Exchange Relations for the $q$-vertex operators of $U_q(\widehat{sl}_2)$

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Abstract

We consider the $q$-deformed Knizhnik-Zamolodchikov equation for the two point function of $q$-deformed vertex operators of $U_q(\widehat{sl}_2)$. We give explicitly the fundamental solutions, the connection matrices and the exchange relations for the $q$-vertex operators of spin $1/2$ and $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Consequently, we confirm that the connection matrices are equivalent to the elliptic Boltzman weights of IRF type obtained by the fusion procedure from ABF models.

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1. Introduction

Recently, the $q$-deformed Knizhnik-Zamolodchikov equation ($q$-KZ eq.), more generally holonomic $q$-difference equation, has been analyzed \cite{1,2,3,4,5,6} and the remarkable relation between the connection matrices of its solutions and the elliptic Boltzman weights was investigated \cite{17}.

To study the $q$-KZ equation \cite{2}, an important and useful concept is the $q$-vertex operators, which are $q$-analogue of the vertex operators in Ref. \cite{10}. First, the solutions of the $q$-KZ equation can be constructed as the correlation functions of the $q$-vertex operators \cite{3,4}. Moreover, we can understand naturally the IRF type Yang-Baxter relation satisfied by the connection matrices of $q$-KZ solutions as the exchange relation for the $q$-vertex operators \cite{11}. So far, explicit calculation of the exchange relation for the $q$-vertex operators has been done in Ref. \cite{4} for the first nontrivial examples and they conjectured that the connection matrices are generally equivalent to the Boltzman weights in Ref. \cite{12,13}. The exchange relations of the $q$-vertex operators of spin $1/2$ or $k/2$ (of level $k$) were calculated and applied to the vertex models in Ref. \cite{8,9}.

The aim of this paper is to generalize these results to arbitrary spins. We consider the $q$-KZ equation for the two point function of the $q$-vertex operators of spin $1/2$ and $j \in \frac{1}{2} \mathbb{Z}_{\geq 0}$ and we give the fundamental solutions and the connection matrices explicitly.

2. Quantum affine algebra $U_q(\widehat{sl_2})$

\S 2.1. First we fix some notation. The algebra $U_q(\widehat{sl_2})$ is generated by $e_i$, $f_i$, invertible $k_i$ ($i = 0, 1$) and $d$ with relations

$$k_i e_j k_i^{-1} = q^{a_{ij}} e_j,$$

$$k_i f_j k_i^{-1} = q^{-a_{ij}} f_j,$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}},$$

$$\sum_{n=0}^{3} (-1)^n \left[ 3 \atop n \right] e_i^{3-n} e_j e_i^n = 0,$$

$$[d, e_i] = \delta_{i,0} e_i,$$

$$\sum_{n=0}^{3} (-1)^n \left[ 3 \atop n \right] f_i^{3-n} f_j f_i^n = 0,$$

$$[d, f_i] = -\delta_{i,0} f_i.$$  (2.1)

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2 The $q$-KZ equation has been applied to the Thirring model in Ref. \cite{3} and to the XXZ model or the vertex model in Ref. \cite{8}.
and \(k_0 k_1 = q^k\) with a level \(k \in \mathbb{C}\), where \(q \in \mathbb{C}\), \(a_{11} = a_{00} = -a_{10} = -a_{01} = 2\) and

\[
\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[n-m]![m]!}, \quad \begin{bmatrix} n \\ 2 \end{bmatrix} = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]  

(2.2)

The algebra \(U_q(\widehat{sl}_2)\) is a Hopf algebra with the comultiplication \(\Delta\), the antipode \(S\) and the co-unit \(\epsilon\)

\[
\begin{align*}
\Delta(e_i) &= e_i \otimes k_i + 1 \otimes e_i, & S(e_i) &= -e_i k_i^{-1}, & \epsilon(e_i) &= 0, \\
\Delta(k_i) &= k_i \otimes k_i, & S(k_i) &= k_i^{-1}, & \epsilon(k_i) &= 1, \\
\Delta(f_i) &= f_i \otimes 1 + k_i^{-1} \otimes f_i, & S(f_i) &= -k_i f_i, & \epsilon(f_i) &= 0, \\
\Delta(d) &= d \otimes 1 + 1 \otimes d, & S(d) &= -d, & \epsilon(d) &= 0.
\end{align*}
\]

(2.3)

§ 2.2. Let \(V_j\) be the Verma module over \(U_q(\widehat{sl}_2)\), generated by the highest weight vector \(|j\rangle\), such that \(e_i |j\rangle = 0, k_1 |j\rangle = q^{2j} |j\rangle\) and \(d |j\rangle = -h_j |j\rangle\) with \(h_j = j(j+1)/\kappa, \kappa = k + 2\). The dual module \(V_j^*\) is generated by \(|j\rangle\) which satisfies \(|j\rangle e_i = 0, \langle j|k_1 = q^{2j} \langle j|\) and \(|j\rangle d = -h_j \langle j|\). The bilinear form \(V_j^* \otimes V_j \to \mathbb{C}\) is uniquely defined by \(|j\rangle \langle j| = 1\) and \(\langle u|a|v\rangle = \langle u|(a|v)\rangle\) for any \(|u\rangle \in V_j^*, |v\rangle \in V_j\) and \(a \in U_q(\widehat{sl}_2)\).

A null vector \(|\chi\rangle \in V_j\) (of grade \(N\) and charge \(Q\)) is defined by \(e_i |\chi\rangle = 0, k_1 |\chi\rangle = q^{2(j+Q)} |\chi\rangle\) and \(d |\chi\rangle = -(h_j + N) |\chi\rangle\). A null vector \(|\chi\rangle \in V_j^*\) is defined in a similar manner.

For \(2j \in \mathbb{Z}_{\geq 0}\), we have a \(2j + 1\) dimensional centerless irreducible representation of \(U_q(\widehat{sl}_2)\), \(V_j(z) = \oplus_{m=0}^{2j} \mathbb{C}(q,z)|j,m\rangle\), which is defined by

\[
\begin{align*}
e_1 |j,m\rangle &= [2j - m + 1] |j,m-1\rangle, & e_0 |j,m\rangle &= z[m+1] |j,m+1\rangle, \\
k_1 |j,m\rangle &= q^{2(j-m)} |j,m\rangle, & k_0 |j,m\rangle &= q^{-2(j-m)} |j,m\rangle, \\
f_1 |j,m\rangle &= [m+1] |j,m+1\rangle, & f_0 |j,m\rangle &= z^{-1}[2j - m + 1] |j,m-1\rangle,
\end{align*}
\]

(2.4)

and \(d\) acts as \(d = -h_j + z \frac{d}{dz}\).

3. The \(q\)-vertex operators and their two point functions

§ 3.1. The \(q\)-vertex operator, \(\Phi_{j_2}(z) : V_{j_1} \to V_{j_3}\), of spin \(j_2 \in \mathbb{C}\) is defined by the transformation property under the adjoint action [4,3,14], or equivalently as an intertwiner (see Appendix A).
**Definition.** For $2j_2 \in \mathbb{Z}_{\geq 0}$, the $q$-vertex operator \( \Phi_{j_2, m_2}(z) \) : \( V_{j_1} \to V_{j_3} \) with \( j_3 = j_1 + j_2 - \alpha_2 \) and \( 0 \leq m_2 \leq 2j_2 \), is defined explicitly as follows:

\[
e_1 \Phi_{j, m}(z) = [2j - m + 1] \Phi_{j, m-1}(z) k_1 + \Phi_{j, m}(z) e_1,
\]

\[
k_1 \Phi_{j, m}(z) = q^{2(j-m)} \Phi_{j, m}(z) k_1,
\]

\[
f_1 \Phi_{j, m}(z) = [m + 1] \Phi_{j, m+1}(z) + q^{-2(j-m)} \Phi_{j, m}(z) f_1,
\]

\[
e_0 \Phi_{j, m}(z) = z[m + 1] \Phi_{j, m+1}(z) k_0 + \Phi_{j, m}(z) e_0,
\]

\[
k_0 \Phi_{j, m}(z) = q^{-2(j-m)} \Phi_{j, m}(z) k_0,
\]

\[
f_0 \Phi_{j, m}(z) = z^{-1}[2j - m + 1] \Phi_{j, m-1}(z) + q^{2(j-m)} \Phi_{j, m}(z) f_0,
\]

and

\[
d \Phi_{j, m}(z) = -z \frac{d}{dz} \Phi_{j, m}(z) + \Phi_{j, m}(z) d.
\]

(3.1)

The existence conditions for the $q$-vertex operator \( \Phi_{j, m}(z) \) with a general spin \( j \in \mathbb{C} \) were analyzed in Ref. [15], and we will review them in Appendix A. In the case that \( q \) is not a root of unity, the existence conditions are essentially the same as those of \( q = 1 \) [16]. For integrable representations of general quantum affine algebras, the complete results on existence and uniqueness were given in Ref. [11].

From the \( k_1 \) and \( d \) commutation relations, the ground state matrix element of the $q$-vertex operator can be determined up to normalization, and we normalize it as

\[
\langle j_3 | \Phi_{j_2, m_2}^{\alpha_2}(z) | j_1 \rangle = \delta^{j_3-j_1}_{j_2-m_2} \delta^{j_3-j_1}_{j_2-\alpha_2} z^{h_3-h_1},
\]

(3.3)

Our $q$-vertex operators correspond to the type-II vertex operators in Ref. [11, §4]. The relation between our $q$-vertex operator \( \Phi_{j_2, m_2}^{j_1+j_2-j_3}(z) : V_{j_1} \to V_{j_3} \) and their one \( \Phi_{j_2, j_1}(z) : V_{j_2}(z) \otimes V_{j_1} \to V_{j_3} \) is

\[
\Phi_{j_2, m_2}^{j_1+j_2-j_3}(z) = \Phi_{j_2, j_1}(z) \left( |j_2, m_2 \rangle, * \right).
\]

The \( \Phi_{j_2, j_1}(z) \) has the intertwining property, \( a \Phi_{j_2, j_1}(z) = \Phi_{j_2, j_1}(z) \Delta(a) \), for all \( a \in U_q(\widehat{sl_2}) \). Note that the coproduct is also slightly different.

\(^3\) Our $q$-vertex operators correspond to the type-II vertex operators in Ref. [11, §4]. The relation between our $q$-vertex operator \( \Phi_{j_2, m_2}^{j_1+j_2-j_3}(z) : V_{j_1} \to V_{j_3} \) and their one \( \Phi_{j_2, j_1}(z) : V_{j_2}(z) \otimes V_{j_1} \to V_{j_3} \) is

\[
\Phi_{j_2, m_2}^{j_1+j_2-j_3}(z) = \Phi_{j_2, j_1}(z) \left( |j_2, m_2 \rangle, * \right).
\]

The \( \Phi_{j_2, j_1}(z) \) has the intertwining property, \( a \Phi_{j_2, j_1}(z) = \Phi_{j_2, j_1}(z) \Delta(a) \), for all \( a \in U_q(\widehat{sl_2}) \). Note that the coproduct is also slightly different.

\(^4\) The upper index \( \alpha \), which specify the modules on which the $q$-vertex operator acts, will be sometimes suppressed.

\(^5\) The sign difference of \( d \) in §2.2 and §3.1 comes from the difference in meaning of \( z \). In §2.2, \( z \) is a generator such that \( [d, z] = 1 \), on the other hand \( z \) is just a variable i.e. \( [d, z] = 0 \), in §3.1.
where \( h_n = h_{j_n} \). The other matrix elements for the descendant fields can be uniquely determined.

We can in principle derive the arbitrary \( N \) point functions by using this one point function and the \( q \)-operator product expansion (\( q \)-OPE). In Appendix B, we will present the \( q \)-OPE of a spin 1/2 \( q \)-vertex operator and the two point functions \( \langle j_1 | \Phi_{j_2, m_2} (z_2) \Phi_{j_2, m_3} (z_3) | j_1 \rangle \) and \( \langle j_1 | \Phi_{j_2, m_2} (z_2) \Phi_{j_2, m_3} (z_3) | j_1 \rangle \).

§ 3.2. As we will discuss in Appendix B, the above two point functions have a complicated form, for example each coefficient of \( z^n \) cannot be factorised. But they can be simplified by dividing by a function

\[
g(x) = (1 - q^{k+2} \frac{[2j]}{[2][k+2]} x + O(x^2)), \quad x = z_2 / z_3.
\]

Here we list the leading terms

\[
\langle j_4 | \Phi_{j_2, 0} (z_3) \Phi_{j_2, M} (z_2) | j_1 \rangle = z_3^{h_4 - h_1} z_2^{h_1 - h_1} \\
\times \{1 + p^{-m_1 - m_2} x \frac{[2m + m_1 + m_2 + 1]_p [2m_1]_p}{[2m + m_1 - m_2 + 1]_p} + O(x^2) \} g(x),
\]

\[
\langle j_4 | \Phi_{j_2, 0} (z_3) \Phi_{j_2, M-1} (z_2) | j_1 \rangle = -z_3^{h_4 - h_1} z_2^{h_1 - h_1} x p^{-m_2} \frac{[2m_1]_p}{[2m + m_1 - m_2 + 1]_p} \\
\times \{1 + p^{-m_1 - m_2} x \frac{[2m + m_1 + m_2 + 1]_p [2m_1 + 1]_p}{[2m + m_1 - m_2 + 2]_p} + O(x^2) \} g(x),
\]

(3.5)

\[
\langle j_4 | \Phi_{j_2, 0} (z_3) \Phi_{j_2, M-1} (z_2) | j_1 \rangle = -z_3^{h_4 - h_1} z_2^{h_1 - h_1} p^{-m_1} \frac{[2m_2]_p}{[2m + m_1 + m_2]_p} \\
\times \{1 + p^{-m_1 - m_2} x \frac{-2m + m_1 + m_2 + 1]_p [2m_2]_p}{[-2m - m_1 + m_2 + 1]_p} + O(x^2) \} g(x),
\]

(3.6)
and

\[
\langle j_4 | \Phi_{j_2, M}(z_2) \Phi_{\frac{1}{2}, 0}(z_3) | j_1 \rangle = z_2^{j_4 - h_1^+ - h_1} \zeta_3^{h_1^+ - h_1} \\
\times \{1 + p^{-m_1 - m_2} x^{-1} \frac{[-2m + m_1 + m_2]_p[2m_1]_p}{[-2m + m_1 - m_2]_p} + O(x^{-2}) \} g(x^{-1}),
\]

(3.7)

\[
\langle j_4 | \Phi_{j_2, M-1}(z_2) \Phi_{\frac{1}{2}, 0}(z_3) | j_1 \rangle = -z_2^{j_4 - h_1^+} \zeta_3^{h_1^+ - h_1} \times \frac{[2m_1]_p}{[-2m + m_1 - m_2]_p} \\
\times \{1 + p^{-m_1 - m_2} x^{-1} \frac{[2m - m_1 + m_2 + 1]_p}{[2m - m_1 + m_2 + 2]_p} + O(x^{-2}) \} g(x^{-1}),
\]

(3.8)

\[
\langle j_4 | \Phi_{j_2, M-1}(z_2) \Phi_{\frac{1}{2}, 1}(z_3) | j_1 \rangle = z_2^{j_4 - h_1^+} \zeta_3^{h_1^+ - h_1} \\
\times \{1 + p^{-m_1 - m_2} x^{-1} \frac{[2m + m_1 + m_2 + 1]_p[2m_2]_p}{[2m - m_1 + m_2 + 1]_p} + O(x^{-2}) \} g(x^{-1}),
\]

here and below we use the following notations, \([n]_p = (p^n - 1)/(p - 1)\), \(p = q^{-2\kappa}\), \(m = -(j_1 + j_4 + 1)/2\kappa\), \(m_1 = (j_1 + j_2 - j_4 + 1/2)/2\kappa = M/2\kappa\), \(m_2 = (-j_1 + j_2 + j_4 + 1/2)/2\kappa = \overline{M}/2\kappa\) and \(h_n^\pm = h_{j_n \pm 1/2}\).

More complete expressions for the two point functions will be given in §5.1 by solving the q-KZ equation.

4. The solutions and their connection formula for the q-KZ equation

§4.1. Arbitrary \(N\) point functions of \(q\)-vertex operators satisfy the q-KZ equation [4]. For the two point function of the \(q\)-vertex operators of spin \(1/2\) and \(j\), \(\langle j_4 | \Phi_{\frac{1}{2}, M}(z_2) \Phi_{j_2, M_2}(z_2) | j_1 \rangle\), the q-KZ equation is written as a \(2 \times 2\) block diagonal \(R\) matrix \(R(z_2/z_3) : V_{j_2}(z_2) \otimes V_{\frac{1}{2}}(z_3) \rightarrow V_{j_2}(z_2) \otimes V_{\frac{1}{2}}(z_3)\), defined by \(R(z_2/z_3) \Delta'_{z_2, z_3} = \Delta_{z_2, z_3} R(z_2/z_3)\) (Appendix C).

Let \(M = M_2 + M_3\), \(M + \overline{M} = 2j_2 + 1\) and \(x = z_2/z_3\).
Up to normalization, this $q$-KZ equation is

$$
\begin{pmatrix}
p^{-m} \tilde{\Psi}_0(px) \\
p^m \tilde{\Psi}_1(px)
\end{pmatrix} = \tilde{R}(x) \begin{pmatrix}
\tilde{\Psi}_0(x) \\
\tilde{\Psi}_1(x)
\end{pmatrix},
$$

$$
\tilde{R}(x) = \begin{pmatrix}
\tilde{R}_0^0(x) & \tilde{R}_0^1(x) \\
\tilde{R}_1^0(x) & \tilde{R}_1^1(x)
\end{pmatrix} = \frac{1}{1-x p^{m_1+m_2}} \begin{pmatrix}
p^{m_1} - x p^{m_2} & p^{m_2} (p^{-m_2} - p^{m_2}) \\
x p^{m_1} (p^{-m_1} - p^{m_1}) & p^{m_2} - x p^{m_1}
\end{pmatrix},
$$

where $p$, $m$ and $m_i$ are the same as in §3.2.

The equation (4.1) can be solved easily, and the solutions are given by the $q$-hypergeometric function

$$
F_p(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n,
$$

where $(a)_n = [a]_p [a + 1]_p \cdots [a + n - 1]_p$.

Two fundamental solutions $\tilde{\Psi}^0_{(+i)}(x)$ and $\tilde{\Psi}^1_{(+i)}(x)$ in the region $x \ll 1$ are

$$
\begin{align*}
\tilde{\Psi}^0_{(+0)}(x) &= -F(-2m + m_1 + m_2, 2m_2 + 1, -2m - m_1 + m_2 + 1; p^{-m_1-m_2} x) \\
&\quad \times p^{-m_1} \frac{[2m_2]_p}{[-2m - m_1 + m_2]_p} x^{m_2-m}, \\
\tilde{\Psi}^0_{(+1)}(x) &= F(-2m + m_1 + m_2, 2m_2, -2m - m_1 + m_2; p^{-m_1-m_2} x) x^{m_2-m}, \\
\tilde{\Psi}^1_{(+0)}(x) &= F(2m + m_1 + m_2 + 1, 2m_1, 2m + m_1 - m_2 + 1; p^{-m_1-m_2} x) x^{m_1+m_1}, \\
\tilde{\Psi}^1_{(+1)}(x) &= -F(2m + m_1 + m_2 + 1, 2m_1 + 1, 2m + m_1 - m_2 + 2; p^{-m_1-m_2} x) \\
&\quad \times p^{-m_2} \frac{[2m_1]_p}{[2m + m_1 - m_2 + 1]_p} x^{m_1+m_1+1}.
\end{align*}
$$

And the other two fundamental solutions $\tilde{\Psi}^0_{(-i)}(x)$ and $\tilde{\Psi}^1_{(-i)}(x)$ in the region $x \gg 1$ are

$$
\begin{align*}
\tilde{\Psi}^0_{(-0)}(x) &= -F(2m + m_1 + m_2 + 1, 2m_2 + 1, 2m - m_1 + m_2 + 2; p^{-m_1-m_2+1} x^{-1}) \\
&\quad \times p^{2m-m_1+1} \frac{[2m_2]_p}{[2m - m_1 + m_2 + 1]_p} x^{-m+m_2-1}, \\
\tilde{\Psi}^0_{(-1)}(x) &= F(2m + m_1 + m_2 + 1, 2m_2, 2m - m_1 + m_2 + 1; p^{-m_1-m_2+1} x^{-1}) x^{-m+m_2}, \\
\tilde{\Psi}^1_{(-0)}(x) &= F(-2m + m_1 + m_2, 2m_1, -2m + m_1 - m_2; p^{-m_1-m_2+1} x^{-1}) x^{-m+m_1}, \\
\tilde{\Psi}^1_{(-1)}(x) &= -F(-2m + m_1 + m_2, 2m_1 + 1, -2m + m_1 - m_2 + 1; p^{-m_1-m_2+1} x^{-1}) \\
&\quad \times p^{-2m-m_2} \frac{[2m_1]_p}{[-2m + m_1 - m_2]_p} x^{-m+m_2}. \tag{4.4}
\end{align*}
$$
§ 4.2. The complete $q$-KZ equation with the correct normalization is

$$\begin{pmatrix} p^{-m}\Psi_0(px) \\ p^m\Psi_1(px) \end{pmatrix} = R(x) \begin{pmatrix} \Psi_0(x) \\ \Psi_1(x) \end{pmatrix}, \quad (4.5)$$

where $R(x)$ is the image of the universal $R$-matrix which satisfies the crossing relation (see Appendix C). This $R(x)$ is given by

$$R(x) = f(x)\tilde{R}(x), \quad f(x) = \prod_{n \geq 0} \frac{(1 - xq^{2j_2+3+4n})(1 - xq^{-2j_2+1+4n})}{(1 - xq^{2j_2+1+4n})(1 - xq^{-2j_2+3+4n})}, \quad (4.6)$$

in the region $x \ll 1$.

The relation of the $\Psi(x)$'s and the previous $\tilde{\Psi}(x)$'s is

$$\Psi_i(x) = g(x)\tilde{\Psi}_i(x), \quad g(xp) = f(x)g(x). \quad (4.7)$$

The factor $g(x)$ in the regions $x \ll 1$ and $x \gg 1$ are

$$g_+(x) = \prod_{n,m \geq 0} \frac{(1 - xq^{2j_2+1+4n}p^m)(1 - xq^{-2j_2+3+4n}p^m)}{(1 - xq^{2j_2+3+4n}p^m)(1 - xq^{-2j_2+1+4n}p^m)},$$

$$g_-(x) = \prod_{n,m \geq 0} \frac{(1 - x^{-1}q^{-2j_2-3-4n}p^{m+1})(1 - x^{-1}q^{2j_2-1-4n}p^{m+1})}{(1 - x^{-1}q^{2j_2-1-4n}p^{m+1})(1 - x^{-1}q^{-2j_2-3-4n}p^{m+1})}, \quad (4.8)$$

respectively.

§ 4.3. The connection formula for the $q$-hypergeometric function $F_p(a, b, c; x)$ is

$$F_p(a, b, c; x) = \frac{\Gamma_p(c)\Gamma_p(b-a)\Theta(p^a x, p)}{\Gamma_p(b)\Gamma_p(c-a)\Theta(x, p)} F_p(a, a - c + 1, a - b + 1, p^{c+1-a-b}x^{-1})$$

$$+ \frac{\Gamma_p(c)\Gamma_p(b)\Theta(p^b x, p)}{\Gamma_p(a)\Gamma_p(c-b)\Theta(x, p)} F_p(b, b + 1, a - b + 1, p^{c+1-a-b}x^{-1}), \quad (4.9)$$

where

$$\Gamma_p(a) = (1 - p)^{1-a} \prod_{n \geq 0} \frac{(1 - p^{n+1})}{(1 - p^{n+a})},$$

$$\Theta(x, p) = \prod_{n \geq 0} (1 - p^{n+1})(1 - xp^n)(1 - x^{-1}p^{n+1}). \quad (4.10)$$

6 For the proof of this formula, see Ref. 2 for example.
Using this formula, we have

**Proposition I.** The connection formulas for the solutions of the $q$-KZ equation (4.1) and for the normalization factor $g(x)$ are

$$
\tilde{\Psi}_{(+)}^{\alpha}(x) = \sum_{\beta} \tilde{\Psi}_{(-)}^{\beta}(x) C_{\beta}^{\alpha}(x),
$$

\begin{align}
C_{0}^{0}(x) &= \frac{\Gamma_{p}(2m + m + 1 - m_2 + 1) \Gamma_{p}(2m - m + m + 1 + 1)}{\Gamma_{p}(2m + m + 1 + 1) \Gamma_{p}(2m + m + 1 + 1)} \frac{\Theta(p^{m_1-m_2})}{\Theta(p^{m_1-m_2})} x^{2m_1},
C_{1}^{0}(x) &= -\frac{\Gamma_{p}(-2m + m + 1 - m_2) \Gamma_{p}(2m + m + 1 + 1)}{\Gamma_{p}(-2m + m + 1 + 1) \Gamma_{p}(2m + m + 1 + 1)} \frac{\Theta(p^{m_1+m_2})}{\Theta(p^{m_1-m_2})} p^{-m_2} x^{2m_1},
C_{0}^{1}(x) &= -\frac{\Gamma_{p}(-2m + m + 1 + m_2) \Gamma_{p}(2m + m + 1 + 1)}{\Gamma_{p}(-2m + m + 1 + 1) \Gamma_{p}(2m + m + 1 + 1)} \frac{\Theta(p^{m_1-m_2})}{\Theta(p^{m_1-m_2})} p^{-m_2} x^{2m_2},
C_{1}^{1}(x) &= \frac{\Gamma_{p}(-2m + m + 1 + m_2) \Gamma_{p}(-2m + m + 1 + m_2)}{\Gamma_{p}(-2m + m + 1 + 1) \Gamma_{p}(-2m + m + 1 + 1)} \frac{\Theta(p^{m_1-m_2})}{\Theta(p^{m_1-m_2})} x^{2m_2},
\end{align}

and

$$
g_{+}(x) = g_{-}(x) C_{g}(x), \quad C_{g}(x) = \prod_{n \geq 0} \frac{\Theta(x q^{2j_2+1+4n}) \Theta(x q^{-2j_2+3+4n})}{\Theta(x q^{2j_2+3+4n}) \Theta(x q^{-2j_2+1+4n})}.
$$

Note that $C_{\alpha}^{\alpha}(x)$ and $C_{g}(x)$ are the pseudo-constant, e.g. $C_{g}(px) = C_{g}(x)$.

5. Exchange relation for the $q$-vertex operators

§5.1. Comparing the two point function in §3.2 and $\tilde{\Phi}(x)$ in §4.1, we have

**Proposition II.** The relations between the solutions of the reduced $q$-KZ equation in §4.1 with the two point functions in §3.2 are

\begin{align}
\left( \begin{array}{c}
\langle j_4 | \Phi_{\frac{M}{z},0}(z_3) \Phi_{j_2,M}(z_2) | j_1 \rangle \\
\langle j_4 | \Phi_{\frac{1}{z},0}(z_3) \Phi_{j_2,M-1}(z_2) | j_1 \rangle
\end{array} \right)
= x^{-j_2/2k} z_2^{-h_2-h_1} g_{+}(x) \left( \begin{array}{c}
\tilde{\Psi}_{(+)}^{0}(x) \\
\tilde{\Psi}_{(+)}^{1}(x)
\end{array} \right),
\end{align}

\begin{align}
\left( \begin{array}{c}
\langle j_4 | \Phi_{j_2,M}(z_2) \Phi_{\frac{1}{z},0}(z_3) | j_1 \rangle \\
\langle j_4 | \Phi_{j_2,M-1}(z_2) \Phi_{\frac{1}{z},1}(z_3) | j_1 \rangle
\end{array} \right)
= x^{j_2/2k} z_2^{-h_2+h_1} g_{-}(px) \left( \begin{array}{c}
p^{-m_1+m_2} \tilde{\Psi}_{(-)}^{0}(px) \\
p^{-m_1+m_2} \tilde{\Psi}_{(-)}^{1}(px)
\end{array} \right).
\end{align}
Proof. Both sides of the above equations satisfy the same $q$-KZ equations. So we need only to compare the leading terms. From $g_+(x) = g(x) + O(x^2)$ and $g_-(px) = g(x^{-1}) + O(x^{-2})$, we obtain the required relation.

Q.E.D.

From the difference equation for $g(x)$ and $\tilde{\Phi}(x)$ and from the Proposition-I,-II, we have

**Theorem.** The exchange relation for the $q$-vertex operators of spin $1/2$ and arbitrary spin $j_2 \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ is as follows

$$
\tilde{R}^{kl}_{ij}(\frac{z}{w}) \Phi_{\frac{1}{2}, k}(w) \Phi_{j_2, l}(z) = \Phi_{j_2, i}(z) \Phi_{\frac{1}{2}, j}(w) \tilde{C}^{\alpha\beta}_{\gamma\delta}(\frac{z}{w}, \lambda),
$$

(5.2)

with

$$
\left( \begin{array}{cc} \tilde{R}^0_{M,0}(x) & R^1_{M,0}(x) \\ R^0_{M-1,1}(x) & \tilde{R}^1_{M-1,1}(x) \end{array} \right) = \frac{f(x)}{x - q^{M+1}} \left( \begin{array}{cc} (xq^M - q^M) & q^M(q^{-M} - q^M) \\ xq^M(q^{-M} - q^M) & (xq^M - q^M) \end{array} \right),
$$

(5.3)

$$
\left( \begin{array}{cc} C^0_{M,0}(x, \lambda) & C^1_{M,0}(x, \lambda) \\ C^0_{M-1,1}(x, \lambda) & \tilde{C}^1_{M-1,1}(x, \lambda) \end{array} \right) = x^{-j_2/\kappa} C_g(x) \left( \begin{array}{cc} p^{-m_1}C_{1}^{0}(x) & p^{-m_1}C_{1}^{1}(x) \\ p^{-m_2}C_{1}^{0}(x) & p^{-m_2}C_{1}^{1}(x) \end{array} \right),
$$

(5.4)

where $M = \alpha + \beta = \gamma + \delta = i + j = k + l$, $\lambda = 2j_4 + 1$ and $f(x)$, $C_\alpha(x)$ and $C_g(x)$ are as given in (1.6), (4.11), and (4.13) respectively.

Proof. From the intertwining property of the $R$-matrix, it is obvious that the Theorem holds not only for the lowest state $|j_4| \in V^*_j$ and the highest state $|j_1\rangle \in V_{j_1}$ but also for arbitrary states in $V^*_j$ and $V_{j_1}$.

Q.E.D.

§ 5.2. If we denote the $q$-vertex operator and the connection matrix as

$$
\Phi \left[ \begin{array}{c} j_2 \\ j_3 \\ j_1 \end{array} \right]_{m_2}(z) = \Phi_{j_2, m_2}(z) : V_{j_1} \rightarrow V_{j_3},
$$

(5.5)

$$
C_{j_2} \left[ \begin{array}{c} j_4 \\ j_1 \end{array} \right]_{j_1}(\frac{z}{w}) = \tilde{C}_{j_2}(z, \lambda),
$$

then the exchange relation (5.2) can be written as

$$
\tilde{R}^{kl}_{ij}(\frac{z}{w}) \Phi \left[ \begin{array}{c} 1 \\ j_4 \\ j \end{array} \right]_k(w) \Phi \left[ \begin{array}{c} j_2 \\ j_4 \\ j_1 \end{array} \right]_l(z) = \Phi \left[ \begin{array}{c} j_2 \\ j_4 \\ j \end{array} \right]_i(z) \Phi \left[ \begin{array}{c} 1 \\ j_4 \\ j_1 \end{array} \right]_j(w) C_{j_2} \left[ \begin{array}{c} j_4 \\ j \end{array} \right]_j(\frac{z}{w}).
$$

(5.6)
From the fact that the $R$-matrix $\tilde{R}_{ij}^{kl}(z,w)$ satisfies the Yang-Baxter equation, the connection matrix $C_{j'j}^{\frac{1}{2},1} j' j_1 (z,w)$ obeys the IRF type Yang-Baxter relation \cite{4,11}. Moreover if we denote $[n]_\vartheta = p^{-n/2\kappa} \Theta(p^n/\kappa)$, $x = q^{2j+1-2u}$, then

$$\left( \begin{array}{c} \tilde{C}_0^{M,0}(x,\lambda) \\ \tilde{C}_1^{M,1}(x,\lambda) \\ \tilde{C}_M^{1,-}(x,\lambda) \end{array} \right) = \frac{C_g(x)x^{-j_2/\kappa}}{[\lambda + M - M]_\vartheta [u - M - \lambda]_\vartheta} \times \left( \begin{array}{c} \tilde{\alpha}_1 \\ \tilde{\alpha}_2 \end{array} \right) \left( \begin{array}{c} [\lambda - M]_\vartheta [u - M]_\vartheta \\ [M]_\vartheta [u - M - \lambda]_\vartheta \end{array} \right) \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right)^{-1},$$

where

$$C_g(x) = \prod_{n \geq 0} \frac{[u - M - M - 2n]_\vartheta [u - 2 - 2n]_\vartheta}{[u - M - M - 1 - 2n]_\vartheta [u - 1 - 2n]_\vartheta},$$

$$\alpha_1 = \frac{\Gamma_p(2m_1) \Gamma_p(2m - m_1 - m_2 + 1)}{\Gamma_p(2m + m_1 - m_2 + 1)} x^{-m-m_1},$$

$$\alpha_2 = \frac{\Gamma_p(2m_2) \Gamma_p(-2m - m_1 - m_2)}{\Gamma_p(-2m - m_1 + m_2)} x^{m-m_2} p^{m_2},$$

$$\tilde{\alpha}_1 = \frac{\Gamma_p(2m_1) \Gamma_p(-2m - m_1 - m_2)}{\Gamma_p(-2m + m_1 - m_2)} x^{-m+m_1} p^{m_1},$$

$$\tilde{\alpha}_2 = \frac{\Gamma_p(2m_2) \Gamma_p(2m - m_1 - m_2 + 1)}{\Gamma_p(2m - m_1 + m_2 + 1)} x^{m+m_2},$$

this connection matrix is equivalent to the elliptic Boltzmann weight of IRF type obtained by the fusion procedure in Ref. \cite{13}.

6. Conclusion

We have given explicitly the two point functions and the exchange relations for the $q$-vertex operators whose spins are $1/2$ and arbitrary $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. We also confirmed that the connection matrix is equivalent to the elliptic Boltzmann weight of IRF type obtained by the fusion procedure in Ref. \cite{13}.

We expect that the exchange relation for the two arbitrary spin $q$-vertex operators will be given by an analogous fusion procedure. Our method essentially relies on the connection formula of $q$-hypergeometric function and it is applicable only to the case when the number
of intermediate channels is at most two. A more promising approach to the connection problem is to use the integral formula.

Recently, a free field realization for $U_q(\widehat{sl}_2)$ was constructed \cite{*[17,18,19]}\{. The free field realization will give a powerful tool \cite{20} to calculate the integral formulas \cite{1,2,5,6} for the $q$-KZ solution and to solve their connection problem.

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Appendix A. The existence condition for the $q$-vertex operator of arbitrary spin

§A.1. Here we briefly review the result of our previous paper \cite{13}. For an arbitrary spin $j \in \mathbb{C}$, the $q$-vertex operator $\Phi_{j_2}(z) : V_{j_1} \rightarrow V_{j_3}$ is defined by the transformation property under the adjoint action of $U_q(\widehat{sl}_2)$ \cite{*[13,14]}, such that \(\text{ad}(a) \Phi(z) \equiv \sum_k a^1_k \Phi(z) S(a^2_k)\) with \(\Delta(a) = \sum_k a^1_k \otimes a^2_k\).

**Definition.** The $q$-vertex operator $\Phi_{j_2}(z) : V_{j_1} \rightarrow V_{j_3}$ is defined as

\[
\text{ad}(a) \Phi(z) = \rho(a) \Phi(z),
\]

for $\forall a \in U_q(\widehat{sl}_2)$, where $\rho$ is a certain representation of $U_q(\widehat{sl}_2)$.

The following intertwining property is equivalent to above definition;

\[
a \Phi(z) = \sum_k \rho(a^1_k) \Phi(z) a^2_k.
\]

For $\rho$, if we take the contravariant difference representation defined by

\[
\rho(e_1) = x[2j - x \frac{d}{dx}], \quad \rho(e_0) = \frac{z}{x}[x \frac{d}{dx}], \\
\rho(k_1) = q^{2(j-x \frac{d}{dx})}, \quad \rho(k_0) = q^{-2(j-x \frac{d}{dx})}, \quad \rho(d) = -z \frac{d}{dz},
\]

\[
\rho(f_1) = \frac{1}{x}[x \frac{d}{dx}], \quad \rho(f_0) = \frac{x}{z}[2j - x \frac{d}{dx}],
\]

\begin{align}
\text{A.1} & \quad \text{for } \forall a \in U_q(\widehat{sl}_2), \text{ where } \rho \text{ is a certain representation of } U_q(\widehat{sl}_2). \\
\text{A.2} & \quad \text{The following intertwining property is equivalent to above definition;} \\
\text{A.3} & \quad \text{For } \rho, \text{ if we take the contravariant difference representation defined by}
\end{align}
then the $q$-vertex operator $\Phi_j(z, x)$ of spin $j$ can be defined explicitly as follows [15]

$$e_i \Phi_j(z, x) = \rho(e_i) \Phi_j(z, x) k_i + \Phi_j(z, x) e_i,$$

$$k_i \Phi_j(z, x) = \rho(k_i) \Phi_j(z, x) k_i,$$

$$f_i \Phi_j(z, x) = \rho(f_i) \Phi_j(z, x) + \rho(k_i^{-1}) \Phi_j(z, x) f_i,$$

and

$$d \Phi_j(z, x) = \rho(d) \Phi_j(z, x) + \Phi_j(z, x) d.$$  \(\text{(A.4)}\)

Here $[n]$ denotes the $q$ integer $[n] = (q^n - q^{-n})/(q - q^{-1})$, so the $\rho(J)$'s are the difference operators, e.g. $q^x \frac{d}{dx} f(x) = f(qx)$ for any function $f(x)$.

And we normalize the ground state matrix element of the $q$-vertex operator as

$$\langle j_3 | \Phi_{j_2}(z, x) | j_1 \rangle = z^{h_3 - h_1} x^{j_1 + j_2 - j_3}. \quad \text{(A.6)}$$

For $2j + 1 \in \mathbb{Z}_{>0}$, if we set $\Phi_j(z, x) = \sum_{m=0}^{2j} \Phi_{j,m}(z) x^m$, then we have the previous definition for the $q$-vertex operators $\Phi_{j,m}(z)$ in §3.1.

§ A.2. The $q$-vertex operator $\Phi_{j,2}(z, x) : V_{j_1} \to V_{j_3}$ exists if and only if

$$\langle j_3 | \Phi_{j,2}(z, x) | \chi_1 \rangle = \langle \chi_3 | \Phi_{j,2}(z, x) | j_1 \rangle = 0,$$  \(\text{(A.7)}\)

for all the null vectors $|\chi_1\rangle \in V_{j_1}$ and $\langle \chi_3| \in V_{j_3}^*$.  

For $r, s \in \mathbb{Z}$ and $j_n \in \mathbb{C}$, let

$$f_{r,s}(j_1, j_2, j_3) = \prod_{n=0}^{r-1} \prod_{m=0}^{s} [j_1 + j_2 - j_3 - n + mk] \prod_{n=1}^{r} \prod_{m=1}^{s} [-j_1 + j_2 + j_3 + n - mk],$$

$$f_{r,s}(j_1, j_2, j_3) = \prod_{n=0}^{r-1} \prod_{m=0}^{s} [-j_1 + j_2 + j_3 - n + mk] \prod_{n=1}^{r} \prod_{m=1}^{s} [j_1 + j_2 - j_3 + n - mk].$$  \(\text{(A.8)}\)

for the case (i) $r > 0$ and $s \geq 0$ or (ii) $r < 0$ and $s < 0$, respectively. Then we have

**Theorem.** [15] *The existence conditions for the $q$-vertex operators $\Phi_{j,2}(z, x) : V_{j_1} \to V_{j_3}$ are given as follows.*
(I). For the rational level \( \kappa = p/q \), with the coprime integers \( p \) and \( q \), and \( 2j_n + 1 = r_n - s_n\kappa \) with \( 0 < r_n < p \) and \( 0 \leq s_n < q \), \((n = 1, 2, 3)\), the \( q \)-vertex operator exists if and only if

\[
f_{r_1,s_1}(j_1,j_2,j_3) = f_{r_1-p,s_1-q}(j_1,j_2,j_3) = 0, \tag{A.9}
\]

(II). For the generic level, the \( q \)-vertex operator exists if and only if

\[
\sum_{r_1,s_1 \in \mathbb{Z}} f_{r_1,s_1}(j_1,j_2,j_3)\delta_{r_1-s_1\kappa}^{2j_1+1} = \sum_{r_3,s_3 \in \mathbb{Z}} f_{r_3,s_3}(j_3,j_2,j_1)\delta_{r_3-s_3\kappa}^{2j_3+1} = 0. \tag{A.10}
\]

Proof is given by using the following Lemma-I and Lemma-II.

**Lemma I** \[21\] For \( \kappa = k + 2 \in \mathbb{C} \setminus \{0\} \) and the highest weight \( j \), parametrized as \( 2j_{r,s} + 1 = r - s\kappa \) with \( r,s \in \mathbb{Z} \), such that (i) \( r > 0 \) and \( s \geq 0 \) or (ii) \( r < 0 \) and \( s < 0 \), there exists a unique null vector \( |\chi_{r,s}\rangle \in V_j \) of grade \( N = rs \) and charge \( Q = -r \). And the null vector in \( V_{j_{r,s}} \) is as follows,

\[
|\chi_{r,s}\rangle = (f_1)^{r+s\kappa}(f_0)^{r+(s-1)\kappa} \cdots (f_0)^{r-(s-1)\kappa}(f_1)^{r-s\kappa}|j_{r,s}\rangle,
\]

\[
|\chi_{r,s}\rangle = (f_0)^{-r-(s+1)\kappa}(f_1)^{-r-(s+2)\kappa} \cdots (f_1)^{-r+(s+2)\kappa}(f_0)^{-r+(s+1)\kappa}|j_{r,s}\rangle, \tag{A.11}
\]

for the cases (i) and (ii) respectively.

**Lemma II.** For the null vectors \( |\chi_{r,s}\rangle \in V_j \) and \( \langle \chi_{r,s}| \in V_j^* \), we have

\[
\langle j_3|\Phi_{j_2}(z,x)|\chi_{r,s}\rangle = f_{r,s}(j_1,j_2,j_3)z^{h_3-h_1-rs}x^{j_1+j_2-j_3-r},
\]

\[
\langle \chi_{r,s}|\Phi_{j_2}(z,x)|j_1\rangle = f_{r,s}(j_3,j_2,j_1)z^{h_3-h_1+rs}x^{j_1+j_2-j_3+r}, \tag{A.12}
\]

up to some non-zero multiple factors.

**Appendix B. Calculation of the two point function by the q-OPE**

§B.1. We calculate the image \( \Phi_{j_2,m_2}(z)|j_1\rangle \in V_{j_3} \) of the highest weight vector \( |j_1\rangle \). Let

\[
\Phi_{j_2,m_2}(z)|j_1\rangle = \delta_{m_2,j_1}^{j_2-j_1+1+N}z^{h_3-h_1+Nq^{kN+2j_1}}|N,Q\rangle_{j_3}, \tag{B.1}
\]

\[
\Phi_{j_2,m_2}(z)|j_1\rangle = \delta_{m_2,j_1}^{j_2-j_1+1+N}z^{h_3-h_1+Nq^{kN+2j_1}}|N,Q\rangle_{j_3}, \tag{B.1}
\]
where \(|N, Q\rangle_{j_3}\) is the homogeneous components of grade \(N\) and charge \(-Q\), such that

\[
|N, Q\rangle_{j_3} = \sum_{(\alpha_1, \ldots, \alpha_n)} \beta_{\alpha_1, \ldots, \alpha_n} f_{\alpha_1 \ldots \alpha_n} |j_3\rangle, \tag{B.2}
\]

\(N = \sum_i \bar{\alpha}_i, Q = \sum_i (\alpha_i - \bar{\alpha}_i), \alpha_i = 0, 1, \bar{\alpha} = 1 - \alpha\) and \(f_{\alpha_1 \ldots \alpha_n} = f_{\alpha_1} \cdots f_{\alpha_n}\). From the definition of the \(q\)-vertex operator \((3.1)\), we have the descent equations for \(|N, Q\rangle_{j_3}\) \([15]\)

\[
e_1|N, Q\rangle_{j_3} = [-j_1 + j_2 + j_3 - Q + 1] |N, Q - 1\rangle_{j_3},
\]

\[
e_0|N, Q\rangle_{j_3} = [j_1 + j_2 - j_3 + Q + 1] |N - 1, Q + 1\rangle_{j_3}. \tag{B.3}
\]

From these descent equations, we can calculate the expansion coefficients \(\beta_{\alpha_1, \ldots, \alpha_n}\).

**Example.** The \(q\)-OPE of spin 1/2 \(q\)-vertex operator is

\[
\Phi_{\frac{1}{2}, 0}^0(z)|j\rangle = z^{h^+ - h}(1 + q^k z(\beta_{01}^+ f_{01} + \beta_{10}^+ f_{10}) + O(z^2))|j + \frac{1}{2}\rangle,
\]

\[
\Phi_{\frac{1}{2}, 1}^0(z)|j\rangle = z^{h^+ - h} q^{2j}(\beta_1^+ f_1 + q^k z(\beta_{011}^+ f_{011} + \beta_{101}^+ f_{101} + \beta_{110}^+ f_{110}) + O(z^2))|j + \frac{1}{2}\rangle,
\]

\[
\Phi_{\frac{1}{2}, 0}^1(z)|j\rangle = z^{h^+ - h}(1 + q^k z(\beta_{10}^- f_{10} + \beta_{01}^- f_{01}) + O(z^2))|j - \frac{1}{2}\rangle,
\]

\[
\Phi_{\frac{1}{2}, 0}^1(z)|j\rangle = z^{1 + h^+ - h} q^{2j}(\beta_0^- f_0 + q^k z(\beta_{100}^- f_{100} + \beta_{010}^- f_{010} + \beta_{001}^- f_{001}) + O(z^2))|j - \frac{1}{2}\rangle, \tag{B.4}
\]

where \(h = h_j, h^\pm = h_j \pm \frac{1}{2}\),

\[
\beta_1^+ = \frac{1}{[2j + 1]}, \quad \beta_0^{+1} = \frac{[2j + 3]}{[2][k + 2][2j + 1]}, \quad \beta_{10}^+ = \frac{-1}{[2][k + 2]}, \quad \beta_{011}^+ = \frac{-[k + 2][2j + 2]}{[k + 2j + 3][2][k + 2][2j + 1]},
\]

\[
\beta_{010}^+ = \frac{-[k + 3]}{[k + 2j + 3][2][k + 2][2j + 1]} \tag{B.5}
\]

and

\[
\beta_{\alpha_1, \ldots, \alpha_n}^- (2j) = \beta_{\bar{\alpha}_1, \ldots, \bar{\alpha}_n}^+(k - 2j). \tag{B.6}
\]

\(\S\) **B.2.** By using the \(q\)-OPE and one point function, we can in principle derive the arbitrary \(N\) point functions.
Example. The two point functions for the spin $\frac{1}{2}$ and arbitrary spin $j_2$ q-vertex operators are

\[
\langle j_4 | \Phi_{\frac{1}{2},0}(z_3) \Phi_{j_2, M}(z_2) | j_1 \rangle = z_3^{h_4-h_1} z_2^{h_4-h_2} q^{k+2} \times \left\{ 1 + \frac{q^{k+2}x}{[2][k+2][k-2j_4+1]} \left( \frac{[k-2j_4+3][M][2j_2-M+1]}{[k-2j_4+1][M+1][2j_2-M]} \right) + O(x^2) \right\},
\]

(B.7)

\[
\langle j_4 | \Phi_{\frac{1}{2},0}(z_3) \Phi_{j_2, M-1}(z_2) | j_1 \rangle = -z_3^{h_4-h_1} z_2^{h_4-h_2} x q^{k-2j_1+1} \frac{[M]}{[k-2j_4+1]}
\]

\[
\times \left\{ 1 + \frac{q^{k+2}x}{[2][k+2][k-2j_4+3]} \left( \frac{[k+2][k-2j_4+2][M-1][2j_2-M+2]}{[k-2j_4+1][M][2j_2-M+1]} \right) + O(x^2) \right\},
\]

(B.8)

and

\[
\langle j_4 | \Phi_{j_2, M}(z_2) \Phi_{\frac{1}{2},0}(z_3) | j_1 \rangle = z_2^{h_4-h_1} z_3^{h_4-h_2} q^{k+2} \times \left\{ 1 + \frac{q^{k+2}x^{-1}}{[2][k+2][k-2j_1+1]} \left( \frac{[2j_1+3][M][2j_2-M+1]}{[2j_1+1][M+1][2j_2-M]} \right) + O(\frac{1}{x^2}) \right\},
\]

(B.9)
\[ \langle j_4 | \Phi_{j_2, M-1}^{1/2} (z_2) \Phi_{j_4, 1}^{1/2} (z_3) | j_1 \rangle = -z_2^{h_4-h_1} z_3^{h_1-h_4} x^{1} q^{k-2j_4+1} [2j_2 - M + 1] / [k - 2j_1 + 1] \]

\[ \times \left\{ 1 + \frac{q^{k+2} x^{-1}}{[2][k+2][2k - 2j_1 + 3]} \right\} \left( \begin{array}{cc}
-\left[ k + 2 \right] [k - 2j_1 + 2] [2j_2 - M] [M + 1] \\
\left[ k - 2j_1 \right] \left[ 2j_2 - M + 1 \right] [M] \\
\left[ k + 3 \right] [k - 2j_1 + 3] [2j_2 - M + 1] [M] \\
\left[ k - 2j_1 + 1 \right] [2j_2 - M + 2] [M - 1]
\end{array} \right) + O\left( \frac{1}{x^2} \right), \]

\[ \langle j_4 | \Phi_{j_2, M-1}^{1/2} (z_2) \Phi_{j_4, 1}^{1/2} (z_3) | j_1 \rangle = z_2^{h_4-h_1} z_3^{h_1-h_4} 
\times \left\{ 1 + \frac{q^{k+2} x^{-1}}{[2][k+2][2k - 2j_1 + 3]} \right\} \left( \begin{array}{cc}
\left[ k - 2j_1 + 3 \right] [2j_2 - M + 1] [M] \\
\left[ k - 2j_1 + 1 \right] [2j_2 - M + 2] [M - 1]
\end{array} \right) + O\left( \frac{1}{x^2} \right), \]

where \( x = z_2 / z_3 \), \( h_n^\pm = h_{j_n} \pm \frac{1}{2} \) and \( M = j_1 + j_2 - j_4 + \frac{1}{2} \).

**Appendix C. The \( R \)-matrix**

For the finite dimensional representation \( V_j (z) \) in §2.2, let \( |M_1, M_2\rangle = |1/2, M_1\rangle \otimes |j, M_2\rangle \in V_{1/2}(z) \otimes V_j(w) \). The \( R \)-matrix, \( R(z/w) : V_{1/2}(z) \otimes V_j(w) \rightarrow V_{1/2}(z) \otimes V_j(w) \), defined by \( R(z/w) \Delta_{z,w}' = \Delta_{z,w} R(z/w) \), is a \( 2 \times 2 \) block diagonal form in each sector \( C|0, M\rangle \oplus C|1, M-1\rangle \), with \( M \in \{1, \ldots, 2j-1\} \).

The \( R \)-matrix \( \tilde{R}(x) \), which is normalized as \( \tilde{R}(x)|0,0\rangle = |0,0\rangle \), is given explicitly as follows

\[ \left( \begin{array}{cc}
\tilde{R}(x)|0, M\rangle \\
\tilde{R}(x)|1, M-1\rangle
\end{array} \right) = \left( \begin{array}{cc}
a_M(x) & b_M(x) \\
c_M(x) & d_M(x)
\end{array} \right) \left( \begin{array}{c}
|0, M\rangle \\
|1, M-1\rangle
\end{array} \right), \] (C.1)

where

\[ \left( \begin{array}{cc}
a_M(x) & b_M(x) \\
c_M(x) & d_M(x)
\end{array} \right) = \frac{1}{x - q^{M+M}} \left( \begin{array}{cc}
x q^M - q^M & q^M (q^{-M} - q^M) \\
x q^M (q^{-M} - q^M) & x q^M - q^M
\end{array} \right), \] (C.2)

with \( M = 2j + 1 - M \). They obey the following crossing relation

\[ \left( (\tilde{R}(x)^{-1} t_1)^{-1} \right)^{t_1} = \left( 1 - x q^{-2j+3} \right) \left( 1 - x q^{2j+3} \right) K^{-1} \tilde{R}(x q^{-4}) K, \] (C.3)

where \( t_1 \) means a transpose on the first component.
The $R$-matrix $R(x)$ which is the image of the universal $R$-matrix can be determined by the crossing relation \[ (((R(x)^{-1})^t)^{-1})^t = K^{-1}R(xq^{-4})K. \] (C.4)

The relation between $R$ and $\tilde{R}$ is then given as

\[ R(x) = f(x)\tilde{R}(x), \quad f(x) = \frac{(1 - xq^{2j_z+3})(1 - xq^{-2j_z+1})}{(1 - xq^{2j_z+1})(1 - xq^{-2j_z+3})}f(xq^4). \] (C.5)

The solution for $f(x)$ in the region $x \ll 1$ is

\[ f(x) = \prod_{n=0}^{\infty} \frac{(1 - xq^{2j_z+3+4n})(1 - xq^{-2j_z+1+4n})}{(1 - xq^{2j_z+1+4n})(1 - xq^{-2j_z+3+4n})}. \] (C.6)

This solution can be analytically continued to the whole $x \in \mathbb{C}$ uniquely.
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