Tensor network trial states for chiral topological phases in two dimensions

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Trial wavefunctions that can be represented by summing over locally-coupled degrees of freedom are called tensor network states; they have seemed difficult to construct for two-dimensional topological phases that possess protected gapless edge excitations. We show it can be done for chiral states of free fermions, using a Gaussian Grassmann integral, yielding $p_x \pm ip_y$ and Chern insulator states. We show that any strictly short-range quadratic parent Hamiltonian for these states is gapless. Further examples are analogs of fractional (including non-Abelian) quantum Hall phases.

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Our theoretical understanding of quantum phases of matter frequently relies on the use of trial ground state wavefunctions whose properties serve as the paradigm for an entire phase, such as the Bardeen-Cooper-Schrieffer (BCS) wavefunction in the theory of superconductivity [1], and the Laughlin wavefunction in the fractional quantum Hall effect [2]; both examples have generalizations that describe distinct phases. In recent work, there has been progress in understanding the structure of ground states of generic short-range lattice Hamiltonians, especially for “topological” phases in which there is a gap in the bulk energy spectrum above the ground state energy. For many ground states, a representation as a tensor network state (TNS), in which the amplitude of a basis configuration of the local degrees of freedom is expressed as a product of tensors that involve auxiliary degrees of freedom which are then summed over, can be found. In our definition of a TNS, we further impose that the auxiliary degrees of freedom live in finite-dimensional local spaces, with only short-range couplings; when they live on the links of a lattice and couple only to the physical degrees of freedom on the end of that link, the state is known as a projected entangled pair state (PEPS) [3], or in one-dimensional space as a matrix product state (MPS) [4].

In one dimension it is known that MPSs provide enough variational freedom to approximate the ground state in any topological phase [5]. In more than one dimension, it is known that many trial states and exact ground states of special lattice Hamiltonians that lie in a topological phase that does not exhibit topologically-protected (e.g. chiral) gapless edge excitations can be viewed as TNSs, and it seems natural to expect the approximation results from one dimension to generalize. But for chiral topological phases in two dimensions, such as quantum Hall states and $p \pm ip$ BCS paired states, it has seemed difficult to obtain a TNS in the phase, even as a trial state. Most attempts did not impose locality [6], while Béri and Cooper [7], by truncating a flat-band Hamiltonian, obtained some local tensor networks that approximate expectation values in certain trial states. However, such a procedure does not in general produce a TNS, that is, a definite pure quantum state.

In this Letter we exhibit some fairly simple trial TNSs that belong to chiral topological phases in two dimensions. We begin by constructing explicit examples of translationally-invariant Gaussian (i.e. free-) fermionic TNSs, in each of which the TNS is defined locally by a few tensors. The basic examples are TNS versions of a topologically-nontrivial $p \pm ip$ BCS paired state of fermions [8], and of a filled band with Chern number equal to one [9]. In each of these, there is a Hamiltonian with the TNS as ground state and a gapped but dispersionless (“flat-band”) fermion excitation-energy spectrum; these Hamiltonians have power-law-decaying matrix elements in position space [10]. Moreover, a consequence of the ground state being a free-fermion TNS is that there are single-fermion destruction operators that annihilate the TNS, and are strictly short-range, in the sense that their support is compact (i.e. a bounded region). Using these, there are many ways to construct a “parent” Hamiltonian that has uniformly bounded support for its terms, each of which annihilates the TNS; we can show that, in the examples, such Hamiltonians always have a gapless bulk energy dispersion relation. The constructions used in these examples generalize to other free-fermion phases, including other symmetry classes, and to higher dimensions. For a one-dimensional lattice, they yield the matrix-product ground state of the Kitaev chain [11]. The gaplessness result does not apply to all these cases.

Back in two dimensions, we go further by utilizing the free-fermion TNSs and (similarly to earlier authors [6,7]) imposing local constraints on a system of several copies of a TNS, producing further TNSs. We argue that such constructions produce the chiral topological phases associated with a variety of Chern-Simons theories [12] or fractional quantum Hall states, including non-Abelian topological phases [13]. We do not at present have parent Hamiltonians for these TNSs, though it is likely these exist based on general principles of TNSs.

Gaussian fermionic TNS—the square lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ is generated by the two vectors $i = (1,0)$ and $j = (0,1)$. Our physical degrees of freedom are fermions, with $n$
orbits per site; the creation/annihilation modes obey the canonical anti-commutation relations \{c_{x,\nu}^\dagger, c_{x',\alpha'}\} = \delta_{\alpha,\alpha'} \delta_{x, x'}.

The fermion vacuum \(|0\rangle\) is annihilated by all the \(c_{x,\alpha}\). A class of translation-invariant Gaussian TNSs is constructed as follows: to every site \(x\), we associate a set of (real) Grassmann variables \(\xi_p^x, p \in \{1, \ldots, P\}\). To every edge \((x' x)\) (\(x' \in \{x+1, x+3\}\)), we associate the expression

\[
e^{-\sum p,q \xi_{p+1}^x A_{pq}^h \xi_q^x} \text{ or } \sum \xi_{p+1}^x A_{pq}^v \xi_q^x, (1)
\]

with \(A_{pq}^h, A_{pq}^v \in \mathbb{C}\). The superscripts \(h, v\) stand for “horizontal” and “vertical”; in what follows, we will leave these superscripts implicit. To every site \(x\), we associate a weight and a generating function of physical particles onsite:

\[
e^{-\sum p,q \xi_{p+1}^x B_{pq} \xi_q^x} \text{ or } \sum \xi_{p+1}^x B_{pq} \xi_q^x, (2)
\]

with \(B_{pq} = -B_{qp} \in \mathbb{C}\) and \(\kappa_q \in \mathbb{C}\). Grassmann variables anticommute with the physical creation/annihilation operators; thus, the two exponentials in (1) and those in (2) all commute, both onsite and at different sites. After taking the product, we integrate out the Grassmann variables to obtain a translation-invariant Gaussian TNS:

\[
|\psi\rangle \propto \int [d\xi] \prod_{\text{edges } (xy)} e^{-\sum x \xi_{x+1}^x A^x \xi_x} \prod_{\text{sites } x} e^{-\sum x \xi_{x+1}^x B \cdot \xi_x} |0\rangle.
\]

(3)

Here we have used the compact notation \(\xi_x = (\xi_1^x, \ldots, \xi_P^x)\), \(c_{x,\alpha}^\dagger = (c_{x,1,\alpha}, \ldots, c_{x,n,\alpha})\), and matrices \(A, B, \kappa\). In [3], some ordering of the Grassmann variables must be chosen to define the “measure” \(\int [d\xi]\). By construction, the state \(|\psi\rangle\) is always a free-fermion BCS paired state. It is possible to write the state in a form that is closer to the usual form of TNS [2], with the Grassmann variables living on the edges rather than on the sites [14].

Example in \(p_x - ip_y\) phase—our first example has \(n = 1\) orbital per site; we use \(P = 2\) Grassmann variables on each site. The \(A, B, \kappa\) matrices are:

\[
\xi_{x+1}^\dagger \cdot A^x \cdot \xi_x = \left(\begin{array}{c} \xi_{x+1}^1 \\ \xi_{x+1}^2 \end{array}\right) \left(\begin{array}{cc} -i & \lambda \\ -\lambda & -i \end{array}\right) \left(\begin{array}{c} \xi_x^1 \\ \xi_x^2 \end{array}\right), (4a)
\]

\[
\xi_{x+1}^\dagger \cdot B^x \cdot \xi_x = \left(\begin{array}{c} \xi_{x+1}^1 \\ \xi_{x+1}^2 \end{array}\right) \left(\begin{array}{cc} 1 & \lambda \\ -\lambda & 1 \end{array}\right) \left(\begin{array}{c} \xi_x^1 \\ \xi_x^2 \end{array}\right), (4b)
\]

\[
\xi_{x}^\dagger \cdot B \cdot \xi_x = \left(\begin{array}{c} \xi_x^1 \\ \xi_x^2 \end{array}\right) \left(\begin{array}{cc} 0 & -2\lambda \\ 2\lambda & 0 \end{array}\right) \left(\begin{array}{c} \xi_x^1 \\ \xi_x^2 \end{array}\right), (4c)
\]

\[
\xi_{x}^\dagger \cdot \kappa \cdot c_{x}^\dagger = \left(\begin{array}{c} \xi_x^1 \\ \xi_x^2 \end{array}\right) \left(\begin{array}{c} \kappa_1 \\ 0 \end{array}\right) \left(\begin{array}{c} c_{x,1}^\dagger \\ c_{x,2}^\dagger \end{array}\right), (4d)
\]

\(\lambda \in \mathbb{R}\) and \(\kappa_1 \in \mathbb{C}\) are two variational parameters.

With these matrices, [3] gives a state \(|\psi_D\rangle\); its behavior is easily analyzed in momentum space. The Fourier modes of the particle creation operator are defined by \(c_{x,\alpha}^\dagger = \int \frac{d^2k}{(2\pi)^2} e^{-ikx} c_{x,\alpha}^\dagger\), where the integral is over the first Brillouin zone \([-\pi, \pi]^2\).

Similarly, for the Grassmann variables \(\xi_x = \int \frac{d^2k}{(2\pi)^2} e^{-ik\cdot x} \xi_{k,x}\) and \(e^S = \prod_{(xy)} e^{\xi_{k,x}^\dagger A-\xi_{k,x}} \prod_{k} e^{\xi_{k,x}^\dagger B \cdot \xi_{k,x}}\) in [4] become respectively:

\[
\exp\left(\int \frac{d^2k}{(2\pi)^2} \xi_{k,x}^\dagger \kappa \cdot c_{k}^\dagger\right), (5a)
\]

\[
\exp\left[\int \frac{d^2k}{(2\pi)^2} \left(\xi_{k,x}^\dagger \xi^2_{-k,x} - \kappa \cdot c_{k}^\dagger c_{-k}^\dagger \right) S_k \left(\xi_{k,x}^\dagger \xi_{-k,x}^\dagger\right)\right], (5b)
\]

where \(S_k\) is the \(2 \times 2\) matrix

\[
\begin{pmatrix}
\sin k_x + i \sin k_y & -\lambda (2 - \cos k_x - \cos k_y) \\
\lambda (2 - \cos k_x - \cos k_y) & \sin k_x - i \sin k_y
\end{pmatrix}.
\]

(6)

The integral over all the Fourier modes of the Grassmann variables is easily performed. It yields the familiar BCS form

\[
|\psi_D\rangle \propto \exp\left(\frac{1}{2} \int \frac{d^2k}{(2\pi)^2} g_k c_{k,x}^\dagger c_{-k,x}^\dagger \right) |0\rangle, (7)
\]

with a pairing function \(g_k\) which is the \((1, 1)\) matrix element of the inverse matrix of \(S_k\), and so is a component of the propagator of the Grassmann variables:

\[
g_k \langle k_1 | c_{k,x}^\dagger | k_1 \rangle = \int [dk] e^{S_{k_0,k_1} c_{k,x}^\dagger c_{-k,x}^\dagger} = [S_{k_1}^{-1}]_{11}. (8)
\]

Explicitly,

\[
g_k \langle k_1 | c_{k,x}^\dagger | k_1 \rangle = \frac{\sin k_x - i \sin k_y}{(\sin k_x)^2 + (\sin k_y)^2 + \lambda^2 [2 - \cos k_x - \cos k_y]^2},
\]

(9)

and \(g_{-k} = -g_k\) for all \(k\). As \(k \to 0\), \(g_k\) diverges as \(g_k \sim k_1^2/(k_0^2 + ik_0)\), and is non-diverging at other \(k\). Hence, in position space, \(g(x) \sim k_1^2/(x + iy)\) as \(|x| \to \infty\), where \(x = x_i - x_j\) represents the separation of the members of \(i, j\) of a pair. These properties are sufficient to show the state is in the non-trivial \(p - ip\) phase [5] in symmetry class D; they can also be related to a Chern number in \(k\)-space (see below) [14]. One can show that the average density of particles scales as \(\sim |k_1|^2 \ln(1/|k_1|)\) as \(k_1 \to 0\).

Annihilation operators and parent Hamiltonians—A state of the Gaussian (or BCS) form in eq. (7) is annihilated by (unnormalized) “destruction” mode operators \(c_{k} - g_k c_{-k}^\dagger\) for all \(k\). In our example, \(g_k = v_k/\cdot \omega_k\) is clearly a ratio of two “trigonometric polynomials” (polynomials in \(\sin k_x, \cos k_x, \sin k_y, \cos k_y\)) \(u_k\) and \(v_k\), which appear to have no common trigonometric-polynomial factor other than a constant times an integer power of \(e^{ikx}\) and another of \(e^{iky}\). Then we define

\[
d_k = u_k c_{k} - v_k c_{-k}^\dagger.
\]

(10)

If we normalize the destruction operators (10) as

\[
d_k = \hat{u}_k c_{k} - \hat{v}_k c_{-k}^\dagger,
\]

(11)
where \( \tilde{u}_k = u_k / \sqrt{|u_k|^2 + |v_k|^2} \), \( \tilde{v}_k = v_k / \sqrt{|u_k|^2 + |v_k|^2} \),
then \( \tilde{a}_k, \tilde{d}_k \) obey canonical anticommutation relations,
\( \{ \tilde{a}_k, \tilde{d}^\dagger_{k'} \} = (2\pi)^2 \delta(k - k') \). For the Hamiltonian
\[
\hat{H}_D = \int \frac{d^2 k}{(2\pi)^2} \tilde{d}^\dagger_k \tilde{a}_k,
\]
the fermion excitations \( \tilde{d}_k |\psi_D\rangle \) have nonzero and \( k \)-independent energy for all \( k \). Expanded in \( c_k, c_k^\dagger \),
the coefficients in \( \hat{H}_D \) are ratios of trigonometric polynomials,
and are not analytic in \( k_x, k_y \) at \( k = 0 \), but are elsewhere in the Brillouin zone. Hence in position space,
\( \hat{H}_D \) contains terms that decay as powers of distance \([10]\).

On the other hand, the operators \( d_k \) are Grassmann variables, that are trigonometric polynomials, and so the inverse Fourier transform gives operators \( d_k \) that annihilate the TNS and are truly local—they have compact support that surrounds \( x \). The existence of such operators is not an accident. Any TNS has by construction the property
that if the sites of the system are bipartitioned into two sets, \( A \) and \( B \), and \( A \) is finite, then the rank of the Schmidt decomposition of the TNS can be bounded by some constant to the power of the surface area or perimeter of the region \( A \). For free fermions, the reduced density matrix again has the form of the exponential of a free-fermion Hamiltonian, and so for a free-fermion TNS the number of fermion modes that can appear in the entanglement Hamiltonian is some constant times the surface area, and so much smaller than the volume of region \( A \) in general. It follows that for such regions \( A \), there must be linear combinations of fermion operators, supported in region \( A \), that annihilate the reduced density matrix, when acting on it from the left. The operators \( d_k \) with support in region \( A \) are a basis set for these operators in our case. We note that these operators anticommute with one another, but do not in general anticommute with the operators \( \tilde{d}_k \) at \( x' \neq x \), except when their supports are disjoint. By contrast, the \( \tilde{d}_k \) operators obey canonical anticommutation relations and are true Wannier functions, but are not supported locally near \( x \) instead they have long power-law tails due to non-analytic behavior in \( k \)-space at \( k = 0 \) \([13]\).

Using the operators \( d_k \), we can form other Hamiltonians that annihilate the TNS, for example:
\[
H_D = \int \frac{d^2 k}{(2\pi)^2} d^\dagger_k d_k = \sum_x d^\dagger_x d_x.
\]
This Hamiltonian is a sum of terms that have compact support, and each annihilates \( |\psi_D\rangle \), so it is a parent Hamiltonian, however from its \( k \)-space form we can see that it is gapless at \( k = 0 \) : by expressing it in terms of \( d_k, \tilde{d}_k \) we find that the energy of a fermion excitation is \( |u_k|^2 + |v_k|^2 \), which is \( \propto k^2 \) near \( k = 0 \).

Chern band example—The preceding construction can be generalized to include more orbitals, more singularities in \( g_k \), more dimensions, or more symmetry (e.g. time reversal); we describe one more example. We take two copies of the previous example, by using \( n = 2 \) orbitals per site; we view particles occupying either type of orbital as two distinct types of particles, that have opposite charges under a U(1) symmetry. In the TNS construction we use \( P = 4 \) Grassmann variables, or \( P = 2 \) complex Grassmann variables, with the matrices constructed to respect the U(1) charge mentioned. Then we can obtain \([13]\) a TNS that is a BCS state with pairing only between opposite particle types,
\[
|\psi_A\rangle \propto \exp \left( \int \frac{d^2 k}{(2\pi)^2} g_{k1}^1 c_k^\dagger c_{-k,1}^\dagger \right) |0\rangle,
\]
which conserves the U(1) symmetry; here \( g_{k1}^1 \) is the same as \( g_k \) above. If we now perform a particle-hole transformation on the type-1 fermions, so that \( c_{k,1}^\dagger \rightarrow c_{-k,1} \), and \( |0\rangle \rightarrow |1,0\rangle \) which is annihilated by all \( c_{k,1} \) and \( c_{k,2} \), we arrive at a particle-number conserving state \( |\tilde{\psi}_A\rangle \) that represents a filled band (symmetry class A). There are now two types of operator \( d_{k,1}, d_{k,2} \) that annihilate \( |\tilde{\psi}_A\rangle \); one of these is
\[
d_{-k,1} = u_k c_{k,1}^\dagger + v_k c_{k,2}^\dagger;
\]
which is a creation operator associated with states in the filled band. The state \( |1,0\rangle \) (i.e. for \( g_k = 0 \) for all \( k \)) describes topologically-trivial bands. For general \( g_k \) (which here does not have to be an odd function of \( k \)), the filled band is non-trivial when its Chern number is nonzero; the other band has opposite Chern number \([9]\). The Chern number can be obtained (up to a choice of sign convention) from the (generically isolated) points at which \( g_k \) diverges, as the sum of the winding numbers of each: the winding number can be defined as the winding of \( g_k/|g_k| \) as \( k \) traverses a small circle about the point of divergence in the counterclockwise direction. Thus in our case, the Chern number is 1. There are Hamiltonians \( H_A \) and \( \hat{H}_A \) with similar properties as \( H_D, \hat{H}_D \).

A no-go theorem—In states of either of the above two forms, we can let \( g_k = v_k/|u_k| \) be any ratio of trigonometric polynomials \( u_k, v_k \), and view these states morally as TNSs \([14]\). In order to obtain non-zero Chern number, there must be some points \( k \) at which both \( u_k \) and \( v_k \) vanish, because a zero of \( u_k \) at which \( u_k/|u_k| \) winds must have a counterpart with opposite sign of the winding, since the total winding around the boundary of the Brillouin zone must vanish; zeroes of \( v_k \) at some of these \( k \)'s are then required in order to remove some of these non-zero-winding poles in \( g_k \). (Alternatively, because a line bundle has a continuous global nowhere-zero section if and only if the Chern number is zero.) This implies that the energy dispersion \( |u_k|^2 + |v_k|^2 \) of the corresponding \( H_D \) or \( H_A \) must vanish somewhere. More generally, we can consider such a TNS with more orbitals per site and any Hamiltonian, quadratic in \( c_{k\alpha}, c_{k\alpha}^\dagger \), that is a
sum over translations of a single operator with compact support that annihilates the TNS; the existence of such Hamiltonians follows from the TNS form. We can prove that the energy spectrum of any such parent Hamiltonian is gapless whenever the filled bands form a complex vector bundle that is topologically non-trivial \cite{13}. This allows a gapped parent Hamiltonian to exist in one dimension, as for the Kitaev chain \cite{11}. It is not clear whether a similar statement still holds for a Chern band \cite{11}.

Non-free-fermion phases— In the wavefunction in the free Chern-band example, without the particle-hole transformation, the component that has \( N_+ \) particles of \( U(1) \) charge +1 (say, those in the 1 orbital), and \( N_- \) of charge −1 (say, those in the 2 orbital) has \( N_+ = N_- \), and is a determinant that when all particles are well-separated has the Cauchy form,

\[
\det \left( \frac{1}{z_i - w_j} \right) = \prod_{i<j}(z_i - z_j) \prod_{k<l}(w_k - w_l) \prod_{m,n}(z_m - w_n).
\]

where we use complex coordinates [e.g. \( z = x + iy \) in place of \( x, y \)] \( z_i, i = 1, \ldots, N_+ \) for charge +1, \( w_k, k = 1, \ldots, N_- \) for charge −1 particles. The norm-square of the state \( |\psi\rangle \) is thus the partition function of a Coulomb plasma, with fugacity \( |\kappa_1|^2 \) for either type of particle. We will assume \( |\kappa_1| \) is small so that particles are well-separated. Then it is known that this plasma is in a screening phase, which confirms the topological identification of the state.

Next we will take \( Q \) copies of this TNS, and impose the constraint that the number of particles in each orbital must be either 0 or \( Q \) (one from each copy) \cite{6,7}. Then the composite of \( Q \) fermions (with \( U(1) \) charge either +1 or −1) will be regarded as the physical particle of the state, and is a boson for \( Q \) even, fermion for \( Q \) odd. The resulting wavefunction for these particles is then the \( Q \)th power of that for the one-copy case. The important point is that such a “product” construction yields a state that is again a TNS, but not Gaussian/free-fermion for \( Q > 1 \).

The constraints are imposed locally and simply change the form of the tensors. The norm-square of the product wavefunction is (at long distances) again a plasma, but with exponent \( Q \) times larger. A renormalization-group (RG) analysis applies to these plasmas \cite{16}, and shows that for asymptotically small positive fugacity, screening occurs for \( Q \leq 2 \).

For \( Q = 2 \), the screening length is exponentially large as \( |\kappa_1| \to 0 \). In this case, the TNS is in the same topological phase as the Laughlin \( \nu = 1/2 \) state for bosons, referred to as the semion, or \( SU(2) \) level 1, phase (the wavefunctions look more similar if we again apply the particle-hole transformation on the 1 orbitals). This can be understood, as the fields in two-dimensional conformal field theory that represent the bosons \cite{13} correspond to the \( S^+ \) and \( S^- \) currents of \( SU(2) \) current algebra, with conformal weight 1. The operator product of these generates also the \( S^\pm \) or \( U(1) \) current. The state possesses fractional-statistics vortex excitations (semions) that carry fractional \( U(1) \) charge, and there are two ground states for the system on a finite torus with periodic boundary conditions. For \( Q > 2 \), the plasma does not screen at small \( |\kappa_1| \), but does at sufficiently large \( |\kappa_1| \), and then corresponds to the \( 1/Q \) Laughlin state.

This construction can be extended in various ways. One is to take now \( k \) copies of the previous \( Q = 2 \) state, and this time use the “sum” construction: physical particles can be any one of the \( k \) different types, so that the creation operator for physical particles is the sum of those in each copy \cite{6,7}. At long distance at small \( |\kappa_1| \), the sums of current operators are \( SU(2) \) level \( k \) currents. The resulting system should again screen (the RG flows are the same at leading order), and the TNS is in the same phase as the \( SU(2) \) level \( k \) Chern-Simons theory \cite{12}, or Read-Rezayi state for bosons \cite{17,18}. These phases have non-Abelian quasiparticles. Further variations include taking \( N \) copies of the free-fermion states, constraining so that a physical particle is represented by a pair of fermion on the same site, and using the sum construction over \( k \) copies of this. In this fashion we can obtain \( SO(N) \) or \( SU(N) \) level \( k \) chiral topological phases, with manifest \( SO(N) \) or \( SU(N) \) symmetry; these are always non-Abelian for \( N \) and \( k \) sufficiently large.

For these TNSs, we do not at present have parent Hamiltonians. There are many local operators that annihilate a given TNS, so it seems likely that a parent Hamiltonian for the TNS should exist; vortex or quasi-particle excitations will have non-zero finite energy. On a torus, such a Hamiltonian will have the number of \textit{exactly} degenerate ground states expected for the topological phase.

Conclusion—it is possible to construct wavefunctions that are TNSs for phases of matter that possess chiral gapless edge modes. This may have important implications for numerical simulation or theoretical analysis of such phases. The parent Hamiltonians for the free-fermion TNSs are gapless for topological reasons; it is not clear if the same will be true for the non-free-fermion chiral TNSs.

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### SUPPLEMENTARY MATERIAL

Transforming a TNS with auxiliary variables on sites to a TNS with auxiliary variables on edges

![Diagram](attachment:image.png)

**FIG. 1.** Particles on the square lattice $\mathbb{Z}^2$ (in green) are created by operators $c_{x,\alpha}$ acting on the vacuum $|0\rangle$, generating a Hilbert (Fock) space of physical states for each site, $\mathcal{H}_x$. One associates each tensor $T_x \in \mathcal{H}_x \otimes V^h \otimes (V^h)^* \otimes V^v \otimes (V^v)^*$ per site $x$. The spaces $V^h, V^v$ are finite-dimensional vector spaces, and the two of these and their duals can be obtained from a general Gaussian TNS; and a straightforward proof of the no-go theorem.

The Gaussian TNS we construct in the main text, with auxiliary variables living on the sites, as in the right figure; it can however be reformulated as a usual TNS.

The construction parallels the one for bosons, but $\mathcal{H}_x, V^h$ and $V^v$ must be...
$\mathbb{Z}_2$-graded vector spaces, while $(V^h)^*$ and $(V^v)^*$ are their $\mathbb{Z}_2$-graded duals, and the tensor product $\otimes$ is $\mathbb{Z}_2$-graded (see appendix A in [19] for definitions). It is customary to represent $\mathbb{Z}_2$-graded vector spaces as spaces of polynomials of Grassmann variables. Indeed, one can view $V^h$ as (isomorphic to) the $\mathbb{Z}_2$-graded vector space generated by a set of $P$ Grassmann variables \( \eta_{x+1/2} \) on the edge at position \( x+1/2 \):

$$\text{span}\{ (\eta^{x+1/2})^{n_1} \cdots (\eta^{x+1/2})^{n_P} \mid n_1, \ldots, n_P = 0, 1 \}. \quad (18)$$

The $\mathbb{Z}_2$-grading is then the number (mod 2) of Grassmann variables in each Grassmann monomial. Similarly, one can think of $V^v$ as being (isomorphic to) the $\mathbb{Z}_2$-graded vector space generated by $P$ Grassmann variables on the edge at position $x+j/2$. The (left) dual $V^h$ is then generated by another set of $P$ Grassmann variables:

$$\text{span}\{ (\eta^{x+j/2})^{n_1} \cdots (\eta^{x+j/2})^{n_P} \mid n_1, \ldots, n_P = 0, 1 \}. \quad (19)$$

With these notations, the canonical evaluation map is nothing but the Berezinski integral:

$$v^* \otimes u \in V^* \otimes V \mapsto \int [d\eta^1 d\eta^2 \cdots d\eta^P d\eta^P] e^{\sum_r \eta^r v^r v^r} v^* u. \quad (20)$$

The tensor $T_x$ must have degree 0, namely it must be a sum of terms with an even total number of Grassmann variables and physical fermions. Finally, the physical state, which is a state in $\bigotimes_{x \in \mathbb{Z}} \mathcal{H}_x$, is defined as $\text{Tr} [\bigotimes_x T_x]$, where one traces over all the auxiliary spaces $V^h,v$ using the canonical evaluation map (see also Fig. 1 left).

The Gaussian TNS that we exhibit in the main text have not been expressed in the form of local tensors attached to the sites $x \in \mathbb{Z}^2$. Instead, they are translation-invariant Gaussian states of the form

$$\int d\mathbf{k} \prod_{\text{edges } (xy)} e^{x \cdot A \cdot \xi_x} \prod_{\text{sites } z} e^{x \cdot B \cdot \xi_z} e^{x \cdot \kappa \cdot c^*_z} |0\rangle \quad (21)$$

where $\xi_x = (\xi_x^1, \ldots, \xi_x^P), A^{h,v}, B$ and $\kappa$ are $P \times P$ matrices. To recast this expression in the form of a $\mathbb{Z}_2$-graded TNS, we introduce new Grassmann variables $\eta_{x+i/2} = (\eta_{x+i/2}^1, \ldots, \eta_{x+i/2}^P), \eta_{x+j/2} = (\eta_{x+j/2}^1, \ldots, \eta_{x+j/2}^P), \eta_{x+1/2} = (\eta_{x+1/2}^1, \ldots, \eta_{x+1/2}^P), \eta_{x-1/2} = (\eta_{x-1/2}^1, \ldots, \eta_{x-1/2}^P)$, and

$$W_x = \xi_x^1 \cdot A^h \cdot \eta_{x+i/2} + \xi_x^1 \cdot A^v \cdot \eta_{x+j/2} + \xi_x^1 \cdot B \cdot \xi_x - \xi_x^1 \cdot \eta_{x-1/2} + \xi_x^1 \cdot \eta_{x-1/2}, \quad (22)$$

such that the integration over the $\eta$-variables on each edge gives back the exponential weights in (21). But, instead of tracing out the $\eta$-variables, we now integrate out the onsite variables $\xi_x$. This gives a $\mathbb{Z}_2$-graded tensor for each site, as we want:

$$T_x = \int d\mathbf{k} e^{W_x e^{x \cdot \kappa \cdot c^*_z}} \in \mathcal{H}_x \otimes (V^h)^* \otimes V^v \otimes (V^v)^*. \quad (23)$$

### Entanglement spectrum

For chiral phases of matter, one expects the entanglement spectrum to exhibit gapless chiral modes [20]. We have checked this for some of the Gaussian TNS constructed in the main text. Here we illustrate this with our $p-i\rho$ example, which is a translation-invariant BCS state $|\psi_D\rangle$ defined by its pairing function

$$g_k = \frac{(\kappa_1)^2(\sin k_x - i \sin k_y)}{(\sin k_x)^2 + (\sin k_y)^2 + \lambda^2(2 - \cos k_x - \cos k_y)^2}. \quad (24)$$

We put this Gaussian TNS on an infinite cylinder $(x, y) \in \mathbb{Z} \times [0, L]$, and chose the bipartition $A \cup B = \{(x, y) \mid x < 0\} \cup \{(x, y) \mid x \geq 0\}$. Notice that the bipartition is translation-invariant in the $y$-direction. Since we are dealing with a Gaussian state $|\psi_D\rangle$, its reduced density matrix $\rho_A = \text{Tr}_B |\psi_D\rangle \langle \psi_D|$ is the exponential of a sum of fermion bilinears. In other words, $-\log \rho_A$ is a quadratic operator. The entanglement spectrum (i.e. the spectrum of $\rho_A$) is a free fermion spectrum, generated by a set of single-particle pseudo-energies $\epsilon_j$.

Finally, $\rho_A$ (and thus $-\log \rho_A$) commutes with $T_y$, the generator of translations $(x, y) \mapsto (x, y + 1)$. Every single-particle pseudo-energy is associated to a fixed momentum sector $k_y$. We plot the single-particle entanglement spectrum $\epsilon_p(k_y)$, which in general has more than one branch (hence the subscript $p$).

By construction, the number of branches in the single-particle entanglement spectrum must be bounded by the number of Grassmann variables per site that we use to define our TNS. Here there are two Grassmann variables per site, so there are at most two branches in the single-particle spectrum (see Fig. 2). Since $|\psi_D\rangle$ only depends on $\lambda^2$, we can focus on $\lambda \geq 0$. For generic values of $\lambda$, there are two branches in the single-particle spectrum, but when $\lambda = 0$ or $\lambda = 1$, one of the two branches disappears (it goes to infinity, meaning that the rank of $\rho_A$ is smaller for these values of $\lambda$), and one is left with a single branch. The reduction of the rank can also be traced back to the properties of the pairing function $g_k$ [21]. Indeed, $g_k$ may be viewed as a rational function of the variable $e^{ik_x}$, which has generically four simple poles, but has only two when $\lambda = 0$ or $\lambda = 1$. This modifies the form of the Fourier transform $g_k(x) = \int \frac{dx}{2\pi} e^{ik_x x} g_k$, which determines the single-particle entanglement spectrum [21].

We clearly observe that, as long as $\lambda \neq 0$, there is one chiral gapless edge mode starting at $k_y = 0$, as expected. At $\lambda = 0$, this branch disappears; this is also expected, since at $\lambda = 0$ the degree of the mapping $k \mapsto g_k$ suddenly changes, and the state $|\psi_D\rangle$ does not belong to a chiral topological phase anymore.
Chern-band example: explicit form

In the main text, we sketch the construction of a Gaussian TNS which corresponds to a filled band with Chern number 1. Here we give a few more details about this state. We start with a BCS state of the form \( |\psi_D\rangle \), with \( P = 4 \) Grassmann variables (equivalently, one could write this state with \( P = 2 \) complex Grassmann variables). The matrices \( A, B \) and \( \kappa \) are chosen such that:

\[
\begin{align*}
\xi_{x+1}^t A^t \cdot \xi_x &= (\xi_{x+1}^2 \xi_{x+1}^4) \left( \begin{array}{cc} -i & \lambda \\ -\lambda & -i \end{array} \right) (\xi_x^1 \xi_x^3) \\
\xi_{x+1}^t A^t \cdot \xi_x &= (\xi_{x+1}^2 \xi_{x+1}^4) \left( \begin{array}{cc} 1 & -\lambda \\ -\lambda & 1 \end{array} \right) (\xi_x^1 \xi_x^3) \\
\xi_x^t B \cdot \xi_x &= (\xi_x^2 \xi_x^4) \left( \begin{array}{cc} 0 & -2\lambda \\ 2\lambda & 0 \end{array} \right) (\xi_x^1 \xi_x^3) \\
\xi_x^t \cdot \kappa \cdot c_x^\dagger &= \kappa_1 \xi_x^1 c_{x,1}^\dagger + \kappa_2 \xi_x^2 c_{x,2}^\dagger.
\end{align*}
\]

Here, \( \lambda \in \mathbb{R} \), and \( \kappa_1, \kappa_2 \in \mathbb{C} \) are free parameters. In momentum space, this state takes the form

\[
\exp \left( \int \frac{d^2k}{(2\pi)^2} g_k c_{k,2}^\dagger c_{-k,1} \right) |1\rangle,
\]

where the function \( g_k \) is the following propagator in the auxiliary theory of the Grassmann variables:

\[
g_k = \frac{\int \prod d^p \exp \left( \xi_{k,x}^2 \xi_{k,x}^4 \right) d\xi_k}{\int \prod d^p d\xi_k} \left( \begin{array}{c} -\sin k_x + i \sin k_y \\ (\sin k_x)^2 + (\sin k_y)^2 + \lambda^2 [2 - \cos k_x - \cos k_y]^2 \end{array} \right).
\]

We see that our BCS state depends only on the product \( \kappa_1 \kappa_2 \), so we can take \( \kappa_2 = \kappa_1 \) without loss of generality. Finally, performing the particle-hole transformation mentioned in the main text, namely \( |0\rangle \rightarrow |1,0\rangle = \prod_{x \in \mathbb{Z}^2} c_{x,1}^\dagger |0\rangle \), and \( c_{x,1} \rightarrow \kappa_1 c_{x,1} \), we obtain the state

\[
\exp \left( \int \frac{d^2k}{(2\pi)^2} g_k c_{k,2}^\dagger c_{k,1} \right) |1,0\rangle,
\]
which corresponds to a filled band; one can check that it has Chern number 1 as soon as \( \lambda \neq 0 \).

**Alternative way to express a paired state with any \( g_k = v_k/u_k \) as a Gaussian TNS**

For any \( g_k = v_k/u_k \), where \( u_k \) and \( v_k \) are trigonometric polynomials, we can obtain the state from a Gaussian TNS that is slightly more general in form. For simplicity, we consider only the \( n = 1 \) examples. First write \( g_k \) as \( g_k = v_k u_{-k}/(v_{-k} u_k) \). We will use a single Grassmann variable \( \xi_x \) on each site. For the \( \kappa \) matrix, we write in \( \mathbf{k} \) space \( v_k \xi_{-\mathbf{k}} c_{\mathbf{k}}^\dagger \) for each \( \mathbf{k} \), and for the \( A \) and \( B \) terms in \( \mathbf{k} \) space \( \xi_{-\mathbf{k}} v_{-\mathbf{k}} u_{\mathbf{k}} \xi_\mathbf{k} \). Note that in real space, both kinds of terms are strictly short range. Then clearly integrating out the \( \xi \) variables produces the desired form. Notice that if \( g_k \) is odd under \( \mathbf{k} \to -\mathbf{k} \), then so is \( v_{-\mathbf{k}} u_{\mathbf{k}} \), as required in the Grassmann bilinear form.

This form differs from our original TNS expression in that the coupling of \( c_{x,\mathbf{a}} \) to \( \xi_x \) is no longer just on site. Moreover, there is now only one variable \( \xi \) per site \((P = 1)\). The first change compensates the second, so that when both forms exist, the results, including the entanglement spectrum, must be the same. Then the rank of the entanglement spectrum cannot be read off simply from the number of \( \xi \) variables in the present form.

**Proof of no-go theorem**

We must show that, given a translation-invariant free-fermion TNS \( |\psi\rangle \) in a topologically-non-trivial phase and any single-particle self-adjoint positive operator with compact support that annihilates it, the sum of this operator over translations has a gapless spectrum. The gapless modes occur at \( \mathbf{k} \) values at which both \( u_k \) and \( v_k \) are zero; such values must occur when the Chern number is non-zero, as shown in the main text. We extend this result to include all multi-band cases.

We will consider the particle-number conserving (particle-hole transformed) version of the class A example, for \( g_k = v_k/u_k \) an arbitrary ratio of trigonometric polynomials. For the class D examples of paired states, such a form can be reached by doubling (taking two copies), and particle-hole transforming; these steps do not change the energy spectrum. We will use the real-space operators \( c_{x,\mathbf{a}} \) and \( c_{x,\mathbf{a}}^\dagger \) obtained after the particle-hole transformation, where \( \mathbf{a} = 1, 2 \). Let \( O \) be the operator; it must be a Hermitian form

\[
O = \sum_{\mathbf{x},\mathbf{a} ; \mathbf{x}',\mathbf{a}'} A_{\mathbf{x},\mathbf{a} ; \mathbf{x}',\mathbf{a}'} c_{\mathbf{x},\mathbf{a}}^\dagger c_{\mathbf{x}',\mathbf{a}'} + C, \tag{29}
\]

where \( A_{\mathbf{x},\mathbf{a} ; \mathbf{x}',\mathbf{a}'} \) are the elements of a Hermitian matrix, and \( C \) is a constant to be determined later. This form preserves the \( U(1) \) symmetry, as required. The sums over \( \mathbf{x} \) and \( \mathbf{x}' \) are over (the same) bounded region. We can diagonalize the matrix \( A \), and obtain eigenoperators \( c_{r}, c_{r}^\dagger \) which have support contained within that of \( A \), and real eigenvalues.

FIG. 3. Same as in Fig. 2 with \( \lambda \) fixed (\( \lambda = 0.8 \)). We vary \(|\kappa_1|\).
In general an eigenmode operator $c_{r}$ could be a mixture of both the $1$ (filled) and $2$ (unfilled) bands (these should not be confused with the original orbitals labeled $1, 2$). But then the corresponding term in $O$ would contain terms that do not annihilate the given TNS. Hence the operator can be written as a sum of two parts, each acting within the subspace of single-particle states in one of the bands. For the filled band, we can reverse the order of operators, so that we obtain two terms $O = O_1 + O_2$, and each $O_{\alpha}$ ($\alpha = 1, 2$) has the form

$$O_{\alpha} = \sum \lambda_{r,\alpha} \bar{\alpha}_{r,\alpha} \alpha_{r,\alpha},$$

(30)

where $\bar{\alpha}_{r,\alpha}$ is a sum (or integral, in an infinite system) of the operators $\hat{d}_{k,\alpha}$, which are a complete set of destruction operators for each band of excitations over the TNS, $-\lambda_{r,1,\alpha}$ are the eigenvalues of the band $1$ part of $A$ before, $\lambda_{r,2,\alpha}$ are those for the band $2$ part; the sign change is due to reversing the order of fermion operators, and we have now chosen the previous constant $C$ to eliminate any constant in this expression. All $\lambda_{r,\alpha}$ are required to be positive.

The operators $\bar{\alpha}_{r,\alpha}$ have compact support. Consequently, and by definition $\bar{\alpha}_{r,\alpha}$ have no trigonometric-polynomial common factor (other than a constant times a power of $e^{i k_x}$ and of $e^{i k_y}$), the $\bar{\alpha}_{r,\alpha}$'s must be the inverse Fourier transform of $\hat{d}_{k,\alpha}$ (not $\hat{\alpha}_{k,\alpha}$) times a trigonometric polynomial. Now summing $O$ over translations, each term in the resulting parent Hamiltonian contains $|u_k|^2 + |v_k|^2$ when written in terms of $\bar{\alpha}_{r,\alpha} \alpha_{r,\alpha}$, where $\alpha$ is either $1$ or $2$, and so for a Chern band must be gapless because of the common zero, as explained in the main text.

For a more general result, we first point out that in the preceding $n = 2$ example, $(\bar{u}_k, \bar{v}_k)$ is a basis vector in the one-dimensional vector space of states in the filled band for each $k$, and sits inside an $n = 2$ dimensional vector space for each $k$. (There may be a $k$ at which this is not true, such as at $k = 0$ in our main example, where $u_k$ and $v_k$ have a common zero; at this $k$ we can use $(0, 1)$ instead. The fact that this basis is not continuous illustrates the following discussion.) This forms a vector bundle (in fact in this case, since it is one dimensional, a line bundle) over the Brillouin zone; the two-dimensional bundle of which it is a sub-bundle is topologically trivial. More generally, we could have (again, after doubling, for BCS states) a sub-bundle of an $n$-dimensional bundle; the sub-bundle dimension $n'$ would be $1 \leq n' \leq n$. There are then operators $\hat{d}_{k,\alpha}$ ($\alpha = 1, \ldots, n'$) that create a particle in one of the filled bands (and so annihilate the TNS), and are given by expressions similar to eq. (15), but $u_k$ and $v_k$ are now matrices of trigonometric polynomials; $u_k$ is $n' \times n'$, while $v_k$ is $n' \times (n - n')$. This form is again assumed to be “reduced” as far as possible, meaning that for any factorization $u_k = w_k u'_k$, $v_k = w_k v'_k$, where $w_k$ is an $n' \times n'$ matrix of trigonometric polynomials, and $u'_k$, $v'_k$ has the same form as $u_k$, $v_k$ must have an inverse that is also a matrix of trigonometric polynomials (if $w_k$ has no such inverse, then we can reduce further by replacing $u_k$, $v_k$ by $u'_k$, $v'_k$). There may be values of $k$ at which the states created by $\hat{d}_{k,\alpha}$ are not linearly-independent, generalizing the common zeroes of $u_k$ and $v_k$ in dimension-one case above. After resolving this issue if it occurs (as before), these expressions define the vector bundle corresponding to the filled bands; there are similar expressions for the unfilled bands, $\alpha = n'+1, \ldots, n$. In symmetry classes other than class A, there are additional “symmetry” properties that should hold for the filled and unfilled bands, which may include particle-hole symmetries.

In the argument for $n' = 1$, $n = 2$ above, it is straightforward to generalize to allow for $n$ bands, of which $n'$ are filled, the remainder empty. We recall that a section of a bundle consists of a choice of a vector in the space over each $k$, for all $k$, that varies continuously with $k$. Given that the ground state is a TNS, the argument shows that the parent Hamiltonian can be gapped at all $k$ if and only if the set of operators $\hat{d}_{k,\alpha}$ defined above are linearly independent (in particular, nonvanishing) for each and every $k$. The coefficient functions for each operator form a section of the trivial $n$-dimensional bundle, but for our purposes must, as before, be composed either only of filled or only of unfilled band states, and so each $d_{k,\alpha}$ determines a section of either the filled or unfilled bundle. But for complex vector bundles over a space (here the Brillouin zone), a complete set of linearly-independent sections cannot exist if the bundle is topologically non-trivial, by the definition of triviality. We conclude that we have a Theorem: for a free-fermion TNS in any number of orbitals and for any dimension of space, if the filled bands (and hence also the unfilled bands) are non-trivial when viewed as a complex vector bundle, then any parent Hamiltonian must be gapless. We note that the conditions are the same as the sufficient conditions for the non-existence of exponentially-localized Wannier states (in general bands, not only a TNS), though they are stated here more generally than in Ref. [15].

Note that in the statement and proof we made no reference to any additional symmetries the bands might have. In some symmetry classes, the filled band could be trivial as a complex vector bundle, but still non-trivial in the classification of free-fermion systems under the conditions of symmetry. (This occurs for the Kitaev chain.) Then the gapped parent Hamiltonian may not respect the symmetry, while an invariant parent Hamiltonian may necessarily be gapless; there are similar statements for exponentially-localized Wannier states [22]. However, a full analysis of these cases is beyond the scope of this paper.