Quantum Field Theory on Noncommutative 
Space-Times and the Persistence of 
Ultraviolet Divergences

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Abstract

We study properties of a scalar quantum field theory on two-dimensional noncommutative space-times. Contrary to the common belief that noncommutativity of space-time would be a key to remove the ultraviolet divergences, we show that field theories on a noncommutative plane with the most natural Heisenberg-like commutation relations among coordinates or even on a noncommutative quantum plane with $E_q(2)$-symmetry have ultraviolet divergences, while the theory on a noncommutative cylinder is ultraviolet finite. Thus, ultraviolet behaviour of a field theory on noncommutative spaces is sensitive to the topology of the space-time, namely to its compactness. We present general arguments for the case of higher space-time dimensions and as well discuss the symmetry transformations of physical states on noncommutative space-times.

PACS: 03.70

Keywords: Quantum field theory, ultraviolet divergences, regularization, 
noncommutative space-times.

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1 Introduction

The standard concept of a geometric space is based on the notion of a manifold $\mathcal{M}$ whose points $x \in \mathcal{M}$ are locally labelled by a finite number of real coordinates $x^\mu \in \mathbb{R}^4$. However, it is generally believed that the picture of space-time as a manifold $\mathcal{M}$ should break down at very short distances of the order of the Planck length $\lambda_P \approx 1.6 \times 10^{-33} \text{cm}$. This implies that the mathematical concepts for high energy (small distance) physics have to be changed, or more precisely, our classical geometrical concepts may not be well suited for the description of physical phenomena at small distances. No convincing alternative description of physics at very short distances is known, though different routes to progress have been proposed. One such direction is to try to formulate physics on some noncommutative space-time. There appear to be too many possibilities to do this, and it is difficult to see what the right choice is. There have been investigations in the context of Connes’ approach \cite{1} to gravity and the Standard Model of electroweak and strong interactions \cite{4, 5}, relation between measurements at very small distances and black hole formations \cite{1} and string theory \cite{2}. One more possibility is based on quantum group theory (see, e.g., \cite{6}). It is worth to note that the generalization of commutation relations for the canonical operators (coordinate-momentum or creation-annihilation operators) was suggested long ago by Heisenberg \cite{7} in attempts to achieve regularization for his (nonrenormalizable) nonlinear spinor field theory (see also Wigner \cite{8}, and for a review \cite{6}).

The essence of the noncommutative geometry consists in reformulating first the geometry in terms of commutative algebras and modules of smooth functions, and then generalizing them to their noncommutative analogs. If the notions of the noncommutative geometry are used directly for the description of the space-time, the notion of points as elementary geometrical entity is lost and one may expect that an ultraviolet (UV) cut-off appears. The simplest model of this kind is the fuzzy sphere (see \cite{9, 10, 11} and refs. therein), i.e. the noncommutative analog of a two-dimensional sphere. As is well known from the standard quantum mechanics, a quantization of any compact space, in particular a sphere, leads to finite-dimensional representations of the corresponding operators, so that in this case any calculation is reduced to manipulations with finite-dimensional matrices and thus there is simply no place for UV-divergences \cite{11}. It is worth to mention also that the parameter of noncommutativity (analog of the Planck constant) in the case of quantization of compact manifolds is related to dimensionality of a matrix representation \cite{9}. After removing the noncommutative parameter, the known results of the classical geometry are recovered.

Things are not so easy in the case of non-compact manifolds. The quantization leads to infinite-dimensional representations and we have no guarantee that noncommutativity of the space-time coordinates removes the UV-divergences. A relatively simple type of noncommutative geometry of non-compact Minkowski space with Heisenberg-like commutation relations among coordinates, has been considered in \cite{4}. Later it was shown \cite{12} that this model has the same UV-behaviour as an ordinary field theory on the commutative space-time. In the next section we shall use this example, reduced to two-dimensional space, for a general discussion of construction of a quantum field theory on a noncommutative space (NC-QFT) and argue that the underlying reason for such a UV-behaviour of the model is that the corresponding algebra of quantum mechanical operators is isomorphic to the usual one and that the momenta degrees of freedom are associated to the
In discussing this example, we shall see that physically meaningful quantities in NC-QFT are the correlation functions (Green functions) of mean values of quantum fields on a noncommutative space-time in localized states from the Hilbert space of a representation of the corresponding coordinate algebra. These localized states are the best counterparts of points on an ordinary commutative space, which "label" (infinite number of) degrees of freedom of the QFT. Thus we must consider a quantum field in NC-QFT as a map from the set of states on the corresponding noncommutative space into the algebra of secondary quantized operators. We also use this example for the study of symmetry transformations of noncommutative space-times with Lie algebra commutation relations for coordinates. The noncommutative coordinates prove to be tensor operators, and we consider concrete examples of the corresponding transformations of localized states (analog of space-time point transformations).

In section 3 we consider more complicated example of NC-QFT, defined on a noncommutative cylinder. The latter is obtained via quantization of its classical counterpart, by considering it as a co-adjoint orbit of the two-dimensional Euclidean group $E(2)$ with the uniquely defined Lie-Kirillov-Kostant Poisson structure. As is well known, a scalar two-dimensional QFT on a commutative cylinder has the same UV-properties as QFT on a plane (in particular, divergent tadpole Feynman diagram): UV-behaviour is related to small distances and, hence, insensitive to the topology of a coordinate manifold. On the contrary, quantization procedure is very sensitive to the topology and this leads to essentially different properties of NC-QFT on the noncommutative cylinder and on the noncommutative plane. While the latter retains the divergent tadpoles (as an ordinary QFT), the former proves to be UV-finite.

It is known that a cylinder is homeomorphic to a plane with one punctured point. In the noncommutative case, this map corresponds to transition to a quantum plane. In section 4 we proceed to the study of a quantum plane with $E_q(2)$-symmetry. Now the coordinate and momentum operators form a subalgebra of ordinary quantum mechanical algebra, so that some operators have discrete spectra. This fact could inspire the belief that the corresponding NC-QFT has an improved UV-behaviour. However, the explicit calculation shows that this is not the case and the QFT on the quantum plane has the usual UV-divergences.

Section 5 is devoted to the summary of the results. In particular, the compactness of a noncommutative space-time is crucial for the UV-behaviour of NC-QFT, namely, at most one dimension (time) is allowed to be noncompact, in order to achieve the removal of UV-divergences of a quantum field theory formulated in a noncommutative space-time of arbitrary dimensions.

2 Two-dimensional quantum field theory on noncommutative space-time with Heisenberg-like commutation relations

As an introductory example, we consider the simplest case of a two-dimensional Euclidean field theory with the noncommutativity of the Heisenberg type. However, for the convenience of exposition we start with a brief summary of the standard commutative case.
2.1 Two-dimensional fields on a Euclidean commutative plane

The complex scalar field \( \phi(x) \) on a Euclidean plane \( P^{(2)} = \mathbb{R}^2 \) is a prescription

\[
x = (x_1, x_2) \in P^{(2)} \rightarrow \phi(x) \in \mathbb{C} ,
\]

which assigns to any point \( x \) of the plane the complex number \( \phi(x) \).

There is an equivalent description of functions on \( P^{(2)} \) in terms of the Fourier transform:

\[
f(x) = \frac{1}{(2\pi)^2} \int d^2 k \tilde{f}(k)e^{ikx} .
\]

The integral of \( f(x) \) over \( P^{(2)} \) can be expressed as

\[
I_0[f] = \int d^2 x f(x) = \tilde{f}(0) ,
\]

where \( d^2 x \) is the standard Lebesgue measure on \( P^{(2)} \). The point-wise multiplication of fields can be expressed as a convolution of their Fourier transforms:

\[
f(x)\phi(x) = \int d^2 k (\tilde{f} \circ \tilde{\phi})(k)e^{ikx} ,
\]

\[
(\tilde{f} \circ \tilde{\phi})(k) \equiv \frac{1}{(2\pi)^2} \int d^2 q d^2 q' \delta(k - q - q')\tilde{f}(q)\tilde{\phi}(q') .
\]

The Euclidean action of self-interacting scalar field usually has the form

\[
S[\phi, \phi^*] = S_0[\phi, \phi^*] + S_{int}[\phi, \phi^*] ,
\]

where

\[
S_0[\phi, \phi^*] = \int d^2 x [\partial_i \phi^*(\partial_i \phi) + m^2 \phi^* \phi] = \frac{1}{(2\pi)^2} \int d^2 k \phi^*(k)[k^2 + m^2]\phi(k) ,
\]

is the free field part of the action, and

\[
S_{int}[\phi, \phi^*] = \int d^2 x V(\phi^* \phi) ,
\]

where \( V(\cdot) \) is a positive definite polynomial describing the self-interaction of the field.

2.2 Transition to noncommutative plane

There exists a possibility to introduce in the two-dimensional Euclidean coordinate plane \( P^{(2)} \) an additional Poisson structure. It is defined by the elementary bracket

\[
\{x_i, x_j\} = \varepsilon_{ij} , \quad i, j = 1, 2 ,
\]

and extended by Leibniz rule to all smooth functions on \( P^{(2)} \) (here \( \varepsilon_{ij} \) is the antisymmetric tensor, \( \varepsilon_{12} = 1 \)). The brackets are invariant with respect to canonical transformations \( x_i \rightarrow M_{ij}x_j + a_i \), where \( M_{ij} \) is an unimodular matrix, \( a_1, a_2 \) are arbitrary constants. In
particular, the brackets are invariant with respect to the two-dimensional group $E(2)$ of isometries of $P^{(2)}$ formed by

\[(i)\] rotations: $x_1 \to x_1 \cos \phi + x_2 \sin \phi$, $x_2 \to x_2 \cos \phi - x_1 \sin \phi$,  

\[(ii)\] translations: $x_1 \to x_1 + a_1$, $x_2 \to x_2 + a_2$. 

The noncommutative version $P^{(2)}_{\lambda}$ of the plane is obtained by the deformation (quantization) of this Poisson structure. In the noncommutative approach we replace the commuting parameters by the hermitian operators $\hat{x}_i$, $(i,j) = 1,2$ satisfying commutation relations

\[ [\hat{x}_i, \hat{x}_j] = i\lambda^2 \varepsilon_{ij}, \quad i,j = 1,2, \]

where $\lambda$ is a positive constant of the dimension of length. We realize the operators $\hat{x}_i$, $i,j = 1,2$ in a suitable Fock space $\mathcal{F}$ introducing the annihilation and creation operators

\[ \hat{\alpha} = \frac{1}{\lambda \sqrt{2}} (\hat{x}_1 + i \hat{x}_2), \quad \hat{\alpha}^\dagger = \frac{1}{\lambda \sqrt{2}} (\hat{x}_1 - i \hat{x}_2), \]

and putting

\[ \mathcal{F} = \{ |n\rangle = \frac{1}{\sqrt{n!}} \hat{\alpha}^n |0\rangle; \quad n = 0,1,\ldots \} . \]

Here $|0\rangle$ is a normalized state satisfying $\hat{\alpha} |0\rangle = 0$. This realization of operators $\hat{x}_i$, $i = 1,2$ (or, equivalently of $\hat{\alpha}$ and $\hat{\alpha}^*$) corresponds to the unitary irreducible representation of the Heisenberg-Weyl group $H(2)$.

For all operators of the form

\[ \hat{f} = \frac{\lambda^2}{(2\pi)^2} \int d^2 k \tilde{f}(k)e^{ik\hat{x}} \]

(with a suitable smooth decreasing $\tilde{f}(k)$) we introduce the integral (linear functional) $I_{\lambda}[\hat{f}]$ as follows:

\[ I_{\lambda}[\hat{f}] = \text{Tr} \hat{f} = \tilde{f}(0) . \]

Here Tr denotes the trace in the Fock space and $k\hat{x} = k_1 \hat{x}_1 + k_2 \hat{x}_2$. The last equality in (11) indicates that $I_{\lambda}[\cdot]$ is the noncommutative analog of the usual integral $I_0[\cdot]$ on $P^{(2)}$.

The noncommutative analogs of field derivatives $\partial_i \hat{\phi}$, $i = 1,2$ are defined as

\[ \partial_i \hat{\phi} = \varepsilon_{ij} \frac{i}{\lambda^2} [\hat{x}_j, \hat{\phi}], \quad i,j = 1,2 . \]

They satisfy the Leibniz rule and reduce to the usual derivatives in the commutative limit.

In the noncommutative case we define the Euclidean action of self-interacting scalar noncommutative quantum field theory (NC-QFT) similarly as before:

\[ S^{(\lambda)}[\hat{\phi}, \hat{\phi}^\dagger] = S_0^{(\lambda)}[\hat{\phi}, \hat{\phi}^\dagger] + S_{\text{int}}^{(\lambda)}[\hat{\phi}, \hat{\phi}^\dagger] , \]

with the free part of the action

\[ S_0^{(\lambda)}[\hat{\phi}, \hat{\phi}^\dagger] = I_{\lambda}[ (\partial_i \hat{\phi})^\dagger (\partial_i \hat{\phi}) + m^2 \hat{\phi}^\dagger \hat{\phi} ] . \]

The interaction part $S_{\text{int}}$ of the action we shall discuss later (see (32) and below).
2.3 Quantum mechanics with the Heisenberg-like relation for the coordinates on a quantum plane

The existence of the fields $\tilde{\varphi}(k_1, k_2)$ in the momentum representation with *commuting* variables $k_1, k_2$ implies that there exist commutative operators of momentum components for a particle on the quantum plane $P^{(2)}_\lambda$. This is indeed the case. The definition (12) of the derivatives shows that the momenta operators should be defined as follows (only in this subsection we use explicit dependence on the Planck constant $\hbar$, outside it we put $\hbar = 1$):

$$\hat{p}_i = i\hbar \lambda^{-2} \varepsilon_{ij} \text{ad}_{x_j},$$

$$\text{ad}_{x_i} \hat{A} \equiv [\hat{x}_i, \hat{A}] \quad \text{for any operator } \hat{A}$$

and since the right-hand side of (8) is a constant, we have

$$[\hat{p}_1, \hat{p}_2] \sim [\text{ad}_{x_2}, \text{ad}_{x_1}] = 0.$$ (17)

Moreover, since the coordinate operators satisfy commutation relations of the Heisenberg-Lie algebra, the form of the "Fourier transform" (10) shows that the momentum variables $p_1, p_2$ play the role of parameters of the corresponding Heisenberg-Weyl group. Thus Quantum Mechanics which corresponds to NC-QFT is defined by the following commutation relations

$$[\hat{x}_1, \hat{x}_2] = i\lambda^2, \quad [\hat{p}_1, \hat{p}_2] = 0,$$

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad i, j = 1, 2.$$ (18)

Introduction of the operators

$$\hat{X}_i = \hat{x}_i + \frac{\lambda^2}{2\hbar} \varepsilon_{ij} \hat{p}_j,$$ (19)

with the commutation relations

$$[\hat{X}_i, \hat{p}_j] = i\hbar \delta_{ij}$$

$$[\hat{X}_i, \hat{X}_j] = [\hat{p}_i, \hat{p}_j] = 0,$$ (20)

shows that the quantum mechanical algebra of the phase space operators on the considered quantum space (as well as the algebra in the model considered in [4,12]) is the same as in the ordinary quantum mechanics. This simple observation has two implications:

- the very existence of noncommutativity, at least of the type considered in this section, does not imply necessarily UV-finiteness (contrary, for instance, to the case of Quantum Mechanics on a lattice); one needs some additional *dynamical* principle to achieve this aim; in particular, one may hope to construct a QFT Lagrangian in terms of noncommutative objects resulting in the essentially nonlocal action and UV-finite QFT.
• the relation \( [19] \) show that for small values of the momentum the noncommutative coordinates are close to usual commutative coordinates:

\[
\frac{\hat{x}_i}{\lambda} = \frac{\hat{X}_i}{\lambda} - \frac{1}{2} \varepsilon^{ij} \frac{\lambda \hat{p}_j}{\hbar},
\]

so that if for a given state \( \langle \psi | \hat{p}_i/\lambda^{-1} | \psi \rangle \ll 1 \), the two types of coordinates have close values. This remark may provide a basis for a more detailed explanation of the fact that particles with relatively low energy (i.e. in any present experiment) do not feel the space-time noncommutativity.

2.4 Calculations by the method of operator symbols

The calculation of Green functions and other quantities in NC-QFT can be carried out in simple and natural way by the use of operator symbols. An operator symbol (see, e.g., \([14]\) and refs. therein), as it is defined in the ordinary Quantum Mechanics, is a function on the phase space which is constructed by a definite rule from a given operator. Different rules for the construction produce different symbols (e.g., Weyl, normal symbols etc.). Sets of such functions endowed with a so-called star-product form algebras which are isomorphic to the initial operator algebra. Knowledge of explicit forms of star-products (⋆-products) allows to make calculations in an easy and short way. In fact, transition to the momentum representation \( \varphi(\tilde{x}) \rightarrow \tilde{\varphi}(k) \) in eq. \([10]\) is the first step in the construction of the corresponding Weyl symbol. If, in addition, we make now inverse ordinary Fourier transformation

\[
\varphi_W(x) = \frac{\lambda^2}{(2\pi)^2} \int d^2k \tilde{\varphi}(k)e^{ikx},
\]

we obtain just what is called the Weyl symbol \( \varphi_W(x) \) of the operator \( \varphi(\tilde{x}) \). Notice that the symbol is defined on the classical (commutative) analog of the noncommutative space. Operator product corresponds to the ⋆-product in the set of the symbols: \( \varphi(\tilde{x})\psi(\tilde{x}) \rightarrow \varphi_W \ast \psi_W \). Thus, the Weyl symbols or their Fourier transforms (which plays the role of the momentum representation for a field on the noncommutative plane \( P^{(2)}_\lambda \)) are in one-to-one correspondence with the set of fields (operators) \( \varphi(\tilde{x}) \) on the noncommutative space. This correspondence is based on the relation \( \text{Tr} \exp \{ i k \cdot \tilde{x} \} = 2\pi \lambda^{-2} \delta^{(2)}(k) \). The trace of an operator \( f(\tilde{x}) \) is needed, in particular, for construction of the action. It is expressed via its Weyl symbol as follows:

\[
\text{Tr} f(\tilde{x}) = \frac{1}{2\pi\lambda^2} \int d^2x f_W(x) = I_\lambda[f(\tilde{x})].
\]

Now any action for NC-QFT can be obtained from the corresponding classical action by the substitution of the ordinary point-wise function multiplication by the ⋆-product. For example,

\[
\text{Tr} \sum_i [\tilde{x}_i, \varphi(\tilde{x})]^2 = \int d^2x \{x_i, \varphi_W(x)\}_M \ast \{x_i, \varphi_W(x)\}_M
\]

\[
= \int d^2x (\partial_i \varphi_W \ast \partial_i \varphi_W)(x),
\]
where \( \{\cdot, \cdot\}_M \) is the Moyal bracket:

\[
\{\varphi, \psi\}_M \equiv \frac{1}{\lambda^2}(\varphi \star \psi - \psi \star \varphi) .
\]

Equivalently, one may use the Fourier transform \( \tilde{\varphi}(p) \) of the Weyl symbol (momentum representation). The existence of this field, depending on the commutative variables \( p_1 \) and \( p_2 \), corresponds to the commutativity of the momentum operators (cf. (17)) of the considered system.

The “second quantization” in the Euclidean case amounts to calculation of the path integral over set of operator symbols which gives the generating functional \( Z[J] \) for Green functions

\[
Z[J] = N^{-1} \int D\varphi_W(x) \exp \{-S[\varphi_W, J, \star]\} ,
\]

where \( S[\varphi_W, J, \star] \) is the operator action (13) on the noncommutative space expressed in terms of symbols, or, in other words, the usual classical action in which the ordinary pointwise multiplication of the fields is substituted by the star-product (\( N \) is the normalization constant).

The Weyl symbol has some special properties which makes it convenient for the calculations. In particular:

i) The explicit form of the \( \star \)-product which makes the algebra of Weyl symbols isomorphic to the operator algebra is defined by the expression

\[
(\varphi_W \star \psi_W)(x) = \varphi_W(x) \exp \left\{ \frac{i\lambda^2}{2} \partial_i \varepsilon^{ij} \partial_j \right\} \psi_W(x)
\]

\[
= \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{i\lambda^2}{2} \right)^m \varepsilon^{i_1j_1} \cdots \varepsilon^{i_mj_m} (\partial_{i_1} \cdots \partial_{i_m} \varphi_W)(\partial_{j_1} \cdots \partial_{j_m} \psi_W)
\]

\[
= \varphi_W(x)\psi_W(x) + O(\eta) .
\]

This immediately shows that a quadratic term in the NC-QFT action, written in terms of the Weyl symbols, has the same form as that on the classical space:

\[
\int d^2x \ (\varphi_W \star \psi_W)(x) = \int d^2x \varphi_W(x) \exp \left\{ -i\frac{\lambda^2}{2} \partial_i \varepsilon^{ij} \partial_j \right\} \psi_W(x)
\]

\[
= \int d^2x \varphi_W(x)\psi_W(x) ,
\]

because \( \varepsilon^{ij} \) is antisymmetric. Therefore, the free action of the NC-QFT in terms of the Weyl symbols has the same form as usual QFT on commutative space. Higher order (interaction) terms contain non-locality, but Filk’s analysis in [12] shows that they do not remove UV-divergences.

ii) One more property of the Weyl symbols is their nice behaviour with respect to linear canonical transformations: if one considers transformations (cf. (6))

\[
\hat{x}' = M_{ij} \hat{x}_j + b_i ,
\]

the corresponding Weyl symbol transforms as follows

\[
\varphi'_W(x) = \varphi_W(Mx + b) ,
\]
i.e. it is transformed as an ordinary scalar field. This essentially simplify study of invariance properties of NC-QFT.

When considering the Green functions $G_W(x, y) \equiv \langle \varphi_W(x) \varphi_W(y) \rangle$, one should take into account that a value of an operator symbol at a point on the classical counterpart of a noncommutative space has not direct physical and even mathematical meaning. Only the total function can be considered as a symbol and defines the corresponding operator. Thus the function $G_W(x, y)$ has the meaning of an operator symbol acting in the direct product $\mathcal{H} \otimes \mathcal{H}$ of two copies of the Hilbert space in which a representation of the coordinate algebra is realized. Now let us remember that in the standard QFT, the points of a commutative space (labelled by values of the coordinates $x_1, x_2$) are considered to “enumerate” different degrees of freedom of the field system. In noncommutative geometry there are no points anymore but there are states in the Hilbert space of representations of a coordinate algebra instead. Thus we must consider a quantum field in NC-QFT as a map from set of states on the corresponding noncommutative space into the algebra of secondary quantized operators, so that the physically meaningful object in NC-QFT is the mean values of the field operators: $\langle \Psi | \hat{\varphi} | \Psi \rangle$, $|\Psi\rangle \in \mathcal{H}$. Of course, we can choose any complete set of states, but for clear physical interpretation and comparing with the commutative limit, the set should satisfy the following requirements:

\begin{enumerate}
  \item the states must be localized in space-time;
  \item as the parameter of noncommutativity $\lambda$ goes to zero, the states must shrink to a point.
\end{enumerate}

This consideration shows that in order to convert the Green function $G_W(x, y)$ into the physically meaningful object, we must average it over some localized state. States which correspond to optimal rotationally invariant localization around the point $x = (x_1, x_2)$ of the plane are uniquely (up to a phase factor) given as the (non-normalized) coherent states $|\xi\rangle$ for the operators $\hat{\alpha}$:

$|\xi\rangle = \exp \left\{ \xi \hat{\alpha}^\dagger \right\} |0\rangle$ ($|0\rangle$ is the vacuum state in the Fock space $\mathcal{F}$; $\xi = (x_1 + ix_2)/(\sqrt{2}\lambda)$).

It can be shown that

$$\varphi_N(x) \equiv \frac{\langle \xi | \hat{\varphi} | \xi \rangle}{\langle \xi | \xi \rangle} = \int d^2 x' \omega_\lambda(x - x') \varphi_W(x') ,$$

with the smearing function

$$\omega_\lambda(x - x') = \frac{1}{\pi \lambda^2} \exp \left\{ -\frac{(x - x')^2}{\lambda^2} \right\} .$$

In fact, $\varphi_N(x)$ is the normal symbol of the operator $\hat{\varphi}$. The physical Green function $G_\lambda(x, y)$ is therefore given as

$$G_\lambda(x, y) = \langle \varphi_N(x) \varphi_N(y) \rangle = \int d^2 x d^2 y \omega_\lambda(x - x') \omega_\lambda(y - y') G_W(x', y') ,$$

and it represents a quantum average of the true field functional $\varphi_N(x) \varphi_N(y) = \langle \xi | \hat{\varphi} | \xi \rangle \langle \zeta | \hat{\varphi} | \zeta \rangle$ (where $\xi = (x_1 + ix_2)/\lambda \sqrt{2}$, $\zeta = (y_1 + iy_2)/\lambda \sqrt{2}$). Similarly, any higher Green functions $G_\lambda(x_1, \ldots, x_n)$ are obtained by smearing the corresponding Green functions $G_W(x_1, \ldots, x_n)$.

A few remarks are in order:
1) The formal Green functions $G_W(x_1, ..., x_n)$ are, as a matter of rule, singular if some arguments coincide. However, the physical Green functions $G_\lambda(x_1, ..., x_n)$ are regular due to intrinsic effective smearing induced by the noncommutativity of the coordinates. We would like to stress that this is not any artificial external smearing, and that no better localized Green functions as $G_\lambda(x_1, ..., x_n)$ can be constructed on the noncommutative plane. We illustrate this assertion on the example of a free field Green function. In this case the formal Green function $G_W^{(0)}(x, y) = \langle \phi_W(x) \phi_W(y) \rangle_0$ is given by the standard formula

$$G_W^{(0)}(x, y) = \frac{1}{(2\pi)^2} \int d^2k \frac{e^{ik(x-y)}}{k^2 + m^2}.$$  

According to (26) the corresponding physical Green function $G_\lambda^{(0)} = \langle \phi_N(x) \phi_N(y) \rangle_0$ can be straightforwardly calculated with the result

$$G_\lambda^{(0)}(x, y) = \frac{1}{(2\pi)^2} \int d^2k \frac{e^{ik(x-y)-\lambda^2k^2/2}}{k^2 + m^2}.$$  

This can be easily derived by use of the normal symbols. In this case the star-product for normal symbols has the form

$$\varphi_N(\bar{\xi}, \xi) \star \phi_N(\bar{\xi}, \xi) = \varphi_N(\bar{\xi}, \xi) \exp \left\{ \lambda^2 \frac{\partial}{\partial \bar{\xi}} \frac{\partial}{\partial \xi} \right\} \phi_N(\bar{\xi}, \xi).$$  

The free action in terms of the normal symbols takes the form

$$S_0^{(N)} = \int d^2\xi \left[ \partial_i \varphi_N(\bar{\xi}, \xi) \exp \left\{ \lambda^2 \frac{\partial}{\partial \bar{\xi}} \frac{\partial}{\partial \xi} \right\} \partial_i \phi_N(\bar{\xi}, \xi) 
+ m^2 \varphi_N(\bar{\xi}, \xi) \exp \left\{ \lambda^2 \frac{\partial}{\partial \bar{\xi}} \frac{\partial}{\partial \xi} \right\} \phi_N(\bar{\xi}, \xi) \right]$$

$$= \frac{1}{(2\pi)^2} \int d^2\kappa \bar{\varphi}_N(-\kappa)(2\bar{\kappa} + m^2) e^{\lambda^2\kappa\kappa} \tilde{\varphi}_N(\kappa).$$  

Here $\tilde{\varphi}_N(\kappa)$ is the Fourier transform of the normal symbol

$$\varphi_N(\bar{\xi}, \xi) = \frac{1}{(2\pi)^2} \int d^2\kappa e^{i(\kappa\xi + \bar{\kappa}\bar{\xi})} \tilde{\varphi}_N(\kappa),$$

and $\kappa = \lambda(k_1 + ik_2)/\sqrt{2}$.

Whereas $G_W^{(0)}(x, y)$ is logarithmically divergent for $x \to y$, the physical Green function is finite

$$\left| G_\lambda^{(0)}(x, y) \right| \lesssim G_\lambda^{(0)}(x, x) = \frac{1}{(2\pi)^2} \int d^2k \frac{e^{-\lambda^2k^2/2}}{k^2 + m^2},$$  

depending only on a dimensionless parameter $a = \lambda m$ characterizing the non-commutativity.

2) If the interaction is switched on, there naturally appears the problem of a perturbative determination of the full Green function $G_\lambda = \langle \phi_N(x) \phi_N(y) \rangle$. Within the perturbation theory the problem is reduced to the calculations of free field averages
of the type $\langle \varphi_N(x)\varphi_N(y)S_{int}^N \rangle_0$. However, now the problem of a noncommutative generalization of the interaction term arises. If we choose as a commutative prototype the $(\varphi^*\varphi)^2$-interaction, the most direct noncommutative generalization is

$$S^\Lambda_{int}[\hat{\varphi},\hat{\varphi}^\dagger] = gI[\hat{\varphi}^\dagger \hat{\varphi} \hat{\varphi}^\dagger \hat{\varphi}] = g\int d^2x \varphi_N^*(x) \star \varphi_N(x) \star \varphi_N^*(x) \star \varphi_N(x).$$

This action produces vertices containing factors $e^{\lambda^2k^2/2}$ on each leg with the momentum $k_i$, $i = 1, 2, 3, 4$, plus additional phase factors $\exp\{\pm i\lambda^2(k_1 \times k_2 + k_3 \times k_4)/2\}$ (here $k \times p \equiv \varepsilon_{ij}k_ip_j$). The Gaussian factor $e^{-\lambda^2k^2/2}$ from the propagators are cancelled in Feynman diagrams and the UV-divergences appear. This confirm Filk’s analysis [12] in terms of Weyl symbols. Of course, calculations with different types of operator symbols, being different at intermediate steps, give the same physical results. Notice, however, that the normal symbols of the field operators on the noncommutative plane have much more clear physical interpretation since they are related (in fact, equal) to mean values over localized coherent states.

3) However, this is not the only possibility. Insisting only on a commutative limit condition $\lim_{\lambda \to 0} S^\Lambda_{int}[\varphi,\varphi^*] = S_{int}[\varphi,\varphi^*]$, the integrand in the noncommutative integral $I[\hat{\varphi}^\dagger \hat{\varphi} \hat{\varphi}^\dagger \hat{\varphi}]$ is defined up to the operator ordering. There is no problem to modify the operator ordering of the generators $\hat{x}_1$ and $\hat{x}_2$ in the integrand $\hat{\varphi}^\dagger \hat{\varphi} \hat{\varphi}^\dagger \hat{\varphi}$ in such a way that the vertices will not contain the exponential factors $\exp\{\lambda^2k_i^2/2\}$ on legs. For example, one can use the normal symbols for the construction of the free action but the Weyl symbols for the interaction part. The resulting action will lead to UV-regular Feynman diagrams. However, besides this pragmatic point of view, we have not been able to find any deeper principle preferring such different ordering.

2.5 Symmetry transformations on the quantum plane

Some subgroup of the group of the canonical transformations of the commutation relations for the coordinate operators can be considered as a group of space-time symmetry for NC-QFT. As we discussed in the preceding subsection, the degrees of freedom of NC-QFT correspond to a set of localized (e.g., coherent) states. Thus there appears the natural question about behaviour of such states under the quantum space-time symmetry transformations.

The fact that a linear transformations preserves commutation relations for a set of some operators means that the latter are tensor operators. It is worth to separate the case of commuting and noncommuting operators:

1) A set of commutative operators. For the general linear transformation of commutative operators $x_i \rightarrow x'_i = M_{ij}x_j + b_i$, where $M_{ij}$, $b_i$ are ordinary c-number group parameters, the vector $|\psi_x\rangle$ remains an eigenvector of the transformed operator $x'$ but with shifted eigenvalue $x'_i = M_{ij}x_j + b_i$.

2) A set of noncommutative operators: tensor operator. A tensor operator $\hat{A}_i$ acting in some Hilbert space $\mathcal{H}$, has, by the definition, the property

$$\hat{A}'_i \equiv M_{ij}(g)\hat{A}_j = \hat{U}(g)\hat{A}_i\hat{U}^{-1}(g),$$

(33)
where $M_{ij}(g) \ (i, j = 1, ..., d)$ is a matrix finite-dimensional representation of a Lie group $G$, $g \in G$ and $U(g)$ is a unitary operator in the Hilbert space $\mathcal{H}$. In general, the components $A_i \ (i = 1, ..., d)$ of a tensor operator do not commute with each other. Consider an eigenvector $|\lambda\rangle_A$ of one component, say $A_d$, of the tensor operator. After the transformation, the eigenvector $|\lambda\rangle_{A'}$ of the transformed component $A'_d$ is related to $|\lambda\rangle_A$ by the operator $U(g)$:

$$|\lambda\rangle_{A'} = \tilde{U}(g) |\lambda\rangle_A = \sum_{\lambda'} \lambda'_{A'} \langle \lambda' | \tilde{U}(g) |\lambda\rangle_A |\lambda'\rangle_A .$$

Of course, if we transform both operators (as in $|\psi\rangle \rightarrow \tilde{U}(g) |\psi\rangle$, nothing changes (equal shifts of both a system and measurement devices lead to the same values of measurements). To study behaviour of a system under the group transformations, we should transform either observables (operators) or states of the system. Then considering the action of transformed component $\tilde{A}'_d$ on the initial eigenstate $|\lambda\rangle_A$, we obtain

$$\tilde{A}'_d |\lambda\rangle_A = \tilde{U}(g) \tilde{A}_d \tilde{U}^{-1}(g) |\lambda\rangle_A$$

$$= \sum_{\lambda'} \lambda'_{A'} |\lambda\rangle_A \langle \lambda' | \tilde{U}(g) |\lambda\rangle_A |\lambda'\rangle_{A'} .$$

Let us apply this consideration (well known in the standard Quantum Mechanics) to the examples of Euclidean and pseudo-Euclidean quantum planes. While UV-behaviour in these cases are the same, their properties with respect to the symmetry transformations are quite different.

We shall consider only homogeneous part of the transformations. In the case of the Euclidean plane, these are rotations $[\phi]$ (one-dimensional subgroup of the group $Sp(2) \sim SL(2, \mathbb{R})$ of the canonical transformations). The corresponding creation and annihilation operators $[\alpha]$ are transformed separately

$$\hat{\alpha} \rightarrow e^{i\phi} \hat{\alpha} , \quad \hat{\alpha}^\dagger \rightarrow e^{-i\phi} \hat{\alpha}^\dagger ,$$

so that the corresponding localized (coherent) states $|\xi\rangle$ are transformed in very simple way:

$$|\xi\rangle \longrightarrow |e^{i\phi} \xi\rangle .$$

Thus the localized coherent states are transformed in the simple and physically transparent way. On the contrary, coordinate eigenstates are transformed non-locally according to $[\phi]$. Indeed, the coordinates are transformed under Euclidean rotations by the formula

$$\hat{x}_1 \rightarrow \hat{x}'_1 = (\cos \phi) \hat{x}_1 + (\sin \phi) \hat{x}_2 = \tilde{U}_\phi \hat{x}_1 \tilde{U}^{-1}_\phi ,$$

$$\hat{x}_2 \rightarrow \hat{x}'_2 = -(\sin \phi) \hat{x}_1 + (\cos \phi) \hat{x}_2 = \tilde{U}_\phi \hat{x}_2 \tilde{U}^{-1}_\phi .$$

The explicit form of the operator $\tilde{U}_\phi$ is easily found and proves to be

$$\tilde{U}_\phi = \exp \left\{ -\frac{i}{2} \phi (\hat{x}_1^2 + \hat{x}_2^2) \right\} .$$
Formally this operator coincides with the evolution operator for a particle in the harmonic potential, so that its kernel is well-known (see, e.g., [18])

\[
\langle x'_1 | \hat{U}_\phi | x_1 \rangle = \frac{1}{\sqrt{2\pi} \lambda^2} \frac{1}{\sin \phi} \exp \left\{ \frac{1}{4\lambda^2} \frac{1}{\sin \phi} \left[ \left( (x'_1)^2 + x_1^2 \right) \cos \phi - 2x'_1 x_1 \right] \right\} .
\]

(38)

Insertion of this kernel into the formulas (34), (35) leads to the nonlocal transformation of eigenstates of the operator \( \hat{x}_1 \) (eigenstates of \( \hat{x}_2 \) are transformed quite similarly).

The situation is opposite in the case of pseudo-Euclidean (Minkowski) plane. Now we have to use another subgroup of the canonical group \( SL(2, \mathbb{R}) \): two-dimensional Lorentz group \( SO(1, 1) \)

\[
x_0 \rightarrow x'_0 = (\cosh \eta) x_0 + (\sinh \eta) x_1 ,
\]

\[
x_1 \rightarrow x'_1 = (\sinh \eta) x_0 + (\cosh \eta) x_1 ,
\]

(39)

It is convenient to use the light-front variables

\[
x_\pm = \frac{1}{\sqrt{2}} (x_0 \pm x_1) .
\]

The boosts (39) now have the simple form

\[
x_\pm \rightarrow e^{\pm \eta} x_\pm
\]

(\( \eta \) has the meaning of rapidity). On the noncommutative plane the coordinates satisfy the commutation relations

\[
[\hat{x}_+, \hat{x}_-] = i\lambda^2 .
\]

(40)

The corresponding annihilation and creation operators

\[
\hat{\alpha} = \frac{1}{\lambda \sqrt{2}} (\hat{x}_+ + i\hat{x}_-) , \quad \hat{\alpha}^\dagger = \frac{1}{\lambda \sqrt{2}} (\hat{x}_+ - i\hat{x}_-) ,
\]

(41)

are transformed non-trivially

\[
\hat{\alpha} \rightarrow \hat{\alpha}_\eta = (\cosh \eta) \hat{\alpha} + (\sinh \eta) \hat{\alpha}^\dagger = \hat{U}_\eta \hat{\alpha} \hat{U}_\eta^{-1} ,
\]

\[
\hat{\alpha}^\dagger \rightarrow \hat{\alpha}^\dagger_\eta = (\sinh \eta) \hat{\alpha} + (\cosh \eta) \hat{\alpha}^\dagger = \hat{U}_\eta \hat{\alpha}^\dagger \hat{U}_\eta^{-1} ,
\]

(42)

The explicit form of the operator \( \hat{U}_\eta \) is easily found and proves to be

\[
\hat{U}_\eta = \exp \left\{ -\frac{1}{2} \eta \left( (\hat{\alpha}^\dagger)^2 - \hat{\alpha}^2 \right) \right\} .
\]

Thus now the corresponding localized coherent states are transformed as:

\[
|\xi\rangle \rightarrow |\xi_\eta\rangle = \hat{U}_\eta |\xi\rangle .
\]

The calculation of the matrix elements \( \langle \zeta | \hat{U}_\eta |\xi\rangle \) can be done, e.g., with the help of the path integral

\[
\mathcal{U}(\tilde{\zeta}, \xi) = \int \prod_\tau \frac{d\tilde{z}(\tau) dz(\tau)}{2\pi} \exp \left\{ \tilde{\zeta} z(\eta) + \int_0^\eta d\tau \left[ -\tilde{z}(\tau) \dot{z}(\tau) - \frac{1}{2} (\tilde{z}'(\tau) - z'(\tau))^2 \right] \right\} .
\]
This is Gaussian integral and, hence, its value is given by the integrand at the extremum trajectory of the exponent with the boundary conditions: \( z(0) = \xi, \quad \bar{z}(\eta) = \bar{\zeta} \). The result is:

\[
\langle \zeta | \hat{U}_\eta | \xi \rangle = \exp \left\{ -\frac{|\zeta|^2}{2} - \frac{|\xi|^2}{2} + \frac{\bar{\zeta} \xi}{\cosh \eta} - \frac{\bar{\zeta}^2 - \xi^2}{2} \tanh \eta \right\} .
\]

Now to realize properties of the transformed state \(| \xi_\eta \rangle = \hat{U}_\eta | \xi \rangle\) we can calculate it in the coordinate representation (either \( x_+ \) or \( x_- \)). For example,

\[
|\langle x_+ | \xi_\eta \rangle|^2 = |\langle x_+ | \hat{U}_\eta | \xi \rangle|^2 = \frac{1}{\sqrt{\pi \lambda e^{-\eta}}} \exp \left\{ -\frac{(x_+ - \lambda e^{-\eta} \xi_1)^2}{\lambda^2 e^{-2\eta}} \right\}
\]

(here \( \xi = (\xi_1 + i \xi_2)/\sqrt{2} \)). This expression shows that \(| \xi_\eta \rangle\) is also localized state and with respect to the coordinate \( x_+ \) it is located around the point \( e^{-\eta} \sqrt{2\lambda \Re \xi} \) with the dispersion \((\lambda e^{-\eta})^2\) (while \(| \xi \rangle\) is located around \( \sqrt{2\lambda \Re \xi} \) with the dispersion \( \lambda^2 \)). Similarly, with respect to the coordinate \( x_- \) the state \(| \xi_\eta \rangle\) is located around the point \( e^{\eta} \sqrt{2\lambda \Im \xi} \) with the dispersion \((\lambda e^{\eta})^2\).

### 3 Quantum field theory on a noncommutative cylinder

The space-time (without dynamical gravitational fields) usually possesses some space-time symmetry described by a Lie group \( G \) of isometries which transitively acts on the space-time manifold. The group action of \( G \) on a scalar field \( \varphi(x) \) is defined by the equality

\[
T_g \varphi(x) \equiv \varphi(g^{-1} x), \quad g \in G, \quad x \in \mathcal{M} .
\]

The right hand side of this definition can be considered as a function on \( G \otimes \mathcal{M} \). Let \( g = \text{Lie}(G) \) denotes the Lie algebra of the group in question, so that for any \( \hat{X} \in g \) there corresponds the group element \( g = e^{t \hat{X}} \in G \). Inserting this into (43) we can assign to any \( \hat{X} \in g \) the left-invariant vector field \( \hat{X} \) on \( \mathcal{M} \) defined by

\[
\hat{X} \varphi(x) \equiv \lim_{t \to 0} \frac{1}{t} \left[ \varphi \left( e^{-t \hat{X}} x \right) - \varphi(x) \right] .
\]

In this way one obtains a representation of the Lie algebra \( g \) (which, of course, depends on the choice of \( \mathcal{M} \) and representation \( T_g \) used in (43)).

#### 3.1 Fields on noncommutative co-adjoint orbits of Lie groups

Let \( T_g \) be a unitary irreducible representation of \( G \) in the Hilbert space \( \mathcal{H} \). The corresponding Lie algebra generators \( \hat{X}_1, \ldots, \hat{X}_n \) of \( g \) satisfy the commutation relations

\[
[\hat{X}_i, \hat{X}_j] = i \hbar_{ij} \hat{X}_k , \quad i, j, k = 1, \ldots, n .
\]

In a given representation these generators satisfy also the polynomial Casimir operator relations:

\[
C_1(\hat{X}_i) = \lambda_1 I , \quad \cdots , \quad C_d(\hat{X}_i) = \lambda_d I ,
\]

(here \( \lambda_1, \ldots, \lambda_d \) are positive numbers).
where \( C_1(\hat{X}_i), \ldots, C_d(\hat{X}_i) \) is the complete set of independent Casimir operators, and the numbers \( \lambda_1, \ldots, \lambda_d \) characterize the representation in question: \( T = T_{\lambda_1, \ldots, \lambda_d} \).

Let us now introduce the vector space \( g^* \) dual to the vector space \( g \) of the Lie algebra (i.e. \( g^* \) is the space of linear functionals on \( g \): \( X \in g, Y \in g^* \rightarrow \langle \langle Y|X \rangle \rangle \in \mathcal{O} \)). The basis \( Y^i \) of the space \( g^* \) is defined by the standard relation: \( \langle \langle Y^i|X_j \rangle \rangle = \delta^i_j \). Now the generators \( X_i \) can be considered as coordinate functions on the dual space \( g^* \). Suppose that the standard commutative space-time \( M \) can be identified with a maximal dimension coadjoint orbit of the group \( G \) in the space \( g^* \) (considered as a usual vector space). Such an orbit is defined by a set of values of the Casimir polynomials

\[
C_1(x_i) = r_1, \quad \ldots, \quad C_d(x_i) = r_d,
\]

the Casimir polynomials depending now on \textit{commutative} variables \( x_1, \ldots, x_n \). Thus, functions on the commutative space-time (associated to the orbit) are generated by the commuting coordinates \( x_1, \ldots, x_n \) satisfying the conditions (47).

The advantage of the interpretation of a space-time manifold as a coadjoint orbit is that on the latter there exists the Lie-Kirillov-Kostant symplectic structure (degenerated on the whole \( g^* \) but well defined on a coadjoint orbit), which leads to a Poisson brackets of the form

\[
\{ x_i, x_j \} = C_{ij}^k x_k.
\]

The standard procedure of quantization, i.e. replacement of the Poisson brackets (43) by a commutator according to the rule \( \{ \cdot, \cdot \} \rightarrow (1/i\lambda)[\cdot, \cdot] \), allows to associate to the commutative manifold \( M \) defined by (47), the noncommutative space \( M_{NC} \). Functions on this noncommutative space are generated by the operators \( \hat{x}_i = \lambda \hat{X}_i \) with the commutation relations

\[
[\hat{x}_i, \hat{x}_j] = i\lambda C_{ij}^k \hat{x}_k,
\]

satisfying the Casimir relations (44), i.e. generated by the Lie algebra operators in the representation \( T_{\lambda_1, \ldots, \lambda_d} \). The parameter \( \lambda \) plays the role of the Planck constant and in the properly defined limit one recovers the commutative limit [9].

Expressing the relevant differential operator \( \Delta \) entering (4) in terms of the Poisson brackets (43), it is then straightforward to define its noncommutative analog by replacing brackets by the commutator (49).

In the commutative case the fields on \( M \) can be expanded as

\[
\varphi(x) = \sum_k a_{k_1 \cdots k_n} x^{k_1} \cdots x^{k_n},
\]

and are transformed according (43). In the noncommutative case the fields \( \varphi(\hat{x}) \) have analogous expansion

\[
\varphi(\hat{x}) = \sum_k a_{k_1 \cdots k_n} \hat{x}^{k_1} \cdots \hat{x}^{k_n},
\]

\textit{i.e.} they are operators in the Hilbert space \( \mathcal{H} \). Consequently, they transforms according to the rule

\[
\varphi(\hat{x}) \rightarrow T_g \varphi(\hat{x}) T^\dagger_g,
\]

as we discussed in the preceding section. Therefore, as a noncommutative generalization of the integral on \( M \), one can take a properly normalized trace

\[
I_\lambda[\varphi(\hat{x})] = N(\lambda)\text{Tr} \left( \varphi(\hat{x}) \right),
\]

(53)
which is obviously $G$-invariant. The noncommutative analog of the action $S[\varphi, \varphi^*]$ given by (3) with the integral over $\mathcal{M}$ is replaced by its noncommutative analog (53), the field $\varphi(x)$ being replaced by the operators $\varphi(\tilde{x})$.

It is well known that an orbit of a Lie group $G$ is in one-to-one correspondence with a coset space $G/G_0$, where $G_0 \subset G$ is the stability subgroup for a point on the orbit (see e.g., [16]). Then it is reasonable to associate to momenta degrees of freedom of the corresponding NC-QFT the coset space $G/G_0$.

Below we shall consider UV-properties of the field theory defined on a noncommutative cylinder which can be considered as a quantization of a co-adjoint orbit of the Euclidean group $E(2)$ in the vector space $e^*(2)$ (dual to the Lie algebra $e(2)$).

### 3.2 Fields on a commutative cylinder

A standard cylinder with radius $\rho$ can be identified with the set of points

$$C_\rho = \{(x,t), \ t \in \mathbb{R}, \ x = \rho e^{i\phi}, \ \rho = \text{const}\}. \quad (54)$$

We shall interpret $C_\rho$ as a space-time manifold. Any function (field) on $C_\rho$ can be expanded as

$$\varphi(x,t) = \sum_{k=-\infty}^{\infty} a_k(t) e^{ik\phi}$$

$$= a_0(t) + \sum_{k=1}^{\infty} [a^{(+)}_k(t)x^k_+ + a^{(-)}_k(t)x^k_-], \quad (55)$$

where $x_+ = x, \ x_- = x^*, \ a^{(\pm)}_k(t) = a_{\pm k}(t)\rho^{-k}, \ k = 1,2,....$ As $I_0[\varphi]$ we denote the usual integral on $C_\rho$

$$I_0[\varphi] \overset{\text{def}}{=} \frac{1}{2\pi \rho} \int_{C_\rho} dx \ dt \ \varphi(x,t) = \int_{\mathbb{R}} dt \ a_0(t). \quad (56)$$

The action for the field $\varphi(x,t)$ we choose as follows

$$S[\varphi, \varphi^*] = I_0[\varphi^*(- \partial_t^2 + \partial_\phi^2 - m^2)\varphi - V(\varphi^*\varphi)]. \quad (57)$$

Since the cylinder $C_\rho$ can be interpreted as the co-adjoint orbit of the two-dimensional Euclidean group $E(2)$, one can introduce on $C_\rho$ the (Lie-Kirillov-Kostant) Poisson bracket

$$\{\varphi_1, \varphi_2\} = \frac{\partial \varphi_1}{\partial x_{\mu}} \frac{\partial \varphi_2}{\partial x_{\nu}} \{x_{\mu}, x_{\nu}\}, \quad (58)$$

where the indices take values $\mu, \nu = 0, +, -$ and we put $x_0 = t$. The elementary brackets are given as

$$\{x_0, x_\pm\} = \pm i x_\pm, \quad \{x_+, x_-\} = 0. \quad (59)$$

These are exactly $e(2)$ Lie algebra relations. The point is that the function $x_+ x_-$ is central: $\{x_0, x_+ x_-\} = \{x_\pm, x_+ x_-\} = 0$. Therefore, the constraint $x_+ x_- = \rho^2$ is consistent with the $e(2)$ Lie algebra structure (58). It can be straightforwardly shown that

$$\partial_t^2 \varphi = \rho^2 \{x_+, \{x_-, \varphi\}\}, \quad \partial_\phi^2 \varphi = -\{x_0, \{x_0, \varphi\}\}. \quad (60)$$
Therefore the free field action can be represented in terms of the Poisson brackets as
\[ S_0[\phi, \phi^*] = I_0[\phi^*(\Box + m^2)\phi] , \] (61)
where
\[ \Box \phi = \{x_0, \{x_0, \phi\}\} + \rho^2\{x_+, \{x_-, \phi\}\} . \] (62)
Inserting expansion (55) into (57) we obtain the free field action in the form
\[ S_0[\phi, \phi^*] = \int d\mathcal{R} \sum_{k=-\infty}^{\infty} \left[ \dot{a}_k^*(t)\dot{a}_k(t) - (k^2 + m^2)a_k^*(t)a_k(t) \right] . \] (63)
Similarly, \( S_{int}[\phi, \phi^*] \) is a higher order polynomial in \( a_k^*(t) \) and \( a_k(t) \).
Performing the Fourier transform
\[ \tilde{a}_k(\omega) = \frac{1}{2\pi} \int d\mathcal{R} \dot{a}_k(t)e^{-i\omega t} , \] (64)
we obtain the free field action in the diagonal form
\[ S_0[\phi, \phi^*] = \int d\omega \sum_{k=-\infty}^{\infty} \tilde{a}_k^*(\omega)(\omega^2 - k^2 - m^2)\tilde{a}_k(\omega) . \] (65)
It follows then straightforwardly, that the free field Green function is
\[ G_0(x', t'; x, t) = \frac{1}{2\pi} \int d\omega \sum_{k=-\infty}^{\infty} e^{i\omega(t-t')} \frac{\delta^{i\omega(t'-t)}}{\omega^2 - k^2 - m^2 + i\varepsilon} \left( x'x^* \right)^k . \] (66)
The Green function is diagonal in the energy-momentum representation, i.e.
\[ \tilde{G}_0(\omega, k) = \frac{1}{\omega^2 - k^2 - m^2 + i\varepsilon} . \] (67)
The vertex contribution in the momentum representation, e.g., for the potential term \( V = g(\phi^*\phi)^2 \), has the form
\[ g \delta(\omega_1 - \omega_2 + \omega_3 - \omega_4) \delta_{k_1+k_3,k_2+k_4} . \] (68)
Here, \( (\omega_i, k_i) \) with \( i = 1, 3, \) or \( i = 2, 4, \) correspond to two fields \( \phi \), or \( \phi^* \) respectively, in the vertex. This leads to a divergent tadpole diagram which, up to the interaction constant \( g \), are given as
\[ G_0(x, t; x, t) = \frac{1}{2\pi} \int d\omega \sum_{k=-\infty}^{\infty} \frac{1}{\omega^2 - k^2 - m^2 + i\varepsilon} \]
\[ = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k^2 + m^2}} = \infty . \] (69)
This leads to UV-divergences in Feynman diagrams calculated within perturbation theory for the model defined by the action (57).
3.3 Field theory on a noncommutative cylinder

The starting point is the replacement of commutative variables \(x_0, x_\pm\) by the noncommutative ones \(\hat{x}_0, \hat{x}_\pm\), satisfying the \(e(2)\) Lie algebra commutation relations
\[
[\hat{x}_0, \hat{x}_\pm] = \pm \lambda \hat{x}_\pm , \quad [\hat{x}_+, \hat{x}_-] = 0 . \tag{70}
\]
The \(e(2)\) possesses one series of infinite dimensional unitary representations \(\pi_\rho\) in which
\[
\hat{x}_+ \hat{x}_- = \rho^2 , \quad \rho > 0 , \tag{71}
\]
(to simplify notation we use the same symbol for generators and their representatives). This representation can be realized in the Hilbert space \(L^2(S^1, d\phi/2\pi)\) as follows
\[
\hat{x}_0 = -i\lambda \partial_\phi , \quad \hat{x}_\pm = \rho e^{\pm i\phi} . \tag{72}
\]
In the basis \(|n\rangle = e^{i n\phi}, \ n \in \mathbb{Z}\), the operator \(\hat{x}_0\) is diagonal
\[
\hat{x}_0 |n\rangle = \lambda n |n\rangle . \tag{73}
\]
Notice, that there exist a set of inequivalent representations \(|n\rangle_\alpha = e^{i(n+\alpha)\phi}, \ 0 \leq \alpha < 1\). For simplicity, we shall use only the representation (73) which corresponds to \(\alpha = 0\). The field \(\hat{\phi} = \phi(\hat{x}_0, \hat{x}_\pm)\) is an operator in \(L^2(S^1, d\phi/2\pi)\) and we suppose that it possesses the expansion
\[
\hat{\phi} = a_0(\hat{t}) + \sum_{k=1}^{\infty} \left[ a_+^{(k)}(\hat{t}) \hat{x}_+^k + a_-^{(k)}(\hat{t}) \hat{x}_-^k \right] , \tag{74}
\]
The noncommutative integral \(I_\lambda[\hat{\phi}]\) we define by
\[
I_\lambda[\hat{\phi}] \overset{\text{def}}{=} \lambda \text{Tr} \hat{\phi} = \lambda \sum_{n \in \mathbb{Z}} a_0(\lambda n) . \tag{75}
\]
We see that the usual measure on \(\mathbb{R}\) in (50) is replaced by a uniform step measure in points \(\lambda n, \ n \in \mathbb{Z}\) with the height \(\lambda\).

All dynamical information which is contained in the eigenvalues \(a_k(\lambda n)\) of operators \(a_k(\hat{t})\) is contained in the Fourier transformed coefficients
\[
\tilde{a}_k(\omega) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} a_k(\lambda n) e^{-i\omega \lambda n} , \tag{76}
\]
defined on the interval \(\omega \in (-\pi/\lambda, +\pi/\lambda)\).

The free field action has the form similar to that in the commutative case:
\[
S_0[\hat{\phi}, \hat{\phi}^*] = I_\lambda[ - \varphi^*(\Box_\lambda + m^2)\varphi] , \tag{77}
\]
where the noncommutative analog \(\Box_\lambda\) of the operator \(\Box\) is defined as (cf. 12)
\[
\Box_\lambda \varphi = -\frac{1}{\lambda^2} [x_0, [x_0, \varphi]] + \frac{1}{\lambda^2 \rho^2} [x_+, [x_-, \varphi]] . \tag{78}
\]
Taking \(\hat{\phi}\) in the form
\[
\hat{\phi} = \sum_{k=\infty}^{\infty} a_k(\hat{t}) \hat{x}^{(k)} , \tag{79}
\]
(here \(a_{\pm k}(\hat{t}) = a_k^{(\pm)}(t)\), \(\hat{x}(\pm k) = \hat{x}_k^{\pm}\), \(k = 1, 2...\)), we obtain the analog of the free field action (63), however, with a discrete time evolution

\[
S_0[\hat{\varphi}, \hat{\varphi}^\dagger] = \lambda \sum_{n,k} \left[ -\frac{1}{\lambda^2} a_k^\dagger(n\lambda)[a_k(n\lambda + \lambda) - 2a_k(n\lambda) + a_k(n\lambda - \lambda)] - a_k^\dagger(n\lambda)(k^2 + m^2)a_k(n\lambda) \right].
\]

This action can be diagonalized by performing the Fourier transformation (inverse to (76))

\[
a_k(\lambda n) = \frac{\lambda}{2\pi} \int_{-\pi/\lambda}^{\pi/\lambda} d\omega \, \tilde{a}_k(\omega)e^{i\omega \lambda n},
\]

The free field action on a noncommutative cylinder in the momentum representation takes the form

\[
S_0[\hat{\varphi}, \hat{\varphi}^*] = \frac{\lambda}{2\pi} \int_{-\pi/\lambda}^{\pi/\lambda} d\omega \sum_{k=-\infty}^{\infty} \tilde{a}_k(t)[\Omega^2_\lambda(\omega) - k^2 - m^2]\tilde{a}_k(\omega),
\]

where \(\Omega_\lambda(\omega) = \frac{2}{\lambda} \sin \frac{\omega \lambda}{2}\). The corresponding free field Green function is

\[
G_0^{(\lambda)}(x', t'; x, t) = \frac{1}{2\pi} \int_{-\pi/\lambda}^{\pi/\lambda} d\omega \sum_{k=-\infty}^{\infty} \frac{e^{i\omega(t'-t)}}{\Omega^2_\lambda(\omega) - k^2 - m^2 + i\varepsilon} \left(\frac{x'x^*}{\rho^2}\right)^k,
\]

with the time variables taking discrete values: \(t = \lambda n\), \(t' = \lambda n'\). Notice, that the energy integral is carried over finite interval. In the energy-momentum representation the Green function is diagonal

\[
\tilde{G}_0^{(\lambda)}(\omega, k) = \frac{1}{\Omega^2_\lambda(\omega) - k^2 - m^2 + i\varepsilon}.
\]

In the noncommutative case the interaction term we take in the form \(V = \frac{g}{4} : (\varphi^* \varphi)^2 :\), where the ordering \(\cdots\) means that all expansion coefficients of fields \(a_k(\hat{t}), a_k^\dagger(\hat{t})\) are collected as a right (or equivalently left) factor. This guarantees that there are no artificial time derivatives in the potential: all operators \(a_k(\hat{t}), a_k^\dagger(\hat{t})\) in the potential term appear at the same time \(t = \lambda n\). The vertex contribution has the same form (68) as in the commutative case with the only difference that the symbol \(\delta(\omega - \omega')\) now refers to the periodic \(\delta\)-function on the interval \([-\pi/\lambda, +\pi/\lambda]\). The tadpole contribution is finite

\[
G_0(x, t; x, t) = \frac{1}{2\pi} \int_{-\pi/\lambda}^{\pi/\lambda} d\omega \sum_{k=-\infty}^{\infty} \frac{1}{\Omega^2_\lambda(\omega) - k^2 - m^2 + i\varepsilon} < \infty.
\]

This can be seen most directly using the well-known summation formula (see \cite{17}, p.36)

\[
\sum_{k=-\infty}^{\infty} \frac{1}{x^2 - k^2} = \frac{\pi}{x} \cot(x).
\]

Thus,

\[
G_0(x, t; x, t) = \int_0^{\pi/\lambda} d\omega \frac{\cot \left( \sqrt{\Omega^2_\lambda(\omega) - m^2 + i\varepsilon} \right)}{\sqrt{\Omega^2_\lambda(\omega) - m^2 + i\varepsilon}}.
\]
The integrand in (86) has one simple pole at \( \omega = 2 \arcsin \frac{\lambda m}{2} \), which due to the \((-i\varepsilon)\)-prescription leads to a finite contribution. The rest of integrand possesses the integrable singularities \((\omega - \omega_k)^{-1/2}\) at points

\[
\omega_k = \frac{2}{\lambda} \arcsin \left( \frac{\lambda}{2} \sqrt{m^2 + k^2 \pi^2} \right),
\]

\( k - \text{integers}, \quad 0 < k \leq \frac{1}{\pi} \sqrt{4/\lambda^2 - m^2} \).

Since the integration range is restricted, this means that the tadpole integral is finite.

In the two-dimensional scalar field theory on commutative space the tadpole is the only divergent contribution to Feynman diagrams. We have shown that transition to the noncommutative cylinder leads to \textit{UV-finite field theory}.

4 Quantum mechanics and quantum field theory on a noncommutative quantum plane with \(E_q(2)\)-symmetry

In this section we consider Quantum Mechanics induced by a quantum group structure. Recall that in the case of ordinary Lie group \(G\), the group structure defines a unique symplectic structure on the cotangent bundle \(T^*_G\) to the group manifold \(G\) and, hence, the corresponding canonical quantization (via substitution of Poisson brackets by the corresponding commutators). A similar construction with necessary generalizations, can be carried out for Lie-Poisson groups, which after the quantization procedure become quantum groups (see, e.g., review in [6] and refs. therein).

Thus instead of starting from ordinary \(E(2)\), we shall use below its quantum version \(E_q(2)\) [14, 21]. Though in the case of quantum groups and corresponding quantum homogeneous spaces (definition of the latter see, e.g., in [21]) group parameters (coordinates) become noncommutative, the general scheme of quantization still can be applied. The role of momentum operators is now attributed the \(q\)-deformed left- (or right-) invariant generalizations of vector fields. Thus the Planck constant \(\hbar\) enters, as usual, the commutation relations for momenta and coordinates, while the group deformation parameter \(q\) governs non-trivial coordinate-coordinate and momentum-momentum commutation relations. Therefore, first of all we have to construct possible representations of this \textit{combined} \(q\)-deformed algebra of noncommuting coordinates and momenta. For the particular case which we consider in this paper (\(q\)-deformed quantum Euclidean plane \(P_q^{(2)}\)) this is not a very complicated problem.

At the next step we proceed to the construction of NC-QFT on the \(q\)-plane \(P_q^{(2)}\) and to the study of its UV-properties.

4.1 Algebra of coordinates and momenta on \(P_q^{(2)}\) and its representations

In order to simplify notation, we shall omit, in what follows, the sign “hat” over operators.
We start from the quantum group \( E_q(2) \) generated by elements \( \bar{v}, v, \bar{t}, t \) with the defining relations
\[
\begin{align*}
\bar{v}v &= v\bar{v} = 1, & t\bar{t} &= q^2\bar{t}t, \\
vt &= q^2tv, & \bar{v}t &= q^{-2}\bar{t}\bar{v} \\
q &\in \mathbb{R}.
\end{align*}
\] (87)
Other commutation relations follow from the involution: \( v^\dagger = \bar{v}, \ t^\dagger = \bar{t} \). The comultiplication has the form
\[
\Delta v = v \otimes v, \quad \Delta \bar{v} = \bar{v} \otimes \bar{v}, \quad \Delta t = v \otimes t + t \otimes 1 \mathbb{I}, \quad \Delta \bar{t} = \bar{v} \otimes \bar{t} + \bar{t} \otimes 1 \mathbb{I}.
\] (88)
The explicit form of other basic maps for \( E_q(2) \) (antipode, counity) will not be used in what follows.

The unitary element \( v \) can be parameterized with the help of the symmetric element \( \theta \):
\[
v = e^{i\theta}, \quad \theta^\dagger = \theta, \quad \Delta \theta = \theta \otimes 1 \mathbb{I} + 1 \mathbb{I} \otimes \theta.
\] (89)

The quantum universal enveloping algebra (QUEA) \( U_q e(2) \) is generated by the elements \( J, \bar{T}, T \) which are dual to the generators \( \theta, \bar{t}, t \) of the algebra \( E_q(2) \) and, as a result of the duality, satisfy the following commutation relations
\[
[J,T] = iT, \quad [J,\bar{T}] = -i\bar{T},
\] (90)
(91)
where \( X \in U_q L, f,f^i \in G_q, \langle \cdot, \cdot \rangle \) denotes the duality contraction, \( S(X) \) is antipode and \( \Delta f = \sum_i f^i_1 \otimes f^i_2 \). An explicit calculation of this left action in the case of \( E_q(2) \) shows that the operators \( \bar{T}, T \in U_q e(2) \) act on elements of \( E_q(2) \) generated by \( \bar{t}, t \) exactly in the same way as the q-deformed derivatives \( \bar{\partial}_q, \partial_q \). In fact, the elements \( \bar{t}, t \) generates q-deformed analog of homogeneous space \( P_q^{(2)} = E_q(2)/U_q(1) \), i.e. algebra of functions on quantum Euclidean plane \([21]\). We shall denote elements of the algebra \( P_q^{(2)} \) by \( \bar{z}, z \) to distinguish them from elements \( \bar{t}, t \) of the algebra \( E_q(2) \).

The elements \( \bar{z}, z \) and \( \bar{\partial}_q, \partial_q \) defines the q-deformed algebra of functions on \( P_q^{(2)} \) together with the q-deformed left-invariant vector fields (derivatives). Its defining relations read as
\[
\begin{align*}
\bar{z}z &= q^2\bar{z}z, & \partial_q \bar{\partial}_q &= q^2 \bar{\partial}_q \partial_q, \\
\partial_q z &= 1 + q^{-2}z\partial_q, & \bar{\partial}_q \bar{z} &= 1 + q^2\bar{z}\partial_q, \\
\bar{\partial}_q z &= q^2z\bar{\partial}_q, & \partial_q \bar{z} &= q^{-2}\bar{z}\partial_q.
\end{align*}
\] (93)
(the commutation relations for the $q$-derivatives is just rewritten commutation relations for $T$, $T$ (90) and those for the $q$-derivatives and coordinates are derived from (91)). If we put $q = 1$ and define $p = -i\hbar \partial$, $\bar{p} = -i\hbar \bar{\partial}$, the relations (93) become the usual canonical commutation relations for a particle in two-dimensional space. The requirement of consistency with antipode dictates the following conjugation rule for the $q$-derivatives

$$\partial_q^\dagger = -q^2 \bar{\partial}_q, \quad \bar{\partial}_q^\dagger = -q^{-2} \partial_q .$$

We consider the relations (93) as a $q$-deformation of the canonical commutation relations and is going to construct their representation in a Hilbert space.

To this aim let us introduce the operators $N$ and $\bar{N}$ defined by the relations

$$[N; q^{-2}] = z \partial_q , \quad [\bar{N}; q^2] = \bar{z} \bar{\partial}_q ,$$

$$[X; q^\alpha] \equiv q^{\alpha X - 1} / q^\alpha - 1 .$$

These operators have simple commutation relations

$$q^\alpha N z = q^\alpha q^\alpha N , \quad q^\alpha \bar{N} \bar{z} = q^\alpha \bar{z} q^\alpha \bar{N} ,$$

$$q^\alpha N \partial_q = q^{-\alpha} \partial_q q^\alpha N , \quad q^\alpha \bar{N} \bar{\partial}_q = q^{-\alpha} \bar{\partial}_q q^\alpha \bar{N} .$$

Using (94) and $z^\dagger = \bar{z}$, we find

$$N^\dagger = -\bar{N} - 1 , \quad \bar{N}^\dagger = -N - 1 .$$

With the help of operators $N$ and $\bar{N}$ we can construct two operators, a hermitian $J$ and an antihermitian $D$ given as

$$J = (\bar{N} - N) , \quad D = N + \bar{N} + 1 ,$$

which correspond to rotations and dilatations on the quantum plane. Their action on polynomial functions in $z$ and $\bar{z}$ is given as

$$q^J z^n \bar{z}^m = q^{m-n} z^n \bar{z}^m ,$$

$$q^D z^n \bar{z}^m = q^{m+n+1} z^n \bar{z}^m .$$

We stress that only $J$ represents a well-defined $U_q e(2)$ action on a quantum plane, when supplementing $J$ by $T = \partial_q$ and $\bar{T} = \bar{\partial}_q$:

$$T z^n \bar{z}^m = [n, q^{-2}] z^{n-1} \bar{z}^m ,$$

$$\bar{T} z^n \bar{z}^m = q^{2n} [n, q^{-2}] z^{n} \bar{z}^{m-1} .$$

The dilatation operator $D$, however, corresponds to a well defined transformations within the $q$-deformed Heisenberg algebra (93) (in (93) the operators $q^{-2N} = z \partial_q$ and $q^{2N} = \bar{z} \bar{\partial}_q$ appear independently).
The operators \( q^{2N}, q^{2\tilde{N}} \) allow to construct commuting pairs of conjugate operators:

\[
\begin{align*}
\bar{Z} &= q^{N-\tilde{N}} \bar{z} , \\
Z &= zq^{N-\tilde{N}} ,
\end{align*}
\]

(102)

with the commutation relations

\[
\begin{align*}
P\bar{Z} &= \bar{Z}P , \\
\bar{Z}Z &= ZZ , \\
ZP &= 1 + q^2 PZ ,
\end{align*}
\]

(103)

If we were given only the algebra of the operators \( \bar{z}, z, \partial_{\bar{q}}, \partial_q \), we would reasonably name the commuting operators \( \bar{Z}, Z \) by coordinates and \( \bar{P}, P \) by the corresponding lattice momenta and then deal with two independent (commuting with each other) one-dimensional algebras on the \( q \)-lattice (for a one-dimensional Quantum Mechanics on \( q \)-lattice see, e.g., [6] and refs therein). However, fields in NC-QFT depend on noncommutative \( (q-\text{commuting}) \) coordinates \( \bar{z}, z \) which are more suitable to trace a result of coaction by \( E_q(2) \). We have found convenient to use the hermitian and unitary combination of the coordinate operators:

\[
r^2 \equiv \bar{z}z \quad \text{(hermitian)}, \quad u \equiv \bar{z}z^{-1} \quad \text{(unitary)},
\]

(104)

together with \( q^{2(N-\tilde{N})}, q^{2(\tilde{N}+N+1)} \) as a basic set of the phase space operators. The commutation relations for this set of operators read as

\[
\begin{align*}
[q^{2(N-\tilde{N})}, r^2] &= 0 , \\
[q^{2(N+\tilde{N}+1)}, u] &= 0 , \\
r^2u &= q^4ru^2 , \\
[q^{2(N-\tilde{N})}, q^{2(\tilde{N}+N+1)}] &= 0 ,
\end{align*}
\]

(105)

Now we are ready to construct a representation of this algebra in the space \( \ell^2 \) (i.e. infinite dimensional matrix representation):

\[
\begin{align*}
\langle n, m | r^2 | n, m \rangle_{r_0, l_0} &= r_0^2q^{4n} | n, m \rangle_{r_0, l_0} \\
\langle n, m | q^{2(N-\tilde{N})} | n, m \rangle_{r_0, l_0} &= l_0^2q^{4m} | n, m \rangle_{r_0, l_0} .
\end{align*}
\]

(106)

\[
\begin{align*}
| n, m \rangle_{r_0, l_0} &= | n+1, m \rangle_{r_0, l_0} , \\
| n, m \rangle_{r_0, l_0} &= | n+1, m \rangle_{r_0, l_0} .
\end{align*}
\]

(107)

The constants \( r_0 \) and \( l_0 \) mark different representations and from the eigenvalues of \( r^2 \) and \( q^{2(N-\tilde{N})} \) it follows that in the ranges \( [r_0, q^4 l_0) \) and \( [l_0, q^4 l_0) \) the representations are inequivalent. The matrices \( r^2, q^{2(N-\tilde{N})} \) are hermitian and \( u, q^{2(N+\tilde{N}+1)} \) are unitary with respect to the scalar product defined by

\[
| n, m \rangle_{r_0, l_0} \langle n, m | n', m' \rangle_{r_0, l_0} = \delta_{nn'}\delta_{mm'} .
\]
Thus we have obtained that states of particle on a quantum plane are characterized by discrete values of its radius-vector and discrete values of the operator \( q^{2(N-N)} \) which is obviously related to deformation of the angular momentum operator. Indeed, from (99) we conclude that the operator

\[ J_q \equiv [\bar{N} - N; q^2] = q^{2(N-N)} - \frac{1}{q^2 - 1}, \]

(which differs from \( q^{2(N-N)} \) by multiplication and shifting by the constants) in the continuum limit \( q \to 1 \) becomes the ordinary angular momentum operator. Therefore it is natural to consider \( J_q \) as an appropriate deformation of the latter. Of course, discreteness of values of an angular momentum operator is not peculiar feature of \( q \)-deformed systems but general property of all quantum systems.

It is worth to mention that the coordinate algebra of the quantum plane \( P_q^{(2)} \) is closely related to the coordinate algebra (70) of the quantum cylinder. Indeed, if operators \( x_0, x_+ \) obey the commutation relations (70), the operators \( r^2 = e^{x_0}, u = x_+ \) serve as the quantum coordinates (104), (103) on the quantum plane \( P_q^{(2)} \). This correspondence of the algebras is a non-commutative analog of the well-known exponential map of a cylinder to a plane with one punctured point. One can check that the exponentiation of the operator \( x_0 : x_0 \to r^2 = e^{x_0} \), does not change properties of the algebra representations. However, this relation between the algebras does not mean that properties of NC-QFT constructed on the quantum cylinder and on \( P_q^{(2)} \) are the same. Different symmetry properties and different structure of phase spaces lead to quite different field theories on these closely related non-commutative spaces. From technical point of view, the corresponding symmetry principles induce different constructions of the derivatives (the commutators in the case of the cylinder (cf. (78)) and \( q \)-derivatives on the \( q \)-plane) and, hence, different form of free actions in the two cases. While NC-QFT on the non-commutative cylinder has no divergences, the theory on the \( q \)-plane \( P_q^{(2)} \), as we shall see later, has the usual UV-divergences (similar to the case of the field theory on a commutative plane).

In the case of quantum group symmetries there appears a new type of noncommutativity: parameters of the group of symmetry also become noncommutative. Thus there are no usual group of transformations, but instead there exists only \( E_q^{(2)} \)-group coaction on the algebra of \( q \)-coordinates. For clear physical interpretation of quantum symmetries, it is desirable to derive transformations of states of a physical system under a coaction of a quantum group. Recall, that in the case of Lie algebra-like spaces the coordinates form a tensor operator \( \hat{x}_i \to \hat{x}_i' = M_{ij} \hat{x}_j + b_i = \hat{U}_g \hat{x}_i \hat{U}_g^{-1} \) and states of the field system are transformed by the operator \( \hat{U}_g \) (we considered the examples of such transformations for the case of noncommutative Euclidean and Minkowski planes in section 2.3). In the \( q \)-deformed case, these transformations are defined by the generalized Clebsch-Gordan coefficients, describing decomposition of tensor products of representations of algebras of functions on quantum spaces and representations of the corresponding quantum group. In other words, the coaction defines (by duality) a map \( \mathcal{S} \) which puts in correspondence to a pair of states on a group algebra \( G_q \) and on a quantum space \( X_q \) some new state on the quantum space:

\[ \mathcal{S}(|\Psi\rangle, |\psi\rangle) = |\psi'\rangle \quad |\Psi\rangle \in \mathcal{H}_{G_q}, \; |\psi\rangle, |\psi'\rangle \in \mathcal{H}_{X_q}. \]

Matrix elements of the map \( \mathcal{S} \) are the generalized Clebsch-Gordan coefficients. We shall consider this new type of symmetry transformations in full detail in future communication,
while in the rest of the present paper we consider ultraviolet behaviour of a field theory on the $q$-plane.

### 4.2 Distorted plane waves on a noncommutative quantum plane

It is well-known that the Casimir operator generating the centrum of the enveloping algebra $U_q e(2)$ is given by the formula

$$C = \bar{T}e^{2J}\bar{T}. \quad (110)$$

On a quantum plane it is the Laplace operator $\Delta = \lambda(C)$ (cf. (12)), or more explicitly,

$$\Delta = \bar{\partial}_q q^{2N-2N} \partial_q = -q^2 \bar{P}P. \quad (111)$$

Our aim in this subsection is to find eigenfunctions of $\Delta$ which further will be used for construction of a field theory on the $q$-plane.

We search for the functions $\Phi_{kk'}(z, \bar{z})$ on the $q$-plane which are common eigenfunctions of two commuting operators $\Delta$ and $\partial_q$:

$$\Delta \Phi_{kk'}(z, \bar{z}) = kk' \Phi_{kk'}(z, \bar{z}), \quad (112)$$

$$\partial_q \Phi_{kk'}(z, \bar{z}) = k \Phi_{kk'}(z, \bar{z}). \quad (113)$$

The solutions of this eigenvalue problem have the form

$$\Phi_{kk'} = e^{kz}q^{-2} e^{-k'\bar{z}}q^{2}, \quad k, k' \in \mathbb{C}. \quad (114)$$

The first factor is the usual $q$-exponential

$$e^{kz}_{q^{-2}} = \sum_{n=0}^{\infty} \frac{(kz)^n}{[n; q^{-2}]}, \quad (115)$$

and it guarantees the validity of (113). The explicit form of the second factor is obtained by inserting (114) into (113):

$$\bar{e}^{k'\bar{z}}_{q^{2}} = \sum_{n=0}^{\infty} \frac{(k'\bar{z})^n q^{-n(n+1)}}{[n; q^{2}!]} \quad (116)$$

We shall show (cf. (129)) that the eigenfunctions form a basis of the Hilbert space $\mathcal{H}_q$ of square integrable (in an appropriate sense, see below) “functions” on the $q$-plane $P_q^{(2)}$. The Laplacian $\Delta$ is the hermitian operator (see (114), (118)) and, moreover, the expression in terms of the conjugate operators $\bar{P}$ and $P$ in (111) shows that $(-\Delta)$ is positive semidefinite. Therefore, we must put in (114) $k' = -\bar{k}$. Thus, the (properly normalized, see below) distorted plane waves are

$$\Phi_k(z, \bar{z}) = q^{-1/2} e^{kz}_{q^{-2}} e^{-\bar{k}\bar{z}}_{q^{2}}, \quad k \in \mathbb{C}. \quad (117)$$

They are solutions of the following eigenvalue problem:

$$\Delta \Phi_k(z, \bar{z}) = -k\bar{k} \Phi_k(z, \bar{z}), \quad (118)$$

$$\partial \Phi_k(z, \bar{z}) = k \Phi_k(z, \bar{z}).$$
We stress that they are not eigenfunctions of $\partial_q$.

Let us now restrict to the case $q > 1$ (the case $q < 1$ will be described later). We shall interpret the plane waves $\Phi_k(z, \bar{z})$, $k \in \mathbb{C}$, as distributions acting on a suitable set of test functions on a quantum plane $f(z, \bar{z})$ according to the rule:

$$ (\Phi, f) = \int d\bar{z}dz \Phi(z, \bar{z})f(z, \bar{z}) . $$

The integral introduced here is the $E_q(2)$ invariant integral which for $q > 1$ can be defined in the following way (see, e.g., [1]):

$$ \int d\bar{z}dz e_q^{-\bar{z}z}z^m\bar{z}^n = \delta_{mn}[n; q^2]! , \ q > 1 . $$

As for the set of test functions we take the space of functions of the form

$$ f(z, \bar{z}) = e_q^{-\bar{z}z} \sum_{m,n=0}^{\infty} a_{mn}z^m\bar{z}^n , \ q > 1 , $$

with rapidly decreasing coefficients for $n, m \to \infty$.

The Fourier transform of any test function (121) we define by

$$ \tilde{f}(k, \bar{k}) = \int d\bar{z}dz f(z, \bar{z})\Phi_k(z, \bar{z}) . $$

In particular, the Fourier transform of a $q$-Gaussian packet can be calculated easily, and we recover a $q$-analog of the well-known formula

$$ \alpha^{-2}e_q^{-kq^{-2}\alpha^{-2}} = \int d\bar{z}dz e_q^{-\bar{z}z}\Phi_k(z, \bar{z}) . $$

So far we considered all $k \in \mathbb{C}$. On the other hand, using the set of the operators $\hat{P}$, $\hat{P}$, $q^{2(N-N)}$, $q^{2(N+N+1)}$, one can easily find (similar to (106), (107)) that the operator $\Delta = -q^2\hat{P}\hat{P}$ has a discrete spectrum. Thus, the eigenfunctions $\Phi_k$, $k \in \mathbb{C}$ form an overcomplete set. To use only complete set of eigenfunctions, we have to reduce consideration to an appropriate discrete subset, e.g., $\Phi_{q^n}$, $n = 0, \pm 1, \pm 2, \ldots$. Correspondingly, an integration over variables $k, \bar{k}$ should be defined in a sense of Jackson-like integral, with the values of basic integrals for $q > 1$ being as follows (see [3], sect. 2.2):

$$ \int d\bar{k}dk e_q^{-kq^{-2}k^\alpha m} = q\delta_{mn}[n; q^2]! , \ q > 1 . $$

The inverse Fourier transform to (122) is given as

$$ f(z, \bar{z}) = \int d\bar{k}dk \tilde{f}(k, \bar{k})\Phi_k^\dagger(z, \bar{z}) . $$

In particular, the validity of (125) can be shown easily for the $q$-Gaussian packet:

$$ e_q^{-\bar{z}z} = \int d\bar{k}dk \alpha^{-2}e_q^{-kq^{-2}\alpha^{-2}}\Phi_k^\dagger(z, \bar{z}) . $$

Analogously, the validity of (126) can shown straightforwardly for any test function of the form $e_q^{\bar{z}z}z^m\bar{z}^m$. This proves that the system of distorted plane waves $\Phi_k(z, \bar{z})$, $k = q^{2n}$, $n$-integers, is a complete orthogonal set of functions on the quantum plane normalized to the $\delta$-function:

$$ \int d\bar{z}dz \Phi_k^\dagger(z, \bar{z})\Phi_k(z, \bar{z}) = \delta_q^{(2)}(k, k') . $$

25
The symbol on the right hand side is defined as an analog of $\delta$-function on the quantum plane by the relation

$$
\int_q d\bar{k}' k' \tilde{f}(k', \bar{k}') \delta_q^{(2)}(k, k') = \tilde{f}(k, \bar{k}) ,
$$

(128)

which should be valid for all Fourier transforms $\tilde{f}(k, \bar{k})$ of any test function.

By $\mathcal{H}_q$ we denote the Hilbert space of functions on a scalar plane for which there exists the scalar product

$$(f_1, f_2) = \int^q d\bar{z} dz f_1^\dagger(z, \bar{z}) f_2(z, \bar{z}) .$$

(129)

It can be shown straightforwardly that the Laplacian $\Delta$ is symmetric on a subspace of $\mathcal{H}_q$ formed by test functions (121). Then by the construction presented above, the operator $\Delta$ can be extended to a self-adjoint operator, with plane waves $\Phi_k(z, \bar{z}), k \in \mathbb{C}$.

Let us now discuss briefly the case $q < 1$. In this case we do not take the test functions in the form (121) but as the series

$$f(z, \bar{z}) = e^{-\bar{z}z} \sum_{m,n=0}^\infty a_{mn} z^m \bar{z}^n , \quad q < 1 ,
$$

(130)

with rapidly decreasing coefficients. Formulas (122) and (125) defining the direct and inverse Fourier transforms remain valid but we have to properly modified the definition of integrals. Namely, eqs. (120) and (124) should be replaced respectively by

$$
\int^q d\bar{z} dz e^{-\bar{z}z} \cdot z^m \bar{z}^n = \delta_{mn}[m; q^2]! , \quad q < 1 ,
$$

(131)

and

$$
\int_q d\bar{k} dk e^{-k\bar{k}q} \cdot k^m \bar{k}^n = q^{-1} \delta_{mn}[n; q^2]! , \quad q < 1 .
$$

(132)

We do not introduce new symbols for the integrals (131) and (132), since they correspond to the same Jackson integrals as (120) and (124), respectively. This can be seen from the matching formulas between both pairs of integrals. The matching formula between (120) and (131) is:

$$
\int^q d\bar{z} dz e^{-\bar{z}z} e^{-a^2z^2} e^{-a^2\bar{z}z} = \frac{1}{\alpha^2 + \beta^2} = \int^q d\bar{z} dz e^{-\alpha^2z^2} e^{-\beta^2\bar{z}z} .
$$

(133)

We have expanded in both sides the second exponent and used the definitions (120) and (121). The result on the left hand side is justified for $\alpha < \beta$, whereas on the right hand side for $\alpha > \beta$. A similar matching formula is valid between integrals (124) and (132):

$$
\int_q d\bar{k} dk e^{-\beta^2k\bar{k}q^2} e^{-\alpha^2k\bar{k}q^2} = \frac{1}{q\alpha^2 + q^{-1}\beta^2} = \int_q d\bar{k} dk e^{-\alpha^2k\bar{k}q^2} e^{-\beta^2k\bar{k}q^2} .
$$

(134)

Both sides have been evaluated in the similar way as before, however now we have used (124) and (132). Again, left hand side is valid for $\alpha < \beta$ and the right hand side for $\beta > \alpha$.

Notice, that the whole analysis presented above can be repeated with the second system of distorted plane waves given as

$$
\Psi_k(z, \bar{z}) = q^{-1/2} e_{q^2 - k\bar{k}} e_{q^2 - k\bar{k}} , \quad k \in \mathbb{C} ,
$$

(135)
which solve the eigenvalue problem:

\[
\Delta \Psi_k(z, \bar{z}) = -k \bar{k} \Psi_k(z, \bar{z}) ,
\]
\[
\bar{\partial}_q \Psi_k(z, \bar{z}) = k \Psi_k(z, \bar{z}) .
\] (136)

There is also an alternative possibility to use $q$-deformed spherical waves on a quantum plane which are eigenfunctions of the Laplacian $\Delta$ and the angular momentum operator $J$.

### 4.3 The field action and the tadpole contribution

We define on the quantum plane the field action for a complex field $\varphi(z, \bar{z})$ in a standard way

\[
S[\varphi, \varphi^\dagger] = \int^q d\bar{z}dz [\varphi^\dagger(z, \bar{z})(-\Delta + m^2)\varphi(z, \bar{z}) - V(\varphi, \varphi^\dagger)] .
\] (137)

We expand the field $\varphi(z, \bar{z})$ into the plane waves (117)

\[
\varphi(z, \bar{z}) = \int_q d\bar{k}dk a(k, \bar{k}) \Phi_k(z, \bar{z}) .
\] (138)

Inserting this into (137) we can diagonalize the free field action:

\[
S_0[\varphi, \varphi^\dagger] = \int^q d\bar{z}dz \varphi^\dagger(z, \bar{z})(-\Delta + m^2)\varphi(z, \bar{z})
\]
\[
= \int_q d\bar{k}dk \bar{a}(k, \bar{k})(k\bar{k} + m^2)a(k, \bar{k}) .
\] (139)

The last formula is valid due to the completeness and orthogonality properties of the distorted plane waves.

From (139) we see that the free field propagator in the $k$-representation is given as

\[
\hat{G}(k, \bar{k}) = \frac{1}{kk + m^2} .
\] (140)

Since, the propagator $\hat{G}(k, \bar{k})$ behaves like $(kk)^{-1}$ for large $k\bar{k}$, the UV-divergencies has to be expected. Indeed, let us consider the scalar field on the $q$-plane in a constant external field $F$, so that the interaction term has the simple form

\[
V(\varphi, \varphi^\dagger) = F \varphi^\dagger(z, \bar{z})\varphi(z, \bar{z}) , \quad F = \text{const} .
\]

Then the tadpole contribution is proportional to the divergent Jackson integral

\[
\Gamma = \int_q d\bar{k}dk \frac{1}{kk + m^2} = \infty .
\] (141)

In order to prove (141) we shall perform the evaluation of the tadpole contribution with the regularized propagator

\[
\tilde{G}_{reg} = m^2 \int^q d\bar{z}dz e^{-m^2 \bar{z}z} e^{-k\bar{k}z\bar{z}}(zz)^s .
\] (142)
For $s > -1$, this is a well-defined Jackson integral behaving asymptotically as $(k\bar{k})^{1-s}$. For $s = 0$ it reduces to (140) as can be seen from (133). Inserting (142) into the tadpole integral and interchanging the order of integrations, the regularized tadpole contribution can be calculated straightforwardly,

$$\Gamma_{s}^{\text{reg}} = m^{2s} \int d\tilde{z} dze^{-m^{2}\tilde{z}z}(\bar{z}\bar{z})^{s} \int_{q} d\bar{k} dk e^{-kk\bar{z}z}$$

$$= \int d\tilde{z} dze^{-\tilde{z}z}(\bar{z}\bar{z})^{s-1} = \frac{1}{2} \Gamma_{q^{2}}(s). \quad (143)$$

Here, $\Gamma_{q^{2}}(s)$ is a $q$-deformed $\Gamma$-function which is singular at $s = 0$, similarly to the usual $\Gamma$-function (repeating the indicated calculations in the standard case we recover (143) with the ordinary $\Gamma$-function).

In the case of the external field $F(z, \bar{z}) = \sum_{n,m=0}^{\infty} a_{mn}z^{n}\bar{z}^{m}$ depending on coordinates, for commutative plane there appears a divergent tadpole contribution of the form $\tilde{F}(0,0)\Gamma_{\text{com}}$, where $\tilde{F}(k, \bar{k})$ is the Fourier transform of the external field and $\Gamma_{\text{com}}$ is the tadpole integral on the commutative plane. On the noncommutative plane we have the interaction action as follows

$$S_{\text{int}} = \int d\tilde{z} dz \varphi^{\dagger}(z, \bar{z})F(\bar{z}, z)\varphi(z, \bar{z})$$

$$= \int_{q} d\tilde{k} dk g(k_{1}, \bar{k}_{1}, k_{2}, \bar{k}_{2})\varphi^{\dagger}(k_{1}, \bar{k}_{2})\varphi(k_{2}, \bar{k}_{2}).$$

Taking into account the explicit form (117) of plane waves and their orthogonality (127), the vertex $g(k_{1}, \bar{k}_{1}, k_{2}, \bar{k}_{2})$ can be cast into the form

$$g(k_{1}, \bar{k}_{1}, k_{2}, \bar{k}_{2}) = \int d\tilde{z} dz \Phi_{k_{1}}^{\dagger}(z, \bar{z})F(z, \bar{z})\Phi_{k_{2}}(z, \bar{z})$$

$$= F(\partial_{k_{1}}, \partial_{\bar{k}_{2}})\delta_{q}^{(2)}(k_{1}, k_{2})$$

(where the derivatives are given by $\partial_{k}k^{n} \equiv [n; q^{-2}]k^{n-1}$ and $\partial_{\bar{k}}\bar{k}^{n} \equiv [n; q^{-2}]\bar{k}^{n-1}$). The tadpole contribution is given as

$$\int_{q} d\tilde{k} dk \frac{g(k, \bar{k}, k, \bar{k})}{kk + m^{2}}.$$  

In the commutative case the numerator $g(k, \bar{k}, k, \bar{k}) = \tilde{F}(0,0)$ is constant leading to the divergent tadpole contribution. In the noncommutative case we obtain even more divergent contributions which cannot cancel each other for a general field $F(z, \bar{z})$ (due to arbitrariness of the expansion coefficients $a_{mn}$). They are proportional to $\log q$ and disappear in the commutative limit.

We conclude that the $q$-deformation of a plane does not lead to the desired UV-regularization.

## 5 Conclusion

We have shown that transition to a noncommutative space-time does not necessarily lead to an ultraviolet regularization of the quantum field theory constructed in this space, at
least in the most natural way of introducing noncommutativity as we have performed in this paper. In particular, QFT on noncommutative planes with the Heisenberg-like commutation relations for coordinates and the deformed plane with the quantum $E_q(2)$-symmetry still contain divergent tadpoles. However, in general, theories which have the same UV-behaviour on classical spaces may acquire essentially different properties after the quantization. The reason is that quantization procedure is highly sensitive to the topology of the manifold under consideration. Thus, while in the case of classical space-time the theories on a sphere, cylinder or plane have UV-divergences, in the case of noncommutative space-time the two-dimensional theories on the fuzzy sphere and on the quantum cylinder do not have divergences at all. This can be traced to the compactness properties of the space-time in question:

- In the case of a fuzzy sphere, models contain a finite number of modes and thus all the usual integrations are replaced by final sums and, consequently, no UV-divergences can appear.

- In the case of a cylinder, a priori one can not claim whether the quantum field theory is finite. However, the non-commutativity of the space-time together with the compactness of the space (circle) lead to the intrinsic cut-off in the energy modes. This guarantees the removal of UV-divergences in the two-dimensional case.

- On a noncommutative plane (whose commutative limit is noncompact in both directions) with Heisenberg-like or even with deformed commutation relations, the noncommutativity of the space-time does not lead to an UV-regular theory.

We conclude with the following general picture:

- the noncommutativity itself does not guarantee the removal of UV-divergences;

- global topological restrictions are needed - namely, at most one dimension (time) is allowed to be noncompact, in order to achieve the removal of UV-divergences of a quantum field theory formulated in a noncommutative space-time of arbitrary dimensions.

It is of great interest to investigate the problem further in order to find out whether there exists any other, than the one presented in this paper, way of introducing noncommutativity so that it would remove the UV-divergences even in the case of fully noncompact space-times.

**Acknowledgements**

The financial support of the Academy of Finland under the Projects No. 37599 and 44129 is greatly acknowledged. A.D.’s work was partially supported also by RFBR-98-02-16769 grant and P.P.’s work by VEGA project 1/4305/97.

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