Finite Dimensional Pointed Hopf Algebras
with Abelian Coradical and Cartan matrices

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Abstract. In a previous work [AS2] we showed how to attach to a pointed Hopf algebra $A$ with coradical $k\Gamma$, a braided strictly graded Hopf algebra $R$ in the category $\mathcal{YD}$ of Yetter-Drinfeld modules over $\Gamma$. In this paper, we consider a further invariant of $A$, namely the subalgebra $R'$ of $R$ generated by the space $V$ of primitive elements. Algebras of this kind are known since the pioneering work of Nichols. It turns out that $R'$ is completely determined by the braiding $c : V \otimes V \rightarrow V \otimes V$. We denote $R' = \mathfrak{B}(V)$. We assume further that $\Gamma$ is finite abelian. Then $c$ is given by a matrix $(b_{ij})$ whose entries are roots of unity; we also suppose that they have odd order. We introduce for these braidings the notion of braiding of Cartan type and we attach a generalized Cartan matrix to a braiding of Cartan type. We prove that $\mathfrak{B}(V)$ is finite dimensional if its corresponding matrix is of finite Cartan type and give sufficient conditions for the converse statement. As a consequence, we obtain many new families of pointed Hopf algebras. When $\Gamma$ is a direct sum of copies of a group of prime order, the conditions hold and any matrix is of Cartan type. We apply this result to show that $R' = R$, in the case when $\Gamma$ is a group of prime exponent. In other words, we show that a finite dimensional pointed Hopf algebra whose coradical is the group algebra of an abelian group of exponent $p$ is necessarily generated by group-like and skew-primitive elements. As a sample, we classify all the finite dimensional coradically graded pointed Hopf algebras whose coradical has odd prime dimension $p$. We also characterize coradically graded pointed Hopf algebras of order $p^4$.

§1. Introduction.

Let $k$ denote an algebraically closed field of characteristic 0. To motivate the results of the present paper, we have to recall a strategy proposed in [AS2]. Let $A$ be a Hopf algebra, let $A_0 \subseteq A_1 \subseteq \ldots$ be its coradical filtration and let $gr\, A$ be the associated graded coalgebra. If $A_0$ is a Hopf subalgebra of $A$ (for instance, if $A$ is pointed, that is, its simple subcoalgebras are one-dimensional) then $gr\, A = \bigoplus_{n \geq 0} gr\, A(n)$ is a graded Hopf algebra; $A_0 \simeq gr\, A(0)$ is a Hopf subalgebra and the projection $\pi : gr\, A \rightarrow gr\, A(0)$ with kernel $\bigoplus_{n > 0} gr\, A(n)$, is a Hopf algebra map and a retraction of the inclusion. In this situation, a general technique due to Radford [Ra] and explained in categorical terms by Majid [Mj1] applies. Let $R$ be the algebra of coinvariants of $\pi$. Then $R$ is a braided Hopf algebra in the category $A_0 \mathcal{YD}$ of Yetter-Drinfeld modules over $A_0$ and $A$ can be reconstructed by from $R$ and $A_0$. This reconstruction is called bosonization or biproduct.

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The principle proposed in [AS2] to treat questions about $A$ was to solve the corresponding problem for $R$, to pass to $\text{gr } A$ by bosonization and finally to lift the information to $A$. This principle was applied successfully to the simplest possible braided Hopf algebras $R$, namely (finite dimensional) quantum linear spaces, in [AS2].

In the present paper we are concerned with braided Hopf algebras $R$. It turns out that a braided Hopf algebra $R$ arising from a coalgebra filtration as above satisfies the following conditions:

(1.1) $R = \oplus_{n \geq 0} R(n)$ is a graded braided Hopf algebra.

(1.2) $R(0) = \mathbb{k}V$ (hence the coradical is trivial, cf. [Sw, Chapter 11]).

(1.3) $R(1) = P(R)$ (the space of primitive elements of $R$).

In particular, $R$ is a strictly graded coalgebra in the sense of [Sw, Chapter 11]. These conditions imply that the coalgebra filtration of $R$ itself coincides with the filtration associated to the gradation; that is, $R$ coradically graded, see [CM] and also [AS2, Section 2]. However, the notion of "coradically graded" pass to $\text{gr } A$ through bosonization but "strictly graded" not.

A first rough invariant of such $R$ is the dimension of $P(R)$ and we call this the rank of $R$.

It is in general not true that a braided Hopf algebra $R$ satisfying (1.1), (1.2) and (1.3) also verifies

(1.4) $R$ is generated as algebra over $\mathbb{k}$ by $R(1)$.

See Section 8. Note that the subalgebra $R'$ of $R$ generated by $R(1)$ is a Hopf subalgebra of $R$ and satisfies (1.1), (1.2), (1.3) and (1.4). We shall say in general that a graded algebra $R = \oplus_{n \geq 0} R(n)$, with $R(0) = \mathbb{k}V$, is generated in degree one if (1.4) holds.

We are specifically interested in braided Hopf algebras satisfying (1.1), (1.2), (1.3) and (1.4); or in other words, which are strictly graded and generated in degree one. We propose to call them Nichols algebras since they appear for the first time in the article [N]. Without pretending to give an exhaustive historical analysis, let us mention that this notion was considered by several authors under various presentations; see [L3], [Mü], [Ro2], [Rz], [Sbg], [W]. Let us explain the reasons of our interest in this special kind of braided Hopf algebras. We refer to the survey article [AG] for more details.

First, there are alternative descriptions of Nichols algebras. Let $R$ be a braided Hopf algebra satisfying (1.1) and (1.2). Let $V = R(1)$; it is a Yetter-Drinfeld submodule of $R$ contained in $P(R)$. The tensor algebra $T(V)$ is also a Yetter-Drinfeld module over $A_0$ and in fact a braided Hopf algebra with comultiplication determined by declaring primitive any element of $V$. Hence the algebra morphism $p : T(V) \to R$ given by the inclusion of $V$ on $R$ is a Hopf algebra morphism. Now $R$ is a Nichols algebra if and only if

(i) $p$ is surjective and

(ii) $\text{ker } p$ is the direct sum of the kernels of the "quantum antisymmetrizers".

We recall that the "quantum antisymmetrizers" are constructed from the braiding $c : V \otimes V \to V \otimes V$. However, an efficient description of the relations does not follow easily from (ii) and it is in general very difficult to decide when $B(V)$...
is finite dimensional. Conversely, given a Yetter-Drinfeld module $V$, the tensor algebra $T(V)$ divided out by the direct sum of the kernels of the ”quantum antisymmetrizers” is a Nichols algebra with $P(R) = V$. This line of argument also shows that a Nichols algebra with prescribed space of primitive elements is unique up to isomorphisms; that is, the notions of ”Nichols algebra” and ”Yetter-Drinfeld module” are naturally equivalent. We shall denote by $\mathfrak{B}(V)$ the Nichols algebra built from the Yetter-Drinfeld module $V$.

It follows also from the description above that $\mathfrak{B}(V)$ is not only completely determined by the Yetter-Drinfeld module structure of $V$, but even more, it is completely determined by the braiding $c : V \otimes V \rightarrow V \otimes V$. This is a solution of the braid equation; conversely, given a solution of the braid equation $c : V \otimes V \rightarrow V \otimes V$, where $V$ is an arbitrary vector space, we can build a braided Hopf algebra $\mathfrak{B}(V)$ in a suitable braided category. In this sense, Nichols algebras seem like symmetric algebras; however, as we shall comment below, they look more like enveloping algebras of Lie algebras. But there is no evident way (for us) to define a ”braided Lie algebra” and, in fact, there is no need (for our purposes) because all the information is in the map $c$.

Let us mention another description of Nichols algebras, at least when the Yetter-Drinfeld module $V$ carries an invariant non-degenerate symmetric bilinear form. (See [AG] for the general case). Then there is a unique extension of this bilinear form to $T(V)$ that transforms multiplication and unit in comultiplication and counit; the kernel of $p$ is exactly the radical of this bilinear form. Through this description, the positive part of the Borel-like subalgebra of a quantized enveloping algebra or of a Frobenius-Lusztig kernel appears as a Nichols algebra. See [L3], [Mü], [Ro2], [Sbg]; a brief account is given in Section 3.

Finally, there is another reason supporting our interest in Nichols algebras; it could be possible that a finer understanding of them gives a way to decide whether in characteristic 0 a finite dimensional braided Hopf algebra $R$ satisfying (1.1), (1.2) and (1.3) also satisfies (1.4). See Section 8.

The main question we address in the present paper is the following: given a finite abelian group $\Gamma$, characterize all the finite dimensional Nichols algebras $R$ in $\mathcal{YD}$. We split this question into two parts. In the first part, we consider braidings $c : V \otimes V \rightarrow V \otimes V$ arising from some finite abelian group and discuss under which conditions the algebra $\mathfrak{B}(V)$ is finite dimensional. We give several necessary and sufficient conditions; ultimately they connect the theory of pointed Hopf algebras, through Lusztig’s work, to Lie theory. A similar point of view is present in [Ro2], though with different methods and results. In the second part, it is necessary to show that a $V$ with $\mathfrak{B}(V)$ finite dimensional actually is realizable over the fixed finite abelian group $\Gamma$. This is a problem of a somewhat different, arithmetic nature; we completely solve it for $\Gamma = \mathbb{Z}/(p)$, $p$ an odd prime.

Let us assume from now on that $A_0$ is the group algebra of our fixed finite abelian group $\Gamma$. Then a finite dimensional Yetter-Drinfeld module $V$ admits a basis $x_1, \ldots, x_\theta$ such that, for some elements $g(1), \ldots, g(\theta) \in \Gamma, \chi(1), \ldots, \chi(\theta) \in \hat{\Gamma}$, the action and coaction of $\Gamma$ are given by

\begin{align*}
  h.x_j &= \chi(j)(h)x_j, \\
  \delta(x_j) &= g(j) \otimes x_j, \\
  j &= 1, \ldots, \theta.
\end{align*}
In other words, the isomorphism class of the Yetter-Drinfeld module $V$ is determined (up to permutation of the index set) by the sequences $g(1), \ldots, g(\theta) \in \Gamma$, $\chi(1), \ldots, \chi(\theta) \in \hat{\Gamma}$. Since we are interested in finite dimensional braided Hopf algebras $R$, we can assume that

$$\langle \chi(i), g(i) \rangle \neq 1,$$

cf. [AS2, Lemma 4.1]. As we said, the fundamental piece of information is the braiding $c$. Under the present hypothesis, it is given with respect to the basis $x_i \otimes x_j$ by

$$c(x_i \otimes x_j) = b_{ij} x_j \otimes x_i,$$

where $(b_{ij})_{1 \leq i,j \leq \theta} = ((\chi(j), g(i)))_{1 \leq i,j \leq \theta}$.

For convenience, we shall say a braiding is represented by $(b_{ij})$ if (1.7) holds for some basis of a fixed vector space $V$; we shall name in this case $(b_{ij})$ the braiding since we reserve ”matrix” for the data $(a_{ij})$ below.

The first main question we want to consider in this paper is: if the braiding is given by (1.7), when is $B(V)$ finite dimensional? We want to make use of the theory of Frobenius-Lusztig kernels (see [L1], [L2], [L3]).

**Definition.** We shall say that a braiding given by a matrix $b = (b_{ij})_{1 \leq i,j \leq \theta}$ whose entries are roots of unity is of Cartan type if for all $i,j$, $b_{ii} \neq 1$ and there exists $a_{ij} \in \mathbb{Z}$ such that

$$b_{ij} b_{ji} = b_{ii}^{a_{ij}}.$$

The integers $a_{ij}$ are completely determined once they are chosen in the following way:

(1.9) If $i = j$ we take $a_{ii} = 2$;
(1.10) if $i \neq j$, we select the unique $a_{ij}$ such that $-\ord b_{ii} < a_{ij} \leq 0$.

Then $a_{ij} = 0$ if and only if $a_{ji} = 0$, so that $(a_{ij})$ is a generalized Cartan matrix [K]. We transfer the terminology from generalized Cartan matrices to braidings of Cartan type: we say that a braiding $b = (b_{ij})_{1 \leq i,j \leq \theta}$ of Cartan type is indecomposable (resp., of finite type, symmetrizable) if $(a_{ij})$ is. We shall also make free use of Dynkin diagrams and refer to connected Cartan matrices for indecomposable ones. Finally, we shall say that a Yetter-Drinfeld module $V$ is of Cartan type (resp., connected, ...) if the matrix (1.7) is of Cartan type (resp., connected, ...).

We remark that there are examples of finite dimensional braided Hopf algebras $\mathcal{B}(V)$ of rank 2 which are not of Cartan type, see [N. pp. 1540 ff.].

The main known examples of braidings of Cartan type are given by $b = (q^{i,j})_{i,j \in I}$, where $q$ is a root of unity and $(I, \cdot)$ is a Cartan datum as in [L3, Chapter 1]. The braiding $(b_{ij})$ is in this case symmetric. Our first reduction is to pass from a general braiding of Cartan type to a symmetric one; this is possible, at least when all the $b_{ij}$’s have odd order, by the twisting operation of Drinfeld. The new braiding is still of Cartan type, and neither the elements $b_{ii}$ nor the Cartan matrix $(a_{ij})$.
change. As we said, the algebra \( B(V) \) depends only on \( c \) and not on the group; therefore, we have freedom to choose an appropriate group to perform the twisting. See Lemma 4.1.

**Remark.** There is an abuse of notation that should cause no confusion: a symmetric braiding means that \( b_{ij} = b_{ji} \) for all \( i, j \) but neither that \( a_{ij} \) is symmetric, nor that the braiding is a symmetry \( (c^2 = 1) \).

Even if the braiding \( (b_{ij}) \) is symmetric, this does not guarantee the existence of \( q \) as above.

**Definition.** Let \( b \) be a braiding of Cartan type with associated Cartan matrix \((a_{ij})\) as in (1.9), (1.10). We say that \( b \) is of FL-type if there exists positive integers \( d_1, \ldots, d_\theta \) such that for all \( i, j \),

\[
\begin{align*}
(1.11) & \quad d_i a_{ij} = d_j a_{ji} \text{ (hence \((a_{ij})\) is symmetrizable).} \\
(1.12) & \quad \text{There exists } q \in \mathbb{k} \text{ such that } b_{ij} = q^{d_i a_{ij}}.
\end{align*}
\]

Furthermore, we shall say that a braiding \( b \) is locally of FL-type if any \( 2 \times 2 \) submatrix of \( b \) gives a braiding of FL-type.

Let \((b_{ij})\) be a braiding of Cartan type. Let \( \mathcal{X} \) be the set of connected components of the Dynkin diagram corresponding to it. For each \( I \in \mathcal{X} \), we let \( N_I \) be the least common multiple of the orders of all the \( b_{ii} \)'s \((i \in I)\), \( g_I \) be the Kac-Moody Lie algebra corresponding to the generalized Cartan matrix \((a_{ij})_{i,j \in I} \) and \( n_I \) be the Lie subalgebra of \( g_I \) spanned by all its positive roots. Here is the main result of this article:

**Theorem 1.1.** Let \( b = (b_{ij}) \) be a braiding of Cartan type, corresponding to \( c : V \otimes V \to V \otimes V \). We also assume that \( b_{ij} \) has odd order for all \( i, j \).

1. If \( b \) is of finite type, then \( B(V) \) is finite dimensional. In fact

\[
\dim B(V) = \prod_{I \in \mathcal{X}} N_I^{\dim n_I}.
\]

2. Let \( B(V) \) be finite dimensional and \( b \) locally of FL-type. Let us assume that for all \( i \), the order of \( b_{ii} \)

   (a) is relatively prime to 3 whenever \( a_{ij} = -3 \) for some \( j \),

   (b) is different from 3, 5, 7, 11, 13, 17.

   Then \( b \) is of finite type.

Let \( p \) be an odd prime number. Assume that \( b = (b_{ij}) \) satisfies \( b_{ii} \neq 1 \) and the order of \( b_{ij} \) is either \( p \) or 1 for all \( i \) and \( j \). Then \( b \) is of Cartan type. It is also not difficult to see that it is locally of FL-type, see Lemma 4.3, using (1.10) for \( p = 3 \).

We conclude from Theorem 1.1:

**Corollary 1.2.** Let \( p \) be an odd prime number, \( \Gamma \) a finite direct sum of copies of \( \mathbb{Z}/(p) \) and \( V \) a finite dimensional Yetter-Drinfeld module over \( \Gamma \) with braiding \( b \). We assume (1.6). Then \( b \) is of Cartan type and

1. If \( b \) is of finite type, then \( B(V) \) is finite dimensional, and \( \dim B(V) = p^M \), \( M = \sum_{I \in \mathcal{X}} \dim n_I \).

2. If \( B(V) \) is finite dimensional and \( p > 17 \), then \( b \) is of finite type.
Now we discuss the second part of our initial question. Theorem 1.1 allows to construct new finite dimensional pointed Hopf algebras over our fixed group $\Gamma$. But we have first to determine which matrices $b$ of finite Cartan type actually appear over $\Gamma$ for some data as in (1.5).

We illustrate this in the case $\Gamma \simeq \mathbb{Z}/(p)$, where $p$ is a prime number. We assume that $p$ is odd; the case $p = 2$ is considered in [N, Th. 4.2.1].

**Theorem 1.3.** Let $\Gamma \simeq \mathbb{Z}/(p)$, where $p$ is an odd prime number. The following list contains all possible Nichols algebras of finite dimension over $\Gamma$ and finite dimensional coradically graded Hopf algebras with coradical isomorphic to $k\Gamma$.

1. The quantum lines and planes discussed in [AS2]. By bosonization we get respectively Taft algebras and book Hopf algebras cf. [AS1].

2. There exists a Nichols algebra with Dynkin diagram $A_2$ if and only if $p = 3$ or $p - 1$ is divisible by 3. For $p = 3$, we obtain by bosonization from Nichols algebras of dimension 27 exactly 2 non-isomorphic pointed Hopf algebras of dimension 81 with coradical of dimension 3. For $p \equiv 1 \mod 3$, we obtain by bosonization from Nichols algebras of dimension $p^3$ exactly $p - 1$ non-isomorphic pointed Hopf algebras of dimension $p^4$ with coradical of dimension $p$.

3. There exists a Nichols algebra with Dynkin diagram $B_2$ if and only if $p \equiv 1 \mod 4$. For each such prime, we obtain by bosonization from Nichols algebras of dimension $p^4$ exactly $2(p - 1)$ non-isomorphic pointed Hopf algebras of dimension $p^5$ with coradical of dimension $p$.

4. There exists a Nichols algebra with Dynkin diagram $G_2$ if and only if $p \equiv 1 \mod 3$. For each such prime, we obtain by bosonization from Nichols algebras of dimension $p^6$ exactly $2(p - 1)$ non-isomorphic pointed Hopf algebras of dimension $p^7$ with coradical of dimension $p$.

5. There exist Nichols algebras with finite Dynkin diagram of rank $\geq 3$ if and only if $p = 3$ and the corresponding Dynkin diagram is $A_2 \times A_1$ or $A_2 \times A_2$. We obtain respectively braided Hopf algebras of dimension $3^4$ and braided Hopf algebras of dimension $3^6$ over $\mathbb{Z}/(3)$; hence, we get exactly 4 Hopf algebras of dimension $3^5$ and 4 Hopf algebras of dimension $3^7$ with coradical of dimension $3$.

These Hopf algebras are new; but the Hopf algebras of order 81 ($p = 3$, type $A_2$) was known [N].

Theorem 1.3 opens the way to classify all finite dimensional pointed Hopf algebras with coradical of dimension $p$ via lifting as in [AS2]. An application of Theorem 1.3 is the determination of all possible finite dimensional pointed coradically graded Hopf algebras of dimension $p^4$. See Theorem 7.1. Again, the classification of all possible finite dimensional pointed Hopf algebras of dimension $p^4$ would follow by lifting to the filtered Hopf algebra.

Finally, we present a striking application of Corollary 1.2. Theorem 1.1 would have a wider range of applications if $R$ were necessarily be generated in degree one (i.e. $R = R'$). We see easily that this question is related, via bosonization and a standard argument in filtered algebras, see e.g. [AS2, Lemma 2.2], to the following:
Question. Is a finite dimensional pointed Hopf algebra necessarily generated by group-like and skew-primitive elements?

By Theorem 1.3, the answer is yes if the coradical has prime order. As an application of Corollary 1.2, we prove:

**Theorem 1.4.** Let $A$ be a finite dimensional pointed Hopf algebra whose coradical is the group algebra of an abelian group of prime exponent $p > 17$. Then $A$ is generated as an algebra by group-like and skew-primitive elements.

We observe that this result implies that for any fixed dimension, there are only finitely many isomorphism classes of pointed coradically graded Hopf algebras whose coradical is the group algebra of an abelian group of prime exponent $p > 17$.

The paper is organized as follows: We discuss the necessary facts concerning the twisting operation (resp., Frobenius-Lusztig kernels) in Section 2 (resp., Section 3); we prove Theorem 1.1 in Section 4. Section 5 is devoted to Nichols algebras over $\mathbb{Z}/(p)$ and the existence part of Theorem 1.3. We discuss isomorphisms between different bosonizations in Section 6; this concludes the proof of Theorem 1.3. Section 7 is devoted to pointed Hopf algebras of order $p^4$. In Section 8, we prove Theorem 1.4.

Our references for the theory of Hopf algebras are [Mo], [Sw]. Our conventions are mostly standard and were already used in [AS2].

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§2. Twisting.

The idea of twisting was introduced by Drinfeld [Dr2], see also [Dr1], [Dr3]. General facts are discussed in [Mj2], [Mj3]; for applications to quantized enveloping algebras see [Re], for applications to semisimple Hopf algebras see [Nk]. We briefly discuss the general idea referring to the literature for more details and work out accurately the case of our interest.

Let $A$ be a Hopf algebra. Given an invertible $F \in A \otimes A$, we can consider the "twisted" comultiplication $\Delta_F$ given by

$$\Delta_F(x) = F \Delta(x) F^{-1}.$$

Under certain conditions on $F$, $A_F$ (the same algebra $A$ but with comultiplication $\Delta_F$) is again a Hopf algebra. We are interested in the particular case when $F$ belongs indeed to $H \otimes H$, where $H$ is a Hopf subalgebra of $A$. We further assume that the inclusion $H \hookrightarrow A$ has a Hopf algebra retraction $\pi : A \to H$; so $A$ is the bosonization of $R$ by $H$ where $R$ is a braided Hopf algebra in the category $\mathcal{H} \otimes \mathcal{YD}$ of Yetter-Drinfeld modules over $H$. Then $H_F$ is a Hopf subalgebra of $A_F$ and $\pi$ is again a Hopf algebra map. If $H_F \simeq H$ (this happens for instance if $H$ is abelian), then $F$ induces an autoequivalence $V \mapsto V_F$ of the braided category $\mathcal{H} \otimes \mathcal{YD}$ and $A_F$ is in fact the bosonization of $R$ by $H_F$. See [Mi2].
We now assume that $H = k\Gamma$ where $\Gamma$ is a finite abelian group. We fix non-zero elements $y(1), \ldots, y(M)$ in $\Gamma$ such that $\text{ord } y(\ell) = E_\ell$,

\begin{equation}
\Gamma = \langle y(1) \rangle \oplus \cdots \oplus \langle y(M) \rangle, \quad \text{and } E_\ell | E_{\ell+1} \text{ for } \ell = 1, \ldots, M - 1.
\end{equation}

Let $q_i$ be root of unity of order $E_i$; we abbreviate $q = q_M$. Let $D_i$ be the smallest positive integer such that $q_i = q_i^{D_i}$.

Let $\gamma(i) \in \hat{\Gamma}$ be the unique character such that $\langle \gamma(i), y(j) \rangle = q_i^{\delta_{ij}}$, for $1 \leq i, j \leq N$. Then $\text{ord } \gamma(\ell) = E_\ell$ and

\begin{equation}
\hat{\Gamma} = \langle \gamma(1) \rangle \oplus \cdots \oplus \langle \gamma(M) \rangle.
\end{equation}

Given $g \in \Gamma$, $\chi \in \hat{\Gamma}$ we shall use the notation $g_i, \chi_j$ meaning that

\begin{equation}
g = g_1 y(1) + \cdots + g_M y(M), \quad \chi = \chi_1 \gamma(1) + \cdots + \chi_M \gamma(M).
\end{equation}

We identify $H$ with the Hopf algebra $k\hat{\Gamma}$ of functions on the group $\hat{\Gamma}$; we denote by $\delta_\tau \in H$ the function given by $\delta_\tau(\zeta) = \delta_{\tau,\zeta}$, $\tau, \zeta \in \hat{\Gamma}$. Then

$$
\delta_\tau = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \langle \tau, g^{-1} \rangle g.
$$

We shall describe the datum for twisting $F$ in terms of 2-cocycles, following [Nk], [Mv]. The theory of the cocycles we are interested in goes back to Schur and there is an extensive literature on it; we refer to [BT].

We recall that $\omega : \hat{\Gamma} \times \hat{\Gamma} \to k^\times$ is a 2-cocycle if $\omega(\tau, 1) = \omega(1, \tau) = 1$ and

\begin{equation}
\omega(\tau, \zeta) \omega(\tau\zeta, \eta) = \omega(\tau, \zeta \eta) \omega(\zeta, \eta).
\end{equation}

It is not difficult to deduce that

\begin{equation}
\omega(\tau\zeta, \chi) \omega(\chi, \tau\zeta)^{-1} \omega(\zeta, \chi)^{-1} \omega(\chi, \zeta) = \omega(\tau, \chi) \omega(\chi, \tau)^{-1}.
\end{equation}

**Lemma 2.1.** Let $c_{ij}$, $1 \leq i < j \leq M$, be integers such that $0 \leq c_{ij} < E_i$. Let $\omega : \hat{\Gamma} \times \hat{\Gamma} \to k^\times$ be the map defined by

\begin{equation}
\omega(\tau, \chi) = \prod_{1 \leq i < j \leq M} q_i^{c_{ij}\tau_j \chi_i} = q_1^{\sum_{1 \leq i < j \leq M} D_i c_{ij} \tau_j \chi_i}.
\end{equation}

Then $\omega$ is a 2-cocycle. Any 2-cocycle can be presented in this way up to a coboundary.

**Proof.** The first statement follows by a direct computation that we leave to the reader. The second is not needed in the remaining of this paper, but follows from [BT, Th. 4.7, p. 44]. \qed
Lemma 2.2 [Mv], [Nk]. Let \( \omega : \hat{\Gamma} \times \hat{\Gamma} \to k^\times \) be a 2-cocycle and let \( F \in H \otimes H \) be given by

\[
F = \sum_{\tau, \zeta \in \hat{\Gamma}} \omega(\tau, \zeta) \delta_\tau \otimes \delta_\zeta.
\]

Let \( A \) be a Hopf algebra containing \( H \) as a Hopf subalgebra. Then \( A_F \), the same algebra \( A \) but with the comultiplication \( \Delta_F \), is again a Hopf algebra. If \( \tilde{\omega} \) is another 2-cocycle and \( \tilde{F} \) is its corresponding twisting datum, then the Hopf algebras \( A_F \) and \( A_{\tilde{F}} \) are isomorphic whenever \( \omega \) and \( \tilde{\omega} \) differ by a coboundary. \( \square \)

We recall that a Yetter-Drinfeld module over \( H \) is a vector space \( V \) provided with structures of left \( H \)-module and left \( H \)-comodule such that

\[
\delta(h.v) = h(1)v(-1)S(h(3)) \otimes h(2).v(0).
\]

This category is semisimple; given \( g \in \Gamma \) and \( \gamma \in \hat{\Gamma} \), we denote by \( V^\gamma_g \) the isotypic component of a Yetter-Drinfeld module \( V \) corresponding to \( g, \gamma \). That is,

\[
(2.6) \quad V_g = \{ v \in V : \delta(v) = g \otimes v \}, \quad V^\gamma = \{ v \in V : h.v = \gamma(h)v, \quad \forall h \in \Gamma \}.
\]

Conversely, a vector space \( V \) provided with a direct sum decomposition

\[
V = \bigoplus_{g \in \Gamma, \gamma \in \hat{\Gamma}} V^\gamma_g
\]

is a Yetter-Drinfeld module with structures defined by (2.6). (This is a reformulation of the fact that the double of \( H \) is the group algebra of \( \Gamma \times \hat{\Gamma} \)).

We now consider a braided Hopf algebra \( R \) in the category \( \mathcal{H} \mathcal{YD} \) of Yetter-Drinfeld modules over \( H \) and its bosonization \( A = R \# H \); then the evident maps \( \pi : A \to H \) and \( \iota : H \to A \) are Hopf algebra homomorphisms, and \( R = \{ a \in A : (\text{id} \otimes \pi)\Delta(a) = a \otimes 1 \} \). The multiplication in \( A \) is determined by the rule \( gr = \chi(g)rg \), whenever \( r \in R^\chi \), \( g \in \Gamma \); hence

\[
\delta_r r = r \delta_{r \chi^{-1}}, \quad \text{if } r \in R^\chi.
\]

It follows from the definitions that \( \pi \) and \( \iota \) are Hopf algebra homomorphisms when replacing \( A \) by its twisting \( A_F \). Hence

\[
R_F := \{ a \in A_F : (\text{id} \otimes \pi)\Delta_F(a) = a \otimes 1 \}
\]

is a braided Hopf algebra in the category \( \mathcal{H} \mathcal{YD} \).

**Lemma 2.3.** (i). The linear map \( \psi : R \to R_F \) defined by

\[
\psi(r) = \sum_{\tau \in \hat{\Gamma}} \omega(\chi, \tau)^{-1} r \# \delta_\tau, \quad r \in R^\chi,
\]

is an isomorphism of \( \Gamma \)-modules. If \( r \in R^\chi \) and \( g \in R^\Gamma \), then...
\[ (2.7) \quad \psi(rs) = \omega(\chi, \tau)\psi(r)\psi(s). \]

If \( r \in R^\chi_g \), then \( \psi(r) \in (R_F)^\chi_{\psi(g)} \), where

\[
\psi(g) = \sum_{\tau \in \hat{\Gamma}} \omega(\tau, \chi)\omega(\chi, \tau)^{-1}(\tau, g)\delta_\tau
\]

\[ (2.8) \quad = \sum_{1 \leq j \leq M} \left( \sum_{1 \leq i < j} \frac{D_i}{D_j} c_{ij} \chi_i + g_j - \sum_{j < h \leq M} c_{jh} \chi_h \right) y(j). \]

Here we have used notations (2.2) and (2.5). (The first equality of (2.8) is in \( \mathbb{k}\Gamma \), the second in \( \Gamma \)).

(ii). If \( R \) is a graded braided Hopf algebra, then \( R_F \) also is and \( \psi \) is a graded map. If \( R \) is a coradically graded braided Hopf algebra (resp. a Nichols algebra), then \( R_F \) also is.

Proof. We first check that \( \psi \) is well-defined:

\[
(id \otimes \pi)\Delta_F(\psi(r)) = F(r \otimes 1) \sum_{\tau, \sigma \in \hat{\Gamma}} \omega(\chi, \tau)\omega(\chi, \tau)^{-1}\delta_\tau \otimes \delta_\sigma F^{-1}
\]

\[
= \sum_{\mu, \eta \in \hat{\Gamma}} \omega(\mu, \eta)\delta_\mu \otimes \delta_\eta (r \otimes 1) \sum_{\tau, \sigma \in \hat{\Gamma}} \omega(\chi, \tau)\omega(\chi, \tau)^{-1}\delta_\tau \otimes \delta_\sigma
\]

\[
= (r \otimes 1) \sum_{\tau, \sigma \in \hat{\Gamma}} \omega(\tau, \chi, \sigma)\omega(\chi, \tau)\omega(\tau, \sigma)^{-1}\delta_\tau \otimes \delta_\sigma
\]

\[
= (r \otimes 1) \sum_{\tau, \sigma \in \hat{\Gamma}} \omega(\chi, \tau)^{-1}\delta_\tau \otimes \delta_\sigma
\]

\[
= \psi(r) \otimes 1.
\]

It is easy to see that \( \psi \) preserves the action of \( \Gamma \). It follows that \( \psi \) is invertible with inverse \( \psi^{-1}(r) = \sum_{\tau \in \hat{\Gamma}} \omega(\chi, \tau) r \# \delta_\tau \), for \( r \in (R_F)^\chi \).

Now we prove (2.7):

\[
\psi(r)\psi(s) = \sum_{\sigma, \eta \in \hat{\Gamma}} \omega(\chi, \sigma)^{-1}\omega(\tau, \eta)^{-1}r \delta_\sigma s \delta_\eta = \sum_{\eta \in \hat{\Gamma}} \omega(\chi, \tau)\eta)\omega(\tau, \eta)^{-1} r s \delta_\eta
\]

\[
= \omega(\chi, \tau)^{-1} \sum_{\eta \in \hat{\Gamma}} \omega(\chi, \tau, \eta)^{-1} r s \delta_\eta = \omega(\chi, \tau)^{-1} \psi(rs). \]

We still assume that \( r \in R^\chi_g \). Then the coaction on \( \psi(g) \) is given by
\[ \delta_F(\psi(r)) = (\pi \otimes \text{id}) \Delta_F(\psi(r)) \]
\[ = \left( \sum_{\mu, \eta} \omega(\mu, \eta) \delta_\mu \otimes \delta_\eta \right) \left( g \otimes r \right) \Delta \left( \sum_{\tau} \omega(\chi, \tau)^{-1} \delta_\tau \right) F^{-1} \]
\[ = \sum_{\tau, \zeta} \omega(\tau, \zeta) \omega(\chi, \tau \zeta)^{-1} \omega(\tau, \zeta)^{-1} g \delta_\tau \otimes r \delta_\zeta \]
\[ = \sum_{\tau, \zeta} \omega(\chi, \tau \zeta)^{-1} \omega(\zeta, \chi)^{-1} \omega(\tau, \chi) g \delta_\tau \otimes r \delta_\zeta \]
\[ = \sum_{\tau} \omega(\tau, \chi) \omega(\chi, \tau)^{-1} \langle \tau, g \rangle \delta_\tau \otimes \psi(r), \]

where we have used repeatedly the cocycle condition and (2.4).

From the coassociativity of \( \Delta_F \) we conclude that
\[ \psi(g) = \sum_{\tau} \omega(\tau, \chi) \omega(\chi, \tau)^{-1} \langle \tau, g \rangle \delta_\tau \]
is a group-like element of \( H \), i.e. an element of \( \Gamma \). The last equality in (2.8) follows by evaluating at \( \gamma(j) \).

Let us suppose now that \( R = \oplus_{n \geq 0} R(n) \) is a graded braided Hopf algebra. Then \( A = \oplus_{n \geq 0} A(n) \), where \( A(n) = R(n) \# H \), is a graded Hopf algebra. Since \( F \in H \otimes H = A(0) \otimes A(0) \), we see that \( A_F \) is also a graded Hopf algebra with respect to the same grading and \( \pi, \iota \) are graded maps. Hence \( R_F \) inherits the grading of \( A \) and is a graded braided Hopf algebra. It is now evident that \( \psi \) is a graded map, \( R \) and \( R_F \) being both graded subspaces of \( A \).

Let us further suppose that \( R \) is coradically graded; then \( A \) also is. Since \( F \in H \otimes H = A(0) \otimes A(0) \), \( A_F \) has the same coradical filtration than \( A \); hence it is also coradically graded. Therefore \( R_F \) is coradically graded. See [AS2, Section 2]. In particular, \( \psi(P(R)) = P(R_F) \). If in addition \( R \) is generated by \( P(R) \), i.e. is a Nichols algebra, then we conclude from (2.7) that \( R_F \) is generated by \( P(R_F) \), thus \( R_F \) is also a Nichols algebra. \( \square \)

Let now \( R \) be a coradically graded braided Hopf algebra, \( x_1, \ldots, x_\theta \) a basis of \( P(R) \) such that for some \( g(1), \ldots, g(\theta) \in \Gamma \), \( \chi(1), \ldots, \chi(\theta) \in \hat{\Gamma} \) (1.5) holds. Then the braiding (1.7) is given by

\[ (b_{ij})_{1 \leq i, j \leq \theta} = \left( (\chi(i), g(j)) \right)_{1 \leq i, j \leq \theta} \]
\[ = \left( \prod_{1 \leq h \leq M} q_h^{\chi(i)_h g(j)_h} \right)_{1 \leq i, j \leq \theta} = (q^{\alpha_{ij}}), \]

where \( \alpha_{ij} = \sum D_h \chi(i)_h g(j)_h. \)
We keep the same meaning for \( F \) as above. By Lemma 2.3, \( R_F \) is a coradically graded braided Hopf algebra and \( \psi(x_1), \ldots, \psi(x_\theta) \) is a basis of \( P(R_F) \) such that (1.5) holds for \( \psi(g(1)), \ldots, \psi(g(\theta)) \in \Gamma, \chi(1), \ldots, \chi(\theta) \in \widehat{\Gamma} \). By (2.8), the braiding (1.7) for \( R_F \) is given by

\[
(b_F^{ij})_{1\leq i,j\leq \theta} = ((\chi(i), \psi(g(j))))_{1\leq i,j\leq \theta} = (q^{\alpha_{ij}}),
\]

(2.10)

where \( \alpha_{ij} = \sum_{1\leq t<h \leq M} D_{t}c_{th}(\chi(i)_h\chi(j)_t - \chi(i)_t\chi(j)_h) + \alpha_{ij} \).

Remark 2.4. The "symmetrization" of the matrix \( \alpha_{ij} \) remains unchanged under twisting:

\[
\alpha_{ij}^F + \alpha_{ji}^F = \sum_{1\leq t<h \leq M} D_{t}c_{th}(\chi(i)_h\chi(j)_t - \chi(i)_t\chi(j)_h + \chi(j)_h\chi(i)_t - \chi(j)_t\chi(i)_h) + \alpha_{ij}^F + \alpha_{ji}^F = \alpha_{ij} + \alpha_{ji}.
\]

Hence, if \( R \) is of Cartan type, then \( R_F \) also is. Also, \( \alpha_{ii}^F = \alpha_{ii} \) and the diagonal elements \( b_{ii} \) do not change after twisting.

In particular, if \( R \) is a quantum linear space, i.e. \( \alpha_{ij} + \alpha_{ji} = 0 \mod E_M \) whenever \( i \neq j \), then \( R_F \) also is a quantum linear space.

§3. Frobenius-Lusztig kernels.

Let \((I, \cdot)\) be a Cartan datum as in [L3, Chapter 1] and let \((Y, X, \ldots)\) be a root datum of type \((I, \cdot)\). Let \( a_{ij} = \langle i, j \rangle \), \( d_i = \frac{i \cdot \bar{i}}{2} \). Let also \( q \neq 1 \) be a root of 1 of odd order \( N \). We consider in this Section a vector space \( V \) with braiding \( b = (b_{ij}) \) where \( b_{ij} = q^{i \cdot j} \). We also assume that

(3.1) \( N_i > -a_{ij} \), where \( N_i \) is the order of \( b_{ii} \), for all \( i, j \).

Note that \( N_i \) has the meaning with respect to \( N \) as in [L3, 2.2.4].

Now \( b \) is of Cartan type and by (3.1), it is of FL-type and its associated Cartan matrix is \((a_{ij})\), with the \( d_i \)’s as in the preceding sentence. Conversely, any braiding of FL-type arises in this way.

Our aim is to sketch the main ideas of the proof of the following Theorem, which follows from deep results of Lusztig and in addition from the work of Müller [Mü] and Rosso [Ro1].

Theorem 3.1.

(1) Assume that \((I, \cdot)\) is of finite type. Then \( \mathfrak{B}(V) \) is isomorphic to the positive part of the Frobenius-Lusztig kernel \( u \), cf. [L3]. In particular, \( \dim \mathfrak{B}(V) = N^M \), where \( M \) is the dimension of the nilpotent subalgebra of the Lie algebra corresponding to \((I, \cdot)\) spanned by all the positive root vectors.

(2) Assume that \((I, \cdot)\) is not of finite type. Then \( \mathfrak{B}(V) \) is infinite dimensional.

Proof. Part (1) follows from the characterization in [Mü, Section 2] or [Ro1]; the claim on the dimension follows from [L1, L2].

We sketch now the proof of Part (2). We keep the notation from [L3]. It is in principle not always true that the positive part \( u^+ \) of the Frobenius-Lusztig kernel is isomorphic to \( \mathfrak{B}(V) \). We have however an epimorphism \( u^+ \rightarrow \mathfrak{B}(V) \).
Since \((I, \cdot)\) is not of finite type, its Weyl group is not finite. Hence, for any \(M > 0\) there exists an element \(w \in W\) of length \(M\), say with reduced expression \(w = s_{i_1} \ldots s_{i_M}\). We shall show that \(B(V)\) contains \(2^M\) linearly independent elements; this concludes the proof of (2).

We work first in the transcendental case. We use now specifically the notation from \[L3, 38.2.2\]. The sequence \(h = (i_1, \ldots, i_M)\) is admissible because of \[L3, Proposition 40.2.1 (a)\]. It is clear that the element \(x = 1\) is adapted to \((h,0)\).

Let \(c, c'\) be as \[L3, Proposition 38.2.3\] and suppose that their entries are either 0 or 1. Then \(L(h,c,0,1)\) is orthogonal (resp, not orthogonal) to \(L(h,c',0,1)\) if \(c\) is different (resp., equal) to \(c'\). But we can pass from the transcendental case to our case \((q\) a root of unity) thanks to the results in \[M¨ u, Section 2\], notably Lemma 2.2, Proposition 2.3 and Theorem 2.11 (a).

\[\square\]

§4. Proof of Theorem 1.1.

4.1 Sketch of the proof.

Let us first outline the proof of Theorem 1.1. Let \(V\) be a Yetter-Drinfeld module over our fixed group \(\Gamma\) as in (1.5) and let \(R = B(V)\). We begin by passing from a general braiding (1.7) to a symmetric one. For this, it is natural to try to apply a Drinfeld twist to the Hopf algebra \(k\Gamma\# R\) (compare with [Ro2]). Unfortunately this is not always possible. It fails already for quantum linear planes over cyclic groups, since no twist is possible in the cyclic group case. However, we can overcome this difficulty because the algebra and coalgebra structures of \(R\) do not depend on \(\Gamma\) but only on the braiding (1.7). That is, let \(\Upsilon\) be another finite abelian group and let \(h(1), \ldots, h(\theta) \in \Upsilon, \eta(1), \ldots, \eta(\theta) \in \hat{\Upsilon}\) be sequences satisfying

\[
\langle \eta(i), h(j) \rangle = \langle \chi(i), g(j) \rangle, \quad \forall i, j.
\]

Let \(T\) be the Nichols algebra over \(\Upsilon\) such that \(T(1) = V\) with Yetter-Drinfeld module structure given by (1.5) but with \(h(i)\)'s and \(\eta(j)\)'s instead of \(g(i)\)'s and \(\chi(j)\)'s. Then \(T\) is isomorphic to \(R\) both as an algebra and coalgebra. This follows at once from the description of \(T\) and \(R\) in terms of “quantum antisymmetrizers” mentioned in the Introduction; see for instance [Sbg], [Ro2], [AG]. So we change the group, twist and assume that the braiding (1.7) is symmetric.

We then discuss briefly the relations between the notions of Cartan type, FL-type and locally of FL-type. We state the last one in arithmetical terms and describe several instances where it holds. One of them is the following:

Let \(b = (b_{ij})\) be a connected braiding of Cartan type whose entries have odd order. We say that \(b\) satisfies the relative primeness condition if \(a_{ij}\) is 0 or relatively prime to the order of \(b_{ii}\) for all \(i, j\).

We prove then that a braiding of symmetrizable Cartan type satisfying the relative primeness condition is of FL-type. By the results evoked in Section 3, this is enough for part (1) of Theorem 1.1.

For the converse part (2) of Theorem 1.1, we prove that if \(B(V)\) is finite dimensional and \(b\) is locally of FL-type then it is symmetrizable. By the results of Section 3 again, this concludes the proof of the theorem.

4.2 Reduction to the symmetric connected case.

Let \((b_{ij})_{1 \leq i, j \leq \theta}\) be a matrix with entries in \(k^\times\). Let \(V\) be a vector space with a basis \(x_1, \ldots, x_\theta\) and let \(c : V \otimes V \rightarrow V \otimes V\) be the linear map given by
It is easy to verify that \( c \) satisfies the Braid relation:

\[
(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c).
\]

Therefore, it induces representations of the Braid group \( \mathbb{B}_n \) on \( V^{\otimes n} \) for any \( n \geq 2 \) and this in turn allows to define "braided antisymmetrizers" which are linear maps \( \Gamma \) below.

Moreover, \( R \) is a braided Hopf algebra. In our case this will follow from Lemma 4.1 below.

If \( \Gamma \) is an abelian group, \( g(1), \ldots, g(\theta) \in \Gamma, \chi(1), \ldots, \chi(\theta) \in \hat{\Gamma} \) are sequences such that \( b_{ij} = \langle \chi(j), g(i) \rangle \) and we consider \( V \) as a Yetter-Drinfeld module over \( \Gamma \) via (1.5), then \( c \) is the "commutativity isomorphism" in the braided category \( \mathcal{YD} \). Moreover, \( R \) has a braided Hopf algebra structure and is in fact the Nichols algebra associated to \( V \).

**Lemma 4.1.** Let \( (b_{ij}) \) be a braiding of Cartan type and let us assume that \( b_{ij} \) is a root of 1 of odd order for any \( i, j \). Then there exists a finite abelian group \( \Gamma \), sequences \( g(1), \ldots, g(\theta) \in \Gamma, \chi(1), \ldots, \chi(\theta) \in \hat{\Gamma} \) and \( F \in \mathbb{k}\Gamma \otimes \mathbb{k}\Gamma \) such that

(i) for \( 1 \leq i, j \leq \theta \), we have

\[
b_{ij} = \langle \chi(j), g(i) \rangle;
\]

(ii) the braiding (1.7) corresponding to the Nichols algebra \( R_F \) is symmetric and of Cartan type, with the same associated Cartan matrix and diagonal elements.

Here \( R \) is the Nichols algebra associated to the Yetter-Drinfeld module \( V \) with basis \( x_1, \ldots, x_\theta \) and structure defined by (1.5).

**Proof.** (i): We define \( \Gamma := \langle y(1) > \oplus \cdots > y(\theta) > \rangle \), where we impose that the order \( E_j \) of \( y_j \) is the least common multiple of the orders of \( b_{ij} \), for all \( i, \) and for \( \ell \leq j \). Then \( E_j|E_{j+1} \): (2.1) holds. Let \( q \in \mathbb{k} \) be a root of 1 of order \( E_\theta \). Let \( D_i = \frac{E_{\theta i}}{E_i}, 1 \leq i \leq \theta \) and let \( q_i = q^{D_i} \). Then \( q_i \) is a root of 1 of order \( E_i \). Let \( \gamma(i) \in \tilde{\Gamma} \) be the unique character such that \( \langle \gamma(i), y(j) \rangle = q_i^{\delta_{ij}} \), for \( 1 \leq i, j \leq N \). Then \( \text{ord} \gamma(\ell) = E_{\ell} \) and \( \tilde{\Gamma} = \langle \gamma(1) > \oplus \cdots > \gamma(M) > \rangle \). We choose \( \chi(j) = \gamma(j) \) for all \( j \) and define \( g(i) \) by (4.2); it makes sense because the order of \( \langle \chi(j), g(i) \rangle \) divides \( E_j \) for all \( i, j \).

(ii): Let \( \alpha_{ij} \) be given by (2.9). We are looking for integers \( c_{ij} \) such that, if \( F \) is associated to the cocycle \( \omega \) given by (2.5), then the matrix \( \alpha_{ij}^F \) is symmetric. Assume that \( i < j \). By (2.10) and our preceding choices, we have
Clearly, $S$ is easy to conclude from the hypothesis and (i) that it is a braided Hopf algebra. thus

$$\alpha_{ij} = D_i g(j)_i,$$

$$\alpha_{ij}^F = -D_i c_{ij} + \alpha_{ij} = -D_i c_{ij} + D_i g(j)_i,$$

$$\alpha_{ji}^F = D_i c_{ij} + \alpha_{ji} = D_i c_{ij} + D_j g(i)_j;$$

hence $\alpha_{ij}^F = \alpha_{ji}^F$ if and only if

$$2D_i c_{ij} = \alpha_{ij} - \alpha_{ji} = D_i g(j)_i - D_j g(i)_j \mod E_\theta.$$

We solve (4.3). The factor 2 does not trouble because $E_\theta$ is odd. That is, it is enough to solve

$$D_i c_{ij} = D_i g(j)_i - D_j g(i)_j \mod E_\theta.$$  

(4.4)

Now, since $g(j) = \sum_t g(j)_t y(t)$ in $\Gamma$, we have

$$b_{ij} = \langle \chi(i), g(j) \rangle = q^{g(j)_i} = q^{D_i g(j)_i}.$$

As the braiding $(b_{ij})$ is of Cartan type, we see that

$$q^{D_i g(j)_i + D_j g(i)_i} = b_{ij} b_{ji} = b_{ii}^{a_{ij}} = q^{D_i a_{ij}}$$

and therefore

$$D_j g(i)_j = D_i a_{ij} - D_i g(j)_i \mod E_\theta.$$

Thus $\overline{c}_{ij} = 2g(j)_i - a_{ij}$ is a solution of (4.4). The last statement follows from Remark 2.4 □

In the rest of this section, we shall consider matrices $b = (b_{ij})_{1 \leq i,j \leq \theta}$ of Cartan type, such that $b_{ij}$ is a root of 1 of odd order and $b_{ij} = b_{ij}$ for all $i,j$. By Lemma 4.1, we can consider its associated Nichols algebra $R$; the comultiplication is an algebra map where the product in $R \otimes R$ is determined by

$$(x_r \otimes x_i)(x_j \otimes x_s) = b_{ij} x_r x_j \otimes x_i x_s.$$  

If $I \subset \{1,\ldots, \theta\}$ then we denote by $R(I)$ the Nichols algebra corresponding to the braiding $(b_{ij})_{i,j \in I}$. Clearly, there is an injective map of braided Hopf algebras $R(I) \rightarrow R$.

**Lemma 4.2.** (i). If $b_{ij} = 1$ then $x_i x_j = x_j x_i$.

(ii). Assume there is $I \subset \{1,\ldots, \theta\}$ such that $b_{ij} = 1$ for all $i \in I, j \in J := \{1,\ldots, \theta\} - I$. Then $R \simeq R(I) \otimes R(J)$.

**Proof.** By a direct computation, we see that $x_i x_j - x_j x_i$ is primitive; but the primitive elements are concentrated in degree one, so it should be 0. This proves (i). Let $S = R(I) \otimes R(J)$; it is a graded algebra and coalgebra with respect to the grading $S(h) = \bigoplus_{\ell} R(I)(\ell) \otimes R(J)(h-\ell)$ and the usual tensor product multiplication and comultiplication. Then $S$ is a strictly graded coalgebra, i.e. (1.1), (1.2), (1.3) hold, cf. [Sw, p. 240]. On the other hand, $S$ is a Yetter-Drinfeld module and it is easy to conclude from the hypothesis and (i) that it is a braided Hopf algebra.

Clearly, $S$ is generated as algebra by $S(1) \simeq V$, i.e. it satisfies (1.4). Thus $S$ is a Nichols algebra, hence isomorphic to $R$ because their spaces of primitive elements coincide. □
We consider the following equivalence relation on \( \{1, \ldots, \theta\} \): \( i \sim j \) if there exists a sequence of elements \( i = h_0, \ldots, h_P = j \) in \( \{1, \ldots, \theta\} \) such that \( h_\ell \neq h_{\ell+1} \) and \( b_{h_\ell h_{\ell+1}} \neq 1 \). Note that \( P = 0 \) is allowed and gives \( i \sim i \). A class of this relation is called a connected component. This is equivalent to the notion of connected Dynkin diagram or indecomposable matrix as in [K, 1.7.1]. The preceding Lemma 4.2 shows that the Nichols algebra \( R \) is the tensor product of the Nichols algebras corresponding to its connected components. Therefore, to prove Theorem 1.1 it is enough to assume that \( b \) is connected.

4.3 Conditions for FL-type.

**Lemma 4.3.** Let \( b = (b_{ij})_{1 \leq i,j \leq \theta} \) be a braiding of Cartan type such that \( b_{ij} \) is a root of 1 of odd order and \( b_{ij} = b_{ji} \) for all \( i,j \).

If \( b \) is symmetrizable and satisfies the relative primeness condition \( (a_{ij} \text{ is 0 or relatively prime to the order of } b_{ii} \text{ for all } i,j) \) then it is of FL-type.

**Proof.** We can easily reduce to the case when \( b \) is connected.

We claim first that the order of \( b_{kk} \) is the same for all \( k \). Indeed, let us fix \( i \neq j \) such that \( a_{ij} \neq 0 \) (if no such pair \( i,j \) exists, then by connectedness \( \theta = 1 \) and there is nothing to prove). Then, since the orders of the entries are odd,

\[
\text{ord } b_{ij} = \text{ord } b_{ij}^2 = \text{ord } b_{ii}^{a_{ij}} = \text{ord } b_{ii};
\]

so \( \text{ord } b_{ii} = \text{ord } b_{jj} \). The claim follows by connectedness. We call \( N = \text{ord } b_{ii} \).

Let \( d_k \) be integers as in (1.11). We can assume that they are relatively prime, that is, \((d_1, \ldots, d_\theta) = 1\). Then it is easy to see that \( N \) and \( d_k \) are relatively prime for all \( k \). Indeed, if \( t \) divides \( N \) and \( d_k \) then \( t \) divides \( d_j \), since \( t \) divides \( d_t a_{ij} = d_j a_{ji} \) and \((N,a_{ji}) = 1\). Again, the claim follows by connectedness.

In particular, there is a unique root of unity \( q \) of order \( N \) such that \( b_{ii} = q^{2d_i} \).

We claim finally that this \( q \) satisfies (1.12). Indeed, \( b_{ii}^{a_{ij}} = q^{2d_i a_{ij}} = q^{2d_j a_{ji}} \) and \( b_{ii}^{a_{ij}} = b_{ij}^2 = b_{jj}^{a_{ji}} \). Thus, \( b_{ij} = q^{a_{ij} a_{ji}} \) (since \( N \) is odd) and \( b_{jj} = q^{2d_j} \) (since \( (N, 2a_{ji}) = 1 \)). Once more, the claim follows by connectedness.

We now investigate conditions for FL-type in rank 2. Let \( b = (b_{ij})_{1 \leq i,j \leq 2} \) be a braiding of connected Cartan type whose entries have odd order, such that \( b_{12} = b_{21} \). Then it is automatically symmetrizable since the rank is 2. “Connected” means that \( a_{12} \neq 0 \). Let \( d_1, d_2 \) be relatively prime integers such that \( d_1 a_{12} = d_2 a_{21} \). We denote by \( N_i \) the order of \( b_{ii} \).

**Lemma 4.4.** The following are equivalent:

(i) \( b \) is of FL-type.

(ii) There exists \( u \in k \) of odd order such that \( u^{2d_i} = b_{ii} \), \( i = 1, 2 \).

(iii) There exists \( v \in k \) such that \( v^{d_i} = b_{ii} \), \( i = 1, 2 \).

**Proof.** (iii) \( \implies \) (ii): Note that the order of \( v \) divides \( d_1 N_1 \) and \( d_2 N_2 \). Since the \( N_i \)’s are odd and \( d_1, d_2 \) are relatively prime, the order of \( v \) is odd. Hence \( v \) has a square root \( u \) of odd order.

(ii) \( \implies \) (i): We have \( b_{12}^{a_{12}} = b_{11}^{a_{12}} = u^{2d_1 a_{12}} \). Hence \( b_{12} = u^{d_1 a_{12}} \) since both have odd order.

(i) \( \implies \) (iii): If \( q \) satisfies (1.12), take \( v = q^2 \). \( \square \)
We want to give a criterion for the condition (iii) in the preceding Lemma. Let $e_i$, $i = 1, 2$ be non-zero integers such that

$$e_1d_1N_1 = e_2d_2N_2 = r;$$

for instance we could take $r$ the lowest common multiple of $d_1N_1$ and $d_2N_2$. Observe that there exists $s \in \mathbb{Z}$ such that $r = d_1d_2s$.

Let now $\xi \in \mathbb{k}$ be a primitive $r$-th root of 1 and choose integers $k_1$, $k_2$ such that

$$b_{ii} = \xi^{e_id_ik_i};$$

this is possible because $\xi^{e_id_i}$ has order $N_i$.

**Lemma 4.5.** Condition (iii) in Lemma 4.4 is equivalent to

(4.5) $e_1k_1 \equiv e_2k_2 \mod s.$

**Proof.** We claim first that (iii) is equivalent to the following statement:

(iv) There exists $t_i \in \mathbb{Z}$, $i = 1, 2$ such that

(4.6) $e_id_ik_i \equiv e_id_it_i \mod r$, $i = 1, 2$

(4.7) $e_1t_1 \equiv e_2t_2 \mod r$.

Indeed, if (iii) holds then $q^{d_iN_i} = 1$; as $\xi^{e_i}$ has order $d_iN_i$, there exists $t_i \in \mathbb{Z}$ such that $q = \xi^{e_i t_i}$. Then (4.6), (4.7) follow now without difficulty. Conversely, if (iv) holds, take $q = \xi^{e_1t_1} = \xi^{e_2t_2}$, by (4.7). Then (iii) is true by (4.6). The claim is proved.

Now (4.6) is equivalent to $k_i \equiv t_i \mod N_i$, $i = 1, 2$.

Assume (iv). Then there exist $x_i \in \mathbb{Z}$ such that $t_i = k_i + N_ix_i$, $i = 1, 2$. Now $r$ (and a fortiori $s$) divides

$$e_1t_1 - e_2t_2 = e_1k_1 - e_2k_2 + e_1N_1x_1 - e_2N_2x_2;$$

but $e_1N_1 = d_2s$ and $e_2N_2 = d_1s$. So (4.5) holds.

If (4.5) holds, let $y \in \mathbb{Z}$ such that $e_1k_1 - e_2k_2 = -ys$. As $d_1$ and $d_2$ are relatively prime, there exists $x_1, x_2$ such that $y = d_2e_1 - d_1x_2$. If now we take $t_i = k_i + N_ix_i$, $i = 1, 2$, then (4.6) holds by definition; and

$$e_1t_1 - e_2t_2 = e_1k_1 - e_2k_2 + e_1N_1x_1 - e_2N_2x_2 = e_1k_1 - e_2k_2 + d_2sx_1 - d_1sx_2 = 0.$$

So, (4.7) follows. \(\square\)

Lemma 4.5 describes an easy algorithm to decide whether a given $b$ is locally of FL-type. As an example we note:

**Corollary 4.6.** There are braidings of symmetrizable Cartan type which are not of FL-type.

**Proof.** Let $p$ be an odd prime number and take $N_1 = N_2 = p^2$, $a_{12} = a_{12} = -p$. Let $k_1$, $k_2$ be two elements not divisible by $p$ such that

$$k_1 \not\equiv k_2 \mod p^2, \quad k_1 \equiv k_2 \mod p.$$

Let $q$ be a root of unity of order $p^2$ and let $b_{ii} = q^{k_i}$, $i = 1, 2$, $b_{12} = b_{21}$ the unique root of unity of odd order such that $b_{12}^2 = b_{11}^p$. Then $b_{12}^2 = b_{22}^p$, but $s = p^2$ and $e_1 = e_2 = 1$, so (4.5) does not hold. \(\square\)
Example 4.7. There are braidings of Cartan type which are locally of FL-type but not symmetrizable.

Proof. Take \( b = (q^{d_i a_{ij}}) \), where \( q \) is a root of 1 of order 5, \( d_1 = 1, d_2 = 2, d_3 = 3 \) and

\[
(a_{ij}) = \begin{pmatrix}
2 & -3 & -3 \\
-4 & 2 & -3 \\
-1 & -2 & 2
\end{pmatrix}.
\]

Then \( b \) is of Cartan type since \( d_i a_{ij} = d_j a_{ji} \mod 5 \) for all \( i, j \). By Lemma 4.3, \( b \) is locally of FL-type. But \( (a_{ij}) \) is not symmetrizable. \( \square \)

Lemma 4.8. Let \( b \) be a braiding of Cartan type and rank 2. If \( a_{12} = -1 \) and \( a_{21} \) is odd, then \( b \) is of FL-type.

Proof. Let \( n = -a_{21} \). Let \( \tilde{q} \) be a root of 1 such that \( \tilde{q}^n = b_{12}^{-1} \); then \( b_{22} = b_{11} = b_{12}^2 \). Hence there exists an \( n \)-th root of 1 \( \omega \) such that \( b_{22} = \omega^2 \tilde{q}^2 \) since \( n \) is odd. We choose then \( q = \omega \tilde{q} \). \( \square \)

Proof of Theorem 1.1.

By Lemma 4.1 we can assume that \( b \) is symmetric. By Lemma 4.2 we can assume that \( b \) is connected.

Part (1). By Lemma 4.3 (if the type is different from \( G_2 \)) or Lemma 4.8 (if the type is \( G_2 \)), the braiding \( b \) is of FL-type. We now apply Theorem 3.1.

Part (2). For any fixed pair \( i, j \) in \( \{1, \ldots, \theta\} \), the braiding of rank 2 corresponding to the submatrix supported by \( i, j \) is of FL-type and \( \mathfrak{B}(V) \) is finite dimensional. By Theorem 3.1, we conclude that \( a_{ij} \) is of FL-type whenever \( a_{ij} = -3 \) for some \( j \) by hypothesis, we can apply Lemma 4.3 and conclude again from Theorem 3.1 that \( b \) is of finite type whenever it is symmetrizable.

So, it only remains to show that: \( b \) is necessarily symmetrizable.

It is known that a matrix which is either simply-laced or has no cycles is symmetrizable, cf. [K, ex. 2.1]. If the matrix has a cycle, its corresponding Nichols algebra is still finite dimensional. We are reduced to prove that no cycle with a double or triple arrow can arise.

Assume first that \( b \) is a cycle with a triple arrow. If \( \theta \geq 4 \), then we remove a suitable vertex and get a subdiagram with no cycles which is not of finite type. This is a contradiction.

So, assume \( \theta = 3 \). Then there are several possibilities of cycles with (at least) one triple arrow. We can discard two of them which are symmetrizable. We discard the rest because of the restrictions on the orders of \( b_{ii} \).

Consider a Cartan matrix

\[
(a_{ij}) = \begin{pmatrix}
2 & -3 & -b \\
-1 & 2 & -c \\
-d & -e & 2
\end{pmatrix}.
\]

Here \( b, c, d, e \) are positive integers such that \( bd, ce = 1, 2 \) or 3. This means that

\[
b^3 = b, \quad b^c = b^e, \quad b = b^d.
\]
Hence,
\[ b_{11}^b = b_{33}^d, \quad b_{11}^3 = b_{33}^e. \]

We know that \( b = 1 \) or \( d = 1 \). Assume \( b = 1 \). Then \( b_{33}^{3cd-e} = 1 \). We have several subcases according to the values of \( c, d, e \).

(i) If \( e = 3 \), then \( c = 1 \) and \( b_{33}^{3(d-1)} = 1 \). There are three possibilities:

If \( d = 1 \) the matrix is symmetrizable and we can apply Theorem 3.1. If \( d = 2 \), \( b_{33}^3 = 1 \). If \( d = 3 \), \( b_{33}^6 = 1 \). As we assume that the orders are odd, we should have \( b_{33}^3 = 1 \). These two possibilities are not possible by our choice following the rule (1.10).

(ii) If \( e = 2 \), then \( c = 1 \) and \( b_{33}^{3d-2} = 1 \). There are three possibilities:

If \( d = 1 \) \( b_{33} = 1 \), a possibility that we excluded. If \( d = 2 \), \( b_{33}^4 = 1 \), but we assume that the orders are odd. If \( d = 3 \), \( b_{33}^7 = 1 \). For this reason we exclude the order 7.

(iii) If \( e = 1 \), then \( b_{33}^{3cd-1} = 1 \). There are various possibilities:

As \( cd = 1, 2, 3, 4, 6, 9, 3cd - 1 \) could take the values \( 2, 5, 8, 11, 17, 26 \). Since we assume that the orders are odd, we have to exclude the orders 5, 11, 13, 17.

Let us now assume that \( d = 1 \). Then \( b_{11}^{3c-be} = 1 \).

(iv) If \( c = 3 \), then \( e = 1 \) and \( b_{11}^{9-b} = 1 \). There are three possibilities:

If \( b = 1 \), \( b_{11}^7 = 1 \); but the orders are odd. If \( b = 2 \), \( b_{11}^7 = 1 \). This is a new instance where 7 should be excluded. If \( b = 3 \), \( b_{11}^7 = 1 \). As we assume that the orders are odd, we should have \( b_{11}^3 = 1 \). This contradicts the rule (1.10).

(v) If \( c = 2 \), then \( e = 1 \) and \( b_{11}^{6-b} = 1 \). There are three possibilities:

If \( b = 1 \), \( b_{11} = 5 \). This is a new instance where 5 should be excluded. If \( b = 2 \), \( b_{11}^4 = 1 \), but we assume that the orders are odd. If \( b = 3 \), \( b_{11}^3 = 1 \). This contradicts the rule (1.10).

(vi) If \( c = 1 \), then \( b_{11}^{3-be} = 1 \). There are various possibilities:

As \( be = 1, 2, 3, 4, 6, 9, 3 - be \) could take the values \( 2, 1, 0, -1, -3, -6 \). The values \( 2, 1, -1 \) are excluded by hypothesis; the values \(-3, -6 \) are excluded by the rule (1.10). The value 0 arises only when \( b \) is symmetrizable, but then we can apply Theorem 3.1.

We next assume that \( b \) is a cycle with no triple arrow but at least a double one. If \( \theta \geq 6 \), again we have a contradiction by removing a suitable vertex. If \( \theta = 5 \), we remove systematically different vertices. Since the resulting subdiagrams are symmetrizable and hence of finite type, we see that \( b \) has only one double arrow. We can easily get a contradiction.

Now we assume \( \theta = 4 \), with a Cartan matrix

\[
(a_{ij}) = \begin{pmatrix}
2 & -2 & 0 & -b \\
-1 & 2 & -c & 0 \\
0 & -d & 2 & -e \\
-f & 0 & -g & 2
\end{pmatrix}.
\]

Here \( b, c, d, e, f, g \) are positive integers such that \( bf, cd, eg = 1, 2 \). By removing the third, respectively the fourth, vertex, we conclude that \( bf = cd = 1 \). There are three possibilities: \( (e, g) = (1, 1), (1, 2) \) or \( (2, 1) \). The first is not possible by a similar argument as for \( \theta = 5 \). The second corresponds to a symmetrizable matrix and the third is the first instance forcing to exclude the order 3.
We finally assume that $\theta = 3$. We have a 3-cycle with no triple arrow but at least a double one. Arguing as in the case of triple arrows, we eliminate all the possibilities except two; for those we exclude the orders 3 and 7. For instance,

$$(a_{ij}) = \begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & -2 \\ -2 & -1 & 2 \end{pmatrix},$$

then

$$b_{11}^2 = b_{22}, \quad b_{22}^2 = b_{33}, \quad b_{11} = b_{33}^2.$$ 

Hence $b_{11} = b_{11}^8$, which is not possible because the order of $b_{11}$ is not 7. \qed

**Remark.** We observe that the preceding proof of Theorem 1.1 Part (2) only needs the hypothesis "locally of FL-type" to conclude that

(4.8) A finite dimensional Nichols algebra $\mathfrak{B}(V)$ of rank 2 is necessarily of finite type.

If $\Gamma$ is any finite abelian group such that (4.8) holds for $\mathfrak{B}(V)$ in $\Gamma \mathcal{Y} D$ then the conclusion of Theorem 1.1 Part (2) is valid for all $\mathfrak{B}(V)$ in $\Gamma \mathcal{Y} D$.

For further use, we list the Cartan matrices, up to numbering, causing troubles for small values of $p$:

(4.9) For $p = 3$, the matrix

$$(a_{ij}) = \begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}.$$ 

(4.10) For $p = 3$, the matrix

$$(a_{ij}) = \begin{pmatrix} 2 & -2 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$ 

(4.11) For $p = 5$, the matrix

$$(a_{ij}) = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}.$$ 

(4.12) For $p = 5$, the matrix

$$(a_{ij}) = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix}.$$ 

(4.13) For $p = 7$, the matrix

$$(a_{ij}) = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ 3 & -2 & 2 \end{pmatrix}.$$
(4.14) For $p = 7$, the matrix

$$(a_{ij}) = \begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & -2 \\ -2 & -1 & 2 \end{pmatrix}.$$ 

(4.15) For $p = 11$, the matrix

$$(a_{ij}) = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -3 \\ -3 & -1 & 2 \end{pmatrix}.$$ 

(4.16) For $p = 13$, the matrix

$$(a_{ij}) = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -3 \\ -3 & -1 & 2 \end{pmatrix}.$$ 

(4.17) For $p = 17$, the matrix

$$(a_{ij}) = \begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -3 \\ -3 & -1 & 2 \end{pmatrix}.$$ 

§5. Nichols algebras over $\mathbb{Z}/(p)$.

In this section, $\Gamma$ will denote $\mathbb{Z}/(p)$, $p$ an odd prime. We first discuss Nichols algebras of rank 2. If $V$ is a Yetter-Drinfeld module of dimension 2 satisfying (1.6), then there exists a generator $u$ of $\Gamma$, $q \in k^\times$ of order $p$ and integers $b, d$ such that

(5.1) \quad $g(1) = u, \quad g(2) = u^b, \quad \langle \chi(1), u \rangle = q, \quad \langle \chi(2), u \rangle = q^d.$

So $b_{11} = q, \quad b_{22} = q^{bd}$.

Dynkin diagram of type $A_2$. We should have

$$\langle \chi(1), g(2) \rangle \langle \chi(2), g(1) \rangle = b_{11}^{-1} = b_{22}^{-1}.$$ 

This means $b + d \equiv -1 \equiv -bd \mod p$. It follows readily that $b^2 + b + 1 \equiv 0 \mod p$. Hence $p = 3$ and $b \equiv 1 \mod p$, or $b \not\equiv 1 \mod p$ is a cubic root of 1. In the last case, $b$ exists if and only if $p - 1$ is divisible by 3. The corresponding matrices are not symmetric.

Dynkin diagram of type $B_2$. We should have

$$\langle \chi(1), g(2) \rangle \langle \chi(2), g(1) \rangle = b_{11}^{-1} = b_{22}^{-1}.$$ 

This means $b + d \equiv -1 \equiv -2bd \mod p$. Thus $2b^2 + 2b + 1 \equiv 0 \mod p$ and looking at the discriminant of this equation we see that it has a solution if and only if $-1$ is a square, i.e. exactly when $p \equiv 1 \mod 4$. The corresponding matrices are not symmetric.
Dynkin diagram of type $G_2$. By (1.10), we have $p > 3$. We should have

$$\langle \chi(1), g(2) \rangle \langle \chi(2), g(1) \rangle = b_{11}^{-1} = b_{22}^{-3}.$$ 

This means $b + d \equiv -1 \equiv -3bd \mod p$. Thus $3b^2 + 3b + 1 \equiv 0 \mod p$ and looking at the discriminant of this equation we see that it has a solution if and only if $-3$ is a square $\mod p$. By the quadratic reciprocity law, this happens exactly when $p \equiv 1 \mod 3$. The matrices are not symmetric.

We now conclude the proof of the existence part of Theorem 1.3.

**Proposition 5.1.** There is no Nichols algebra of rank $\geq 3$ over $\mathbb{Z}/(p)$ with finite Cartan matrix except when $p = 3$ and the corresponding Dynkin diagram is $A_2 \times A_1$ or $A_2 \times A_2$.

**Proof.** We first consider Dynkin diagrams of rank 3. We have the following cases:

(a) $A_2 \times A_1$,  
(b) $B_2 \times A_1$,  
(c) $G_2 \times A_1$,  
(d) $A_3$,  
(e) $B_3$,  
(f) $C_3$,  
(g) $A_1 \times A_1 \times A_1$.

The case (g) does not arise, as shown in [AS2, Section 4]. Therefore we can fix a numeration of the vertices of the Dynkin diagram such that the vertices 1 and 2 (resp., 1 and 3) are connected (resp., not connected). We also assume that there is only one arrow between 1 and 2 in cases (e) an (f). That is,

(a) $\circ \rightarrow \circ \rightarrow \circ$,  
(b) $\circ \Rightarrow \circ \rightarrow \circ$,  
(c) $\circ \Rightarrow \circ \rightarrow \circ$,  
(d) $\circ \rightarrow \circ \rightarrow \circ$,  
(e) $\circ \rightarrow \circ \Rightarrow \circ$,  
(f) $\circ \rightarrow \circ \leftarrow \circ$.

If $V$ is a Yetter-Drinfeld module of dimension 3 satisfying (1.6), then there exist a generator $u$ of $\Gamma$, $q \in k^\times$ of order $p$ and integers $b, d, e, f$ (none of them divisible by $p$) such that

$$g(1) = u, \quad g(2) = u^b, \quad g(3) = u^e,$$

$$\langle \chi(1), u \rangle = q, \quad \langle \chi(2), u \rangle = q^d, \quad \langle \chi(3), u \rangle = q^f.$$ 

So $b_{11} = q, b_{22} = q^{bd}, b_{33} = q^{ef}$. Also, $\langle \chi(1), g(3) \rangle \langle \chi(3), g(1) \rangle = 1$ means

(5.2) $e + f \equiv 0 \mod p$.

Now, considering the subdiagram supported by the vertices 1 and 2, we conclude from the arguments above for the rank 2 case that

(5.3) In cases (a), (d), (e) and (f), $b^2 + b + 1 \equiv 0 \mod p, bd \equiv 1 \mod p$ and $p = 3$ or $p \equiv 1 \mod 3$.

(5.4) In case (b), $2b^2 + 2b + 1 \equiv 0 \mod p, 2bd \equiv 1 \mod p$ and $p \equiv 1 \mod 4$.

(5.5) In case (c), $3b^2 + 3b + 1 \equiv 0 \mod p, 3bd \equiv 1 \mod p$ and $p \equiv 1 \mod 3$. 

On the other hand, in cases (a), (b) and (c) we have \(\langle \chi(2), g(3)\rangle \langle \chi(3), g(2)\rangle = 1\), i.e.

\[(5.6) \quad ed + bf \equiv 0 \mod p.\]

Now, combining (5.6) with (5.2) we conclude that \(b \equiv d \mod p\). Hence:

In case (a), \(b^2 \equiv 1 \mod p\). But also \(b^3 \equiv 1 \mod p\), so \(b \equiv 1 \mod p\) and \(p = 3\).

In case (b), \(2b^2 \equiv 1 \mod p\) and thus \(b \equiv -1 \mod p\) and \(1 \equiv 0 \mod p\), a contradiction.

In case (c), \(3b^2 \equiv 1 \mod p\). Plugging this into the first equation of (5.5) we easily get a contradiction.

Now we turn to the remaining cases. In case (d), we have

\[
\langle \chi(2), g(3)\rangle \langle \chi(3), g(2)\rangle = b_{22}^{-1} = b_{33}^{-1};
\]

using (5.2), this implies

\[
f(b - d) \equiv -1 \equiv f^2 \mod p.
\]

Thus \(b - d \equiv f \mod p\) and \((b - d)^2 \equiv b^2 + b - 2 \equiv -1 \mod p\); this last equation contradicts the first of (5.3). In case (e), we have \(\langle \chi(2), g(3)\rangle \langle \chi(3), g(2)\rangle = b_{22}^{-1} = b_{33}^{-1}\); using (5.2), this implies

\[
f(b - d) \equiv -1 \equiv 2f^2 \mod p.
\]

Thus \(b - d \equiv 2f \mod p\) and \((b - d)^2 \equiv b^2 + b - 2 \equiv 4f^2 \equiv -2 \mod p\); this last equation contradicts the first of (5.3). In case (f), we have \(\langle \chi(2), g(3)\rangle \langle \chi(3), g(2)\rangle = b_{22}^{-2} = b_{33}^{-1}\); using (5.2), this implies

\[
f(b - d) \equiv -2 \equiv f^2 \mod p.
\]

Thus \(b - d \equiv f \mod p\) and \((b - d)^2 \equiv b^2 + b - 2 \equiv f^2 \equiv -2 \mod p\); this last equation contradicts the first of (5.3).

We have shown that the only possibility in rank 3 is case (a) with \(p = 3\). Then, if \(V\) is a Yetter-Drinfeld module of Cartan type satisfying (1.6) with finite Cartan matrix and rank \(\geq 3\), then \(p = 3\), the corresponding Dynkin diagram should at most 2 connected components—cf. case (g)—and each component is of type \(A_1\) or \(A_2\). On the other hand, let \(u\) be a generator of \(\mathbb{Z}/(3)\) and \(q\) a root of 1 of order 3. It is easy to see that the sequences

\[(5.7) \quad g(1) = u = g(2), \quad g(3) = u^e = g(4), \quad \langle \chi(1), u\rangle = q = \langle \chi(2), u\rangle, \quad \langle \chi(3), u\rangle = q^{-e} = \langle \chi(4), u\rangle\]

define a Yetter-Drinfeld module of rank 4 over \(\mathbb{Z}/(3)\), which is of Cartan type and has Dynkin diagram \(A_2 \times A_2\). \(\Box\)
Lemma 5.2. Let $\mathcal{B}(V)$ be a finite dimensional Nichols algebra over $\mathbb{Z}/(p)$. Then $V$ is of finite Cartan type.

Proof. We know that the Lemma is true whenever $p$ is different from 3,5,7,11,13,17. Assume that $p$ is one of these small primes. For all of them, it would be enough to prove that no Cartan matrix as in (4.9), . . . , (4.17) is possible over $\mathbb{Z}/(p)$. All these matrices have a subdiagram of rank 2 of type $B_2$ or $G_2$. This eliminates all the primes except for 13. We show then that no matrix like (4.16) exists over $\mathbb{Z}/(13)$. Suppose in the contrary that it exists. Then there exists a generator $u$ of $\Gamma$, $q \in k^\times$ of order $p$ and integers $b, c, d, e$ (none of them divisible by 13) such that $g(1) = u, g(2) = u^b, g(3) = u^c, \langle \chi(1), u \rangle = q, \langle \chi(2), u \rangle = q^4, \langle \chi(3), u \rangle = q^e$; and the following equations hold

\begin{align*}
(5.8) \quad & b + d \equiv -3 \equiv -bd \mod 13, \\
(5.9) \quad & c + e \equiv -1 \equiv -3ec \mod 13, \\
(5.10) \quad & be + dc \equiv -ec \equiv -3bd \mod 13.
\end{align*}

Now (5.8) implies $(b, d) = (2, 8)$ or $(8, 2)$; and (5.9) implies $(c, e) = (5, 7)$ or $(7, 5)$. We have four possibilities for $be + dc$ but none of them gives $-9 \mod 13$; that is, (5.10) does not hold. □

Lemma 5.3. Let $R$ be a finite dimensional Hopf algebra braided Hopf algebra in $[1]YD$, $\Gamma = \mathbb{Z}/(p)$. Then $dim P(R) \leq 2$ for $p > 3$; and $dim P(R) \leq 4$ for $p = 3$.

Proof. Consider first the coradical filtration of $R$ and the subalgebra $S$ of the corresponding graded coalgebra generated by $P(R)$. Then $S$ is a Nichols algebra and the claim follows from Proposition 5.1 and Lemma 5.2. □

Proposition 5.4. Let $R$ be a finite dimensional braided Hopf algebra in $[1]YD$, $\Gamma = \mathbb{Z}/(p)$. Assume that (1.1), (1.2) and (1.3) hold. Then $R$ is a Nichols algebra.

Proof. Let $S = R^*$; it is a braided Hopf algebra in $[1]YD$ and satisfies (1.1), (1.2) and (1.4); that is, $S$ is graded, $S_0 = k1$ and is generated in degree 1. We want to show that (1.3) holds for $S$; hence (1.4) holds for $R$. See Lemma 8.1 below.

We know that $S(1) \subseteq P(S)$ and we want to prove the equality. If $dim S(1) = 1$, we are done by [AS2, Th. 3.2]. If $dim S(1) \geq 2$ and $p > 3$ then we are done by Lemma 5.3. We can then assume that $p = 3$; this case is treated in Section 8. □

§6. Isomorphisms between bosonizations.

We begin with a general Lemma whose proof is straightforward. We shall denote the comultiplication of $R$ by $\Delta_R(r) = \sum r^{(1)} \otimes r^{(2)}$; we omit most of the time the summation sign.

Lemma 6.1. Let $H$ be a Hopf algebra, $\psi : H \to H$ an automorphism of Hopf algebras, $V, W$ Yetter-Drinfeld modules over $H$.

(1) Let $V^\psi$ be the same space underlying $V$ but with action and coaction

\begin{align*}
(6.1) \quad & h \cdot^\psi v = \psi(h) v, \quad \delta^\psi(v) = (\psi^{-1} \otimes id) \delta(v), \quad h \in H, v \in V.
\end{align*}

Then $V^\psi$ is also a Yetter-Drinfeld module over $H$. If $T : V \to W$ is a morphism in $[1]YD$, then $T : V^\psi \to W^\psi$ also is. Moreover, the braiding $\sigma : V^\psi \otimes W^\psi \to W^\psi \otimes V^\psi$ coincides with the braiding $\sigma : V \otimes W \to W \otimes V$. 


Lemma 6.2. Let $B$ with coradical denote the inclusion. Let $A$ the restriction of $B$ the same objects as $\iota$. Let $H$ statement (2) follows from (1). Let us check (3). If $r, s \in R$ and $h, g \in H$ then

$$\Psi((r\#h)(s\#g)) = \Psi(rh_{(1)}\psi s\#h_{(2)}g) = r\psi(h_{(1)}).s\#\psi(h_{(2)}g);$$

$$\Psi(r\#h)\Psi(s\#g) = (r\#\psi(h))(s\#\psi(g)) = r\psi(h_{(1)}).s\#\psi(h_{(2)}\psi(g)).$$

That is, $\Psi$ is an algebra map. On the other hand,

$$\Delta\Psi(r\#h) = \Delta(r\#\psi(h)) = r^{(1)}\#(r^{(2)})_{(-1)}\psi(h)_{(1)} \otimes (r^{(2)})_{(0)}\#\psi(h)_{(2)};$$

$$(\Psi \otimes \Psi)\Delta(r\#h) = (\Psi \otimes \Psi) \left( r^{(1)}\#\psi^{-1} \left( (r^{(2)})_{(-1)} \right) h_{(1)} \otimes (r^{(2)})_{(0)}\#h_{(2)} \right)$$

$$= r^{(1)}\#(r^{(2)})_{(-1)}\psi(h_{(1)}) \otimes (r^{(2)})_{(0)}\#\psi(h_{(2)}).$$

Thus $\Psi$ is a bialgebra map and a fortiori a Hopf algebra map. □

Let $H$ be a cosemisimple Hopf algebra. Let $A = \oplus_{n \geq 0}A(n)$ be a coradically graded Hopf algebra with coradical $A_0 = A(0)$ isomorphic to $H$. Let $\iota : A(0) \rightarrow A$ denote the inclusion. Let $\pi : A \rightarrow A(0)$ be the unique graded projection with image $A(0)$ and let $R = \{a \in A : (id \otimes \pi)\Delta(a) = a \otimes 1\}$. Let $B$ be another Hopf algebra with coradical $B_0$ isomorphic to $H$.

Lemma 6.2. Let $\Phi : A \rightarrow B$ be an isomorphism of Hopf algebras. Let $B(n) := \Psi(A(n))$; then $B$ is also coradically graded with respect to the grading $B = \oplus_{n \geq 0}B(n)$. Let $\psi : H \rightarrow H$ denote the isomorphism induced by the restriction of $\Phi$, with respect to fixed identifications $A_0 \simeq H$, $B_0 \simeq H$. Let $\iota, \pi, R$ denote the same objects as $\iota, \pi, R$ but with respect to the grading in $B$. Then $\Psi(R) = S$ and the restriction $\phi : R \rightarrow S$ of $\Psi$ is a morphism $R \rightarrow S^\psi$ of braided Hopf algebras in $H\underline{\text{YD}}$. Moreover, the following diagram commutes

$$\begin{array}{ccc}
A & \xrightarrow{\Psi} & B \\
\simeq \downarrow & & \simeq \downarrow \\
R\#H & \xrightarrow{\phi \otimes \psi} & S\#H
\end{array}$$

where the vertical isomorphisms are the inverses of the bosonization maps.

Note that $R\#H \xrightarrow{\phi \otimes \psi} S\#H$ factorizes as the composition of two Hopf algebra maps: $R\#H \xrightarrow{\phi \otimes \id} S^\psi \#H \xrightarrow{\id \otimes \psi} S\#H$.

Proof. It is clear that $B = \oplus_{n \geq 0}B(n)$ is a grading of Hopf algebras. Since $\Psi$ preserves the terms of the coradical filtration, we have

$$R = \Psi(A_0) = \Psi(\oplus_{n \geq 0}A(n)) = \oplus_{n \geq 0}B(n);$$
that is, $B$ is coradically graded. It follows at once that

\begin{equation}
\delta \phi = \Psi \eta \pi \Psi^{-1}.
\end{equation}

Hence $\Psi(R) = S$. We check that $\phi$ is a morphism from $R$ to $S^\psi$. It preserves the multiplication because is the restriction of the algebra map $\Psi$. It preserves the comultiplication because $\Psi$ is a coalgebra map and by (6.2). It preserves the action and the coaction:

\[
\phi(h.r) = \Psi(h_1 r S(h_2)) = \psi(h_1) \phi(r) \psi(S(h_2))
\]

\[
= (\psi(h))_1 \phi(r) S((\psi(h))_2) = \psi(h) \phi(r) = h_\psi \phi(r);
\]

\[
\delta \phi(r) = (\zeta \otimes \text{id}) \Delta \Psi(r) = (\zeta \otimes \text{id})(\Psi \otimes \Psi) \Delta(r)
\]

\[
= (\text{id} \otimes \phi)(\psi^{-1} \pi \otimes \text{id}) \Delta(r) = (\text{id} \otimes \phi) \delta(r).
\]

Here again we used (6.2). The last statement follows because $\Psi$ is an algebra map. \qed

Let now $H = k\Gamma$, where $\Gamma$ is a finite abelian group. Let $g(1), \ldots, g(\theta) \in \Gamma$, $\chi(1), \ldots, \chi(\theta) \in \hat{\Gamma}$ and let $V$ be the Yetter-Drinfeld module with structure given by (1.5). Let also $h(1), \ldots, h(\theta) \in \Gamma$, $\eta(1), \ldots, \eta(\theta) \in \hat{\Gamma}$ and let analogously $W$ be the Yetter-Drinfeld module with structure given by (1.5) with $g(i)$'s, resp. $\chi(j)$'s, replaced by $h(i)$'s, resp. $\eta(j)$'s. Let $R$, $S$ be the corresponding Nichols algebras and $A = R \# H$, $B = S \# H$.

**Proposition 6.3.** Assume that $R$ and $S$ are finite dimensional. The Hopf algebras $A$ and $B$ are isomorphic if and only if there exist $\varphi \in \text{Aut}(\Gamma)$ and $\sigma \in S_\theta$ such that

\begin{equation}
\psi(j) = \varphi^{-1}(h(\sigma j)) , \quad \chi(j) = \eta(\sigma j) \psi, \quad 1 \leq j \leq \theta.
\end{equation}

**Proof.** Assume that $\Psi : A \to B$ is an isomorphism of Hopf algebras. We claim that $\Psi(A(n)) = B(n)$ for all $n \geq 0$. This is clear for $n = 0$ since $\Psi$ preserves the coradical filtration. For $n = 1$ we argue as in [AS2, Lemma 5.4]. Indeed, let us consider the adjoint action of $\Gamma$ on $A$. The decomposition $A_1 = A(0) \oplus A(1)$ is of $\Gamma$-modules. By [AS2, Lemma 3.1] $A(0)$ is the isotypical component of trivial type of the $\Gamma$-module $A_1$ (here we use that $A$ is finite dimensional). Hence $A(1)$ is the sum of all the other isotypical components. This implies that $\Psi(A(1)) = B(1)$, since $\Psi$ intertwines the respective adjoint actions of $\Gamma$. It is not difficult to see that $A$ is generated by $A_1$ as an algebra, by definition of Nichols algebra and [AS2, Lemma 2.4 (iii)]. Hence $\Psi(A(n)) = B(n)$ for all $n \geq 0$. We can apply now (the second half of) Lemma 6.2. The automorphism $\psi \in \text{Aut}(H)$ is determined by $\varphi \in \text{Aut}(\Gamma)$. We also know that $R \simeq S^\psi$ as braided Hopf algebras; but this is equivalent to $V \simeq W^\psi$ in $\mathcal{YD}$. This is possible if and only if (6.3) holds. If (6.3) holds, we do not need to assume that $R$ and $S$ are finite dimensional. By what we have just said, $R \simeq S^\psi$. Hence $B \# H \simeq S^\psi \# H$. By Lemma 6.1, we infer that $A \simeq B$. \qed
We now finish the proof of Theorem 1.3. We shall apply Proposition 6.3 with \( \Gamma = \mathbb{Z}/(p) \). Let first assume \( \theta = 2 \). Let \( g(1), g(2) \in \Gamma \) and \( \chi(1), \chi(2) \in \hat{\Gamma} \) define a Nichols algebra of finite Cartan type. Let \( u \in \Gamma - 1, q \in k^{\times} \) of order \( p \) and integers \( b, d \) such that the \( g(i) \)'s and \( \chi(j) \)'s are given by (6.1). Note that \( d \) is determined by \( b \). We denote by \( R(q, b) \) the Nichols algebra corresponding to the \( g(i) \)'s and \( \chi(j) \)'s. Let now \( h(1), h(2) \in \Gamma, \eta(1), \eta(2) \in \hat{\Gamma} \) define another Nichols algebra of finite Cartan type. Let also \( v \in \Gamma - 1, \widetilde{q} \in k^{\times} \) of order \( p \), integers \( s, t \) and

\[
h(1) = v, \quad h(2) = v^s, \quad \langle \eta(1), v \rangle = \widetilde{q}, \quad \langle \eta(2), v \rangle = \widetilde{q}^t.
\]

We can assume, by Proposition 6.3, that \( u = v \). Then the Nichols algebra corresponding to the \( h(i) \)'s and \( \eta(j) \)'s is \( R(\widetilde{q}, s) \).

**Lemma 6.4.** The Hopf algebras \( A = R(q, b)\#k\Gamma \) and \( B = R(\widetilde{q}, s)\#k\Gamma \) are isomorphic if and only if

(i) \( q = \widetilde{q} \) and \( b \equiv s \) mod \( p \) when the type is \( B_2 \) or \( G_2 \),

(ii) \( q = \widetilde{q} \) and \( b \equiv s \) or \( \equiv -1 - s \) mod \( p \) when the type is \( A_2 \).

**Proof.** We assume that \( A \) and \( B \) are isomorphic; it is clear that they are of the same type. Let \( \varphi, \sigma \) be as in Proposition 6.3, i.e. (6.3) holds. If \( \sigma = \text{id} \), then \( \varphi(u) = u \) and hence \( q = \widetilde{q}, b = s \). If \( \sigma \neq \text{id} \), then \( \varphi(u) = u^s, \varphi(u^b) = u \),

\[
q = \langle \chi(1), u \rangle = \langle \eta(2), u^s \rangle = \widetilde{q}^{st}, \quad q^d = \langle \chi(2), u \rangle = \langle \eta(1), u^s \rangle = \widetilde{q}^s.
\]

These equalities imply \( bs \equiv 1 \) mod \( p \), \( dt \equiv 1 \) mod \( p \). Hence \( st \equiv (bd)^{-1} \) mod \( p \). We know that \( (bd)^{-1} \equiv (st)^{-1} \equiv 1, 2 \) or \( 3 \) mod \( p \) if the type is respectively \( A_2, B_2 \) or \( G_2 \). This leads immediately to a contradiction unless the type is \( A_2 \). But in this last case, \( s \equiv d \) mod \( p \) and \( t \equiv b \) mod \( p \). Hence \( q = \widetilde{q} \) and \( b \equiv -1 - s \) mod \( p \). □

We now pass to the cases \( \theta = 3 \) and \( \theta = 4 \). We denote by \( R_3(q, e) \), resp. \( R_3(q, e) \), the Nichols algebra corresponding to the sequence in (6.7), resp. the sequence of the first three terms in (6.7). Here \( q \) is a third root of 1 and \( e = 1 \) or \( 2 \). By Proposition 6.3, arguing as for \( \theta = 2 \), we are reduced to prove:

**Lemma 6.5.** (i). The Hopf algebras \( R_3(q, e)\#H, q \) a third root of 1, \( e = 1 \) or \( 2 \), are not isomorphic to each other. (ii). The Hopf algebras \( R_3(q, e)\#H, q \) a third root of 1, \( e = 1 \) or \( 2 \), are not isomorphic to each other.

**Proof.** (i). All the four elements \( g(i) \) are equal for \( e = 1 \) but \( g(1) \neq g(3) \) for \( e = 2 \). All the four elements \( \chi(j) \) are equal for \( e = 2 \) but \( \chi(1) \neq \chi(3) \) for \( e = 1 \). The statement follows from these two observations. (ii). This is completely analogous and is left to the reader. □

§7. Pointed Hopf algebras of order \( p^4 \).

Let \( p \) be a prime number. We now characterize coradically graded pointed Hopf algebras of order \( p^4 \). This result implies the classification of all pointed Hopf algebras of order \( p^4 \) generated by group-like and skew primitive elements. We shall postpone to a separate paper the statement and proof of this last result.
Theorem 7.1. Let $G$ be a coradically graded pointed Hopf algebras of order $p^4$ with coradical $k\Gamma$. Then $G \simeq R\#k\Gamma$, where

1. $R$ is a quantum line if $\Gamma$ has order $p^3$;
2. $R$ is a quantum line or plane if $\Gamma$ has order $p^2$;
3. $R = \mathcal{B}(V)$, where $V$ has Dynkin diagram $A_2$ if $\Gamma$ has order $p$, where $p = 3$ or $p - 1$ is divisible by 3.

Proof. The case (7.2) follows from Proposition 7.5 below; we can not apply directly [AS2] since there are non-abelian groups of order $p^3$. The case (7.3) follows directly from [AS2].

Let us consider the case (7.4). Let $V = P(R)$ as usual. We know that $\dim P(R) > 1$. Let $R' = \mathcal{B}(V)$; it follows from Theorem 1.3 by a dimension argument that $V$ is as claimed and $R = R'$, or $V$ is a quantum plane and $R'$ has dimension $p^2$.

We now discard the case when $V$ is a quantum plane and $R'$ has dimension $p^2$ because of Proposition 5.4.

Proposition 7.5. Let $\Gamma$ be a finite (non-necessarily abelian) group. Let $R$ be a braided Hopf algebra in $\mathcal{YD}$ of prime order such that $R_0 = k1$. Then $R$ is a quantum line.

Proof. As $P(R)$ is a Yetter-Drinfeld submodule of $R$ and it is non-zero by hypothesis, there exists $g \in \Gamma$ such that $P(R)^g \neq 0$. But the Yetter-Drinfeld condition shows that $P(R)^g$ is stable by the action of $g$. Hence, there exists $y \in P(R)$, $y \neq 0$, such that $g.y = qy$, $\delta(y) = g \otimes y$.

The proof follows now exactly as in the final step of [AS1, Thm. B].

Remark. Proposition 7.5 allows the classification of finite dimensional pointed Hopf algebras whose coradical has index $p$. S. Dascalescu told us that he also knows this last result.

§8. Hopf algebras generated in degree one.

Some examples.

Let us first discuss examples of braided Hopf algebras $R$ satisfying (1.1), (1.2) and (1.3) but not (1.4). It is enough to show examples of braided Hopf algebras $S$ satisfying (1.1), (1.2) and (1.4) but not (1.3).

Lemma 8.1. Let $R = \oplus_{n \geq 0} R(n)$ be a graded braided Hopf algebra in $\mathcal{YD}$, $\Gamma$ a finite abelian group such that $R(0) = k1$. Assume that the homogeneous components of $R$ are finite dimensional. Let $S$ be the graded dual of $R$: $S = \oplus_{n \geq 0} R(n)^*$; it is a graded braided Hopf algebra in $\mathcal{YD}$ and $S(0) = k1$. Then $R$ satisfies (1.3) if and only if $S$ satisfies (1.4).

Proof. This is implicit in [AG, Example 3.2.8]. This was also observed by the referee of [AS2].

We show two easy examples:

(8.1). Let $F$ be a field of positive characteristic $p$. Let $S$ be the (usual) Hopf algebra $F[x]/(x^{p^2})$ with $x \in P(S)$. Then $x^p \in P(S)$. Hence $R = S^*$ satisfies (1.1), (1.2) and (1.3) but not (1.4).
(8.2). Let $S = \mathbb{k}[X] = \bigoplus_{n \geq 0} S(n)$ be a polynomial algebra in one variable. We consider $S$ as a braided Hopf algebra in $\mathcal{YD}^H$, where $H = \mathbb{k}\Gamma$, $\Gamma$ an infinite cyclic group with generator $g$, with action, coaction and comultiplication given by

$$\delta(X^n) = g^n \otimes X^n, \quad g.X = qX, \quad \Delta(X) = X \otimes 1 + 1 \otimes X.$$ 

Here $q \in \mathbb{k}$ is a root of 1 of order $N$. That is, $S$ is a so-called quantum line. Then $S$ satisfies (1.1), (1.2) and (1.4) but not (1.3) since $X^N$ is also primitive. Hence the graded dual $R = S^d = \bigoplus_{n \geq 0} S(n)^*$ is a braided Hopf algebra satisfying (1.1), (1.2) and (1.3) but not (1.4). Let us present explicitly $R$. It is a vector space with basis $y_n, n \geq 0$; the structure is given by $y_0 = 1$,

$$y_n y_m = \begin{bmatrix} m+n \quad n \end{bmatrix}_q y_{n+m}, \quad \Delta(y_n) = \sum_{0 \leq i \leq n} y_i \otimes y_{n-i},$$

$$\delta(y_n) = g^{-n} \otimes y_n, \quad g.y_n = q^{-n}y_n.$$

We do not know any finite dimensional counterexample in characteristic zero. Note that in both examples above there is implicit a Frobenius homomorphism. In characteristic zero, Lusztig said that “there are no powers of the quantum Frobenius map” [L4, 8.5, p. 58]; this could be related to our lack of examples. Indeed our argument below for braided Hopf algebras over an abelian group of exponent $p$ is a version of Lusztig’s claim.

**Proof of Theorem 1.4.**

We now prove Theorem 1.4 (see also Proposition 5.4 for $\Gamma = \mathbb{Z}/(p)$, $p > 3$). From now on $\Gamma = (\mathbb{Z}/(p))^s$. However, for any finite abelian group $\Gamma$ such that (4.8) holds the arguments below apply.

By [AS2, Lemma 2.2] it is enough to assume that $A$ is coradically graded. By the formula of the multiplication in the biproduct (see for instance [AS2, Section 2]) it is enough to prove that a finite dimensional braided Hopf algebra $R$ fulfilling (1.1), (1.2) and (1.3) also satisfies (1.4). By Lemma 8.1, we are reduced to prove

**Theorem 8.2.** Let $S = \bigoplus_{n \geq 0} S(n)$ be a graded braided Hopf algebra in $\mathcal{YD}^\Gamma$. We assume that $S$ is finite dimensional and $S(0) = \mathbb{k}1$. If $S$ is generated as an algebra by $S(1)$ then $P(S) = S(1)$.

**Proof.** Let $V = S(1)$; clearly $V \subseteq P(S)$. If $\dim V = 1$, we are done by [AS2, Th. 3.2]. We shall assume that $\dim V > 1$.

It is not difficult to verify the existence of a map $\psi : S \to \mathfrak{B}(V)$ of braided Hopf algebras in $\mathcal{YD}$ which is the identity on $V$ (for instance, use Lemma 8.1). By Corollary 1.2, $V$ is of finite Cartan type. By Lemma 2.3, we can twist $V$ and assume $b_{ij} = b_{ji}$, changing the group if necessary. By Lemma 4.3, we then know that $V$ is of finite FL-type. Let $x_1, \ldots, x_d$ be a basis of $V$ as in (1.5) and let $(a_{ij})$ be the associated Cartan matrix as in (1.9), (1.10).

Let $U_2^+$ be the quotient of the braided Hopf algebra $T(V)$ by the ideal generated by the elements $z_{ij} := (\text{ad}_c x_i)^{1-a_{ij}} x_j, i \neq j$. By Lemma A.1, $z_{ij}$ is primitive in $T(V)$; hence its image in $\mathfrak{B}(V)$ is 0. Note also that $U_2^+$ is a braided Hopf algebra in $\mathcal{YD}$, since the defining ideal is a Yetter-Drinfeld submodule of $T(V)$. Therefore, we have a commutative diagram of braided Hopf algebras in $\mathcal{YD}$. 
The map \(\phi\) factorizes through \(\psi\): \(\phi = \zeta \psi\), where \(\zeta: U^+_2 \to S\) is a morphism of braided Hopf algebras in \(\mathcal{YD}\).

Let \(i \neq j\), \(y_1 = x_i\), \(y_2 = z_{ij}\) and \(T\) the subalgebra of \(S\) generated by \(y_1\) and \(y_2\). Note that

\[
\delta(y_2) = g(i)^{1-a_{ij}}g(j) \otimes y_2, \quad h.(y_2) = \chi(i)^{1-a_{ij}}(h)\chi(j)(h)y_2, \quad \forall h \in \Gamma.
\]

Assume that \(y_2 \neq 0\). Since \(T\) is finite dimensional, passing to the graded Hopf algebra associated to its coradical filtration, we conclude from Corollary 1.2 that the matrix

\[
(8.4) \quad d = \begin{pmatrix}
\langle \chi(i), g(i) \rangle & \langle \chi(i)^{1-a_{ij}}, \chi(j), g(i) \rangle \\
\langle \chi(i), g(i)^{1-a_{ij}}g(j) \rangle & \langle \chi(i)^{1-a_{ij}}, \chi(j), g(i)^{1-a_{ij}}g(j) \rangle
\end{pmatrix}
\]

is of finite Cartan type, say with Cartan matrix \((A_{k\ell})_{1 \leq k, \ell \leq 2}\). Now

\[
\langle \chi(i)^{1-a_{ij}}, \chi(j), g(i) \rangle \langle \chi(i), g(i)^{1-a_{ij}}g(j) \rangle = \langle \chi(i), g(i) \rangle^{2-a_{ij}} = \langle \chi(i), g(i) \rangle^{A_{12}}.
\]

Hence \(2 - a_{ij} = 2, 3, 4, 5\) is congruent to \(A_{12} = 0, -1, -2\) or \(-3\) mod the order of \(\langle \chi(i), g(i) \rangle\). This is a contradiction since \(p > 17 > 7\). The claim is proved.

Claim 8.4. The map \(\psi\) is an isomorphism.

The algebra \(U^+_2\) is the positive part of a quantized algebra at a root of unity as in many places in the literature. We shall follow now the exposition in [AJS], see also [dCP]. Let \(R\) be the root system corresponding to the Cartan matrix \((a_{ij})\), \(R^+\) a system of positive roots, \(\Sigma\) the system of simple roots in \(R^+\). We can numerate \(\Sigma\) by \(\{1, \ldots, \theta\}\). We identify \(x_i\) with the element \(E_\alpha\), with respect to this numeration.

The algebra \(U^0_2\) in [AJS, p. 14], is nothing but the group algebra of the group \(\mathbb{Z}^\theta\). We consider the Hopf algebra \(U^\geq_2 := U^0_2 U^+_2\); it has a PBW basis which is a subset of the basis in [AJS, p. 15, formula (1)]. For each \(\beta \in R^+\), there is a "root vector" \(E_\beta \in U^+_2\). We consider

(a) the ideal \(I^+\) of \(U^+_2\) generated by all \(E^p_\beta\), \(\beta \in R^+\);

(b) the subalgebra \(\Upsilon^0_2\) of \(U^\geq_2\) generated by \(E^p_\beta\), \(\beta \in R^+\), and all the \(K^\pm_\alpha\), \(\alpha \in \Sigma\);

(c) the subalgebra \(\Upsilon^+_2\) of \(U^+_2\) generated by \(E^p_\beta\), \(\beta \in R^+\).

Then we have:

(i) \(U^+_2/I^+ \simeq \mathcal{B}(V)\).

(ii) \(\Upsilon^0_2\) is a central Hopf subalgebra of \(U^\geq_2\).

(iii) \(\Upsilon^+_2\) is a central Hopf subalgebra of \(U^+_2\) in \(U^0_2 \mathcal{YD}\) with trivial braiding.
The statement (i) follows from the considerations in the last paragraph of p. 15 and the first paragraph of p. 16 in [AJS], in view of Theorem 3.1.

The statement (ii) is [dCP, §19, Corollary in p. 120], in presence of the Theorem four lines below the Corollary.

We proceed with the statement (iii). It is clear by (ii) that $\mathfrak{U}_z^+$ is central in $U_z^+$. Now $U_2^0$ is a Hopf subalgebra of $U_2^{\geq 0}$ and its inclusion admits a retraction $U_2^{\geq 0} \rightarrow U_2^0$ sending the (non-trivial) monomials in $E_1^\beta$ to 0. It follows e.g. from the PBW basis that $U_2^+$ is a braided Hopf algebra in $U_2^0\mathcal{YD}$. It is clear that the algebra, coalgebra and braiding in this setting are exactly the same as those from (8.3). By (ii), the action of $U_2^0$ on $\mathfrak{U}_z^+$ is trivial; hence the braiding is trivial. Finally, $\mathfrak{U}_z^+$ is a Hopf subalgebra of $U_2^+$ since $\mathfrak{U}_z^{\geq 0}$ is a Hopf subalgebra of $U_2^{\geq 0}$.

We prove now the Claim 8.4; Theorem 8.2 and Theorem 1.4 follow immediately. Note that $\mathfrak{U}_z^+$ is a connected coalgebra since so is $U_2^+$. Let $S' = \zeta(\mathfrak{U}_z^+)$; this is a connected braided Hopf subalgebra of $S$. If $S' \neq k$ then there exists $x \in P(S')$, $x \neq 0$. But then the powers of $x$ are linearly independent: prove this by induction using the comultiplication and the triviality of the braiding. This is a contradiction; hence $S' = k$ and a fortiori $\zeta(\mathfrak{T}^+) = 0$. The Claim follows now from (i). □

We finally observe that the proof of Theorem 8.2 can be adapted to finish the proof of Proposition 5.4.

**Lemma 8.5.** Let $S$ be a finite dimensional braided Hopf algebra in $\mathcal{YD}$, $\Gamma = \mathbb{Z}/(3)$. Assume that (1.1), (1.2) and (1.4) hold. Then $S$ is a Nichols algebra.

**Proof.** The considerations before Claim 8.3 are still valid, using Lemma 5.2 instead of Corollary 1.2. Note that by (1.10) all the $a_{ij}$’s are 0, -1 or -2. We also observe that Claim 8.4 is also valid. That is, it is enough to prove Claim 8.3 for $\Gamma = \mathbb{Z}/(3)$.

By Proposition 5.1, we can assume that dim $S(1)$ is 2 or 3. Indeed, dim $S(1) = 4$ is maximal, and dim $S(1) = 1$ is treated via [AS2, Thm. 3.2].

We assume first that dim $S(1) = 2$. Then there exists a generator $u$ of $\Gamma$, $q \in k^\times$ of order $p$ and integers $b, d$ such that

\begin{equation}
(5.1) \quad g(1) = u, \quad g(2) = u^b, \quad \langle \chi(1), u \rangle = q, \quad \langle \chi(2), u \rangle = q^d.
\end{equation}

So $b_{11} = q$, $b_{22} = q^{bd}$.

There are two possibilities: a quantum plane or $A_2$.

In the first, we have to show that the skew-commutator $z_{12}$ is 0. But

\[ \delta(z_{12}) = u^{1+b} \otimes z_{12}, \quad u.z_{12} = q^{1+d}z_{12}. \]

Now $u^{1+b}.z_{12} = q^{(1+b)(1+d)}z_{12}$. Since $b = -d$, the exponent $(1 + b)(1 + d) = 1 - b^2$ is 0 mod 3. This is a contradiction unless $z_{12} = 0$.

In the second, we have to show that a "quantum Serre relation"-type element $z_{12}$ is 0. In this situation, $b = d = 1$. But $\delta(z_{12}) = u^{2+1} \otimes z_{12}$. Again, this is a contradiction unless $z_{12} = 0$.

We assume finally that dim $S(1)$ is 3. In this case, $S(1)$ is the direct sum $V_1 \oplus V_2$, where say $V_1$ is of type $A_2$ and $V_2$ is of type $A_1$, cf. Proposition 5.1. Applying the considerations for case 2, we see that $S$ is the twisted tensor product of $\mathfrak{B}(V_1)$ and $\mathfrak{B}(V_2)$. It follows that $S$ is $\mathfrak{B}(V)$, as claimed. □
Appendix. Primitive elements from quantum Serre relations.

Gaussian binomial coefficients.

In the polynomial algebra \( \mathbb{Z}[q] \), we consider the Gaussian, or \( q \)-binomial, coefficients

\[
\binom{n}{i}_q = \frac{(n)_q!}{(n-i)_q! (i)_q!},
\]
where \((n)_q = (n)_q \cdots (2)_q (1)_q\), and \((n)_q = 1 + q + \cdots + q^{n-1}\), for \( n \in \mathbb{N}, 0 \leq i \leq n \).

We have the identity

\[
\binom{n}{h}_q + \binom{n}{h-1}_q = \binom{n+1-h}{h}_q, 1 \leq h \leq n;
\]

This implies that \( \binom{n}{i}_q \in \mathbb{Z}[q] \). If \( A \) is an associative algebra over \( k \) and \( q \in k \), then \( \binom{n}{i}_q \) denotes the specialization of \( \binom{n}{i}_q \) in \( q \). If \( x, y \in A \) are two elements that \( q \)-commute, i.e. \( xy = qyx \), then the quantum binomial formula holds for every \( n \in \mathbb{N} \):

\[
(x+y)^n = \sum_{i=0}^{n} \binom{n}{i}_q y^i x^{n-i}.
\]

Let us also record

\[
(r)_q + q^r (s)_q = (r+s)_q.
\]

There is another version of "\( q \)-binomial coefficients": in the Laurent polynomial algebra \( \mathbb{Z}[q,q^{-1}] \), we consider

\[
\left[ \frac{n}{i} \right]_q = \frac{[n]_q!}{[n-i]_q! [i]_q!}, \quad \text{where} \quad [n]_q = [n]_q \cdots [2]_q [1]_q, \quad \text{and} \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]

Clearly,

\[
(n)_q^2 = q^{n-1} [n]_q,
\]

and hence

\[
\binom{n}{i}_q^2 = q^{i(n-i)} \left[ \frac{n}{i} \right]_q.
\]

In particular, the following equality- see [L3, 1.3.4 (a)]-

\[
\sum_{i=0}^{n} (-1)^i q^{i(1-n)} \left[ \frac{n}{i} \right]_q = 0
\]

translates into

\[
\sum_{i=0}^{n} (-1)^i q^{(i^2+i)/2-ni} \binom{n}{i}_q = 0.
\]

This last equality is also equivalent, by a change of the summation index, to

\[
\sum_{h=0}^{n} (-1)^h q^{(h^2-h)/2} \binom{n}{h}_q = 0.
\]
**Primitive elements.**

Let $R$ be a braided Hopf algebra over our fixed finite abelian group $\Gamma$. We assume that (1.1), (1.2) and (1.3) hold; we also suppose that $\dim P(R) = 2$. But we do not assume that $R$ is finite dimensional, unless explicitly stated. To simplify the notation, we fix a basis $x, y$ of $P(R)$ such that

$$\delta(x) = g \otimes x, \quad h.x = \chi(h)x, \quad \delta(y) = t \otimes y, \quad h.y = \eta(h)y \quad \forall h \in \Gamma,$$

for some $g, t \in \Gamma$, $\chi, \eta \in \hat{\Gamma}$. Let $N, M$ be respectively the orders of $\chi(g), \eta(t)$. We do suppose that they are positive.

**Definition.** Let $S$ be a braided Hopf algebra. The *braided adjoint representation* is the linear map $\text{ad}_c : S \to \text{End} S$ given by

$$\text{ad}_c(u)(v) = \mu(u \otimes v - c(u \otimes v));$$

here $\mu$ is the multiplication map of $S$ and $c$ is the commutativity constraint.

The following property holds:

(A.7) \quad $\text{ad}_c(u)(vw) = \text{ad}_c(u)(v)w + (u_{(-1)}v) \text{ad}_c(u_{(0)})(w)$.

In our situation, we define inductively $z_1 = \text{ad}_c(x)(y) = xy - \eta(g)yx$; $z_{j+1} = \text{ad}_c(x)(z_j)$. For instance,

$$z_2 = x^2y - \eta(g)(1 + \chi(g))xy + \eta(g)^2\chi(g)y^2,$$

$$z_3 = x^3y - \eta(g)(3)x^2yx + \eta(g)^2\chi(g)(3)\chi(g)xy^2 - \eta(g)^3\chi(g)^3yx^3,$$

and in general

(A.8) \quad $z_N = \sum_{i=0}^{N}(-1)^{i}\binom{N}{i} \chi(g)^{i(i-1)/2}\eta(g)^i x^{N-i}yx^i$.

The proof of (A.8) is by induction using (A.7).

We following Lemma is probably well-known. For completeness, we give a direct proof.

**Lemma A.1.** Assume that

(A.9) \quad $\eta(g)\chi(t)\chi(g)^{r-1} = 1$.

Then $(\text{ad}_c x)^r y$ is primitive.

**Proof.** Let $z = \sum_{i=0}^{r} \alpha_i x^iyx^{r-i}$, where $\alpha_i \in k$. Then

$$\Delta(z) = \sum_{i=0}^{r} \sum_{\ell=0}^{r-i} \sum_{h=0}^{r-i} \alpha_i \binom{i}{\ell} \binom{r-\ell}{h} \chi(g)^{\ell} \chi(g)^{h} \eta(g)^{i-\ell} \chi(g)^{h(i-\ell)} x^{\ell}yx^h \otimes x^{r-\ell-h}$$

$$+ \chi(t)^h \chi(g)^{h(i-\ell)} x^{\ell+h} \otimes x^{i-\ell}yx^{r-i-h}$$

$$= z \otimes 1 + 1 \otimes z$$

$$+ \sum_{\ell \geq 0, h \geq 0} \sum_{\ell \leq i \leq r-h} \alpha_i \binom{i}{\ell} \binom{r-\ell}{h} \eta(g)^{i-\ell} \chi(g)^{h(i-\ell)} x^{\ell}yx^h \otimes x^{r-\ell-h}$$

$$+ \sum_{u \geq 0, v \geq 0} \sum_{u \leq i \leq r-v} \alpha_i \binom{i}{u} \binom{r-i-v}{v} \chi(t)^{r-i-v} \chi(g)^u(x^{r-i-v}) x^{r-u-v} \otimes x^u yx^v.$$
Hence, if $\alpha_0, \ldots, \alpha_r$ is a solution of the system of equations

\[(A.10a) \quad \sum_{\ell \leq i \leq r-h} \alpha_i \binom{i}{\ell} \binom{r-i}{h} \eta(g)^{i-\ell} \chi(g)^{h(i-\ell)} = 0,\]

\[(A.10b) \quad \sum_{u \leq i \leq r-v} \alpha_i \binom{i}{u} \binom{r-i}{v} \chi(t)^{r-i-v} \chi(g)^{u(r-i-v)} = 0,\]

then $z$ is primitive.

Now assume that $z = (\text{ad}_c x)^r y$, that is

$$z = \sum_{0 \leq i \leq r} (-1)^{r-i} \binom{r}{i} \chi(g)^{2r-i} \chi(g)^{(r-i)/(r-i-1)/2} \eta(g)^{r-i} x^i y x^{r-i}.$$

To see that $z$ is primitive, we divide $z$ by $(-1)^r \chi(g)^{(r^2-r)/2} \eta(g)^r$ and reduce to check that

$$\alpha_i := (-1)^i \binom{r}{i} \chi(g)^{(i^2+i)/2-ri+2h(i-\ell)}$$

satisfies the system of equations (A.10). We proceed with (A.10a). If $\ell \geq 0$, $h \geq 0$, $\ell + h < r$, then

$$\sum_{\ell \leq i \leq r-h} (-1)^i \binom{r}{i} \chi(g)^{2r-i} \chi(g)^{(r-\ell)/(r-\ell-1)/2} \eta(g)^{r-i} \chi(g)^{(i^2+i)/2} = 0,$$

by (A.5).

Now we pass to (A.10b). For $u \geq 0$, $v v \geq 0$, $u + v < r$, we compute:

$$\sum_{u \leq i \leq r-v} (-1)^i \binom{r}{i} \chi(g)^{2r-i} \chi(g)^{u(r-i-v)} = 0,$$

where we have used (A.6). After dividing by the appropriate factor and changing the summation index, we arrive to

$$(-1)^u \chi(g)^{(u^2+u)/2} \sum_{0 \leq i \leq r-v-u} (-1)^i \binom{r-u-v}{i} \chi(g)^{(i^2-i)/2} = 0,$$

by (A.6).
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