Stability assessment of reaction-diffusion PDEs coupled at the boundaries with an ODE

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Abstract

This paper addresses the derivation of sufficient linear matrix inequality conditions ensuring the stability of a coupled system composed of a reaction-diffusion partial differential equation (PDE), with possibly spatially-varying coefficients, and a finite-dimensional linear time invariant ordinary differential equation (ODE). The coupling of the PDE with the ODE is located at both left and right boundary conditions of the reaction-diffusion equation and takes the form of the input and output of the ODE. We investigate the four possible sets of left/right boundary couplings among Dirichlet and Neumann traces. The adopted approach relies on the spectral reduction of the problem by projecting the trajectory of the PDE into a Hilbert basis composed of the eigenvectors of the underlying Sturm-Liouville operator. We propose numerical examples, consisting of an unstable reaction-diffusion equation and an unstable ODE, such that the application of the derived stability conditions ensure the stability of the resulting coupled PDE-ODE system.

Key words: Coupled PDE-ODE, stability, reaction-diffusion equation, modal decomposition, LMI

1 Introduction

The stability analysis and control of coupled PDE-ODE systems has emerged relatively recently in the literature (and more generally PDEs with dynamical boundary conditions, see e.g. [16]). Such a trend was driven by a certain number of practical applications involving a finite-dimensional dynamics coupled to a phenomenon described by a PDE. This includes, to cite a few, solid–gas interaction of heat diffusion and chemical reaction [20], flexible cranes [13], flexible aircraft [7,15], drilling mechanisms [4], and power converters connected to transmission lines [11]. PDE-ODE coupling can also arise due to feedback control. Indeed, the PDE can represent the open-loop plant to be controlled while the ODE part gathers controller and actuator dynamics, see e.g. [19]. Conversely, the PDE can represent the dynamics of an actuator (e.g., heat or flux sensors) that is embedded into the closed-loop control of a finite-dimensional plant modeled by an ODE.

The most traditional approach for studying the stability of coupled PDE-ODE systems consists of the adequate selection of a Lyapunov functional that is equivalent to the norm in which we intend to prove the stability of the coupled plant. However, when it comes to PDEs, this type of construction heavily depends on the structure of the PDE itself, the considered set of boundary conditions, and the regularity of the norm considered for stability analysis. At a very high level, the general trend is to build the Lyapunov functional by considering terms related to 1) the energy of the PDE (measured via a relevant norm); 2) the energy of the ODE; 3) the coupling of the PDE-ODE system. Such Lyapunov functionals can be built manually [9,10,14] but can also be obtained numerically by considering very general Lyapunov functional candidates while resorting to numerical methods, such as a sum of square procedure, to obtain an admissible suitable set of parameters [1,12].

In the abovementioned context, a number of contributions have been reported in the recent years to study the stability of coupled PDE-ODE systems with couplings occurring at the boundaries of the PDE. One of the most fruitful approaches relies on the use of Legendre polynomials as a basis of projection for the PDE trajectories. In essence, this consists of the construction of a classical Lyapunov functional accounting for the PDE and
ODE parts considered separately while adding a cross quadratic term mixing the state of the ODE with a finite number of coefficients of projection of the PDE trajectory into the basis of Legendre polynomials. Such an approach was reported in [6] for the study of a coupled system composed of a reaction PDE and an ODE. This method was also reported in [5] in the case of a string equation coupled with an ODE, as well as in [2] for the study of input-output stability. Input-output stability properties for coupled PDE-ODE systems using Legendre polynomials-based projections was further investigated in [3]. Recently, the stability of abstract boundary control systems with dynamic boundary conditions and positive underlying \( C_0 \)-semigroups was studied in [8].

In this paper, we propose the study of the stability of a general 1-D reaction diffusion equation coupled at the boundaries with a finite-dimensional ODE. Compared to [6], which was concerned with an open-loop stable constant coefficient diffusion PDE with left and right Dirichlet couplings, our approach allows the consideration of reaction-diffusion PDEs, with spatially varying coefficients and that is possibly open-loop unstable, with the four possible sets of left/right boundary couplings among Dirichlet and Neumann traces. The adopted approach relies on a spectral reduction-based method used to build a suitable Lyapunov functional candidate. We obtain a set of tractable LMI conditions ensuring the exponential stability of the coupled PDE-ODE system. In the process, we also show the exponential decrease to zero of the coupling channels, particularly the ones corresponding to Neumann traces. The relevance of these LMI conditions are assessed based on numerical examples associated with PDEs and ODEs that are all open-loop unstable. In this context, the derived stability conditions, once applied to these examples, succeed to show the stability of the resulting coupled PDE-ODE systems.

The rest of the paper is organized as follows. Section 2 describes the notations and reports a number of basic properties for Sturm-Liouville operators. Then the study is split into two parts. Firstly, the case of a Dirichlet trace used as an input for the ODE is investigated in Section 3. Secondly, the case of a Neumann trace used as an input for the ODE is reported in Section 4. Finally, concluding remarks are formulated in Section 5.

2 Notation and properties

Spaces \( \mathbb{R}^n \) are endowed with the Euclidean norm denoted by \( \| \cdot \| \). The associated induced norms of matrices are also denoted by \( \| \cdot \| \). \( L^2(0,1) \) stands for the space of square integrable functions on \( (0,1) \) and is endowed with the inner product \( \langle f, g \rangle = \int_0^1 f(x)g(x) \, dx \) with associated norm denoted by \( \| \cdot \|_{L^2} \). For an integer \( m \geq 1 \), the \( m \)-order Sobolev space is denoted by \( H^m(0,1) \) and is endowed with its usual norm denoted by \( \| \cdot \|_{H^m} \). For a symmetric matrix \( P \in \mathbb{R}^{n \times n} \), \( P \geq 0 \) (resp. \( P > 0 \)) means that \( P \) is positive semi-definite (resp. positive definite) while \( \lambda_M(P) \) (resp. \( \lambda_m(P) \)) denotes its maximal (resp. minimal) eigenvalue.

Let \( p \in C^1([0,1]) \) and \( q \in C^0([0,1]) \) with \( p > 0 \) and \( q \geq 0 \). For \( n_L, n_R \in \{0,1\} \), let the Sturm-Liouville operator \( A_{n_L,n_R} : D(A_{n_L,n_R}) \subset L^2(0,1) \to L^2(0,1) \) be defined by \( A_{n_L,n_R}f = -(pf')' + qf \) on the domain \( D(A_{n_L,n_R}) = \{ f \in H^2(0,1) : f(n_L)(0) = f(n_R)(1) = 0 \} \). The operator \( A_{n_L,n_R} \) is self-adjoint and its eigenvalues \( \lambda_n \), \( n \geq 1 \), are simple, non-negative, and form an increasing sequence with \( \lambda_n \to +\infty \) as \( n \to +\infty \). Moreover, the associated unit eigenvectors \( \phi_n \in L^2(0,1) \) form a Hilbert basis and we also have \( \langle A_{n_L,n_R}f, \phi_n \rangle = \sum_{n \geq 1} |\lambda_n|^2 \langle f \phi_n \rangle^2 < +\infty \) with \( A_{n_L,n_R}f = \sum_{n \geq 1} \lambda_n \langle f \phi_n \rangle \phi_n \).

Let \( p_*, p^* \in \mathbb{R} \) be such that \( 0 < p_* \leq p(\xi) \leq p^* \) and \( 0 \leq q(\xi) \leq q^* \) for all \( \xi \in [0,1] \), then it holds [17]:

\[
0 \leq \pi^2 (n-1)^2 p_* \leq \lambda_n \leq \pi^2 n^2 p^* + q^*
\]  

for all \( n \geq 1 \). Assuming further than \( p \in C^2([0,1]) \), we have that \( \phi_n(0) = O(1) \) and \( \phi'_n(0) = O(\sqrt{\lambda_n}) \) as \( n \to +\infty \) (see [17]). Finally, one can check that, for all \( f \in D(A_{n_L,n_R}) \),

\[
\sum_{n \geq 1} |\lambda_n| \langle f \phi_n \rangle^2 = \langle A_{n_L,n_R}f, f \rangle = \int_0^1 f(p')^2 + qf^2 \, dx.
\]

This in particular implies that, for any \( f \in D(A_{n_L,n_R}) \) and any \( \xi \in [0,1] \), \( f(\xi) = \sum_{n \geq 1} \langle f \phi_n \rangle \phi_n(\xi) \) and \( f'(\xi) = \sum_{n \geq 1} \langle f \phi_n \rangle \phi'_n(\xi) \). For any \( \alpha \in (0,1) \), we introduce the fractional powers of \( A_{n_L,n_R} \) by defining \( D(A^\alpha_{n_L,n_R}) = \{ f \in L^2(0,1) : \sum_{n \geq 1} |\lambda_n|^{2\alpha} |\langle f \phi_n \rangle|^2 < +\infty \} \) and \( A^\alpha_{n_L,n_R}f = \sum_{n \geq 1} \lambda_n^\alpha \langle f \phi_n \rangle \phi_n \).

For any \( f \in L^2(0,1) \) we define \( \mathcal{P}_N f = f - \sum_{n=1}^N \langle f \phi_n \rangle \phi_n = \sum_{n \geq N+1} \langle f \phi_n \rangle \phi_n \).

3 Dirichlet trace as an input of the ODE

3.1 Coupled PDE-ODE systems

We consider in this section one of the two following PDE-ODE systems:

\[
\begin{align*}
z_t(t, \xi) &= (pz\xi)z(t, \xi) - \bar{q}(\xi)z(t, \xi) \quad (3a) \\
z_t(t, 0) &= 0, \quad z(t, 1) = Cz(t) \quad (3b) \\
x(t) &= Ax(t) + Bz(t, 0) \quad (3c) \\
z(0, \xi) &= \bar{z}_0(\xi), \quad x(0) = x_0 \quad (3d)
\end{align*}
\]
or
\[
\begin{align*}
  z(t, \xi) &= (pz\xi)(t, \xi) - \bar{q}(\xi)z(t, \xi) \\
  z(t, 0) &= 0, \quad z(t, 1) = y(t) = Cx(t) \\
  \dot{x}(t) &= Ax(t) + Bz(t, 0) \\
  z(0, \xi) &= z_0(\xi), \quad x(0) = x_0
\end{align*}
\]
for \( t > 0 \) and \( \xi \in (0, 1) \). Here \( p \in C^2([0, 1]) \) with \( p > 0 \), \( \bar{q} \in C^0([0, 1]) \), \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^n \), and \( C \in \mathbb{R}^{n \times 1} \) are matrices, \( z_0 \in L^2(0, 1) \) and \( x_0 \in \mathbb{R}^n \) are initial conditions, and \( z(t, \cdot) \in L^2(0, 1) \) and \( x(t) \in \mathbb{R}^n \) are the state of the reaction-diffusion PDE and of the ODE at time \( t \), respectively. We introduce without loss of generality a function \( q \in C^0([0, 1]) \) and a constant \( q_c \in \mathbb{R} \) all such that
\[
\bar{q} = q - q_c, \quad q \geq 0.
\]
In the case (4), without loss of generality, we further constrain \( q \) such that \( q > 0 \).

**Remark 1** Systems (4) and \( (q) \), as well as systems (18) and (19) that will be described in the next section, can be used to represent a variety of practical situations. For instance the PDE part can stand for a reaction-diffusion process coupled with a finite-dimensional LTI controller materialized by the ODE. Conversely, the ODE part can merge both finite-dimensional LTI plant along with its associated infinite-dimensional LTI controller while the PDE part describes the sensor dynamics (e.g., thermocouple and heat flux sensors).

**Remark 2** It is worth noting that the PDE-ODE system (3) consisting of the coupling via both left and right boundary Dirichlet traces was studied in [6] in the case of a stable diffusion PDE (i.e., in the absence of reaction term) with a constant diffusion coefficient. The approach reported therein employs the projection of the PDE trajectory into a finite subset of Legendre polynomials which is used to handle the coupling between the PDE and the ODE in the construction of a suitable Lyapunov functional. In this paper, we propose a spectral reduction-based framework for handling general reaction-diffusion PDEs, possibly unstable with spatially varying coefficient, and for various combinations of couplings selected among Dirichlet and Neumann traces.

### 3.2 Preliminary spectral reduction

We start by rewriting (3) and (4) under an equivalent PDE-ODE system with homogeneous boundary conditions. Specifically, introducing for (3) the change of variable
\[
w(t, \xi) = z(t, \xi) - \xi^2 y(t)
\]
we infer that (3) is equivalent to
\[
w(t, \xi) = (pz\xi)(t, \xi) + (q_c - q(\xi))w(t, \xi) + a(\xi)y(t) + b(\xi)\dot{y}(t)
\]
with \( a(\xi) = 2p(\xi) + 2\xi p'(\xi) + (q_c - q(\xi))\xi^2 \), \( b(\xi) = -\xi^2 \), and \( w_0(\xi) = z_0(\xi) - \xi^2 y(0) \). Similarly, we introduce for (4) the change of variable
\[
w(t, \xi) = z(t, \xi) - \xi^2 y(t),
\]
with which we obtain that (4) is equivalent to
\[
w(t, \xi) = (pz\xi)(t, \xi) + (q_c - q(\xi))w(t, \xi) + a(\xi)y(t) + b(\xi)\dot{y}(t)
\]
with \( a(\xi) = p(\xi) + \xi p'(\xi) + (q_c - q(\xi))\xi^2 \), \( b(\xi) = -\xi^2 \), and \( w_0(\xi) = z_0(\xi) - \xi^2 y(0) \). After these change of variable, the well-posedness in terms of classical solutions of the above PDE-ODE systems for initial conditions \( w_0 \in D(A^{1/2}_{1,n_R}) \) and \( x_0 \in \mathbb{R}^n \) is a consequence of [18, Chap. 6.3]. More precisely, setting \( n_R = 0 \) in the case (7) and \( n_R = 1 \) in the case (9), we have for any \( w_0 \in D(A^{1/2}_{1,n_R}) \) and any \( x_0 \in \mathbb{R}^n \) the existence and uniqueness of a classical solution \( (w, x) \in C^0([0, 1] \times \mathbb{R}^n) \cap C^1([0, 1]; L^2(0, 1) \times \mathbb{R}^n) \) with \( w(t, \cdot) \in D(A_{1,n_R}) \) for all \( t > 0 \). Moreover, from the proof of [18, Thm. 6.3.1], we have \( A_{1,n_R} w \in C^0([0, 1]; L^2(0, 1)) \) and \( A_{1/2,n_R} w \in C^0([0, 1]; L^2(0, 1)) \).

We now introduce in the case (7) [resp. (9)] the Hilbert basis \( \{ \phi_i : i \geq 1 \} \) of \( L^2(0, 1) \) formed by eigenvectors of the Sturm-Liouville operator \( A_{1,0} \) [resp. \( A_{1,1} \)]. From now on, the treatment of both (7) and (9) is identical. We define the coefficients of projection:
\[
w_i(t) = \langle w(t, \cdot), \phi_i \rangle, \quad a_i = \langle a, \phi_i \rangle, \quad b_i = \langle b, \phi_i \rangle
\]
for \( i \geq 1 \). Considering classical solutions, we obtain that
\[
\begin{align*}
  \dot{w}_i(t) &= (-\lambda_i + q_c)w_i(t) + a_iCx(t) \\
  &\quad + b_i C \left\{ A x(t) + B \sum_{j \geq 1} w_j(t) \phi_j(0) \right\} \\
  \dot{x}(t) &= A x(t) + B \sum_{j \geq 1} w_j(t) \phi_j(0)
\end{align*}
\]
for \( i \geq 1 \). For an arbitrary integer \( N \geq 1 \), we define
\[
W(t) = \left[\begin{array}{c} w_1(t) \\ \vdots \\ w_N(t) \end{array}\right]^T \in \mathbb{R}^N,
\]
\[ A_N = \text{diag}(-\lambda_1 + q_c, \ldots, -\lambda_N + q_c) \in \mathbb{R}^{N \times N}, \]
\[ B_{a,N} = \begin{bmatrix} a_1 & \ldots & a_N \end{bmatrix}^T \in \mathbb{R}^N, \]
\[ B_{b,N} = \begin{bmatrix} b_1 & \ldots & b_N \end{bmatrix}^T \in \mathbb{R}^N, \]
\[ C_N = \begin{bmatrix} \phi_1(0) & \ldots & \phi_N(0) \end{bmatrix} \in \mathbb{R}^{1 \times N}, \]
and \( R(t) = \sum_{i \geq N+1} w_i(t) \phi_i(0) \). We infer from (10a) that
\[
\dot{W}(t) = (A_N + B_{b,N} C B_{C N}) W(t)
+ (B_{a,N} C + B_{b,N} C A) x(t) + B_{b,N} C B R(t).
\]
Combining this latter identity with (10b) while defining
\[ X(t) = \begin{bmatrix} W(t) \\
\end{bmatrix} \in \mathbb{R}^{N+n}, \]
we infer that
\[ \dot{X}(t) = FX(t) + GR(t) \quad (12) \]
where
\[ F = \begin{bmatrix}
A_N + B_{b,N} C B_{C N} & B_{a,N} C + B_{b,N} C A \\
B C N & A
\end{bmatrix} \]
and
\[ G = \begin{bmatrix}
B_{b,N} C B \\
B
\end{bmatrix}. \]
The residual dynamics, which corresponds to the modes \( i \geq N+1 \), is characterized by
\[ \dot{w}_i(t) = (-\lambda_i + q_c) w_i(t) + a_i C x(t) + b_i C A x(t) + b_i C B C N W(t) + b_i C B R(t). \quad (13) \]
We finally define \( M_{1,\phi} = \sum_{i \geq N+1} \frac{\phi_i(0)^2}{\lambda_i} < +\infty \) and
\[ H = \begin{bmatrix}
H_{1,1} & 0 \\
0 & H_{2,2}
\end{bmatrix}. \]
with \( H_{1,1} = \| P_N b \|_{L_2^\infty} C_N B C N + \| P_N a \|_{L_2^\infty} C^T C B C N \) and \( H_{2,2} = \| P_N b \|_{L_2^\infty} C^T C + \| P_N b \|_{L_2^\infty} A C^T C B C N \).

### 3.3 Main result

We can now introduce the main result of this section.

**Theorem 3** Let \( p \in C^2([0, 1]) \) with \( p > 0, q \in C^0([0, 1]), A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^n, \) and \( C \in \mathbb{R}^{n \times 1} \) be given. Let \( q \in C^0([0, 1]) \) and \( q_c \in \mathbb{R} \) be such that (5) holds with \( q \geq 0 \) in the case (3) or \( q > 0 \) in the case (4). Assume that there exist \( N \geq 1, P > 0, \alpha > 2, \) and \( \beta > 0 \) such that \( \Theta_1, \Theta_2 < 0 \) where
\[
\Theta_1 = \begin{bmatrix}
F^T P + P F + \alpha H & PG \\
G^T P & \alpha \| P_N b \|_{L_2^\infty} (C B)^2 - \beta
\end{bmatrix},
\]
\[
\Theta_2 = \begin{bmatrix}
-\lambda_{N+1} + q_c + \frac{\beta M_1}{2} \sqrt{2\lambda_{N+1}} \\
\sqrt{2\lambda_{N+1}} - \alpha
\end{bmatrix}.
\]
then there exist constants \( \eta, M > 0 \) such that, for any initial conditions \( z_0 \in H^2(0, 1) \) and \( x_0 \in \mathbb{R}^n \) such that \( z_0^T(0) = 0 \) and \( \| z(0) \|_{H^1} = \| z_0 \|_{H^1} \), the classical solution of (3) [resp. (4)] satisfies
\[
\| z(t, \cdot) \|^2_{H^2} + \| x(t) \|^2 \leq M e^{-2\eta t} (\| z_0 \|^2_{H^1} + \| x_0 \|^2) \quad (14)
\]
with coupling channels such that
- for (3): \( z(t, 0)^2 + z(t, 1)^2 \leq M e^{-2\eta t} (\| z_0 \|^2_{H^1} + \| x_0 \|^2) \); 
- for (4): \( z(t, 0)^2 + z(t, 1)^2 \leq M e^{-2\eta t} (\| z_0 \|^2_{H^1} + \| x_0 \|^2) \)
for all \( t > 0 \).

**Remark 4** The conclusions of Theorem 3 actually hold for any initial conditions such that \( w_0 \in D(A_{1,0}^{1/2}) \). In the case (3), it is easy to show based on (2) that \( D(A_{1,0}^{1/2}) = \{ f \in H^1(0, 1) : f(1) = 0 \} \). In this setting the conclusions of Theorem 3 hold for any \( z_0 \in H^1(0, 1) \) and \( x_0 \in \mathbb{R}^n \) such that \( \| z_0 \|_{H^1} = C x_0 \). Similarly, since \( D(A_{1,1}^{1/2}) = H^1(0, 1) \), the conclusions of Theorem 3 in the case (4) hold for any \( z_0 \in H^1(0, 1) \) and \( x_0 \in \mathbb{R}^n \).

**Proof.** Let \( N \geq 1, P > 0, \alpha > 2, \) and \( \beta > 0 \) such that \( \Theta_1, \Theta_2 < 0 \). Hence, there exist \( \eta > 0 \) such that \( \Theta_{1,\eta} < 0 \) and \( \Theta_{2,\eta} < 0 \) where
\[
\Theta_{1,\eta} = \begin{bmatrix}
F^T P + P F + 2\eta P + \alpha H & PG \\
G^T P & \alpha \| P_N b \|_{L_2^\infty} (C B)^2 - \beta
\end{bmatrix},
\]
\[
\Theta_{2,\eta} = \begin{bmatrix}
-\lambda_{N+1} + q_c + \eta + \frac{\beta M_1}{2} \sqrt{2\lambda_{N+1}} \\
\sqrt{2\lambda_{N+1}} - \alpha
\end{bmatrix}.
\]
Define the Lyapunov functionnal candidate
\[
V(X, w) = X^T P X + \sum_{i \geq N+1} \lambda_i \langle w, \phi_i \rangle^2 \quad (15)
\]
with \( X \in \mathbb{R}^{N+n} \) and \( w \in D(A_{1,n,\eta}) \) where \( n_R = 0 \) in the case (7) while \( n_R = 1 \) in the case (9). With the slight abuse of notation \( V(t) = V(X(t), w(t)) \), the computation of the time derivative along the system trajectories
We estimate the four latter series by using Young’s inequality. For instance, the first term is estimated as
\[2 \sum_{i \geq N+1} \lambda_i w_i(t) a_i C x(t)\]
\[\leq \sum_{i \geq N+1} \left\{ \frac{1}{\alpha} \lambda_i^2 w_i(t)^2 +\alpha a_i^2 (C x(t))^2 \right\}\]
\[\leq \frac{1}{\alpha} \sum_{i \geq N+1} \lambda_i^2 w_i(t)^2 +\alpha \|P N a\|^2_{L^2 x(t)} C^T C x(t)\].

Similarly, we obtain that
\[\sum_{i \geq N+1} \lambda_i w_i(t) b_i C A x(t)\]
\[\leq \frac{1}{\alpha} \sum_{i \geq N+1} \lambda_i^2 w_i(t)^2 +\alpha \|P N b\|^2_{L^2 x(t)} A^T C^T C A x(t)\],
\[2 \sum_{i \geq N+1} \lambda_i w_i(t) b_i C B C N W(t) \leq \frac{1}{\alpha} \sum_{i \geq N+1} \lambda_i^2 w_i(t)^2\]
\[+\alpha \|P N b\|^2_{L^2 W(t)} C^T N B^T C^T C B C N W(t)\],
and
\[\sum_{i \geq N+1} \lambda_i w_i(t) b_i C B R(t)\]
\[\leq \frac{1}{\alpha} \sum_{i \geq N+1} \lambda_i^2 w_i(t)^2 +\alpha \|P N b\|^2_{L^2} R(t)^2\].

The use of the four latter estimates implies that
\[\dot{V}(t) \leq \begin{bmatrix} X(t)^	op \newline R(t) \end{bmatrix} \begin{bmatrix} F^T P + PF + \alpha H & PG \\ G^T P & \alpha \|P N b\|^2_{L^2}(CB)^2 \end{bmatrix} \begin{bmatrix} X(t) \newline R(t) \end{bmatrix}\]
\[+ 2 \sum_{i \geq N+1} \lambda_i (-\lambda_i + q_c + \frac{2\lambda_i}{\alpha}) w_i(t)^2\]
for \(t > 0\). Since \(R(t) = \sum_{i \geq N+1} \lambda_i w_i(t)\), we infer that \(R(t)^2 \leq M_1, \phi \sum_{i \geq N+1} \lambda_i w_i(t)^2\). This implies for \(t > 0\) that
\[\dot{V}(t) + 2\eta V(t) \leq \begin{bmatrix} X(t) \newline R(t) \end{bmatrix} \Theta_1, \eta \begin{bmatrix} X(t) \newline R(t) \end{bmatrix} + 2 \sum_{i \geq N+1} \lambda_i \Gamma_i w_i(t)^2\]
where \(\Gamma_i = -\left(1 - \frac{\eta}{t}\right) \lambda_i + q_c + \eta + \frac{(M_1, \phi)}{\alpha}\). Now, since \(\alpha > 2\), we have \(\Gamma_i \leq \Gamma_{N+1}\) for all \(i \geq N+1\). Moreover, combining \(\Theta_2, \eta \leq 0\) and the Schur complement, we infer that \(\Gamma_{N+1} \leq 0\). Using also \(\Theta_1, \eta \leq 0\), we obtain that \(\dot{V}(t) + 2\eta V(t) \leq 0\) for all \(t > 0\). Since \(\lambda_i w_i(t) \in C^0(\mathbb{R}_{\geq 0}; L^2(0, 1))\), the mapping \(t \mapsto \dot{V}(t)\) is continuous for \(t > 0\), implying that \(\dot{V}(t) \leq e^{-2\eta t} V(0)\) for all \(t > 0\). We now note from (15) that \(V(0) \leq \lambda_M(\|P\|X(0)\|)^2 + \sum_{i \geq N+1} \lambda_i (w_i, \phi_i)^2\). Noting on one side that \(\|X(0)\|^2 \leq \|x_0\|^2 + \|w_0\|^2_{L^2}\), while on the other side, from (2),
\[\sum_{i \geq N+1} \lambda_i (w_i, \phi_i)^2 \leq \max (\sup |p|, \sup |q|) \|w_0\|^2_{H^1}\],
we infer the existence of a constant \(M_3 > 0\) such that \(V(0) \leq M_1 (\|x_0\|^2 + \|w_0\|^2_{H^1})\). We now consider separately the cases (3) and (4). In the case (3), since \(w(t, 1) = 0\), Poincaré’s inequality along with (2) and (15) shows the existence of a constant \(M_2 > 0\) such that \(\|V(t, \cdot)\|^2_{H^1} \leq M_2 V(t)\). In the case (4), the extra assumption \(q > 0\) directly implies from (2) and (15) the existence of a constant \(M_4 > 0\) such that \(\|w(t, \cdot)\|^2_{H^1} + \|x(t)\|^2 \leq M_3 (\|x_0\|^2 + \|w_0\|^2_{H^1})\). The claimed conclusion follows from the change of variable given by either (6) or (8) and the continuous embedding \(H^1(0, 1) \subset L^\infty([0, 1])\). □

From the proof of Theorem 3, we deduce the following corollary.

**Corollary 5** In the context of Theorem 3, the decay rate \(\eta > 0\) of the stability estimate (14) is guaranteed provided the LMI conditions \(\Theta_1, \eta \leq 0\) and \(\Theta_2, \eta \leq 0\) are feasible.

### 3.4 Numerical illustration

We illustrate the results of Theorem 3 and Corollary 5 for the coupled PDE-ODE system described by (3) with
\[ p = 1, \, q = 0, \, q_e = 3, \]
\[
A = \begin{bmatrix}
0 & -1/4 & -1/5 & 1/5 & 1/6 \\
1/2 & 1 & -4 & 9/2 & 7/2 \\
-9/4 & -1/2 & -14 & 23 & 16 \\
-1/5 & -1/2 & -11/4 & 1/10 & 5/4 \\
-4/3 & -4/3 & -9 & 9 & 5/2
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
-7/2 & -3/2 & -1/10 & 1/2 & 1
\end{bmatrix}^T,
\]
\[
C = \begin{bmatrix}
1/10 & -1/3 & -4 & 7/8 & 7/8
\end{bmatrix}.
\]

In this case, both PDE and ODE systems are open-loop unstable. Indeed, the dominant eigenvalue of the PDE is located approximately at +0.533 while the matrix \( A \) has two unstable eigenvalues located approximately at +1.046 and +0.247. The application of Theorem 3 with \( N = 6 \) shows the exponential stability of the coupled PDE-ODE system (3). Moreover, the application of Corollary 5 with \( N = 30 \) shows the exponential stability of the coupled PDE-ODE system with decay rate \( \eta = 0.47 \).

We illustrate this result with a numerical simulation. The numerical scheme consists in the modal approximation of the PDE plant by its 100 dominant modes. The initial condition is set as \( u_0(\xi) = -1 + \xi^2 \) and \( x_0 = \begin{bmatrix} -2 & 1 & 2 & 1 & 3 \end{bmatrix}^T \). The obtained results are depicted on Fig. 1, confirming the theoretical predictions of Theorem 3 and Corollary 5.

### 4 Neumann trace as an input of the ODE

#### 4.1 Coupled PDE-ODE systems

We consider in this section one of the two below PDE-ODE systems. The difference comparing to the previously studied coupled systems (3) and (4) is that the PDE enters into the ODE by means of a Neumann trace instead of a Dirichlet trace.

\[ z_t(t, \xi) = (pz_\xi)z(t, \xi) - \tilde{q}(\xi)z(t, \xi) \]  
\[ z(t, 0) = 0, \quad z(t, 1) = y(t) = Cx(t) \]  
\[ x(t) = Ax(t) + Bz(t, 0) \]  
\[ z(0, \xi) = z_0(\xi), \quad x(0) = x_0 \]

or

\[ z_t(t, \xi) = (pz_\xi)z(t, \xi) - \tilde{q}(\xi)z(t, \xi) \]  
\[ z(t, 0) = 0, \quad z(t, 1) = y(t) = Cx(t) \]  
\[ x(t) = Ax(t) + Bz(t, 0) \]  
\[ z(0, \xi) = z_0(\xi), \quad x(0) = x_0 \]

for \( t > 0 \) and \( \xi \in (0, 1) \). Here \( p \in C^2([0, 1]) \) with \( p > 0 \), \( \tilde{q} \in C^0([0, 1], R) \), \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n} \), and \( C \in \mathbb{R}^{n \times 1} \) are matrices, \( z_0 \in L^2(0, 1) \) and \( x_0 \in \mathbb{R}^n \) are initial conditions, and \( z(t, \cdot) \in L^2(0, 1) \) and \( x(t) \in \mathbb{R}^n \) are the state of the reaction-diffusion PDE and of the ODE at time \( t \), respectively. As in the previous section, we introduce without loss of generality a function \( q \in C^1([0, 1]) \) and a constant \( q_e \in \mathbb{R} \) all such that (5) holds.

#### 4.2 Preliminary spectral reduction

We transform (18) and (19) into an equivalent PDE-ODE system with homogeneous boundary conditions. Considering the change of variable

\[ w(t, \xi) = z(t, \xi) - \xi y(t) \]

we infer that (18) is equivalent to

\[ w_t(t, \xi) = (pz_\xi)z(t, \xi) + (q_e - q(\xi))w(t, \xi) \]  
\[ + a(\xi)y(t) + b(\xi)\tilde{y}(t) \]

\[ w(t, 0) = w(t, 1) = 0, \quad y(t) = Cx(t) \]

\[ x(t) = Ax(t) + B(w(t, 0) + y(t)) \]

\[ w(0, \xi) = w_0(\xi), \quad x(0) = x_0 \]
while, for the same change of variable (20), (19) is equivalent to
\[ w_i(t, \xi) = (p z_i) \xi(t, \xi) + (q_c - q(\xi))w(t, \xi) \quad (22a) \]
\[ + a(\xi)g(t) + b(\xi)g(t) \]
\[ w(t, 0) = \omega(t, 1) = 0, \quad y(t) = Cx(t) \quad (22b) \]
\[ x(t) = Ax(t) + B (w_0(t, 0) + y(t)) \quad (22c) \]
\[ w_0(0, \xi) = w_0(0, \xi), \quad x(0) = x_0 \quad (22d) \]
both with \( a(\xi) = p'(\xi) + (q_c - q(\xi))\xi, b(\xi) = -\xi, \) and \( w_0(0, \xi) = \omega_0 - \xi(0). \) Note that, after these change of variables, the well-posedness in terms of classical solutions of the above PDE-ODE systems for initial conditions \( w_0 \in \bigcup_{a_0 \in (3/4, 1)} D(A^0_{0,n_R}) \) and \( x_0 \in \mathbb{R}^n \) is a consequence of [18, Chap. 6.3]. More precisely, for a given \( a_0 \in (3/4, 1) \) and setting \( n_R = 0 \) in the case (21) and \( n_R = 1 \) in the case (22), we have for any \( w_0 \in D(A^0_{0,n_R}) \) and any \( x_0 \in \mathbb{R}^n \) the existence and uniqueness of a classical solution \( (w, x) \in C^0(\mathbb{R}_0; L^2(0, 1) \times \mathbb{R}^n) \cap C^1(\mathbb{R}_0; L^2(0, 1) \times \mathbb{R}^n) \) with \( w(t, \cdot) \in D(A_{0,n_R}) \) for all \( t > 0. \) Moreover, from the proof of [18, Thm. 6.3.1], we have \( A_{0,n_R}w \in C^0(\mathbb{R}_0; L^2(0, 1)) \) and \( A^0_{0,n_R}w \in C^0(\mathbb{R}_0; L^2(0, 1)) \) hence \( A^1_{0,n_R}w \in C^0(\mathbb{R}_0; L^2(0, 1)). \)

We introduce \( \{\phi_i : i \geq 1\} \) a Hilbert basis of \( L^2(0, 1) \) composed of eigenvectors of either \( A_{0,n_R} \) in the case (21) or \( A_{0,1} \) in the case (22). We now treat the two cases simultaneously. To do so, we define the coefficients of projection:
\[ w_i(t) = \langle w(t, \cdot), \phi_i \rangle, \quad a_i = \langle a, \phi_i \rangle, \quad b_i = \langle b, \phi_i \rangle \]
for \( i \geq 1. \) Considering classical solutions, we obtain that
\[ \dot{w}_i(t) = (-\lambda_i + q_c)w_i(t) + a_iCw(t) \quad (23a) \]
\[ + b_iC \left( (A + BC)x(t) + B \sum_{j \geq 1} w_j(t)\phi_j^\prime(0) \right) \]
\[ \dot{x}(t) = (A + BC)x(t) + B \sum_{j \geq 1} w_j(t)\phi_j^\prime(0) \quad (23b) \]
for \( i \geq 1. \) For an arbitrary integer \( N \geq 1, \) we define
\[ W(t) = \begin{bmatrix} w_1(t) & \ldots & w_N(t) \end{bmatrix}^T \in \mathbb{R}^N, \]
\[ A_N = \text{diag}(-\lambda_1 + q_c, \ldots, -\lambda_N + q_c) \in \mathbb{R}^{N \times N}, \]
\[ B_{a,N} = \begin{bmatrix} a_1 & \ldots & a_N \end{bmatrix}^T \in \mathbb{R}^N, \]
\[ B_{b,N} = \begin{bmatrix} b_1 & \ldots & b_N \end{bmatrix}^T \in \mathbb{R}^N, \]
\[ C_N = \begin{bmatrix} \phi_1^\prime(0) & \ldots & \phi_N^\prime(0) \end{bmatrix} \in \mathbb{R}^{1 \times N}, \]
and \( R(t) = \sum_{i \geq N+1} w_i(t)\phi_i^\prime(0). \) We infer from (23a) that
\[ \dot{W}(t) = (A_N + B_{b,N}BC_N)W(t) \]
\[ + (B_{a,N}C + B_{b,N}(A + BC))x(t) + B_{b,N}CBR(t). \]
Combining this latter identity with (23b) while defining
\[ X(t) = \begin{bmatrix} W(t) \\ x(t) \end{bmatrix} \in \mathbb{R}^{N+n}, \]
we infer that
\[ \dot{X}(t) = FX(t) + GR(t) \quad (25) \]
where
\[ F = \begin{bmatrix} A_N + B_{b,N}BC_N & B_{a,N}C + B_{b,N}(A + BC) \\ BC_N & A + BC \end{bmatrix} \]
and
\[ G = \begin{bmatrix} B_{b,N}CB \\ B \end{bmatrix}. \]
The residual dynamics, which corresponds to the modes \( i \geq N + 1, \) is characterized by
\[ \dot{w}_i(t) = (-\lambda_i + q_c)w_i(t) + a_iCw(t) \quad (26) \]
\[ + b_iC(A + BC)x(t) + b_iCBW(t) + b_iCBR(t). \]
We finally define for any \( \epsilon \in (0, 1/2] \) the constant \( M_{2,\epsilon}(\epsilon) = \sum_{i \geq N+1} \phi_i(0)^2 < +\infty \) and the matrix
\[ H = \begin{bmatrix} H_{1,1} & 0 \\ 0 & H_{2,2} \end{bmatrix} \]
with \( H_{1,1} = \|P_Nb\|^2_{L_2^2}C_N^TBC_N \) and \( H_{2,2} = \|P_Na\|^2_{L_2^2}(A + BC)^TCC(A + BC). \)

4.3 Main result

We can now introduce the main result of this section.

**Theorem 6** Let \( p \in C^2([0, 1]) \) with \( p > 0, q \in C^0([0, 1]), \)
\( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^n, \) and \( C \in \mathbb{R}^{n \times 1} \) be given. Let \( q \in C^0([0, 1]) \) and \( q_c \in \mathbb{R} \) be such that (5) holds. Assume that there exist \( N \geq 1, \epsilon \in (0, 1/2], p > 0, \alpha > 2, \) and \( \beta > 0 \) such that \( \Theta_1 < 0, \Theta_2 < 0, \) and \( \Theta_3 > 0 \) where
\[ \Theta_1 = \begin{bmatrix} F^T P + PF + \alpha H & PG \\ G^T P & \alpha \|P_Nb\|^2_{L_2^2}(CB)^2 - \beta \end{bmatrix}, \]
Remark 7 For any fixed \( t > 0 \), \( r \geq 0 \), \( \Gamma_i \leq 0 \) for all \( i \geq N + 1 \). Combining this result with \( \Theta_1, \eta \leq 0 \), we deduce from (17) that \( V(t) + 2\eta \bar{V}(t) \leq 0 \) for all \( t > 0 \). From now on, the proof of (27) follows from the same arguments than the ones reported in the previous section. Noting that \( z(t, 1) = Cx(t) \) in the case (18) while \( z(t, 1) = Cx(t) \) in the case (19), we only need to establish the exponential decrease of the term \( z(t, 0) \) to conclude the proof. This is done in Appendix by invoking a small gain argument. □

Corollary 8 In the context of Theorem 6, the decay rate \( \eta > 0 \) of the stability estimate (27) is guaranteed provided the LMI conditions \( \Theta_1, -\alpha \leq 0 \), \( \Theta_2, -\alpha \leq 0 \), and \( \Theta_3 > 0 \) are feasible.

4.4 Numerical illustration

We illustrate the results of Theorem 6 and Corollary 8 for the coupled PDE-ODE system described by (19) with \( p = 1 \), \( q = 0 \), \( q_c = 3 \),

\[
\begin{bmatrix}
-1/4 & -1/6 & 2 & 1 & 1/12 \\
-3/2 & -3/2 & 5 & 5 & 1/6 \\
3/2 & -4 & -15/2 & -5 & -1/3 \\
-13/2 & 22 & 22 & -14 & -1/2 \\
1/7 & -1/2 & -1/2 & 1/5 & -5/2
\end{bmatrix},
\]

\[
\begin{bmatrix}
-5/4 & 2/3 & 1/6 & -1/6 & 0 \\
-2/5 & -5/4 & 3/2 & 1/3 & 1/40
\end{bmatrix}.
\]

In this case, both PDE and ODE systems are open-loop unstable. Indeed, the dominant eigenvalue of the PDE is located approximately at +0.533 while the matrix \( A \) has one unstable eigenvalue located approximately at +0.393. The application of Theorem 3 with \( N = 2 \) shows the exponential stability of the coupled PDE-ODE system (3). Moreover, the application of Corollary 5 with \( N = 30 \) shows the exponential stability of the coupled PDE-ODE system with decay rate \( \eta = 0.37 \).

We illustrate this result with a numerical simulation. The initial condition is set as \( w_0(\xi) = 5\xi(1-\xi)^2 \cos(3\pi \xi) \) and \( x_0 = [-1 1 -2 2 -1] \). The obtained results are depicted on Fig. 2, confirming the theoretical predictions of Theorem 6 and Corollary 8.

5 Conclusion

This paper has addressed the topic of assessing the stability of coupled systems composed of a reaction-
Fig. 2. Time evolution of the coupled PDE-ODE system (19)
diffusion equation and a finite-dimensional linear time-invariant ordinary differential equation (ODE). The considered coupling channels are located at the boundaries of the PDE and consist of the input and output signals of the ODE. The reported sufficient stability conditions take the form of tractable LMIs and have been derived by adopting a spectral reduction-based method. A distinguished feature is that this approach allows to capture all the possible combinations of either Dirichlet or Neumann boundary conditions in terms of coupling channels. Moreover, we have also assessed the exponential decrease to zero of the aforementioned coupling channels, particularly in the case of Neumann boundary couplings. As illustrated via the reported numerical examples, this method can be successfully applied to assess the exponential stability of coupled PDE-ODE systems for which both the open-loop PDE and ODE plants are exponentially unstable.

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A End of the proof of Theorem 6

We investigate the exponential decrease of $z(t)$ to zero. Using the change of variable (20) and the identity $y(t) = Cx(t)$, we have $z(t) = w(t) + Cx(t)$ for $t > 0$. Hence, based on (27), we only need to study the term $w(t) = \sum_{j=1}^{M} w_j(t)$. Let $\alpha_0 \in (3/4, 1)$ such that $w_0 \in D(A_{0, n R}^\alpha)$. Let $N_0 \geq 1$ and $\kappa > 0$ be such that $-\lambda_n + q_c \leq -\eta - \kappa$ for all $n \geq N_0 + 1$. Consider an arbitrary integer $M \geq N_0$. Then we have

$$|w(t)| \leq \sum_{j=1}^{M} |w_j(t)|$$

$$\leq C_0 \sum_{j=1}^{M} w_j(t)^2 = C_0 \sum_{j=1}^{M} \lambda_j^{2\alpha_0} w_j(t)^2$$

$$\leq C_M \|w(t)\|_{L^2} + C_0 \sum_{j=1}^{M} \lambda_j^{2\alpha_0} w_j(t)^2$$

where $C_M = \sqrt{\sum_{j=1}^{M} \phi_j(t)}$ and $C_0 = \sqrt{\sum_{j=1}^{M} \phi_j(t)} < \infty$. Based again on (20) and (27) we only need to study the term $S_M(t) = \sum_{j=M+1}^{\infty} \lambda_j^{2\alpha_0} w_j(t)^2$. To do so, we integrate (23a) for $i \geq N_0 + 1$ and direct estimations give

$$\|\lambda_i^{2\alpha_0} w_i(t)\| \leq e^{-(\lambda_i - \eta + \kappa)t} \|\lambda_i^{2\alpha_0} w_i(0)\|$$

$$+ |a_i| \|C|| \|b_i || \|CA_i\| \int_0^t \lambda_i^{2\alpha_0} e^{-(\lambda_i + q_c)(t-s)} \|x(s)\| ds$$

$$+ |b_i||CB| \int_0^t \lambda_i^{2\alpha_0} e^{-(\lambda_i + q_c)(t-s)} |w_i(s)| ds$$

with $A_c = A + BC$. For any function $f \in L^\infty_{loc}(\mathbb{R})$, we note that

$$\int_0^t \lambda_i^{2\alpha_0} e^{-(\lambda_i + q_c)(t-s)} f(s) ds$$

$$= e^{-(\lambda_i + q_c)t} \int_0^t \lambda_i^{2\alpha_0} e^{-(\lambda_i + q_c)(t-s)} x(s) ds$$

$$\leq \lambda_i^{\alpha_0} (\lambda_i - q_c - \eta) \sup_{s \in [0, t]} e^{\eta s} |f(s)|.$$