Fatigue tests simulation of materials with a random endurance limit

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Abstract. Fatigue tests of materials are characterized by long duration and high cost. In this regard, it is relevant to develop methods for modeling test results in the widest possible range of loads. Fatigue curve mathematical model includes the equation of the relationship between the amplitude of the stress and the durability of the samples, considering the random nature of the values of durability and the limit of unbounded endurance. At the first stage, the values of the model parameters are determined using the maximum likelihood function using real test data in a limited range of stress amplitudes. At the second stage, the problem solution is found considering the random value of the limit of unbounded endurance. Moreover, the mean estimate is obtained from the solution of the first part of the problem. The estimate of the unbounded endurance limit variance is obtained by calculation from the variance balance conditions. The results of modeling fatigue tests for aluminum alloy samples in a wide range of stress amplitude values are presented. Simulation results for determining the values of the fatigue curve left tolerance are considered.

1. Introduction
Fatigue tests are carried out in order to build a fatigue curve, determine durability, as well as the limit of unbounded endurance of the material. During the tests, the relationship between the amplitude of the stress (deformation) and the durability of the sample is established. In this case, the first value is nonrandom, its values are set in the process of planning the experiment. The value of durability is random and, as a rule, has significant scattering. The scatter of results is associated with a complex of random factors. The long duration and high cost of fatigue tests often make them economically unreasonable, and at stress values close to the limit of unbounded endurance, practically unfeasible. In such cases, it is customary to introduce a threshold for censoring data, i.e. the value of durability, upon reaching which the tests of the sample are terminated. The researcher receives additional uncertainty in this case. At the same time, the magnitude of durability presents the great interest precisely in the low-load region, where the nonlinear properties of materials are manifested, and the variance of durability increases significantly. Here, the forecast of durability is possible by statistical modeling based on the available results of real tests and the selected model of the studied dependence. This article proposes an approach to the analysis and modeling of fatigue test results using the Stromeyer
equation, considering the randomness of the limit of unbounded endurance and variability of the material durability variance.

The main issue in modeling fatigue tests is the choice of the relationship equation between the durability of materials and the amplitude of the stress cycle. This choice regards goes to the practical problem being solved. The monograph [1] provides 12 models of such a process, linear and nonlinear. The equations contain parameters that have physical meaning, for example, unbounded endurance limit, sensitivity threshold, and parameters providing a more accurate «geometric» description of the studied dependencies. For linear or linearizable models, standard methods for statistical analysis of fatigue test results have been developed [2; 3]. The study of such models [4; 5] is reduced to the construction of quantile fatigue curves, as well as to the calculation of confidence and tolerance intervals for these curves, considering censored observations. The use of linear models is limited either by the properties of the materials or by the test conditions. Nonlinear models expand the capabilities of researchers. The works [1; 6] shows fatigue models using a modification of the formula for the well-known three-parameter Weibull distribution, as well as the Gumbel distribution. Articles [7; 8] uses a modified Stremeyer model containing a random limit of unbounded endurance of materials. Much attention is paid to methods for estimating the parameters of equations and distributions of random variables included in the equation of the fatigue curve. To describe the change in durability, the logarithmically normal distribution law is used more often, less often the Weibull, Gumbel distribution, the logarithmic Weibull distribution, or the distribution of extreme values [1; 7]. The limit of unbounded endurance of materials is modeled by the Weibull distribution, normal or lognormal distribution [1; 7; 9; 10; 11; 12]. To evaluate the parameters, it is generally accepted to use the likelihood function [1; 7; 9; 10; 11]. The function contains the desired regression coefficients and distribution parameters of random variables included in the equation of the fatigue curve. From partial derivatives of this function, under the conditions that they equal to zero, the desired model parameters are determined. A separate issue is the methods for estimating the distribution parameters for censored samples [7; 13; 14; 15]. In these works, parameter values are also determined using the likelihood function. A significant limitation in this approach is the durability dispersion invariance over the entire range of stress amplitude changes. Theoretically, this is due to the normal distribution used for the logarithm of durability. But this often contradicts experimental data. For many materials, the conditional variance of durability increases with decreasing load. As a result, neglecting this is not possible. In such conditions, it may be appropriate to combine analytical and numerical approaches for solving the problem, in particular, the organization of a statistical experiment – the simulation of test results.

2. Source data and maximum likelihood estimation
The work [16] presents test results using samples of aluminum alloy in the range of stress amplitudes of 190–330 MPa, for 20–25 samples in a series. For stress levels at which all samples of the series were destroyed, sample means and the variance of the durability logarithm were calculated. The article uses the base 10 logarithm here and below. At stress amplitude of 210 MPa for half of the series of samples, a censorship threshold of cycles was reached, equal to \(10^7\) cycles, and the samples were removed from testing. For lower amplitudes of the stress cycle in most tests, the censorship threshold is reached and samples are removed from the tests. Characteristics of the experimental data are presented in Table 1.

Note that the conditional variances of the durability logarithm increase monotonically with decreasing stress amplitude. We test the hypothesis of variances homogeneity using the Bartlett criterion, distributed according to the «chi-square» law with (k-1) degree of freedom. The calculated value of the criterion is 49.4, which is significantly more than the critical value B (0.05; 4) = 9.49, obtained for a five percent level of significance of the null hypothesis and four degrees of freedom. It can be argued that changing the variance of the durability logarithm is no coincidence, and averaging is unacceptable.
Table 1. Statistical characteristics of experimental data.

| The amplitude of the stress, MPa | Mean values of the durability logarithm | Variance of the durability logarithm |
|----------------------------------|----------------------------------------|--------------------------------------|
| 330                              | 4.528                                  | 0.012                                |
| 285                              | 5.13                                   | 0.025                                |
| 254                              | 5.6                                    | 0.051                                |
| 228                              | 6.255                                  | 0.152                                |
| 210                              | 6.987                                  | 0.27                                 |

Next, consider the dependence

$$\log N = \alpha + \beta \log (S - \gamma) + \varepsilon,$$

(1)

Here, \(N\) is the durability of the material, \(S\) is the stress cycle amplitude, while \(\alpha, \beta, \gamma\) are constants. This equation is called the Stromeyer equation and is used to describe the fatigue curves of some materials and alloys in a symmetric loading cycle [16]. In this case, \(\gamma\) is the limit of unbounded endurance of the material. We describe the value of durability by the logarithmically normal distribution law, which corresponds to experimental data. Then, the regression error \(\varepsilon\) will have a normal distribution.

Let there be a sample of volume \(n\) with censored observations. We pose the problem of determining the parameters of model (1) by the maximum likelihood method. We denote \(W = \log N\) and define the likelihood function for the model parameters:

$$L(\alpha, \beta, \gamma) = \prod_{i=1}^{r} f_w(w_i, S_i, \alpha, \beta, \gamma) \prod_{j=1}^{n-r} [1 - F_w(w_j, S_j, \alpha, \beta, \gamma)]$$

(2)

Here, \((S_i; w_i), \ i = 1; r\) are observations made before the destruction of the sample; \((S_j; w_j), \ j = 1; n - r\) are the censored observations; \(f_w(w_i, S_i, \alpha, \beta, \gamma)\) is the value of the distribution density of the durability logarithm; \(F_w(w_j, S_j, \alpha, \beta, \gamma)\) is the value of the cumulative distribution function of the durability logarithm. Given the assumption about the distribution \(W \approx N(\alpha + \beta \log (S - \gamma); \sigma)\), we get

$$L(S_i; w_i; \alpha, \beta, \gamma) = \prod_{i=1}^{r} \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ - \frac{(w_i - \alpha - \beta \log(S_i - \gamma))^2}{2\sigma^2} \right\}$$

$$\times \prod_{j=1}^{n-r} \left(1 - \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{w_j} \exp \left\{ - \frac{[t - \alpha - \beta \log(S_j - \gamma)]^2}{2\sigma^2} \right\} dt \right)$$

The logarithmic likelihood function will take the form

$$\Lambda(S_i; w_i; \alpha, \beta, \gamma) = - \sum_{i=1}^{r} \ln \sqrt{2\pi} - \sum_{i=1}^{r} \ln \sigma - \sum_{i=1}^{r} \left\{ \frac{(w_i - \alpha - \beta \log(S_i - \gamma))^2}{2\sigma^2} \right\}$$
Next, find the maximum of the logarithmic likelihood function

\[
\frac{\partial A(S_i; w_i; \alpha, \beta, \gamma)}{\partial \alpha} = \sum_{i=1}^{r} \left\{ \frac{[w_i - \alpha - \beta \lg(S_i - \gamma)]}{\sigma^2} \right\} + \sum_{j=1}^{n-r} f(S_j; w_j; \alpha, \beta, \gamma) \\
\frac{\partial A(S_i; w_i; \alpha, \beta, \gamma)}{\partial \beta} = \sum_{i=1}^{r} \left\{ \frac{[w_i - \alpha - \beta \lg(S_i - \gamma)]}{\sigma^2} \right\} \cdot \frac{\beta}{(S_i - \gamma) \ln 10} \\
\frac{\partial A(S_i; w_i; \alpha, \beta, \gamma)}{\partial \gamma} = \sum_{i=1}^{r} \left\{ \frac{[w_i - \alpha - \beta \lg(S_i - \gamma)]}{\sigma^2} \right\} \cdot \left( -\frac{\beta}{(S_i - \gamma) \ln 10} \right) \\
+ \sum_{j=1}^{n-r} f(S_j; w_j; \alpha, \beta, \gamma) \cdot \left( -\frac{\beta}{(S_j - \gamma) \ln 10} \right)
\]

Denoting

\[
A_i = \left\{ \frac{[w_i - \alpha - \beta \lg(S_i - \gamma)]}{\sigma^2} \right\} \\
B_j = \frac{f_w(S_j; w_j; \alpha; \beta; \gamma)}{1 - F_w(S_j; w_j; \alpha; \beta; \gamma)}
\]

and equating the obtained derivatives to zero, we obtain a system whose solutions are the required parameter estimates \(\alpha, \beta, \gamma\):

\[
\begin{align*}
\sum_{i=1}^{r} A_i + \sum_{j=1}^{n-r} B_j &= 0 \\
\sum_{i=1}^{r} A_i \cdot \lg(S_i - \gamma) + \sum_{j=1}^{n-r} B_j \cdot \lg(S_j - \gamma) &= 0 \\
\sum_{i=1}^{r} A_i \cdot \frac{\beta}{S_i - \gamma} + \sum_{j=1}^{n-r} B_j \cdot \frac{\beta}{S_j - \gamma} &= 0
\end{align*}
\]

(3)

An approximate solution to the system can be found using the Newton method.

Denote

\[
G_1(S_i; w_i; \alpha; \beta; \gamma) = \sum_{i=1}^{r} \left\{ \frac{[w_i - \alpha - \beta \lg(S_i - \gamma)]}{\sigma^2} \right\} + \sum_{j=1}^{n-r} \frac{f_w(S_j; w_j; \alpha; \beta; \gamma)}{1 - F_w(S_j; w_j; \alpha; \beta; \gamma)}
\]
\[ G_2(S_i; w_i; \alpha; \beta; \gamma) = \sum_{i=1}^{r} \left[ \frac{w_i - \alpha - \beta \log(S_i - \gamma)}{\sigma^2} \right] \log(S_i - \gamma) \]

\[ + \sum_{j=1}^{n-r} f_W(S_j; w_j; \alpha; \beta; \gamma) \cdot \log(S_j - \gamma) \]

\[ G_3(S_i; w_i; \alpha; \beta; \gamma) = \sum_{i=1}^{r} \left[ \frac{w_i - \alpha - \beta \log(S_i - \gamma)}{\sigma^2} \right] \cdot \frac{\beta}{S_i - \gamma} \]

\[ + \sum_{j=1}^{n-r} f_W(S_j; w_j; \alpha; \beta; \gamma) \cdot \frac{\beta}{S_j - \gamma} \]

Let \( x_k = (\alpha_k, \beta_k, \gamma_k) \) to be \( k \)-th system solution approximation (3). Define matrices

\[ G(x_k) = \begin{pmatrix} G_1(x_k) \\ G_2(x_k) \\ G_3(x_k) \end{pmatrix} \]

\[ W(x_k) = \begin{pmatrix} \frac{\partial G_1}{\partial \alpha}_{x_k} & \frac{\partial G_1}{\partial \beta}_{x_k} & \frac{\partial G_1}{\partial \gamma}_{x_k} \\ \frac{\partial G_2}{\partial \alpha}_{x_k} & \frac{\partial G_2}{\partial \beta}_{x_k} & \frac{\partial G_2}{\partial \gamma}_{x_k} \\ \frac{\partial G_3}{\partial \alpha}_{x_k} & \frac{\partial G_3}{\partial \beta}_{x_k} & \frac{\partial G_3}{\partial \gamma}_{x_k} \end{pmatrix} \]

Then, according to Newton’s method \((k+1)\)-th the approximation of the system solution will be determined by the formula

\[ x_{k+1} = x_k - W^{-1}(x_k) \cdot G(x_k). \]

3. The random endurance limit model

Using the above approach for modeling the available data, we obtain the values of the model parameters (1): \( \alpha = 14.66; \beta = -4.556; \gamma = 160.85 \).

It presents a great interest to solve the problem considering the random nature of the endurance limit. We understand this limit as the level of stress, below which the sample will not collapse in the number of cycles significantly, exceeding the censorship threshold. To determine the law and distribution parameters of materials fatigue limit experimentally is a very expensive task. Based on the experimental data, it is possible to estimate the mean value and variance of the distribution. The values of the asymmetry and excess of the distribution are not justified. Three-parameter distributions, for example, Weibull or lognormal distributions, allow more accurate description of experimental data. But the presence of a limit value of a random variable from them is also unreasonable. For some materials and alloys based on non-ferrous metals, this problem cannot be solved, because the endurance limit of these materials is not obvious. It is possible to select the distribution of the quantity \( \gamma \), which allows one to describe the experimental data with reasonable accuracy. In a substantial work [7], the normal distribution and the distribution of minimum values are used to describe the endurance limit. His article uses the normal and lognormal distributions, the Weibull distribution, as well as the distribution of extreme values: minimum and maximum.
If \( \gamma \) is a random variable, then the second term of equation (1) also becomes a random variable, which mainly determines the variation in the durability of the samples. Then the regression error \( \varepsilon \) will be considered as the result of the interaction of many independent and uncontrolled factors, and therefore having a normal distribution.

We write a mathematical model for calculating the durability of material samples in the form:

\[
\log N_i = \alpha + \beta \log(S_j - \gamma_i) + \varepsilon_i
\]

(4)

Here, \( S_j \), \( j = 1, k \) are adjustable stress amplitude levels; \( \gamma_i \), \( i = 1, n \) are the values of the limit of samples unbounded endurance; \( \varepsilon_i \), \( i = 1, n \) are random components of the samples test conditions, distributed according to the normal law with the parameters \( (0; \sigma^2_j) \); \( \sigma^2_j \) is the variance of random component, depending on amplitude \( S_j \); \( \alpha, \beta \) are statistical estimates of regression coefficients; \( N_i \), \( i = 1, n \) are calculated values of the samples durability.

To start the process of statistical testing, we use the available data in the form of a durability value series at given stress amplitude levels. The regression parameters \( \alpha, \beta \) were determined from solving the previous problem. Estimates of the distribution parameters of the unbounded endurance limit will be obtained using the initial data. The mean value of the endurance limit is taken equal to the value = 160.85. Two ways are possible to estimate the variance of the endurance limit. According to the first, we make a forecast of the variance of the durability logarithm in the field of small loads. Then, we select the distribution parameters from the variance balance condition in equation (4). The disadvantage of this approach is the ambiguity of extrapolating the variance of the durability. The second approach involves the selection of parameters of the endurance limit distribution according to the available experimental data. It also uses the variance balance in equation (4). There is no exact solution to this problem. The variance of the random component \( \sigma^2_j \) allows to balance the variance in equation (4) over the entire range of stress amplitude values. Here, the second approach is used. For all the laws of distribution of the endurance limit, parameters are selected at which the model data correspond to the experimental ones.

An additional criterion is the share of censored data in the volume of the series. This proportion depends on the variance of the limit of unbounded endurance and should be consistent with the experimental results during modeling. In particular, at an amplitude of 190 MPa, the proportion of samples with a long cycle life was 0.88, and at an amplitude of 210 MPa, 0.52 [16].

We determine the range of acceptable values of the stress amplitude in statistical tests. It is known that with a normal distribution of 95.4% of the data lie in the «plus-minus» two RMS from the mean interval. Applying this estimate to the distribution of the limit of unbounded endurance, we restrict ourselves to \( S_{\min} \) MPa. At lower values of the stress amplitude, a significant number of samples of the series will have unlimited durability, which will not allow the necessary calculations.

As an accuracy criterion, we use the sum of the squared differences of the calculated and experimental values of the coefficients of durability logarithm variation. The sum is calculated considering weights, since the closeness of values in the low-stress region is more important. The calculations showed that when the normal and lognormal distributions of the endurance limit are used, the values of the variation coefficient are more accurate (Figure 1).

We present an algorithm for statistical modeling of test results using the lognormal distribution of the endurance limit.

1. Set the amplitude of the stress cycle \( S_j \), \( j = 1, k \);
2. generate arrays of pseudo random numbers \( z_{1i}, z_{2i}, i = 1, n \);
3. simulate quantiles of the standard normal distribution:
   \[
   \mu_{1i} = \sqrt{-2 \ln z_{1i}} \sin 2 \pi z_{2i}
   \]
   \[
   \mu_{2i} = \sqrt{-2 \ln z_{1i}} \cos 2 \pi z_{2i}
   \]
4. simulate quantiles of the lognormal distribution \( \gamma_i \) with parameters \( b \) and \( c \), equal to the mean and standard deviation of the quantity \( \lg \gamma_i \):

\[
\lg \gamma_i = b + c \mu_i
\]

5. simulate \( \varepsilon_i \) - normal random variables with parameters \( (0, \sigma_j^2) \):

\[
\varepsilon_i = \mu_i \sigma_j
\]

6. calculate the material samples durability logarithms \( \lg N_i \) according to formula (4);

7. change the value of the stress amplitude \( S_j \) and repeat the calculations (paragraphs 2-6).

If a situation arises that \( \gamma_i > S_j \), then calculations for such material samples are not performed.

Figure 1. The values of the coefficient of the durability logarithm variation \( v(\lg N) \) depending on the amplitude of the stress (\( S \)) and for different laws of the endurance limit distribution.

4. Results and discussion

Figure 2 shows the results of simulation modeling of fatigue tests of an aluminum alloy for a sample of 100 units from a series of 10000 samples. The lognormal distribution of the endurance limit is used. The quantile fatigue curves for fracture probabilities are also shown here 0.1; 0.5; 0.9.

The graph reflects the main features of the material properties: nonlinearity of dependence and variability of conditional variance. The mean values and variance of the durability logarithm for a series of 10000 samples at small values of the stress cycle amplitude are shown in Table 2.

Figure 2. Model values of the durability logarithm of an aluminum alloy at different levels of stress amplitude and quantile fatigue curves.
The works [17; 18] show the relationship between the mean values and the standard deviation of the durability logarithm:

\[ G = aH^b \]  

(5)

Here, \( G \) is standard deviation of durability logarithm; \( H \) is the mean value of the durability logarithm; \( a, b \) are dependency parameters.

This dependence was obtained from experimental data for materials and alloys in the region of multi-cycle fatigue. The values of the dependency parameters (5) are determined according to the data of Table 1. The corresponding forecast values of the variance are shown in Table 2. We note a few large values of the variance obtained by our approach. Perhaps this is due to the fact that dependence (5) is mean for different materials.

Formula (4) does not impose restrictions on the distribution of durability. The appearance of this distribution becomes important when solving some additional problems. Consider the task of predicting the calculated values of the structural strength characteristics of materials. When tested for fatigue and long-term strength, these values correspond to the left (lower) border of the tolerance interval of the median fatigue curve. Boundaries are calculated for the necessary probabilities of non-destruction, with a given reliability. For example, if the probability of non-destruction is 0.9 and the confidence probability is 0.95, it can be argued that the characteristic value of at least 90% of the samples in the general population will be greater than the value of the left tolerant border.

### Table 2. Statistical characteristics of model data.

| Stress amplitude, MPa | Mean values of durability logarithm | Variance of durability logarithm | Variance of durability logarithm, calculated according to (5) |
|-----------------------|------------------------------------|--------------------------------|---------------------------------------------------------------|
| 200                   | 7.442                              | 0.482                           | 0.449                                                         |
| 190                   | 8.061                              | 0.936                           | 0.816                                                         |
| 180                   | 8.972                              | 1.98                            | 1.797                                                         |

**Figure 3.** Values of the alloy durability logarithm at different levels of stress amplitude for a series of 100 samples and tolerance limits \( L \) of the fatigue curve for different nondestructive probabilities.
The data obtained by simulation allow to calculate the tolerance border at each stress level. We use the normal distribution law to describe the durability logarithm. Figure 3 shows the results of fatigue test simulations of a series of 100 aluminum alloy samples. The tolerance limits of the fatigue curve for non-fracture probabilities of 0.9 and 0.99 are also shown here.

Note the non-monotonic changes in the tolerance boundary for p = 0.99. This result is unrealistic, but explainable. With a large variance of observations, the specified reliability of calculations can be ensured only with significant amounts of experimental data. When using the normal distribution law of the durability logarithm, the non-monotonicity of the change in the tolerance boundary for p = 0.99 is saved even with very large sample sizes. If we use the lognormal distribution of the durability logarithm, we can get a more real change in the tolerance border when sampling as many as 100 samples (Figure 4).

Figure 4. Values of the left tolerance boundary of the durability logarithm L(LgN) with a non-destruction probability of 0.99 depending on the amplitude of the stress cycle S and the sample size for the adopted distribution laws

In conclusion, we note that the approach described in the article can be useful in designing fatigue tests. Opportunities for obtaining large samples will provide reliable estimates of material properties. Traditional methods that use linear models can be expanded considering the non-linear relationships between material durability and stress.

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