Existence of Solutions for Nonconvex Differential Inclusions of Monotone Type

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Abstract

Differential inclusions with compact, upper semi-continuous, not necessarily convex right-hand sides in $\mathbb{R}^n$ are studied. Under a weakened monotonicity-type condition the existence of solutions is proved.

Key words: differential inclusion, nonconvex right-hand side, existence of solutions, weak monotonicity, one-sided Lipschitz condition

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1 Introduction

We study the autonomous differential inclusion:

$$\dot{x}(t) \in F(x(t)), \quad x(0) = x_0 \in \mathbb{R}^n, \quad t \in I = [0, T],$$

where the set-valued mapping $F$ has compact, not necessarily convex values in $\mathbb{R}^n$, and is upper semi-continuous, or equivalently, has a closed graph. We also assume linear growth of $F$, to ensure boundedness of all solutions, and a weakened monotonicity-type condition in the spirit of the strengthened one-sided Lipschitz (S-OSL) condition [1].

The results on existence of solutions of such inclusions are not so numerous. First, one should mention the well-known existence result in the case of maximal monotone right-hand sides [2, Sec. 3.2, Theorem 1]. Maximal monotone set-valued maps, as is well-known, are almost everywhere single-valued [3, 4], and at the points where they are not single-valued, their values are convex sets. Other important existence results for differential inclusions with non-convex right-hand sides are the results of Filippov [5] for Lipschitz $F$, and of Hermes [6], who relaxed the Lipschitz continuity of $F$ to continuity with bounded variation. The result of [7] is for upper semi-continuous and cyclically monotone map $F$, which is a stronger condition than just monotonicity. In [8] the phenomenon of “colliding” on the set of discontinuities of $F$ is studied and conditions to avoid or to escape from this set are investigated.

We prove the existence under another monotonicity-type condition that ensures componentwise monotonicity of the Euler polygons and their derivatives, which is the key for this existence proof. The meaning of this condition is that the set-valued map $-F(\cdot)$ (with images being the pointwise negation of $F(x)$) satisfies the strengthened one-sided Lipschitz condition [1] with a constant zero.

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The latter condition is a weaker form of the S-OSL condition for set-valued maps introduced in \cite{9}, see \cite{1} Remark 2.1.

We give examples that show that our condition, although simple, does not imply monotonicity, hence does not imply cyclical monotonicity.

\section{Main result}

First we introduce some notation. For every notion used in the paper, but not explicitly defined here we refer the reader to \cite{10}.

Let $v \in \mathbb{R}^n$. We denote by $|v|$ the Euclidean norm of the vector $v$ and by $v_j$ its $j$-th coordinate, i.e. $v = (v_1, v_2, \ldots, v_n)$. Denote by $\mathbb{B}$ the unit ball in $\mathbb{R}^n$. For a bounded set $A \subset \mathbb{R}^n$, we denote $\|A\| = \text{sup}\{\|a\| : a \in A\}$.

We impose the following assumptions in order to prove the existence of solution:

\begin{enumerate}
  \item \textbf{A1.} $F : \mathbb{R}^n \Rightarrow \mathbb{R}^n$ has compact, nonempty values and closed graph.
  \item \textbf{A2. Linear growth condition} There exist constants $A$ and $B$ such that $\|F(x)\| \leq A + B|x|$ for any $x \in \mathbb{R}^n$.
  \item \textbf{A3. Weak Componentwise Monotonicity (WCM) Condition:} For every $x, y \in \mathbb{R}^n$ and every $v \in F(x)$ there exists $w \in F(y)$ such that

\begin{equation}
(x_j - y_j)(v_j - w_j) \geq 0, \quad \forall \ j = 1, 2, \ldots, n.
\end{equation}

\end{enumerate}

In other words, (2) means that the negation of the given set-valued map, $-F(\cdot)$ satisfies the S-OSL condition from \cite{1} with a constant zero.

\textbf{Theorem 1.} Under the conditions \textbf{A1, A2, A3} the differential inclusion (1) has a solution.

To prove the theorem, we use the following Euler-Cauchy construction of polygonal approximate solutions. Fix the natural number $N$ and let the mesh size $h = \frac{T}{N}$ be such that $hM < 1$. Denote the mesh points by $t_i = ih$. We define Euler’s polygons $x^N : [0, T] \to \mathbb{R}^n$ in the following way: We set $x^N(0) = x_0$, and for $t \in [0, t_1]$, we construct $x^N(t) = x_0 + tv^0$, where $v^0 \in F(x_0)$ is arbitrary. Further, we construct subsequently the Euler polygons in each subinterval $t \in [t_i, t_{i+1}]$, for $i = 1, \ldots, N - 1$, by $x^N(t) = x^N(t_i) + (t - t_i)v^i$, where the velocity $v^i \in F(x_N(t_i))$ is chosen by the assumption A3, such that

$$(x_j^N(t_i) - x_j^N(t_{i-1}))(v_j^i - v_{j-1}^{i-1}) \geq 0, \quad j = 1, \ldots, n.$$ 

The following lemma and proposition represent the main steps of the proof of Theorem 1.

\textbf{Lemma 2.} The polygonal functions $x_j^N(t)$ and their derivatives $\dot{x}_j^N(t)$ are monotone for every $j \in \{1, 2, \ldots, n\}$.

\textbf{Proof.} Fix a coordinate $j \in \{1, \ldots, n\}$, and suppose that $v_j^i = 0$ for $i < k$ and $v_j^k \neq 0$. Here $k = 0$ is possible, i.e. possibly $v_0^j \neq 0$. Clearly, if $v_j^i = 0$ for all $i \leq N$, then the claim holds trivially. If $v_j^k > 0$, then $x_j^N(\cdot)$ is strictly monotone increasing on the subinterval $[t_k, t_{k+1}]$, and therefore $x_j^N(t_{k+1}) > x_j^N(t_k)$. Again, using the assumption (2), it is easy to see that $v_j^{k+1} > v_j^k > 0$. On the
next subintervals, \([t_i, t_{i+1}], \ i > k\), continuing in the same way, we show that \(\{v^j_i\}_{i=k}^\infty\) is positive and monotone nondecreasing, while \(x^N_j(t)\) is increasing. If for some \(j \in \{1, \ldots, n\}\), \(v^j_i = 0\) for all \(i < k\), and \(v^k_j < 0\), then in a similar way we get that \(\{v^j_i\}_{i=k}^\infty\) is negative and monotone nonincreasing, while \(x^N_j(t)\) is strictly monotone decreasing.

The following proposition is proved using Helly’s selection principle [12, Chap. 10] replacing the Arzelà-Ascoli theorem, which is usually applied in precompactness proofs for continuous functions, and used to prove the existence of solutions for differential inclusions with convex right hand sides (see e.g. [13, Theorem 2.2]).

**Proposition 1.** Under the conditions \(A_1, A_2, A_3\), the sequence \(x^N_N(\cdot)\) has a subsequence converging uniformly on \(I\) to a function \(x^\infty(\cdot)\), with each coordinate \(x^\infty_j(\cdot)\) being monotone. Furthermore, \(x^\infty(\cdot)\) is a solution of the inclusion (1).

Let us recall that a mapping \(F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n\) is **monotone** if
\[
\langle x - y, v - w \rangle \geq 0 \quad \text{for all } x, y \in \mathbb{R}^n, \ v \in F(x), \ w \in F(y).
\] (3)
The map \(F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n\) is **cyclically monotone** if for every cyclic sequence of points \(x_0, x_1, \ldots, x_N = x_0\) and all \(v_i \in F(x_i), i = 1, \ldots, N\),
\[
\sum_{i=1}^N \langle x_i - x_{i-1}, v_i \rangle \geq 0.
\] (4)
It is easy to check that every cyclically monotone map is monotone. The classical monotonicity condition (3) requires that \(F(\cdot)\) is almost everywhere single-valued [3, 4].

In [7] an existence proof for solutions of differential inclusions with compact right-hand side is given which is cyclically monotone. It is also proved that cyclically monotone map have images that are subsets of a subdifferential map of a convex function.

### 3 Examples

We give here examples of set-valued maps which are weakened monotone and fulfill \(A_3\), but are neither monotone nor cyclically monotone.

The mappings of the examples below are not monotone, hence are not cyclically monotone, since they are not single-valued almost everywhere.

The following example is a modification of [14, Example 2.1] in which \(G(x) = -F(x)\) is shown to be OSL. Here, \(F(\cdot)\) satisfies \(A_3\), but is not monotone and is discontinuous.

**Example 1.** Let \(F : \mathbb{R} \rightrightarrows \mathbb{R}\) be defined as
\[
F(t) = \begin{cases} 
[-1, 0] & (t < 0), \\
[-1, 1] & (t \geq 0) .
\end{cases}
\]
Then, \(F(\cdot)\) has convex images and satisfies \(A_3\), but is not monotone in the sense of (3).

There are maps with compact images fulfilling \(A_3\) that are weakened monotone, but not monotone, as the following example shows.

**Example 2.** Let \(F, G : \mathbb{R} \rightrightarrows \mathbb{R}\) be defined as \(F(t) = \{t, t^2\}\), \(G(t) = \{t^2, t + \text{sign}(t)\}\). Then, \(F(\cdot), G(\cdot)\) have compact non-convex images, \(F\) is continuous, while \(G\) is discontinuous at the origin. Both \(F\) and \(G\) satisfy \(A_3\), but are not monotone in the sense of (3).
To construct examples of set-valued maps that satisfy $A3$ in higher dimensions, we may take the Cartesian product of such one-dimensional mappings and use the simple fact that the union of two mappings that satisfy $A3$ also satisfy $A3$.

**Example 3.** Let $F : \mathbb{R}^2 \Rightarrow \mathbb{R}^2$ be defined as

$$F(x) = \begin{cases} 
(\{x_1^+\} + [-1,0]) \times ([-2,-1] \cup [1,2]) & (x_1 < 0), \\
(\{x_1^+\} + [-1,1]) \times ([-2,-1] \cup [1,2]) & (x_1 \geq 0)
\end{cases}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$.

Then, $F(\cdot)$ has compact images and satisfies $A3$, but is not monotone in the sense of $\mathbb{R}$.

**Example 4.** Let $f : \mathbb{R} \Rightarrow \mathbb{R}$ be defined as follows:

$$f(s) = \begin{cases} 
\{\text{sign}(s)\} & s \neq 0, \\
\{-1, 1\} & x = 0.
\end{cases}$$

Define $F : \mathbb{R}^n \Rightarrow \mathbb{R}^n$ by $F(x) = \left\{ \frac{1}{2} (f(x_1), f(x_2), \ldots, f(x_n)), (f(x_1), f(x_2), \ldots, f(x_n)) \right\}$. Clearly, $F(\cdot)$ satisfies all our conditions, but is neither monotone, nor cyclically monotone.

Clearly, there are monotone mappings which do not satisfy $A3$. A simple example is the subdifferential of the Euclidean norm. We believe that there are other classes set-valued maps of monotone-type for which existence of solutions of differential inclusions with non-convex upper semi-continuous right-hand sides can be proved.

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