# Harmonic Gauss maps of submanifolds of arbitrary codimension of the Euclidean space and sphere and some applications

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## Abstract

It is proven results about existence and nonexistence of unit normal sections of submanifolds of the Euclidean space and sphere, which associated Gauss maps, are harmonic. Some applications to constant mean curvature hypersurfaces of the sphere and to isoparametric submanifolds are obtained too.

**KEYWORDS**

harmonic Gauss maps, isoparametric submanifolds, minimal surfaces, surfaces with parallel mean curvature

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## 1 INTRODUCTION

A well-known theorem of Ruh–Vilms [31] establishes that the Grassmanian Gauss map of an orientable submanifold $M^n$ of $\mathbb{R}^m$, $1 \leq n \leq m - 1$, is harmonic if and only if the mean curvature vector of $M$ is parallel in the normal connection. In codimension 1, this result is equivalent to say that the Gauss map

$$\gamma_\eta : M \rightarrow S^{m-1}$$

associated to a unit normal section $\eta$ of $M$ is harmonic if and only if $M$ has constant mean curvature (CMC). This relation in the codimension 1 case provides a strong tool to the study of CMC hypersurfaces of the Euclidean space, in particular their Gauss image, a classical topic of research in differential geometry (see, for instance, [21]). In this paper, we try to extend the codimension 1 case of Ruh–Vilms theorem to submanifolds $M$ of arbitrary codimension by finding geometric conditions on $M$ that guarantee the existence of a unit normal section $\eta$ of $M$ such that the associated Gauss map $\gamma_\eta : M \rightarrow S^{m-1}$ is harmonic.

Considering that the space of unit normal sections has locally higher dimension in codimension bigger than or equal to 2, we should expect the existence of unit normal sections with harmonic Gauss maps under usual geometric assumptions as minimality or parallelism of the mean curvature vector. However, this first intuition is not true in general. Indeed, in Theorem 1.1 below, we prove that the existence of a unit normal section of a surface of $\mathbb{R}^4$ determining a harmonic Gauss map yields strong restrictions on the geometry of the surface. The result is local.

**Theorem 1.1.** Let $M$ be a surface of $\mathbb{R}^4$ with parallel mean curvature vector field such that the second fundamental form of $M$ spans the normal space of $M$ in $\mathbb{R}^4$ at each point. Then, any unit normal section of $M$ in $\mathbb{R}^4$ can be written as $\eta = \alpha v + b\mu$, where...
where \( \nu \) is a unit normal section of \( M \) in \( \mathbb{R}^4 \) tangent to the unit sphere \( S^3 \), \( \mu \) is a unit normal section of \( M \) orthogonal to \( \nu \), and \( a, b \) are functions on \( M \) with \( a^2 + b^2 = 1 \). And, up to isometries of \( \mathbb{R}^4 \),

(1) \( \gamma_\eta \) is a harmonic map and \( a, b \) are constants if and only if one of the following alternatives is satisfied:

(a) \( M \) is an open subset of the product of circles

\[
S^1(r) \times S^1(\sqrt{1-r^2}) \subset S^3,
\]

with \( 0 < r < 1 \).

(b) \( M \) is an open subset of a totally umbilical non-geodesic surface of \( S^3 \).

(c) \( M \) is a minimal surface of \( S^3 \) and up to a reorientation of \( M \) and up to a composition of \( M \) with the antipodal map, \( \eta \) is the position vector of \( M \) in \( \mathbb{R}^4 \) or \( \eta \) is a unit normal section of \( M \) in \( S^3 \).

(2) If \( \gamma_\eta \) is a harmonic map and \( a, b \) are not constants, then \( M \) is a CMC surface of \( S^3 \) with a nonconstant principal curvature, which is constant along a principal direction.

Concerning alternative (2) of Theorem 1.1, CMC surfaces invariant by a one-parameter subgroup of isometries of \( S^3 \), apart from spheres and tori, have a nonconstant principal curvature, which is constant along a principal direction (see [24] for the minimal case). We do not know if these surfaces admit a unit normal section whose Gauss map is harmonic.

As a direct consequence of Theorem 1.1, it follows a nonexistence result of harmonic Gauss maps.

**Corollary 1.2.** Let \( M \) be a minimal surface of \( \mathbb{R}^4 \) such that the second fundamental form of \( M \) spans the normal space of \( M \) at each point. Then, for any unit normal section \( \eta \), the Gauss map

\[
\gamma_\eta : M \to S^3
\]

\[
p \mapsto \eta(p)
\]

is not a harmonic map.

We also get an analogous result of Corollary 1.2 of nonexistence harmonic Gauss maps for minimal surfaces of \( S^4 \).

**Theorem 1.3.** Let \( M \) be a minimal surface of \( S^4 \subset \mathbb{R}^5 \) such that the second fundamental form of \( M \) spans the normal space of \( M \) in \( S^4 \) at each point. Then, for any unit normal section \( \eta \) of \( M \) in \( S^4 \), the Gauss map

\[
\gamma_\eta : M \to S^4
\]

\[
p \mapsto \eta(p)
\]

is not a harmonic map.

It seems that what is behind the Ruh–Vilms result is a fact that is trivial in codimension 1, namely, unit normal sections are parallel in the normal connection. To bring up this relation in arbitrary codimension, we recall some preliminary facts, which are introduced in a more general Riemannian setting for later use.

Let \( M \) be a manifold immersed in a Riemannian manifold \( N \). Since there is no possibility of confusion in our context, we identify, as usual, \( M \) with its image on \( N \) by the immersion. Denote by \( S(M) \) the vector bundle over \( M \) of symmetric linear transformations (s.l.t), that is,

\[
S(M) = \{(p, T) \mid p \in M \text{ and } T : T_pM \to T_pM \text{ is a s.l.t}\}.
\]

Let \( N(M) \) and \( S(M) \) be the vector bundles over \( M \) of the sections of the normal bundle \( TM^\perp \) and the sections of \( S(M) \), respectively.

We denote by \( \hat{B} := B^*B \) the Simons operator [32], a section of the vector bundle \( \text{Hom}(TM^\perp, TM^\perp) \), where \( B : N(M) \to S(M) \) is the vector bundle homomorphism \( B(\eta) = S_\eta \), \( S_\eta \) is the second fundamental form associated to \( \eta \), and \( B^* : S(M) \to N(M) \) is the fiber-wise adjoint map of \( B \) (considering in \( S(M) \) the Hilbert–Schmidt metric, see Section 2 ahead for more details).
Recall that a map from a manifold $M$ to $\mathbb{S}^{m-1}$ is harmonic, in the case that $M$ is compact, if it is a critical point of the functional $H : C^\infty(M, \mathbb{S}^{m-1}) \to \mathbb{R}$

$$H(f) = \int_M \|df\|^2, \ f \in C^\infty(M, \mathbb{S}^{m-1}).$$

If $M$ is only complete, a map is harmonic if it is a critical point of $H$ restricted to compact subdomains of $M$.

Denote by $\mathcal{N}_1(M)$ the subbundle of unit normal sections of $\mathcal{N}(M)$. In the case that $M$ is compact, we say that a unit normal section is harmonic if it is a critical point of the functional $\mathcal{N} : \mathcal{N}_1(M) \to \mathbb{R}$

$$\mathcal{N}(\eta) = \int_M \|\nabla\eta\|^2, \ \eta \in \mathcal{N}_1(M),$$

where $\nabla$ is the Riemannian connection of $\mathcal{N}$. In the case that $M$ is only complete, a unit normal section is harmonic if it is a critical point of $\mathcal{N}$ restricted to compact subdomains of $M$.

Finally, recalling that a normal section $\eta$ of $M$ is parallel (in the normal connection) if $(\nabla_E\eta)^\perp = 0$, where $E$ is any tangent vector field of $M$ and $\perp$ is the orthogonal projection of $TN$ on $TM^\perp$, we prove the following:

**Theorem 1.4.** Let $M$ be a smooth manifold immersed in $\mathbb{R}^m$ with parallel mean curvature vector. Let $\eta$ be a parallel unit normal section of $M$. Then, the following alternatives are equivalent:

1. $\gamma_\eta \in C^\infty(M, \mathbb{S}^{m-1})$ is harmonic.
2. $\eta$ is a harmonic section.
3. $\eta$ is an eigenvector of the Simons operator $\Bar{B}$.

We observe that elementary linear algebra guarantees the existence of eigenvectors of $\Bar{B}$ as well as, in the case that $M$ is compact, standard results from analysis guarantee the existence of a minimizer $\eta$ of the functional $\mathcal{N}$ (if $\mathcal{N}_1(M) \neq \emptyset$ and possibly only in the weak sense). The difficulty to have the harmonicity of the Gauss map $\gamma_\eta$ associated to $\eta$ is to guarantee that $\eta$ is parallel in the normal connection. But existence of parallel unit normal sections in codimension bigger than or equal to 2 seems to be very restrictive. For example, any minimal surface immersed in any space form admitting a parallel unit normal section cannot be substantial. Indeed, if the shape operator of a parallel normal section is zero, then it is easy to see that the surface is contained in a totally geodesic hypersurface. If not, it follows from the Ricci equation and the minimality that the surface has a flat normal bundle and then a parallel one-dimensional first normal bundle. This implies that the surface has actually substantial codimension 1 (see chapter 2 of [13]).

Nevertheless, as explained below, there are some interesting cases where Theorem 1.4 can be applied, as to CMC hypersurfaces of spheres and to isoparametric submanifolds.

Although not being able to prove, we believe that the parallelism of the unit normal section in Theorem 1.4 is necessary to have the equivalences among items (1), (2), and (3).

We observe that a normal section $\eta$ in $\mathbb{R}^{n+2}$ of a hypersurface of $\mathbb{S}^{n+1}$ is parallel (in the normal connection of $M$) if and only if $\eta = a\nu + b\mu$, where $a, b$ are constants, $\nu$ a unit normal section of $M$ in $\mathbb{S}^{n+1}$, and $\mu$ is the position vector of $M$ in $\mathbb{R}^{n+2}$.

**Theorem 1.5.** Let $M$ be an orientable hypersurface of $\mathbb{S}^{n+1}$, $\eta$ a unit normal section of $M$ in $\mathbb{R}^{n+2}$ parallel in the normal connection, and let $\theta \in [0, \pi)$ be the angle between $\eta$ and $\mu$. We have the following:

1. If $\theta = 0$ or $\theta = \frac{\pi}{2}$, then the Gauss map $\gamma_\eta$ is harmonic if and only if $M$ is a minimal hypersurface.
2. If $\theta \neq 0$ and $\theta \neq \frac{\pi}{2}$, then the Gauss map $\gamma_\eta$ is harmonic if and only if $M$ is a CMC hypersurface of $\mathbb{S}^{n+1}$ and the second fundamental form of $M$ in $\mathbb{S}^{n+1}$ has constant length and satisfies

$$\|S_\nu\|^2 = nH(\cot \theta - \tan \theta) + n,$$

where $\nu$ is a unit normal vector of $M$ in $\mathbb{S}^{n+1}$ and $H$ is the mean curvature of $M$ with respect to $\nu$. 
Item (1) of Theorem 1.5 is essentially known. Indeed, the case $\theta = 0$ is included in the Theorem 3 of [33] and the case $\theta = \frac{\pi}{2}$ can be proved from the results of [10] and [23] with the Gauss map as defined by Obata [27]. Also, although not explicitly stated in [32], the case $\theta = \frac{\pi}{2}$ follows from its results too (section 5 of [32]).

In the next result, we apply Theorem 1.5 to characterize the minimal Clifford torus and the $H(r)$-torus in the sphere $S^{n+1}$ in terms of the harmonicity of the Gauss maps associated to unit normal sections in $R^{n+2}$ parallel in the normal connection. We make use of theorems on CMC hypersurfaces in the sphere due to H. Alencar, S. Chern, M. Do Carmo, and S. Kobayashi [1] and [11]. We need to recall a notation of [1].

Let $M$ be a compact and oriented CMC hypersurface of $S^{n+1}$. We assume that $M$ is oriented in such way that the mean curvature $H$ of $M$ is nonnegative. For each $H$, set

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} H x - n(H^2 + 1),$$

and let $B_H$ be the square of the positive root of $P_H(x) = 0$.

We observe that when $H \neq 0$, the equation on $\theta$

$$nH(\cot \theta - \tan \theta) = B_H + nH^2 - n \quad (1.2)$$

has exactly two solutions $\theta_1, \theta_2 \in (0, \pi)$, where $\theta_1 \in (0, \frac{\pi}{2})$ and $\theta_2 = \theta_1 + \frac{\pi}{2}$. These solutions correspond to two parallel unit normal sections of $M$ in $R^{n+2}$ having angles $\theta_1, \theta_2$, with $\mu$.

**Corollary 1.6.** Let $M$ be a compact CMC hypersurface of $S^{n+1} \subset R^{n+2}$. Then,

(1) Case $H = 0$: The following alternatives are equivalent:

(a) For any parallel unit normal section $\eta$ of $M$ in $R^{n+2}$, the Gauss map $\gamma_\eta$ is a harmonic map.

(b) There exists a parallel unit normal section $\eta$ of $M$ in $R^{n+2}$ where the angle between $\eta$ and $\mu$ is neither 0 nor $\frac{\pi}{2}$ and the associated Gauss map $\gamma_\eta$ is a harmonic map.

(c) $M$ is a minimal Clifford torus, that is,

$$M = S^k \left( \left( \frac{k}{n} \right)^{\frac{1}{2}} \right) \times S^{n-k} \left( \left( \frac{n-k}{n} \right)^{\frac{1}{2}} \right).$$

(2) Case $H \neq 0$ and $n > 2$:

(a) $M$ is an $H(r)$-torus with $r^2 < (n-1)/n$ if and only if the only harmonic Gauss maps of $M$ are, up to sign, the two parallel unit normal sections determined by the two solutions of Equation (1.2).

(b) Let $\eta$ be a parallel unit normal section of $M$. If the Gauss map $\gamma_\eta$ is a harmonic map and the angle $\theta$ between $\eta$ and $\mu$ satisfies

$$nH(\cot \theta - \tan \theta) < B_H + nH^2 - n,$$

then $M$ is a totally umbilical hypersurface of $S^{n+1}$ with principal curvature $\lambda > 0$ and

$$\cot \theta - \tan \theta = \frac{\lambda^2 - 1}{\lambda}.$$

(3) Case $H \neq 0$ and $n = 2$:

Let $\eta$ be a parallel unit normal section of $M$ in $R^{n+2}$, and let $\theta$ the angle between $\eta$ and $\mu$. Then, $\gamma_\eta$ is harmonic if and only if $M$ is one of the following two surfaces:

(a) a product of circles $S^1(r) \times S^1(\sqrt{1-r^2})$ with $0 < r < 1$, $r^2 \neq 1/2$ and

$$\cot \theta - \tan \theta = \frac{1 - 2r^2}{r \sqrt{1-r^2}},$$
(b) a totally umbilical hypersurface of the sphere with principal curvature \( \lambda \neq 0 \) and

\[
\cot \theta - \tan \theta = \frac{\lambda^2 - 1}{\lambda}.
\]

A question that arises is if the hypersurfaces of \( \mathbb{S}^{n+1} \) appearing in Corollary 1.6 are the only hypersurfaces of the sphere that admits a parallel unit normal section such that its associated Gauss map is a harmonic map. The answer is no, as one may see in Theorem 1.7 below.

A well-known family of submanifolds of the Euclidean space, studied in the classical and modern theory of submanifolds, where we can apply Theorem 1.4, is the family of isoparametric submanifolds. Recall that a complete submanifold \( M \) of \( \mathbb{R}^m \) or \( \mathbb{S}^m \) is called isoparametric if it has flat normal bundle and its principal curvature in the direction of any parallel normal vector field is constant [34].

The classification of the isoparametric submanifolds of the Euclidean space is a longstanding problem in differential geometry. It can be reduced to the problem of classifying the isoparametric submanifolds in the spheres since any isoparametric submanifold of \( \mathbb{R}^m \) is the product of an isoparametric submanifold of a sphere with an affine subspace of \( \mathbb{R}^m \) [34]. A classification of the compact isoparametric submanifolds of codimension bigger than or equal to 2 in spheres have been obtained during the last decades; however, a classification of the isoparametric hypersurfaces was obtained only recently (see [29]). As the authors know, a complete classification is still an open problem. In the next result, we obtain what we believe to be a strong property of isoparametric submanifolds of the Euclidean space, seemingly not known:

**Theorem 1.7.** Let \( M \) be an isoparametric submanifold of \( \mathbb{R}^m \). Then, the Simons operator \( \tilde{\mathcal{B}} \) has constant nonnegative eigenvalues and there is an orthonormal basis \( \eta_1, \ldots, \eta_r \) of \( \mathcal{N}_1(M) \) of eigenvectors of \( \tilde{\mathcal{B}} \) such that the Gauss maps \( \gamma_{\eta_i} : M \to \mathbb{S}^{m-1}, \ i = 1, \ldots, r \), are harmonic maps.

In Section 6, we extend Theorem 1.4 to more general ambient spaces, which include symmetric spaces, and prove an extension of Theorem 1.7 to minimal isoparametric submanifolds of spheres. A key concept related to the existence of parallel normal sections that allows to obtain part of our extensions, is that of **polar action** (see Section 4). Our results also extend and generalize several results of [4–6, 17, 18, 26], and [30].

The paper is organized as follows. In Section 2, we give basic definitions and set the notation used throughout the paper.

In Section 3, we define and prove several facts about the rough Laplacian of a vector field along a submanifold \( M \) of a Riemannian manifold \( N \). In particular, we obtain an extension, to submanifolds of arbitrary codimension, of Proposition 1 of [18]. This proposition, which gives a formula, in case of hypersurfaces, for the Laplacian of a function of the form \( \langle \eta, V \rangle \), where \( \eta \) is a unit normal section of \( M \) and \( V \) is a Killing vector field of \( N \), is fundamental for proving Theorem 1 of [30]. The extension of this proposition obtained here (Corollary 3.10) is also fundamental for the paper.

In Section 4, we obtain the Euler–Lagrange equation of the critical points of an energy functional and prove the existence of a harmonic unit normal section on any principal orbit of a polar action.

In Section 5, we prove our main results on the Euclidean space stated above. Finally, in Section 6, we extend the results of the Euclidean space to a class of homogeneous space where a Gauss map is naturally associated to a unit normal section of a submanifold of the space. A special interesting case is the sphere \( \mathbb{S}^7 \) with the octonionic multiplication.

### 2 Preliminaries

Let \( M \) be an \( n \)-dimensional smooth manifold immersed in a Riemannian manifold \( N \). Denote by \( T(M) \) and \( T(N) \) the space of smooth sections of \( TM \) and \( TN \), that is, the space of smooth vector fields of \( M \) and \( N \), respectively. Set \( T(M, N) := \mathcal{N}(M) \oplus T(M) \). If \( W \in T(M, N) \), then we say that \( W \) is a vector field along \( M \).

We denote by \( V^\perp := (V)^\perp \) the normal connection of \( M \), acting on \( TM^\perp \), where \( \perp \) is the orthogonal projection of \( TN \) on \( TM^\perp \). The curvature tensor \( R \) and normal curvature tensor \( R^\perp \) are given by

\[
R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z,
\]

\[
R^\perp(X, Y)\eta = \nabla_Y^\perp \nabla_X^\perp \eta - \nabla_X^\perp \nabla_Y^\perp \eta + \nabla_{[X, Y]}^\perp \eta,
\]

for \( X, Y, Z \) vector fields of \( M \) and \( \eta \) a normal vector field of \( M \). We say that the normal bundle of \( M \) is flat if \( R^\perp \equiv 0 \).
The mean curvature vector of the immersion of $M$ in $N$ is defined by $\vec{H} = \frac{1}{n} \text{tr}(\mathbf{F})$, where $\mathbf{F}$ denotes the second fundamental form of the immersion of $M$ in $N$ and $n$ is the dimension of $M$.

Next, we introduce an operator, the normal Ricci operator, that appears frequently in our results. First, consider the symmetric bilinear form $\text{Ric}_{M}$ on $\mathcal{T}(M, N)$ defined as follows: Given $Z, W \in \mathcal{T}(M, N)$, then $\text{Ric}_{M}(Z, W)$ is the trace of $(X, Y) \in \mathcal{T}(M) \times \mathcal{T}(M) \mapsto \langle R(Z, X)W, Y \rangle$.

We then define the normal Ricci operator

$$\text{Ric}^N_M : \mathcal{N}(M) \rightarrow \mathcal{N}(M)$$

of $M$ by the relation

$$\text{Ric}_M(\eta_1, \eta_2) = \langle \text{Ric}^N_M(\eta_1), \eta_2 \rangle,$$

for all $\eta_1, \eta_2 \in \mathcal{N}(M)$. Notice that the normal Ricci operator is self-adjoint. In particular, if $N$ is a connected space of constant curvature $c$, then for any unit normal section $\eta$ of $M$, the normal Ricci operator of $\eta$ is given by

$$\text{Ric}^N_M(\eta) = cn \eta. \quad (2.1)$$

We conclude this section by giving more details for the computation of the Simons Operator $\tilde{\mathbf{F}}$. The vector bundle homomorphism $\mathbf{F} : \mathcal{N}(M) \rightarrow S(M)$ is defined by $\mathbf{F}(\eta) = S_{\eta}$, where $S_{\eta}$ is the second fundamental form associated to $\eta$. The bundle $S(M)$ is provided with the Hilbert–Schimidt metric, that is, for $T_1, T_2 \in S(M)$ we have, at each fiber,

$$\langle T_1, T_2 \rangle = \sum_{i=1}^{n} \langle T_1(E_i), T_2(E_i) \rangle,$$

where $E_1, ..., E_n$ is a local orthonormal basis of $TM$. Since, the Simons operator is defined by $\mathbf{F}^* \mathbf{F}$ where $\mathbf{F}^* : S(M) \rightarrow \mathcal{N}(M)$ is the fiber-wise adjoint map of $\mathbf{F}$, we have

$$\langle \tilde{\mathbf{F}}(\eta_1), \eta_2 \rangle = \langle \mathbf{F}^* \mathbf{F}(\eta_1), \eta_2 \rangle = \langle \mathbf{F}(\eta_1), \mathbf{F}(\eta_2) \rangle = \langle S_{\eta_1}, S_{\eta_2} \rangle, \quad (2.2)$$

for any $\eta_1, \eta_2 \in \mathcal{N}(M)$.

### 3.1 The Rough Laplacian of a Vector Field Along a Submanifold

The notion of the rough Laplacian of a vector field of a Riemannian manifold is well known [35]. In this section, we define and prove several facts about the rough Laplacian of a vector field along a submanifold $M$ of $N$, to be used in the next sections. Some of the results might have independent interest.

The rough Laplacian $\nabla^2 W$ of vector field $W \in \mathcal{T}(M, N)$ along $M$ is also a vector field along $M$ defined by

$$\nabla^2 W = \sum_{i=1}^{n} (\nabla_{E_i} \nabla_{E_i} W - \nabla_{\nabla_{E_i} E_i} W),$$

where $n = \text{dim}(M)$ and $\{E_1, ..., E_n\}$ is a local orthonormal basis of $M$. One may see that this formula is given as the trace of a bilinear form and hence does not depend of the orthonormal basis $\{E_i\}$.

We are mainly interested in the case of the rough Laplacian acting on Killing fields.

**Example 3.1.** Let $M$ be an $n$-dimensional manifold immersed in the Euclidean space $\mathbb{R}^m$ and let $V$ be a Killing vector field of $\mathbb{R}^m$. One may see that

$$\nabla^2 V = n \nabla_{\vec{H}} V,$$

where $\nabla$ denotes the Riemannian connection of $\mathbb{R}^m$. 
Example 3.2. Let $M$ be an $n$-dimensional manifold immersed in the sphere $\mathbb{S}^m$ and let $V$ be a Killing vector field of $\mathbb{S}^m$. We denote by $\nabla$ and $\nabla^\perp$ the Riemannian connections of $\mathbb{S}^m$ and $\mathbb{R}^{m+1}$, respectively. Note that $V$ can be represented as a constant skew-symmetric matrix $A$ of dimension $m + 1$, that is, $V(p) = Ap$, where $p$ is regarded as column vector of $\mathbb{R}^{m+1}$.

Choose $p \in M$ and let $\{E_1, \cdots, E_n\}$ be a local orthonormal basis of $M$ geodesic at $p$. Then, at $p$

$$
\nabla^2V = \sum_i \nabla_{E_i} \nabla_{E_i} V \\
= \sum_i \left( \nabla_{E_i} (AE_i - \langle AE_i, p \rangle p) - \langle \nabla_{E_i} (AE_i - \langle AE_i, p \rangle p), p \rangle p \right) \\
= \sum_i \left( A \nabla_{E_i} E_i - \langle A \nabla_{E_i} E_i, p \rangle p - \langle AE_i, p \rangle E_i \right) \\
= nA\overline{H} - nAp - n\left( A\overline{H}, p \right) p + V^TM \\
= n\nabla^\perp V - nV + V^TM,
$$

where $(\cdot)^TM$ denotes the orthogonal projection on $TM$.

Example 3.3. Let $M$ be an $n$-dimensional manifold immersed in the hyperbolic space $\mathbb{H}^m$ and let $V$ be a Killing vector field of $\mathbb{H}^m$, then

$$
\nabla^2V = n\nabla^\perp V + nV - V^TM, \tag{3.1}
$$

where $(\cdot)^TM$ denotes the orthogonal projection on $M$ and $\nabla$ is the Riemannian connection of $\mathbb{H}^m$. This formula can be obtained as the previous case of the sphere by using the Lorentz model of the hyperbolic space.

The next lemma gives an expression for the tangent part of the rough Laplacian of a normal section.

Lemma 3.4. Let $M$ be an $n$-dimensional manifold immersed in a Riemannian manifold $N$. Then, for all $\eta \in \mathcal{N}(M)$ and $X \in \mathcal{T}(M)$, it holds the equality

$$
\langle \nabla^2 \eta, X \rangle = \text{Ric}_M(\eta, X) - n\langle \text{grad}(H), X \rangle + n\langle H, \nabla_X \eta \rangle - 2\text{tr}\left(S_{V^\perp}(X)\right),
$$

where $\text{grad}$ denotes the gradient on $M$ and $H$ the mean curvature vector of $M$.

Proof. Choose $p \in M$ and let $\{E_1, \cdots, E_n\}$ be a local orthonormal frame of $M$ geodesic at $p$. Then,

$$
(\nabla^2 \eta)(p) = \sum_{i=1}^n \nabla_{E_i} \nabla_{E_i} \eta.
$$

The following equalities are satisfied:

$$
\langle \nabla_{E_i}[E_i, E_j], \eta \rangle = \langle [E_i, [E_i, E_j]], \eta \rangle = 0
$$

and $\langle E_i, \eta \rangle = 0$, along $M$. Then, for each $j \in \{1, \cdots, n\}$, we have at $p$

$$
\langle \nabla^2 \eta, E_j \rangle = -\sum_{i=1}^n \left( \langle \nabla_{E_i} \nabla_{E_i} E_j, \eta \rangle + 2\langle \nabla_{E_i} E_j, \nabla_{E_i} \eta \rangle \right) \\
= -\sum_{i=1}^n \left( \langle \nabla_{E_i}[E_i, E_j], \eta \rangle + \langle \nabla_{E_i} \nabla_{E_i} E_i, \eta \rangle + 2\langle \nabla_{E_i} E_j, \nabla_{E_i} \eta \rangle \right)
$$
\[
= - \sum_{i=1}^{n} \left( \langle \nabla E_i \nabla E_i, \eta \rangle \right) - 2\text{tr} \left( S_{\nabla \nabla} (E_j) \right)
\]
\[
= \sum_{i=1}^{n} \left( \langle R(E_i, E_j) E_i, \eta \rangle - \langle \nabla E_i \nabla E_i, \eta \rangle \right) - 2\text{tr} \left( S_{\nabla \nabla} (E_j) \right)
\]
\[
= \sum_{i=1}^{n} \left( \langle R(E_i, E_j) E_i, \eta \rangle - \langle \nabla E_i \nabla E_i, \eta \rangle + \langle \nabla i E_i, V_{E_i, \eta} \rangle \right) - 2\text{tr} \left( S_{\nabla \nabla} (E_j) \right)
\]
\[
= \text{Ric} M(\eta, E_j) - n \langle \text{grad} (\vec{H}, \eta), E_j \rangle + n \langle \vec{H}, V_{E_i, \eta} \rangle - 2\text{tr} \left( S_{\nabla \nabla} (E_j) \right).
\]

The lemma is then proved by observing that \( p \) is arbitrary and writing \( X \) on the basis \( \{E_j\} \). □

**Corollary 3.5.** Let \( M \) be an \( n \)-dimensional manifold immersed in a Riemannian manifold \( N \) and assume that \( \eta \) is a parallel normal section in the normal connection of \( M \). Then,

\[
\langle \nabla^2 \eta, X \rangle = \text{Ric} M(\eta, X) - n \langle \text{grad} (\vec{H}, \eta), X \rangle,
\]

for all \( X \in T(M) \). In particular, if \( M \) is a two-sided hypersurface of \( N \) and \( \eta \) is a unit normal section, then

\[
\langle \nabla^2 \eta, X \rangle = \text{Ric}(\eta, X) - n \langle \text{grad} H, X \rangle,
\]

where \( H \) is the mean curvature of \( M \) with respect to \( \eta \) and \( \text{Ric}(\cdot, \cdot) \) is the Ricci tensor of \( N \).

In the next lemma, we relate the normal part of the rough Laplacian of a unit normal section \( \eta \) parallel in the normal connection with the Simons operator of \( \eta \).

**Lemma 3.6.** Let \( M \) be a manifold immersed in a Riemannian manifold \( N \) and assume that \( \eta \) is a parallel normal section in the normal connection of \( M \). Then,

\[
(\nabla^2 \eta)^\perp = -\mathcal{B} \eta.
\]

**Proof.** Let \( \{E_1, \ldots, E_n\} \) be a local orthonormal frame of \( M \) and let \( \nu \) be a normal vector to \( M \). Then, differentiating

\[
\langle \nabla E_i \eta, \nu \rangle = \langle \nabla^2 \eta, \nu \rangle = 0
\]

with respect to \( E_i \) and summing up we have

\[
0 = \sum_{i=1}^{n} \left( \langle \nabla E_i \nabla E_i, \nu \rangle + \langle \nabla E_i \eta, \nabla E_i \nu \rangle \right)
\]
\[
= \sum_{i=1}^{n} \left( \langle \nabla E_i \nabla E_i, \nu \rangle + \langle S_\eta (E_i), S_\nu (E_i) \rangle \right)
\]
\[
= \langle \nabla^2 \eta, \nu \rangle + \langle S_\eta, S_\nu \rangle
\]
\[
= \langle \nabla^2 \eta, \nu \rangle + \langle B(\eta), B(\nu) \rangle
\]
\[
= \langle \nabla^2 \eta, \nu \rangle + \langle B^* B(\eta), \nu \rangle
\]
\[
= \langle \nabla^2 \eta, \nu \rangle + \langle \tilde{B}(\eta), \nu \rangle.
\]
that is, \( \langle \nabla^2 \eta, \nu \rangle = -\langle \tilde{B}(\eta), \nu \rangle \), then

\[
(\nabla^2 \eta)^\perp = -\tilde{B}(\eta).
\]

In the next result, we obtain a formula for the normal part of the rough Laplacian of a Killing vector field.

**Lemma 3.7.** Let \( M \) be an \( n \)-dimensional manifold immersed in a Riemannian manifold \( N \). If \( V \) is a Killing vector field of \( N \) and \( \eta \) a normal section of \( M \), then it holds the equality

\[
-\langle \nabla^2 V, \eta \rangle = \text{Ric}_M(\eta, V) + n\langle \tilde{H}, \nabla_\eta V \rangle. \tag{3.3}
\]

**Proof.** Choose \( p \in M \) and let \( \{E_1, \cdots, E_n\} \) be a local orthonormal frame of \( M \) geodesic at \( p \). We extend \( \{E_i\} \) to an open set of \( N \) parallelly along the geodesics orthogonal to \( M \). Since \( V \) is a Killing vector field, we have at \( p \)

\[
\langle \nabla^2 V, \eta \rangle = \sum_{i=1}^n \langle \nabla_{E_i} \nabla_{E_i} V, \eta \rangle
\]

\[
= \sum_{i=1}^n (E_i \langle \nabla_{E_i} V, \eta \rangle - \langle \nabla_{E_i} V, \nabla_{E_i} \eta \rangle)
\]

\[
= -\sum_{i=1}^n (\langle \nabla_{\eta} V, \nabla_{E_i} E_i \rangle + \langle \nabla_{E_i} \nabla_{\eta} V, E_i \rangle + \langle \nabla_{E_i} \eta, \nabla_{E_i} V \rangle).
\]

For the second term of the last expression, we obtain

\[
\sum_{i=1}^n \langle \nabla_{E_i} \nabla_{\eta} V, E_i \rangle = -n \langle \tilde{H}, \nabla_{\eta} V \rangle - \langle \nabla^2 V, \eta \rangle - \sum_{i=1}^n \langle \nabla_{E_i} \eta, \nabla_{E_i} V \rangle. \tag{3.4}
\]

On the other hand, since \( V \) is a Killing vector field of \( N \), we have \( \langle \nabla_{E_i} V, E_i \rangle = 0 \), then

\[
0 = \langle \nabla_{\eta} V, E_i \rangle(p) + \langle \nabla_{E_i} V, \nabla_{\eta} E_i \rangle(p), \tag{3.5}
\]

and noting that \( \nabla_{\eta} E_i(p) = 0 \) since \( E_i \) is parallel along \( \eta \), we have

\[
0 = \langle \nabla_{E_i} V, \nabla_{\eta} E_i \rangle(p),
\]

then from Equation (3.5), we obtain that \( \langle \nabla_{\eta} V, E_i \rangle(p) = 0 \). Thus,

\[
\langle R(\eta, E_i)V, E_i \rangle(p) = \langle \nabla_{E_i} \nabla_{\eta} V, E_i \rangle - \langle \nabla_{\eta} \nabla_{E_i} V, E_i \rangle + \langle \nabla_{[\eta, E_i]} V, E_i \rangle
\]

\[
= \langle \nabla_{E_i} \nabla_{\eta} V, E_i \rangle - \langle \nabla_{E_i} V, [\eta, E_i] \rangle
\]

\[
= \langle \nabla_{E_i} \nabla_{\eta} V, E_i \rangle + \langle \nabla_{E_i} V, \nabla_{E_i} \eta \rangle,
\]

that is,

\[
\langle E_i, \nabla_{E_i} \nabla_{\eta} V \rangle = \langle R(\eta, E_i)V, E_i \rangle - \langle \nabla_{E_i} V, \nabla_{E_i} \eta \rangle. \tag{3.6}
\]

Thus, from (3.4) and (3.6) we obtain (3.3).

**Corollary 3.8.** Let \( M \) be a minimal submanifold of a Riemannian manifold \( N \), \( \eta \) a normal section of \( M \), and \( V \) a Killing vector field of \( N \). Then, it holds the equality

\[
\langle \nabla^2 V, \eta \rangle = -\text{Ric}_M(\eta, V).
\]
Lemma 3.9. Let $M$ be an $n$-dimensional manifold immersed in a Riemannian manifold $N$. Let $V$ be a Killing field of $N$ and $\eta$ a normal section of $M$. Then, one has the following formula for the Laplacian of the function $f = \langle \eta, V \rangle$:

$$
\Delta_M f = \langle (\nabla^2 \eta)^\perp, V \rangle - \langle \text{Ric}_M(\eta), V \rangle - n\langle \nabla H, V \rangle - n\langle H, \nabla V \rangle + 2\langle \nabla V, \nabla V \rangle - 2\text{tr}\left(S_{\perp\eta}((V^T)^T)\right).
$$

(3.7)

Proof. Given $p \in M$, let $\{E_1, \ldots, E_n\}$ be a local orthonormal basis of $M$ geodesic at $p$. Then, at $p$,

$$
\Delta_M f = \sum_{i=1}^n E_i(E_i(f)).
$$

Note that

$$
\Delta_M f = \langle \nabla^2 \eta, V \rangle + 2\langle \nabla \eta, \nabla V \rangle + \langle \eta, \nabla^2 V \rangle.
$$

From this equation and Lemma 3.7, we have

$$
\Delta_M f = \langle \nabla^2 \eta, V \rangle + 2\langle \nabla \eta, \nabla V \rangle - \text{Ric}_M(\eta, V) - n\langle H, \nabla V \rangle.
$$

(3.8)

Writing $V = V^\perp + V^T$, we have that

$$
\langle \nabla^2 \eta, V \rangle = \langle \nabla^2 \eta, V^\perp \rangle + \langle (\nabla^2 \eta)^\perp, \rangle V^T \rangle
$$

(3.9)

and

$$
\text{Ric}_M(\eta, V^T) = \text{Ric}_M(\eta, V) - \langle \text{Ric}^\perp_M(\eta), V \rangle.
$$

(3.10)

Recall from Lemma 3.4 that

$$
\langle \nabla^2 \eta, V^T \rangle = \text{Ric}_M(\eta, V^T) - n\langle \text{grad}(\nabla H, \eta), V^T \rangle + n\langle \nabla V, \nabla V \eta \rangle - 2\text{tr}\left(S_{\perp\eta}((V^T)^T)\right).
$$

(3.11)

Then, from Equations (3.8), (3.9), (3.10), and (3.11), we obtain (3.7).

As a consequence of Lemma 3.9, we obtain the following useful formula.

Corollary 3.10. Let $M$ be an $n$-dimensional manifold immersed in a Riemannian manifold $N$, $V$ a Killing vector field of $M$, and $\eta$ a parallel unit normal section of $M$. If $f : M \to \mathbb{R}$ is given by

$$
f(q) = \langle \eta(q), V(q) \rangle, \quad q \in M,
$$

then

$$
-\Delta_M f = n\langle \text{grad}(\nabla H, \eta), V \rangle + n\langle \nabla H, \nabla V \eta \rangle + \langle \nabla H, V \eta \rangle + \langle \text{Ric}^\perp_M(\eta), V \rangle
$$

(3.12)

where $\nabla H$ is the mean curvature vector of $M$.

Proof. Given $p \in M$, let $\{E_1(p), \ldots, E_n(p)\}$ be a basis of the tangent space $T_pM$ that diagonalizes the second fundamental form of $M$ determined by $\eta$. Then, since $\eta$ is parallel in the normal connection of $M$ and $V$ is a Killing vector field of $N$, the following terms in Equation (3.7) are zero:

$$
\langle \nabla \eta, \nabla V \rangle = \sum_{i=1}^n \langle S_\eta(E_i), V E_i \rangle = 0,
$$

$$
\langle \nabla V, \nabla V^\perp \eta \rangle = \langle \nabla V, \nabla V^\perp \eta \rangle = 0,
$$

$$
\langle \nabla H, \nabla V^\perp \eta \rangle = \langle \nabla H, \nabla V^\perp \eta \rangle = 0,
$$

$$
\langle \text{Ric}^\perp_M(\eta), V \rangle = \langle \text{Ric}^\perp_M(\eta), V \rangle = 0.
$$

□
and

$$\text{tr} \left( S_{\frac{1}{2}g} (V^T) \right) = 0,$$

thus, from Lemma 3.9, we have

$$\Delta_M f = \langle \left( \nabla^2 \eta \right)^\perp, V \rangle - \langle \text{Ric} \left( \frac{1}{2} M(\eta), V \right) - n \langle \text{grad}(H, \eta), V \rangle - n \langle H, \nabla V \rangle,$$

which together with Lemma 3.6 gives (3.12).

\[\square\]

4 HARMONIC UNIT NORMAL SECTIONS

In this section, we introduce the concept of harmonic unit normal sections of a submanifold $M$ of a Riemannian manifold $N$ and prove that unit normal sections of principal orbits of polar actions provide a family of examples of harmonic unit normal sections. These examples are interesting on their own and important in our applications.

Let $M$ be a compact manifold immersed in a Riemannian manifold $N$. Herein, we assume that $\mathcal{N}_1(M)$ is nonempty. We define the energy functional $\mathcal{N} : \mathcal{N}_1(M) \to \mathbb{R}$ by

$$\mathcal{N}(\eta) = \int_M \|\nabla \eta\|^2.$$

We say that a unit normal section of $M$ is harmonic if it is a critical point of $\mathcal{N}$. If $M$ is not compact, a unit normal section $\eta$ of $M$ is harmonic if $\eta$ is a critical point of $\mathcal{N}$ on any compact subdomain of $M$.

Taking variations of a unit normal section $\eta$ on $\mathcal{N}_1(M)$ we obtain the following:

**Proposition 4.1.** Let $M$ be a compact manifold immersed in a Riemannian manifold $N$. A unit normal section $\eta \in \mathcal{N}_1(M)$ is harmonic if and only if

$$\left( \nabla^2 \eta \right)^\perp = -\|\nabla \eta\|^2 \eta.$$

**Proof.** Let $N \in \mathcal{N}(M)$ be a normal vector field of $M$. Consider the variation

$$\eta_N(t) = \frac{\eta + tN}{\|\eta + tN\|},$$

and differentiating the energy $\mathcal{N}(\eta_N(t))$ at $t = 0$, we have

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{N}(\eta_N(t)) = -2 \int_M \langle \|\nabla \eta\|^2, N \rangle + 2 \int_M \langle \nabla \eta, \nabla N \rangle.$$

Then, $\eta \in \mathcal{N}(M)$ is a critical point of $\mathcal{N}$ if and only if

$$\int_M \langle \|\nabla \eta\|^2, N \rangle = \int_M \langle \nabla \eta, \nabla N \rangle.$$

(4.2)

Let $\{E_1, \ldots, E_n\}$ be a geodesic frame in a neighborhood of a fixed point $p \in M$. Defining the tangent vector field

$$X = \sum_{i=1}^n \langle \nabla E_i, N \rangle E_i,$$

we have

$$\text{div} X = \sum_{i=1}^n E_i \langle \nabla E_i, N \rangle$$

$$= \langle \nabla^2 \eta, N \rangle + \langle \nabla \eta, \nabla N \rangle.$$
Noting that this equality is independent of \( p \) and integrating we have

\[
\int_M \langle \nabla^2 \eta, N \rangle = - \int_M \langle \nabla \eta, \nabla N \rangle. \tag{4.3}
\]

Thus, from Equations (4.2) and (4.3), since \( N \) is arbitrary in \( \mathcal{N}(M) \) it follows from (4.1).

**Proposition 4.2.** Let \( \eta \) be a unit normal section of a manifold \( M \) immersed in a Riemannian manifold \( N \) such that \( \eta \) is parallel in the normal connection of \( M \). Then, \( \eta \) is a harmonic normal section of \( M \) if and only if \( \eta \) satisfies \( (\nabla^2 \eta)^\perp = -\|\nabla \eta\|^2 \eta \). Then, the proof is completed noting that from Lemma 3.6, the equation \( (\nabla^2 \eta)^\perp = -\hat{\mathbf{B}}(\eta) \) holds.

An isometric action of a compact Lie group \( \mathbb{G} \) on a complete Riemannian manifold \( N \) is called polar if there is a connected complete Riemannian manifold \( \Sigma \) isometrically immersed in \( N \), which intersects orthogonally all the orbits of \( \mathbb{G} \) (see [2], section 4.1). We prove the following:

**Theorem 4.3.** If an action of a compact Lie group \( \mathbb{G} \) on a Riemannian manifold \( N \) is polar, then the Simons operator \( \hat{\mathbf{B}} \) on the principal orbits of \( \mathbb{G} \) is diagonalizable by parallel unit normal sections with constant eigenvalues. Moreover, these parallel unit eigenvectors of \( \hat{\mathbf{B}} \) are harmonic normal sections.

**Proof.** Let \( \mu : \mathbb{G} \times N \to N \) be a polar action. For each \( g \in \mathbb{G} \), we denote by \( \mu^g : N \to N \) the map defined by \( \mu^g(\cdot) := \mu(g, \cdot) \).

Let \( x \in N \) be such that \( \mathbb{G}(x) \) is a principal orbit of \( \mathbb{G} \). Let \( \eta_1(x), \ldots, \eta_r(x) \) be an orthonormal basis of eigenvectors of the self-adjoint operator \( \hat{\mathbf{B}} \) defined on the normal space \( T_x \mathbb{G}(x) \). For each eigenvector \( \eta_i(x) \), we define a normal vector field \( \hat{\eta}_i \) along \( \mathbb{G}(x) \) by

\[
\hat{\eta}_i(\mu(g, x)) = \frac{d}{dg} \mu^g_{\eta_i(x)}, \quad i = 1, \ldots, r.
\]

It is easy to see that \( \hat{\eta}_i \) is well defined and is parallel in the normal connection (see [2], sections 3.4 and 4.1).

We claim that \( \hat{\eta}_1, \ldots, \hat{\eta}_r \) are eigenvectors of \( \hat{\mathbf{B}} \) of \( \mathbb{G}(x) \). Indeed, let \( \{v_1, \ldots, v_r\} \) and \( \{w_1, \ldots, w_r\} \) be orthonormal basis of \( T_x \mathbb{G}(x) \) that diagonalizes the second fundamental forms \( S_{\eta_i(x)} \) and \( S_{\eta_j(x)} \), respectively, \( l, k \in \{1, \ldots, r\} \). Thus, if the principal curvatures are denoted by \( \lambda_l \) and \( \sigma_k \), we have \( S_{\eta_l(x)}(v_i) = \lambda_i v_i \) and \( S_{\eta_k(x)}(w_j) = \sigma_j w_j \).

Now, it is easy to see that \( S_{\eta_i(\mu(g, x))} = d\mu^g S_{\eta_i(x)} d\mu^{g^{-1}} \) and that the tangent vector fields to \( \mathbb{G}(x) \), \( V_i(\mu(g, x)) = d\mu^g v_i \) and \( W_j(\mu(g, x)) = d\mu^g w_j \), are eigenvectors of \( S_{\hat{\eta}_i} \) and \( S_{\hat{\eta}_k} \) with eigenvalues equal to \( \lambda_l \) and \( \sigma_k \). Hence, if \( (a^i_j) \) is the change basis matrix from \( \{w_1, \ldots, w_r\} \) to \( \{v_1, \ldots, v_r\} \), then \( (a^i_j) \) is the change basis matrix from \( \{W_1, \ldots, W_r\} \) to \( \{V_1, \ldots, V_r\} \) too. As consequence, we have at \( \mu(g, x) \)

\[
\langle S_{\hat{\eta}_i}, S_{\hat{\eta}_k} \rangle = \sum_{i=1}^n \langle S_{\hat{\eta}_i}(V_i), S_{\hat{\eta}_k}(V_i) \rangle
\]

\[
= \sum_{i=1}^n \sum_{j=1}^r \langle S_{\hat{\eta}_i}(V_i), S_{\hat{\eta}_k}(a^i_j W_j) \rangle
\]

\[
= \sum_{j=1}^r \sum_{i=1}^n \langle S_{\hat{\eta}_i}(V_i), S_{\hat{\eta}_k}(W_j) \rangle
\]

\[
= \sum_{j=1}^r \sum_{i=1}^n a^i_j \lambda_i \sigma_j \langle v_i, w_j \rangle.
\]

Therefore, noting that all the terms \( a^i_j, \lambda_i, \sigma_j \), and \( \langle v_i, w_j \rangle \) are constant, we have that \( \langle S_{\hat{\eta}_i}, S_{\hat{\eta}_k} \rangle \) is constant along \( \mathbb{G}(x) \). That is,

\[
\langle \hat{\mathbf{B}}(\hat{\eta}_i), \hat{\eta}_k \rangle = \langle S_{\eta_i(x)}, S_{\eta_k(x)} \rangle = \delta_{lk} \langle S_{\eta_l(x)}, S_{\eta_k(x)} \rangle.
\]
Since \( l \) and \( k \) are arbitrary, it follows that \( \hat{\eta}_1, \ldots, \hat{\eta}_r \) is an eigenbasis of \( \bar{B} \) along \( G(x) \) with constant eigenvalues equal to \( \|S_{\eta(x)}\|^2 \).

Finally, from Proposition 4.2, it follows that the parallel eigenvectors \( \hat{\eta}_1, \ldots, \hat{\eta}_r \) of \( \bar{B} \) are harmonic unit normal sections of \( G(x) \).

\[ \square \]

5 HARMONICITY OF THE GAUSS MAPS OF SUBMANIFOLDS OF THE EUCLIDEAN SPACE

In this section, we prove the results about the harmonicity of the Gauss maps of submanifolds of the Euclidean space. We begin by stating some facts, which are direct consequences of our previous results.

**Proposition 5.1.** Let \( M \) be an \( n \)-dimensional manifold immersed on the Euclidean space \( \mathbb{R}^m \) and let \( \eta \) be a unit normal section parallel in the normal bundle of \( M \). The Laplacian of the Gauss map \( \gamma_\eta \) associated to \( \eta \) is given by

\[ -\Delta \gamma_\eta = n \text{ grad}(\bar{H}, \eta) + \bar{B}(\eta). \]

**Proof.** The Ricci normal operator \( \text{Ric}_M \) is equal to zero since the curvature tensor of \( \mathbb{R}^m \) vanishes identically. Then, denoting by \( \{e_1, \cdots, e_m\} \) the canonical basis of \( \mathbb{R}^m \) we have, from Corollary 3.10,

\[ \Delta \gamma_\eta = \sum_{i=1}^m \Delta_M \langle \eta, e_i \rangle e_i = -n \text{ grad}(\bar{H}, \eta) - \bar{B}(\eta). \]

Recall that a submanifold has parallel normalized mean curvature vector if \( \bar{H} \) is nonzero and the unit normal section \( \eta = \bar{H}/\|\bar{H}\| \) is parallel in the normal connection (see [9]). Then, from Proposition 5.1, we have that an immersed submanifold in the Euclidean space with parallel normalized mean curvature has parallel mean curvature vector if and only if the unit vector \( \eta \) in the direction of the mean curvature vector satisfies:

\[ -\Delta \gamma_\eta = \bar{B}(\eta). \]

As a consequence of Proposition 5.1, we can prove the equivalences relating \( \eta \) and the associated Gauss map \( \gamma_\eta \) of the Theorem 1.4.

**Proof of Theorem 1.4.** From Proposition 5.1 and since \( M \) has parallel mean curvature vector, we have

\[ -\Delta \gamma_\eta = \bar{B}(\eta). \]

From this equality and recalling that the map \( \gamma_\eta \) is harmonic if and only if \( \Delta(\gamma_\eta) = -f \gamma_\eta \), for some function \( f : M \to \mathbb{R} \), we obtain that \( \gamma_\eta \) is harmonic if and only if \( \eta \) is and eigenvector of \( \bar{B} \). This proves the equivalence between (1) and (3).

The equivalence between (2) and (3) is given in Proposition 4.2.

Let \( M \) be an orientable hypersurface of \( \mathbb{S}^{n+1} \). Recall that we denote by \( \nu \) a unit normal section of \( M \) in \( \mathbb{S}^{n+1} \) and by \( \mu \) the position vector of \( M \) in \( \mathbb{R}^{n+2} \).

**Proposition 5.2.** Let \( M \) be an orientable hypersurface of \( \mathbb{S}^{n+1} \), \( \eta \) a unit normal section of \( M \) in \( \mathbb{R}^{n+2} \) parallel in the normal connection, and let \( \theta \) be the angle between \( \eta \) and \( \mu \). Then, the Gauss map \( \gamma_\eta : M \to \mathbb{S}^{n+1} \) satisfies

\[ -\Delta \gamma_\eta = n a \text{ grad}(H) + (a\|S_\eta\|^2 - nbH)\gamma_\nu + (nb - naH)\gamma_\mu, \]

where \( H \) denotes the mean curvature of \( M \) in \( \mathbb{S}^{n+1} \) with respect to \( \nu \), \( a = \sin \theta \), and \( b = \cos \theta \).

**Proof.** The normal vector \( \eta \) parallel in the normal connection is of the form \( \eta = a \nu + b \mu \), with \( a, b \in \mathbb{R} \) where \( a = \sin \theta \) and \( b = \cos \theta \). Then by Proposition 5.1, we have

\[ -\Delta \gamma_\eta = n a \text{ grad}(H) + \bar{B}(\eta). \]
and, noting that,

\[ \hat{B}(\eta) = a \hat{B}(\nu) + b \hat{B}(\mu) = (a \|S_\nu\|^2 - nbH)\nu + (nb - naH)\mu, \]

we conclude the proof. \(\square\)

We are now in position to prove Theorem 1.5.

**Proof of Theorem 1.5.**

(1) Assume \( \theta = 0 \). We then have \( a = 0 \) and \( b = 1 \) in Proposition 5.2. Also from this proposition,

\[ -\Delta \gamma_\eta = -nH\gamma_\nu + n\gamma_\mu. \]

Therefore, \( \gamma_\eta \) is a harmonic map if and only if \( H = 0 \). Analogously, if \( \theta = \frac{\pi}{2} \), then \( a = 1 \) and \( b = 0 \) in Proposition 5.2 and

\[ -\Delta \gamma_\eta = n \text{grad}(H) + \|S_\nu\|^2\gamma_\nu - nH\gamma_\mu. \]

Hence, \( \gamma_\eta \) is harmonic if and only if \( H = 0 \).

(2) Let \( \eta^\perp = -b\nu + a\mu \) be a unit normal section of \( M \) orthogonal to \( \eta \), where \( a = \sin \theta \) and \( b = \cos \theta \). From Proposition 5.2, it follows that \( \gamma_\eta \) is a harmonic map if and only if \( H \) is constant and \( \langle -\Delta \gamma_\eta, \eta^\perp \rangle = 0 \). Also from this proposition, we see that the condition \( \langle -\Delta \gamma_\eta, \eta^\perp \rangle = 0 \) is equivalent to

\[ \|S_\nu\|^2 = nH \left( \frac{b}{a} - \frac{a}{b} \right) + n. \]

This concludes the proof of the theorem. \(\square\)

Following Eells–Lemaire [16], an eigenmap of \( M \) is a map \( f : M \to S^k \) such that \( \Delta f = \lambda f \), \( \lambda \) constant. The eigenmaps are studied in different contexts, mainly between spheres. The eigenmaps are clearly harmonic, their coordinate functions are eigenfunctions of the Laplacian with the same eigenvalue.

An immediate consequence of Theorem 1.5 is as follows:

**Corollary 5.3.** Let \( M \) be an oriented hypersurface of \( S^{n+1} \subset \mathbb{R}^{n+2} \) and \( \eta \) a parallel unit normal section of \( M \) with \( \theta \neq \frac{\pi}{2} \). The Gauss map \( \gamma_\eta \) is harmonic if and only if \( \gamma_\eta \) is an eigenmap.

**Proof of Corollary 1.6.** First, we define a linear map \( \phi \) on the tangent bundle of \( M \) by

\[ \langle \phi(X), Y \rangle = H\langle X, Y \rangle - \langle S_\nu(X), Y \rangle, \]

and note that

\[ \|\phi\|^2 = \|S_\nu\|^2 - nH^2. \quad (5.1) \]

(1) Case \( H = 0 \) : (1a) \( \Rightarrow \) (1b) it is trivial.

(1b) \( \Rightarrow \) (1c) Let \( \eta \) be a unit normal section of \( M \) such that the angle \( \theta \) between \( \eta \) and \( \mu \) is neither 0 nor \( \frac{\pi}{2} \) and \( \gamma_\eta \) is a harmonic map. By item (1) of Theorem 1.5 we have that \( \|S_\nu\|^2 = n \). Then, \( M \) is a minimal Clifford torus of \( S^{n+1} \) since they are the only compact minimal hypersurfaces of \( S^{n+1} \) such that quality \( \|S_\nu\|^2 = n \) is satisfied, see [11].

(1c) \( \Rightarrow \) (1a) Let \( M \) be a minimal Clifford torus and let \( \eta \) a unit normal section of \( M \) in \( \mathbb{R}^{n+2} \). If the angle \( \theta \) between \( \eta \) and \( \mu \) is 0 or \( \frac{\pi}{2} \), then from Theorem 1.5 \( \gamma_\eta \) is a harmonic map since \( H = 0 \). If \( \theta \neq 0 \) and \( \theta \neq \frac{\pi}{2} \), then from Theorem 1.5, \( \gamma_\eta \) is a harmonic map since \( \|S_\nu\|^2 = n \).
(2) Case $H \neq 0$ and $n > 2$:

(a) On one hand, from Theorem 1.5 of [1], the $H(r)$-torus with $r^2 < (n - 1)/n$ and $n > 2$ is the only compact CMC hypersurface of the sphere $\mathbb{S}^{n+1}$ with $\|S_\eta\|^2 = B_H + nH^2$. On the other hand, from Theorem 1.5 above, $\gamma_\eta$ is a harmonic map if and only if $\|S_\eta\|^2 = nH(\cot \theta - \tan \theta) + n$. Therefore, $M$ is the $H(r)$-torus with $r^2 < (n - 1)/n$ if and only if the only harmonic Gauss maps of $M$ are, up to sign, the two ones determined by the solutions of equation $nH(\cot \theta - \tan \theta) = B_H + nH^2 - n$.

(b) From Theorem 1.5, we have that

$$\|S_\eta\|^2 = nH(\cot \theta - \tan \theta) + n \tag{5.2}$$

since $\gamma_\eta$ is harmonic. Then, the inequality (3b) is equivalent to $\|\phi\|^2 < B_H$. From Theorem 1.5 of [1], $M$ is a totally umbilical hypersurface of $\mathbb{S}^{n+1}$. Since $H \neq 0$, it follows from (5.2) that the principal curvature $\lambda$ of $M$ is nonzero and satisfies

$$\cot \theta - \tan \theta = \frac{\lambda^2 - 1}{\lambda}.$$

(3) Case $n = 2$ and $H \neq 0$:

From Theorem 1.5, it follows that $\gamma_\eta$ is harmonic if and only if the mean curvature $H$ and the square of norm of the second fundamental form $S_\eta$ are constant. Hence, since $n = 2$, the principal curvatures of $M$ in $\mathbb{S}^3$ are constant. It follows that $M$ is a product of circles $S^1(r) \times S^1(\sqrt{1 - r^2})$ with $0 < r < 1$ or $M$ is a totally umbilical surface with $\lambda \neq 0$ (see Theorem 3.29 of [12]). Noting that $\|S_\eta\| = nH(\cot \theta - \tan \theta) + n$, we have

$$\cot \theta - \tan \theta = \frac{1 - 2r^2}{r \sqrt{1 - r^2}}$$

when $M$ is the product of circles since the principal curvatures are $\frac{\sqrt{1 - r^2}}{r}$ and $-\frac{r}{\sqrt{1 - r^2}}$ and

$$\cot \theta - \tan \theta = \frac{\lambda^2 - 1}{\lambda}$$

when $M$ is a totally umbilical surface.

Proof of Theorem 1.1. Assume that $M$ admits a unit normal section $\eta$ in $\mathbb{R}^4$ such that $\gamma_\eta$ is a harmonic map.

We claim that $M$ is a not minimal surface of $\mathbb{R}^4$. Indeed let $\nu$ be a local unit normal section of $M$ orthogonal to $\eta$ and let $\{e_1, e_2, e_3, e_4\}$ be a parallel orthonormal basis of $\mathbb{R}^4$ such that $e_1$ and $e_2$ are eigenvectors of $S_\nu$ at a point $p$ of $M$. Then, from Lemma 3.9,

$$\Delta \gamma_\eta(p) = \sum_{i=1}^{4} \langle \Delta \gamma_\eta(p), e_i \rangle e_i$$

$$= \sum_{i=1}^{4} \left( \langle (\nabla^2 \eta)^\perp(p), e_i \rangle - 2 \text{tr} \left( S_{\nu}^\perp \eta^\perp(e_i^T) \right) \right) e_i$$

$$= \sum_{i=3}^{4} \langle (\nabla^2 \eta)^\perp(p), e_i \rangle e_i - 2 \sum_{i=1}^{2} \text{tr} \left( S_{\nu}^\perp \eta^\perp(e_i) \right) e_i.$$

Since $\gamma_\eta$ is harmonic, the tangent part of the Laplacian of $\gamma_\eta$ is zero, thus at $p$ we have, for $i, j = 1, 2$ with $i \neq j$,

$$0 = \text{tr} \left( S_{\nu}^\perp \eta^\perp(e_j) \right)$$

$$= \langle S_{\nu}^\perp \eta^\perp(e_i), e_i \rangle + \langle S_{\nu}^\perp \eta^\perp(e_j), e_j \rangle$$

$$= \langle \nabla^\perp_\nu \eta^\perp, \nu \rangle \langle S_\nu(e_i), e_i \rangle + \langle \nabla^\perp_\nu \eta^\perp, \nu \rangle \langle S_\nu(e_j), e_j \rangle$$

$$= \langle \nabla^\perp_\nu \eta^\perp, \nu \rangle \langle S_\nu(e_i), e_i \rangle.$$
Note that \( \langle S_\nu(e_i), e_i \rangle \neq 0 \) since the trace of \( S_\nu \) is zero and the second fundamental form of \( M \) spans the normal space of \( S \). Therefore, \( \langle (\nabla^\perp_\nu \eta)(p), \nu(p) \rangle = 0 \) for \( j = 1, 2 \), that is, \( \eta \) is parallel in the normal connection.

Since the codimension of \( M \) is 2, its normal bundle is flat. This contradicts the fact that the second fundamental form of \( M \) spans the normal bundle of \( \mathbb{R}^m \) with flat normal bundle is contained in a \( \mathbb{R}^3 \) (see chapter 2 of [13]). Then, \( M \) is not a minimal surface of \( \mathbb{R}^4 \) with substantial codimension 2.

Therefore, from the classification of surfaces with parallel mean curvature vector of \( \mathbb{R}^4 \) given by B. Chen [8] and S. Yau [36], we have, up to isometries of \( \mathbb{R}^4 \), that \( M \) is a CMC surface of \( S^3 \subset \mathbb{R}^4 \). As a consequence \( \eta \) can be written (locally) in the form

\[
\eta = a\nu + b\mu,
\]

where \( \nu \) is a local unit normal section of \( M \) in \( S^3 \), \( \mu \) the position vector of \( M \), and \( a, b \) are differentiable functions defined in \( M \) such that \( a^2 + b^2 = 1 \).

We now consider the case (2) of the theorem, that is, \( a \) and \( b \) are not constants. Let \( \{e_1, e_2, e_3, e_4\} \) be a parallel orthonormal basis of \( \mathbb{R}^4 \) such that \( e_1 \) and \( e_2 \) are eigenvectors of \( S_\nu \) at a point \( p \in M \).

We claim that: for \( i = 1, 2 \), it holds

\[
\langle \nabla(\bar{H}, \eta), e_i \rangle = \langle \bar{H}, \nabla e_i \eta \rangle,
\]

where \( \bar{H} \) is the mean curvature of \( M \) in \( S^3 \) with respect to \( \nu \). Indeed,

\[
\langle \nabla(\bar{H}, \eta), e_i \rangle = \langle \nabla(\bar{H}\nu + \mu, f_1\nu + f_2\mu), e_i \rangle
\]

\[
= \langle \nabla(f_1\nu + f_2), e_i \rangle
\]

\[
= e_i(f_1)\bar{H} + e_i(f_2)
\]

\[
= \langle \bar{H}\nu + \mu, e_i(f_1)\nu + e_i(f_2)\mu \rangle
\]

\[
= \langle \bar{H}, \nabla e_i \eta \rangle.
\]

Then, from Equation (5.3) and Lemma 3.9, we have for \( i = 1, 2 \)

\[
\Delta\langle \eta, e_i \rangle(p) = -2\operatorname{tr}(S_{\nu}^\perp \eta)(p)e_i.
\]

From the harmonicity of \( \gamma_\eta \), we have that the Laplacian \( \Delta \gamma_\eta(p) \) has no tangent part. Since the Laplacian of \( \gamma_\eta \) at \( p \) is

\[
\Delta \gamma_\eta(p) = \sum_{i=1}^{4} \Delta \langle \eta, e_i \rangle(p)e_i,
\]

we have

\[
\operatorname{tr}(S_{\nu}^\perp \eta)(p)e_i) = 0, \text{ for } i = 1, 2.
\]

The last equation implies a condition on the second fundamental determined by \( \eta^\perp = -f_2\nu + f_1\mu \). Indeed, first note that \( \eta^\perp \) is a unit normal section of \( M \) orthogonal to \( \eta \). We then have, for \( i, j = 1, 2 \) with \( i \neq j \)

\[
0 = \operatorname{tr}(S_{\nu}^\perp \eta)(p)e_i)
\]

\[
= \langle S_{\nu}^\perp \eta(e_i), e_i \rangle + \langle S_{\nu}^\perp \eta(e_i), e_j \rangle
\]

\[
= \langle \nabla^\perp_\nu \eta, \eta^\perp \rangle \langle S_{\nu}^\perp \eta(e_i), e_i \rangle + \langle \nabla^\perp_\nu \eta, \eta^\perp \rangle \langle S_{\nu}^\perp \eta(e_i), e_j \rangle
\]

\[
= \langle \nabla^\perp_\nu \eta, \eta^\perp \rangle \langle S_{\nu}^\perp \eta(e_i), e_i \rangle,
\]

as claimed.
We now observe that none of the following two equalities can occur:

\[ \langle \nabla_{\mathbf{e}_1} \mathbf{\eta}, \mathbf{\eta} \rangle = 0 \quad \text{and} \quad \langle \nabla_{\mathbf{e}_2} \mathbf{\eta}, \mathbf{\eta} \rangle = 0, \]

\[ \langle S_{\mathbf{\eta}}(\mathbf{e}_1), \mathbf{e}_1 \rangle = 0 \quad \text{and} \quad \langle S_{\mathbf{\eta}}(\mathbf{e}_2), \mathbf{e}_2 \rangle = 0. \]

Indeed, the first equality implies that \( \mathbf{\eta} \) is parallel in the normal connection and then \( a \) and \( b \) are constants, a contradiction! The second equality contradicts the hypothesis that the second fundamental form of \( M \) spans the normal space of \( M \).

Therefore, we have

\[ \langle \nabla_{\mathbf{e}_i} \mathbf{\eta}, \mathbf{\eta} \rangle = 0 \quad \text{and} \quad \langle S_{\mathbf{\eta}}(\mathbf{e}_j), \mathbf{e}_j \rangle = 0, \]

for \( i, j = 1, 2 \) and \( i \neq j \).

From \( \langle S_{\mathbf{\eta}}(\mathbf{e}_j), \mathbf{e}_j \rangle = 0 \) we obtain \( a = \lambda_j b \), where \( \lambda_j \) is the principal curvature of \( M \) with respect to \( e_j \). It follows that

\[ a = \pm \frac{\lambda_j}{\sqrt{1 + \lambda_j^2}} \quad \text{and} \quad b = \pm \frac{1}{\sqrt{1 + \lambda_j^2}}. \]

Moreover, for \( i \neq j \), one has \( \langle \nabla_{\mathbf{e}_i} \mathbf{\eta}, \mathbf{\eta} \rangle = 0 \) and hence \( e_i(\lambda_j) = 0 \), that is, \( \lambda_j \) is a nonconstant principal curvature, which is constant along the principal direction \( e_i \). We have then proved item (2) of the corollary.

It remains to consider the case when \( a \) and \( b \) are constants. If \( a = 0 \) or \( b = 0 \) then, from Theorem 1.5, \( M \) is a minimal surface of \( S^3 \). It is the alternative (1c).

Assume that \( a, b \) are nonzero constants. Then, from Theorem 1.5, the second fundamental form of \( M \) has constant length. Since \( M \) is a CMC surface of \( S^3 \), it follows that the principal curvatures of \( M \) are constant. Then, \( M \) is contained in a totally umbilical nontotally geodesic surface of \( S^3 \) or \( M \) is contained in an isoparametric surface of \( S^3 \) with two principal curvatures, that is, \( M \) is an open subset of the product of circles \( S^1(r) \times S^1(\sqrt{1 - r^2}) \subset S^3 \subset \mathbb{R}^4 \) with \( 0 < r < 1 \) (see Theorem 3.29 of [12]).

We now prove the converse of items (1a), (1b), and (1c). If \( M \) is a minimal surface and \( \mathbf{\eta} \) is the position vector or \( M \) or \( \mathbf{\eta} = \mathbf{v} \) then from Theorem 1.5 \( \gamma_{\mathbf{\eta}} \) is a harmonic map. This proves the converse of item (1c).

If \( M \) is an open subset of an umbilical surface of \( S^3 \) or \( M \) is an open subset of \( S^1(r) \times S^1(\sqrt{1 - r^2}) \). Then, from Theorem 1.5, the harmonic Gauss maps are given by the solutions for \( \theta \) of

\[ \| S_{\mathbf{v}} \| = nH(\cot \theta - \tan \theta) + n, \]

where \( H \) is the mean curvature of \( M \) with respect to \( \mathbf{v} \). This proves the converse of items (1a) and (1b).

\[ \square \]

**Proof of Theorem 1.3.** The proof is analogous to the first part of proof of Theorem 1.1. Assume by contradiction that the minimal surface \( M \) of \( S^4 \subset \mathbb{R}^5 \) admits a unit normal section \( \mathbf{\eta} \) of \( M \) such that \( \gamma_{\mathbf{\eta}} \) is a harmonic map.

Let \( \mathbf{v} \) be a local unit normal section of \( M \) in \( S^4 \) such that \( \langle \mathbf{v}, \mathbf{\eta} \rangle = 0 \). We consider a parallel orthonormal basis \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\} \) of \( \mathbb{R}^5 \) such that \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) are eigenvectors of \( S_{\mathbf{v}} \) at \( p \). Then, by Lemma 3.9 we have

\[ \Delta \gamma_{\mathbf{\eta}}(p) = \sum_{i=1}^{5} \langle \Delta \gamma_{\mathbf{\eta}}(p), \mathbf{e}_i \rangle \mathbf{e}_i \]

\[ = \sum_{i=1}^{5} \left( \langle (\nabla^2 \mathbf{\eta})(p), \mathbf{e}_i \rangle - 2 \langle \mu, \nabla_{\mathbf{e}_i} \mathbf{\eta} \rangle - 2 \text{tr} \left( S_{(\mathbf{\eta})}(\mathbf{e}_i) \right) \right) \mathbf{e}_i \]

\[ = \sum_{i=3}^{5} \langle (\nabla^2 \mathbf{\eta})(p), \mathbf{e}_i \rangle \mathbf{e}_i - 2 \sum_{i=1}^{2} \text{tr} \left( S_{(\mathbf{\eta})}(\mathbf{e}_i) \right) \mathbf{e}_i. \]
From the harmonicity of \( \gamma \), we have at \( p \), for \( i, j = 1, 2 \) with \( i \neq j \) that

\[
0 = \text{tr} \left( S_{\gamma}^\perp (e_i) \right) = \langle S_{\gamma}^\perp (e_i), e_i \rangle + \langle S_{\gamma}^\perp (e_j), e_j \rangle = \langle \nabla^\perp \gamma, \nu \rangle \langle S_{\gamma} (e_i), e_i \rangle + \langle \nabla^\perp \gamma, \mu \rangle \langle S_{\gamma} (e_j), e_j \rangle = \langle \nabla^\perp \gamma, \nu \rangle \langle S_{\gamma} (e_i), e_i \rangle.
\]

If the second fundamental form of \( M \) spans the normal space at each point, then \( \gamma \) is parallel in the normal connection. The minimality of \( M \) then implies that its substantial codimension can be reduced, contradiction! This concludes the proof of the theorem.

We state the next result, we recall the classical notion of isoparametric submanifold, see [34]:

**Definition 5.4.** A complete submanifold \( M \) of \( N (N = \mathbb{R}^m, \mathbb{S}^m \text{ or } \mathbb{H}^m) \) is called isoparametric if the normal bundle of \( M \) is flat and the principal curvatures of \( M \) in the directions of any parallel normal vector field are constant.

**Proof of Theorem 1.7.** Choose \( p \in M \) and let \( \{\eta_1 (p), \cdots, \eta_r (p)\} \) be an orthonormal basis of \( T_p M \) of eigenvectors of \( \widetilde{\mathcal{B}} \). Since the normal bundle of \( M \) is globally flat (see [34]), we can extend \( \{\eta_1 (p), \cdots, \eta_r (p)\} \) to an orthonormal basis \( \{\eta_1, \cdots, \eta_r\} \) of the normal bundle of \( M \) such that each vector field of this basis is parallel in the normal bundle. Moreover, the principal curvatures of the shape operators \( S_{\eta_l} \), \( l = 1, \cdots, r \), are constant. Then, each function \( \langle S_{\eta_l}, S_{\eta_k} \rangle \) is constant and hence

\[
\langle \widetilde{\mathcal{B}} (\eta_l), \eta_j \rangle = \langle \widetilde{\mathcal{B}} (\eta_i), \eta_j \rangle_p = \delta_{ij}.
\]

Therefore, \( \eta_1, \cdots, \eta_r \) are eigenvectors of \( \widetilde{\mathcal{B}} \). By Theorem 1.4, each \( \gamma_{\eta_l} \) is a harmonic map of \( M \).

The influence of the image of the Gauss map of a minimal or a CMC hypersurface \( M \) of \( \mathbb{R}^m \) on the geometry and topology of \( M \) is a classical topic of study in differential geometry. We obtain here the following:

**Theorem 5.5.** Let \( M \) be a compact \( n \)-dimensional manifold immersed in \( \mathbb{R}^{n+r} \) with parallel mean curvature vector. Let \( \eta_1, \cdots, \eta_k \) be parallel unit normal vector fields in the normal connection that are linearly independent. If \( \eta_1, \cdots, \eta_k \) are eigenvectors of \( \mathcal{B} \) and the images of the associated Gauss maps \( \gamma_{\eta_1}, \cdots, \gamma_{\eta_k} \) lie each in an open hemisphere of the sphere \( \mathbb{S}^{n+r-1} \), then \( k < r \) and the codimension of \( M \) can be reduced to \( r - k \), that is, \( M \) is contained in a totally geodesic submanifold of dimension \( n + r - k \) of \( \mathbb{R}^{m+r} \).

**Proof.** From Theorem 1.4, it follows that the Gauss maps \( \gamma_{\eta_1}, \cdots, \gamma_{\eta_k} : M \to \mathbb{S}^{n+r-1} \) are harmonic and satisfy

\[
\Delta \gamma_{\eta_l} = -\|S_{\eta_l}\|^2 \gamma_{\eta_l}, \tag{5.5}
\]

\( i = 1, \cdots, n \). Equation (5.5) and the hypothesis on the images of the Gauss maps imply that for some vectors \( v_1, \cdots, v_k \), the functions \( \langle \eta_1, v_1 \rangle, \cdots, \langle \eta_k, v_k \rangle \) are positive and superharmonic. Thus, each \( \langle \eta_l, v_l \rangle \) is constant and nonvanishing. Then, from (5.5), the shape operators \( S_{\eta_l} \) vanishes identically, \( i = 1, \cdots, k \). Consequently, the parallel subbundle \( L = (\text{Span} \{\eta_1, \cdots, \eta_k\})^\perp \) contains, for all \( p \in M \), the first normal space of the immersion \( N_1 (p) := \text{Span} \{B(X, Y) \mid X, Y \in T_p M\} \).

Therefore, from Proposition 2.1 of [13], we conclude that the codimension of \( M \) can be reduced to the rank of the subbundle \( L \). Clearly \( k < r \) otherwise \( M \) should be totally geodesic, which is not possible since \( M \) is compact.
6 | HARMONICITY OF THE GAUSS MAPS OF SUBMANIFOLDS OF HOMOGENEOUS SPACES

Gauss maps of oriented hypersurfaces have been defined and studied in several ambient spaces (e.g., [4–7, 14, 15, 17, 19, 22, 25, 26], and [30]), and different versions of the Ruh–Vilms theorem were obtained.

The main goal of this section is to extend to symmetric spaces the results of the previous section and several results about the harmonicity of the Gauss maps of some of the references cited above. In Sections 6.1 and 6.2, we recall and adapt to our context the constructions of [5, 6], and [30].

6.1 | Homogeneous spaces

Let $N$ be a homogeneous space and let $G$ be the connected component containing the identity of the isometry group of $N$. Then, $N$ is isometric to

$$G/K = \{ xK \mid x \in G \},$$

where $K$ is the isotropy subgroup of $G$ at some point of $N$ and the metric of $G/K$ comes from a metric in $G$ such that the projection $\pi : G \to G/K$ is a pseudo Riemannian submersion. We assume that the metric in $G$ is a pseudo bi-invariant metric.

The set of horizontal vectors of the projection $\pi : G \to G/K$, at each $x \in G$ is the subspace $(T_x(xK))^{\perp}$ of $T_xG$. The linear isometry defined on horizontal vectors at $x$ by

$$d\pi_x \mid_{(T_x(xK))^{\perp}} : (T_x(xK))^{\perp} \to T_{\pi(x)}G/K,$$

we shall denote by $l_x$.

Now consider the map $\Gamma : T(G/K) \to \mathfrak{g}$ between the tangent bundle $T(G/K)$ and the Lie algebra $\mathfrak{g}$ of $G$, defined at each $p \in G/K$ by

$$\Gamma_p : T_pG/K \to \mathfrak{g}$$

$$u \mapsto (dR_{x^{-1}})_x l_x^{-1}(v),$$

where $R$ denotes the right translation on $G$ and $x$ is any point in the fiber $\pi^{-1}(p)$.

**Lemma 6.1.** For each $p \in G/K$, the linear transformation $\Gamma_p : T_p(G/K) \to \mathfrak{g}$ is well defined.

**Proof.** If $x, y \in \pi^{-1}(p)$, then $y = R_k(x)$ for some $k \in K$. Then, for any horizontal vector $v \in T_xG$, we observe that $l_x(v) = (d\pi)_y((dR_k)_x(v)) = l_y((dR_k)_x(v))$, since $R_k$ preserves horizontality. Note that $[(dR_k)_x]^{-1} = (dR_{k^{-1}})_x = (dR_{y^{-1}x})_y$, then,

$$(dR_{x^{-1}})_x l_x^{-1}(v) = (dR_{x^{-1}})_x [(dR_k)_x]^{-1} l_y^{-1}(v)$$

$$= (dR_{x^{-1}})_x (dR_{y^{-1}x})_y l_y^{-1}(v)$$

$$= (dR_y^{-1})_y l_y^{-1}(v).$$

Therefore, $\Gamma_p$ is well defined. $\square$

**Proposition 6.2.** For each $v \in \mathfrak{g}$ is possible to associate a Killing vector field $\xi(v)$ of $N$ such that for all $p \in M$

$$\langle \Gamma_p(u), v \rangle = \langle u, \xi(v)(p) \rangle, \ u \in T_pN.$$

**Proof.** First note that the left multiplication of the Lie Group $G$ on $G/K$, define an isometric action

$$\mu : G \times G/K \to G/K$$
given for each \( g \in G \) by

\[
\mu(g, x\mathbb{K}) = \pi(L_g(x)), \quad x \in G,
\]

where \( L_g \) denotes the left translation by \( g \) on \( G \). Then, for each \( v \in g \) the vector field \( \zeta(v) \) of \( G/\mathbb{K} \) is determined by

\[
\zeta(v)(p) := \left. \frac{d}{dt} \mu(\exp(tv), p) \right|_{t=0}, \quad p \in G/\mathbb{K},
\]

where \( \exp \) is the Lie exponential map of \( G \), is a Killing vector field. Now note that, given \( p \in G/\mathbb{K} \) and \( x \in \pi^{-1}(p) \), we have that \( \zeta(v)(p) = (d\pi)_x(dR_x)_e(v) \) since

\[
\zeta(v)(p) = \left. \frac{d}{dt} \pi(L_{\exp(tv)}(x)) \right|_{t=0} = \left. \frac{d}{dt} (R_x(exp(tv))) \right|_{t=0}.
\]

Thus, for each \( u \in T_pG/\mathbb{K} \), we then have

\[
\langle u, \zeta(v)(p) \rangle = \langle u, (d\pi)_x(dR_x)_e(v) \rangle = \langle \Gamma_p^{-1}(u), (dR_x)_e(v) \rangle = \langle (dR_x^{-1})_x\Gamma_x^{-1}(u), v \rangle = \langle \Gamma_p(u), v \rangle.
\]

This concludes the proof the proposition. \( \Box \)

An important family of examples of homogeneous spaces that can be represented as a quotient as above are the symmetric spaces \([30]\). A simple example is the trivial quotient \( G/\mathbb{K} \), where \( G \) is a lie group with a bi-invariant metric and \( \mathbb{K} = \{0\} \). The map \( \Gamma : TG \to \mathbb{R}^m := T_eG, m = \dim G \), is given by \( \Gamma(p, v) = d(L_p)^{-1}(v) \) where \( L_p \) is the left translation on \( G, L_p(x) = px \).

**Definition 6.3** (Gauss map). Let \( M \) be an \( n \)-dimensional manifold immersed in a homogeneous space \( G/\mathbb{K} \), such that \( G \) has dimension \( m + k \) and \( \mathbb{K} \) has dimension \( k \), and assume that \( \mathcal{N}_1(M) \) is nonempty. For each \( \eta \in \mathcal{N}_1(M) \), we say that the map

\[
\gamma_\eta : M \to S^{m+k-1} \subset \mathfrak{g},
\]

\[
p \mapsto \Gamma_p(\eta(p)).
\]

is the **Gauss map** associated to \( \eta \).

The Gauss map of hypersurfaces of connected spaces of constant curvature and of the spaces \( S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R} \) is computed explicitly in \([30]\) and \([25]\), respectively.

In the following result, we calculate the Laplacian of \( \gamma_\eta \).

**Theorem 6.4.** Let \( M \) be an \( n \)-dimensional manifold immersed in a homogeneous space \( G/\mathbb{K} \). Then, the Gauss map

\[
\gamma_\eta : M \to S^{m+k-1} \subset \mathfrak{g}, \quad \dim \mathbb{K} = k, \quad \dim \mathfrak{g} = m + k,
\]

associated to a parallel unit normal vector \( \eta \) of \( M \) satisfies

\[
-\Delta \gamma_\eta = \Gamma \left( n \text{grad}(\mathcal{H}, \eta) + \mathcal{B}(\eta) + \mathcal{R}_{M} \right) + \sum_{i=1}^{m+k} \left( \mathcal{H} \cdot \nabla \gamma_\eta(v_i) \right) v_i,
\]

where \( \{v_1, \cdots, v_{m+k}\} \) is a fixed orthonormal basis of \( \mathfrak{g} \).
Proof. Let $\eta$ be a unit parallel normal vector field of $M$ and let $v_1, \ldots, v_{m+k}$ be an orthonormal basis of $\mathfrak{g}$. Thus, from Corollary 3.10,

\[-\Delta \gamma_\eta = - \sum_{i=1}^{m+k} \Delta_M \langle \gamma_\eta, v_i \rangle v_i \]

\[= - \sum_{i=1}^{m+k} \Delta_M \langle \eta, \zeta(v_i) \rangle v_i \]

\[= \sum_{i=1}^{m+k} \left( n \langle \text{grad} \langle \overrightarrow{H}, \eta \rangle, \zeta(v_i) \rangle + \langle \text{Ric}_M^\perp(\eta), \zeta(v_i) \rangle + n \langle \overrightarrow{H}, \nabla_\eta \zeta(v_i) \rangle \right) v_i \]

\[= \Gamma \left( n \text{grad} \langle \overrightarrow{H}, \eta \rangle + \text{Ric}_M^\perp(\eta) \right) + n \sum_{i=1}^{m+k} \left( \langle \overrightarrow{H}, \nabla_\eta \zeta(v_i) \rangle \right) v_i. \]

\[\Box\]

Corollary 6.5. Let $M$ be an $n$-dimensional manifold immersed in an $m$-dimensional homogeneous space $G/K$ and let $\eta$ be a parallel unit normal vector of $M$ such that $\overrightarrow{H} = \|\overrightarrow{H}\|\eta$. Then, the Gauss map $\gamma_\eta$ associated to $\eta$ satisfies

\[\Delta \gamma_\eta = -\Gamma \left( n \text{grad} \|\overrightarrow{H}\| + \overrightarrow{B}(\eta) + \text{Ric}_M^\perp(\eta) \right).\]

In particular, if $M$ is an orientable hypersurface of $N$, then

\[-\Delta \gamma_\eta = n\Gamma(\text{grad}(H)) + \left( \|S_\eta\|^2 + \text{Ric}(\eta) \right)\gamma_\eta, \tag{6.1}\]

where Ric is the Ricci curvature of $N$.

Considering the extension $\overrightarrow{B} + \text{Ric}_M^\perp$ of the Simons operator to homogeneous spaces, we obtain as a direct consequence of Corollary 6.5 the following extension, to homogenous spaces, in the minimal case, of Theorem 1.4:

Theorem 6.6. Let $M$ be an $n$-dimensional manifold immersed in an $m$-dimensional homogeneous space $G/K$. Assume that $M$ is minimal and let $\eta$ be a parallel unit normal vector of $M$. Then, The Gauss map $\gamma_\eta$ is harmonic if and only if $\eta$ is an eigenvector of the operator $\overrightarrow{B} + \text{Ric}_M^\perp$. Moreover, if $G/K$ is a simply connected complete space form $Q_n^c$ with constant sectional curvature $c \in \{-1, 0, 1\}$. We have the equivalences:

1. $\gamma_\eta$ is harmonic.
2. $\eta$ is a harmonic section.
3. $\eta$ is an eigenvector of the operator $\overrightarrow{B} + \text{Ric}_M^\perp$.

The round sphere $S^n$ can be represented as a quotient $G/K$ of Lie subgroups $G$ and $K$ of $O(n+1)$ in many ways, depending on the dimension $n$ (e.g., $S^n = O(n+1)/O(n)$ for any $n$, $S^n = U(n+1)/U(n)$ for $n$ odd, etc.), and the metric of $S^n$ is the Riemannian projection of a bi-invariant metric in $G$. In any such representation, we can consider the Gauss map as in Definition 6.3.

In the next result, we prove the existence of harmonic unit normal sections on isoparametric submanifolds of the sphere. These sections determine harmonic Gauss maps, as defined in Section 6.3, for each Lie group quotient representation of $S^n$ as described above.

We prove that if the isoparametric submanifold is also minimal, then the associated Gauss maps are eigenmaps of the Laplacian. This result is an extension of Theorem 1.7 and a spherical version of the main result of [4].

We note that any isoparametric submanifold of $S^m$ is a leaf of a singular foliation of $S^m$ by isoparametric submanifolds and any of these foliations contains a leaf, which is regular and minimal (see [28], pp. 138–139). We have the following:

Theorem 6.7. Let $M$ be an isoparametric submanifold of $S^m$. Then,
(1) there exists an orthonormal basis of parallel unit normal sections \( \eta_1, \ldots, \eta_r \), \( r = m - \dim(M) \), which are eigenvectors of \( B + \text{Ric}_M^\perp \) with constant eigenvalues. Moreover, the sections \( \eta_1, \ldots, \eta_r \), are harmonic unit normal sections of \( M \).

(2) If \( M \) is a minimal isoparametric submanifold then the Gauss maps
\[
\gamma_{\eta_j} : M \to \mathbb{S}^{m+k-1},
\]
\( 1 \leq j \leq r \) associated to any homogeneous representation \( \mathbb{G}/\mathbb{K} \) of \( \mathbb{S}^m \) are eigenmaps, where \( \mathbb{S}^{m+k-1} \) is the unit sphere of the Lie algebra \( \mathfrak{g} \) of \( \mathbb{G} \).

Proof.

(1) The proof is analogous to the one of Theorem 1.7 since the isoparametric submanifolds of \( \mathbb{S}^n \) have globally flat normal bundle.

(2) It is a direct consequence of Theorem 6.7 since from Equation (2.1), we have \( \text{Ric}_M^\perp(\eta) = \dim(M)\eta \).

\[ \square \]

We can use polar actions (see Section 4) and Theorem 4.3 to extend Theorem 6.7 to more general ambient spaces. For example, the left action
\[ \mathbb{K} \times \mathbb{G}/\mathbb{K} \]
of \( \mathbb{K} \) on a symmetric space \( N = \mathbb{G}/\mathbb{K}, \ k(g\mathbb{K}) = (kg)\mathbb{K} \) (see [2]).

In the following theorem, assuming that the Ricci normal operator \( \text{Ric}_M^\perp \) is a multiple of the identity, we obtain harmonic Gauss maps on principal orbits of polar actions. We note that the normal Ricci curvature \( \text{Ric}_M^\perp \) of submanifold of a space form is always a multiple of the identity. If \( M \) is a codimension 2 manifold of an Einstein manifold \( N \), then the Ricci normal operator is given by
\[
\text{Ric}_M^\perp(\eta) = (\text{Ric}(\eta) - K(\eta, \eta^\perp))\eta
\]
and then is also a multiple of the identity, where \( \eta^\perp \) is a normal vector to \( M \) and orthogonal to \( \eta \), \( K(\eta, \eta^\perp) \) the sectional curvature on \( N \) and \( \text{Ric}(\eta) \) is the Ricci curvature of \( N \). We mention [24] where it is classified isometric compact group actions in \( \mathbb{R}^m \) and \( \mathbb{S}^m \) whose principal orbits have codimension 2.

**Theorem 6.8.** If a principal orbit \( M \) of a polar action on a symmetric space \( N \) is minimal and the Ricci normal operator of \( M \) is a multiple of the identity, then the associated Gauss maps \( \gamma_{\eta_1}, \ldots, \gamma_{\eta_k} \) of an eigenbasis \( \eta_1, \ldots, \eta_k \) of \( B \), as in Theorem 4.3, are harmonic maps.

We note that if an orbit of an action of a compact Lie group on a manifold \( N \) is a local maximum for the volume, then this orbit is a critical point of the area functional and hence is minimal (see [24]). Therefore, if \( N \) is compact, there always exists an orbit, which is minimal.

### 6.2 | The sphere \( \mathbb{S}^7 \) with the octonionic multiplication

In this section, we apply the computations of Section 3 to constructions of [5]. It allows us to obtain a version of the Ruh–Vilms Theorem of hypersurfaces of \( \mathbb{S}^k \) with \( 3 \leq k \leq 7 \), where the Gauss map depends of the octonionic structure of \( \mathbb{R}^8 \). The results in this section extend Corollary 1.2 and Theorem 1.4 of [5]. We begin recalling some basic facts about octonionic geometry.

Given \( n \in \{0, 1, 2, \ldots\} \), the Cayley–Dickson algebra \( \mathbb{C}_n \) is a division algebra structure on \( \mathbb{R}^{2^n} \) defined inductively by \( \mathbb{C}_0 = \mathbb{R} \) and by the following formulas: If \( x = (x_1, x_2), \ y = (y_1, y_2) \) are in \( \mathbb{R}^{2^n} = \mathbb{R}^{2^{n-1}} \times \mathbb{R}^{2^{n-1}}, \ n \geq 1 \), then
\[
x \cdot y = (x_1 y_1 - \bar{y}_2 x_2, y_2 x_1 + x_2 \bar{y}_1),
\]
where
\[
\bar{x} = (\bar{x}_1, -x_2),
\]
with \( \bar{x} = x \) if \( x \in \mathbb{R} \) (see [3]).
The Cayley–Dickson algebra $C_3$ of $\mathbb{R}^8$ is called the octonions and denoted by $\mathbb{O}$. We next mention some well-known facts about the octonions, of which proofs can be found in [3]. Let $1$ be the neutral element of $\mathbb{O}$. Besides being a division algebra, $\mathbb{O}$ is normed: $\|x \cdot y\| = \|x\| \|y\|$, for any $x, y \in \mathbb{O}$, where $\|\|$ is the usual norm of $\mathbb{R}^8$, and $\|x\| = \sqrt{x \cdot \overline{x}}$. Setting $\text{Re}(x) = (x + \overline{x})/2$ we have

$$T_1 \mathbb{S}^7 = \{x \in \mathbb{R}^8 \mid \text{Re}(x) = 0\}.$$ 

The right and left translations $R_x, L_x : \mathbb{O} \to \mathbb{O}, R_x(v) = v \cdot x, L_x(v) = x \cdot v, v \in \mathbb{O}$, are orthogonal maps if $\|x\| = 1$ and are skew-symmetric if $\text{Re}(x) = 0$. In particular, the unit sphere $\mathbb{S}^7$ centered at the origin of $\mathbb{R}^8$ is preserved by left and right translation of unit vectors and, moreover, any $v \in T_1 \mathbb{S}^7$ determines a Killing vector field $V$ of $\mathbb{S}^7$ given by the left translation, $V(x) = x \cdot v, x \in \mathbb{S}^7$.

Defining a translation on $\mathbb{S}^7$ by

$$\Gamma : T \mathbb{S}^7 \to T_1 \mathbb{S}^7$$

$$(x, v) \mapsto L_{x^{-1}}(v),$$

it follows immediately from the properties of the octonions described above that $\Gamma$ is a linear transformation and the Killing vector field $V$ of $\mathbb{S}^7$ determined by $v \in T_1 \mathbb{S}^7$ by left multiplication satisfies

$$\langle \Gamma(x, u), v \rangle = \langle u, V(x) \rangle, \quad u \in T_x \mathbb{S}, \quad x \in \mathbb{S}.$$ 

Herein, we identify a $k$-dimensional unit sphere $\mathbb{S}^k, k = 3, 4, 5, 6$, as a totally geodesic sphere of $\mathbb{S}^7$. Then, we may define the octonionic Gauss map of an oriented hypersurface $M$ of $\mathbb{S}^k$, with $3 \leq k \leq 7$, by

$$\gamma_\eta : M \to \mathbb{S}^6 \subset T_1 \mathbb{S}^7$$

$$x \mapsto x^{-1} \cdot \eta,$$ 

where $\eta$ is a unit normal vector of $M$ in $\mathbb{S}^k$.

Regarding $M$ as a submanifold of the sphere $\mathbb{S}^7$ it is easy to see that $\eta$ is a parallel unit normal vector $\eta$ and an eigenvector of $B + \text{Ric}_M^\perp$. Therefore, as a consequence of Corollary 3.10, we have the following version of the Ruh–Vilms Theorem:

**Theorem 6.9.** Let $M$ be an oriented immersed hypersurface of $\mathbb{S}^k, 3 \leq k \leq 7$. Then, the octonionic Gauss map of $M$ in $\mathbb{S}^k$ is harmonic if and only if $M$ has CMC.

**Proof.** Let $M$ be an oriented immersed hypersurface of $\mathbb{S}^k$. If $\eta$ is a unit normal vector of $M$ in $\mathbb{S}^k$, then $\eta$ is a parallel vector in the normal bundle of $M$ in $\mathbb{S}^7$ and

$$(B + \text{Ric}_M^\perp)(\eta) = (\|B\|^2 + (k - 1))\eta,$$ 

where $B$ is the second fundamental form of $M$ in $\mathbb{S}^k$. Then, from Corollary 3.10, we have

$$-\Delta \gamma_\eta = (k - 1)\Gamma(\text{grad}(H)) + (\|B\|^2 + (k - 1))\gamma_\eta,$$ 

(6.3)

where $H$ is the mean curvature of $M$ in $\mathbb{S}^k$. Therefore, $\gamma_\eta$ is a harmonic map if and only if $H$ is constant. □

A Gauss map of an orientable hypersurface $M$ of $\mathbb{S}^n, n \geq 3$, that have been often studied in the literature associates, to each $x \in M$, the vector $\eta(x) \in \mathbb{S}^n$, where $\eta$ is a unit normal vector field along $M$. E. De Giorgi [20] and J. Simons [32] proved that if $M$ is compact, has CMC, and the image of the Gauss map is contained in an open hemisphere of $\mathbb{S}^n$, then $M$ must be a totally geodesic hypersphere of $\mathbb{S}^n$. Using the octonionic Gauss map, we obtain the following:

**Theorem 6.10.** Let $M$ be a compact and oriented hypersurface of $\mathbb{S}^k, 3 \leq k \leq 7$. If $M$ has CMC, then the image of the octonionic Gauss map of $M$ is not contained in an open hemisphere of $\mathbb{S}^6 \subset T_1 \mathbb{S}^7$.
Proof. Assume that the image of \( \gamma := \gamma \eta \) is contained in an open hemisphere of \( \mathbb{S}^6 \) having \( v \in \mathbb{S}^6 \) as pole. This means that \( \langle \gamma(x), v \rangle > 0 \) for all \( x \in M \). We claim that one may find seven linearly independent vectors \( v_1, \ldots, v_7 \in T_x\mathbb{S}^7 \) such that \( \langle \gamma(x), v_i \rangle > 0 \), for all \( x \in M \) and \( 1 \leq i \leq 7 \). Indeed, note that

\[
\langle \gamma(x), v \rangle > 0 \iff d(\gamma(x), v) < \frac{\pi}{2}
\]

for all \( x \in M \), where \( d \) is the distance in \( \mathbb{S}^6 \). By compactness there is \( \varepsilon > 0 \) such that

\[
d(\gamma(x), v) < \varepsilon < \frac{\pi}{2}
\]

for all \( x \in M \). Let \( U \) be a neighborhood of \( v \) in \( \mathbb{S}^6 \) such that

\[
d(v, u) < \frac{\pi}{2} - \varepsilon
\]

for all \( u \in U \). Then,

\[
d(\gamma(x), u) \leq d(\gamma(x), v) + d(v, u) < \frac{\pi}{2},
\]

that is, the inequality

\[
\langle \gamma(x), u \rangle > 0
\]

holds for all \( x \in M \) and all \( u \in U \). Since \( \mathbb{R}^7 \) is the smallest linear subspace of \( \mathbb{R}^7 \) containing \( U \) the claim is proved.

From (6.3), we obtain

\[
\Delta_M \langle \gamma, v_i \rangle = -(\|B\|^2 + (k - 1))\langle \gamma, v_i \rangle < 0. \tag{6.4}
\]

The functions \( \langle \gamma, v_i \rangle \) being superharmonic and not changing sign must be constant. Hence, \( \gamma \) must be a constant map and then \( \Delta \gamma = 0 \). From (6.4), \( \gamma \) must be zero, contradiction! This proves the theorem. \( \square \)

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