STRONG EXTENSIONS FOR $q$-SUMMING OPERATORS ACTING IN $p$-CONVEX BANACH FUNCTION SPACES FOR $1 \leq p \leq q$

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Abstract. Let $1 \leq p \leq q < \infty$ and let $X$ be a $p$-convex Banach function space over a $\sigma$-finite measure $\mu$. We combine the structure of the spaces $L^p(\mu)$ and $L^q(\xi)$ for constructing the new space $S_{X,p}^q(\xi)$, where $\xi$ is a probability Radon measure on a certain compact set associated to $X$. We show some of its properties, and the relevant fact that every $q$-summing operator $T$ defined on $X$ can be continuously (strongly) extended to $S_{X,p}^q(\xi)$. This result turns out to be a mixture of the Pietsch and Maurey-Rosenthal factorization theorems, which provide (strong) factorizations for $q$-summing operators through $L^q$-spaces when $1 \leq q \leq p$. Thus, our result completes the picture, showing what happens in the complementary case $1 \leq p \leq q$, opening the door to the study of the multilinear versions of $q$-summing operators also in these cases.

1. Introduction

Fix $1 \leq p \leq q < \infty$ and let $T: X \to E$ be a Banach space valued linear operator defined on a saturated order semi-continuous Banach function space $X$ related to a $\sigma$-finite measure $\mu$. In this paper we prove an extension theorem for $T$ in the case when $T$ is $q$-summing and $X$ is $p$-convex. In order to do this, we first define and analyze a new class

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of Banach function spaces denoted by $S^q_{X_p}(\xi)$ which have some good properties, mainly order continuity and $p$-convexity. The space $S^q_{X_p}(\xi)$ is constructed by using the spaces $L^p(\mu)$ and $L^q(\xi)$, where $\xi$ is a finite positive Radon measure on a certain compact set associated to $X$.

Corollary 5.2 states the desired extension for $T$. Namely, if $T$ is $q$-summing and $X$ is $p$-convex then $T$ can be strongly extended continuously to a space of the type $S^q_{X_p}(\xi)$. Here we use the term “strongly” for this extension to remark that the map carrying $X$ into $S^q_{X_p}(\xi)$ is actually injective; as the reader will notice (Proposition 3.1), this is one of the goals of our result. In order to develop our arguments, we introduce a new geometric tool which we call the family of $p$-strongly $q$-concave operators. The inclusion of $X$ into $S^q_{X_p}(\xi)$ turns out to belong to this family, in particular, it is $q$-concave.

If $T$ is $q$-summing then it is $p$-strongly $q$-concave (Proposition 5.1). Actually, in Theorem 4.4 we show that in the case when $X$ is $p$-convex, $T$ can be continuously extended to a space $S^q_{X_p}(\xi)$ if and only if $T$ is $p$-strongly $q$-concave. This result can be understood as an extension of some well-known relevant factorizations of the operator theory:

(I) Maurey-Rosenthal factorization theorem: If $T$ is $q$-concave and $X$ is $q$-convex and order continuous, then $T$ can be extended to a weighted $L^q$-space related to $\mu$, see for instance [3, Corollary 5]. Several generalizations and applications of the ideas behind this fundamental factorization theorem have been recently obtained, see [1, 2, 4, 5, 9].

(II) Pietsch factorization theorem: If $T$ is $q$-summing then it factors through a closed subspace of $L^q(\xi)$, where $\xi$ is a probability Radon measure on a certain compact set associated to $X$, see for instance [6, Theorem 2.13].

In Theorem 4.4 the extreme case $p = q$ gives a Maurey-Rosenthal type factorization, while the other extreme case $p = 1$ gives a Pietsch type factorization. We must say also that our generalization will allow to face the problem of the factorization of several $p$-summing type of multilinear operators from products of Banach function spaces—a topic of current interest—, since it allows to understand factorization
of $q$-summing operators from $p$-convex function lattices from a unified point of view not depending on the order relation between $p$ and $q$.

As a consequence of Theorem 4.4, we also prove a kind of Kakutani representation theorem (see for instance [7, Theorem 1.b.2]) through the spaces $S^q_{X_p}(\xi)$ for $p$-convex Banach function spaces which are $p$-strongly $q$-concave (Corollary 4.5).

2. Preliminaries

Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space and denote by $L^0(\mu)$ the space of all measurable real functions on $\Omega$, where functions which are equal $\mu$-a.e. are identified. By a Banach function space (briefly B.f.s.) we mean a Banach space $X \subset L^0(\mu)$ with norm $\| \cdot \|_X$, such that if $f \in L^0(\mu)$, $g \in X$ and $|f| \leq |g|$ $\mu$-a.e. then $f \in X$ and $\|f\|_X \leq \|g\|_X$. In particular, $X$ is a Banach lattice with the $\mu$-a.e. pointwise order, in which the convergence in norm of a sequence implies the convergence $\mu$-a.e. for some subsequence. A B.f.s. $X$ is said to be saturated if there exists no $A \in \Sigma$ with $\mu(A) > 0$ such that $f\chi_A = 0$ $\mu$-a.e. for all $f \in X$, or equivalently, if $X$ has a weak unit (i.e. $g \in X$ such that $g \geq 0$ $\mu$-a.e.).

Lemma 2.1. Let $X$ be a saturated B.f.s. For every $f \in L^0(\mu)$, there exists $(f_n)_{n \geq 1} \subset X$ such that $0 \leq f_n \uparrow |f|$ $\mu$-a.e.

Proof. Consider a weak unit $g \in X$ and take $g_n = ng/(1 + ng)$. Note that $0 < g_n < ng \mu$-a.e., so $g_n$ is a weak unit in $X$. Moreover, $(g_n)_{n \geq 1}$ increases $\mu$-a.e. to the constant function equal to 1. Now, take $f_n = g_n|f|\chi_{\{\omega \in \Omega : |f| \leq n\}}$. Since $0 \leq f_n \leq ng_n \mu$-a.e., we have that $f_n \in X$, and $f_n \uparrow |f|$ $\mu$-a.e. □

The Köthe dual of a B.f.s. $X$ is the space $X'$ given by the functions $h \in L^0(\mu)$ such that $\int |hf|d\mu < \infty$ for all $f \in X$. If $X$ is saturated then $X'$ is a saturated B.f.s. with norm $\|h\|_{X'} = \sup_{f \in B_X} \int |hf|d\mu$ for $h \in X'$. Here, as usual, $B_X$ denotes the closed unit ball of $X$. Each function $h \in X'$ defines a functional $\zeta(h)$ on $X$ by $\langle \zeta(h), f \rangle = \int hf \, d\mu$ for all $f \in X$. In fact, $X'$ is isometrically order isomorphic (via $\zeta$) to a closed subspace of the topological dual $X^*$ of $X$.

From now and on, a B.f.s. $X$ will be assumed to be saturated. If for every $f, f_n \in X$ such that $0 \leq f_n \uparrow f$ $\mu$-a.e. it follows that $\|f_n\|_X \uparrow$
∥f∥_X, then X is said to be order semi-continuous. This is equivalent to ζ(X') being a norming subspace of X*, i.e. ∥f∥_X = \sup_{h\in B_{X'}} |fh| d\mu for all f ∈ X. A B.f.s. X is order continuous if for every f, f_n ∈ X such that 0 ≤ f_n ↑ f µ-a.e., it follows that f_n → f in norm. In this case, X' can be identified with X*.

For general issues related to B.f.s.’ see [7], [8] and [10, Ch. 15] considering the function norm ρ defined as ρ(f) = ∥f∥_X if f ∈ X and ρ(f) = ∞ in other case.

Let 1 ≤ p < ∞. A B.f.s. X is said to be p-convex if there exists a constant C > 0 such that

\[ \left\| \left( \sum_{i=1}^{n} |f_i|^p \right)^{1/p} \right\|_X \leq C \left( \sum_{i=1}^{n} \|f_i\|_X^p \right)^{1/p} \]

for every finite subset (f_i)_{i=1}^{n} ⊂ X. In this case, M^p(X) will denote the smallest constant C satisfying the above inequality. Note that M^p(X) ≥ 1. A relevant fact is that every p-convex B.f.s. X has an equivalent norm for which X is p-convex with constant M^p(X) = 1, see [7, Proposition 1.d.8].

The p-th power of a B.f.s. X is the space defined as

\[ X_p = \{ f ∈ L^0(\mu) : |f|^{1/p} ∈ X \}, \]

endowed with the quasi-norm ∥f∥_{X_p} = ∥|f|^{1/p}∥_X^p, for f ∈ X_p. Note that X_p is always complete, see the proof of [8, Proposition 2.22]. If X is p-convex with constant M^p(X) = 1, from [3] Lemma 3], ∥·∥_{X_p} is a norm and so X_p is a B.f.s. Note that X_p is saturated if and only if X is so. The same holds for the properties of being order continuous and order semi-continuous.

3. The space S_{X_p}^q(ξ)

Let 1 ≤ p ≤ q < ∞ and let X be a saturated p-convex B.f.s. We can assume without loss of generality that the p-convexity constant M^p(X) is equal to 1. Then, X_p and (X_p)' are saturated B.f.s.’. Consider the topology σ((X_p)', X_p) on (X_p)' defined by the elements of X_p. Note that the subset B_i^+(X_p)' of all positive elements of the closed unit ball of (X_p)' is compact for this topology.
Let $\xi$ be a finite positive Radon measure on $B_{(X_p)^\prime}^+$. For $f \in L^0(\mu)$, consider the map $\phi_f : B_{(X_p)^\prime}^+ \to [0, \infty]$ defined by

$$\phi_f(h) = \left( \int_{\Omega} |f(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p}$$

for all $h \in B_{(X_p)^\prime}^+$. In the case when $f \in X$, since $|f|^p \in X_p$, it follows that $\phi_f$ is continuous and so measurable. For a general $f \in L^0(\mu)$, by Lemma 2.1, we can take a sequence $(f_n)_{n \geq 1} \subset X$ such that $0 \leq f_n \uparrow |f| \mu$-a.e. Applying monotone convergence theorem, we have that $\phi_{f_n} \uparrow \phi_f$ pointwise and so $\phi_f$ is measurable. Then, we can consider the integral

$$\int_{B_{(X_p)^\prime}^+} \phi_f(h) \, d\xi(h) \in [0, \infty]$$

and define the following space:

$$S_{X_p}^q(\xi) = \left\{ f \in L^0(\mu) : \int_{B_{(X_p)^\prime}^+} \left( \int_{\Omega} |f(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p} \, d\xi(h) < \infty \right\}.$$

Let us endow $S_{X_p}^q(\xi)$ with the seminorm

$$\|f\|_{S_{X_p}^q(\xi)} = \left( \int_{B_{(X_p)^\prime}^+} \left( \int_{\Omega} |f(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p} \, d\xi(h) \right)^{1/q} = \left\| h \rightarrow \left\| f|h|^{1/p} \right\|_{L^q(\mu)} \right\|_{L^q(\xi)}.$$

In general, $\| \cdot \|_{S_{X_p}^q(\xi)}$ is not a norm. For instance, if $\xi$ is the Dirac measure at some $h_0 \in B_{(X_p)^\prime}^+$ such that $A = \{ \omega \in \Omega : h_0(\omega) = 0 \}$ satisfies $\mu(A) > 0$, taking $f = g\chi_A \in X$ with $g$ being a weak unit of $X$, we have that

$$\|f\|_{S_{X_p}^q(\xi)} = \left( \int_A |g(\omega)|^p h_0(\omega) \, d\mu(\omega) \right)^{1/p} = 0$$

and

$$\mu(\{ \omega \in \Omega : f(\omega) \neq 0 \}) = \mu(A \cap \{ \omega \in \Omega : g(\omega) \neq 0 \}) = \mu(A) > 0.$$ 

**Proposition 3.1.** If the Radon measure $\xi$ satisfies

$$\int_{B_{(X_p)^\prime}^+} \left( \int_A h(\omega) \, d\mu(\omega) \right)^{q/p} \, d\xi(h) = 0 \Rightarrow \mu(A) = 0 \quad (3.1)$$

then, $S_{X_p}^q(\xi)$ is a saturated B.f.s. Moreover, $S_{X_p}^q(\xi)$ is order continuous, $p$-convex (with constant 1) and $X \subset S_{X_p}^q(\xi)$ continuously.
Proof. It is clear that if $f \in L^0(\mu)$, $g \in S_{X_p}^q(\xi)$ and $|f| \leq |g|$ $\mu$-a.e. then $f \in S_{X_p}^q(\xi)$ and $\|f\|_{S_{X_p}^q(\xi)} \leq \|g\|_{S_{X_p}^q(\xi)}$. Let us see that $\|\cdot\|_{S_{X_p}^q(\xi)}$ is a norm. Suppose that $\|f\|_{S_{X_p}^q(\xi)} = 0$ and set $A_n = \{\omega \in \Omega : |f(\omega)| > \frac{1}{n}\}$ for every $n \geq 1$. Since $\chi_{A_n} \leq n|f|$ and

$$\int_{B_{(X_p)}^+}(\int_{A_n} h(\omega) \, d\mu(\omega))^{q/p} \, d\xi(h) = \|\chi_{A_n}\|_{S_{X_p}^q(\xi)}^q \leq n^q \|f\|_{S_{X_p}^q(\xi)} = 0,$$

from (3.1) we have that $\mu(A_n) = 0$ and so

$$\mu(\{\omega \in \Omega : f(\omega) \neq 0\}) = \lim_{n \to \infty} \mu(A_n) = 0.$$

Now we will see that $S_{X_p}^q(\xi)$ is complete by showing that $\sum_{n \geq 1} f_n \in S_{X_p}^q(\xi)$ whenever $(f_n)_{n \geq 1} \subset S_{X_p}^q(\xi)$ with $C = \sum \|f_n\|_{S_{X_p}^q(\xi)} < \infty$. First let us prove that $\sum_{n \geq 1} |f_n| < \infty$ $\mu$-a.e. For every $N, n \geq 1$, taking $A_N^n = \{\omega \in \Omega : \sum_{j=1}^n |f_j(\omega)| > N\}$, since $\chi_{A_N^n} \leq \frac{1}{N} \sum_{j=1}^n |f_j|$, we have that

$$\int_{B_{(X_p)}^+}(\int_{A_N^n} h(\omega) \, d\mu(\omega))^{q/p} \, d\xi(h) = \|\chi_{A_N^n}\|_{S_{X_p}^q(\xi)}^{q} \leq \frac{1}{N^q} \sum_{j=1}^n |f_j| \|f_j\|_{S_{X_p}^q(\xi)}^{q} \leq \frac{C^n}{N^q}.$$

Note that, for $N$ fixed, $(A_N^n)_{n \geq 1}$ increases. Taking limit as $n \to \infty$ and applying twice the monotone convergence theorem, it follows that

$$\int_{B_{(X_p)}^+}(\int_{\bigcup_{n \geq 1} A_N^n} h(\omega) \, d\mu(\omega))^{q/p} \, d\xi(h) \leq \frac{C^n}{N^q}.$$

Then,

$$\int_{B_{(X_p)}^+}(\int_{\bigcap_{N \geq 1} \bigcup_{n \geq 1} A_N^n} h(\omega) \, d\mu(\omega))^{q/p} \, d\xi(h) \leq \lim_{N \to \infty} \frac{C^n}{N^q} = 0,$$

and so, from (3.1),

$$\mu(\left\{\omega \in \Omega : \sum_{n \geq 1} |f_n(\omega)| = \infty\right\}) = \mu(\bigcap_{N \geq 1} \bigcup_{n \geq 1} A_N^n) = 0.$$
Hence, $\sum_{n \geq 1} f_n \in L^0(\mu)$. Again applying the monotone convergence theorem, it follows that

$$
\int_{B_{(X_p)'}^+} \left( \int_{\Omega} \left| \sum_{n \geq 1} f_n(\omega) \right|^p h(\omega) \, d\mu(\omega) \right)^{q/p} \, d\xi(h) \leq \int_{B_{(X_p)'}^+} \left( \int_{\Omega} \left| \sum_{n \geq 1} |f_n(\omega)|^p h(\omega) \, d\mu(\omega) \right|^{q/p} \, d\xi(h) =
\lim_{n \to \infty} \int_{B_{(X_p)'}^+} \left( \int_{\Omega} \left| \sum_{j=1}^n |f_j(\omega)|^p h(\omega) \, d\mu(\omega) \right|^{q/p} \, d\xi(h) =
\lim_{n \to \infty} \left\| \sum_{j=1}^n |f_j| \right\|_{S_{X_p}^q(\xi)}^q \leq C^q
$$

and thus $\sum_{n \geq 1} f_n \in S_{X_p}^q(\xi)$.

Note that if $f \in X$, for every $h \in B_{(X_p)'}^+$, we have that

$$
\int_{\Omega} |f(\omega)|^p h(\omega) \, d\mu(\omega) \leq \|f\|_{X_p}^p \|h\|_{(X_p)'} \leq \|f\|_{X_p}^p
$$

and so

$$
\int_{B_{(X_p)'}^+} \left( \int_{\Omega} |f(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p} d\xi(h) \leq \|f\|_{X_p}^q \xi\left(B_{(X_p)'}^+\right).
$$

Then, $X \subset S_{X_p}^q(\xi)$ and $\|f\|_{S_{X_p}^q(\xi)} \leq \xi\left(B_{(X_p)'}^+\right)^{1/q} \|f\|_X$ for all $f \in X$. In particular, $S_{X_p}^q(\xi)$ is saturated, as a weak unit in $X$ is a weak unit in $S_{X_p}^q(\xi)$.

Let us show that $S_{X_p}^q(\xi)$ is order continuous. Consider $f, f_n \in S_{X_p}^q(\xi)$ such that $0 \leq f_n \uparrow f$ $\mu$-a.e. Note that, since

$$
\int_{B_{(X_p)'}^+} \left( \int_{\Omega} |f(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p} d\xi(h) < \infty,
$$

there exists a $\xi$-measurable set $B$ with $\xi(B_{(X_p)'} \setminus B) = 0$ such that $\int_{\Omega} |f(\omega)|^p h(\omega) \, d\mu(\omega) < \infty$ for all $h \in B$. Fixed $h \in B$, we have that $|f - f_n|^p h \downarrow 0$ $\mu$-a.e. and $|f - f_n|^p h \leq |f|^p h$ $\mu$-a.e. Then, applying the dominated convergence theorem, $\int_{\Omega} |f(\omega) - f_n(\omega)|^p h(\omega) \, d\mu(\omega) \downarrow 0$. 
Consider the measurable functions \( \phi, \phi_n : B^+_{(X_p)'} \to [0, \infty] \) given by
\[
\phi(h) = \left( \int_{\Omega} |f(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p}
\]
\[
\phi_n(h) = \left( \int_{\Omega} |f(\omega) - f_n(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p}
\]
for all \( h \in B^+_{(X_p)'} \). It follows that \( \phi_n \downarrow 0 \) \( \xi \)-a.e. and \( \phi_n \leq \phi \) \( \xi \)-a.e. Again by the dominated convergence theorem, we obtain
\[
\|f - f_n\|_{S^q_{X_p}(\xi)} = \int_{B^+_{(X_p)'}} \phi_n(h) \, d\xi(h) \downarrow 0.
\]

Finally, let us see that \( S^q_{X_p}(\xi) \) is \( p \)-convex. Fix \( (f_i)_{i=1}^n \subset S^q_{X_p}(\xi) \) and consider the measurable functions \( \phi_i : B^+_{(X_p)'} \to [0, \infty] \) \( (1 \leq i \leq n) \) defined by
\[
\phi_i(h) = \int_{\Omega} |f_i(\omega)|^p h(\omega) \, d\mu(\omega).
\]
for all \( h \in B^+_{(X_p)'} \). Then,
\[
\left\| \left( \sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_{S^q_{X_p}(\xi)}^q = \int_{B^+_{(X_p)'}} \left( \int_{\Omega} \sum_{i=1}^n |f_i(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p} \, d\xi(h)
\]
\[
= \int_{B^+_{(X_p)'}} \left( \sum_{i=1}^n \phi_i(h) \right)^{q/p} \, d\xi(h)
\]
\[
\leq \left( \sum_{i=1}^n \|\phi_i\|_{L^{q/p}(\xi)} \right)^{q/p}.
\]
Since \( \|\phi_i\|_{L^{q/p}(\xi)} = \|f_i\|_{S^q_{X_p}(\xi)} \) for all \( 1 \leq i \leq n \), we have that
\[
\left\| \left( \sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_{S^q_{X_p}(\xi)} \leq \left( \sum_{i=1}^n \|f_i\|_{S^q_{X_p}(\xi)}^p \right)^{1/p}.
\]

\[\blacksquare\]

**Example 3.2.** Take a weak unit \( g \in (X_p)' \) and consider the Radon measure \( \xi \) as the Dirac measure at \( g \). If \( A \in \Sigma \) is such that
\[
0 = \int_{B^+_{(X_p)'}} \left( \int_A h(\omega) \, d\mu(\omega) \right)^{q/p} \, d\xi(h) = \left( \int_A g(\omega) \, d\mu(\omega) \right)^{q/p}
\]
then, \( g\chi_A = 0 \) \( \mu \)-a.e. and so, since \( g > 0 \) \( \mu \)-a.e., \( \mu(A) = 0 \). That is, \( \xi \) satisfies \((3.1)\). In this case, \( S^q_{X_p}(\xi) = L^p(gd\mu) \) with equal norms, as
\[
\int_{B^+(X_p)'} \left( \int_{\Omega} |f(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p} \, d\xi(h) = \left( \int_{\Omega} |f(\omega)|^p g(\omega) \, d\mu(\omega) \right)^{q/p}
\]
for all \( f \in L^0(\mu) \).

**Example 3.3.** Write \( \Omega = \bigcup_{n \geq 1} \Omega_n \) with \((\Omega_n)_{n \geq 1}\) being a disjoint sequence of measurable sets and take a sequence of strictly positive elements \((\alpha_n)_{n \geq 1} \in \ell^1\). Let us consider the Radon measure \( \xi = \sum_{n \geq 1} \alpha_n \delta_{g\chi_{\Omega_n}} \) on \( B^+(X_p)' \), where \( \delta_{g\chi_{\Omega_n}} \) is the Dirac measure at \( g\chi_{\Omega_n} \) with \( g \in (X_p)' \) being a weak unit. Note that for every positive function \( \phi \in L^0(\xi) \), it follows that \( \int_{B^+(X_p)'} \phi \, d\xi = \sum_{n \geq 1} \alpha_n \phi(g\chi_{\Omega_n}) \). If \( A \in \Sigma \) is such that \( 0 = \int_{B^+(X_p)'} \left( \int_A h(\omega) \, d\mu(\omega) \right)^{q/p} \, d\xi(h) = \sum_{n \geq 1} \alpha_n \left( \int_{A \cap \Omega_n} g(\omega) \, d\mu(\omega) \right)^{q/p} \)
then, \( \int_{A \cap \Omega_n} g(\omega) \, d\mu(\omega) = 0 \) for all \( n \geq 1 \). Hence,
\[
\int_A g(\omega) \, d\mu(\omega) = \sum_{n \geq 1} \int_{A \cap \Omega_n} g(\omega) \, d\mu(\omega) = 0
\]
and so \( g\chi_A = 0 \) \( \mu \)-a.e., from which \( \mu(A) = 0 \). That is, \( \xi \) satisfies \((3.1)\). For every \( f \in L^0(\mu) \) we have that
\[
\int_{B^+(X_p)'} \left( \int_{\Omega} |f(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p} \, d\xi(h) = \sum_{n \geq 1} \alpha_n \left( \int_{\Omega_n} |f(\omega)|^p g(\omega) \, d\mu(\omega) \right)^{q/p}.
\]
Then, the B.f.s. \( S^q_{X_p}(\xi) \) can be described as the space of functions \( f \in \bigcap_{n \geq 1} L^p(g\chi_{\Omega_n} \, d\mu) \) such that \( \left( \alpha_n^{1/q} \|f\|_{L^p(g\chi_{\Omega_n} \, d\mu)} \right)_{n \geq 1} \in \ell^q \). Moreover, \( \|f\|_{S^q_{X_p}(\xi)} = \left( \sum_{n \geq 1} \alpha_n \|f\|^q_{L^p(g\chi_{\Omega_n} \, d\mu)} \right)^{1/q} \) for all \( f \in S^q_{X_p}(\xi) \).

4. \( p \)-Strongly \( q \)-Concave Operators

Let \( 1 \leq p \leq q < \infty \) and let \( T \colon X \to E \) be a linear operator from a saturated B.f.s. \( X \) into a Banach space \( E \). Recall that \( T \) is said to be
\( q \)-concave if there exists a constant \( C > 0 \) such that
\[
\left( \sum_{i=1}^{n} \| T(f_i) \|_E^q \right)^{1/q} \leq C \left( \sum_{i=1}^{n} |f_i|^q \right)^{1/q}
\]
for every finite subset \( (f_i)_{i=1}^{n} \subset X \). The smallest possible value of \( C \) will be denoted by \( M_q(T) \). For issues related to \( q \)-concavity see for instance [7, Ch. 1.d]. We introduce a little stronger notion than \( q \)-concavity: \( T \) will be called \( p \)-strongly \( q \)-concave if there exists \( C > 0 \) such that
\[
\left( \sum_{i=1}^{n} \| T(f_i) \|_E^q \right)^{1/q} \leq C \sup_{(\beta_i)_{i \geq 1} \in B_{\ell^r}} \left( \sum_{i=1}^{n} |\beta_i f_i|^p \right)^{1/p}
\]
for every finite subset \( (f_i)_{i=1}^{n} \subset X \), where \( 1 < r \leq \infty \) is such that \( \frac{1}{r} = \frac{1}{p} - \frac{1}{q} \). In this case, \( M_{p,q}(T) \) will denote the smallest constant \( C \) satisfying the above inequality. Noting that \( \frac{q}{p} \) and \( \frac{p}{q} \) are conjugate exponents, it is clear that every \( p \)-strongly \( q \)-concave operator is \( q \)-concave and so continuous, and moreover \( \| T \| \leq M_q(T) \leq M_{p,q}(T) \).

As usual, we will say that \( X \) is \( p \)-strongly \( q \)-concave if the identity map \( I : X \rightarrow X \) is so, and in this case, we denote \( M_{p,q}(X) = M_{p,q}(I) \).

Our goal is to get a continuous extension of \( T \) to a space of the type \( S_{X_p}^q(\xi) \) in the case when \( T \) is \( p \)-strongly \( q \)-concave and \( X \) is \( p \)-convex. To this end we will need to describe the supremum on the right-hand side of the \( p \)-strongly \( q \)-concave inequality in terms of the Köthe dual of \( X_p \).

**Lemma 4.1.** If \( X \) is \( p \)-convex and order semi-continuous then
\[
\sup_{(\beta_i)_{i \geq 1} \in B_{\ell^r}} \left( \sum_{i=1}^{n} |\beta_i f_i|^p \right)^{1/p} \leq M_{p,q}(T) \sup_{h \in B^{+}_{(X_p)'}} \left( \int |f_i|^p h \, d\mu \right)^{q/p} \]
for every finite subset \( (f_i)_{i=1}^{n} \subset X \), where \( 1 < r \leq \infty \) is such that \( \frac{1}{r} = \frac{1}{p} - \frac{1}{q} \) and \( B^{+}_{(X_p)'} \) is the subset of all positive elements of the closed unit ball \( B_{(X_p)'} \) of \( (X_p)' \).
Proof. Given \((f_i)_{i=1}^n \subset X\), since \(X_p\) is order semi-continuous, as \(X\) is so, and \((\ell^{q/p})^* = \ell^{r/p}\), as \(\frac{r}{p}\) is the conjugate exponent of \(\frac{q}{p}\), we have that

\[
\sup_{(\beta_i) \in B_{X^{p'}}} \left\| \left( \sum_{i=1}^n |\beta_i f_i|^p \right)^{1/p} \right\|_X = \sup_{(\beta_i) \in B_{X^{p'}}} \left\| \sum_{i=1}^n |\beta_i f_i|^p \right\|_X
\]

\[
= \sup_{(\beta_i) \in B_{X^{p'}}} \sup_{h \in B_\Omega \ell^q} \int \sum_{i=1}^n |\beta_i f_i|^p |h| d\mu
\]

\[
= \sup_{h \in B_\Omega \ell^q} \sup_{(\beta_i) \in B_{X^{p'}}} \int \sum_{i=1}^n |\beta_i f_i|^p h d\mu
\]

\[
= \sup_{h \in B_\Omega \ell^q} \sup_{(\alpha_i) \in B_{\ell^r/p}} \int \sum_{i=1}^n |\beta_i f_i|^p h d\mu
\]

\[
= \sup_{h \in B_\Omega \ell^q} \left( \sum_{i=1}^n \left( \int |f_i|^p h d\mu \right)^{q/p} \right)^{p/q}.
\]

In the following remark, from Lemma 4.1, we obtain easily an example of \(p\)-strongly \(q\)-concave operator.

**Remark 4.2.** Suppose that \(X\) is \(p\)-convex and order semi-continuous. For every finite positive Radon measure \(\xi\) on \(B_\Omega^{\ell^q}\) satisfying (3.1), it follows that the inclusion map \(i: X \to S_{X_p}^q(\xi)\) is \(p\)-strongly \(q\)-concave. Indeed, for each \((f_i)_{i=1}^n \subset X\), we have that

\[
\sum_{i=1}^n \|f_i\|_{S_{X_p}^q(\xi)}^q = \sum_{i=1}^n \int_{B_\Omega^{\ell^q}} \left( \int_{\Omega} |f_i(\omega)|^p h(\omega) d\mu(\omega) \right)^{q/p} d\xi(h)
\]

\[
\leq \xi(B_\Omega^{\ell^q}) \sup_{h \in B_\Omega^{\ell^q}} \sum_{i=1}^n \left( \int_{\Omega} |f_i(\omega)|^p h(\omega) d\mu(\omega) \right)^{q/p}
\]

and so, Lemma 4.1 gives the conclusion for \(M_{p,q}(i) \leq \xi(B_\Omega^{\ell^q})^{1/q}\).

Now let us prove our main result.
Theorem 4.3. If $T$ is $p$-strongly $q$-concave and $X$ is $p$-convex and order semi-continuous, then there exists a probability Radon measure $\xi$ on $B^+_{(X_p)'}$ satisfying (3.1) such that

$$
\|T(f)\|_E \leq M_{p,q}(T) \left( \int_{B^+_{(X_p)'}} \left( \int_{\Omega} |f(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p} d\xi(h) \right)^{1/q}
$$

(4.1)

for all $f \in X$.

Proof. Recall that the stated topology on $(X_p)'$ is $\sigma((X_p)', X_p)$, the one which is defined by the elements of $(X_p)'_0$ (with possibly repeated elements). For each finite subset $M = \{f_i\}_{i=1}^m \subset X$, consider the map $\psi_M : B^+_{(X_p)'} \to [0, \infty]$ defined by $\psi_M(h) = \sum_{i=1}^m \left( \int_{\Omega} |f_i|^p h \, d\mu \right)^{q/p}$ for $h \in B^+_{(X_p)'}$. Note that $\psi_M$ attains its supremum as it is continuous on a compact set, so there exists $h_M \in B^+_{(X_p)'}$ such that $\sup_{h \in B^+_{(X_p)'}} \psi_M(h) = \psi_M(h_M)$. Then, the $p$-strongly $q$-concavity of $T$, together with Lemma 4.1 gives

$$
\sum_{i=1}^m \|T(f_i)\|_E^q \leq M_{p,q}(T)^q \sup_{h \in B^+_{(X_p)'}} \sum_{i=1}^m \left( \int_{\Omega} |f_i|^p h \, d\mu \right)^{q/p} \\
\leq M_{p,q}(T)^q \sup_{h \in B^+_{(X_p)'}} \psi_M(h) \\
= M_{p,q}(T)^q \psi_M(h_M).
$$

(4.2)

Consider now the continuous map $\phi_M : B^+_{(X_p)'} \to \mathbb{R}$ defined by

$$
\phi_M(h) = M_{p,q}(T)^q \psi_M(h) - \sum_{i=1}^m \|T(f_i)\|_E^q
$$

for $h \in B^+_{(X_p)'}$. Take $B = \{\phi_M : M \text{ is a finite subset of } X\}$. Since for every $M = \{f_i\}_{i=1}^m$, $M' = \{f'_i\}_{i=1}^k \subset X$ and $0 < t < 1$, it follows that $t\phi_M + (1-t)\phi_{M'} = \phi_{M''}$ where $M'' = \{(t^{1/q}f_i)\}_{i=1}^m \cup ((1-t)^{1/q}f'_i)_{i=1}^k$, we have that $B$ is convex. Denote by $\mathcal{C}(B^+_{(X_p)'})$ the space of continuous real functions on $B^+_{(X_p)'}$, endowed with the supremum norm, and by $A$ the open convex subset $\{\phi \in \mathcal{C}(B^+_{(X_p)}') : \phi(h) < 0 \text{ for all } h \in B^+_{(X_p)'}\}$. By (4.2) we have that $A \cap B = \emptyset$. From the Hahn-Banach separation theorem, there exist $\xi \in \mathcal{C}(B^+_{(X_p)'})^*$ and $\alpha \in \mathbb{R}$ such that $\langle \xi, \phi \rangle < \alpha \leq \langle \xi, \phi_M \rangle$ for all $\phi \in A$ and $\phi_M \in B$. Since every negative constant
function is in \( A \), it follows that \( 0 \leq \alpha \). Even more, \( \alpha = 0 \) as the constant function equal to 0 is just \( \phi_{\{0\}} \in B \). It is routine to see that \( \langle \xi, \phi \rangle \geq 0 \) whenever \( \phi \in \mathcal{C}(B^+_{(X_p)'} \rangle \) is such that \( \phi(h) \geq 0 \) for all \( h \in B^+_{(X_p)'} \). Then, \( \xi \) is a positive linear functional on \( \mathcal{C}(B^+_{(X_p)'} \rangle \) and so it can be interpreted as a finite positive Radon measure on \( B^+_{(X_p)'} \). Hence, we have that

\[
0 \leq \int_{B^+_{(X_p)'} \rangle} \phi_M \, d\xi
\]

for all finite subset \( M \subset X \). Dividing by \( \xi(B^+_{(X_p)'} \rangle) \), we can suppose that \( \xi \) is a probability measure. Then, for \( M = \{ f \} \) with \( f \in X \), we obtain that

\[
\|T(f)\|_E^q \leq M_{p,q}(T)^q \int_{B^+_{(X_p)'} \rangle} \left( \int_{\Omega} |f(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p} \, d\xi(h)
\]

and so (4.1) holds. \( \square \)

Actually, Theorem 4.3 says that we can find a probability Radon measure \( \xi \) on \( B^+_{(X_p)'} \rangle \) such that \( T : X \to E \) is continuous when \( X \) is considered with the norm of the space \( S^q_{X_p}(\xi) \). In the next result we will see how to extend \( T \) continuously to \( S^q_{X_p}(\xi) \). Even more, we will show that this extension is possible if and only if \( T \) is \( p \)-strongly \( q \)-concave.

**Theorem 4.4.** Suppose that \( X \) is \( p \)-convex and order semi-continuous. The following statements are equivalent:

\( (a) \) \( T \) is \( p \)-strongly \( q \)-concave.

\( (b) \) There exists a probability Radon measure \( \xi \) on \( B^+_{(X_p)'} \rangle \) satisfying (3.1) such that \( T \) can be extended continuously to \( S^q_{X_p}(\xi) \), i.e. there is a factorization for \( T \) as

\[
\begin{array}{ccc}
X & \xrightarrow{T} & E \\
\downarrow{i} & & \downarrow{\tilde{T}} \\
S^q_{X_p}(\xi) & \xrightarrow{\tilde{T}} & E
\end{array}
\]

where \( \tilde{T} \) is a continuous linear operator and \( i \) is the inclusion map.

If \( (a)-(b) \) holds, then \( M_{p,q}(T) = \|\tilde{T}\| \).
Proof. (a) ⇒ (b) From Theorem 4.3, there is a probability Radon measure $\xi$ on $B^+_p(X_p)$ satisfying (3.1) such that $\|T(f)\|_E \leq M_{p,q}(T)\|f\|_{S^q_p(X_p)}$ for all $f \in X$. Given $0 \leq f \in S^q_p(X_p)$, from Lemma 2.1, we can take $(f_n)_{n \geq 1} \subset X$ such that $0 \leq f_n \uparrow f$ $\mu$-a.e. Then, since $S^q_p(X_p)$ is order continuous, we have that $f_n \to f$ in $S^q_p(X_p)$ and so $(T(f_n))_{n \geq 1}$ converges to some element $e$ of $E$. Define $\tilde{T}(f) = e$. Note that $\tilde{T}$ is well defined, since if $(g_n)_{n \geq 1} \subset X$ is such that $0 \leq g_n \uparrow f$ $\mu$-a.e., then

$$\|T(f_n) - T(g_n)\|_E \leq M_{p,q}(T)\|f_n - g_n\|_{S^q_p(X_p)} \to 0.$$ 

Moreover,

$$\|\tilde{T}(f)\|_E = \lim_{n \to \infty} \|T(f_n)\|_E \leq M_{p,q}(T)\lim_{n \to \infty} \|f_n\|_{S^q_p(X_p)} = M_{p,q}(T)\|f\|_{S^q_p(X_p)}.$$ 

For a general $f \in S^q_p(X_p)$, writing $f = f^+ - f^-$ where $f^+$ and $f^-$ are the positive and negative parts of $f$ respectively, we define $\tilde{T}(f) = \tilde{T}(f^+) - \tilde{T}(f^-)$. Then, $\tilde{T}: S^q_p(X_p) \to E$ is a continuous linear operator extending $T$. Moreover $\|\tilde{T}\| \leq M_{p,q}(T)$. Indeed, let $f \in S^q_p(X_p)$ and take $(f^+_n)_{n \geq 1}$, $(f^-_n)_{n \geq 1} \subset X$ such that $0 \leq f^+_n \uparrow f^+$ and $0 \leq f^-_n \uparrow f^-$ $\mu$-a.e. Then, $f^+_n - f^-_n \to f$ in $S^q_p(X_p)$ and

$$T(f^+_n - f^-_n) = T(f^+_n) - T(f^-_n) \to \tilde{T}(f^+) - \tilde{T}(f^-) = \tilde{T}(f)$$

in $E$. Hence,

$$\|\tilde{T}(f)\|_E = \lim_{n \to \infty} \|T(f^+_n - f^-_n)\|_E \leq M_{p,q}(T)\lim_{n \to \infty} \|f^+_n - f^-_n\|_{S^q(X_p)} = M_{p,q}(T)\|f\|_{S^q(X_p)}.$$
Given \((f_i)_{i=1}^n \subset X\), we have that

\[
\sum_{i=1}^n \|T(f_i)\|_E^q = \sum_{i=1}^n \|\tilde{T}(f_i)\|_E^q \leq \|\tilde{T}\|^q \sum_{i=1}^n \|\tilde{T}(f_i)\|_{S_q^{X_p}(\xi)}^q \\
= \|\tilde{T}\|^q \sum_{i=1}^n \int_{B_{(X_p)^\prime}} \left( \int_{\Omega} |f_i(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p} \, d\xi(h) \\
\leq \|\tilde{T}\|^q \sup_{h \in B_{(X_p)^\prime}} \sum_{i=1}^n \left( \int_{\Omega} |f_i(\omega)|^p h(\omega) \, d\mu(\omega) \right)^{q/p}.
\]

That is, from Lemma 4.1, \(T\) is \(p\)-strongly \(q\)-concave with \(M_{p,q}(T) \leq \|\tilde{T}\|\).

A first application of Theorem 4.4 is the following Kakutani type representation theorem (see for instance [7, Theorem 1.b.2]) for B.f.s. being order semi-continuous, \(p\)-convex and \(p\)-strongly \(q\)-concave.

**Corollary 4.5.** Suppose that \(X\) is \(p\)-convex and order semi-continuous. The following statements are equivalent:

(a) \(X\) is \(p\)-strongly \(q\)-concave.

(b) There exists a probability Radon measure \(\xi\) on \(B_{(X_p)^\prime}^+\) satisfying (3.1), such that \(X = S_q^{X_p}(\xi)\) with equivalent norms.

**Proof.** (a) \(\Rightarrow\) (b) The identity map \(I : X \rightarrow X\) is \(p\)-strongly \(q\)-concave as \(X\) is so. Then, from Theorem 4.4, there exists a probability Radon measure \(\xi\) on \(B_{(X_p)^\prime}^+\) satisfying (3.1), such that \(I\) factors as

\[
\begin{array}{ccc}
X & \xrightarrow{I} & X \\
\downarrow{\tilde{I}} & & \downarrow{\tilde{I}} \\
S_q^{X_p}(\xi) & \xrightarrow{i} & X
\end{array}
\]

where \(\tilde{I}\) is a continuous linear operator with \(\|\tilde{I}\| = M_{p,q}(X)\) and \(i\) is the inclusion map. Since \(\xi\) is a probability measure, we have that \(\|f\|_{S_q^{X_p}(\xi)} \leq \|f\|_X\) for all \(f \in X\), see the proof of Proposition 3.1. Let \(0 \leq \tilde{f} \in S_q^{X_p}(\xi)\). By Lemma 2.1, we can take \((f_n)_{n \geq 1} \subset X\) such that \(0 \leq f_n \uparrow \tilde{f}\) \(\mu\)-a.e. Since \(S_q^{X_p}(\xi)\) is order continuous, it follows that \(f_n \rightarrow \tilde{f}\) in \(S_q^{X_p}(\xi)\) and so \(f_n = \tilde{I}(f_n) \rightarrow \tilde{I}(\tilde{f})\) in \(X\). Then, there is a
subsequence of \((f_n)_{n \geq 1}\) converging \(\mu\)-a.e. to \(\tilde{I}(f)\) and hence \(f = \tilde{I}(f) \in X\). For a general \(f \in S^q_{X_p}(\xi)\), writing \(f = f^+ - f^-\) where \(f^+\) and \(f^-\) are the positive and negative parts of \(f\) respectively, we have that \(f = \tilde{I}(f^+) - \tilde{I}(f^-) = \tilde{I}(f) \in X\). Therefore, \(X = S^q_{X_p}(\xi)\) and \(\tilde{I}\) is the identity map. Moreover, \(\|f\|_X = \|\tilde{I}(f)\|_X \leq \|\tilde{I}\| \|f\|_{S^q_{X_p}(\xi)} = M_{p,q}(X) \|f\|_{S^q_{X_p}(\xi)}\) for all \(f \in X\).

(b) \(\Rightarrow\) (a) From Remark 4.2 it follows that the identity map \(I : X \to X\) is \(p\)-strongly \(q\)-concave. \(\square\)

Note that under conditions of Corollary 4.5, if \(X\) is \(p\)-strongly \(q\)-concave with constant \(M_{p,q}(X) = 1\), then \(X = S^q_{X_p}(\xi)\) with equal norms.

5. \(q\)-summing operators on a \(p\)-convex B.f.s.

Recall that a linear operator \(T : X \to E\) between Banach spaces is said to be \(q\)-summing \((1 \leq q < \infty)\) if there exists a constant \(C > 0\) such that

\[
\left( \sum_{i=1}^{n} \|Tx_i\|^q_E \right)^{1/q} \leq C \sup_{x^* \in B_{X^*}} \left( \sum_{i=1}^{n} |\langle x^*, x_i \rangle|^q \right)^{1/q}
\]

for every finite subset \((x_i)_{i=1}^{n} \subset X\). Denote by \(\pi_q(T)\) the smallest possible value of \(C\). Information about \(q\)-summing operators can be found in [6].

One of the main relations between summability and concavity for operators defined on a B.f.s. \(X\), is that every \(q\)-summing operator is \(q\)-concave. This is a consequence of a direct calculation which shows that for every \((f_i)_{i=1}^{n} \subset X\) and \(x^* \in X^*\) it follows that

\[
\left( \sum_{i=1}^{n} |\langle x^*, f_i \rangle|^q \right)^{1/q} \leq \|x^*\|_{X^*} \left( \sum_{i=1}^{n} |f_i|^q \right)^{1/q} \left\| X^* \right\|_X, \tag{5.1}
\]

see for instance [7, Proposition 1.d.9] and the comments below. However, this calculation can be slightly improved to obtain the following result.

**Proposition 5.1.** Let \(1 \leq p \leq q < \infty\). Every \(q\)-summing linear operator \(T : X \to E\) from a B.f.s. \(X\) into a Banach space \(E\), is \(p\)-strongly \(q\)-concave with \(M_{p,q}(T) \leq \pi_q(T)\).
Proof. Let $1 < r \leq \infty$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ and consider a finite subset $(f_i)_{i=1}^n \subset X$. We only have to prove

$$\sup_{x^* \in B_{X^*}} \left( \sum_{i=1}^n |\langle x^*, f_i \rangle|^q \right)^{1/q} \leq \sup_{(\beta_i)_{i \geq 1} \in B_{\ell^r}} \left\| \left( \sum_{i=1}^n |\beta_i f_i|^p \right)^{1/p} \right\|_X.$$ 

Fix $x^* \in B_{X^*}$. Noting that $\frac{2}{p}$ and $\frac{2}{q}$ are conjugate exponents and using the inequality (5.1), we have

$$\left( \sum_{i=1}^n |\langle x^*, f_i \rangle|^q \right)^{1/q} = \sup_{(\alpha_i)_{i \geq 1} \in B_{\ell^p}} \left( \sum_{i=1}^n |\alpha_i||\langle x^*, f_i \rangle|^p \right)^{1/p} = \sup_{(\alpha_i)_{i \geq 1} \in B_{\ell^p}} \left\| \left( \sum_{i=1}^n |\alpha_i|^p \right)^{1/p} \right\|_{\ell^q} \leq \sup_{(\beta_i)_{i \geq 1} \in B_{\ell^r}} \left\| \left( \sum_{i=1}^n |\beta_i|^p \right)^{1/p} \right\|_X.$$

Taking supremum in $x^* \in B_{X^*}$ we get the conclusion. □

From Proposition 5.1, Theorem 4.4 and Remark 4.2, we obtain the final result.

**Corollary 5.2.** Set $1 \leq p \leq q < \infty$. Let $X$ be a saturated order semi-continuous $p$-convex B.f.s. and consider a $q$-summing linear operator $T: X \to E$ with values in a Banach space $E$. Then, there exists a probability Radon measure $\xi$ on $B_{(X_p)^+}^*$ satisfying (3.1) such that $T$ can be factored as

$$\begin{tikzcd}
X \arrow{r}{T} & E \arrow{dr}{\tilde{T}} \\
& & S_{X_p}^q(\xi) \arrow{ul}{\iota}
\end{tikzcd}$$

where $\tilde{T}$ is a continuous linear operator with $\|\tilde{T}\| \leq \pi_q(T)$ and $\iota$ is the inclusion map which turns out to be $p$-strongly $q$-concave, and so $q$-concave.

Observe that what we obtain in Corollary 5.2 is a proper extension for $T$, and not just a factorization as the obtained in the Pietsch theorem for $q$-summing operators through a subspace of an $L^q$-space.
REFERENCES

[1] J. M. Calabuig, O. Delgado and E. A. Sánchez Pérez, Factorizing operators on Banach function spaces through spaces of multiplication operators, J. Math. Anal. Appl. 364 (2010), 88-103.

[2] J. M. Calabuig, J. Rodríguez and E. A. Sánchez Pérez, Strongly embedded subspaces of p-convex Banach function spaces, Positivity 17 (2013), 775-791.

[3] A. Defant, Variants of the Maurey-Rosenthal theorem for quasi Köthe function spaces, Positivity 5 (2001), 153-175.

[4] A. Defant and E. A. Sánchez Pérez, Maurey-Rosenthal factorization of positive operators and convexity, J. Math. Anal. Appl. 297 (2004), 771-790.

[5] O. Delgado and E. A. Sánchez Pérez, Summability properties for multiplication operators on Banach function spaces, Integr. Equ. Oper. Theory 66 (2010), 197-214.

[6] J. Diestel, H. Jarchow and A. Tonge, Absolutely Summing Operators, Cambridge University Press, Cambridge, 1995.

[7] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces II, Springer-Verlag, Berlin, 1979.

[8] S. Okada, W. J. Ricker and E. A. Sánchez Pérez, Optimal Domain and Integral Extension of Operators acting in Function Spaces, Operator Theory: Adv. Appl., vol. 180, Birkhäuser, Basel, 2008.

[9] E. A. Sánchez Pérez, Factorization theorems for multiplication operators on Banach function spaces, Integr. Equ. Oper. Theory 80 (2014), 117-135.

[10] A. C. Zaanen, Integration, 2nd rev. ed., North-Holland, Amsterdam, 1967.

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