CHAOTIC DISTRIBUTIONS FOR RELATIVISTIC PARTICLES

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ABSTRACT. We study a modified Kac model where the classical kinetic energy is replaced by an arbitrary energy function $\phi(v), v \in \mathbb{R}$. The aim of this paper is to show that the uniform distribution with respect to the microcanonical measure is $Ce^{-z_0\phi(v)}$-chaotic, $C, z_0 \in \mathbb{R}^+$. The kinetic energy for relativistic particles is a special case. A generalization to the case $v \in \mathbb{R}^d$ which involves conservation momentum is also formally discussed.

1. INTRODUCTION

In 1956, Mark Kac published a paper [6], in which he answered some fundamental questions concerning the derivation of the spatially homogeneous Boltzmann equation based on a probabilistic interpretation. Kac considered a stochastic model consisting of $N$ identical particles, and obtained an equation like the spatially homogeneous Boltzmann equation as a mean-field limit when the numbers of particles tend to infinity. The key ingredient in his paper was the notion of chaos. Loosely speaking, chaos is related to asymptotic independence. We give a precise mathematical definition later. He showed that if the probability distribution of the initial state of the particles is chaotic then this property is propagated in time, i.e., the probability distribution at time $t > 0$ is also chaotic. This is referred to as propagation of chaos. He also gave a description of how one can construct chaotic probability distributions.

In the Kac model the total classical kinetic energy of the particles is conserved. Inspired by the theory of relativity, where the kinetic energy of a particle has a different expression, we modify the Kac model to a case where the particles have an energy given by function $\phi(v)$ with $v \in \mathbb{R}$ representing the velocity of a particle. The function $\phi$ has to satisfy some technical conditions but is otherwise arbitrary. In a one-dimensional relativistic Kac model where $v$ represents the momentum rather than the velocity and with normalized mass and normalized speed of light

$$\phi(v) = \sqrt{v^2 + 1} - 1.$$

The main goal of this paper is to show that the uniform distribution on the manifold represented by the total kinetic energy of the particles (depends on $\phi$) with respect to the microcanonical measure is chaotic by using the approach of Kac.
In mathematical kinetic theory of gases, one of the most influential equations is the Boltzmann equation. The Boltzmann equation describes the time evolution of the density of a single particle in a gas consisting of a large number of particles, and in the spatially homogeneous form is given by

\[
\begin{align*}
\frac{\partial}{\partial t} f(v,t) &= Q(f,f)(v,t), \quad v \in \mathbb{R}^3, t > 0, \\
f(v,0) &= f_0, 
\end{align*}
\]

where \(Q\) is a quadratic collision operator, given by

\[
Q(f,f) = \int_{\mathbb{R}^3} \int_{S^2} (f(v',t)f(v_*,t) - f(v,t)f(v_*,t)) B(v - v_*,\sigma) dv_* d\sigma.
\]

The pair \((v,v_*)\) represents the velocities of two particles before a collision, and \((v',v'_*)\) the velocities of these particles after the collision. In each collision, the energy and momentum are conserved. This leads to

\[
v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad \sigma \in S^2.
\]

The non-negative function \(B\) is called the Boltzmann collision kernel and its structure depends on physical model considered. The derivation of the Boltzmann equation from a classical (Newtonian) many particle system is still an open problem. The classical result in this direction is by Lanford, [7]. The drawback with this result is that it only holds for a very short time. Up to this date the result by Lanford has only been improved upon in details, see e.g., [10] and the references therein. For the study of the relativistic Boltzmann equation, see e.g., [4].

To obtain an equation like (1.1) as a mean-field limit, Kac introduced a particle system consisting of \(N\) identical particles where each particle has a one-dimensional velocity and the total energy in the system is conserved. The state space of the system is chosen to be the sphere

\[
S^{N-1}(\sqrt{N}) = \{(v_1, \ldots, v_N) \mid v_1^2 + \cdots + v_N^2 = N\}.
\]

Assuming that the state the particles have a probability distribution \(F_0\) initially, its time evolution is given by the following linear master equation

\[
\begin{align*}
\frac{\partial}{\partial t} F_N(v_1, \ldots, v_N,t) &= \mathcal{K} F_N(v_1, \ldots, v_N,t), \\
F_N(v_1, \ldots, v_N,0) &= F_0(v_1, \ldots, v_N).
\end{align*}
\]

where \(\mathcal{K}\) is a collision operator of the form

\[
\mathcal{K} F_N(v_1, \ldots, v_N) = N[Q - I] F_N(v_1, \ldots, v_N),
\]

with \(I\) being the identity operator and

\[
Q F_N(v_1, \ldots, v_N) = \frac{2}{N(N-1)} \sum_{i<j} \int_{-\pi}^{\pi} F_N(v_1, \ldots, v_i(\theta), \ldots, v_j(\theta), \ldots, v_N) \frac{d\theta}{2\pi}.
\]
The pair of velocities $v_i(\theta), v_j(\theta)$ is the outcome of the velocities $v_i, v_j$ undergoing a collision, and in the classical case, as proposed by Kac are given by

$$v_i(\theta) = v_i \cos \theta + v_j \sin \theta, \quad v_j(\theta) = -v_i \sin \theta + v_j \cos \theta.$$ \hfill (1.3)

The energy of a pair of particles is always conserved in a collision, i.e.,

$$v_i(\theta)^2 + v_j(\theta)^2 = v_i^2 + v_j^2.$$ \hfill (1.4)

In a realistic model the momentum should also be conserved, but since the particles have one-dimensional velocities, imposing conservation of momentum would lead to either the particles keep the velocities during a collision or exchange velocities. Momentum conservation is therefore sacrificed in this case.

The fact that particles are assumed to be indistinguishable corresponds to the initial distribution $F_0$ being symmetric with respect to permutations, which in turn implies that $F_N$ is symmetric at time $t > 0$.

In order to obtain a mean-field limit of the master equation (1.2) as $N \to \infty$, Kac introduced the notion of chaos (in [6], he referred to it as the Boltzmann property).

**Definition 1.1.** Let $f$ be probability density on $\mathbb{R}$. For each $N \in \mathbb{N}$, let $F_N$ be a probability distribution on $\mathbb{R}^N$ with respect to a measure $m^{(N)}$. The sequence $\{F_N\}_{N \in \mathbb{N}}$ of probability distributions on $\mathbb{R}^N$ is said to be $f$–chaotic if the following two conditions are satisfied:

1. Each $F_N$ is a symmetric function of $v_1, \ldots, v_N$.
2. For each fixed $k \in \mathbb{N}$, the $k$-th marginal $f_k^{(N)}(v_1, \ldots, v_k)$ of $F_N$ converges to $\prod_{i=1}^k f(v_i)$, as $N \to \infty$, where $f(v) = \lim_{N \to \infty} f_N^{(N)}(v)$.

The convergence is in the weak sense, that is, if $\varphi(v_1, \ldots, v_k)$ is a bounded continuous function of $k$ variables, $v_1, \ldots, v_k \in \mathbb{R}$, then

$$\lim_{N \to \infty} \int_{\mathbb{R}^N} \varphi(v_1, \ldots, v_k) F_N(v_1, \ldots, v_N) \, dm^{(N)} = \int_{\mathbb{R}^k} \varphi(v_1, \ldots, v_k) \prod_{i=1}^k f(v_i) \, dv_1 \cdots dv_k.$$ \hfill (1.4)

This definition can be generalized to more general spaces than $\mathbb{R}^N$, but is enough for the purpose this paper, we are considering chaotic probability distributions on subspaces of $\mathbb{R}^N$.

A chaotic family of probability distributions on $S^{N-1}(\sqrt{N})$ is the following:

**Example 1.2.** It is a well known fact that the surface area of the sphere $S^{N-1}(\sqrt{N})$ in $\mathbb{R}^N$ is given by

$$|S^{N-1}(\sqrt{E})| = \frac{2\pi^{\frac{N}{2}} E^{\frac{N-1}{2}}}{\Gamma\left(\frac{N}{2}\right)}.$$
Let
\[ F_N(v_1, \ldots, v_N) = \frac{1}{|S^{N-1}(\sqrt{N})|}, \]
be the symmetric uniform distribution on \( S^{N-1}(\sqrt{N}) \) with respect to surface measure.

Let \( \varphi \) be continuous function on \( \mathbb{R} \) and \( \sigma^{(E)} \) the surface measure on \( S^{N-1}(\sqrt{E}) \). It is easy to see that
\[
\int_{S^{N-1}(\sqrt{E})} \varphi(v_1) d\sigma^{(E)} = \int_{\sqrt{E} - \sqrt{E}} \varphi(v_1) |S^{N-2}(\sqrt{E} - v_1^2)| dv_1.
\]
Replacing \( E \) with \( N \) we have
\[
\lim_{N \to \infty} \frac{\int_{S^{N-1}(\sqrt{N})} \varphi(v_1) d\sigma^{(N)}}{|S^{N-1}(\sqrt{N})|} = \lim_{N \to \infty} \frac{\Gamma \left( \frac{N}{2} \right)}{\pi^{\frac{N}{2}} \Gamma \left( \frac{N-1}{2} \right)} \int_{\sqrt{N} - \sqrt{N}} \varphi(v_1) \left( 1 - \frac{v_1^2}{N} \right)^{\frac{N-2}{2}} dv_1
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(v_1) e^{-\frac{v_1^2}{2}} dv_1.
\]
Taking \( \varphi \) to be a function on \( \mathbb{R}^k \), the same calculations shows that, the uniform distribution on \( S^{N-1}(\sqrt{N}) \) with respect to the surface measure is \( \frac{1}{\sqrt{2\pi}} e^{-\frac{v_1^2}{2}} \)-chaotic.

Kac showed by using a combinatorial argument that the master equation (1.2) propagates chaos, that is, if \( \{F_N(v_1, \ldots, v_N, 0)\}_{N \in \mathbb{N}} \) is \( f_0 \)-chaotic, then the solution to (1.2), \( \{F_N(v_1, \ldots, v_N, t)\}_{N \in \mathbb{N}} \) is \( f(v, t) \)-chaotic. Moreover, the density \( f(v, t) \) satisfies the Boltzmann-Kac equation
\[
\frac{\partial}{\partial t} f(v, t) = 2 \int_{\mathbb{R}} \int_{-\pi}^{\pi} \left[ f(v(\theta), t) f(w(\theta), t) - f(v, t) f(w, t) \right] \frac{d\theta}{2\pi} dw,
\]
\[
f(v, 0) = f_0,
\]
with \( v(\theta), w(\theta) \) given by (1.3).

Kac described also a method to construct chaotic probability distributions on \( S^{N-1}(\sqrt{N}) \): Let
\[
F_N(v_1, \ldots, v_N) = \frac{\prod_{i=1}^{N} f(v_i)}{Z(\sqrt{N})},
\]
where
\[
Z(E) = \int_{S^{N-1}(\sqrt{E})} \prod_{i=1}^{N} f(v_i) d\sigma^{(E)},
\]
with $\sigma^{(E)}$ being the surface measure on $S^{N-1}(\sqrt{E})$. Kac showed that, under some conditions on the function $f$, the family of probability distributions $\{F_N\}_{N \in \mathbb{N}}$ is $h(v)$–chaotic, where

$$h(v) = \frac{e^{-z_0v^2}f(v)}{\int_{\mathbb{R}} e^{-z_0v^2}f(v)dv},$$

and $z_0$ is a positive constant. Note that the case $f(v) = 1$ corresponds to the example above.

The purpose of the present article is to study chaotic probability distributions on other surfaces than $S^{N-1}(\sqrt{N})$. Define

$$\Omega^{N-1}(\sqrt{E}) = \left\{ (v_1, \ldots, v_N) \mid \sum_{i=1}^N \phi(v_i) = E \right\},$$

where $\phi(v)$ is the energy of a particle with velocity $v$. It is assumed that $\phi$ is even, increasing and $\phi(0) = 0$. We show that the uniform distribution on $\Omega^{N-1}(\sqrt{N})$ with respect to the microcanonical measure is $C e^{-z_0 \phi(v)}$–chaotic, where $C$ is a normalisation constant and $z_0 > 0$ is the unique solution to a specific equation.

Investigation of chaotic probability distributions have been considered by many authors. In a paper by Carlen, Carvahlo, Le Roux, Loss and Villani, [1], by a different method to Kac, the authors showed the following:

**Theorem 1.3.** Let $f$ be a probability density on $\mathbb{R}$ such that $f \in L^\infty(\mathbb{R})$, $\int_{\mathbb{R}} v^2 f(v)dv = 1$ and $\int_{\mathbb{R}} v^4 f(v)dv < \infty$. Then the family of distributions $\{F_N\}_{N \in \mathbb{N}}$, where $F_N$ is defined by (1.6) with $\sigma^{(N)}$ now being the normalized surface measure on $S^{N-1}(\sqrt{N})$ is $f$–chaotic.

In [11], Sznitman described the following method to construct chaotic probability distributions: Let $v_1, \ldots, v_N$ be i.i.d random variables with law $\mu(dv) = f(v)dv$, where for any $\lambda$

$$\int (f(v) + |\nabla f(v)|) e^{\lambda|v|}dv < \infty.$$

Then, the conditional distribution $\mu^N$ of $(v_1, \ldots, v_N)$ subject to the mean $\frac{v_1 + \cdots + v_N}{N} = a$ is $Y$–chaotic, where

$$Y = \frac{1}{Z_\lambda} e^{\lambda a} \mu,$$

with $\lambda_a$ determined by the equation $\int v dY(v) = a$ and $Z_\lambda$ is a normalisation constant.

The organization of this paper is as follows. In section 2 we define the microcanonical measure on $\Omega^{N-1}(\sqrt{N})$ and show that the uniform distribution on $\Omega^{N-1}(\sqrt{N})$ with respect to this measure is chaotic. In section 3, we introduce the modified Kac model with particles having the energy $\phi(v)$
and the corresponding master equation. We also show that this master equation propagates chaos and obtain the limiting equation. In section 4, we discuss how to generalize the results in section 2 to the case \( v \in \mathbb{R}^d, \ d > 1, \) in which case the momentum is also conserved. An Appendix is included to introduce the methods that are used in the paper.

### 2. Chaotic distributions

This section is devoted to showing that the family of uniform probability distributions on \( \Omega^{N-1}(\sqrt{N}) \) with respect to the microcanonical measure is chaotic. By Example 1.2, another approach would be show that the uniform distribution on \( \Omega^{N-1}(\sqrt{N}) \) with respect to surface measure is chaotic. However, since we consider a particle system where the total energy is conserved it is natural to use the microcanonical measure on \( \Omega^{N-1}(\sqrt{N}) \). We show that the family of uniform probability distributions on \( \Omega^{N-1}(\sqrt{N}) \) with respect to the microcanonical measure is \( C e^{-z_0 \phi(v)} \)-chaotic where \( z_0 > 0 \) is the solution to a specific equation and \( C \) is a normalisation constant.

The microcanonical measure on \( \Omega^{N-1}(\sqrt{N}) \) is defined as follows: Let

\[
H(v_1, \ldots, v_N) = \sum_{i=1}^{N} \phi(v_i).
\]

**Definition 2.1.** Provided that \( ||\nabla H|| \neq 0 \) the microcanonical measure \( \eta^{(E)} \) on \( \Omega^{N-1}(\sqrt{E}) \) is defined by

\[
\eta^{(E)} = \frac{\sigma_{\Omega}}{||\nabla H||},
\]

where \( \sigma_{\Omega} \) is the surface measure on \( \Omega^{N-1}(\sqrt{E}) \).

For a detailed discussion about the microcanonical measure, see [8]. One intuition behind the definition above is the following: A measure concentrated on concentrated on \( \Omega^{N-1}(\sqrt{E}) \) is given by

\[
\delta(H(v_1, \ldots, v_N) - E),
\]

where \( \delta \) is the Dirac delta function. For an integrable function \( g : \mathbb{R}^N \rightarrow \mathbb{R} \), the coarea formula implies

\[
\int_{\mathbb{R}^N} g(v_1, \ldots, v_N) dv_1 \ldots dv_N = \int_{\mathbb{R}} \int_{\Omega^{N-1}(\sqrt{y})} \frac{g}{||\nabla H||} dv_1 \ldots dv_N dy.
\]

Then

\[
\int_{\mathbb{R}^N} g(v_1, \ldots, v_N) \delta(H(v_1, \ldots, v_N) - E) dv_1 \ldots dv_N
\]

\[
= \int_{\mathbb{R}} \int_{\Omega^{N-1}(\sqrt{y})} g\delta(y - E) \frac{d\sigma_{\Omega}}{||\nabla H||} dy = \int_{\Omega(E)} g \frac{d\sigma_{\Omega}}{||\nabla H||}.
\]
This shows that
\[ \delta(H(v_1, \ldots, v_N) - E) = \frac{\sigma_{\Omega}}{||\nabla H||}. \]

Remark 2.2. On \( S^{N-1}(\sqrt{E}) \), we have that \( ||\nabla H|| = 2\sqrt{E} \). This implies that the microcanonical measure is up to a constant factor equal to the surface measure on \( S^{N-1}(\sqrt{E}) \).

Using the equality \( H(v_1, \ldots, v_N) = E \), we have (\( \phi \) is even)
\[ v_N = \pm \phi^{-1}(E - \sum_{i=1}^{N-1} \phi(v_i)) := U_\pm(v_1, \ldots, v_{N-1}). \]

By this parametrization, the surface \( \Omega^{N-1}(\sqrt{E}) \) can be represented as the graph of \( U_\pm : \mathbb{R}^{N-1} \to \mathbb{R} \). The surface measure \( \sigma_{\Omega} \) on \( \Omega^{N-1}(\sqrt{E}) \) is now given by
\[ d\sigma_{\Omega} = \sqrt{1 + ||\nabla U_\pm||^2} \, dv_1 \ldots dv_{N-1}. \]

By the implicit function theorem it follows that
\[ \frac{\partial U_\pm}{\partial v_k} = -\frac{\partial H}{\partial v_k} \frac{\partial H}{\partial v_N}^{-1} \quad k = 1, \ldots, N - 1. \]

Thus
\[ \frac{d\sigma_{\Omega}}{||\nabla H||} = \frac{1}{||\nabla H||} dv_1 \ldots dv_{N-1}. \]

To carry out integration on \( \Omega^{N-1}(\sqrt{E}) \) with respect to the microcanonical measure \( \eta^{(E)} \) we use the last equality:
(2.3)
\[ \int_{\Omega^{N-1}(\sqrt{E})} g(v_1, \ldots, v_N) d\eta^{(E)} = \sum_{\epsilon=\pm} \int_{\sum_{i=1}^{N-1} \phi(v_i) \leq E} g(v_1, \ldots, \epsilon v_N) \frac{1}{||\nabla H||} \, dv_1 \ldots dv_{N-1}, \]
where
(2.4)
\[ \left| \frac{\partial H}{\partial v_N} \right|_{\epsilon} = \left| \frac{\partial H}{\partial v_N} \left( v_1, \ldots, \epsilon \phi^{-1} \left( E - \sum_{i=1}^{N-1} \phi(v_i) \right) \right) \right|. \]

The uniform distribution \( F(v_1, \ldots, v_N) \) on \( \Omega^{N-1}(\sqrt{N}) \) with respect to the microcanonical measure \( \eta^{(N)} \) is
(2.5)
\[ F_N(v_1, \ldots, v_N) = \frac{1}{Z_\phi(\sqrt{N})}, \]
where
\[ Z_\phi(\sqrt{E}) = \int_{\Omega^{N-1}(\sqrt{E})} d\eta^{(E)}. \]
To show that the uniform distribution on $\Omega^{N-1}(\sqrt{N})$ with respect to the microcanonical measure $\eta^{(N)}$ is $Ce^{-20\phi(y^{2})}$-chaotic we follow Kac [6], and start by determining the asymptotic behaviour of

$$Z_{\phi}(\sqrt{E}) = \sum_{\epsilon=+,-} \int_{\sum_{i=1}^{N-1} \phi(v_{i}) \leq E} \frac{1}{\partial H/\partial v_{\epsilon}} dv_{1} \ldots dv_{N-1}$$

with $E = N$ for large $N$. Since

$$\frac{\partial H}{\partial v_{\epsilon}} = |\phi'(v_{\epsilon})|$$

and $v_{N} = \pm \phi^{-1} \left( E - \sum_{i=1}^{N-1} \phi(|v_{i}|) \right)$,

we have

$$Z_{\phi}(\sqrt{E}) = 2 \int_{\sum_{i=1}^{N-1} \phi(v_{i}) \leq E} \frac{1}{|\phi'(\phi^{-1}(E - \sum_{i=1}^{N-1} \phi(v_{i})))|} dv_{1} \ldots dv_{N-1}.$$ 

To write $Z_{\phi}(\sqrt{E})$ as an integral over the sphere $S^{N-1}(\sqrt{E})$, we make the change of variables $y_{i}^{2} = \phi(v_{i})$ with respect to sign of $v_{i}, i = 1, \ldots, N - 1$.

This leads to

$$Z_{\phi}(\sqrt{E}) =$$

$$2^{N} \int_{\sum_{i=1}^{N-1} y_{i}^{2} \leq E} \frac{1}{\phi'(\phi^{-1}(E - \sum_{i=1}^{N-1} y_{i}^{2}))} \prod_{i=1}^{N-1} \frac{|y_{i}|}{|\phi'(\phi^{-1}(y_{i}^{2}))|} dy_{1} \ldots dy_{N-1}.$$ 

Let

$$f(y) := \frac{|y|}{|\phi'(\phi^{-1}(y^{2}))|}.$$ 

The integrand in (2.7) is almost a product of $N$ copies of $f(y)$. Multiply and divide the integrand by $|y_{N}| = \sqrt{E - \sum_{i=1}^{N-1} y_{i}^{2}}$. Recall the following formula for integration over a sphere

$$\int_{S^{N-1}(E)} g(y_{1}, \ldots, y_{N}) d\sigma(y^{2}) = \sum_{\epsilon=+,-} \int_{\sum_{i=1}^{N-1} y_{i}^{2} \leq E} g(y_{1}, \ldots, y_{N}) \frac{E dy_{1} \ldots dy_{N-1}}{\sqrt{E^{2} - \sum_{i=1}^{N-1} y_{i}^{2}}}.$$ 

We now get

$$Z_{\phi}(\sqrt{E}) = \frac{2^{N-1}}{\sqrt{E}} \int_{S^{N-1}(\sqrt{E})} \prod_{i=1}^{N} f(y_{i}) d\sigma(y^{2}).$$

Having $Z_{\phi}$ given by (2.9) is convenient in sense that, in [6], Kac determined the asymptotic behaviour of $Z_{\phi}(\sqrt{N})$ for large $N$ by using the saddle point method (see e.g. [5]). For completeness, we present each step of the result with rigorous justification with $f(y)$ given by (2.8). A short description of the saddle point method is given in the Appendix.
We start by computing the Laplace transform of \( E \mapsto Z_{\varphi}(\sqrt{E}) \). The Laplace transform of \( Z_{\varphi}(\sqrt{E}) \) is defined provided that \( Z_{\varphi}(\sqrt{E}) \) grows at most exponentially. Since the behaviour of \( Z_{\varphi}(\sqrt{E}) \) depends on the function \( f(y) \) defined by (2.8) we assume that \( \phi(y) \) is such that

\[
(2.10) \quad f(y) \leq Ke^{by^2},
\]

for some \( K \geq 0 \) and \( b > 0 \). This condition ensures that \( Z_{\varphi}(\sqrt{E}) \) grows at most exponentially.

Taking the Laplace transform of \( Z_{\varphi}(\sqrt{E}) \), making the change of variables \( r = \sqrt{E} \), we have, for \( w \in \mathbb{C} \) where \( \Re(w) > b \)

\[
\int_0^\infty e^{-wE}Z_{\varphi}(\sqrt{E})dE = 2 \int_0^\infty e^{-wr^2}rZ_{\varphi}(r)dr.
\]

Using (2.9), the last equality equals

\[
2^N \left( \int_{-\infty}^{\infty} e^{-wy^2} f(y)dy \right)^N.
\]

From condition (2.10) and since \( \Re(w) > b \) it follows that

\[
(2.11) \quad \Phi(w) := \int_{-\infty}^{\infty} e^{-wy^2} f(y)dy
\]

is an analytic function of \( w \) for \( \Re(w) > b \). By applying the inverse of the Laplace transform, we get

\[
Z_{\varphi}(\sqrt{N}) = \frac{2N}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zE} \left( \int_{-\infty}^{\infty} e^{-zy^2} f(y)dy \right)^N dz,
\]

where \( \gamma > b \) is smallest number for which the line \( \gamma = \Re(z) \) lies in the half-plane where \( \Phi(z) \) is analytic. Replacing \( E \) with \( N \), we get

\[
Z_{\varphi}(\sqrt{N}) = \frac{2^{N-1}}{\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} (e^z \int_{-\infty}^{\infty} e^{-zy^2} f(y)dy)^N dz.
\]

By comparing with the saddle point integral (5.1) in the Appendix, we set

\[
(2.12) \quad q(z) = 1 \quad \text{and} \quad S(z) = z + \log \Phi(z),
\]

and hence

\[
(2.13) \quad Z_{\varphi}(\sqrt{N}) = \frac{2^{N-1}}{\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{NS(z)} dz.
\]

The asymptotic behaviour of \( Z_{\varphi}(\sqrt{N}) \) for large \( N \) is determined by the saddle points of \( S(z) \). The following lemma concerns the saddle points of \( S(z) \).
Lemma 2.3. Assume that there exists a $\gamma > 0$ such that
\begin{equation}
\int_{-\infty}^{\infty} e^{-\gamma y^2} f(y) dy < \infty.
\end{equation}
Moreover, assume that
\begin{equation}
\int_{-\infty}^{\infty} (1 - y^2) f(y) dy < 0,
\end{equation}
and
\begin{equation}
\int_{|y| \leq 1} (1 - y^2) f(y) dy > 0.
\end{equation}
Let $S(z)$ be given by (2.12) with $\Phi(z)$ by (2.11). For $\Re(z) \geq \gamma$, the function $S(z)$ is analytic and there exists a unique saddle point $z_0$ to $S(z)$ such that $z_0$ is real and $z_0 \geq \gamma$ and $S''(z_0) > 0$. Moreover
\begin{equation}
Z_{\Phi}(\sqrt{N}) \sim \frac{2^{N-1} e^{Nz_0}}{\sqrt{NS''(z_0)}} \left( \int_{-\infty}^{\infty} e^{-z_0 y^2} f(y) dy \right)^N.
\end{equation}

Proof. Note that, for $z = \xi + i\eta$, for all $\xi$
\[
\text{arg max}_{\eta} |e^{\xi+i\eta} \Phi(\xi + i\eta)| = \{0\}.
\]
Hence, we only need to find saddle points on the real line.

Claim 1: The function $S(z)$ has a unique saddle point $z_0$ where $z_0 \geq \gamma$.

Proof. For $\Re(z) \geq \gamma$, we have
\begin{equation}
S'(z) = 1 - \frac{\int_{-\infty}^{\infty} y^2 e^{-y^2} f(y) dy}{\int_{-\infty}^{\infty} e^{-y^2} f(y) dy}.
\end{equation}
For $z = \xi + i\eta$ where $\xi \geq \eta$, solving the equation $S'(\xi) = 0$ is equivalent to
\[
\int_{-\infty}^{\infty} e^{-\xi y^2} (1 - y^2) f(y) dy = 0.
\]
Multiplying the last equality by $e^{\xi}$, we see that, solving the equation $S'(\xi) = 0$ is equivalent to
\[
\int_{-\infty}^{\infty} e^{-\xi (y^2-1)} (1 - y^2) f(y) dy = 0.
\]
Let
\[
A(\xi) = \int_{-\infty}^{\infty} e^{-\xi (y^2-1)} (1 - y^2) f(y) dy.
\]
Since
\[
A'(\xi) = \int_{-\infty}^{\infty} e^{-\xi (y^2-1)} (1 - y^2)^2 f(y) dy > 0,
\]
it follows that $A(\xi)$ is an increasing function of $\xi$. Moreover, by (2.15) we have

$$A(0) < 0.$$  

Note that

$$\lim_{{\xi \to \infty}} \int_{{|y| > 1}} e^{-\xi(y^2 - 1)} (1 - y^2) f(y) dy = 0.$$  

Hence

$$\int_{-\infty}^{\infty} e^{-\xi(y^2 - 1)} (1 - y^2) f(y) dy \sim \int_{{|y| \leq 1}} e^{-\xi(y^2 - 1)} (1 - y^2) f(y) dy$$

$$= \int_{{|y| \leq 1}} e^{\xi(1 - y^2)} (1 - y^2) f(y) dy.$$  

The last integral goes to infinity by (2.16) as $\xi \to \infty$. Hence, there exists a unique $z_0 \geq \gamma$ such that $S'(z_0) = 0$.  

Claim 2: The second derivative of $S(z)$ at $z_0$ is positive.

Proof. We have

$$S''(z) = \frac{\Phi''(z)}{\Phi(z)} - \frac{\Phi'(z)^2}{\Phi(z)^2}.$$  

For $z = z_0$, using the Jensen inequality, we get

$$\frac{\Phi''(z_0)}{\Phi(z_0)} = \frac{\int_{-\infty}^{\infty} y^4 e^{-z_0y^2} f(y) dy}{\int_{-\infty}^{\infty} e^{-z_0y^2} f(y) dy} > \left( \frac{\int_{-\infty}^{\infty} y^2 e^{-z_0y^2} f(y) dy}{\int_{-\infty}^{\infty} e^{-z_0y^2} f(y) dy} \right)^2 = \frac{\Phi'(z_0)^2}{\Phi(z_0)^2}.$$  

This proves the claim.  

We now turn to the proof of (2.17). We can write

$$\Phi(z) = \int_{-\infty}^{\infty} e^{-z_0y^2} f(y) dy = \int_{0}^{\infty} e^{-z_0y^2} (f(y) + f(-y)) dy.$$  

By a change of variables, the last integral equals

$$\int_{0}^{\infty} e^{-z_0y^2} f(\sqrt{y}) + f(-\sqrt{y}) \frac{dy}{2\sqrt{y}}.$$  

Let $z = \xi + i\eta$, with $\xi \geq \gamma$. The last integral can be written as

$$\int_{-\infty}^{\infty} e^{-\xi y} \frac{f(\sqrt{y}) + f(-\sqrt{y})}{2\sqrt{y}} 1_{y \geq 0} e^{-i\eta y} dy.$$  

The last integral is the Fourier transform of the function $\Phi_\eta$ at the point $\eta$, where

$$\Phi_\eta(y) = e^{-\xi y} \frac{f(\sqrt{y}) + f(-\sqrt{y})}{2\sqrt{y}} 1_{y \geq 0}.$$  

For $\xi = z_0$, it follows that $\Phi_\eta \in L^1(\mathbb{R})$, and $|\mathcal{F}(\Phi_\eta)(\eta)| < |\mathcal{F}(\Phi_\eta)(0)|$. Moreover, By the Riemann Lebesgue lemma, it follows that

$$|\mathcal{F}(\Phi_\eta)(\eta)| \to 0 \text{ when } |\eta| \to \infty.$$
Let \( C_+ T \) and \( C_- T \) be the curves in the complex plane given \([\gamma + iT, z_0 + iT]\) and \([\gamma - iT, z_0 - iT]\), respectively.

We have

\[
\left| \int_{C_+ T} e^{NS(z)} dz \right| = \left| \int_{\gamma}^{z_0} e^{N(\xi + iT)} F(\Phi_\xi(T))^N d\xi \right|
\leq \int_{\gamma}^{z_0} |F(\Phi_\xi(T)|^N d\xi
\leq |z_0 - \gamma| \max_{\xi \in [\gamma, z_0]} e^{N|\xi|} |F(\Phi_\xi(T)|^N \to 0, \text{ as } |T| \to \infty.
\]

By Cauchy’s theorem, we can deform the contour in (2.13) to a contour passing through the saddle point \( z_0 \). Hence

\[
Z_\phi(\sqrt{N}) = \frac{2N}{\pi i} \int_{z_0 - i\infty}^{z_0 + i\infty} e^{NS(z)} dz.
\]

By the saddle point method, we obtain

\[
Z_\phi(\sqrt{N}) = \frac{2N}{\pi i} \sqrt{-\frac{2\pi}{NS''(z_0)}} \left( 1 + O\left( \frac{1}{N} \right) \right) e^{NS(z_0)}
\sim \frac{2N e^{Nz_0}}{\sqrt{NS''(z_0)}} \left( \int_{-\infty}^{\infty} e^{-z_0 y^2} f(y) dy \right)^N.
\]

This finishes the proof of the lemma. \( \square \)

The main theorem of this section is:

**Theorem 2.4.** Let \( f(y) \) defined by (2.8) satisfy the conditions (2.10), (2.15) and (2.16). Then, the uniform distribution on \( \Omega_{N^{-1}}(\sqrt{N}) \) with respect to the microcanonical is \( Ce^{-z_0\phi(v)} \) chaotic. The positive constant \( z_0 \) is the unique real solution to the following equation

\[
\int_{-\infty}^{\infty} (1 - \phi(v)) e^{-z_0\phi(v)} dv = 0,
\]

and \( C \) is a normalisation constant given by

\[
C = \int_{-\infty}^{\infty} e^{-z_0\phi(v)} dv.
\]
Proof. Let $\varphi$ be a bounded continuous function on $\mathbb{R}^k$. For each fixed $k \in \mathbb{N}$, a modification of Lemma 2.3 shows that, for large $N$,

$$
\sum_{\epsilon = +, -} \frac{1}{\sum_{i=1}^{N-1} \varphi(v_i) \leq E} \frac{\varphi(v_1, \ldots, v_k)}{\partial v_1 \cdots \partial v_{N-1}} d\epsilon_1 \cdots d\epsilon_{N-1}
$$

$$
\sim \int_{\mathbb{R}^k} \varphi(\varphi^{-1}(y_1^2), \ldots, \varphi^{-1}(y_k^2)) \prod_{i=1}^{k} e^{-z_0 y_i^2} f(y_i) dy_1 \cdots dy_k
$$

$$
\times \frac{2^N \varphi(N-k)z_0}{\sqrt{NS''(z_0)}} \left( \int_{-\infty}^{\infty} e^{-z_0 y^2} f(y) dy \right)^{N-k}.
$$

Using Lemma 2.3 and making the change of variable $y_i^2 = \varphi(v_i)$, where $dv_i = f(y_i)dy_i$, $i = 1, \ldots, k$ leads to

$$
\lim_{N \to \infty} \frac{\int_{\Omega_{N-1}(\sqrt{N})} \varphi(v_1, \ldots, v_k) d\eta(N)}{Z_\varphi(\sqrt{N})}
$$

$$
= \int_{\mathbb{R}^k} \varphi(\varphi^{-1}(y_1^2), \ldots, \varphi^{-1}(y_k^2)) \prod_{i=1}^{k} e^{-z_0 y_i^2} f(y_i) dy_1 \cdots dy_k
$$

$$
= \int_{-\infty}^{\infty} \varphi(v_1, \ldots, v_k) \prod_{i=1}^{k} e^{-z_0 \varphi(v_i)} dv_1 \cdots dv_k
$$

$$
= \int_{\mathbb{R}^k} \prod_{i=1}^{k} e^{-z_0 \varphi(v_i)} dv_1 \cdots dv_k.
$$

This is what we wanted to prove. □

We end this section by considering two examples where Theorem 2.4 applies:

Example 2.5. In the classical Kac model $\varphi(v) = v^2$. We get that $f(y) = \frac{1}{\pi}$ and thus, the conditions (2.14), (2.15) and (2.16) are fulfilled. To determine $z_0$ we need solve the equation

$$
\int_{-\infty}^{\infty} (1 - v^2) e^{-z_0 v^2} dv = 0.
$$

Direct calculations shows that $z_0 = \frac{1}{2}$ is the unique real solution. Hence, the uniform distribution on $(\Omega^{N-1}(\sqrt{N}) = S^{N-1}(\sqrt{N}))$ with respect to the microcanonical measure is $\frac{1}{\sqrt{2\pi}}$-chaotic. This has already been discussed in Example 1.2. Note that, on $S^{N-1}(\sqrt{N})$, the microcanonical measure up to a constant is equal to the surface measure.
Example 2.6. For a relativistic energy function \( \phi(v) = \sqrt{v^2 + 1} - 1 \), it follows that

\[
f(y) = \frac{(y^2 + 1)|y|}{\sqrt{(y^2 + 1)^2 - 1}}.
\]

It is easy to check that the conditions (2.14), (2.15) and (2.16) are satisfied. To find \( z_0 \) we solve the equation

\[
\int_{-\infty}^{\infty} (1 - (\sqrt{v^2 + 1} - 1))e^{-z_0(\sqrt{v^2 + 1} - 1)}dv = 0.
\]

By using numerical integration, we get that \( z_0 \approx 0.734641 \). Hence, the family of uniform distributions on \( \Omega_{N-1}(\sqrt{N}) \) with \( \phi(v) = \sqrt{v^2 + 1} - 1 \) in (1.9) is \( C e^{-z_0(\sqrt{v^2 + 1} - 1)} \)-chaotic, where \( C \approx 4.082 \).

3. Particle dynamics and master equation for general energy functions

In this section we follow [2] and [6] to introduce dynamics between particles having the energy given by the function \( \phi \) and obtain the corresponding master equation. By similar arguments as in [6], we will see that the master equation propagates chaos.

To introduce dynamics between the particles. Let the master vector

\[
V = (v_1, \ldots, v_N) \in \Omega_{N-1}(\sqrt{N}).
\]

The master vector \( V \) makes a jump on \( \Omega_{N-1}(\sqrt{N}) \) according to the following steps:

1. Pick a pair \( (i, j), i < j \) according to the uniform distribution

\[
P_{ij} = \frac{2}{N(N-1)}.
\]

2. The pair of velocities \( (v_i, v_j) \) satisfies

\[
\phi(v_i) + \phi(v_j) = h, \quad h > 0.
\]

Let

\[
y_i = \text{sign}(v_i) \sqrt{\phi(v_i)} \quad \text{and} \quad y_i = \text{sign}(v_j) \sqrt{\phi(v_j)}.
\]

Then \( (y_i, y_j) \) is a point on the circle, and as in the original Kac model, the collision may be preformed there.

Pick an angle \( \theta \) uniformly on \( (0, 2\pi] \) and let

\[
y_i(\theta) = y_i \cos \theta + y_j \sin \theta \quad \text{and} \quad y_j(\theta) = -y_i \sin \theta + y_j \cos \theta.
\]

The pair \( (v_i, v_j) \) is transformed on \( \Omega^1(\sqrt{h}) \) to \( (v_i(\theta), v_j(\theta)) \) according to

\[
v_i(\theta) = \text{sign}(y_i(\theta)) \phi^{-1}(y_i(\theta)^2),
\]

\[
v_j(\theta) = \text{sign}(y_j(\theta)) \phi^{-1}(y_j(\theta)^2),
\]

(3.1)
where

\[ \text{sign}(v) = \begin{cases} 1, & \text{if } v \geq 0 \\ -1, & \text{if } v < 0. \end{cases} \]

Note that

\[ \phi(v_i(\theta)) + \phi(v_j(\theta)) = \phi(v_i) + \phi(v_j). \]

(3) Update the master vector \( V \) and denote the new master vector by \( T_{i,j}(\theta)V \). By step 2, it follows that \( T_{i,j}(\theta)V \in \Omega^{N-1}(\sqrt{N}) \). Repeat step 1, 2 and 3.

This is only a generalization of the dynamics in [6] where \( \phi(v) = v^2 \).

The steps above describe a random walk on \( \Omega^{N-1}(\sqrt{N}) \). As in [2], we define its Markov transition operator \( Q_\phi \). If \( V_k \) is the state of the particles after the \( k \)-th step of the walk, for a continuous function \( \phi \) on \( \Omega^{N-1}(\sqrt{N}) \), the operator \( Q_\phi \) is defined by

\[ Q_\phi \phi(y) = E[\phi(V_{k+1})|V_k = y]. \]

Writing out the expectation above, we get

\[ Q_\phi \phi(V) = \frac{2}{N(N-1)} \sum_{i<j} \int_0^{2\pi} \phi(T_{i,j}(\theta)V) \frac{d\theta}{2\pi}. \]

If \( F_k \) is probability density of \( V_k \) with respect to the microcanonical measure \( \eta^{(N)} \) on \( \Omega^{N-1}(\sqrt{N}) \), we have

\[ \int_{\Omega^{N-1}(\sqrt{N})} \phi F_{k+1} d\eta^{(N)} = E[\phi(V_{k+1})] = E[E[\phi(V_{k+1}|V_k)]] \]

\[ = \int_{\Omega^{N-1}(\sqrt{N})} Q_\phi \phi F_k d\eta^{(N)}. \]

By definition, the microcanonical measure is invariant under the transformation \( V \rightarrow T_{i,j}(\theta)V \). It follows that \( Q_\phi \) is self-adjoint and

\[ F_{k+1} = Q_\phi F_k. \]

So far, the process defined above is discrete in time. To obtain a time continuous process, we let the master vector \( V \) be a function of time, and the times between the jumps (collisions) exponentially distributed. In this way, if \( F_N(V,0) \) is the probability distribution of the \( N \) particles on \( \Omega^{N-1}(\sqrt{N}) \) at time 0, the time evolution of \( F_N(V,t) \) is described by the following master equation

\[ \frac{\partial F_N(V,t)}{\partial t} = \mathcal{K}_\phi F_N(V,t), \]

where

\[ \mathcal{K}_\phi = N[Q_\phi - I] \]
and $I$ is the identity operator. A more complete discussion of master equations of this kind may be found in [2].

Note that, for $\phi(v) = v^2$, the collision operator $K_{\phi}$ is the same as the collision operator $K$ in the Kac model. We have propagation of chaos for the master equation (3.3):

**Theorem 3.1.** Assume that the family of distributions $\{F_N(V, 0)\}_{N \in \mathbb{N}}$ on $\Omega^{N-1}(\sqrt{N})$ is $f(v, 0)$-chaotic. Then, the family of distributions $\{F_N(V, t)\}_{N \in \mathbb{N}}$, where $F_N(V, t)$ is the solution to (3.3), is $f(v, t)$-chaotic. Moreover, the density $f(v, t)$ satisfies the following equation

$$\frac{\partial f(v, t)}{\partial t} = \int_{\mathbb{R}} \int_{0}^{2\pi} [f(v(\theta), t)f(w(\theta), t) - f(v, t)f(w, t)] d\theta dw,$$

with $f(v, 0) = f_0(v)$ and $v(\theta)$, $w(\theta)$ given by (3.1).

**Proof.** The proof follows by the same arguments as in [6]. For a more detailed proof where propagation of chaos is shown for more general master equations, we refer to [2].

\[\square\]

4. CHAOTIC DISTRIBUTIONS IN HIGHER DIMENSIONS

So far we have considered a particle system where the velocity of a particle is one-dimensional and only satisfies the conservation of energy. In Section 2 we proved that the uniform distribution with respect to the microcanonical measure on $\Omega^{N-1}(\sqrt{N})$ is $C e^{-\frac{1}{2}E\phi(v)}$-chaotic, $v \in \mathbb{R}$. The goal of this section is to generalize this to the case $v \in \mathbb{R}^2$ where now both the energy and momentum are conserved; the generalization to $\mathbb{R}^d, d > 2$ may be treated in the same way. The calculations here are formal. For $p \in \mathbb{R}^2$, define

$$\Gamma^N(\sqrt{E}, p) = \left\{ (v_1, \ldots, v_N) \in \mathbb{R}^{2N} \mid \sum_{i=1}^{N} \phi(v_i) = 2E, \sum_{i=1}^{N} v_i = p \right\}.$$

We assume that $E$ and $p$ are chosen such that $\Gamma^N(\sqrt{E}, p)$ is non-empty. The classical case when $\phi(v) = |v|^2$ has been thoroughly investigated in [3].

For $p = (p_1, p_2)$ and $v_i = (v_{i1}, v_{i2}), i = 1, \ldots, N$, a measure $\mu_{E, p_1, p_2}$ concentrated on $\Gamma^N(\sqrt{E}, p)$ is defined by

$$\mu_{E, p_1, p_2} = \delta(2E - \sum_{i=1}^{N} \phi(v_i))\delta(p_1 - \sum_{i=1}^{N} v_{i1})\delta(p_2 - \sum_{i=1}^{N} v_{i2}).$$

The product of the Dirac measures is well defined since the hypersurfaces defined by setting the arguments of the Dirac measures to zero are mutually transversal.

$$Z(E, p_1, p_2) = \int_{\mathbb{R}^{2N}} \delta(2E - \sum_{i=1}^{N} \phi(v_i))\delta(p_1 - \sum_{i=1}^{N} v_{i1})\delta(p_2 - \sum_{i=1}^{N} v_{i2})dv_1 \ldots dv_N.$$
As in Section 2, we need to determine the asymptotic behaviour of $Z(E, p_1, p_2)$ with $E = N$ for large $N$. We note that in the case of relativistic collisions, with $\phi(\nu) = \sqrt{|\nu|^2 + 1} - 1$, the measure $\mu_{E,p_1,p_2}$ is Lorentz invariant (see [12]).

In the sense of distributions, the $\delta$ function is the inverse Fourier transform of the function 1 and formally can be written as

$$\delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} d\xi.$$  

For $z = (z_1, z_2, z_3)$, we can formally write

$$Z(E, p_1, p_2) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i(2E - \sum_{i=1}^{N} \phi(\nu_i))z_3} e^{i(p_1 - \sum_{i=1}^{N} v_{i1})z_1} e^{i(p_2 - \sum_{i=1}^{N} v_{i2})z_2} dz_1 dz_2 dz_3 \, dv_1 \ldots dv_N.$$  

With $E = N$, the last equality is

$$Z(N, p_1, p_2) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ip_{1z_1} + ip_{2z_2}} \left( e^{2iz_3} \int_{\mathbb{R}^2} e^{-i\phi(\nu_1)z_3 - i\nu_{11}z_1 - i\nu_{12}z_2}dv_{11}dv_{12} \right)^N dz_1 dz_2 dz_3.$$  

Note that here $p_1$ and $p_2$ are assumed to be independent of $N$. A natural and straightforward variation is to replace $p_1$ by $Np_1$ and $p_2$ by $Np_2$.

Let

$$q(z_1, z_2, z_3) = e^{ip_{1z_1} + ip_{2z_2}},$$  

$$S(z_1, z_2, z_3) = 2iz_3 + \log \int_{\mathbb{R}^2} e^{-i\phi(\nu_1)z_3} - i\nu_{11}z_1 - i\nu_{12}z_2 \, dv_{11}dv_{12}.$$  

where $q : \mathbb{C}^3 \to \mathbb{R}, S : \mathbb{C}^3 \to \mathbb{R}$. With these notations, we can write

$$Z(N, p_1, p_2) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} q(z_1, z_2, z_3)e^{NS(z_1, z_2, z_3)} dz_1 dz_2 dz_3.$$  

The right hand side of the last equality is a saddle point integral in dimension 3. The asymptotic behaviour of $Z(N, p_1, p_2)$ for large $N$ is determined by the saddle points of $S(z_1, z_2, z_3)$, i.e., points $(\bar{z}_1, \bar{z}_2, \bar{z}_3)$ such that

$$\nabla S(z_1, z_2, z_3) = 0.$$  

Using (4.5), we need to solve the following system of equations:

$$2i - \frac{i \int_{\mathbb{R}^2} \phi(\nu_1) e^{-i\phi(\nu_1)z_3} - i\nu_{11}z_1 - i\nu_{12}z_2 \, dv_{11}dv_{12}}{\int_{\mathbb{R}^2} e^{-i\phi(\nu_1)z_3} - i\nu_{11}z_1 - i\nu_{12}z_2 \, dv_{11}dv_{12}} = 0,$$

$$-\frac{i \int_{\mathbb{R}^2} \nu_{11} e^{-i\phi(\nu_1)z_3} - i\nu_{11}z_1 - i\nu_{12}z_2 \, dv_{11}dv_{12}}{\int_{\mathbb{R}^2} e^{-i\phi(\nu_1)z_3} - i\nu_{11}z_1 - i\nu_{12}z_2 \, dv_{11}dv_{12}} = 0,$$
\[ -i \int_{\mathbb{R}^2} v_{12} e^{-i\phi(v_1)z_3 - iv_1z_1 - iv_2z_2} dv_{11} dv_{12} = 0. \]

Since \( \phi \) is even, it follows that \( \bar{z}_1, \bar{z}_2 = 0 \) are the unique solutions to equations (4.9) and (4.10). We can now obtain \( \bar{z}_3 \) by solving the following equation:

\[ \int_{\mathbb{R}^2} (2 - \phi(v_1)) e^{-i\phi(v_1)z_3} dv_1 = 0. \]

Assuming that \((0, 0, \bar{z}_3)\) is the unique solution to (4.7), and we can deform the integration domain in (4.6) to contain \((0, 0, \bar{z}_3)\), then

\[ Z(N, p_1, p_2) \sim \frac{1}{(2\pi)^3} \left( \frac{2\pi}{N} \right)^{3/2} \frac{1}{(\det S''((0, 0, \bar{z}_3)))^{1/2}} e^{NS(0,0,\bar{z}_3)} q(0,0,\bar{z}_3) \]

\[ = \frac{1}{(2\pi N)^{3/2}} \left( \det S''((0,0,\bar{z}_3)) \right)^{1/2} 2Ni\bar{z}_3 \left( \int_{\mathbb{R}^2} e^{-i\phi(v_1)z_3} dv_1 \right)^N. \]

By the discussion in Section 2, this procedure shows formally that the uniform distributions on \( \Gamma^N(\sqrt{N}, p) \) with respect to the measure \( \mu_{N,p_1,p_2} \) is \( Ce^{-z_0\phi(v)} \)-chaotic, where \( z_0 = i\bar{\xi}_1 \) and \( \bar{z}_3 \) the unique solution to (4.11) and

\[ C = \int_{\mathbb{R}^2} e^{-z_0\phi(v)} dv. \]

5. Appendix

The saddle point method is used to determine the asymptotic behaviour of integrals depending on a parameter. For a detailed description we refer to [5]. Without proof we only state below the saddle point method which is concerned with this paper.

**One-dimensional saddle point method**

Let \( \gamma \) be a contour in the complex plane. Assume that \( f \) and \( S \) are analytic functions in a neighborhood of the contour \( \gamma \). Consider the following integral

\[ F(\lambda) = \int_{\gamma} q(z) e^{\lambda S(z)} dz. \]

A point \( z_0 \in \mathbb{C} \) is called a simple saddle point of the function \( S : \mathbb{C} \rightarrow \mathbb{C} \) if \( S'(z_0) = 0 \) and \( S''(z_0) \neq 0 \). Assume that \( z_0 \in \gamma \) is the unique simple saddle point of \( S \). Then as \( \lambda \to \infty \)

\[ f(\lambda) = \sqrt{-\frac{2\pi}{\lambda S''(z_0)}} e^{\lambda S(z_0)} \left( q(z_0) + \mathcal{O}\left( \frac{1}{\lambda} \right) \right). \]
Many-dimensional saddle point method
Let $\gamma$ be an $N$-dimensional smooth compact manifold. Consider the following integral

$$F(\lambda) = \int_{\gamma} q(z) e^{\lambda S(z)} dz.$$  (5.3)

where $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ and the functions $q(z)$ and $S(z)$ are assumed to be analytic in a domain $D$ containing $\gamma$. A point $z_0$ is called a simple saddle point of $S(z)$ if $\nabla S(z_0) = 0$ and $det S''(z_0) \neq 0$. Assume that $z_0 \in \gamma$ is the unique simple saddle point of $S$. Then as $\lambda \to \infty$

$$f(\lambda) = \left(\frac{2\pi}{\lambda}\right)^{N/2} \frac{1}{(det S''(z_0))^{1/2}} e^{\lambda S(z_0)} \left(q(z_0) + O\left(\frac{1}{\lambda}\right)\right).$$  (5.4)

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