Existence of Smooth Solutions of the Navier-Stokes Equations

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Abstract. In this paper, we prove existence of smooth solutions of the Navier-Stokes equations that gives a positive answer to the problem proposed by Fefferman [3].

Key Words. Navier-Stokes equations, existence, smooth solutions

1 Introduction and the main results

The Navier-Stokes equations are given by

\[
\begin{align*}
    u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f, \\
    \text{div} u &= 0,
\end{align*}
\]

where \( \nu \) is a positive constant. The existence of smooth solutions of (1.1) is an open problem standing for a long time. Here we only mention some remarkable works that tried to solve it. Leray [4] showed existence of weak solutions. Under additional assumptions of more integrability of \( u \), Serrin [7] proved existence of smooth solutions. In [2], Constantin-Fefferman showed smoothness of solutions with a constraint on vorticity. Caffarelli-Kohn-Nirenberg [1] gave a partial regularity result of the Navier-Stokes equations that the dimension of the set of singular points is at most one, which improved the results of Scheffer [6]. Later Lin [5] simplified the proof. In this paper, we will solve this long standing problem. We state our main results as the following, which are corresponding to the statements (A) and (B) in [3] respectively.

Theorem 1.1. Let \( u_0 \) be any smooth, divergence-free vector field in \( \mathbb{R}^3 \) satisfying

\[
|\partial_x^\alpha u_0(x)| \leq C_\alpha K (1 + |x|)^{-K} \text{ on } \mathbb{R}^3, \text{ for any } \alpha \text{ and } K.
\]

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Then there exist smooth functions $p(x,t)$ and $u(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1.1) with $u(x,0) = u_0(x)$, $f \equiv 0$ and

$$\int_{\mathbb{R}^3} |u(x,t)|^2 dx < C \text{ for all } t \geq 0.$$  \hfill (1.3)

**Theorem 1.2.** Let $u_0$ be any smooth, divergence-free vector field in $\mathbb{R}^3$ satisfying

$$u_0(x + e_j) = u_0(x) \text{ for } 1 \leq j \leq 3. \hfill (1.4)$$

Then there exist smooth functions $p(x,t)$ and $u(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1.1) with $u(x,0) = u_0(x)$, $f \equiv 0$ and

$$u(x,t) = u(x + e_j, t) \text{ on } \mathbb{R}^3 \times [0,\infty) \text{ for } 1 \leq j \leq 3. \hfill (1.5)$$

**Remark 1.3.** (i) To prove Theorem 1.1 and 1.2, it is the key to get an a priori estimate with sufficient regularity. In order to express our idea clearly and neatly, we only focus on solving the problems proposed in [3] which are essential as considering this kind of a priori estimates of regularity of the Navier-Stokes equations. However, using our method, it is not hard to obtain the regularity of Navier-Stokes equations in higher spatial dimensions including interior estimates and boundary estimates, and the regularity of steady-states, where the righthand term $f$ need not to be vanishing.

(ii) In this paper, we only prove Theorem 1.1, while Theorem 1.2 can be proved similarly. \hfill \Box

The difficulty of proving the existence of smooth solutions of the Navier-Stokes equations arises from the following fact: When we multiply $F$ on both sides of the equation, where $F$ may contain $u$ and (or) derivative of $u$, and then integrate over $\mathbb{R}^3$, the bad terms coming from $(u \cdot \nabla)u$ and $\nabla p$ can not be controlled by the good terms coming from $u_t$ and $-\Delta u$ (except $F = u$).

To overcome this difficulty, we multiply a series of $\{F_k\}$ to the equation and take integral over $\mathbb{R}^3$. Then we have infinitely many inequalities and the bad terms in the former inequalities can be controlled by the good terms in the later inequalities. Therefore if we add all the inequalities together, all the bad terms can be controlled. To dealt with the nonlinear term with differential operators, it is convenient to use the Littlewood-Paley projections. Actually, we will arrive at a series of the form

$$\sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} \frac{||D^a P_j u(t)||^k_{B_k}}{2^{B_k}}, \hfill (1.6)$$
where $\sigma$ is a fixed integer and $B_k$ is given by (5.1).

Our first step is to show that if (1.6) is convergent, then it is bounded by a constant $C$ depending only on $u_0$, $\nu$ and $T$. We call this uniform bound estimate. Now, to obtain the a priori estimate, we only need to show that (1.6) is always convergent. Using the uniform bound estimate, we see (1.6) is always convergent on a closed time interval and then it is left to show that if (1.6) is convergent at $T'$, then it is convergent on $[T', T' + \delta]$ with some $\delta > 0$. To do this, we separate (1.6) into low frequency part (finite $j$) and high frequency part (infinite $j$).

Our second step is to show the convergence of the low frequency part, that is,

$$
\sum_{k=k_0}^{\infty} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u(t)||^k_{L^2}}{2^{B_k}}
$$

is convergent on $[T', T' + \delta]$, where $J_0$ is a large integer. This is hard. We design a different series

$$
\sum_{k=k_0}^{\infty} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u(t)||^k_{L^2}}{2^{\hat{B}_k}}
$$

and show that (1.8) can not blow up before any given time $T$ if its the initial value is small enough which can be satisfied by choosing $\hat{B}_k$ (defined by (6.2)) to be large enough. Then (1.8) is convergent on $[T', T' + \delta]$ which implies the convergence of (1.7). Devising suitable $B_k$ and $\hat{B}_k$ is the key to these two steps.

Our third step is to show a regularity improving result, from which it follows easily the convergence of the high frequency part, that is, the convergence of

$$
\sum_{k=k_0}^{\infty} \sum_{j=j_0+1}^{\infty} \frac{||D^\sigma P_j u(t)||^k_{L^2}}{2^{B_k}}
$$

on $[T', T' + \delta]$. This kind of regularity improving is not new essentially, but we need a special form. These three steps are the scheme of our proof of the new a priori estimates.

We organize the paper as the following. In Section 2, we study the Littlewood-Paley projections. In Section 3, some well known results of the Navier-Stokes equations are stated. In Section 4, we give some preparations for our attack. We demonstrate the above three steps of the proof of the new a priori estimates in Section 5, 6 and 7 respectively. Then in section 8, we show our new a priori estimates of the Navier-Stokes equations. The proof of Theorem 1.1 is given in the last section.

Throughout of this paper, the spatial dimension is confined to be 3 although all the main results can be extended to higher dimensions. We will use standard notations in this paper.

$[x]$: the maximal integer less than or equal to $x$;
\[ \|f\|_p: L^p \text{ norm in } \mathbb{R}^3 \text{ of function } f; \]
\[ \|f(t)\|_p: L^p \text{ norm in } \mathbb{R}^3 \text{ of function } f(x,t) \text{ defined on } \mathbb{R}^3 \times [0,T], \text{ if } t \text{ is clear from the context, it is simplified to be } \|f\|_p; \]
\[ D^\sigma f: \text{ all the } \sigma\text{-th order derivatives of } f \text{ with respect to space variables for any integer } \sigma \geq 0; \]
\[ P_j: \text{ Littlewood-Paley projection; } \]
\[ \otimes: \text{ tensor product; } \]
\[ C: \text{ universal constants which may be different at different occurrence.} \]

2 The Littlewood-Paley projections

In this section, we will study the Littlewood-Paley decomposition which is an important tool for analysis (cf.[8] for more discussion). For any test function \( f \), define the Fourier transformation and the inverse Fourier transformation by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{-2\pi ix \cdot \xi} \, dx
\]

and

\[
\check{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{2\pi ix \cdot \xi} \, dx
\]

respectively. Then we have

\[
(\hat{\check{f}}) = (\check{\hat{f}}) = f.
\]

Let \( \phi \in C^\infty(\mathbb{R}^3) \) be a real radial function supported on \( B_2 \) such that \( \phi \equiv 1 \) on \( B_1 \) and \( 0 \leq \phi \leq 1 \) in \( \mathbb{R}^3 \). Let

\[
\psi(\xi) = \phi(\xi) - \phi(2\xi)
\]

and

\[
\psi_j(\xi) = \psi(2^{-j}\xi) \text{ for } j = \ldots, -3, -2, -1, 0, 1, 2, 3, . . .
\]

Then we have

\[
\sum_{j=-\infty}^{+\infty} \psi_j(\xi) = 1 \text{ in } \mathbb{R}^3 \setminus \{0\}.
\]

We now define the Littlewood-Paley projection \( P_j \) by

\[
\hat{P_j}f(\xi) = \psi_j(\xi)\hat{f}(\xi).
\]

Then we have

\[
f = \sum_{j=-\infty}^{+\infty} P_j f. \quad (2.1)
\]
For simplicity, we denote
\[
P_{\leq j} = \sum_{k=-\infty}^{j} P_k \quad \text{and} \quad P_{\geq j} = \sum_{k=j}^{+\infty} P_k.
\]

**Lemma 2.1.** Let \(1 \leq q \leq +\infty\) and \(-\infty < j < +\infty\). Then
\[
||P_j f||_q \quad \text{and} \quad ||P_{\leq j} f||_q \leq C||f||_q,
\]
where \(C\) is a universal constant.

**Proof.** From
\[
(P_j f)(x) = (\check{\psi}_j * f)(x) = \int_{\mathbb{R}^3} \check{\psi}_j(y)f(x-y)dy
\]
and Minkowski’s inequality, we have
\[
||P_j f||_q \leq \int_{\mathbb{R}^3} |\check{\psi}_j(y)||f||_q dy = ||\check{\psi}_j||_1||f||_q.
\]
Since
\[
||\check{\psi}_j||_1 = ||\check{\psi}_1||_1,
\]
we see that
\[
||P_j f||_q \leq ||\check{\psi}_1||_1||f||_q.
\]
Similarly, we have
\[
||P_{\leq j} f||_q \leq ||(\hat{\phi}(2^{-j} \cdot))||_1||f||_q = ||\hat{\phi}||_1||f||_q.
\]
Let
\[
C = \max\{||\check{\psi}_1||_1, ||\hat{\phi}||_1\}
\]
and the proof of Lemma 2.1 is complete. \(\square\)

From (2.1) and Lemma 2.1, we have the following so called cheap Littlewood-Paley inequality (cf.[8]).

**Theorem 2.2.** Let \(1 \leq q \leq +\infty\). Then
\[
C \sup_j ||P_j f||_q \leq ||f||_q \leq \sum_{j=-\infty}^{+\infty} ||P_j f||_q,
\]
where \(C\) is a universal constant.
Lemma 2.3 (Bernstein’s inequality)\[8\]. Let $1 \leq q \leq q' \leq +\infty$ and $j$ be an integer. Then
\[
\|P_j f\|_{q'} \leq C 2^j \left(\frac{3}{q} - \frac{3}{q'}\right) \|P_j f\|_q,
\]
where $C$ is a universal constant.

**Proof.** Let $\frac{1}{q} = \frac{\lambda}{q}$ or $\lambda = \frac{2}{q} + \lambda$ and then
\[
\|P_j f\|_{q'} \leq \|P_j f\|_q \|P_j f\|_{1-\lambda}^{1-\lambda}.
\tag{2.2}
\]
From $P_j f = \psi_j \hat{f} = \phi(2^{-j-1}) \psi_j \hat{f}$, we deduce
\[
P_j f = (\phi(2^{-j-1})) \ast P_j f.
\]
It follows that
\[
\|P_j f\|_\infty \leq \|\phi(2^{-j-1})\|_{r} \|P_j f\|_q = (2^{j+1})^{\frac{r-1}{r}} \|\phi\|_r \|P_j f\|_q,
\]
where $\frac{1}{r} + \frac{1}{q} = 1$ or $r = \frac{q}{q-1}$. It is easy to see that
\[
\frac{r-1}{r} (1-\lambda) = \frac{1}{q} - \frac{1}{q'}.
\]
In view of (2.2),
\[
\|P_j f\|_{q'} \leq \|P_j f\|_q \|P_j f\|_{\infty-\lambda} \leq 2^{(j+1)(\frac{3}{q} - \frac{3}{q'})} \|\phi\|_r \|P_j f\|_q.
\]
Since
\[
\|\phi\|_r = \left( \int_{\mathbb{R}^3} \phi(x) \frac{1+|x|^4}{1+|x|^4} dx \right)^{\frac{1}{r}} \leq \sup_{x \in \mathbb{R}^3} (|\phi(x)|(1+|x|^4)) \left( \int_{\mathbb{R}^3} \left( \frac{1}{1+|x|^4} \right)^r dx \right)^{\frac{1}{r}}
\]
\[
\leq \sup_{x \in \mathbb{R}^3} (|\phi(x)|(1+|x|^4)) \left( \int_{\mathbb{R}^3} \frac{1}{1+|x|^4} dx \right)^{\frac{1}{r}}
\]
which can be bounded by a universal constant, we have the conclusion. $\square$

**Lemma 2.4.** Let $j$ be an integer, $f$ and $g$ be two test functions. We have the
following product inequality,

\[ |P_j(fg)| \leq \]

\[
\left| P_j \left\{ \left( \sum_{m=-\infty}^{j-3} P_m f \right) \left( \sum_{m'=j-2}^{j+2} P_{m'} g \right) \right\} \right| + 
\left| P_j \left\{ \left( \sum_{m=-\infty}^{j+2} P_m f \right) \left( \sum_{m'=j-2}^{j+2} P_{m'} g \right) \right\} \right| + 
\left| P_j \left\{ \left( \sum_{m=j+3}^{\infty} P_m f \right) \left( \sum_{m'=m-3}^{m'+3} P_{m'} g \right) \right\} \right|
\]

\[
\left| P_j \left\{ \left( \sum_{m=-\infty}^{j-3} P_m f \right) \left( \sum_{m'=m-3}^{\infty} P_{m'} g \right) \right\} \right| + 
\left| P_j \left\{ \left( \sum_{m=j+3}^{\infty} P_m f \right) \left( \sum_{m'=m-3}^{m'+3} P_{m'} g \right) \right\} \right|.
\]

**Proof.** From (2.1), it follows that

\[
P_j(fg) = P_j \left\{ \left( \sum_{m=-\infty}^{j-3} P_m f \right) \left( \sum_{m'=m-3}^{\infty} P_{m'} g \right) \right\}
\]

\[
= P_j \left\{ \left( \sum_{m=-\infty}^{j-3} P_m f \right) + \sum_{m=j-2}^{j+2} P_m f + \sum_{m=j+3}^{\infty} P_m f \right\}
\]

\[
\times \left( \sum_{m=-\infty}^{j-3} P_m g + \sum_{m'=m-3}^{m'+3} P_{m'} g \right) \].
\]

Since

\[
P_j (P_{\leq j-3} f P_{\leq j-3} g) = 0
\]

and

\[
P_j (P_m f P_{m'} g) = 0
\]
as \( m \geq j + 3 \) and \( |m - m'| > 3 \), we have the conclusion clearly. \( \square \)

**Lemma 2.5.** Let \( 1 = j_0 \leq j \) be two integers, \( 1 \leq q \leq \infty, 2 \leq q_0, q_1 \leq \infty \) be three real numbers, and \( f \) and \( g \) be two test functions. If \( \frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1} \), then

\[
||P_j(fg)||_q \leq C \left\{ \alpha_j ||f||_2 ||g||_2 + \sum_{m=\max\{j_0, j-2\}}^{\infty} ||P_m f||_{q_1} ||g||_{q_0} + \sum_{m'=\max\{j_0, j-2\}}^{\infty} ||P_{m'} g||_{q_1} ||f||_{q_0} \right\},
\]
where \( C \) is a universal constant and
\[
\alpha_j = \begin{cases} 
1 & \text{as } j = 1, 2; \\
0 & \text{as } j \geq 3.
\end{cases}  \tag{2.3}
\]

**Proof.** From Lemma 2.4 and Hölder’s inequality, it follows that
\[
\|P_j(fg)\|_q \leq \|P_{\leq j-3}f\|_{q_0} \sum_{m'=j-2}^{j+2} \|P_{m'}g\|_{q_1} + \|P_{\leq j-3}g\|_{q_0} \sum_{m=j-2}^{j+2} \|P_mf\|_{q_1}
\]
\[
+ \sum_{m=j-2}^{j+2} \sum_{m'=j-2}^{j+2} \|P_mf\|_{q_1} \|P_{m'}g\|_{q_1} + \sum_{m=j+3}^{\infty} \sum_{m'=m+3}^{m+3} \|P_mf\|_{q_1} \|P_{m'}g\|_{q_1}.
\tag{2.4}
\]

From Lemma 2.1 and 2.3, we have
\[
\|P_{\leq j-3}f\|_{q_0} \sum_{m'=j-2}^{j+2} \|P_{m'}g\|_{q_1} = \|P_{\leq j-3}f\|_{q_0} \left( \sum_{m=j-2}^{j+2} \|P_{m'}g\|_{q_1} \right) \leq C \left( \alpha_j \|f\|_2 \|g\|_2 + \|f\|_{q_0} \sum_{m'=\max\{j_0, j-2\}}^{j+2} \|P_{m'}g\|_{q_1} \right),
\]
where \( C \) is a universal constant and \( \alpha_j \) is given by (2.3). Similarly,
\[
\|P_{\leq j-3}g\|_{q_0} \sum_{m=j-2}^{j+2} \|P_mf\|_{q_1} \leq C \left( \alpha_j \|f\|_2 \|g\|_2 + \|f\|_{q_0} \sum_{m=\max\{j_0, j-2\}}^{j+2} \|P_mg\|_{q_1} \right)
\]
and
\[
\sum_{m=j-2}^{j+2} \sum_{m'=j-2}^{j+2} \|P_mf\|_{q_1} \|P_{m'}g\|_{q_1} \leq C \left( \alpha_j \|f\|_2 \|g\|_2 + \|f\|_{q_0} \sum_{m'=\max\{j_0, j-2\}}^{j+2} \|P_{m'}g\|_{q_1} \right)
\]
where \( C \) are universal constants and \( \alpha_j \) is given by (2.3). From Lemma 2.1, we have
\[
\sum_{m=j+3}^{\infty} \sum_{m'=m+3}^{m+3} \|P_mf\|_{q_1} \|P_{m'}g\|_{q_0} + \sum_{m'=j+3}^{\infty} \sum_{m=j+3}^{m+3} \|P_mf\|_{q_0} \|P_{m'}g\|_{q_1}
\leq C \left( \sum_{m=j+3}^{\infty} \|P_mf\|_{q_1} \|g\|_{q_0} + \sum_{m'=j+3}^{\infty} \|f\|_{q_0} \|P_{m'}g\|_{q_1} \right),
\]

where $C$ is a universal constant. Plug the above inequalities into (2.4) and then we have the conclusion. □

The following lemma gives one of the key reason why the Littlewood-Paley projection is useful. We refer to [8] for its proof.

**Lemma 2.6.** Let $1 \leq q \leq \infty$ be a real number and $j$ be an integer. Then we have
\[
C_1 2^j \|P_j f\|_q \leq \|\nabla P_j f\|_q \leq C_2 2^j \|P_j f\|_q,
\]
where $C_1$ and $C_2$ are universal constants.

**Lemma 2.7.** Let $1 = j_0 \leq j$, $0 \leq \sigma$ be three integers, $1 \leq q \leq \infty, 2 \leq q_0, q_1 \leq \infty$ be three real numbers, and $f$ and $g$ be two test functions. If $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$, then
\[
\|D^\sigma P_j (fg)\|_q \leq C(\sigma) \left\{ \alpha_j \|f\|_2 \|g\|_2 + \sum_{m = \max\{j_0, j-2\}}^{+\infty} 2^{(j-m)\sigma} \|D^\sigma P_m f\|_{q_1} \|g\|_{q_0} \right. + \left. \sum_{m' = \max\{j_0, j-2\}}^{+\infty} 2^{(j-m')\sigma} \|D^\sigma P_{m'} g\|_{q_1} \|f\|_{q_0} \right\},
\]
where $C(\sigma)$ is a constant depending only on $\sigma$ and $\alpha_j$ is given by (2.3).

**Proof.** From Lemma 2.6, we have
\[
\|D^\sigma P_j (fg)\|_q \leq C(\sigma) 2^{j\sigma} \|P_j (fg)\|_q,
\]
where $C(\sigma)$ is a constant depending only on $\sigma$. From Lemma 2.5 and 2.6, we have
\[
\|P_j (fg)\|_q \leq C \left\{ \alpha_j \|f\|_2 \|g\|_2 + \sum_{m = \max\{j_0, j-2\}}^{\infty} \|P_m f\|_{q_1} \|g\|_{q_0} \right. + \left. \sum_{m' = \max\{j_0, j-2\}}^{\infty} \|P_{m'} g\|_{q_1} \|f\|_{q_0} \right\},
\]
where $C$ is a universal constant, $C(\sigma)$ is a constant depending only on $\sigma$, and $\alpha_j$ is given by (2.3). Combining it with (2.5), we see the conclusion clearly. □
Corollary 2.8. Suppose all the assumptions of Lemma 2.7 hold. Let \( q > 1 \) be a real number. Then
\[
||D^\sigma P_j(fg)||_q^q \leq C(\sigma)^q \left\{ \alpha_j ||f||_2 ||g||_2 + \sum_{m = \max\{j_0, j-2\}}^{+\infty} 2^{(j-m)\sigma} ||D^\sigma P_m f||_{q_1} ||g||_{q_0} \right. \\
+ \left. \sum_{m' = \max\{j_0, j-2\}}^{-\infty} 2^{(j-m')\sigma} ||D^\sigma P_{m'} g||_{q_1} ||f||_{q_0} \right\},
\]
where \( C(\sigma) \) is a constant depending only on \( \sigma \) and \( \alpha_j \) is given by (2.3).

Proof. From Lemma 2.7, we have
\[
||D^\sigma P_j(fg)||_q^q \leq C(\sigma)^q \left\{ \alpha_j ||f||_2 ||g||_2 + \sum_{m = \max\{j_0, j-2\}}^{+\infty} 2^{(j-m)\sigma} ||D^\sigma P_m f||_{q_1} ||g||_{q_0} \right. \\
+ \left. \sum_{m' = \max\{j_0, j-2\}}^{-\infty} 2^{(j-m')\sigma} ||D^\sigma P_{m'} g||_{q_1} ||f||_{q_0} \right\}^q,
\]
where \( C(\sigma) \) is a constant depending only on \( \sigma \) and \( \alpha_j \) is given by (2.3). From Hölder’s inequality, we have
\[
\alpha_j ||f||_2 ||g||_2 + \sum_{m = \max\{j_0, j-2\}}^{+\infty} 2^{(j-m)\sigma} ||D^\sigma P_m f||_{q_1} ||g||_{q_0} \]
\[
+ \sum_{m' = \max\{j_0, j-2\}}^{-\infty} 2^{(j-m')\sigma} ||D^\sigma P_{m'} g||_{q_1} ||f||_{q_0} \]
\[
\leq \left\{ \alpha_j + \sum_{m = \max\{j_0, j-2\}}^{+\infty} 2^{(j-m)\sigma} + \sum_{m' = \max\{j_0, j-2\}}^{-\infty} 2^{(j-m')\sigma} \right\}^\frac{1}{q'},
\]
\[
\times \left\{ \alpha_j ||f||_2 ||g||_2 + \sum_{m = \max\{j_0, j-2\}}^{+\infty} 2^{(j-m)\sigma} ||D^\sigma P_m f||_{q_1} ||g||_{q_0} \right. \\
+ \left. \sum_{m' = \max\{j_0, j-2\}}^{-\infty} 2^{(j-m')\sigma} ||D^\sigma P_{m'} g||_{q_1} ||f||_{q_0} \right\}^\frac{1}{q'},
\]
\[
\leq (1 + 8^\sigma) \left\{ \alpha_j ||f||_2 ||g||_2 + \sum_{m = \max\{j_0, j-2\}}^{+\infty} 2^{(j-m)\sigma} ||D^\sigma P_m f||_{q_1} ||g||_{q_0} \right. \\
+ \left. \sum_{m' = \max\{j_0, j-2\}}^{-\infty} 2^{(j-m')\sigma} ||D^\sigma P_{m'} g||_{q_1} ||f||_{q_0} \right\}^\frac{1}{q'},
\]
for some \( \sigma \) depending only on \( \sigma \) and \( \alpha_j \).
where $\frac{1}{q} + \frac{1}{q'} = 1$. Combining it with (2.6), we have the conclusion. □

3 Classical results

In this section, we state some classical results of Navier-Stokes equations, which are all well known. Although we can not find exact references for some of them, we still omit the proofs here. We refer to [9] for the definition of Leray’s weak solutions and strong solutions of Navier-Stokes equations. Here the domain of the space is $R^3$. We should point out that any strong solutions will be Leray’s weak solutions.

**Theorem 3.1.** Let $u$ be a Leray’s weak solution of (1.1) with $f = 0$ and $u(0) = u_0$ ($\text{div} u_0 = 0$). Then

$$||u(t)||_2 \leq ||u_0||_2$$

and

$$\int_0^t ||\nabla u(s)||_2^2 ds \leq \frac{1}{2\nu} ||u_0||_2^2$$

for any $t \geq 0$.

**Corollary 3.2.** Let $u$ be given by Theorem 3.1. Then

$$\int_0^t ||u(s)||_{1/4}^\frac{8}{3} ds \leq \frac{C}{\nu} ||u_0||_2^\frac{8}{3}$$

for any $t \geq 0$, where $C$ is a universal constant.

**Proof.** From interpolation inequality and Sobolev’s embedding inequality,

$$||u||_4 \leq ||u||_2^\frac{1}{3} ||u||_0^\frac{2}{3} \leq C ||u||_2^\frac{1}{3} ||\nabla u||_2^\frac{2}{3},$$

where $C$ is a universal constant. Therefore

$$\int_0^t ||u(s)||_{1/4}^\frac{8}{3} ds \leq C \int_0^t ||u(s)||_2^\frac{8}{3} ||\nabla u(s)||_2^2 ds.$$

From Theorem 3.1, we deduce the conclusion easily. □

**Theorem 3.3 (Uniqueness).** Let $u_0 \in H^1(R^3)$ be a divergence-free vector field. Then the strong solution of (1.1) with $u(0) = u_0$ and $f \equiv 0$ is unique.

**Theorem 3.4 (Short time existence).** Let $u_0 \in H^1(R^3)$ be a divergence-free vector field. Then there exists $T^* > 0$ depending only on $\nu$ and $||\nabla u_0||_2$ such that the strong solution of (1.1) exists on the time interval $[0, T^*]$ with $u(0) = u_0$ and $f \equiv 0$. 

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Theorem 3.5 (Regularity). Suppose $u_0$ is a smooth, divergence-free vector field in $R^3$ satisfying (1.2). Let $u$ and $p$ be the strong solution of (1.1) on the time interval $[0, T]$ with $u(0) = u_0$ and $f \equiv 0$. Then $u$ and $p$ are smooth and $u, p \in C^1([0, T], W^{m, 2}(R^3) \cap W^{m, \infty}(R^3))$ for any integer $m \geq 1$.

Theorem 3.6 (Blow up). Let $u_0 \in H^1(R^3)$ be a divergence-free vector field. Suppose $0 < T < \infty$ and $[0, T)$ is the largest time interval that (1.1) has strong solution with $u(0) = u_0$ and $f \equiv 0$. Then

$$\limsup_{t \to T^-} ||u(t)||_\infty = \infty.$$ 

Remark 3.7. In Theorem 3.5, if we do not assume that $u_0$ is smooth and satisfies (1.2), then we will have for any $0 < \hat{T} < T$, $u$ and $p$ are smooth on the time interval $[\hat{T}, T]$ and $u, p \in C^1([\hat{T}, T], W^{m, 2}(R^3) \cap W^{m, \infty}(R^3))$ for any integer $m \geq 1$. \hfill \square

Finally, we prove a simple lemma.

Lemma 3.8. Let $q \geq 2$ be a real number and $\sigma$ and $j$ be two positive integers. Suppose $u$ and $p$ satisfy (1.1) with $f \equiv 0$. Then

$$||D^\sigma P_j p||_q \leq Cq ||D^\sigma P_j (u \otimes u)||_q,$$

where $C$ is a universal constant.

Proof. Take divergence on the both sides of the first equation of (1.1) and then

$$\sum_{i,j=1}^3 \partial_i \partial_j (u_i u_j) + \Delta p = 0.$$

Now take the operator $D^\sigma P_j$ on the both sides and then

$$\sum_{i,j=1}^3 D^\sigma P_j \partial_i \partial_j (u_i u_j) + D^\sigma P_j \Delta p = 0.$$

From Calderón-Zygmund’s estimate, we have (3.1). \hfill \square
4 Some lemmas

In this section, we will show some lemmas which will be used to prove our new a priori estimates.

Lemma 4.1. Let $k \geq 2$ and suppose $f \in W^{2,2}(\mathbb{R}^3) \cap W^{2,\infty}(\mathbb{R}^3)$. Then we have
\[ ||f||_{3k}^k \leq C k^2 \int_{\mathbb{R}^3} |f|^{k-2} |\nabla f|^2, \]
where $C$ is a universal constant.

Proof. From Sobolev’s embedding inequality, we have
\[ ||f||_{3k}^k = |||f||_6^2 \leq C |||\nabla f||_6^2 \leq C k^2 \int_{\mathbb{R}^3} |f|^{k-2} |\nabla f|^2, \]
where $C$ is a universal constant. □

Lemma 4.2. Let $T > 0$ be a real number and \( \{f_k(t)\}_{k=1}^\infty \) and \( \{g_k(t)\}_{k=1}^\infty \subset C^1[0,T] \) be two nonnegative function sequences. Suppose
\[ \sum_{k=1}^\infty f_k(t) \leq B \text{ and } \frac{d}{dt} f_k(t) \leq g_k(t), \forall 0 \leq t \leq T \text{ and } k \geq 1, \quad (4.1) \]
where $B$ is a constant. Then we have
\[ \frac{d}{dt} \sum_{k=1}^\infty f_k(t) \leq \sum_{k=1}^\infty g_k(t) \]
for any $0 \leq t \leq T$.

Proof. From Lebesgue’s dominated convergence theorem, (4.1) and Fatou’s Lemma, we deduce
\[ \int_0^T \phi(t) \frac{d}{dt} \sum_{k=1}^\infty f_k(t) = - \int_0^T \sum_{k=1}^\infty f_k(t) \frac{d}{dt} \phi(t) = - \sum_{k=1}^\infty \int_0^T f_k(t) \frac{d}{dt} \phi(t) \]
\[ = \sum_{k=1}^\infty \int_0^T \phi(t) \frac{d}{dt} f_k(t) \leq \sum_{k=1}^\infty \int_0^T \phi(t) g_k(t) \leq \int_0^T \phi(t) \sum_{k=1}^\infty g_k(t) \]
for any test function $\phi \geq 0$. This implies the conclusion clearly. □

Lemma 4.3. Let $\epsilon, T, B > 0$ and $M > 1$ be real numbers. Let $0 \leq F \in C^1[0,T]$ and $0 \leq g \in C[0,T]$. Suppose
\[ \int_0^T g(t) \leq B, \quad F(0) \leq \epsilon \]
\[
\frac{d}{dt} F(t) \leq g(t) \left( \epsilon + F(t) + F^M(t) \right).
\] (4.2)

If
\[
\epsilon \leq \frac{1}{(3\epsilon^B)^{\frac{1}{2}}},
\] (4.3)
then we have
\[
F(t) \leq 3\epsilon e^B
\] (4.4)
for any \(0 \leq t \leq T\).

**Proof.** Suppose by the contradiction that (4.4) is not true. Then since
\[
F(0) \leq \epsilon < 3\epsilon e^B,
\]
there exists \(T' \in (0, T)\) such that (4.4) holds for \(t \in [0, T']\) and
\[
F(T') = 3\epsilon e^B.
\] (4.5)

It is easy to see that (4.3) and (4.4) which we assume holds for \(t \in [0, T']\) imply that
\[
F^M(t) \leq \epsilon \text{ for } t \in [0, T'].
\]

From (4.2), it follows that
\[
\frac{d}{dt} F(t) \leq g(t) (2\epsilon + F(t)) \text{ for } t \in [0, T']
\]
or
\[
\ln \frac{2\epsilon + F(t)}{2\epsilon + F(0)} \leq \int_0^{T'} g(t) \leq B \text{ for } t \in [0, T'].
\]

Therefore
\[
F(T') \leq (2\epsilon + F(0)) e^B - 2\epsilon < 3\epsilon e^B.
\]
This contradicts with (4.5). \(\Box\)

**Lemma 4.4**\(^1\). Let \(k \geq 2\) and \(\sigma \geq 1\) be integers. Suppose \(f \in W^{2,2}(\mathbb{R}^3) \cap W^{2,\infty}(\mathbb{R}^3)\). Then we have
\[
\int_{\mathbb{R}^3} |D^\sigma f|^k \leq C(\sigma)k^2 \left( \int_{\mathbb{R}^3} |D^\sigma f|^{k-2} |D^{\sigma+1} f|^2 \right)^{\frac{k}{2}} \left( \int_{\mathbb{R}^3} |D^{\sigma-1} f|^k \right)^{\frac{1}{k}},
\] (4.6)
where \(C(\sigma)\) is a constant depending only on \(\sigma\).

\(^1\)This lemma is given by Prof. Lihe Wang.
Proof. We only need to prove (4.6) with the assumption that $f \in C^\infty_c(R^3)$. From Green's and Hölder's formula, we see

\[
\int_{R^3} |D^\sigma f|^k = - \int_{R^3} D^{\sigma-1} f \, D\left(|D^\sigma f|^{k-2} D^\sigma f\right)
\leq C(\sigma)(k-1) \int_{R^3} |D^{\sigma-1} f||D^\sigma f|^{k-2}|D^{\sigma+1} f|
\leq C(\sigma)(k-1) \left(\int_{R^3} |D^{\sigma-1} f|^k\right)^{\frac{k}{k+2}} \left(\int_{R^3} |D^\sigma f|^{k-2}|D^{\sigma+1} f|^2\right)^{\frac{1}{2}},
\]

where $C(\sigma)$ is a constant depending only on $\sigma$. It follows (4.6) clearly. \(\Box\)

Lemma 4.5. Let $k \geq 2$, $\sigma \geq 1$ and $j$ be integers. Suppose $f \in W^{2,2}(R^3) \cap W^{2,\infty}(R^3)$. Then we have

\[
\int_{R^3} |D^\sigma P_j f|^k \leq C(\sigma)k^2 2^{-2j} \int_{R^3} |D^\sigma P_j f|^{k-2}|D^{\sigma+1} P_j f|^2,
\]

where $C(\sigma)$ is a constant depending only on $\sigma$.

Proof. From Lemma 4.4, we have

\[
\int_{R^3} |D^\sigma P_j f|^k \leq C(\sigma)k^2 \left(\int_{R^3} |D^\sigma P_j f|^{k-2}|D^{\sigma+1} P_j f|^2\right)^{\frac{k}{k+2}} \left(\int_{R^3} |D^{\sigma-1} P_j f|^k\right)^{\frac{k+2}{k}}.
\]

From Lemma 2.6, we have

\[
\left(\int_{R^3} |D^{\sigma-1} P_j f|^k\right)^{\frac{k}{k+2}} \leq C2^{-\frac{2k}{k+2}} \left(\int_{R^3} |D^\sigma P_j f|^k\right)^{\frac{k}{k+2}},
\]

where $C$ is a universal constant. Then we see (4.7) clearly. \(\Box\)

Lemma 4.6. Let $k_0 \geq 1$, $\sigma \geq 0$ be two integers, $T > 0$ be a real number and $u \in C([0,T], W^{\sigma+1,k}(R^3))$ for any $k \geq k_0$. Then for any $k \geq k_0$,

\[
\sum_{j=j_0}^\infty \|D^\sigma P_j u(t)\|_k^k
\]

is continuous as a function on $[0,T]$, and there exists a constant $B > 0$ such that

\[
\sum_{j=j_0}^\infty \|D^\sigma P_j u\|_k^k \leq B
\]

for any $0 \leq t \leq T$.  

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Proof. From Lemma 2.6, we have
\[ ||D^\sigma P_j u(t)||_k \leq C2^{-j} ||D^{\sigma+1} P_j u(t)||_k \leq C2^{-j} \sup_{t \in [0,T]} ||D^{\sigma+1} u(t)||_k. \]

Since
\[ \sum_{j=j_0}^{\infty} \left( C2^{-j} \sup_{t \in [0,T]} ||D^{\sigma+1} u(t)||_k \right)^k \]
is convergent, we see the conclusion clearly. \( \square \)

Lemma 4.7. Let \( k_0 \geq 10, j_0 \geq 1 \) and \( \sigma \geq 1 \) be three integers, \( B > 0 \) be a real number and \( u_0 \) be a function satisfying (1.2). Then there exists \( \tilde{B}_0 > 0 \) depending only on \( u_0 \) and \( \sigma \) such that if \( B \geq \tilde{B}_0 \), then
\[ \sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} \frac{||D^\sigma P_j u_0||^k_2}{2^{B(1-\frac{2}{\sqrt{\sigma}})k}} \leq 2^{-4}. \]

Proof. From Lemma 2.6, we have
\[ \sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} \frac{||D^\sigma P_j u_0||^k_2}{2^{B(1-\frac{2}{\sqrt{\sigma}})k}} \leq \sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} \left( C2^{-j} ||D^{\sigma+1} P_j u_0||_k \right)^k \]
\[ \leq \sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} 2^{-jk} \left( \frac{C||D^{\sigma+1} u_0||_k}{2^{B(1-\frac{2}{\sqrt{\sigma}})k}} \right)^k \leq \sum_{k=k_0}^{\infty} \left( \frac{C||D^{\sigma+1} u_0||_k}{2^{B(1-\frac{2}{\sqrt{\sigma}})k}} \right)^k. \]

where \( C \) is universal constant. Let
\[ \tilde{B}_0 = 2C \left( ||D^{\sigma+1} u_0||_2 + ||D^{\sigma+1} u_0||_\infty \right) + 4. \]

By interpolation inequality,
\[ ||D^{\sigma+1} u_0||_k \leq ||D^{\sigma+1} u_0||_2^\beta ||D^{\sigma+1} u_0||_\infty^{1-\beta} \leq ||D^{\sigma+1} u_0||_2 + ||D^{\sigma+1} u_0||_\infty \]
\[ \leq \frac{\tilde{B}_0 - 4}{2C}, \]
we see that if \( B \geq \tilde{B}_0 \),
\[ \sum_{k=k_0}^{\infty} \frac{\left( C||D^{\sigma+1} u_0||_k \right)^k}{2^{B(1-\frac{2}{\sqrt{\sigma}})k}} \leq \sum_{k=k_0}^{\infty} \frac{1}{2^{Bk}} \left( \frac{C||D^{\sigma+1} u_0||_k}{2^{B(1-\frac{2}{\sqrt{\sigma}})k}} \right)^k \]
\[ \leq \sum_{k=k_0}^{\infty} \frac{1}{2^{Bk}} \left( \frac{B/2}{2^{B/2}} \right)^k \leq 2^{-4}, \]
where \( B \geq \tilde{B}_0 \geq 4 \) is used. \( \square \)
5 Uniform bound estimate

In this section, we will prove the following uniform bound estimate, Theorem 5.1, which is the first key step to show our new a priori estimates. We design the following series

$$\sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} \frac{||D^\sigma P_j u(t)||^k_{L^2}}{2^{B_k}},$$

where \( j_0 = 1, k_0 = 100 \) and the power

$$B_k = \left( B + 1 + \frac{1}{\sqrt{k}} \right) k$$

with positive constant \( B \) to be given later.

For convenience, we define the following Condition (S) of \( u \) and \( p \):

\[
(S) \begin{cases}
  (i) & u \text{ and } p \in C^1([0, T], W^{m,2}(R^3) \cap W^{m,\infty}(R^3)) \\
  \quad \text{for any integer } m \geq 1; \\
  (ii) & u \text{ and } p \text{ satisfy (1.1) with } f \equiv 0 \text{ and } u(x, 0) = u_0(x),
\end{cases}
\]

for the given real number \( T > 0 \) and the given suitable function \( u_0 \).

Our main theorem of this section is:

**Theorem 5.1 (Uniform bound).** Let \( \sigma = 2, j_0 = 1, k_0 = 100, 0 < T' \leq T \) and \( B > 0 \) be real numbers, \( B_k \) be given by (5.1) for any \( k \geq k_0 \) and \( u_0 \) be a function satisfying (1.2). Suppose \( u \) and \( p \) satisfy Condition (S). If

$$\sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} \frac{||D^\sigma P_j u(t)||^k_{L^2}}{2^{B_k}} \leq B$$

for any \( 0 \leq t \leq T' \), where \( B \) is a constant, then we have

$$\sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} \frac{||D^\sigma P_j u(t)||^k_{L^2}}{2^{B_k}} \leq C \sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} \frac{||D^\sigma P_j u_0||^k_{L^2}}{2^{B_k}} + C - 1$$

with

$$C = e^{C(1+\nu^{-2})(1+||u_0||^2_2)T+\nu^{-1}||u_0||^2_2}$$

for any \( 0 \leq t \leq T' \), where \( C > 0 \) is a universal constant.

**Proof.** Let \( j \geq j_0, k \geq k_0 \) and \( 0 \leq t \leq T' \). We divide the proof into 6 steps.
Step 1. We first take the Littlewood-Paley projection on both sides of Navier-Stokes equation:

\[ P_j u_t - \nu P_j \Delta u + P_j (u \cdot \nabla) u + P_j \nabla p = 0 \]

and then use the differential operator \( D^\sigma \) on both sides:

\[ D^\sigma P_j u_t - \nu D^\sigma P_j \Delta u + D^\sigma P_j ((u \cdot \nabla) u) + D^\sigma P_j \nabla p = 0. \]

Finally, multiply \( |D^\sigma P_j u|^{k-2} D^\sigma P_j u \) and then integrate over \( \mathbb{R}^3 \) on both sides:

\[
\int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} D^\sigma P_j u D^\sigma P_j u_t \\
- \nu \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} D^\sigma P_j u D^\sigma P_j \Delta u \\
+ \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} D^\sigma P_j u D^\sigma P_j ((u \cdot \nabla) u) \\
+ \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} D^\sigma P_j u D^\sigma P_j \nabla p = 0.
\]

It is easy to see that

\[
\int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} D^\sigma P_j u D^\sigma P_j u_t = \frac{1}{k} \frac{d}{dt} \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k}.
\]

(5.6)

From Condition (S), by Green’s formula,

\[
- \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} D^\sigma P_j u D^\sigma P_j \Delta u = \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |D^\sigma P_j \nabla u|^2 \\
+ (k - 2) \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla| D^\sigma P_j u||^2.
\]

(5.7)

Similarly, we also have

\[
\int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} D^\sigma P_j u D^\sigma P_j (u \cdot \nabla) u = \\
\sum_{\alpha, \beta = 1}^{3} \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} D^\sigma P_j u \partial_\alpha D^\sigma P_j (u_\alpha u_\beta) = \\
-(k - 2) \sum_{\alpha, \beta = 1}^{3} \int_{\mathbb{R}^3} D^\sigma P_j (u_\alpha u_\beta) |D^\sigma P_j u|^{k-3} \partial_\alpha |D^\sigma P_j u| D^\sigma P_j u_\beta \\
- \sum_{\alpha, \beta = 1}^{3} \int_{\mathbb{R}^3} D^\sigma P_j (u_\alpha u_\beta) |D^\sigma P_j u|^{k-2} D^\sigma P_j \partial_\alpha u_\beta
\]

(5.8)
and
\[
\int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} D^\sigma P_j u D^\sigma P_j \nabla p
\]
\[
= -\sum_{\alpha}^3 \int_{\mathbb{R}^3} \partial_\alpha (|D^\sigma P_j u|^{k-2} D^\sigma P_j u_{\alpha}) D^\sigma P_j p
\]
\[
= -(k-2) \sum_{\alpha}^3 \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-3} D^\sigma P_j u_{\alpha} \partial_\alpha D^\sigma P_j u |D^\sigma P_j p|
\]
(5.10)

where \( u = (u_1, u_2, u_3) \) and \( \text{div}u = 0 \) is used.

Plug (5.7)-(5.10) into (5.6) and we have
\[
\frac{1}{k} \frac{d}{dt} \int_{\mathbb{R}^3} |D^\sigma P_j u|^k + \nu \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |D^\sigma P_j \nabla u|^2
\]
\[
+ (k-2) \nu \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla|D^\sigma P_j u|^2
\]
\[
= (k-2) \sum_{\alpha,\beta}^3 \int_{\mathbb{R}^3} D^\sigma P_j (u_{\alpha} u_{\beta}) |D^\sigma P_j u|^{k-3} \partial_\alpha D^\sigma P_j u |D^\sigma P_j u| D^\sigma P_j u_{\beta}
\]
\[
+ \sum_{\alpha,\beta}^3 \int_{\mathbb{R}^3} D^\sigma P_j (u_{\alpha} u_{\beta}) |D^\sigma P_j u|^{k-2} |D^\sigma P_j \partial_\alpha u_{\beta}|
\]
\[
+ (k-2) \sum_{\alpha,\beta}^3 \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-3} D^\sigma P_j u_{\alpha} \partial_\alpha D^\sigma P_j u |D^\sigma P_j P_j p
\]
\[
:= (k-2)I_1 + I_2 + (k-2)I_3.
\]

**Step 2. Estimates of \( I_1, I_2 \) and \( I_3 \) in (5.11).** From Hölder’s inequality,
\[
|I_1| \leq \int_{\mathbb{R}^3} |D^\sigma P_j (u \otimes u)| |D^\sigma P_j u|^{k-2} |\nabla|D^\sigma P_j u|
\]
\[
\leq ||D^\sigma P_j (u \otimes u)|| \frac{2^k}{\mathbb{K}} \left( ||D^\sigma P_j u||^k_k \right)^{k-2} \left( ||D^\sigma P_j u||^5_{5k} \right)^{k-2}
\]
\[
\times \left( \frac{\int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla|D^\sigma P_j u|^2}{2} \right)^{\frac{1}{2}}.
\]
From Lemma 4.5, it follows that
\[
||D^\sigma P_j u||^k_k \leq C k^2 2^{-2j} \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^2.
\]
And then

\[ |I_1| \leq Ck^{\frac{k-2}{5} - \frac{k-2}{6} - \frac{k-2}{6(k+1)}} \left( \|D^\sigma P_j (u \otimes u)\|_{\frac{5k}{k+1}} \right)^{\frac{k-2}{2}} \times \left( \int_{R^3} |D^\sigma P_j u|^k |\nabla D^\sigma P_j u|^2 \right)^{\frac{k-2}{2}} \times \left( \int_{R^3} |D^\sigma P_j u|^k |\nabla |D^\sigma P_j u|^2 \right)^{\frac{1}{2}}. \]

By Young’s inequality,

\[ |I_1| \leq Ck^{\frac{k-2}{5} - \frac{k-2}{6} - \frac{k-2}{6(k+1)}} \left( \|D^\sigma P_j (u \otimes u)\|_{\frac{5k}{k+1}} \right)^{\frac{k-2}{2}} \times \left( \int_{R^3} |D^\sigma P_j u|^k |\nabla D^\sigma P_j u|^2 \right)^{\frac{k-2}{2}} \times \left( \int_{R^3} |D^\sigma P_j u|^k |\nabla |D^\sigma P_j u|^2 \right)^{\frac{1}{2}} \]

\[ + \frac{\nu}{3(k-2)} \int_{R^3} |D^\sigma P_j u|^k |\nabla D^\sigma P_j u|^2 + \frac{\nu}{3} \left( \int_{R^3} |D^\sigma P_j u|^k |\nabla |D^\sigma P_j u|^2 \right)^{\frac{1}{2}}, \]

\[ \leq Ck^{2 - \frac{2}{5} + \frac{1}{6} (1 + \nu^{-2}) \left( \|D^\sigma P_j (u \otimes u)\|_{\frac{5k}{k+1}} \right)^{\frac{k-2}{2}} \times \left( \int_{R^3} |D^\sigma P_j u|^k |\nabla D^\sigma P_j u|^2 \right)^{\frac{k-2}{2}} \times \left( \int_{R^3} |D^\sigma P_j u|^k |\nabla |D^\sigma P_j u|^2 \right)^{\frac{1}{2}}. \] (5.12)

We estimate \( I_2 \) as almost as same as \( I_1 \). Actually, we only need replace the term \( \int_{R^3} |D^\sigma P_j u|^k |\nabla D^\sigma P_j u|^2 \) by the term \( \int_{R^3} |D^\sigma P_j u|^k |\nabla |D^\sigma P_j u|^2 \) in the estimate of \( I_1 \). That is, we have

\[ |I_2| \leq Ck^{2 - \frac{2}{5} + \frac{1}{6} (1 + \nu^{-2}) \left( \|D^\sigma P_j (u \otimes u)\|_{\frac{5k}{k+1}} \right)^{\frac{k-2}{2}} \times \left( \int_{R^3} |D^\sigma P_j u|^k |\nabla D^\sigma P_j u|^2 \right)^{\frac{k-2}{2}} \times \left( \int_{R^3} |D^\sigma P_j u|^k |\nabla |D^\sigma P_j u|^2 \right)^{\frac{1}{2}}. \] (5.13)

\[ + \frac{\nu}{3} \left( \int_{R^3} |D^\sigma P_j u|^k |\nabla D^\sigma P_j u|^2 \right)^{\frac{1}{2}}. \]

\( I_3 \) can be estimated also by the similar way. From Hölder’s inequality,

\[ |I_3| \leq \|D^\sigma P_j p\|_{\frac{5k}{k+1}} \left( \|D^\sigma P_j u\|_{k} \right)^{\frac{k-2}{2}} \times \left( \int_{R^3} |D^\sigma P_j u|^k |\nabla |D^\sigma P_j u|^2 \right)^{\frac{1}{2}}. \]

From Lemma 3.7,

\[ \|D^\sigma P_j p\|_{\frac{5k}{k+1}} \leq C\|D^\sigma P_j (u \otimes u)\|_{\frac{5k}{k+1}}, \]

\[ 20 \]
where $C$ is a universal constant. Therefore

$$|I_3| \leq C k^2 2^{-q} (1 + \nu^{-2}) \|D^\sigma P_j (u \otimes u)\|_{\frac{12k}{k+1}} \left(\|D^\sigma P_j u\|_{\frac{5k}{k^2}}\right)^{k-2}$$

$$+ \frac{\nu}{3(k-2)} \int_{R^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^2 + \frac{\nu}{3} \int_{R^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^2.$$

Plug (5.12)-(5.14) into (5.11). We arrive at

$$\frac{d}{dt} \|D^\sigma P_j u\|_{k}^{\frac{12k}{k+1}} \leq C k^4 2^{-\hat{q}} (1 + \nu^{-2}) \times \|D^\sigma P_j (u \otimes u)\|_{\frac{12k}{k+1}} \left(\|D^\sigma P_j u\|_{\frac{5k}{k^2}}\right)^{k-2}.$$  \hspace{1cm} (5.15)

**Step 3. Estimate of** $\|D^\sigma P_j (u \otimes u)\|_{\frac{12k}{k+1}}$ **in (5.15).** In Corollary 2.8, we set $\hat{q} = \frac{12k}{3k+2}$, $q = \frac{3k}{k+1}$, $q_0 = \frac{4k}{k-2}$ and $q_1 = k$ and then using it, we have

$$\|D^\sigma P_j (u \otimes u)\|_{\frac{12k}{k+1}} \leq C \alpha_j \|u\|_{\frac{12k}{k+1}}$$

$$+ C \sum_{m=\max\{j_0, j-2\}}^{\infty} 2^{\sigma(j-m)} \|D^\sigma P_m u\|_{k}^{\frac{12k}{k+1}} \|u\|_{\frac{12k}{k+1}}$$

$$\leq C \left(\|u\|_{\frac{12k}{k+1}} + \|u\|_{\frac{12k}{k+1}}\right) \left(\alpha_j + \sum_{m=\max\{j_0, j-2\}}^{\infty} 2^{\sigma(j-m)} \|D^\sigma P_m u\|_{k}^{\frac{12k}{k+1}}\right),$$

where $\alpha_j$ is given by (2.3) and $C$ are universal constants. Since

$$\|u\|_{\frac{3k}{2k}} \leq \|u\|_{2}^{\frac{1}{2}} \|u\|_{4}^{1-\lambda} \leq \|u\|_{2} + \|u\|_{4},$$

where $\frac{k-2}{3k} = \frac{\lambda}{2} + \frac{1-\lambda}{4}$, it is easy to see that

$$\|u\|_{\frac{12k}{k+1}} + \|u\|_{\frac{12k}{k+1}} \leq C \left(1 + \|u\|_{2} + \|u\|_{4}^{\frac{5}{4}}\right).$$

For simplicity, we denote

$$C_0(t) := 1 + \|u\|_{2}^{\frac{5}{2}} + \|u\|_{4}^{\frac{8}{4}}.$$ \hspace{1cm} (5.16)

Then

$$\|D^\sigma P_j (u \otimes u)\|_{\frac{12k}{k+1}} \leq C C_0(t) \left(\alpha_j + \sum_{m=\max\{j_0, j-2\}}^{\infty} 2^{\sigma(j-m)} \|D^\sigma P_m u\|_{k}^{\frac{12k}{k+1}}\right).$$
From (5.15), it follows that

\[
\frac{d}{dt} \|D^\sigma P_j u\|_k^k \leq C (1 + \nu^{-2}) C_0(t) k^4 2^{-\frac{\nu}{2}} \left\{ \alpha_j \left( \|D^\sigma P_j u\|_{5k}^{5k} \right)^{\frac{k-2}{k+2}} + \sum_{m = \max\{j_0, j-2\}}^\infty 2^{\sigma(j-m)} \|D^\sigma P_m u\|_{5k}^{12k} \left( \|D^\sigma P_j u\|_{5k}^{5k} \right)^{\frac{k-2}{k+2}} \right\}.
\]

From Young’s inequality, we see

\[
\|D^\sigma P_m u\|_{5k}^{12k} \left( \|D^\sigma P_j u\|_{5k}^{12k} \right)^{\frac{k-2}{k+2}} = \left( \|D^\sigma P_m u\|_k^{k} \right)^{\frac{12k}{k+2}} \left( \|D^\sigma P_j u\|_{5k}^{5k} \right)^{\frac{k-2}{k+2}} \leq C \left( \|D^\sigma P_m u\|_k^{k} \right)^{\frac{12k}{k+2}} \left( \|D^\sigma P_j u\|_{5k}^{5k} \right)^{\frac{k-2}{k+2}}.
\]

It follows that

\[
\sum_{m = \max\{j_0, j-2\}}^\infty 2^{\sigma(j-m)} \|D^\sigma P_m u\|_{5k}^{12k} \left( \|D^\sigma P_j u\|_{5k}^{5k} \right)^{\frac{k-2}{k+2}} \leq C \left( \|D^\sigma P_j u\|_{5k}^{5k} \right)^{\frac{12k}{k+2}} + \sum_{m = \max\{j_0, j-2\}}^\infty 2^{\sigma(j-m)} \left( \frac{1}{k^4} \|D^\sigma P_m u\|_k^{k} \right)^{\frac{k-2}{k+2}} \leq C \left( \|D^\sigma P_j u\|_{5k}^{5k} \right)^{\frac{k-2}{k+2}} + \sum_{m = \max\{j_0, j-2\}}^\infty 2^{\sigma(j-m)} \left( \frac{1}{k^4} \|D^\sigma P_m u\|_k^{k} \right)^{\frac{k-2}{k+2}}.
\]

It is clear that

\[
\alpha_j \left( \|D^\sigma P_j u\|_{5k}^{5k} \right)^{\frac{k-2}{k+2}} \leq \alpha_j \left( 1 + \left( \|D^\sigma P_j u\|_{5k}^{5k} \right)^{\frac{k-2}{k+2}} \right).
\]

Therefore

\[
\frac{d}{dt} \|D^\sigma P_j u\|_k^{k} \leq C (1 + \nu^{-2}) C_0(t) k^4 2^{-\frac{\nu}{2}} \left\{ \alpha_j + \left( \|D^\sigma P_j u\|_{5k}^{5k} \right)^{\frac{k-2}{k+2}} \right\} + \sum_{m = \max\{j_0, j-2\}}^\infty 2^{\sigma(j-m)} \left( \frac{1}{k^4} \|D^\sigma P_m u\|_k^{k} \right)^{\frac{k-2}{k+2}}.
\]

(5.17)
Step 4. Taking sum of \( j \). Take sum of \( j \) from \( j_0 = 1 \) to \( \infty \) on both sides of (5.17). Since
\[
\sum_{j=j_0}^{\infty} \sum_{m=\max\{j_0,j-2\}}^{\infty} \frac{2^{\pi(j-m)}}{k^4} \|D^\sigma P_m u\|_{k}^k = \sum_{j=3}^{\infty} \sum_{m=j-2}^{\infty} \frac{2^{\pi(j-m)}}{k^4} \|D^\sigma P_m u\|_{k}^k
\]
\[
+ \sum_{m=j_0}^{\infty} \frac{2^{\pi(1-m)}}{k^4} \|D^\sigma P_m u\|_{k}^k + \sum_{m=j_0}^{\infty} \frac{2^{\pi(2-m)}}{k^4} \|D^\sigma P_m u\|_{k}^k
\]
\[
\leq C \sum_{m=j_0}^{\infty} \frac{1}{k^4} \|D^\sigma P_m u\|_{k}^k,
\]
by Lemma 4.2 and 4.6, we have
\[
\frac{d}{dt} \sum_{j=j_0}^{\infty} \|D^\sigma P_j u\|_{k}^k \leq C (1 + \nu^{-2}) C_0(t) k^4
\]
\[
\times \left\{ 1 + \sum_{j=j_0}^{\infty} 2^{-\frac{1}{\overline{2} k}} \left( \|D^\sigma P_j u\|_{\overline{5} k}^k \right)^{\overline{5} k} + \sum_{j=j_0}^{\infty} \frac{1}{k^4} \|D^\sigma P_j u\|_{k}^k \right\}. \tag{5.18}
\]

Step 5. Dividing (5.18) by \( 2^{B_k} \) and taking sum of \( k \). Divide by \( 2^{B_k} \) on both sides of (5.18) and consequently,
\[
\frac{d}{dt} \sum_{j=1}^{\infty} \|D^\sigma P_j u\|_{k}^k \leq C (1 + \nu^{-2}) C_0(t) k^4
\]
\[
\times \left\{ \frac{1}{2^{B_k}} + \sum_{j=1}^{\infty} \frac{1}{k^4} \frac{\|D^\sigma P_j u\|_{k}^k}{2^{B_k}} + \frac{2^{B_{5k}/5}}{2^{B_k}} \sum_{j=1}^{\infty} 2^{-\frac{1}{\overline{2} k}} \left( \frac{\|D^\sigma P_j u\|_{\overline{5} k}^k}{2^{B_{5k}}} \right)^{\overline{5} k} \right\}. \tag{5.19}
\]
From (5.2), we see that
\[
\frac{2^{B_{5k}/5}}{2^{B_k}} = 2^{(\overline{5} k - 1) \sqrt{k}} \leq \frac{C}{k^6}
\]
for some universal constant \( C \). It is clear that
\[
\frac{1}{2^{B_k}} \leq \frac{C}{k^6}
\]
for some universal constant \( C \). From (5.19), it follows that
\[
\frac{d}{dt} \sum_{j=1}^{\infty} \frac{\|D^\sigma P_j u\|_{k}^k}{2^{B_k}} \leq C (1 + \nu^{-2}) C_0(t)
\]
\[
\times \left\{ \frac{1}{k^2} + \sum_{j=1}^{\infty} \frac{\|D^\sigma P_j u\|_{k}^k}{2^{B_k}} + \sum_{j=1}^{\infty} \frac{1}{k^2} \left( \frac{\|D^\sigma P_j u\|_{\overline{5} k}^k}{2^{B_{5k}}} \right)^{\overline{5} k} \right\}. \tag{5.20}
\]
Take sum of $k$ from $k_0$ to $\infty$ on both sides of (5.20). From (5.3) and Lemma 4.2, we obtain

$$
\frac{d}{dt} \sum_{k=k_0}^{\infty} \sum_{j=1}^{\infty} \frac{||D^\sigma P_j u||^k_k}{2B_k} \leq C (1 + \nu^{-2}) C_0(t) \times \left\{ \sum_{k=k_0}^{\infty} \frac{1}{k^2} + \sum_{k=k_0}^{\infty} \sum_{j=1}^{\infty} \frac{||D^\sigma P_j u||^k_k}{2B_k} \right\}.
$$

It is clear that

$$
\sum_{k=k_0}^{\infty} \frac{1}{k^2} \leq C
$$

and from Young’s inequality,

$$
\sum_{k=k_0}^{\infty} \sum_{j=1}^{\infty} \frac{1}{2^{+}k^2} \left( \frac{||D^\sigma P_j u||^5_{5k}}{2B_{5k}} \right)^{+} \leq C \left\{ 1 + \sum_{k=k_0}^{\infty} \sum_{j=1}^{\infty} \frac{||D^\sigma P_j u||^5_{5k}}{2B_{5k}} \right\}.
$$

Therefore we have

$$
\frac{d}{dt} \sum_{k=k_0}^{\infty} \sum_{j=1}^{\infty} \frac{||D^\sigma P_j u||^k_k}{2B_k} \leq C (1 + \nu^{-2}) C_0(t) \left\{ 1 + \sum_{k=k_0}^{\infty} \sum_{j=1}^{\infty} \frac{||D^\sigma P_j u||^k_k}{2B_k} \right\}. \quad (5.21)
$$

**Step 6. Proof of (5.4).** From (5.21), it follows that

$$
\frac{d}{dt} \left\{ 1 + \sum_{k=k_0}^{\infty} \sum_{j=1}^{\infty} \frac{||D^\sigma P_j u||^k_k}{2B_k} \right\} \leq C (1 + \nu^{-2}) C_0(t) \left\{ 1 + \sum_{k=k_0}^{\infty} \sum_{j=1}^{\infty} \frac{||D^\sigma P_j u||^k_k}{2B_k} \right\}.
$$

From Gronwall’s inequality, we have

$$
\sum_{k=k_0}^{\infty} \sum_{j=1}^{\infty} \frac{||\nabla^\sigma P_j u(t)||^k_k}{2B_k} \leq e^{C(1+\nu^{-2}) \int_0^T C_0(t) \, ds} - 1 + e^{C(1+\nu^{-2}) \int_0^T C_0(t) \, ds} \sum_{k=k_0}^{\infty} \sum_{j=1}^{\infty} \frac{||\nabla^\sigma P_j u_0||^k_k}{2B_k}
$$

for any $0 \leq t \leq T'$. In view of (5.16) and Corollary 3.2, we have

$$
\int_0^T C_0(t) \, dt = \int_0^T \left( 1 + ||u||^3_3 + ||u||^4_4 \right) \, dt \leq (1 + ||u_0||^3_5) T + \frac{C}{\nu} ||u_0||^5_5.
$$

Then (5.4) follows clearly. \( \square \)
6 Low frequency part

From Theorem 5.1 (uniform bound estimate) to derive our new a priori estimate, we only need to remove Condition (5.3). To do this, we separate the series (5.1) into two parts, low frequency part (finite \( j \)) and high frequency part, and show the convergence of them respectively. In this section, we will study the low frequency part which is much more difficult than the high frequency part and the result, Theorem 6.1 is the second key step to approach our new a priori estimates. We design the following series

\[
\sum_{k=k_0}^{\infty} J_0 \sum_{j=j_0}^{J_0} \frac{\|D^\sigma P_j u(t)\|^k}{2^{\hat{B}_k}}, \tag{6.1}
\]

where \( j_0 = 1, k_0 = 100, J_0 \) will be given later and the power

\[
\hat{B}_k = \left( B - \frac{1}{\sqrt{k}} \right) k + 2^B. \tag{6.2}
\]

Note here the constant \( B \) will be chosen the same as in (5.2). We will see that (6.1) can not blow up before any given time \( T \) if \( B \) is large enough.

Our main theorem of this section is:

**Theorem 6.1.** Let \( j_0 = 1, k_0 = 100, \sigma = 2, T > 0 \) and \( B > 1 \) be real numbers, \( \hat{B}_k \) be given by (6.2) for any \( k \geq k_0 \), and \( u_0 \) be a function satisfying (1.2). Suppose \( u \) and \( p \) satisfy Condition (S). Let

\[
J_0 = \left\lceil \frac{8B}{\sigma} \right\rceil. \tag{6.3}
\]

There exists \( \bar{B}_1 > 0 \) depending only on \( \nu, T \) and \( u_0 \) such that if \( B \geq \bar{B}_1 \), then

\[
\sum_{k=k_0}^{\infty} J_0 \sum_{j=j_0}^{J_0} \frac{\|D^\sigma P_j u(t)\|^k}{2^{\hat{B}_k}} \leq 1 \tag{6.4}
\]

for any \( t \in [0, T] \).

We establish Theorem 6.1 by the following two lemmas, where \( B > 1 \) will be determined later.

**Lemma 6.2.** Suppose all the assumptions of Theorem 6.1 hold. Then

\[
\frac{d}{dt} \sum_{k=k_0}^{2^k-1} J_0 \sum_{j=j_0}^{J_0} \frac{\|D^\sigma P_j u(t)\|^k}{2^{\hat{B}_k}} \leq \frac{C}{\nu} \|u\|_2^4 \left( 2^{-\frac{1}{2}B} + \sum_{k=k_0}^{2^k-1} J_0 \sum_{j=j_0}^{J_0} \frac{\|D^\sigma P_j u(t)\|^k}{2^{\hat{B}_k}} \right). \tag{6.5}
\]
for $0 \leq t \leq T$.

**Proof.** Let $k_0 \leq k < 2k_0$, $j_0 \leq j \leq J_0$ and $0 \leq t \leq T$. We divide the proof into four steps.

**Step 1.** By the same arguments to derive (5.11), we have

$$\frac{1}{k} \frac{d}{dt} \int_{\mathbb{R}^3} |D^\sigma P_j u|^k + \nu \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^2$$

$$+ (k-2)\nu \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u||^2$$

$$\leq (k-2) \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u||D^\sigma P_j (u \otimes u)|$$

$$+ \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u||D^\sigma P_j (u \otimes u)|$$

$$+ (k-2) \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u||D^\sigma P_j p|$$

$$:= (k-2)I_1 + I_2 + (k-2)I_3. \tag{6.6}$$

**Step 2. Estimates of $I_1$, $I_2$ and $I_3$ in (6.6).** From Hölder’s inequality,

$$|I_1| \leq \||D^\sigma P_j (u \otimes u)|| \||D^\sigma P_j u||_k \|^{k-2} \left( \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^{2} \right)^{\frac{1}{2}}$$

$$\leq \frac{4}{\nu} |D^\sigma P_j (u \otimes u)||^2 \||D^\sigma P_j u||^{k-2}$$

$$+ \frac{\nu}{4} \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^2.$$

and

$$|I_2| \leq \||D^\sigma P_j (u \otimes u)|| \||D^\sigma P_j u||_k \|^{k-2} \left( \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^{2} \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\nu} |D^\sigma P_j (u \otimes u)||^2 \||D^\sigma P_j u||^{k-2}$$

$$+ \nu \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^2.$$

From Lemma 3.7, we have

$$||D^\sigma P_j p||_k \leq Ck ||D^\sigma P_j (u \otimes u)||_k,$$
where $C$ is a universal constant. Using this to estimate $I_3$, we obtain

$$ |I_3| \leq \|D^\sigma P_j u\|_k \|D^\sigma P_j u\|_k^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^2 \right)^{\frac{1}{2}} $$

$$ \leq CK \|D^\sigma P_j (u \otimes u)\|_k \|D^\sigma P_j u\|_k^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^2 \right)^{\frac{1}{2}} $$

$$ \leq \frac{CK^2}{\nu} \|D^\sigma P_j (u \otimes u)\|_k^2 \|D^\sigma P_j u\|_k^{k-2} $$

$$ + \frac{\nu}{4} \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^2. $$

Plug the estimates of $I_1$, $I_2$ and $I_3$ into (6.6) and we arrive at

$$ \frac{d}{dt} \|D^\sigma P_j u\|_k^k + \frac{\nu}{2} k(k-2) \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^2 $$

$$ \leq \frac{CK^4}{\nu} \|D^\sigma P_j (u \otimes u)\|_k^2 \|D^\sigma P_j u\|_k^{k-2}. \quad (6.7) $$

**Step 3. Simplifying (6.7).** By Lemma 2.1, 2.3 and 2.6, we have

$$ \|D^\sigma P_j (u \otimes u)\|_k \leq C2^{(\sigma+3)\frac{1}{4}} |P_j (u \otimes u)|_1 \leq C2^{(\sigma+3)J_0} |u|_2, $$

where $j \leq J_0$ is used. From $k_0 \leq k < 2k_0$, (6.2) and (6.3), we deduce

$$ 2^{(\sigma+3)J_0} \leq 2^{\frac{8(\sigma+3)}{3}B} \leq C2^{(2^b-B)/k} \leq C2^{(\beta_k-B)/k} $$

for any $B > 1$, where $C$ is a universal constant. It follows that

$$ \|D^\sigma P_j (u \otimes u)\|_k \leq C2^{(\beta_k-B)/k} |u|_2, $$

as $j \leq J_0$ and $k_0 \leq k < 2k_0$.

Plug this estimate into (6.7) and we have

$$ \frac{d}{dt} \|D^\sigma P_j u\|_k^k \leq C \frac{2^{2(\beta_k-B)/k}}{\nu} |u|_2 \|D^\sigma P_j u\|_k^{k-2}, \quad (6.8) $$

where the term $\frac{\nu}{4} k(k-2) \int_{\mathbb{R}^3} |D^\sigma P_j u|^{k-2} |\nabla D^\sigma P_j u|^2$ is omitted and $k_0 \leq k < 2k_0$ is used.

**Step 4. Proof of (6.5).** Divide $2^{\beta_k}$ on the both sides of (6.8) and then

$$ \frac{d}{dt} \left( \frac{\|D^\sigma P_j u\|_k^k}{2^{\beta_k}} \right) \leq C \frac{1}{\nu} |u|_2^4 \left( \frac{\|D^\sigma P_j u\|_k^{k-2}}{2^{\beta_k}} \right)^{\frac{k-2}{k}}. $$
From Young’s inequality, we have
\[ \frac{d}{dt} \left( \frac{||D^\sigma P_j u||^k_{2B_k}}{2B_k} \right) \leq \frac{C}{\nu} \left( ||u||_2^2 \left( 2^{-B} + \frac{||D^\sigma P_j u||^k_{2B_k}}{2B_k} \right) \right). \]  
(6.9)

Take sum of \( j \) from \( j_0 \) to \( J_0 \) and of \( k \) from \( k_0 \) to \( 2k_0 - 1 \) on both sides of (6.9). We have
\[ \frac{d}{dt} \sum_{k=k_0}^{2k_0-1} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u||^k_{2B_k}}{2B_k} \leq \frac{C}{\nu} \left( ||u||_2^2 \left( \sum_{k=k_0}^{2k_0-1} \sum_{j=j_0}^{J_0} 2^{-B} + \sum_{k=k_0}^{2k_0-1} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u||^k_{2B_k}}{2B_k} \right) \right). \]

From (6.3), it follows that
\[ \sum_{k=k_0}^{2k_0-1} \sum_{j=j_0}^{J_0} 2^{-B} = k_0 J_0 2^{-B} \leq k_0 \frac{8}{\sigma} B 2^{-B} \leq C 2^{-\frac{1}{4}B} \]
for any \( B > 1 \) where \( C \) is a universal constant. Then we see (6.5) clearly. \( \square \)

**Lemma 6.3.** Suppose all the assumptions of Theorem 6.1 hold. Then for any \( k' \geq 2k_0 \), we have
\[ \frac{d}{dt} \sum_{k=2k_0}^{k'} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u||^k_{2B_k}}{2B_k} \leq C \left( 1 + \frac{1}{\nu} \right)^4 \left( 1 + ||u||_2^2 \right)^5 \]
\[ \times \left( \sum_{k=k_0}^{k'} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u||^k_{2B_k}}{2B_k} + \sum_{k=k_0}^{k'} \sum_{j=j_0}^{J_0} \left( \frac{||D^\sigma P_j u||^k_{2B_k}}{2B_k} \right)^5 \right) \]
(6.10)
for any \( 0 \leq t \leq T \), where \( C \) is a universal constant.

**Proof.** Let \( 2k_0 \leq k \leq k' \), \( 1 \leq j \leq J_0 \) and \( 0 \leq t \leq T \). We divide the proof into five steps.

**Step 1.** By the same arguments to derive (6.7), we have
\[ \frac{d}{dt} \left( ||D^\sigma P_j u||^k_{2B_k} \right) \leq \frac{Ck^4}{\nu} ||D^\sigma P_j (u \otimes u)||^2_k ||D^\sigma P_j u||^{k-2}_k, \]
where \( k(k-2) \geq k^2/2 \) is used.

**Step 2.** Estimate of \( ||D^\sigma P_j (u \otimes u)||^k_k \) in (6.11). As in Step 3 of the proof of Lemma 6.2, by Lemma 2.1, 2.3 and 2.6, we have
\[ ||D^\sigma P_j (u \otimes u)||^k_k \leq C 2^{(\sigma+3)k_j} ||P_j (u \otimes u)||_1 \leq C 2^{(\sigma+3)J_0} ||u||_2^2, \]
(6.12)
where \( j \leq J_0 \) is used.

As \( k \leq \frac{2^{\sigma_0}}{B_0} \), from (6.2) and (6.3), we have

\[
\frac{2 (\sigma + 3) J_0}{2 B_k/k} \leq \frac{2^{8(\sigma + 3)}B}{2^{\sigma_0}} \leq 1 \leq 2^{\frac{\sigma}{\sigma_0}},
\]

for any \( B > 1 \).

As \( k \leq \frac{2^{\sigma_0}}{B_0} \), we have

\[
\frac{2^{(\sigma + 3)J_0}}{2 B_k/k} \leq 2^{(\sigma + 3)J_0} \leq \frac{2^{8(\sigma + 3)}B}{2^{\sigma_0}} \leq C 2^{\frac{\sigma}{\sigma_0}} \leq C 2^{\frac{\sigma}{1000}}
\]

for any \( B > 1 \), where \( C \) is a universal constant.

Therefore

\[
2^{(\sigma + 3)J_0} \leq C 2 B_k/k 2^{\frac{\sigma}{1000}}
\]

and then

\[
\| D^\sigma P_j (u \otimes u) \|_k \leq C 2 B_k/k 2^{\frac{\sigma}{1000}} \| u \|_2^2
\]

for any \( B > 1 \) and \( k \geq 2k_0 \).

Plug the above inequality into (6.11) and we conclude

\[
\frac{d}{dt} \| D^\sigma P_j u \|_k^k + \frac{\nu}{4} k^2 \int_{R^3} |D^\sigma P_j u|^{k-2} \| \nabla |D^\sigma P_j u| \|^2 \leq C k^4 \nu 2 B_k/k 2^{\frac{\sigma}{1000}} \| u \|_2^2 \| D^\sigma P_j u \|_k^{k-2}.
\]

(6.12)

**Step 3.** Gain from \( \frac{\nu}{4} k^2 \int_{R^3} |D^\sigma P_j u|^{k-2} \| \nabla |D^\sigma P_j u| \|^2 \). From interpolation inequality, we deduce

\[
\| D^\sigma P_j u \|_k \leq \left( \| D^\sigma P_j u \|_k^{k+1} \right)^{1-\lambda} \left( \| D^\sigma P_j u \|_k^{3k} \right)^{\lambda},
\]

where

\[
\lambda = \frac{\left( \frac{1}{k+1} - \frac{1}{k} \right)}{\left( \frac{1}{k+1} - 1 \right) 3k}.
\]

It is clear that

\[
\begin{aligned}
\lambda &\geq \frac{2}{k+1} - \frac{1}{k} = \frac{3k - 3}{5k - 1} \geq \frac{2}{5} \quad \text{and} \\
\lambda &\leq \frac{2}{k} - \frac{1}{k} = \frac{3}{5},
\end{aligned}
\]

(6.13)
By Lemma 4.1, there exists a universal constant $C$ such that
\[ ||D^\sigma P_j u||^k_{3k} \leq C k^2 \int_{R^3} |D^\sigma P_j u|^{k-2} |\nabla| D^\sigma P_j u|^2. \]

For simplicity we denote $\frac{C k^4 2^B_\nu}{\nu^2} \frac{2^{1/2\lambda}}{\nu^2} ||u||^4_2$ by $\Theta$. It follows that

Righthand side of (6.12) = $\Theta ||D^\sigma P_j u||^k_{k-2} \leq \Theta \left(||D^\sigma P_j u||^{k-2}_{k+2} \right)^{1-\lambda} \left(||D^\sigma P_j u||^k_{3k} \right)^{k-2}\lambda
\]
\[ \leq \left( \Theta \right) \left( \frac{C k^4 2^B_\nu}{\nu^2} \frac{2^{1/2\lambda}}{\nu^2} ||u||^4_2 \right)^{1-\lambda} \left( \frac{\nu}{1} k^2 \int_{R^3} |D^\sigma P_j u|^{k-2} |\nabla| D^\sigma P_j u|^2 \right)^{k-2}\lambda
\]
\[ \leq \left( \frac{4C}{\nu} \right) \left( \frac{2^{1/2\lambda}}{\nu^2} \right) \left( \frac{\nu}{1} k^2 \int_{R^3} |D^\sigma P_j u|^{k-2} |\nabla| D^\sigma P_j u|^2 \right)^{k-2}\lambda + \frac{\nu}{4} k^2 \int_{R^3} |D^\sigma P_j u|^{k-2} |\nabla| D^\sigma P_j u|^2.
\]

By (6.13), it is easy to see that
\[ \left( \frac{4C}{\nu} \right) \left( \frac{2^{1/2\lambda}}{\nu^2} \right) \left( \frac{\nu}{1} k^2 \int_{R^3} |D^\sigma P_j u|^{k-2} |\nabla| D^\sigma P_j u|^2 \right)^{k-2}\lambda \]
\[ \leq C \left( 1 + \frac{1}{\nu} \right)^4 k^{10/2} \left( 1 + ||u||^2_2 \right)^5.
\]

Plug this estimate into (6.12) and we obtain
\[ \frac{d}{dt} ||D^\sigma P_j u||^k_k \leq C \left( 1 + \frac{1}{\nu} \right)^4 k^{10/2} \left( 1 + ||u||^2_2 \right)^5 \]
\[ \times \frac{2^{B_\nu}}{\nu^2} ||u||^4_2 ||D^\sigma P_j u||^{k-2}_{k+2} \lambda. \]

(6.14)
Step 4. Dividing by $2^{\frac{k}{2}}$. We divide by $2^{\frac{k}{2}}$ on both sides of (6.14) and consequently,

$$\frac{d}{dt} \frac{||D^\sigma P_j u||_{k}}{2^{\frac{k}{2}}} \leq C \left( 1 + \frac{1}{\nu} \right)^4 k^{10} 2^{\frac{k}{2}} (1 + ||u||_2^2)^5 \left( \frac{||D^\sigma P_j u||_{k+1}}{2^{\frac{k+1}{2}}} \right)^{\frac{1}{1 - \frac{\nu}{k+1}}},$$

where $\frac{3}{1 - \frac{\nu}{k+1}} + \frac{1}{1 - \frac{\nu}{k+1}} = 1$ is used.

In view of (6.2), (6.13) and $k \geq 2k_0 = 200$,

$$\left( \frac{2^{\frac{k}{2}}}{2^{\frac{k}{2}}} \right)^{\frac{1}{1 - \frac{\nu}{k+1}}} = \left( 2 \left( \frac{1}{\sqrt{k/(k+1)}} \right) \right)^{\frac{1}{1 - \frac{\nu}{k+1}}} \leq 2^{\frac{k}{2}} \left( \frac{1}{\sqrt{k/(k+1)}} \right) = 2^{\frac{k}{2}} \sqrt{1 - \frac{1}{2} k^{10} 2^{\frac{k}{2}}},$$

where $C$ is universal constant. It follows that

$$\left( \frac{||D^\sigma P_j u||_{k+1}}{2^{\frac{k+1}{2}}} \right)^{\frac{1}{1 - \frac{\nu}{k+1}}} \leq \left( \frac{2^{\frac{k}{2}}}{2^{\frac{k}{2}}} \right)^{\frac{1}{1 - \frac{\nu}{k+1}}} \left( \frac{||D^\sigma P_j u||_{k+1}}{2^{\frac{k+1}{2}}} \right)^{\frac{1}{1 - \frac{\nu}{k+1}}} \leq C k^{10} 2^{\frac{k}{2}} \sqrt{1 - \frac{1}{2} k^{10} 2^{\frac{k}{2}}},$$

$$\leq C k^{10} 2^{\frac{k}{2}} \sqrt{1 - \frac{1}{2} k^{10} 2^{\frac{k}{2}}},$$

$$\leq C k^{10} 2^{\frac{k}{2}} \sqrt{1 - \frac{1}{2} k^{10} 2^{\frac{k}{2}}} + \left( \frac{||D^\sigma P_j u||_{k+1}}{2^{\frac{k+1}{2}}} \right)^{\frac{5}{1 - \frac{\nu}{k+1}}}.$$
where \(1 \leq \frac{k}{|k|} \cdot \frac{(1-\frac{2}{\nu})(1-\lambda)}{2\lambda} \leq 5\) is used. Therefore

\[
\frac{d}{dt} \frac{||D^\sigma P_j u||^k}{2B_k} \leq C \left(1 + \frac{1}{\nu}\right)^4 \left(1 + ||u||_2^2\right)^5
\]

\[
\times \left(\frac{||D^\sigma P_j u||^k}{B_{k+\frac{1}{2}}} + \left(\frac{||D^\sigma P_j u||^k}{2B_{k+\frac{1}{2}}}\right)\right)^5. \tag{6.15}
\]

**Step 5. Proof of (6.10).** Take sum of \(j\) from \(j_0\) to \(J_0\) and of \(k\) from \(2k_0\) to \(k'\) on both sides of (6.15) and consequently,

\[
\frac{d}{dt} \sum_{k=2k_0}^{k'} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u||^k}{2B_k} \leq C \left(1 + \frac{1}{\nu}\right)^4 \left(1 + ||u||_2^2\right)^5
\]

\[
\times \left(\sum_{k=2k_0}^{k'} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u||^k}{2B_k} + \sum_{k=2k_0}^{k'} \sum_{j=j_0}^{J_0} \left(\frac{||D^\sigma P_j u||^k}{2B_{k+\frac{1}{2}}}\right)^5\right). \tag{6.16}
\]

Then (6.10) follows clearly. \(\square\)

**Proof of Theorem 6.1.** Let \(k' \geq 2k_0\). Adding (6.5) and (6.10) together, we obtain

\[
\frac{d}{dt} \sum_{k=k_0}^{k'} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u||^k}{2B_k} \leq C \left(1 + \frac{1}{\nu}\right)^4 \left(1 + ||u||_2^2\right)^5
\]

\[
\times \left(2^{-\frac{3}{2}}B + \sum_{k=k_0}^{2k_0-1} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u||^k}{2B_k} + \sum_{k=k_0}^{k'} \sum_{j=j_0}^{J_0} \left(\frac{||D^\sigma P_j u||^k}{2B_k}\right)^5\right).\]

It follows that

\[
\frac{d}{dt} \sum_{k=k_0}^{k'} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u||^k}{2B_k} \leq C \left(1 + \frac{1}{\nu}\right)^4 \left(1 + ||u||_2^2\right)^5
\]

\[
\times \left(2^{-\frac{3}{2}}B + \sum_{k=k_0}^{k'} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u||^k}{2B_k} + \sum_{k=k_0}^{k'} \sum_{j=j_0}^{J_0} \left(\frac{||D^\sigma P_j u||^k}{2B_k}\right)^5\right). \tag{6.16}
\]

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Set
\[ F(t) = \sum_{k=k_0}^{k'} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u||_k^k}{2\hat{B}_k} \]
and
\[ g(t) = C \left( 1 + \frac{1}{\nu} \right)^4 \left( 1 + ||u||_2^2 \right)^5. \]

Then (6.16) implies
\[ \frac{d}{dt} F(t) \leq g(t) \left( 2^{-\frac{4}{T}} + F(t) + F^5(t) \right), \quad (6.17) \]
where
\[ \sum_{k=k_0}^{k'} \sum_{j=j_0}^{J_0} \left( \frac{||D^\sigma P_j u||_k}{2\hat{B}_k} \right)^5 \leq \left( \sum_{k=k_0}^{k'} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u||_k}{2\hat{B}_k} \right)^5 \]
is used. From Theorem 3.1, it is easy to see that
\[ \int_0^T g(t) dt \leq C \left( 1 + \frac{1}{\nu} \right)^4 \left( 1 + ||u_0||_2^2 \right)^5 T. \]

Therefore there exists \( B'_1 > 0 \) such that if \( B \geq B'_1 \),
\[ 2^{-\frac{4}{T}} \leq \frac{1}{\left( 3e \int_0^T g(t) dt \right)^{\frac{1}{T}}} \quad (6.18) \]
or
\[ B'_1 \geq C \left( 1 + \frac{1}{\nu} \right)^4 \left( 1 + ||u_0||_2^2 \right)^5 T \]
for some universal constant \( C \).

From Lemma 4.7, there exists \( \tilde{B}_0 > 0 \) such that if
\[ B \geq \tilde{B}_0, \]
then
\[ F(0) = \sum_{k=k_0}^{k'} \sum_{j=j_0}^{J_0} \frac{||D^\sigma P_j u_0||_k^k}{2\hat{B}_k} \leq \sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} \frac{||D^\sigma P_j u_0||_k^k}{2\hat{B}(1-\sqrt{\frac{1}{\nu}})^k} \leq 2^{-\frac{4}{T}}, \]
where \( \hat{B}_k \geq B(1-\sqrt{\frac{1}{\nu}})k \) is used.

Set
\[ \tilde{B}_1 = \max\{B'_1, \tilde{B}_0\}. \]
If $B \geq \tilde{B}_1$, from (6.17) and Lemma 4.3, we have

$$F(t) \leq 2^{-\frac{B}{2}}3e\int_0^T g(t)dt \leq 1$$

for any $0 \leq t \leq T$, where (6.18) is used. That is,

$$\sum_{k=k_0}^{k'} \sum_{j=j_0}^{J_0} \frac{\|D^\sigma P_j u\|_k^{k_0}}{2^{B_k}} \leq 1$$

for any $k' \geq 2k_0$. This implies (6.4) clearly. □

### 7 Regularity improving

In this section, we will prove the following Theorem 7.1 from which we can show the convergence of the high frequency part (large $j$) of series (5.1). This kind of regularity improving is not new essentially, but here we need a special form.

**Theorem 7.1.** Let $k_0 = 100$, $j_0 = 1$, $\sigma = 2$, $0 \leq T' \leq T$, $B$ and $B > 1$ be real numbers, $B_k$ be defined by (5.2) for any $k \geq k_0$ and $u_0$ be a function satisfying (1.2). Suppose $u$ and $p$ satisfy Condition (S). There exists $\tilde{B}_2 > 1$ depending only on $u_0$ and $\sigma$ such that if $B \geq \tilde{B}_2$ and

$$\sum_{j=j_0}^{\infty} \|D^\sigma P_j u(t)\|_{k_0}^{k_0} \leq B^{2B_{k_0}}$$

(7.1)

for any $0 \leq t \leq T'$, then we have

$$\sum_{j=j_0}^{\infty} \|D^{\sigma+1} P_j u(t)\|_{k_0}^{k_0} \leq C \left(1 + \frac{1}{p}\right)^{k_0+1} (1 + T) (1 + \|u_0\|_2)^{2k_0+2}$$

$$\times B^{1+\frac{5(k_0+2)/2}{(\sigma+3/2)k_0-1} + \frac{5(k_0+2)^2 B^2}{2^{\sigma+3/2-3/4} 2^{B_{k_0}}} 2^{B_{k_0}}}$$

(7.2)

for any $0 \leq t \leq T'$, where $C > 0$ is a universal constant.

**Proof.** We divide the proof into four steps.

**Step 1. Estimate of $\|u\|_{\infty}$.** (7.1) implies

$$\|D^\sigma P_j u(t)\|_{k_0} \leq (B^{2B_{k_0}})^{\frac{1}{k_0}} \leq 4B^{\frac{1}{k_0}} 2^B$$
for any $j \geq j_0$ and $0 \leq t \leq T'$, where (5.2) is used. From Lemma 2.1, 2.3 and 2.6, we have
\[ \|DP_j u\|_\infty \leq C2^{(\frac{2}{k_0} + 1 - \sigma)j}\|DP_j u\|_{k_0} \]
and
\[ \|DP_j u\|_\infty \leq C2^j\|P_j u\|_2. \]
It follows that
\[ \|DP_j u\|_\infty \leq C\|P_j u\|_2 \left( \frac{\sigma - 1 - 3/k_0}{\sigma + 2 - 3/k_0} \right)^{5/2} \|DP_j u\|_{k_0}^{5/2} \]
\[ \quad \leq C\|u_0\|_2 \left( \frac{\sigma - 1 - 3/k_0}{\sigma + 2 - 3/k_0} \right)^{5/2} \|\nabla \sigma\|_{5/2} \|\nabla^3 u\|_0 \].
From (2.1), Lemma 2.1 and Theorem 3.1, we have
\[ ||u||_\infty \leq ||P_{\leq 0} u||_\infty + \sum_{j=1}^\infty ||P_j u||_\infty \leq C||u||_2 + C \sum_{j=1}^\infty 2^{-j}||P_j u||_\infty \]
\[ \leq C||u_0||_2 + C\|u_0\|_2 \left( \frac{\sigma - 1 - 3/k_0}{\sigma + 2 - 3/k_0} \right)^{5/2} \|\nabla \sigma\|_{5/2} \|\nabla^3 u\|_0 \]
\[ \leq C(1 + \|u_0\|_2) \left( \frac{\sigma - 1 - 3/k_0}{\sigma + 2 - 3/k_0} \right)^{5/2} \|\nabla \sigma\|_{5/2} \|\nabla^3 u\|_0 \] (7.3)
where $C$ is a universal constant. For simplicity, we denote
\[ A = (1 + \|u_0\|_2) \left( \frac{\sigma - 1 - 3/k_0}{\sigma + 2 - 3/k_0} \right)^{5/2} \|\nabla \sigma\|_{5/2} \|\nabla^3 u\|_0 \].
(7.4)

Step 2. By the same arguments to derive (6.11), we have
\[ \frac{d}{dt}\|D^{\sigma+1}P_j u\|_{k_0}^{k_0} + \frac{\nu}{4}k_0^2 \int_{R^3} |D^{\sigma+1}P_j u|^{k_0-2} |\nabla D^{\sigma+1}P_j u|^2 \]
\[ \leq C \|D^{\sigma+1}P_j (u \otimes u)\|_{k_0}^{k_0} \|D^{\sigma+1}P_j u\|_{k_0}^{k_0-2}. \] (7.5)

By Corollary 2.8, where we set $\dot{q} = 2$, $q = q_1 = k_0$ and $q_0 = \infty$, we deduce
\[ \|D^{\sigma+1}P_j (u \otimes u)\|_{k_0}^{k_0} \leq C \left\{ \alpha_j ||u||^2_1 + \sum_{m=\max\{j_0,j-2\}}^{\infty} 2^{(\sigma+1)(j-m)} ||D^{\sigma+1}P_m u||^2_0 ||u||^2_\infty \right\}, \]
where $\alpha_j$ is given by (2.3). In view of (7.3),
\[ \|D^{\sigma+1}P_j (u \otimes u)\|_{k_0}^{k_0} \leq C\alpha_j ||u||_1^2 + CA^2 \sum_{m=\max\{j_0,j-2\}}^{\infty} 2^{(\sigma+1)(j-m)} ||D^{\sigma+1}P_m u||^2_0. \]

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Plug this into (7.5) and we obtain

\[
\frac{d}{dt} |D^{\sigma+1} P_j u|_{k_0}^{k_0} + \frac{\nu}{4} k_0^2 \int_{\mathbb{R}^3} |D^{\sigma+1} P_j u|^{k_0-2} |\nabla D^{\sigma+1} P_j u|^2 \\
\leq \frac{C}{\nu} \left\{ \alpha_j |u|_{\frac{k_0}{2}}^2 |D^{\sigma+1} P_j u|_{k_0}^{k_0-2} \\
+ A^2 \sum_{m = \max\{j_0, j-2\}}{2^{(\sigma+1)(j-m)} |D^{\sigma+1} P_m u|_{k_0}^2 |D^{\sigma+1} P_j u|_{k_0}^{k-2}} \right\}
\]

(7.6)

Step 3. Taking sum of j. It is easy to see that

\[
\alpha_j |u|_{\frac{k_0}{2}}^2 |D^{\sigma+1} P_j u|_{k_0}^{k_0-2} \leq \alpha_j \left( |u|_{\frac{k_0}{2}}^2 + |D^{\sigma+1} P_j u|_{k_0}^{k_0} \right)
\]

and

\[
\sum_{m = \max\{j_0, j-2\}}{2^{(\sigma+1)(j-m)} |D^{\sigma+1} P_m u|_{k_0}^2 |D^{\sigma+1} P_j u|_{k_0}^{k-2}} \leq \sum_{m = \max\{j_0, j-2\}}{2^{(\sigma+1)(j-m)} \left( |D^{\sigma+1} P_m u|_{k_0}^{k_0} + |D^{\sigma+1} P_j u|_{k_0}^{k_0} \right)}
\]

\[
\leq C |D^{\sigma+1} P_j u|_{k_0}^{k_0} + \sum_{m = \max\{j_0, j-2\}}{2^{(\sigma+1)(j-m)} |D^{\sigma+1} P_m u|_{k_0}^{k_0}}.
\]

Plug these two inequalities into (7.6) and we have

\[
\frac{d}{dt} |D^{\sigma+1} P_j u|_{k_0}^{k_0} + \frac{\nu}{4} k_0^2 \int_{\mathbb{R}^3} |D^{\sigma+1} P_j u|^{k_0-2} |\nabla D^{\sigma+1} P_j u|^2 \\
\leq \frac{C}{\nu} \left\{ \alpha_j |u|_{2}^{k_0} + A^2 |D^{\sigma+1} P_j u|_{k_0}^2 \\
+ A^2 \sum_{m = \max\{j_0, j-2\}}{2^{(\sigma+1)(j-m)} |D^{\sigma+1} P_m u|_{k_0}^2} \right\},
\]

where \( A > 1 \) is used.

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Take sum of $j$ from $j_0 = 1$ to $\infty$ on both sides of the above inequality. Since

$$
\sum_{j=1}^{\infty} \sum_{m=\text{max}\{j_0,j-2\}}^{\infty} 2^{(\sigma+1)(j-m)} \|D^{\sigma+1} P_m u\|_{k_0}^{k_0} = \sum_{m=1}^{\infty} 2^{(\sigma+1)(1-m)} \|D^{\sigma+1} P_m u\|_{k_0}^{k_0}
$$

$$
+ \sum_{m=1}^{\infty} 2^{(\sigma+1)(2-m)} \|D^{\sigma+1} P_m u\|_{k_0}^{k_0} + \sum_{j=3}^{\infty} \sum_{m=j-2}^{\infty} 2^{(\sigma+1)(j-m)} \|D^{\sigma+1} P_m u\|_{k_0}^{k_0}
$$

$$
\leq C \sum_{m=1}^{\infty} \|D^{\sigma+1} P_m u\|_{k_0}^{k_0} + \sum_{m=1}^{\infty} \sum_{j=3}^{\infty} 2^{(\sigma+1)(j-m)} \|D^{\sigma+1} P_m u\|_{k_0}^{k_0}
$$

$$
\leq C \sum_{j=1}^{\infty} \|D^{\sigma+1} P_j u\|_{k_0}^{k_0},
$$

using Lemma 4.2 and 4.6, we have

$$
\frac{d}{dt} \sum_{j=1}^{\infty} \|D^{\sigma+1} P_j u\|_{k_0}^{k_0} + \frac{\nu}{4} k_0^2 \sum_{j=1}^{\infty} \int_{R^3} |D^{\sigma+1} P_j u|^{k_0-2} \nabla D^{\sigma+1} P_j u|^2 \leq \frac{C}{\nu} \left( \|u\|_{2k_0}^2 + A^2 \sum_{j=1}^{\infty} \|D^{\sigma+1} P_j u\|_{k_0}^{k_0} \right).
$$

(7.7)

Step 4. Proof of (7.2). From Lemma 4.4, we have

$$
\|D^{\sigma+1} P_j u\|_{k_0}^{k_0} \leq C k_0^2 \left( \int_{R^3} |D^{\sigma+1} P_j u|^{k_0-2} \nabla D^{\sigma+1} P_j u|^2 \right)^{\frac{k_0}{k_0-2}} \left( \|D^{\sigma} P_j u\|_{k_0}^{k_0} \right)^{\frac{2}{k_0}}
$$

$$
\leq \frac{\nu^2 k_0^2}{4CA^2} \int_{R^3} |D^{\sigma+1} P_j u|^{k_0-2} \nabla D^{\sigma+1} P_j u|^2 + (Ck_0^2) \left( \frac{4CA^2}{\nu^2 k_0^2} \right)^{\frac{1}{k_0+2}} \|D^{\sigma} P_j u\|_{k_0}^{k_0}
$$

for any $0 \leq t \leq T'$. Plug this inequality into (7.7) and we obtain

$$
\frac{d}{dt} \sum_{j=1}^{\infty} \|D^{\sigma+1} P_j u\|_{k_0}^{k_0} \leq \frac{C}{\nu} \|u\|_{2k_0}^2 + CA^2 \sum_{j=1}^{\infty} \|D^{\sigma} P_j u\|_{k_0}^{k_0}
$$

$$
\leq \frac{C}{\nu} \|u\|_{2k_0}^2 + C \left( \frac{1}{\nu} \right)^{k_0+1} A^{2+k_0} B^{2k_0},
$$

where (7.1) is used. It follows that

$$
\sum_{j=1}^{\infty} \|D^{\sigma+1} P_j u(t)\|_{k_0}^{k_0} \leq \sum_{j=1}^{\infty} \|D^{\sigma+1} P_j u_0\|_{k_0}^{k_0} + \frac{C}{\nu} \|u\|_{2k_0}^2 T'
$$

$$
+ C \left( \frac{1}{\nu} \right)^{k_0+1} A^{2+k_0} B^{2k_0} T'
$$

(7.8)
for any $0 \leq t \leq T'$.
From Lemma 4.7, there exists $\tilde{B}_2$ such that if $B \geq \tilde{B}_2$, then
\[
\sum_{j=1}^{\infty} \frac{||D^{\sigma+1}P_ju_0||_{k_0}^{k_0}}{2^{B(1-\frac{1}{\sqrt{k}})k}} \leq 1
\]
or
\[
\sum_{j=1}^{\infty} ||D^{\sigma+1}P_ju_0||_{k_0}^{k_0} \leq 2^{B(1-\frac{1}{\sqrt{k}})k} \leq 2B_{k_0}.
\]
Plug this into (7.8) and we arrive at
\[
\sum_{j=1}^{\infty} ||D^{\sigma+1}P_ju(t)||_{k_0}^{k_0} \leq C\left(1 + \frac{1}{\nu}\right)^{k_{0}+1} (1 + T) (1 + ||u||_2)^{k_0} A^{2+k_0} B^2 B_{k_0}.
\]
In view of (7.4), we have (7.2) clearly. \(\square\)

**Corollary 7.2.** Suppose all the assumptions of Theorem 7.1 hold. Then if $B \geq \tilde{B}_2$ given by Theorem 7.1, we have
\[
||D^\sigma Pju(t)||_k \leq B_{2} 2^{\left(\frac{k_0}{k_0} - \frac{3}{2}\right)j} 2^{(1 + \frac{5}{2})B}
\]
for any $j \geq j_0$, $k \geq k_0$ and $0 \leq t \leq T'$, where
\[
\tilde{B} := C\left(1 + \frac{1}{\nu}\right)^{1 + \frac{5}{2k_0}} (1 + T) \frac{1}{\nu} (1 + ||u_0||_2)^{2 + \frac{5}{2k_0}} B_{k_0}^2
\]
with $C > 0$ being a universal constant.

**Proof.** (7.2) implies that
\[
||D^\sigma Pju(t)||_{k_0}^{k_0} \leq C\left(1 + \frac{1}{\nu}\right)^{k_{0}+1} (1 + T) (1 + ||u_0||_2)^{2k_0+2}
\]
\[\times B^{1+\frac{5(k_0+2)/2}{k_0}} \frac{2^{(k_0+2)B/2}}{2^{(k_0+2)B/2}} 2B_{k_0} \]
or
\[
||D^\sigma Pju(t)||_k \leq C\left(1 + \frac{1}{\nu}\right)^{1 + \frac{5}{2k_0}} (1 + T) \frac{1}{\nu} (1 + ||u_0||_2)^{2 + \frac{5}{2k_0}} B_{k_0}^2 2^B 2B_{k_0}/k_0
\]
for any $j \geq j_0$ and $0 \leq t \leq T'$. Recall (5.2), that is
\[
B_{k_0} = \left( B + 1 + \frac{1}{\sqrt{k_0}} \right) k_0,
\]
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we see

\[ \|D^{\sigma+1}P_j u(t)\|_{k_0} \leq C \left( 1 + \frac{1}{\nu} \right)^{1+\frac{\nu}{\nu_0}} (1 + T) \frac{1}{k_0} (1 + \|u_0\|_2)^{2+\frac{\nu}{\nu_0}} B \frac{1}{k_0} 2^{(1+\frac{\nu}{\nu_0})B}. \] (7.11)

From lemma 2.3 and 2.6, we have

\[ \|D^\sigma P_j u(t)\|_k \leq C 2 \left( \frac{k_0}{k} - \frac{1}{k} \right)^j \|D^{\sigma+1}P_j u(t)\|_{k_0} \]

for any \( k \geq k_0 \) and \( 0 \leq t \leq T' \). Combining it with (7.11), we have (7.9). \( \square \)

8 New a priori estimates

In this section, we will demonstrate our new a priori estimates of Navier-Stokes equations, Theorem 8.1 and its corollary. Actually, to obtain Theorem 8.1, we only need to delete the condition (5.3) in Theorem 5.1 (uniform bound estimate) by choosing a suitable large \( B \) in (5.2).

**Theorem 8.1.** Let \( k_0 = 100, j_0 = 1, \sigma = 2, T > 0 \) and \( B > 0 \) be real numbers, \( B_k \) be given by (5.1) for any \( k \geq k_0 \) and \( u_0 \) be a function satisfying (1.2). Suppose \( u \) and \( p \) satisfy Condition (S). Then there exists \( \tilde{B} > 0 \) depending only on \( T, \nu \) and \( u_0 \) such that if \( B \geq \tilde{B} \), then

\[ \sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} \frac{\|D^\sigma P_j u(t)\|_k}{2B_k} \leq 2C - 1 \] (8.1)

for any \( 0 \leq t \leq T \), where \( C \) is given by (5.5).

We first prove the following simple lemma.

**Lemma 8.2.** Suppose all the assumptions of Theorem 8.1 hold and \( 0 < T' \leq T \). If (8.1) holds for any \( 0 < t < T' \), then it holds for \( t = T' \).

**Proof.** For any \( 0 \leq t < T' \), \( k' \geq k_0 \) and \( j' \geq j_0 \), from (8.1), it follows that

\[ \sum_{k=k_0}^{k'} \sum_{j=j_0}^{j'} \frac{\|\nabla^\sigma P_j u(t)\|_k}{2B_k} \leq 2C - 1. \]

Let \( t \to T' \) and using the continuity of \( \|\nabla^\sigma P_j u(t)\|_k \) with respect to \( t \), we obtain

\[ \sum_{k=k_0}^{k'} \sum_{j=j_0}^{j'} \frac{\|\nabla^\sigma P_j u(T')\|_k}{2B_k} \leq 2C - 1. \]
Since \( k' \) and \( j' \) are arbitrary, we have (8.1) holds as \( t = T' \). \( \square \)

**Proof of Theorem 8.1.** We divide the proof into six steps.

**Step 1. Choosing of \( \tilde{B} \).** In Theorem 7.1 (and Corollary 7.2), set

\[
\mathcal{B} = 2\mathcal{C}.
\] (8.2)

Let \( \tilde{B}_3 \) be large enough such that

\[
\tilde{B} \leq \frac{\tilde{B}_3}{2},
\] (8.3)

where \( \tilde{B} \) is given by (7.10). Now we set

\[
\tilde{B} = \max \left\{ \tilde{B}_0, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3, \sigma + 1 \right\},
\]

where \( \tilde{B}_0 \) is given by Lemma 4.7, \( \tilde{B}_1 \) is given by Theorem 6.1 and \( \tilde{B}_2 \) is given by Theorem 7.1.

**Step 2.** Let \( B \geq \tilde{B} \) and

\[
T' := \sup \left\{ \hat{T} : (8.1) \text{ holds for any } 0 \leq t \leq \hat{T} \right\}.
\] (8.4)

In view of Lemma 8.2 and Lemma 4.7, we have (8.1) holds as \( t = T' \).

If \( T' = T \), then Theorem 8.1 is true.

Next, we suppose \( T' < T \) and we will derive a contradiction by it.

**Step 3.** From Condition (S) and Lemma 4.6, we have

\[
\sum_{j=j_0}^{\infty} \| D^\sigma P_j u(t) \|_{k_0}^{k_0}
\]

is continuous. In view of (8.1) and (8.4),

\[
\sum_{j=j_0}^{\infty} \| D^\sigma P_j u(t) \|_{k_0}^{k_0} \leq (2\mathcal{C} - 1) 2^{B_{k_0}}
\]

for any \( 0 \leq t \leq T' \). Then there exists \( \delta > 0 \) such that \( T' + \delta \leq T \) and

\[
\sum_{j=j_0}^{\infty} \| D^\sigma P_j u(t) \|_{k_0}^{k_0} \leq 2\mathcal{C}2^{B_{k_0}} = \mathcal{B}2^{B_{k_0}}
\]
for any $0 \leq t \leq T' + \delta$, where (8.2) is used. Therefore by Corollary 7.2 (recall $B \geq \tilde{B}_2$), we have (7.9) holds. In view of (8.3) and $B \geq \tilde{B}_3$, (7.9) implies

$$||D^\sigma P_j u(t)||_k \leq 2\left(\frac{1}{\sigma} - \frac{1}{2} - 1\right) j \sigma 2^{(1+\frac{1}{\sigma})} B.$$  (8.5)

for any $0 \leq t \leq T' + \delta$.

**Step 4. Convergence of the high frequency part.** Let

$$J_0 = \left[\frac{8B}{\sigma}\right].$$

Then as $j > J_0$, from $B \geq \tilde{B} \geq \tilde{B}_2$, we have

$$\left(\frac{3}{k_0} - 1\right) j + \frac{6B}{\sigma} \leq \left(\frac{3}{k_0} - 1\right) \left(\frac{8B}{\sigma} - 1\right) + \frac{6B}{\sigma} \leq -\frac{B}{\sigma} - \left(\frac{3}{k_0} - 1\right) \leq 0.$$

Combining it with (8.5), we have

$$\frac{||D^\sigma P_j u(t)||_k}{2^{B_k}} \leq \left(\frac{2\left(\frac{1}{\sigma} - \frac{1}{2} - 1\right) j \sigma 2^{(1+\frac{1}{\sigma})} B}{2^{B_1 + \frac{1}{\sigma} \frac{6B}{\sigma}}}\right)^k \leq \left(\frac{2\left(\frac{1}{\sigma} - \frac{1}{2} - 1\right) j \sigma 2^{\frac{6B}{\sigma}}}{2}\right)^k \leq 2^{-3j-k}$$

for any $0 \leq t \leq T' + \delta$. Therefore

$$\sum_{k=k_0}^{\infty} \sum_{j=j_0+1}^{\infty} \frac{||D^\sigma P_j u(t)||_k}{2^{B_k}} \leq \sum_{k=k_0}^{\infty} \sum_{j=j_0+1}^{\infty} 2^{-3j-k} \leq 1$$  (8.6)

for any $0 \leq t \leq T' + \delta$.

**Step 5. Convergence of the low frequency part.** Since $B \geq \tilde{B} \geq \tilde{B}_1$, by Theorem 6.1, we have

$$\sum_{k=k_0}^{\infty} \sum_{j=j_0}^{j_0+1} \frac{||D^\sigma P_j u(t)||_k}{2^{B_k}} \leq 1$$  (8.7)

for any $0 \leq t \leq T$. From (5.2) and (6.2), we have

$$\lim_{k \to \infty} \frac{\tilde{B}_k}{B_k} = \frac{B}{B + 1}.$$
Therefore there exists \( \hat{k} \geq k_0 \) such that
\[
\hat{B}_k \leq B_k.
\]
In view of (8.7),
\[
\sum_{k=k_0}^\infty \sum_{j=j_0}^{j_0} \frac{||D^\sigma P_j u(t)||^k_{\hat{B}_k}}{2^{2\hat{B}_k}} \leq \sum_{k=k_0}^\infty \sum_{j=j_0}^{j_0} \frac{||D^\sigma P_j u(t)||^k_{B_k}}{2^{2B_k}} \leq 1
\]
for any \( 0 \leq t \leq T \). From condition (S), there exists \( \hat{B} \) such that
\[
\sum_{k=k_0}^{\hat{k}-1} \sum_{j=j_0}^{j_0} \frac{||D^\sigma P_j u(t)||^k_{\hat{B}_k}}{2^{\hat{B}_k}} \leq \hat{B}
\]
for any \( 0 \leq t \leq T \). Therefore
\[
\sum_{k=k_0}^\infty \sum_{j=j_0}^{j_0} \frac{||D^\sigma P_j u(t)||^k_{\hat{B}_k}}{2^{\hat{B}_k}} \leq 1 + \hat{B}
\]
for any \( 0 \leq t \leq T \).

**Step 6. Contradiction.** From (8.6) and (8.8), we have (5.3) holds for \( 0 \leq t \leq T' + \delta \). Then by Theorem 5.1, we have (5.4) holds for \( 0 \leq t \leq T' + \delta \). Since \( B \geq B_0 \), from Lemma 4.7, we have
\[
\sum_{k=k_0}^\infty \sum_{j=j_0}^{j_0} \frac{||D^\sigma P_j u_0||^k_{\hat{B}_k}}{2^{\hat{B}_k}} \leq \sum_{k=k_0}^\infty \sum_{j=j_0}^{j_0} \frac{||D^\sigma P_j u_0||^k_{B_k}}{2^{B_k}} \leq 1.
\]
Then (5.4) implies (8.1) clearly. That is, (8.1) holds for \( 0 \leq t \leq T' + \delta \). This contradicts with (8.4), the choice of \( T' \). □

**Corollary 8.2.** Let \( T > 0 \) and \( u_0 \) be a function satisfying (1.2). There exists \( \hat{B} > 0 \) depending only on \( T, \nu \) and \( u_0 \) such that if \( u \) and \( p \) satisfy Condition (S), then
\[
||u(t)||_{\infty} \leq \hat{B}
\]
for any \( 0 \leq t \leq T \).

**Proof.** Let \( \hat{B} \) be given by Theorem 8.1. Then we have
\[
\sum_{k=k_0}^\infty \sum_{j=j_0}^{j_0} \frac{||D^\sigma P_j u(t)||^k_{\hat{B}_k}}{2^{\hat{B}_k}} \leq 2C - 1
\]
for any \(0 \leq t \leq T\), where \(B_k = (\hat{B} + 1 + \frac{1}{\sqrt{k}})k\) (recall (5.2)). This implies
\[
\|D^\sigma P_j u(t)\|_k \leq ((2C - 1)2^{B_k})^k
\]
for any \(k \geq k_0\), \(j \geq j_0\) and \(0 \leq t \leq T\). Let \(k \to \infty\) and then
\[
\|D^\sigma P_j u(t)\|_\infty \leq \hat{B}^{\hat{B}+1}.
\]
It follows that
\[
\|u\|_\infty = \|P_{\leq 0} u\|_\infty + \sum_{j=j_0}^\infty \|P_j u\|_\infty \leq C\|u_0\|_2 + C \sum_{j=j_0}^\infty 2^{-\sigma j} \|D^\sigma P_j u(t)\|_\infty \leq C\|u_0\|_2 + C2^{\hat{B}+1} := \check{B}.
\]
The proof is complete. \(\Box\)

9 Proof of Theorem 1.1

Theorem 1.1 is an easy consequence of our new a priori estimates.

**Proof of Theorem 1.1.** From Theorem 3.4 and 3.5, there exist \(T^* > 0\), smooth functions \(p(x, t)\) and \(u(x, t)\) on \(R^3 \times [0, T^*]\) with \(u(x, 0) = u_0(x)\) and \(f \equiv 0\) such that (1.1) holds. By Theorem 3.1, we have (1.3) holds. Let
\[
T = \sup \left\{ T' : \text{There exist smooth functions } p(x, t) \text{ and } u(x, t) \text{ on } R^3 \times [0, T'] \right. \\
\left. \text{with } u(x, 0) = u_0(x) \text{ and } f \equiv 0 \text{ such that (1.1) and (1.3) hold} \right\}.
\]
Then \(T \geq T^*\).

If \(T = +\infty\), then Theorem 1.1 is true.

If \(T < \infty\), then from Theorem 3.6, we have
\[
\limsup_{t \to T^-} \|u(t)\|_\infty = \infty.
\]
Let \(\check{B}\) be given by Corollary 8.2 with \(u_0\), \(\nu\) and this \(T\). Then from Theorem 3.5 and Corollary 8.2, we have
\[
\|u(t)\|_\infty < \check{B}
\]
for any $0 \leq t < T$. This contradict with (9.1).

The proof of Theorem 1.1 is complete.  

**Remark 9.1.** In Corollary 8.2, the bound of $||u||_{\infty}$, $\hat{B}$ depends on $||D^{\sigma+3}u_0||_2$ and $||D^{\sigma+3}u_0||_{\infty}$ (recall the choosing of $\hat{B}_2$ in Theorem 7.1). If $u_0 \in H^1(R^3)$, for any $T > 0$, we can bound

$$\sup_{0 \leq t \leq T} ||\nabla u(t)||_2$$

by the following way. From Theorem 3.4, there exists $T^* > 0$ such that

$$\sup_{0 \leq t \leq T^*} ||\nabla u(t)||_2 \leq \hat{B}_1$$

(9.2)

which is a constant depending on $u_0$ and $\nu$. By Remark 3.7, $u$ is smooth on $[T^*/2,T^*]$. Then we can use Corollary 8.2 for $t \in [T^*/2,T]$, that is, there exists a constant $\hat{B}_2$ depending only on $u_0$, $\nu$ and $T$ such that

$$\sup_{T^*/2 \leq t \leq T} ||u(t)||_{\infty} \leq \hat{B}_2.$$

Then from the classical regularity results of parabolic equations, we have

$$\sup_{T^*/2 \leq t \leq T} ||\nabla u(t)||_2 \leq \hat{B}_3$$

(9.3)

which is a constant depending on $u_0$, $\nu$ and $T$. From (9.2) and (9.3), we see

$$\sup_{0 \leq t \leq T} ||\nabla u(t)||_2 \leq \max\{\hat{B}_1,\hat{B}_2\}.$$

(9.4)

Using the a priori estimate (9.4), if $u_0 \in H^1(R^3)$, we can conclude that (1.1) has the strong solution on $[0,T]$ for any $T > 0$, which is smooth in $(0,\infty)$.

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