THE MYSTERY OF THE BRANE RELATION

STAVROS GAROUFALIDIS

Dedicated to the memory of M. Goussarov.

Abstract. The purpose of the present paper is to introduce and explore two surprises that arise when we apply a standard procedure to study the number of finite type invariants of 3-manifolds introduced independently by M. Goussarov and K. Habiro based on surgery on claspers, Y-graphs or clovers, [Gu, Ha, GGP]. One surprise is that the upper bounds depend on a bit more than a choice of generators for $H_1$. A complementary surprise a curious brane relation (in two flavors, open and closed) which shows that the upper bounds are in a certain sense independent of the choice of generators of $H_1$.

1. Introduction

1.1. Motivation. It is well-known that starting from a move (often described in terms of surgery on a link) on a set of knotted objects (such as knots, links, braids, tangles, 3-manifolds, graphs), one can define a notion of finite type invariants. The question of how many invariants are there in any degree gets divided into two separate questions: one that provides upper bounds for the number of invariants, and one that provides lower bounds. Traditionally, upper bounds are obtained by providing a set of topological relations among the moves, whereas lower bounds are obtained by constructing (by quite different means) invariants. The purpose of the present paper is to introduce and explore two surprises that arise when we study the first question regarding the finite type invariants of 3-manifolds due, independently, to M. Goussarov and K. Habiro based on surgery on clovers, see [Gu, Ha]. One surprise is that the upper bounds depend on a bit more than a choice of generators for $H_1$. A complementary surprise a curious brane relation which shows that the upper bounds are in a certain sense independent of the choice of generators of $H_1$.

Let us briefly recall that Goussarov and Habiro studied the notion of surgery $M_G$ of a 3-manifold $M$ along an (embedded) clasper or Y-graph $G$, see [Gu, Ha, GGP]. For a detailed definition of clovers and their associated surgery see [Gu, Ha] or [GGP] the notation of which we follow here. This notion of surgery allows one to consider the abelian group $\mathcal{F}^Y(M)$ freely generated by (isomorphism classes of) 3-manifolds obtained surgery on $M$, and define a decreasing filtration on it where $\mathcal{F}^Y_m(M)$ is the subgroup generated by $\left[ M, G \right] = \sum_{G' \subset \{G_1, \ldots, G_n\}} (-1)^{|G'|} M_{G'}$ for all clovers $G$ in $M$ partitioned into $n \geq m$ blocks $G_1, \ldots, G_n$, where for $G' \subset \{G_1, \ldots, G_n\}$, $M_{G'}$ denotes the result of surgery on the graph $\cup_{i \in G' \setminus G_i} G_i$, and where $|G'|$ denotes the cardinality of the set $G'$.

Dually, and perhaps more naturally, this filtration allows us to call a $\mathbb{Z}$-valued function $\lambda$ on the set of 3-manifolds obtained by surgery on clovers a finite type invariant of type $m$ if $\lambda(\mathcal{F}_m^{Y+1}(M)) = 0$. Thus, the question of how many finite type invariants of type $m$ are there translates into a question about the structure of the (graded quotient) abelian groups $\mathcal{G}^Y_m(M) = \mathcal{F}^Y_m(M)/\mathcal{F}^Y_{m+1}(M)$. For the case of $M = S^3$ (or any other integral homology 3-sphere), it is well-known that the topological calculus of clovers developed independently by Goussarov and Habiro, implies the existence of upper bounds of $\mathcal{G}^Y(M)$ in terms of an abelian group $\mathcal{A}(\phi)$ generated by (abstract) trivalent graphs, modulo the well known antisymmetry AS and IHX relations, see for instance [GGP, Section 4]. The case of arbitrary 3-manifolds $M$ (needed for instance in [GL, Theorems 5,6,7]) seems to be missing from the literature, even though the main tools are the same as in the case of...
M = S^3. There are, however, two surprises in extending the above upper bound to all closed 3-manifolds, which are the main point of this paper: one is that the upper bound for \( \mathcal{G}^X(M) \) is given in terms of a finitely generated (in each degree) abelian group \( \mathcal{A}^0(b) \) defined below, where \( b \) is a \( H_1 \)-spanning link i.e., an oriented framed link in \( M \) that generates (possibly with redundances) \( H_1(M, \mathbb{Z}) \), see Theorem \[3\]. In other words, the generators of \( \mathcal{A}^0(b) \) depend on just a bit more than a choice of generators for \( H_1(M, \mathbb{Z}) \), they depend on a choice of 1-cycles. The other surprise is the existence of a new relation in \( \mathcal{A}^0(b) \), the open brane (OBR) and the closed brane (BR) relation, which is also given in terms of a choice of embedded 2-cycles in \( M \).

Of course the choice of \( b \) is not unique, and the choice of cycles in the OBR relation is not unique, however the OBR and BR relations imply that any two such choices \( b \) and \( b' \) lead to rather canonical isomorphisms between \( \mathcal{A}^0(b) \) and \( \mathcal{A}^0(b') \) as well as commutative diagrams, see Theorem \[3\].

If one is willing to work with rational coefficients, then the above upper bound \( \mathcal{A}^0(b) \) can be identified with an invariant \( \mathcal{A} \)-group \( \mathcal{A}(H(M)) \) that depends only on the cohomology ring \( H^*(M, \mathbb{Q}) \) of \( M \), see Corollary \[3\] (although the map \( \mathcal{A}(H(M)) \to \mathcal{G}^X(M) \) still depends on a choice of a \( H_1 \)-spanning link \( b \).

As a final comment before the details, we should mention that for finite type invariants of integral homology 3-spheres, or for \( \mathbb{Q} \)-valued finite type invariants of rational homology spheres the above mentioned choices of 1-cycles and 2-cycles are invisible, which partly explains why they were not discovered despite the success of constructing and studying theories of finite type invariants of rational homology spheres.

1.2. Statement of the results. Throughout, by graph we mean we mean one with (symmetric) univalent and trivalent vertices, together with a choice of cyclic order on each trivalent vertex. Note that graphs that contain struts, i.e., an interval with two univalent vertices and no trivalent ones, will not be allowed here. Univalent vertices of graphs will often be called legs or leaves. Given a set \( X \), an \( X \)-colored graph is a graph \( G \) together with a function \( c : \text{Legs}(G) \to X \). This assignment can be extended linearly to include graphs each univalent vertex of which is assigned a nonzero formal linear combination of elements of \( X \).

Let \( B(X) \) denote the abelian group spanned by \( X \)-colored graphs modulo the well-known AS, IHX and LOOP relations shown in Figure 1. \( B(X) \) is graded, by declaring the degree of a graph to be the number of its trivalent vertices.

Notice that the group \( B(X) \) is closely related to a group that appears when one studies finite type invariants of \( X \)-component links in \( S^3 \), with some notable differences: one is that we do not allow struts, another is that we do not grade by half the number of vertices, and the third is that we allow graphs with no legs.

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{Figure 1. The AS, IHX (all trivalent vertices oriented counterclockwise), and LOOP relations. In the LOOP relation, the appearing loop is an edge and not a leaf of the graph.}
\end{array}
\end{array}
\end{align*}
\]

Given a \( H_1 \)-spanning link \( b \), we now define two important relations on \( B(b) \). Let \( \cdot : H_2(M, \mathbb{Z}) \otimes H_1(M, \partial M, \mathbb{Z}) \to \mathbb{Z} \) be the intersection pairing.

**Definition 1.1.** Fix a closed surface \( \Sigma \) in \( M \). Let \( (G, \ast) \) be \( b \)-colored graph, which contains a special leg colored by the special symbol \( \ast \) (disjoint from the alphabet \( b \)). Let

\[
\langle G, \Sigma \rangle := \sum_l [\Sigma] \cdot [c_l]G_l \in B(b)
\]

where the summation is over all legs of \( G \) except \( \ast \) and where \( G_l \) is the result of gluing the \( \ast \)-leg of \( G \) to a \( c_l \)-colored leg of \( G \), as shown in the following example

\[
\langle \begin{array}{c}
\begin{array}{c}
\text{Figure 2. The OBR and BR relations.}
\end{array}
\end{array}\rangle = [\Sigma] \cdot [x] + [\Sigma] \cdot [y] + [\Sigma] \cdot [x] + [\Sigma] \cdot [y]
\]
By convention, the summation over the empty set equals to zero. The BR (closed brane) relation is the subgroup of $B(b)$ generated by $\langle G, \Sigma \rangle = 0$ for all surfaces $\Sigma$, or really, only a generating set for $H_2(M, \mathbb{Z})$ and all graphs $(G, *)$ as above. Let $A(b) = B(b)/(BR)$.

**Definition 1.2.** Fix a $b$-colored graph that contains a distinguished leg * colored by a nullhomologous label $c_0$ which bounds a surface $\Sigma_0$ in $M$. Let

$$\langle G, \Sigma_0 \rangle := G + \sum_l [\Sigma_0] \cdot [c_l] G_l \in B(b)$$

where the summation is over all legs of $G$ except * and where $G_l$ is the result of gluing the *-leg of $G$ to a $c_l$-colored leg of $G$. The OBR (open brane) relation is the subgroup of $B(b)$ generated by $\langle G, \Sigma_0 \rangle = 0$ for all graphs $G$ as above and all surfaces $\Sigma_0$. Note that the OBR subgroup of $B(b)$ includes the BR subgroup if we assume that one of the components of $b$ is the boundary of an embedded disk disjoint from the rest of the components of $b$. Let $A^0(b) = A(b)/(OBR)$.

**Theorem 1.** (i) For every $H_1$-spanning link $b$ in a manifold $M$, there is a group homomorphism

$$W_{M, b} : A^0(b) \to G^Y(M)$$

which is onto, once tensored with $\mathbb{Z}[1/2]$.

(ii) For every two $H_1$-spanning links $b$ and $b'$ in $M$, there are isomorphisms $W_{M, b, b'} : A^0(b) \to A^0(b')$ such that over $\mathbb{Z}[1/2]$:

$$W_{M, b} = W_{M, b'} \circ W_{M, b, b'}.$$

1.3. **The size of $A^0(b)$.** It is natural to ask how big is the (finitely generated in each degree) abelian group $A^0(b)$ which bounds from above $G^Y(M)$.

**Corollary 1.3.** (i) If $H_1(M, \mathbb{Z})$ is torsion-free and $b$ is a basis of $H_1$, then $A(b) \cong A^0(b)$.

(ii) If $b$ is $H_1, \mathbb{Q}$-basis and $b'$ is $H_1$-spanning then $A(b) \cong Q A^0(b')$.

(iii) If $H_1(M, \partial M, \mathbb{Q}) = 0$, then for every $H_1$-spanning $b$ we have $B(b) \cong A(b)$.

(iv) In particular, for $M$ a rational homology 3-sphere, we have that $A^0(b) \cong Q A(b) \cong Q A(\phi)$.

(v) For $M$ a homology-cylinder (i.e., a manifold with the same integer homology as that of $\Sigma \times I$ for a surface $\Sigma$ with one boundary component) and a $H_1, \mathbb{Q}$-basis $b$, we have that $B(b) \cong Q A(b) \cong Q A^0(b)$.

If we are willing to work with rational coefficients, then one can define in an invariant way a group of graphs, that depends only on the cohomology ring $H^*(M, \mathbb{Q})$ as follows: $A(H(M))$ is generated by graphs colored by nonzero elements of $H_1(M, \mathbb{Q})$, modulo the AS, IHX, LOOP and BR relations.

**Corollary 1.4.** For every manifold $M$, there is a map

$$A(H(M)) \to G^Y(M),$$

onto over $\mathbb{Q}$.

For manifolds $M$ with $b_1(M) = 0$, i.e., for rational homology 3-spheres, we show a promised universal property of the LMO invariant restricted to the set of rational homology spheres $[\text{LMO}]$, or of its cousin, the Aarhus integral $[A]$:

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1. which does not seem to be related in any meaningful way to the wonderful (mem)branes of string theory.
Theorem 2. The LMO invariant is the universal \(\mathbb{Q}\)-valued finite type invariant of rational homology spheres. In particular, for \(M\) a rational homology 3-sphere and \(b\) \(H_1\)-spanning, we have \(\mathcal{A}(b) \cong_{\mathbb{Q}} \mathcal{A}(b) \cong_{\mathbb{Q}} \mathcal{A}(\phi) \cong_{\mathbb{Q}} \mathcal{G}^Y(M)\). 

With regards to the size of \(\mathcal{A}(\phi)\), it is well-known that Lie algebras and their representation theory provides lower bounds for the abelian groups \(\mathcal{A}(\phi)\). In the case of manifolds \(M\) with positive betti number, we do not know yet of lower bounds for \(\mathcal{A}(b)\). The little we know at present is the following:

Corollary 1.5. Let \(b\) be a \(H_1\)-spanning link in a closed manifold \(M\) and \(G\) be a graph colored by a sublink \(b'\) of \(b\). Assume that \(G\) has an internal edge, that is an edge between two trivalent vertices of \(G\). If \(b'\) is not \(H_1\)-spanning (over \(\mathbb{Q}\)), then \(G = 0 \in \mathcal{A}(b)\). In particular, if \(b_1(M) > 0\), then every graph without legs vanishes in \(\mathcal{A}(b)\).

Corollary 1.6. Let \(b\) be a \(H_1\)-spanning link in a closed manifold \(M\) and \(G\) be a graph whose \(r + 1\) legs are colored by \(x, y_1, \ldots, y_r\) so that \(x\) is primitive and linearly independent from \(\{y_1, \ldots, y_r\}\). For \(k = 1, \ldots, r\), let \(G^{(k)}\) denote the sum of all ways of replacing \(k\) many \(y_i\) by \(x\). Assume that \(G\) contains an internal edge. Then \(G^{(k)} = 0 \in \mathcal{A}(b)\) for all \(k\).

2. Proofs

The proofs of the theorem and its corollaries involve standard alternations of the topological calculus of clovers as appears in [Ha, Gu]; the uninitiated reader may also look at [GGP, Section 3].

Before we prove the theorem, it will be important to state some lemmas of which follows by applying to the topological calculus of clovers elementary alternations, see for example [GGP, Section 4.1]:

Lemma 2.1. \([\text{Gu}, \text{Ha}]\) (Cutting a Leaf) Let \(G\) be a clover of degree \(m\) in a manifold \(M\) and \(L\) be a leaf of \(G\). An arc \(a\) starting in the external vertex incident to \(L\) and ending in other point of \(L\), splits \(L\) into two arcs \(L'\) and \(L''\). Denote by \(G'\) and \(G''\) the graphs obtained from \(G\) by replacing the leaf \(L\) with \(L' \cup a\) and \(L'' \cup a\) respectively, see Figure 2. Then \([M, G] = [M, G'] + [M, G'']\) in \(\mathcal{G}_m^Y(M)\).

![Figure 2. Splitting a leaf.](image)

Lemma 2.2. \([\text{Gu}, \text{Ha}]\) (Sliding an Edge) Let \(G\) be a clover of degree \(m\) in a manifold \(M\), and let \(G'\) be obtained from \(G\) by sliding an edge of \(G\) along a tube in \(M\). Then \([M, G] = [M, G']\) in \(\mathcal{G}_m^Y(M)\).

Lemma 2.3. \([\text{Gu}, \text{Ha}]\) For all \(\varepsilon = \pm 1\), we have the following identities in \(\mathcal{G}^Y(M)\):

\[
2[M, \begin{array}{c} \circ \\ \circ \end{array}] \varepsilon = 0 \quad \text{and} \quad [M, \begin{array}{c} \circ \\ \circ \end{array}] \varepsilon = -\varepsilon [M, \begin{array}{c} \circ \\ \circ \end{array}].
\]

Lemma 2.4. Let \(G\) be a clover with \(r + 1\) leaves \(l_i\) for \(i = 0, \ldots, r\) in a manifold \(M\). Assume that \(l_0\) bounds an embedded surface \(\Sigma_0\) in \(M\). Then

\[
G + \sum_{i=1}^r [\Sigma_0] \cdot [l_i]G_i = 0 \in \mathcal{G}^Y(M)
\]

where \(G_i\) is the result of gluing the 0-th leg of \(G\) to its \(i\)-th leg.
Proof. Consider a graph $G$ and a surface $\Sigma_0$ as above. $\Sigma_0$ can be thought of as an embedded disk with bands. We can assume that $G$ is disjoint from the (interiors of the bands) of $\Sigma_0$ and thus $G$ intersects the (interior of) $\Sigma_0$ only in the embedded disk. Cut each band along arcs (in the normal direction to the core of the band) using the Cutting and Sliding Lemmas 2.1 and 2.2 as shown

(where $\Sigma_0$ is a surface of genus 1, and the solid arcs represent arbitrary tubes in the 3-manifold). The above calculation reduces to the case of a surface $\Sigma_0$ with no bands, i.e., a disk. Using the Cutting and Sliding Lemmas 2.1 and 2.3 once again, we may assume that the leaf $l_0$ of $G$ is zero-framed and that the disk $\Sigma_0$ intersects geometrically once a leaf of $G$ and is otherwise disjoint from $G$. The following equality

$$[M, \gamma_0] = [M, \gamma_1] + [M, \gamma_3] = [M, \gamma_1] + [M, \gamma_3]$$

which follows by Lemma 2.1, concludes our proof.

Lemma 2.5. Let $(G, \gamma)$ be a clover in a manifold $M$ together with a distinguished leaf $\gamma$ that bounds two surfaces $\Sigma_0$ and $\Sigma_1$ in $M$. Then,

$$\langle G, \Sigma_0 - \Sigma_1 \rangle = 0 \in G^Y(M).$$

Proof. It follows from two applications of the OBR relation that

$$-G = \langle G, \Sigma_0 \rangle = \langle G, \Sigma_1 \rangle \in G^Y(M).$$

Proof. (of Theorem 1) First we construct the map $W_{M, b}$. Let $b = (b_1, \ldots, b_r)$ be a $H_1$-spanning link in $M$. Given a graph $G$ with colored legs choose an arbitrary embedding of it in $M$. For every coloring $\sum_i a_i b_i$ of each of its univalent vertices (where $a_i$ are integers), push $|a_i|$ disjoint copies of $b_i$ (using the framing of $b_i$), orient them the same (resp. opposite) way from $b_i$ if $a_i \geq 0$ (resp. $a_i < 0$), and finally take an arbitrary band sum of them. We can arrange the resulting knots, one for each univalent vertex of the embedding of $G$, to be disjoint from each other, and together with the embedding of $G$ to form an embedded graph with leaves in $M$. Although the isotopy class of the embedded graph with leaves is not unique, the image of $[M, G] \in G^Y_m(M)$ is well-defined. This follows from Lemma 2.1. We need to show that the relations AS, IHX, OBR, BR and LOOP are mapped to zero, which will define our map $W_{M, b}$. For the AS and IHX relations, see for example [GGP, Section 4.1]. The LOOP relation follows from Figure 2.

The OBR and BR relations follow from Lemmas 2.4 and 2.5.

We now show that $W_{M, b}$ is onto, over $\mathbb{Z}[1/2]$. Note first that $G^Y_m(M)$ is generated by $[M, G]$ for all simple graphs of degree $m$, where a simple graph is a disjoint union of graphs of degree 1. Each of the leaves of $G$ are isotopic to some connected sum of (possibly orientation reversed) components of $b$ and contractible knots. Using the Cutting Lemma 2.1, we may assume that each leaf is isotopic to one of the components of $b$ (with possibly reversed orientation) or is contractible in $M$. From this point on, the proof is analogous to the case of $M = S^3$. Let $L$ be the link consisting of all contractible leaves of $G$. There exists a trivial,
Lemma now follows by induction on the number of components of $b$ orientation) so that $b$ point and intersects no other components of $G$. It suffices to show that under these moves $b$ remaining the relations in $A$. Lemma 2.6. imply that we can assume each of the disks $D_i$ are disjoint from $G$ and intersect $C$ at most two points $C$. Lemma 2.3 imply that $W_{M,b}$ is onto, over $\mathbb{Z}[1/2]$.

In order to show that $A^o(b)$ is independent of $b$, up to isomorphism, we need the following:

Lemma 2.6. Every two $H_1$-spanning links $b$ and $b'$ in $M$ are equivalent by a sequence of moves

M1: Add one component (after possibly changing its orientation) of $b$ to another.
M2: Change the framing of a component of $b$.
M3: Insert or delete a null-homologous zero-framed component of $b$.

Proof. It suffices to show that under these moves $b$ is equivalent to $b \cup b'$. Consider a component $b_i'$ of $b'$. Since $b$ is a basis of $H_1(M,\mathbb{Z})$, we can add a multiple of components of $b$ (after perhaps changing their orientation) so that $b_i'$ is nullhomologous, in which case we can change its framing to zero, and erase it. The lemma now follows by induction on the number of components of $b'$.

If $b'$ is obtained from $b$ by applying one of the three moves above, we will now define $W_{M,b,b'} : A^o(b) \rightarrow A^o(b')$ (abbreviated by $W_{b,b'}$ in what follows) and show that Equation (1) holds.

For the first move, if $b = (b_1, b_2, \ldots, b_r)$ and $b' = (b_1b_2b_3, b_2, \ldots, b_r)$ (where $b_1b_2$ is an arbitrary oriented band sum of $b_1$ with $b_2$) then $W_{b,b'}$ sends a $b_1$ colored vertex of an abstract graph $G$ to a $b_1b_2b_3$ colored vertex of $G$. It is easy to see that this defines a map $A^o(b) \rightarrow A^o(b')$ whose inverse sends a $b_1b_2b_3$ colored vertex of $G$ to a $b_1 + b_2$ colored vertex of $G$. Similarly, one can define a map $W_{b,b'}$ where $b' = (b_1b_2, b_2, \ldots, b_r)$. Equation (1) follows from Lemma 2.7.

For the second move, let $b = (b_1, b_2, \ldots, b_r)$ and $b' = (b_1', b_2, \ldots, b_r)$ where $b_i'$ is a knot whose framing differs from that of $b_1$ by $\varepsilon = \pm 1$. For graph $G$ with $n$ legs colored by $b_1$ we define

$$W_{b,b'}(G) = \sum_{I : |I| = \text{even}} \varepsilon^{|I|/2} G_I'$$

where the summation is over all functions $I : \{1, \ldots, n\} \rightarrow \{0, 1\}$ such that the cardinality $I$ of $I^{-1}(1)$ is even and $G_I'$ is the result of gluing the $b_1$ colored legs $l_i$ of $G$ for which $I(i) = 1$ pairwise and recoloring the remaining $b_1$ colored legs with $b_i'$ colored legs. It is easy to see that $W_{b,b'}$ is well-defined (i.e., that it respects the relations in $A^o(b)$) and that its inverse is given by

$$W_{b,b'}(G') = \sum_{I : |I| = \text{even}} (-\varepsilon)^{|I|/2} G_I.$$

Let $C$ denotes a $(-\varepsilon)$-framed unknot in $M$ which bounds a disk that geometrically intersects $b_1$ in one point and intersects no other components of $b$. Then $M_C$ is diffeomorphic to $M$ under a diffeomorphism that sends the image of $b$ in $M_C$ to $b'$ in $M$. Since $W_{M,b'}(G) = [M_C, G]$ and $W_{M,b}(G_I) = [M, G_I]$, Equation (1) (or rather, its equivalent form $W_{b'} = W_b \circ W_{b',b}$) follows from the following:
Lemma 2.7. For a graph $G$ of degree $m$ as above, we have in $G^V_{m}(M)$:

$$[M_C, G] = \sum_{l_i : |I| = \text{even}} (-\varepsilon)^{|I|/2}[M, G_I].$$

Proof. Using the Cutting Lemma 2.1 each $b_1$-colored leaf $l_i$ of $G$ can be split along an arc in two leaves; one that bounds a disk $D_i$ intersecting $C$ once and disjoint from $b$, and another that is isotopic to $b_1$ but disjoint from $C$. For $I : \{1, \ldots, n\} \to \{0, 1\}$, let $G_I$ denote the graph $(G \setminus (b_1 \text{ colored leaves of } G)) \cup \cup_{l_i \in I} D_i$. Lemma 2.1 implies that $[M_C, G] = \sum_I [M_C, G_I]$. Let $G_I''$ denote the graph in $M$ that corresponds to $G_I'$ under the diffeomorphism $M = M_C$; we obviously have $[M_C, G_I'] = [M, G_I'']$. Note that $G_I''$ has a collection of $|I|$ leaves each of which is unknotted bounding a disk with linking number $\varepsilon$ with every other leaf of this collection. An application of Lemma 2.4 $|I|$ times together with Lemma 2.3 implies that $[M_C, G_I''] = (-\varepsilon)^{|I|/2}[M, G_I]$ (resp. 0) for even (resp. odd) $|I|$. \hfill \Box

For the third move, let $b = (b_1, b_2, \ldots, b_r)$ and $b' = (b_0, b_1, b_2, \ldots, b_r)$ where $b_0$ is a null-homologous zero-framed knot, and consider the natural map $W_{b,b'} : A^o(b) \to A^o(b')$. Choose a surface $\Sigma_0$ that $b_0$ bounds. The OBR relation in $A^o(b')$ for $b_0$ colored vertices defines a map $W_{b_0,b} : A^o(b') \to A^o(b)$; this map is independent of $\Sigma_0$ since the difference between two choices of $\Sigma_0$ equals to a choice of a closed surface and the resulting difference vanishes due to the BR relation on $A^o(b)$. It is easy to see that $W_{b_0,b}$ is inverse to $W_{b,b'}$. Equation (3) follows essentially by definition. This completes the proof of Theorem 3. \hfill \Box

Proof. (of Corollary 1.3) The first statement follows immediately from the fact that if $b$ is a basis then no nontrivial linear combination is nullhomologous, thus the OBR relation is vacuous.

For the second statement, since we are using $\mathbb{Q}$ coefficients, we may assume that the link $b$ is a basis for $H_1(M, \mathbb{Z})/(\text{torsion}$), and choose $b^i$ to span the torsion part of $H_1(M, \mathbb{Z})$. Then, we have that $A(b) = A^o(b) \to A^o(b \cup b^i)$. There are integers $n_i$ and surfaces $\Sigma_i$ such that $n_i b_i = \partial \Sigma_i$ for all components of $b^i$. The OBR relation for $b^i$ colored legs gives a map $A^o(b \cup b^i) \to A^o(b)$ which is independent of the choices of $\{n_i, \Sigma_i\}$ and is inverse to the map $A^o(b) \to A^o(b \cup b^i)$. Thus, $A(b) \cong_b A^o(b \cup b^i)$. Since $A^o(b \cup b^i) \cong A^o(b^i)$ for every $H_1$-spanning link $b^i$, the result follows.

The third statement follows immediately from the fact that if the intersection form on $M$ vanishes, then the BR relation is vacuous.

The forth and fifth statements are immediate consequences of those above. \hfill \Box

Proof. (of Corollary 1.4) Let $b'$ be a $H_1$-spanning link and $b$ be a $H_1, \mathbb{Q}$-basis. Then, we have over $\mathbb{Q}$

$$A(H(M)) \cong_\mathbb{Q} A(b) \cong_\mathbb{Q} A^o(b') \to G^V(M)$$

which concludes the proof of the corollary. \hfill \Box

Proof. (of Theorem 3) The proof is a simple application of the locality property of the Kontsevich integral, as explained leisurely in [A II, Section 4.2], and a simple counting argument.

We now give the details. We need to show that

- The part of the LMO=Arhus integral $Z \in A^o(\phi)$ of degree at most $n$ is an invariant of type $n$.
- For a trivalent graph $G$ of degree $n$ in a rational homology 3-sphere $M$, we have that

$$Z(M_G) = G + \text{ higher degree diagrams } \in A^o(\phi).$$

For the first claim, recall that a degree 1 clover $G$ in a manifold $M$ is the image of an embedding $V \to M$ of a neighborhood $V$ of the standard (framed) graph $\Gamma$ of $\mathbb{R}^3$, and that surgery of $M$ along $G$ can be described as the result of Dehn surgery on the six component link $L$ in $V$ shown below
Let $L$ be partitioned in three blocks $L_1, L_2, L_3$ of two component links each. We call each block an arm of $G$. Alternating a rational homology 3-sphere $M$ with respect to surgery on $G$ equals to alternating $M$ with respect to all nine subsets of the set of arms of $G$.

Recall also that the Kontsevich integral of a framed link $L$ in a 3-manifold $M$ $Z(M, L)$ (defined by Kontsevich for links in $S^3$ and extended by Le-Murakami-Ohtsuki for links in arbitrary 3-manifolds [LMO, Section 6.2]) takes values in linear combinations of $L$-colored uni-trivalent graphs.

Recall also that the LMO=Aarhus integral of a rational homology 3-sphere $M_L$ (obtained by surgery on a framed link $L$ in a rational homology 3-sphere $M$) is obtained by considering the Kontsevich integral $Z(M, L)$, splitting it in a quadratic $Z^q$ and trivalent (a better name would be “other”) part $Z^t$, and gluing the $L$-colored legs of $Z^t$ using the inverse linking matrix of $L$.

Given a clover $G = \cup_{i=1}^n G_i$ in a rational homology 3-sphere $M$, (where $G_i$ are of degree 1), let $L^{\text{act}}$ denote the link that consists of the $3n$ arms of $G$. When we compute $Z([M, G]) = Z([M, L^{\text{act}}])$, we need to concentrate on all $L^{\text{act}}$-colored uni-trivalent graphs that have at least one univalent vertex on each block of $G$. Such graphs will have at least $3n$ univalent vertices. Since at most three univalent vertices can share a trivalent vertex, it follows that the above considered graphs will have at least $n$ trivalent vertices; in other words it follows that $Z([M, G]) \in A_{2n}(\phi)$.

The second claim is best shown by example. Recall that surgery on the (generic trivalent graph) $\Theta$ shown below corresponds to surgery on two clovers $G_1$ and $G_2$, each with arms $\{E_{ij}, L_{ij}\}$ for $i = 1, 2$ and $j = 1, 2, 3$. The linking matrix of the 12 component link $L^{\text{act}} = E_{ij} \cup L_{ij}$ and its inverse are given by

\[
\begin{pmatrix}
0 & I \\
I & I
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
-I & I \\
I & 0
\end{pmatrix}
\]

where $I$ is the identity $6 \times 6$ matrix. The relevant trivalent part $Z^t(M, L^{\text{act}})$ is shown schematically in four cases here, where the graphs on the left terms of each case come from $G_1$ and the graphs on the right terms of each case come from $G_2$ and the dashed lines correspond to gluings of the univalent vertices:

\[
\begin{array}{cccc}
E & \cdots & E \\
E & \cdots & E \\
E & \cdots & E \\
\end{array}
\quad
\begin{array}{cccc}
E & \cdots & L \\
E & \cdots & L \\
E & \cdots & L \\
\end{array}
\quad
\begin{array}{cccc}
L & \cdots & E \\
L & \cdots & E \\
L & \cdots & E \\
\end{array}
\quad
\begin{array}{cccc}
L & \cdots & L \\
L & \cdots & L \\
L & \cdots & L \\
\end{array}
\]

However, the last three cases all contribute zero, since $LLL$ is a 3-component unlink whose coefficient in $Z^t$ is a multiple of the triple Milnor invariant and thus vanishes. Thus, we are only left to glue terms in the first case, and this is summarized in the following figure

which concludes the proof. \hfill \square

Proof. (of Corollary 1.3) If $G$ is as in the statement of the corollary, colored by a sublink $b'$ of $b$ which is not $H_1$-spanning , then we can find an $x \in b \setminus b'$, and a closed surface $x^*$ such that $[x^*][y] = \delta_{y,x}$ for all components $y$ of $b$. Cut $G$ along an edge, and color the two new leaves $x$ and $*$ to obtain a graph $(G, *)$. By definition, we have $(G, x^*) = G$, thus the result follows from the BR relation. \hfill \square

Proof. (of Corollary 1.4) We will first show the result for $k = 0$. Let $G$ be as in the statement of the corollary and let $G'$ be the graph with two more leaves than $G$, colored by $x$ and $*$ respectively as shown:

\[
\begin{array}{c}
G \\
\circ \\
x
\end{array}
\quad
\begin{array}{c}
G' \\
\circ \\
*
\end{array}
\]

The BR relation implies that

\[0 = (G', x^*) = G + \text{LOOP} = G = G^{(0)}.
\]

Now, we will show the result for all $k$. Let $G(n)$ be the same graph as $G$ with $r + 1$ leaves colored by $x, x + ny_1, \ldots, x + ny_r$, for $n \in \mathbb{N}$. Since $x$ is primitive and linearly independent from $\{x + ny_1, \ldots, x + ny_r\}$, the $k = 0$ case for $G(n)$ shown above implies that $G(n) = 0$ for all $n$. Since $G(n) = \sum_k n^k G^{(k)}$, the result follows. \hfill \square
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School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA.

E-mail address: stavros@math.gatech.edu