The non-compact XXZ spin chain as stochastic particle process

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Abstract

In this note we relate the Hamiltonian of the integrable non-compact spin $s$ XXZ chain to the Markov generator of a stochastic particle process. The hopping rates of the continuous-time process are identified with the ones of a q-Hahn asymmetric zero range model. The main difference with the asymmetric simple exclusion process (ASEP), which can be mapped to the ordinary XXZ spin chain, is that multiple particles can occupy one and the same site. For the non-compact spin $\frac{1}{2}$ XXZ chain the associated stochastic process reduces to the multiparticle asymmetric diffusion model introduced by Sasamoto-Wadati.

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1 Introduction

The XXZ spin chain remains one of the most studied integrable models solvable by Bethe ansatz [1]. Still it seems that there are many corners to be yet discovered and open problems to be solved, see e.g. [2–5] for some excellent literature on the topic. The bulk model is described by the nearest neighbor Hamiltonian

$$H = \sum_{i=1}^{N-1} H_{i,i+1}$$

which is given in terms of the Hamiltonian density $H$ and the number of spin chain sites $N$. Since the Leningrad school of the quantum inverse scattering method it is known that the Hamiltonian density of the Heisenberg spin chain admits a concise expression in terms of the digamma function which only depends on the representations in the irreducible tensor product decomposition of two neighboring sites [4, 6]. This formula arises as the logarithmic derivative of the corresponding R-matrix which is written in terms of gamma functions. The q-analogs of the Hamiltonian density and the R-matrix were given in [7], see also [8, 9] for alternative presentations of the R-matrix. In analogy to the rational case, the Hamiltonian density is written in terms of the q-analog of the digamma function and the R-matrix via the q-analog of the gamma function. Despite the beauty of this expression it is not immediately obvious how the Hamiltonian density acts on the tensor product of two sites which is often what one would like to know when studying a physical problem. The change of basis can be fulfilled using Clebsch-Gordan coefficients which can however be quite cumbersome, especially when looking at non-compact spin chains.

Stochastic particle processes of continuous time are usually defined through the master equation which is given in terms of the Markov matrix. The latter contains the rates with which the particles of the system move. In the following we restrict ourselves to one-dimensional chains where particles can “hop” only to the neighboring sites. In this case the Markov matrix can be written in terms of local Markov generators as

$$M = \sum_{i=1}^{N-1} M_{i,i+1}$$

similar to the Hamiltonian in (1.1). It is well known that the Hamiltonian of the ordinary finite-dimensional XXZ chain can be related to the Markov matrix of the asymmetric simple exclusion process (ASEP), see e.g. [10, 11] for an overview. As a consequence the ASEP can be solved using Bethe ansatz methods. The Hamiltonian density of the ordinary finite-dimensional XXZ chain is usually written in terms of Pauli matrices at two neighboring sites. It belongs to the family of integrable XXZ chains mentioned above and can thus be written in terms of the digamma function, see [7]. One may expect that all XXZ spin chains that arise for different representations of $U_q(sl_2)$ from the integrable Hamiltonian density can be mapped to a stochastic particle system. This however does not seem to be the case. Furthermore, as the mapping depends on the choice of the basis, the stochastic process related to a Hamiltonian may be difficult to identify.

Recently, a relation between rational non-compact spin chains and stochastic processes was pointed out in [12]. The types of spin chains considered appeared previously in relation to high-energy QCD and $\mathcal{N} = 4$ super Yang-Mills theory in the context of the AdS/CFT correspondence, see [13] for further references as well as [14] and references thereof. It was found in [12] that the Hamiltonian density of such spin chains yields the rational limits of a class of stochastic transport models that were known to be integrable. More precisely, the multiparticle asymmetric diffusion model (MADM) and a more general q-Hahn process that were defined in [15] and [16] respectively. Both arise from the zero range chipping model proposed in [17] which has been studied using commuting transfer matrices of the so-called stochastic R-matrix in [18].

Here we focus on connecting q-deformed non-compact (XXZ) spin chains with stochastic particle processes. From the many aspects that were studied in [12], which presumably do have a q-analog, we restrict ourselves to setting the stage and relate the Hamiltonian of the non-compact XXZ chain to the Markov matrix of a stochastic q-Hahn process.
The paper is organised as follows. First we give some background about the relevant representations of \( U_q(sl_2) \) and the non-compact XXZ spin chain, see Section 2. In Section 3 we define the Markov process via the local Markov matrix which is given in terms of the rates of a q-Hahn process. Section 4 contains the relation between the Hamiltonian of non-compact XXZ chain and the Markov matrix of the stochastic process. This relation is proven, following [19], in the subsections. Further we study some limiting cases including the MADM and rational case in Section 5 and elaborate on an apparent connection between the TAZRP and Baxter Q-operators. Finally we conclude in Section 6. In Appendix A we collected the definition of some special functions.

2 Hamiltonian density of the XXZ spin chain

The non-compact XXZ spin chain can be defined through its Hamiltonian density acting on two neighboring sites of the spin chain

\[
\mathcal{H} : D_s \otimes D_s \rightarrow D_s \otimes D_s .
\]  

(2.1)

Here \( D_s \) denotes a spin \( s \) lowest weight \( U_q(sl_2) \) module. It is spanned by linearly independent vectors \(|m\rangle\) with \( m=0,1,\ldots \) on which the action of the q-deformed spin generators \( S_\pm \) and \( S_0 \) is defined as

\[
S_+|m\rangle = (m+2s)|m+1\rangle , \quad S_-|m\rangle = |m| |m-1\rangle , \quad S_0|m\rangle = (m+s)|m\rangle .
\]  

(2.2)

For simplicity we assume that the spin variable is a positive half integer, i.e. \( 2s \in \mathbb{N} \). The spin generators satisfy the standard \( U_q(sl_2) \) commutation relations

\[
[S_+, S_-] = -[2S_0] , \quad [S_0, S_\pm] = \pm S_\pm ,
\]  

(2.3)

with the q-number

\[
[x] = \frac{q^x - q^{-x}}{q - q^{-1}}.
\]  

(2.4)

The Hamiltonian density is known to be invariant under \( U_q(sl_2) \), i.e. it satisfies the commutation relations

\[
[\Delta(S_0), \mathcal{H}] = 0 , \quad [\Delta(S_\pm), \mathcal{H}] = 0 .
\]  

(2.5)

Here we have defined the co-product \( \Delta \) which is given via

\[
\Delta(S_0) = S_0 \otimes 1 + 1 \otimes S_0 , \quad \Delta(S_\pm) = S_\pm \otimes q^{-S_0} + q^{S_0} \otimes S_\pm .
\]  

(2.6)

It can be shown to be compatible with the commutation relations (2.3). The Hamiltonian density can be derived purely algebraically, as done for the rational XXX spin chain in [4, 6], by solving the Yang-Baxter equation for certain R-operators. The logarithmic derivative then yields the Hamiltonian density. This has been done for the trigonometric XXZ chain in [7]. The result can be written as

\[
\mathcal{H}(S) = \frac{\psi_q(S) - \psi_q(2s)}{-q^{4s} \log(q)}
\]  

(2.7)

with the q-analog of the digamma function \( \psi_q(x) \) which we define following Bytsko as the logarithmic derivative \( \psi_q(x) = \partial_x \log \Gamma_q(x) \) of the q-analog of the Gamma function

\[
\Gamma_q(x) = q^{x(1-x)} (q^{-1} - q)^{1-x} (q^2; q^2)_{\infty} (q^{2x}; q^2)_{\infty} ,
\]  

(2.8)

for \( |q| < 1 \) which we assume throughout the paper. Further we introduced \( (a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k) \).

The operator \( S \) in (2.7) is related to the two site Casimir via

\[
\Delta(C) = [S][S - 1] .
\]  

(2.9)
It can be computed explicitly using $C = [S_0][S_0 - 1] - S_+ S_-$ and the action of the co-product defined in (2.6). The action of the operator $S$ is diagonal on each module in the irreducible tensor product decomposition

$$D_s \otimes D_s = \bigoplus_{j=0}^{\infty} D_{2s+j},$$

and coincides with the action of $\Delta(S_-)$ on the corresponding lowest weight state. More precisely, given a lowest weight state $|\Phi_{2s+j}\rangle$ we have

$$\Delta(S_-)|\Phi_{2s+j}\rangle = 0, \quad \Delta(S_0)|\Phi_{2s+j}\rangle = S|\Phi_{2s+j}\rangle = (2s+j)|\Phi_{2s+j}\rangle.$$  

As it has been discussed at various points in the literature, the form of Hamiltonian density in (2.7) is not very convenient to study its action on the tensor product of two sites. In principle it can be obtained using Clebsch-Gordan coefficients, see e.g. [20]. The change of basis is however quite involved. We will come back to this problem later in Section 4.

### 3 Markov generator and the q-Hahn process

In the following we give the local Markov matrix that is obtained from the transfer rates of the q-Hahn process. These arise from the model introduced by Povolotsky [17] and were given in [16].

Let $(m,m')$ denote a configuration at two neighboring sites with $m$ particles at the first and $m'$ particles at the second site. The local generator of the Markov process $\mathcal{M}$ describes the transition rates to a configuration $(\tilde{m},\tilde{m}')$

$$\mathcal{M} : (m,m') \mapsto (\tilde{m},\tilde{m}'),$$

where the particle number $n$ is conserved, i.e. we have $n = m + m' = \tilde{m} + \tilde{m}'$. For our purposes it is convenient to represent the Markov generator as an $(n+1) \times (n+1)$ matrix acting on the basis defined via the identification

$$(n-i,i) \longleftrightarrow e^n_{i+1}.$$  

Here $e^n_{i+1}$ denotes the basis vector of size $n+1$ such that $(e_i)_j = \delta_{ij}$. Thus for a given number of particles $n$ the local Markov matrix is of the form

$$\mathcal{M}_n = \begin{pmatrix}
    \alpha_+(n) + \alpha_-(0) & -\beta_-(1,1) & -\beta_-(2,2) & \cdots & -\beta_-(n,n) \\
    -\beta_+(n,1) & \alpha_+(n-1) + \alpha_-(1) & -\beta_-(2,1) & \cdots & -\beta_-(n,n-1) \\
    -\beta_+(n,2) & -\beta_+(n-1,1) & \alpha_+(n-2) + \alpha_-(2) & \cdots & -\beta_-(n,n-2) \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    -\beta_+(n,n) & -\beta_+(n-1,n-1) & -\beta_+(n-2,n-2) & \cdots & \alpha_+(0) + \alpha_-(n)
\end{pmatrix}.  \quad (3.3)$$

Here $\beta_+(m,k)$ ($\beta_-(m,k)$) denotes the rate with which $k$ of $m$ particles at the left (right) site are moved to the right (left) site. The diagonal entries are given in terms of the rates $\beta_\pm(m,k)$ via

$$\alpha_+(m) = \sum_{k=1}^{m} \beta_+(m,k), \quad \alpha_-(m) = \sum_{k=1}^{m} \beta_-(m,k). \quad (3.4)$$

This ensures that the sum over all columns vanishes for any $n$ which is a necessary requirement of the local Markov matrix. The hopping rates for particles moving to the right take the form

$$\beta_+(m,k) = \frac{\mu^k}{\mu(1-\gamma^k)} \frac{\langle \gamma; \gamma \rangle_m \langle \mu; \gamma \rangle_{m-k}}{\langle \gamma; \gamma \rangle_{m-k} \langle \mu; \gamma \rangle_m}, \quad (3.5)$$

Note that we label non-compact representations with positive half-integers which would correspond to negative spin labels in [7]. The Hamiltonian density (2.7) can be related to the one in [7] using the reflection property of $\psi_n$ as given in (4.51) and taking $S \rightarrow -S$. 

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and the rates for particles moving to the left are given by

\[
\beta_-(m, k) = \frac{1}{\mu (1 - \gamma^k)} (\gamma; \gamma)_m (\mu; \gamma)_{m-k}.
\] (3.6)

The parameters take the values \(0 < \gamma, \mu < 1\). Further we introduced the q-Pochhammer symbol \((a; \gamma)_m = \prod_{j=0}^{m-1} (1 - a \gamma^j)\). The diagonal entries of the local Markov matrix (3.3) can then be computed. We find

\[
\alpha_+(m) = \sum_{k=0}^{m-1} \frac{\gamma^k}{1 - \mu \gamma^k}, \quad \alpha_-(m) = \frac{1}{\mu} \sum_{k=0}^{m-1} \frac{1}{1 - \mu \gamma^k}.
\] (3.7)

The transition rates of the q-Hahn process were given in [16] and coincide with the the \(U_\gamma(A_1^{(1)})\) zero range process in [21, equation (36) for \(a = 1\) and \(b = \mu^{-1}\)] which was obtained from the stochastic R-matrix [18]. We remark that the left and right hopping rates introduced there are weighted by an extra parameter which is fixed in our setup, see also the discussion on the totally asymmetric zero range process (TAZRP) in Section [5].

In the next section we will show that the Markov process defined via the local Markov matrix (3.3) is directly related to the Hamiltonian density of the non-compact spin \(s\) XXZ chain.

4 From the q-Hahn process to the XXZ spin chain

In order to connect the local Markov generator of the q-Hahn process in (3.3) with the Hamiltonian density of the XXZ chain in (2.7) we first identify

\[
\gamma = q^2, \quad \mu = q^{4s}.
\] (4.1)

Here \(q\) denotes the deformation parameter and \(s\) the spin label that were introduced in Section [2]. For a given number of particles \(n\) we then define the matrix

\[
\mathcal{H}_n = S_n M_n S_n^{-1},
\] (4.2)

where the similarity transformation for fixed \(n\) only depends on the spin variable. It reads

\[
S_n = \text{diag} \left( 1, q^{-2s}, q^{-4s}, \ldots, q^{-2ns} \right).
\] (4.3)

As we will see this is the Hamiltonian density of the integrable non-compact spin \(s\) XXZ chain for a given magnon block of \(n\) magnons. In order to reveal the algebraic structure we write down how the Hamiltonian density acts on the tensor product of two \(U_q(sl_2)\) modules as defined in (2.2). We find that it can be written as

\[
\mathcal{H}|m\rangle \otimes |m'\rangle = (\alpha_+(m) + \alpha_-(m')) |m\rangle \otimes |m'\rangle - \sum_{k=1}^{m} \rho(m, k) |m - k\rangle \otimes |m' + k\rangle - \sum_{k=1}^{m'} \rho(m', k) |m + k\rangle \otimes |m' - k\rangle,
\] (4.4)

with the off-diagonal coefficients

\[
\rho(m, k) = \frac{q^{2ks}}{q^{4s} (1 - q^{2k}) (q^2; q^2)_m (q^{4s}; q^2)_{m-k}}.
\] (4.5)

We note that the similarity transformation symmetrised the hopping coefficients (3.5) and (3.6) as \(q^{-2ks} \beta_+(m, k) = q^{2ks} \beta_-(m, k)\) under the identification (4.1). As the diagonal terms remain unchanged by the similarity transformation, \(\mathcal{H}_n\) is not a Markov matrix as the sum over its columns
is non-vanishing. Using the variables $q$ and $s$ as given in (4.1) the terms on the diagonal can nicely be written in terms of the q-analog of the digamma function, cf. Appendix A as

$$\alpha_{\pm}(m) = \frac{\psi_q(m+2s) - \psi_q(2s) \pm m \log(q)}{-2q^{4s}\log(q)}.$$  \hspace{1cm} (4.6)

We note that the Markov process can be written in the form (4.4) using the relation

$$\text{Verifying the commutation relations involving the creation and annihilation operators where the coefficients are given by we then find}$$

$$\text{To verify that it indeed holds, we explicitly evaluate the action of the commutator on the states as well as the action of the Hamiltonian density given in (4.4). For the annihilation operators using}$$

$$\text{involved, i.e.}$$

$$\text{In the remaining part of this section we show that the Hamiltonian density defined via (4.2) or equivalently by the action (4.4) can be identified with the Hamiltonian density (2.7) of the non-compact XXZ spin chain. Following [19], we first show that}$$

$$\text{coincide with the Hamiltonian density in (2.7) by acting on lowest weight states }|\Phi_{2s+j}\rangle \text{ of the irreducible tensor product decomposition (2.10), i.e.}$$

$$\text{In this subsection we show that the Hamiltonian density defined by (4.4) is } U_q(sl_2) \text{ invariant or in other words commutes with the co-product of the generators as described in Section 2.}$$

$$\text{The relation involving the Cartan generator } S_0 \text{ is easily verified. As the particle number is conserved in every magnon block we find}$$

$$\text{Verifying the commutation relations involving the creation and annihilation operators } S_{\pm} \text{ is more involved, i.e.}$$

$$\text{To verify that it indeed holds, we explicitly evaluate the action of the commutator on the states using}$$

$$\text{and}$$

$$\Delta(S_{+})m) \otimes |m\rangle = |m + 2s|q^{-m - s}|m + 1) \otimes |n\rangle + q^{m+s}|m'|m) \otimes |m' + 1\rangle,$$  \hspace{1cm} (4.13)

$$\text{as well as the action of the Hamiltonian density given in (4.4). For the annihilation operators } S_{-} \text{ we then find}$$

$$[\Delta(S_{-}), \mathcal{H}]|m\rangle \otimes |m\rangle = - \sum_{k=0}^{m-2} A_k (m,m')|k\rangle \otimes |m' + m - k - 1\rangle + B^{-} (m,m')|m - 1\rangle \otimes |m'\rangle$$

$$- \sum_{k=0}^{m'-2} C_k (m,m')|m + m' - k - 1\rangle \otimes |k\rangle + D^{-} (m,m')|m\rangle \otimes |m' - 1\rangle,$$  \hspace{1cm} (4.14)

$$\text{where the coefficients are given by}$$
\[ A_k^-(m, m') = \rho(m, m - k - 1)|k + 1\rangle q^{-m'-m+k+1-s} + \rho(m, m - k)q^{k+s}|m' + m - k\rangle \]
\[ B^-(m, m') = |m\rangle q^{-m'-s}\alpha_+(m) - \rho(m, 1)q^{m-1+s}|m' + 1\rangle \]
\[ C_k^+(m, m') = \rho(m', m' - k)|m + m' - k\rangle q^{k+s} + \rho(m', m' - k - 1)q^{m+m'-k+1+s}|k + 1\rangle \]
\[ D^+(m, m') = q^{m+s}|m\rangle \alpha_-(m') - \rho(m', 1)|m + 1\rangle q^{-m'+1-s} \]

For the creation operator \( S_+ \) we obtain
\[ [\Delta(S_+), \mathcal{H}]|m\rangle \otimes |m'\rangle = -\sum_{k=0}^{-m-1} A_k^+(m, m')|k\rangle \otimes |m' + m - k + 1\rangle + B^+(m, m')|m + 1\rangle \otimes |m'\rangle \]
\[ - \sum_{k=0}^{-m'-1} C_k^+(m, m')|m + m' - k + 1\rangle \otimes |k\rangle + D^+(m, m')|m\rangle \otimes |m' + 1\rangle \]

where \( A^+, B^+, C^+ \) and \( D^+ \) are given by
\[ A_k^+(m, m') = \rho(m, m - k + 1)|k - 1 + 2s\rangle q^{-m'-m+k-1-s} + \rho(m, k)q^{k+s}|m' + m - k + 2s\rangle \]
\[ - |m + 2s\rangle q^{-m'-s}\rho(m + 1, m - k + 1) - q^{m+s}|m' + 2s\rangle \rho(m, m - k) \]
\[ B^+(m, m') = |m + 2s\rangle q^{-m'-s}\alpha_+(m) - \rho(m', 1)q^{m+1+s}|m' + 2s - 1\rangle \]
\[ - |m + 2s\rangle q^{-m'-s}\alpha_+(m + 1) + q^{m+s}|m' + 2s\rangle \rho(m' + 1, 1) \]
\[ C_k^+(m, m') = \rho(m', m' - k)|m + m' - k + 2s\rangle q^{k+s} + \rho(m', m' - k + 1)q^{m+m'-k+1+s}|k + 1 + 2s\rangle \]
\[ - |m + 2s\rangle q^{-m'-s}\rho(m', m' - k) - q^{m+s}|m' + 2s\rangle \rho(m' + 1, m' - k + 1) \]
\[ D^+(m, m') = q^{m+s}|m' + 2s\rangle \alpha_-(m') - \rho(m, 1)|m + 2s - 1\rangle q^{-m'-1-s} \]
\[ - q^{m+s}|m' + 2s\rangle \alpha_-(m' + 1) + |m + 2s\rangle q^{-m'-s}\rho(m + 1, 1) \].

All coefficients \( A^\pm, B^\pm, C^\pm \) and \( D^\pm \) vanish. This can be shown using
\[ \alpha_-(m + 1) - \alpha_-(m) = \frac{1}{q^{4s} - q^{2m+8s}} \] 
\[ \alpha_+(m) - \alpha_+(m + 1) = \frac{1}{q^{4s} - q^{-2m}} \]

which follow from the definition of \( \alpha_\pm \) in (4.6), and the relations
\[ \rho(m, k) = \frac{q^{2m - 1} \left(q^{2(m+2s)} - q^{2k+2}\right)}{(q^{2m} - q^{2k}) \left(q^{2(m+2s)} - q^{2}\right)} \rho(m - 1, k), \]
\[ \rho(m, k) = \frac{q^{2k} - q^{2}s}{q^{2k} - q^{2s}} \rho(m - 1, k - 1), \]

that arise from the formula for \( \rho \) in (4.5).
We thus found that the Hamiltonian density defined in (4.4) is $U_q(sl_2)$ invariant. As a consequence the eigenvalues of $\mathcal{H}$ are degenerate and can be obtained by only considering the lowest weight states of the representations of the irreducible tensor product decomposition in (2.10). In the next section we verify that the corresponding lowest weights states are eigenstates of the Hamiltonian density and compute the eigenvalue.

### 4.2 Eigenvalues of the Hamiltonian density on the lowest weight state

To identify the Hamiltonian density in (4.4) with the one given in (2.7) it remains to compare their action on the lowest weight states of the irreducible tensor product decomposition in (2.10).

The lowest weight states can be determined from the condition (2.11) by making the ansatz

$$|\Phi_{2s+j}\rangle = \sum_{k=0}^{j} c_{j,k} |k\rangle \otimes |j-k\rangle .$$

(4.27)

The conditions in (2.11) then yield the difference equation for the coefficients in (4.27). It reads

$$[k + 1]q^{-s-j+k+1}c_{j,k+1} + [j-k]q^{k+s}c_{j,k} = 0 .$$

(4.28)

Thus up to a normalisation we take

$$c_{j,k} = q^{2k(j+s)} \frac{(q^{2j};q^2)_{k-1}}{(q^2;q^2)_{k-1}}$$

(4.29)

which can be shown to satisfy (4.28).

Now, using (4.4), we compute the action of the Hamiltonian density on the lowest weight states. First we note that after exchanging the sums, that appear in the lowest weight state and in the definition of the Hamiltonian density (4.4), the action of the latter can be written as

$$\mathcal{H}|\Phi_{2s+j}\rangle = \sum_{k=0}^{j} \mathcal{C}(j,k)|k\rangle \otimes |j-k\rangle$$

(4.30)

where the coefficients read

$$\mathcal{C}(j,k) = c_{j,k} (\alpha_+(k) + \alpha_-(j-k)) - \sum_{l=k+1}^{j} c_{j,l} \rho(l,l-k) - \sum_{l=0}^{k-1} c_{j,l} \rho(j-l,k-l) .$$

(4.31)

It thus remains to show that

$$\mathcal{C}(j,k) = \lambda_j c_{j,k} ,$$

(4.32)

where

$$\lambda_j = \alpha_-(j) + \alpha_+(j) = \frac{\psi_q(j+2s) - \psi_q(2s)}{-q^s \log(q)} ,$$

(4.33)

cf. (2.7). This can be done as follows. First we note that

$$\frac{\mathcal{C}(j,k)}{c_{j,k}} = \alpha_+(k) + \alpha_-(j-k) - \sum_{l=1}^{j-k} q^{2l(j+s)} \frac{(q^{2k-2l};q^2)_l}{(q^{2k+2};q^2)_l} \rho(l+k,l) - \sum_{l=1}^{k} q^{-2l(s-1)} \frac{(q^{-2k};q^2)_l}{(q^{2j-2k+2};q^2)_l} \rho(j-k+l,l) ,$$

(4.34)

after shifting the boundaries of the sum and using the relations

$$\frac{c_{j,k+n}}{c_{j,k}} = q^{2n(j+s)} \frac{(q^{2k-2j};q^2)_n}{(q^{2k+2};q^2)_n} , \quad \frac{c_{j,k-n}}{c_{j,k}} = q^{-2n(s-1)} \frac{(q^{-2k};q^2)_n}{(q^{2j-2k+2};q^2)_n} .$$

(4.35)
As a final step, the relation (4.32) then arises from the identities

\[ \alpha_+(j) - \alpha_+(k) = -\sum_{l=1}^{j-k} \frac{q^{2l(j+s)} (q^{2k-2j}; q^2)_l}{(q^{2k+2}; q^2)_l} \rho(l + k, l) \tag{4.36} \]

and

\[ \alpha_-(j) - \alpha_-(j - k) = -\sum_{l=1}^{k} \frac{q^{-2l(s-1)} (q^{-2k}; q^2)_l}{(q^{2k-2k+2}; q^2)_l} \rho(j - k + l, l) \tag{4.37} \]

Both of them are shown in the following by taking a little journey through the land of special functions, see Section 4.2.1 and 4.2.2 respectively.

### 4.2.1 Proof of relation (4.36)

In order to show (4.36) we define

\[ \mathcal{F}(j, k) = \sum_{l=1}^{j-k} \frac{q^{2l(j+s)} (q^{2k-2j}; q^2)_l}{(q^{2k+2}; q^2)_l} \rho(l + k, l). \tag{4.38} \]

As \(2k - 2j \leq 0\) we can extend the sum to infinity such that

\[ \mathcal{F}(j, k) = \frac{q^{2j} (q^{4s}; q^2)_k}{(q^{2}; q^2)_k} \sum_{l=0}^{\infty} \frac{q^{2l(j+2s)} (q^{2k-2j}; q^2)_{l+1}}{(q^{2k+2}; q^2)_{l+1}} \frac{(q^2; q^2)_{l+k+1}}{(q^{4s}; q^2)_{l+k+1}} \tag{4.39} \]

\[ = q^{2j} \frac{1 - q^{2(k-j)}}{1 - q^{2(k+2s)}} \sum_{l=0}^{\infty} \frac{q^{2l(j+2s)} (q^{2k-2j+2}; q^2)_l}{(q^{2k+2+4s}; q^2)_l}. \]

Here we used \((a, q^2)_{n+k} = (a, q^2)_n (aq^{2n}, q^2)_k\) in the second step. Next we write \(\mathcal{F}\) as a basic hypergeometric function

\[ \mathcal{F}(j, k) = \frac{q^{2j}}{1 - q^2} \frac{1 - q^{2(k-j)}}{1 - q^{2k+2s}} \sum_{l=0}^{\infty} \frac{\left(q^{2(j+2s)}\right)_l (q^2; q^2)_{l+1} (q^{2k-2j+2}; q^2)_{l+k+1}}{(q^{2k+2}; q^2)_l (q^{4s}; q^2)_{l+k+1}} \tag{4.40} \]

\[ = q^{2j} \frac{1 - q^{2(k-j)}}{1 - q^{2(k+2s)}} 3\Phi_2 \left( q^2, q^2, q^{2(1-j+k)}; q^4, q^{2+2k+4s}; q^2, q^{2+4s} \right) \]

cf. (A.7). Serendipitously, such function was studied before in [22]. Here the following formula was given

\[ 3\Phi_2 \left( q^2, q^2, q^{2(a+1)}; q^4, q^{2(b+1)}; q^2, q^{2(b-a)} \right) = \frac{1 - q^{-2b}}{1 - q^{-2a}} \frac{1 - q^{-2}}{2 \log(q)} \left( \psi_q(b - a) - \psi_q(b) - a \log(q) \right), \tag{4.41} \]

see [22] equation (3.4)]. Taking \(a = k - j\) and \(b = k + 2s\), where \(b - a = 2s + j\), we obtain

\[ \mathcal{F}(j, k) = \frac{1}{2q^{4s} \log(q)} \left( \psi_q(2s + j) - \psi_q(2s + k) + (j - k) \log(q) \right) = \alpha_+(k) - \alpha_+(j). \tag{4.42} \]

This proves (4.36).
4.2.2 Proof of relation (4.37)

To show (4.37) we define

$$G(j, k) = \sum_{l=1}^{k} \frac{q^{-2l(s-1)} \left(q^{-2k}; q^2\right)_l}{(q^{2j+2k+2}; q^2)_l} \rho(j - k + l, l).$$  \hspace{1cm} (4.43)

Again we can extend the sum to infinity as $-2k \leq 0$ and write

$$G(j, k) = \frac{(q^{4s}; q^2)_{j-k}}{q^{4s-2(2q^2)}(q^2; q^2)_{j-k}} \sum_{l=0}^{\infty} \frac{q^{2l}}{1 - q^{2(l+1)}} \left(q^{-2k}; q^2\right)_{l+1} \left(q^2; q^2\right)_{j-k+l+1}.$$

$$= q^{2-4s} \frac{1 - q^{-2k}}{1 - q^{4s+2(j-k)}} \sum_{l=0}^{\infty} \frac{q^{2l}}{1 - q^{2(l+1)}} \left(q^2; q^2\right)_l.$$

(4.44)

Then as before we write the sum as a basic hypergeometric function

$$G(j, k) = q^{2-4s} \frac{1 - q^{-2k}}{1 - q^2} \frac{1 - q^{4s+2(j-k)}}{1 - q^{4s+2(j-k)}} \sum_{l=0}^{\infty} \left(q^2; q^2\right)_l (q^2; q^2)_{l+1} (q^{-2k}; q^2)_{l+1} (q^2; q^2)_{j-k+l+1}.$$

(4.45)

This is not yet of the form to apply (4.41). However, the latter basic hypergeometric function can be written as

$$3\Phi_2 \left(q^2, q^2, q^{2(1-k)}; q^4, q^{2(2s+j-k+1)}; q^2, q^2\right) = q^{2(k-1)} \frac{(q^2; q^2)_{(2s+j-k)} (q^2; q^2)_{(2s+j-k+1)}}{(q^2; q^2)_{(2s+j-k)} (q^2; q^2)_{(2s+j-k+1)}} \times 3\Phi_2 \left(q^2, q^2, q^{2(1-k)}; q^4, q^{2(2s+j-k)}; q^2, q^2\right).$$

(4.46)

see [23] equation (III.12)]. Now we can use (4.41) again and arrive at

$$G(j, k) = \frac{1}{2q^{4s} \log(q)} \left(\psi_q(1 - 2s - j) - \psi_q(1 - 2s - j + k) - k \log(q)\right).$$

(4.47)

This is not quite the result we expected but one may hope for the existence of the q-analog of the reflection formula of the psi-function such that

$$\psi_q(1 - 2s - j) - \psi_q(1 - 2s - j + k) = \psi_q(2s + j) - \psi_q(2s + j - k).$$

(4.48)

This equation can be derived using the relation

$$\frac{\Gamma_q \left(\frac{1}{2} + x\right) \Gamma_q \left(\frac{1}{2}\right)}{\Gamma_q \left(\frac{1}{2} + x\right) \Gamma_q \left(\frac{1}{2} - x\right)} = \cos(\pi x) \sum_{n=1}^{\infty} \frac{1 + 2r^{2n}\cos(2\pi x) + r^{4n}}{1 + 2r^{2n} + r^{4n}}.$$

(4.49)

where $\log(q) \log(r) = \pi^2$, see [24] equation (5.24)]. As a consequence we find that

$$(-1)^k \Gamma_q \left(\frac{1}{2} + x\right) \Gamma_q \left(\frac{1}{2} - x\right) = \Gamma_q \left(\frac{1}{2} + x + k\right) \Gamma_q \left(\frac{1}{2} - x - k\right).$$

(4.50)

with $k \in \mathbb{Z}$. Finally, taking the logarithmic derivative yields

$$\psi_q \left(\frac{1}{2} + x\right) - \psi_q \left(\frac{1}{2} - x\right) = \psi_q \left(\frac{1}{2} + x + k\right) - \psi_q \left(\frac{1}{2} - x - k\right).$$

(4.51)

We then find that

$$G(j, k) = \frac{1}{2q^{4s} \log(q)} \left(\psi_q(2s + j) - \psi_q(2s + j - k) - k \log(q)\right) = \alpha_-(j - k) - \alpha_-(j).$$

(4.52)

which concludes the proof of (4.37).
5 Limiting cases

The Hamiltonian of the non-compact XXZ spin chain depends on the parameters $q$ and $s$. In the rational limit $q \to 1$ we recover the non-compact XXX chain, cf. [12]. The similarity transformation relating the spin chain and the stochastic process becomes trivial in this limit. The process corresponding to the case $s = \frac{1}{2}$ with parameter $q$ has been studied in [15] and is known as the MADM. Its relation in the rational limit with the non-compact Heisenberg chain [19] has only recently been pointed out in [12]. See also Figure 1 where the relations between the different models are summarised.

Spin $\frac{1}{2}$ and MADM For $s = \frac{1}{2}$ the Pochhammer symbols in the hopping rates disappear and so does the dependence on the number of particles at the initial site. For the spin chain Hamiltonian density one finds

$$\rho_{s=\frac{1}{2}}(m,k) = -\frac{1}{q^2(q^k-q^{-k})} = -\frac{q-q^{-1}}{q^2} \frac{1}{|k|},$$

and

$$\alpha_{\pm}(m) = \frac{\psi_q(m+1) - \psi_q(1) \pm m \log(q)}{-2q^2 \log(q)},$$

For the stochastic process setting $s = \frac{1}{2}$ is equivalent to $\mu \to \gamma$. Here we obtain the rates

$$\beta_{\pm}(m,k) = \frac{\gamma^k}{\gamma(1-\gamma^k)}, \quad \beta_{\mp}(m,k) = \frac{1}{\gamma(1-\gamma^k)}.$$  

They can be identified with the hopping rates of the MADM.

Totally asymmetric limit Like in the finite dimensional ASEP the Markov generator reduces to the one of a left or right moving process for $\gamma \to 0$ and $\gamma \to \infty$ respectively. Namely for $\gamma \to 0$ we find that

$$\lim_{\gamma \to 0} \gamma^{2s} \beta_+(m,k)\big|_{\mu=\gamma^{2s}} = 0, \quad \lim_{\gamma \to 0} \gamma^{2s} \beta_-(m,k)\big|_{\mu=\gamma^{2s}} = 1,$$

and

$$\lim_{\gamma \to 0} \gamma^{2s} \alpha_+(m)\big|_{\mu=\gamma^{2s}} = 0, \quad \lim_{\gamma \to 0} \gamma^{2s} \alpha_-(m)\big|_{\mu=\gamma^{2s}} = m,$$

for $2s \in \mathbb{N}$. While for the case $\gamma \to \infty$ we get

$$\lim_{\gamma \to \infty} \gamma^{2s} \beta_+(m,k)\big|_{\mu=\gamma^{2s}} = 1, \quad \lim_{\gamma \to \infty} \gamma^{2s} \beta_-(m,k)\big|_{\mu=\gamma^{2s}} = 0,$$

Figure 1: Stochastic hopping models in green boxes and corresponding spin chains in yellow boxes. The q-Hahn process reduces to the multi particle asymmetric diffusion model (MADM) for $\mu \to \gamma$. The identification with the non-compact spin chains is realised by taking $\mu = q^{4s}$ and $\gamma = q^2$. Here $s$ denotes the spin label and the deformation parameter $q$ is related to the anisotropy in the spin chain Hamiltonian. Sending the latter to $q \to 1$ we recover the rational limit of the trigonometric spin chains.
while
\[
\lim_{\gamma \to \infty} \gamma^{2s} \alpha_+(m)\big|_{\mu=\gamma^{2s}} = m, \quad \lim_{\gamma \to \infty} \gamma^{2s} \alpha_-(m)\big|_{\mu=\gamma^{2s}} = 0. \tag{5.7}
\]

**Rational limit** In the rational limit we recover the non-compact XXX spin chain. The hopping rates are of the form
\[
\lim_{q \to 1} \log(q^{-1})\rho(m, k) = \frac{\Gamma(m + 1)\Gamma(m - k + 2s)}{2k\Gamma(m - k + 1)\Gamma(m + 2s)} \tag{5.8}
\]
and
\[
\lim_{q \to 1} \log(q^{-1})\alpha_{\pm}(m) = \frac{\psi(m + 2s) - \psi(2s)}{2}. \tag{5.9}
\]
We stress again that the similarity transformation which relates the local Markov generator to the Hamiltonian density becomes trivial in this case.

**TAZRP** Taking a limit that can be related to a single totally asymmetric zero range process (TAZRP), cf. [18, 21], does not seem to be straightforward at the level of the Hamiltonian density. However, as a consequence of Baxter’s TQ-equation which relates the transfer matrix and the Q-operator, the generators of the two TAZRP’s may arise from the logarithmic derivative of the Q-operator of the non-compact XXZ chain at two special points of the spectral parameter. Such mechanism was demonstrated for the non-compact spin $\frac{1}{2}$ XXX chain in [25, Appendix C] using an oscillator construction of Q-operators going back to [26–28].

### 6 Conclusion

In this note we gave an explicit expression for the action of the Hamiltonian density of the spin $s$ non-compact XXZ chain on the tensor product of two sites of the spin chain (4.4). Further we showed that it directly relates to the local generator of a continuous-time Markov process (3.3) via (4.7). The rates of the particles hopping are identified with a $q$-Hahn process studied previously in [15–17].

The identification of the integrable spin chain and the stochastic particle process allows to describe the system in the standard framework of the quantum inverse scattering method. The latter immediately provides a huge variety of integrability methods, like the algebraic Bethe ansatz, separation of variables and functional methods, to study the model and its limits. Here in particular it might be instructive to study the large spin limit which may be related to the $q$-Boson totally asymmetric process [29] whose Lax matrices are known to arise in that limiting case. Having understood the algebraic structure of the $q$-Hahn process for the bulk may further allow to derive the appropriate stochastic boundary conditions from the boundary Yang-Baxter equation as done for the rational limit in [12]. Further, one may expect that duality and the limit of fluctuating hydrodynamics which were studied in the previous reference do have a $q$-analog. It would be natural to study whether the Hamiltonian density of non-compact spin chains with higher rank allows a similar identification, see e.g. [28] where the analogs of (2.7) were discussed in the rational limit.

Our findings suggest that the local charges which arise from the Q-operator of the non-compact XXZ spin chain are directly connected to the Markov generator of the totally asymmetric zero range process (TAZRP) which arises from the transfer matrix constructed from the stochastic R-matrix for $U_q(A^{(1)}_1)$. It would be interesting to study this relation in detail. The oscillator type construction of Q-operators, mentioned in Section 5 has been carried out for non-compact rational spin chains in [28, 30], their trigonometric counterpart was so far only studied for the fundamental representation, cf. [31, 32], and deserves further investigation. We refer the reader to [33, 34] and references therein for alternative approaches.
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A Special functions

In this appendix we collect some definitions of the q-analogs of some special functions used in the main text. As mentioned before we were using the conventions of [7] such that

$$\Gamma_q(x) = \left(q^{-1} - q\right)^{1-x} q^{-\frac{1}{2}(x-1)x} \frac{(q^2; q^2)_\infty}{(q^{2x}; q^2)_\infty},$$

(A.1)

for $|q| < 1$. The q-analog of the $\psi$-function can then be written as

$$\psi_q(z) = \frac{\partial}{\partial z} \log \Gamma_q(z) = -\log(1 - q^2) + 2 \log(q) \sum_{k=0}^{\infty} \frac{q^{2(k+z)}}{1 - q^{2(k+z)}} + \frac{1}{2} (3 - 2z) \log(q).$$

(A.2)

These conventions differ from the definitions which seem to be commonly used. Alternatively the q-analog of the $\Gamma$-function is defined as

$$\tilde{\Gamma}_\gamma(z) = (1 - \gamma)^{1-z} \frac{\gamma(z; \gamma)}{(\gamma^2; \gamma)_\infty},$$

(A.3)

for $|\gamma| < 1$. The $\psi$-function then reads

$$\tilde{\psi}_\gamma(z) = \frac{\partial}{\partial z} \log \tilde{\Gamma}_\gamma(z) = -\log(1 - \gamma) + \log(\gamma) \sum_{k=0}^{\infty} \frac{\gamma^{k+z}}{1 - \gamma^{k+z}}.$$ (A.4)

The two conventions are related via

$$\Gamma_q(x) = q^{-\frac{1}{2}(x-2)(x-1)} \tilde{\Gamma}_{q^2}(x),$$

(A.5)

and

$$\psi_q(x) = \tilde{\psi}_{q^2}(x) + \frac{1}{2} (3 - 2x) \log(q).$$

(A.6)

Finally we give the definition of the basic hypergeometric function

$$\Phi_3 (a, b, c; d, e; q; z) = \sum_{l=0}^{\infty} \frac{(a; q)_l (b; q)_l (c; q)_l}{(d; q)_l (e; q)_l} \frac{z^l}{(q; q)_l}.$$ (A.7)

References

[1] H. Bethe, “Zur Theorie der Metalle,” Zeitschrift für Physik 71 no. 3, (Mar, 1931) 205–226
[2] R. Baxter, Exactly solved models in statistical mechanics, vol. 1. Academic press London, 1982.
[3] M. Gaudin, La fonction d’onde de Bethe. Masson, 1983.
[4] L. D. Faddeev, “How algebraic Bethe ansatz works for integrable model,” in Relativistic gravitation and gravitational radiation. Proceedings, School of Physics, Les Houches, France, September 26-October 6, 1995, pp. pp. 149–219. 1996. arXiv:hep-th/9605187 [hep-th]
[5] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, Quantum inverse scattering method and correlation functions, vol. 3. Cambridge university press, 1997.
[6] P. P. Kulish, N. Yu. Reshetikhin, and E. K. Sklyanin, “Yang-Baxter Equation and Representation Theory. 1.”, *Lett. Math. Phys.* 5 (1981) 393–403.

[7] A. G. Bytsko, “On integrable Hamiltonians for higher spin XXZ chain,” *J. Math. Phys.* 44 (2003) 3698–3717, arXiv:hep-th/0112163 [hep-th].

[8] D. Chicherin, S. E. Derkachov, and V. P. Spiridonov, “From Principal Series to Finite-Dimensional Solutions of the Yang-Baxter Equation,” *SIGMA* 12 (2016) 028, arXiv:1411.7595 [math-ph].

[9] V. V. Mangazeev, “On the Yang-Baxter equation for the six-vertex model,” *Nucl. Phys.* B882 (2014) 70–96, arXiv:1401.6494 [math-ph].

[10] K. Mallick, “Some exact results for the exclusion process,” *Journal of Statistical Mechanics: Theory and Experiment* no. 01, (Jan, 2011) P01024, arXiv:1101.2849 [cond-mat].

[11] G. Schutz, “Exactly solvable models for many-body systems far from equilibrium,” *Phase transitions and critical phenomena* (2000).

[12] R. Frassek, C. Giardinà, and J. Kurchan, “Non-compact quantum spin chains as integrable stochastic particle processes,” arXiv:1904.01048 [math-ph].

[13] N. Beisert *et al.*, “Review of AdS/CFT Integrability: An Overview,” *Lett. Math. Phys.* 99 (2012) 3–32, arXiv:1012.3982 [hep-th].

[14] S. E. Derkachov, “Baxter’s Q-operator for the homogeneous XXX spin chain,” *J. Phys.* A32 (1999) 5299–5316, arXiv:solv-int/9902015 [solv-int].

[15] T. Sasamoto and M. Wadati, “One-dimensional asymmetric diffusion model without exclusion,” *Phys. Rev. E* 58 (Oct, 1998) 4181–4190.

[16] G. Barraquand and I. Corwin, “The $q$-Hahn asymmetric exclusion process,” *Ann. Appl. Probab.* 26 no. 4, (08, 2016) 2304–2356, arXiv:1501.03445 [math.PR].

[17] A. M. Povolotsky, “The integrability of zero-range chipping models with factorized steady states,” *Journal of Physics A: Mathematical and Theoretical* 46 no. 46, (Nov, 2013) 465205, arXiv:1308.3250 [math-ph].

[18] A. Kuniba, V. V. Mangazeev, S. Maruyama, and M. Okado, “Stochastic R matrix for $U_q(A^{(1)}_n)$,” *Nucl. Phys.* B913 (2016) 248–277, arXiv:1604.08304 [math.QA].

[19] N. Beisert, “The complete one loop dilatation operator of N=4 super Yang-Mills theory,” *Nucl. Phys.* B676 (2004) 3–42, arXiv:hep-th/0307015 [hep-th].

[20] H. T. Koelink and J. Van der Jeugt, “Convolutions for orthogonal polynomials from Lie and quantum algebra representations,” *SIAM journal on mathematical analysis* 29 no. 3, (1998) 794–822, arXiv:q-alg/9607010 [math.QA].

[21] A. Kuniba and M. Okado, “A $q$-boson representation of Zamolodchikov-Faddeev algebra for stochastic R matrix of $U_q(A^{(1)}_n)$,” *Lett. Math. Phys.* 107 no. 6, (2017) 1111–1130, arXiv:1610.00531 [math-ph].

[22] C. Krattenthaler and H. Srivastava, “Summations for basic hypergeometric series involving a $q$-analogue of the digamma function,” *Computers and Mathematics with Applications* 32 no. 3 (1996) 73–91.

[23] G. Gasper, M. Rahman, and G. George, *Basic hypergeometric series*, vol. 96. Cambridge university press, 2004.
[24] R. Askey, “The q-Gamma and q-Beta Functions,” *Applicable Analysis* 8 no. 2, (1978) 125–141.

[25] R. Frassek and C. Meneghelli, “From Baxter Q-Operators to Local Charges,” *J. Stat. Mech.* 1302 (2013) P02019 [arXiv:1207.4513 [hep-th]].

[26] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, “Integrable structure of conformal field theory. 3. The Yang-Baxter relation,” *Commun. Math. Phys.* 200 (1999) 297–324 [arXiv:hep-th/9805008 [hep-th]].

[27] V. V. Bazhanov, T. Lukowski, C. Meneghelli, and M. Staudacher, “A Shortcut to the Q-Operator,” *J. Stat. Mech.* 1011 (2010) P11002 [arXiv:1005.3261 [hep-th]].

[28] R. Frassek, T. Lukowski, C. Meneghelli, and M. Staudacher, “Baxter Operators and Hamiltonians for ’nearly all’ Integrable Closed gl(n) Spin Chains,” *Nucl. Phys.* B874 (2013) 620–646 [arXiv:1112.3600 [math-ph]].

[29] T. Sasamoto and M. Wadati, “Exact results for one-dimensional totally asymmetric diffusion models,” *Journal of Physics A: Mathematical and General* 31 no. 28, (Jul, 1998) 6057–6071.

[30] R. Frassek, C. Marboe, and D. Meidinger, “Evaluation of the operatorial Q-system for non-compact super spin chains,” *JHEP* 09 (2017) 018 [arXiv:1706.02320 [hep-th]].

[31] V. V. Bazhanov and Z. Tsuboi, “Baxter’s Q-operators for supersymmetric spin chains,” *Nucl. Phys.* B805 (2008) 451–516 [arXiv:0805.4274 [hep-th]].

[32] H. Boos, F. Göhmann, A. Klümper, K. S. Nirov, and A. V. Razumov, “Universal R-matrix and functional relations,” *Rev. Math. Phys.* 26 no. 06, (2014) 1430005 [arXiv:1205.1631 [math-ph]].

[33] D. Chicherin, S. Derkachov, D. Karakhanyan, and R. Kirschner, “Baxter operators with deformed symmetry,” *Nucl. Phys.* B868 (2013) 652–683 [arXiv:1211.2965 [math-ph]].

[34] V. V. Mangazeev, “Q-operators in the six-vertex model,” *Nucl. Phys.* B886 (2014) 166–184 [arXiv:1406.0662 [math-ph]].

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