Collective excitations of a BEC under anharmonic trap position jittering

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Abstract

Collective excitations of a Bose–Einstein condensate under periodic oscillations of the position of a quadratic plus quartic trap have been studied. A coupled set of variational equations is derived for the width and the centre of the condensate wavefunction. Analytical expressions for the growth of oscillation amplitudes in the resonance case are derived. It is shown that jittering of the position of an anharmonic trap can cause double resonance in oscillations of the BEC width and centre of mass in a wide range of the BEC parameter values. The predictions of the variational approach are confirmed by full numerical simulations of the 1D Gross–Pitaevskii equation.

1. Introduction

The investigation of low energy collective excitations is important for understanding of dynamics of atomic quantum fluids (see the review [1]). Most of these theoretical and experimental studies have been performed for a condensate trapped in a harmonic (parabolic) trap. The description of the wavefunction dynamics in such a trap has many simplifying properties both for repulsive and for attractive interactions between atoms. The theory is based on the Gross–Pitaevskii equation, which is the nonlinear Schrödinger equation with a linear oscillator potential. The analysis for the repulsive condensate shows that in such a potential the motion of the centre of mass of the condensate is decoupled from the oscillations of the condensate width. This observation is also valid for the case of an attractive BEC, where in a quasi-1D geometry the matter wave soliton can exist. For the solitonic wave in a parabolic potential, the centre-of-mass motion is well known to be completely decoupled from the internal excitations and it represents the analogue of the Kohn theorem for a solitonic wave packet [2]. It can be shown both on the level of the symmetries of 1D GP equation and using the moments method [3]. The separate resonances in oscillations of the soliton width and the position have been investigated in [4]. Resonances under periodic variations of the scattering length have been considered for a 1D Bose gas in [5].

For the case of an anharmonic trap potential the evolution of the translational mode (motion of the centre of mass) and the internal mode (oscillation of the width) becomes coupled. It gives rise to the possibility of controlling internal modes by manipulating the position of the trap. Such a possibility may be useful also in creation of new technological devices, including quantum computers [6] and ultra-sensitive interferometers [7]. Frequencies for low-energy excitation modes of a one-dimensional Bose–Einstein condensate with repulsive interaction between atoms in a quadratic plus quartic trap have been calculated in [8]. An approximate solution to describe the dynamics of Bose–Einstein condensates in anharmonic trapping potentials based on scaling solutions for the Thomas–Fermi radii has been presented in [9].

In this work, we study the effect of periodic oscillation of the position of an anharmonic elongated trap potential on the dynamics of a BEC confined in this trap. We will consider the case when the anharmonicity and the oscillation amplitude are small.

2. The model

The dynamics of trapped quasi-one-dimensional Bose gases elongated in the longitudinal direction of an anharmonic trap can be described in the framework of the 1D Gross–Pitaevskii equation

$$i\hbar \partial_t \phi = -\frac{\hbar^2}{2m} \phi_{xx} + V(x,t)\phi + g_{1D}|\phi|^2\phi,$$  \hspace{1cm} (1)
with the total number of atoms $N = \int |\psi|^2 dx$. This equation is obtained in the case of a highly anisotropic external potential under the assumption that the transversal trapping potential is harmonic: $V(y, z) = m\omega^2(y^2 + z^2)/2$ and $\omega \gg \omega_x$. Under such conditions we can seek the solution of the 3D equation in the form $U(x, y, z; t) = \Psi(x, t)\phi(y, z)$, where $R^2_0 = m\omega^2 \exp(-m\omega^2 \rho^2/\hbar)/(\pi \hbar)$. Averaging in the radial direction (i.e., integrating over the transversal variables) we have equation (1) for the dynamics of the gas in the longitudinal direction. The effective one-dimensional mean field nonlinearity coefficient $g_1 = 2\hbar a/\omega^2_\perp$, where $a_\perp$ is the atomic scattering length. $a_\parallel > 0$ corresponds to the Bose gas with a repulsive interaction between atoms and $a_\parallel < 0$ to an attractive interaction.

The dimensionless form of equation (1)

$$i\psi_t + \frac{1}{2}\psi_{xx} - V(x, t)\psi - g|\psi|^2\psi = 0$$

(2)
can be obtained by setting

$$t = \omega_0 t, \quad l = \frac{\hbar}{m\omega_0}, \quad x = \frac{x}{l}, \quad \psi = \sqrt{2|a_\parallel|\omega_\perp/\omega_0}\phi,$$

with $g = \pm 1$ for the repulsive and attractive two-body interactions, respectively.

3. Variational analysis

To describe collective oscillations of a Bose gas in an anharmonic trap we employ the variational approach. For this purpose, we use the Gaussian trial function for the wavefunction $\psi(x, t)$

$$\psi(x, t) = A(t)\exp\left(-\frac{(x-x_0(t))^2}{2\eta^2(t)}\right) + k(t)(x-x_0(t))$$

$$+ \frac{ib(t)(x-x_0(t))^2}{2} + i\phi(t),$$

(3)

where $A$, $\eta$, $b$, $x_0$ and $\phi$ are the amplitude, width, chirp, centre of mass and linear phase, respectively. The trap potential is chosen of the form $V(x) = V_2(x - c(t))^2 + V_4(x - c(t))^4$, where $c(t)$ is an external parameter describing the forced motion of the centre of the trap.

Using this ansatz in obtaining the Euler–Lagrange equations, we come to the following system of equations for the width and the centre of mass of the wave packet

$$\eta_{tt} = \frac{1}{\eta^3} - 2\eta V_2 - 6\eta^3 - 12V_4\eta(x_0 - c)^2 + \frac{gN}{\sqrt{2\pi}\eta^2},$$

$$x_{0tt} = -2V_2(x_0 - c) - 6V_4\eta^2(x_0 - c)^2 - 4V_4(x_0 - c)^3.$$  

(4)

(5)

Linearizing (4) and (5) around the equilibrium points ($\eta_{tt} = 0, x_{0tt} = 0$) we get

$$\delta_{tt} = -w_0^2\delta - 12V_4\eta_s(x_0 - c)^2, \quad \delta_{0tt} = -w_s^2(x_0 - c)$$

(6)

(7)

where $\eta_s$ is the equilibrium point of the width, $\delta = \eta - \eta_s$ is the deviation from the equilibrium point, $w_0$ and $w_s$ are determined by the expressions

$$w_0^2 = 2V_2 + 18V_4\eta_s^2 + \frac{3}{\eta_s^2} + \sqrt{\frac{2gN}{\pi\eta_s^2}} + 12V_4x_s^2,$$

$$w_s^2 = 2V_2 + 6V_4\eta_s^2 + 12V_4x_s^2.$$  

(8)

For the excitation frequencies we have

$$w_{1,2} = \left(\frac{w_0^2 + w_s^2 \pm \sqrt{(w_0^2 - w_s^2)^2 + 4k_1k_2}}{2}\right)^{1/2},$$

(9)

where

$$k_1 = -24V_4\eta_s x_s,$$

$$k_2 = -12V_4\eta_s x_s.$$  

(10)

Taking into account that at the equilibrium point $x_s = 0$ and $w_0 > w_s$ we get

$$w_1 = w_0 = \sqrt{\frac{\sqrt{2gN}}{\sqrt{\pi\eta_s^2}} + 2V_2 + 18V_4\eta_s^2 + \frac{3}{\eta_s^2}},$$

$$w_2 = w_s = \sqrt{2V_2 + 6V_4\eta_s^2}.$$  

(11)

4. Resonance

Let us suppose oscillations of the trap position to be periodical, namely $c(t) = c_0 \sin(wt)$, where $w$ is the oscillation frequency. As easily seen from the linearized equations, the centre of mass and the width oscillations behave like periodically driven oscillator with the ‘external forces’ $c^2(t)$ in equation (6) and $c(t)$ in equation (7). This means that the frequency of the ‘external force’ is equal to $2w$ in the first equation and to $w$ in the second. Then double resonance in oscillations of the centre of mass and the width is possible when $w = w_s = w_0/2$.

To describe the resonant growth of oscillations of the width and the centre of mass in equations (6) and (7), we seek $\delta$ and $x_0$ as $\delta = A(t)\sin(2wt + \phi_0)$ and $x_0 = B(t)\sin(wt + \phi_2)$. Supposing that $A(t)$ and $B(t)$ depend weakly on time and substituting these expressions into (6) and (7) and assembling coefficients of $\sin(2wt + \phi_0)$, $\cos(2wt + \phi_0)$, $\sin(wt + \phi_2)$ and $\cos(wt + \phi_2)$, we come to differential equations for $A(t)$ and $B(t)$:

$$A_t = \frac{3V_4\eta_s}{w}B^2,$$

$$B_t = \frac{c_0w}{4} + \frac{12V_4\eta_s}{w}AB.$$  

(12)

For small amplitudes $|A| \ll 1$ and $|B| \ll 1$ we have

$$A(t) = \frac{1}{16}V_4\eta_s c_0^2 w t^3,$$

$$B(t) = \frac{c_0w}{4}t.$$  

(13)
The parameters are $w$ rise to the following relation between them, the system evolution based on the variational approach using 5. Numerical simulations

The obtained expressions for amplitudes which describe the growth of the amplitudes of oscillations. In a wide range of parameters of the trap potential are $V_2 = 0.5$ and $V_4 = 0.0005$. The value of the forced oscillation frequency $w = w_3$ under the condition $w_3 = w_4/2$. The necessary value of the nonlinearity coefficient $g$, providing this condition is obtained by solving a set of equations

$$w_3 = w_4/2,$$

$$\frac{1}{\eta^3} - 2\eta V_2 - 6V_4\eta^3 + \frac{gN}{\sqrt{2\pi}\eta} = 0.$$

Here, the second equation determines the equilibrium point of equation (4).

As seen, unlike the harmonic case, in an anharmonic trap potential the forced oscillations of the trap centre position induce oscillations not only in the condensate centre of mass but also in the condensate width. In figure 1 for comparison full GPE and ODE simulations of the width and the centre-of-mass oscillations are shown. Theoretical prediction is shown by envelope lines described by equation (13).

It should be noted that equation (13) is obtained from the linearized equations (6) and (7) and, therefore, describes a linear resonance with infinitely increasing amplitudes. Difference between numerical simulations and the theory is explained by the fact that in the numerical simulations we deal with a nonlinear resonance. In the beginning stage of the resonance the numerical simulations coincide with the theory. But then, due to detuning of the oscillation frequency from the resonant, one can observe beatings in amplitudes of the oscillations.

Double resonance presented in figure 1 occurs under the condition $w_3 = w_4/2$. It corresponds to the particular value of the nonlinearity coefficient $g = 0.015$ (repulsive BEC). However, here we meet with an interesting fact that in a wide range of the nonlinearity values the ratio $w_3/w_4$ is close to 2. For example, when $g = 1.0$ the ratio $w_4/w_3 = 1.924$.

Closeness of this ratio to 2 can be proved directly from expressions (11) for $w_4$ and $w_3$. Let us write out an explicit form of the ratio $w_4^2/w_3^2$:

$$w_4^2/w_3^2 = \frac{\sqrt{2\pi}N}{\sqrt{\pi}\eta} + 2V_2 + 18V_4\eta^2 + \frac{3}{\eta^2}.$$

Using an algebraic equation for the equilibrium point $\eta$ obtained from equation (4) under the condition $\eta = 0$, expression (16) can be transferred into the following form:

$$w_4^2/w_3^2 = \frac{8V_2 + 36V_4\eta^2 - \frac{gN}{\sqrt{2\pi}V_2\eta^2}}{2V_2 + 6V_4\eta^2}.$$

Finally, neglecting small anharmonic terms containing $V_4$, we get the following expression for the ratio $w_4^2/w_3^2$:

$$w_4^2/w_3^2 = 4 - \frac{gN}{2\sqrt{2\pi} V_2\eta^2}.$$

In a wide range of parameters $V_2$ and $g$ the second term

$$\frac{gN}{2\sqrt{2\pi} V_2\eta^2} < 1$$
Figure 2. Ratio of eigenfrequencies of the BEC width and centre-of-mass oscillations versus the nonlinearity $g$ for different values of $V_2$. Solid, dashed, dot-dashed and dot-dot-dashed lines stand for the cases $V_2 = 0.9, 0.5, 0.3, 0.1$, respectively. Field I corresponds to attractive BEC and field II to repulsive one.

Figure 3. The width and centre-of-mass oscillations of the repulsive BEC in an anharmonic trap with the forced periodical oscillation of the trap centre with the frequency $w = w_x$ and $w_\eta/w_x = 1.966$. Nonlinear coefficient $g$ equals 0.4. The parameters are $V_2 = 0.5$, $V_4 = 0.0005$, $c_0 = 0.1$. Solid and dotted lines stand for PDE and ODE simulations, respectively.

Figure 4. The width and centre-of-mass oscillations of the attractive BEC in an anharmonic trap with the forced periodical oscillation of the trap centre with the frequency $w = w_x$ and $w_\eta/w_x = 2.046$. Nonlinear coefficient $g$ equals $-0.4$. The parameters are $V_2 = 0.5$, $V_4 = 0.0005$, $c_0 = 0.1$. Solid and dotted lines stand for ODE and PDE simulations, respectively.

As an illustration of this fact, in figure 2 the ratio of the eigenfrequencies of the width and the mass centre oscillations versus the nonlinearity coefficient $g$ is shown for several values of the quadratic part of potential $V_2$. When $g$ ranges from $-0.4$ (attractive BEC) to 0.4 (repulsive BEC) the ratio changes from 2.084 to 1.964. Closeness of the ratio value $w_\eta/w_x$ to 2 makes possible existence of the oscillations evolution close to a double resonance in a very wide range of the Bose–Einstein condensate parameters under forced oscillations of the anharmonic trap potential position at the frequency $w = w_x$.

To check this assertion we carried out ODE and PDE simulations when $w_\eta/w_x \neq 2$. In figure 3, oscillations of the repulsive condensate width and centre of mass are presented when the nonlinearity coefficient $g = 0.4$. In this case, the ratio $w_\eta/w_x = 1.966$. Here, the parameters are $V_2 = 0.5$, $V_4 = 0.0005$, $c_0 = 0.1$. The value of the forced oscillation is taken $w = w_x$. In spite of the fact that the ratio of eigenfrequencies of the BEC width and centre-of-mass oscillation is not equal to 2, one can observe oscillations close to double resonance.

Simulations of a double nonlinear resonance presented in figures 1 and 3 relate to the case of the repulsive condensate. Let us now consider an attractive condensate where the matter wave solitons can exist. As seen from figure 2 for the ratio $w_\eta/w_x$ can be kept close to 2. Thus, by an appropriate choice of parameters of the trap potential and nonlinearity, one can keep the value of the eigenfrequencies ratio close to the harmonic value 2.
the attractive BEC \( (g < 0) \) the ratio \( w/\omega_x \) is not equal to 2 and exact resonance is impossible in this case (when \( V_4 > 0 \)). Nevertheless, the ratio \( w/\omega_x \) remains to be close to 2 (in considered range of the BEC parameters) and one can expect resonant behaviour of the width and the centre-of-mass oscillations of the attractive BEC under forced oscillations of the trap potential position with the frequency \( \omega = \omega_x \).

In figure 4, resonant behaviour of oscillations of the attractive condensate width and centre of mass is depicted when the nonlinearity coefficient \( g = -0.4 \). In this case, the ratio \( w/\omega_x = 2.046 \). Here, the parameters are \( V_2 = 0.5, V_4 = 0.0005, \omega_0 = 0.1 \). The value of the forced oscillation is taken \( \omega = \omega_x \). As in the case of repulsive BEC one can observe that in the case of attractive BEC the oscillations close to double resonance too.

As seen the results of ODE simulations of the double resonance are in a good agreement with full PDE ones at times \( t < 100 \) and then begin to differ at larger times. At these times, the amplitude of oscillations of the BEC centre of mass \( x_0 \) is great and the anharmonic part of the trap potential becomes noticeable that leads to the difference between ODE and PDE simulations.

6. Conclusion

In this paper, we have studied collective oscillations of a quasi-one-dimensional Bose gas in an anharmonic trap under periodic oscillations of the trap position in time. To describe evolution of oscillations we use the variational approach with the Gaussian ansatz. Double resonance in the condensate oscillations has been studied. Analytical expressions have been derived for the growth of oscillation amplitudes in the resonance.

Analysis of the variational equations has shown existence of a double resonance in oscillations of the centre of mass and the width under forced oscillations of the trap centre position, provided that the ratio of the eigenfrequencies \( w/\omega_x = 2 \) and the forced trap position oscillation frequency \( \omega = \omega_x \). It is shown that for a wide range of values of the BEC parameters the ratio \( w/\omega_x \) is close to 2 and the behaviour of the oscillations is close to resonant both for repulsive and for attractive Bose gases.

Theoretical predictions are confirmed by full numerical simulations of the 1D GP equation.

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