K-THEORY OF ADMISSIBLE ZARISKI-RIEMANN SPACES

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ABSTRACT. We study relative algebraic K-theory of admissible Zariski-Riemann spaces and prove that it is equivalent to relative G-theory and satisfies homotopy invariance. Moreover, we provide an example of a non-noetherian abelian category whose negative K-theory vanishes.

KEYWORDS. K-Theory, Zariski-Riemann spaces.

MATHEMATICAL SUBJECT CLASSIFICATION 2010. 19E08, 19D35.

CONTENTS

1. Introduction 1
2. Admissible Zariski-Riemann spaces 2
3. Modules on admissible Zariski-Riemann spaces 3
4. K-theory of admissible Zariski-Riemann spaces 9
Appendix A. Limits of locally ringed spaces 14
References 16

1. INTRODUCTION

Under the assumption of resolution of singularities, one can obtain for a non-regular scheme $X$ a regular scheme $X'$ which admits a proper, birational morphism $X' \to X$. For many purposes $X'$ behaves similarly to $X$. Unfortunately, resolution of singularities is not available at the moment in positive characteristic. From the perspective of K-theory, a good workaround for this inconvenience is to work with a Zariski-Riemann type space which is not a scheme anymore, but behaves almost as good as a regular model does. For instance, for a regular noetherian scheme $X$ one has equivalences $K(\text{Vec}(X)) \simeq K(\text{Coh}(X))$ and $K(X) \simeq K(X \times A^1)$ for Quillen’s K-theory. For the Zariski-Riemann type space $\langle X \rangle$ defined to be the limit of all schemes which are projective and birational over a scheme $X$ within the category of locally ringed spaces, Kerz-Strunk-Tamme [KST18] established that $K(\text{Vec}(\langle X \rangle)) \simeq K(\text{Coh}(\langle X \rangle))$ and $K(\text{Vec}(\langle X \rangle)) \simeq K(\text{Vec}(X \times A^1))$. The purpose of this note is to prove the analogous statement for admissible Zariski-Riemann spaces $\langle X \rangle_U$ which are limits of projective morphisms to a scheme $X$ which are isomorphisms over an open subscheme $U$ (Definition 2.1). As one only modifies something outside $U$, one has to pass to the relative K-theory. The main results are the following ones, see Theorem 4.8 and Theorem 4.15 as well as their corollaries.

**Theorem.** Let $X$ be a reduced, divisorial, and noetherian scheme and let $U$ be a dense open subset of $X$. Denote by $\tilde{Z}$ the complement of $U$ in $\langle X \rangle_U$. Then

(i) $K(\langle X \rangle_U$ on $\tilde{Z}) \simeq K(\text{Coh}\langle (X)_U \rangle) \simeq G(\langle X \rangle_U$ on $\tilde{Z})$ and

(ii) $K(\langle X \rangle_U$ on $\tilde{Z}) \simeq K(\langle X \rangle_U \times A^1$ on $\tilde{Z} \times A^1)$.

Moreover, if $U$ is regular, then

(iii) $K(\langle X \rangle_U \simeq G(\langle X \rangle_U$ and

(iv) $K(\langle X \rangle_U \simeq K(\langle X \rangle_U \times A^1)$.

where $G(\langle X \rangle_U) := K(\text{Mod}^f(\langle X \rangle_U))$ (Definition 4.4).

This work was supported by the Swiss National Science Foundation (grant number 184613). The author is supported by Deutsche Forschungsgemeinschaft (DFG) through the Collaborative Research Centre TRR 326 "Geometry and Arithmetic of Uniformized Structures" (project number 444845124).
The notion of Zariski-Riemann spaces goes back to Zariski [Zar44] who called them “Riemann manifolds” and was further studied by Temkin [Tem11]. Recently, Kerz-Strunk-Tamme [KST18] used them to prove that homotopy algebraic K-theory [Wei89] is the cdh-sheafification of algebraic K-theory, and Elmanto-Hoyois-Iwasa-Kelly applied them to prove a bound on the cdh-cohomological dimension [EHIK21].

Combining part (i) of the theorem with a result of Kerz about the vanishing of negative relative K-theory, we provide an example of a non-noetherian abelian category whose negative K-theory vanishes (Example 4.19). This gives evidence to a conjecture by Schlichting (which was shown to be false at the generality it was stated), see section 4.3.

**Notation.** A scheme is said to be divisorial iff it admits an ample family of line bundles [TT90, 2.1.1]; such schemes are quasi-compact and quasi-separated.

**Acknowledgements.** I thank Andrew Kresch for providing a fruitful research environment at the University of Zurich where most of this paper was written as well as Georg Tamme, Matthew Morrow, and Moritz Kerz for helpful conversations. Furthermore, I want to thank Marco Schlichting for pointing out the reference to Hiranouchi-Mochizuki in the proof of Theorem 4.8, Quentin Guignard for pointing out a mistake in a previous version as well as Elden Elmanto and Katharina Hübner for some helpful remarks. Last, but not least, I thank the referee for their very detailed reports which pointed out some gaps in my proofs and led to a much improved presentation.

## 2. Admissible Zariski-Riemann Spaces

**Notation.** In this section let $X$ be a reduced quasi-compact and quasi-separated scheme and let $U$ be a quasi-compact open subscheme of $X$.

**Definition 2.1.** A $U$-modification of $X$ is a proper morphism $X' \to X$ of schemes which is an isomorphism over $U$. The category of $U$-modifications of $X$ is defined with morphisms over $X$ and denoted by $\text{Mdf}(X, U)$. The **U-admissible Zariski-Riemann space** of $X$ is defined as the limit

$$\langle X \rangle_U = \lim_{X' \in \text{Mdf}(X, U)} X'$$

in the category of locally ringed spaces. This limit exists, its underlying topological space is coherent and sober, and for any $X' \in \text{Mdf}(X, U)$ the projection map $\langle X \rangle_U \to X'$ is quasi-compact; see Proposition A.7 in the appendix.

**Example 2.2.** Let $V$ be a valuation ring with fraction field $K$. Then the canonical projection $\langle \text{Spec}(V) \rangle_{\text{Spec}(K)} \to \text{Spec}(V)$ is an isomorphism as every $\text{Spec}(K)$-modification of $\text{Spec}(V)$ is split according to the valuative criterion for properness.

**Lemma 2.3.** The following are cofinal subcategories of $\text{Mdf}(X, U)$:

(i) The full subcategory spanned by morphisms $X' \to X$ where $X'$ is reduced.

(ii) If $U$ is schematically dense in $X$: the full subcategory spanned by $U$-admissible blow-ups, i.e. blow-ups whose centre does not meet $U$.

(iii) If $U$ is schematically dense in $X$: the full subcategory spanned by projective morphisms $X' \to X$.

In particular, under these assumptions the respective limits over these subcategories agree with the $U$-admissible Zariski-Riemann space.

**Proof.** (i) is [Dah19 3.5], (ii) is [Dah19 3.4], and (iii) follows from (ii). \qed

**Remark 2.4.** For the choice of a scheme-structure $Z$ on the closed complement $X \setminus U$, the complement $\langle X \rangle_U \setminus U$ comes equipped with the structure of a locally ringed space, namely

$$\langle X \rangle_U \setminus U = \lim_{X' \in \text{Mdf}(X, U)} X'_Z$$

where $X'_Z := X' \times_X Z$. 

Remark 2.5. Let $Z$ be a scheme-structure on $X \setminus U$. One can define the Zariski-Riemann space $\langle Z \rangle$ to be the limit over all projective and birational morphism $Z' \to Z$ in the category of locally ringed spaces. In general, there does not exist a canonical morphism $\langle Z \rangle \to (X)_U \setminus U$ of locally ringed spaces. Although for $X' \in \text{Mdf}(X, U)$ and a $U$-admissible blow up $\text{Bl}_Y (X') \to X'$ there exists a canonical morphism $\text{Bl}_Y (X'_Z) \to \text{Bl}_Y (X')$, the problem is that the blow-up $\text{Bl}_Y (X'_Z) \to X'_Z$ needs not to be birational in case that $X'_Z \setminus Y$ is not dense in $X'_Z$.

Comparison to Temkin’s relative Riemann-Zariski spaces.

Remark 2.6. Temkin [Tem11] introduced the notion of a relative Riemann-Zariski space as-sociated with any separated morphism $f : Y \to X$ between quasi-compact and quasi-separated schemes. In loc. cit. a $Y$-modification is a factorisation $Y \xrightarrow{f_i} X_i \xrightarrow{g_i} X$ of $f$ where $f_i$ is schematically dominant and $g_i$ is proper. The family of these $Y$-modifications is cofiltered and the relative Riemann-Zariski space of the morphism $f : Y \to X$ is defined as the cofiltered limit $\text{RZ}_Y (X) := \lim_{i} X_i$ which is indexed by all $Y$-modifications.

Lemma 2.7. If $U$ is a dense in $X$, then there exists a canonical morphism $\text{RZ}_U (X) \to (X)_U$ which is an isomorphism.

Proof. If $X$ is reduced, then $U$ is dense in $X$ if and only if it is schematically dense in $X$ [Sta21, Tag 056D]. Hence every $U$-modification in our sense is also a $U$-modification in Temkin’s sense with respect to the inclusion $U \to X$. Given a $U$-modification $U \xrightarrow{g} X_i \xrightarrow{p} X$ in Temkin’s sense, we take a Nagata compactification $U \xhookrightarrow{j} \bar{X}_i \xrightarrow{q} X'$, i.e. $j$ is an open immersion, $q$ is proper, and $g = q \circ j$ [Sta21, Tag 0F41]. Then the base change $(p \circ q) \times_X U : \bar{X}_i \times_X U \to U$ is bijective and split by an open immersion, hence an isomorphism. Thus the $U$-modifications in our sense are cofinal within the $U$-modifications in Temkin’s sense. □

3. Modules on admissible Zariski-Riemann spaces

The following results depend on Raynaud-Gruson’s platification par éclatement [RG71, 5.2.2]. These results and their proofs are modified versions of results of Kerz-Strunk-Tamme who considered birational and projective schemes over $X$ instead of $U$-modifications, cf. Lemma 6.5 and Proof of Proposition 6.4 in [KST18].

Notation. In this section let $X$ be a reduced quasi-compact and quasi-separated scheme and let $U$ be a quasi-compact open subscheme of $X$. For any locally ringed space $(Y, \mathcal{O}_Y)$ we denote by $\text{Mod}^{fp}(Y)$ the full subcategory of all $\mathcal{O}_Y$-modules spanned by finitely presented objects, i.e. modules locally of finite presentation.

Lemma 3.1. The canonical functor $\colim_{X' \in \text{Mdf}(X, U)} \text{Mod}^{fp}(X') \to \text{Mod}^{fp}((X)_U)$. is an equivalence (within the 2-category of categories).

1The author apologises for the switched order of the names “Zariski” and “Riemann” which tries to be coherent with the different sources.
Proof. This is a general fact about limits of coherent and sober locally ringed spaces with quasi-compact transition maps, see [PK18 ch. 0, 4.2.1–4.2.3] and Proposition A.7. □

This equivalence restricts to the full subcategories of vector bundles, i.e. locally free modules of finite rank.

Lemma 3.2. The canonical functor

$$\colim_{X' \in \text{Mfd}(X,U)} \text{Vec}(X') \to \text{Vec}((X)_U)$$

is an equivalence (within the 2-category of categories).

Proof. Clearly, the pullback of a vector bundle is again a vector bundle. Hence fully faithfulness follows from the corresponding statement for finitely presented modules. It remains to show that the functor is essentially surjective, so let $F$ be a locally free $O_{(X)_U}$-module of finite rank. Since the topological space $(X)_U$ is coherent, we may assume that there exists a finite cover $(X)_U = V_1 \cup \ldots \cup V_k$ of quasi-compact open subsets such that $F|_{V_i} \cong O_{V_i}^{p_i}$ for all $i$ and suitable natural numbers $n_i$. Hence we can argue by induction on $k$ and reduce to the case when $k = 2$. There exists a $U$-modification $X_0$ and quasi-compact open subsets $V'_1$ and $V'_2$ of $X_0$ such that $V_i = p_0^{-1}(V'_i)$ for $i = 1, 2$ and $V_1 \cap V_2 = p_0^{-1}(V'_1 \cap V'_2)$ and there exists $O_{V'_i}$-modules $G_i$ such that $p_0^*G_i \cong F|_{V_i}$ where $p_0^*:(X)_U \to X_0$ denotes the canonical projection. By general properties of sheaves on limits of locally ringed spaces we have a bijection

$$\colim_{X \in \text{Mfd}(X_0, U)} \text{Hom}_{X_a}(p_{a, 0}^*G_1, p_{a, 0}^*G_2) \cong \text{Hom}_{(X)_U}(F|_{V_1}, F|_{V_2}),$$

where $p_{a, 0}: X_a \to X_0$ is the transition map [PK18 ch. 0, Thm. 4.2.2]. Hence there exists an $a$ such that $p_{a, 0}^*G_1$ and $p_{a, 0}^*G_2$ glue to a locally free sheaf $G$ on $X_a$ which further pulls back to $F$ which shows that the functor in question is essentially surjective. □

Definition 3.3. Let $(Y, O_Y)$ be a locally ringed space and let $n \geq 0$. An $O_Y$-module $F$ is said to be pseudo-coherent of Tor-dimension $\leq n$ iff there exists an exact sequence

$$0 \to E_n \to \ldots \to E_1 \to E_0 \to F \to 0$$

where $E_n, \ldots, E_1, E_0$ are vector bundles (i.e. locally free $O_Y$-modules of finite rank). Denote by $\text{Mod}^{\leq n}(Y)$ and $\text{Coh}^{\leq n}(Y)$ the full subcategories of $\text{Mod}(Y)$ resp. $\text{Coh}(Y)$ spanned by pseudo-coherent $O_Y$-modules of Tor-dimension $\leq n$.

Lemma 3.4 (cf. [KST18 6.5 (i)]). Assume that $U$ is dense in $X$. Then:

(i) For every $U$-modification $p:X' \to X$ the pullback functor preserves modules of Tor-dimension $\leq 1$ and the restricted functor

$$p^*:\text{Mod}^{\leq 1}(X) \to \text{Mod}^{\leq 1}(X')$$

is exact.

(ii) For every morphism $\varphi:F \to G$ in $\text{Mod}^{\leq 1}(Y)$ such that $F, G,$ and $\text{coker}(\varphi)$ all lie in $\text{Mod}^{\leq 1}(Y)$ and for every $U$-modification $p:X' \to X$ with $X'$ reduced the canonical maps

$$q^*(\ker(\varphi)) \to \ker(q^*\varphi) \text{ and } q^*(\text{im}(\varphi)) \to \text{im}(q^*\varphi)$$

are isomorphisms.

Proof. (i) If $F$ is a $O_X$-module of Tor-dimension $\leq 1$, there exists an exact sequence $0 \to E_1 \xrightarrow{\varphi} E_0 \to F \to 0$ where $E_1$ and $E_0$ are $O_X$-vector bundles. Then the pulled back sequence

$$0 \to p^*E_1 \xrightarrow{p^*\varphi} p^*E_0 \to p^*F \to 0$$

is exact at $p^*E_0$ and $p^*F$. We claim that the map $p^*\varphi$ is injective. The $O_{X'}$-modules $p^*E_1$ and $p^*E_0$ are locally free, say of rank $n$ and $m$, respectively. Let $\eta$ be a generic point of an irreducible component of $X'$. Since $U$ is dense, the map $(p^*\varphi)_\eta: O^n_{X', \eta} \to O^m_{X', \eta}$ is injective since it
identifies with the injective map \( \varphi_{\eta(p)}: (E_1)_{\eta(p)} \hookrightarrow (E_0)_{\eta(p)} \). By Lemma 2.3(i), we may assume that \( X' \) is reduced. For a general point \( x \in X' \), the stalk \( O_{X',x} \) embeds into \( \prod_{\eta} O_{X',\eta} \) where the product is over the generic points of irreducible components of \( X' \). Hence the induced map \((p^* \varphi)_x: O_{X',x}^m \to O_{X',x} \) is injective. Thus \( p^* \varphi \) is injective at every point of \( X' \), hence injective. Now the exactness of \( p^* \) follows from the nine lemma.

(ii) This follows directly from (i). \( \Box \)

**Definition 3.5.** Let \((Y,O_Y)\) be a locally ringed space, \( Z \) a closed subset of \( Y \), and \( j:(V,O_V) \hookrightarrow (Y,O_Y) \) the inclusion of the open complement. An \( O_Y \)-module \( F \) has **support in \( Z \)** if \( j^*F \) vanishes. Denote with a “\( \cdot \)” in the lower index the full subcategory of those \( O_Y \)-modules which have support in \( Z \), e.g. \( \text{Mod}^{fp}_Z(Y) \subset \text{Mod}^{fp}(Y) \).

**Definition 3.6.** Let \( C \) be a pointed category. We denote by \( \text{Nil}(C) \) the category whose objects are pairs \((F,v)\) where \( F \) is an object of \( C \) and \( v:F \to F \) is a nilpotent endomorphism, i.e. there exists a \( k \) such that \( v^k = 0 \) in the point set \( \text{Hom}_C(F,F) \). The morphisms are given by those morphisms in \( C \) which are compatible with the respective endomorphisms.

**Lemma 3.7 (cf. [KST18, 6.5]).** Assume that \( X \) is divisorial and that \( U \) is dense in \( X \). Let \( f:Y \to X \) be a quasi-projective morphism of finite presentation and let \( Z = Y \setminus (Y \times_X U) \). Given a \( U \)-modification \( p:X' \to X \), we denote by \( q:Y' = Y \times_X X' \to Y \) the pullback of \( p \) along \( f \).

(i) For every \( F \in \text{Mod}^{fp}_Z(Y) \) there exists a \( U \)-modification \( p:X' \to X \) such that \( q^*F \) lies in \( \text{Mod}^{fp,\leq 1}(Y') \).

(ii) For every morphism \( \varphi:F \to G \) in \( \text{Mod}^{fp}_Z(Y) \) there exists a \( U \)-modification \( p:X' \to X \) such that \( q^*F,q^*G,\ker(q^*\varphi),\text{im}(q^*\varphi) \) and \( \text{coker}(q^*\varphi) \) all lie in \( \text{Mod}^{fp,\leq 1}(Y') \).

(iii) For every \((F,v) \in \text{Nil}(\text{Coh}_Z(Y)) \) there exists a \( U \)-modification such that there exists a finite resolution

\[
0 \to (E_k,v_k) \to \ldots \to (E_0,v_0) \to (q^*F,q^*v) \to 0
\]

where all \((E_i,v_i)\) are locally free of finite rank.

**Proof.** (i) Since \( X \) has an ample family of line bundles and the map \( Y \to X \) is quasi-projective, also \( Y \) has an ample family of line bundles [Tt90, 2.1.2.(h)]. Hence there exists an exact sequence \( E_1 \xrightarrow{\varphi} E_0 \to F \to 0 \) where \( E_1,E_0 \) are \( O_Y \)-vector bundles. By our assumptions,

\[
\text{im}(\varphi)|_U = \ker(E_0 \to F)|_U = \ker(E_0|_U \to 0) = E_0|_{U \times_X Y}
\]

is flat. By **platification par éclatement** [RG71, 5.2.2] there exists a \( U \)-admissible blow-up \( p:X' \to X \) such that the strict transform \( q^*\text{im}(\varphi) \) is flat, i.e. locally free. Furthermore, \( q^*\text{im}(\varphi) = \text{im}(q^*\varphi) \), cf. Remark 3.9 below. Hence we obtain an exact sequence

\[
0 \to \text{im}(q^*\varphi) \to q^*E_0 \to q^*F \to 0,
\]

hence \( q^*F \in \text{Mod}^{fp,\leq 1}(Y') \) by Lemma 3.4(ii).

(ii) By (i), there is a \( U \)-modification \( p:X' \to X \) such that both \( q^*F \) and \( q^*G \) have Tor-dimension \( \leq 1 \). Since \( q^* \) is a left-adjoint, \( q^*(\text{coker}(\varphi)) = \text{coker}(q^*\varphi) \) so that we may assume that also \( \text{coker}(q^*\varphi) \) has Tor-dimension \( \leq 1 \). Hence we can apply Lemma 3.4(ii) and are done.

(iii) We argue by induction on \( k > 0 \) with \( v^k = 0 \). The case \( k = 1 \) is follows from (i) and we assume that \( k \geq 2 \). By (ii) we find a \( U \)-modification \( p:X' \to X \) such that \( q^*F,\ker(q^*v^{k-1}),\text{im}(q^*v^{k-1}) \), and \( \text{coker}(q^*v^{k-1}) \) all lie in \( \text{Mod}^{fp,\leq 1}(Y') \). By induction assumption, there exists after further \( U \)-modification a finite resolution

\[
0 \to (E'_k,v'_k) \to \ldots \to (E'_0,v'_0) \to (\ker(q^*v^{k-1}),v') \to 0
\]

where all \( E'_i \) are vector bundles and where \( v' \) is the restriction of \( q^*v \) to \( \ker(q^*v^{k-1}) \). Similarly, there exists a finite resolution of \( (\text{im}(q^*v^{k-1}),v'') \) where \( v'' \) is the restriction of \( q^*v \) to \( \text{im}(q^*v^{k-1}) \). These two finite resolutions now can be patched together to a finite resolution of \( q^*F \). \( \Box \)
Lemma 3.8. Assume that $X$ is divisorial and that $U$ is dense in $X$. Then the inclusion

$$\text{Mod}^{\text{fp}, \leq 1}_{\tilde{Z}}(\langle X \rangle_U) \longrightarrow \text{Mod}^{\text{fp}}_{\tilde{Z}}(\langle X \rangle_U)$$

is an equivalence of categories where $\tilde{Z} := \langle X \rangle_U \setminus U$.

Proof. It suffices to show that the functor is essentially surjective. Given a module $F$ in the target, we find by Lemma 3.1 an $\tilde{Z}$ that $p^*F$ is a vector bundle, hence $F$ has Tor-dimension $\leq 1$, thus also $F$ has Tor-dimension $\leq 1$ by Lemma 3.4.

Remark 3.9. In the situation of the proofs of Lemma 3.7 and Lemma 3.8 we used that $\text{im}(q^*\varphi)$ is the strict transform of $\text{im}(\varphi)$ along the $U$-admissible blow-up $q: X'' \to X'$. By definition, $q^*\text{im}(\varphi)$ is the quotient of $q^*\text{im}(\varphi)$ by its submodule of sections whose support is contained in the centre of the blow-up $q$. Since the surjective map $q^*\text{im}(\varphi) \to q^*\text{im}(\varphi)$ factors over the map $\text{im}(q^*\varphi) \subset p^*E'_0$, the following commutative diagram has exact rows and exact columns.

\[
\begin{array}{ccccccccc}
0 & 0 \\
\downarrow & & \\
0 & \ker(\sigma) & \text{im}(q^*\varphi) & \text{im}(q^*\text{im}(\varphi)) & 0 \\
\downarrow & & \downarrow & \downarrow & \\
0 & \ker(\tau) & q^*E'_0 & q^*\text{im}(\varphi) & 0
\end{array}
\]

Since $E'_0$ is a vector bundle, $q^*E'_0 = q^*\text{im}(\varphi)$ which implies the claim.

If a locally ringed space is cohesive, i.e., its structure sheaf is coherent (Definition A.1), then a module is coherent if and only if it is finitely presented (Lemma A.3). Unfortunately, we do not know whether or not this is also true for the Zariski-Riemann space $\langle X \rangle_U$. But passing to the complement $\langle X \rangle_U \setminus U$ we have the following.

Proposition 3.10. Assume that $X$ is divisorial and that $U$ is dense in $X$. Let $\tilde{Z}$ be the complement of $U$ in $\langle X \rangle_U$. An $O_{\langle X \rangle_U}$-module with support on $\tilde{Z}$ is coherent if and only if it is finitely presented.

Proof. We have to show that every finitely presented $O_{\langle X \rangle_U}$-module with support in $\tilde{Z}$ is coherent. Let $F$ be a finitely presented $O_{\langle X \rangle_U}$-module with support on $\tilde{Z}$. By definition, $F$ is of finite type. Let $V$ be an open subset of $\langle X \rangle_U$ and let $\varphi: O^m_V \to F|_V$ be a morphism. We need to show that $\ker(\varphi)$ is of finite type. Since this is a local property and $\langle X \rangle_U$ is coherent, we may assume that $V$ is quasi-compact. By passing iteratively to another $U$-modification, there exists an $X' \in \text{Mfd}(X, U)$ and a canonical projection $p_X: \langle X \rangle_U \to X'$ such that

- $F = (p_X)^*F_X$ for some $F_X \in \text{Mod}^{\text{fp}}_{\langle X \rangle_U}(X')$ (Proposition A.7 (iv)),
- $F_X$ has Tor-dimension $\leq 1$ (Lemma 3.8),
- $V = (p_X)^{-1}(V')$ for some open subset $V'$ of $X'$ (Proposition A.6 (i)),
- $\varphi$ is induced by a morphism $\varphi': O^m_V \to F_X|_V'$ (Proposition A.7 (v)), and
- $\text{coker}(\varphi)$ has Tor-dimension $\leq 1$ (Lemma 3.3).

Since $\ker(\varphi')$ is of finite type we may assume that there exists a surjection $O^m_{V'} \to \ker(\varphi')$ for some $m \in \mathbb{N}$ (otherwise we have to shrink $V'$). By Lemma 3.3, $(p_X)^*(\ker(\varphi')) = \ker(\varphi)$, hence it is of finite type. □
Theorem 3.11. Assume that $X$ is divisorial and that $U$ is dense in $X$. Let $\tilde{Z}$ be the complement of $U$ in $\langle X \rangle_U$. Then the canonical functors

$$
\begin{align*}
\text{Coh}_{\tilde{Z}}^1(\langle X \rangle_U) &\longrightarrow \text{Coh}_{\tilde{Z}}(\langle X \rangle_U) \\
\text{Mod}_{\tilde{Z}}^{\text{fp}, \leq 1}(\langle X \rangle_U) &\longrightarrow \text{Mod}_{\tilde{Z}}^{\text{fp}}(\langle X \rangle_U)
\end{align*}
$$

are equivalences. In particular, the canonical functor

$$
\text{colim}_{X' \in \text{Mfd}(X_U)} \text{Coh}_{\tilde{Z}'}(X') \longrightarrow \text{Coh}_{\tilde{Z}}(\langle X \rangle_U)
$$

is an equivalence of categories where $Z' = X' \setminus U$ and the colimit is taken in the 2-category of categories.

Proof. The lower horizontal functor is an equivalence by Lemma 3.8 and the right vertical functor is an equivalence by Lemma 3.10. The other two functors are equivalence since the square is a pullback square. The equivalence from Proposition A.7 restricts to an equivalence

$$
\text{colim}_{X' \in \text{Mfd}(X_U)} \text{Mod}_{\tilde{Z}'}^\text{fp}(X') \longrightarrow \text{Mod}_{\tilde{Z}}^\text{fp}(\langle X \rangle_U)
$$

which yields the desired statement. □

Corollary 3.12. Assume that $X$ is divisorial and that $U$ is dense in $X$. Let $\tilde{Z}$ be the complement of $U$ in $\langle X \rangle_U$ with induced structure of a locally ringed space (cf. Remark 2.4). Then $\mathcal{O}_{\tilde{Z}}$ is a coherent $\mathcal{O}_X$-module, i.e., the locally ringed space $\tilde{Z}$ is cohesive.

Proof. By Lemma 3.13 below it is sufficient to show that $i_* \mathcal{O}_{\tilde{Z}}$ is a coherent $\mathcal{O}_{\langle X \rangle_U}$-module where $i: \tilde{Z} \hookrightarrow \langle X \rangle_U$ denotes the inclusion map. This follows from Proposition 3.10 since $i_* \mathcal{O}_{\tilde{Z}}$ is finitely presented by Lemma 3.14 below.

Lemma 3.13. Let $i: Z \hookrightarrow Y$ be a closed immersion of locally ringed spaces and let $F \in \text{Mod}(\mathcal{Z})$ of finite type. If $i_* F \in \text{Coh}(Y)$, then $F \in \text{Coh}(\mathcal{Z})$.

Proof. Let $\varphi: \mathcal{O}_W^m \rightarrow F$ be a morphism for an open subset $W \subset Z$. Choose an open subset $V \subset Y$ such that $W = V \cap Z$. Then the kernel of the morphism $\psi: \mathcal{O}_V^m \rightarrow i_* \mathcal{O}_W^m \rightarrow i_* (F|_W)$ is of finite type if $i_* F$ is coherent so that we get an epimorphism $\mathcal{O}_V^m \rightarrow \ker(\psi)$. Then the induced morphism $\mathcal{O}_W^m \rightarrow i_* \ker(\psi)$ witnesses $\ker(\varphi)$ to be of finite type. □

Lemma 3.14. Assume that $X$ is noetherian and that $U$ is dense in $X$. Let $i: \tilde{Z} \hookrightarrow \langle X \rangle_U$ be the inclusion of the complement with induced structure of a locally ringed space (cf. Remark 2.4). Then the functor $i_*: \text{Mod}(\tilde{Z}) \rightarrow \text{Mod}(\langle X \rangle_U)$ restricts to a functor $i_*: \text{Mod}^{\text{fp}}(\tilde{Z}) \rightarrow \text{Mod}^{\text{fp}}(\langle X \rangle_U)$.

Proof. Let $F = p^* F'$ with $F' \in \text{Mod}^{\text{fp}}(X')$ for some $X' \in \text{Mfd}(X, U)$ with projection $p: \langle X \rangle_U \rightarrow X'$. Then $i^* F \cong q^* k^* F'$ for the induced morphisms $q: \tilde{Z} \rightarrow Z':= Z \times_X X'$ and $k: Z' \rightarrow X'$. Note that the functor $k_*$ preserves finitely presented objects since $X'$ is noetherian. We claim that $i_* i^* F \cong p_* k_* k^* F'$ which implies the assertion of the lemma. The unit of the adjunction $q^* \rightarrow q_*$ induces a morphism $k_* k^* F' \rightarrow k_* q_* q^* k^* F' = p_* i_* q^* k^* F'$ which induces a morphism $\varphi: p_* k_* k^* F' \rightarrow i_* q^* k^* F'$ by the adjunction $p^* \rightarrow p_*$. Applying the functor $i^*$ to $\varphi$ yields the identity map on $q^* k^* F'$, hence $\varphi$ is an isomorphism since both its source and its target are supported on $\tilde{Z}$. □

Reminder 3.15. Recall that an exact category is an additive category $A$ together with a class of conflations

$$
A \rightarrowtail B \twoheadrightarrow C
$$

satisfying certain axioms, cf. [Sch04, 1.1]. The morphism $A \rightarrowtail B$ appearing in conflations are called inflations and the morphisms $B \twoheadrightarrow C$ are called deflations. Every abelian category is
an exact category whose conflations are the the short exact sequences; hence the inflations are the monomorphisms and the deflations are the epimorphisms.

**Definition 3.16.** Let \( \mathcal{A} \) be an exact subcategory of an exact category \( \mathcal{B} \). We say that \( \mathcal{A} \subset \mathcal{B} \) is...

(i) a **Serre subcategory** iff for every conflation \( X' \rightarrow X \rightarrow X'' \) in \( \mathcal{B} \) we have that \( X \in \mathcal{A} \) if and only if both \( X' \) and \( X'' \) lie in \( \mathcal{A} \).

(ii) **right-filtering** iff it is a Serre subcategory and if for every morphism \( f: B \rightarrow A \) with \( B \in \mathcal{B} \) and \( A \in \mathcal{A} \) there exists an object \( A' \in \mathcal{A} \) such that \( f \) can be factored as a composition \( B \rightarrow A' \rightarrow A \) where the morphism \( B \rightarrow A' \) is a deflation.

(iii) **right-s-filtering** iff it is right-filtering and if for every inflation \( A \rightarrow B \) in \( \mathcal{B} \) with \( A \in \mathcal{A} \) there exists a deflation \( B \rightarrow A' \) with \( A' \in \mathcal{A} \) such that the composition \( A \rightarrow B \rightarrow A' \) is an inflation.

**Lemma 3.17.** Let \( Y \) be a locally ringed space, \( j: V \rightarrow Y \) be the inclusion of an open subspace, and \( i: Z \rightarrow Y \) be the inclusion of its closed complement. For a full exact subcategory \( \mathcal{M} \) of \( \text{Mod}(Y) \) denote by \( \mathcal{M}_Z \) the full subcategory of \( \mathcal{M} \) which is spanned by modules with support on \( Z \). Assume that \( Z \) has the structure of a locally ringed space such that for every \( M \in \mathcal{M} \) the object \( i_*i^*M \) lies in \( \mathcal{M} \) (and hence in \( \mathcal{M}_Z \)). Then \( \mathcal{M}_Z \) is a right-s-filtering subcategory of \( \mathcal{M} \).

**Proof.** The inclusion \( \mathcal{M}_Z \rightarrow \mathcal{M} \) is the kernel of the exact restriction functor \( j^*: \mathcal{M} \rightarrow \text{Mod}(U) \), hence it is closed under subobjects, quotients, and extension, thus a Serre subcategory. Given a morphism \( f: B \rightarrow A \) in \( \mathcal{M} \) with \( A \in \mathcal{M}_Z \), we have a factorisation of \( f \) as

\[
B \xrightarrow{\epsilon_B} i_*i^*B \rightarrow i_*i^*A \xrightarrow{(\epsilon_A)^{-1}} A.
\]

and the composition \( (\epsilon_A)^{-1} \circ (i_*i^*f) \) is a deflation. Thus \( \mathcal{M}_Z \) is right-filtering. Now let \( A \rightarrow B \) be an inflation (i.e. a monomorphism whose cokernel lies in \( \mathcal{M} \)) with \( A \in \mathcal{M}_Z \). Then the unit \( B \rightarrow i_*i^*B \) is a deflation and the composition \( A \rightarrow B \rightarrow i_*i^*B \) is an inflation as it equals the composition \( A \xrightarrow{\eta_A} i_*i^*A \rightarrow i_*i^*B \).

**Proposition 3.18.** Assume that \( X \) is noetherian. Denote by \( \tilde{Z} \) the closed complement of \( U \) in \( (X)_U \). Then \( \text{Mod}^\text{fp}_Z((X)_U) \) is a right-s-filtering subcategory of the exact category \( \text{Mod}^\text{fp}_Z((X)_U) \) and the inclusion \( j^*: \text{Mod}^\text{fp}_Z((X)_U) \rightarrow (X)_U \) induces an equivalence of categories

\[
\text{Mod}^\text{fp}_Z((X)_U) \rightarrow \text{Mod}^\text{fp}(U).
\]

**Proof.** Let \( i: \tilde{Z} \rightarrow (X)_U \) be the inclusion map. For \( F \in \text{Mod}^\text{fp}_Z((X)_U) \) we have that \( i_*i^*F \in \text{Mod}^\text{fp}_Z((X)_U) \) by Lemma 3.14. Applying Lemma 3.17 with \( Y := (X)_U \), \( V := U \), \( Z := \tilde{Z} \), and \( \mathcal{M} := \text{Mod}^\text{fp}_Z((X)_U) \) we obtain that \( \text{Mod}^\text{fp}_Z((X)_U) \) is a right-s-filtering subcategory of \( \text{Mod}^\text{fp}_Z((X)_U) \). Hence the quotient category

\[
\text{Mod}^\text{fp}_Z((X)_U)/\text{Mod}^\text{fp}_Z((X)_U)
\]

is defined \([\text{Sch04}], 1.12, 1.14\). By definition, the restriction \( j^*: \text{Mod}^\text{fp}_Z((X)_U) \rightarrow \text{Mod}^\text{fp}(U) \) factors through the canonical functor \( \text{Mod}^\text{fp}_Z((X)_U) \rightarrow \text{Mod}^\text{fp}(\text{Mod}^\text{fp}_Z((X)_U) \rightarrow \text{Mod}^\text{fp}(U) \). We will show that the induced functor

\[
j^*: \text{Mod}^\text{fp}_Z((X)_U)/\text{Mod}^\text{fp}_Z((X)_U) \rightarrow \text{Mod}^\text{fp}(U)
\]

is fully faithful and essentially surjective, following the lines of the classical proof for noetherian schemes, cf. \([\text{Sch11}], 2.3.8\).

For essential surjectivity let \( F \in \text{Mod}^\text{fp}(U) = \text{Coh}(U) \). The inclusion \( j_X: U \rightarrow X \) induces an equivalence of categories

\[
(j_X)^*: \text{Coh}(X)/\text{Coh}_Z(X) \rightarrow \text{Coh}(U)
\]

where \( Z := X \setminus U \) \([\text{Sch11}], 2.3.8\). Thus there exists an \( F_X \in \text{Coh}(X) = \text{Mod}^\text{fp}(X) \) such that \( (j_X)^*F_X \cong F \). Since \( j_X \) factors as \( p_X \circ j \), it follows that

\[
F \cong j_X^*F_X \cong (p_X \circ j)^*F_X \cong j^*(p_X^*F_X),
\]
i.e. $F$ comes from a finitely presented module $p^*_X F_X \in \text{Mod}_{fp}^{\text{gp}}(\langle X \rangle_U)$.

For fullness let $\varphi : j^* F \to j^* G$ be a morphism of $\mathcal{O}_U$-modules with $F$ and $G$ in $\text{Mod}_{fp}^{\text{gp}}(\langle X \rangle_U)$. The quotient of $\text{Mod}_{fp}^{\text{gp}}(\langle X \rangle_U)$ by $\text{Mod}_{Z}^{\text{gp}}(\langle X \rangle_U)$ is the localisation of $\text{Mod}_{fp}^{\text{gp}}(\langle X \rangle_U)$ along those morphisms which are sent to isomorphisms by $j^*$. Consider the pullback diagram

$$
\begin{array}{ccc}
H & \rightarrow & G \\
\downarrow \phi & & \downarrow \beta \\
F & \rightarrow & j_+ j^* F \\
\end{array}
$$

in $\text{Mod}(\langle X \rangle_U)$. Since $j^*$ is exact, the square $j^*(\Delta)$ is also a pullback; hence $j^*(\alpha)$ is an isomorphism (as $j^*(\beta)$ is one). Thus the span $F \leftarrow H \rightarrow G$ represents a morphism $\varphi$ in the quotient category such that $\varphi = j^*(\varphi)$, hence $j^*$ is full.

For faithfullness let $F' \xrightarrow{\eta} F \xrightarrow{\varphi} G$ be a span representing a morphism in the quotient category $\text{Mod}_{fp}^{\text{gp}}(\langle X \rangle_U)/\text{Mod}_{Z}^{\text{gp}}(\langle X \rangle_U)$ which becomes zero when restricted to $U$. Let $\varphi' : F' \to G'$ be a morphism in $\text{Mod}_{fp}^{\text{gp}}(X')$ for some $X' \in \text{Mdf}(X, U)$ with $\varphi = p^*(\varphi')$ for the projection $p : \langle X \rangle_U \to X'$. Then $\varphi'|_U = 0$ by design so that $\varphi' = 0$ in the quotient category $\text{Mod}_{fp}^{\text{gp}}(X')/\text{Mod}_{Z}^{\text{gp}}(X')$, where $Z' = X' \setminus U$, by the classical case for noetherian schemes. Hence $\varphi$ and also the morphism represented by $F' \xrightarrow{\eta} F \xrightarrow{\varphi} G$ are zero in $\text{Mod}_{fp}^{\text{gp}}(\langle X \rangle_U)/\text{Mod}_{Z}^{\text{gp}}(\langle X \rangle_U)$.

\[ \square \]

4. K-theory of admissible Zariski-Riemann spaces

In this section we prove for the K-theory of the Zariski-Riemann space a comparison with G-theory (Theorem 4.15, Corollary 4.18) and homotopy invariance (Theorem 4.15, Corollary 4.18). Finally, we provide an example of a non-noetherian abelian category whose negative K-theory vanishes (Example 4.19).

**Notation.** In this section let $X$ be a reduced quasi-compact and quasi-separated scheme and let $U$ be a quasi-compact open subscheme of $X$. Denote by $\tilde{Z}$ the complement of $U$ within the $U$-admissible Zariski-Riemann space $\langle X \rangle_U$ (Definition 2.1). We equip the closed complement $X \setminus U$ with the reduced scheme structure so that $\tilde{Z}$ has the structure of a locally ringed space (Remark 2.4).

**Definition 4.1.** For an exact category $\mathcal{E}$ we denote by $K(\mathcal{E})$ its non-connective K-theory spectrum as defined by Schlichting [Sch06]. If $X$ is a divisorial scheme, then $K(\text{Vec}(X))$ is equivalent to the non-connective algebraic K-theory spectrum of $X$ à la Thomason-Trobaugh [TT90].

We define the **K-theory of the Zariski-Riemann space** as

$$K(\langle X \rangle_U) := K(\text{Vec}(\langle X \rangle_U))$$

and the **K-theory with support** as

$$K(\langle X \rangle_U \text{ on } \tilde{Z}) := \text{fib}(K(\langle X \rangle_U) \to K(U)).$$

**Lemma 4.2.** The canonical maps

$$\colim_{X' \in \text{Mdf}(X, U)} K(X') \to K(\langle X \rangle_U)$$

$$\colim_{X' \in \text{Mdf}(X, U)} K(X' \text{ on } Z') \to K(\langle X \rangle_U \text{ on } \tilde{Z})$$

are equivalences where $Z' := Z \times_X X'$.

**Proof.** The first equivalence follows from Lemma 3.2 and the fact that K-theory commutes with filtered colimits of exact functors; in non-negative degrees this is due to Quillen [Qui73, p. 20] and in negative degrees due to Schlichting [Sch06, §7, Cor. 5]. The second one follows since filtered colimits commute with finite limits.

\[\text{2 Beware that in loc. cit. this object is denoted by } \ast_k.\]
Theorem 4.3 (Kerz). If $X$ is reduced and divisorial, then
\[ K_i((X)_U \text{ on } \tilde{Z}) = 0 \quad \text{for } i < 0. \]
Furthermore, if $U$ is additionally regular, then we have $K_i((X)_U) \simeq 0$ for every $i < 0$.

Proof. The first statement follows from Lemma 4.2 and a result of Kerz saying that every negative relative K-theory class vanishes after some admissible blow-up [Ker18 Prop. 7]. The second statement is an immediate consequence using that negative K-theory of regular schemes vanishes.

4.1. Comparison with G-theory. For divisorial noetherian schemes, G-theory is the K-theory of the abelian category of coherent modules (which is the same as the category of finitely presented modules). Over an arbitrary scheme (or locally ringed space) a finitely generated module may not be coherent. Since we want the category of vector bundles to be included, we work with finitely presented modules for defining G-theory for admissible Zariski-Riemann spaces.

Definition 4.4. We define the G-theory of the Zariski-Riemann space as
\[ G((X)_U) := K(\text{Mod}^\text{fp}((X)_U)). \]
and the G-theory with support as
\[ G((X)_U \text{ on } \tilde{Z}) := \text{fib}(G((X)_U) \to G(U)). \]

Proposition 4.5. Assume that $X$ is divisorial and noetherian and that $U$ is dense in $X$. Then there is a fibre sequence
\[ K(\text{Coh}_{\tilde{Z}}((X)_U)) \to G((X)_U) \to G(U). \]
In other words, the canonical map
\[ K(\text{Coh}_{\tilde{Z}}((X)_U)) \to G((X)_U \text{ on } \tilde{Z}) \]
is an equivalence.

Proof. By Proposition 3.18 the category $\text{Mod}^\text{fp}_{\tilde{Z}}((X)_U)$ is a right-s-filtering subcategory of the exact category $\text{Mod}^\text{fp}((X)_U)$ and the quotient is equivalent to $\text{Mod}^\text{fp}(U)$. Applying Schlichting’s localisation theorem for additive categories [Sch04 2.10] yields the desired fibre sequence.

Lemma 4.6. The canonical map
\[ K_{\geq 0}(\text{Coh}(\tilde{Z})) \to K_{\geq 0}(\text{Coh}_{\tilde{Z}}((X)_U)) \]
between the connective covers of K-theory spectra is an equivalence.

Proof. We claim that that the fully faithful pushforward functor $\text{Coh}(\tilde{Z}) \hookrightarrow \text{Coh}_{\tilde{Z}}((X)_U)$ satisfies the conditions of the Dévissage Theorem [Wei13 V.4.1]. Let $F \in \text{Coh}_{\tilde{Z}}((X)_U)$. Using iteratively Lemma 3.4 and Lemma 3.7, there exist $X' \in \text{Mdf}(X,U)$ and a filtration
\[ F_0' \supset F_1' \supset \ldots \supset F_n' = 0 \]
with $F_i' \in \text{Coh}_{\tilde{Z}}^1(X')$ such that $F \simeq p^*F_0'\text{ and such that the quotients } F_i'/F_{i-1}' \text{ lie in the essential image of the functor } \text{Coh}(Z') \to \text{Coh}_{\tilde{Z}}(X') \text{ where } Z' = Z \times_X X'$. Then pulling back to $(X)_U$ yields a filtration of $F$ such that the successive quotients lie in $\text{Coh}(\tilde{Z})$ as desired.

Remark 4.7. Combining Theorem 4.3 above with Theorem 4.8 below we see that the connective cover $K_{\geq 0}(\text{Coh}_{\tilde{Z}}((X)_U)) \to K(\text{Coh}_{\tilde{Z}}((X)_U))$ is an equivalence. On the hand, the author does not know whether or not the connective cover $K_{\geq 0}(\text{Coh}(\tilde{Z})) \to K(\text{Coh}(\tilde{Z}))$ is an equivalence as well.

Theorem 4.8. Assume that $X$ is divisorial and noetherian and that $U$ is dense in $X$. Then we have an equivalences of spectra
\[ K((X)_U \text{ on } \tilde{Z}) \simeq K(\text{Coh}_{\tilde{Z}}((X)_U)) \simeq G((X)_U \text{ on } \tilde{Z}). \]
Proof. Every $U$-modification of $X$ has an ample family of line bundles since it is quasi-projective over a scheme admitting such a family \cite[2.1.2 (h)]{TT90}. Moreover, those $X' \in \text{Mdf}(X,U)$ such that $X' \setminus U \to X'$ is a regular closed immersion form a cofinal subsystem (since $U$-admissible blow-ups are cofinal). Thus we have $K(X' \to Z') \simeq K(\text{Coh}_Z^1(X'))$ for every $X' \in \text{Mdf}(X,U)$; in the connective case this is due to Thomason \cite[5.7 (e)]{TT90} and the general case is due to Hiranouchi-Mochizuki \cite[Thm. 3.3]{HM10}. Hence we have that

$$K((X)_{\text{U}} \text{ on } \tilde{Z}) \simeq \text{colim}_{X' \in \text{Mdf}(X,U)} K(X' \text{ on } Z') \simeq \text{colim}_{X' \in \text{Mdf}(X,U)} K(\text{Coh}_Z^1(X')) \simeq K((\text{Coh}_Z^1((X)_{\text{U}}))).$$

By Theorem \ref{thm:equivalence} we have an equivalence $K((\text{Coh}_Z^1((X)_{\text{U}})) \to K((\text{Coh}_Z((X)_{\text{U}}))$ so that we are done by Proposition \ref{prop:equivalence}

\begin{corollary}
Assume that $X$ is divisorial and noetherian and that $U$ is regular and dense in $X$. Then the canonical map

$$K((X)_{\text{U}}) \to G((X)_{\text{U}})$$

is an equivalence.
\end{corollary}

\begin{proof}
We have a commutative diagram

\[
\begin{array}{ccc}
K((X)_{\text{U}} \text{ on } \tilde{Z}) & \to & K((X)_{\text{U}}) \to K(U) \\
\downarrow & & \downarrow \downarrow & \downarrow \\
K(\text{Coh}_Z((X)_{\text{U}})) & \to & G((X)_{\text{U}}) \to G(U)
\end{array}
\]

where the upper line is a fibre sequence by design and the lower line is a fibre sequence by Proposition \ref{prop:equivalence}. Furthermore, the first vertical map is an equivalence by Theorem \ref{thm:equivalence} and the third vertical map is an equivalence since $U$ is regular.
\end{proof}

4.2. Homotopy invariance.

\begin{definition}
For an integer $n \geq 1$ we define

$$\langle X \rangle_{\text{U}}[t_1,\ldots,t_n] := \lim_{X' \in \text{Mdf}(X,U)} X'[t_1,\ldots,t_n].$$

\end{definition}

\begin{lemma}
There exists a canonical isomorphism

$$\langle X \rangle_{\text{U}}[t_1,\ldots,t_n] \xrightarrow{\simeq} \langle X \rangle_{\text{U}} \times_X \langle X \rangle_{[t_1,\ldots,t_n]}.$$

\end{lemma}

\begin{proof}
This follows since different limits commute among each other, namely:

$$\langle X \rangle_{\text{U}}[t_1,\ldots,t_n] = \lim_{X' \in \text{Mdf}(X,U)} X'[t_1,\ldots,t_n]$$

$$= \lim_{X' \in \text{Mdf}(X,U)} \lim_{X' \in \text{Mdf}(X,U)} (X' \to X \leftarrow X'[t_1,\ldots,t_n])$$

$$= \lim_{X' \in \text{Mdf}(X,U)} (X' \to X \leftarrow X'[t_1,\ldots,t_n])$$

$$= \langle X \rangle_{\text{U}} \times_X \langle X \rangle_{[t_1,\ldots,t_n]}.$$
\end{proof}

\begin{definition}
For a locally ringed space $Y$ we define

$$\text{Nil}(Y) := \text{fib}(K(\text{Nil}(\text{Vec}(Y))) \to K(Y))$$

where the map is induced by the forgetful functor $\text{Nil}(Y) \to \text{Vec}(Y)$ (cf. Definition \ref{def:definition}).
\end{definition}

\footnote{In loc. cit. the category $\text{Coh}_Z^1(X')$ is denoted by $\text{Wt}^1(X' \text{ on } Z')$ and it is shown that the category of bounded complexes in this category is equivalent to the derived category $\text{Perf}(X' \text{ on } Z')$.}

\footnote{The author apologises for the possibly confusing notation which tries to be coherent with existing literature.
Lemma 4.13. Let $Y$ be a quasi-compact and quasi-separated scheme. Then the category $\text{Nil}(\text{Coh}(Y))$ is an abelian category and $\text{Nil}(\text{Vec}(Y))$ is an exact subcategory. Furthermore the fibre sequence

$$\text{Nil}(Y) \to \text{K}(\text{Nil}(\text{Vec}(Y))) \to \text{K}(Y)$$

splits so that we get a decomposition

$$\text{K}(\text{Nil}(\text{Vec}(Y))) \simeq \text{K}(Y) \times \text{Nil}(Y).$$

Proof. The first part follows straightforwardly from the fact that $\text{Vec}(Y)$ is an exact subcategory of the abelian category $\text{Coh}(Y)$. The forgetful functor $\text{Nil}(\text{Vec}(Y)) \to \text{Vec}(Y)$ is split by the functor sending a vector bundle $E$ to the pair $(E, 0)$ so that the claim follows. \qed

Remark 4.14. There exists a split fibre sequence

$$\text{Nil}(\langle X \rangle_U) \to \text{K}(\text{Nil}(\langle X \rangle_U)) \to \langle X \rangle_U$$

defined as the colimit of the respective fibre sequences for every $X' \in \text{Mdf}(X, U)$ coming from Lemma 4.13. Hence there exists a split fibre sequence

$$\text{Nil}(\langle X \rangle_U \text{ on } \tilde{Z}) \to \text{K}(\text{Nil}(\langle X \rangle_U \text{ on } \tilde{Z})) \to \langle X \rangle_U \text{ on } \tilde{Z}$$

which is defined to be the fibre of the map

$$\text{Nil}(\langle X \rangle_U) \to \text{K}(\text{Nil}(\langle X \rangle_U)) \to \langle X \rangle_U$$

of fibre sequences.

Theorem 4.15. Assume that $X$ is divisorial and noetherian and that $U$ is dense in $X$. Denote $\tilde{Z} := \langle X \rangle_U \setminus U$. Then the canonical map

$$\text{K}(\langle X \rangle_U \text{ on } \tilde{Z}) \to \text{K}(\langle X \rangle_U[t] \text{ on } \tilde{Z}[t])$$

is an equivalence.

Proof. First note that the map in question identifies with the induced map

$$\text{K}_{20}(\langle X \rangle_U \text{ on } \tilde{Z}) \to \text{K}_{20}(\langle X \rangle_U[t] \text{ on } \tilde{Z}[t])$$

of the respective connective covers due to [Ker18 Prop 7], cf. Theorem 4.3; for the target use the relative version with respect to the morphism $X[t] \to X$ which is smooth of finite presentation.

We will show that the cofibre $\text{NK}(\langle X \rangle_U \text{ on } \tilde{Z})$ of the desired equivalence vanishes. For every $n \in \mathbb{Z}$ we have that

$$\text{Nil}_n(\langle X \rangle_U \text{ on } \tilde{Z}) \cong \text{NK}_{n+1}(\langle X \rangle_U \text{ on } \tilde{Z})$$

by the corresponding statement for schemes [Wei13 V.8.1] since K-theory commutes with filtered colimits of exact categories with exact functors. Thus it suffices to show that $\text{Nil}_n(\langle X \rangle_U \text{ on } \tilde{Z})$ vanishes, i.e. that the map

$$(\star) \quad \text{K}_{20}(\text{Nil}(\langle X \rangle_U \text{ on } \tilde{Z})) \to \text{K}_{20}(\langle X \rangle_U \text{ on } \tilde{Z})$$

is an equivalence. In order to see this, consider the following diagram of connective K-theory spectra.

$$\begin{array}{ccc}
\text{K}_{20}(\text{Ch}^b(\text{Nil}(\langle X \rangle_U))) & \xrightarrow{\epsilon} & \text{K}_{20}(\text{Nil}(\langle X \rangle_U \text{ on } \tilde{Z})) & \xrightarrow{\ast} & \text{K}_{20}(\langle X \rangle_U \text{ on } \tilde{Z}) \\
\text{K}_{20}(\langle X \rangle_U) & \xrightarrow{\delta} & \text{K}_{20}(\text{Nil}(\text{Coh}_Z(\langle X \rangle_U))) & \xrightarrow{\gamma} & \text{K}_{20}(\text{Nil}(\text{Coh}_Z(\langle X \rangle_U))) \\
\end{array}$$

The map $\alpha$ is an equivalence due to Theorem 4.8. The map $\beta$ is an equivalence due to the Dévissage Theorem [Wei13 V.4.1] since every object $(E, v)$ in $\text{Nil}(\text{Coh}(\tilde{Z}))$ has a filtration

$$(E, v) \supset (\ker(v), v) \supset (\ker(v^2), v) \supset \ldots \supset 0$$
whose quotients have trivial endomorphisms. The map $\gamma$ is an equivalence by the Gillet-Waldhausen Theorem [Wei13, V.2.2]. The map $\delta$ is a composition of equivalences by Lemma 3.7 (iii) and the Thomason-Trobaugh Resolution Theorem [Wei13, V.3.9], see Lemma 4.16 below. The map $\epsilon$ is an equivalence by combining the Waldhausen Localisation Theorem and the Waldhausen Approximation Theorem [Wei13, V.2.5], see Lemma 4.17 below. Thus the map $\zeta$ is an equivalence and we are done.

**Lemma 4.16.** The canonical maps
\[ K_{\geq 0}(\text{Ch}^b_Z(\text{Nil}(\langle X \rangle_U))) \to K_{\geq 0}(\text{Ch}^b_Z(\text{Nil}(\text{Mod}^\partial_{\langle X \rangle_U}))) \leftarrow K_{\geq 0}(\text{Ch}^b_Z(\text{Nil}(\text{Coh}^Z_2(\langle X \rangle_U)))) \]
are equivalences.

**Proof.** For each of the desired equivalences we want to invoke the Waldhausen Approximation Theorem [Wei13, V.2.5]. In loc. cit. we set $\mathcal{M} := \text{Nil}(\text{Mod}^\partial_{\langle X \rangle_U})$.

First we treat the right for which we set $\mathcal{B} := \text{Ch}^b_Z(\mathcal{M})$ and $\mathcal{A} := \text{Ch}^b_Z(\text{Nil}(\text{Coh}^Z_2(\langle X \rangle_U)))$ in loc. cit. We claim that every complex in $\mathcal{B}$ is quasi-isomorphic to a complex in $\mathcal{A}$. If the complex is concentrated in one degree, then this follows from the equivalence $\text{Coh}^Z_2(\langle X \rangle_U) \to \text{Mod}^\partial_{\langle X \rangle_U}$ of Theorem 3.11. For a general complex $0 \to F_m \to \ldots \to F_{n+1} \xrightarrow{\partial} F_n \to 0$ in $\mathcal{A}$ the complexes $0 \to F_m \to \ldots \to F_{n+1} \to 0$ and $0 \to \text{coker}(\partial) \to 0$ are (by induction) quasi-isomorphic to complexes in $\mathcal{A}$ which can be put together so that the initial complex is quasi-isomorphic to a complex in $\mathcal{A}$.

For the left equivalence we keep $\mathcal{B} := \text{Ch}^b_Z(\mathcal{M})$ and reset $\mathcal{A} := \text{Ch}^b_Z(\text{Nil}(\langle X \rangle_U)))$ in loc. cit. For a complex in $\mathcal{B}$ we may assume that it lies in $\text{Ch}^b_Z(\text{Nil}(\text{Coh}^Z_2(\langle X \rangle_U)))$ by the previous argument. In this case, Lemma 3.7 (iii) and an induction argument as above yield a quasi-isomorphism to a complex in $\mathcal{A}$.

**Lemma 4.17.** There is a canonical equivalence
\[ K_{\geq 0}(\text{Ch}^b_Z(\text{Nil}(\langle X \rangle_U))) \to K_{\geq 0}(\text{Nil}(\langle X \rangle_U \text{ on } \bar{Z})). \]

**Proof.** By definition we have a fibre sequence
\[ K_{\geq 0}(\text{Nil}(\langle X \rangle_U \text{ on } \bar{Z})) \to K_{\geq 0}(\text{Nil}(\langle X \rangle_U)) \to K_{\geq 0}(\text{Nil}(U)) \]
so that is suffices to show that the sequence
\[ K_{\geq 0}(\text{Ch}^b_Z(\text{Nil}(\langle X \rangle_U))) \to K_{\geq 0}(\text{Nil}(\langle X \rangle_U)) \to K_{\geq 0}(\text{Nil}(U)) \]
is a fibre sequence as well. Using the Gillet-Waldhausen Theorem [Wei13, V.2.2] we see that it suffices to show that the sequence
\[ K_{\geq 0}(\text{Ch}^b_Z(\text{Nil}(\langle X \rangle_U))) \to K_{\geq 0}(\text{Ch}^b_Z(\text{Nil}(\langle X \rangle_U))) \to K_{\geq 0}(\text{Ch}^b_Z(\text{Nil}(U))) \]
is a fibre sequence. The latter follows by combining the Waldhausen Localisation Theorem and the Waldhausen Approximation Theorem [Wei13, V.2.5]. In order to see that the assumptions of the cited theorem are satisfied, note that the assumptions are satisfied without the “Nil”, that the operators “Ch” and “Nil” commute with each other, and that “Nil” preserves the assumptions (as it is basically a diagram category).

**Corollary 4.18.** Assume that $X$ is divisorial and noetherian and that $U$ is regular and dense in $X$. Then the canonical projection $\langle X \rangle_U[t] \to \langle X \rangle_U$ induces an equivalence
\[ K(\langle X \rangle_U[t]) \to K(\langle X \rangle_U[\bar{t}]). \]

**Proof.** We have a commutative diagram of fibre sequences
\[
\begin{array}{ccc}
K(\langle X \rangle_U) & \to & K(\langle X \rangle_U[t]) & \to & K(U[t]) \\
\downarrow & & \downarrow & & \downarrow \\
K(\langle X \rangle_U[\bar{t}]) & \to & K(\langle X \rangle_U[\bar{t}]) & \to & K(U[\bar{t}])
\end{array}
\]
where the left map is an equivalence by Theorem 4.15 and the right map is an equivalence by homotopy invariance of algebraic K-theory for regular noetherian schemes [TT90 6.8].

4.3. Vanishing of negative K-theory. There was a conjecture of Schlichting whereby the negative K-theory of an abelian category vanishes [Sch06 9.7]. Schlichting himself proved the vanishing in degree -1 and subsequently the conjecture in the noetherian case [Sch06 9.1, 9.3]. A generalised conjecture about stable ∞-categories with noetherian heart was studied by Antieau-Gepner-Heller [AGH19]. Recently, Neeman [Nee21] gave a counterexample to both conjectures, Schlichting’s and Antieau-Gepner-Heller’s. Nevertheless, the following example provides a non-noetherian abelian category whose negative K-theory vanishes.

Example 4.19. Let $k$ be a discretely valued field with valuation ring $k^\circ$, uniformiser $\pi$, and residue field $\bar{k} = k^\circ/(\pi)$. Consider the scheme $X_0 := \text{Spec}(k^\circ(t))$, its special fibre $X_0/\pi = \text{Spec}(\bar{k}(t))$ and the open complement $U := \text{Spec}(k(t))$. Let $x_0 : \text{Spec}(\bar{k}) \to X_0/\pi$ be the zero section. Then we define the blow-up $X_1 := \text{Bl}_{x_0}(X_0)$. Its special fibre $X_1/\pi$ is an affine line over $\bar{k}$ with an $\mathbb{P}^1_{\bar{k}}$ attached to the origin. We get a commutative diagram

\[ \begin{array}{ccc}
E_1 & \to & X_1/\pi \\
\downarrow & & \downarrow \\
\{x_0\} & \to & X_0/\pi & \to & X_0 \end{array} \]

where the two squares are cartesian, all horizontal maps are closed immersions, the blow-up $\text{Bl}_{x_0}(X_0/\pi) \to X_0/\pi$ is an isomorphism, and the map $\text{Bl}_{x_0}(X_0/\pi) \to X_1/\pi$ is also closed immersion. Hence there is a closed immersion $X_0/\pi \hookrightarrow X_1/\pi$ which splits the canonical projection. Now choose a closed point $x_1 : \text{Spec}(\bar{k}) \to E_1 \setminus X_0/\pi$ and define $X_2 := \text{Bl}_{x_1}(X_1)$ and iterate this construction. Thus we obtain a strictly increasing chain of closed immersions

$$ X_0/\pi \hookrightarrow X_1/\pi \hookrightarrow \ldots \hookrightarrow X_n/\pi \hookrightarrow \ldots $$

and we can consider $X_i/\pi$ as a closed subscheme of $X_n/\pi$ for $i < n$. In each step, $X_{n+1}/\pi$ is obtained from $X_n/\pi$ by attaching a $\mathbb{P}^1_{\bar{k}}$ to a closed point not contained in $X_{n-1}/\pi$. Passing to the Zariski-Riemann space $(X_0)_U$, its special fibre $(X_0)_U/\pi$ admits a strictly decreasing chain of closed subschemes

$$ (X_0)_U/\pi = p_0^*(X_0/\pi) \supseteq p_1^*(X_1/\pi \setminus X_0/\pi) \supseteq \ldots \supseteq p_n^*(X_n/\pi \setminus X_{n-1}/\pi) \supseteq \ldots $$

where $p_n : (X_0)_U/\pi \to X_n/\pi$ denotes the canonical projection. Thus we obtain a strictly increasing chain of ideals

$$ 0 = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n \subseteq \ldots $$

in $\mathcal{O}_{(X_0)_U/\pi}$ where $I_n$ denotes the ideal sheaf corresponding to $p_n^*(X_n/\pi \setminus X_{n-1}/\pi)$. By construction, they all have support in $\bar{Z} = (X_0)_U/\pi$. Thus the category $\text{Coh}_{(X_0)_U/\pi}(X_0)_U$ is a non-noetherian abelian category and whose negative K-theory vanishes due to Theorem 4.3 and Theorem 4.8.

APPENDIX A. LIMITS OF LOCALLY RINGED SPACES

In this section we collect for the convenience of the reader some facts about locally ringed spaces and filtered limits of those. This is based on the exposition by Fujiwara-Kato [FK18 ch. 0, §4.2].

Definition A.1. We say that a locally ringed space $(X, \mathcal{O}_X)$ is cohesive iff its structure sheaf $\mathcal{O}_X$ is coherent.
Example A.2. For a locally noetherian scheme \( X \), an \( O_X \)-module is coherent if and only if it is finitely presented \([\text{Sta21 Tag 01XZ}]\). Thus a locally noetherian scheme is a cohesive locally ringed space.

**Lemma A.3** ([FK18 ch. 0, 4.1.8, 4.1.9]).

(i) If \( (X, O_X) \) is cohesive, then an \( O_X \)-module \( F \) is coherent if and only if it is finitely presented.

(ii) If \( f: (X, O_X) \to (Y, O_Y) \) is a morphism of locally ringed spaces and \( (X, O_X) \) is cohesive, then \( f^*: \text{Mod}(Y) \to \text{Mod}(X) \) restricts to a functor \( f^*: \text{Coh}(Y) \to \text{Coh}(X) \).

**Definition A.4.** A topological space is said to be coherent iff it is quasi-compact, quasi-separated, and admits an open basis of quasi-compact subsets. A topological space is called sober iff it is a \( T_0 \)-space and any irreducible closed subset has a (unique) generic point.

Example A.5. The underlying topological space of a quasi-compact and quasi-separated scheme is coherent and sober.

**Proposition A.6** ([FK18 ch. 0, 2.2.9, 2.2.10]). Let \( (X_i, (p_{ij})_{j \in J})_{i \in I} \) be a filtered system of topological spaces. Denote by \( X \) its limit and by \( p_i: X \to X_i \) the projection maps.

(i) Assume that the topologies of the \( X_i \) are generated by quasi-compact open subsets and that the transition maps \( p_{ij} \) are quasi-compact. Then every quasi-compact open subset \( U \subset X \) is the preimage of a quasi-compact open subset \( U_i \subset X_i \) for some \( i \in I \).

(ii) Assume that all the \( X_i \) are coherent and sober and that the transition maps \( p_{ij} \) are quasi-compact. Then \( X \) is coherent and sober and the \( p_i \) are quasi-compact.

**Proposition A.7.** Let \( (X_i, O_{X_i}, (p_{ij})_{j \in J})_{i \in I} \) be a filtered system of locally ringed spaces. Then its limit \( (X, O_X) \) in the category of locally ringed spaces exists. Let \( p_i: X \to X_i \) be the canonical projections.

(i) The underlying topological space \( X \) is the limit of \( (X_i)_{i \in I} \) in \( \text{Top} \).

(ii) \( O_X = \colim_{i \in I} p_i^{-1} O_{X_i} \) in \( \text{Mod}(X) \).

(iii) For every \( x \in X \) have \( O_{X,x} = \colim_{i \in I} O_{X_i, p_i(x)} \) in \( \text{Ring} \).

Assume additionally that every \( X_i \) is coherent and sober and that all transitions maps are quasi-compact.

(iv) The canonical functor

\[
\colim_{i \in I} \text{Mod}^b(X_i) \to \text{Mod}^b(X)
\]

is an equivalence in the 2-category of categories. In particular, for any finitely presented \( O_X \)-module \( F \) there exists an \( i \in I \) and a finitely presented \( O_{X_i} \)-module \( F_i \) such that \( F \cong p_i^* F_i \).

(v) For any morphism \( \varphi: F \to G \) between finitely presented \( O_X \)-modules there exists an \( i \in I \) and a morphism \( \varphi_i: F_i \to G_i \) between finitely presented \( O_{X_i} \)-modules such that \( \varphi \cong p_i^* \varphi_i \). Additionally, if \( \varphi \) is an isomorphism or an epimorphism, then one can choose \( \varphi_i \) to be an isomorphism or an epimorphism, respectively.

(vi) For every \( i \in I \) let \( F_i \) be an \( O_{X_i} \)-module and for every \( i \leq j \) in \( I \) let \( \varphi_{ij}: p_{ij}^* F_i \to F_j \) be a morphism of \( O_{X_j} \)-modules such that \( \varphi_{ik} = \varphi_{jk} \circ p_{jk}^* \varphi_{ij} \) whenever \( i \leq j \leq k \) in \( I \). Denote by \( F \) the \( O_X \)-module \( \colim_{i \in I} p_i^* F_i \). Then the canonical map

\[
\colim_{i \in I} H^*(X_i, F_i) \to H^*(X, F)
\]

is an isomorphism of abelian groups.

**Proof.** The existence, (i), (ii), and (iii) are \([\text{FK18 ch. 0, 4.1.10}]\) and (iv) and (v) are \([\text{FK18 ch. 0, 4.2.1–4.2.3}]\). Finally, (vi) is \([\text{FK18 ch. 0, 4.4.1}]\). \( \square \)
References

[AGH19] Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. (2019).

[Dah19] Christian Dahlhausen, *Continuous K-Theory and Cohomology of Rigid Spaces*, arXiv:1910.10437, 2019.

[EHIK21] Elden Elmanto, Marc Hoyois, Ryomei Iwasa, and Shane Kelly, *Cdh descent, cdarc descent, and Milnor excision*, Math. Ann. 379 (2021), no. 3-4, 1011–1045.

[FK18] Kazuhiro Fujiwara and Fumiharo Kato, *Foundations of Rigid Geometry I*, Monographs in Mathematics, vol. 7, EMS, 2018.

[HM10] Toshiro Hiranouchi and Satoshi Mochizuki, *Pure weight perfect modules on divisorial schemes*, Deformation spaces, Aspects Math., E40, Vieweg + Teubner, Wiesbaden, 2010, pp. 75–89.

[Ker18] Moritz Kerz, *On negative algebraic K-groups*, Proc. Int. Cong. of Math. 1 (2018), 163–172.

[KST18] Moritz Kerz, Florian Strunk, and Georg Tamme, *Algebraic K-theory and descent for blow-ups*, Invent. Math. 211 (2018), no. 2, 523–577.

[Nee21] Amnon Neeman, *A counterexample to vanishing conjectures for negative K-theory*, Invent. Math. 225 (2021), no. 2, 427–452.

[Qui73] Daniel Quillen, *Higher algebraic K-theory. I*, 85–147. Lecture Notes in Math., Vol. 341.

[RG71] Michel Raynaud and Laurent Gruson, *Critères de platitude et de projectivité. Techniques de “platification” d’un module*, Invent. Math. 13 (1971), 1–89.

[Sch04] Marco Schlichting, *Delooping the K-theory of exact categories*, Topology 43 (2004), no. 5, 1089–1103.

[Sch06] _______, *Negative K-theory of derived categories*, Math. Z. 253 (2006), no. 1, 97–134.

[Sch11] _______, *Higher algebraic K-theory*, Topics in algebraic and topological K-theory, Lecture Notes in Math., vol. 2008, Springer, Berlin, 2011, pp. 167–241.

[Sta21] The Stacks Project Authors, *Stacks Project*, http://stacks.math.columbia.edu, 2021.

[Tem11] Michael Temkin, *Relative Riemann-Zariski spaces*, Israel J. Math. 185 (2011), 1–42.

[TT90] Robert W. Thomason and Thomas Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435.

[Wei89] Charles A. Weibel, *Homotopy algebraic K-theory*, Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), Contemp. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1989, pp. 461–488.

[Wei13] _______, *The K-book*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013, An Introduction to Algebraic K-theory.

[Zar44] Oscar Zariski, *The compactness of the Riemann manifold of an abstract field of algebraic functions*, Bull. Amer. Math. Soc. 50 (1944), 683–691.

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