Abstract

We study the possible existence of charged black holes in the Bergmann-Wagoner class of scalar-tensor theories (STT) of gravity in four dimensions. The existence of black holes is shown for anomalous versions of these theories, with a negative kinetic term in the Lagrangian. The Hawking temperature $T_H$ of these holes is zero, while the horizon area is (in most cases) infinite. As a special case, the Brans-Dicke theory is studied in more detail, and two kinds of infinite-area black holes are revealed, with finite and infinite proper time needed for an infalling particle to reach the horizon; among them, analyticity properties select a discrete subfamily of solutions, parametrized by two integers, which admit an extension beyond the horizon. The causal structure and stability of these solutions with respect to small radial perturbations is discussed. As a by-product, the stability properties of all spherically symmetric electrovacuum STT solutions are outlined.

1. Introduction

This study was to a certain extent stimulated by a controversy in the recent literature: the paper by Campanelli and Lousto \[1\] asserts that in the well-known family of static, spherically symmetric vacuum solutions of the Brans-Dicke (BD) theory there exists a subfamily which possesses all properties of black hole (BH) solutions, but (i) these solutions exist only for negative values of the coupling constant $\omega$ and (ii) the horizons have an infinite area. These authors argue that large negative $\omega$ are compatible with modern observations and that such black holes may be of astrophysical relevance. On the other hand, H. Kim and Y. Kim \[2\], agreeing that there are non-Schwarzschild black holes in the Brans-Dicke theory, claim that such black holes have unacceptable thermodynamical and geometric properties and are therefore physically irrelevant; meanwhile, they ascribe such solutions to positive values of $\omega$.

The aim of this study is not only to make the situation clear, but a bit wider: to reveal possible vacuum and electrically charged black hole solutions among static, spherically symmetric solutions of the general (Bergmann-Wagoner) class of scalar-tensor theories (STT) of gravity, which may be described in terms of the coupling function $\omega(\phi)$; the BD theory ($\omega = \text{const}$) will be used as the most well-known example. One of the reasons for such an approach is that, by modern views, it is rather probable that $\omega$ could have been sufficiently small and could appreciably affect the physical processes in the early Universe, but by now became large, making the theory very close to general relativity (GR) in observational predictions \[3\].

We show, in the framework of the general STT, that nontrivial BH solutions can exist for the coupling function $\omega(\phi) + 3/2 < 0$, and that only in exceptional cases these BHs have a finite horizon area.

The case of the BD theory is studied in more detail. Various types of geometry are indicated, including BHs, wormholes and “hornlike” structures, all of them existing in the anomalous case $\omega < -3/2$. All nontrivial (with the scalar field $\phi \neq \text{const}$) BHs have infinite horizon areas and zero Hawking temperature (“cold BHs”), thus confirming the conclusions of \[1\]. These BHs in turn split into two subclasses: B1, where horizons are attained by infalling particles in a finite proper time $\tau$, and B2, for which $\tau$ is infinite.

The static region of a type B2 BH is geodesically complete since its horizon is infinitely remote and actually forms a second spatial asymptotic. For type B1 BHs the global picture is more complex and is discussed in some

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5 For brevity, we call BHs with infinite horizon areas type B BHs \[1\], to distinguish them from the conventional ones, with finite horizon areas, to be called type A.
detail. It turns out that the horizon is generically singular due to violation of analyticity, despite the vanishing curvature invariants. Only a discrete set of B1-solutions, parametrized by two integers $m$ and $n$, admits a Kruskal-like extension, and, depending on their parity, four different global structures are distinguished. Two of them, where $m - n$ is even, are globally regular, in two others the region beyond the horizon contains a spacelike or null singularity.

All BHs under consideration turn out to be stable under small radial perturbations.

Since the vacuum case has been described in detail in our previous papers [5, 6], we concentrate here on the solutions with nonzero electric charge, only referring to vacuum configurations as a limiting case of the charged ones.

The paper is organized as follows.

In Sec. 2 we discuss the criteria used to single out black hole metrics among other static, spherically symmetric metrics. In Sec. 3 we present the (well-known) electrovacuum solution of the general STT and its vacuum counterpart. Sec. 4 is devoted to a search for possible black holes in the framework of the general STT. In Sec. 5 we outline the properties of various electrovacuum BD solutions, paying special attention to black hole ones. The properties of electrovacuum BD black holes are discussed in Sec. 6. In Sec. 7 we investigate the stability of the above STT solutions under radial perturbations and, in particular, show that the BH and wormhole solutions are stable. Sec. 8 contains some concluding remarks.

In the Appendix it is explicitly shown that, in an arbitrary static, spherically symmetric space-time, an infinite Hawking temperature can occur only at a curvature singularity, and the regularity requirement implies the invisibility of a horizon for an observer at rest.

2. Criteria for black hole selection

We will deal with static, spherically symmetric space-times, whose metric in a general may be written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{2\gamma}dt^2 - e^{2\alpha}du^2 - e^{2\beta}d\Omega^2$$  \hspace{1cm} (1)

where $\gamma$, $\alpha$ and $\beta$ are functions of $u$ only and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

Black hole (BH) solutions with the metric (1) are conventionally singled out among other solutions by the following criteria: at some surface $u = u^*$ (horizon)

C1. $e^{\gamma} \to 0$ (the timelike Killing vector becomes null).

C2. $e^{\beta}$ is finite (finite horizon area).

C3. The integral $t^* = \int e^{\alpha-\gamma} \to \infty$ as $u \to u^*$ (invisibility of the horizon for an observer at rest).

The evident requirement that a horizon must be a regular surface (otherwise we deal with a singularity rather than a horizon) creates two more criteria:

C4. The Hawking temperature $T_H$ is finite;

C5. The Kretschmann scalar $K$ is finite at $u = u^*$.

As shown in the Appendix, the conditions C3 and C4 are necessary but not sufficient for regularity of a candidate horizon (a surface where $e^{\gamma} = 0$), so they will be used as convenient selection tools. As for C5, the scalar $K$, due to its structure, is the most reliable probe for space-time regularity.

The condition C2 is apparently less essential than the others. In principle, C2 can be cancelled, leading to a generalized notion of a BH, that with a horizon having an infinite area, as described in [5]. We will call the BHs satisfying all the criteria C1–C5 type A black holes, and those with an infinite horizon — type B black holes.

We shall see that in the general scalar-tensor theory (STT) most of nontrivial (non-Schwarzschild and non-Reissner-Nordström) black holes are type B. In particular, in the BD theory all of them are type B, while all configurations satisfying C1–C3 turn out to be singular.
3. The generalized Reissner-Nordström solution

A general Lagrangian describing the interaction between gravity and a scalar field in the presence of an electromagnetic field in four dimensions can be written as

\[ L = \sqrt{-g} \left( f(\phi)R + \frac{\omega(\phi)}{\phi} \phi_{,\mu} \phi^{,\mu} - F_{\mu\nu} F^{\mu\nu} \right) \] (2)

where \( f(\phi) \) and \( \omega(\phi) \) are, in principle, arbitrary functions of the scalar field \( \phi \) (the so-called Bergmann-Wagoner class of STT). Reparametrization of \( \phi \) makes it possible to leave only one arbitrary function; a conventional choice is such that \( f = \phi \) and \( \omega(\phi) \) remains arbitrary. We will use it here as well.

The formulation (3) (the so-called Jordan conformal frame) is commonly considered to be fundamental since just in this frame the matter energy-momentum tensor \( T^\mu_\nu \) obeys the conventional conservation law \( \nabla_\alpha T^\alpha_\nu = 0 \), leading to the usual equations of motion (the so-called atomic system of measurements). In particular, free particles move along geodesics of the Jordan-frame metric. Therefore, in what follows we discuss the geometry and causal structure of the solutions in the Jordan frame.

The field equations are easier to deal with in the Einstein conformal frame

\[ L = \sqrt{-\bar{g}} \left( \bar{R} + \bar{c}\bar{g}^{\alpha\beta} \bar{\varphi}_{,\alpha} \bar{\varphi}_{,\beta} - F_{\mu\nu} F^{\mu\nu} \right), \] (3)

\[ \epsilon = \text{sign}(\omega + 3/2), \quad \frac{d\varphi}{d\phi} = \left| \frac{\omega + 3/2}{\phi^2} \right|^{1/2} \] (4)

where bars mark quantities defined in the Einstein frame; the field \( F_{\mu\nu} \) is not transformed and the indices in (3) are raised using \( \bar{g}^{\mu\nu} \). The value \( \epsilon = +1 \) corresponds to normal STT, with positive scalar field energy density in the Einstein frame; the choice \( \epsilon = -1 \) is anomalous. When \( \phi = \text{const} \), the theory reduces to GR.

With the aid of the above transformation, the following form of the exact static, spherically symmetric solution to the field equations due to (2), containing a nonzero electric charge \( q \), has been obtained [7] (the notations are here slightly changed):

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \phi^{-1} \bar{g}_{\mu\nu} dx^\mu dx^\nu = \frac{1}{\phi} \left( \frac{1}{q^2 s^2(h, u + u_1)} \right) dt^2 - \frac{q^2 s^2(h, u + u_1)}{s^2(k, u)} \left[ \frac{du^2}{s^2(k, u)} + d\Omega^2 \right], \] (5)

\[ F_{\mu\nu} = (\delta_{\mu0}\delta_{\nu1} - \delta_{\nu0}\delta_{\mu1}) q e^{\alpha + \gamma - 2\beta} = (\delta_{\mu0}\delta_{\nu1} - \delta_{\nu0}\delta_{\mu1}) \frac{1}{q s^2(h, u + u_1)}; \] (6)

\[ \varphi = Cu, \quad \frac{\omega + 3/2}{\phi^2} \left( \frac{d\phi}{du} \right)^2 = \epsilon C^2 \] (7)

where the constant \( C \) has the meaning of a scalar charge. The integration constants \( C, k, h \) are related by

\[ 2k^2 \text{sign} k = \epsilon C^2 + 2h \text{sign} h. \] (8)

The function \( s(k, u) \) is defined as follows:

\[ s(k, u) = \begin{cases} k^{-1} \sinh ku, & k > 0 \\ u, & k = 0 \\ k^{-1} \sin ku, & k < 0. \end{cases} \] (9)

Here \( u \) is a convenient radial variable (it is a harmonic coordinate in the Einstein frame). The range of \( u \) is \( 0 < u < u_{\text{max}} \), where \( u = 0 \) corresponds to spatial infinity, while \( u_{\text{max}} \) may be finite or infinite depending on the constants \( k, h, u_1 \) and the behaviour of \( \phi(\varphi) \) described in (3).

Without losing generality we can normalize \( \phi \) to unity at spatial infinity (\( u = 0 \)), so that \( \phi(0) = 1 \), and require \( g_{00}(0) = 1 \). The integration constant \( u_1 \) then satisfies the condition

\[ s^2(h, u_1) = 1/q^2 \] (10)
(preserving some discrete arbitrariness of \(u_1\)). We thus have three essential integration constants: \(k\) or \(h\) and the charges \(q\) and \(C\). An expression for the mass \(M\) of the configuration is obtained by comparing the asymptotic of (11) with the Schwarzschild metric and depends on the asymptotic behaviour of \(\omega(\phi)\):

\[
GM = \frac{\phi'}{2\phi|_{u=0}} + \frac{s'(h, u + u_1)}{s(h, u + u_1)}|_{u=0} = \frac{\pm C}{\sqrt{|\omega(1) + 3/2|}} \pm \sqrt{q^2 + h^2 \text{sign } h}
\]

where \(G\) is Newton’s gravitational constant. The first “±” sign reflects the arbitrariness in the sign of \(C\) while the second one depends on the choice of \(u_1\) among the variants admitted by (11).

The Reissner-Nordström solution of GR is a special case obtained herefrom by putting \(\phi \equiv 1\), whence \(C = 0\) and \(\varphi \equiv 0\). Then from (8), it follows \(h = k\), and the familiar form of the Reissner-Nordström metric is recovered after the coordinate transformation

\[
r = |q| \frac{s(k, u + u_1)}{s(k, u)} \Rightarrow e^{2ku} = \frac{r + k - GM}{r - k - GM}.
\]

To obtain another limiting case \(q = 0\) (the scalar-vacuum solution), one should consider the limit \(q \to 0\) preserving the boundary condition (11). This is only possible for \(h \geq 0\) and \(u_1 \to \infty\). The resulting metric is

\[
ds^2 = \frac{1}{\phi} \left\{ e^{-2hu} dt^2 - \frac{e^{2hu}}{s^2(k, u)} \left[ \frac{du^2}{s^2(k, u)} + dt^2 \right] \right\}.
\]

The scalar field is determined, as before, from (7), and the integration constants are related by

\[
2k^2 \text{sign } k = 2h^2 + \epsilon C^2
\]

This is just the scalar-vacuum solution in the form obtained in (5), which has been studied in detail in (1) and (3).

It should be noted that in (13), (14) the constant \(h\) can have any sign but in (11) in the vacuum case the second term is just \(+h\).

4. Possible black holes in the general scalar-tensor theory

Let us analyze the possible existence of nontrivial (i.e. non-Schwarzschild and non-Reissner-Nordström) BHs in the general STT, i.e. with variable \(\omega = \omega(\phi)\), using Criteria C1–C5. We assume \(C \neq 0\).

Recalling that the range of \(u\) is \(0 < u < u_{\text{max}}\), we can assume in our search for black holes that \(u_{\text{max}}\) is specified by the behaviour of \(s(k, u)\) and \(s(h, u + u_1)\). The opportunity of \(\phi \to 0\) or \(\phi \to \infty\) at some \(u = u_0\) inside this range must be rejected since in this case, e.g., Criterion C3 is necessarily violated at such \(u_0\); moreover, when \(\phi \to \infty\), it is a singular centre, and when \(\phi \to 0\), we have \(g_{tt} \to \infty\).

The solutions belonging to the family (5)–(7) may be then classified as follows:

\[
\begin{align*}
[1+] & \quad \epsilon = +1, \quad k > h > 0; \\
[2+] & \quad \epsilon = +1, \quad k > h = 0; \\
[3+] & \quad \epsilon = +1, \quad h < 0; \\
[1-] & \quad \epsilon = -1, \quad h > k > 0; \\
[2-] & \quad \epsilon = -1, \quad h > k = 0; \\
[3-] & \quad \epsilon = -1, \quad h \geq 0, \quad k < 0; \\
[4-] & \quad \epsilon = -1, \quad 0 > h > k.
\end{align*}
\]

For the vacuum solution (5), (6) there exist only four classes:

\[
\begin{align*}
[1+] & \quad \epsilon = +1, \quad k > 0; \\
[1-] & \quad \epsilon = -1, \quad k > 0; \\
[2-] & \quad \epsilon = -1, \quad k = 0; \\
[3-] & \quad \epsilon = -1, \quad k < 0;
\end{align*}
\]

here \(h \in \mathbb{R}\).

One can verify that C3 (whose formulation does not depend on \(\phi(u)\) and hence on the choice of \(\omega(\phi)\)) is violated for all solutions with \(\epsilon = +1\). From the viewpoint of Criteria C1–C3, for both vacuum (5) and electrovacuum solutions, there are four opportunities of BH existence:
Let us consider, for \( q \neq 0 \), each case separately, except the second one, since it is hard to handle in a general form due to the continuation. We will try first to apply the requirements C1–C3 and after that C4 and C5. One can notice that in all cases to be considered the theory is anomalous.\(^6\)

\[ [1^-] : k > 0, \ u^* = \infty; \]
\[ [1^+] : u = \infty \] is a regular sphere and a horizon may be found beyond it by proper continuation (example: a BH with a conformal scalar field);
\[ [2^-] : k = 0, \ u^* = \infty; \]
\[ [3^-], [4^-] : k < 0, \ u^* = \pi/|k|. \]

In (1) both \( s(.,.) \) are hyperbolic sines and \( u_1 > 0 \). Criteria C1–C3 are satisfied when, as \( u \to \infty \),
\[ \phi \sim e^{2(h-k)u}, \quad e^{2\gamma} \sim e^{(2k-4h)u}. \] (17)
The Hawking temperature and the term \( K_1 \) in the Kretschmann scalar (see the Appendix) behave as follows:
\[ T_H = \lim_{u \to \infty} e^{2(k-h)u} = 0; \quad K_1 \sim e^{2ku} \to \infty. \] (18)

\[ [2^-]. \] Criteria C1–C3 are satisfied when, as \( u \to \infty \),
\[ \phi \sim e^{2hu}/u^2, \quad e^{2\gamma} \sim u^2 e^{-4hu}. \] (19)

Similarly to (18), we obtain
\[ T_H = \lim_{u \to \infty} u^2 e^{-2hu} = 0; \quad K_1 \sim 2hu^2 \to \infty. \] (20)

\[ [3^-], [4^-]. \] A possible horizon is at \( u^* = \pi/|k| \), and in its vicinity all functions depend on \( \Delta u \equiv |u - u^*| \). A calculation shows that all the requirements C1–C5 can be satisfied:
\[ \phi \sim (\Delta u)^{-2}, \quad e^{2\gamma} \sim (\Delta u)^2, \]
\[ T_H = \lim_{u \to u^*} \Delta u = 0; \quad K_1 < \infty. \]
\[ \frac{\phi_u}{\phi} \sim \frac{1}{\Delta u} \to \infty \implies \omega + \frac{3}{2} \to -0. \] (21)
The behaviour of \( g_{00} \) and \( g_{11} \) near the horizon is in this case similar to that in the extreme Reissner-Nordström solution. This is the only case when a type A BH can appear among the STT solutions under consideration. As follows from (21), this opportunity can be realized only in a theory with \( \omega(\phi) \neq \text{const} \), so the BD theory is excluded.

The qualitative picture for \( q = 0 \) (vacuum) \([3], [3]\) is reproduced if we exclude the class \([4^-]\).

We notice that in all these solutions the Hawking temperature calculated for the assumed horizons is zero. Meanwhile, \( T_H < \infty \) is only necessary but not sufficient for regularity. In all cases with \( k \geq 0 \), the Kretschmann scalar tends to infinity, so that the surfaces of finite area, satisfying the conventional criteria C1–C3 of an event horizon, turn out to be singular. Therefore only type B black holes can exist for \( k \geq 0 \). This will be demonstrated explicitly for the BD theory, that is, in cases \([1^-]\) and \([2^-]\). As for \( k < 0 \), type A black holes are possible, but such examples are yet to be found and must be sought for in theories other than BD.

### 5. Electrovacuum and vacuum Brans-Dicke solutions

Consider the solution \([3], [3]\) and put \( \omega = \text{const} \). Then we can explicitly write
\[ \phi = e^{-2\sigma u}, \quad 2\sigma = C/\sqrt{\omega + 3/2}. \] (22)

Let us briefly characterize the properties of the solution using the classification \([3]\) and single out possible BH solutions, which, as we already know, can be found in the classes \([1^-] \) and \([2^-] \).
[1+] : $k > h > 0$
Suppose $u_1 > 0$, then $u_{\text{max}} = \infty$. A positive mass (11) is obtained when $h > \sigma$, and then, as $u \to \infty$, 
\[ e^\gamma \sim e^{(\sigma-h)u} \to 0, \quad e^\beta \sim e^{(\sigma+h-k)u}. \] (23)
One can easily verify that $T_\text{H} = \infty$. So we have at $u = \infty$ an attracting (due to $e^\gamma \to 0$) singularity having a zero, finite or infinite area depending on $\text{sign}(\sigma + h - k)$.

[2+] : $k > h = 0$
For $u_1 > 0$ we have again a violation of the requirements C3 and C4, hence a singularity at $u = u_{\text{max}} = \infty$. In its neighbourhood the metric functions behave as follows:
\[ e^\gamma \sim e^{\sigma u}/(u + u_1), \quad e^\beta \sim (u + u_1)e^{(\sigma-k)u}. \] (24)
The singularity is attracting for $\sigma < 0$ and repelling for $\sigma > 0$; it has a zero area (i.e. it is a centre) if $\sigma < k$ and an infinite area if $\sigma \geq k$. The requirement $M > 0$ leads to just $\sigma < 1/u_1$, which does not forbid any of the qualitatively different variants of behaviour.

[3+] : $h < 0$
In this case $k$ can have either sign but the qualitative behaviour of the solution is in all cases governed by the function $\sin[\sigma h(u + u_1)]$ and $u_{\text{max}}$ is its smallest positive zero. The value $u = u_{\text{max}}$ corresponds to a Reissner-Nordström-like naked repelling central singularity: $e^{\gamma} \to \infty, \ e^{\beta} \to 0$.

The solutions $[1 \pm]$ and $[2 \pm]$ with $u_1 < 0$ have $u_{\text{max}} = -u_1$ and behave qualitatively in the same way as $[3+]$. Therefore in what follows we will omit the opportunity $u_1 < 0$ when treating the classes $[1 \pm]$ and $[2 \pm]$.

[1−] : $h > k > 0$
Now $u_{\text{max}} = \infty$ and, as $u \to \infty$, the metric functions and the term $K_1$ of the Kretschmann scalar behave like
\[ e^\gamma \sim e^{(\sigma-h)u}; \quad e^\beta \sim e^{(\sigma+h-k)u}; \quad K_1 \sim e^{2(2k-h-\sigma)u}, \] (25)
so that the regularity condition is $h + \sigma \geq 2k$. On the other hand, the condition $e^\gamma \to 0$ (criterion C1) implies $h > \sigma$. Combined, these two conditions give the following allowed range of $\sigma$:
\[ h > \sigma \geq 2k - h. \] (26)
As follows from (25), in this range the area of the surface $u = \infty$ is infinite, and, as is easily verified, all the criteria C1, C3, C4, C5 are satisfied, so that we have a type B BH with zero Hawking temperature $T_\text{H}$.

Outside the allowed range $[2k]$, when $\sigma \geq h$, there is a nonsingular wormhole-like structure where $e^\beta \to \infty$ as $u \to \infty$, while $e^\gamma \to \infty$ if $\sigma > h$, so that the second spatial infinity is repulsive, and $\gamma$ tends to a finite limit giving a usual spatial infinity if $\sigma = h$.

For $\sigma < 2k - h$, the surface $u = \infty$ is singular, attracting ($e^\gamma \to 0$), and can have a zero, finite or infinite area, depending on $\text{sign}(\sigma + h - k)$.

[2−] : $h > k = 0$
Again $u_{\text{max}} = \infty$ and instead of (25) we have
\[ e^\gamma \sim e^{(\sigma-h)u}, \quad e^\beta \sim \frac{1}{u} e^{(\sigma+h)u}, \quad K_1 \sim u^4 e^{-2(\sigma+h)u}. \] (27)
The resulting regularity condition $\sigma + h > 0$, combined with the requirement $h > \sigma$ that follows from BH criterion C1, yield the allowed range of $\sigma$ in the form
\[ h > \sigma \geq -h \] (28)
and, in full similarity to class $[1−]$ with (26), we have a cold ($T_\text{H} = 0$) type B black hole.

Outside the allowed range $[2k]$, for $\sigma \geq h$ we find a wormhole-type regular structure similar to the one in the case $[1−]$, $\sigma \geq 2h - k$, while for $\sigma < -h$ we find an attracting singular centre.
Figure 1: Solutions [4−]: different positions (a,b,c) of the curve \( \sin(|h|(u + u_1)) \) with respect to \( \sin(|k|u) \) determine different behaviours of the solution.

[3−] : \( h \geq 0, \ k < 0 \)

In this case the solution behaviour is entirely governed by the function \( \sin |k|u \) and \( u_{\text{max}} = \pi/|k| \), while other functions entering into the solution are finite and smooth over the whole range of \( u \). The two asymptotics, \( u = 0 \) and \( u = u_{\text{max}} \), are both flat and are connected by a regular bridge, so that we are dealing with a static, traversable wormhole.

[4−] : \( k < h < 0 \)

The qualitative nature of the metric is unaffected by the factor \( 1/\phi = e^{2\sigma u} \) and is determined by the interplay of the two sines: \( \sin |k|u \) and \( \sin [h |(u + u_1)] \). Namely, depending on the positions of their zeros, three cases are possible, as shown in Fig. 1:

- [4–a]: \( u_{\text{max}} = \pi/|h| - u_1 \) (without loss of generality): the solution behaves like that of class [3+].
- [4–b]: \( u_{\text{max}} = \pi/|k| \): a behaviour like that of class [3−].
- [4–c]: \( u_{\text{max}} = \pi/|h| - u_1 = \pi/|k| \). As \( u \to u_{\text{max}} \), \( e^\gamma \to \infty \), while \( e^\delta \) and the Kretschmann scalar tend to finite limits. So we obtain a singularity-free hornlike structure (like the ones obtained in some solutions of dilaton gravity [9]), where the infinitely remote (since \( l = \int e^\alpha du \) diverges) “end of the horn”, whose radius \( e^\beta \) is asymptotically constant, repels test particles.

In the vacuum case \([13], [22]\) we are left with the classes \([16]\), for which the estimates \([23],[25]−[28]\) remain valid.

As has been expected, there are type B black holes among the solutions of classes \([1−]\) and \([2−]\). However, the family of charged BD solutions is richer in variants of behaviour as compared with the vacuum family: in addition to analogues of vacuum structures, we now find repelling Reissner-Nordström-like singularities and, in the special case \([4−c]\), a nonsingular hornlike structure.

6. Charged and neutral Brans-Dicke black holes

6.1. Preliminaries

We will study in some detail the cases \([1−]\) and \([2−]\) of the BD solution of the previous section, when one can indeed find some black hole type configurations.

In the case \([1−]\) the metric has the form

\[
ds^2 = e^{2\sigma u} \left\{ \frac{h^2 dt^2}{q^2 \sinh^2[h(u + u_1)]} - \frac{q^2 k^2 \sinh^2[h(u + u_1)]}{h^2 \sinh^2 ku} \left[ \frac{k}{\sinh^2 ku} \right]^2 \right\}.
\]

For this metric, the allowed \( \sigma \) range \([26]\) naturally splits into two parts: \( \sigma < k \) and \( \sigma \geq k \). One can easily show that for \( \sigma < k \) particles moving along geodesics can arrive at the horizon \( u = \infty \) in a finite proper time and
may eventually (if geodesics can be extended) cross it, entering the BH interior (type B1 BHs \[\text{[4]}\]). When, on the contrary, \(\sigma \geq k\), the sphere \(u = \infty\) turns out to be infinitely far and it takes an infinite proper time for a particle to reach it. Since in the same limit \(g_{22} \to \infty\), this configuration (a type B2 BH \[\text{[4]}\]) resembles a wormhole.

In the case \([2-]\), \(k = 0\), the BD metric containing a nonzero electric charge \(q\) has the form

\[
ds^2 = e^{2\sigma u} \left\{ \frac{h^2 dt^2}{q^2 \sin^2[h(u + u_1)]} - \frac{q^2 \sin^2[h(u + u_1)]}{h^2 u^2} \left[ \frac{du^2}{u^2} + d\Omega^2 \right] \right\}.
\]

The allowed range of the integration constants \((28)\) again splits into two halves: for \(\sigma < 0\) we deal with a type B1 BH, for \(\sigma > 0\) with that of type B2 (\(\sigma = 0\) is excluded since it leads to \(\phi = \text{const.}\), hence to GR).

The properties of B1 and B2 structures are quite different and will be discussed separately.

### 6.2. Type B1 black holes: analytical extensions

Consider type B1 BHs for \(k > 0\). To obtain a Kruskal-like extension, we introduce, as usual, the null coordinates \(v\) and \(w\):

\[
v = t + x, \quad w = t - x, \quad x = -\int \frac{\sinh^2[h(u + u_1)]}{\sinh^2 ku} du
\]

where \(x \to \infty\) as \(u \to 0\) and \(x \to -\infty\) as \(u \to \infty\). The asymptotic behaviour of \(x\) as \(u \to \infty\) is \(x \sim e^{2(h-k)u}\), and in a finite neighbourhood of the horizon \(u = \infty\) one can write

\[
x \equiv \frac{1}{2} (v - w) = -\frac{k^2 q^2 e^{2hu_1}}{2h^2(h-k)} e^{2(h-k)u} f(u)
\]

where \(f(u)\) is an analytic function of \(u\), with \(f(\infty) = 1\).

To regularize the metric at the horizon, let us define new null coordinates \(V < 0\) and \(W > 0\) related to \(v\) and \(w\) by

\[
-v = (-V)^{-n-1}, \quad w = W^{-n-1}, \quad n = \text{const.}
\]

The mixed coordinate patch \((V, W)\) is defined for \(v < 0\) \((t < -x)\) and covers the whole past horizon \(v = -\infty\). Similarly, the patch \((v, W)\) is defined for \(w > 0\) \((t > x)\) and covers the whole future horizon \(w = +\infty\). So these patches can be used to extend the metric through any of the horizons.

As is easily verified, a finite value of the metric coefficient \(g_{VW}\) at the future horizon \(W = 0\) is achieved if we take

\[
n + 1 = (h - k)/(k - \sigma),
\]

which is positive for \(h > k > \sigma\). This provides as well a finite value of \(g_{VW}\) at the past horizon \(V = 0\).

(If we tried to do the same for the B2 structure, we would find that the regularization is only achieved at \(W = \infty\) and there are no values of \(W\) where to continue the manifold.)

One can now study the conditions of crossing, say, \(W = 0\) from positive to negative values of \(W\), since \(W\) is an admissible coordinate on the future horizon. This coordinate is, however, defined explicitly only in the close neighbourhood of the horizon. To study the geometry in a finite or infinite region beyond the horizon, it is helpful to introduce a new radial coordinate \(\rho(u)\) behaving like \(W\) near \(W = 0\). Indeed, let us introduce the coordinate \(\rho\) for the metric \((29)\) by

\[
e^{-2ku} = \rho^m
\]

where

\[
m = (h - k + \sigma)/(k - \sigma).
\]

As a result, the solution \((29)\), defined originally in the static region \((\rho > 0)\), takes the form

\[
ds^2 = \frac{h^2}{q^2 P_1(\rho)} \rho^{n+2} dt^2 - 4q^2 \left( \frac{m-n}{m+1} \right)^2 P_2(\rho) [\rho^{-n-2} d\rho^2 + \rho^{-m} d\Omega^2],
\]

\[
\phi = \rho^{m-n-1},
\]

\[
P_1(\rho) \overset{\text{def}}{=} \frac{1}{2} e^{hu_1} \left[ 1 - e^{-2hu_1} \rho^{m+1} \right], \quad P_2(\rho) \overset{\text{def}}{=} 1 - \rho^{m-n}.
\]
Due to (33), $\rho$ is related to the mixed null coordinates $(v, W)$ by

$$\rho(v, W) = W \left( 1 - v W^{n+1} \right)^{-1/(n+1)} \left( 2f \right)^{1/(n+1)}$$

This relation and a similar one giving $\rho(V, w)$ show that when the future (past) horizon is crossed, $\rho$ varies smoothly, behaving like and $W$ or $V$ and changing its sign simultaneously with them. For $\rho < 0$ the metric (37) describes the space-time regions beyond the horizons if the latter are regular.

However, the metric (37) makes sense at $\rho < 0$ only if the numbers $m$ and $n$ are both integers since otherwise fractional powers of negative numbers violate the analyticity as soon as the horizon is crossed. This leads to a discrete set of ratios of the integration constants $h/k$ and $\sigma/k$:

$$\frac{h}{k} = \frac{m + 1}{m - n}, \quad \frac{\sigma}{k} = \frac{m - n - 1}{m - n}.$$  

where, according to the regularity conditions (20), $m > n \geq 0$. Excluding the case $m = n + 1$ that leads to $\sigma = 0$, we see that BD BHs with regular horizons correspond to integers $m$ and $n$ such that

$$m - 2 \geq n \geq 0.$$  

We conclude that, although the curvature scalars are regular on the Killing horizon $u = \infty$, the metric cannot be extended beyond it unless the ratios $h/k$ and $\sigma/k$ obey the “quantization condition” (39), and is generically singular. The Killing horizon, which is at a finite affine distance, is part of the boundary of the space-time, where geodesics and other possible trajectories terminate. Similar properties have been obtained in our previous papers (1), (2) for the vacuum case and earlier in a $(2+1)$–dimensional model with exact power–law metric functions (12) and in the case of black $p$–branes (13).

We have obtained a discrete family of BH solutions whose parameters depend on the two integers $m$ and $n$. This does not mean, however, that the observable parameters of the solution, the mass and the electric and scalar charges are “quantized”. Indeed, the electric charge remains to be an independent integration constant, the scalar field $\phi$ is well characterized by the the constant $\sigma$ given in (39) as a multiple of $k$, and the mass $M$ [cf. (13)] in the present case reads:

$$GM = \sqrt{h^2 + q^2} - \sigma = \sqrt{k^2 + q^2 + \sigma^2 |2\omega + 3|} - \sigma.$$  

Thus two constants, $k > 0$, specifying the length scale of the solution, and the charge $q \neq 0$, remain arbitrary, and other constants are expressed in terms of them and the integers $m$ and $n$. On the other hand, the coupling constant $\omega$ takes, according to (11) and (23), discrete values:

$$|2\omega + 3| = \frac{(2m - n + 1)(n + 1)}{(m - n - 1)^2}.$$  

Notably the Reissner-Nordström solution cannot be obtained from the present discrete family as a special case. Indeed, putting $m = n + 1$, we obtain $\sigma = 0$ and $\omega = \infty$, that is, we abandon the BD theory; however, as follows from (12) and (23), an expression for the scalar charge $C$ corresponding to the Einstein-frame field $\varphi$, remains finite: $C^2 = 2k^2(n + 1)(n + 3)$, and we have still $h > k$, whereas for the Reissner-Nordström solution one must have $h = k$. We thus arrive at a subfamily of solutions of GR with an electric field and a minimally coupled scalar field. The Reissner-Nordström solution can be only recovered if we admit $n = -1$ and $m = 0$.

The solution (2) $(k = 0)$ of the BD theory also has a Killing horizon $(u \to \infty)$ at a finite geodesic distance provided $\sigma < 0$. However, this space-time, as well as its vacuum counterpart, does not admit a Kruskal–like extension and is therefore singular. The reason is that in this case the relation giving the tortoise–like coordinate $x$,

$$x = - \int \frac{q^2}{h^2} \sinh^2 \left[ h(u + u_1) \right] du = - \frac{q^2 e^{2hu_1}}{4h^2} \frac{e^{2hu_1}}{u^2} [1 + o(1)]$$

(where $o(1)$ corresponds to the asymptotic $u \to \infty$) cannot be used to obtain $u$ as an analytic function of $x$ near $u = \infty$. The same happens to the coordinate $\rho$ which might be introduced in the above manner to describe the region beyond the horizon since here, as $u \to \infty$, $\rho = \text{const} \times u^{-2} e^{2\sigma u} [1 + o(1)]$. 

6.3. Type B1 black holes: causal structures

Let us return to the case \([1−].\) One can notice that by (32) the radial coordinate \(x\) is related to \(\rho\) by

\[ x = -(\rho f)^{-n-1}, \]  

so that for odd \(n\) the horizon as seen from region II \((\rho < 0)\) also corresponds to \(x \to -\infty\). For even \(n\), in region II \(\rho\) is also negative but remains to be a spatial coordinate, while the horizon corresponds to \(x \to \infty\). These observations are helpful in constructing the Penrose diagrams.

The resulting causal structures depend on the parities of \(m\) and \(n\).

\([1–a].\) Both \(m\) and \(n\) are even, so \(P_2(\rho)\) is an even function vanishing at \(\rho = \pm 1\), where \(\rho = +1\) is the "old" spatial infinity and \(\rho = -1\) is a new one. The only feature that makes the two regions \(\rho > 0\) and \(\rho < 0\) different is the function \(P_1(\rho)\) which is everywhere regular and finite. The resulting Penrose diagram is similar to that for the extreme Kerr space-time, an infinite tower of alternating regions I and II (Fig. 2). All points of the diagram, except the boundary and the horizons, correspond to usual 2-spheres.

\([1–b].\) Both \(m\) and \(n\) are odd; then both \(P_1(\rho)\) and \(P_2(\rho)\) are even functions and \(\rho\) ranges from \(+1\) to \(-1\). The regions I and II are now anti-isometric \((g_{\mu\nu}(-\rho) = -g_{\mu\nu}(\rho))\); the metric tensor in region II \((\rho < 0)\) has the signature \((-+++)\) instead of \((+−−−)\) in region I. Nevertheless, the Lorentzian nature of the space-time is preserved. The Penrose diagram is shown in Fig. 3. The apparently acausal behaviour of geodesics like E1 can be avoided by assuming helicoidal space-time extension, see \([3]\).

In both cases \([1−a,b]\) the maximally extended space-times are globally regular\(\footnote{A globally regular extension of an extreme dilatonic black hole, with the same Penrose diagram as in our case \([1–a]\), was discussed in \([3]\).}\)

\([1–c].\) \(m\) even, \(n\) odd. In region II \((\rho < 0)\) both \(P_1\) and \(P_2\) are positive, and the metric is regular up to \(\rho \to -\infty\). In this limit the metric functions behave as follows:

\[ g_{tt} \sim |\rho|^{n-2m}, \quad g_{\rho\rho} \sim |\rho|^{-2m+3n}, \quad g_{\theta\theta} \sim |\rho|^{2n+2-m}. \]  

Evidently, as \(\rho \to -\infty\), in all cases \(g_{tt} \to 0\), while the area function \(g_{\theta\theta}\) tends either to zero (for \(m > 2n + 2\)), or to a finite limit \((m = 2n + 2)\), or to infinity \((m < 2n + 2)\). In the first case this is a central singularity like Schwarzschild's. If \(m \leq 2n + 2\), we can suspect one more new horizon and apply the above methodology to study its nature and to try to cross it.

In short, one can introduce one more radial coordinate \(\eta(\rho)\) which behaves in the same way as a null coordinate providing a finite metric coefficient at a possible horizon. Such a coordinate can be determined from the asymptotic condition \(|g_{tt}| \sim |g_{\eta\eta}|^{-1}\) as \(\rho \to -\infty\). This is achieved by substituting

\[ \rho = -\eta^{-1/p}, \quad p = 2m - 2n - 1 > 0, \]  

Figure 2: The causal structure of a BH with \(m\) and \(n\) both even. The curves E1–E6 depict various geodesics possible with this metric.
the limit \( \rho \to -\infty \) corresponds to \( \eta \to +0 \), and a transition across a possible horizon should be described by passing from positive to negative \( \eta \). In terms of \( \eta \) the metric coefficients behave in the following way:

\[
g_{tt} \sim (g_{\eta\eta})^{-1} \sim \eta^{a_1}, \quad a_1 = 1 + (n + 1)/p,
\]

\[
g_{\theta\theta} \sim \eta^{a_2}, \quad a_2 = 1 - (m + 1)/p.
\]

(47)

Just as before, a transition to negative \( \eta \) makes sense only if both exponents \( a_1 \) and \( a_2 \) are integers. One can, however, notice that the sum

\[
a_1 + a_2 = 2 - m - n/p = \frac{3}{2} - \frac{1}{2p}
\]

is non-integer as long as \( p = 2m - 2n - 1 > 1 \). For the relevant \( m \) and \( n \) the number \( p \) can be equal to 5, 9, 13,... We conclude that in all nontrivial cases at least one of the exponents \( a_1 \) and \( a_2 \) is a fraction and hence there is no analytic extension beyond \( \rho = -\infty \).

The Kretschmann scalar tends to a finite limit as \( \rho \to -\infty \) if \( m \leq \frac{3}{2}n + 1 \), otherwise it diverges. However, even for \( m \) and \( n \) such that it is finite, the space-time terminates due to analyticity violation.

In all cases \([1 - c]\) the singularity at \( \rho = -\infty \) is null, therefore the Penrose diagram does not repeat that for the Schwarzschild metric, but, instead, coincides with that of Fig. 3, where now the outer boundaries of regions II depict singularities.

It is of interest that the Reissner-Nordström solution, described in this scheme by the values \( m = 0 \), \( n = -1 \) outside the allowed range \([10] \), also belongs to class \([1 - c]\); in this case the surface \( \rho = -\infty \) is regular and corresponds to the well-known Cauchy horizon of the Reissner-Nordström metric, and a further continuation proceeds as described in the textbooks.

\([1-d]\): \( m \) odd, \( n \) even. In region II the range of \( \rho \) terminates at a zero of the function \( P_1(\rho) \) where one finds a Reissner-Nordström-like repulsive \((g_{tt} \to \infty)\) central \((g_{\theta\theta} \to 0)\) singularity. The resulting Penrose diagram is similar to that of the extreme \((q^2 = GM^2)\) Reissner-Nordström space-time, with the difference that now the 4-dimensional metric changes its signature when crossing the horizon, similarly to case \([1 - b]\), therefore the singularity should be interpreted as a spacelike one.

### 6.4. Type B2 structures

For \( k > 0 \), a type B2 structure occurs when \( h > \sigma > k \). As before, the metric is transformed according to \([31], [33]\) and at the future null limit (now infinity rather than a horizon, so we avoid the term “black hole”) where now \( W \to \infty \) the asymptotic form of the metric is

\[
\begin{align*}
\text{ds}^2 &= -C_1 dv dW - C_2 W^{-m} d\Omega^2
\end{align*}
\]

(49)
where $C_{1,2}$ are some positive constants, while the constant $m$, defined in (38), is now negative. A further application of the $v$-transformation (38) at the same asymptotic, valid for any finite $v < 0$, leads to

$$ds^2 = -C_1(-V)^{(h-s)/(\sigma-k)}dVdW - C_2W^{-m}d\Omega^2.$$  \hspace{1cm} (50)

If we now introduce new radial ($R$) and time ($T$) coordinates by $T = V + W$ and $R = V - W$, in a spacelike section $T = \text{const}$ the limit $R \rightarrow -\infty$ corresponds to simultaneously $V \rightarrow -\infty$ and $W \rightarrow +\infty$, with $|V| \sim W$, and the metric (50) turns into

$$ds^2 = 4C_1(-R)^{(h-s)/(\sigma-k)}(dT^2 - dR^2) - C_2(-R)^{-m}d\Omega^2.$$  \hspace{1cm} (51)

This asymptotic is a nonflat spatial infinity, with infinitely growing coordinate spheres and also $g_{00} \rightarrow \infty$, i.e., this infinity repels test particles.

A Penrose diagram of a B2 type configuration coincides with a single region I in any of the above diagrams; all its sides depict null infinities, its right corner corresponds to the usual spatial infinity and its left corner to the unusual one, represented by the metric (51).

A similar picture is obtained for type B2 structures in the case $[2-]$ ($k = 0$, $\sigma > 0$).

6.5. Comparison with Brans-Dicke vacuum

Vacuum BD configurations are easily obtained from the charged ones in the limit $q \rightarrow 0$, as outlined at the end of Sec. 3. The analysis of the case $[1-]$ was a little more transparent in Refs. 4, 5, 6 because for $k > 0$, after the coordinate transformation

$$e^{-2ku} = 1 - 2k/r \equiv P(r)$$

the solution took the form

$$ds^2 = P^{-\xi}\left(P^a dt^2 - P^{-a} dr^2 - P^{1-a}r^2d\Omega^2\right),$$

$$\phi = P^\xi$$

with the constants related by

$$(2\omega + 3)\xi^2 = 1 - a^2, \quad a = h/k, \quad \xi = \sigma/k.$$  \hspace{1cm} (54)

and the further consideration was conducted in terms of $P$; the horizon took place at $P = 0$. One can easily verify that our present conditions for the occurence of type B1 and B2 structures are still valid for the vacuum case and reduce to those of 4, 5, 6 with the notations (54). The same applies to the parities of $m$ and $n$ in the classification $[1-a] - [1-d]$. There are only some differences in the description of particular cases. Thus, in the vacuum case $[1-a]$ not only the qualitative behaviour of the solution is symmetric with respect to $\rho = 0$, but even the transition $\rho \rightarrow -\rho$ is an isometry. A description of the $[1-b]$ case is unchanged. For $[1-c]$ the vacuum solution behaves simpler: at $\rho = -\infty$ there is always a central spacelike singularity and the Penrose diagram repeats that for the Schwarzschild metric. Lastly, for $[1-d]$, the vacuum solution is singular at $\rho = -\infty$ (as for $[1-c]$ and unlike charged $[1-d]$) and the singularity is again central and spacelike; the Penrose diagram coincides with that of charged $[1-d]$, and there is the same signature change when crossing the horizon.

7. Stability

A study of small (linear) spherically symmetric perturbations of the above static solutions (or static regions of the charged BHs) is to a large extent similar to that of vacuum systems described in 5, 6, therefore we here omit some details of the method but give the results completely.

We now consider, instead of $\varphi(u)$, a perturbed unknown function

$$\varphi(u, t) = \varphi(u) + \delta \varphi(u, t)$$

and similarly for the metric functions $\alpha, \beta, \gamma$, where $\varphi(u)$, etc., are taken from the static solutions of Sec. 2. The electromagnetic field is, by assumption, still governed by the potential component $A_t$, therefore it does not invoke a new dynamical degree of freedom as compared with the vacuum case. We are working in the Einstein conformal
frame. The consideration applies to the whole class of STT; its different members can differ only in boundary conditions to be satisfied by the perturbations.

We use the gauge freedom existing in the perturbation analysis (a choice of the frame of reference and the coordinates in the perturbed space-time) by putting

$$\delta \alpha = 2 \delta \beta + \delta \gamma,$$

thus extending to perturbations the harmonic coordinate condition of the static system in the Einstein conformal frame. In this and only in this case the scalar equation due to (3) for $\delta \varphi (\varphi = 0)$ decouples from the other perturbation equations and reads

$$e^{4 \beta} \delta \ddot{\varphi} - \delta \varphi'' = 0.$$

Here $e^{2 \beta} = -g^E_{\theta \theta}$ is the area function of the unperturbed solution in the Einstein frame, dots denote $d/dt$ and primes, as before, $d/du$. Since the scalar field is the only dynamical degree of freedom, Eq. (56) can be used as the master one, while other equations due to (3) only express the metric and electromagnetic variables in terms of $\delta \varphi$, provided the whole set of field equations is consistent. That it is indeed the case, is directly verified.

The static nature of the background solution makes it possible to separate the variables in Eq. (56),

$$\delta \varphi = \psi(u) e^{i \omega t},$$

and to reduce the stability problem to a boundary-value problem for $\psi(u)$. Namely, if there exists a nontrivial solution to (56) with $\omega^2 < 0$, satisfying some physically reasonable conditions at the ends of the range of $u$, then the static system is unstable since $\delta \varphi$ can exponentially grow with $t$. Otherwise it is stable in the linear approximation.

Suppose $-\omega^2 = \Omega^2$, $\Omega > 0$. In what follows we use two forms of the radial equation (56): the one directly following from (57),

$$\psi'' - \Omega^2 e^{4 \beta(u)} \psi = 0,$$

and the normal Liouville (Schrödinger-like) form

$$d^2 y/dx^2 - [\Omega^2 + V(x)] y(x) = 0,$$

$$V(x) = e^{-4 \beta} (\beta'' - \beta'^2).$$

obtained from (58) by the transformation

$$\psi(u) = y(x) e^{-\beta}, \quad x = - \int e^{2 \beta(u)} du.$$

Here, as before, a prime denotes $\partial/\partial u$. It is of interest to note that $x$ is the same “tortoise” coordinate that was used for continuing the black hole metrics through horizons, see Eq. (31).

The boundary condition at spatial infinity ($u \to 0$, $x \approx 1/u \to +\infty$) is evident: $\delta \varphi \to 0$, or $\psi \to 0$. For our metric (5) the effective potential $V(x)$ has the asymptotic form

$$V(x) \approx 2h/x^3, \quad \text{as} \quad x \to +\infty,$$

hence the general solutions to (59) and (58) have the asymptotic form

$$y \sim c_1 e^{\Omega x} + c_2 e^{-\Omega x} (x \to +\infty),$$

$$\psi \sim u (c_1 e^{\Omega/u} + c_2 e^{-\Omega/u}) (u \to 0),$$

with arbitrary constants $c_1$, $c_2$. Our boundary condition leads to $c_1 = 0$.

For $u \to u_{\text{max}}$, where in many cases the background field $\varphi$ tends to infinity, the boundary condition is not so evident. Refs. [15, 16] and others, dealing with minimally coupled or dilatonic scalar fields, used the minimal requirement providing the validity of the perturbation scheme in the Einstein frame:

$$|\delta \varphi/\varphi| < \infty.$$  

In STT, where Jordan-frame and Einstein-frame metrics are related by $g^J_{\mu \nu} = (1/\phi) g^E_{\mu \nu}$, it seems reasonable to require that the perturbed conformal factor $1/\phi$ behave no worse than the unperturbed one, i.e.

$$|\delta \phi/\phi| < \infty.$$
An explicit form of this requirement depends on the specific STT and can differ from the BD theory, where $\phi$ and $\varphi = Cu$ are connected by (62), the requirement (64) leads to $|\delta \varphi| < \infty$. We will refer to (64) and (65) as to the “weak” and “strong” boundary condition, respectively. For systems where both $\phi$ and $\varphi$ are regular at $u \to u_{\text{max}}$ these conditions coincide and both give $|\delta \varphi| < \infty$.

Let us now discuss different cases of the STT solutions under study. We will suppose that the scalar field $\phi$ is regular for $0 < u < u_{\text{max}}$, so that the conformal factor $\delta^{-1}$ in (61) does not affect the range of the $u$ coordinate.

[1–], the singular solution of normal STT. As $u \to +\infty$, $\beta \sim (h-k)u \to -\infty$, so that $x$ tends to a finite limit and it is convenient to suppose $x \to 0$. The effective potential $V(x)$ then behaves as $V \sim -1/(4x^2)$, and the general asymptotic solution to (62) leads to
\[
\psi(u) \approx y(x)/\sqrt{x} \approx (c_3 + c_4 \ln x) \quad (x \to 0).
\]

The weak boundary condition leads to the requirement $|\delta \varphi/\varphi| \approx |y|/(\sqrt{x} \ln x)) < \infty$, met by the general solution (65) and consequently by its special solution that joins the allowed case ($c_1 = 0$) of the solution (62) at the spatial asymptotic. We then conclude that the static field configuration is unstable, in agreement with the previous work [1–].

As for the strong boundary condition (65), probably more appropriate in STT, its explicit form varies from theory to theory, and a general conclusion is impossible. In the special case of the BD theory the condition (65) means $|\varphi| < \infty$ as $u \to +\infty$. Such an asymptotic behavior is forbidden by Eq. (63), according to which $\psi''/\psi > 0$, i.e. the function $\psi(u)$ is convex and so cannot be bounded as $u \to \infty$ for an initial value $\psi(0) = 0$ ($c_1 = 0$). We conclude that the BD static system is stable.

Thus in this singular case the choice of a boundary condition is crucial for the stability conclusion. In GR with a minimally coupled scalar field [13] there is no reason to “strengthen” the weak condition that leads to the instability. In the BD case the strong condition seems more reasonable and implies stability. For any other STT the situation must be considered separately.

[2–]. With slightly more effort, the results of item [1+] are reproduced, and a stability conclusion again depends on the boundary conditions.

[3–], the case of Reissner-Nordström-like central singularities in normal theories. Here $u_{\text{max}} < \infty$, and we assume that $|\phi|(u_{\text{max}})| < \infty$. (We exclude possible pathological cases of zero or infinite $\phi$ at $u = u_{\text{max}} < \infty$ which can be considered specially if necessary.) Then the weak and strong conditions coincide. As $u \to u_{\text{max}}$, we can put again $x \to 0$ and it appears then that $V(x) \approx -2/(9x^2)$. The general solution to Eq. (59) behaves as
\[
y = c_1 x^{1/3} + c_2 x^{2/3}
\]

near $x = 0$, whereas the boundary condition is $yx^{-1/3} < \infty$. Since this condition is satisfied for any constants $c_1$, $c_2$, we conclude that (generically) this type of solution is unstable in any STT.

[1–], [2–]. This case includes singular solutions and cold black holes as exemplified above for the BD theory.

As $u \to +\infty$, $\beta \to +\infty$, so that $x \to -\infty$ and $V(x) \to 0$. The general solution to Eq. (54) again has the asymptotic form (62) for $x \to -\infty$. The weak condition (64) leads, as in the previous case, to the requirement $|y|/(\sqrt{|x|} \ln |x|) < \infty$, and, applied to (62), to $c_2 = 0$. This means that the function $\psi$ must tend to zero for both $u \to 0$ and $u \to \infty$, which is impossible due to $\psi''/\psi > 0$. Thus the static system is stable. Obviously the more restrictive strong condition (65) can only lead to the same conclusion.

[3–]. In the generic case the solution describes a wormhole, and in the exceptional case (21) there is a cold black hole with a finite horizon area. In all such cases, as $u \to u_{\text{max}} = \pi/|k|$, one has $x \to -\infty$ and $V \sim 1/|x|^3 \to 0$, so that the stability is concluded just as in the cases [1–], [2–].

[4–]. The results differ for different cases a,b,c described in Sec. 5 (and this description applies to all STT under our assumptions). Thus, in the singular case [4–a] we repeat the instability conclusion made for 3+. In the wormhole case [4–b] we obtain stability just as for [3–]. Lastly, for the “horn” [4–c] we have a finite potential at $x = 0$ corresponding to $u = u_{\text{max}}$ and a finite general solution for $\psi(u)$, hence instability.

In the vacuum case we are restricted to the above variants [1+], [1–], [2–], [4–b] with their corresponding stability conclusions.
8. Concluding remarks

We can conclude by the following observations.

1. Black holes do exist in anomalous scalar-tensor theories, i.e., when the kinetic term of the scalar field is negative, contrary to what was sometimes claimed \[2\].

2. For \( k \geq 0 \), there are no conventional (type A) BHs, but there exist BHs with an infinite area (type B), as confirmed explicitly for a special case — the BD theory. They in turn split into two classes, B1 and B2, with, respectively, finite and infinite proper time needed for an infalling particle to reach a horizon. Type B2 structures do not need an analytic extension and resemble wormholes in that they possess another spatial asymptotic.

3. In the case \( k < 0 \) type A BHs can exist, but only in theories with variable \( \omega \), and such explicit examples are yet to be found.

4. Type 1 Brans-Dicke BHs generically possess singular horizons, the singularity being caused by analyticity violation. Only a discrete family of solutions, parametrized by two integers, \( m \) and \( n \), describes BHs with traversable horizons.

5. From the above relations one can observe that at the second asymptotic of all type B2 configurations the BD scalar field \( \phi \to 0 \), i.e., the effective gravitational coupling tends to infinity. The same happens at traversable horizons of type B1 BHs, and, moreover, the effective coupling \( \phi^{-1} \) is negative in regions II when \( m \) and \( n \) have equal parity, i.e. in the cases \([1 - a] \) and \([1 - b] \). At singularities of B1 configurations with \( \sigma < 0 \), on the contrary, \( \phi \to \infty \) and the gravitational coupling vanishes.

6. The electric charge adds some kinds of solution behaviour as compared with the vacuum case but does not drastically change the situation with BHs.

7. Despite their exotic properties, the BH solutions found here are stable, at least with respect to small radial perturbations.

8. For non-BH solutions in normal STT stability conclusions crucially depend on the boundary condition adopted for perturbations at singularities. Old results on the instability of solutions with scalar fields in GR \[15\] are confirmed.

9. The Brans-Dicke BHs under consideration have infinite horizon areas and zero Hawking temperature. This suggest an infinite entropy, consistently with the fact that BHs have negative specific heat. However, a precise calculation of the entropy requires the determination of the surface term in the gravitational action. In the case of an STT, this surface term differs from the usual one by the presence of the scalar field, making the usual expression for the entropy inappropriate. Therefore such a calculation requires a separate study.

10. Tidal forces become infinite at horizons with infinite areas. Hence, only a point particle can cross such a horizon without being destroyed, just as in the vacuum case \[6\].

Appendix. On horizon regularity conditions

An event horizon is, by definition, a regular surface, which implies finite values of all curvature invariants. The finiteness of the Kretschmann scalar \( R^{\mu \nu \lambda \gamma} R_{\mu \nu \lambda \gamma} \) is known to be the most efficient criterion of regularity.

Using it, we will prove that (at least for static, spherically symmetric space-times) an infinite Hawking temperature \( T_H \) of an assumed horizon indicates that it is a curvature singularity rather than a horizon (Lemma 1).\footnote{Although Lemma 1 has been proved \[17\] in a more general \( D \)-dimensional setting, it seems useful to present it here for \( D = 4 \). Besides, the expressions for \( K_i \) and \( T_H \) are used in the text of the paper.}

Another simple result (Lemma 2) is that \( T_H = \infty \) — hence there is a singularity — if an assumed horizon is visible for a static observer, i.e., the integral \( t^* = \int e^{a-u} du \) converges.

Thus Criteria C3 and C4 from Sec. 2 are simple and convenient necessary conditions of horizon regularity.
The Kretschamnn scalar for the metric (A.1) may be written as

\[ K = 4K_1^2 + 8K_2^2 + 8K_3^2 + 4K_4^2 \]  \hspace{1cm} (A.1)

where

\[ K_1 = R_{0101}^{01} = -e^{-\alpha}\gamma \left( \gamma' e^{\gamma-\alpha} \right)', \]
\[ K_2 = R_{0202}^{02} = R_{0303}^{03} = -e^{-2\alpha} \beta' \gamma', \]
\[ K_3 = R_{1212}^{12} = R_{1313}^{13} = -e^{-\alpha-\beta} \left( \beta' e^{\beta-\alpha} \right)', \]
\[ K_4 = R_{2323}^{23} = e^{-2\beta} - e^{-2\alpha} \beta^2 \]  \hspace{1cm} (A.2)

where a prime denotes \( d/du \). The structure of Eq. (A.1) indicates that an infinite value of any \( K_i \) implies the presence of a singularity at a given point of the space-time.

On the other hand, using e.g. formulae from the book [14], one finds for static metrics written in the form (4) the following expression for the Hawking temperature of a surface \( u = u^* \) where \( e^{\gamma} = 0 \), assumed to be a horizon:

\[ T_H = \frac{\kappa^*}{2\pi}, \quad \kappa^* \equiv \lim_{u \to u^*} \kappa(u), \quad \kappa(u) \equiv e^{\gamma - |\gamma'|} \]  \hspace{1cm} (A.3)

where we have put the Boltzmann constant \( k_B \) and the Planck constant \( \hbar \) equal to 1. (The same expression can be obtained using other methods, such as Euclidean continuation of the metric).

We are now ready to prove the following two lemmas.

**Lemma 1.** If, at a certain surface \( u = u^* \) of a static, spherically symmetric space-time with the metric (4), \( e^{\gamma} = 0 \) (a candidate horizon) and the Hawking temperature \( T_H \) calculated for \( u = u^* \), is infinite, this surface is a curvature singularity.

By assumption, \( e^{\gamma} \to 0 \) when \( u \to u^* \). Assume, in addition, that \( \kappa^* = \infty \), while both functions \( \gamma(u) \) and \( \kappa(u) \) are monotonic in some neighbourhood of \( u^* \). Let us show that then the Kretschamnn scalar \( K \to \infty \) as \( u \to u^* \).

It is sufficient to prove that \( K_1 \to \infty \).

Let us use the fact that the expressions \( K_1 \) (as well as other \( K_i \)) and \( \kappa(u) \) are unaffected by reparametrizations of the radial coordinate \( u \). With this invariance, any coordinate condition for \( u \) may be chosen without loss of generality. Let us choose the following one:

\[ \gamma + \alpha = 0. \]  \hspace{1cm} (A.4)

Then

\[ K_1 = -\frac{1}{2} [2\gamma e^{2\gamma}]' = -\frac{1}{2} [e^{2\gamma}]''. \]

By our assumptions we have \( e^{2\gamma} \to 0 \) and \( (e^{2\gamma})' \to \infty \) as \( u \to u^* \).

Let us denote \( g(u) = e^{2\gamma} \), \( 1/g'(u) = G(g) \). Then \( G(g) \to 0 \) as \( g \to 0 \). On the other hand, one can write:

\[ \frac{dg}{du} = \frac{1}{G(g)} \quad \implies \quad u = \int G(g) dg. \]

This integral is evidently finite, hence \( u^* \) is finite in the coordinates (A.4). Thus, for a finite value of \( u \), we have \( g' = dg/du \to \infty \), therefore

\[ g'' \to \infty \quad \implies \quad |K_1| \to \infty, \]

which proves Lemma 1.

**Lemma 2.** If, at a candidate horizon \( u = u^* \) (\( e^{\gamma(u^*)} = 0 \)) of a static, spherically symmetric space-time with the metric (4), the integral \( t^* = \int e^{\gamma} du \) converges, then at \( u = u^* \) the temperature \( T_H = \infty \).

Let us again use the coordinate freedom and put \( \alpha = \gamma \). Then we have simply

\[ t^* = \int du, \quad \kappa(u) = |\gamma'(u)|. \]
So the convergence of $t^\ast$ means just $|u^\ast| < \infty$, which is compatible with $\gamma(u^\ast) = -\infty$ only if $\gamma'(u^\ast) = \infty$, whence $\kappa^\ast = \infty$. Lemma 2 is proved.

**Comment.** The very notion of a horizon implies that it must be in absolute past or future for an observer at rest in a static space-time, and, moreover, it is physically clear that $T_H = \infty$ must mean that such a configuration immediately evaporates and actually cannot exist. These considerations, however, rest on physical interpretations, whereas Lemmas 1 and 2 are of purely geometric nature and provide certain mathematical grounds for such interpretations.

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