String Field Theory in Rindler Space-Time
and String Thermalization

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Quantization of free string field theory in the Rindler space-time is studied by using the covariant formulation and taking the center-of-mass value of the Rindler string time-coordinate \( \eta(\sigma) \) as the time variable for quantization. We construct the string Rindler modes which vanish in either of the Rindler wedges \( \pm \) defined by the Minkowski center-of-mass coordinate of the string. We then evaluate the Bogoliubov coefficients between the Rindler string creation/annihilation operators and the Minkowski ones, and analyze the string thermalization. An approach to the construction of the string Rindler modes corresponding to different definitions of the wedges is also presented toward a thorough understanding of the structure of the Hilbert space of the string field theory on the Rindler space-time.

§1. Introduction

From various viewpoints including the problem of information loss and thermalization in the black hole space-time, of much importance is to study string theory near the event horizon of a black hole.\(^1\),\(^2\) Instead of a real black hole, it is simpler and instructive to consider the Rindler space-time, a flat space-time \( (ds)^2 = -\xi^2(\eta)^2 + (d\xi)^2 + (dx_\perp)^2 \) with an event horizon \( (x_\perp \) denotes the transverse space coordinates). In fact, in the large mass limit of a black hole, space-time near the event horizon approaches Rindler space-time.

The purpose of this paper is to study string theory in the Rindler space-time on the basis of string field theory (SFT), which is the most natural and convenient framework for investigating problems concerning string creations and annihilations. In particular, we employ the covariant SFT based on the BRST formulation,\(^3\)\(-6\) which is more suitable for the present subject than the light-cone SFT.\(^7\) Although the Rindler space-time is flat and the quantization of local (particle) field theory on it is well known,\(^8\),\(^9\) the quantization of SFT on Rindler space-time is a non-trivial matter due to the fact that string is an extended object and the Rindler space-time has a horizon. (The Rindler quantization of string theory using the first quantized formalism was previously considered in Refs. 10) and 11).) In this paper, we consider the free open bosonic SFT in 26 dimensions, and ignore the problem related to the presence of the tachyon mode.

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In the Rindler quantization of SFT, the string field is a functional of the Rindler string coordinates \((\xi(\sigma), \eta(\sigma), X^\perp(\sigma))\) related to the Minkowski string coordinates \(X^\mu(\sigma)\) by

\[
X^0(\sigma) = \xi(\sigma) \sinh \eta(\sigma), \quad X^1(\sigma) = \xi(\sigma) \cosh \eta(\sigma) .
\]  

The SFT action expressed in terms of \((\xi(\sigma), \eta(\sigma), X^\perp(\sigma))\) (see Eq. (4.1) in §4.1) has an invariance under the \(\sigma\)-independent shift of the \(\eta(\sigma)\) coordinate, and we shall take the center-of-mass (CM) coordinate of \(\eta(\sigma)\),

\[
\eta_0 = \int_0^{\pi} \frac{d\sigma}{\pi} \eta(\sigma) ,
\]  
as the Rindler time for quantization.

The most important problem in the Rindler quantization of SFT is the definition of Hilbert spaces of states on each Rindler wedge. The whole Rindler space-time can be divided into two Rindler wedges \(\pm\) depicted in Fig. 1, where the Rindler wedge \(+\) \((-\) is defined by \(x^1 > |x^0|\) \((x^1 < -|x^0|)\). In a local (particle) field theory, a point in one Rindler wedge is not causally connected to a point in the other wedge, so the whole Hilbert space on the Rindler space-time can be considered a product of two Hilbert spaces on the Rindler wedges \(\pm\). In the case of SFT, however, there exist configurations of a string ((B) in Fig. 1) which crosses the origin and lives in both Rindler wedges, and this makes the definition of the Hilbert spaces on the Rindler wedges a difficult and obscure problem. In fact, it is not clear whether the string (B) in Fig. 1 is causally connected to strings (A) and (C) in Fig. 1. (It is evident that the string (A) in the \(+\) wedge is causally disjoint from the string (C) in the \(-\) wedge.)

To completely clarify the structure of the Hilbert space of SFT on the Rindler space-time, we need a deep understanding of the causality of strings in the Rindler space-time. In spite of recent investigations on the causality of strings, \(^{12} - 14\) we do not yet have a complete solution. In this paper, we first adopt one way of defining the Hilbert spaces on each Rindler wedge. This is to define the division of the whole Hilbert space into two by the criterion of whether the Minkowski CM coordinate of the string, \(x^\mu_{CM} = \int_0^{\pi} (d\sigma/\pi) X^\mu(\sigma)\), is in the Rindler wedge \(+\) or \(-\). In §§3 and 4, we shall apply this definition to determine a complete orthonormal system of the string Rindler modes on each Rindler wedge and evaluate the Bogoliubov coefficients between the Minkowski string creation/annihilation operators and the Rindler ones, and study the string thermalization.

This division of the Hilbert space, however, is not a fully satisfactory one in the point that the division is through the Minkowski CM coordinate of the string.

![Fig. 1. Three strings (A), (B) and (C). They are confined in the Rindler wedges (shaded region), and string (B) crosses the origin.](https://example.com/fig1.png)
Division by the Rindler CM coordinate, $\xi_0 = \int_0^\pi (d\sigma/\pi) \xi(\sigma)$, would be more natural. Although the complete analysis of the construction of the string Rindler modes corresponding to various ways of dividing the Hilbert space is still beyond our technical ability, in §5 we present a construction of the string Rindler modes corresponding to different divisions from that adopted in §§3 and 4. We hope that this construction will shed light on the complete understanding of the wedge problem.

Before closing the Introduction, let us mention another approach to the Rindler quantization of SFT. This is to first regard the SFT on Minkowski space-time as a set of the infinite number of component (particle) field theories with masses and spins lying on the Regge trajectories, and then to carry out the Rindler quantization of the respective component fields. Since a component field $\varphi(x_{\text{CM}}^\mu)$ is a function of the Minkowski CM coordinate $x_{\text{CM}}^\mu$ of the string, it is this CM string coordinate $x_{\text{CM}}^\mu$ rather than the whole string coordinate $X^\mu(\sigma)$ that we put on the Rindler space-time in this approach. The Rindler time for this quantization is $\tilde{\eta}$ defined by

$$x_{\text{CM}}^0 = \xi \sinh \tilde{\eta}, \quad x_{\text{CM}}^1 = \xi \cosh \tilde{\eta},$$

and is different from $\eta_0$ of Eq. (1.2). Note that a choice of the Rindler time for quantization leads to the corresponding definition of the positive frequency modes of the string wave function and consequently determines the structure of the Rindler ground state. Therefore, string thermalizations could differ between the Rindler quantization using the $\eta_0$ time and that using the $\tilde{\eta}$ time.

However, this approach is unsatisfactory and unnatural since the “Rindler string” here is not always confined in Rindler space-time. For example, for the “Rindler string” depicted in Fig. 2, the CM coordinate $x_{\text{CM}}^\mu$ is within Rindler space-time but a part of the “Rindler string” is outside it. That is, this “Rindler string” crosses the Rindler horizon and hence cannot be described in terms of the Rindler string coordinates $(\xi(\sigma), \eta(\sigma))$ via Eq. (1.1). Although we do not adopt this approach in this paper, we shall in some places make comparison with this other “Rindler quantization” method by referring to it as the “Minkowski component field (MCF) approach”.

The organization of the remainder of this paper is as follows. First summarizing in §2 the quantization of free SFT on the Minkowski space-time, in §3 we construct the string Rindler modes corresponding to the above explained division of the Hilbert space. Using this Rindler modes, we carry out the Rindler quantization of free SFT and study the string thermalization in §4. In §5, an attempt to the construction of the string Rindler modes different from those of §3 is presented. The final section (§6) is devoted to a summary of the remaining problems. This paper contains four appendices. In three of them (A, B and D), various formulas used in the text are
In Appendix E, we present the proof of the BRST invariance of the Rindler vacuum defined in §4.

§2. Free string field theory on Minkowski space-time

Before starting the construction of SFT on Rindler space-time, we shall recapitulate the elements of the free SFT covariantly quantized on Minkowski space-time. In this paper we shall confine ourselves to opening bosonic SFT.

First, the string field $\Phi$ is a real ($\Phi^I = \Phi$) functional of the space-time string coordinate $X^\mu(\sigma)$ as well as the ghost and anti-ghost coordinates, $c(\sigma)$ and $\bar{c}(\sigma)$ ($0 \leq \sigma \leq \pi$):

$$\Phi = \Phi [X^\mu(\sigma), c(\sigma), \bar{c}(\sigma)]. \quad (2.1)$$

In the following we shall consider the SFT which is gauge-fixed in the Siegel gauge, and therefore $\bar{c}(\sigma)$ in (2.1) (and in all the equations below) should be understood to be free from the zero-mode. Then the BRST invariant action for $\Phi$ reads

$$S = -\frac{1}{2} \int \mathcal{D}X^\mu(\sigma) \mathcal{D}c(\sigma) \mathcal{D}\bar{c}(\sigma) \Phi L\Phi, \quad (2.2)$$

and the corresponding equation of motion is given by

$$L\Phi = 0. \quad (2.3)$$

In Eqs. (2.2) and (2.3), $L$ is the kinetic operator,

$$L = \pi \int_0^\pi d\sigma \left\{ -\eta^{\mu\nu} \frac{\delta}{\delta X^\mu} \frac{\delta}{\delta X^\nu} + \eta_{\mu\nu} X^\mu X^\nu + \text{(ghost part)} \right\}$$

$$= -\eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} + 2 \sum_{n=1}^\infty \eta^{\mu\nu} \left\{ -\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} + \left(\frac{n\pi}{2}\right)^2 x^\mu_n x^\nu_n \right\} + \text{(ghost part)}$$

$$= -\eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} + \left[\text{mass}^2\text{-operator}\right], \quad (2.4)$$

where $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1, 1, \cdots, 1)$ is the flat Minkowski metric, and we define the Fourier expansion coefficient $(x^\mu, x^\mu_n)$ of $X^\mu(\sigma)$ by

$$X^\mu(\sigma) = x^\mu + \sum_{n=1}^\infty x^\mu_n \cos n\sigma, \quad (2.5)$$

$$\frac{\delta}{\delta X^\mu(\sigma)} = \frac{1}{\pi} \left\{ \frac{\partial}{\partial x^\mu} + 2 \sum_{n=1}^\infty \cos(n\sigma) \frac{\partial}{\partial x^\mu_n} \right\}. \quad (2.6)$$

The coordinate $x^\mu$ is the same as $x^\mu_{\text{CM}}$ in §1. We omit the subscript CM for simplicity.

The Minkowski quantization of this SFT system is carried out by taking the CM coordinate $x^0$ of $X^\mu=0(\sigma)$ as the time coordinate. Namely, defining the canonical momentum $\Pi_M$ conjugate to $\Phi$ by

$$\Pi_M [X^\mu(\sigma), c(\sigma), \bar{c}(\sigma)] \equiv \frac{\delta S}{\delta (\partial \Phi / \partial x^0)} = \frac{\partial}{\partial x^0} \Phi [X^\mu(\sigma), c(\sigma), \bar{c}(\sigma)], \quad (2.7)$$
we impose the equal $x^0$ canonical commutation relation

$$\left[ \Phi[X, c, \bar{c}], \Pi_M[\tilde{X}, \tilde{c}, \tilde{\bar{c}}] \right]_{x^0 = \tilde{x}^0} = i \prod_\sigma \delta \left( X(\sigma) - \tilde{X}(\sigma) \right) \delta \left( c(\sigma) - \tilde{c}(\sigma) \right) \delta \left( \bar{c}(\sigma) - \tilde{\bar{c}}(\sigma) \right),$$

where the hat on $\delta(X - \tilde{X})$ implies that the delta function for the time variable, $\delta(x^0 - \tilde{x}^0)$, is missing.

Then we expand the string field $\Phi$ in terms of the complete set of the Minkowski modes $\{U_{k(M,N)}A, U^*_{k(M,N)A}\}$:

$$\Phi = \int_{-\infty}^{\infty} dk \sum_{M_n,N_n=0}^{\infty} \sum_A \left( U_{k(M,N)}A a_{k(M,N)}^A + a_{k(M,N)}^{A*} U^*_{k(M,N)A} \right).$$

This manifestly respects the hermiticity of $\Phi$. $U_{k(M,N)A}$ is the normalized positive frequency solution (with respect to the Minkowski time $x^0$) of the wave equation,

$$LU_{k(M,N)A} = 0,$$

and the complex conjugate $U^*_{k(M,N)A}$ is the negative frequency one. The composition of our Minkowski mode $U_{k(M,N)A}$ is rather unconventional. However, it is convenient for the analysis of the Rindler quantization in later sections. The meaning of the indices of $U_{k(M,N)A}$ is as follows: $k$ is the momentum conjugate to the CM coordinate $x^1$, $\{M, N\}$ denotes the set of the level numbers $(M_n, N_n)$ of the oscillators $(x_n, x_n') (n = 1, 2, \cdots)$, and the index $A$ represents collectively the transverse momentum $k_\perp$ and the level numbers of the $(X^\perp, c, \bar{c})$ oscillators ($X^\perp$ denotes $X^\mu$ with $\mu = 2, \cdots, 25$). Therefore, the symbol $\sum_A$ implies integrations and summations over the momentum variables and the level numbers contained in $A$.

Corresponding to the three sets of the quantum numbers $k, \{M, N\}$ and $A$, the Minkowski mode $U_{k(M,N)A}$ is given as the product of three components:

$$U_{k(M,N)A} = U_k(x^0, x^1) \cdot \Psi_{\{M,N\}}(x^0_n, x^1_n) \cdot \phi_A,$$

where

$$U_k(x^0, x^1) = \frac{1}{\sqrt{2\omega_k \cdot 2\pi}} \exp \left( -i\omega_k x^0 + ik x^1 \right),$$

$$\Psi_{\{M,N\}}(x^0_n, x^1_n) = \prod_{n=1}^{\infty} \psi_{M_n, N_n}(x^0_n, x^1_n),$$

$$\Psi_{M_n, N_n}(x^0_n, x^1_n) = \mathcal{I}^{M_n} \psi_{M_n}^{(n)}(ix^0_n) \psi_{N_n}^{(n)}(x^1_n),$$

and $\phi_A$ is the wave function for the transverse and the ghost coordinates $(X^\perp, c, \bar{c})$. Various quantities appearing in (2.12) and (2.13) are as follows:

$$\omega_k = \sqrt{k^2 + \mu^2},$$
\[ \mu^2 = k_\perp^2 + m^2, \]  
\[ m^2 = m^2 (\{M,N\}, A) = 2\pi \sum_{n=1}^{\infty} n (M_n + N_n) + m_A^2, \]  
\[ \psi_{M}^{(n)}(x) = \sqrt{\frac{1}{M!}} \sqrt{\frac{n!}{2}} e^{-n\pi x^2/4} H_M (\sqrt{n\pi x}), \]  
\[ \frac{\mu^2}{M} = \frac{m^2}{2} \{M,N\}, A \]

where \( m_A^2 \) is the (mass)² contributing from the transverse and ghost oscillators \( m_A^2 \) also contains the intercept term \(-2\pi\), and \( H_M(x) \) is the Hermite polynomial defined by \( H_M(x) = e^{x^2/2} (-d/dx)^M e^{-x^2/2} \). Due to the factor \( i^{M_n} \) in Eq. (2·14), \( \Psi_{M_n,N_n}^{(n)} \) is real when \( x_n^0 \) is real. In the above definitions we have used some abbreviations for the indices for the sake of simplicity. First, although \( U_{k}(X_0, x^1) \) depends on \( \{M,N\} \) and \( A \) through \( \omega_k \) and therefore should carry them as its index, we have omitted this dependence. Second, we have omitted the superscript \((n)\) which \( \Psi_{M_n,N_n}^{(n)} \) should carry as \( \psi_{M}^{(n)} \) since it can be specified by the subscript \( n \) of \( (M_n, N_n) \).

For the normalization of the Minkowski modes, we define the Minkowski inner product for the string wave functions \( \Phi_1 \) and \( \Phi_2 \) by

\[ \langle \Phi_1, \Phi_2 \rangle = i \int \frac{D\mu}{\mu} d\mu d\sigma d\bar{\sigma} \frac{\partial^+}{\partial x^0} \Phi^*_1 d\Phi_2 \]

\[ = i \prod_{k=1}^{D-1} \int_{-\infty}^{\infty} dx_k \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{\infty} i dx_n' \int Dc(\sigma) D\bar{c}(\sigma) \]

\[ \times \Phi^*_1 (x_n^0) \frac{\partial^+}{\partial x^0} \Phi_2 (x_n^0), \]

where we have made explicit only the \( x_n^0 \)-dependence of the wave functions. The integration in (2·19) is carried out over the string coordinates \( (X^\mu(\sigma), c(\sigma), \bar{c}(\sigma)) \) with a fixed Minkowski CM time coordinate \( x^0 \). If both \( \Phi_1 \) and \( \Phi_2 \) satisfy the wave equation \( L\Phi_{1,2} = 0 \), the inner product (2·19) is conserved. Namely, it is independent of the choice of \( x^0 \).

We need to explain the prescription concerning the variables \( x_n^0 \) in the inner product (2·19). First, the complex conjugation operation on \( \Phi_1 \) in Eq. (2·19) should be carried out by regarding the \( x_n^0 \) as real variables. Then, the integrations over \( x_n^0 \) are carried out in the pure-imaginary direction. This prescription is due to the fact in the first quantized string theory that \( x_n^0 \) is a negative-norm harmonic oscillator with pure-imaginary eigenvalues, and hence the completeness relation for the eigenstates of \( x_n^0 \) reads \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_n^0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_n^0 \) \( \langle x_n^0 | x_n^0 \rangle = 1 \). This is consistent with the inner product using the indefinite metric creation/annihilation operators in the first quantized string theory. The complex conjugation operation in all places other than the inner product should also be done by regarding the \( x_n^0 \) as real variables.

Using the above inner product (2·19), the Minkowski modes \( \{U,U^*\} \) are nor-
malized as follows: * \]
\[
(U_k{M,N}A, U_{k'}{M',N'}A') = \delta(k - k') \delta_{M,N,M',N'}(-)^{\sum_n M_n} \eta_{AA'} ,
\]
\[
(U_k^*{M,N}A, U_{k'}^*{M',N'}A') = -(-)^{|A|} \delta(k - k') \delta_{M,N,M',N'}(-)^{\sum_n M_n} \eta_{A'A} ,
\]
\[
(U_k{M,N}A, U_{k'}^*{M',N'}A') = 0 ,
\]
\[
(2·20)
\]
where we have used the abbreviation \( \delta_{M,N,M',N'} \equiv \prod_n \delta_{M_n,M_n'} \delta_{N_n,N_n'} \), and \( \eta_{AA'} \)
\[
\text{is the inner product of the transverse wave functions,}
\]
\[
\eta_{AA'} = (\phi_A, \phi_{A'}) \equiv \int D\bar{c} Dc D\bar{c}' Dc' \phi_A^* \phi_{A'} ,
\]
\[
(2·21)
\]
and \(|A|\) in \((-)^{|A|}\) is defined by
\[
|A| = \begin{cases} 
0 & \text{if } \phi_A \text{ is Grassmann-even} , \\
1 & \text{if } \phi_A \text{ is Grassmann-odd} .
\end{cases}
\]
\[
(2·22)
\]
Equation (2·20) is the product of the inner products of the factor wave functions \( U_k, \Psi_{M_n,N_n} \) and \( \phi_A \). The inner product for \( U_k(x^0, x^1) \), \( (U_k, U_k') \)
\[
\equiv i \int_{-\infty}^{\infty} dx^1 U_k^*(x^0, x^1)(\partial/\partial x^0) U_k'(x^0, x^1) ,
\]
given by
\[
(U_k, U_k') = - (U_k^*, U_k'^*) = \delta(k - k') ,
\]
\[
(U_k, U_{k}') = (U_k^*, U_{k'}^*) = 0 ,
\]
\[
(2·23)
\]
where \( U_k \) and \( U_{k'} \) should have a common \( \mu^2 \) of Eq. (2·15). As for the inner product
\[
of \Psi_{M_n,N_n} \text{ defined by}
\]
\[
(\Psi, \bar{\Psi}) = \int_{-\infty}^{\infty} dx_n^1 \int_{-\infty}^{\infty} i dx_n^0 \Psi^*(x_n^0, x_n^1) \bar{\Psi}(x_n^0, x_n^1) ,
\]
\[
(2·24)
\]
we have
\[
(\Psi_{M_n,N_n}, \Psi_{M'_n,N'_n}) = \delta_{M_n,M'_n} \delta_{N_n,N'_n} (-)^{M_n} .
\]
\[
(2·25)
\]
In Eq. (2·24), the meaning of the complex conjugation is the same as that explained below Eq. (2·19).

Let us return to Eq. (2·9). Using the orthonormality (2·20), \( a^A_{k\{M,N\}} \) and \( a^{A\dagger}_{k\{M,N\}} \)
\[
\text{are expressed in terms of } \Phi \text{ as}
\]
\[
a^A_{k\{M,N\}} = (-)^{\sum_n M_n} \sum_{A'} \eta^{AA'}(U^*_{k\{M,N\}A'}, \Phi) ,
\]
\[
a^{A\dagger}_{k\{M,N\}} = (-)^{\sum_n M_n} \sum_{A'} (U^*_{k\{M,N\}A'}, \Phi) \eta^{A'A} .
\]
\[
(2·26)
\]
The original string field \( \Phi \) is Grassmann-even, and hence \( a^A_{k\{M,N\}} \) is Grassmann-even (odd) if \( \phi_A \) and hence \( U_{k\{M,N\}} \) are Grassmann-even (odd). Then using the

* The following two formulas are useful: \((\Phi_1, \Phi_2) = (\bar{\Phi}_1, \bar{\Phi}_2)^* = -(-)^{|\Phi_1|} (\bar{\Phi}_1, \bar{\Phi}_2) \) and \( \eta_{\Phi A} = \eta_{\bar{\Phi} A} \) where \(|\Phi| = 0 \) (1) if \( \Phi \) is Grassmann-even (odd). We assume that the measure \( \int Dc D\bar{c} \) is hermitian.
canonical commutation relations (2.8), we can show the (anti-)commutation relation between the creation and annihilation operators:

\[
\left[ a_k^{A\{M,N\}}, a_{k'}^{A'\{M',N'\}} \right] = a_k^{A\{M,N\}} a_{k'}^{A'\{M',N'\}} - (-)^{|A||A'|} a_k^{A'\{M',N'\}} a_{k'}^{A\{M,N\}} = \delta(k - k') \delta_{\{M,N\},\{M',N'\}} (-) \sum_{n} M_n \eta^{AA'},
\]

(2.27)

where \( \eta^{AA'} \) is the inverse of \( \eta_{AA'} \):

\[
\sum_B \eta_{AB} \eta_{BA'} = \sum_B \eta_{AB} \eta_{BA'} = \delta_{A'}^A.
\]

(2.28)

Finally, the Minkowski vacuum state \(|0\rangle_M\) is defined as usual by

\[
a_k^{A\{M,N\}} |0\rangle_M = 0.
\]

(2.29)

This is the lowest eigenstate of the Minkowski Hamiltonian \(H_M\),

\[
H_M = \int_{-\infty}^{\infty} dk \sum_{\{M,N\}} \sum_{A,B} \omega_k \cdot a_{k\{M,N\}}^A a_{k\{M,N\}}^B (-) \sum_{n} M_n \eta_{AB},
\]

(2.30)

which is the generator of the \(x^0\) translation:

\[
[H_M, \Phi] = -i \frac{\partial}{\partial x^0} \Phi.
\]

(2.31)

\section*{§3. String Rindler modes}

As a first step toward the Rindler quantization of SFT, we shall construct the string Rindler modes, namely, a complete set of the solution \(u_\sigma^{(\sigma)}\) of the string field equation in the Rindler wedge \(\sigma (\sigma = \pm)\) having the Rindler energy \(\Omega (\Omega > 0)\). Since the string field \(\Phi\) is a space-time scalar, the Rindler wave equation for \(u_\sigma^{(\sigma)}\) is given simply by

\[
Lu_\sigma^{(\sigma)} = 0,
\]

(3.1)

using the same kinetic operator \(L\) as given by (2.4) in the Minkowski quantization (c.f., Appendix A). The Rindler mode \(u_\sigma^{(\sigma)}\) is required to satisfy the following three conditions:

\begin{enumerate}
  \item \(u_\sigma^{(\sigma)}\) is the positive frequency mode with energy \(\Omega\) with respect to the Rindler time \(\eta_0\) (1.2). Namely, we have \(u_\sigma^{(\sigma)} \propto \exp (-i \sigma \Omega \eta_0)\), and hence
  \[
  \frac{\partial}{\partial \eta_0} u_\sigma^{(\sigma)} = -i \sigma \Omega u_\sigma^{(\sigma)}.
  \]
  (3.2)
  \item \(u_\sigma^{(\sigma)}\) is normalized (with respect to the Rindler inner product).
  \item \(u_\sigma^{(\pm)}\) vanishes in the \(\mp\)-wedge (Wedge condition).
\end{enumerate}
We should add a few comments on these conditions. First, as stated in §1, we take as the Rindler time for quantization the CM coordinate \(\eta_0\) given by (1·2). The reason \(\sigma\) is multiplied on the RHS of Eq. (3·2) is that we go to the past as \(\eta_0\) is increased in the negative Rindler wedge. Second, we can (formally) define the conserved inner product \((\ast, \ast)_R\) for the Rindler quantization as given by Eq. (A·13) in Appendix A, and the inner product for the condition (II) should be this one. However, we assume throughout this paper that the Rindler inner product is identical to the Minkowski inner product (2·19). This assumption is fairly reasonable, as is explained in Appendix A. Among the above three conditions, the last one (III), which we call the \textit{wedge condition} hereafter, is the most difficult and obscure condition in the SFT case, as explained in §1. Note that the wedge condition of the Rindler modes determines the way of dividing the Hilbert space of states into those of the + and − wedges.

At present we are omitting possible additional indices (quantum numbers) that the Rindler mode \(u^{(\sigma)}_{\Omega,A}\) should carry. These quantum numbers will become clear in the course of the construction of the Rindler modes.

3.1. \textit{Condition (I)}

In this paper, we shall not attempt constructing the string Rindler modes directly as functions of the string Rindler coordinate \((\zeta(\sigma), \eta(\sigma), X^\perp(\sigma))\). Rather, we shall express the Rindler modes as linear combinations of the Minkowski modes of §2. This is possible since the kinetic operator \(L\) for the Rindler wave equation (3·1) is actually the same as for the Minkowski modes. Taking as the quantum numbers for the transverse and the ghost coordinates the same \(A\) as for the Minkowski modes, let us express \(u^{(\sigma)}_{\Omega,A}\) (carrying the index \(A\)) as

\[
u^{(\sigma)}_{\Omega,A} = \int_{-\infty}^{\infty} dk \sum_{M,N} \left( \alpha^{k(\sigma)}_{\Omega}(M,N) u_k + \beta^{k(\sigma)}_{\Omega}(M,N) u_k^* \right) \Psi_{(M,N)} \cdot \phi_A , \quad (3·3)
\]

where \(U_k\) and \(\Psi_{(M,N)}\) are given by Eqs. (2·12) and (2·13), and \(\alpha^{k(\sigma)}_{\Omega}(M,N)\) and \(\beta^{k(\sigma)}_{\Omega}(M,N)\) are the coefficients to be determined below. \(u^{(\sigma)}_{\Omega,A}\) given by (3·3) obviously satisfies the wave equation (3·1), and we next impose the condition (3·2). For this purpose, we note first that

\[
\frac{\partial}{\partial \eta_0} = \int_0^\pi d\sigma \frac{\delta}{\delta \eta(\sigma)} = \int_0^\pi d\sigma \left( X^0(\sigma) \frac{\delta}{\delta X^1(\sigma)} + X^1(\sigma) \frac{\delta}{\delta X^0(\sigma)} \right) = x^0 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^0} + \sum_{n=1}^\infty \left( x_n^0 \frac{\partial}{\partial x_n^1} + x_n^1 \frac{\partial}{\partial x_n^0} \right) , \quad (3·4)
\]

and then use the following properties for \(U_k\) (2·12) and \(\Psi_{M_n,N_n}\) (2·14):

\[
\left( x^0 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^0} \right) U_k(x^0, x^1) = -\sqrt{\omega_k} \frac{\partial}{\partial k} \left( \sqrt{\omega_k} U_k(x^0, x^1) \right) , \quad (3·5)
\]

\[
\left( x_n^0 \frac{\partial}{\partial x_n^1} + x_n^1 \frac{\partial}{\partial x_n^0} \right) \Psi_{M_n,N_n}(x_n^0, x_n^1) = -\sqrt{M_n(N_n + 1)} \Psi_{M_n-1,N_n+1} - \sqrt{(M_n + 1)N_n} \Psi_{M_n+1,N_n-1} , \quad (3·6)
\]
where Eq. (3·6) is a consequence of the two formulas for the Hermite polynomials:

\[ \frac{d}{dx} H_n(x) = nH_{n-1}(x) , \]
\[ xH_n(x) = H_{n+1}(x) + nH_{n-1}(x) . \]

(3·7)

Making the \( k \)-integration by parts for the term arising from Eq. (3·5),\(^*\) the condition (3·2) is reduced to the following equation for the coefficient \( \alpha_k^{(M,N)}(\sigma) \) and exactly the same one for \( \beta_k^{(M,N)}(\sigma) \):

\[
\left( \sqrt{\omega_k} \frac{\partial}{\partial k} \sqrt{\omega_k + i\sigma \Omega} \right) \alpha_k^{(M,N)}(\sigma) = \sum_{n=1}^{\infty} \left( \sqrt{(M_n + 1)N_n} \alpha_k^{(M_n+1,N_n-1)}(\sigma) + \sqrt{M_n(N_n + 1)} \alpha_k^{(M_n-1,N_n+1)}(\sigma) \right) ,
\]

(3·8)

where the meaning of \( M \pm 1_n \) is

\[
(M \pm 1_n)_m = \begin{cases} M_n \pm 1, & (m = n) \\ M_m, & (m \neq n) \end{cases}
\]

(3·9)

In deriving Eq. (3·8) we have used the fact that \( U_k(x^0, x^1) \) depends on \( \{M, N\} \) only through the combination \( \sum_n n (M_n + N_n) \), and hence the \( U_k \) associated with the three terms of Eq. (3·8) in the original equation (3·2) are common.

To solve the differential-recursion equation (3·8), we should note that

\[
T_n \equiv M_n + N_n
\]

(3·10)

is common for the three terms of (3·8) and hence we can consider the solution with a fixed \( T_n \). Then, Eq. (3·8) is solved by the separation of variables. Namely, we assume that the dependences of \( \alpha_k^{(M,N)}(\sigma) \) on \( k \) and \( M_n \) \( (n = 1, 2, \ldots) \) are factorized,

\[
\alpha_k^{(M,N)}(\sigma) = \alpha_0(k) \prod_{n=1}^{\infty} \alpha_n(M_n) , \quad (M_n + N_n = T_n)
\]

(3·11)

where the dependence on \( (\Omega, \sigma) \) is omitted on the RHS for the sake of simplicity. Then, since \( \omega_k \) depends on \( \{M, N\} \) only through \( \{T\} \),

\[
\omega_k = \sqrt{k^2 + k^2_\perp + 2\pi \sum_n nT_n + m^2_A} ,
\]

(3·12)

\( \alpha_0 \) and \( \alpha_n \) should satisfy the following equations:

\[
\left( \sqrt{\omega_k} \frac{\partial}{\partial k} \sqrt{\omega_k + i\sigma \Omega} + \sum_{n=1}^{\infty} \lambda_n \right) \alpha_0(k) = 0 ,
\]

(3·13)

\[
\sqrt{(M_n + 1)(T_n - M_n)} \alpha_n(M_n + 1) + \sqrt{M_n(T_n - M_n + 1)} \alpha_n(M_n - 1)
\]

\[ + \lambda_n \alpha_n(M_n) = 0 ,
\]

(3·14)

\(^*\) See §3.3 for the validity of discarding the surface term.
where the constants $\lambda_n$ ($n = 1, 2, \cdots$) are to be determined as eigenvalues of the recursion equation (3.14). Our solution is now characterized by the set of quantum numbers $\{T, \lambda\}$.

The solution to the differential equation (3.13) is given by

$$\alpha_0(k) = \frac{1}{\sqrt{2\pi\omega_k}} \left( \frac{\omega_k + k}{\omega_k - k} \right)^{-\frac{(i\sigma\Omega + \sum_n \lambda_n)}{2}} ,$$

(3.15)

up to an overall constant. As for Eq. (3.14), the solution $\alpha_n(M_n)$ and the allowed values of $\lambda_n$ and $T_n$ are found from the angular momentum formula in three dimensions:

$$2\hat{J}_x |j, j_x; z\rangle = \sqrt{(j - j_x)(j + j_x + 1)} |j, j_x + 1; z\rangle + \sqrt{(j + j_x)(j - j_x + 1)} |j, j_x - 1; z\rangle ,$$

(3.16)

where the ket $|j, j_x; z\rangle$ is the eigenstate of $\hat{J}^2$ and $\hat{J}_x$ with eigenvalues $j(j+1)$ and $j_x$, respectively. In fact, by taking the inner product between Eq. (3.16) and the bra-state $\langle j, j_x; x|$, which is the eigenstate of $\hat{J}_x$ with eigenvalue $j_x$, we find the correspondence:

$$T = 2j, \quad M = j + j_x, \quad N = j - j_x,$$

(3.17)

$$\lambda = -2j_x ,$$

(3.18)

$$\alpha_{T,\lambda}(M) = \langle \frac{T}{2}, \frac{-\lambda}{2}; x | \frac{T}{2}, M - \frac{T}{2}; z \rangle ,$$

(3.19)

where we have omitted the subscript $n$ and attached the index $(T, \lambda)$ to $\alpha(M)$. Therefore the allowed values of $(T_n, \lambda_n)$ are

$$T_n = 0, 1, 2, \cdots$$

(3.20)

$$\lambda_n = T_n, T_n - 2, T_n - 4, \cdots, -T_n + 2, -T_n .$$

(3.21)

The $\alpha_{T,\lambda}(M)$ of Eq. (3.19) can be chosen to be real for all $M,$ and they are orthonormal with respect to $\lambda$ for a common $T$:

$$\sum_{M=0}^{T} \alpha_{T,\lambda}(M)\alpha_{T,\lambda'}(M) = \delta_{\lambda,\lambda'} .$$

(3.22)

Another useful formula for the $\alpha_{T,\lambda}(M)$ is

$$\alpha_{T,\lambda}(M) = (-)^M \alpha_{T,\lambda}(M) .$$

(3.23)

Recalling that $\beta^k_{\Omega}(M, N)(\sigma)$ should satisfy exactly the same equation (3.8) as $\alpha^k_{\Omega}(M, N)(\sigma)$, the string Rindler mode satisfying Eq. (3.2) and labeled by $\{T, \lambda\}$ is

$\alpha^k_{\Omega}(M = 0)$ to be real, $\alpha_{T,\lambda}(M)$ ($M \geq 1$) determined from the recursion relation (3.14) are all real.
given as

\[
    u^{(\sigma)}_{\Omega(T,\lambda)A} = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi\omega_k}} \left( \frac{\omega_k + k}{\omega_k - k} \right)^{(i\sigma\Omega + \sum_n \lambda_n)/2} 
    \times \left( \alpha^{(\sigma)}_{\Omega(T,\lambda)} U_k + \beta^{(\sigma)}_{\Omega(T,\lambda)} U_k^* \right) \Phi_{(T,\lambda)} \cdot \phi_A ,
\]

where \(\alpha^{(\sigma)}_{\Omega(T,\lambda)}\) and \(\beta^{(\sigma)}_{\Omega(T,\lambda)}\) are constants undetermined at this stage, and \(\Phi_{(T,\lambda)}\) is given by

\[
    \Phi_{(T,\lambda)} = \prod_{n=1}^{\infty} \Phi_{T_n,\lambda_n} ,
\]

\[
    \Phi_{T_n,\lambda_n} = \sum_{M_n=0}^{T_n} \alpha_{T_n,\lambda_n}(M_n)\Psi_{M_n,T_n-M_n} ,
\]

with \(\Psi_{M_n,N_n}\) of Eq. (2.14).

Looking back at the above derivation of \(u^{(\sigma)}_{\Omega(T,\lambda)A}\), we find that the following two equations hold:

\[
    \left( x^0 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^0} + i\sigma \Omega + \sum_n \lambda_n \right) u^{(\sigma)}_{\Omega(T,\lambda)A} = 0 ,
\]

\[
    \left( x_n^0 \frac{\partial}{\partial x_n^1} + x_n^1 \frac{\partial}{\partial x_n^0} - \lambda_n \right) \Phi_{T_n,\lambda_n} = 0 .
\]

In particular, since we have \(\partial/\partial \tilde{\eta} = x^0 (\partial/\partial x^1) + x^1 (\partial/\partial x^0)\) for the string Rindler time \(\tilde{\eta}\) of Eq. (1.3) in the MCF approach mentioned in §1, Eq. (3.27) implies that our \(u^{(\sigma)}_{\Omega(T,\lambda)A}\) carries the complex energy \(\Omega - i\sigma \sum_n \lambda_n\) with respect to the \(\tilde{\eta}\) time.

The Rindler modes \(u^{(\sigma)}_{\Omega(T,\lambda)A}\) (3.24) are specified by the index \(\{T, \lambda\}\). However, the quantum number \(\{T, \lambda\}\) may not be a good one when we take into account the wedge condition (condition (III)). Then we would have to consider a more general wave function,

\[
    u^{(\sigma)}_{\Omega,Z,A} = \sum_{\{T,\lambda\}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi\omega_k}} \left( \frac{\omega_k + k}{\omega_k - k} \right)^{(i\sigma\Omega + \sum_n \lambda_n)/2} 
    \times \left( \alpha^{(\sigma)}_{\Omega,Z,T_n,\lambda_n} U_k + \beta^{(\sigma)}_{\Omega,Z,T_n,\lambda_n} U_k^* \right) \Phi_{(T,\lambda)} \cdot \phi_A ,
\]

obtained by summing over \(\{T, \lambda\}\) and labeled by a new quantum number \(Z\). We shall try to construct this type of Rindler modes in §5.

### 3.2. Normalization of the Rindler modes

Next we impose the condition that the Rindler modes \(u^{(\sigma)}_{\Omega(T,\lambda)A}\) as given by (3.24) be properly normalized (condition (II)). As stated at the beginning of this section, we shall employ the Minkowski inner product, which we assume to be identical to
the Rindler inner product. For calculating the inner product between two \( u_{\Omega\{T,\lambda\}A}^{(\sigma)} \), note first that \( \Phi\{T,\lambda\} \) satisfies the normalization:

\[
\left( \Phi\{T,\lambda\}, \Phi\{T',\lambda'\} \right) = \delta_{\{T,\lambda\},\{T',-\lambda'\}} \prod_{n=1}^{\infty} \delta_{T_n, T'_n} \delta_{\lambda_n + \lambda'_n, 0} , \tag{3.30}
\]

which follows from Eqs. (2.25), (3.22), (3.23) and (3.25). The non-diagonal form of (3.30) with respect to \( \lambda_n \) originates from the fact that \( x^0_n \) is a negative norm oscillator. Then using Eqs. (2.21), (2.23) and (3.30), we find that the inner product between two \( u_{\Omega\{T,\lambda\}A}^{(\sigma)} \) is given by

\[
\left( u_{\Omega\{T,\lambda\}A}^{(\sigma)}, u_{\Omega\{T',\lambda'\}A'}^{(\sigma')} \right) = \int_{-\infty}^{\infty} \frac{dk}{2\pi \omega_k} \left( \frac{\omega_k + k}{\omega_k} \right)^{i(\sigma \Omega - \sigma' \Omega')/2 - \sum_{n}(\lambda_n + \lambda'_n)/2} \times \left( \alpha_{\Omega\{T,\lambda\}}^{(\sigma)} \alpha_{\Omega\{T',\lambda'\}}^{(\sigma')} - \beta_{\Omega\{T,\lambda\}}^{(\sigma)} \beta_{\Omega\{T',\lambda'\}}^{(\sigma')} \right) \delta_{\{T,\lambda\},\{T',-\lambda'\}} \cdot \eta_{AA'}
\]

\[
\times \left( \alpha_{\Omega\{T,\lambda\}}^{(\sigma)} \alpha_{\Omega\{T,-\lambda\}}^{(\sigma)} - \beta_{\Omega\{T,\lambda\}}^{(\sigma)} \beta_{\Omega\{T,-\lambda\}}^{(\sigma)} \right) , \tag{3.31}
\]

where the \( k \)-integration has been carried out by the change of the integration variables from \( k \) to \( y \) through \( k = \mu \sinh y \), which implies \( \omega_k = \mu \cosh y \), \( (\omega_k + k)/(\omega_k - k) = e^{2y} \), and \( dk/\omega_k = dy \):

\[
\int_{-\infty}^{\infty} \frac{dk}{2\pi \omega_k} \left( \frac{\omega_k + k}{\omega_k} \right)^{i(\sigma \Omega - \sigma' \Omega')/2} = \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{i(\sigma \Omega - \sigma' \Omega')y} = \delta (\sigma \Omega - \sigma' \Omega') . \tag{3.32}
\]

Similarly for \( (u^*, u) \), we have

\[
\left( u^{(\sigma)}_{\Omega\{T,\lambda\}A}, u^{(\sigma')}_{\Omega\{T',\lambda'\}A'} \right) = \delta_{\sigma,-\sigma'} \delta(\Omega - \Omega') \delta_{\{T,\lambda\},\{T',-\lambda'\}} \cdot (\phi_A^*, \phi_{A'})
\]

\[
\times \left( \beta_{\Omega\{T,\lambda\}}^{(\sigma)} \alpha_{\Omega\{T,-\lambda\}}^{(-\sigma)} - \alpha_{\Omega\{T,\lambda\}}^{(\sigma)} \beta_{\Omega\{T,-\lambda\}}^{(-\sigma)} \right) . \tag{3.33}
\]

Therefore, the present Rindler modes satisfy the orthonormality,

\[
\left( u_{\Omega\{T,\lambda\}A}^{(\sigma)}, u_{\Omega\{T',\lambda'\}A'}^{(\sigma')} \right) = \delta_{\sigma,\sigma'} \delta(\Omega - \Omega') \delta_{\{T,\lambda\},\{T',-\lambda'\}} \cdot \eta_{AA'} ,
\]

\[
\left( u^{(\sigma)}_{\Omega\{T,\lambda\}A}, u^{(\sigma')}_{\Omega\{T',\lambda'\}A'} \right) = -(-1)^{\lvert A \rvert} \delta_{\sigma,\sigma'} \delta(\Omega - \Omega') \delta_{\{T,\lambda\},\{T',-\lambda'\}} \cdot \eta_{A'A'} ,
\]

\[
\left( u^{(\sigma)}_{\Omega\{T,\lambda\}A}, u^{(\sigma')}_{\Omega\{T',\lambda'\}A'} \right) = (u^{(\sigma)}_{\Omega\{T,\lambda\}A}, u^{(\sigma')}_{\Omega\{T',\lambda'\}A'}) = 0 , \tag{3.34}
\]

if the following conditions hold for \( \alpha_{\Omega\{T,\lambda\}}^{(\sigma)} \) and \( \beta_{\Omega\{T,\lambda\}}^{(\sigma)} \):

\[
\alpha_{\Omega\{T,\lambda\}}^{(\sigma)} \alpha_{\Omega\{T,-\lambda\}}^{(\sigma)} - \beta_{\Omega\{T,\lambda\}}^{(\sigma)} \beta_{\Omega\{T,-\lambda\}}^{(\sigma)} = 1 , \tag{3.35}
\]

\[
\beta_{\Omega\{T,\lambda\}}^{(\sigma)} \alpha_{\Omega\{T,-\lambda\}}^{(-\sigma)} - \alpha_{\Omega\{T,\lambda\}}^{(\sigma)} \beta_{\Omega\{T,-\lambda\}}^{(-\sigma)} = 0 . \tag{3.36}
\]
3.3. Wedge condition of the Rindler modes

To complete constructing the string Rindler modes, we must impose the wedge condition (condition (III)), the vanishing of the modes in one of the Rindler wedges ±. This is, however, a difficult task since a string is an extended object, as we explained in detail in §1. In this section we fix this problem of imposing the wedge condition by taking the most naive definition of the wedges. This is to define the ± wedges according to whether the Minkowski CM coordinate \( x^\mu \) is in the + wedge or the − wedge:

\[
\left. u^{(\pm)}_{\Omega \{T, \lambda\}A} \right|_{\pm} = 0 \quad \text{if } \pm x^1 < 0 . \tag{3.37}
\]

This is the same wedge condition as in the MCF approach mentioned at the end of §1.

Before imposing the wedge condition on \( u^{(\sigma)}_{\Omega \{T, \lambda\}A} \), we must first introduce a regularization factor into (3.24) necessary to make the \( k \)-integration well-defined. The same regularization also justifies the discarding of the surface term which we did in deriving Eq. (3.8) using integration by parts. Consider the \( k \)-integration in (3.24):

\[
\int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi} \omega_k} \left( \frac{\omega_k + k}{\omega_k - k} \right)^{-(i\sigma\Omega + \sum_n \lambda_n)/2} U_k(x^0, x^1) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} dy e^{-(i\sigma\Omega + \sum_n \lambda_n)y + i\mu \xi \sinh(y - \tilde{\eta})} \nonumber
\]

\[
= \frac{1}{2\sqrt{2\pi}} e^{-(i\sigma\Omega + \sum_n \lambda_n)\tilde{\eta}} \int_{0}^{\infty} du \left( \frac{1}{u} \right)^{(i\sigma\Omega + \sum_n \lambda_n)} \exp \left\{ \frac{i}{2} \mu \xi \left( u - \frac{1}{u} \right) \right\} , \tag{3.38}
\]

where \((\tilde{\xi}, \tilde{\eta})\) is related to the Minkowski CM coordinate \((x^0, x^1)\) by

\[
x^0 = \tilde{\xi} \sinh \tilde{\eta}, \quad x^1 = \tilde{\xi} \cosh \tilde{\eta}, \tag{3.39}
\]

(this is the same as Eq. (1.3)), and the three integration variables \( k, y \) and \( u \) are related by \( k = \mu \sinh y \) and \( u = e^y \) together with the shift \( y \to y + \delta \). The integrations of (3.38) are, however, well-defined only when \( \text{Re} (i\sigma\Omega + \sum_n \lambda_n) < 1 \), which implies \( \sum_n \lambda_n = 0 \). In order to make the integration (3.38) well-defined for any \( \sum_n \lambda_n \), we multiply the integrand by the regularization factor,

\[
\exp (-\epsilon \omega_k) = \exp (-\epsilon \mu \cosh y) = \exp \left\{ -\frac{1}{2} \epsilon \mu \left( u + \frac{1}{u} \right) \right\} , \tag{3.40}
\]

and take the limit \( \epsilon \to 0 \) after the integration. Then, the integral is reduced to the modified Bessel function by the formula:

\[
Q_\nu(x) \equiv \lim_{\epsilon \to +0} \frac{1}{2} \int_{0}^{\infty} \frac{du}{u} u^{-\nu} \exp \left\{ \frac{i}{2} x \left( u - \frac{1}{u} \right) - \epsilon \left( u + \frac{1}{u} \right) \right\} .
\]

\(^{*}\) Precisely speaking, we impose the condition (3.37) only for the string configurations whose Minkowski CM coordinate is within the Rindler wedge, i.e., \( |x^1| \geq |x^0| \). Although the Minkowski CM coordinate is not always confined within the Rindler wedge for the Rindler strings which cross the origin (for example, string (B) in Fig. 1), our condition is sufficient to determine \( u^{(\pm)}_{\Omega \{T, \lambda\}A} \) as we shall see below.
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\[ e^{-i\pi v/2}K_\nu(x) \quad x > 0 \]
\[ e^{i\pi v/2}K_\nu(-x) \quad x < 0 , \]

where the modified Bessel function \( K_\nu(x) \) is defined for \( x > 0 \) and an arbitrary complex \( \nu \) by

\[ K_\nu(x) = \frac{1}{2} \int_0^\infty \frac{du}{u} u^{-\nu} \exp \left\{ -\frac{1}{2} x \left( \frac{1}{u} + \frac{1}{u} \right) \right\} \quad (x > 0, \forall \nu \in \mathbb{C}) \]

(3.42)

(see Chapter 6 of Ref. 16 for details). Owing to the regularization factor, the \( u \)-integration along the positive real axis in (3.41) can be deformed to integration along the imaginary axis: \( u \rightarrow i \text{sgn}(x)u \). Without the regularization, this contour deformation is allowed only when \( |\text{Re } \nu| < 1 \) due to the singularities at \( |u| \rightarrow 0 \) and \( \infty \). Note also that \( Q_\nu(x) \) satisfies the relation,

\[ Q_{-\nu}(x) = Q_\nu(-x) , \]

as is seen by the change of the integration variables \( u \rightarrow 1/u \).

Having completed the mathematical preparation, the Rindler wave function (3.24) with the regularization is rewritten using (3.41) as

\[ u^{(\sigma)}_{\Omega(T,\lambda)} = \frac{e^{-\Delta T}}{\sqrt{2\pi}} \left( \alpha^{(\sigma)}_{\Omega(T,\lambda)} Q_A (\mu \tilde{\xi}) + \beta^{(\sigma)}_{\Omega(T,\lambda)} Q_A (-\mu \tilde{\xi}) \right) \Phi_{\Omega(T,\lambda)} \cdot \phi_A \]

\[ \propto \left( \alpha^{(\sigma)}_{\Omega(T,\lambda)} e^{-i\pi \lambda^A \text{sgn}(\tilde{\xi})/2} + \beta^{(\sigma)}_{\Omega(T,\lambda)} e^{i\pi \lambda^A \text{sgn}(\tilde{\xi})/2} \right) K_A (\mu |\tilde{\xi}|) , \]

(3.44)

with

\[ \Lambda \equiv i \sigma \Omega + \sum_n \lambda_n . \]

(3.45)

Now we impose the wedge condition defined by the sign of \( \tilde{\xi} \): \( u^{(+)}_{\Omega(T,\lambda)} \) \( (u^{(-)}_{\Omega(T,\lambda)} \) should vanish when \( \tilde{\xi} < 0 \) \( (\tilde{\xi} > 0) \). Equation (3.44) implies that this condition is satisfied if \( \alpha^{(\sigma)}_{\Omega(T,\lambda)} \) and \( \beta^{(\sigma)}_{\Omega(T,\lambda)} \) are related by

\[ \alpha^{(\sigma)}_{\Omega(T,\lambda)} e^{i\pi \Lambda/2} + \beta^{(\sigma)}_{\Omega(T,\lambda)} e^{-i\pi \Lambda/2} = 0 , \]

(3.46)

and hence

\[ \beta^{(\sigma)}_{\Omega(T,\lambda)} = -(-)^n \sum_n \lambda_n e^{-i\pi \Omega} \alpha^{(\sigma)}_{\Omega(T,\lambda)} . \]

(3.47)

Equation (3.47) together with the normalization condition (3.35) determines \( \alpha^{(\sigma)}_{\Omega(T,\lambda)} \) and \( \beta^{(\sigma)}_{\Omega(T,\lambda)} \) as

\[ \alpha^{(\sigma)}_{\Omega(T,\lambda)} = \sqrt{N(\Omega) + 1} , \quad \beta^{(\sigma)}_{\Omega(T,\lambda)} = -(-)^n \sum_n \lambda_n \sqrt{N(\Omega)} , \]

(3.48)

where \( N(\Omega) \) is the Bose distribution function at temperature \( T = 1/2\pi k_B \) (\( k_B \) is the Boltzmann's constant):

\[ N(\Omega) = (e^{2\pi \Omega} - 1)^{-1} . \]

(3.49)
The other condition (3·36) is automatically satisfied by Eq. (3·48). \( \alpha_{\Omega(T,\lambda)}^{(\sigma)} \) and \( \beta_{\Omega(T,\lambda)}^{(\sigma)} \) as given by (3·48) are not the unique solution of (3·35); there is an arbitrariness of multiplying them by a factor \( C(\{\lambda\}) \) with the property \( C^*(\{\lambda\})C(\{-\lambda\}) = 1 \). However, this arbitrariness has no physical meaning since the whole wave function \( u_{\Omega(T,\lambda)}^{(\sigma)} \) is multiplied by \( C(\{\lambda\}) \), and hence can be absorbed into the definition of the creation/annihilation operators in the expansion of the string field \( \Phi \) in terms of the Rindler modes (c.f., Eq. (4·4)). Substituting Eq. (3·48) into (3·24), our string Rindler mode is finally given by

\[
\begin{align*}
  u_{\Omega(T,\lambda)}^{(\sigma)} &= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi \omega_k}} e^{-\omega_k} \left( \frac{\omega_k + k}{\omega_k - k} \right)^{-i(\sigma\Omega + \sum_n \lambda_n)/2} \\
  &\quad \times \left( \sqrt{N(\Omega) + 1} U_k - (\sum_n \lambda_n \sqrt{N(\Omega) U_k^*}) \Phi_{\Omega(T,\lambda)} \cdot \phi_A \right). 
\end{align*}
\]

(3·50)

3.4. Comparison with the Rindler mode in the MCF approach

We finish this section by comparing our Rindler mode \( u_{\Omega(T,\lambda)}^{(\sigma)} \) (3·50) with another Rindler mode in the MCF approach which treats \( \tilde{\eta} \) of (3·39) as the Rindler time (see §1). Let us denote the latter mode by \( v_{\Omega(M,N)}^{(\sigma)} \). Corresponding to Eq. (3·50) for \( u_{\Omega(T,\lambda)}^{(\sigma)} \), \( v_{\Omega(M,N)}^{(\sigma)} \) is given in terms of the components of the string Minkowski mode as

\[
\begin{align*}
  v_{\Omega(M,N)}^{(\sigma)} &= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi \omega_k}} \left( \frac{\omega_k + k}{\omega_k - k} \right)^{-i\sigma/2} \\
  &\quad \times \left( \sqrt{N(\tilde{\eta}) + 1} U_k - \sqrt{N(\tilde{\eta}) U_k^*} \right) \Phi_{\Omega(M,N)} \cdot \phi_A. 
\end{align*}
\]

(3·51)

In addition to the requirement that \( v_{\Omega(M,N)}^{(\sigma)} \) satisfy the string wave equation \( L v_{\Omega(M,N)}^{(\sigma)} = 0 \), \( v_{\Omega(M,N)}^{(\sigma)} \) has been constructed by imposing the following three conditions. First it carries the Rindler energy \( \Omega \) with respect to the \( \tilde{\eta} \) time:

\[
\frac{\partial}{\partial \tilde{\eta}} v_{\Omega(M,N)}^{(\sigma)} = -i\sigma \Omega v_{\Omega(M,N)}^{(\sigma)}. 
\]

(3·52)

Second, it satisfies the normalization condition:

\[
\begin{align*}
  \left( v_{\Omega(M,N)}^{(\sigma)} , v_{\Omega(M',N')}^{(\sigma')} \right) &= \delta_{\sigma,\sigma'} \delta (\Omega - \Omega') \delta_{M,N,M',N'} (-)^{\sum_n M_n \eta_{AA'}}, \\
  \left( v_{\Omega(M,N)}^{(\sigma)} , v_{\Omega(M,N)}^{(\sigma')} \right)^* &= -(-)^{|A| \delta_{\sigma,\sigma'} \delta (\Omega - \Omega') \delta_{M,N,M',N'} (-)^{\sum_n M_n \eta_{AA'}}}, \\
  \left( v_{\Omega(M,N)}^{(\sigma)} , v_{\Omega(M,N)}^{(\sigma')} \right)^* &= 0. 
\end{align*}
\]

(3·53)

Finally, \( v_{\Omega(M,N)}^{(\sigma)} \) is subject to the wedge condition specified by the Minkowski CM coordinate \( x^\mu \):

\[
v_{\Omega(M,N),A}^{(\sigma)} = 0 \quad \text{if} \quad \pm x^1 < 0. 
\]

(3·54)
The wedge condition (3·54) is the same as (3·37) which we imposed on $u^{(\sigma)}_{\Omega(T,\lambda)A}$ in the last subsection.

Then, let us consider the inner product between $u^{(\sigma)}_{\Omega(T,\lambda)A}$ and the present $v^{(\sigma)}_{\Omega(M,N)A}$. However, this inner product is ill-defined:

$$
\langle v^{(\sigma)}_{\Omega(M,N)A}, u^{(\sigma)}_{\Omega(T,\lambda)A} \rangle = \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega_k} \frac{i(\sigma'\Omega' - \sigma\Omega)/2 - \sum_n \lambda_n/2}{(\omega_k + k)/(\omega_k - k)}
\times \left( \sqrt{(N(\Omega') + 1)(N(\Omega) + 1)} - (-)^{\sum_n \lambda_n} \sqrt{N(\Omega')N(\Omega)} \right)
\times \prod_n \alpha_{T_n,-\lambda_n(M_n)} \delta_{M_n + N_n,T_n} \cdot \eta_{A'A} .
$$

This quantity is divergent at either $k = \infty$ or $-\infty$ unless $\sum_n \lambda_n = 0$. Therefore, we conclude that the Rindler quantization of SFT using $u^{(\sigma)}_{\Omega(T,\lambda)A}$ and that using $v^{(\sigma)}_{\Omega(M,N)A}$ are not equivalent.

§4. Rindler quantization of SFT and string thermalization

In the previous section we have constructed the string Rindler mode $u^{(\sigma)}_{\Omega(T,\lambda)A}$, Eq. (3·50). Though this Rindler mode is not fully satisfactory since the wedge condition is specified by the Minkowski $\epsilon^M$ coordinate, we shall in this section carry out the Rindler quantization of free SFT using this Rindler mode and then study the string thermalization.

4.1. Rindler quantization of free SFT

The Rindler quantization starts by defining the canonical momentum $\Pi_R$ conjugate to $\Phi$ and then imposing the canonical commutation relation. The action of free SFT expressed in terms of the Rindler string coordinate reads (c.f., Eqs. (A·5), (A·11) and (A·12) in Appendix A),

$$
S = \frac{\pi}{2} \int \mathcal{D}\xi(\sigma) \mathcal{D}\eta(\sigma) \mathcal{D}X(\sigma) \mathcal{D}(\text{ghosts}) \prod_{\sigma'} |\xi(\sigma')| 
\times \int_{0}^{\pi} d\sigma \left\{ \frac{1}{\xi(\sigma)^2} \left( \frac{\delta \Phi}{\delta \eta(\sigma)} \right)^2 - \left( \frac{\delta \Phi}{\delta \xi(\sigma)} \right)^2 + \left[ (\xi(\sigma))^2 (\eta'(\sigma))^2 - (\xi'(\sigma))^2 \right] \Phi^2 
+ (\text{transverse and ghost coordinates part}) \right\} .
$$

Taking $\eta_0$ of Eq. (1·2) as the time variable for quantization, the canonical momentum $\Pi_R$ is given by

$$
\Pi_R = \frac{\delta S}{\delta (\partial_\Phi/\partial \eta_0)} = \prod_{\sigma'} |\xi(\sigma')| \int_{0}^{\pi} d\sigma \frac{1}{\xi(\sigma)^2} \frac{\delta \Phi}{\delta \eta_0(\sigma)} .
$$
Then we impose the equal-\(\eta_0\) commutation relation
\[
\left[\Phi[\eta, \xi, X, c, \bar{c}], \Pi_R[\bar{\eta}, \bar{\xi}, \bar{X}, \bar{c}, \bar{\bar{c}}]\right]_{\eta_0 = \bar{\eta}_0} = i \prod_{\sigma} \delta(\eta(\sigma) - \bar{\eta}(\sigma)) \delta(\xi(\sigma) - \bar{\xi}(\sigma)) \delta(X, c, \bar{c} \text{-part}),
\]
where the hat on \(\hat{\delta}(\eta(\sigma) - \bar{\eta}(\sigma))\) implies that the delta function for the time variable, \(\delta(\eta_0 - \bar{\eta}_0)\), is omitted.

Corresponding to Eq. (2.9) in the Minkowski quantization, let us expand the string field \(\Phi\) in terms of the string Rindler modes \(u_{\Omega(T, \lambda)}^{(\sigma)}\) and its complex conjugate:
\[
\Phi = \sum_{\sigma = \pm} \int_0^\infty d\Omega \sum_{\{T, \lambda\}} \left( u_{\Omega(T, \lambda)}^{(\sigma)} A b_{\Omega(T, \lambda)}^{(\sigma) A} + b_{\Omega(T, \lambda)}^{(\sigma) A \dagger} u_{\Omega(T, \lambda)}^{(\sigma) *} \right).
\]
The orthonormality of the Rindler modes (3.34) allows us to express \(b\) and \(b^\dagger\) in terms of \(\Phi\):
\[
b_{\Omega(T, \lambda)}^{(\sigma) A} = \sum_B \eta^{AB} \left( u_{\Omega(T, -\lambda)}^{(\sigma)} A, \Phi \right),
b_{\Omega(T, \lambda)}^{(\sigma) A \dagger} = -\sum_B \left( u_{\Omega(T, -\lambda)}^{(\sigma) *} A, \Phi \right) \eta^{BA}.
\]
The (anti-)commutation relations between \(b\) and \(b^\dagger\) are obtained by identifying the inner product in Eq. (4.5) as the Rindler one (A.13), and using the commutation relation (4.3). Then we find
\[
\left[ b_{\Omega(T, \lambda)}^{(\sigma) A}, b_{\Omega(T', \lambda')}^{(\sigma') A \dagger} \right] = \sum_{B, B'} \eta^{AB} \left( u_{\Omega(T, -\lambda)}^{(\sigma)} A B', \Phi \right) \eta^{B' A'}
\]
\[
= \delta_{\sigma, \sigma'} \delta(\Omega - \Omega') \delta_{\{T, \lambda\}, \{T', -\lambda'\}} \eta^{AA'},
\]
and
\[
\left[ b_{\Omega(T, \lambda)}^{(\sigma) A}, b_{\Omega(T', \lambda')}^{(\sigma') A \dagger} \right] = \left[ b_{\Omega(T, \lambda)}^{(\sigma) A \dagger}, b_{\Omega(T', \lambda')}^{(\sigma') A \dagger} \right] = 0.
\]
The Rindler Hamiltonian which is the generator of the \(\eta_0\) translation and hence satisfies
\[
[H_R, \Phi] = -i \frac{\partial}{\partial \eta_0} \Phi,
\]
is given as
\[
H_R = H_R^{(+)} - H_R^{(-)},
\]
where \(H_R^{(\pm)}\) is the Hamiltonian in the \(\pm\)-wedges:
\[
H_R^{(\sigma)} = \int_0^\infty d\Omega \sum_{\{T, \lambda\}} b_{\Omega(T, \lambda)}^{(\sigma) A \dagger} \eta_{AB} b_{\Omega(T, -\lambda)}^{(\sigma) B}.
\]
The Rindler vacuum state $|0\rangle_R$ is defined as the state annihilated by both $b^{(+)}$ and $b^{(-)}$:

$$b^{(\pm)A}_{\Omega(T,\lambda)}|0\rangle_R = 0 .$$

(4-11)

This is the ground state of the Hamiltonians $H^{(\pm)}_R$. In Appendix C, we show that the Rindler vacuum defined by Eq. (4-11) is a BRST invariant physical state.

4.2. Bogoliubov coefficients and string thermalization

Our task in this subsection is first to obtain the relationship between the Minkowski creation/annihilation operators $\{a, a^\dagger\}$ and the Rindler ones $\{b, b^\dagger\}$. The relationship between these two sets of creation/annihilation operators is obtained by equating the two expressions of the string field $\Phi$, Eqs. (2-9) and (4-4), and using the orthonormality of the string modes, either Eqs. (2-26) or (4-5). Therefore, the Rindler annihilation operator expressed in terms of the Minkowski set $\{a, a^\dagger\}$ is given as

$$b^{(\sigma)A}_{\Omega(T,\lambda)} = \sum_{\{M,N\}} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega_k} \left( \frac{\omega_k + k}{\omega_k - k} \right)^{(i\sigma\Omega + \sum_n \lambda_n)/2} \prod_n \alpha_{T_n,\lambda_n}(M_n)$$

$$\times \left( \sqrt{N(\Omega)} + 1 \right)^A_{k(M,N)} + (-)^{\sum_n \lambda_n} \sqrt{N(\Omega)} \sum_{B,C} \eta^{AB}(\phi_B, \phi_C^*)(-)^{|C|} a^{C^\dagger}_{k(M,N)} ,$$

(4-12)

where we have used the formulas

$$\left( u^{(\sigma)A}_{\Omega(T,\lambda)} A, U^{A}_{k(M,N)} B \right) = \sqrt{\frac{N(\Omega) + 1}{2\pi\omega_k}} \left( \frac{\omega_k + k}{\omega_k - k} \right)^{(i\sigma\Omega - \sum_n \lambda_n)/2} \prod_n \alpha_{T_n,\lambda_n}(M_n) \cdot \eta_{AB} ,$$

(4-13)

$$\left( u^{(\sigma)A}_{\Omega(T,\lambda)} A, U^{B^\dagger}_{k(M,N)} B \right) = (-)^{\sum_n \lambda_n} \sqrt{\frac{N(\Omega)}{2\pi\omega_k}} \left( \frac{\omega_k + k}{\omega_k - k} \right)^{(i\sigma\Omega - \sum_n \lambda_n)/2} \prod_n \alpha_{T_n,\lambda_n}(M_n) \cdot (\phi_A, \phi_B^*) ,$$

(4-14)

obtainable from Eqs. (3-50), (3-25), (2-25) and (3-23). As in particle field theory,\(^8,9\) it is convenient to define another set of Minkowski annihilation operators $d^{(\sigma)A}_{\Omega(T,\lambda)}$,

$$d^{(\sigma)A}_{\Omega(T,\lambda)} = \sum_{\{M,N\}} \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega_k} \left( \frac{\omega_k + k}{\omega_k - k} \right)^{(i\sigma\Omega + \sum_n \lambda_n)/2} \prod_n \alpha_{T_n,\lambda_n}(M_n) \cdot a^{A}_{k(M,N)} ,$$

(4-15)

which annihilate the Minkowski vacuum,

$$d^{(\sigma)A}_{\Omega(T,\lambda)} |0\rangle_M = 0 .$$

(4-16)
and satisfy the (anti-)commutation relations,
\[
\left\{ d^{(σ)}_{\Omega(T, λ)}, d^{(σ')\dagger}_{\Omega'(T', λ')} \right\} = δ_{σ, σ'} δ(Ω - Ω') δ_{\{T, λ\}, \{T', λ'\}} \cdot η^{AA'},
\]
\[
\left\{ d^{(σ)}_{\Omega(T, λ)}, d^{(σ')\dagger}_{\Omega'(T', λ')} \right\} = \left\{ d^{(σ')\dagger}_{\Omega(T, λ)}, d^{(σ)}_{\Omega'(T', λ')} \right\} = 0.
\] (4.17)

Then, the Rindler operators \{b, b^{\dagger}\} expressed in terms of the new Minkowski operators \{d, d^{\dagger}\} are given by
\[
b^{(σ)}_{\Omega(T, λ)} = \sqrt{N(Ω)} + 1 d^{(σ)}_{\Omega(T, λ)} + (-)^{n} \sum_{B, C} η^{AB} (φ_B, φ_C^*)(-)^{|C|} d^{(σ)C}_{\Omega(T, λ)},
\]
\[
b^{(σ)\dagger}_{\Omega(T, λ)} = \sqrt{N(Ω)} + 1 d^{(σ)\dagger}_{\Omega(T, λ)} + (-)^{n} \sum_{B, C} η^{AB} (φ_C^*, φ_B)(-)^{|C|} d^{(-σ)C}_{\Omega(T, λ)}. \] (4.18)

Therefore, the Minkowski vacuum \(|0\rangle_M\) is formally expressed in terms of the Rindler vacuum \(|0\rangle_R\) defined by (4.11) and the Rindler creation operators \(b^{(±)\dagger}_{\Omega(T, λ)}\) as
\[
|0\rangle_M \propto \exp \left( \int_{0}^{∞} dΩ \sum_{\{T, λ\}, A, B} e^{-πΩ} (-)^{n} η^{AB}(φ_A, φ_B^*)(-)^{|B|} b^{(±)\dagger}_{\Omega(T, λ)} \right) |0\rangle_R,
\] (4.19)
up to the normalization.

As an application of the Bogoliubov transformation relation (4.18), let us consider the Minkowski vacuum expectation value of the Rindler Hamiltonian \(H_R^{(+)}\) in the + wedge, namely, \(\langle M|H_R^{(+)}|0\rangle_M\). Only the last terms of Eq. (4.18) contribute to this expectation value, and we obtain
\[
\langle M|H_R^{(+)}|0\rangle_M = \sum_{\{T, λ\}, A} (-)^{|A|} δ_A^A \cdot δ(Ω - Ω) \int_{0}^{∞} dΩ Ω N(Ω),
\] (4.20)
where in the course of the calculation we have used the formulas
\[
φ = \sum_{A, B} φ_A η^{AB} (φ_B, φ) \quad \text{for} \quad φ, \quad (4.21)
\]
\[
φ = \sum_{A, B} φ_A^* (-)^{|A|} η^{BA} (φ_B^*, φ) \quad \text{for} \quad φ, \quad (4.22)
\]
\[
(φ_A^*, φ_B^*) = (-)^{|A|} η^{BA} \quad (4.23)
\]

*) More generally, we can show the thermalization formula from Eq. (4.19). For an operator \(O^{(+)}\) consisting of \(b^{(+)}_{\Omega(T, λ)}\) and \(b^{(+)\dagger}_{\Omega(T, λ)}\) in the + wedge, we have
\[
\langle M|O^{(+)}|0\rangle_M = \text{Tr} (\langle O^{(+)} e^{-2πH_R^{(+)} + iπN_{gh}} \rangle) / \text{Tr} (e^{-2πH_R^{(+)} + iπN_{gh}}),
\]
where \(\text{Tr}\) denotes the trace operation over the complete set of states associated to the + wedge, and \(N_{gh}\) is the ghost number operator, which is an even (odd) integer for a Grassmann-even (odd) state. This is the standard statistical mechanics formula for a gauge theory with temperature \(T = 1/2πk_B\). See Ref. 17) for the factor \(e^{iπN_{gh}}\).
Equations (4·21) and (4·22) are the completeness relations of the transverse wave functions \{\phi_A\} and \{\phi'_A\}.

The last part \(\delta(\Omega = 0) \int_0^\infty d\Omega N(\Omega)\) of Eq. (4·20) is the same as \(\mathcal{M}\langle 0|H_{R^+}|0\rangle_{\mathcal{M}}\) calculated in the free scalar particle field theory,\(^8\),\(^9\) and shows the standard Bose distribution at temperature \(T = 1/2\pi k_B\). The interpretation of the other part on the RHS of Eq. (4·20) is as follows. First, since the index \(A\) specifies the transverse momentum \(k_\perp\) as well as the discrete level numbers, \(\delta^A\) for a fixed \(A\) is given as

\[
\delta^A = \delta^{24}(k_\perp - k'_\perp) = V_\perp/(2\pi)^{24},
\]

with \(V_\perp \equiv \int d^24x\) being the transverse volume. Then the summation of the sign factor \((-)^{1A}\) over \(\{T,\lambda\}\) and the transverse index \(A\) (except \(k_\perp\)) counts the number of physical component fields contained in the free string field \(\Phi\) in the Minkowski quantization. Therefore, we have

\[
\sum_{\{T,\lambda\}} \sum_A (-)^{1A} \delta^A = (\# \text{ physical component fields}) \times V_\perp \int \frac{d^24k_\perp}{(2\pi)^{24}}, \tag{4·24}
\]

and hence (4·20) is equal to \(\mathcal{M}\langle 0|H_{R^+}|0\rangle_{\mathcal{M}}\) in the free scalar field theory multiplied by the number of the physical component fields in \(\Phi\). If we consider the same expectation value \(\mathcal{M}\langle 0|H_{R^+}|0\rangle_{\mathcal{M}}\) in the MCF approach mentioned at the end of §1, we obtain the same result (4·20) with \(\sum_{\{T,\lambda\}}\) replaced by \(\sum_{\{M,N\}}\).

The (ultraviolet) divergent factor \(\delta(\Omega - \Omega)\) in Eq. (4·20), which also appears in particle field theory, is due to the continuum nature of the Rindler energy \(\Omega\). To regularize this factor in the present SFT case, we need to introduce something like the horizon regularization of Refs. 10) and 2). The physical difference between the Rindler quantization of this paper and the MCF approach may be exposed by such detailed analysis. In relation to this, it was argued by several authors that there is a limiting acceleration and hence is a natural horizon cutoff for the Rindler string due to the Hagedorn transition.\(^18\) Since these arguments are based on the MCF approach, it is interesting to reexamine their conclusion in our present formulation. These are the subjects of our future investigation.

§5. Rindler modes with different wedge conditions

The wedge condition (3·37) of the Rindler mode \(u^{(\sigma)}_{\Omega,T,\lambda} A\) (3·50) which we constructed in §3.3 was specified by the Minkowski CM coordinate \(x^\mu\). In this section we present an approach to the construction of the string Rindler mode which is subject to different wedge conditions. Although our consideration here is still incomplete, we hope that the analysis in this section will give a hint for a thorough understanding of the Rindler quantization of SFT.

The Rindler mode we construct here is given in the form of Eq. (3·29), namely, by summing over the \(\{T,\lambda\}\) quantum numbers with a suitable weight. Using the function \(Q_\nu(x)\) defined by Eq. (3·41), \(u^{(\sigma)}_{\Omega,Z,A}\) (3·29) is expressed as

\[
u^{(\sigma)}_{\Omega,Z,A} = \frac{1}{\sqrt{2\pi}} \sum_{\{T,\lambda\}} e^{-\Delta'_{\Omega,Z,A}(T,\lambda)} Q_\nu(\mu \xi) \Phi_{\{T,\lambda\}} \cdot \phi_A,
\]

(5·1)
with $\Lambda$ of Eq. (3-45). We carry out the summation over $\{T, \lambda\}$ in (5-1) by assuming a certain $\{T, \lambda\}$ dependence of the coefficients $\alpha_{\Omega, \pm T, \lambda}$ and $\beta_{\Omega, \pm T, \lambda}$. For this purpose we need a number of mathematical preliminaries. The first is the following expansion formula for $Q_\nu(x)^*$:

$$Q_\nu(\sqrt{1 + ax}) = (1 + a)^{\nu/2} \sum_{\ell=0}^{\infty} \frac{(-)^{\ell}}{\ell!} \left( \frac{i ax}{2} \right)^\ell Q_{\nu+\ell}(x),$$

$$= (1 + a)^{-\nu/2} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( \frac{i ax}{2} \right)^\ell Q_{\nu-\ell}(x),$$

(5-2)

for a real $a$. Expressing $\mu\xi$ as

$$\mu_{\xi} = \left( 1 + \frac{2\pi}{\mu_\perp} \sum_n n T_n \right)^{1/2} \mu_\perp \xi,$$

(5-3)

with $\mu_\perp = \sqrt{k^2 + m_A^2}$, we can apply Eq. (5-2) to extract the $\{T\}$ dependence of $Q_{\pm \Lambda}(\mu_{\xi})$:

$$\left( \begin{array}{c} Q_{\Lambda}(\mu_{\xi}) \\ Q_{-\Lambda}(\mu_{\xi}) \end{array} \right) = \left( 1 + \frac{2\pi}{\mu_\perp} \sum_n n T_n \right)^{-\Lambda/2}$$

$$\times \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( \frac{i \pi}{\mu_\perp} \sum_n n T_n \right)^\ell \left( \begin{array}{c} Q_{\Lambda-\ell}(\mu_\perp \xi) \\ (-)^\ell Q_{-\Lambda+\ell}(\mu_\perp \xi) \end{array} \right).$$

(5-4)

The second formula we need is the expression of $\Phi_{\{T, \lambda\}} = \prod_n \Phi_{T_n, \lambda_n}$ in terms of the coordinate $(\hat{\xi}_n, \tilde{\eta}_n)$ defined by

$$x_n^{\mu=0} = \hat{\xi}_n \sinh \tilde{\eta}_n, \quad x_n^{\mu=1} = \hat{\xi}_n \cosh \tilde{\eta}_n.$$  

(5-5)

The desired expression of $\Phi_{T_n, \lambda_n}$ is given by Eq. (B-17) in Appendix B using the Laguerre polynomial. For our present purpose, observe that

$$\sqrt{\frac{(T - |\lambda|)}{2}}! \frac{y^{|\lambda|} L_{\frac{1}{2}(T + |\lambda|)}(y^2)}{(T + |\lambda|)!} \times \sum_{p=0}^{N(T, \lambda)} (-)^N(T, \lambda - p) \left[ p! \left( \frac{T + \lambda}{2} - p \right)! \left( \frac{T - \lambda}{2} - p \right)! \right]^{-1} y^{T - 2p},$$

(5-6)

*) This formula can be understood from the same expansion formula for the Bessel functions (see Chapter 5.22 of Ref. 16) and Eq. (3-41) which relates $Q_\nu(x)$ to $K_\nu(\pm x)$. Although $a$ is restricted to lie in the range $|a| < 1$, we employ Eq. (5-2) beyond this restriction since the purpose of this section is merely to give a hint for a more complete analysis of the wedge condition of the string Rindler modes.
where \( N(T, \lambda) \) is defined by

\[
N(T, \lambda) = \frac{1}{2} (T - |\lambda|) ,
\]

(5.7)

and the summation variable \( p \) is related to \( r \) in Eq. (B.18) by \( p = N(T, \lambda) - r \).

Taking into account Eqs. (5.4) and (5.6), let us assume the following form for the coefficients \( \alpha^{(\sigma)}_{\Omega, Z, \{T, \lambda\}} \) and \( \beta^{(\sigma)}_{\Omega, Z, \{T, -\lambda\}} \) in Eq. (5.1):

\[
\begin{align*}
\alpha^{(\sigma)}_{\Omega, Z, \{T, \lambda\}} &= \sqrt{2\pi} \left( 1 + \frac{2\pi}{\mu^2} \sum \frac{nT_n}{n} \right)^{-\lambda/2} \\
\beta^{(\sigma)}_{\Omega, Z, \{T, -\lambda\}} &= \prod_n (-)^n \left( \frac{T_n + \lambda_n}{2} \right) \left( \frac{T_n - \lambda_n}{2} \right) \right]^{-1/2} C_n^T D_n^\lambda \left( \alpha^{(\sigma)}_{\Omega} \right) ,
\end{align*}
\]

(5.8)

where \( C_n, D_n, \alpha^{(\sigma)}_{\Omega} \) and \( \beta^{(\sigma)}_{\Omega} \) will be determined later. Note the minus sign of the index \( \lambda \) for \( \beta^{(\sigma)}_{\Omega, Z, \{T, -\lambda\}} \) on the LHS. Most of the factors on the RHS of Eq. (5.8) are introduced to cancel the front factors on the RHSs of Eqs. (5.4) and (5.6). Our assumption in Eq. (5.8) besides these factors is that the \( \{T, \lambda\} \) dependence is factorized and is given simply by \( \prod_n C_n^T D_n^\lambda \). Surprisingly, for this particular form of \( \alpha^{(\sigma)}_{\Omega, Z, \{T, \lambda\}} \) and \( \beta^{(\sigma)}_{\Omega, Z, \{T, -\lambda\}} \), we can explicitly carry out the summations in Eq. (5.1). Namely, we substitute Eqs. (5.4), (B.17), (5.6) and (5.8) into \( u^{(\sigma)}_{\Omega, Z, A} \) of Eq. (5.1) and carry out the summations over \( \{T, \lambda\} \), \( \ell \) in Eq. (5.4), and \( p_n \) for the Laguerre polynomial in Eq. (5.6). After a tedious but straightforward calculation which is summarized in Appendix D, we obtain

\[
\begin{align*}
u^{(\sigma)}_{\Omega, Z, A} (\xi, \eta, \xi_n, \eta_n) &= \exp \left( -i \sigma T \Omega - \sum \frac{n \pi}{4} \xi_n^2 \right) \\
\times \frac{1}{2} \int_0^\infty \frac{du}{u} \left\{ \frac{i}{2} \mu u \left( \frac{1}{u} \right) \right\} \\
\times \left\{ \alpha^{(\sigma)}_{\Omega} u^{-i\sigma T} \prod_n \exp \left[ \left( \frac{u}{D_n e^{\Omega - \eta}} + \frac{D_n e^{\Omega - \eta}}{u} \right) C_n \sqrt{\frac{n \pi}{2}} \xi_n e^{\frac{ix_n}{\mu} u} - C_n^2 e^{\frac{2ix_n}{\mu} u} \right] \\
+ \beta^{(\sigma)}_{\Omega} u^{-i\sigma T} \prod_n \exp \left[ \left( \frac{u}{D_n e^{\Omega - \eta}} + \frac{D_n e^{\Omega - \eta}}{u} \right) C_n \sqrt{\frac{n \pi}{2}} \xi_n e^{\frac{ix_n}{\mu} u} - C_n^2 e^{\frac{2ix_n}{\mu} u} \right] \right\} ,
\end{align*}
\]

(5.9)

where the regularization necessary to make the \( u \)-integration well-defined will be taken into account later, and we have omitted the transverse wave function \( \phi_A \) for simplicity.

We have not yet succeeded in giving the full analysis of the wave function \( u^{(\sigma)}_{\Omega, Z, A} \) of Eq. (5.9) for a general string configuration specified by \( (\xi, \eta, \xi_n, \eta_n) \). Here we shall content ourselves with the analysis for a simple string configuration satisfying

\[
\hat{\eta}_n = \eta ,
\]

(5.10)
for all \( n = 1, 2, \ldots \). This implies that we are considering the string configuration which does not fluctuate in the \( \eta \) direction:

\[
\eta(\sigma) = \tilde{\eta} = \sigma\text{-independent},
\]

\[
\xi(\sigma) = \hat{\xi} + \sum_{n=1}^{\infty} \xi_n \cos n\sigma\quad .\tag{5.11}
\]

For such a string configuration, let us adopt the following \( C_n \) and \( D_n \):

\[
C_n = \frac{\mu_1}{\sqrt{2\pi n}} \cos n\theta, \quad D_n = -i .\tag{5.12}
\]

Then, making the change of integration variables \( u \to 1/u \) for the \( \beta^{(\sigma)}_\Omega \) term, our Rindler mode (5.9), which we denote as \( u^{(\sigma)}_{\Omega, \theta, A} \) with the index \( \theta \), is reduced to

\[
u^{(\sigma)}_{\Omega, \theta, A}(\xi, \tilde{\eta}, \tilde{\xi}_n, \tilde{\eta}_n = \tilde{\eta})
= \left\{ \alpha^{(\sigma)}_\Omega R(\xi, \tilde{\xi}_n, \theta) + \beta^{(\sigma)}_\Omega R(-\hat{\xi}, -\tilde{\xi}_n, \theta) \right\} e^{-i\sigma\Omega\tilde{\eta} - \sum_n (n\pi/4)(\xi_n)^2} ,\tag{5.13}
\]

using \( R(\xi, \tilde{\xi}_n, \theta) \) defined by

\[
R(\xi, \tilde{\xi}_n, \theta) = \frac{1}{2} \int_{0}^{\infty} \frac{du}{u} u^{-i\sigma\Omega}
\times \exp \left\{ \frac{i}{2} \mu_1 \left[ \hat{\xi} + \sum_{n} \xi_n \cos n\theta e^{i\mu_1 u} \right] \left( u - \frac{1}{u} \right) \right\} F(\hat{\xi}, \theta; u) ,\tag{5.14}
\]

with

\[
F(\hat{\xi}, \theta; u) = \left[ 1 - e^{2i\theta \mu_1 u} \right]^2 \left( 1 - e^{2i\theta + 2i\xi \mu_1 u} \right) \left( 1 - e^{-2i\theta + 2i\xi \mu_1 u} \right) \mu_1^{1/8\pi} .\tag{5.15}
\]

To make the \( u \)-integration of Eq. (5.14) well-defined for any \((\xi, \tilde{\xi}_n)\), we multiply its integrand by \( \exp \{-\epsilon (u + (1/u))\} \) (which corresponds to the original regularization (3.40) for \( Q_\nu(x) \)), and in addition replace \( \exp\{(i\pi n\xi/\mu_1)u\} \) in Eqs. (5.14) and (5.15) by the regularized ones, \( \exp\{(i\pi n\xi/\mu_1) - \epsilon\} \) \((n = 2 \text{ for Eq. (5.15)})\), and take the limit \( \epsilon \to +0 \) after the integration.

Then, as we did for the function \( Q_\nu(x) \) in §3.3, let us consider the \textquotedblleft Wick rotation\textquotedblright\, of the \( u \)-integration of \( R(\xi, \tilde{\xi}_n, \theta) \), namely, the deformation of the \( u \)-integration contour from the positive real axis to the imaginary one: \( u \to \pm iu \). In the present case, the direction of the \textquotedblleft Wick rotation\textquotedblright\, depends on the signs of \( \xi(\theta) = \xi + \sum_n \xi_n \cos n\theta \): the behavior of the integrand around \( u = \infty \) requires the rotation \( u \to i \text{sgn}(\xi)u \) \((u > 0)\), while the behavior around \( u = 0 \) requires \( u \to i \text{sgn}(\xi(\theta))u \). Therefore, defining the condition \((r, s) \text{ (} r = \pm, s = \pm)\) concerning the signature of \( \xi \) and \( \xi(\theta) \) by

\[
(r, s) : \text{sgn}(\xi) = r \quad \text{and} \quad \text{sgn}(\xi(\theta)) = s ,\tag{5.16}
\]

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we see that a consistent “Wick rotation” is possible only for $(+, +)$ and $(-, -)$. Corresponding to the cases $(\pm, \pm)$, we have

$$u^{(\sigma)}_{\Omega, \theta, A} \propto (\alpha^{(\sigma)}_\Omega e^{\pm i \pi \sigma \Omega/2} + \beta^{(\sigma)}_\Omega e^{\mp i \pi \sigma \Omega/2}) R_E(\pm \tilde{\xi}, \pm \tilde{\xi}_n, \theta),$$

where $R_E$ is defined by

$$R_E(\tilde{\xi}, \tilde{\xi}_n, \theta) = \frac{1}{2} \int_0^\infty \frac{du}{u} u^{-i \sigma \Omega}$$

$$\times \exp \left\{ -\frac{1}{2} \mu_\perp \left[ \tilde{\xi} + \sum_n \tilde{\xi}_n \cos n \theta e^{-\frac{\pi n \xi}{\mu_\perp}} \left( u + \frac{1}{u} \right) \right] \right\} F_E(\xi, \theta; u),$$

with

$$F_E(\xi, \theta; u) = \left[ \left( 1 - e^{-\frac{2i \xi}{\mu_\perp}} u \right)^2 \left( 1 - e^{2i \theta - \frac{2i \xi}{\mu_\perp}} u \right) \left( 1 - e^{-2i \theta - \frac{2i \xi}{\mu_\perp}} u \right) \right]^{1/8\pi}.$$ 

Therefore, if $\alpha^{(\sigma)}_\Omega$ and $\beta^{(\sigma)}_\Omega$ are related by

$$\beta^{(\sigma)}_\Omega = -e^{-\pi \sigma} \alpha^{(\sigma)}_\Omega,$$

the present $u^{(+)}_{\Omega, \theta, A}$ ($u^{(-)}_{\Omega, \theta, A}$) satisfies the wedge condition that it vanishes when both the CM coordinate $\xi$ and the string point $\xi(\theta)$ at a particular parameter value are negative (positive). Our $u^{(\pm)}_{\Omega, \theta, A}$ does not vanish when $\xi$ and $\xi(\theta)$ are in different wedges.

The analysis in this section shows that, by summing over $\{T, \lambda\}$, we can construct string Rindler modes with wedge conditions different from that for $u^{(\sigma)}_{\Omega(T, \lambda), A}$ of §3. Although it is not clear whether the present wedge condition itself has any physical significance, our result is expected to give a clue to the construction of the string Rindler modes with more natural wedge conditions. As regards the analysis in this section, there are a number of points to be clarified. First, we have restricted our consideration only to the string configurations with $\eta^{(\sigma)} = \sigma$-independent. For a general string configurations with a non-trivial $\eta^{(\sigma)}$, our Rindler mode $u^{(\sigma)}_{\Omega, \theta, A}$ does not seem to obey a simple wedge condition. It is also an open question whether we can construct a complete basis of Rindler modes containing the present $u^{(\sigma)}_{\Omega, \theta, A}$.

§6. Remaining problems

In this paper we have learned many things about the quantization of SFT in the Rindler space-time. However, there remain a number of problems to be clarified.

* The inner product of our $u^{(\sigma)}_{\Omega, \theta, A}$ is proportional to

$$\left( u^{(\sigma)}_{\Omega, \theta, A}, u^{(\sigma')}_{\Omega', \theta', A'} \right) \propto \delta_{\sigma, \sigma'} \delta (\Omega - \Omega') \eta_{AA'} | \cos \theta - \cos \theta' |^{1/2\pi}.$$
before reaching a complete understanding. In the following we shall summarize these problems, although some of them were already mentioned in the text.

The first problem is concerned with the wedge condition of the string Rindler mode, or the division of the Hilbert space of states. In §§3 and 4, we adopted the wedge condition (3·37) specified by the Minkowski CM coordinate \( x^\mu \). This choice, however, is mainly due to the technical reason that our string Rindler mode is constructed using the components of the Minkowski modes. Another wedge condition specified by the Rindler CM coordinate \( \xi_0 \) should also be considered seriously. Furthermore, as stated in §1, this problem of the wedge condition of the Rindler mode is related to the causality problem in SFT, which has been recently investigated in the Minkowski space-time using the light-cone gauge SFT.\(^{12)}\) - \(^{14)}\) The causality relation in SFT on the Rindler space-time needs to be studied in detail. In particular, it is crucial to understand the causality structure for the string which crosses the origin (i.e., the string (B) depicted in Fig. 1) for solving this problem. For this purpose, the techniques used in constructing the Rindler modes with an interesting wedge condition in §5 may be helpful. In any case, it is an open question to determine the causality structure of SFT in the Rindler space-time and at the same time to find the most natural wedge condition for the Rindler modes.

The second problem we must clarify is to present the physical observable quantities in the Rindler quantization of SFT. In §4.2, we calculated the Minkowski vacuum expectation value of the Rindler Hamiltonian, Eq. (4·20). As stated there, we need a more detailed analysis by introducing the regularization for \( \delta(\Omega = 0) \). In Ref. 2), it was argued that the black hole entropy per unit area is finite to all orders in superstring perturbation theory, though it is divergent in the case of a free scalar field. It is interesting to examine this observation from the viewpoint of SFT which we have presented here. In addition, it would be necessary to study whether we could devise a "detector" in string theory as in particle field theory.\(^{8)}\), \(^{9)}\)

As our third problem, we should examine the inner product for the string modes more precisely. In this paper, we have assumed that the Rindler inner product (A·13) is identical to the Minkowski inner product (2·19), and this is crucial for our analysis in this paper. Although the equality of the two inner products holds if certain surface terms vanish as is explained in Appendix A, mathematically detailed examination is necessary.

**Appendix A**

---

**Inner Product for String Wave Functions**

---

In this appendix we shall explain the conserved inner product for the string wave functions. Let us consider SFT (2·2) described by a string coordinate \( Y^\mu(\sigma) \) related to the Minkowski coordinate \( X^\mu(\sigma) \) by the coordinate transformation:

\[
X^\mu(\sigma) = f^\mu(Y(\sigma)) . \tag{A·1}
\]

The SFT action rewritten in terms of the new string coordinate \( Y^\mu(\sigma) \) reads
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\[ S = -\frac{1}{2} \int D\mathcal{Y}^\mu(\sigma) \mathcal{D}\text{(ghosts)} \prod_{\sigma'} \sqrt{-G(Y(\sigma'))} \Phi L \Phi, \quad (A.2) \]

with \( L \) reexpressed using \( Y^\mu \) as

\[ L = \pi \int_0^\pi d\sigma \left\{ -\frac{1}{\sqrt{-G}} \frac{\delta}{\delta Y^\mu} \left( \sqrt{-G} G_{\mu\nu}^{\alpha\beta} \frac{\delta}{\delta Y^\nu} \right) + G_{\mu\nu} Y^\mu Y^\nu + (\text{ghost part}) \right\}. \quad (A.3) \]

In Eqs. (A.2) and (A.3), the metric \( G_{\mu\nu}(Y(\sigma)) \) is given by

\[ G_{\mu\nu}(Y(\sigma)) = \eta_{\alpha\beta} \left. \frac{\partial f^\alpha(y)}{\partial y^\mu} \frac{\partial f^\beta(y)}{\partial y^\nu} \right|_{y=Y(\sigma)}, \quad (A.4) \]

and \( G \equiv \det G_{\mu\nu} \). Functional integration by parts gives another expression of \( S \),

\[ S = -\frac{\pi}{2} \int D\mathcal{Y}^\mu(\sigma) \mathcal{D}\text{(ghosts)} \prod_{\sigma'} \sqrt{-G(Y(\sigma'))} \]

\[ \times \int_0^\pi d\sigma \left\{ G_{\mu\nu}(Y(\sigma)) \frac{\delta \Phi}{\delta Y^\mu(\sigma)} \frac{\delta \Phi}{\delta Y^\nu(\sigma)} + G_{\mu\nu}(Y(\sigma)) Y^\mu(\sigma) Y^\nu(\sigma) \Phi^2 \right. \]

\[ + (\text{ghost part}) \right\}. \quad (A.5) \]

We assume that a mathematically awkward expression,

\[ \prod_{\sigma} \sqrt{-G(\sigma)} = \exp \left\{ \delta(\sigma = 0) \int_0^\pi d\sigma \ln \sqrt{-G(\sigma)} \right\}, \quad (A.6) \]

is defined by means of a suitable regularization, which allows formal and naive functional manipulations.

Let \( \Phi_1 \) and \( \Phi_2 \) be string wave functions satisfying the equation of motion \( L \Phi_{1,2} = 0 \). The current \( J^\mu_{12} \) defined by

\[ J^\mu_{12}(Y;\sigma) = G^{\mu\nu}(Y(\sigma)) \Phi_1^* \delta_{\nu} \Phi_2, \quad (A.7) \]

is conserved in the following sense:

\[ \int_0^\pi d\sigma \frac{1}{\sqrt{-G(\sigma)}} \frac{\delta}{\delta Y^\mu(\sigma)} \left( \sqrt{-G(\sigma)} J^\mu_{12}(Y;\sigma) \right) = 0, \quad (A.8) \]

with \( G(\sigma) \) representing \( G(Y(\sigma)) \). Integrating Eq. (A.8) over some (functional) region \( \mathcal{V} \), we obtain

\[ 0 = \int_\mathcal{V} D\mathcal{Y} \mathcal{D}\text{(ghost)} \prod_{\sigma'} \sqrt{-G(\sigma')} \times \text{Eq. (A.8)} \]

\[ = \int_\mathcal{V} D\mathcal{Y} \mathcal{D}\text{(ghost)} \int_0^\pi d\sigma \frac{\delta}{\delta Y^\nu(\sigma)} \left( \prod_{\sigma'} \sqrt{-G(\sigma')} \right) \left( J_{12}^\nu(Y;\sigma) \right), \quad (A.9) \]
where the easiest way to understand the last equality is to regard $\prod_{\sigma'}$ as a discrete product.

In the case of the Rindler coordinate,

\[
Y^\mu(\sigma) = \left( \eta(\sigma), \xi(\sigma), X^\perp(\sigma) \right), \tag{A.10}
\]

we have

\[
G_{\eta\eta} = -(\xi(\sigma))^2, \quad G_{\xi\xi} = 1, \quad G_{\perp\perp} = \text{diag}(1, \cdots, 1), \quad \text{others} = 0, \tag{A.11}
\]

and

\[
\sqrt{-G} \equiv \sqrt{-\det G_{\mu\nu}} = |\xi(\sigma)|. \tag{A.12}
\]

Taking as $\mathcal{V}$ in Eq. (A.9) the region bounded by two different values of $\eta_0$, and assuming that the surface terms vanish except for the one for the zero-mode $\eta_0$, we see that the Rindler inner product defined by

\[
(\Phi_1, \Phi_2)_R = i \int \frac{d\eta_0 D\xi D\xi Dx^\perp D(\text{ghost})}{d\eta_0} \prod_{\sigma'} |\xi(\sigma')| \int_0^\pi d\sigma \frac{1}{\xi(\sigma)^2} \Phi^*_1 \frac{\delta}{\delta \eta(\sigma)} \Phi_2 \bigg|_{\eta_0=\text{const.}} \tag{A.13}
\]

is independent of the choice of $\eta_0$. On the other hand, taking as $Y^\mu(\sigma)$ the original Minkowski string coordinate $X^\mu(\sigma)$, Eq. (A.9) leads to the conserved Minkowski inner product (2.19). Finally, by considering in Eq. (A.9) the region $\mathcal{V}$ bounded on the one hand by $\eta_0 = \text{const}$ and on the other by $x^0 = \text{const}$ and assuming again that the other surface terms vanish, it follows that the Minkowski inner product (2.19) and the Rindler inner product (A.13) are identical.

---

**Appendix B**

--- Two Other Expressions of $\Phi_{T_n,\lambda_n}$ ---

In §3, the oscillator part wave function $\Phi_{T_n,\lambda_n}$ (3.26) of the Rindler mode $u^{(T,\lambda)}_{n,R}$ was given as a linear combination of the Minkowski oscillator mode $\Psi_{M_n,N_n}$ (2.14). In this appendix, we present two other expressions of $\Phi_{T_n,\lambda_n}$ which are equivalent to Eq. (3.26).

In the following we use the coordinates $(\hat{\xi}_n, \hat{\eta}_n)$ $(n \geq 1)$ of Eq. (5.5), or equivalently,

\[
x_n^\pm \equiv x_n^1 \pm x_n^0 = \hat{\xi}_n \exp(\pm \hat{\eta}_n). \tag{B.1}
\]

Note that

\[
\frac{\partial}{\partial \hat{\eta}_n} = x_n^+ \frac{\partial}{\partial x_n^+} - x_n^- \frac{\partial}{\partial x_n^-} = x_n^0 \frac{\partial}{\partial x_n^0} + x_n^1 \frac{\partial}{\partial x_n^1}. \tag{B.2}
\]

**Fock space expression**

The first expression is based on the Fock space representation of the oscillator modes. Let us define the creation/annihilation operators $\alpha_n^\mu (n = \pm 1, \pm 2, \cdots)$ in
terms of $x_n^\mu$ and $\partial/\partial x_n^\mu$ of (2·5) and (2·6) as

$$\alpha_n^\mu = -\frac{i}{\sqrt{\pi}} \left( \frac{n\pi}{2} x_n^\mu + \eta^{\mu\nu} \frac{\partial}{\partial x_n^\nu} \right), \tag{B·3}$$

which satisfies the commutation relation,

$$[\alpha_n^\mu, \alpha_m^\nu] = n\delta_{n+m,0} \eta^{\mu\nu}. \tag{B·4}$$

The $x_n^\mu$-oscillator part of the (mass)$^2$ operator in Eq. (2·4) is given in terms of $\alpha_n^\mu$ as

$$(\text{mass})^2_{n,\mu} \equiv 2\eta^{\mu\nu} \left\{ -\frac{\partial}{\partial x_n^\mu} \frac{\partial}{\partial x_n^\nu} + \left( \frac{n\pi}{2} \right)^2 x_n^\mu x_n^\nu \right\} = 2\pi \left( \eta^{\mu\nu} \alpha_n^\mu \alpha_n^\nu + \frac{n}{2} \right). \tag{B·5}$$

Then the light-cone component $\alpha_n^\pm$,

$$\alpha_n^\pm \equiv \frac{1}{\sqrt{2}} \left( \alpha_n^1 \pm \alpha_n^0 \right) = -i \sqrt{\frac{2}{\pi}} \left( \frac{n\pi}{4} x_n^\pm + \frac{\partial}{\partial x_n^\pm} \right), \tag{B·6}$$

satisfies the commutation relations,

$$[\alpha_n^\pm, \alpha_m^\pm] = n\delta_{n+m,0}, \quad [\alpha_n^\pm, \alpha_m^\pm] = 0. \tag{B·7}$$

We also need the commutation relation between $\partial/\partial \eta_n$ and $\alpha_n^\pm$:

$$\left[ \frac{\partial}{\partial \eta_n}, \alpha_n^\pm \right] = \pm \delta_{n,m} \alpha_n^\pm. \tag{B·8}$$

Now recall that the wave function $\Phi_{T_n,\lambda_n}$ is characterized by the two conditions:

$$\frac{\partial}{\partial \eta_n} \Phi_{T_n,\lambda_n} = \lambda_n \Phi_{T_n,\lambda_n}, \tag{B·9}$$

$$\hat{T}_n \Phi_{T_n,\lambda_n} = T_n \Phi_{T_n,\lambda_n}, \tag{B·10}$$

where the first equation (B·9) is nothing but Eq. (3·28), and $\hat{T}_n$ is the level number operator of the $n$-th mode of the $\mu = 0$ and 1 oscillators:

$$n\hat{T}_n = \alpha_{-n}^1 \alpha_n^1 - \alpha_n^0 \alpha_n^0 = \alpha_n^+ \alpha_n^- + \alpha_n^- \alpha_n^+. \tag{B·11}$$

The creation operators $\alpha_n^\pm (n \geq 1)$ shift $\lambda_n$ by $\pm 1$ (see Eq. (B·8)), and they both raise $T_n$ by one; namely, we have

$$\alpha_n^\pm \Phi_{T_n,\lambda_n} \propto \Phi_{T_n+1,\lambda_n\pm 1}. \quad (n \geq 1) \tag{B·12}$$

The wave function $\Phi_{T_n,\lambda_n}$ with the normalization condition

$$(\Phi_{T_n,\lambda_n}, \Phi_{T_n',\lambda_n'}) = \delta_{T_n,T_n'} \delta_{\lambda_n,\lambda_n'}, \tag{B·13}$$

is therefore given as

$$\Phi_{T_n,\lambda_n} = \frac{1}{\sqrt{N_+!N_-!n^{N_+}N_-}} \left( \alpha_{-n}^+ \right)^{N_+} \left( \alpha_n^- \right)^{N_-} \Phi_{0,0}, \tag{B·14}$$
where $\Phi_{0,0}$ is the Fock space vacuum satisfying $\alpha_n^+ \Phi_{0,0} = 0$ ($n \geq 1$), and $N_\pm$ is given by

$$N_\pm = \frac{1}{2} (T_n \pm \lambda_n) . \quad (B\cdot15)$$

In this construction, it is evident that $\lambda_n$ should lie in the range of (3·21) since $N_\pm$ are non-negative integers.

Expression by the coordinate $(\xi_n, \eta_n)$

Next we shall express $\Phi_{T_n, \lambda_n}$ as a function of the variables $(\xi_n, \eta_n)$ of Eq. (5·5). The operator $T_n$ in (B·10) is expressed in terms of $(\xi_n, \eta_n)$ as

$$n\pi (\xi_n^2 + 1) = -\left( \frac{\partial}{\partial \xi_n} \right)^2 - \frac{1}{2} \frac{\partial}{\partial \xi_n} \frac{\partial}{\partial \xi_n} + \frac{1}{2} \frac{\partial}{\partial \eta_n} \frac{\partial}{\partial \eta_n} + \left( \frac{n\pi}{2} \right)^2 \xi_n^2 , \quad (B\cdot16)$$

where $1 = (1/2) \times 2$ on the LHS is the zero-point energy of the two oscillators. The condition (B·10) is now reduced to the differential equation for the variable $\xi_n$ (the $\eta_n$ dependence is fixed by another condition (B·9) to be $\Phi_{T_n, \lambda_n} \propto \exp (\lambda_n \eta_n)$). This differential equation has a solution regular at $\xi_n = 0$ only when $T_n$ is a non-negative integer and $\lambda_n$ is in the range of (3·21). For such $(T_n, \lambda_n)$, the normalized solution is given by

$$\Phi_{T_n, \lambda_n} (\xi_n, \eta_n) = \sqrt{\frac{n!}{2^n (T_n + |\lambda_n|)!}} \left( \frac{n\pi}{2} \xi_n^2 |\lambda_n|/2 \right) \frac{\lambda_n}{(T_n + |\lambda_n|)} \exp \left( -\frac{n\pi}{4} \xi_n^2 + \lambda_n \eta_n \right) , \quad (B\cdot17)$$

where $L_m^{(\alpha)}(x)$ is the Laguerre polynomial:

$$L_m^{(\alpha)}(x) = \sum_{r=0}^m (-)^r \binom{m+\alpha}{m-r} \frac{x^r}{r!} . \quad (B\cdot18)$$

Namely, $\Phi_{T_n, \lambda_n} \exp \left( \frac{n\pi}{4} \xi_n^2 - \lambda_n \eta_n \right)$ is a polynomial of $\xi_n$ of the form,

$$\xi_n^{T_n} + \xi_n^{T_n-2} + \cdots + \xi_n^{|\lambda_n|+2} + \xi_n^{|\lambda_n|} , \quad (B\cdot19)$$

up to the coefficient of each term. Corresponding to the fact that the inner product integration over $x_n^0$ should be carried out in the pure-imaginary direction (see Eq. (2·19)), the inner product integration over the present variables $(\xi_n, \eta_n)$ should be defined by regarding $\xi_n$ as the radius variable and $i\eta_n$ as the (real) angle variable. Therefore, $\Phi_{T_n, \lambda_n}$ is normalized in the following sense:

$$\int_0^\infty \xi_n d\xi_n \int_0^{2\pi} d(i\eta_n) \Phi_{T_n, \lambda_n}^* (\xi_n, \eta_n) \Phi_{T_n, \lambda_n} (\xi_n, \eta_n) = \delta_{T_n, T_n} \delta_{\lambda_n, \lambda_n} = 0 , \quad (B\cdot20)$$
where the complex conjugation of $\hat{\Phi}_{\alpha}(\xi_n, \bar{\eta}_m)$ should be done by regarding $\bar{\eta}_m$ as a real variable. The integration of (B.20) can be carried out by using the formula for the Laguerre polynomial:

$$
\int_0^\infty dx e^{-x} x^n \Phi_m^{(\alpha)}(x) L_n^{(\alpha)}(x) = \frac{\Gamma(\alpha + m + 1)}{m!} \delta_{m,n} .
$$

(B.21)

**Appendix C**

**BRST Invariance of the Rindler Vacuum**

In this appendix we show that the Rindler vacuum state $|0\rangle_R$ defined by Eq. (4.11) is BRST invariant and hence is a physical state. Namely, we show that

$$
Q_B |0\rangle_R = 0 ,
$$

(C.1)

where $Q_B$ is the BRST operator of the free SFT generating the BRST transformation $\delta_B$:

$$
[i Q_B, \Phi] = \delta_B \Phi = \tilde{Q}_B \Phi .
$$

(C.2)

In Eq. (C.2), $\tilde{Q}_B$ is the BRST charge in the first quantized string theory with terms containing the ghost zero-modes $\bar{c}_0$ and $c_0 = \partial / \partial \bar{c}_0$ omitted. $Q_B$ and $\tilde{Q}_B$ should not be confused: the former is the operator in the second quantized string theory (SFT), while the latter is that in the first quantized string theory.

To show the BRST invariance of $|0\rangle_R$, it is sufficient to show that

$$
\left( u^{(\sigma)}_{\Omega(T,\lambda)B}, \tilde{Q}_B u^{(\sigma)^*}_{\Omega(T',\lambda')A'} \right) = \left( u^{(\sigma)}_{\Omega(T,\lambda)B}, \tilde{Q}_B u^{(\sigma)^*}_{\Omega(T',\lambda')A'} \right) = 0 .
$$

(C.4)

To clarify the reason why Eq. (C.4) is sufficient, note first that

$$
\delta_B u^{(\sigma)A}_{\Omega(T,\lambda)} = (-)^{|A|} \sum_B \eta^{AB} \left( u^{(\sigma)}_{\Omega(T,\lambda)B}, \tilde{Q}_B \Phi \right)
$$

$$
= (-)^{|A|} \sum_B \eta^{AB} \int d\Omega' \sum_{\sigma'} \left\{ \left( u^{(\sigma)}_{\Omega(T,\lambda)B}, \tilde{Q}_B u^{(\sigma')}_{\Omega(T',\lambda')A'} \right) b^{(\sigma')A'}_{\Omega(T',\lambda')} \right. 
$$

$$
\left. + (-)^{|A'|} \left( u^{(\sigma)}_{\Omega(T,\lambda)B}, \tilde{Q}_B u^{(\sigma')*}_{\Omega(T',\lambda')A'} \right) b^{(\sigma')A'^*}_{\Omega(T',\lambda')} \right\} ,
$$

(C.5)

and a similar equation for $\delta_B u^{(\sigma)A'^*}_{\Omega(T,\lambda)}$. Therefore, if Eq. (C.4) is satisfied, $Q_B$ is expressed in terms of the Rindler creation/annihilation operators as

$$
Q_B = i \sum_{\sigma, \sigma'} \int d\Omega \int d\Omega' \sum_{\{T, \lambda\}} \sum_{\{T', \lambda'\}} b^{(\sigma)A'^*}_{\Omega(T,\lambda)} \left( u^{(\sigma)}_{\Omega(T,\lambda)A}, \tilde{Q}_B u^{(\sigma')}_{\Omega(T',\lambda')A'} \right) b^{(\sigma')A'}_{\Omega(T',\lambda')} .
$$

(C.6)
which implies Eq. (C·1). There is no ordering ambiguity in (C·6) since $b^\dagger$ and $b$ there have opposite statistics.

In the following we show the vanishing of the second term of Eq. (C·4) (the first term is in fact equal to the second term due to the hermiticity of $Q_B$). For this purpose we separate $\tilde{Q}_B$ (C·3) into three parts:

$$\tilde{Q}_B = \tilde{Q}_\perp + \tilde{Q}_{ck\alpha} + \tilde{Q}_{ca\alpha}, \quad \text{(C·7)}$$

where $\tilde{Q}_\perp$ is the part of $\tilde{Q}_B$ which contains no $\alpha^\mu_n$ with $\mu = 0, 1$, and the other two terms are defined by

$$\tilde{Q}_{ck\alpha} = -\frac{1}{2} \sum_{n\neq 0} c_{-n} (\alpha^+_n k^- + \alpha^-_n k^+) \quad \text{(C·8)}$$
$$\tilde{Q}_{ca\alpha} = -\sum_{n,m} c_{-n} \alpha^+_n \alpha^-_m \quad \text{(C·9)}$$

using the light-cone components of Appendix B. Note that the summation $\sum_{n,m}$ in Eq. (C·9) is subject to the constraint $nm(n-m) \neq 0$. In Eq. (C·8) we have replaced $\alpha^\mu_0$ by $k^\pm$. This is valid when $\tilde{Q}_{ck\alpha}$ acts on $U_k$ (2·12).

Recalling the derivation of (3·33), it is evident that the condition (C·4) is satisfied for the $\tilde{Q}_\perp$ part since the effect of the insertion of $\tilde{Q}_\perp$ is only to change $(\phi^*_A, \phi_A')$ in (3·33) into $(\phi^*_A, \tilde{Q}_\perp \phi_A')$.

Next, let us consider the contribution of the $\tilde{Q}_{ck\alpha}$ term (C·8). Owing to Eq. (B·12), the inner product of (C·4) vanishes unless

$$\sum_n (\lambda_n + \lambda'_n) = \mp 1 \quad \text{(C·10)}$$

with $\mp$ corresponding to the $\alpha^\pm_n k^\mp$ term of $\tilde{Q}_{ck\alpha}$. In addition, note that under the change of variables $k = \mu \sinh y$ we have

$$k^\pm = \frac{1}{\sqrt{2}} (k \pm \omega_k) = \pm \frac{\mu}{\sqrt{2}} e^{\pm y}. \quad \text{(C·11)}$$

Therefore, the $k$-integration relevant to the inner product of (C·4) with $\tilde{Q}_{ck\alpha}$ is proportional to (c.f., Eq. (3·32))

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi \omega_k} k^\mp (\omega_k + k)^{-i(\sigma \Omega + \sigma' \Omega')/2 \pm 1/2}$$
$$\propto \int_{-\infty}^{\infty} dy e^{i(\sigma \Omega + \sigma' \Omega')y} = \delta (\sigma \Omega + \sigma' \Omega') = \delta_{\sigma_- \sigma_+} \delta (\Omega - \Omega'). \quad \text{(C·12)}$$

Namely, the contributions of $k^\pm$ and the extra exponent $-\sum_n (\lambda_n + \lambda'_n)/2 = \pm 1/2$ of $(\omega_k + k)/(\omega_k - k)$ just cancel to give the same integration as without $Q_{ck\alpha}$. Since the $k$-integration implies $\Omega = \Omega'$, the present matrix element of $\tilde{Q}_{ck\alpha}$ is proportional to

$$\beta^{(\sigma)}_{\Omega[T,\lambda]} \alpha^{(-\sigma)}_{\Omega[T',\lambda']} + \alpha^{(\sigma)}_{\Omega[T,\lambda]} \beta^{(-\sigma)}_{\Omega[T',\lambda']}. \quad \text{(C·13)}$$
Note that the two terms of Eq. (C·13) have relatively the same sign contrary to the case of Eq. (3·33): this is due to the fact that $\alpha^{\pm}_{\Omega(T,\lambda)} = -k^{\pm}U_k$ while $\alpha^{0}_{\Omega(T,\lambda)} = k^{\mu}U_k$.

Using the form of $\alpha^{(s)}_{\Omega(T,\lambda)}$ and $\beta^{(s)}_{\Omega(T,\lambda)}$ given by Eq. (3·48) and the constraint (C·10), we see that the contribution of the $Q_{c\alpha\alpha}$ term to (C·4) vanishes.

Finally, let us consider the contribution from $\bar{Q}_{c\alpha\alpha}$ (C·9). Since we have $\alpha^{\pm}_{n-m} \alpha^{0}_{\Omega(T,\lambda)} \propto \Phi_{\Omega(T,\lambda)}$ with $\sum_n \lambda_n = \sum_n \lambda_n$, the $k$-integration is again reduced to $\delta_{\sigma,-\sigma}\delta(\Omega' - \Omega)$. In this case the matrix element is proportional to $\beta^{(s)}_{\Omega(T,\lambda)} \alpha^{(-s)}_{\Omega(T',\lambda')} - \alpha^{(s)}_{\Omega(T,\lambda)} \beta^{(s)}_{\Omega(T',\lambda')}$, which is also seen to vanish. This completes the proof of the BRST invariance of the Rindler vacuum.

Quite similarly but much more easily, we see that the BRST operator $Q_B$ is expressed in terms of the Minkowski creation/annihilation operator as

$$Q_B = i \int dk \int dk' \sum_{\{M,N\},A} \sum_{\{M',N'\},A'} a^A_{k\{M,N\}A} \left( U_{k\{M,N\}A}, \bar{Q}_B U_{k'\{M',N'\}A'} \right) a^A_{k'\{M',N'\},A'},$$

which implies the BRST invariance of the Minkowski vacuum,

$$Q_B |0\rangle_M = 0.$$  \hfill (C·15)

### Appendix D

Derivation of Eq. (5·9)

In this appendix, we summarize the derivation of Eq. (5·9). Here we must substitute Eqs. (5·4), (B·17), (5·6) and (5·8) into $u^{(s)}_{\Omega(T,\lambda)}$ of Eq. (5·1) and carry out the summations over $(T_n, \lambda_n)$, $\ell$ in Eq. (5·4), and $p_n$ for the Laguerre polynomial in Eq. (5·6). For this purpose it is convenient to make a change of summation variables from $(T_n, \lambda_n, p_n)$ to $(L_n, m_n, p_n)$ defined by

$$L_n = T_n - 2p_n, \quad m_n = \frac{1}{2} (T_n - \lambda_n) - p_n = \frac{1}{2} (L_n - \lambda_n).$$  \hfill (D·1)

The range of the new summation variables $(T_n, \lambda_n, p_n)$ is

$$0 \leq m_n \leq L_n, \quad p_n \geq 0.$$  \hfill (D·2)

Then using $y_n \equiv \sqrt{n\pi/2\xi_n}$ instead of $\xi_n$, we have

$$\exp \left( i \sigma \Omega \tilde{\eta} + \sum_n \frac{n\pi}{4} \xi_n^2 \right) \cdot u^{(s)}_{\Omega(Z,A)} \left( \xi, \tilde{\eta}, \tilde{\xi}, \tilde{\eta} \right) = \prod_{n=0}^{\infty} \sum_{\lambda_n} \sum_{p_n=0}^{T_n} \left( \frac{(-)^{p_n}}{T_n - 2p_n} \right) \frac{T_n - 2p_n}{(T_n - \lambda_n)/2 - p_n} y_n^{T_n - 2p_n} \left( D_ne^{\tilde{\eta} - \tilde{\eta}} \right)^{\lambda_n} C_n^{T_n} \times \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left( \frac{\mu}{\mu_\perp} \right)^{\ell} \left( \alpha^{(s)}_{\Omega} Q_{i\sigma \Omega + \sum_n \lambda_n - \ell} \left( \mu_\perp \tilde{\xi} \right) + (-)^{\ell} \beta^{(s)}_{\Omega} Q_{-i\sigma \Omega + \sum_n \lambda_n + \ell} \left( \mu_\perp \tilde{\xi} \right) \right)$$

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\begin{align*}
\frac{1}{2} \int_{0}^{\infty} \frac{du}{u} \exp \left\{ i \mu_\perp \hat{\xi} \left( u - \frac{1}{u} \right) \right\} \\
\times \prod_{n} \frac{L_n}{N_n} \sum_{m_n=0}^{L_n} \sum_{p_n} \frac{(-)^{p_n}}{p_n! L_n!} \left( \frac{L_n}{m_n} \right) y_{n}^{L_n} \left( \frac{D_n e^{\hat{\eta} \Delta \eta}}{u} \right)^{L_n-2m_n} C_n^{L_n+2p_n} \\
\times \sum_{l=0}^{\infty} \frac{1}{l!} \left( \frac{i \pi}{\mu_\perp} \right) n (L_n + 2p_n) \hat{\xi} \left( a^{(\sigma)}_n u^{-i \sigma \Omega + l} + (-)^l b^{(\sigma)}_n u^{i \sigma \Omega - l} \right) \\
= \frac{1}{2} \int_{0}^{\infty} \frac{du}{u} \exp \left\{ i \mu_\perp \hat{\xi} \left( u - \frac{1}{u} \right) \right\} \\
\times \prod_{n} \frac{1}{L_n!} \left[ \left( \frac{u}{D_n e^{\hat{\eta} \Delta \eta}} + \frac{D_n e^{\hat{\eta} \Delta \eta}}{u} \right) C_n y_n \right]^{L_n} \sum_{p_n=0}^{\infty} (-)^{p_n} \frac{C_n^{2p_n}}{p_n!} \\
\times \left\{ a^{(\sigma)}_n u^{-i \sigma \Omega} \exp \left( n (L_n + 2p_n) \frac{i \pi \hat{\xi}}{\mu_\perp} u \right) + b^{(\sigma)}_n u^{i \sigma \Omega} \exp \left( -n (L_n + 2p_n) \frac{i \pi \hat{\xi}}{\mu_\perp} u \right) \right\} \\
= \frac{1}{2} \int_{0}^{\infty} \frac{du}{u} \exp \left\{ i \mu_\perp \hat{\xi} \left( u - \frac{1}{u} \right) \right\} \\
\times \left\{ a^{(\sigma)}_n u^{-i \sigma \Omega} \prod_{n} \exp \left[ \left( \frac{u}{D_n e^{\hat{\eta} \Delta \eta}} + \frac{D_n e^{\hat{\eta} \Delta \eta}}{u} \right) C_n y_n e^{\frac{i \pi \hat{\xi}}{\mu_\perp} u} - C_n^2 e^{2 \frac{i \pi \hat{\xi}}{\mu_\perp} u} \right] \right. \\
+ b^{(\sigma)}_n u^{i \sigma \Omega} \prod_{n} \exp \left[ \left( \frac{u}{D_n e^{\hat{\eta} \Delta \eta}} + \frac{D_n e^{\hat{\eta} \Delta \eta}}{u} \right) C_n y_n e^{-\frac{i \pi \hat{\xi}}{\mu_\perp} u} - C_n^2 e^{-2 \frac{i \pi \hat{\xi}}{\mu_\perp} u} \right] \right\}, 
\end{align*}
\]

(D-3)

where the summation over \( m_n \) was carried out using the binomial theorem, and we removed the regularization factor for \( Q_\nu(x) \) (3-41).

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