On \( k \)-Column Sparse Packing Programs

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Abstract

We consider the class of packing integer programs (PIPs) that are column sparse, where there is a specified upper bound \( k \) on the number of constraints that each variable appears in. We give an improved \((ek + o(k))\)-approximation algorithm for \( k \)-column sparse PIPs. Our algorithm is based on a linear programming relaxation, and involves randomized rounding combined with alteration. We also show that the integrality gap of our LP relaxation is at least \( 2k - 1 \); it is known that even special cases of \( k \)-column sparse PIPs are \( \Omega(\frac{k}{\log k}) \)-hard to approximate.

We generalize our result to the case of maximizing monotone submodular functions over \( k \)-column sparse packing constraints, and obtain an \((e^{2k} + o(k))\)-approximation algorithm. In obtaining this result, we prove a new property of submodular functions that generalizes the fractionally subadditive property, which might be of independent interest.

When the capacities of all constraints are large relative to the sizes, we obtain substantially better guarantees for these \( k \)-column sparse packing problems; again our result is tight (up to constant factors) relative to the natural LP relaxation.

1 Introduction

Packing integer programs (PIPs) are those of the form:

\[
\max \left\{ w^T x \mid Sx \leq c, \ x \in \{0,1\}^n \right\}, \quad \text{where} \ w \in \mathbb{R}_+^n, \ c \in \mathbb{R}_+^m \ \text{and} \ S \in \mathbb{R}_+^{m \times n}.
\]

Above, \( n \) is the number of variables/columns, \( m \) is the number of rows/constraints, \( S \) is the matrix of sizes, \( c \) is the capacity vector, and \( w \) is the weight vector. In general, PIPs are very hard to approximate: a special case is the classic independent set problem, which is NP-Hard to approximate within a factor of \( n^{1-\epsilon} \) \([30]\), whereas an \( n \)-approximation is trivial. Thus, various special cases of PIPs are often studied. Here, we consider \( k \)-column sparse PIPs (denoted \( k \)-CS-PIP), which are PIPs where the number of non-zero entries in each column of matrix \( S \) is at most \( k \). This is a fairly general class and models several basic problems such as \( k \)-set packing \([18]\) and independent set in graphs with degree at most \( k \).

Recently, in a somewhat surprising result, Pritchard \([25]\) gave an algorithm for \( k \)-CS-PIP where the approximation ratio only depends on \( k \); this is useful when \( k \) is small. This result is surprising because in contrast, no such guarantee is possible for \( k \)-row sparse PIPs. In particular, the independent set problem on general graphs is a 2-row sparse PIP, but is \( n^{1-o(1)} \)-hard to approximate. Pritchard’s algorithm \([25]\) had an approximation ratio of \( 2k \cdot k^2 \). Subsequently, an improved \( O(k^2) \) approximation algorithm was obtained independently by Chekuri et al. \([13]\) and Chakrabarty-Pritchard \([10]\).

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Our Results: In this paper, we first consider the $k$-CS-PIP problem and obtain an $(ek + o(k))$-approximation algorithm for it. Our algorithm is based on solving a strengthened version of the natural LP relaxation of $k$-CS-PIP, and then performing randomized rounding followed by suitable alterations. In the randomized rounding step, we pick each variable independently (according to its LP value) and obtain a set of variables with good expected weight; however, some constraints may be violated. Then in the alteration step, we drop some variables so as to satisfy all constraints, while still having good expected weight. A similar approach can be used with the natural relaxation for $k$-CS-PIP obtained by simply dropping the integrality constraints on the variables; this gives a slightly weaker $8k$-approximation bound. However, the analysis of this weaker result is much simpler and we thus present it first. To obtain the $ek + o(k)$ bound, we construct a stronger LP relaxation by adding additional valid constraints to the natural relaxation for $k$-CS-PIP. The analysis of our rounding procedure is based on exploiting these additional constraints and using the positive correlation between various probabilistic events via the FKG inequality.

Our result is almost the best possible that one can hope for using the LP based approach. We show that the integrality gap of the strengthened LP is at least $2k - 1$, so our analysis is tight up to a small constant factor $e/2 \approx 1.36$ for large values of $k$. Even without restricting to LP based approaches, an $O(k)$ approximation is nearly best possible since it is NP-Hard to obtain an $o(k/\log k)$-approximation for the special case of $k$-set packing [17]. We also obtain improved results for $k$-CS-PIP when capacities are large relative to the sizes. In particular, we obtain an $O(k^{1/|B|})$-approximation algorithm for $k$-CS-PIP, where $B := \min_{c \in [n], j \in [m]} c_j/S_j$ measures the relative slack between the capacities $c$ and sizes $S$. We also show that this result is tight up to constant factors relative to its LP relaxation.

Our second main result is for the more general problem of maximizing a monotone submodular function over packing constraints that are $k$-column sparse. This problem is a common generalization of maximizing a submodular function over (a) a $k$-dimensional knapsack [21], and (b) the intersection of $k$ partition matroids [24]. Here, we obtain an $(\epsilon_k^2 k + o(k))$-approximation algorithm for this problem. Our algorithm uses the continuous greedy algorithm of Vondrak [29] in conjunction with our randomized rounding plus alteration based approach. However, it turns out that the analysis of the approximation guarantee is much more intricate: In particular, we need a generalization of a result of Feige [15] that shows that submodular functions are also fractionally subadditive. See Section 3 for a statement of the new result, Theorem 3.3, and related context. This generalization is based on an interesting connection between submodular functions and the FKG inequality. We believe that this result and technique might be of further use in the study of submodular optimization.

Related Previous Work: Various special cases of $k$-CS-PIP have been extensively studied. An important special case is the $k$-set packing problem, where given a collection of sets of cardinality at most $k$, the goal is to find the maximum weight sub-collection of mutually disjoint sets. This is equivalent to $k$-CS-PIP where the constraint matrix $S$ is 0-1 and the capacity $c$ is all ones. Note that for $k = 2$ this is maximum weight matching which can be solved in polynomial time, and for $k = 3$ the problem becomes APX-hard [17]. After a long line of work [18] [2] [11] [8], the best-known approximation ratio for this problem is $k + 1/2 + \epsilon$ obtained using local search techniques [3]. An improved bound of $k^2 + \epsilon$ is also known [18] for the unweighted case, i.e., the weight vector $w = 1$. It is also known that the natural LP relaxation for this problem has integrality gap at least $k - 1 + 1/k$, and in particular this holds for the projective plane instance of order $k - 1$. Hazan et al. [17] showed that $k$-set packing is $\Omega(k/\log k)$-hard to approximate.

Another special case of $k$-CS-PIP is the independent set problem in graphs with maximum degree at most $k$. This is equivalent to $k$-CS-PIP where the constraint matrix $S$ is 0-1, capacity $c$ is all ones, and each row is 2-sparse. This problem has an $O(k \log \log k/\log k)$-approximation [16], and is $\Omega(k/\log^2 k)$-hard to approximate [3], assuming the Unique Games Conjecture [19].

Shepherd and Vetta [26] studied the demand matching problem on graphs, which is $k$-CS-PIP with
A celebrated result of Beck-Fiala [7] shows that the capacity violation is at most $O(k)$, and showed that the natural LP relaxation for this problem has integrality gap at least $3$. They also showed the demand matching problem to be APX-hard even on bipartite graphs. For larger values of $k$, problems similar to demand matching have been studied under the name of column-restricted PIPs [20], which arise in the context of routing flow unsplittably (see also [5,6]). In particular, an $1.54k$-approximation algorithm was known [13] where (i) in each column all non-zero entries are equal, and (ii) the maximum entry in $S$ is at most the minimum entry in $c$ (this is also known as the no bottle-neck assumption); later, it was observed in [12] that even without the second of these conditions, one can obtain an $8k$ approximation. The literature on unsplittable flow is quite extensive; we refer the reader to [4,12] and references therein.

For the general $k$-CS-PIP, Pritchard [24] gave a $2^k k^2$-approximation algorithm, which was the first result with approximation ratio depending only on $k$. Pritchard’s algorithm was based on solving an iterated LP relaxation, and then applying a randomized selection procedure. Independently, [13] and [10] showed that this final step could be derandomized, yielding an improved bound of $O(k^2)$. All these previous results crucially use the structural properties of basic feasible solutions of the LP relaxation.

However, as stated above, our result is based on randomized rounding with alterations and does not use properties of basic solutions. This is crucial for the submodular maximization version of the problem, as a solution to the fractional relaxation there does not have these properties.

We remark that randomized rounding with alteration has also been used earlier by Srinivasan [28] in the context of PIPs. However, the focus of this paper is different from ours; in previous work [27], Srinivasan had bounded the integrality gap for PIPs by showing a randomized algorithm that obtained a “good” solution (one that satisfies all constraints) with positive — but perhaps exponentially small — probability. In [23], he proved that rounding followed by alteration leads to an efficient and parallelizable algorithm; the rounding gives a “solution” of good value in which most constraints are satisfied, and one can alter this solution to ensure that all constraints are satisfied. (We note that [27,28] also gave derandomized versions of these algorithms.)

Related issues have been considered in discrepancy theory, where the goal is to round a fractional solution to a $k$-column sparse linear program so that the capacity violation for any constraint is minimized. A celebrated result of Beck-Fiala [7] shows that the capacity violation is at most $O(k)$. A major open question in discrepancy theory is whether the above bound can be improved to $O(\sqrt{k})$, or even $O(k^{1-\epsilon})$ for some $\epsilon > 0$. While the result of [25] uses techniques similar to that of [7], a crucial difference in our problem is that no constraint can be violated at all. In fact, at the end of Section 2 we show another crucial qualitative difference between discrepancy and $k$-CS-PIP.

There is a large body of work on constrained maximization of submodular functions; we only cite the relevant papers here. Calinescu et al. [9] introduced a continuous relaxation (called the multi-linear extension or extension-by-expectation) of submodular functions and subsequently Vondrak [29] gave an elegant $\frac{e}{e-1}$-approximation algorithm for solving this continuous relaxation over any “downward monotone” polytope $P$, as long as there is a polynomial-time algorithm for optimizing linear functions over $P$. We use this continuous relaxation in our algorithm for submodular maximization over $k$-sparse packing constraints. As noted earlier, $k$-sparse packing constraints generalize both $k$-partition matroids and $k$-dimensional knapsacks. Nemhauser et al. [24] gave a $(k+1)$-approximation for submodular maximization over the intersection of $k$ partition matroids; when $k$ is constant, Lee et al. [22] improved this to $k + \epsilon$. Kulik et al. [21] gave an $\left(\frac{1}{e-1} + \epsilon\right)$-approximation for submodular maximization over $k$-dimensional knapsacks when $k$ is constant; if $k$ is part of the input, the best known approximation bound is $O(k)$.

**Problem Definition and Notation:** Before we begin, we formally describe the $k$-CS-PIP problem and fix some notation. Let the items (i.e., columns) be indexed by $i \in [n]$ and the constraints (i.e., rows)
be indexed by \( j \in [m] \). We consider the following packing integer program.

\[
\max \left\{ \sum_{i=1}^{n} w_i x_i \mid \sum_{i=1}^{n} s_{ij} \cdot x_i \leq c_j, \forall j \in [m]; \; x_i \in \{0, 1\}, \; \forall i \in [n] \right\}
\]

We say that item \( i \) participates in constraint \( j \) if \( s_{ij} > 0 \). For each \( i \in [n] \), let \( N(i) := \{ j \in [m] \mid s_{ij} > 0 \} \) be the set of constraints that \( i \) participates in. In a \( k \)-column sparse PIP, we have \( |N(i)| \leq k \) for each \( i \in [n] \). The goal is to find the maximum weight subset of items such that all the constraints are satisfied.

We define the slack as \( B := \min_{i \in [n], j \in [m]} c_j / s_{ij} \). By scaling the constraint matrix, we may assume that \( c_j = 1 \) for all \( j \in [m] \). We also assume that \( s_{ij} \leq 1 \) for each \( i, j \); otherwise, we can just fix \( x_i = 0 \).

Finally, for each constraint \( j \), we let \( P(j) \) denote the set of items participating in this constraint. Note that \(|P(j)|\) can be arbitrarily large.

**Organization:** In Section 2, we begin with the natural LP relaxation, and describe a simple algorithm with approximation ratio \( 8k \). We then present a stronger relaxation, and use it to obtain an \((e + o(1))k\)-approximation. We also present the integrality gap of \( 2k - 1 \) for this strengthened LP, implying that our result is almost tight. In Section 3, we describe the \((\frac{e^2}{4} + o(1))k\)-approximation for \( k \)-column sparse packing problems over a submodular objective. Finally, in Section 4, we deal with the \( k \)-CS-PIP problem when the capacities of all constraints are large relative to the sizes, and obtain significantly better approximation ratios. Again there is a matching integrality gap up to a constant factor.

## 2 Approximation Algorithms for \( k \)-CS-PIP

Before presenting our algorithm, we describe a (seemingly correct) algorithm that does not quite work. Understanding why this easier algorithm fails gives useful insight into the design for the correct algorithm.

### A strawman Algorithm:

Consider the following algorithm. Let \( x \) be some optimum solution to the natural LP relaxation of \( k \)-CS-PIP (i.e. dropping integrality). For each element \( i \in [n] \), select it independently at random with probability \( x_i / (2k) \). Let \( S \) be the chosen set of items. For any constraint \( j \in [m] \), if it is violated, then discard all items \( S \cap P(j) \), i.e. items \( i \in S \) for which \( s_{ij} > 0 \).

Since the probabilities are scaled down by \( 2k \), by Markov’s inequality any constraint \( j \) is violated with probability at most \( 1/(2k) \). Hence, any constraint will discard its items with probability at most \( 1/2k \). By the \( k \)-sparse property, each element can be discarded by at most \( k \) constraints, and hence by union bound over those \( k \) constraints, it is discarded with probability at most \( k \cdot (1/2k) = 1/2 \). Since an element is chosen in \( S \) with probability \( x_i / 2k \), this implies that it lies in the overall solution with probability at least \( x_i / (4k) \), implying that the proposed algorithm is a \( 4k \) approximation.

However, the above argument is not correct. Consider the following example. Suppose there is a single constraint (and so \( k = 1 \)),

\[
Mx_1 + x_2 + x_3 + x_4 + \ldots + x_M \leq M
\]

where \( M \gg 1 \) is a large integer. Clearly, setting \( x_i = 1/2 \) for \( i = 1, \ldots, M \) is a feasible solution. Now consider the execution of the strawman algorithm. Note that whenever item 1 is chosen in \( S \), it is very likely that some item other than 1 will also be chosen (since \( M \gg 1 \) and we pick each item independently with probability \( x_1 / 2k = 1/4 \)); in this case, item 1 would be discarded. Thus the final solution will almost always not contain item 1, violating the claim that it lies in the final solution with probability at least \( x_1 / 4k = 1/8 \).
Thus every constraint is satisfied by the solution set. We will show that this algorithm gives an 8k-approximation.

### 2.1 A Simple Algorithm for k-CS-PIP

In this subsection, we use the obvious LP relaxation for k-CS-PIP (i.e., dropping the integrality condition) and obtain an 8k-approximation algorithm. An item \(i \in [n]\) is called *big* for constraint \(j \in [m]\) iff \(s_{ij} > \frac{1}{2}\); and *small* for constraint \(j\) iff \(0 < s_{ij} \leq \frac{1}{2}\). The algorithm first solves the LP relaxation to obtain an optimal fractional solution \(x\). Then we round to an integral solution as follows. With foresight, set \(\alpha = 4\).

1. Sample each item \(i \in [n]\) independently with probability \(x_i/(\alpha k)\). Let \(S\) denote the set of chosen items. We call an item in \(S\) an \(S\)-item.
2. For each item \(i\), mark \(i\) (for deletion) if, for any constraint \(j \in N(i)\), either:
   - \(S\) contains some other item \(i' \in [n] \setminus \{i\}\) which is big for constraint \(j\) or
   - The sum of sizes of \(S\)-items that are small for \(j\) exceeds 1. (i.e., the capacity).
3. Delete all marked items, and return \(S'\), the set of remaining items.

#### Analysis:
We will show that this algorithm gives an 8k approximation.

**Lemma 2.1.** Solution \(S'\) is feasible with probability one.

**Proof.** Consider any fixed constraint \(j \in [m]\).

1. Suppose there is some \(i' \in S'\) that is big for \(j\). Then the algorithm guarantees that \(i'\) will be the only item in \(S'\) (either small or big) that participates in constraint \(j\): Consider any other \(S\)-item \(i\) participating in \(j\); \(i\) must have been deleted from \(S\) because \(S\) contains another item (namely \(i'\)) that is big for constraint \(j\). Thus, \(i'\) is the only item in \(S'\) participating in constraint \(j\), and so the constraint is trivially satisfied, as all sizes \(\leq 1\).
2. The other case is when all items in \(S'\) are small for \(j\). Let \(i \in S'\) be some item that is small for \(j\) (if there are none such, then constraint \(j\) is trivially satisfied). Since \(i\) was not deleted from \(S\), it must be that the total size of \(S\)-items that are small for \(j\) did not exceed 1. Now, \(S' \subseteq S\), and so this condition is also true for items in \(S'\).

Thus every constraint is satisfied by solution \(S'\) and we obtain the lemma. \(\square\)

We now prove the main theorem.

**Theorem 2.2.** For any item \(i \in [n]\), the probability \(\Pr[i \in S' \mid i \in S] \geq 1 - \frac{2}{\alpha k}\). Equivalently, the probability that item \(i\) is deleted from \(S\) conditional on it being chosen in \(S\) is at most \(2/\alpha\).

**Proof.** For any item \(i\) and constraint \(j \in N(i)\), let \(B_{ij}\) denote the event that \(i\) is marked for deletion from \(S\) because there is some other \(S\)-item that is big for constraint \(j\). Let \(G_j\) denote the event that the total size of \(S\)-items that are small for constraint \(j\) exceeds 1. For any item \(i \in [n]\) and constraint \(j \in N(i)\), we will show that:

\[
\Pr[B_{ij} \mid i \in S] + \Pr[G_j \mid i \in S] \leq \frac{2}{\alpha k}
\]  

(1)
We prove (1) using the following intuition: The total extent to which the LP selects items that are big for any constraint cannot be more than 2 (each big item has size at least 1/2); therefore, $B_{ij}$ is unlikely to occur since we scaled down probabilities by factor $\alpha k$. Ignoring for a moment the conditioning on $i \in S$, event $G_j$ is also unlikely, by Markov’s Inequality. But items are selected for $S$ independently, so if $i$ is big for constraint $j$, then its presence in $S$ does not affect the event $G_j$ at all. If $i$ is small for constraint $j$, then even if $i \in S$, the total size of $S$-items is unlikely to exceed 1.

We now prove (1) formally, using some care to save a factor of 2. Let $B(j)$ denote the set of items that are big for constraint $j$, and $Y_j := \sum_{\ell \in B(j)} x_\ell$. By the LP constraint for $j$, it follows that $Y_j \leq 2$ (since each $\ell \in B(j)$ has size $s_\ell j > \frac{1}{2}$). Now by a union bound,

$$\Pr[B_{ij} \mid i \in S] \leq \frac{1}{\alpha k} \sum_{\ell \in B(j) \setminus \{i\}} x_\ell \leq \frac{Y_j}{\alpha k} \leq \frac{2}{\alpha k}. \quad (2)$$

Now, let $G_{-i}(j)$ denote the set of items that are small for constraint $j$, not counting item $i$, even if it is small. Using the LP constraint $j$, we have:

$$\sum_{\ell \in G_{-i}(j)} s_{\ell j} \cdot x_\ell \leq 1 - \sum_{\ell \in B(j)} s_{\ell j} \cdot x_\ell \leq 1 - \frac{Y_j}{2}. \quad (3)$$

Since each item $i'$ is chosen into $S$ with probability $x_{i'}/(\alpha k)$, inequality (3) implies that the expected total size of $S$-items in $G_{-i}(j)$ is at most $\frac{1}{\alpha k} (1 - Y_j/2)$. By Markov’s inequality, the probability that the total size of these $S$-items exceeds 1/2 is at most $\frac{1}{\alpha k} (1 - Y_j/2)$. Since items are chosen independently and $i \notin G_{-i}(j)$, we obtain this probability even conditioned on $i \in S$.

If $i$ is big for $j$, event $G_j$ occurs only if the total size of $S$-items in $G_{-i}(j)$ exceeds 1. If $i$ is small for $j$, event $G_j$ occurs only if the total size of small $S$-items participating in $j$ exceeds 1; as $s_{ij} \leq 1/2$, the total size of $S$-items in $G_{-i}(j)$ must exceed $1/2$. Thus, whether $i$ is big or small,

$$\Pr[G_j \mid i \in S] \leq \frac{2}{\alpha k} \left(1 - \frac{Y_j}{2}\right) = \frac{2}{\alpha k} - \frac{Y_j}{\alpha k}.$$

Combined with inequality (2) we obtain (1):

$$\Pr[B_{ij} \mid i \in S] + \Pr[G_j \mid i \in S] \leq \frac{Y_j}{\alpha k} + \Pr[G_j \mid i \in S] \leq \frac{Y_j}{\alpha k} + \frac{2}{\alpha k} - \frac{Y_j}{\alpha k} = \frac{2}{\alpha k}.$$

To see that (1) implies the theorem, for any item $i$, simply take the union bound over all $j \in N(i)$. Thus, the probability that $i$ is deleted from $S$ conditional on it being chosen in $S$ is at most $2/\alpha$. Equivalently, $\Pr[i \in S' \mid i \in S] \geq 1 - 2/\alpha$.

We are now ready to prove the final result.

**Theorem 2.3.** There is a randomized $8k$-approximation algorithm for $k$-CS-PIP.

**Proof.** First observe that our algorithm always outputs a feasible solution (Lemma 2). To bound the objective value, recall that $\Pr[i \in S] = \frac{x_i}{\alpha k}$ for all $i \in [n]$. Hence Theorem 2.2 implies that

$$\Pr[i \in S'] \geq \Pr[i \in S] \cdot \Pr[i \in S' \mid i \in S] \geq \frac{x_i}{\alpha k} \cdot \left(1 - \frac{2}{\alpha}\right)$$

for all $i \in [n]$. Finally using linearity of expectation and $\alpha = 4$, we obtain the theorem. \qed
**Remark:** We note that the analysis above only uses Markov’s inequality conditioned on a single item being chosen in set $S$. Thus a pairwise independent distribution suffices to choose the set $S$, and hence the algorithm can be easily derandomized.

**General upper bounds:** The $k$-CS-PIP problem as defined assumes all variables to be 0-1. We note that our result easily extends to the $k$-CS-PIP problem with general upper bounds on variables. Assuming an LP-based $\rho$-approximation algorithm for $k$-CS-PIP with unit upper-bounds, it is straightforward to obtain a $(\rho + 1)$-approximation for $k$-CS-PIP with general upper-bounds. The algorithm first solves the natural LP relaxation to obtain fractional solution $y \in \mathbb{R}_+^n$. Let $z \in \mathbb{Z}_+^n$ and $x \in [0, 1]^n$ be defined as: $z_i = \lfloor y_i \rfloor$ and $x_i = y_i - \lfloor y_i \rfloor$ for all $i \in [n]$; note that $w^T y = w^T z + w^T x$. Clearly $z$ is a feasible integral solution. Moreover $x$ is a feasible fractional solution to the same $k$-CS-PIP instance even with unit upper-bounds. Hence using the rounding algorithm of this subsection, we obtain a feasible integral solution. Hence using the rounding algorithm of this subsection, we obtain a feasible integral solution $z$ and $x$ is a $(\rho + 1)$-approximate solution relative to the natural LP relaxation for $k$-CS-PIP with general upper-bounds.

### 2.2 A Stronger LP, and Improved Approximation

We now present our strengthened LP and the $(ek + o(k))$-approximation algorithm for $k$-CS-PIP.

**Stronger LP relaxation.** Recall that entries are scaled so that all capacities are one. An item $i$ is called big for constraint $j$ iff $s_{ij} > 1/2$. For each constraint $j \in [m]$, let $B(j) = \{i \in [n] \mid s_{ij} > \frac{1}{2}\}$ denote the set of big items. Since no two items that are big for some constraint can be chosen in an integral solution, the inequality $\sum_{i \in B(j)} x_i \leq 1$ is valid for each $j \in [m]$. The strengthened LP relaxation that we consider is as follows.

$$\max \sum_{i=1}^{n} w_i x_i$$

s.t. $\sum_{i=1}^{n} s_{ij} \cdot x_i \leq c_j, \quad \forall j \in [m]$  

$$\sum_{i \in B(j)} x_i \leq 1, \quad \forall j \in [m].$$

$$0 \leq x_i \leq 1, \quad \forall i \in [n].$$

**Algorithm:** The algorithm obtains an optimal solution $x$ to the LP relaxation $\{4, 5, 6, 7\}$, and rounds it to an integral solution $S'$ as follows (parameter $\alpha$ will be set to 1 later).

1. Pick each item $i \in [n]$ independently with probability $x_i/(\alpha k)$. Let $S$ denote the set of chosen items.

2. For any item $i$ and constraint $j \in N(i)$, let $E_{ij}$ denote the event that the items $\{i' \in S \mid s_{i'j} \geq s_{ij}\}$ have total size (in constraint $j$) exceeding one. Mark $i$ for deletion if $E_{ij}$ occurs for any $j \in N(i)$.

3. Return set $S' \subseteq S$ consisting of all items $i \in S$ not marked for deletion.

Note the rule for deleting an item from $S$. In particular, whether item $i$ is deleted from constraint $j$ only depends on items that are at least as large as $i$ in $j$. 

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Analysis: It is clear that \( S' \) is feasible with probability one. The main lemma is the following, where we show that each item appears in \( S' \) with good probability.

**Lemma 2.4.** For every item \( i \in [n] \) and constraint \( j \in N(i) \), we have \( \Pr[|E_{ij}| i \in S] \leq \frac{1}{ak} \left( 1 + \left( \frac{2}{\alpha k} \right)^{1/3} \right) \).

**Proof.** Let \( \ell := (4\alpha k)^{1/3} \). We classify items in relation to constraints as:

- Item \( i \in [n] \) is big for constraint \( j \in [m] \) if \( s_{ij} > \frac{1}{\ell} \).
- Item \( i \in [n] \) is medium for constraint \( j \in [m] \) if \( \frac{1}{\ell} \leq s_{ij} \leq \frac{1}{\ell} \).
- Item \( i \in [n] \) is tiny for constraint \( j \in [m] \) if \( s_{ij} < \frac{1}{\ell} \).

For any constraint \( j \in [m] \), let \( B(j), M(j), T(j) \) respectively denote the set of big, medium, tiny items for \( j \). In the next three claims, we bound \( \Pr[|E_{ij}| i \in S] \) when item \( i \) is big, medium, and tiny respectively.

**Claim 2.5.** For any \( i \in [n] \) and \( j \in [m] \) s.t. item \( i \) is big for constraint \( j \), \( \Pr[|E_{ij}| i \in S] \leq \frac{1}{\alpha k} \).

**Proof.** The event \( E_{ij} \) occurs if some item that is at least as large as \( i \) for constraint \( j \) is chosen in \( S \). Since \( i \) is big in constraint \( j \), \( E_{ij} \) occurs only if some big item other than \( i \) is chosen for \( S \). Now by the union bound, the probability that some item from \( B(j) \setminus \{i\} \) is chosen into \( S \) is:

\[
\Pr \left[ \left( B(j) \setminus \{i\} \right) \cap S \neq \emptyset \mid i \in S \right] \leq \sum_{i' \in B(j) \setminus \{i\}} \frac{x_{i'}}{\alpha k} \leq \frac{1}{\alpha k} \sum_{i' \in B(j)} x_{i'} \leq \frac{1}{\alpha k},
\]

where the last inequality follows from the new LP constraint (6) on big items for \( j \).

**Claim 2.6.** For any \( i \in [n] \), \( j \in [m] \) s.t. item \( i \) is medium for constraint \( j \), \( \Pr[|E_{ij}| i \in S] \leq \frac{1}{\alpha k} \left( 1 + \left( \frac{2}{\alpha k} \right)^{1/3} \right) \).

**Proof.** Here, if event \( E_{ij} \) occurs then it must be that either some big item is chosen or (otherwise) at least two medium items other than \( i \) are chosen, i.e. \( E_{ij} \) implies that either \( S \cap B(j) \neq \emptyset \) or \( |S \cap (M(j) \setminus \{i\})| \geq 2 \). This is because \( i \) together with any one other medium item is not enough to reach the capacity of constraint \( j \). (Since \( i \) is medium, we do not consider tiny items for constraint \( j \) in determining whether \( i \) should be deleted.)

Just as in Claim 2.5, we have that the probability some big item for \( j \) is chosen is at most \( 1/\alpha k \), i.e. \( \Pr[|S \cap B(j)| \neq \emptyset \mid i \in S] \leq \frac{1}{\alpha k} \).

Now consider the probability that \( |S \cap (M(j) \setminus \{i\})| \geq 2 \), conditioned on \( i \in S \). We will show that this probability is much smaller than \( 1/\alpha k \). Since each item \( h \in M(j) \setminus \{i\} \) is chosen independently with probability \( \frac{x_h}{\alpha k} \) (even given \( i \in S \)):

\[
\Pr \left[ |S \cap (M(j) \setminus \{i\})| \geq 2 \mid i \in S \right] \leq \frac{1}{2} \cdot \left( \sum_{h \in M(j)} \frac{x_h}{\alpha k} \right)^2 \leq \frac{\ell^2}{2 \alpha^2 k^2}
\]

where the last inequality follows from the fact that

\[
1 \geq \sum_{h \in M(j)} s_{hj} \cdot x_h \geq \frac{1}{\ell} \sum_{h \in M(j)} x_h
\]

(recall each item in \( M(j) \) has size at least \( \frac{1}{\ell} \)). Combining these two cases, we have the desired upper bound on \( \Pr[|E_{ij}| i \in S] \).
Claim 2.7. For any $i \in [n]$, $j \in [m]$ s.t. item $i$ is tiny for constraint $j$, $\Pr[E_{ij} \mid i \in S] \leq \frac{1}{\alpha k} \left(1 + \frac{2}{\ell} \right)$.

Proof. Since $i$ is tiny, if event $E_{ij}$ occurs then the total size (in constraint $j$) of items $S \setminus \{i\}$ is greater than $1 - \frac{1}{\ell}$. So,

$$\Pr[E_{ij} \mid i \in S] \leq \Pr \left[ \sum_{h \in S \setminus \{i\}} s_{hj} > 1 - \frac{1}{\ell} \right] \leq \frac{1}{\alpha k} \cdot \ell \cdot \frac{\ell}{\ell - 1} - 1 \leq \frac{1}{\alpha k} \left(1 + \frac{2}{\ell}\right)$$

where the first inequality follows from the above observation and the fact that $S \setminus \{i\}$ is independent of the event $i \in S$, the second is Markov’s inequality, and the last uses $\ell \geq 2$. □

Thus, for any item $i$ and constraint $j \in N(i)$, $\Pr[E_{ij} \mid i \in S] \leq \frac{1}{\alpha k} \max\{(1 + \frac{2}{\ell}), (1 + \frac{\ell^2}{2\alpha k})\}$. From the choice of $\ell = (4\alpha k)^{1/3}$, which makes the probability in Claims 2.6 and 2.7 equal, we obtain the lemma. □

We now prove the main result of this section

**Theorem 2.8.** For each $i \in [n]$, probability $\Pr[i \in S' \mid i \in S] \geq \left(1 - \frac{1}{\alpha k} \left(1 + \left(\frac{2}{\alpha k}\right)^{1/3}\right)\right)^k$.

Proof. For any item $i$ and constraint $j \in N(i)$, the conditional event $(\neg E_{ij} \mid i \in S)$ is a decreasing function over the choice of items in set $[n] \setminus \{i\}$. Thus, by the FKG inequality [1], for any fixed item $i \in [n]$, the probability that no event $(E_{ij} \mid i \in S)$ occurs is:

$$\Pr \left[ \bigwedge_{j \in N(i)} \neg E_{ij} \mid i \in S \right] \geq \prod_{j \in N(i)} \Pr[\neg E_{ij} \mid i \in S]$$

From Lemma 2.4, $\Pr[\neg E_{ij} \mid i \in S] \geq 1 - \frac{1}{\alpha k} \left(1 + \left(\frac{2}{\alpha k}\right)^{1/3}\right)$. As each item is in at most $k$ constraints, we obtain the theorem. □

Now, by setting $\alpha = 1$\footnote{Note that this is optimal only asymptotically; in the case of $k = 2$, for instance, it is better to choose $\alpha \approx 2.8$.} we have $\Pr[i \in S] = 1/k$, and $\Pr[i \in S' \mid i \in S] \geq \frac{1}{e + o(1)}$, which immediately implies:

**Theorem 2.9.** There is a randomized $(\epsilon k + o(k))$-approximation algorithm for $k$-CS-PIP.

Remark: We note that this algorithm can be derandomized using conditional expectation and pessimistic estimators, since we can compute exactly estimates of the relevant probabilities. Also, using ideas from [28] the algorithm can be implemented in RNC. We defer details to the full version.

**Integrality Gap of LP [21,7].** Recall that the LP relaxation for the $k$-set packing problem has an integrality gap of $k - 1 + 1/k$, as shown by the instance given by the projective plane of order $k - 1$. If we have the same size-matrix and set each capacity to $2 - \epsilon$, this directly implies an integrality gap arbitrarily close to $2(k - 1 + 1/k)$ for the (weak) LP relaxation for $k$-CS-PIP. This is because the LP can set each $x_i = (2 - \epsilon)/k$ hence obtaining a profit of $(2 - \epsilon)(k - 1 + 1/k)$, while the integral solution can only choose one item. However, for our stronger LP relaxation [4,7] used in this section, this example does not work and the projective plane instance only implies a gap of $k - 1 + 1/k$ (note that here each item is big in every constraint that it appears in).

However, using a different instance of $k$-CS-PIP, we show that even the stronger LP relaxation has an integrality gap at least $2k - 1$. Consider the instance on $n = m = 2k - 1$ items and constraints...
defined as follows. We view the indices \([n] = \{0, 1, \ldots, n-1\}\) as integers modulo \(n\). The weights \(w_i = 1\) for all \(i \in [n]\). The sizes are:

\[
s_{ij} := \begin{cases} 
1 & \text{if } i = j \\
\epsilon & \text{if } j \in \{i+1, \ldots, i+k-1 \text{ (mod } n)\} \\
0 & \text{otherwise}
\end{cases}, \quad \forall i, j \in [n],
\]

where \(\epsilon > 0\) is arbitrarily small, in particular \(\epsilon \ll \frac{1}{nk}\).

Observe that setting \(x_i = 1 - k\epsilon\) for all \(i \in [n]\) is a feasible fractional solution to the strengthened LP (4-7); each constraint has only one big item and so the new constraint (6) is satisfied. Thus the optimal LP value is at least \((1 - k\epsilon) \cdot n \approx n = 2k - 1\).

On the other hand, we claim that the optimal integral solution can only choose one item and hence has value 1. For the sake of contradiction, suppose that it chooses two items \(i, h \in [n]\). Then there is some constraint \(j\) (either \(j = i\) or \(j = h\)) that implies either \(x_i + \epsilon \cdot x_h \leq 1\) or \(x_h + \epsilon \cdot x_i \leq 1\); in either case constraint \(j\) would be violated.

Thus the integrality gap of the LP we consider is at least \(2k - 1\), for every \(k \geq 1\).

**Bad example for possible generalization.** A natural extension of the \(k\)-CS-PIP result is to consider PIPs where the \(\ell_1\)-norm of each column is upper-bounded by \(k\) (when capacities are all-ones). We observe that unlike \(k\)-CS-PIP, the LP relaxation for this generalization has an \(\Omega(n)\) integrality gap. The example has \(m = n\); sizes \(s_{ii} = 1\) for all \(i \in [n]\), and \(s_{ij} = \frac{1}{n}\) for all \(i \neq j\); and all weights one. The \(\ell_1\)-norm of each column is at most 2. Clearly, the optimal integral solution has value one. On the other hand, picking each column to the extent of \(1/2\) is a feasible LP solution of value \(n/2\).

This integrality gap is in sharp contrast to the results on discrepancy of sparse matrices, where the classic Beck-Fiala bound of \(O(k)\) applies also to matrices with entries in \([-1, 1]\), just as well as \(\{-1, 0, 1\}\) entries; here \(k\) denotes an upper-bound on the \(\ell_1\)-norm of the columns.

## 3 Submodular Objective Functions

We now consider the more general case when the objective we seek to maximize is an arbitrary monotone submodular function \(f : 2^{[n]} \rightarrow \mathbb{R}_+\). The problem we consider is:

\[
\max \left\{ f(T) \mid \sum_{i \in T} s_{ij} \leq c_j, \forall j \in [m]; \ T \subseteq [n] \right\} \tag{8}
\]

As is standard when dealing with submodular functions, we only assume value-oracle access to the function: i.e. the algorithm can query any subset \(T \subseteq [n]\), and it obtains the function value \(f(T)\) in constant time. Again, we let \(k\) denote the column-sparness of the underlying constraint matrix. Observe that this problem is a common generalization of maximizing submodular functions over: \(k\) partition matroids, and \(k\) knapsack constraints. In this section we obtain an \(O(k)\)-approximation algorithm for Problem (8). The algorithm is similar to that for \(k\)-CS-PIP (where the objective was additive), and involves the following two steps.

1. We first solve (approximately) a suitable continuous relaxation of (8). This step follows directly from the algorithm of Vondrák [29].
2. Then, using the fractional solution, we perform the randomized rounding with alteration described in Section [2]. Although the algorithm is the same as for additive functions, the analysis requires considerably more work. In the process, we also establish a new property of submodular functions that generalizes fractional subadditivity [15].
Solving the Continuous Relaxation. The extension-by-expectation (also called the multi-linear extension) of a submodular function $f$ is a continuous function $F : [0, 1]^n \rightarrow \mathbb{R}_+$ defined as follows:

$$F(x) := \sum_{T \subseteq [n]} \Pi_{i \in T} x_i \cdot \Pi_{j \notin T} (1 - x_j) \cdot f(T)$$

Note that $F(x) = f(x)$ for $x \in \{0, 1\}^n$ and hence $F$ is an extension of $f$. Even though $F$ is a non-linear function, using the continuous greedy algorithm from Vondrák [29], we can obtain a $(1 - \frac{1}{e})$-approximation algorithm to the following fractional relaxation of (8).

$$\max \left\{ F(x) \mid \sum_{i=1}^n s_{ij} \cdot x_i \leq c_j, \forall j \in [m]; 0 \leq x_i \leq 1, \forall i \in [n] \right\} \tag{9}$$

In order to apply the algorithm from [29], one needs to solve in polynomial time the problem of maximizing a linear objective over the constraints \{\(\sum_{i=1}^n s_{ij} \cdot x_i \leq c_j, \forall j \in [m]; 0 \leq x_i \leq 1, \forall i \in [n]\). This is indeed possible since it is a linear program on $n$ variables and $m$ constraints.

The Rounding Algorithm. The rounding algorithm is identical to that for $k$-CS-PIP. Let $x$ denote any feasible solution to Problem (9). We apply the rounding algorithm for the additive case (from the previous section), to first obtain a (possibly infeasible) solution $S \subseteq [n]$ and then feasible integral solution $S' \subseteq [n]$. In the rest of this section, we prove the performance guarantee of this algorithm.

Fractional Subadditivity. The following is a useful lemma (see Feige [15]) showing that submodular functions are also fractionally subadditive.

Lemma 3.1 ([15]). Let $U$ be a set of elements and \{\(A_i \subseteq U\)\} be a collection of subsets with non-negative weights \{\(\lambda_i\)\} such that \(\sum_{i \in A_i} \lambda_i \geq 1\) for all elements $i \in U$. Then, for any submodular function $f$, we have $f(U) \leq \sum_{i} \lambda_i f(A_i)$.

The above result can be used to show that (the infeasible solution) $S$ has good profit in expectation.

Lemma 3.2. For any $x \in [0, 1]^n$ and $0 \leq p \leq 1$, let set $S$ be constructed by selecting each item $i \in [n]$ independently with probability $p \cdot x_i$. Then, $E[f(S)] \geq p F(x)$. In particular, this implies that our rounding algorithm that forms set $S$ by independently selecting each element $i \in [n]$ with probability $x_i/(\alpha k)$ satisfies $E[f(S)] \geq \frac{1}{\alpha k} F(x)$.

Proof. Consider the following equivalent procedure for constructing $S$: First, construct $S_0$ by selecting each item $i$ with probability $x_i$. Then construct $S$ by retaining each element in $S_0$ independently with probability $p$.

By definition $E[f(S_0)] = F(x)$. For any fixed set $T \subseteq [n]$, consider the outcomes for set $S$ conditioned on $S_0 = T$; the set $S \subseteq S_0$ is a random subset such that $\Pr[i \in S \mid S_0 = T] = p$ for all $i \in T$. Thus by Lemma 3.1 we have $E[f(S) \mid S_0 = T] \geq p \cdot F(T)$. Hence:

$$E[f(S)] = \sum_{T \subseteq [n]} \Pr[S_0 = T] \cdot E[f(S) \mid S_0 = T] \geq \sum_{T \subseteq [n]} \Pr[S_0 = T] \cdot p F(T) = p E[f(S_0)] = p \cdot F(x).$$

Thus we obtain the lemma. \[\square\]

However, the analysis approach in Theorem 2.8 does not work. The problem is that even though $S$ (which is chosen by random sampling) has good expected profit, i.e. $E[f(S)] = \Omega(\frac{1}{k}) F(x)$ (from Lemma 3.2 above), it may happen that the alteration step used to obtain $S'$ from $S$ may end up throwing away essentially all the profit. This was not an issue for linear objective functions since our alteration
procedure guarantees that $\Pr[i \in S'|i \in S] = \Omega(1)$ for each $i \in [n]$, and if $f$ is linear, this implies $E[f(S)] = \Omega(1) E[f(S')]$. However, this property is not enough for general monotone submodular functions. Consider the following:

Example: Let set $S \subseteq [n]$ be drawn from the following distribution:

- With probability $1/2n$, $S = [n]$.
- For each $i \in [n]$, $S = \{i\}$ with probability $1/2n$.
- With probability $1/2 - 1/2n$, $S = \emptyset$.

Now define $S' = S$ if $S = [n]$, and $S' = \emptyset$ otherwise. Note that for each $i \in [n]$, we have $\Pr[i \in S' | i \in S] = 1/2 = \Omega(1)$. However, consider the profit with respect to the “coverage” submodular function $f$, where $f(T) = 1$ if $T \neq \emptyset$ and is 0 otherwise. We have $E[f(S)] = 1/2 + 1/2n$, but $E[f(S')]$ is only $1/2n \ll E[f(S)]$.

Remark: Note that if $S'$ itself was chosen randomly from $S$ such that $\Pr[i \in S'|S = T] = \Omega(1)$ for every $T \subseteq [n]$ and $i \in T$, then we would be done by Lemma 3.1. Unfortunately, this is too much to hope for. In our rounding procedure, for any particular choice of $S$, set $S'$ is a fixed subset of $S$; and there could be (bad) sets $S$, where after the alteration step we end up with sets $S'$ such that $|S'| \ll |S|$.

However, it turns out that we can use the following two additional properties beyond just marginal probabilities to argue that $S'$ has reasonable profit. First, the sets $S$ constructed by our algorithm are drawn from a product distribution on the items; in contrast, the example above does not have this property. Second, our alteration procedure has the following ‘monotonicity’ property: Suppose $T_1 \subseteq T_2 \subseteq [n]$, and $i \in T_2$ when $S = T_2$. Then we are guaranteed that $i \in S'$ when $S = T_1$. (That is, if $S$ contains additional items, it is more likely that $i$ will be discarded by some constraint it participates in.) The above example does not satisfy this property either. That these properties suffice is proved in Corollary 3.4. Roughly speaking, the intuition is that, since $f$ is submodular, the marginal contribution of item $i$ to $S$ is largest when $S$ is “small”, and this is also the case when $i$ is most likely to be retained for $S'$. That is, for every $i \in [n]$, both $\Pr[i \in S'|i \in S]$ and the marginal contribution of $i$ to $f(S)$ are decreasing functions of $S$. To show Corollary 3.4 we need the following generalization of Feige’s Subadditivity Lemma.

Theorem 3.3. Let $[n]$ denote a groundset, $x \in [0,1]^n$, and for each $B \subseteq [n]$ define $p(B) = \Pi_{i \in B} x_i \cdot \Pi_{j \not\in B} (1 - x_j)$. Associated with each $B \subseteq [n]$, there is an arbitrary distribution over subsets of $B$, where each set $A \subseteq B$ has probability $q_B(A)$; so $\sum_{A \subseteq B} q_B(A) = 1$ for all $B \subseteq [n]$. That is, we choose $B$ from a product distribution, and then retain a subset $A$ of $B$ by applying a randomized alteration.

Suppose that the system satisfies the following conditions.

Marginal Property:

$$\forall i \in [n], \quad \sum_{B \subseteq [n]} p(B) \sum_{A \subseteq B: i \in A} q_B(A) \geq \beta \cdot \sum_{B \subseteq [n]: i \in B} p(B). \quad (10)$$

Monotonicity: For any two subsets $B \subseteq B' \subseteq [n]$ we have,

$$\forall i \in B, \quad \sum_{A \subseteq B: i \in A} q_B(A) \geq \sum_{A' \subseteq B': i \in A'} q_{B'}(A') \quad (11)$$

Then, for any monotone submodular function $f$, 

$$\sum_{B \subseteq [n]} p(B) \sum_{A \subseteq B} q_B(A) \cdot f(A) \geq \beta \cdot \sum_{B \subseteq [n]} p(B) \cdot f(B). \quad (12)$$


Clearly both inequality [1] on the right hand side of (12), which will imply the result by (13). We now focus on proving (15). To see that this suffices, observe that upon adding (14) to (15) we obtain that the right hand side of (13) and

\[ q \]

This inequality actually follows by induction, by applying (12) to suitably constructed distributions and

\[ p \]

Moreover function

\[ f \]

Proof. \[ (A) \sum_{B \subseteq [n]} p(B) \sum_{A \subseteq B, n \in A} q_B(A) f(A) = \sum_{B \subseteq [n]} p(B) \sum_{A \subseteq B, n \in A} q_B(A) (f(B) - f(B \setminus \{n\})) + \sum_{B \subseteq [n]} p(B) \sum_{B \subseteq A, n \in A} q_B(A) f(A) \]

(13)

Next, we need the following inequality,

\[ \sum_{B \subseteq [n]} \sum_{A \subseteq B} p(B) \cdot q_B(A) \cdot f(A \setminus \{n\}) \geq \beta \sum_{B \subseteq [n]} p(B) \cdot f(B \setminus \{n\}) \] (14)

This inequality actually follows by induction, by applying (12) to suitably constructed distributions and probability on subsets of \([n - 1]\). We first complete the proof of the theorem using (14), and prove (14) later.

We now claim that it suffices to show the following.

\[ \sum_{B \subseteq [n]} p(B) \sum_{A \subseteq B, n \in A} q_B(A) (f(B) - f(B \setminus \{n\})) \geq \beta \sum_{B \subseteq [n]} p(B) (f(B) - f(B \setminus \{n\})) \] (15)

To see that this suffices, observe that upon adding (14) to (15) we obtain that the right hand side of (13) is at least the right hand side of (12), which will imply the result by (13). We now focus on proving (15).

Firstly, note that if \( x_n = 0 \) then (15) is trivially true. In the following we assume \( x_n > 0 \).

For any set \( Y \subseteq [n - 1] \), define the following two functions:

\[ g(Y) := f(Y \cup \{n\}) - f(Y), \quad \text{and} \quad h(Y) := \sum_{A \subseteq Y} q_{Y \cup \{n\}}(A \cup \{n\}). \]

Clearly both \( g \) and \( h \) are non-negative. Note that \( g \) is a decreasing function due to submodularity of \( f \). Moreover function \( h \) is also decreasing: for any \( Y \subseteq Y' \subseteq [n - 1] \),

\[ h(Y) = \sum_{A \subseteq Y} q_{Y \cup \{n\}}(A \cup \{n\}) \geq \sum_{A' \subseteq Y'} q_{Y' \cup \{n\}}(A' \cup \{n\}) = h(Y'), \]

where the inequality is by the monotonicity condition with \( i = n, B = Y \cup \{n\} \) and \( B' = Y' \cup \{n\} \).

Consider the product probability space on \( 2^{[n - 1]} \) with marginal probabilities given by \( \{x_i\}_{i=1}^{n-1}. \) For any \( Y \subseteq [n - 1] \), let \( p'(Y) = \Pi_{i \in Y} x_i \cdot \Pi_{j \in [n-1] \setminus Y} (1 - x_j) \) denote its probability. Applying the FKG inequality [11] on the decreasing functions \( g \) and \( h \), it follows that

\[ \sum_{Y \subseteq [n-1]} p'(Y) \cdot g(Y) \cdot h(Y) \geq \left( \sum_{Y \subseteq [n-1]} p'(Y) g(Y) \right) \cdot \left( \sum_{Y \subseteq [n-1]} p'(Y) h(Y) \right). \] (16)

Observe that

\[ \sum_{Y \subseteq [n-1]} p'(Y) \cdot h(Y) = \sum_{Y \subseteq [n-1]} \frac{p(Y \cup \{n\})}{x_n} \sum_{A \subseteq Y} q_{Y \cup \{n\}}(A \cup \{n\}) = \frac{1}{x_n} \sum_{B \subseteq [n]} p(B) \sum_{A \subseteq B, n \in A} q_B(A), \]

13
which by the Marginal property with \( i = n \) is at least \( \frac{1}{x_n} \cdot \beta x_n = \beta \). Combining this with (16),

\[
\sum_{Y \subseteq [n-1]} p'(Y) \cdot g(Y) \cdot h(Y) \geq \beta \sum_{Y \subseteq [n-1]} p'(Y) \cdot g(Y).
\]

(17)

Using the definitions of \( g \) and \( h \) (and multiplying both sides by \( x_n \)), we obtain (15).

**Proof of Inequality (14).** We show that this follows by applying Theorem (3.3.3) suitably on the ground-set \([n - 1]\), with marginal probabilities \( \{x_i\}_{i=1}^{n-1} \). For each \( C \subseteq [n - 1] \) define \( p'(C) = \Pi_i \in C x_i \cdot \Pi_{j \in [n-1] \setminus C} (1 - x_j) \). Associated with each \( C \subseteq [n - 1] \), let us define the distribution \( \{q_C(A) : A \subseteq C\} \) over subsets of \( C \) as follows:

\[
q_C' (A) := x_n \cdot q_{C \cup \{n\}} (A) + x_n \cdot q_{C \cup \{n\}} (A \cup \{n\}) + (1 - x_n) \cdot q_C (A), \quad \text{for all } A \subseteq C \subseteq [n-1].
\]

Note that \( \sum_{A : A \subseteq C \subseteq [n-1]} q_C' (A) = 1 \) for every \( C \subseteq [n - 1] \), since

\[
\sum_{A : A \subseteq C \subseteq [n-1]} q_C' (A) = x_n \cdot \sum_{A : A \subseteq C \subseteq [n-1]} q_{C \cup \{n\}} (A) + (1-x_n) \cdot \sum_{A : A \subseteq C \subseteq [n-1]} q_C (A) = x_n + (1-x_n) = 1,
\]

using the fact that \( \sum_{A : A \subseteq C \subseteq [n-1]} q_{C \cup \{n\}} (A) = \sum_{A : A \subseteq C} q_C (A) = 1. \)

To see that the **Marginal property** holds, for any \( i \in [n - 1] \), we have:

\[
\sum_{C \subseteq [n-1]} p'(C) \sum_{A : A \subseteq C : i \in A} q_C (A) = \sum_{C \subseteq [n-1]} p'(C) \sum_{A : A \subseteq C \subseteq [n] : i \in A} q_C (A) = \sum_{A : A \subseteq C : i \in A} \left( x_n \cdot q_{C \cup \{n\}} (A) + x_n \cdot q_{C \cup \{n\}} (A \cup \{n\}) + (1-x_n) \cdot q_C (A) \right) = x_n \cdot \sum_{A : A \subseteq C : i \in A} q_C (A) + (1-x_n) \sum_{A : A \subseteq C : i \in A} q_C (A) = \sum_{B \subseteq [n]} p(B) \sum_{A : A \subseteq B : i \in A} q(A) \geq \beta \sum_{B \subseteq [n]} p(B) = \beta \sum_{C \subseteq [n-1]} p'(C).
\]

Above, the inequality is by the Marginal property for the original instance on \([n] \).

To show the **Monotonicity property** for any subsets \( C \subseteq C' \subseteq [n - 1] \), observe that:

\[
\sum_{A : A \subseteq C : i \in A} q_C' (A) = x_n \sum_{A : A \subseteq C : i \in A} \left( q_{C \cup \{n\}} (A) + q_{C \cup \{n\}} (A \cup \{n\}) \right) + (1-x_n) \sum_{A : A \subseteq C : i \in A} q_C (A) = x_n \sum_{A' : A' \subseteq C \subseteq [n] : i \in A'} q_{C \cup \{n\}} (A') + (1-x_n) \sum_{A : A \subseteq C : i \in A} q_C (A) \geq x_n \sum_{A' : A' \subseteq C : i \in A'} q_{C' \cup \{n\}} (A') + (1-x_n) \sum_{A : A \subseteq C : i \in A} q_{C'} (A) = \sum_{A : A \subseteq C : i \in A} q_{C'} (A).
\]

Again, the inequality is by the monotonicity property on the original instance (on groundset \([n]\)) for the pairs \( C \subseteq C' \) and \( C \cup \{n\} \subseteq C' \cup \{n\} \).
Finally, we can express the left-hand-side of inequality (14) as:

$$\sum_{B \subseteq [n]} p(B) \sum_{A \subseteq B} q_B(A) f(A \setminus \{n\})$$

$$= \sum_{C \subseteq [n-1]} \left( p(C \cup \{n\}) \sum_{A \subseteq C} q_{C \cup \{n\}}(A) f(A \setminus \{n\}) + p(C) \sum_{A \subseteq C} q_C(A) f(A) \right)$$

$$= \sum_{C \subseteq [n-1]} \left( x_n \cdot p'(C) \sum_{A' \subseteq C} \left( q_{C \cup \{n\}}(A') + q_{C \cup \{n\}}(A' \cup \{n\}) \right) f(A') + (1-x_n)p'(C) \sum_{A' \subseteq C} q_{C}(A') f(A') \right)$$

$$= \sum_{C \subseteq [n-1]} p'(C) \sum_{A' \subseteq C} q_C(A') f(A')$$

$$\geq \beta \sum_{C \subseteq [n-1]} p'(C) f(C) = \beta \sum_{C \subseteq [n-1]} (p(C) + p(C \cup \{n\})) f(C) = \beta \sum_{B \subseteq [n]} p(B) f(B \setminus \{n\}),$$

which equals the right-hand-side of (14). Above, the inequality is by the induction hypothesis on the instance on \([n-1]\). This completes the proof of Inequality (14), and Theorem 3.3. □

**Remark:** It is easy to see that Theorem 3.3 generalizes Lemma 3.1. Let \(x_i = 1\) for each \(i \in [n]\). The distribution \(\\{A_t, \frac{1}{\lambda_i}p(A_t)\}\) is associated with \(B = [n]\). For all other \(B' \subseteq [n]\), its distribution has \(q_{B'}(B') = 1\). The monotonicity condition is trivially satisfied. By the assumption in Lemma 3.1, the Marginal property holds with \(\beta = 1/\sum_t \lambda_t\). Thus Theorem 3.3 applies and yields the conclusion in Lemma 3.1.

**Corollary 3.4.** Let \(S\) be a random set drawn from a product distribution on \([n]\). Let \(S'\) be another random set where for each choice of \(S\), set \(S'\) is an arbitrary subset of \(S\). Suppose that for each \(i \in [n]\) the following hold:

- \(\Pr_S[i \in S' \mid i \in S] \geq \beta, \text{ and} \)
- For all \(T_1 \subseteq T_2\) with \(T_1 \ni i\), if \(i \in S'\) when \(S = T_2\) then \(i \in S'\) when \(S = T_1\).

Then \(E[f(S')] \geq \beta E[f(S)]\).

**Proof.** This is immediate from Theorem 3.3, we simply associate the single set distribution (i.e. \(A = S'\)) for each choice \(B\) of \(S\). The two conditions stated above on the construction of \(S'\) imply the Marginal and Monotonicity properties respectively; and inequality (12) translates to \(E[f(S')] \geq \beta E[f(S)]\). □

We are now ready to prove the performance guarantee of our algorithm. Observe that our rounding algorithm satisfies the hypothesis of Corollary 3.4 with \(\beta = \frac{1}{e+o(1)}\), when parameter \(\alpha = 1\). Moreover, by Lemma 3.2, it follows that \(E[f(S)] \geq F(x)/(\alpha k)\). Thus,

$$E[f(S')] \geq \frac{1}{e + o(1)} E[f(S)] \geq \frac{1}{ek + o(k)} \cdot F(x),$$

Combined with the fact that \(x\) is an \(\frac{e^2}{e-1}\)-approximate solution to the continuous relaxation (9), we have proved our main result.

**Theorem 3.5.** There is a randomized algorithm for maximizing any monotone submodular function over \(k\)-column sparse packing constraints achieving approximation ratio \(\frac{e^2}{e-1}k + o(k)\).
4  \(k\)-CS-PIP Algorithm for general \(B\)

In this section, we obtain substantially better approximation guarantees for \(k\)-CS-PIP when the capacities are large relative to the sizes. A useful parameter that measures this is the following (see eg. [27]).

\[
B := \min_{i \in [n], j \in [m]} \frac{c_j}{s_{ij}}.
\]

We consider the \(k\)-CS-PIP problem as a function of both \(k\) and \(B\), and obtain an improved approximation ratio of \(O(k^{1/[B]}); \) we also give a matching integrality gap (for every \(k\) and \(B \geq 1\)) for the natural LP relaxation. Previously, Pritchard [25] studied \(k\)-CS-PIP when \(B > k\) and obtained a ratio of \((1 + k/B)/(1 - k/B); \) in contrast, we obtain improved approximation ratios even when \(B = 2\).

**Theorem 4.1.** There is a \(\left(4e \cdot \left((e + o(1)) \left[ B \right] k^{1/[B]} \right)\right)\)-approximation algorithm for \(k\)-CS-PIP, and a \(\left(\frac{4e^2}{(e + o(1))} \left[ B \right] k^{1/[B]} \right)\)-approximation algorithm for maximizing any monotone submodular function over \(k\)-column sparse packing constraints.

It will be convenient to assume that the entries are scaled so that for every constraint \(j \in [m],\)

\[
\max_{i \in P(j)} s_{ij} = 1. \text{ So } B = \min_{j \in [m]} c_j \geq 1.
\]

Set \(\alpha := 4e \cdot \left(\left[ B \right] k^{1/[B]} \right). \) The algorithm first solves the natural LP relaxation for \(k\)-CS-PIP to obtain fractional solution \(x. \) Then it proceeds as follows.

1. Sample each item \(i \in [n]\) independently with probability \(x_i/\alpha. \)

   Let \(S\) denote the set of chosen items.

2. Define new sizes as follows: for every item \(i \) and constraint \(j \in N(i),\) round up \(s_{ij}\) to \(t_{ij} \in \{2^{-a} | a \in \mathbb{Z}_+\},\) the next larger power of \(2.\)

3. For any item \(i\) and constraint \(j \in N(i),\) let \(E_{ij}\) denote the event that the items \(\{i' \in S | t_{ij} \geq t_{ij}\}\) have total \(t\)-size (in constraint \(j\)) exceeding one. Mark \(i\) for deletion if \(E_{ij}\) occurs for any \(j \in N(i).\)

4. Return set \(S' \subseteq S\) consisting of all items \(i \in S\) not marked for deletion.

Note the differences from the algorithm in Section 2: the scaling factor for randomized rounding is smaller, and the alteration step is more intricate (it uses slightly modified sizes). It is clear that \(S'\) is a feasible solution with probability one, since the original \(s\)-sizes are at most the new \(t\)-sizes.

The approximation guarantee is proved using the following theorem.

**Theorem 4.2.** For each \(i \in [n],\) probability \(\Pr[i \in S' | i \in S] \geq \left(1 - \frac{1}{k[B]}\right)^k.\)

**Proof.** Fix any \(i \in [n]\) and \(j \in N(i).\) Recall that \(E_{ij}\) is the event that items \(\{i' \in S | t_{ij} \geq t_{ij}\}\) have total \(t\)-size (in constraint \(j\)) greater than \(c_j.\)

We first bound \(\Pr[E_{ij} | i \in S].\) Let \(t_{ij} = 2^{-\ell}, \) where \(\ell \in \mathbb{N}.\) Observe that all the \(t\)-sizes that are at least \(2^{-\ell}\) are actually integral multiples of \(2^{-\ell}\) (since they are all powers of two). Let \(I_{ij} = \{i' \in [n] | t_{ij} \geq t_{ij}\} \setminus \{i\},\) and \(Y_{ij} := \sum_{i' \in I_{ij}} t_{ij} \cdot 1_{i' \in S}\) where \(1_{i' \in S}\) are indicator random variables. The previous observation implies that \(Y_{ij}\) is always an integral multiple of \(2^{-\ell}.\) Note that

\[
\Pr[E_{ij} | i \in S] = \Pr \left[Y_{ij} > c_j - 2^{-\ell} | i \in S\right] \leq \Pr \left[Y_{ij} > |c_j| - 2^{-\ell} | i \in S\right] = \Pr[Y_{ij} \geq |c_j| | i \in S],
\]
where the last equality uses the fact that $Y_{ij}$ is always a multiple of $2^{-\ell}$. Since each item is included into $S$ independently, we also have $\Pr[Y_{ij} \geq [c_j] \mid i \in S] = \Pr[Y_{ij} \geq (c_j)]$. Now $Y_{ij}$ is the sum of independent $[0, 1]$ random variables with mean:

$$E[Y_{ij}] = \sum_{i' \in C_{ij}} t_{i'i} \cdot \Pr[i' \in S] \leq \sum_{i' = 1}^{n} t_{i'i} \cdot \frac{x_{i'}}{\alpha} \leq \frac{2}{\alpha} \sum_{i' = 1}^{n} s_{i'i} \cdot x_{i'} \leq \frac{2}{\alpha} c_j.$$ 

Choose $\delta$ such that $(\delta + 1) \cdot E[Y_{ij}] = [c_j]$, i.e. (using $c_j \geq 1$),

$$\delta + 1 = \frac{[c_j]}{E[Y_{ij}]} \geq \frac{\alpha [c_j]}{2 \cdot c_j} \geq \frac{\alpha}{4}.$$

Now using Chernoff Bound \cite{23}, we have:

$$\Pr[Y_{ij} \geq [c_j]] = \Pr[Y_{ij} \geq (1 + \delta) \cdot E[Y_{ij}]] \leq \left(\frac{e}{\delta + 1}\right)^{[c_j]} \leq \left(\frac{4 e}{\alpha}\right)^{[c_j]} \leq \left(\frac{4 e}{\alpha}\right)^{|B|}.$$

The last inequality uses the fact that $c_j \geq B$. Finally, by the choice of $\alpha = 4e \cdot (|B| k)^{1/|B|}$,

$$\Pr[E_{ij} \mid i \in S] \leq \Pr[Y_{ij} \geq [c_j]] \leq \frac{1}{k |B|}.$$

As in the proof of Theorem \ref{2.3} for any fixed item $i \in [n]$, the conditional events $\{E_{ij} \mid i \in S\}_{j \in N(i)}$ are positively correlated. Thus using \cite{13} and the FKG inequality \cite{11},

$$\Pr[i \in S' \mid i \in S] = \Pr\left[\bigwedge_{j \in N(i)} \neg E_{ij} \mid i \in S\right] \geq \prod_{j \in N(i)} \Pr[\neg E_{ij} \mid i \in S] \geq \left(1 - \frac{1}{k |B|}\right)^k.$$

This completes the proof of the theorem. \hfill \square

As a function of $k$, we obtain that $\Pr[i \in S' \mid i \in S] \geq (e + o(1))^{-1/|B|}$. Since $\Pr[i \in S] = x_i/\alpha$, we obtain the first part of Theorem \ref{4.1}

This algorithm can also be used for maximizing monotone submodular functions over such packing constraints (parameterized by $k$ and $B$). Again we would first (approximately) solve the continuous relaxation using \cite{29}, and perform the above randomized rounding and alteration. Corollary \ref{4.4} can be used with Theorem \ref{4.2} to obtain a $\left(\frac{4e^2}{e-1} \cdot ((e + o(1)) |B| k)^{1/|B|}\right)$-approximation algorithm.

This completes the proof of Theorem \ref{4.3}.

### 4.1 Integrality Gap for General $B$

We show that the natural LP relaxation for $k$-CS-PIP has an $\Omega(k^{1/|B|})$ integrality gap for every $B \geq 1$, matching the above approximation ratio up to constant factors.

For any $B \geq 1$, let $t := \lfloor B \rfloor$. We construct an instance of $k$-CS-PIP with $n$ columns and $m = \binom{n}{t+1}$ constraints. For all $i \in [n]$, weight $w_i = 1$.

For every $(t + 1)$-subset $C \subseteq [n]$, there is a constraint $j(C)$ involving the variables in $C$: set $s_{i,j(C)} = 1$ for all $i \in C$, and $s_{i,j(C)} = 0$ for $i \not\in C$. For each constraint $j \in [m]$, the capacity $c_j = B$. Note that the column sparsity $k = \binom{n-1}{t} \leq (ne/t)^t$.

Setting $x_i = \frac{1}{t}$ for all $i \in [n]$ is a feasible fractional solution. Indeed, each constraint is occupied to extent $\frac{t+1}{t} \leq \frac{B+1}{t+1} \leq B$ (since $B \geq 1$). Thus the optimal LP value is at least $\frac{n}{t}$.

On the other hand, the optimal integral solution has value at most $t$. Suppose for contradiction that the solution contains some $t + 1$ items, indexed by $C \subseteq [n]$. Then consider the constraint $j(C)$, which is occupied to extent $t + 1 = \lfloor B \rfloor + 1 > B$, this contradicts the feasibility of the solution! Thus the integral optimum is $t$, and the integrality gap for this instance is at least $\frac{n}{2t} \geq \frac{1}{2e} k^{1/|B|}$. 

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