Joint Space Decomposition-and-Synthesis Theory for $K$-User MIMO Channels: Interference Alignment and DoF Region

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Abstract

The degree-of-freedom (DoF) region for a general interference-aligned multi-user multiple-input multiple-output (MIMO) channel is of theoretic importance. Yet, its exact characterization is not available in the literature, due to the lack of an appropriate theoretical framework. A joint space decomposition-and-synthesis theory is developed in the paper. The joint decomposition is done on the receive spaces to uncover the geometric mechanism behind interference formation and alignment which, alongside the parametrization of the augmented interference channel matrices, forms a foundation to synthesize the required precoders at transmitters. The new framework leads to exact DoF regions for the 3-user channel with arbitrary number of antennas at receivers and transmitters, and an inner and an outer bound for the DoF region of $K$-user ($K > 3$) MIMO interference channels. It also covers the existing results for the 2-user MIMO channel as a special case.

Index Terms

DoF region, Geometric mechanism for MIMO interference, Interference alignment, Multiuser MIMO channels, MIMO interference channels, Precoder synthesis, Quotient singular value decomposition.

I. INTRODUCTION

Interference alignment is an efficient interference-elimination technique for wireless multi-user communication systems [1]–[3]. The basic idea is to separate, through appropriate use of

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precoders, the projection of the desired and interfering signals over their respective disjoint receive subspaces while keeping the dimension of the resulting interference subspace as small as possible. Interference alignment was originally intended for a higher multiplexing gain in multiple-input-multiple-output (MIMO) X channels [1], [2], but was shortly demonstrated to achieve the optimal $\frac{1}{2}$ degree-of-freedom (DoF) per user in the $K$-user single-input-single-output (SISO) interference channel subject to time varying or frequency selective fading [3]. The latter result is somewhat surprising, thereby stimulating the application of interference alignment to MIMO interference channels [4]–[13]. The motivation of this paper is to investigate the achievable DoF region of interference alignment when applied to a general $K$-user constant MIMO interference channel.

A. Related Work

The problem of characterizing the DoF of $K$-user MIMO interference channels has attracted intensive effort in the past several years, yet remaining unsolved except for some special cases. Indeed, the DoF of the 2-user MIMO interference channel was completely characterized in [4]. For the 3-user case, however, only the exact achievable sum DoF by interference alignment was given in [3, Section V-A] by assuming that each transmitter and each receiver are equipped with the same number of antennas. For the general $K$-user MIMO interference channels with time-varying channel coefficients, the best known results were upper and lower bounds obtained, again, for the achievable sum DoF [5].

Given the lack of powerful mathematical tools of relevance, many efforts adopted indirect approaches by tackling other aspects of the DoF regions than an exact characterization [6]–[13]. The work along this line can be classified into two categories according to the approach used. The first category resorted to optimization tools as exemplified in [6], where the interference alignment problem was formulated as searching for the optimized linear precoders and receive filters that maximize the sum DoF under the constraint of nullifying interfering signals. However, the nonlinearity and non-convexity nature of the resultant optimization problem makes it nearly impossible to derive closed-form expressions for the achievable DoF, forcing the use of numerical algorithms for its solution [6], [7]. In [8], the interference alignment problem was reformulated as a rank-constrained rank minimization problem, to take advantage of the latter which was convertible to a convex optimization by tight-convex relaxing a rank operator [14], [15].
resulting convex optimization problem could be solved by off-the-shelf numerical algorithms. In [9], the formulated problem was to minimize the interference power that leaks into the signal subspace while preserving the desired signal dimensions. The solution, again, relied on numerical algorithms.

The second category of research work investigated the feasibility of interference alignment in $K$-user MIMO interference channels. A feasible solution of interference alignment is a set of precoders and receive filters that nullify interferences while preserving the dimension of desired signal at all receivers. Given a DoF tuple, the feasibility issue can be transformed to the solvability of a set of (nonlinear) equations. According to the theory of algebraic geometry, the solvability of a system of equations depends on whether the number of variables exceeds the number of equations. In [10], the authors classified the interference alignment problems as either proper or improper based on the numbers of variables and equations, and then established a connection between the feasibility and a proper system which employs only a single beamforming vector for each user. A more general result was subsequently derived in [11] giving a condition that any feasible DoF tuple must satisfy. The authors of [13] proposed a feasible condition for the case where all transmitters and receivers have the same number of antennas. In [12], generalized feasibility conditions were presented for interference alignment in MIMO channels with constant coefficients, complemented by a simple algorithm for feasibility test. The researches along this direction do provide insights into the DoF problem, but are short of exact characterization of the achievable DoF region in general.

B. Main Contributions and Organization of the Paper

In this paper, we tackle the DoF problem of MIMO interference channels in a new general framework, with three main contributions.

The nature of multi-user MIMO interference channels is that the desired and interfering signals propagate through their respective MIMO channels, projecting onto the same receive space of each user and thus, unavoidably causing interference among users. The first contribution of this paper is to develop a systematic theoretic framework to fully understand the nature and formation-mechanism of interference in a general multiuser MIMO channel and to synthesize the precoders needed at the transmitters. The second contribution is the exact characterization of DoF region, in closed form, for $K = 2$ and $K = 3$ users. The former result is equivalent to the
one previously derived in [4], but in different form. The latter result for 3-user MIMO channels with arbitrary number of antennas at transmitters and receivers, to the best of our knowledge, is the first of the same kind in the literature. The third contribution is made for the case of $K > 3$ users by establishing an inner and an outer bound for its DoF region.

The rest of this paper is organized as follows. In Section II, the system model is described. In Section III, the interference mechanism of multi-user interference channels is investigated utilizing the quotient singular value decomposition. Based on the intuition obtained in this investigation, we introduce the joint space-decomposition approach in Section III-B. By this approach, we derive the DoF regions for the 2-user, 3-user and $K$-user MIMO interference channels in Section IV. Section V concludes this paper.

Notation: The block matrices of a partitioned matrix are represented by appending a subscript to the original one. For example, the block matrices of a matrix $O_{ij}$ first partitioned by row and then by column are denoted as $O_{ij} = \begin{bmatrix} O_{i1} & O_{i2} \\ O_{j1} & O_{j2} \end{bmatrix} = \begin{bmatrix} O_{i11} & O_{i12} \\ O_{j11} & O_{j12} \\ O_{i21} & O_{i22} \\ O_{j21} & O_{j22} \end{bmatrix}$. The sizes of the identity matrix $I$ and the all zero matrix $0$ are implicit in the context, if not explicitly indicated.

All boldface letters represent vectors (lower case) or matrices (upper case). For ease of reading, we list the notations and variables used throughout this paper in following table.

**Symbols and Notation**
$(\cdot)^T$, $(\cdot)^\dagger$, $(\cdot)^-$ Transpose, conjugate transpose, and generalized inverse, respectively

$\mathbb{N}^+$ The set of nonnegative integers

$\mathbb{R}^+$ The set of nonnegative real numbers

$\mathbb{C}$ The field of complex numbers

$\mathbb{R}_{K}^+$ The set of $K$-tuples of nonnegative real numbers

$\mathbb{C}^{M \times N}$ The set of $M \times N$ matrices in the complex field

$\mathcal{CN}(\mathbf{v}, \mathbf{R})$ complex Gaussian distribution with mean $\mathbf{v}$ and covariance $\mathbf{R}$

$\dim(\cdot)$ The dimension of a space

$\text{span}(\cdot)$ The column space of a matrix

$\text{null}(\cdot)$ The null space of a matrix

$\text{rank}(\cdot)$ The rank of a matrix

$\text{diag}(\mathbf{A}_1, \cdots, \mathbf{A}_K)$ Block diagonal matrix with elements $\mathbf{A}_i$, $i = 1, \cdots, K$

$\text{Conv}(\mathcal{S}_K)$ The convex hull of the set $\mathcal{S}_K$

$(a)^+$ The nonnegative value $\max(0, a)$

$[a]$ The largest integer not exceeding $a$

$T_i$, $R_i$ Transmitter and receiver of user $i$, respectively

$N_i$, $M_i$ Number of antennas at $T_i$ and $R_i$, respectively

$\tilde{N}_i$ $\triangleq \sum_{j \neq i} N_j$

$\mathbf{H}_{ij}$ Channel matrix from $T_j$ to $R_i$

$\mathbf{H}_{ii}$ Alternatively called the signal transmission matrix to $R_i$

$\mathbf{V}_i$ Precoder matrix at $T_i$

$s_i$ Column signal vector from $T_i$

$\tilde{\mathbf{H}}_{ii}$ $\triangleq [\mathbf{H}_{i1}, \cdots, \mathbf{H}_{i,i-1}, \mathbf{H}_{i,i+1}, \cdots, \mathbf{H}_{iK}]$, the augmented interference matrix to $R_i$

$\tilde{\mathbf{V}}_i$ $\triangleq \text{diag}(\mathbf{V}_1, \cdots, \mathbf{V}_{i-1}, \mathbf{V}_{i+1}, \cdots, \mathbf{V}_K)$, block diagonal interference precoding matrix to $R_i$

$s_i^T$ $\triangleq [s_1^T, \cdots, s_{i-1}^T, s_{i+1}^T, \cdots, s_K^T]$, the total row signal vector excluding $s_i$

$r_{i1}$ $\triangleq \text{rank}(\mathbf{H}_{ii})$

$r_{i2}$ $\triangleq \text{rank}(\tilde{\mathbf{H}}_{ii})$

$r_{i3}$ $\triangleq \text{rank}([\mathbf{H}_{ii} \  \tilde{\mathbf{H}}_{ii}])$

$d_i$ DoF for user $i$
II. System Model

Consider a $K$-user MIMO interference channel comprised of $K$ pairs of transmitters and receivers, where the $i$-th ($i = 1, \cdots, K$) transmitter $T_i$ and the $i$-th receiver $R_i$ are equipped with $N_i$ and $M_i$ antennas, respectively. The transmitter $T_i$ first precodes its $d_i$-by-$1$ vector signal $s_i$ using precoder $V_i \in \mathbb{C}^{N_i \times d_i}$ before transmitting it to the target receiver $R_i$. The $M_i$-by-$1$ received signal $y_i$ at $R_i$ is then given by

$$y_i = H_{ii}V_is_i + \sum_{1 \leq j \leq K, j \neq i} H_{ij}V_js_j + n_i = H_{ii}V_is_i + \tilde{H}_{ii}\tilde{V}_i\tilde{s}_i + n_i$$

where the matrix $H_{ij}$ represents the MIMO fading channel from $T_j$ to $R_i$ and the $M_i \times 1$ additive white Gaussian noise (AWGN) vector is assumed to be $n_i \sim \mathcal{CN}(0, I_{M_i})$. In this interference channel, vector $s_i$ is the desired signal at $R_i$ but an interfering signal to $R_j$ ($j = 1, \cdots, K, j \neq i$).

To simplify notation in (1), we collectedly denote the channel matrices, the precoders, and the interfering signals to form the augmented interference matrix $\tilde{H}_{ii}$, the block diagonal interference precoding matrix $\tilde{V}_i$, and interference signal vector $\tilde{s}_i$ at $R_i$, as given by

$$\tilde{H}_{ii} \triangleq [H_{i1}, \cdots, H_{i,i-1}, H_{i,i+1}, \cdots, H_{iK}],$$
$$\tilde{V}_i \triangleq \text{diag}(V_1, \cdots, V_{i-1}, V_{i+1}, \cdots, V_K),$$
$$\tilde{s}_T \triangleq [s_{T1}^T, \cdots, s_{Ti-1}^T, s_{Ti+1}^T, \cdots, s_{TK}^T],$$

respectively.

Assume that all channel matrices are independent of each other and their elements are i.i.d. and drawn from a continuous distribution. Hence, the channel matrices are full rank almost surely. For $R_i$, denote the ranks of its signal transmission matrix, augmented interference matrix and total matrix, respectively, by

$$r_{i1} \triangleq \text{rank}(H_{ii}) = \min (M_i, N_i),$$
$$r_{i2} \triangleq \text{rank}(\tilde{H}_{ii}) = \min \left( M_i, \sum_{j \neq i} N_j \right),$$
$$r_{i3} \triangleq \text{rank}(|H_{ii} \tilde{H}_{ii}|) = \min \left( M_i, \sum_{j=1}^{K} N_j \right).$$

We also assume perfect channel information at all transmitters and receivers in the network.
To define the DoF region for the $K$-user MIMO interference channel, let $\rho$ denote its signal to noise ratio (SNR) and let $C(\rho)$ be its capacity region. The capacity region is the set of all possible achievable rate tuples $\mathcal{R}(\rho) = (\mathcal{R}_1(\rho), \mathcal{R}_2(\rho), \ldots, \mathcal{R}_K(\rho)) \in \mathbb{R}_+^K$, where $\mathcal{R}_i$ is the data rate associated with link $T_i - R_i$. The DoF of a single link is its asymptotic data rate as SNR approaches infinity. For a $K$-user MIMO interference channel, its DoF region $\mathcal{D}_K$ encompasses all the $K$-tuples of DoFs achievable by the $K$ users in the network and is defined as \[ \mathcal{D}_K = \{(d_1, d_2, \ldots, d_K) \in \mathbb{R}_+^K : \forall (w_1, w_2, \ldots, w_K) \in \mathbb{R}_+^K; \sum_{i=1}^K w_i d_i \leq \limsup_{\rho \to \infty} \left[ \sup_{\mathcal{R}(\rho) \in C(\rho)} \left( \sum_{i=1}^K w_i \mathcal{R}_i(\rho) \right) \frac{1}{\log(\rho)} \right] \} \]. The maximum sum DoF is given by \[ d_{\text{max},K} = \max_{\mathcal{D}_K} \sum_{i=1}^K d_i. \] \[ \text{(6)} \]

III. NOVEL METHODOLOGY

To gain inspiration for interference alignment, it is necessary to understand how the interferers spatially clash with the desired signal. To this end, we temporally remove the precoders so that the received signal in (1) becomes \[ y_i = H_{ii} s_i + \tilde{H}_{ii} \tilde{s}_i + n_i. \] Geometrically, the desired signal $s_i$ projects onto the $M_i$-dimensional receive space at $R_i$ via channel $H_{ii}$ whereas the interfering signals $s_j, j \neq i$ do the same thing through their own interference channels $H_{ij}$. The collision of these projections in the receive space unavoidably incurs interference. The important thing to the precoder designer is to thoroughly understand to what extent or over which subspace, the interference collides with the signal. Clearly, the interference behavior is dictated by the nature of matrices $H_{ii}$ and $\tilde{H}_{ii}$.

A. Joint Channel Decomposition

We need a mathematical tool, called the quotient singular value decomposition (QSVD) [16]–[20] to proceed, which is restated here as a lemma for subsequent use.
Lemma 1 (QSVD): For matrices \( H_{ii} \) and \( \tilde{H}_{ii} \), there exist unitary matrices \( \Phi_i \) and \( \Psi_i \) and nonsingular square matrix \( X_i \) such that

\[
H_{ii} = X_i \Sigma_{i1} \Phi_i, \quad \tilde{H}_{ii} = X_i \Sigma_{i2} \Psi_i, \tag{9}
\]

where

\[
\Sigma_{i1} = \begin{bmatrix}
 r_{i3} - r_{i2} & r_{i1} + r_{i2} - r_{i3} & N_i - r_{i1} \\
 0 & 0 & D_{i1} \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
 r_{i3} - r_{i2} \\
 r_{i1} + r_{i2} - r_{i3} \\
 r_{i3} - r_{i1} \\
 M_i - r_{i3} \\
\end{bmatrix}, \tag{10a}
\]

\[
\Sigma_{i2} = \begin{bmatrix}
 \tilde{N}_i - r_{i2} & r_{i1} + r_{i2} - r_{i3} & r_{i3} - r_{i1} \\
 0 & 0 & 0 \\
 0 & 0 & D_{i2} \\
 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
 r_{i3} - r_{i2} \\
 r_{i1} + r_{i2} - r_{i3} \\
 r_{i3} - r_{i1} \\
 M_i - r_{i3} \\
\end{bmatrix}, \tag{10b}
\]

and \( \tilde{N}_i = \sum_{j \neq i} N_j \), and \( D_{i1} \) and \( D_{i2} \) are square diagonal matrices with positive diagonal elements, satisfying

\[
D_{i1}^2 + D_{i2}^2 = I.
\]

This lemma provides a framework enabling the representation of both signal and augmented interference channel matrices in a common space spanned by the column vectors of \( X_i \), a set of usually non-orthogonal basis functions.

We further partition \( \Phi_i \), \( \Psi_i \), and \( X_i \) into blocks compatible with \( \Sigma_{i1} \) and \( \Sigma_{i2} \), as shown by

\[
\Phi_i = \begin{bmatrix}
 \Phi_{i1} \\
 \Phi_{i2} \\
 \Phi_{i3} \\
\end{bmatrix}, \quad \Psi_i = \begin{bmatrix}
 \Psi_{i1} \\
 \Psi_{i2} \\
 \Psi_{i3} \\
\end{bmatrix}, \quad X_i = \begin{bmatrix}
 X_{i1} \\
 X_{i2} \\
 X_{i3} \\
 X_{i4} \\
\end{bmatrix}.
\]

\[
X_i = \begin{bmatrix}
 X_{i1} & X_{i2} & X_{i3} & X_{i4} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
 r_{i3} - r_{i2} & r_{i1} + r_{i2} - r_{i3} & r_{i3} - r_{i1} & M_i - r_{i3} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
 X_{i1} & X_{i2} & X_{i3} & X_{i4} \\
\end{bmatrix}
\]
and rewrite (9) as

\[ H_{ii} = X_{i1} \Phi_{i1} + X_{i2} D_{i1} \Phi_{i2}, \]  

(13a)

\[ \tilde{H}_{ii} = X_{i2} D_{i2} \Psi_{i2} + X_{i3} \Psi_{i3}. \]  

(13b)

These expressions indicate that the receive space spanned by the columns of \( X_i \) can be partitioned into four disjoint subspaces. The subspace span(\( X_{i1} \)) of dimension \( r_{i3} - r_{i2} \) uniquely belongs to \( \text{span}(H_{ii}) \), whereas the subspace span(\( X_{i3} \)) of dimension \( r_{i3} - r_{i1} \) uniquely belongs to \( \text{span}(\tilde{H}_{ii}) \). The two channel matrices, \( H_{ii} \) and \( \tilde{H}_{ii} \), overlap over the subspace span(\( X_{i2} \)) of dimension \( r_{i1} + r_{i2} - r_{i3} \). The subspace that is not covered in \( \text{span}(H_{ii}) \) or \( \text{span}(\tilde{H}_{ii}) \) is denoted by \( \text{span}(X_{i4}) \).

It is the spatial overlap of the signal and interference channel matrices that causes interference among signals from different transmitters. To see this, we use Lemma 1 and the block matrices defined above to rewrite (1) yielding

\[ y_i = X_i \Sigma_1 \Phi_i V_i s_i + X_i \Sigma_2 \Psi_i \tilde{V}_i \tilde{s}_i + n_i \]

\[ = X_{i1} \Phi_{i1} V_i s_i + X_{i2} (D_{i1} \Phi_{i2} V_i s_i + D_{i2} \Psi_{i2} \tilde{V}_i \tilde{s}_i) + X_{i3} \Psi_{i3} \tilde{V}_i \tilde{s}_i + n_i \]

\[ = \begin{bmatrix} X_{i1} & X_{i2} & X_{i3} \end{bmatrix} \begin{bmatrix} \Phi_{i1} \\ D_{i1} \Phi_{i2} \\ 0 \end{bmatrix} V_i s_i + \begin{bmatrix} X_{i1} & X_{i2} & X_{i3} \end{bmatrix} \begin{bmatrix} 0 \\ D_{i2} \Psi_{i2} \\ \Psi_{i3} \end{bmatrix} \tilde{V}_i \tilde{s}_i + n_i. \]  

(14)

After projection, the signal \( s_i \) uniquely occupies the subspace \( \text{span}(X_{i1}) \) of dimensions \( r_{i3} - r_{i2} \), whereas the interfering signals \( \tilde{s}_i \) uniquely occupies the subspace \( \text{span}(X_{i3}) \) of dimensions \( r_{i3} - r_{i1} \). The subspace \( \text{span}(X_{i2}) \) of dimension \( r_{i1} + r_{i2} - r_{i3} \) represents the intersection of \( \text{span}(H_{ii}) \) and \( \text{span}(\tilde{H}_{ii}) \), over which the interference collides with the signal. The space geometry is heuristically sketched in Fig.1 where the disjoint subspace defined by \( X_{i4} \) is also included for completeness.

**B. Interference-Free Conditions for Precoders**

In the traditional channel equalization theory, inter-symbol interference over the temporal domain is caused by bandlimited channels and can be removed by a channel equalizer. The
spatial interference caused by an interference MIMO channel can be removed by following a similar philosophy, but by virtue of a set of spatial precoders $V_i$ and $\tilde{V}_i$.

The challenge is to determine the corresponding interference-free conditions for the precoders. The representation of $H_{ii}$ and $\tilde{H}_{ii}$ in terms of the same set of basis functions $X_i$ paves a path to success.

**Lemma 2:** Let matrices $X \in \mathbb{C}^{m \times n}$, $P \in \mathbb{C}^{n \times n_1}$ and $Q \in \mathbb{C}^{n \times n_2}$ be full column rank and $n_1 + n_2 \leq n$. Then, the subspaces of $\text{span}(XP)$ and $\text{span}(XQ)$ are disjoint, i.e., $\text{span}(XP) \cap \text{span}(XQ) = \{0\}$, if and only if, $\text{span}(P) \cap \text{span}(Q) = \{0\}$, or equivalently, $\text{rank}([P \ Q]) = n_1 + n_2$.

**Proof:** See Appendix A.

Directly applying Lemma 2 to (14), we obtain a set of simultaneous equations for the desired precoders:

$$\text{span} \left\{ \begin{bmatrix} \Phi_{i1} \\ D_{i1} \Phi_{i2} \\ 0 \end{bmatrix} V_i \right\} \cap \text{span} \left\{ \begin{bmatrix} 0 \\ D_{i2} \Psi_{i2} \\ \Psi_{i3} \end{bmatrix} \tilde{V}_i \right\} = \{0\}$$

for $i = 1, \cdots, K$. It demonstrates that the use of appropriately designed precoders $V_i$ and $\tilde{V}_i$ can clearly cut the $(r_{i1} + r_{i2} - r_{i3})$-dimensional common subspace $X_{i2}$ into two disjoint parts for signal and interference. There are sufficient degrees of freedom in the design of $V_i$ and $\tilde{V}_i$ to meet the interference-free condition (15), thus allowing for varying dimensions in the two
disjoint parts. In particular, if \(0 \leq a_i \leq r_{i1} + r_{i2} - r_{i3}\) dimensions are assigned to the interference part, then we have

\[
\begin{align*}
\text{rank}(& D_{i2} \Psi_{i2} \tilde{V}_i) = a_i, \\
\text{rank}(& D_{i1} \Phi_{i2} V_i) = r_{i1} + r_{i2} - r_{i3} - a_i,
\end{align*}
\]

which, alongside (15), allow us to determine the dimension of signal subspace as

\[
\text{rank}(V_i) = \text{rank}(\Phi_{i1} V_i) + \text{rank}(D_{i1} \Phi_{i2} V_i) = r_{i1} - a_i.
\]

The interference \(\tilde{s}_i\) can transmit through the \((\tilde{N}_i - r_{i2})\)-dimensional null-space of \(\tilde{H}_{ii}\), beside the interference-alone subspace \(\text{span}(X_{i3})\) of dimension \((r_{i3} - r_{i1})\) and \(a_i\)-dimensional subspace of \(\text{span}(X_{i2})\) allocated to the interference. Hence, we obtain

\[
\begin{align*}
\text{rank}(\tilde{V}_i) &= \text{rank}(\Psi_{i3} \tilde{V}_i) + \text{rank}(D_{i2} \Psi_{i2} \tilde{V}_i) + \text{rank}(\Psi_{i3} \tilde{V}_i) \\
&= \tilde{N}_i - r_{i2} + r_{i3} - r_{i1} + a_i.
\end{align*}
\]

Expressions (15-18) constitute the basis for precoders determination.

C. Synthesis of Precoders and DoF Region

From (15), it is clear that the channel pair \((H_{ii}, \tilde{H}_{ii})\) of the communication link \(T_i-R_i\) imposes an interference-free constraint on the signal and the combined interference precoders, i.e., \(V_i\) and \(\tilde{V}_i\). The signal and the combined interference precoders satisfying the alignment requirement only for a single \(T_i-R_i\) link are denoted by \(V^{[i]}_i\) and \(\tilde{V}^{[i]}_i\), referred to as possible precoders hereafter for ease of description. The latter matrix \(\tilde{V}^{[i]}_i\), in turn, allows us to determine the constraint on the precoders for transmitters other than \(T_i\), imposed by \((H_{ii}, \tilde{H}_{ii})\); symbolically, we write

\[
\tilde{V}^{[i]}_i \longrightarrow \{V^{[i]}_k, k = 1, \cdots, K \text{ and } k \neq i\}
\]

Clearly, each channel pair determines a set of possible precoders, as shown by

\[
(H_{ii}, \tilde{H}_{ii}) \longrightarrow \{V^{[i]}_k, k = 1, \cdots, K\}
\]

where \(i = 1, \cdots, K\). The source \(s_i\), as the desired signal, requires its precoder to meet the constraint of the \(T_i-R_i\) link and, as an interferer, requires its precoder to satisfy the constraints by all the \(T_k-R_k\) link for \(k \neq i\). As such, a feasible precoder \(V_i\) for \(T_i\) should be the one, for
which the transmit space is defined by the intersection of the spaces corresponding to the $K$ possible precoders of relevance, i.e.,

\[ \text{span}(V_i) = \cap_{j=1}^{K} \text{span}(V_i^{[j]}). \tag{21} \]

Note that $V_i$ and $\tilde{V}_i$ is a function of $a_i$; such dependence can be written explicitly as $V_i(a_i)$ and $\tilde{V}_i(a_i)$, respectively, wherever necessary. Thus, for a given set of $a \triangleq [a_1, \cdots, a_K]$, the dimension of the achievable DoF for $T_i$ is given by

\[ d_i(a) = \dim \left( \cap_{j=1}^{K} \text{span}(V_i^{[j]}(a_j)) \right), \tag{22} \]

and the corresponding vector

\[ \mathbf{d}(a) = [d_1(a), \cdots, d_K(a)] \tag{23} \]

defines a point in the achievable DoF region for a particular $a$. The point set obtained for all $a's$ ($0 \leq a_i \leq r_1 + r_2 - r_3$) constitutes the entire DoF region. In what follows, the index $a$ will be dropped for notational simplicity if no ambiguity introduces.

To determine the DoF, we need to retrieve $\text{span}(V_i^{[j]}), j \neq i$ first from the higher dimensional space $\text{span}(\tilde{V}_i^{[i]})$, a key step to the precoder design for interference alignment in multi-user MIMO channels. The detailed design for the cases of $K = 2$ and $K = 3$ is given in Appendix C. In the subsequent sections, we will focus on the DoF-region analysis for various cases with different numbers of users.

\section*{IV. DoF Region}

In this section, we derive the possible and feasible transmit spaces for all the transmitters, and thus obtain the DoF regions for the 2-user, 3-user and $K$-user MIMO interference channels.

\subsection*{A. 2-user MIMO Interference Channel}

In this case, there is only one interferer at each receiver, which implies that $V_i^{[2]} = \tilde{V}_2$ and $V_2^{[1]} = \tilde{V}_1$, making the retrieval of possible precoders from the latter extremely simple. Specifically, it follows from from (17) and (18) that the dimensions of the two possible transmit spaces for $T_1$ are equal to

\[ \dim(\text{span}(V_1^{[1]})) = r_{11} - a_1, \tag{24a} \]

\[ \dim(\text{span}(V_1^{[2]})) = N_1 - r_{22} + r_{23} - r_{21} + a_2. \tag{24b} \]
while their counterparts for $T_2$ are given by

$$\dim(\text{span}(V_2^{[2]})) = r_{21} - a_2, \quad (25a)$$

$$\dim(\text{span}(V_2^{[1]})) = N_2 - r_{12} + r_{13} - r_{11} + a_1. \quad (25b)$$

With these expressions, we can write an arbitrary point $(d_1, d_2)$ in the two-dimensional DoF region where

$$d_1 = \dim(\text{span}(V_1^{[1]}) \cap \text{span}(V_2^{[2]})) \leq \min(r_{11} - a_1, N_1 - r_{22} + r_{23} - r_{21} + a_2), \quad (26a)$$

$$d_2 = \dim(\text{span}(V_2^{[1]}) \cap \text{span}(V_2^{[2]})) \leq \min(r_{21} - a_2, N_2 - r_{12} + r_{13} - r_{11} + a_1). \quad (26b)$$

The upper bound in (26) can be achieved by appropriately designing the precoders, as shown in Appendices C-A. It can be also shown that the DoFs are convex combinations of the points given in (26), achievable through time sharing between the points [21], [22]. In summary, we can make the following assertion for 2-user channels.

**Theorem 1:** The achievable DoF region for the 2-user constant MIMO interference channel is given by

$$D_2 = \left\{ (\eta_1, \eta_2) \in \mathbb{R}_2^+ : \eta_1 \leq \mu_1, \eta_2 \leq \mu_2, \forall (\mu_1, \mu_2) \in \text{Conv}(S_2) \right\}, \quad (27)$$

where

$$S_2 = \left\{ (d_1, d_2) : d_1 = \min(r_{11} - a_1, N_1 - r_{22} + r_{23} - r_{21} + a_2), \right.\left. d_2 = \min(r_{21} - a_2, N_2 - r_{12} + r_{13} - r_{11} + a_1), \right.\left. \forall a_1, a_2 \in \mathbb{N}_+, 0 \leq a_1 \leq r_{11} + r_{12} - r_{13}, 0 \leq a_2 \leq r_{21} + r_{22} - r_{23} \right\}. \quad (28)$$

Based on the above theorem, we can calculate the maximum sum DoF as given by the following corollary.

**Corollary 1:** The maximum sum DoF for the 2-user constant MIMO interference channel is given by

$$d_1 + d_2 \leq \min(r_{11} + r_{21}, N_2 - r_{12} + r_{13}, N_1 - r_{22} + r_{23}, N_1 + N_2). \quad (29)$$

**Proof:** See Appendix B.

**Remark 1:** The possible values of the maximum sum DoF shown in (29) are enumerated in Table I, which, when compared with that obtained in [4], enables the assertion that our derived result is equivalent to that in [4].
TABLE I
THE DOF ACHIEVED BY INTERFERENCE ALIGNMENT FOR 2-USER INTERFERENCE CHANNEL IN ALL CASES

| $N_1 > (N_2, M_1, M_2)$ | $M_1 > (N_1, N_2, M_2)$ |
|-------------------------|-------------------------|
| $M_1 > N_2$             | $M_1 < N_2$             |
| DoF = $M_1$             | $N_2 \leq M_1 + M_2$    |
| DoF = $N_2$             | $N_2 > M_1 + M_2$       |
|                         | DoF = $M_2$             |
|                         | $M_2 \geq N_1 + N_2$    |
|                         | $M_2 < N_1 + N_2$       |

Remark 2: Expressions (27) and (28) present a parametric description of the DoF region. In [4], the DoF region for the 2-user constant MIMO interference channel is given by (also ref. to [23])

$$\tilde{D}_2 = \left\{ (\eta_1, \eta_2) \in \mathbb{R}^2_+ : \eta_1 \leq \min(M_1, N_1), \eta_2 \leq \min(M_2, N_2), \eta_1 + \eta_2 \leq \min(N_1 + N_2, M_1 + M_2, \max(N_1, M_2), \max(N_2, M_1)) \right\},$$

which is an inequality description. It is difficult at first sight to decide whether these two results are equivalent. However, as shown in Remark 1, the maximum sum DoF achieved in $\tilde{D}_2$ is the same as $\eta_1 + \eta_2 \leq \min(N_1 + N_2, M_1 + M_2, \max(N_1, M_2), \max(N_2, M_1))$ in $\tilde{D}_2$. This assertion, alongside the fact that the other two inequalities (i.e., $\eta_1 \leq \min(M_1, N_1)$ and $\eta_2 \leq \min(M_2, N_2)$) in $\tilde{D}_2$ are implied in the expressions of $d_1$ and $d_2$ in $D_2$, allows us to conclude that the DoF region characterizations in (27) and in (30) are equivalent.

B. 3-user MIMO Interference Channel

1) Possible transmit spaces: In this section, we derive the possible transmit spaces $\text{span}(V_j^{[i]})$.

For $K = 3$, the received signal at $R_i$ is expressible as

$$y_i = [X_{i1} \ X_{i2} \ X_{i3}] \text{diag}(I, D_{i1}, 0) \Phi_i V_i s_i + [X_{i1} \ X_{i2} \ X_{i3}] \text{diag}(0, D_{i2}, I) \Psi_i \begin{bmatrix} v_j^{[i]} \\ 0 \\ v_i^{[i]} \end{bmatrix} [s_j] + n_i,$$

where $i, j, l = 1, 2, 3$, $i \neq j \neq l$ and $j < l$. From the previous section, the signal and interference overlap in $\text{span}(X_{i2})$. Suppose an $a_i$-dimensional subspace of $\text{span}(X_{i2})$ is allocated for accommodating interference. Then, the signal subspace is of $r_{i1} - a_i$ dimensions and the interference subspace $r_{i3} - r_{i1} + a_i$ dimensions. From Lemma 2, to align the signal $s_i$ and the
interference \([\Psi_i^\dagger, \Psi_i^\dagger, \Psi_i^\dagger]\) with these two disjoint subspaces, their possible transmit precoders \(V_i[i]\) and \([V_j[i], 0, 0, V_i[i]]\) must be designed to meet the following disjoint condition:

\[
\text{span}\left(\text{diag}(I, D_{i1}, 0)\Phi_i V_i[i]\right) \cap \text{span}\left(\text{diag}(0, D_{i2}, I)\Psi_i \begin{bmatrix} V_j[i] & 0 & 0 & V_i[i] \end{bmatrix}\right) = \{0\}
\]  

(32)

where dimension of the two subspaces is \(r_{i1} - a_i\) and \(r_{i3} - r_{i1} + a_i\), respectively. This is a loose condition and there are many possible solutions. However, directly seeking the possible interference precoder matrix of a block diagonal structure is quite difficult. We, therefore, concentrate on parameterizing the augmented interference transmit space \(\text{diag}(V_j[i], V_i[i])\) whereby to facilitate the determination of the dimension of \(V_j[i]\) and \(V_i[i]\) on one hand, and to simplify their construction on the other. The detailed construction of precoders is left in Appendix C-B.

Possible interference transmit subspaces are contained in the space spanned by unitary matrix \(\Psi_i\). It is, therefore, natural to use it to parameterize the possible interference transmit subspace of a block-diagonal structure to yield

\[
\text{span}\left(\begin{bmatrix} V_j[i] & 0 & 0 & V_i[i] \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} \Psi_{i1}^\dagger & \Psi_{i2}^\dagger & \Psi_{i3}^\dagger \end{bmatrix} F_{i2}\right)
\]  

(33)

with \(F_{i2}\) denoting the parameter matrix. The block-diagonal structure of \(\tilde{V}_i[i] = \text{diag}(V_j[i], V_i[i])\) determines that the parametric matrix \(F_{i2}\) should have a special structure, which is exploited in order. The signal transmit space \(\text{span}(V_i)\) is simple and there is no need for parametrization. Yet, we still parameterize it in terms of a full column-rank matrix \(F_{i1}\) for notational consistence, obtaining

\[
\text{span}(V_i) = \text{span}(\Phi_i^\dagger F_{i1})
\]  

(34)

By left-multiplying the arguments in the both sides of (33) by \(\text{diag}(0, D_{i2}, I)\Psi_i\) and left-multiplying their counterparts in (34) by \(\text{diag}(I, D_{i1}, 0)\Phi_i\), it follows from [24, Corollary 4.2.4] that the disjoint condition in (32) is reducible to

\[
\text{span}(\text{diag}(I, D_{i1}, 0)F_{i1}) \cap \text{span}(\text{diag}(0, D_{i2}, I)F_{i2}) = \{0\}.
\]  

(35)

Following the above dimension analysis, \(F_{i1}\) has \(r_{i1} - a_i\) columns and \(F_{i2}\) has \(r_{i3} - r_{i1} + a_i\) columns. In addition, \(F_{i2}\) can have another \(N_j + N_l - r_{i2}\) columns, taking the form of

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]  

(the partition of the row is the same as the partition of the columns of \(\begin{bmatrix} \Psi_{i1}^\dagger & \Psi_{i2}^\dagger & \Psi_{i3}^\dagger \end{bmatrix}\)), since \(\text{diag}(0, D_{i2}, I)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\). Such setting corresponds to transmitting interference along the \((N_j + N_l - r_{i2})\)-dimensional null space of the composite interference channel, i.e., \(\text{span}(\Psi_{i1}^\dagger)\), as
expounded in Section III. We assume, without loss of the generality, that the first \( N_j + N_l - r_{i2} \) columns have such a fixed structure. It follows that the dimension of the composite possible transmit space for the interference is equal to \( r_{i3} - r_{i1} + a_i + N_j + N_l - r_{i2} \).

Finding the possible transmit spaces for \( T_j \) and \( T_l \), is, mathematically, to diagonalize \( \text{span}(\Psi_i F_{i2}) \), a job equivalent to block-diagonalizing \( \Psi_i \) through \( F_{i2} \). There are \( r_{i3} - r_{i1} + a_i \) undetermined columns in \( F_{i2} \); each of them can contribute to diagonalize \( \Psi_i \) either for \( \text{span}(\begin{bmatrix} V_i^i \\ 0 \end{bmatrix}) \) or for \( \text{span}(\begin{bmatrix} 0 \\ V_i^i \end{bmatrix}) \). Assume \( b_{ij} \) columns are set in favor of \( T_j \) and \( b_{il} \) columns in favor of \( T_l \), with their range given by

\[
0 \leq b_{ij} \leq N_j, \quad 0 \leq b_{il} \leq N_l, \quad b_{ij} + b_{il} = r_{i3} - r_{i1} + a_i.
\]  

Accordingly, we can partition \( F_{i2} \) into three column submatrices of \( N_j + N_l - r_{i2}, b_{ij} \) and \( b_{il} \) columns, respectively, yielding

\[
F_{i2} = [F_{i21} F_{i22} F_{i23}].
\]  

The first submatrix is a fixed one whereas the remaining two are undetermined, requiring further parametrization:

\[
F_{i21} = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, \quad F_{i22} = \begin{bmatrix} \Psi_{i11} \\ \Psi_{i12} \end{bmatrix}, \quad F_{i23} = \begin{bmatrix} \tilde{F}_{i2j} \\ \Psi_{i22} \end{bmatrix} \begin{bmatrix} \tilde{F}_{i2l} \end{bmatrix}.
\]  

Here, \( \tilde{F}_{i2j} \) and \( \tilde{F}_{i2l} \) are \( N_j \times b_{ij} \) and \( N_l \times b_{il} \) full column-rank undetermined matrices, and the two column matrices of relevance to \( \Psi \) result from row partitioning the unitary matrix \( \Psi_i = \begin{bmatrix} \Psi_{i1}^+ & \Psi_{i2}^+ & \Psi_{i3}^+ \end{bmatrix} \) into two rows, as shown by

\[
\begin{bmatrix} \Psi_{i1}^+ & \Psi_{i2}^+ & \Psi_{i3}^+ \end{bmatrix} = \begin{bmatrix} \Psi_{i11}^+ & \Psi_{i12}^+ & \Psi_{i13}^+ \\ \Psi_{i21}^+ & \Psi_{i22}^+ & \Psi_{i23}^+ \end{bmatrix}.
\]  

where the upper block has \( N_j \) rows and the lower block has \( N_l \) rows. Substituting (38) into (33), we have \( \text{span}(\Psi_i F_{i22}) = \text{span}(\begin{bmatrix} \tilde{F}_{i2j} \\ 0 \end{bmatrix}) \subseteq \text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) \) (the unity matrix is of order \( N_j \) and the zero matrix is of size \( N_l \times N_j \)) and \( \text{span}(\Psi_i F_{i23}) = \text{span}(\begin{bmatrix} 0 \\ \tilde{F}_{i2l} \end{bmatrix}) \subseteq \text{span}(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) \) (the unity matrix is of order \( N_l \) and the zero matrix is of size \( N_j \times N_l \)).

In addition to \( b_{ij} \) dimensions, the possible transmit space \( \text{span}(\begin{bmatrix} V_i^i \\ 0 \end{bmatrix}) \) for \( T_j \) can have another

\[
c_{ijl} = (N_j - r_{i2} + b_{il})^+
\]  

dimensions, resulted from the intersection subspace \( \text{span}\left( \Psi_i^\dagger [F_{i21} F_{i23}] \right) \cap \text{span}\left( [I 0] \right) \). This common subspace can be obtained by zero-forcing the lower part of the matrix \( \Psi_i^\dagger [F_{i21} F_{i23}] \) to give

\[
\text{span}\left( \Psi_i^\dagger [F_{i21} F_{i23}] (I - Q_{ii} Q_{i1}) \right) = \text{span}\left( \begin{bmatrix} 0 & 0 \\ 0 & I - Q_{i1} Q_{i1} \end{bmatrix} \right),
\]

where \( (I - Q_{i1} Q_{i1}) \) is the orthogonal projection operator with \( Q_{i1} \) defined by

\[
Q_{i1} = \begin{bmatrix} \Psi_{i12}^\dagger \\ \Psi_{i22}^\dagger \end{bmatrix} [F_{i21} F_{i23}] = \begin{bmatrix} \Psi_{i12}^\dagger F_{i2l} \end{bmatrix},
\]

and \( Q_{i1} \) denoting the generalized inverse of \( Q_{i1} \) [24] Ch9 and Ch11]. The dimension \( c_{ij} \) is obtained by [25] Fact 2.10.13]

\[
\text{rank}\left( \Psi_i^\dagger [F_{i21} F_{i23}] (I - Q_{i1} Q_{i1}) \right)
= \text{rank}(I - Q_{i1} Q_{i1}) - \text{dim} \left( \text{null}(\Psi_i^\dagger [F_{i21} F_{i23}] \cap \text{span}(I - Q_{i1} Q_{i1}) \right)
= \text{dim} (\text{null}(Q_{i1}))
= (N_j - r_{i2} + b_{il})^+,
\]

where the third equality is obtained because \( \Psi_i^\dagger [F_{i21} F_{i23}] \) is a full column rank matrix and do not have null space. Similarly, \( \text{span}\left( \begin{bmatrix} 0 \\ V^{[i]}_l \end{bmatrix} \right) \) for \( T_l \) can have another

\[
c_{ij} = (N_i - r_{i2} + b_{ij})^+
\]

dimensions resulted from intersection subspace

\[
\text{span}\left( \Psi_i^\dagger [F_{i21} F_{i22}] \right) \cap \text{span}\left( [I 0] \right) = \text{span}\left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right),
\]

where

\[
Q_{i2} = \begin{bmatrix} \Psi_{i12}^\dagger \\ \Psi_{i21}^\dagger \\ \Psi_{i31}^\dagger \end{bmatrix} [F_{i21} F_{i22}] = \begin{bmatrix} \Psi_{i11}^\dagger \tilde{F}_{i2j} \end{bmatrix}.
\]

Hence, the possible transmit spaces for \( T_j \) and \( T_l \) are

\[
\text{span}(V^{[j]}_j) = \text{span}\left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (I - Q_{i1} Q_{i1}) \end{bmatrix}, \tilde{F}_{i2j} \right),
\]

\[
\text{span}(V^{[i]}_l) = \text{span}\left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (I - Q_{i2} Q_{i2}) \end{bmatrix}, \tilde{F}_{i2j} \right),
\]

with dimension \( c_{ij} + b_{ij} \) and \( c_{il} + b_{il} \), respectively.
2) Feasible Transmit Spaces: The feasible transmit space span($V_i$) is the one that simultaneously satisfies alignment conditions at all three transceivers, i.e.,

$$\text{span}(V_i) = \text{span}(V_i^{[1]}) \cap \text{span}(V_i^{[2]}) \cap \text{span}(V_i^{[3]}),$$

with $\text{span}(V_i^{[k]}) \ (k = 1, 2, 3)$ explicitly given in (34), (46) and (47). Let us consider $T_1$ as an example for which, we have

$$\text{span}(V_1^{[1]}) = \text{span}(\Phi_1^1 F_1),$$
$$\text{span}(V_1^{[2]}) = \text{span}(\Omega_2, \tilde{F}_{221}),$$
$$\text{span}(V_1^{[3]}) = \text{span}(\Omega_3, \tilde{F}_{321}),$$

where

$$\Omega_2 = [\Psi_{211}^+ 0] (I - Q_{21}^- Q_{21}), \quad \Omega_3 = [\Psi_{311}^+ 0] (I - Q_{31}^- Q_{31}).$$

Note that both span($\Omega_2$) and span($\Omega_3$) are unadjustable in the precoder design for $T_1$ because $\tilde{F}_{223}$ and $\tilde{F}_{322}$ explicitly involved therein (see (42)) are intended exclusively for $T_3$ and $T_2$, respectively. Furthermore, $\Phi_1$ is also unadjustable in the design of the $T_1$ precoder. Thus, the only room left is to adjust $F_{11}, \tilde{F}_{221}$ and $\tilde{F}_{321}$ whereby $T_1$ can transmit through as many DoF’s as possible. For ease of description, the unadjustable span($\Omega_i$) and adjustable span($\tilde{F}_{21}$) are referred to as the fixed and undetermined subspaces, respectively, of span($V_i^{[i]}$), $i = 2, 3$.

Let us first show how to maximize the intersection span($V_1^{[2]}$) and span($V_1^{[3]}$). Before proceeding, we assume, for simplicity, that the column dimension of the joint space $[\Omega_2, \Omega_3]$ exceeds its row size, i.e., $c_{213} + c_{312} > N_1$, or more specifically, $(N_1 - r_{22} + b_{23})^+ + (N_1 - r_{32} + b_{32})^+ > N_1$ according to (43). In fact, the results so obtained cover their counterpart under the reverse assumption by introducing the notation $(a)^+ = \max\{0, a\}$. Observe that both $\Omega_2$ and $\Omega_3$ are random, being an algebraic function of random matrices $\Psi_{k11} \ (k = 2, 3)$ and $Q_{k1}$, which are generated either from the decomposition of random channel matrices or from the construction of precoders as detailed in Appendix C-B. The randomness of $\Omega_2$ and $\Omega_3$ enables the assertion that their joint space is almost surely of full rank (equal to $N_1$) which, in turn, implies that the two fixed subspaces share a common portion of $(c_{213} + c_{312} - N_1)^+$ dimensions.

To maximize span($V_1^{[2]}$) $\cap$ span($V_1^{[3]}$), our strategy is to align the undetermined subspace in span($V_1^{[3]}$), defined by the column vectors of $\tilde{F}_{321}$, with the $c_{213} - (c_{213} + c_{312} - N_1)^+$ remaining
dimensions in the fixed subspace of $\text{span}(V_1^{[2]})$, thereby creating the second common subspace of dimension $\min(c_{213} - (c_{213} + c_{312} - N_i)^+, b_{31})$. By the same token, we align the subspace spanned by $\tilde{F}_{221}$ with the remaining $c_{312} - (c_{213} + c_{312} - N_i)^+$ dimensional fixed subspace in $\text{span}(\Omega_2)$. This results in the third common subspace of dimension $\min(c_{312} - (c_{213} + c_{312} - N_i)^+, b_{21})$. Finally, we align the unused subspace in $\text{span}(\tilde{F}_{221})$ with the unused subspace in $\text{span}(\tilde{F}_{321})$ yielding the fourth common subspace of dimension

\[
\min\{b_{31} - \min(c_{213} - (c_{213} + c_{312} - N_i)^+, b_{31}), b_{21} - \min(c_{312} - (c_{213} + c_{312} - N_i)^+, b_{21})\}.
\]

(51)

Summing the dimensions of all the common subspaces obtained above, it follows that the dimension of the maximized $\text{span}(V_1^{[2]}) \cap \text{span}(V_1^{[3]})$:

\[
f_1 = (c_{213} + c_{312} - N_i)^+ + \min(c_{213} - (c_{213} + c_{312} - N_i)^+, b_{31})
+ \min(c_{312} - (c_{213} + c_{312} - N_i)^+, b_{21})
+ \min(b_{31} - \min(c_{213} - (c_{213} + c_{312} - N_i)^+, b_{31}), b_{21} - \min(c_{312} - (c_{213} + c_{312} - N_i)^+, b_{21})\).
\]

(52)

Next take $\text{span}(V_1^{[1]})$ into consideration. All of its $r_{11} - a_1$ dimensions are free, and employable to align with the common subspace produced above, producing the final dimension of the common subspace for $T_1$:

\[
d_1 \leq \min(r_{11} - a_1, f_1).
\]

(53)

Following the same suit, we obtain the DoF for $T_2$ and $T_3$ as

\[
d_i \leq \min(r_{i1} - a_i, f_i),
\]

(54)

where

\[
f_i = (c_{jil} + c_{lij} - N_i)^+ + \min(c_{jil} - (c_{jil} + c_{lij} - N_i)^+, b_{li})
+ \min(c_{lij} - (c_{jil} + c_{lij} - N_i)^+, b_{ji})
+ \min(b_{li} - \min(c_{jil} - (c_{jil} + c_{lij} - N_i)^+, b_{li}), b_{ji} - \min(c_{lij} - (c_{jil} + c_{lij} - N_i)^+, b_{ji})\),
\]

(55)

with $c_{jil}$ given by

\[
c_{jil} = (N_i - r_{jl} + b_{jl})^+, \quad i, j, l = 1, 2, 3; \quad i \neq j \neq l.
\]

(56)
The upper bounds in (53-56) are achievable by appropriately designing the precoders. One design scheme is presented in Appendix C-B. Thus, we obtain the DoF region for the 3-user channel as shown in the following subsection.

3) DoF Region:

**Theorem 2:** The DoF region achieved by interference alignment in the 3-user constant MIMO interference channel is given by

\[ D_3 = \left\{ (\eta_1, \eta_2, \eta_3) \in \mathbb{R}_3^+ : \eta_i \leq \mu_i, i = 1, 2, 3; \forall (\mu_1, \mu_2, \mu_3) \in \text{Conv}(S_3) \right\}, \tag{57} \]

where

\[ S_3 = \left\{ (d_1, d_2, d_3) : d_i = \min(r_{i1} - a_i, f_i), f_i = (c_{jit} + c_{tij} - N_i)^+, \right. \]
\[ + \min \left( c_{jit} - (c_{jit} + c_{tij} - N_i)^+, b_{ji} \right) \]
\[ + \min \left( b_{li} - \min(c_{jit} - (c_{jit} + c_{tij} - N_i)^+, b_{ji}), b_{ji} - \min(c_{jit} - (c_{jit} + c_{tij} - N_i)^+, b_{ji}) \right), \]
\[ c_{jit} = (N_i - r_{j2} + b_{ji})^+, 0 \leq a_i \leq r_{i1} + r_{i2} - r_{i3}, 0 \leq b_{ji} \leq N_j, 0 \leq b_{il} \leq N_l, \]
\[ b_{ij} + b_{il} = r_{i3} - r_{i1} + a_i; \forall i, j, l = 1, 2, 3; i \neq j \neq l \}. \tag{58} \]

In the existing literature, there is no exact characterization of the achievable DoF region for the general 3-user constant MIMO interference channel with arbitrary number of antennas at transmitters and receivers. The only exception is a special case considered in [3], where the authors propose a zero forcing scheme to achieve \( M/2 \) DoF per user assuming that all the transmitters and receivers have the same number of antennas (i.e., \( M \)). In this scheme, no symbol extension is required for an even \( M \), and two-symbol extension is required for an odd \( M \). It is easy to examine that for an even \( M \), the DoF of \( d_i = M/2 \) (\( i = 1, 2, 3 \)) achievable therein by the three users is a point in the region \( D_3 \) given in (57). This assertion can be verified by setting the parameters

\[ a_1 = a_2 = a_3 = M/2, b_{12} = b_{13} = b_{21} = b_{23} = b_{31} = b_{32} = M/4, \]

if \( M/2 \) is even; and by setting

\[ a_1 = a_2 = a_3 = M/2, b_{12} = b_{23} = b_{31} = \left\lfloor \frac{M}{4} \right\rfloor + 1, b_{13} = b_{21} = b_{32} = \left\lfloor \frac{M}{4} \right\rfloor, \]

otherwise. If \( M \) is odd, the maximum sum rate in \( D_3 \), which is achieved without symbol extension, is fractionally smaller than \( 3M/2 \).
To visualize the DoF region in (57), several examples are provided here for illustration.

**Example 1:** \((N_i, M_i) = (6, 5), i = 1, 2, 3\). The DoF region with such setting is shown in Fig. 2. We see that the region is a symmetric polyhedron and each user can at most attain the DoF \(d_i \leq \min(6, 5)\). The maximum sum DoF is 7 and it is achieved on the triangular facet with vertices \((1, 1, 5)\), \((1, 5, 1)\) and \((5, 1, 1)\). At these vertices, one transmitter can send five independent signals while the other two send one each.

**Example 2:** \((N_i, M_i) = (5, 5), i = 1, 2, 3\). For this case, the DoF region is depicted in Fig. 3. The maximum sum DoF is 7, which is the same as that in Example 1 but is achieved with less transmit antennas. By comparing Fig. 3 with Fig. 2 it is observed that the reduction in antenna number, though retaining the maximum sum DoF, shrinks the DoF region thus reducing the operating space in practical applications, e.g., scheduling or rate allocation.

**Example 3:** \((N_1, M_1) = (4, 3), (N_2, M_2) = (4, 4)\) and \((N_3, M_3) = (4, 5)\). The DoF attainable by each user is limited by \(d_i \leq \min(N_i, M_i)\). The maximum sum DoF is achieved at the vertex \((2, 2, 2)\). In this example, we cannot get the same maximum sum DoF by reducing the antenna numbers at any transmitter or receiver.
Fig. 3. The DoF region for the 3-user MIMO interference channel with \((N_i, M_i) = (5, 5)\) \((i = 1, 2, 3)\). The maximum sum DoF is \(7\). ‘o’ represents points in the set \(\mathcal{S}_3\).

C. \(K\)-user \((K > 3)\) MIMO Interference Channel

The methodology used to determine the DoF-region for the case \(K = 3\) is equally well applicable to the the general case of \(K > 3\), at least in principle. The difficulty, however, lies in the last step to synthesize the feasible precoders. Recall that the procedure of maximizing the intersection dimension of three possible precoders is already very sophisticated for the case of \(K = 3\). The dimension maximization becomes intractable for the case \(K > 3\). We, therefore, adopt an alternative strategy to bound the DoF region instead.

Suppose we have obtained the possible transmit spaces \(\text{span}(V_k^{[i]})\) for \(T_k\), \(i, k = 1, \cdots, K\). For \(i \neq k\), \(V_k^{[i]}\) is a \(N_k \times \left( (N_k - r_{i2} + \sum_{j \neq k, i} b_{ij})^+ + b_{ik} \right)\) full column rank matrix with \(\left( N_k - r_{i2} + \sum_{j \neq k, i} b_{ij} \right)^+\) fixed columns and \(b_{ik}\) undetermined columns; for \(i = k\), the \(N_k \times (r_{k1} - a_k)\) matrix \(V_k^{[k]}\) has all of its columns adjustable for space alignment. Note that, as in the 3-user channel, \(a_i\) is the subspace dimension of \(\text{span}(X_{i2})\) that is assigned to the interference subspace at \(R_i\), and

\[
\sum_{j \neq i} b_{ij} = r_{i3} - r_{i1} + a_i, \ 0 \leq b_{ij} \leq N_j. \tag{59}
\]
Fig. 4. The DoF region for the 3-user MIMO interference channel with \((N_1, M_1) = (4, 3), (N_2, M_2) = (4, 4)\) and \((N_3, M_3) = (4, 5)\). The maximum sum DoF is 6. ‘o’ represents points in the set \(S_3\).

The achievable DoF for \(T_k\) is given by

\[
d_k = \dim \left( \bigcap_{j=i}^{K} \text{span}(V_k^{[j]}) \right) .
\] (60)

As all the \((r_{k1} - a_k)\) dimensions of \(\text{span}(V_k^{[i]})\) are undetermined, (60) can be equivalently written as

\[
d_k = \min(r_{k1} - a_k, f_k),
\] (61)

where \(f_k = \dim \left( \bigcap_{i \neq k} \text{span}(V_k^{[i]}) \right)\). To compute the dimension of intersection subspace of \(K-1\) subspaces, we invoke the following lemma.

**Lemma 3:** \([26]\) Let \(A_i \in \mathbb{C}^{m \times n_i}, i = 1, 2, \ldots, k\). Then

\[
\dim \left( \bigcap_{i=1}^{k} \text{span}(A_i) \right) = \sum_{i=1}^{k} \text{rank}(A_i) - \text{rank}(W),
\] (62)
where

\[
W = \begin{bmatrix}
A_1 & A_2 \\
A_1 & A_3 \\
\vdots & \ddots \\
A_1 & A_k
\end{bmatrix}.
\] (63)

From this lemma, it follows that

\[
f_k = \dim \left( \bigcap_{i \neq k} \text{span}(V^{[i]}_k) \right) = \sum_{i \neq k} \text{rank}(V^{[i]}_k) - \text{rank}(W_k),
\] (64)

where

\[
W_k = \begin{bmatrix}
V^{[i_1]}_k & V^{[i_2]}_k \\
V^{[i_1]}_k & V^{[i_3]}_k \\
\vdots & \ddots \\
V^{[i_1]}_k & V^{[i_{K-1}]}_k
\end{bmatrix},
\]

and \(i_1, \cdots, i_{K-1}\) are distinct indexes and none of them equal to \(k\). Now the problem is to seek the maximum value of \(f_k\) by designing the undetermined columns of \(V^{[i]}_k\), a job equivalent to minimizing the rank of \(W_k\) according to (64). Inspecting the structure of \(W_k\), we see that the more dimensions the common subspace of \(\text{span}(V^{[i]}_k)\) \((l = 1, \cdots, K-1)\) has, the less rank does the matrix \(W_k\). At least, we can have \(\min_{i \neq k} b_{ik}\) dimensions of common subspace by aligning the undetermined subspaces of dimension \(\min_{i \neq k} b_{ik}\) in \(\text{span}(V^{[i]}_k)\) \((i \neq k)\) with a common one. Thus, we have

\[
\text{rank}(W_k) \leq \min \left( (K-2)N_k, \sum_{i \neq k} \left( N_k - r_{i2} + \sum_{j \neq k, i} b_{ij} \right) + b_{ik} \right) - \min_{i \neq k} b_{ik}.
\] (65)

At most, we can have a \(\min_{i \neq k} \left( (N_k - r_{i2} + \sum_{j \neq k, i} b_{ij}) + b_{ik} \right)\) dimensional common subspace of \(\text{span}(V^{[i]}_k)\) \((i \neq k)\), which result in

\[
\text{rank}(W_k) \geq \min \left( (K-2)N_k, \sum_{i \neq k} \left( N_k - r_{i2} + \sum_{j \neq k, i} b_{ij} \right) + b_{ik} \right) - \min_{i \neq k} \left( (N_k - r_{i2} + \sum_{j \neq k, i} b_{ij}) + b_{ik} \right),
\] (66)
By substituting (65) and (66) into (64), we have the upper bound \( f^u_k \) and lower bound \( f^l_k \) which, when inserted into (61), establish the following outer set and inner set for the achievable DoFs.

\[
\mathcal{S}^{out}_K = \left\{ (d_1, \cdots, d_K) : d_k = \min(r_{k1} - a_k, f^out_k), \ 0 \leq a_k \leq r_{k1} + r_{k2} - r_{k3} \right\}
\]

\[
f^u_k = \sum_{i \neq k} (N_k - r_{i2} + \sum_{j \neq k, i} b_{ij})^+ + b_{ik} - \min \left((K-2)N_k, \sum_{i \neq k} (N_k - r_{i2} + \sum_{j \neq k, i} b_{ij})^+ + b_{ik}\right)
\]

\[
+ \min_{i \neq k} \left( (N_k - r_{i2} + \sum_{j \neq k, i} b_{ij})^+ + b_{ik} \right), \sum_{j \neq i} b_{ij} = r_{i3} - r_{i1} + a_i, \ 0 \leq b_{ij} \leq N_j \right\}.
\]

\[
\mathcal{S}^{in}_K = \left\{ (d_1, \cdots, d_K) : d_k = \min(r_{k1} - a_k, f^in_k), \ 0 \leq a_k \leq r_{k1} + r_{k2} - r_{k3} \right\}
\]

\[
f^l_k = \sum_{i \neq k} (N_k - r_{i2} + \sum_{j \neq k, i} b_{ij})^+ + b_{ik} - \min \left((K-2)N_k, \sum_{i \neq k} (N_k - r_{i2} + \sum_{j \neq k, i} b_{ij})^+ + b_{ik}\right) + \min_{i \neq k} b_{ik}, \sum_{j \neq i} b_{ij} = r_{i3} - r_{i1} + a_i, \ 0 \leq b_{ij} \leq N_j \right\}.
\]

Then, the outer and the inner regions for the achievable DoF region are given by

\[
\mathcal{D}^{out}_K = \left\{ (\eta_1, \cdots, \eta_K) \in \mathbb{R}^+_K : \eta_k \leq \mu_k, k = 1, \cdots, K; \ \forall (\mu_1, \cdots, \mu_K) \in \text{Conv}(\mathcal{S}^{out}_K) \right\},
\]

and

\[
\mathcal{D}^{in}_K = \left\{ (\eta_1, \cdots, \eta_K) \in \mathbb{R}^+_K : \eta_k \leq \mu_k, k = 1, \cdots, K; \ \forall (\mu_1, \cdots, \mu_K) \in \text{Conv}(\mathcal{S}^{in}_K) \right\}.
\]

Thus, we have the following theorem.

**Theorem 3:** (Inner and outer bounds on DoF region) For the \( K \)-user \((K > 3)\) constant MIMO interference channel, the DoF region \( \mathcal{D}_K \) achieved by interference alignment satisfies

\[
\mathcal{D}^{in}_K \subseteq \mathcal{D}_K \subseteq \mathcal{D}^{out}_K.
\]

**V. Conclusion**

In this paper, we have derived a joint space decomposition-and-synthesis theory for the achievable DoF region by interference alignment of a general \( K \)-user constant MIMO channel,
without symbol extension and assuming that the channel coefficients are independently drawn from a continuous distribution, and are known at all transmitters and receivers. The decomposition is done at the receive spaces whereas the synthesis is done at the transmitters.

In the $K$-user MIMO channel, signal and interference collision at each user’s receiver is uniquely determined by its signal and augmented interference channel matrices. Joint space decomposition of these channel matrices in terms of a common set of non-orthogonal basis functions provides an insightful collision picture whereby the conditions for interference-free (IF) precoders can be established. Each user’s signal requires its precoder to simultaneously satisfy the IF condition at its target receiver when functioning as the desired signal, and satisfy the IF condition at other receivers when functioning as an interferer. The precoder satisfying the IF requirement at a single receiver is called a possible precoder. The diagonal structure of the augmented interference channel matrix enables the parametrization of its column space into a fixed and an adjustable subspace, thereby facilitating the retrieval of the $K - 1$ possible precoders. A feasible precoder is the intersection of all $K$ possible precoders of relevance, and the dimensions of all $K$ feasible precoders define the DoF region. The synthesis of the DoF region is, in essence, a discrete optimization problem, for which we obtained a closed form solution for $K = 2$, 3, and inner/outer bounds for the case of $K > 3$, as summarized in Theorem 1, Theorem 2, and Theorem 3 respectively. The derivation of the exact DoF region for $K > 3$ is worth further efforts in the future.

**APPENDIX A**

**Proof of Lemma 2**

_Proof:_ If $\text{span}(XP) \cap \text{span}(XQ) = \{0\}$, then we have $\text{span}(P) \cap \text{span}(Q) = \{0\}$. Otherwise, there exist some vector $y_1 \in \mathbb{C}^{n_1 \times 1}$ and $y_2 \in \mathbb{C}^{n_2 \times 1}$ such that $Py_1 = Qy_2 \neq 0$. As a consequence, we have $XP\!y_1 = XQy_2 \neq 0$, which is in contradiction with the assumption. The condition $\text{span}(P) \cap \text{span}(Q) = \{0\}$ is equivalent to that the columns of $P$ and that of $Q$ are linear independent, i.e., $\text{rank}[P \ Q] = n_1 + n_2$.

If $\text{span}(P) \cap \text{span}(Q) = \{0\}$, then we have $\text{rank}[XP \ XQ] = \text{rank}[P \ Q] - \text{dim}(\text{null}(X) \cap \text{span}([P \ Q])) = n_1 + n_2$, by virtue of [25] Fact 2.10.13. Thus, we get $\text{span}(XP) \cap \text{span}(XQ) = \{0\}$. 


APPENDIX B

PROOF OF COROLLARY

Proof: From Theorem, we have

\[ d_1 + d_2 \leq \max_{D_2} \, \eta_1 + \eta_2 \]

\[ = \max_{a_1, a_2} \min(r_{11} + r_{21} - a_1 - a_2, \, N_2 - r_{12} + r_{13}, \, N_1 - r_{22} + r_{23}, \, a_1 + a_2 + N_1 + N_2 + r_{13} + r_{23} - r_{12} - r_{22} - r_{11} - r_{21}). \]  \hspace{1cm} (73)

Because the convex hull operation in \( D_2 \) does not increase the sum DoF, it is equivalent to calculate the maximum sum DoF over \( S_2 \), which justifies the second step in (72).

By inspection, we can rewrite (72) as

\[ d_1 + d_2 \leq \max_{a_1, a_2} \min(r_{11} + r_{21} - a_1 - a_2, \, N_2 - r_{12} + r_{13}, \, N_1 - r_{22} + r_{23}, \, a_1 + a_2 + N_1 + N_2 + r_{13} + r_{23} - r_{12} - r_{22} - r_{11} - r_{21}). \]  \hspace{1cm} (73)

In (73), the max-min value can be any one of the four items in the right hand side. Only the first and the fourth term are functions of \( a_i \) \( (i = 1, 2) \). In what follows, we will find out what values of \( a_i \) are when either of these two terms achieves the max-min value. First consider the first term, it cannot achieve the max-min value with \( a_i > 0 \), except when the first term equal to the fourth term, which, by simple algebraic operation, is equal to \( (N_1 + N_2 + r_{13} + r_{23} - r_{12} - r_{22})/2 \).

Otherwise, the sum DoF can be retained or increased by decreasing \( a_i \). Hence, when the first term achieves the max-min value, it is equal to \( r_{11} + r_{21} \) or \( (N_1 + N_2 + r_{13} + r_{23} - r_{12} - r_{22})/2 \). Similarly, when the fourth term achieves the max-min value, it is equal to \( N_1 + N_2 \) \( (a_i = r_{i1} + r_{i2} - r_{i3}) \) or \( (N_1 + N_2 + r_{13} + r_{23} - r_{12} - r_{22})/2 \). In summary, The maximum sum DoF can only be one of the five terms, i.e., \( d_1 + d_2 \leq \min(r_{11} + r_{21}, \, N_2 - r_{12} + r_{13}, \, N_1 - r_{22} + r_{23}, \, N_1 + N_2, \, [(N_1 + N_2 + r_{13} + r_{23} - r_{12} - r_{22})/2]) \). Note that \( (N_1 + N_2 + r_{13} + r_{23} - r_{12} - r_{22})/2 = [(N_2 - r_{12} + r_{13}) + (N_1 - r_{22} + r_{23})]/2 \geq \min(N_2 - r_{12} + r_{13}, \, N_1 - r_{22} + r_{23}) \). So the fifth term can be omitted and the expression for the maximum sum DoF can be simplified to

\[ d_1 + d_2 \leq \min(r_{11} + r_{21}, \, N_2 - r_{12} + r_{13}, \, N_1 - r_{22} + r_{23}, \, N_1 + N_2). \]
Appendix C

Construction of Precoders

A. 2-user Case

We construct the precoder matrices \( V_i \in \mathbb{C}^{M_i \times d_i} \) that achieve the DoFs given in (28). Let \( \tilde{g}_i \) be the dimension of the subspaces that are in \( \text{null}(H_{i(3-i)}) \) but disjoint to \( \text{null}(H_{ii}) \). Choose a \( g_i = \min(d_i, \tilde{g}_i) \) dimensional subspace that satisfy the above condition and randomly pick a basis for this subspace. Assign this basis as the first \( g_i \) columns of the precoder matrix \( V_i \). The remaining \( d_i - g_i \) columns of \( V_i \) can be randomly generated from a continuous distribution, so that \( V_i \) is full column rank and \( \text{span}(V_i) \cap \text{null}(H_{ii}) = \{0\} \) almost surely. We now only need to prove that \( \text{span} \left( \begin{bmatrix} X_{i1} & X_{i2} & X_{i3} \end{bmatrix} \begin{bmatrix} \Phi_{i1} & D_{i1} & 0 \end{bmatrix} \right) \) and \( \text{span} \left( \begin{bmatrix} X_{i1} & X_{i2} & X_{i3} \end{bmatrix} \begin{bmatrix} 0 & D_{i2} & \Phi_{i3} \end{bmatrix} \right) \) are disjoint. By Lemma 2, it is equivalent to prove that \( \text{rank} \left( \begin{bmatrix} \Phi_{i1} & D_{i1} & 0 \end{bmatrix} \begin{bmatrix} V_i & D_{i2} & \Phi_{i3} \end{bmatrix} \right) = d_1 + d_2 - g_{3-i} \), where the subtraction of \( g_{3-i} \) is due to the precoder \( V_{3-i} \) containing \( g_{3-i} \) columns that are in \( \text{null}(H_{i(3-i)}) \) according to the above construction. By [25] Fact 2.10.13, we have

\[
\text{rank} \left( \begin{bmatrix} \Phi_{i1} & D_{i1} & 0 \\ 0 & D_{i2} & \Phi_{i3} \end{bmatrix} \right) \left[ \begin{bmatrix} V_i \\ 0 \end{bmatrix} \right] = \text{rank} \left( \begin{bmatrix} V_i \\ 0 \end{bmatrix} \right) - \text{dim} \left( \text{null} \left( \begin{bmatrix} \Phi_{i1} & D_{i1} & 0 \\ 0 & D_{i2} & \Phi_{i3} \end{bmatrix} \right) \right) \cap \text{span} \left( \begin{bmatrix} V_i \\ 0 \end{bmatrix} \right) = d_1 + d_2 - g_{3-i}.
\]

B. 3-user Case

In the following, we construct precoders that achieve the DoF derived in Section IV-B2.

1) We first design the precoder for \( T_1 \). Let the first \( c_{213} \) columns of \( \Psi_{211}^\dagger \) being denoted by \( \Psi_{2111}^\dagger \). Then, due to the non-uniqueness of the QSVD [20], we can introduce an unitary matrix \( O_2 \) of size \( (N_1+N_3-r_{22}) \times (N_1+N_3-r_{22}) \) to make sure that \( \text{span} \left( \begin{bmatrix} \Psi_{211}^\dagger & 0 \end{bmatrix} (I - Q_{21} Q_{21}) \right) = \text{span}(\Psi_{2111}^\dagger) \) for \( \forall Q_{21} \) and \( X_2 \) and \( O_{21} \) do not alter. ( We can construct \( O_2 \) as shown in the following. By a decomposition of an idempotent matrix in [27] P 7.4.1), we have \( I - Q_{21} Q_{21} = L \Lambda L^{-1} \), where \( L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \) is a \( (N_1+N_3-r_{22}) \times (N_1+N_3-r_{22}) \) non-singular matrix with the block matrix \( L_{11} \) of size \( (N_1+N_3-r_{22}) \times c_{213} \) and \( \Lambda = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \) with \( I \) is of size \( c_{213} \). Partition \( O_2 \) as \( \begin{bmatrix} O_{21} \\ O_{22} \end{bmatrix} \) with the upper block matrix having \( c_{213} \) rows and the lower one \( N_1+N_3-r_{22} - c_{213} \) rows, and let columns \( O_{22}^\dagger \) be an orthogonal basis of
null($L_{11}^\dagger$) and let that of $O_{21}^\dagger$ be an orthogonal basis of $\text{span}(L_{11})$. Similarly, we can construct $O_3$ to let $\text{span}\left(\left[\Psi_{311}^\dagger O_3 \ 0\right](I - Q_{31}^\dagger Q_{31})\right) = \text{span}(\Psi_{311}^\dagger)$ for $\forall Q_{31}$, where $\Psi_{311}^\dagger$ is the first $c_{312}$ columns of $\Psi_{311}^\dagger$. Thus, $\text{span}(V_1) = \text{span}\left(\Phi_1^\dagger F_{11}\right) \cap \text{span}\left(\left[\Psi_{311}^\dagger \tilde{F}_{321}\right]\right) \cap \text{span}\left(\left[\Psi_{311}^\dagger \tilde{F}_{321}\right]\right)$. The columns of the $N_1 \times \min(r_{11} - a_1, f_1)$ precoder $V_1$ can be obtained as follows. Independent vectors in $\text{span}(\Psi_{2111}^\dagger) \cap \text{span}(\Psi_{3111}^\dagger)$ form the first $(c_{213} + c_{312} - N_1)^+$ columns. Excluding the common subspace $\text{span}(\Psi_{2111}^\dagger) \cap \text{span}(\Psi_{3111}^\dagger)$, the left subspace in $\text{span}(\Psi_{2111}^\dagger)$ can be aligned with $\tilde{F}_{321}$ to give another $\min(c_{213} - (c_{213} + c_{312} - N_1)^+, b_{31}) (\min(c_{312} - (c_{213} + c_{312} - N_1)^+, b_{21})$) columns. The left min $\min(b_{31} - \min(c_{213} - (c_{213} + c_{312} - N_1)^+, b_{31}) \cdot b_{21} - \min(c_{312} - (c_{213} + c_{312} - N_1)^+, b_{21})$) columns in both $\tilde{F}_{321}$ and $\tilde{F}_{221}$ are aligned with each other and their values are drawn from a continuous distribution.

2) For user 2, $\text{span}(V_2) = \text{span}(V_2^{[1]}) \cap \text{span}(V_2^{[2]}) \cap \text{span}(V_2^{[3]}), \text{where}$

\[ V_2^{[1]} = \left[\Psi_{111}^\dagger O_3 \ 0\right](I - Q_{11}^\dagger Q_{11}) \tilde{F}_{122}, \]
\[ V_2^{[2]} = \Phi_2^\dagger F_{21}, \]
\[ V_2^{[3]} = \left[\Psi_{312}^\dagger O_3 \ 0\right](I - Q_{32}^\dagger Q_{32}) \tilde{F}_{322}. \]

Since $\tilde{F}_{321}$ has been determined in 1), $Q_{32}$ is known. We only have to determine the fixed subspace in $\text{span}(V_2^{[1]})$. Introducing a unitary matrix $O_1$ as in 1), we can have $\text{span}\left(\left[\Psi_{111}^\dagger O_1 \ 0\right](I - Q_{11}^\dagger Q_{11})\right) = \text{span}(\Psi_{111}^\dagger)$ for $\forall Q_{11}$, where $\Psi_{111}^\dagger$ is the first $c_{123}$ columns of $\Psi_{111}^\dagger$. Then, we can follow the same suit as in 1) to produce the $\min(r_{21} - a_2, f_2)$ columns of the precoder $V_2$.

3) For user 3, $\text{span}(V_3) = \text{span}(V_3^{[1]}) \cap \text{span}(V_3^{[2]}) \cap \text{span}(V_3^{[3]}), \text{where}$

\[ V_3^{[1]} = \left[\Psi_{112}^\dagger O_1 \ 0\right](I - Q_{12}^\dagger Q_{12}) \tilde{F}_{123}, \]
\[ V_3^{[2]} = \left[\Psi_{212}^\dagger O_2 \ 0\right](I - Q_{22}^\dagger Q_{22}) \tilde{F}_{223}, \]
\[ V_3^{[3]} = \Phi_3^\dagger F_{31}. \]

From 1) and 2), $Q_{12}$ and $Q_{22}$ can be readily calculated. The fixed subspace in $\text{span}(V_3^{[1]})$ and $\text{span}(V_3^{[2]})$ is known. We can follow the same suit as in 1) to get the $\min(r_{31} - a_3, f_3)$ columns of the precoder $V_3$.
From the above construction, we see that the precoder $V_i$ is determined by the values of $\Psi^\dagger j_1$ and $\Psi^\dagger l_1$ ($i \neq j \neq l$) and randomly generated matrix, which are independent of the channel matrices $[H_{ij} \ H_{il}]$. Thus, it is almost surely that the signal subspace $\text{span}(H_{ii}V_i)$ and the interference subspace $\text{span}\left([H_{ij} \ H_{il}] \begin{bmatrix} V_j & 0 \\ 0 & V_l \end{bmatrix}\right)$ disjoint.

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