Derivation of the Konishi anomaly relation from Dijkgraaf-Vafa with (Bi-)fundamental matters

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Abstract: We explicitly write down the Feynman rules following the work of Dijkgraaf, Vafa and collaborators for $\mathcal{N} = 1$ super Yang-Mills having products of $SU$ groups as the gauge group and matter chiral superfields in adjoint, fundamental, and bi-fundamental representations without baryonic perturbations. As an application of this, we show expectation values calculated by these methods satisfy the Konishi anomaly relation.
1 Introduction

Dijkgraaf and Vafa proposed a duality between four dimensional $\mathcal{N} = 1$ super Yang-Mills theories and the old matrix model, which states that the effective superpotential of the super Yang-Mills written as a function of the gaugino condensate $S$ and coupling constants, can be obtained from a matrix model free energy. This important proposal is checked by many papers.

Originally the proposal were made in the context of the deformation of $\mathcal{N} = 2$ theories, recent papers showed that it can be generalized to include models with fundamental matter, and successfully reproduced the Affleck-Dine-Seiberg superpotential by this method. But it seems to us there is some confusion about what is the correct prescription. This can be determined if one follows the argument in the paper by Dijkgraaf, Grisaru, Lam, Vafa and Zanon (DGLVZ), in which they derive the Dijkgraaf-Vafa proposal by integrating out matter superfields in the presence of an external gauge field. Although the analysis made in DGLVZ is quite clear for diagrams containing at least one vertex, it needs some care in evaluating the contribution from one loop diagram without any vertex.

The purpose of this short note is to write down the precise rules for calculating the effective superpotential and expectation values for models having products of $SU(N_i)$ as the gauge groups and matters in the fundamental, adjoint, or bi-fundamental representation and without baryonic perturbations. The restriction to the case without baryonic perturbations comes from the fact that they cannot be depicted by 't Hooft’s double-line notation. For the case with only quadratic mass perturbations, this method reduces to that presented in the section three in Intriligator’s ‘integrating in’ paper.

As an application we show the expectation values calculated by the prescription of this paper satisfies important identities coming from the Konishi anomaly. The importance of the Konishi anomaly in the framework of Dijkgraaf-Vafa was first pointed out by Gorsky.

In the following, we follow the conventions of [1].

Note Added: After completion of this work, we have noticed a new preprint hep-th/0211170 by Cachazo, Douglas, Seiberg and Witten, which has some overlap with this article.

2 Prescription

First, recall that DGLVZ explicitly states the following Feynman rules for the contribution of a diagram to the effective superpotential of the gaugino condensate:
1. Write the diagram following ’t Hooft’s double-line notation.

2. Assign on each index loop the gaugino condensate $S$ or the dimension of the fundamental representation $N$.

3. The contribution to the superpotential comes only from those diagrams, in which the number of $S$ assigned is equal to the number of independent loop momenta.

4. The propagator for each matter superfield is the inverse of its mass.

5. Interaction vertices come from the cubic and higher order terms in the tree-level superpotential.

6. Finally, multiply them together.

The restriction 3 ensures the correct number of Grassmann integrals, because there are two Grassmann integrations for each momentum loop and each insertion of $S$ contains two Grassmann variables. The same condition restricts the topology of the diagrams that can contribute to the effective superpotential to be a sphere or a disk.

Let $W_{\text{eff}}$ denote the non-perturbative superpotential calculated following Dijkgraaf-Vafa.

The expectation value of the lowest component of a gauge-invariant operator can be calculated using the above rules and finally substituting $S$ by the value which extremizes $W_{\text{eff}}$.

Now let us determine the contribution of one-loop graphs with no insertion.

- For $Q$ in the fundamental and $\tilde{Q}$ in the antifundamental of $SU(N)$ with a mass term $mQ\tilde{Q}$, the expectation value of the bilinear satisfies $\langle Q\tilde{Q} \rangle = S/m$. This is equal to $m(\partial/\partial m)W_{\text{eff}}$, so that $W_{\text{eff}}$ contains $S\log(m/\Lambda)$, where the integration constant $\Lambda$ is some multiple of the dynamically generated scale for pure $\mathcal{N} = 1$ super Yang-Mills. We adopt a prescription that the proportionality constant is unity.

- The case with $\Phi$ in the adjoint. The argument goes essentially the same with that given above, except that the diagrams are now double-lined and so the loop contributes $NS\log(m/\Lambda)$.

- A bi-fundamental $\Phi$ for $SU(N_1) \times SU(N_2)$. Its effect for $SU(N_1)$ is the same as introducing $N_2$ fundamentals of $SU(N_1)$.
And finally, as noted by Dijkgraaf and Vafa, we must add the Veneziano-Yankielowicz term
\( NS(1 - \log(S/A^3)) \) for each gauge group. The total effective superpotential can be written as
\[
W_{\text{eff}} = W_{\text{VY}} + W_{\text{one loop}} + W_{\text{higher}},
\]
where \( W_{\text{VY}} \) is the Veneziano-Yankielowicz piece, \( W_{\text{one loop}} \) is what are discussed in the preceding paragraph, and \( W_{\text{higher}} \) comes from diagrams containing at least one vertex. With only quadratic perturbations, this formula reduces to that mentioned in \([3]\) section 3.

As a consistency check, one can verify that the prescription given here correctly reproduces for example the decoupling equation for fundamental flavors
\[
\Lambda_3^{3N - N_f + 1} = m \Lambda_3^{3N - N_f}.
\]

3 An Example

As an example, consider the \( \mathcal{N} = 1 \) supersymmetric \( SU(2)_1 \times SU(2)_2 \) theory with two fundamentals \( L_1, L_2 \) of the second \( SU(2) \) and one bi-fundamental \( Q \). (This model is the one given in \([10]\) section 4.1 and the superpotential \([1]\) as a function of \( S \) and \( m \) is essentially given in \([3]\) section 3, so this is not essentially new. It is just for an illustrative purpose.) Let us denote the gaugino condensates and the dynamically generated scales of each group as \( S_1, S_2, \Lambda_1, \Lambda_2 \), respectively. Then include mass perturbations \( mQ^2 + \mu L_1L_2 \) to the tree level superpotential. As the bifundamental is \((2,2)\), we can write down immediately the effective superpotential of the gaugino condensate as
\[
W_{\text{eff}} = 2S_1(1 - \log \frac{S_1}{A_1^3}) + 2S_2(1 - \log \frac{S_2}{A_2^3}) + S_1 \log \frac{m}{A_1} + S_2 \log \frac{m}{A_2} + S_2 \log \frac{\mu}{A_2},
\]
where the first and the second terms are the Veneziano-Yankielowicz superpotential for each of the gauge groups, and the third and the fourth are the contribution of the bi-fundamental, and the last comes from two fundamentals. The expectation value of \( X = \langle Q^2 \rangle \) and \( Y = \langle L_1L_2 \rangle \) can also be calculated to be \( \langle Q^2 \rangle = (S_1 + S_2)/m \) and \( \langle L_1L_2 \rangle = S_2/\mu \). Integrating out \( S_i \)’s and writing \( W_{\text{eff}} = W_{\text{non-perturbative}} + m\langle Q^2 \rangle + \mu \langle L_1L_2 \rangle \), we immediately obtain
\[
W_{\text{non-perturbative}} = \langle S_2 \rangle = \frac{\Lambda_3^3 Y}{X Y - \Lambda_2^4}.
\]

4 Derivation of the Konishi relation.

As remarked by a recent paper by Gorsky\([7]\), the Konishi anomaly relation, when evaluated at some supersymmetric vacuum, amounts in the Old Matrix model lan-
guage to the Virasoro $L_0$ condition supplanted by an anomaly of entropy terms. He also checked the relation with several examples already appeared on the literature as the test for the Dijkgraaf-Vafa proposal.

Here we derive, as an easy application of the rules stated in the previous section, the Konishi relation from the Dijkgraaf-Vafa prescription. What we want to prove is that

$$2N\langle S\rangle = \langle \phi \frac{\partial}{\partial \phi} W_{\text{tree}} \rangle$$

for each adjoint chiral superfield $\phi$ and

$$\langle S\rangle = \langle Q \frac{\partial}{\partial Q} W_{\text{tree}} \rangle$$

for each fundamental $Q$. We present a derivation for the case of an $SU(N)$ adjoint. The proof for other cases is essentially the same.

Decompose $W_{\text{tree}}$ and $W_{\text{eff}}$ as

$$W_{\text{tree}} = \frac{1}{2} m \text{tr} \phi^2 + \text{(other mass terms)} + W_{\text{vertices}},$$

where $W_{\text{vertices}}$ is the interaction terms in the tree-level superpotential, and

$$W_{\text{eff}} = W_{\text{VY}} + W_{\text{one loop}} + W_{\text{higher}}$$

where

$$W_{\text{VY}} = NS(1 - \log(S/\Lambda^3)) + \text{(V-Y terms for other gauge groups)},$$

$$W_{\text{one loop}} = NS \log(m/\Lambda) + \text{(one loop terms for other flavors)},$$

and $W_{\text{higher}}$ comes from the diagrams which contains at least one interaction vertex.

Now, note that $\langle \phi \partial/\partial \phi \rangle$ counts the number of the $\phi$ legs of each diagram in $W_{\text{tree}},$

$$\langle \phi \frac{\partial}{\partial \phi} W_{\text{vertices}} \rangle = \sum_{\text{diagram } D} \text{(the number of legs of } \phi \text{ in } D) \times \text{(the value of } D)$$

But the number of $\phi$ legs is twice the number of the $\phi$ propagators, so that

$$\langle \phi \frac{\partial}{\partial \phi} W_{\text{vertices}} \rangle = 2m^{-1} \frac{\partial}{\partial m^{-1}} W_{\text{higher}}.$$

On the other hand, the definition of $W_{\text{eff}}$ shows

$$\langle m \text{ tr } \phi^2 \rangle = 2m \frac{\partial}{\partial m} W_{\text{eff}}.$$

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Here all the partial derivatives on the RHS are taken first, and then $S$ is substituted by the value which minimizes the $W_{\text{eff}}$, so that
\[
\langle \phi \frac{\partial}{\partial \phi} W_{\text{tree}} \rangle = 2m \frac{\partial}{\partial m} (W_{\text{eff}} - W_{\text{higher}}) \\
= 2m \frac{\partial}{\partial m} (W_{\text{VY}} + W_{\text{one loop}}) \\
= 2N \langle S \rangle.
\]
This is what we want to derive.

5 Conclusion and Outlook

In this short note we explicitly write down the perturbative rules of the Dijkgraaf-Vafa proposal for classical gauge groups with fundamental, adjoint, or bifundamental matter fields, and showed that the Konishi anomaly relation is indeed satisfied by the expectation values calculated from these rules.

One of the biggest remaining problems is the incorporation of baryonic perturbations which involve the invariant tensor $\epsilon_{ijk}$. And another is the extension of this framework to the chiral matter contents. In this case, propagators cannot be easily given because no gauge-invariant mass deformations can be introduced. Really interesting, and also phenomenologically important examples of $\mathcal{N} = 1$ super Yang-Mills theory are usually of this kind. So this well deserves a study.

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