Model-Assisted Estimators under Nonresponse in Sample Surveys

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Abstract

In the presence of auxiliary information, model-assisted estimators use a working model that links the variable of interest and the auxiliary variables in order to improve the Horvitz-Thompson estimator. The resulting estimators are asymptotically design unbiased and asymptotically more efficient than the Horvitz-Thompson estimator under some regularity conditions and for a wide range of working models. In this work, we adapt model-assisted total estimators to missing at random data building on the idea of nonresponse weighting adjustment. We consider nonresponse as a second phase of the survey and reweight the units in model-assisted estimators using the inverse of estimated response probabilities in order to compensate for the nonrespondents. We develop the asymptotic properties and discuss calibration of the weights of our proposed estimators. We provide formulae for asymptotic variance and variance estimators. We conduct a simulation study that describes the behavior of the proposed estimators.

Keywords: Auxiliary Information, Horvitz-Thompson Estimator, Missing Data, Response Probabilities, Superpopulation Model, Weighting Adjustment.
1 Introduction

In surveys with complete response, the Horvitz-Thompson (HT) estimator is a design-unbiased estimator of population totals (Horvitz and Thompson, 1952). In the presence of auxiliary information, its efficiency can be improved by incorporating into the estimator a working model that links the variable of interest and the auxiliary variables. The resulting estimators are called model-assisted estimators. Such estimators are asymptotically unbiased and asymptotically more efficient than the HT estimator regardless of whether the working model is correctly specified.

Särndal (1980), Robinson and Särndal (1983), and Särndal and Wright (1984) are, to the best of our knowledge, some of the first papers that study model-assisted estimators. They are based on generalized linear regression as working model. Särndal et al. (1992) extend traditional sampling theory to the model-assisted approach. More recent works study model-assisted survey estimation with modern and flexible prediction techniques such as Breidt and Opsomer (2000), Breidt et al. (2005), Breidt et al. (2007), McConville and Breidt (2013), Breidt and Opsomer (2017), and Dagdoug et al. (2021).

Survey data generally suffer from nonresponse which can be seen as a second phase of the survey. In this second phase, a sample of survey respondents is selected from the sampled units. This results in a partition of the sample into two subsamples: the respondents for which the value of the variable of interest is observed, and the nonrespondents for which this value is missing. The probability that a sampled unit is a survey respondent is called response probability. It represents the inclusion probability of the second phase and is unknown (Särndal and Swensson, 1987).

One approach to handle nonresponse consists of reweighting survey respondents to compensate for nonrespondents. By considering nonresponse as a second phase of the survey, the HT estimator can be adapted to two-phase sampling by increasing the weight of respondents using the inverse of the response probabilities. However, the response probabilities
are unknown in practice. One solution is to estimate the response probabilities and use the estimated probabilities instead of the true response probabilities in the two-phase estimator. The resulting estimator is called nonresponse weighting adjusted (NWA) estimator, or empirical double expansion estimator. An overview of NWA methods is available in Lundström and Särndal (1999) and Lee et al. (2002).

In presence of nonresponse, the aforementioned model-assisted estimators are unavailable. In this article, we propose a model-assisted estimator adapted for nonresponse. This estimator is a blend between a model-assisted estimator and a NWA estimator. We reweight the respondents in a model-assisted estimator using the inverse of the estimated response probabilities to compensate for the nonrespondents. To the best of our knowledge, Kim and Haziza (2014) are the only other authors who provide a model-assisted estimator that handles nonresponse. In their approach, both the working model and the nonresponse model are parametric. The models are estimated simultaneously using maximum likelihood. They show that the resulting estimator is doubly-robust. Our proposed approach is more flexible: it allows for these models to be parametric and non-parametric and to be estimated separately.

Different working models are studied. Asymptotic design unbiasedness and efficiency of the proposed estimator are studied and proven for some working and nonresponse models. We show that the proposed estimator can be viewed as a reweighted estimator and that the resulting weights are calibrated to the totals of the auxiliary variables for some working models. We provide a formula for the asymptotic variance and a variance estimator of the proposed total estimator. We also conduct a simulation study that shows that our proposed estimator performs well in terms of bias and variance, even when one of the two models, nonresponse model or working model, is misspecified.

The article is organized as follows. In Section 2, we provide an introduction to the context and some notations. Our proposed estimator is introduced in Section 3. We study different statistical learning techniques as working models in Section 4. We develop the asymptotic properties of our proposed estimator in Section 5. In section 6, we discuss the variance and
its estimator. A simulation study is presented in Section 7. It confirms the performance of our estimator. The main part of this article closes with a short discussion in Section 8. Supplementary material is presented in the Appendices.

2 Context

We consider a finite population $U = \{1, 2, ..., N\}$. Let $s \subset U$ be a sample of size $n$ selected in $U$ according to a sampling design $p(.)$. The first and second order inclusion probabilities are denoted by $\pi_k$ and $\pi_{k\ell} = \text{pr}(k, \ell \in s)$ for generic units $k$ and $\ell$. Consider the sample membership indicator $a_k$ of a unit $k$. We have $\text{pr}(a_k = 1) = \pi_k$, $\text{pr}(a_k = 0) = 1 - \pi_k$, and $\mathbb{E}_p(a_k) = \pi_k$, where subscript $p$ means that the expectation is computed with respect to the sampling design $p(.)$. The covariance between the sample membership indicators is $\Delta_{k\ell} = \text{cov}_p(a_k, a_\ell) = \pi_{k\ell} - \pi_k \pi_\ell$.

The goal is to estimate the population total $t$ of some variable of interest $y$ with values $\{y_k\}$ known only for those units in the sample. With no additional information, the total $t$ can be estimated by the expansion estimator or Horvitz-Thompson (HT) estimator (Horvitz and Thompson, 1952)

$$\hat{t}_{HT} = \sum_{k \in s} \frac{y_k}{\pi_k}.$$  

The HT estimator is design-unbiased, that is $\mathbb{E}_p(\hat{t}_{HT}) = t$, provided that the inclusion probabilities are all strictly larger than 0. Under additional assumptions detailed in Section 5, the estimator

$$\text{var}(\hat{t}_{HT}) = \sum_{k,\ell \in s} \frac{y_k y_\ell}{\pi_k \pi_\ell} \frac{\Delta_{k\ell}}{\pi_k \pi_\ell}$$  

is design-unbiased and consistent for the variance of the HT estimator.

Suppose that a vector of auxiliary variables $x_k = (x_{k1}, \ldots, x_{kp})^T$ is known for each population unit $k \in U$ or at least each sampled unit $k \in s$. We consider a working model $\xi$
that links the $x_k$’s and $y_k$’s as follows

$$\xi : y_k = m(x_k) + \varepsilon_k,$$  \hspace{1cm} (2)

where $m(.)$ is an unknown function, $E_{\xi}(\varepsilon_k) = 0$, $\text{var}_{\xi}(\varepsilon_k) = \sigma_k^2$, and subscript $\xi$ indicates that the expectation and variance are computed under model $\xi$. This working model may be used to improve the efficiency of the HT estimator while maintaining, or almost, its design unbiasedness. Such methods are called *model-assisted*. The *difference estimator*

$$\hat{t}_m = \sum_{k \in U} m(x_k) + \sum_{k \in s} \frac{y_k - m(x_k)}{\pi_k}$$

is an estimator of total $t$. It is design-unbiased and has less variability that the HT estimator provided that the “residuals” $\{y_k - m(x_k)\}$ have less variability than the “raw values” $\{y_k\}$. This holds regardless of the quality of model $\xi$ (Breidt and Opsomer, 2017). Since function $m(.)$ is unknown, we may estimate it from values $\{(x_k, y_k)\}, k \in U$ based on a standard estimation method. The estimate is written $m_U(.)$. Substituting $m_U(.)$ for $m(.)$ yields the *pseudo-generalized difference estimator*

$$\hat{t}_{mU} = \sum_{k \in U} m_U(x_k) + \sum_{k \in s} \frac{y_k - m_U(x_k)}{\pi_k}. \hspace{1cm} (3)$$

It is a typical model-assisted estimator. Breidt and Opsomer (2017) show that it is 1) design-unbiased and 2) more efficient than the HT estimator provided that the “residuals” $\{y_k - m_U(x_k)\}$ have less variability than the “raw values” $\{y_k\}$. This holds regardless of the quality of working model $\xi$.

The population estimator $m_U(.)$ is unavailable and can be estimated by $m_s(.)$ based on the known sample values $\{(x_k, y_k)\}, k \in s$. Substituting in (3) yields the *model-assisted estimator*

$$\hat{t}_{m_s} = \sum_{k \in U} m_s(x_k) + \sum_{k \in s} \frac{y_k - m_s(x_k)}{\pi_k}.$$ 

Breidt and Opsomer (2017) show that, under some regularity conditions and for some specific working models including heteroscedastic multiple regression, linear mixed models, and
some statistical learning techniques, the model-assisted estimator $\hat{t}_{m_s}$ based on $m_s(\cdot)$ is 1) asymptotically design unbiased and 2) asymptotically more efficient than the HT estimator provided that the “residuals” $\{y_k - m_s(x_k)\}$ have less variability than the “raw values” $\{y_k\}$. This holds regardless of the quality of working model $\xi$.

In practice, some values $\{y_k\}$ may be missing because they are collected incorrectly or some units refrain from responding. In this case, the HT estimator and all aforementioned estimators are unavailable. Let $p_k$ and $r_k$ denote, respectively, the response probability and response indicator to variable $y$ of a unit $k \in U$. These are related via $\text{pr}(r_k = 1) = p_k$ and $\text{pr}(r_k = 0) = 1 - p_k$. Consider $s_r = \{k \in U \mid a_k = 1, r_k = 1\}$, the set of $n_r$ units in $s$ for which variable $y$ is known. The units in $s_r$ are called respondents. The process that generates the respondents is called the nonresponse mechanism. In the main part of this paper, we suppose that the nonresponse mechanism satisfies the following conditions:

(NR1): The data is missing at random (see Rubin, 1976, for a detailed definition).

(NR2): The response indicators are independent of one another and of the selected sample $s$.

This means that the values $\{r_k\}$ are obtained from a Poisson sampling design, i.e. the $\{r_k\}$ are generated from independent Bernoulli random variables with $\text{E}_q(r_k \mid s) = \text{E}_q(r_k) = p_k$, where $\text{E}_q(\cdot)$ is the expectation under the nonresponse mechanism.

(NR3): The response probabilities are bounded below, i.e. there exists a constant $c > 0$ so that $p_k > c$ for all $k \in s$.

(NR4): The response probabilities are

$$p_k = \frac{1}{F(x_k^\top \lambda_0)} = \frac{\exp(x_k^\top \lambda_0)}{1 + \exp(x_k^\top \lambda_0)} = \frac{1}{1 + \exp(-x_k^\top \lambda_0)},$$

for some true unknown parameter vector $\lambda_0$.

Assumption (NR4) is relaxed in Appendix C, where a general nonresponse function is assumed.
Nonresponse can be seen as a second phase of the survey, where the nonresponse mechanism is unknown (Särndal and Swensson, 1987). In the first phase, a sample $s$ is selected from population $U$ according to a sampling design $p(\cdot)$. In the second phase, a sample $s_r$ is selected from $s$ according to a Poisson sampling design with unknown inclusion probabilities $\{p_k\}$. Under nonresponse, all aforementioned estimators are unavailable. An approach to control nonresponse bias consists of increasing the weights of the respondents in order to compensate for the nonrespondents. If nonresponse is seen as a second phase of the survey, the design weights are multiplied by the inverse of the response probabilities. This yields the two-phase estimator or double expansion estimator

$$\hat{t}_{2HT} = \sum_{k \in s_r} \frac{y_k}{\pi_k p_k}.$$ 

Since the response probabilities $\{p_k\}$ are unknown in practice, they must be estimated. The estimated response probabilities are denoted by $\hat{p}_k$. Using the estimated response probabilities in the two-phase estimator yields the Nonresponse Weighting Adjusted (NWA) estimator or empirical double expansion estimator

$$\hat{t}_{NWA} = \sum_{k \in s_r} \frac{y_k}{\pi_k \hat{p}_k}.$$ 

3 NWA model-assisted estimator

In this paper, we introduce a model-assisted estimator adapted to nonresponse. It is a blend between a model-assisted estimator and a NWA estimator. It is constructed as follows. We replace the estimated function $m_s(\cdot)$, unavailable with nonresponse, by an estimator $m_r(\cdot)$ constructed from the respondents in the model-assisted estimator and we see nonresponse as a second phase of the survey. This yields

$$\hat{t}_{m_r,p} = \sum_{k \in U} m_r(x_k) + \sum_{k \in s_r} \frac{y_k - m_r(x_k)}{\pi_k \hat{p}_k}.$$
We call this estimator the \textit{two-phase model-assisted estimator}. It is unknown in practice since it contains the unknown response probabilities \( \{p_k\} \). We borrow the idea of the NWA estimation and obtain

\[
\hat{t}_{m_r,\hat{p}} = \sum_{k \in U} m_r(x_k) + \sum_{k \in S_r} \frac{y_k - m_r(x_k)}{\pi_k \hat{p}_k}.
\] (4)

We call this estimator the \textit{NWA model-assisted estimator} as it corresponds to a model-assisted estimator where the weights are adjusted for nonresponse. This estimator covers a wide range of estimators depending on the chosen working model \( \xi \) and the chosen nonresponse model. The first term of this estimator is the population total of the predicted values \( \{m_r(x_k)\} \). For most working models, this requires the values \( \{x_k\} \) to be known for all population units. If this population total is unavailable, we may use a HT-type estimator of this sum, see Appendix A.

The NWA model-assisted estimator in (4) contains two estimators: the response probabilities \( \{\hat{p}_k\} \) and the function \( m_r(\cdot) \). Depending on both these choices, we obtain a different estimator. The response probabilities are \( p_k = 1/F(x_k^T\lambda_0) \) for some unknown parameter vector \( \lambda_0 \). The estimated response probabilities are \( \hat{p}_k = 1/F(x_k^T\hat{\lambda}) \) for some estimator \( \hat{\lambda} \) of \( \lambda_0 \). Unless otherwise specified, we estimate the response probabilities via calibration. The estimator \( \hat{\lambda} \) is the solution to the estimating equation

\[
Q(\lambda) = \sum_{k \in U} x_k - \sum_{k \in S_r} \frac{x_k}{\pi_k} F(x_k^T\lambda) = 0.
\]

We present NWA model-assisted estimators with response probabilities estimated via two alternate techniques, generalized calibration and maximum likelihood, in Appendices B and C, respectively.
4 Statistical Learning Techniques

4.1 Generalized Regression

Consider the working model

\[ \xi : y_k = x_k^\top \beta + \varepsilon_k, \]

where the \( \varepsilon_k \) are uncorrelated with mean \( E_\xi(\varepsilon_k) = 0 \) and variance \( \text{var}_\xi(\varepsilon_k) = \sigma_k^2 \). The finite population regression coefficient is

\[ B_U = \left( \sum_{k \in U} x_k x_k^\top \right)^{-1} \sum_{k \in U} x_k y_k. \]

If parameter \( \beta \) is estimated based on \( s_r \) we use

\[ B_r = \left( \sum_{k \in s_r} x_k x_k^\top \right)^{-1} \sum_{k \in s_r} x_k y_k, \]

where \( m_r(x_k) = x_k^\top B_r \) and \( c_k \) is any of \( 1, \sigma_k^2, \pi_k \hat{p}_k, \pi_k \hat{p}_k \sigma_k^2 \).

The NWA model-assisted estimator can be written in weighted form

\[ \hat{\tau}_{m_r, \hat{p}} = \sum_{k \in U} x_k^\top B_r + \sum_{k \in s_r} \frac{y_k - x_k^\top B_r}{\pi_k \hat{p}_k} \]

\[ = \sum_{k \in s_r} \frac{y_k}{\pi_k \hat{p}_k} + \left( \sum_{k \in U} x_k - \sum_{k \in s_r} \frac{x_k}{\pi_k \hat{p}_k} \right)^\top \left( \sum_{k \in s_r} \frac{x_k x_k^\top}{c_k} \right)^{-1} \sum_{k \in s_r} \frac{x_k y_k}{c_k} \]

\[ = \sum_{k \in s_r} \left\{ \frac{1}{\pi_k \hat{p}_k} + \left( t^x - \hat{\tau}^x_{\text{NWA}} \right)^\top \left( \sum_{k \in s_r} \frac{x_k x_k^\top}{c_k} \right)^{-1} \frac{x_k}{c_k} \right\} y_k \]

\[ = \sum_{k \in s_r} w_{k, s_r} y_k, \]

where \( t^x \) is the vector of population total of the auxiliary variables and \( \hat{\tau}^x_{\text{NWA}} \) its NWA estimator. The weights \( w_{k, s_r} \) are those of the NWA estimator \( 1/(\pi_k \hat{p}_k) \) plus a corrective term induced by the working model. The second term cancels when calibration is applied.
to estimate the response probabilities. The NWA model-assisted estimator is the NWA estimator in this case. The weights are free from values \( \{x_k\} \) in \( U \setminus s_r \) except through the population totals \( t^X \). Only the values \( \{x_k\} \) on \( s_r \) and the population totals \( t^X \) are needed to compute the NWA model-assisted estimator, unless some other values are needed to estimate the response probabilities. The weights are free from \( \{y_k\} \). They can therefore be used for several variables of interest provided that they have observed values on \( s_r \). In particular, the weights can be applied to the auxiliary variables. It comes

\[
\hat{t}_{m_r, \hat{p}}^X = \sum_{k \in s_r} \left\{ \frac{1}{\pi_k \hat{p}_k} + \left( t^X - \hat{t}_{\text{NWA}}^X \right) \left( \sum_{k \in s_r} \frac{x_k x_k^\top c_k}{c_k} \right)^{-1} \frac{x_k}{c_k} \right\} x_k^\top = t^X.
\]

This means that the weights of the NWA model-assisted estimator are calibrated to the totals of the auxiliary variables when calibration is applied to estimate the response probabilities.

### 4.2 \( K \)-Nearest Neighbor

Consider the working model where the prediction for a nonrespondent is obtained by averaging the \( y \)-values of the closest respondents. A predicted value \( m_r(x_k) \) is obtained by

\[
m_r(x_k) = \frac{1}{K} \sum_{\ell \in L_k} y_{\ell},
\]

where \( L_k \) is the set of the \( K \) nearest respondents of unit \( k \). The neighborhood is determined based on the auxiliary variables and a distance measure such as the Euclidean distance. Consider \( \alpha_{k\ell} \) an indicator that takes value 1 if respondent \( \ell \in s_r \) is in the neighborhood \( L_k \) of unit \( k \in U \). We have \( \alpha_{k\ell} = 0 \) if \( \ell \in U \setminus s_r \). A prediction can be written

\[
m_r(x_k) = \frac{1}{K} \sum_{\ell \in s_r} \alpha_{k\ell} y_{\ell}.
\]
The NWA model-assisted estimator can be written in weighted form

\[ \hat{\tau}_{m_r, \hat{p}} = \sum_{k \in U} m_r(x_k) + \sum_{k \in s_r} \frac{y_k - m_r(x_k)}{\pi_k \hat{p}_k} \]

\[ = \sum_{k \in U} \frac{1}{K} \sum_{\ell \in s_r} \alpha_{k\ell} y_{\ell} + \sum_{k \in s_r} \frac{y_k}{\pi_k \hat{p}_k} - \sum_{k \in s_r} \frac{1}{\pi_k \hat{p}_k} K \sum_{\ell \in s_r} \alpha_{k\ell} y_{\ell} \]

\[ = \sum_{\ell \in s_r} \left\{ \frac{1}{\pi_{\ell \hat{p}_\ell}} + \frac{1}{K} \left( \sum_{k \in U} \alpha_{k\ell} - \sum_{k \in s_r} \frac{1}{\pi_k \hat{p}_k} \alpha_{k\ell} \right) \right\} y_{\ell}. \]

The weights are the ones of the NWA estimator \(1/(\pi_k \hat{p}_k)\) plus a corrective term induced by the working model. The second term cancels when the response probabilities are calibrated on variables \((\alpha_{1\ell}, \alpha_{2\ell}, \ldots, \alpha_{N\ell})^T, \ell \in s_r\). The NWA model-assisted estimator is the NWA estimator in this case. The weights depend on the values of the auxiliary variables through the distance measure applied to construct the neighborhoods. They are free from values \\{\{y_k\}\} and could therefore be used for several variables of interest provided that they have observed values on \(s_r\). In particular, they can be applied to \(\{x_k\}\). This yields

\[ \hat{\tau}_{m_r, \hat{p}}^X = \sum_{k \in s_r} \frac{x_k}{\pi_k \hat{p}_k} + \frac{1}{K} \sum_{k \in U} \sum_{\ell \in s_r} \alpha_{k\ell} x_{\ell} - \sum_{k \in s_r} \frac{1}{\pi_k \hat{p}_k} \frac{1}{K} \sum_{\ell \in s_r} \alpha_{k\ell} x_{\ell} \]

\[ = \sum_{k \in s_r} \frac{x_k}{\pi_k \hat{p}_k} + \frac{1}{K} \sum_{k \in U} \left( \sum_{k \in s_r} \frac{1}{\pi_k \hat{p}_k} \right) \sum_{\ell \in s_r} \alpha_{k\ell} x_{\ell}. \]

The weights are calibrated to the totals of the auxiliary variables when \(K^{-1} \sum_{\ell \in s_r} \alpha_{k\ell} x_{\ell} = x_k\) for all \(k \in U\). This is for instance the case when the neighborhoods \(L_k\) are disjoint and have constant values \(\{x_k\}\). In practice, we can reasonably assume that this holds at least approximately for large populations and samples.

### 4.3 Local Polynomial Regression

Local polynomial regression is studied in the context of model-assisted survey estimation in Breidt and Opsomer (2000). Consider a working model in which \(x_k\) is a scalar, i.e. \(x_k = x_k, x_k \in \mathbb{R}\). Function \(m(\cdot)\) is approximated locally at \(x_k\) by \(q\)-th order polynomial regression. The
model is fitted via weighted least squares with weights based on a kernel function centered at \( x_k \). \text{Breidt and Opsomer} (2000) propose and study the model-assisted estimator with a survey weighted estimator of \( m(\cdot) \) fitted at the sample level. Adapting their estimator to the context of nonresponse yields

\[
m_r(x_k) = e_1 \cdot (X_{rk}^T W_{rk} X_{rk})^{-1} X_{rk}^T W_{rk} Y_{rk} = \omega_{rk}^T Y_{rk},
\]

where \( e_j \) is a vector with 1 at the \( j \)-th coordinate and 0 otherwise,

\[
X_{rk} = [1 \ x_j - x_k \ \cdots \ (x_j - x_k)^q]_{j \in s_r},
\]

\[
W_{rk} = \text{diag}\left\{ \frac{1}{k_j h} K\left( \frac{x_j - x_k}{h} \right) \right\}_{j \in s_r},
\]

and

\[
y_{rk}^T = [y_j]_{j \in s_r},
\]

and \( k_j \) is either 1 for all \( j \in s_r \) or \( \pi_j \hat{p}_j \), \( K(\cdot) \) is a continuous kernel function, and \( h \) a bandwidth. The NWA model-assisted estimator can be written in weighted form

\[
\hat{t}_{mr,\hat{p}} = \sum_{k \in U} m_r(x_k) + \sum_{k \in s_r} \frac{y_k - m_r(x_k)}{\pi_k \hat{p}_k}
\]

\[
= \sum_{k \in U} \omega_{rk}^T Y_{rk} + \sum_{k \in s_r} \frac{y_k}{\pi_k \hat{p}_k} - \sum_{k \in s_r} \frac{1}{\pi_k \hat{p}_k} \omega_{rk}^T Y_{rk}
\]

\[
= \sum_{k \in s_r} \left\{ \frac{1}{\pi_k \hat{p}_k} + \sum_{\ell \in U} \left( 1 - \frac{a_{rl} r_l}{\pi_l \hat{p}_l} \right) \omega_{rl}^T e_k \right\} y_k.
\]

The weights are the weights of the NWA estimator \( 1/(\pi_k \hat{p}_k) \) plus a corrective term induced by the working model. They are free from values \( \{y_k\} \) and could therefore be used for several variables of interest provided that they have observed values on \( s_r \). In particular, they can be applied to \( \{x_k\} \). This yields

\[
\hat{t}_{mr,\hat{p}}^X = \sum_{k \in s_r} \frac{x_k}{\pi_k \hat{p}_k} + \sum_{\ell \in U} \omega_{r\ell}^T \sum_{k \in s_r} e_k x_k + \sum_{\ell \in s_r} \omega_{r\ell}^T \sum_{k \in s_r} e_k x_k = \sum_{k \in U} x_k,
\]

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where we used
\[ \omega_{r\ell}^T \sum_{k \in s_r} \mathbf{e}_k x_k = x_\ell. \]

The weights are calibrated to the totals of the auxiliary variables.

5 Asymptotics

5.1 Preliminaries

In this section we develop the asymptotic properties of the two-phase model-assisted estimator and of the NWA model-assisted estimator. We build on the asymptotic framework of Isaki and Fuller (1982). Consider a sequence \( U_N \) of embedded finite populations of size \( N \) where \( N \) grows to infinity. A sample \( s_N \) of size \( n_N \) is selected from \( U_N \) with sampling design \( p_N(\cdot) \). The associated first- and second-order inclusion probabilities are \( \pi_{k(N)} \) and \( \pi_{k\ell(N)} \), respectively, for some generic units \( k \) and \( \ell \). A subsample \( s_{rN} \) is obtained from \( s_N \) with Poisson sampling design with unknown inclusion probabilities \( p_k(N) \). We consider the following common regularity conditions on the sequence of sampling designs.

(A1): \( \lim_{N \to +\infty} n_N / N = \pi \in (0, 1) \),

(A2): For all \( N \), there exists \( \lambda_1 \in \mathbb{R} \) such that \( \pi_{k(N)} > \lambda_1 > 0 \) for all \( k \in U_N \),

(A3): For all \( N \), there exists \( \lambda_2 \in \mathbb{R} \) such that \( \pi_{k\ell(N)} > \lambda_2 > 0 \), for all \( k, \ell \in U_N \),

(A4): \( \limsup_{N \to +\infty} n_N \max_{k, \ell \in U_N, k \neq \ell} |\Delta_{k\ell(N)}| < +\infty \).

For a sampling design with random sample size, \( n_N \) in Assumption (D1) is the expected sample size. We also consider the following condition on the sequence of finite populations.
(A5): The study variable has finite second and fourth moments, i.e.

$$\limsup_{N \to +\infty} N^{-1} \sum_{k \in U_N} u_k < +\infty$$

for $u_k = y_k^2, y_k^4$.

Conditions (A1)-(A5) ensure consistency of the HT estimator and its variance estimator in (1). Finally we consider the following regularity conditions on the sequence of Poisson sampling designs that generate the sets of respondents.

(A6): $\lim_{N \to +\infty} \sum_{k \in U_N} \pi_k(N)p_k(N)/n_N = \pi \in (0,1)$,

(A7): For all $N$, there exists $\lambda_3 \in \mathbb{R}$ such that $p_k(N) > \lambda_3 > 0$, for all $k \in U_N$.

Assumption (A6) states that the fraction of respondents to sampled units does not increase or decrease as $N$ grows to infinity. Assumption (A7) states that each unit has a strictly positive probability of responding. In what follows, we will omit the subscript $N$ whenever possible to simplify notation.

### 5.2 Two-Phase Difference Estimator

To study the asymptotic properties of the proposed estimator, it is useful to introduce a sampling design $p^*(.)$ that selects the sample $s_r$ directly in population $U$. The associated first- and second-order inclusion probabilities are, respectively, $\pi^*_k = \pi_k p_k$ and $\pi^*_{k\ell} = \pi_k p_k p_{\ell}$, if $k \neq \ell$;

$$\pi^*_k = \begin{cases} \pi_k p_k p_{\ell}, & \text{if } k \neq \ell; \\ \pi_k p_k, & \text{if } k = \ell. \end{cases}$$

The membership indicator of a unit $k \in U$ in the set of respondents $s_r$ is $a^*_k = a_k r_k$. The membership indicator of two different units $k, \ell \in U, k \neq \ell$ in $s_r$ is $a^*_{k\ell} = a_k a_\ell r_k r_\ell$. Given that the nonresponse process is independent from the selected sample, we have $E_{p^*}(a^*_k) = E_p E_q(a_k r_k) = \pi_k p_k$ and $E_{p^*}(a^*_{k\ell}) = \pi_{k\ell} p_k p_\ell$, where the subscript $p^*$ means that the expectation
is computed with respect to the two-phase sampling design \( p^*(\cdot) \). The covariance between the membership indicators \( \{a_k^*\} \) is

\[
\Delta_k^* = \begin{cases} 
\Delta_k P_k P_\ell = (\pi_k - \pi_k \pi_\ell) P_k P_\ell, & \text{if } k \neq \ell; \\
\pi_k P_k (1 - \pi_k P_k), & \text{if } k = \ell.
\end{cases}
\]

**Result 1.** Suppose that the sequence of sampling designs, populations, and response mechanisms satisfy Assumptions (A1)-(A5). Consider a working model \( \xi \) in (2) for which

\[
\hat{t}_{ms} = \hat{t}_{mu} + R_{ms}
\]

where the remainder term divided by the population size \( R_{ms}/N \) converges in probability to 0 under the aforementioned assumptions. The reference probability distribution is here the design \( p(\cdot) \).

Suppose moreover that the sequence of response mechanisms satisfies Assumptions (NR1), (NR2), (A6), and (A7). Then the two-phase model-assisted estimator can be written

\[
\hat{t}_{mr,p} = \hat{t}_{mu} + R_{mr,p}
\]

where the remainder term divided by the population size \( R_{mr,p}/N \) converges in probability to 0. The reference probability distribution is here the two-phase design \( p^*(\cdot) \).

**Proof.** It follows directly from the fact that when the sampling design \( p(\cdot) \) satisfies Assumptions (A1)-(A4) and the response mechanism satisfies Assumptions (NR1), (NR2), (A6), and (A7), the sampling design \( p^*(\cdot) \) satisfies Assumptions (A1)-(A4).

### 5.3 NWA Model-Assisted Estimator

We now turn to the NWA model-assisted estimator adapted for nonresponse \( \hat{t}_{mr,\hat{p}} \). We can write

\[
\hat{t}_{mr,\hat{p}} = \hat{t}_{mu,\hat{p}} + \sum_{k \in U} \{m_r(x_k) - m_U(x_k)\} \left(1 - \frac{a_k r_k}{\pi_k P_k}\right)
\]

\[
= \hat{t}_{mu,\hat{p}} + R_{mr,\hat{p}}
\]
where
\[ \hat{t}_{m_r, \hat{p}} = \sum_{k \in U} m_U(x_k) + \sum_{k \in s_r} \frac{y_k - m_U(x_k)}{\pi_k \hat{p}_k}. \]

Estimator \( \hat{t}_{m_r, \hat{p}} \) is unknown in practice but useful to derive some asymptotic properties of \( \hat{t}_{m_r, \hat{p}} \). The idea is to first study the asymptotic properties of \( \hat{t}_{m_r, \hat{p}} \), then show that the remainder \( R_{m_r, \hat{p}} \) is negligible. This allows us to conclude that \( \hat{t}_{m_r, \hat{p}} \) inherits the asymptotic properties of \( \hat{t}_{m_U, \hat{p}} \). The asymptotic properties of \( \hat{t}_{m_U, \hat{p}} \) and whether the remainder \( R_{m_r, \hat{p}} \) is negligible depends on the working model and the method applied to estimate the response probabilities.

**Result 2.** Under the aforementioned assumptions and regularity conditions in Hasler (2022), the NWA model-assisted estimator \( \hat{t}_{m_r, \hat{p}} \) can be written
\[ \hat{t}_{m_r, \hat{p}} = \hat{t}_{m_U, \hat{p}, \ell} + R \]
where \( R \) is negligible provided that the remainder \( R_{m_r, \hat{p}} \) is negligible, and where estimator \( \hat{t}_{m_U, \hat{p}, \ell} \) is asymptotically unbiased and has a variance expected to be lower than that of the HT estimator.

**Proof.** From Result 1 of Hasler (2022), when the response probabilities are estimated via calibration,
\[ \hat{t}_{m_U, \hat{p}} = \hat{t}_{m_U, \hat{p}, \ell} + O_p(N n^{-1}), \] (6)
where
\[ \hat{t}_{m_U, \hat{p}, \ell} = \sum_{k \in U} \left[ m_U(x_k) + x_k^\top \gamma + \frac{a_k r_k}{\pi_k p_k} \left\{ y_k - m_U(x_k) - x_k^\top \gamma \right\} \right], \]
\[ \gamma = \left\{ \sum_{k \in U} (1 - p_k) x_k x_k^\top \right\}^{-1} \sum_{k \in U} (1 - p_k) x_k \left\{ y_k - m_U(x_k) \right\}. \]
Estimator $\hat{t}_{m_U,p,t}$ is unbiased for $t$ and we expect its variance to be lower than that of $\hat{t}_{m_U,p}$ provided that the “residuals” $\{y_k - m_U(x_k) - x_k^\top \gamma\}$ have less variability than values $\{y_k - m_U(x_k)\}$ (Hasler, 2022). Moreover we expect the variance of $\hat{t}_{m_U,p}$ to be lower than that of the HT estimator provided that the “residuals” $\{y_k - m_U(x_k)\}$ have less variability than the “raw values” $\{y_k\}$ (Breidt and Opsomer, 2017, p.192).

### 5.4 Generalized Regression Estimator

Consider the generalized regression (GREG) estimator described in Section 4.1. Suppose that $(B_U - B_r) = O_p(1)$. This is the case for most sequences of populations, sampling designs, and response mechanisms. For instance, if $c_k = 1$, this equality holds when the respondent moments of $x_k y_k$ and $x_k x_k^\top$ converge to their population moments via

$$
\frac{1}{N} \sum_{k \in U} x_k y_k - \frac{1}{N} \sum_{k \in s_r} x_k y_k = o_p^* \left(n^{-1/2}\right),
$$

$$
\frac{1}{N} \sum_{k \in U} x_k x_k^\top - \frac{1}{N} \sum_{k \in s_r} x_k x_k^\top = o_p^* \left(n^{-1/2}\right),
$$

$x_i y_i$ is bounded,

$$
\limsup_{N \to +\infty} \frac{1}{N} \sum_{k \in U} x_k y_k < +\infty,
$$

and $X_r$ is of full rank.

Let us first study the asymptotic properties of the two-phase model assisted estimator $\hat{t}_{m_r,p}$, the model-assisted estimator with the true response probabilities. For this working model, the remainder $R_{m_s}$ in Result 1 is

$$
R_{m_r,p} = (B_r - B_U)^\top \left(t^X - \hat{t}_{2HT}^X\right),
$$

with $\hat{t}_{2HT}^X$ the vector of Horvitz-Thompson estimators of these variables under the two-phase sampling $p^*$. This reminder is negligible as the first term is $O_{p^*}(1)$ and the second
$O_{p^*}(Nn_r^{-1/2})$. It follows that the two-phase model-assisted estimator $\hat{t}_{m_r,p}$ behaves asymptotically like the population pseudo-generalized difference estimator $\hat{t}_{m_U}$. It is 1) asymptotically design-unbiased and 2) asymptotically more efficient than the HT provided that the finite population “residuals” $\{y_k - x_k^\top B_U\}$ have less variability than the “raw values” $\{y_k\}$. This holds regardless of the quality of the working model.

Let us now turn to the NWA model-assisted estimator $\hat{t}_{m_r,\hat{p}}$. From Equation (5)

$$\hat{t}_{m_r,\hat{p}} = \hat{t}_{m_U,\hat{p}} + R_{m_r,\hat{p}},$$

where

$$R_{m_r,\hat{p}} = (B_r - B_U)^\top \left( t^\times - \hat{t}^\times_{NWA} \right),$$

(7)

where $\hat{t}^\times_{NWA}$ is the NWA estimator of the auxiliary variables. By assumption the first term is $O_{p^*}(1)$. When calibration is applied, the second term is 0. Hence, the remainder $R_{m_r,\hat{p}}$ is negligible. Estimator $\hat{t}_{m_r,\hat{p}}$ is asymptotically unbiased for $t$ and we expect its variance to be smaller than that of the HT estimator. This is true even if the working model is misspecified.

6 Variance and Variance Estimation

Under nonresponse, we can write the variance of a generic estimator $\hat{t}_g$ as

$$\text{var} (\hat{t}_g) = \text{var}_{\text{sam}} (\hat{t}_g) + \text{var}_{\text{nr}} (\hat{t}_g),$$

where the two terms are the sampling variance and the nonresponse variance, respectively, and are given by

$$\text{var}_{\text{sam}} (\hat{t}_g) = \text{var}_p \left\{ \mathbb{E}_q (\hat{t}_g | s) \right\},$$

and

$$\text{var}_{\text{nr}} (\hat{t}_g) = \mathbb{E}_p \left\{ \text{var}_q (\hat{t}_g | s) \right\}.$$

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Using the approximations in Equation (5) and (6) and results of Section 5 of Hasler (2022), the variance of the NWA model-assisted estimator \( \hat{t}_{m,r} \) can be approximated by

\[
\text{var}(\hat{t}_{m,r}) \approx \text{var}_{\text{sam}}(\hat{t}_{mU,\hat{p},\ell}) + \text{var}_{\text{nr}}(\hat{t}_{mU,\hat{p},\ell}),
\]

where

\[
\text{var}_{\text{sam}}(\hat{t}_{mU,\hat{p},\ell}) = \text{var}_p \left[ \sum_{k \in s} \frac{1}{\pi_k} \left\{ y_k - m_U(x_k) - x_k^\top \hat{\gamma} \right\} \right],
\]

\[
\text{var}_{\text{nr}}(\hat{t}_{mU,\hat{p},\ell}) = E_p \left[ \sum_{k \in s} \frac{1}{\pi_k^2} \frac{1}{\hat{p}_k} \left\{ y_k - m_U(x_k) - x_k^\top \hat{\gamma} \right\}^2 \right],
\]

and

\[
\hat{\gamma} = \left\{ \sum_{k \in U} (1 - p_k)x_kx_k^\top \right\}^{-1} \sum_{k \in U} (1 - p_k)x_k \left\{ y_k - m_U(x_k) \right\}.
\]

The first term is the variance of the full sample HT estimator of the differences \( \{ y_k - m_U(x_k) - x_k^\top \hat{\gamma} \}. Based on this approximation, a variance estimator is

\[
\hat{\text{var}}(\hat{t}_{m,r}) = \hat{\text{var}}_{\text{sam}}(\hat{t}_{mU,\hat{p},\ell}) + \hat{\text{var}}_{\text{nr}}(\hat{t}_{mU,\hat{p},\ell}),
\]

where

\[
\hat{\text{var}}_{\text{sam}}(\hat{t}_{mU,\hat{p},\ell}) = \sum_{k \in s_r} \frac{1 - \pi_k}{\pi_k^2} \frac{e_k^2}{\hat{p}_k} + \sum_{k,\ell \in s_r, k \neq \ell} \frac{\pi_k \pi_\ell}{\pi_k^2 \pi_\ell} \frac{e_k e_\ell}{\hat{p}_k \hat{p}_\ell},
\]

\[
\hat{\text{var}}_{\text{nr}}(\hat{t}_{mU,\hat{p},\ell}) = \sum_{k \in s_r} \frac{1}{\pi_k^2} \frac{1}{\hat{p}_k} e_k^2,
\]

\[
e_k = y_k - m_r(x_k) - x_k^\top \hat{\gamma},
\]

and

\[
\hat{\gamma} = \left( \sum_{k \in s_r} \frac{1}{\pi_k} \frac{1}{\hat{p}_k} x_kx_k^\top \right)^{-1} \sum_{k \in s_r} \frac{1}{\pi_k} \frac{1}{\hat{p}_k} x_k \left\{ y_k - m_r(x_k) \right\},
\]

where we substituted \( \hat{p}_k \) for the unknown \( p_k \) and \( m_r(x_k) \) for \( m_U(x_k) \).
Table 1: Correlations of the values \( \{y_k\} \) of the survey variable with the response probabilities \( \{p_k\} \) and values \( \{x_{k1}\}, \{x_{k2}\}, \{x_{k3}\} \) and \( \{x_{k4}\} \) of the auxiliary variables on population \( U \).

| \{p_k\} | \{x_{k1}\} | \{x_{k2}\} | \{x_{k3}\} | \{x_{k4}\} |
|-----------|-----------|-----------|-----------|-----------|
| \{y_k\}  | 0.55      | 0.69      | 0.65      | 0.00      | 0.027     |

7 Simulations

Let us consider a population \( U \) of size \( N = 1000 \). For each unit \( k \) of \( U \), a vector \( x_k = (x_{k1}, x_{k2}, x_{k3}, x_{k4})^\top \) is generated from independent and identically distributed random variables. Values \( \{x_{k1}\} \) are realisations of a gaussian random variable with mean 1 and variance 0.25, \( \{x_{k2}\} \) of a mixture of two gaussian random variables with respective means 6 and 10 and variances 0.25 with mixing proportions 0.5, \( \{x_{k3}\} \) of a gamma distribution with shape 2 and rate 3, and \( \{x_{k4}\} \) of a mixture of a gaussian distribution of mean 2 and variance 16 and a gamma distribution with shape 3 and rate 3 with mixing proportions 0.5. The goal is to estimate the total \( t \) on population \( U \) of a survey variable \( y \) generated as

\[
y_k = 6 \cdot x_{k1} + 4 \cdot x_{k2} + \cos(x_{k3}) + \sqrt{|x_{k4} - \bar{x}_4|} + \varepsilon_k,
\]

where \( \bar{x}_4 \) is the mean of values \( \{x_{k4}\} \) in population \( U \) and \( \varepsilon_k \) the realisation of a normal distribution of mean 0 and variance 1. Each unit \( k \) of the population has a probability

\[
p_k = \left\{1 + \exp\left[-\lambda^\top (x_{k1}, x_{k2})^\top\right]\right\}^{-1}
\]

of responding to variable \( y \), with \( \lambda = (0.46, -0.06)^\top \). Value \( \lambda \) is set so that the expected rate of missing values, i.e. the mean of the \( \{p_k\} \) on the population, is 50%. The correlations between the variable of interest and the other variables are given in Table 1.

A comparison between some aforementioned total estimators is performed in different scenarios: when the nonresponse model is correctly versus incorrectly specified, when the working model is correctly versus incorrectly specified. The couple \( \{x_{k1}, x_{k2}\} \) is strongly
related to \{y_k\} and \{p_k\}. The couple \{x_{k3}, x_{k4}\} is weakly related to \{y_k\} and unrelated to \{p_k\}.

Four different scenarios are considered in which different couples of variables are used to fit the response model and the working model. In scenarios 1 and 2, the working model fits well the data, whereas in scenarios 3 and 4 it fits poorly. In scenarios 1 and 3, the nonresponse model fits well the data, whereas in scenarios 2 and 4 it fits poorly. Note that in scenario 1 both models fit well the data, in scenario 2 and 3 only one of the two models fits well, and in scenario 4 both models fit poorly. Table 2 shows which couple of variables, i.e. \{x_{k1}, x_{k2}\} or \{x_{k3}, x_{k4}\}, is used to fit the models.

We compare five estimators: \(\hat{t}_{HT}\), \(\hat{t}_{NW A}\), and \(\hat{m}_{r, \hat{p}}\), defined in Section 2, the imputed estimator \(\hat{t}_{imp} = \sum_{k \in s_r} y_k/\pi_k + \sum_{k \not\in s_r} m_r(x_k)/\pi_k\), and the naive estimator \(\hat{t}_{naive} = N n_r^{-1} \sum_{k \in s} y_k\). Estimator \(\hat{t}_{HT}\) is unavailable in practice with nonresponse. It serves here as comparison point. Estimators \(\hat{t}_{imp}\) and \(\hat{m}_{r, \hat{p}}\) depend on the estimated function \(m_r(\cdot)\). Three different prediction methods are used to obtain \(m_r(\cdot)\): generalized regression, local polynomial regression, and \(K\)-nearest neighbors. Estimators \(\hat{t}_{NW A}\) and \(\hat{m}_{r, \hat{p}}\) depend on the estimated response probabilities.

We select \(I = 10'000\) samples denoted by \(s^{(i)}, i = 1, \ldots, I\), of size \(n = 200\) from population \(U\) using simple random sampling without replacement. For each sample, we randomly generate missing values in the survey variable using the response probabilities \(\{p_k\}\) and a Poisson sampling design. The expected number of observed values \(n_r\) in each sample \(s^{(i)}\) is \(n/2 = 100\). We can then define the sub-sample \(s_{r}^{(i)} \subset s^{(i)}\) containing the units for which \(y_k\) is observed at simulation run \(i\).

In order to evaluate the quality of the nonresponse model and of the working model at simulation run \(i, i \in \{1, \ldots, I\}\), two quantities are computed: the mean absolute error of the estimated response probabilities \(\{\hat{p}_k\}\)

\[
\text{MAE}(\hat{p}_k) = \frac{1}{n_r} \sum_{k \in s_{r}^{(i)}} |\hat{p}_k - p_k|,
\]
and the mean relative prediction error

$$\text{MRPE}(m_r(z_k)) = \frac{1}{N} \sum_{k \in U} \left| \frac{m_r(z_k) - y_k}{\sum_{k \in U} y_k} \right|, $$

where $z_k = (x_{k1}, x_{k2})^\top$ or $z_k = (x_{k3}, x_{k4})^\top$, depending on the scenario. The goodness of fit of the working and nonresponse models is assessed by averaging, for each scenario, the MAE and MRPE over the simulation runs. Table 3 contains these averages.

Table 2: Couple of variables used to obtain the estimated response probabilities $\hat{p}_k$ and the estimated function $m_r(.)$ for four scenarios.

| $\hat{p}_k$ | $m_r(.)$ | Scenario 1 | Scenario 2 | Scenario 3 | Scenario 4 |
|--------------|----------|------------|------------|------------|------------|
| $\{x_{k1}, x_{k2}\}$ | $\{x_{k3}, x_{k4}\}$ | $\{x_{k1}, x_{k2}\}$ | $\{x_{k3}, x_{k4}\}$ | $\{x_{k3}, x_{k4}\}$ | $\{x_{k1}, x_{k2}\}$ |

For each samples $s^{(i)}$ and $s_r^{(i)}$, we estimate the population total with the five aforementioned total estimators. For a generic total estimator $\hat{t}$, we compute the Monte Carlo bias relative to the true total

$$\text{RB}(\hat{t}) = \frac{1}{t} \sum_{i=1}^{t} (\hat{t}^{(i)} - t)$$

and the Monte Carlo standard deviation relative to the true total

$$\text{RSd}(\hat{t}) = \sqrt{\frac{1}{t-1} \sum_{i=1}^{t} (\hat{t}^{(i)} - t)^2},$$

where $\hat{t}^{(i)}$ is the value of $\hat{t}$ obtained at simulation run $i \in \{1, \ldots, I\}$. We compare the total estimators for each scenario. Figure 1 summaries the results. More details of the results are in Appendices in Tables 4-7 for scenarios 1-4 respectively. Note that only the first three considered estimators are available in practice with nonresponse. The HT estimator $\hat{t}_{HT}$ is unavailable and serves as comparison point.
Table 3: Average over the simulation runs of the mean absolute error (MAE) of the estimated response probabilities $\hat{p}_k$ and of the mean relative prediction error (MRPE) of the estimation $m_r(z_k)$ in four scenarios.

| Scenario | 1   | 2   | 3   | 4   |
|----------|-----|-----|-----|-----|
| MAE($\hat{p}_k$) | 0.046 | 0.136 | 0.046 | 0.136 |
| MRPE($m_r(z_k)$) | GREG 0.025, 0.015 | 0.240 | 0.240 |
|          | poly 0.028 | 0.028 | 0.252 | 0.252 |
|          | K-nn 0.058 | 0.058 | 0.252 | 0.252 |

In scenario 1, both the nonresponse and working models fit well the data. Our proposed NWA model-assisted estimator $\hat{t}_{m_r,\hat{p}}$ and $\hat{t}_{NWA}$ perform the best in this scenario. They have a RB close to that of the unbiased estimators $\hat{t}_{HT}$ and have the lowest relative standard deviation. In scenario 2, our proposed estimator $\hat{t}_{m_r,\hat{p}}$ shows the best results of all available estimators even if the nonresponse model fits poorly. It has a bias of the same order as $\hat{t}_{HT}$, which is an unbiased estimator, for the first two prediction methods. Its bias is smaller than that of the other three available estimators for $K$-nearest neighbors. It shows the best results in term of standard deviation and is more efficient than the NWA and HT estimator. It confirms that the working model allows to improve the efficiency of the total estimator. In scenario 3, the working model is misspecified. The NWA estimator $\hat{t}_{NWA}$ provides the best results followed by our proposed model-assisted estimator $\hat{t}_{m_r,\hat{p}}$. The reason is that the response model is correctly specified in this scenario. Finally, in scenario 4, both the nonresponse model and the working model fit poorly the data. In this case, the performance
of $\hat{t}_{mr,\hat{p}}$ is comparable to that of $\hat{t}_{NW,A}$ and $\hat{t}_{imp}$ that rely on only one of the two models.

The general conclusion of the simulation study is that the proposed estimator $\hat{t}_{mr,\hat{p}}$ globally performs as well or better than estimators $\hat{t}_{NW,A}$, $\hat{t}_{imp}$, and $\hat{t}_{naive}$ even when one or both of the working model and the nonresponse model is or are misspecified. Our estimators hence provides security against model misspecification and greater confidence in the total estimator.

8 Discussion

We adapt model-assisted total estimators to missing at random data building on the idea of nonresponse weighting adjustment. We consider nonresponse as a second phase of the survey and reweight the units using the inverse of estimated response probabilities in model-assisted estimators in order to compensate for the nonrespondents. We develop the asymptotic properties of our proposed estimator and show conditions under which it is asymptotically unbiased. Our proposed estimator can be written as a weighted estimator. We show cases in which the resulting weights are calibrated to the total of the auxiliary variables. We conduct a simulation study to empirically study the performance of our estimator. The results of this study confirm that our estimator generally outperforms the competing estimators, even when the underlying models are misspecified. Further work includes the study of our estimator under other working models as well as the extension to non-missing at random data.

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Figure 1: Bias and standard deviation of total estimators relative to the true total for scenarios 1 to 4 with three prediction methods: generalized regression (GREG), polynomial regression (poly) and $K$-nearest neighbors ($K$-nn) with $K = 5$. Estimator $\hat{t}_{HT}$ is a comparison point and is unavailable with nonresponse.
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A Auxiliary Variables Known at the Sample Level Only

The first term of the NWA model-assisted estimator in (4) is the population total of the predicted values $m_r(x_k)$. For most working models, this requires the values $\{x_k\}$ to be known for all population units. If this population total is unavailable, we may use a Horvitz-Thompson-type estimator of this sum which yields estimator

$$\hat{t}_{m_r,s,p} = \sum_{k \in s} \frac{m_r(x_k)}{\pi_k} + \sum_{k \in s_r} \frac{y_k - m_r(x_k)}{\pi_k \hat{p}_k}.$$ 

This estimator is equivalent to that of Kim and Haziza (2014), Equation (3.2). The authors suppose parametric models for the nonresponse model and working model. They estimate the parameter vectors of these two models simultaneously based on a system of estimating equations. This results in doubly robust point and variance estimators. Our approach is different and more general. The working model may be nonparametric and both models may be estimated separately.

B Response Probabilities Estimated via Generalized Calibration

Throughout the paper, we suppose that the response probabilities are estimated via calibration. Another approach consists of estimating the response probabilities via generalized calibration, hence allowing for the variables in the response model to differ from the variables on which we calibrate. The parameter vector $\lambda_0$ is then estimated via $\hat{\lambda}$ which is the solution to the estimating equation

$$Q^{cal}(\lambda) = \sum_{k \in U} z_k - \sum_{k \in s_r} \frac{z_k}{\pi_k} F(x_k^\top \lambda)$$  (8)
where \( z_k \) is a vector of dimension equal to that of \( x_k \). Variables \( z_k \) are the calibration variables and variables \( x_k \) the response model variables. The calibration variables are ideally highly correlated to \( y \). This approach is preferred over calibration when the variables that predict the nonresponse differ from the variables that predict the variable of interest.

Now suppose that the \( m_r(\cdot) \) is free from \( \hat{p}_k \). It is for instance the case for the GREG with weights \( c_k = 1 \) or \( \sigma_k^2 \) and the working models based on statistical learning techniques presented in Section 4. Consider that the response probabilities are estimated via generalized calibration where one of the calibration variables \( z_k \) is \( m_r(x_k) \). Because \( z_k \) includes \( m_r(x_k) \), Equation (8) implies

\[
\sum_{k \in U} m_r(x_k) = \sum_{k \in s_r} \frac{m_r(x_k)}{\pi_k} F(x_k^\top \hat{\lambda}).
\]

The NWA model-assisted estimator can then be written

\[
\hat{t}_{m_r,\hat{p}} = \sum_{k \in s_r} \frac{y_k}{\pi_k \hat{p}_k}.
\]

This estimator is an instrumental calibration estimator. This estimator is consistent for the population total under some regularity conditions, see Lesage et al. (2019), Section 3.

C Response Probabilities Estimated via Maximum Likelihood Estimation

A third approach consists of estimating the response probabilities via maximum likelihood. The estimation of parameter vector \( \lambda_0 \) is then \( \hat{\lambda} \) which is the solution to the estimating equation

\[
Q_{mle}(\hat{\lambda}) = \sum_{k \in s} c_k \left\{ r_k - F^{-1}(x_k^\top \hat{\lambda}_0) \right\} x_k = 0,
\]

\( 30 \)
for some weights $c_k$, Kim and Kim (2007). Common choices for the weights are 1 or $\pi_k^{-1}$. When $c_k = 1$ usual maximum likelihood estimation is applied and from Theorem 1 of Kim and Kim (2007)

$$\hat{t}_{m_U, \hat{p}, \ell} = \hat{t}_{m_U, \hat{p}, \ell} + O_p(Nn^{-1}),$$

where

$$\hat{t}_{m_U, \hat{p}, \ell} = \sum_{k \in U} m_U(x_k) \sum_{k \in s} \frac{1}{\pi_k} \left[ \pi_k p_k x_k^\top \gamma^{mle} + \frac{r_k}{p_k} \left\{ y_k - m_U(x_k) - \pi_k p_k x_k^\top \gamma^{mle} \right\} \right],$$

$$\gamma^{mle} = \left\{ \sum_{k \in s} p_k (1 - p_k) x_k x_k^\top \right\}^{-1} \sum_{k \in s} \frac{1 - p_k}{\pi_k} x_k \left\{ y_k - m_U(x_k) \right\}.$$ 

Estimator $\hat{t}_{m_U, \hat{p}, \ell}$ is unbiased for $t$ and at least as efficient as $\hat{t}_{m_U, \hat{p}}$ (Kim and Kim, 2007, p.505) which in turn we expect to be more efficient than the HT estimator (Breidt and Opsomer, 2017, p.192).

From Equation (5) it comes that the NWA model-assisted estimator $\hat{t}_{m_U, \hat{p}}$ behaves asymptotically like an unbiased estimator which we expect to be more efficient then the HT estimator provided that the remainder $R_{m_U, \hat{p}}$ is negligible. For GREG, this is the case when $B_r - B_U$ is $O_{p^*}(1)$ and $t_X - t_X^{NW_A}$ is $o_{p^*}(Nn^{-1})$, see Equation (7).

## D Variance and Variance Estimation Under General Response and Working Models

In Section 6 we discuss variance and variance estimation of the NWA estimator when calibration is used to estimate the response probabilities and the nonresponse model is as in Assumption (NR (4)). In this section, we generalize to any nonresponse model and any working model. We first consider the case where the link function $m(.)$ in the working model and the response probabilities are known. Then we relax to the case where only the response probabilities are unknown, and finally when both are unknown.
D.1 When the link function $m(\cdot)$ and the response probabilities $\{p_k\}$ are known

In this case, the NWA model-assisted estimator is

$$\hat{t}_{m,p} = \sum_{k \in U} m(x_k) + \sum_{k \in s_r} \frac{y_k - m(x_k)}{\pi_k p_k} = \sum_{k \in U} m(x_k) + \sum_{k \in s_r} \frac{y_k - m(x_k)}{\pi_k^*}$$

Based on formula (1) and the two-phase design described in Section 5, the variance of this estimator is

$$\text{var}(\hat{t}_{m,p}) = \sum_{k,\ell \in U} \frac{y_k - m(x_k)}{\pi_k^*} \frac{y_\ell - m(x_\ell)}{\pi_\ell^*} \Delta_{k\ell}^*$$

It can be estimated unbiasedly via

$$\hat{\text{var}}(\hat{t}_{m,p}) = \sum_{k,\ell \in U} \frac{y_k - m(x_k)}{\pi_k^*} \frac{y_\ell - m(x_\ell)}{\pi_\ell^*} \Delta_{k\ell}^* \pi_k^* \pi_\ell^* \pi_k \pi_\ell.$$

D.2 When the link function $m(\cdot)$ is unknown and the response probabilities $\{p_k\}$ are known

When the link function $m(\cdot)$ is unknown it is estimated from $s_r$ which provides $m_r(\cdot)$. Suppose in the remaining of this section that $m_r$ can be written in terms of $a^*_k = a_k r_k$ without any further $a_k$ or $r_k$. This is for instance the case of the working models considered in Section 4. Our estimator is

$$\hat{t}_{m_r,p} = \sum_{k \in U} m_r(x_k) + \sum_{k \in U} \frac{y_k - m_r(x_k)}{\pi_k p_k} a^*_k.$$

We use the approach of Vallée and Tillé (2019) to linearize our estimator with respect to the $a^*_k$'s and obtain

$$\hat{t}_{m_r,p} = \hat{t}_{m_r,p} \big|_{a^* = \pi_p} + \sum_{k \in U} v_k (a^*_k - \pi_k p_k) + R(\tau), \quad (9)$$

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where
\[ v_k = \frac{\partial \tilde{t}_{m_p}}{\partial a_k} \bigg|_{a^* = \pi p}, \]
\[ R(\tau) = \frac{1}{2} \sum_{k \in U} \sum_{j \in U} \frac{\partial^2 \tilde{t}_{m_p}}{\partial a_k^* \partial a_j^*} \bigg|_{a^* = \tau a^* + (1-\tau)\pi p}, \]
for some \( \tau \in (0, 1) \). Straightforward computations yield
\[ \tilde{t}_{m_p} \bigg|_{a^* = \pi p} = \sum_{k \in U} y_k \]
\[ v_k = \frac{y_k - m_r(x_k)}{\pi p_k} \bigg|_{a^* = \pi p}. \]

To approximate the variance of \( \tilde{t}_{m_p} \), we use the linearized estimator based on Equation (9) which yields
\[ \text{var}(\tilde{t}_{m_p}) \approx \text{var} \left( \sum_{k \in U} v_k a_k^* \right) = \sum_{k,\ell \in U} v_k v_\ell \Delta_{k\ell}^*. \]
The \( v_k \)’s may be unknown, in which case we use the variance estimator
\[ \widehat{\text{var}}(\tilde{t}_{m_r}) = \sum_{k,\ell \in s_r} \widehat{v}_k \widehat{v}_\ell \frac{\Delta_{k\ell}^*}{\pi_k \pi_\ell}, \]
for some estimator \( \widehat{v}_k \) of \( v_k \). See Vallée and Tillé (2019) for more details.

**Example 1.** Consider the generalized regression (GREG) estimator (Cassel et al., 1976; Särndal et al., 1992) estimators associated with the superpopulation model
\[ m : y_k = x_k^\top \beta + \varepsilon_k, \]
where the \( \varepsilon_k \) are uncorrelated with \( E_m(\varepsilon_k) = 0 \) and \( \text{var}_m(\varepsilon_k) = \sigma_k^2 \). As stated in Breidt and Opsomer (2017), if the \( y_k \)’s were observed on the entire population, \( \beta \) would be estimated via least squared which would yield predictors \( m_U(x_k) = x_k^\top B_U \) where
\[ B = \left( \sum_{k \in U} \frac{x_k x_k^\top}{\sigma_k^2} \right)^{-1} \sum_{k \in U} \frac{x_k y_k}{\sigma_k^2}. \]
In our case, $\beta$ is estimated based on $s_r$ and the associated predictors are $m_r(x_k) = x_k^T B_r$, where

$$B_r = \left( \sum_{i \in U} \frac{x_k x_k^T}{\pi_k p_k \sigma_k^2} \right)^{-1} \sum_{k \in U} \frac{x_k y_k}{\pi_k p_k \sigma_k^2} a_k^*.$$

The model assisted estimator becomes

$$\hat{t}_{m_r, p} = \sum_{k \in U} x_k^T B_r + \sum_{k \in U} \frac{y_k - x_k^T B_r}{\pi_k p_k} x_k^T B_r a_k^*.$$

Its variance is approximated by

$$\text{var}(\hat{t}_{m_r, p}) \approx \sum_{k, \ell \in U} v_k v_{\ell} \Delta_{k\ell}^*, \quad (10)$$

where

$$v_k = \frac{y_k - x_k^T B_r}{\pi_k p_k} \bigg|_{a^* = \pi_p} = \frac{y_k - x_k^T B_U}{\pi_k p_k}$$

and a natural estimator is

$$\hat{\text{var}}(\hat{t}_{m_r, p}) = \sum_{k, \ell \in s_r} \frac{y_k - x_k^T B_r}{\pi_k p_k} y_{\ell} - x_{\ell}^T B_r \frac{\Delta_{k\ell}^*}{\pi_{k\ell}}.$$

### D.3 When both the link function $m(\cdot)$ and the response probabilities $\{p_k\}$ are unknown

When it is unknown, an estimator of $p_k$ is $\hat{p}_k = 1/\hat{F}_k$ where $\hat{F}_k = F(x_k^T \hat{\lambda})$ for some estimator $\hat{\lambda}$ of $\lambda$. The NWA model-assisted estimator is

$$\hat{t}_{m_r, \hat{p}} = \sum_{k \in U} m_r(x_k) + \sum_{k \in U} \frac{y_k - m_r(x_k)}{\pi_k} F(x_k^T \hat{\lambda}) a_k^*.$$

We use the approach of Vallée and Tillé (2019) to linearize our estimator with respect to the $a_k^*$'s and obtain

$$\hat{t}_{m_r, \hat{p}} = \hat{t}_{m_r, \hat{p}} \bigg|_{a^* = \pi_p/\hat{F}} + \sum_{k \in U} v_k (a_k^* - \pi_k/F(x_k^T \hat{\lambda})) + R(\tau), \quad (11)$$
where

\[ v_k = \frac{\partial \hat{t}_{m_r, \hat{p}}}{\partial a_k^*} \bigg|_{a^* = \pi/\hat{F}}, \]

\[ R(\tau) = \frac{1}{2} \sum_{k \in U} \sum_{\ell \in U} \frac{\partial^2 \hat{t}_{m_r, \hat{p}}}{\partial a_k^* \partial a_\ell^*} \bigg|_{a^* = \tau a^* + (1-\tau)\pi/\hat{F}}, \]

for some \( \tau \in (0, 1). \) Straightforward computations yield

\[ \hat{t}_{m_r, \hat{p}} \bigg|_{a^* = \pi/\hat{F}} = \sum_{k \in U} y_k \]

\[ v_k = \sum_{k \in U} \frac{(y_k - m_r(x_k))}{F(x_k)} \frac{\partial F(x_k^\top \hat{\lambda})}{\partial a_k^*} + \frac{(y_k - m_r(x_k))}{\pi_k} \frac{F(x_k^\top \hat{\lambda})}{\pi_k} \bigg|_{a^* = \pi/\hat{F}}. \]

To approximate the variance of \( \hat{t}_{m_r, \hat{p}} \), we use the linearized estimator based on Equation (11) which yields

\[ \text{var}(\hat{t}_{m_r, \hat{p}}) \approx \text{var} \left( \sum_{k \in U} v_k (a_k^* - \pi_k/F(x_k^\top \hat{\lambda})) \right) \approx \sum_{k, \ell \in U} v_k u_{k\ell} \Delta_{k\ell}^*, \]

where we used that \( \sum_{k \in U} v_k \pi_k/F(x_k^\top \hat{\lambda}) \approx \sum_{k \in U} v_k \pi_k \) when the weights \( \pi_k/F(x_k^\top \hat{\lambda}) \) and \( \pi_k \) are close to one another. We can decompose this formula as follows

\[ \text{var}(\hat{t}_{m_r, \hat{p}}) \approx \sum_{k, \ell \in U} v_k u_{k\ell} \Delta_{k\ell}^* = V_1 + V_2, \]

where

\[ V_1 = \sum_{k, \ell \in U} u_{k\ell} \Delta_{k\ell}^*, \]

\[ u_k = \left\{ y_k - m_r(x_k) \right\} \frac{F(x_k^\top \hat{\lambda})}{\pi_k} \bigg|_{a^* = \pi/\hat{F}}, \]

and \( V_2 \) is defined accordingly. Comparing with Formula (10), we can see that the approximated variance of \( \hat{t}_{m_r, \hat{p}} \) can be decomposed as the variance if the response probabilities \( p_k \) were known \( (V_1) \) plus the variance due to the estimation of these \( (V_2). \)
The $u_k$ may be unknown, in which case we use the variance estimator

$$\hat{\text{var}}(\hat{t}_{m_r, \hat{p}}) = \sum_{k, \ell \in s_r} \hat{u}_k \hat{u}_\ell \frac{\Delta^*_k \Delta^*_\ell}{\pi^*_k \pi^*_\ell},$$

for some estimator $\hat{u}_k$ of $u_k$. See Vallée and Tillé (2019) for more details.

## E Results of the simulations

The results of the simulations presented in Tables 4 to 7.

Table 4: Bias and standard deviation of the total estimator relative to the true total for scenario 1 with three prediction methods: generalized regression (GREG), polynomial regression (poly) and $K$-nearest neighbors ($K$-nn) with $K = 5$. Estimator $\hat{t}_{HT}$ is a comparison point and is unavailable with nonresponse.

| Estimators       | $\hat{t}_{m_r, \hat{p}}$ | $\hat{t}_{NWA}$ | $\hat{t}_{imp}$ | $\hat{t}_{naive}$ | $\hat{t}_{HT}$ |
|------------------|--------------------------|----------------|-----------------|------------------|----------------|
| RB               |                          |                |                 |                  |                |
| GREG             | <0.001                   | <0.001         | 0.035           | 0.061            | <0.001         |
| poly             | 0.002                    | <0.001         | 0.035           | 0.061            | <0.001         |
| $K$-nn           | 0.004                    | <0.001         | 0.044           | 0.061            | <0.001         |
| RSd              |                          |                |                 |                  |                |
| GREG             | 0.003                    | 0.003          | 0.039           | 0.067            | 0.019          |
| poly             | 0.004                    | 0.003          | 0.040           | 0.067            | 0.019          |
| $K$-nn           | 0.007                    | 0.003          | 0.048           | 0.067            | 0.019          |
Table 5: Bias and standard deviation of the total estimator relative to the true total for scenario 2 with three prediction methods: generalized regression (GREG), polynomial regression (poly) and $K$-nearest neighbors ($K$-nn) with $K = 5$. Estimator $\hat{t}_{HT}$ is a comparison point and is unavailable with nonresponse.

| Estimators | $\hat{t}_{m.r}$ | $\hat{t}_{NW.A}$ | $\hat{t}_{imp}$ | $\hat{t}_{naive}$ | $\hat{t}_{HT}$ |
|------------|-----------------|-----------------|----------------|----------------|----------------|
| RB         |                 |                 |                 |                 |                |
| GREG       | $<0.001$        | $0.057$         | $0.035$         | $0.061$         | $<0.001$       |
| poly       | $0.002$         | $0.057$         | $0.035$         | $0.061$         | $<0.001$       |
| $K$-nn     | $0.019$         | $0.057$         | $0.044$         | $0.061$         | $<0.001$       |
| RSd        |                 |                 |                 |                 |                |
| GREG       | $0.003$         | $0.080$         | $0.039$         | $0.067$         | $0.019$        |
| poly       | $0.004$         | $0.080$         | $0.040$         | $0.067$         | $0.019$        |
| $K$-nn     | $0.020$         | $0.080$         | $0.048$         | $0.067$         | $0.019$        |
Table 6: Bias and standard deviation of the total estimator relative to the true total for scenario 3 with three prediction methods: generalized regression (GREG), polynomial regression (poly) and $K$-nearest neighbors ($K$-nn) with $K = 5$. Estimator $\hat{t}_{HT}$ is a comparison point and is unavailable with nonresponse.

| Estimators         | $\hat{t}_{mr,\bar{p}}$ | $\hat{t}_{NW,\bar{A}}$ | $\hat{t}_{imp}$ | $\hat{t}_{naive}$ | $\hat{t}_{HT}$ |
|--------------------|--------------------------|--------------------------|-----------------|-------------------|---------------|
| RB                 |                          |                          |                 |                   |               |
| GREG               | 0.001                    | <0.001                   | 0.060           | 0.061             | <0.001        |
| poly               | 0.007                    | <0.001                   | 0.064           | 0.061             | <0.001        |
| $k$nn              | 0.011                    | <0.001                   | 0.061           | 0.061             | <0.001        |
| RSd                |                          |                          |                 |                   |               |
| GREG               | 0.025                    | 0.003                    | 0.067           | 0.067             | 0.019         |
| poly               | 0.029                    | 0.003                    | 0.071           | 0.067             | 0.019         |
| $K$-nn             | 0.030                    | 0.003                    | 0.068           | 0.067             | 0.019         |
Table 7: Bias and standard deviation of the total estimator relative to the true total for scenario 4 with three prediction methods: generalized regression (GREG), polynomial regression (poly) and $K$-nearest neighbors ($K$-nn) with $K = 5$. Estimator $\hat{t}_{HT}$ is a comparison point and is unavailable with nonresponse.

| Estimators | $\hat{t}_{mr,\hat{p}}$ | $\hat{t}_{NW,\hat{A}}$ | $\hat{t}_{imp}$ | $\hat{t}_{naive}$ | $\hat{t}_{HT}$ |
|------------|------------------------|------------------------|----------------|------------------|---------------|
| RB         |                        |                        |                |                  |               |
| GREG       | 0.060                  | 0.057                  | 0.060          | 0.061            | $<0.001$      |
| poly       | 0.060                  | 0.057                  | 0.064          | 0.061            | $<0.001$      |
| $K$-nn     | 0.061                  | 0.057                  | 0.061          | 0.061            | $<0.001$      |
| RSd        |                        |                        |                |                  |               |
| GREG       | 0.067                  | 0.080                  | 0.067          | 0.067            | 0.019         |
| poly       | 0.068                  | 0.080                  | 0.071          | 0.067            | 0.019         |
| $K$-nn     | 0.068                  | 0.080                  | 0.068          | 0.067            | 0.019         |