This paper is the first in the series where we treat both almost contact and almost paracontact metric \((\kappa, \mu)\)-manifolds. General point of view is to consider almost (para-)contact metric manifold as local line bundle and distribution \(\{\eta = 0\}\) as horizontal connection. We introduce some integrability condition for almost contact structure. Under assumption the integrability condition is satisfied we provide full classification of 3-dimensional almost contact metric manifolds. These manifolds all appear to be \((\kappa, \mu)\)-manifolds. The class contains both contact metric and almost cosymplectic \((\kappa, \mu)\)-manifolds.

1. **Introduction**

Note classes of \((\kappa, \mu)\)-manifolds both almost contact and para-contact metric admit linear differential system of the first order in terms of Lie derivative \(\mathcal{L}_\xi\) for structure tensor fields \(\phi, h = \frac{1}{2}\mathcal{L}_\xi \phi\) and \(h' = \phi h\). Constants in the system usually depends on parameters \(\kappa\) and \(\mu\). For example for almost cosymplectic \((\kappa, \mu)\)-manifold

\[
\mathcal{L}_\xi \phi = 2h, \quad \mathcal{L}_\xi h = -2\kappa \phi - \mu h', \quad \mathcal{L}_\xi h' = \mu h.
\]

For other manifolds there are similar identities.

Having in mind these systems there comes general idea from following considerations. Let take local moving frame on \(\mathcal{M}\) in the form \((\partial_t, V_i), \mathcal{L}_\xi = \partial_t, V_i\) are spanning \(\{\eta = 0\}\) and

\[
\mathcal{L}_\xi V_i = 0,
\]

For example for contact form \(\eta = dt - \sum_{i=1}^{n} y^i dx^i\), such local frame is given by

\[
\partial_t, y^i \partial_t + \partial_x^i, \partial_y^i, \quad i = 1, \ldots n.
\]

Now think about \(\xi\) as time arrow, hence linear systems, as above for example, can be treated as evolution equations. Moreover if we assume coefficients of \(\phi\), etc. are only time dependent, system like above turns into systems of ordinary differential equations with constant coefficients. So solutions exists, they unique given initial values. Knowing solutions and frame allows to describe structure completely.

The concept can be applied for almost contact metric manifolds as almost para-contact metric. Giving unified approach.

Our study is divided into three separate papers. The first addresses almost contact metric 3-dimensional manifolds. This part serves two purposes. The first is to illustrate that it is possible to reach common point of view for structures usually being treated as very different, standing on opposite end-points. The second to obtain common local classification for both contact metric and almost cosymplectic manifolds \((\kappa, \mu)\)-manifolds.
There is 1-parameter family of almost contact metric structures on \( \mathbb{R}^3 \), for particular values of parameter we obtain contact metric structure or almost cosymplectic structure.

The second part essentially is very similar to the first. There are studied 3-dimensional para-contact metric and almost para-cosymplectic 3-manifolds - as elements of 1-parameter family of almost para-contact metric structures. The family satisfies the same analytically integrability condition.

In the last part we describe a construction how to extend almost (para)-contact metric manifolds. The extension posses nice properties. For example extension of \((\kappa, \mu)\) manifold by 3-dimensional \((\kappa \mu)\)-manifold is again \((\kappa, \mu)\)-manifold. The procedure works for almost contact metric manifolds and almost para-contact metric manifolds. As side effect it is possible to consider mixing these classes to obtain pseudo-Riemannian manifolds with \(\phi^4\) structure: 
\[
\phi^4 = \text{Id} - \eta \otimes \xi.
\]
Such manifold is equipped with corresponding fundamental form and usual classes can be defined: contact metric with pseudo-metric, almost cosymplectic with pseudo-metric, etc. For example we can equip odd-dimensional Lorentzian manifold with structure of contact metric manifold with Lorentzian metric.

2. Preliminaries

All manifolds considered are smooth and connected. Also tensor fields on manifold are considered to be smooth. Let \(\mathcal{M}\) be \((2n+1)\)-dimensional manifold, \(n \geq 1\). Almost contact metric structure is a quadruple of tensor fields \((\phi, \xi, \eta, g)\), where \(\phi\) is \((1,1)\)-tensor field, \(\xi\) a vector field, \(\eta\) a 1-form and \(g\) - a Riemannian metric. By definition there are following identities
\[
\phi^2 = -\text{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]
\(X, Y \in \Xi(\mathcal{M})\), \(\Xi(\mathcal{M})\) denotes module of vector fields on \(\mathcal{M}\).

Tensor field \(\Phi(X, Y) = g(X, \phi Y)\) is skew-symmetric, \(\Phi(X, Y) + \Phi(Y, X) = 0\). It determines a 2-form on \(\mathcal{M}\). We will use \(\Phi\) to denote this 2-form.

The field \(\xi\) and form \(\eta\) will be referred as structure vector field and form, or as characteristic vector field and form. The 2-form \(\Phi\) is customary called fundamental form. From definition of \(\Phi\), there is
\[
\eta \wedge \Phi^n \neq 0,
\]
at every point of \(\mathcal{M}\). In particular \(\mathcal{M}\) is orientable.

Manifold equipped with some fixed almost contact metric structure is called almost contact metric manifold.

Denote by \(N_S\) Nijenhuis torsion of a \((1,1)\) tensor field \(S\). Almost contact metric manifold is called normal if \(N_{\phi} + 2d\eta \otimes \xi = 0\). Normality is related to the existence of complex structure on product \(S^1 \times \mathcal{M}\).

Let \(\mathcal{M}\) be an almost contact metric manifold: \(\mathcal{M}\) is called contact metric if \(d\eta = \Phi\), almost cosymplectic (or almost coKähler) if \(d\eta = 0\), \(d\Phi = 0\) and almost Kenmotsu if \(d\eta = 0\), \(d\Phi = 2\eta \wedge \Phi\). Assuming normality we obtain respectively: Sasakian (contact metric and normal), cosymplectic (or coKähler) and Kenmotsu manifolds. Tanno proved that almost contact metric manifold with maximal isometry group is locally isometric either to Sasakian of constant sectional curvature \(c = +1\), Kenmotsu of constant sectional curvature \(c = -1\), or cosymplectic of constant sectional curvature \(c = 0\). General literature on the subject can be found eg. in \[4\], \[7\], \[10\], \[19\], \[25\], \[27\].

Let \(\mathcal{M}\) be almost contact metric manifold, let \(\nabla\) Levi-Civita connection of the metric, 
\[
R(X, Y) = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \text{ curvature of } \nabla.
\]
Define $h = \frac{1}{2}\phi$. Let $\kappa, \mu, \nu$ be real constants. Manifold $\mathcal{M}$ is called $(\kappa, \mu, \nu)$-manifold or $(\kappa, \mu, \nu)$-space if by definition
\begin{equation}
R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)\phi hX - \eta(X)\phi hY), \quad \kappa, \mu, \nu \in \mathbb{R}.
\end{equation}

In case $\nu = 0$ manifold is called $(\kappa, \mu)$-manifold. Note if $h = 0$ it is not possible to determine $\mu$ or $\nu$. But condition is still formally valid for any possible values $\mu, \nu$. Ambiguity also arrives if
\begin{equation}
R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y).
\end{equation}
e.g., $\mu = \nu = 0$, but such manifolds are called $\kappa$-manifolds.

We provide fundamental results concerning contact metric, almost symplectic and almost Kenmotsu $(\kappa, \mu, \nu)$-manifolds. By $(\kappa, \mu)$ manifold we understand manifold which satisfies (2.2) but without the last sum term.

We denote $\mathcal{D} = \{\eta = 0\}$. $\mathcal{D}$-homotety of $\mathcal{M}$ is deformation of structure
\[ \mathbb{R}^+ \ni \alpha, \quad \mathcal{D}_{\text{hom}} : (\phi, \xi, \eta, g) \mapsto (\phi', \xi', \eta', g'), \]
where
\[ \phi' = \phi, \quad \xi' = \alpha^{-1}\xi, \quad \eta' = \alpha \eta, \quad g' = \alpha g + \alpha(\alpha^2 - 1)\eta \otimes \eta. \]

For some classes of manifolds condition (2.2) is $\mathcal{D}$-homotetic invariant.

**Theorem 1** (Blair, Koufogiorgos, Papantoniou, 1995, [6]). Let $\mathcal{M}$ be contact metric $(\kappa, \mu)$-manifold. Then $\kappa \leq 1$. The following relations hold
\begin{equation}
(\nabla_X \phi)Y = g(X, Y + hY)\xi - \eta(Y)(X + hX),
\end{equation}
\begin{equation}
(\nabla_X h)Y = ((1 - \kappa)g(X, \phi Y) + g(X, \phi hY))\xi + \eta(Y)h(\phi X + \phi hX) - \mu \phi hY.
\end{equation}

**Theorem 2** (Boeckx, 2000, [6]). Let $\mathcal{M}$ be non-Sasakian contact metric $(\kappa, \mu)$-manifold. Define $I_M = (1 - \mu/2)/\sqrt{1 - \kappa}$. $I_M$ is $\mathcal{D}$-homotetit invariant. If $I_{M_1} = I_{M_2}$, manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are locally isometric up to $\mathcal{D}$-homotety as almost contact metric manifolds.

Theorem is base for classification of non-Sasakian contact metric $(\kappa, \mu)$-manifolds. It is enough to provide an example of manifold $\mathcal{M}$, for every allowable value $I$ of Boeckx invariant, such that $I_M = I$.

For almost symplectic manifold distribution $\{\eta = 0\}$ is completely integrable. Let $\mathcal{F}$ denote leaf passing through some point $\in \mathcal{M}$. Then $\mathcal{F}$ inherits structure of almost Kähler manifold. Assuming structure is Kähler for every leaf manifold is called almost symplectic with Kähler leaves.

**Theorem 3** (Olszak, 1987, [29]). Let define $A = -\nabla \xi$. Almost symplectic manifold has Kählerian leaves if and only if
\begin{equation}
(\nabla_X \phi)Y = -g(\phi AX, Y) + \eta(Y)\phi AX.
\end{equation}

**Theorem 4** (Dacko, Olszak, 2005, [15]). Let $\mathcal{M}$ be non-symplectic almost symplectic $(\kappa, \mu)$-manifold. Then $\kappa \leq 0$. If $\kappa = 0$ $\mathcal{M}$ is locally isometric to product of real line and almost Kähler manifold. For $\kappa < 0$ $\mathcal{M}$ has Kähler leaves and each leaf is locally flat Kähler manifold. There is following identity
\begin{equation}
\mathcal{L}_\xi \phi = 2h, \quad \mathcal{L}_\xi h = -2\kappa \phi - \mu \phi h, \quad \mathcal{L}_\xi (\phi h) = \mu h,
\end{equation}
The theorem allows to classify, by analytic solution, almost cosymplectic \((\kappa, \mu)\)-manifolds in terms of so-called models. For every \(\mu\) there is almost cosymplectic \((-1, \mu)\)-manifold - called model - and every other \((\kappa, \mu)\)-manifold is locally isometric up to \(D\)-homotety to particular model, \[16\]. The value \(\frac{\mu}{\sqrt{-\kappa}}\), \(\kappa < 0\) is \(D\)-homotety invariant. We set \(C_M = \frac{\mu}{\sqrt{-\kappa}}\), \(\kappa < 0\) and call \(C_M\) Dacko-Olszak invariant of almost cosymplectic \((\kappa, \mu)\) -manifold.

For almost Kenmotsu manifolds there are following basic results.

**Theorem 5** (Dileo, Pastore, 2009). Let \(M\) be almost Kenmotsu \((\kappa, \mu)\)-manifold. Then \(\kappa = -1\), \(h = 0\) and \(M\) is locally warped product of an almost Kähler manifold and open interval. If \(M\) is locally symmetric then \(M\) is locally isometric to the hyperbolic space \(\mathbb{H}(-1)\) of constant sectional curvature \(-1\).

**Theorem 6** (Dileo, Pastore, 2009). Let \(M\) be almost Kenmotsu manifold such that \(h \neq 0\) and

\[
R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h\phi X - \eta(X)h\phi Y),
\]

then \(M\) is locally isometric to warped products

\[
\mathbb{H}^{n+1}(\kappa - 2\lambda) \times_f \mathbb{R}^{n}, \quad B^{n+1}(\kappa + 2\lambda) \times_f \mathbb{R}^{n},
\]

where \(H^{n+1}(\kappa - 2\lambda)\) is the hyperbolic space of constant sectional curvature \(k - 2\lambda \leq -1\), \(B^{n+1}(\kappa + 2\lambda)\) is a space of constant sectional curvature \(\kappa + 2\lambda \leq 0\), \(f = ce^{(1-\lambda)t}\), \(f' = c'e^{(1+\lambda)t}\), \(\lambda = \sqrt{1 + \kappa}\).

Compare differences between (2.8) and our definition (2.2). It is known that almost Kenmotsu manifold as Riemannian manifold is locally conformal to almost cosymplectic manifold. For this point of view see [24], [29]. In [31] authors study generalized nullity distribution on almost Kenmotsu manifold, i.e. in terminology we use in this paper almost Kenmotsu \((\kappa, 0, \mu)\)-manifolds where in general \(\kappa\) and \(\mu\) are functions.

By stationary vector field, tensor field, or other geometric objects like connection eg, it is understood that equation \(\mathcal{L}_\xi A = 0\) is satisfied, whether it is possible to define Lie derivative for geometric object \[1\] \(A\). Of course for metric \(A\), \(\xi\) is just Killing vector field.

If \(A\) is an affine connection \(\xi\) is an affine motion, i.e. local diffeomorphisms group of \(\xi\) are affine maps. Here we think about \(\xi\) as time arrow. Stationary object is time-independent which is expressed by \(\mathcal{L}_\xi A = 0\).

For Lie differential there is following useful expression. Let \(V_{\perp} \omega\) denote the inner product of vector field \(V\) and \(p\)-form \(\omega\). As usually \(d\omega\) denotes exterior derivative. Let \(\nabla\) be affine torsion-less connection on a manifold. Then

\[
\mathcal{L}_V = V_{\perp} \circ d + d \circ V_{\perp}, \quad \mathcal{L}_V = \nabla_V + A, \quad A = -\nabla V,
\]

in the second equation it is understood that \(A\) acts as tensor algebra derivative. It is known that such derivative trivial on smooth functions, is determined uniquely by tensor field of type \((1,1)\), cf. Kobayashi, Nomizu. Probably it would be more correct to write \(\mathcal{L}_V = \nabla_V + D_A\). Thus indicating the role plays \(A\) in decomposition. For example

\[
(D_A \phi)X = D_A \phi Y - \phi D_A X = A\phi X - \phi AX = [A, \phi] X.
\]

\(^1\)Things which appear studying geometry
3. MANIFOLD WITH STATIONARY FRAMES OF HORIZONTAL CONNECTION, 
TIME-DEPENDENT ONLY STRUCTURE AND WITH $\mathcal{L}_\xi(L_\xi h) = 4r(\kappa, \mu)h$

In this section we establish following fact: if there is stationary frame so coefficients
of $\phi$ are only time-dependent, and $\mathcal{L}_\xi^2 h = 4r(\kappa, \mu)h, \ h = \frac{1}{4}L_\xi \phi$, $r(\kappa, \mu)$ is arbitrary
function of free real parameters, then there exists very particular frame - non-stationary - within $\phi$ has constant coefficients, and in the same time $h$ is diagonal also with constant
coefficients. The existence of such frame is rather evident, yet our method do not employs
directly metric: so it can be used to an almost contact structure only. Of course by
assumption of only time-dependence. This section is technical in nature - but its contents
is important to understand examples at the end of section. Examples which are crucial
for our further study.

Note by 2.4, 2.5

\begin{equation}
(3.1) \quad \nabla \xi = \phi(\nabla \phi) \xi = -\phi - \phi h, \quad \nabla \xi h = -\mu \phi h.
\end{equation}

Therefore, if we set $A = -\nabla \xi$ in 2.10 for contact metric $(\kappa, \mu)$-manifold there is

\begin{equation}
(3.2) \quad \mathcal{L}_\xi h = \nabla \xi h + [\phi, h] + [\phi h, h] = 2(1 - \kappa)\phi + (2 - \mu)\phi h,
\end{equation}

\begin{equation}
(3.3) \quad \mathcal{L}_\xi(\phi h) = (\mathcal{L}_\xi \phi)h + \phi(\mathcal{L}_\xi h) = -(2 - \mu)h.
\end{equation}

Second differential

\begin{equation}
(3.4) \quad \mathcal{L}_\xi^2 h = c(\kappa, \mu)h, \quad c(\kappa, \mu) = 4(1 - \kappa) - (2 - \mu)^2,
\end{equation}

For almost cosymplectic $(\kappa, \mu)$-manifold by 2.4

\begin{equation}
(3.5) \quad \mathcal{L}_\xi^2 h = b(\kappa, \mu)h, \quad b(\kappa, \mu) = -4(\kappa + \mu^2).
\end{equation}

These examples illustrate that considerations both contact metric and almost cosymplectic manifolds lead to similar problem. General idea is to solve equation in Lie derivative of the form

\begin{equation}
(3.6) \quad \mathcal{L}_\xi^2 h = 4r(\kappa, \mu)h, \quad \kappa, \mu \in \mathbb{R},
\end{equation}

$\kappa$ and $\mu$ are real parameters, $r(\kappa, \mu)$ is real function, and scalar 4 in equation is normalization constant. A priori function $r$ is arbitrary. It is important to realize that in this equation $\kappa, \mu$ are free parameters. They do not have a priori geometric interpretation. It is in contrast to eqs. 3.3 5.5 where both $\kappa, \mu$ are coming from identity 2.2

Assume $(\xi = \frac{\partial}{\partial \eta}, V_1, V_2)$ is stationary frame on $\mathcal{M}$, $(V_1, V_2)$ are spanning horizontal connection $\{\eta = 0\}$. Let $\phi(t)$ be matrix of coefficients of $\phi$ in this frame. Assume

\begin{equation}
(3.7) \quad \phi(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{tA} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & F_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-tA} \end{pmatrix}.
\end{equation}

Denote $F(t) = e^{tA}F_0e^{-tA}, F_0$ constant $(2 \times 2)$-matrix, $F_0^2 = -Id$. Subsequent derivatives of $F(t)$ there are

\begin{equation}
(3.8) \quad \dot{F} = e^{tA}[A, F_0]e^{-tA} = 2H(t), \quad [A, F_0] = AF_0 - F_0A,
\end{equation}

\begin{equation}
(3.9) \quad \dot{H} = e^{tA}[A, [A, H_0]]e^{-tA}, \quad H_0 = \frac{1}{2}[A, F_0],
\end{equation}

Taking into account $\dot{H} = r(\kappa, \mu)H$, it is necessary that

\begin{equation}
(3.10) \quad [A, [A, H_0]] = 4r(\kappa, \mu)H_0.
\end{equation}

For $F_0^2 = -Id$, equation $F_0[A, F_0] + [A, F_0]F_0 = 0$, is identity. Matrix $B$ satisfying $F_0B + BF_0 = 0$ is symmetric, its eigenvalues has to be equal in absolute value and opposite in signs. Therefore $B = H_0 = \frac{1}{2}[A, F_0]$ is symmetric with opposite eigenvalues
\(\lambda, -\lambda\). There is matrix \(P\) so \(PF_0P^{-1} = (0 - 1), \ PH_0P^{-1} = (\lambda 0 \quad 0), \ \lambda \in \mathbb{R}\) are in form. we call canonical. By
\[
\frac{1}{2}P[A, F_0]P^{-1} = \frac{1}{2}[PAP^{-1}, PF_0P^{-1}] = PH_0P^{-1},
\]
up to adjoint \(A\) solves \(\frac{1}{2}[A, F_0] = H_0\), where both \(F_0, H_0\) are in canonical form. From now on let \(F_0 = (0 - 1), \ H_0 = (\lambda 0 \quad 0 - \lambda)\). Denote by \(MT\) matrix transposition
\[
F_0^T = -F_0, \quad H_0^T = H_0, \quad [A, F_0]^T = [A^T, F_0].
\]
If \(A\) is solution, then also its transposition \(A^T\) and its symmetric part
\[
\frac{1}{2}[A^*, F_0] = H_0, \quad A^* = \frac{1}{2}(A + A^T).
\]
It can be seen that \(A^* = (\begin{smallmatrix} a & \lambda \\ \lambda & a \end{smallmatrix})\), hence \(A = (\begin{smallmatrix} a & \lambda + c \\ \lambda - c & a \end{smallmatrix})\), \(a, c \in \mathbb{R}\), in general. By
\[
F(t) = e^{tA}F_0e^{-tA} = e^{t(A-aId)}F_0e^{-t(A-aId)},
\]
we may assume \(2a = tr(A) = 0\). Set \(H_0 = (\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix})\), \(A = (\begin{smallmatrix} 0 & \lambda + c \\ \lambda - c & 0 \end{smallmatrix})\) in (3.10). The equation is satisfied if and only if \(\lambda^2 - c^2 = r(\kappa, \mu)\). To avoid unimportant ambiguity let \(\lambda > 0\) always. For example sign reversal \((b, c) \rightarrow (-b, -c)\) is equivalent to time reversal \(F(t) \rightarrow F(-t)\).

Explicit solutions for \(e^{tA}\), depending on sign of \(r(\kappa, \mu)\), there are
\[
e^{tA} = \begin{cases} \begin{pmatrix} \cos(t\sqrt{-r}) & \frac{\lambda + c}{\sqrt{r}} \sin(t\sqrt{-r}) \\ \frac{\lambda - c}{\sqrt{r}} \sin(t\sqrt{-r}) & \cos(t\sqrt{-r}) \end{pmatrix}, & r = r(\kappa, \mu) = \lambda^2 - c^2 < 0, \\ 
\begin{pmatrix} 1 & 2t\lambda \\ 0 & 1 \end{pmatrix}, & r = r(\kappa, \mu) = \lambda^2 - c^2 = 0, \\ 
\begin{pmatrix} \cosh(\sqrt{rt}) & \frac{\lambda + c}{\sqrt{rt}} \sinh(\sqrt{rt}) \\ \frac{\lambda - c}{\sqrt{rt}} \sinh(\sqrt{rt}) & \cosh(\sqrt{rt}) \end{pmatrix}, & r = r(\kappa, \mu) = \lambda^2 - c^2 > 0, \end{cases}
\]
(3.11)

Let define new non-stationary frame treating \((1 0 e^A)\) as transition matrix \((\xi, V_1, V_2) \mapsto (\xi, E_1, E_2)\). By (3.7) in this new frame coefficients of \(e\) and \(h\) are now constants
\[
[\phi] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad [h] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}
\]
(3.12)

**Remark 1.** The frame \((\xi, E_1, E_2)\) is frame of orthogonal vector fields. We cannot claim these fields \(E_1, E_2\) are normalized

**Remark 2.** Procedure to obtain frame \((\xi, E_1, E_2)\) do not explicitly employs metric. And in fact we do not have to require metric exists. In other words procedure works if we drop metric, for almost contact structure \((\phi, \xi, \eta)\) only.

**Remark 3.** On a base of our solution tensor fields \(\phi, h\) and \(\phi h\) satisfy system of linear equations with constant coefficients \((\xi \lambda = 0, \xi c = 0)\), in Lie derivative:
\[
\mathcal{L}_\xi \phi = 2h, \quad \mathcal{L}_\xi h = 2\lambda^2 \phi - 2c \phi h, \quad \mathcal{L}_\xi (\phi h) = 2c h.
\]
(3.13)

Let recall orthonormal frame \((\xi, E_1, E_2)\) is called \(\phi\)-basis if \(\phi E_1 = E_2\).
Example 1 (stationary contact metric). Contact metric manifold with \( \phi \) stationary coefficients are constant. Metric is given by \( ds^2 = (dt - ydx)^2 + \frac{1}{2}(dx^2 + dy^2) \). Frame

\[
(3.14) \quad \xi = \partial_t, \quad V_1 = \sqrt{2}\partial_y, \quad V_2 = \sqrt{2}(y\partial_t + \partial_x),
\]

is both stationary and it is \( \phi \)-basis, \( A = 0 \).

Example 2 (stationary almost cosymplectic). Almost cosymplectic manifold with \( \phi \) stationary. Metric is given by \( ds^2 = dt^2 + \frac{1}{4}(dx^2 + dy^2) \), evidently manifold is locally flat. Stationary \( \phi \)-basis there is

\[
(3.15) \quad \xi = \partial_t, \quad V_1 = \sqrt{2}\partial_y, \quad V_2 = \sqrt{2}\partial_x.
\]

Example 3 (non-stationary contact metric). \( M = \mathbb{R}^3, p = (t, x, y) \in M \) be contact manifold with contact form \( \eta = dt - ydx \). Stationary frame is as in \( [3,4] \). Define new frame \( (\xi, V_1, V_2) \to (\xi, E_1, E_2) \) using \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), \( A \neq 0 \), as transition matrix. Let define contact metric structure where \( (\xi, E_1, E_2) \) is \( \phi \)-basis.

Example 4 (non-stationary almost cosymplectic). \( M = \mathbb{R}^3, p = (t, x, y) \in M, \eta = dt \). Stationary frame is as in \( [3,4] \). As in previous example we switch to \( (\xi, E_1, E_2) \). Again almost contact metric structure is defined in the way that \( (\xi, E_1, E_2) \) is \( \phi \)-basis. Manifold equipped with this structure is almost cosymplectic.

Recall conflation on 3-dimensional manifold, is non-zero everywhere 1-form \( \omega \), such that \( \omega \wedge d\omega \geq 0 \). If \( \omega \wedge d\omega > 0 \), everywhere \( \omega \) is contact form. If \( \omega \wedge d\omega = 0 \), everywhere, distribution \( \{ \omega = 0 \} \) determines foliation.

Example 5 (conflation). Let \( M = \mathbb{R}^3, p = (t, x, y) \in M, 0 \leq e_r(x, y) \leq 1, 0 < r < R, \) be smooth real function such that

\[
(3.16) \quad e_{r,R} = \begin{cases} 
1, & 0 \leq x^2 + y^2 \leq r, \\
0, & x^2 + y^2 > R,
\end{cases}
\]

now \( \eta = dt - (e_{r,R})ydx, f : \mathbb{R}^3 \to \mathbb{R} \). Stationary frame of \( \{ \eta = 0 \} \) is

\[
(3.17) \quad \xi = \partial_t, \quad V_1 = \sqrt{2}\partial_y, \quad V_2 = \sqrt{2}((e_{r,R})y\partial_t + \partial_x),
\]

we define \( (\xi, E_1, E_2) \) and almost contact metric structure as in previous examples. Manifold equipped with this structure globally is neither contact metric nor almost cosymplectic, yet which is clear from definition: it is contact metric manifold inside disk radius \( \leq r \) and almost cosymplectic outside disk radius \( > R \). Point is that this manifold as a whole satisfies corresponding differential system \( [3,4] \).

4. Riemann connection, curvature.

In this section we provide detailed study of Riemann geometry of manifolds described in examples \( [3,4] \). It is convenient to encase these two examples into 1-parameter family of almost contact metric structures.

Example 6. Let \( M = \mathbb{R}^3, p = (t, x, y, z) \in M \). We define one parameter family of contact metric manifolds \( (M, \phi_k, \xi_k, \eta_k, g_k), k \in \mathbb{R}, \) \( \eta = \eta_k = dt - kydx \), stationary frame

\[
(4.1) \quad \xi = \partial_t, \quad V_1 = \sqrt{2}\partial_y, \quad V_2 = \sqrt{2}(ky\partial_t + \partial_x),
\]
and as in examples frame $(\xi, E_1, E_2)$ is determined. By definition it is $\phi_k$-basis. Note $d\eta_k = k\Phi$, hence structure is contact metric for $k = 1$ and almost cosymplectic for $k = 0$. Fundamental form is given by $\Phi = dx \wedge dy$.

**Remark 4.** Our considerations are strictly local. We cannot claim structures $(\phi_k, \xi_k, \eta_k, g_k)$ live on the same manifold if there are made additional global assumptions. For example that $(M, g)$ is metrically complete. Indeed by Myers theorem $M_k$ is compact with compact universal cover if $\text{Ric}_k > 0$. From other hand if sectional curvatures everywhere $< 0$, by Hadamard theorem universal cover of $M_k$ is diffeomorphic to $\mathbb{R}^3$. There are compact 3-dimensional manifolds which can be equipped both with contact metric and almost cosymplectic structures. However we cannot claim that these structures share the same characteristic vector field. Compact 3-dimensional solvmanifolds are classified in [3].

**Proposition 1.** Jacobi commutators of $\xi, E_1, E_2$ there are

$$[E_1, E_2] = 2\mu_1, \quad [E_2, \xi] = -(\lambda + c)E_1, \quad [\xi, E_1] = (\lambda - c)E_2,$$

**Proof.** Direct verification.

Curvature of left-invariant Riemann metric on 3-dimensional Lie group was described in simple and intuitive way by John Milnor in his genuine paper.

**Theorem 7** (Milnor). Let $\mathcal{G}$ be 3-dimensional Lie group, equipped with left-invariant Riemann metric. Let $\mathfrak{g}$ be Lie algebra of $\mathcal{G}$. There is orthonormal frame $(e_1, e_2, e_3)$ of left-invariant vector fields, there are constants $\lambda_1, \lambda_2, \lambda_3$, such that

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3.$$

Signs of $\lambda_i$ up to the order determine $\mathcal{G}$ uniquely if $\mathcal{G}$ is connected and simply connected. Define $\mu_1, \mu_2, \mu_3$

$$\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i.$$

The orthonormal basis $(e_1, e_2, e_3)$ diagonalizes Ricci quadratic form, the principal Ricci curvatures being given by

$$r(e_1) = 2\mu_2 \mu_3, \quad r(e_2) = 2\mu_1 \mu_3, \quad r(e_3) = 2\mu_1 \mu_2.$$

Let $v \times w$ be vector product determined by $e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2$. On Lie algebra connection maps are given by $x \mapsto \nabla_x e_i = \mu_i (x \times e_i), x \mapsto \nabla_x e_i = \mu_i (e_i \times x)$.

**Proposition 2.** Almost contact metric manifold $(M, \phi_k, \xi_k, \eta_k, g_k)$ is $(k^2 - \lambda^2, 2(k - c))$-manifold, ie.

$$R_{XY}\xi = (k^2 - \lambda^2)(\eta(Y)X - \eta(X)Y) + 2(k - c)(\eta(Y)hX - \eta(X)hY).$$

**Proof.** We employ Milnor’s method to find Levi-Civita connection coefficients and then directly compute:

$$R_{E_i E_2} = (k^2 - \lambda^2 + 2\lambda(k - c))E_1, \quad R_{E_2 E_3} = (k^2 - \lambda^2 - 2\lambda(k - c))E_2,$$

from the second identity it follows that

$$R_{XY}\xi = \eta(Y)R_{X\xi}\xi - \eta(X)R_{Y\xi}\xi,$$

Jacobi operator $J_\xi X = R_{X\xi}\xi$ by the first set of identities has decomposition

$$J_\xi X = (k^2 - \lambda^2)(Id - \eta(X)\xi) + 2(c - k)h,$$
therefore
\[ R_{XY} \xi = \eta(Y)R_{X\xi} \xi - \eta(X)R_{Y\xi} \xi \]
\[ = (k^2 - \lambda^2)(\eta(Y)X - \eta(X)Y) + 2(k - c)(\eta(Y)hX - \eta(X)hY). \]

Following Milnor’s paper we obtain

**Proposition 3.** Principal Ricci curvatures of \( M_k \) there are
\[ r_1 = \text{Ric}(\xi, \xi) = 2(k^2 - \lambda^2), \quad r_2 = \text{Ric}(E_1, E_1) = -2(c + k)(k - \lambda), \]
\[ r_3 = \text{Ric}(E_2, E_2) = -2(c + k)(k + \lambda), \]
scalar curvature
\[ s = \sum r_i = -(k^2 + \lambda^2) - 2kc. \]

Interesting classes are manifolds with positive, negative Ricci tensor, manifolds of constant sectional curvature, ie. Einstein manifolds - as in dimension three Einstein manifold has constant sectional curvature. For example if \( M_k \) is locally flat then it is either cosymplectic \( k = \lambda = c = 0 \) or \( k = \pm \lambda = c \). In particular for \( k = 1, M_k \) is locally flat contact metric manifold. Note according to Blair’s theorem in dimensions \( \geq 5 \) do not exist locally flat contact metric manifolds, [4].

**Remark 5 (Classification - contact metric case).** Manifold \( M_1 \) is contact metric \((\kappa, \mu)\)-manifold with \( \kappa = 1 - \lambda^2, \mu = 2(1-c) \). Resolving these equations \( \lambda = \sqrt{1 - \kappa}, c = 1 - \mu/2, \)
we obtain explicit form
\[ [E_1, E_2] = 2\xi, \quad [E_2, \xi] = -(\sqrt{1 - \kappa} - \mu/2 + 1)E_1, \]
\[ [\xi, E_1] = (\sqrt{1 - \kappa} + \mu/2 - 1)E_2, \]
This is 2-parameter family of contact metric structures - in these expressions parameters are geometric quantities. In terms of Boeckx invariant for non-Sasakian manifolds resp. coefficients can be expressed as
\[ I_M - 1 \]
\[ \sqrt{1 - \kappa}, \quad I_M + 1 \]
\[ \sqrt{1 - \kappa}. \]

For \( \kappa = 1 \) equiv. \( \lambda = 0 \) tensor field \( h = 0 \). Due to our remarks concerning definition of \((\kappa, \mu)\)-manifold \( \mu \) is not determined. In this particular case manifold is Sasakian. Note for \( \lambda = 0 \) we can directly find that \( M_1 \) is Sasakian. For \( \lambda \neq 0 \) we have obtained classification of all non-Sasakian contact metric 3-dimensional \((\kappa, \mu)\)-manifolds. By [4,10] for non-Sasakian manifold
\[ r_1 = \text{Ric}(\xi, \xi) = 2\kappa, \quad r_2 = \text{Ric}(E_1, E_1) = (\mu - 4)(1 - \sqrt{1 - \kappa}), \]
\[ r_3 = \text{Ric}(E_2, E_2) = (\mu - 4)(1 + \sqrt{1 - \kappa}), \]
scalar curvature
\[ s = \sum r_i = 2(\kappa + \mu) - 8. \]

For example \( \text{Ric} > 0 \) if and only if \( 0 < \kappa < 1, \mu > 4 \). Manifold is compact with compact covering. By list below universal cover is \( S^3 \).
Remark 6 (Classification - almost cosymplectic case). Manifold $\mathcal{M}_0$ is almost cosymplectic. As in previous example we find $\lambda = \sqrt{\kappa}$, $\mu = -2c$, explicit form

\[(4.16) \quad [E_1, E_2] = 0, \quad [E_2, \xi] = (\mu/2 - \sqrt{-\kappa})E_1, \quad [\xi, E_1] = (\mu/2 + \sqrt{-\kappa})E_2,\]

In terms of Dacko-Olszak invariant $C_\mathcal{M}$ corresponding coefficients there are

\[(4.17) \quad C_\mathcal{M} - \frac{1}{\sqrt{-\kappa}}, \quad C_\mathcal{M} + \frac{1}{\sqrt{-\kappa}},\]

For $\lambda = 0$, $h = 0$ again $\mu$ is undetermined. In general such manifold is a local product or real line and almost Kähler manifold. Dimension three is special case due to every 2-dimensional almost Kähler manifold is Kähler so for $\lambda = 0$, $\mathcal{M}_0$ is cosymplectic. For $\lambda \neq 0$ we have obtained classification of all 3-dimensional non-cosymplectic almost cosymplectic $(\kappa, \mu)$-manifolds. By \cite{4,10} principal curvatures of Ricci tensor being given by

\[(4.18) \quad r_1 = \text{Ric}(\xi, \xi) = 2\kappa, \quad r_2 = \text{Ric}(E_1, E_1) = -\mu\sqrt{-\kappa},\]

\[r_3 = \text{Ric}(E_2, E_2) = \mu\sqrt{-\kappa},\]

scalar curvature

\[(4.19) \quad s = \sum r_i = 2\kappa.\]

For example we easily see that there are only two possibilities for signature of Ricci tensor in case manifold is non-cosymplectic: $(-1, 0, 0)$ or $(-1, -1, +1)$.

| Lie group | Description | $I_\mathcal{M} = \frac{1-\mu/2}{\sqrt{-\kappa}}$ |
|-----------|-------------|---------------------------------|
| $SO(3)$ or $SU(2)$ | simple, compact, $S^3$ or $S^3/\{\pm1\}$ | $I_\mathcal{M} > 1$ |
| $SL(2, \mathbb{R})$ or $O(1, 2)$ | simple, $\mathbb{R}^3$, compact quotients, $[3]$ | $|I_\mathcal{M}| < 1$, $I_\mathcal{M} < -1$ |
| $E(2)$ | solvable, $\mathbb{R}^3$, compact quotients, $[3]$ | $|I_\mathcal{M}| = 1$ |

Table 1. Complete list of Lie groups with left-invariant, non-Sasaki contact metric $(\kappa, \mu)$-structures

Table 2. Complete list of Lie groups with left-invariant, non-cosymplectic almost cosymplectic $(\kappa, \mu)$-structures

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