On the Regularity Theory for Mixed Local and Nonlocal Quasilinear Elliptic Equations

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Abstract

We consider a combination of local and nonlocal $p$-Laplace equations and discuss several regularity properties of weak solutions. More precisely, we establish local boundedness of weak subsolutions, local Hölder continuity of weak solutions, Harnack inequality for weak solutions and weak Harnack inequality for weak supersolutions. We also discuss lower semicontinuity of weak supersolutions as well as upper semicontinuity of weak subsolutions. Our approach is purely analytic and it is based on the De Giorgi-Nash-Moser theory, the expansion of positivity and estimates involving a tail term. The main results apply to sign changing solutions and capture both local and nonlocal features of the equation.

Keywords: Regularity, mixed local and nonlocal $p$-Laplace equation, local boundedness, Hölder continuity, Harnack inequality, weak Harnack inequality, lower semicontinuity, energy estimates, De Giorgi-Nash-Moser theory, expansion of positivity.

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1 Introduction

In this article, we develop regularity theory of weak solutions for the problem

$$-\Delta_p u + L(u) = 0 \text{ in } \Omega, \quad 1 < p < \infty,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$. The local $p$-Laplace operator is defined by

$$\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u),$$

and $L$ is the nonlocal $p$-Laplace operator given by

$$L(u)(x) = \text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))K(x, y)}{|x - y|^{n+ps}} dy,$$

where P.V. denotes the principal value. Here $K$ is a symmetric kernel in $x$ and $y$ such that

$$\frac{\Lambda^{-1}}{|x - y|^{n+ps}} \leq K(x, y) \leq \frac{\Lambda}{|x - y|^{n+ps}},$$

where $\Lambda$ is a constant.
for some $\Lambda \geq 1$ and $0 < s < 1$. Note that the $p$-Laplace operator $\Delta_p$ reduces to the classical Laplace operator $\Delta$ for $p = 2$. When $K(x, y) = |x - y|^{-(n+p)}$, the operator $\mathcal{L}$ is the fractional $p$-Laplace operator $(-\Delta_p)^s$, which further reduces to the fractional Laplacian $(-\Delta)^s$ for $p = 2$.

A prototype problem of type (1.1) is

$$-\Delta_p u + (-\Delta_p)^s u = 0, \quad 1 < p < \infty, \quad 0 < s < 1. \tag{1.5}$$

Before describing our contribution, let us discuss some of the known results. In the local case, the regularity theory for the $p$-Laplace equation $\Delta_p u = 0$ has been studied extensively, for example, see Lindqvist [31], Malý-Ziemer [32] and references therein. For the nonlocal $p$-Laplace equation

$$(-\Delta_p)^s u = 0, \tag{1.6}$$
a scale invariant Harnack inequality holds for globally nonnegative solutions. However, such inequality fails when the solution changes sign as shown by Kassmann [27] in the case $p = 2$, see also Dipierro-Savin-Valdinoci [21]. These results have been extended to $1 < p < \infty$ by Di Castro-Kuusi-Valanatucci [14]. In addition, a weak Harnack inequality for supersolutions of (1.6) has been discussed in [14]. They introduced a nonlocal tail term to compensate for the sign change in the Harnack estimates. Di Castro-Kuusi-Valanatucci [15] has studied local boundedness estimate along with Hölder continuity of solutions for (1.6). See also Brasco-Lindgren-Scikorra [8] for higher regularity results. For lower semicontinuity results of supersolutions, we refer to Korvenpää-Kuusi-Lindgren [29].

For the mixed local and nonlocal case with $p = 2$, i.e.

$$-\Delta u + (-\Delta)^s u = 0, \tag{1.7}$$

Foondun [24] has proved Harnack inequality and local Hölder continuity for nonnegative solutions. Barlow-Bass-Chen-Kassmann [3] has obtained a Harnack inequality for the parabolic problem related to (1.7). Chen-Kumagai [13] has proved Harnack inequality and local Hölder continuity for the parabolic problem of (1.7). Such a parabolic Harnack estimate has been used to prove elliptic Harnack inequality for (1.7) by Chen-Kim-Song-Vondraček in [12]. For more regularity results related to (1.7), we refer to Athreya-Ramachandran [1], Chen-Kim-Song [10] and Chen-Kim-Song-Vondraček [11]. The arguments in these articles combine probability and analysis. Moreover, the Harnack inequality is proved only for globally nonnegative solutions. Recently, an interior Sobolev regularity, a strong maximum principle and a symmetry property among many other qualitative properties of solutions to (1.7) has been studied by Biagi-Dipierro-Valdinoci-Vecchi [13, 15], Dipierro-Proietti Lippi-Valdinoci [18, 19] and Dipierro-Ros-Oton-Serra-Valdinoci [20]. There also exist regularity results for a nonhomogeneous analogue of (1.7). More precisely, Athreya-Ramachandran [1] has proved Harnack inequality by probabilistic and analytic methods and authors in [1, 18] has obtained boundedness, interior as well as boundary regularity results by analytic techniques. Biagi-Dipierro-Valdinoci-Vecchi [6] has obtained interior regularity results for a nonhomogeneous version of (1.5).
We establish the following regularity results for weak solutions (Definition 2.5) of (1.1) with $1 < p < \infty$ and $0 < s < 1$.

- **Local boundedness of weak subsolutions (Theorem 4.2).** The argument is based on an energy estimate (Lemma 3.1), the Sobolev inequality and an iteration technique (Lemma 4.1).

- **Local Hölder continuity of weak solutions (Theorem 5.1).** Local Hölder continuity is not a direct consequence of the Harnack inequality in the nonlocal case, see [3, 23]. We follow the approach of Di Castro-Kuusi-Palatucci [15] in which the local boundedness estimate and the logarithmic energy estimate (Lemma 3.4) play an important role.

- **Harnack inequality (Theorem 8.3) for weak solutions and weak Harnack inequality for weak supersolutions (Theorem 8.4).** The expansion of positivity (Lemma 7.1), the local boundedness result and a tail estimate (Lemma 6.1) are crucial here.

- **Lower and upper semicontinuity of weak supersolutions and subsolutions, respectively (Theorem 9.2 and Corollary 9.3).** This result is an adaptation to the mixed local and nonlocal case of a measure theoretic approach (Lemma 9.1) in Liao [30]. We refer to Banerjee-Garain-Kinnunen [2] for an adaptation of this approach to a class of doubly nonlinear parabolic nonlocal problems.

In contrast to the techniques from probability and analysis introduced in [1, 3, 10, 11, 12, 13, 24], our approach is purely analytic and based on the De Giorgi-Nash-Moser theory. To the best of our knowledge, all of our main results are new for $p \neq 2$. Moreover, some of our main results (Theorem 4.2, Theorem 8.4, Theorem 9.2 and Corollary 9.3) seem to be new even for $p = 2$. Furthermore, our approach applies to sign changing solutions. In this respect, our Harnack estimate (Theorem 8.3) also extends the result of Chen-Kim-Song-Vondraček [12] and Foondun [24] to sign changing solutions. We introduce a tail term (Definition 3.2), that differs from the one discussed in [14], and a tail estimate (Lemma 6.1) that capture both local and nonlocal features of (1.1). Technical novelties include an adaptation of the expansion of positivity technique (Lemma 7.1) for the mixed problem.

This article is organized as follows. In Section 2, we discuss some definitions and preliminary results. Necessary energy estimates are proved in Section 3. In Sections 4 and 5, we establish the local boundedness and Hölder continuity results. In Sections 6 and 7, we obtain a tail estimate and the expansion of positivity property. In Section 8, we prove Harnack and weak Harnack estimates. Finally, in Section 9, we establish the lower and upper semicontinuity results.

## 2 Preliminaries

In this section, we present some known results for fractional Sobolev spaces, see Di Nezza-Palatucci-Valdinoci [16] for more details.
Definition 2.1 Let $1 < p < \infty$ and $0 < s < 1$. Assume that $\Omega$ is a domain in $\mathbb{R}^n$. The fractional Sobolev space $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n + ps}} \, dx \, dy < \infty \right\}$$

and endowed with the norm

$$\| u \|_{W^{s,p}(\Omega)} = \left( \int_{\Omega} |u(x)|^p \, dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n + ps}} \, dx \, dy \right)^{\frac{1}{p}}.$$

The fractional Sobolev space with zero boundary value $W^{s,p}_0(\Omega)$ consists of functions $u \in W^{s,p}(\mathbb{R}^n)$ with $u = 0$ on $\mathbb{R}^n \setminus \Omega$.

Both $W^{s,p}(\Omega)$ and $W^{s,p}_0(\Omega)$ are reflexive Banach spaces, see [16]. The space $W^{s,p}_0(\Omega)$ is defined by requiring that a function belongs to $W^{s,p}(\Omega')$ for every $\Omega' \subset \subset \Omega$, where $\Omega' \subset \subset \Omega$ denotes that $\overline{\Omega'}$ is a compact subset of $\Omega$. Throughout, we write $c$ or $C$ to denote a positive constant which may vary from line to line or even in the same line. The dependencies on parameters are written in the parentheses.

The next result asserts that the standard Sobolev space is continuously embedded in the fractional Sobolev space, see [16, Proposition 2.2]. The argument applies a smoothness property of $\Omega$ so that we can extend functions from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^n)$ and that the extension operator is bounded.

Lemma 2.2 Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$, $1 < p < \infty$ and $0 < s < 1$. There exists a positive constant $C = C(n, p, s)$ such that $\| u \|_{W^{s,p}(\Omega)} \leq C\| u \|_{W^{1,p}(\Omega)}$ for every $u \in W^{1,p}(\Omega)$.

The following result for the fractional Sobolev spaces with zero boundary value follows from [9, Lemma 2.1]. The main difference compared to Lemma 2.2 is that the result holds for any bounded domain, since for the Sobolev spaces with zero boundary value, we always have zero extension to the complement.

Lemma 2.3 Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $1 < p < \infty$ and $0 < s < 1$. There exists a positive constant $C = C(n, p, s, \Omega)$ such that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n + ps}} \, dx \, dy \leq C \int_{\Omega} |\nabla u(x)|^p \, dx$$

for every $u \in W^{1,p}_0(\Omega)$. Here we consider the zero extension of $u$ to the complement of $\Omega$.

The following version of the Gagliardo-Nirenberg-Sobolev inequality will be useful for us, see [32, Corollary 1.57].
Lemma 2.4 Let $1 < p < \infty$, $\Omega$ be an open set in $\mathbb{R}^n$ with $|\Omega| < \infty$ and

$$\kappa = \begin{cases} \frac{n}{n-p}, & \text{if } 1 < p < n, \\ 2, & \text{if } p \geq n. \end{cases}$$

(2.1)

There exists a positive constant $C = C(n,p)$ such that

$$\left( \int_{\Omega} |u(x)|^p \, dx \right)^\frac{1}{p} \leq C|\Omega|^{\frac{1}{p} - \frac{1}{n} + \frac{1}{\kappa p}} \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^\frac{1}{p}$$

(2.2)

for every $u \in W^{1,p}_0(\Omega)$.

Next, we define the notion of weak solution to (1.1).

Definition 2.5 A function $u \in L^\infty(\mathbb{R}^n)$ is a weak subsolution of (1.1) if $u \in W^{1,p}_{\text{loc}}(\Omega)$ and for every $\Omega' \subset \subset \Omega$ and nonnegative test functions $\phi \in W^{1,p}_0(\Omega')$, we have

$$\int_{\Omega'} |\nabla u|^{p-2}\nabla u \cdot \nabla \phi \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(u(x,y))(\phi(x) - \phi(y)) \, d\mu \leq 0,$$

(2.3)

where

$$A(u(x,y)) = |u(x) - u(y)|^{p-2}(u(x) - u(y)) \quad \text{and} \quad d\mu = K(x,y) \, dx \, dy.$$

Analogously, a function $u$ is a weak supersolution of (1.1) if the integral in (2.3) is nonnegative for every nonnegative test functions $\phi \in W^{1,p}_0(\Omega')$. A function $u$ is a weak solution of (1.1) if the equality holds in (2.3) for every $\phi \in W^{1,p}_0(\Omega')$ without a sign restriction.

Remark 2.6 The boundedness assumption, together with Lemma 2.2 and Lemma 2.3, ensures that Definition 2.5 is well stated and the tail that will be defined in (3.5) is finite. Under the assumption that the tail in (3.5) is bounded, our main results Theorem 4.2, Theorem 5.1, Theorem 8.3 and Theorem 8.4 hold true without the a priori boundedness assumption on the function. In such a case, the local boundedness follows from Theorem 4.2.

It follows directly from Definition 2.5 that $u$ is a weak subsolution of (1.1) if and only if $-u$ is a weak supersolution of (1.1). Moreover, for any $c \in \mathbb{R}$, $u+c$ is a weak solution of (1.1) if and only if $u$ is a weak solution of (1.1). We discuss some further structural properties of weak solutions below. We denote the positive and negative parts of $a \in \mathbb{R}$ by $a_+ = \max\{a,0\}$ and $a_- = \max\{-a,0\}$, respectively. Also, the barred integral sign denotes the corresponding integral average.

Lemma 2.7 A function $u$ is a weak solution of (1.1) if and only if $u$ is a weak subsolution and a weak supersolution of (1.1).

Proof. It follows immediately from Definition 2.5 that a weak solution $u$ of (1.1) is a weak subsolution and a weak supersolution of (1.1). Conversely, assume that $u$ is both weak
subsolution and weak supersolution of (1.1). Let \( \Omega' \subseteq \Omega \) and \( \phi \in W_0^{1,p}(\Omega') \). Then \( \phi_+ \) and \( \phi_- \) belong to \( W_0^{1,p}(\Omega') \). Since \( u \) is a weak subsolution, we have
\[
\int_{\Omega'} |\nabla u|^{p-2}\nabla u \cdot \nabla \phi_+ \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(u(x,y)) (\phi_+(x) - \phi_+(y)) \, d\mu \leq 0. \tag{2.4}
\]
Analogously, since \( u \) is a weak supersolution, we have
\[
\int_{\Omega'} |\nabla u|^{p-2}\nabla u \cdot \nabla \phi_- \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(u(x,y)) (\phi_-(x) - \phi_-(y)) \, d\mu \geq 0. \tag{2.5}
\]
Subtracting (2.4) and (2.5) and using \( \phi = \phi_+ - \phi_- \), we obtain
\[
\int_{\Omega'} |\nabla u|^{p-2}\nabla u \cdot \nabla \phi \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(u(x,y)) (\phi(x) - \phi(y)) \, d\mu \leq 0.
\]
The reverse inequality holds by replacing \( \phi \) with \( -\phi \). Hence, \( u \) is a weak solution of (1.1).

Next, we show that the property of being a weak subsolution is preserved under taking the positive part. Then, it follows immediately that \( u_- \) is a weak subsolution of (1.1), whenever \( u \) is a weak supersolution of (1.1).

Lemma 2.8 Assume that \( u \) is a weak subsolution of (1.1). Then \( u_+ \) is a weak subsolution of (1.1).

Proof. Consider functions \( u_k = \min\{ku_+, 1\} \), \( k = 1, 2, \ldots \). Then \( (u_k)_{k=1}^\infty \) is an increasing sequence of functions in \( W^{1,p}_{\text{loc}}(\Omega) \) and \( 0 \leq u_k \leq 1 \) for every \( k \in \mathbb{N} \). Let \( \phi \in C_c^\infty(\Omega') \) be a nonnegative function. By choosing \( u_k \phi \in W_0^{1,p}(\Omega') \) as a test function in (2.3), we obtain
\[
0 \geq \int_{\Omega'} |\nabla u|^{p-2}\nabla u \cdot \nabla (u_k \phi) \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(u(x,y)) (u_k(x)\phi(x) - u_k(y)\phi(y)) \, d\mu = I_1 + I_2. \tag{2.6}
\]
Estimate of \( I_1 \): We observe that
\[
I_1 = \int_{\Omega'} |\nabla u|^{p-2}\nabla u \cdot \nabla (u_k \phi) \, dx = k \int_{\Omega' \cap \{0 < u < 1\}} \phi |\nabla u|^{p-2} \, dx + \int_{\Omega'} u_k |\nabla u|^{p-2}\nabla u \cdot \nabla \phi \, dx. \tag{2.7}
\]
Estimate of \( I_2 \): Let \( x, y \in \mathbb{R}^n \). First, we consider the case when \( u(x) > u(y) \).

If \( u_k(x) = 0 \), then \( u_k(y) = 0 \). Hence, we have
\[
(u(x) - u(y))^{p-1}(u_k(x)\phi(x) - u_k(y)\phi(y)) = 0. \tag{2.8}
\]
If \( u_k(y) > 0 \), then \( u(y) = u_+(y) \). Under the assumption \( u(x) > u(y) \), it follows that \( u(x) = u_+(x) \) and \( u_k(x) > u_k(y) \). This implies that
\[
(u(x) - u(y))^{p-1}(u_k(x)\phi(x) - u_k(y)\phi(y)) = (u_+(x) - u_+(y))^{p-1}(u_k(x)\phi(x) - u_k(y)\phi(y)) \geq (u_+(x) - u_+(y))^{p-1}u_k(x)(\phi(x) - \phi(y)). \tag{2.9}
\]
If \( u_k(y) = 0 \) and \( u_k(x) > 0 \), then \( u(x) > 0 \geq u(y) \) and hence
\[
(u(x) - u(y))^{p-1}(u_k(x)\phi(x) - u_k(y)\phi(y)) = (u(x) - u(y))^{p-1}u_k(x)\phi(x) \\
\geq (u_+(x) - u_+(y))^{p-1}u_k(x)\phi(x) \\
\geq (u_+(x) - u_+(y))^{p-1}u_k(x)(\phi(x) - \phi(y)).
\]
(2.10)
Therefore, from (2.8), (2.9) and (2.10) we have
\[
A(u(x, y))(u_k(x)\phi(x) - u_k(y)\phi(y)) \geq (u_+(x) - u_+(y))^{p-1}u_k(x)(\phi(x) - \phi(y)).
\]
(2.11)
When \( u(x) = u(y) \), the estimate (2.11) holds true. In case of \( u(x) < u(y) \), by interchanging the roles of \( x \) and \( y \) in the above estimates, we arrive at
\[
A(u(x, y))(u_k(x)\phi(x) - u_k(y)\phi(y)) \geq (u_+(y) - u_+(x))^{p-1}u_k(y)(\phi(y) - \phi(x)).
\]
(2.12)
Combining the estimates (2.7), (2.11) and (2.12) in (2.6) and letting \( k \to \infty \), along with an application of the Lebesgue dominated convergence theorem, we obtain
\[
\int_{\Omega} |\nabla u_+|^{p-2}\nabla u_+ \cdot \nabla \phi \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(u_+(x, y))(\phi(x) - \phi(y)) \, d\mu \leq 0.
\]
(2.13)
By a density argument (2.13) holds for every \( \phi \in W^{1,p}_0(\Omega') \). This shows that \( u_+ \) is a weak subsolution of (1.1).

### 3 Energy estimates

The following energy estimate will be crucial for us. We denote an open ball with center \( x_0 \in \mathbb{R}^n \) and radius \( r > 0 \) by \( B_r(x_0) \).

**Lemma 3.1** Let \( u \) be a weak subsolution of (1.1) and denote \( w = (u - k)_+ \) with \( k \in \mathbb{R} \). There exists a positive constant \( C = C(p, \Lambda) \) such that
\[
\int_{B_r(x_0)} \psi^p|\nabla w|^p \, dx + \int_{B_r(x_0)} \int_{B_r(x_0)} |w(x)\psi(x) - w(y)\psi(y)|^p \, d\mu \\
\leq C \left( \int_{B_r(x_0)} w^p|\nabla \psi|^p \, dx + \int_{B_r(x_0)} \int_{B_r(x_0)} \max\{w(x), w(y)\}^p|\psi(x) - \psi(y)|^p \, d\mu \\
+ \sup_{x \in \text{supp} \psi} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{w(y)^{p-1}}{|x - y|^{n+p\alpha}} \, dy \cdot \int_{B_r(x_0)} w^p \, dx \right),
\]
whenever \( B_r(x_0) \subset \Omega \) and \( \psi \in C^\infty_c(B_r(x_0)) \) is a nonnegative function. If \( u \) is a weak supersolution of (1.1), the estimate in (3.1) holds with \( w = (u - k)_- \).

**Proof.** Let \( u \) be a weak subsolution of (1.1). For \( w = (u - k)_+ \), by choosing \( \phi = w\psi^p \) as a test function in (2.3), we obtain
\[
0 \geq \int_{B_r(x_0)} |\nabla u|^{p-2}\nabla u \cdot \nabla (w\psi^p) \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(u(x, y))(w(x)\psi(x)^p - w(y)\psi(y)^p) \, d\mu \\
= I + J.
\]
(3.2)
Proceeding as in the proof of [7] Page 14, Proposition 3.1], for some constants \( c = c(p) > 0 \) and \( C = C(p) > 0 \), we have

\[
I = \int_{B_r(x_0)} |\nabla u|^{p-2} \nabla u \cdot \nabla (w\psi^p) \, dx \\
\geq c \int_{B_r(x_0)} \psi^p |\nabla w|^p \, dx - C \int_{B_r(x_0)} w^p |\nabla \psi|^p \, dx. \tag{3.3}
\]

Moreover, from the lines of the proof of [15 Pages 1285–1287, Theorem 1.4], for some constants \( c = c(p, \lambda) > 0 \) and \( C = C(p, \Lambda) > 0 \), we have

\[
J = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(u(x, y))(w(x)\psi(x)^p - w(y)\psi(y)^p) \, d\mu \\
\geq c \int_{B_r(x_0)} \int_{B_r(x_0)} |w(x)\psi(x) - w(y)\psi(y)|^p \, d\mu \\
- C \int_{B_r(x_0)} \int_{B_r(x_0)} \max\{w(x), w(y)\}^p |\psi(x) - \psi(y)|^p \, d\mu \\
- C \text{ess sup}_{x \in \text{supp } \psi} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{w(y)^{p-1}}{|x - y|^{n+ps}} dy \cdot \int_{B_r(x_0)} w\psi^p \, dx. \tag{3.4}
\]

By applying (3.3) and (3.4) in (3.2), we obtain (3.1). In the case of a weak supersolution, the estimate in (3.1) follows by applying the obtained result to \(-u\).

Next we define a tail which appears in estimates throughout the article.

**Definition 3.2** Let \( u \) be a weak subsolution or a weak supersolution of (1.1) as in Definition 2.5. The tail of \( u \) with respect to a ball \( B_r(x_0) \) is defined by

\[
\text{Tail}(u; x_0, r) = \left( \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|u(y)|^{p-1}}{|y - x_0|^{n+ps}} \, dy \right)^{\frac{1}{p-1}}. \tag{3.5}
\]

We prove an energy estimate which will be crucial to obtain a reverse Hölder inequality for weak supersolutions of (1.1).

**Lemma 3.3** Let \( q \in (1, p) \) and \( d > 0 \). Assume that \( u \) is a weak supersolution of (1.1) such that \( u \geq 0 \) in \( B_R(x_0) \subset \Omega \) and denote by \( w = (u + d)^{\frac{q-d}{q}} \). There exists a positive constant \( c = c(p, \lambda) \) such that

\[
\int_{B_r(x_0)} \psi^p |\nabla w|^p \, dx \leq c \left( \frac{(p - q)^p}{(q - 1)^p} \right) \int_{B_r(x_0)} w^p |\nabla \psi|^p \, dx \\
+ \frac{(p - q)^p}{(q - 1)^p} \int_{B_r(x_0)} \int_{B_r(x_0)} \max\{w(x), w(y)\}^p |\psi(x) - \psi(y)|^p \, d\mu \\
+ \frac{(p - q)^p}{(q - 1)^p} \left( \text{ess sup}_{z \in \text{supp } \psi} \int_{\mathbb{R}^n \setminus B_r(x_0)} K(z, y) \, dy + d^{1-p} R^{-p} \text{Tail}(u_-, x_0, R)^{p-1} \right) \int_{B_r(x_0)} w^p \psi^p \, dx, \tag{3.6}
\]

whenever \( B_r(x_0) \subset B_{3R}(x_0) \) and \( \psi \in C_c^\infty(B_r(x_0)) \) is a nonnegative function. Here \( \text{Tail}(-) \) is defined in (3.5).
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Proof. Let $d > 0$, $v = u + d$ and $q \in [1 + \epsilon, p - \epsilon]$ for $\epsilon > 0$ small enough. Then $v$ is a weak supersolution of (1.1). By choosing $\phi = v^{1-q} \psi^p$ as a test function in [2,3], we obtain

$$0 \leq \int_{B_r(x_0)} |\nabla v|^{p-2} \nabla v \cdot \nabla (v^{1-q} \psi^p) \, dx$$
$$+ \int_{B_r(x_0)} \int_{B_r(x_0)} A(v(x,y))(v(x)^{1-q} \psi(x)^p - v(y)^{1-q} \psi(y)^p) \, d\mu$$
$$+ 2 \int_{\mathbb{R}^n \setminus B_r(x_0)} v^{1-q} |\nabla v| \psi^p \, d\mu$$
$$= I_1 + I_2 + 2I_3. \tag{3.7}$$

Estimate of $I_1$: We observe that

$$I_1 = \int_{B_r(x_0)} |\nabla v|^{p-2} \nabla v \cdot \nabla (v^{1-q} \psi^p) \, dx$$
$$\leq (1-q) \int_{B_r(x_0)} v^{-q} |\nabla v| \psi^p \, dx + p \int_{B_r(x_0)} v^{1-q} |\nabla \psi| |\nabla v|^{p-1} \psi^{p-1} \, dx \tag{3.8}$$
$$= (1-q)J_1 + J_2,$$

where

$$J_1 = \int_{B_r(x_0)} v^{-q} |\nabla \psi| \psi^p \, dx$$

and

$$J_2 = p \int_{B_r(x_0)} v^{1-q} |\nabla \psi| |\nabla v|^{p-1} \psi^{p-1} \, dx.$$

Estimate of $J_2$: By Young’s inequality, we obtain

$$J_2 = p \int_{B_r(x_0)} v^{1-q} |\nabla \psi| |\nabla v|^{p-1} \psi^{p-1} \, dx \leq \frac{q-1}{2} J_1 + \frac{c(p)}{(q-1)^{p-1}} \int_{B_r(x_0)} |\nabla \psi|^{p} \psi^{p-q} \, dx. \tag{3.9}$$

By applying (3.9) in (3.8), for some constant $c = c(p) > 0$, we have

$$I_1 \leq \frac{1-q}{2} \int_{B_r(x_0)} v^{-q} |\nabla v| \psi^p \, dx + \frac{c}{(q-1)^{p-1}} \int_{B_r(x_0)} |\nabla \psi|^{p} \psi^{p-q} \, dx$$
$$= -\frac{q-1}{2} \left( \frac{p}{p-q} \right)^p \int_{B_r(x_0)} |\nabla \left( \frac{v^{1-q}}{\psi^p} \right)|^p \psi^p \, dx + \frac{c}{(q-1)^{p-1}} \int_{B_r(x_0)} |\nabla \psi|^{p} \psi^{p-q} \, dx. \tag{3.10}$$

Estimate of $I_2$: Following the lines of the proof of [14] Pages 1830–1833, Lemma 5.1] for $w = v^{\frac{p}{p-q}}$, with some positive constants $c(p, q)$ and $c(p)$, we obtain

$$I_2 = \int_{B_r(x_0)} \int_{B_r(x_0)} A(v(x,y))(v(x)^{1-q} \psi(x)^p - v(y)^{1-q} \psi(y)^p) \, d\mu$$
$$\leq -c(p, q) \int_{B_r(x_0)} \int_{B_r(x_0)} |w(x) - w(y)|^p \psi(y)^p \, d\mu$$
$$+ \frac{c(p)}{(q-1)^{p-1}} \int_{B_r(x_0)} \int_{B_r(x_0)} \max\{w(x), w(y)\}^p |\psi(x) - \psi(y)|^p \, d\mu. \tag{3.11}$$
**Estimate of $I_3$:** Following the lines of the proof of [14] Page 1830, Lemma 5.1 for $w = \frac{p-q}{p}$, we obtain

\[
I_3 = 2 \int_{\mathbb{R}^n \setminus B_r(x_0)} \int_{B_r(x_0)} A(v(x,y))v(x)^{1-q} \psi(x)^p d\mu
\]

\[
\leq c \left( \text{ess sup}_{\psi \in \sup \psi} \int_{\mathbb{R}^n \setminus B_r(x_0)} K(z,y) dy + d^{1-p} \int_{\mathbb{R}^n \setminus B_R(x_0)} (u(y))^{p-1} |y-x|^{-n-ps} dy \right) \int_{B_r(x_0)} w^p \psi^p dx,
\]

with $c = c(p, \Lambda) > 0$. By applying (3.10), (3.11) and (3.12) in (3.7), we obtain (3.6).

Next, we obtain a logarithmic energy estimate.

**Lemma 3.4** Assume that $u$ is a weak supersolution of (1.1) such that $u \geq 0$ in $B_R(x_0) \subset \Omega$.

There exists a positive constant $c = c(n, p, s, \Lambda)$ such that

\[
\int_{B_r(x_0)} |\nabla \log(u+d)|^p dx + \int_{B_r(x_0)} \int_{B_r(x_0)} \left| \log \left( \frac{u(x)+d}{u(y)+d} \right) \right|^p d\mu \leq c r^n \left( r^{-p} + r^{-ps} + d^{1-p} R^{-p} \text{Tail}(u_-, x_0, R)^{p-1} \right),
\]

whenever $B_r(x_0) \subset B_{R \over 2}(x_0)$ and $d > 0$. Here Tail(·) is given by (3.5).

**Proof.** Let $\psi \in C^\infty_c(B_{3r \over 2}(x_0))$ be such that $0 \leq \psi \leq 1$ in $B_{3r \over 2}(x_0)$, $\psi = 1$ in $B_r(x_0)$, and $|\nabla \psi| \leq \frac{2}{r}$ in $B_{2r \over 2}(x_0)$. By choosing $\phi = (u+d)^{-1-p} \psi^p$ as a test function in (2.3), we obtain

\[
0 \leq \int_{B_{2r}(x_0)} \int_{B_{2r}(x_0)} A(u(x,y))((u(x)+d)^{-1-p} \psi(x)^p - (u(y)+d)^{-1-p} \psi(y)^p) d\mu
\]

\[
+ 2 \int_{\mathbb{R}^n \setminus B_{2r}(x_0)} \int_{B_{2r}(x_0)} A(u(x,y))(u(x)+d)^{-1-p} \psi(x)^p d\mu
\]

\[
+ \int_{B_{2r}(x_0)} |\nabla u|^{p-2} \nabla u \cdot \nabla ((u+d)^{-1-p} \psi^p) dx
\]

\[
= I_1 + I_2 + I_3.
\]

**Estimate of $I_1$:** Following the lines of the proof of [15] Pages 1288–1289, Lemma 1.3 and using the properties of $\psi$, for some positive constant $c = c(n, p, s, \Lambda)$, we obtain

\[
I_1 = \int_{B_{2r}(x_0)} \int_{B_{2r}(x_0)} A(u(x,y))((u(x)+d)^{-1-p} \psi(x)^p - (u(y)+d)^{-1-p} \psi(y)^p) d\mu
\]

\[
\leq - \frac{1}{c} \int_{B_{2r}(x_0)} \int_{B_{2r}(x_0)} K(x,y) \left| \log \left( \frac{u(x)+d}{u(y)+d} \right) \right|^p \psi^p dx dy + cr^{n-ps}.
\]

**Estimate of $I_2$:** Following the lines of the proof of [15] Page 1290, Lemma 1.3 and using the properties of $\psi$, for some positive constant $c = c(n, p, s, \Lambda)$, we obtain

\[
I_2 = 2 \int_{\mathbb{R}^n \setminus B_{2r}(x_0)} \int_{B_{2r}(x_0)} A(u(x,y))(u(x)+d)^{-1-p} \psi(x)^p d\mu
\]

\[
\leq cd^{1-p} r^n R^{-p} \text{Tail}(u_-, x_0, R)^{p-1} + cr^{n-ps}.
\]
Estimate of $I_3$: Arguing similarly as in the proof of [28, Pages 717-718, Lemma 3.4] and using the properties of $\psi$, for some positive constant $c = c(p)$, we have

$$I_3 = \int_{B_{2r}(x_0)} |\nabla u|^{p-2} \nabla u \cdot \nabla ((u + d)^{1-p} \psi^p) \, dx \leq -c \int_{B_r(x_0)} |\nabla \log(u + d)|^p \, dx + cr^{n-p} \tag{3.17}$$

Hence using (3.15), (3.16) and (3.17) in (3.14) along with the fact that $\psi \equiv 1$ in $B_r(x_0)$, the estimate (3.13) follows.

As a consequence of Lemma 3.4, we have the following result.

Corollary 3.5 Assume that $u$ is a weak solution of (1.1) such that $u \geq 0$ in $B_R(x_0) \subset \Omega$. Let $a,d > 0$, $b > 1$ and denote

$$v = \min \left\{ \left( \frac{\log(a + d)}{u + d} \right)_+, \log b \right\}.$$  

There exists a positive constant $c = c(n, p, s, \Lambda)$ such that

$$\int_{B_r(x_0)} |v - (v)_{B_r(x_0)}|^p \, dx \leq c \left( 1 + d^{1-p} \left( \frac{r}{R} \right)^p \text{Tail}(u_-; x_0, R)^{p-1} \right), \tag{3.18}$$

whenever $B_r(x_0) \subset B_{2r}(x_0)$ with $r \in (0,1]$. Here $(v)_{B_r(x_0)} = \frac{1}{B_r(x_0)} \int_{B_r(x_0)} v \, dx$ and $\text{Tail}(\cdot)$ is given by (3.5).

Proof. By the Poincaré inequality from [22, Theorem 2], for a constant $c = c(n, p) > 0$, we have

$$\int_{B_r(x_0)} |v - (v)_{B_r(x_0)}|^p \, dx \leq c r^{p-n} \int_{B_r(x_0)} |\nabla v|^p \, dx. \tag{3.19}$$

Now since $v$ is a truncation of the sum of a constant and $\log(u + d)$, we have

$$\int_{B_r(x_0)} |\nabla v|^p \, dx \leq \int_{B_r(x_0)} |\nabla \log(u + d)|^p \, dx. \tag{3.20}$$

The estimate in (3.18) follows by employing (3.13) in (3.20) along with (3.19) and the fact that $r \in (0,1]$.

4 Local boundedness

We apply the following real analysis lemma. For the proof of Lemma 4.1 see [17, Lemma 4.1].

Lemma 4.1 Let $(Y_j)_{j=0}^{\infty}$ be a sequence of positive real numbers such that $Y_0 \leq c_0 \frac{\beta}{b^\beta} \frac{1}{\beta}$ and $Y_{j+1} \leq c_0 b^\beta Y_j^{1+\frac{\beta}{b}}$, $j = 0, 1, 2, \ldots$, for some constants $c_0, b > 1$ and $\beta > 0$. Then $\lim_{j \to \infty} Y_j = 0$.

Our first main result shows that weak subsolutions of (1.1) are locally bounded. This result comes with a useful estimate.
Theorem 4.2 (Local boundedness). Let $u$ be a weak subsolution of (1.1). There exists a positive constant $c = c(n,p,s,\Lambda)$, such that

$$\text{ess sup } u \leq \delta \text{Tail}(u_+; x_0, \frac{r}{2}) + c\delta \left( \frac{(p-1)\kappa}{\kappa(n-1)} \left( \int_{B_r(x_0)} u^p \, dx \right)^{\frac{1}{p}} \right), \tag{4.1}$$

whenever $B_r(x_0) \subset \Omega$ with $r \in (0,1]$ and $\delta \in (0,1]$. Here $\kappa$ and $\text{Tail}(\cdot)$ are given by (2.1) and (3.5), respectively.

Proof. Let $B_r(x_0) \subset \Omega$ with $r \in (0,1]$. For $j = 0,1,2,\ldots$, we denote $r_j = \frac{r}{2}(1 + 2^{-j})$, $\tilde{r}_j = \frac{r_j + r_{j+1}}{2}$, $B_j = B_{r_j}(x_0)$ and $B_j = B_{\tilde{r}_j}(x_0)$. Let $(\psi_j)_j \subset C^\infty_c(\bar{B}_j)$ be a sequence of cutoff functions such that $0 \leq \psi_j \leq 1$ in $\bar{B}_j$, $\psi_j = 1$ in $B_{j+1}$ and $|\nabla \psi_j| \leq \frac{2j+3}{j}$ for every $j = 0,1,2,\ldots$. For $j = 0,1,2,\ldots$ and $k, \tilde{k} \geq 0$, we denote $k_j = k + (1 - 2^{-j})\tilde{k}$, $\bar{k}_j = \frac{k_j + k_{j+1}}{2}$, $w_j = (u - k_j)_+$ and $\bar{w}_j = (u - \bar{k}_j)_+$. Then there exists a constant $c = c(n,p) > 0$ such that

$$\left( \frac{k_j}{2^{j+1} + 2} \right)^{\frac{p(n-1)}{\kappa}} \left( \int_{B_{j+1}} u^p_{j+1} \, dx \right)^{\frac{1}{p}} = (k_{j+1} - \bar{k}_j)^{\frac{p(n-1)}{\kappa}} \left( \int_{B_{j+1}} u^p_{j+1} \, dx \right)^{\frac{1}{p}} \leq c \left( \int_{B_j} |\bar{w}_j \psi_j|^p \, dx \right)^{\frac{1}{p}}, \tag{4.2}$$

where $\kappa$ is given by (2.1). By the Sobolev inequality in (2.2), with $c = c(n,p,s) > 0$, we obtain

$$\left( \int_{B_j} |\bar{w}_j \psi_j|^p \, dx \right)^{\frac{1}{p}} \leq cr^{-n} \int_{B_j} |\nabla (\bar{w}_j \psi_j)|^p \, dx 
\leq cr^{-n} \left( \int_{B_j} \bar{w}_j^p |\nabla \psi_j|^p \, dx + \int_{B_j} \bar{w}_j^p |\nabla \bar{w}_j|^p \, dx \right) = I_1 + I_2. \tag{4.3}$$

Estimate of $I_1$: Using the properties of $\psi_j$, for some $c = c(n,p,s) > 0$, we have

$$I_1 = cr^{-n} \int_{B_j} \bar{w}_j^p |\nabla \psi_j|^p \, dx \leq c2^ip \int_{B_j} \bar{w}_j^p \, dx. \tag{4.4}$$

Estimate of $I_2$: By Lemma 3.1 with $c = c(n,p,s)$ and $C = (n,p,s,\Lambda)$ positive, we obtain

$$I_2 = cr^{-n} \int_{B_j} \psi_j^p |\nabla \bar{w}_j|^p \, dx 
\leq C r^{-n} \left( \int_{B_j} \bar{w}_j^p |\nabla \psi_j|^p \, dx + \int_{B_j} \int_{B_j} \max \{\bar{w}_j(x), \bar{w}_j(y)\}^p |\psi_j(x) - \psi_j(y)|^p \, d\mu \right) 
+ \int_{B_j} \bar{w}_j(y) \psi_j(y)^p \, dy \cdot \text{ess sup} \psi_j \int_{\mathbb{R}^n \setminus B_j} \bar{w}_j(x)^{p-1} K(x,y) \, dx \tag{4.5}$$

$$= J_1 + J_2 + J_3.$$

Estimates of $J_1$ and $J_2$: To estimate $J_1$, we use the estimate of $I_1$ in (4.4) above and to estimate $J_2$, proceeding similarly as in the proof of the estimate (4.5) in [15], Page 1292 and
again using the properties of $\psi_j$, for every $r \in (0, 1]$, we obtain

$$J_i \leq c(n, p, s, \Lambda)2i^p \int_{B_j} w_j^p \, dx, \quad j = 1, 2.$$  \hfill (4.6)

**Estimate of $J_3$:** We observe that $w_j^p \geq (\bar{k}_j - k_j)^{p-1} \bar{w}_j$. For any $\delta \in (0, 1]$, we have

$$J_3 = C(n, p, s, \Lambda)r^{p-n} \int_{B_j} \bar{w}_j(y)\psi_j(y) \, dy \cdot \text{ess sup}_{y \in \text{supp } \psi_j} \int_{\mathbb{R}^n \setminus B_j} \bar{w}_j(x)^{p-1} K(x, y) \, dx$$

$$\leq c2^{j(n+ps+p-1)} \int_{B_j} \frac{w_j(y)}{(k_j - \bar{k}_j)^{p-1}} \, dy \int_{\mathbb{R}^n \setminus B_j} \frac{w_j(x)^{p-1}}{|x - x_0|^{n+ps}} \, dx$$

$$\leq c \frac{2^{j(n+ps+p-1)}}{k^{p-1}} \text{Tail}(w_0; x_0, \frac{r}{2})^{p-1} \int_{B_j} w_j(y)^p \, dy$$

with $c = c(n, p, s, \Lambda) > 0$, whenever $\bar{k} \geq \delta \text{Tail}(w_0; x_0, \frac{r}{2})$. Here we used the fact that

$$\frac{|x - x_0|}{|x - y|} \leq \frac{|x - y| + |y - x_0|}{|x - y|} \leq 1 + \frac{\bar{r}}{r - \bar{r}} \leq 2^{j+4},$$

which holds for $x \in \mathbb{R}^n \setminus B_j$ and $y \in \text{supp } \psi_j = \bar{B}_j$.

By applying (4.6) and (4.7) in (4.5), we obtain

$$I_2 \leq c(n, p, s, \Lambda)2^{j(n+ps+p-1)} \delta^{1-p} \int_{B_j} w_j^p \, dx$$

for every $\delta \in (0, 1]$. Inserting (4.4) and (4.8) into (4.3) we have

$$\left( \int_{B_j} |\bar{w}_j\psi_j|^{p\kappa} \, dx \right) \frac{1}{k} \leq c(n, p, s, \Lambda)2^{j(n+ps+p-1)} \delta^{1-p} \int_{B_j} w_j^p \, dx.$$  \hfill (4.9)

Setting

$$Y_j = \left( \int_{B_j} w_j^p \, dx \right)^{\frac{1}{p}},$$

and

$$\bar{k} = \delta \text{Tail}(w_0; x_0, \frac{r}{2}) + c_0 \frac{1}{b} \left( \int_{B_r(x_0)} w_0^p \, dx \right)^{\frac{1}{p}},$$

where

$$c_0 = c(n, p, s, \Lambda)\delta^{\frac{(1-p)\kappa}{p}}, \quad b = 2^{\left(\frac{n+ps+p-1}{p} + \frac{\kappa-1}{p}\right)\kappa} \quad \text{and} \quad \beta = \kappa - 1.$$

From (4.2) and (4.9) we obtain

$$\frac{Y_{j+1}}{k} \leq c(n, p, s, \Lambda)2^{j(n+ps+p-1) + \frac{\kappa-1}{p}\kappa} \delta^{\frac{(1-p)\kappa}{p}} \left( \frac{Y_j}{k} \right)^{\kappa}.$$  \hfill (4.10)

Moreover, by the definition of $\bar{k}$ above we have

$$\frac{Y_0}{k} \leq c_0 \frac{1}{\bar{k}^{\frac{1}{p}}} \frac{1}{b} \frac{1}{\bar{k}^{\frac{1}{p}}}.$$
Thus from Lemma 4.1 we obtain $Y_j \to 0$ as $j \to \infty$. This implies that
\[
\text{ess sup}_{B_2(x_0)} u \leq k + \tilde{k},
\]
which gives (4.1) by choosing $k = 0$.

5 Oscillation estimates

The following local Hölder continuity result for weak solutions of (1.1) follows from Lemma 5.2 below.

Theorem 5.1 (Hölder continuity) Let $u$ be a weak solution of (1.1). Then $u$ is locally Hölder continuous in $\Omega$. Moreover, there exist constants $\alpha \in (0, \frac{p}{p-1})$ and $c = c(n, p, s, \Lambda)$, such that
\[
\text{osc}_{B_\rho(x_0)} u = \text{ess sup}_{B_\rho(x_0)} u - \text{ess inf}_{B_\rho(x_0)} u \leq c \left( \frac{\rho}{r} \right)^\alpha \left( \text{Tail}(u; x_0, r) + \left( \int_{B_{2r}(x_0)} |u|^p \, dx \right)^{\frac{1}{p}} \right),
\]
whenever $B_{2r}(x_0) \subset \Omega$ with $r \in (0, 1)$ and $\rho \in (0, r]$. Here $\text{Tail}(\cdot)$ is given by (3.5).

We prove the next result by arguing similarly as in the proof of [15, Lemma 5.1].

Lemma 5.2 Let $u$ be a weak solution of (1.1) and $0 < r < \frac{R}{2}$ for some $R$ such that $B_R(x_0) \subset \Omega$ with $r \in (0, 1]$. For $\eta \in (0, \frac{1}{4}]$, we set $r_j = \eta^j \frac{r}{2}$ and $B_j = B_{r_j}(x_0)$ for $j = 0, 1, 2, \ldots$. Denote
\[
\frac{1}{2} \omega(r_0) = \text{Tail}(u; x_0, \frac{r}{2}) + c \left( \int_{B_r(x_0)} |u|^p \, dx \right)^{\frac{1}{p}},
\]
where $\text{Tail}(\cdot)$ is given by (3.5), $c = c(n, p, s, \Lambda)$ is the constant in (4.1) and let
\[
\omega(r_j) = \left( \frac{r_j}{r_0} \right)^\alpha \omega(r_0), \quad j = 1, 2, \ldots,
\]
for some $\alpha \in (0, \frac{p}{p-1})$. Then
\[
\text{osc}_{B_j} u \leq \omega(r_j), \quad j = 0, 1, 2, \ldots
\]

Proof. Lemma 2.7 and Lemma 2.8 imply that $u_+$ and $(-u)_+$ are weak subsolutions of (1.1). By applying Theorem 4.2 with $u_+$ and $(-u)_+$, we observe that (5.4) holds true for $j = 0$.

Suppose (5.4) holds for every $i = 0, \ldots, j$ for some $j \in \{0, 1, 2, \ldots \}$. To obtain (5.4), by induction, it is enough to deduce (5.4) for $i = j + 1$. We prove it in two steps below. In Step 1, we obtain the estimate (5.9) below related to $u_j$, where $u_j$ will be defined in (5.7). In Step 2, we use the estimate (5.9) along with an iteration argument to conclude the proof of (5.4).
Mixed local and nonlocal quasilinear equations

We observe that either

$$
\frac{|B_{2r_j+1}(x_0) \cap \{ u \geq \text{ess inf}_{B_j} + \frac{\omega(r_j)}{2} \}}{|B_{2r_j+1}(x_0)|} \geq \frac{1}{2} \quad (5.5)
$$

or

$$
\frac{|B_{2r_j+1}(x_0) \cap \{ u \leq \text{ess inf}_{B_j} + \frac{\omega(r_j)}{2} \}}{|B_{2r_j+1}(x_0)|} \geq \frac{1}{2} \quad (5.6)
$$

holds. Let

$$
u_j = \begin{cases} 
\text{ess inf}_{B_j} - (u - \text{ess inf}_{B_j}), & \text{if } (5.6) \text{ holds,} \\
\omega(r_j) - (u - \text{ess inf}_{B_j}), & \text{if } (5.5) \text{ holds.}
\end{cases} \quad (5.7)
$$

Then \(u_j\) is a weak solution of (1.1). Also, in both cases of (5.5) and (5.6), \(u_j \geq 0\) in \(B_j\) and

$$
\frac{|B_{2r_j+1}(x_0) \cap \{ u_j \geq \frac{\omega(r_j)}{2} \}}{|B_{2r_j+1}(x_0)|} \geq \frac{1}{2} \quad (5.8)
$$

Step 1: We claim that

$$
\frac{|B_{2r_j+1}(x_0) \cap \{ u_j \leq 2 \varepsilon \omega(r_j) \}}{|B_{2r_j+1}(x_0)|} \leq \frac{\hat{C}}{\log(\frac{2}{\varepsilon})} \quad (5.9)
$$

where \(\varepsilon = \eta \frac{\varepsilon_{-1}}{r} - \alpha\) for some positive constant \(\hat{C}\) depending only on \(n, p, s, \Lambda\) and the difference between \(\frac{p}{p-1}\) and \(\alpha\) via the definition of \(\varepsilon\). To this end, we will apply the logarithmic estimate from Corollary 3.5, where a tail quantity appears. We set

$$
\mu = \log \left( \frac{\omega(r_j)}{3 \varepsilon \omega(r_j)} \right) = \log \left( \frac{1}{2 + \varepsilon} \right) \approx \log \left( \frac{1}{\varepsilon} \right) \quad (5.10)
$$

and define

$$
\Theta = \min \left\{ \left( \log \left( \frac{\omega(r_j)}{u_j + \varepsilon \omega(r_j)} \right) \right) + \mu \right\} \quad (5.11)
$$

By (5.8) we have

$$
\mu = \frac{1}{|B_{2r_j+1}(x_0) \cap \{ u_j \geq \frac{\omega(r_j)}{2} \}|} \int_{B_{2r_j+1}(x_0) \cap \{ u_j \geq \frac{\omega(r_j)}{2} \}} \mu \, dx
$$

$$
= \frac{1}{|B_{2r_j+1}(x_0) \cap \{ u_j \geq \frac{\omega(r_j)}{2} \}|} \int_{B_{2r_j+1}(x_0) \cap \{ \Theta = 0 \}} \mu \, dx \quad (5.12)
$$

$$
\leq \frac{2}{|B_{2r_j+1}(x_0)|} \int_{B_{2r_j+1}(x_0)} (\mu - \Theta) \, dx = 2(\mu - (\Theta)_{B_{2r_j+1}(x_0)}),
$$

where \((\Theta)_{B_{2r_j+1}(x_0)} = \frac{1}{|B_{2r_j+1}(x_0)|} \int_{B_{2r_j+1}(x_0)} \Theta \, dx\). Integrating (5.12) over the set \(B_{2r_j+1}(x_0) \cap \Theta = \mu\) we get

$$
\frac{|B_{2r_j+1}(x_0) \cap \{ \Theta = \mu \}|}{|B_{2r_j+1}(x_0)|} \mu \leq \frac{2}{|B_{2r_j+1}(x_0)|} \int_{B_{2r_j+1}(x_0)} |\Theta - (\Theta)_{B_{2r_j+1}(x_0)}| \, dx \quad (5.13)
$$
Applying Corollary \[5.5\] with \(a = \frac{\omega(r_j)}{2}\), \(d = \varepsilon\omega(r_j)\) and \(b = e^\mu\) for some constant \(c = c(n, p, s, \Lambda)\) we obtain
\[
\int_{B_{2r_j+1}(x_0)} |\Theta - (\Theta)_{B_{2r_j+1}(x_0)}|^p dx \leq c \left( \varepsilon\omega(r_j) \right)^{1-p} \left( \frac{r_{j+1}}{r_j} \right)^p \text{Tail}(u_j; x_0, r_j)^{p-1} + 1). \tag{5.14}
\]
Noting that \(\eta \in (0, \frac{1}{4}]\), \(\alpha \in (0, \frac{p}{p-1})\) along with \(r \in (0, 1]\) and following the lines of the proof of the estimate (5.6) in [15] Pages 1294–1295], we obtain
\[
\text{Tail}(u_j; x_0, r_j)^{p-1} \leq c \eta^{-\alpha(p-1)} \omega(r_j)^{p-1} \tag{5.15}
\]
for some positive constant \(c\) depending only on \(n, p, s\) and the difference between \(\frac{p}{p-1}\) and \(\alpha\), but independent of \(\eta\). Therefore, using the estimate (5.15) in (5.14) we obtain
\[
\int_{B_{2r_j+1}(x_0)} |\Theta - (\Theta)_{B_{2r_j+1}(x_0)}| dx \leq C \tag{5.16}
\]
for some positive constant \(C\) depending only on \(n, p, s, \Lambda\) and the difference between \(\frac{p}{p-1}\) and \(\alpha\). The estimate (5.14) follows by employing (5.16) in (5.13).

**Step 2:** Now we use an iteration argument to obtain (5.4) for \(i = j + 1\). To this end, for every \(i = 0, 1, 2, \ldots\), let \(\rho_i = (1 + 2^{-i})r_{j+1}\), \(\hat{\rho}_i = \frac{\rho_i + \rho_{i+1}}{2}\), \(B^i = B_{\rho_i}(x_0)\) and \(\tilde{B}^i = B_{\hat{\rho}_i}(x_0)\). Recalling that \(\varepsilon = \eta^{p-1-\alpha}\), we denote \(k_i = (1 + 2^{-i})\varepsilon\omega(r_j)\) and
\[
A^i = B^i \cap \{u_j \leq k_i\}, \quad i = 0, 1, 2, \ldots.
\]
Let \(w_i = (k_i - u_j)_+\) and \((\psi_i)_{i \geq 0} \subset C_c^\infty(\tilde{B}^i)\) be such that \(0 \leq \psi_i \leq 1\) in \(\tilde{B}^i\), \(\psi_i = 1\) in \(B^{i+1}\) and \(|\nabla \psi_i| \leq \frac{\kappa}{\rho_i}\) in \(\tilde{B}^i\), with \(c = c(n, p) > 0\). By applying the Sobolev inequality in (2.2), for \(\kappa\) as defined in (2.1), we obtain a constant \(c = c(n, p, s) > 0\) such that
\[
(k_i - k_{i+1})^p \left( \frac{A_{i+1}}{|B^{i+1}|} \right)^{\frac{1}{p}} \leq \left( \int_{B^{i+1}} w_i^{kp} dx \right)^{\frac{1}{p}} \leq c \left( \int_{B^i} w_i^{kp} \psi_i^{kp} dx \right)^{\frac{1}{p}} \leq c r_j^{p} \int_{B^i} |\nabla(w_i \psi_i)|^p dx \leq c r_j^{p}(I + J), \tag{5.17}
\]
where
\[
I = \int_{B^i} w_i^p |\nabla \psi_i|^p dx \quad \text{and} \quad J = \int_{B^i} |\nabla w_i|^p \psi_i^p dx.
\]

**Estimate of \(I\):** Since \(u_j \geq 0\) in \(B_j\), we have \(w_i \leq k_i \leq 2\varepsilon\omega(r_j)\) in \(B^i\). Thus, using the properties of \(\psi_i\) above, for some constant \(c = c(n, p) > 0\), we have
\[
I = \int_{B^i} w_i^p |\nabla \psi_i|^p dx \leq c r_j^{p}(\varepsilon\omega(r_j))^{p+2p} \frac{|A_i|}{|B^i|}. \tag{5.18}
\]

**Estimate of \(J\):** By Lemma 3.1 we obtain a constant \(C = C(p, \Lambda)\) such that
\[
\int_{B^i} |\nabla w_i|^p \psi_i^p dx \leq C(J_1 + J_2 + J_3), \tag{5.19}
\]
where
\[ J_1 = \int_{B^i} w_i^p |\nabla \psi_i|^p \, dx, \quad J_2 = \int_{B^i} \int_{B^i} \max\{w_i(x), w_i(y)\}^p |\psi_i(x) - \psi_i(y)|^p \, d\mu \]
and
\[ J_3 = \operatorname{ess\, sup}_{x \in B^i} \int_{\mathbb{R}^n \setminus B^i} \frac{w_i(y)^{p-1}}{|x - y|^{n+ps}} \, dy \cdot \int_{B^i} w_i \psi_i^p \, dx. \]
From (5.18) we have
\[ J_1 \leq c r_j^{-p} (\varepsilon \omega_j)^p 2^{i_p^j} |A_i|, \quad (5.20) \]
with \( c = c(n,p) > 0 \). For \( x \in \hat{B}^i \) and \( y \in \mathbb{R}^n \setminus B^i \), we have
\[ \frac{1}{|y - x|} = \frac{1}{|y - x_0|} \frac{|y - x|}{|y - x|} \leq \frac{1}{|y - x_0|} \left( 1 + \frac{|x - x_0|}{|y - x|} \right) \]
\[ \leq \frac{1}{|y - x_0|} \left( 1 + \frac{\hat{\rho}_i}{\rho_i - \hat{\rho}_i} \right) \leq \frac{2^{i_p^j}}{|y - x|}. \quad (5.21) \]
By applying (5.21), (5.15), the properties of \( \psi_i, r \in (0,1] \) and proceeding along the lines of the proof of the estimates (5.12) and (5.15) in [15, Page 1297], we obtain
\[ J_m \leq Cr_j^{-p} (\varepsilon \omega_j)^p 2^{i_p^j(n+p)} |A_i|, \quad m = 2, 3, \quad (5.22) \]
for some positive constant \( C \) depending on \( n, p, s, \Lambda \) and the difference between \( \frac{p}{p-1} \) and \( \alpha \). Using (5.20) and (5.22) in (5.19), we obtain
\[ J = \int_{B^i} |\nabla w_i|^p \psi_i^p \, dx \leq Cr_j^{-p} (\varepsilon \omega_j)^p 2^{i_p^j(n+p)} \frac{|A_i|}{|B^i|}, \quad (5.23) \]
for some positive constant \( C \) depending only on \( n, p, s, \Lambda \) and the difference between \( \frac{p}{p-1} \) and \( \alpha \). Let
\[ Y_i = \frac{|A_i|}{|B^i|}, \quad i = 0, 1, 2, \ldots. \]
Noting that \( k_i - k_{i+1} = 2^{-i-1} \varepsilon \omega_j \) and applying (5.18) and (5.23) in (5.17), we get
\[ Y_{i+1} \leq C 2^{\lambda(2p+n)\kappa} Y_i^\kappa, \]
for some constant \( C \) depending only on \( n, p, s, \Lambda \) and the difference between \( \frac{p}{p-1} \) and \( \alpha \). From Step 1, by (5.9), we have
\[ Y_0 \leq \frac{\hat{C}}{\log(\frac{1}{\eta_j})}, \]
for some positive constant \( \hat{C} \) depending only on \( n, p, s, \Lambda \) and the difference between \( \frac{p}{p-1} \) and \( \alpha \). Let
\[ c_0 = C, \quad b = 2^{(2p+n)\kappa}, \quad \beta = \kappa - 1 \quad \text{and} \quad \eta_1 = c_0^{-\frac{1}{p}} b^{-\frac{1}{p-1}}. \]
By choosing \( \eta = \frac{1}{2} \min \left\{ \frac{1}{4}, e^{-n} \right\} \) we have \( Y_0 \leq \eta_1 \). Thus by Lemma 4.1 we deduce that

\[
\lim_{i \to \infty} Y_i = 0 \quad \text{and therefore, } u_j \geq \varepsilon \omega(r_j) \text{ in } B_{j+1}.
\]

Using the definition of \( u_j \) from (5.7), we obtain

\[
\text{osc}_{B_{j+1}} u \leq (1 - \varepsilon)\omega(r_j) = (1 - \varepsilon)\eta^{-\alpha} \omega(r_{j+1}) \leq \omega(r_{j+1}), \quad (5.24)
\]

where we have chosen \( \alpha \in (0, \frac{p}{p-1}) \) (depending on \( n, p, s, \Lambda \)) small enough such that

\[
\eta^{\alpha} \geq 1 - \varepsilon = 1 - \eta^{\frac{p}{p-1} - \alpha}.
\]

Thus (5.24) proves the induction estimate (5.4) for \( i = j + 1 \). Hence the result follows.

6 Tail estimate

The following tail estimate will be useful for us.

**Lemma 6.1** Let \( u \) be a weak solution of (1.1) such that \( u \geq 0 \) in \( B_R(x_0) \subset \Omega \). There exists a positive constant \( c = c(n, p, s, \Lambda) \) such that

\[
\text{Tail}(u_+; x_0, r) \leq c \text{ess sup}_{B_r(x_0)} u + c \left( \frac{r}{R} \right)^{\frac{p}{p-1}} \text{Tail}(u_-; x_0, R), \quad (6.1)
\]

whenever \( 0 < r < R \) with \( r \in (0, 1] \). Here \( \text{Tail}(\cdot) \) is given by (3.5).

**Proof.** Let \( M = \text{ess sup}_{B_r(x_0)} u \) and \( \psi \in C^\infty_c(B_r(x_0)) \) be a cutoff function such that \( 0 \leq \psi \leq 1 \) in \( B_r(x_0), \psi = 1 \) in \( B_{\frac{r}{2}}(x_0) \) and \( |\nabla \psi| \leq \frac{2}{r} \) in \( B_r(x_0) \). By letting \( w = u - 2M \) and choosing \( \phi = w\psi^p \) as a test function in (2.3) we obtain

\[
0 = \int_{B_r(x_0)} |\nabla u|^{p-2} \nabla u \cdot \nabla (w\psi^p) \, dx
+ \int_{B_r(x_0)} \int_{B_r(x_0)} A(u(x, y))(w(x)\psi(x)^p - w(y)\psi(y)^p) \, d\mu
+ 2 \int_{B_r(x_0)} \int_{\mathbb{R}^n \setminus B_r(x_0)} A(u(x, y))w(x)\psi(x)^p \, d\mu
= I_1 + I_2 + I_3. \quad (6.2)
\]

**Estimate of \( I_1 \):** By Young’s inequality, the estimate

\[
|\nabla w|^{p-2} \nabla w \cdot \nabla (w\psi^p) = |\nabla w|^p \psi^p + p\psi^{p-1}w|\nabla w|^{p-2} \nabla w \cdot \nabla \psi
\geq \frac{1}{2} |\nabla w|^p \psi^p - c(p)|w|^p|\nabla \psi|^p - c(p)M^p|\nabla \psi|^p,
\]

holds in \( B_r(x_0) \). By the properties of \( \psi \), we have

\[
I_1 = \int_{B_r(x_0)} |\nabla u|^{p-2} \nabla u \cdot \nabla (w\psi^p) \, dx \geq -c(p)M^p r^{-p} |B_r(x_0)|, \quad (6.3)
\]
7 Expansion of positivity

The following lemma shows that the expansion of positivity technique applies to mixed problems.

Lemma 7.1 Let $u$ be a weak supersolution of (1.1) such that $u \geq 0$ in $B_R(x_0) \subset \Omega$. Assume $k \geq 0$ and there exists $\tau \in (0, 1)$ such that

$$|B_r(x_0) \cap \{u \geq k\}| \geq \tau |B_r(x_0)|,$$

(7.1)

for some $r \in (0, 1]$ with $0 < r < \frac{R}{4}$. There exists a constant $\delta = \delta(n, p, s, \Lambda, \tau) \in (0, \frac{1}{4})$ such that

$$\text{ess inf}_{B_{4r}(x_0)} u \geq \delta k - \left( \frac{r}{R} \right)^{\frac{n}{p-1}} \text{Tail}(u_+; x_0, R),$$

(7.2)

where Tail(\cdot) is given by (5.5).

Proof. We prove the lemma in two steps.

Step 1. Under the assumption in (7.1), we claim that there exists a positive constant $c_1 = c(n, p, s, \Lambda)$ such that

$$|B_{6r}(x_0) \cap \left\{ u \leq 2\delta k - \frac{1}{2} \left( \frac{r}{R} \right)^{\frac{n}{p-1}} \text{Tail}(u_-; x_0, R) - \epsilon \right\}| \leq \frac{c_1}{\tau \log \frac{1}{rs}} |B_{6r}(x_0)|$$

(7.3)

for every $\delta \in (0, \frac{1}{4})$ and for every $\epsilon > 0$.

Let $\epsilon > 0$ and $\psi \in C_c^\infty(B_{7r}(x_0))$ be a cutoff function such that $0 \leq \psi \leq 1$ in $B_{7r}(x_0)$, $\psi = 1$ in $B_{6r}(x_0)$ and $|\nabla \psi| \leq \frac{n}{r}$ in $B_{7r}(x_0)$. We denote $w = u + t_\epsilon$, where

$$t_\epsilon = \frac{1}{2} \left( \frac{r}{R} \right)^{\frac{n}{p-1}} \text{Tail}(u_-; x_0, R) + \epsilon.$$
Since \( w \) is a weak supersolution of (1.1), we can choose \( \phi = w^{1-p}\psi^p \) as a test function in (2.3) to obtain

\[
0 \leq \int_{B_{6\epsilon}(x_0)} |\nabla w|^{p-2}\nabla w \cdot \nabla (w^{1-p}\psi^p) \, dx \\
+ \int_{B_{6\epsilon}(x_0)} \int_{B_{6\epsilon}(x_0)} A(w(x,y))(w(x)^{1-p}\psi(x)^p - w(y)^{1-p}\psi(y)^p) \, d\mu \\
+ 2 \int_{\mathbb{R}^n \setminus B_{6\epsilon}(x_0)} \int_{B_{6\epsilon}(x_0)} A(w(x,y))w(x)^{1-p}\psi(x)^p \, d\mu \\
= I_1 + I_2 + I_3.
\]

**Estimate of \( I_1 \):** Proceeding similarly as in the proof of (28) Pages 717–718, Lemma 3.4 and using the properties of \( \psi \), we obtain a constant \( c = c(p) > 0 \) such that

\[
I_1 = \int_{B_{6\epsilon}(x_0)} |\nabla w|^{p-2}\nabla w \cdot \nabla (w^{1-p}\psi^p) \, dx \leq -c \int_{B_{6\epsilon}(x_0)} |\nabla \log w|^p \, dx + cr^{n-p}. \tag{7.5}
\]

**Estimate of \( I_2 \):** Arguing as in the proof of the estimate of \( I_1 \) in (14) page 1817 and using the fact that \( r \in (0, 1) \), we obtain a constant \( c = c(n, p, s, \Lambda) > 0 \) such that

\[
I_2 = \int_{B_{6\epsilon}(x_0)} \int_{B_{6\epsilon}(x_0)} A(w(x,y))(w(x)^{1-p}\psi(x)^p - w(y)^{1-p}\psi(y)^p) \, d\mu \\
\leq -\frac{1}{c} \int_{B_{6\epsilon}(x_0)} \int_{B_{6\epsilon}(x_0)} \left| \log \frac{w(x)}{w(y)} \right|^p \, d\mu + cr^{n-p}. \tag{7.6}
\]

**Estimate of \( I_3 \):** Here we follow the proof of the estimate of \( I_2 \) in (14) Pages 1817–1818. To this end, we write

\[
I_3 = 2 \int_{\mathbb{R}^n \setminus B_{6\epsilon}(x_0)} \int_{B_{6\epsilon}(x_0)} A(w(x,y))w(x)^{1-p}\psi(x)^p \, d\mu = 2(I_3^1 + I_3^2), \tag{7.7}
\]

where

\[
I_3^1 = \int_{\mathbb{R}^n \setminus B_{6\epsilon}(x_0) \cap \{w(y) < 0\}} \int_{B_{6\epsilon}(x_0)} A(w(x,y))w(x)^{1-p}\psi(x)^p \, d\mu
\]

and

\[
I_3^2 = \int_{\mathbb{R}^n \setminus B_{6\epsilon}(x_0) \cap \{w(y) \geq 0\}} \int_{B_{6\epsilon}(x_0)} A(w(x,y))w(x)^{1-p}\psi(x)^p \, d\mu.
\]

**Estimate of \( I_3^1 \):** Using the definitions of \( w \) and \( t_\epsilon \), the assumption on the kernel \( K \) and the fact that support of \( \psi \) is inside \( B_{7\epsilon}(x_0) \), we get

\[
I_3^1 \leq cr^n \int_{\mathbb{R}^n \setminus B_{6\epsilon}(x_0)} \left( 1 + \frac{(w(y) - t_\epsilon)}{y - x_0} \right)^{p-1} |y - x_0|^{-n-ps} \, dy \\
\leq cr^{n-ps} + cr^n t_\epsilon^{1-p} R^{-p} \mathrm{Tail}(u_-, x_0, R)^{p-1} \leq cr^{-p}, \tag{7.8}
\]

with \( c = c(n, p, s, \Lambda) \). Here we also used the hypothesis that \( u \geq 0 \) in \( B_R(x_0) \) and \( r \in (0, 1) \).

**Estimate of \( I_3^2 \):** Let \( x \in B_{6\epsilon}(x_0) \). Suppose \( y \in \mathbb{R}^n \setminus B_{6\epsilon}(x_0) \) such that \( w(y) \geq 0 \). If
By using (7.5), (7.6) and (7.10) in (7.4), we obtain
\[ I_3^2 \leq c \int_{\mathbb{R}^n \setminus B_{6r}(x_0)} \int_{B_{7r}(x_0)} |y - x_0|^{-n-ps} \, dx \, dy \leq cr^{-n-p}, \tag{7.9} \]
with \( c = c(n, p, s, \Lambda). \) By applying (7.8) and (7.9) in (7.7), we have
\[ I_3 \leq cr^{-n-p}, \tag{7.10} \]
for some constant \( c = c(n, p, s, \Lambda). \)

By using (7.5), (7.6) and (7.10) in (7.4), we obtain
\[ \int_{B_{6r}(x_0)} |\nabla \log w|^p \, dx + \int_{B_{6r}(x_0)} \int_{B_{6r}(x_0)} \left| \log \left( \frac{w(x)}{w(y)} \right) \right|^p \, d\mu \leq cr^{-n-p}, \tag{7.11} \]
for some constant \( c = c(n, p, s, \Lambda). \) For \( \delta \in (0, \frac{1}{4}), \) we denote
\[ v = \left( \min \left\{ \log \frac{1}{2\delta}, \log \frac{k + t_\varepsilon}{w} \right\} \right)_+. \]

By (7.11), we have
\[ \int_{B_{6r}(x_0)} |\nabla v|^p \, dx \leq \int_{B_{6r}(x_0)} |\nabla \log w|^p \, dx \leq cr^{-n-p}. \tag{7.12} \]

From (7.12), by Hölder’s inequality and Poincaré inequality (see [22, Theorem 2]), we obtain
\[ \int_{B_{6r}(x_0)} |v - (v)_{B_{6r}(x_0)}| \, dx \leq cr^{1+ \frac{p}{p'}} \left( \int_{B_{6r}(x_0)} |\nabla v|^p \, dx \right)^{\frac{1}{p}} \leq c|B_{6r}(x_0)|, \tag{7.13} \]
where \( p' \frac{p}{p-1} \) and \( (v)_{B_{6r}(x_0)} = \int_{B_{6r}(x_0)} v \, dx. \) We observe that \( \{ v = 0 \} = \{ w \geq k + t_\varepsilon \} = \{ u \geq k \}. \) By the assumption (7.1), it follows that
\[ |B_{6r}(x_0) \cap \{ v = 0 \}| \geq \frac{\tau}{6^n} |B_{6r}(x_0)|. \tag{7.14} \]

Following the proof of [14] Page 1819, Lemma 3.1 and using (7.14), we obtain
\[ \log \frac{1}{2\delta} = \frac{1}{|B_{6r}(x_0) \cap \{ v = 0 \}|} \int_{B_{6r}(x_0) \cap \{ v = 0 \}} \left( \log \frac{1}{2\delta} - v(x) \right) \, dx \]
\[ \leq \frac{6^n}{\tau} \left( \log \frac{1}{2\delta} - (v)_{B_{6r}} \right). \tag{7.15} \]

Now integrating (7.15) over the set \( B_{6r}(x_0) \cap \{ v = \log \frac{1}{2\delta} \} \) and using (7.13), we obtain a constant \( c_1 = c_1(n, p, s, \Lambda) \) such that
\[ \left| \left\{ v = \log \frac{1}{2\delta} \right\} \cap B_{6r}(x_0) \right| \log \frac{1}{2\delta} \leq \frac{6^n}{\tau} \int_{B_{6r}(x_0)} |v - (v)_{B_{6r}(x_0)}| \, dx \leq \frac{c_1}{\tau} |B_{6r}(x_0)|. \]
Hence, for any \( \delta \in (0, \frac{1}{4}) \), we have

\[
|B_{6r}(x_0) \cap \{ w \leq 2\delta(k + t_\epsilon) \}| \leq \frac{c_1}{r} \log \frac{1}{25} |B_{6r}(x_0)|.
\]

This implies (7.3).

**Step 2.** We claim that, for every \( \epsilon > 0 \), there exists a constant \( \delta = \delta(n, p, s, \Lambda, \tau) \in (0, \frac{1}{4}) \) such that

\[
\text{ess inf } u \geq \delta k - (\frac{r}{R})^\frac{p}{p-1} \text{Tail}(u_-; x_0, R) - 2\epsilon.
\]  
(7.16)

As a consequence of (7.16), the property (7.2) follows.

To prove (7.16), without loss of generality, we may assume that

\[
\delta k \geq (\frac{r}{R})^\frac{p}{p-1} \text{Tail}(u_-; x_0, R) + 2\epsilon.
\]  
(7.17)

Otherwise (7.16) holds true, since \( u \geq 0 \) in \( B_R(x_0) \).

Let \( \rho \in [r, 6r] \) and \( \psi \in C^\infty_c(B_\rho(x_0)) \) be a cutoff function such that \( 0 \leq \psi \leq 1 \) in \( B_\rho(x_0) \). For any \( l \in (\delta k, 2\delta k) \), from Lemma 3.11 and the proof of [14, Pages 1820–1821, Lemma 3.2] for \( w = (l - u)_+ \), for some constant \( c = c(n, p, s, \Lambda) \), we obtain

\[
\int_{B_\rho(x_0)} \psi^p |\nabla w|^p \, dx + \int_{B_\rho(x_0)} \int_{B_\rho(x_0)} |w(x)\psi(x) - w(y)\psi(y)|^p \, d\mu
\leq c \int_{B_\rho(x_0)} w^p |\nabla \psi|^p \, dx + c \int_{B_\rho(x_0)} \max\{w(x), w(y)\}^p |\psi(x) - \psi(y)|^p \, d\mu
+ \epsilon \text{ ess sup}_{x \in \text{supp} \psi} \int_{\mathbb{R}^n \setminus B_\rho(x_0)} (1 + (u(y)_-)_)^{p-1} K(x, y) \, dy \cdot |B_\rho(x_0) \cap \{ u < l \}|
= J_1 + J_2 + J_3.
\]  
(7.18)

We apply Lemma 14.1 to conclude the proof. For \( j = 0, 1, 2, \ldots \), we denote

\[
l = k_j = \delta k + 2^{-j-1} \delta k, \quad \rho = \rho_j = 4r + 2^{1-j} r, \quad \hat{\rho}_j = \frac{\rho_j + \rho_{j+1}}{2}.
\]  
(7.19)

Then \( l \in (\delta k, 2\delta k) \), \( \rho_j, \hat{\rho}_j \in (4r, 6r) \) and

\[
k_j - k_{j+1} = 2^{-j-2} \delta k \geq 2^{-j-3} k_j
\]
for every \( j = 0, 1, 2, \ldots \). Set \( B_j = B_{\rho_j}(x_0) \), \( \hat{B}_j = B_{\hat{\rho}_j}(x_0) \) and we observe that

\[
w_j = (k_j - u)_+ \geq 2^{-j-3} k_j \chi_{\{u < k_j\}}.
\]

Let \( (\psi_j)_{j=0}^{\infty} \subset C^\infty_c(\hat{B}_j) \) be a sequence of cutoff functions such that \( 0 \leq \psi_j \leq 1 \) in \( \hat{B}_j \), \( \psi_j = 1 \) in \( B_{j+1} \) and \( |\nabla \psi_j| \leq \frac{2^{j+3}}{r} \). We choose \( \psi = \psi_j \), \( w = w_j \) in (7.18). By the properties of \( \psi_j \), we obtain

\[
J_1 = \int_{B_j} w_j^p |\nabla \psi_j|^p \, dx \leq c(p) 2^{j+3} k_j^p r^{-p} |B_j \cap \{ u < k_j \}|.
\]  
(7.20)
Now proceeding along the lines of the proof of [14] Page 1822, Lemma 3.2, for any \( r \in (0, 1] \), we get
\[
J_2 = \int_{B_j} \int_{B_j} \max\{w_j(x), w_j(y)\}^p |\psi_j(x) - \psi_j(y)|^p \, d\mu \\
\leq c(n, p, s, \Lambda)2^{jp}k_j^pr^{-p}|B_j \cap \{u < k_j\}|.
\] (7.21)

To estimate \( J_3 \), we follow the proof of [14] Page 1823, Lemma 3.2. To this end, we observe that, for any \( x \in \text{supp} \psi_j \subset B_j \) and \( y \in \mathbb{R}^n \setminus B_j \), we have
\[
\frac{|y-x_0|}{|y-x|} = \frac{|y-x+x-x_0|}{|y-x|} \leq 1 + \frac{|x-x_0|}{|y-x|} \leq 1 + \frac{\hat{\rho}_j}{\rho_j - \hat{\rho}_j} = 2^{j+4}.
\] (7.22)

Using (7.22) and the properties of the kernel \( K \), we have
\[
\text{ess sup}_{x \in \text{supp} \psi_j} \int_{\mathbb{R}^n \setminus B_j} (k_j + (u(y))_-)^{p-1} K(x, y) \, dy \\
\leq c2^{j(n+ps)} \int_{\mathbb{R}^n \setminus B_j} (k_j + (u(y))_-)^{p-1} |y-x_0|^{-n-ps} dy \\
\leq c2^{j(n+ps)} \left( k_j^{p-1}r^{-p} + \int_{\mathbb{R}^n \setminus B_R(x_0)} (u(y))^{p-1} |y-x_0|^{-n-ps} dy \right) \\
= c2^{j(n+ps)} \left( k_j^{p-1}r^{-p} + r^{-p} \left( \frac{T}{R} \right)^p \text{Tail}(u_-, x_0, R)^{p-1} \right) \\
\leq c2^{j(n+ps)}k_j^{p-1}r^{-p},
\]
with \( c = c(n, p, s, \Lambda) \). Here we also used the fact that \( r \in (0, 1] \) along with (7.17), \( \delta k < k_j \) and the fact that \( u \geq 0 \) in \( B_R(x_0) \). Therefore, from (7.23), we obtain
\[
J_3 = c k_j \text{ess sup}_{x \in \text{supp} \psi_j} \int_{\mathbb{R}^n \setminus B_j} (k_j + (u(y))_-)^{p-1} K(x, y) \, dy \cdot |B_j \cap \{u < k_j\}| \\
\leq c2^{j(n+ps)}k_j^{p-1}r^{-p}|B_j \cap \{u < k_j\}|,
\] (7.24)

with \( c = c(n, p, s, \Lambda) \). By using (7.20), (7.21) and (7.24) in (7.18), we obtain
\[
\int_{B_j} \psi_j^p |\nabla w_j|^p \, dx \leq c2^{j(n+ps+p)}k_j^pr^{-p}|B_j \cap \{u < k_j\}|,
\] (7.25)

with \( c = c(n, p, s, \Lambda) \). By applying the Sobolev inequality in (2.2) along with (7.20) and (7.25), for \( \kappa \) defined in (2.1) and using the constant \( c = c(n, p, s, \Lambda) \) such that
\[
(k_j - k_{j+1})^p \left( \frac{|B_{j+1} \cap \{u < k_{j+1}\}|}{|B_{j+1}|} \right)^{\frac{1}{p}} \leq \left( \int_{B_{j+1}} w_j^{ep} \psi_j^{ep} \, dx \right)^{\frac{1}{\kappa}} \leq c \left( \int_{B_j} w_j^{ep} \psi_j^{ep} \, dx \right)^{\frac{1}{k}} \\
\leq cr^p \int_{B_j} |\nabla (w_j \psi_j)|^p \, dx \leq c2^{j(n+ps+p)}k_j^pr^{-p}|B_j \cap \{u < k_j\}|,
\] (7.26)
Let
\[ Y_j = \frac{|B_j \cap \{u < k_j\}|}{|B_j|}, \quad j = 0, 1, 2, \ldots. \]

From (7.26) we have
\[ Y_{j+1} \leq c_2 2^{j(n+2p+ps)\kappa} \kappa Y_{j+1}, \quad j = 0, 1, 2, \ldots, \]
for some constant \( c_2 = c_2(n, p, s, \Lambda) \). We choose \( c_0 = c_2, b = 2^{(n+2p+ps)\kappa} > 1 \) and \( \beta = \kappa - 1 > 0 \) in Lemma 4.1. By (7.17), we have
\[ k_0 = \frac{3}{2} \delta k \leq 2 \delta k - \frac{1}{2} \left( \frac{r}{R} \right)^{\frac{p-1}{p}} \text{Tail}(u_-; x_0, R) - \epsilon. \]

By (7.20) we have
\[ Y_0 \leq \frac{|B_{6r}(x_0) \cap \{u \leq 2 \delta k - \frac{1}{2} \left( \frac{r}{R} \right)^{\frac{p-1}{p}} \text{Tail}(u_-; x_0, R) - \epsilon\}|}{|B_{6r}(x_0)|} \leq \frac{c_1}{\tau \log \frac{1}{2^\beta}} \]
for some constant \( c_1 = c_1(n, p, s, \Lambda) \) and for every \( \delta \in (0, \frac{1}{4}) \). Using (7.28) we choose \( \delta = \delta(n, p, s, \Lambda) \in (0, \frac{1}{4}) \) such that
\[ 0 < \delta = \frac{1}{4} \exp \left( -c_1 c_0 b^{\frac{1}{\beta}} \right) < \frac{1}{4}, \]
so that the estimate \( Y_0 \leq c_0^{-\frac{1}{\beta}} b^{-\frac{1}{\beta}} \) holds. By Lemma 4.1 we conclude that \( Y_j \to 0 \) as \( j \to \infty \).

Therefore, we have
\[ \text{ess inf}_{B_{4r}(x_0)} u \geq \delta k, \]
which gives (7.16) and so (7.2) holds.

## 8 Harnack inequalities

Proceeding similarly as in the proof of [14, Lemma 4.1], along with an application of Lemma 7.1, we obtain the following preliminary version of the weak Harnack inequality, compared to Theorem 8.4.

**Lemma 8.1** Let \( u \) be a weak supersolution of (1.1) such that \( u \geq 0 \) in \( B_R(x_0) \subset \Omega \). There exist constants \( \eta = \eta(n, p, s, \Lambda) \in (0, 1) \) and \( c = c(n, p, s, \Lambda) \geq 1 \) such that
\[ \left( \int_{B_r(x_0)} u^n \, dx \right)^{\frac{1}{n}} \leq c \text{ess inf}_{B_r(x_0)} u + c \left( \frac{r}{R} \right)^{\frac{p-1}{p}} \text{Tail}(u_-; x_0, R), \]
whenever \( B_r(x_0) \subset B_R(x_0) \) with \( r \in (0, 1] \). Here \( \text{Tail}() \) is defined in (3.5).

The following Harnack inequality follows with a similar argument as in the proof of [14, Theorem 1.1]. For convenience of the reader, we give a proof here in the mixed case. To this end, the following iteration lemma from [25, Lemma 1.1] will be useful for us.
Lemma 8.2  Let $0 \leq T_0 \leq t \leq T_1$ and assume that $f : [T_1, T_2] \to [0, \infty)$ is a nonnegative bounded function. Suppose that for $T_0 \leq t < s \leq T_1$, we have

$$f(t) \leq A(s - t)^{-\alpha} + B + \theta f(s),$$

(8.2)

where $A, B, \alpha, \theta$ are nonnegative constants and $\theta < 1$. Then there exists a constant $c = c(\alpha, \theta)$ such that for every $\rho, R$ and $T_0 \leq \rho < R \leq T_1$, we have

$$f(\rho) \leq c(A(R - \rho)^{-\alpha} + B).$$

(8.3)

**Theorem 8.3 (Harnack inequality).** Let $u$ be a weak solution of (1.1) such that $u \geq 0$ in $B_R(x_0) \subset \Omega$. There exists a positive constant $c = c(n, p, s, \Lambda)$ such that

$$\text{ess sup}_{B_{\frac{R}{2}}(x_0)} u \leq c \text{ess inf}_{B_r(x_0)} u + c\left(\frac{r}{R}\right)^{\frac{p}{p-1}}\text{Tail}(u_-; x_0, R),$$

(8.4)

whenever $B_r(x_0) \subset B_{\frac{R}{2}}(x_0)$ and $r \in (0, 1)$. Here Tail($\cdot$) is given by (3.3).

**Proof.** Let $0 < \rho < r$. Then by Lemma 2.7 and Theorem 4.2 for every $\delta \in (0, 1]$, there exists a positive constant $c = c(n, p, s, \Lambda)$ such that

$$\text{ess sup}_{B_{\frac{R}{2}}(x_0)} u \leq \delta\text{Tail}(u_+; x_0, \frac{\rho}{2}) + c\delta^{-\gamma}\left(\int_{B_{\rho}(x_0)} u^p \, dx\right)^{\frac{1}{p}},$$

(8.5)

where $\gamma = \frac{(p-1)\varsigma}{p(\kappa-1)}$ and $\kappa$ as in (2.1). By Lemma 6.1 and (8.5), we obtain

$$\text{ess sup}_{B_{\frac{R}{2}}(x_0)} u \leq c\delta\left(\text{ess sup}_{B_{\rho}(x_0)} u + \left(\frac{\rho}{R}\right)^{\frac{p}{p-1}}\text{Tail}(u_-; x_0, R)\right) + c\delta^{-\gamma}\left(\int_{B_{\rho}(x_0)} u^p \, dx\right)^{\frac{1}{p}},$$

(8.6)

for some constant $c = c(n, p, s, \Lambda)$. Let $\frac{1}{2} \leq \sigma' < \sigma \leq 1$ and $\rho = (\sigma - \sigma')r$. Using a covering argument, it follows that

$$\text{ess sup}_{B_{\frac{\rho}{2}}(x_0)} u \leq c\delta^{-\gamma}\left(\text{ess sup}_{B_{\sigma r}(x_0)} u + c\delta\left(\frac{r}{R}\right)^{\frac{p}{p-1}}\text{Tail}(u_-; x_0, R)\right)$$

$$\leq c\delta^{-\gamma}\left(\frac{r}{\sigma - \sigma'}\right)^{\frac{p}{p-1}}\left(\text{ess sup}_{B_{\sigma r}(x_0)} u\right)^{\frac{p}{p-1}} + c\delta\left(\frac{r}{R}\right)^{\frac{p}{p-1}}\text{Tail}(u_-; x_0, R)$$

(8.7)

for every $t \in (0, p)$ with a constant $c = c(n, p, s\Lambda)$. Young’s inequality with exponents $\frac{p}{t}$ and $\frac{p}{p-t}$ and choosing $\delta = \frac{1}{4c}$ in (8.7) implies that

$$\text{ess sup}_{B_{\frac{\rho}{2}}(x_0)} u \leq \frac{1}{2} \text{ess sup}_{B_{\rho}(x_0)} u + c \left(\frac{r}{\sigma - \sigma'}\right)^{\frac{p}{p-1}}\left(\int_{B_{\sigma r}(x_0)} u^t \, dx\right)^{\frac{1}{t}} + c\left(\frac{r}{R}\right)^{\frac{p}{p-1}}\text{Tail}(u_-; x_0, R)$$

(8.8)
for every $t \in (0,p)$ with a constant $c = c(n,p,s,t,\Lambda)$. Using Lemma 8.2 in [8.8], we have

$$\esssup_{B_{2r}(x_0)} u \leq c \left( \frac{1}{r} \int_{B_r(x_0)} u^t \, dx \right)^{\frac{1}{t}} + c \left( \frac{r}{R} \right)^{\frac{p}{p-t}} \text{Tail}(u_-; x_0, R), \quad (8.9)$$

for every $t \in (0,p)$ with a positive constant $c = c(n,p,s,t,\Lambda)$. Combining the above estimate with Lemma 8.1 and choosing $t = \eta \in (0,1)$, the result follows.

We have the following weak Harnack inequality for supersolutions of (1.1).

**Theorem 8.4 (Weak Harnack inequality).** Let $u$ be a weak supersolution of (1.1) such that $u \geq 0$ in $B_R(x_0) \subset \Omega$. There exists a positive constant $c = c(n,p,s,\Lambda)$ such that

$$\left( \frac{1}{r} \int_{B_r(x_0)} u^t \, dx \right)^{\frac{1}{t}} \leq c \text{ess inf }_{B_{r}(x_0)} u + c \left( \frac{r}{R} \right)^{\frac{p}{p-t}} \text{Tail}(u_-; x_0, R), \quad (8.10)$$

whenever $B_r(x_0) \subset B_{\frac{3}{2}}(x_0)$, $r \in (0,1)$ and $0 < l < \kappa(p-1)$. Here $\kappa$ and Tail($\cdot$) are given by (2.1) and (3.5), respectively.

**Proof.** We prove the result for $1 < p < n$. For $p \geq n$, the result follows in a similar way. Let $r \in (0,1)$, $\frac{1}{2} < \tau' < \tau \leq \frac{3}{4}$ and we choose $\psi \in C_0^\infty(B_{\tau r}(x_0))$ such that $0 \leq \psi \leq 1$ in $B_{\tau r}(x_0)$, $\psi = 1$ in $B_{\tau r}(x_0)$ and $|\nabla \psi| \leq \frac{4}{(\tau - \tau')^p}$. For $d > 0$ and $q \in (1,p)$, we set

$$v = u + d \quad \text{and} \quad w = (u + d)^{\frac{q}{p}}.$$ 

Noting $r \in (0,1)$, the property of $\psi$ above and the proof of the estimate (5.11) in [14], Page 1834, there exists a constant $c = c(n,p,s,\Lambda)$ such that

$$I_1 = \int_{B_r(x_0)} w^p |\nabla \psi|^p \, dx \leq \frac{c r^{-p}}{(\tau - \tau')^p} \int_{B_{\tau r}(x_0)} w^p \, dx, \quad (8.11)$$

$$I_2 = \int_{B_r(x_0)} \int_{B_r(x_0)} \max\{w(x), w(y)\}^p |\psi(x) - \psi(y)|^p \, d\mu \leq \frac{c r^{-p}}{(\tau - \tau')^p} \int_{B_{\tau r}(x_0)} w^p \, dx. \quad (8.12)$$

Assume that Tail($u_-; x_0, R$) is positive. Then for any $\epsilon > 0$ and $r \in (0,1]$ choosing

$$d = \frac{1}{2} \left( \frac{\tau}{R} \right)^{\frac{p}{p-t}} \text{Tail}(u_-; x_0, R) + \epsilon > 0,$$

and noting that

$$\esssup_{z \in \text{supp } \psi} \int_{\mathbb{R}^n \setminus B_r(x_0)} K(z, y) \, dy \leq c(n,p,s,\Lambda) r^{-p}, \quad (8.13)$$

we obtain

$$I_3 = \left( \esssup_{z \in \text{supp } \psi} \int_{\mathbb{R}^n \setminus B_r(x_0)} K(z, y) \, dy + c r^{-p} R^{-p} \text{Tail}(u_-; x_0, R)^{p-1} \right) \int_{B_r(x_0)} w^p \psi^p \, dx \leq \frac{c(n,p,s,\Lambda) r^{-p}}{(\tau - \tau')^p} \int_{B_{\tau r}(x_0)} w^p \, dx. \quad (8.14)$$
If \( \text{Tail}(u_{-}; x_0, R) = 0 \), we can choose an arbitrary \( d = \epsilon > 0 \) and again using (8.13) the estimate in (8.14) follows. Now using Sobolev inequality in (2.2) and the fact that \( \psi \equiv 1 \) in \( B_{r'} \), \( r \in (0, 1] \), along with Lemma 3.3 and the estimates (8.11), (8.12) and (8.14), we have for \( p^* = \frac{n p}{n - p} \) with \( 1 < p < n \),

\[
\left( \int_{B_{r'}(x_0)} u^{n(p-q)/(n-p)} \, dx \right)^{\frac{n}{p^*}} = \left( \int_{B_{r'}(x_0)} w^{p^*} \, dx \right)^{\frac{n}{p^*}} \leq \left( \int_{B_{r'}(x_0)} |w\psi|^{p^*} \, dx \right)^{\frac{n}{p^*}} \leq (\tau r)^{p-n} \int_{B_{r'}(x_0)} |\nabla (w\psi)|^p \, dx \leq \frac{c}{(\tau - \tau')^p} \int_{B_{r'}(x_0)} w^p \, dx,
\]

with \( c = c(n, p, s, q, \Lambda) \). Using \( q \in (1, p) \) and the Moser iteration technique as in [26, Theorem 8.18] and [33, Theorem 1.2], we get

\[
\left( \int_{B_{\tau}(x_0)} u^{l'} \, dx \right)^{\frac{1}{l'}} \leq c \left( \int_{B_{\tau}(x_0)} v'' \, dx \right)^{\frac{1}{l'}}, \quad 0 < l' < l < \frac{n(p-1)}{n-p}.
\]

Let \( \eta \in (0, 1) \) be given by Lemma 8.1 and then choosing \( l' = \eta \in (0, 1) \) and observing that

\[
\left( \int_{B_{\tau}(x_0)} u^{l} \, dx \right)^{\frac{1}{l}} \leq \left( \int_{B_{\tau}(x_0)} v^{l} \, dx \right)^{\frac{1}{l}},
\]

we obtain from (8.16)

\[
\left( \int_{B_{\tau}(x_0)} u^{l} \, dx \right)^{\frac{1}{l}} \leq \text{ess inf}_{B_{\tau}(x_0)} v + c \left( \frac{r}{R} \right)^{p-n} \text{Tail}(u_{-}; x_0, R),
\]

for all \( 0 < l < \frac{n(p-1)}{n-p} \). For any \( \epsilon > 0 \), choosing

\[
d = \frac{1}{2} \left( \frac{\tau}{R} \right)^{p-n} \text{Tail}(u_{-}; x_0, R) + \epsilon,
\]

in (8.17) and letting \( \epsilon \to 0 \), the result follows.

## 9 Semicontinuity

Before stating our results on pointwise behavior, we discuss a result from Liao [30]. Let \( u \) be a measurable function that is locally essentially bounded below in \( \Omega \). Let \( \rho \in (0, 1] \) be such that \( B_{\rho}(y) \subset \Omega \). Assume that \( a, c \in (0, 1), M > 0 \) and \( \mu_{-} \leq \text{ess inf}_{B_{\rho}(y)} u \). Following [30], we say that \( u \) satisfies the property (\( D \)), if there exists a constant \( \tau \in (0, 1) \) depending on \( a, M, \mu_{-} \) and other data (may depend on the partial differential equation and will be made precise in Lemma 9.4), but independent of \( \rho \), such that

\[
|\{ u \leq \mu_{-} + M \} \cap B_{\rho}(y)| \leq \tau |B_{\rho}(y)|,
\]

implies that \( u \geq \mu_{-} + aM \) almost everywhere in \( B_{c\rho}(y) \).
Moreover, for $u \in L^1_{\text{loc}}(\Omega)$, we denote the set of Lebesgue points of $u$ by

$$\mathcal{F} = \left\{ x \in \Omega : |u(x)| < \infty, \lim_{r \to 0} \int_{B_r(x)} |u(x) - u(y)| \, dy = 0 \right\}.$$  

Note that, by the Lebesgue differentiation theorem, $|\mathcal{F}| = |\Omega|$. 

The following result follows from [30, Theorem 2.1].

**Lemma 9.1** Let $u$ be a measurable function that is locally integrable and locally essentially bounded below in $\Omega$. Assume that $u$ satisfies the property $(D)$. Then $u(x) = u_+(x)$ for every $x \in \mathcal{F}$, where

$$u_+(x) = \lim_{r \to 0} \text{ess inf}_{y \in B_r(x)} u(y).$$  

In particular, $u_+$ is a lower semicontinuous representative of $u$ in $\Omega$.

Since $u$ is assumed to be locally essentially bounded below, the lower semicontinuous regularization $u_+(x)$ is well defined at every point $x \in \Omega$. Our final regularity results stated are consequences of Lemma 9.1 and Lemma 9.4 below.

**Theorem 9.2** (Lower semicontinuity). Let $u$ be a weak supersolution of (1.1). Then

$$u(x) = u_+(x) = \lim_{r \to 0} \text{ess inf}_{y \in B_r(x)} u(y)$$

for every $x \in \mathcal{F}$. In particular, $u_+$ is a lower semicontinuous representative of $u$ in $\Omega$.

As a Corollary Theorem 9.2, we have the following result.

**Corollary 9.3** (Upper semicontinuity). Let $u$ be a weak subsolution of (1.1). Then

$$u(x) = u^-(x) = \lim_{r \to 0} \text{ess sup}_{y \in B_r(x)} u(y)$$

for every $x \in \mathcal{F}$. In particular, $u^-$ is an upper semicontinuous representative of $u$ in $\Omega$.

We prove a De Giorgi type lemma for weak supersolutions of (1.1).

**Lemma 9.4** Let $u$ be a weak supersolution of (1.1). Let $M > 0$, $a \in (0, 1)$, $B_r(x_0) \subset \Omega$ with $r \in (0, 1]$ and $\mu_- \leq \text{ess inf}_{B_r(x_0)} u$, $\lambda_- \leq \text{ess inf}_{\mathbb{R}^n} u$. There exists a constant $\tau = \tau(n, p, s, \Lambda, a, M, \mu_-, \lambda_-) \in (0, 1)$ such that if

$$|\{u \leq \mu_- + M \} \cap B_r(x_0)| \leq \tau |B_r(x_0)|,$$

then $u \geq \mu_- + aM$ almost everywhere in $B_{\frac{r}{2}}(x_0)$. 
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Proof. Without loss of generality, we may assume that \( x_0 = 0 \). For \( j = 0, 1, 2, \ldots \), we denote

\[
k_j = \mu_+ + aM + \frac{(1 - a)M}{2^j}, \quad \bar{k}_j = \frac{k_j + k_{j+1}}{2},
\]

\[
r_j = \frac{3r}{4} + \frac{r}{2^{j+2}}, \quad \bar{r}_j = \frac{r_j + r_{j+1}}{2},
\]

\( B_j = B_{r_j}(0), \quad \bar{B}_j = B_{\bar{r}_j}(0), \quad w_j = (k_j - u)_+ \) and \( \bar{w}_j = (\bar{k}_j - u)_+ \). We observe that \( B_{j+1} \subset \bar{B}_j \subset B_j, \quad \bar{k}_j < k_j \) and hence \( \bar{w}_j \leq w_j \). Let \( (\psi_j)_{j=0}^\infty \subset C_c^\infty(\bar{B}_j) \) be a sequence of cutoff functions satisfying \( 0 \leq \psi_j \leq 1 \) in \( B_j, \quad \psi_j = 1 \) in \( B_{j+1}, \quad |\nabla \psi_j| \leq \frac{2^{j+3}}{r_j} \).

By applying Lemma 3.1 to \( w_j \), we obtain

\[
\int_{B_j} |\nabla w_j|^p \psi_j^p \, dx \leq C(n, p, s, \Lambda)(I_1 + I_2 + I_3),
\]

where

\[
I_1 = \int_{B_j} \int_{B_j} \max\{w_j(x), w_j(y)\}^p |\psi_j(x) - \psi_j(y)|^p \, d\mu, \quad I_2 = \int_{B_j} w_j^p |\nabla \psi_j|^p \, dx
\]

and

\[
I_3 = \text{ess sup}_{x \in \text{supp } \psi_j} \int_{\mathbb{R}^n \setminus B_j} \frac{w_j(y)^{p-1}}{|x - y|^{n+ps}} \, dy \cdot \int_{B_j} w_j \psi_j^p \, dx.
\]

Since \( u \geq \lambda_- \) in \( \mathbb{R}^n \), noting the definition of \( k_j \) from above, we have

\[
w_j = (k_j - u)_+ \leq (M + \mu_- - \lambda_-)_+ = L \text{ in } \mathbb{R}^n.
\]

Let \( A_j = \{ u < k_j \} \cap B_j \). We estimate the terms \( I_j \), for \( j = 1, 2, 3 \), separately.

**Estimate of \( I_1 \):** Using \( w_j \leq L \) from (9.3) along with \( \frac{r}{2} < r_j < r, \quad r \in (0, 1] \) and the properties of \( \psi_j \), we obtain

\[
I_1 = \int_{B_j} \int_{B_j} \max\{w_j(x), w_j(y)\}^p |\psi_j(x) - \psi_j(y)|^p \, d\mu \leq C \frac{2^{j+2}p}{r^p} L^p |A_j|,
\]

with \( C = C(n, p, s, \Lambda) \).

**Estimate of \( I_2 \):** Using the properties of \( \psi_j \) and the fact that \( w_j \leq L \) from (9.3), we have

\[
I_2 = \int_{B_j} w_j^p |\nabla \psi_j|^p \, dx \leq C \frac{2^{j+2}p}{r^p} L^p |A_j|,
\]

with \( C = C(n, p, s, \Lambda) \).

**Estimate of \( I_3 \):** For every \( x \in \text{supp } \psi_j \) and every \( y \in \mathbb{R}^n \setminus B_j \), we observe that

\[
|\frac{1}{|x - y|} - \frac{1}{|y|}| \leq \frac{1}{|y|} \left( 1 + \frac{2^j + 5}{|y|} \right) \leq \frac{2^j + 5}{|y|}.
\]

Then using \( r_j > \frac{r}{2}, \quad w_j \leq L, \quad 0 \leq \psi_j \leq 1 \), we obtain

\[
I_3 = \text{ess sup}_{x \in \text{supp } \psi_j} \int_{\mathbb{R}^n \setminus B_j} \frac{w_j(y)^{p-1}}{|x - y|^{n+ps}} \, dy \cdot \int_{B_j} w_j \psi_j^p \, dx \leq C(n, p, s) \frac{2^{j(n+ps)}}{r^p} L^p |A_j|,
\]
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for every $r \in (0,1]$. By using (9.4), (9.5) and (9.7) in (9.2), we have

$$
\int_{B_j} |\nabla w_j|^p \psi_j^p \, dx \leq C \frac{2^{j(n+ps+p)}}{r^p} L^p |A_j|,
$$

(9.8)

with $C = C(n,p,s,\Lambda)$. Noting that $B_{j+1} \subset B_j \subset B_j$, $\bar{w}_j \leq w_j$ and using the Sobolev inequality in (2.2), we obtain

$$
(1-a)M_{j+1} = \int_{A_{j+1}} (k_j - k_{j+1}) \, dx \leq \int_{B_{j+1}} \bar{w}_j \, dx
$$

$$
\leq \int_{B_{j+1}} w_j \, dx \leq |A_j|^{1 - \frac{1}{pn}} \left( \int_{B_j} w_j^{p\kappa} \psi_j^{p\kappa} \, dx \right)^{\frac{1}{p\kappa}}
$$

$$
\leq Cr^{1 + \frac{n}{pn} - \frac{n}{p\kappa}} |A_j|^{1 - \frac{1}{pn}} \left( \int_{B_j} |\nabla (w_j \psi_j)|^p \, dx \right)^{\frac{1}{p}},
$$

(9.9)

with $C = C(n,p,s)$ and $\kappa$ as given in (2.1). By using (9.8) together with the fact that $w_j \leq L$ and the properties of $\psi_j$ in (9.9), we get

$$
(1-a)M_{j+1} \leq C(n,p,s,\Lambda)Lr^{\frac{n}{pn} - \frac{n}{p\kappa}} 2^{j(n+ps+p)} |A_j|^{1 - \frac{1}{pn} + \frac{1}{p}}.
$$

(9.10)

Hence, we obtain from (9.10) that

$$
|A_{j+1}| \leq \frac{C(n,p,s,\Lambda)Lr^{\frac{n}{pn} - \frac{n}{p\kappa}} 2^{j(n+ps+p)}}{(1-a)M} |A_j|^{1 - \frac{1}{pn} + \frac{1}{p}}.
$$

(9.11)

By dividing both sides of (9.11) with $|B_{j+1}|$ and noting that $|B_j| < 2^n |B_{j+1}|$ together by $r_j < r$, we obtain

$$
Y_{j+1} \leq \frac{C(n,p,s,\Lambda)Lr^{\frac{n}{pn} - \frac{n}{p\kappa}} 2^{j(n+ps+p)}}{(1-a)M} Y_j^{1 + \frac{1}{p} (1 - \frac{1}{\kappa})},
$$

(9.12)

where we denoted $Y_j = \frac{|A_j|}{|B_j|}$, $j = 0,1,2,\ldots$. By choosing

$$
c_0 = \frac{C(n,p,s,\Lambda)L}{(1-a)M}, \quad b = 2^{\left(2 + \frac{n+ps}{p}\right)}, \quad \beta = \frac{1}{p} \left(1 - \frac{1}{\kappa}\right), \quad \tau = c_0^{-\frac{1}{\beta}} b^{-\frac{1}{\beta^2}} \in (0,1)
$$

in Lemma 4.1 gives $Y_j \to 0$ as $j \to \infty$, if $Y_0 \leq \tau$. This implies that $u \geq \mu_- + aM$ almost everywhere in $B_{r/4}(0)$.

References

[1] Siva Athreya and Koushik Ramachandran. Harnack inequality for non-local Schrödinger operators. Potential Anal., 48(4):515–551, 2018.

[2] Agnid Banerjee, Prashanta Garain, and Juha Kinnunen. Lower semicontinuity and pointwise behavior of supersolutions for some doubly nonlinear nonlocal parabolic $p$-Laplace equations. arXiv e-prints, page arXiv:2101.10042 January 2021.
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[3] Martin T. Barlow, Richard F. Bass, Zhen-Qing Chen, and Moritz Kassmann. Non-local Dirichlet forms and symmetric jump processes. *Trans. Amer. Math. Soc.*, 361(4):1963–1999, 2009.

[4] Stefano Biagi, Serena Dipierro, Enrico Valdinoci, and Eugenio Vecchi. Mixed local and nonlocal elliptic operators: regularity and maximum principles. *arXiv e-prints*, page arXiv:2005.06907, May 2020.

[5] Stefano Biagi, Serena Dipierro, Enrico Valdinoci, and Eugenio Vecchi. Semilinear elliptic equations involving mixed local and nonlocal operators. *arXiv e-prints*, page arXiv:2006.05830, June 2020.

[6] Stefano Biagi, Serena Dipierro, Enrico Valdinoci, and Eugenio Vecchi. A Hong-Krahn-Szegö inequality for mixed local and nonlocal operators. *arXiv e-prints*, page arXiv:2110.07129, October 2021.

[7] Verena Bögelein, Frank Duzaar, and Naian Liao. On the Hölder regularity of signed solutions to a doubly nonlinear equation. *J. Funct. Anal.*, 281(9):Paper No. 109173, 58, 2021.

[8] Lorenzo Brasco and Erik Lindgren. Higher Sobolev regularity for the fractional $p$-Laplace equation in the superquadratic case. *Adv. Math.*, 304:300–354, 2017.

[9] S. Buccheri, J. V. da Silva, and L. H. de Miranda. A System of Local/Nonlocal $p$-Laplacians: The Eigenvalue Problem and Its Asymptotic Limit as $p \to \infty$. *arXiv e-prints*, page arXiv:2001.05985, January 2020.

[10] Zhen-Qing Chen, Panki Kim, and Renming Song. Heat kernel estimates for $\Delta + \Delta^{\alpha/2}$ in $C^{1,1}$ open sets. *J. Lond. Math. Soc. (2)*, 84(1):58–80, 2011.

[11] Zhen-Qing Chen, Panki Kim, Renming Song, and Zoran Vondraček. Sharp Green function estimates for $\Delta + \Delta^{\alpha/2}$ in $C^{1,1}$ open sets and their applications. *Illinois J. Math.*, 54(3):981–1024 (2012), 2010.

[12] Zhen-Qing Chen, Panki Kim, Renming Song, and Zoran Vondraček. Boundary Harnack principle for $\Delta + \Delta^{\alpha/2}$. *Trans. Amer. Math. Soc.*, 364(8):4169–4205, 2012.

[13] Zhen-Qing Chen and Takashi Kumagai. A priori Hölder estimate, parabolic Harnack principle and heat kernel estimates for diffusions with jumps. *Rev. Mat. Iberoam.*, 26(2):551–589, 2010.

[14] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci. Nonlocal Harnack inequalities. *J. Funct. Anal.*, 267(6):1807–1836, 2014.

[15] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci. Local behavior of fractional $p$-minimizers. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(5):1279–1299, 2016.
[16] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[17] Emmanuele DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993.

[18] Serena Dipierro, Edoardo Proietti Lippi, and Enrico Valdinoci. Linear theory for a mixed operator with Neumann conditions. *arXiv e-prints*, page arXiv:2006.03850, June 2020.

[19] Serena Dipierro, Edoardo Proietti Lippi, and Enrico Valdinoci. (Non)local logistic equations with Neumann conditions. *arXiv e-prints*, page arXiv:2101.02315, January 2021.

[20] Serena Dipierro, Xavier Ros-Oton, Joaquim Serra, and Enrico Valdinoci. Non-symmetric stable operators: regularity theory and integration by parts. *arXiv e-prints*, page arXiv:2012.04833, December 2020.

[21] Serena Dipierro, Ovidiu Savin, and Enrico Valdinoci. All functions are locally $s$-harmonic up to a small error. *J. Eur. Math. Soc. (JEMS)*, 19(4):957–966, 2017.

[22] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[23] Matthieu Felsinger and Moritz Kassmann. Local regularity for parabolic nonlocal operators. *Comm. Partial Differential Equations*, 38(9):1539–1573, 2013.

[24] Mohammud Foondun. Heat kernel estimates and Harnack inequalities for some Dirichlet forms with non-local part. *Electron. J. Probab.*, 14:no. 11, 314–340, 2009.

[25] Mariano Giaquinta and Enrico Giusti. On the regularity of the minima of variational integrals. *Acta Math.*, 148:31–46, 1982.

[26] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[27] Moritz Kassmann. A new formulation of Harnack’s inequality for nonlocal operators. *C. R. Math. Acad. Sci. Paris*, 349(11-12):637–640, 2011.

[28] Juha Kinnunen and Tuomo Kuusi. Local behaviour of solutions to doubly nonlinear parabolic equations. *Math. Ann.*, 337(3):705–728, 2007.

[29] Janne Korvenpää, Tuomo Kuusi, and Erik Lindgren. Equivalence of solutions to fractional $p$-Laplace type equations. *J. Math. Pures Appl. (9)*, 132:1–26, 2019.

[30] Naian Liao. Regularity of weak supersolutions to elliptic and parabolic equations: lower semicontinuity and pointwise behavior. *J. Math. Pures Appl. (9)*, (to appear), page arXiv:2011.03944, November 2020.
[31] Peter Lindqvist. *Notes on the stationary p-Laplace equation*. SpringerBriefs in Mathematics. Springer, Cham, 2019.

[32] Jan Malý and William P. Ziemer. *Fine regularity of solutions of elliptic partial differential equations*, volume 51 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[33] Neil S. Trudinger. On Harnack type inequalities and their application to quasilinear elliptic equations. *Comm. Pure Appl. Math.*, 20:721–747, 1967.

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