On rate optimal local estimation in functional linear model

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Abstract

We consider the problem of estimating for a given representer $h$ the value $\ell_h(\beta)$ of a linear functional of the slope parameter $\beta$ in functional linear regression, where scalar responses $Y_1,\ldots,Y_n$ are modeled in dependence of random functions $X_1,\ldots,X_n$. The proposed estimators of $\ell_h(\beta)$ are based on dimension reduction and additional thresholding. The minimax optimal rate of convergence of the estimator is derived assuming that the slope parameter and the representer belong to some ellipsoid which are in a certain sense linked to the covariance operator associated to the regressor. We illustrate these results by considering Sobolev ellipsoids and finitely or infinitely smoothing covariance operator.

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1 Introduction

Functional data analysis (Ramsay and Silverman [2005] and Ferraty and Vieu [2006]) has become very important in a diverse range of disciplines including chemometrics (Frank and Friedman [1993]), econometrics (Forni and Reichlin [1998] and Preda and Saporta [2005]), biometry or climatology (Besse et al. [2000]). Roughly speaking, in all these applications the dependence of a scalar response variable, say $Y \in \mathbb{R}$, on the variation of an explanatory random function $X$ is modeled by

$$Y = \int_0^1 \beta(t)X(t)dt + \sigma \varepsilon, \quad \sigma > 0,$$

for some slope function $\beta$ and error term $\varepsilon$. In recent years the nonparametric estimation of the slope function $\beta$ given a sample of $(Y,X)$ has been of growing interest in the literature. For example, Bosq [2000], Cardot et al. [2007] or Müller and Stadtmüller [2005] consider a functional principal components regression, while a penalized least squares approach combined with projection onto some basis (such as splines) is studied in Ramsay and Dalzell [1991], Eilers and Marx [1996], Cardot et al. [2003], Hall and Horowitz [2007] or Crambes et al. [2009]. However, the nonparametric estimation of $\beta$ leads in general to

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an ill-posed inverse problem and hence all the proposed estimators have under reasonable assumptions very poor rates of convergence. In other words, even relatively large sample sizes may not be of much help in accurately estimating \( \beta \). In contrast, it might be possible to estimate certain local features of \( \beta \), such as the value of a linear functional \( \ell_h(\beta) := \int_0^1 \beta(t)h(t)dt \) with respect to some given representer \( h \), at the usual parametric rate of convergence. For example, rather than estimating the slope parameter \( \beta \) itself one may be interested in its average value \( \int_0^1 \beta(t)dt \) over a certain interval \([a,b]\). Then it is of interest to characterize the attainable accuracy of any estimator, for example, in terms of the mean squared error (MSE), which obviously depends on the representer \( h \) and the conditions imposed on \( \beta \). It is worth noting, that the nonparametric estimation of the value of a linear functional from Gaussian white noise observations is a subject of considerable literature (c.f. Speckman [1979], Li [1982] or Ibragimov and Has’minskii [1984] in case of direct observations, while in case of indirect observations we refer to Donoho and Low [1992], Donoho [1994] or Goldenshluger and Pereverzev [2000] and references therein). However, as far as we know this question has not yet been addressed in functional linear regression, which in general is not a Gaussian white noise model. The objective of this paper is the nonparametric estimation of the value \( \ell_h(\beta) \) of a linear functional based on an independent and identically distributed (i.i.d.) sample of \((Y,X)\) obeying (1.1).

In this paper we suppose that the random function \( X \) is taking its values in \( L^2[0,1] \), which is endowed with the usual norm \( \| \cdot \| \) and inner product \( \langle \cdot, \cdot \rangle \), and that \( X \) has a finite second moment, i.e., \( E\|X\|^2 < \infty \). In order to simplify notations we assume that the mean function of \( X \) is zero. Moreover, the random function \( X \) and the error term \( \varepsilon \) are uncorrelated, where \( \varepsilon \) has mean zero and variance one. This situation has been considered, for example, in Bosq [2000], Cardot et al. [2003] or Cardot et al. [2007]. Then multiplying both sides in (1.1) by \( X(s) \) and taking the expectation leads to the continuous equivalent of the normal equation in a classical multivariate linear model. That is,

\[
g(s) := E[YX(s)] = \int_0^1 \beta(t) \text{Cov}(X(t), X(s))dt =: [T_{\text{cov}} \beta](s), \quad s \in [0,1],
\]

where \( g \) belongs to \( L^2[0,1] \) and \( T_{\text{cov}} \) denotes the covariance operator associated to the random function \( X \). Estimation of \( \beta \) is thus linked with the inversion of the covariance operator \( T_{\text{cov}} \) of \( X \) and, hence called an inverse problem. Moreover, due to the finite second moment of the regressor \( X \) the associated covariance operator \( T_{\text{cov}} \) is a non negative nuclear operator (c.f. Dauxois et al. [1982]). Consequently, unlike in a multivariate linear model, a continuous generalized inverse of \( T_{\text{cov}} \) does not exist as long as the range of the operator \( T_{\text{cov}} \) is an infinite dimensional subspace of \( L^2[0,1] \). This corresponds to the setup of ill-posed inverse problems (with the additional difficulty that \( T_{\text{cov}} \) is unknown and, hence has to be estimated). In what follows we always assume that there exists a unique solution \( \beta \in L^2[0,1] \) of equation (1.2), i.e., \( g \) belongs to the range \( \mathcal{R}(T_{\text{cov}}) \) of \( T_{\text{cov}} \), and that the null space \( \mathcal{N}(T_{\text{cov}}) \) of \( T_{\text{cov}} \) is trivial or equivalently \( T_{\text{cov}} \) is strictly positive (for a detailed discussion in the context of inverse problems see Chapter 2.1 in Engl et al. [2000], while in the special case of a functional linear model we refer to Cardot et al. [2003]). Furthermore, we suppose that the representer \( h \) of the linear functional \( \ell_h \) of interest is an element of \( L^2[0,1] \) as well. Then it is straightforward to see, that the value of the linear functional \( \ell_h(\beta) \) is identified if and only if \( h \) belongs to the orthogonal complement \( \mathcal{N}(T_{\text{cov}})^\perp \) of the null space \( \mathcal{N}(T_{\text{cov}}) \). Hence, for all \( h \in L^2[0,1] \) the identification is in particular guaranteed under the assumption of a strictly positive covariance operator \( T_{\text{cov}} \).
In this paper we follow an often in the literature used approach to construct an estimator of the value of a linear functional. That is, we replace in $\ell_h(\beta)$ the unknown slope function $\beta$ by an estimator. In the particular case of second order stationary regressors (defined below) the considered estimator of $\beta$ is just an orthogonal series estimator with an additional thresholding in the Fourier domain (Johannes [2009b]). Note that over relatively short periods of time, the assumption of second order stationarity is in many situations realistic and moreover it can be checked from the data by estimating the covariance function using the multiple realizations of $X$. It is remarkable, that in this situation under mild moment assumptions the obtained plug-in estimator of $\ell_h(\beta)$ attains minimax-optimal rates of convergence in terms of the MSE over a wide range of ellipsoids (defined below) characterizing the prior information about slope parameter and representer respectively, and which are linked (defined below) to the covariance operator $T_{\text{cov}}$. In particular, we illustrate these results by considering Sobolev ellipsoids and finitely or infinitely smoothing covariance operator. However, in a second step we drop this assumption and we no longer suppose that the regressor is second order stationary. In this general setting the estimator of $\beta$ is based on a dimension reduction together with an additional thresholding (Cardot and Johannes [2008]). Then we show under stronger moment assumptions that the plug-in estimator of $\ell_h(\beta)$ still attains minimax-optimal rates of convergence in terms of the MSE but only over a more restrictive range of ellipsoids for $\beta$ and $h$ respectively.

The paper is organized in the following way. In Section 2 we introduce our basic assumptions and derive a lower bound for estimating the value of a linear functional based on an i.i.d. sample obeying the functional linear model (1.1). In Section 3 under the assumption of second order stationarity we show first consistency of the proposed estimator and second its minimax-optimality. The general case without the second order stationarity assumption is then considered in Section 4. All proofs can be found in the Appendix.

2 Complexity of local estimation: a lower bound.

2.1 Notations and assumptions.

In this section we show that the obtainable accuracy of any estimator of the value $\ell_h(\beta)$ of a linear functional can be essentially determined by additional regularity conditions imposed on the slope parameter $\beta$, the representer $h$ and the covariance operator $T_{\text{cov}}$. In this paper these conditions are characterized through different weighted norms in $L^2[0,1]$ with respect to a pre-specified orthonormal basis $\{\psi_j, j \in \mathbb{N}\}$ in $L^2[0,1]$, which we formalize now. We shall stress that this basis corresponds not necessarily to the eigenfunctions of $T_{\text{cov}}$. Then given a strictly positive sequence of weights $w := (w_j)_{j \geq 1}$ and a constant $c > 0$ denote for all $r \in \mathbb{R}$ by $F_{w,r}$ the ellipsoid given by

$$F_{w,r} := \left\{ f \in L^2[0,1] : \sum_{j=1}^{\infty} w_j |\langle f, \psi_j \rangle|^2 =: \|f\|_{w,r}^2 \leq c \right\}. \quad (2.1)$$

Furthermore let $F_{w,r} := \{ f \in L^2[0,1] : \|f\|_{w,r}^2 < \infty \}$. It is worth to note, that in case $w \equiv 1$ the set $F_{w}^{c}$ denotes an ellipsoid in $L^2[0,1]$ and hence does not impose additional restrictions.

Minimal regularity conditions. Let $\gamma := (\gamma_j)_{j \geq 1}$ and $\omega := (\omega_j)_{j \geq 1}$ denote two sequences of weights. Then we suppose, here and subsequently, that the slope function $\beta$
belonging to the ellipsoid $\mathcal{F}_\rho^\ell$ for some $\rho > 0$ and that the representor $h$ of the linear functional $\ell_h$ is an element of the ellipsoid $\mathcal{F}_\tau^\omega$ for some $\tau > 0$. The ellipsoids $\mathcal{F}_\rho^\ell$ and $\mathcal{F}_\omega^\tau$ capture all the prior information (such as smoothness) about the unknown slope function $\beta$ and the given representor $h$ respectively. Furthermore, as usual in the context of ill-posed inverse problems, we link the mapping properties of the covariance operator $T_{\text{cov}}$ and the regularity conditions on $\beta$ and $h$. Therefore, consider the sequence $((T_{\text{cov}}\psi_j, \psi_j))_{j \geq 1}$, which is summable and hence converges to zero since $T_{\text{cov}}$ is nuclear. In what follows we impose restrictions on the decay of this sequence. Denote by $\mathcal{N}_\nu^d$ the set of all strictly positive nuclear operator defined on $L^2(0, 1)$. Then given a sequence of weights $v := (v_j)_{j \geq 1}$ and $d \geq 1$ define the subset $\mathcal{N}_\nu^d$ of $\mathcal{N}$ by

$$\mathcal{N}_\nu^d := \left\{ T \in \mathcal{N} : \|\psi\|_{L^2}^2 / d^2 \leq \|T \psi\|^2 \leq d^2 \|\psi\|^2_{L^2}, \quad \forall \psi \in L^2(0, 1) \right\}. \quad (2.2)$$

Notice that for all $T \in \mathcal{N}_\nu^d$ it follows that $\langle T \psi_j, \psi_j \rangle \asymp_d v_j$. Hence, the sequence $(v_j)_{j \geq 1}$ has to be strictly positive and summable since $T$ is strictly positive and nuclear. Moreover, if $\{\lambda_j, \psi_j, j \geq 1\}$ is a spectral decomposition of $T \in \mathcal{N}$. Then the condition $T \in \mathcal{N}_\nu^d$ is satisfied if and only if $\lambda_j \asymp_d v_j$. In what follows the results are derived under regularity conditions on the slope parameter $\beta$, the representer $h$ and the covariance operator $T_{\text{cov}}$ described through the sequence $\gamma$, $\omega$ and $v$ respectively. However, we provide below illustrations of these conditions by assuming a “regular decay” of these sequences. The next assumption summarizes our minimal regularity conditions on these sequences.

**Assumption 2.1.** Let $\gamma := (\gamma_j)_{j \geq 1}$, $\omega := (\omega_j)_{j \geq 1}$ and $v := (v_j)_{j \geq 1}$ be strictly positive sequences of weights with $\gamma_1 = 1$, $\omega_1 = 1$ and $v_1 = 1$ such that $\gamma$ and $\omega$ are nondecreasing and $v$ is nonincreasing with $\Lambda := \sum_j v_j < \infty$. Furthermore, there exists a constant $D \geq 1$ such that $\sup_{1 \leq j \leq m} \{1/(v_j^k \omega_j)\} \leq D \max(1/(\omega_{m+1}^k), 1)$ for all $m \in \mathbb{N}$ and $k = 1, 2$.

We shall stress that $\mathcal{F}_\rho^\ell$ is just an ellipsoid in $L^2(0, 1)$ in case $\gamma \equiv 1$, hence in this situation there is not an additional regularity condition on the slope parameter $\beta$ imposed. Furthermore, the last condition in Assumption 2.1 is obviously satisfied with $D = 1$ if the sequence $(v_j^k \omega_j)$ is either monotonically decreasing or increasing.

**Matrix and operator notations.** Given $m \geq 1$, $\Psi_m$ denotes the subspace of $L^2(0, 1)$ spanned by the functions $\{\psi_1, \ldots, \psi_m\}$. $\Pi_m$ and $\Pi_m^\perp$ denote the orthogonal projections on $\Psi_m$ and its orthogonal complement $\Psi_m^\perp$, respectively. Given an operator (matrix) $K$, $\|K\|$ denotes its operator norm. The inverse operator (matrix) of $K$ is denoted by $K^{-1}$. The identity operator (matrix) is denoted by $I$ and the diagonal matrix with vector of entries $v$ is denoted by Diag($v$). $[f]$ and $[K]$ denote the (infinite) vector and matrix of the function $f$ and the operator $K$ with entries $[f]_j = \langle f, \psi_j \rangle$ and $[K]_{j,l} = \langle K \psi_j, \psi_l \rangle$ respectively. The upper $m$ subvector and $m \times m$ submatrix of $[f]$ and $[K]$ is denoted by $[f]^m$ and $[K]_{m,m}$, respectively. Clearly, $[\Pi_m f]^m = [f]^m$ and if we restrict $\Pi_m K \Pi_m$ to an operator from $\Psi_m$ into itself, then it has the matrix $[K]_{m,m}$.

Consider the covariance operator $T_{\text{cov}}$ given in (1.2). We assume throughout the paper that $T_{\text{cov}}$ is strictly positive definite and hence the matrix $[T_{\text{cov}}]^m_{m}$ is nonsingular for all $m \in \mathbb{N}$, so that $[T_{\text{cov}}]^{-1}_{m}$ always exists. Under this assumption the notation $(T_{\text{cov}})^{-1}$ is used for the operator from $L^2(0, 1)$ into itself, whose matrix in the basis $\{\psi_j\}$ has the entries $([T_{\text{cov}}]^{-1})_{j,l}$ for $1 \leq j, l \leq m$ and zeros otherwise.

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1We write $a \asymp_d b$ if $d^{-1} \leq b/a \leq d$. 

Moment assumptions. The results derived below involve additional conditions on the moments of the random function \(X\), which we formalize now. Since \([T_{\text{cov}}]_m\) is nonsingular it follows that the random vector \([T_{\text{cov}}]_m^{-1/2}[X]_m\) has uncorrelated entries \(([T_{\text{cov}}]_m^{-1/2}[X]_m)_j, j = 1, \ldots, m\), with mean zero and variance one. Thereby, for all \(m \in \mathbb{N}\) and \(z \in \mathbb{S}^m := \{z \in \mathbb{R}^m : z^\top z = 1\}\) the centered random variable \(\sum_{j=1}^m z_j([T_{\text{cov}}]_m^{-1/2}[X]_m)_j\) has variance one too. Furthermore, \([T_{\text{cov}}]_j, j\) is the variance of the centered random variable \([X]_j, j \in \mathbb{N}\), and \(\text{tr}(T_{\text{cov}}) = \sum_{j \in \mathbb{N}}[T_{\text{cov}}]_{j,j} = \mathbb{E}\|X\|^2 < \infty\). Let \(\mathcal{X}\) be the set of all centered random functions \(X\) with finite second moment, i.e., the associated covariance operator \(T_{\text{cov}}\) satisfies \(\text{tr}(T_{\text{cov}}) < \infty\). Here and subsequently, we denote by \(\mathcal{X}_k^\eta, k \in \mathbb{N}, \eta \geq 1\), the subset of \(\mathcal{X}\) containing only random functions \(X\) such that the \(k\)-th moment of the corresponding random variables \([X]_j/[T_{\text{cov}}]_{j,j}, j \in \mathbb{N}\), and \(\sum_{j=1}^m z_j([T_{\text{cov}}]_m^{-1/2}[X]_m)_j\) are uniformly bounded in \(z \in \mathbb{S}^m\) and \(m \in \mathbb{N}\), that is

\[
\mathcal{X}_k^\eta := \left\{X \in \mathcal{X} \text{ such that } \sup_{j \in \mathbb{N}} \mathbb{E}\left|\frac{[X]_j}{[T_{\text{cov}}]_{j,j}}\right|^k \leq \eta \right\},
\]

and

\[
\sup_{m \in \mathbb{N}} \sup_{z \in \mathbb{S}^m} \mathbb{E}\left|\sum_{j=1}^m z_j([T_{\text{cov}}]_m^{-1/2}[X]_m)_j\right|^k \leq \eta.
\]

(2.3)

It is worth noting that in case \(X \in \mathcal{X}\) is a Gaussian random function the corresponding random variables \([X]_j/[T_{\text{cov}}]_{j,j}, j \in \mathbb{N}\), and \(\sum_{j=1}^m z_j([T_{\text{cov}}]_m^{-1/2}[X]_m)_j\), \(z \in \mathbb{S}^m, m \in \mathbb{N}\), are Gaussian with mean zero and variance one. Hence, for each \(k \in \mathbb{N}\) there exists \(\eta\) such that any Gaussian random function \(X \in \mathcal{X}\) belongs also to \(\mathcal{X}_k^\eta\). In what follows, \(\mathcal{E}_\eta^k\) stands for the set of all centered error terms \(\varepsilon\) with variance one and finite \(k\)-th moment, i.e., \(\mathbb{E}|\varepsilon|^{k} \leq \eta\).

2.2 The lower bound.

In the proof of the next theorem we show that a one-dimensional subproblem captures the full difficulty in estimating a linear functional \(\ell_h(\beta)\) of the slope parameter \(\beta\). In other words, there exist two sequences of slope functions \(\beta_{1,n}, \beta_{2,n} \in \mathcal{F}_n\), which are statistically not consistently distinguishable, and a sequence of representers \(h_n \in \mathcal{F}_n\) such that \(|\ell_h_n(\beta_{1,n}) - \ell_h_n(\beta_{2,n})|^2 \geq C\delta_n^*\), where \(\delta_n^*\) is the optimal rate of convergence. Moreover, we obtain the following lower bound under the additional assumption that the error term \(\varepsilon\) is standard normal distributed, i.e., \(\varepsilon \sim \mathcal{N}(0,1)\), and independent of the regressor \(X\).

**Theorem 2.1.** Assume an \(n\)-sample of \((Y,X)\) obeying (1.1) with \(\sigma > 0\). Suppose that the error term \(\varepsilon \sim \mathcal{N}(0,1)\) and the regressor \(X \in \mathcal{X}_k^\eta, \eta \geq 1, k \in \mathbb{N}\), with associated covariance operator \(T_{\text{cov}} \in \mathcal{N}_d^q, d \geq 1\), are independent. Let \(m_* := m_*(n) \in \mathbb{N}\) and \(\delta_n^* := \delta_n^*(m_*) \in \mathbb{R}^+\) be such that for some \(\Delta \geq 1\) hold

\[
1/\Delta \leq \frac{\gamma m_*}{n v_m} \leq \Delta \quad \text{and} \quad \delta_n^* := \gamma^{-1}_{m_*} \omega^{-1}_{m_*}.
\]

(2.4)

If the sequences \(\gamma, \omega\) and \(\nu\) satisfy in addition the Assumption 2.1, then for any estimator \(\hat{\ell}\) we have

\[
\sup_{\beta \in \mathcal{F}_n} \sup_{h \in \mathcal{F}_n} \left\{\mathbb{E}|\hat{\ell} - \ell_h(\beta)|^2\right\} \geq \frac{\tau}{4\Delta} \min\left(\frac{\sigma^2}{2d}, \frac{\rho}{\Delta}\right) \max\left(\delta_n^*, n^{-1}\right).
\]
The normality and independence assumption of the error term in the last theorem is only used to simplify the calculation of the distance between distributions corresponding to different slope functions. Below we derive an upper bound assuming that the error term $\varepsilon \in \mathcal{E}_n^{\mathbb{K}}$ and the regressor $X$ are uncorrelated. Obviously in this situation Theorem 2.1 provides a lower bound for any estimator as long as the moment restrictions $\mathcal{E}_n^{\mathbb{K}}$ do not exclude a Gaussian error. Furthermore, the lower bound tends only to zero if $(\omega^j \gamma^j)_{j \geq 1}$ is a divergent sequence. In other words, in case $\gamma \equiv 1$, i.e., without any additional restriction on $\beta$, consistency of an estimator of $\ell_h(\beta)$ uniform for all $\beta \in L^2[0, 1]$ is only possible under restrictions on the representer, that is, $\omega$ is a divergent sequence. This obviously reflects the ill-posedness of the underlying inverse problem. Finally, it is worth to note that independent of the regularity condition on the slope parameter, i.e., even $\gamma \equiv 1$ is possible, the lower bound tends to zero with parametric rate $1/n$ if and only if the sequence $(\omega^j \nu^j)_{j \geq 1}$ is bounded away from zero. \hfill $\square$

3 The case of second order stationary regressors.

We assume in this section that the regressor $X$ is second order stationary, i.e., there exists a positive definite function $c : [-1, 1] \to \mathbb{R}$ such that $\text{Cov}(X(t), X(s)) = c(t - s), s, t \in [0, 1]$. Then it is shown in Johannes [2009b] that the eigenfunctions of the covariance operator $T_{\text{cov}}$ associated to $X$ are given by the trigonometric basis

$$\psi_1 := 1, \; \psi_{2j}(s) := \sqrt{2} \cos(2\pi js), \; \psi_{2j+1}(s) := \sqrt{2} \sin(2\pi js), \; s \in [0, 1], \; j \in \mathbb{N} \quad (3.1)$$

and that the corresponding strictly positive, possibly not ordered eigenvalues satisfy

$$\lambda_1 = \int_{-1}^{1} c(s)ds, \; \lambda_{2j} = \lambda_{2j+1} = \int_{-1}^{1} \cos(2\pi js)c(s)ds, \; j \in \mathbb{N}. \quad (3.2)$$

The eigenfunctions are thus known to the statistician and only the eigenvalues depend on the unknown covariance function $c(\cdot)$, i.e., have to be estimated. Therefore, we suppose in this section that the pre-specified basis $\{\psi_j, j \geq 1\}$ is given by the trigonometric functions. Notice that in this situation for each $m \geq 1$ the matrix $[T_{\text{cov}}]_m$ is diagonalized with diagonal entries $[T_{\text{cov}}]_{j,j} = \lambda_j, 1 \leq j \leq m$.

**Definition of the estimator.** Since $\{[T_{\text{cov}}]_{j,j}, \psi_j, j \geq 1\}$ provides a spectral decomposition of the covariance operator $T_{\text{cov}}$ defining the normal equation (1.2). It follows that the linear functional $\ell_h(\beta)$ with given representer $h$ can be rewritten as follows

$$\ell_h(\beta) = \langle \beta, h \rangle = \sum_{j=1}^{\infty} [h]_j [T_{\text{cov}}]_{j,j}^{-1} [g]_j \quad \text{with} \; [g]_j = \langle g, \psi_j \rangle \; \text{and} \; [h]_j = \langle h, \psi_j \rangle, \; j \geq 1. \quad (3.3)$$

It is well-known that even in case of an a-priori known sequence of eigenvalues $([T_{\text{cov}}]_{j,j})_{j \geq 1}$ replacing in (3.3) the unknown function $g$ by a consistent estimator $\hat{g}$ does in general not lead to a consistent estimator of $\ell_h(\beta)$. Therefore, a regularization step is necessary. We follow the approach presented in Johannes [2009b] (there the objective has been the estimation of $\beta$ itself). That is we introduce a dimension reduction together with an additional thresholding in the Fourier domain. To be more precise, we replace the unknown quantities $[g]_j$ and
\[ [T_{\text{cov}}]_{j,j} \text{ in equation (3.3)} \] by their empirical counterparts. That is, if \((Y_1, X_1), \ldots, (Y_n, X_n)\) denotes an i.i.d. sample of \((Y, X)\), then we consider the estimator

\[
\hat{g} := \frac{1}{n} \sum_{i=1}^{n} Y_i X_i, \quad \text{and} \quad \hat{T}_{\text{cov}} := \frac{1}{n} \sum_{i=1}^{n} \langle ., X_i \rangle X_i
\]

(3.4)

for \(g\) and \(T_{\text{cov}}\) respectively. The orthogonal series estimator of the linear functional \(\ell_h(\beta)\) with given representer \(h\) is then defined by

\[
\hat{\ell}_h := \sum_{j=1}^{m} [\hat{h}]_j \cdot [\hat{T}_{\text{cov}}]_{j,j}^{-1} \cdot [\hat{g}]_j \cdot 1 \{ [\hat{T}_{\text{cov}}]_{j,j} \geq 1/\alpha \}
\]

(3.5)

where the dimension parameter \(m = m(n)\) and the threshold \(\alpha = \alpha(n)\) have to tend to infinity as the sample size \(n\) increases. Note that we introduce an additional threshold \(\alpha\) on each estimated eigenvalue \([\hat{T}_{\text{cov}}]_{j,j}\), since it could be arbitrarily close to zero even in case that the true eigenvalue \([T_{\text{cov}}]_{j,j}\) is sufficiently far away from zero. Thresholding in the Fourier domain has been used, for example, in a deconvolution problem in Mair and Ruymgaart [1996], Neumann [1997] or Johannes [2009a] and coincides with an approach called spectral cut-off in the numerical analysis literature (c.f. Tautenhahn [1996]).

**Consistency.** The next assertion summarizes minimal conditions to ensure consistency of the estimator \(\hat{\ell}_h\) defined in (3.5).

**Proposition 3.1.** Assume an \(n\)-sample of \((Y, X)\) satisfying (1.1) with \(\sigma > 0\). Consider the estimator \(\hat{\ell}_h\) with threshold \(m := m(n)\) and parameter \(\alpha := \alpha(n)\) satisfying \(1/m = o(1)\), \(1/\alpha = o(1)\) and \(\alpha^2/n = o(1)\) as \(n \to \infty\). If in addition \(X \in \mathcal{X}_\eta^4\) and \(\varepsilon \in \mathcal{E}_\eta^4\), \(\eta \geq 1\), then we have for all \(h, \beta \in L^2[0,1]\) that \(\mathbb{E}[\hat{\ell}_h - \ell_h(\beta)]^2 = o(1)\) as \(n \to \infty\).

**Remark 3.1.** It is worth noting that the last result states consistency of the estimator \(\hat{\ell}_h\) without any additional restriction than square integrability on both the slope parameter and the representer.

**The upper bound.** In the last assertion we have shown that the estimator \(\hat{\ell}_h\) defined in (3.5) is consistent without additional regularity conditions. However, if these conditions are given through ellipsoids \(\mathcal{F}_\gamma^\delta\) and \(\mathcal{F}_\omega^T\) for the slope function and the representer respectively and a link condition \(\mathcal{N}_v^\delta\) for the covariance operator. Then the next theorem states that the rate \(\max(\delta^*_n, 1/n)\) of the lower bound given in Theorem 2.1 provides up to a constant also an upper bound of the risk of the estimator \(\hat{\ell}_h\). Therefore the rate \(\max(\delta^*_n, 1/n)\) is optimal and hence \(\hat{\ell}_h\) is minimax-optimal.

**Theorem 3.2.** Assume an \(n\)-sample of \((Y, X)\) satisfying (1.1) with \(\sigma > 0\). Suppose that the regressor \(X\) is second order stationary with associated covariance operator \(T_{\text{cov}} \in \mathcal{N}_v^\delta\), \(d \geq 1\). Let \(m_* := m_*(n)\) and \(\delta^* := \delta^*_n(m_*)\) such that (2.4) holds for some \(\Delta \geq 1\). Consider the estimator \(\hat{\ell}_h\) with \(m := m_*\) and \(\alpha := n \max(1, 2d\Delta/\gamma m_*)\). If in addition \(X \in \mathcal{X}_\eta^4\) and \(\varepsilon \in \mathcal{E}_\eta^k\), \(k \geq 8\) then for some generic constant \(C > 0\) we have

\[
\sup_{\beta \in \mathcal{F}_\gamma^\delta} \sup_{h \in \mathcal{F}_\omega^T} \mathbb{E}[\hat{\ell}_h - \ell_h(\beta)]^2 \leq C D d^3 \Delta^2 \Lambda^2 \rho \tau \eta [\rho d \Lambda + \sigma^2] \max(\delta^*_n, n^{-1})
\]

for all sequences \(\gamma, \omega\) and \(v\) satisfying Assumption 2.1.
Remark 3.2. It is worth to note that the bound in the last result is nonasymptotic. Furthermore, from Theorem 2.1 and 3.2 follows that for all sequences $\gamma$, $\omega$ and $\nu$ satisfying the minimal regularity conditions summarized in Assumption 2.1 the estimator $\tilde{f}_n$ attains the optimal rate $\max(\delta_n^s, n^{-1})$ and hence is minimax-optimal. We shall emphasize the interesting influence of the sequences $\gamma$, $\omega$ and $\nu$. As we see from Theorem 2.1 and 3.2, if the sequence $\nu$ decreases more quickly to zero then the obtainable optimal rate of convergence decreases. On the other hand, a faster increasing sequence $\gamma$ or $\omega$ leads to a faster optimal rate. In other words, as expected, values of a linear functional given by a slope function or representer satisfying a stronger regularity condition can be estimated faster. Moreover, independent of the imposed regularity assumption on the slope parameter (even $\gamma \equiv 1$ is possible) the parametric rate $n^{-1}$ is obtained if and only if the sequence $(\omega_j \nu_j)_{j \geq 1}$ is bounded away from zero. Note further if the sequence $\gamma$ increases then in Theorem 3.2 for all large enough $n$ the threshold $\alpha = n$ is used to construct the estimator $\tilde{f}_h$. On the other hand the choice of the dimension $m$ depends on the sequences $\gamma$ and $\nu$ characterizing the regularity conditions imposed on the slope parameter and the covariance operator respectively which are in practise not known. Building data driven rules that can permit to choose automatically the value of $m$ is certainly a topic that deserves further attention and one promising direction is to adapt the selection technique proposed in Efroimovich and Koltchinskii [2001], Goldenshluger and Pereverzev [2000] or Tsybakov [2000]. $\square$

3.1 The finitely and infinitely smoothing case.

In the rest of this section we shall describe the prior information about the unknown slope function $\beta$ and the given representer $h$ by their level of smoothness. Therefore, let us introduce the Sobolev space of periodic functions $W_r$, $r \geq 0$, which for integer $r$ is given by

$$W_r = \left\{ f \in H_p : f^{(j)}(0) = f^{(j)}(1), \ j = 0, 1, \ldots, r - 1 \right\},$$

where $H_p := \{ f \in L^2[0,1] : f^{(r-1)} \text{ absolutely continuous}, f^{(r)} \in L^2[0,1] \}$ is a Sobolev space.

Furthermore, consider $F_{w^r}$ given in (2.1) with weight sequence $w_1 = 1$, $w_j = |j|^2$, $j \geq 2$.

Then it is well-known that the subset $F_{w^r}$ coincides with the Sobolev space of periodic functions $W_r$ (c.f. Neubauer [1988a,b], Mair and Ruymgaart [1996] or Tsybakov [2004]).

Therefore, let us denote by $W_r^c := F_{w^r}$, $c > 0$ an ellipsoid in the Sobolev space $W_r$. We use in case $r = 0$ again the convention that $W_r^o$ denotes an ellipsoid in $L^2[0,1]$. In the rest of this section we consider the Sobolev ellipsoid $W_r^o$, $p \geq 0$, and $W_r^s$, $s \geq 0$, as class of slope parameter and representer respectively. To illustrate the previous results we consider two special cases describing a “regular decay” of the unknown eigenvalues of $T_{cov}$. Precisely, we assume in the following the sequence $\nu$ to be either polynomially decreasing, i.e., $\nu_1 = 1$ and $\nu_j = |j|^{-2a}$, $j \geq 2$, for some $a > 0$ or exponentially decreasing, i.e., $\nu_1 = 1$ and $\nu_j = \exp(-|j|^{2a})$, $j \geq 2$, for some $a > 0$. In the polynomial case easy calculus shows that a covariance operator $T_{cov} \in N_{\nu}^d$ acts like integrating $(2a)$-times and hence it is called \textit{finitely smoothing} (c.f. Natterer [1984]). Furthermore, since the eigenfunctions of $T_{cov}$ are $\{\psi_j\}$ it follows that $T_{cov} \in N_{\nu}^d$ holds if and only if the eigenvalues $[T_{cov}]_{j,j}$ of $T_{cov}$ satisfy $[T_{cov}]_{j,j} \asymp_d |j|^{-2a}$, which is the usual case considered in the literature (c.f. Crambes et al. [2009] or Hall and Horowitz [2007]). On the other hand in the exponential case it can easily be seen that the link condition $T_{cov} \in N_{\nu}^d$ implies $R(T_{cov}) \subset W_p$ for all $p > 0$, therefore the operator $T_{cov}$ is called \textit{infinitely smoothing} (c.f. Mair [1994]). Moreover, $T_{cov} \in N_{\nu}^s$ holds if and only if the eigenvalues $[T_{cov}]_{j,j}$ of $T_{cov}$ satisfy $[T_{cov}]_{j,j} \asymp_d \exp(-j^{2a})$ by using that $\{\psi_j\}$
are the eigenfunctions of $T_{\text{cov}}$. Since in both cases the minimal regularity conditions given in Assumption 2.1 are satisfied, the lower bounds presented in the next assertion follow directly from Theorem 2.1. Here and subsequently, we write $a_n \lesssim b_n$ when there exists $C > 0$ such that $a_n \lesssim C b_n$ for all sufficiently large $n \in \mathbb{N}$ and $a_n \sim b_n$ when $a_n \lesssim b_n$ and $b_n \lesssim a_n$ simultaneously.

**Proposition 3.3.** Under the assumptions of Theorem 2.1 we have for any estimator $\tilde{\ell}$

(i) in the polynomial case, i.e. $v_1 = 1$ and $v_j = |j|^{-2a}$, $j \geq 2$, for some $a > 1/2$, that
\[
\sup_{\beta \in W^p_1} \sup_{h \in W_1} \{ E[|\tilde{\ell} - \ell_h(\beta)|^2] \} \gtrsim \max(n^{-1/p+1/(2a)}, n^{-1}),
\]

(ii) in the exponential case, i.e. $v_1 = 1$ and $v_j = \exp(-|j|^{2a})$, $j \geq 2$, for some $a > 0$, that
\[
\sup_{\beta \in W^p_1} \sup_{h \in W_1} \{ E[|\tilde{\ell} - \ell_h(\beta)|^2] \} \gtrsim (\log n)^{-1/(p+1/a)}.
\]

On the other hand, if the dimension $m$ and the threshold $\alpha$ in the definition of the estimator $\tilde{\ell}_h$ given in (3.5) are chosen appropriate, then by applying Theorem 3.2 the rates of the lower bound given in the last assertion provide up to a constant also the upper bound of the risk of the estimator $\tilde{\ell}_h$, which is summarized in the next proposition. We have thus proved that these rates are optimal and the proposed estimator $\tilde{\ell}_h$ is minimax-optimal in both cases.

**Proposition 3.4.** Under the assumptions of Theorem 3.2 consider the estimator $\tilde{\ell}_h$

(i) in the polynomial case, i.e. $v_1 = 1$ and $v_j = |j|^{-2a}$, $j \geq 2$, for some $a > 1/2$, with dimension $m \sim n^{1/(2p+2a)}$ and threshold $\alpha \sim n$. Then
\[
\sup_{\beta \in W^p_1} \sup_{h \in W_1} \{ E[|\tilde{\ell}_h - \ell_h(\beta)|^2] \} \lesssim \max(n^{-1/p+1/(2a)}, n^{-1}),
\]

(ii) in the exponential case, i.e. $v_1 = 1$ and $v_j = \exp(-|j|^{2a})$, $j \geq 2$, for some $a > 0$, with dimension $m \sim (\log n)^{1/(2a)}$ and threshold $\alpha \sim n$. Then
\[
\sup_{\beta \in W^p_1} \sup_{h \in W_1} \{ E[|\tilde{\ell}_h - \ell_h(\beta)|^2] \} \lesssim (\log n)^{-1/(p+1/a)}.
\]

**Remark 3.3.** We shall emphasize the interesting influence of the parameters $p$, $s$ and $a$ characterizing the smoothness of $\beta$, $h$ and the decay of the eigenvalues of $T_{\text{cov}}$ respectively. As we see from Proposition 3.3 and 3.4, if the value of $a$ increases the obtainable optimal rate of convergence decreases. Therefore, the parameter $a$ is often called degree of ill-posedness (c.f. Natterer [1984]). On the other hand, an increasing of the value $p + s$ leads to a faster optimal rate. In other words, as expected, values of a linear functional given by a smoother slope function or representer can be estimated faster. Moreover, in the polynomial case independent of the imposed smoothness assumption on the slope parameter (even $p = 0$ is possible) the parametric rate $n^{-1}$ is obtained if and only if the representer is smoother than the degree of ill-posedness of $T_{\text{cov}}$, i.e., $s \geq a$. The situation is different in the exponential case. As long as the representer $h$ is only finitely times differentiable, then due to Proposition 3.3 and 3.4 the optimal rate of convergence is logarithmic. However, if we restrict the class of representers even more, e.g. by considering $F_\omega$ with weights $\omega_1 := 1$, $\omega_j := \exp(|j|^{2q}), j \geq 2$, which contains only analytic functions given $q > 1$ (c.f. Kawata [1972]). Then faster rates are possible. Again independent of the imposed smoothness assumption on the slope parameter (again $p = 0$ is possible) the parametric rate $n^{-1}$ is obtained if and only if the representer $h$ is smoother than the degree of ill-posedness of $T_{\text{cov}}$, e.g., $q \geq a$. Finally, in opposite to the polynomial case in the exponential case the smoothing parameter $m$ does not depend on the value of $p$. It follows that the proposed estimator is automatically adaptive, i.e., it does not depend on an a-priori knowledge of
the degree of smoothness of the slope function $\beta$. However, the choice of the smoothing parameter depends on the smoothing properties of $T_{\text{cov}}$, i.e., the value of $a$. □

**Remark 3.4.** There is an interesting issue hidden in the parametrization we have chosen. Consider a classical indirect regression model given by the covariance operator $T_{\text{cov}}$ and Gaussian white noise $\hat{W}$, i.e., $g_n = T_{\text{cov}}\beta + n^{-1/2}\hat{W}$ (for details see e.g. Hoffmann and Reiß [2008]). If $T_{\text{cov}}$ is finitely smoothing, i.e., $\nu_1 = 1$ and $\nu_j = |j|^{-2\alpha}$, $j \geq 2$, then it is shown in Johannes and Kroll [2009] that the optimal rate of convergence over the classes $W_p^\rho$ and $W_\infty^\rho$ of any estimator of $\ell_h(\beta)$ is of order $\max(n^{-(p+\alpha)/(p+2\alpha)}, n^{-1})$. However, from Proposition 3.3 and 3.4 follows that in a functional linear model the optimal rate is of order $\max(n^{-(p+\alpha)/(p+\alpha)}, n^{-1})$. Thus comparing both rates we see that in a functional linear model the covariance operator $T_{\text{cov}}$ has the degree of ill-posedness $\alpha$ while the same operator has in the indirect regression model a degree of ill-posedness $(2\alpha)$. In other words in a functional linear model we do not face the complexity of an inversion of $T_{\text{cov}}$ but only of its square root $T_{\text{cov}}^{1/2}$. The same remark holds true in the exponential case. But, the rate of convergence is the same as in an indirect regression model with Gaussian white noise (c.f. Johannes and Kroll [2009]). This, however, is due to the fact that in case $\nu_j \asymp \exp(-r|j|^{2\alpha})$, $j \in \mathbb{N}$, for some $r > 0$, the dependence of the rate of convergence on the value $r$ is hidden in the constant (a more detailed discussion can be found in Johannes [2009b]). □

### 4. Optimal local estimation in the general case.

In this section the pre-specified basis $\{\psi_j\}$ corresponds not necessarily to the set of eigenfunctions of $T_{\text{cov}}$. In this situation for each $m \geq 1$ the matrix $[T_{\text{cov}}]_{m,m}$ will be in general no longer diagonalized. Nevertheless, the estimator proposed below is also based on a dimension reduction together with an additional thresholding. That is, if $\hat{g}$ and $\hat{T}_{\text{cov}}$ denote the estimator of $g$ and $T_{\text{cov}}$ respectively given in (3.4), then the general estimator of the linear functional $\ell_h(\beta)$ is now defined by

$$
\hat{\ell}_h := \begin{cases} 
[\hat{h}]_{m}[T_{\text{cov}}]_{m}^{-1}[^{\hat{g}}]_{m}, & \text{if } [\hat{T}_{\text{cov}}]_{m,m} \text{ is nonsingular and } \|[T_{\text{cov}}]_{m}^{-1}\| \leq \alpha, \\
0, & \text{otherwise,}
\end{cases}
$$

(4.1)

where the dimension parameter $m = m(n)$ and the threshold $\alpha = \alpha(n)$ again have to tend to infinity as the sample size $n$ increases. In fact, the general estimator $\hat{\ell}_h$ is obtained from the linear functional $\ell_h(\beta)$ by replacing the unknown slope parameter $\beta$ by an estimator proposed by Cardot and Johannes [2008], which takes its inspiration in the linear Galerkin approach coming from the inverse problem community (c.f. Efroimovich and Koltchinskii [2001] or Hoffmann and Reiß [2008]).

**Consistency.** The next assertion summarizes minimal conditions to ensure consistency of the estimator $\hat{\ell}_h$ introduced in (4.1).

**Proposition 4.1.** Assume an $n$-sample of $(Y, X)$ satisfying (1.1) with $\sigma > 0$. Let $X \in X^n_\eta$ and $\varepsilon \in E^n_\eta$, $\eta \geq 1$. Consider $\hat{\ell}_h$ defined with dimension $m := m(n)$ and threshold $\alpha := \alpha(n)$ satisfying $\alpha \geq 2\|[T_{\text{cov}}]_{m,m}^{-1}\|$ as $n \to \infty$ that $1/m = o(1)$, $\alpha/n = o(1)$, $m^3/n = O(1)$ and $(m^2\alpha^2)/n = O(1)$. If in addition $\sup_{m \in \mathbb{N}}\|[T_{\text{cov}}]_{m}^{-1}[\Pi_m T_{\text{cov}}]\Pi_m^{\perp}\| < \infty$, then we have $\mathbb{E}\|\hat{\ell}_h - \ell_h(\beta)\|^2 = o(1)$ as $n \to \infty$. 
In Proposition 4.1 consistency is only obtained under the additional condition $\sup_{m \in \mathbb{N}} \| (T_{\text{cov}})^{-1} \Pi_m T_{\text{cov}} \Pi_m^2 \| < \infty$, which is known to be sufficient to ensure $L^2$-convergence of the Galerkin solution given by $\beta_m = \sum_{j=1}^m [\beta_m]_j \psi_j$ with $[\beta_m]_m = (T_{\text{cov}})^{-1} [g]_m$ to the slope parameter $\beta$ as $m \to \infty$. However, this condition is automatically fulfilled if the operator $T_{\text{cov}}$ satisfies a link condition, i.e., $T_{\text{cov}} \in \mathcal{N}_d^\ell$, which is summarized in the next assertion.

**Corollary 4.2.** Assume an $n$-sample of $(Y, X)$ satisfying (1.1) with $\sigma > 0$ and associated covariance operator $T_{\text{cov}} \in \mathcal{N}_d^\ell$, $d \geq 1$. Consider the estimator $\hat{\ell}_h$ with threshold $\alpha = 8d^3/\nu_m$ and dimension $m := m(n)$ chosen such that $1/m = o(1)$, $1/(\nu_m) = o(1)$, $(m^3/n) = O(1)$ and $m^2/(\nu_m^2) = O(1)$ as $n \to \infty$. If in addition $X \in \mathcal{X}_n^8$ and $\varepsilon \in \mathcal{E}_n^8$, $\eta \geq 1$, then we have $\mathbb{E}(\hat{\ell}_h - \ell_h(\beta))^2 = o(1)$ as $n \to \infty$.

**The upper Bound.** The last assertions show that the estimator $\hat{\ell}_h$ defined in 4.1 is consistent without any additional regularity conditions on slope function and representer. The following theorem provides an upper bound if these conditions are given again through ellipsoids $\mathcal{F}_n^\ell$ and $\mathcal{F}_n^\omega$ for the slope function and the representer respectively together with a link condition $\mathcal{N}_d^\ell$ for the covariance operator. But in contrast to the case of second order stationary regressor considered in the last section the following additional properties of the sequences $\gamma$, $\omega$ and $\nu$ are needed. We suppose that for some $k \in \mathbb{N}$

$$\frac{m^{2k}n^{-k}}{\max(\delta_*^n, 1/n)} = O(1), \quad \frac{m_*^{1+k/2}}{\nu_{k/2}} = O(1), \quad \frac{m_*^{3+k}}{\nu_{k/2-1}} = O(1), \quad \frac{m_*^3}{n} = O(1)$$

as $n \to \infty$.

where $m_* := m_*(n)$ and $\delta_* := \delta_*(m_*)$ are given by (2.4). The next theorem states that in this situation the rate $\max(\delta_*^n, 1/n)$ of the lower bound given in Theorem 2.1 provides up to a constant also an upper bound of the general estimator $\hat{\ell}$ defined in (4.1). Thus we have proved that the rate $\max(\delta_*^n, 1/n)$ is optimal and hence $\hat{\ell}$ is minimax-optimal.

**Theorem 4.3.** Assume an $n$-sample of $(Y, X)$ satisfying (1.1) with $\sigma > 0$ and associated covariance operator $T_{\text{cov}} \in \mathcal{N}_d^\ell$, $d \geq 1$. Suppose that the sequences $\gamma$, $\omega$ and $\nu$ satisfy Assumption 2.1 and condition (4.2) for some $k \geq 3$. Let $m_* := m_*(n)$ and $\delta_* := \delta_*(m_*)$ such that (2.4) holds for some $\Delta \geq 1$. Consider $\hat{\ell}_h$ defined with dimension $m := m_*$ and threshold $\alpha := n \max(1, 8d^3 \Delta/\gamma m_*)$. If in addition $X \in \mathcal{X}_n^{4k}$ and $\varepsilon \in \mathcal{E}_n^{4k}$, then for some generic constant $C > 0$ we have

$$\sup_{\beta \in \mathcal{F}_n^\ell} \sup_{h \in \mathcal{F}_n^{\omega}} \mathbb{E}(\hat{\ell}_h - \ell_h(\beta))^2 \leq C \Delta^3 d^{11} \rho \sigma \rho \tau \left\{ \gamma^2 + \rho \delta_{\omega} \Lambda / \gamma m_* + 1 \right\} \max(\delta_*^n, n^{-1}).$$

**Remark 4.1.** We shall stress that the bound in the last theorem is again nonasymptotic. Moreover, it is worth to note that if the sequence $\gamma$ increases then the condition on the threshold writes $\alpha = n$ for all sufficiently large $n$. Therefore, also in the general case only the dimension $m$ has to be chosen data-driven in order to build an adaptive estimation procedure. Furthermore, even in case $\gamma \equiv 1$, i.e., $\beta$ is only assumed to be square integrable, the upper bound still tends to zero as long as the sequence $\omega$ is increasing. Moreover, the proposed plug-in estimator attains again the parametric rate under the conditions of Theorem 4.3 if and only if the sequence $(\omega_j)_{j \geq 1}$ is bounded away from zero. Note furthermore, if the eigenfunctions of the operator $T_{\text{cov}}$ are given by $\{\psi_j\}$, then $T_{\text{cov}} \in \mathcal{N}_d^\ell$ holds if and only if the corresponding eigenvalues $\lambda_j = (T_{\text{cov}} \psi_j, \psi_j)$, $j \geq 1$, satisfy $\lambda_j \geq \sqrt{d} \, \nu_j$. Hence, in this situation the optimal rate obtained in the last assertion equals the rate in...
Theorem 3.2. However, the set \( \mathcal{N}_v^d \) contains also operators with eigenfunctions not given by \( \{ \psi_j \} \). Then their corresponding eigenvalues may decay far slower than the sequence of weights \( v \). Hence, by using a projection onto the basis \( \{ \psi_j \} \) instead of their eigenfunctions, the obtainable rate of convergence given in Theorem 4.3 may be far slower than the rate given in Theorem 3.2. However, the rate in Theorem 4.3 is optimal, and thus cannot be improved without additional information.

The finitely and infinitely smoothing case. In the rest of this section the basis \( \{ \psi_j \} \) is again given by the trigonometric functions defined in (3.1). But in opposite to Section 3.1 this basis corresponds not necessarily to the set of eigenfunctions of \( T_{\text{cov}} \). Furthermore, we consider also the Sobolev ellipsoids \( W_p \), \( p \geq 0 \), and \( W_s \), \( s \geq 0 \), for the slope parameter and the representer respectively. In the following theorem we illustrate the general result obtained for the estimator \( \hat{\ell}_h \) defined in (4.1) by considering again the finitely and infinitely smoothing case presented in Section 3.1.

**Proposition 4.4.** Under the assumptions of Theorem 4.3 consider the estimator \( \hat{\ell}_h \)

(i) in the polynomial case, i.e. \( v_1 = 1 \) and \( v_j = |j|^{-2a} \), \( j \geq 2 \), for some \( a > 1/2 \), with \( m \sim n^{1/(2p+2a)} \) and threshold \( \alpha \sim n \). If in addition \( k \geq 12 \) and \((p+a) \geq 3/2\) then

\[
\sup_{\beta \in W_p^\alpha, h \in W_s^\alpha} \{ \mathbb{E} |\hat{\ell}_h - \ell_h(\beta)|^2 \} \lesssim \max(n^{-\{(p+s)/(p+a)\}}, n^{-1}),
\]

(ii) in the exponential case, i.e. \( v_1 = 1 \) and \( v_j = \exp(-|j|^{2a}) \), \( j \geq 2 \), for some \( a > 0 \), with \( m \sim (\log n)^{1/(2a)} \) and threshold \( \alpha \sim n \). If \( k \geq 8 \) then

\[
\sup_{\beta \in W_p^\alpha, h \in W_s^\alpha} \{ \mathbb{E} |\hat{\ell}_h - \ell_h(\beta)|^2 \} \lesssim (\log n)^{-\{(p+s)/a\}}.
\]

**Remark 4.2.** The last assertion shows that under stronger moment conditions, e.g. \( k \geq 12 \) in case (i), and additional restrictions on the parameter \( a \) and \( p \), e.g. \((p+a) \geq 3/2\) in case (i), the rate of the lower bound over the Sobolev ellipsoids \( W_p^\alpha \) and \( W_s^\alpha \) (see Proposition 3.3) provides up to a constant also an upper bound of the estimator \( \hat{\ell}_h \) for both a finitely and an infinitely smoothing covariance operator. Thereby, this rate is also in case of unknown eigenfunctions optimal and hence \( \hat{\ell}_h \) is minimax-optimal. Furthermore, the findings discussed in Remark 3.3 and 3.4 still apply here.

**A Appendix**

**A.1 Proofs of Section 2.**

Consider the covariance operator \( T_{\text{cov}} \) associated to the regressor \( X \), then \( \mathbb{E}[X]_j^2 = \langle T_{\text{cov}} \psi_j, \psi_j \rangle \), \( j \in \mathbb{N} \). Therefore, if the link condition (2.2), i.e., \( T_{\text{cov}} \in \mathcal{N}_v^d \), is satisfied, then it follows that \( \mathbb{E}[X]_j^2 \asymp_d v_j \), for all \( j \in \mathbb{N} \). This result will be used below without further reference. We shall prove at the end of this section the technical Lemma A.1 used in the next proof.

**Proof of the lower bound.**

**Proof of Theorem 2.1.** Let \( X_i, i \in \mathbb{N} \), be i.i.d. copies of \( X \). Consider independent error terms \( \varepsilon_i \sim \mathcal{N}(0,1) \), \( i \in \mathbb{N} \), which are independent of the random functions \( \{X_i, i \in \mathbb{N} \} \).
Then we prove for any estimator $\hat{\ell}$ the following lower bounds:

$$\sup_{\beta \in \mathcal{F}_c \setminus \mathcal{F}_l} \mathbb{E}[\hat{\ell} - \ell_h(\beta)]^2 \geq \frac{\tau}{4\Delta} \min \left\{ \frac{\sigma^2}{2d}, \frac{\rho}{\sqrt{d}} \right\} \delta_n^*, \tag{A.1}$$

$$\sup_{\beta \in \mathcal{F}_c \setminus \mathcal{F}_l} \mathbb{E}[\hat{\ell} - \ell_h(\beta)]^2 \geq \frac{\tau}{4} \min \left\{ \frac{\sigma^2}{2d}, \rho \right\} \frac{1}{n}. \tag{A.2}$$

The result follows then by combination of these two lower bounds.

Proof of (A.1). Define the slope function $\beta_\ell := [\beta_\ell]_m \psi_m$, where $[\beta_\ell]_m$ is given in (A.7) (Lemma A.1) and $m_* := m_*(n) \in \mathbb{N}$ satisfies (2.4) for some $\Delta \geq 1$. Then by using (A.9) in Lemma A.1 we have $\beta_\ell \in \mathcal{F}_\ell^p$. Consider the two slope functions $\beta_\theta := \theta \beta_\ell \in \mathcal{F}_\theta^p$, $\theta \in \{-1, 1\}$. Then, for each $\theta$ the random variables $(Y_i, X_i)$ with $Y_i := \int_0^1 \beta_\theta(s)X_i(s)ds + \sigma \varepsilon_i$, $i = 1, \ldots, n$, form a sample of the model (1.1) and we denote its joint distribution by $\mathbb{P}_\theta$. In case of $\mathbb{P}_\theta$ the conditional distribution of $Y_i$ given $X_i$ is Gaussian with mean $\theta[\beta_\ell]_m[X_i]_m$ and variance $\sigma^2$. Thereby, it is easily seen that the log-likelihood of $\mathbb{P}_{-1}$ with respect to $\mathbb{P}_1$ is given by

$$\log \left( \frac{d\mathbb{P}_{-1}}{d\mathbb{P}_1} \right) = -\frac{1}{\sigma^2} \sum_{i=1}^n (Y_i - [\beta_\ell]_m[X_i]_m)[\beta_\ell]_m[X_i]_m - \frac{2n}{\sigma^2} \sum_{i=1}^n [\beta_\ell]_m^2 [X_i]_m^2$$

and hence its expectation with respect to $\mathbb{P}_1$ satisfies

$$\mathbb{E}_{\mathbb{P}_1}[\log(d\mathbb{P}_{-1}/d\mathbb{P}_1)] = -(2n/\sigma^2)[\beta_\ell]_m^2 \mathbb{E}_{[X]_m}^2 \geq -(2dn/\sigma^2)[\beta_\ell]_m^2 v_m.$$  

In terms of Kullback-Leibler divergence this means $\text{KL}(\mathbb{P}_1, \mathbb{P}_{-1}) \leq (2dn/\sigma^2)[\beta_\ell]_m^2 v_m$. Since the Hellinger distance $H(\mathbb{P}_1, \mathbb{P}_{-1})$ between $\mathbb{P}_1$ and $\mathbb{P}_{-1}$ satisfies $H^2(\mathbb{P}_1, \mathbb{P}_{-1}) \leq \text{KL}(\mathbb{P}_1, \mathbb{P}_{-1})$ it follows from (A.9) in Lemma A.1 that

$$H^2(\mathbb{P}_1, \mathbb{P}_{-1}) \leq \frac{2dn}{\sigma^2} [\beta_\ell]_m^2 v_m \leq 1. \tag{A.3}$$

Consider the Hellinger affinity $\rho(\mathbb{P}_1, \mathbb{P}_{-1}) = \int \sqrt{d\mathbb{P}_1d\mathbb{P}_{-1}}$ then we obtain for any estimator $\hat{\ell}$ and for all $h \in \mathcal{F}_\theta^p$ that

$$\rho(\mathbb{P}_1, \mathbb{P}_{-1}) \leq \int \frac{[\hat{\ell} - \ell_h(\beta_1)]^2}{2\ell_h(\beta_1)} \sqrt{d\mathbb{P}_1d\mathbb{P}_{-1}} + \int \frac{[\hat{\ell} - \ell_h(\beta_1)]^2}{2\ell_h(\beta_1)} \sqrt{d\mathbb{P}_1d\mathbb{P}_{-1}}$$

$$\leq \left( \frac{1}{4} \frac{[\hat{\ell} - \ell_h(\beta_1)]^2}{\ell_h(\beta_1)^2} \sqrt{d\mathbb{P}_{1}} \right)^{1/2} + \left( \frac{1}{4} \frac{[\hat{\ell} - \ell_h(\beta_1)]^2}{\ell_h(\beta_1)^2} \sqrt{d\mathbb{P}_{-1}} \right)^{1/2}. \tag{A.4}$$

Due to the identity $\rho(\mathbb{P}_1, \mathbb{P}_{-1}) = 1 - \frac{1}{2} H^2(\mathbb{P}_1, \mathbb{P}_{-1})$ combining (A.3) with (A.4) yields

$$\left\{ \mathbb{E}_{\mathbb{P}_1}[\hat{\ell} - \ell_h(\beta_1)]^2 + \mathbb{E}_{\mathbb{P}_{-1}}[\hat{\ell} - \ell_h(\beta_{-1})]^2 \right\} \geq \frac{1}{2} \ell_h(\beta_1)^2. \tag{A.5}$$

Consider now the representer $h_\ast := [h_\ast]_m \psi_m$, where $[h_\ast]_m$ is given in (A.7) (Lemma A.1). Then by construction $h_\ast \in \mathcal{F}_\ell^p$ and $|\ell_h(\beta_\ast)|^2 = [h_\ast]_m^2 [\beta_\ast]_m^2$. From (A.5) we conclude then for each estimator $\hat{\ell}$ that

$$\sup_{\beta \in \mathcal{F}_c \setminus \mathcal{F}_l} \mathbb{E}[\hat{\ell} - \ell_h(\beta)]^2 \geq \sup_{\theta \in \{-1, 1\}} \mathbb{E}_{\mathbb{P}_\theta}[\hat{\ell} - \ell_h(\beta_\theta)]^2$$

$$\geq \frac{1}{4} \left\{ \mathbb{E}_{\mathbb{P}_1}[\hat{\ell} - \ell_h(\beta_1)]^2 + \mathbb{E}_{\mathbb{P}_{-1}}[\hat{\ell} - \ell_h(\beta_{-1})]^2 \right\}$$

$$\geq \frac{1}{4} [\beta_\ast]_m^2 [h_\ast]_m^2 \geq \frac{\tau}{4\Delta} \min \left\{ \frac{\sigma^2}{2d}, \frac{\rho}{\sqrt{d}} \right\} \delta_n^*.$$
where the last inequality follows from (A.9) in Lemma A.1, which proves (A.1).

The proof of (A.2) is similar to the proof of (A.1), but uses (A.8) in Lemma A.1 rather than (A.9). Precisely, we consider the slope function \( \beta_a := [h_a]_1 \psi_1 \), and the representer \( \hat{h}_a := [h_a]_1 \psi_1 \) with \( [\beta_a]_1 \) and \( [h_a]_1 \) given in (A.6) (Lemma A.1). Then by following along the same lines as in the proof of (A.1) we obtain (A.2), which completes the proof. \( \square \)

Technical assertion.

**Lemma A.1.** Let \( m_* := m_*(n) \) and \( \delta^n_\alpha := \delta^n_\alpha(m_*) \) be given by (2.4) with \( \Delta \geq 1 \). If we define

\[
[h_*]_1 := \tau, \quad [\beta_*]_1 := \frac{\zeta}{n}, \quad \text{with} \quad \zeta := \min \left\{ \frac{\sigma^2}{2d}, \rho \right\}, \tag{A.6}
\]

\[
[h_*]_{m_*} := \frac{-\tau}{\omega_{m_*}} \quad \text{and} \quad [\beta_*]_{m_*} := \frac{\xi}{n \nu_{m_*}}, \quad \text{where} \quad \xi := \min \left\{ \frac{\sigma^2}{2d}, \frac{\rho}{\Delta} \right\}. \tag{A.7}
\]

Then under the Assumption 2.1, i.e., \( \gamma_1 = \omega_1 = v_1 = 1 \), we have

\[
\begin{align*}
\frac{2d n^2}{\sigma^2} [\beta_*]_1^2 v_1 \leq 1, & \quad [\beta_*]_1^2 \gamma_1 \leq \rho, \quad [h_*]_1^2 [\beta_*]_1^2 \geq \tau \min \left\{ \frac{\sigma^2}{2d}, \rho \right\} \frac{1}{n}, \tag{A.8}
\end{align*}
\]

\[
\begin{align*}
2d n^2 [\beta_*]_{m_*}^2 \nu_{m_*} \leq 1, \quad & \quad [\beta_*]_{m_*}^2 \gamma_{m_*} \leq \rho \quad \text{and} \quad [h_*]_{m_*}^2 [\beta_*]_{m_*}^2 \geq \tau \min \left\{ \frac{\sigma^2}{2d}, \frac{\rho}{\Delta} \right\} \delta^n_\alpha. \tag{A.9}
\end{align*}
\]

**Proof.** We only prove (A.9). The proof of (A.8) follows in analogy and we omit the details. The first inequality in (A.9) follows trivially by using the definition of \( \xi \), while the definition of \( m_* \) given in (2.4) implies the second, i.e., \( [\beta_*]_{m_*}^2 \gamma_{m_*} \leq \xi \gamma_{m_*/(n \nu_{m_*})} \leq \xi \Delta \leq \rho \). To deduce the third estimate from the definition of \( m_* \) and \( \delta^n_\alpha \) observe that \( [h_*]_{m_*}^2 [\beta_*]_{m_*}^2 = \tau \delta^n_\alpha \xi \gamma_{m_*/(n \nu_{m_*})} \geq \tau \delta^n_\alpha \xi / \Delta \), which proves the lemma. \( \square \)

### A.2 Proofs of Section 3.

We begin by defining and recalling notations to be used in the proofs of this section. Since the eigenfunctions of \( T_{\text{cov}} \) associated to the regressor \( X \) are given by the basis \( \{ \psi_j \} \) it follows that the values \( \lambda_j := \mathbb{E}[X]^2_j = [T_{\text{cov}}]_{j,j} \) are the corresponding eigenvalues. Moreover, \( T_{\text{cov}} \) satisfies the link condition (2.2), i.e., \( T_{\text{cov}} \in \mathcal{N}^d_{\nu} \), if and only if \( \lambda_j \asymp_n v_j \), for all \( j \in \mathbb{N} \). Thus, if \( T_{\text{cov}} \in \mathcal{N}^d_{\nu} \), then \( \mathbb{E}[X]^2 \leq d \Delta \) by using Assumption 2.1. These results will be used below without further reference. Furthermore, given independent and identically distributed (i.i.d.) copies \( (Y_i, X_i), 1 \leq i \leq n \), of \( (Y, X) \) we use for all \( j \in \mathbb{N} \) the notations

\[
[X_i]_j = \langle X_i, \psi_j \rangle, \quad [\beta]_j = \langle \beta, \psi_j \rangle, \quad [h]_j = \langle h, \psi_j \rangle, \quad [Z_i]_j := [X_i]_j / \sqrt{\lambda_j}
\]

\[
\hat{\lambda}_j := [\hat{T}_{\text{cov}}]_{j,j}, \quad T_{n,j} := \frac{1}{n} \sum_{i=1}^n \langle Y_i - [X_i]_j [\beta]_j ] [Z_i]_j = ( [\beta]_j - \hat{\beta}_j \hat{\lambda}_j ) / \sqrt{\lambda_j}. \tag{A.10}
\]

We shall prove in the end of this section a technical Lemma A.2 used in the following proofs.

**Proof of the consistency.**

**Proof of Proposition 3.1.** Let \( \tilde{\ell}_h^n := \sum_{j=1}^n [h]_j [\beta]_j \mathbb{1}\{[\hat{T}_{\text{cov}}]_{j,j} \geq 1/\alpha \} \). Then the proof is based on the decomposition

\[
\mathbb{E}[\tilde{\ell}_h - \ell_h(\beta)^2] \leq 2\{\mathbb{E}[\tilde{\ell}_h - \tilde{\ell}_h^n]^2 + \mathbb{E}[\tilde{\ell}_h^n - \ell_h(\beta)]^2\}. \tag{A.11}
\]

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We show below for some generic constant $C > 0$ the following bound
\[
E[\tilde{\ell}_h - \tilde{\ell}_h^m]^2 \leq C \eta (\alpha^2/n) \|h\|^2 E\|X\|^2 \{\sigma^2 + \|\beta\|^2 E\|X\|^2\}, \tag{A.12}
\]
while we conclude from Lebesgue’s dominated convergence theorem
\[
E[\tilde{\ell}_h^m - \ell_h(\beta)]^2 = o(1) \text{ provided } 1/m = o(1) \text{ and } 1/\alpha = o(1) \text{ as } n \to \infty. \tag{A.13}
\]
Thereby, the conditions on $\alpha$ and $m$ ensure the convergence to zero as $n \to \infty$ of the two terms on the right hand side in (A.11), which gives the result.

Proof of (A.12). By making use of the notations given in (A.10) it follows that
\[
E[\tilde{\ell}_h - \tilde{\ell}_h^m]^2 = \|h\|^2 \sum_{j=1}^m E((|\beta|_j - [\beta]_j) \lambda_j^2 \mathbb{1}\{\lambda_j \geq 1/\alpha\}) \leq \|h\|^2 \alpha^2 \sum_{j=1}^m \lambda_j E[T_{n,j}]^2
\]
and hence (A.12) follows by using (A.18) in Lemma A.2 together with $\sum_{j=1}^m \lambda_j \leq E\|X\|^2$.

Proof of (A.13). By making use of the relation
\[
\tilde{\ell}_h - \ell_h(\beta) = - \sum_{j=1}^m [h]_j [\beta]_j \mathbb{1}\{\lambda_j < 1/\alpha\} - \sum_{j>m} [h]_j [\beta]_j
\]
where $|\sum_{j>m} [h]_j [\beta]_j = o(1)$ as $m \to \infty$ due to Lebesgue’s dominated convergence theorem and
\[
E\left(\sum_{j=1}^m [h]_j [\beta]_j \mathbb{1}\{\lambda_j < 1/\alpha\}\right) = \|h\|^2 \cdot \sum_{j=1}^\infty [\beta]_j^2 \cdot \mathbb{P}(\lambda_j < 1/\alpha) \leq \|h\|^2 \cdot \|\beta\|^2 < \infty.
\]
Thus Lebesgue’s dominated convergence theorem implies the result since for each $j \in \mathbb{N}$ \( \mathbb{P}(\lambda_j < 1/\alpha) = o(1) \) as $n \to \infty$, which can be realized as follows. By using that $1/\alpha = o(1)$ there exists $n_j > 0$ such that for all $n \geq n_j$ it holds $\lambda_j \geq 2/\alpha$ and hence \( \mathbb{P}(\lambda_j < 1/\alpha) \leq \mathbb{P}(\lambda_j/\lambda_j < 1/2) \) together with (A.20) in Lemma A.2 implies the assertion, which completes the proof. \( \square \)

Proof of the upper bound.

**Proof of Theorem 3.2.** Consider the decomposition (A.11), then we show below under the condition $\varepsilon \in \mathcal{E}_n^s$, $X \in \mathcal{X}_n^s$ and $\lambda_j \geq 2/\alpha$, $1 \leq j \leq m$, for some generic constant $C > 0$ the following two bounds
\[
E[\tilde{\ell}_h - \tilde{\ell}_h^m]^2 \leq C \left\{ \sum_{j=1}^m [h]_j^2 \right\} d \Lambda \left\{ \sigma^2 + \eta \|\beta\|^2 d \Lambda + \sigma^2 \right\}, \tag{A.15}
\]
\[
E[\tilde{\ell}_h^m - \ell_h(\beta)]^2 \leq C \|h\|^2 \|\beta\|^2 \eta \|\beta\| \|h\|^2. \tag{A.16}
\]
Consider $m_* := m_* (n)$ given in (2.4), i.e., $\gamma_{m*}/(n v_{m*}) \leq \Delta$ with $\Delta \geq 1$. Then under Assumption 2.1 the conditions $m = m_*$ and $\alpha = n \max(1, 2d/\gamma_{m*})$ imply together $\alpha/n \leq 2d\Delta, 1/(nv_{m*}) \leq \Delta$, $\lambda_j \geq 2/\alpha$, $1 \leq j \leq m$ by using $\lambda_j \asymp d v_j$. Consequently, for all $\beta \in \mathcal{F}_\gamma^n$ and $h \in \mathcal{F}_\beta^n$ we have
\[
E[\tilde{\ell}_h - \ell_h(\beta)]^2 \leq C \left\{ n^{-1} \sup_{1 \leq j \leq m_*} \{\omega_j^{-1} v_j^{-1}\} d \rho \Delta^2 \Lambda^2 + \omega_{m*}^{-1} \gamma_{m*}^{-1} + 1/n \right\} \rho \tau \eta [\rho d \Lambda + \sigma^2].
\]
Now under Assumption 2.1, i.e., \(\sup_{1 \leq j \leq m_s} \omega_j^{-1} v_j^{-1} \leq D v_m^{-1} \max(\omega_m^{-1}, v_m)\), follows \(n^{-1} \sum_{1 \leq j \leq m_s} \omega_j^{-1} v_j^{-1} \leq D \Delta \max(\delta_n^{-1}, 1/n)\) by using the definition of \(m_s\) given in (2.4) and \(\delta_n^{-1} = \omega_m^{-1} \gamma_m^{-1}\), which implies the result.

Proof of (A.15). By using the notations introduced in (A.10) we obtain the identity

\[
\tilde{\ell}_h - \tilde{\ell}_h^m = \sum_{j=1}^{m} \frac{[h_j]}{\lambda_j^{3/2}} \cdot T_{n,j} \cdot \{\lambda_j/\lambda_j - 1\} \cdot 1\{\lambda_j \geq 1/\alpha\}
\]

\[
- \sum_{j=1}^{m} \frac{[h_j]}{\lambda_j^{3/2}} \cdot T_{n,j} \cdot 1\{\lambda_j < 1/\alpha\} + \sum_{j=1}^{m} \frac{[h_j]}{\lambda_j} \cdot T_{n,j} =: S_1 + S_2 + S_3,
\]

where we show below that each term \(E|S_1|^2, E|S_2|^2\) and \(E|S_3|^2\) is bounded by

\[
C \left\{ \sum_{j=1}^{m} \frac{[h_j]}{n \lambda_j} \left\{ \sum_{j=1}^{m} \frac{\lambda_j^2 \alpha^2}{n^2} + \sum_{j=1}^{m} \frac{1}{n} \right\} \cdot \eta \cdot \{||\beta||^2 E\|X\|^2 + \sigma^2\} \right\}.
\]

(A.17)

for some constant \(C > 0\). Consequently, the inequality (A.15) follows from (A.17) by using \(v_j \downarrow \lambda_j\) for all \(j \in \mathbb{N}\) and \(\sum_j v_j = \Lambda\). Consider \(S_1\). First by using the Cauchy-Schwarz inequality together with the elementary inequality \(1/2 \leq \lambda_j/\lambda_j - 1/2^2 + \lambda_j/\lambda_j\) we obtain

\[
E|S_1|^2 \leq 4 \left\{ \sum_{j=1}^{m} \frac{[h_j]}{\lambda_j} (E|T_{n,j}|^4)^{1/2} \right\} \left\{ \sum_{j=1}^{m} \frac{\lambda_j^2 \alpha^2}{n^2} (E|\lambda_j/\lambda_j - 1|^4)^{1/2} + \sum_{j=1}^{m} (E|\lambda_j - 1|^4)^{1/2} \right\}.
\]

Thereby, the bound (A.17) follows from (A.18) and (A.19) in Lemma A.2. Let us evaluate \(S_2\). The Cauchy-Schwarz inequality together with \(\lambda_j \geq 2/\alpha\) for all \(j = 1, \ldots, m\) implies

\[
E|S_2|^2 \leq 2 \left\{ \sum_{j=1}^{m} \frac{[h_j]}{\lambda_j} (E|T_{n,j}|^4)^{1/2} \right\} \left\{ \sum_{j=1}^{m} |P\lambda_j/\lambda_j < 1/2)|^{1/2} \right\}.
\]

We thus get the bound (A.17) by using (A.18) and (A.20) in Lemma A.2. Consider \(S_3\). Define \(V^2 := \sum_{j=1}^{m} [h_j]/\lambda_j\) and \(s \in \mathbb{R}^m\) with \(s_j := [h_j]/(V \sqrt{\lambda_j})\). Clearly \(s \in \mathbb{S}^m\) and \(S_3 = V \sum_{j=1}^{m} s_j T_{n,j}\). Consequently, the bound (A.17) follows from (A.18) in Lemma A.2.

The proof of (A.16) is based on the identity (A.14) where we again bound each summand separately. First, by using the Cauchy-Schwarz inequality we conclude \(\sum_{j>m} [h_j] \beta_j^2 \leq \omega_m^{-1} \gamma_m^{-1} ||\beta||^2 \|h\|^2\). On the other hand, applying the Cauchy-Schwarz inequality together with \(\lambda_j \geq 2\alpha\) for all \(j = 1, \ldots, m\) implies

\[
E\left\{ \sum_{j=1}^{m} [h_j] \beta_j 1\{\lambda_j < 1/\alpha\} \right\} \leq \left\{ \sum_{j=1}^{m} [h_j] \right\} \left\{ \sum_{j=1}^{m} \beta_j^2 P\lambda_j/\lambda_j < 1/2 \right\} \leq C ||h||^2 ||\beta||^2 n^{-1} \eta
\]

for some \(C > 0\), where the last inequality follows from (A.20) in Lemma A.2. Combining the two bounds we obtain (A.16), which completes the proof.

\[\square\]

The finitely and infinitely smoothing case.

**Proof of Proposition 3.3.** Observe that \(W^p_p = F^p_p\) and \(W^p_p = F^p_p\) with \(\gamma = (\gamma_j)_{j \geq 1}\) and \(\omega = (\omega_j)_{j \geq 1}\) given by \(\gamma_1 := 1, \gamma_j := |j|^{2p}\) and \(\omega_1 := 1, \omega_j := |j|^{2s}, j \geq 2\), respectively. Obviously, the sequences \(\gamma, \omega\) and \(v\) given in (i) by \(v = 1, v_j = |j|^{-2\alpha}\) and (ii) by \(v = 1, v_j = |j|^{-2\alpha}\)
exp(−|j|^{2a}), j ≥ 2, satisfy Assumption 2.1. Furthermore, in case (i) we have 1/(γ_m, v_m,) = m^{2a+2p}. It follows that m_\ast and δ_\ast given in (2.4) of Theorem 2.1 satisfies m_\ast \sim n^{1/(2p+2a)} and δ_\ast \sim n^{-(p+s)/(p+a)} respectively. On the other hand, 1/(γ_m, v_m,) = m^{2p} \exp(m^{2a}) implies in case (ii) that m_\ast \sim (log n)^{1/(2a)} and δ_\ast \sim (log n)^{(p-s)/a}. Consequently, the lower bounds in Proposition 3.3 follow by applying Theorem 2.1.

\textbf{Proof of Proposition 3.4.} Since the condition on m and α ensures in both cases that m \sim m_\ast and α \sim n (see proof of Proposition 3.3) the result follows from Theorem 3.2.

\textbf{Technical assertion.}

\textbf{Lemma A.2.} Suppose X ∈ \mathcal{X}_\eta^{4k} and ε ∈ \mathcal{E}_\eta^{4k}, k ∈ \mathbb{N}. Then for some constant C > 0 only depending on k we have

\begin{align}
\sup_{m \in \mathbb{N}} \sup_{s \in \mathbb{S}^m} \mathbb{E} \left[ \sum_{j=1}^m s_j \cdot T_{n,j} \right]^{2k} &\leq C n^{-k} \left( \|\beta\|^2 \mathbb{E} \|X\|^2 + \sigma^2 \right)^k \eta, \tag{A.18} \\
\sup_{j \in \mathbb{N}} \mathbb{E}\left[ \widehat{\lambda}_j / \lambda_j - 1 \right]^{2k} &\leq C n^{-k} \eta, \tag{A.19} \\
\sup_{j \in \mathbb{N}} \mathbb{P}(\widehat{\lambda}_j / \lambda_j < 1/2) &\leq C n^{-k} \eta. \tag{A.20}
\end{align}

\textbf{Proof.} Let s \in \mathbb{S}^m and denote \zeta_{i,j} := \langle \beta, X_i \rangle - [\beta]_j[i]. Then by the definition of T_{n,j} given in (A.10) we have

\[ \sum_{j=1}^m s_j \cdot T_{n,j} = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^m s_j[Z_i,j] \{ \zeta_{i,j} + \sigma \varepsilon_i \} =: S_1 + S_2, \]

where we bound below each summand separately, that is

\begin{align}
\mathbb{E}|S_1|^{2k} &\leq C \cdot n^{-k} \cdot \|\beta\|^{2k} \cdot (\mathbb{E} \|X\|^2)^k \cdot \eta, \tag{A.21} \\
\mathbb{E}|S_2|^{2k} &\leq C \cdot n^{-k} \cdot \sigma^{2k} \cdot \eta. \tag{A.22}
\end{align}

for some C > 0 only depending on k. Consequently, the inequality (A.18) follows from (A.21) and (A.22). Consider S_1. The random variables (\sum_j s_j[Z_1,j] \zeta_{1,j})_{1 \leq i \leq n}, are i.i.d. with mean zero. From Theorem 2.10 in Petrov [1995] we conclude \mathbb{E}|S_1|^{2k} ≤ C n^{-k} \mathbb{E} \left\| \sum_j s_j[Z_1,j] \zeta_{1,j} \right\|^{2k}

for some constant C > 0 only depending on k. Then we claim that (A.21) follows in case of S_1 from the Cauchy-Schwarz inequality together with X_1 ∈ \mathcal{X}_\eta^{4k}, i.e., \mathbb{E} \left\| \sum_j s_j[Z_1,j] \right\|^{4k} ≤ \eta and \sup_{i \in \mathbb{N}} \mathbb{E}|Z_1|^{4k} \leq \eta. Indeed, we have

\[ \mathbb{E}\left\| \sum_{j=1}^m s_j[Z_1,j] \zeta_{1,j} \right\|^{2k} \leq 2^{2k-1} \left\{ (\mathbb{E} |\langle \beta, X_1 \rangle|^{4k})^{1/2} (\mathbb{E} \left\| \sum_{j=1}^m s_j[Z_1,j] \right\|^{4k})^{1/2} \right. \\
\left. + (\mathbb{E} \left\| \sum_{j=1}^m s_j[Z_1,j] \right\|^{2k})^{1/2} (\mathbb{E} \left\| \sum_{j=1}^m s_j[Z_1,j] \right\|^{2k})^{1/2} \right\} \]

where \mathbb{E}|\langle \beta, X_1 \rangle|^{4k} ≤ \|\beta\|^{4k} \sum_j \lambda_{j_1} \cdots \sum_{j_{2k}} \lambda_{j_{2k}} \prod_{i=1}^{2k} |Z_{i,j_i}|^2 ≤ \|\beta\|^{4k} (\mathbb{E} \|X\|^2)^{2k} \eta by using \mathbb{E} \|X\|^2 = \sum_j \lambda_j \) and in an analogous manner \mathbb{E} \left\| \sum_{j=1}^m s_j[Z_1,j] \right\|^{2k} ≤ \|\beta\|^{4k} (\mathbb{E} \|X\|^2)^{2k} \eta.
and $\mathbb{E} |\sum_{j=1}^{m} s_j^2 [Z_j^2]^{2k}| \leq \eta$. Consider $S_2$. (A.22) follows in analogy to the case of $S_1$, because $\{\sum_{j=1}^{m} s_j [Z_j^2] \epsilon_j \}$ are i.i.d. with mean zero. Note that $X_1 \in X^{4k}$ and $\epsilon \in \mathcal{E}^{4k}$ imply together $\mathbb{E} |\sum_{j=1}^{m} s_j [Z_j^2] \epsilon_j|^{2k} \leq \sigma^{2k} \eta$.

Proof of (A.19). From the identity $\hat{\lambda}_j / \lambda_j = (1/n) \sum_i |Z_i|^2$ the result follows by applying Theorem 2.10 in Petrov [1995], because $\{[Z_i]^2 - 1\}_{1 \leq i \leq n}$ are i.i.d. with mean zero, and $\mathbb{E} |[Z_i]|^{4k} \leq \eta$.

To deduce (A.20) from (A.19) by applying Markov’s inequality, take $\mathbb{P}(\hat{\lambda}_j / \lambda_j \leq 1/2) \leq \mathbb{P}(\hat{\lambda}_j / \lambda_j - 1 \geq 1/2)$, which proves the lemma.

### A.3 Proofs of Section 4

We begin by defining and recalling notations to be used in the proofs of this section. Given $m > 0, \beta_m \in \Psi_m$ denotes a Galerkin solution of $g = T_{\text{cov}} \beta$, i.e.,

$$\|g - T_{\text{cov}} \beta_m\| \leq \|g - T_{\text{cov}} \beta\|, \quad \forall \beta \in \Psi_m.$$  \hspace{1cm} (A.23)

Since $T_{\text{cov}}$ is strictly positive it follows that $\beta_m = [T_{\text{cov}}^{-1}]_m \mathbf{g}_m$ is the unique Galerkin solution satisfying $[T_{\text{cov}} (\beta - \beta_m)]_m = 0$. Furthermore, we use the notations

$$[\hat{T}_{\text{cov}}]_m = \frac{1}{n} \sum_{i=1}^{n} \langle X_i \rangle_m [X_i]^4_m, \quad \langle X_i \rangle_m := [T_{\text{cov}}]_m^{-1/2} [X_i]_m, \quad [\tilde{T}_{\text{cov}}]_m := \frac{1}{n} \sum_{i=1}^{n} [X_i]_m [\hat{X}_i]_m,$$

$$[\Xi]_m := [\hat{T}_{\text{cov}}]_m - I_m, \quad [V]_m := \frac{1}{n} \sum_{i=1}^{n} \langle X_i, \beta - \beta_m \rangle [X_i]_m, \quad [W]_m := \frac{1}{n} \sum_{i=1}^{n} \epsilon_i [X_i]_m,$$

and $[Z]_m := [V]_m + [W]_m = \mathbf{g} - [\tilde{T}_{\text{cov}}]_m \beta_m$.  \hspace{1cm} (A.24)

where $\mathbb{E} [V]_m = [T_{\text{cov}} (\beta - \beta_m)]_m = 0$, $\mathbb{E} [W]_m = 0$, hence $\mathbb{E} [Z]_m = 0$, and furthermore $\mathbb{E} [\hat{T}_{\text{cov}}]_m = [T_{\text{cov}}]_m, [\tilde{T}_{\text{cov}}]_m = [T_{\text{cov}}]_m^{-1/2} [T_{\text{cov}}]_m [T_{\text{cov}}]_m^{-1/2}$, thus $\mathbb{E} [\Xi]_m = 0$. Moreover, let us introduce the events

$$\Omega := \\{\|T_{\text{cov}}^{-1}\| \leq \alpha\}, \quad \Omega_{1/2} := \\{\|\Xi\|_m \leq 1/2\}$$

$$\Omega^c := \\{\|T_{\text{cov}}^{-1}\| > \alpha\} \quad \text{and} \quad \Omega^c_{1/2} = \\{\|\Xi\|_m > 1/2\}. \hspace{1cm} (A.25)$$

Observe that $\Omega_{1/2} \subset \Omega$ in case $\alpha \geq 2\|T_{\text{cov}}^{-1}\|$. Indeed, if $\|\Xi\|_m \leq 1/2$ then the identity $[\hat{T}_{\text{cov}}]_m = [T_{\text{cov}}]_m^{-1/2} \{I + [\Xi]_m\} [T_{\text{cov}}]_m^{1/2}$ implies by the usual Neumann series argument that $\|T_{\text{cov}}^{-1}\| \leq 2\|T_{\text{cov}}^{-1}\|$. Thereby, if $\alpha \geq 2\|T_{\text{cov}}^{-1}\|$, then we have $\Omega_{1/2} \subset \Omega$. These results will be used below without further reference.

We shall prove in the end of this section two technical Lemmata (A.3 and A.4) which are used in the following proofs.

**Proof of the consistency.**

**Proof of Proposition 4.1.** Let $\hat{\ell}_h^m := \ell_h(\beta_m) 1\{\|T_{\text{cov}}^{-1}\| \leq \alpha\}$. Then the proof is based on the decomposition

$$\mathbb{E} |\hat{\ell}_h - \ell_h(\beta)|^2 \leq 2 \{\mathbb{E} |\hat{\ell}_h - \hat{\ell}_h^m|^2 + \mathbb{E} |\hat{\ell}_h^m - \ell_h(\beta)|^2\}. \hspace{1cm} (A.26)$$

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Since \( \alpha \geq 2 \| [T_{\text{cov}}^{-1}] \| \) it follows that \( \Omega^c \subset \Omega_{1/2}^c \) and hence

\[
\mathbb{E}[\ell_h^n - \ell_h(\beta)]^2 \leq 2\{ |\ell_h(\beta - \beta_m)|^2 + |\ell_h(\beta_m)|^2 \mathbb{P}(\Omega_{1/2}) \}. \tag{A.27}
\]

On the other hand we show below for some constant \( C > 0 \) the following bound

\[
\mathbb{E}[\hat{\ell}_h - \hat{\ell}_h(\beta)]^2 \leq C \cdot (1/n) \| [h]_m^T [T_{\text{cov}}]^{-1/2} \|^2 \eta \left\{ \sigma^2 + \| \beta - \beta_m \|^2 \mathbb{E}[|X|^2] \right\}
\]

\[
\left\{ m \eta^{-1/2}(P(\Omega_{1/2}^c))^{1/2} + \alpha^2 m^3 n^{-1} \eta^{-1/2}(P(\Omega_{1/2}^c))^{1/2} \| [T_{\text{cov}}]^{-1/2} + m^3/n \right\}. \tag{A.28}
\]

where by applying Markov’s inequality (A.33) in Lemma A.3 implies \( P(\Omega_{1/2}^c) \leq C \eta m^4/n^2 \) for some \( C > 0 \). Moreover, \( \| [T_{\text{cov}}]^{-1/2} \|^2 \| [h]_m[T_{\text{cov}}]^{-1/2} \|^2 \leq \alpha \| h \|^2 \) since \( \alpha \geq 2 \| [T_{\text{cov}}]^{-1/2} \|^2 \), which by combination of (A.27) and (A.28) leads to the estimate

\[
\mathbb{E}[\hat{\ell}_h - \ell_h(\beta)]^2 \leq C \left\{ |\ell_h(\beta - \beta_m)|^2 + |\ell_h(\beta_m)|^2 (m^4/n^2) \eta \right. 
\]

\[
+ \left( \alpha/n \right) \| h \|^2 \eta \left\{ \sigma^2 + \| \beta - \beta_m \|^2 \mathbb{E}[|X|^2] \right\} \left\{ (\alpha^2 m^2/n) \| T_{\text{cov}} \|^2 + 1 \right\} (m^3/n) \right\}. \tag{A.29}
\]

for some \( C > 0 \). Furthermore, for each \( \beta \in L^2[0, 1] \), we have \( \| \beta - \beta_m \| = o(1) \) as \( m \to \infty \), which can be realized as follows. Since \( \| \Pi_{\alpha} \beta \| = o(1) \) as \( m \to \infty \) by using Lebesgue’s dominated convergence theorem, the assertion follows from the identity \( [\Pi_{\alpha} \beta - \beta_m] = -[T_{\text{cov}}]^{-1/2}[T_{\text{cov}} \Pi_{\alpha} \beta - \beta_m] \) by using that \( \| \Pi_{\alpha} \beta - \beta_m \| \leq \| \Pi_{\alpha} \beta \| \| \Pi_{\alpha} \beta - \beta_m \| = O(\| \Pi_{\alpha} \beta \|) \). Consequently, the conditions on \( m \) and \( \alpha \) ensure the convergence to zero as \( n \to \infty \) of the bound given in (A.29), which proves the result.

Proof of (A.28). From the identity \( \hat{[g]}_m - [\hat{T}_{\text{cov}}]_m[\beta m] = [Z]_m \) it follows that

\[
\mathbb{E}[\hat{\ell}_h - \ell_h(\beta)]^2 = \mathbb{E}[h]_m^T \{ [T_{\text{cov}}]^{-1} + [\hat{T}_{\text{cov}}]_m^{-1}([\hat{T}_{\text{cov}}]_m - [\hat{T}_{\text{cov}}]_m)\} [Z]_m^T 1\Omega.
\]

Since \( 2 \| [T_{\text{cov}}]^{-1/2} \| \leq \alpha \) we have \( \Omega_{1/2} \subset \Omega \), and hence by using \( \| [\hat{T}_{\text{cov}}]^{-1/2} \| \leq \alpha^2 \) and \( \| [I + [\Xi]_m]^{-1} \| \leq \alpha^2 \) we obtain

\[
\mathbb{E}[\hat{\ell}_h - \ell_h(\beta)]^2 \leq 4 \left\{ \mathbb{E}[h]_m^T [T_{\text{cov}}]^{-1} [Z]_m^T \| + [h]_m^T [T_{\text{cov}}]^{-1} \right\} \left\{ \mathbb{E}[h]_m^T [T_{\text{cov}}]^{-1/2} [Z]_m^T \|^{1/2}(P(\Omega_{1/2}^c))^{1/2} \right. 
\]

\[
+ \alpha^2 \| [T_{\text{cov}}]^{-1/2} \| \mathbb{E}[[Z]_m^T \|^{1/2}(P(\Omega_{1/2}^c))^{1/2} \right. 
\]

\[
+ 4(\mathbb{E}[\Xi]_m^T \|^{1/2}(P(\Omega_{1/2}^c))^{1/2} \right\}^2) \right\},
\]

where \( \mathbb{E}[h]_m^T [T_{\text{cov}}]^{-1} [Z]_m^T \leq \| [h]_m^T [T_{\text{cov}}]^{-1} \| \sup_{x \in \Xi} \mathbb{E}|x| [T_{\text{cov}}]^{-1} [Z]_m^T \|^2 \). From (A.31) - (A.33) in Lemma A.3 follows then (A.28), which completes the proof.

**Proof of Corollary 4.2.** Due to (A.37) in Lemma A.4 the link condition \( T_{\text{cov}} \in \mathcal{N}_v^d \) implies \( 2 \| [T_{\text{cov}}]^{-1} \| \leq 8d^3/v_m = \alpha \). Thus, from (A.29) in the proof of Proposition 4.1 follows

\[
\mathbb{E}[\hat{\ell}_h - \ell_h(\beta)]^2 \leq C \left\{ |\ell_h(\beta - \beta_m)|^2 + |\ell_h(\beta_m)|^2 (m^4/n^2) \eta \right. 
\]

\[
+ d^3/(v_m n) \| h \|^2 \eta \left\{ \sigma^2 + \| \beta - \beta_m \|^2 \mathbb{E}[|X|^2] \right\} \left\{ (d^6 m^2/(v_m n)) \| T_{\text{cov}} \|^2 + 1 \right\} (m^3/n) \right\}. \tag{A.30}
\]
for some $C > 0$. Furthermore, the link condition $T_{\text{cov}} \in \mathcal{N}_d$ implies $\| (T_{\text{cov}})^{-1} \Pi_m T_{\text{cov}} \Pi_m^\perp \| = \sup_{\|\beta\|_2 = 1} \| \Pi_m \beta - \beta_m \| ^2 \leq 2(1 + d^2)$ for all $m \in \mathbb{N}$ by using the identity $[\Pi_m \beta - \beta_m]_{m}^{-1} = -[T_{\text{cov}}]_{m}^{-1} [T_{\text{cov}} \Pi_m \beta]_{m}$ and the estimate (A.40) in Lemma A.4. Thus, $\| \beta - \beta_m \| = o(1)$ as $m \to \infty$ for all $\beta \in L^2[0,1]$. Consequently, the conditions on $m$ and $\alpha$ ensure the convergence to zero of the bound given in (A.30) as $n \to \infty$, which proves the result. 

\[ \square \]

**Proof of the upper bound.**

**Proof of Theorem 4.3.** Our proof starts with the observation that the link condition $T_{\text{cov}} \in \mathcal{N}_d$ implies $2\| [T_{\text{cov}}]_{m}^{-1} \| \leq 8d^3 / \nu_m$, $\| [\text{Diag}(\omega)]_{m}^{1/2} [T_{\text{cov}}]_{m}^{-1/2} \| \leq 4d^3 \sup_{1 \leq j \leq m} \{ \omega_j / \nu_j \}$ and $\| [T_{\text{cov}}]_{m}^{-1} \| ^2 \leq d^3$ by using the estimates (A.37), (A.38) and (A.39) in Lemma A.4 respectively. Hence, under Assumption 2.1 we have $\| [h]_{m}^{2/3} [T_{\text{cov}}]_{m}^{-1/2} \| ^2 \leq 4d^3 D \tau \nu_m \max(\omega_m^{-1}, \nu_m)$ for all $h \in \mathcal{F}_\gamma$. Moreover, since $X \in \mathcal{X}_{\eta}^{4k}$ it follows that $\mathbb{E} \| X \| ^2 \leq d \Lambda$ and from (A.33) in Lemma A.3 by applying Markov’s inequality that $P(\Omega_{1/2}^\nu) \leq C \eta m^{2k} / n^k$ for some $C > 0$. Furthermore, by using the definition of $m_*$, i.e., $1 / \nu_m = n / \gamma_m$, the condition $m = m_*$ implies $\alpha = n \max(1, 8d^3 / \gamma_m) \geq 2\| [T_{\text{cov}}]_{m}^{-1} \|$ and $\alpha / n \leq 8d^3 \Delta$. Therefore, from (A.27) and (A.28) in the proof of Proposition 4.1 follows

\[
\mathbb{E}[\delta_h - \ell_h(\beta)]^2 \leq C \left\{ [\ell_h(\beta - \beta_m)]^2 + [\ell_h(\beta_m)]^2 \eta \frac{m_{2k}}{n^k} + \frac{1}{m \nu_m} \max(\omega_m^{-1}, \nu_m) \right. \\
\left. d^3 D \tau \eta \{ \sigma^2 + \| \beta - \beta_m \| ^2 \} \, dA \right\}
\]

for some $C > 0$. Moreover, the definition of $m_*$ and $\delta_n^*$ implies $1 / n \nu_m \max(\omega_m^{-1}, \nu_m) \leq \Delta \max(\delta_n^*, 1 / n)$ and together with (A.41) and (A.42) in Lemma A.4 $\| \beta - \beta_m \| ^2 \leq 10d^4 \rho / \gamma_m$ and $[\ell_h(\beta - \beta_m)]^2 \leq 10 D d^4 \rho \tau \delta_n^*$ for all $\beta \in \mathcal{F}_\gamma$ and $h \in \mathcal{F}_{\nu}$. Consequently, we have

\[
\mathbb{E}[\delta_h - \ell_h(\beta)]^2 \leq C \max(\delta_n^*, 1 / n) \Delta^3 d^{11} D \eta \rho \tau \{ \sigma^2 + \rho d^5 \Lambda / \gamma_m + 1 \} \\
\left[ 1 + \frac{m_{2k} n^{k} - k}{\max(\delta_n^*, 1 / n)} + \frac{m_{1+k/2} + m_{3+k/2}}{n^{k/2-1}} + \frac{m_3}{n} \right]
\]

Thereby, the result follows from the condition (4.2) which ensures that the factor in brackets is bounded as $n \to \infty$, which completes the proof. 

\[ \square \]

**The finitely and infinitely smoothing case.**

**Proof of Proposition 4.4.** Observe that both cases the condition (4.2) is satisfied, where in part (i) it follows from the additional assumption $p + a > 3 / 2$. Since the condition on $m$ and $\alpha$ ensures again in both cases that $m \sim m_*$ and $\alpha \sim n$ the result follows also from Theorem 3.2. 

\[ \square \]

**Technical assertions.**

The following two lemmata gather technical results used in the proof of Proposition 4.1 and Theorem 4.3.
Lemma A.3. Suppose \( X \in X_{\eta}^{4k} \) and \( \varepsilon \in \mathcal{E}_{\eta}^{4k} \), \( k \in \mathbb{N} \). Then for some constant \( C > 0 \) only depending on \( k \) we have

\[
\sup_{z \in S^m} \mathbb{E} \left[ \sum_{j=1}^{m} z_j \left( [T_{\text{cov}}]^{-1/2} T_{n,m} \right)_j \right]^{2k} \leq C n^{-k} \left( \|\beta - \beta_m\| \mathbb{E} \|X\|^2 + \sigma^2 \right)^k \eta, \tag{A.31}
\]

\[
\mathbb{E} \| [T_{\text{cov}}]^{-1/2} T_{n,m} \|^2 \leq C \frac{m^k}{n^k} \left\{ \|\beta - \beta_m\|^{2k} (\mathbb{E} \|X\|^2)^k + \sigma^{2k} \right\} \eta, \tag{A.32}
\]

\[
\mathbb{E} \| \Xi_{n,m} \|^{2k} \leq C \cdot \eta \cdot \frac{m^{2k}}{n^k}, \tag{A.33}
\]

\[
\mathbb{E} \left\{ \left[ [T_{\text{cov}}] - \left[ \hat{T}_{\text{cov}} \right] \right]^{-1/2} \left[ T_{\text{cov}} \right]^{-1/2} \right\}^{2k} \leq C \cdot \eta \cdot \frac{m^{2k}}{n^k} \left\{ \mathbb{E} \|X\|^2 \right\}^k \tag{A.34}
\]

Proof. Let \( z \in S^m \) and denote \( \zeta_{i,j} := \langle \beta - \beta_m, X_i \rangle X_j \). Then by the definition of \( T_{n,m} \) given in (A.24) we have

\[
\sum_{j=1}^{m} z_j \cdot \left[ [T_{\text{cov}}]^{-1/2} T_{n,m} \right]_j = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} z_j \{ \zeta_{i,j} + \sigma \varepsilon_i X_j \} =: S_1 + S_2,
\]

where we bound below each summand separately, that is

\[
\mathbb{E} [S_1]^{2k} \leq C \cdot n^{-k} \cdot \|\beta - \beta_m\|^{2k} \cdot (\mathbb{E} \|X\|^2)^k \cdot \eta, \tag{A.35}
\]

\[
\mathbb{E} [S_2]^{2k} \leq C \cdot n^{-k} \cdot \sigma^{2k} \cdot \eta \tag{A.36}
\]

for some \( C > 0 \) only depending on \( k \). Consequently, the inequality (A.31) follows from (A.35) and (A.36). Consider \( S_1 \). Since \( \mathbb{E} \langle \beta - \beta_m, X_i \rangle X_j \) \( \|X\|_{\text{cov}}(\beta - \beta_m)\|_{\text{cov}} = [g]_{\text{cov}}[\beta]_{\text{cov}} = 0 \) for all \( 1 \leq i \leq n \), it follows that the random variables \( \sum_j z_j \zeta_{i,j} \), \( i = 1, \ldots, n \), are i.i.d. with mean zero. From Theorem 2.10 in Petrov [1995] we conclude \( \mathbb{E} [S_1]^{2k} \leq C n^{-k} \mathbb{E} \sum_{j=1}^{m} |z_j \zeta_{1,j}|^{2k} \) for some constant \( C > 0 \) only depending on \( k \). Then we claim that (A.35) follows from the Cauchy-Schwarz inequality together with \( X_1 \in X_{\eta}^{4k} \), i.e., \( \mathbb{E} \sum_{j=1}^{m} |z_j \hat{X}_j|^{4k} \leq \eta \) and \( \sup_{j \in \mathbb{N}} \mathbb{E} |X_j|^{1/2} \mathbb{E} |X_j|^{1/2} \leq \eta \). Indeed, we have

\[
\mathbb{E} \langle \beta - \beta_m, X_1 \rangle^{4k} \leq \|\beta - \beta_m\|^{4k} \mathbb{E} \sum_{j_1} \sum_{j_2} \sum_{j_3} \sum_{j_4} \cdots \sum_{j_{2k}} [T_{\text{cov}}]_{j_1,j_1} [T_{\text{cov}}]_{j_2,j_2} \cdots \sum_{j_{2k}} [T_{\text{cov}}]_{j_{2k},j_{2k}} \mathbb{E} \prod_{l=1}^{2k} |X_{j_l}|^{1/2} \mathbb{E} [T_{\text{cov}}]_{j_l,j_l}^{1/2} \mathbb{E} [T_{\text{cov}}]_{j_l,j_l}^{1/2}
\]

\[
\leq \|\beta - \beta_m\|^{4k} (\mathbb{E} \|X\|^2)^{2k} \eta.
\]

and hence

\[
\mathbb{E} \sum_{j=1}^{m} |z_j \zeta_{1,j}|^{2k} \leq (\mathbb{E} \langle \beta - \beta_m, X_1 \rangle^{4k})^{1/2} (\mathbb{E} \sum_{j=1}^{m} |z_j \hat{X}_1|^{4k})^{1/2}
\]

\[
\leq \|\beta - \beta_m\|^{2k} (\mathbb{E} \|X\|^2)^{k} \eta.
\]

Consider \( S_2 \). Since \( \sigma \varepsilon \sum_j z_j \hat{X}_j \) are i.i.d. with mean zero, (A.36) follows in analogy to the proof of (A.35) by using \( \mathbb{E} \varepsilon_1 \sum_j z_j \hat{X}_j \leq \eta \) for all \( X_1 \in X_{\eta}^{4k} \) and \( \varepsilon_1 \in \mathcal{E}_{\eta}^{4k} \).

To deduce (A.32) from (A.31) we use that

\[
\mathbb{E} \| [T_{\text{cov}}]^{-1/2} T_{n,m} \|^{2k} \leq m^{k-1} \sum_{j=1}^{m} \mathbb{E} \| [T_{\text{cov}}]^{-1/2} T_{n,m} \|^{2k} \leq m^{k-1} \sup_{z \in S^m} \mathbb{E} \left[ \sum_{j=1}^{m} z_j \left( [T_{\text{cov}}]^{-1/2} T_{n,m} \right)_j \right]^{2k}.
\]

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Proof of (A.33). From the identity \((\Xi_{n,m})_{j,l} = (1/n) \sum_{i=1}^{n} [\tilde{X}_{i}]_{j} [\tilde{X}_{i}]_{l} - \delta_{j,l}\) with \(\delta_{j,l} = 1\) if \(j = l\) and zero otherwise, we conclude \(E(\Xi_{n,m})_{j,l}^{2k} \leq C n^{-k} E(\|X_1\|_j |X_1| - \delta_{j,l})^{2k}\). Thus \(X \in \mathcal{X}_{n}^{2k}\) implies \(E(\Xi_{n,m})_{j,l}^{2k} \leq C m^{2k} n^{-k} \eta\).

The estimate (A.34) follows by using \([\hat{T}_{cov}]_{m} - [T_{cov}]_{m}] \leq \frac{1}{\sqrt{m}} [T_{cov}]_{m}^{1/2} \Xi_{n,m}\) from (A.33), which completes the proof.

The next Lemma is partially shown in Cardot and Johannes [2008].

**Lemma A.4.** Suppose the sequences \(\gamma, \omega\) and \(v\) satisfy Assumption 2.1. Let \(T \in \mathcal{N}_{0}^{d}\). Then

\[
\sup_{m \in \mathbb{N}} \left\{ v_m \|T_{m}^{-1/2}\|^2 \right\} \leq \{2d^2 (2d^4 + 3)\}^{1/2} \leq 4d^3, \tag{A.37}
\]

\[
\sup_{m \in \mathbb{N}} \|T_{m}^{-1/2} [\text{Diag}(v)]_{m}^{1/2}\|^2 \leq \{2d^2 (2d^4 + 3)\}^{1/2} \leq 4d^3, \tag{A.38}
\]

\[
\sup_{m \in \mathbb{N}} \|T_{m}^{-1/2} [\text{Diag}(v)]_{m}^{1/2}\|^2 \leq d. \tag{A.39}
\]

If in addition \(\beta_m\) denotes a Galerkin solution of \(g = T \beta\) then

\[
\sup_{m \in \mathbb{N}} \left\{ \sup_{\|\beta\|=1} \|\Pi_m \beta - \beta_m\|^2 \right\} \leq 2(1 + d^2), \tag{A.40}
\]

and in case \(\beta \in \mathcal{F}_0^d\) is additionally satisfied then

\[
\sup_{m \in \mathbb{N}} \|\beta_m - \beta_m\|^2 \leq 2(2d^4 + 3) \rho \leq 10d^4 \rho. \tag{A.41}
\]

Suppose additionally that \(h \in \mathcal{F}_0^d\), then we have

\[
\sup_{m \in \mathbb{N}} \{ \gamma_m [\max(\omega_m^{-1}, v_m^2)]^{-1} |\langle h, \beta - \beta_m \rangle|^2 \} \leq 2D (3 + 2d^4) \rho \tau \leq 10Dd^4 \rho \tau. \tag{A.42}
\]

**Proof.** The estimates (A.37) - (A.39) are given in Lemma A.3 in Cardot and Johannes [2008]. Furthermore, from (A.19) and (A.20) in Lemma A.3 in Cardot and Johannes [2008] follows (A.40) and (A.41). We start our proof of (A.42) with the observation that the link condition \(T \in \mathcal{N}_{0}^{d}\) implies that \(T\) is strictly positive and that for all \(|s| \leq 1\) by using the inequality of Heinz [1951]

\[
d^{-2|s|} \|f\|^2_{\mathcal{L}^{2s}} \leq \|T^s f\|^2 \leq d^{2|s|} \|f\|^2_{\mathcal{L}^{2s}}. \tag{A.43}
\]

Thus, by using successively the first inequality of (A.43), the Galerkin condition (A.23) and the second inequality of (A.43), we obtain

\[
\|\beta - \beta_m\|^2 \leq d^2 \|T(\beta - \beta_m)\|^2 \leq d^4 \|\beta - \Pi_m \beta\|^2 \leq d^4 \|\beta - \beta_m\|^2 \tag{A.44}
\]

Since \(\beta \in \mathcal{F}_0^d\) and \((\gamma_m^{-1} v_m^2)_{j \in \mathbb{N}}\) is monotonically decreasing, (A.44) implies \(\|\beta - \beta_m\|^2 \leq d^4 \|\beta - \Pi_m \beta\|^2 \leq d^4 \|\gamma_m^{-1} v_m^2\| \|\beta\|^2 \) and hence,

\[
\|\Pi_m \beta - \beta_m\|^2 \leq 2\{\|\beta - \beta_m\|^2 + \|\beta - \Pi_m \beta\|^2\} \leq 2(1 + d^4) \gamma_m^{-1} v_m^2 \|\beta\|^2. \tag{A.45}
\]

Finally, by applying the Cauchy-Schwarz inequality we have

\[
|\langle h, \beta - \Pi_m \beta \rangle|^2 \leq \omega_m^{-1} \gamma_m^{-1} \|h\|^2_{\mathcal{L}^{2s}} \|\beta\|^2 \tag{A.46}
\]
and by using (A.45) it follows
\[
|\langle h, \Pi_m \beta - \beta_m \rangle|^2 \leq \|h\|_2^2 \|\text{Diag}(\omega)\|_m^{-1/2} \|\text{Diag}(\nu)\|_m^{-1} \|\Pi_m \beta - \beta_m\|_2^2 \\
\leq 2(1 + d^4) \gamma_m^{-1} v_m^2 \{\sup_{1 \leq j \leq m} 1/(\omega_j v_j^2)\} \|h\|_2^2 \|\beta\|_2^2, \quad (A.47)
\]

The estimate (A.42) follows now from (A.46) and (A.47) since under Assumption 2.1 there exists a constant \(D\) such that
\[
v_m^2 \{\sup_{1 \leq j \leq m} 1/(\omega_j v_j^2)\} \leq D \max(\omega_m^{-1}, v_m^2) \text{ for all } m \in \mathbb{N},
\]
which completes the proof.

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