Quantum entanglement in spinor Bose-Einstein condensates

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We propose a scheme to generate and detect various kinds of quantum entanglement in a spin-1 Bose-Einstein condensate. It is shown that substantial many-particle entanglement can be generated directly in the spin-1 condensate by free dynamical evolution with a properly prepared initial state. The scheme also provides a simple method to generate three-mode entanglement in the second-quantization picture and to detect the continuous variable type of entanglement between two effective modes in the spin-1 condensate.

The achievement of atomic Bose-Einstein condensate (BEC) [1] has raised extensive experimental and theoretical studies in this area, and these studies have opened up some possibilities for important applications. An impressive example of these applications is to use BECs with internal degrees of freedom for generation of quantum entanglement [2-7], which is the essential ingredient of many quantum information protocols [8]. To generate entanglement, one requires an experimental system which can be prepared in a pure state, and there should be significant interactions between the relevant particles which can be coherently controlled. BECs fulfill these requirements, and therefore they could provide an ideal experimental system for studying quantum entanglement. It has been shown that with coherent interaction in BECs, it is possible to generate substantial many-particle entanglement in a two-component condensate [2], or to generate entangled atomic beams from the outputs of a single condensate prepared in a specific internal level which enables spin-exchange collisions [3,4].

A good experimental system for demonstration of these proposals is the Sodium condensate confined in an optical trap [9-11]. Recently, Stamper-Kurn et al. have realized an optically trapped BEC in which all the three ground Zeeman levels of sodium atoms are involved [9]. In such a three-component (so-called spin-1) condensate, there are spin-exchange collisions, which constantly mix the different spin components, and other kinds of internal spin dynamics, which result in complex nonlinear collision phase shifts [12-17]. In this paper, we will investigate whether one can directly prepare and detect quantum entanglement in such a spin-1 condensate. The motivation of this work is two-folds: firstly, direct detection of quantum entanglement in the realized spin-1 condensate eliminates the requirements of the level shifting and the output techniques in previous works [2-4], and thus allows for an easier experiment; and secondly, the spin-1 condensate with the presence of both the spin-exchange collisions and the nonlinear phase shifts is a more complicated system, and opens up possibilities for richer physical behaviors. We will show that various kinds of entanglements are available in this system. We can get substantial many-particle entanglement (the spin squeezing type of entanglement [18,19]) as well as three-mode and two-mode entanglement (the continuous-variable type of entanglement [20]). All these entanglements are useful for different purposes of applications.

The spin-1 BEC has been studied both theoretically [12-17] and experimentally [9-11], with the emphasis on the ground-state structure, the spin-mixing dynamics, and the response to the external magnetic field. Here, we will focus on its entanglement property. In Ref. [15], it was shown as a by-product that the spin ground state of the sodium condensate with the anti-ferromagnetic interaction should be entangled. Unfortunately, this ground-state entanglement was not confirmed by the experiment [10] because it is very hard to cool the system to the spin ground state (the energy gap between the spin ground and the first excited states is many orders smaller than the chemical potential), and the spin ground state is very fragile under the influence of the external magnetic field [17]. Here, we describe a different method to generate and detect quantum entanglement by free dynamical evolution with a proper initial state. As we will see, this provides a feasible method of detecting entanglement in the spin-1 BEC with the current technology.

Before we enter the discussion of the spin-1 condensate, let us first clarify different kinds of entanglements available in a bosonic many-particle system. Consider \( N \) bosonic identical particles with each particle having three internal levels \(|+1\rangle, |-1\rangle, \) and \(|0\rangle\). First, the \( N \) particles are said to be entangled in the first quantization picture if their internal density matrix \( \rho \) cannot be decomposed into

\[ \rho = \sum_k p_k \rho_i^k \otimes \rho_2^k \otimes \cdots \otimes \rho_N^k, \]

(1)

where the coefficients \( p_k \) are positive real numbers satisfying \( \sum_k p_k = 1 \), and \( \rho_i^k \) is a \( 3 \times 3 \) density matrix of the \( i \)th particle. To check whether an \( N \)-particle system is entangled, a good experimentally detectable criterion is the squeezing parameter. For particles with only two internal levels \(|a\rangle \) and \(|b\rangle \), one can construct a collective spin operator \( J_{ab} = \sum_{i=1}^N j_i \), with the individual spin-1/2 operators \( j_i^x = (|a\rangle_i \langle b|_i + |b\rangle_i \langle a|_i)/2 \), \( j_i^y = i (|a\rangle_i \langle b|_i - |b\rangle_i \langle a|_i)/2 \), and \( j_i^z = (|b\rangle_i \langle b|_i - |a\rangle_i \langle a|_i)/2 \). The squeezing parameter \( \xi_{ab} \) is defined directly from the variance and mean values of the \( J_{ab} \) operator.
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\[ |\alpha\rangle \] in Eq. (2). Either of
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isentangled state [8,20]). For later applications in the spin-1 condensate, we are more interested in two effective

bosonic modes \( a_{\pm1} \), defined as
\[ a_{\pm1} = a_{\pm1}a_0^\dagger/\sqrt{\langle a_0^\dagger a_0 \rangle}. \]
The modes \( a_{\pm1} \) are approximately bosonic if nearly all the atomic populations remain in the component \( a_0 \), i.e.,
\[ \langle a_0^\dagger a_0 \rangle \approx N. \]
The definition of the new modes \( a_{\pm1} \), which are proportional to \( a_{\pm1} \), is to remove the phase randomness in the condensate by phase locking both of the modes \( a_{\pm1} \) to the condensate mode \( a_0 \),
for the modes \( a_{\pm1} \), the quadrature phase amplitudes defined above reduce to
\[ \langle (\Delta X_\pm^0)^2 \rangle < \xi_\pm^0/2. \]
the spin squeezing parameters thus becomes a sufficient criterion for entanglement between the modes \( a_{\pm1} \). As a result, a measurement of the spin squeezing parameters \( \xi_\pm^0 \) provides criteria for both many-particle entanglement and two-mode entanglement.

We will see that three-mode entanglement is also produced in a spin-1 condensate. A general criterion for the three-mode entanglement is still absent, but our system to a good approximation will be in a pure three-mode state \( |\phi\rangle_{+1,-1} \) and in this case the three-mode entanglement can be easily characterized by the quantity
\[ E_3 = \sqrt{E(\rho_{+1})E(\rho_{-1})E(\rho_0)}, \]
where \( E(\rho_\alpha) = -tr(\rho_\alpha \log_2 \rho_\alpha) \) denotes the von Neuman entropy of \( \rho_\alpha \), and \( \rho_\alpha \) is the reduced density operator of the mode \( \alpha \) \( (\alpha = +1,-1,0) \). It is obvious that for pure states \( E_3 \) is non-increasing under local operations and positive if and only if the system state is intrinsically three-mode entangled. It will be shown that in a spin-1 condensate \( E_3 \) can be significantly larger than 0. Note that different to the spin squeezing parameters it is not easy to directly measure \( E_3 \).

Now we turn to the physical system, which is a dilute gas of trapped bosonic atoms with hyperfine spin \( f = 1 \).
Assume that the system is very cold so that the collision interaction between atoms is effectively described by a pair-wise pseudo-\( \delta \)-potential, which preserves the hyperfine spin of individual atoms and is rotationally invariant in the hyperfine spin space [12]. With this symmetry assumption, the most general form of the Hamiltonian can be dived into two parts \( H = H_s + H_a \), with the symmetric and asymmetric Hamiltonians \( H_s \) and \( H_a \) respectively given by [12,15] (setting \( \hbar = 1 \))
\[ H_s = \sum_\alpha \int d^3r \frac{\lambda_\alpha}{2} \left( -\frac{\nabla^2}{2m} + V \right) \Psi_\alpha \]
\[ + \frac{\lambda_\beta}{2} \sum_{\alpha,\beta} \int d^3r \Psi_\alpha^\dagger \Psi_\beta \Psi_\alpha \Psi_\beta, \]

\[ c_{ab}^n = \frac{N \langle (\Delta J_{ab}^n)^2 \rangle}{\langle J_{ab}^n \rangle^2 + \langle J_{ab}^n \rangle^2}, \]
where \( J_{ab}^n = n \cdot J_{ab} \), and the \( n \)s are mutually orthogonal unit vectors. For two-level particles, the spin squeezing parameter (2) with a value less than 1 provides a sufficient condition for many-particle entanglement [2] (such a case is also referred as spin squeezing [18,19]). For three-level particles, we can construct two orthonormal states \( |a\rangle \) and \( |b\rangle \) from arbitrary superpositions of the three levels \( |+1\rangle, |-1\rangle, \) and \( |0\rangle \), and define the \( J_{ab} \) and \( c_{ab}^n \) in the same way. Then, \( c_{ab}^n \) < 1 assures that the projected \( N \)-particle density matrix \( \rho_{PP} \) onto the subspace spanned by the states \( |a\rangle_i \) and \( |b\rangle_i \) \( (i = 1, 2, \ldots, N) \) is entangled. Since the projection \( P \) is a local operator which cannot increase entanglement, the original density matrix \( \rho \) for the \( N \) three-level particles should also be entangled if \( c_{ab}^n \) < 1. For our later applications, we define the spin operators \( J_\pm \) and the squeezing parameters \( \xi_\pm^0 \) respectively by choosing \( |a\rangle = (|+1\rangle \pm |-1\rangle)/\sqrt{2}, \)

\( |b\rangle = |0\rangle \), and \( n_1 = [\cos \theta, \sin \theta, 0] \) in Eq. (2). Either of
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\[ H_a = \frac{\lambda_a}{2} \sum_{\alpha, \beta} \int dr \psi^\dagger_\alpha S_{\alpha \beta} \cdot S_{\alpha \beta} \psi \psi^\dagger_\beta, \quad (4) \]

where \( \psi_\alpha (\alpha = -1, 0, +1) \) is the atomic field operator associated with atoms in the spin state \( |f = 1, m_f = \alpha \rangle \). The first and the second terms of the Hamiltonian \( H_a \) denote the kinetic energy and the potential energy, with \( m \), the mass of the atom, and \( V \), the trap potential, which has been assumed to be the same for all the three components. The \( 3 \times 3 \) spin matrices \( S \) in \( H_a \) denote the conventional 3-dimensional representation (corresponding to the spin \( f = 1 \)) of the angular momentum operator, with \( S^\alpha_\beta = (\delta_{\alpha, \beta - 1} + \delta_{\alpha, \beta + 1})/\sqrt{2}, S^y_\beta = i (\delta_{\alpha, \beta - 1} - \delta_{\alpha, \beta + 1})/\sqrt{2}, S^z_\beta = a \delta_{\alpha, \beta}. \) The interaction energy has been divided into the symmetric and the asymmetric parts, represented respectively by the last term of \( H_a \) and the Hamiltonian \( H_s \). The symmetric interaction remains the same if we exchange two arbitrary spin components, whereas the asymmetric one is spin-dependent. The symmetric and the asymmetric collision coefficients \( \lambda_\alpha \) and \( \lambda_s \) are determined respectively by the average and the difference of the scattering lengths in two different collision channels. For sodium atoms, \( \lambda_s \) is positive and about 25 times smaller than \( \lambda_a \) [11,12]. Thus, the Hamiltonian \( H_s \) is the dominant one for determining the dynamics of the motional state. However, the internal spin dynamics which we are interested in is completely determined by \( H_a \) since the symmetric part \( H_s \) is spin-independent.

For generating entanglement, we start with a BEC prepared in the internal level \( |m_f = 0 \rangle \). To quantify the amount of various kinds of squeezing and entanglement obtainable in this system, we need to calculate the time evolution of the entanglement criteria \( \xi^2 \) and \( E_3 \) defined before. For this purpose, we make use of the single motional mode approximation to the Hamiltonian \( H_a \) following Refs. [15-17]. Note that the Hamiltonian \( H_a \) is the dominant one for determining the motional state. When the system is at a very low temperature, the motional state is frozen to be approximately the ground state \( \varphi (r) \) (normalized as \( \int dr |\varphi (r)|^2 = 1 \)) of the spin-independent Hamiltonian \( H_s \), and we can factorize the field operators as \( \psi_\alpha \approx a_\alpha \varphi (r) \) (\( \alpha = 0, \pm 1 \)), where \( a_\alpha \) is the usual bosonic annihilation operator. With this approximation, one can introduce a spin operator \( L \) with \( L = \sum_{\alpha, \beta} a^\dagger_\alpha S_{\alpha \beta} a_\beta \), and the spin-dependent Hamiltonian \( H_a \) is simplified to \( H_a = \lambda_a \left( L^2 - 2N \right) \), where \( \lambda_a = \lambda_\alpha \int dr |\varphi (r)|^4 / 2. \) This Hamiltonian has been used for studying the spin ground state structure [15]. We use it here for calculating squeezing and entanglement in this system. For this purpose, it is convenient to express the Hamiltonian \( H_a \) by the more relevant spin operators \( J_\pm \) defined before. For the specific system configuration considered here, \( L^2 = 0 \) for the initial state, and it will remain zero since \( L^2 \) is conserved. On the other hand, by definition we have \( L^x = 2J_+^2 \) and \( L^y = 2J_-^2 \). Therefore, the Hamiltonian \( H_a \) expressed by \( J_\pm \) has the form

\[ H_a = 4\lambda_a \left( J_+^2 + J_-^2 \right), \quad (5) \]

where we have dropped the irrelevant constant \( 2\lambda_a N \). The only non-zero commutation relation between \( J_+ \) and \( J_- \) is \( [J_+^2, J_-^2] = [J_+^2, J_-^2] = \left( a_+ a_{+1} a_{+1} a_+ - a_+ a_{+1} a_{+1} a_+ \right)/4 \), which remains very small if the population of the atoms is still dominantly in the level \( |m_f = 0 \rangle \), as it is in our scheme. In this case, as an approximation we can consider separately the dynamics of \( J_+ \) and \( J_- \), respectively with the effective Hamiltonian \( J_+^2 \) or \( J_-^2 \). It is well known that the Hamiltonian \( J_+^2 \) or \( J_-^2 \) will produce a spin squeezing state, with the minimum squeezing parameter about \( (3/2)^{2/3} \) after an evolution time proportional to \( N^{-2/3} / \lambda_a \) [18]. This is the intuitive reason why we can get substantial spin squeezing and entanglement in this system. To be more quantitative, we have calculated exactly the time evolution of the squeezing parameter \( \xi^2 \) under the Hamiltonian (5) using the numerical approach. The system spin state \( |\phi \rangle \) under the Hamiltonian (5) can be expanded as \( |\phi \rangle = \sum_n c_n |N - 2n, n, n \rangle_0,_{+1, -1} \) in the number basis, where the subscript \( \alpha = (\alpha = 0, \pm 1) \) represents the corresponding internal mode \( a_\alpha \). One can solve numerically the time evolution of the expansion coefficients \( c_n \), from which the spin squeezing parameters \( \xi^2 \) are exactly obtainable. We find that the squeezing magnitudes \( \xi_\pm = \min_{\theta} \xi^2_\theta \) evolve in the same way for \( J_+ \) and \( J_- \), while their squeezing directions corresponding to the optimal angle \( \theta \) are always orthogonal to each other. This means that the criterion \( \min_{\theta} \left( \xi^2_+ + \xi^2_- + \pi/2 \right) / 2 \) for entanglement between the two effective modes \( a_+ a_1 \) evolves in the same way as the criterion \( \xi_\pm \) for many-particle entanglement. The resultant time evolution of \( \xi_\pm \) is shown in Fig. 1a, together with a prediction from the simple Hamiltonian \( J_+^2 \) (or \( J_-^2 \)). One can see that the results from the two approaches agree quite well, with only slight difference in the maximum obtainable squeezing and the corresponding evolution time. For \( 10^7 \) atoms, about three orders reduction is obtainable in \( \xi_\pm \), which is a clear demonstration that substantial many-particle entanglement and two-mode entanglement have been generated. Fig. 1b shows the time evolution of the three-mode entanglement \( E_3 \). One can see that significant three-mode entanglement can be generated with only \( 10^7 \) atoms.

Finally, we would like to address some imperfections and approximations used in this scheme. First, in a practical case there would be some remaining external magnetic field due to the special way of preparing the condensate. It is remarkable that the linear Zeeman shift from a known homogeneous magnetic field has no influence on our scheme since it commutes with the spin-dependent Hamiltonian \( H_a \) and will be eliminated if one
transforms to the interaction picture. As a result, only quadratic Zeeman shift and linear Zeeman shift from an inhomogeneous field have influence. These high-order effects can be easily controlled very small in the relevant time scale. For instance, one can see from Fig. 1 that the time scale in this scheme is roughly set by the inverse of $N\lambda^2$, which is about $2\pi \times 10^{12}$ Hz for a typical density $3 \times 10^{14}$ cm$^{-3}$ of sodium atoms [11]. Within this time, the quadratic and the inhomogeneous Zeeman shifts have negligible effects if the magnetic field and its gradient are respectively smaller than 1 mG and 1 mG/cm. Secondly, we have assumed that the three components with $m_f = 0, \pm 1$ have the same spatial wave function. This is a good approximation as long as there is no demixing instability leading to phase separation of different components [2,22]. For the two-component condensate with components $\alpha$ and $\beta$ having equal macroscopic populations, a mean-field theory has shown that there is no demixing instability if the inter-component collision coefficient $\lambda_{\alpha\beta}$ is smaller than the geometric average of the intra-component collision coefficients $\sqrt{\lambda_{\alpha\alpha}\lambda_{\beta\beta}}$ [22,11]. For the relevant dynamics in our scheme with only the level $|m_f = 0\rangle$ macroscopically populated, Fig. 1 shows that this system is well approximated by two independent sets of two-component condensates. For each set of two-component condensate, we can apply the above result, and find there is no demixing instability in this scheme if the asymmetric collision coefficient $\lambda_\alpha$ is positive, as it is for sodium atoms. This is different to the case where both of the levels with $m_f = 0, \pm 1$ are equally populated, which leads to demixing instabilities [10,11]. Finally, we assumed the single-mode approximation for the atomic motional state. This approximation is supported by the following arguments: first, in the two-component condensate more complete numerical simulations give a squeezing which agrees quite well with the calculation from the single-mode approximation [2,6,23], and the present spin-1 system is well approximated by two independent sets of two-component condensates; and second, a direct energy analyses in Ref. [16] also shows the validity of the single-mode approximation for the spin-1 condensate with only the level $|m_f = 0\rangle$ initially populated.

In summary, we have shown how to directly generate and detect various kinds of squeezing and entanglement in the experimentally realized spin-1 condensate. This helps for the final observation of quantum entanglement in Bose-Einstein condensates.

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**FIG. 1.** a. Time evolution of the squeezing magnitudes for $10^3$ atoms. The solid curve shows the squeezing from an exact numerical solution to the Hamiltonian (5), and the dashed curve is the result from the Hamiltonian $4\lambda^2 \hat{J}_+^2$. The upper dotted curve shows the ratio of the atomic population in the level $|m_f = 0\rangle$, which is basically 1. b. Time evolution of the three-mode entanglement $E_3$ for $10^3$ atoms.