DYNAMICAL INVARIANTS AND BERRY’S PHASE FOR GENERALIZED DRIVEN HARMONIC OSCILLATORS

BARBARA SANBORN, SERGEI K. SUSLOV, AND LUC VINET

Abstract. We present quadratic dynamical invariant and evaluate Berry’s phase for the time-dependent Schrödinger equation with the most general variable quadratic Hamiltonian.

1. Introduction

In the previous Letter [24], the exact wave functions for generalized (driven) harmonic oscillators [2], [4], [20], [26], [29], [53], [55], [56] have been constructed in terms of Hermite polynomials by transforming the time-dependent Schrödinger equation into an autonomous form [57]. Relationships with certain Ermakov and Riccati-type systems have been investigated. A goal of this Letter is to find the corresponding dynamical invariants and to evaluate Berry’s phase [1], [2], [40], [52] for quantum systems with general variable quadratic Hamiltonians as an extension of the works [3], [12], [18], [21], [22], [26], [36], [35], [46] (see also references therein).

2. Generalized Driven Harmonic Oscillators

We consider the one-dimensional time-dependent Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = H \psi,$$

(2.1)

where the variable Hamiltonian $H = Q(p, x)$ is an arbitrary quadratic of two operators $p = -i \partial / \partial x$ and $x$, namely,

$$i \psi_t = -a(t) \psi_{xx} + b(t) x^2 \psi - ic(t) x \psi_x - id(t) \psi - f(t) x \psi + ig(t) \psi_x$$

(2.2)

(a, b, c, d, f and g are suitable real-valued functions of time only). We shall refer to these quantum systems as the generalized (driven) harmonic oscillators. A general approach and known elementary solutions can be found in Refs. [4], [5], [6], [7], [10], [14], [15], [16], [24], [29], [30], [33], [42], [53] and [56]. In addition, a case related to Airy functions is discussed in [25] and Ref. [8] deals with another special case of transcendental solutions.

In this Letter, we shall use the following result established in [24].

Lemma 1. The substitution

$$\psi = \frac{e^{i(\alpha(t) x^2 + \beta(t) x + c(t))}}{\sqrt{\mu(t)}} \chi(\xi, \tau), \quad \xi = \beta(t) x + \varepsilon(t), \quad \tau = \gamma(t)$$

(2.3)

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transforms the non-autonomous and inhomogeneous Schrödinger equation (2.2) into the autonomous form

\[-i\chi_t = -\chi_{xx} + c_0\xi^2\chi \quad (c_0 = 0, 1)\]  

(2.4)

provided that

\[\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = c_0a\beta^4,\]  

(2.5)

\[\frac{d\beta}{dt} + (c + 4a\alpha)\beta = 0,\]  

(2.6)

\[\frac{d\gamma}{dt} + a\beta^2 = 0\]  

(2.7)

and

\[\frac{d\delta}{dt} + (c + 4a\alpha)\delta = f + 2g\alpha + 2c_0a\beta^3\varepsilon,\]  

(2.8)

\[\frac{d\varepsilon}{dt} = (g - 2a\delta)\beta,\]  

(2.9)

\[\frac{d\kappa}{dt} = g\delta - a\delta^2 + c_0a\beta^3\varepsilon^2.\]  

(2.10)

Here

\[\alpha = \frac{1}{4a} \frac{\mu'}{\mu} - \frac{d}{2a}.\]  

(2.11)

The substitution (2.11) reduces the inhomogeneous equation (2.5) to the second order ordinary differential equation

\[\mu'' - \tau (t) \mu' + 4\sigma (t) \mu = c_0 (2a)^2 \beta^4 \mu,\]  

(2.12)

that has the familiar time-varying coefficients

\[\tau (t) = \frac{a'}{a} - 2c + 4d, \quad \sigma (t) = ab - cd + d^2 + \frac{d}{2} \left( \frac{a'}{a} - \frac{d'}{d} \right).\]  

(2.13)

When \(c_0 = 0\), equation (2.5) is called the \textit{Riccati nonlinear differential equation} [50], [51] and the system (2.5)–(2.10) shall be referred to as a \textit{Riccati-type system}. (Similar terminology is used in [44], [45] for the corresponding parabolic equation.) If \(c_0 = 1\), equation (2.12) can be reduced to a generalized version of the \textit{Ermakov nonlinear differential equation} (see, for example, [6], [13], [27], [46] and references therein regarding Ermakov’s equation) and we shall refer to the corresponding system (2.5)–(2.10) with \(c_0 \neq 0\) as an \textit{Ermakov-type system}. Throughout this Letter, we use the notations from Ref. [24] where a more detailed bibliography on the quadratic systems can be found.

Using standard oscillator wave functions for equation (2.4) when \(c_0 = 1\) (for example, [17], [23] and/or [34]) results in the solution

\[\psi_n (x, t) = e^{i(\alpha x^2 + \delta x + \kappa) + i(2n+1)\gamma} \frac{e^{-(\beta x + \varepsilon)^2/2}}{\sqrt{2^n n! \mu \sqrt{\pi}}} H_n (\beta x + \varepsilon),\]  

(2.14)

where \(H_n (x)\) are the Hermite polynomials [39] and the general real-valued solution of the Ermakov-type system (2.5)–(2.10) is available in Ref. [24] — Lemma 3, Eqs. (42)–(48).

The Green function of generalized harmonic oscillators has been constructed in Ref. [41]. (See also important previous works [11], [31], [53], [56], [57] and references therein for more details.)
The corresponding Cauchy initial value problem can be solved (formally) by the superposition principle:

$$\psi(x, t) = \int_{-\infty}^{\infty} G(x, y, t) \psi(y, 0) \, dy$$

(2.15)

for some suitable initial data $$\psi(x, 0) = \varphi(x)$$ (see Refs. [4], [42] and [46] for further details). The corresponding eigenfunction expansion can be written in terms of the wave functions (2.14) as follows

$$\psi(x, t) = \sum_{n=0}^{\infty} c_n \psi_n(x, t),$$

(2.16)

where the time-independent coefficients are given by

$$c_n = \frac{\int_{-\infty}^{\infty} \psi_n^*(x, t) \psi(x, 0) \, dx}{\int_{-\infty}^{\infty} |\psi_n(x, 0)|^2 \, dx}.$$

(2.17)

This expansion complements the integral form of solution (2.15).

The maximum symmetry group of the autonomous Schrödinger equation (2.4) is studied in [37] and [38] (see also [49] and references therein).

3. Dynamical Invariants for Generalized Driven Harmonic Oscillators

A concept of dynamical invariants for generalized harmonic oscillators has been recently revisited in Refs. [6] and [46] (see [9], [10], [11], [31], [32] and references therein for classical works). In this Letter, we would like to point out a simple extension of the quadratic dynamical invariant to the case of driven oscillators:

$$E(t) = \lambda(t)^2 \left[ \hat{a}(t) \hat{a}^\dagger(t) + \hat{a}^\dagger(t) \hat{a}(t) \right]$$

(3.1)

$$= \frac{\lambda(t)}{2} \left[ \frac{(p - 2\alpha x - \delta)^2}{\beta^2} + (\beta x + \varepsilon)^2 \right], \quad \frac{d}{dt}(E) = 0.$$

(See also [12], [18], [35] and [55].) Here, $$\lambda(t) = \exp\left(-\int_0^t (c(s) - 2d(s)) \, ds\right)$$ and the corresponding time-dependent annihilation $$\hat{a}(t)$$ and creation $$\hat{a}^\dagger(t)$$ operators are explicitly given by

$$\hat{a}(t) = \frac{1}{\sqrt{2}} \left( \beta x + \varepsilon + i \frac{p - 2\alpha x - \delta}{\beta} \right),$$

(3.2)

$$\hat{a}^\dagger(t) = \frac{1}{\sqrt{2}} \left( \beta x + \varepsilon - i \frac{p - 2\alpha x - \delta}{\beta} \right)$$

(3.3)

with $$p = i^{-1} \partial / \partial x$$ in terms of solutions of the Ermarov-type system (2.5)–(2.10). These operators satisfy the canonical commutation relation:

$$\hat{a}(t) \hat{a}^\dagger(t) - \hat{a}^\dagger(t) \hat{a}(t) = 1.$$

(3.4)
The oscillator-type spectrum of the dynamical invariant $E$ can be obtained in a standard way by using the Heisenberg–Weyl algebra of the raising and lowering operators (a “second quantization” [28], the Fock states):

$$\hat{a} (t) \Psi_n (x,t) = \sqrt{n} \Psi_{n-1} (x,t), \quad \hat{a}^\dagger (t) \Psi_n (x,t) = \sqrt{n+1} \Psi_{n+1} (x,t), \quad (3.5)$$

$$E (t) \Psi_n (x,t) = \lambda (t) \left( n + \frac{1}{2} \right) \Psi_n (x,t). \quad (3.6)$$

The corresponding orthogonal time-dependent eigenfunctions are given by

$$\Psi_n (x,t) = e^{i \left( \frac{\alpha x^2 + \delta x + \kappa}{2} \right) - \frac{\beta x + \epsilon}{2}} H_n (\beta x + \epsilon), \quad \langle \Psi_m, \Psi_n \rangle = \delta_{mn} \lambda^{-1} \quad (3.7)$$

(provided that $\beta (0) \mu (0) = 1$, when $\beta \mu = \lambda$ [24]) in terms of Hermite polynomials [39] and

$$\psi_n (x,t) = e^{i (2n+1) \gamma (t)} \Psi_n (x,t) \quad (3.8)$$

is the relation to the wave functions (2.14) with

$$\varphi_n (t) = - (2n + 1) \gamma (t) \quad (3.9)$$

being the Lewis phase [18], [26], [28].

The dynamic invariant operator derivative identity [6], [46]:

$$\frac{dE}{dt} = \frac{\partial E}{\partial t} + i^{-1} (EH - H^\dagger E) = 0 \quad (3.10)$$

can be verified in the following fashion. Introducing new linear momentum and coordinate operators in the form

$$P = \frac{\lambda}{\beta} (p - 2\alpha x - \delta), \quad Q = \lambda (\beta x + \epsilon), \quad (3.11)$$

when $[Q, P] = i\lambda^2$ (a generalized canonical transformation), one can derive the simple differentiation rules

$$\frac{dP}{dt} = -2c_0 a \lambda^2 Q, \quad \frac{dQ}{dt} = 2a \lambda^2 P. \quad (3.12)$$

(It is worth noting that if $c_0 = 0$, the operator $P$ becomes the linear invariant of Dodonov, Malkin, Manko and Trifonov [10], [11], [32], [55] for generalized driven harmonic oscillators.)

Then

$$E = \frac{\lambda^{-1}}{2} \left( P^2 + c_0 Q^2 \right) \quad (c_0 = 0, 1) \quad (3.13)$$

and it is useful to realise that $E$ is just the original Hamiltonian $H$ after the canonical transformation [26]. The required operator identity (3.10) can be formally derived with the aid of product rule (3.7) of Ref. [46] (quantum calculus):

$$2 \frac{dE}{dt} = \frac{d}{dt} \left( \lambda^{-1} P^2 \right) + c_0 \frac{d}{dt} \left( \lambda^{-1} Q^2 \right) \quad (3.14)$$

$$= \lambda^{-1} \left( \frac{dP}{dt} P + P \frac{dP}{dt} \right) + c_0 \lambda^{-1} \left( \frac{dQ}{dt} Q + Q \frac{dQ}{dt} \right)$$

and by (3.12):

$$\lambda \frac{dE}{dt} = c_0 a \lambda^2 (-QP - PQ + PQ + QP) = 0, \quad (3.15)$$

which completes the proof.
Remark 1. The kernel
\[ K(x, y, t) = \frac{1}{\sqrt{\mu}} e^{i \left( \alpha x^2 + \beta xy + \gamma y^2 + \delta x + \epsilon y + \kappa \right)} \] (3.16)
is a particular solution of the Schrödinger equation (2.2) for any solution of the Riccati-type system (2.5)–(2.11) with \( c_0 = 0 \) [4]. A direct calculation shows that this kernel is an eigenfunction of the linear dynamical invariant [46].

4. Evaluation of Berry’s Phase

The holonomic effect in quantum mechanics known as Berry’s phase ([1], [2]) had received considerable attention over the years (see, for example, [3], [12], [19], [20], [21], [22], [26], [35], [36], [35], [40], [48], [52], [54] and references therein). The solution of the time-dependent Schrödinger equation (2.2) has the form (2.16) with the oscillator-type wave functions (2.14) [24]:
\[ \psi_n(x, t) = e^{-i \phi_n(t)} \Psi_n(x, t) \] (4.1)
where \( \phi_n(t) \) is the Lewis (or dynamical) phase and \( \Psi_n(x, t) \) is the eigenfunction of quadratic invariant (3.6). (In the self-adjoint case, one chooses \( c = 2d \) when \( \lambda = 1 \).)

Then
\[ i \int_{\mathbb{R}} \psi_n^* \frac{\partial \psi_n}{\partial t} \, dx = \lambda^{-1} \frac{d \phi_n}{dt} + i \int_{\mathbb{R}} \Psi_n^* \frac{\partial \Psi_n}{\partial t} \, dx \] (4.2)
and Berry’s phase \( \theta_n \) is given by
\[ \lambda^{-1} \frac{d \theta_n}{dt} = \text{Re} \left( i \int_{\mathbb{R}} \Psi_n^* \frac{\partial \Psi_n}{\partial t} \, dx \right) = \text{Re} \left( i \left\langle \Psi_n, \frac{\partial \Psi_n}{\partial t} \right\rangle \right). \] (4.3)
Here, the eigenfunction \( \Psi_n \) is a \( \gamma \)-free part [26] of the wave function (2.14), namely
\[ \Psi_n = \lambda^{-1/2} e^{i \left( \alpha x^2 + \delta x + \kappa \right)} \Phi_n(x, t), \] (4.4)
and \( \Phi_n \) is, essentially, the real-valued stationary orthonormal wave function for the simple harmonic oscillator with respect to the new variable \( \xi = \beta x + \varepsilon \) (see (3.7) and (4.5)). The integral (4.3) can be evaluated as in Refs. [22] and [26]:
\[ \lambda \left\langle \Psi_n, \frac{\partial \Psi_n}{\partial t} \right\rangle = i \left\langle \Phi_n, \left( \frac{d\alpha}{dt} x^2 + \frac{d\delta}{dt} x + \frac{d\kappa}{dt} \right) \Phi_n \right\rangle + \frac{1}{2} (c - 2d) + \left\langle \Phi_n, \frac{\partial \Phi_n}{\partial t} \right\rangle \]
\[ = i \frac{d\alpha}{dt} \left\langle \Phi_n, x^2 \Phi_n \right\rangle + i \frac{d\delta}{dt} \left\langle \Phi_n, x \Phi_n \right\rangle + i \frac{d\kappa}{dt} \left\langle \Phi_n, \Phi_n \right\rangle + \frac{1}{2} (c - 2d) + \left\langle \Phi_n, \frac{\partial \Phi_n}{\partial t} \right\rangle, \]
where the last term is zero due to the normalization condition
\[ \int_{-\infty}^{\infty} \Phi_n^2 \, dx = 1. \] (4.5)
Moreover,
\[ \left\langle \Phi_n, x^2 \Phi_n \right\rangle = \beta^{-3} \int_{-\infty}^{\infty} \left( \xi^2 + \varepsilon^2 \right) \Phi_n^2 \, d\xi = \beta^{-2} \left( \varepsilon^2 + n + \frac{1}{2} \right), \]
\[ \left\langle \Phi_n, x \Phi_n \right\rangle = -\varepsilon \beta^{-2} \int_{-\infty}^{\infty} \Phi_n^2 \, d\xi = -\varepsilon \beta^{-1}. \]
with the help of
\[ \beta^{-1} \int_{-\infty}^{\infty} \xi \Phi_n^2 \, d\xi = 0, \quad \beta^{-1} \int_{-\infty}^{\infty} \xi^2 \Phi_n^2 \, d\xi = n + \frac{1}{2}. \] (4.6)

As a result,
\[ \frac{d\theta_n}{dt} = -\beta^{-2} \left( \varepsilon^2 + n + \frac{1}{2} \right) \frac{d\alpha}{dt} + \varepsilon \beta^{-1} \frac{d\delta}{dt} - \frac{d\kappa}{dt} \] (4.7)

and the phase \( \theta_n \) can be obtained by integrating (4.7). Our observation reveals the connection of Berry’s phase with the Ermakov-type system (2.5)–(2.11), whose general solution is found in Ref. [24].

When \( c - 2d = f = g = 0 \), one may choose \( \delta = \varepsilon = \kappa = 0 \) and our expression (4.7) simplifies to
\[ \frac{d\theta_n}{dt} = -\mu^2 \left( n + \frac{1}{2} \right) \frac{d\alpha}{dt} \] (4.8)

with the help of (2.11). The function \( \mu \) is a solution of the Ermakov equation (2.12)–(2.13) with \( c_0 = 1 \) and \( \beta = \mu^{-1} \). This result is consistent with Refs. [12] and [26], where the original expression of Ref. [36] has been corrected.

5. An Alternative Derivation of Berry’s Phase

In view of (2.1) and (4.1)–(4.3), we get
\[ \lambda^{-1} \left( \frac{d\theta_n}{dt} + \frac{d\varphi_n}{dt} \right) = \text{Re} \langle \psi_n, H \psi_n \rangle = \text{Re} \langle \Psi_n, H \Psi_n \rangle, \] (5.1)

because the Hamiltonian in (2.1)–(2.2) does not involve time differentiation. Here,
\[ H = \alpha p^2 + bx^2 + \frac{c}{2} (px + xp) + \frac{i}{2} (c - 2d) - fx - gp \] (5.2)

and the position and linear momentum operators are given by
\[ x = \frac{1}{\beta} \left[ \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger) - \varepsilon \right], \] (5.3)
\[ p = \frac{\beta}{i \sqrt{2}} (\hat{a} - \hat{a}^\dagger) + \frac{\sqrt{2} \alpha}{\beta} (\hat{a} + \hat{a}^\dagger) + \delta - \frac{2 \alpha \varepsilon}{\beta} \] (5.4)
in terms of the creation and annihilation operators (3.2)–(3.3). After the substitution, the Hamiltonian takes the form

\[ H = \left[ \frac{a}{2} \left( \frac{4 \alpha^2}{\beta^2} - \beta^2 \right) + \frac{b + 2 \alpha \varepsilon}{\beta^2} - \frac{i}{2} (c + 4 a \alpha) \right] (\hat{a})^2 + \left[ \frac{a}{2} \left( \frac{4 \alpha^2}{\beta^2} - \beta^2 \right) + \frac{b + 2 \alpha \varepsilon}{\beta^2} + \frac{i}{2} (c + 4 a \alpha) \right] (\hat{a}^\dagger)^2 + \frac{1}{2} \left[ a \left( \beta^2 + \frac{4 \alpha^2}{\beta^2} \right) + \frac{b + 2 \alpha \varepsilon}{\beta^2} \right] (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) + \frac{i}{2} (c - 2d) \] (5.5)
\[ + \sqrt{2} \left[ \frac{4a\alpha + c}{2\beta} \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right) - \frac{\varepsilon}{\beta^2} \left( b + c\alpha \right) - \frac{f + 2g\alpha}{2\beta} \right. \\
+ i \left( \frac{\beta}{2} (g - 2a\delta) + \frac{\varepsilon}{2} (c + 4a\alpha) \right) \right] \hat{a} \]
\[ + \sqrt{2} \left[ \frac{4a\alpha + c}{2\beta} \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right) - \frac{\varepsilon}{\beta^2} \left( b + c\alpha \right) - \frac{f + 2g\alpha}{2\beta} \right. \\
- i \left( \frac{\beta}{2} (g - 2a\delta) + \frac{\varepsilon}{2} (c + 4a\alpha) \right) \right] \hat{a}^\dagger \]
\[ + a \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right)^2 + \left( \frac{\varepsilon}{\beta} \left( f + \frac{b\varepsilon}{\beta} \right) - \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right) \left( g + \frac{c\varepsilon}{\beta} \right) \right). \]

Here,

\[ J_+ = \frac{1}{2} (\hat{a}^\dagger)^2, \quad J_- = \frac{1}{2} (\hat{a})^2, \quad J_0 = \frac{1}{4} (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \] (5.6)

are the generators of a non-compact $SU(1,1)$ algebra:

\[ [J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0 \] (5.7)

and, therefore, a use can be made of the group properties of the corresponding discrete positive series $D_\pm^1$ for further investigation of Berry’s phase. (This is a ‘standard procedure’ for quadratic Hamiltonians — more details can be found in Refs. [3], [19], [29], [31], [33], [39], [41], [48] and/or elsewhere.) Together, the linears and bilinears in $\hat{a}$ and $\hat{a}^\dagger$ realize the semi-direct sum of the $SU(1,1)$ and the Heisenberg algebra (3.4) (see Ref. [49] for more details).

Thus

\[ \lambda \text{Re} \langle \Psi_n, H\Psi_n \rangle = \left( n + \frac{1}{2} \right) \left[ a \left( \beta^2 + \frac{4\alpha^2}{\beta^2} \right) + \frac{b + 2c\alpha}{\beta^2} \right] \]
\[ + a \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right)^2 + \left( \frac{\varepsilon}{\beta} \left( f + \frac{b\varepsilon}{\beta} \right) - \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right) \left( g + \frac{c\varepsilon}{\beta} \right) \right) \] (5.8)

by (3.5)–(3.6).

Finally, from (3.9) and (5.1) we arrive at a different formula for Berry’s phase

\[ \frac{d\theta_n}{dt} = \left( n + \frac{1}{2} \right) \left[ a \left( \frac{4\alpha^2}{\beta^2} - \beta^2 \right) + \frac{b + 2c\alpha}{\beta^2} \right] \]
\[ + a \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right)^2 + \left( \frac{\varepsilon}{\beta} \left( f + \frac{b\varepsilon}{\beta} \right) - \left( \delta - \frac{2\alpha\varepsilon}{\beta} \right) \left( g + \frac{c\varepsilon}{\beta} \right) \right), \] (5.9)

which is consistent with the previous expression (4.7) for any solution of the Ermakov-type system (2.5)–(2.10) ($c_0 = 1$).

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Department of Mathematics, Western Washington University, Bellingham, WA 98225-9063, U.S.A.

E-mail address: barbara.sanborn@wwu.edu

School of Mathematical and Statistical Sciences & Mathematical, Computational and Modeling Sciences Center, Arizona State University, Tempe, AZ 85287–1804, U.S.A.

E-mail address: sks@asu.edu

URL: http://hahn.la.asu.edu/~suslov/index.html

Centre de Recherches Mathématiques, Université de Montréal, Montréal, Québec, Centre-ville Station, P.O. Box 6128, Canada H3C 3J7.

E-mail address: luc.vinet@umontreal.ca