Some questions related to fractals

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Classes of fairly smooth functions

A fundamental aspect of harmonic analysis on Euclidean spaces deals with various classes of real or complex-valued functions, such as the $C^k$ classes of functions which are continuous and continuously differentiable up to order $k$, where $k$ is a nonnegative integer. If $k = 0$, this simply means that the function is continuous. Of course continuity of a function makes sense on any metric space, or on topological spaces more generally, while the notion of derivatives entails more structure. Compare with [43].

We can be more precise and consider $C^{k,\alpha}$ classes of functions, where $k$ is a nonnegative integer and $\alpha$ is a real number, $0 \leq \alpha \leq 1$. If $\alpha = 0$, then $C^{k,\alpha}$ is taken to mean the same as $C^k$; when $\alpha > 0$, $C^{k,\alpha}$ consists of the $C^k$ functions with the extra property that their $k$th order derivatives are locally Hölder continuous of order $\alpha$. For this let us recall that a function $h(x)$ on a metric space $(M, d(x,y))$ is Hölder continuous of order $\alpha$ if there is a nonnegative real number $A$ such that

$$|h(x) - h(y)| \leq A d(x,y)^\alpha \quad \text{for all } x, y \in M$$

(1)

This makes sense on any metric space, so that $C^{0,\alpha}$ classes of functions make sense on any metric space, but $C^{k,\alpha}$ involves more structure when $k \geq 1$.

The condition (1) also makes sense for $\alpha > 1$, but on a Euclidean space it would then imply that the function is constant, because the first derivatives would be identically equal to 0. One can also check this fact more directly. On a general metric space, this is not true in general. However, it is true under fairly mild conditions, e.g., if the metric space is connected and there are enough rectifiable curves around, because such a function would have to be constant on each one of them. Cantor sets and snowflake curves are basic examples of metric spaces in which there are a lot of functions which satisfy (1) with $\alpha > 1$.  

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Another smoothness class for functions on $\mathbb{R}^n$ is given by the Zygmund condition

\begin{equation}
|f(x+v) + f(x-v) - 2f(x)| \leq L|v|
\end{equation}

for all $x, v \in \mathbb{R}^n$,

where $f$ is a continuous function on $\mathbb{R}^n$ and $L$ is a nonnegative real number. This condition is implied by (1) when $\alpha = 1$, with $L = 2A$, but the converse does not work, even locally. It can be shown that if a continuous function $f$ satisfies (2), then it is locally Hölder continuous of order $\alpha$ for each $\alpha < 1$.

Note that the Zygmund condition is not defined on general metric spaces. Here is a nice reformulation of it, although not exactly with the same constant, in terms of affine functions on $\mathbb{R}^n$. A function $f$ on $\mathbb{R}^n$ satisfies the Zygmund condition for some $L$ if and only if there is a nonnegative real number $L'$ such that for every ball $B$ in $\mathbb{R}^n$ there is an affine function $a(x)$ such that

\begin{equation}
\sup_{x \in B} |f(x) - a(x)| \leq L' \text{radius}(B).
\end{equation}

It is very easy to go from this to (2) with $L = L'$, but the other direction is more complicated.

If we replace $|v|$, radius$(B)$ on the right sides of (2), (3) with $|v|^{1+\alpha}$, radius$(B)^{1+\alpha}$, respectively, where $0 < \alpha \leq 1$, then the corresponding conditions can be shown to be equivalent to $f$ being continuously differentiable with first derivatives Hölder continuous of order $\alpha$. To go from Hölder continuous first derivatives to the other conditions is basically a matter of calculus, but the other direction is again more complicated. If one replaces $|v|$, radius$(B)$ on the right sides of (2), (3) with $|v|^{1+\alpha}$, radius$(B)^{1+\alpha}$ when $\alpha > 1$, then the resulting conditions imply that the second derivatives of $f$ are equal to 0, and $f$ is affine. To accommodate higher powers of $|v|$ or radius$(B)$, one can use differences of higher order in place of the second difference in (2), or polynomials of higher degree in place of affine functions in (3).

Some basic references concerning these matters are [13, 31, 37, 45, 47, 48, 50, 53, 55].

All of this uses the special structure of Euclidean spaces, to define differences of order at least 2, or to have affine functions and polynomials of higher degree, rather than just constants. What about other metric spaces, and the structure that they might have? This leads to a lot of questions.

In the case of Heisenberg groups, other nilpotent Lie groups, and sub-Riemannian spaces more generally, a lot of theory has been developed along
these lines. There are various smoothness classes analogous to classical ones on Euclidean spaces which are adapted to the new geometry. A basic notion is that $C^\infty$ functions or polynomials in general might be the same as before, but more precise degrees of smoothness are now different, and different kinds of degrees of polynomials are used. See [18, 32], for instance.

What about fractal sets sitting inside of Euclidean spaces? For these the usual affine functions and polynomials can be used, just restricted to the fractal set under consideration. One can also define classes like $C^{k,\alpha}$ on the set by taking restrictions of $C^{k,\alpha}$ functions from the ambient Euclidean space. Well-known extension theorems of Whitney characterize functions on a closed set in a Euclidean space which can be extended to $C^{k,\alpha}$ functions on the whole Euclidean space, as in [33, 48].

Smoothness inherited from the ambient space in this manner is quite different from the kind of regularity discussed in [29, 52], for instance.

Other very interesting examples with their own special behavior can be found in [5, 6, 35, 36].

### Semi-Markovian spaces

Gromov [21, 9] introduced a remarkable notion of “semi-Markovian spaces” for which there are combinatorial “presentations”, and which includes a number of standard fractals. This appears to have a lot of room for interesting analysis of functions.

### Families of fractals

It seems to me that there are a lot of issues around the general theme of “families of fractals”. A basic set-up would be to have two metric spaces $T$ and $P$ and a mapping $\pi : T \to P$, where the fibers $\pi^{-1}(p), p \in P$, would be the fractals in the family. It is customary to call $T$ the “total space” for the family of fractals, and $P$ the “space of parameters”. One might ask that $\pi$ be Lipschitz, which is to say Hölder continuous of order 1. One could also ask for some nondegeneracy conditions, along the lines of the nonlinear quotient mappings discussed in [1, 4, 27, 28], or $(\tau, \rho)$-regular mappings or noncollapsing mappings as in [11]. For $(\tau, \rho)$-regular mappings, $\tau, \rho$ could be some kind of dimensions for $T, P$, respectively.

Here is a somewhat more specific type of situation. Suppose that $n$ is a positive integer, and consider the Cartesian product $\mathbf{R}^n \times \mathbf{R}$. Let $\lambda :
$\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the standard projection onto the last coordinate, so that $\lambda(x, t) = t$ for all $x$ in $\mathbb{R}^n$ and $t$ in $\mathbb{R}$. Let $E$ be a subset of $\mathbb{R}^n \times \mathbb{R}$, and assume for instance that $E$ is compact and that $\lambda(E) = [0, 1]$. The sets $E(t) = \lambda^{-1}(t)$, $0 \leq t \leq 1$, would be the fractals in the family. Let $d$ be a positive real number, and suppose that $E(t)$ has Hausdorff dimension $d$ for all $t$ in $[0, 1]$, or almost all $t$. Then $E$ itself should have Hausdorff dimension at least $d+1$, and if the Hausdorff dimension of $E$ is equal to $d+1$, then that reflects a kind of regularity in the situation. Of course there are a number of variants of this.

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