[
\begin{align*}
[D, P^\pm] &= iP^\pm, \\
[D, P] &= iP, \\
[D, K^\pm] &= -iK^\pm, \\
[D, K] &= -iK, \\
[K^\pm, P^\mp] &= i(D \mp J), \\
[K^\pm, P] &= -[K, P^\pm] = \pm iJ^\pm, \\
[K, P] &= iD, \\
[K^\pm, J^\mp] &= \pm iK, \\
[K^\pm, J] &= \mp iK^\pm, \\
[K, J^\pm] &= \pm iK^\pm,
\end{align*}
\]

other commutators vanish.

The traceless part of \([K^1, J^{-K}]\) is anomalous in \(D>3\),
but the traceless part of its counterpart in \(D=3\) doesn’t exist
and the trace part defines \(K^-, \hat{K}^- \equiv i[K, J^-]\).
3. New dangerous commutator

For Hermitian R-ordered generators of the 3D spacetime conformal symmetry in light-cone gauge,

- Poincaré group ← easy
- Commutators without superscript “-” ← easy
- Most of other commutators ←
  - Definition of $\mathcal{K}^-$
  - Constraint $M_0 = 0$
  - Jacobi identities

The last commutator we must investigate is

$$\left[\hat{\mathcal{K}}^-, \hat{J}^-\right] = 0.$$

- Quintic term of $X_n$ and $P_n$
- Many divergent terms
- Ordering-dependent

New Dangerous commutator!

We need a very lengthy, hard calculation!
3. Calculation of $[\hat{K}^-, J^-]$

The commutator of two Hermitian ops. are anti-Hermite,

$$\Rightarrow [\hat{K}^-, J^-] = \frac{1}{2} \left( [\hat{K}^{-}_R, J^-] - [\hat{K}^{-}_R, J^-]^\dagger \right).$$

Into R-order

$$J^- = \frac{1}{2} (J^-_R + J^-_R^\dagger) = J^-_R + \frac{1}{2} i \frac{p}{p_-}.$$

$$x^-x^- (\ldots) + x^- \frac{1}{p_-} (\ldots) + \frac{1}{p_-^2} (\ldots)$$

Easy = 0

Easy = 0

Ordering independent!

$$\sum P \propto p \Rightarrow 0$$

Anti-Hermitian version of cubic-part recollects the form with a commutator $[X,P]=i$ and becomes linear term.
3. Conclusion of the first half

For Hermitian R-ordered generators of the 3D spacetime conformal symmetry in light-cone gauge,

- Poincaré group \(\leftarrow\) easy
- Commutators without “-” \(\leftarrow\) easy

- Definition of \(\mathcal{K}^-\): \(\hat{\mathcal{K}}^- \equiv i[\mathcal{K}, J^-]\)
- \(\hat{\mathcal{K}}^-, J^- = 0\)
- Constraint: \(M_0 = 0\)
- Jacobi identities

All commutators have no anomaly.

A quantum 3D tensionless closed string has the spacetime conformal symmetry.

※ Conformal symmetry may have anomaly in other ordering.
  e.g. Normal order of \(X\) and \(P\)
3. Spectrum

- **Expected spectrum**
  - Conformal (scale) symmetry
  - Mass eigenvalues are zero or continuous.

\[ X \rightarrow \lambda^{-1} X, \ P \rightarrow \lambda P \ \Rightarrow \ M^2 \rightarrow \lambda^2 M^2 \]

- **Investigation of mass eigenstates**
  - \[ M^2 = 2 \sum_{n>0} P_n P_{-n} \]
  - Constraint: \[ 0 = M_0 \equiv -i \sum_n nX_n P_{-n} \]
  - R-order string ground state \[ |0\rangle_R: \ P_n |0\rangle_R = 0 \]
  - X-representation: \[ P_n = -i \frac{\partial}{\partial X_{-n}}, |\Psi\rangle = \Psi(\{X_n\})|0\rangle_R \]

We first consider eigenstates \[ \Psi(\{X_n\}) = \prod_{n>0} \psi^{(n)}(X_n, X_{-n}) \]
in each \( n>0 \) and put a constraint on eigenstates.
3. Coordinate transformation

\[ X_n = r_n e^{i\theta_n}, X_{-n} = r_n e^{-i\theta_n} \quad \text{for } n > 0 \]

\[(X_n)^\dagger = X_{-n} \rightarrow r_n, \theta_n : \text{real} \]

Mass operator:

\[ \mathcal{M}^2 = -2 \sum_{n>0} \frac{\partial}{\partial X_n} \frac{\partial}{\partial X_{-n}} = -\frac{1}{2} \sum_{n>0} \left[ \frac{\partial^2}{\partial r_n^2} + \frac{1}{r_n} \frac{\partial}{\partial r_n} + \frac{1}{r_n^2} \frac{\partial^2}{\partial \theta_n^2} \right] \]

Assume \( \theta_n \)-dependence: \( s_n \) integer

\[ \psi^{(n)}(r_n, \theta_n) = \phi^{(n)}(r_n) e^{i s_n \theta_n} \quad \text{in } \Psi(\{X_n\}) = \prod_{n>0} \psi^{(n)}(r_n, \theta_n) \]

\[ \frac{\partial^2}{\partial r_n^2} + \frac{1}{r_n} \frac{\partial}{\partial r_n} + \frac{1}{r_n^2} \frac{\partial^2}{\partial \theta_n^2} \rightarrow \frac{\partial^2}{\partial r_n^2} + \frac{1}{r_n} \frac{\partial}{\partial r_n} - \frac{s_n^2}{r_n^2} \]

Constraint: \( M_0 = -\sum_n n X_n \frac{\partial}{\partial X_n} = i \sum_n n \frac{\partial}{\partial \theta_n} = 0 \)

\[ \Rightarrow \sum_n n s_n = 0 \quad \text{←simple} \]
3. Eigenfunctions in each $n$

Mass op. : $\mathcal{M}^2 = -\frac{1}{2} \sum_{n>0} \left[ \frac{\partial^2}{\partial r_n^2} + \frac{1}{r_n} \frac{\partial}{\partial r_n} + \frac{1}{r_n^2} \frac{\partial^2}{\partial \theta_n^2} \right]$

We consider eigenvalue equation in each $n$.

$$-\frac{1}{2} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] \psi_m(r, \theta) = 2m^2 \psi_m(r, \theta)$$

$$\downarrow$$

$$\psi_m(r, \theta) = \phi_{m,s}(r)e^{is\theta}$$

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + 4m^2 - \frac{s^2}{r^2} \right] \phi_{m,s}(r) = 0$$

- $m^2 > 0$ case
  
  $\phi_{m,s}(r, \theta) = N_m J_{|s|}(2mr)$

  $$J_v(z) = \left(\frac{z}{2}\right)^v \sum_{l=0}^{\infty} \frac{(-1)^l}{l! \Gamma(l+v+1)} \left(\frac{z}{2}\right)^{2l}$$

  : Bessel function

- $m^2 = 0$ case
  
  $\phi_{0,s}(r, \theta) = r^{|s|}$ or $r^{-|s|}$ for $s \neq 0$

  $\phi_{0,0}(r, \theta) =$ constant or $\log r$ for $s = 0$
3. Scalar product  

[J. Gamboa, C. Ramirez & M. Rizaltaba]

**Definition:** Scalar product of two functions \( \psi_1 \) and \( \psi_2 \) is

\[
(\psi_1, \psi_2) = \int_0^\infty dr \int_0^{2\pi} d\theta \ r \ \psi_1(r, \theta)^* \psi_2(r, \theta)
\]

**Formulation:**

\[
\int_0^\infty dx \ x J_l(ax)^* J_l(bx) = \lim_{\Lambda \to \infty} \frac{\Lambda}{a^2-b^2} \left[ aJ_{l+1}(a\Lambda)J_l(b\Lambda) - bJ_l(a\Lambda)J_{l+1}(b\Lambda) \right]
\]

\[
= \frac{1}{\pi \sqrt{ab}} \lim_{\Lambda \to \infty} \left[ \frac{\sin(a-b)\Lambda}{a-b} - (-1)^l \frac{\cos(a+b)\Lambda}{a+b} \right]
\]

\[
\cdot J_l(0) = 0 \text{ for } l > 0
\]

\[
\cdot \text{ Hankel asymptotic form } (x \to \infty): \quad J_v(x) = \sqrt{\frac{2}{\pi x}} \left[ \cos \left( x - \frac{2v+1}{4} \pi \right) + O(x^{-1}) \right]
\]

**Scalar product of** \( \psi_{m,s} = N_m J_{|s|} (2mr)e^{is\theta} \) **and** \( \psi_{m',s'} \):

\[
(\psi_{m,s}, \psi_{m',s'}) = \frac{\pi}{2} \delta_{s,s'} \frac{|N_m|^2}{m} \delta(m - m')
\]

By a similarly calculation, we get eigenfunctions of \( m^2 = 0 \).

\[
(\psi_{0,s}, \psi_{m,s}) = 0 \Rightarrow \psi_{0,s} = r^{|s|} \text{ for } s \neq 0 \text{ and } \psi_{0,0} = \text{const.}
\]
3. Total eigenfunction

The eigenfunctions of $\mathcal{M}^2$ are product.

$$\Psi = C \prod_{n>0} \psi_{m_n, s_n} (X_n, X_{-n})$$

Normalization constant

$$\begin{cases} 
\psi_{m_n, \pm |s_n|} = (m_n X_{\pm n})^{\pm s_n} \sum_{l=0}^{\infty} \frac{(-m_n^2)^l}{l! (l+|s_n|)!} (X_n X_{-n})^l \\
\psi_{0, \pm |s_n|} = (X_{\pm n})^{|s_n|} 
\end{cases}$$

Mass eigenvalue: $\mathcal{M}^2 = 2 \sum_{n>0} m_n^2$

Constraint: $\sum_{n>0} ns_n = 0$

Massless states ($m_n = 0$ for all $n > 0$):

$$X_{n_1} X_{n_2} \cdots X_{n_L} |0\rangle_R \text{ with } \sum_{i=1}^{L} n_i = 0 \text{ and } n_i + n_j \neq 0$$

Simple example: $X_2 X_{-1} X_{-1} |0\rangle_R$

$\times X_n |0\rangle \quad \leftarrow \sum_{n>0} ns_n \neq 0$

$\times X_n X_{-n} |0\rangle \quad \leftarrow \mathcal{M}^2 \neq 0$

※ $(X_n X_{-n} - X_m X_{-m}) |0\rangle$-type states are created by acting $\Lambda$ on above states
3. Massless sector

Massless states:
\[ X_{n_1} X_{n_2} \cdots X_{n_L} |0\rangle_R \text{ with } \sum_{i=1}^{L} n_i = 0 \text{ and } n_i + n_j \neq 0 \]

Massless sector is spacetime-conformal invariant.

\[ \rightarrow \text{Some 3D CFT corresponds?} \]

- infinite elementary operators \( \{X_n\} \)
- Even the number of lowest states (\( \Delta = 3 \)) is infinite! \( X_{n+m}X_{-n}X_{-m} \)

The important quantum number in 3D conformal group

is conformal dimension.

Conformal dimension: \( \Delta \equiv iD_R, [\Delta, X_n] = X_n \)

\[ \Rightarrow \Delta \text{ counts the number of } X\text{-operator.} \]

\[ \Delta = 0 \text{ for } |0\rangle_R \text{ and } \Delta = L \text{ for the above states.} \]

Detail investigation of another quantum number “spin”

\[ \Rightarrow \text{future work} \]
4. Summary & Outlook

Summary

- 3D tensionless closed string in light-cone gauge has no anomaly of the spacetime-conformal symmetry.
- There are massless and massive states in its spectrum.
- Massless sector corresponds to some 3D CFT?

Outlook

- Other tensionless ver.: open, supersymmetric, p-brane
- The detail investigation from the perspective of 3D conformal group → What kind of CFT?
- Explanation for the difference between light-cone gauge quantization and BRST method → Is there covariant quantization method which reproduces results in light-cone gauge?
- Relation with Higher spin theories?