Competition between superfluid order and charge-density-wave order in SU($N$) disordered attractive Fermi systems

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Abstract. SU($N$) Fermi systems with disorder are studied within the Bogoliubov-de Gennes method for the $N = 3$ and $N = 4$ cases. In our previous work[1], we focused on the properties of systems in the weakly-disordered region and found that the spatial nonuniformity caused by disorder induces the charge-density-wave(CDW)-superfluid(SF) transition. Though it was also suggested that a transition between the SF and the Anderson localized state(AL) for $N = 3$ and the CDW-AL transition for $N = 4$ would occur even in the strongly-disordered region, the nature of each phase transition was not discussed. In this paper, we present the results which were not addressed in Ref.[1] and clarify whether each of these phase transitions is of first-order or of second-order.

1. Introduction
Marked progress in cooling techniques in cold atomic systems provides us with many opportunities to investigate multicomponent Fermi systems[2]. Quantum degenerate states of many species of atoms ($^{40}$K[3], $^{6}$Li[4], $^{173}$Yb[5], $^{87}$Sr[6], etc.) have been observed. The atoms mentioned here have multiple low-lying states. Thereby, the quantum systems including these atoms are expected to play a role as a quantum simulator of multicomponent Fermi systems. Especially, in this study we consider the Fermi systems with SU($N>2$) symmetry, which are an example of multicomponent Fermi systems, and more recently have been realized for $N=6$[7]. The SU($N>2$) systems have been analyzed in previous studies[8, 9, 10, 11], and it has been suggested that novel phases and phenomena, which are not reported in usual solid state physics, would be observed in the SU($N>2$) systems. In addition to the development of cooling techniques, optical lattices with disorder which induces Anderson localization have been realized in cold atoms[12, 13, 14, 15, 16]. An important property of these optical lattices is that the disorder strength can be finely tuned. Thanks to this controllability, one can experimentally investigate the effects of disorder in a much wider region. We expect for disordered optical lattices to offer new insights into the effects of disorder which have been studied in solid state physics for several decades[17, 18, 19, 20, 21, 22]. Disordered multicomponent Fermi systems can be realized by loading the atoms mentioned above into disordered optical lattices.

Here, we would like to comment on the phases which can be realized as the ground state in the SU($N$) Fermi systems. For $N=2$, the CDW state and the s-wave singlet SF state are degenerate in a two-dimensional square lattice without disorder at zero temperature and half-filling[8, 20]. The spatial nonuniformity lifts this degeneracy, and therefore stabilizes the SF
state as the ground state[20]. By contrast, the CDW state is stabilized as the ground state in SU(N>2) systems without disorder. This is because the SU(N) symmetry is spontaneously broken when the SF order with even-parity pairs is realized and thereby the energy gained by the Cooper-pair formation is smaller than that gained by the CDW order[8]. In our previous work[1], we studied the SU(N>2) systems with disorder and determined the ground state phase diagram. We focused on the weakly-disordered region and discussed the CDW-SF transition induced by the spatial nonuniformity in detail. It was also suggested that in the strongly-disordered region the SF-AL transition occurs for N=3, and the CDW state directly undergoes a transition to the AL state for N=4 because the CDW state is rather stable in the SU(4) systems and thus the energy gained by the condensation of Cooper-pairs is not large enough to stabilize the superfluidity in between the CDW state and the Anderson localized state. However, several important properties of these transitions were not investigated. In particular, it remains open whether these transitions are of first-order or of second-order. In this paper, we present the results which were not addressed in the previous work, and discuss the properties of the systems in the strongly disordered region in more detail.

2. Model and Method
2.1. Model
To describe the disordered SU(N) Fermi systems, we employ the SU(N) Anderson-Hubbard model, which can be written as

\[
H = -t \sum_{\langle i,j \rangle, \sigma} \{ \hat{c}_i^{\dagger} \hat{c}_j + h.c. \} - \sum_{i, \sigma} (\mu - \epsilon_i) \hat{n}_{i\sigma} - \frac{U}{2} \sum_{i, \sigma \neq \bar{\sigma}} \hat{n}_{i\sigma} \hat{n}_{i\bar{\sigma}},
\]

where \(\hat{c}_i^{\dagger}(\hat{c}_i)\) denotes the annihilation (creation) operator of a fermion on the \(i\)-th site with spin \(\sigma\), and \(\hat{n}_{i\sigma} = \hat{c}_i^{\dagger} \hat{c}_i\) is the number operator. The sum \(\langle i, j \rangle\) is over all pairs of nearest neighbor lattice sites. \(\mu\) is the chemical potential of the whole system, \(t\) the hopping amplitude which we take as a unit of energy, and \(U(>0)\) the on-site attractive interaction leading to the s-wave SF order and the CDW order. The random site energies \(\epsilon_i\) are independent random variables with a box-type distribution over \([-\Delta/2, \Delta/2]\). \(\Delta\) shows the disorder strength which indicates both of the impurity density and the strength of randomness. It is noted here that the number of spin degrees of freedom is equal to \(N\). In this paper, we use the notations \(\sigma = a, b, c, \cdots\) and \(i = 1, 2, 3, \cdots\).

2.2. Method
We use the Bogoliubov-de Gennes(BdG) method to analyze the above Hamiltonian. Within the BdG method, the s-wave pairing mean-field \(\Phi_i^{\sigma\bar{\sigma}} = \langle \hat{c}_{i\sigma} \hat{c}_{i\bar{\sigma}} \rangle\) and the density mean-field \(n_{i\sigma} = \langle \hat{c}_{i\sigma} \hat{c}_{i\sigma} \rangle\) are introduced, and the interaction term which is quartic in the fermion operator is changed to quadratic terms in the fermion operator. Therefore, the effective mean-field Hamiltonian \(H_{\text{eff}}\) is written as

\[
H_{\text{eff}} = -t \sum_{\langle i,j \rangle, \sigma} \{ \hat{c}_i^{\dagger} \hat{c}_j + h.c. \} - \sum_{i, \sigma} (\bar{\mu}_{i\sigma} - \epsilon_i) \hat{n}_{i\sigma} - \frac{U}{2} \sum_{i, \sigma \neq \bar{\sigma}} \{ \Phi_i^{\sigma\sigma} \hat{c}_{i\sigma}^{\dagger} \hat{c}_{i\sigma} + h.c. \},
\]

where \(\bar{\mu}_{i\sigma} = \mu + U \sum_{\sigma \neq \sigma} n_{i\sigma}\). Note that \(H_{\text{eff}}\) is quadratic in \(\hat{c}_{i\sigma}\), and thus can be diagonalized via the transformation

\[
\hat{c}_{i\sigma} = \sum_m \{ u_m (r_i) \hat{a}_m + v_m^{\ast} (r_i) \hat{a}_m^{\dagger} \},
\]

\[
\hat{c}_{i\bar{\sigma}} = \sum_m \{ u_m (r_{i+V}) \hat{a}_m + v_m^{\ast} (r_{i+V}) \hat{a}_m^{\dagger} \},
\]

(3)
Here, $V$ is the number of sites, $\alpha_m$ and $\alpha_m^\dagger$ are the operators for quasiparticles. We obtain the BdG equations for the components of the normalized eigenvector $u_m(r_j)$ and $v_m(r_j)$,

$$
\begin{pmatrix}
\hat{K} & \hat{\Phi} \\
\hat{\Phi}^\dagger & -\hat{K}
\end{pmatrix}
\begin{pmatrix}
\hat{u}_m(r_j) \\
\hat{v}_m(r_j)
\end{pmatrix} = E_m
\begin{pmatrix}
\hat{u}_m(r_j) \\
\hat{v}_m(r_j)
\end{pmatrix},
$$

where $E_m$ is the $m$-th eigenvalue, $\hat{K}$ the single-particle part, and $\hat{\Phi}$ the pairing-part. We iterate Eq.(4) until $\Phi^{\sigma\sigma}$ and $n_{i\sigma}$ are self-consistently determined. We select the simplest SF state in which $\Phi^{ba}_i \neq 0$ and $\Phi^{cb}_i = \Phi^{ac}_i = 0$ for $N = 3$, and $\Phi^{ba}_i = \Phi^{dc}_i \neq 0$ and $\Phi^{cb}_i = \Phi^{ad}_i = \Phi^{ca}_i = \Phi^{db}_i = 0$ for $N = 4$, as an initial state of the self-consistent loop in this study (for the validity of this choice see Refs.[1, 9]).

3. Results

In this section, the results obtained using the BdG method at zero temperature and half-filling are presented. The calculation is performed for the lattice sizes up to $V=32\times32$ and a periodic boundary condition is imposed on the lattices. It should be noted that we also perform the calculation for different lattice sizes. Thereby, it is confirmed that the results presented in this paper do not qualitatively depend on the lattice size. Hence, we expect that the effects of the edge of the lattice and the boundary condition would not qualitatively change results obtained in our study. To eliminate the sample dependence, we average the results over 12-40 different configurations. Note here that the error bars plotted in the figures of this paper represent the statistical errors caused by averaging over disorder-configurations. Before presenting the results, we would like to comment on the definition of the phases which can be realized in disordered SU($N$) Fermi systems and are discussed in this paper. These phases are the CDW, the SF phase, and the Anderson localized phase, distinguished by the site average of the local s-wave superfluid order parameter

$$
\Phi^{\sigma\sigma} = \frac{1}{V} \sum_i \Phi^{\sigma\sigma}_i.
$$

and the spectral gap width $E_g$. The system is in a CDW phase if $\Phi^{\sigma\sigma} = 0$ and $E_g \neq 0$, in a SF phase if $\Phi^{\sigma\sigma} \neq 0$ and $E_g = 0$, or in a Anderson localized phase if $\Phi^{\sigma\sigma} = 0$ and $E_g = 0$.

3.1. SU(3) systems

In the SU(3) systems, the CDW state is stabilized as the ground state if the systems are clean[8, 9]. As the strength of disorder increases, the CDW-SF transition is induced and thereby the SF state is realized as the ground state[1]. Further increase of disorder triggers the SF-AL transition. The SF-AL transition is characterized by $\Phi^{\sigma\sigma}_b$. We plot $\Phi^{ba}$ as a function of the disorder strength in Fig.1. As disorder increases, $\Phi^{ba}$ decreases both for $U=0.4$ and for $U=0.6$. This result demonstrates that the effects of disorder make the SF state unstable. $\Phi^{ba}$ suddenly becomes zero at critical disorder $\Delta_{AL}$. At $\Delta > \Delta_{AL}$, $\Phi^{ba}$ always takes zero. It is confirmed that the spectral gap width $E_g$ is equal to zero in the strongly disordered region where $\Phi^{ba} = 0$. These results imply that the SF phase undergoes a transition to the Anderson localized phase and the SF-AL transition is of first-order. Fig.1 also shows that the strength of critical disorder increases with an increase in the strength of the attractive interaction. This increase of $\Delta_{AL}$ would be explained from the fact that the increase of the attractive interaction between atoms leads to the increase of the energy gained by the Cooper-pairs condensation. At the end of this paragraph, it is noted that we can not completely eliminate the dependence of our results on
disorder configurations. This dependence has an especially strong influence on the results near the transition points. Owing to this dependence, the results near the SF-AL transition points are not sharply-defined. It is also observed that $\Phi^{ba}$ seems to exhibit a slight increase near the SF-AL transition points, however, it may be possible that this tendency is just caused by a noise because statistical errors are too large. Thereby, we should analyze this point in detail by using a more accurate method such as quantum Monte Carlo simulations. This important issue is under consideration. However, the jump in the SF order parameter at critical disorder is larger than the numerical errors caused by the sample dependence, and thereby we believe that the conclusion that the SF-AL transition is of first-order is reliable.

3.2. SU(4) systems
The SF state is not stabilized as the ground state at half-filling in the disordered SU(4) systems[1]. Thus, the CDW phase directly undergoes a transition to the Anderson localized phase. There is a spectral gap caused by the density-density correlation at the Fermi level in the CDW phase though the spectral gap is closed in the Anderson localized phase. Therefore, the spectral gap width $E_g$ provides a good criterion for the CDW-AL transition. We plot $E_g$ as a function of the disorder strength in Fig.2. As the strength of disorder increases, the spectral gap monotonically decreases and becomes zero at critical disorder. These results demonstrate that disorder suppresses the CDW order and induces the CDW-AL transition. The critical disorder at which the CDW-AL transition is triggered for the $U=0.6$ case is larger than that of the $U=0.4$ case. This is because the spectral gap width in the CDW state is larger as the attractive interaction gets stronger. Furthermore, we find from Fig.2 that the CDW-AL transition is of second-order since the spectral gap continuously decreases and becomes zero. It is instructive to compare the results of $E_g$ with those of the charge correlation function for $N=4$ which were discussed in Ref.[1]. The decrease of $E_g$ associated with the increase of the disorder strength is monotonic and well reflects the response of the systems to disorder. By contrast, the amplitude of the charge correlation function is smaller with increasing the strength of disorder but seems to be hardly affected by the effects of disorder away from the transition points. It should be noted that the amplitude of the charge correlation function corresponds to the spectral gap, and therefore the response of these two physical quantities to disorder is considered to be similar. However, this correspondence seems not to be established in our study. This would be because

Figure 1. (color online). $\Phi^{ba}$ as a function $\Delta$ for two different values of the attractive interaction strength, $U = 0.4$ (red solid line with crosses), $U = 0.6$ (blue dotted line with open squares).
the charge correlation function is strongly affected by configurations of disorder in space, and hence the effects of disorder are misestimated when the charge correlation function is calculated and the disorder average of it is taken. Thereby, we can not definitely determine the CDW-AL transition points only from the results of the charge correlation function or the charge structure factor associated with it. These arguments suggest that the spectral gap width is more suitable for determining critical disorder at which the CDW-AL transition occurs, as has been done in our studies.

4. Summary
In this study, we have studied SU($N$) Fermi systems with disorder within the Bogoliubov-de Gennes method for $N=3$ and $N=4$. In particular, we have focused on the properties of the systems in the strongly-disordered region. The site average of the $s$-wave superfluid order parameter for the $N=3$ case has suggested that the $s$-wave superfluid order parameter suddenly changes at the SF-AL transition points. This result demonstrates that the SF-AL transition which is induced by the spatial inhomogeneity is of first-order. For $N=4$, the result of the spectral gap has been presented. It has been found from this result that the spectral gap width continuously decreases as disorder increases and becomes zero at the CDW-AL transition points. This result suggests that the CDW-AL transition is of second-order. In the previous study[21], the SU(2) attractive Fermi systems with disorder have been investigated at finite temperatures, and it has been suggested that the pseudogap state, which arises from the fluctuations of the formation of Cooper-pairs, is observed. As discussed in our studies, the CDW phase is stabilized as the ground state at zero temperature in the two-dimensional square lattices with disorder, and thereby the fluctuations of the density-density correlation would be strong in the SU($N>2$) Fermi systems with disorder at finite temperatures. It may be possible for the fluctuations of the density-density correlation to affect the pseudogap state. This important problem is under consideration.

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