Existence of solution for a class of heat equation involving the $p(x)$ Laplacian with triple regime

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Abstract. In this paper, we study the local and global existence of solution and the blow-up phenomena for a class of heat equation involving the $p(x)$-Laplacian with triple regime.

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1. Introduction

In this paper, we study the local and global existence of solution for the following class of heat equation

$$
\begin{cases}
  u_t - \Delta_{p(x)} u = |u|^{q(x)-2} u & \text{in } \Omega \times (0, +\infty), \\
  u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\
  u(x, 0) = u_0(x) & \text{in } \Omega,
\end{cases}
$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain and $u_0 \in W^{1,p(x)}_0(\Omega)$ and $p, q : \overline{\Omega} \to \mathbb{R}$ are continuous functions satisfying some conditions that will be mentioned later on.

When $p$ and $q$ are constant functions, the problem above becomes

$$
\begin{cases}
  u_t - \Delta_p u = |u|^{q-2} u & \text{in } \Omega \times (0, +\infty), \\
  u(x, 0) = u_0(x) & \text{in } \Omega.
\end{cases}
$$

The methods used to solve that problem are developed in relationship with the values of $q$ with respect to the Sobolev critical exponent $p^*$ of $p$, which is defined by

$$
p^* = \begin{cases} Np \\ N-p \\ +\infty \end{cases} \begin{array}{l} \text{if } 1 < p < N \\
\text{if } p \geq N, \end{array}
$$

and only one of the situations below can occur:

(i) $q < p^*$ (subcritical case).
(ii) $q = p^*$, provided that $1 < p < N$ (critical case).
(iii) $q > p^*$, provided that $1 < p < N$ (supercritical case).

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In the case of variable exponents, the problem (1.2) becomes more rich, because it can fulfill even a “subcritical–critical–supercritical” triple regime, in the sense that we can have \( \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \) with

\[
\begin{align*}
q(x) &< p^*(x) \quad \text{if } x \in \Omega_1; \\
q(x) & = p^*(x) \quad \text{if } x \in \Omega_2; \\
q(x) &> p^*(x) \quad \text{if } x \in \Omega_3.
\end{align*}
\]

(1.3)

In the last few decades, special attention has been paid to the study of partial differential equations involving \( p(x) \)-growth conditions. The interest in studying such problems is motivated by their applications in image restoration, nonlinear elasticity theory, electrorheological fluids, and so forth. In particular, parabolic equations involving the \( p(.) \)-Laplacian are related to the field of electrorheological fluids which are characterized by their ability to change the mechanical properties under the influence of the exterior electromagnetic; for a more physical motivation we refer the reader to [13,22] and [23] and their references. The rigorous study for these physical problems has been facilitated by the development of Lebesgue and Sobolev spaces with variable exponents.

It is important to point out that many results have obtained on parabolic equations with nonlinearities of variable exponent where the authors have studied the global existence in the subcritical case \((q<p^*)\), for example, see ([5,6,24,27–30,32]) and the references therein. In [33], by using the Galerkin method, the authors have studied the global existence and asymptotic behavior of global weak solutions for the following class of problem

\[
\begin{cases}
  u_t - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \left| \frac{\partial u}{\partial x_i} \right|^{p(x) - 2} \frac{\partial u}{\partial x_i} \right) + |u|^\sigma(x) = 0, & x \in \Omega, \ t > 0, \\
  u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), & x \in \Omega
\end{cases}
\]

(M)

for \( p^+ < \sigma^- + 1 \leq \sigma(x) + 1 \leq \sigma^+ + 1 < p^*(x) \). In [31], by using the sub-differential approach, Akagi and Matsuura obtained the well-posedness of solutions for problem (1.1) with \( f(x,t) \) instead of \( |u|^{q(x)-2}u \). Moreover, the large time behavior of solutions also is considered. The asymptotic stability for Kirchhoff systems with variable exponent growth conditions has been discussed by G. Autuori and P. Pucci in [34]. We recall that in the case when \( p(x) \) and \( q(x) \) are constant functions, the global existence and blow-up of solutions for the problem (1.1) have been studied by many authors; we refer to ([4,9–11,15,16,18]) for interested reader. Finally, for the existence theorem of solutions for elliptic problems with variable exponents we refer to ([1–3,35,36]) and the references therein.

Motivated by the above works, the main goal this paper is to study problem (1.1) by consideration of the triple regime mentioned in (1.3), which is a novelty for this class of problem.

The plan of the paper is as follows: In Sect. 2, we recall some facts involving the variable exponent Lebesgue and Sobolev spaces. In Sect. 3, we assume that \( \Omega \) is a ball centered at origin and show the existence local and global of solution for (1.1) with triple regime. In Sect. 4, we work with the non-radial case; here \( \Omega \) is not a ball centered at origin. Again, it is showed the existence local and global of solution for (1.1) with triple regime. Finally, in Sect. 5, we establish the blow-up phenomena for a general situation that also involves a triple regime.

2. Variable exponent Lebesgue and Sobolev spaces

In this section, we recall some results on variable exponent Lebesgue and Sobolev spaces. For more details, we refer to [8,12,17] and their references.
Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $p \in L^\infty(\Omega)$ with $p_- := \text{essinf}_{x \in \mathbb{R}^N} p(x) > 1$. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \left| u \text{ is measurable and } \int_{\Omega} |u|^{p(x)} \, dx < \infty \right\}$$

endowed with the norm

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 \left| \int_{\Omega} \frac{|u|^{p(x)}}{\lambda} \, dx \leq 1 \right\}.$$ 

The variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \left| \nabla u \in L^{p(x)}(\Omega) \right\}$$

with the norm

$$||u||_{1,p(x)} = |u|_{p(x)} + \left| \nabla u \right|_{p(x)}$$

With these norms, the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are reflexive and separable Banach spaces.

The space $W^{1,p(x)}_0(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the above norm. Moreover, by Poincaré inequality, if $p \in C(\overline{\Omega})$, then $||u|| = |\nabla u|_{p(x)}$ is a norm in $W^{1,p(x)}_0(\Omega)$, and it is equivalent to norm $||u||_{1,p(x)}$.

**Proposition 2.1.** The functional $\rho : W^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho(u) = \int_{\Omega} \left( |\nabla u|^{p(x)} + |u|^{p(x)} \right) \, dx,$$ 

has the following properties:

(i) If $||u||_{1,p(x)} \geq 1$, then $||u||^p_{1,p(x)} \leq \rho(u) \leq ||u||^p_{1,p(x)}$. 

(ii) If $||u||_{1,p(x)} \leq 1$, then $||u||^p_{1,p(x)} \leq \rho(u) \leq ||u||^p_{1,p(x)}$.

In particular, $\rho(u) = 1$ if and only if $||u||_{1,p(x)} = 1$ and if $(u_n) \subset W^{1,p(x)}(\Omega)$, then $||u_n||_{1,p(x)} \to 0$ if and only if $\rho(u_n) \to 0$.

**Remark 1.** For the functional $\xi : L^{p(x)}(\Omega) \to \mathbb{R}$ given by

$$\xi(u) = \int_{\Omega} |u|^{p(x)} \, dx,$$

the conclusion of Proposition 2.1 also holds; for example, if $(u_n) \subset L^{p(x)}(\Omega)$, then $|u_n|_{p(x)} \to 0$ if and only if $\xi(u_n) \to 0$. Moreover, from (i) and (ii),

$$|u|_{p(x)} \leq \max \left\{ \left( \int_{\Omega} |u|^{p(x)} \, dx \right)^{1/p_-}, \left( \int_{\Omega} |u|^{p(x)} \, dx \right)^{1/p_+} \right\}.$$ 

Related to the Lebesgue space $L^{p(x)}(\Omega)$, we have the following generalized Hölder-type inequality.

**Proposition 2.2.** ([19] [p. 9]) For $p \in L^\infty(\Omega)$ with $p_- > 1$, let $p' : \Omega \to \mathbb{R}$ be such that

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \ a.e. \ x \in \mathbb{R}^N.$$
Then, for any \( u \in L^{p(x)}(\Omega) \) and \( v \in L^{p'(x)}(\Omega) \),
\[
\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p_-} + \frac{1}{p'_-} \right) |u|_{p(x)} |v|_{p'(x)}.
\tag{2.3}
\]

**Proposition 2.3.** ([12] [Theorems 1.1, 1.3]) Let \( p : \Omega \to \mathbb{R} \) be a Lipschitz continuous satisfying \( 1 < p_- \leq p_+ < N \) and \( t : \Omega \to \mathbb{R} \) be a continuous function.

(i) If \( 1 \leq t \leq p^* \), the embedding \( W^{1,p(x)}(\Omega) \hookrightarrow L^t(\Omega) \) is continuous.

(ii) If \( 1 \leq t < p^* \), the embedding \( W^{1,p(x)}(\Omega) \hookrightarrow L^t(\Omega) \) is compact.

Here \( t < p^* \) means \( \inf_{x \in \Omega} (p^*(x) - t(x)) > 0 \).

### 3. Problem 1: the radial case

In this section, \( \Omega = B_R(0) \) and \( p, q : \overline{B}_R(0) \to \mathbb{R} \) are continuous functions satisfying:
\[
2 \leq p_- = \min_{x \in B_R(0)} p(x) \leq \max_{x \in \overline{B}_R(0)} p(x) = p_+ < N \quad \text{and} \quad 1 < q_- = \min_{x \in B_R(0)} q(x).
\tag{H_1}
\]

\[
p(x) = p(|x|) \quad \text{and} \quad q(x) = q(|x|), \quad \forall x \in \overline{B}_R(0).
\tag{H_2}
\]

Our intention is to prove the existence of local and global solution for (1.1).

#### 3.1. Existence of local solution in \( W^{1,p(x)}_0(B_R) \)

In this subsection, we will show the local existence of solution for problem (1.1) with triple regime. In addition to conditions \((H_1) - (H_2)\), we assume that there is \( r \in (0, R) \) such that
\[
1 < q_- \leq q(x) \leq \min \left\{ p^*(x), 1 + \frac{p^*(x)}{2} \right\}, \quad \forall x \in \overline{B}_r(0).
\tag{H_3}
\]

Note that there is a control on the function \( q \) in the ball \( \overline{B}_r(0) \), but there are no hypotheses on the function \( q \) in the annulus \( A_{R,r} = \overline{B}_R(0) \setminus B_r(0) \); hence, close to the boundary can exist there subsets \( \Omega_1, \Omega_2 \) and \( \Omega_3 \) such that \( q \) is subcritical in \( \Omega_1 \), critical in \( \Omega_2 \) and supercritical in \( \Omega_3 \), which characterizes a triple regime.

Note that for any \( r_0 \in (0, R) \) we have the continuous embedding
\[
W^{1,p(x)}(B_R(0)) \hookrightarrow W^{1,p_-}(A_{R,r_0})
\]
and the compact embedding
\[
W^{1,p_-}(A_{R,r_0}) \hookrightarrow C(\overline{A}_{R,r_0}),
\]
which is due to Strauss [20]. Therefore, the embedding
\[
W^{1,p(x)}_0(B_R(0)) \hookrightarrow C(\overline{A}_{R,r_0}),
\tag{3.1}
\]
is compact, where
\[
W^{1,p(x)}_0(B_R(0)) = \{ u \in W^{1,p(x)}(B_R(0)) : u(x) = u(|x|) \quad \text{a.e} \quad x \in B_R(0) \}.
\]
Hence, it follows that the embedding
\[
W^{1,p(x)}_0(B_R(0)) \hookrightarrow L^{q(x)}(B_R(0)),
\tag{3.2}
\]
is also compact, which is crucial in our approach. From this, the energy functional
\[ E(u) = \int_{B(0)} \frac{1}{p(x)} |\nabla u|^p(x) \, dx - \int_{B(0)} \frac{1}{q(x)} |u|^q(x) \, dx \]
belongs to \( C^1(W^{1,p(x)}_0(B_R(0)), \mathbb{R}) \) with
\[ E'(u)v = \int_{B_R(0)} |\nabla u|^p(x) - 2\nabla u \nabla v \, dx - \int_{B_R(0)} |u|^q(x) - 2uv \, dx, \quad \forall u, v \in W^{1,p(x)}_0(B_R(0)). \]

Hereafter, we endow \( W^{1,p(x)}_0(B_R(0)) \) with the norm
\[ ||u|| = ||\nabla u||_{p(x)} \]
and \( B_R \) denotes the ball \( B_R(0) \).

The main result in this section has the following statement:

**Theorem 3.1.** (Local weak solution) Assume \((H_1) - (H_3), \Omega = B_R \) and \( u_0 \in W^{1,p(x)}_0(B_R(0)) \). Then, there exist \( T > 0 \) and a function \( u \in L^\infty([0,T], W^{1,p(x)}_0(B_R)) \) with \( u_t \in L^2((0,T), L^2(B_R)) \) such that \( u(0) = u_0 \in W^{1,p(x)}_0(B_R) \), and for each \( v \in W^{1,p(x)}_0(B_R) \)
\[ \int_{B_R} u_t(t)v \, dx + \int_{B_R} |\nabla u(t)|^{p(x)} - 2\nabla u \nabla v \, dx = \int_{B_R} |u(t)|^{q(x)} - 2u(t)v \, dx, \quad a.e. \ in \ (0,T). \]

In order to prove Theorem 3.1, let us introduce some useful properties of subdifferentials of proper, convex and lower semicontinuous functional on a Hilbert space.

Let \( H \) be a Hilbert space with the inner product \((.,.)_H\) and the norm \( ||.||_H \). For a functional \( \varphi \) from \( H \) to \((-\infty, +\infty)\), we shall write
\[ D(\varphi, \delta) = \{ u \in H, \varphi(u) \leq \delta \} \quad \text{for} \quad \delta \in \mathbb{R} \quad \text{and} \quad D(\varphi) = \bigcup_{\delta \in \mathbb{R}} D(\varphi, \delta). \]

If \( \varphi : H \to (-\infty, +\infty) \) is a convex functional, the subdifferential \( \partial \varphi \) of \( \varphi \) is defined by
\[ \partial \varphi(u) = \{ w \in H, \varphi(v) - \varphi(u) \geq (w, v-u)_H, \quad \forall v \in H \}. \]
It is well known that the subdifferential \( \partial \varphi \) is a maximal monotone operator and \( D(\partial \varphi) \subset D(\varphi) \). The following type chain rule for subdifferentials is taken from [7][Lemma 3.3, p. 73].

**Lemma 3.2.** Let \( \varphi : H \to (-\infty, +\infty) \) be a proper, convex and lower semicontinuous functional. For some \( T > 0 \), let \( u \in W^{1,1}(0,T,H) \) and \( u(t) \in D(\partial \varphi) \) a.e in \([0,T] \). If there exists a function \( f \in L^2(0,T,H) \) such that \( f(t) \in \partial \varphi(u(t)) \) a.e in \([0,T] \), then the function \( t \to \varphi(u(t)) \) is absolutely continuous on \([0,T]\) and
\[ \frac{d}{dt} \varphi(u(t)) = (f(t), u'(t))_H, \quad a.e. \ in \ [0,T]. \]

In what follows, let us recall a very important result found in [15][Theorem 3.4, p. 297].

**Theorem 3.3.** Let \( \varphi, \phi : H \to (-\infty, +\infty) \) be a proper, convex and lower semicontinuous functionals. Under the following assumptions :
(1) the set \( D(\varphi, \delta) \) is compact in \( H \) for any \( \delta \in \mathbb{R}, \)
(2) \( D(\varphi) \subset D(\phi), \)
Proof. It is enough to consider
\[
\{ \frac{du(t)}{dt} + \partial \varphi(u(t)) - \partial \phi(u(t)) \geq 0, \text{ in } H, \ 0 < t < T, 
\]
\[
\varphi(u) = w, 
\]

Proof of Theorem 3.1. Hereafter, we will assume that \( u_0 \in W_{0,\text{rad}}^{1,p(x)}(B_R) \) and \( H = L^2(B_R) \); then, by Proposition 2.3, the embedding
\[
W_{0,\text{rad}}^{1,p(x)}(B_R) \hookrightarrow L^2(B_R) 
\]
is compact. In what follows, let us consider the operators \( \varphi : L^2(B_R) \to (-\infty, +\infty) \) and \( \phi : L^2(B_R) \to (-\infty, +\infty) \) given by
\[
\varphi(u) = \begin{cases} 
\int_{B_R} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx, & \text{if } u \in W_{0,\text{rad}}^{1,p(x)}(B_R) \\
+\infty, & \text{if } u \in L^2(B_R) \setminus W_{0,\text{rad}}^{1,p(x)}(B_R) 
\end{cases}
\]
and
\[
\phi(u) = \begin{cases} 
\int_{B_R} \frac{1}{q(x)} |u|^{q(x)} \, dx, & \text{if } u \in L^{q(x)}(B_R) \\
+\infty, & \text{if } u \in L^2(B_R) \setminus L^{q(x)}(B_R). 
\end{cases}
\]

Lemma 3.4. The set \( D(\varphi, \delta) \) is compact in \( L^2(B_R) \) for any \( \delta \in \mathbb{R} \). Moreover, \( D(\varphi) \subset D(\phi) \).

Proof. This is an immediate consequence of (3.3) and (3.4). \( \square \)

Lemma 3.5. The set \( \{ (\partial \phi)^0(u), \ u \in D(\varphi, \delta) \} \) is bounded in \( L^2(B_R) \) for any \( \delta \in \mathbb{R} \).

Proof. It is enough to consider \( \delta > 0 \). Since \( \phi \in C^1(L^{q(x)}(B_R), \mathbb{R}) \), a simple computation gives that there is unique \( h_u \in L^2(B_R(0)) \) such that \( (\partial \phi)(u) = \{ h_u \} \), and so, \( (\partial \phi)^0(u) = \{ h_u \} \). Moreover,
\[
\int_{B_R} |u(x)|^{q(x)-2} u(x) \, dx = (h_u, v)_{L^2(B_R)}, \quad \forall v \in C_0^\infty(B_R).
\]
The above equality yields \( |u(x)|^{q(x)-2} u(x) = h_u(x) \) a.e. in \( B_R \). From this,
\[
|(h_u, v)_{L^2(B_R)}| \leq \|u(x)|^{q(x)-1}\|L^2(B_R)||v||L^2(B_R), \quad \forall v \in C_0^\infty(B_R).
\]
A simple computation shows that
\[
\int_{B_R(0)} |u(x)|^{2(q(x)-1)} \, dx \leq C \left( \|u\| + \|u\|_{p^*(x)}^\gamma + \|u\|_{p^*(x)}\gamma \right),
\]
where
\[
(p^*)^\gamma = \max_{x \in \overline{B_R(0)}} p^*(x) \quad \text{and} \quad (p^*)^\gamma = \min_{x \in \overline{B_R(0)}} p^*(x).
\]
Using the fact that \( u \in D(\varphi, \delta) \), there is \( C = C(\delta) > 0 \) such that
\[
\|u\| + \|u\|_{p^*(x)}^\gamma + \|u\|_{p^*(x)}^\gamma \leq C, \quad \forall u \in D(\varphi, \delta).
\]
This together with Proposition 2.1 yields
\[
||u(x)||^{q(x)-1}_{L^2(B_R)} \leq C, \quad \forall u \in D(\varphi, \delta).
\]
Hence

\[ \|h_u\|_{L^2(B_R)} = \sup_{\|v\|_{L^2(B_R)} \leq 1} |(h_u, v)|_{L^2(B_R)} \leq C, \quad \forall u \in D(\varphi, \delta), \]

showing the lemma.

From Theorem 3.3, for each \( u_0 \in W^{1,p(x)}_{0,\text{rad}}(B_R) \) there exist \( T > 0 \) and a strong solution \( u \in D(\varphi) \) of the value problem

\[
\begin{cases}
\frac{du(t)}{dt} + \partial\varphi(u(t)) - \partial\phi(u(t)) \geq 0, & \text{in } L^2(B_R), \ 0 < t < T,
\end{cases}
\]

Since

\[ \partial\phi(u(t)) = \{|u(t)|^{q(x)-2}u(t)\} \]

and

\[ \partial\varphi(u(t)) = \{\xi_u\}, \]

that is,

\[ \langle -\Delta_{p(x)}u(t), v \rangle = \int_{B_R} |\nabla u(t)|^{p(x)} \nabla u \nabla v \, dx = (\xi_u, v)_{L^2(B_R)}), \quad \forall v \in W^{1,p(x)}_{0,\text{rad}}(B_R), \]

the solution \( u \) must verify the equality below for all \( v \in W^{1,p(x)}_{0,\text{rad}}(B_R) \)

\[ \int_{B_R} u_t(t)v \, dx + \int_{B_R} |\nabla u(t)|^{p(x)-2} \nabla u(t) \nabla v \, dx = \int_{B_R} |u(t)|^{q(x)-2} u(t)v \, dx, \quad (3.5) \]

a.e. in \( t \in [0, T] \). Moreover, from Lemma 3.2,

\[ \int_0^t \|u_t(s)\|_{L^2(\Omega)}^2 \, ds + E(u(t)) = E(u_0), \quad \text{a.e. in } \ t \in [0, T]. \]

Here we will apply the Palais principle of symmetric criticality developed by Kobayashi and Otani [14][Theorem 2.2] to conclude that the function \( u \) is in fact a local weak solution in \( W^{1,p(x)}_{0,\text{rad}}(B_R) \). Hereafter, we set \( J \subset [0, T] \) with \( \text{med}(J^c) = 0 \) such that for all \( \varphi \in W^{1,p(x)}_{0,\text{rad}}(B_R) \)

\[ \int_{B_R} u_t(t)\varphi \, dx + \int_{B_R} |\nabla u(t)|^{p(x)-2} \nabla u(t) \nabla \varphi \, dx = \int_{B_R} |u(t)|^{q(x)-2} u(t)\varphi \, dx, \quad t \in J. \quad (3.6) \]

Fixing \( t \in J \) and defining \( v = u(t) \in W^{1,p(x)}_{0,\text{rad}}(B_R) \) and \( f = u_t(t) \in L^2(B_R) \), we derive that

\[ \int_{B_R} |\nabla v|^{p(x)-2} \nabla v \nabla \varphi \, dx = \int_{B_R} |v|^{q(x)-2} v\varphi - \int_{B_R} f\varphi \, dx, \quad \forall \varphi \in W^{1,p(x)}_{0,\text{rad}}(B_R). \quad (3.7) \]

Our intention is to prove that the equality above holds in whole \( W^{1,p(x)}_{0,\text{rad}}(B_R) \), and an important tool in this direction is the principle of symmetric criticality developed by Kobayashi and Otani [14][Theorem 2.2]. First of all, it is very important point out that (3.7) ensures that \( u \) is a critical point of the energy functional

\[ E_1(w) = \int_{B_R} \frac{1}{p(x)} |\nabla w|^{p(x)} \, dx - \int_{B_R} \frac{1}{q(x)} |w|^{q(x)} \, dx - \int_{B_R} f w \, dx, \quad w \in W^{1,p(x)}_{0,\text{rad}}(B_R), \]
which is not well defined in whole $W^{1, p(x)}_0(B_R)$. However, we cannot use directly the principle of symmetric
criticality to guarantee that $u$ is a critical point of $E_1$ in $W^{1, p(x)}_0(B_R)$. In order to overcome this difficulty,
we will use the following idea: Consider the function

$$g(x, t) = \xi(|x|) |t|^{q(x) - 2}t + (1 - \xi(|x|)) |u(x)|^{q(x) - 2}u(x), \quad \forall x \in B_R,$$

where $\xi \in C^{\infty}([0, R], \mathbb{R})$ satisfies

$$\xi(x) = \begin{cases} 1, & x \in \overline{B}_{\frac{R}{2}}(0) \\ 0, & x \in \overline{B}_R(0) \setminus \overline{B}_{\frac{3R}{4}}(0). \end{cases}$$

Since $u \in C(\overline{A}_{R, \xi})$ [see (3.1)], it follows from $(H_3)$ that

$$|g(x, t)| \leq C(|t|^{p^*(x) - 1} + 1), \quad \forall (x, t) \in B_R \times \mathbb{R}. \quad (3.8)$$

Associated with the function $g$, we have the problem

$$\begin{cases} -\Delta_{p(x)} w = g(x, w) - f & \text{in } B_R, \\ w = 0 & \text{on } \partial B_R, \end{cases} \quad (P_g)$$

whose the associated energy is given by

$$I(w) = \int_{B_R} \frac{1}{p(x)} |\nabla w|^{p(x)} \, dx - \int_{B_R} G(x, w) \, dx - \int_{\partial B_R} f w \, d\sigma,$$

where $G(x, t) = \int_0^t g(x, s) \, ds$. From (3.8), $I$ is well defined in the whole space $W^{1, p(x)}_0(B_R)$, $I \in C^1(W^{1, p(x)}_0(B_R), \mathbb{R})$ and

$$I'(w)\psi = \int_{B_R} |\nabla w|^{p(x) - 2} \nabla w \nabla \psi \, dx - \int_{B_R} g(x, w) \psi \, dx - \int_{\partial B_R} f \psi \, d\sigma, \quad \forall w, \psi \in W^{1, p(x)}_0(B_R).$$

Since

$$g(x, u(x)) = |u|^{q(x) - 2}u(x), \quad \forall x \in B_R,$$

we see that $u$ is a critical point of $I$ restricted to $W^{1, p(x)}_0(B_R)$. Now we can apply the Palais principle of symmetric
criticality developed by Kobayashi and Otani [14][Theorem 2.2] to conclude that $u$ is a nontrivial critical point of $I$ in the whole $W^{1, p(x)}_0(B_R)$. This shows that (3.7) holds for all $\varphi \in W^{1, p(x)}_0(B_R)$; then, for all $\varphi \in W^{1, p(x)}_0(B_R)$

$$\int_{B_R} u(t) \varphi \, dx + \int_{B_R} |\nabla u(t)|^{p(x) - 2} \nabla u(t) \nabla \varphi \, dx = \int_{B_R} |u(t)|^{q(x) - 2} u(t) \varphi \, dx, \quad t \in J,$$

finishing the proof of Theorem 3.1.

### 3.2. Existence of global weak solution in $W^{1, p(x)}_0(B_R)$

In order to prove the existence of a global solution, we replaced $(H_3)$ by the following condition:

There exists $0 < r < R$ such that

$$p_+ < q_+ = \min_{x \in \overline{B}_r(0)} q(x) \leq q_+ = \min_{x \in \overline{B}_r(0)} q(x) \leq \max_{x \in \overline{B}_r(0)} q(x) = q^*_+ < \min_{x \in \overline{B}_r(0)} p^*(x). \quad (H_4)$$
Arguing as in the last section, it is possible to prove that the hypotheses \((H_1) - (H_2)\) and \((H_4)\) are enough to conclude that the compact embedding (3.2) still holds with these assumptions. Associated with the functional \(E\), we have the Nehari manifold given by

\[
\mathcal{N} = \left\{ u \in W_{0,\text{rad}}^{1,p(x)}(B_R) \setminus \{0\} : \int_{B_R} |\nabla u|^{p(x)} \, dx = \int_{B_R} |u|^{q(x)} \, dx \right\}
\]

and the real number

\[
d = \inf \{ E(u) : u \in \mathcal{N} \}.
\]

Hereafter, let us assume that \(u_0 \in W \setminus \{0\}\) where

\[
W := \{ u \in W_{0,\text{rad}}^{1,p(x)}(B_R), \ E(u) < d, \ J(u) > 0 \} \cup \{0\},
\]

with \(J : W_{0,\text{rad}}^{1,p(x)}(B_R) \to \mathbb{R}\) given by

\[
J(u) = \int_{B_R} |\nabla u|^{p(x)} \, dx - \int_{B_R} |u|^{q(x)} \, dx.
\]

One can show easily that the functional \(E\) restricted to \(\mathcal{N}\) is bounded from below, and the assumption \(u_0 \in W \setminus \{0\}\) leads to

\[
d > E(u_0) > \left( \frac{1}{p_+} - \frac{1}{q_-} \right) \int_{\Omega} |\nabla u_0|^{p(x)} \, dx > 0.
\]

**Theorem 3.6. (Global weak solution)** Assume \((H_1) - (H_2), (H_4), \Omega = B_R\) and \(u_0 \in W \setminus \{0\}\). Then, there is a function \(u \in L^\infty((0, +\infty), W_{0,\text{rad}}^{1,p(x)}(B_R))\) with \(u_t \in L^2((0, +\infty), L^2(B_R))\) such that

\[
u(0) = u_0 \in W_{0,\text{rad}}^{1,p(x)}(B_R),
\]

and for each \(v \in W_{0,\text{rad}}^{1,p(x)}(B_R)\)

\[
\int_{B_R} u_t(t)v \, dx + \int_{B_R} |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla v \, dx = \int_{B_R} |u(t)|^{q(x)-2}u(t)v \, dx \quad \text{a.e.} \ t \in (0, +\infty).
\]

**Proof.** Here we are going to use the Galerkin method to prove firstly the existence of global weak solutions for problem (1.1) in \(W_{0,\text{rad}}^{1,p(x)}(B_R)\). After that, we will show that the solution obtained is in fact a global weak solutions for problem (1.1) in \(W_{0,\text{rad}}^{1,p(x)}(B_R)\). Since \(W_{0,\text{rad}}^{1,p(x)}(B_R)\) is separable, and \(W_{0,\text{rad}}^{1,p(x)}(B_R)\) is dense in \(L^2_{\text{rad}}(B_R)\), we have a base \(\mathcal{V} = \{ w_j, \ j \in \mathbb{N} \} \) in \(W_{0,\text{rad}}^{1,p(x)}(B_R)\) and also in \(L^2_{\text{rad}}(B_R)\) such that

\[
\mathcal{V} \equiv \{ w_j, \ j \in \mathbb{N} \} \quad (w_i, w_j)_{L^2_{\text{rad}}(B_R)} = \delta_{i,j}, \ i, j = 1, 2, \ldots
\]

Thereby, as \(u_0 \in W_{0,\text{rad}}^{1,p(x)}(B_R)\), there exists \(\{ a_{im}, \ i = 1, \ldots, m \} \) such that

\[
u_{im}(0) = \sum_{j=1}^{m} a_{im} w_j \to u_0 \text{ in } W_{0,\text{rad}}^{1,p(x)}(B_R).
\]

For each \(m\), we look for the approximate solutions \(u_m(x,t) = \sum_{i=1}^{m} g_{im}(t)w_i(x)\) satisfying the following identities:

\[
\int_{B_R} (u_m)_t w_j \, dx + \int_{B_R} |\nabla u_m|^{p(x)-2}\nabla u_m \nabla w_j \, dx = \int_{B_R} |u_m|^{q(x)-2}u_m w_j \, dx, \ j \in \{1, \ldots, m\}, \tag{3.13}
\]
with the initial condition
\[ u_m(0) = u_{0m}. \]  
(3.14)

Then, (3.13) and (3.14) is equivalent to the following initial value problem for a system of nonlinear ordinary differential equations on \( g_{jm} \):
\[
\begin{cases}
g'_{jm}(t) = H_j(g(t)), & j = 1, 2, \ldots, m, \ t \in [0, t_0], \\
g_{jm}(0) = a_{jm}, & j = 1, 2, \ldots, m,
\end{cases}
\]  
(3.15)

where \( H_j(g(t)) = - \int_{B_R} |\nabla u_m|^{p(x)} - 2 \nabla u_m \nabla \varphi_j \, dx + \int_{B_R} |u_m|^{q(x)} - 2u_m \varphi_j \, dx \). By the Picard iteration method, there is \( t_{0,m} > 0 \) depending on \( |a_{jm}| \) such that problem (3.15) admits a unique local solution \( g_{jm} \in C^1([0, t_{0,m}]) \).

Hereafter, the letters \( c, c_i, C, C_i, i = 1, 2, \ldots, \) denote positive constants which vary from line to line, but they are independent of \( m \).

Multiplying the \( j^{th} \) equation in (3.13) by \( g'_{jm}(t) \) and summing over \( j \) from 1 to \( m \), afterward integrating over \((0, t)\), we find
\[
\int_{0}^{t} \| (u_m)_s(s) \|_2^2 \, ds + E(u_m(t)) = E(u_{0m}), \ t \in [0, t_{0,m}].
\]  
(3.16)

Since \( u_{0m} \) converges to \( u_0 \) strongly in \( W^{1,p(x)}(B_R) \), the continuity of \( E \) on \( W^{1,p(x)}(B_R) \) ensures that
\[ E(u_{0m}) \to E(u_0), \text{ as } m \to +\infty. \]

From the assumption \( E(u_0) < d \), we have \( E(u_{0m}) < d \) for sufficiently large \( m \). This combined with (3.16) yields
\[ E(u_m(t)) < d, \quad t \in [0, t_{0,m}], \]  
(3.17)

for sufficiently large \( m \). We will show that \( t_{0,m} = +\infty \) and
\[ u_m(t) \in W, \quad \forall t \geq 0, \]  
(3.18)

for sufficiently large \( m \). Suppose by contradiction that \( u_m(t_1) \notin W \) for some \( t_1 \in [0, t_{0,m}] \). Let \( t_* \in [0, t_{0,m}] \) be the smallest time for which \( u_m(t_*) \notin W \). Then, by continuity of \( u_m(t) \), we get \( u_m(t_*) \in \partial W \). Hence, it turns out that
\[ E(u_m(t_*)) = d \]  
(3.19)

or
\[ J(u_m(t_*)) = 0. \]  
(3.20)

It is clear that (3.19) could not occur by (3.17) while if (3.20) holds, then by the definition of \( d \),
\[ E(u_m(t_*)) \geq \inf_{u \in \mathcal{N}} E(u) = d, \]
which also is a contradiction with (3.17). Consequently, (3.18) is true.

Combining (3.17) and (3.18),
\[ d > E(u_m(t)) > \left( \frac{1}{p_+} - \frac{1}{q_-} \right) \int_{B_R} |\nabla u_m(t)|^{p(x)} \, dx, \quad t \in [0, t_{0,m}], \]
for sufficiently large \( m \). Moreover, by (3.16),
\[
\int_{0}^{t} \| (u_m)_s(s) \|_2^2 \, ds + \left( \frac{1}{p_+} - \frac{1}{q_-} \right) \int_{B_R} |\nabla u_m(t)|^{p(x)} \, dx < d, \quad t \in [0, t_{0,m}],
\]  
(3.21)
It is well known that \( t_{0,m} = +\infty \). Thus, (3.21) ensures the existence of a function \( u \in L^\infty((0, +\infty), W^{1,p}(B_R)) \) with \( u_t \in L^2((0, +\infty), L^2(B_R)) \) such that for a subsequence of \( \{u_m\}_{m=1}^\infty \), still denoted by \( \{u_m\}_{m=1}^\infty \),

\[
\begin{align*}
&\left\{ \begin{array}{l}
u_m \rightarrow u \\
(u_m)_t \rightharpoonup u_t
\end{array} \right. \\
&\text{in } L^\infty((0, +\infty), W^{1,p}(B_R)),
&(u_m)_t \rightarrow u_t
&\text{in } L^2((0, +\infty), L^2(B_R)),
&-\text{div}(\sqrt[p]{|x|} \nabla u_m) \rightarrow \chi \text{ in } L^\infty((0, +\infty), W^{-1,p}(B_R)).
\end{align*}
\]

(3.22)

Moreover, from [21] [Proposition 1.3] for all \( T > 0 \) we have

\[
u_m \rightarrow u \text{ in } L^2(0,T,L^2(B_R))
\]

(3.23)

and

\[
u_m \rightarrow u \text{ a.e. } (x,t) \in B_R \times (0,T).
\]

(3.24)

The next step is to prove that

\[-\text{div}(\sqrt[p]{|x|} \nabla u) = \chi.\]

(3.25)

It is well known that \( Au = -\text{div}(\sqrt[p]{|x|} \nabla u) \) is bounded, monotone and hemicontinuous from \( W^{1,p}(B_R) \rightarrow W^{-1,p}(B_R) \) (see [8] [Section 13.4, p. 421]). In addition,

\[
\langle Au, u \rangle = \int_{B_R} |\nabla u|^p \, dx, \forall u \in W^{1,p}(B_R).
\]

For this purpose, for each \( T > 0 \) fixed, we can apply [3] [Proposition 2.6] to get

\[
\int_0^T \int_{B_R} |u_m|^q |x|^{-2} u_m v \, dx \, dt \rightarrow \int_0^T \int_{B_R} |u|^q |x|^{-2} u v \, dx \, dt, \forall v \in L^q(B_R).
\]

(3.26)

Moreover, we also claim that

\[
\limsup_{m \rightarrow \infty} \int_0^T \langle Au_m, u_m \rangle \, dt = \int_0^T \langle \chi, u \rangle \, dt.
\]

(3.27)

Indeed, integrating (3.13) over \( (0,T) \),

\[
\int_0^T \int_{B_R} (u_m)_s w_j \, dx \, ds + \int_0^T \langle Au_m, w_j \rangle \, ds = \int_0^T \int_{B_R} |u_m|^q |x|^{-2} u_m w_j \, dx \, ds, \quad j \in \{1, \ldots, m\}.
\]

(3.28)

Therefore, for any fixed \( j \), (3.22), (3.26) and (3.28) give

\[
\int_0^T \int_{B_R} u_s w_j \, dx \, ds + \int_0^T \langle \chi, w_j \rangle \, ds = \int_0^T \int_{B_R} |u|^q |x|^{-2} u w_j \, dx \, ds, \quad j \in \{1, \ldots, m\}.
\]

Accordingly, from the density of \( \mathcal{V} \) in \( W^{1,p}(B_R) \), it follows that for all \( \varphi \in W^{1,p}(B_R) \)

\[
\int_0^T \int_{B_R} u_s \varphi \, dx \, ds + \int_0^T \langle \chi, \varphi \rangle \, ds = \int_0^T \int_{B_R} |u|^q |x|^{-2} u \varphi \, dx \, ds.
\]

(3.29)
On the other hand, multiplying the $j$th equation in (3.13) by $g_{jm}(t)$ and summing over $j$ from 1 to $m$, afterward integrating over $(0, T)$, we get

$$
\frac{1}{2}\|u_m(T)\|_2^2 - \frac{1}{2}\|u_{0m}\|_2^2 + \int_0^T \langle Au_m, u_m \rangle \, dt = \int_0^T \int_{B_R} |u_m|^q(x) \, dx \, dt.
$$

(3.30)

From (3.2), (3.22) and [26][Corollary 4, p. 85],

$$
u_m \to u \text{ in } C([0, T], L^q(x)(B_R)),
$$

and so,

$$
\limsup_{m \to \infty} \int_0^T \langle Au_m, u_m \rangle = -\frac{1}{2}\|u(T)\|_2^2 + \frac{1}{2}\|u_0\|_2^2 + \int_0^T \int_{B_R} |u|^q(x) \, dx \, dt.
$$

(3.31)

Taking $u = \varphi$ in (3.29) and using [21][Proposition 1.2], we obtain

$$
\int_0^T \langle \chi, u \rangle \, dt = -\frac{1}{2}\|u(T)\|_2^2 + \frac{1}{2}\|u_0\|_2^2 + \int_0^T \int_{B_R} |u|^q(x) \, dx \, dt.
$$

(3.32)

Combining (3.31) and (3.32), it follows that (3.27) holds. Now, [25][Lemma 3.2.2, p. 117] together with (3.27) yields (3.25) occurs. Hence, from (3.29),

$$
\int_0^T \int_{B_R} u(t) \varphi \, dx \, dt + \int_0^T \int_{B_R} |\nabla u(t)|^{p(x)-2} \nabla u(t) \nabla \varphi \, dx \, ds = \int_0^T \int_{B_R} |u(t)|^{q(x)-2} u(t) \varphi \, dx \, ds,
$$

for all $\varphi \in W^{1,p(x)}(B_R)$. Therefore,

$$
\int_{B_R} u(t) \varphi \, dx + \int_{B_R} |\nabla u(t)|^{p(x)-2} \nabla u(t) \nabla \varphi \, dx = \int_{B_R} |u(t)|^{q(x)-2} u(t) \varphi \, dx, \text{ a.e. in } t \in (0, +\infty),
$$

(3.33)

for all $\varphi \in W^{1,p(x)}(B_R)$. From (3.22) and [25][Lemma 3.1.7],

$$
u_m(0) \to u(0) \text{ weakly in } L^2_{rad}(B_R).
$$

However, by (3.12), we know that $u_m(0) \to u_0$ in $W^{1,p(x)}_{0,rad}(B_R)$, in particular $u_m(0) \to u_0$ in $L^2_{rad}(B_R)$, and so, $u(0) = u_0$ in $L^2_{rad}(B_R)$, from where it follows that $u(0) = u_0$. This shows that $u$ satisfies the initial condition.

Finally, arguing as in Sect. 3.1, we must have that for all $\varphi \in W^{1,p(x)}(B_R)$

$$
\int_{B_R} u(t) \varphi \, dx + \int_{B_R} |\nabla u(t)|^{p(x)-2} \nabla u(t) \nabla \varphi \, dx = \int_{B_R} |u(t)|^{q(x)-2} u(t) \varphi \, dx, \text{ a.e. in } t \in (0, +\infty).
$$

(3.34)

\[\square\]

4. Existence of solution for the non-radial case

In this section, we will assume the following condition on $\Omega$:

There are $0 < r < R$ such that $A_{R,r} \subset \Omega$ where $A_{R,r} = \overline{B}_R(0) \setminus B_r(0)$. $((\Omega)_{R,r})$

It is easy to see that $\Omega$ is not necessarily a ball centered at origin.
Related to the functions $p,q: \overline{\Omega} \rightarrow \mathbb{R}$, we assume that they are continuous functions and satisfy the following conditions:

$$2 \leq p_- = \min_{x \in \overline{\Omega}} p(x) \leq \max_{x \in \overline{\Omega}} p(x) = p_+ < N \quad \text{and} \quad 1 < q_- = \min_{x \in \overline{\Omega}} q(x) \quad (H_5)$$

and

$$p(x) = p(|x|) \quad \text{and} \quad q(x) = q(|x|), \quad \forall x \in \overline{A}_{R,r}, \quad (H_6)$$

4.1. Existence of local solution in $W_{0}^{1,p(x)}(\Omega)$

In order to show the existence of local solution, in addition to conditions ($H_5$) – ($H_6$), we also assume

$$1 < q_- \leq q(x) \leq \min \left\{ p'(x), 2 + \frac{p'(x)}{2} \right\}, \quad \forall x \in \overline{\Omega} \setminus A_{R,r}. \quad (H_7)$$

We point out that we are not assuming any growth condition on $q$ in the annulus $A_{R,r}$; hence, as in the previous section $q$ can have a triple regime.

Hereafter, we will consider the following subspace of $W_{0}^{1,p(x)}(\Omega)$ given by

$$X = \{ u \in W_{0}^{1,p(x)}(\Omega) : u(x) = u(|x|) \ \text{a.e.} \ \ x \in \overline{A}_{R,r} \}.$$ 

Arguing as in Sect. 2, the compact embedding (3.2) still holds with $W_{0}^{1,p(x)}(B_{R})$ replaced by $X$.

**Theorem 4.1.** (Local weak solution) Assume ($H_5$) – ($H_7$), $(\Omega)_{R,r}$ and $u_0 \in X$. Then, there exist $T > 0$ and a function $u \in L^{\infty}([0,T],W_{0}^{1,p(x)}(\Omega))$ with $u_0 \in L^{2}((0,T),L^{2}(\Omega))$ that satisfies the initial condition

$$u(0) = u_0 \ \text{and} \ W_{0}^{1,p(x)}(\Omega),$$

and for each $v \in W_{0}^{1,p(x)}(\Omega)$

$$\int_{\Omega} u_t(t)v \, dx + \int_{\Omega} |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla v \, dx = \int_{\Omega} |u(t)|^{q(x)-2}u(t)v \, dx, \quad \text{a.e.} \ t \in (0,T). \quad (4.1)$$

**Proof.** The existence of local solution in the present case follows as in Sect. 3.1, because $W_{0}^{1,p(x)}(B_{R})$ and $X$ have the same continuous and compact embedding. Therefore, there is a $u$ that verifies the equality below

$$\int_{\Omega} u_t(t)v \, dx + \int_{\Omega} |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla v \, dx = \int_{\Omega} |u(t)|^{q(x)-2}u(t)v \, dx, \quad \text{a.e. in } t \in (0,T) \ \text{for all } v \in X.$$ 

Our goal is proving that

$$\int_{\Omega} u_t(t)v \, dx + \int_{\Omega} |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla v \, dx = \int_{\Omega} |u(t)|^{q(x)-2}u(t)v \, dx, \quad \text{a.e. in } t \in (0,T) \ \text{for all } v \in W_{0}^{1,p(x)}(\Omega).$$

For this purpose, we cannot use the Palais principle used in Sect. 3, because $\Omega$ is not a ball. Hereafter, we will use the approach found in Alves and Rădulescu [1] [Section 3]. Hereafter, we fix $J \subset [0,T]$ with $med(J^c) = 0$ such that (4.3) holds for all $t \in J$.

To begin with, for all $\varphi \in X_{0}(A_{R,r}) = \{ u \in X : u = 0 \ \text{on} \ \partial(A_{R,r}) \}$ we have

$$\int_{A_{R,r}} u_t(t)\varphi \, dx + \int_{A_{R,r}} |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla \varphi \, dx = \int_{A_{R,r}} |u(t)|^{q(x)-2}u(t)\varphi \, dx.$$
By using regularity theory, it is possible to show that for all \( \psi \in E_0(A_{R,r}) \)
\[
\int_{A_{R,r}} u_t(t)\varphi \, dx + \int_{A_{R,r}} |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla \varphi \, dx = \int_{A_{R,r}} |u(t)|^{q(x)-2}u(t)\varphi \, dx,
\]
(4.4)
where \( E_0(A_{R,r}) = \{ u \in W^{1,p(x)}(\Omega) : u = 0 \text{ on } \partial(A_{R,r}) \} \).

Using the above information, we are ready to prove (4.3). Have this in mind, let us set an even function \( \phi \in C^\infty(\mathbb{R}, \mathbb{R}) \) satisfying
\[ 0 \leq \phi(s) \leq 1 \quad \forall s \in \mathbb{R}, \quad \phi(s) = 0 \quad \forall s \in [-1, 1] \quad \text{and} \quad \phi(s) = 1 \quad \forall s \in (-\infty, -2] \cup [2, +\infty). \]

Thereby, for \( \epsilon > 0 \) small enough and \( v \in C^\infty_0(B_R) \subset E_0(B_R) \), where
\[ E_0(B_R) = \{ u \in W^{1,p(x)}(\Omega) : u = 0 \text{ on } \partial(B_R) \}, \]
we define the function
\[ v_\epsilon(x) = \phi\left(\frac{|x| - r}{\epsilon}\right)v(x), \quad \forall x \in \Omega. \]

From the definition of \( v_\epsilon \), we see that \( v_\epsilon \in E_0(B_r(0)) \subset X \) and \( v_\epsilon \in E_0(A_{R,r}) \). Thus, by (4.2) and (4.4),
\[
\int_{B_r} u_t(t)v_\epsilon \, dx + \int_{B_r} |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla v_\epsilon \, dx = \int_{B_r} |u(t)|^{q(x)-2}u(t)v_\epsilon \, dx
\]
and
\[
\int_{A_{R,r}} u_t(t)v_\epsilon \, dx + \int_{A_{R,r}} |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla v_\epsilon \, dx = \int_{A_{R,r}} |u(t)|^{q(x)-2}u(t)v_\epsilon \, dx,
\]
which leads to
\[
\int_\Omega u_t(t)v_\epsilon \, dx + \int_\Omega |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla v_\epsilon \, dx = \int_\Omega |u(t)|^{q(x)-2}u(t)v_\epsilon \, dx.
\]

After some calculus, taking the limit \( \epsilon \to 0 \), we get
\[
\int_\Omega u_t(t)v \, dx + \int_\Omega |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla v \, dx + \int_\Omega u(t)v \, dx = \int_\Omega |u(t)|^{q(x)-2}u(t)v \, dx.
\]

Now, we are going to show that the above equality holds for any \( w \in C^\infty_0(\Omega) \subset W^{1,p(x)}_0(\Omega) \). The idea is as above; we set the function
\[ w_\epsilon(x) = \phi\left(\frac{|x| - R}{\epsilon}\right)w(x), \quad \forall x \in \Omega, \]
which belongs to \( E_0(B_R) \) and \( E_0(\Omega \setminus \overline{B_R}) \subset W^{1,p(x)}(\Omega) \). Since \( w_\epsilon|_{\Omega \setminus \overline{B_R}} \in W^{1,p(x)}(\Omega) \),
\[
\int_\Omega u_t(t)w_\epsilon \, dx + \int_\Omega |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla w_\epsilon \, dx = \int_\Omega |u(t)|^{q(x)-2}u(t)w_\epsilon \, dx.
\]
Taking the limit \( \epsilon \to 0 \), we obtain
\[
\int_\Omega u_t(t)w \, dx + \int_\Omega |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla w \, dx = \int_\Omega |u(t)|^{q(x)-2}u(t)w \, dx.
\]
Again by density,
\[
\int_\Omega u_t(t)w \, dx + \int_\Omega |\nabla u(t)|^{p(x)-2}\nabla u(t)\nabla w \, dx = \int_\Omega |u(t)|^{q(x)-2}u(t)w \, dx,
\]
for all \( w \in W^{1,p(x)}_0(\Omega) \). This shows that \( u \) satisfies (4.3). \( \square \)
Before concluding this subsection, we would like point out that by Lemma 3.2,

\[ u \in C([0, T], W_0^{1,p(x)}(\Omega)) \cap C([0, T], L^{q(x)}(\Omega)) \]

and

\[
\int_0^t \|u_s(s)\|_{L^2(\Omega)}^2 + E(u(t)) = E(u_0), \quad t \in [0, T].
\]

These information will be used later on.

### 4.2. Existence of global solution

In order to prove the existence of global solution, we replaced \((H_7)\) by

\[
p_+ < q_-^A = \min_{x \in \overline{\Omega} \setminus A_{R,r}} q(x) \leq q_+^A = \max_{x \in \overline{\Omega} \setminus A_{R,r}} q(x) < \min_{x \in \overline{\Omega}} p^*(x). \tag{H_8}
\]

Hereafter, \(W\) and \(J\) are defined as in (3.9) and (3.10), respectively, with \(W_0^{1,p(x)}(B_R)\) replaced by \(X\).

**Theorem 4.2.** (Global weak solution) Assume \((H_5) - (H_6), (H_8), (\Omega_{R,r})\) and \(u_0 \in W \setminus \{0\}\). Then, there is a function \(u \in L^\infty((0, +\infty), W_0^{1,p(x)}(\Omega))\) with \(u_t \in L^2((0, +\infty), L^2(\Omega))\) that satisfies the initial condition

\[ u(0) = u_0 \in W_0^{1,p(x)}(\Omega), \]

and for each \(v \in W_0^{1,p(x)}(\Omega)\)

\[
\int_\Omega u_t(t)v \, dx + \int_\Omega \nabla u(t) \nabla v \, dx = \int_\Omega |u(t)|^{q(x)-2}u(t)v \, dx \quad \text{a.e.} \quad t \in (0, +\infty).
\]

**Proof.** Since \(W_0^{1,p(x)}(B_R)\) and \(X\) have the same compact embedding, we can repeat the arguments explored in Sect. 3.2 to guarantee the existence of a function \(u\) such that \(u(0) = u_0\) in \(W_0^{1,p(x)}(\Omega)\) and for all \(\varphi \in W_0^{1,p(x)}(\Omega)\)

\[
\int_\Omega u_t(t)\varphi \, dx + \int_\Omega |\nabla u(t)|^{p(x)-2}\nabla u(t) \nabla \varphi \, dx = \int_\Omega |u(t)|^{q(x)-2}u(t)\varphi \, dx, \quad \text{a.e.} \quad t \in (0, +\infty).
\]

Now, we repeat the same approach used in Sect. 4.1 to conclude that for all \(\varphi \in W_0^{1,p(x)}(\Omega)\)

\[
\int_\Omega u_t(t)\varphi \, dx + \int_\Omega |\nabla u(t)|^{p(x)-2}\nabla u(t) \nabla \varphi \, dx = \int_\Omega |u(t)|^{q(x)-2}u(t)\varphi \, dx, \quad \text{a.e.} \quad t \in (0, +\infty). \tag{4.6}
\]

\[ \square \]

### 5. Blow-up of the solution

In order to prove the blow-up phenomena, we assume that \((\Omega)_{R,r}\) holds and that the functions \(p\) and \(q\) satisfy \((H_5) - (H_6)\) and the following condition

\[ 2 \leq p_+ < q_- \leq q(x) \leq \min \left\{ p^*(x), 1 + \frac{p^*(x)}{2} \right\}, \quad \forall x \in \overline{\Omega} \setminus A_{R,r}. \tag{H_9} \]

The main result this section is the following
Theorem 5.1. \textbf{(Blow-up phenomena)} Assume \((H_5) - (H_6), (H_0), (\Omega)_{\mathcal{R},r}\) and \(u_0 \in X\) with \(E(u_0) < 0\). Then, there exist \(T_{\text{max}} > 0\) and a function \(u \in L^\infty([0,T],W^{1,p(x)}_0(\Omega))\) with \(u_t \in L^2((0,T),L^2(\Omega))\) for all \(0 < T < T_{\text{max}}\) that satisfies the initial condition

\[ u(0) = u_0 \in W^{1,p(x)}_0(\Omega), \]

and for each \(v \in W^{1,p(x)}_0(\Omega)\)

\[
\int_\Omega u_t(t)v \, dx + \int_\Omega |\nabla u(t)|^{p(x)-2}\nabla u \nabla v \, dx = \int_\Omega |u(t)|^{q(x)-2}u(t)v \, dx, \quad \text{a.e. } t \in (0,T).
\]

Moreover, \(\lim_{t \to T_{\text{max}}} \|u(t)\|_{L^2(\Omega)} = +\infty\).

\textbf{Proof.} Let \(u\) be the local solution of problem (1.1). Set

\[ g(t) = \frac{1}{2} \|u(t)\|_2^2. \]

By [21][Proposition 1.2],

\[ g'(t) = \int_\Omega u_t(t)u(t) \, dx. \]

Taking \(u = v\) in (4.1) and \(\mu \in (p_-, q_-)\), it follows that

\[
g'(t) = - \int_\Omega |\nabla u|^{p(x)} \, dx + \int_\Omega |u|^{q(x)} \, dx \geq -\mu E(u) + \left(1 - \frac{\mu}{q_-}\right) \int_\Omega |u|^{q(x)} \, dx \geq G(t),
\]

where \(G(t) = -\mu E(u)\). From (4.5),

\[ G'(t) = \mu \|u_t(t)\|_{L^2(\Omega)}^2, \]

then by Hölder inequality

\[
g(t)G'(t) = \frac{\mu}{2} \|u_t(t)\|_2^2 \|u(t)\|_2^2 \geq \frac{\mu}{2} \left( \int_\Omega u_t(t)u(t) \, dx \right)^2 = \frac{\mu}{2} |g'(t)|^2.
\]

Since \(E(u_0) < 0\) and \(E(u(t)) \leq E(u_0)\) for all \(t \geq 0\), it follows that \(G(t) > G(0) > 0\) for all \(t > 0\). Hence, from (5.1),

\[
g(t)G'(t) \geq \frac{\mu}{2} g'(t)G(t),
\]

or equivalently

\[
\frac{G'(t)}{G(t)} \geq \frac{\mu}{2} \frac{g'(t)}{g(t)}.
\]

Integrating over \((0,t)\) and using (5.1), we get

\[
g'(t)|g(t)|^{-\frac{q}{2}} \geq \frac{G(0)}{|g(0)|^\frac{q}{2}}. \tag{5.3}
\]

Since \(\mu > 2\), inequality (5.3) implies that

\[
g(t) \geq \left( |g(0)|^{1-\frac{q}{2}} - \left( \frac{\mu - 2}{2} \frac{G(0)}{|g(0)|^\frac{q}{2}} \right) t \right)^{-\frac{q}{2-\mu}}.
\]
Therefore,

\[
\lim_{t \to T_{\max}} \| u(t) \|_2 = +\infty, \quad \text{with} \quad T_{\max} = \frac{2g(0)}{(\mu - 2)G(0)},
\]

showing the desired result. \(\square\)

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