Spectroscopy of $B_c$ meson in a semi-relativistic quark model using the shifted large-N expansion method

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Abstract

We calculate the $c\bar{b}$ mass spectrum, the splitting values and some other properties in the framework of the semi-relativistic equation by applying the shifted large-N expansion technique. We use seven different central potentials together with an improved QCD-motivated interquark potentials calculated to two loops in the modified minimal-subtraction ($\overline{MS}$) scheme. The parameters of these potentials are fitted to generate the semi-relativistic bound states of $c\bar{b}$ quarkonium system in close conformity with the experimental and the present available calculated center-of-gravity (c.o.g.) data. Calculations of the energy bound states are carried out up to third order. Our results are in excellent fit with the results of the other works.

I. INTRODUCTION

The spectroscopy of the $c\bar{b}$ system have already been widely studied before in the framework of the heavy quarkonium theory [1]. After discovery of the $B_c$ was reported in 1998 by the (CDF) collaboration [2], the observed mass $M_{B_c} = 6.40^{+0.39}_{-0.13}$ GeV has inspired new theoretical interest [1,3-9]. Bound state masses have been estimated for the $B_c$ system which consists of heavy quarks [1,3-9]. Hence, it can be reliably described by the use of the methods developed for the $c\bar{c}$ and the $b\bar{b}$ spectra.

Quite recently, the revised analysis of the $B_c$ spectroscopy has been performed in the framework of the potential approach [1,4,7,9] and QCD sum rule [3,6]. Kwong and Rosner [7] predicted the masses of the lowest $c\bar{b}$ vector (triplet) and pseudoscalar (singlet) states using an empirical mass formula and a logarithmic potential. Eichten and Quigg [1] gave
a more comprehensive account of the energies and properties of the $c\bar{b}$ system that was based on the QCD-motivated potential of Buchmüller and Tye [10]. Gershtein et al. [8] also published a detailed account of the energies and decays of the $c\bar{b}$ system using a QCD sum-rule calculation. Baldicchi and Prosperi have computed the $c\bar{b}$ spectrum based on an effective mass operator with full relativistic kinematics [6]. Moreover, they have fitted the entire quarkonium spectrum. Fulcher et al. [4] extended the treatment of the spin-dependent potentials to the full radiative one-loop level and thus included effects of the running coupling constant in these potentials. He also used the renormalization scheme developed by Gupta and Radford [11]. Very recently, we have applied the shifted large-N expansion technique (SLNET) [12-14] to study the $c\bar{b}$ system in the context of Schrödinger equation [9].

In this work, we choose a group of seven central potentials [1,13-24] with a strong coupling constant $\alpha_s$ to fit the spectroscopy of the present spin-averaged energies of the existing quarkonium systems. We insist upon strict flavor-independence of their parameters. We also extend this study to an improved QCD-motivated interquark potential calculated to two loops in the modified minimal-subtraction ($\overline{MS}$) scheme [25,26]. Such a potential contains an expansion to $\alpha_s$ to at least 2-loop order and use of nontrivial methods in the interpolation between long-distance (string constant) and short-distance behaviors [10].

Since one would expect the average values of the momentum transfer in the various quark-antiquark states to be different, some variation in the values of the strong coupling constant and the normalization scale in the spin-dependent should be expected.

In order to minimize the role of flavor-dependence, we use the same values for the coupling constant and the normalization scale for each of the levels in a given system and require that these values be consistent with a universal QCD scale.

The contents of this article will be as follow: In section II, we present the solution of the spinless Salpeter equation using the SLNET for the $c\bar{f}$ Salpeter spin-averaged binding energies. Section III presents the proposed phenomenological central and the QCD-motivated potentials. Summary and conclusions are presented in section IV.
The relativistic wave Salpeter equation [14] is constructed by considering the kinetic energies of the constituents and the interaction potential. The spinless Salpeter (SS) for the case of two particles with unequal masses \( m_q \) and \( m_Q \), interacting via a spherically symmetric potential \( V(r) \) in the center-of-momentum system of the two particles is given by

\[
\sum_{i=q,Q} \sqrt{-\Delta_N + m_i^2} + V(r) - M(q\overline{Q}) \right] \psi(r) = 0, \tag{1}
\]

where the kinetic terms involving the operation \( \sqrt{-\Delta_N + m_i^2} \) are nonlocal operators and \( \psi(r) = Y_{\ell,m}(\theta, \phi)R_{n,l}(r) \) denotes the Salpeter’s wave function. For heavy quarks, the kinetic energy operators in Eq. (1) can be approximated, (cf. e.g., Jaczko and Durand of Ref. [27]), as

\[
\sum_{i=q,Q} \sqrt{-\Delta + m_i^2} = m_q + m_Q - \frac{\Delta_N}{2\mu} - \frac{\Delta_N^2}{8\eta^3} + \cdots, \tag{2}
\]

where \( \mu = \frac{m_q m_Q}{m_q + m_Q} \) denotes the reduced mass and \( \eta = \left( \frac{m_q m_Q}{m_q + m_Q - \mu} \right)^{1/3} \). \( \mu \) is a useful mass parameter. This SS-type equation retains its relativistic kinematics and is suitable for describing the spin-averaged spectrum of two bound quarks of masses \( m_q \) and \( m_Q \) and total binding mass \( M(q\overline{Q}) \). The Hamiltonian containing the relativistic corrections up to order \( (v^2/c^2) \) is called as the generalized Breit-Fermi Hamiltonian (cf. e.g., Lucha et al. of Ref. [21]). Therefore, the spinless Salpeter equation can be written (in units \( \hbar = c = 1 \)) [14]

\[
\left\{ -\frac{\Delta_N}{2\mu} - \frac{\Delta_N^2}{8\eta^3} + V(r) \right\} R_{n,l}(r) = E_{n,l} R_{n,l}(r), \tag{3}
\]

where \( E_{n,l} = M_{n,l}(q\overline{Q}) - m_q - m_Q \) refers to the Salpeter quark binding energy with \( M_{n,l}(q\overline{Q}) \) is the semirelativistic-bound-state masses of the non-self-conjugate atomlike meson such as \( c\overline{b} \) meson and \( \Delta_N = \nabla_N^2 \).\(^1\) Furthermore, in order to obtain a Schrödinger-like equation, the perturbed term in Eq. (3) is treated using the reduced Schrödinger equation [29]

\(^1\)This approximation is correct to \( O(v^2/c^2) \). The \( \Delta_N^2 \) term in (3) should be properly treated as a perturbation by using trial wavefunctions [28].
\[ p^4 = 4\mu^2 [E_{n,l} - V(r)]^2, \quad (4) \]

with \( p^4 = \Delta^2_N \), and consequently one would reduce Eq. (3) to the Schrödinger-type form [14]

\[
\left\{ -\frac{\Delta_N}{2\mu} - \frac{\mu^2}{2\eta^2} \left[ E_{n,l}^2 + V^2(r) - 2E_{n,l}V(r) \right] + V(r) \right\} R_{n,l}(r) = E_{n,l} R_{n,l}(r). \quad (5)
\]

Further, the \( N \)-dimensional space operator in the spherical polar coordinates is

\[
\Delta_N = \frac{\partial^2}{\partial r^2} + \frac{N - 1}{r} \frac{\partial}{\partial r} - \frac{L_{N-1}^2}{r^2}, \quad (6)
\]

with \( L_{N-1}^2 = l(l + N - 2) \). After employing the following transformation

\[ R_{n,l}(r) = \frac{u_{n,l}(r)}{r^{(N-1)/2}}, \quad (7) \]

and proposing the shifting parameter \( \bar{k} = k - a \) with \( k = N + 2l \), one obtains [14]

\[
\Delta_N = \frac{\partial^2}{\partial r^2} - \frac{(k - 1)(k - 3)}{4r^2}, \quad (8)
\]

and

\[
\Delta_N^2 = \frac{\partial^4}{\partial r^4} - \frac{(k - 1)(k - 3)}{2r^2} \frac{\partial^2}{\partial r^2} + \frac{(k - 1)(k - 3)}{r^3} \frac{\partial}{\partial r} \]

\[ + \frac{(k - 1)^2(k - 3)^2 - 24(k - 1)(k - 3)}{16r^4}. \quad (9) \]

Thus, using Eqs (7) and (8) and after a lengthy manipulation but straightforward, we may write Eq.(5) in more simple and explicit form as

\[
\left[ -\frac{1}{2\mu} \frac{d^2}{dr^2} + \frac{\bar{k} - (1-a)}{8\mu r^2} \frac{[\bar{k} - (3-a)]}{2} + W_{n,l}(r) - \frac{W_{n,l}(r)^2}{2m^*} \right] u_{n,l}(r) = 0, \quad (10)
\]

with

\[ W_{n,l}(r) = V(r) - E_{n,l}, \quad (11) \]

and the effective mass
\[m' = \eta^3/\mu^2 = (m_q m_Q \mu)/(m_q m_Q - 3\mu^2).\]  \hfill (12)

It is worthwhile to note that the expression (10) is in complete agreement with the expansion formula made by Durand and Durand in Ref. [27]. The perturbation term, \(W_{n,l}(r)^2\), that is, \((v^2/c^2)\) term in Eq. (10) is significant only when it is small (i.e., \(W_{n,l}(r)/m' \ll 1\)). This condition is verified by the confining potentials used to describe heavy–quark systems except near the color–Coulomb singularity at the origin, and for \(r \to \infty\). However, it is always satisfied on the average as stated by Durand and Durand [27]. We now proceed to solve Eq. (10) by applying the SLNET, obtaining an exact result in series for the energy [12-14]. This is a fundamental feature that allows one to attack problems that do not involve small coupling constant or Hamiltonians without a solvable strong term. Such an approximation has been used for the solution of nonrelativistic as well as relativistic wave equations with spherically symmetric potentials [12-14] yielding sufficiently accurate results. Imbo et al. [12] showed the high accuracy and the applicability of this method to a large number of spherically symmetric potentials. In the SLNET [9] it is convenient to shift the origin of coordinates to \(r = r_0\) (or \(x = 0\)) by defining

\[x = \bar{k}^{1/2}(r - r_0)/r_0.\]  \hfill (13)

Expansions about this point in powers of \(x\) and \(\bar{k}\) yield [9,13,14]

\[\frac{1}{r^2} = \frac{1}{r_0^2} \sum_{j=0}^{\infty} (-x)^j \frac{(1 + j)}{\bar{k}^{j/2}},\]  \hfill (14)

\[V(x(r_0)) = \frac{1}{Q} \sum_{j=0}^{\infty} \left( \frac{d^j V(r_0)}{dr_0^j} \right) \frac{(r_0 x)^j}{j!} \frac{1}{\bar{k}^{(4-j)/2}},\]  \hfill (15)

and also

\[E_{n,l} = \frac{1}{Q} \sum_{j=0}^{\infty} \bar{k}^{(2-j)} E_j.\]  \hfill (16)

Further, by substituting Eqs. (14) through (16) into Eq. (10), one gets

\[
\left\{- \frac{1}{4\mu} \frac{d^2}{dx^2} + \left[ \frac{\bar{k}}{16\mu} - \frac{(2 - a)}{8\mu} + \frac{(1 - a)(3 - a)}{16\mu \bar{k}} \right] \times \sum_{j=0}^{\infty} (-x)^j \frac{(1 + j)}{\bar{k}^{j/2}} \right\} \quad \text{or}
\]
\[
+ \frac{r_0^2}{Q} \sum_{j=0}^{\infty} \left( \frac{d^j V(r_0)}{dr_0^j} \right) \frac{(r_0 x)^j}{j!} \left( \begin{array}{c} \infty \\ j \end{array} \right) - \frac{r_0^2}{m'Q} \left[ \sum_{j=0}^{\infty} \left( \frac{d^j V(r_0)}{dr_0^j} \right) \frac{(r_0 x)^j}{j!} \right]^2
\]

\[+ \frac{2r_0^2}{m'Q} \sum_{j=0}^{\infty} \left( \begin{array}{c} \infty \\ j \end{array} \right) E_j x \sum_{j=0}^{\infty} \left( \frac{d^j V(r_0)}{dr_0^j} \right) \frac{(r_0 x)^j}{j!} \left( \begin{array}{c} \infty \\ j \end{array} \right) \chi_n(x) = \mathcal{E}_n \chi_n(x), \quad (17)\]

with the eigenvalues

\[
\mathcal{E}_n = \frac{r_0^2}{Q} \left\{ k \left( E_0 + \frac{E_0^2}{m'} \right) + \left( E_1 + \frac{2E_0 E_1}{m'} \right) + \left( E_2 + \frac{2E_0 E_2}{m'} + \frac{E_2^2}{m'} \right) \frac{1}{k} \right. \\
+ \left. \left( E_3 + \frac{2E_0 E_3}{m'} + \frac{2E_1 E_2}{m'} \right) \frac{1}{k^2} + \cdots \right\}. \quad (18)
\]

The parameter \( Q \) is an arbitrary scale, but is to be set equal to \( \kappa^2 \) to rescale potential after the problem is solved formally in the \( 1/\kappa \) expansion. Thus comparing Eqs. (17) and (18) with its counterpart Schrödinger–like equation for the one–dimensional anharmonic oscillator problem which has been investigated in detail for spherically symmetric case by Imbo et al. [12] . The final analytic expression in the \( 1/\kappa \) expansion of the energy eigenvalues appropriate to the SS particle is

\[
\mathcal{E}_n = \hat{k} \left[ \frac{1}{16 \mu} + \frac{r_0^2 V(r_0)}{Q} - \frac{r_0^2 V(r_0)^2}{m' Q} + \frac{2r_0^2 E_0 V(r_0)}{m' Q} \right] \\
+ \left[ 1 + 2n_r \right] \omega \left( \frac{2 - a}{8 \mu} \right) \frac{1}{k} \left[ \frac{2r_0^2 E_2 V(r_0)}{m' Q} + \beta^{(1)} \right]
\]

\[+ \frac{1}{k^2} \left[ \frac{2r_0^2 E_3 V(r_0)}{m' Q} + \beta^{(2)} \right] + O \left( \frac{1}{k^3} \right), \quad (19)\]

but \( n_r \) is to be set equal to 0, 1, 2, \( \cdots \). The quantities \( \beta^{(1)} \) and \( \beta^{(2)} \) appearing in the correction to the leading order of the energy expression are defined and listed in the Appendix A.

Thus, comparing the terms of Eq. (18) with their counterparts in Eq. (19) and equating terms of same order in \( \kappa \) yields the leading energy [14]

\[
E_0 = V(r_0) + \frac{m'}{2} \left[ \sqrt{1 + \frac{Q}{4r_0^2 m_\mu m'^2}} - 1 \right], \quad (20)
\]
where \( m_\mu = \mu/m' \) and \( r_0 \) is chosen to minimize the leading energy, Eq. (20), that is, [9,13-14]

\[
\frac{dE_0}{dr_0} = 0 \quad ; \quad \frac{d^2E_0}{dr_0^2} > 0.
\]  

(21)

Therefore, \( r_0 \) satisfies the following expression

\[
r_0^3V''(r_0)\left(\frac{m'^2}{4} + \frac{Q}{16r_0^2m_\mu}\right)^{1/2} = \frac{Q}{16m_\mu}.
\]  

(22)

Further, to solve for the shifting parameter \( a \), the next contribution to the energy eigenvalue is chosen to vanish, (i.e., \( E_1 = 0 \)), (cf. e.g. Refs. [9,12-14]), which gives

\[
a = 2 - 4\mu(1 + 2n_r)\omega,
\]  

(23)

where \( \omega \) is given by

\[
\omega = \frac{1}{4\mu} \left[ 3 + r_0V''(r_0)/V'(r_0) - 16r_0^4m_\mu V''(r_0)^2/Q \right]^{1/2},
\]  

(24)

and \( Q \) in Eq. (22) can be rewritten in a more convenient form as

\[
Q = 8m_\mu \left[ r_0^2V'(r_0) \right]^2(1 + \lambda),
\]  

(25)

with

\[
\lambda = \sqrt{1 + \left( \frac{m'}{r_0V''(r_0)} \right)^2}.
\]  

(26)

Therefore, solving Eqs. (23) through (26) together with \( Q = \bar{k}^2 \), yields

\[
1 + 2l + 4\mu(2n_r + 1)\omega = 2r_0^2V'(r_0) (2m_\mu + 2m_\mu \lambda)^{1/2},
\]  

(27)

which is an explicit equation in \( r_0 \). Once \( r_0 \) is determined via Eq. (27), \( E_0 \) can be obtained via Eq. (20), \( E_2 \) and \( E_3 \) are also obtained by solving Eqs. (18) and (19). Consequently, the general expression for the quark binding energy is

\[
E_{n,l} = E_0 + \frac{1}{r_0^2 \left( 1 - \frac{2W_{n,l}(r_0)}{m'} \right)} \left\{ \beta^{(1)} + \frac{\beta^{(2)}}{\bar{k}} + O \left[ \frac{1}{\bar{k}^2} \right] \right\},
\]  

(28)

which is an elegant algebraic expression that gives a rapidly convergent binding energy value with high accuracy. Now, in the framework of our semi-relativistic independent particle
model the mass levels for an atomlike $q\bar{Q}$ meson where a light quark $q$ is moving around an almost fixed heavy anti-quark $\bar{Q}$, the Salpeter bound-state mass can also be easily obtained from the expression of quark binding energy (28) as

$$M_{n,t}(q\bar{Q}) = m_q + m_Q + 2E_{n,t}. \quad (29)$$

Here, of course, we do not consider the recoil effects of the heavy anti-quark $\bar{Q}$. Finally, making use of Eqs. (27) through (29), one can resolve Eq. (10) for different types of central potentials taking into account the spin dependent terms $V_{SD}$ as correction terms to the static potential. We also consider a common parameter set for the upsilon and charmonium mass spectra in a flavour-independent case.

III. SOME POTENTIAL MODELS

A. Static potentials

The $c\bar{b}$ system that we investigate in the context of SS wave equation is often considered as nonrelativistic system, and consequently our treatment is based upon Salpeter equation with a Hamiltonian

$$H = H_0 + H_1 + V_{SD}, \quad (30)$$

where $H_0 = -\Delta_N/2\mu + V(r)$ denotes the Schrödinger Hamiltonian, $H_1 = -\Delta^2_N/8\eta^3$ denotes the perturbation term and $V_{SD}$ is the spin-dependent term [1,4,9,21] given by

$$V_{SD} \rightarrow V_{SS} = \frac{32\pi\alpha_s}{9m_qm_Q}\delta^3(r)s_1.s_2. \quad (31)$$

\[\text{To the moment, the only measured splitting of } nS\text{-levels is that of } \eta_c \text{ and } J/\psi, \text{ which allows us to evaluate the so-called (c.o.g.) data using } \overline{M}_\psi(1S) = (3M_{J/\psi} + M_{\eta_c})/4 \text{ and also } \overline{M}(nS) = M_V(nS) - (M_{J/\psi} - M_{\eta_c})/4n \text{ [15,21].}\]
Spin-independent relativistic corrections are not included. For the purpose of making some preliminary estimates of the energies of the lowest two S-states of the $B_c$ system, it is necessary to consider only the spin-spin part of the spin-dependent potential since these are S-states. Our solutions to the SS equation are generated numerically for the central potentials. The effects of the spin-dependent parts are added as a perturbation as improvement for splitting the lowest S-state of the spin-averaged energies. The potential parameters in this section are all strictly flavor-independent. The potential parameters and the constituent mass parameters are fitted to the low-lying energy levels of charmonium and upsilon systems. So, like most authors (cf. e.g. [1,4,8,9,21]), we determine the coupling constant $\alpha_s(m_c)$ from the well measured experimental charmonium hyperfine splitting of the $1S(c\bar{c})$ state value of $\Delta E_{\text{HF}} = M_{J/\psi} - M_{\eta_c} = 117 \pm 2$ MeV. The numerical value of $\alpha_s$ is dependent on the potential form and found to be compatible to the other measurements [1,3-4,6-9,15]. The observed $M_{J/\psi} - M_{\eta_c}$ hyperfine splitting fixes $\alpha_s$ for each potential. The perturbative part of such a quantity was evaluated at the lowest order in $\alpha_s$. Baldicchi and Prosperi [6] used the standard running QCD coupling expression

$$\alpha_s(Q) = \frac{4\pi}{(11 - \frac{2}{3} n_f) \ln \left( \frac{Q^2}{\Lambda^2} \right)},$$

with $n_f = 4$ and $\Lambda = 200$ MeV cut at a maximum value $\alpha_s(0) = 0.35$ to get the right $M_{J/\psi} - M_{\eta_c}$ splitting and to treat properly the infrared region [6]. Whereas Brambilla and Vairo [3] took in their perturbative analysis $0.26 \leq \alpha_s(\mu = 2 \text{ GeV}) \leq 0.30$.

We consider now a class of static potentials of general form

$$V(r) = -Ar^{-\gamma} + Br^\gamma + C_0; \quad 0 < \gamma \leq 1, \quad A, B > 0$$

had been previously proposed by Lichtenberg et al. [17] in which the additive constant $C_0$ may be of either sign. It was necessary to add a substantial flavor-independent constant to each potential for each quark-antiquark system. The constant $C_0$ in Eq. (33) is determined from the fit to the spin-averaged data (SAD) or center-of-gravity (c.o.g.) mass of the 1S state (i.e., $\overline{M}(1S) = 3068$ MeV) from the PDG [30]. Since the model is spin independent
and as the energies of the singlet states of quarkonium families have not been measured, a theoretical estimate of these unknown levels introduces uncertainty into the calculated SAD [9,31]. The best parameter fittings to (c.o.g.) data are obtained via the chi-square value or the least-square fit

\[ \chi^2 = \sum_i \left[ \frac{M_{\text{th}}^i - M_{\text{exp}}^i}{\Delta M_i} \right]^2, \]

where \( \Delta M_i \) refer to the errors in the experimental data. The static quarkonium potentials are monotone nondecreasing, and concave functions which satisfy the condition

\[ V'(r) > 0 \quad \text{and} \quad V''(r) \leq 0. \quad (34) \]

This class of potentials, in the generality (33), comprises the following family of potentials:

1. **Cornell potential**

   The QCD-motivated Coulomb-plus-linear potential (Cornell potential) has the form [24]

   \[ V_C(r) = -\frac{\alpha_c}{r} + \kappa r + C_0, \quad (35) \]

   The main drawback of this potential is that the \( c\bar{c} \) and \( b\bar{b} \) states lie in an intermediate region of quark separation where neither limiting forms of (34) should be valid.

   a. Coulomb and linear potential of Eichten et al. [1,22,24]: with \( \alpha_c = 0.52, \kappa = 0.1756 \, \text{GeV}^2, C_0 = -0.8138 \, \text{GeV}, m_c = 1.840 \, \text{GeV}, \) and \( m_b = 5.232 \, \text{GeV} \). We label this potential by Cornell 1.

   b. Coulomb and linear potential (35) of Hagiwara et al. [18,22]: with \( \alpha_c = 0.47, \kappa = 0.19 \, \text{GeV}^2, C_0 = 51 \, \text{MeV}, m_c = 1.32 \, \text{GeV} \) and \( m_b = 4.729 \, \text{GeV} \). We label this potential by Cornell 2.

2. **Song-Lin potential**

   This phenomenological potential was proposed by Song and Lin [16]

   \[ V_{SL}(r) = -Ar^{-1/2} + Br^{1/2} + C_0, \quad (36) \]
with $A = 0.923 \text{ GeV}^{1/2}$, $B = 0.511 \text{ GeV}^{3/2}$, $C_0 = -0.760 \text{ GeV}$, $m_c = 1.820 \text{ GeV}$, and $m_b = 5.190 \text{ GeV}$. The characteristic features of this potential may be traced in Ref. [16].

3. Turin potential

Lichtenberg et al. [17] suggested such a potential which is an intermediate between the Cornell and Song-Lin potentials. It has the simple form

$$V_T (r) = -Ar^{-3/4} + Br^{3/4} + C_0,$$

with $A = 0.620 \text{ GeV}^{1/4}$, $B = 0.304 \text{ GeV}^{7/4}$, $C_0 = -0.783 \text{ GeV}$, $m_c = 1.790 \text{ GeV}$ and $m_b = 5.166 \text{ GeV}$. The characteristic feature of this potential can be traced in Ref. [17].

4. Power-law potential

a. Power-law potential of Martin [19,22]: The phenomenological power-law potential

$$V_M (r) = -8.056 \text{ GeV} + (6.898 \text{ GeV})(r \times 1 \text{ GeV})^{0.1},$$

is labeled as Martin’s potential [19] with $m_c = 1.80 \text{ GeV}$ and $m_b = 5.162 \text{ GeV}$.

b. Power-law potential of Rosner et al. [20,22]:

$$V_R (r) = -0.726 \text{ GeV} + (0.801 \text{ GeV}) [(r \times 1 \text{ GeV})^\gamma - 1]/\gamma,$$

with $\gamma = -0.12$, $m_c = 1.56 \text{ GeV}$ and $m_b = 4.96 \text{ GeV}$.

5. Logarithmic potential of Quigg and Rosner [21,22]

A Martin’s power-law potential reduces [19] into

$$V_L (r) = -0.6161 \text{ GeV} + (0.733 \text{ GeV}) \ln(r \times 1 \text{ GeV}),$$

with $m_c = 1.50 \text{ GeV}$ and $m_b = 4.890 \text{ GeV}$. The potential forms in (38), and (40) were used by Eichten and Quigg [1,21]. Further, all of these potential forms were also used for $\psi$ and $\Upsilon$ data probing $0.1 \text{ fm} < r < 1 \text{ fm}$ region [15].
B. QCD-motivated potentials

1. Igi-Ono potential

The Cornell, logarithmic and power law potentials for some systems should account properly for the running $\alpha_s$ at the different distances. The $\alpha_s$ and the string constant $\kappa$ that describes the long-distance behavior in (35) can vary independently. The $\Lambda_{QCD}$ used to evaluate $\alpha_s$ is related to a specific renormalization scheme so that comparison with other calculations is possible. This leads to expanding $\alpha_s$ to at least 2-loop order and use of nontrivial methods in the interpolation between long- and short-distance behaviors. The interquark potential at short distances has been calculated to 2-loop calculations in the modified minimal-subtraction ($\overline{MS}$) scheme [25]. Together with the 2-loop expression for $\alpha_s$, one has [10,25]

$$V(r) = -\frac{4\alpha_s(\mu)}{3r} + \left[1 + \frac{\alpha_s(\mu)}{2\pi}(b_0 \ln \mu r + A)\right],$$

(41)

and

$$\alpha_s(\mu) = \frac{4\pi}{b_0 \ln(\mu^2/\Lambda_{MS}^2)} \left[1 - b_1 \frac{\ln(\mu^2/\Lambda_{MS}^2)}{b_0^2 \ln(\mu^2/\Lambda_{MS}^2)}\right],$$

(42)

with

$$[b_0, b_1, A] = \left[11 - \frac{2}{3} n_f, 102 - \frac{38}{3} n_f, b_0 \gamma_E + \frac{31}{6} - \frac{5}{9} n_f\right].$$

(43)

Here $n_f$ is the number of flavors with mass below $\mu$ and $\gamma_E = 0.5772$ is the Euler’s number. The renormalization scale $\mu$ is usually chosen to be $1/r$ to obtain a simple form for $V(r)$. The singularity in $\alpha_s(1/r)$ at $r = 1/\Lambda_{MS} = 5 \text{ GeV}^{-1} \approx 1 \text{ fm}$ for $\Lambda_{MS} = 200 \text{ MeV}$. This singularity can be removed by the substitution

$$\ln \frac{1}{r^2 \Lambda_{MS}^2} \rightarrow f(r) = \ln \left[\frac{1}{r^2 \Lambda_{MS}^2} + b_0\right].$$

(44)

The constant $b_0$ is an adjustable parameter of the potential and will not affect the perturbative part of the potential. Hence, setting $n_f = 4$ and $n_f = 5$ in Eq. (43), the one-gluon exchange part of the interquark potential simply take the following two forms
\[ V^{(n_f=4)}_{OGE}(r) = -\frac{16\pi}{25} \frac{1}{r f(r)} \left[ 1 - \frac{462 \ln f(r)}{625 f(r)} + \frac{2\gamma_E + \frac{53}{75}}{f(r)} \right], \quad (45) \]

and

\[ V^{(n_f=5)}_{OGE}(r) = -\frac{16\pi}{23} \frac{1}{r f(r)} \left[ 1 - \frac{348 \ln f(r)}{529 f(r)} + \frac{2\gamma_E + \frac{43}{69}}{f(r)} \right], \quad (46) \]

respectively, where \( f(r) \) is the function given in Eq. (44). Further, the long distance interquark potential grows linearly leading to confinement as

\[ V_L(r) = a_0 r. \quad (47) \]

Igi and Ono [25] proposed a potential whose general form

\[ V^{(n_f=4,5)}(r) = V^{(n_f=4,5)}_{OGE} + a_0 r + d_0 r e^{-g_0 r} + C_0, \quad (48) \]

so as to interpolate smoothly between the two parts. They added phenomenologically a term \( d_0 r e^{-g_0 r} \) to the potential so that to adjust the intermediate range behavior by which the range of \( \Lambda_{\overline{MS}} \) is extended keeping linearly rising confining potential. Hence, the \( \Lambda_{\overline{MS}} = (100, 500) \) MeV keeping a pretty good fit to the \( c\bar{c} \) and \( b\bar{b} \) data.

2. Improved Chen-Kuang potential

Chen and Kuang [26] proposed two improved potential models so that the parameters in Ref. [26] all vary explicitly with \( \Lambda_{\overline{MS}} \) so that these parameters can only be given numerically for several values of \( \Lambda_{\overline{MS}} \). Such potentials have the natural QCD interpretation and explicit \( \Lambda_{\overline{MS}} \) dependence both for giving clear link between QCD and experiment and for convenience in practical calculation for a given value of \( \Lambda_{\overline{MS}} \). This potential takes the general form

\[ V^{(n_f=4)}(r) = -\frac{16\pi}{25} \frac{1}{r f(r)} \left[ 1 - \frac{462 \ln f(r)}{625 f(r)} + \frac{2\gamma_E + \frac{53}{75}}{f(r)} \right] + \kappa r, \quad (49) \]

where the string tension is related to Regge slope by \( \kappa = \frac{1}{2\pi \alpha} \). The function \( f(r) \) in Eq. (49), takes the general form
\[ f(r) = \ln \left[ \frac{1}{\Lambda_{\text{MS}}^2} + 4.62 - A(r) \right]^2, \]  

(50)

and

\[ A(r) = \left[ 1 - \frac{1 - \frac{\Lambda_{\text{MS}}}{4 \Lambda_{\text{MS}}^2}}{1 - \exp \left\{ - \left[ 15 \left[ 3 \frac{\Lambda_{\text{MS}}}{\Lambda_{\text{MS}}^2} - 1 \right] \Lambda_{\text{MS}}^2 \right] \right\} \right], \]  

(51)

with the following set of parameters fit

\[ [\kappa, \alpha', \Lambda_{\text{MS}}^f] = [0.1491 \text{ GeV}^2, 1.067 \text{ GeV}^{-2}, 180 \text{ MeV}] \]  

(52)

together with \( m_c = 1.478 \text{ GeV}, m_b = 4.876 \text{ GeV}, \Lambda_{\text{MS}} = (100, 600) \text{ MeV} \) and \( \alpha_s = 0.235 \).

The details of this potential can be traced in Ref. [26].

**IV. SUMMARY AND CONCLUSIONS**

We have solved numerically SS wave equation, Eq. (3), for various potentials to determine the position of the spin-averaged ground-state masses of \( M_{1S}(J/\psi \text{ and } \eta_c) = 3068 \text{ MeV}, M_{1S}(\Upsilon \text{ and } \eta_b) = 9447 \text{ MeV} \), and also \( M_{1S}(B_c) \) with the help of Eq. (29). For simplicity and for the sake of comparison, we have neglected the variation of \( \alpha_s \) with momentum in (32) to have a common spectra for all states and scale the splitting of \( c\bar{c} \) and \( b\bar{b} \) from the charmonium value. In our calculations for the hyperfine nsplitting values of the \( c\bar{c} \) spectrum, we have followed the same approach of Ref. [9] in fitting the QCD coupling constant \( \alpha_s \).

Table I reports the SS SAD mass spectrum and hyperfine splitting values of the vector and pseudoscalar masses of the \( c\bar{c} \) for a family of seven central potentials. Masses and splittings lie within the ranges quoted by Kwong and Rosner [7], Eichten and Quigg [1] and also Fulcher et al. [4]. The value of the coupling constant \( \alpha_s \), is obtained from the proper splitting values of the \( c\bar{c} \) and \( b\bar{b} \) quarkonium systems [1,4,6-9].

It has been stated the possibility of producing \( c\bar{c} \) mesons in \( e^+e^- \) and hadron-hadron colliders [31-34]. We have used the coupling constant in our analysis in the range \( 0.185 \leq \alpha_s \leq 0.293 \) for all central potentials and \( 0.190 \leq \alpha_s \leq 0.448 \) for the QCD-motivated potentials as shown in Tables I and IV. Here, we have followed most authors in fixing the coupling
constant $\alpha_s(m_c)$ in reproducing all the spectra, [c.f. e.g. Refs. [1,4,8,9]), although slight changes are made by other authors [15,35].

Our results for SAD $c\bar{b}$ masses together with their lowest S-state splittings for both Schrödinger and Salpeter models are presented in Table II and compared with the other estimates of Refs. [4,24,32,36]. The values of the $\chi^2$ fits are listed in Table II, they show that the Salpeter results yield $\chi^2 = 447$ in the present model and give better agreement with experiment, PDG [30], than the Schrödinger results of $\chi^2 = 836$. Thus, the $\chi^2$ of the overall fit increases by about a factor 2. Hence, we conclude that extending the contest between the Salpeter equation and Schrödinger’s [9] to the charmonium and bottomonium systems provides additional evidence of an experimental signature for the semi-relativistic kinetic energy correction operator (second term) of Eq. (3). We have found that the measured values for the SAD of charmonium and the upsilon system are in better agreement with the Salpeter equation results than the Schrödinger equation results, (cf. e.g. Ref. [9]) and Table II of the present work, which is the same conclusion reached by Jacobs et al. [37] and Fulcher et al. [4]. Therefore, minimizing the quantity $\chi^2$ is a stringent argument that indicates the accuracy of the semirelativistic model over the previous model [9]. Overmore, in Table II, we have considered two different fitted sets of parameters for reproducing the $c\bar{b}$ masses. We noted that the values of quark masses increase if we are allowed to add an additional constant $C_0$ as given in the potential generality (33) and decrease if we drop it out. Hence, the overall fit increases by nearly a factor 4 in this trend.

Bambilla and Vairo [3] have calculated the maximum final result of $(M_{B^*})_{\text{pert}} = 6326^{+29}_{-29}$ MeV, the upper limit corresponds to the choice of parameters $\Lambda_{MS}^{n_f=3} = 350$ MeV and $\mu = 1.2$ GeV, while the lower limit to $\Lambda_{MS}^{n_f=3} = 250$ MeV and $\mu = 2.0$ GeV as the best approximation to their perturbative calculation. Our predictions for the $c\bar{b}$ fine and hy-

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3Kiselev et al. [15] have taken into account that $\Delta M_{\Upsilon}(1S) = \frac{\alpha_s(\Upsilon)}{\alpha_s(\psi)} \Delta M_\psi(1S)$ with $\alpha_s(\Upsilon)/\alpha_s(\psi) \simeq 3/4$. On the other hand, Motyka and Zalewski [35] also found $\frac{\alpha_s(m_\Upsilon^2)}{\alpha_s(m_\psi^2)} \simeq 11/18$. 
perfine splittings together with the ones estimated by other authors are listed in Table III. The SLNET fine and hyperfine splitting estimations for a group potentials are all fall in the range demonstrated by other authors. Larger discrepancies among the various methods occur for the ground and excited states [6].

The fitted set of parameters and SAD mass spectrum and their splittings of the Igi-Ono potential [25] labeled by type 1, 2 and 3, are listed in Table IV, V and VI. It is clear that the overall study seems likely to be pretty good and the reproduced fine and hyperfine splittings of the states are also reasonable. Further, it is worthwhile to note that the quark masses $m_c$ and $m_b$ increase as the values of $\Lambda_{\overline{MS}}$ increase. This is explained in the following way. The absolute value of the short-range asymptotic behavior of the potential in Eqs. (45) and (46) decreases with increasing $\Lambda_{\overline{MS}}$. In order to reproduce the SAD masses we need larger quark masses for larger values of $\Lambda_{\overline{MS}}$, (cf. e.g. Table III of Ref. [9]). Clearly, this is well noted in the $m_c$ values.

We have also calculated the predicted $c\bar{b}$ spectrum obtained from the Chen-Kuang potential in Table VII. We see that for states below the threshold, the deviations of the predicted spectra from the experimental SAD are within several MeV. We find that $m_c$ and $m_b$ are insensitive to the variation of $\Lambda_{\overline{MS}}$ for this potential. The $\Lambda_{\overline{MS}}$ dependence of the Chen-Kuang potential is given in Eq. (49). The potential found to be more sensitive especially for the lowest states and found not sensitive for higher states. The effect of $\Lambda_{\overline{MS}}$ is clearly on the Coulombic part of Eq. (49).

In Table VIII, in order to get some idea of an error estimate (precision) for our model, we have calculated the low-lying two S-state masses of the singlet and triplet $B_c$ system which are completely dependent on the running coupling constant $\alpha_s$ values, (cf. e.g. Table I), determined in charmonium and the upsilon systems. Using the largest and smallest values of $\alpha_s$ in Table I to determine the errors, we have calculated $M_{B_c} = 6234^{+10}_{-14}$ MeV, $M_{B_c^*} = 6310^{+3}_{-6}$ MeV, and $69 \text{ MeV} \leq \Delta_{1S} \leq 80 \text{ MeV}$. Our hyperfine splitting value is in excellent agreement with the other estimates made by other authors. Clearly, the precision of the experiments [2] requires a very substantial improvement to be sensitive to the bound-state mass differences.
between the various calculations listed in Table VIII.

We study some properties of the $B_c$ system like the pseudoscalar decay constant given by the Van Royen-Weisskopf formula modified for color [4,38], that is,

$$f_{B_c} = \sqrt{\frac{3}{\pi M_{B_c}}} |R_{1S}(0)|,$$

with the nonrelativistic radial wavefunction at the origin [21]

$$|R_{1S}(0)| = \sqrt{4\pi} |\psi_{1S}(0)|.$$  

Moreover, in Eq. (53), the Salpeter bound-state mass of the low-lying ($n = 1, l = 0$) pseudoscalar $B_{c}$-state can be found via

$$M_{B_{c}}(0^-) = m_c + m_b + 2E_{1,0} - 3\Delta E_{HF}/4,$$  

and also the vector $B_{c}^*$-state via

$$M_{B_{c}^*}(1^-) = m_c + m_b + 2E_{1,0} + \Delta E_{HF}/4.$$  

whereas the square-mass difference can be simply found via

$$\Delta M^2 = M_{B_{c}^*}^2(1^-) - M_{B_{c}}^2(0^-) = 2\Delta E_{HF} \left[m_c + m_b + 2E_{1,0} - \Delta E_{HF}/4\right],$$  

with

$$\Delta E_{HF} = \frac{8\alpha_s(\mu)}{9m_cm_b} |R_{1S}(0)|^2.$$  

Using our estimates of $M_{B_c}$ and $M_{B_{c}^*}$ in Table II, we find decay constant for SS and for S models. They are listed in Table IX together with the results of other authors [1,4,39-41]. All of these results are also in reasonable agreement with the lattice results [5].

The empirical result obtained by Collins et al. [42] for potential model wave functions at the origin, that is,

$$|R_{B_c}(0)|^2 \simeq |R_{J/\psi}(0)|^{1.3} |R_{Y}(0)|^{0.7},$$

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provides another touchstone for our numerical work. From charmonium and upsilon calculations, we get wave function at the origin for SS and for S as reported in Table IX which are in excellent agreement with the result of Fulcher [4], in which $|R_{B_c}(0)|^2 \simeq 1.81 \text{ GeV}^3$. Moreover, the empirical relationship for the ground state hyperfine splittings, that is,

$$M_{B'_c} - M_{B_c} \simeq 0.7 \left( M_{J/\psi} - M_{\eta_c} \right)^{0.65} \left( M_{\Upsilon} - M_{\eta_b} \right)^{0.35}, \quad (60)$$

yields a splitting of 63 MeV, about 1% and 5% lower than our estimations for S and SS equations, respectively, as shown in Table II. Both of these results are in reasonable agreement with the results of Collins et al. [42], Fulcher [4] and FCY [4] and other authors [39-41].

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Here we list the analytic expressions of $\beta^{(1)}$, $\beta^{(2)}$, $\epsilon_j$, and $\delta_j$ for the spinless Salpeter equation

$$\beta^{(1)} = \frac{(1 - a)(3 - a)}{16\mu} + \left[(1 + 2n_r)\bar{\varepsilon}_2 + 3(1 + 2n_r + 2n_r^2)\bar{\varepsilon}_4\right]$$

$$- \omega^{-1}\left[\bar{\varepsilon}_1^2 + 6(1 + 2n_r)\bar{\varepsilon}_1\bar{\varepsilon}_3 + (11 + 30n_r + 30n_r^2)\bar{\varepsilon}_3^2\right], \quad (A1)$$

$$\beta^{(2)} = \left[(1 + 2n_r)\bar{\delta}_2 + 3(1 + 2n_r + 2n_r^2)\bar{\delta}_4 + 5(3 + 8n_r + 6n_r^2 + 4n_r^3)\bar{\delta}_6\right]$$

$$- \omega^{-1}(1 + 2n_r)\bar{\varepsilon}_2^2 + 12(1 + 2n_r + 2n_r^2)\bar{\varepsilon}_2\bar{\varepsilon}_4 + 2\bar{\varepsilon}_1\bar{\delta}_1$$

$$+ 2(21 + 59n_r + 51n_r^2 + 34n_r^3)\bar{\varepsilon}_4^2 + 6(1 + 2n_r)\bar{\varepsilon}_1\bar{\delta}_3$$

$$+ 30(1 + 2n_r + 2n_r^2)\bar{\varepsilon}_1\bar{\delta}_5 + 2(11 + 30n_r + 30n_r^2)\bar{\varepsilon}_3\bar{\delta}_3$$

$$+ 10(13 + 40n_r + 42n_r^2 + 28n_r^3)\bar{\varepsilon}_3\bar{\delta}_5 + 6(1 + 2n_r)\bar{\varepsilon}_3\bar{\delta}_1]$$

$$+ \omega^{-2}\left[4\bar{\varepsilon}_1^2\bar{\varepsilon}_2 + 36(1 + 2n_r)\bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_3 + 8(11 + 30n_r + 30n_r^2)\bar{\varepsilon}_2\bar{\varepsilon}_3^2\right]$$

$$+ 24(1 + 2n_r)\bar{\varepsilon}_1^2\bar{\varepsilon}_4 + 8(31 + 78n_r + 78n_r^2)\bar{\varepsilon}_1\bar{\varepsilon}_3\bar{\varepsilon}_4$$

$$+ 12\left(57 + 189n_r + 225n_r^2 + 150n_r^3\right)\bar{\varepsilon}_3^2\bar{\varepsilon}_4\right]$$

$$- \omega^{-3}\left[8\bar{\varepsilon}_1^2\bar{\varepsilon}_3 + 108(1 + 2n_r)\bar{\varepsilon}_1^2\bar{\varepsilon}_3^2 + 48(11 + 30n_r + 30n_r^2)\bar{\varepsilon}_1\bar{\varepsilon}_3^3\right]$$

$$+ 30(31 + 109n_r + 141n_r^2 + 94n_r^3)\bar{\varepsilon}_3^3\right], \quad (A2)$$

where

$$\bar{\varepsilon}_i = \frac{\varepsilon_i}{(4\mu\omega)_i^{1/2}}, \quad i = 1, 2, 3, 4. \quad (A3)$$

and

$$\bar{\delta}_j = \frac{\delta_j}{(4\mu\omega)_j^{1/2}}, \quad j = 1, 2, 3, 4, 5, 6. \quad (A4)$$

$$\varepsilon_1 = \frac{(2 - a)}{4\mu}, \quad \varepsilon_2 = -\frac{3}{8\mu}(2 - a), \quad (A5)$$

$$\varepsilon_3 = -\frac{1}{4\mu} + \frac{r_0^5}{6Q} \left[V''(r_0) - \frac{2V(r_0)V'''(r_0)}{m'}\right]$$
\(- \frac{6V'(r_0)V''(r_0)}{m'} + \frac{2V'''(r_0)E_0}{m'} \bigg), \quad (A6)\)

\(\varepsilon_4 = \frac{5}{16\mu} + \frac{r_0^6}{24Q} \left[ V'''(r_0) - \frac{2V(r_0)V'''(r_0)}{m'} \right], \quad (A7)\)

\(\delta_1 = -(1 - a)(3 - a) \frac{2r_0^3E_2V'(r_0)}{8\mu m'Q}, \quad (A8)\)

\(\delta_2 = \frac{3(1 - a)(3 - a)}{16\mu} + \frac{r_4^4E_2V''(r_0)}{m'Q}, \quad (A9)\)

\(\delta_3 = \frac{(2 - a)}{2\mu}; \quad \delta_4 = -\frac{5(2 - a)}{8\mu}, \quad (A10)\)

\(\delta_5 = -\frac{3}{8\mu} + \frac{r_0^7}{120Q} \left[ V'''(r_0) - \frac{2V(r_0)V'''(r_0)}{m'} \right] \quad - \frac{10V'(r_0)V'''(r_0)}{m'} - \frac{20V''(r_0)V'''(r_0)}{m'} + \frac{2V'''(r_0)E_0}{m'} \bigg], \quad (A11)\)

\(\delta_6 = \frac{7}{16\mu} + \frac{r_0^8}{720Q} \left[ V''''(r_0) - \frac{2V(r_0)V''''(r_0)}{m'} - \frac{12V'(r_0)V''''(r_0)}{m'} \right] \quad - \frac{20V''(r_0)V''(r_0)}{m'} - \frac{30V''(r_0)V'''(r_0)}{m'} + \frac{2V'''(r_0)E_0}{m'} \bigg], \quad (A12)\)
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TABLE I. $c\bar{b}$ quarkonium masses and hyperfine splittings $\Delta_{nS}^*$ (in MeV) of the lowest two S-states calculated for a group of static potentials (GSP).

| States          | $\alpha_s(m_c)$ | Cornell 1 | Cornell 2 | Song-Lin | Turin | Martin | Rosner | Logarithmic |
|-----------------|-----------------|-----------|-----------|----------|-------|--------|--------|-------------|
| $M(1S)$         | 6289            | 6296      | 6288      | 6291     | 6286  | 6294   | 6294   |             |
| $M(1^3S_1)$     | 6327            | 6312      | 6308      | 6311     | 6304  | 6312   | 6311   |             |
| $M(1^1S_0)$     | 6220            | 6244      | 6230      | 6231     | 6232  | 6238   | 6241   |             |
| $\Delta_{1S}$   | 77              | 91.7      | 69.1      | 77.9     | 79.6  | 72.2   | 73.7   | 69.4        |
| $M(2S)$         | 6877            | 6876      | 6853      | 6859     | 6865  | 6882   | 6870   |             |
| $M(2^3S_1)$     | 6906            | 6886      | 6861      | 6868     | 6874  | 6889   | 6878   |             |
| $M(2^1S_0)$     | 6850            | 6848      | 6827      | 6832     | 6836  | 6859   | 6846   |             |
| $\Delta_{2S}$   | 42              | 36.1      | 37.0      | 34.1     | 35.8  | 37.6   | 29.7   | 32.4        |
| $M(3S)$         | 7247            | 7267      | 7181      | 7214     | 7202  | 7185   | 7190   |             |
| $M(4S)$         | 7545            | 7586      | 7422      | 7491     | 7444  | 7387   | 7412   |             |
| $M(1P)$         | 6754            | 6759      | 6741      | 6731     | 6735  | 6721   | 6770   | 6744        |
| $M(2P)$         | 7154            | 7133      | 7140      | 7089     | 7109  | 7102   | 7113   | 7104        |
| $M(3P)$         | 7395            | 7470      | 7347      | 7398     | 7367  | 7334   | 7347   |             |
| $M(1D)$         | 7028            | 7018      | 7014      | 6998     | 7002  | 7003   | 7043   | 7021        |
| $M(2D)$         | 7367            | 7332      | 7353      | 7272     | 7304  | 7290   | 7282   | 7284        |
| $M(3D)$         | 7562            | 7649      | 7488      | 7556     | 7509  | 7454   | 7479   |             |

* $\Delta_{nS} = M(n^3S_1) - M(n^1S_0)$. 
TABLE II. $c\bar{b}$ masses and $\Delta_{nS}$ (in MeV) of the using Coulomb-plus-linear potential (CPLP), in the Schrödinger and Salpeter models.

| States | CPLP$^a$ | CPLP$^b$ | [4]$^a$ | [4]$^b$ | [32] | [36] | [24] |
|--------|----------|----------|--------|--------|------|------|------|
| $\alpha_s(m_c)$ | 0.276 | 0.257 | 0.322 |
| $\overline{M}(1S)$ | 6339 | 6303 | 6297 | 6338 | 6314 | 6301 | 6317 | 6315 |
| $M(1^3S_1)$ | 6355 | 6319 | 6317 | 6332 |
| $M(1^1S_0)$ | 6291 | 6253 | 6235 | 6270 |
| $\Delta_{1S}$ | 63.7 | 66.2 | 82.2 | 62 |
| $\overline{M}(2S)$ | 6930 | 6885 | 6852 | 6918 | 6872 | 6893 | 6870 | 7009 |
| $M(2^3S_1)$ | 6940 | 6894 | 6862 | 6881 |
| $M(2^1S_0)$ | 6902 | 6856 | 6823 | 6835 |
| $\Delta_{2S}$ | 38 | 38.5 | 39.6 | 46 |
| $\overline{M}(3S)$ | 7352 | 7285 | 7220 | 7336 | 7270 | 7237 | 7225 |
| $\overline{M}(4S)$ | 7707 | 7615 | 7520 |
| $\overline{M}(1P)$ | 6756 | 6741 | 6725 | 6755 | 6745 | 6728 | 6735 |
| $\overline{M}(2P)$ | 7195 | 7152 | 7100 | 7193 | 7157 | 7122 |
| $\overline{M}(3P)$ | 7563 | 7493 | 7410 |
| $\overline{M}(1D)$ | 7036 | 7019 | 6980 | 7145 |
| $\overline{M}(2D)$ | 7418 | 7371 | 7299 |
| $\overline{M}(3D)$ | 7755 | 7677 | 7577 |

Parameters

| | $\alpha_c$ | $\kappa(GeV^2)$ | $C_0(GeV)$ | $m_c(GeV)$ | $m_b(GeV)$ | $\chi^2$ |
| | 0.472 | 0.191 | 0.0 | 1.3205 | 4.7485 | 836$^c$ |
| | 0.437 | 0.203 | 0.0 | 1.321 | 4.731 | 447$^c$ |
| | 0.457 | 0.182 | $-0.790$ | 1.79 | 5.17 | 2011$^c$ |

$^a$In the Schrödinger model, $^b$In the Salpeter model, $^c$Fitting to $c\bar{c}$ and $b\bar{b}$ spectra.
| States                      | Latt.[6] | Qua.[6] | Lin.[6] [4] | IOP* | GSP* | CKP* |
|-----------------------------|----------|---------|-------------|------|------|------|
| Fine Splitting              |          |         |             |      |      |      |
| $M(2S) - M(1S)$             | 672 ± 120| 558     | 533         | 579  | 577 ± 25 | 578±^{10}_{-13} |
| $M(3S) - M(1S)$             | 931      | 899     |             | 916±^{30}_{-13} | 921±^{50}_{-30} | 866±^{45}_{-43} |
| $M(4S) - M(1S)$             |          |         |             | 1176±^{29}_{-31} | 1178±^{112}_{-85} | 1149±^{23}_{-31} |
| $M(2P) - M(1P)$             | 382      | 376     |             | 375 ± 8 | 370±^{29}_{-27} | 389 ± 1 |
| $M(3P) - M(1P)$             |          |         |             | 657±^{32}_{-10} | 637±^{92}_{-73} | 695±^{3}_{-1} |
| $M(2D) - M(1D)$             | 324      | 321     |             | 296±^{21}_{-12} | 288±^{51}_{-49} | 318±^{2}_{-1} |
| $M(3D) - M(1D)$             |          |         |             | 547±^{42}_{-18} | 514±^{121}_{-103} | 586 ± 1 |
| Hyperfine Splitting         |          |         |             |      |      |      |
| $\Delta_{1S}$              | 41 ± 20  | 77      | 62          | 55  | 63 ± 1 | 76±^{16}_{-7} |
| $\Delta_{2S}$              | 30 ± 8   | 42      | 33          | 32  | 32    | 35±^{3}_{-5} |

* The averaged splittings in our work for different potentials.

| Type 1 ($n_f = 5$)          |          |         |             |      |      |      |
|----------------------------|----------|---------|-------------|------|------|------|
| $a_0$ (GeV$^2$)            | 0.1587   | 0.3436  | 0.2550      | 0.190| 1.38 | 4.782 |
| $g_0$ (GeV)                |          |         |             |      |      |      |
| $d_0$ (GeV$^2$)            |          |         |             |      |      |      |
| $\alpha_s(m_c)$            |          |         |             |      |      |      |
| $m_c$ (GeV)                |          |         |             |      |      |      |
| $m_b$ (GeV)                |          |         |             |      |      |      |

| Type 2 ($n_f = 4$)          |          |         |             |      |      |      |
|----------------------------|----------|---------|-------------|------|------|------|
| $a_0$ (GeV$^2$)            | 0.1587   | 0.3436  | 0.2550      | 0.403| 1.353| 4.746 |
| $g_0$ (GeV)                |          |         |             |      |      |      |
| $d_0$ (GeV$^2$)            |          |         |             |      |      |      |
| $\alpha_s(m_c)$            |          |         |             |      |      |      |
| $m_c$ (GeV)                |          |         |             |      |      |      |
| $m_b$ (GeV)                |          |         |             |      |      |      |

| Type 3 ($n_f = 4$)          |          |         |             |      |      |      |
|----------------------------|----------|---------|-------------|------|------|------|
| $a_0$ (GeV$^2$)            | 0.1585   | 0       | 0           | 0.448| 1.494| 4.885 |
| $g_0$ (GeV)                |          |         |             |      |      |      |
| $d_0$ (GeV$^2$)            |          |         |             |      |      |      |
| $\alpha_s(m_c)$            |          |         |             |      |      |      |
| $m_c$ (GeV)                |          |         |             |      |      |      |
| $m_b$ (GeV)                |          |         |             |      |      |      |
TABLE V. $c\bar{b}$ quarkonium mass spectrum and $\Delta_{nS}$ (in MeV) predicted by using IOP of types 1, 2, and 3 in our work.

| States       | $\Lambda_{\overline{MS}}$(MeV) | 200 | 200 | 250 | 300 | 400 |
|--------------|---------------------------------|-----|-----|-----|-----|-----|
|              |                                 | $b$ | $a$ | $b$ | $c$ | $c$ |
| $\overline{M}(1S)$ | 6295 6294 6364 6324 6292 6245
| $M(1^3S_1)$  | 6311 6310 6377 6340 6310 6267 |     |     |     |     |     |
| $M(1^1S_0)$  | 6247 6247 6324 6277 6239 6179 |     |     |     |     |     |
| $\Delta_{1S}$ | 64.6 63.5 53.3 62.1 70.9 87.9  |     |     |     |     |     |
| $\overline{M}(2S)$ | 6897 6872 6899 6876 6857 6829 |     |     |     |     |     |
| $M(2^3S_1)$  | 6905 6880 6907 6884 6865 6837 |     |     |     |     |     |
| $M(2^1S_0)$  | 6873 6848 6876 6852 6832 6803 |     |     |     |     |     |
| $\Delta_{2S}$ | 32.2 32.2 31.2 32.3 33.1 34  |     |     |     |     |     |
| $\overline{M}(3S)$ | 7231 7197 7249 7233 7222 7210 |     |     |     |     |     |
| $\overline{M}(4S)$ | 7474 7439 7539 7529 7523 7516 |     |     |     |     |     |
| $\overline{M}(1P)$ | 6757 6733 6755 6741 6731 6719 |     |     |     |     |     |
| $\overline{M}(2P)$ | 7132 7099 7134 7124 7117 7108 |     |     |     |     |     |
| $\overline{M}(3P)$ | 7404 7368 7437 7430 7425 7420 |     |     |     |     |     |
| $\overline{M}(1D)$ | 7041 7008 7019 7013 7009 7005 |     |     |     |     |     |
| $\overline{M}(2D)$ | 7328 7292 7335 7330 7327 7323 |     |     |     |     |     |
| $\overline{M}(3D)$ | 7570 7532 7606 7602 7599 7597 |     |     |     |     |     |

$a$ Type 1 with $C_0 = -28$ MeV to fit (c.o.g.) value.

$b$ Type 2 with $C_0 = -28$ MeV to fit (c.o.g.) value

$c$ Type 3 with $C_0 = -29$ MeV to fit (c.o.g.) value.
TABLE VI. $c\bar{b}$ quarkonium mass spectrum and $\Delta M^a_n$ (in MeV) predicted by using IOP.$^b$

| States | $\Lambda_{\overline{MS}}(MeV)$ | 200 | 250 | 300 |
|--------|-------------------------------|-----|-----|-----|
| $M(1S)$ | 6382 6342 6311 |     |     |     |
| $M(2S)$ | 6918 6894 6875 |     |     |     |
| $M(3S)$ | 7267 7251 7241 |     |     |     |
| $M(4S)$ | 7558 7548 7542 |     |     |     |
| $M(1P)$ | 6773 6759 6750 |     |     |     |
| $M(2P)$ | 7153 7142 7135 |     |     |     |
| $M(3P)$ | 7456 7448 7443 |     |     |     |
| $M(1D)$ | 7037 7031 7027 |     |     |     |
| $M(2D)$ | 7353 7348 7345 |     |     |     |
| $M(3D)$ | 7625 7620 7618 |     |     |     |
| $\Delta M_{2S}$ | 536 552 564 | 585$^c$ |     |     |
| $\Delta M_{3S}$ | 349 357 366 |     |     |     |
| $\Delta M_{4S}$ | 291 297 301 |     |     |     |
| $\Delta M_{2P}$ | 380 383 385 |     |     |     |
| $\Delta M_{3P}$ | 303 306 308 |     |     |     |
| $\Delta M_{2D}$ | 316 317 318 |     |     |     |
| $\Delta M_{3D}$ | 272 272 273 |     |     |     |

$^a$Fine splitting is $\Delta M_{(n+1)S} = M((n + 1)S) - M(nS)$.

$^b$Type 3 fitted to the experimental $c\bar{c}$ and $b\bar{b}$ spectra with $m_b = 4.874$ GeV.

$^c$Here we cite GKLT in Ref. [32].
TABLE VII. $c\bar{b}$ quarkonium mass spectrum and $\Delta_{nS}$
(in MeV) predicted by using CKP in our work.

| States     | $\Lambda_{MS}$ (MeV) | 100 - 270 | 275 - 350 | 350 - 400 | 400 - 450 | 450 - 500 | 500 - 600 |
|------------|-----------------------|-----------|-----------|-----------|-----------|-----------|-----------|
| $\bar{M}$ (1S) | 6250 6308 6348 6278 6318 6277 |
| $M(1^3S_1)$ | 6267 6325 6365 6293 6331 6293 |
| $M(1^1S_0)$ | 6197 6255 6296 6232 6279 6231 |
| $\Delta_{1S}$ | 69.9 69.9 68.9 68.9 51.8 61.6 |
| $\bar{M}$ (2S) | 6808 6808 6808 6808 6841 6835 |
| $M(2^3S_1)$ | 6817 6817 6817 6817 6850 6844 |
| $M(2^1S_0)$ | 6782 6782 6782 6782 6816 6809 |
| $\Delta_{2S}$ | 35.2 35.2 35.2 35.2 33.8 35.1 |
| $\bar{M}$ (3S) | 7171 7171 7171 7171 7229 7173 |
| $\bar{M}$ (4S) | 7466 7466 7466 7466 7490 7466 |
| $\bar{M}$ (1P) | 6674 6674 6674 6674 6690 6678 |
| $\bar{M}$ (2P) | 7062 7062 7062 7062 7080 7065 |
| $\bar{M}$ (3P) | 7368 7368 7368 7368 7388 7368 |
| $\bar{M}$ (1D) | 6953 6953 6953 6953 6957 6955 |
| $\bar{M}$ (2D) | 7270 7270 7270 7270 7277 7270 |
| $\bar{M}$ (3D) | 7540 7540 7540 7540 7540 7540 |

$^a$ Type 1 with $C_0 = -28$ MeV to agree with (c.o.g.) value.
TABLE VIII. The predicted $\bar{c}b$ masses of the lowest S-wave and its splitting (all in MeV) compared with the other authors.

| work                  | $\mathcal{M}(1S)$ | $M_{B_c}(1^1S_0)^*$ | $M_{B_c}(1^3S_1)$ | $\Delta_{1S}$ |
|-----------------------|-------------------|---------------------|-------------------|---------------|
| Eichten et al. [1]    | 6258 ± 20         |                     |                   |               |
| Colangelo et al. [3]  | 6280              | 6350                | 70                |               |
| Baker et al. [33]     | 6287              | 6372                | 85                |               |
| Roncaglia et al. [33] |                   | 6320 ± 10           |                   |               |
| Godfrey et al. [1]    | 6270              | 6340                |                   |               |
| Bagan et al. [1,33]   | 6255 ± 20         | 6330 ± 20           | 75                |               |
| Bambilla et al. [3]   |                   | 6326$^{+29}_{-9}$   |                   |               |
| Baldicchi et al. [6]  | 6194 ~ 6292       | 6284 ~ 6357         | 65 ≤ $\Delta_{1S}$ ≤ 90 |               |
| SLNET (GSP) †         | 6291 ± 5          | 6234$^{+10}_{-14}$  | 6310$^{+3}_{-6}$  | 69 ≤ $\Delta_{1S}$ ≤ 80 |
| SLNET (IOP) ‡         | 6324              | 6277                | 6340              | 62            |
| SLNET                 | 6291†             | 6234                | 6310              | 66            |

* Experimental mass of such a singlet state is presented in [2,6].
† Averaging over the seven values in Table I.
‡ An estimation using type 3 with $\Lambda_{\overline{MS}} = 250$ MeV.
† Best estimation to the (c.o.g.) lowest S-state.

TABLE IX. The leptonic $B_c$ meson constant and radial wave function at the origin calculated in our model and by the other authors.

| Level ($1^1S_0$) | Martin | Coulomb | EQ[1] | F[4] | [39] | [40] | [41] | S* | SS* |
|------------------|--------|---------|-------|------|------|------|------|----|-----|
| $f_{B_c}$ (MeV)  | 510 ± 80 | 456 ± 70 | 495   | 517  | 410  | 600  | 500  | 495 | 524 |
| $|R_{B_c}(0)|^2$ (GeV$^3$) | 1.716 | - | 1.638 | 1.81 | - | - | - | 1.85 | 1.90 |

*Present work.