Reflection matrices for the
$U_q[sl(m|n)^{(1)}]$ vertex model

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Abstract. We investigate the possible regular solutions of the boundary Yang–Baxter equation for the vertex models associated with the graded version of the $A_{n-1}^{(1)}$ affine Lie algebra, the $U_q[sl(m|n)^{(1)}]$ vertex model, also known as the Perk–Schultz model.

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1. Introduction

The present work is the fourth of the four papers devoted to the classification of the integrable reflection $K$-matrices for the vertex models associated with superalgebras. We already have considered the vertex models associated with the $U_q[sl(r|2m)^{(2)}]$ [1], $U_q[osp(r|2m)^{(1)}]$ [2] and $U_q[spo(2m|2m)]$ [3] superalgebras. In this paper we have presented the general set of regular solutions of the graded reflection equation for the $U_q[sl(m|n)^{(1)}]$ vertex model.

Our findings can be summarized into four types of solutions: diagonal solutions with one free parameter, quasi-diagonal solutions with only two non-diagonal entries and three free parameters, quasi-diagonal solution with $2 + 2\alpha$ non-diagonal entries in the same secondary diagonal and $3 + \alpha$ free parameters and, one special type of quasi-diagonal solutions with $4 + 2\alpha + 2\beta$ non-diagonal entries in two secondary diagonals and with $4 + \alpha + \beta$ free parameters.

This paper is organized as follows. In section 2 we present the $R$-matrix of the $U_q[sl(m|n)^{(1)}]$ vertex model in terms of standard Weyl matrices. In the section 3 we present the solutions of the reflection equations. In that way we hope that they are the most general set of $K$-matrices for the vertex model here considered. Concluding remarks are discussed in the section 4. The models with the first values of $m$ and $n$ have their $K$-matrices written explicitly in the appendix.

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2. The $U_q[sl(m|n)^{(1)}]$ reflection equations

The $R$-matrix associated with the $U_q[sl(m|n)^{(1)}]$ superalgebra [4–6], whose matrix elements are the statistical weights of the Perk–Schultz vertex model [7] has the form

\[
R(x) = \sum_{i=1}^{N} (-1)^{p_i} a_i(x) E_{ii} \otimes E_{ii} + b(x) \sum_{i,j=1}^{N} E_{ii} \otimes E_{jj}
\]

\[
+ c_2(x) \sum_{i<j}^{N} (-1)^{p_i} E_{ji} \otimes E_{ij} + c_1(x) \sum_{i>j}^{N} (-1)^{p_i} E_{ij} \otimes E_{ji}
\]

(2.1)

where $N = n + m$ is the dimension of the graded space with $n$ fermionic and $m$ bosonic degrees of freedom and $E_{ij}$ refers to the $N$ by $N$ Weyl matrix with only one non-null entry with value 1 in row $i$ and column $j$.

In what follows we shall adopt the grading structure

\[
p_i = \begin{cases} 
0, & i = 1, 2, \ldots, m \\
1, & i = m + 1, \ldots, N
\end{cases}
\]

(2.2)

and the corresponding Boltzmann weights with functional dependence on the spectral parameter $u = \ln x$ are given by

\[
a_i(x) = (x^{1-p_i} - q^2 x^{p_i}), \quad b(x) = q(x - 1),
\]

\[
c_1(x) = (1 - q^2), \quad c_2(x) = x(1 - q^2).
\]

(2.3)

Here $q$ denotes an arbitrary parameter.

The $R$-matrix (2.1) satisfies symmetry relations, besides the standard properties of regularity and unitarity, namely:

- PT invariance

\[
P_{12} R_{12}(x) P_{12} \equiv R_{21}(x) = R_{12}(x)^{st_1st_2}.
\]

(2.4)

- Weaker property [8, 9]:

\[
\left\{ \left\{ R_{12}(x)^{st_2} \right\}^{-1} \right\}^{st_2} = \frac{\zeta(x)}{\zeta(x^{-1} \eta^{-1})} M_2 R_{12}(x^{-1} \eta^{-1}) M_2^{-1},
\]

(2.5)

where $\zeta(x) = a_1(x) a_1(x^{-1})$ and $M$ is a symmetry of the $R$-matrix

\[ [R(u), M \otimes M] = 0, \quad M_{ij} = \delta_{ij} (-1)^{p_i} q^{n+m+1-2i}, \quad \eta = q^{n+m}. \]

(2.6)

The matrix $K_-(u)$ satisfies the left boundary Yang–Baxter equation [10], also known as the reflection equation [11],

\[
R_{12}(x/y) K^-_1(x) R_{21}(xy) K^-_2(y) = K^-_2(y) R_{12}(xy) K^-_1(x) R_{21}(x/y),
\]

(2.7)

which governs the integrability at the boundary for a given bulk theory. A similar equation should also hold for the matrix $K_+(u)$ at the opposite boundary. However, one can see from [12] that the corresponding quantity

\[
K^+(x) = K^-(x^{-1} \eta^{-1})^{st} M
\]

(2.8)

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satisfies the right boundary Yang–Baxter equation. Here \( st = st_1st_2 \) and \( st_i \) stands for transposition taken in the \( i \)th superspace.

Therefore, we can start for searching the matrices \( K^- (x) \). In this paper only regular solutions will be considered. Regular solutions mean that the matrix \( K^- (x) \) has the form

\[
K^- (x) = \sum_{i,j=1}^{N} k_{i,j}(x) E_{ij}
\]

(2.9)

and satisfies the condition

\[
k_{i,j}(1) = \delta_{i,j}, \quad i,j = \{1, 2, \ldots, N\}.
\]

(2.10)

Substituting (2.1) and (2.9) into (2.7), we will get \( N^4 \) functional equations for the \( k_{ij} \) matrix elements, many of which are dependent. In order to solve them, we shall proceed in the following way. First we consider the \((i, j)\) component of the matrix equation (2.7). By differentiating it with respect to \( y \) and taking \( y = 1 \), we get algebraic equations involving the single variable \( x \) and \( N^2 \) parameters

\[
\beta_{i,j} = \left. \frac{d k_{i,j}(y)}{dy} \right|_{y=1} \quad i,j = 1,2,\ldots,N.
\]

(2.11)

Second, these algebraic equations are denoted by \( E[i,j] = 0 \) and collected into blocks \( B[i,j] \), \( i = 1, \ldots, M - 1 - i \) and \( j = i, i+1,\ldots, M - 1 - i \), defined by

\[
B[i,j] = \begin{cases} 
E[i,j] = 0, & E[j,i] = 0, \\
E[M-i,M-j] = 0, & E[M-j,M-i] = 0 
\end{cases}
\]

(2.12)

where \( M = N^2 + 1 \).

For a given block \( B[i,j] \), the equation \( E[M-i,M-j] = 0 \) can be obtained from the equation \( E[i,j] = 0 \) by interchanging

\[
k_{i,j} \leftrightarrow k_{N+1-i,N+1-j}, \quad \beta_{i,j} \leftrightarrow \beta_{N+1-i,N+1-j}, \quad c_1(x) \leftrightarrow c_2(x)
\]

(2.13)

and the equation \( E[j,i] = 0 \) is obtained from the equation \( E[i,j] = 0 \) by the interchanging

\[
k_{i,j} \leftrightarrow k_{j,i}, \quad \beta_{i,j} \leftrightarrow \beta_{j,i}.
\]

(2.14)

In this way, we can control all the equations and a particular solution is simultaneously connected with at least four equations.

3. The \( U_q[sl(m|n)^{(1)}]K \)-matrix solutions

Analyzing the \( U_q[sl(m|n)^{(1)}] \) reflection equations one can see that they possess a very special structure. The simplest equations are

\[
b(x) \beta_{i,j} k_{i,j}(x)(a_i(x) - a_j(x)) = 0 \quad (i \neq j).
\]

(3.1)

From (2.3), \( a_i(x) \neq a_j(x) \) when the labels \( i \) and \( j \) are different types of degree of freedom. It means that all \( U_q[sl(m|n)^{(1)}] \) reflection matrices have the following block diagonal
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structure

$$K^-(x) = \begin{pmatrix} K^b & 0_{m \times n} \\ 0_{n \times m} & K^f \end{pmatrix}$$  \hspace{1cm} (3.2)

where $K^b$ is an $m$ by $m$ matrix with entries $k_{i,j}$ for $i,j = \{1, 2, \ldots, m\}$ and $K^f$ is an $n$ by $n$ matrix with entries $k_{r,s}$ for $r,s = \{m+1, m+2, \ldots, N\}$.

Now, by direct inspection of the equations (2.12), one can see that the diagonal equations $B[i, i]$ are uniquely solved by the relations

$$\beta_{i,j} k_{j,i}(x) = \beta_{j,i} k_{i,j}(x), \hspace{1cm} \forall i \neq j.$$  \hspace{1cm} (3.3)

It means that we only need to find the $m(m-1)/2$ and $n(n-1)/2$ elements $k_{i,j}$ with $i < j$. Now we choose a particular $k_{i,j}(i < j)$ to be different from zero, with $\beta_{i,j} \neq 0$, and try to express all the remaining non-diagonal matrix elements in terms of this particular element. We have verified that this is possible provided that

$$k_{r,s}(x) = \begin{cases} x \beta_{r,s} k_{i,j}(x) & \text{if } r > i \text{ and } s > j \\ \beta_{r,s} k_{i,j}(x) & \text{if } r > i \text{ and } s < j \end{cases} \hspace{1cm} (r \neq s).$$  \hspace{1cm} (3.4)

Combining (3.3) with (3.4) we will obtain a very strong entail for the elements out of the diagonal

$$k_{i,j}(x) \neq 0 \Rightarrow \begin{cases} k_{p,q}(x) = 0 & \text{for } p \neq i \\ k_{i,q}(x) = 0 & \text{for } q \neq j. \end{cases}$$  \hspace{1cm} (3.5)

It means that for a given $k_{i,j}(x) \neq 0$, the only elements different from zero in the $i$th-row and in the $j$th-column are $k_{i,i}(x), k_{i,j}(x), k_{j,j}(x)$.

Analyzing more carefully these equations with the conditions (3.3) and (3.5), we have found from the $m(m-1)/2$ elements $k_{i,j}(i < j) \in K^b$ and $n(n-1)/2$ elements $k_{i,j}(i < j) \in K^f$ that there are three possibilities to choose a particular $k_{i,j}(x) \neq 0$:

- Only one non-diagonal element and its symmetric are allowed to be different from zero. Thus, we have $m(m-1)/2$ reflection $K$-matrices with $N+2$ non-zero elements and $n(n-1)/2$ reflection $K$-matrices with $N+2$ non-zero elements. These solutions will be denoted by $K^{(0)}_{[ij]}$ and named Type-I solutions.

- For each $k_{i,j}(x) \neq 0$, additional non-diagonal elements and its asymmetric are allowed to be different from zero provided they satisfy the equations

$$k_{i,j}(x) k_{j,i}(x) = k_{r,s}(x) k_{s,r}(x), \hspace{1cm} i + j = r + s \hspace{1cm} \text{with } \{i,j,r,s\} \in K^b \text{ or } K^f.$$  \hspace{1cm} (3.6)

It means that we will get a $K$-matrix with entries of the principal diagonal and the entries of a secondary diagonal with the element $k_{i,j}(x)$ on the top. These solutions will be denoted by $K^{(a)}_{[ij]}$ and named Type-II solutions.
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- For each $k_{i,j}(x) \neq 0$, additional non-diagonal elements and its asymmetric are allowed to be different from zero provided they satisfy the equations

$$k_{i,j}(x)k_{j,i}(x) = k_{r,s}(x)k_{s,r}(x), \quad i + j = r + s \mod N \quad \text{with} \quad \{i, j\} \in K^b \quad \text{and} \quad \{r, s\} \in K^f. \quad (3.7)$$

It means that we will get a $K$-matrix with the principal diagonal elements and the elements of two secondary diagonals with the top elements $k_{i,j}(x) \in K^b$ and $k_{r,s}(x) \in K^f$. These solutions will be denoted by $K^{(\alpha)(\beta)}$ and named Type-III solutions.

Here the symbols $\alpha$ and $\beta$ mean the number of additional pairs of non-zero entries $(k_{a,b}(x), k_{b,a}(x))$ on the secondary diagonals.

For example, the $U_q[sl(4|2)^{(1)}]$ model has the following $K$-matrix

$$K = \begin{pmatrix}
  k_{1,1} & k_{1,2} & k_{1,3} & k_{1,4} \\
  k_{2,1} & k_{2,2} & k_{2,3} & k_{2,4} \\
  k_{3,1} & k_{3,2} & k_{3,3} & k_{3,4} \\
  k_{4,1} & k_{4,2} & k_{4,3} & k_{4,4} \\
  k_{5,5} & k_{5,6} & \end{pmatrix}$$

where we identify 7 Type-I solutions

$$K_{[12]}^{(0)} = \begin{pmatrix}
  k_{1,1} & k_{1,2} \\
  k_{2,1} & k_{2,2} \\
  k_{3,3} & k_{4,4} \\
  k_{5,5} \\
  k_{6,6} \\
\end{pmatrix}, \ldots,$$

$$K_{[56]}^{(0)} = \begin{pmatrix}
  k_{1,1} \\
  k_{2,2} \\
  k_{3,3} \\
  k_{4,4} \\
  k_{5,5} & k_{5,6} \\
  k_{6,5} & k_{6,6} \\
\end{pmatrix}. $$

In addition to $K_{[14]}^{(0)}$ we have the Type-II solution

$$K_{[14]}^{(1)} = \begin{pmatrix}
  k_{1,1} & k_{1,2} & k_{1,3} & k_{1,4} \\
  k_{2,2} & k_{2,3} & k_{2,4} \\
  k_{3,3} & k_{3,4} \end{pmatrix}$$
with the constraint equation \( k_{1,4}k_{4,1} = k_{2,3}k_{3,2} \), (the pairs of entries of the same secondary diagonal) and the Type-III solution

\[
K^{(1)}_{[14][56]} = \begin{pmatrix}
  k_{1,1} & k_{2,2} & k_{2,3} & k_{1,4} \\
  k_{2,2} & k_{3,3} & k_{4,4} & \ \\
  k_{3,3} & k_{4,4} & k_{5,5} & k_{5,6} \\
  k_{4,1} & k_{5,5} & k_{6,5} & k_{6,6}
\end{pmatrix}
\]

with the constraint equation \( k_{1,4}k_{4,1} = k_{2,3}k_{3,2} = k_{5,6} \) since \((1 + 4) = (5 + 6)\) mod 6, \( k_{1,4} \in K^b \) and \( k_{5,6} \in K^f \).

Although we already know the counting for the \( K \)-matrices, for the \( U_q[sl(m|n)]^{(1)} \) models we still have to identify among them which are similar. Indeed we can see a \( \mathbb{Z}_N \) similarity transformation which maps their matrix elements positions:

\[
K_a = g_a K_0 g_{N-a}, \quad a = 0, 1, 2, \ldots, N - 1
\]  

(3.8)

where \( g_a \) are the \( \mathbb{Z}_N \) matrices

\[
(g_a)_{i,j} = \delta_{i,i+a} \mod N.
\]  

(3.9)

In order to do this we can choose \( K_0 \) as \( K^{(a)}_{[12]} \) and the similarity transformations (3.8) give us the \( K_a \) matrices whose matrix elements are in the same positions of the matrix elements of the \( K^{(a)}_{[12]} \) and \( K^{(a)}_{[m]} \) matrices. However, due to the fact that the relations (3.4) involve the ratio \( c_2(x)/c_1(x) = x \), as well as the additional constraints (3.6), we could not find a similarity transformation among these \( K \)'s matrices, even after a gauge transformation. Even for the Type-I solutions the similarity account is not simple due to the presence of three types of scalar functions and the constraint equations for the parameters \( \beta_{i,j} \). Nevertheless, as we have found a way to write all solutions, we can leave the similarity account to the reader.

Having identified these possibilities we may proceed in order to find the \( N \) diagonal elements \( k_{i,i}(x) \) in terms of the non-diagonal elements \( k_{i,j}(x) \) for each \( K^{(a)}_{i,j} \) matrix. This procedure is now standard [13]. For instance, if we are looking for \( K^{(1)[0]}_{[14][56]} \), the non-diagonal elements \( k_{i,j}(x) \), \((i + j = 5 \text{ mod } 6)\) in terms of \( k_{1,4}(x) \neq 0 \) are given by

\[
k_{2,3}(x) = \frac{\beta_{2,3}}{\beta_{1,4}} k_{1,4}(x), \quad k_{3,2}(x) = \frac{\beta_{3,2}}{\beta_{1,4}} k_{1,4}(x), \quad k_{4,1}(x) = \frac{\beta_{4,1}}{\beta_{1,4}} k_{1,4}(x),
\]

(3.10)

\[
k_{5,6}(x) = \frac{\beta_{5,6}}{\beta_{1,4}} x k_{1,4}(x), \quad k_{6,5}(x) = \frac{\beta_{6,5}}{\beta_{1,4}} x k_{1,4}(x).
\]

Substituting (3.10) into the reflection equations we can now easily find the \( k_{i,i}(x) \) elements up to an arbitrary function, in this example identified as \( k_{1,4}(x) \). Moreover, their consistency relations will yield us some constraint equations for the parameters \( \beta_{i,j} \).

After we have found all diagonal elements in terms of \( k_{i,j}(x) \), we can, without loss of generality, choose the arbitrary functions as

\[
k_{i,j}(x) = \frac{1}{2} \beta_{i,j}(x^2 - 1), \quad i < j.
\]  

(3.11)
This choice allows us to work out the solutions in terms of the functions $f_{i,i}(x)$ and $h_{i,j}(x)$ defined by

$$f_{i,i}(x) = \beta_{i,i}(x-1) + 1 \quad \text{and} \quad h_{i,j}(x) = \frac{1}{2}\beta_{i,j}(x^2 - 1), \quad (3.12)$$

for $i, j = 1, 2, \ldots, N$.

Now, we will simply present the general solutions and write them explicitly for the first values of $N$ in the appendices.

### 3.1. The quasi-diagonal $K$-matrices

For Type-I and Type-II solutions we have the same general $K$-matrix form

$$K_{[i,j]}^{[\alpha]} = \sum_{k=0}^{\alpha} \{ f_{i,i}(x)E_{i+ki+k} + h_{i+ki+k}(x)E_{i+ki+k} + h_{j+ki+k}(x)E_{j+ki+k} \}

+ x^2 f_{i,i}(x^{-1})E_{j-kj-k} + \mathcal{Z}_i(x) \sum_{l=1}^{i-1} E_{ll} + \mathcal{Y}^{(i)}_{i+1+\alpha}(x) \sum_{l=i+1+\alpha}^{j-1-\alpha} E_{ll}

+ (1 - \delta_{i,j}) x^2 \mathcal{Z}_i(x) \sum_{l=j+1}^{N} E_{ll} + \delta_{i,j} \mathcal{X}_{j+1}(x) \sum_{l=j+1}^{N} E_{ll}

1 \leq i < j \leq m \quad \text{or} \quad m + 1 \leq i < j \leq m + n. \quad (3.13)$$

For the Type-III solutions we have matrices with non-diagonal entries into two secondary diagonals with different degree of freedom but related by $Z_N$ symmetry

$$K_{[i,j]}^{[\alpha]} = \sum_{k=0}^{\alpha} \{ f_{i,i}(x)E_{i+ki+k} + h_{i+ki+k}(x)E_{i+ki+k} + h_{j+ki+k}(x)E_{j+ki+k} \}

+ x^2 f_{i,i}(x^{-1})E_{j-kj-k} + \mathcal{Y}^{(i)}_{i+1+\alpha}(x) \sum_{l=i+1+\alpha}^{j-1-\alpha} E_{ll}

+ \sum_{k=0}^{\beta} \{ x^2 f_{i,i}(x^{-1})E_{r+kr+k} + xh_{r+ks-k}(x)E_{r+ks-k} + xh_{s-kr+k}(x)E_{r-ks+k} \}

+ x^2 f_{i,i}(x^{-1})E_{s-ks-k} + \mathcal{X}_{r+1+\alpha}(x) \sum_{l=r+1+\alpha}^{s-1-\alpha} E_{ll}

1 \leq i < j \leq m \quad \text{and} \quad m + 1 \leq r < s \leq m + n

i + j = r + s \mod N. \quad (3.14)$$

Note that for $\alpha, \beta \neq 0$ we can use $\alpha = [(j - i - 1)/2]$ and $\beta = [(r - s - 1)/2]$. Moreover, we have defined three more types of scalar functions

$$\mathcal{X}_{j+1}(x) = f_{11}(x^{-1}) + \frac{1}{2} (\beta_{j+1,j+1} + \beta_{1,1} - 2) x^{-1}(x^2 - 1),$$

$$\mathcal{Y}^{(i)}_{i}(x) = f_{ii}(x) + \frac{1}{2} (\beta_{i,i} - \beta_{i,1}) (x^2 - 1),$$

$$\mathcal{Z}_i(x) = f_{ii}(x^{-1}) + \frac{1}{2} (\beta_{i,i} + \beta_{1,1}) x^{-1}(x^2 - 1). \quad (3.15)$$
The number of free parameters is fixed by the constraint equations which depend on the presence of these scalar functions: when \( Y^{(i)}_t(x) \) is present in the \( K \)-matrix we have constraint equations of the type

\[
\beta_{i,j} \beta_{j,i} = (\beta_{i,i} + \beta_{i,i} - 2)(\beta_{i,i} - \beta_{i,i}),
\]

(3.16)

but when \( Z_i(x) \) is present the corresponding constraints are of the type

\[
\beta_{i,j} \beta_{j,i} = (\beta_{i,1} + \beta_{i,1})(\beta_{i,1} - \beta_{i,1}).
\]

(3.17)

The presence of at least one \( X_{j+1}(x) \) yields a third type of constraint,

\[
\beta_{i,j} \beta_{j,i} = (\beta_{j+1,j+1} + \beta_{j,1} - 2)(\beta_{j+1,j+1} - \beta_{j,1} - 2).
\]

(3.18)

Here we recall again that \( i + j = r + s \mod N \).

From (3.13) and (3.14) we can see that for each solution we have at most two scalar functions in addition to the \( f_i(x) \), and \( \alpha + \beta \) pairs of the \( h(x) \) functions in addition to \( h_{i,j}(x) \) and \( h_{i,s}(x) \) functions. It means that our Type-I matrices are 3-parameter solutions. The Type-II matrices have \( 3 + \alpha \) free parameters and the Type-III matrices have \( 4 + \alpha + \beta \) free parameters.

### 3.2. The diagonal \( K \)-matrices

For diagonal solution we have \( \beta_{i,j} = 0 \). It means that all scalar functions \( h_{i,j}(x) \) are equal to zero and we have to solve the constraint equations (3.16)–(3.18). Now, we can recall (3.13) and (3.14) and replace the scalar function \( X_{j+1}(x) \) by \( x^2 f_{i1}(x^{-1}) \) or by \( x^2 f_{i1}(x) \), the scalar function \( Y^{(i)}_t(x) \) by \( f_{i1}(x) \) or by \( x^2 f_{i1}(x^{-1}) \) and the scalar function \( Z_i(x) \) by \( f_{i1}(x^{-1}) \) or by \( f_{i1}(x) \) in order to get the diagonal solutions. It follows due to the substitution of the solutions of (3.16)–(3.18) into (3.15)

\[
\lim_{\beta_{i,j} \to \pm \beta_{i,i} + 2} X_j(x) = x^2 f_{i1}(x^{+1})
\]

\[
\lim_{\beta_{i,i} \to \beta_{i,i}} Y^{(i)}_t(x) = f_{i1}(x)
\]

and

\[
\lim_{\beta_{i,i} \to \pm \beta_{i,i}} Z_i(x) = f_{i1}(x^{\pm 1}).
\]

(3.19)

This reduction procedure gives us the diagonal solutions:

\[
\mathbb{D}_{ij} = Z_i(x) \sum_{l=1}^{i-1} E_{il} + f_{i1}(x) E_{ii} + Y^{(i)}_{i+1}(x) \sum_{l=i+1}^{j-1} E_{il} + x^2 f_{i1}(x^{-1}) E_{jj}
\]

\[
+ (1 - \delta_{i,i}) x^2 Z_i(x) \sum_{l=j+1}^{N} E_{il} + \delta_{i,i} X_{j+1}(x) \sum_{l=j+1}^{N} E_{il}
\]

\[1 \leq i < j \leq m \quad \text{or} \quad m + 1 \leq i < j \leq m + n
\]

(3.20)
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and

$$D_{ij} [rs] = f_{ii}(x)E_{ii} + x^2f_{ii}(x^{-1})E_{jj} + Y_{i+1}^{(i)}(x)\sum_{l=i+1}^{j-1}E_{ll}$$

$$+ x^2f_{ii}(x^{-1})E_{rr} + x^2f_{ii}(x)E_{ss} + X_{r+1}(x)\sum_{l=r+1}^{s-1}E_{ll}$$

$$1 \leq i < j \leq m \quad \text{and} \quad m+1 \leq r < s \leq n+m$$

(3.21)

where

$$X_{j+1}(x) = \{x^2f_{11}(x^{-1}), x^2f_{11}(x)\},$$

$$Y_{i+1}^{(i)}(x) = \{x^2f_{ii}(x^{-1}), f_{ii}(x)\}, \quad Z_i(x) = \{f_{ii}(x^{-1}), f_{ii}(x)\}.$$  

(3.22)

From these results we can see that the $U_q[sl(m|n)^{(1)}]$ model has many diagonal solutions. In particular, the substitution $Z_i(x) = f_{ii}(x)$, $Y_{i+1}^{(i)}(x) = f_{ii}(x)$ and $X_{j+1} = x^2f_{11}(x^{-1})$ into (3.20) yields the diagonal solutions already derived in [9] and used in the study of the nested Bethe ansatz for the Perk–Schultz model with open boundary condition [14]. Moreover, these diagonal solutions have been used recently in [15] for the study of the nested Bethe ansatz for ‘all’ open chain with diagonal boundary conditions.

4. Conclusion

After a systematic study of the functional equations we find that there are three types of solutions for the $U_q[sl(m|n)^{(1)}]$ model. We call as Type-I the $K$-matrices with three free parameters and $n+m+2$ non-zero matrix elements. These solutions were denoted by $R_{(0)}^{(ij)}$ to emphasize the non-zero element out of the diagonal and its symmetric, which results in $n(n-1)/2$ and $m(m-1)/2$ reflection $K$-matrices.

The Type-II and Type-III solutions are more interesting because they have many free parameters. We also have used a reduction procedure to obtain the diagonal solutions. However, we could not derive a similar procedure in order to obtain the Type-I solutions from the Type-II solutions or the Type-II solutions from the Type-III solutions. Thus, we believe that they are independent.

The corresponding $K^+(x)$ are obtained from the isomorphism (2.8). Out of this classification we have the trivial solution ($K^- = 1, K^+ = M$) for these models.

Before the end of our discussion on the $U_q[sl(m|n)^{(1)}]$ reflection matrices, we will make (by a referee suggestion), the comparison with the $sl(m+n)$ reflection matrices [13]. The diagonal solutions and the Type-I solutions are the same for the both models. The Type-III solutions of the $U_q[sl(m|n)^{(1)}]$ model are identified with the Type-II of the $sl(m+n)$ model. However, the Type-II solutions of the $U_q[sl(m|n)^{(1)}]$ model are different because in the graded case, the $Z_N$ symmetry (3.7) is lost when the labels have the same degree of freedom (3.6).

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Appendix. Some examples

In this appendix some $K$-matrices are written explicitly only for the cases with $m \geq n$. The cases $m < n$ are easily deduced from the $U_q[sl(m|n)^{(1)}]$ solutions with $m > n$ using (3.8).

For the $U_q[sl(1|1)^{(1)}]$ model there is only one diagonal $K$-matrix

$$D_{[12]} = \begin{pmatrix} f_{11}(x) & 0 \\ 0 & x^2 f_{11}(x^{-1}) \end{pmatrix}.$$

(A.1)

It follows from (3.13) that we have only one Type-I solution $K_{[12]}^{(0)}$ for the $U_q[sl(2|1)^{(1)}]$ model:

$$D_{[12]} = f_{11}(x)E_{11} + h_{12}(x)E_{12} + h_{21}(x)E_{21} + x^2 f_{11}(x^{-1})E_{22} + \mathcal{X}_3(x)E_{33}$$

$$= \begin{pmatrix} f_{11}(x) & h_{12}(x) & 0 \\ h_{21}(x) & x^2 f_{11}(x^{-1}) & 0 \\ 0 & 0 & \mathcal{X}_3(x) \end{pmatrix},$$

(A.2)

where four parameters $\beta_{11}$, $\beta_{12}$, $\beta_{21}$ and $\beta_{33}$ satisfy the constraint equation

$$\beta_{12}\beta_{21} = (\beta_{33} - \beta_{11} - 2)(\beta_{33} + \beta_{11} - 2).$$

(A.3)

Two diagonal solutions are derived from (A.2) due to the constraint equation (A.3)

$$\lim_{\beta_{33} \to \pm \beta_{11} + 2} \mathcal{X}_3(x) = x^2 f_{11}(x^{+1})$$

$$\Rightarrow D_{[12]} = \text{diag}(f_{11}(x), x^2 f_{11}(x^{-1}), x^2 f_{11}(x^{-1}))$$

$$\Rightarrow D_{[12]} = \text{diag}(f_{11}(x), x^2 f_{11}(x^{-1}), x^2 f_{11}(x)).$$

(A.4)

The solution $D_{[12]}$ is the diagonal solution derived by the first time in [9].

For the $U_q[sl(3|1)^{(1)}]$ model we have three Type-I matrices:

$$D_{[12]}^{(0)} = \begin{pmatrix} f_{11}(x) & h_{12}(x) & 0 \\ 0 & x^2 f_{11}(x^{-1}) & 0 \\ 0 & 0 & \mathcal{X}_3(x) \end{pmatrix}$$

$$= \begin{pmatrix} f_{11}(x) & h_{12}(x) & 0 \\ 0 & x^2 f_{11}(x^{-1}) & 0 \\ 0 & 0 & \mathcal{X}_3(x) \end{pmatrix},$$

(A.5)

with two diagonals

$$D_{[12]}^{(1)} = \text{diag}(f(x), x^2 f(x^{-1}), x^2 f(x^{-1}), x^2 f(x^{-1})), \quad D_{[12]}^{(2)} = \text{diag}(f(x), x^2 f(x^{-1}), x^2 f(x), x^2 f(x)).$$

(A.6)

where $f(x) = \beta(x - 1) + 1$ and $\beta$ is a free parameter,

$$K_{13}^{(0)} = \begin{pmatrix} f_{11}(x) & 0 & h_{13}(x) & 0 \\ 0 & \mathcal{Y}_2^{(1)}(x) & 0 & 0 \\ h_{31}(x) & 0 & x^2 f_{11}(x^{-1}) & 0 \\ 0 & 0 & 0 & \mathcal{X}_3(x) \end{pmatrix}$$

$$= \begin{pmatrix} f_{11}(x) & 0 & h_{13}(x) & 0 \\ 0 & \mathcal{Y}_2^{(1)}(x) & 0 & 0 \\ h_{31}(x) & 0 & x^2 f_{11}(x^{-1}) & 0 \\ 0 & 0 & 0 & \mathcal{X}_3(x) \end{pmatrix},$$

(A.7)

$$\beta_{13}\beta_{31} = (\beta_{44} + \beta_{11} - 2)(\beta_{44} - \beta_{11} - 2) = (\beta_{22} + \beta_{11} - 2)(\beta_{22} - \beta_{11}),$$

(A.8)

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with four diagonals
\[
\mathbb{D}_{[13],a} = \mathbb{D}_{[12],a},
\]
\[
\mathbb{D}_{[13],b} = \text{diag}(f(x), f(x), x^2 f(x^{-1}), x^2 f(x^{-1})),
\]
\[
\mathbb{D}_{[13],c} = \text{diag}(f(x), x^2 f(x^{-1}), x^2 f(x^{-1}), x^2 f(x)),
\]
\[
\mathbb{D}_{[13],d} = \text{diag}(f(x), f(x), x^2 f(x^{-1}), x^2 f(x)).
\]

and
\[
\mathbb{K}^{(0)}_{[23]} = \begin{pmatrix}
Z_2(x) & 0 & 0 & 0 \\
0 & f_{22}(x) & h_{23}(x) & 0 \\
0 & h_{32}(x) & x^2 f_{22}(x^{-1}) & 0 \\
0 & 0 & 0 & x^2 Z_2(x)
\end{pmatrix}
\]
(A.9)

\[
\beta_{23}\beta_{32} = (\beta_{11} + \beta_{22})(\beta_{11} - \beta_{22}),
\]

with two diagonals
\[
\mathbb{D}_{[23],a} = \mathbb{D}_{[13],d},
\]
\[
\mathbb{D}_{[23],b} = \text{diag}(f(x^{-1}), f(x), x^2 f(x^{-1}), x^2 f(x^{-1})).
\]
(A.11)

For the \( U_q[sl(2|2)^{(1)}] \) model we have the same Type-I \( \mathbb{K}^{(0)}_{[34]} \) \( \mathbb{K}^{(0)}_{[12]} \) matrices written above and one Type-III matrix with four free parameters \( \beta_{1,2}, \beta_{2,1}, \beta_{3,4} \) and \( \beta_{1,1} \):
\[
\mathbb{K}^{(0)}_{[12][34]} = \begin{pmatrix}
f_{11}(x) & h_{12}(x) & 0 & 0 \\
h_{21}(x) & x^2 f_{11}(x^{-1}) & 0 & 0 \\
0 & 0 & x^2 f_{11}(x^{-1}) & x h_{34}(x) \\
0 & 0 & x h_{43}(x) & x^2 f_{11}(x)
\end{pmatrix}
\]
(A.12)

\[
\beta_{12}\beta_{21} = \beta_{34}\beta_{43}
\]

with one diagonal equal to \( \mathbb{D}_{[13],c} \). The diagonals with only entries of the types \( f(x) \) and \( x^2 f(x^{-1}) \) are the solutions obtained in [9].

References

[1] Lima-Santos A and Gallens W, Reflection matrices for the \( U_q[sl(r|2m)^{(2)}] \) vertex model, 2008 arXiv:0806.3659 [nlin.SI]
[2] Lima-Santos A, Reflection matrices for the \( U_q[osp(r|2m)^{(1)}] \) vertex model, 2008 arXiv:0809.0421 [nlin.SI]
[3] Lima-Santos A, 2009 J. Stat. Mech. P04005 [arXiv:0810.1766] [nlin.SI]
[4] Kac V G, 1977 Adv. Math. 26 8
[5] Frappat L, Sciarrino A and Sorba P, 1989 Commun. Math. Phys. 121 457
[6] Chaichian M and Kulish P P, 1990 Phys. Lett. B 234 72
[7] Perk J H H and Schultz C L, 1981 Phys. Lett. A 84 407
[8] Reshetikhin N Yu and Semenov-Tian-Shansky M, 1990 Lett. Math. Phys. 19 133
[9] Yue R H and Liang H, 1996 High Energy Phys. Nucl. Phys. 20 514
[10] Cherednik I V, 1984 Theor. Math. Phys. 61 977
[11] Sklyanin E K, 1988 J. Phys. A: Math. Gen. 21 2375
[12] Mezincescu L and Nepomechie R I, 1992 Int. J. Mod. Phys. A 7 5657
[13] Lima-Santos A, 2002 Nucl. Phys. B 644 [FS] 568
[14] Li G-L, Yue R-H and Hou B-Y, 2000 Nucl. Phys. B 586 [FS] 711
[15] Belliard S and Ragoucy E, 2009 J. Phys. A: Math. Theor. 42 205203