Inversion of hyperelliptic integrals of arbitrary genus with application to particle motion in general relativity

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\section*{A R T I C L E I N F O}

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\section*{A B S T R A C T}

The description of many dynamical problems like the particle motion in higher dimensional spherically and axially symmetric space–times is reduced to the inversion of a holomorphic hyperelliptic integral. The result of the inversion is defined only locally, and is done using the algebro-geometric techniques of the standard Jacobi inversion problem and the foregoing restriction to the \(\theta\)-divisor. For a representation of the hyperelliptic functions the Klein–Weierstraß multivariable sigma function is introduced. It is shown that all parameters needed for the calculations like period matrices and Abelian images of branch points can be expressed in terms of the periods of holomorphic differentials and theta-constants. The cases of genus 2 and genus 3 are considered in detail. The method is exemplified by particle motion associated with a genus 3 hyperelliptic curve.

\section*{1. Introduction}

In a wide range of dynamical problems of classical systems one faces the problem of the inversion of integrals of the type [1]

\[ t - t_0 = \int_{x_0}^x \frac{\xi^k}{\sqrt{\mathcal{P}_n(\xi)}} \, d\xi \]  

(1.1)

where \(\mathcal{P}_n(\xi)\) is a polynomial of order \(n\). Such so-called hyperelliptic integrals of genus \(g, n = 2g + 1\), are found, for example, for the particle motion in higher dimensional axially symmetric space–times. The solution of the inversion is the function \(x = x(t)\). If the genus is \(g = 1\) then the result of the inversion is an elliptic function, which is a doubly periodic function of one complex variable. For higher genera \(g > 1\) such an inversion becomes impossible because, as already recognized by Jacobi (see e.g. [2]), \(2g\)-periodic functions of one variable do not exist. However, Jacobi was able to resolve this contradiction by formulating his celebrated Jacobi inversion problem that involves \(g\) hyperelliptic integrals. The problem was solved in terms of so-called hyperelliptic functions which are indeed \(2g\)-periodic functions for \(g > 1\) which depend on \(g\) variables while the periods (also called moduli) are \(g \times g\) matrices. The domain of these hyperelliptic functions – the Jacobi variety – is thus the \(g\)-dimensional complex space \(\mathbb{C}^g\) factorized by the period lattice.

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The Jacobi inversion problem stimulated the development of algebraic geometry and in particular led to the discovery of solutions of many classical mechanical systems like Neumann’s geodesic on an ellipsoid, the spinning top of Kowalewska, Kirchhoff’s motion of a rigid body in a fluid, and others that were integrated using Jacobi’s procedure. This special type of integrability that might be called algebro-geometric integrability has been receiving much attention owing to the discovery of the vast class of partial differential equations of Korteweg–de Vries type that admits this type of integrability.

In this paper we consider the problem of the inversion of one hyperelliptic integral on the basis of the well developed algebro-geometric technique for the standard Jacobi inversion problem. The results of such inversions can be obtained by a restriction of hyperelliptic functions to special subsets of the Jacobi variety – the \( \theta \)-divisor, that is given as a solution of the equation including the Riemann \( \theta \)-function. Such restrictions of hyperelliptic functions can be defined only locally; alternatively they can be realized on an infinitely sheeted Riemann surface [3]. These functions inherit a number of properties of standard elliptic functions, like the addition formulae of the Frobenius–Stickelberger type [4].

Our development starts with the standard Jacobi inversion problem that involves \( g \) hyperelliptic integrals with variable bounds called divisors and describes a dynamic system with \( g \) degrees of freedom. Then we are fixing \( 1 < m < g \) points of the divisor making it special or a divisor with deficiency. In the context of this paper that means that we are considering the case when the genus of the underlying algebraic curve exceeds the number of degrees of freedom of the system. Although different values of \( m \) appear in various problems, we are concentrating here on the case of maximal deficiency \( m = g - 1 \), i.e., on the inversion of one hyperelliptic integral.

The Jacobi inversion problem for a divisor with deficiency has a long history that includes Baker’s consideration [5], Grant’s [6] and Jorgenson’s [7] treatment of the genus 2 case, Ônishi’s consideration [8] of the genus 3 case, the description of certain dynamic systems with separable variables [9], the treatment of the weak Kowalewski–Painlevé property [10], the integration of Somos sequences [11–13], reductions of Benney hierarchies [14,15] and other approaches. Here we will consider the problem of inversion in a systematic way within the Klein–Weierstraß realization of the theory of Abelian functions that is documented in the book of Baker [16] (see also the review [17] and the more recent developments in Buchstaber and Leykin [18], and also Nakayashiki [19], and Matsutani and Previat [20]). The inversion formulae discussed here result in the restriction of the solution of the standard Jacobi inversion problem written in terms of the Kleinian \( \wp \)-functions to the corresponding stratum of the \( \theta \)-divisor. In this context there appears the problem of a suitable parametrization of the \( \theta \)-divisor in the case of higher genera. We solve this problem on the basis of Newton’s method for the approximation of multivariable functions. We note that in this paper we are discussing the inversion of holomorphic integrals only; similar considerations can be undertaken for meromorphic integrals and integrals of the third kind; see e.g. [21].

In order to elaborate an effective calculation procedure for the inversion of one holomorphic hyperelliptic integral, we are solving a problem of general interest, that is, the calculation of the period matrix of meromorphic differentials that is sometimes called the second period matrix. Existing Maple codes contain the evaluation of the Riemann period matrix using the given curve, i.e., period matrices of holomorphic differentials, only. However, the calculation of the periods of meromorphic differentials is unavoidable in certain problems and in particular in our inversion problem. Therefore we show that it is possible to express this second period matrix in terms of the first period matrix and theta-constants. In this way we reduce the number of complete integrals, which are necessary to calculate, to the Riemann period matrix given by Maple codes. Another result states that the characteristics of the Abelian images of branch points can be constructed from the given holomorphic period matrices and, thus, that the homology basis can be reconstructed from these data. That permits us to find the vector of Riemann constants and to carry out the whole calculation without referring to the homology basis. While \( \theta \)-functional calculations are usually considered as technically complicated, here we describe an easily algorithmized scheme that turns such calculations into routine procedures at least in the hyperelliptic case.

Our paper is organized as follows. In Section 2 we briefly describe the problem of particle motion in higher dimensional spherically and axially symmetric space–times. This problem will serve as a laboratory for the approbation of the methods developed. In Section 3 we recall the known facts from the theory of hyperelliptic functions and develop a realization of these functions in terms of Klein–Weierstraß multivariable \( \sigma \)-function. Special attention is focused on the effective calculation of the moduli of the system. We show in particular how to express periods of meromorphic differentials in terms of the theta-constants and periods of holomorphic differentials. In Section 4 we consider the stratification of the \( \theta \)-divisor and show how to single out the stratum that is the image of the curve inside the Jacobian in terms of conditions on the \( \sigma \)-function. In Sections 5 and 6 we consider the application of the method developed to the cases of genus 2 and genus 3 hyperelliptic curves. Finally in the last Section 7 we come back to the initial physical problem and demonstrate how to compute the trajectories of test particles in higher dimensions by the method of restriction to the \( \theta \)-divisor. We explicitly calculate orbits in a nine-dimensional Reissner–Nordström–de Sitter space–time, which is characterized by its mass, electric charge and the cosmological constant. The underlying polynomial is of degree 7 which corresponds to a genus 3 hyperelliptic curve. This is the generalization of the examples considered in [22], where orbits with underlying hyperelliptic curves of genus 2 were calculated.

We believe that our method has much wider applications than the special physical problem considered here and that it can be used in other problems that need the inversion of a hyperelliptic integral. The same approach works for the inversion of meromorphic integrals, that we will consider elsewhere. Some of our results can be used for the Jacobi inversion problem on the strata with smaller deficiency.
2. Particle motion in general relativity

Ordinary differential equations of the form
\[
\frac{dx}{dt} = f(x, \sqrt{P_3(x)}),
\]  
(2.1)
where \( f \) is a rational function of \( x \) and \( P_3(x) \) is a polynomial of order 3 are solved by using elliptic integrals introduced by Jacobi and Weierstraß.\(^1\) The corresponding equations of that form with a fourth-order polynomial \( P_4(x) \) can be reduced to one with a third-order polynomial. As an example, we mention the motion of a point particle or light ray given by the geodesic equation
\[
\frac{d^2x^\mu}{ds^2} + \left\{ \frac{\mu}{\rho\sigma} \right\} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} = 0
\]  
(2.2)
with the Christoffel symbol
\[
\left\{ \frac{\mu}{\rho\sigma} \right\} := \frac{1}{2} g^{\mu\nu} \left( \partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma} \right),
\]  
(2.3)
where \( g_{\mu\nu} \) is the space–time metric, and \( ds \) is the proper time defined by \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \). For a light-like particle (photon) the parameter \( s \) is replaced by some affine parameter.

For a Schwarzschild metric
\[
ds^2 = g_{tt} dt^2 - g_{rr} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]  
(2.4)
with
\[
g_{tt} = \frac{1}{g_{rr}} = 1 - \frac{2M}{r},
\]  
(2.5)
where \( M \) is the mass of the gravitating body (we choose units such that the Newtonian gravitational constant and the velocity of light are unity, \( G = c = 1 \)), this geodesic equation with a substitution \( x = f(r) \) yields an equation describing the dependence of the radial coordinate \( r \) on the azimuthal angle \( \varphi \):
\[
\left( \frac{dx}{d\varphi} \right)^2 = 4x^3 - g_2 x - g_3,
\]  
(2.6)
where the Weierstraß invariants \( g_2 \) and \( g_3 \) depend on \( M \) and the energy and angular momentum of the particle. The complete set of solutions in terms of the Weierstraß \( \wp \)-function have been given and extensively discussed by Hagihara [24]. The corresponding periods of the \( \wp \)-functions are directly related to observable effects like the perihelion shift and the deflection (scattering) angle of massive bodies and of light.

Geodesic equations of neutral test particles or photons in Taub–NUT space–times [25] and of charged test particles in Reissner–Nordström space–times [26] are also solved in terms of elliptic Weierstraß functions. In the case of a Schwarzschild–(anti-)de Sitter or Kerr–(anti-)de Sitter metric we encounter similar equations but with a polynomial of fifth or sixth order (where the sixth-order polynomial can be reduced to a fifth-order one):
\[
\left( \frac{dx}{dt} \right)^2 = P_5(x), \quad i = 0, 1, \quad \text{or,} \quad \left( \frac{dx}{dt} \right)^2 = (x - c)^2 P_5(x),
\]  
(2.7)
where \( c \) is a constant. The corresponding equations have been solved explicitly in [27–30].

These examples can be generalized further to include polynomials of even higher orders as outlined in the following.

2.1. Geodesic equations in higher dimensional spherically symmetric space–times

The metric of a spherically symmetric Reissner–Nordström–(anti-)de Sitter space–times in \( d \) dimensions is given by
\[
ds^2 = g_{tt} dt^2 - g_{rr} dr^2 - r^2 d\Omega_{d-2}^2,
\]  
(2.8)
with
\[
g_{tt} = \frac{1}{g_{rr}} = 1 - \left( \frac{r_S}{r} \right)^{d-3} - \frac{2AR^2}{(d-1)(d-2)} + \left( \frac{q}{r} \right)^{2(d-3)},
\]  
(2.9)
\(^1\) Here and below we will closely follow the standard notation of the theory of elliptic functions fixed in [23].
where $\Lambda$ is the cosmological constant, $q$ the charge of the gravitating mass $M$, $r_5 = 2M$, and $d\Omega_{d-2}^2$ is the surface element of the $d-2$-dimensional unit sphere. The geodesic equation then leads to

$$
\left( \frac{dr}{d\varphi} \right)^2 = \frac{r^4}{L^2} \frac{1}{g_{tt}} \left( E^2 - g_{tt} \left( \delta + \frac{l^2}{r^2} \right) \right)
$$

(2.10)

$$
= \frac{r^4}{L^2} \left( E^2 - \left( 1 - \left( \frac{\rho}{r} \right)^{d-3} - \frac{2\Lambda r^2}{(d-1)(d-2)} + \left( \frac{q}{r} \right)^{2(d-3)} \right) \left( \delta + \frac{l^2}{r^2} \right) \right),
$$

(2.11)

where $E$ and $L$ are two conserved quantities: the dimensionless energy $E$, and the angular momentum $L$ with the dimension of length (both are normalized to the mass of a test particle)

$$
E = g_{tt} \frac{dt}{d\lambda}, \quad L = r^2 \frac{d\varphi}{d\lambda},
$$

(2.12)

where $\lambda$ is an affine parameter along the geodesic. $\delta = 1$ for massive test particles and $\delta = 0$ for massless particles. A substitution $x = f(r)$ gives equations of the form

$$
\left( x^i \frac{dx}{d\varphi} \right)^2 = P_n(x)
$$

(2.13)

for some $0 \leq i < g$ where $P_n$ denotes a polynomial of order $n$ and $g = \left[ \frac{n+1}{2} \right]$ is the genus of a curve $w^2 = P_n(x)$. In some cases, through appropriate substitutions, the order of the polynomial can be reduced [22]. However, in general we have $n \geq 7$, as in the example in Section 7.

2.2. The effective one-body problem

Another example of spherically symmetric problems is the relativistic effective one-body problem in four dimensions. While in Newtonian gravity the two-body problem can be exactly reduced to a one-body problem, this is not possible in Einstein’s general relativity. The relativistic two-body problem can be reduced to a one-body problem only in terms of a series expansion. In this framework the relative coordinate of two bodies with masses $M_1$ and $M_2$ formally fulfills a geodesic equation in a space–time with the effective metric

$$
ds^2 = -g_{tt}(r, v)dt^2 + g_{rr}(r, v)dr^2 + r^2(d\varphi^2 + \sin^2\vartheta d\varphi^2),
$$

(2.14)

where $u = 2(M_1 + M_2)/r$, $v = M_1M_2/(M_1 + M_2)^2$ and

$$
g_{tt}(r, v) = 1 - 2u + 2vu^2 + 3a^2u^4 + o(u^5),
$$

$$
g_{rr}(r, v) = 1 + 6vu^2 + 2(26 - 3v)vu^3 + o(u^4).
$$

(2.15)

The corresponding effective one-body equation of motion is then given by [31,32]

$$
\left( \frac{dr}{d\varphi} \right)^2 = \frac{r^4}{L^2} \frac{1}{g_{tt}} \left( E^2 - g_{tt} \left( 1 + \frac{l^2}{r^2} \right) \right),
$$

(2.16)

where $E$ and $L$ are again the conserved energy and angular momentum. This is a series expansion which can be expanded to arbitrary order. Although this is only a series expansion, analytic methods are helpful for the purpose of having a complete discussion of the possible kinds of orbits of a binary system.

2.3. Geodesic equations in higher dimensional axially symmetric space–times

As an example of $d$-dimensional axially symmetric space–times we consider the simplest one, namely the Myers–Perry [33] space–times with only one rotation parameter $a$ given by [34,35]

$$
ds^2 = \frac{1}{\rho^2} \left( \frac{2M}{r_{n-1}} - \rho^2 \right) dt^2 - 4aM \sin^2\theta \frac{dt d\varphi}{\rho^2 r_{n-1}} + \frac{\sin^2\theta}{\rho^2} \left( (r^2 + a^2) \rho^2 + 2a^2M \frac{r}{r_{n-1}} \sin^2\theta \right) d\varphi^2
$$

(2.17)

$$
+ \rho^2 \frac{\Delta}{\Delta} dr^2 + \rho^2 d\theta^2 + r^2 \cos^2\theta d\Omega_n^2,
$$

where $\Delta = r^2 + a^2 \cos^2\theta$, $\Delta = (r^2 + a^2) - \frac{2M}{r_{n-1}}$, and $n = d - 4$. Here, $M$ is the mass of the black hole and the surface element of the unit $n$-sphere is $d\Omega_n^2 = \sum_{i=1}^{n} \prod_{k=1}^{i-1} \sin^2\psi_k d\psi_k^2$. 


Owing to the conservation laws related to the symmetries of the underlying space–time there are a conserved energy $E$, an angular momentum $L_\varphi$ and $n$ further constants $\Psi^2_i, i = 1, \ldots, n$. As a consequence, it is possible to separate the Hamilton–Jacobi equation \[ E \text{L}, an angular momentum } (r^2 + a^2)E - aL_\varphi \text{ or } a [L_\varphi - a \sin^2 \theta E] \frac{d \theta}{\sqrt{\rho}}. \] (2.19)

\[
\frac{dv}{d\lambda} = a \left[(r^2 + a^2)E - aL_\varphi\right] \frac{1}{\sqrt{\rho}} \frac{dr}{d\lambda} + \frac{1}{\sin^2 \theta} \left[L_\varphi - a \sin^2 \theta E\right] \frac{1}{\sqrt{\rho}} \frac{d\theta}{d\lambda}. \] (2.20)

\[
\frac{d\psi_i}{d\lambda} = \frac{\sqrt{\lambda_i}}{\prod_{k=1}^{n-1} \sin^2 \psi_k} \frac{1}{r^2 \cos^2 \theta}, \quad i = 1, \ldots, n, \] (2.22)

with $A_i = \Psi^2_i - \frac{\psi^2_{i+1}}{\sin^2 \psi_i}$ and

\[
R(r) = \left[(r^2 + a^2)E - aL_\varphi\right] - \Delta \left[K + \delta r^2 + \frac{\Psi^2_i}{r^2} \right], \] (2.23)

\[
\Theta(\theta) = K - \delta a^2 \cos^2 \theta - \frac{\Psi^2_i}{\cos^2 \theta} - \frac{1}{\sin^2 \theta} \left[L_\varphi - a \sin^2 \theta E\right]^2, \] (2.24)

where $K$ is a further constant, called the Carter constant, which emerges from the separation process. It is obvious that $\mathcal{P}(r)$ in \[ R(r) = \frac{\mathcal{P}(r)}{\mathcal{P}_{g=2}} \] is a polynomial whose order increases with the dimension of the space–time. (In some cases the order of $\mathcal{P}(r)$ can be reduced by a substitution.)

In the following sections we will explain the theory and the details of how to analytically solve the geodesic equations in the above cases and for similar physical problems.

### 3. Hyperelliptic functions

The solutions of the differential equations given in the foregoing section can be considered as points on a hyperelliptic curve $X_g$ of genus $g$ given by the equation

\[
w^2 = \mathcal{P}_{2g+1}(z) = \sum_{i=0}^{2g+1} \lambda_i z^i = 4 \prod_{k=1}^{2g+1} (z - e_k), \] (3.1)

and realized as a two-sheeted covering over the Riemann sphere branched at the points $(e_k, 0), k \in \mathcal{G} = \{1, \ldots, 2g + 1\}$, with $e_j \neq e_k$ for $j \neq k$, and at infinity, $e_{2g+2} = \infty$. Notice that we do not require the $e_k$ to be real. However, when they are real, we find it convenient to order them according to $e_1 < e_2 < \cdots < e_{2g+1}$, i.e., in the opposite way as compared to the Weierstraß ordering; see Fig. 1. Define $P(z, u)$, a coordinate of the curve. The factor 4 in (3.1) is introduced to preserve the resemblance with the Weierstraß cubic for $g = 1$:

\[
w^2 = 4z^3 - 2r^2z - 3 \equiv 4(z - e_1)(z - e_2)(z - e_3). \] (3.2)

As shown in Fig. 1 we equip the hyperelliptic curve $X_g$ with a canonical homology basis

\[
(a_1, \ldots, a_g; b_1, \ldots, b_g), \quad a_i \cap b_j = -h_i \cap a_i = \delta_{ij}, \quad a_i \cap a_j = b_i \cap b_i = 0, \] (3.3)

where $\delta_{ij}$ is the Kronecker symbol and where the $\cap$ denotes the intersection of cycles.

#### 3.1. Canonical differentials

We choose canonical holomorphic differentials (of the first kind) $d\omega = (du_1, \ldots, du_g)$ and associated meromorphic differentials (of the second kind) $d\omega' = (dr_1, \ldots, dr_g)$ in such a way that their $g \times g$ period matrices

\[
2\omega = \left(\begin{array}{c} \phi \quad du_i \\ a_k \quad \end{array} \right)_{i,k=1,\ldots,g}, \quad 2\omega' = \left(\begin{array}{c} \phi \quad du_i \\ b_k \quad \end{array} \right)_{i,k=1,\ldots,g}, \] (3.4)

\[
2\eta = \left(\begin{array}{c} -\phi \quad dr_i \\ a_k \quad \end{array} \right)_{i,k=1,\ldots,g}, \quad 2\eta' = \left(\begin{array}{c} -\phi \quad dr_i \\ b_k \quad \end{array} \right)_{i,k=1,\ldots,g}.
\]
In the context of our consideration we take the divisor $D$ in

$$\Theta$$

We will use both versions: the first one

$$\text{(3.14)}$$

The corresponding Jacobian is introduced as

$$\text{Jac}(X_g):=(2\omega)^{-1}\text{Jac}(X_g) = \mathbb{C}^g/\Gamma_1 \oplus \tau.$$  

We will use both versions: the first one ($2\omega$, $2\omega'$) in the context of $\sigma$-functions, and the second one ($1_g$, $\tau$) in the case of $\theta$-functions.

The Abel map $\mathcal{A} : (X_g)^n \rightarrow \mathbb{C}^g$ with the base point $P_0$ relates the set of points $(P_1, \ldots, P_n)$ (which are called the divisor $D$) with a point in the Jacobian $\text{Jac}(X_g)$:

$$\mathcal{A}(P_1, \ldots, P_n) := \sum_{k=1}^{n} \int_{P_0}^{P_k} \text{d}u.$$  

The divisor $D$ in (3.14) can also be denoted as $P_1 + \cdots + P_n - nP_0$.

Analogously we define

$$\widetilde{\mathcal{A}}(P_1, \ldots, P_n) = \sum_{k=1}^{n} \int_{P_0}^{P_k} \text{d}v = (2\omega)^{-1}\mathcal{A}(P_1, \ldots, P_n).$$  

In the context of our consideration we take $P_0$ as infinity, $P_0 = (\infty, \infty)$. 

Fig. 1. A homology basis on a Riemann surface of the hyperelliptic curve of genus $g$ with real branch points $e_1, \ldots, e_{2g+2} = \infty$ (upper sheet). The cuts are drawn from $e_{2i-1}$ to $e_i$ for $i = 1, \ldots, g+1$. The $b$-cycles are completed on the lower sheet (the picture on the lower sheet is just flipped horizontally). 

satisfy the generalized Legendre relation

$$MJM^T = -\frac{i\pi}{2}J$$  

with

$$M = \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix},$$  

with $0_g$ and $1_g$ as the zero and unit $g \times g$ matrices. Such a basis of differentials can be realized as follows (see [16], p. 195):

$$\text{d}u(z, w) = \frac{\mathcal{U}(z)\text{d}z}{w}, \quad \mathcal{U}(z) = z^{i-1}, \quad i = 1, \ldots, g, \quad \mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_g)^T$$  

$$\text{d}r(z, w) = \frac{\mathcal{R}(z)\text{d}z}{4w}, \quad \mathcal{R}(z) = \sum_{k=1}^{2g+1-i} (k+1-i)\lambda_{k+1}z^k, \quad i = 1, \ldots, g, \quad \mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_g)^T$$  

where the coefficients $\lambda_i$ are given by (3.1).

We denote by $\text{Jac}(X_g)$ the Jacobian of the curve $X_g$, i.e., the factor $\mathbb{C}^g/\Gamma'$, where $\Gamma' = 2\omega \oplus 2\omega'$ is the lattice generated by the periods of the canonical holomorphic differentials. Any point $u \in \text{Jac}(X_g)$ can be presented in the form

$$u = 2\omega e + 2\omega' e'$$  

where $\epsilon, \epsilon' \in \mathbb{R}^g$. The vectors $\epsilon$ and $\epsilon'$ combine to a $2 \times g$ matrix and form the characteristic of the point $u$,

$$[u] := \begin{pmatrix} \epsilon^T \\ -\epsilon'^T \end{pmatrix} = \begin{pmatrix} \epsilon_1 & \cdots & \epsilon_g \\ -\epsilon'_1 & \cdots & -\epsilon'_g \end{pmatrix} =: \epsilon.$$  

If $u$ is a half-period, then all entries of the characteristic $\epsilon$ are equal to $\frac{1}{2} \lambda$ or $0$.

Beside the canonical holomorphic differentials $\text{d}u$ we will also consider normalized holomorphic differentials defined by

$$\text{d}v = (2\omega)^{-1}\text{d}u.$$  

Their corresponding holomorphic periods are $1_g$ and $\tau$, where the Riemann period matrix $\tau := \omega^{-1}\omega'$ is in the Siegel upper half-space $\mathfrak{H}_g$ of $g \times g$ matrices (or half-space of degree $g$),

$$\mathfrak{H}_g = \left\{ \tau \text{ is } g \times g \text{ matrix} \mid \tau^T = \tau, \Im \tau \text{ positive definite} \right\}.$$  

The corresponding Jacobian is introduced as

$$\widetilde{\text{Jac}}(X_g) := (2\omega)^{-1}\text{Jac}(X_g) = \mathbb{C}^g/\Gamma_1 \oplus \tau.$$  

We will use both versions: the first one ($2\omega$, $2\omega'$) in the context of $\sigma$-functions, and the second one ($1_g$, $\tau$) in the case of $\theta$-functions.

The Abel map $\mathcal{A} : (X_g)^n \rightarrow \mathbb{C}^g$ with the base point $P_0$ relates the set of points $(P_1, \ldots, P_n)$ (which are called the divisor $D$) with a point in the Jacobian $\text{Jac}(X_g)$:

$$\mathcal{A}(P_1, \ldots, P_n) := \sum_{k=1}^{n} \int_{P_0}^{P_k} \text{d}u.$$  

Analogously we define

$$\widetilde{\mathcal{A}}(P_1, \ldots, P_n) = \sum_{k=1}^{n} \int_{P_0}^{P_k} \text{d}v = (2\omega)^{-1}\mathcal{A}(P_1, \ldots, P_n).$$  

In the context of our consideration we take $P_0$ as infinity, $P_0 = (\infty, \infty)$. 

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3.2. $\theta$-functions

The hyperelliptic $\theta$-function with characteristics $[\varepsilon]$ is a mapping $\theta : \widetilde{\text{Jac}}(X_g) \times \mathfrak{S}_g \to \mathbb{C}$ defined through the Fourier series

$$
\theta[\varepsilon](\mathbf{v} | \tau) := \sum_{m \in \mathbb{Z}} e^{\pi i \left\{ (m + \varepsilon)^2 r(m + \varepsilon) + 2(\varepsilon + m)^2 r(m + \varepsilon) \right\}}.
$$

(3.16)

It possesses the periodicity property

$$
\theta[\varepsilon](\mathbf{v} + \mathbf{n} + \tau \mathbf{m}' | \tau) = e^{-2\pi i \varepsilon^T \mathbf{m}' | \tau} e^{2\pi i \varepsilon^T \mathbf{m} | \tau} \theta[\varepsilon](\mathbf{v} | \tau).
$$

(3.17)

For vanishing characteristic we use the abbreviation $\theta(\mathbf{v}) := \theta(0)(\mathbf{v} | \tau)$. In the following, the values $\varepsilon_k, \varepsilon'_k$ will either be 0 or $\frac{1}{2}$. The equality (3.17) implies

$$
\theta[\varepsilon](\mathbf{v} | \tau) = e^{-4\pi i \varepsilon^T \mathbf{r}'} \theta[\varepsilon](\mathbf{v} | \tau),
$$

(3.18)

so the function $\theta[\varepsilon](\mathbf{v} | \tau)$ with characteristics $[\varepsilon]$ of only half-integers is even if $4\varepsilon^T \mathbf{r}'$ is an even integer, and odd otherwise. Correspondingly, $[\varepsilon]$ is called even or odd, and among the $4^g$ half-integer characteristics there are $\frac{1}{2}(4^g + 2^g)$ even and $\frac{1}{2}(4^g - 2^g)$ odd characteristics.

The non-vanishing values of the $\theta$-functions with half-integer characteristics and their derivatives are called $\theta$-constants and are denoted as

$$
\theta[\varepsilon] := \theta[\varepsilon](0; \tau), \quad \theta_{i,j}[\varepsilon] := \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \theta[\varepsilon](\mathbf{z}; \tau) \bigg|_{z=0}, \quad \text{etc. for even } [\varepsilon];
$$

$$
\theta_i[\varepsilon] := \frac{\partial}{\partial z_i} \theta[\varepsilon](\mathbf{z}; \tau) \bigg|_{z=0}, \quad \theta_{i,j,k}[\varepsilon] := \frac{\partial^3}{\partial z_i \partial \bar{z}_j \partial z_k} \theta[\varepsilon](\mathbf{z}; \tau) \bigg|_{z=0}, \quad \text{etc. for odd } [\varepsilon].
$$

Even characteristics $[\varepsilon]$ are called nonsingular if $\theta[\varepsilon] \neq 0$, and odd characteristics $[\varepsilon]$ are called nonsingular if $\theta_i[\varepsilon] \neq 0$ for at least one index $i$.

We identify each branch point $e_j$ of the curve $X_g$ with a vector

$$
\mathfrak{A}_j := \int_{e_j}^{e_j} \mathbf{u} := 2\omega e_j + 2\omega' e'_j \in \text{Jac}(X_g), \quad j = 1, \ldots, 2g + 2,
$$

(3.19)

what defines the two vectors $\mathbf{e}_j$ and $\mathbf{e}'_j$. Evidently, $[\mathfrak{A}_{2g+2}] = [0] = 0$.

In terms of the $2g + 2$ characteristics $[\mathfrak{A}_j]$, all $4^g$ half-integer characteristics $[\varepsilon]$ can be constructed as follows. There is a one-to-one correspondence between these $[\varepsilon]$ and partitions of the set $\mathcal{G} = \{1, \ldots, 2g + 2\}$ of indices of the branch points ([37], p. 13, [16] p. 271). The partitions of interest are

$$
\mathcal{I}_m \cup \mathcal{J}_m = \{i_1, \ldots, i_{g+1-2m}\} \cup \{j_1, \ldots, j_{g+1+2m}\},
$$

(3.20)

where $m$ is any integer between 0 and $[\frac{g+1}{2}]$. The corresponding characteristic $[\varepsilon_m]$ is defined by the vector

$$
\Delta_m = \sum_{k=1}^{g+1-2m} \widetilde{\mathfrak{A}}_{i_k} + \mathbf{K}_\infty =: \varepsilon_m + \tau \varepsilon'_m,
$$

(3.21)

where $\mathbf{K}_\infty \in \text{Jac}(X_g)$ is the vector of Riemann constants with base point $\infty$, which will always be used in the argument of the $\theta$-functions, and which is given as a vector in $\text{Jac}(X_g)$ by

$$
\mathbf{K}_\infty := \sum_{\text{all odd } [\mathfrak{A}_j]} \widetilde{\mathfrak{A}}_j
$$

(3.22)

(see e.g. [38], p. 305, for a proof).

It can be seen that characteristics with even $m$ are even, and those with odd $m$ are odd. There are $\frac{1}{2} \left( \frac{2^g+2}{g+1} \right)$ different partitions with $m = 0$, $\frac{2^g+2}{g-1}$ different partitions with $m = 1$, and, in general, $\left( \frac{2^g+2}{g+1-2m} \right)$ down to $\left( \frac{2^g+2}{g+1} \right) = 2g + 2$ partitions if $g$ is even and $m = g/2$, or $\left( \frac{2^g+2}{g+1-2m} \right) = 1$ partitions if $g$ is odd and $m = (g + 1)/2$. One may check that the total number of even (odd) characteristics is indeed $2^{2g-1} \pm 2^{g-1}$. According to the Riemann theorem on the zeros of $\theta$-functions [37], $\theta(\Delta_m + \mathbf{v})$ vanishes to order $m$ at $\mathbf{v} = 0$ and in particular, the function $\theta(\mathbf{K}_\infty + \mathbf{v})$ vanishes to order $\left[ \frac{g+1}{2} \right]$ at $\mathbf{v} = 0$. 
Let us demonstrate, following [38], p. 303, what the set of characteristics \([A_k] \equiv [A_k], k = 1, \ldots, 2g + 2\), looks like in the homology basis shown in Fig. 1. Using the notation \(f_k = \frac{1}{2}(\delta_{1k}, \ldots, \delta_{2k})^t\) and \(\tau_k\) for the kth column vector of the matrix \(r\), we find

\[
\tilde{A}_{2g+1} = \tilde{A}_{2g} + \sum_{k=1}^g \int_{\varphi_{z_k-1}}^{\varphi_{z_k}} 2d\varphi = \sum_{k=1}^g f_k, \quad [A_{2g+1}] = \frac{1}{2} \begin{pmatrix} 1 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}.
\]

Continuing in the same manner, we get for arbitrary \(1 \leq k < g\)

\[
[A_{2k+2}] = \frac{1}{2} \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 & 0 & 0 \end{pmatrix},
\]

\[
[A_{2k+1}] = \frac{1}{2} \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 & 0 & 0 \end{pmatrix},
\]

and finally

\[
[A_2] = \frac{1}{2} \begin{pmatrix} 0 & 0 & \ldots & 0 \end{pmatrix}, \quad [A_1] = \frac{1}{2} \begin{pmatrix} 0 & 0 & \ldots & 0 \end{pmatrix}.
\]

The characteristics with even indices, corresponding to the branch points \(e_{2n}, n = 1, \ldots, g\), are odd (except for \([A_{2g+2}]\) which is zero); the others are even. Therefore in the basis drawn in Fig. 1 we get

\[
K_{\infty} = \sum_{k=1}^g \tilde{A}_{2k}.
\]  

The formula (3.26) is in accordance with the classical theory where the vector of Riemann constants is defined as (see Fay [37], Eq. (14))

\[
\text{Divisor } K_{p_0} = \Delta - (g - 1)p_0,
\]

where \(\Delta\) is divisor of degree \(g - 1\) that is the Riemann divisor. In the case considered, \(p_0 = \infty\) and \(\Delta = e_2 + e_4 + \cdots + e_{2g} - \infty\). The calculation of the divisor of the differential \(\prod_{\delta}(x - e_2)dx/y\) leads to the required conclusion \(2\Delta = K_{X_g}\) where \(K_{X_g}\) is canonical class.

### 3.3. \(\sigma\)-functions

The Kleinian \(\sigma\)-function of the hyperelliptic curve \(X_g\) is defined over the Jacobian \(\text{Jac}(X_g)\) as

\[
\sigma(u; M) := C \theta([K_{\infty}]((2\omega)^{-1}u; \tau) \exp \left\{ u^T \mathcal{K} x \right\},
\]

where \(x = \eta(2\omega)^{-1}\), the constant

\[
C = \sqrt{\frac{\pi^g}{\det(2\omega)}} \left( \prod_{1 \leq i < j \leq 2g+1} (e_i - e_j) \right)^{-1/4},
\]

and \(M\) defined in (3.6) contains the set of all moduli \(2\omega, 2\omega'\) and \(2\eta, 2\eta'\). In what follows we will use the shorter notation \(\sigma(u; M) = \sigma(u)\). Sometimes the \(\sigma\)-function (3.28) is called the fundamental \(\sigma\)-function.

The multivariable \(\sigma\)-function (3.28) represents a natural generalization of the Weierstraß \(\sigma\)-function given by

\[
\sigma(u) = \sqrt[\pi \sqrt{(e_1 - e_2)(e_1 - e_3)(e_2 - e_3)}} \vartheta_1 \left( \frac{u}{2\omega} \right) \exp \left\{ \frac{\eta u^2}{2\omega} \right\}, \quad e^8 = 1,
\]

where \(\vartheta_1\) is the standard \(\theta\)-function.
The fundamental $\sigma$-function introduced by the formula (3.28) respects the following properties:

- It is an entire function on $\text{Jac}(X_g)$.
- It satisfies the two sets of functional equations

$$
\sigma(u + 2\omega k + 2\omega' k; M) = e^{2\pi i (k + k')^{T} (u + \omega k + \omega' k)} \sigma(u; M),
\sigma(u; (\gamma M^{T})^{\gamma}) = \sigma(u; M),
$$

(3.31)

where $\gamma \in \text{Sp}(2g, \mathbb{Z})$, that is, $\gamma^{T} \gamma = I$, and $M^{T}$ is the matrix $M$ with interchanged submatrices $\omega'$ and $\eta$. The first of these equations displays the periodicity property, and the second one the modular property.

- In the vicinity of the origin the power series of $\sigma(u)$ is of the form

$$
\sigma(u) = S_{n}(u) + \text{higher order terms},
$$

(3.32)

where $S_{n}(u)$ are the Schur–Weierstraß functions whose definition we will recall in the next subsection.

### 3.4. Schur–Weierstraß functions

The Schur function is a polynomial in the variables $u_1, u_2, \ldots$ built via a partition $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of weight $|\lambda| = \sum_{i=1}^{n} \lambda_i$. Any partition $\lambda$ can be written in the Frobenius notation $\lambda = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_s)$, where the number $r$ is the rank of the partition and the integers $(\alpha_j, \beta_i)$ are the numbers of nodes in the Young diagram to the left from the $j$th diagonal node and down to it (see [39]).

In what follows we will deal with the special kind of Schur functions that are related to the hyperelliptic curve and that we will call Schur–Weierstraß functions $S_{n}(u)$ following [40]. The associated partition $\pi$ is defined by the Weierstraß gap sequence $w = (w_1, \ldots, w_{2g})$ at the infinite branch point, $w = (1, 3, \ldots, 2g - 1)$, by the formula

$$
\pi_i = \omega_{k-i+1} + k - g, \quad i = 1, \ldots, g.
$$

(3.33)

In the case considered the associated Young diagrams are therefore symmetric and satisfy the constraint $\pi_k - \pi_{k+1} = 1$, i.e., in the cases $g = 5$ and $g = 6$ we have the diagrams

and

corresponding to a partition of rank 3. In general, for arbitrary $g$ the rank of the partition is the integer part $[\frac{g+1}{2}]$.

For the polynomials $S_{n}(u)$ the following representation is valid:

$$
S_{n} = \det (c_{\pi_{j+i}})_{1 \leq i, j \leq g}
$$

(3.34)

where $c_{k}$ is given by the determinant

$$
c_{k} = \frac{1}{k!} \begin{vmatrix}
1 & 1 & 0 & \cdots & 0 \\
p_1 & p_2 & p_1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
p_{k-1} & p_{k-2} & p_{k-3} & \cdots & k-1 \\
p_k & p_{k-1} & p_{k-2} & \cdots & p_1
\end{vmatrix}
$$

(3.35)

and the quantities $p_k$ are related to the Jacobian variables $(u_1, \ldots, u_{2g})$ as

$$
p_k = k u_{g-[k/2]}, \quad k = 1, \ldots, 2g.
$$

(3.36)

For example,

$$
g = 1 : \quad S_{1}(u_1) = u_1,
$$

(3.37)

$$
g = 2 : \quad S_{2,1}(u_1, u_2) = \frac{1}{3} u_3^2 - u_1,
$$

(3.38)

$$
g = 3 : \quad S_{3,2,1}(u_1, u_2, u_3) = \frac{1}{45} u_3^6 - \frac{1}{3} u_2 u_3^3 - u_2^2 + u_1 u_3,
$$

(3.39)

$$
g = 4 : \quad S_{4,3,2,1}(u_1, u_2, u_3, u_4) = \frac{1}{4725} u_3^{10} - \frac{1}{105} u_2^2 u_3^4 + \frac{1}{15} u_2 u_4^2 - u_4 u_3^3 - \frac{1}{3} u_3^2 u_1 + u_2 u_3 u_4^2 - u_2^2 + u_1 u_3.
$$

(3.40)
3.5. Kleinian \( \wp \)-functions

The \( \wp \)-functions are a natural generalization of the corresponding Weierstraß functions and given as logarithmic derivatives of \( \sigma \):

\[
\wp_j(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(u), \quad \wp_{ijk}(u) = -\frac{\partial^3}{\partial u_i \partial u_j \partial u_k} \ln \sigma(u), \quad \text{etc.,}
\]

(3.41)

where \( i, j, k \in \{1, \ldots, g\} \). In this notation the Weierstraß \( \wp \)-function is \( \wp_{11}(u) \). For convenience, we will also denote the derivatives of the \( \sigma \)-function by

\[
\sigma_i(u) = \frac{\partial}{\partial u_i} \sigma(u), \quad \sigma_{ij}(u) = \frac{\partial^2}{\partial u_i \partial u_j} \sigma(u), \quad \text{etc.}
\]

(3.42)

The Jacobi inversion problem is the problem of the inversion of the Abel map and can be formulated as follows: for an arbitrary vector \( u \in \text{Jac}(X_g) \) find the symmetric functions of \( g \) points \( P_1, \ldots, P_g \in X_g \) from Eq. (3.14), that is

\[
u = \sum_{k=1}^{g} \int_{P_0}^{P_k} du.
\]

In the case of a hyperelliptic curve considered, Jacobi's inversion problem is written in coordinate notation as

\[
\begin{align*}
\int_{P_0}^{P_1} \frac{dz}{w} + \cdots + \int_{P_0}^{P_g} \frac{dz}{w} &= u_1, \\
\int_{P_0}^{P_1} \frac{dz}{w} + \cdots + \int_{P_0}^{P_g} \frac{dz}{w} &= u_2, \\
&\vdots \\
\int_{P_0}^{P_1} \frac{z^{\ell-1}dz}{w} + \cdots + \int_{P_0}^{P_g} \frac{z^{\ell-1}dz}{w} &= u_\ell,
\end{align*}
\]

(3.43)

where \( P_k = (z_k, w_k) \). It can be solved in terms of Kleinian \( \wp \)-functions, i.e., \( z_1, \ldots, z_g \) are given by the \( g \) solutions of

\[
z^g - \wp_{gg}(u)z^g - \cdots - \wp_{g1}(u) = 0,
\]

and \( w_k = -\wp_{ggk}(u)z^g_k - \cdots - \wp_{g1k}(u), \quad k = 1, \ldots, g \).

(3.44)

Other expressions for the symmetric functions \( \sum_{k=1}^{g} \chi_k \), with \( l = 1, 2, 3, \ldots \), are given in [41].

Among the various differential relations between the Kleinian \( \wp \)-functions we quote the representation of the Jacobi variety as an algebraic variety in \( \mathbb{C}^g + \mathbb{Z}_{2g} \), obtained in [17] and used in the foregoing development. It is described using the set of cubic relations that can also be represented as minors of a certain matrix [17],

\[
\begin{align*}
\wp_{ggk} &= 4\wp_{gg} \wp_{gk} - 2(\wp_{gg} \wp_{k-1} + \wp_{gk} \wp_{g-1} + 4(\wp_{gk} \wp_{g-1} + \wp_{gk} \wp_{g-1} + 2) \wp_{gk} \wp_{g-1} + \lambda_{2g-1} \wp_{gk} \wp_{g-1} + 2(\delta_{gk} \wp_{gk} + 2 \wp_{gk} \wp_{g-1}) \\
&+ 2(\delta_{gk} \wp_{gk} + 2 \wp_{gk} \wp_{g-1} + \lambda_{2g-1} \delta_{gk} \wp_{gk} + 2 \delta_{gk} \wp_{gk}) \quad 1 \leq i, k \leq g.
\end{align*}
\]

(3.45)

For \( g = 1 \), Eq. (3.45) reduces to the Weierstraß cubic \( \wp^2 = 4\wp^3 - g_2 \wp - g_3 \) if we set \( \lambda_2 = 0, \wp_{111} = \wp', \wp_{111} = \wp \).

3.6. Period matrices

The construction of the fundamental \( \sigma \)-functions requires the knowledge of the first period matrix \( (2\omega, 2\omega') \) and the second period matrix \( (2\eta, 2\eta') \) which satisfy the generalized Legendre relation (3.5). We will show that the second period matrix can be constructed in terms of the first period matrix and even \( \theta \)-constants.

For that purpose we consider any even nonsingular half-period of the \( \theta \)-function \( (2\omega)^{-1} \mathcal{A}_{2\omega} + \mathcal{K}_\infty \) where the half-period

\[
\mathcal{A}_{2\omega} = \int_{-\infty}^{z_1} du + \cdots + \int_{-\infty}^{z_g} du
\]

(3.46)

corresponds to the partition of the branch points

\[
\{i_1, \ldots, i_g, 2g + 2\} \cup \{j_1, \ldots, j_{g+1}\} = \mathcal{J}_0 \cup \mathcal{J}_0.
\]

(3.47)
Then from (3.44) with \( z_k = \epsilon_k, u = \mathcal{A}_{I_0} \), there follows an expression for \( \varphi_{\mathcal{A}_{I_0}} \) in terms of symmetric functions of the elements \( e_i, \ i \in I_0 \):

\[
e_1 + \cdots + e_g = \varphi_{\mathcal{A}_{I_0}},
\]

\[
e_1e_2 + \cdots + e_{g-1}e_g = -\varphi_{g-1,g}(\mathcal{A}_{I_0}),
\]

\[
\vdots
\]

\[
e_1 \cdots e_g = (-1)^{g-1} \varphi_{g}(\mathcal{A}_{I_0}).
\]

(3.48)

One can see that all two-index symbols \( \varphi_{ij}(\mathcal{A}_{I_0}) \) can be expressed in terms of symmetric functions of two sets of variables \( e_i, \ i \in I_0 \) and \( e_j, \ j \in I_0 \). That follows from the fundamental cubic relation (3.45). Indeed \( \varphi_{g\mathcal{A}_{I_0}} = 0 \) for all \( i = 1, \ldots, g \) and we get \( \frac{g(g+1)}{2} \) relations to which (3.48) are substituted as well as expressions for the \( \lambda_i \)’s in terms of \( e_i, i = 1, \ldots, 2g + 1 \). These equations can always be solved with respect of the remaining \( \varphi_{ij}(\mathcal{A}_{I_0}) \). Therefore, all \( \varphi_{ij}(\mathcal{A}_{I_0}) \) are known in terms of the branch points \( \epsilon_i \).

**Proposition 3.1.** Let \( \mathcal{A}_{I_0} + 2\omega K_\infty \) be an arbitrary even nonsingular half-period corresponding to the \( g \) branch points of the set of indices \( I_0 = \{i_1, \ldots, i_g\} \). We define the symmetric \( g \times g \) matrices

\[
\mathcal{P}(\mathcal{A}_{I_0}) := (\varphi_{ij}(\mathcal{A}_{I_0}))_{i,j=1,\ldots,g}
\]

(3.49) and

\[
\mathcal{T}(\mathcal{A}_{I_0}) := \left( -\frac{\partial^2}{\partial z_i \partial z_j} \log \theta(K_\infty)(z; \tau)|_{z=(2\omega)^{-1}\mathcal{A}_{I_0}} \right)_{i,j=1,\ldots,g}.
\]

(3.50)

Then the \( \kappa \)-matrix is given by

\[
\kappa = -\frac{1}{2} \mathcal{P}(\mathcal{A}_{I_0}) - \frac{1}{2} ((2\omega)^{-1})^T \mathcal{T}(\mathcal{A}_{I_0})(2\omega)^{-1}
\]

(3.51)

and the half-periods \( \eta \) and \( \eta' \) of the meromorphic differentials can be represented as

\[
\eta = 2\kappa \omega, \quad \eta' = 2\kappa \omega' - \frac{i\pi}{2} (\omega^{-1})^T.
\]

(3.52)

We remark that (3.51) represents the natural generalization of the Weierstraß formulæ

\[
2\eta \omega = -2e_1\omega^2 - \frac{1}{2} \theta''(0) \frac{1}{\theta'(0)}; \quad 2\eta \omega = -2e_2\omega^2 - \frac{1}{2} \theta''(0) \frac{1}{\theta'(0)}; \quad 2\eta \omega = -2e_3\omega^2 - \frac{1}{2} \theta''(0) \frac{1}{\theta'(0)};
\]

(3.53)

see e.g. the Weierstraß–Schwarz lectures, [42] p. 44. Therefore Proposition 3.1 allows the reduction of the variety of moduli necessary for the calculation of the \( \sigma \)- and \( \varphi \)-functions to the first period matrix.

4. Inversion of one hyperelliptic integral

Classically it is known that only symmetric functions of \( g \) points of algebraic curves of genus \( g \) can be presented as single-valued functions over the Jacobi variety of the curve. Nevertheless, one hyperelliptic integral can also be inverted analytically in terms of \( \sigma \)-functions restricted to the \( \theta \)-divisor.

4.1. Stratification of the \( \theta \)-divisor

The \( \theta \)-divisor \( \tilde{\Theta} \) is defined as the subset of \( \tilde{\text{Jac}}(X_g) \) that nullifies \( \theta \) and, therefore, the \( \sigma \)-function, i.e.

\[
\tilde{\Theta} = \{ \nu \in \tilde{\text{Jac}}(X_g) \mid \theta(\nu) \equiv 0 \}.
\]

(4.1)

The subset \( \tilde{\Theta}_k \subset \tilde{\Theta}, 0 \leq k < g \), is called the \( k \)th stratum if each point \( \nu \in \tilde{\Theta} \) admits a parametrization

\[
\tilde{\Theta}_k := \left\{ \nu \in \tilde{\Theta} \mid \nu = \sum_{j=1}^{k} \int_{P_j} \omega + K_\infty \right\},
\]

(4.2)

where \( \tilde{\Theta}_0 = \{K_\infty\} \) and \( \tilde{\Theta}_{k-1} = \tilde{\Theta} \). We furthermore define \( \tilde{\Theta}_g = \tilde{\text{Jac}}(X_g) \). The following natural embedding is valid:

\[
\tilde{\Theta}_0 \subset \tilde{\Theta}_1 \subset \cdots \subset \tilde{\Theta}_{k-1} \subset \tilde{\Theta}_g = \tilde{\text{Jac}}(X_g).
\]

(4.3)
Table 1

Orders $m(\tilde{\sigma}_k)$ of zeros $\theta(\tilde{\sigma}_k + \psi)$ at $\psi = 0$ on the strata $\tilde{\sigma}_k$.

| $g$ | $m(\tilde{\sigma}_0)$ | $m(\tilde{\sigma}_1)$ | $m(\tilde{\sigma}_2)$ | $m(\tilde{\sigma}_3)$ | $m(\tilde{\sigma}_4)$ | $m(\tilde{\sigma}_5)$ | $m(\tilde{\sigma}_6)$ |
|-----|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 1   | 1                    | 1                    | 1                    | 1                    | 1                    | 0                    | 0                    |
| 2   | 1                    | 1                    | 1                    | 2                    | 0                    | 0                    | 0                    |
| 3   | 2                    | 2                    | 2                    | 3                    | 0                    | 0                    | 0                    |
| 4   | 3                    | 3                    | 2                    | 4                    | 1                    | 1                    | 0                    |
| 5   | 3                    | 3                    | 2                    | 4                    | 1                    | 1                    | 0                    |
| 6   | 3                    | 3                    | 2                    | 5                    | 1                    | 1                    | 0                    |

We define the $\theta$-function to be vanishing to the order $m(\tilde{\sigma}_k)$ along the stratum $\tilde{\sigma}_k$ if for all sets $\alpha_j, j = 1, \ldots, g$ with $0 \leq \alpha_1 + \cdots + \alpha_g < m$,

$$\frac{\partial^{\alpha_1 + \cdots + \alpha_g}}{\partial u_1^{\alpha_1} \cdots \partial u_g^{\alpha_g}} \theta(\psi) \equiv 0, \quad \forall \psi \in \tilde{\sigma}_k,$$

(4.4)

while there is a certain set of $\alpha_j$, with $\alpha_1 + \cdots + \alpha_g = m$, such that (4.4) does not hold. The orders $m(\tilde{\sigma}_k)$ of the vanishing of $\theta(\tilde{\sigma}_k + \psi)$ along the stratum $\tilde{\sigma}_k$ for the first genera are given in Table 1.

In the following we focus on the stratum $\theta_1$ corresponding to the variety $\theta_1 \subset Jac(X_\infty)$, which is the image of the curve inside the Jacobian,

$$\theta_1 = \left\{ u \in \Theta \left| u = \int_{-\infty}^{\psi_0} du + K_\infty \right. \right\}, \quad \text{and} \quad \tilde{\theta}_k = (2\omega)^{-1} \theta_k + K_\infty, \quad k = 0, \ldots, g.$$

(4.5)

We remark that another stratification was introduced in [41] for hyperelliptic curves of even order with two infinite points $\infty_1$ and $\infty_2$ that was implemented for studying the poles of functions on Jacobians of these curves. The same problem relevant to strata of the $\theta$-divisor was studied in [10].

4.2. Inversion formulae

For the case of genus 2 the inversion of a holomorphic hyperelliptic integral by the method of restriction to the $\theta$-divisor was obtained independently by Grant [6] and Jorgenson [7] in the form

$$x = -\frac{\sigma_1(u)}{\sigma_2(u)} \sigma(u) = 0, \quad u = (u_1, u_2)^T.$$

(4.6)

This result was implemented in [21], and explicitly worked out in the series of publications [27,28,22,29,30], and others.

The case of genus 3 was studied by Onishi [8], where the inversion formula is given in the form

$$x = -\frac{\sigma_{12}(u)}{\sigma_{23}(u)} \sigma(u) = \sigma_3(u) = 0, \quad u = (u_1, u_2, u_3)^T.$$

(4.7)

Formula (4.7) is based on the detailed analysis of the genus 3 KdV hierarchy and its restriction to the $\theta$-divisor. Below we will present the generalization of (4.6) and (4.7) to higher genera. For doing so we first analyze the Schur–Weierstraß polynomials that represent the first term of the expansion of $\sigma(u)$ in the vicinity of the origin $u \sim 0$.

4.3. Schur–Weierstraß polynomials on $\Theta_1$-strata

The Schur–Weierstraß polynomials associated with the curve $X_\infty$ are defined in the space $C^g \supset (u_1, \ldots, u_g)$ by the Weierstraß gap sequence at the infinite branch point that for $g > 1$ is always a Weierstraß point. The $\theta$-divisor $\Theta$ and its strata $\tilde{\sigma}_k$ in the vicinity of the origin $u \sim 0$ are given as polynomials in $u$. An analysis of the Schur–Weierstraß polynomials leads to the following:

**Proposition 4.1.** The following statements are valid for the Schur–Weierstraß polynomials $S_\pi(u)$ associated with a partition $\pi$:

1. In the vicinity of the origin, an element $u$ of the first stratum $\Theta_1 \subset \Theta$ is singled out by

$$S_\pi(u) = 0, \quad \frac{\partial^j}{\partial u_g^j} S_\pi(u) = 0 \quad \forall j = 1, \ldots, g - 2.$$

(4.8)

2. The derivatives fulfill

$$\frac{\partial^j}{\partial u_g^j} S_\pi(u) \begin{cases} 
= 0 & \text{if } 1 \leq j < \frac{g(g - 1)}{2} \\
\neq 0 & \text{if } j \geq \frac{g(g - 1)}{2}
\end{cases} \quad \text{with } u \in \Theta_1.$$

(4.9)
(3) The following equalities are valid for $u \in \Theta_1$:
\[
x = -\frac{1}{U_g^2} = -\frac{\partial^M \partial_{u_1 u_2} S_\pi(u)}{\partial^M \partial_{u_2 u_2} S_\pi(u)} = -\frac{\partial^{M+1} \partial_{u_1 u_2} S_\pi(u)}{\partial^{M+1} \partial_{u_2 u_2} S_\pi(u)},
\]
where $M = \frac{1}{2}(g - 2)(g - 3) + 1$.

(4) The order of vanishing of $S_\pi$ restricted to $\Theta_1$ is the rank of the partition $\pi$.

It was noted in [40] that the Schur–Weierstraß polynomials respect all statements of the Riemann singularity theorem. In particular, if
\[
Z = \left(\left(\frac{2g-1}{2g-1}, \cdots, \frac{2k-1}{2k-1}, \cdots, \frac{3}{3}, z\right)\right)
\]
and if $\pi$ is the partition at the infinite Weierstraß point of the hyperelliptic curve $X_g$ of genus $g$, then the function
\[
G(z) := S_\pi(Z - u)
\]
either has $g$ zeros or vanishes identically. This result was extended in [43,20]. Moreover we will conjecture here that the properties of the Schur–Weierstraß polynomials given in Proposition 4.1 can be “lifted” to the fundamental $\sigma$-function (3.28).

### 4.4. Inversion for higher genera

The above analysis permits us to conjecture the following inversion formula for the general case of hyperelliptic curves of genus $g > 2$:
\[
\left.x \equiv -\frac{\partial^{M+1} \partial_{u_1 u_2} \sigma(u)}{\partial^{M+1} \partial_{u_2 u_2} \sigma(u)} \right|_{u \in \Theta_1}, \quad M = \frac{(g - 2)(g - 3)}{2} + 1
\]
and
\[
\Theta_1 = \left\{ u \in \text{Jac}(X_g) \left| \sigma(u) = 0, \quad \frac{\partial^j}{\partial u_g^j} \sigma(u) = 0, \quad \forall j = 1, \ldots, g - 2 \right. \right\}.
\]

The analog of this formula for strata $\Theta_k$, $1 < k < g$ and $(n, s)$-curves in the terminology of [40] was recently considered by Matsutani and Previato [43,20].

We remark that the half-periods associated with one of the branch points $e_1, \ldots, e_{2g-1}$ are elements of the first stratum and, therefore, the following proposition is valid:

**Proposition 4.2.** Let the curve $X_g$ be of genus $g > 2$ and let $\mathfrak{A}_i$ be the half-period that is the Abelian image with the base point $P_0 = (\infty, \infty)$ of a branch point $e_i$. Then
\[
e_i = -\frac{\partial^{M+1} \partial_{u_1 u_2} \sigma(\mathfrak{A}_i)}{\partial^{M+1} \partial_{u_2 u_2} \sigma(\mathfrak{A}_i)}, \quad M = \frac{1}{2}(g - 2)(g - 3) + 1.
\]
The formula (4.15) can be considered as an equivalent of the Thomae formulae [44]. Similar formulae can be written for other strata $\Theta_k$, but we only mention here the set of formulae that we are using in our construction.

**Proposition 4.3.** Let $X_g$ be a hyperelliptic curve of genus $g$ and consider a partition
\[
\mathcal{I}_1 \cup \mathcal{J}_1 = \{i_1, \ldots, i_{g-1}\} \cup \{j_1, \ldots, j_{g+3}\}
\]
of branch points such that the half-periods
\[
(2\omega)^{-1} \mathfrak{A}_{\mathcal{I}_1} + K_\infty \in \tilde{\Theta}_{g-1} \cup \tilde{\Theta}_{g-2}
\]
are nonsingular odd half-periods. Consider two cases:
\[
\mathcal{I}'_1 := \{i_1, \ldots, i_{g-1}\} \nless \geq 2g + 2,
\]
\[
\mathcal{I}'_1 := \{i_1, \ldots, i_{g-1}\} \nless \geq 2g + 2, \quad \text{i.e. } i_{g-1} = 2g + 2.
\]
Denote by $s_k(I_i')$ and $s_k(I_i'')$ the elementary symmetric function of order $k$ built by the branch points $e_{i_1}, \ldots, e_{i_{g-1}}$ and $e_{i_1}, \ldots, e_{i_{g-2}}$ respectively. Then the following formulae are valid:

$$
\begin{align*}
s_k(I_i') &= (-1)^k \frac{\sigma_{g-k}}{\sigma_g} (\mathfrak{A}_{I_i'}), \quad k = 1, \ldots, g - 1, \\
s_{g-1}(I_i'') &= (-1)^k \frac{\sigma_{g-k}}{\sigma_{g-1}} (\mathfrak{A}_{I_i''}), \quad k = 2, \ldots, g - 1,
\end{align*}
$$

(4.20)

where we used for typographic convenience the notation $\frac{s_k(\mathfrak{A})}{\sigma(\mathfrak{A})} =: \frac{\mathfrak{A}}{\sigma(\mathfrak{A})}$.

The following corollary follows immediately from (4.20):

**Corollary 4.4.** Let $g > 3$ and $\kappa' = \kappa'' \cup \{i\}$ where $\kappa'' = \{i_1, \ldots, i_{g-2}\}$ and $i \neq 2g + 2, i \notin \kappa''$. Then the representation for the branch points $e_i$

$$
e_i = -\frac{\sigma_{g-1}}{\sigma_g} (\mathfrak{A}_{\kappa''}) + \frac{\sigma_{g-2}}{\sigma_{g-1}} (\mathfrak{A}_{\kappa''})
$$

(4.21)

is valid.

The formulae (4.21) can be understood as a generalization of those given in Bolza [45]. We also remark that a comparison of the two representations (4.15) and (4.21) of the branch point $e_i$ leads to an interesting $\theta$-constant relation.

4.5. Calculation of moduli

Calculations in terms of $\theta$- or $\sigma$-functions are usually considered as technically cumbersome which prevents wide applications of algebro-geometric methods. In particular, the procedure of the evaluation of the period matrix in the given homology basis is technically complicated even in the case of a hyperelliptic curve. On the basis of the above analysis we show that modern software like the “Maple/algcurves” package now allows the calculation of the $\theta$- or $\sigma$-functions without drawing and even without knowledge of the homology basis, at least in the hyperelliptic case. The calculation scheme is given by the following steps:

**Step 1.** For the given curve, compute first the period matrices $(2\omega, 2\omega')$ and $\tau = \omega^{-1}\omega'$ by means of the “Maple/algcurves” code. Compute then the winding vectors, i.e., the columns of the inverse matrix

$$
(2\omega)^{-1} = (U_1, \ldots, U_g).
$$

(4.22)

**Step 2.** We then find all nonsingular odd characteristics by direct computation of all odd $\theta$-constants. According to Table 1 we have two sets $B_1 \subset \tilde{\Theta}_{g-1}$ and $B_2 \subset \tilde{\Theta}_{g-2}$ of nonsingular odd half-periods. For each element of $b_1 \in B_1$ there are $e_{i_1}, \ldots, e_{i_{g-1}} \neq \infty$ such that

$$
b_1 = \int_{\infty}^{e_{i_1}} dv + \cdots + \int_{\infty}^{e_{i_{g-1}}} dv + K_\infty \in \tilde{\Theta}_{g-1}
$$

(4.23)

and for each element of $b_2 \in B_2$ there are $e_{i_1}, \ldots, e_{i_{g-2}} \neq \infty$ such that

$$
b_2 = \int_{\infty}^{e_{i_1}} dv + \cdots + \int_{\infty}^{e_{i_{g-2}}} dv + K_\infty \in \tilde{\Theta}_{g-2}.
$$

(4.24)

By using (4.20) and the known values of the winding vectors one can find the correspondence between the sets $\{e_{i_1}, \ldots, e_{i_{g-1}}\}$ and $\{e_{i_1}, \ldots, e_{i_{g-2}}\}$ of branch points and the nonsingular odd characteristics $\{(2\omega)^{-1} (\mathfrak{A}_{i_1, \ldots, i_{g-1}}) + K_\infty\}$ and $\{(2\omega)^{-1} (\mathfrak{A}_{i_1, \ldots, i_{g-2}}) + K_\infty\}$. Then one can add these characteristics and find the one-to-one correspondence

$$
\int_{\infty}^{e_{i_{g-1}}} dv = [\mathfrak{A}_{i_{g-1}}], \quad i = 1, \ldots, 2g + 2.
$$

(4.25)

**Step 3.** Among the $2g + 2$ characteristics (4.25) there should be precisely $g$ odd and $g + 2$ even characteristics. The sum of all odd characteristics gives the vector of Riemann constants with the base point at infinity. Check that this characteristic is of order $\frac{g+1}{2}$.

**Step 4.** Calculate the symmetric matrix $\chi$ by using (3.51) and then the second period matrices $2\eta, 2\eta'$ according to Proposotion 3.1.

We add two remarks:

**Remark 1.** The proposed way of computing the $[\mathfrak{A}_i]$ is not the only possible way. One can also use the standard Thomae formulae for the $\theta$-constants with even characteristics while we used Bolza-type formulae for odd characteristics. Also “Maple/algcurves” contains expressions for homology cycles written in terms of paths connecting branch points and it could be possible in principle to find the $[\mathfrak{A}_i]$ by solving the system of equations.
Remark 2. One could imagine a case where the drawing of the homology basis is nevertheless important for the problem, e.g., in the case when the curve possesses additional symmetry. In this case one can use Northower’s program [46,47] to find the symplectic transformation between the required basis and the one given by “Maple/algcurves”.

4.6. The procedure for the inversion

We are now in the position to calculate the inversion, that is \( x(t) \), of the hyperelliptic integral

\[
\int_{\infty}^{x} \frac{z^k dz}{\sqrt{P_{2g+1}(z)}} = t, \quad 0 \leq k < g. \tag{4.26}
\]

This will be carried out through the following steps.

**Step 1.** We first fix the homology basis, e.g., the basis of “Maple/algcurves”, and compute all moduli of the curve according to Section 4.5.

**Step 2.** The dynamic system considered is evaluated between two branch points of the polynomial \( P_{2g+1}(z) \) that defines the curve (3.1). Therefore fix a branch point, say \( e_i \), that is the starting point of the system evolution, and find the half-period \( \mathfrak{A}_i = \int_{e_i}^{u} du \).

**Step 3.** Use the formula (4.15) for \( g > 2 \) or (4.6) for \( g = 2 \) with the argument \( u \in \Theta_1 \) of the \( \sigma \)-function given as

\[
u = \mathfrak{A}_i + \begin{pmatrix} f_1(t) \\ \vdots \\ f_{k-1}(t) \\ t \\ f_{k+1}(t) \\ \vdots \\ f_g(t) \end{pmatrix} \quad \text{with} \quad f(0) = 0, \tag{4.27}
\]

where \( f(t) = (f_1(t), \ldots, f_g(t))^T \) are locally given functions that resolve the conditions (4.14) of the restriction to the stratum \( \Theta_1 \). The vector function \( f(t) \) can be obtained from \( f(0) \) using the Newton method. In this case the approximation process should be carried through for the real and imaginary parts of each function \( f_i(t) \) separately.

We emphasize that the inversion procedure given above carries local character. But we believe that our method elucidates the geometric structure of the object and leads to exact calculations in contrast to numerically solving ordinary differential equations.

Below we will consider the case \( g = 2 \) and \( g = 3 \) and demonstrate in more detail how this scheme can be applied.

5. A hyperelliptic curve of genus 2

We consider a hyperelliptic curve \( X_2 \) of genus 2

\[
w^2 = 4(z - e_1)(z - e_2)(z - e_3)(z - e_4)(z - e_5) = 4z^5 + \lambda_4z^4 + \lambda_3z^3 + \lambda_2z^2 + \lambda_1z + \lambda_0. \tag{5.1}
\]

From (3.7) and (3.8), the basic holomorphic and meromorphic differentials are

\[
\begin{align*}
\frac{du_1}{w} &= \frac{dz}{w}, \\
\frac{dr_1}{w} &= \frac{12z^3 + 2\lambda_4z^2 + \lambda_3z}{4w} dz,
\end{align*}
\tag{5.2}
\]

\[
\begin{align*}
\frac{du_2}{w} &= \frac{zdz}{w}, \\
\frac{dr_2}{w} &= \frac{z^2}{w} dz.
\end{align*}
\tag{5.3}
\]

Then the Jacobi inversion problem for the equations

\[
\begin{align*}
\int_{\infty}^{(z_1, w_1)} \frac{dz}{w} + \int_{\infty}^{(z_2, w_2)} \frac{dz}{w} &= u_1, \\
\int_{\infty}^{(z_1, w_1)} \frac{zdz}{w} + \int_{\infty}^{(z_2, w_2)} \frac{zdz}{w} &= u_2
\end{align*}
\tag{5.4}
\]

is solved in the form

\[
\begin{align*}
z_1 + z_2 &= \varphi_{22}(u), \\
z_1z_2 &= -\varphi_{12}(u),
\end{align*}
\tag{5.5}
\]

\[
w_k = -\varphi_{22}(u)z_k - \varphi_{12}(u), \quad k = 1, 2.
\]
5.1. Characteristics in genus 2

The homology basis of the curve is fixed by defining the set of half-periods corresponding to the branch points. The characteristics of the Abelian images of the branch points are defined as

\[ [\mathcal{A}_i] = \left[ \int_0^{\xi_i} d\mathcal{U} \right] = \left( \begin{array}{c} \xi_{i,1}' \\ \xi_{i,1} \end{array} \right), \]

which can also be written as

\[ \mathcal{A}_i = 2\omega_1 + 2\omega_1' \xi_i, \quad i = 1, \ldots, 6. \]

In the homology basis given in Fig. 2 we have

\[ [\mathcal{A}_1] = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \quad [\mathcal{A}_2] = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array} \right), \quad [\mathcal{A}_3] = \frac{1}{2} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \]

\[ [\mathcal{A}_4] = \frac{1}{2} \left( \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right), \quad [\mathcal{A}_5] = \frac{1}{2} \left( \begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array} \right), \quad [\mathcal{A}_6] = \frac{1}{2} \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right). \]

The characteristic of the vector of Riemann constants \( K_\infty \) is

\[ [K_\infty] = [\mathcal{A}_2] + [\mathcal{A}_4] = \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right). \]

From the above characteristics, 16 half-periods can be built as follows. Denote the 10 half-periods for \( 1 \leq i < j \leq 5 \) that are images of two branch points as

\[ \Omega_{ij} = 2\omega(e_i + e_j) + 2\omega'(e_i' + e_j'). \]

Then the characteristics of the 6 half-periods

\[ \left[ (2\omega)^{-1} \mathcal{A}_i + K_\infty \right] = : \delta_i, \quad i = 1, \ldots, 6 \]

are nonsingular and odd, whereas the characteristics of the 10 half-periods

\[ \left[ (2\omega)^{-1} \Omega_{ij} + K_\infty \right] = : \epsilon_{ij}, \quad 1 \leq i < j \leq 5 \]

are nonsingular and even.

Odd characteristics correspond to partitions \( \{6\} \cup \{1, \ldots, 5\} \) and \( \{k\} \cup \{i_1, \ldots, i_4, 6\} \) for \( i_1, \ldots, i_4 \neq k \). The first partition from these two corresponds to \( \Theta_0 \) and the second to \( \Theta_1 \).

From the solution of the Jacobi inversion problem we obtain for any \( i, j = 1, \ldots, 5, i \neq j \),

\[ \epsilon_i + \epsilon_j = \varphi_{22}(\Omega_{ij}), \quad -\epsilon_i \epsilon_j = \varphi_{12}(\Omega_{ij}). \]

Using the relation (see [48,17])

\[ \varphi_{22}^2 = 4\varphi_{12}^3 + 4\varphi_{12}\varphi_{22} + 4\varphi_{11} + \lambda_3 \varphi_{22} + \lambda_4 \varphi_{22}^2 + \lambda_2 \]

one can also find

\[ \epsilon_i \epsilon_j (e_p + e_q + \epsilon_i) + e_p e_q \epsilon_i = \varphi_{11}(\Omega_{ij}), \]

where \( i, j, p, q, \) and \( r \) are mutually different.

From (5.12) and (5.14) we obtain an expression for the matrix \( \kappa \) that is useful for numeric calculations because it reduces the second period matrix to an expression in the first period matrix and \( \theta \)-derivatives, namely, in the case \( e_i = e_1, e_j = e_2 \),

\[ \kappa = -\frac{1}{2} \left( \begin{array}{cc} e_1 e_2 (e_3 + e_4 + e_5) + e_3 e_4 e_5 - e_1 e_2 \\ -e_1 e_2 & e_1 + e_2 \end{array} \right) - \frac{1}{2} (2\omega)^{-1} \mathcal{I}(\Omega_{1,2})(2\omega)^{-1}, \]

where \( \mathcal{I} \) is the \( 2 \times 2 \) matrix defined in Proposition 3.1.
5.2. Inversion of a holomorphic integral

Taking the limit $z_2 \to \infty$ in the Jacobi inversion problem (5.4) we obtain

$$\int_{\infty}^{(2, w)} \frac{dz}{w} = u_1, \quad \int_{\infty}^{(2, w)} \frac{zdz}{w} = u_2. \tag{5.16}$$

The same limit in the ratio

$$\frac{\varphi_{12}(u)}{\varphi_{22}(u)} = \frac{-z_1z_2}{z_1 + z_2} \tag{5.17}$$

leads to the Grant–Jorgenson formula (4.6). In terms of $\theta$-functions this can be given the form

$$z = \frac{\partial u}{\partial \theta(K)} \left( (2\omega)^{-1} u; \tau \right) \bigg|_{\theta(2\omega)^{-1} u; \tau} = 0, \tag{5.18}$$

where here and below $\partial u = \sum_{j=1}^{g} U_j \frac{\partial}{\partial \tau_j}$ is the derivative along the direction $U$. Here we introduced the “winding vectors” $U, V$ as column vectors of the inverse matrix

$$(2\omega)^{-1} = (U, V). \tag{5.19}$$

From (4.6) we obtain for all finite branch points

$$e_i = -\frac{\sigma_1(\mathcal{A}_i)}{\sigma_2(\mathcal{A}_i)}, \quad i = 1, \ldots, 5 \tag{5.20}$$

or, equivalently,

$$e_i = -\frac{\partial \theta(\delta_i)}{\partial \theta(\delta_1)}, \quad i = 1, \ldots, 5. \tag{5.21}$$

This formula was mentioned by Bolza [45] (see his Eq. (6)) for the case of genus 2 curve with finite branch points (Fig. 2).

6. A hyperelliptic curve of genus 3

As the next case we consider the hyperelliptic curve $X_3$ of genus 3 with seven real zeros as a model problem in advance of the real physical problems studied in the next section. Let the curve $X_3$ be given by

$$w^2 = 4(z-e_1)(z-e_2)(z-e_3)(z-e_4)(z-e_5)(z-e_6)(z-e_7) = 4z^7 + \lambda_6z^6 + \cdots + \lambda_1z + \lambda_0. \tag{6.1}$$

The complete set of holomorphic and meromorphic differentials with a unique pole at infinity is

$$du_1 = \frac{dz}{w}, \quad dr_1 = z(20z^4 + 4\lambda_6z^3 + 3\lambda_5z^2 + 2\lambda_4z + \lambda_3) \frac{dz}{4w},$$

$$du_2 = \frac{zdz}{w}, \quad dr_2 = z^2(12z^2 + 2\lambda_6z + \lambda_5) \frac{dz}{4w}, \tag{6.2}$$

$$du_3 = \frac{z^2dz}{w}, \quad dr_3 = z^3 \frac{dz}{w}.$$

Again we introduce the winding vectors

$$(2\omega)^{-1} = (U, V, W). \tag{6.3}$$

The Jacobi inversion problem for the equations

$$\int_{\infty}^{z_1} \frac{dz}{w} + \int_{\infty}^{z_2} \frac{dz}{w} + \int_{\infty}^{z_3} \frac{dz}{w} = u_1,$$

$$\int_{\infty}^{z_1} \frac{zdz}{w} + \int_{\infty}^{z_2} \frac{zdz}{w} + \int_{\infty}^{z_3} \frac{zdz}{w} = u_2, \tag{6.4}$$

$$\int_{\infty}^{z_1} \frac{z^2dz}{w} + \int_{\infty}^{z_2} \frac{z^2dz}{w} + \int_{\infty}^{z_3} \frac{z^2dz}{w} = u_3$$

is solved by

$$z_1 + z_2 + z_3 = \varphi_{23}(u), \quad z_1z_2 + z_1z_3 + z_2z_3 = -\varphi_{23}(u), \quad z_1z_2z_3 = \varphi_{13}(u), \quad w_k = -\varphi_{333}(u)z_k^2 - \varphi_{233}(u)z_k - \varphi_{133}(u), \quad k = 1, 2, 3. \tag{6.5}$$
We finally introduce the 35 half-periods that are images of three branch points: a nonsingular and odd as are the characteristics of the 21 half-periods and the $\theta$-function $\theta(K_\infty + \nu)$ vanishes at the origin $\nu = 0$ to the order $m = 2$. The half-periods $X_1 = (2\omega)^{-1}\mathcal{A}_k + K_\infty \in \Theta_1$ correspond to partitions $T_1 \cup T_2 = \emptyset \cup \{1, 2, \ldots, 8\}$ and the $\theta$-function $\theta(X_1 + \nu)$ vanishes at the origin $\nu = 0$ to the order $m = 1$. Also denote the 21 half-periods that are images of two branch points $\Omega_{i,j} = 2\omega(e_i + e_j) + 2\omega'(e_i' + e_j')$, $i, j = 1, \ldots, 7, i \neq j$. The half-periods $\hat{X}_1 = (2\omega)^{-1}\Omega_{i,j} + K_\infty \in \Theta_2$ correspond to the partitions $T_1 \cup T_2 = \{i, j\} \cup \{j_1, \ldots, j_4\}$, $j_1, \ldots, j_4 \notin \{i, j\}$ and the $\theta$-function $\theta(\hat{X}_1 + \nu)$ vanishes at the origin, $\nu = 0$, as before, to the order $m = 1$. Therefore the characteristics of the 7 half-periods

$$[(2\omega)^{-1}\mathcal{A}_i + K_\infty] = \delta_i, \quad i = 1, \ldots, 7$$

are nonsingular and odd as are the characteristics of the 21 half-periods

$$[(2\omega)^{-1}\Omega_{i,j} + K_\infty] = \delta_{ij}, \quad 1 \leq i < j \leq 7.$$ 

We finally introduce the 35 half-periods that are images of three branch points:

$$\Omega_{i,j,k} = 2\omega(e_i + e_j + e_k) + 2\omega'(e_i' + e_j' + e_k') \in \text{Jac}(X_8), \quad 1 \leq i < j < k \leq 7.$$ 

### 6.1. Characteristics in genus 3

Let $\mathcal{A}_k$ be the Abelian image of the $k$th branch point, namely

$$\mathcal{A}_k = \int_{\infty}^{e_k} du = 2\omega e_k + 2\omega' e_k', \quad k = 1, \ldots, 8,$$

where $e_k$ and $e_k'$ are column vectors whose entries $e_{k,j}, e_{k,j}'$ are $\frac{1}{2}$ or zero for all $k = 1, \ldots, 8, j = 1, 2, 3$. The correspondence between branch points and characteristics in the fixed homology basis (see Fig. 3) is given as

$$[\mathcal{A}_1] = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\mathcal{A}_2] = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad [\mathcal{A}_3] = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. $$

The vector of Riemann constant $K_\infty$ with the base point at infinity is given in the above basis by the even singular characteristics

$$[K_\infty] = [\mathcal{A}_2] + [\mathcal{A}_4] + [\mathcal{A}_6] = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

From the above characteristics, 64 half-periods can be built as follows. Start with singular even characteristics; there should be only one such characteristic that corresponds to the vector of Riemann constants $K_\infty$. The corresponding partition reads $T_2 \cup T_3 = \{1, 2, \ldots, 8\}$ and the $\theta$-function $\theta(K_\infty + \nu)$ vanishes at the origin $\nu = 0$ to the order $m = 2$. The half-periods $X_1 = (2\omega)^{-1}\mathcal{A}_k + K_\infty \in \Theta_1$ correspond to partitions $T_1 \cup T_2 = \emptyset \cup \{1, 2, \ldots, 8\}$ and the $\theta$-function $\theta(X_1 + \nu)$ vanishes at the origin $\nu = 0$ to the order $m = 1$. Also denote the 21 half-periods that are images of two branch points

$$\Omega_{i,j} = 2\omega(e_i + e_j) + 2\omega'(e_i' + e_j'), \quad i, j = 1, \ldots, 7, i \neq j.$$ 

The half-periods $X_1 = (2\omega)^{-1}\Omega_{i,j} + K_\infty \in \Theta_2$ correspond to the partitions $T_1 \cup T_2 = \{i, j\} \cup \{j_1, \ldots, j_4\}$ and the $\theta$-function $\theta(X_1 + \nu)$ vanishes at the origin, $\nu = 0$, as before, to the order $m = 1$. Therefore the characteristics of the 7 half-periods

$$[(2\omega)^{-1}\mathcal{A}_i + K_\infty] = \delta_i, \quad i = 1, \ldots, 7$$

are nonsingular and odd as are the characteristics of the 21 half-periods

$$[(2\omega)^{-1}\Omega_{i,j} + K_\infty] = \delta_{ij}, \quad 1 \leq i < j \leq 7.$$ 

We finally introduce the 35 half-periods that are images of three branch points:

$$\Omega_{i,j,k} = 2\omega(e_i + e_j + e_k) + 2\omega'(e_i' + e_j' + e_k') \in \text{Jac}(X_8), \quad 1 \leq i < j < k \leq 7.$$ 

Fig. 3. Homology basis on the Riemann surface of the curve $X_3$ with real branch points $e_1 < e_2 < \ldots < e_8 = \infty$ (upper sheet). The cuts are drawn from $e_{2i-1}$ to $e_{2i}, i = 1, 2, 4$. The $b$-cycles are completed on the lower sheet (dotted lines).
The half-periods \( \widehat{\Delta}_1 = (2\omega)^{-1}\Omega_{i,j,k} + K_\infty \) correspond to the partitions
\[
\mathcal{I}_0 \cup \mathcal{J}_0 = \{i, j, k, 8\} \cup \{j_1, \ldots, j_4\}, \quad j_1, \ldots, j_4 \not\in \{i, j, k, 8\}.
\]
(6.15)

The \( \theta \)-function \( \theta(\widehat{\Delta}_1 + \nu) \) does not vanish at the origin \( \nu = 0 \).
Furthermore, the 35 characteristics
\[
[e_{i,j,k}] = [(2\omega)^{-1}\Omega_{i,j,k} + K_\infty], \quad 1 \leq i < j < k \leq 7
\]
are even and nonsingular while the characteristic \([K_\infty]\) is even and singular. Altogether, we have got all \( 64 = 4^3 \) characteristics classified by the partitions of the branch points.

### 6.2. Inversion of a holomorphic integral

All three holomorphic integrals
\[
\int_\infty^x \frac{dz}{w} = u_1, \quad \int_\infty^x \frac{zdz}{w} = u_2, \quad \int_\infty^x \frac{z^2dz}{w} = u_3
\]
(6.17)
are inverted by the same formula (4.7). Nevertheless, there are three different cases for which one of the variables \( u_1, u_2, u_3 \) is considered as independent while the remaining two result from solving the divisor conditions \( \sigma(u) = \sigma_3(u) = 0 \).

Formula (4.7) can be rewritten in terms of \( \theta \)-functions as
\[
x = -\frac{\partial^2}{\partial \nu_1^2} \theta[K_\infty]((2\omega)^{-1}u) + 2(\partial_u \theta[K_\infty]((2\omega)^{-1}u))e_j^1 x u
\]
\[
\frac{\partial^2}{\partial \nu_2^2} \theta[K_\infty]((2\omega)^{-1}u) + 2(\partial_u \theta[K_\infty]((2\omega)^{-1}u))e_j^1 x u.
\]
(6.18)
where \( e_j = (0, 0, 1)^T \). This represents the solution of the inversion problem.

From the solution of the Jacobi inversion problem it follows that for any \( 1 \leq i < j < k \leq 7 \),
\[
e_i + e_j + e_k = \wp_{33}(\Omega_{i,j,k}), \quad -e_i e_j - e_i e_k - e_j e_k = \wp_{23}(\Omega_{i,j,k}), \quad e_i e_j e_k = \wp_{13}(\Omega_{i,j,k}).
\]
(6.19)

From (3.45) we get the relations
\[
\wp_{33}^2 = 4\wp_{33}^2 + \lambda_5 \wp_{33}^2 + 4\wp_{23} \wp_{33} + \lambda_5 \wp_{33} + 4\wp_{22} \wp_{23} + 4\wp_{13} + \lambda_4,
\]
\[
\wp_{23}^2 = 4\wp_{23} \wp_{33} + \lambda_6 \wp_{23} - 4\wp_{22} \wp_{23} + 8\wp_{13} \wp_{23} + 4\wp_{11} + \lambda_2,
\]
\[
\wp_{13}^2 = 4\wp_{13} \wp_{33} + \lambda_6 \wp_{13} - 4\wp_{12} \wp_{13} + \lambda_0
\]
(6.20)

and also find
\[
\wp_{12}(\Omega_{i,j,k}) = -s_i S_1 - S_4, \quad \wp_{11}(\Omega_{i,j,k}) = s_3 S_2 + s_1 S_4, \quad \wp_{22}(\Omega_{i,j,k}) = s_2 + s_3 + s_2 S_1,
\]
(6.21)
where \( s_i \) are the elementary symmetric functions of order \( l \) of the branch points \( e_i, e_j, e_k \) and \( S_l \) are the elementary symmetric functions of order \( l \) of the remaining branch points \( \{1, \ldots, 7\} \setminus \{i, j, k\} \).

From (6.19) and (6.21) one can find the following expression for the matrix \( x \):
\[
x = -\frac{1}{2} \wp((\Omega_{i,j,k})) - \frac{1}{2} (2\omega)^{-1} \Sigma(\Omega_{i,j,k})(2\omega)^{-1},
\]
(6.22)
where \( i, j, k \) are arbitrary and \( \Sigma(\Omega_{i,j,k}) \) is the \( 3 \times 3 \) matrix defined in Proposition 3.1,
\[
\Sigma(\Omega_{i,j,k}) = \left(-\frac{\partial^2}{\partial z_m \partial \bar{z}_n} \log \theta[K_\infty]((2\omega)^{-1}\Omega_{i,j,k}; \tau)\right)_{m,n=1,2,3}
\]
(6.23)

For the branch points \( e_1, \ldots, e_8 \) the expression
\[
e_i = -\frac{\partial y}{\partial \nu} \left[ \wp_{12}(\nu) \right] \theta[K_\infty]((2\omega)^{-1}\Omega_{i,j}; \tau)
\]
\[
\frac{\partial y}{\partial \nu} \left[ \wp_{12}(\nu) \right] \theta[K_\infty]((2\omega)^{-1}\Omega_{i,j}; \tau)
\]
is valid. Furthermore we have for \( i, j = 1, \ldots, 8, i \neq j \),
\[
e_i + e_j = \frac{\sigma_2(\Omega_{i,j})}{\sigma_3(\Omega_{i,j})} \equiv -\frac{\partial y [\delta_{ij}]}{\partial \nu [\delta_{ij}]},
\]
\[
e_i e_j = \frac{\sigma_1(\Omega_{i,j})}{\sigma_3(\Omega_{i,j})} \equiv \frac{\partial y [\delta_{ij}]}{\partial \nu [\delta_{ij}]},
\]
(6.25)
and for $i = 1, \ldots, 7$,
\[
e_i = -\frac{\sigma_1(\mathcal{A}_i)}{\sigma_2(\mathcal{A}_i)} = -\frac{\partial_\mathcal{U}[\delta_i]}{\partial \varphi[\delta_i]}.
\] (6.26)

Using these data one can reconstruct using the known matrices $2\omega$, $2\omega'$ all the characteristics $[\mathcal{A}_i], j = 1, \ldots, 2g + 2$, as well as the vector of Riemann constants following the procedure given in Section 4.5. For example, for the genus 3 curve
\[
w^2 = 4z(z - 1)(z - 2)(z - 3)(z - 4)(z - 5)(z - 6)
\]
(6.27)
we construct the "Maple/algcurves" homology basis and compute the Riemann period matrices $(2\omega, 2\omega')$. We order the branch points according to
\[
e_1 = 0, \quad e_2 = 1, \quad e_3 = 2, \quad e_4 = 3, \quad e_5 = 4, \quad e_6 = 5, \quad e_7 = 6, \quad e_8 = \infty.
\] (6.28)

Then we find the Abelian images of the branch points
\[
[\mathcal{A}_1] = \begin{pmatrix} 1/2 & 0 & 0 \\ 1 & 1/2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad [\mathcal{A}_2] = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad [\mathcal{A}_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},
\]
\[
[\mathcal{A}_4] = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\mathcal{A}_5] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\mathcal{A}_6] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},
\]
\[
[\mathcal{A}_7] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad [\mathcal{A}_8] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.
\] (6.29)

There are three odd ones among these characteristics and therefore
\[
[K_\infty] = \begin{pmatrix} 1/2 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.
\] (6.30)

One can check that $[K_\infty]$ is the only even characteristic such that $\theta([K_\infty]) = 0$ as follows from Table 1. Now one can construct arbitrary nonsingular even characteristics and calculate $x$ and then the second period matrix.

7. An application: nine-dimensional Reissner–Nordström–de Sitter space–time

Armed with the method developed, we now come back to one of the physical model systems presented in Section 2. We consider here the geodesic equation in a Reissner–Nordström–de Sitter space–time of dimension 9. Setting $d = 9$ and introducing a new dimensionless coordinate $\tilde{r} = r/r_5$ and the dimensionless parameters $\tilde{L} = L/r_5$, $\tilde{q} = q/r_5$, and $\tilde{\Lambda} = \Lambda r_5^2$, we obtain from (2.11)
\[
\left(\frac{d\tilde{r}}{d\varphi}\right)^2 = \frac{R_{16}(\tilde{r})}{\tilde{r}^{10}},
\] (7.1)
where
\[
\tilde{L}^2 R_{16}(\tilde{r}) = \tilde{r}^{16} \frac{\tilde{\Lambda}}{28} \left( E^2 - \delta + \frac{\tilde{\Lambda}}{28} \tilde{L}^2 \right) \tilde{r}^{14} - \tilde{r}^{12} \tilde{L}^2 + \tilde{r}^8 \tilde{\delta} + \tilde{r}^6 \tilde{L}^2 - \tilde{q}^2 \tilde{r}^2 - \tilde{q}^2 \tilde{L}^2.
\] (7.2)

Through the substitution $\tilde{r} = 1/\sqrt{u}$ we have the order of the polynomial and by using $u = x^{-1} + u_8$, where $u_8$ is any root of the polynomial $R_8(u)$, we obtain a polynomial of seventh order such that Eq. (7.1) reduces to
\[
\left( x^2 \frac{dx}{d\varphi} \right)^2 = \mathcal{P}_7(x) \equiv \sum_{i=0}^{7} b_i x^i = b_7 \sum_{i=1}^{7} (x - e_i).
\] (7.3)

Thus, solving the differential equation is reduced to the inversion of a genus 3 holomorphic hyperelliptic integral:
\[
\varphi - \varphi_{\text{in}} = \int_{x_{\text{in}}}^{x} \frac{x^2 dx'}{\sqrt{\mathcal{P}_7(x')}}.
\] (7.4)

where $\varphi_{\text{in}}$ denotes the initial conditions for $\varphi$ and $x_{\text{in}}$ is the starting point of the integration.

From (4.7) we find the solution of the equation of motion (7.1):
\[
\tilde{r}(\varphi) = \frac{1}{\sqrt{-\frac{\sigma_3(\mathcal{W})}{\sigma_1(\mathcal{W})} + u_8}}.
\] (7.5)
Fig. 4. Examples of test particle motion in a nine-dimensional Reissner–Nordström–de Sitter space–time. The chosen parameters for the cosmological constant and the charge are \( \Lambda = 8.7 \cdot 10^{-5}, \bar{q} = 0.4 \). We also consider massive particles, \( \delta = 1 \). For these parameters (a) shows the \( E^2-L^2 \) parameter plot where the dark area indicates three positive real zeros and the bright area indicates one real positive zero of \( P_7 \). For the chosen values \( E^2 = 1.045 \) and \( L = 0.5 \), two orbits are calculated analytically and plotted in (b) and (c). The black circles are the event and Cauchy horizons. In this example \( e_1 < e_2 < e_3 < e_4 < \text{Re}(e_5) < e_5 \), and \( e_5 = \bar{e}_5 \). For the many-world periodic bound orbit (b) we choose a starting point \( e_2 \), which corresponds to the half-period \( [\bar{q}]_2 \) in (6.7), and the motion is bounded by \( e_1 \) and \( e_5 \). For the escape orbit (c) the starting point is \( e_3 \), which corresponds to the half-period \( [\bar{q}]_3 \) in (6.7), and the test particle moves to infinity.

where

\[
\phi = \Omega \sum_i + \left( f_1(\phi - \phi_{in}) \right), \quad f_1(0) = f_2(0) = 0,
\]

and where the functions \( f_1(\phi - \phi_{in}) \) and \( f_2(\phi - \phi_{in}) \) can be found from the conditions \( \sigma(\phi) = 0 \) and \( \sigma_3(\phi) = 0 \). Also \( \phi_{in} \) is chosen as the branch point of the polynomial \( P_7(x) \) which defines the half-integer characteristic \( \Omega \). The homology basis is fixed by the characteristics (6.7) and the \( \sigma \)-quotient is computed according to the formula (6.18) while the genus 3 \( \phi \)-functions were calculated with the aid of the “Maple/RiemannTheta” code.

The structure of the orbits is given by the number of zeros of the polynomial \( P_7 \). This depends on the choice of the parameters \( E, L, \Lambda, \) and \( \bar{q} \). We choose a certain nontrivial value for \( \Lambda \) and \( \bar{q} \) and draw the \( E^2-L^2 \) parameter plot presented in Fig. 4(a). In the dark area the polynomial \( P_7 \) possesses three positive zeros resulting in one bound and one escape orbit, and in the brighter area it possesses one positive zero, yielding one two-world escape orbit. For certain values for \( E^2 \) and \( L^2 \) we obtain analytically the orbits shown in Fig. 4. From the causal structure of the metric encoded in the zeros of \( g_{\mu\nu} = g_{\mu\nu}^{-1} \), one concludes that after traversing both horizons a second time the test body enters a new universe (see, e.g., the discussion in [22]).

Furthermore, the observable perihelion shift \( \Omega_{\text{perihelion}} \) can be given by a complete hyperelliptic integral

\[
\Omega_{\text{perihelion}} = 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{\xi^2}{\sqrt{P_7(\xi)}} d\xi - 2\pi.
\]

The perihelion shift compares the period of the radial motion with \( 2\pi \).

8. Conclusion

In this paper we presented the analytical solution of geodesic equations in general relativistic models, containing a hyperelliptic curve of arbitrary genus. On the basis of the solutions for genus 2 and genus 3 and the investigation of the Schwar–Weierstrass polynomials, we present the solution (4.15) for the geodesic equations with an underlying curve of arbitrary genus which is of the form or can be reduced to the form

\[
\varphi - \varphi_{in} = \int_{\varphi_{in}}^{\phi} \frac{d\xi}{\sqrt{P_{2g+1}(\xi)}}, \quad i \in \{0, \ldots, g - 1\},
\]

where \( g \) is the genus of the curve \( \varphi^2 = P_{2g+1}(x) \). As an example we integrate the geodesic equations with an underlying hyperelliptic curve of genus 3 in the Reissner–Nordström–de Sitter space–time in nine dimensions.
One major task in the calculation of the analytical solution is the calculation of the first and second period matrices. In Proposition 3.1 we provide a convenient and quick method for the calculation of the matrix $\sigma$ and the second period matrix from the first period matrix. The matrix $\sigma$ appears in the $\sigma$-function and, thus, in the general solution (4.15). In this method no integration of meromorphic differentials is required and the result is given in terms of the holomorphic period matrix and $\theta$-constants.

In Section 4.5 we also propose a step by step algorithm for the calculation of the characteristics of the branch points and the vector of Riemann constants which appear in the $\theta$- and $\sigma$-functions and which are required for the calculation of the matrix $\sigma$. We note that all the calculations can be done with the "Maple/algcurves" package, working in the homology basis automatically chosen by the computer program.

The proposed solution (4.15) for the curves of $g \geq 3$ together with the solution for genus 2 curves has a wide spectrum of applications. These range from the geodesic equations in a wide class of black hole space–times, like Schwarzschild–de Sitter [27,28], Kerr–de Sitter [30], NUT–de Sitter [49] and general type D Plebański–Demiański [29] space–times in four dimensions, to the higher dimensional spherically symmetric Schwarzschild–(anti–)de Sitter space–times [22], higher dimensional axially symmetric Myers–Perry space–times [50] discussed in Section 2, and general Kerr–NUT–(anti–)de Sitter metrics in all dimensions introduced in [51]. This might be extended to space–times with cosmic strings [52], even in higher dimensions. The mathematical methods described here are also applicable to such problems as the motion of test particles with spin [53] and mass multipoles in various gravitational fields, in particular in the gravitational field given by gravitating mass multiple. This has applications in astrophysics, satellite dynamics, and geodesy. The motion of test particles in the gravitational field of a black hole with disturbances related to mass multipoles has been discussed as test of the no-hair theorem [54]. As a consequence, there is a wide range of applications of the formalism developed in the article to problems in the area of general relativity.

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References

[1] H. Goldstein, C. Poole, J. Safko, Classical Mechanics, 3rd ed., Addison–Wesley, 2002.
[2] A.L. Markushevich, Introduction to the Classical Theory of Abelian Functions, AMS, Providence, 1992.
[3] Yu.N. Fedorov, D. Gómes-Ulate, Dynamical systems on infinitely sheeted Riemann surfaces, Physica D, 227, 120.
[4] Y. Ōnishi, Determinant expressions for abelian functions in genus two, Glasg. Math. J. 44 (2002) 353.
[5] H.F. Baker, Multiple Periodic Functions, Cambridge Univ. Press, Cambridge, 1907.
[6] D. Grant, Formal groups in genus two, J. Reine Angew. Math. 411 (1990) 96–121.
[7] J. Jorgenson, On directional derivatives of the theta function along its divisor, Israel J. Math. 77 (1992) 274.
[8] V. Ōnishi, Complex multiplication formulae for hyperelliptic curve of genus three, Tokyo J. Math. 21 (1998) 381.
[9] J.C. Eilbeck, V.Z. Enol’skii, A.B. Kuznetsov, D.V. Leykin, Linear $rr$-matrix algebra for systems separable in parabolic coordinates, Phys. Lett. A 180 (1993) 208.
[10] S. Abenda, Yu. Fedorov, On the weak Kowalevski–Painlevé property for hyperelliptically separable systems, Acta Appl. Math. 60 (2000) 137.
[11] S. Matsutani, Recursion relation of hyperelliptic PSI-functions of genus two, Int. Transforms Spec. Func. 14 (2003) 517.
[12] H.W. Braden, V.Z. Enol’skii, A.N.W. Hone, Bilinear recurrences and addition formulae for hyperelliptic sigma functions, J. Nonlinear Math. Phys. 12 (suppl. 2) (2005) 46.
[13] A.N.W. Hone, Sigma function solution of the initial value problem for Somos 5 sequences, Trans. Amer. Math. Soc. 359 (2007) 5019 (electronic).
[14] S. Baldwin, J. Gibbons, Higher genus hyperelliptic reductions of the Benney equations, J. Phys. A 37 (2004) 5341.
[15] S. Baldwin, J. Gibbons, Genus 4 trigonal reduction of the Benney equations, J. Phys. A 39 (2006) 3607.
[16] H.F. Baker, Abel’s Theorem and the Allied Theory of Theta Functions, Cambridge Univ. Press, Cambridge, 1897. Reprinted in 1995.
[17] V.M. Buchstaber, V.Z. Enol’skii, D.V. Leykin, Kleinian functions, hyperelliptic Jacobians and applications, in: I.M. Krichever, S.P. Novikov (Eds.), Reviews in Mathematics and Math. Physics, vol. 10, Gordon and Breach, London, 1997, part 2, p. 3.
[18] V.M. Buchstaber, D.V. Leykin, Addition laws on Jacobian variety of plane algebraic curves, Proceedings of the Steklov Institute of Mathematics 251 (2005) 1.
[19] A. Nakayashiki, Algebraic expression of sigma functions of $(n, s)$ curves, 2008. arXiv:0803.2083.
[20] S. Matsutani, E. Previato, Jacobi inversion on strata of the Jacobian of $C_n$, curve $y' = f(x)$ II, arXiv:1006.1090v1 [math.AG] 6 Jun. 2010.
[21] V.Z. Enol’skii, M. Pronine, P. Richter, Double pendulum and theta-divisor, J. Nonlinear Sci. 13 (2003) 157.
[22] E. Hackmann, V. Kagramanova, J. Kunz, C. Lämmerzahl, Analytic solutions of the geodesic equation in higher dimensions, Phys. Rev. D 78 (2008) 124018.
[23] H. Bateman, A. Erdelyi, Higher Transcendental Functions, vol. 2, McGraw-Hill, New York, 1955.
[24] Y. Hagiwara, Theory of the relativistic trajectories in a gravitational field of Schwarzschild, Japan. J. Astron. Geophys. 8 (1931) 67.
[25] V. Kagramanova, J. Kunz, E. Hackmann, C. Lämmerzahl, Analytic treatment of complete and incomplete geodesics in Taub–NUT space–times, Phys. Rev. D 81 (2010) 124044.
[26] S. Grunau, V. Kagramanova, Geodesics of electrically and magnetically charged test particles in the Reissner–Nordström space–time: analytical solutions, Phys. Rev. D (2011) (in press) arXiv:1011.5399 [gr-qc].
[27] E. Hackmann, C. Lämmerzahl, Complete analytic solution of the geodesic equation in Schwarzschild–(anti–)de Sitter space–times, Phys. Rev. Lett. 100 (2008) 171101.
[28] E. Hackmann, C. Lämmerzahl, Geodesic equation in Schwarzschild–(anti–)de Sitter space–time: Analytic solutions and applications, Phys. Rev. D 78 (2008) 024035.
[29] E. Hackmann, V. Kagramanova, J. Kunz, C. Lämmerzahl, Analytic solutions of the geodesic equation in axially symmetric space–times, Europhys. Lett. 88 (2009) 30008.
[30] E. Hackmann, C. Lämmerzahl, V. Kagramanova, J. Kunz, Analytical solution of the geodesic equation in Kerr-(anti) de Sitter space–times, Phys. Rev. D 81 (2010) 044020.
[31] A. Buonanno, T. Damour, Effective one-body approach to general relativistic two-body dynamics, Phys. Rev. D 59 (1999) 084006.
[32] T. Damour, P. Jaranowski, G. Schäfer, On the determination of the last stable orbit for circular general relativistic binaries at the third post-Newtonian approximation, Phys. Rev. D 62 (2000) 084011.
[33] R.C. Myers, M.J. Perry, Black holes in higher dimensional space–times, Ann. Phys. (NY) 172 (1986) 304.
[34] H. Kodama, R.A. Konoplya, A. Zhidenko, Gravitational stability of simply rotating Myers–Perry black holes: Tensorial perturbations, Phys. Rev. D 81 (2010) 044007.
[35] R. Emparan, Black holes galore in $D > 4$, Fortschr. Phys. 56 (2008) 723.
[36] M. Vasudevan, K.A. Stevens, D.N. Page, Separability of the Hamilton–Jacobi and Klein–Gordon equations in Kerr–de Sitter metrics, Classical Quantum Gravity 22 (2005) 339.
[37] J.D. Fay, Theta functions on Riemann surfaces, in: Lectures Notes in Mathematics, vol. 352. Springer, Berlin, 1973.
[38] H.M. Farkas, I. Kra, Riemann Surfaces, Springer, New York, 1980.
[39] B.E. Sagan, The symmetric group, in: Representations, Combinatorial Algorithms and Symmetric Function, 2nd ed., in: Graduate texts in Mathematics, vol. 203, Springer, New York, 2001.
[40] V.M. Buchstaber, V.Z. Enolskii, D.V. Leykin, Rational analogues of Abelian functions, Funk. Anal. Appl. 33 (1999) 1.
[41] P. Vanhaecke, Stratification of hyperelliptic Jacobians and the Sato Grassmanian, Acta. Appl. Math. 40 (1995) 143.
[42] K. Weierstraß, Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen, bearbeitet und herausgegeben von H. A. Schwarz (Springer, 1893).
[43] S. Matsutani, E. Previato, Jacobi inversion on strata of the Jacobian of the $C_v$ curve $y^d = f(x)$. J. Math. Soc. Japan. 60 (2008) 1009–1044. http://www.mittag-leffler.se/preprints/0607/files/IML-0607-41.pdf.
[44] J. Thomae, Beitrag zur Bestimmung von $\Theta(0, 0, \ldots, 0)$ durch die Klassenmoduln algebraischer Functionen. J. Reine Angew. Math. 71 (1870) 201.
[45] O. Bolza, Ueber die Reduction hyperelliptischer Integrale erster Ordnung und erster Gattung auf elliptische durch eine Transformation vierten Grades, Math. Ann. XXVIII (1886) 447.
[46] T. Northover, Riemann surfaces with symmetry: algorithms and applications, Ph.D. Dissertation, University of Edinburgh, 2010.
[47] T. Northover, Software excurves and cyclerpainter. http://gitorious.org/riemanncycles.
[48] H.F. Baker, On a system of differential equations leading to periodic functions, Acta. Math. 27 (1903) 135.
[49] E. Hackmann, V. Kagramanova, J. Kunz, C. Lämmerzahl, Analytic solution of geodesic equations in the Taub–NUT–de Sitter space–times (in preparation).
[50] E. Hackmann, V. Kagramanova, J. Kunz, C. Lämmerzahl, Analytic solution of geodesic equations in the Myers–Perry space–time (in preparation).
[51] W. Chen, H. Lü, C.N. Pope, General Kerr–NUT–AdS metrics in all dimensions. Classical Quantum Gravity 23 (2006) 5323.
[52] E. Hackmann, B. Hartmann, C. Lämmerzahl, P. Sirimanchan, Test particle motion in the space–time of a Kerr black hole pierced by a cosmic string, Phys. Rev. D 82 (2010) 044024.
[53] E. Hackmann, C. Lämmerzahl, I. Schaffer, Analytical solutions of the motion of test particles with spin (in preparation).
[54] C. Will, Testing the general relativistic no-hair theorems using the galactic center black hole SgrA*, Astrophys. J. Lett. 674 (2008) L25.