GRAVITATING VORTICES AND THE EINSTEIN–BOGOMOLOV’NYI EQUATIONS

LUIS ÁLVAREZ-CÓNSUL, MARIO GARCIA-FERNANDEZ, OSCAR GARCÍA-PRADA, AND VAMSI PRITHAM PINGALI

Abstract. We consider the moduli space parametrizing Riemann surfaces equipped with an effective divisor, and propose an analytic parametrization in terms of the gravitating vortex equations. These equations couple a metric over a compact Riemann surface with a Hermitian metric over a holomorphic line bundle equipped with a fixed global section — the Higgs field —, and have a symplectic interpretation as moment-map equations. Applying the Kähler-quotient procedure, the corresponding moduli space acquires a generalized Weil–Petersson metric. As a particular case of the gravitating vortex equations on $\mathbb{P}^1$, we find the Einstein–Bogomol’nyi equations, previously studied in the theory of cosmic strings in physics. We prove two main results in this paper. Our first main result gives a converse to an existence theorem of Y. Yang for the Einstein–Bogomol’nyi equations, establishing in this way a correspondence with Geometric Invariant Theory for these equations. In particular, we prove a conjecture by Y. Yang about the non-existence of cosmic strings on $\mathbb{P}^1$ superimposed at a single point. Our second main result is an existence and uniqueness result for the gravitating vortex equations in genus greater than one.

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1. Introduction

The classical Teichmüller spaces and the closely related moduli of Riemann surfaces can be interpreted as spaces parametrizing various types of geometric data, such as surface group representations, complex structures, and algebraic curves. Essential ties that bind these interpretations together are the Uniformization Theorem, and the intimately related result that a compact Riemann surface admits a metric of constant curvature with fixed volume, unique up to biholomorphisms. Ever since the pioneering work of Fricke and Teichmüller, it has been convenient to enhance the geometric data with marked points or other types of decoration on the surface. This paper paves the way to extend the complex analytic approach to one of the simplest decorations for which this space has not been so well studied yet, namely an effective divisor on the surface. Specifically, we consider the analytic moduli problem of Riemann surfaces Σ equipped with an effective divisor $D$ (cf., e.g., [33, §2.1.3], for the corresponding algebro-geometric moduli problem).

The compact Riemann surface Σ and the effective divisor $D$ canonically define a holomorphic line bundle $L = \mathcal{O}(D)$ and a section $\phi \in H^0(\Sigma, L)$, and our analytic parametrization is through pairs of metrics $(g_\Sigma, h)$ satisfying the gravitating vortex equations originally introduced in [2]. For given $\Sigma, L$ and $\phi$, and constant parameters $\alpha, \tau \in \mathbb{R}$, these equations are

$$i \Lambda F + \frac{1}{2}(|\phi|^2 - \tau) = 0,$$

$$S + \alpha(\Delta + \tau)(|\phi|^2 - \tau) = c.$$  \hspace{1cm} (1.1)

They involve two unknowns: a Kähler metric $g_\Sigma$ on Σ and a hermitian metric $h$ on $L$. Here, $F$ is the curvature of the Chern connection of $h$, $\Lambda F$ is its contraction by the Kähler form $\omega$ of $g_\Sigma$, $|\phi|$ is the pointwise norm of $\phi$ with respect to $h$, $S$ is the scalar curvature of $g_\Sigma$, and $\Delta$ is the Laplacian of the metric on the surface acting on functions. The constant $c \in \mathbb{R}$ is topological, as it can be obtained by integrating (1.1) over Σ. Explicitly, it is

$$c = \frac{2\pi(\chi(\Sigma) - 2\alpha \tau c_1(L))}{\text{Vol}_\omega(\Sigma)}. \hspace{1cm} (1.2)$$

Solutions of the first equation in (1.1), known as the vortex equation (also known as Bogomol’nyi equations in the abelian Higgs model) are called vortices, and have been extensively studied in the literature after the seminal work of Jaffe and Taubes [32, 52] on the Euclidean plane, and Witten [59] on the 2-dimensional Minkowski spacetime. It is known [10, 26, 45] that the existence of solutions (with $\phi \neq 0$) is equivalent to the inequality

$$c_1(L) < \frac{\tau \text{Vol}(\Sigma)}{4\pi}. \hspace{1cm} (1.3)$$

As proved by the third author [27], this result follows the same principles as the Theorem of Donaldson, Uhlenbeck and Yau [19, 57] — relating the existence of solutions of the Hermitian–Yang–Mills equations with an algebraic numerical condition —, and indeed can be obtained as a corollary of this correspondence [26]. Since the scalar equation in (1.1) couples the vortices to a Riemannian metric on $\Sigma$, it seems reasonable to refer to the solutions of the system (1.1) as gravitating vortices. In fact, this system ties up with the physics of cosmic strings when $c = 0$, $c_1(L) > 0$, and $\alpha > 0$ (see [2] for background). In
this case, it becomes equivalent to the Einstein–Bogomol’nyi equations on a Riemann surface \[ (60, 61) \] and, as observed by Yang \[ (63, 64) \], if they have solutions, then our assumption that \( \Sigma \) is compact implies that it is the Riemann sphere \( \mathbb{P}^1 \); see Section 2.2 for details.

To explain the geometric meaning of the gravitating vortex equations and construct our moduli spaces, we fix a symplectic form \( \omega \) on the \( C^\infty \) compact surface \( S_g \) of genus \( g \) underlying \( \Sigma \) and a hermitian metric \( h \) on the \( C^\infty \) complex line bundle underlying \( L \), and regard (1.1) as equations involving three unknowns: a complex structure \( J \) on \( S_g \) compatible with \( \omega \), a unitary connection \( A \) on the hermitian line bundle \( (L, h) \), and a section \( \phi \) of \( L \) such that \( \bar{\partial}_A \phi = 0 \). Using methods of \[ (1, 24) \], in Section 3 we construct a Kähler structure (for positive \( \alpha \)) on the infinite-dimensional space \( \mathcal{T} \) of triples \( (J, A, \phi) \), with a Hamiltonian action of an infinite-dimensional group \( \tilde{G} \), and in this guise, the solutions of the gravitating vortex equations are the zeros of a moment map \( \mu_\alpha: \mathcal{T} \to (\text{Lie } \tilde{G})^\ast \), where \( \text{Lie } \tilde{G} \) is the Lie algebra of \( \tilde{G} \). By considering a larger symmetry group \( \tilde{G}_\omega \) of \( \mathcal{T} \), consisting of the unitary automorphisms of the line bundle that cover symplectomorphisms of \( S_g \), we can define the vortex Riemann moduli space \( \mathcal{M}_\alpha \), the vortex Teichmüller space \( \text{Teich}_\alpha \), and the moduli space \( \mathcal{M}_\alpha \) of gravitating vortices, as spaces of solutions \( (J, A, \phi) \in \mathcal{T} \) of (1.1), modulo suitable group actions. Then \( \mathcal{M}_\alpha = \mu_\alpha^{-1}(0)/\tilde{G} \) is a Kähler quotient, \( \text{Teich}_\alpha \) is its orbit space for an induced torus action, and \( \mathcal{M}_\alpha \) is a further quotient by the action of the discrete group \( \pi_0(\tilde{G}_\omega) \). We expect that \( \mathcal{M}_\alpha \) is isomorphic to the complex analytic space underlying the algebro-geometric moduli space of smooth complex projective curves equipped with an effective divisor; see Section 3.

The core of this paper (Sections 4–7) is devoted to address questions related to the existence and uniqueness of solutions of (1.1). As in the classical case, they should be relevant to compare our complex analytic moduli spaces with corresponding algebro-geometric moduli spaces. And again, as in the classical Uniformization Theorem, the simplest answers are in genus \( g \geq 2 \), so for the sake of clarity, let us temporarily skip Sections 4–6 and start with the main result in higher genus (see Section 7 for details).

**Theorem 1.1.** Let \( \Sigma \) be a compact Riemann surface of genus \( g \geq 2 \), and \( L \) a holomorphic line bundle over \( \Sigma \) of degree \( N > 0 \) equipped with a holomorphic section \( \phi \neq 0 \). Let \( \tau \) be a real constant such that \( 0 < N < \tau/2 \). Define

\[
\alpha_* := \frac{2g - 2}{2\tau(\tau/2 - N)} > 0.
\]

Then the set of \( \alpha \geq 0 \) for which (2.1) has smooth solutions of volume \( 2\pi \) is open and contains the closed interval \([0, \alpha_*]\). Furthermore, the solution is unique for \( \alpha \in [0, \alpha_*] \).

The proof of Theorem 1.1 involves the continuity method, where openness is proven using the moment-map interpretation given in Section 3, while closedness needs a priori estimates as usual. As in the Kähler–Einstein situation, the hardest part is the \( C^0 \) estimate, and in fact it is for this estimate that the value of \( \alpha \) should not be too large. Once these estimates are proven, they also help with uniqueness, wherein, akin to a corresponding uniqueness result of Bando and Mabuchi in the Kähler–Einstein situation \[ (5) \], we run the continuity path backwards. An interesting open question is to see what the largest value of \( \alpha \) is, for which solutions exist.
Turning now to the case of surfaces of lower genus, we observe that in genus \( g = 1 \), the gravitating vortex equations (1.1) (with \( \phi \neq 0 \)) always have a solution in the weak coupling limit \( 0 < |\alpha| \ll 1 \) (see [2, Theorem 4.1] for a precise formulation), and it is an interesting open problem to find effective bounds for \( \alpha \) for which (1.1) admit solutions.

Unlike the cases of genus \( g \geq 1 \), we show in Sections 4–6 that in genus \( g = 0 \), new phenomena arise that did not appear in the classical situation, namely there exist obstructions to the existence of solutions of the gravitating vortex equations. This may be interpreted as saying that our problem is comparatively closer to the more sophisticated problem of Calabi on the existence of Kähler–Einstein metrics, where algebro-geometric stability obstructions appear on compact Kähler manifolds with \( c_1 > 0 \).

Based on our moment-map interpretation of the gravitating vortex equations (1.1), in genus \( g = 0 \) we pursue an analogue for them of the theorem of Donaldson, Uhlenbeck and Yau [19, 57], in the case \( \alpha > 0 \). The first clue pointing out to such a correspondence for \( \Sigma = \mathbb{P}^1 \) lies in Yang’s existence result [63, 64], reformulated more elegantly in the language of Mumford’s Geometric Invariant Theory (GIT) [42] (it is perhaps worth emphasizing that physicists did not have a moment-map interpretation of the Einstein–Bogomol’nyi equations, and this result was not formulated in the language of GIT; see [2] for details).

**Theorem 1.2** (Yang’s existence theorem). Suppose \( c = 0 \) and (1.3) is satisfied. Let \( D = \sum n_j p_j \) be an effective divisor on \( \mathbb{P}^1 \) corresponding to a pair \((L, \phi)\). Then, the Einstein–Bogomol’nyi equations on \((\mathbb{P}^1, L, \phi)\) have solutions, provided that the divisor \( D \) is GIT polystable for the canonical linearized \( SL(2, \mathbb{C}) \)-action on the space of effective divisors.

Our main result in genus \( g = 0 \) is the following converse to Theorem 1.2, for the more general gravitating vortex equations (see Section 6.2).

**Theorem 1.3.** If \((\mathbb{P}^1, L, \phi)\) admits a solution of the gravitating vortex equations with \( \alpha > 0 \), then (1.3) holds and the divisor \( D \) is polystable for the \( SL(2, \mathbb{C}) \)-action.

The key idea for its proof comes from the observation that the powerful methods of the theory of symplectic and GIT quotients are ideally suited to analyze the gravitating vortex equations.

Combining now Theorems 1.2 and 1.3 we obtain a correspondence theorem for the Einstein–Bogomol’nyi equations.

**Theorem 1.4.** A triple \((\mathbb{P}^1, L, \phi)\) with \( \phi \neq 0 \) admits a solution of the Einstein–Bogomol’nyi equations if and only (1.3) holds and the divisor \( D \) is polystable for the \( SL(2, \mathbb{C}) \)-action.

Note that Theorem 1.4 does not claim uniqueness of solutions modulo automorphisms of \((\mathbb{P}^1, L, \phi)\). However this should be expected on general grounds, as our methods rely on appropriate versions of techniques developed over the years for the recently solved Kähler–Einstein problem [13] (see also [8, 18, 15, 55]). In fact, Theorem 1.4 can be seen as a 2-dimensional toy model for this problem. As in this case, uniqueness is a delicate issue, related to the geodesic equation in the space of Kähler potentials [20, 39, 50]; see Section 6.3 for details and a discussion of the relevance of the homogeneous complex Monge–Ampère equation in this problem.
Theorem 1.4 clarifies Yang’s guess [64] that the location of the zeros of $\phi$ should “play an important role to global existence” and his observation that the condition corresponding to strict polystability is a “borderline situation” (with solutions preserved by an $S^1$-action). By comparison with the case $D = np$ (where he proved solutions cannot be $S^1$-symmetric [64, Theorem 1.1(ii)]), he stated the following (see also [65, p. 437]).

**Conjecture 1.5** (Yang’s conjecture). *There is no solution of the Einstein–Bogomol’nyi equations for $N$ strings superimposed at a single point, that is, when $D = Np$.*

Using our approach to the Einstein–Bogomol’nyi equations via symplectic and algebraic geometry, and the general theory for coupled equations developed in [1], in Sections 4–6 we construct obstructions to the existence of solutions for the gravitating vortex equations on $\mathbb{P}^1$, and apply them to settle Conjecture 1.5 in the affirmative. Actually we provide two different proofs of this conjecture. The first one (Corollary 4.8) follows from an analogue of the Matsushima–Lichnerowicz Theorem [40, 36] (see Theorem 4.1). The second proof is obtained by construction and direct evaluation of a Futaki invariant for the gravitating vortex equations on $\mathbb{P}^1$, and leads to a stronger result (Theorem 4.7).

We finish this section by observing that the gravitating vortex equations were obtained in [2] as a dimensional reduction of the Kähler–Yang–Mills equations introduced in [1], and indeed our original aim was to obtain new solutions of the latter equations (see [2, Section 1]). With the same motivation, we can use Theorem 1.1 to construct a new class of non-trivial solutions of the Kähler–Yang–Mills equations. In the context of dimensional reduction considered in [2], they are defined on a pair $(X, E)$ consisting of the product $X = \Sigma \times \mathbb{P}^1$ with Kähler form $\omega_\tau = p^*\omega + \frac{1}{2}q^*\omega_{FS}$, where $\omega_{FS}$ is the Fubini–Study metric on the complex projective line $\mathbb{P}^1$, $p: X \to \Sigma$ and $q: X \to \mathbb{P}^1$ are the canonical projections, and $E$ is the rank 2 vector bundle over $X$ given by the holomorphic extension in $\text{Ext}^1_X(q^*\mathcal{O}_{\mathbb{P}^1}(2), p^*L) \cong H^0(\Sigma, L)$ determined by $\phi$ (cf. [2, Theorem 1.1]).

**Theorem 1.6.** Let $\Sigma, L, \phi, N, \tau$ and $\alpha_*$ be as in Theorem 1.1 (so $\Sigma$ has genus $g \geq 2$). Then the set of $\alpha \geq 0$ for which the Kähler–Yang–Mills equations on $(\Sigma \times \mathbb{P}^1, E)$, with Kähler form $\omega_\tau$, admit an SU(2)-invariant solution of fixed volume, is open and contains the closed interval $[0, \alpha_*]$. Furthermore, the SU(2)-invariant solution is unique for $\alpha \in [0, \alpha_*]$.

**Proof.** Immediate by Theorem 1.1 and [2] Proposition 3.4. \qed

2. The gravitating vortex equations

In this section we introduce the gravitating vortex equations and reformulate Yang’s Existence Theorem in the language of Geometric Invariant Theory.

2.1. Gravitating vortices. Let $\Sigma$ be a compact connected Riemann surface of arbitrary genus, $L$ a holomorphic line bundle over $\Sigma$, and $\phi$ a global holomorphic section of $L$. Fix $\tau, \alpha \in \mathbb{R}$, respectively called the symmetry breaking parameter and the coupling constant.

**Definition 2.1.** The *gravitating vortex equations*, for a Kähler metric on $\Sigma$ with Kähler form $\omega$ and a hermitian metric $h$ on $L$, are

\[
\begin{align*}
 i\Lambda_\omega F_h + \frac{1}{2}(|\phi|^2_h - \tau) &= 0, \\
 S_\omega + \alpha(\Delta_\omega + \tau)(|\phi|^2_h - \tau) &= c.
\end{align*}
\]

(2.1)
In \( (2.1) \), \( F_h \) is the curvature 2-form of the Chern connection of \( h \), \( \Lambda_\omega F_h \in C^\infty(\Sigma) \) is its contraction with \( \omega \), \( |\phi|_h \in C^\infty(\Sigma) \) is the pointwise norm of \( \phi \) with respect to \( h \), \( S_\omega \) is the scalar curvature of \( \omega \) (as usual, Kähler metrics will be identified with their associated Kähler forms), and \( \Delta_\omega \) is the Laplace operator for the metric \( \omega \), given by
\[
\Delta_\omega f = 2i \Lambda_\omega \bar{\partial} \partial f,
\]
for \( f \in C^\infty(\Sigma) \). The constant \( c \in \mathbb{R} \) is topological, and is explicitly given by
\[
c = \frac{2\pi(\chi(\Sigma) - 2\alpha_1(L))}{\text{Vol}_\omega(\Sigma)}, \tag{2.2}
\]
with \( \text{Vol}_\omega(\Sigma) := \int_\Sigma \omega \), as can be deduced by integrating \( (2.1) \) over \( \Sigma \).

Given a fixed Kähler metric \( \omega \), the first equation in \( (2.1) \), that is,
\[
i \Lambda_\omega F_h + \frac{1}{2}(|\phi|_h^2 - \tau) = 0 \tag{2.3}
\]
is the vortex equation for a hermitian metric \( h \) on \( L \), also known as the Bogomol’nyi equations in the abelian-Higgs model. The solutions of \( (2.3) \) are called vortices and correspond to the absolute minima of an energy functional \([10, 27]\).

If \( \phi = 0 \), then the existence of solutions of \( (2.3) \) is equivalent by the Hodge Theorem to the numerical condition \( c_1(L) = \tau \text{Vol}_\omega(\Sigma)/4\pi \). For \( \phi \neq 0 \), Noguchi [45], Bradlow [10] and the third author [26, 27] gave independently and with different methods the following characterization of the existence of vortices.

**Theorem 2.2.** Assume that \( \phi \neq 0 \). Then, for every fixed Kähler form \( \omega \), there exists a unique solution \( h \) of the vortex equation \( (2.3) \) if and only if
\[
c_1(L) < \frac{\tau \text{Vol}_\omega(\Sigma)}{4\pi}. \tag{2.4}
\]

Finding a solution of the vortex equation \( (2.3) \) is not enough to solve the more complicated system of equations in Definition \( 2.1 \). As explained in Section \( 1 \) the gravitating vortex equations \( (2.1) \) describe vortices on a Riemann surface coupled with the Kähler metric \( \omega \), as the second equation in \( (2.1) \) intertwines the scalar curvature of \( \omega \) with the function \( |\phi|_h^2 \). The gravitating vortices, that is, the solutions of \( (2.1) \), are the main subject of the present paper. An important goal of our study, partially achieved in Theorem \( 1.4 \), is to find an analogue of Theorem \( (2.2) \) for gravitating vortices.

We discuss next two simple examples, where it is straightforward to characterize the existence of gravitating vortices.

**Example 2.3.** If \( \phi = 0 \), then the existence of solutions of \( (2.3) \) is equivalent to the numerical condition \( c_1(L) = \tau \text{Vol}_\omega(\Sigma)/4\pi \). This follows from the fact that for \( \phi = 0 \), the first and second equations in \( (2.1) \) reduce to the conditions that \( h \) is a Hermite–Einstein metric on \( L \) and \( \omega \) is a constant scalar curvature Kähler metric on \( \Sigma \), respectively. Therefore, the equivalence is a consequence of the Hodge Theorem applied to the equation in \( u \in C^\infty(\Sigma) \) obtained from \( (2.3) \) by making a conformal transformation \( h' = e^{2u}h \) to a fixed \( h \), and the Uniformization Theorem for Riemann surfaces.

**Example 2.4.** If \( \phi \neq 0 \) and \( c_1(L) = 0 \), the gravitating vortex equations \( (2.1) \) always have a solution for \( \tau > 0 \). This follows from the fact that if \( c_1(L) = 0 \) and \( H^0(\Sigma, L) \neq 0 \), then \( L \cong \mathcal{O}_\Sigma \) (see e.g. [31] Ch. IV]). By Theorem \( (2.2) \) for any choice of Kähler metric \( \omega \) on \( \Sigma \),
the unique solution of (2.3) is the constant hermitian metric $h$ on the trivial line bundle $L$, satisfying $|φ|^2_h = τ$. We conclude that the unique solutions of (2.1) in this case are pairs $(ω, h)$ such that $h$ is constant, $|φ|^2_h = τ$, and $ω$ has constant scalar curvature.

The conditions $φ ≠ 0$ and $c_1(L) > 0$ will be assumed throughout the rest of this paper. With these assumptions, the existence of a gravitating vortex implies $τ > 0$, by (2.4) in Theorem 2.2, and thus we fix $τ > 0$ in the sequel. We will also impose the condition $α > 0$, as this enables one to apply methods of the theory of Kähler quotients (see Section 3).

The sign of $c$ plays an important role in the problem of existence of gravitating vortices. The dependence of the gravitating vortex equations (2.7) on the topological constant $c$ is better observed using the following Kähler–Einstein type formulation of (2.1), where $ρ_ω$ is the Ricci form of $ω$:

$$iΛ_ωF_h + \frac{1}{2}(|φ|^2_h - τ) = 0,$$

$$ρ_ω - αdd^c(|φ|^2_h) - 2ατiF_h = cω.$$

Using that $Σ$ is compact, this system reduces to a second-order system of PDE. To see this, we fix a constant scalar curvature metric $ω_0$ on $Σ$ and the unique hermitian metric $h_0$ on $L$ with constant $Λω_0F_{h_0}$, and apply a conformal change to $h$ while changing $ω$ within its Kähler class. The equations (2.5) for $ω = ω_0 + dd^cv, h = e^{2f}h_0$, with $v, f ∈ C^∞(Σ)$, are equivalent to the following semi-linear system of partial differential equations (cf. [2, Lemma 4.3])

$$Δf + \frac{1}{2}(e^{2f}|φ|^2 - τ)e^{4ατf + 2αe^{2f}|φ|^2 + 2cv} = -c_1(L),$$

$$Δv + e^{4ατf + 2αe^{2f}|φ|^2 + 2cv} = 1.$$

Here, $Δ$ is the Laplacian of the fixed metric $ω_0$, normalized to have volume $2\pi$ and $|φ|$ is the pointwise norm with respect to the fixed metric $h_0$ on $L$. Note that $ω = (1 - Δv)ω_0$ implies $1 - Δv > 0$, which is compatible with the last equation in (2.6).

For $c ≥ 0$, the existence of gravitating vortices forces the topology of the surface to be that of the 2-sphere, because $c_1(L) > 0$ implies $χ(Σ) > 0$ by (2.2). The important case $c = 0$, for which the system (2.6) reduces to a single PDE, is treated in Section 2.2. The genus zero case of the gravitating vortex equations is studied in Section 4, Section 5, and Section 6. The condition $c < 0$ has important consequences for the analysis of (2.6), and is considered in genus $≥ 2$ in Section 7.

2.2. The Einstein-Bogomol’nyi equations. When $c$ in (2.2) is zero, the gravitating vortex equations (2.1) turn out to be a system of partial differential equations that have been extensively studied in the physics literature. Following Yang [60, 62], we will refer to them as the Einstein–Bogomol’nyi equations:

$$iΛ_ωF_h + \frac{1}{2}(|φ|^2_h - τ) = 0,$$

$$S_ω + α(Δ_ω + τ)(|φ|^2_h - τ) = 0.$$

(2.7)
As observed by Yang [63, Section 1.2.1], the existence of solutions of (2.7) constrains the topology of \( \Sigma \) to be the complex projective line (or 2-sphere) \( \mathbb{P}^1 \), since \( c = 0 \) if and only if
\[
\chi(\Sigma) = 2\alpha \tau c_1(L).
\]
The particular features of the Einstein–Bogomol’nyi equations (2.7) are better observed using the Kähler–Einstein type formulation of the gravitating vortex equations (2.1), given by (2.6). In the case \( c = 0 \), for \( L = O_{\mathbb{P}^1}(N) \) and
\[
e^{2u} = 1 - \Delta v
\]
the system (2.6) reduces to a single partial differential equation
\[
\Delta f + \frac{1}{2} e^{2u} (e^{2f}|\phi|^2 - \tau) = -N,
\]
for a function \( f \in C^\infty(\mathbb{P}^1) \), where
\[
u = 2\alpha \tau f - \alpha e^{2f}|\phi|^2 + c',
\]
and \( c' \) is a real constant that can be chosen at will. By studying the Liouville type equation (2.8) on \( \mathbb{P}^1 \), Yang [63, 64] proved the existence of solutions of the Einstein–Bogomol’nyi equations under certain numerical conditions on the zeros of \( \phi \), to which he refers as “technical restriction” [63, Section 1.3]. It turns out that these conditions have a precise algebro-geometric meaning in the context of Mumford’s Geometric Invariant Theory (GIT) [42], as a consequence of the following standard result.

**Proposition 2.5 ([42, Ch. 4, Proposition 4.1]).** Consider the space of effective divisors on \( \mathbb{P}^1 \) with its canonical linearised \( SL(2, \mathbb{C}) \)-action. Let \( D = \sum_j n_j p_j \) be an effective divisor, for finitely many different points \( p_j \in \mathbb{P}^1 \) and integers \( n_j > 0 \) such that \( N = \sum_j n_j \). Then

1. \( D \) is stable if and only if \( n_j < \frac{N}{2} \) for all \( j \).
2. \( D \) is strictly polystable if and only if \( D = \frac{N}{2} p_1 + \frac{N}{2} p_2 \), where \( p_1 \neq p_2 \) and \( N \) is even.
3. \( D \) is unstable if and only if there exists \( p_j \in D \) such that \( n_j > \frac{N}{2} \).

Using Proposition 2.5, Yang’s existence theorem has the following reformulation, where “GIT polystable” means either conditions (1) or (2) of Proposition 2.5 are satisfied, and
\[
D = \sum_j n_j p_j
\]
is the effective divisor on \( \mathbb{P}^1 \) corresponding to a pair \( (L, \phi) \), with \( N = \sum_j n_j = c_1(L) \).

**Theorem 2.6 (Yang’s Existence Theorem).** Assume that (2.4) holds. Then, there exists a solution of the Einstein–Bogomol’nyi equations (2.7) on \( (\mathbb{P}^1, L, \phi) \) if \( D \) is GIT polystable for the canonical linearised \( SL(2, \mathbb{C}) \)-action of the space of effective divisors.

For the benefit of the reader, we comment briefly on the proof. If \( D \) is stable, then the existence of solutions of the Einstein–Bogomol’nyi equations follows by Yang’s result [63 Theorem 1.2] and part (1) of Proposition 2.5. Yang also proved [63, Theorem 1.1(i)] that the Einstein–Bogomol’nyi equations have a solution if \( D = \frac{N}{2} p + \frac{N}{2} \overline{p} \), for \( N \) even and antipodal points \( p, \overline{p} \) on \( \mathbb{P}^1 \), and that this solution admits an \( S^1 \)-symmetry given by rotation along the \( \{ p, \overline{p} \} \) axis. If \( D \) is an arbitrary strictly polystable effective divisor, so \( D = \frac{N}{2} p_1 + \frac{N}{2} p_2 \) as in part (2) of Proposition 2.5, then the existence of solutions of the
The gravitating vortex equations were first obtained [2] by dimensional reduction of the Kähler–Yang–Mills equations [1, 24], whereby the solutions acquired an interpretation as the zeros of a moment map in the general theory of symplectic quotients, for suitable infinite-dimensional manifolds. A direct approach to this moment-map interpretation, as described in this section, is however better suited to prove obstructions for the existence of gravitating vortices in the next sections (it will rely on previous calculations [21, 25]), and, furthermore to, construct the vortex Teichmüller space and vortex Riemann moduli space referred to in the Introduction (see Section 3.3).

3.1. A hamiltonian action on the space of sections of a line bundle. Let $S$ be a compact connected oriented smooth surface and $L$ a $C^\infty$ line bundle over $S$, respectively endowed with a symplectic form $\omega$ and a hermitian metric $h$. The group of symmetries relevant for our moment-map construction is the (Hamiltonian) extended gauge group $\tilde{G}$ of $(L, h)$ and $(S, \omega)$, given by an extension

$$1 \to G \longrightarrow \tilde{G} \overset{p}{\longrightarrow} \mathcal{H} \to 1,$$

of the group $\mathcal{H}$ of Hamiltonian symplectomorphisms of $(S, \omega)$ by the unitary gauge group $G$ of $(L, h)$. More precisely, $\tilde{G}$ is the group of automorphisms of the hermitian line bundle $(L, h)$ that cover elements of the group $\mathcal{H}$, and $p$ maps any element of $\tilde{G}$ into the element of $\mathcal{H}$ that it covers.

For each unitary connection $A$ on $(L, h)$, $A\zeta$ denotes the corresponding vertical component of a vector field $\zeta$ on the total space of $L$, and $A^\perp y$ denotes the horizontal lift of a vector field $y$ on $S$ to a vector field on the total space of $L$. Then the decompositions $\zeta = A\zeta + A^\perp y$, with $y = p(\zeta)$, determine a vector-space splitting of the Lie-algebra short exact sequence

$$0 \to \text{Lie } G \longrightarrow \text{Lie } \tilde{G} \overset{p}{\longrightarrow} \text{Lie } \mathcal{H} \to 0,$$

associated to (3.1), because $A^\perp \eta \in \text{Lie } \tilde{G}$ for all $\eta \in \text{Lie } \mathcal{H}$. Note also that the equation

$$\eta_f \omega = df$$

(3.3)
determines an isomorphism between the space $\text{Lie } \mathcal{H}$ of Hamiltonian vector fields $\eta = \eta_f$ on $S$, and the space $C^\infty_0(S, \omega)$ of smooth functions $f$ on $S$ such that $\int_S f \omega = 0$.

We start describing a Hamiltonian $\tilde{G}$-action on the space $\Omega^0(L)$ of smooth global sections of $L$ over $S$. This vector space has a symplectic form $\omega_\Omega$ determined by $\omega$ and $h$, given by

$$\omega_\Omega(\phi_1, \phi_2) = -\text{Im} \int_S (\phi_1, \phi_2) h \omega,$$
where \( \dot{\phi}_1, \dot{\phi}_2 \in \Omega^0(L) \) are regarded as tangent vectors at any \( \phi \in \Omega^0(L) \). The 2-form \( \omega_\Omega \) is exact, that is,
\[
\omega_\Omega = d\sigma,
\]
where the 1-form \( \sigma \) on \( \Omega^0(L) \) is given by
\[
\sigma_\phi(\dot{\phi}) = -\text{Im} \int_S (\dot{\phi}, \phi)_h \omega,
\]
for all \( \phi \in \Omega^0(L) \) and \( \dot{\phi} \in \Omega^0(L) \). Furthermore, \( \omega_\Omega \) is a Kähler 2-form with respect to the canonical complex structure on \( \Omega^0(L) \) given by multiplication by \( i = \sqrt{-1} \).

We observe now that \( \tilde{G} \) has a canonical action on \( \Omega^0(L) \), defined by
\[
(g \cdot \phi)(x) := g(\phi(p(g)^{-1}x)),
\]
for all \( g \in \tilde{G}, \phi \in \Omega^0(L), x \in S \), where \( p \) is the map in (3.1).

**Lemma 3.1.** The \( \tilde{G} \)-action on \( \Omega^0(L) \) is Hamiltonian, with equivariant moment map
\[
\mu : \Omega^0(L) \to (\text{Lie } \tilde{G})^*
\]
given by
\[
\langle \mu, \zeta \rangle = -\sigma(Y_\zeta),
\]
where \( Y_\zeta \) denotes the infinitesimal action of \( \zeta \in \text{Lie } \tilde{G} \) on \( \Omega^0(L) \). For any choice of unitary connection \( A \) on \( L \), the moment map is given explicitly by
\[
\langle \mu(\phi), \zeta \rangle = \frac{i}{2} \int_S A\zeta |\phi|^2 \omega - \frac{i}{2} \int_S f(dA\phi, \phi)_h
\]
(3.5)
for all \( \phi \in \Omega^0(L) \) and \( \zeta \in \text{Lie } \tilde{G} \) covering \( \eta_f \in \text{Lie } H \), with \( f \in C^\infty_0(S) \).

**Proof.** The first part follows trivially because \( \omega_\Omega = d\sigma \) and \( \sigma \) is \( \tilde{G} \)-invariant. To prove (3.5), we fix a unitary connection \( A \), so the infinitesimal action of \( \text{Lie } \tilde{G} \) on \( \Omega^0(L) \) is given by [25]
\[
Y_{\zeta}\phi = -\zeta \cdot dA\phi + A\zeta \cdot \phi,
\]
for all \( \zeta \in \text{Lie } \tilde{G} \) and \( \phi \in \Omega^0(L) \), with \( \zeta := p(\zeta) \). Then \( \zeta = \eta_f \), where \( f \in C^\infty_0(S) \), so
\[
\langle \mu(\phi), \zeta \rangle = \frac{i}{2} \int_S (-\zeta \cdot dA\phi + A\zeta \cdot \phi)_h \omega
\]
\[
= \frac{i}{2} \int_S (A\zeta \cdot \phi, \phi)_h \omega - \frac{i}{2} \int_S f(dA\phi, \phi)_h,
\]
where we have used the identity
\[
(\zeta \cdot dA\phi)_\omega = -df \wedge dA\phi.
\]

### 3.2. From Kähler reduction to gravitating vortices.

Let \( J \) and \( A \) be the spaces of almost complex structures on \( S \) compatible with \( \omega \) and unitary connections on \( (L, h) \), respectively; their respective elements will usually be denoted \( J \) and \( A \). The spaces \( J \) and \( A \) have a natural action by \( \tilde{G} \) and admit \( \tilde{G} \)-invariant symplectic structures \( \omega_J \) and \( \omega_A \) induced by \( \omega \). Consider now the space of triples
\[
J \times A \times \Omega^0(L),
\]
endowed with the symplectic structure
\[
\omega_J + 4\alpha \omega_A + 4\alpha \omega_\Omega
\]
(3.7)
GRAVITATING VORTEXES AND THE EINSTEIN–BOGOMOLOVNYI EQUATIONS

(for any non-zero coupling constant $\alpha$). By Lemma 3.1 combined with [24, Proposition 2.3.1], the diagonal action of $\tilde{G}$ on this space is Hamiltonian, with equivariant moment map $\mu_\alpha: J \times A \times \Omega^0(L) \to (\text{Lie } \tilde{G})^*$ given by

$$\langle \mu_\alpha(J, A, \phi), \zeta \rangle = 4i\alpha \int_S \tr A\zeta \wedge \left( i\Lambda F_A + \frac{1}{2} |\phi|_h^2 - \frac{\tau}{2} \right) \omega$$

$$- \int_M f \left( S_J + 2i\alpha (d(d_A\phi, \phi)_h - \tau\Lambda F_A) \right) \omega,$$

for any choice of a parameter $\tau \in \mathbb{R}$.

To make the link with the gravitating vortex equations (2.1), consider the space of ‘integrable triples’

$$T \subset J \times A \times \Omega^0(L)$$

defined by

$$T := \{ (J, A, \phi) \mid \bar{\partial}A\phi = 0 \}.$$

(3.9)

Then $T$ is a complex submanifold (away from its singularities) for the product formally integrable almost complex structure on the space (3.6) (see [11, (2.45)]). Moreover, it is preserved by the $\tilde{G}$-action and, by the condition $\alpha > 0$, it inherits a Hamiltonian action for the Kähler form induced by (3.7).

**Proposition 3.2.** The $\tilde{G}$-action on $T$ is Hamiltonian with $\tilde{G}$-equivariant moment map $\mu_\alpha: T \to (\text{Lie } \tilde{G})^*$ given by

$$\langle \mu_\alpha(J, A, \phi), \zeta \rangle = 4i\alpha \int_S A\zeta \left( i\Lambda F_A + \frac{1}{2} |\phi|_h^2 - \frac{\tau}{2} \right) \omega$$

$$- \int_M f \left( S_J + \alpha \Delta_\omega |\phi|_h^2 - 2\alpha \tau i\Lambda F_A \right) \omega,$$

for all $(J, A, \phi) \in T$ and $\zeta \in \text{Lie } \tilde{G}$ covering $\eta f \in \text{Lie } \mathcal{H}$, where $f \in C_0^\infty(S)$.

Proof. Since $(J, A, \phi) \in T$, we have $\bar{\partial}A\phi = 0$ and hence

$$\Delta_\omega |\phi|_h^2 = 2i\Lambda \bar{\partial}\partial |\phi|_h^2 = 2i\Lambda d(\partial A\phi, \phi)_h = 2i\Lambda d(d_A\phi, \phi)_h.$$

The statement follows now from (3.8). \Box

It can be readily checked that the zeros of the moment map $\mu_\alpha$, restricted to the space of integrable pairs, correspond to solutions of the gravitating vortex equations

$$i\Lambda F_A + \frac{1}{2} (|\phi|_h^2 - \tau) = 0,$$
$$\bar{\partial}A\phi = 0,$$

$$S_J + \alpha (\Delta_\omega + \tau)(|\phi|_h^2 - \tau) = c,$$

where the topological constant $c \in \mathbb{R}$ is explicitly given by

$$c = \frac{2\pi (\chi(S) - 2\alpha \tau c_1(L))}{\text{Vol}_\omega(S)}.$$

(3.12)

Given a solution of (3.11), considering the complex structure on $S$ given by $J$, the holomorphic structure on the line bundle $L$ given by $A$ and the holomorphic section $\phi$, we can regard $(\omega, h)$ as a solution of the gravitating vortex equations (2.1) as originally stated in Section 2. Conversely, any solution $(\omega, h)$ of (2.1), for a holomorphic line bundle with a
global section over a compact Riemann surface, determines a solution of (3.11) by taking $A$ to be the Chern connection of $h$.

3.3. Moduli spaces. We define the moduli space of gravitating vortices $\mathcal{M}_\alpha$ as the space of solutions $(J, A, \phi)$ of the gravitating vortex equations (3.11), modulo the action of the Hamiltonian extended gauge group $\tilde{\mathcal{G}}$ defined in (3.1). By Proposition 3.2 this is a symplectic quotient

$$\mathcal{M}_\alpha = \mu^{-1}_\alpha(0)/\tilde{\mathcal{G}},$$

(3.13)

so away from its singularities, it is a Kähler quotient for the action of $\tilde{\mathcal{G}}$ on the smooth part of $\mathcal{T}$, equipped with a Kähler structure induced by the restriction of (3.7), that may be interpreted as a generalized Weil–Petersson metric (see, e.g., [22, Theorem 5.7] for a similar construction). Note that the moduli space of solutions of the vortex equations (2.3) also has a Kähler-reduction interpretation, and can be identified as a complex manifold with the $N$-th symmetric product $S^N\Sigma$ of the Riemann surface, where $N = c_1(L)$ (see [10, Theorem 4.7], [26], and [27, p. 92]).

Recall now that the link between the Teichmüller space and the moduli space of Riemann surfaces involves the mapping class group $\Gamma = \pi_0(Diff_\omega(S)) = Diff_\omega(S)/Diff_\omega,0(S)$, where $Diff_\omega(S)$ is the group of symplectomorphisms of $(S, \omega)$, and $Diff_\omega,0(S)$ is its identity component. Likewise, to construct moduli spaces parametrizing complex structures together with effective divisors, we consider a ‘bundle-automorphism class group’ $\tilde{\Gamma} := \pi_0(\tilde{\mathcal{G}}_\omega) = \tilde{\mathcal{G}}_\omega/\tilde{\mathcal{G}}_\omega,0$, where the ‘large’ extended gauge group $\tilde{\mathcal{G}}_\omega$ of $S$ and $(L, h)$ is the group of automorphisms of the hermitian line bundle $(L, h)$ that cover elements of $Diff_\omega(S)$, and $\tilde{\mathcal{G}}_\omega,0$ is its identity component. It turns out that an element of $\tilde{\mathcal{G}}_\omega$ covers an element of $Diff_\omega,0(S)$ if and only if it is in $\tilde{\mathcal{G}}_\omega,0 \subset \tilde{\mathcal{G}}_\omega$ (see, e.g., [6, p. 280]), that is, we have a group extension

$$1 \rightarrow \mathcal{G} \rightarrow \tilde{\mathcal{G}}_\omega,0 \rightarrow Diff_\omega,0(S) \rightarrow 1,$$

so the quotient of $\tilde{\mathcal{G}}_\omega,0$ by the Hamiltonian extended gauge group $\tilde{\mathcal{G}} \subset \tilde{\mathcal{G}}_\omega,0$ defined in (3.1) is a $2g$-torus (where $g$ is the genus of $S$), namely

$$A_S := H^1(S, \mathbb{R})/H^1(S, \mathbb{Z}) \cong Diff_\omega,0(S)/\mathcal{H} \cong \tilde{\mathcal{G}}_\omega,0/\tilde{\mathcal{G}}.$$

Hence the $\tilde{\mathcal{G}}_\omega,0$-action on the space $\mathcal{T}$ of triples $(J, A, \phi)$ given by pull-back induces an action of the torus $A_S$ on $\mathcal{M}_\alpha$, and we define the vortex Teichmüller space as the orbit space

$$\text{Teich}_\alpha := \mathcal{M}_\alpha/A_S = \mu^{-1}_\alpha(0)/\tilde{\mathcal{G}}_\omega,0.$$

Finally we define the vortex Riemann moduli space as

$$\tilde{\mathcal{M}}_\alpha := \mu^{-1}_\alpha(0)/\tilde{\mathcal{G}}_\omega = \mathcal{M}_\alpha/(\tilde{\mathcal{G}}_\omega/\tilde{\mathcal{G}}) = \text{Teich}_\alpha/\tilde{\Gamma},$$

where $\tilde{\Gamma}$ plays the role of the mapping class group $\Gamma = \pi_0(Diff_\omega(S))$ in the standard construction of the moduli of Riemann surfaces as the orbit space of the Teichmüller space by the $\Gamma$-action. As in the classical connection between the complex analytic and algebro-geometric descriptions of the moduli space of Riemann surfaces, we expect that $\tilde{\mathcal{M}}_\alpha$ is isomorphic to the complex analytic space underlying the algebro-geometric moduli space of smooth complex projective curves equipped with an effective divisors (see, e.g., [33].
§2.1.3]. In fact, Theorem [13] should be a basic ingredient to obtain such an isomorphism in genus $g \geq 2$.

4. Reductive Lie algebras and gravitating vortices

In this section we give an affirmative answer to Yang’s Conjecture [13]. For this, we consider triples $(\Sigma, L, \phi)$, given by a Riemann surface $\Sigma$, a line bundle $L$ over $\Sigma$ and a holomorphic section $\phi$ of $L$, and study the complex Lie algebra of the corresponding group of automorphisms $\text{Aut}(\Sigma, L, \phi)$. The proof of Yang’s Conjecture follows from an analogue of Matsushima–Lichnerowicz Theorem [36, 40] for the gravitating vortex equations, which relates the existence of a solution on $(\Sigma, L, \phi)$ with the reductivity of $\text{Lie Aut}(\Sigma, L, \phi)$.

4.1. Matsushima-Lichnerowicz for gravitating vortices. Let $\Sigma$ be a compact Riemann surface of arbitrary genus $g(\Sigma)$, $L$ a holomorphic line bundle over $\Sigma$ with $c_1(L) > 0$, and $\phi \in H^0(\Sigma, L)$ a non-zero section. The automorphism group of the pair $(\Sigma, L)$ is the group $\text{Aut}(\Sigma, L)$ of $C^\ast$-equivariant automorphisms of the total space of the holomorphic line bundle $L$, with the $C^\ast$-action on $L$ given by fibrewise scalar multiplication. As $C^\ast$-equivariant automorphisms of $L$ preserve the zero section, there is a canonical exact sequence

$$1 \to C^\ast \longrightarrow \text{Aut}(\Sigma, L) \overset{p}{\longrightarrow} \text{Aut}(\Sigma),$$

where $\text{Aut}(\Sigma)$ is the automorphism group of $\Sigma$ and the right-hand arrow maps any $g \in \text{Aut}(\Sigma, L)$ into the unique $p(g) = \hat{g} \in \text{Aut}(\Sigma)$ covered by $g$. The automorphism group of the triple $(\Sigma, L, \phi)$ is the isotropy subgroup

$$\text{Aut}(\Sigma, L, \phi) \subset \text{Aut}(\Sigma, L)$$

of $\phi$ for the induced action of $\text{Aut}(\Sigma, L)$ on $H^0(\Sigma, L)$. By a result of Morimoto [31, p. 158], $\text{Aut}(\Sigma, L)$ (with the compact-open topology) is a complex Lie group, and the action of $\text{Aut}(\Sigma, L)$ on $L$ and the right-hand map in (4.1) are both holomorphic [31, Section 7]. Let

$$\text{Lie Aut}(\Sigma, L, \phi) \subset \text{Lie Aut}(\Sigma, L)$$

be the Lie algebras of $\text{Aut}(\Sigma, L, \phi) \subset \text{Aut}(\Sigma, L)$, respectively. By definition of $\text{Aut}(\Sigma, L)$, $\text{Lie Aut}(\Sigma, L)$ consists of the $C^\ast$-invariant holomorphic vector fields on the total space of $L$, and so $\text{Lie Aut}(\Sigma, L, \phi)$ consists of those vector fields with zero infinitesimal action on $\phi \in H^0(\Sigma, L)$.

In this section, we obtain a first class of obstructions to the existence of gravitating vortices on $(\Sigma, L, \phi)$, in terms of the complex Lie algebra $\text{Lie Aut}(\Sigma, L, \phi)$.

**Theorem 4.1.** If $(\Sigma, L, \phi)$ admits a solution of the gravitating vortex equations, then the Lie algebra of $\text{Aut}(\Sigma, L, \phi)$ is reductive.

This result should be compared with the Matsushima–Lichnerowicz Theorem [36, 40], which states that the Lie algebra of holomorphic vector fields of a compact Kähler manifold with constant scalar curvature is reductive. Our proof breaks up into two separate cases, depending on whether $\Sigma$ has positive or zero genus, respectively. The positive genus case is a formality, and follows from the fact that $\text{Aut}(\Sigma, L)$ is discrete in this case. We give a proof of this basic fact in Proposition [43, 3] for the benefit of the reader. The genus zero case is discussed in Section 4.2.

We will use the following description of $\text{Lie Aut}(\Sigma, L)$ (see e.g. [21, p. 490]).
Lemma 4.2. A $\mathbb{C}^*$-invariant vector field $y$ on the total space of $L$ belongs to Lie Aut($\Sigma, L$) if and only if for any choice of hermitian metric $h$ on $L$ the following equation is satisfied:

$$\bar{\partial}(A_h y) + i_{g^1,0} F_h = 0. \tag{4.4}$$

In (4.4) $A_h$ is the Chern connection of the hermitian metric $h$ and $\bar{y}$ denotes the unique holomorphic vector field on $\Sigma$ covered by $y$.

Proposition 4.3. If $g(\Sigma) > 0$, then the group Aut($\Sigma, L, \phi$) is discrete.

Proof. Let $y \in$ Lie Aut($\Sigma, L, \phi$). We will show that $y$ is vertical, that is, the holomorphic vector field $\bar{y} \in$ Lie Aut($\Sigma$) covered by $y$ is zero. The result will then follow because the only holomorphic vertical vector fields are $y = t1$, for constant $t \in \mathbb{C}$, where 1 is the tautological vector field on the fibres, and so the condition that $y$ fixes $\phi \neq 0$ implies $y = 0$. If $g(\Sigma) > 1$, the fact that $\bar{y} = 0$ is deduced, e.g., because there exists a negative curvature Kähler metric on $\Sigma$. Suppose now $g(\Sigma) = 1$. Then, there exists a flat Kähler metric on $\Sigma$, so either $\bar{y}$ has no zeros or it vanishes identically, as it is necessarily parallel with respect to the flat Kähler metric (see e.g. [28]). Now, by assumption $c_1(L) > 0$, so $L$ is ample [31, Ch IV, Cor. 3.3], and therefore there exists a hermitian metric $h$ on $L$ such that $\omega = i F_h$ is a Kähler metric on $\Sigma$. But Lemma 4.2 applied to $y$ and the hermitian metric $h$ imply

$$- i\bar{\partial}(A_h y) = i_{g^1,0} \omega, \tag{4.5}$$

that is, $A_h y$, identified with a complex function on $\Sigma$, is a complex potential for $\bar{y}$. Therefore, $\bar{y}$ vanishes somewhere on $\Sigma$ (see [34]) and hence it identically vanishes. □

Remark 4.4. Theorem 4.1 can be extended to cover the cases in Example 2.3 and Example 2.4 for which the Lie algebra of Aut($\Sigma, L, \phi$) is always reductive. To illustrate this, consider the seconde case, corresponding to $\phi \neq 0$ and $c_1(L) = 0$. Then, it is easy to check that Aut($\Sigma, L, \phi$) $\cong$ Aut($\Sigma$), and thus Lie Aut($\Sigma, L, \phi$) is isomorphic to $\mathfrak{gl}(2, \mathbb{C})$, $H^1(\Sigma, \mathbb{C})$ or is trivial, if, respectively, $g(\Sigma) = 0$, $g(\Sigma) = 1$ or $g(\Sigma) > 1$.

4.2. Genus zero. Our proof of Theorem 4.1 in the remaining case $\Sigma = \mathbb{P}^1$, whereby we provide a first obstruction to the existence of gravitating vortices, relies on the moment-map interpretation of the gravitating vortex equations, following closely Donaldson–Wang’s abstract proof [58, Theorem 38] of the Matsushima–Lichnerowicz Theorem.

Let $L$ be a holomorphic line bundle over $\mathbb{P}^1$ and $\phi \in H^0(\mathbb{P}^1, L)$. To simplify the notation, $G$ and $\mathfrak{g}$ will denote the complex Lie group Aut($\mathbb{P}^1, L, \phi$) and its Lie algebra Lie Aut($\mathbb{P}^1, L, \phi$), respectively. Consider the smooth manifold underlying $\mathbb{P}^1$, namely the 2-sphere $S^2$, and the corresponding almost complex structure $J$ on $S^2$.

Let $\omega$ be a Kähler form on $\mathbb{P}^1$ and $h$ a hermitian metric on $L$. In Lemma 4.5, we will not suppose that $(\omega, h)$ is a solution of the gravitating vortex equations. This lemma gives a convenient formula for the elements of Lie Aut($\mathbb{P}^1, L$) adapted to the pair $(\omega, h)$, and is reminiscent of the Hodge-theoretic description of holomorphic vector fields on compact Kähler manifolds (see e.g. [28, Ch. 2]). As in (3.2), Lie $\mathcal{G}$ will denote the Lie algebra of the extended gauge group of $(L, h)$ and $(S^2, \omega)$.

Lemma 4.5. For any $y \in$ Lie Aut($\mathbb{P}^1, L$), there exist $\zeta_1, \zeta_2 \in$ Lie $\mathcal{G}$ such that

$$y = \zeta_1 + I \zeta_2.$$
Proof. Let $A$ be the Chern connection of $h$ on $L$. We will use the decomposition of
\[ y = Ay + A^\perp \tilde{y} \] into its vertical and horizontal components $Ay, A^\perp \tilde{y}$, where $\tilde{y}$ is the unique holomorphic vector field on $\mathbb{P}^1$ covered by $y$. Since $L$ is a line bundle, we can make the identification
\[ Ay = f^1, \] where $f^1 \in C^\infty(S^2, \mathbb{C})$ is a smooth complex function and $1$ is the tautological vector field on the fibres. Furthermore, as $S^2$ is simply connected,
\[ \tilde{y} = \tilde{y}_1 + J \tilde{y}_2, \] where $\tilde{y}_1$ and $\tilde{y}_2$ are Hamiltonian vector fields on $(S^2, \omega)$. Hence, defining the vector fields
\[ \zeta_j = if^j 1 + A^\perp \tilde{y}_j, \] for $j = 1, 2$, where $f_1 = \text{Im } f$ and $f_2 = -\text{Re } f$, we obtain the required result. \qed

We will now apply the decomposition of Lemma 4.5 to elements of $g \subset \text{Lie Aut}(\mathbb{P}^1, L)$.

Lemma 4.6. Let $y \in g$. If $(\omega, h)$ is a solution of the gravitating vortex equations, then $\zeta_1, \zeta_2 \in g$.

Proof. By the results of Section 3, if $(\omega, h)$ is a solution of the gravitating vortex equations, then the triple $t := (J, A, \phi)$ is a zero of a moment map
\[ \mu_\alpha : \mathcal{T} \to \text{Lie} \tilde{G}^* \] for the action of $\tilde{G}$ on the space of 'integrable triples'
\[ \mathcal{T} \subset \mathcal{J} \times \mathcal{A} \times \Omega^0(L) \] defined in (3.9). Recall that $\mathcal{T}$ is endowed with a (formally) integrable almost complex structure $I$, and Kähler metric
\[ g_\alpha = \omega_\alpha(\cdot, I \cdot) \] (as $\alpha > 0$), where the compatible symplectic structure $\omega_\alpha$ is as in (3.7). Given a $C^*$-invariant smooth vector field $v$ on the total space of $L$, we denote by $Y_{v|t}$ the infinitesimal action of $v$ on $t = (J, A, \phi)$. Then the proof reduces to show that $Y_{\zeta_1|t} = Y_{\zeta_2|t} = 0$. To prove this, we note that since the almost complex structure $I$ on $L$ determined by $J$ and $A$ is integrable, we have (see [1, Section 3.2])
\[ 0 = Y_{g|t} = Y_{\zeta_1 + I\zeta_2|t} \] (3.9). Considering now the norm $\| \cdot \|_\alpha$ on $\mathcal{T}$ induced by the metric $g_\alpha$, we obtain
\[ 0 = \| Y_{g|t} \|_\alpha^2 = \| Y_{\zeta_1|t} \|_\alpha^2 + \| Y_{\zeta_2|t} \|_\alpha^2 - 2\omega_\alpha(Y_{\zeta_1|t}, Y_{\zeta_2|t}). \]
Now, $\mu_\alpha(t) = 0$ and the moment map $\mu_\alpha$ is equivariant, so
\[ \omega_\alpha(Y_{\zeta_1|t}, Y_{\zeta_1|t}) = d\langle \mu_\alpha(\zeta_1), Y_{\zeta_1|t} \rangle = \langle \mu_\alpha(t), [\zeta_1, \zeta_2] \rangle = 0, \] and therefore
\[ \| Y_{\zeta_1|t} \|_\alpha^2 = \| Y_{\zeta_2|t} \|_\alpha^2 = 0, \] so we conclude that $\zeta_1, \zeta_2 \in g$, as required. \qed

Theorem 4.1 is now a formal consequence of Lemma 4.6.
Proof of Theorem 4.1 in genus zero. Considering the $\widetilde{G}$-action on $T$, we note that the Lie algebra $\mathfrak{k} = \text{Lie} \widetilde{G}_t$ of the isotropy group $\widetilde{G}_t$ of the triple $t = (J, A, \phi) \in T$ satisfies

$$\mathfrak{k} \oplus I \mathfrak{k} \subset \mathfrak{g}.$$ 

Furthermore, the Lie group $\widetilde{G}_t$ is compact, because it can be regarded as a closed subgroup of the isometry group of a Riemannian metric on the total space of the frame $U(1)$-bundle of $L$ (see [1, Section 2.3]). Now, Lemma 4.6 implies that

$$\mathfrak{g} = \mathfrak{k} \oplus I \mathfrak{k},$$ 

so $\mathfrak{g}$ is the complexification of the Lie algebra $\mathfrak{k}$ of a compact Lie group, and hence $\mathfrak{g}$ is a reductive complex Lie algebra. □

4.3. Proof of Yang’s Conjecture. To apply Theorem 4.1 we will now consider the gravitating vortex equations (3.11) on $\Sigma = \mathbb{P}^1$, with fixed $\alpha > 0$ and $\tau > 0$.

Theorem 4.7. If $\phi$ has only one zero, then there are no solutions of the gravitating vortex equations for $(\mathbb{P}^1, L, \phi)$.

Proof. We make the identification $L = \mathcal{O}_{\mathbb{P}^1}(N)$, with $N := c_1(L) > 0$, and fix homogeneous coordinates $[x_0, x_1]$ on $\mathbb{P}^1$. Then $H^0(\Sigma, L) \cong S^N(\mathbb{C}^2)^*$ is the space of degree $N$ homogeneous polynomials in the coordinates $x_0, x_1$, so it is a $\text{GL}(2, \mathbb{C})$-representation, where $g \in \text{GL}(2, \mathbb{C})$ maps a polynomial $p(x_0, x_1)$ into the polynomial $p(g^{-1}(x_0, x_1))$. Furthermore, $\text{Aut}(\mathbb{P}^1) = \text{PGL}(2, \mathbb{C})$ and the sequence (4.1) is a short exact sequence

$$1 \rightarrow \mathbb{C}^* \rightarrow \text{Aut}(\mathbb{P}^1, L) \xrightarrow{p} \text{PGL}(2, \mathbb{C}) \rightarrow 1.$$

Here, the third arrow is surjective, since it is the horizontal arrow in a commutative diagram

$$\text{GL}(2, \mathbb{C}) \xrightarrow{\rho} \text{Aut}(\mathbb{P}^1, L) \xrightarrow{\rho} \text{PGL}(2, \mathbb{C}),$$

where the diagonal arrow is the canonical surjective morphism, and the vertical arrow $\rho$ is the canonical $\text{GL}(2, \mathbb{C})$-linearization of $L$, that is, it is the surjective morphism induced by the $\text{GL}(2, \mathbb{C})$-representation $H^0(\mathbb{P}^1, L)$. Note that an element in the centre, $\lambda \in \mathbb{C}^* \subset \text{GL}(2, \mathbb{C})$, acts via $\rho$ on $L$ by fibrewise multiplication by $\lambda^{-N}$.

Suppose now $\phi \in H^0(\Sigma, L)$ vanishes at a single point, with multiplicity $N$, so it can be identified, up to the $\text{GL}(2, \mathbb{C})$-action, with the degree $N$ homogeneous polynomial

$$\phi \cong x_0^N.$$

Computing its isotropy group for the $\text{GL}(2, \mathbb{C})$-action on the space of degree $N$ homogeneous polynomials, it is straightforward to see that

$$\text{Aut}(\mathbb{P}^1, L, \phi) \cong \mathbb{C}^* \rtimes \mathbb{C},$$

or more explicitly, $\text{Aut}(\mathbb{P}^1, L, \phi)$ the image under $\rho$ (see (4.7)) of the subgroup

$$\begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix} \subset \text{GL}(2, \mathbb{C}).$$

(4.8)

Consequently, $\text{Aut}(\mathbb{P}^1, L, \phi)$ is not reductive, and thus the result follows from Theorem 4.1. □
Since the Einstein–Bogomol’nyi equations (2.7) are a particular case of the gravitating vortex equations (2.1) on $\mathbb{P}^1$, Theorem 4.7 settles Yang’s Conjecture 1.5 (cf. Remark 5.5).

**Corollary 4.8** (Yang’s conjecture). There is no solution of the Einstein–Bogomol’nyi equations for $N$ strings superimposed at a single point, that is, when $(L, \phi)$ corresponds to a divisor $D = Np$.

5. A Futaki invariant for gravitating vortices in $g = 0$

Relying on the moment map interpretation of the gravitating vortex equations (3.11) provided in Section 3, we apply now the general method in [1, Section 3] to construct a Futaki invariant for the gravitating vortex equations.

5.1. Definition of the Futaki invariant. Let $\Sigma$ be a compact Riemann surface, $L$ a holomorphic line bundle over $\Sigma$ with $c_1(L) > 0$, and $\phi \in H^0(\Sigma, L)$ a non-zero section. The Futaki invariant is a character of the Lie algebra Lie Aut($\Sigma, L, \phi$) of infinitesimal automorphisms of $(\Sigma, L, \phi)$ (see (4.3)). By Proposition 4.3, this Lie algebra is trivial if $g(\Sigma) > 0$, and therefore we assume $\Sigma = \mathbb{P}^1$ throughout this section.

Fix $\alpha, \tau$, and $\text{Vol}(\mathbb{P}^1)$ positive real numbers. We denote by $B$ the space of pairs $(\omega, h)$ consisting of a Kähler form $\omega$ on $\mathbb{P}^1$ with volume $\text{Vol}(\mathbb{P}^1)$, and a hermitian metric $h$ on $L$. Throughout Section 5, we will view the gravitating vortex equation (3.11) as equations where the unknowns belong to the space $B$. Define a map

$$\mathcal{F}_{\alpha, \tau}: \text{Lie Aut}(\mathbb{P}^1, L, \phi) \longrightarrow \mathbb{C},$$

by the following formula, for all $y \in \text{Lie Aut}(\mathbb{P}^1, L, \phi)$, where $(\omega, h) \in B$:

$$\langle \mathcal{F}_{\alpha, \tau}, y \rangle = 4i\alpha \int_{\mathbb{P}^1} A_h y \left( i\Lambda_\omega F_h + \frac{1}{2}|\phi|^2_h - \frac{\tau}{2} \right) \omega - \int_{\mathbb{P}^1} \varphi \left( S_\omega + \alpha \Delta_\omega |\phi|^2_h - 2i\alpha \tau \Lambda_\omega F_h \right) \omega.$$  

(5.2)

Here, $A_h$ is the Chern connection of $h$ on $L$, $A_h y \in C^\infty(\mathbb{P}^1, i\mathbb{R})$ is the vertical projection of $y$ with respect to $A_h$, and the complex valued function $\varphi$ on $\mathbb{P}^1$ is defined as follows. Let $\tilde{y}$ be the holomorphic vector field on $\mathbb{P}^1$ covered by $y$ and $A^\perp \tilde{y}$ its horizontal lift to a vector field on the total space of $L$ given by the connection $A_h$, so $y$ has a decomposition

$$y = Ay + A^\perp \tilde{y}$$  

(5.3)

(see (4.6)). Then $\varphi := \varphi_1 + i\varphi_2 \in C^\infty(\mathbb{P}^1, \mathbb{C})$ is determined by the unique decomposition

$$\tilde{y} = \eta_{\varphi_1} + J\eta_{\varphi_2}$$

associated to the Kähler form $\omega$ (see (3.4)), where $\eta_{\varphi_j}$ is the Hamiltonian vector field of the function $\varphi_j \in C^\infty(\mathbb{P}^1, \omega)$ on $(\mathbb{P}^1, \omega)$ (see (3.3)), for $j = 1, 2$, and $J$ is the almost complex structure of $\mathbb{P}^1$. Note that the previous decomposition uses the fact that $\mathbb{P}^1$ is simply connected.

The non-vanishing of $\mathcal{F}_{\alpha, \tau}$ is our second obstruction to the existence of gravitating vortices.

**Proposition 5.1.** The map $\mathcal{F}_{\alpha, \tau}$ is independent of the choice of $(\omega, h) \in B$. It is a character of the Lie algebra Lie Aut$(\mathbb{P}^1, L, \phi)$, that vanishes identically if there exists a solution $(\omega, h)$ of the gravitating vortex equations (3.11) on $(\mathbb{P}^1, L, \phi)$, with volume $\text{Vol}(\mathbb{P}^1)$.
By analogy with Futaki’s obstruction to the existence of Kähler–Einstein metrics [23], \( F_{\alpha, \tau} \) will be called the Futaki invariant for the gravitating vortex equations (3.11), with symmetry breaking parameter \( \tau \), coupling constant \( \alpha \), and volume \( \text{Vol}(\Sigma) \). Note that the term \( \int_{\mathbb{P}^1} \varphi S_\omega \omega \) in (5.2) is in fact the original Futaki character.

**Proof.** In the framework of Section 3, we consider the \( C^\infty \) manifold \( S^2 \) underlying the Riemann sphere \( \mathbb{P}^1 \). For \( b = (\omega, h) \in B \), let \( T_b \) be the associated space of ‘integrable triples’

\[
T_b \subset J_\omega \times A_h \times \Omega^0(L)
\]

defined in (3.9), with a distinguished point \( t_b = (J, A, \phi) \) given by the triple \( (\mathbb{P}^1, L, \phi) \). Recall that \( T_b \) is endowed with a (formally) integrable almost complex structure \( I \), and a Kähler metric (as \( \alpha > 0 \)), with compatible symplectic structure \( \omega_\alpha \) as in (3.7). Furthermore, there is a Hamiltonian action of the extended gauge group \( \tilde{G}_b \) on \( T_b \) such that if \( b = (\omega, h) \) is a solution of the gravitating vortex equations, then the triple \( t_b = (J, A, \phi) \) is a zero of a moment map (3.10). Then, we can construct a \( \mathbb{C} \)-linear map

\[
F_b : \text{Lie Aut}(\mathbb{P}^1, L, \phi) \rightarrow \mathbb{C}
\]
as in [1] (3.108). The explicit formula for this map is obtained as in [1] (3.126), replacing the moment map formula [1] (2.44) by (3.10). The proof now follows as for [1] Theorem 3.9.

An alternative proof can be given using [1] Theorem 3.9 and the relation of the gravitating vortex equations with the Kähler–Yang–Mills equations via dimensional reduction [2]. □

5.2. **An application of the Futaki character.** The following result illustrates the non-vanishing of the Futaki character as an obstruction to the existence of gravitating vortices. Contrary to the case of Theorem 4.7 and Yang’s Conjecture (Corollary 4.8), it corresponds to a situation in which the automorphism group is reductive (see Lemma 5.3) and so Theorem 4.1 cannot be applied.

**Theorem 5.2.** There is no solution of the gravitating vortex equations for \( (\mathbb{P}^1, L, \phi) \) with \( \phi \) vanishing exactly at two points with different multiplicities.

The proof of Theorem 5.2 follows from Proposition 5.1 and a direct calculation of the Futaki invariant on a holomorphic line bundle \( L = \mathcal{O}_{\mathbb{P}^1}(N) \) over \( \mathbb{P}^1 \). In order to show this, we fix homogeneous coordinates \( [x_0, x_1] \) and follow the notation of Section 4.3 so in particular \( H^0(\mathbb{P}^1, L) \cong S^N(\mathbb{C}^2)^* \) is the space of degree \( N \) homogeneous polynomials in \( x_0, x_1 \). We wish to evaluate the Futaki invariant for \( (\mathbb{P}^1, L, \phi) \), when \( L = \mathcal{O}_{\mathbb{P}^1}(N) \) and

\[
\phi \cong x_0^{N-\ell} x_1^\ell, \quad (5.4)
\]

with \( 0 \leq \ell < N \) (the case \( \ell = 0 \) corresponds to a Higgs field \( \phi \) that has only one zero).

**Lemma 5.3.** If \( \ell \neq 0 \), the group of automorphisms of \( (\mathbb{P}^1, L, \phi) \) is given by the image of the standard maximal torus \( \mathbb{C}^* \times \mathbb{C}^* \subset \text{GL}(2, \mathbb{C}) \) under the morphism

\[
\rho : \mathbb{C}^* \times \mathbb{C}^* \rightarrow \text{Aut}(\mathbb{P}^1, L)
\]
defined by

\[
\rho(\lambda_0, \lambda_1) = \lambda_0^{N-\ell} \lambda_1^\ell \rho(\lambda_0, \lambda_1)
\]

where \( \lambda_0^{N-\ell} \lambda_1^\ell \) acts on \( L \) by multiplication on the fibres.
The proof follows from the surjectivity of \( \rho \) in (4.7). Using this lemma for \( 0 < \ell < N \), and (4.8) for \( \ell = 0 \), the Lie algebra element

\[
y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{C})
\]  

(5.5)
can be identified with an element in Lie Aut(\( \mathbb{P}^1, L, \phi \)) for any \( 0 \leq \ell < N \).

**Lemma 5.4.**

\[
\langle F_{\alpha, \tau}, y \rangle = 2\pi i \alpha (2N - \tau)(2\ell - N)
\]  

(5.6)

**Proof.** Without loss of generality, we assume \( \text{Vol}(\mathbb{P}^1) = 2\pi \) in the definition of the Futaki invariant. We will apply formula (5.2) to the pair \((\omega_{FS}, h_{FS}^N)\) consisting of the Fubini–Study metric \(\omega_{FS}\) on \(\mathbb{P}^1\), normalized so that \(\int_{\mathbb{P}^1} \omega_{FS} = 2\pi\), and the Fubini–Study metric \(h_{FS}^N\) on \(L = \mathcal{O}_{\mathbb{P}^1}(N)\). We choose coordinates \(z = \frac{x_1}{x_0}\), so that the vector field on \(\mathbb{P}^1\) induced by \(y\) and the holomorphic section \(\phi\) are

\[
y^{1,0} = z \frac{\partial}{\partial z}, \quad \phi = z^\ell.
\]

In these coordinates, we also have

\[
\omega_{FS} = \frac{idz \wedge d\bar{z}}{(1 + |z|^2)^2}, \quad h_{FS}^N = \frac{1}{(1 + |z|^2)^N},
\]

and \(y = J\eta_{\varphi_2}\), with global complex potential \(\varphi = i\varphi_2\) given by

\[
\varphi = \frac{i |z|^2 - 1}{2 |z|^2 + 1}.
\]

Hence, by Lemma 5.3, the infinitesimal action of \(y\) induces a vector field on the total space of \(L\), also denoted \(y\), with vertical part

\[
A_{h_{FS}^N} y = \ell + \iota_y \iota^{1,0} \partial \log h_{FS}^N
\]

\[
= \ell - N \frac{|z|^2}{1 + |z|^2}.
\]

Applying these formulae in (5.2), we obtain

\[
\langle F_{\alpha, \tau}, y \rangle = 4i\alpha \int_{\mathbb{P}^1} A_{h_{FS}^N} y \left( N + \frac{1}{2} |\varphi|_{h_{FS}^N}^2 - \frac{\tau}{2} \right) \omega_{FS} - \alpha \int_{\mathbb{P}^1} \varphi \Delta_{\omega_{FS}} |\varphi|_{h_{FS}^N}^2 \omega_{FS}
\]

\[
= 2i\alpha (2N - \tau) \int_{\mathbb{P}^1} (A_{h_{FS}^N} y) \omega_{FS} + 2\alpha \int_{\mathbb{P}^1} (iA_{h_{FS}^N} y - 2\varphi) |\varphi|_{h_{FS}^N}^2 \omega_{FS}
\]

where we have used the facts that \(i\Lambda_{\omega_{FS}} F_{h_{FS}^N} = N, S_{\omega_{FS}}\) is constant, and \(\varphi\) is normalised so that

\[
\Delta_{\omega_{FS}} \varphi = 4\varphi,
\]

so in particular \(\int_{\mathbb{P}^1} \varphi \omega_{FS} = 0\). Using now the explicit formula

\[
|\varphi|_{h_{FS}^N}^2 = \frac{|z|^{2\ell}}{(1 + |z|^2)^N},
\]
we have
\[ \int_{\mathbb{P}^1} (A_{h_{FS}^N} y) \omega_{FS} = \int \left( \ell - N \frac{|z|^2}{1 + |z|^2} \right) \frac{1}{(1 + |z|^2)^2} idz \wedge d\overline{z} \]
\[ = 4\pi \int_0^\infty \left( \ell - N \frac{r^2}{1 + r^2} \right) \frac{r}{(1 + r^2)^2} dr \]
\[ = 4\pi \left[ \frac{2Nr^2 + N - 2(r^2 + 1)\ell}{4(r^2 + 1)^2} \right]_0^\infty \]
\[ = \pi(2\ell - N) \]
and also, using that \( \ell < N \),
\[ \int_{\mathbb{P}^1} (iA_{h_{FS}^N} y - 2\varphi) |\phi|_{h_{FS}^N}^2 \omega_{FS} = i \int \left( \ell - N \frac{|z|^2}{1 + |z|^2} + \frac{1 - |z|^2}{1 + |z|^2} \right) \frac{|z|^{2\ell}}{(1 + |z|^2)^{N+2}} idz \wedge d\overline{z} \]
\[ = 4\pi i \int_0^\infty \left( \ell - N \frac{r^2}{1 + r^2} + \frac{1 - r^2}{1 + r^2} \right) \frac{r^{2\ell+1}}{(1 + r^2)^{N+2}} dr \]
\[ = 2\pi i \left[ \frac{r^{2\ell+2}}{(1 + r^2)^2N+2} \right]_0^\infty \]
\[ = 0, \]
which completes the proof. □

Proof of Theorem 5.2. This is now a direct consequence of Proposition 5.1 and Lemma 5.4 if there exists a solution of the gravitating vortex equations for \((\mathbb{P}^1, L, \phi)\), then \(F_{\alpha,\tau} = 0\) and therefore \(2\ell = N\) or \(\tau = 2N\). The second case is excluded by Theorem 2.2. □

Remark 5.5. Lemma 5.4 combined with Proposition 5.1 provide an alternative proof of Theorem 4.7 and Corollary 4.8. This follows from the fact that if \(\phi\) has only one zero, so it is given by (5.4) with \(\ell = 0\), then \(\langle F_{\alpha,\tau}, y \rangle \neq 0\) by Lemma 5.4 with \(y\) given by (5.5). Note also that in the same case \(\ell = 0\), then we could have chosen another Lie algebra element (see (4.8))
\[ y' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]
but then \(\langle F_{\alpha,\tau}, y' \rangle = 0\), since \([y, y'] = y'\) and \(F_{\alpha,\tau}\) is a character by Proposition 5.1.

5.3. Relation with extremal pairs. In the case \(N = 1\) and \(\ell = 0\), there is a simpler proof of the non-vanishing of the Futaki invariant, which is related to a suitable notion of extremal pair (cf. [11, Definition 4.1]). Let \(\omega\) be a Kähler form on \(\mathbb{P}^1\) and \(h\) a hermitian metric on \(L\). Associated with the pair \((\omega, h)\) and a constant \(a \in \mathbb{R}\), we consider a vector field
\[ \zeta_{a,\alpha,\tau}(\omega, h) := ai(iA_F + \frac{1}{2} |\phi|^2 - \frac{\tau}{2}) 1 + A_{h}^1 \eta_{a,\tau} \]
on the total space of \(L\), where \(\eta_{a,\tau}\) is the Hamiltonian vector field of the smooth function
\[ S_\omega + a\Delta_\omega |\phi|^2 - 2a\tau iA_\omega F_h. \]
Note that the vector field \(\zeta_{a,\alpha,\tau}(\omega, h)\) is \(\mathbb{C}^*\)-invariant (actually it belongs to the extended gauge group determined by \((\omega, h))\).
**Definition 5.6.** The pair \((\omega, h)\) is extremal if there exists \(a \in \mathbb{R}_{>0}\) such that

\[ \zeta_{a,\alpha,\tau}(\omega, h) \in \text{Lie Aut}(\mathbb{P}^1, L, \phi), \]

that is, the vector field \(\zeta_{a,\alpha,\tau}(\omega, h)\) is holomorphic and preserves \(\phi\).

The existence of a non-trivial extremal pair \((\omega, h)\) with a fixed volume \(\text{Vol}(\mathbb{P}^1)\) is an obstruction to the existence of solutions of the gravitating vortex equations with the same volume, where non-triviality means \(\zeta_{a,\alpha,\tau}(\omega, h) \neq 0\) for some \(a \in \mathbb{R}_{>0}\). This follows from Proposition 5.1, because \(\zeta_{a,\alpha,\tau}(\omega, h) \neq 0\) implies

\[ \langle F_{\alpha,\tau}, \zeta_{a,\alpha,\tau}(\omega, h) \rangle < 0, \]

as can be shown by applying formula (5.2) to \(y = \zeta_{a,\alpha,\tau}(\omega, h)\), using \((\omega, h)\) (cf. [1, Proposition 4.2]).

**Proposition 5.7.** The pair \((\omega_{FS}, h_{FS})\) is an extremal pair for \((\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), \phi)\), with \(\phi = x_0\).

**Proof.** We note that

\[ \Delta_{\omega_{FS}}|\phi|_{h_{FS}}^2 = 2 \frac{1 - |z|^2}{1 + |z|^2}, \]

which is the Hamiltonian for the vector field

\[ v = -4iy_1 = 4iz\frac{\partial}{\partial z} \in \text{Lie Aut}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), \phi). \]

Furthermore, we have

\[ \partial|\phi|_{h_{FS}}^2 = \frac{1}{4}i_{\omega_{FS}}\omega_{FS}, \]

and hence the result holds for the choice \(a = 8\), by Lemma 4.2. \(\square\)

**Remark 5.8.** One can compare the definition of extremal pair for the Kähler–Yang–Mills equations in [1, Definition 4.1] with Definition 5.6 via the process of dimensional reduction described in [2, §3.2]. Under this comparison, the former definition corresponds to the latter only for \(a = 4\), but clearly the notion of extremal pair for the Kähler–Yang–Mills equations can be generalized to arbitrary \(a \in \mathbb{R}_{>0}\) by considering a modification of the vector field \(\zeta_\alpha\) (see [1, (4.136)]), with the Hermite–Yang–Mills term multiplied by \(a\), as in Definition 5.6.

6. FROM GRAVITATING VORTICES IN \(g = 0\) TO POLYSTABILITY

In this section, we introduce a notion of geodesic stability for the gravitating vortex equations, which is valid for a surfaces of arbitrary genus \(g\) (Section 6.1), prove our main Theorem 1.3 in \(g = 0\) (Section 6.2), and discuss the role of the Homogeneous Complex Monge–Ampère equation in the problem of existence and uniqueness of gravitating vortices (Section 6.3). Key tools are our description of the Lie algebra of automorphisms of a triple \((\Sigma, L, \phi)\) carrying gravitating vortices (Section 4) and the Futaki invariant for the gravitating vortex equations (Section 5).
6.1. Geodesic stability. Let $\Sigma$ be a compact connected Riemann surface of arbitrary genus, $L$ a holomorphic line bundle over $\Sigma$, $\phi$ a global holomorphic section of $L$, and $\tau, \alpha \in \mathbb{R}$, where $\alpha > 0$. Fix $\text{Vol}(\Sigma) > 0$. In this section we construct an obstruction to the existence of solutions of the gravitating vortex equations that is intimately related to the geometry of the infinite-dimensional space $B$ consisting of pairs $(\omega, h)$, where $\omega$ is a Kähler form on $\Sigma$ with volume $\text{Vol}(\Sigma)$ and $h$ is a hermitian metric on $L$. This space has a structure of symmetric space [1, Theorem 3.6], that is, it has a torsion-free affine connection $\nabla$, with holonomy group contained in the extended gauge group (each point of $B$ determines one such group) and covariantly constant curvature. The partial differential equations that define the geodesics $(\omega_t, h_t)$ on $B$, with respect to the connection $\nabla$, are [1, Proposition 3.17]

$$
\dd^c(\dot{\phi}_t - (d\phi_t, d\phi_t)_{\omega_t}) = 0,
\tilde{h}_t - 2J\eta_{\dot{\phi}_t} d\dot{h}_t + iF_{h_t}(\eta_{\dot{\phi}_t}, J\eta_{\dot{\phi}_t}) = 0.
$$

(6.1)

Here, $\omega_t = \omega + \dd^c \phi_t$ with $\phi_t \in C^\infty(\Sigma)$, and $\eta_{\dot{\phi}_t}$ is the Hamiltonian vector field of $\dot{\phi}_t$ with respect to $\omega_t$, that is, given by

$$
d\dot{\phi}_t = \eta_{\dot{\phi}_t} \omega_t.
$$

The long-time existence of smooth geodesics on $B$ — a very hard analytical open problem — has strong consequences for the problem of the gravitating vortex equations (3.11). Assuming existence of smooth geodesic rays, that is, smooth solutions $(\omega_t, h_t)$ of (6.1) defined on an infinite interval $0 \leq t < \infty$, with prescribed boundary condition at $t = 0$, one can define a stability condition for the triple $(\Sigma, L, \phi)$. Define a 1-form $\sigma_{\alpha, \tau}$ on $B$ by

$$
\sigma_{\alpha, \tau}(\omega, h) = -4\alpha \int_{\Sigma} \text{tr} h^{-1} h \wedge (i\Lambda_\omega F_h + \frac{1}{2}|\phi|^2_h - \frac{\tau}{2})\omega
- \int_{\Sigma} \dot{\phi} \left(S_\omega + \alpha \Delta_\omega |\phi|^2_h - 2\alpha \tau i\Lambda_\omega F_h\right) \omega,
$$

(6.2)

where $(\omega, h)$ is a tangent vector to $B$ at $(\omega, h)$, so $\dot{\omega} = \dd^c \dot{\phi}$ with $\dot{\phi} \in C_0^\infty(\Sigma, \omega)$, that is, $\dot{\phi}$ is normalised by the condition $\int_{\Sigma} \dot{\phi} \omega = 0$. Then $\sigma_{\alpha, \tau}$ vanishes precisely at the pairs $(\omega, h) \in B$ that are solutions of the gravitating vortex equations. As in [1, Proposition 3.8],

$$
\frac{d}{dt} \sigma_{\alpha, \tau}(\omega_t, h_t) \geq 0
$$

(6.3)

along a geodesic ray (cf. [1, Proposition 3.10]), with speed controlled by the infinitesimal action of the extended gauge group on the space $T$ in (3.9) (cf. the proof of [1, Proposition 3.14]), and hence it makes sense to evaluate the maximal weight

$$
w(\Sigma, L, \phi) := \lim_{t \to +\infty} \sigma_{\alpha, \tau}(\omega_t, h_t).
$$

(6.4)

Definition 6.1 (cf. [1, Definition 3.13]). The triple $(\Sigma, L, \phi)$ is geodesically semi-stable if $\nu(\Sigma, L, \phi) \geq 0$ for every smooth geodesic ray $(\omega_t, h_t)$ on $B$. It is geodesically stable if this inequality is strict whenever $(\omega_t, h_t)$ is non-constant.

Under the assumption that $B$ is geodesically convex, that is, any two points in $B$ can be joined by a smooth geodesic segment, geodesic semi-stability provides an obstruction to the existence of solutions of the gravitating vortex equations (2.1). Furthermore, this assumption has strong consequences for the uniqueness of solutions (cf. Section 6.3).
The next proposition follows from the fact that the quantity \( \sigma_{a,\tau}(\hat{b}_t) \) is increasing along geodesics in \( B \) (see (6.3)), and \( \sigma_{a,\tau} \) vanishes at the solutions \((\omega,h)\in B\) of the gravitating vortex equations.

**Proposition 6.2** (cf. [1] Corollary 3.11). Assume that \( B \) is geodesically convex. If there exists a solution of the gravitating vortex equations in \( B \), then \((\Sigma, L, \phi)\) is geodesically semi-stable. Furthermore, such a solution is unique modulo the action of \( \text{Aut}(\Sigma, L, \phi) \).

### 6.2. The converse of Yang’s Existence Theorem

We are now in a position to prove Theorem 1.3. We start with the observation that the geodesic equation (6.1) is independent of the global section \( \phi \) (it is the geodesic equation already considered in the Kähler–Yang–Mills problem [1]), so one obtains a wealth of geodesic rays starting at any point of \( B \).

**Lemma 6.3.** Let \((\omega, h)\in B\). Then, any \( \zeta \in \text{Lie Aut}(\Sigma, L) \) determines a smooth geodesic ray in \( B \) starting at \((\omega, h)\), given by

\[
\hat{b}_t = (\omega_t, h_t) = (g_t^*\omega, g_t^*h),
\]

where \( g_t \in \text{Aut}(\Sigma, L) \) is the flow of \( \zeta \), with initial condition \( g_0 = \text{Id} \).

We now restrict ourselves to the case \( \Sigma = \mathbb{P}^1 \), and evaluate the maximal weight (6.4) on a geodesic ray of the form (6.5). Since \( \Sigma = \mathbb{P}^1 \) and \( L \) is fixed, throughout Section 6.2 we will denote by \( \mathcal{F}_{a,\tau}(\phi) \) the Futaki invariant defined in Section 5, corresponding to a triple \((\mathbb{P}^1, L, \phi)\).

**Lemma 6.4.** Let \((\omega, h)\in B\) and \( \zeta \in \text{Lie Aut}(\Sigma, L) \). Assume that the limit

\[
\phi_0 := \lim_{t \to +\infty} g_t \cdot \phi
\]

exists, where \( g_t \in \text{Aut}(\Sigma, L) \) is the flow of \( \zeta \), with initial condition \( g_0 = \text{Id} \). Then the maximal weight of \((\mathbb{P}^1, L, \phi)\), evaluated at the geodesic ray (6.5) starting at \((\omega, h)\), is

\[
w(\mathbb{P}^1, L, \phi) = \text{Im} \langle \mathcal{F}_{a,\tau}(\phi_0), \zeta \rangle.
\]

Note that the right-hand side of (6.6) is well defined, that is, \( \zeta \in \text{Lie Aut}(\mathbb{P}^1, L, \phi_0) \), because by the hypothesis of the lemma, \( \phi_0 \) is fixed by the one-parameter subgroup induced by \( \zeta \).

**Proof.** By Lemma 6.6 any \( \zeta \in \text{Lie Aut}(\Sigma, L) \) admits a unique decomposition

\[
\zeta = \zeta_1 + I\zeta_2,
\]

where \( \zeta_1, \zeta_2 \) are in the Lie algebra of the extended gauge group of \((\omega, h)\). Furthermore, any \((\omega, h)\in B\) determines a space \( \mathcal{T} \) with a moment map \( \mu_a \) as in Proposition 3.2, and using the decomposition (6.7) and a change of variable in (6.2) (cf. [1] (3.104)), we obtain

\[
\sigma_{a,\tau}(\hat{b}_t) = \langle \mu_a(J, A, g_t \cdot \phi), \zeta_2 \rangle,
\]

where \( g_t \cdot (J, A, \phi) = (J, A, g_t \cdot \phi) \in \mathcal{T} \), as \( g_t \in \text{Aut}(\Sigma, L) \). We observe now that the proof of [1] Proposition 3.8 works for the 1-form \( \sigma_{a,\tau} \) (which does depend on \( \phi \)), so

\[
\frac{d}{dt} \sigma_{a,\tau}(\hat{b}_t) = \| Y_{\xi_2|J, A, g_t \cdot \phi} \|^2 \geq 0
\]

(cf. [1] (3.113)), where \( Y_{\xi_2|(J, A, g_t \cdot \phi)} \) denotes the infinitesimal action of \( \zeta_2 \) on \((J, A, g_t \cdot \phi)\in \mathcal{T} \), and the norm is given by the Kähler form on \( \mathcal{T} \) described in Section 3 — it is a positive definite norm precisely because \( \alpha > 0 \) (note that (6.3) follows from (6.9)). The proof of the lemma is now straightforward from (6.8) and the definition of the Futaki invariant (5.2). □
Proof of Theorem 6.3. As in the proof of Theorem 4.7, we make the identification $L = \mathcal{O}_{\mathbb{P}^1}(N)$, with $N := c_1(L) > 0$, and use homogeneous coordinates $[x_0, x_1]$ on $\mathbb{P}^1$, so $H^0(\Sigma, L)$ is the space of degree $N$ homogeneous polynomials in the coordinates $x_0, x_1$. Assume that $\phi$ is not polystable. Then there exists a 1-parameter subgroup

$$\lambda : \mathbb{C}^* \longrightarrow \text{SL}(2, \mathbb{C}) \subset \text{Aut}(\mathbb{P}^1, L),$$

and a suitable choice of homogeneous coordinates, such that (up to rescaling)

$$\lim_{t \to +\infty} \lambda(e^{-t}) \cdot \phi = x_0^{N-\ell} x_1^\ell,$$

with non-positive Hilbert–Mumford weight, that is,

$$N - 2\ell \leq 0 (6.10)$$

(cf. e.g. the proof of [54, Theorem 3.10]). Consider the geodesic ray

$$b_t = \lambda(e^t)^* (\omega, h).$$

The corresponding maximal weight is

$$w(\mathbb{P}^1, L, \phi) = \text{Im} \langle \mathcal{F}_{\alpha, \tau}(\phi_0), \zeta \rangle,$$

by Lemma 6.4, with

$$\zeta = \begin{pmatrix} N - 2\ell - 1 & 0 \\ 0 & N - 2\ell + 1 \end{pmatrix},$$

and therefore Lemma 5.4 implies

$$w(\mathbb{P}^1, L, \phi) = 4\pi \alpha(\tau - 2N)(N - 2\ell). (6.11)$$

Assume now $(\mathbb{P}^1, L, \phi)$ admits a solution $(\omega, h)$ of the gravitating vortex equations, that is, $\sigma_{\alpha, \tau}$ vanishes at $(\omega, h)$. Then $\sigma_{\alpha, \tau}(b_t) = 0$ and (6.3) imply that for any geodesic ray,

$$w(\mathbb{P}^1, L, \phi) \geq 0. (6.12)$$

Moreover, Theorem 2.2 implies

$$\tau - 2N > 0. (6.13)$$

However, if the inequality (6.10) is strict, i.e. $N - 2\ell < 0$, then $w(\mathbb{P}^1, L, \phi) < 0$, by (6.11) and (6.13), contradicting (6.12). In the remaining case $N - 2\ell = 0$, $w(\mathbb{P}^1, L, \phi) = 0$ by (6.11), which combined with (6.9) and $\sigma_{\alpha, \tau}(b_0) = 0$, imply that $\zeta$ fixes $\phi$, so $\phi = \phi_0$, but this cannot happen because $\phi_0$ is polystable.

6.3. A conjecture about uniqueness and the moduli space of gravitating vortices. The contents of Sections 6.1 and 6.2, especially Proposition 6.2, suggest an approach to the uniqueness problem for the gravitating vortex equations on a compact Riemann surface $\Sigma$ of arbitrary genus. To the knowledge of the authors, this problem has not been explored so far, even for the Einstein–Bogomol’nyi equations (for which $\Sigma = \mathbb{P}^1$). This approach rests on the geometry of the infinite-dimensional space $B$, and the closely related space $K$ of Kähler forms on $\Sigma$ with fixed volume $\text{Vol}(\Sigma)$. The space $B$ is a symmetric space, as briefly reviewed in Section 6.1, and the space $K$ is a Riemannian symmetric space, as shown by Semmes [50] and rediscovered by Mabuchi [39] and Donaldson [20]. Since the geodesic equation on $K$ is the first equation in (6.1), i.e. the map

$$B \longrightarrow K,$$

given by $(\omega, h) \longmapsto \omega$, is a geodesic submersion, one cannot expect in general existence of smooth geodesic segments on $B$ with arbitrary boundary conditions, by results of Lempert and Vivas [35] about the geometry of $K$. Hence one cannot expect either that a direct
application of Proposition 6.2 will work in the uniqueness problem for the gravitating vortex equations. Here we propose a possible way to circumvent this difficulty.

As shown by Donaldson [20] and Semmes [50], for a suitable choice of Riemann surface $D$, the geodesic equation on $\mathcal{K}$ reduces to a homogeneous complex Monge–Ampère equation on the complex surface $\Sigma \times D$. This method has been fruitfully applied in the context of the problem for constant scalar curvature Kähler metrics [9, 12, 16, 47]. We expect that these results, and in particular the recent proof of the uniqueness of constant scalar curvature Kähler metrics by Berman and Berndtsson [7], can be adapted to the context of the geodesic equation (5.1). In light of this, the following conjecture seems reasonable.

**Conjecture 6.5.** Given $\alpha > 0$, if $\tau$ satisfies (2.4) and $\phi$ is polystable, then there exists a unique solution of the gravitating vortex equations on $(\mathbb{P}^1, L, \phi)$ modulo automorphisms.

A proof of Conjecture 6.5 combined with Theorem 1.4 would lead to the following explicit description of the moduli space of solutions of the Einstein–Bogomol’nyi equations.

**Conjecture 6.6.** The moduli space of solutions of the degree-$N$ Einstein–Bogomol’nyi equations is biholomorphic to the GIT quotient

$$S^N \mathbb{P}^1 / \text{SL}(2, \mathbb{C}).$$

Furthermore, it is reasonable to hope that Yang’s Existence Theorem 1.2 for the Einstein–Bogomol’nyi equations holds for the more general gravitating vortex equations on $\mathbb{P}^1$ in the case $\alpha > 0$. This result, combined with Conjecture 6.5, would provide an explicit description of the moduli space of gravitating vortices on $\mathbb{P}^1$, exactly as in Conjecture 6.6.

Note that the biholomorphism of Conjecture 6.6 would show an intriguing link between the physics of cosmic strings and the classical theory of binary quantics [42, 51].

### 7. Existence and uniqueness of gravitating vortices in $g \geq 2$

In this section we prove that the gravitating vortex equations (2.1) have a unique solution in genus $g \geq 2$, assuming a suitable effective bound on the coupling constant $\alpha > 0$ (provided that the inequality (2.4) is satisfied). The main result of this section, combined with Theorem 1.4, draws a parallel between the existence problem for the gravitating vortex equations and the Kähler–Einstein problem, where stability only plays a role in the Fano case [3, 13, 66, 67] (i.e. positive canonical bundle).

#### 7.1. Statement of the result and the continuity method.

**Theorem 7.1.** Let $\Sigma$ be a compact Riemann surface of genus $g \geq 2$, and $L$ a holomorphic line bundle over $\Sigma$ of degree $N > 0$ equipped with a holomorphic section $\phi \neq 0$. Let $\tau$ be a real constant such that $0 < N < \tau/2$. Define

$$\alpha_* := \frac{2g - 2}{2\tau(\tau/2 - N)} > 0.$$  

Then the set of $\alpha \geq 0$ for which (2.1) has smooth solutions of volume $2\pi$ is open and contains the closed interval $[0, \alpha_*]$. Furthermore, the solution is unique for $\alpha \in [0, \alpha_*]$.

For the purposes of the proof, we fix a metric $\omega_0$ of constant curvature $-1$ on $\Sigma$ with volume $2\pi$, a Hermite–Einstein metric $h_0$ on $L$ (with respect to $\omega_0$), and define

$$c = \chi - 2\alpha \tau N < 0,$$
as in (2.2), where \( \chi = 2 - 2g < 0 \) is the Euler characteristic of \( \Sigma \), so (7.1) is equivalent to
\[
0 = c + \alpha \tau - = \chi + \alpha \tau (\tau - 2N).
\]
Furthermore, we will consider the equations (2.6), with unknowns \( u, f \in C^\infty(\Sigma) \), which are equivalent to the gravitating vortex equations (2.1), for \( \omega \) with the coupling constant \( \alpha \).

To prove this fact, we use the properties of the moment map of Proposition 3.2. Consider\( u, f \), and therefore there exist unique constants \( c_0, c_1 \) (that depend continuously on \( f, v \) in \( C^0 \)-norm), such that \( f + c_0 \) and \( v + c_1 \) solve (7.3). Hence it suffices to prove that the set of \( \alpha \in \mathbb{R} \) for which there exists a solution of (2.1) is open.

**Remark 7.2.** Note that the second equation of (7.3) automatically implies that \( \omega = \omega_0 + 2\sqrt{-1} \partial \bar{\partial} v \) is Kähler.

**Lemma 7.3.** The subset \( S \subset \mathbb{R} \) is open.

**Proof.** As observed in Section 2.1 if \((v, f)\) is a solution of (7.3), then \((\omega_0 + d\bar{\partial} v, e^{2f} h_0)\) is a solution of (2.5) (and hence of (2.1)) for \( \alpha = t \), because (2.5) is equivalent to
\[
\Delta f + \frac{1}{2}(e^{2f}|\phi|^2 - \tau)(1 - \Delta v) = -N, \tag{7.4}
\]
\[
d\bar{\partial} (\log(1 - \Delta v) - 4a \tau f + 2ae^{2f}|\phi|^2 + 2cv) = 0.
\]
Conversely, for any solution \((\omega_0 + d\bar{\partial} v, e^{2f} h_0)\) of (2.1) with \( \alpha = t \), it follows that
\[
\log(1 - \Delta v) - 4a \tau f + 2ae^{2f}|\phi|^2 + 2cv = c'
\]
by (7.4), for some constant \( c' \in \mathbb{R} \), and therefore there exist unique constants \( c_0, c_1 \) (that depend continuously on \( f, v \) in \( C^0 \)-norm), such that \( f + c_0 \) and \( v + c_1 \) solve (7.3). Consider the linear differential operator
\[
L_\alpha = (L^0, L^1): C^\infty(X) \times C^\infty(X) \longrightarrow C^\infty(X) \times C^\infty(X),
\]
such that the value at \((\dot{v}, \dot{f})\) is given by the linearization
\[
(L^0, L^1) = \delta \left( i\Lambda_\omega F_h + \frac{1}{2}|\phi|^2 - \frac{\tau}{2}, -S_\omega - \alpha \Delta_\omega |\phi|^2 + 2a \tau i\Lambda_\omega F_h + c \right)
\]
of the moment map at \((J, A, \phi)\) along the infinitesimal action of the vector field
\[
A^i J_{\eta^0} - \dot{f}.
\]
Here, $J$ is the almost complex structure on $\Sigma$, $A$ is the Chern connection of $h$ on the line bundle $L$ and $\eta_0$ is the Hamiltonian vector field of $v$ with respect to $\omega$. More explicitly,

$$
L^0_\alpha(v, \dot{f}) = d^* (d\dot{f} + \eta_0 \cdot iF_h) + (\phi, -J\eta_0 \cdot dA\phi + 2f\phi)_h,
$$

$$
L^1_\alpha(v, \dot{f}) = P^* P \dot{v} - 4\alpha id(dA\phi, -J\eta_0 \cdot dA\phi + 2f\phi)_h
- 2\alpha id((d\dot{f} + \eta_0 \cdot iF_h)\phi)_h + 2\alpha \tau d^* (d\dot{f} + \eta_0 \cdot iF_h),
$$

where $P^* P$ is, up to a multiplicative constant factor, the Lichnerowicz operator of the Kähler manifold $(\Sigma, J, \omega)$ \[30\]. Consider now the moment-map operator

$$
T_\alpha = (T^0_\alpha, T^1_\alpha) : C^\infty(X) \times C^\infty(X) \to C^\infty(X) \times C^\infty(X),
$$

given by

$$
T^0_\alpha(v, f) = i\Lambda_\omega F_h + \frac{1}{2} |\phi|^2_h - \frac{\tau}{2},
$$

$$
T^1_\alpha(v, f) = -S_\omega - \alpha \Delta_\omega |\phi|^2_h + 2\alpha \tau i\Lambda_\omega F_h + c,
$$

where $(\omega, h) = (\omega_0 + dd^c v, e^{2f} h_0)$. Arguing as in \[1\] Proposition 4.7 for our moment map \[38\], the linearization of $T_\alpha$ at $(v, f)$ satisfies

$$
\delta T^0_\alpha(v, f) = L^0_\alpha(\dot{v}, \dot{f}) + J\eta_0 \cdot d(T^0_\alpha(v, f)),
$$

$$
\delta T^1_\alpha(v, f) = L^1_\alpha(\dot{v}, \dot{f}) + (d(T^1_\alpha(v, f)), d\dot{v})_\omega,
$$

and so $\delta T_\alpha = L_\alpha$ if $(\omega, h)$ is a solution of \[2.1\]. Moreover, as in \[1\] Lemma 4.5, we have

$$
\| (\dot{v}, 4\alpha \dot{f}), L_\alpha(\dot{v}, \dot{f}) \|_{L^2} = \| L_{\eta_0} J \|_{L^2}^2 + 4\alpha \| d\dot{f} + \eta_0 \cdot iF_h \|_{L^2}^2 + 4\alpha \| J\eta_0 \cdot dA\phi - 2f\phi \|_{L^2}^2,
$$

$$
+ 4\alpha (\| J\eta_0 \cdot (id\dot{f} + \eta_0 \cdot iF_h), T^0_\alpha(v, f) \|_{L^2}^2.
$$

Assuming that $(v, f)$ is a solution of \[2.1\], that is, $T_\alpha(v, f) = 0$, the operator $\delta T_\alpha$ is self-adjoint and the condition

$$
\delta T_\alpha(\dot{v}, \dot{f}) = 0
$$

implies that $A^\perp \eta_0 + i\dot{f}$ is an infinitesimal automorphism of $(\Sigma, L, \phi)$. By assumption, $g \geq 2$ and $\phi \neq 0$, and by Proposition \[4.3\] it follows that $\dot{v}, \dot{f}$ must be constant functions on $\Sigma$. The result now follows by application of the implicit function theorem in a Sobolev completion of $C^\infty(X) \times C^\infty(X)$. \hfill \Box

7.2. Closedness, existence, and uniqueness. In this section, we prove that the set of $t \in [0, \alpha]$ for which the system admits a smooth solution is also closed. By the previous section, this means that the set is all of $[0, \alpha]$, thus proving the existence part of Theorem \[7.4\]. To do so, we prove $C^{2, \gamma}$ a priori estimates for $(f, v)$ independent of $t$. The Arzela–Ascoli theorem with usual elliptic bootstrapping implies closedness.

Firstly, we reduce the problem to proving a $C^0$ estimate.

**Proposition 7.4.** Suppose $(f, v)$ is a smooth solution of \[7.3\]. Assume that there exists $C > 0$ independent of $t$ such that $\|v\|_{C^0} \leq C, \|f\|_{C^0} \leq C$. Then, for some $\gamma > 0$, $\|v\|_{C^{2, \gamma}}$ and $\|f\|_{C^{2, \gamma}}$ are bounded independently of $t$.

**Proof.** Indeed, rewriting equations \[7.3\] as

$$
\Delta_0 f = -\frac{1}{2} (|\phi|^2_h - \tau)e^{4\tau f - 2t|\phi|^2_h - 2cv} - N,
$$

$$
\Delta_0 v = -e^{4\tau f - 2t|\phi|^2_h - 2cv} + 1,
$$

(7.8)
we see that the right-hand sides have $L^p$-norm bounds independent of $t$ for all $p > 0$. Therefore by elliptic regularity we see that $v$ and $f$ have $W^{2,p}$-norm bounded independently of $t$ and hence $C^{1,\gamma}$-norm bounded independently of $t$. This means that the right-hand sides have $C^0,\gamma$-norm bounds independent of $t$ and the Schauder estimates allow us to conclude the proof. □

Secondly, we prove the following useful inequality for solutions of the vortex equation (2.3).

**Lemma 7.5.** Assume that $(\omega, h)$ is a solution of the vortex equation (2.3). Then,

$$|\phi|^2_h - \tau \leq 0.$$  

**Proof.** Let $b = |\phi|^2_h - \tau$. At the maximum of $b$, $\nabla b = 0$. Computing the laplacian at the maximum, by the Weitzenböck formula we see that

$$0 \leq \Delta_0 b \leq i \Lambda_0 F_h |\phi|^2_h.$$  

(7.9)

Using now (2.3), we see that indeed $|\phi|^2_h - \tau \leq 0$. □

From now onwards we assume that $(f, v)$ is a smooth solution to (7.3) unless specified otherwise. We also denote all constants independent of $t$ as $C$ by default. Note that the equations (7.3) force a normalisation condition on $(f, v)$ via integration. This allows us to prove an integral bound:

**Lemma 7.6.**

$$\int_{\Sigma} ((2 + 4t\tau)f - 2cv)\omega_0 \leq C,$$

$$\int_{\Sigma} (4t\tau f - 2cv)\omega_0 \leq C.$$  

(7.10)

**Proof.** Integrating (7.3) after multiplying by $\omega_0$, we see that

$$\int_{\Sigma} (|\phi|^2_h - \tau)e^{4t\tau f - 2t|\phi|^2_h - 2cv}\omega_0 = -4\pi N,$$

$$\int_{\Sigma} e^{4t\tau f - 2t|\phi|^2_h - 2cv}\omega_0 = 2\pi.$$  

(7.11)

Using the second equation in the first, we get

$$\int_{\Sigma} e^{\ln(|\phi|^2_{h_0}) + (2 + 4t\tau)f - 2t|\phi|^2_h - 2cv}\omega_0 = 2\pi(\tau - 2N),$$

$$\int_{\Sigma} e^{4t\tau f - 2t|\phi|^2_h - 2cv}\omega_0 = 2\pi.$$  

We use Jensen’s inequality $e^{\int f} \leq \int e^{f}$ to conclude

$$e^{\int_{\Sigma} \ln(|\phi|^2_{h_0}) + (2 + 4t\tau)f - 2t|\phi|^2_h - 2cv} \leq C,$$

$$e^{\int_{\Sigma} 4t\tau f - 2t|\phi|^2_h - 2cv} \leq C.$$  

Since $\phi$ is locally of the form $z^k$, $\ln |\phi|^2_{h_0}$ is integrable. Moreover, $0 \leq |\phi|^2_h \leq \tau$. Hence this implies the desired result. □

Next we consider the function $y = e^{4t\tau f - 2cv}$ and prove a bound on it.
Lemma 7.7. Assuming that \( \chi < 0 \), then \( -C \leq \ln(y) \). Furthermore, if \( c + \tau r^2 \leq 0 \), then \( -C \leq \ln(y) \leq C \), and therefore \( \|4\tau f - 2cv\|_{C^0} \leq C \).

Proof. At an extremum of \( y \), \( \nabla y = 0 \). Let \( p \) be such a point of extrema. We compute

\[
\Delta_0 y(p) = y(p)[4\tau \Delta_0 f(p) - 2c \Delta_0 v(p)]
\]

\[
\Rightarrow \frac{\Delta_0 y(p)}{y(p)} = 4\tau \left( -\frac{1}{2}(\phi|n|^2 - \tau)y(p)e^{-2t|\phi|_h^2} - N \right) - 2c \left( 1 - y(p)e^{-2t|\phi|_h^2} \right)
\]

\[
\Rightarrow \frac{\Delta_0 y(p)}{2e^{-t|\phi|_h^2} y(p)} = y(p)[c - \tau(\phi|n|^2 - \tau)] - (c + 2\tau N)e^{2t|\phi|_h^2}.
\]

(7.12)

At a point of minimum, \( \Delta_0 y(p) \leq 0 \). Hence,

\[
y_{\min}[c - \tau(\phi|n|^2 - \tau)] \leq (c + 2\tau N)e^{2t|\phi|_h^2} \leq (c + 2\tau N) = \chi < 0
\]

\[
\Rightarrow y_{\min}c \leq \chi.
\]

(7.13)

Thus \( y_{\min} \geq \frac{1}{C} \). This means that \( \ln(y) \geq -C \).

Now, the Green representation formula (see [4], Theorem 4.13, p. 108) is

\[
u(Q) = \int_{\Sigma} u(P)\omega_0(Q) + \int_{\Sigma} G(P, Q)\Delta_0 u(Q)\omega_0(Q),
\]

(7.14)

where \( 0 \leq G(P, Q) \leq C + C|\ln(d_\omega(P, Q))| \). Applying this to \( u = \ln(y) = 4\tau f - 2cv \), we see that

\[
\ln(y) = \int_{\Sigma} (4\tau - 2cv)\omega_0 + \int_{\Sigma} G(P, Q)(2y[c - \tau(\phi|n|^2 - \tau)]e^{-t|\phi|_h^2} - 2(c + 2\tau N))\omega_0
\]

\[
\leq C + C \int_{\Sigma} G(P, Q)y[c + \tau r^2]\omega_0 \leq C,
\]

(7.15)

where the last equality follows from the assumption \( c + \tau r^2 \leq 0 \) (cf. (7.2)). Thus \( -C \leq \ln(y) \leq C \).

Next we prove one-sided bounds.

Lemma 7.8. \( f \leq C \) and \( v \geq -C \).

Proof. The estimate in Lemma 7.7 shows that \( -C \leq 4\tau f - 2cv \leq C \). This means that

\[
-C \leq \int_{\Sigma} (4\tau f - 2cv)\omega_0 \leq C.
\]

(7.16)

This in conjunction with the first inequality in (7.10) implies that \( \int_{\Sigma} f \leq C \). Using this along with inequality (7.16) implies that \( \int_{\Sigma} v \geq -C \). By the Green representation formula (7.14), this means that \( v \geq -C \) and hence \( f \leq C \).

This means that the function \( 0 < U = e^{2f} \leq C \). Define \( \tilde{f} = f - \int_{\Sigma} f \) and \( \tilde{v} = v - \int_{\Sigma} v \). We now prove bounds on these two functions.

Lemma 7.9. \( \|\tilde{f}\|_{C^{1,\gamma}} \leq C \), \( \|\tilde{v}\|_{C^{1,\gamma}} \leq C \).
Proof. Writing equations (7.3) in terms of \( \tilde{f}, \tilde{v} \), we get
\[
\Delta_0 \tilde{f} = \Delta_0 f = -\frac{1}{2}(\|\phi\|_h^2 - \tau)e^{4\tau f - 2t\|\phi\|_h^2 - 2cv} - N = \eta_1, \\
\Delta_0 \tilde{v} = \Delta_0 v = -e^{4\tau f - 2t\|\phi\|_h^2 - 2cv} + 1 = \eta_2.
\]
By previous arguments, \( \eta_1 \) and \( \eta_2 \) are bounded above and below independent of \( t \). Multiplying the first equation of (7.17) by \( \tilde{f} \), the second one by \( \tilde{v} \), and integrating by parts, we get
\[
\int_{\Sigma} |\nabla_0 \tilde{f}|^2 \omega_0 = \int_{\Sigma} \eta_1 \tilde{f} \omega_0 \leq \int_{\Sigma} \frac{1}{2} \eta_1^2 \omega_0 + \frac{\varepsilon}{2} \int_{\Sigma} \tilde{f}^2 \omega_0, \\
\int_{\Sigma} |\nabla_0 \tilde{v}|^2 \omega_0 = \int_{\Sigma} \eta_2 \tilde{v} \omega_0 \leq \int_{\Sigma} \frac{1}{2} \eta_2^2 \omega_0 + \frac{\varepsilon}{2} \int_{\Sigma} \tilde{v}^2 \omega_0.
\]
(7.18) Since \( \tilde{f} \) and \( \tilde{v} \) have zero average, one can use Poincaré's inequality \( \|\tilde{f}\|_{L^2} \leq C\|\nabla_0 \tilde{f}\|_{L^2} \) in (7.18) to conclude that
\[
\int_{\Sigma} \tilde{f}^2 \omega_0 + \int_{\Sigma} |\nabla_0 \tilde{f}|^2 \omega_0 \leq C, \\
\int_{\Sigma} \tilde{v}^2 \omega_0 + \int_{\Sigma} |\nabla_0 \tilde{v}|^2 \omega_0 \leq C.
\]
(7.19) Using (7.19) and the Sobolev embedding theorem, we conclude that the \( L^p \) norms of \( \tilde{f} \) and \( \tilde{v} \) are bounded for all \( p \). Going back to (7.17) we see using \( L^p \) elliptic regularity that \( \|\tilde{f}\|_{W^{2,p}} \leq C \) and \( \|\tilde{v}\|_{W^{2,p}} \leq C \). Thus the Sobolev embedding theorem gives the desired result. \( \square \)

In the next result we prove the desired \( C^0 \)-estimate for \( f, v \). By Proposition 7.4 this implies \( C^{2,\gamma} \) bounds on \( f, v \).

Proposition 7.10. Suppose \((f, v)\) is a smooth solution of (7.3). Then \( \|v\|_{C^0} \leq C, \|f\|_{C^0} \leq C \) independent of \( t \).

Proof. Since \( |\nabla f| = |\nabla \tilde{f}| \leq C \), this means that \( 0 \leq \max f - \min f \leq C \). If \( \max f \) is not bounded below, then there exists a sequence \( t_n \to t \) such that \( f_n \to -\infty \) everywhere. This would mean that \( \|\phi\|_h^2 \to 0 \) everywhere. By the dominated convergence theorem applied to the first equality in (7.11), we find a contradiction because \( \tau > 2N \) by assumption. Therefore, \(-C \leq f \leq C\). This implies that \(-C \leq v \leq C\) because \(-C \leq \ln y \leq C\). \( \square \)

Proof of Theorem 7.4. The existence part follows from Proposition 7.10 and Proposition 7.4. As for uniqueness, suppose there are two solutions \((f_1, v_1)\) and \((f_2, v_2)\). We will use a method inspired by Bando and Mabuchi [5]. Run the continuity method backwards starting at \( t = \alpha \) for both of these solutions. Openness holds as long as \( t \geq 0 \). The \textit{a priori} estimates also hold as long as \( t \geq 0 \). We know that at \( t = 0 \), the system decouples and the solution is unique. Suppose \( T \) is the supremum of all \( t \) such that there is a unique solution on \([0, T] \), i.e., both continuity paths agree on \([0, T] \). By the \textit{a priori} estimates, the continuity paths of \((f_1, v_1)\) and \((f_2, v_2)\) are defined on \( t \in (T - \varepsilon, T + \varepsilon) \) and their points lie near the unique solution for \( t = T \) (in the \( C^{2,\gamma} \) topology). Furthermore, by openness, there is a unique solution near the unique solution for \( t = T \) (in the \( C^{2,\gamma} \) topology). Thus, uniqueness holds beyond \( T \) and that is a contradiction unless \((f_1, v_1) = (f_2, v_2)\).
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**Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM)**, Nicolás Cabrera 13–15, Cantoblanco, 28049 Madrid, Spain  
*E-mail address*: l.alvarez-consul@icmat.es

Dep. Matemáticas, Universidad Autónoma de Madrid, and Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Ciudad Universitaria de Cantoblanco, 28049 Madrid, Spain  
*E-mail address*: mario.garcia@icmat.es

**Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM)**, Nicolás Cabrera 13–15, Cantoblanco, 28049 Madrid, Spain  
*E-mail address*: oscar.garcia-prada@icmat.es

**Department of Mathematics, Indian Institute of Science**, Bangalore, India - 560012  
*E-mail address*: vamsipingali@iisc.ac.in