Abstract

Recently it has been demonstrated by Dienes and Mafi that the physics of toroidal compactified models of extra dimensions can depend on the shape angle of the torus. Toroidal compactification has also recently been used as a regulator for numerical solutions of supersymmetric field theories in 2+1 dimensions. The question is: does the shape angle of the torus also affect the physics in this situation? Clearly a numerical solution should be independent of the shape of the space on which we compactify, at least when the regulator is removed. We show that, for standard discrete light-cone quantization with transverse parity invariance, toroidal compactification is only allowed for a specific set of shape angles and for that set of shape angles the numerical solutions are unchanged.
1 Introduction

Using a method we call Supersymmetric Discrete Light-Cone Quantization (SDLCQ) we have been able to solve a number of supersymmetric theories in 2+1 dimensions [1, 2, 3]. This method, which is an extension of DLCQ [4], exactly preserves supersymmetry and requires no renormalization in 2 + 1 dimensions.

DLCQ is a numerical method for solving quantum field theory that is actually the combination of two very well known ideas. The first idea, light-cone (LC) quantization, originally proposed by Dirac in 1949 [5], points out that one can evolve a system with operators other than \( P^0 \). When the system is evolved with the operator \( P^- = (P^0 - P^1)/\sqrt{2} \) this leads, when quantized, to LC quantization. In LC quantization one replaces \((x^0, x^1, x^\perp)\) by \((x^+, x^-, x^\perp)\) where \(x^\pm = (x^0 \pm x^1)/\sqrt{2} \). The metric is implicitly defined by \(x^\pm = x_\pm^* \) and \(x^\perp = -x^\perp \). In general \(x_\perp\) can have any number of components, but here we will be considering only one transverse coordinate \(x_2\). In this system \(x_-\) is the LC time, and \(P^-\) is the LC Hamiltonian. In DLCQ one regulates the system by putting it in a LC spacial box with boundary conditions on \(x_\perp\) and \(x^+\), which gives rise to discrete momentum modes in \(P^+\) and \(P^\perp\). The modes are formulated in Fock space. Truncation then turns the quantum field theory into a finite-dimensional numerical problem. A detailed review of DLCQ can be found in [6].

In the context of extra-dimensional physics, Dienes and Mafi [7, 8] considered compactification on a generalized torus, shown in Fig. 1, which contains a shape angle \(\theta\). Dienes and Mafi showed that the properties of the Kaluza–Klein particle in an extra-dimensional field theory depend on this shape angle [7, 8].

This naturally leads to the question: when we formulate a (2+1)-dimensional supersymmetric field theory on a torus and solve it using DLCQ, will the physics depend on the shape angle \(\theta\) as well? The difference between a truly extra-dimensional theory and DLCQ is that the cylinder in DLCQ is introduced as a regulator for the field theory rather than as a fundamental part of the theory. One could think of the compactification in \(x_2\) as a model for a true extra dimension, but we will not consider that here. If we were to find that the results depend on the shape angle, it would surely put in question this method of regulating (2+1)-dimensional DLCQ theories.

The torus with shape angle \(\theta\) is shown in Fig. 1. The periodicities of the torus take the form

\[
\begin{align*}
\{ & x_+ \rightarrow x_+ + 2\pi R_+ \\
& x_2 \rightarrow x_2 , \\
& x_\perp \rightarrow x_\perp + 2\pi R_2 \cos\theta \\
& x_2 \rightarrow x_2 + 2\pi R_2 \sin\theta .
\end{align*}
\]

(1)

The conventional or rectangular torus corresponds to \(\theta = \pi/2\). In discussing the generalized torus it is conventional to introduce the complex quantity \(\tau\),

\[
\tau \equiv \frac{R_2}{R_+} e^{i\theta} = \tau_1 + i\tau_2 = \frac{R_2}{R_+} \cos\theta + i \frac{R_2}{R_+} \sin\theta .
\]

(2)
It is also conventional within this context to normalize the scale by taking the side of the torus along the horizontal axis to be of length one, i.e. $2\pi R_+ = 1$. In this form the torus is represented by $\tau$ in the complex $\tau$ plane. It can be shown that the torus has an invariance, generally called a modular invariance [9]. One of these modular transformations, which will play a key role here, is $\tau_1' = \tau_1 + 1$.

The periodic functions that replace the simple exponential are

$$\exp \left( i \frac{n_+}{R_+} \left[ x_+ - \frac{x_2}{\tan \theta} \right] + i \frac{n_2}{R_2} x_2 \sin \theta \right) . \quad (3)$$

For the case where $n_+$ and $n_2$ are (odd half) integers, the expression in Eq. (3) is a (anti-)periodic function. We will focus on periodic boundary conditions because those are required in SDLCQ, but the conclusions are also valid for anti-periodic boundary conditions.

In Section 2 we will carefully define the standard DLCQ formulation of a free scalar field theory in 2+1 dimensions. We will then ask if this theory with the same cutoffs can be defined on the torus with a shape angle. We will find that the theory can only be defined on a subset of tori. That is, only some shape angles are allowed. We will then show that for this subset of allowed shape angles the physics is unchanged. In Section 3 we will show that this result carries over to the SDLCQ formulation of supersymmetric Yang–Mills (SYM) theory in 2+1 dimensions.

## 2 DLCQ

We start by considering the DLCQ formulation of the theory for a free massive boson in 2 + 1 dimensions. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 . \quad (4)$$

The LC Hamiltonian for this theory is

$$P^- = \int_{-\infty}^{\infty} dx_+ dx_2 \left( \frac{1}{2} \partial_2 \phi \partial^2 \phi + \frac{1}{2} m^2 \phi^2 \right) . \quad (5)$$
By the phrase “standard DLCQ” we mean precisely this set of interactions, with all of its symmetries, and a discrete set of longitudinal momentum modes, with a longitudinal resolution $K$ and cutoffs that respect the symmetries.

We now want to quantize this theory on a torus with a shape angle but otherwise follow this standard DLCQ procedure. It is straightforward to quantize this theory using the periodic functions in Eq. 3. We define creation and annihilation operators that satisfy the standard commutation relations

$$ [A(n_+, n_2), A^\dagger(m_+, m_2)] = \delta_{n_+, m_+} \delta_{n_2, m_2}. $$

In terms of these operators the second-quantized field takes the form

$$ \phi(x) = \frac{1}{2\pi \sqrt{R_2 \sin \theta}} \sum_{n_+ = 0}^{\infty} \sum_{n_2 = -\infty}^{\infty} \left[ e^{-i \left( \frac{n_+}{R_+} Z_+ + \frac{n_2}{R_2} Z_2 \right)} A(n_+, n_2) + e^{i \left( \frac{n_+}{R_+} Z_+ + \frac{n_2}{R_2} Z_2 \right)} A^\dagger(n_+, n_2) \right], $$

where

$$ Z_2 = \frac{x_2}{\sin \theta}, $$

$$ Z_+ = x_+ - \frac{x_2}{\tan \theta}. $$

We define the Hamiltonian in momentum space by integrating over the torus. To actually do the integrals it is convenient to change to the variables $Z_+ \text{ and } Z_2$, because

$$ \int_\text{torus} dx_+ dx_2 = \sin \theta \int_0^{2\pi R_+} dZ_+ \int_0^{2\pi R_2} dZ_2. $$

The Hamiltonian then takes the form

$$ P^- = \sum_{n_+ = 1}^{\infty} \sum_{n_2 = -\infty}^{\infty} \frac{R_+}{2n_+} \left[ m_+^2 + \frac{n_2^2}{R_2^2 \sin \theta^2} \left( 1 - \frac{n_+}{n_2} \right) \right] A^\dagger(n_+, n_2) A(n_+, n_2). $$

Following the standard DLCQ procedure, we will now define the Fock basis on a torus with a shape angle. In standard DLCQ we use transverse boost invariance to work in the frame where the total transverse momentum is zero. We will refer to this frame as the total transverse momentum center of momentum (TMCM) frame. In standard DLCQ the total longitudinal momentum $P^+$ is given by $P^+ = \frac{K}{R_+}$, the $i$-th particle has a longitudinal momentum $n_{(i)+}/R_+$, and the sum of the integers $n_{(i)+}$ is just $K$. We follow the same procedure on the torus with a shape angle. The single-particle Fock state is

$$ |\psi_1\rangle = A^\dagger(n_+, n_2) |0\rangle. $$

For this state we have $n_+ = K$ and a total transverse momentum

$$ P_2 = \frac{1}{R_2 \sin \theta} \left( n_2 - n_+ \frac{R_2}{R_+} \cos \theta \right). $$
In the TMCM frame this transverse momentum is zero, and therefore

\[ n_2 = K \tau_1 . \] (13)

It will be convenient to define the integer \( n \equiv K \tau_1 \). Then the above equation can be simply written \( n_2 = n \), and we conclude that \( \tau_1 = \frac{n}{K} \) must be a rational number. This is the first restriction we find on the allowed tori. Applying the Hamiltonian to this state we find

\[ 2 \frac{K}{R^+} P^- |\psi_1\rangle = m^2 |\psi_1\rangle , \] (14)

as expected. The physics of this one-particle state is unchanged on a torus with a shape angle, provided that \( \tau_1 \) is a rational number.

Now let us consider a two-particle Fock state. A general state with longitudinal resolution \( K \) has the form

\[ |\psi_2\rangle = \sum_{n, n_2, n_2'} f_2(n_+, n_2, n_2') A^\dagger(n_+, n_2) A^\dagger(K - n_+, n_2') |0\rangle . \] (15)

As we will see, the sums on \( n_2 \) and \( n_2' \) are not independent. The transverse momentum of the two particles in this state are

\[ k_2 = \left( \frac{n_2}{R^2 \sin \theta} - \frac{n_+}{R^+ \tan \theta} \right) , \] (16)

\[ k_2' = \left( \frac{n_2'}{R^2 \sin \theta} - \frac{K - n_+}{R^+ \tan \theta} \right) . \] (17)

In the TMCM frame the sum \( k_2 + k_2' \) is zero, and we find

\[ n_2 + n_2' = K \frac{R_2}{R^+} \cos \theta = K \tau_1 = n . \] (18)

Again the restriction that \( \tau_1 \) must be a rational number appears. It is straightforward to generalize this to higher Fock states, and we find that \( \sum_i n_{(i)2} = n \). The general form of the two-particle state is now

\[ |\psi_2\rangle = \sum_{n_+ = 1}^{K-1} \sum_{n_2} f_2(n_+, n_2) A^\dagger(n_+, n_2) A^\dagger(K - n_+, n - n_2) |0\rangle . \] (19)

This theory is invariant with respect to transverse parity, \( k_2 \rightarrow -k_2 \). It is easy to see that in terms of \( n_2 \) this transformation is \( n_2 \rightarrow -n_2 + n_+ 2 \tau_1 \). It can also be shown that this transformation works for all higher states as well. For this to be a symmetry of the discrete theory, we conclude that \( 2n_+ \tau_1 \) must be an integer.

Also in DLCQ we must truncate the sum on \( n_2 \) to make the problem numerically solvable. We impose the conventional DLCQ cutoff, a symmetric cutoff in \( k_2 \), to preserve transverse parity [1, 2, 3], which can be imposed independently of the shape
The creation operators with \( p_k \) translate into upper and lower cutoffs on \( k_2 \), and for these to be symmetric we must have

\[
T_u = T_l + 2n_+ \tau_1 .
\] (20)

Again we find that \( 2n_+ \tau_1 \) must be an integer.

This condition on \( 2n_+ \tau_1 \) leaves us with two allowed cases to consider, that \( \tau_1 \) is an integer or a half integer. Using the modular transformation \( \tau'_1 = \tau_1 + 1 \), we see that all tori are equivalent to tori with \( -1/2 \leq \tau_1 \leq 1/2 \). Therefore the case where \( \tau_1 \) is equal to an integer is equivalent to \( \tau_1 = 0 \), and the case where \( \tau_1 \) is equal to a half integer is equivalent \( \tau_1 = 1/2 \).

Now, since \( \tau_1 = n/K \), we find that \( \tau_1 = 1/2 \) implies \( K = n/\tau_1 = 2n \), where \( n \) is an integer since \( n_2 + n'_2 = n \). We conclude that if \( \tau_1 = 1/2 \) we cannot formulate a two-particle state for all integer values of \( K \). It is unacceptable to forbid some basis states at some resolutions; therefore, we conclude that we cannot formulate standard DLCQ on a torus with \( \tau_1 = 1/2 \). Thus we find that it is only possible to form two-particle basis states on a torus with a shape angle if \( \tau_1 \) is an integer. This is equivalent by modular invariance to \( \tau_1 = 0 \), which is standard DLCQ without a shape angle. We conclude that for the allowed torus with a shape angle the physics is equivalent to standard DLCQ.

It is interesting to take an explicit look at modular invariance and show that the free energy of the two-particle state with \( \tau_1 = 1 \) is equivalent to the case \( \tau_1 = 0 \). The free energy of a general two-particle state is obtained by applying the Hamiltonian to \( |\psi_2\rangle \). We find

\[
P^-|\psi_2\rangle = \sum_{n=1}^{K-1} \sum_{n_2=-T_l}^{T} \frac{R_+}{2} \left( \frac{1}{n_+} + \frac{1}{K-n_+} \right) \left( m^2 + \frac{1}{R_2^2 \sin^2 \theta^2} (n_2 - n_+ \tau_1)^2 \right) \times f_2(n_+, n_2)A^\dagger(n_+, n_2)A^\dagger(K-n_+, n-n_2)|0\rangle .
\] (21)

Now, if we take \( \tau_1 = 1 \) and therefore \( n = K \), we find from Eq. (17) that

\[
k_2 = \frac{1}{R_2 \sin \theta} (n_2 - n_+) , \quad k'_2 = -\frac{1}{R_2 \sin \theta} (n_2 - n_+).
\] (22)

We next make a change of variables to \( p_2 \equiv n_2 - n_+ \). It is appropriate to also relabel the creation operators with \( p_2 \) and \( -p_2 \) according to Eq. (22). We define a new integer, \( T \equiv T_l + n_+ \), to be used in the limits of the transverse sum. We then obtain

\[
P^-|\psi_2\rangle = \sum_{n=1}^{K-1} \sum_{p_2=-T}^{T} \frac{R_+}{2} \left( \frac{1}{n_+} + \frac{1}{K-n_+} \right) \left( m^2 + \frac{1}{R_2^2 \sin^2 \theta^2 p_2^2} \right) \times f_2(n_+, p_2)A^\dagger(n_+, p_2)A^\dagger(K-n_+, -p_2)|0\rangle .
\] (23)

After rescaling \( R_2 \) by \( \sin \theta \), we find as expected the standard DLCQ result for the free energy of a two-particle system. This argument can be extended to systems with higher numbers of free particles.
3 SDLCQ

Now let us consider the interacting theory $\mathcal{N} = 1$, SYM theory in 2+1 dimensions. This is a theory we that have solved on a rectangular torus using SDLCQ [1, 2, 3]. SDLCQ is a numerical method that exactly preserves the supersymmetry and therefore renders this theory totally finite. The only real difference between DLCQ and SDLCQ is that the supercharge $Q^-$ is constructed in the Fock basis and the Hamiltonian is constructed by squaring the supercharge. The Fock basis is the same as DLCQ, and the arguments in the previous section follow essentially unchanged.

The supercharge for $\mathcal{N} = 1$ SYM theory has a rather simple form. It is

$$Q^- = Q^\parallel + \sum_{n_2} \frac{n_2}{R_2} Q_2^- .$$

The transverse momentum explicitly appears in only one location. When we quantize the theory on the torus with a shape angle, we find

$$Q^- = Q^\parallel + \sum_{n_2} \frac{(n_2 - n \tau_1)}{R_2 \sin \theta} Q_2^- .$$

The operators $Q^\parallel$ and $Q_2^-$ have the same form as on the rectangular lattice, except that they are written in term of the Fock operators of the lattice with the shape angle.

As in the free case, a symmetric cutoff on the transverse momentum in the TMCM frame requires that $\tau_1$ be an integer. From modular invariance this is of course equivalent to the rectangular torus of standard SDLCQ. We can explicitly demonstrate this equivalence by shifting the transverse sum and rescaling $R_2$, exactly as we did in the free DLCQ case, to reproduce the quantized supercharge found for the rectangular torus.

4 Summary

We considered standard DLCQ for a free scalar theory and standard SDLCQ for $\mathcal{N} = 1$ SYM theory, in 2+1 dimensions, compactified on a rectangular torus and on a torus with a shape angle $\theta$. The “standard” definition uniquely defines these cutoff theories so that in the comparison we can be assured that we are only looking at the effect of the shape angle of the torus. We find that it is only possible to formulate these theories with transverse parity as a symmetry on tori with $\tau_1$ equal to an integer. Modular invariance then shows that tori with integer $\tau_1$ are equivalent to the rectangular torus, and therefore the physics of standard DLCQ and SDLCQ are unchanged on the allowed tori with a shape angle.

A possible direction for future work would be to consider the implications of longitudinal parity[10] on this problem.
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