Biquandles for Virtual Knots

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Virtual knot theory can be viewed as the theory of abstract Gauss codes. A Gauss code for a classical knot is obtained by walking along the knot diagram and recording the names and “states” of the crossings in the order that the knot is traversed. This abstract information can be used as a substitute for the knot. Arbitrary Gauss codes are not necessarily planar, but can be used to generalize knots. That generalization is virtual knot theory. We recommend [14] as an introduction for the reader who is not already familiar with virtual knot theory.

This paper studies an algebraic invariant of virtual knots called the biquandle. The biquandle generalizes the fundamental group and the quandle of virtual knots. The approach taken in this paper to the biquandle emphasizes understanding its structure in terms of compositions of morphisms, where elementary morphisms are associated to oriented classical and virtual crossings in the diagram.

Section 1 gives definitions of virtual knots and reminds the reader of the basic result that classical knot theory embeds in virtual knot theory. Section 2 defines the biquandle. Section 3 shows how to associate morphisms to the crossings. Section 4 gives our basic formulae for inverting the morphism corresponding to a crossing. We use this method to show that the biquandle descriptions for the virtual Hopf link and its mirror image are identical. Of course, this shows that biquandles do not classify virtual links. We also use this method to show that the biquandle is invariant under AD inversion of the diagram (see Definition 4.2). This method of inverting a diagram is defined in the paper. Section 5 explains how to use braids to compute the biquandle. We also prove that $BQ(K) \cong BQ(K^\uparrow)$. The vertical mirror image $K^\uparrow$ is one of various kinds of mirror images that can be defined for virtual knots. The section ends with an example (see Figure 2) showing the difference between two such types of mirror images. Section 6 applies our methods to calculations of the Alexander biquandle. Section 7 discusses the structure of the biquandle description of the Kishino knot, and uses our framework to give a proof of its non-triviality.
1 Definitions

We define a knot $K$ in the combinatorial sense, as a class of diagrams which represent a generic projection of an embedding $S^1 \to S^3$ or $S^1 \sqcup \cdots \sqcup S^1 \to S^3$. Each circle represented in such a diagram is called a component, and if there is more than one component, the diagram (or related class) is sometimes referred to as a link. A strand in a knot diagram is the projection of a connected interval in the embedded curve. In a knot diagram, Certain arcs of the projection of the curve embedded in space are eliminated to form a knot diagram, creating broken strands that indicate the over and under crossings. At each crossing in the diagram there is an indicated over strand and a broken under strand. In this sense, there are two local strands at any crossing in a diagram. Whenever we refer to a strand of a crossing, we mean one of these two local strands.

**Definition 1.1.** Two knot diagrams $K$ and $K'$ are said to be equivalent when there is a finite sequence of the following (Reidemeister) moves which transform $K$ into $K'$:

(R1) ![Diagram](R1.png)

(R2) ![Diagram](R2.png)

(R3) ![Diagram](R3.png)

A knot is an equivalence class of knot diagrams under the Reidemeister moves.

A knot can have an orientation. This is indicated on a knot diagram by drawing an arrowhead on one or more strands in such a way that each component has a consistent labeling. The Reidemeister moves for an oriented knot are the same as for unoriented knots. We are free to apply the moves without paying attention to the particular orientations of the strands involved.

There is also the notion of more than one kind of Reidemeister equivalence. Regular isotopy is equivalence under only the R2 and R3 moves. Ambient isotopy is equivalence under all three moves. It is useful to distinguish between the two because there are invariants only of regular isotopy. For references on knot theory, see [13], [22] and [19].

**Definition 1.2.** A virtual knot diagram is a generic immersion $S^1 \sqcup \cdots \sqcup S^1 \to \mathbb{R}^2$ such that each double point is labelled with either a real (over or under) or a virtual (circled) crossing. Thus a virtual knot is a class of equivalent virtual diagrams, where two virtual knot diagrams are said to be equivalent when one can be transformed into the other by a finite sequence of real Reidemeister moves (R1), (R2) and (R3), along with the following virtual moves:

(V1) ![Diagram](V1.png)

(V2) ![Diagram](V2.png)
As in the classical case, a virtual knot with multiple components is sometimes called a virtual link. For this paper, we will use the term ‘virtual knot’ to refer to both virtual knots and links. When we do not wish to consider virtual crossings, we will use the term ‘classical knot’ to mean knots and links without virtual crossings.

An alternate definition to virtual Reidemeister equivalence is to allow the classical Reidemeister moves with the addition of a more general “detour move” (See [14, 16]):

\[ m \quad \rightarrow \quad n \]

In the detour move, any number of strands may emanate from the top and bottom of the tangle (represented by a box). The idea is that in a virtual diagram, if we have an arc with any number of consecutive virtual crossings, then we can cut that arc out and replace it with another arc connecting the same points, provided that any crossings on the new arc are also virtual. It is easily seen that this yields the same equivalence.

Notice that there are some mixed moves which are not allowed. Consider the following moves:

\[ \neq \quad \text{and} \quad \neq \]

These are forbidden because they change the underlying Gauss code of a knot, and the only changes we allow on the Gauss codes are the Reidemeister moves.

However, inclusion of the above left (overstrand) version of this move has been studied in the form of welded braids [4, 6] – a generalization of braid theory preceding virtual knot theory.

There has been progress in applying well known knot invariants to the virtual theory. In [14], Kauffman extended the fundamental group, the Jones polynomial and classes of quantum link invariants to virtual knots and gave examples of non-trivial virtual knots with trivial Jones Polynomial and trivial fundamental group.

One of the results we will assume is the following, proved by Kauffman and by Goussarov, Polyak and Viro:

1by trivial fundamental group, we mean a group isomorphic to the integers
Lemma 1.3. If $K$ and $K'$ are classical knots which are equivalent under virtual Reidemeister equivalence, then they are also equivalent under classical Reidemeister equivalence.

An immediate question that arises is how to determine when a virtual knot is classical. Many virtual knots are indeed non-classical, and in some cases it not possible to determine this property using extensions of classical invariants. In this paper, we will discuss the biquandle, a recent invariant, which turns out to be surprisingly useful in detecting non-classical knots.
2 The Biquandle

The biquandle has a non-associative structure, and in order to handle it, we will use a special operator notation system, whose symbols are \( \uparrow, \downarrow, \updownarrow, \) and \( \dagger, \). The notation allows us to write expressions in a non-associative setting without requiring lots of parentheses. Suppose we have an algebraic system with four binary operations, \( *, \bar{*}, \# \) and \( \bar{\#} \).

If \( X \) is any symbol-string representing an expression in the algebra, and \( Y \) is another such string, then the binary operations on \( X \) and \( Y \) are expressed by

\[
\begin{align*}
X * Y &= X \uparrow Y \\
X \bar{*} Y &= X \downarrow Y \\
X \# Y &= X \updownarrow Y \\
X \bar{\#} Y &= X \dagger Y
\end{align*}
\]

It then follows that the associations between these symbols are expressed by:

\[
\begin{align*}
(X * Y) * Z &= X \uparrow Y \downarrow Z \\
X * (Y * Z) &= X \uparrow Y \downarrow Z
\end{align*}
\]

Use the same convention for all operators. Here are some mixed expressions:

\[
\begin{align*}
(X \bar{*} Y) * Z &= X \downarrow Y \uparrow Z \\
(X \bar{\#} (Y \bar{*} (Z \bar{\#} W))) \bar{\#} T &= X \dagger Y \updownarrow Z \dagger W \uparrow T
\end{align*}
\]

In the operator formalism, the simplest expressions correspond to the left-associated expressions in the binary algebra. Note that when an expression is associated differently, the over and under bars serve as parentheses by extending over or under entire sub-expressions.

The biquandle \( \{13, 8, 5\} \) is a generalization of the quandle \( \{7, 10\} \), which in turn is a generalization of the fundamental group of a classical knot.

In the following, when we speak of an \emph{algebra}, we mean a set that is closed under certain binary operations. The rules specifying the properties of those binary operations will be part of the definition of that algebra.

**Definition 2.1 (Biquandle Axioms).** A \emph{biquandle} \( BQ \) is a non-associative algebra with four binary operations represented through the symbols \( \uparrow, \downarrow, \updownarrow, \) and \( \dagger, \) which satisfy following axioms:

1. For every element \( a \in BQ \), there is an element \( x \in BQ \) such that \( a \uparrow x \updownarrow a = a \).
   
   The left/right variant of this statement must also be true, that is:
   
   For every element \( a \in BQ \), there is an element \( x \in BQ \) such that \( a \downarrow x \updownarrow a = a \).
2. For every element \( a \in BQ \), there is an element \( x \in BQ \) such that \( x = a \overline{x} \) and \( a = x \overline{a} \). The left/right variant of this statement must also be true.

3. For any \( a, b \in BQ \), we have
\[
a = a \overline{b} \overline{a} = a \overline{b} \overline{a} = a \overline{b} \overline{a} = a \overline{b} \overline{a}
\]

4. For any \( a, b \in BQ \), there is an element \( x \in BQ \) such that
\[
x = a \overline{b} \overline{x}, \quad a = x \overline{b} \quad \text{and} \quad b = b \overline{x} \overline{a}
\]
The left/right variant of this statement must also be true.

5. For any \( a, b, c \in BQ \), the following equations hold, as well as the left/right variants:
\[
a \overline{c} \overline{c} = a \overline{b} \overline{c} \overline{c} = a \overline{b} \overline{c} \overline{c} = a \overline{b} \overline{c} \overline{c}
\]

We define a biquandle \( BQ(K) \) associated with a knot \( K \) as follows. Start with an oriented knot diagram \( K \). Assign one generator to each arc. For each classical crossing, assign two relations according the equations given in the diagrams below:
\[
c = b \overline{a} \overline{b} \quad d = a \overline{b} \overline{a} \quad c = b \overline{a} \overline{b} \quad d = a \overline{b} \overline{a}
\]
\[\text{(2)}\]

A diagram with \( n \) real crossings gives rise to \( 2n \) generators and \( 2n \) relations.

An algebra \( A \) with four binary operations and \( 2n \) generators that satisfies the biquandle axioms along with these \( 2n \) relations is said to be a biquandle for the knot \( K \). In practice, it will often suffice to analyze the structure of any hypothetical biquandle algebra that satisfies these generators and relations.

Our aim is to improve the description of the biquandle of a knot or link. We will describe a simpler way to construct generators and relations.

### 3 A Simplified Approach to Biquandle Computation

View each crossing (both real and virtual types) as a morphism on an ordered pair of biquandle elements. We will use the standard diagrammatic convention...
that inputs are ordered in a counterclockwise direction, and outputs are ordered clockwise. When we define a morphism related to a crossing, it will act vertically up the page, mapping the inputs at the bottom of the crossing to the outputs at the top.

Once we define the morphisms, we do not require that the crossings be aligned upwards in the page. Instead, we place a mapping direction directly on the tangles within a diagram. The direction indicates input and output strands, as well as which morphisms to apply. This will be made clear in the context of each diagram.

For example, let $\phi_u$ be the automorphism which describes how the crossing acts on $(a, b)$:

\[
\begin{align*}
(\phi_u)_1(a, b) &= b \quad a \\
(\phi_u)_2(a, b) &= a \quad b
\end{align*}
\]

or

\[
\phi_u(a, b) = (b \quad a), (a \quad b)
\]

The subscript of $u$ indicates that the direction is upwards, or coinciding with the direction of the strands in the crossing.

Since many knots contain multiple “twists” in succession, we view tangle sums as acting on the two generators $(a, b)$ through compositions of the elementary morphisms. The following tangle becomes the morphism $\phi_u^3$ when viewed as mapping from left to right:

\[
(\phi_u^3)_1(a, b) \quad (\phi_u^3)_2(a, b)
\]

This example is a special case of working with a braid. Every virtual knot is equivalent to the closure of some virtual braid, proved in Theorem 5.4, a generalization of Alexander’s theorem for classical knots and links.

This point of view helps in many ways:

1. It decreases the number of generators and relations in a presentation by taking compositions of the morphisms related to a string of crossings.
2. It suggests that we might also consider the inverse automorphism associated with each crossing.
3. In examples such as the Alexander biquandle (see Section 6) where the crossings are linear maps, we may be able to define morphisms for every possible crossing rotation.
In this paper, a presentation of a biquandle in terms of generators and relations is a description of some biquandle generated by these generators, which satisfies the additional equations given by the crossings. To discriminate such a structure from a universal algebraic presentation, we shall refer to it as a \textit{description} of a biquandle.

In this paper, we have not discussed the universal algebra for biquandles. In order to define a free biquandle, or a biquandle with generators and relations, one needs to modify the axioms to accommodate the logic of this level of construction. The details of this will be taken up elsewhere.

4 Inverting the Biquandle Automorphisms

There are inverse equations for the biquandle.

\textbf{Theorem 4.1.} When assigning equations for the biquandle from any crossing in a virtual diagram, we may instead use the following “inverse equations”:

\begin{align*}
    & \begin{array}{c}
    a \\
    b\overline{a}
    \end{array} \quad \begin{array}{c}
    b \\
    a\overline{b}
    \end{array} \quad \begin{array}{c}
    a \\
    b\overline{a}
    \end{array} \quad \begin{array}{c}
    b \\
    a\overline{b}
    \end{array}
\end{align*}

(4)

\textbf{Proof.} These equations result from the axioms given in [14], which are required for the biquandle to be invariant under Reidemeister II:

\begin{align*}
    x &= x\overline{y}\overline{y} \quad y = y\overline{x}\overline{x} = x\overline{y} \quad y = y\overline{y} \quad x = y\overline{y} \quad x (5)
\end{align*}

To derive the “inverse equations” for the right handed crossing, start with the original biquandle equations given in (2), and solve for \( a \) and \( b \) in terms of \( X = c \) and \( Y = d \):

\begin{align*}
    b\overline{a} = X \quad \text{and} \quad a\overline{b} = Y .
\end{align*}

(6)

Operate on each of these equations appropriately to get

\begin{align*}
    b\overline{a} + a\overline{b} = X\overline{a}b \quad \text{and} \quad a\overline{b} + b\overline{a} = Y\overline{b}a ,
\end{align*}

which by (5), simplifies to

\begin{align*}
    b = X\overline{a}b \quad \text{and} \quad a = Y\overline{b}a .
\end{align*}

Finally, backsubstitute from (6) to get the desired result:

\begin{align*}
    b = X\overline{Y} \quad \text{and} \quad a = Y\overline{X} .
\end{align*}

(7)

To arrive at the “inverse equations” for the left handed crossing, apply the same argument. \qed
Table 1: Morphisms used to compute a knot biquandle

The five basic morphisms used to compute a knot biquandle are summarized in Table 1.

One result of the inverse equations is that biquandles will never be able to distinguish the virtual Hopfs (and many other virtual knots) regardless of how we define the operations. To see this, view the downward pointing negative crossing as an automorphism, $\phi_d^{-1}$:

We’ve swapped the variables above to demonstrate the similarities between $\phi_d^{-1}$ and $\phi_u$:

This time, the subscript of $d$ indicates that the direction of the morphism is downwards, or against the orientation of the strands. It is indicated as an inverse morphism because the crossing orientation is negative.
Let's compute the description for a biquandle associated with $V_{Hopf^+}$. The composition $\phi_u \tau$ (see 1) represents the tangle of a positive crossing placed above a virtual crossing.

\[
\begin{array}{c}
\text{a} & \text{b} \\
\downarrow & \downarrow \\
\text{b} & \text{a}
\end{array}
\]

Connecting the inputs below to the outputs above gives us a diagram of $V_{Hopf^+}$ whose associated biquandles have the following description:

\[\langle a, b \mid a = a \overline{a}, b = b \overline{a} \rangle.\]

To compute a biquandle of $V_{Hopf^-}$, label the strands across the top with the generators $a$ on the left and $b$ on the right. Apply the morphism $\tau \phi_d^{-1}$ on $(b, a)$ to get the description:

\[\langle b, a \mid a = a \overline{a}, b = b \overline{a} \rangle.\]

Both virtual Hopf links have the same biquandle descriptions.

**Definition 4.2.** If $K$ is a virtual knot diagram, then the **AD inversion of $K$** is obtained by replacing all classical crossings by their switched virtualizations and reversing the orientation of the resulting diagram, as shown in Figure 1.

**Theorem 4.3.** The biquandle is invariant under AD inversion.

**Proof.** Consider equation (8). We modify it slightly to get

\[
\begin{align*}
\phi_u &= \tau \phi_d^{-1} \tau \\
\phi_u^{-1} &= \tau \phi_d \tau.
\end{align*}
\]

The example crossing replacement in Figure 1 is the personification of (9a). Note that if we perform this replacement on all classical crossings in a given diagram, then the orientations will all be compatible, and the result will follow.
Figure 1: Equivalence Under AD Inversion

5 Using Braids to Compute the Biquandle

Recall the braid group on $n$ strings.

**Theorem 5.1** (Artin). The $n$-string braid group $B_n$ has the following presentation:
\[
\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_k = \sigma_k \sigma_i, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle
\]

(10)

For a proof, see [2].

Each of the generators $\sigma_i$ along with their inverses $\sigma_i^{-1}$ can be represented by the following diagrams:

For a further generator $\nu_i$, corresponding to a virtual crossing between the $i^{th}$
and \((i + 1)^{th}\) strands to generalize the \(n\)-strand Braid group to virtual braids.

In this way, the \(n\) string braid group extends to virtual theory (see [11, 12, 17, 15]).

**Definition 5.2.** The \(n\)-string virtual braid group \(\mathcal{VB}_n\) has the following presentation:

\[
\langle \sigma_1, \ldots, \sigma_{n-1}, \nu_1, \ldots, \nu_{n-1} \mid \sigma_i \sigma_k = \sigma_k \sigma_i, \quad |i - k| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \nu_i^2 = 1, \\
\nu_i \nu_k = \nu_k \nu_i, \quad |i - k| > 1, \\
\nu_i \nu_{i+1} \nu_i = \nu_{i+1} \nu_i \nu_{i+1}, \quad \sigma_i \nu_k = \nu_k \sigma_i, \quad |i - k| > 1, \\
\sigma_i \nu_{i+1} \nu_i = \nu_{i+1} \nu_i \sigma_{i+1} \rangle
\]

In [4, 6], Fenn, Rimányi, and Rourke prove that the welded braid group on \(n\) strands is isomorphic to the automorphism group of the free quandle\(^2\) on \(n\) generators. The morphisms described above make it possible to analyze the automorphism group of a biquandle. A virtual braid determines a morphism of biquandle elements. Their result motivated the approach in this paper.

**Definition 5.3.** For any braid \(\beta \in \mathcal{VB}_n\), the closure \(cl(\beta)\) is a knot obtained by identifying initial points and end points of each of the braid strings.

We will need a generalization of Alexander’s Theorem to virtual braids, proved independently by the authors and by Kamada [11].

**Theorem 5.4** (Alexander’s Theorem for Virtual Braids). If \(K\) is a virtual knot, then for some \(n\), there exists a braid \(\beta \in \mathcal{VB}_n\) such that the closure of \(\beta\) is \(K\).

**Proof.** Suppose \(K\) is a virtual knot. Choose a representative knot diagram and set a braid axis in an arbitrary region in the diagram. Now, choose an orientation for the knot, and rotate each real crossing via planar isotopy so that both the over and under crossings align in a clockwise direction around the axis. Note that this leaves a finite number of virtual arcs connecting each real crossing to

\(^2\)Recall that welded braids are virtual braids with the addition of an extra mixed move.
another, and it is only these connecting arcs which might not travel counterclockwise around the axis. One by one, we can perform a detour move on each of these arcs so that they travel clockwise between each real crossing. After performing all of these detour moves, we will have a virtual knot diagram for which all the real crossings are aligned clockwise around the braid axis and every connecting path between these crossings is also aligned clockwise around the axis. Since this new diagram has only a finite number of real and virtual crossings, there exists a ray from the axis to the exterior of the diagram which avoids every crossing. Cut along this ray to get the braid $B$ whose closure is equivalent to $K$.

**Theorem 5.5.** If $BQ$ is a biquandle, then for every $\beta \in \mathcal{VB}_n$, $BQ(cl(\beta)) \cong BQ(cl(\beta^{-1}))$.

**Proof.** Let $\beta = x_1x_2 \cdots x_k \in \mathcal{VB}_n$, where

$$x_i \in \{\sigma_1^{\epsilon_1}, \ldots, \sigma_{(n-1)}^{\epsilon_{(n-1)}}, \nu_1, \ldots, \nu_{(n-1)} \mid \epsilon_i = \pm 1\} \subset \mathcal{VB}_n.$$  

Define a map $f_u^{(n)} : \mathcal{VB}_n \to \text{Aut}(BQ^n)$ by setting

$$f_u^{(n)}(\sigma_i^\epsilon) = \phi_{u,i}^{(n)} = 1_{i-1} \oplus \phi_u^\epsilon \oplus 1_{n-1-i}$$

$$f_u^{(n)}(\nu_i^\epsilon) = \tau_i^{(n)} = 1_{i-1} \oplus \tau \oplus 1_{n-1-i}$$

and extending $f_u^{(n)}$ to all of $\mathcal{VB}_n$ using the multiplication reversing rule:

$$f_u^{(n)}(xy) = f_u^{(n)}(y)f_u^{(n)}(x). \quad (11)$$

We will use $f_u^{(n)}$ to construct a description for $BQ(cl(\beta))$. Label the bottom strands of $\beta$ with the generators $a_1, a_2, \ldots, a_n$ ordered from left to right. To get the equations, apply the composition of upward morphisms described by $\beta$ and equate the resulting labels on the upper strands with the corresponding generators below. In terms of $f_u^{(n)}$, the description is

$$BQ(cl(\beta)) = \langle a_1, a_2, \ldots, a_n \mid f_u^{(n)}(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n) \rangle \quad (12)$$

Define the map $f_d^{(n)} : \mathcal{VB}_n \to \text{Aut}(BQ^n)$ by setting

$$f_d^{(n)}(\sigma_i^\epsilon) = \phi_{d,i}^{(n)} = 1_{n-1-i} \oplus \phi_d^{-\epsilon} \oplus 1_{i-1}$$

$$f_d^{(n)}(\nu_i^\epsilon) = \tau_i^{(n)} = 1_{n-1-i} \oplus \tau \oplus 1_{i-1}$$

and extending $f_d^{(n)}$ to all of $\mathcal{VB}_n$ using the multiplication preserving rule:

$$f_d^{(n)}(xy) = f_d^{(n)}(x)f_d^{(n)}(y). \quad (13)$$
We use $f_d^{(n)}$ to construct a description for $BQ(cl(\beta^{-1}))$. Label the top strands of $\beta^{-1}$ with the generators $a_1, a_2, \ldots, a_n$ ordered from left to right. Recall this is the reverse of the counter-clockwise convention for inputs. The description is:

$$BQ(cl(\beta^{-1})) = \langle a_1, a_2, \ldots, a_n \mid f_d^{(n)}(a_n, \ldots, a_2, a_1) = (a_n, \ldots, a_2, a_1) \rangle$$  \hfill (14)

Express $f_u^{(n)}(\beta)$ and $f_d^{(n)}(\beta^{-1})$ in terms of the $x_i$'s:

$$f_u^{(n)}(\beta) = f_u^{(n)}(x_1 x_2 \cdots x_k) = f_u^{(n)}(x_k) \cdots f_u^{(n)}(x_2) f_u^{(n)}(x_1)$$

$$f_d^{(n)}(\beta^{-1}) = f_d^{(n)}(x_k^{-1} \cdots x_2^{-1} x_1^{-1}) = f_d^{(n)}(x_k^{-1}) \cdots f_d^{(n)}(x_2^{-1}) f_d^{(n)}(x_1^{-1})$$

We claim that these compositions generate the same set of equations for the descriptions given in (12) and in (14). Start with the virtual morphisms:

$$f_u^{(n)}(\nu_i)(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n)$$

$$f_d^{(n)}(\nu_i^{-1})(a_n, \ldots, a_{i+1}, a_i, \ldots, a_1) = (a_n, \ldots, a_i, a_{i+1}, \ldots, a_1).$$

Both of the above permute $a_i$ with $a_{i+1}$. Moreover they are reversals of each other,

$$f_u^{(n)}(\nu_i) = T_n f_d^{(n)}(\nu_i^{-1})$$  \hfill (15)

where $T_n$ is the map which reverses $n$-tuples: $T_n(b_1, b_2, \ldots, b_n) = (b_n, \ldots, b_2, b_1)$.

Recall the relationship between $\phi_u$ and $\phi_d^{-1}$. Equation (8) can be generalized for $\phi_{(u,i)}$ and $\phi_{(d,i)}$. Consider how it works for a specific example: the braids $\sigma_2$ and $\sigma_2^{-1}$ in $VB_3$. The morphism $f_u^{(3)}(\sigma_2) = \phi_{(u,i)}$ mapping upwards on the page looks like

```
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (0.5,0.5);
\draw (0,1) -- (0.5,0.5);
\draw (1,1) -- (0.5,0.5);
\draw (0,0) -- (0.5,0.5);
\draw (0,0) -- (0.5,0.5);
\draw (1,0) -- (0.5,0.5);
\node at (0,0) {$a_1$};
\node at (1,0) {$a_2$};
\node at (0,1) {$a_3$};
\node at (0.5,0.5) {$(\phi_u)_1(a_2, a_3)$};
\end{tikzpicture}
```

and the morphism $f_d^{(3)}(\sigma_2^{-1}) = \phi_{(d,i)}^{-1}$ mapping downwards on the page looks like

```
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\draw (1,0) -- (1,1);
\draw (0,0) -- (0.5,0.5);
\draw (0,1) -- (0.5,0.5);
\draw (1,1) -- (0.5,0.5);
\draw (0,0) -- (0.5,0.5);
\draw (0,0) -- (0.5,0.5);
\draw (1,0) -- (0.5,0.5);
\node at (0,0) {$a_1$};
\node at (1,0) {$a_2$};
\node at (0,1) {$a_3$};
\node at (0.5,0.5) {$(\phi_u)_2(a_2, a_3)$};
\end{tikzpicture}
```
The outputs of both maps read the same from left to right. This means that the 3-tuples $\phi_{u,2}(a_1, a_2, a_3)$ and $\phi_{d,1}(a_3, a_2, a_1)$ are reversals of each other. More generally,

$$f_{u}^{(n)}(\sigma^i_e)(a_1, a_2, \ldots, a_n) = T_n f_{d}^{(n)}(\sigma^{-e}_i)(a_n, \ldots, a_2, a_1).$$

Equations (15) and (16) imply that $f_{u}^{(n)}(x_i) = T_n f_{d}^{(n)}(x_i^{-1})$ for each $i$. The result carries through to compositions. If we equate

$$(a_1, a_2, \ldots, a_n) = f_{u}^{(n)}(x_k) \cdots f_{u}^{(n)}(x_2) f_{u}^{(n)}(x_1)(a_1, a_2, \ldots, a_n)$$

then it must also be true that

$$T_n(a_n, \ldots, a_2, a_1) = T_n f_{d}^{(n)}(x_k^{-1}) \cdots f_{d}^{(n)}(x_2^{-1}) f_{d}^{(n)}(x_1^{-1})(a_n, \ldots, a_2, a_1).$$

Therefore, the descriptions in (12) and (14) are the same. This concludes our proof of the theorem.

**Remark.** We can summarize this argument as follows. Consider two braids, $\beta$ and $\beta^{-1}$:

If we view $\beta$ as an upward morphism, and $\beta^{-1}$ as a downward morphism, then it is clear that the two morphisms give rise to identical biquandle descriptions. The crucial step in seeing this is illustrated in the final two diagrams in the above proof.

Recall that the fundamental group $\pi_1(K)$ is a quandle. One consequence of Theorem 5.5 is the following well known result, generalized to virtual braids.
Corollary 5.6. If $\beta \in VB_n$, $\pi_1(cl(\beta)) = \pi_1(cl(\beta^{-1}))$

Corollary 5.7. If $K$ is a virtual knot diagram and $K^\uparrow$ is the vertical mirror image of $K$, then $BQ(K) \cong BQ(K^\uparrow)$.

Proof. By following the proof of Theorem 5.4, we can simultaneously convert a knot and its vertical mirror image to closed braid forms that are vertical mirror images of one another. 

Remark. Corollary 5.7 shows that the biquandle is the same for a virtual link and its vertical mirror image. In Figure 2, we give an example of virtual knot $K$ and two mirror images: the vertical mirror image $K^\uparrow$, and the mirror image $K^*$, obtained by switching all the crossings in the plane. By the Corollary, the biquandle of $K^\uparrow$ is equal to the biquandle of $K$. However, the biquandle of $K^*$ is not isomorphic to the biquandle of $K$. In particular, calculation shows that $\pi_1(K)$ is the same as the fundamental group of the classical trefoil knot, and hence the biquandle is non-trivial. Calculation also shows that the fundamental group of $K^*$ is trivial. Hence, $K$ and $K^*$ have non-isomorphic biquandles. Furthermore, one sees that $K$ and $K^*$ have the same Jones polynomial, whereas the Jones polynomial of $K^\uparrow$ is different.

6 The Alexander Biquandle

Consider any module $M$ over the ring $R = \mathbb{Z}[s, s^{-1}, t, t^{-1}]$. Defining the binary operations with the following equations provides us with a biquandle structure on $M$:

\[
\begin{align*}
\overline{ab} &= ta + (1 - st)b \\
\underline{ab} &= \frac{1}{t}a + (1 - \frac{1}{st})b
\end{align*}
\]

(17)

\[
\begin{align*}
\overline{ab} &= sa \\
\underline{ab} &= \frac{1}{s}a
\end{align*}
\]

(18)
\[
\begin{align*}
  c &= tb + (1 - st)a \\
  d &= sa \\
  c &= \frac{1}{t}b \\
  d &= \frac{1}{s}a + (1 - \frac{1}{st})b
\end{align*}
\]

Figure 3: The Alexander biquandle relations

If \( M \) is a free module, we call this a free Alexander biquandle.

We associate a specific biquandle to a virtual knot diagram by taking the free Alexander module obtained by assigning one generator for each arc and factoring out by the submodule generated by the relations given in (17). We call the resulting biquandle \( ABQ(K) \) the Alexander biquandle of the virtual knot \( K \).

The biquandle morphisms are linear maps, listed in [2]. In the context of the Alexander biquandle, the morphisms from before become

\[
\begin{align*}
  \phi_u &= A \\
  \phi_d &= \hat{B} \\
  \phi_u^{-1} &= B \\
  \phi_d^{-1} &= \hat{A} \\
  \tau &= V
\end{align*}
\]

We have included four more matrices in Table 2 in addition to the morphisms given in Table 1. They are the linear maps which result when a change of basis transforms the upward and downward morphisms into left and right morphisms.

There are advantages to considering the Alexander biquandle. The binary operations are fairly simple, which allows more flexibility when choosing the morphisms. In addition, the set of relations for a presentation of \( ABQ(K) \) contains a generalization the Alexander polynomial (see [9, 18, 23, 24]).

**Definition 6.1.** The Generalized Alexander Polynomial of \( K \), \( G_K(s, t) \) is the determinant of the relation matrix from a presentation of \( ABQ(K) \). Up to multiples of \( \pm s^i t^j \) for \( i, j \in \mathbb{Z} \), it is an invariant of \( K \).

A significant problem to study comes from our general results on biquandle equivalent pairs of knots, or rather pairs of knots such as those arising from Theorem 4.3 and Theorem 5.5. We show two such pairs in Figure 4. These pairs both come from an application of Theorem 4.3. Relation matrices for the Alexander modules of each pair are also given in the Figure, and we remark that the generalized Alexander polynomials are the same for all four knots, not just between the pairs: \( (s - 1)(t - 1)(st - 1) \).

It is unclear why \( G_K(s, t) \) cannot distinguish between the two pairs. In fact, by adding more twists to the vertical tangle, we can extend the two virtual knot pairs to an infinite set of virtual knot pairs with the same generalized Alexander polynomials. Note that it still is possible for the Alexander modules of the resulting pairs to be different. We conjecture that this is the case.
\[
A = \begin{pmatrix} 1 - st & t \\ s & 0 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 0 & s \\ t & 1 - st \end{pmatrix}
\]

\[
B = \begin{pmatrix} 0 & \frac{1}{s} \\ \frac{1}{t} & 1 - \frac{1}{st} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 1 - \frac{1}{st} & \frac{1}{t} \\ \frac{1}{s} & 0 \end{pmatrix}
\]

\[
C = \begin{pmatrix} 0 & \frac{1}{s} \\ t & \frac{1}{s} - t \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} \frac{1}{s} - t & t \\ \frac{1}{s} & 0 \end{pmatrix}
\]

\[
D = \begin{pmatrix} 0 & s \\ \frac{1}{t} & s - \frac{1}{t} \end{pmatrix}, \quad \hat{D} = \begin{pmatrix} s - \frac{1}{t} & \frac{1}{t} \\ \frac{1}{s} & 0 \end{pmatrix}
\]

\[
V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

Table 2: Matrices for the Alexander biquandle

Figure 4: Virtual knot pairs with identical Alexander biquandles.
7 The Biquandle of the Kishino Knot

We leave the reader with some open questions which remain.

There is an interesting knot discovered by Kishino (see 5) which is nontrivial, but many attempts using known virtual knot invariants fail to detect it. In fact, it has been shown to be nontrivial by a variety of methods [1, 3, 20, 21]. One of the more intriguing methods is via the use of the quaternionic biquandle, discovered by Bartholomew and Fenn [1].

Here we give a version of that technique, by first obtaining a presentation of the biquandle of the Kishino knot using our methods and then showing that the resulting biquandle has a non-trivial quaternionic representation.

Using biquandle morphisms, any biquandle associated with Kishino’s knot can be described with 3 generators, along with the following relations which are computed from the diagram in 5:

\[
\phi_u^{-1} \tau \phi_u(a, b) = (d, b) \quad (19) \\
\phi_d^{-1} \tau \phi_d(c, a) = (c, d) \quad (20)
\]

Calculating further,

\[
\phi_u^{-1} \tau \phi_u (a, b) = \phi_u^{-1} \tau ( b \overline{a} , a b ) \\
= \phi_u^{-1} ( a b \overline{a} , b \overline{b} ) \\
= ( b \overline{a} \overline{b} , a b \overline{b} )
\]
and

\[
\phi_d^{-1} \tau \phi_d (c, a) = \phi_d^{-1} \tau (a \overline{c}, c \overline{a}) \\
= \phi_d (c \overline{a}, a \overline{c}) \\
= (a \overline{c}, c \overline{a}, c \overline{a}, a \overline{c})
\]

so that the relations for the biquandle of the virtual knot diagram in 5 are:

\[
\begin{align*}
  b &= a \overline{b} \\
  c &= a \overline{c} \\
  d &= b \overline{a} \\
  c \overline{a} &= a \overline{c} \overline{d}
\end{align*}
\]

This gives an associated biquandle via a description which has 3 generators and 3 relations:

\[
\langle a, b, c \mid b = a \overline{b}, c = a \overline{c}, b \overline{a} = c \overline{a} \overline{d} \rangle
\]  

(22)

This description for a biquandle associated with Kishino’s knot is much simpler than the description obtained from the original definition in Section 2.

**Theorem 7.1.** The description of the biquandle associated with Kishino’s knot is non-trivial.

The calculations in our proof are simplified by starting with the description in (22). This demonstrates the efficiency the morphism approach given in this paper.

The quaternionic biquandle is defined by the following rules:

\[
\begin{align*}
  a \overline{b} &= ia + (i + j)b \\
  a \overline{b} &= ia + (1 - j)b \\
  a \overline{b} &= -ia + (i + j)b \\
  a \overline{b} &= -ia + (1 - j)b
\end{align*}
\]

Applying this representation to the biquandle description in (22), the 3 equations become the following relations:

\[
\begin{align*}
  (-3a + b) - 2ib - 2kb &= 0 \\
  -(a + c) + i(a + c) + k(a + c) &= 0 \\
  (3b - 3c) - 4ia &= 0
\end{align*}
\]

(23) (24) (25) (26)
Observe that by tensoring with the integers modulo three, denoted by $\mathbb{Z}_3$, we obtain a module over the group ring of the quaternion group with mod 3 coefficients. Note that by direct calculation, this is a non-trivial module, for by direct reduction, we see that the equations above become

\begin{align*}
b - 2ib - 2kb &= 0 \\
-(a + c) + i(a + c) + k(a + c) &= 0 \\
-ia &= 0.
\end{align*}

Hence, $a = 0$, giving us

\begin{align*}
(1 - 2i - 2k)b &= 0 \\
(-1 + i + k)c &= 0.
\end{align*}

This module is clearly non-trivial. Hence, the Kishino knot is non-trivial.
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