Superfluid and Pseudo-Goldstone Modes in Three Flavor Crystalline Color Superconductivity

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We study the bosonic excitations in the favorite cubic three flavor crystalline LOFF phases of QCD. We calculate in the Ginzburg-Landau approximation the masses of the eight pseudo Nambu-Goldstone Bosons (NGB) present in the low energy theory. We also compute the decay constants of the massless NGB Goldstones associated to superfluidity as well as those of the eight pseudo NGB. Differently from the corresponding situation in the Color-Flavor-Locking phase, we find that meson condensation phases are not expected in the present scenario.

I. INTRODUCTION

The last few years have witnessed a conspicuous research activity on Quantum Chromodynamics (QCD) in extreme conditions of baryon density and/or temperature. Though mostly performed by model calculations, these investigations suggest a rich structure for the QCD phase diagram in the high density and low temperature regime. As a matter of fact, in these conditions a phase transition from hadrons to deconfined quark matter is expected to occur. Because of the attractive interaction in the antisymmetric color channel the ground state of deconfined quark matter is reorganized to be a color superconductor (for reviews on color superconductivity see ).

It is well accepted that at asymptotic densities the ground state of three flavor quark matter is the color-flavor-locked (CFL) phase. In this exotic state of matter both color and flavor symmetries are spontaneously broken because of the non-vanishing of the expectation value of a diquark operator. Nevertheless, a residual subgroup linking color and flavor is left unbroken. At lower densities, as probably important for the cores of neutron stars, less symmetric pairing patterns have to be considered. This is so because effects of the quark masses, as well as of electrical and color neutrality conditions, cause a mismatch of the Fermi surfaces of the pairing quarks. Examples include spin one pairing, homogeneous gapless two flavor, and three flavor superconductivity, and crystalline color superconductivity with two and three flavors. Crystaline superconductors are known as Larkin-Ovchinikov-Fulde-Ferrell (LOFF) states, from studies in condensed matter superconductors with magnetic impurities. Homogeneous gapless phases are affected by chromo-magnetic instability. This means that the screening masses of some of the gluons are imaginary. On the other hand, it has been shown that this instability can be cured either by the crystalline color superconductivity or by quenched condensed phases, as well as condensed meson current states.

It has been found in Refs. that there exists a window of values of the baryon chemical potential $\mu$ where the three flavor crystalline color superconductor is the most favorable candidate to represent the ground state of high density QCD. This fact, together with the possibility of the crystalline phases in the core of a compact star, makes interesting the study of the quantum excitations of the ground state. This is the aim of our work. In particular we study the quantum excitations (Goldstone or pseudo-Goldstone modes) in the three flavor LOFF phase of QCD, arising from the spontaneous breaking of the global symmetries. The spontaneous breaking of $SU(3)_A$ implies the existence of eight pseudo-Goldstone modes, while that of $U(1)_V$ (superfluid mode) entails a massless Goldstone mode. The superfluid mode is massless even in presence of massive quarks and/or differences of the quark chemical potentials. On the other hand, finite quark masses and chemical potential differences cause non-vanishing masses for the $SU(3)_A$ pseudo-Goldstone modes. In the CFL case it has been found that even if one takes $m_u = m_d = 0$ and $m_s \neq 0$, a finite...
mass is found for the excitations with the quantum numbers of the kaons \(39, 40, 41, 42, 43, 44, 45, 46, 47\). Since, in the case \(m_u = m_d = 0\), the squared masses of these excitations are found to be negative, a kaon condensation may occur. Therefore we are led to investigate the possibility of meson condensation in the three flavor LOFF phase as well. To do that we compute in this paper the masses of the pseudo Goldstone modes. Our main result is the absence of meson condensation and the stability of the LOFF phase at least to the order \(\Delta^2/\delta \mu^2\).

The plan of the paper is as follows. In Section II we derive the effective quark lagrangian in the crystalline LOFF phase. In Section III we discuss the coupling of the quarks to the Goldstones and we derive the effective action of the scalar excitations and their parameters. Finally, in Section IV we draw our conclusions and discuss possible prosecutions of this work.

## II. THE EFFECTIVE QUARK LAGRANGIAN

In this paper we deal with three flavor quark matter, whose interaction is modeled by a local Nambu-Jona Lasinio (NJL) lagrangian, evaluated in the high density effective theory as in Ref. [25] (see below Eq. (8)). At finite chemical potential and in presence of color condensation the quark lagrangian is given by

\[
L = \bar{\psi} (i\not{D} + \mu \not{c}) \psi - M_f \bar{\psi}_f \psi_f + L_\Delta. 
\]  

In the above equation \(\hat{\mu}\) is the quark chemical potential matrix, with color and flavor indices. It depends on \(\mu\) (the average quark chemical potential), \(\mu_e\) (the electronic chemical potential), and \(\mu_3, \mu_8\) (color chemical potentials) [22]. To implement color and electric neutrality it is sufficient to consider only these chemical potentials, related to the charge matrix \(Q\) and to the diagonal color operators \(T_3 = \frac{1}{2} \text{diag}(1,-1,0)\) and \(T_8 = \frac{1}{2\sqrt{3}} \text{diag}(1,1,-2)\). In general one should introduce a color chemical potential for each \(SU(3)\) color charge; however, as shown in [18], for the condensate with the color-flavor structure considered in this paper it is enough to consider only \(\mu_3\) and \(\mu_8\), since the charges related to the other color generators automatically vanish. Therefore the matrix \(\hat{\mu}\) is written as follows

\[
\hat{\mu}_{ij} = (\mu \delta_{ij} - \mu_e Q_{ij}) \delta^{\alpha\beta} + \delta_{ij} \left( \mu_3 T_3^{\alpha\beta} + \frac{2}{\sqrt{3}} \mu_8 T_8^{\alpha\beta} \right) \]  

with \(Q = \text{diag}(2/3, -1/3, -1/3)\) (1, 1, 3 flavor indices; \(\alpha, \beta = 1, 3\) colour indices).

The term \(L_\Delta\) is responsible for color condensation, and in the mean field approximation it is given by [28, 29]

\[
L_\Delta = -\frac{1}{2} \sum_{I=1}^{3} \left( \epsilon_{3t} \gamma_5 \alpha \beta \mu_3 \epsilon_{ij} C \psi_{ij} \right) \frac{\Delta_I(r) \Delta_I^\dagger(r)}{3G}. 
\]  

Eq. (3) describes the fact that in the ground state one has a non-vanishing expectation value of the diquark field operator

\[
\langle \psi(r) \gamma_5 \psi(r) \rangle \propto \Delta_I(r) \epsilon_{ij} \epsilon_{3t} \neq 0. 
\]  

As discussed in the Introduction we will consider kinematical conditions favoring the LOFF phase [25, 26, 28]. In this case the \(r\)-dependence of the gap parameters is given by a linear combination of plane waves,

\[
\Delta_I(r) = \sum_{a=1}^{P_I} e^{2i q^a \cdot r}. 
\]  

In [28] several crystalline structures have been considered, and their free energy has been computed in the Ginzburg-Landau approximation. All the structures considered in [28] have \(\Delta_1 = 0\), \(\Delta_2 = \Delta = \Delta_3 \equiv \Delta\), \(P_2 = P_3 \equiv P\). It was found that a crystalline color superconductive phase exists in the following interval:

\[
2.88 \Delta_0 \leq \frac{M_2^2}{\mu} \leq 10.36 \Delta_0, 
\]  

where \(\Delta_0\) is the CFL gap in the chiral limit and \(M_2\) is the in-medium strange quark mass. In more detail, for \(2.88 \Delta_0 \leq M_2^2/\mu \leq 6.20 \Delta_0\) the ground state of three flavor quark matter is the CubeX. In this structure \(P = 4\); for each pairing channel the wave vectors \(\{q_I\}\) form a square, and the two squares are arranged in such a way that they point to the vertices of a cube. In the remaining region the favored structure is the 2Cube45z in which \(P = 8\); each
wave vector set \( \{ \mathbf{q} \} \) forms a cube, and the two cubes are rotated by 45 degrees around an axes perpendicular to one of the faces of the cube. In the following we shall concentrate on these two crystalline structures.

Finally \( M_f \) denote the in-medium quark mass of the flavor \( f \). In the crystalline superconductive phases the in-medium quark masses have been evaluated self-consistently in \cite{29}. It was found that for values of \( \mu \) high enough for the condensation in the three flavor case to occur, the constituent \( u \) and \( d \) quark masses numerically coincide with their bare values \( M_u \sim m_u, M_d \sim m_d \). In Ref. \cite{28} it was found the LOFF window

\[
442 \text{ MeV} < \mu < 515 \text{ MeV},
\]

and correspondingly \( M_s \) belongs to the interval \( 270 \text{ – } 463 \text{ MeV} \).

In this paper we adopt the high density effective description of QCD \cite{11, 49, 50}. This approximation amounts to consider only the quarks with momenta close to the Fermi surface and it is justified since in the weak coupling regime we are interested in here the quarks living in the depth of the Fermi sphere are Pauli blocked and irrelevant for the dynamics. Furthermore the antiparticle poles can be neglected in the quark propagator, as they give rise to operators that are formally suppressed by inverse powers of \( \mu \).

The high density effective lagrangian of the quarks in the three flavor LOFF phase of QCD, derived from Eq. (1), is obtained in Ref. \cite{28}; therefore here we simply quote the result in the momentum space, namely

\[
\mathcal{L} = \frac{1}{2} \int \frac{d \mathbf{n}}{4\pi} \lambda_A^\dagger \left( V \cdot \ell + \mu \delta_{AB} + \delta_{\mu AB} - \Delta_{AB} \right) \lambda_B + L \rightarrow R.
\]

Here \( A = 1, \ldots, 9 \) is a color-flavor index; the rotation to the new basis is performed by means of the matrices \( F_A \) defined in \cite{32}. The quark momenta are measured as \( \mathbf{p} = \mu \mathbf{n} + \ell, p_0 = \ell_0 \), with \( \mathbf{n} \) a unit vector denoting the Fermi velocity of the quarks and \( \mu \) is a reference large momentum (usually one takes \( \mu \) equal to the baryon chemical potential, but in the LOFF phase it is more convenient to measure the momenta with respect to the \( u \) Fermi momentum, see below). The chemical potential of the quark with index \( A \) is written as \( \mu_A = \mu + \delta \mu_A \) and \( \delta \mu_A = \delta \mu_A \). In the three flavor LOFF phase one can assume \( \mu_3 = 0 \) with \( \mu_c = M_c^2/4\mu \) \cite{27}, therefore the quark chemical potential matrix can be written as \( \text{diag}(\mu_u, \mu_u + 2\delta \mu, \mu_u - 2\delta \mu) \) with

\[
\delta \mu = \mu_c/2 = M_c^2/8\mu.
\]

The gap matrix is given by \( \Delta_{AB} = \Delta_I(r) \text{Tr}[\epsilon_I \epsilon_I F_A^F \epsilon_I F_B] \); the explicit form is in \cite{32}. Finally, we have introduced the Nambu-Gorkov doublet

\[
\chi = \begin{pmatrix} \psi(n) \\ C\psi^*(-n) \end{pmatrix}.
\]

Here \( \psi(n) \) is a positive energy field with velocity \( n \); the projection is achieved by the projectors \( P_\pm = (1 \pm \gamma_0 \gamma \cdot n)/2 \).

### III. GOLDSTONE MODES

The symmetry group of three massless flavor QCD is

\[
G = \text{SU}(3)_c \otimes \text{SU}(3)_V \otimes \text{SU}(3)_A \otimes U(1)_V \otimes U(1)_A.
\]

Here and in the following we assume that the baryon chemical potential is high enough to restore the \( U_A(1) \) symmetry.

In the neutral unpaired quark matter with a massive strange quark the flavor symmetries of \( G \) are also explicitly broken by mass terms. Even with vanishing \( M_u \) and \( M_d \) the chemical potential matrix \( \text{diag}(\mu_u, \mu_u + 2\delta \mu, \mu_u - 2\delta \mu) \otimes \delta_{\alpha\beta} \) is invariant only under the transformations of \( SU(3)_V \) and \( SU(3)_A \) generated by \( \lambda_3 \) and \( \lambda_8 \). In the sequel we will keep track not only of \( M_s \neq 0 \), but also of the small corrections due to \( M_u \) and \( M_d \).

In the CFL case \cite{15} the condensates leave unbroken the global symmetry group \( SU(3)_c \otimes U(1)_L \otimes U(1)_R \), which is explicitly broken in the present case by mass terms and chemical potential differences. On the other hand the color gauge group is spontaneously broken, which gives masses to the eight gluons \cite{34}. We therefore expect nine NGB due to spontaneous breaking of the axial group \( SU(3) \) and \( U(1)_V \). The eight bosons associated to \( SU(3) \) have small masses while the \( U(1)_V \) boson (superfluid mode) is massless.
A. Effective lagrangian of the superfluid mode

The superfluid mode is relevant for the transport properties of the three flavor LOFF phase. Since its lagrangian and parameters have not been derived before we give here a brief account of this calculation, even though its derivation is similar to that presented in [53] for the phonons associated to breaking of the rotational invariance.

The field $\phi$ is introduced as an external field by means of the transformation $\psi \to U^\dagger \psi$ with $U = \exp \{ i \phi / f \}$ [11, 51, 52]. The quark lagrangian after the rotation reads

$$L = \int \frac{dn}{8\pi} \chi_A^\dagger \left( \begin{array}{cc} V \cdot \ell \delta_{AB} + \delta_{\mu AB} & -\Xi_{BA} \\ -\Xi_{AB} & -V \cdot \ell \delta_{AB} - \delta_{\mu AB} \end{array} \right) \chi_B \ ,$$

(12)

where

$$\Xi_{AB} = \Delta^*_I(r) \text{Tr}[\epsilon_I(F_A U^\dagger)^T \epsilon_I F_B U^\dagger] \ .$$

(13)

From Eqs. (12) and (13) it is clear that the field $\phi$ enters in the model as the phase of the order parameter. The expansion of Eq. (12) at the lowest order in $\phi$ gives rise to a three body and a four body interaction lagrangians, namely

$$\mathcal{L}_{XX\pi} = \frac{2i\phi}{f} \int \frac{dn}{8\pi} \chi_A^\dagger \left( \begin{array}{cc} 0 & -\Delta_{AB} \\ \Delta_{AB}^* & 0 \end{array} \right) \chi_B \ ,$$

(14)

$$\mathcal{L}_{XX\pi\pi} = -\frac{2\phi^2}{f^2} \int \frac{dn}{8\pi} \chi_A^\dagger \left( \begin{array}{cc} 0 & -\Delta_{AB} \\ -\Delta_{AB}^* & 0 \end{array} \right) \chi_B \ .$$

(15)

Next we integrate on the quark fields in the generating functional of the model [11],

$$W[\eta, \eta^\dagger] = \int D\chi D\phi \exp \left\{ i \int \mathcal{L} + \mathcal{L}_{XX\pi} + \mathcal{L}_{XX\pi\pi} + \eta^\dagger \chi + \chi^\dagger \eta \right\} \ ,$$

(16)

with $\mathcal{L}$ defined in Eq. (3). The integration procedure has been discussed in the literature both in the homogeneous [11, 33, 44, 52, 54] and in inhomogeneous cases [53]; in particular, in Ref. [53] it was shown that in the LOFF phase, where gapless quark excitations belong to the spectrum, one has to introduce an infrared cutoff on the quark momenta, and integrate over the fields with momenta greater than the cutoff; eventually one sends the cutoff to zero. We apply the same procedure here. Once the integration over the quark fields is performed one is left with the effective action of $\phi$, $S = S_{s.e.} + S_{tad}$ with [11, 33, 44, 52, 53, 54]:

$$S_{s.e.} = \frac{i}{2} \left( \frac{2i\phi}{f} \right)^2 \text{Tr} \left[ S \left( \begin{array}{cc} 0 & -\Delta \\ \Delta^* & 0 \end{array} \right) S \left( \begin{array}{cc} 0 & -\Delta \\ \Delta^* & 0 \end{array} \right) \right] \ ,$$

(17)

$$S_{tad} = -i \left( \frac{-2\phi^2}{f^2} \right) \text{Tr} \left[ S \left( \begin{array}{cc} 0 & -\Delta \\ -\Delta^* & 0 \end{array} \right) \right] \ .$$

(18)

In the above equations $S$ is the quark propagator, and $\Delta$ is the $9 \times 9$ gap matrix $\Delta_{AB}$; we use the notation $S_{tad}$ because the corresponding Feynman diagram has a tadpole shape; on the other hand $S_{s.e.}$ corresponds to a self energy diagram. The trace is on space coordinates and over all the internal degrees of freedom of the quarks, namely color, flavor and Nambu-Gorkov.

In general the fermion propagator cannot be computed exactly (an exception is the single plane wave structure). Therefore one has to employ some approximation; in this paper we use the Ginzburg-Landau (GL) approximation, based on an expansion of $S$ in powers of $\Delta / \delta_{\mu I}$. This approximation has been used in [53] for the computation of the shear modulus in the three flavor LOFF phase of QCD, evaluating the low energy parameters in the phonon lagrangian at the second order in $\Delta_I / \delta_{\mu I}$. We work here at the same order.

To begin with we consider the contribution (17). Evaluation of the traces gives

$$S_{s.e.} = -\frac{2}{f^2} \sum_{i=1}^3 \Delta_i^2 \sum_{a=1}^{P_1} \int \frac{d^3k}{(2\pi)^3} \phi(-k)\phi(k) \mathcal{P}_i(k_0, k) \ ,$$

(19)
with \( k = (k_0, \mathbf{k}) \) and

\[
\mathcal{P}^I_\ell(k_0, \mathbf{k}) = -2 \int \frac{dn}{4\pi} \int \frac{d^4 \ell}{(2\pi)^4} \left[ \frac{1}{(V \cdot \ell + \delta \mu - q_\ell \cdot n)(V \cdot (\ell + k) + \delta \mu - q_\ell \cdot n)} \right. \\
+ \frac{1}{(V \cdot \ell - \delta \mu - q_\ell \cdot n)(V \cdot (\ell + k) - \delta \mu - q_\ell \cdot n)} \right] + \delta \mu \rightarrow -\delta \mu .
\]

(20)

On the other hand from [13] one gets

\[
\mathcal{S}_{\text{tad}} = \frac{i}{2} \frac{f^2}{2\pi} \sum_{I=2}^3 \Delta_i^2 \sum_{P_I} \int \frac{d^4 k}{(2\pi)^4} \phi(-k) \phi(k) \mathcal{P}^I_\ell(k_0 = 0, \mathbf{k} = 0) .
\]

(21)

From Eqs. (19) and (21) it is easily recognized that one needs to evaluate \( \mathcal{P}^I_\ell(k_0, \mathbf{k}) - \mathcal{P}^I_\ell(0, \mathbf{0}) \), which is well-behaved in the ultraviolet. The loop integral is evaluated in the usual way by Wick rotating to imaginary energies \( \ell_0 \rightarrow i\lambda_4 \). Since the integral is convergent one can send the ultraviolet cutoff on \( \ell_\parallel \) to infinity, and perform the integral over \( \ell_\parallel \) by residues, followed by integration over \( \ell_4 \). This calculation is similar to the one of Refs. [35, 53], therefore we simply quote here the final result of the lagrangian at small external momenta, namely

\[
\mathcal{L}(k) = \frac{1}{2} \phi(-k) \left[ k_0^2 \mathcal{I}_0 - k_i k_j V_{ij} \right] \phi(k) ,
\]

(22)

where

\[
\mathcal{I}_0 = -\frac{\mu^2}{\pi^2} f^2 \sum_{I=2}^3 \Delta_i^2 P_I \Re \int \frac{dn}{4\pi} \left( \delta \mu - q_\ell \cdot n + i0^+ \right)^2 + \delta \mu \rightarrow -\delta \mu ,
\]

(23)

\[
V_{ij} = -\frac{\mu^2}{\pi^2} f^2 \sum_{I=2}^3 \Delta_i^2 P_I \Re \int \frac{dn}{4\pi} \left( \delta \mu - q_\ell \cdot n + i0^+ \right)^2 + \delta \mu \rightarrow -\delta \mu .
\]

(24)

We specialize the results to the symmetric case \( \Delta_2 = \Delta_3 \equiv \Delta, |q_2^\parallel| = |q_3^\parallel| \equiv q, P_2 = P_3 \equiv P \). At the minimum one has

\[
q = \eta \delta \mu , \quad \eta \approx 1.1997 .
\]

(25)

Requiring canonical normalization of the Lagrangian in Eq. (22) implies \( \mathcal{I}_0 = 1 \), that is

\[
f^2 = \frac{4P \mu^2}{\pi^2} \frac{\Delta^2}{\delta \mu^2(\eta^2 - 1)} .
\]

(26)

The squared velocity tensor \( V_{ij} \) can be computed following the same steps as in [35]. For \( P = 1 \) we find \( V_{ij} = [\text{diag}(0,0,1)]_{ij} \), where we have chosen the \( q_2 \) and \( q_3 \) along the positive \( z \)-axis. For the two cubic structures corresponding to the values \( P = 4 \) and \( P = 8 \) we find \( V_{ij} = \delta_{ij}/3 \), ie the velocity is isotropic and has the value \( 1/\sqrt{3} \).

**B. Parameters of the SU(3)_A Goldstone bosons**

The quark coupling to the octet of pseudo-Goldstones can be introduced in a similar way [11, 51]:

\[
\psi_{ai} \rightarrow \psi_{ak} \left( U^I \right)_{ki} , \quad U \equiv \exp \left\{ \frac{i \lambda_a \lambda_b \lambda_c}{2 F_a} \right\} ,
\]

(27)

where \( a = 1, \ldots, 8 \), \( \lambda_a \) are the Gell-Mann matrices, normalized as \( \text{Tr}\{\lambda_a \lambda_b \lambda_c\} = 2 \delta_{ab} \), and \( F_a \) are the decay constants relative to \( \pi_a \). We remind that Latin indices denote flavor, while Greek indices stand for color.

In order to describe the flavor excitations we promote the chemical potential matrix to a spurion field with definite transformation property under flavor transformation, namely \( \mu \rightarrow L \mu L^\dagger \) with \( L \in SU(3)_L \) (analogously \( \mu \rightarrow R \mu R^\dagger \))
with \( R \in SU(3)_{R} \). This transformation leaves invariant a chemical potential term under an \( SU(3)_{A} \) transformation, and thus under \( \mathcal{U} \) in Eq. (27). The quark lagrangian after the rotation reads

\[
\mathcal{L} = \int \frac{dn}{8\pi} \chi_{A}^{\dagger}(V \cdot \ell \delta_{AB} + \delta_{i\mu} - \Xi_{AB} - \Xi_{BA}) \chi_{B},
\]

where

\[
\Xi_{AB} = \Delta_{i}^{j}(r)\text{Tr}[\epsilon_{j}(F_{A}^{T}U_{i})^{T}F_{B}^{U}_{i}] .
\]

By expanding \( \mathcal{U} \) in Eq. (27) up to the second order in the meson fields, we have a three body and a four body interaction Lagrangians as in Eqs. (14), (15) (the difference arising here from the non-trivial flavor structure of \( \mathcal{U} \)):}

\[
\mathcal{L}_{\chi x} = +\int \frac{dn}{8\pi} \chi_{A}^{\dagger}(G_{3})_{AB} \chi_{B} ,
\]

\[
\mathcal{L}_{\chi x x} = +\int \frac{dn}{8\pi} \chi_{A}^{\dagger}(G_{4})_{AB} \chi_{B} .
\]

The expressions of \( G_{3}, G_{4} \) are as follows:

\[
G_{3} = \begin{pmatrix} 0 & -(K_{2a}^{3} \pi) \\ K_{2a}^{3} & 0 \end{pmatrix} ,
\]

\[
G_{4} = \begin{pmatrix} 0 & (K_{2a}^{4} \pi)^{\star} \\ K_{2a}^{4} & 0 \end{pmatrix} ,
\]

and the off-diagonal entries are defined as

\[
K_{2a}^{3} = \Delta_{i}^{j}(r)\text{Tr}[\epsilon_{j}(F_{2}^{T}U_{i})^{T}F_{2}^{U}_{i}] + \epsilon_{i}F_{2}^{T}F_{2}^{i} \lambda_{a} ,
\]

\[
K_{2a}^{4} = \Delta_{i}^{j}(r)\text{Tr}[\epsilon_{j}(F_{2}^{T}U_{i})^{T}F_{2}^{U}_{i}] + \epsilon_{i}F_{2}^{T}F_{2}^{i} \lambda_{a} + 2\epsilon_{i}F_{2}^{T}F_{2}^{i} \epsilon_{j}F_{2}^{j} \lambda_{b} .
\]

From now on the steps leading to the effective lagrangian are analogous to the previous case. However a complication arises from the non-trivial flavor-color structure of the interaction vertices. Integrating out the fermion fields in the generating functional one gets the effective lagrangian in momentum space, \( \mathcal{L}(p) = \mathcal{L}_{\text{s.e.}}(p) + \mathcal{L}_{\text{tad}} \), with

\[
i\mathcal{L}_{\text{tad}} = +\left(\frac{\pi_{a} \pi_{b}}{2F_{a} F_{b}}\right) \frac{\mu^{2}}{4\pi^{3}} \int \frac{dn}{4\pi} \int d^{2}\ell \text{ Tr}[S(\ell)G_{4}] ,
\]

\[
i\mathcal{L}_{\text{s.e.}}(p) = -\frac{1}{2} \left(\frac{\pi_{a}}{2F_{a}}\right) \left(\frac{\pi_{b}}{2F_{b}}\right) \frac{\mu^{2}}{4\pi^{3}} \int \frac{dn}{4\pi} \int d^{2}\ell \text{ Tr}[S(\ell + p)G_{3}S(\ell)G_{3}] .
\]

We have already kept into account the \( L + R \) contribution, and the trace is on Nambu-Gorkov and color-flavor indices. One gets

\[
\mathcal{L}_{\text{s.e.}}(p = 0) = 8\Delta_{2}^{2}I_{2}(\pi_{2} + \pi_{2}^{2}) + 8\Delta_{2}^{3}I_{2}(\pi_{1} + \pi_{2}^{3}) + 8(\Delta_{3}^{2} + \Delta_{3}^{3})(-I_{1})(\pi_{2}^{2} + \pi_{2}^{2})
\]

\[
+ 8\Delta_{2}^{3}(-I_{1})\pi_{2}^{3} + 8\left(\frac{\Delta_{3}^{2}}{3} + 4\Delta_{3}^{3}\right)(-I_{1})\pi_{2}^{8}
\]

\[
+ \frac{16}{\sqrt{3}}\Delta_{2}^{3}I_{1}\pi_{3}^{\pi_{8}} ,
\]

\[
\mathcal{L}_{\text{tad}} = 8\Delta_{2}^{2}I_{2}(\pi_{2} + \pi_{2}^{2}) + 8\Delta_{3}^{2}I_{1}(\pi_{1} + \pi_{2}^{3}) + 8(\Delta_{3}^{2} + \Delta_{3}^{3})I_{1}(\pi_{2}^{2} + \pi_{2}^{2})
\]

\[
+ 8\Delta_{2}^{3}I_{1}\pi_{2}^{3} + 8\left(\frac{\Delta_{3}^{2}}{3} + 4\Delta_{3}^{3}\right)I_{1}\pi_{8}^{2}
\]

\[
- \frac{16}{\sqrt{3}}\Delta_{2}^{3}I_{1}\pi_{3}^{\pi_{8}} ,
\]

where

\[
I_{2} = \frac{-i}{8F^{2}} \frac{\mu^{2}}{4\pi^{3}} \sum_{a=1}^{P} \int \frac{dn}{4\pi} \int d^{2}\ell \frac{(-1)}{(\ell_{0} + \ell_{i} - 2\delta\mu - q^{a} \cdot n)(\ell_{0} - \ell_{i} - 2\delta\mu - q^{a} \cdot n)}
\]

\[
= \frac{\mu^{2}}{16\pi^{2} F^{2}} \left(1 - \frac{1}{\eta} \log \frac{2 + \eta}{2 - \eta} + \frac{1}{2} \log \frac{\Lambda^{2}}{\delta\mu^{2}(4 - \eta^{2})}\right)
\]
isospin symmetry in the light quark sector is explicitly broken by we would put

\[ \mathcal{I}_{1} = -\frac{i}{8F^{2}} \mu^{2} \sum_{a=1}^{P} \int \frac{dn}{4\pi} \int d^{2}\ell \left( \frac{1}{(\ell_{0} + \ell_{1} + 2\delta\mu)(\ell_{0} - \ell_{1} - 2q^{a} \cdot n)} \right) \]

\[ = -P \frac{\mu^{2}}{16\pi^{2}F^{2}} \left( 1 - \frac{1}{2\eta} \log \frac{\eta + 1}{\eta - 1} + \frac{1}{2} \log \frac{\Lambda^{2}}{\delta\mu^{2}(\eta^{2} - 1)} \right). \]  

(40)

In these equations \( \Lambda \) is an ultraviolet cutoff, needed since both contributions are ultraviolet divergent; \( \eta \) and \( \delta\mu \) have been defined above.

We can take into account the effect of the light quark masses by adding an anti-gap term coupling two antiquarks \([39]\). According to \([39, 41, 54]\) we can write the gap plus the anti-gap lagrangian in the form

\[ \mathcal{L} = \frac{\Delta_{\gamma}(r)}{2} \psi_{\gamma}^{T} C \psi_{\delta} \epsilon^{\alpha\beta\gamma} \epsilon_{ij} \]

\[ - \frac{\Delta_{\gamma}(r)}{2} \frac{1}{4\mu^{2}} \psi_{\gamma}^{T} C \psi_{\delta} M_{ij}^{FI} M_{ij}^{FI} \epsilon^{\alpha\beta\gamma} \epsilon_{ij} - L + R + h.c., \]  

(41)

where \( \psi \) are left-handed and positive energy fields, and \( M \) is the quark mass matrix in flavor space. In the basis spanned by the \( F_{A} \) matrices the coupling of the Goldstones to the anti-gap is obtained by rotating the quark field according to Eq. (27),

\[ \mathcal{L} = \int \frac{dn}{8\pi} \chi_{A}^{\dagger} \left( \begin{array}{cc} 0 & \Upsilon_{AB}^{*} \\ \Upsilon_{AB} & 0 \end{array} \right) \chi_{B}, \]

(42)

where

\[ \Upsilon_{AB} = \frac{\Delta_{\gamma}(r)}{4\mu^{2}} \text{Tr}[\epsilon_{1}(F_{A}U^{\dagger}M)^{T}F_{B}U^{\dagger}M]. \]

(43)

Since the anti-gap gives contributions of the order of \( M_{u,d}M_{s}/\mu^{2} \), we treat it as an insertion. On the other hand the corrections due to \( \delta\mu = M_{s}^{2}/8\mu \) are computed exactly in the GL approach, because in the kinematical region where this expansion is valid, and the LOFF phase is favored, \( \delta\mu \) is rather large and the small parameter is \( \Delta/\delta\mu \). Considering all the contributions we get the following results for the boson masses (we put \( \Delta_{2} = \Delta_{3} = \Delta \) and \( P_{2} = P_{3} = P \)):

\[ m_{\pi_{+}}^{2} = m_{K_{+}}^{2} = c \frac{P\Delta^{2} \mu^{2}}{\pi^{2}F^{2}}, \]

\[ m_{\pi_{0}}^{2} = m_{K_{0}}^{2} = \frac{P\Delta \Delta}{8\pi^{2}F^{2}} M_{u}(M_{d} + M_{s}) \log \frac{\mu^{2}}{\delta\mu^{2}(\eta^{2} - 1)}, \]

\[ m_{\pi_{3}}^{2} = \frac{P\Delta \Delta}{8\pi^{2}F^{2}} M_{u}M_{s} \log \frac{\mu^{2}}{\delta\mu^{2}(\eta^{2} - 1)}, \]

\[ m_{\pi_{8}}^{2} = \frac{P\Delta \Delta}{24\pi^{2}F^{2}} (M_{u}M_{s} + 4M_{u}M_{d}) \log \frac{\mu^{2}}{\delta\mu^{2}(\eta^{2} - 1)}, \]

\[ m_{\pi_{8}}^{2} = \frac{P\Delta \Delta}{8\sqrt{3}\pi^{2}F^{2}} M_{u}M_{s} \log \frac{\mu^{2}}{\delta\mu^{2}(\eta^{2} - 1)}. \]  

(44)

The numerical value of \( c \) is \( c \approx 1.03 \), obtained using the numerical value of \( \eta \) in the expressions of \( \mathcal{I}_{1} \) and \( \mathcal{I}_{2} \); moreover we have introduced the fields \( \pi_{\pm} = (\pi_{1} \pm i\pi_{2})/\sqrt{2}, K_{\pm} = (\pi_{4} \pm i\pi_{5})/\sqrt{2}, K^{0}/\bar{K}^{0} = (\pi_{6} \mp i\pi_{7})/\sqrt{2}. \) In the above mass formulæ we have neglected light quark mass effects in the case of the charged pions, since they are suppressed in comparison to the leading order result, see Eqs. \([38, 39]\).

Several comments are in order. First, we find \( m_{K_{0}} \neq m_{K_{+}} \). This is easily explained by noticing that the \( SU(2) \) isospin symmetry in the light quark sector is explicitly broken by \( \mu_{u} \neq \mu_{d} \). As a matter of fact if in the quark loops we would put \( \mu_{u} = \mu_{d} \), thus restoring isospin symmetry, then we would obtain \( m_{K_{0}} = m_{K_{+}} \). Second, in the limit \( \Delta_{2} = \Delta_{3} \) we find \( m_{\pi_{+}} = m_{K_{+}} \). From the diagrammatic point of view this equality is explained in the following way. The tadpole diagram of \( K^{+} \) is obtained from the \( \pi^{+} \) tadpole by replacing \( \Delta_{2} \rightarrow \Delta_{3} \) and \( \mu_{+} \rightarrow \mu_{d} \), which is equivalent to \( \delta\mu \rightarrow -\delta\mu \). Since the tadpole diagram does not depend on the sign of \( \delta\mu \), then the equality of the two diagrams follows. Similarly, the \( K^{+} \) self-energy diagram is obtained from the \( \pi^{+} \) one by replacing \( \Delta_{2} \rightarrow \Delta_{3} \); also in this case the equality is achieved since \( \Delta_{2} = \Delta_{3} \).

The mass formulæs of the three flavor LOFF phase depend on the light quark masses similarly to the CFL phase, see for example \([39]\). The differences arise because here we have adopted the expansion of the quark propagator in powers
of $\Delta/\delta\mu$, resulting in a different argument in the logarithm (in the CFL one has $\log(\mu/\Delta)$ instead of $\log(\mu/\delta\mu)$); moreover, in the LOFF phase terms containing the product $M_d M_s$ are not present since they are proportional to $\Delta$, which is zero in our approximation.

Finally, we find that all the squared masses of the pseudo-Goldstone modes are positive. Therefore, at least to the order $O(\Delta^2/\delta\mu^2)$, there is not meson condensation in the three flavor LOFF phase.

By the same procedure we can compute the decay constants of the octet. Expanding the self-energy lagrangian up to the second order in the external momentum of the bosons, and requiring the lagrangian to be canonically normalized, we find

$$F_{\pi^\pm}^2 = \frac{P \Delta^2 \mu^2}{8\pi^2 \delta\mu^2 (4 - \eta^2)},$$
$$F_{K^\pm}^2 = \frac{P \Delta^2 \mu^2}{8\pi^2 \delta\mu^2 (4 - \eta^2)},$$
$$F_{K^0}^2 = F_{\bar{K}^0}^2 = \frac{2P \mu^2 \Delta^2}{8\pi^2 \delta\mu^2 (\eta^2 - 1)},$$
$$F_{33}^2 = \frac{P \Delta^2 \mu^2}{8\pi^2 \delta\mu^2 (\eta^2 - 1)},$$
$$F_{88}^2 = \frac{5P \mu^2 \Delta^2}{24\pi^2 \delta\mu^2 (\eta^2 - 1)},$$
$$F_{38}^2 = \frac{P \mu^2 \Delta^2}{8\sqrt{3}\pi^2 \delta\mu^2 (\eta^2 - 1)}. \quad (45)$$

IV. CONCLUSIONS

In this paper we have computed the parameters of the low energy effective action of the meson excitations (Goldstone modes) in the cubic structures of the three flavor LOFF phase of QCD. Since in the LOFF state we are not able to write exactly the full quark propagator we use an approximation, obtained by the expansion in $\Delta/\delta\mu$, to the order $\Delta^2/\delta\mu^2$. We consider the mode corresponding to the breaking of $U(1)_V$ (superfluid mode) and the octet of scalar fields related to $SU(3)_A$. The motivation of this work was twofold. First, since the superfluid mode is massless even in presence of finite quark masses, it is relevant for the low energy dynamics of the LOFF phase. Thus, if LOFF quark matter is present in the core of a compact star, the superfluid mode should have a role for quark transport properties. Therefore the computation of its low energy parameter can be of interest for applications. Preliminary investigations of the astrophysical effects of the LOFF state are in [55].

The second motivation was to study the possibility of pseudoscalar field condensation in the octet sector, in order to see if, similarly to the CFL phase, one has such effect also the three flavor LOFF phase. To this end we have evaluated the octet mass matrix. We have found that the squared mass tensor is positive defined, hence excluding the possibility of scalar condensation (at least to the order $\Delta^2/\delta\mu^2$). Since all the masses are non-vanishing, the octet is not expected to play an important role in the low energy dynamics. As a conclusion, the low energy effective theory for the three flavor LOFF phase should include the gapless fermions, the superfluid mode and the phonon fields related to the deformations of the crystal lattice [53].

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