Asymptotics of Studentized U-type processes for changepoint problems

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Dedicated to the memory of Tibor Nemetz

ABSTRACT

This paper investigates weighted approximations for studentized $U$-statistics type processes, both with symmetric and antisymmetric kernels, only under the assumption that the distribution of the projection variate is in the domain of attraction of the normal law. The classical second moment condition $E|h(X_1, X_2)|^2 < \infty$ is also relaxed in both cases. The results can be used for testing the null assumption of having a random sample versus the alternative that there is a change in distribution in the sequence.

Key Words and Phrases: Weighted approximations in probability, functional limit theorems, $U$-statistics type processes, Studentization, change in distribution, symmetric and antisymmetric kernels, Gaussian processes.

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Running Head: Studentized U-type processes

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1 Introduction and main results: the case of symmetric kernels

Let $X, X_1, X_2, \ldots$ be a sequence of non-degenerate independent real-valued random variables with distribution function $F$. Suppose we are interested in testing the null hypothesis:

$$H_0: \ X_i, \ 1 \leq i \leq n, \ have \ the \ same \ distribution,$$

against the one change in distribution alternative:

$$H_A: \ there \ is \ an \ integer \ k, \ 1 \leq k < n, \ such \ that$$

$$P(X_1 \leq t) = \cdots = P(X_k \leq t), \ P(X_{k+1} \leq t) = \cdots = P(X_n \leq t)$$

for all $t$ and $P(X_k \leq t_0) \neq P(X_{k+1} \leq t_0)$ for some $t_0$.

Testing for this kind of a change in distribution has been studied extensively in the literature by using parametric as well as non-parametric methods. One of the non-parametric methods was proposed by Csörgő and Horváth (1988a, b), who used functionals of a $U$-statistic type ($U$-type, from now on) process to test $H_0$ against $H_A$. Let $h(x, y)$ be a measurable real valued symmetric function, i.e. $h(x, y) = h(y, x)$. The $U$-type process of Csörgő and Horváth (1988a, b) is defined by

$$U_n(t) = Z_{[(n+1)t] - n^2t(1-t)\theta}, \quad 0 \leq t \leq 1,$$

where $\theta = Eh(X_1, X_2)$, and

$$Z_k = \sum_{i=1}^{k} \sum_{j=k+1}^{n} h(X_i, X_j), \quad 1 \leq k < n.$$

While $Z_k$ itself is not a $U$-statistic, it can be written as the sums of three $U$-statistics [cf. Csörgő and Horváth (1988a, b, 1997)]. The rational behind the definition of $Z_k$ is comparing the first $k$ observations to the remaining $(n-k)$ ones for $k=1, \ldots, n-1$, via an appropriate bivariate kernel function $h(x, y)$ for the sake of capturing the possibility of having a change in distribution at an unknown time $k$ as postulated in $H_A$. Typical choices of symmetric kernel $h$ are $xy, (x-y)^2/2$ (the sample variance), $|x-y|$ (Gini’s mean difference), and sign$(x+y)$ (Wilcoxon’s one-sample statistic).

Throughout the paper, we write $g(t) = E(h(X, t) - \theta), \sigma^2 = Eg^2(X_1)$ and, for later use, we define a Gaussian process $\Gamma$ by

$$\Gamma(t) = (1-t) W(t) + t [W(1) - W(t)], \quad 0 \leq t \leq 1, \quad (1)$$

where $\{W(t), 0 \leq t < \infty\}$ is a standard Wiener process. Furthermore, let $Q$ be the class of positive functions $q$ on $(0, 1)$, i.e., $\inf_{\delta \leq t \leq 1-\delta} g(t) > 0$ for $0 < \delta < 1$, which are nondecreasing in a neighbourhood of zero and nonincreasing in a neighbourhood of one, and let

$$I(q, c) = \int_{0+}^{1-} \frac{1}{t(1-t)} \exp \left(- \frac{ct^2(t)}{t(1-t)}\right) dt, \quad 0 < c < \infty.$$
In terms of these notations, Csörgő and Horváth (1988a, b), Szyszkowicz (1991, 1992) established the following result [cf. Theorem 2.4.2 in Csörgő and Horváth (1997)].

**Theorem A** Assume $H_0$, $0 < \sigma^2 < \infty$ and $E|h(X_1,X_2)|^2 < \infty$. Then, on an appropriate probability space for $X, X_1, X_2, \cdots$, we can define a sequence of Gaussian processes \{\Gamma_n(t), 0 \leq t \leq 1\} such that the equality in distribution

$$\{\Gamma_n(t), 0 \leq t \leq 1\} =_d \{\Gamma(t), 0 \leq t \leq 1\}$$

holds for each $n \geq 1$, and as $n \to \infty$,

$$\sup_{0 < t < 1} \left| n^{-3/2} \sigma^{-1} U_n(t) - \Gamma_n(t) / q(t) \right| = o_P(1). \quad (3)$$

if and only if $I(q,c) < \infty$ for all $c > 0$.

**Remark 1** The condition $E|h(X_1,X_2)|^2 < \infty$ implies that $Eg^2(X_1) < \infty$, and we assume that $\sigma^2 = Eg^2(X_1) > 0$. This is the so-called non-degenerate case when studying $U$-statistics via the function $g(t) = E(h(X,t) - \theta)$ that induces the projection of $U$-statistics into sums of i.i.d. random variables, the so-called Hoeffding (1948) projection principle that, in part, rests on a paper of Halmos (1946).

For functions $x, y$ in $D[0,1]$ and $q \in \mathbb{Q}$, we define the weighted sup-norm metric $||/q||$ by

$$||/(x-y)/q|| = \sup_{0 < t < 1} ||(x(t) - y(t))/q(t)||,$$

whenever this is well defined, i.e., when $\lim \sup ||(x(t) - y(t))/q(t)||$ is finite for $t \downarrow 0$ and $t \uparrow 1$.

In view of (2) and this terminology, (3) of Theorem A implies the following weak convergence, a functional limit theorem.

**Corollary A** With $q \in \mathbb{Q}$, and $\rightarrow_d$ standing for convergence in distribution as $n \to \infty$, we have

$$h\{n^{-3/2} \sigma^{-1} U_n(\cdot)/q(\cdot)\} \rightarrow_d h\{\Gamma(\cdot)/q(\cdot)\}$$

for all $h : D = D[0,1] \to \mathbb{R}$ that are $(D,D)$ measurable and $||/q||$-continuous, or $||/q||$-continuous except at points forming a set of measure zero on $(D,D)$ with respect to the measure generated by the Gaussian $\Gamma(\cdot)$ process, if and only if $I(q,c) < \infty$ for all $c > 0$, where $D$ denotes the $\sigma$-field of subsets of $D$ generated by the finite dimensional subsets of $D$.

**Remark A** For further use the statement of Corollary A will be summarized by writing, as $n \to \infty$,

$$n^{-3/2} \sigma^{-1} U_n(\cdot)/q(\cdot) \Rightarrow \Gamma(\cdot)/q(\cdot) \quad \text{on} \quad (D[0,1], D, ||/q||).$$

For a summary of notions of convergence and weak convergence in general along these lines, we refer to pages 26–28 and Remarks 2 and 3 on page 49 of Shorack and Wellner (1986), and to Sections 3.3 and 3.4 of Csörgő (2002).
Thus Theorem A provides a basic tool for investigating the asymptotic behaviour of many test statistics for testing $H_0$ versus $H_A$ via corresponding functionals of $\Gamma(\cdot)/q(\cdot)$ for appropriate choices of the kernel $h(x, y)$. This, in turn, motivates the establishment of our first result, in which we reduce the moment conditions related to the kernel $h(x, y)$. It reads as follows.

**Theorem 1** Assume $H_0$, $0 < \sigma^2 < \infty$ and $E|h(X_1, X_2)|^{4/3} < \infty$. Then, on an appropriate probability space for $X, X_1, X_2, \ldots$, we can define a sequence of Gaussian processes $\{\Gamma_n(t), 0 \leq t \leq 1\}$ such that (2) holds true, and if $I(q,c) < \infty$ for some $c > 0$, then as $n \to \infty$,

$$\sup_{1/n \leq t \leq (n-1)/n} \left| n^{-3/2} \sigma^{-1} U_n(t) - \Gamma_n(t) \right|/q(t) = o_P(1). \quad (4)$$

In addition to reducing the moment conditions required in Theorem A, the result (4) of Theorem 1 generalizes (3) as well. Namely, as a direct consequence of Theorem 1, we have the following corollary.

**Corollary 1** Assume $H_0$, $0 < \sigma^2 < \infty$ and $E|h(X_1, X_2)|^{4/3} < \infty$. If $q \in Q$, then

(a) we still have the conclusion of Theorem A, i.e., (3) holds true if and only if $I(q, c) < \infty$ for all $c > 0$;

(b) as $n \to \infty$,

$$n^{-3/2} \sigma^{-1} U_n(\cdot) / q(\cdot) \Rightarrow \Gamma(\cdot)/q(\cdot) \text{ on } (D[0,1], D, ||/q||) \quad (5)$$

if and only if $I(q, c) < \infty$ for all $c > 0$;

(c) as $n \to \infty$,

$$n^{-3/2} \sigma^{-1} \sup_{0 < t < 1} |U_n(t)|/q(t) \rightarrow_d \sup_{0 < t < 1} |\Gamma(t)|/q(t) \quad (6)$$

if and only if $I(q, c) < \infty$ for some $c > 0$.

We note in passing that (a) implies (b), just like (3) implies Corollary A (cf. (a) of Lemma 3). However, (a) does not imply (c) (cf. (b) of Lemma 3).

In view of the definition of $Z_k$, and hence also that of $U_n(t)$, when $\theta$ and $\sigma$ are known, large values of the statistic on the left hand sides in (6) for example, indicate a change in the distribution, and hence, based on Corollary 1, rejection of $H_0$ can be quantified accordingly. Otherwise $\theta$ and $\sigma$ need to be estimated. A natural estimate of $\theta$ is

$$\hat{\theta} = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} h(X_i, X_j),$$

and that of $\sigma^2$ is

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{j=1}^{n} \left( g(X_j) - \frac{1}{n} \sum_{j=1}^{n} g(X_j) \right)^2.$$
According to the definition of $g(x)$, $g(X_j)$ still depends on the usually unknown distribution function $F$ of $X$, and hence it then can not be computed explicitly. Since we have that $g(x) + \theta = \int h(x, y) dF(y)$, we can replace $F$ by the empirical distribution function $F_n$ of $X_1, X_2, \ldots, X_n$ under $H_0$. Consequently, we may for example estimate $\sigma^2$ by

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{n-1} \sum_{i=1, i\neq j}^{n} h(X_i, X_j) - \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} h(X_i, X_j) \right)^2.$$ 

We note that this estimate is in fact the jackknife estimator of $Var(\hat{\theta})$. Now we may introduce a studentized U-type process as follows:

$$\hat{U}_n(t) = n^{-3/2}(\hat{\sigma})^{-1} \left( Z_{\lfloor (n+1)t \rfloor} - n^2 t (1 - t) \hat{\theta} \right), \quad 0 \leq t \leq 1.$$ 

This process does not depend on the unknown parameters $\theta$ and $\sigma$ and we now state the following main result of this paper, in which we replace the assumption that $0 < \sigma^2 < \infty$ by assuming only that $g(X_1)$ is in the domain of attraction of the normal law, written $g(X_1) \in$ DAN throughout.

**Theorem 2** Let $q \in Q$. Assume $H_0$, $E|h(X_1, X_2)|^{5/3} < \infty$ and that $g(X_1) \in$ DAN. Then, on an appropriate probability space for $X, X_1, X_2, \ldots$, we can define a sequence of Gaussian processes $\{\Gamma_n(t), 0 \leq t \leq 1\}$ such that $(2)$ holds true and, as $n \to \infty$,

$$\sup_{0 < t < 1} \left| \hat{U}_n(t) - \Gamma_n(t) \right| / q(t) = o_P(1), \quad (7)$$

if and only if $I(q, c) < \infty$ for all $c > 0$. Consequently, as $n \to \infty$,

$$\hat{U}_n(\cdot) / q(\cdot) \Rightarrow \Gamma(\cdot) / q(\cdot), \quad on \ (D[0, 1], \mathcal{D}, ||q||) \quad (8)$$

if and only if $I(q, c) < \infty$ for all $c > 0$. Furthermore, as $n \to \infty$, we also have

$$\sup_{0 < t < 1} |\hat{U}_n(t)| / q(t) \to_d \sup_{0 < t < 1} |\Gamma(t)| / q(t) \quad (9)$$

if and only if $I(q, c) < \infty$ for some $c > 0$.

**Remark 2** It is interesting and also of interest to note that the class of the weight functions in $(9)$ is bigger than that in $(8)$ [also compare (6) with (5)]. Such a phenomenon was first noticed and proved for weighted empirical and quantile processes by Csörgő, Csörgő, Horváth and Mason [CsCsHM] (1986) and then by Csörgő and Horváth (1988b) for partial sums on assuming $E|X|^v < \infty$ for some $v > 2$. For more details along these lines, we refer to Szyszkowicz (1991, 1996, 1997), and to Csörgő, Norvaiša and Szyszkowicz (1999).

**Remark 3** As we noted already in Remark 1, the condition that $0 < \sigma^2 = Eg^2(X_1) < \infty$ is the so-called non-degenerate case when studying $U-$ statistics. In Theorem 1 it is a
necessary condition, while assuming \( E|h(X_1, X_2)|^{4/3} < \infty \) is close to being necessary, on account of the central limit theorem for \( U \)-statistics (see Borovskikh (2002), for example).

Theorem 2 puts a totally new countenance on the classical theory of weak convergence for standardized \( U \)-type process as in Theorem 1 [cf. also Theorem A, Section 2.2.4 of Csörgő and Horváth (1997), Gombay and Horváth (1995, 2002)] in that here we derive results assuming only \( g(X_1) \in \text{DAN} \) and, consequently, we may have \( \sigma^2 = E g^2(X_1) = \infty \).

The price we pay for this is the somewhat higher moment condition \( E|h(X_1, X_2)|^{5/3} < \infty \) than that of the corresponding one with exponent \( 4/3 \) in Theorem 1. What is crucial in Theorem 2 in this regard is that the existence of the second moment of \( h(X_1, X_2) \) is not assumed, for assuming the latter would exclude the possibility of having \( \sigma^2 = E g^2(X_1) = \infty \) (cf. Remark 1).

This paper is organized as follows. In the next section we provide the proofs of main results. Then, in Section 3, we investigate the asymptotic behaviour of the \( U \)-type process \( U_n(\cdot) \) when it is based on kernels that are antisymmetric, i.e., \( h(x, y) \) in such that \( h(x, y) = -h(y, x) \). Throughout the paper \( A, A_1, \ldots \) will denote constants which may be different in each appearance.

## 2 Proofs of main results

We need some preliminaries to proving our main theorems. The following lemma constitutes the key step. We note in passing that the three basic relations (11), (12), (13) of Lemma 1 are of interest on their own in studying \( U \)-statistics type processes, independently of their kernel function \( h(\cdot, \cdot) \) being symmetric, or antisymmetric.

**Lemma 1** Let \( \psi(x, y) \) be a measurable real valued symmetric function for which we have

\[
\int \psi(x, y) dF(y) = 0 \tag{10}
\]

and \( E|\psi(X_1, X_2)|^{4/3} < \infty \). Then, as \( n \to \infty \),

\[
\frac{1}{n} \max_{1 \leq k \leq n-1} k^{-1/2} \left| \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i, X_j) \right| = O_P(1), \tag{11}
\]

\[
\frac{1}{n} \max_{1 \leq k \leq n-1} (n-k)^{-1/2} \left| \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i, X_j) \right| = O_P(1), \tag{12}
\]

\[
\frac{1}{n^{3/2}} \max_{1 \leq k \leq n-1} \left| \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i, X_j) \right| = o_P(1). \tag{13}
\]

**Proof.** We only prove (11) and (13). By virtue of the symmetry of \( \psi(x, y) \) and the i.i.d. properties of \( X_i \), the proof of (12) is similar to that of (11). We omit the latter details.
In order to prove (11), write

$$
\psi^*(x, y) = \psi(x, y)I_{\{|\psi(x, y)| \leq \alpha/2\}} - \int \int \psi(u, v)I_{\{|\psi(u, v)| \leq \alpha/2\}}dF(u)dF(v),
$$

$$
g^*(x) = \int \psi^*(x, y)dF(y) \quad \text{and} \quad \psi^{**}(x, y) = \psi^*(x, y) - g^*(x) - g^*(y).
$$

It is readily seen that

$$
E[\psi^*(X_i, X_j)] = 0 \quad \text{and} \quad E[\psi^{**}(X_i, X_j) | X_i] = 0, \quad \text{for all } i \neq j. \quad (14)
$$

Having $E[\psi(X_i, X_j) | X_i] = 0$ by (10), we also have

$$
g^*(X_i) = E[\psi^*(X_i, X_j) | X_i]
$$

$$
= E[\psi(X_i, X_j)I_{\{|\psi(X_i, X_j)| > \alpha/2\}} | X_i] - E[\psi(X_1, X_2)I_{\{|\psi| \geq \alpha/2\}}]. \quad (15)
$$

We now turn to the proof of (11). We have

$$
\frac{1}{n} \max_{1 \leq k \leq n-1} k^{-1/2} \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i, X_j) \leq I_1(n) + I_2(n) + I_3(n), \quad (16)
$$

where

$$
I_1(n) = \frac{1}{n} \max_{1 \leq k \leq n-1} k^{-1/2} \sum_{i=1}^{k} \sum_{j=1, j \neq i}^{k} \psi^{**}(X_i, X_j),
$$

$$
I_2(n) = \frac{1}{n} \max_{1 \leq k \leq n-1} k^{-1/2} \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi^{**}(X_i, X_j),
$$

$$
I_3(n) = \frac{1}{n} \max_{1 \leq k \leq n-1} k^{-1/2} \sum_{i=1}^{k} \sum_{j=k+1}^{n} (\psi(X_i, X_j) - \psi^*(X_i, X_j) + g^*(X_i) + g^*(X_j)).
$$

We next prove $I_t(n) = O_P(1)$ for $t = 1, 2, 3$ and then (11) follows accordingly.

First consider $t = 1$. Write $Y_t = \sum_{j=1}^{t-1} \psi^{**}(X_i, X_j)$. Note that $E(Y_i Y_k) = 0$ for all $i \neq k$ by (14). It is readily seen that

$$
E \left[ \sum_{i=2}^{\infty} i^{-3/2} Y_i \right]^2 = \sum_{i=2}^{\infty} i^{-3} EY_i^2 \leq A \sum_{i=2}^{\infty} i^{-2} E[\psi^2(X_1, X_2)I_{|\psi| \leq \alpha/2}] \leq A \sum_{k=1}^{\infty} E[\psi^2(X_1, X_2)I_{(k-1)\alpha/2 < |\psi| \leq k\alpha/2}] \sum_{i=k}^{\infty} i^{-2} \leq A E|\psi(X_1, X_2)|^{4/3} < \infty. \quad (17)
$$

This, together with the Kronecker lemma, implies that $k^{-3/2} \sum_{i=1}^{k} Y_i \to 0$, a.s., and hence $I_1(n) = O_P(1)$, since $I_1(n) \leq 2 \max_{1 \leq k \leq n-1} k^{-3/2} \sum_{i=2}^{k} Y_i$. 

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Indeed, the result (19) implies that

\[ I_2(n) \leq \frac{1}{n^{1/2}} \max_{1 \leq k \leq n-1} k^{-1/2} \sum_{i=1}^{k} Z_{i,n} \leq \frac{2}{n^{1/2}} \max_{1 \leq k \leq n-1} \left| \sum_{i=1}^{k} \frac{1}{i} Z_{i,n} \right|. \]

Equation (18)

Therefore, it only needs to be shown that, uniformly in \( n \geq 1 \),

\[ \frac{1}{n^{1/2}} E \left[ \sum_{i=1}^{\infty} \frac{1}{i} Z_{i,n} \right] \leq A < \infty. \]

Indeed, the result (19) implies that \( \frac{1}{n^{1/2}} \sum_{i=1}^{\infty} \frac{1}{i} Z_{i,n} \leq A < \infty \) a.s., and \( \frac{1}{n^{1/2}} \sum_{i=k}^{\infty} \frac{1}{i} Z_{i,n} \to 0 \), a.s., as \( k \to \infty \), uniformly in \( n \geq 1 \). This, together with (18), yields

\[ I_{2n} \leq \frac{2}{n^{1/2}} \max_{1 \leq k \leq N} \left| \sum_{i=1}^{k} \frac{1}{i} Z_{i,n} \right| \leq 2A + \max_{k \geq 1} \frac{2}{n^{1/2}} \sum_{i=k}^{\infty} \frac{1}{i} Z_{i,n} = O_P(1). \]

The proof of (19) follows from a similar argument as in the proof of (17). In fact, for all \( n \geq 1 \), we have

\[
\frac{1}{n^{1/2}} E \left[ \sum_{i=1}^{\infty} \frac{1}{i} Z_{i,n} \right] \leq \frac{1}{n^{1/2}} \left[ E \left( \sum_{i=1}^{\infty} \frac{1}{i} Z_{i,n} \right)^2 \right]^{1/2} \\
= \frac{1}{n^{1/2}} \left[ \sum_{i=1}^{\infty} \left( \frac{1}{i^2} E \left( Z_{i,n} \right)^2 \right)^{1/2} \right] \\
\leq A \left[ \sum_{i=1}^{\infty} \frac{1}{i^2} E \psi^2(X_1, X_2) I_{(|\psi| \leq \varepsilon/2)} \right]^{1/2} \\
< A [E|\psi(X_1, X_2)|^{4/3}]^{1/2} < \infty,
\]

which yields (19).

Finally we prove \( I_3(n) = O_P(1) \). Recalling (15) and \( E\psi(X_1, X_2) = 0 \), we have

\[
\Lambda_{i,j} := |\psi(X_i, X_j) - \psi^*(X_i, X_j) + g^*(X_i) + g^*(X_j)| \\
\leq |\psi(X_i, X_j)| I_{(|\psi| \geq \varepsilon/2)} + E[|\psi(X_i, X_j)| I_{(|\psi| > \varepsilon/2)} | X_i] \\
+ E[|\psi(X_i, X_j)| I_{(|\psi| > \varepsilon/2)} | X_j] + E[|\psi(X_1, X_2)| I_{(|\psi| > \varepsilon/2)}].
\]

This implies that \( E(\Lambda_{i,j}) \leq 4E[|\psi(X_1, X_2)| I_{(|\psi| > \varepsilon/2)}], \) and hence

\[
E I_3(n) \leq \frac{1}{n} E \left[ \max_{1 \leq k \leq n-1} k^{-1/2} \sum_{i=1}^{k} \sum_{j=1}^{n} \Lambda_{i,j} \right] \\
\leq \frac{1}{n} \sum_{i=1}^{\infty} i^{-1/2} \sum_{j=1}^{n} E(\Lambda_{i,j})
\]

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uniformly for all $n \geq 1$. By Markov’s inequality, we obtain $I_3(n) = O_P(1)$. The proof of (11) is now complete.

The proof of (13) is similar to that of (11), but we have to use a different truncation. In the following, we let

$$\psi^*(x, y) = \psi(x, y)I_{|\psi(x, y)| \leq n^{3/2}} - \int \int \psi(u, v)I_{|\psi(u, v)| \leq n^{3/2}}dF(u)dF(v),$$

$$g^*(x) = \int \psi^*(x, y)dF(y) \text{ and } \psi^{**}(x, y) = \psi^*(x, y) - g^*(x) - g^*(y).$$

It follows easily that

$$\frac{1}{n^{3/2}} \max_{1 \leq k \leq n-1} \left| \sum_{i=1}^k \sum_{j=k+1}^n \psi(X_i, X_j) \right| \leq \frac{1}{2} \left[ I_0^*(n) + I_1^*(n) + I_2^*(n) + I_3^*(n) \right], \quad (21)$$

where

$$I_0^*(n) = \frac{1}{n^{3/2}} \sum_{i=1}^n \sum_{j=1}^n \psi^{**}(X_i, X_j),$$

$$I_1^*(n) = \frac{1}{n^{3/2}} \max_{1 \leq k \leq n-1} \left| \sum_{i=1}^k \sum_{j=1}^n \psi^{**}(X_i, X_j) \right|,$$

$$I_2^*(n) = \frac{1}{n^{3/2}} \max_{1 \leq k \leq n-1} \left| \sum_{i=k+1}^n \sum_{j=k+1}^n \psi^{**}(X_i, X_j) \right|,$$

$$I_3^*(n) = \frac{1}{n^{3/2}} \max_{1 \leq k \leq n-1} \left| \sum_{i=1}^k \sum_{j=k+1}^n (\psi(X_i, X_j) - \psi^{**}(X_i, X_j) + g^*(X_i) + g^*(X_j)) \right|.$$

It is readily seen that

$$E[I_0^*(n)]^2 \leq A n^{-1} E\psi^2(X_1, X_2)I_{|\psi| \leq n^{3/2}} \leq A \left[ \epsilon^{-2} n^{-1/3} E|\psi(X_1, X_2)|^{4/3} + E|\psi(X_1, X_2)|^{4/3} I_{|\psi| \geq n} \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This yields $I_0^*(n) = O_P(1)$. Noting that $\{\sum_{j=2}^k Y_j, \mathcal{F}_k, 2 \leq k \leq n\}$ is a martingale, where $Y_j = \sum_{i=1}^{j-1} \psi^{**}(X_i, X_j)$ and $\mathcal{F}_k = \sigma\{X_1, \ldots, X_k\}$, it follows from the well-known Maximum inequality for martingales that, for any $\epsilon > 0$,

$$P(I_1^*(n) \geq \epsilon) \leq 4 \epsilon^{-2} n^{-3} E \max_{1 \leq k \leq n-1} \left| \sum_{j=2}^k Y_j \right|^2 \leq A \epsilon^{-2} n^{-3} \sum_{j=2}^n EY_j^2 \leq A \epsilon^{-2} n^{-1} E\psi^2(X_1, X_2)I_{|\psi| \leq n^{3/2}} \leq A \left[ \epsilon^{-2} n^{-1/3} E|\psi(X_1, X_2)|^{4/3} + E|\psi(X_1, X_2)|^{4/3} I_{|\psi| \geq n} \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$
This yields $I_1^*(n) = o_P(1)$. By a similar argument as in the proof for $I_1^*(n) = o_P(1)$, we have $I_2^*(n) = o_P(1)$. As for $I_3^*(n)$, by using a similar argument as in the proof of (20), we obtain

$$E|I_3^*(n)| \leq \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{j=1}^{n} E\left| \psi(X_i, X_j) - \psi^*(X_i, X_j) + g^*(X_i) + g^*(X_j) \right|$$

$$\leq 4 n^{1/2} E[|\psi(X_1, X_2)| I_{|\psi| \geq n^{1/2}}]$$

$$\leq 4 E[|\psi(X_1, X_2)|]^{1/3} I_{|\psi| \geq n^{1/2}} \to 0,$$

as $n \to \infty$, which implies that $I_3^*(n) = o_P(1)$. Taking all the respective estimates for $I_i^*(n), i = 1, 2, 3$ into (21), we obtain the required (13). The proof of Lemma 1 is now complete.

The next two lemmas are due to CsCsHM (1986) [cf. Lemma A.5.1 and Theorem A.5.1 respectively in Csörgő and Horváth (1997)]. Proofs of Lemmas 2 and 3 can also be found in Section 4.1 of Csörgő and Horváth (1993).

**Lemma 2** Let $q(t) \in Q$. If $I(q, c) < \infty$ for some $c > 0$, then

$$\lim_{t \uparrow 0} t^{1/2} / q(t) = 0 \quad \text{and} \quad \lim_{t \uparrow 1} (1 - t)^{1/2} / q(t) = 0.$$ 

**Lemma 3** Let $\{W(t), 0 \leq t < \infty\}$ be a standard Wiener process and $q(t) \in Q$. Then,

(a) $I(q, c) < \infty$ for all $c > 0$ if and only if

$$\limsup_{t \downarrow 0} |W(t)| / q(t) = 0, \ a.s. \quad \text{and} \quad \limsup_{t \uparrow 1} |W(1) - W(t)| / q(t) = 0, \ a.s.$$

(b) $I(q, c) < \infty$ for some $c > 0$ if and only if

$$\limsup_{t \downarrow 0} |W(t)| / q(t) < \infty, \ a.s. \quad \text{and} \quad \limsup_{t \uparrow 1} |W(1) - W(t)| / q(t) < \infty, \ a.s.$$

We are now ready to prove our main theorems.

**Proof of Theorem 1.** Together with the notation as in Section 1, we write $\psi(x, y) = h(x, y) - \theta - g(x) - g(y)$ and $T_n(t) = W_{\lfloor (n+1)q \rfloor}, 0 \leq t \leq 1$, where

$$W_k = (n - k) \sum_{j=1}^{k} g(X_j) + k \sum_{j=k+1}^{n} g(X_j).$$

Noting that $g(X_j)$ are i.i.d. random variables with $Eg(X_1) = 0$ and $\sigma^2 = Eg^2(X_1) < \infty$, along the lines of the proof of (2.1.45) in Csörgő and Horváth (1997), on an appropriate probability space for $X, X_1, X_2, \cdots$ we can define a sequence of Gaussian processes $\{\Gamma_n(t), 0 \leq t \leq 1\}$ such that, for each $n \geq 1$,

$$\{\Gamma_n(t), 0 \leq t \leq 1\} =_{d} \{\Gamma(t), 0 \leq t \leq 1\},$$

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and if \( q \in Q \) and \( I(q, c) < \infty \) for some \( c > 0 \), then, as \( n \to \infty \),
\[
\sup_{1/n \leq t \leq (n-1)/n} \left| n^{-3/2} \sigma^{-1} T_n(t) - \Gamma_n(t) \right| / q(t) = o_P(1). \tag{22}
\]

By virtue of (22), Theorem 1 will follow if we prove
\[
J_n := \sup_{1/n \leq t \leq (n-1)/n} \left| n^{-3/2} U_n(t) - n^{-3/2} T_n(t) \right| / q(t) = o_P(1). \tag{23}
\]

In order to prove (23), write \( V_n(t) = W^*_n(t) \), where \( W^*_n = \sum_{j=1}^{k} \sum_{j=k+1}^{n} \psi(X_i, X_j) \).

Note that \( E(\psi(X_1, X_2) \mid X_1) = E(\psi(X_1, X_2) \mid X_2) = 0 \) and
\[
E|\psi(X_1, X_2)|^{4/3} \leq A E|h(X_1, X_2)|^{4/3} < \infty.
\]

It follows from (13) that
\[
J_n^{(1)} := \sup_{0 < t \leq \delta} \left| n^{-3/2} V_n(t) \right| / q(t)
\leq \frac{1}{n^{3/2}} \max_{1 \leq k \leq n-1} \left| \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i, X_j) \right| \sup_{\delta \leq t \leq 1-\delta} t^{-1/2} / q(t) = o_P(1),
\]

for any \( \delta \in (0, 1) \) and \( q \in Q \). Let \( \delta > 0 \) be so small that \( q(t) \) is already nondecreasing on \( (0, \delta) \) and nonincreasing on \( (1-\delta, 1) \) and let \( n \) be so large such that \( 1/n \leq \delta \). It follows from (11) and Lemma 2 that
\[
J_n^{(2)} := \sup_{0 < t \leq \delta} \left| n^{-3/2} V_n(t) \right| / q(t)
\leq \frac{1}{n} \max_{1 \leq k \leq n-1} (n-k)^{-1/2} \left| \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i, X_j) \right| \sup_{0 < t \leq \delta} t^{1/2} / q(t) = o_P(1),
\]

when \( n \to \infty \) and then \( \delta \to 0 \). Similarly, we have also
\[
J_n^{(3)} := \sup_{1-\delta \leq t < 1} \left| n^{-3/2} V_n(t) \right| / q(t)
\leq \frac{1}{n} \max_{1 \leq k \leq n-1} (n-k)^{-1/2} \left| \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i, X_j) \right| \sup_{1-\delta \leq t < 1} (1-t)^{1/2} / q(t)
= o_P(1),
\]

when \( n \to \infty \) and then \( \delta \to 0 \). By virtue of these estimates, it is readily seen that
\[
J_n \leq J_n^{(1)} + J_n^{(2)} + J_n^{(3)} + A n^{-1/2} \sup_{1/n \leq t \leq (n-1)/n} 1/q(t) = o_P(1), \tag{24}
\]

which yields (23). The proof of Theorem 1 is now complete.
**Proof of Corollary 1.** Having Theorem 1, Lemmas 2-3 and the result (23), the proof of Corollary 1 is the same as that given in the proof of Theorem 2.4.2 in Csörgő and Horváth (1997), and hence the details are omitted.

**Proof of Theorem 2.** We first prove (7). It is readily seen that

\[
\hat{U}_n(t) = \frac{n^{-3/2}(\hat{\sigma})^{-1}}{n} \left\{ \sum_{j=1}^{n} g^2(X_j) \right\}^{1/2} n^{-1} \left\{ \sum_{j=1}^{n} g^2(X_j) \right\}^{1/2} U_n(t) + t(1-t)n^{1/2}(\hat{\sigma})^{-1}(\hat{\theta} - \theta).
\]  

(25)

Furthermore, \(U_n(t) = T_n(t) + V_n(t)\), where \(T_n(t)\) and \(V_n(t)\) are defined as in the proof of Theorem 1. Recalling that \(g(X_1)\) is in the domain of attraction of the normal law, as in the proof of Theorem 5.2 of Csörgő, Szyszkowicz and Wang [CsSzW] (2004) with minor modifications, we have that on an appropriate probability space for \(X, X_1, X_2, \ldots\), we can define a sequence of Gaussian processes \(\{\Gamma_n(t), 0 \leq t \leq 1\}\) such that (2) holds true, and as \(n \to \infty\),

\[
\sup_{0 < t < 1} \left| n^{-1} \left\{ \sum_{j=1}^{n} g^2(X_j) \right\}^{1/2} T_n(t) - \Gamma_n(t) \right|/q(t) = o_P(1),
\]

if and only if \(I(q,c) < \infty\) for all \(c > 0\). Therefore, to prove (7), it suffices to show that

\[
\frac{1}{n} \left\{ \sum_{j=1}^{n} g^2(X_j) \right\}^{1/2} \sup_{0 < t < 1} |V_n(t)|/q(t) = o_P(1),
\]  

(26)

\[
\left\{ n^{-1} \sum_{j=1}^{n} g^2(X_j) \right\}^{-1} \hat{\sigma}^2 - 1 = o_P(1),
\]  

(27)

and

\[
n^{1/2}(\hat{\sigma})^{-1}(\hat{\theta} - \theta) = o_P(1).
\]  

(28)

The proof of (26) is simple and in fact (26) holds true if \(q(x)\) satisfies \(I(q,c) < \infty\) for some \(c > 0\). Indeed, since \(g(X_1)\) is in the domain of attraction of the normal law, we have \(\frac{1}{b_n} \sum_{j=1}^{n} g^2(X_j) \to_P 1\), where \(b_n = n l(n)\) with that \(l(n) = Eg^2(X_1)\) if \(Eg^2(X_1) < \infty\) or \(l(n) \to \infty\) if \(Eg^2(X_1) = \infty\). On the other hand, as in the proof of (23), \(n^{-3/2} \sup_{0 < t < 1} |V_n(t)|/q(t) = o_P(1)\) even when \(q(x)\) satisfies \(I(q,c) < \infty\) for some \(c > 0\), and hence (26) follows immediately from these facts.

We next prove (27). The claim (28) follows by using (27), and hence the details are omitted. Without loss of generality, we assume \(\theta = 0\). We may rewrite \(\hat{\sigma}^2\) as

\[
\hat{\sigma}^2 = \frac{1}{n(n-1)} \sum_{i \neq j \neq k} h(X_i, X_j) h(X_i, X_k) + \frac{1}{n(n-1)} \sum_{i \neq j} h^2(X_i, X_j) - \hat{\theta}^2 \\
:= W_{n1} + W_{n2} - \hat{\theta}^2.
\]
Recalling $E|h(X_1, X_2)|^{5/3} < \infty$, it follows from a Marcinkiewicz type strong law for $U$-statistics that $W_{n_2} - \theta^2 \to 0$, a.s. [see Gine and Zinn (1992), for example]. Therefore (27) will follow if we prove

$$\left\{ n^{-1} \sum_{j=1}^{n} g_j^2(X_j) \right\}^{-1} W_{n_1} - 1 = o_P(1). \quad (29)$$

Write, for $i \neq j \neq k$,

$$h_{ij}^{(1)} = h(X_i, X_j)I_{|h| \leq n^{6/5}}, \quad g^{(1)}(X_i) = E(h_{ij}^{(1)}|X_i),$$

$$\psi_{ijk} = h_{ij}^{(1)} h_{ik}^{(1)} - E h_{ij}^{(1)} h_{ik}^{(1)},$$

$$\varphi_i^{(1)} = E(\psi_{ijk}|X_i), \quad \varphi_j^{(2)} = E(\psi_{ijk}|X_j), \quad \varphi_k^{(3)} = E(\psi_{ijk}|X_k).$$

Noting that $E\{h_{ij}^{(1)} h_{ik}^{(1)}|X_i\} = \{g^{(1)}(X_i)\}^2$, it is readily seen that $\varphi_i^{(1)} = \{g^{(1)}(X_i)\}^2 - E[h_{ij}^{(1)} h_{ik}^{(1)}]$, and

$$\sum_{i \neq j \neq k} h_{ij}^{(1)} h_{ik}^{(1)} = \sum_{i \neq j \neq k} \psi_{ijk} + \sum_{i \neq j \neq k} E[h_{ij}^{(1)} h_{ik}^{(1)}]$$

$$= \sum_{i \neq j \neq k} \{g^{(1)}(X_i)\}^2 + \sum_{i \neq j \neq k} \{\varphi_j^{(2)} + \varphi_k^{(3)}\}$$

$$+ \sum_{i \neq j \neq k} (\psi_{ijk} - \varphi_i^{(1)} - \varphi_j^{(2)} - \varphi_k^{(3)})$$

$$:= V_{n_1} + V_{n_2} + V_{n_3}.$$

In the next paragraph, we will show that

$$\left\{ n^{-1} \sum_{j=1}^{n} g_j^2(X_j) \right\}^{-1} \left( n^{-3} V_{n_1} \right) - 1 = o_P(1), \quad (30)$$

$$n^{-3} (V_{n_2} + V_{n_3}) = o_P(1). \quad (31)$$

It follows from (30) and (31) that

$$\left\{ n^{-1} \sum_{j=1}^{n} g_j^2(X_j) \right\}^{-1} n^{-3} \sum_{i \neq j \neq k} h_{ij}^{(1)} h_{ik}^{(1)} - 1 = o_P(1), \quad (32)$$

and then (29) follows from (32) and

$$P \left( \sum_{i \neq j \neq k} h_{ij} h_{ik} \neq \sum_{i \neq j \neq k} h_{ij}^{(1)} h_{ik}^{(1)} \right) \leq n^2 P( |h(X_1, X_2)| \geq n^{6/5} )$$

$$\leq E|h(X_1, X_2)|^{5/3} I_{|h| \geq n^{6/5}} \to 0.$$
We are to prove (30) and (31) now. Consider (30) first. By noting that $g^{(1)}(X_1) = g(X_1) - g^*(X_j)$, where $g^*(X_j) = E\{h(X_1, X_2)I_{|h| \geq n^{6/5}}|X_1\}$, we have

$$\begin{align*}
\left| \sum_{j=1}^{n} \left[ (g^{(1)}(X_j))^2 - g^2(X_j) \right] \right| & \leq \sum_{j=1}^{n} \left[ 2|g(X_j)||g^*(X_j)| + |g^*(X_j)|^2 \right] \\
& \leq 2 \left[ \sum_{j=1}^{n} g^2(X_j) \right]^{1/2} \left[ \sum_{j=1}^{n} |g^*(X_j)|^2 \right]^{1/2} + \sum_{j=1}^{n} |g^*(X_j)|^2.
\end{align*}$$

Now, since $g(X_1)$ is in the domain of attraction of the normal law [which implies that $\frac{1}{n} \sum_{j=1}^{n} g^2(X_j) \rightarrow P C > 0$, where $C$ may be $\infty$], simple calculations show that (30) will follow if we prove

$$\frac{1}{n} \sum_{j=1}^{n} (g^*(X_j))^2 = o_P(1). \quad (33)$$

In fact, for any $\epsilon > 0$, we have

$$P\left( \sum_{j=1}^{n} (g^*(X_j))^2 \geq \epsilon n \right) \leq \epsilon^{-1/2} n^{-1/2} \sum_{j=1}^{n} E|g^*(X_j)|$$

$$\leq \epsilon^{-1/2} n^{-1/2} E|h(X_1, X_2)|I_{|h| \geq n^{6/5}}$$

$$\leq \epsilon^{-1/2} E|h(X_1, X_2)|^{5/3} I_{|h| \geq n^{6/5}} \rightarrow 0,$$

as $n \rightarrow \infty$. This implies (33) and hence completes the proof of (30).

We next prove (31). By noting that $n^{-3}V_{n3}$ is a degenerate $U$-statistic of order 3, it follows from moment inequality for degenerate $U$-statistics (see, Borovskikh (1996), for example) that, for any $\epsilon > 0$,

$$P(|V_{n3}| \geq \epsilon n^3) \leq \epsilon^{-5/3} n^{-5} E|V_{n3}|^{5/3}$$

$$\leq A \epsilon^{-5/3} n^{-2} E\left[ \psi_{123} - \varphi^{(1)}_1 - \varphi^{(2)}_2 - \varphi^{(3)}_3 \right]^{5/3}$$

$$\leq A \epsilon^{-5/3} n^{-2} E|h(X_1, X_2)|^{10/3} I_{|h| \leq n^{6/5}}$$

$$\leq A \epsilon^{-5/3} \left[ n^{-1/3} + E|h(X_1, X_2)|^{5/3} I_{|h| \geq n^{1/2}} \right] \rightarrow 0, \quad (34)$$

as $n \rightarrow \infty$. On the other hand, by noting that

$$E\left\{ E\left[ h_{12}^{(1)} h_{13}^{(1)} X_2 \right] \right\}^2 = E\left\{ h_{12}^{(1)} h_{13}^{(1)} E\left[ h_{42}^{(1)} h_{45}^{(1)} X_2 \right] \right\}$$

$$= E\left[ h_{12}^{(1)} h_{13}^{(1)} h_{42}^{(1)} h_{45}^{(1)} \right]$$

$$\leq \left[ Eh^2(X_1, X_2)I_{|h| \leq n^{6/5}} \right]^2 \leq n^{4/5} \left\{ E|h(X_1, X_2)|^{5/3} \right\}^2,$$

it is readily seen that, for any $\epsilon > 0$,

$$P(|V_{n3}| \geq \epsilon n^3) \leq \epsilon^{-2} E\left( n^{-3}V_{n2} \right)^2$$

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\[ \leq A \epsilon^{-2} n^{-1} E \left( \varphi_1^{(2)} + \varphi_1^{(3)} \right)^2 \]
\[ \leq A \epsilon^{-2} n^{-1} \left[ E \left\{ E \left( h_{12}^{(1)} h_{13}^{(1)} | X_2 \right) \right\}^2 + \left( E h_{12}^{(1)} \right)^2 \right] \]
\[ \leq A \epsilon^{-2} n^{-1/5} \left\{ E \left| h(X_1, X_2) \right|^{5/3} \right\}^2 \to 0, \] (35)

as \( n \to \infty \). By virtue of (34) and (35), we obtain (31). The proof of (7) is now complete.

The result (8) is a direct consequence of (7). As for (9), by virtue of (25)-(28) (recalling that (26) still holds true for \( q(x) \) satisfying \( I(q, c) < \infty \) for some \( c > 0 \), as explained in its proof), it suffices to show that

\[ \sup_{0 < t < 1} \left| n^{-1} \left\{ \sum_{j=1}^{n} g^2(X_j) \right\}^{-1/2} T_n(t) \right| \to_d \sup_{0 < t < 1} |\Gamma(t)|/q(t) \] (36)

if and only if \( I(q, c) < \infty \) for some \( c > 0 \), where \( T_n(t) = W_{[(n+1)t], 0 \leq t \leq 1} \), with

\[ W_k = (n - k) \sum_{j=1}^{k} g(X_j) + k \sum_{j=k+1}^{n} g(X_j). \]

This follows from the same arguments as in the proof of Corollary 5.2 in CsSzW (2004), and hence the details are omitted. This also completes the proof of Theorem 2.

### 3 Antisymmetric kernel

In this section we consider the asymptotics of \( U \)-type processes with antisymmetric kernel \( h(x, y) \), i.e., \( h(x, y) = -h(y, x) \). This kind of kernels can not be symmetrized, but they are especially useful to check the equality of distributions for different groups of random variables since \( \theta = Eh(X_1, X_2) = 0 \) whenever \( X_1 =_d X_2 \), if \( E|h(X_1, X_2)| < \infty \). Consequently, for antisymmetric kernels, \( U_n(t) = Z_{[(n+1)t], 0 \leq t \leq 1} \) under \( H_0 \). An example is given in Pettitt (1979), who used functions of the Mann-Whitney type statistics

\[ (12)^{1/2} n^{-3/2} \sum_{1 \leq i \leq n} \sum_{n < j \leq n} \text{sign} \left( X_i - X_j \right) \]

to detect possible changes in distribution. Another important example is given by taking \( H(x, y) = x - y \) for studying the probable error of a change in a mean. We will say more about that in Remark 5.

For the anti-symmetric kernel \( h(x, y) \), by letting \( g(t) = Eh(X_1, t) \), i.e., keeping our earlier notation with \( \theta = 0 \), we may write

\[ Z_k = \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i, X_j) + n \left[ \sum_{i=1}^{k} g(X_i) - \frac{k}{n} \sum_{i=1}^{n} g(X_i) \right], \]

where \( \psi(x, y) = h(x, y) + g(x) - g(y) \) with

\[ E [\psi(X_1, X_2) | X_1] = E [\psi(X_1, X_2) | X_2] = 0. \]
Since Lemma 1 does not depend on the symmetry of the kernel, similarly to the proofs of Theorems 1 and 2, we have the following results for \( U \)-type processes with antisymmetric kernel \( h(x, y) \), which improve and generalize the similar earlier results of Csörgő and Horváth (1988a, b), Szyszkowicz (1991, 1992) and those given in Section 2.4 of Csörgő and Horváth (1997) along these lines. It is interesting to note that the Gaussian limit process that is shared by Theorems 1 and 2 and that shared by Theorems 3 and 4 are different, although they are of equal variance. For further related results, we refer to Janson and Wichura (1983), and Gombay (2000a, b, 2001, 2004).

We continue to use the notations introduced in Section 1, but \( U_n(t) \) and \( \hat{U}_n(t) \) are now defined in terms of antisymmetric kernel \( h(x, y) = -h(y, x) \). Consequently, under \( H_0 \), \( \theta \) and \( \hat{\theta} \) are both zero now.

**Theorem 3** Let \( q \in Q \). Assume \( H_0 \), \( 0 < \sigma^2 < \infty \) and \( E|h(X_1, X_2)|^{4/3} < \infty \). Then, on an appropriate probability space for \( X, X_1, X_2, \ldots \), we can define a sequence of Brownian bridges \( \{B_n(t), 0 \leq t \leq 1\} \) such that if \( I(q, c) < \infty \) for some \( c > 0 \), then as \( n \to \infty \),

\[
\sup_{1/n \leq t \leq (n-1)/n} \left| n^{-3/2} \sigma^{-1} U_n(t) - B_n(t) \right| / q(t) = o_P(1). \tag{37}
\]

Consequently,

(a) as \( n \to \infty \),

\[
\sup_{0 < t < 1} \left| n^{-3/2} \sigma^{-1} U_n(t) - B_n(t) \right| / q(t) = o_P(1) \tag{38}
\]

if and only if \( I(q, c) < \infty \) for all \( c > 0 \);

(b) as \( n \to \infty \),

\[
n^{-3/2} \sigma^{-1} U_n(\cdot) / q(\cdot) \Rightarrow B(\cdot) / q(\cdot) \text{ on } (D[0,1], \mathcal{D}, ||/q||) \tag{39}
\]

if and only if \( I(q, c) < \infty \) for all \( c > 0 \);

(c) as \( n \to \infty \),

\[
n^{-3/2} \sigma^{-1} \sup_{0 < t < 1} |U_n(t)| / q(t) \to_d \sup_{0 < t < 1} |B(t)| / q(t) \tag{40}
\]

if and only if \( I(q, c) < \infty \) for some \( c > 0 \), where, in (b) and (c), \( \{B(t), 0 \leq t \leq 1\} \) is a Brownian bridge.

Theorem 3 is to be compared to Szyszkowicz (1991, Theorem 2.1) [cf. Theorem 2.4.1 in Csörgő and Horváth (1997)].

**Theorem 4** Let \( q \in Q \). Assume \( H_0 \), \( E|h(X_1, X_2)|^{5/3} < \infty \) and that \( g(X_1) \in DAN \). Then, on an appropriate probability space for \( X, X_1, X_2, \ldots \), we can define a sequence of Brownian bridges \( \{B_n(t), 0 \leq t \leq 1\} \) such that, as \( n \to \infty \),

\[
\sup_{0 < t < 1} |\hat{U}_n(t) - B_n(t)| / q(t) = o_P(1), \tag{41}
\]
if and only if \( I(q, c) < \infty \) for all \( c > 0 \). Consequently, as \( n \to \infty \),

\[
\hat{U}_n(\cdot)/q(\cdot) \Rightarrow B(\cdot)/q(\cdot), \quad \text{on } (D[0, 1], \mathcal{D}, ||/||)
\]  

(42)

if and only if \( I(q, c) < \infty \) for all \( c > 0 \), where \( \{B(t), 0 \leq t \leq 1\} \) is a Brownian bridge. Furthermore, as \( n \to \infty \), we also have

\[
\sup_{0 < t < 1} |\hat{U}_n(t)|/q(t) \to_d \sup_{0 < t < 1} |B(t)|/q(t)
\]  

(43)

if and only if \( I(q, c) < \infty \) for some \( c > 0 \).

**Remark 4** As compared to Theorem 3, where it is assumed that \( 0 < \sigma^2 = Eg^2(X_1) < \infty \), in Theorem 4 we assume only that \( g(X_1) \) is in the domain of attraction of the normal law and, consequently, we may have \( \sigma^2 = Eg^2(X_1) = \infty \), just like in Theorem 2 (cf. Remark 3).

**Remark 5** On taking \( h(x, y) = x - y \), Theorem 4 essentially extends Corollary 2.1.1 of Csörgő and Horváth (1997) [cf. Theorem 5.1 in CsSzW (2004)] and rhymes with Theorem 5.2 and Corollaries 5.1 and 5.2 of CsSzW (2004) [cf. also Theorem 2.1 and Corollaries 2.1 and 2.2 of CsSzW (2006)], where we study the problem of change in the mean in DAN directly via Theorem 2 and Corollaries 3 and 4 of CsSzW (2007), quoting these results without proof for the sake of studying the probable error of a change in a mean in the domain of attraction of the normal law. In this regard our present Theorems 2 and 4 can be viewed in as extensions of the initial scope of our research in CsSzW (2007) on weighted approximations of self-normalized partial sum processes to those of Studentized U-statistics type processes with symmetric and antisymmetric kernel functions \( h(\cdot, \cdot) \), whose respective projections \( g(X_1) \) are in DAN.

**REFERENCES**

Borovskikh, Yu. V. (1996). *U-statistics in Banach spaces.* VSP, Utrecht.
Borovskikh, Yu. V. (2002). On the normal approximation of U-statistics. *Theory Probab. Appl.* 45, 406–423.
Csörgő, M. (2002). A glimpse of the impact of Pál Erdős on probability and statistics. *The Canadian Journal of Statistics* 30, 493–556.
Csörgő, M., Csörgő, S., Horváth, L. and Mason, D. (1986). Weighted empirical and quantile processes, *Ann. Probab.* 14, 31-85.
Csörgő, M. and Horváth, L. (1988a). Invariance principles for changepoint problems. *J. Multivariate Anal.* 27, 151–168.
Csörgő, M. and Horváth, L. (1988b). Nonparametric methods for changepoint problems, In *Handbook of Statistics*, Elsevier Science Publisher B.V., 403-425, North-Holland, Amsterdam.
Csörgő, M. and Horváth, L. (1993). *Weighted Approximations in Probability and Statistics*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, Wiley, Chichester.

Csörgő, M., and Horváth, L. (1997). *Limit Theorems in Change-Point Analysis*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, Wiley, Chichester.

Csörgő, M., Norvaisa, R. and Szyszkowicz, B. (1999). Convergence of weighted partial sums when the limiting distribution is not necessarily Radon, *Stochastic Process. Appl.* 81, 81-101.

Csörgő, M., Szyszkowicz, B. and Wang, Q. (2004). On weighted approximations and strong limit theorems for self-normalized partial sums processes. In *Asymptotic methods in stochastic*, 489–521, Fields Inst. Commun. 44, Amer. Math. Soc., Providence, RI.

Csörgő, M., Szyszkowicz, B. and Wang, Q. (2006). Change in the mean in the domain of attraction of the normal law. *Austrian Journal of Statistics*, 35, 93-103.

Csörgő, M., Szyszkowicz, B. and Wang, Q. (2007). Weighted approximations in $D[0,1]$ with applications to self-normalized partial sum processes. *Preprint*.

Gine, E., and Zinn, J. (1992). Marcinkiewicz type laws of large numbers and convergence of moments for U-statistics. In *Probability in Banach Spaces* (R. Dudley, M. Hahn and J. Kuelbs, eds) 8 273-291, Birkhauser, Boston.

Gombay, E. (2000a). Comparison of U-statistics in the change-point problem and in sequential change detection. Endre Csáki 65. *Period. Math. Hungar.* 41, 157–166.

Gombay, E. (2000b). U-statistics for sequential change detection. *Metrika* 52, 133–145.

Gombay, E. (2001). U-statistics for change under alternatives. *J. Multivariate Anal.* 78, 139–158.

Gombay, E. (2004). U-statistics in sequential tests and change detection. Abraham Wald centennial celebration: invited papers. Part II. *Sequential Anal.* 23, 257–274.

Gombay, E. and Horváth, L. (1995). An application of U-statistics to change-point analysis. *Acta Sci. Math.* (Szeged) 60, 345–357.

Gombay, E. and Horváth, L. (2002). Rates of convergence for U-statistic processes and their bootstrapped versions. *J. Statist. Plann. Inference* 102, 247–272.

Halmos, P.R. (1946). The theory of unbiased estimation. *Ann. Math. Statist.* 17, 34–43.

Hoeffding, W. (1948). A class of statistics with asymptotically normal distribution. *Ann. Math. Statist.* 19, 293–325.

Janson, S. and Wichura, M. J. (1984). Invariance principles for stochastic area and related stochastic integrals. *Stochastic Process. Appl.* 16, 71–84.

Pettitt, A. N. (1979). A nonparametric approach to the change-point problem. *J. Roy. Statist. Soc. Ser. C* 28, 126–135.

Shorack, G.R. and Wellner, J.A. (1986). *Empirical Processes with Applications to Statistics*, Wiley, New York.

Szyszkowicz, B. (1991). Weighted stochastic processes under contiguous alternatives, *C.R. Math. Rep. Acad. Sci. Canada* 13, 211-216.
Szyszkowicz, B. (1992). *Weak Convergence of Stochastic Processes in Weighted Metrics and their Applications to Contiguous Changepoint Analysis*. Ph. D. Dissertation, Carleton University.

Szyszkowicz, B. (1996). Weighted approximations of partial sum processes in $D[0, \infty)$. I, *Studia Sci. Math. Hungar.* 31, 323-353.

Szyszkowicz, B. (1997). Weighted approximations of partial sum processes in $D[0, \infty)$. II, *Studia Sci. Math. Hungar.* 33, 305-320.

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