Poisson equation for the Mercedes diagram in string theory at genus one

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Abstract
The Mercedes diagram has four trivalent vertices which are connected by six links such that they form the edges of a tetrahedron. This three-loop Feynman diagram contributes to the $\mathcal{D}^{12}R^4$ amplitude at genus one in type II string theory, where the vertices are the points of insertion of the graviton vertex operators, and the links are the scalar propagators on the toroidal worldsheet. We obtain a modular invariant Poisson equation satisfied by the Mercedes diagram, where the source terms involve one- and two-loop Feynman diagrams. We calculate its contribution to the $\mathcal{D}^{12}R^3$ amplitude.

Keywords: string amplitude, supersymmetry, modular invariance

1. Introduction
Perturbative amplitudes in superstring theory in a certain background contain invaluable information about string interactions. When expanded around weak string coupling for a certain compactification, this expansion is an asymptotic expansion which at genus $g$ is of the form $(e^{-2\phi}V)^{1-gf}(\lambda)$, where $\phi$ is the dilaton, $V$ is the volume of the internal manifold in the string frame metric, and $\lambda$ are the various other moduli of the compactification which show up in the perturbative amplitudes. Along with non-perturbative corrections, these amplitudes yield exact S-matrix elements leading to various interactions in the low-energy effective action. Although these terms in the effective action are difficult to determine in general, they can be determined in some cases exactly. These include BPS interactions in toroidally compactified type II string theory which preserve maximal supersymmetry [1–29]. In the Einstein frame, this leads to U-duality covariant equations of motion. Hence, these perturbative amplitudes are useful not only to evaluate perturbative parts of the various S-matrices, but also to understand the role of the non-perturbative U-duality symmetries of the theory.
Let us consider the perturbative amplitude at genus one in type II string theory in ten-dimensional flat spacetime, where the external states involve four on-shell graviton vertex operators. This amplitude is the same in the type IIA and type IIB theories. This has analytic as well as non-analytic terms in the $\alpha'$ expansion. The total amplitude is given by an integral over the fundamental domain of $SL(2, \mathbb{Z})$. While the analytic terms have polynomial dependence on the Mandelstam variables, the non-analytic ones have logarithmic dependence on them. The analytic contribution at every order in the $\alpha'$ expansion involves an integral of the form $[30–32]\[
abla\int_{\mathcal{F}_L} \frac{d^2\tau}{ \tau_2} f(\tau, \bar{\tau}),
abla\]
(1.1)
where
\[
\mathcal{F}_L = \left\{ -\frac{1}{2} \leq \gamma_1 \leq \frac{1}{2}, \quad |\gamma_2| \leq L \right\},
(1.2)
\]
and one takes $L \to \infty$. On the other hand, the non-analytic contribution involves an integral over $\mathcal{R}_L$ defined by
\[
\mathcal{R}_L = \left\{ -\frac{1}{2} \leq \gamma_1 \leq \frac{1}{2}, \quad |\gamma_2| > L \right\},
(1.3)
\]
with appropriate integrands depending on the amplitude. Note that $\mathcal{F}_L \oplus \mathcal{R}_L$ is the fundamental domain of $SL(2, \mathbb{Z})$.

Thus, the analytic contributions are obtained by integrating over the truncated fundamental domain of $SL(2, \mathbb{Z})$, and the integral yields terms finite as well as divergent in this limit. The non-analytic contributions are obtained from the boundary of moduli space as $L \to \infty$ in the integral over $\mathcal{R}_L$. Both the integrals over $\mathcal{F}_L$ and $\mathcal{R}_L$ have terms that diverge as $L \to \infty$, but the total divergence cancels at each order in the $\alpha'$ expansion. The remaining finite contributions are the contributions to the one-loop amplitude.

We shall be concerned with analytic terms obtained by integrating over $\mathcal{F}_L$ for the four-graviton amplitude. They involve modular invariant integrands which satisfy Poisson equations. These integrands are completely determined by the topology of the Feynman diagrams resulting from joining the positions of the vertex operators by scalar propagators on the toroidal worldsheet in various ways. Using these Poisson equations, the contribution of the four-graviton amplitude up to the $D^{10}\mathcal{R}^4$ interaction in the low-energy expansion has been worked out $[32,33]$. Among the several worldsheet Feynman diagrams that contribute to the $D^{12}\mathcal{R}^4$ interaction, we consider the contribution of the Mercedes diagram to the ten-dimensional amplitude. Note that these amplitudes, beyond the $D^{10}\mathcal{R}^4$ interaction, contribute to non-BPS terms in the effective action.

We begin with a brief review of the genus one four-graviton amplitude in the type II theory in ten dimensions. This is followed by a general discussion of the moduli dependence of the integrand, and how we propose to analyze it using Beltrami differentials. We then derive the Poisson equation satisfied by the Mercedes diagram, which involves four vertices and six propagators connecting them, such that every vertex is trivalent. This is a three-loop Feynman diagram. We show that the Poisson equation for this diagram involves source terms with one- and two-loop Feynman diagrams. Finally, we calculate its contribution to the $D^{12}\mathcal{R}^4$ amplitude by integrating over $\mathcal{F}_L$ and keeping the finite terms as $L \to \infty$. 

2. The general structure of the type II one-loop four-graviton amplitude

The one-loop four-graviton amplitude in type II superstring theory is given by
\[ A_4^{(1)} = 2\pi \mathcal{I}(s, t, u) \mathcal{R}^4, \] (2.4)
where
\[ \mathcal{I}(s, t, u) = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} F(s, t, u; \tau, \bar{\tau}), \] (2.5)
where we have integrated over \( \mathcal{F} \), the fundamental domain of \( SL(2, \mathbb{Z}) \). The Mandelstam variables \( s, t, u \) satisfy the on-shell condition
\[ s + t + u = 0. \] (2.6)

We have defined \( d^2 \tau = d\tau_1 d\tau_2 \). The factor \( F(s, t, u; \tau, \bar{\tau}) \) which encodes the moduli dependence is given by
\[ F(s, t, u; \tau, \bar{\tau}) = \prod_{i=1}^{4} \int_{\Sigma} \frac{d^2 z(i)}{\tau_2} e^{D}. \] (2.7)

Here \( z(i) (i = 1, 2, 3, 4) \) are the positions of insertions of the four vertex operators on the toroidal worldsheet \( \Sigma \). Hence \( d^2 z(i) = d(Re z(i)) d(Im z(i)) \), where
\[ -\frac{1}{2} \leq Re z(i) \leq \frac{1}{2}, \quad 0 \leq Im z(i) \leq \tau_2 \] (2.8)
for all \( i \). In (2.7), the expression for \( D \) is given by
\[ 4D = \alpha s (\hat{G}_{12} + \hat{G}_{34}) + \alpha t (\hat{G}_{14} + \hat{G}_{23}) + \alpha u (\hat{G}_{13} + \hat{G}_{24}), \] (2.9)
where \( \hat{G}_{ij} \) is the scalar Green function on the torus with complex structure \( \tau \) between points \( z(i) \) and \( z(j) \), and so
\[ \hat{G}_{ij} \equiv \hat{G}(z(i) - z(j); \tau). \] (2.10)

In particular, it is defined as [30, 31]
\[ \hat{G}(z; \tau) = -\ln \left| \frac{\theta_1(z|\tau)}{\theta_1(0|\tau)} \right| + \frac{2\pi (Im z)^2}{\tau_2} \] (2.11)
\[ -\frac{1}{\pi} \sum_{(m,n) \neq (0,0)} \frac{\tau_2}{|m\tau + n|^2} e^{\pi i (m\tau + n - z(m\tau + n))/\tau_2} + 2\ln |\sqrt{2\pi \eta(\tau)}|^2. \]

Now the \( z \)-independent zero mode part given by the second term in the second line of (2.11) cancels in the whole amplitude, which follows from the expression for \( D \) in (2.9) on using \( s + t + u = 0 \). Thus, in the expression for \( D \) we simply replace \( \hat{G}(z; \tau) \) by \( G(z; \tau) \) where
\[ G(z; \tau) = \frac{1}{\pi} \sum_{(m,n) \neq (0,0)} \frac{\tau_2}{|m\tau + n|^2} e^{\pi i (m\tau + n - z(m\tau + n))/\tau_2}. \] (2.12)

Note that \( G(z; \tau) \) is modular invariant, and single valued. Thus
\[ G(z; \tau) = G(z + 1; \tau) = G(z; \tau). \] (2.13)
As explained before, in (2.5), \( \mathcal{F} \) is split into
\[ \mathcal{F} = \mathcal{F}_L + \mathcal{R}_L, \] (2.14)
where $\mathcal{F}_L$ is defined for $\tau_2 \leq L$, and $\mathcal{R}_L$ is defined for $\tau_2 > L$. Thus the analytic part of the amplitude is given by

$$I_m(s, t, u) = \sum_{n=0}^{\infty} \int_{\mathcal{F}_L} \frac{d^2 \tau}{\tau_2^2} \prod_{i=1}^{4} \int_{\Sigma} \frac{d^2 z^{(i)}}{\tau_2} \cdot \frac{\mathcal{D}^n}{n!}. \quad (2.15)$$

Hence, performing an $\alpha'$ expansion we get that

$$I_m(s, t, u) = \sum_{p-q} \sigma_p^e \sigma_q^f J^{p,q}, \quad (2.16)$$

where

$$J^{p,q} = \int_{\mathcal{F}_L} \frac{d^2 \tau}{\tau_2^2} j^{p,q}(\tau, \bar{\tau}), \quad (2.17)$$

and

$$\sigma_2 = \alpha^2 (s^2 + t^2 + u^2), \quad \sigma_3 = \alpha^3 (s^3 + t^3 + u^3). \quad (2.18)$$

Here $j^{p,q}(\tau, \bar{\tau})$ is obtained after integrating over the insertion points of the vertex operators and encodes the topologically distinct ways the scalar propagators are connected on the toroidal worldsheet. These contributions up to the $D^{10}{\mathcal{R}}^4$ term in the low-energy expansion have been considered in detail [32, 33]. The integrands satisfy Poisson equations with specific source terms. The structure of several of these equations was conjectured in [32], while one of which was proven in [33] using elaborate calculations. We shall consider a particularly simple diagram at $O(D^{12}{\mathcal{R}}^4)$ in the derivative expansion, for which we derive the Poisson equation it satisfies. This is given by the Mercedes diagram in figure 1. This is a three-loop Feynman diagram where the six scalar propagators connect all the four-graviton vertices such that each vertex is trivalent. Note that the six links of the diagram form the edges of a tetrahedron.

Thus, we have that

$$\mathcal{M} = \int_{\Sigma} \frac{d^2 z^{(1)}}{\tau_2} \int_{\Sigma} \frac{d^2 z^{(2)}}{\tau_2} \int_{\Sigma} \frac{d^2 z^{(3)}}{\tau_2} \int_{\Sigma} \frac{d^2 z^{(4)}}{\tau_2} G_{12}G_{13}G_{14}G_{23}G_{34}. \quad (2.19)$$

Its contribution to the $D^{12}{\mathcal{R}}^4$ interaction is given by [30]

$$j^{(0,2)} = \frac{80}{6!} \mathcal{M} + \ldots. \quad (2.20)$$

In order to find the Poisson equation satisfied by $\mathcal{M}$, we find it useful to define the non-holomorphic Eisenstein series

$$E_s(\tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^s}{\pi^s |m + n\tau|^2}. \quad (2.21)$$
which satisfies the Laplace equation

\[ \Delta E_s(\tau, \bar{\tau}) = s(s - 1)E_s(\tau, \bar{\tau}). \tag{2.22} \]

Here, the \( SL(2, \mathbb{Z}) \) invariant Laplacian is defined by

\[ \Delta = 4\tau_2^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}}. \tag{2.23} \]

3. The systematics of the moduli dependence

In order to obtain the differential equation satisfied by \( M \), we shall use the relations satisfied by the Green function \( G_{ij} \) in (2.12) under variations of the Beltrami differentials. To obtain them, we first write down these relations satisfied by \( \tilde{G}_{ij} \) at arbitrary genus and specialize to the case of genus one. Then, on using (2.11) this gives us the required relations for \( G_{ij} \).

In general, the scalar Green function on the genus \( g \) Riemann surface is given by (see [33] for a detailed discussion)

\[ \tilde{G}(z, w) = -\ln |E(z, w)|^2 + 2\pi Y_{ij}^{-1}\left( \operatorname{Im} \int_{\Sigma} E_z^w \right) \left( \operatorname{Im} \int_{\Sigma} E_w^z \right). \tag{3.24} \]

Here \( E(z, w) \) is the prime form, \( \omega \) is the Abelian differential one form, and the period matrix \( \Omega \) is defined as \( \Omega_{ij} = X_{ij} + iY_{ij} \), where \( X \) and \( Y \) are matrices with real entries, and \( I, J = 1, \ldots, g \). We also have that \( Y^{-1}Y = Y^{-1} \).

Now, for the Beltrami differential \( \mu \), the holomorphic variation \( \delta_{\mu} \Psi \) for any \( \Psi \) is given by

\[ \delta_{\mu} \Psi = \frac{1}{\pi} \int_{\Sigma} d^2z \mu \delta_{zz} \Psi. \tag{3.25} \]

Using the general expressions for the variation [33, 34]

\[ \delta_{\mu wz} \omega(z) = \omega(w) \partial_w \partial_z \ln E(z, w), \]

\[ \delta_{\mu wz} \Omega_{ij} = 2\pi i \omega_j(w) \omega_i(w), \]

\[ \delta_{\mu wz} \ln E(z_1, z_2) = -\frac{1}{2} (\partial_w \ln E(w, z_1) - \partial_u \ln E(w, z_2))^2, \]

\[ \delta_{\mu wz} \partial_z = \pi \delta^2(z - w) \partial_z, \tag{3.26} \]

we get that [20]

\[ \delta_{\mu wz} \tilde{G}(z_1, z_2) = \frac{1}{2} (\partial_w \tilde{G}(w, z_1) - \partial_u \tilde{G}(w, z_2))^2, \]

\[ \delta_{\mu wz} (\partial_z \tilde{G}(z, w)) = -\frac{\pi}{2} \partial_z \delta^2(z - u) + \pi Y_{ij}^{-1} \omega_j(w) (\partial_u \tilde{G}(u, w) - \partial_u \tilde{G}(u, z)). \tag{3.27} \]

Now, on the torus, the Beltrami differential \( \mu^z \) is unity, \( \omega(z) = dz \), and \( \Omega = \tau \). The Green function satisfies

\[ \partial_u \partial_z G(z, w) = \pi \delta^2(z - w) - \frac{\pi}{\tau_2}, \]

\[ \partial_z \partial_u G(z, w) = -\pi \delta^2(z - w) + \frac{\pi}{\tau_2}. \tag{3.28} \]
Now, consider (3.25) when $\Psi = \hat{G}(z_1, z_2)$. From (2.11) we get that
\[
\delta_\mu G(z_1, z_2) + 4i\tau_2 \frac{\partial \ln \eta(\tau)}{\partial \tau} = \frac{1}{2\pi} \int_{\Sigma} d^2z \left( \partial_\mu G(z, z_1) - \partial_\mu G(z, z_2) \right)^2,
\]  
(3.29)
where we have used
\[
\int_{\Sigma} d^2w \delta_{\mu\nu} \ln \eta(\tau) = 2\pi i \tau^2 \frac{\partial \ln \eta(\tau)}{\partial \tau}.
\]  
(3.30)
We also see that
\[
\int_{\Sigma} d^2z \left( \partial_\mu G(z, w) \right)^2 = -\tau_2 G_2(\tau),
\]  
(3.31)
where $G_2(\tau)$ is the holomorphic Eisenstein series defined by
\[
G_2(\tau) = \sum_{(m,n)\neq (0,0)} \frac{1}{(m\tau + n)^2}.
\]  
(3.32)
This leads to the simple relation
\[
\delta_\mu G(z_1, z_2) = -\frac{1}{\pi} \int_{\Sigma} d^2z \partial_\mu G(z, z_1) \partial_\mu G(z, z_2),
\]  
(3.33)
on using the identity
\[
\frac{\partial \ln \eta(\tau)}{\partial \tau} = \frac{iG_2(\tau)}{4\pi}.
\]  
(3.34)
Also from (3.33) we get that
\[
\delta_\mu \delta_\nu G(z_1, z_2) = -\frac{1}{2\pi} \int_{\Sigma} d^2z \int_{\Sigma} d^2u \delta_{\mu\nu} \delta^2(z-u) \partial_\mu G(z, z_1)
- \frac{1}{\pi \tau_2} \int_{\Sigma} d^2z \int_{\Sigma} d^2u \partial_\mu G(z, z_1) \partial_\mu (G(u, z_2) - G(u, z)) + 1 \leftrightarrow 2
\]  
(3.35)
on using the second equation in (3.27). The contribution from the second line vanishes as it is a total derivative in $u$ and $G(u, z)$ is single valued. The contribution from the first line vanishes, and it is a total derivative in $u$ again, and the boundary contributions cancel by relabeling $z$ by $z + 1$ and $z + \tau$ in the delta function and using the single valuedness of $G(z, z_1)$. Thus, we get that
\[
\delta_\mu \delta_\nu G(z_1, z_2) = 0.
\]  
(3.36)
We shall use (3.33) and (3.36) repeatedly in our analysis below. Finally, the $SL(2, \mathbb{Z})$ invariant Laplacian is given by
\[
\Delta = \delta_\mu \delta_\mu.
\]  
(3.37)

4. The Poisson equation satisfied by the Mercedes diagram

We now obtain the Poisson equation satisfied by $M$ given in (2.19) using the analysis of the section above. In the various manipulations that are needed, we often obtain expressions involving $\partial_\mu G(z, w)$ where $z$ is integrated over $\Sigma$. We then integrate by parts without picking up boundary contributions on $\Sigma$ as $G(z, w)$ is single valued. Also, we readily use $\partial_\mu G(z, w) = -\partial_\mu G(z, w)$ using the translational invariance of the Green function. Finally,
we have that
\[ \int_{\Sigma} d^2z G(z, w) = 0 \] (4.38)
which follows from (2.12).

In the analysis below, for brevity we write
\[ \int_{\Sigma} d^2z \int_{\Sigma} d^2w \cdots \equiv \int_{2w} \cdots . \] (4.39)

4.1. Deriving the Poisson equation

From (3.37) we have that (\( \mathcal{M} \) is referred to as \( D_{1,1,1,1,1,1} \) in [30].)
\[ \Delta \mathcal{M} = \delta_\beta \delta_\rho \mathcal{M} = 24\mathcal{M}_1 + 6\mathcal{M}_2, \] (4.40)

where \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are defined by
\[ \mathcal{M}_1 = \frac{1}{r_2^2} \int_{1234} \delta_\mu G_{12} \delta_\rho G_{13} G_{14} G_{23} G_{34} G_{42}, \]
\[ \mathcal{M}_2 = \frac{1}{r_2^2} \int_{1234} \delta_\mu G_{12} \delta_\rho G_{34} G_{13} G_{14} G_{23} G_{24}. \] (4.41)

These two topologically distinct contributions are given in figure 2. Here \( \mu \) along a link stands for \( \delta_\mu \), while \( \bar{\mu} \) stands for \( \delta_\bar{\mu} \).
In our analysis, it shall be very convenient to depict the various relations using diagrams.

The convention for holomorphic and antiholomorphic derivatives acting on the Green function are given in figure 3.

We shall also need the expressions for the various diagrams listed below. One of them is (in the convention of [31])

\[ C_{3,2,1} = \frac{1}{\tau^2} \int_{12345} G_{23} G_{34} G_{24} G_{12} G_{16} G_{36} \]  

as depicted by figure 4. This is a two-loop Feynman diagram involving integration over five vertices. Note that this diagram does not involve any derivatives acting on Green functions.

We also need the diamond diagrams \( D_1, D_2 \) and \( D_3 \) in the intermediate steps, defined by

\[ D_1 = \frac{1}{\tau^2} \int_{12345} \partial_2 G_{12} \bar{\partial}_1 G_{13} G_{35} G_{23} G_{34} G_{14} G_{45}, \]

\[ D_2 = \frac{1}{\tau^2} \int_{12345} \partial_1 G_{15} \bar{\partial}_2 G_{23} G_{12} G_{35} G_{14} G_{45} G_{34}, \]

\[ D_3 = \frac{1}{\tau^2} \int_{12345} \partial_3 G_{35} \bar{\partial}_4 G_{45} G_{14} G_{13} G_{12} G_{23}, \]  

and depicted by figure 5. These are three-loop diagrams that involve integrals over five vertices. Each diagram has one holomorphic and one antiholomorphic derivative acting on distinct Green functions.

Next, we list the fan diagrams \( F_1, F_2, F_3 \) and \( F_4 \) which also arise in the intermediate steps, which are defined by
and depicted by figure 6. Again, these are three-loop diagrams that involve integrals over five vertices. Each diagram has one holomorphic and one antiholomorphic derivative acting on distinct Green functions.

Finally, we consider the ladder diagrams $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ as they are needed in the intermediate steps, which are defined by

\[
\mathcal{L}_1 = \frac{1}{7/2} \int_{12345} \partial_1 G_{35} \partial_1 G_{12} G_{23} G_{24} G_{25} G_{45},
\]
\[
\mathcal{L}_2 = \frac{1}{7/2} \int_{12345} \partial_1 G_{14} \partial_5 G_{25} G_{12} G_{23} G_{24} G_{35} G_{45},
\]
\[
\mathcal{L}_3 = \frac{1}{7/2} \int_{12345} \partial_1 G_{12} \partial_4 G_{24} G_{23} G_{25} G_{35} G_{14} G_{45},
\]
\[
\mathcal{L}_4 = \frac{1}{7/2} \int_{12345} \partial_1 G_{12} \partial_2 G_{23} G_{14} G_{24} G_{25} G_{35} G_{45}
\]

and depicted by figure 7. These are two-loop diagrams that involve integrals over six vertices. Each diagram has one holomorphic and one antiholomorphic derivative acting on distinct Green functions.

Figure 7. The ladder diagrams (i) $\mathcal{L}_1$, (ii) $\mathcal{L}_2$, (iii) $\mathcal{L}_3$, (iv) $\mathcal{L}_4$.

\[
\frac{\tau_s}{\pi} \quad \frac{1}{2} \quad \frac{\tau_f}{\pi}
\]

Figure 8. The relation $\tau_s D_s/\pi = -\tau_f D_f/\pi = -\mathcal{M}/2$.

\[
\begin{align*}
\mathcal{F}_1 &= \frac{1}{7/2} \int_{12345} \partial_3 G_{35} \partial_1 G_{12} G_{13} G_{23} G_{24} G_{25} G_{45}, \\
\mathcal{F}_2 &= \frac{1}{7/2} \int_{12345} \partial_1 G_{14} \partial_5 G_{25} G_{12} G_{23} G_{24} G_{35} G_{45}, \\
\mathcal{F}_3 &= \frac{1}{7/2} \int_{12345} \partial_1 G_{12} \partial_4 G_{24} G_{23} G_{25} G_{35} G_{14} G_{45}, \\
\mathcal{F}_4 &= \frac{1}{7/2} \int_{12345} \partial_1 G_{12} \partial_2 G_{23} G_{14} G_{24} G_{25} G_{35} G_{45}
\end{align*}
\]
Now let us first consider the contribution coming from $M_1$ in (4.41). We have that
\[
M_1 = -\frac{\tau_2}{\pi} (D_1 + D_2) + C_{3,2,1} - \frac{\tau_2}{\pi} F_1.
\] (4.46)

Using the identity\(^1\)
\[
\frac{\tau_2}{\pi} D_1 = -\frac{\tau_2}{\pi} D_2 = -\frac{M}{2}
\] (4.47)
depicted by figure 8, this gives us
\[
M_1 = C_{3,2,1} - \frac{\tau_2}{\pi} F_1,
\] (4.48)
as the sum of the two diamond diagrams cancel.

Next, let us consider the contribution from $M_2$. We have that
\[
\frac{M_2}{2} = \frac{\tau_2}{\pi} (L_1 + L_2 + L_3 + F_1 + F_2 - F_3 - D_3).
\] (4.49)

On using the identities for the ladder diagrams
\[
\frac{\tau_2}{\pi} L_1 = C_{3,2,1}, \quad L_2 = L_3
\] (4.50)
depicted by figures 9 and 10, respectively, we get that
\[
\frac{\tau_2}{\pi} (L_1 + L_2 + L_3) = C_{3,2,1} + 2L_2 = C_{3,2,1} + 2L_3.
\] (4.51)

Now to solve for $L_3$ we note that
\[
\frac{\tau_2}{\pi} L_3 = \frac{\tau_2}{\pi} (L_1 - L_4) = C_{3,2,1} - \frac{\tau_2}{\pi} L_4.
\] (4.52)

We directly solve for $L_4$ to obtain
\[
\frac{2\tau_2}{\pi} L_4 = E_7^2 - E_6,
\] (4.53)

\(^1\) It is easy to see that $D_1$ remains the same when $\partial$ and $\bar{\partial}$ are interchanged in the integrand.
as depicted by figure 11. This leads to
\[ \frac{\tau_2}{\pi} \mathcal{L}_3 = C_{3,2,1} - \frac{E_3^2}{2} + \frac{E_6}{2}, \]  
(4.54)
as depicted by figure 12. Thus, we finally get that
\[ \frac{\tau_2}{\pi} (\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3) = 3C_{3,2,1} - E_3^2 + E_6. \]  
(4.55)

Let us consider the remaining contributions to \( \mathcal{M}_2/2 \) in (4.49). We have that
\[ \frac{\tau_2}{\pi} (\mathcal{F}_2 - \mathcal{F}_3) = -D_{2,1,1,1,1} - \frac{\tau_2}{\pi} \mathcal{F}_4, \]  
(4.56)
as depicted by figure 13. Here, we have a new diagram \( D_{2,1,1,1,1} \) defined by
\[ D_{2,1,1,1,1} = \frac{1}{\tau_2^2} \int_{1234} G_{13} G_{23} G_{12} G_{24} G_{i4}^2 \]  
(4.57)
as depicted by figure 14. This diagram also arises in the expression for the $D^{12}R^4$ amplitude at genus one. Finally, we use the relation

$$a = -4.5841$$

as depicted by figure 15, to obtain

$$a = -4.5921$$

Thus, putting all the contributions together, we get that

$$a = -4.6021$$

Now the expression (4.60) simplifies further by using the relation

$$C_{3,2,1} - D_{2,1,1,1,1} = \frac{72}{5} D_3 = \frac{72}{5} D_3 = -M/2.$$

as depicted by figure 16, leading to

$$M_2 = 3C_{3,2,1} - E^2 + E_6 - D_{2,1,1,1,1} + \frac{72}{5} D_3 = \frac{M}{5}$$

as depicted by figure 17, to obtain

$$M_2 = 2C_{3,2,1} - E^2 + E_6 - \frac{M}{5} + \frac{72}{5} D_3.$$

Now from (4.48) and (4.62) we see that the total contribution involving $F_1$ exactly cancels in $24M_0 + 6M_2$, resulting in considerable simplification.

This leads to the Poisson equation

$$(\Delta + 6)M = 48C_{3,2,1} + 12(E_6 - E^2)$$

satisfied by the Mercedes diagram, as depicted in figure 17. Thus, the source terms involve simple one- and two-loop Feynman diagrams.

4.2. An elementary consistency check

We now perform an elementary consistency check of (4.63). We show that the non-vanishing contributions as $\tau_2 \to \infty$ match on both sides of the equation.
In order to obtain these contributions for $C_{3,2,1}$, we note that $C_{3,2,1}$ and $C_{2,2,2}$ satisfy the coupled Poisson equations \[ (\Delta - 8)C_{3,2,1} = -C_{2,2,2} - 4(E_3^2 - 4E_6), \]
\[ (\Delta - 6)C_{2,2,2} = -24C_{3,2,1} + 12(E_3^2 - E_6). \] (4.64)
Here $C_{2,2,2}$ is a three-loop Feynman diagram with five vertices defined by (in the convention of [31])

\[ C_{2,2,2} = \frac{1}{\tau_2^4} \int G_{13}G_{23}G_{14}G_{24}G_{15}G_{25} \] (4.65)

as given in figure 18.

Thus, from (4.64) we get that
\[ (\Delta - 2)(4C_{3,2,1} + C_{2,2,2}) = 52E_6 - 4E_3^2, \]
\[ (\Delta - 12)(6C_{3,2,1} - C_{2,2,2}) = 108E_6 - 36E_3^2. \] (4.66)

Now let us consider the contributions from terms that diverge or are constant as $\tau_2 \to \infty$, where we make use the expressions

\[ E_6 = \frac{2}{\pi^6} \zeta(12)\tau_2^6 + \ldots, \]
\[ E_3^2 = \frac{4}{\pi^6} \zeta(6)^2\tau_2^6 + \frac{3}{\pi^3} \zeta(5)\zeta(6)\tau_2^6 + \ldots, \] (4.67)

where we have ignored terms that vanish as $\tau_2 \to \infty$, which is also true of the various expressions below. Solving (4.66) we get that

\[ 4C_{3,2,1} + C_{2,2,2} = \frac{2158}{691\pi^6} \zeta(12)\tau_2^6 + c_1\tau_2^2 + \frac{6}{\pi^3} \zeta(5)\zeta(6)\tau_2^2 + \ldots, \]
\[ 6C_{3,2,1} - C_{2,2,2} = \frac{2572}{691\pi^6} \zeta(12)\tau_2^6 + c_2\tau_2^6 + \frac{9}{\pi^3} \zeta(5)\zeta(6)\tau_2^6 + \ldots, \] (4.68)

where $c_1$ and $c_2$ are undetermined constants, as they are the zero modes of (4.66). This leads to the expression for $C_{3,2,1}$ given by

\[ 10C_{3,2,1} = \frac{4730}{691\pi^6} \zeta(12)\tau_2^6 + c_2\tau_2^4 + c_1\tau_2^2 + \frac{15}{\pi^3} \zeta(5)\zeta(6)\tau_2^2 + \ldots \] (4.69)

However $c_1$ and $c_2$ are easily seen to vanish as discussed in appendix D.1 of [30]. We also get that

\[ C_{2,2,2} = \frac{266}{691\pi^6} \zeta(12)\tau_2^6 + \ldots, \] (4.70)

**Figure 18.** The diagram $C_{2,2,2}$. 
as the $O(\tau_2)$ term cancels. Thus, from (4.67) and (4.69) we get that

$$48C_{3,2,1} + 12(E_6 - E_3^2) = \frac{4968}{691\pi^6} \zeta(12)\tau_2^6 + \frac{36}{\pi^2} \zeta(5)\zeta(6)\tau_2 + \ldots$$  \hspace{1cm} (4.71)

which appears on the right-hand side of the Poisson equation (4.63). This precisely agrees with $(\Delta + 6)M$ which appears on the left-hand side of (4.63) on using the large $\tau_2$ expansion [31]

$$M = \frac{138}{691\pi^6} \zeta(12)\tau_2^6 + \frac{6}{\pi^2} \zeta(5)\zeta(6)\tau_2 + \ldots$$  \hspace{1cm} (4.72)

5. The contribution of the Mercedes diagram to the genus one $D^{12}R^4$ amplitude

The Mercedes diagram contributes [31]

$$I^{D^{12}R^4} = \frac{2\pi \cdot 80}{6!} \int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2} M$$  \hspace{1cm} (5.73)

to the genus one $D^{12}R^4$ amplitude which follows from (2.20), which we now evaluate. Using (4.63) and the eigenvalue equation (2.22) for $E_6$, we get that

$$\int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2} M = \frac{1}{6} \int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2} \left( -\Delta M + \frac{2}{5} \Delta E_6 - 12E_3^2 + 48C_{3,2,1} \right).$$  \hspace{1cm} (5.74)

From (4.66) we also obtain that

$$C_{3,2,1} = \frac{1}{24} \Delta(6C_{3,2,1} + C_{2,2,2}) - \frac{7}{2} E_6 + \frac{1}{2} E_3^2.$$  \hspace{1cm} (5.75)

Thus, we get that

$$\int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2} M = \int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2} \Delta \left( -\frac{M}{6} + 2C_{3,2,1} + \frac{1}{3} C_{2,2,2} - \frac{13}{15} E_6 \right)$$

$$+ \int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2} E_3^2,$$  \hspace{1cm} (5.76)

on using (2.22) again for $E_6$. The first integral on the right-hand side of (5.76) is a boundary term which receives a contribution only from $\tau_2 \rightarrow \infty$. This evaluates to

$$\int_{-\frac{1}{2}}^{1/2} d\tau_1 \frac{\partial}{\partial \tau_2} \left( -\frac{M}{6} + 2C_{3,2,1} + \frac{1}{3} C_{2,2,2} - \frac{13}{15} E_6 \right) \big|\big|_{\tau_2 = L \rightarrow \infty},$$  \hspace{1cm} (5.77)

which we now obtain using (4.72), (4.69), (4.70) and (4.67). Apart from an $O(L^5)$ contribution which must cancel from the integral over $R_L$, there is a finite contribution equal to

$$\frac{2}{\pi^2} \zeta(5)\zeta(6).$$  \hspace{1cm} (5.78)

The second integral on the right-hand side of (5.76) can be directly evaluated using [30, 31, 36]

$$\frac{\pi^2 s}{4\zeta(2s)^2} \int_{\mathcal{F}_L} \frac{d^2\tau}{\tau_2^2} E_3^2 = \frac{L^{2s-1}}{2s-1} + 2\phi(s)\ln \left( \frac{L}{\mu_{2s}} \right) + \ldots$$  \hspace{1cm} (5.79)
for $s > 1/2$, where we have dropped terms that vanish as $L \to \infty$. This can be obtained using the explicit expression for the non-holomorphic Eisenstein series $E_s$. In (5.79) we have that

$$\phi(s) = \sqrt{\pi} \frac{\Gamma(s - 1/2) \zeta(2s - 1)}{\Gamma(s) \zeta(2s)},$$

$$\ln \mu_{2s} = \frac{\zeta'(2s - 1)}{\zeta(2s - 1)} - \frac{\zeta'(2s)}{\zeta(2s)} + \frac{\Gamma'(s - 1/2)}{2 \Gamma(s - 1/2)} - \frac{\Gamma'(s)}{2 \Gamma(s)}. \quad (5.80)$$

Thus, apart from terms that diverge as $L^s$ and $\ln L$, we get a finite contribution from the scale of the logarithm. This finite contribution to

$$\int_{\mathcal{F}_L} \frac{d^2 \tau}{\tau_2^2} E_3^2 \quad (5.81)$$

is equal to

$$- \frac{3}{\pi^2} \zeta(5) \zeta(6) \ln \mu_6, \quad (5.82)$$

where

$$\ln \mu_6 = \frac{\zeta'(5)}{\zeta(5)} - \frac{\zeta'(6)}{\zeta(6)} + \frac{7}{12} - \ln 2, \quad (5.83)$$

on using

$$\frac{\Gamma'(5/2)}{\Gamma(5/2)} = \frac{8}{3} - \gamma - \ln 4, \quad \frac{\Gamma'(3)}{\Gamma(3)} = \frac{3}{2} - \gamma. \quad (5.84)$$

Thus, adding the contributions (5.78) and (5.82), we get a finite non-vanishing contribution given by

$$\int_{\mathcal{F}_L} \frac{d^2 \tau}{\tau_2^2} \mathcal{M} = - \frac{6 \zeta(5) \zeta(6)}{\pi^3} \left( \frac{\zeta'(5)}{\zeta(5)} - \frac{\zeta'(6)}{\zeta(6)} + \frac{1}{4} - \ln 2 \right), \quad (5.85)$$

leading to

$$I^{D^{12}R^4} = - \frac{4}{3 \pi^3} \zeta(5) \zeta(6) \left( \frac{\zeta'(5)}{\zeta(5)} - \frac{\zeta'(6)}{\zeta(6)} + \frac{1}{4} - \ln 2 \right) \sigma_3^2 R^4. \quad (5.86)$$

In particular, note that there is a term which diverges as $\ln L$ in (5.81). This must be cancelled by a term schematically of the form $\sigma_3^2 R^4 \ln(\alpha' L \mu)$ coming from the integral over $\mathcal{R}_L$, where $\mu$ is a constant. This leads to a non-analytic term in the external momenta in the effective action.

### 6. Discussion

We have considered a particularly simple Feynman diagram which contributes to the $D^{12}R^4$ amplitude at genus one, and showed that it satisfies a modular invariant Poisson equation, with a very specific structure of source terms. Clearly, it would be interesting to generalize the analysis and obtain Poisson equations for the other Feynman diagrams at this order, and at higher orders in the low-momentum expansion, which are needed to obtain their contribution to the string amplitude at genus one. Along with a similar analysis for other amplitudes, this will give us a detailed understanding of non-BPS interactions at genus one. Such interactions
are not well understood, apart from some analysis based on constraints due to supersymmetry, and multi-loop supergravity [14, 15, 37, 38]. Also, it would be interesting to generalize the analysis to amplitudes at higher genus in string theory. Explicit expressions for the four-graviton amplitude at three and four loops in maximal supergravity [22, 39–42] might provide useful hints about the structure of the corresponding string amplitudes.

References

[1] Green M B and Gutperle M 1997 Effects of D-instantons Nucl. Phys. B 498 195–227
[2] Green M B, Gutperle M and Vanhove P 1997 One loop in eleven dimensions Phys. Lett. B 409 177–84
[3] Kiritis E and Pioline B 1997 On R^4 threshold corrections in type IIB string theory and (p, q) string
instantons Nucl. Phys. B B508 509–34
[4] Green M B and Sethi S 1999 Supersymmetry constraints on type IIB supergravity Phys. Rev. D 59 046006
[5] Green M B, Kwon H-h and Vanhove P 2000 Two loops in eleven dimensions Phys. Rev. D D61 104010
[6] D’Hoker E and Phong D H 2005 Two-loop superstrings VI: non-renormalization theorems and the
4-point function Nucl. Phys. B 715 3–90
[7] D’Hoker E, Gutperle M and Phong D H 2005 Two-loop superstrings and S-duality Nucl. Phys. B 722 81–118
[8] Matone M and Volpato R 2006 Higher genus superstring amplitudes from the geometry of moduli
space Nucl. Phys. B 732 321
[9] Berkovits N and Mafra C R 2006 Equivalence of two-loop superstring amplitudes in the generalised
spinor and RNS formalisms Phys. Rev. Lett. 96 011602
[10] Green M B and Vanhove P 2006 Duality and higher derivative terms in M theory J. High Energy
Phys. JHEP01(2006)093
[11] Berkovits N 2007 New higher-derivative R^4 theorems Phys. Rev. Lett. 98 211601
[12] Basu A 2008 The D^6R^4 term in type IIB string theory on T^5 and U-duality Phys. Rev. D 77 106003
[13] Basu A 2008 The D^6R^4 term in type IIB string theory on T^2 and U-duality Phys. Rev. D D77 106004
[14] Green M B, Russo J G and Vanhove P 2008 Modular properties of two-loop maximal supergravity
and connections with string theory J. High. Energy Phys. JHEP07(2008)126
[15] Basu A and Sethi S 2008 Recursion Relations from Space-time Supersymmetry J. High Energy
Phys. JHEP09(2008)081
[16] Green M B, Miller S D, Russo J G and Vanhove P 2010 Eisenstein series for higher-rank groups
and string theory amplitudes Commun. Num. Theor. Phys. 4 551
[17] Green M B, Russo J G and Vanhove P 2010 Automorphic properties of low energy string
amplitudes in various dimensions Phys. Rev. D 81 086008
[18] Basu A 2011 Supersymmetry constraints on the R^4 multiplet in type IIB on T^2 Class. Quant. Grav. 28 225018
[19] D’Hoker E and Green M B 2014 Zhang-Kawazumi invariants and superstring amplitudes J. Number Theory 144 111
[20] Gomez H and Mafra C R 2013 The closed-string 3-loop amplitude and S-duality J High Energy
Phys. JHEP10(2013)217
[21] D’Hoker E, Green M B, Pioline B and Russo R 2015 Matching the D^8R^4 interaction at two-loops
J. High Energy Phys. JHEP01(2015)031
[22] Basu A 2014 The D^8R^4 term from three loop maximal supergravity Class. Quant. Grav. 31 245002
[23] Bossard G and Verschinin V 2015 C^4V^4R^4 type invariants and their gradient expansion J High
Energy Phys. JHEP03(2015)089
[24] Pioline B 2015 D^8R^4 amplitudes in various dimensions J. High Energy Phys. JHEP04(2015)057
[25] Bossard G and Verschinin V 2015 The two D^8R^4 type invariants and their higher order
generalisation J. High Energy Phys. JHEP07(2015)154
26. Bossard G and Kleinschmidt A 2015 Supergravity divergences, supersymmetry and automorphic forms J High Energy Phys. JHEP08(2015)102
27. Basu A 2016 Perturbative type II amplitudes for BPS interactions Classical Quantum Gravity 33 045000
28. Pioline B and Russo R 2015 Infrared divergences and harmonic anomalies in the two-loop superstring effective action J. High. Energy Phys. 1512 102
29. Bossard G and Kleinschmidt A Loops in exceptional field theory 1510.07859
30. Green M B and Vanhove P 2000 The low-energy expansion of the one-loop type II superstring amplitude Phys. Rev. D 61 104011
31. Green M B, Russo J G and Vanhove P 2008 Low energy expansion of the four-particle genus-one amplitude in type II superstring theory J High Energy Phys. JHEP02(2008)020
32. D’Hoker E, Green M B and Vanhove P 2015 On the modular structure of the genus-one Type II superstring low energy expansion J High Energy Phys. JHEP08(2015)041
33. D’Hoker E, Green M B and Vanhove P Proof of a modular relation between 1-2- and 3-loop Feynman diagrams on a torus 1509.00363
34. D’Hoker E and Phong D H 1988 The Geometry of String Perturbation Theory Rev. Mod. Phys. 60 917
35. Verlinde E P and Verlinde H L 1987 Chiral Bosonization, Determinants and the String Partition Function Nucl. Phys. B 288 357
36. Zagier D 1982 The Rankin-Selberg method for automorphic functions which are not of rapid decay J. Fac. Sci. Tokyo 28 415–38
37. Basu A 2013 The structure of the $\mathcal{R}^{\text{th}}$ term in type IIB string theory Class. Quant. Grav. 30 235028
38. Basu A 2014 Constraining gravitational interactions in the M theory effective action Class. Quant. Grav. 31 165007
39. Bern Z, Carrasco J, Dixon L J, Johansson H and Roiban R 2008 Manifest Ultraviolet Behavior for the Three-Loop Four-Point Amplitude of N = 8 Supergravity Phys. Rev. D 78 105019
40. Bern Z, Carrasco J, Dixon L J, Johansson H and Roiban R 2009 The Ultraviolet Behavior of N = 8 Supergravity at Four Loops Phys. Rev. Lett. 103 081301
41. Basu A 2015 Constraining non-BPS interactions from counterterms in three loop maximal supergravity Class. Quant. Grav. 32 045012
42. Basu A 2015 Some finite terms from ladder diagrams in three and four loop maximal supergravity Class. Quant. Grav. 32 195023