OBSTRUCTIONS TO COUNTABLE SATURATION IN CORONA ALGEBRAS

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Abstract. We study the extent of countable saturation for coronas of abelian C*-algebras. In particular, we show that the corona algebra of $C_0(\mathbb{R}^n)$ is countably saturated if and only if $n = 1$.

1. Introduction

Saturated models are an essential tool in model theory. For example, sufficiently saturated ‘monster’ models are universal and homogeneous, and as such provide the ambient for stability-theoretic considerations. Our motivation for studying saturation of C*-algebras (viewed as metric structures, as in [2], [15], or [14]) derives from the following two closely related facts. First, an infinite (noncompact) saturated model of cardinality (or, in the case of continuous model theory, density character $\kappa$) has $2^\kappa$ automorphisms. Second, saturated models of the same cardinality (or of the same density character) are isomorphic if and only if they are elementarily equivalent.

The corona algebra (see §1.1 below for the definition) of any $\sigma$-unital C*-algebra satisfies a weakening of countable saturation, called countable degree-1 saturation ([13], also [10, §15]). This property implies virtually all saturation-like properties of coronas used in the literature (such as the properties of ultrapowers isolated by Kirchberg in [23], and many more, see [10, §15.3 and §15.4], or [12, §5.2]). However, it is not clear whether countable degree-1 saturation suffices for construction of isomorphisms or automorphisms. Also, the Calkin algebra is not countably quantifier-free saturated, and not even countably homogeneous ([10, Example 16.1.1], [15, §4], and [13]). This makes constructing outer automorphisms of the Calkin algebra rather complicated (but not impossible, granted the Continuum Hypothesis (CH)—see [26] and [10, §17.1]). We do not know whether there exists a simple nonunital separable C*-algebra whose corona is countably saturated or at least countably homogeneous (see Question 6.2).

In [5] it was conjectured that if $A$ is a separable, nonunital C*-algebra then CH implies that the corona $Q(A)$ has $2^{2^\omega}$ automorphisms. (The other conjecture of this paper, dual to this one, according to which forcing axioms imply that $Q(A)$ has only (appropriately defined) trivial automorphisms, has been confirmed in [33].)

If $A$ is a separable nonunital C*-algebra whose corona is countably saturated, then $A$ clearly satisfies the first conjecture, but very few coronas of separable C*-algebras are known to be countably saturated. By [17, Theorem 1.5], if $A_n$ is a
sequence of unital C*-algebras then the corona algebra of $\bigoplus_n A_n$ is countably saturated. This, together with a generalization of the Feferman–Vaught theorem, was used in [19] and [18] (see also [25] and [33]) to find sequences of pairwise nonisomorphic simple unital separable C*-algebras $A_n$ and $B_n$ such that the isomorphism of coronas of $\bigoplus A_n$ and $\bigoplus B_n$ is independent from ZFC; see [12] for the current state of the art on the conjectures of [5]. Under CH, a countably saturated structure of density character $2^{\aleph_0}$ is isomorphic to an ultrapower. Thus countable saturation of reduced products was used in [11] to prove that the quotient map from $\ell_\infty(A)/c_0(A)$ to a (norm) ultrapower $A_U$ associated with a nonprincipal ultrafilter $U$ on $\mathbb{N}$ has a right inverse if CH holds. This implies (in ZFC) a transfer theorem for realization of morphisms with respect to classification functors ([11, Theorem A]).

Our main interest in the present paper is in the saturation of coronas of abelian C*-algebras, a study initiated in [6]. If $X$ is a locally compact1, noncompact, space that is an increasing union of compact subsets $K_n$ such that $\sup_n |\partial K_n| < \infty$, where $\partial$ denotes the topological boundary, then the corona algebra of $C_0(X)$ is countably saturated ([17, Theorem 2.5]). This covers all known examples of countably saturated coronas of abelian C*-algebras. The coronas of the C*-algebras of the form $A = C_0([0, \infty)), B)$ for a separable, unital, and infinite-dimensional C*-algebra $B$ need not be countably quantifier-free saturated. (See Question 6.1, also [10, Exercise 16.8.36]; the point of the latter is that the Murray-von Neumann semigroup of $Q(A)$ is isomorphic to that of $B$, and it can therefore be countably infinite.)

Since the corona algebra of $C_0(\mathbb{R})$ is countably saturated, CH implies that it has $2^{2^{\aleph_0}}$ automorphisms. By an intricate construction in [32] it was proved that under CH the corona algebra of $C_0(\mathbb{R}^n)$, for $n \geq 2$ (as well as many other coronas with similar properties) also has $2^{2^{\aleph_0}}$ automorphisms. The question whether these coronas are countably saturated remained open until now.

**Theorem 1.** Suppose that $D$ is an infinite discrete space and that $Y$ is locally compact, $\sigma$-compact, noncompact, and connected. If $D \times Y$ embeds as a closed subspace in a locally compact space $X$, then the corona algebra of $C_0(X)$ is not countably quantifier-free saturated.

**Corollary 2.** Let $X$ be a locally compact space in which $\mathbb{R}^2$ embeds as a closed subspace. Then $C_0(X)$ is not countably saturated. In particular, for every $n \geq 1$ the following are equivalent.

1. The corona algebra of $C_0(\mathbb{R}^n)$ is countably saturated.
2. The corona algebra of $C_0(\mathbb{R}^n)$ is countably quantifier-free saturated.
3. $n = 1$.

The question whether, for C*-algebras, countable saturation is equivalent to countable quantifier-free saturation was raised in [6, Question 6.5]. While Corollary 2 gives some support for the positive answer, by Proposition 6.3 below, the answer is in general negative.

The paper is organised as follows. In §2 we give a rather elementary topological proof of Theorem 1 and in §3 we state some corollaries and limiting examples. In §4 we introduce the notion of definable homotopy and give another proof of the same theorem. This, a bit more involved proof, is more likely to be adaptable to a more

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1 All spaces in this note are Hausdorff.
general case. In §3 we show that the corona algebra of a direct sum \( \bigoplus_n A_n \) of a sequence of nonunital \( C^* \)-algebras is not necessarily countably saturated, contrast-
ing the result of [17] Theorem 1.5] mentioned earlier. In §4 we answer a question of Eagle and the second author (§3 Question 6.5]), by finding a family of \( C^* \)-algebras that are countably quantifier-free saturated but not countably saturated.

We assume familiarity with logic of metric structures, as applied to \( C^* \)-algebras ([14]: the relevant definitions of a condition, type, and saturation can be found in

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\section*{2. An Elementary Proof}

The proof of Theorem 2.1 is based on an idea of the proof of [17] Theorem 2.2].

\textbf{Theorem 2.1.} Suppose that \( X \) is a locally compact and \( \sigma \)-compact space such that there are infinitely many closed disjoint connected subsets \( Y_n \) of \( X \) with none of them compact and such that each compact subset of \( X \) intersects only finitely many \( Y_n \)’s. Then the corona algebra of \( C_0(X) \) is not countably quantifier-free saturated.

\textbf{Proof.} Since \( X \) has a noncompact closed subset, it is not compact either. Let \( C_b(X) \) denote the \( C^* \)-algebra of bounded continuous complex-valued functions on \( X \). We can identify \( \mathcal{Q}(C_0(X)) \) with \( C_b(X)/C_0(X) \). Let \( \pi: C_b(X) \to \mathcal{Q}(C_0(X)) \) denote the quotient map. For brevity, for \( c \in C_b(X) \) we write \( \hat{c} = \pi(c) \).

Write \( X \) as an increasing union of compact subspaces, \( X_n = \bigcup_n K_n \). First find open sets \( U_n, V_n, W_n \) such that for all \( n \) the following conditions hold.

1. \( Y_n \subseteq U_n \subseteq \overline{V_n} \subseteq V_n \subseteq \overline{W_n} \subseteq W_n \),
2. all the sets \( \overline{W_n} \) are disjoint,
3. \( K_m \cap W_n = \emptyset \) if \( K_m \cap Y_n = \emptyset \) for all \( m \) and \( n \),
4. the sets \( U_n \) are connected.

For this, first use Uryshon’s Lemma to obtain \( U_n', V_n' \) and \( W_n \) satisfying (1), (2), and (3). Then let \( U_n \) be the connected component of \( U_n' \) that contains \( Y_n \).

We will need positive contractions \( a_n \in C_b(X) \) satisfying the following.

5. \( a_n(t) = 1 \) for \( t \in Y_n \),
6. \( a_n(t) \geq \frac{2}{3} \) for \( t \in \overline{Y_n} \),

\[ \text{An ideal } A \subseteq B \text{ is essential if } A^\perp = \{ b \in B \mid bA = Ab = 0 \} \text{ is 0} \]
7. \( a_n(t) \leq \frac{2}{3} \) for \( t \in X \setminus U_n \), and
8. \( \text{supp}(a_n) \subseteq V_n \).

In order to find \( a_n \), using the Tietze extension theorem first fix a continuous \( a_0^1: X \to [0, \frac{2}{3}] \) such that \( \text{supp}(a_0^1) \subseteq V_n \) and \( a_n^0(t) = \frac{2}{3} \) for \( t \in U_n \). Then find continuous \( a_n^1: X \to [0, \frac{1}{3}] \) such that \( \text{supp}(a_n^1) \subseteq U_n \) and \( a_n^1(t) = \frac{1}{3} \) for all \( t \in Y_n \) and let \( a_n = a_n^0 + a_n^1 \).

Since on every compact subset of \( X \) at most finitely many of the summands are nonempty and all the \( a_i \)'s are orthogonal, \( a = \sum_j a_j \) is in \( C_0(X) \). Let \( b_n \in C_0(X) \) be a positive contraction such that \( b_n(t) = 1 \) for all \( t \in V_n \) and \( \text{supp}(b_n) \subseteq W_n \). Notice that \( a_n b_n = a_n \) and \( a_m b_n = 0 \) for all \( n \) and all \( m \neq n \). In particular \( b_n a = b_n^0 a = a_n \) for all \( n \).

Consider the type \( t(x) \) over \( Q(X) \) with the conditions
\[
\|x \hat{a}\| = 1, x \geq 0, (x - x^2) \hat{a} = 0, \text{ and } x \hat{a}_n = 0 \text{ for all } n.
\]

It will suffice to show that \( t(x) \) is consistent, but not realized in \( Q(X) \).

If \( t_0 \) is a finite subset of \( t \), fix \( n \) such that \( \hat{a}_n \) does not appear in \( t_0 \). Then \( b_n \) realizes \( t_0 \) (it realizes every condition of \( t \) except \( x \hat{a}_n = 0 \)), therefore \( t \) is consistent.

It remains to prove that \( t \) is not realized in \( Q(X) \). Assume otherwise, fix \( c \in C_0(X) \) such that \( \hat{c} \) realizes \( t \). Since \( (\hat{c} - c^2)\hat{a} = 0 \), we can fix \( m \) such that \( ((\hat{c} - c^2)\hat{a})(t) < \frac{1}{6} \) for all \( t \in X \setminus K_m \). Fix for a moment \( t \in \bigcup_n U_n \setminus K_m \). Then \( \frac{1}{6} > (\hat{c} - c^2)(t) \geq \frac{1}{3}(c - c^2)(t) \). Therefore \( (c - c^2)(t) < 1/4 \) and so
\[
c(t) \neq 1/2 \text{ for all } t \in \bigcup_n U_n \setminus K_m.
\]

In addition, by the assumption that each \( K_m \) intersects only finitely many \( Y_n \) nontrivially and \( \{0\} \), the set
\[
F = \{ n | U_n \cap K_m \neq \emptyset \}
\]
is finite. Since \( \hat{c} \hat{a}_n = 0 \) for all \( n \in F \), there is \( m' \geq m \) such that for all \( n \in F \) we have \( (\hat{c} \hat{a}_n)(t) < 1/2 \) for all \( t \in X \setminus K_{m'} \). Since \( \|\hat{c} \hat{a}\| = 1 \), we can find \( s_0 \in X \setminus K_{m'} \) such that \( \frac{2}{3} < (\hat{c} \hat{a})(s_0) \). This implies \( \frac{2}{3} < (c \hat{a})(s_0) \), and since \( \{ s | c \hat{a}(s) > \frac{2}{3} \} \subseteq \{ s | a(s) > \frac{2}{3} \} \subseteq \bigcup U_n \) we have \( s_0 \in \bigcup U_n \). Fix \( n \) such that \( s_0 \in U_n \). Then \( n \notin F \) and therefore \( U_n \cap K_m = \emptyset \).

Since \( \hat{c} \hat{a}_n = 0 \), some \( s_1 \in Y_n \) satisfies \( \frac{1}{3} > c(s_1) \); if not, \( c \hat{a}_n(s) \geq \frac{1}{3} \) for all \( s \in Y_n \), and therefore, as \( Y_n \) is not compact, \( \|c \hat{a}_n\| \geq \frac{1}{2} \). But \( Y_n \subseteq U_n \), hence \( c \) attains values both above and below \( \frac{1}{2} \) on the connected space \( U_n = U_n \setminus K_m \). Hence some \( t \in U_n \) satisfies \( c(t) = \frac{1}{2} \); contradiction.

\[ \square \]

3. Corollaries and a Limiting Example

Theorem 2.1 resolves the question of countable saturation for many, but not all, coronas of abelian \( C^* \)-algebras.

**Proof of Theorem 3.1.** Suppose that \( D = \{ d_n \} \) is an infinite discrete set, that \( Y \) is locally compact, \( \sigma \)-compact, noncompact, and connected, and that \( D \times Y \) embeds as a closed subspace of a locally compact space \( X \). By discarding the extras, we can assume \( D \) is countable and enumerate it as \( d_n \), for \( n \in \mathbb{N} \). Let \( Y_n = \{ d_n \} \times Y \). To check the assumptions of Theorem 2.1, suppose \( K \) is a compact subset of \( X \). Then \( K \cap (D \times Y) \) is compact, and therefore \( K \cap Y_n \) is nonempty for at most finitely many \( n \). By Theorem 2.1, the corona algebra of \( C_0(X) \) is not countably quantifier-free saturated. \[ \square \]
**Proof of Corollary 3.2.** The corona algebra of $C_0(\mathbb{R})$ is countably saturated by [17, Theorem 2.5].

It remains to prove that the corona algebra of $C_0(\mathbb{R}^n)$ for $n \geq 2$ is not countably quantifier-free saturated. Since $\mathbb{N} \times \mathbb{R}$ is a closed subset of $\mathbb{R}^2$ (and in turn of $\mathbb{R}^n$ for all $n \geq 2$), if a locally compact space $X$ has $\mathbb{R}^2$ as a closed subspace then Theorem 1 implies that the corona of $C_0(X)$ is not countably quantifier-free saturated. □

The proof of Corollary 2 may seem to suggest that if $X$ and $X'$ are locally compact, noncompact spaces such that $X$ is a closed subspace of $X'$ and the corona algebra $C_0(X)$ is not countably quantifier-free saturated, then the corona algebra of $C_0(X')$ is not countably quantifier-free saturated. It is however not clear whether this is true in general.

If $Y$ is sufficiently connected at infinity, the space $D$ in Theorem 1 can be assumed to be locally compact instead of discrete.

**Corollary 3.1.** Suppose that $D$ is an infinite locally compact space and that $Y$ is locally compact and noncompact. Suppose furthermore that $Y$ can be written as an increasing union of compact sets $K_n$ such that $Y \setminus K_n$ is connected for all $n$, and that $D \times Y$ embeds as a closed subspace in a locally compact space $X$. Then the corona algebra of $C_0(X)$ is not countably quantifier-free saturated.

**Proof.** Choose an infinite discrete subset $\{d_n \mid n \in \mathbb{N}\}$ of $D$. Then the subspaces $Y_n = \{d_n\} \times (Y \setminus K_n)$ of $X$ are closed, connected, and disjoint, and $\bigcup_n Y_n$ is a closed subspace of $X$. If $K \subset X$ is compact, then so is $K \cap (D \times Y)$ and therefore the latter set is included in some $D \times K_n$. Hence $K$ intersects at most finitely many $Y_n$ nontrivially. By Theorem 2.1, the corona algebra of $C_0(X)$ is not countably quantifier-free saturated. □

The conclusion of Corollary 3.1 may fail if $Y$ is not assumed to be $\sigma$-compact. This is Corollary 3.3, an easy consequence of the following.

**Proposition 3.2.** There is a locally compact connected space $Y$ such that $B \cong Q(C_0(Y,B))$ for every unital C*-algebra $B$.

**Proof.** Ignoring the connectedness requirement for a moment, consider $\aleph_1$ with the order topology. Then the multiplier algebra of $A = C_0(\aleph_1, B)$ can be identified with $C_0(\aleph_1, B)$ (the C*-algebra of bounded continuous functions from $\aleph_1$ into $B$); this is a very easy case of [1, Theorem 3.3]. Every continuous function $f$ from $\aleph_1$ into a metric space is eventually constant. We include a proof of this well-known fact. For $\alpha < \aleph_1$ let $\varepsilon(\alpha)$ denote the diameter of $\{f(\beta) \mid \alpha \leq \beta < \aleph_1\}$. Since $\alpha \mapsto \varepsilon(\alpha)$ is a nonincreasing function, if $f$ is not eventually constant, then there is $\delta > 0$ such that $\varepsilon(\alpha) \geq \delta$ for all $\alpha$. We can now recursively find two increasing sequences of ordinals $\alpha_n$ and $\beta_n$ such that $\sup_n \alpha_n = \sup_n \beta_n$ and $d(f(\alpha_n), f(\beta_n)) > \delta/2$. Thus $f$ is discontinuous at $\sup_n \alpha_n$; contradiction.

Therefore the diagonal map of $B$ into $C_0(\aleph_1, B)/C_0(\aleph_1, B)$ is surjective. Since it is clearly injective, $B$ is isomorphic to the corona.

In order to prove that $Y$ can be chosen to be connected, let $Y$ be the so-called long line, defined as follows. To the space $\aleph_1$, between every ordinal $\alpha$ and its successor $\alpha + 1$, attach a copy of $[0,1]$ (with the endpoints identified with $\alpha$ and $\alpha + 1$). Every $f : Y \to B$ is eventually constant, via a proof analogous to the corresponding fact for $f : \aleph_1 \to B$. As before, for $A = C_0(Y,B)$ this implies that the diagonal map of $B$ into $Q(A)$ is an isomorphism. □
Corollary 3.3. There are an infinite compact $K$ and a connected, locally compact, noncompact $Y$ such that the corona algebra of $C_0(K \times Y)$ is countably saturated.

Proof. Take $Y$ to be the long line. For every unital abelian $C^*$-algebra $C(K)$, the corona algebra of $C_0(Y, C(K)) \cong C_0(Y \times K)$ is by Proposition 3.2 isomorphic to $C(K)$. If $K = \beta\mathbb{N} \setminus \mathbb{N}$ then $C(K)$ is countably saturated by [6], or by the fact that it is the reduced product $\ell_\infty/c_0$ and therefore countably saturated by [10, Theorem 16.5.1].

One can ask for the space $K$ in Corollary 3.3 to be connected. For this, just consider the spectrum of any nontrivial ultrapower of $C([0, 1])$, as in this case $C(K)$ is countably saturated.

An open problem due to Sakai related to Proposition 3.2 is whether every unital (abelian) $C^*$-algebra is isomorphic to the corona of a simple $C^*$-algebra (see [28, Questions 5–7], also [12, Question 3.21]). The simplicity requirement suggests that some form of Proposition 3.2 was known to Sakai, although it does not appear in [28] explicitly. In [28] Sakai shows that if $L$ is a maximal left ideal in a II$_1$ factor then the corona of $L \cap L^*$ is one-dimensional. An extension of $K(\ell_2(2^{\aleph_0}))$ by $K(\ell_2)$ whose corona is one-dimensional was constructed in [20].

If a locally compact, noncompact, space $X$ can be written as an increasing union of compact subsets $K_n$ such that $\sup_n |\partial K_n| < \infty$ then (by [17, Theorem 2.5]) $C_0(X)/C_0(\mathbb{R}^3)$ is countably saturated. There is however an example of a locally compact subspace of $\mathbb{R}^3$ without this property that does not satisfy the assumptions of Theorem 2.1 either, communicated to us by Logan Hoehn.

4. A HOMOTOPIY PROOF

In this section we combine model-theoretic definability (see [2, §9] and [13, §3]) with some homotopy considerations. This is rather unusual, because in general it is impossible to describe the existence of a continuous path by a first-order formula. For example, the unitary group of $C([0, 1])$ is path-connected, while the unitary group of its ultrapower has $2^{\aleph_0}$ connected components. The proof of Corollary 3.3 obtained here is a bit longer than the one in [2], but its ideas are likely to be more relevant to analyzing the saturation of coronas of simple $C^*$-algebras.

On the set of $k$-tuples of a $C^*$-algebra $A$ we consider the max norm and write

$$
(4.1) \quad \|a - b\| = \max_{j<k} \|a_j - b_j\|.
$$

A modulus of uniform continuity is a nondecreasing $\Delta: (0, 1) \to (0, 1]$ such that $\lim_{t \to 0^+} \Delta(t) = 0$. If $A$ is a $C^*$-algebra, by $Z^A(\varphi)$ we denote the zero-set of $\varphi$ as computed in $A$. Given a theory $T$, we will say that the zero-set of $\varphi(\bar{x})$ is definable in $T$ if there exist $\delta > 0$ and a modulus of uniform continuity $\Delta_\varphi$ such that $\varphi^A(\bar{b}) < \delta$ implies $\text{dist}(\bar{a}, Z(\varphi)) < \Delta_\varphi(\varphi^A(\bar{a}))$ for every model $A$ of $T$ ([2, §9]). If $\mathcal{E}$ is the class of all models of $T$, we say that the zero-set of $\varphi(\bar{x})$ is definable relative to $\mathcal{E}$. In our applications $\mathcal{E}$ will be the class of abelian $C^*$-algebras.

Two elements of $Z^A(\varphi)$ are said to be homotopic if there is a continuous function $g: [0, 1] \to Z^A(\varphi)$ such that $g(0) = \bar{a}$ and $g(1) = \bar{b}$. The homotopy classes in $Z^A(\varphi)$ are denoted $[\bar{a}]_h$, and we write

$$
Z^A(\varphi)/\sim_h
$$

for the quotient space.
**Definition 4.1.** Let $\mathcal{E}$ be an axiomatizable class of $C^*$-algebras, and let $\varphi(\bar{x})$ be a nonnegative formula in variables $\bar{x} = (x_0, \ldots, x_{k-1})$ whose zero-set is a definable predicate in $\mathcal{E}$, witnessed by the modulus of uniform continuity $\Delta_{\varphi}$. The formula $\varphi$ is said to admit a continuous Skolem function in $\mathcal{E}$ if there exist a term $g_{\varphi}(\bar{x})$ in the language of $C^*$-algebras and $\delta > 0$ such that, for every $B \in \mathcal{E}$,

$$
\varphi^B(\bar{b}) < \delta \implies \varphi^B(g_{\varphi}(\bar{b})) = 0 \quad \text{and} \quad \|\bar{b} - g_{\varphi}(\bar{b})\| < \Delta_{\varphi}(\varphi^B(\bar{b})).
$$

The term $g_{\varphi}$ is said to be the Skolem function for $\varphi$.

Some formulas that admit continuous Skolem functions are the natural formulas whose zero-sets are the set of projections and the set of unitaries (proofs use the continuous functional calculus, see [14, §3]), as well as the example from Lemma 4.6 below.

**Proposition 4.2.** Suppose that $\mathcal{E}$ is an axiomatizable class of $C^*$-algebras, and that $\varphi$ admits a continuous Skolem function in $\mathcal{E}$. Then there exists $\varepsilon > 0$ such that for every $B \in \mathcal{E}$ and any two $\bar{a}$ and $\bar{b}$ in $Z^B(\varphi)$, $\|\bar{a} - \bar{b}\| < \varepsilon$ implies that $\bar{a}$ and $\bar{b}$ are homotopic.

**Proof.** Let $g_{\varphi}$ be the Skolem function for $\varphi$ as in Definition 4.1. As $\varphi$, being a formula, has its own modulus of uniform continuity (Theorem 3.5), we can fix $\varepsilon > 0$ small enough so that for every structure $B$ in $\mathcal{E}$ and all $\bar{x}$ and $\bar{y}$ in $B$ of the appropriate sort we have that $\|\bar{x} - \bar{y}\| < \varepsilon$ implies $|\varphi^B(\bar{x}) - \varphi^B(\bar{y})| < \delta$.

Fix $B \in \mathcal{E}$. First, notice that by the properties of $g_{\varphi}$, $\varphi^B(\bar{a}) = 0$ implies $g_{\varphi}(\bar{a}) = \bar{a}$ for all $\bar{a}$ of the proper arity. Fix $\bar{a}$ and $\bar{b}$ in $Z^B(\varphi)$ such that $\|\bar{a} - \bar{b}\| < \varepsilon$. For $t \in [0, 1]$ define $c_{t,j} := t a_j + (1-t)b_j$, for $j < k$. Let $\bar{c}_t = (c_{t,0}, \ldots, c_{t,k-1})$. Then $\|\bar{a} - \bar{c}_t\| < \|\bar{a} - \bar{b}\| < \varepsilon$, and therefore $\varphi^B(\bar{c}_t) < \delta$ for all $t$. The map $t \mapsto g_{\varphi}(\bar{c}_t)$ is a continuous path in $Z^B(\varphi)$ connecting $\bar{a}$ and $\bar{b}$. \(\square\)

For a $\bar{a} = (a_0, \ldots, a_{k-1})$ in a unital $C^*$-algebra $A$, by $\text{jsp}(\bar{a})$ we denote its joint spectrum. This is the set of tuples $\lambda$ in $C^k$ such that $\bar{a} - \lambda$ generates a proper ideal in $A^k$. If $\bar{a}$ is a tuple of commuting normal operators, then $C^*(\bar{a}, 1) \cong C(\text{jsp}(\bar{a}))$. The following definition generalizes that of universal $C^*$-algebra given by generators and relations (see [4, §§II.8.3] or [10, §2.3]), in the abelian setting.

**Definition 4.3.** Let $k \in \mathbb{N}$ and let $X \subseteq \mathbb{R}^k$ be compact. Let $\varphi(\bar{x})$ be a formula in $\bar{x} = (x_0, \ldots, x_{k-1})$. We say that $\varphi$ is controlled by $X$ (or that $X$ controls $\varphi$) if for every $C^*$-algebra $B$ and every $k$-tuple $\bar{a}$ in $B$ we have $\varphi^B(\bar{a}) = 0$ if and only if $\bar{a}$ is a $k$-tuple of commuting self-adjoint operators such that $\text{jsp}(\bar{a}) \subseteq X$.

In the following Lemma and elsewhere, an *-homomorphism $\Phi: A \to B$ is canonically extended to all finite powers $A^n$ and if $\bar{a} = (a_0, \ldots, a_{n-1})$ then we write $\Phi(\bar{a})$ for $(\Phi(a_0), \ldots, \Phi(a_{n-1}))$.

**Lemma 4.4.** If $\varphi(\bar{x})$ is a formula controlled by a compact $X_\varphi$ and $\Phi: A \to B$ is an *-homomorphism between $C^*$-algebras, then $\Phi[Z^A(\varphi)] \subseteq Z^B(\varphi)$.

**Proof.** Fix $\bar{a} \in Z^A(\varphi)$. Since $\bar{a}$ is a tuple of commuting self-adjoints, so is $\Phi(\bar{a})$. By the Spectral Mapping Theorem, for every *-homomorphism $\Phi$ and every $f \in C(X_\varphi)$ we have $\Phi(f(\bar{a})) = f(\Phi(\bar{a}))$ (see e.g. [4, II.1.5.2]). In particular, since $\bar{r} \in \text{jsp}(\bar{a})$ if and only if $\sum_j (a_j - r_j)^2$ is not invertible, we have that $\text{jsp}(\Phi(\bar{a})) \subseteq \text{jsp}(\bar{a})$, which gives the thesis. \(\square\)

Before we continue the general analysis, an example could be helpful.
Lemma 4.5. For $n \geq 1$ consider the formula

$$\varphi_n = \max \left( \max_{j<n} \|x_j - x_j^*\|, \left\| \sum_{j<n} x_j^2 - 1 \right\| \right).$$

Then the following holds.

1. $\varphi_n$ admits a continuous Skolem function in the class of abelian $C^*$-algebras and

2. $\varphi_n$ is controlled by $S^{n-1}$, the $n-1$-dimensional unit sphere.

Proof. Fix $n \geq 1$. We first prove (1). Let $\varepsilon > 0$ be small enough ($\varepsilon < 1/(5n)$) and suppose that in some abelian $C^*$-algebra $B$ we have a tuple $\tilde{a}$ such that $\varphi_n(B)(\tilde{a}) < \varepsilon$. Set $b_j = (a_j + a_j^*)/2$ and $b = \sum b_j^2$. Since $\|\sum a_j\| \leq 1 + \varepsilon$, we have $\|b\| \leq 1 + \varepsilon$. Furthermore, since $\|1 - b\| < 1$, $b$ is invertible. With $c_j = b_j b^{-1}/2$ we have $\varphi_n(B)(\tilde{c}) = 0$. Also, $\|a - c\| \leq \|a - b\| + \|b\|1 - b^{-1}\|$ can be made arbitrarily small by choosing a small $\varepsilon > 0$.

Let $f_j(\bar{x}) := \frac{1}{2}(x_j + x_j^*)$, and let

$$\tilde{f}_{n,j}(\bar{x}) := \frac{1}{2}(x_j + x_j^*) \left( \sum_{j < n} f_j(\bar{x})^2 \right)^{-1}$$

The argument above shows that the function $g_n$ given by

$$g_n(\bar{x}) := (\tilde{f}_{n,0}(\bar{x}), \ldots, \tilde{f}_{n,n-1}(\bar{x}))$$

is a Skolem function for $\varphi_n$.

2 Let $B$ be a $C^*$-algebra, and let $\tilde{a}$ be a tuple in $B$ (of the appropriate length). Then $\varphi_n(B)(\tilde{a}) = 0$ asserts exactly that $\tilde{a}$ is a tuple of commuting self-adjoints and that the joint spectrum of $\tilde{a}$ is contained in the unit sphere $S^{n-1}$. This is precisely stating that $\varphi_n$ is controlled by $S^{n-1}$. □

The reason for restricting our attention to abelian $C^*$-algebras is that, rather inconveniently for our purposes, the zero-set of $\varphi_n$ is not definable in the class of all $C^*$-algebras. This is because the algebra $C(S^{n-1})$, for $n \geq 3$, is not weakly semiprojective (see [29] and the introduction to [8]). This means that there is an ultraproduct of $C^*$-algebras $\prod_{j} A_j$ and an embedding of $C(S^{n-1})$ into it that cannot be lifted to an embedding of $C(S^{n-1})$ into any of the $A_j$. However, the machinery of this section may be applicable to other weakly stable formulas (see [24]).

We are ready to introduce the main technical tool of this section. Let $\text{End}^1(A)$ denote the semigroup of unital endomorphisms of a unital $C^*$-algebra $A$, taken with respect to composition. Suppose that $A$ is finitely generated and fix a tuple of generators $\tilde{a}$. Thus $\alpha \in \text{End}^1(A)$ is uniquely determined by $\alpha(\tilde{a})$. Equip $\text{End}^1(A)$ with the metric defined by (using (4.1))

$$d(\alpha, \beta) = \|\alpha(\tilde{a}) - \beta(\tilde{a})\|$$

This metric endows $\text{End}^1(A)$ with a natural homotopy relation $\sim_h$.

Suppose that a formula $\varphi$ is controlled by a compact $X_\varphi \subseteq \mathbb{R}^k$ for some $k \geq 1$. Write, for brevity,

$$A_\varphi := C(X_\varphi).$$

Let $\tilde{a}$ denote the canonical generators of the universal $C^*$-algebra $A_\varphi$, corresponding to the free variables of $\varphi$. Suppose that $B$ is a unital $C^*$-algebra such that $Z^B(\varphi)$ is nonempty. Fix $\tilde{b} \in Z^B(\varphi)$. Since $X_\varphi$ controls $\varphi$ we have $\text{jsp}(\tilde{b}) \subseteq X_\varphi$. This
inclusion induces a \( \ast \)-homomorphism \( \Psi_b : A_\varphi \to C^*(\tilde{b}) \) such that \( \Psi_b(\tilde{a}) = \tilde{b} \). For \( \alpha \in \text{End}^1(A_\varphi) \) let
\[
(4.2) \quad \alpha\tilde{b} = \Psi_b(\alpha(\tilde{a})).
\]
Applying Lemma 4.4 twice, we have \( \varphi(\alpha\tilde{b}) = \varphi(\alpha(\tilde{a})) = 0 \). Thus \( (4.2) \) defines a continuous action of \( \text{End}^1(A_\varphi) \) on \( Z^B(\varphi) \). We call this action natural.

An action of \( \text{End}^1(A_\varphi) \) on \( Z^B(\varphi) \) is said to be continuously implemented if there is a continuous map
\[
(4.3) \quad \Phi : \text{End}^1(A_\varphi) \times Z^B(\varphi) \to Z^B(\varphi)
\]
such that \( \alpha\tilde{b} = \Phi(\alpha, \tilde{b}) \). Our interest in continuously implemented actions stems from the fact that they preserve homotopy in \( Z^B(\varphi) \).

The natural action of \( \text{End}^1(A_\varphi) \) on some \( Z^B(\varphi) \) is standard if it is continuously implemented and there is a surjection
\[
(4.4) \quad \Psi : Z^B(\varphi)/\sim_h \to Z^A(\varphi)/\sim_h
\]
which is equivariant, i.e., \( \alpha \Psi(\tilde{b}) = \Psi(\alpha\tilde{b}) \).

If \( \text{End}^1(A) \) is infinite, then its natural action on the zero-set of \( \varphi \) in the ultrapower of \( A \) is typically nonstandard, by countable saturation. The contrapositive of this fact serves as the main idea of the proofs in this section.

**Definition 4.6.** Suppose that \( \mathcal{E} \) is an axiomatizable class of \( C^* \)-algebras, that \( \varphi(\bar{x}) \) is a nonnegative formula in variables \( \bar{x} = (x_0, \ldots, x_{k-1}) \). We say that \( \varphi \) admits definable homotopy in \( \mathcal{E} \) if
- \( \varphi \) is controlled by some compact \( X_\varphi \subseteq \mathbb{R}^k \),
- \( A_\varphi = C(X_\varphi) \) is in \( \mathcal{E} \), and
- \( \varphi \) admits a continuous Skolem function in \( \mathcal{E} \).

**Proposition 4.7.** Suppose that a formula \( \varphi \) admits definable homotopy in an axiomatizable class of \( C^* \)-algebras \( \mathcal{E} \). If \( B \) is a \( C^* \)-algebra in \( \mathcal{E} \), then the natural action of \( \text{End}^1(A_\varphi) \) on \( Z^B(\varphi) \) is continuously implemented.

**Proof.** Let \( \bar{a} = (a_0, \ldots, a_{k-1}) \) be the generators of \( A_\varphi \), so that \( X_\varphi \) is homeomorphic to \( \text{jsp}(\bar{a}) \) and \( A_\varphi \cong C(\text{jsp}(\bar{a})) \). Fix for a moment \( \alpha \in \text{End}^1(A_\varphi) \). Then \( \alpha \) induces a continuous map \( f_\alpha : X_\varphi \to X_\varphi \) naturally identified with a \( k \)-tuple \( \bar{f}_\alpha = (f_{\alpha,j} : j < k) \) in \( C(X_\varphi) \) such that \( \alpha(a_j) = f_{\alpha,j} \) for \( j < k \). We will summarize this situation by
\[
\alpha(\bar{a}) = \bar{f}_\alpha.
\]
The correspondence \( \alpha \mapsto \bar{f}_\alpha \) is a homeomorphism between \( \text{End}^1(A_\varphi) \) and the space of continuous functions from \( X_\varphi \) into \( X_\varphi \). Lemma 4.4 implies that if \( \beta \in \text{End}^1(A_\varphi) \) then \( \beta(f_\alpha) = \bar{f}_{\beta\alpha} \). Therefore the natural action of \( \text{End}^1(A_\varphi) \) on \( Z^B(\varphi) \) (see \( (4.2) \)) satisfies
\[
\alpha\tilde{b} = \bar{f}_{\alpha}(\tilde{b}).
\]
This action is clearly continuous, and it therefore preserves homotopy classes. Thus \( \Phi(\alpha, b) = \bar{f}_\alpha(\tilde{b}) \) is the required continuous functions. \( \square \)

For a unital \( C^* \)-algebra \( A \) we write
\[
(4.5) \quad A_\infty := C_b([0, \infty), A)/C_0([0, \infty), A).
\]

**Lemma 4.8.** Suppose that a formula \( \varphi \) admits definable homotopy in an axiomatizable class of \( C^* \)-algebras \( \mathcal{E} \). Then
1. the natural action of \( \text{End}^1(A_\varphi) \) on \( Z^{A_\varphi}(\varphi) \) is standard;
2. if \( B \) is a unital \( C^* \)-algebra in \( \mathcal{E} \) and the action of \( \text{End}^1(A_\varphi) \) on \( Z^B(\varphi) \) is standard, then the natural action of \( \text{End}^1(A_\varphi) \) on \( Z^{B=}(\varphi) \) is standard; and
3. the action of \( \text{End}^1(A_\varphi) \) on \( Z^{(A_\varphi)=}(\varphi) \) is standard.

**Proof.** By Proposition 4.7 each of the actions in (1)–(3) is continuously implemented. We fix \( \Phi \) as in (4.3) for each one of them and prove that \( \Psi \) as in (4.3) exists.

For (1), take \( \Psi = \text{id}_{Z^{A_\varphi}(\varphi)/\sim_h} \).

2. Fix \( \Phi : \text{End}^1(A_\varphi) \times Z^B(\varphi) \to Z^B(\varphi) \) and \( \Psi : Z^B(\varphi)/\sim_h \to Z^{A_\varphi}(\varphi)/\sim_h \) as in (4.3) and (4.4).

Let \( \delta > 0 \) and a continuous Skolem function \( g_\varphi \) for \( \varphi \) be as in Definition 4.1. By Proposition 4.2 there is \( \varepsilon > 0 \) such that for every unital abelian \( C^* \)-algebra \( D \) and any two \( \bar{a} \) and \( \bar{b} \) in \( Z^B(\varphi) \), \( \| \bar{a} - \bar{b} \| < \varepsilon \) implies that \( \bar{a} \) and \( \bar{b} \) are homotopic. Fix \( 0 < \delta_0 < \delta \) such that \( \varphi(\bar{a}) < \delta_0 \) implies \( \| g_\varphi(\bar{a}) - \bar{a} \| < \varepsilon/3 \).

Let \( C := B_{\infty, \bar{c}} \) and denote by \( \pi \) the canonical quotient map \( \pi : C_{\bar{c}}([0, \infty), B) \to C \).

Fix \( \bar{b} \in C_{\bar{c}}([0, \infty), B) \) such that \( \pi(\bar{b}) \in Z^C(\varphi) \). Hence for some \( t_0 \) and all \( t \geq t_0 \) we have \( \varphi(\bar{b}(t)) = \delta_0 \). Let \( \bar{b}'(t) = g_\varphi(\bar{b}(\min(t, t_0))) \). Then \( \bar{b}' \in C_{\bar{c}}([0, \infty), B) \) and \( \pi(\bar{b}') \in Z^C(\varphi) \). Also, \( \pi(\bar{b}) = \pi(\bar{b}') \), and for all \( t, \bar{b}'(t) \) belongs to the homotopy class of \( \bar{b}'(0) \); for a moment let \( \Theta_0(\bar{b}) = \bar{b}'(0) \).

**Claim 4.9.** If \( \pi(\bar{b}) \sim_h \pi(\bar{c}) \), then \( \Theta_0(\bar{b}) \sim_h \Theta_0(\bar{c}) \).

**Proof.** If \( t \) is large enough to have \( \max(\varphi(b(t)), \varphi(c(t))) < \delta_0 \) and \( |b(t) - c(t)| < \varepsilon/3 \), then (by the choice of \( \delta_0 \)) we have \( \| b'(t) - c'(t) \| < \varepsilon \) and therefore \( b'(t) \sim_h c'(t) \).

By the continuity of \( b' \) and \( c' \), \( b'(s) \sim_h c'(s) \) for all \( s \) and \( s' \), and the conclusion follows.

By the Claim, we can define \( \Theta : Z^C(\varphi) \to Z^B(\varphi) \) by

\[
\Theta([\bar{b}]_h) = \Theta_0(\bar{b}).
\]

By considering constant paths in \( B_{\infty, \bar{c}} \), one sees that \( \Theta \) is surjective.

Therefore, \( \Psi \circ \Theta \) is the required equivariant surjection of \( Z^C(\varphi)/\sim_h \) onto \( Z^{A_\varphi}(\varphi)/\sim_h \).

(3) is an immediate consequence of (1) and (2). \( \square \)

**Proposition 4.10.** Suppose that a formula \( \varphi \) admits definable homotopy in an axiomatizable class of \( C^* \)-algebras \( \mathcal{E} \). Moreover assume that there is \( \alpha \in \text{End}^1(A_\varphi) \) such that for every \( \bar{b} \in Z^{A_\varphi}(\varphi) \) the set

\[
I_\alpha(\bar{b}) := \{ n \in \mathbb{N} \mid \bar{b} \sim_h \alpha^n(\bar{a}) \text{ for some } \bar{a} \in Z^{A_\varphi}(\varphi) \}
\]

is finite. If \( D \) is a unital abelian \( C^* \)-algebra such that the action of \( \text{End}^1(A_\varphi) \) on \( D \) is standard, then \( D \) is not countably saturated.

**Proof.** As before, let \( X_\varphi \) denote the spectrum of \( A_\varphi \), so that Lemma 4.4 implies that for an abelian \( C^* \)-algebra \( B \) we have \( X_\varphi \supseteq \text{jsp}(\bar{a}) \) if and only if \( \bar{a} \in Z^B(\varphi) \).

As in the proof of Proposition 4.7 for every \( \alpha \in \text{End}^1(A_\varphi) \) there exists a function \( f_\alpha \) such that for all \( \bar{a} \) in \( A_\varphi \) we have

\[
\alpha(\bar{a}) = f_\alpha(\bar{a}).
\]

Fix \( \alpha \in \text{End}^1(A) \) such that \( I_\alpha(\bar{b}) \) is finite for all \( \bar{b} \) in \( A_\varphi \). For \( m \geq 1 \) write \( \bar{f}_m := \bar{f}_\alpha^m \), hence \( f_{m,j} = f_{\alpha^m,j} \) for \( j < k \).
Since the zero-set of $\varphi$ is definable, applying quantification over it to definable predicates results in definable predicates (see [14, Definition 3.2.3]). We will write $\inf_{\bar{x}, \varphi(\bar{x})=0}$ and $\sup_{\bar{x}, \varphi(\bar{x})=0}$ for the corresponding quantifiers.

Consider the $k$-type $(\bar{x})$ whose conditions are $\varphi(\bar{x}) = 0$, together with

$$(4.6) \quad \inf_{\bar{y}, \varphi(\bar{y})=0} \|\bar{f}_m(\bar{y}) - \bar{x}\| = 0$$

for $m \geq 1$. We claim that every finite subset of $t$ is satisfied. Fix $\bar{a} \in Z^D(\varphi)$. For $n \geq 1$ let $\bar{b}$ in $D$ be defined by

$$\bar{b} := \bar{f}_{n!}(\bar{a})$$

Note that $\varphi^D(\bar{b}) = 0$. For $1 \leq m \leq n$, with $\bar{c} := \bar{f}_{n!/(m)}(\bar{a})$, we have $\bar{f}_m(\bar{c}) = \bar{f}_{n!}(\bar{a}) = \bar{b}$, thus $\bar{b}$ satisfies all conditions as in (4.6) for $m \leq n$; since $n \geq 1$ was arbitrary, the type $t$ is finitely satisfiable.

Assume, towards obtaining a contradiction, that $t$ is satisfied by some $\bar{b} \in D$. Then $\bar{b} \in Z^D(\varphi)$. Since the action is standard, there is a surjective equivariant $\Psi: Z^D(\varphi)/\sim_h \to Z^A(\varphi)/\sim_h$. Fix $\bar{c} \in Z^A(\varphi)$ such that $\Psi(\bar{b})[h] = [\bar{c}]_h$. By the assumption, the set $I_0(\bar{c})$ is finite. Fix $m$ that does not belong to this set.

Let $\varepsilon > 0$ be as in the conclusion of Proposition 4.10. Since $\bar{b}$ satisfies $t$, there exists $\bar{d} \in Z^D(\varphi)$ such that $\|\bar{f}_m(\bar{d}) - \bar{b}\| < \varepsilon$. Therefore $\bar{f}_m(\bar{d}) \sim_h \bar{b}$, and $\alpha^\varepsilon \cdot \Psi(d)[h] = \Psi(\bar{b})[h] = [\bar{c}]_h$. This contradicts the choice of $m$, and completes the proof. \qed

**Corollary 4.11.** For $n \geq 2$, the corona algebra of $C_0(\mathbb{R}^n)$ is not countably saturated.

**Proof.** We claim that the corona algebra of $C_0(\mathbb{R}^n)$ is isomorphic to $C(S^{n-1})_\infty$ (see [14,5] for the definition of $A_\infty$). To see this, let $\mathbb{D}^n$ denote the $n$-dimensional open unit ball. Since the closure of $\mathbb{D}^n$ inside $\mathbb{R}^n$ is compact, we have

$$Q(C_0(\mathbb{R}^n)) \cong Q(C_0(\mathbb{R}^n \setminus \mathbb{D}^n)).$$

Since every compact Hausdorff space $K$ satisfies

$$C(K)_\infty \cong Q(C_0(K \times [0, \infty))),$$

it remains to see that $\mathbb{R}^n \setminus \mathbb{D}^n$ is homeomorphic to $S^{n-1} \times [0, \infty)$. The map $S^{n-1} \times [0, \infty) \to \mathbb{R}^n \setminus \mathbb{D}^n$ that sends $(x_0, \ldots, x_n, t)$ to $(tx_0, tx_1, \ldots, tx_n)$ is clearly a homeomorphism.

To conclude the proof, we just have to compute the homotopy classes of maps $S^n \to S^n$. This is the $n$-th homotopy group of the sphere $S^n$, $\pi_n(S^n)$. It is well known (see e.g., [22, Corollary 4.25]) that $\pi_n(S^n) \cong \mathbb{Z}$. With such association, fix an endomorphism $\alpha$ of $C(S^n)$ that corresponds to the class of $1$. Then $\alpha$ has an infinite order. Recall that $\varphi_n$ was defined in Lemma 3.5 and that it is controlled by $S^{n-1}$. Every $\bar{b} \in Z^C(S^n)(\varphi_n)$ belongs to some homotopy class, and the set $I(\bar{b})$ as in Proposition 4.10 consists of its divisors. It is therefore finite, and by Proposition 4.11 the conclusion follows. \qed

5. Reduced Products

By [17, Theorem 1.5] (also [10, Theorem 16.5.1]), the reduced product $\prod A_n/\bigoplus A_n$ of $C^*$-algebras (or other metric structures of the same language) is countably saturated. If all $A_n$ are unital, then this reduced product is isomorphic to the corona of $\bigoplus A_n$. Even if the $A_n$s are not unital, the quotient $\mathcal{M}(\bigoplus A_n)/\bigoplus A_n \mathcal{M}(A_n)$ is also countably saturated. This is because we have $\mathcal{M}(\bigoplus A_n) \cong \prod \mathcal{M}(A_n)$ (see [4].
Consider the type \( t \) that \( \dot{\text{Theorem 5.2.}} \).

\[ \text{Corollary 5.1.} \quad \text{There is a sequence of separable abelian C}^*\text{-algebras } A_n \text{ such that the corona of } \bigoplus_n A_n \text{ is not countably saturated.} \]

**Proof.** Suppose that \( X_n \) is a locally compact connected noncompact space, and let \( X = \bigsqcup X_n \) (the disjoint union of the spaces \( X_n \)). This space satisfies the hypotheses of \( \text{Theorem 2.1} \), therefore the corona algebra of \( C_0(X) \) is not quantifier-free saturated. Since \( C_0(X) \cong \bigoplus C_0(X_n) \), by \( [5, \text{II.8.1.3}] \), \( \mathcal{M}(C_0(X)) \) is isomorphic to \( \prod C_0(X_n) \), and therefore by \( \text{Theorem 2.1} \) the algebra \( \prod C_0(X_n)/\bigoplus C_0(X_n) \) is not quantifier-free countably saturated.

(\( \square \)

\[ \text{A C}^*\text{-algebra is called projectionless if it has no projections other than 0 and (possibly) 1.} \]

**Theorem 5.2.** Let \( A_n \) be a sequence of nonunital C\(^*\)-algebras infinitely many of which are projectionless. Then the corona of \( \bigoplus A_n \) is not quantifier-free saturated.

In particular, there is a sequence of simple separable C\(^*\)-algebras \( A_n \) such that the corona of \( \bigoplus A_n \) is not quantifier-free saturated.

**Proof.** Let \( A = \bigoplus A_n \). By \( [5, \text{II.8.1.3}] \), we can identify \( \mathcal{M}(A) \) with \( \prod \mathcal{M}(A_n) \). Again, if \( a \in \mathcal{M}(A) \) we denote by \( \dot{a} \) its image in \( \mathcal{Q}(A) \).

We first prove the theorem in case all \( A_n \) are projectionless. Let \( a_n \in \mathcal{M}(A) \) be the sequence which is the identity of \( \mathcal{M}(A_n) \) in the \( n \)-th entry and 0 otherwise. Consider the type \( t(x) \) given by \( \dot{x} \geq 0, \parallel \dot{x} \parallel = 1, \dot{x}^2 - \dot{x} = 0 \) and \( \dot{x}a_n = 0 \).

If \( t_0 \subseteq t \) is finite, fix \( n \) such that \( x\dot{a}_n = 0 \) does not appear in \( t_0 \). Then \( \dot{a}_n \) satisfies \( t_0 \), so \( t_0 \) is consistent.

We now show that \( t \) is not realized. Suppose that \( c = (c_n) \in \prod \mathcal{M}(A_n) \) is such that \( \dot{c} \) realizes \( t \). Since \( \dot{c}\dot{a}_n = 0 \), we have \( c_n \in A_n \) for all \( n \). In addition \( c^2 - \dot{c} = 0 \), hence \( c^2 - c \in \bigoplus A_n \), and in particular \( \parallel c^2 - c_n \parallel \to 0 \) as \( n \to \infty \). Furthermore, since \( \parallel c \parallel = 1 \), we have that \( c \notin \bigoplus A_n \), and in particular \( \parallel c_n \parallel \to 1 \), so \( \lim\sup_n \parallel c_n \parallel \neq 0 \). Therefore, for \( n \) large enough, by usual functional calculus, we can pick a nontrivial projection \( d_n \in A_n \) such that \( \parallel c_n - d_n \parallel < \frac{1}{4} \). Since \( A_n \) is projectionless, this is a contradiction.

In case not all the algebra \( A_n \) are projectionless, let

\[ X = \{ n \mid A_n \text{ is projectionless} \}. \]

Notice that \( X \) is infinite. Denote by \( p_X \) the sequence in \( \prod \mathcal{M}(A_n) \) which is \( 1_{\mathcal{M}(A_n)} \) in case \( n \in X \) and 0 otherwise. Adding to the type the requirement that the realizing element is orthogonal to \( 1 - p_X \) gives a consistent not realized type.

For the second part, choose any sequence of projectionless, nonunital, simple C\(^*\)-algebras from \( [27] \).

(\( \square \)

The conclusion of \( \text{Theorem 5.2} \) may fail if the technical condition that (infinitely many of) the algebras \( A_n \) in are projectionless is dropped. For example, if each \( A_n \) equals \( c_0(\mathbb{N}) \), then the corona of \( \bigoplus A_n \) is naturally isomorphic to \( \ell_\infty/c_0 \), a countably saturated C\(^*\)-algebra (see the proof of \( \text{Corollary 3.3} \)).

6. Saturation of non-abelian coronas

If \( A \) is a C\(^*\)-algebra then \( Z(A) \) denotes its center. Since the center is the zero-set of the formula \( \varphi(x) = \sup_y \|xy - yx\| \), if \( Z(A) \) is not countably saturated, neither
is $A$. Furthermore, if the center is not quantifier-free saturated, then the failure of saturation of $A$ is witnessed by a universal type (this is optimal, see Proposition 6.3 below). Suppose that $A$ is a separable nonunital $C^*$-algebra such that $Z(A) = C_0(X)$ for some locally compact space $X$. What can we say in general about $Z(\mathcal{Q}(A))$? An ability to control $Z(\mathcal{Q}(A))$ would, together with the results of this note, imply the failure of countable saturation of some corona algebras. A test case: fix a unital $C^*$-algebra $B$ with trivial center, and let $X$ be a locally compact noncompact space. Consider $A = C_0(X, B)$. By [1], in this case $\mathcal{M}(A) \cong C_0(X, B)$, and therefore $Z(\mathcal{M}(A)) \cong C(\beta X)$. By the main result of [31], we then have $\pi[Z(\mathcal{M}(A))] = Z(\mathcal{Q}(A))$, and consequently $Z(\mathcal{Q}(A)) \cong C(\beta X \setminus X)$, if and only if the maximal ideal space of $Z(\mathcal{M}(A))$ separates that of $\mathcal{M}(A)$. This reflects in a technical condition on the $C^*$-algebra $B$, which we leave open for future investigations.

In the following ‘$\aleph_2$-saturated’ means that every consistent type of cardinality $\aleph_1$ is realized. (According to this convention, ‘countable saturation’ is called ‘$\aleph_1$-saturation’ but we find the former terminology more appealing.) By the classical Hausdorff’s gap construction, $\ell_\infty/c_0$ is not quantifier-free $\aleph_2$-saturated. Hausdorff’s gap has been transferred to the Calkin algebra in [34] and to the corona of every $C^*$-algebra with a sequential approximate unit consisting of projections in [10, Theorem 14.2.1]. By a standard argument (e.g., [10, Lemma 15.3.3]), this implies that the Calkin algebra is not degree-1 $\aleph_2$-saturated. Similar phaemenomena (such as constructions of certain Luzin families) were analyzed for coronas of simple $\sigma$-unital $C^*$-algebras in [50]. Luciano Salvetti proved that the corona of every separable, nonunital, $C^*$-algebra is not degree-1 $\aleph_2$-saturated, answering a question from the original version of this note.

If $A$ is finite-dimensional, then $A_\infty = C_b([0, \infty), A)/C_0([0, \infty), A)$ is countably saturated. The reason for this is that for every $C^*$-algebra $B$, $A \otimes B$ is definable in the theory of $B$ (the case when $A$ is a full matrix algebra is [21, Appendix C], also [14, Lemma 4.2.4], and the general case follows easily since $A$ is a direct sum of finitely many full matrix algebras). This implies that every type over $A \otimes B$ can be rewritten as a type over $B$, and the new type is satisfiable (realized) if, and only if, the original one is. (See the discussion of eq in [14, §3]).) Thus countable saturation of $A_\infty$ follows by [17, Theorem 1]. These algebras play an important role in the E-theory of Connes and Higson (3, §25) and in the Phillips–Weaver construction of an outer automorphism of the Calkin algebra (26).

**Question 6.1.** Is there an infinite-dimensional $C^*$-algebra $A$ such that $A_\infty$ is countably saturated?

In case $A$ is unital, $A_\infty$ is the corona of a $\sigma$-unital $C^*$-algebra, and therefore countably degree-1 saturated. By Corollary 6.1, an infinite-dimensional $C^*$-algebra $A$ such that $A_\infty$ is countably saturated cannot be abelian.

We suspect that the following question has a negative answer.

**Question 6.2.** Is there a separable, simple, and nonunital $C^*$-algebra $A$ such that $\mathcal{Q}(A)$ is countably saturated, or at least countably homogeneous? What about the stabilization of the Cuntz algebra $\mathcal{O}_2$?

We conclude the paper with an observation loosely related to Corollary 2, which answers [6, Question 6.5] and a question.
Proposition 6.3. There exists a $C^*$-algebra that is countably quantifier-free saturated, but not countably saturated.

Proof. Let $A$ be the CAR algebra (or any other separable, simple, unital $C^*$-algebra) and let $D$ be the $C^*$-algebra of continuous functions on the Cantor space (or any other infinite-dimensional, unital, abelian subalgebra of $A$). Fix a nonprincipal ultrafilter $U$ on $\mathbb{N}$. Then the norm ultrapower of $A$, $A_U$, is countably saturated (e.g., [10, Theorem 16.4.1]). Identify $D$ with its diagonal image in $A_U$. By [9, Corollary 2], the bicommutant of $D$ in $A_U$ is equal to $(A_U \cap D')^\prime = D$. Since $D$ is separable and infinite-dimensional, the type of a central element of $A_U \cap D'$ is countable and approximately satisfiable, but not realized, in $A_U \cap D'$. Therefore the $C^*$-algebra $A_U \cap D'$ is not countably saturated. However, it is easily seen to be countably quantifier-free saturated (see e.g., [10, Corollary 16.5.3]). □

Question 6.4. Is there an abelian $C^*$-algebra that is countably quantifier-free saturated, but not countably saturated?

Notably, the only unital abelian $C^*$-algebras that admit elimination of quantifiers are $\mathbb{C}$, $\mathbb{C}^2$, and $C(\text{Cantor space})$ ([7]).

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