THE THEORY OF DYNAMICAL RANDOM SURFACES WITH EXTRINSIC CURVATURE

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Abstract

We analyze numerically the critical properties of a two-dimensional discretized random surface with extrinsic curvature embedded in a three-dimensional space. The use of the toroidal topology enables us to enforce the non-zero external extension without the necessity of defining a boundary and allows us to measure directly the string tension. We show that a phase transition from the crumpled phase to the smooth phase observed earlier for a spherical topology appears also for a toroidal surface for the same finite value of the coupling constant of the extrinsic curvature term. The phase transition is characterized by the vanishing of the string tension. We discuss the possible non-trivial continuum limit of the theory, when approaching the critical point. Numerically we find a value of the critical exponent $\nu$ to be between .38 and .42. The specific heat, related to the extrinsic curvature term seems not to diverge (or diverge slower than logarithmically) at the critical point.

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1 Introduction

The theory of random walks has provided us with a powerful link between statistical mechanics and euclidean field theory. An euclidian field theory can be expanded in a series of intersecting random walks and a number of rigorous inequalities can be proven using the random walk representation. In addition various aspects of universality in regularized field theory can easily be understood from the corresponding universality of the random walks. It has further been possible to develop a theory of random walks, which allows the description of fermions in a geometrical way.

The theory of random surfaces ought to provide us with even stronger tools in the study of string theories. String theory in its first quantized version is nothing but the theory of a specific kind of free random surfaces. This has been substantiated during the last couple of years, but many questions remain unanswered. One of the greatest puzzles is that the formulas derived by Knizhnik, Polyakov and Zamolodchikov \[1\] using conformal field theory seem to make no sense for strings embedded in "physical" dimensions \((d > 1)\). In more general terms we do not know how to couple two-dimensional gravity to conformal field theories with central charge \(c > 1\). On the other hand it seems to be no problem to formulate statistical theories of random surfaces in physical dimensions. The use of these random surface theories in the context of strings is hampered by our lack of understanding of scaling and universality properties of the random surface theories themselves. The situation here is quite different from the random walk case. The present work is an attempt to clarify the situation for a particular class of random surface models. Let us briefly motivate why this class is interesting.

The class of random surface theories we have in mind is one of freely intersecting surfaces with an action having an area term and an extrinsic curvature term. Such a model can be viewed as a simplified model of physical membranes. The statistical aspect enters due to thermal fluctuations and the statistical fluctuations may drive the system to a point where the effective surface tension vanishes and the membranes are dominated by their curvature energy rather than their surface tension. This is the case for fluid/fluid interfaces and also for the so-called amphiphilic membranes, which are membranes formed when amphiphilic molecules are brought into contact with water and form bilayers by orienting their polar parts towards water and their oily hydrophobic tails away from the water. Of course such surfaces are not allowed to self-intersect and from this point of view our models can only be viewed as toy models for real membranes. However, since the seminal work of Helfrich \[2\] such toy models have received a lot of attention \[3\].

The random surface models with extrinsic curvature are also interesting as effective theories of strings. In this context they were first suggested by Polyakov \[4\].
He had in mind an effective string theory which could be equivalent (at least at long distances) to QCD with heavy quarks. However, it is now clear that fermionic string theories can give rise to effective bosonic string theories which have extrinsic curvature terms, and in this context the surfaces should be allowed to self-intersect. In the case of the superstring it is possible to integrate out the world-sheet fermions. After this integration two types of terms are produced, which both depend on the extrinsic geometry of the world sheet:

\[ S_{\text{eff}} = S_{\text{bosonic}} + W_k(A^{(n)}) + \frac{\tau}{8} \int d^2 \xi \sqrt{g_{\text{ind}}} \left\{ (e^\mu_\alpha \partial_\beta e^\mu_\gamma)^2 + (D_\alpha n^\mu_i)^2 \right\} \]  

(1.1)

Here \( g_{\text{ind}} \) refers to the metric induced by target space, \( n^\mu_i \), \( i = 1, \ldots, d - 2 \) are normals to the surface, \( e^\mu_\alpha \) \( \alpha = 1, 2 \) are the tangents and \( D_\alpha \) denotes the covariant derivative with respect to the connection \( A^{(n)} \) in the normal bundle:

\[ A^{(n)}_{ij} = n^\mu_i \partial_\alpha n^\mu_j - n^\mu_j \partial_\alpha n^\mu_i. \]  

(1.2)

\( \tau \) is a Dynkin factor coming from the fermionic representation and finally \( W_k(A^{(n)}) \) denotes the Wess-Zumino action:

\[ W_k(A^{(n)}) = \frac{ik}{8} \text{Tr} \left( \frac{1}{2} \int d^2 \xi A \wedge A + \frac{1}{3\pi} \int_D d^3 x A \wedge A \wedge A \right) \]  

(1.3)

where we have used the notation \( A^\alpha = A^{(n)}_{ij} M_{ij} \), \( M_{ij} \) being the generators of \( \text{SO}(d - 2) \) and \( D \) a three-dimensional disc bounded by the world sheet.

One class of terms depends very explicitly on the extrinsic geometry of the world sheet and is minimized by smooth, flat surfaces. If \( K(\xi) \) denotes the extrinsic curvature of the world sheet we have

\[ K(\xi) = \frac{1}{r_1(\xi)} + \frac{1}{r_2(\xi)} \]  

(1.4)

where \( r_1(\xi) \) and \( r_2(\xi) \) are the principal curvatures of the surface. Since

\[ \int d^2 \xi \sqrt{g_{\text{ind}}} (D_\alpha n^\mu_i)^2 = \int d^2 \xi \sqrt{g_{\text{ind}}} K^2(\xi) \]  

(1.5)

this term clearly favours smooth surfaces. The same is the case for the term involving the tangents.

As for the Wess-Zumino-like term \( W(A) \), the effect is not so clear. In euclidean space-time the term is purely imaginary. It is interesting to compare with the situation in the case of random walks. What we have in mind is a “supersymmetric” random walk, where we have introduced world-line fermions. In this case it is again possible to integrate out the fermions. The result is a Wess-Zumino-like term on the world-line and it is the analogue of the term \( W_k(A) \) defined in eq. (1.3). In the random walk case the effect of the Wess-Zumino term can be analyzed in detail (see
for a continuum treatment and \([7]\) for a random walk approach). Heuristically the result can be described as follows: The amplitudes of back-tracking random walks tend to cancel due to the phase factor coming from the Wess-Zumino term and we are effectively left with a class of random walks which are much smoother than the generic random walks of a scalar particle. In effect the short distance Hausdorff dimension of the random walk is reduced from two to one, and the corresponding short distance behaviour of the corresponding propagator changes from being like \(1/k^2\) to \(1/k\). As first noticed by Polyakov and proven in the context of random walks in [7], the scaling limit of this fermionic random walk leads to a representation of the massive Dirac propagator. In this case the effect of world-line supersymmetry and the corresponding Wess-Zumino term is clearly to favour “smooth” curves. It is natural to conjecture that the two-dimensional Wess-Zumino term (1.3) has a similar effect.

As we see all additional terms in (1.1) act in favour of smoother surfaces as compared to the class of surfaces singled out by the standard bosonic term \(S_{\text{bosonic}}\). This opens up the possibility of an intuitive and simple understanding of the tachyon problem of the bosonic string and its cure in the case of the superstring. The appearance of tachyons in a bosonic random surface theory is somewhat different from that based on formal continuum manipulations. By definition there can be no tachyons in a theory of random surfaces where each surface is given a positive weight and which satisfy the principle of reflection positivity. However, the scaling limit of such discretized, regularized theories might be pathological from the point of view of string theory. This is precisely what happens for the ordinary bosonic string theory where the dimension \(d\) of target space is larger than one. One can prove that the string tension does not scale \([12]\) for such theories and as a consequence the scaling limit is not really that of a “surface” theory, but rather that of a theory of so-called branched polymers consisting of a minimal surface (depending on the boundary conditions) from which the only allowed fluctuations are thin “branches” which carry no area. Maybe somewhat contrary to intuition the entropy of such surfaces is large compared to that of “smooth” surfaces and completely determines the scaling properties of the “surface” theory. An obvious cure is to put in by hand additional terms in the action which suppress the “spikes” and favour a smoother class of surfaces and this is precisely what is done by imposing world-sheet supersymmetry and integrating out the world-sheet fermions as indicated in (1.1).

We know there are no tachyons in the superstring theories. However, we also know that in the usual continuum description the absence is due to a delicate cancellation between bosonic and fermionic degrees of freedom. From this point of view it is intriguing how a discretization of the action \((1.1)\) and a subsequent scaling
limit is able to capture this. The obvious answer would be: by universality. But although it makes no sense to derive an effective action like (1.1), except in the special dimensions \( d = 3, 4, 6 \) and 10, where a classical superstring theory can be defined, the effective action itself makes sense in any dimension and one can consider the corresponding discretized theory and possible scaling limits. Appealing to analytic results there should be a difference between the theories, depending on the dimension of target space. At the moment we have no understanding of the physics leading to such a difference. The concept of universality triggers a related question: How far are we able to modify a discretized version of the effective action (1.1) and still stay in the same universality class. Due to the imaginary Wess-Zumino term we are clearly discussing a rather unconventional class of theories from the point of view of statistical mechanics. Nevertheless it does not necessarily mean that one cannot apply the conventional machinery of statistical mechanics. The above mentioned example of the fermionic random walk provides documentation for this and shows that one can indeed talk about universality classes \( \mathcal{W} \). It would be very convenient if we were able to drop completely the imaginary Wess-Zumino term and still stay in the same universality class. At a first glance it looks unlikely, again appealing to the fermionic random walk example. One can analyse the random walk where one has both an extrinsic curvature term like (1.5) and the Wess-Zumino term. The result is as follows: if we choose the coefficient of the Wess-Zumino term to be zero we will in general be in the universality class of the ordinary random walk (and the corresponding scaling limit will be that of a free scalar particle\(^2\). As soon as we take the coefficient of the Wess-Zumino term to be non-zero we are driven to the fermionic random walk, where the scaling limit leads to a massive Dirac propagator. The same could be true for the effective action (1.1). However, since the world-sheet is two-dimensional while the world-line is one-dimensional we are discussing vastly different theories from the point of view of statistical mechanics. The structure of phase transitions is much richer in two dimensions, and although a one loop calculation seems to support the conclusion reached in the random walk case \( \mathcal{W} \), this is a purely perturbative argument, based on assumptions which will not be satisfied in case we have a non-perturbative phase transition. A priori it is possible that one could drop the Wess-Zumino term and still stay in the same universality class.

The above discussion is closely linked to another discussion of possible critical behaviour of membranes (or random surfaces) with extrinsic curvature terms. It is believed that one has a transition from so-called “crumpled” surfaces to smooth surfaces in the case of crystalline surfaces. Crystalline surfaces can be viewed as

\(^2\)The only exception is if we take a scaling limit where we scale the bare coupling constant of the extrinsic curvature term to infinity. It is possible to do this in such a way that we get what can be called a rigid random walk\(^3\).
membranes where the individual molecules have a fixed connectivity, i.e. their neighbours are fixed, contrary to the situation for fluid membranes. If the coefficient of the extrinsic curvature term is zero such surfaces will be “crumpled” (the Hausdorff dimension for the statistical ensemble of surfaces is infinite), at least in the idealized situation where they are allowed to intersect freely. However, for a finite value of the extrinsic curvature coupling there seems to be a phase transition, and for couplings above this critical value the surfaces are smooth (the Hausdorff dimension for the statistical ensemble of surfaces is two). No rigorous proof of this phase transition exists, but numerous numerical results seem to confirm the existence of such a transition, and it seems to be a second order transition for the kind of extrinsic curvature term we are going to use in this work. In the beginning some confusion surrounded this transition. Depending on the details of the discretized version of the extrinsic curvature term used, the transition was classified as first, second or third order, respectively. It is now understood that only the discretization which seems to lead to a second order transition is not “pathological”. The other actions used, although formally equivalent for smooth surfaces, led to singular surface configurations if the surface was allowed to fluctuate wildly, as in fact happened in the “crumpled phase”. This complicated situation highlights the possibility of a nontrivial phase structure in random surface theories and the care one has to exercise in order to choose a correct discretized action. One can now imagine a situation where the attachment of the molecules to specific neighbour molecules in the crystalline surface is gradually decreased. The crystalline structure is allowed to “melt”. Ultimately one will end up with a fluid membrane where the individual molecules can move freely. Will the second order phase transition extend all the way to a fluid membrane or can one trust the one loop calculation [14, 2] done for the fluid membrane case, which indicated that there is no transition? Computer simulations first started by Catterall and later repeated by other groups point to the existence of a “crumpling” transition even in the case of fluid membranes, but contrary to the situation for crystalline surfaces numerical simulations and analytic arguments contradict each other in the fluid membrane case.

We hope to have convinced the reader that the critical properties of random surfaces are a fascinating topic of importance in vastly different areas of physics, but that it is an area where only little is known at present. The aim of the present article is to develop some theoretical concepts, adequate for the description of the critical phenomena of random surfaces, and by extensive numerical simulations try to answer some of the questions raised above.

The paper is organized as follows: In section 2 we discuss the model and define

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3We thank John Wheater and François David, who independently suggested this interpolating scenario to us.
the observables, which we use in the canonical numerical simulations. We also relate them to the standard grand canonical definitions of the mass gap and string tension. Scaling properties of these observables are discussed in section 3. In section 4 we describe the system with twisted boundary conditions, which was used in all our numerical work to measure the string tension and the mass gap. Numerical results are presented in section 5, which is divided into four sub-sections, concerning the specific heat, the mass gap, the string tension and the radius of gyration. We conclude the paper with a discussion in section 6.

2 The action and observables

Let us in this section define the action which we are going to use and discuss the observables which will allow us to investigate the critical properties of the system. We want to approximate the continuum surfaces discussed in the introduction with piecewise flat surfaces. The intrinsic parameter space is then identified with an abstract triangulation of appropriate topology, defined entirely by its vertices $i$ and links $<ij>$. The embedding in a $D$-dimensional euclidean target space is a map assigning to each link $<ij>$ a vector $X^\mu_{ij}, \mu = 1...D$ in the target space. The numerical results presented in this paper are related solely to the case $D = 3$. Three vectors forming a triangle $i, j, k$ should satisfy

$$X^\mu_{ij} + X^\mu_{jk} + X^\mu_{ki} = 0. \quad (2.1)$$

If the topology is spherical, (2.1) would mean that

$$X^\mu_{ij} = X^\mu_j - X^\mu_i, \quad (2.2)$$

where $X^\mu_i$ denotes coordinates of a vertex $i$ in the target space and a sum of vectors $X^\mu_{ij}$ along any closed path would be zero. For surfaces with handles the embedding may be non-trivial and we will discuss this further in section 4. In the rest of this section we will assume that the topology is trivial.

A configuration of the surface is defined by specifying a triangulation (links joining the vertices) and by assigning values to vectors $X^\mu_{ij}$ satisfying (2.1). For each configuration we define the action $S = \beta S_G + \lambda S_C$. $S_G$ is the gaussian part of the action

$$S_G = \frac{1}{2} \sum_{<ij>} (X^\mu_{ij})^2 \quad (2.3)$$

and $S_C$ is the extrinsic curvature part

$$S_C = \sum_{<ij>} (1 - \cos \theta_{ij}), \quad (2.4)$$
\( \theta_{ij} \) denoting the angle between the normals to two oriented triangles \(< ijk >\) and \(< jil >\) with a common link \(< ij >\). The coupling constant \( \beta \) of the gaussian part of the action plays the role of the unit length.

This action is intended to serve as a discretized version of the terms

\[
\int d^2 \xi \sqrt{g} \partial_{\alpha} X^\mu \partial^\alpha X^\mu + \lambda \int d^2 \xi \sqrt{g_{\text{ind}}} K^2(\xi) \tag{2.5}
\]

in (1.1), where \( g_{\text{ind}} \) refers to the metric induced by target space. Strictly speaking \( S_C \) given by (2.4) is not a direct translation of the last term in (2.3), since we are not using the induced metric. However, we do not expect there will be any significant difference (i.e. the action (2.3) should belong to the correct universality class), since both the order of vertices and the shape of triangles in target space are smooth functions.

Two statistical ensembles have our interest: the canonical ensemble where the number triangles is kept fixed, and the grand canonical ensemble where the number of triangles is allowed to fluctuate.

The canonical partition function for a closed surface of trivial topology is given by

\[
Z_N(\beta, \lambda) = \sum_{T \in T} \int \prod_{i=1}^{N} dX^\mu_i \delta(\sum_{i} X^\mu_i) \exp(-S), \tag{2.6}
\]

where the first sum is over all possible triangulations of the surface with \( N \) vertices and the delta function is necessary because of the translational invariance of the action. We can always scale out \( \beta \) since \( S_C \) is scale invariant and we are left only with one coupling constant \( \lambda \). The partition function defines Helmholtz free energy

\[
\mathcal{F}(N, \beta, \lambda) = -\log Z_N(\beta, \lambda). \tag{2.7}
\]

Using again the scale invariance of \( S_C \) we have the following transformation under scaling \( X^\mu_i \rightarrow yX^\mu_i \):

\[
S(X^\mu_i) = y^2 \beta S_G(X^\mu_i) + \lambda S_C(X^\mu_i) \tag{2.8}
\]

and Helmholtz free energy will scale as

\[
\mathcal{F}(N, \beta, \lambda) = \mathcal{F}(N, y^2 \beta, \lambda) - (N - 1)D \log y. \tag{2.9}
\]

The grand canonical ensemble will be defined by

\[
Z(\mu, \beta, \lambda) = \sum_{N} e^{-\mu N} Z_N(\beta, \lambda) \tag{2.10}
\]

where \( \mu \) is a chemical potential for the number of vertices \( N \) (or equivalently the number of triangles \( N_2 \)). \( \beta \) was not a dynamical coupling constant for the canonical
ensemble. The same is true in the grand canonical ensemble where it is clear from (2.8)-(2.10) that
\[ Z(\mu, \beta, \lambda) = Z(\mu - D \log \beta, 1, \lambda) \]  
(2.11)

It is now obvious that we can always take \( \beta = 1 \), and we will do that in the following except when explicitly stated differently. The Gibbs free energy can now be defined by
\[ G(\mu, \lambda) = -\log Z(\mu, \lambda) \]  
(2.12)

where we have taken \( \beta = 1 \) as announced.

In the case of the the canonical ensemble we reach the thermodynamic limit by taking the size of the system \( N \to \infty \). This thermodynamic limit will in general depend on the coupling constant \( \lambda \) and for certain values of \( \lambda \) there might be phase transitions which, in case they are second order, might serve as points where we can define a continuum field theory.

In the case of a grand canonical ensemble we will have a critical line \( \mu = \mu_c(\lambda) \) in the \((\lambda, \mu)\) coupling constant plane, such that the theory is defined for \( \mu > \mu_c \). The thermodynamic limit is obtained when number of vertices \( N \) diverges, and \( \langle N \rangle \to \infty \) corresponds in the language of the grand canonical ensemble to moving close to the curve \( \mu = \mu_c(\lambda) \) along the \( \mu \)-axis, starting at large \( \mu \). The possibility of phase transitions for certain values of \( \lambda \), as described above in the language of the canonical ensemble can be addressed in the grand canonical ensemble too. In this ensemble such a transition will manifest itself as points on the critical line, which separate two different types of critical behaviour when we move along the critical line \( \mu = \mu_c(\lambda) \).

For numerical purposes it is much more convenient to work with a canonical ensemble. Certain observables are however naturally defined in the grand canonical ensemble, and in the following we will show how it is possible to extract information about them using only the canonical ensemble.

The simplest observable which gives us information about the nature of the phase transition is the specific heat with respect to \( \lambda \). It can be defined directly in the canonical ensemble:
\[ C(\lambda) \equiv \frac{\lambda^2}{N} \frac{\partial^2 F(N, \lambda)}{\partial \lambda^2} = \frac{\lambda^2}{N} \left( \langle S_C^2 \rangle - \langle S_C \rangle^2 \right) \]  
(2.13)

In case there is a second order transition at a finite \( \lambda_c \) we should see a singularity of \( C(\lambda) \) as \( N \to \infty \) as will be discussed later.

Even if we observe a second order transition at a finite \( \lambda_c \), it does not ensure that this transition has anything to do with a string theory. A minimal requirement is that the string tension and possible mass excitations scale to zero when \( \lambda \to \lambda_c \). The following discussion is stimulated by the work of David and Leibler [15]. In order to
define the string tension we introduce the grand canonical ensemble of open surfaces where the boundary is kept fixed. We imagine the surface will enclose a large area \( A \) and denote the corresponding partition function \( Z(\mu, \lambda; A) \). The partition function will behave as

\[
Z(\mu, \lambda; A) \sim \exp(-\sigma(\mu, \lambda)A), \quad \text{for} \quad A \to \infty
\]  

and we can define the string tension as

\[
\sigma(\mu, \lambda) = \lim_{A \to \infty} \frac{G(\mu, \lambda; A)}{A}.
\]  

(2.15)

In the same way we can define the mass gap in the theory as the exponential decay of the grand canonical partition function of surfaces with a boundary consisting of two points separated by a distance \( L \)

\[
Z(\mu, \lambda; L) \sim \exp(-m(\mu, \lambda)L), \quad \text{for} \quad L \to \infty
\]  

i.e.

\[
m(\mu, \lambda) = \lim_{L \to \infty} \frac{G(\mu, \lambda; L)}{L}.
\]  

(2.17)

In the thermodynamic limit (where \( \mu(\lambda) \approx \mu_c(\lambda) \) we have (for any boundary conditions “B”) the usual relation between Helmholtz free energy \( F \) and the Gibbs free energy \( \mathcal{G} \)

\[
\mathcal{G}(\mu, \lambda; “B”) = \mu N + \mathcal{F}(N, \lambda; “B”)
\]  

(2.18)

\[
N = \frac{\partial \mathcal{G}(\mu, \lambda; “B”)}{\partial \mu}
\]  

(2.19)

If we apply the formula to the case where the boundary condition “B” is the one which is used in the definition of the string tension we get

\[
\sigma(\mu, \lambda)A = \mu N + \mathcal{F}(N, \lambda; A)
\]  

(2.20)

from which we conclude that

\[
\sigma(\mu, \lambda) = \frac{\partial \mathcal{F}(N, \lambda; A)}{\partial A} = \sigma(\lambda, N, A)
\]  

(2.21)

with \( N \) expressed in terms \( \mu \) and \( A \) by (2.19)

\[
N = \frac{\partial \sigma(\mu, \lambda)}{\partial \mu} A.
\]  

(2.22)

In the same way we get where the boundary condition “B” is the one used in the definition of the mass gap

\[
m(\mu, \lambda)L = \mu N + \mathcal{F}(N, \lambda; L)
\]  

(2.23)
from which we conclude that
\[ m(\mu, \lambda) = \frac{\partial F(N, \lambda; L)}{\partial L} = m(\lambda, N, L), \]  
with \( N \) expressed in terms of \( \mu \) and \( L \) by (2.19):
\[ N = \frac{\partial m(\mu, \lambda)}{\partial \mu} L. \] (2.25)

The advantage of (2.21) and (2.24) is that they allow us to measure the string tension and the mass gap using only canonical ensembles. Further, with the action in question it follows by a simple scaling argument using (2.8) that
\[ \frac{\partial F(N, \lambda; A)}{\partial A} = \langle s_G \rangle - \frac{1}{2} D(N - 1), \] (2.26)
\[ \frac{\partial F(N, \lambda; L)}{\partial L} = 2 \langle s_G \rangle - D(N - 1). \] (2.27)

In this way we see that we can measure string tension and mass gap by a simple measurement of the local observable \( S_G \). To study the critical behaviour it is convenient to consider an analogue of the specific heat (2.13)
\[ \rho(\lambda, N; A) = \frac{A^2}{N} \frac{\partial^2 F(N, \lambda; A)}{\partial A^2} = - \frac{\langle s_G^2 \rangle - \langle s_G \rangle^2 - \frac{1}{2} D(N - 1)}{N}. \] (2.28)

In deriving (2.27) and (2.28) we have used only scaling arguments, so the string tension should not depend on the shape of the fixed boundary loop. This will only be true when the two linear dimensions of the enclosed area are comparable. For elongated loops the behaviour will eventually change, since it will be difficult to distinguish the physics associated with the area from the (different) physics associated with the perimeter.

### 3 Scaling properties of the observables

The aim of the numerical study is to find and understand the critical properties of the system. For a canonical system with a fixed number of points \( N \to \infty \) the system is believed to undergo a phase transition for a finite value of the coupling constant \( \lambda \). For the target space dimension \( D = 3 \) this phase transition seems to be of the second order. A standard method to localize the critical point is to study the specific heat \( C(\lambda) \) (2.13). For a second order phase transition the function \( C(\lambda) \) has a singularity at \( \lambda = \lambda_c \) in the thermodynamic limit \( N \to \infty \). For finite values of \( N \), \( C_N(\lambda) \) is a continuous function of \( \lambda \), which develops a maximum at some \( \lambda = \lambda_c^N \). Let the maximum of \( C_N(\lambda) \) be
\[ C_{max}^N = C_N(\lambda_c^N). \] (3.1)
From the standard finite-size scaling arguments (e.g., [30]) we expect for $N \to \infty$ that $\lambda_c^N \to \lambda_c$ with

$$|\lambda_c^N - \lambda_c| \sim \left(\frac{1}{N}\right)^\varepsilon,$$  

$$C_{\max}^N = A + BN^\omega,$$  

where $\varepsilon$ and $\omega$ are some critical exponents. The increase of $C_{\max}^N$ with growing $N$ is considered to be a signal of the second order phase transition unless $\omega = 1$, in which case the transition could be first order. We will measure $C_N^N(\lambda)$ by means of (2.13). However, it is important to stress, as is apparent from (2.13), that $C_N(\lambda)$ is not the integral of a simple normal-normal correlation function and consequently there is no a priori reason that $C_N(\lambda)$ should diverge even if the normal-normal correlation length diverges and in fact our numerical results are compatible with no or less than a logarithmic growth of the specific heat with $N$.

For the string tension measurement we can use the canonical ensemble of open surfaces with a fixed boundary which we will usually take to be a rectangular $L_1 \times L_2$ loop enclosing a given area $A$, which is not to be confused with the area of the surface. As explained in the next section we actually implement the boundary conditions differently, but for the discussion of scaling properties the above definition will be sufficient.

The string tension is known not to scale to zero for $\lambda = 0$ and $\mu \to \mu_c(\lambda = 0)$ [12]. The thermodynamic limit in the grand canonical ensemble is obtained when

$$\mu_R \equiv \mu - \mu_c(\lambda) \to 0 \quad \text{(3.3)}$$

and it is therefore natural to expect a general behaviour

$$\sigma(\lambda, \mu) = \sigma_0(\lambda) + d(\lambda)\mu^\nu(\lambda) R A.$$  

(3.4)

with some critical exponent $\nu(\lambda)$. From (2.22), which tells us that $N/A = \partial \sigma(\mu, \lambda)/\partial \mu$ we then deduce

$$N \sim \mu_R^{2\nu(\lambda) - 1} A.$$  

(3.5)

It implies that the quantity $r = A/N$ can be expressed in terms of $\mu_R$ and using eq. (2.21) we see that the string tension $\sigma(\lambda, N, A)$, defined in the canonical ensemble, in the critical region for a fixed $\lambda$ should depend on $N$ and $A$ only through $r$:

$$\sigma(\lambda, N, A) = \sigma(\lambda, r) = \sigma_0(\lambda) + d(\lambda)r^{2\nu(\lambda)/(1-2\nu(\lambda))}.$$  

(3.6)

In the derivation of (3.6) we assumed of course that the finite-size effects can be neglected. Formula (3.6) will be used to extract from the numerical simulations the critical behaviour of $\sigma_0(\lambda)$ and the value of the critical index $\nu(\lambda)$. The continuum
limit is related to the small-$r$ behaviour of (3.4), where the finite-size corrections may be important. This problem will be discussed in the section 5.

Numerical results on $\sigma(\lambda, r)$ can be used to get an estimate of the critical exponent $\nu$, but the behaviour of $\sigma(\lambda, \mu)$ given by (3.4) does not guarantee that the continuum theory will have a finite physical string tension. As the thermodynamic limit for a fixed $\lambda$ is approached for $\mu_R \to 0$ we expect that the possible scaling behaviour is extracted from

$$\sigma(\lambda, \mu) A = \sigma_{phys}(\lambda) A_{phys}$$

where the physical string tension $\sigma_{phys}(\lambda)$ and the physical area $A_{phys}$ are kept fixed for $\mu_R \to 0$. We further want the “bare” area $A$ to go to infinity in order not to deal with lattice artifacts. These requirements clearly demand that $\sigma_0(\lambda)$ in (3.4) goes to zero for $\lambda \to \lambda_c$. Phrased differently the physical string tension will be infinite, except at special critical points $\lambda_c$ where the coefficient $\sigma_0(\lambda_c)$ vanishes. Assume that

$$\sigma_0(\lambda) \sim (\lambda_c - \lambda)^\alpha.$$  \hspace{1cm} (3.8)

(The exponent $\alpha$ can be directly measured in our experiment by fitting the behaviour of $\sigma_0(\lambda)$ for $\lambda \to \lambda_c$ (3.8)). We do not expect a scaling of $d(\lambda)$ for $\lambda \to \lambda_c$. In the continuum limit we have $\mu_R \to 0$ and $\lambda \to \lambda_c$ with

$$\lambda_c - \lambda \sim \mu_R^\rho.$$  \hspace{1cm} (3.9)

The critical exponents $\alpha$ and $\rho$ satisfy

$$\alpha \rho = 2 \nu,$$  \hspace{1cm} (3.10)

In (3.10) we assumed that both terms in (3.12) contribute at the same order and we now see that the physical area $A_{phys}$ in (3.7) will be related to the “bare” area $A$ by

$$A_{phys} \sim \mu_R^{2\nu} A$$  \hspace{1cm} (3.11)

and the physical string tension to the “bare” string tension as

$$\sigma_{phys} \sim \frac{\sigma(\lambda, \mu)}{\mu_R^{2\nu}} = \frac{\sigma_0(\lambda)}{\mu_R^{2\nu}} + d(\lambda)$$  \hspace{1cm} (3.12)

showing explicitly that the physical string tension would be infinite except at the critical points where $\sigma_0(\lambda)$ vanishes.

The scaling assignments (3.11) and (3.5) are in agreement with the general scaling law relation of mass critical exponents like $\nu$ to the Hausdorff dimension $d_H$ of the surfaces in target space:

$$\nu = \frac{1}{d_H}.$$  \hspace{1cm} (3.13)
In fact one would define the Hausdorff dimension of the present ensemble of surfaces by

\[ N \sim A^{d_H/2} \]  

in the limit where \( A_{phys} \) is kept fixed but the “bare” \( A \) goes to infinity. As we see \( A_{phys} \) sets the scale for the divergence of \( A \) and \( N \) for \( \mu_R \to 0 \) (under the assumption that \( \sigma_0(\lambda) = 0 \)):

\[ A \sim A_{phys} / \mu_R, \quad N \sim A_{phys} / \mu_R, \]  

which implies (3.13):

\[ N \sim (A)^{1/(2\nu)}. \]  

Eq. (3.16) is not in contradiction to (3.5), but describes how the limit \( r \to 0 \) should be taken in order to reach the continuum limit.

As we mentioned in the introduction an additional requirement for an interesting scaling limit is the correct scaling of other physical observables. An observable independent of the string tension is the mass gap, defined in the grand canonical ensemble by the exponential decay of the two-point function. In the standard approach one fixes two points, separated by a distance \( y \), and sums over all surfaces passing through these points. The transformation of the mass gap definition from the grand canonical to the canonical ensemble was discussed in the preceding section (equations (2.23) to (2.25)) and we have the relation

\[ N = \frac{\partial G}{\partial \mu} = \frac{\partial m(\lambda, \mu)}{\partial \mu} y. \]  

Let us now assume that the mass scales at the critical point \( \mu_c(\lambda) \). Opposite to the situation for the string tension we expect the bare mass \( m(\lambda, \mu) \) to scale for \( \mu_R = \mu - \mu_c(\lambda) \to 0 \) for all \( \lambda \). Assume the scaling is of the form

\[ m(\lambda, \mu) \sim c(\lambda) \mu^{\nu(\lambda)} \]  

Using (3.18) one shows that \( m(\lambda, N, y) \) defined by (2.23) to (2.25) has a scaling behaviour

\[ m(\lambda, N, y) \sim D(\lambda) t^{\nu(\lambda)/(1-\nu(\lambda))}, \]  

where \( t = y/N \) is a scaling variable for the mass gap (analogous the \( r \) variable for the string tension). This formula allows us to determine the critical exponent \( \nu(\lambda) \) from a canonical simulation.

### 4 The twisted boundary conditions

In numerical simulations the systems necessarily have finite dimensions and all measurements are subject to finite-size effects. In the ideal situation one would like to
use these effects to gain additional information about the critical properties of the system (like a finite-size scaling analysis). For the string tension measurements one may however meet effects of this kind, which are very difficult to estimate and which can strongly bias the measured critical behaviour. Following the discussion of the previous section, measurements of the string tension would require introducing a boundary to the system. For a finite system this may present a serious problem: the number of vertices belonging to the boundary may be, and in practice is, a sizeable fraction of all vertices. One has also to provide some method of assigning vertices to the boundary. No obvious solution seems to be at hand and any solution chosen may in fact influence the results.

In this section we propose a method of avoiding these problems with the help of twisted boundary conditions imposed on a surface with the topology of a torus. In this case the parameter space of the surface with $N$ vertices can be visualized as a plane, periodic in two non-parallel directions. In each elementary cell we have $N$ vertices, each vertex having infinitely many periodic copies. The elementary cells can be numbered by two integers $k_1$ and $k_2$. Links can connect points both inside a single cell and in the neighbouring cells. As before we assign to each link $<ij>$ of a lattice in the parameter space a vector $X_{ij}^\mu, \mu = 1...D$, in the target space. We assume the map to be periodic. On a torus the embedding may be nontrivial in the following sense. As a consequence of (2.1) a sum of vectors along a closed path can take values

$$E^\mu(n_1, n_2) = n_1E_1^\mu + n_2E_2^\mu$$

(4.1)

with two constant vectors $E_1^\mu$ and $E_2^\mu$ and integers $n_1$ and $n_2$ denoting the number of times the path winds around the two axes of the torus. The fact that the vector $E^\mu$ is non-zero does not contradict the periodicity of $X_{ij}^\mu$, because on a torus the closed loops with non-zero values of $n_1$ and $n_2$ can not be contracted to a point. We can therefore define non-trivial boundary conditions by choosing two arbitrary vectors $E_1^\mu$ and $E_2^\mu$. Such a choice of boundary conditions on a torus is the analogue of the twisted boundary conditions used in the case of the gauge theories, where the gauge fields are periodic up to non-trivial gauge transformations. The vectors $E_k^\mu$ are topological invariants, they remain unchanged if one or more points along the path are shifted in the target space or if the internal geometry of the surface is changed by a flip of some of its links. The vector $E_k^\mu$ does not depend on the point index $i$, but only on the integer distance between the initial and final periods of the path. This is a simple consequence of the periodicity of $X_{ij}^\mu$. For nonzero vectors $E_1^\mu$ and $E_2^\mu$, (4.1) means that the coordinates $X_i^\mu$ of the vertex $i$ are not strictly periodic but depend also on the integer coordinates $k_1, k_2$ numbering the particular
elementary cell and are given by

\[ X_\mu^i(k_1, k_2) = X_\mu^i + k_1 E_\mu^1 + k_2 E_\mu^2. \] (4.2)

This again does not contradict the periodicity of the dynamical quantities, which are functions of \( X_{ij}^\mu \).

One can check that this choice of boundary conditions corresponds to spanning of the system on a frame \( E_1 \times E_2 \). This can most easily be seen by considering a minimum action configuration of the model. In the embedding space all triangles lie in the two-plane spanned by the vectors \( E_1 \) and \( E_2 \) with the lengths of the links minimizing the Gaussian part of the action. Due to (4.2) these lengths are non-zero for a non-zero frame. The advantage, as compared to the situation where some points are assigned to the boundary is that now the boundary is not present, it becomes a translationally invariant concept and the distribution of points is purely dynamical.

In our computation we have chosen the vectors \( E_1 \) and \( E_2 \) to be perpendicular with the lengths \( L_1 \) and \( L_2 \). For the string tension measurements we used the square frame \( L_1 = L_2 \). For large \( L_1 \) and \( L_2 \) we expect the leading behaviour to be on \( A = L_1 L_2 \). Translational invariance suggests that subleading terms (\( \propto L_1 + L_2 \)) are absent and we should observe only small finite-size corrections of the form \( L_1 / L_2 \). Another possible choice can be \( L_1 \neq 0 \) and large and \( L_2 = 0 \). This clearly corresponds to a completely different physical situation, where we measure the analogue of the point-point correlation function (or rather loop-loop correlation function) with the two loops kept at a distance \( L_1 \).

The implementation of the twisted boundary conditions in the numerical simulations can present some practical problems. In accordance with the discussion above, one could be tempted to store the lattice configurations, using the link vectors \( X_{ij}^\mu \) rather than the vertex coordinates \( X_i^\mu \). Such a parametrization is however dangerous since it allows rounding errors to accumulate. In particular, this can lead to a numerical violation of the boundary conditions initially imposed, since both the vertex coordinates and the triangulations are changed many times. To avoid this problem we decomposed, following (4.2), the vector \( X_{ij}^\mu \) as

\[ X_{ij}^\mu = X_j^\mu - X_i^\mu + E_{ij}^\mu, \] (4.3)

where \( E_{ij}^\mu = n_{1ij}^E E_1^\mu + n_{2ij}^E E_2^\mu \) and the integers \( n_{1ij}^E \) are non-zero when the link \( < ij > \) connects points in different elementary cells. To store the configuration we need the positions \( X_j^\mu \) of all points and two integers \( n_{1ij}^E, n_{2ij}^E \) for every link. Vectors \( E_{ij} \) are additive, \( i.e. E_{ij} + E_{jk} + E_{ki} = 0 \) for every triangle \( < ijk > \). This implies that also the \( n_{ij} \) are additive. The decomposition (4.3) is however not unique. For arbitrary
integers $l_1^i$ and $l_2^i$ one can perform a transformation

$$
X^\mu_i \rightarrow X^\mu_i + l_1^i E_1^\mu + l_2^i E_2^\mu,
$$

$$
E_{ij} \rightarrow E_{ij} + l_1^i E_1^\mu + l_2^i E_2^\mu - l_1^j E_1^\mu - l_2^j E_2^\mu,
$$

which leaves (4.3) invariant. This transformation can be used to keep the $E_{ij}$ vectors bounded.

In the last section we defined the mass gap by the exponential decay of the two-point function, the marked points on the surface separated by a distance $y$ in target space. Here we shall consider a different definition, making again use of the twisted boundary conditions. This time we consider a system with a zero projected area $A_p$, with only one non-zero vector $E_i$. We take $E_1 = (y, 0, ...), E_2 = 0$. These boundary conditions permit to measure a two-loop function, where two identical loops are separated in a translationally invariant way by a distance $y$ and where we sum over the loop size. The two-loop function should have the same scaling behaviour as the two point function.

## 5 Numerical results

Let us now turn to the numerical simulations of the system. For the measurements of the string tension and mass gap we will use the twisted boundary conditions described in the last section. For measurements of the specific heat where no extended frame in target space is needed these reduce to ordinary periodic boundary conditions of surfaces with toroidal topology. We are going to use essentially standard Monte Carlo techniques, but before presenting the results, let us collect various computational aspects.

The degrees of freedom in the model are the vertex coordinates and those describing the connectivity of the triangulation. Both kinds we update by using a local Metropolis algorithm. For the triangulation we use the familiar flip procedure, which is known to be ergodic in the class of all triangulations with the same topology. We use the standard restriction that we do not allow non-trivial loops of order one and two. In the graph dual to a given triangulation, i.e. a $\phi^3$ graph, this restriction corresponds to considering graphs without tadpoles and self-energy parts. New values for the coordinates are proposed by performing a heatbath or an $\omega = 2$ overrelaxation step with respect to the gaussian part of the action. A Metropolis accept/reject step then takes the curvature term into account. We found this coordinate update slightly more efficient than the one in which the change in coordinates is generated from a fixed distribution. The gain we measured in decorrelation time was somewhat less than a factor two. The flip rate was close
to 33% in all our calculations. The acceptance rate for the coordinate update varied with the simulation parameters and was typically similar to the flip rate. Its lowest value was 20%, at $\lambda = 1.5$ with periodic boundary conditions.

The dynamical nature of the triangulation restricts the possibilities to vectorize the simulation program. We have therefore developed a program in which vectorization is achieved by simulating several systems in parallel. Our results have been obtained by simulating 64 parallel systems on a CRAY-YMP. The CPU time required per sweep per bond was $5.7 \mu s$. For more details on the algorithm and the program, see ref. \[26, 32\].

While the simultaneous simulation of several systems significantly reduces the CPU time needed per sweep, it does, of course, not bring down the number of sweeps needed to decorrelate configurations. In fig. 1 we show estimates of the integrated autocorrelation time, $\tau_{int}$, for the most non-local quantity measured, the radius of gyration. $\tau_{int}$ refers to the autocorrelation in one system, and the values given are averages over the 64 systems considered. We may note that the increase in $\tau_{int}$ between $N = 144$ and 256, at fixed coupling, corresponds to $\tau_{int} \propto N^{z'}$ with $z'$ roughly 1.6. In our calculations, the number of sweeps used for thermalization corresponds in most cases to $20\tau_{int}$ and always to more than $10\tau_{int}$. Measurements were typically taken over $40\tau_{int}$. This length of the runs is not very long so one might worry about insufficient thermalization. As a check, we therefore carried out a few additional much longer runs. The longest one, at $(N, \lambda) = (144, 1.5)$ with periodic boundary conditions, was two million sweeps, corresponding to $700\tau_{int}$. No significant change in the observables was observed in these extended runs, and the statistical errors scaled approximately as they should with increasing number of iterations. The integrated autocorrelation time did however increase by 15%. The numbers given above for $\tau_{int}$ and its $N$ dependence should therefore be used only as a rough guide.

In the analysis of the specific heat below we use the multi-histogram technique by Ferrenberg and Swendsen \[29\]. This method allows continuation of results obtained at one or more couplings to, in principle, arbitrary couplings. With limited statistics this is, of course, not true, and care is needed in selecting the range for the continuation. We made the selection in the following way, similar to that used in ref. \[28\]. We start by determining at each simulated coupling two numbers $S_1$ and $S_2$ such that the probabilities that $S_C < S_1$ and that $S_C > S_2$ are both 25%. We then perform a single-histogram continuation, restricted to the coupling interval where the resulting value of $S_C$ lies between $S_1$ and $S_2$. With this restriction we found that single-histogram results for the specific heat agreed whenever the couplings overlapped. Final numbers were obtained by combining the histograms according
to the method in ref. [29]. The continuation was carried out over a coupling range that could be covered by the single-histogram intervals defined earlier. The observed consistency between different single-histogram results we take as another indication that thermalization effects are under control.

Finally, we mention that the statistical errors quoted below are jackknife errors \[33\], obtained by taking the results from the different systems as 64 independent measurements.

## 5.1 Specific heat

As observed first by Catterall [16], using spherical topology, the specific heat has a maximum at a finite value of the coupling, $\lambda \approx 1.5$. We have performed a detailed study of the specific heat for the case of toroidal topology, with periodic boundary conditions. To accurately determine the location and height of the maximum we use the multi-histogram technique by Ferrenberg and Swendsen [29], as described above. Fig. 2 shows the specific heat as a function of $\lambda$ for system sizes up to $N = 576$. The curves are the results from multi-histogram analysis, and the points show the results at simulated couplings. In agreement with the results from earlier studies using spherical topology, we find a maximum in the specific heat near $\lambda = 1.5$, the height of which increases with increasing $N$.

The dependence of the position of the maximum, $\lambda^N_c$, on the system size is shown in fig. 3. The three data points with the largest values of $N$ are well described by the form $\lambda^N_c = \lambda_c - \text{const} N^{-\alpha}$ with $\alpha = 1/2$. However, acceptable fits to this form can be obtained for a fairly wide range of $\alpha$ values. As a result of this uncertainty about the form of the finite-size corrections, we cannot get a very precise estimate of the $N \to \infty$ limit $\lambda_c$. Fits with $\chi^2 < 1$ are obtained for $0.35 < \alpha < 1.2$. The corresponding bounds on the critical coupling are $1.47 < \lambda_c < 1.53$.

How the height of the maximum, $C^N_{\max}$, varies with the system size is shown in fig. 4. The data strongly suggest that $C^N_{\max}$ remains finite in the limit $N \to \infty$. The increase in $C^N_{\max}$ with increasing $N$ becomes slower at large $N$. The finite-size correction to the $N \to \infty$ value appears to vanish faster than $1/N$ at large $N$.

## 5.2 Mass gap

In section 2 we saw how one can define a mass gap measurement $m(\lambda, N, L)$ in the canonical ensemble. In the thermodynamic limit $N \to \infty$, this measurement gives information about the mass gap $m(\lambda, \mu)$, as defined from the exponential decay of the grand canonical partition function. We have

$$m_{\text{can}}(\lambda, t) \equiv \lim_{N \to \infty, L/N = t} m(\lambda, N, L) = m(\lambda, \mu(t)) ,$$

(5.1)
where $\mu(t)$ is the solution to

$$\frac{\partial m(\lambda, \mu(t))}{\partial \mu} = \frac{1}{t}. \quad (5.2)$$

Corresponding to the expected scaling $m(\lambda, \mu) \sim c(\lambda)\mu_\nu(\lambda)$ at small $\mu_R$, we have (eq. 3.19) $m_{\text{can}}(\lambda, t) \sim D(\lambda)t^{-\nu(\lambda)/1-\nu(\lambda)}$ at small $t \sim d(\lambda)\mu_\nu(\lambda)$. By verifying this scaling of $m_{\text{can}}(\lambda, t)$, we can compute the critical exponent $\nu(\lambda)$.

The numerical calculations are restricted to finite $N$ and it is therefore essential to keep finite-size effects under control in the estimate of $m_{\text{can}}(\lambda, t)$. Let us give an estimate of how large $N$ has to be taken at a given value of $t$. The finite-size effects occur when the asymptotic relation $G(\lambda, \mu, L) \sim m(\lambda, \mu)L$ is not fulfilled. This relation is valid if terms of lower order in $L$ are negligible, which should be the case when $m(\lambda, \mu)L$ is large. We thus expect finite-size corrections to be small if $m_{\text{can}}(\lambda, t)L$ is large. This condition can in the scaling regime be written as $t^{\nu(\lambda)/1-\nu(\lambda)}L \gtrsim 1$, or

$$tN \gtrsim N^{\nu(\lambda)}. \quad (5.3)$$

Let us briefly discuss the most likely finite-size effect to the mass gap. The generic form of the two point function will be (in the following we will suppress the dependence on the coupling constant $\lambda$)

$$Z(\mu, y) \sim y^{-\alpha}e^{-m(\mu)y} \quad (5.4)$$

This means that Gibbs free energy will be of the form

$$G(\mu, y) = m(\mu)y + \alpha \ln y \quad (5.5)$$

and we would expect corrections of the form

$$m_{\text{eff}}(\mu) = \frac{\partial G}{\partial y} = m(\mu) + \frac{\alpha}{y}. \quad (5.6)$$

We can now translate this to the canonical ensemble and get

$$m_{\text{can}}(t, N) = m_{\text{can}}(t) + \frac{\alpha}{tN}. \quad (5.7)$$

Our calculations were performed at two different couplings, one in the crumpled

phase, $\lambda = 1.25$, and one close to the maximum in the specific heat, $\lambda = 1.5$. Fig. 5 shows the results for $m(\lambda, N, L)$, plotted against $t = L/N$. In order to check the finite-size dependence, we carried out simulations for different $N$ at some fixed values of $t$. As expected from the discussion above, the finite-size corrections turn out to be largest at small $t$. The corrections are large for small $N$ ($N = 64$ data are omitted in the plot), but seem to rapidly become smaller with increasing $N$. The
results for the largest values of $N$ agree well, which suggests that these can be taken as reasonable approximations to the thermodynamic limit.

At large $t$ we find that the results at the two couplings are similar, which is expected due to the dominance of the gaussian term in the action. The interesting region from the point of view of scaling is, however, at small $t$, and there we find a clear coupling dependence. The results at the three smallest values of $t$ are at both couplings consistent with scaling, but with different exponents $\nu(\lambda)$. Fitting the data points with largest $N$ to eq. (3.19) we get $\nu(\lambda = 1.25) = 0.279(7)$ and $\nu(\lambda = 1.5) = 0.417(7)$, with $\chi^2$ near one in both cases. The errors are statistical only. Clearly, one would like to confirm that these results reflect the true scaling behaviour by going to much smaller values of $t$. Increasing finite-size effects and decorrelation times have prevented us from doing so. In the limited regime probed so far, we find that the scaling assumption gives a good description of fairly accurate data.

5.3 String tension

Our method to calculate the string tension is very similar to that for the mass gap. Corresponding to eq. (5.1), we have

$$\sigma_{\text{can}}(\lambda, r) \equiv \lim_{N \to \infty, A/N = r} \sigma(\lambda, N, A) = \sigma(\lambda, \mu(r)),$$  

(5.8)

where $\sigma(\lambda, N, A)$ is the measurement defined in section 2 and $\mu(r)$ is given by

$$\frac{\partial \sigma(\lambda, \mu(r))}{\partial \mu} = \frac{1}{r}.$$  

(5.9)

The major difference is in the expected scaling behaviour. As explained in section 3, we expect $\sigma_{\text{can}}(\lambda, r) = \sigma_0(\lambda) + \sigma(\lambda)r^{2\nu/(1-2\nu)}$ with an in general non-zero $r$ independent term $\sigma_0(\lambda)$.

In the same way as for the mass gap, we can estimate for which parameter values one should expect significant finite-size effects. They should be small if $\sigma_{\text{can}}(\lambda, r)A$ is large. If we assume the scaling form and that $\sigma_0(\lambda) = 0$, then we can write this condition as $r^{2\nu/(1-2\nu)}A \gtrsim 1$, or

$$r^N \gtrsim N^{2\nu}.$$  

(5.10)

Repeating the arguments above we expect the finite-size scaling corrections to be of the form

$$\sigma_{\text{can}}(\lambda, r, N) = \sigma_{\text{can}}(\lambda, r) + \frac{\alpha(\lambda)}{r^N}.$$  

(5.11)

We have carried out simulations at three couplings $\lambda = 1.25, 1.4$ and 1.5, for system sizes up to $N = 576$. The results are summarized in fig. 6, where $\sigma(\lambda, N, A)$
is plotted against \( r = A/N \). Finite-size effects are large for \( r \to 0 \) and \( \lambda \to \lambda_c \), as expected. Therefore, we have not been able to measure the string tension for very small \( r \) at \( \lambda = 1.5 \). It seems, however, that \( \sigma_0(\lambda) \) could vanish near \( \lambda = 1.5 \), as required for an interesting scaling behaviour. For this we need, in addition, that the exponent in

\[
\sigma_{\text{can}}(\lambda, r) = \sigma(\lambda) r^\omega
\]

(5.12)
satisfies \( \omega = 2\nu(\lambda)/(1 - 2\nu(\lambda)) \), where \( \nu(\lambda) \) is the mass gap exponent. The value at \( \lambda = 1.5 \) obtained above, \( \nu(\lambda = 1.5) = 0.417(7) \), corresponds to \( \omega = 4.7 \pm 0.5 \).

Let us check if the measured string tension at \( \lambda = 1.5 \) shows this scaling behaviour. To do so we first extrapolate the results to the thermodynamic limit. In fig. 7 we show \( \sigma(\lambda = 1.5, A, N) \) at three fixed values of \( r \) for \( N = 144, 256, 400 \) and 576. For all three \( r \), we find that the \( 1/N \) term gives a statistically acceptable description of the finite-size effects, in agreement with (5.11). The lines shown are fits to such a form. We can now test for scaling by fitting the extrapolated values to eq. (5.12). We find that the data indeed are in good agreement with this behaviour. This is illustrated in fig. 8. The exponent obtained, \( \omega = 3.94 \pm 0.06 \), is, moreover, close to the value from the mass gap measurement.

This analysis suggests that there is a coupling close to \( \lambda = 1.5 \) where the scaling form eq. (5.12) is valid. It is however likely that it is not exactly valid at \( \lambda = 1.5 \). We therefore want to check how the estimate of \( \omega \) is affected if we allow for a non-zero value of \( \sigma_0(\lambda = 1.5) \). To this end we performed a sequence of fits for fixed values of \( \omega \), having \( \sigma_0 \) as a free parameter. Fits with \( \chi^2 < 1 \) were obtained for \( 3.4 < \omega < 4.5 \).

The determination of \( \omega \) also involves an assumption about the precise form of the finite-size dependence. To get an idea of the importance of this assumption we considered the \( N = 576 \) data directly, without any extrapolation procedure. Repeating the type of fit with \( \omega \) fixed and \( \sigma_0 \) as a free parameter, we obtained this time \( \chi^2 < 1 \) for \( 3.4 < \omega < 4.0 \). We take this to suggest that the uncertainty in \( \omega \) arising from the \( N \to \infty \) extrapolation is smaller than 0.5. We then arrive at the estimate \( 3.4 < \omega < 5.0 \), corresponding to \( 0.38 < \nu < 0.42 \).

The results presented above were all obtained in the crumpled phase or close to the phase transition. Unfortunately, we have no conclusive results for the flat phase. For a flat surface the minimum in the free energy \( \mathcal{F} \) is at a finite value of \( r \). The canonical string tension is therefore negative for sufficiently small \( r \). We tried to verify this by performing some simulations at \( \lambda = 1.75 \). Due to large finite-size effects and decorrelation times we could however not estimate values for the limit of large \( N \).
5.4 Radius of gyration

Another important quantity for the characterization of the surfaces is the radius of gyration $R_g(\lambda, N)$, which we, in the case of periodic boundary conditions, define as

$$R_g(\lambda, N) = \frac{1}{N} \sum_i <\sqrt{(X_i^\mu)^2}>. \quad (5.13)$$

The behaviour at large $N$,

$$R_g(\lambda, N) \sim N^{1/d_H(\lambda)}, \quad (5.14)$$

is taken to define the Hausdorff dimension $d_H(\lambda)$. $d_H(\lambda)$ is said to be infinite if the growth with $N$ is logarithmic.

The Hausdorff dimension is expected to decrease with increasing $\lambda$, from infinity at $\lambda = 0$ to $d_H = 2$ at large $\lambda$. We here want to estimate the value at the phase transition, $d_H(\lambda_c)$. To this end we consider the radius of gyration at the maximum in the specific heat, $R_g(\lambda_c^N, N)$, which we obtain by using again the Ferrenberg-Swendsen method. The dependence of $R_g(\lambda_c^N, N)$ on $N$ is shown in the log-log plot fig. 9. If we assume that $d_H(\lambda)$ varies smoothly across the transition, then the slope at large $N$ should give us $1/d_H(\lambda_c)$. Over the range of $N$ studied, we find that the slope keeps decreasing with increasing $N$. The two data points with largest $N$ correspond to a value $d_H=3.41(7)$. We take this to suggest the lower bound $d_H(\lambda_c) > 3.4$.

6 Discussion

We have shown that our data are compatible with a second order transition at $\lambda = \lambda_c \approx 1.5$. This transition is characterized by a vanishing string tension $\sigma(\lambda)$ for $\lambda \to \lambda_c$ and a vanishing mass gap $m(\lambda)$ of the two point function for $\lambda \to \lambda_c$. Furthermore

$$\frac{\sigma(\lambda)}{m^2(\lambda)} \approx \text{const.} \quad \text{for} \quad \lambda \to \lambda_c. \quad (6.1)$$

This is precisely what we would expect in a string theory without tachyons. It is not clear to us what is the effective continuum theory which is compatible with such behaviour in three-dimensional target space. As we explained in the introduction one part of the motivation for investigating the critical behaviour of strings with extrinsic curvature was the relation to fermionic strings where extrinsic curvature terms arise in a natural way. From this point of view the results are encouraging: Models with extrinsic curvature seem to exhibit non-trivial critical behaviour. From the point of view of the superstring it would be interesting to verify that this non-trivial behaviour persists for strings embedded in higher dimensions. A further,
most interesting problem to address would be the inclusion of the Wess-Zumino-like terms mentioned in the introduction. Such terms also arise naturally for fermionic strings and could change the critical behaviour even further. It is unfortunately not clear to us how these terms can be a part of a numerical study like the one performed here since the Wess-Zumino term is complex.

In the context of conformal field theory the existence of a second order transition gives rise to further interesting questions. For the crystalline surfaces we have a well defined regular two-dimensional lattice structure, and to each lattice site we have associated $d$ real variables $x_i$ if target space is $d$-dimensional. The action between these variables is local, although it is not polynomial due to the extrinsic curvature term. In case there is a second order transition for a finite value $\lambda_c$ of the extrinsic curvature coupling one would be tempted to conjecture the existence of an associated conformal field theory, characterized by a central charge $c$. If the extrinsic curvature coupling is put to zero we just have a standard gaussian field theory on the lattice and it has a gaussian fixed point corresponding to $d$ free fields and a total central charge $d$. In the present context the nature of this fixed point is a little unusual, since it is an infrared fixed point and not the usual ultraviolet gaussian fixed point known for field theories in less than four dimensions. In fact it seems as if the following effective action describes the behaviour of the system for $\lambda < \lambda_c$:

$$S_{eff} = \int d^2 \xi \ x_\mu(\xi) \left[ -m^2(\lambda) \partial^2(\xi) + \lambda \partial^4(\xi) \right] x_\mu(\xi)$$  \hspace{1cm} (6.2)

where

$$m(\lambda) = m_0(\lambda_c - \lambda).$$  \hspace{1cm} (6.3)

Eqs. (6.2) and (6.3) mean that the long range properties of the system are determined by the gaussian part and for this reason one would classify the first fixed point away from $\lambda = 0$ as ultra-violet stable. If we naively apply Zamolodchikov's $c$-theorem we can conclude that the central charge of the conformal theory associated with "crumpling" transition has to be larger than or equal $d$. In fact it is easy to calculate the central charge at the fixed point, either analytically or numerically, using the effective action (6.2)-(6.3). One gets $c = 2d$. The numerical results for $d = 3$ using the full action for crystalline surfaces, not the approximation (1.2), indicate that the (effective) central charge is between zero and one [18], in contradiction with the $c$-theorem. Most likely we are in fact considering a non-unitary conformal field theory in which case the $c$-theorem is formally invalidated. However, one expects a $c_{eff}$-theorem to apply even in that case and since the central charge measured from the partition function in [18] is the effective central charge $c_{eff}$ it does not change the apparent contradiction. At least the situation calls for further investigations and one should keep in mind that the action involving extrinsic curvature is non-polynomial.
and it might be that we in this case are mislead by the general principles, which do not necessarily apply. Whatever the effective central charge is one can couple the lattice system to two-dimensional quantum gravity and ask for the properties of the corresponding random surface theory. This is what we have done, and since the situation already without coupling to gravity (i.e. the crystalline surfaces) appears to be non-trivial, the same might be the case when coupled to gravity. In fact the connection between the conformal field theory on the world-sheet and the scaling properties of the correlation functions in target space has to be non-trivial if the model without the Wess-Zumino term will describe a genuine surface theory which is euclidean invariant and where the string tension scales. Recent analytic results of Polchinski and Strominger indicate that such non-trivial situations can indeed occur precisely when the world-sheet variables $x_i$ contain non-polynomial terms [35]. Whether our effective string theory with a non-trivial scaling of string tension and mass gap has any relation to the scenario suggested in [35] is presently under investigation.

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Fig.1 Integrated autocorrelation time for the radius of gyration against system size. The boundary conditions are periodic, \( y_1 = y_2 = 0 \).

Fig.2 The specific heat as a function of \( \lambda \). The bands give the results with errors of the multi-histogram analysis.

Fig.3 \( \lambda_c^N \) against \( N^{-1/2} \).

Fig.4 \( C_{\text{max}}^N \) against \( 1/N \).

Fig.5 The mass gap measurement \( m(\lambda, N, L) \) plotted against \( t = L/N \). The straight lines are fits to eq. (3.19).

Fig.6 The string tension measurement \( \sigma(\lambda, N, A) \) plotted against \( r = A/N \). The lines connect points with the same \( \lambda \) and largest \( N \).

Fig.7 \( \sigma(\lambda, N, A) \) against \( 1/N \) at three fixed values of \( r \). The lines are fits to the form \( \sigma(\lambda, N, A) = \sigma_{\text{can}}(\lambda, r) + c(r)/N \).

Fig.8 \( \sigma(\lambda, N, A) \) against \( r^\omega \) at \( \lambda = 1.5 \). The full circles are extrapolations to \( N \to \infty \), as explained in the text. For the exponent we have used the value \( \omega = 3.94 \), obtained from a fit to eq. (5.12).

Fig.9 \( R_g(\lambda_c^N, N) \) against \( N \).