Reasoning about exceptions in ontologies: from the lexicographic closure to the skeptical closure

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Abstract. Reasoning about exceptions in ontologies is nowadays one of the challenges the description logics community is facing. The paper describes a preferential approach for dealing with exceptions in Description Logics, based on the rational closure. The rational closure has the merit of providing a simple and efficient approach for reasoning with exceptions, but it does not allow independent handling of the inheritance of different defeasible properties of concepts. In this work we outline a possible solution to this problem by introducing a variant of the lexicographical closure, that we call skeptical closure, which requires to construct a single base. We develop a bi-preference semantics for defining a characterization of the skeptical closure.

1 Introduction

Reasoning about exceptions in ontologies is nowadays one of the challenges the description logics community is facing, a challenge which is at the very roots of the development of non-monotonic reasoning in the 80s. Many non-monotonic extensions of Description Logics (DLs) have been developed incorporating non-monotonic features from most of the non-monotonic formalisms in the literature \cite{20,22,35,11,28,15,42,21,18,36,14,34,29,30}, or defining new constructions and semantics such as in \cite{7}.

We focus on the rational closure for DLs \cite{15,18,44,30} and, in particular, on the construction developed in \cite{30}, which is semantically characterized by minimal (canonical) preferential models. While the rational closure provides a simple and efficient approach for reasoning with exceptions, exploiting polynomial reductions to standard DLs \cite{24,41}, the rational closure does not allow an independent handling of the inheritance of different defeasible properties of concept\textsuperscript{3} so that, if a subclass of $C$ is exceptional for a given aspect, it is exceptional tout court and does not inherit any of the typical properties of $C$. This problem was called by Pearl \cite{44} “the blocking of property inheritance problem”, and it is an instance of the “drowning problem” in \cite{6}.

To cope with this problem Lehmann \cite{39} introduced the notion of the lexicographic closure, which was extended to Description Logics by Casini and Straccia \cite{17}, while in \cite{18} the same authors develop an inheritance-based approach for defeasible DLs. Other proposals to deal with this “all or nothing” behavior in the context of DLs are the the logic of overriding, $DL^N$, by Bonatti, Faella, Petrova and Sauro \cite{7}, a nonmonotonic

\textsuperscript{3} By \textit{properties} of a concept, here we generically mean characteristic features of a class of objects (represented by a set of inclusion axioms) rather than roles (properties in OWL \cite{43}).
description logic in which conflicts among defaults are solved based on specificity, and the work by Gliozzi [33], who develops a semantics for defeasible inclusions in which models are equipped with several preference relations.

In this paper we will consider a variant of the lexicographic closure. The lexicographic closure allows for stronger inferences with respect to rational closure, but computing the defeasible consequences in the lexicographic closure may require to compute several alternative bases [39], namely, consistent sets of defeasible inclusions which are maximal with respect to a (so called seriousness) ordering. We propose an alternative notion of closure, the skeptical closure, which can be regarded as a more skeptical variant of the lexicographic closure. It is a refinement of rational closure which allows for stronger inferences, but it is weaker than the lexicographic closure and its computation does not require to generate all the alternative maximally consistent bases. Roughly speaking, the construction is based on the idea of building a single base, i.e. a single maximal consistent set of defeasible inclusions, starting with the defeasible inclusions with highest rank and progressively adding less specific inclusions, when consistent, but excluding the defeasible inclusions which produce a conflict at a certain stage without considering alternative consistent bases. Our construction only requires a polynomial number of calls to the underlying preferential $\mathcal{ALC} + \mathbf{T_R}$ reasoner to be computed.

To develop a semantic characterization of the skeptical closure, we introduce a bi-preference semantics (BP-semantics), which is still in the realm of the preferential semantics for defeasible description logics [27,11,28], developed along the lines of the preferential semantics introduced by Kraus, Lehmann and Magidor [37,38]. The BP-semantics has two preference relations and is a refinement of the rational closure semantics. We show that the BP semantics provides a characterization of the MP-closure, a variant of the lexicographic closure introduced in [25], and exploit it to build a semantics for the Skeptical closure.

Schedule of the paper is the following. In Section 2 we recall the definition of the rational closure for $\mathcal{ALC}$ in [30] and of its semantics. In Section 3 we define the Skeptical closure. In Section 4 we introduce the bi-preference semantics and, in Section 5 we show that it provides a semantic characterization of the MP-closure, a sound approximation of a multipreference semantics in [25]. In Section 6 the BP-semantics is used to define a semantic characterization for the skeptical closure. Finally, in Section 7 we compare with related work and conclude the paper.

This work is based on the extended abstract presented at CILC/ICTCS 2017 [23], where the notion of skeptical closure were first introduced.

2 The rational closure for $\mathcal{ALC}$

We briefly recall the logic $\mathcal{ALC} + \mathbf{T_R}$ which is at the basis of a rational closure construction proposed in [30] for $\mathcal{ALC}$. The idea underlying $\mathcal{ALC} + \mathbf{T_R}$ is that of extending the standard $\mathcal{ALC}$ with concepts of the form $\mathbf{T}(C)$, whose intuitive meaning is that $\mathbf{T}(C)$ selects the typical instances of a concept $C$, to distinguish between the properties that hold for all instances of concept $C$ ($C \subseteq D$), and those that only hold for the typical such instances ($\mathbf{T}(C) \subseteq D$). The $\mathcal{ALC} + \mathbf{T_R}$ language is defined as follows:

$$
\begin{align*}
C_R & := A \mid \top \mid \bot \mid \neg C_R \mid C_R \cap C_R \mid C_R \cup C_R \mid \forall R.C_R \mid \exists R.C_R \\
C_L & := C_R \mid \mathbf{T}(C_R),
\end{align*}
$$

where $A := \{C \mid \mathbf{T}(C) \subseteq D\}$.
where $A$ is a concept name and $R$ a role name. A knowledge base $K$ is a pair $(\mathcal{T}, A)$, where the TBox $\mathcal{T}$ contains a finite set of concept inclusions $C_L \subseteq C_R$, and the ABox $A$ contains a finite set of assertions of the form $C_R(a)$ and $R(a, b)$, for $a, b$ individual names, and $R$ role name.

The semantics of $\mathcal{ALC} + \mathcal{T}_R$ is defined in terms of rational models, extending to $\mathcal{ALC}$ the preferential semantics by Kraus, Lehmann and Magidor in [37,38]: ordinary models of $\mathcal{ALC}$ are equipped with a preference relation $<$ on the domain, whose intuitive meaning is to compare the “typicality” of domain elements: $x < y$ means that $x$ is more typical than $y$. The instances of $\mathcal{T}(C)$ are the instances of concept $C$ that are minimal with respect to $<$. In rational models, which characterize $\mathcal{ALC} + \mathcal{T}_R$, $<$ is further assumed to be modular (i.e., for all $x, y, z \in \Delta$, if $x < y$ then either $x < z$ or $z < y$) and well-founded (i.e., there is no infinite $<$-descending chain, so that, if $S \neq \emptyset$, also $\min_{<}(S) \neq \emptyset$). Ranked models characterize $\mathcal{ALC} + \mathcal{T}_R$. Let us shortly recap their definition.

**Definition 1 (Semantics of $\mathcal{ALC} + \mathcal{T}_R$ [30]).** An interpretation $M$ of $\mathcal{ALC} + \mathcal{T}_R$ is any structure $(\Delta, <, I)$ where: $\Delta$ is the domain; $<$ is an irreflexive, transitive, modular and well-founded relation over $\Delta$. $I$ is an interpretation function that maps each concept name $C \in R_C$ to $C^I \subseteq \Delta$, each role name $R \in R_R$ to $R^I \subseteq \Delta^I \times \Delta^I$ and each individual name $a \in N_I$ to $a^I \in \Delta$. For concepts of $\mathcal{ALC}$, $C^I$ is defined in the usual way in $\mathcal{ALC}$ interpretations [32]. In particular:

\[ T^I = \Delta \]

\[ \Delta^I = \emptyset \]

\[ (\neg C)^I = \Delta \setminus C^I \]

\[ (C \land D)^I = C^I \cap D^I \]

\[ (C \lor D)^I = C^I \cup D^I \]

\[ (\forall R.C)^I = \{ x \in \Delta \mid \text{for all } y, (x, y) \in R^I \text{ implies } y \in C^I \} \]

\[ (\exists R.C)^I = \{ x \in \Delta \mid \text{for some } y, (x, y) \in R^I \text{ and } y \in C^I \} \]

For the $\mathcal{T}$ operator, we have $(\mathcal{T}(C))^I = \min_{<}(C^I)$.

The notion of satisfiability of a KB in an interpretation is defined as usual. Given an $\mathcal{ALC}$ interpretation $I = (\Delta, <, I)$:

- $I$ satisfies an inclusion $C \subseteq D$ if $C^I \subseteq D^I$;
- $I$ satisfies an assertion $C(a)$ if $a^I \in C^I$;
- $I$ satisfies an assertion $R(a, b)$ if $(a^I, b^I) \in R^I$.

A model $\mathcal{M}$ satisfies a knowledge base $K = (\mathcal{T}, A)$ if it satisfies all the inclusions in its TBox $\mathcal{T}$ and all the assertions in its ABox $A$. A query $F$ (either an assertion $C_L(a)$ or an inclusion relation $C_L \subseteq C_R$) is logically (rationally) entailed by a knowledge base $K$ ($K \models_{\mathcal{ALC} + \mathcal{T}_R} F$) if $F$ is satisfied in all the models of $K$.

As shown in [30], the logic $\mathcal{ALC} + \mathcal{T}_R$ enjoys the finite model property and finite $\mathcal{ALC} + \mathcal{T}_R$ models can be equivalently defined by postulating the existence of a function $k_M : \Delta \rightarrow \mathbb{N}$, where $k_M$ assigns a finite rank to each world: the rank $k_M$ of a domain element $x \in \Delta$ is the length of the longest chain $x_0 \prec \cdots \prec x$ from $x$ to a minimal

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4 Since $\mathcal{ALC} + \mathcal{T}_R$ has the finite model property, this is equivalent to having the Smoothness Condition, as shown in [30]. We choose this formulation because it is simpler.
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Instead,

WStudent

has rank 1, as it is exceptional for

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and has rank 0. In fact, such a

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and hence, it would be an instance of

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by the second inclusion. But, as a

WStudent

is a Student as well, it should satisfy the first defeasible inclusion as well and be an instance of

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which is impossible. Hence, any instance

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of

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cannot have rank 0.
It is easy to see that the rank of the concepts $\text{Student} \sqcap \text{Italian}$, and $\text{Student} \sqcap \text{Italian} \sqcap \neg \text{Pay}_n \sqcap \text{Taxes}$ is 0; that the rank of concepts $\text{Student} \sqcap \text{Italian} \sqcap \neg \text{Pay}_n \sqcap \text{Taxes}$, $\text{WStudent} \sqcap \text{Italian}$ and $\text{WStudent} \sqcap \text{Italian} \sqcap \neg \text{Pay}_n \sqcap \text{Taxes}$ is 1; and that the rank of concept $\text{WStudent} \sqcap \text{Italian} \sqcap \neg \text{Pay}_n \sqcap \text{Taxes}$ is 2.

Rational closure builds on this notion of exceptionality:

**Definition 3 (Rational closure of TBox).** Let $K = (\mathcal{T}, \mathcal{A})$ be a DL knowledge base. The rational closure of $\mathcal{T}$ is defined as:

$$\text{RC}(\mathcal{T}) = \{ T(C) \sqsubseteq D \in \mathcal{T} \mid \text{either rank}(C) < \text{rank}(C \sqcap \neg D) \text{ or } \text{rank}(C) = \infty \} \cup \{ C \sqsubseteq D \in \mathcal{T} \mid \text{KB} \models_{\text{ALC} + \text{T}^n} C \sqsubseteq D \}$$

where $C$ and $D$ are ALC concepts.

In Example 1, $\text{T}(\text{Student} \sqcap \text{Italian}) \sqsubseteq \neg \text{Pay}_n \sqcap \text{Taxes}$ is in the rational closure of the TBox, as $\text{rank}(\text{Student} \sqcap \text{Italian}) < \text{rank}(\text{Student} \sqcap \text{Italian} \sqcap \neg \text{Pay}_n \sqcap \text{Taxes})$; so is $\text{T}(\text{WStudent} \sqcap \text{Italian}) \sqsubseteq \neg \text{Pay}_n \sqcap \text{Taxes}$.

Exploiting the fact that entailment in $\text{ALC} + \text{T}^n$ can be polynomially encoded into entailment in $\text{ALC}$, it is easy to see that deciding if an inclusion $T(C) \sqsubseteq D$ belongs to the rational closure of TBox is a problem in EXPTIME and requires a polynomial number of entailment checks to an $\text{ALC}$ knowledge base. In [30] it is also shown that the semantics corresponding to rational closure can be given in terms of minimal canonical $\text{ALC} + \text{T}^n$ models. In such models the rank of domain elements is minimized to make each domain element to be as typical as possible. Furthermore, canonical models are considered in which all possible combinations of concepts are represented. This is expressed by the following definitions.

**Definition 4 (Minimal models of $K$).** Given $\mathcal{M} = (\Delta, <, I)$ and $\mathcal{M}' = (\Delta', <', I')$, we say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ ($\mathcal{M} \prec \mathcal{M}'$) if: $\Delta = \Delta'$, $C^I = C'^I$ for all (non-extended) concepts $C$, for all $x \in \Delta$, it holds that $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$ whereas there exists $y \in \Delta$ such that $k_{\mathcal{M}}(y) < k_{\mathcal{M}'}(y)$.

Given a knowledge base $K = (\mathcal{T}, \mathcal{A})$, we say that $\mathcal{M}$ is a minimal model of $K$ (with respect to TBox) if it is a model satisfying $K$ and there is no $\mathcal{M}'$ model satisfying $K$ such that $\mathcal{M}' \prec \mathcal{M}$.

The models corresponding to rational closure are required to be canonical. This property, expressed by the following definition, is needed when reasoning about the (relative) rank of the concepts: it is important to have them all represented by some instance in the model.

**Definition 5 (Canonical model).** Given $K = (\mathcal{T}, \mathcal{A})$, a model $\mathcal{M} = (\Delta, <, I)$ satisfying $K$ is canonical if for each set of concepts $\{ C_1, C_2, \ldots, C_n \}$ consistent with $K$, there exists (at least) a domain element $x \in \Delta$ such that $x \in (C_1 \cap C_2 \cap \cdots \cap C_n)^I$.

**Definition 6 (Minimal canonical models (with respect to TBox)).** $\mathcal{M}$ is a minimal canonical model of $K$, if it is a canonical model of $K$ and it is minimal with respect $\prec$ (see Definition 4) among the canonical models of $K$. 

The correspondence between minimal canonical models and rational closure is established by the following key theorem.

**Theorem 1** ([30]). Let \( K = (T, A) \) be a knowledge base and \( C \subseteq D \) a query. Let \( TBox \) be the rational closure of \( K \) w.r.t. \( TBox \). We have that \( C \subseteq D \in TBox \) if and only if \( C \subseteq D \) holds in all minimal canonical models of \( K \) with respect to \( TBox \).

Furthermore: the rank of a concept \( C \) in any minimal canonical model of \( K \) is exactly the rank \( rank(C) \) assigned by the rational closure construction, when \( rank(C) \) is finite. Otherwise, the concept \( C \) is not satisfiable in any model of the \( TBox \).

**Example 2.** Considering again the KB in Example 1, we can see that defeasible inclusions \( T(Student \sqcap Italian) \subseteq \neg Pay\_Taxes \) and \( T(WStudent \sqcap Italian) \subseteq Pay\_Taxes \) are satisfied in all the minimal canonical models of \( K \). In fact, for the first inclusion, in all the minimal canonical models of \( K \), \( Student \sqcap Italian \) has rank 0, while \( Student \sqcap Italian \sqcap Pay\_Taxes \) has rank 1. Thus, in all the minimal canonical models of \( K \) each typical Italian student must be an instance of \( \neg Pay\_Taxes \).

Instead, the defeasible inclusion \( T(WStudent) \subseteq Smart \) is not minimally entailed from \( K \) and, consistently, this inclusion does not belong to the rational closure of \( T \). Indeed, the concept \( WStudent \) is exceptional for \( E_0 \), as it violates the defeasible property of students that, normally, they do not pay taxes \( (T(Student) \subseteq \neg Pay\_Taxes) \).

For this reason, \( WStudent \) does not inherit "any" of the defeasible properties of \( Student \). This problem is a well known problem of rational closure, called by Pearl [44] "the blocking of property inheritance problem", and it is an instance of the "drowning problem" in [6].

To overcome this weakness of the rational closure, Lehmann introduced the notion of lexicographic closure [39], which strengthens the rational closure by allowing, roughly speaking, a class to inherit as many as possible of the defeasible properties of more general classes, giving preference to the more specific properties. The lexicographic closure has been extended to the description logic \( ALC \) by Casini and Straccia in [17]. In the example above, the property of students of being smart would be inherited by working students, as it is consistent with all other (strict or defeasible) properties of working students. In the general case, however, there may be exponentially many alternative bases to be considered, which are all maximally preferred, and the lexicographic closure has to consider all of them to determine which defeasible inclusions can be accepted. In the next section we propose an approach weaker than the lexicographic closure, which leads to the construction of a single base.

### 3 The Skeptical Closure

Given a concept \( B \), one wants to identify the defeasible properties of the \( B \)-elements (if any). Assume that the rational closure of the knowledge base \( K \) has already been constructed and that \( k \) is the (finite) rank of concept \( B \) in the rational closure.\(^5\) The typical

\(^5\) When \( rank(B) = \infty \), the defeasible inclusion \( T(B) \subseteq D \) belongs to the rational closure of \( TBox \) for any \( D \). Hence, we assume \( T(B) \subseteq D \) also belongs to the skeptical closure, and we defer considering this case until Definition 9. So far, we always assume \( k \) to be finite.
\( B \) elements are clearly compatible (by construction) with all the defeasible inclusions in \( E_k \), but they might satisfy further defeasible inclusions with lower rank, i.e. those included in \( E_0, E_1, \ldots, E_{k-1} \).

For instance, in the example above, concept \( W_{\text{Student}} \) has rank 1, and for working students all the defeasible inclusions in set \( E_1 \) above apply (in particular, that typical working students pay taxes). As for \( E_0 \), the defeasible inclusion \( T(\text{Student}) \sqsubseteq \neg \text{PayTaxes} \) is not compatible with this property of typical students, while the defeasible property \( T(\text{Student}) \sqsubseteq \text{Smart} \) is, as there may be typical students which are Smart.

In general, there may be alternative maximal sets of defeasible inclusions compatible with \( B \), among which one would prefer those that maximize the sets of defeasible inclusions with higher rank. This is indeed what is done by the lexicographic closure \([39]\), which considers alternative maximally preferred sets of defaults called “bases”, which, roughly speaking, maximize the number of defaults of higher ranks with respect to those with lower ranks (degree of seriousness), and where situations which violate a number of defaults with a certain rank are considered to be less plausible than situations which violate a lower number of defaults with the same rank. In general, there may be exponentially many alternative sets of defeasible inclusions (called bases in \([39]\)) which are maximal and consistent for a given concept, and the lexicographic closure has to consider all of them to determine if a defeasible inclusion is to be accepted or not. As a difference, in the following, we define a construction which skeptically builds a single set of defeasible inclusions compatible with \( B \). The advantage of this construction is that it only requires a polynomial number of calls to the underlying preferential \( \mathcal{ALC} + \mathcal{T} \mathcal{R} \) reasoner.

Let \( B \) a concept with rank \( k \) in the rational closure. In order to see which are the defeasible inclusions compatible with \( B \) (beside those in \( E_k \)), we first single out the defeasible inclusions which are individually consistent with \( B \) and \( E_k \). This is done while building the set \( S_B \) of the defeasible inclusions which are not overridden by those in \( E_k \). As the set \( S_B \) might not be globally consistent with \( B \), for the presence of conflicting defaults, we will consider the sets of defaults in \( S_B \) with the same rank, going from \( k - 1 \) to \( 0 \) and we will add them to \( E_k \), if consistent. When we find an inconsistent subset, we stop. In this way, we extend \( E_k \) with all the defeasible inclusions which are not conflicting and can be inherited by \( B \) instances, even though the construction of rational closure has excluded them from \( E_k \).

Let \( S_B \) be the set of typicality inclusions \( T(C) \sqsubseteq D \) in the TBox \( \mathcal{T} \) which are individually compatible with \( B \) (with respect to \( E_k \)), that is

\[
S_B = \{ T(C) \sqsubseteq D \in \mathcal{T} \mid E_k \cup \{ T(C) \sqsubseteq D \} \not\models_{\mathcal{ALC} + \mathcal{T} \mathcal{R}} T(\top) \sqsubseteq \neg B \}
\]

For instance, in Example[1] for \( B = W_{\text{Student}} \), which has rank 1, we have that

\[
S_{W_{\text{Student}}} = \{ T(\text{Student}) \sqsubseteq \text{Smart}, T(W_{\text{Student}}) \sqsubseteq \text{PayTaxes} \}
\]

is the set of defeasible inclusions compatible with \( W_{\text{Student}} \) and \( E_1 \). The defeasible inclusion \( T(\text{Student}) \sqsubseteq \neg \text{PayTaxes} \) is not included in \( S_{W_{\text{Student}}} \) as it is not (individually) compatible with \( W_{\text{Student}} \).
Clearly, although each defeasible inclusion in \( S^B \) is compatible with \( B \), it might be the case that overall set \( S^B \) is not compatible with \( B \), i.e.,

\[
E_k \cup S^B \not\models_{\mathcal{A}C^C + \mathbf{TR}} T(\top) \subseteq \neg B.
\]

Let us consider the following variant of Example 1.

**Example 3.** Let \( K' \) be the knowledge base with the TBox:

\[
\begin{align*}
T(\text{Student}) & \subseteq \text{Young} \\
T(\text{Student}) & \subseteq \neg \text{PayTaxes} \\
T(\text{Employee}) & \subseteq \text{PayTaxes} \\
T(\text{Student} \cap \text{Employee}) & \subseteq \neg \text{Young}
\end{align*}
\]

Let \( B = \text{Student} \cap \text{Employee} \). While concepts \( \text{Student} \) and \( \text{Employee} \) have rank 0, concept \( \text{Student} \cap \text{Employee} \) has rank 1. In this example:

\[
\begin{align*}
E_0 &= T \\
E_1 &= \text{Strict}_T \cup \{ T(\text{Student} \cap \text{Employee}) \subseteq \neg \text{Young} \}
\end{align*}
\]

where \( \text{Strict}_T \) is the set of strict inclusions in \( T \). The property that typical employed students are not young, overrides the property that students are typically young. Indeed the default \( T(\text{Student}) \subseteq \text{Young} \) is not individually compatible with \( \text{Student} \cap \text{Employee} \). Instead, the defeasible properties \( T(\text{Student}) \subseteq \neg \text{PayTaxes} \) and \( T(\text{Employee}) \subseteq \text{PayTaxes} \) are both individually compatible with \( \text{Student} \cap \text{Employee} \), and

\[
S^B = \{ T(\text{Student}) \subseteq \neg \text{PayTaxes}, \ T(\text{Employee}) \subseteq \text{PayTaxes} \}.
\]

Nevertheless, the overall set \( S^B \) is not compatible with \( \text{Student} \cap \text{Employee} \). In fact, the two defeasible inclusions in \( S^B \) are conflicting.

When compatible with \( B \), \( S^B \) is the unique maximal basis with respect to the seriousness ordering in [39] (as defined for constructing the lexicographic closure).

When \( S^B \) is not compatible with \( B \), we cannot use all the defeasible inclusions in \( S^B \) to derive conclusions about typical \( B \) elements. In this case, we can either just use the defeasible inclusions in \( E_k \), as in the rational closure, or we can additionally use a subset of the defeasible inclusions \( S^B \). This is essentially what is done in the lexicographic closure, where (in essence) the most preferred subsets of \( S^B \) are selected according to a lexicographic order, which prefers defaults with higher ranks to defaults with lower ranks. In our construction instead, we consider the subsets \( S^B_0, S^B_1 \ldots S^B_{k-1} \) of the set \( S^B \) defined above, by adding to \( E_k \) all the defeasible inclusions in \( S^B \) with rank \( k - 1 \) (let us call this set \( S^B_{k-1} \)), provided they are (altogether) compatible with \( B \) and \( E_k \). Then, we can add all the defeasible inclusions with rank \( k - 2 \) which are individually compatible with \( B \) w.r.t. \( E_k \cup S^B_{k-1} \) (let us call this set \( S^B_{k-2} \)), provided they are altogether compatible with \( B \), \( E_k \) and \( S^B_{k-1} \), and so on and so forth, for lower ranks. This leads to the construction below.

**Definition 7.** Given two sets of defeasible inclusions \( S \) and \( S' \), \( S \) is globally compatible with \( B \) w.r.t. \( E_k \cup S' \) if

\[
E_k \cup S \cup S' \not\models_{\mathcal{A}C^C + \mathbf{TR}} T(\top) \subseteq \neg B
\]
Definition 8. Let $B$ be a concept such that $\text{rank}(B) = k$ ($k$ finite). The skeptical closure of $K$ with respect to $B$ is the set of inclusions $S^{sk,B} = E_k \cup S^B_{k-1} \cup S^B_{k-2} \cup \ldots \cup S^B_h$ where:

- $S^B_i \subseteq E_i - E_{i+1}$ is the set of defeasible inclusions with rank $i$ which are individually compatible with $B$ w.r.t. $E_k \cup S^B_{k-1} \cup S^B_{k-2} \cup \ldots \cup S^B_{i+1}$ (for each finite rank $i < k$);
- $h$ is the least $j$ (for $0 \leq j < k$) such that $S^B_j$ is globally compatible with $B$ w.r.t. $E_k \cup S^B_{k-1} \cup S^B_{k-2} \cup \ldots \cup S^B_{j+1}$, if such a $j$ exists; $S^{sk,B} = E_k$, otherwise.

Intuitively, $S^{sk,B}$ contains, for each rank $j$, all the defeasible inclusions having rank $j$ which are compatible with $B$ and with the more specific defeasible inclusions (having rank $> j$). As $S^B_{h-1}$ is not included in the skeptical closure, it must be that $E_k \cup S^B_{k-1} \cup S^B_{k-2} \cup \ldots \cup S_h \cup S^B_{k-1} \models_{ALC+T\alpha} (\top) \subseteq \neg B$ i.e., the set $S^B_{h-1}$ contains conflicting defeasible inclusions which are not overridden by more specific ones. In this case, the inclusions in $S^B_{h-1}$ (and, similarly, all the defeasible inclusions with rank lower than $h - 1$) are not included in the skeptical closure w.r.t. $B$.

Example 4. For the knowledge base $K$ in Example 1 where $B = \text{WStudent}$ has rank 1, we have $S^B_0 = \{ \text{T(Student) } \sqsubseteq \text{Smart} \}$, which is compatible with $\text{WStudent}$ and $E_1$. Hence, $S^{sk,B} = E_1 \cup S^B_0$.

When a defeasible inclusion belongs to the skeptical closure of a TBox is defined as follows.

Definition 9. Let $K = (\mathcal{T}, \mathcal{A})$ be a knowledge base and $\text{T}(B) \sqsubseteq D$ a query. $\text{T}(B) \sqsubseteq D$ is in the skeptical closure of $\mathcal{T}$ if either $\text{rank}(B) = \infty$ in the rational closure of $\mathcal{T}$ or $S^{sk,B} \models_{ALC+T\alpha} (\top) \subseteq (\neg B \sqcup D)$.

Once the rational closure of TBox has been computed, the identification of the defeasible inclusions in $S^{sk,B}$ requires a number of entailment checks which is linear in the number of defeasible inclusions in TBox. First, the compatibility of each defeasible inclusion in TBox with $B$ has to be checked to compute all the $S^B_j$'s. Then, a compatibility check for each rank of the rational closure is needed, to verify the compatibility of $S^B_j$, for each $j$ from $k - 1$ to 0 in the worst case. The maximum number or ranks in the rational closure is bounded by the number of defeasible inclusions in TBox (but it might be significantly lower in practical cases). Hence, computing the skeptical closure for $B$ requires a number of entailment checks which is, in the worst case, $O(2 \times |\mathcal{T}|)$.

Example 5. For the knowledge base $K$ in Example 1 we have seen that, for $B = \text{WStudent}$ (with rank 1), $S^B_0 = \{ \text{T(Student) } \sqsubseteq \text{Smart} \}$ is (globally) compatible with $\text{WStudent}$ w.r.t. $E_1$, and $S^{sk,B} = E_1 \cup S^B_0$. It is easy to see that $S^{sk,B} \models_{ALC+T\alpha} (\top) \subseteq (\neg \text{WStudent } \sqcup \text{Smart})$, and that $\text{T(WStudent) } \sqsubseteq \text{Smart}$ is in the skeptical closure of TBox. In this case, the typical property of students of being Smart is inherited by working students.

Example 6. For the knowledge base $K'$ in Example 3 as we have seen, $B = \text{Student } \sqcap \text{Employee}$ has rank 1, $E_1 = \{ \text{T(Student } \sqcap \text{Employee) } \sqsubseteq \text{Young} \}$, and $S^B = \ldots$
\{ T(\text{Student}) \sqsubseteq \neg \text{PayTaxes}, T(\text{Employee}) \sqsubseteq \text{PayTaxes} \}. In this case, as \( S^B_0 = S^B \) contains conflicting defaults about tax payment, \( S^B_0 \) is not (globally) compatible with \( \text{Student} \sqcap \text{Employee} \) and \( E_1 \), so that \( S^{sk,B} = E_1 \).

Let us consider the following knowledge base from \[25\] to see that, in the skeptical closure, inheritance of defeasible properties, when not overridden for more specific concepts, applies to concepts of all ranks.

**Example 7.** Consider a knowledge base \( K = (T,A) \), where \( A = \emptyset \) and \( T \) contains the following inclusions:

\[
\begin{align*}
& T(\text{Bird}) \sqsubseteq \text{Fly} \\
& T(\text{Bird}) \sqsubseteq \text{NiceFeather} \\
& \text{Penguin} \sqsubseteq \text{Bird} \\
& T(\text{Penguin}) \sqsubseteq \neg \text{Fly} \\
& T(\text{Penguin}) \sqsubseteq \text{BlackFeather} \\
& \text{BabyPenguin} \sqsubseteq \text{Penguin} \\
& T(\text{BabyPenguin}) \sqsubseteq \neg \text{BlackFeather}. 
\end{align*}
\]

Here, we expect that the defeasible property of birds having a nice feather is inherited by typical penguins, even though penguins are exceptional birds regarding flying. We also expect that typical baby penguins inherit the defeasible property of penguins that they do not fly, although the defeasible property \text{BlackFeather} is instead overridden for typical baby penguins, and that they inherit the typical property of birds of having nice feather. We have that \( \text{rank}(\text{Bird}) = 0 \), \( \text{rank}(\text{Penguin}) = 1 \), and \( \text{rank}(\text{BabyPenguin}) = 2 \) as, in the rational closure construction:

\[
\begin{align*}
E_0 &= \text{Strict}_T \cup \{ T(\text{Bird}) \sqsubseteq \text{Fly}, \ T(\text{Bird}) \sqsubseteq \text{NiceFeather} \} \\
E_1 &= \text{Strict}_T \cup \{ T(\text{Penguin}) \sqsubseteq \neg \text{Fly}, \ T(\text{Penguin}) \sqsubseteq \text{BlackFeather} \} \\
E_2 &= \text{Strict}_T \cup \{ T(\text{BabyPenguin}) \sqsubseteq \neg \text{BlackFeather} \}
\end{align*}
\]

In particular, for \( B = \text{BabyPenguin} \), we get

\[
\begin{align*}
S^B_1 &= \{ T(\text{Penguin}) \sqsubseteq \neg \text{Fly} \} \\
S^B_0 &= \{ T(\text{Bird}) \sqsubseteq \text{NiceFeather} \}
\end{align*}
\]

Also, \( S^B_1 \) is (globally) consistent with \( E_2 \), and \( S^B_0 \) is (globally) consistent with \( E_2 \sqcup S^B_1 \). Hence, \( S^{sk,B} = E_2 \sqcup S^B_1 \sqcup S^B_0 \) = \{ T(\text{BabyPenguin}) \sqsubseteq \neg \text{BlackFeather}, \ T(\text{Penguin}) \sqsubseteq \neg \text{Fly}, \ T(\text{Bird}) \sqsubseteq \text{NiceFeather} \}. Furthermore,

\[
T(\text{BabyPenguin}) \sqsubseteq \text{NiceFeather} \sqcap \neg \text{Fly} \sqcap \neg \text{BlackFeather}
\]

is in the skeptical closure of \( T \) as \( S^{sk,B} \models_{A_{\text{CC}}+T_\text{r}} T(\top) \sqsubseteq (\neg \text{BabyPenguin} \sqcup (\text{NiceFeather} \sqcap \neg \text{Fly} \sqcap \neg \text{BlackFeather})). \)

To see that the notion of skeptical closure is rather weak, let us slightly modify the KB in Example 3 (removing the last inclusion).

**Example 8.** Consider the TBox

\[
\begin{align*}
& T(\text{Student}) \sqsubseteq \text{Young} \\
& T(\text{Student}) \sqsubseteq \neg \text{PayTaxes} \\
& T(\text{Employee}) \sqsubseteq \text{PayTaxes}
\end{align*}
\]
As in Example 3, the rational closure assigns rank 0 to concepts \textit{Student} and \textit{Employee} and rank 1 to \textit{Student} \sqcap \textit{Employee}. In this case, 
\[ E_0 = \{ T(\text{Student}) \sqsubseteq \neg \text{PayTaxes}, \; T(\text{Student}) \sqsubseteq \text{Young}, \; T(\text{Employee}) \sqsubseteq \text{PayTaxes} \}; \]
\[ E_1 = \emptyset; \]
\[ S_B^0 = \{ T(\text{Student}) \sqsubseteq \neg \text{PayTaxes}, \; T(\text{Student}) \sqsubseteq \text{Young}, \; T(\text{Employee}) \sqsubseteq \text{PayTaxes} \}. \]

As \( S_B^0 \) is not (globally) compatible with \textit{Student} \sqcap \textit{Employee} and \( E_1 \), again \( S_{sk,B} = E_1 \). Therefore, the defeasible property that typical students are young is not inherited by typical employed students.

The skeptical closure is a weak construction: in Example 8 due to the conflicting defaults concerning tax payment for \textit{Employee} and \textit{Student} (both with rank 0) also the property that typical students are young is not inherited by the typical employed students. Notice that, the property that typical working students are young would be accepted in the lexicographic closure of \( K' \), as there are two bases, the one including \( T(\text{Student}) \sqsubseteq \neg \text{PayTaxes} \) and the other including \( T(\text{Employee}) \sqsubseteq \text{PayTaxes} \), both containing \( T(\text{Student}) \sqsubseteq \text{Young} \). The skeptical closure is indeed weaker than the lexicographic closure (and, in particular, \( T(\text{Student} \sqcap \text{Employee}) \sqsubseteq \text{Young} \) would be in the lexicographic closure as defined in [17]).

In the next section, we introduce a semantics based on two preference relations. We will show that this semantics characterizes a variant of the lexicographic closure introduced in [25] and exploit it to define a semantic construction for the weaker skeptical closure.

### 4 Refined, bi-preference Interpretations

To capture the semantics of the skeptical closure, we build on the preferential semantics for rational closure of \( \text{ALC} + \text{T} \neg \), introducing a notion of refined, bi-preference interpretation (for short, BP-interpretation), which contains an additional notion of preference with respect to an \( \text{ALC} + \text{T} \neg \) interpretation. We let an interpretation to be a tuple \( M = (\Delta, \prec_{rc}, \prec, I) \), where the triple \( (\Delta, \prec_{rc}, I) \) is a ranked interpretation as defined in Section 2 and \( \prec \) is an additional preference relation over \( \Delta \), with the properties of being irreflexive, transitive and well-founded (but we do not require modularity of \( \prec \)). In BP-interpretations, \( \prec \) represents a refinement of \( \prec_{rc} \).

**Definition 10 (BP-interpretation).** Given a knowledge base \( K \), a bi-preference interpretation (or BP-interpretation) is a structure \( M = (\Delta, \prec_{rc}, \prec, I) \), where \( \Delta \) is a domain, \( I \) is an interpretation function as defined in Definition 7 where, in particular, \( (T(C))^I = \min_{\prec}(C^I) \), and \( \prec_{rc}, \prec \) are preference relations over \( \Delta \), with the properties of being irreflexive, transitive, well-founded. Furthermore \( \prec_{rc} \) is modular.

The bi-preference semantics, builds on a ranked semantics for the preference relation \( \prec_{rc} \), providing a characterization of the rational closure of \( K \), and exploits it to
While the satisfiability conditions (1), (2) and (4) are the same as in Section 2 for the x which is defined as the ranking function k satisfies some defeasible inclusion M < k

Proof. We show that x < r y implies if x < y. If x < r y, then for some r, kM,rc(x) = r < kM,rc(y). As M is a minimal canonical BP-model of K, by the correspondence with the rational closure, x satisfies all the defeasible inclusions in E_r. Instead, y falsifies some defeasible inclusion T(C_k) ⊆ D_k with rank(C_k) = r. As x can only falsify
Corollary 1. Given a knowledge base \( K \) and a BP-model \( M = \langle \Delta, <_{rc}, <, I \rangle \) of \( K \), for all inclusions \( T(C)^I \subseteq D^I \), \( \min_{<}(C^I) \subseteq D^I \) holds.

Proof. From item (2) in Definition 11 we know that \( \min_{<}(C^I) \subseteq D^I \). By Proposition 11 \( <_{rc} \subseteq < \), from which it follows that \( \min_{<}(C^I) \subseteq \min_{<}(C^I) \). Hence, the thesis follows.

We define logical entailment under the BP-semantics as follows: a query \( F \) (of the form \( C_L(a) \) or \( C_L \subseteq C_R \)) is logically entailed by \( K \) in \( ALC^{+\text{T}_{BP}} \) (written \( K \models_{ALC^{+\text{T}_{BP}}} F \)) if \( F \) holds in all BP-models of \( K \).

The following result can be easily proved for BP-entailment:

Theorem 2. If \( K \models_{ALC^{+\text{T}_{R}}} F \) then also \( K \models_{ALC^{+\text{T}_{BP}}} F \). If \( \text{T} \) does not occur in \( F \) the other direction also holds: If \( K \models_{ALC^{+\text{T}_{BP}}} F \) then also \( K \models_{ALC^{+\text{T}_{R}}} F \).

To define a notion of minimal canonical BP-model for \( K \), we proceed as in the semantic characterization of the rational closure in Section 2. Let the a function \( d_M \) associated with the preference relation \( < \) be such that, for any element \( x \in \Delta \): if \( x \in \min_{<}(\Delta) \), then \( d_M(x) = 0 \); otherwise, \( d_M(x) \) is the length of the longest path \( x_0 < x_1 < \ldots < x \) from \( x \) to an element \( x_0 \) such that \( d_M(x) = 0 \).

Although \( < \) is not assumed to be modular, for each domain element \( x \), \( d_M(x) \) represents the distance of \( x \) from the most preferred elements in the model, and can be used for defining the notion of preference \( \sim_{BP} \) among BP-models of \( K \). Let \( \text{Min}_{RC}(K) \) be the set of all BP-models \( M = \langle \Delta, <_{rc}, <, I \rangle \) of \( K \) such that \( \langle \Delta, <_{rc}, I \rangle \) is a minimal canonical model of \( K \) according to the semantics of rational closure in Section 2 (Definition 6). Thus, the models in \( \text{Min}_{RC}(K) \) are those built from the minimal canonical models of the rational closure of \( K \). The minimal (canonical) BP-models of \( K \) will be the models in \( \text{Min}_{RC}(K) \) which also minimize the distance \( d_M(x) \) of each domain element \( x \).

Definition 12 (Minimal canonical BP-Models). Given two BP-models of \( K \), \( M = \langle \Delta, <_{rc}, <, I \rangle \) and \( M' = \langle \Delta', <'_{rc}, <', I' \rangle \) in \( \text{Min}_{RC}(K) \). \( M' \) is preferred to \( M \) (written \( M' \prec_{BP} M \)) if

\[
\Delta = \Delta', I = I', \text{ and }
\]

\[- \text{ for all } x \in \Delta, d_{M'}(x) \leq d_M(x);
\]

\[- \text{ for some } y \in \Delta, d_{M'}(y) < d_M(y)
\]

A BP-interpretation \( M \) is a minimal canonical BP-model of \( K \) if \( M \) is a model of \( K \), \( M \in \text{Min}_{RC}(K) \) and there is no \( M' \in \text{Min}_{RC}(K) \) such that \( M' \prec_{BP} M \).

We denote by \( K \models_{BP}^{\text{min}} F \) entailment with respect to minimal canonical BP-models: for a query \( F \), \( K \models_{BP}^{\text{min}} F \) if \( F \) is satisfied in all the minimal canonical BP-models of \( K \).

Observe that, according to this definition, for computing the minimal (canonical) BP-models of \( K \) one first needs to compute the set of the minimal (canonical) models
of $K$ which characterize rational closure of $K$. Then, among such models, one has to select those which are minimal with respect to $\prec_{BP}$.

Clearly, as minimal canonical BP-models of a KB are minimal ranked models as defined in Section 2, $\prec_{rc}$ corresponds to the preference relation in minimal canonical models of the rational closure, and the rank $k_{M,rc}(x)$ of domain elements will be the same as in the minimal models of rational closure. Thus, by Theorem 1, the value of $k_{M,rc}(C)$, for any concept $C$, in a minimal canonical BP-model is equal to $\text{rank}(C)$, the rank assigned to $C$ by the rational closure construction in Section 2.

The rank of domain elements with respect to $<$ is used to determine the preference relation $<$ on domain elements, according to condition (3). Minimization with respect to $<$ is needed to guarantee that $<$ is minimal, among all the reflexive and transitive preference relations $<$ satisfying condition (3).

Let us consider again Examples 1 and 3 above.

**Example 9.** Let us consider the TBox in Example 1:

$$
\begin{align*}
T(\text{Student}) & \sqsubseteq \neg \text{PayTaxes} \\
T(\text{WStudent}) & \sqsubseteq \text{PayTaxes} \\
T(\text{Student}) & \sqsubseteq \text{Smart} \\
\text{WStudent} & \sqsubseteq \text{Student}
\end{align*}
$$

In all minimal canonical BP-models $M$, $k_{M,rc}(\text{Student}) = 0$, while $k_{M,rc}(\text{WStudent}) = k_{M,rc}(\text{WStudent} \cap \text{Smart}) = k_{M,rc}(\text{WStudent} \cap \neg \text{Smart}) = 1$, as in the model of the rational closure. Let $x$ and $y$ be two elements in the domain of $M$ such that: $k_{M,rc}(x) = k_{M,rc}(y) = 1$, $x \in \text{WStudent} \cap \text{PayTaxes} \cap \text{Smart}$, and $y \in \text{WStudent} \cap \text{PayTaxes} \cap \neg \text{Smart}$. Such elements $x$ and $y$ exist in $M$ as $M$ is canonical. As $y$ violates the typicality inclusion $T(\text{Student}) \sqsubseteq \text{Smart}$, which is satisfied by $x$, and there is no typicality inclusion which is satisfied by $y$ and violated by $x$, by condition (3) in Definition 11, it must be that $x < y$.

Hence, in all the minimal canonical models $M$ of the KB, the domain elements $z$ which are instances of $T(\text{WStudent})$ (and hence must have rank $k_{M,rc}(z) = 1$), not only must be instances of $\text{WStudent} \cap \text{PayTaxes}$ (as the defeasible inclusion $T(\text{WStudent}) \sqsubseteq \text{PayTaxes}$ must be satisfied by all the typical working student), but also must be instances of $\text{WStudent} \cap \text{PayTaxes} \cap \text{Smart}$, as they are preferred in $M$ to $\text{WStudent} \cap \text{PayTaxes} \cap \neg \text{Smart}$ elements. Therefore, $T(\text{WStudent}) \sqsubseteq \text{Smart}$ holds in $M$.

In Example 9, entailment in minimal canonical BP-models captures the defeasible inclusions which belong to the skeptical closure. However, this is not the case in general.

**Example 10.** Let us consider, as a variant of Example 3, a knowledge base $K = (T, A)$ with $A = \emptyset$ and the following TBox $T$:

$$
\begin{align*}
T(\text{Student}) & \sqsubseteq \text{Young} \\
T(\text{Student}) & \sqsubseteq \neg \text{PayTaxes} \cap \exists \text{hasSSN}. \top \\
T(\text{Employee}) & \sqsubseteq \text{PayTaxes} \cap \exists \text{hasSSN}. \top \\
T(\text{Student} \cap \text{Employee}) & \sqsubseteq \neg \text{Young}
\end{align*}
$$
stating that typical students (and typical employee) have a social security number. As in Example 3 in all the minimal canonical BP-model $M$ of $K$, we have $k_{M,rc}(\text{Student}) = k_{M,rc}(\text{Employee}) = 0$ and $k_{M,rc}(\text{Student} \cap \text{Employee}) = 1$, as in the rational closure. As $E_1 = \text{Strict}_T \cup \{T(\text{Student} \cap \text{Employee}) \subseteq \neg \text{Young}\}$, in the skeptical closure construction:

$$S^B_0 = \{ T(\text{Student}) \subseteq \neg \text{PayTaxes} \sqcap \exists \text{hasSSN}.T, \quad T(\text{Employee}) \subseteq \text{PayTaxes} \sqcap \exists \text{hasSSN}.T \}$$

and the set $S^B_0$ is not (globally) compatible with $\text{Student} \cap \text{Employee}$ and $E_1$, so that $S^{sk,B}_0 = E_1$. Hence, $T(\text{Student} \cap \text{Employee}) \subseteq \exists \text{hasSSN}.T$ is not in the skeptical closure of the KB. However, it is easy to see that this defeasible inclusion is satisfied in all the minimal canonical BP-models $M$ of $K$, i.e., $K \models_{BP}^\text{min} T(\text{Student} \cap \text{Employee}) \subseteq \exists \text{hasSSN}.T$.

To see why $K \models_{BP}^\text{min} T(\text{Student} \cap \text{Employee}) \subseteq \exists \text{hasSSN}.T$, let $M = (\Delta, <_{rc}, \prec, I)$ be a minimal canonical BP-model of $K$ and let $\forall y \in T((\text{Student} \cap \text{Employee}))^I = \text{min}_< \forall (\text{Student} \cap \text{Employee})^I$.

(one inclusion holds by Corollary 1). We show that $y \in (\exists \text{hasSSN}.T)^I$. By contradiction, suppose that $y \notin (\exists \text{hasSSN}.T)^I$. As $y \in (\text{Student} \cap \exists \neg \text{hasSSN}.T)^I$ $y$ violates both the second and the third defeasible inclusions in $T$. In the canonical model $M$ there must be an element $x \in \text{min}_{<_{rc}}(\text{Student} \cap \text{Employee})^I$ such that $x \in (\text{PayTaxes} \sqcap \exists \text{hasSSN}.T)^I$, so that $x$ does not violate the second defeasible inclusion $T(\text{Student}) \subseteq \neg \text{PayTaxes} \sqcap \exists \text{hasSSN}.T$, which is violated by $y$. Also, $x$ satisfies the inclusions in $E_1$, so that there is no inclusion which is violated by $x$ and not by $y$. Hence, $x < y$ must hold in $M$, by condition (3) of Definition 11, which contradicts the hypothesis that $y \in T((\text{Student} \cap \text{Employee}))^I$.

The example above shows that entailment in minimal canonical BP-models is too strong for providing a characterization of the skeptical closure: $T(\text{Student} \cap \text{Employee}) \subseteq \exists \text{hasSSN}.T$ is minimally entailed by $K$, but it is not in the skeptical closure of $K$. In the next section we consider a stronger closure construction, which is characterized by minimal canonical BP-models and, from this result, in Section 6 we can provide a semantics for the skeptical closure.

5 Correspondence between BP-models and a variant of lexicographic closure

In this section we show that the semantics of minimal canonical BP-models introduced in the previous section provides a characterization of the multipreference closure (MP-closure, for short), introduced in [25] as a variant of the lexicographic closure [39,17]. More precisely, the MP-closure has been show to provide a sound approximation of the multipreference semantics introduced in [25], a refinement of the rational closure semantics to cope with the “all or nothing” problem.
In the following we recap the definition of MP-closure and we prove that the typicality inclusions which hold in the MP-closure are those entailed from the KB under the minimal canonical BP-models semantics defined in section 4, which thus provides a sound and complete characterization of the MP-closure.

Let $B$ be a concept with rank $k$. Informally, we want to consider all the possible maximal sets of typicality inclusions $S$ which are compatible with $E_k$ and with $B$, i.e. the maximal sets of defeasible properties that a $B$ element can enjoy besides those in $E_k$. For instance, in Example 3 if $B = \text{Student} \cap \text{Employee}$, with rank$(B) = 1$, we have two possible alternative ways of maximally extending the set $E_1$, containing the defeasible inclusion $\text{T(Student) } \subseteq \neg \text{PayTaxes}$ or with the defeasible inclusion $\text{T(Employee) } \subseteq \neg \text{PayTaxes}$. As we have seen in Example 3 these two defeasible inclusions are conflicting, and in the skeptical closure we do not accept any of them. However, here we consider all alternative maximally consistent scenarios, compatible with the fact that the concept $B = \text{Student} \cap \text{Employee}$ is nonempty. In none of these scenarios the defeasible property that normally students are young can be accepted, as it is incompatible with the more specific property that normally employees are not young.

Let $\delta(E_i)$ be the set of typicality inclusions contained in $E_i$ (i.e. those defeasible inclusions with rank $\geq i$) and let $D_i = \delta(E_i) - \delta(E_{i+1})$ be the set of typicality inclusions with rank $i$. Observe that $\delta(E_0) = \delta(T)$. Given a set $S$ of typicality inclusions of the TBox, we let: $S_i = S \cap D_i$, for all ranks $i = 0, \ldots, n$ in the rational closure, thus defining a partition of the typicality inclusions with finite rank in $S$, according to their rank. We introduce a preference relation among sets of typicality inclusions as follows: $S' \prec S$ ($S'$ is preferred to $S$) if and only if there is an $h$ such that, $S_h \subset S'_h$ and, for all $j > h$, $S'_j = S_j$. The meaning of $S' \prec S$ is that, considering the highest rank $h$ in which $S$ and $S'$ do not contain the same defeasible inclusions, $S'$ contains more defeasible inclusions in $D_h$ than $S$.

The preference relation $\prec$ introduced above differs from the one used in the lexicographic closure as the lexicographical order in $[39][17]$ considers the size of the sets of defaults for each rank. Here, the comparison of the sets of defeasible inclusions with the same rank is based on subset inclusion ($S_h \subset S'_h$) and on equality among sets ($S'_j = S_j$) rather than on comparing the size of the sets (e.g., $|S_h| < |S'_h|$ or $|S'_j| = |S_j|$), as in the lexicographic closure. For this reason, the partial order relation $\prec$ is not necessarily modular, which fits with the fact that in BP-interpretations, the partial order relation $\prec$ is not required to be modular.

**Definition 13 ([23]).** Let $B$ be a concept such that rank$(B) = k$ and let $S \subseteq \delta(T \Box)$, $S \cup E_k$ is a maximal set of defeasible inclusions compatible with $B$ in $K$ if:

- $E_k \not\models_{ALC+T_R} T(\top) \land \tilde{S} \subseteq \neg B$
- and there is no $S' \subseteq \delta(T \Box)$ such that $E_k \not\models_{ALC+T_R} T(\top) \land \tilde{S'} \subseteq \neg B$ and $S' \prec S$ ($S'$ is preferred to $S$).

---

6 Observe that, we can ignore the defeasible inclusions with infinite rank when we consider a set of defaults maximally compatible with a concept $B$ (with rank $k$) and with $E_K$, as all the defeasible inclusions with infinite rank already belong to $E_k$. 

where $\tilde{S}$ is the materialization of $S$, i.e., $\tilde{S} = \cap \{(\neg C \sqcup D) \mid T(C) \sqsubseteq D \in S\}$.

Informally, $S$ is a maximal set of defeasible inclusions compatible with $B$ and $E_k$ if there is no set $S'$ which is consistent with $E_k$ and $B$ and is preferred to $S$ since it contains more specific defeasible inclusions. The construction is similar to the lexicographic closure \[39,17\], although, in this case, the lexicographic order $\prec$ is different, and it is easy to see that the MP-closure is weaker than the lexicographic closure (see Example 12 below).

To check if a subsumption $T(B) \sqsubseteq D$ is derivable from the MP-closure of TBox we consider all the maximal sets of defeasible inclusions $S$ that are compatible with $B$.

**Definition 14** (\[25\]). A query $T(B) \sqsubseteq D$ follows from the MP-closure of $T$ if either the rank of concept $B$ in the rational closure of $T$ is infinite or rank($B$) = $k$ is finite and for all the maximal sets of defeasible inclusions $S$ that are compatible with $B$ in $K$, we have:

$$E_k \models_{A_{CC} + T_n} T(\top) \sqcap \tilde{S} \sqsubseteq (\neg B \sqcup D)$$

Verifying whether a query $T(B) \sqsubseteq D$ is derivable from the MP-closure of the TBox in the worst case requires to consider a number of maximal subsets $S$ of defeasible inclusions compatible with $B$ and $E_k$, which is exponential in the number of typicality inclusions in $K$. As entailment in $A_{CC} + T_n$ can be computed in ExPTIME \[30\], this complexity is still in ExPTIME. However, in practice, it is clearly less effective than computing subsumption in the skeptical closure of TBox, which only requires a polynomial number of calls to entailment checks in $A_{CC} + T_n$, which can be computed by a linear encoding of an $A_{CC} + T_n$ KB into $A_{CC}$ \[24\].

**Example 11.** Let us consider again the knowledge base $K = (T, A)$ of Example 11 with $A = \emptyset$ and the following TBox $T$:

- $T(Student) \sqsubseteq Young$
- $T(Student) \sqsubseteq \neg PayTaxes \sqcap \exists hasSSN . \top$
- $T(Employee) \sqsubseteq PayTaxes \sqcap \exists hasSSN . \top$
- $T(Student \sqcap Employee) \sqsubseteq \neg Young$

We have seen that the typicality inclusion $T(Student \sqcap Employee) \sqsubseteq \exists hasSSN . \top$ is not in the skeptical closure of $T$, but it holds in all the minimal canonical BP-models of $K$. We can see that $T(Student \sqcap Employee) \sqsubseteq \exists hasSSN . \top$ follows from the MP-closure of TBox $T$. In fact, in this example there are two maximal sets of defeasible inclusions compatible with $B = Student \sqcap Employee$ (where rank($B$) = 1):

- $S = \{T(Student) \sqsubseteq \neg PayTaxes \sqcap \exists hasSSN . \top, T(Student \sqcap Employee) \sqsubseteq \neg Young\}$
- $S' = \{T(Employee) \sqsubseteq PayTaxes \sqcap \exists hasSSN . \top, T(Student \sqcap Employee) \sqsubseteq \neg Young\}$

where $S$ is partitioned, according to the ranks of defaults, as follows:

- $S_0 = \{T(Student) \sqsubseteq \neg PayTaxes \sqcap \exists hasSSN . \top\}$
- $S_1 = \{T(Student \sqcap Employee) \sqsubseteq \neg Young\}$
- $S_2 = \emptyset$
and $S'$ is partitioned as follows:

$$
S'_0 = \{ T(\text{Employee}) \sqsubseteq \text{PayTaxes} \cap \exists \text{hasSSN}. \top \}
$$

$$
S'_1 = \{ T(\text{Student} \cap \text{Employee}) \sqsubseteq \neg \text{Young} \}
$$

$$
S'_2 = \emptyset
$$

Observe that neither $S \prec S'$ nor $S' \prec S$ and hence both $S$ and $S'$ are maximal sets of defeasible inclusions compatible with $B$. In this case, $S$ and $S'$ would also correspond to the bases of the lexicographic closure of the KB.

We refer to [25] for further examples concerning the MP-closure. Before showing the correspondence between the MP-closure and BP-semantics, let us show an example in which the lexicographic closure allows conclusions which are not in the MP-closure.

**Example 12.** If we modify the knowledge base in Example 11 above, by adding to the TBox the typicality inclusion $T(\text{Student}) \sqsubseteq \neg \text{PayTaxes} \cap \text{Smart}$ we would get again two maximal sets of defeasible inclusions compatible with $B = \text{Student} \cap \text{Employee}$ in the MP-closure construction:

$$
S = \{ T(\text{Student}) \sqsubseteq \neg \text{PayTaxes} \cap \exists \text{hasSSN}. \top, 
\quad T(\text{Student} \cap \text{Employee}) \sqsubseteq \neg \text{Young} \}
$$

$$
S' = \{ T(\text{Employee}) \sqsubseteq \text{PayTaxes} \cap \exists \text{hasSSN}. \top, 
\quad T(\text{Student} \cap \text{Employee}) \sqsubseteq \neg \text{Young} \}
$$

However, only $S$ corresponds to a base in the lexicographic closure, as $S$ contains two defaults with rank 1, while $S'$ contains just one default of rank 1 (and both $S$ and $S'$ contain the same number of defaults of rank 2).

To show that the typicality inclusions derivable form the MP-closure of the KB are exactly those that hold in all the minimal canonical BP-models of the KB, we prove the following two propositions. The next one shows that the MP-closure is sound with respect to the minimal canonical BP-semantics: If $T(B) \sqsubseteq D$ follows from the MP-closure of TBox, then $TBox \models_{\text{min BP}} T(B) \sqsubseteq D$. Let us prove the contrapositive.

**Proposition 2.** Let $T$ be a TBox and $B$ a concept with rank $(B) = k$ a finite rank in the rational closure construction. If there is a minimal canonical BP-model $M = \langle \Delta, <_{rc}, <, I \rangle$ of $T$ and an element $x \in \Delta$ such that $x \notin min_{<} (B^I) \cap \neg D$, then there is a maximal set of defeasible inclusions $S$ compatible with $B$ in $T$, such that

$$
E_k \not\models_{\text{ACC} + T_{\text{RC}}} T(\top) \cap \tilde{S} \sqsubseteq (\neg B \sqcup D)
$$

**Proof.** Assume that for some minimal canonical BP-model $M = \langle \Delta, <_{rc}, <, I \rangle$ of $K$ there is an element $x \in \Delta$ such that $x \notin min_{<} (B^I) \cap \neg D$. Let us define $S$ as the set of all the defeasible inclusions in TBox which are satisfied in $x$, i.e.

$$
S = \{ T(C) \sqsubseteq E \in \text{TBox} \mid x \in (\neg C \sqcup E) \}.
$$

We show that, $E_k \not\models_{\text{ACC} + T_{\text{RC}}} T(\top) \cap \tilde{S} \sqsubseteq (\neg B \sqcup D)$.

Let $M^{RC} = \langle \Delta, <_{rc}, I \rangle$. By construction, $M^{RC}_k$ is a minimal canonical model of the rational closure of $K$. By a property of $M^{RC}$ (Proposition 12 in [30]), $M^{RC}_k$
(i.e., the model obtained by \( M \) by collapsing all the element with rank \( \leq k \) to rank 0) satisfies \( E_k : M_k^{RC} \models_{ACC + T_k} E_k \). Also, as \( \text{rank}(B) = k \) and \( x \in T(B)^I \), \( x \) must have rank \( k \) in \( M_k^{RC} \), and hence rank 0 in \( M_k^{RC} \) (and, clearly, \( k_{M,rc}(x) = k \) in \( M \)). Thus, \( x \in T(T)^I \) holds in \( M_k^{RC} \), but also \( x \in (B \cap \tilde{S})^I \) (by definition of \( S \)). Therefore \( M_k^{RC} \nmid_{ACC + T_k} T(T) \cap \tilde{S} \subseteq \neg B \). Hence, \( E_k \nmid_{ACC + T_k} T(T) \cap \tilde{S} \subseteq \neg B \), i.e., \( S \) is a set of defeasible inclusions compatible with \( B \).

Furthermore, as \( x \in \neg D \), \( E_k \nmid_{ACC + T_k} T(T) \cap \tilde{S} \cap \neg D \subseteq \neg B \), and hence, \( E_k \nmid_{ACC + T_k} T(T) \cap S \subseteq (\neg B \cup D) \), i.e., \( T(B) \subseteq D \) does not follow from the MP-closure of TBox.

To show that \( S \) is a maximal set of defeasible inclusions compatible with \( B \), we have still to show that \( S \) is maximal. Suppose, by contradiction, it is not. Then there is a set \( S' \) such that \( S' < S \) and \( E_k \nmid_{ACC + T_k} T(T) \cap S' \subseteq \neg B \). Therefore, there must be a \( \mathcal{ALC} + T_k \) model \( \mathcal{N} = (\Delta', \prec_{rc}, I') \) of \( E_k \) and an element \( y \in \Delta' \), having rank 0 in \( \mathcal{N} \) such that: \( y \in (S' \cap B)^I' \).

As \( M \) is canonical, then \( M_k^{RC} \) is canonical as well. Hence, there must be an element \( z \in \Delta \) such that \( z \in (S' \cap B)^I \) (i.e., the interpretation of all non-extended concepts in \( z \) is the same as in \( y \) in \( \mathcal{N} \)). As \( y \) has rank 0 in \( \mathcal{N} \), \( y \) satisfies all the defeasible inclusions in \( E_k \). Hence, the concept \( S' \cap B \) must have rank \( k \) in the rational closure and, therefore, \( z \) must have rank \( k \) in \( M_k^{RC} \). Thus, \( z \in (T(T) \cap S' \cap B)^I \) in \( M_k^{RC} \), and, clearly, \( k_{M,rc}(z) = k \) in \( M \).

Since \( S' < S \) there must be some \( h \) such that \( S_h \subseteq S'_h \), and, for all \( j > h \), \( S'_j = S_j \). Thus, there is some defeasible inclusion \( T(C') \subseteq E' \subseteq S' \) such that \( T(C') \subseteq E' \notin S \). so that \( z \) satisfies \( T(C') \subseteq E' \) (i.e., \( z \in (\neg C' \cup E')^I \), while \( x \) violates it (i.e., \( x \in (C' \cap \neg E')^I \)). On the other hand, all the defeasible inclusion violated by \( x \) and not by \( z \) cannot have rank \( \geq h \), as \( x \) satisfies only the inclusions \( S \) (by definition of \( S \)) and, for all \( j \geq h \), \( S'_j = S_j \) (the typicality inclusions with infinite rank are trivially satisfied both in \( x \) and in \( z \)).

Therefore, \( z < x \) holds in \( M \) by condition (3), and \( x \) cannot be a typical \( B \) element, thus contradicting the hypothesis.

The next proposition shows that the MP-closure is complete with respect to the minimal canonical BP-semantics: If \( TBox \models_{min} T(B) \subseteq D \), then \( T(B) \subseteq D \) follows from the MP-closure of TBox. Let us prove the contrapositive.

**Proposition 3.** \( T \) be a TBox and \( T(B) \subseteq D \) a defeasible inclusion such that \( \text{rank}(B) = k \) is a finite rank in the rational closure. If \( T(B) \subseteq D \) does not follow from the MP-closure of \( T \), then there is a minimal canonical MP model \( M = (\Delta, \prec_{rc}, \prec, I) \) of \( T \) and an element \( x \in \Delta \) such that \( x \in \text{min}_{<}(B^I) \cap \neg D \).

**Proof.** If \( T(B) \subseteq D \) does not follow from the MP-closure of \( T \), then there is a maximal set of defeasible inclusions \( S \) compatible with \( B \) in \( K \), such that \( E_k \nmid_{ACC + T_k} T(T) \cap \tilde{S} \subseteq (\neg B \cup D) \).

Then
\[
E_k \nmid_{ACC + T_k} T(T) \cap (\neg (\tilde{S} \cap B \cap \neg D))
\]
and concept $\tilde{S} \cap B \cap \neg D$ is not exceptional with respect to $E_k$ and, in the rational closure, it must have rank less or equal to $k$. As $\text{rank}(B) = k$, it must be $\text{rank}(\tilde{S} \cap B \cap \neg D) = k$.

Let us consider any minimal canonical $\mathcal{ALC} + \mathcal{T}_r$ model $\mathcal{N} = (\Delta', <_{\text{rc}}, I')$ of $K$. As $\text{rank}(\tilde{S} \cap B \cap \neg D) = k$, by Proposition 13 in [30], the concept $\tilde{S} \cap B \cap \neg D$ must have rank $k$ in any minimal canonical model of $K$. Therefore, $k_{\mathcal{N}}(\tilde{S} \cap B \cap \neg D) = k$, and there is an element $y \in \Delta$ such that $y \in (\tilde{S} \cap B \cap \neg D)^I'$ and $k_{\mathcal{N}}(y) = k$.

From $\mathcal{N}$ we build a minimal canonical MP model $\mathcal{M} = (\Delta, <_{\text{rc}}, <, I)$ falsifying $T(B) \subseteq D$ as follows. We let $\Delta = \Delta'$, $I = I'$ and $<_{\text{rc}} = <_{\text{rc}}$. We define $<$ as the transitive closure of $<^1$, where $x <^1 y$ is true if and only if the antecedent of condition (3) in Definition [11] holds, that is:

$$x <^1 y \text{ if and only if}$$
there is some $T(C) \subseteq D \subseteq K$ which is violated by $y$ and,
for all $T(C_j) \subseteq D_j \subseteq K$, which is violated by $x$ and not by $y$,
there is a $T(C_k) \subseteq D_k \subseteq K$, which is violated by $y$ and not by $x$, and
$k_{\mathcal{M},rc}(C_j) < k_{\mathcal{M},rc}(C_k)$.

Observe that, for all concepts $C$, $k_{\mathcal{M},rc}(C) = k_{\text{rc}}(C) = \text{rank}(C)$, the rank of $C$ in the rational closure. We have to show that $\mathcal{M}$ is a minimal canonical MP model of $K$ and that $y \in (T(B) \cap \neg D)^I$.

We first show that $\mathcal{M}$ is an MP model of $K$, that it is canonical and that it is minimal among the canonical MP models of $K$. To show that $\mathcal{M}$ is an MP model of $K$, we observe that, by definition of $<$, condition (3) in Definition [11] holds for $\mathcal{M}$ by construction.

It can be easily seen that $\mathcal{M}$ satisfies the assertions in ABox and the strict inclusions $C \subseteq E$ in TBox, since $\mathcal{N}$ does, $\Delta = \Delta'$ and $I = I'$. To show that $\mathcal{M}$ is an MP model of $K$, we have also to show that for all $T(C) \subseteq E$ in TBox, $\min_{\text{rc}}(C^I) \subseteq E^I$ holds. It follows from the fact that $\min_{\text{rc}}(C^I) \subseteq E^I$ holds in $\mathcal{N}$ and that, by definition of $\mathcal{M}$, $<_{\text{rc}} = <_{\text{rc}}$ and $I = I'$.

We show that $\mathcal{M}$ is a canonical BP model of $K$: If not, there are $C_1, C_2, \ldots, C_n$ such that $K \not \models_{\mathcal{ALC} + \mathcal{T}_r} C_1 \cap C_2 \cap \cdots \cap C_n \subseteq \bot$, but there is no $x \in \Delta$ such that $x \in (C_1 \cap C_2 \cap \cdots \cap C_n)^I$. By Theorem [2] $K \not \models_{\mathcal{ALC} + \mathcal{T}_r} C_1 \cap C_2 \cap \cdots \cap C_n \subseteq \bot$. This would contradict the hypothesis that $\mathcal{N}$ is an $\mathcal{ALC} + \mathcal{T}_r$ canonical model of $K$.

We have to show that $\mathcal{M}$ is minimal among the canonical BP models of $K$. If, by absurdum, $\mathcal{M}$ were not a minimal canonical BP model, then there would be a BP model $\mathcal{M}'' = (\Delta'', <''_{\text{rc}}, <''''_{\text{rc}}, I'')$ in $\text{Min}_{\text{rc}}(K)$, such that $\Delta'' = \Delta$, $I'' = I$, and $\mathcal{M}'' <_{\text{rc}} \mathcal{M}$. Observe that the relation $<''_{\text{rc}}$ in $\mathcal{M}''$ must be equal to $<_{\text{rc}}$, as it is determined by a minimal canonical $\mathcal{ALC} + \mathcal{T}_r$ model (and hence by the rational closure of TBox).

Concerning $<''$, as $\mathcal{M}''$ is an BP interpretation, $<''$ must be transitive and contain $<^1$. Hence, $<''$ must contain the transitive closure of $<^1$. As $<^1$ is defined as the transitive closure of $<^1$, it must be $< \subseteq <''$, which contradicts the hypothesis that that $\mathcal{M}'' <_{\text{rc}} \mathcal{M}$.

Finally, we want to show that $y \in (T(B) \cap \neg D)^I$. We have seen that in $\mathcal{N}$ there is an element $y \in \Delta$ such that $y \in (\tilde{S} \cap B \cap \neg D)^I'$ and $k_{\mathcal{N}}(y) = k$. By construction of $\mathcal{M}$, $I = I'$ and then $y \in (B \cap \neg D)^I$. Furthermore, $<_{\text{rc}} = <_{\text{rc}}$ and, hence, $k_{\mathcal{M},rc}(y) = k_{\mathcal{N}}(y) = k$ and, also, $k_{\mathcal{M},rc}(B) = k_{\mathcal{N}}(B) = \text{rank}(B) = k$. 

To see that $y \in min_{\prec}(B)$, we need to show that there is no $z \in \Delta$ such that $z \in B^I$ and $z < y$. Suppose by contradiction that there is such a $z$. As $z$ is a $B$-element, it cannot have rank less than $k$ in the rational closure. Hence, it must be $k_{M,rc}(z) = k$.

Let $S'$ be the set of defeasible inclusions satisfied by $z$, i.e., $S' = \{T(C) \subseteq E \in \text{TBox} \mid z \in (\neg C \sqcup E)\}$. Then $z \in (S' \cap B)^I$. Let $M^{RC} = (\Delta, <_{rc}, I)$ be the $\text{ALC} + \text{T}_h$ model obtained from $M$, ignoring the preference relation $\prec$. By Proposition 12 in [20], $M^{RC}_k \models_{\text{ACC} + \text{T}_h} E_k$ and, as $k_{M,rc}(z) = k$, $z$ must have rank 0 in $M^{RC}_k$. Therefore,

$$E_k \not\models_{\text{ACC} + \text{T}_h} T(\Delta) \cap S' \subseteq \neg B.$$  

As $z < y$, for all defeasible inclusions $T(C_j) \subseteq A_j \in K$ violated by $z$ and satisfied by $y$, there is a more specific defeasible inclusion $T(C_j') \subseteq A_j \in K$ violated by $y$ and satisfied by $z$ (that is $k_{M,rc}(C_j') < k_{M,rc}(C_j)$). Suppose that $j$ is the rank of the defeasible inclusion with highest rank violated by $z$ and that $h$ is the rank of the defeasible inclusion with highest rank violated by $y$. It must be $j < h$. Therefore, $S_h \subseteq S'_h$ (as $z$ satisfies all the defeasible inclusions of rank $h$). Therefore, $S'$ is preferred to $S$, $S' \prec S$. However, this contradicts the hypothesis that $S$ is a maximal set of defeasible inclusions compatible with $B$ in $K$. Therefore, $z$ with $z < y$ cannot exist and $y \in T(B)^I$, so that $y \in (T(B) \cap \neg D)^I$.

We can now establish a correspondence between the minimal canonical MP models semantics and the MP closure.

**Theorem 3.** Given a knowledge base $K = (T, A)$ and a query $T(B) \subseteq D$, $T \models_{\text{MP}} T(B) \subseteq D$ if and only if $T(C) \subseteq D$ follows from the MP-closure of the TBox $T$.

**Proof.** The proof of this result can be done by contraposition and is an easy consequence of Proposition 1 and Proposition 2. Just observe that, for the “If” part, when $T \not\models_{\text{MP}} T(B) \subseteq D$, concept $B$ must have a finite rank, otherwise $T(B) \subseteq D$ would be a logical consequence of $T$, for any concept $D$. For the “Only if” part, when $T(C) \subseteq D$ does not follow from the MP-closure of the TBox $T$, the rank of $B$ in the rational closure must be finite.

In [25] we have shown that the MP-closure provides a sound approximation of a multi-preference semantics, the $S$-enriched semantics. From the correspondence result above (Theorem 3), it also follows that entailment with respect to the minimal canonical BP-models (as defined in Section 4) is strictly weaker than entailment with respect to the minimal canonical S-enriched models defined in [25].

### 6 A semantic characterization for the skeptical closure

First we show that we can equivalently reformulate the notion of global compatibility of a set of defeasible inclusions (Definition 7), as stated by the following property:

**Proposition 4.** Let $T$ be a TBox and $B$ be a concept with finite rank $\text{rank}(B) = k$. Given two sets of defeasible inclusions $S$ and $S'$, $S$ is (globally) compatible with $B$ w.r.t. $E_k \cup S'$ if and only if

$$E_k \not\models_{\text{ACC} + \text{T}_h} T(\Delta) \cap S \cap S' \subseteq \neg B$$
where $\tilde{S}$ is the materialization of $S$, i.e., $\tilde{S} = \cap \{ (\neg C \cup D) \mid T(C) \subseteq D \in S \}$.

Proof. Remember that $E_k \subseteq T$ is the set of defeasible inclusion having rank $\geq k$ in the rational closure construction. We show that, for any set $H$ of defeasible inclusions in $T$:

$$E_k \cup H \models_{ACC+T_n} T(\top) \subseteq \neg B \iff E_k \models_{ACC+T_n} T(\top) \cap \tilde{H} \subseteq \neg B.$$  

(=) By contraposition, suppose $E_k \cup H \not\models_{ACC+T_n} T(\top) \subseteq \neg B$. Then, there is an $ALC + T_n$ model $M = \langle \Delta, \prec, I \rangle$ of $E_k \cup H$, and a domain element $x \in \Delta$ such that $k_M(x) = 0$ and $x \in B^1$.

We show that $x \in \tilde{H}^I$. Let us consider any typicality inclusion $T(C) \subseteq D$ in $H$. We show that $x$ is an instance of its materialization $\neg C \cup D$, i.e., $x \in (\neg C \cup D)^I$. If $x \not\in C^I$, the conclusion follows trivially. If $x \in C^I$, the considering that $x$ has rank 0 in $\tilde{M}$ and that $M$ satisfies $T(C) \subseteq D$, $x$ is a typical element and hence it must be $x \in D^I$. Therefore, $x \in (\neg C \cup D)^I$. As this holds for all the typicality inclusion in $H$, $x \in \tilde{H}^I$ and, hence, $x \in (T(\top) \cap \tilde{H} \cap B)^I$, which proves the thesis.

(⇒) By contraposition, let $E_k \not\models_{ACC+T_n} T(\top) \cap \tilde{H} \subseteq \neg B$. Then, there is a model of $M_1 = \langle \Delta_1, <, I_1 \rangle$ of $E_k$, and a domain element $x \in \Delta_1$ such that $x \in (T(\top) \cap \tilde{H} \cap B)^I$, i.e., $k_{M_1}(x) = 0$, $x \in \tilde{H}^{I_1}$ and $x \in B^{I_1}$.

The model $M_1$ might not satisfy all the typicality inclusions $T(C) \subseteq D$ in $H$. Let us consider a model of $E_k \cup H$. Such a model must exist, otherwise, the TBox $T$ would be unsatisfiable and any concept would have an infinite rank in the rational closure of $T$. Conversely, we know that $B$ has a finite rank $k$. Hence, let $M = \langle \Delta, \prec, I \rangle$ be a finite minimal canonical model of $E_k \cup H$. Existence of a finite, minimal, canonical models of a consistent TBox in $ALC + T_n$ is guaranteed by Theorem 7 in [30]. Suppose that $\Delta$ and $\Delta_1$ are disjoint. We build from $M$ and $M_1$ a new model $M'$ of $E_k \cup H$ in which the concept $T(\top) \cap \tilde{H} \cap B$ is satisfiable.

Let us define $M' = \langle \Delta', \prec', I' \rangle$ as follows: $\Delta' = \Delta \cup \Delta_1$; $I'$ is defined on the elements of $\Delta$ as $I$ in $M$, and on the elements of $\Delta_1$ as $I_1$ in $M_1$. For the interpretation of concepts: for $x \in \Delta$, $x \in C'^I$ if and only if $x \in C^I$; for $x \in \Delta_1$, $x \in C'^I$ if and only if $x \in C^I$. For the interpretation of roles: for $(x, y) \in \Delta$, $(x, y) \in R'$ if and only if $(x, y) \in R$; for $(x, y) \in \Delta_1$, $(x, y) \in R'$ if and only if $(x, y) \in R_1$; for any two elements $x \in \Delta$ and $y \in \Delta_1$, $(x, y) \not\in R'$ and $(y, x) \not\in R'$. For all individual constants $a \in O$, we let $a'= a^I$. Finally, for all $w \in \Delta$, we let $k_{M'}(w) = k_M(w)$, for the element $x \in \Delta_1$ which is an instance of $T(\top) \cap \tilde{H} \cap B$, we let $k_{M'}(x) = 0$; finally, for all $y \in \Delta_1 (y \neq x)$, we let $k_{M'}(y) = n + 1 + k_{M_1}(y)$, where $n$ is the highest value of $k_M$ in $M$.

It is easy to show that by construction the resulting model $M'$ satisfies $E_k \cup H$. Let $C \subseteq D$ be strict inclusion in $E_k \cup H$. In the first case, $C \subseteq D$ is a strict inclusion. Let $x \in C'^I$ and $x \in C^I$. There are two cases: either $x \in \Delta$ or $x \in \Delta_1$. In the first case, $x \in C'^I$ in $M$. As $M$ satisfies $K$, $x \in D^I$ and, by definition of $M'$, $x \in D'^I$. In the second case, $x \in C^I$. As $M_1$ satisfies all the strict inclusions in $T$ (which belong to $E_k$), $x \in D'^I$ and, by definition of $M'$, $x \in D'^I$.

Let $T(C) \subseteq D$ be a defeasible inclusion in $E_k \cup H$. If $\text{rank}(C) \geq k$, then by the construction of the rational closure $T(C) \subseteq D$ is in $E_k$ and hence is satisfied both in $M$ and in $M_1$. Let $z \in (T(C))^I$, then either $z \in \Delta$ or $z \in \Delta_1$. In the first case, $z$
is $C$-minimal in $\mathcal{M}$ and $z \in D^I$. Hence, by definition of $\mathcal{M}'$, $z \in D^{I'}$. In the second case, $z$ is $C$-minimal in $\mathcal{M}_1$ and $z \in D^{I_1}$. Hence, by definition of $\mathcal{M}'$, $z \in D^{I'}$.

If $\text{rank}(C) = j < k$, then $\mathbf{T}(C) \subseteq D$ is in $H$ but not in $E_k$. As the rank of $C$ in the rational closure is finite, by Proposition 13 in [30], $C$ has finite rank $j$ in any minimal canonical model of the TBox $\mathcal{T}$. Hence, $C$ is consistent with the TBox $\mathcal{T}$, as well as with its subset $\mathcal{E}_k \cup H \subseteq \mathcal{T}$. As $\mathcal{M}$ is a canonical model of $\mathcal{E}_k \cup H \subseteq \mathcal{T}$, there must be an element in $w \in \Delta$ such that $w \in C^I$. Therefore, each minimal $C$ element in $\mathcal{M}$ either is $x$ (and, in this case, $x$ is in $(\neg C \cap D)^I$ and hence in $D'^I$), or it is an element $z \in \Delta$. As $\mathcal{M}$ satisfies $H$, it satisfies $\mathbf{T}(C) \subseteq D$ and, hence, $z \in D$.

From this, we can conclude that $\mathcal{M}'$ is a model satisfying $\mathcal{E}_k \cup H$, which contains an element $x$ with rank $k_{\mathcal{M}'}(x) = 0$ such that $x \in B$. Therefore, $\mathcal{E}_k \cup H \not\models_{\mathbf{ALC}+\mathbf{T}_N} \mathbf{T}(\top) \subseteq \neg B$, which concludes the proof.

The above reformulation of the notion of global compatibility makes the relationship between the notion of skeptical closure and the notion of MP-closure more evident.

In particular, for a concept $B$ with $\text{rank}(B) = k$, when (in the MP-closure construction) there is a single maximal set of defeasible inclusions $S$ compatible with $B$ in $\mathcal{T}$, i.e., such that $\mathcal{E}_k \not\models_{\mathbf{ALC}+\mathbf{T}_N} \mathbf{T}(\top) \cap S \subseteq \neg B$, then $\mathcal{E}_k \cup S$ corresponds to the skeptical closure $S^{k,B}$ of $\mathcal{T}$ with respect to $B$.

When in the MP-closure there are different maximal sets of defeasible inclusions $S^1, \ldots, S^r$ compatible with $B$ in $\mathcal{T}$, the skeptical closure is defined to contain, in addition to $\mathcal{E}_k$, the defeasible inclusions with rank $j$ in $S^1, \ldots, S^r$, for those ranks $j$ from $h$ to $k-1$ on which $S^1, \ldots, S^r$ exactly agree (i.e., $S^1_j = \ldots = S^r_j$), where $h-1$ is the highest rank on which $S^1, \ldots, S^r$ disagree (i.e., $S^l_{h-1} \neq S^m_{h-1}$, for some $l$ and $m$). If the sets $S^1, \ldots, S^r$ disagree on some defeasible inclusion with rank $j$, no defeasible inclusion with rank $j$ or lower is included in the skeptical closure.

Based on the reformulation above and on the correspondence between the MP-closure of a knowledge base and its minimal canonical BP-models, we are now able to provide a semantic characterization of the skeptical closure.

Given a TBox $\mathcal{T}$, let $DI(B)$ be the set of the defeasible inclusions $\mathbf{T}(C) \subseteq D \in \mathcal{T}$ which are satisfied by all the minimal $B$ elements in any the minimal canonical BP-models of $\mathcal{T}$:

$$DI(B) = \{ \mathbf{T}(C) \subseteq D \in K \mid x \in (\neg C \cup D)^I \text{, for any } x \in \text{min}_{<}(B^I) \text{ in any minimal canonical BP-model } \mathcal{M} = \langle \Delta, <_{\text{rc}}, <_{\cdot}, \cdot ^{\cdot} \rangle \text{ of } \mathcal{T} \}$$

Let $Conf_{\mathbf{DI}}(B)$ be the set of the conflicting defeasible inclusions for $B$ in $\mathcal{T}$, defined as the typicality inclusions which are satisfied in some minimal $B$ element in a minimal canonical BP-model of $\mathcal{T}$, but not in all of them:

$$Conf_{\mathbf{DI}}(B) = \{ \mathbf{T}(C) \subseteq D \in K \mid x \in (\neg C \cup D)^I \text{ and } y \in (C \cap \neg D)^I \text{ for some minimal canonical BP-model } \mathcal{M} = \langle \Delta, <_{\text{rc}}, <_{\cdot}, \cdot ^{\cdot} \rangle \text{ of } \mathcal{T} \text{ and for some } x, y \in \text{min}_{<}(B^I) \}$$

They are the defaults on which there is no agreement among minimal $B$ elements in at least some minimal canonical BP-model of $\mathcal{T}$. Let $S$ be all the concepts occurring in
the knowledge base or in the query, and let $C_j$ be the set of all the concepts with rank $j$:

$$C_j = \{ C \in S \mid k_M(C^I) = j \} \text{ in any minimal canonical BP-model}$$

$M = \langle \Delta, <_{rc}, <, \cdot^I \rangle$ of $T$

We identify the defeasible inclusions with rank $j$ in $DI(B)$ and in $Conf_D(IN(B))$, respectively:

$$DI_j(B) = DI(B) \cup C_j$$

$$Conf_D(IN(B)) = Conf_D(IN(B)) \cup C_j$$

We can now define the set of defeasible inclusions which are included in the skeptical closure of $B$, $S_{sk,B}$, as follows:

$$DI_{Sk}(B) = \bigcup_{j=h,k-1} DI_j(B)$$

where $h$ is the lowest integer, form 0 to $k - 1$, such that, for all $j > h$, $Conf_D(IN(B)) = \emptyset$.

$DI_{Sk}(B)$ is the set of defeasible inclusions which are included in the skeptical closure of $B$, $S_{sk,B}$. Essentially, $DI_{Sk}(B)$ contains the defeasible inclusions on which all the minimal canonical models agree, in the following sense: for each rank $j$, from $h$ to $k - 1$, $DI_j(B)$ is the set of all the defeasible inclusions of rank $j$ which are satisfied by all the minimal $B$-elements in all the minimal canonical BP-model of $T$. Also, the minimal $B$ elements of the minimal canonical BP-models of $T$ must agree on accepting or not the defeasible inclusions with rank $\geq h$ (as there are no conflicting defeasible inclusions of rank $\geq h$ for $B$). Instead, they disagree on accepting or not some defeasible inclusion with rank $h - 1$.

**Proposition 5.** Let $T(B) \sqsubseteq D$ be a query and $\mathcal{T}$ a TBox. The defeasible inclusion $T(B) \sqsubseteq D$ is in the skeptical closure of $TBox$ if and only if

$$Strict(\mathcal{T}) \cup DI_{Sk}(B) \models_{ALC + TR} T(\mathcal{T}) \sqsubseteq (\neg B \sqcup D)$$

where $Strict(\mathcal{T})$ is the set of strict inclusions in $\mathcal{T}$.

### 7 Conclusions and related work

We have introduced the skeptical closure which is a weaker variant of the lexicographic closure [39][17], which deals with the problem of “all or nothing” affecting the rational closure without generating alternative “bases”. Its computation only requires a polynomial number of calls to the underlying preferential $ALC + TR$ reasoner.

Other refinements of the rational closure, which also deal with this limitation of the rational closure, are the relevant closure [12] and the inheritance-based rational closure [16][18]. In particular, in [16][18], a new closure construction is defined by combining the rational closure with defeasible inheritance networks. This inheritance-based rational closure, in Example 8, is able to conclude that typical working students are young,
relying on the fact that only the information related to the connection of \textit{WStudent} and \textit{Young} (and, in particular, only the defeasible inclusions occurring on the routes connecting \textit{WStudent} and \textit{Young} in the corresponding net) are used in the rational closure construction for answering the query.

Another approach which deals with the above problem of inheritance blocking has been proposed by Bonatti et al. in \cite{7}, where the logic $\mathcal{DL}^N$ captures a form of “inheritance with overriding”: a defeasible inclusion is inherited by a more specific class if it is not overridden by more specific (conflicting) properties. In Example\ref{ex:8} our construction behaves differently from $\mathcal{DL}^N$, as in $\mathcal{DL}^N$ the concept \textit{WStudent} has an inconsistent prototype, as working students inherit two conflicting properties by superclasses: the property of students of non paying taxes and the property of workers of paying taxes. Instead, in the skeptical closure one cannot conclude that $\top \models \text{\textit{WStudent}}$ and, using the terminology in \cite{7}, the conflict is “silently removed”. In this respect, the skeptical closure appears to be weaker than $\mathcal{DL}^N$, although it shares with $\mathcal{DL}^N$ (and with the lexicographic closure) a notion of overriding.

Bozzato et al. in \cite{10} present an extension of the CKR framework in which defeasible axioms are allowed in the global context and can be overridden by knowledge in a local context. Exceptions have to be justified in terms of semantic consequence. A translation of extended CHRs (with knowledge bases in $\mathcal{SROIQ}$-\textit{RL}) into Datalog programs under the answer set semantics is also defined.

Concerning the multipreference semantics introduced in \cite{33} (and further refined in \cite{31}) to provide a semantic strengthening of the rational closure, we have shown in \cite{31} that the MP-semantics, a variant of Lehmann’s lexicographic closure which does not take into account the number of defaults within the same rank, but only their subset inclusion (as recalled in Section \ref{sec:5}), provides a sound approximation of the multipreference semantics. In this paper we have given a semantic characterization of the MP-closure by bi-preference minimal entailment. As a consequence, BP-minimal entailment is weaker that the multipreference semantics and, furthermore, the skeptical closure introduced in Section \ref{sec:5} is still a sound, weaker, approximation for the multipreference semantics in \cite{31}.

The relationships among the above variants of rational closure for DLs and the notions of rational closure for DLs developed in the contexts of fuzzy logic \cite{19} and probabilistic logics \cite{40} are worth of being investigated. As it has been show in \cite{14} for the propositional logic case, KLM preferential logics and the rational closure \cite{37,38}, the probabilistic approach \cite{1}, the system \textit{Z} \cite{44} and the possibilistic approach \cite{54} are all related with each other, and similar relations might be expected to hold for the non-monotonic extensions of description logics as well. Although the skeptical closure has been defined based on the preferential extension of \textit{ALC}, the same construction could be adopted for more expressive description logics, provided the rational closure can be consistently defined \cite{26}, as well as for the propositional case.

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