Abstract

The Abelian Higgs model with or without external particles is considered in curved space. Using the dual transformation, we rewrite the model in terms of dual gauge fields and derive the Bogomol’nyi-type bound. We find all possible cylindrically symmetric vortex solutions and vortex-particle composites by examining the Einstein equations and the first-order Bogomol’nyi equations. The underlying spatial manifold of these objects comprises a cylinder asymptotically and a two sphere in addition to the well-known cone.
I. Introduction

A classic example of solitons in field theories is the topologically stable vortex solution in Abelian Higgs model \([1, 2]\). Such vortex-like excitations have been used to describe the system of flux-tubes in type-II superconductors \([1]\). These objects have played considerable roles also as cosmic strings. For example, phase transitions in early universe give rise to string-like defects and they generate the density fluctuation which leads to a possible explanation of galaxy formation \([3, 4]\).

In this paper, we consider \(U(1)\) local strings of Einstein Maxwell Higgs theory. However, the model may be too complex to handle analytically if one considers arbitrary shaped cosmic strings coupled to gravity. Thus one may adopt local cosmic strings which are infinitely straight in one direction and remain in equilibrium, which describes an idealized but a physically relevant situation. This assumption simplifies the system to that of Nielsen-Olesen vortices coupled to Einstein gravity in \((2+1)\) spacetime dimensions, and the zero size limit of such objects can be identified with point particles on a plane \([5]\). Then \((2+1)\) dimensional Einstein Maxwell Higgs theory with or without the coupling of external point particles is of our interest, and we examine possible vortex configurations whose centers point particles may stick to. Furthermore, if we choose a specific form of scalar potential, static local vortex configurations can be solutions of a first-order differential equations, which satisfies second-order Euler-Lagrange equation automatically \([6, 7]\). These Bogomol’nyi equations have been established in the theory coupled to gravity despite the difficulty of constructing the gravitational energy \([8, 9, 10]\). For such vortex solutions to this Bogomol’nyi equation, we show that they are exact solutions of Einstein equation and that the cosmological constant is zero for a general stationary metric and arbitrary distribution of point particles. Specifically, the Bogomol’nyi-type bound is saturated when the metric is static \([8]\).

Another way of understanding the role of vortices in \(U(1)\) gauge theory is to recapitulate it in dual formulation and elicit the physically relevant aspects such as phase transitions
induced by vortices and classical dynamics of vortices \cite{11,12}. Here we derive the dual-
transformed version of Einstein Maxwell Higgs theory in (2+1) dimensions, and show that
the Bogomol’nyi-type bound can be attained within this formulation. It is found that the
extended objects corresponding to the vortices in the original theory do not carry dual
magnetic flux but dual electric field, and the energy which is proportional to Euler invariant
is expressed by the sum of the spatial integral of electrostatic potential and the total mass
of point particles in the Bogomol’nyi limit.

Bogomol’nyi equations being set up, we investigate cylindrically symmetric configurations
and find all possible vortex-particle solutions by explicitly proving their existence. Interest-
ingly, there are solutions whose two manifold constitutes a cylinder in its asymptotic form
or a two sphere, in addition to those solutions whose two manifold constitutes a cone.

The rest of this paper is organized as following. In section 2 we introduce the model
and derive Bogomol’nyi-type bound in a more detailed way by solving Einstein equations
under general stationary metric. Section 3 is devoted to the dual transformation of Einstein
Maxwell Higgs theory within path-integral formalism and to the study of various aspects
of the dual formulation, including the derivation of Bogomol’nyi-type bound. In section 4
we analyze cylindrically symmetric solutions and their global geometrical structures. We
conclude in section 5 with some remarks.

II. Einstein Maxwell Higgs Theory and Bogomol’nyi Bound

We consider the system composed of vortices in Abelian Higgs model and the massive
point particles in the presence of gravity. Einstein Maxwell Higgs theory coupled to a set of
$N$ external particles in (2+1) spacetime dimensions is described by the action

$$
S = S_{\text{gravity}} + S_{\text{matter}} + S_{\text{point particle}}
= \int d^3x \sqrt{g} \left\{ -\frac{1}{16\pi G} (R + 2\Lambda) - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \frac{1}{2} g^{\mu\nu} D\mu \phi D\nu \phi - V(|\phi|) \right\}
$$
\[ + \sum_{a=1}^{N} m_a \int_{-\infty}^{\infty} ds \sqrt{g^{\mu\nu} \frac{dx^a_{\mu}}{ds} \frac{dx^a_{\nu}}{ds}}. \]  

(2.1)

where \( \phi = e^{i\Omega} |\phi| \), \( D_\mu \phi = (\partial_\mu - ieA_\mu)\phi \), and \( \Lambda \) the cosmological constant.

Equations of motion read

\[ \frac{1}{\sqrt{g}} \partial_\nu (\sqrt{g} F^{\mu\nu}) = e j^\mu \]  

(2.2)

\[ \frac{1}{2 \sqrt{g}} D_\mu (\sqrt{g} g^{\mu\nu} D_\nu \phi) = - \frac{\partial V}{\partial \phi} \]  

(2.3)

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R + 2\Lambda) = 8\pi G T_{\mu\nu}, \]  

(2.4)

where \( j^\mu \) is the conserved U(1) current,

\[ j^\mu = - i g^{\mu\nu} \left( \overline{\phi} D_\nu \phi - \overline{\phi} \phi D_\nu \phi \right), \]  

(2.5)

and \( T_{\mu\nu} \) is the energy-momentum tensor,

\[ T_{\mu\nu} = - g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + \frac{1}{2} \left( \overline{\phi} D_\nu \phi + D_\nu \overline{\phi} D_\nu \phi \right) - g_{\mu\nu} \mathcal{L}_{\text{matter}} \]

\[ + \frac{1}{\sqrt{g}} \sum_{a=1}^{N} m_a \int_{-\infty}^{\infty} ds_a \delta(x_a(s_a) - x) \frac{dx^a_{\mu}}{ds_a} \frac{dx^a_{\nu}}{ds_a}. \]  

(2.6)

We are interested in static soliton solutions of the equations of motion, specifically the Nielsen-Olesen vortices which are electrically-neutral and are characterized by the magnetic flux

\[ \Phi = - \int d^2x \frac{1}{2} \epsilon^{ij} F_{ij}, \]  

(2.7)

where \( \epsilon^{ij} \) two dimensional Levi-Civita tensor density of \( \epsilon^{12} = \epsilon_{12} = 1 \). With an appropriate choice of the potential \( V(\psi) \), let us look for multivortex-particle configurations in curved spacetime which satisfy Bogomol’nyi-type equations. If we assume that the vortices satisfy such equations, Einstein equations will turn out to be solved under the general stationary metric of the form

\[ ds^2 = N^2 (dt + K_i dx^i)^2 - \gamma_{ij} dx^i dx^j, \]  

(2.8)
where the functions \( N(x), K_i(x), \) and \( \gamma_{ij}(x), (i, j = 1, 2), \) are independent of time.

Under \( A_0 = 0 \) gauge condition, 0i-components of energy-momentum tensor \( T_{i0} \) vanish, so the corresponding solution of Einstein equations is

\[
K_{ij} = \frac{\kappa \epsilon_{ij}}{N^3 \sqrt{\gamma}},
\]

where \( K_{ij} = \partial_i K_j - \partial_j K_i, \) and \( \kappa \) is an undetermined constant.

Taking \( N = 1 \) and rearranging the terms in the spatial integration of the 00-component of energy-momentum tensor, we have

\[
\int d^2x \sqrt{g} T_{00} = \int d^2x \sqrt{\gamma} \left\{ \frac{1}{4} \gamma^{ik} \gamma^{jl} (F_{ij} \mp \sqrt{\gamma} \epsilon_{ij} \sqrt{2V}) (F_{kl} \mp \sqrt{\gamma} \epsilon_{kl} \sqrt{2V}) \\
+ \frac{1}{4} \gamma^{ij} (D_i \phi \mp i \sqrt{\gamma} \epsilon_{ik} \gamma^{kl} D_l \phi) (D_j \phi \mp i \sqrt{\gamma} \epsilon_{jm} \gamma^{mn} D_n \phi) \\
\pm \frac{e \epsilon_{ij}}{\sqrt{\gamma}} F_{ij} \left( \sqrt{2V} - \frac{e}{2} \left( |\phi|^2 - v^2 \right) \right) \right\} \mp \frac{e v^2}{2} \Phi + \sum_{a=1}^{N} m_a \\
\pm \frac{1}{2e} \int d^2x \partial_i (\sqrt{\gamma} \epsilon^{ij} j_j). \tag{2.10}
\]

Now we choose the scalar potential as

\[
V = \frac{e^2}{8} (|\phi|^2 - v^2)^2, \tag{2.11}
\]

then the first two terms in the braces of Eq. (2.10) are nonnegative definite and we obtain the so called Bogomol'nyi bound. The bound is attained for configurations satisfying the Bogomol'nyi equations

\[
D_i \phi \mp i \sqrt{\gamma} \epsilon_{ij} \gamma^{jk} D_k \phi = 0 \tag{2.12}
\]

\[
\frac{1}{2} \frac{e \epsilon_{ij}}{\sqrt{\gamma}} F_{ij} = \pm \sqrt{2V}. \tag{2.13}
\]
In this Bogomol’nyi limit, the $ij$-components of energy-momentum tensor vanish and then Einstein equations force both an integration constant $\kappa$ and the cosmological constant $\Lambda$ to be zero. Thus we have no off-diagonal components of metric $K_i$ up to a gauge. The 00-component of Einstein equations relates the topological quantities, i.e. the geometric part gives the Euler number and the matter part, the sum of magnetic flux from vortices and total mass from point particles

$$-\frac{1}{16\pi G} \int d^2x \sqrt{\gamma} \, 2R = \frac{v^2}{2} |e\Phi| + \sum_{a=1}^{N} m_a.$$

(2.14)

After eliminating the gauge field $A_i$ by use of a Bogomol’nyi equation (2.12)

$$A_i = \frac{1}{e} (\partial_i \Omega \mp \sqrt{\gamma} \varepsilon_{ij} \gamma^{jk} \partial_k \ln |\phi|),$$

(2.15)

and fixing the gauge for the spatial components of metric

$$\gamma_{ij} = -\delta_{ij} b(x^i),$$

(2.16)

we solve 00-component of Einstein equations

$$b(x^i) = e^{h(\tilde{z}) + \tilde{h}(\tilde{z})} \left( \frac{f^2 e^{-(f^2 - 1)}}{\prod_{p=1}^{n} |\tilde{z} - \tilde{z}_p|^2 \prod_{a=1}^{N} |\tilde{z} - \tilde{z}_a|^{2\tilde{m}_a}} \right) \tilde{G},$$

(2.17)

where $h(\tilde{z})$ ($\tilde{h}(\tilde{z}_p)$) is a holomorphic (antiholomorphic) function and the tilded variables are dimensionless quantities

$$\tilde{z} = \tilde{x}^1 + i\tilde{x}^2 = ev(x^1 + ix^2), \quad f = \frac{|\phi|}{v}, \quad \tilde{G} = 4\pi G v^2, \quad \tilde{m}_a = \frac{m_a}{\pi v^2}. \quad (2.18)$$

If the spacetime of the vacuum (no particle ($N = 0$) and no vortex ($n = 0$)) is to be Minkowski, the harmonic function should be chosen to be zero, $h(\tilde{z}) + \tilde{h}(\tilde{z}) = 0$. Substituting Eq.(2.17) and Eq.(2.15) into Eq.(2.13), we obtain a single equation for the amplitude of Higgs field

$$\tilde{\partial}^2 \ln f^2 = e^{h + \tilde{h}} \left( \frac{f^2 e^{-(f^2 - 1)}}{\prod_{p=1}^{n} |\tilde{z} - \tilde{z}_p|^2 \prod_{a=1}^{N} |\tilde{z} - \tilde{z}_a|^{2\tilde{m}_a}} \right) (f^2 - 1) + 4\pi \sum_{p=1}^{n} \delta^{(2)}(\tilde{z} - \tilde{z}_p), \quad (2.19)$$
where $\hat{\partial}^2$ is a Laplacian in flat two-dimensional space. Returning to the expression of Euler number in Eq. (2.14) and inserting Eq. (2.17) into it, we have

$$\frac{1}{16\pi G} \int d^2x \gamma^{2R},$$

$$= \frac{v^2}{4} \left\{ \int d^2x \ \partial^2 \ln \left( \prod_{p=1}^{n} |z - z_p|^2 \prod_{a=1}^{N} |z - z_a|^{2m_a} \right) - \int d^2x \ \partial^2 \ln |\phi|^2 + \int d^2x \ \partial^2 |\phi|^2 \right\}.$$  

(2.20)

From the above expression one may notice that, in addition to the contribution from the first two terms for $n \neq 0$ or $m_a \neq 0$, there can exist contribution from the second term if the Bogomol'nyi equation (2.13) contains the finite energy solution which behaves as $|\phi| \sim |\vec{x}|^{-\varepsilon} \ (\varepsilon > 0)$ for large $|\vec{x}|$. We shall show that it is indeed the case and present the detailed analysis for the existence of such solutions in section 4.

### III. Dual Formulation

In this section we reformulate the theory by use of the dual transformation and re-derive the Bogomol'nyi limit in this formulation. The path integral for the system we are considering is given by

$$Z = \left\langle F | e^{iHT} | I \right\rangle$$

$$= \int [dg_{\mu\nu}][dA_{\mu}][|\phi|d|\phi|][d\Omega]$$

$$\times \exp i \left\{ \int d^3x \sqrt{g} \left[ -\frac{1}{16\pi G}(R + 2\Lambda) - \frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + \frac{1}{2}g^{\mu\nu}\partial_{\mu}|\phi|\partial_{\nu}|\phi| \right.$$

$$+ \frac{1}{2}g^{\mu\nu}|\phi|^2(\partial_{\mu}\Omega - eA_{\mu})(\partial_{\nu}\Omega - eA_{\nu}) - V(|\phi|) \right]$$

$$+ \sum_{a=1}^{N} m_a \int_{-\infty}^{\infty} ds \sqrt{g_{\mu\nu}dx_{\mu}^{a}dx_{\nu}^{a}} \right\}.$$  

(3.1)
Introducing an auxiliary vector field $C_\mu$, we rewrite the interaction term between scalar field and gauge field as the following

$$
\exp \left\{ i \int d^3 x \sqrt{g} \frac{1}{2} g^{\mu\nu} |\phi|^2 (\partial_\mu \Omega - e A_\mu) (\partial_\nu \Omega - e A_\nu) \right\}
$$

$$
= \int [dC_\mu] \prod_x \frac{g^x}{|\phi|^3} \exp i \int d^3 x \sqrt{g} \left\{ -\frac{g^{\mu\nu}}{2|\phi|^2} C_\mu C_\nu + g^{\mu\nu} C_\mu (\partial_\nu \Omega - e A_\nu) \right\}.
$$

(3.2)

Classically the auxiliary field $C_\mu$ is nothing but the conserved $U(1)$ current.

Since the theory contains vortex and antivortex configurations, the phase of the scalar field need not be single-valued and then can be split into two parts

$$
\Omega(t, \vec{x}) = \Theta(t, \vec{x}) + \eta(t, \vec{x}).
$$

(3.3)

The first term $\Theta$ which describes a configuration of vortices and antivortices is defined by a multi-valued function

$$
\Theta(t, \vec{x}) = \sum_p (\mp) \tan^{-1} \frac{x^2 - x_p^2(t)}{x^1 - x_p^1(t)},
$$

(3.4)

where $(x_p^1(t), x_p^2(t))$ is the position of a (anti-)vortex and $(\mp)$ denotes $-1$ for vortex and 1 for anti-vortex. The single-valued function $\eta$ represents the fluctuation around a given vortex sector. Hence the path integral measure is divided into two contributions

$$
[d\Omega] = [d\Theta][d\eta],
$$

(3.5)

i.e. the one for the sum over single-valued fluctuation around a given configuration of vortices and the other for that over all possible configurations of vortices, including the annihilation and creation of vortex-antivortex pairs.

After $\eta$-integration,

$$
\int [d\eta] \exp \left\{ i \int d^3 x \sqrt{g} \ g^{\mu\nu} C_\mu \partial_\nu \eta \right\} \approx \frac{1}{\sqrt{g}} \delta(\nabla_\mu C^\mu),
$$

(3.6)

we can rewrite a part of the path integral by introducing dual gauge field $H_\mu$,

$$
\int [dC_\mu] \frac{1}{\sqrt{g}} \delta(\nabla_\mu C^\mu) \cdots = \int [dH_\mu][dC_\mu] \delta(\sqrt{g} C_\mu - \frac{1}{e} \epsilon^{\mu\nu\rho} \partial_\nu H_\rho) \cdots.
$$

(3.7)
Substituting Eq. (3.7) into the path integral in Eq. (3.2) and integrating out the auxiliary field $C_\mu$, we obtain an effective theory described by the action

$$S' = \int d^3x \sqrt{g} \left\{ -\frac{1}{16\pi G}(R+2\Lambda) - \frac{1}{2}g^{\mu\nu}\tilde{F}_\mu \tilde{F}_\nu + \frac{1}{2}g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi - V(|\phi|) \\
- \frac{1}{4e^2}g^{\nu\sigma}H_{\mu\nu}H_{\rho\sigma} + \frac{1}{2}\sqrt{g}H_{\mu\nu}(\frac{1}{e}\partial_\rho \Theta - A_\rho) \right\}$$

(3.8)

$$+ \sum_{a=1}^{N} m_a \int_{-\infty}^{\infty} ds \sqrt{g^{\mu\nu}} \frac{dx_a^\mu}{ds} \frac{dx_a^\nu}{ds}$$

where $H_{\mu\nu} = \partial_\mu H_\nu - \partial_\nu H_\mu$ and $\tilde{F}^\mu = \frac{\epsilon^{\mu\nu\rho}}{2\sqrt{g}}F_{\nu\rho}$, the dual of field strength tensor. For performing $A_\mu$ integration, let us regard $A_\mu$ and $\tilde{F}^\mu$ as independent variables and rewrite the path integral measure by introducing another auxiliary field $N_\mu$

$$\int [dA_\mu][d\tilde{F}^\mu] \delta(\tilde{F}^\mu - \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}}\partial_\nu A_\rho) \ldots$$

$$= \int [dA_\mu][d\tilde{F}^\mu][dN_\mu] \prod_x \sqrt{g} \exp \left\{ i \int d^3x \sqrt{g} N_\mu (\tilde{F}^\mu - \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}}\partial_\nu A_\rho) \right\} \ldots$$

(3.9)

Putting the above equation (3.9) into the path integral and then integrating $\tilde{F}^\mu$ and $A_\mu$ fields out, we have

$$\int [dA_\mu] \exp i \int d^3x \sqrt{g} \left\{ -\frac{1}{2}g^{\mu\nu}\tilde{F}_\mu \tilde{F}_\nu + H_\mu \tilde{F}^\mu \right\}$$

$$= \int [dN_\mu] \prod_x g^{\frac{1}{4}} \delta(\epsilon^{\mu\nu\rho}\partial_\nu N_\rho) \exp i \int d^3x \sqrt{g} \frac{g^{\mu\nu}}{2}(H_\mu + N_\mu)(H_\nu + N_\nu).$$

(3.10)

The delta functional in Eq. (3.10) implies $N_\mu = -\partial_\mu \chi$ for a single-valued scalar field $\chi$.

Finally the path integral of dual transformed theory becomes

$$Z = \int [gdg_{\mu\nu}][dH_\mu][|\phi|-2d|\phi|][d\Theta][d\chi] \exp \left\{ iS_D \right\}$$

(3.11)

where the action of dual transformed theory is

$$S_D = \int d^3x \sqrt{g} \left\{ -\frac{1}{16\pi G}(R+2\Lambda) + \frac{1}{2}g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi - V(|\phi|) \right\}$$
In Higgs phase this dual-transformed theory describes a gauge boson of mass $ev$ and a neutral Higgs. Gauge coupling $e$ is inversely multiplied to the interaction term between the gauge field and the Higgs field, which looks like the strong coupling expansion being done. However, when Higgs effects are important, one must take into account the nonpolynomial interaction in the Maxwell-like term and the Jacobian in the measure of the Higgs field. Though the classical gravity is not affected by the dual transformation, the Jacobian in the measure of gravitational field is introduced and this induced Jacobian factor depends on both the gauge dynamics and the dimension of spacetime [13].

The Euler-Lagrange equations are

$$
\frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu \nu} \partial_{\nu} \phi) = \frac{1}{2e^2 |\phi|^3} g^{\mu \nu} g^{\rho \sigma} H_{\mu \nu} H_{\rho \sigma} - \frac{dV}{d|\phi|} \tag{3.13}
$$

$$
\frac{1}{\sqrt{g}} \partial_{\nu} \left( \sqrt{g} e^2 |\phi|^2 H_{\mu \nu} \right) - g^{\mu \nu} (H_{\nu} - \partial_{\nu} \chi) = \frac{1}{e} \frac{e^{\mu \nu \rho}}{\sqrt{g}} \partial_{\nu} \partial_{\rho} \Theta \tag{3.14}
$$

$$
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} (R + 2\Lambda) = T_{\mu \nu}^{D}, \tag{3.15}
$$

where the energy-momentum tensor of dual-transformed theory is

$$
T_{\mu \nu}^{D} = \frac{1}{4e^2 |\phi|^2} g^{\rho \sigma} (g_{\mu \nu} g^{\tau \kappa} - 4g_{\mu \nu} g^{\kappa \tau}) H_{\rho \tau} H_{\sigma \kappa} + \left( g_{\mu}^{\rho} g_{\nu}^{\sigma} - \frac{1}{2} g_{\mu \nu} g^{\rho \sigma} \right) (H_{\rho} - \partial_{\rho} \chi) (H_{\sigma} - \partial_{\sigma} \chi)
$$

$$
\left. + (g_{\mu}^{\rho} g_{\nu}^{\sigma} - \frac{1}{2} g_{\mu \nu} g^{\rho \sigma}) \partial_{\rho} |\phi| \partial_{\sigma} |\phi| \right| + g_{\mu \nu} V \tag{3.16}
$$

$$
+ \frac{1}{\sqrt{g}} \sum_{a=1}^{N} m_{a} \int_{-\infty}^{\infty} ds_a \delta(x_a(s_a) - x) \frac{dx_a}{ds_a} \frac{dx_a}{ds_a}.
$$

From now on let us look for the counterpart of the vortices in the original theory through the derivation of Bogomol’nyi-type bound in the dual-transformed theory. From Eq. [3.2]
and Eq. (3.7), we notice that the classical configurations of both formulations are related by

$$|\phi|^2(\partial_\mu \Omega - e A_\mu) = \frac{1}{\epsilon} \sqrt{g} \epsilon_{\mu\nu\rho} \partial^\nu H^\rho. \quad (3.17)$$

Together with Eq. (3.14), we obtain

$$\tilde{F}^\mu = -g^{\mu\nu}(H_\nu - \partial_\nu \chi). \quad (3.18)$$

For the static counterpart of vortices, Eq. (3.18) and the condition that static vortices do not carry electric field imply

$$\tilde{F}_i = H_i - K_i H_0 - \partial_i \chi = 0. \quad (3.19)$$

These solitons are characterized by dual electrostatic potential, when $N = 1$

$$\Phi_D = \int d^2 x \sqrt{\gamma} H_0. \quad (3.20)$$

The spatial components of Eq. (3.14) are solved with the help of Eq. (3.19),

$$K^{ij} = \kappa_D e^2 \frac{\epsilon^{ij}|\phi|^2}{N \sqrt{\gamma} H_0}, \quad (3.21)$$

where $\kappa_D$ is a constant. Inserting this result into 0i-components of Einstein equations (3.15), we obtain

$$\kappa_D = 0 \quad \text{or} \quad \frac{N^2 |\phi|^2}{2 H_0} - 8\pi G H_0 = C, \quad (3.22)$$

where $C$ is an integration constant. Since the vortex solutions of our interest do not satisfy the second condition in the sequel, we take the first one from now on. Then $T^{D}_0$ vanishes, and the vortices in dual formulation are also spinless as expected.

The Bogomol’nyi-type bound of the dual-transformed Einstein Maxwell Higgs theory is obtained as follows.

$$\int d^2 x \sqrt{\gamma} T^{D}_{00}$$
To get the last expression we have set \( N = 1 \). Hence, if we choose the scalar potential as Eq.(2.11), \( H_0 \) becomes

\[
H_0 = \pm \frac{1}{2}(v^2 - |\phi|^2).
\]

(3.24)

Since \( T_{ij} = 0 \) in this Bogomol’nyi limit and \( K_{ij} = 0 \) from Eq.(3.21) and Eq.(3.22), \( ij \)-components of Einstein equations are solved by \( \Lambda = 0 \). The 00-component of Einstein equations is exactly solved as that of the original theory and the solution is Eq.(2.17).

Substituting this result into Gauss’ law (3.14), we obtain a single Bogomol’nyi equation, Eq.(2.19).

**IV. Cylindrically Symmetric Solution**

At the outset of our consideration in Einstein Maxwell Higgs model, we had ten second-order differential equations (2.2)∼(2.4) (or (3.13)∼(3.15)) for scalar, gauge, and gravitational fields even though the system was introduced in (2+1) dimensions after the reduction of a dimension along \( z \)-axis. However, once we have limited our interest to the static vortex
solutions under a specific $\phi^4$ scalar potential which saturate the Bogomol’nyi-type bound, three components of the gauge field and six components of metric tensor have been expressed by the scalar field and now there remains only one second-order equation (2.19) to solve. In this section let us look for regular and finite-energy vortex solutions of the Bogomol’nyi equation of which the base manifold is smooth except for the positions where the point particles lie. Let $\Sigma$ be the spatial part of the (2+1) dimensional spacetime. It will be seen that there exist solutions such that the space $\Sigma$ is a cone or a cylinder asymptotically, or a two sphere.

For the sake of tractability while keeping the main physical properties, let us begin with examining the cylindrically symmetric solutions of Bogomol’nyi equation (2.19). The metric which is compatible with cylindrically symmetric configurations is of the form

$$ds^2 = dt^2 - \frac{1}{(ev)^2}b(r)(dr^2 + r^2d\theta^2),$$

where $r = ev\sqrt{x^ix^i}$ and $0 \leq \theta < 2\pi$. Since we consider the regular static solutions of Einstein equations, the space $\Sigma$ described by metric $b(r)$ is smooth except for the points where there are massive point particles. The global geometry of $\Sigma$ may be characterized by the area

$$A = \frac{2\pi}{e^2v^2}\int_0^\infty dr\ r\ b(r),$$

the radial distance from the origin $\rho(r) = \frac{1}{ev}\int_0^r dr'\ \sqrt{b(r')},$ and the circumference $l(r) = \frac{2\pi}{ev}\ r\ \sqrt{b(r)}$.

With the aid of gauge transformation, any static cylindrically symmetric field configuration can be brought into the following ansatz

$$\phi = vf(r)e^{in\theta} \equiv v\ \exp\left(\frac{u(r)}{2} + in\theta\right).$$

Substituting the ansatz into the spatial integral of 00-component of the Einstein equations (2.4) or (3.15), we obtain

$$\frac{1}{16\pi G} \int d^2x\sqrt{\gamma} \ 2R = \frac{2\pi}{e^2}\int_0^\infty dr\ r\ \left\{ \left(\frac{de^2}{dr}\right)^2 + \frac{F}{4r^2G(\nu+M)} e^{-G(e^u-u-1)(e^u-1)^2}\right\} + \tilde{M},$$

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where $F$ is the harmonic function factor in Eq. (2.17) which is reduced to a constant in the cylindrically symmetric case and $\tilde{M} = \sum_{a=1}^{N} \tilde{m}_a$ is the total mass of point particles which are now superimposed at the origin.

For nonsingular solutions, $f = |\phi|/v$ should behave as

$$f \sim f_0 r^n$$  \hspace{1cm} (4.4)

near the origin. Further conditions are obtained by the requirement that Eq. (4.3) is finite. Analyzing the behavior of the integrand of Eq. (4.3) near the origin by use of Eq. (4.4), we find a condition that

$$\tilde{G}\tilde{M} < 1,$$  \hspace{1cm} (4.5)

for finite energy solutions to exist except the trivial one $u = 0$; analysis in the asymptotic region gives the boundary condition for $u$,

$$u(r = \infty) = \begin{cases} 
0 \text{ or } -\infty, & \text{if } 0 < \tilde{G}(n + \tilde{M}) \leq 1 \\
\text{arbitrary number between } -\infty \text{ and } 0, & \text{if } \tilde{G}(n + \tilde{M}) > 1.
\end{cases}$$  \hspace{1cm} (4.6)

which is enlarged in compared with that in flat space-time. Now let us examine soliton solutions of the Bogomol’nyi equation for the cases of $e^{u(r = \infty)/2} \neq 0$ and $e^{u(r = \infty)/2} = 0$ separately.

(a) $e^{u(r = \infty)/2} \neq 0$

At first we consider the case that the scalar field $e^{u/2}$ does not vanish at spatial infinity. As we see in Eq. (4.6), when $\tilde{G}(n + \tilde{M})$ is smaller than one the boundary value of $u$ at $r = \infty$ has to be zero for finite-energy solutions. In this case, it is convenient to introduce a variable $R$ such that

$$R = \frac{r^{1-\tilde{G}(n + \tilde{M})} - 1}{1 - \tilde{G}(n + \tilde{M})}.$$  \hspace{1cm} (4.7)
Then the Bogomol’nyi equation (2.19) is rewritten as

$$\frac{d^2 u}{dR^2} = -\frac{dV_{\text{eff}}}{du} - \frac{1}{R} \frac{du}{dR},$$

(4.8)

where $V_{\text{eff}}$ is defined by

$$V_{\text{eff}} = \frac{F}{G} \exp[-\tilde{G}(e^u - u - 1)],$$

(4.9)

which is an increasing function of $u$ for $-\infty < u \leq 0$ and has a maximum at $u = 0$. (See Fig. 1.) If we interpret $u$ as a particle position and $R$ as time, Eq. (4.8) is nothing but the Newton’s equation for a particle of unit mass moving in a potential $V_{\text{eff}}$ and subject to a friction. The particle also receives an impact at $R = 0$ from the delta function term in Eq.(2.19). When $n = 0$, $u = 0$ is the unique solution and it describes two dimensional flat space when there is no particle and a cone when there is a massive particle at the origin.

When $n \neq 0$, we now show that there always exists a finite energy solution whose base manifold $\Sigma$ is a cone asymptotically. For this we have to show that, if we suitably choose the initial parameter $f_0$ in Eq. (4.4), we can obtain the motion of the hypothetical particle such that it starts at negative infinity with the initial velocity given by Eq. (4.4) and stops at $u = 0$ at $R = \infty$.

First, the behavior of $f = e^{u/2}$ near the origin is

$$f \approx f_0 r^n \left(1 - \frac{F e^{\tilde{G} f_0^2 G}}{8(1 - GM)^2} r^{2(1 - \tilde{G} n)} + \ldots\right).$$

(4.10)

Let $R = R_0$ be an arbitrary large number. If we choose $f_0$ sufficiently small, then the higher order term in (4.10) can be neglected for $R \leq R_0$ and the energy of the particle $E(R_0)$ at $R = R_0$ is given by

$$E(R_0) = \frac{1}{2} \left(\frac{du}{dR}\right)^2 + V_{\text{eff}} \bigg|_{R=R_0} \approx \frac{2n^2}{[1 - G(n + M)]^2 R_0^2}.$$  

(4.11)

Let us choose $R_0$ (and $f_0$, correspondingly) such that $E(R_0) < V_{\text{eff}}(0)$. Then since the particle energy decreases as the particle moves due to friction, $E(R) < V_{\text{eff}}(0)$ for $R > R_0$.  

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In other words, for sufficiently small $f_0$, the hypothetical particle turns back at a point and goes to negative infinity as $R \to \infty$. Next, we choose $f_0$ arbitrarily large and $R_0$ sufficiently small so that (4.10) is good at $R = R_0$. Then it is not difficult to show that for $R > R_0$,

$$u(R) > 2 \ln f_0 r^n + \frac{V'_M}{4}(R_0^2 - R^2) + \frac{V'_M}{2} R_0^2 \ln R_0/R,$$

where $V'_M = \max_{-\infty < w \leq 0} \frac{dV_{eff}}{du}$. The right hand side of Eq. (4.12) has the maximum value at $R = R_1 \equiv \left[ R_0^2 + \frac{4n}{V'_M[1 - \tilde{G}(n + \tilde{M})]} \right]^{1/2}$ and then $u(R_1)$ satisfies, for small $R_0$,

$$u(R_1) > \frac{n}{1 - \tilde{G}(n + \tilde{M})} \left( \ln \frac{4n[1 - \tilde{G}(n + \tilde{M})]}{V'_M} - 1 \right) + 2 \ln f_0.$$

Therefore if we choose $f_0$ sufficiently large, then $u(R_1) > 0$, i.e. the particle goes over the hilltop of the potential. From these results, continuity now guarantees the existence of the vortex solution connecting the boundary values, $u(0) = -\infty$ and $u(\infty) = 0$, for an appropriate $f_0$. This completes the proof.

The geometry of $\Sigma$ for this solution can be read from the behavior of the metric $b(r)$: it is singular at the origin when $\tilde{M} \neq 0$ and the space $\Sigma$ has an apex there due to the point particle. As $r$ goes to infinity, the radial distance $\rho$ and the circumference $l$ diverge. The asymptotic region in terms of $R$ and $\theta'$ ($\theta' \equiv (1 - \tilde{G}(n + \tilde{M}))\theta$) is flat since the solutions approach their boundary values exponentially (see Fig. 2-(a),(b))

$$f \approx 1 - f_\infty K_0((1 - \tilde{G}(n + \tilde{M}))/R),$$

where $f_\infty$ is a constant determined by the proper behavior of the fields near the origin. As shown in Fig. 2-(c), the asymptotic structure of $\Sigma$ is a cone with deficit angle $\delta = 2\pi \tilde{G}(n + \tilde{M})$. The solution for $M = 0$ has been found in Ref. [8].

Now we consider the case $\tilde{G}(n + \tilde{M}) = 1$. In this case we define $R$ as $R = \ln r \ (-\infty < R < \infty)$ which reflects the scale symmetry ($r \to \lambda r$) of the Bogomol’nyi equation (2.19)
at this critical value, and then it takes the same form as Eq.(4.8) with no friction term. Therefore in this case the hypothetical particle moves under a conservative force only and hence the Bogomol’nyi equation can be integrated to the first order equation

\[ \frac{1}{2} \left( \frac{du}{dR} \right)^2 + V_{\text{eff}}(u) = \mathcal{E}, \tag{4.15} \]

where \( \mathcal{E} \) is a constant which is interpreted as the energy of the hypothetical particle which is conserved in this case. The particle energy \( \mathcal{E} \) is also determined by the initial behavior as

\[ \mathcal{E} = \left. \frac{1}{2} \left( \frac{du}{dR} \right)^2 \right|_{R=-\infty} = 2n^2. \tag{4.16} \]

In terms of classical mechanics the vortex solution is described as follows: a hypothetical particle of unit mass with energy \( \mathcal{E} = 2n^2 \) starts at position \( u = \infty \) at time \( R = -\infty \), climbs the hill of the potential \( V_{\text{eff}} \), and finally stops at the top of hill \( (u = 0) \) at \( R = \infty \). For such solutions we must have \( V_{\text{eff}}(0) = \mathcal{E} = 2n^2 \). From the definition of \( V_{\text{eff}} \), \( V_{\text{eff}}(0) = F/\tilde{G} \) and the constant \( F \) is determined for each \( n \) as \( F = 2\tilde{G}n^2 \). Thus we have completely determined free parameters in this case and obtain the solution as the form

\[ \frac{1}{2n} \int \frac{du}{\sqrt{1 - e^{-\tilde{G}(u-u-1)}}} = \int dR. \tag{4.17} \]

which is, unfortunately, not integrable to a closed form. It is amusing to note that this kind of analysis is not possible in flat case. The behavior of solution near the origin is the same as that in Eq.(4.10), so that the space \( \Sigma \) also has a apex at the origin for \( \tilde{M} \neq 0 \) solutions. However, since \( b(r) \sim r^{-2} \) for large \( r \), the radial distance from the origin \( \rho \) diverges but the circumference \( l \) approaches a finite value \( l = \frac{2\sqrt{2\pi n \sqrt{G}}}{ev} \) though \( r \) goes to infinity. Then the space \( \Sigma \) comprises asymptotically a cylinder as shown in Fig. 3.

Lastly, let us consider the case that \( \tilde{G}(n + \tilde{M}) \) is larger than one. If \( n = 0, \tilde{G}\tilde{M} > 1 \) and the only allowed finite energy solution is the trivial one \( u = 0 \) as we have seen in Eq.
(1.5). Suppose that there exists a solution when $\tilde{G}(n + \tilde{M}) > 1$ with $n \neq 0$. According to the similar argument given before, the space $\Sigma$ also has an apex if $\tilde{G}\tilde{M}$ is not zero. Since the metric $b(r)$ decreases more rapid than $1/r^2$ for large $r$, the radial distance $\rho(r = \infty)$ is finite and the circumference $l(r = \infty)$ vanishes. Then $\Sigma$ is compact and, since the Euler number given in Eq. (2.14) must be nonnegative, two dimensional sphere is the unique candidate. From now on let us call the point which corresponds to $r = 0$ “the south pole” on $S^2$ and that which corresponds to $r = \infty$ “the north pole” on $S^2$. Then, at the north pole, $\phi$ does not vanish and is not well-defined; $\phi = \phi(r = \infty) e^{in\theta}$. Therefore there is no regular solution in this case.

(b) $e^{u(r=\infty)/2} = 0$

Now let us consider the case that the scalar field $e^{u/2}$ vanishes at $r = \infty$ and suppose that it behaves like $e^{u/2} \approx r^{-\varepsilon}$ for large $r$. Then the finite energy condition from Eq. (4.3) forces $\varepsilon$ to satisfy $\tilde{G}(n + \varepsilon + \tilde{M}) > 1$. Examining the asymptotic behavior of the metric $b(r)$, we find that the radial distance $\rho(\infty)$ and the area $A$ of the manifold are finite, and the circumference $l$ vanishes at $r = \infty$. Therefore the space $\Sigma$ should form a two dimensional sphere $S^2$ and it is described by our coordinate $(r, \theta)$ except the north pole where a point particle may sit; let the mass be $M_n$. The Euler invariant given in Eq. (2.20) then should be equal to that of $S^2$,

$$\tilde{G}(n + \varepsilon + \tilde{M}_s + \tilde{M}_n) = 2,$$  \hspace{1cm} (4.18)

where $\tilde{M}_s = \tilde{M}$ and the subscript $s$ is attached as an indication that it represents the mass of particles at the south pole. Now we show that $\varepsilon$ can not be arbitrary by imposing the regularity condition. From the behavior of the metric $b(r)$ at large $r$, the radial distance $\rho(r)$ behaves near the north pole as $\rho - \rho(\infty) \sim r^{-\tilde{G}(n+\varepsilon+\tilde{M}_s)+1}$. On the other hand, if we choose a coordinate around the north pole rather than around the south pole, we can do all the analysis we have done by replacing $M_s$ by $M_n$. For example, regularity requires that
the scalar field behaves as \( f \sim s^n \) for \( s \sim 0 \), where \( s \) is the radial coordinate in the new coordinate whose origin is at the north pole; it means that \( r^{-\varepsilon} \sim s^n \). Also, the radial distance near the north pole will behave as \( \rho - \rho(\infty) = s^{\tilde{G}\tilde{M}_n+1} \). Comparing these with Eq. (4.18), we get the consistency condition \( r \propto s^{-1} \) and thereby \( \varepsilon = n \).

Now we discuss the existence of solutions. At first if \( n = 0 \), there is no nontrivial solution because \( \tilde{G}\tilde{M} > 1 \). Next let us consider the case \( \tilde{G}(n + \tilde{M}_s) \neq 1 \), i.e. \( \tilde{M}_s \neq \tilde{M}_n \). Without loss of generality we may assume that \( \tilde{G}(n + \tilde{M}_s) < 1 \). With \( \tau \equiv \ln R (-\infty \leq \tau \leq \infty) \), the Bogomol’nyi equation (4.8) is rewritten as

\[
\frac{d^2 u}{d\tau^2} = -e^{2\tau} \frac{dV_{eff}}{du}. \tag{4.19}
\]

Integrating over \( \tau \) from \(-T\) to \( T\),

\[
\frac{1}{2} \left( \frac{du}{d\tau} \right)^2 \bigg|_{\tau=-T}^{\tau=T} = -e^{2\tau} V_{eff} \bigg|_{\tau=-T}^{\tau=T} + 2 \int_{-T}^{T} d\tau e^{2\tau} V_{eff}. \tag{4.20}
\]

From the behavior near the poles, \( f \sim r^n (r \approx 0) \) and \( f \sim r^{-\varepsilon} (r \to \infty) \), it is easy to check that the left hand side becomes

\[
\frac{1}{2} \left( \frac{du}{d\tau} \right)^2 \bigg|_{\tau=-T}^{\tau=T} = \frac{2}{[1 - \tilde{G}(n + \tilde{M}_s)]^2} \left( \varepsilon^2 - n^2 + O(e^{-T}) \right), \quad T \to \infty. \tag{4.21}
\]

On the other hand, the first term of the right hand side in Eq. (4.20) is \( O(e^{-T}) \) while the second term goes to a positive definite and finite limit as \( T \to \infty \). Therefore, in this case, \( \varepsilon^2 - n^2 > 0 \), which is contradictory to the aforementioned condition \( \varepsilon = n \), i.e. there is no regular solution if \( \tilde{M}_s \neq \tilde{M}_n \).

The only remaining case is that with \( \tilde{M}_s = \tilde{M}_n (= \tilde{M}) \), or \( \tilde{G}(n + \tilde{M}) = 1 \). But in this case the second-order Bogomol’nyi equation reduces to the first-order equation (4.15) and we can simply extend the discussion below Eq. (4.15). For sphere solutions to exist the maximum of the potential \( V_{eff}(0) \) have only to be larger than the energy of the hypothetical
particle. [The condition \( \varepsilon = n \) is automatically satisfied.] That is, if \( F > 2n^2\tilde{G} \) for a given \( n \), there always exists a unique solution which supports two sphere. If we look at the shape of scalar field in \( r \)-coordinate, it resembles \( n \neq 0 \) nontopological vortex solution: scalar field vanishes both at \( r = 0 \) and at \( r = \infty \). However, since this solution comprises two sphere, it can be interpreted as a configuration that two vortices with vorticity \( n \) lie both at the south pole and at the north pole, and two particles make an apex at each pole if \( M \neq 0 \) (see Fig. 4). Similar to the cylinder case where \( F = 2n^2\tilde{G} \), the maximum circumference of two sphere along the tropical line is \( \frac{2\sqrt{2n}\sqrt{\tilde{G}}}{ev} \) which is independent of \( F \). If we regard the point particles as Planck scale strings \(^{[14]}\) parallel to strings of which the symmetry breaking scale is lower than the Planck scale, e.g. GUT scale, it may imply a possibility of compactification of spatial manifold in the lower symmetry breaking scale.

V. Conclusion

In this paper we studied a self-dual system of (2+1) dimensional Einstein Maxwell Higgs theory with or without external particles. Bogomol’nyi-type bound for the original and the dual-transformed theory has been derived under a specific condition for the form of \( \phi^4 \) scalar potential. Then, using cylindrical symmetry ansatz, we found all possible soliton solutions of the equation. One type of solutions has a Higgs vacuum value as the boundary value and the underlying spatial manifold of these solutions is an asymptotic cylinder or a cone. The other, which is absent in the flat spacetime case, has a symmetry-restored local maximum value as the boundary value and the spatial manifold constitutes two sphere with two vortices or vortex-particle composites. These solutions exist when the gravitational constant and the mass of external particles satisfy the relation \( \tilde{G}(n + \tilde{M}) = 1 \).

While the existence of cylindrically symmetric solutions of Bogomol’nyi equation has been rigorously demonstrated, the stability of those solutions, multi-soliton solutions with zero-mode counting, the existence of such solutions away from the Bogomol’nyi limit remain
to be clarified. Some aspects of classical self-dual solitons were discussed using the dual-transformed form of the model, and the quantum field theoretic issues such as the phase transition structure need further study. As the particular choice we made for $\phi^4$ scalar potential was understood by a consideration of supersymmetric models in flat spacetime [13], a supergravity version of the model may be interesting to investigate.

**Note Added**

After writing this paper, we became aware of other references closely related with the present paper. The possibility of cosmic strings of cylinder type or 2-sphere type was first proposed by Gott [16]. In Abelian Higgs model without point particles, Linet [17] found cylinder and sphere configurations by solving the Bogomol’nyi equations under the choice of parameters $n\tilde{G} = 1$. (But he misinterpreted the sphere solution as representing a vortex-antivortex pair; it should be interpreted as representing a vortex-vortex pair as we have seen in Sec. 4.) Ortiz [18] studied the same model out of Bogomoln’yi limit including solutions with singularity. We thank Ortiz for bringing these references to our attention.

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Figure Captions

Figure 1: A typical shape of $V_{eff}(u)$. $V_{eff}(u)$ has an asymmetric bell shape with the maximum at $u = 0$.

Figure 2: Plot of cylindrically symmetric solution with or without point particles at the origin for which the underlying space form a cone at asymptotic region with deficit angle $\delta = 2\pi \tilde{G}(n + \tilde{M})$. Parameters chosen in the figures are: $\tilde{M} = 0$, $\tilde{G} = 1/2$ and $F = 1$ (no particle); $\tilde{M} = 1/2$, $\tilde{G} = 1/2$ and $F = 1$ (particles at the origin). (a) $|\phi|/v$ vs $\rho$, (b) $B/b$ vs $\rho$ and (c) shape of the underlying space $\Sigma$ when embedded in three dimensional Euclidean space with vertical coordinate denoted as $Z$.

Figure 3: Plot of cylindrically symmetric solution with or without point particles at the origin for which the underlying space form a cylinder at asymptotic region. Parameters chosen in the figures are: $\tilde{M} = 0$, $\tilde{G} = 1$ and $F = 2$ (no particle); $\tilde{M} = 1$, $\tilde{G} = 1/2$ and $F = 1$ (particles at the origin). (a) $|\phi|/v$ vs $\rho$, (b) $B/b$ vs $\rho$ and (c) shape of the underlying space $\Sigma$ when embedded in three dimensional Euclidean space with vertical coordinate denoted as $Z$.

Figure 4: Plot of cylindrically symmetric solution with or without point particles at the origin for which the underlying space form a two sphere. Parameters chosen in figures are: $\tilde{M} = 0$, $\tilde{G} = 1$ and $F = 3$ (no particle); $\tilde{M} = 1$, $\tilde{G} = 1/2$ and $F = 3$ (particles at the origin). (a) $|\phi|/v$ vs $\rho$, (b) $B/b$ vs $\rho$ and (c) shape of the underlying space $\Sigma$ when embedded in three dimensional Euclidean space with vertical coordinate denoted as $Z$. 

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Figure 1
Figure 2-(a)
Figure 2-(b)
Figure 2-(c)
Figure 3-(a)
Figure 3-(b)
Figure 3-(c)
Figure 4-(a)
Figure 4-(b)
Figure 4-(c)