QUANTIZATION OF BRANCHED COVERINGS

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Abstract. We identify branched coverings (continuous open surjections \( p : Y \to X \) of Hausdorff spaces with uniformly bounded number of pre-images) with Hilbert \( C^* \)-modules \( C(Y) \) over \( C(X) \) and with faithful unital positive conditional expectations \( E : C(Y) \to C(X) \) topologically of index-finite type. The case of non-branched coverings corresponds to projective finitely generated modules and expectations (algebraically) of index-finite type. This allows to define non-commutative analogues of (branched) coverings.

1. Introduction

The purpose of the present paper is to obtain an appropriate description of branched coverings in terms of (commutative) \( C^* \)-algebras and their modules in such a way that it admits a natural generalization to a non-commutative setting. In fact, we will obtain two (closely related to each other) descriptions.

A branch covering (in this paper) is a closed and open continuous surjection of compact Hausdorff spaces \( p : Y \to X \) with a finite bounded number of pre-images \#\( p^{-1}(x) \), \( x \in X \). (In Section 2 we describe some properties of branch coverings and their equivalent descriptions.)

The main result of the present paper is the following theorem.

Theorem 1.1. Suppose, \( i : C(X) \to C(Y) \) is an inclusion, where \( X \) and \( Y \) are compact Hausdorff spaces. Let \( p = i^* \) be its Gelfand dual surjection \( p : Y \to X \). Then the following properties are equivalent:

1) The surjection \( p \) is a branched covering.
2) Consider \( C(Y) \) as a \( C(X) \)-module with respect to the natural action induced by \( i \). Then \( C(Y) \) can be equipped with an inner \( C(X) \)-product in such a way that it becomes a (complete) Hilbert \( C(X) \)-module.
3) It is possible to define a positive unital conditional expectation \( E : C(Y) \to C(X) \) topologically of index-finite type (in the sense of [2]).

Proof. The implication 1) \( \Rightarrow \) 2) will be proved in Theorem 4.3. The implication 3) \( \Rightarrow \) 1) will be proved in Theorem 5.6. The equivalence 2) \( \iff \) 3) is known (see [5] and Proposition 5.4 below).

This theorem suggests how to quantize branched coverings. More precisely we can introduce the following definition.

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Definition 1.2. A non-commutative branched covering is a pair \((B, A)\) consisting of a 
\(C^*\)-algebra \(B\) and its \(C^*\)-subalgebra \(A\) with common unity, such that one of the following equivalent (by [3, Theorem 1]) conditions holds.

1) The algebra \(B\) may be equipped with an inner \(A\)-valued product in such a way that it 
becomes a (complete) Hilbert \(A\)-module.

2) There exists a positive conditional expectation \(E : B \to A\) topologically of index-finite 
type.

The above theorem and definition can be specialized to the case of (non-singular finite-fold) coverings in the following way. The most part of the next theorem is known.

Theorem 1.3. Suppose, \(i : C(X) \to C(Y)\) is an inclusion, where \(X\) and \(Y\) are compact 
Hausdorff spaces. Let \(p = i^*\) be its Gel
dorf dual surjection \(p : Y \to X\). Then the following 
properties are equivalent:

1) The surjection \(p\) is a finite-fold covering.

2) The module \(C(Y)\) may be equipped with an inner \(C(X)\)-product in such a way that it 
becomes a finitely generated projective Hilbert \(C(X)\)-module.

3) It is possible to define a positive unital conditional expectation \(E : C(Y) \to C(X)\) 
(algebraically) of index-finite type (in the sense of [25]).

Proof. The implication 1)\(\Rightarrow\)3) is proved in [25, Proposition 2.8.9]. The implication 2)\(\Rightarrow\)1) 
will be proved in Theorem 4.4. The equivalence 2)\(\Leftrightarrow\)3) can be extracted from [25, pp. 
92–93] (see Theorem 5.7 below). \(\Box\)

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type.

Our research continues the research on spaces and modules arising from discrete group 
actions (cf. [9, 24]). Apart from the mentioned above papers let us indicate the research of 
Buchstaber, Rees, Gugnin and others on algebraic definition and topological applications 
of Dold-Smith ramified coverings (see, e.g., [3, 4, 12]).

A number of known as well as of new facts about branched coverings are collected in 
Section 2 Preliminaries on Hilbert modules, basic lemmas and examples are contained in 
Section 3. Section 4 deals mostly with Hilbert module aspects of proofs of the main theorems. At its end a couple of related statements concerning other types of Hilbert modules 
is proved. Section 5 is devoted to conditional expectations and to the corresponding parts 
of proofs. At the end of the section the role of index elements is discussed.

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present paper.
2. Branched coverings

In this section we present (mostly known) statements about continuous surjections of Hausdorff spaces. Let

\[ p : Y \to X \]

be a continuous surjection of compact Hausdorff spaces, in particular, a closed map.

**Definition 2.1.** Let us consider the map (1) and a certain point \( x \) of \( X \), which has a finite number of pre-images \( y_1, \ldots, y_m \). Then a neighborhood \( U \) of \( x \) is said to be regular if

\[ p^{-1}(U) = V_1 \sqcup \cdots \sqcup V_m, \]

where \( V_i \) are some neighborhoods of \( y_i, i = 1, \ldots, m \).

**Lemma 2.2.** Let \( p : Y \to X \) be a continuous closed map of Hausdorff spaces. Then any point \( x \) of \( X \) with a finite number of pre-images has a regular neighborhood.

**Proof.** Suppose the pre-image of \( x \) consists of points \( y_1, \ldots, y_m \). These points can be separated by pairwise disjoint neighborhoods \( V_{1}', \ldots, V_{m}' \). Then the set \( U = X \setminus p(Y \setminus \sqcup_{i=1}^{m} V_i') \) is an open neighborhood of \( x \), because the map \( p \) is closed. Now one can set \( V_i = p^{-1}(U) \cap V_i' \).

**Lemma 2.3.** Let \( p : Y \to X \) be a continuous closed map of Hausdorff spaces. Suppose \( U \) is a regular neighborhood of a point \( x \in X \) satisfying (2) and \( U' \) is an open set satisfying the condition: \( x \in U' \subset \overline{U'} \subset U \). Put \( V_i' = p^{-1}(U') \cap V_i \).

**Proof.** The first statement is obvious. The second is true because the map \( p \) is closed. And the third immediately follows from the second.

Given the covering (1). Denote by \( X_j \) the subset (stratum) of \( X \) consisting of points that have exactly \( j \) pre-images and reserve the notation \( \tilde{X}_j \) for the union \( \bigcup_{i=0}^{j} X_i \), \( j \geq 0 \).

Now for any point \( x \) of \( X \) consider the collection (2) of neighborhoods \( U_i, V_i, \ldots, V_m \), where \( m = \#p^{-1}(x) \).

Then

\[ X(x, U)^{(k)}_j = \{ z \in U : \#p^{-1}(z)_k = j \}, \]

where \( p^{-1}(z)_k = p^{-1}(z) \cap V_k \), and \( \tilde{X}(x, U)^{(k)}_j \) stands for the union \( \bigcup_{i=0}^{j} X(x, U)^{(k)}_i \).

**Definition 2.4.** The map (1) is said to be a branched covering if both \( X \) and \( Y \) are compact Hausdorff spaces, \( p \) is a continuous surjective map (in particular, closed) and the following conditions hold:

(i) \( \tilde{X}(x, U)^{(k)}_j \) is a closed subset of \( \tilde{X}(x, U)^{(k)}_{j+1} \) for any point \( x \) of \( X \), some its neighborhood \( U \) satisfying (2) and for all \( k = 1, \ldots, m, j = 0, 1, 2, \ldots \),

(ii) the cardinalities of the pre-images \( p^{-1}(x) \) are uniformly bounded over \( x \in X \).

A finite-fold covering \( p : Y \to X \) of connected compact spaces, obviously, satisfies the conditions (i), (ii) of Definition 2.4, so it is a particular case of a branched covering.
Proposition 2.5. Let \( p : Y \to X \) be a branched covering. Then the stratum \( \hat{X}_j \) is closed in the next stratum \( \hat{X}_{j+1} \) for all \( j \geq 0 \).

Proof. Consider any point \( x \in X \) and its regular neighborhood \( U \) satisfying (2). Then for any \( j \) the set \( \hat{X}_j \cap U \) coincides with the following finite intersection of the sets
\[
\hat{X}(x,U)^{(1)}_{j_1} \cap \cdots \cap \hat{X}(x,U)^{(m)}_{j_m}
\]
over \( j_1 + \cdots + j_m \leq j \). In particular, the set \( \hat{X}_j \cap U \) is closed in \( \hat{X}_{j+1} \cap U \). Since \( X \) is compact, \( \hat{X}_j \) is closed in \( \hat{X}_{j+1} \) as well. \( \square \)

Proposition 2.6. Let \( p : Y \to X \) be a branched covering. Then for any point \( x \) of \( X \) there is its regular neighborhood such that the restriction of \( p \) on \( V_k \) is surjective for any \( k = 1, \ldots, m \).

Proof. Let us suppose the opposite is true. Then for some point \( x \) of \( X \) and for some its regular neighborhood (2) one can find a net \( \{x_\alpha\} \) converging to \( x \) and such that the intersection of its pre-image with \( V_k \) is empty for some \( k \). In the other words, \( \{x_\alpha\} \) belongs to \( \hat{X}(x,U)^{(k)}_0 \), whereas \( x \) lies in \( X(x,U)^{(k)}_1 \). This contradicts to the condition (i) of Definition 2.7. \( \square \)

Definition 2.7. A map \( f : Y \to X \) is said to be a local epimorphism if for any \( y \in Y \) and any its neighborhood \( U \ni y \) there exists another neighborhood \( U_y \subset U \) such that \( V_x := f(U_y) \) is an (open) neighborhood of \( x = f(y) \).

Lemma 2.8. A map \( f : Y \to X \) is a local epimorphism if and only if it is an open map.

Proof. ‘If’ is evident: take \( U_y = U \).

Now let \( f \) be a local epimorphism and \( U \subset Y \) be an arbitrary open set. For each \( y \in U \) find \( U_y \) and \( V_x \) in accordance with Definition 2.7. Then
\[
f(U) = f(\cup_{y \in U} U_y) = \cup_{y \in U} f(U_y) = \cup_{y \in U} V_{f(y)}
\]
is open. \( \square \)

Theorem 2.9. Consider a surjective map \( p : Y \to X \) of compact Hausdorff spaces with uniformly bounded number of pre-images, i.e.
\[
\sup_{x \in X} \#p^{-1}(x) = m < \infty.
\]
Then \( f \) is a branched covering if and only if it is open.

Proof. In fact the proof of Proposition 2.6 may be slightly changed to obtain the ‘only if’ statement. Indeed, for any point \( y \in Y \) with \( x = p(y) \) we consider a regular neighborhood \( U \) of \( x \) as in Proposition 2.6. Let \( y \) belong to \( V_k \) and \( V \) be an arbitrary neighborhood of \( y \). Set \( H = p(V \cap V_k) \subset U \), \( x \in H \). Now to make sure that \( f \) is a local epimorphism we have only to verify that \( H \) contains some (open) neighborhood of \( x \). But otherwise there is a net \( \{x_i\} \) of \( U \setminus H \) converging to \( x \). Then the net \( y_i := p^{-1}(x_i) \cap V_k \) converges to \( y \) and does not belong to the (open) neighborhood \( V_k \cap V \) of \( y \). We come to a contradiction. Thus \( p \) is a local epimorphism and by Lemma 2.8 it is open.

Now let \( p \) be open, hence be a local epimorphism. Suppose, the first item of Definition 2.7 does not hold. This means that for some point \( x' \in U \) (may be \( x' \neq x \)), any its regular
neighborhood $U'$ and some its pre-image $y'_k$ we can find a point $x'' \in U$ with no pre-image in $V'_k$. Thus, $p$ is not a local epimorphism. A contradiction. \qed

Remark 2.10. If we replace the first condition of Definition 2.4 by the condition that $\hat{X}_j$ is closed in $\hat{X}_{j+1}$ for all $j \geq 0$, then the statement of Proposition 2.6 will not be true. The corresponding example is given by Figure 1 that differs from Figure 2 by one additional (closed) interval ending over the branch point.

3. Projective and finitely generated Hilbert $C^*$-modules. Examples

For facts on Hilbert $C^*$-modules we refer the reader to [16, 17, 22]. We recall here just the most important for us statements. For a Hilbert $C^*$-module $M$ over a $C^*$-algebra $A$ (it always is supposed to be unital in the present paper, unless otherwise is explicitly stated) the $A$-dual module $M'$ is the module of all bounded $A$-linear maps from $M$ to $A$. $M$, equipped with an $A$-inner product $\langle \cdot, \cdot \rangle$, is called self-dual if the map $\& : M \to M'$, $x \& (\cdot) = \langle x, \cdot \rangle$ is an isomorphism, and $M$ is called reflexive if the map $\cdot : M \to M''$, $x(\cdot) = \langle x, \cdot \rangle^* (f \in M')$ is an isomorphism. Unlike the Banach space situation the third dual $M'''$ for $M$ is always isomorphic to $M'$, whereas the modules $M$, $M'$ and $M''$ may be pairwise non-isomorphic in particular situations (cf. [19]).

$M$ is called finitely generated Hilbert $C^*$-module if it is an $A$-span of a finite system of its vectors, $M$ is called finitely generated projective if it is a direct summand of $A^n$ for some $n$. It is easy to see that a finitely generated projective module over a unital $C^*$-algebra is always self-dual. $M$ is called countably generated if it is a norm-closure of an $A$-span of a countable system of its vectors. Kasparov’s stabilization theorem asserts that any countably generated Hilbert $A$-module can be represented as a direct orthogonal summand of the standard module $l_2(A)$ [15].

**Theorem 3.1.** Any finitely generated Hilbert module over a unital $C^*$-algebra is a projective one.

**Proof.** By Kasparov’s stabilization theorem a finitely generated module is an orthogonal direct summand of the standard module $l_2(A)$. Therefore it is projective by [18 Theorem 1.3]. \qed

Now we will prove more statements about (not) finitely generated and (not) finitely generated projective modules over commutative $C^*$-algebras. Some related examples will be used in the sequel.

The next statement is well known.
Lemma 3.2. Let $X$ be a compact Hausdorff space and $x_0$ be its non-isolated point. Then the module $C(X)_0 := \{ f \in C(X) : f(x_0) = 0 \}$ is not finitely generated over $C(X)$.

Proof. Assume there is a finite number of generators $f_1, \ldots, f_s$ of $C(X)_0$ over $C(X)$ and consider the function

$$f = |f_1|^{1/2} + \cdots + |f_s|^{1/2}.$$ 

Obviously it can not vanish on an entire neighborhood of $x_0$. Under our assumptions

$$f = g_1 \cdot f_1 + \cdots + g_s \cdot f_s$$

for some $g_i \in C(X)$. Suppose $I$ is the subset of the set $\{1, \ldots, s\}$ such that $g_i$ is not the zero function if and only if $i \in I$. Let us put

$$m_i = \|g_i\| = \max_{x \in X} |g_i(x)|, \quad i \in I.$$ 

There is an open neighborhood $U$ of $x_0$ such that the following inequalities

$$|f_i(x)|^{1/2} \leq \frac{1}{2 \cdot m_i}$$

hold for any $x \in U$, $i \in I$. Hence

$$|f(x)| \leq \sum_{i \in I} |f_i(x)|^{1/2} \cdot |f_i(x)|^{1/2} m_i \leq \frac{1}{2} \sum_{i \in I} |f_i(x)|^{1/2} \leq \frac{1}{2} \sum_{i=1}^s |f_i(x)|^{1/2} = \frac{1}{2} |f(x)|.$$ 

A contradiction. \hfill \Box

Let $p : Y \to X$ be a continuous map of Hausdorff topological spaces. Then $C(Y)$ is a Banach $C(X)$-module with respect to the action:

$$\tag{3} (f \xi)(y) = f(y) \xi(p(y)), \quad f \in C(Y), \xi \in C(X).$$

Lemma 3.3. Let $X' \subset X$ be a directed set $\{x_\alpha\}$ together with a unique limit point $x$. Let $Y' \subset Y$ be equal to the union of directed sets $\{y_\alpha^0\}$ and $\{y_\alpha^1\}$ with a common limit point $y$, and

$$p(y_\alpha^0) = p(y_\alpha^1) = x_\alpha, \quad p(y) = x.$$ 

Then $C(Y')$ is not a finitely generated module over $C(X')$.

Proof. Consider two $C^*$-subalgebras $C(Y')_1$ and $C(Y')_0$ of the $C^*$-algebra $C(Y')$, where $C(Y')_1$ consists of those continuous functions, which are constant on $\{y_\alpha^0\} \cup y$, and $C(Y')_0$ consists of those continuous functions, which are zero on $\{y_\alpha^1\} \cup y$. Then any continuous function $f$ on $Y'$ can be represented in a unique way as the sum $f = f_1 + f_0$ of the function $f_1 \in C(Y')_1$, which is equal to $f$ on $\{y_\alpha^1\} \cup y$, and the function $f_0 = f - f_1 \in C(Y')_0$. Hence, $C(Y') = C(Y')_1 \oplus C(Y')_0$. Clearly, $C(Y')_1$ is isomorphic to $C(X')$, and $C(Y')_0$ is isomorphic to $C(X')_0$ as Hilbert $C(X')$-modules, where $C(X')_0$ consists of continuous functions vanishing at $x$. Thus, if $C(Y')$ is finitely generated, then $C(X')_0$ is finitely generated too. A contradiction with Lemma 3.2. \hfill \Box

Lemma 3.4. Given the map $[1]$, where $X, Y$ are normal Hausdorff spaces. Suppose $X' \subset X$ is a closed subset, $Y' = p^{-1}(X')$, and $C(Y)$ is a finitely generated $C(X)$-module. Then

(i) $C(Y')$ is a finitely generated $C(X')$-module.

(ii) $C(Y'')$ is a finitely generated $C(X')$-module for any closed subset $Y'' \subset Y'$. 
Proof. (i) Consider generators \( \{f_1, \ldots, f_n\} \) of \( C(Y) \) over \( C(X) \) and put \( f'_i = f_i|_{Y'} \), \( i = 1, \ldots, n \). Then by the Tietze theorem for any \( h' \in C(Y') \) there is \( h \in C(Y) \) satisfying \( h|_{Y'} = h' \). Since \( h = f_1 g_1 + \cdots + f_n g_n \) for some \( g_1, \ldots, g_n \in C(X) \), one has \( h' = f'_1 g'_1 + \cdots + f'_n g'_n \), where \( g'_i = g_i|_{X'} \).

(ii) Given a closed subset \( Y'' \subset Y' \). For any function \( f'' \in C(Y'') \) it is possible to construct its extension \( f' \in C(Y') \), which may be decomposed as \( f' = f'_1 \alpha_1 + \cdots + f'_n \alpha_n \) with \( \alpha_i \in C(X') \). Then \( f'' = f''_1 \alpha_1 + \cdots + f''_n \alpha_n \) with \( f''_1 = f''|_{Y''} \in C(Y'') \) as required. \( \square \)

**Lemma 3.5.** Given the map (1), where \( X \) and \( Y \) are compact Hausdorff spaces. Let \( X' \subset X \) be a closed subset and \( Y' = p^{-1}(X') \). If \( C(Y) \) is a finitely generated projective \( C(X) \)-module, then \( C(Y') \) is a finitely generated projective \( C(X') \)-module.

**Proof.** Let \( M = C(Y), M' = C(Y') \). Then one has two \(*\)-epimorphisms

\[
\varphi : M \to M' \quad \text{and} \quad \psi : C(X) \to C(X')
\]

given by \( \varphi(f) = f|_{Y'} \), and \( \psi(\alpha) = \alpha|_{X'} \), respectively, and satisfying the conditions

\[
\varphi(f \alpha) = \varphi(f) \psi(\alpha), \quad \varphi(1) = 1, \quad \psi(1) = 1,
\]

where \( f \in C(Y), \alpha \in C(X) \). There is an injection \( i : M \hookrightarrow C(X)^n \) and a surjection \( s : C(X)^n \to M \), such that \( s \circ i = \text{Id}_M \). In particular, \( i \circ s \circ i \circ s = i \circ s = \pi \) for some idempotent \( \pi \) on \( C(X)^n \). Obviously, \( i \) is also topologically injective, i.e. \( \|i(f)\| \geq k \|f\| \) for a certain \( k > 0 \) and for any \( f \in C(Y) \). Define \( i' : C(Y') \hookrightarrow C(X)^n \) in the following way. Take \( f' \in C(Y') \), extend it by Tietze’s lemma to a continuous function \( f \) on \( Y \), apply \( i \) and then \( \psi_n \). Evidently, the result does not depend on the choice of extensions, because of modularity and topological injectivity of \( i \). Moreover, \( i' \) is a module map and \( i' \) is an injection. To verify the last statement let us take any function \( f' \in C(Y') \) for which \( i'(f') = \psi_n i(f) = 0 \), where \( f \) is a certain continuous extension of \( f' \) to \( Y \). This implies \( i(f)|_{X'} = 0 \). Then for any \( \varepsilon > 0 \) there exists an open neighborhood \( U \) of \( X' \) such that \( \|i(f)|_U\| < \varepsilon \). Consider a function \( 0 \leq \gamma \leq 1 \) of \( C(X) \), which is 1 on \( X' \) and 0 outside of \( U \). Assume that \( f' \neq 0 \), then \( \|f'(y)\| = C > 0 \) for a certain point \( y \in Y' \). Consequently, \( \|\gamma f\| \geq \|\gamma f(y)\| = C \). On the other hand, one has \( \|i(\gamma f)|_U\| = \|\gamma f|_{X'}\| < \varepsilon \) due to modularity of \( i \). But it contradicts to topological injectivity of \( f \). Define in a similar way \( i'' : C(Y')^n \to C(Y') \): take \( (f'_1, \ldots, f'_n) \), extend them by Tietze’s lemma to \( (f_1, \ldots, f_n) \) on \( X \), apply \( s \) and then \( \varphi \). It is well defined. Varying functions \( f'_i \) and their extensions we obtain all elements of \( C(Y) \) as \( (f_1, \ldots, f_n) \). This implies surjectivity of \( i'' \). Evidently, \( i'' \circ i' = \text{Id}_M' \). \( \square \)

**Lemma 3.6.** Let \( X = \{x\} \cup \{x_\alpha\} \), where \( x_\alpha \) is a net, which converges to the point \( x \), \( Y = \{y\} \), \( p(y) = x \). Then \( C(Y) \) is finitely generated but not projective Banach module over \( C(X) \) with respect to the action (3).

**Proof.** Evidently, \( C(Y) \) is finitely (namely, one) generated over \( C(X) \). If it is finitely generated projective, then there exists a \( C(X) \)-valued inner product \( \langle \cdot, \cdot \rangle \) on \( C(Y) \). For any \( f \neq 0 \) on \( Y \) and any \( x_\alpha \) consider a continuous function \( \varphi : X \to [0, 1], \varphi(x) = 1, \varphi(x_\alpha) = 0 \). Then \( f = 1 \cdot f = \varphi : f \) and

\[
\langle f, f \rangle(x_\alpha) = \langle \varphi f, \varphi f \rangle(x_\alpha) = \varphi(x_\alpha) \langle f, f \rangle(x_\alpha) \varphi(x_\alpha) = 0.
\]

Since \( \alpha \) is an arbitrary index, \( \langle f, f \rangle \equiv 0 \). \( \square \)
Example 3.7. Let $X = S^1 = [0, 1]/\sim$ be a circle, which is thought as the interval $[0, 1]$ whose end points are identified, $Y = [0, 1]$, and $p : Y \to X$ is defined by the formula $p(t) = [t]$, where $[t] \in [0, 1]/\sim$ means the equivalence class of $t$. Then $C(Y)$ is a Banach $C(X)$-module with respect to the action (3). We claim that this module is finitely generated but not projective. Indeed, consider the sets $Y_1 = [0, 1/3]$, $Y_2 = (1/3, 2/3)$, $Y_3 = [2/3, 1]$ and functions $\psi_1, \psi_2 \in C(Y)$ such that $\psi_1|Y_1 = 1$, $\psi_1|Y_3 = 0$, $\psi_2|Y_1 = 0$, $\psi_2|Y_3 = 1$ and $\psi_1, \psi_2$ are linear on $Y_2$. Then $\psi_1 + \psi_2 = 1$ and for any $f \in C(Y)$ the equality $f = f\psi_1 + f\psi_2$ takes place. So $\psi_1, \psi_2$ are generators of $C(Y)$ over $C(X)$ and the module is finitely generated. By Lemmas 3.5 and 3.6 it is not projective finitely generated.

Example 3.8. Let $x_0$ be a point of the circle $S^1$ and $X = S^1 \times \{x_0\} \cup \{x_0\} \times S^1$ be a union of two circles (i.e. “8”). Let $Y$ be a disjoint union $S^1 \sqcup S^1$ and the natural surjective map $p : Y \to X$ has one pre-image for all points except of $x_0$. Then it immediately follows from Lemma 3.5 and Example 3.7 that $C(Y)$ is a finitely generated but not projective $C(X)$-module.

4. Branched coverings and Hilbert $C^*$-modules

We start this section with a couple of observations.

Lemma 4.1. Consider the map (1), where $X, Y$ are compact and $Y$ has a countable base. Then $C(Y)$ is a countably generated module over $C(X)$ with respect to the action (3).

Proof. Under our assumptions the $C^*$-algebra $C(Y)$ is separable [10, 1.6.9], [11, Prop. 1.11], so it is a countably generated module over $C(X)$. □

Now we would like to describe an example of a countably, but not finitely generated Hilbert $C^*$-module arising from the simplest branched covering. In addition, this example illustrates some ideas of the proof of Theorem 4.3.

Example 4.2. Consider the map $p : Y \to X$ of Figure 2, where $X$ is an interval, say $[0,1]$, and $Y$ is the topological union of one interval with two copies of another half-interval with a branch point at $1/2$. Then $C(Y)$ is a Banach $C(X)$-module for the action (3). Define a $C(X)$-valued inner product on $C(Y)$ by the formula

$$\langle f, g \rangle(x) = \frac{1}{\#p^{-1}(x)} \sum_{y \in p^{-1}(x)} f(y)g(y),$$

where $\#p^{-1}(x)$ is the cardinality of the pre-image $p^{-1}(x)$. The obvious inequality

$$\frac{\|f\|^2}{2} \leq \|\langle f, f \rangle\| \leq \|f\|^2, \quad f \in C(Y)$$

Figure 2. Example 4.2
implies that the $C^*$-Hilbert norm $\|\langle f, f \rangle\|$ is equivalent to the $C^*$-norm on $C(Y)$. Therefore $C(Y)$ is a Hilbert $C(X)$-module with respect to the inner product (4) and this module is countably generated by Lemma 4.1. Moreover, this module is reflexive by [17] Theorem 4.4.2. But this module is not self-dual. Indeed, by Lemmas 3.3, 3.4 it is not a finitely generated projective one. Recall (cf. [24]) that a unital $C^*$-algebra is said to be MI (module infinite) if each countably generated Hilbert module over it is projective finitely generated if and only if it is self-dual. The $C^*$-algebra $C(X)$ of this example is MI by [24, Theorem 33], therefore $C(Y)$ is not a self-dual module over it.

**Theorem 4.3.** Let $p : Y \to X$ be a branched covering. Then $C(Y)$ may be equipped with a $C(X)$-valued inner product in such a way that it becomes a $C(X)$-Hilbert module, whose norm is equivalent to the $C^*$-norm of $C(Y)$.

**Proof.** Given any functions $f$, $g$ of $C(Y)$. We will construct their $C(X)$-valued inner product by induction over the sets $\hat{X}_j$, $j = 0, 1, \ldots, N$. Suppose $X_j$, is the first non-empty stratum. Then the formula

$$\langle f, g \rangle (x) = \frac{1}{\#p^{-1}(x)} \sum_{y \in p^{-1}(x)} \overline{f(y)} g(y)$$

provides the base of induction. Now suppose the inner product is defined on the strata $X_1, \ldots, X_j$ and the next non-empty set is $X_{j+k}$, $k > 0$.

By Proposition 2.6 for any point $x \in \hat{X}_j$ there exists its regular neighborhood $U$ satisfying (2) such that the restriction of $p$ on $V_k$ is surjective for any $k = 1, \ldots, m$. We will define the inner product $\langle f, g \rangle$ at any point $z$ of $U \cap X_{j+k}$ as follows. Let

$$p^{-1}(z) \cap V_k = \{u_1^{(k)}, \ldots, u_{i_k}^{(k)}\},$$

where $i_1 + \cdots + i_m = j + k$ and $i_k \neq 0$ for any 0. Denote

$$f_k := f|_{V_k}, \quad g_k := g|_{V_k}$$

and define a function $\langle f_k, g_k \rangle : U \cap X_{j+k} \to \mathbb{C}$ by the formula:

$$\langle f_k, g_k \rangle (z) = \frac{1}{i_k} \sum_{t=1}^{i_k} \overline{f_k(u_t^{(k)})} g_k(u_t^{(k)}).$$

Then

$$\langle f, g \rangle_{U \cap X_{j+k}} (z) = \frac{1}{m} \sum_{k=1}^{m} \langle f_k, g_k \rangle (z).$$

Consider such a regular neighborhood $U = U(x)$ for each point $x \in \hat{X}_j$. Extend the system $\{U(x) : x \in \hat{X}_j\}$ up to a cover of $\hat{X}_{j+k}$ by open sets $O_i$ satisfying $O_i \cap \hat{X}_j = \emptyset$. Let $\{U_1, \ldots, U_K, O_1, \ldots, O_M\} = \{W_1, \ldots, W_{K+M}\}$ be a finite subcovering of the compact space $\hat{X}_{j+k}$ and $\{\varphi_i(x)\}_{i=1}^{K+M}$ be a partition of unity subordinated to this subcovering. Define $\langle f, g \rangle_{W_i}$ over $W_i$ by the formulas (6), (7) if $i \leq K$ and by the formula (5) otherwise. Define an inner product on $C(p^{-1}(\hat{X}_{j+k}))$ in the following way:

$$\langle f, g \rangle (x) = \sum_{i=1}^{K+M} \langle f, g \rangle_{W_i}(x) \varphi_i(x),$$
where \( f, g \in C(Y), x \in \widehat{X}_{j+k} \). The inductive step is complete.

We claim that \( \langle f, g \rangle \) is continuous on \( X \). Indeed, consider any point \( x \in X \) and any net \( \{x_{\alpha}\} \) converging to \( x \). Then \( x \in X_i \) for some \( j \). Denote \( \{x_{\alpha}^{(i)}\} = \{x_{\alpha}\} \cap X_i \). By Proposition 2.5 we can assume that \( i \geq j \). It remains to verify that for any \( i \) the difference \( |\langle f, g \rangle(x) - \langle f, g \rangle(x_{\alpha}^{(i)})| \) goes to zero when \( x_{\alpha}^{(i)} \) goes to \( x \). But it directly follows from the definition of the inner product, namely from the continuity of \( \langle f, g \rangle \).

Thus \( \langle f, f \rangle(x) \) is a convex combination of not more than \( N = \max_{x \in X} \# \pi^{-1}(x) < \infty \) numbers \( |f(y_i)|^2 \), where \( p(y_i) = x \). Hence we obtain the following inequality

\[
\frac{\|f\|^2}{N} \leq \|\langle f, f \rangle\| \leq \|f\|^2, \quad f \in C(Y).
\]

Thus the Hilbert norm \( \|\langle f, f \rangle\| \) is equivalent to the \( C^* \)-norm of \( C(Y) \).

**Theorem 4.4.** Suppose \( X \) and \( Y \) are compact Hausdorff connected spaces and \( p : Y \to X \) is a continuous surjection. If \( C(Y) \) is a projective finitely generated Hilbert module over \( C(X) \) with respect to the action \( \pi \), then \( p \) is a finite-fold covering.

**Proof.** Let functions \( g_1, \ldots, g_n \) generate the projective module \( C(Y) \) over \( C(X) \). Then we claim that the cardinality of the pre-image of any point \( x \in X \) does not exceed \( n \). Indeed, assume there is a point \( x \in X \), whose pre-image is \( \{y_1, \ldots, y_m\} \) and \( m > n \). By the Urysohn’s lemma there are continuous functions \( f_1, \ldots, f_m \in C(Y) \) such that \( f_i(y_j) = 1 \) and \( f_i(y_j) = 0 \) whenever \( i \neq j \). The functions \( f_1, \ldots, f_m \) can be expressed as linear combinations of the generators \( g_1, \ldots, g_n \) with coefficients from \( C(X) \). Let us denote by \( \widehat{f}_i \) and \( \widehat{g}_j \) the restrictions of \( f_i \) and \( g_j \) onto \( \{y_1, \ldots, y_m\} \). Then both \( \widehat{f}_i \) and \( \widehat{g}_j \) belong to the vector space

\[ C\{y_1, \ldots, y_m\} \cong \mathbb{C}^m. \]

The vectors \( \widehat{f}_1, \ldots, \widehat{f}_m \) form a base of this vector space and, consequently, they can not be represented as linear combinations of the vectors \( \widehat{g}_1, \ldots, \widehat{g}_n \), when \( m > n \). Thus, \( m \) does not exceed \( n \).

Assume \( k_x \) denotes the cardinality of the pre-image of a point \( x \in X \) and \( k \) is a minimal value of \( k_x \)'s over \( x \in X \). Firstly, we claim that the set \( X_k = \{x \in X : k_x = k\} \) is open. Indeed, in the opposite case there is a net \( \{x_{\alpha}\} \) in \( \{x_{\alpha}\} \) converging to a certain point \( x \) of \( X_k \). By Lemma 2.2 one can find a regular neighborhood \( U \) of \( x \) satisfying the condition \( 2 \) with \( m = k \). Moreover, one can assume (passing to a sub-net of \( \{x_{\alpha}\} \) if it is necessary) that the net \( \{x_{\alpha}\} \) belongs to \( U \) and there is a number \( i \) such that the neighborhood \( V_i \) has at least two points \( y'_{\alpha} \) and \( y''_{\alpha} \) from the pre-image of \( x_{\alpha} \) for any \( \alpha \). Put \( X' = \{x \in \{x_{\alpha}\} \} \) and \( Y' = \{y \in \{y'_{\alpha}\} \} \), where \( y = p^{-1}(x) \cap V_i \). Then \( C(Y') \) is a finitely generated module over \( C(X') \) by Lemma 3.4. But this contradicts to Lemma 3.3.

Secondly, let us show that \( X_k \) is closed. In the opposite case there is a net \( \{x_{\alpha}\} \) of \( X_k \) converging to some point \( x_{\beta} \) with \( j > k \). Denote \( X' = \{x \cup \{x_{\alpha}\} \} \), \( Y' = p^{-1}(X') \) and choose neighborhoods \( U, U' \) of the point \( x \) and \( V_i \), \( V'_{ij} \) \( (i = 1, \ldots, j) \) as in Lemmas 2.2 2.3. Then \( C(\overline{U'}) = \bigoplus C(\overline{V_{ij}}) \) is a finitely generated projective \( C(U) \)-module by Lemma 3.5. Therefore, obviously, each \( C(U) \)-module \( C(\overline{V_{ij}}) \) is finitely generated too. We can assume (passing to a sub-net of \( \{x_{\alpha}\} \) if it is necessarily) that the intersection of the set \( p^{-1}(\{x_{\alpha}\}) \) with a neighborhood \( V_i \) is empty for some number \( i \). Now consider
the submodule \( C(p^{-1}(V')) = C(\{y_i\}) \) of the module \( C(V_i') \), where \( y_i = p^{-1}(x) \cap V_i \).

It has to be finitely generated projective by Lemmas 3.4 and 3.5 but it is impossible by Lemma 3.6.

So we have proved that the set \( X \setminus X_k \) is both open and closed and, consequently, it has to be empty, because \( X \) is supposed to be connected. Thus, all points of \( X \) have the same number of pre-images.

Now for an arbitrary point \( x \in X \) let us choose its regular neighborhood \( U \) satisfying the condition (2) with \( m = k \). Then \( p \) is a (local) bijection, which is closed and open (by our argument for branched coverings). Thus it is a local homeomorphism. \(\square\)

We complete this section with a couple of statements relating coverings to some other classes of Hilbert \( C^* \)-modules.

**Theorem 4.5.** Consider the map \((1)\), where \( X \) and \( Y \) are compact spaces, \( X \) is connected and \( Y \) has a countable base. Then the following conditions are equivalent:

(i) \( C(Y) \) is a self-dual module with respect to the action \((3)\);

(ii) the map \((1)\) is a finite-fold covering.

**Proof.** The implication \((ii) \Rightarrow (i)\) follows from [25, Proposition 2.8.9] and [25, pp. 92–93] (see also Theorem 5.7 below). To prove the inverse implication let us remark that \( X \) does not have isolated points because it is connected, so the \( C^* \)-algebra \( C(X) \) is MI by [24, Theorem 33]. According to our assumptions and Lemma 4.1 the \( C(X) \)-module \( C(Y) \) has to be both countably generated and self-dual. Therefore it is a finitely generated projective module. Then by Theorem 4.4 the map \((1)\) defines a finite-fold covering. \(\square\)

**Theorem 4.6.** Let \( p : Y \rightarrow X \) be a branched covering over a compact metric space \( X \). Then the \( C(X) \)-Hilbert module \( C(Y) \) is \( C(X) \)-reflexive.

**Proof.** It follows from Theorem 4.3 and [8, Theorem 4.1]. \(\square\)

5. Branched coverings and conditional expectations

In this section we complete proofs of Theorems 1.1 and 1.3 working with conditional expectations. Recall briefly some necessary facts from [25] (see also [20, 23]).

**Definition 5.1.** Suppose, \( B \) is a \( C^* \)-algebra and \( i : A \hookrightarrow B \) is its \( C^* \)-subalgebra. A conditional expectation \( E : B \rightarrow A \) is a surjective projection of norm one satisfying the following conditions:

\[
E(i(a) \cdot b) = aE(b), \quad E(b \cdot i(a)) = E(b)a, \quad E(i(a)) = a,
\]

for \( a \in A, b \in B \). We deal with unital \( C^* \)-algebras and we will always assume, that

(i) \( E \) is positive: \( E(b^*b) \geq 0 \) for any \( b \in B \);

(ii) \( E \) is unital, i.e. \( i \) is unital, or \( A \) and \( B \) have a common unity.

**Definition 5.2.** A family \( \{u_1, \ldots, u_n\} \subset B \) is called a quasi-basis for \( E \) if

\[
b = \sum_j u_j E(u_j^*b) \quad \text{for} \ b \in B.
\]

A conditional expectation \( E : B \rightarrow A \) is algebraically of index-finite type if there exists a finite quasi-basis for \( E \). In this case the index of \( E \) is defined by: \( \text{Index}(E) = \sum_j u_j u_j^* \),
which is a positive invertible element in the center of $B$ and it does not depend on the choice of the quasi-basis $\{u_1, \ldots, u_n\}$.

**Definition 5.3.** Given a $C^*$-algebra $B$ and its $C^*$-subalgebra $A$. A conditional expectation $E : B \to A$ is topologically of index-finite type [2] (see also [3]) if the mapping $(K : E - id_{B})$ is positive for some real number $K \geq 1$.

We need the following result [3, Theorem 1] (see also [2, Proposition 3.3], [14, Theorems 3.4, 3.5], [11, Proposition 2.1, Corollary 2.4], [21, Theorem 1.1.6, Remark 1.1.7], [9, Proposition 1.1]):

**Proposition 5.4.** Let $E : B \to A$ be a conditional expectation. Then the following conditions are equivalent:

(i) $E$ is topologically of index-finite type;

(ii) $E$ is faithful and the pre-Hilbert $A$-module $\{B, E(\langle \cdot, \cdot \rangle_B)\}$ is complete with respect to the norm $\|E(\langle \cdot, \cdot \rangle_B)\|_A^{1/2}$.

**Proof.** The second condition means that the original norm and the Hilbert module norm on $B$ are equivalent, in particular,

$$K\|E(x^*x)\| \geq \|x^*x\|$$

for some constant $K > 0$ and for any $x \in B$. Consider an element $x = b(\varepsilon + E(b^*b))^{-\frac{1}{2}}$ for $\varepsilon > 0$. Then, obviously,

$$(\varepsilon + E(b^*b))^{-\frac{1}{2}}E(b^*b)(\varepsilon + E(b^*b))^{-\frac{1}{2}} \leq 1_B,$$

what exactly means that $E(x^*x) \leq 1_B$. Hence, $\|x^*x\| \leq K$, or, equivalently, $x^*x \leq K \cdot 1_B$. In other words, one has

$$(\varepsilon + E(b^*b))^{-\frac{1}{2}}b^*b(\varepsilon + E(b^*b))^{-\frac{1}{2}} \leq K \cdot 1_B,$$

which may be rewritten as $K(\varepsilon + E(b^*b)) \geq b^*b$. Since $\varepsilon > 0$ is arbitrary, we are done. The converse is immediate.

Let $i : A \to B$ be a unital inclusion of commutative $C^*$-algebras $A = C(X)$, $B = C(Y)$. Then its Gelfand dual $p = i^* : Y \to X$ is an epimorphism.

**Definition 5.5.** A conditional expectation $E : C(Y) \to C(X)$ is said to be fiber-wise if for any $x \in X$ and $f \in C(Y)$ such that $f|_{p^{-1}(x)} = 0$ one has $E(\overline{f}f)(x) = 0$, $p = i^*$, $i : C(X) \subset C(Y)$.

Any unital $E$ is fiber-wise. Indeed, up to re-denoting it is sufficient to prove that $E(f)(x) = 0$. For any $\varepsilon > 0$ there is a neighborhood $U_\varepsilon$ of $p^{-1}(x)$ such that $|f(y)| < \varepsilon$ for any $y \in U_\varepsilon$. Choose a neighborhood $V_\varepsilon$ of $x$ such that $p^{-1}(V_\varepsilon) \subset U_\varepsilon$ and $a_\varepsilon \in C(X) = A$ such that $\|a_\varepsilon\| = 1$, $a_\varepsilon(x) = 0$, $a_\varepsilon(x') = 1$ for any $x' \notin V_\varepsilon$. Then $\|E(f - a_\varepsilon f)\| \leq \|f - a_\varepsilon f\| < \varepsilon$ and $E(a_\varepsilon f)(x) = a_\varepsilon(x)E(f)(x) = 0$. Since $\varepsilon$ is arbitrary, we are done.

Note, that the conditional expectation related to the inner product constructed in Theorem 4.3 is, obviously, fiber-wise.

**Theorem 5.6.** Let $X$, $Y$ be compact spaces, $i : C(X) \to C(Y)$ be a unital $*$-inclusion of $C^*$-algebras and $E : C(Y) \to C(X)$ be a (unital positive) conditional expectation topologically of index-finite type. Then the map $p = i^* : Y \to X$ is a branched covering.
Proof. The map $p$ is surjective and continuous. The number of pre-images of $p$ is uniformly bounded over $X$. Indeed, suppose a point $x \in X$ has $n$ pre-images $\{y_1, \ldots, y_n\}$. Consider non-negative functions $f_k \in C(Y)$, $k = 1, \ldots, n$ such that $f_k(y_j) = \delta_{kj}$, $f_k : Y \to [0, 1]$ and $f_kf_j = 0$ if $k \neq j$. By Definition 5.3 (i), $\langle E(f_k)(y_k) \rangle > \frac{1}{K}f_k(y_k)$, i.e., $E(f_k)(x) > \frac{1}{K}$. Let $1 \in C(Y)$ be the unity element. Then by positivity of $E$ we have

$$E(1)(x) \geq E\left(\sum_{k=1}^{n} f_k\right)(x) = \sum_{k=1}^{n} E(f_k)(x) > \frac{n}{K}.$$ 

Thus, if $n$ is not bounded, then $E(1) = 1$ is not bounded. A contradiction.

Now let us verify the item (i) of Definition 5.3. By Theorem 1.3 it is sufficient to verify that $p$ is an open map. Suppose that $p$ is not open, i.e., there is an open set $V \subset Y$ such that $p(V) \subset X$ is not open. Let $x \in p(V)$ be a limit point of $X \setminus p(V)$ and $y \in V$ its pre-image. Consider a positive function $f : Y \to [0, 1]$ such that $f(y) = 1$ and $f$ vanishes outside $V$. Then $E(f)(x) > 1/K$ while $E(f)$ vanishes on $X \setminus p(V)$. Thus $E(f)$ is not continuous. A contradiction. \hfill \Box

This completes the proof of Theorem 1.1. The next statement completes the proof of Theorem 1.3.

**Theorem 5.7.** Let $E : B \to A$ be a conditional expectation, where $C^*$-algebras $A$ and $B$ have a common unity. Then the following conditions are equivalent:

1. $E$ is algebraically of index-finite type;
2. $B$ is a finitely generated projective $A$-module.

**Proof.** The statement [25, Corollary 3.1.4] differs from our theorem only by one additional condition, which may be omitted in the unital $C^*$-case, because in this situation all finitely generated projective modules are self-dual. \hfill \Box

The remaining part of the section is devoted to a clarifying of the role of the index of $E$ in our theory. The main idea of [2] concerning the definition of the index element is the following. Given a $W^*$-algebra $B$ and its $W^*$-subalgebra $A$. Consider a conditional expectation $E : B \to A$ topologically of index-finite type, then it defines an $A$-Hilbert module structure on $B$. Choose any quasi-orthonormal basis $\{x_i\}$ (relating to this inner structure) in $B$ and define the index of $E$ as the sum $\sum x_i^*x_i$ with respect to the ultra-weak topology. Actually, this definition is very close to the frame approach elaborated in [17]. The index of $E$ provided both $B$ and $A$ are $C^*$-algebras was defined in [3], but in this situation it is an element of the enveloping von Neumann algebra $B^{**}$ of $B$.

We have constructed in the proof of Theorem 1.3 a function $\mu : Y \to [0, 1]$, such that $\sum_{p(y) = x} \mu(y) = 1$ for any $x \in X$. This function (not uniquely determined!) was used to define a $C(X)$-valued inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_\mu$ in such a way that

$$\langle f, f \rangle_\mu(x) = \sum_{p(y) = x} f^*(y)f(y)\mu(y).$$

Similarly for the induced conditional expectation $E = E_\mu$:

$$E_\mu(f)(x) = \sum_{p(y) = x} f(y)\mu(y).$$
As we have explained, this expectation $E_\mu$ is topologically of index-finite type. Thus, by [5], its index element $\text{Index}(E_\mu) \in B^{**}$ is defined, valued in the enveloping von Neumann algebra of $B = C(Y)$.

In the remaining part of the section all spaces are supposed to be \textit{second countable}.

Choose a countable partition of $X_j$,

$$X_j = X_1^j \sqcup \cdots \sqcup X_r^j \sqcup \cdots$$

in such a way that $X^s_j$ is open in $X^{s+1}_j \sqcup \cdots$ and $X^s_j$ is inside of some regular neighborhood of a point of $X_j$. For this purpose we take a countable covering $U_1, U_2, \ldots$ of $X_j$ and take

$$X^1_j := U_1 \cap X_j, \quad X^2_j := (U_2 \setminus U_1) \cap X_j, \quad \ldots \quad X^r_j := (U_r \setminus (U_1 \cup \cdots \cup U_{r-1})) \cap X_j, \quad \ldots$$

Let $p^{-1}(X^k_j) = Y^k_1 \sqcup \cdots \sqcup Y^k_{r,j}$ be its “regular” decomposition. Thus, $Y$ is a disjoint union of countably many Borel sets $Y^s,t_j$. Define

$$m_{jkt}(y) = \begin{cases} \frac{1}{\sqrt{\mu(y)}}, & y \in Y^{k,t}_j \setminus \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

Then $m_{jkt}$ are pairwise orthogonal, $\langle m_{jkt}, m_{jkt} \rangle_\mu(x) = 0$, if $x \not\in X^k_j$, and

$$\langle m_{jkt}, m_{jkt} \rangle_\mu(x) = \sum_{y \in Y^{k,t}_j} m_{jkt}^*(x) m_{jkt}(y) \mu(y) = 1$$

if $x \in X^k_j$. Let $M := \sum_{j,k,t} m_{jkt}^* m_{jkt}$. It is a bounded function, which is continuous (being $\frac{1}{\mu}$) on each component of the disjoint union

$$Y = \bigsqcup Y^{k,t}_j$$

of Borel sets. Also, it is bounded (by the maximal number of pre-images under $p$).

\textbf{Theorem 5.8.} In this situation

$$M = \text{Index}(E_\mu).$$

\textit{Proof.} In fact (cf. [2], [14], [5]) it is sufficient to verify that for any $y \in Y$ and any $f \in C(Y)$

$$f(y) = \sum_{j,k,t} m_{jkt}^*(i E_\mu)(m_{jkt}^* f)(y)$$

(in our notation with the inclusion $i$). We have $y \in Y^{p,s}_t$ for some (uniquely defined) indices $l$, $p$, and $s$. Then

$$\sum_{j,k,t} m_{jkt}^*(i E_\mu)(m_{jkt}^* f)(y) = m_{tps}(y) \sum_{y' \in p(y')} m_{tps}^*(y') f(y') \mu(y')$$

$$= m_{tps}(y) m_{bps}^*(y) f(y) \mu(y) = f(y).$$

$\square$
Remark 5.9. The equality in Theorem 5.8 should be considered in $B^{**}$. In particular, $M$ is an element of $B^{**}$ in the following sense. We approximate $M$ (in fact each $m^* m_i$) by a sequence of continuous functions point-wise. In fact it is sufficient to approximate the characteristic function of a set of the form $K \setminus K'$, where $K$ and $K'$ are compacts. For each of them it is well known how to find such a sequence, and then we take the difference. Finally, we apply the Egoroff theorem (see e.g. [13, Sect. 21]) to see that the sequence converges ultraweakly, i.e. the values on each regular positive measure converge. See also [23, III.1 and III.2].

Let us illustrate the above general considerations by the following simple example (see [5, Example 3.3] for a similar situation).

Example 5.10. Consider the branched covering of Example 4.2 defined by Figure 2. This covering is equipped with a conditional expectation $E : C(Y) \to C(X)$ of index-finite type given by the formula
\[
E(f)(x) = \frac{1}{\# p^{-1}(x)} \sum_{y \in p^{-1}(x)} f(y)
\]
and the inner product [4] satisfies $\langle f, g \rangle := \langle f, g \rangle_E = E(f^* g)$. In this example the weight function $\mu$ is 1 over $[0, 1/2]$ and 1/2 over $(1/2, 1]$. Now we enumerate three intervals forming $Y$ in the following way: by 1 for the horizontal interval and by 2 and 3 for two others. Define three functions $e_1, e_2, e_3$ of $C(X)$ such that $e_1$ is equal to 1 over the first interval of $Y$ and 0 otherwise, and $e_i$ equals to $\sqrt{2}$ over the $i$-th interval and 0 otherwise for $i = 2, 3$. Then obviously the vectors $\{e_i\}$ form a quasi-orthonormal basis (i.e. an orthogonal system, where inner squares of all vectors are projections) of the $C(X)$-Hilbert module $(C(Y)^{**}, \langle \cdot, \cdot \rangle_E)$. Thus the index $\text{Index}(E)$ coincides with the sum $\sum e_i^* e_i$. This function is equal to 1 over the first subinterval of $Y$, and to 2 over two other subintervals of $Y$. Its value in the branching point defines an element of the discrete part of $C(Y)^{**}$.

Example 5.11. Let $X$ be a unit circle and $Y$ consists of two disjoint copies of $X$, in which the zero-point below is connected by an interval with the $\pi/2$-point above as it is shown by Figure 3. Obviously, $p$ is open and by Theorem 2.9 it is a branched covering. The weight function $\mu$, constructed by the formulas [5] – [8] of Theorem 4.3, is equal to 1/2 on fibers of cardinality 2, is equal identically to 1/4 on the line between the circles, is the function $f_1(x) := \frac{x}{2\pi} + \frac{1}{4}$ over the interval $(0, \pi/2)$ of the circle below, and is the function

![Figure 3. Example 5.11](image-url)
\( f_2(x) := -\frac{1}{2\pi} + \frac{1}{2} \) over the interval \((0, \pi/2)\) of the circle above. Then the index element of the corresponding conditional expectation is a function of \(C(Y)^*\), which equals \(\sqrt{2}\) on fibers, whose cardinality is 2, equals 2 on the line between the circles, is the function \(1/\sqrt{f_1(x)}\) over the interval \((0, \pi/2)\) of the circle below, and is the function \(1/\sqrt{f_2(x)}\) over the interval \((0, \pi/2)\) of the circle above.

**References**

[1] E. Andruchow and D. Stojanoff, *Geometry of conditional expectations of finite index*, Internat. J. Math. 5 (1994), 169–178.

[2] Michel Baillet, Yves Denizeau, and Jean-François Havet, *Indice d’une espérance conditionnelle*, Compositio Math. 66 (1988), no. 2, 199–236. MR MR945550 (90e:46050)

[3] V.M. Buchstaber and E.G. Rees, *The Gelfand map and symmetric products*, Selecta Math. 8 (2002), no. 4, 523–535.

[4] Frobenius \(n\)-homomorphisms, transfers and branched coverings, Math. Proc. Camb. Philos. Soc. 144 (2008), no. 1, 1–12.

[5] M. Frank and E. Kirchberg, *On conditional expectations of finite index*, J. Oper. Theory 40 (1998), no. 1, 87–111.

[6] M. Frank and D.R. Larson, *A module frame concept for Hilbert \(C^*\)-modules*, Contemp. Math. 247 (1999), 207–233. The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999). American Mathematical Society, Providence, RI. MR MR1738091

[7] *Frames in Hilbert \(C^*\)-modules and \(C^*\)-algebras*, J. Oper. Theory 48 (2002), no. 2, 273–314.

[8] M. Frank, V. M. Manuilov, and E. V. Troitsky, *Hilbert \(C^*\)-modules from group actions: beyond the finite orbits case*, E-print arxiv:math.OA/0903.1741.

[9] *On conditional expectations arising from group actions*, Zeitschr. Anal. Anwendungen 16 (1997), 831–850.

[10] D. B. Fuks and V. A. Rokhlin, *Beginner’s course in topology. geometric chapters.*, Springer Series in Soviet Mathematics. Universitext., Springer-Verlag, Berlin, 1984.

[11] J. M. Gracia-Bondía, J. C. Varilly, and H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser, Boston, 2001.

[12] D. V. Gugnin, *On continuous and irreducible Frobenius \(m\)-homomorphisms*, Russian Math. Surveys 60 (2005), no. 5, 967–969.

[13] P. R. Halmos, *Measure theory*, D. Van Nostrand Company, Inc., Princeton, N.J., 1950.

[14] P. Jolissaint, *Indice d’esperances conditionnelles et algèbres de von neumann finies*, Math. Scand. 68 (1991), 221–246.

[15] G. G. Kasparov, *Hilbert \(C^*\)-modules: theorems of Stinespring and Voiculescu*, J. Operator Theory 4 (1980), 133–150.

[16] E. C. Lance, *Hilbert \(C^*\)-modules - a toolkit for operator algebraists*, London Mathematical Society Lecture Note Series, vol. 210. Cambridge University Press, England, 1995.

[17] V. M. Manuilov and E. V. Troitsky, *Hilbert \(C^*\)-modules*, Translations of Mathematical Monographs, vol. 226, American Mathematical Society, Providence, RI, 2005. MR MR2125398

[18] A. S. Mishchenko and A. T. Fomenko, *The index of elliptic operators over \(C^*\)-algebras*, Izv. Akad. Nauk SSSR, Ser. Mat. 43 (1979), 831–859, English translation, Math. USSR-Izv. 15, 87-112, 1980.

[19] W. L. Paschke, *The double \(B\)-dual of an inner product module over a \(C^*\)-algebra \(B\)*, Can. J. Math. 26 (1974), no. 5, 1272–1280.

[20] G. K. Pedersen, *\(C^*\)-algebras and their automorphism groups*, London Mathematical Society Monographs, vol. 14, Academic Press, London, 1979.

[21] Sorin Popa, *Classification of subfactors and their endomorphisms*, CBMS Regional Conference Series in Mathematics, vol. 86, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1995. MR MR1339767 (96d:46085)

[22] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace \(C^*\)-algebras*, Mathematical Surveys and Monographs, vol. 60, Amer. Math. Soc., Providence, RI, 1998.

[23] M. Takesaki, *Theory of operator algebras*, vol. I, Springer-Verlag, New York, 1979.
[24] E. V. Troitsky, *Discrete groups actions and corresponding modules*, Proc. Amer. Math. Soc. **131** (2003), no. 11, 3411–3422.

[25] Y. Watatani, *Index for C*-subalgebras*, Memoirs Amer. Math. Soc., vol. 424, AMS, Providence, 1990.

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