\textbf{$L^2$-blowup estimates of the wave equation and its application to local energy decay}

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Abstract

We consider the Cauchy problems in $\mathbb{R}^n$ for the wave equation with a weighted $L^1$-initial data. We derive sharp infinite time blowup estimates of the $L^2$-norm of solutions in the case of $n=1$ and $n=2$. Then, we apply it to the local energy decay estimates for $n=2$, which is not studied so completely when the 0-th moment of the initial velocity does not vanish. The idea to derive them is strongly inspired from a technique used in \cite{12, 15}.

1 Introduction

We consider the Cauchy problem of the wave equation:

\begin{equation}
\begin{aligned}
&u_{tt} - \Delta u = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n.
\end{aligned}
\end{equation}

Here, we assume, for the moment, $[u_0, u_1] \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

Concerning the existence of a unique energy solution to problem (1.1)-(1.2), by the Lumer-Phillips Theorem one can find that the problem (1.1)-(1.2) has a unique weak solution

\[ u \in C([0, \infty); H^1(\mathbb{R}^n)) \cap C^1([0, \infty); L^2(\mathbb{R}^n)) \]

satisfying the energy conservation law such that

\[ E(t) = E(0), \quad t \geq 0, \quad (1.3) \]

where the total energy $E(t)$ for the solution to problem (1.1)-(1.2) can be defined by

\[ E(t) := \frac{1}{2} \left( \|u(t, \cdot)\|^2 + \|\nabla u(t, \cdot)\|^2 \right). \]

Here, $\|u\|$ denotes the usual $L^2$-norm of $u \in L^2(\mathbb{R}^n)$.

Our main purpose is to establish the sharp estimates in time of the factor $M(t) := \|u(t, \cdot)\|$. This observation is important from the viewpoint of the local energy decay, which is one of main topics in the wave equation field. As is known from the work of C. Morawetz \cite{22}, the main task to get the local energy decay is to control the behavior of the function $M(t)$. By the way, as in a series of papers \cite{11, 112} and so on, in the exterior mixed problems on a star-shaped compliment (or more generally just on exterior...
domains) one can control $M(t)$ for all $n \geq 2$. While if one considers the local energy decay problems to (1.1) in the Euclidean space $\mathbb{R}^n$ (no obstacle case!) one can proceed the same argument as in the exterior mixed problem case when $n \geq 3$ (see [3]), however, in the case of $n = 2$, one has to impose stronger assumption such that $I_0 := \int_{\mathbb{R}^2} u_1(x)dx = 0$, which means the vanishing condition of the 0th moment of the initial data. Note that in these problems, the author images that one never assumes the compactness of the support of the initial data, and low regularity on them to control the function $M(t)$, and to get the local energy decay. Therefore, it is highly desirable to remove the stronger condition $I_0 = 0$ in the two dimensional case for the completeness of the theory. In this connection, for $n \geq 3$ $L^p$-decay problems of the solution to problem (1.1)-(1.2) including the Klein-Gordon type are already discussed by [32] and [33], however, there the compactness of the support of the initial data seems to be conscious, and their main concern is not in local energy decay itself, but an application to nonlinear problems of the $L^p$-boundedness.

Now, in this paper we study the asymptotic behavior of the function $M(t)$ as $t \to \infty$ under the non-compactly supported initial data framework. It seems that this problem does not studied so far. We employ the simple Fourier analysis to get the estimates of the function $M(t)$ by dividing the Fourier space of the integrand of the Fourier transformed solution $\hat{u}(t, \xi)$ into two parts: one is low frequency zone ($|\xi| \leq q(t)$), and the other is high frequency zone ($|\xi| \geq q(t)$), where $q(t)$ is a t-variable function chosen suitably. So, one never relies on the Hardy type inequality, which never holds in the low dimensional case (cf. [25]) because a core of the idea in a series of papers [1, 5, 9, 10, 13, 16] is that the boundedness of the solution satisfies the following properties under the additional regularity on the initial data:

$$[u_0, u_1] \in L^1(\mathbb{R}) \times L^1(\mathbb{R}) \quad \Rightarrow \quad \|u(t, \cdot)\| \leq C_1 I_{0,1} \sqrt{t},$$

$$[u_0, u_1] \in L^1(\mathbb{R}) \times L^{1,1}(\mathbb{R}) \quad \Rightarrow \quad C_2 \left| \int_{\mathbb{R}} u_1(x)dx \right| \sqrt{t} \leq \|u(t, \cdot)\|$$

for $t \gg 1$, where $C_j > 0$ ($j = 1, 2$) are constants depending only on the space dimension.

Our next result is the case of $n = 2$.

**Theorem 1.2** Let $n = 2$. Let $[u_0, u_1] \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. Then, the solution $u(t, x)$ to problem (1.1)-1.2 satisfies the following properties under the additional regularity on the initial data:

$$[u_0, u_1] \in L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2) \quad \Rightarrow \quad \|u(t, \cdot)\| \leq C_1 I_{0,2} \sqrt{\log t},$$

$$[u_0, u_1] \in L^1(\mathbb{R}^2) \times L^{1,1}(\mathbb{R}^2) \quad \Rightarrow \quad C_2 \left| \int_{\mathbb{R}^2} u_1(x)dx \right| \sqrt{\log t} \leq \|u(t, \cdot)\|$$

for $t \gg 1$, where $C_j > 0$ ($j = 1, 2$) are constants depending only on the space dimension.

**Remark 1.1** Surprisingly if one imposes the additional $L^{1,1}$-regularity on the initial velocity, one can get the sharp infinite time blowup result in the one and two dimensional free waves. It seems that this crucial phenomenon is not so known. Generally, people assume a vanishing moment condition $\int_{\mathbb{R}^n} u_1(x)dx = 0$ when one studies the $L^2$-boundedness of the solution $u(t, x)$, for the low dimensional case $n = 1, 2$ (cf., [11]).

**Remark 1.2** From the proof done in Section 3 for the upper bound estimates one can see that $L^{1,1}$-assumption is no need to get upper bound estimates, and only $L^1$-assumption works well. On the other hand, to obtain lower bound estimates one needs to impose the $L^{1,1}$ assumption on the initial velocity $u_1$ (see [2,14] and [21,15]).
Let us introduce one example to make sure the reliability of Theorems in the paper about only in the one dimensional case.

**Example.** Let \( n = 1 \), \( u_0 \equiv 0 \), and we choose \( u_1 \in (L^1(R) \cap L^2(R)) \) as follows:

\[
u_1(x) = \begin{cases} 
2 & (|x| \leq 1), \\
0 & (|x| > 1).
\end{cases}
\]

Then, one easily gets the formula:

\[
u(t, x) = \frac{1}{2} \int_{x-t}^{x+t} u_1(r)dr = -\chi(x-t) + \chi(x+t),
\]

where

\[
\chi(x) := \frac{1}{2} \int_0^x u_1(r)dr.
\]

One can soon check that

\[
-\chi(x-t) = \begin{cases} 
-(x-t) & (|x-t| \leq 1), \\
-1 & (x-t > 1), \\
1 & (x-t < -1),
\end{cases}
\]

\[
\chi(x+t) = \begin{cases} 
(x+t) & (|x+t| \leq 1), \\
1 & (x+ t > 1), \\
-1 & (x+t < -1).
\end{cases}
\]

Therefore, if one fixes a sufficiently large \( t_0 > 1 \), then one can finally arrive at the following situation for the solution \( u(t_0, x) \):

\[
u(t_0, x) = \begin{cases} 
2 & (1 - t_0 \leq x \leq t_0 - 1), \\
x + t_0 + 1 & (-t_0 - 1 < x < 1 - t_0), \\
-x + t_0 + 1 & (t_0 - 1 < x < t_0 + 1), \\
0 & (x < -t_0 - 1, t_0 + 1 < x).
\end{cases}
\]

Therefore, one can compute

\[
\int_{\mathbb{R}} |u(t_0, x)|^2dx = 8(t_0 - 1) + \frac{16}{3},
\]

which implies the credibility of the result

\[
\|u(t_0, \cdot)\| \sim \sqrt{t_0}, \quad t_0 \gg 1.
\]

In this connection, one naturally has the condition \( \int_{\mathbb{R}} u_1(x)dx = 4 \neq 0 \). This example can be stated in the fundamental text book of PDE (e.g. [17].)

**Remark 1.3** In a sense, the infinite time blowup results in \( L^2 \)-sense may express quantitatively failure of the Huygens principle.

**Notation.** Throughout this paper, \( \| \cdot \|_q \) stands for the usual \( L^q(\mathbb{R}^n) \)-norm. For simplicity of notation, in particular, we use \( \| \cdot \| \) instead of \( \| \cdot \|_2 \). We also introduce the following weighted functional space.

\[
L^{1,\gamma}(\mathbb{R}^n) := \left\{ f \in L^1(\mathbb{R}^n) \mid \|f\|_{1,\gamma} := \int_{\mathbb{R}^n} (1 + |x|^\gamma)|f(x)|dx < +\infty \right\}.
\]

One denotes the Fourier transform \( \mathcal{F}_{x \to \xi}(f)(\xi) \) of \( f(x) \) by

\[
\mathcal{F}_{x \to \xi}(f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x)dx, \quad \xi \in \mathbb{R}^n,
\]
as usual with $i := \sqrt{-1}$, and $F_{\xi \to x}^{-1}$ expresses its inverse Fourier transform. Finally, we denote the surface area of the $n$-dimensional unit ball by $\omega_n := \int_{|\omega|=1} d\omega$.

The paper is organized as follows. In Section 2 we derive the lower bound estimates of the $L^2$-norm of solutions, and in Section 3 we obtain the upper bound estimates of the $L^2$-norm of solutions, and by combining the results obtained in Sections 2 and 3 one can prove Theorems 1.1 and 1.2 at a stroke. Section 4 is devoted to apply Theorem 1.2 to the local energy decay results of the wave equation (1.1) in the two dimensional Euclidean space $\mathbb{R}^2$.

## 2 \( L^2 \)-lower bound estimates of the solution

In this section, let us derive the lower bound estimates of the $t$-function $\|u(t, \cdot)\|$ by using the Plancherel Theorem and low frequency estimates. This technique is well-developed in the damped wave equation field (cf. [12]).

We first prepare two basic facts which will be frequently used in the proof.

Set

$$L := \sup_{\theta \neq 0} \left| \frac{\sin \theta}{\theta} \right| < +\infty. \quad (2.1)$$

Furthermore, since

$$\lim_{\theta \to +0} \frac{\sin \theta}{\theta} = 1,$$

one can find a real number $\delta_0 \in (0, 1)$ such that

$$\left| \frac{\sin \theta}{\theta} \right| \geq \frac{1}{2} \quad (2.2)$$

for all $\theta \in (0, \delta_0]$. One also prepares the fundamental inequality:

$$|a + b|^2 \geq \frac{1}{2} |a|^2 - |b|^2 \quad (2.3)$$

for all $a, b \in \mathbb{C}$.

In order to get the lower bound estimate for the quantity $\|u(t, \cdot)\|$, it suffices to treat $\|w(t, \cdot)\|$ with $w(t, \xi) := F_{x \to \xi}(u(t, \cdot))(\xi) = \hat{u}(t, \xi)$ because of the Plancherel Theorem.

Now we decompose the quantity $\|w(t, \cdot)\|$ as follows: for each $n \geq 1$

$$\|w(t, \cdot)\|^2 = \left( \int_{|\xi| \leq \frac{\delta_0}{2t}} + \int_{|\xi| \geq \frac{\delta_0}{2t}} \right) |w(t, \xi)|^2 d\xi = I_{low}^{(n)}(t) + I_{high}^{(n)}(t). \quad (2.4)$$

Here we have just chosen $t > 0$ large enough such that

$$\frac{\delta_0}{t} \leq 1.$$

By the way, as is well-known in the text book of PDE, in the Fourier space $\mathbb{R}^n_\xi$, the problem (1.1) and its solution $u(t, x)$ can be transformed into the following ODE with parameter $\xi \in \mathbb{R}^n_\xi$:

$$w_{tt} + |\xi|^2 w = 0, \quad t > 0, \quad \xi \in \mathbb{R}^n_{\xi}, \quad (2.5)$$

$$w(0, \xi) = w_0(\xi), \quad w_t(0, \xi) = w_1(\xi), \quad \xi \in \mathbb{R}^n, \quad (2.6)$$

where $w_0(\xi) := \hat{u}_0(\xi)$ and $w_1(\xi) := \hat{u}_1(\xi)$. Moreover, one can easily solve the problem (2.5)-(2.6) as follows:

$$w(t, \xi) = \frac{\sin(t|\xi|)}{|\xi|} w_1(\xi) + \cos(t|\xi|)w_0(\xi). \quad (2.7)$$
Now, let us give the lower bound estimates for the quantity $I_{low}(t)$ because of $||w(t, \cdot)||^2 \geq I_{low}^{(n)}(t)$. The technique developed below is strongly inspired from the idea in [12]. Indeed, it follows from (2.7) and (2.8) that

$$I_{low}^{(n)}(t) = \int_{|\xi| \leq \frac{\omega}{2n}} \left| \frac{\sin(t|\xi|)}{|\xi|} w_1(\xi) + \cos(t|\xi|)w_0(\xi) \right|^2 d\xi$$

$$\geq \frac{1}{2} \int_{|\xi| \leq \frac{\omega}{2n}} \frac{\sin^2(t|\xi|)}{|\xi|^2} |w_1(\xi)|^2 d\xi - \int_{|\xi| \leq \frac{\omega}{2n}} \cos^2(t|\xi|)|w_0(\xi)|^2 d\xi$$

$$=: \frac{1}{2} J_1(t) - J_2(t). \quad (2.8)$$

Let us first estimate $J_1(t)$ by using the decomposition of the initial data $w_1(\xi)$ in the Fourier space:

$$w_1(\xi) = P + (A(\xi) - iB(\xi)), \quad \xi \in \mathbb{R}^n,$$

where

$$P := \int_{\mathbb{R}^n} u_1(x) dx,$$

$$A(\xi) := \int_{\mathbb{R}^n} (\cos(x\xi) - 1)u_1(x) dx, \quad B(\xi) := \int_{\mathbb{R}^n} \sin(x\xi)u_1(x) dx.$$ 

It is known (see [3]) that with some constant $M > 0$ one has

$$|A(\xi) - iB(\xi)| \leq M||\xi||u_1||_{L,1}, \quad \xi \in \mathbb{R}^n,$$ 

in the case when $u_1 \in L^{1,1}(\mathbb{R}^n)$. Then, it follows from (2.3) that

$$J_1(t) = \int_{|\xi| \leq \frac{\omega}{2n}} \frac{\sin^2(t|\xi|)}{|\xi|^2} |w_1(\xi)|^2 d\xi$$

$$\geq \frac{P^2}{2} \int_{|\xi| \leq \frac{\omega}{2n}} \frac{\sin^2(t|\xi|)}{|\xi|^2} d\xi - \int_{|\xi| \leq \frac{\omega}{2n}} |A(\xi) - iB(\xi)|^2 \frac{\sin^2(t|\xi|)}{|\xi|^2} d\xi$$

$$= \frac{P^2}{2} K_1(t) - K_2(t). \quad (2.10)$$

$K_2(t)$ can be estimated from above as follows by using (2.7):

$$K_2(t) \leq M^2||u_1||^2_{L,1,1} \int_{|\xi| \leq \frac{\omega}{2n}} \sin^2(t|\xi|) d\xi = M^2||u_1||^2_{L,1,1,\omega_n} \int_0^{\frac{\omega n}{2}} r^{n-1} dr$$

$$= \frac{M^2}{n} \omega_n \delta_0^n ||u_1||^2_{L,1,1,t^{-n}}, \quad t \gg 1. \quad (2.11)$$

While, one can get the lower bound estimate for $K_1(t)$ because of (2.2):

$$K_1(t) = t^2 \int_{|\xi| \leq \frac{\omega}{2n}} \frac{\sin^2(t|\xi|)}{(t|\xi|)^2} d\xi \geq \frac{t^2}{4} \int_{|\xi| \leq \frac{\omega}{2n}} d\xi$$

$$= \frac{\omega_n \delta_0^n}{4n} t^{2-n}, \quad t \gg 1. \quad (2.12)$$

Therefore from (2.10), (2.11) and (2.12) one can get the estimate from below for $J_1(t)$:

$$J_1(t) \geq \frac{P^2}{2} \frac{\omega_n \delta_0^n}{4n} t^{2-n} - \frac{M^2}{n} \omega_n \delta_0^n ||u_1||^2_{L,1,1,t^{-n}} \gg 1. \quad (2.13)$$

Since the upper bound estimate of $J_2(t)$ can be easily obtained as follows:

$$J_2(t) \leq \int_{|\xi| \leq \frac{\omega}{2n}} |w_0(\xi)|^2 d\xi \leq ||u_0||^2_{L,1,1,\omega_n} \int_0^{\frac{\omega n}{2}} r^{n-1} dr = \frac{\omega_n \delta_0^n}{n} ||u_0||^2_{L,1,1,t^{-n}},$$

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Lemma 2.2

Let \( n \) follows from (2.3) and a similar argument to the one dimensional case that 

\[
\|w(t, \cdot)\|^2 \geq I_{low}(t) \geq \frac{P^2}{4n} \omega_n \delta_0^2 t^{-n} - \frac{M^2}{n} \omega_n \delta_0 \|u_1\|_{L^{1,1}}^2 t^{-n} - \frac{\omega_n \delta_0^2}{n} \|u_0\|^2_{L^{1,1}} t^{-n} \quad t \gg 1.
\]  

(2.14)

Thus, there exists a positive real number \( t_0 \) such that 

\[
\|w(t, \cdot)\|^2 \geq \frac{P^2}{32n} \omega_n \delta_0^2 t^{-n}
\]  

(2.15)

for all \( t \geq t_0 \). It should be mentioned that \( t_0 > 0 \) depends on \( n \) and the quantities \( \|u_1\|_{L^{1,1}} \) and \( \|u_0\|_{L^{1,1}} \). By choosing \( n = 1 \) in (2.15) one has the following blowup property of the solution when \( P \neq 0 \).

**Lemma 2.1** Let \( n = 1 \), and \([u_0, u_1] \in L^1(\mathbb{R}^n) \times L^{1,1}(\mathbb{R}^n) \). Then, it holds that 

\[
\|w(t, \cdot)\|^2 \geq CP^2 t, \quad t \gg 1.
\]

Let us prove the statement for \( n = 2 \) at a stroke by using a trick with a function \( e^{-r^2} \). Indeed, it follows from (2.13) and a similar argument to the one dimensional case that 

\[
\|w(t, \cdot)\|^2 \geq \frac{1}{2} \int_{\mathbb{R}^2} \frac{\sin^2(tr)}{r^2} |w_1(\xi)|^2 d\xi - \int_{\mathbb{R}^2} \cos^2(tr)|w_0(\xi)|^2 d\xi \\
\geq \frac{1}{2} \int_{\mathbb{R}^2} e^{-r^2 \sin^2(tr)/r^2} |P + (A(\xi) - iB(\xi))|^2 d\xi - \|u_0\|^2 \\
\geq \frac{1}{4} P^2 \int_{\mathbb{R}^2} e^{-r^2 \sin^2(tr)/r^2} d\xi - \frac{1}{2} \int_{\mathbb{R}^2} e^{-r^2 \sin^2(tr)/r^2} \left( M^2 \|u_1\|^2_{L^{1,1}} - \|u_0\|^2 \right) d\xi \\
\geq \frac{1}{4} P^2 \int_{\mathbb{R}^2} e^{-r^2 \sin^2(tr)/r^2} d\xi - \frac{1}{2} M^2 \|u_1\|^2_{L^{1,1}} \int_{\mathbb{R}^2} e^{-r^2} d\xi - \|u_0\|^2 \\
= \frac{1}{4} P^2 T(t) - \frac{\omega_2}{4} M^2 \|u_1\|^2_{L^{1,1}} - \|u_0\|^2. 
\]

(2.16)

Now, we apply an idea which has its origin in [13]. For this purpose, we set 

\[
\nu_j := \left( \frac{3}{4} + j \right) \pi, \quad \mu_j := \left( \frac{3}{4} + j \right) \pi 
\]

\( j = 0, 1, 2, \ldots \),

and choose \( t > 1 \) large enough such that \( \nu_1 = \frac{5\pi}{4} < 1 \). Then, since 

\[
|\sin(tr)| \geq \frac{1}{\sqrt{2}}
\]

for \( r \in [\nu_j, \mu_j] \) and for each \( j = 0, 1, 2, \ldots \), one has the estimate:

\[
T(t) = \int_{\mathbb{R}^2} e^{-r^2 \sin^2(tr)/r^2} d\xi \geq \frac{1}{2} \sum_{j=0}^{\infty} \int_{\nu_j \leq r \leq \mu_j} e^{-r^2} d\xi = \frac{\omega_2}{2} \left( \sum_{j=0}^{\infty} \int_{\nu_j}^{\mu_j} e^{-r^2} \frac{dr}{r} \right) 
\]

(2.17)

\[
\geq \frac{\omega_2}{2} \left( \frac{1}{2} \int_{\nu_1}^{\infty} e^{-r^2} \frac{dr}{r} \right) \geq \frac{\omega_2}{4} \int_{\nu_1}^{1} e^{-r^2} \frac{dr}{r} 
\]

(2.18)

\[
\geq \frac{\omega_2}{4} \int_{\nu_1}^{1} \frac{1}{r} dr = \omega_2 \frac{e^{-1}}{4} (\log t + \log 4 - \log 5\pi),
\]

(2.19)

where in the inequality from (2.17) to (2.18) one has just used the monotone decreasing property of the function \( r \mapsto e^{-r^2} \), and the fact that \( \frac{\pi}{2} = \mu_j - \nu_j = \nu_{j+1} - \mu_j \) (\( \forall j \)). Therefore, by (2.16) and (2.19) one has the following estimates for \( n = 2 \) for large \( t > 1 \).

**Lemma 2.2** Let \( n = 2 \), and \([u_0, u_1] \in L^1(\mathbb{R}^n) \times L^{1,1}(\mathbb{R}^n) \). Then, it holds that 

\[
\|w(t, \cdot)\|^2 \geq CP^2 \log t, \quad t \gg 1.
\]
3 \( L^2 \)-upper bound estimates of the solution

In this section, one derives upper bound estimates of the quantity \( \| u(t, \cdot) \| \) as \( t \to \infty \) by calculating the function \( w(t, \xi) \) in both high and low frequency region.

From (2.4) one first shows the upper bound estimate for \( I_{low}^{(n)}(t) \) for all \( n \geq 1 \). Indeed, by (2.3) one has

\[
I_{low}^{(n)}(t) \leq \int_{|\xi| \leq \frac{\omega_0}{n}} \left\{ \frac{\sin(t|\xi|)}{|\xi|} w_1(\xi) + \cos(t|\xi|)w_0(\xi) \right\}^2 d\xi
\]

\[
\leq 2 \int_{|\xi| \leq \frac{\omega_0}{n}} \frac{\sin^2(t|\xi|)}{|\xi|^2} |w_1(\xi)|^2 d\xi + 2 \int_{|\xi| \leq \frac{\omega_0}{n}} \cos^2(t|\xi|) |w_0(\xi)|^2 d\xi
\]

\[
= 2L_1(t) + 2L_2(t). \tag{3.1}
\]

To begin with, it is easy to get the estimate for \( L_2(t) \) as in the estimate for \( J_2(t) \) in Section 2:

\[
L_2(t) \leq \int_{|\xi| \leq \frac{\omega_0}{n}} |w_0(\xi)|^2 d\xi \leq \| u_0 \|^2_{L^1} \omega_n \int_0^t r^{n-1} dr = \frac{\omega_n \delta_0^n}{n} \| u_0 \|^2_{L^1} t^{-n}. \tag{3.2}
\]

Let us estimate \( L_1(t) \) as follows:

\[
L_1(t) = \int_{|\xi| \leq \frac{\omega_0}{n}} \frac{\sin^2(t|\xi|)}{|\xi|^2} |w_1(\xi)|^2 d\xi
\]

\[
\leq \| u_1 \|^2_{L^1} \int_{|\xi| \leq \frac{\omega_0}{n}} \frac{\sin^2(t|\xi|)}{|\xi|^2} d\xi. \tag{3.3}
\]

As in (3.2) one has

\[
\int_{|\xi| \leq \frac{\omega_0}{n}} d\xi \leq \frac{\omega_n \delta_0^n}{n} t^{-n}, \quad t \gg 1. \tag{3.4}
\]

Therefore, if one uses (3.1), (3.3) and (3.4) then it holds that

\[
L_1(t) \leq \| u_1 \|^2_{L^1} L^2 t^2 \frac{\omega_n \delta_0^n}{n} t^{-n} = \| u_1 \|^2_{L^1} L^2 \frac{\omega_n \delta_0^n}{n} t^{2-n}, \quad t \gg 1. \tag{3.5}
\]

Thus, from (3.1), (3.2) and (3.5) one can obtain the low-frequency estimate

\[
I_{low}^{(n)}(t) \leq C \left( \| u_0 \|^2_{L^1} t^{2-n} + \| u_1 \|^2_{L^1} t^{2-n} \right), \quad t \gg 1, \tag{3.6}
\]

with some constant \( C > 0 \).

Next, one treats \( I_{high}^{(n)}(t) \) to get the upper bound estimate. Indeed, similar to the above estimate one stars with the following inequalities.

\[
I_{high}^{(n)}(t) \leq \int_{|\xi| \geq \frac{\omega_0}{n}} \left\{ \frac{\sin(t|\xi|)}{|\xi|} w_1(\xi) + \cos(t|\xi|)w_0(\xi) \right\}^2 d\xi
\]

\[
\leq 2 \int_{|\xi| \geq \frac{\omega_0}{n}} \frac{\sin^2(t|\xi|)}{|\xi|^2} |w_1(\xi)|^2 d\xi + 2 \int_{|\xi| \geq \frac{\omega_0}{n}} \cos^2(t|\xi|) |w_0(\xi)|^2 d\xi
\]

\[
= 2N_1^{(n)}(t) + 2N_2^{(n)}(t). \tag{3.7}
\]

It is easy to treat \( N_2^{(n)}(t) \) as follows. This can hold for all \( n \geq 1 \).

\[
N_2^{(n)}(t) \leq \int_{|\xi| \geq \frac{\omega_0}{n}} |w_0(\xi)|^2 d\xi \leq \| u_0 \|^2. \tag{3.8}
\]
Let us estimate $N_1^{(n)}(t)$ when $n = 1, 2$. First, for all $n \geq 1$ one has

$$N_1^{(n)}(t) = \int_{|\xi| \geq \frac{\delta_0}{\sqrt{n}}} \frac{\sin^2(t|\xi|)}{|\xi|^2} |w_1(\xi)|^2 d\xi + \int_{\frac{\delta_0}{\sqrt{n}} \leq |\xi| \leq \frac{\omega}{\sqrt{n}}} \frac{\sin^2(t|\xi|)}{|\xi|^2} |w_1(\xi)|^2 d\xi$$

$$=: O_1^{(n)}(t) + O_2^{(n)}(t). \tag{3.9}$$

To begin with, we consider the case $n = 1$. In this case, for $O_1^{(1)}(t)$ one can get the estimate

$$O_1^{(1)}(t) \leq \frac{t}{\delta_0} \int_{|\xi| \geq \frac{\delta_0}{\sqrt{n}}} \sin^2(t|\xi|) |w_1(\xi)|^2 d\xi$$

$$\leq \frac{t}{\delta_0} \int_{|\xi| \geq \frac{\delta_0}{\sqrt{n}}} |w_1(\xi)|^2 d\xi \leq \frac{t}{\delta_0} \|u_1\|^2, \quad t \gg 1. \tag{3.10}$$

While, $O_2^{(1)}(t)$ can be treated as follows:

$$O_2^{(1)}(t) \leq \|u_1\|_1^2 \int_{\frac{\delta_0}{\sqrt{n}} \leq |\xi| \leq \frac{\omega}{\sqrt{n}}} \frac{1}{|\xi|^2} d\xi$$

$$= \|u_1\|^2_1 \omega_1 \int_{\frac{\delta_0}{\sqrt{n}} r}^{\frac{\omega}{\sqrt{n}}} \frac{1}{r^2} d\xi = \frac{\omega_1}{\delta_0} \|u_1\|^2_1 (t - \sqrt{t}), \quad t \gg 1. \tag{3.11}$$

Thus, from (3.7), (3.8), (3.9), (3.10) and (3.11) one has the estimate for $I_{\text{high}}^{(n)}(t)$ in the case of $n = 1$:

$$I_{\text{high}}^{(1)}(t) = C \left( \|u_0\|^2 + \|u_1\|^2_2 t + \|u_1\|^2_2 (t - \sqrt{t}) \right), \quad t \gg 1. \tag{3.12}$$

Next, let us give sharp estimates for $N_1^{(2)}(t)$ in the case $n = 2$ at a stroke by using a more precise task. For this purpose, by choosing $t > 1$ large enough to realize the relation $1 \leq \log t \leq t \leq t^2$ one has a decomposition of the desired integrand:

$$N_1^{(2)}(t) = \int_{|\xi| \geq \frac{\delta_0}{\sqrt{n}}} \frac{\sin^2(t|\xi|)}{|\xi|^2} |w_1(\xi)|^2 d\xi$$

$$+ \int_{\frac{\delta_0}{\sqrt{n}} \leq |\xi| \geq \frac{\omega}{\sqrt{n}}} \frac{\sin^2(t|\xi|)}{|\xi|^2} |w_1(\xi)|^2 d\xi + \int_{\frac{\omega}{\sqrt{n}} \leq |\xi| \geq \frac{\omega}{\sqrt{n}}} \frac{\sin^2(t|\xi|)}{|\xi|^2} |w_1(\xi)|^2 d\xi$$

$$=: O_1(t) + O_2(t) + O_3(t). \tag{3.13}$$

Let us estimate them in order.

First, one has

$$O_1(t) \leq \frac{\log t}{\delta_0^2} \int_{|\xi| \geq \frac{\delta_0}{\sqrt{n}}} |w_1(\xi)|^2 d\xi$$

$$\leq \frac{\log t}{\delta_0^2} \|u_1\|^2, \quad t \gg 1. \tag{3.14}$$

For $O_2(t)$ one can proceed as follows:

$$O_2(t) \leq \|u_1\|^2_2 \omega_2 \int_{\frac{\omega}{\sqrt{n}}}^{\frac{\delta_0}{\sqrt{n}}} \frac{1}{r} dr$$

$$\leq \omega_2 \|u_1\|^2_2 \left( \log t - \log(\log t) \right) \quad t \gg 1. \tag{3.15}$$

Lastly, we treat $O_3(t)$ to give the following estimate such that

$$O_3(t) \leq \|u_1\|^2_2 \int_{\frac{\omega}{\sqrt{n}} \leq |\xi| \leq \frac{\omega}{\sqrt{n}}} \frac{1}{r^2} d\xi$$
Thus, it follows from (3.15) and (3.16) that for large $t > 1$
\[
N_1^2(t) \leq C \left( \|u_1\|_{L_1}^2 \log t + \|u_1\|^2 \log t + \|u_1\|_{L_1}^2 \log (t - \log \log t) \right), \quad t > 1
\] (3.17)
with some constant $C > 0$. Therefore, by combining (3.3), (3.5) and (3.17) one can get the high-frequency estimate for $n = 2$:
\[
l^{(2)}_{high}(t) \leq C \left( \|u_0\|^2 + \|u_1\|^2 + \|u_1\|_{L_1}^2 \right) \log t, \quad t > 1.
\] (3.18)

Finally, it follows from (2.4), (3.6), (3.12) and (3.18) one can get the crucial upper bound estimates

Finally, the proofs of Theorems 1.1 and 1.2 are direct consequence of Lemmas 2.1, 2.2, 3.1 and 3.2.

4 Application to local energy decay problems

In this section we apply Theorem 1.2 to the local energy decay problems. This problem is very important problem in the wave equation field, and one can observe many important previous papers (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40]), and a recent interesting application to inverse scattering problem of the local energy decay can be found in [19]. In particular, Morawetz [22] studied such problems by constructing the so-called Morawetz identity to develop the multiplier method (see [7] for its text book). The author in [22] considered the following exterior mixed problems:

\[
u_{tt} - \Delta u = 0, \quad (t, x) \in (0, \infty) \times \Omega,
\] (4.1)

\[
u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \Omega,
\] (4.2)

\[
u(t, \sigma) = 0, \quad \sigma \in \partial \Omega,
\] (4.3)

where $\Omega \subset \mathbb{R}^n$ is a smooth exterior domain with bounded boundary $\partial \Omega$ and star-shaped compliment $\Omega^c$. In [22] the author derived the crucial estimate:

\[
\int_{B_R \cap \Omega} \left( |u_t(t, x)|^2 + |\nabla u(t, x)|^2 \right) dx \leq C_R t^{-1}, \quad t > 1,
\]
the author solved a corresponding Poisson equation to get $L^2$-boundedness of the solution using CSI. These methods can be also applied to the Cauchy problem (1.1)-(1.2) in $\mathbb{R}^n$, and in fact, without CSI condition Charão-Ikehata [5] have derived the local energy decay results by using the method due to [13] to problem (1.1)-(1.2). However, it should be remarked that in [5] only the two dimensional case is not studied yet. This is because in the two dimensional case, one can not obtain the Hardy type inequality as was pointed out by [25] (1.14) of Theorem 5. One of useful ideas to treat the two dimensional case is to impose the extra assumption $\int_{\mathbb{R}^2} u_1(x)dx = 0$ to get $L^2$-boundedness of the solution, and this is tried in [11] recently. The method in [11] is independent from establishing the Hardy-type inequality, and the Fourier transform is suitably used like in this paper. From this observation, in the whole space case one can apply the Fourier transform, and it suffices to analyze the $L^2$-behavior of the Fourier transformed solution $\hat{u}(t, \xi)$ through the Plancherel Theorem. At present, in the two dimensional case, under the non-vanishing condition of the moment $\int_{\mathbb{R}^2} u_1(x)dx$ it seems open to get the local energy decay because in general, the $L^2$-boundedness of the solution is not trivial.

Now, from Theorem 1.2 one can get the upper bound estimate with log-order under the loose condition on the initial data (not necessarily $\int_{\mathbb{R}^2} u_1(x)dx = 0$). Although this is not the boundedness, it may work nicely to get the local energy decay in the two dimensional whole space case. It should be mentioned once more that one does not rely on any Hardy type inequalities.

Let us mention our story. First the Morawetz identity in $\mathbb{R}^n$-version can be stated as follows. Let $u \in C([0, \infty); H^1(\mathbb{R}^n)) \cap C^1([0, \infty); L^2(\mathbb{R}^n))$ be the weak solution to problem (1.1)-(1.2). Then, one has the following identity.

$$tE(t) = \frac{n - 1}{2} \int_{\mathbb{R}^n} u_1(x)u_0(x)dx + \int_{\mathbb{R}^n} u_1(x)(x \cdot \nabla u_0(x))dx - \frac{n - 1}{2} \int_{\mathbb{R}^n} u_1(t, x)u(t, x)dx + \int_{\mathbb{R}^n} u_1(t, x)(x \cdot \nabla u(t, x))dx, \quad t \geq 0. \quad (4.4)$$

$L^2$-boundedness directly affects the part $\int_{\mathbb{R}^n} u_1(t, x)u(t, x)dx$ because of the Schwarz inequality and energy identity (1.3):

$$\left| \int_{\mathbb{R}^n} u_1(t, x)u(t, x)dx \right| \leq \left( \int_{\mathbb{R}^n} |u(t, x)|^2dx \right)^{1/2} \left( \int_{\mathbb{R}^n} |u_1(t, x)|^2dx \right)^{1/2} = \|u(t, \cdot)\| \|u_1(t, \cdot)\| \leq \|u(t, \cdot)\| \sqrt{2E(0)}, \quad t \geq 0. \quad (4.5)$$

So, it suffices to control the quantity $\|u(t, \cdot)\|$ in order to get some good estimate of the term

$$F(t) := \int_{\mathbb{R}^n} u_1(t, x)u(t, x)dx.$$

**Remark 4.1** In the case when $n = 1$, the term $F(t)$ vanishes in (4.4), so in the one dimensional case we only control the term

$$G(t) := \int_{\mathbb{R}^n} u_1(t, x)(x \cdot \nabla u(t, x))dx.$$

The analysis on $G(t)$ can be done in [5] [13] [16] by the (modified) weighted energy method due to [34] in order to absorb this term to a part of the total energy under the non-compact support assumption of the initial data. So, this topic on $G(t)$ is out of scope of this paper.

Now, one introduces the midway formula derived in [5] [13] to get the local energy decay in the case when the CSI condition is not assumed. For each $R > 0$, we define the local energy in the region $B_R \subset \mathbb{R}^2$:

$$E_R(t) := \int_{B_R} (|u_1(t, x)|^2 + |\nabla u(t, x)|^2) dx.$$

Then, the result in [13] (2.13) at p. 272 and/or [5] (3.5) with $C(x) \equiv 1$ and $\eta = 0$ tell us the following fact.
Proposition 4.1 Let $n = 2$, $R > 0$, and assume that $[u_0, u_1] \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ further satisfies
\[
\int_{\mathbb{R}^2} |x| (|u_1(x)|^2 + |\nabla u_0(x)|^2) \, dx < +\infty.
\]
Then, the weak solution $u(t, x)$ to problem (1.1)-(1.2) satisfies
\[
(t - R) E_R(t) \leq K_0 + \frac{1}{2} \left| \int_{\mathbb{R}^2} u_1(t, x) u(t, x) \, dx \right|, \quad \forall t > R,
\]
where
\[
K_0 := \int_{\mathbb{R}^2} u_1(x) (x \cdot \nabla u_0(x)) \, dx + \frac{1}{2} \int_{\mathbb{R}^2} u_1(x) u_0(x) \, dx + E(0).
\]
Therefore, from this Proposition 4.1 one can observe that if one can control the function $F(t)$ then one has a possibility to get the local energy decay. In [5] the authors have imposed rather stronger condition such that
\[
\int_{\mathbb{R}^2} u_1(x) \, dx = 0
\]
to get the "boundedness" of the function $F(t)$. This is their weak point in [5] in the two dimensional case.

Let us apply Theorem 2 to control (4.5). Indeed, by applying Theorem 2 to (4.5) one can get
\[
|F(t)| \leq C \sqrt{2} E(0) I_{0.2} \sqrt{\log t}, \quad t \gg 1,
\]
where the constant $C > 0$ does not depend on any $R > 0$. By combining with Proposition 4.1 one has arrived at the following crucial result.

Theorem 4.1 Let $n = 2$, $R > 0$, and assume that $[u_0, u_1] \in (H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \times (L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2))$ further satisfies
\[
\int_{\mathbb{R}^2} |x| (|u_1(x)|^2 + |\nabla u_0(x)|^2) \, dx < +\infty.
\]
Then, the weak solution $u(t, x)$ to problem (1.1)-(1.2) satisfies
\[
E_R(t) \leq \frac{K_0}{t - R} + \frac{C}{2} \sqrt{2} E(0) I_{0.2} \sqrt{\log t}, \quad \forall t > R.
\]

Remark 4.2 It should be strongly mentioned that Theorem 4.1 can be derived without assuming CSI. For the one dimensional case, the effect of the function $F(t)$ is nothing, so one has no any good applications of Theorem 1.1 yet.

Remark 4.3 Shapiro [30] has also obtained the local energy decay estimates for $n = 2$ with $(\log t)^{-2}$ order for more general wave equations with variable coefficients including (1.1), however, the work of [30] has been done under stronger assumptions such as CSI and high regularity on the initial data. On the other hand, by [27] Theorem 2.1 applied to the case for $n = 2$ one can get the local energy decay with rate $t^{-1}$ under high regularity assumptions on the initial data (see also [20] (22)).

Remark 4.4 By combining our method with the Morawetz identity, in the forthcoming project we will study similar local energy decay estimates for the linear Klein-Gordon equation in the framework of non-compactly supported initial data.

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