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An example of a Brauer–Manin obstruction to weak approximation at a prime with good reduction

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Abstract
Following Bright and Newton, we construct an explicit K3 surface over the rational numbers having good reduction at 2, and for which 2 is the only prime at which weak approximation is obstructed.

Keywords: Brauer–Manin obstruction to weak approximation, Rational points, K3 surfaces, Good ordinary reduction

1 Introduction
Let $k$ be a number field and $A_k$ be the ring of adèles of $k$, i.e. the restricted product of $k_v$ for all places $v$ of $k$, taken with respect to the rings of integers $O_v \subseteq k_v$. Let $X$ be a smooth, proper, geometrically irreducible variety over $k$. In order to study the rational points on $X$ it is useful to look at the image of $X(k)$ in the set of the adèlic points $X(A_k)$. More precisely, Manin [14] has shown that there exists a pairing

$$\text{Br}(X) \times X(A_k) \rightarrow \mathbb{Q}/\mathbb{Z}$$

such that the rational points of $X$ lie in the image of the right kernel of the pairing, denoted by $X(A_k)^{\text{Br}}$. If $X(A_k)^{\text{Br}}$ is not equal to the whole $X(A_k)$ we say that there is a Brauer–Manin obstruction to weak approximation on $X$.

In this paper we follow the ideas presented in [4] to construct an example of a K3 surface $X$ over the rational numbers with a Brauer–Manin obstruction to weak approximation arising from a prime of good ordinary reduction. More precisely, there exist an element $A \in \text{Br}(X)$ and a prime $p$ of good ordinary reduction such that the evaluation map $|A|: X(Q_p) \rightarrow \text{Br}(Q_p)$ is non-constant.

Let $X \subseteq \mathbb{P}^3_Q$ be the projective K3 surface defined by the equation

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0. \quad (1)$$

Theorem 1 The class of the quaternion algebra

$$A = \left( \frac{z^3 + w^3x + xzw}{x^3}, - \frac{z}{x} \right) \in \text{Br}(Q(X))$$

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defines an element in \( \text{Br}(X) \). The evaluation map \( \varphi_A : X(\mathbb{Q}_2) \rightarrow \text{Br}(\mathbb{Q}_2) \) is non-constant, and therefore gives an obstruction to weak approximation on \( X \). Finally, \( X(\mathbb{Q}) \) is not dense in \( X(\mathbb{Q}_2) \), with respect to the 2-adic topology.

This theorem shows, with a concrete example, what was already predicted by Bright and Newton in [4, Theorem C]. Indeed, they prove that, for a smooth, proper variety \( V \) over a number field \( L \) such that \( H^0(V, \Omega^2_V) \neq 0 \), every prime of good ordinary reduction is involved in a Brauer–Manin obstruction over an extension of the base field. The example provided in this article is optimal, in the following sense: we build an element \( A \in \text{Br}(X)[2] \) such that the evaluation map associated to it is non-constant without the need to take an algebraic extension of \( \mathbb{Q} \). Moreover, as pointed out in [4, Remark 11.5], when we are dealing with K3 surfaces defined over the rational numbers, the only prime with good reduction that can play a role in the obstruction to weak approximation is the prime 2.

The following example and, more generally, the result proven by Bright and Newton give a negative answer to the following question, asked by Swinnerton-Dyer ([6] Question 1).

**Question 1** Let \( k \) be a number field and let \( S \) be a finite set of places of \( k \) containing the Archimedean places. Let \( \mathcal{V}_S \) be a smooth projective \( \mathcal{O}_S \)-scheme with geometrically integral fibres, and let \( V/k \) be the generic fibre. Assume that \( \text{Pic}(\mathcal{V}) \) is finitely generated and torsion-free. Swinnerton-Dyer asks if there is an open and closed \( Z \subseteq \prod_{\nu \in S} V(k_\nu) \) such that

\[
V(A_k)^{\text{Br}} = Z \times \prod_{\nu \in S} V(k_\nu).
\]

Roughly speaking, is it true that the Brauer–Manin obstruction involves only the places of bad reduction and the Archimedean places?

Finally, we point out that the element \( A \) defined in Theorem 1 has to be a transcendental element in \( \text{Br}(X) \). Indeed, Colliot-Thélène and Skorobogatov proved [6, Lemma 2.2] that for every element in the algebraic Brauer group the associated evaluation map at a prime with good reduction has to be constant.

In general, let \( V \) be a variety over a field \( k \), \( \bar{k} \) an algebraic closure of \( k \) and \( \mathcal{V} \) the base change of \( V \) to \( \bar{k} \), i.e. \( \mathcal{V} := V \times_k \bar{k} \); the algebraic and transcendental Brauer groups of \( V \) are defined, respectively, as the kernel and the image of the natural map \( \text{Br}(V) \rightarrow \text{Br}(\mathcal{V}) \). The algebraic Brauer group of \( V \) is denoted by \( \text{Br}_1(V) \).

For curves and surfaces with negative Kodaira dimension we have \( \text{Br}(V) = \text{Br}_1(V) \). Hence, K3 surfaces are among the first example of varieties where the transcendental Brauer group is potentially non-trivial. However, this is not always the case: for example in [11] they show that, under certain conditions, the whole Brauer group of a diagonal quartic surface over \( \mathbb{Q} \) is algebraic. The first example of a transcendental element in the Brauer group of a K3 surface defined over a number field was given by Wittenberg in [19]. In particular, Wittenberg constructed a 2-torsion transcendental element that obstructs weak approximation on the surface. Other examples of 2-torsion transcendental elements that obstruct weak approximation can be found in [9, 10]. In all these articles, the obstruction to weak approximation comes from the fact that the transcendental quaternion algebra has non-constant evaluation at the place at infinity. With a construction similar to the
one used in [9], Hassett and Várilly-Alvarado [8] have also built an example of a 2-torsion element on a K3 surface that obstructs the Hasse principle.

Furthermore, there are examples of transcendental elements of order 3 on K3 surfaces that obstruct the Hasse principle or weak approximation (for example, see [1, 15, 16]). In all these cases, the evaluation map at the place at infinity has to be trivial, since $\text{Br}(\mathbb{R})$ does not contain elements of order 3, and the obstruction to weak approximation comes from the evaluation map at the prime 3, which in every example is a prime of bad reduction for the K3 surface taken into account. Therefore, none of the examples mentioned above can be used to give a negative answer to Question 1.

Outline of the paper. Section 2 contains the proof of Theorem 1. In Sect. 3 we show that the K3 surface $X$ has good ordinary reduction at the prime 2. Moreover, we explain the ideas behind the construction of the quaternion algebra $\mathcal{A}$ of Theorem 1 and why we could expect a priori that it obstructs weak approximation on $X$. Finally, in Sect. 4 we slightly generalise the result, by exhibiting a family of K3 surfaces for which there exists a 2-torsion element in the Brauer group whose evaluation map on $\mathbb{Q}_2$-points is non-constant.

The computations in Theorems 1 and 2 were done using Magma [3].

2 Proof of the main theorem
In the first part of the proof we will show that the element $\mathcal{A} \in \text{Br}(\mathbb{Q}(X))$ lies in $\text{Br}(X)$. Next, we will exhibit two points $P_1, P_2 \in X(\mathbb{Q}_2)$ such that

$$\mathcal{A}(P_1) \neq \mathcal{A}(P_2).$$

Finally, we will prove that, for every place $\nu$ different from 2, the evaluation map

$$|\mathcal{A}| : X(\mathbb{Q}_\nu) \rightarrow \text{Br}(\mathbb{Q}_\nu)$$

is constant.

Proof of Theorem 1 Let $f := z^3 + w^2x + xyz$ and $C_x, C_z, C_f$ be the closed subsets of $X$ defined by the equations $x = 0, z = 0$ and $f = 0$ respectively. The quaternion algebra $\mathcal{A}$ defines an element in $\text{Br}(U)$, where $U := X \setminus (C_x \cup C_z \cup C_f)$. The purity theorem for the Brauer group [5, Theorem 3.7.2], assures us of the existence of the exact sequence

$$0 \rightarrow \text{Br}(X)[2] \rightarrow \text{Br}(U)[2] \xrightarrow{\text{Br}(\mathbb{Q}_\nu)} \bigoplus_D \text{H}^1(k(D), \mathbb{Z}/2) \hspace{1cm} (2)$$

where $D$ ranges over the irreducible divisors of $X$ with support in $X \setminus U$ and $k(D)$ denotes the residue field at the generic point of $D$.

In order to use the exact sequence (2) we need to understand what the prime divisors of $X$ with support in $X \setminus U = C_x \cup C_z \cup C_f$ look like. It is possible to check the following:

- $C_x$ has as irreducible components $D_1$ and $D_2$, defined by the equations $x = 0, z = 0$ and $y^3 + z^2w = 0$ respectively;
- $C_z$ has as irreducible components $D_1$ and $D_3$, where $D_3$ is defined by the equations $z = 0, x^2y + w^3 = 0$;
- $C_f$ has as irreducible components $D_1, D_4$ and $D_5$, where $D_4$ and $D_5$ are defined by the equations $z^3 + xw^2 = 0, y = 0$ and $y^3z - x^2z^2 + y^2w^2 = 0, x^3 + y^2z = 0, xyz + z^3 + xw^2 = 0$ respectively.
Therefore, we can rewrite (2) in the following way:

\[ 0 \rightarrow \text{Br}(\mathcal{X})[2] \rightarrow \text{Br}(U)[2] \xrightarrow{\oplus \partial_{D_i}} \bigoplus_{i=1}^{5} H^1(k(D_i), \mathbb{Z}/2). \]  

(3)

Moreover, we have an explicit description of the residue map on quaternion algebras: for an element \((a, b) \in \text{Br}(U)[2]\) we have

\[ \partial_{D_i}(a, b) = \left[ (-1)^{v_i(a) - v_i(b)} a^{v_i(b)} b^{v_i(a)} \right] \in \frac{k(D_i)^x}{k(D_i)^{x^2}} \cong H^1(k(D_i), \mathbb{Z}/2) \]  

(4)

where \(v_i\) is the valuation associated to the prime divisor \(D_i\). This follows from the definition of the tame symbols in Milnor \(K\)-theory together with the compatibility of the residue map \(\partial_i\) with the tame symbols given by the Galois symbols (see [7], Proposition 7.5.1).

We can proceed with the computation of the residue maps \(\partial_{D_i}\) for \(i = 1, \ldots, 5\):

1. \(v_1(f) = v_1(x) = v_1(z) = 1\). Hence,

\[ \partial_{D_1} \left( \frac{f}{x^3}, -\frac{z}{x} \right) = \left[ \left( -\frac{z}{x} \right)^2 \right] = 1 \in \frac{k(D_1)^x}{k(D_1)^{x^2}}. \]

2. \(v_2(x) = 1\) and \(v_2(f) = v_2(z) = 0\). Hence,

\[ \partial_{D_2} \left( \frac{f}{x^3}, -\frac{z}{x} \right) = \left[ -\left( \frac{f}{x^3} \right)^{-1} \left( -\frac{z}{x} \right)^3 \right] = \left[ \frac{z^3}{f} \right] = 1 \in \frac{k(D_2)^x}{k(D_2)^{x^2}} \]

where the last equality follows from the fact that \(x = 0\) on \(D_2\), thus \(f \mid_{D_2} = z^3\).

3. \(v_3(x) = v_3(f) = 0\) and \(v_3(z) = 1\). Hence,

\[ \partial_{D_3} \left( \frac{f}{x^3}, -\frac{z}{x} \right) = \left[ \left( \frac{f}{x^3} \right) \right] = \left[ \left( \frac{w}{x} \right)^2 \right] = 1 \in \frac{k(D_3)^x}{k(D_3)^{x^2}} \]

where the last equality follows from the fact that \(z = 0\) on \(D_3\), thus \(f \mid_{D_3} = w^2 x\).

4. \(v_4(x) = v_4(z) = 0\) and \(v_4(f) = 1\). Hence,

\[ \partial_{D_4} \left( \frac{f}{x^3}, -\frac{z}{x} \right) = \left[ \left( -\frac{x}{z} \right) \right] = \left[ \left( \frac{z}{w} \right)^2 \right] = 1 \in \frac{k(D_4)^x}{k(D_4)^{x^2}} \]

where the last equality follows from the fact that \(z^3 + w^2 x = 0\) on \(D_4\), thus \(-\frac{x}{z} = \left( \frac{z}{w} \right)^2\).

5. \(v_5(x) = v_5(z) = 0\) and \(v_5(f) = 1\). Hence,

\[ \partial_{D_5} \left( \frac{f}{x^3}, -\frac{z}{x} \right) = \left[ \left( -\frac{x}{z} \right) \right] = \left[ \left( \frac{y}{x} \right)^2 \right] = 1 \in \frac{k(D_5)^x}{k(D_5)^{x^2}} \]

where the last equality follows from the fact that \(x^3 + y^2 z = 0\) on \(D_5\), thus \(-\frac{x}{z} = \left( \frac{y}{x} \right)^2\).

Therefore, \(\partial_{D_i}(\mathcal{A}) = 0\) for all \(i \in \{1, \ldots, 5\}\), hence \(\mathcal{A} \in \text{Br}(\mathcal{X})\).

We now show that the element \(\mathcal{A}\) obstructs weak approximation on \(\mathcal{X}\). Let \(\mathcal{X} \subseteq \mathbb{P}^3_{\mathbb{Z}}\) be the projective scheme defined by the equation

\[ x^3 y + y^3 z + z^3 w + w^3 x + x y z w = 0. \]  

(5)

\(\mathcal{X}\) is a \(\mathbb{Z}\)-model for \(\mathcal{X}\) and has good reduction at the prime 2.

Let \(P_1 := (1 : 0 : 1 : 0) \in \mathcal{X}(\mathbb{Z}_2)\), then \(P_1\) is such that \(\mathcal{A}(P_1) = (1, -1)\). Therefore \(\mathcal{A}(P_1)\) is the trivial class in \(\text{Br}(\mathbb{Q}_2)\). On the other hand, Hensel’s lemma assures us of the existence of a solution \(P_2 = (1 : 2 : 1 : d) \in \mathcal{X}(\mathbb{Z}_2)\) whose reduction modulo 8 is \((1 : 2 : 1 : 2)\). Hence,

\[ \mathcal{A}(P_2) = (f(P_2), -1) \quad \text{with} \quad f(P_2) \equiv 7 \pmod{8}. \]
Therefore, we get that $\mathcal{A}(P_2)$ defines a non-trivial element in the Brauer group of $\mathbb{Q}_2$ [17, Theorem 3.1]. The existence of the points $P_1$ and $P_2$ implies that there is a Brauer–Manin obstruction to weak approximation arising from $\mathcal{A}$. Indeed,

$$X(\mathbb{A}_Q)^{\mathcal{A}} = \left\{ (x_p)_p \in X(\mathbb{A}_Q) : \sum_p \text{inv}_p \mathcal{A}(x_p) = 0 \right\} \subseteq X(\mathbb{A}_Q).$$

In order to conclude the proof of the theorem we investigate the behaviour of the evaluation map at the other primes and at infinity. For every prime $p$ let $X'_p$ be the base change of $X$ to $\mathbb{Z}_p$. We distinguish the following cases.

**Case $p \notin \{3, 5, 17, \infty\}$.** In this case, $X'$ has good reduction at $p$. Therefore, we can use [6, Proposition 2.4] to conclude that the evaluation map

$$|\mathcal{A}| : X'(\mathbb{Z}_p) \to \text{Br}(\mathbb{Q}_p)$$

is constant. Moreover, $P = (1 : 0 : 1 : 0) \in X'(\mathbb{Z}_p)$ and

$$\mathcal{A}(P) = (1, -1)$$

which is trivial in $\text{Br}(\mathbb{Q}_p)$; hence the evaluation map is trivial on the whole $X'(\mathbb{Z}_p) = X(\mathbb{Q}_p)$.

**Case $p \in \{3, 5, 17\}$.** Under this assumption, $X'_p/\mathbb{Z}_p$ is not smooth. In these three cases, we want to show that the evaluation map is trivial on $X'(\mathbb{Z}_p)$ by showing that it factors through $\text{Br}(\mathbb{Z}_p)$.

The special fibre $Y_p := X'_p \times_{\mathbb{Z}_p} \text{Spec}(\mathbb{F}_p)$ is a non-smooth $\mathbb{F}_p$-scheme. However, $Y_p$ is an irreducible $\mathbb{F}_p$-scheme, with just isolated singularities. The $\mathbb{Z}_p$-points of $X'_p$ are all smooth. In fact, $Y_p$ contains just one singular point defined over $\mathbb{F}_p$ and that does not even lift to a $\mathbb{Z}/p^2\mathbb{Z}$-point. Let $\mathcal{V}$ be the smooth locus of $X'_p$; because of what we have just said we have

$$X(\mathbb{Q}_p) = X(\mathbb{Z}_p) = \mathcal{V}(\mathbb{Z}_p).$$

Let $V$ be the base change of $\mathcal{V}$ to $\text{Spec}(\mathbb{Q}_p)$. The purity theorem on $\mathcal{V}$ [6, Theorem 3.7.1] gives us the exact sequence

$$\text{Br}(V)[2] \to \text{Br}(V)[2] \mathrel{\overset{\partial_{D_p}}{\longrightarrow}} H^1(k(D_p), \mathbb{Z}/2\mathbb{Z})$$

where $D_p$ is the divisor associated to the special fibre ($D_p$ is the smooth locus of $Y_p$). We just need to show that $\partial_{D_p}(\mathcal{A}) = 0$. Let $v_p$ be the valuation corresponding to the prime divisor $D_p$; then

$$v_p \left( \frac{f}{x^3} \right) = 0 \quad \text{and} \quad v_p \left( -\frac{z}{x} \right) = 0.$$

Indeed, the point $(1 : 0 : 1 : 0) \in Y_p(\mathbb{F}_p)$ is smooth, hence it lies in $D_p$. Moreover

$$\frac{f}{x^3}(1 : 0 : 1 : 0) = 1 \quad \text{and} \quad -\frac{z}{x}(1 : 0 : 1 : 0) = -1.$$

Therefore both $\frac{f}{x^3}$ and $-\frac{z}{x}$ do not vanish identically on $D_p$, which implies that $\partial_{D_p}(\mathcal{A}) = 0$. Therefore, $\mathcal{A}$ lies in $\text{Br}(V) \subseteq \text{Br}(X_p)$ and the evaluation map factors as

$$\xymatrix{ \mathcal{V}(\mathbb{Z}_p) \ar[r]^-{|\mathcal{A}|} & \text{Br}(\mathbb{Q}_p) \ar@/_/[l]_-{\text{Br}(\mathbb{Z}_p)}}.$$
Since $\text{Br}(\mathbb{Z}_p)$ is trivial, the evaluation map has to be constant and trivial.

**Case $p = \infty$.** The evaluation map

$$|A| : X(\mathbb{R}) \to \text{Br}(\mathbb{R})$$

is constant and equal to 0.

We will show that it is constant on the dense open subset

$$W := \{ P \in X(\mathbb{R}) : x(P), z(P), f(P) \neq 0 \} \subseteq X(\mathbb{R}).$$

Then, from the continuity of the evaluation map it will follow that it has to be constant also on the whole of $X(\mathbb{R})$. Let $P = (\alpha : \beta : \gamma : \delta) \in W$, thus $\gamma \neq 0$. First, assume that $-\frac{\gamma}{\alpha} > 0$. Then

$$A(P) = \left( \frac{f(P)}{x(P)^3} - \frac{z(P)}{x(P)} \right) = \left( \frac{f(P)}{\alpha^3}, -\frac{\gamma}{\alpha} \right)$$

is trivial in $\text{Br}(\mathbb{R})$. Now, suppose that $-\frac{\gamma}{\alpha} < 0$. Without loss of generality, we can assume that both $\alpha$ and $\gamma$ are positive. We want to show that in this case $f(P)$ has to be positive:

- if $\delta = 0$, then $P \in X(\mathbb{R})$ implies that $\beta(\alpha^3 + \beta^2 \gamma) = 0$. Therefore $\beta = 0$, since $\alpha^3 + \beta^2 \gamma \geq \alpha^3 > 0$. Hence, $f(P) = \gamma^3 > 0$;
- if $\delta \neq 0$ then $P \in X(\mathbb{R})$ implies

$$f(P) = -\frac{\beta}{\delta}(\alpha^3 + \beta^2 \gamma).$$

Hence, since $\alpha^3 + \beta^2 \gamma > 0$,

$$f(P) > 0 \quad \text{if and only if} \quad -\frac{\beta}{\delta} > 0.$$

Equivalently, $\beta, \delta$ do not have the same sign. Hence, we just need to show that there is no point $P \in W$ with $\alpha, \gamma$ positive and $\beta, \delta$ with the same sign. First, we observe that $\beta, \delta$ cannot both be positive, since otherwise

$$\alpha^3 \beta + \beta^3 \gamma + \gamma^3 \delta + \delta^3 \alpha + \alpha \beta \gamma \delta > 0.$$ 

On the other hand $\beta, \delta$ cannot both be negative. Indeed, we have that $P \in X(\mathbb{R})$ if and only if

$$\alpha^3(-\beta) + (-\beta)^3 \gamma + \gamma^3(-\delta) + (-\delta)^3 \alpha = \alpha(-\beta)\gamma(-\delta).$$

Without loss of generality we may assume that $\alpha \geq \max\{-\beta, \gamma, -\delta\}$; but if $\alpha, -\beta, \gamma, -\delta$ are all positive, then

$$\alpha^3(-\beta) + (-\beta)^3 \gamma + \gamma^3(-\delta) + (-\delta)^3 \alpha > \alpha(-\beta)\gamma(-\delta).$$

Hence, $(\alpha : \beta : \gamma : \delta) \notin X(\mathbb{R})$. 

\[\Box\]

### 3 Construction of the quaternion algebra $\mathcal{A}$

The aim of this section is to give a glance at the ideas behind the construction of the quaternion algebra $\mathcal{A}$.

Through this section we will indicate by $X_2$ the base change of the $\mathbb{Z}$-model $X'$ to $\mathbb{Z}_2$, with $X_2$ the base change of $X_2$ to $\mathbb{Q}_2$ and with $Y$ the reduction of $X_2$ at the prime 2.
In the previous section we already mentioned that the K3 surface $X$ has good reduction at the prime 2 (i.e. $\mathcal{X}_2$ is a smooth $\mathbb{Z}_2$-scheme); actually $X$ has good ordinary reduction at the prime 2, as we will show in the following lemma.

**Lemma 1** $Y$ is an ordinary K3 surface over $\mathbb{F}_2$.

*Proof* It is enough to show that the cardinality of $Y(\mathbb{F}_2)$ is even (see [18]). $Y$ is the projective $\mathbb{F}_2$-variety defined by the equation

$$ax^3 + y^3z + z^3w + w^3x + xyzw = 0.$$ 

Therefore, it is possible to compute $|Y(\mathbb{F}_2)|$ directly, which turns out to be equal to 10. $\square$

Before proceeding with the actual construction, we mention a property of the K3 surface $Y$, related to the fact that it is ordinary. For every $q \geq 0$, let $\Omega^q_{Y/\mathbb{F}_2}$ be the sheaf of differential $q$-forms on $Y$. We define

$$Z^q_{Y/\mathbb{F}_2} := \ker \left( d : \Omega^q_{Y/\mathbb{F}_2} \to \Omega^{q+1}_{Y/\mathbb{F}_2} \right) \quad \text{and} \quad B^q_{Y/\mathbb{F}_2} := \text{im} \left( d : \Omega^{q-1}_{Y/\mathbb{F}_2} \to \Omega^q_{Y/\mathbb{F}_2} \right).$$

Let

$$C_{Y/\mathbb{F}_2} : Z^q_{Y/\mathbb{F}_2} \to \Omega^q_{Y/\mathbb{F}_2}$$

be the Cartier operator on $Y$ (for the construction of the Cartier operator see [12, §2]). For every $q$ the sheaf of logarithmic differential $q$-forms $\Omega^q_{Y/\mathbb{F}_2,\log}$ is defined as the kernel of the morphism

$$1 - C_{Y/\mathbb{F}_2} : Z^q_{Y/\mathbb{F}_2} \to \Omega^q_{Y/\mathbb{F}_2}.$$ 

Hence, $\Omega^q_{Y/\mathbb{F}_2,\log}$ fits in the following exact sequence:

$$0 \to \Omega^q_{Y/\mathbb{F}_2,\log} \to Z^q_{Y/\mathbb{F}_2} \xrightarrow{1 - C_{Y/\mathbb{F}_2}} \Omega^q_{Y/\mathbb{F}_2}.$$ 

(6)

**Remark 1** The Cartier operator is defined, more generally, for smooth $S$-schemes $X$ in characteristic $p$. The Cartier operator $C_{X/S}$ goes from $Z^\bullet_{X/S}$ to $\Omega^\bullet_{X^{(p)}/S}$, where $X^{(p)}$ is the base change of $X$ by the Frobenius morphism $F_S : S \to S$. In this setting $\Omega^q_{X^{(p)}\log}$ is defined as the kernel of the morphism

$$W^* - C_{X/S} : Z^q_{X/S} \to \Omega^q_{X^{(p)}/S}$$

where $W^*$ is the map induced on the differential forms by the natural projection

$$W : X^{(p)} \to X.$$ 

However, since in our case $S = \mathbb{F}_2$, we have $Y^{(2)} = Y$ and $W = id$.

The sheaf $\Omega^q_{Y/\mathbb{F}_2,\log}$ is a sheaf of $\mathbb{F}_2$-vector spaces on the Zariski site of $Y$. Moreover, if we look at it on $Y_{\text{ét}}$, then its formation is compatible with étale base change [12, 2.1.8].

Let $k$ be an algebraic closure of $\mathbb{F}_2$ and $\overline{Y}$ the base change of $Y$ to $k$. Bloch and Kato proved [2, Proposition 7.3] that $Y$ ordinary implies that the natural map

$$H^0 \left( \overline{Y}, \Omega^2_{\overline{Y}/k,\log} \right) \otimes_{\mathbb{F}_2} k \to H^0 \left( \overline{Y}, \Omega^2_{\overline{Y}/k} \right)$$

is an isomorphism. For K3 surfaces, $H^0 \left( \overline{Y}, \Omega^2_{\overline{Y}/k} \right)$ is a one-dimensional $k$-vector space. Hence $H^0 \left( \overline{Y}, \Omega^2_{\overline{Y}/k,\log} \right)$ has to be a one-dimensional $\mathbb{F}_2$-vector space. Let $\omega$ be the only
non-trivial element in $H^0 \left( \tilde{Y}, \Omega^2_{Y/k\log} \right)$. Since $C_{Y/k}$ respects the Galois action, we get that the element $\omega$ is Galois fixed, hence it comes from $H^0 \left( Y, \Omega^2_{Y/k} \right)$. Therefore, the unique non-trivial element of $H^0 \left( Y, \Omega^2_{Y/k} \right)$ must be logarithmic.

**Lemma 2** Let $F$ be the function field of $Y$. The image of $\omega \in H^0 \left( Y, \Omega^2_{Y/k} \right)$ in $\Omega^2_{F/k}$ can be written as

$$\frac{d\eta_1}{\eta_1} \wedge \frac{d\eta_2}{\eta_2}, \quad \text{where} \quad \eta_1 = \frac{z^3 + w^2x + xyz}{x^3} \quad \text{and} \quad \eta_2 = \frac{z}{x}.$$ 

**Proof** Let $\xi \in Y$ be the generic point of $Y$ and $\omega_\xi$ the image of $\omega$ in $\Omega^2_{F/k}$ under the inclusion

$$H^0 \left( Y, \Omega^2_{Y/k} \right) \to \Omega^2_{F/k}.$$ 

For convenience, to give an explicit description of a non-zero element in $H^0 \left( Y, \Omega^2_{Y/k} \right)$, we will use the following notation: instead of the variables $x, y, z, w$ defining $Y$, we define $T = \partial x, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}, \frac{\partial w}{\partial x}$ does not vanish. Moreover, we set

$$\omega_{p,q} := \frac{d\left( \frac{x}{y} \right)}{1} \wedge \frac{d\left( \frac{y}{z} \right)}{\frac{\partial G}{\partial x}_p} \in H^0(W_{p,q}, \Omega^2_{Y/k}).$$

Since $Y$ is smooth, the open sets $\{ W_{p,q} \}$ cover it. It is easy to check that, since we are working over the field $\mathbb{F}_2$, for every $(p, q) \neq (p', q')$

$$\omega_{p,q} |_{W_{p,q} \cap W_{p',q'}} = \omega_{p',q'} |_{W_{p,q} \cap W_{p',q'}}.$$ 

Therefore, there exists $\omega \in H^0 \left( Y, \Omega^2_{Y/k} \right)$ such that for every $p, q$ as above

$$\omega |_{W_{p,q}} = \omega_{p,q} \in H^0 \left( W_{p,q}, \Omega^2_{Y/k} \right).$$

Now, going back to the usual notation with which we denote our variables by $\{ x, y, z, w \}$, let $G_w$ be the partial derivative of the polynomial defining $Y$ with respect to the variable $w$. We have

$$\omega_\xi = (\omega_{0,3})_\xi = \frac{d \left( \frac{G_w}{x^3} \right) \wedge d \left( \frac{x}{z} \right)}{G_w / x^3} = \frac{d \left( \frac{G_w}{x^3} \right) \wedge d \left( \frac{y}{x} \right)}{G_w / x^3}.$$ 

Indeed,

$$d \left( \frac{G_w}{x^3} \right) \wedge d \left( \frac{z}{x} \right) = d \left( \frac{z^3 + w^2x + xyz}{x^3} \right) \wedge d \left( \frac{z}{x} \right) = d \left( \frac{z^3}{x^3} + \frac{w^2}{x^2} + \frac{yz}{x^2} \right) \wedge d \left( \frac{z}{x} \right).$$

Using that $d \left( \frac{z}{x} \right) \wedge d \left( \frac{y}{x} \right) = 0$, together with the fact that we are working over a field of characteristic 2, we get

$$d \left( \frac{G_w}{x^3} \right) \wedge d \left( \frac{z}{x} \right) = \frac{z}{x} \cdot d \left( \frac{y}{x} \right) \wedge d \left( \frac{z}{x} \right).$$
We can finally proceed with the construction of the quaternion algebra \( A \) and explain why we could expect a priori that \( A \) gives an obstruction to weak approximation on \( X \).

Let \( R \) be the henselisation of the discrete valuation ring \( \mathcal{O}_{X_2, \mathbb{Q}_2} \) and \( K^h \) be the fraction field of \( R \). Bloch and Kato \([2]\) introduced a decreasing filtration

\[
\left\{ U^m H^2 \left( K^h, \mu_2^{\otimes 2} \right) \right\}_{m \geq 0}
\]
on \( H^2 \left( K^h, \mu_2^{\otimes 2} \right) \) as follows. For \( a_1, a_2 \in K^h \), let \((a_1, a_2)_2\) denote the class \( \delta(a_1) \cup \delta(a_2) \in H^2 \left( K^h, \mu_2^{\otimes 2} \right) \) where \( \delta : (K^h)^\times \to H^1 \left( K^h, \mu_2 \right) \) is the connecting map coming from the Kummer sequence. Let

\[
\text{gr}^m := \frac{U^m H^2 \left( K^h, \mu_2^{\otimes 2} \right)}{U^{m+1} H^2 \left( K^h, \mu_2^{\otimes 2} \right)}.
\]

In \([2, \S 5]\), Bloch and Kato proved that the map

\[
\rho_0 : \Omega^2_{F, \log} \oplus \Omega^1_{F, \log} \to \text{gr}^0 := \frac{H^2 \left( K^h, \mu_2^{\otimes 2} \right)}{U^1 \left( H^2 \left( K^h, \mu_2^{\otimes 2} \right) \right)}
\]

\[
\left( \frac{d\eta_1}{\eta_1} \wedge \frac{d\eta_2}{\eta_2}, 0 \right) \mapsto (\tilde{\eta}_1, \tilde{\eta}_2)_2
\]

\[
\left( 0, \frac{d\eta_1}{\eta_1} \right) \mapsto (\tilde{\eta}_1, 2)_2
\]

is an isomorphism, where \( \tilde{\eta}_1, \tilde{\eta}_2 \) are arbitrary lifts of \( \eta_1, \eta_2 \) to \( K^h \).

**Remark 2** We are working over \( \mathbb{Q}_2 \), which contains a primitive second root of unity, hence we have an isomorphism \([7, \text{Proposition 4.7.1}]\)

\[
H^2 \left( K^h, \mu_2^{\otimes 2} \right) \simeq \text{Br} \left( K^h \right)[2]
\]

which sends \((a, b)_2\) to the class of the quaternion algebra \((a, b)\).

Let \( B \in \text{Br}(X_2)[2] \); we can always look at \( B \) as an element in \( H^2 \left( K^h, \mu_2^{\otimes 2} \right) \). Hence, there exists an \( m \geq 0 \) such that \( B \) gives a non-zero element in \( \text{gr}^m \). Bright and Newton proved \([4]\) that knowing such an \( m \) gives information about the behaviour of the evaluation map

\[
|B| : \mathcal{X}_2(\mathcal{O}_L) \to \text{Br}(L)
\]

\[
P \mapsto B(P)
\]

for finite field extensions \( L/\mathbb{Q}_2 \). We will make this sentence more precise in the following subsection.

### 3.1 The evaluation filtration

Bright and Newton define an evaluation filtration on the Brauer group of \( X_2 \) in the following way. Given a finite field extension \( L \) of \( \mathbb{Q}_2 \), let \( e_L/\mathbb{Q}_2 \) be the ramification index of \( L \), \( \mathcal{O}_L \) its ring of integers and \( \pi \) a uniformiser. For all positive integers \( r \) and \( P \in \mathcal{X}_2(\mathcal{O}_L) \), let

Let $B(P, r)$ be the set of points $Q \in \mathcal{X}_2(\mathcal{O}_L)$ such that $Q$ has the same image as $P$ in $\mathcal{X}_2(\mathcal{O}_L/\pi^r)$ (equivalently, we will say that $Q \equiv P \pmod{\pi^r}$). We define

$$2 \operatorname{Ev}_n \operatorname{Br} X_2 := \{ B \in \operatorname{Br}(X_2) | \forall L/\mathbb{Q}_2 \text{ finite}, \forall P \in X_2(\mathcal{O}_L)$$

$|B|$ is constant on $B(P, e_{L/\mathbb{Q}_2} n + 1)$, $(n \geq 0)$

$$\operatorname{Ev}_{-1} \operatorname{Br} X_2 := \{ B \in \operatorname{Br}(X_2) | \forall L/\mathbb{Q}_2 \text{ finite}, |B|$ is constant on $X_2(\mathcal{O}_L)\}$

$$\operatorname{Ev}_{-2} \operatorname{Br} X_2 := \{ B \in \operatorname{Br}(X_2) | \forall L/\mathbb{Q}_2 \text{ finite}, |B|$ is zero on $X_2(\mathcal{O}_L)\}$

In order to compare the evaluation filtration with the filtration $U^m H^2 (K^h, \mu_2^{\otimes 2})$, Bright and Newton used the filtration $[\operatorname{fil}_{n \geq 0} \operatorname{Br}(X_2)[2]]$ on $\operatorname{Br}(X_2)[2]$ given by Kato’s Swan conductor [4, §2]. Briefly, using the Swan conductor, Kato [13] defined an increasing filtration on $H^2 (K^h, \mathbb{Z}/2\mathbb{Z}(1))$. From the Kummer sequence, $H^2 (K^h, \mathbb{Z}/2\mathbb{Z}(1))$ is isomorphic to $\operatorname{Br}(K^h)[2]$. Therefore, we get a filtration $[\operatorname{fil}_{n \geq 0} \operatorname{Br}(K^h)[2]]_{n \geq 0}$ on $\operatorname{Br}(K^h)[2]$; the pullback of this filtration gives us a filtration $[\operatorname{fil}_{n \geq 0} \operatorname{Br}(X_2)[2]]_{n \geq 0}$ on $|\operatorname{Br}(X_2)[2]|$.

Moreover, since $\mathbb{Q}_2$ contains a primitive second root of unity, we have that $\mathbb{Z}/2\mathbb{Z}(1) \simeq \mu_2^{\otimes 2}$, thus we can identify

$$H^2(K^h, \mathbb{Z}/2\mathbb{Z}(1)) \simeq H^2(K^h, \mu_2^{\otimes 2}).$$

(7)

Kato [13, Lemma 4.3], proved that in our setting the isomorphism of Eq. (7) induces isomorphisms

$$U^m H^2(K^h, \mu_2^{\otimes 2}) \simeq \operatorname{fil}_{2^m} \operatorname{Br}(K^h)[2], \quad 0 \leq m \leq 2.$$ 

(8)

Moreover, for $m > 2$ we have that $U^m H^2(K^h, \mu_2^{\otimes 2})$ vanishes.

In particular, using the filtration $U^m$ on $H^2(K^h, \mu_2^{\otimes 2})$ we can describe $\operatorname{fil}_0 \operatorname{Br}(X_2)[2]$ as

$$\{ B \in \operatorname{Br}(X_2)[2] \text{ such that the image of } B \text{ in } H^2(K^h, \mu_2^{\otimes 2}) \text{ lies in } U^2 H^2(K^h, \mu_2^{\otimes 2}) \}.$$

Finally, Bright and Newton proved [4, Theorem A] that there is an equality

$$\operatorname{Ev}_0 \operatorname{Br}(X_2) = \operatorname{fil}_0 \operatorname{Br}(X_2).$$

### 3.2 The quaternion algebra $\mathcal{A}$

Summing up, the construction of $\mathcal{A}$ goes through the following steps. By Lemma 2 we know that $\omega \in H^0(Y, \Omega^2_{Y/\mathbb{Z}_2})$ is such that

$$\omega = \frac{d\eta_1}{\eta_1} \wedge \frac{d\eta_2}{\eta_2} \in \Omega^2_{F/\log}$$

is a non-trivial logarithmic 2-form of $F$, where $\eta_1 := \frac{Gx}{x^2}, \eta_2 := \frac{z}{x}$. The isomorphism $\rho_0$ assures us that for every choice of lifts $\tilde{\eta}_1, \tilde{\eta}_2$ in $K^h$, the element $(\tilde{\eta}_1, \tilde{\eta}_2) \in \operatorname{Br}(K^h)[2]$ has non-trivial image in $\operatorname{gr}^0$, that is $(\tilde{\eta}_1, \tilde{\eta}_2)[2]$ lies in $H^2(K^h, \mu_2^{\otimes 2}) \setminus U^2 H^2(K^h, \mu_2^{\otimes 2})$. At this point, the idea behind the construction of $\mathcal{A}$ is to find lifts $\tilde{\eta}_1, \tilde{\eta}_2$ such that $(\tilde{\eta}_1, \tilde{\eta}_2)$ defines an element in $\operatorname{Br}(X)$. Indeed, if such lifts exist, then the image of $(\tilde{\eta}_1, \tilde{\eta}_2)$ in $\operatorname{Br}(X_2)$ does not lie in $\operatorname{fil}_0 \operatorname{Br}(X_2)$. Rather surprisingly, as proven in Theorem 1, it turns out that the choice of lifts

$$\tilde{\eta}_1 := \frac{x^3 + w^2 x + x y z}{x^3} \quad \text{and} \quad \tilde{\eta}_2 := -\frac{z}{x}$$

defines an element

$$\mathcal{A} := (\tilde{\eta}_1, \tilde{\eta}_2) \in \operatorname{Br}(X).$$
By construction, using Eq. (8), the image of $\mathcal{A}$ in $\text{Br}(K^h)[2]$ is not in $\text{fil}_1 \text{Br}(K^h)[2]$. Hence, if we look at $\mathcal{A}$ in $\text{Br}(X_2)$, then it does not lie in $\text{fil}_1 \text{Br}(X_2) \supseteq \text{fil}_0 \text{Br}(X_2)$. In particular, by [4, Theorem A(3)] we have that there exists a finite field extension $L/\mathbb{Q}_2$ and two points $P, Q \in X_2(O_L)$ such that $P$ and $Q$ have the same image in $X_2(O_L/\pi_L)$ and $A(P) \neq A(Q)$.

We saw in the proof of Theorem 1 that there exist two points $P_1$ and $P_2$ defined over $\mathbb{Z}_2$, with the same reduction modulo 2 and whose evaluation map is different. Namely, there is no need to take a field extension of $\mathbb{Q}_2$ in our case.

**Remark 3** From the identification

$$\text{Br}(K^h)[2] \simeq U_0(H^2(K^h, \mu \otimes 2)) \simeq \text{fil}_2 \text{Br}(K^h)[2]$$

we get that

$$\text{Br}(X_2)[2] = \text{fil}_2 \text{Br}(X_2)[2].$$

Hence, clearly, $A \in \text{fil}_2 \text{Br}(X_2)[2]$.

[4, Theorem A(4)] tells us that

$$\text{Ev}_{1 \text{Br}(X_2)[2]} = \{ B \in \text{fil}_2 \text{Br}(X_2)[2] \mid \text{rs}_{2,2}(A) \in \Omega^F_2 \oplus 0 \}.$$ (9)

Hence, if $A$ is such that $\text{rs}_{2,2}(A) = (\alpha, 0)$, then $A$ lies in $\text{Ev}_1 \text{Br}(X_2)[2]$. The notation $\text{rs}_{2,2}$ denotes the refined Swan conductor (for the definition of it see [13, §5]). In particular, in our case

$$\text{rs}_{2,2} : \frac{\text{Br}(K^h)[2]}{\text{fil}_1 \text{Br}(K^h)[2]} \to \Omega^F_2 \oplus \Omega^1_F.$$  

Furthermore, the refined Swan conductor morphism is strictly related to the morphism $\rho_0$. Indeed, by Kato [13, Lemma 4.3], we know that

$$\frac{\text{Br}(K^h)[2]}{\text{fil}_1 \text{Br}(K^h)[2]} \simeq \frac{\text{Br}(K^h)[2]}{U^1 \text{Br}(K^h)[2]} = \text{gr}^0$$

and

$$\text{rs}_{2,2}(\rho_0(\alpha, \beta)) = (\alpha, \beta).$$ (10)

By construction, in our case, the image of $\mathcal{A}$ in $\text{gr}^0$ is of the form

$$\rho_0 \left( \frac{d\eta_1}{\eta_1} \wedge \frac{d\eta_2}{\eta_2}, 0 \right).$$

Therefore $\text{rs}_{2,2}(\mathcal{A})$ belongs to $\Omega^2_F \oplus 0$ and because of Eq. (9) it holds that $\mathcal{A}$ lies in $\text{Ev}_1 \text{Br}(X_2)[2]$. Thus, by the definition of the last object, the evaluation map from $X_2(\mathbb{Z}_2)$ to $\text{Br}(\mathbb{Q}_2)$ depends only on the reduction of the points modulo 4.

**Remark 4** Using Theorem B(4) in [4] we could already predict that the evaluation map attached to $\mathcal{A}$ is non-constant on the 2-adic points. By Eq. (10), we have that

$$\text{rs}_{2,2} = (\omega, 0).$$

Let $P_0 \in Y(\mathbb{F}_2)$ be the point $(1 : 0 : 1 : 0)$; locally in a neighbourhood of $P_0$, $\omega$ is of the form

$$\frac{x^3}{G_w} \cdot d \left( \frac{y}{x} \right) \wedge d \left( \frac{z}{x} \right) = \frac{x^3}{G_w} \cdot d \left( \frac{y}{x} \right) \wedge d \left( \frac{z}{x} - 1 \right).$$
with \( G_w = z^3 + w^2x + xyz \) (see the proof of Lemma 2). The functions \( \frac{y}{x} \) and \( \frac{z}{w} - 1 \) constitute a system of parameters for the local ring \( \mathcal{O}_{Y,P_0} \). Hence, we have that

\[
\omega_{P_0} = \left( \frac{y}{x} \right) \wedge \left( \frac{z}{w} - 1 \right) \neq 0 \in \Omega^2_{Y,P_0}.
\]

By [4, Theorem B(4)] there exists a point \( Q \in X(\mathbb{Z}_2) \) whose reduction modulo 2 coincides with \( P_0 \) and such that the evaluation map attached to \( A \) maps \( B(\mathbb{Q}, 1) \) surjectively to \( \text{Br}(\mathbb{Q}_2)[2] \).

### 4 A family of K3 surfaces with the same property

In this section we will show that the first part of Theorem 1 can be easily generalised to a family of K3 surfaces that share some properties with our K3 surface \( X \).

Let \( a, b, c, d, e \) be odd integers, \( \alpha = (a, b, c, d, e) \) and \( X_\alpha \) be the K3 surface in \( \mathbb{P}^3_\mathbb{Q} \) associated to the equation

\[
a \cdot x^3y + b \cdot y^3z + c \cdot z^3w + d \cdot w^3x + e \cdot xyzw = 0. \tag{11}
\]

Let \( X_\alpha \) be the projective scheme over \( \mathbb{Z} \) defined by the polynomial Eq. (11). Then \( X_\alpha \) is a \( \mathbb{Z} \)-model of \( X_\alpha \). Moreover, since \( a, b, c, d, e \) are all odd integers, all these varieties have the same reduction, which we will denote by \( Y \), modulo the prime 2. Hence, by Lemma 1, all these K3 surfaces have ordinary good reduction at the prime 2. Therefore, as already pointed out in Sect. 3, the unique non-trivial element \( \omega \) in \( H^0(Y, \Omega^2_Y) \) must be logarithmic.

A natural question that arises at this point is if also for all these K3 surfaces there exists an element \( A \in \text{Br}(X_\alpha)[2] \) such that

\[
\rho_0(\omega_0) = [A] \in \frac{\text{Br}(K^h)[2]}{U_1 \text{Br}(K^h)[2]}.
\]

Indeed, by Sect. 3.2 this would imply that, at least after taking a finite field extension of \( \mathbb{Q}_2 \), the quaternion algebra \( A \) gives an obstruction to weak approximation. The following theorem gives a partial answer to the question.

**Theorem 2** Assume that \( abcd \in \mathbb{Q}^{\times 2} \). Then, the class of the quaternion algebra

\[
A = \left( \frac{d \cdot c \cdot z^3 + d \cdot w^3x + e \cdot xyz}{x^3}, -(cd) \cdot \frac{z}{x} \right) \in \text{Br}(\mathbb{Q}(X_\alpha))
\]

defines an element in \( \text{Br}(X_\alpha) \). The evaluation map \( [A] : X_\alpha(\mathbb{Q}_2) \to \text{Br}(\mathbb{Q}_2) \) is non-constant, and therefore gives an obstruction to weak approximation on \( X \).

**Proof** The proof is very similar to the first part of the proof of Theorem 1. We denote by \( f \) the polynomial \( c \cdot z^3 + d \cdot w^3x + e \cdot xyz \). Also in this case, let \( C_x, C_z, C_f \) be the closed subsets of \( X_\alpha \) defined by the equations \( x = 0, z = 0 \) and \( f = 0 \) respectively. Let \( U \) be the open subset of \( X \) defined as the complement of \( C_x \cup C_z \cup C_f \). Clearly, \( A \in \text{Br}(U) \). Moreover,

- \( C_x \) consists of two irreducible components, \( D_1 = \{ x = 0, z = 0 \} \) and \( D_2 = \{ x = 0, b \cdot y^3 + c \cdot z^2w = 0 \} \).
- \( C_z \) consists of two irreducible components, \( D_1 \) and \( D_3 = \{ z = 0, a \cdot x^2y + d \cdot w^3 = 0 \} \).
- \( C_f \) consists of three irreducible components, \( D_1, D_4 = \{ y = 0, c \cdot z^3 + d \cdot w^3x = 0 \} \) and \( D_5 = \{ f = 0, a \cdot x^3 + b \cdot y^2z = 0, be \cdot y^3z - ac \cdot x^2z^2 + bd \cdot y^2w^2 = 0 \} \).

In order to show that the quaternion algebra \( A \) lies in the Brauer group of \( X \) we will use the exact sequence (3) coming from the purity theorem and the explicit description of the
residue map given in Eq. (4). We will denote by \( v_i \) the valuation associated to the prime divisor \( D_i \).

1. \( v_1(f) = v_1(x) = v_1(z) = 1 \), and so \( v_1 \left( \frac{f}{x^3} \right) = -2 \) and \( v_1 \left( \frac{x}{z} \right) = 0 \). Hence

\[
\partial_{D_1}(A) = \left( -1 \right)^0 \left( \frac{d \cdot f}{x^3} \right) \left( -\frac{1}{cd} \cdot \frac{x^2}{z} \right) = \left[ \left( \frac{1}{cd} \cdot \frac{x}{z} \right)^2 \right] = 1 \in \frac{k(D_1)^\times}{k(D_1)^\times/2}.
\]

2. \( v_2(f) = v_2(z) = 0 \) and \( v_2(x) = 1 \), and so \( v_2 \left( \frac{f}{x^3} \right) = -3 \) and \( v_2 \left( \frac{x}{z} \right) = -1 \). Hence

\[
\partial_{D_2}(A) = \left( -1 \right)^3 \left( \frac{d \cdot f}{x^3} \right)^{-1} \left( -\frac{1}{cd} \cdot \frac{x^3}{z^3} \right) = \left[ \left( \frac{(cd)^3 \cdot x^3}{d \cdot z^3} \right)^{-3} \right] = 1 \in \frac{k(D_2)^\times}{k(D_2)^\times/2},
\]

where the second equality follows from the fact that \( f \mid D_2 = c \cdot z^3 \).

3. \( v_3(f) = v_3(x) = 0 \) and \( v_3(z) = 1 \), and so \( v_3 \left( \frac{f}{x^3} \right) = 0 \) and \( v_3 \left( \frac{x}{z} \right) = 1 \). Hence

\[
\partial_{D_3}(A) = \left( -1 \right)^0 \left( \frac{d \cdot f}{x^3} \right)^1 \left( -\frac{1}{cd} \cdot \frac{x^3}{z^3} \right) = \left[ d \cdot \frac{x}{z} \right] = 1 \in \frac{k(D_3)^\times}{k(D_3)^\times/2}
\]

where the last equality follows from the fact that \( f \mid D_3 = d \cdot w^2x \).

4. \( v_4(f) = 1 \) and \( v_4(x) = v_4(z) = 0 \), and so \( v_4 \left( \frac{f}{x^3} \right) = 1 \) and \( v_4 \left( \frac{x}{z} \right) = 0 \). Hence

\[
\partial_{D_4}(A) = \left( -1 \right)^0 \left( \frac{d \cdot f}{x^3} \right) \left( -\frac{1}{cd} \cdot \frac{x^3}{z^3} \right) = \left[ \frac{1}{cd} \cdot \frac{c}{d} \right] = 1 \in \frac{k(D_4)^\times}{k(D_4)^\times/2}
\]

where the second equality follows from the fact that \( -\frac{x}{z} = \frac{d}{c} \left( \frac{x}{y} \right)^2 \) on \( D_4 \).

5. \( v_5(f) = 1 \) and \( v_5(x) = v_5(z) = 0 \), and so \( v_5 \left( \frac{f}{x^3} \right) = 1 \) and \( v_5 \left( \frac{x}{z} \right) = 0 \). Hence

\[
\partial_{D_5}(A) = \left( -1 \right)^0 \left( \frac{d \cdot f}{x^3} \right) \left( -\frac{1}{cd} \cdot \frac{x^3}{z^3} \right) = \left[ \frac{b}{acd} \right] = 1 \in \frac{k(D_5)^\times}{k(D_5)^\times/2}
\]

where the last equality follows from the fact that \( -\frac{x}{z} = \frac{a}{b} \left( \frac{y}{x} \right)^2 \) on \( D_5 \) and the assumption that \( abcd \) is a square in \( \mathbb{Q} \).

The above computations together with the purity theorem show indeed that \( \mathcal{A} \) lies in \( \text{Br}(X_{\mathbb{Q}}) \). Finally, we need to show that the evaluation map on the \( \mathbb{Q}_2 \)-points of \( X_{\mathbb{Q}} \) is non-constant. Let

\[
P_1 := (1 : 0 : 1 : 0) \in X_{\mathbb{Q}}(\mathbb{Q}).
\]

Then, \( P_1 \) is such that \( \mathcal{A}(P_1) = (dc_0 - dc) \), which is trivial in \( \text{Br}(\mathbb{Q}_2) \). Furthermore, let

\[
P_2 := \left( cd : y : 1 : -2 \cdot \frac{acde}{2c + cd} \right) \in X_{\mathbb{Q}}(\mathbb{Q}_2)
\]

be such that the reduction modulo 8 of \( y \) is equal to \( 2 \cdot de \). Then

\[
f(P_2) \equiv c + d \cdot 4 \cdot (cd) + e \cdot (cd) \cdot 2 \cdot de \equiv 7 \cdot c \mod 8
\]

and therefore evaluation of \( \mathcal{A} \) at \( P_2 \) is

\[
\mathcal{A}(P_2) = \left( \frac{d \cdot f(P_2)}{(cd)^3}, -cd \cdot \frac{1}{cd} \right) = (g(P_2), -1) \text{ with } g(P_2) \equiv 7 \mod 8.
\]
Thus, $A(P_2)$ defines a non-trivial element in $\text{Br}(\mathbb{Q}_2)$. Hence, the element $A \in \text{Br}(X_\alpha)$ gives an obstruction to weak approximation on $X_\alpha$. □

A natural question that arises at this point is what happens if $\Delta := abcd$ is not a square in $\mathbb{Q}$. Note that, in this case we can repeat the same computations that we did in the proof of Theorem 2. That is, for every divisor $D \neq D_5$ we get $\partial_D(A) = 1$, while for $D_5$ we have

$$\partial_{D_5}(A) = \lfloor \Delta \rfloor \in \frac{k(D_5)^x}{k(D_5)^{x/2}}.$$  

Hence, in this case, $A$ defines an element in the Brauer group of the base change of $X_\alpha$ to $\mathbb{Q}(\sqrt{\Delta})$. With an argument similar to the one of Remark 4, also in this case we expect to be able to find two points $P_1, P_2$ defined over $\mathbb{Q}_2(\sqrt{\Delta})$ such that $A(P_1) \neq A(P_2)$.

4.1 Final considerations

As already mentioned in the introduction, this paper was strongly inspired by the following result proven by Bright and Newton.

**Theorem 3** [4, Theorem C] Let $V$ be a smooth, proper variety over a number field $L$ with $H^0(V, \Omega^2_V) \neq 0$. Let $p$ be a prime of $L$ at which $V$ has good ordinary reduction, with residue characteristic $p$. Then there exists a finite field extension $L'/L$, a prime $p'$ of $L'$ lying over $p$, and an element $A \in \text{Br}_{L'/p}$ such that the evaluation map $|A| : V(L'_{p'}) \to \text{Br}(L'_{p'})$ is non-constant. In particular, if $V(A_{L'}) \neq \emptyset$ then $A$ obstructs weak approximation on $V_{L'}$.

In our example,

$$V = X = \text{Proj} \left( \frac{\mathbb{Q}[x, y, z, w]}{x^3y + y^3z + z^3w + w^3x + xyz^2} \right) \subseteq \mathbb{P}^3_{\mathbb{Q}}$$

is a smooth projective variety defined over the number field $\mathbb{Q}$.

Since $X$ is a K3 surface, the hypotheses of Theorem 3 are satisfied. In this example, we were able to construct an element $A$ that satisfies Theorem 3 which is already defined over the rational numbers and the corresponding evaluation map is non-constant on the 2-adic points. Moreover, $A$ does not just lie in the 2-primary part of the Brauer group of $X$, it has order exactly 2.

It is still unclear whether one can hope to extend this strategy to a more general setting.

**Acknowledgements**

I am deeply grateful to Martin Bright for introducing me to the topic and for the ideas he shared with me that were very helpful in writing this paper. I thank Francesco Viganò for the useful conversation that helped me in finding the equation defining $X$. I am grateful to the anonymous referees for helpful comments.

**Funding**

Not applicable.

**Data Availability**

Not applicable.

**Declarations**

**Conflict of interest**

Not applicable.

**Code availability**

Not applicable.

Received: 9 July 2021  Accepted: 15 June 2022  Published online: 1 September 2022
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