A Note on Mutually Unbiased Unextendible Maximally Entangled Bases in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$

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Abstract

We systematically study the construction of mutually unbiased bases in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$, such that all the bases are unextendible maximally entangled ones. Necessary conditions of constructing a pair of mutually unbiased unextendible maximally entangled bases in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$ are derived. Explicit examples are presented.

Mutually unbiased bases (MUBs) play important roles in many quantum information processing such as quantum state tomography [1, 2, 3], crypticographic protocols [4, 5], and the mean kings problem [6]. They are also useful in the construction of generalized Bell states. Let $B_1 = \{|\phi_i\rangle\}$ and $B_2 = \{|\psi_i\rangle\}$, $i = 1, 2, \ldots, d$, be two orthonormal bases of a $d$-dimensional complex vector space $\mathbb{C}^d$, $\langle \phi_j | \phi_i \rangle = \delta_{ij}$, $\langle \psi_j | \psi_i \rangle = \delta_{ij}$. $B_1$ and $B_2$ are said to be mutually unbiased if and only if

$$|\langle \phi_i | \psi_j \rangle| = \frac{1}{\sqrt{d}} \quad \forall \; i, j = 1, 2, \ldots, d. \quad (1)$$

Physically if a system is prepared in an eigenstate of basis $B_1$ and is measured in basis $B_2$, then all the measurement outcomes have the same probability.

A set of orthonormal bases $\{B_1, B_2, \ldots, B_m\}$ in $\mathbb{C}^d$ is called a set of MUBs if every pair of bases in the set is mutually unbiased. For given dimensional $d$, the maximum number of MUBs is no more than $d + 1$. It has been shown that there are $d + 1$ MUBs when $d$ is a prime power [1, 7, 8]. However, for general $d$, e.g. $d = 6$, it is a formidable problem to determine the maximal numbers of MUBs [9, 10, 11, 12, 13, 14, 15, 16, 17].
When the vector space is a bipartite system $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ of composite dimension $dd'$, there are different kinds of bases in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ according to the entanglement of the basis vectors. The unextendible product basis (UPB) is a set of incomplete orthonormal product basis whose complementary space has no product states [18]. It is shown that the mixed state on the subspace complementary to a UPB is a bound entangled state. Moreover, the states comprising a UPB are not distinguishable by local measurements and classical communication.

The unextendible maximally entangled basis (UMEB) is a set of orthonormal maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ consisting of less than $d^2$ vectors which have no additional maximally entangled vectors that are orthogonal to all of them [19]. Recently, the UMEB in arbitrary bipartite spaces $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ has been investigated in [20]. A systematic way in constructing $d^2$-member UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ ($\frac{d^2}{2} < d < d'$) is presented. It is shown that the subspace complementary to the $d^2$-member UMEB contains no states of Schmidt rank higher than $d - 1$. From the approach of constructing UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, two mutually unbiased UMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^3$ are constructed in [20].

In this note, we systematically study the UMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^3$ and present a generic way in constructing a pair of UMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^3$ such that they are mutually unbiased. The special example given in [20] can be easily obtained from our approach.

A set of states $\{|\phi_i\rangle\}$ in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, $i = 1, 2, \cdots, n < dd'$, is called an $n$-member UMEB if and only if

(i) all the states $|\phi_i\rangle$ are maximally entangled;

(ii) $\langle\phi_i|\phi_j\rangle = \delta_{i,j}$;

(iii) if $\langle\phi_i|\psi\rangle = 0$, $\forall i = 1, 2, \cdots, n$, then $|\psi\rangle$ cannot be maximally entangled.

Here a state $|\psi\rangle$ is said to be a $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ maximally entangled state if and only if for an arbitrary given orthonormal complete basis $\{|i_A\rangle\}$ of the subsystem $A$, there exist an orthonormal basis $\{|i_B\rangle\}$ of the subsystem $B$ such that $|\psi\rangle$ can be written as $|\psi\rangle = \frac{1}{\sqrt{d'}} \sum_{i=0}^{d-1} |i_A\rangle \otimes |i_B\rangle$ [21].

Let $\{|0\rangle, |1\rangle\}$ and $\{|0'\rangle, |1'\rangle |2'\rangle\}$ be the computational bases in $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively. To construct a pair of MUBs which are both UMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^3$, we start with the first
UMEB in \( \mathbb{C}^2 \otimes \mathbb{C}^3 \) given by

\[
|\phi_i\rangle = \frac{1}{\sqrt{2}} (\sigma_i \otimes I_3) (|00\rangle + |11\rangle), \\
|\phi_4\rangle = |0\rangle \otimes |2\rangle', \\
|\phi_5\rangle = |1\rangle \otimes |2\rangle',
\]

where \( \sigma_0 \) denotes the \( 2 \times 2 \) identity matrix, \( \sigma_i, i = 1, 2, 3, \) are the Pauli matrices, \( I_3 \) stands for the \( 3 \times 3 \) identity matrix, \( |\alpha\beta\rangle \equiv |\alpha\rangle \otimes |\beta\rangle \).

If we choose \( \{|a\rangle, |b\rangle\} \) and \( \{|x'\rangle, |y'\rangle, |z'\rangle\} \) to be another two bases of \( \mathbb{C}^2 \) and \( \mathbb{C}^3 \) respectively, then we have the second UMEB in \( \mathbb{C}^2 \otimes \mathbb{C}^3 \),

\[
|\psi_i\rangle = \frac{1}{\sqrt{2}} (\sigma_i \otimes I_3) (|0x'\rangle + |1y'\rangle), \\
|\psi_4\rangle = |a\rangle \otimes |z'\rangle, \\
|\psi_5\rangle = |b\rangle \otimes |z'\rangle.
\]

The bases \( \{|\phi_i\rangle\} \) and \( \{|\psi_i\rangle\} \) are mutually unbiased if and only if they satisfy the relations \eqref{mutually_unbiased_relations},

\[
|\langle \phi_i | \psi_j \rangle| = \frac{1}{\sqrt{6}}, \quad \forall \ i, j = 0, 1, \cdots, 5.
\]

Let \( S \) and \( W \) be the unitary matrices that transforms the bases \( \{|0\rangle, |1\rangle\} \) and \( \{|0', 1', |2'\} \) to \( \{|a\rangle, |b\rangle\} \) and \( \{|x'\rangle, |y'\rangle, |z'\rangle\} \) respectively,

\[
S(|0\rangle, |1\rangle) = (|a\rangle, |b\rangle), \quad W(|0'\rangle, |1'\rangle, |2'\rangle) = (|x'\rangle, |y'\rangle, |z'\rangle).
\]

Correspondingly we have the relations between \( |\phi_i\rangle \) and \( |\psi_j\rangle \),

\[
|\psi_j\rangle = (I_2 \otimes W)|\phi_j\rangle, \quad \forall \ j = 0, 1, 2, 3, \\
|\psi_j\rangle = (S \otimes W)|\phi_j\rangle, \quad \forall \ j = 4, 5.
\]

From \eqref{mutually_unbiased_relations} one gets,

\[
|\langle \phi_i | J_2 \otimes W |\phi_j \rangle| = \frac{1}{\sqrt{6}}, \quad \forall \ i = 0, 1, \ldots, 5, \ j = 0, 1, 2, 3, \\
|\langle \phi_i | S \otimes W |\phi_j \rangle| = \frac{1}{\sqrt{6}}, \quad \forall \ i = 0, 1, \ldots, 5, \ j = 4, 5.
\]
As \{\ket{\phi_i}\} forms a base in $\mathbb{C}^2 \otimes \mathbb{C}^3$, the relations in (7) imply that the absolute values of the entries of the matrices $I \otimes W$ and $S \otimes W$ under the base \{\ket{\phi_i}\} have the following forms:

\[
\begin{pmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \\
\end{pmatrix}, \tag{8}
\]

\[
\begin{pmatrix}
X & X & X & X & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
X & X & X & X & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
X & X & X & X & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
X & X & X & X & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
X & X & X & X & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
X & X & X & X & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\end{pmatrix}, \tag{9}
\]

where $X$ denotes any numbers.

Let

\[
S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \quad W = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix} \tag{10}
\]

be the matrices of $S$ and $W$ in the computational product basis \{|0\rangle, |1\rangle \otimes |0', 1', 2'\rangle\}. Let $F$ be the unitary matrix that transforms the computational product basis to the basis \{\ket{\phi_i}\}, i.e., $F(|00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle) = (|\phi_0\rangle, ..., |\phi_5\rangle)$. Form (2), one can easily get

\[
F = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}. \tag{11}
\]
Therefore the matrices of $I_2 \otimes W$ and $S \otimes W$ under the basis $\{|\phi_i\}\,$ are given by

$$F^\dagger(I_2 \otimes W)F, \quad (12)$$

and

$$F^\dagger(S \otimes W)F, \quad (13)$$

respectively.

Comparing (12) and (13) with (8) and (9), by straightforward calculations, we have

(i) The absolute values of the entries of $w$ are $1/\sqrt{3}$. Moreover, in the complex plane, $w_{11} \perp w_{22}$ and $w_{21} \perp w_{12}$.

(ii) The absolute values of the entries of $S$ is $1/\sqrt{2}$. In the complex plane, $w_{13}s_{11} \perp w_{23}s_{21}$, $w_{13}s_{12} \perp w_{23}s_{22}$ and $w_{23}s_{12} \perp w_{13}s_{22}$.

From the condition (i), for simplification, we can set

$$W = 1/\sqrt{3} \begin{pmatrix}
  e^{i\theta_1} & e^{i(\theta_2 + \pi/2)} & e^{i\theta_4} \\
  e^{i\theta_2} & e^{i(\theta_1 + \pi/2)} & e^{i\theta_5} \\
  e^{i\theta_3} & e^{i(\theta_3 - \pi/2)} & e^{i\theta_6}
\end{pmatrix}, \quad (14)$$

where, due the properties of unitary matrix, $\theta_i$ satisfy the following conditions,

$$|\theta_1 - \theta_2| = \frac{\pi}{3}, \quad |\theta_4 - \theta_5| = \pi, \quad e^{i(\theta_1 - \theta_4)}e^{-i\pi/3} + e^{i(\theta_3 - \theta_6)} = 0. \quad (15)$$

From equation (14), (15) and condition (ii), we find $s_{11}$ and $s_{21}$ are orthogonal, $s_{12}$ and $s_{22}$ are orthogonal. Then we can simply set

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix}
  e^{i\theta'_1} & e^{i\theta'_2} \\
  \pm e^{i(\theta'_1 + \pi/2)} & \mp e^{i(\theta'_2 + \pi/2)}
\end{pmatrix}, \quad (16)$$

where $\theta'_1, \theta'_2$ can be any real numbers.

Therefore, for any $\theta_i$s and $\theta'_i$s satisfying (15) and (16) respectively, one has a $W$ and a $S$. Then from (6) one gets the UMEB $\{|\psi_i\}\,$ that is mutually unbiased with the UMEB $\{|\phi_i\}\,$.

We next give some concrete examples of mutually unbiased UMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^3$. 

5
The UMEB \{\phi_i\} presented in \[20\] is of the form,

\[ |\phi_0\rangle = \frac{1}{\sqrt{2}}(|00'\rangle + |11'\rangle), \]
\[ |\phi_i\rangle = \frac{1}{\sqrt{2}}(\sigma_i \otimes I_3)(|00'\rangle + |11'\rangle), \quad i = 1, 2, 3, \]
\[ |\phi_4\rangle = |c\rangle \otimes |2'\rangle, \]
\[ |\phi_5\rangle = |d\rangle \otimes |2'\rangle, \quad (17) \]

where \(|c\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{\sqrt{2}}|1\rangle, |d\rangle = \frac{\sqrt{3}}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\). This example corresponds to a different transformation matrix \(F\),

\[ F = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}. \]

From our approach, \(w_{31} \perp w_{32}\) should be added to the condition (i). With respect to the condition (ii), the orthogonal relation becomes \((s_{11} + \sqrt{3}s_{12}) \perp (s_{21} + \sqrt{3}s_{22})\) and \((\sqrt{3}s_{11} - s_{12}) \perp (\sqrt{3}s_{21} - s_{22})\). However, since we have already set \(w_{31} \perp w_{32}\) in (14), (15) can be also used for this example.

We choose \(\{\theta_i\}\) to be

\[ \{\theta_1 = 0, \theta_2 = \frac{\pi}{3}, \theta_3 = 0, \theta_4 = \pi, \theta_5 = 0, \theta_6 = \frac{\pi}{3}\}, \quad (18) \]

which satisfy the condition (15). From (14) we have

\[ W = 1/\sqrt{3} \begin{pmatrix}
1 & -\sqrt{3} & -1 \\
\frac{1+\sqrt{3}i}{2} & i & 1 \\
1 & -i & \frac{1+\sqrt{3}}{2}
\end{pmatrix}. \quad (19) \]

The unitary matrix \(W\) transforms the basis \(\{|0',|1',|2'\rangle\}\) to basis \(\{|x',|y',|z'\rangle\}\).
From (5) we have

\[ |x\rangle = \frac{1}{\sqrt{3}} (|0\rangle + \frac{1+i\sqrt{3}}{2} |1\rangle + |2\rangle), \]
\[ |y\rangle = \frac{1}{\sqrt{3}} (\frac{-\sqrt{3}+i}{2} |0\rangle + i|1\rangle - i|2\rangle), \]
\[ |z\rangle = \frac{1}{\sqrt{3}} (-|0\rangle + |1\rangle + \frac{1+i\sqrt{3}}{2} |2\rangle). \tag{20} \]

We have the unitary operator \( S \),
\[ S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ \frac{\sqrt{3}+i}{2} & \frac{1-i\sqrt{3}}{2} \end{pmatrix}. \tag{21} \]

The corresponding operator \( S(|c\rangle, |d\rangle) = (|a\rangle, |b\rangle) \), give rise to

\[ |a\rangle = \frac{1}{\sqrt{2}} (\frac{1+i\sqrt{3}}{2} |0\rangle + \frac{\sqrt{3}-i}{2} |1\rangle), \]
\[ |b\rangle = \frac{1}{\sqrt{2}} (\frac{\sqrt{3}-i}{2} |0\rangle + \frac{1+i\sqrt{3}}{2} |1\rangle). \tag{22} \]

Therefore, the second UMEB that is mutually unbiased to (17) is given by

\[ |\psi_j\rangle = \frac{1}{\sqrt{2}} ((\sigma_i \otimes I_3)(|x\rangle + |y\rangle)), \quad j = 1, 2, 3, \]
\[ |\psi_4\rangle = \frac{1}{\sqrt{2}} (\frac{1+i\sqrt{3}}{2} |0\rangle + \frac{\sqrt{3}-i}{2} |1\rangle) \otimes |z\rangle, \]
\[ |\psi_5\rangle = \frac{1}{\sqrt{2}} (\frac{\sqrt{3}-i}{2} |0\rangle + \frac{1+i\sqrt{3}}{2} |1\rangle) \otimes |z\rangle. \tag{23} \]

(17) and (23) are exactly the ones presented in [20].

Now we give a new example by choosing other values of \( \{\theta_i\} \) and \( \{\theta'_i\} \). Let the first UMEB in \( \mathbb{C}^2 \otimes \mathbb{C}^3 \) be the one given in (2). Taking into the condition (15), we set

\[ \theta_1 = \pi, \theta_2 = \frac{2\pi}{3}, \theta_3 = \theta_4 = 0, \theta_5 = \pi, \theta_6 = \frac{\pi}{3}. \tag{24} \]

From (14), we get
\[ W = 1/\sqrt{3} \begin{pmatrix} -1 & -\sqrt{3} & 1 \\ -\frac{1+i\sqrt{3}}{2} & -i & -1 \\ 1 & -i & \frac{1+i\sqrt{3}}{2} \end{pmatrix}, \tag{25} \]
and

\[ |x'\rangle = \frac{1}{\sqrt{3}} (-|0'\rangle + \frac{1 + \sqrt{3}i}{2} |1'\rangle + |2'\rangle), \]

\[ |y'\rangle = \frac{1}{\sqrt{3}} \left( \frac{\sqrt{3} - i}{2} |0'\rangle - i |1'\rangle - i |2'\rangle \right), \]

\[ |z'\rangle = \frac{1}{\sqrt{3}} \left( |0'\rangle - |1'\rangle + \frac{1 + \sqrt{3}i}{2} |2'\rangle \right). \quad (26) \]

Taking \( \theta_1' = 0 \) and \( \theta_2' = \frac{\pi}{2} \), we have

\[ S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad (27) \]

and

\[ |a\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i |1\rangle), \quad |b\rangle = \frac{1}{\sqrt{2}} (i |0\rangle + |1\rangle). \quad (28) \]

From (3) we obtain the second UMEB that is mutually unbiased to the UMEB given by Eq. (2),

\[ |\psi_j\rangle = \frac{1}{\sqrt{2}} (\sigma_i \otimes I_3)(|0x'\rangle + |1y'\rangle), \quad j = 1, 2, 3, \]

\[ |\psi_4\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i |1\rangle) \otimes |z'\rangle, \]

\[ |\psi_5\rangle = \frac{1}{\sqrt{2}} (i |0\rangle + |1\rangle) \otimes |z'\rangle. \quad (29) \]

It can be directly verified that the two UMEBs (17) and (29) satisfy the condition (4).

As another example we choose

\[ \theta_1 = \frac{4\pi}{3}, \theta_2 = \pi, \theta_3 = 0, \theta_4 = \pi, \theta_5 = 0, \theta_6 = \pi, \]

\[ \theta_1' = \frac{\pi}{3}, \theta_2' = \frac{\pi}{6}. \quad (30) \]

The corresponding unitary matrix \( W \) and \( S \) are of the form,

\[ W = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1 - \sqrt{3}i}{2} & -i & -1 \\ -1 & \frac{\sqrt{3} - i}{2} & 1 \\ 1 & -i & -1 \end{pmatrix}, \quad (31) \]

\[ S = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1 + \sqrt{3}i}{2} & \frac{\sqrt{3} + i}{2} \\ -\frac{\sqrt{3} + i}{2} & \frac{1 - \sqrt{3}i}{2} \end{pmatrix}. \quad (32) \]
The basis $\{|a\rangle, |b\rangle\}$ in $\mathbb{C}^2$ and the basis $\{|x'\rangle, |y'\rangle, |z'\rangle\}$ in $\mathbb{C}^3$ are given by

$$|x'\rangle = \frac{1}{\sqrt{3}}(- \frac{1-\sqrt{3} i}{2}|0'\rangle - |1'\rangle + |2'\rangle),$$
$$|y'\rangle = \frac{1}{\sqrt{3}}(-i|0'\rangle + \frac{\sqrt{3} - i}{2}|1'\rangle - i|2'\rangle),$$
$$|z'\rangle = \frac{1}{\sqrt{3}}(-|0'\rangle + |1'\rangle - |2'\rangle),$$

and

$$|a\rangle = \frac{1}{\sqrt{2}}\left(\frac{1 + \sqrt{3} i}{2}|0\rangle + \frac{-\sqrt{3} + i}{2}|1\rangle\right),$$
$$|b\rangle = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3} + i}{2}|0\rangle + \frac{1 - \sqrt{3} i}{2}|1\rangle\right).$$

Therefore, another UMEB that is mutually unbiased to the UMEB given by (2) is of the form,

$$|\psi_j\rangle = \frac{1}{\sqrt{2}}(\sigma_i \otimes I_3)(|0x'\rangle + |1y'\rangle), \quad j = 1, 2, 3,$$
$$|\psi_4\rangle = \frac{1}{\sqrt{2}}\left(\frac{1 + \sqrt{3} i}{2}|0\rangle + \frac{-\sqrt{3} + i}{2}|1\rangle\right) \otimes |z'\rangle,$$
$$|\psi_5\rangle = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3} + i}{2}|0\rangle + \frac{1 - \sqrt{3} i}{2}|1\rangle\right) \otimes |z'\rangle.$$

We have presented a general way in constructing UMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^3$ such that they are mutually unbiased. Explicit examples are given for constructing a pair of mutually unbiased unextendible maximally entangled bases, including the one in [20] as a special case. Our approach may shed light in constructing more UMEBs that are pairwise mutually unbiased in $\mathbb{C}^2 \otimes \mathbb{C}^3$ or higher dimensional bipartite systems.

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