LONG TIME EXISTENCE FOR THE BOUSSINESQ-FULL DISPERSION SYSTEMS

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Abstract.
We establish the long time existence of solutions for the "Boussinesq-Full dispersion" systems modeling the propagation of internal waves in a two-layer system. For the two-dimensional Hamiltonian case $b = d > 0, a \leq 0, c < 0$, we study the global existence of small solutions of the corresponding system.

Keywords : Internal waves. Boussinesq-Full dispersion systems. Long time existence.

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1. Introduction

This paper is concerned with a class of asymptotic models of internal waves arising in the so-called two-layer system. This idealized system, when it is at rest, consists of a homogeneous fluid of depth $d_1$ and density $\rho_1$ lying over another homogeneous fluid of depth $d_2$ and density $\rho_2 > \rho_1$. The bottom on which both fluids rest is presumed to be horizontal and featureless while the top of fluid 1 is restricted by the rigid lid assumption, which is to say, the top is viewed as an impenetrable, bounding surface. Both of these require that the deviation of the interface be a graph over the flat bottom, actually parametrized by a scalar function $\zeta$, see Figure below.
The mathematical analysis of the full two-layer system displays tremendous difficulties due in particular to the possible appearance of Kelvin-Helmholtz instabilities. We refer to [25] for a deep analysis and far reaching results.

By expanding two non-local operators and for various ranges of parameters related to wave lengths, amplitudes, depths, densities,.., asymptotic models were rigorously (in the sense of consistency) derived in [14] and [11]. We will follow the framework of [11].

More precisely, denoting \( \rho_1, \rho_2 \) the densities, \( a \) a typical amplitude of the wave and \( \lambda \) a typical wave length in the horizontal direction, we define the dimensionless parameters

\[
\gamma := \frac{\rho_1}{\rho_2}, \quad \delta := \frac{d_1}{d_2}, \quad \epsilon := \frac{a}{d_1}, \quad \mu := \frac{d_1^2}{\lambda^2}.
\]

Though they are redundant, it is also notationally convenient to introduce two other parameter's \( \epsilon_2 \) and \( \mu_2 \) defined as

\[
\epsilon_2 = \frac{a}{d_2} = \epsilon \delta, \quad \mu_2 = \frac{d_2^2}{\lambda^2} = \frac{\mu}{\delta^2}.
\]

**Remark 1.1.** The parameters \( \epsilon_2 \) and \( \mu_2 \) correspond to \( \epsilon \) and \( \mu \) with \( d_2 \) rather than \( d_1 \) taken as the unit of length in the vertical direction.

The Boussinesq-Full dispersion regime corresponds to \( \mu \sim \epsilon \ll 1, \mu_2 \sim 1 \) so that the wave amplitude is small only with respect to the upper layer.

It is shown in [11] that in this Boussinesq-Full dispersion regime and in the absence of surface tension, the two-layers system is consistent with the three-parameter family of Boussinesq/FD systems

\[
\begin{align*}
(1 - b\mu_2)\partial_t \zeta + \frac{1}{\gamma} \nabla \cdot ((1 - \epsilon_2) v) &
\end{align*}
\]

\[
\begin{align*}
&- \frac{\sqrt{\gamma}}{\gamma} |D\coth(\sqrt{\mu_2}|D|)\nabla \cdot v + \frac{\epsilon_2}{\gamma} \left(a - \frac{1}{\gamma^2} \coth^2(\sqrt{\mu_2}|D|)\right) \nabla \cdot v = 0
\end{align*}
\]

\[
\begin{align*}
&\nabla \cdot (1 - d\mu_2) \partial_t v + (1 - \gamma) \nabla \zeta - \frac{\delta}{\gamma^2} \nabla (|v|^2) + c\mu(1 - \gamma) \Delta \nabla \zeta = 0,
\end{align*}
\]

where \( \zeta \) is the elevation of the wave, \( \gamma \in (0, 1) \), \( v_\beta = (1 - \beta \mu_2)^{-1} v \) (\( v \) being the horizontal velocity) and the constants \( a, b, c \) and \( d \) are defined as

\[
\begin{align*}
a = \frac{1}{3}(1 - \alpha_1 - 3\beta), \quad b = \frac{1}{3} \alpha_1, \quad c = \beta \alpha_2, \quad d = \beta(1 - \alpha_2),
\end{align*}
\]

with \( \alpha_1 \geq 0, \beta \geq 0 \) and \( \alpha_2 \leq 1 \).

Note that the parameters \( a, b, c, d \) are constrained by the relation \( a + b + c + d = \frac{1}{3} \).

The initial condition for (1.1) is imposed as follows

\[
\zeta|_{t=0} = \zeta_0, \quad v_\beta|_{t=0} = v_0,
\]

It is easily checked that (1.1) is linearly well posed when

\[
a \leq 0, c \leq 0, b \geq 0, d \geq 0.
\]

The local well-posedness of the Cauchy problem for (1.1) was established in [15] in the following cases

(1) \( b > 0, d > 0, a \leq 0, c < 0 \):
Moreover,\(\mu\) systems at the end of the paper. We will go back to this issue for the two-dimensional Hamiltonian systems in [9]. We will go back to this issue for the two-dimensional Hamiltonian systems at the end of the paper.

The existence of one dimensional solitary waves for the Boussinesq -Full Dispersion systems in the Hamiltonian case was proven in [5]. No such result seems to be known in the non-hamiltonian case.

In the present paper we will prove the long time existence for (1.1)-(1.2) that is existence on time scales of order \(1/\epsilon\) for all cases stated in (1.3). This time scale is the one on which the Boussinesq-Full Dispersion systems are “good” approximations of the two-layer system in the relevant regime.

Similar results for the “abcd” class of Boussinesq systems were established in [28, 30, 31, 12, 13]. As in [30, 31] the proof of our main result is based on the derivation of a suitable symmetrizer.

In the two-dimensional Hamiltonian case \(b = d > 0, a \leq 0, c < 0\), we shall moreover establish the global existence of small solutions of (1.1)-(1.2) when \(\epsilon = 1\). This is as far as we know the first global existence result for this type of systems in the two-dimensional case. Similar results in the non-Hamiltonian case are not known, even in the one-dimensional case.

Before presenting the main results of this paper, we give the following definition of the functional spaces that will be used:

**Definition 1.1.** For any \(s \in \mathbb{R}, k \in \mathbb{N}, \mu \in (0, 1), \) the Banach space \(X^s_{\mu,k}(\mathbb{R}^n)\) is defined as \(H^{s+k}(\mathbb{R}^n)\) equipped with the norm:

\[
\|u\|_{X^s_{\mu,k}}^2 = \|u\|_{H^s}^2 + \mu^k \|\nabla^k u\|_{H^s}^2.
\]

The solutions to the Cauchy problem of (1.1) will belong to some space \(X^s_{\mu,k}(\mathbb{R}^n) \times X^{s'}_{\mu,k'}(\mathbb{R}^n)\) with \(k\) and \(k'\) determined by \(a, b, c, d\) as follows:

**Definition 1.2.** For any \(a, b, c, d\) satisfying (1.3), we define a pair of numbers \((k,k') \overset{\text{def}}{=} (k(a, b, c, d), k'(a, b, c, d))\) according to the admissible sets of \((a, b, c, d)\) ’s as follows:

1. \((k,k') = (3,3)\) for \(b \neq d, b, d > 0, a \leq 0, c < 0;\)
2. \((k,k') = (2,2)\) for \(b = d > 0, a \leq 0, c < 0\) or \(b > 0, d = 0, a \leq 0, c = 0;\)
3. \((k,k') = (4,3)\) for \(b > 0, d = 0, a \leq 0, c < 0;\)
4. \((k,k') = (1,2)\) for \(b > 0, d > 0, a \leq 0, c = 0;\)
5. \((k,k') = (3,4)\) for \(b = 0, d > 0, a \leq 0, c < 0;\)
6. \((k,k') = (1,3)\) for \(b = 0, d > 0, a \leq 0, c = 0;\)
7. \((k,k') = (1,1)\) for \(b = d = 0, a \leq 0, c < 0;\)
8. \((k,k') = (0,1)\) for \(b = d = 0, a \leq 0, c = 0.\)

**Remark 1.2.** The cases (7) and (8) cannot occur for internal waves without surface tension but may occur for internal waves with a sufficiently large surface tension parameter.

We now state the main results of this paper. The first theorem concerns the long time existence for (1.1)-(1.2).

**Theorem 1.1.** Let \(t_0 > \frac{n}{2}, n = 1, 2, s \geq t_0 + 2\) and \(a, b, c, d\) satisfy the condition (1.3). Assume that \(\zeta_0 \in X^s_{\mu,k}(\mathbb{R}^n), v_0 \in X^{s'}_{\mu,k'}(\mathbb{R}^n)\) satisfy the (non-cavitation) condition

\[
1 - \varepsilon \zeta_0 \geq H > 0, \quad H \in (0,1),
\]

where \((k,k')\) is defined in Definition 1.2. Then there exist positive constants \(\tilde{c}\) and \(\tilde{\mu}\) (maybe depending on \(|\zeta_0|_{X^s_{\mu,k}} + |v_0|_{X^{s'}_{\mu,k'}}\) and \(H\)), such that for any \(\varepsilon \leq \tilde{c} \) and \(\mu \leq \tilde{\mu}\), there exists \(T > 0\) independent of \(\varepsilon\) and \(\mu\), such that (1.1)-(1.2) has a unique solution \((\zeta, v_\beta)\) with \((\zeta, v_\beta) \in C([0,T/\epsilon]; X^s_{\mu,k}(\mathbb{R}^n) \times X^{s'}_{\mu,k'}(\mathbb{R}^n))\). Moreover,

\[
\max_{t \in [0,T/\epsilon]} (||\zeta||_{X^s_{\mu,k}} + ||v_\beta||_{X^{s'}_{\mu,k'}}) \leq \tilde{c}(||\zeta_0||_{X^s_{\mu,k}} + ||v_0||_{X^{s'}_{\mu,k'}}).
\]

Here \(\tilde{c} = C(H^{-1})\) are nondecreasing functions of their argument.
Remark 1.3. In Theorem 1.1 we only use the condition \( \mu \ll 1, \epsilon \ll 1, \mu_2 \sim 1 \). We do not need the restriction \( \mu \sim \epsilon \).

The second theorem is about the global existence for (1.1)-(1.2) in the Hamiltonian case \( b = d > 0, a \le 0, c < 0 \).

Theorem 1.2. Let \( b = d > 0, a \le 0, c < 0 \). Assume that \( \xi_0 \in X^0_\mu(\mathbb{R}^2), v_0 \in X^0_\mu(\mathbb{R}^2) \). Then there exist a positive constant \( c_0 \) (maybe depending on \( \|\xi_0\|_{X^s_{\mu_0}} + \|v_0\|_{X^s_{\mu_0}} \)), such that for any \( \epsilon \le c_0 \) and \( \mu \sim \epsilon \), (1.1)-(1.2) has a unique solution \((\xi, v_\beta)\) with \((\xi, v_\beta) \in C([0, \infty) ; X^0_\mu(\mathbb{R}^2) \times X^0_\mu(\mathbb{R}^2))\). Moreover,

\[
\max_{t \in [0,\infty)} (\|\xi\|_{X^s_{\mu_0}} + \|v_\beta\|_{X^s_{\mu_0}}) \le C(\|\xi_0\|_{X^0_{\mu_0}} + \|v_0\|_{X^0_{\mu_0}}) .
\]

Here \( C \) is a universal constant which may change from line to line.

Corollary 1.1. Theorem 1.2 is in fact a global existence result for small solutions of (1.1) with \( \epsilon \sim \mu \sim 1 \) when \( b = d > 0, a \le 0, c < 0 \). Actually one reduces to this modified system with \( \epsilon = \mu = 1 \) by the change of variables

\[
\xi(t, X) = \epsilon^{-1} \zeta(\sqrt{\mu}^{-1} t, \sqrt{\mu}^{-1} X), \quad v_\beta(t, X) = \epsilon^{-1} v_\beta(\sqrt{\mu}^{-1} t, \sqrt{\mu}^{-1} X), \quad X = (x, y).
\]

2. Preliminary

2.1. Notations. Vectors will be denoted in bold letters, e.g. \( \mathbf{v} \). When \( B \) is a Banach space, \( \mathbf{v} \in B \) means that each component of \( \mathbf{v} \) belongs to \( B \). For any \( s \in \mathbb{R} \), we denote by \( H^s(\mathbb{R}^n) \) the classical \( L^2 \) based Sobolev spaces with the norm \( \| \cdot \|_{H^s} \). The notation \( \| \cdot \|_{L^p} \) stands for the \( L^p(\mathbb{R}^n) \) norm, \( 1 \le p \le \infty \). The \( L^2(\mathbb{R}^n) \) inner product is denoted by \( (\mathbf{u} \mid \mathbf{v})_2 = \int_{\mathbb{R}^n} \mathbf{u} \cdot \mathbf{v} dx \). For any \( k \in \mathbb{N} \), we denote

\[
\|f\|_{W^{k, \infty}} = \sum_{j=0}^{k} \|\nabla^j f\|_{L^\infty} .
\]

The notation \( f \sim g \) means that there exists a constant \( C \) such that \( \frac{1}{C} f \le g \le Cf \). \( f \lesssim g \) and \( g \gtrsim f \) means that there exists a constant \( C \) such that \( f \le Cg \). The condensed notation \( A_s = B_s + (C_n)_{s \ge s} \), is to say that \( A_s = B_s \) if \( s \le s \) and \( A_s = B_s + C_s \) if \( s > s \).

The Fourier transform of a tempered distribution \( u \in S' \) is denoted by \( \hat{u} \), which is defined as follows

\[
\hat{u}(\xi) \overset{\text{def}}{=} \mathcal{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} u(x) dx .
\]

We use \( \mathcal{F}^{-1}(f) \) to denote the inverse Fourier transform of \( f(\xi) \).

If \( f \) and \( u \) are two functions defined on \( \mathbb{R}^n \), the Fourier multiplier \( f(D)u \) is defined in term of Fourier transforms, i.e.,

\[
\widehat{f(D)u}(\xi) = f(\xi) \hat{u}(\xi) .
\]

We shall use notations

\[
\langle \xi \rangle \overset{\text{def}}{=} (1 + |\xi|^2)^{\frac{1}{2}}, \quad \Lambda \overset{\text{def}}{=} (1 - \Delta)^{\frac{1}{2}} .
\]

If \( A, B \) are two operators, \([A, B] = AB - BA\) denotes their commutator.

Throughout the paper, \( a, b, c, d \in \mathbb{R}, \gamma \in (0, 1), \mu_2 \sim 1 \) are given parameters. We shall not show the dependence on such given parameters. \( C \) will always denote a universal constant which may be different from line to line but is independent of the parameters involved (say, \( \mu \) and \( \gamma \)). Otherwise, one uses the notation \( C(\lambda_1, \lambda_2, \ldots) \) to denote a nondecreasing function of the parameters \( \lambda_1, \lambda_2, \ldots \).

The paper is organized as follows. In the remaining part of this Section we prove technical lemmas that will be used in the proofs of the main theorems. Section 3 is devoted to the proof of Theorem 1.1 which involves the symmetrization techniques used in our previous work [30] (see also [31] on the Boussinesq \((abcd)\) systems). In Section 4, we prove Theorem 1.2 by adapting the proof of a similar result for the Hamiltonian Boussinesq systems (see [9], [16]). Finally an Appendix is devoted to the proof of the equivalence of norms (3.8), (3.37) and (3.54).
2.2. Symmetrizer of \((1.1)\). Here and in the following sections, we shall only deal with the two-dimensional case, since the one-dimensional case is very similar and actually much simpler. For simplicity, we shall use \(v\) instead of \(v_\beta\) and use the following notation

\[
\sigma(D) \overset{\text{def}}{=} \sqrt{\mu_2} D \coth(\sqrt{\mu_2} D), \quad A(D) \overset{\text{def}}{=} 1 + a\mu A + \frac{1}{\gamma} \sqrt{\frac{\mu}{\mu_2}} \sigma(D) + \frac{1}{\gamma^2} \frac{\mu}{\mu_2} \sigma(D)^2. \tag{2.1}
\]

With such notations, we rewrite \((1.1)\) as

\[
\begin{cases}
(1 - b\mu A) \partial_t \xi + \frac{1}{\gamma} \nabla \cdot ((A(D) - \epsilon \zeta)v) = 0,

(1 - d\mu A) \partial_t v + (1 - \gamma)(1 + c\mu A) \nabla \xi - \frac{\epsilon}{2\gamma} \nabla (|v|^2) = 0.
\end{cases} \tag{2.2}
\]

If \(b > 0, d \geq 0\) or \(b = d = 0\), letting \(g(D) = (1 - b\mu A)(1 - d\mu A)^{-1}\), setting \(V = (\zeta, v)^T = (\zeta, v_1, v_2)^T\), then \((2.2)\) is equivalent after applying \(g(D)\) to the second equation to the condensed system

\[
(1 - b\mu A) \partial_t V + M(V, D) V = 0, \tag{2.3}
\]

where

\[
M(V, D) = \begin{pmatrix}
-\frac{\gamma}{\gamma^2 - 1} (A(D) - \epsilon \zeta) \partial_t v & \frac{1}{\gamma} (A(D) - \epsilon \zeta) \partial_t v \\
\frac{1}{\gamma} (A(D) - \epsilon \zeta) \partial_t v & \frac{1}{\gamma} (A(D) - \epsilon \zeta) \partial_t v \\
\end{pmatrix}.
\]

When \(a, b, c, d\) satisfies the condition \((1.3)\), the system \((2.3)\) could be treated similarly to a symmetrizable hyperbolic system under some smallness assumption on \(\epsilon\) and \(\mu\). The key point to solve \((2.3)\) is to search a symmetrizer \(S_V(D)\) of \(M(V, D)\) such that the principal part of \(iS_V(\xi)M(V, \xi)\) is self-adjoint, and that of \(S_V(\xi)\) is positive and self-adjoint under a smallness assumption on \(\epsilon\) and \(\mu\). It is not difficult to find that:

(i) if \(b = d, g(D) = 1, S_V(D)\) is defined by

\[
\left( \begin{array}{ccc}
\gamma(1 - \gamma)(1 + c\mu A) & -\epsilon v_1 & -\epsilon v_2 \\
-\epsilon v_1 & A(D) - \epsilon \zeta & 0 \\
-\epsilon v_2 & 0 & A(D) - \epsilon \zeta
\end{array} \right).
\]

(ii) if \(b \neq d, S_V(D)\) is defined by

\[
\gamma(1 - \gamma) \begin{pmatrix}
\gamma(1 - \gamma)(1 + c\mu A)^2 g(D) & -\epsilon g(D)(v^1(1 + c\mu A)) & -\epsilon g(D)(v^2(1 + c\mu A)) \\
-\epsilon g(D)(v^1(1 + c\mu A)) & (A(D) - \epsilon \zeta)(1 + c\mu A) & 0 \\
-\epsilon g(D)(v^2(1 + c\mu A)) & 0 & (A(D) - \epsilon \zeta)(1 + c\mu A)
\end{pmatrix} + \epsilon^2 \begin{pmatrix}
0 & 0 & 0 \\
0 & v^1 v_1 & v^1 v_2 \\
0 & v^1 v_2 & v^2 v_2
\end{pmatrix} (g(D) - 1). \tag{2.6}
\]

Note that \(S_V(D)\) is not self-adjoint since at least its diagonal part is not.

Next we define the energy functional associated to \((2.3)\) as

\[
E_s(V) = \langle (1 - b\mu A)^* V, S_V(D)A^* V \rangle_2. \tag{2.7}
\]

We shall show (see Appendix) that \(E_s(V)\) defined in \((2.7)\) is actually a true energy functional equivalent to some \(X^*_{\mu e}(\mathbb{R}^2)\) norm.

**Remark 2.1.** When \(b = 0, d > 0\), \((2.2)\) is equivalent after applying \((1 - d\mu A)\) to the first equation to the condensed system

\[
(1 - d\mu A) \partial_t V + M(V, D) V = 0, \tag{2.8}
\]

with \(M(V, D)\) defined by

\[
\left( \begin{array}{ccc}
-\frac{\gamma}{\gamma^2 - 1} (1 - d\mu A)(v \cdot \nabla) & \frac{1}{\gamma}(1 - d\mu A)((A(D) - \epsilon \zeta) \partial_t v) & \frac{1}{\gamma}(1 - d\mu A)((A(D) - \epsilon \zeta) \partial_t v) \\
(1 - \gamma)(1 + c\mu A) \partial_t v & -\frac{\gamma}{\gamma^2 - 1} v^1 \partial_t v & -\frac{\gamma}{\gamma^2 - 1} v^2 \partial_t v \\
(1 - \gamma)(1 + c\mu A) \partial_t v & -\frac{\gamma}{\gamma^2 - 1} v^2 \partial_t v & -\frac{\gamma}{\gamma^2 - 1} v^2 \partial_t v
\end{array} \right). \tag{2.9}
\]
Lemma 2.1. The symmetrizer $S_{\mathbf{V}}(D)$ of $M(\mathbf{V}, D)$ is defined by
\[
\gamma(1-\gamma)(1+c\mu\Delta)^2 \quad -ev^3(1+c\mu\Delta) \quad -ev^2(1+c\mu\Delta) \\
-ev^3(1+c\mu\Delta) \quad (1+c\mu\Delta)((A(D)-\epsilon\zeta)(1-d\mu\Delta)) \quad 0 \\
-ev^2(1+c\mu\Delta) \quad 0 \quad (1+c\mu\Delta)((A(D)-\epsilon\zeta)(1-d\mu\Delta))
\]

We could also have defined the energy functional associated to (2.5) as
\[
E_s(\mathbf{V}) = ((1-d\mu\Delta)\Lambda^s \mathbf{V}, S_{\mathbf{V}}(D)\Lambda^s \mathbf{V})_{2}. 
\]
As for the previous choice, we shall show (see Appendix) that $E_s(\mathbf{V})$ defined in (2.14) is actually a true energy functional equivalent to some $X^s_{\mu k}(\mathbb{R}^2)$ norm.

2.3. Technical lemmas. We complete this section by recalling some useful well-known results. Firstly, we recall the tame product estimates in Sobolev spaces: if $t_0 > \frac{n}{2}$, we have (see Theorem 3 and 6 in [24])
\[
\|fg\|_{H^s} \lesssim \|f\|_{H^{t_0}}\|g\|_{H^s} + (\|f\|_{H^s}\|g\|_{H^{t_0}})_{s > t_0}, \quad \forall f, g \in H^s \cap H^{t_0}(\mathbb{R}^n). \tag{2.12}
\]

The following interpolation inequality will be also used frequently
\[
\mu^\frac{s}{k}\|f\|_{H^{s+r}} \lesssim \|f\|_{H^s}^{1-\frac{r}{k}}\left(\mu^\frac{s}{k}\|f\|_{H^{s+r}}\right)^{\frac{r}{k}} \lesssim \|f\|_{X^s_{\mu k}}, \tag{2.13}
\]
where $0 < \theta < k$ and $s \geq 0$.

We now present commutator estimates (see Theorems 3 and 6 in [24]).

Lemma 2.1. Let $t_0 > \frac{n}{2}$, $-t_0 < r \leq t_0 + 1$. Then for all $s \geq 0$, $f \in H^{t_0+1} \cap H^{s+r}(\mathbb{R}^n)$ and $u \in H^{s+r-1}(\mathbb{R}^n)$, there holds
\[
\|([\Lambda^s, f]u)\|_{H^s} \lesssim \|\nabla f\|_{H^{t_0}}\|u\|_{H^{s+r-1}} + (\|\nabla f\|_{H^{s+r-1}}\|u\|_{H^{t_0}})_{s > t_0 + 1 - r}. \tag{2.14}
\]

Concerning the Fourier multiplier $g(D)$ for $b, d > 0, b \neq d$, we have the following lemma (see Lemma 2.3 in [39]).

Lemma 2.2. Let $b,d > 0$ and $b \neq d$, $s \in \mathbb{R}$, $\theta \geq 0$. Then
(i) for all $f \in H^{s}(\mathbb{R}^n)$, there hold
\[
\min\{1, \left(\frac{b}{d}\right)^\theta\}\|f\|_{H^s} \leq \|g(D)^{\theta}f\|_{H^s} \leq \max\{1, \left(\frac{b}{d}\right)^\theta\}\|f\|_{H^s}, \tag{2.15}
\]
\[
\|(g(D) - 1)f\|_{H^s} \leq \frac{|b-d|}{d}\|f\|_{H^s}; \tag{2.16}
\]
(ii) let $t_0 > \frac{n}{2}$, $-t_0 < r < t_0 + 1$, for all $f \in H^{t_0+1}(\mathbb{R}^n)$ and $u \in H^{r-1}(\mathbb{R}^n)$, there holds
\[
\|\|g(D)^{\theta}f\|_{H^s} \leq C\|f\|_{H^{t_0+1}}\|u\|_{H^{r-1}}, \tag{2.17}
\]
where $C$ is a constant independent of $\mu$.

We now state a useful lemma for the Fourier multiplier $\sigma(D)$.

Lemma 2.3. Let $\theta \geq 0$. We have
(i) for all $f \in H^{s}(\mathbb{R}^n)$, there holds
\[
\mu^\frac{s}{k}\|g(D)^{\theta}f\|_{L^2}^2 \leq \|\sigma(D)^{\theta}f\|_{L^2}^2 \lesssim \|f\|_{H^{\theta}}^2, \tag{2.18}
\]
(ii) let $t_0 > \frac{n}{2}$, $-t_0 < r + \theta \leq t_0 + 1$, for all $f \in H^{t_0+1}(\mathbb{R}^n)$ and $u \in H^{r-1}(\mathbb{R}^n)$, there holds
\[
\|\sigma(D)^{\theta}f\|_{H^s} \lesssim \|f\|_{H^{t_0+1}}\|u\|_{H^{r-1}}, \tag{2.19}
\]
(iii) let $-t_0 < r \leq t_0$, $\theta = \frac{1}{2}$, $1$, we have
\[
\|\sigma(D)^{\theta}g(D)^{\frac{1}{2}}f\|_{H^s} \lesssim \|f\|_{H^{t_0+1}}\|u\|_{H^{r-1}}. \tag{2.20}
\]
Proof. (i). Recalling the definition of \( \sigma(D) \) in \((2.1)\), by Plancherel theorem, using the fact that \( \coth(s) \geq 1 \), we have

\[
\| \sigma(D)^\theta f \|_{L^2}^2 = (2\pi)^{-n} \int_{\mathbb{R}^n} (\sqrt{\mu_2|\xi|} \coth(\sqrt{\mu_2|\xi|}))^{2\theta} |\hat{f}(\xi)|^2 d\xi \geq (2\pi)^{-n} \int_{\mathbb{R}^n} (\sqrt{\mu_2|\xi|})^{2\theta} |\hat{f}(\xi)|^2 d\xi,
\]

which implies

\[
\| \sigma(D)^\theta f \|_{L^2}^2 \geq \mu_2^\theta \| D^\theta f \|_{L^2}^2. \tag{2.21}
\]

This is the first part of \((2.18)\).

Since \( \lim_{s \to 0} s \coth(s) = 1 \) and \( \lim_{s \to \pm \infty} \coth(s) = 1 \), it is easy to get the second part of \((2.18)\).

(ii). Recalling the Definition 9 of \([24]\), one could check that \( \sigma(\xi)^\theta \) is a pseudo-differential operator of order \( \theta \). Indeed, on one hand, for \( |\xi| \leq 1 \), we have

\[
\sigma(\xi)^\theta = \left( \sqrt{\mu_2|\xi|} + \frac{2\sqrt{\mu_2|\xi|}}{e^{2\sqrt{\mu_2|\xi|}} - 1} \right)^\theta \leq (1 + \sqrt{\mu_2})^\theta,
\]

which gives rise to

\[
\sup_{|\xi| \leq 1} |\sigma(\xi)^\theta| \leq (1 + \sqrt{\mu_2})^\theta. \tag{2.22}
\]

On the other hand, for \( |\xi| \geq \frac{1}{4} \), \( \mu_2 \sim 1 \) and \( \beta \in \mathbb{Z}_{\geq 0} \), it is easy to check

\[
\sup_{|\beta| \leq 2 + |\xi| + n \geq 4} |\partial^\beta \sigma(\xi)^\theta| \lesssim 1. \tag{2.23}
\]

Due to \((2.22)\) and \((2.23)\), we have \( \sigma(D)^\theta \in \Gamma^\theta_{\infty} \) (see Definition 9 of \([24]\)). Then Theorems 3 and 6 yield \((2.19)\).

(iii) Since

\[
|\sigma(D)^\theta g(D)^{\frac{1}{2}}, f|u = |\sigma(D)^\theta ([g(D)^{\frac{1}{2}}, f]u) + [\sigma(D)^\theta, f]g(D)^{\frac{1}{2}}u|,
\]

using \((2.15), (2.17), (2.18)\) and \((2.19)\), we have

\[
\| \sigma(D)^\theta g(D)^{\frac{1}{2}}, f|u \|_{H^{r+\theta}} \lesssim \| [g(D)^{\frac{1}{2}}, f]u \|_{H^r} + \| f \|_{H^{r+\theta}} \| g(D)^{\frac{1}{2}}u \|_{H^{r+\theta}} \lesssim \| f \|_{H^{r+\theta}} \| u \|_{H^{r+\theta}}.
\]

This is \((2.20)\). We complete the proof of Lemma. \(\square\)

3. LONG TIME EXISTENCE FOR \((1.1)-(1.2)\)

The goal of this section is to prove Theorem 1.1 that is the long time existence of solutions of \((1.1)-(1.2)\). The proof follows the same approach used in \([30]\) that is to derive energy estimates on suitable symmetrizable linearized system and then use an iterative scheme.

3.1. Proof of Theorem 1.1. The proof of Theorem 1.1 relies heavily on the \textit{a priori} energy estimates for \((1.1)\). To do so, we assume

\[
1 - \epsilon \| \zeta(t) \|_{L^\infty} \geq \frac{H}{2}, \quad \sqrt{\epsilon} \| (\zeta(t), v(t)) \|_{W^{1,\infty}} \leq 1 \quad \text{for any } t \in (0, t^*), \tag{3.1}
\]

where \(t^* \) will be taken at the end of the proof. Then we have the following \textit{a priori} energy estimates.

**Proposition 3.1.** Let \( s \geq t_0 + 2 \) and \( t_0 > \frac{n}{2} \) with \( n = 1, 2 \). Assume that \( (\zeta, v) \) are smooth solutions to \((1.1)\). Then under the assumption \((3.1)\), there exist small constants \( \bar{\epsilon}, \bar{\mu} > 0 \) such that for all \( \epsilon \leq \bar{\epsilon}, \mu \leq \bar{\mu}, \)

\[
\frac{d}{dt} E_s(V) \lesssim \epsilon (1 + \epsilon^2 E_s(V))(E_s(V))^{\frac{1}{2}}, \tag{3.2}
\]

where \( E_s(V) \) is defined in \((2.7)\) or \((2.11)\).

**Remark 3.1.** Under the assumption \((3.1)\) taking \( \bar{\epsilon} > 0 \) and \( \bar{\mu} > 0 \) sufficiently small, there will hold for any \( \epsilon \leq \bar{\epsilon}, \mu \leq \bar{\mu} \)

\[
E_s(V) \sim E_s(t) \overset{\text{def}}{=} \| \zeta(t) \|_{X^{s,k}}^2 + \| v(t) \|_{X^{s,k}}^2, \tag{3.3}
\]

where \( (k, k') \) is defined in Definition 9. We shall use \((3.3)\) to derive \((3.2)\). The proof of \((3.3)\) will be postponed to the Appendix for only three typical cases in two dimensional space.
Proof of Theorem 1.1. Assume that
\[ E_s(V) \leq 16E_s(V_0), \quad \text{for any } t \in [0, t^*], \] (3.4)
where \( t^* = \frac{T}{\epsilon} \) will be determined later on. Taking \( \tilde{\epsilon}_2 = \frac{1}{4(\epsilon, (0))^{\frac{3}{2}}} \), due to \( \{3.2\}, \{3.3\} \) and \( \{3.4\} \), for any \( \epsilon \leq \tilde{\epsilon}_2 \), there exists a constant \( C_1 > 0 \) such that
\[ \frac{d}{dt}(E_s(V)) \frac{1}{2} \leq C_1 \epsilon E_s(V), \]
which gives rise to
\[ (E_s(V)) \frac{1}{2} \leq \frac{(E_s(V_0)) \frac{1}{2}}{1 - C_1 \epsilon E_s(V_0)} \leq 2(E_s(V_0)) \frac{1}{2}, \] (3.5)
for any \( t \leq \frac{T}{\epsilon} \) with \( T = \frac{1}{2C_1C_2(E_s(0))} \). On the other hand, \( \{3.3\} \) implies there exists a constant \( C_2 > 0 \) such that \( (E_s(V_0)) \frac{1}{2} \leq C_2(E_s(0)) \frac{1}{2} \). Taking
\[ T = \frac{1}{2C_1C_2(E_s(0))} \leq \frac{T}{\epsilon}, \quad t^* = T/\epsilon, \]
we have that \( \{3.5\} \) holds for any \( t \leq T/\epsilon \) which improves the ansatz \( \{3.4\} \). Moreover, using \( \{3.3\} \) again, we deduce from \( \{3.5\} \) that for some \( C_3 > 0 \),
\[ \sup_{(0,T/\epsilon)} E_s(t) \leq C_3 E_s(0). \] (3.6)

By virtue of Sobolev inequality and \( \{3.6\} \), noticing that \( s \geq t_0 + 2 > 3 \), there exists a constant \( C_4 > 0 \) such that
\[ ||(\zeta(t), v(t))||_{W^{1,\infty}} \leq C_4||\zeta(t), v(t)||_{H^s} \leq C_4 C_3^2 (E_s(0)) \frac{1}{2}. \]
Taking \( \tilde{\epsilon}_3 = \min\{\frac{1-H}{C_1C_3(\epsilon, (0))}, \frac{1}{2C_2^2C_3E_s(0)}\} \), we have for any \( \epsilon \leq \min\{\tilde{\epsilon}_2, \tilde{\epsilon}_3\} \),
\[ 1 - \epsilon \|\zeta\|_{L^\infty} \geq H > \frac{H}{2}, \quad \sqrt{\epsilon}\|\zeta(t), v(t)\|_{W^{1,\infty}} \leq \frac{1}{2}, \] (3.7)
which improves the ansatz \( \{3.1\} \). Then taking \( \tilde{\epsilon} = \min\{\tilde{\epsilon}_1, \tilde{\epsilon}_2, \tilde{\epsilon}_3\} \), we have for any \( \epsilon \leq \tilde{\epsilon} \) and \( \mu \leq \tilde{\mu} \), energy estimate \( \{3.6\} \) holds for any \( t \in [0, T/\epsilon] \). Thus, \( \{1.5\} \) is proved.

The existence and uniqueness of the solution can be verified by standard mollification method and the Cauchy-Lipschitz theorem. One could refer to [30]. Now, we complete the proof of Theorem 1.1. \( \square \)

The rest of this section is devoted to prove Proposition 3.1. We only sketch the proof of three typical cases in two dimensional space, since the others could be treated in a similar way.

3.2. A priori estimates for the "general case": \( b \neq d, d > 0, a > 0, a \leq 0, c < 0 \). In this case, one could check that
\[ E_s(V) \sim E_s(t) \overset{\text{def}}{=} \|\zeta(t)\|_{X^s_{\mu}}^2 + \|v(t)\|_{X^s_{\mu}}^2 \] (3.8)
for any \( \epsilon \leq \tilde{\epsilon}_1 \) and \( \mu \leq \tilde{\mu} \) with \( \tilde{\epsilon}_1 \) and \( \tilde{\mu} \) being sufficiently small. We postpone the proof of \( \{3.8\} \) to Appendix.

A direct energy estimate shows that
\[ \frac{d}{dt}E_s(V) = ((1 - b\mu\Delta)\Lambda^s \partial_t V | (S_V(D) + S_V(D^*)^\Lambda^s V)_2 \]
\[ - b\mu[(S_V(D)^*, \Delta)\Lambda^s V | \Lambda^s \partial_t V)_2 + ((1 - b\mu\Delta)\Lambda^s V | \partial_t S_V(D)\Lambda^s V)_2 \]
\[ \overset{\text{def}}{=} I + II + III, \]
where \( S_V(D)^* \) is the adjoint operator of \( S_V(D) \).

**Step 1. Estimate on I.** Using \( \{2.3\} \), we have
\[ I = -((\Lambda^s, M(V, D)V) | (S_V(D) + S_V(D^*)^\Lambda^s V)_2 \]
\[ - ((S_V(D) + S_V(D^*)^*(M(V, D)\Lambda^s V) | \Lambda^s V)_2 \overset{\text{def}}{=} I_1 + I_2. \] (3.10)
Step 1.1. Estimate on $I_1$. Using \ref{eq:2.1} and \ref{eq:2.6}, a direct calculation yields
\begin{equation}
[(\Lambda^s, M(V, D))V | S_V(D)\Lambda^s V]_2 \\
= -(1 - \gamma)^2 (g(D)([\Lambda^s, v] \cdot \nabla \zeta + [\Lambda^s, \zeta] \nabla \cdot v) \| (1 + c\mu \Delta)^2 \Lambda^s \zeta) \\
+ c^2 (1 - \gamma) (g(D)([\Lambda^s, v] \cdot \nabla \zeta + [\Lambda^s, \zeta] \nabla \cdot v) \| v \cdot (1 + c\mu \Delta) \Lambda^s v)_2 \\
+ c^2 (1 - \gamma) \sum_{j=1,2} (g(D)([\Lambda^s, v] \cdot \partial_j v) \| g(D)(v^j (1 + c\mu \Delta) \Lambda^s \zeta + \zeta (1 + c\mu \Delta) \Lambda^s v^j)_2 \\
- \epsilon (1 - \gamma) \sum_{j=1,2} (g(D)^j (1 + c\mu \Delta)([\Lambda^s, v] \cdot \partial_j v) \| g(D)^j A(D) \Lambda^s v^j)_2 \\
- \frac{c^3}{\gamma} \sum_{j=1,2} (g(D)([\Lambda^s, v] \cdot \partial_j v) \| v^j (g(D) - 1) \Lambda^s v^j)_2 \equiv I_{11} + I_{12} + I_{13} + I_{14} + I_{15}.
\end{equation}

For $I_{11}$, integration by parts yields
\begin{equation}
|I_{11}| \lesssim \epsilon \| g(D)([\Lambda^s, v] \cdot \nabla \zeta + [\Lambda^s, \zeta] \nabla \cdot v) \| L^2 \| (1 + c\mu \Delta) \Lambda^s \zeta \| L^2 \\
+ |c| \epsilon \| g(D) \nabla([\Lambda^s, v] \cdot \nabla \zeta + [\Lambda^s, \zeta] \nabla \cdot v) \| L^2 \| (1 + c\mu \Delta) \nabla^s \zeta \| L^2,
\end{equation}

By virtue of \ref{eq:2.15} and \ref{eq:2.14}, noticing that $s \geq t_0 + 2 > 3$, we have
\begin{align*}
\| g(D)([\Lambda^s, v] \cdot \nabla \zeta) \| L^2 & \lesssim \| [\Lambda^s, v] \cdot \nabla \zeta \| L^2 \lesssim \| v \| H^{t_0+1} \| \zeta \| H^s + \| v \| H^{t_0+1} \| \zeta \| H^{t_0+1} \leq \| \zeta \| H^s \| v \| H^s, \\
\| g(D) \nabla([\Lambda^s, v] \cdot \nabla \zeta) \| L^2 & \lesssim \| v \| H^{t_0+1} \| \zeta \| H^{t_0+1} + \| v \| H^{t_0+1} \| \zeta \| H^{t_0+1} \leq \| \zeta \| H^s \| v \| H^{t_0+1} + \| v \| H^{t_0+1} \| \zeta \| H^s.
\end{align*}

Similar estimates hold for $\| g(D)([\Lambda^s, \zeta] \nabla \cdot v) \| L^2$ and $\| g(D) \nabla([\Lambda^s, \zeta] \nabla \cdot v) \| L^2$. Since
\begin{equation}
\|(1 + c\mu \Delta) \Lambda^s \zeta \| L^2 \lesssim \| \zeta \| H^s + \| \zeta \| H^{t_0+1} + \| \nabla (1 + c\mu \Delta) \Lambda^s \zeta \| L^2 \lesssim \| \zeta \| H^{t_0+1} + \mu \| \zeta \| H^{t_0+1},
\end{equation}

we have
\begin{equation}
|I_{11}| \lesssim \epsilon \| v \| X^s_{\mu^2} \| \zeta \| X^s_{\mu^2}.
\end{equation}

For $I_{14}$, we first have
\begin{equation}
|I_{14}| \lesssim \epsilon \sum_{j=1,2} \| (g(D)^j (1 + c\mu \Delta)([\Lambda^s, v] \cdot \partial_j v) \| L^2 \| g(D)^j A(D) \Lambda^s v^j \| L^2.
\end{equation}

Recalling that $A(D) = 1 + a\mu \Delta + \frac{1}{7} \sqrt{\frac{\mu}{\mu^2}} \sigma(D) + \frac{1}{7^2} \frac{\mu}{\mu^2} \sigma(D)^2$, using \ref{eq:2.18} and \ref{eq:2.13}, we have
\begin{equation}
\| A(D) f \| L^2 \lesssim \| f \| L^2 + \mu \| \nabla f \| L^2 + \mu \| \nabla^2 f \| L^2 \lesssim \| f \| X^s_{\mu^2}.
\end{equation}

Following a similar derivation as \ref{eq:3.12}, using \ref{eq:2.14}, \ref{eq:2.15}, \ref{eq:13.13} and \ref{eq:2.13}, we arrive at
\begin{equation}
|I_{14}| \lesssim \epsilon \| v \| X^s_{\mu^2}.
\end{equation}

Similar estimates as \ref{eq:3.12} and \ref{eq:3.14} hold for $I_{12}$, $I_{13}$ and $I_{15}$. Then we get
\begin{equation}
\| ([\Lambda^s, M(V, D)]V | S_V(D)\Lambda^s V) \| L^2 \lesssim (1 + \epsilon \| v \| L^\infty)^2 (\| v \| X^s_{\mu^2} + \| v \| X^s_{\mu^2})^2.
\end{equation}

The same estimate holds for $([\Lambda^s, M(V, D)]V | S_V(D)^s \Lambda^s V) \| L^2$. Using \ref{eq:3.8}, we obtain
\begin{equation}
|I_1| \lesssim \epsilon (1 + \epsilon^2 E_s(V)) \left( E_s(V) \right)^{\frac{3}{2}}.
\end{equation}
Step 1.2. Estimate on $I_2$. In order to estimate $I_2$, we first calculate $S_V(D)M(V, D) \overset{\text{def}}{=} A_V(D) \overset{\text{def}}{=} (a_{ij})_{i,j=1,2,3}$ as follows:

\[ a_{11} = -c_\gamma (1 - \gamma)^2 [(1 + c\mu \Delta)^2 g(D)(v \cdot \nabla) + g(D)(v \cdot \nabla(1 + c\mu \Delta)^2 g(D))] \]

\[ = -c_\gamma (1 - \gamma)^2 (a_{111} + a_{112}), \]

\[ a_{12} = c_\gamma (1 - \gamma)^2 [(1 + c\mu \Delta)^2 g(D)((A(D) - \epsilon \gamma)\partial_1) + c^2 (1 - \gamma)g(D)(v \cdot (1 + c\mu \Delta)g(D)(v^1 \nabla))] \]

\[ = a_{121} + a_{122}, \]

\[ a_{13} = c_\gamma (1 - \gamma)^2 (1 + c\mu \Delta)^2 g(D)((A(D) - \epsilon \gamma)\partial_2) + c^2 (1 - \gamma)g(D)(v \cdot (1 + c\mu \Delta)g(D)(v^2 \nabla)) \]

\[ = a_{121} + a_{122} + a_{123}, \]

\[ a_{21} = c_\gamma (1 - \gamma)^2 (A(D) - \epsilon \gamma)(1 + c\mu \Delta)^2 g(D)\partial_1 + c^2 (1 - \gamma)g(D)(v^1 (1 + c\mu \Delta)(v \cdot \nabla)) \]

\[ + c^2 (1 - \gamma)v^1 v \cdot \nabla (g(D) - 1)g(D)(1 + c\mu \Delta) \overset{\text{def}}{=} a_{211} + a_{212} + a_{213}, \]

\[ a_{22} = -c_\gamma (1 - \gamma)g(D) [v^1 (1 + c\mu \Delta)((A(D) - \epsilon \gamma)\partial_1)] - c_\gamma (1 - \gamma)(A(D) - \epsilon \gamma)(1 + c\mu \Delta)g(D)(v^1 \partial_1) \]

\[ - \frac{\epsilon^3}{\gamma} v^1 v \cdot (g(D) - 1)g(D)(v^1 \nabla) \overset{\text{def}}{=} a_{221} + a_{222} + a_{223}, \]

\[ a_{23} = -c_\gamma (1 - \gamma)g(D) [v^1 (1 + c\mu \Delta)((A(D) - \epsilon \gamma)\partial_2)] - c_\gamma (1 - \gamma)(A(D) - \epsilon \gamma)(1 + c\mu \Delta)g(D)(v^2 \partial_1) \]

\[ - \frac{\epsilon^3}{\gamma} v^1 v \cdot (g(D) - 1)g(D)(v^2 \nabla) \overset{\text{def}}{=} a_{231} + a_{232} + a_{233}, \]

\[ a_{31} = c_\gamma (1 - \gamma)^2 (A(D) - \epsilon \gamma)(1 + c\mu \Delta)^2 g(D)\partial_2 + c^2 (1 - \gamma)g(D)(v^2 (1 + c\mu \Delta)(v \cdot \nabla)) \]

\[ + c^2 (1 - \gamma)v^2 v \cdot \nabla (g(D) - 1)g(D)(1 + c\mu \Delta), \]

\[ a_{32} = -c_\gamma (1 - \gamma)(A(D) - \epsilon \gamma)(1 + c\mu \Delta)g(D)(v^1 \partial_2) - c_\gamma (1 - \gamma)g(D) [v^2 (1 + c\mu \Delta)((A(D) - \epsilon \gamma)\partial_1)] \]

\[ - \frac{\epsilon^3}{\gamma} v^2 v \cdot (g(D) - 1)g(D)(v^1 \nabla) \overset{\text{def}}{=} a_{321} + a_{322} + a_{323}, \]

\[ a_{33} = -c_\gamma (1 - \gamma)g(D) [v^2 (1 + c\mu \Delta)((A(D) - \epsilon \gamma)\partial_2)] - c_\gamma (1 - \gamma)(A(D) - \epsilon \gamma)(1 + c\mu \Delta)g(D)(v^2 \partial_2) \]

\[ - \frac{\epsilon^3}{\gamma} v^2 v \cdot (g(D) - 1)g(D)(v^2 \nabla). \]

The expression of $A_V(D)$ shows that the principal part of $iA_V(D)$ is symmetric. Now, we estimate $(S_V(D)M(V, D)\Lambda^\star V | \Lambda^\star V)_2 = (A_V(D)\Lambda^\star V | \Lambda^\star V)_2$ term by term.

For $a_{11}$, we have

\[ (a_{111}\Lambda^\star \xi | \Lambda^\star \xi)_2 = -c_\gamma (1 - \gamma)^2 ((a_{111}\Lambda^\star \xi | \Lambda^\star \xi)_2 + (a_{112}\Lambda^\star \xi | \Lambda^\star \xi)_2). \]

Using the expression of $a_{111}$, integrating by parts, we have

\[ (a_{111}\Lambda^\star \xi | \Lambda^\star \xi)_2 = (g(D) \frac{1}{2} (c\mu \Delta, v) \cdot \nabla \Lambda^\star \xi) | g(D) \frac{1}{2} (1 + c\mu \Delta) \Lambda^\star \xi)_2 \]

\[ + [(g(D) \frac{1}{2}, v) \cdot \nabla (1 + c\mu \Delta) \Lambda^\star \xi | g(D) \frac{1}{2} (1 + c\mu \Delta) \Lambda^\star \xi)_2 \]

\[ + (v \cdot \nabla g(D) \frac{1}{2} (1 + c\mu \Delta) \Lambda^\star \xi | g(D) \frac{1}{2} (1 + c\mu \Delta) \Lambda^\star \xi)_2. \]

Integration by parts yields that the last term in (3.17) equals

\[ - \frac{1}{2} (\nabla \cdot g(D) \frac{1}{2} (1 + c\mu \Delta) \Lambda^\star \xi | g(D) \frac{1}{2} (1 + c\mu \Delta) \Lambda^\star \xi)_2. \]

which along with (3.17) and (2.15) implies

\[ |(a_{111}\Lambda^\star \xi | \Lambda^\star \xi)_2| \overset{\mu \|[\Delta, v] \cdot \nabla \Lambda^\star \xi\|_{L^2} \|\xi\|_{X_{\mu^2}} + \|g(D) \frac{1}{2}, v\cdot \nabla (1 + c\mu \Delta) \Lambda^\star \xi\|_{L^2} \|\xi\|_{X_{\mu^2}} + \|\nabla v\|_{L^\infty} \|\xi\|_{\tilde{X}_{\mu^2}}. \]

Thanks to (2.17), we have

\[ \|(g(D) \frac{1}{2}, v) \cdot \nabla (1 + c\mu \Delta) \Lambda^\star \xi\|_{L^2} \|\xi\|_{X_{\mu^2}}. \]

Since

\[ \mu \|[\Delta, v] \cdot \nabla \Lambda^\star \xi\|_{L^2} \overset{\mu \|v\|_{H^{\mu+2}} \|\xi\|_{H^{\mu+1}} + \mu \|v\|_{H^{\mu+1}} \|\xi\|_{H^{\mu+2}} \]

using (2.13), we get

\[ |(a_{111}\Lambda^\star \xi | \Lambda^\star \xi)_2| \overset{\mu \|v\|_{H^{\mu+2}} \|\xi\|_{\tilde{X}_{\mu^2}} \overset{\|v\|_{H^{\mu}} \|\xi\|_{\tilde{X}_{\mu^2}}. \]

}
The same estimate holds for \( (a_{112}\Lambda^s \zeta | \Lambda^s \zeta)^2 \). Then we obtain
\[
|a_{11} \Lambda^s \zeta | \Lambda^s \zeta^2 \| \lesssim \epsilon \|v\|_{H^{1/2 + \epsilon}} \|\zeta\|_{X_{\mu^3}} \lesssim \epsilon \|v\|_{H^1} \|\zeta\|_{X_{\mu^3}}^2. \tag{3.18}
\]

For \( a_{12} \) and \( a_{21} \), we have
\[
(a_{12} \Lambda^s v^1 | \Lambda^s \zeta^2) + (a_{21} \Lambda^s \zeta | \Lambda^s v^1)^2 = ((a_{12} + a_{21}) \Lambda^s \zeta | \Lambda^s v^1)^2,
\]
where \( a_{12}^* \) is the adjoint operator of \( a_{12} \). By the expression of \( a_{12} \), we first have
\[
a_{121}^* = -\gamma(1 - \gamma)^2 \partial_1 \left[ (A(D) - \epsilon \zeta)(1 + \mu\Delta)^2 g(D) \right]
\]
\[
a_{122}^* = -\epsilon^2(1 - \gamma) \nabla \cdot \left[ v^1(1 + \mu\Delta)g(D)(v g(D)) \right].
\]
Due to the expression of \( a_{21} \), we have
\[
a_{121} + a_{211} = \epsilon \gamma(1 - \gamma)^2 \partial_1 \zeta(1 + \mu\Delta)^2 g(D),
\]
which implies
\[
((a_{121} + a_{211}) \Lambda^s \zeta | \Lambda^s v^1)^2 = \epsilon \gamma(1 - \gamma)^2 (1 + \mu\Delta)g(D) \Lambda^s \zeta | \Lambda^s v^1)^2. \tag{3.19}
\]
Noticing that \( s \geq t_0 + 2 > 3 \), using (2.15) and (2.13), we have
\[
|((a_{121} + a_{211}) \Lambda^s \zeta | \Lambda^s v^1)^2| \lesssim \epsilon \|\zeta\|_{X_{\mu^3}^{t_0 + 1}} \|\zeta\|_{X_{\mu^3}} \|v\|_{X_{\mu^3}} \lesssim \epsilon \|\zeta\|_{X_{\mu^3}}^2 \|v\|_{X_{\mu^3}}. \tag{3.20}
\]
Since
\[
a_{122}^* = -\epsilon^2(1 - \gamma) \nabla \cdot \left[ v^1(1 + \mu\Delta)g(D)(v) - \epsilon^2(1 - \gamma) \nabla \cdot \left[ v^1(1 + \mu\Delta)g(D)(v(g(D) - 1)) \right] \right]
\]
\[
\defeq a_{122,1}^* + a_{122,2}^*,
\]
we have
\[
\frac{1}{\epsilon^2(1 - \gamma)} (a_{122,1}^* + a_{122,2}^*) = -\nabla v^1 \cdot (1 + \mu\Delta)g(D)(v) - v^1(1 + \mu\Delta)g(D) (\nabla \cdot v^1)
\]
\[
+ [g(D), v^1]((1 + \mu\Delta)(v \cdot \nabla)), \tag{3.21}
\]
and
\[
\frac{1}{\epsilon^2(1 - \gamma)} (a_{122,2}^* + a_{122,3}^*) = -\nabla v^1 \cdot (1 + \mu\Delta)g(D)(v(g(D) - 1))
\]
\[
- v^1(1 + \mu\Delta)g(D) (\nabla \cdot v(g(D) - 1)) - v^1(1 + \mu\Delta)g(D)(v) \cdot \nabla (g(D) - 1), \tag{3.22}
\]
which long with (2.21), (2.22), (2.26), and (2.13) implies
\[
\|a_{122,1}^* + a_{212} \Lambda^s \zeta \|_{L^2} + \|a_{122,2}^* + a_{213} \Lambda^s \zeta \|_{L^2} \lesssim \epsilon^2 \|v^1\|_{H^{1/2 + \epsilon}} \|v\|_{X_{\mu^3}^{t_0 + 1}} \|\zeta\|_{X_{\mu^3}},
\]
where we used the formula
\[
((1 + \mu\Delta)g(D), v) = (1 + \mu\Delta)((g(D), v)) + \mu\Delta [v, g(D)].
\]
Since \( s \geq t_0 + 2 > 3 \), using (2.21) again, we have
\[
|((a_{12} + a_{21} + a_{213}) \Lambda^s \zeta | \Lambda^s v^1)^2| \lesssim \epsilon^2 \|v^1\|_{X_{\mu^3}^{t_0 + 1}} \|\zeta\|_{X_{\mu^3}} \|v\|_{X_{\mu^3}} \lesssim \epsilon^2 \|\zeta\|_{X_{\mu^3}} \|v\|_{X_{\mu^3}}^2. \tag{3.23}
\]
Thanks to (3.20) and (3.23), we have
\[
\|a_{12} \Lambda^s v | \Lambda^s \zeta^2\| + (a_{21} \Lambda^s \zeta | \Lambda^s v^1)^2 \| \lesssim \epsilon (1 + \epsilon \|v\|_{X_{\mu^3}}) \|\zeta\|_{X_{\mu^3}} \|\zeta\|_{X_{\mu^3}} \|v\|_{X_{\mu^3}} \|v\|_{X_{\mu^3}}^2. \tag{3.24}
\]
The same estimate holds for \( (a_{13} \Lambda^s v^2 | \Lambda^s \zeta^2)^2 + (a_{23} \Lambda^s \zeta | \Lambda^s v^2)^2 \).

For \( a_{22} \), we first estimate \( (a_{22} \Lambda^s v^1 | \Lambda^s v^1)^2 \). Using the expression of \( a_{22} \), we have
\[
- \frac{1}{1 - \gamma} (a_{22} \Lambda^s v^1 | \Lambda^s v^1)^2 = \epsilon (g(D)(v^1 A(D) \partial_1 \Lambda^s v^1) | \Lambda^s v^1)^2 + \epsilon (\mu \epsilon (g(D)(v^1 \Delta A(D) \partial_1 \Lambda^s v^1) | \Lambda^s v^1)^2
\]
\[
- \epsilon^2 (g(D)(v^1 \partial_1 \Lambda^s v^1) | \Lambda^s v^1)^2 - \epsilon^2 \mu \epsilon (g(D)(v^1 \Delta (\zeta \partial_1 \Lambda^s v^1))) | \Lambda^s v^1)^2 \defeq B_{11} + B_{12} + B_{13} + B_{14}.
\]
For $B_{11}$, using the expression of $A(D)$ in (2.15), we have

$$B_{11} = \epsilon (g(D) \frac{1}{2} (v^1 \partial_1 \Lambda^s v^1) | g(D) \frac{1}{2} \Lambda^s v^1)_2 + a \epsilon \mu (g(D) \frac{1}{2} (v^1 \Delta \partial_1 \Lambda^s v^1) | g(D) \frac{1}{2} \Lambda^s v^1)_2$$

$$+ \frac{\epsilon}{\gamma} \frac{\mu}{\mu_2} (g(D) \frac{1}{2} (\sigma(D) \partial_1 \Lambda^s v^1) | g(D) \frac{1}{2} \Lambda^s v^1)_2 + \frac{\epsilon \mu}{\gamma_2^2 \mu_2} (g(D) \frac{1}{2} (v^1 \partial_1^2 \Lambda^s v^1) | g(D) \frac{1}{2} \Lambda^s v^1)_2$$

$$\overset{\text{def}}{=} B_{11,1} + B_{11,2} + B_{11,3} + B_{11,4}.$$

A direct calculation shows that

$$\gamma^2 \frac{\mu_2}{\epsilon \mu} B_{11,4} = \left( [g(D) \frac{1}{2} (v^1) | g(D) \frac{1}{2} \Lambda^s v^1)_2 \right)$$

$$- \left( [(\sigma(D), v^1) | g(D) \frac{1}{2} \Lambda^s v^1 | g(D) \frac{1}{2} \Lambda^s v^1)_2 + (v^1 \partial_1 \sigma(D) | g(D) \frac{1}{2} \Lambda^s v^1 | \sigma(D) g(D) \frac{1}{2} \Lambda^s v^1)_2.$$

Integrating by parts for the last term of $\gamma^2 \frac{\mu_2}{\epsilon \mu} B_{11,4}$, we see that it equals

$$- \frac{1}{2} (\partial_1 v^1 | \sigma(D) g(D) \frac{1}{2} \Lambda^s v^1 | \sigma(D) g(D) \frac{1}{2} \Lambda^s v^1)_2.$$

Using (2.15) and (2.18), we have

$$| B_{11,4} | \lesssim \epsilon \mu \left( \| g(D) \frac{1}{2}, v^1 | \sigma(D) g(D) \frac{1}{2} \Lambda^s v^1 | L^2 \| \| \sigma(D) v^1 | \sigma(D) g(D) \frac{1}{2} \Lambda^s v^1 | L^2 \| \right) \| v^1 \|_{L^2}$$

$$+ \epsilon \mu \| \partial_1 v^1 \|_{L^2} \| v^1 \|_{L^2}^{\frac{1}{2}}.$$  \hfill (3.25)

Thanks to (2.17), (2.19), (2.15) and (2.18), we have

$$\| [g(D) \frac{1}{2}, v^1] | \sigma(D) g(D) \frac{1}{2} \Lambda^s v^1 | L^2 \| \lesssim \| v^1 \|_{H^0} \| \sigma(D) g(D) \frac{1}{2} \Lambda^s v^1 \|_{L^2} \lesssim \| v^1 \|_{H^2} \| v^1 \|_{H^{+2}},$$

$$\| [\sigma(D), v^1] | \sigma(D) g(D) \frac{1}{2} \Lambda^s v^1 | L^2 \| \lesssim \| v^1 \|_{H^0} \| \sigma(D) g(D) \frac{1}{2} \Lambda^s v^1 \|_{L^2} \lesssim \| v^1 \|_{H^2} \| v^1 \|_{H^{+2}}$$

which along with (2.13) and (3.25) imply

$$| B_{11,4} | \lesssim \epsilon \mu \| v^1 \|_{H^{+2}} \cdot \| v^1 \|_{\dot{H}^2} + \epsilon \mu \| v^1 \|_{H^{+2}} \cdot \| v^1 \|_{\dot{H}^2} \lesssim \epsilon \| v^1 \|_{X_{\mu,3}}^3.$$  \hfill (3.26)

Similar estimates hold for $B_{11,1}, B_{11,2}$ and $B_{11,3}$. Then we obtain

$$| B_{11} | \lesssim \epsilon \| v^1 \|_{H^2} \| v^1 \|_{X_{\mu,3}}^3 \lesssim \epsilon \| v^1 \|_{X_{\mu,3}}^3.$$  \hfill (3.27)

Following similar derivation as (3.26), we have

$$| B_{12} | \lesssim \epsilon \| v^1 \|_{X_{\mu,2}}^3 \| v^1 \|_{X_{\mu,3}}^3 \lesssim \epsilon \| v^1 \|_{X_{\mu,3}}^3,$$

$$| B_{13} | + | B_{14} | \lesssim \epsilon \| v^1 \|_{H^{0+2}} \| \sigma(D) g(D) \frac{1}{2} \Lambda^s v^1 \|_{L^2} \| v^1 \|_{X_{\mu,2}}^2 \lesssim \epsilon \| v^1 \|_{H^2} \| v^1 \|_{H^{+2}} \| v^1 \|_{X_{\mu,3}}^2,$$

which along with (3.26) imply

$$| (a_{221} \Lambda^s v^1 | \Lambda^s v^1)_2 | \lesssim \epsilon (1 + \epsilon \| \sigma(D) g(D) \|_{L^2}) \| v^1 \|_{X_{\mu,3}}^3.$$  \hfill (3.27)

Similarly, we have

$$| (a_{222} \Lambda^s v^1 | \Lambda^s v^1)_2 | \lesssim \epsilon (1 + \epsilon \| \sigma(D) g(D) \|_{L^2}) \| v^1 \|_{X_{\mu,3}}^3,$$

$$| (a_{233} \Lambda^s v^1 | \Lambda^s v^1)_2 | \lesssim \epsilon (1 + \epsilon \| \sigma(D) g(D) \|_{L^2}) \| v^1 \|_{X_{\mu,3}}^3,$$

which along with (3.27) implies

$$| (a_{22} \Lambda^s v^1 | \Lambda^s v^1)_2 | \lesssim \epsilon (1 + \epsilon \| \sigma(D) g(D) \|_{L^2}) \| v^1 \|_{X_{\mu,3}}^3.$$  \hfill (3.28)

The same estimate holds for $(a_{33} \Lambda^s v^2 | \Lambda^s v^2)_2$.

For $a_{23}$ and $a_{32}$, we have

$$(a_{23} \Lambda^s v^2 | \Lambda^s v^2)_2 + (a_{32} \Lambda^s v^1 | \Lambda^s v^2)_2 = (a_{23} + a_{32}) \Lambda^s v^1 | \Lambda^s v^2)_2,$$

where $a_{23}^*$ is the adjoint operator of $a_{23}$. By the expression of $a_{23} = a_{231} + a_{232} + a_{233}$, we first have

$$a_{231} = \epsilon (1 - \gamma) \partial_2 \left[ (A(D) - \epsilon \zeta) \| (1 + \epsilon) \zeta \| (A(D) - \epsilon \zeta) \right],$$

$$a_{232} = \epsilon (1 - \gamma) \partial_1 \left[ v^2 g(D) \| (1 + \epsilon) \zeta \| (A(D) - \epsilon \zeta) \right],$$

$$a_{233} = \frac{\epsilon \gamma^3}{2} \nabla \cdot [ v^2 g(D) (A(D) - \epsilon \zeta) - (A(D) - \epsilon \zeta) \partial_1].$$
which along with the expression of $a_{321}$ imply
\[
\begin{align*}
a_{321}^* + a_{321} &= -c^2(1 - \gamma)\partial_2\zeta(1 + c\mu\Delta)(v^*g(D)) + \epsilon(1 - \gamma)(A(D) - \epsilon\zeta)(1 + c\mu\Delta)(\partial_2 v^*g(D)) \\
&- \epsilon(1 - \gamma)(A(D) - \epsilon\zeta)(1 + c\mu\Delta)(g(D), v^2)\partial_2,
\end{align*}
\]
\[
\begin{align*}
a_{322}^* + a_{322} &= \epsilon(1 - \gamma)\partial_2 v^2 g(D)(1 + c\mu\Delta)((A(D) - \epsilon\zeta)) - c^2(1 - \gamma)v^2(1 + c\mu\Delta)g(D)(\partial_1\zeta) \\
&- \epsilon(1 - \gamma)g(D), v^2)(1 + c\mu\Delta)((A(D) - \epsilon\zeta)\partial_1,
\end{align*}
\]
\[
\begin{align*}
a_{323}^* + a_{323} &= \frac{\epsilon^3}{\gamma}(\nabla v^2 \cdot g(D)(g(D) - 1)(v^2) + v^2 g(D)(g(D) - 1)(\nabla \cdot (v^2)) \\
&+ v^2 g(D)(g(D) - 1), v \cdot (v^2))
\end{align*}
\]

Thanks to (2.11), (2.15), (3.19), (2.17) and (2.13), we have
\[
\begin{align*}
\|((a_{321}^* + a_{321})\Lambda^* v^1 | \Lambda^* v^2)^2_2 \| &\leq \epsilon\|A(D) - \epsilon\zeta\Lambda^* v^1\|_{L^2}\|1 + c\mu\Delta(\partial_1 v^2 \Lambda^* v^2)\|_{L^2} \\
&+ \epsilon^2\|v^2(1 + c\mu\Delta)(\partial_1 \zeta \Lambda^* v^1)\|_{L^2}\|\Lambda^* v^2\|_{L^2} \\
&+ \epsilon\|((A(D) - \epsilon\zeta)\partial_1 \zeta v^1)_{H^1}((1 + c\mu\Delta)(g(D), v^2)\Lambda^* v^2)\|_{H^1} \\
&\leq \epsilon(1 + \epsilon\|\zeta\|_{H^1} + \epsilon\|v\|_{X^{\mu_1}} + \epsilon\|\zeta\|_{X^{\mu_2}} + \epsilon^2\|v\|_{X^{\mu_2}}^2)\|\zeta\|_{X^{\mu_3}}^3.
\end{align*}
\]

where we also used the fact $s \geq t_0 + 2 > 3$. Similar estimates hold for $((a_{321}^* + a_{321})\Lambda^* v^1 | \Lambda^* v^2)^2_2$ and $((a_{323}^* + a_{323})\Lambda^* v^1 | \Lambda^* v^2)^2_2$. Using (2.13), we have
\[
\begin{align*}
\|((a_{323}^* + a_{323})\Lambda^* v^2 | \Lambda^* v^2)^2_2 \| &\leq \epsilon(1 + \epsilon\|\zeta\|_{X^{\mu_1}} + \epsilon\|v\|_{X^{\mu_1}} + \epsilon\|\zeta\|_{X^{\mu_2}} + \epsilon^2\|v\|_{X^{\mu_2}}^2)\|\zeta\|_{X^{\mu_3}}^3.
\end{align*}
\]

Due to (2.11), (2.15) and (3.8), we obtain the estimate for $\dot{S}_V(D) M(V, D) \Lambda^* V | \Lambda^* V \|_{H^0} 2$. Since the same estimate holds for $\dot{S}_V(D)^* M(V, D) \Lambda^* V | \Lambda^* V \|_{H^0} 2$, using (3.8), we arrive at
\[
|I_2| \leq \epsilon(1 + \epsilon^2 E_s(V))(E_s(V))\frac{3}{2}.
\]

**Step 1.3. Estimate on I.** Due to (3.15) and (3.31), we obtain
\[
|I| \leq \epsilon(1 + \epsilon^2 E_s(V))(E_s(V))\frac{3}{2}.
\]

**Step 2. Estimate on II.** Thanks to the expression of $S_V(D)$, we have
\[
\begin{align*}
|II| &\leq \mu\|\Delta, v \|g(D)\Lambda^* V | H^0\|\|1 + c\mu\Delta\Lambda^* \partial V\|_{H^{-1}} + \mu\|\Delta, \zeta\|\Lambda^* V | H^0\|\|1 + c\mu\Delta\Lambda^* \partial V\|_{H^{-1}} \\
&+ \mu\epsilon^2 \sum_{i,j=1,2} \|\Delta, v^i v^j| \Lambda^* V | H^0\|\|(g(D) - 1)\Lambda^* \partial V\|_{H^{-1}},
\end{align*}
\]

which along with (2.15), (2.13) and (3.8), noticing that $s \geq t_0 + 2 > 3$, we have
\[
\begin{align*}
|II| &\leq \epsilon(1 + \epsilon\|\Lambda^* V\|_{H^0})\|\Lambda^* V\|_{X^{\mu_2}}\|\partial V\|_{X^{\mu_2}}\|_{X^{\mu_2}} \\
&\leq \epsilon(1 + \epsilon^2 E_s(V))\frac{3}{2} E_s(V)\|\partial V\|_{X^{\mu_2}}\|_{X^{\mu_2}}.
\end{align*}
\]

**Step 3. Estimate on III.** Thanks to the expression of $S_V(D)$, using (2.15) and (2.16), we have
\[
\begin{align*}
|III| &\leq \epsilon(1 + \epsilon\|\Lambda^* V\|_{L^2})\|\partial S_V(D)\Lambda^* V\|_{L^2} \|\partial V\|_{H^{-1}}\|\Lambda^* V\|_{H^0} \|\Lambda^* V\|_{X^{\mu_2}}. \\
\end{align*}
\]

Due to (2.13) and (3.8), noticing that $s \geq t_0 + 2 > 3$, we have
\[
|III| \leq \epsilon(1 + \epsilon^2 E_s(V))\frac{3}{2} E_s(V)\|\partial V\|_{H^{-2}}.
\]

**Step 4. The a priori energy estimate.** Thanks to (3.32), (3.33) and (3.34), we deduce from (3.9) that
\[
\frac{d}{dt} E_s(V) \leq \epsilon(1 + \epsilon^2 E_s(V))(E_s(V))\frac{3}{2} + \epsilon(1 + \epsilon^2 E_s(V))\frac{3}{2} E_s(V)\|\partial V\|_{X^{\mu_2}}.
\]

Going back to the equation (2.3), using (2.15), (2.12), (3.13), (2.13) and (3.8), we have
\[
\begin{align*}
\|\partial V\|_{X^{\mu_2}} \sim \epsilon(1 + \epsilon\|\Lambda^* V\|_{H^0})\|\Lambda^* V\|_{X^{\mu_2}} \leq \epsilon(1 + \epsilon^2 E_s(V))E_s(V)\frac{3}{2}.
\end{align*}
\]
which along with (3.35) implies
\[
\frac{d}{dt} E_s(V) \lesssim (1 + \epsilon^2 E_s(V)) \left( E_s(V) \right)^{\frac{3}{2}}.
\] (3.36)

This is exactly (3.2).

3.3. A priori estimates for the case: \( b > 0, d = 0, a \leq 0, c < 0 \). In this case, one could check that
\[
E_s(V) \sim E_s(t) \overset{\text{def}}{=} ||\zeta||_{X_{\mu_1}^4}^2 + ||v(t)||_{X_{\mu_3}^2}^2
\] (3.37)

for any \( \epsilon \leq \epsilon_1 \) and \( \mu \leq \tilde{\mu} \) with \( \epsilon_1 \) and \( \tilde{\mu} \) being sufficiently small. We postpone the proof of (3.37) to the Appendix.

Since the proof of (3.2) of this case is similar to that of case \( b \neq d, b > 0, d > 0, a \leq 0, c < 0 \), we only sketch it.

A direct energy estimate shows that
\[
\frac{d}{dt} E_s(V) = ( (1 - b\mu\Delta)\Lambda^s \partial_t V | (S_V(D) + S_V(D)^*)\Lambda^s V )_2
\]
\[
- b\mu((S_V(D)^*)\Lambda^s \partial_t V) + (1 - b\mu\Delta)\Lambda^s | \partial_t S_V(D)\Lambda^s V )_2
\]
\[
\overset{\text{def}}{=} I + II + III,
\]
where \( S_V(D)^* \) is the adjoint operator of \( S_V(D) \).

**Step 1. Estimate on I.** Using (2.3), we have
\[
I = -((\Lambda^s, M(V, D)|V | (S_V(D) + S_V(D)^*)\Lambda^s V )_2
\]
\[
- ((S_V(D) + S_V(D)^*)(M(V, D)|\Lambda^s V )_2 \overset{\text{def}}{=} I_1 + I_2.
\] (3.39)

**Step 1.1. Estimate on \( I_1 \).** Firstly, notice that (3.11) also holds for the present case with \( g(D) = 1 - b\mu\Delta \). Similar derivation as (3.15), we have
\[
|I_1| \lesssim \epsilon(1 + \epsilon ||\zeta||_{X_{\mu_1}^4}^2 + \epsilon ||v||_{X_{\mu_3}^2})^2 ||v||_{X_{\mu_3}^2}^2 + ||\zeta||_{X_{\mu_1}^4}^2 + \epsilon ||v||_{X_{\mu_3}^2}^2 \lesssim \epsilon(1 + \epsilon^2 E_s(V)) \left( E_s(V) \right)^{\frac{3}{2}}.
\] (3.40)

**Step 1.2. Estimate on \( I_2 \).** In order to estimate \( I_2 \), we first calculate \( S_V(D) M(V, D) \overset{\text{def}}{=} \Lambda V(D) = (a_{ij})_{i,j=1,2,3} \). We point out that \( a_{ij} \) has the same expression as that in (3.16) with \( g(D) = 1 - b\mu\Delta \).

Now, we estimate \( (S_V(D) M(V, D)|\Lambda^s V )_2 = (\Lambda^s V )_2 \) term by term. Following similar derivation as that of case \( b \neq d, b > 0, d > 0, a \leq 0, c < 0 \), integrating by parts, we first have
\[
|((a_{11}\Lambda^s \zeta | \Lambda^s \zeta )_2| \lesssim \epsilon ||v||_{X_{\mu_3}^2}^2 ||\zeta||_{X_{\mu_3}^4}^2.
\]
\[
|((a_{22}\Lambda^s v^1 | \Lambda^s v^1 )_2| + |(a_{33}\Lambda^s v^2 | \Lambda^s v^2 )_2| \lesssim \epsilon(1 + \epsilon ||\zeta||_{X_{\mu_3}^4}^2 + \epsilon ||v||_{X_{\mu_3}^2}^2)^2 ||v||_{X_{\mu_3}^2}^3.
\] (3.41)

**For \( a_{12} \) and \( a_{21} \),** firstly, noticing that (3.19) holds for \( a_{12} + a_{21} \) with \( g(D) = 1 - b\mu\Delta \), we have similar estimate as (3.20) as follows
\[
|((a_{12} + a_{21})\Lambda^s \zeta | \Lambda^s v^1 )_2| \lesssim \epsilon ||v||_{X_{\mu_3}^2}^2 ||\zeta||_{X_{\mu_3}^4}^2 ||v||_{X_{\mu_3}^2} \lesssim \epsilon ||\zeta||_{X_{\mu_3}^4}^2 ||v||_{X_{\mu_3}^2}.
\] (3.42)

For \( a_{12} + a_{21} \), there hold (3.21) and (3.22) with \( g(D) = 1 - b\mu\Delta \). Similarly as (3.23), we have
\[
|((a_{12} + a_{21})\Lambda^s \zeta | \Lambda^s v^1 )_2| \lesssim \epsilon^2 ||v||_{H^{2,1}}^2 ||v||_{X_{\mu_3}^2}^2 ||\zeta||_{X_{\mu_3}^4}^2 \lesssim \epsilon ||v||_{X_{\mu_3}^2}^2 ||\zeta||_{X_{\mu_3}^4}^2.
\]
\[
|((a_{22} + a_{21})\Lambda^s \zeta | \Lambda^s v^1 )_2| \overset{\text{def}}{=} \epsilon ||v||_{X_{\mu_3}^2}^2 ||\zeta||_{X_{\mu_3}^4}^2.
\]
\[
|((a_{22} + a_{21})\Lambda^s \zeta | \Lambda^s v^1 )_2| \lesssim \epsilon(1 + \epsilon ||\zeta||_{X_{\mu_3}^4}^2 + \epsilon ||v||_{X_{\mu_3}^2}^2)^2 ||v||_{X_{\mu_3}^2} ||\zeta||_{X_{\mu_3}^4}^2 + ||v||_{X_{\mu_3}^2}^2.
\] (3.43)

The same estimate holds for \( (a_{13}\Lambda^s v^2 | \Lambda^s \zeta )_2 + (a_{31}\Lambda^s \zeta | \Lambda^s v^2 )_2 \).

**For \( a_{23} \) and \( a_{32} \),** there also holds (3.29) with \( g(D) = 1 - b\mu\Delta \). Then we get
\[
|((a_{23}\Lambda^s v^2 | \Lambda^s \zeta )_2 + (a_{32}\Lambda^s \zeta | \Lambda^s v^2 )_2| \lesssim \epsilon(1 + \epsilon ||\zeta||_{X_{\mu_3}^4}^2 + \epsilon ||v||_{X_{\mu_3}^2}^2)^2 ||v||_{X_{\mu_3}^2} ||\zeta||_{X_{\mu_3}^4}^2 + ||v||_{X_{\mu_3}^2}^2.
\] (3.44)

Thanks to (3.42), (3.43) and (3.44), we obtain the bound of \( (S_V(D) M(V, D)|\Lambda^s V )_2 \). The same estimate holds for \( (S_V(D)^* M(V, D)|\Lambda^s V )_2 \). Using (3.37), we arrive at
\[
|I_2| \lesssim \epsilon(1 + \epsilon^2 E_s(V)) \left( E_s(V) \right)^{\frac{3}{2}}.
\] (3.45)
Due to \((3.40)\) and \((3.45)\), we obtain

\[
|I| \lesssim \epsilon (1 + c^2 E_s(V)) (E_s(V))^{\frac{3}{2}}.
\]  

(3.46)

**Step 2. Estimate on \(II\).** Thanks to the expression of \(S_V(D)\), we have

\[
II = b \epsilon (1 - \gamma) \gamma \mu ([\Delta, v] \cdot (1 - b \mu \Delta) \Lambda^s v | (1 + c \mu \Delta) \Lambda^s \partial_t \zeta) \| (1 - b \mu \Delta) \Lambda^s \partial_t v \|_2
\]

\[
+ b \epsilon (1 - \gamma) \gamma \mu ([\Delta, \zeta] \Lambda^s v | (1 + c \mu \Delta) \Lambda^s \partial_t \zeta) \| (1 + c \mu \Delta) \Lambda^s \partial_t v \|_2
\]

\[
+ b^2 \epsilon^2 \mu^2 \sum_{i,j=1,2} ([\Delta, v^i v^j] \Lambda^s v^j | \Delta \Lambda^s \partial_t v^j) \|_2,
\]

which along with \((2.13)\) implies

\[
|II| \lesssim \epsilon \|v\|_{X_{\mu,3}^{1,0}} \|v\|_{X_{\mu,3}^{1,0}} \|\partial_t \zeta\|_{X_{\mu,3}^{2,1}} + \epsilon \|\partial_t v\|_{X_{\mu,3}^{2,1}} \|\partial_t \zeta\|_{X_{\mu,3}^{2,1}}
\]

\[
+ \epsilon \|\partial_t \zeta\|_{X_{\mu,3}^{2,1}} \|\partial_t v\|_{X_{\mu,3}^{2,1}} + \epsilon^2 \|v\|_{H_{\mu,1}^{0,1}} \|\partial_t v\|_{X_{\mu,3}^{2,1}} \|\partial_t \zeta\|_{X_{\mu,3}^{2,1}}.
\]

Since \(s \geq t_0 + 2 > 3\), using \((2.13)\) and \((3.37)\), we obtain

\[
|II| \lesssim \epsilon (1 + \epsilon (E_s(V))^{\frac{3}{2}} E_s(V) (\|\partial_t \zeta\|_{X_{\mu,3}^{2,1}} + \|\partial_t v\|_{X_{\mu,3}^{2,1}}).
\]

(3.47)

**Step 3. Estimate on \(III\).** Thanks to the expression of \(S_V(D)\), we have

\[
III = -\epsilon (1 - \gamma) \gamma \mu ((1 - b \mu \Delta)^2 \Lambda^s \partial_t \zeta \cdot (1 + c \mu \Delta) \Lambda^s v) - \epsilon (1 - \gamma) \gamma \mu ((1 - b \mu \Delta) \Lambda^s \partial_t \zeta | (1 - b \mu \Delta) \Lambda^s \partial_t v) \|_2
\]

\[
- \epsilon (1 - \gamma) \gamma (1 - b \mu \Delta) \Lambda^s \partial_t \zeta (1 + c \mu \Delta) \Lambda^s v) - \epsilon (1 - \gamma) \gamma (1 - b \mu \Delta) \Lambda^s \partial_t \zeta | (1 + c \mu \Delta) \Lambda^s \partial_t v) \|_2
\]

\[
- \epsilon (1 - \gamma) \gamma \mu \sum_{i,j=1,2} ([\Lambda, v^i v^j] \Lambda^s v^j | \partial_t (v^i v^j) (1 + c \mu \Delta) \Lambda^s v^j) \|_2,
\]

which along with \((2.13)\) implies

\[
|III| \lesssim \epsilon \|\partial_t \zeta\|_{X_{\mu,3}^{2,1}} \|v\|_{X_{\mu,3}^{2,1}} \|\partial_t v\|_{H_{\mu,1}^{0,1}} + \epsilon (1 + \epsilon \|v\|_{H_{\mu,1}^{0,1}}) \|v\|_{X_{\mu,3}^{2,1}} (\|\partial_t \zeta\|_{H_{\mu,1}^{0,1}} + \|\partial_t v\|_{H_{\mu,1}^{0,1}}).
\]

Since \(s \geq t_0 + 2 > 3\), using \((2.13)\) and \((3.37)\), we obtain

\[
|III| \lesssim \epsilon (1 + \epsilon (E_s(V))^{\frac{3}{2}} E_s(V) (\|\partial_t \zeta\|_{H_{\mu,1}^{0,1}} + \|\partial_t v\|_{H_{\mu,1}^{0,1}}).
\]

(3.48)

**Step 4. The a priori energy estimate.** Thanks to \((3.46), (3.47)\) and \((3.48)\), we deduce from \((3.38)\) that

\[
\frac{d}{dt} E_s(V) \lesssim \epsilon (1 + c^2 E_s(V)) (E_s(V))^{\frac{3}{2}} + \epsilon (1 + \epsilon (E_s(V))^{\frac{3}{2}}) E_s(V) (\|\partial_t \zeta\|_{X_{\mu,3}^{2,1}} + \|\partial_t v\|_{X_{\mu,3}^{2,1}}).
\]

(3.49)

Going back to the equation \((2.2)\), using \((2.15), (2.12), (3.13)\) and \((2.13)\), we have

\[
\|\partial_t \zeta\|_{X_{\mu,3}^{2,1}} \lesssim \|\Lambda(D) - \epsilon \zeta\|_{X_{\mu,3}^{2,1}} \lesssim (1 + \epsilon \|\zeta\|_{X_{\mu,3}^{2,1}}) \|v\|_{X_{\mu,3}^{2,1}},
\]

\[
\|\partial_t v\|_{X_{\mu,3}^{2,1}} \lesssim \|1 + c \mu \Delta\|_{X_{\mu,3}^{2,1}} + \|v \cdot v\|_{X_{\mu,3}^{2,1}} \lesssim \|\zeta\|_{X_{\mu,3}^{2,1}} + \|v\|_{X_{\mu,3}^{2,1}}^2,
\]

which along with \((2.13)\) and \((3.37)\) implies

\[
\|\partial_t \zeta\|_{X_{\mu,3}^{2,1}} + \|\partial_t v\|_{X_{\mu,3}^{2,1}} \lesssim (1 + \epsilon (E_s(V))^{\frac{3}{2}}) (E_s(V))^{\frac{3}{2}}.
\]

(3.50)

Due to \((3.49)\) and \((3.50)\), we get

\[
\frac{d}{dt} E_s(V) \lesssim \epsilon (1 + c^2 E_s(V)) (E_s(V))^{\frac{3}{2}}.
\]

(3.51)

This is exactly \((3.2)\).
3.4. **A priori estimates for the case:** \( b = d = 0, \, a \leq 0, \, c < 0. \) In this case, the equation (2.2) is equivalent to the following condensed system

\[
\partial_t \mathbf{V} + M(\mathbf{V}, D)\mathbf{V} = 0, \tag{3.52}
\]

where \( M(\mathbf{V}, D) \) is defined in (2.4) with \( g(D) = 1. \) The symmetrizer \( S_\mathbf{V}(D) \) of \( M(\mathbf{V}, D) \) is defined by (2.5).

Defining the associated energy functional as

\[
E_\mathbf{s}(\mathbf{V}) \overset{\text{def}}{=} (\Lambda^*\mathbf{V} \mid S_\mathbf{V}(D)\Lambda^*\mathbf{V})_2, \tag{3.53}
\]

one could check that

\[
E_\mathbf{s}(\mathbf{V}) \sim E_\mathbf{s}(t) \overset{\text{def}}{=} \|\zeta(t)\|_{X_2^\mu}^2 + \|\mathbf{v}(t)\|_{X_2^\mu}^2, \tag{3.54}
\]

for any \( \epsilon \leq \tilde{\epsilon}_1 \) and \( \mu \leq \tilde{\mu} \) with \( \tilde{\epsilon}_1 \) and \( \tilde{\mu} \) being sufficiently small. We postpone the proof of (3.54) to the Appendix.

A direct energy estimate shows that

\[
\frac{d}{dt} E_\mathbf{s}(\mathbf{V}) = (\Lambda^*\partial_t \mathbf{V} \mid (S_\mathbf{V}(D) + S_\mathbf{V}(D)^*)\Lambda^*\mathbf{V})_2 + (\Lambda^*\mathbf{V} \mid \partial_t S_\mathbf{V}(D)\Lambda^*\mathbf{V})_2 \overset{\text{def}}{=} I + II. \tag{3.55}
\]

**Step 1. Estimate on I.** Thanks to (3.52), we have

\[
I = -([\Lambda^*, M(\mathbf{V}, D)]\mathbf{V} \mid (S_\mathbf{V}(D) + S_\mathbf{V}(D)^*)\Lambda^*\mathbf{V})_2 \\
- (S_\mathbf{V}(D) + S_\mathbf{V}(D)^*) (M(\mathbf{V}, D)\Lambda^*\mathbf{V}) \mid \Lambda^*\mathbf{V})_2 \overset{\text{def}}{=} I_1 + I_2. \tag{3.56}
\]

**Step 1.1. Estimate on I_1.** By the expressions of \( M(\mathbf{V}, D) \) and \( S_\mathbf{V}(D) \) in (2.4) and (2.5) with \( g(D) = 1, \) we first have

\[
([\Lambda^*, M(\mathbf{V}, D)]\mathbf{V} \mid S_\mathbf{V}(D)\Lambda^*\mathbf{V})_2 \\
= -\epsilon(1 - \gamma)([\Lambda^*, \mathbf{v}] \cdot \nabla \zeta \mid (1 + c_1 \Delta)\Lambda^*\zeta)_2 \tag{3.57}
\]

\[
+ \frac{\epsilon^2}{\gamma} ([\Lambda^*, \mathbf{v}] \cdot \nabla \zeta + [\Lambda^*, \zeta] \nabla \cdot \mathbf{v} \mid \mathbf{v} \cdot \Lambda^*\mathbf{v})_2 \\
+ \epsilon \gamma \sum_{j=1,2} ([\Lambda^*, \mathbf{v}] \cdot \partial_j \mathbf{v} \mid v_j^3 \Lambda^*\zeta)_2 \overset{\text{def}}{=} I_{11} + I_{12} + I_{13} + I_{14} + I_{15}.
\]

For \( I_{11}, \) integration by parts gives rise to

\[
|I_{11}| \lesssim \epsilon \|\Lambda^*\mathbf{v} \cdot \nabla \zeta\|_{L^2} \|\Lambda^*\zeta\|_{L^2} + \epsilon \mu \|\nabla([\Lambda^*, \mathbf{v}] \cdot \nabla \zeta)\|_{L^2} \|\nabla \Lambda^*\zeta\|_{L^2},
\]

which along with (2.14) implies

\[
|I_{11}| \lesssim \epsilon \|\mathbf{v}\|_{X_2^\mu} \|\zeta\|_{X_2^\mu}^2. \tag{3.58}
\]

Similar estimates hold for \( I_{12}, \) \( I_{13}, \) \( I_{14} \) and \( I_{15}. \) Then using (3.54), we obtain

\[
|I_1| \lesssim (1 + \epsilon \|\mathbf{v}\|_{X_2^\mu}) \|\mathbf{v}\|_{X_2^\mu} (\|\zeta\|_{X_2^\mu}^2 + \|\mathbf{v}\|_{X_2^\mu}^2) \lesssim (1 + \epsilon (E_\mathbf{s}(\mathbf{V}))^\frac{1}{2}) (E_\mathbf{s}(\mathbf{V}))^\frac{3}{2}.
\]
Step 1.2. Estimate on $I_2$. By the expressions of $M(V, D)$ and $S_V(D)$ in (2.4) and (2.5) with $g(D) = 1$, we calculate $S_V(D) M(V, D) = (a_{ij})_{i,j=1,2}$ as follows:

$$
\begin{align*}
a_{11} &= -\frac{\epsilon}{\gamma}(1 - \gamma)(1 + c\mu \Delta)(v \cdot \nabla) + v \cdot \nabla(1 + c\mu \Delta), \\
a_{12} &= (1 - \gamma)(1 + c\mu \Delta)((A(D) - \epsilon\zeta)\partial_1) + \frac{\epsilon^2}{\gamma} v^1 v \cdot \nabla, \\
a_{13} &= (1 - \gamma)(1 + c\mu \Delta)((A(D) - \epsilon\zeta)\partial_2) + \frac{\epsilon^2}{\gamma} v^2 v \cdot \nabla, \\
a_{21} &= (1 - \gamma)(A(D) - \epsilon\zeta)(1 + c\mu \Delta)\partial_1 + \frac{\epsilon^2}{\gamma} v^1 v \cdot \nabla, \\
a_{22} &= -\frac{\epsilon}{\gamma}[v^1(A(D) - \epsilon\zeta)\partial_1 + (A(D) - \epsilon\zeta)(v^1 \partial_1)], \\
a_{23} &= -\frac{\epsilon}{\gamma}[v^1(A(D) - \epsilon\zeta)\partial_2 + (A(D) - \epsilon\zeta)(v^2 \partial_1)], \\
a_{31} &= (1 - \gamma)(A(D) - \epsilon\zeta)(1 + c\mu \Delta)\partial_2 + \frac{\epsilon^2}{\gamma} v^2 v \cdot \nabla, \\
a_{32} &= -\frac{\epsilon}{\gamma}[v^2(A(D) - \epsilon\zeta)\partial_1 + (A(D) - \epsilon\zeta)(v^1 \partial_2)], \\
a_{33} &= -\frac{\epsilon}{\gamma}[v^2(A(D) - \epsilon\zeta)\partial_2 + (A(D) - \epsilon\zeta)(v^2 \partial_2)].
\end{align*}
$$

Now, we calculate $(S_V(D)(M(V, D)\Lambda^s V) | \Lambda^s V)_2$.

For $a_{11}$, integration by parts gives rise to

$$
(a_{11}\Lambda^s \zeta | \Lambda^s \zeta)_2 = \epsilon(1 - \gamma)(\nabla \cdot v) \Lambda^s \zeta | \Lambda^s \zeta)_2 - c\mu \epsilon(1 - \gamma)(\nabla \cdot v) \nabla \Lambda^s \zeta | \nabla \Lambda^s \zeta)_2
$$

$$
+ c\mu \epsilon(1 - \gamma) \sum_{j=1,2} \{((\partial_j v \cdot \nabla) \Lambda^s \zeta | \partial_j \Lambda^s \zeta)_2 - (\nabla \cdot (\partial_j v) \Lambda^s \zeta | \partial_j \Lambda^s \zeta)_2\},$$

which implies

$$
|(a_{11}\Lambda^s \zeta | \Lambda^s \zeta)_2| \leq \epsilon \|v\|_{X_{\mu}} \|\zeta\|_{X_{\mu}^s}^2. \quad (3.59)
$$

For $a_{22}$, we first deal with one term involving $A(D)$ as follows

$$
-\frac{\epsilon}{\gamma \mu^2} \mu^2 (v^1 \sigma(D)^2 \partial_1 \Lambda^s v^1 | \Lambda^s v^1)_2
$$

$$
= \frac{\epsilon}{\gamma \mu^2} \{ \sigma(D)^2 \partial_1 \Lambda^s v^1 | \partial_1 (\sigma(D), v^1 \Lambda^s v^1)_2 + \frac{1}{2} (v^1 \partial_1 \sigma(D) \Lambda^s v^1 | \sigma(D) \Lambda^s v^1)_2 \},
$$

which along with (2.18) and (2.19) implies

$$
\frac{\epsilon}{\gamma \mu^2} \|v^1 \sigma(D)^2 \partial_1 \Lambda^s v^1 | \Lambda^s v^1)_2 \| \leq \epsilon \|v\|_{X_{\mu}}^3.
$$

Similar estimate holds for the other terms in $(a_{22}\Lambda^s v^1 | \Lambda^s v^1)_2$. Then we obtain

$$
|(a_{22}\Lambda^s v^1 | \Lambda^s v^1)_2| \leq \epsilon(1 + \epsilon \|\zeta\|_{X_{\mu}^s}) \|v\|_{X_{\mu}}^3. \quad (3.60)
$$

The same estimate holds for $(a_{33}\Lambda^s v^2 | \Lambda^s v^2)_2$.

For $a_{12}$ and $a_{21}$, it is easy to check that

$$
a_{12} + a_{21} = \epsilon(1 - \gamma)\partial_1 \zeta(1 + c\mu \Delta) - \frac{\epsilon^2}{\gamma} \nabla \cdot (v^1 v),
$$

which implies

$$
|(a_{12}\Lambda^s v^1 | \Lambda^s \zeta)_2 + (a_{21}\Lambda^s \zeta | \Lambda^s v^1)_2| \leq \epsilon \|\zeta\|_{X_{\mu}^{p+1}} \|v\|_{X_{\mu}^s} \|\zeta\|_{X_{\mu}^s} + \epsilon^2 \|v\|_{H^{p-1}} \|\zeta\|_{H^s} \|v\|_{H^s}.
$$

(3.61)

The same estimate holds for $(a_{13}\Lambda^s v^2 | \Lambda^s \zeta)_2 + (a_{31}\Lambda^s \zeta | \Lambda^s v^2)_2$.

For $a_{23}$ and $a_{32}$, it is easy to check that

$$
a_{23} + a_{32} = \frac{\epsilon}{\gamma} \partial_1 v^2 A(D) - \frac{\epsilon^2}{\gamma} \partial_1 (v^2 \zeta) \cdot + \frac{\epsilon}{\gamma} A(D)(\partial_2 v^1) - \frac{\epsilon^2}{\gamma} \partial_2 (v^1 \zeta).
$$
Thanks to the expression of \( A(D) \) in (2.1), using (2.18) and (2.13), we get
\[
| (a_{23} \Lambda^s v^2 | \Lambda^s v^1) + (a_{22} \Lambda^s v^1 | \Lambda^s v^2) | \lesssim (1 + \epsilon \| \zeta \|_{X^s_\mu}) \| v \|_{X^s_{\mu}}^3. \tag{3.62}
\]

Combining (3.59), (3.60), (3.61) and (3.62), we obtain the estimate for \((S_V(D) (M(V, D) \Lambda^s V) | \Lambda^s V \big)_2\). The same estimate holds for \((S_V(D)^* (M(V, D) \Lambda^s V) | \Lambda^s V \big)_2\). Then using (3.54), we obtain
\[
| I_2 | \lesssim (1 + \epsilon (E_s(V))^{1/2}) (E_s(V))^{3/2},
\]
which along with (3.58) implies
\[
| I | \lesssim (1 + \epsilon (E_s(V))^{1/2}) (E_s(V))^{3/2}. \tag{3.64}
\]

**Step 2. Estimate on \( II \).**

Thanks to the expression of \( S_V(D) \) in (2.5), we have
\[
II = - \epsilon (\Lambda^s \zeta | \partial_t v \cdot \Lambda^s v)_2 - \epsilon (\Lambda^s v | \partial_t v \Lambda^s \zeta)_2,
\]
which along with (3.54) implies
\[
| II | \lesssim \epsilon (\| \zeta \|_{H^s}^2 + \| v \|_{H^s}^2) (\| \partial_t \zeta \|_{H^{s-2}} + \| \partial_t v \|_{H^{s-2}}) \lesssim \epsilon E_s(V) \| \partial_t V \|_{H^{s-2}}, \tag{3.65}
\]
where we used the fact that \( s \geq t_0 + 2 \).

**Step 3. The a priori energy estimate.**

Thanks to (3.52), we have
\[
\| \partial_t V \|_{H^{s-2}} \lesssim (1 + \epsilon \| \zeta \|_{H^s} + \epsilon \| v \|_{H^{s}}) \| V \|_{X^s_\mu},
\]
which along with (3.64), (3.65), (3.55) and (3.54) implies
\[
\frac{d}{dt} E_s(V) \lesssim \epsilon (1 + \epsilon (E_s(V))^{1/2}) (E_s(V))^{3/2} \lesssim \epsilon (1 + \epsilon^2 E_s(V)) (E_s(V))^{3/2}. \tag{3.66}
\]
This is exactly (4.2).

**Remark 3.2.** The a priori estimate (3.20) for the remain cases in Definition 1.2 can be treated in a similar way as the cases in this section.

4. **Global existence for the Hamiltonian case \( b = d > 0, a \leq 0, c < 0 \)**

In this section, we shall prove Theorem 1.2 that is the global existence of solutions of (1.1) with \( b = d > 0, a \leq 0, c < 0 \). We only discuss the two-dimensional case. The one-dimensional case follows in a similar way and actually it is considered in [5].

4.1. **Hamiltonian structure for the Boussinesq-Full dispersion system when \( b = d \).**

Recalling (2.2), we search a function \( \mathcal{H} = \mathcal{H} (\zeta, v) \) satisfying
\[
\frac{\delta \mathcal{H}}{\delta \zeta} = (1 - \gamma)(1 + \epsilon \mu \Delta) \zeta - \frac{\epsilon}{2 \gamma} | v |^2, \\
\frac{\delta \mathcal{H}}{\delta v} = \frac{1}{\gamma} (1 - \epsilon \zeta) v + \frac{a \mu}{\gamma} \Delta v + \frac{1}{\gamma^2} \sqrt{\frac{\mu}{\mu_2} \sigma(D) v} + \frac{1}{\gamma^3} \frac{\mu}{\mu_2} \sigma(D)^2 v.
\]
Then we have
\[
\mathcal{H}(\zeta, v) = \frac{1}{2} \int_{\mathbb{R}^2} \left( (1 - \gamma) | \zeta |^2 + \frac{1}{\gamma} (1 - \epsilon \zeta) | v |^2 - (1 - \gamma) \epsilon c \mu | \nabla \zeta |^2 - \frac{a \mu}{\gamma} | \nabla v |^2 \\
+ \frac{1}{\gamma^2} \sqrt{\frac{\mu}{\mu_2} | \sigma(D) |^2 v} + \frac{1}{\gamma^3} \frac{\mu}{\mu_2} | \sigma(D) | v^2 \right) dx.
\]

**Remark 4.1.** By the expression of \( \mathcal{H}(\zeta, v) \), and assuming that
\[
1 - \epsilon \zeta \geq H > 0,
\]
we have for \( a \leq 0, c < 0 \)
\[
\mathcal{H}(\zeta, v) \sim \| \zeta \|_{X^0_\mu}^2 + \| v \|_{X^2_\mu}^2.
\]

However, condition (4.3) could not be conserved in \( X^0_\mu \) since \( H^1(\mathbb{R}^2) \) is not embedding in \( L^{\infty}(\mathbb{R}^2) \) contrary the one-dimensional case. Thus, the Hamiltonian is not obviously positive.
Thanks to (2.2) and (4.1), we have
\[
(1 - b\mu\Delta)\partial_t \zeta = -\nabla \cdot \frac{\delta H}{\delta \zeta},
\]
\[
(1 - d\mu\Delta)\partial_t v = -\nabla \frac{\delta H}{\delta \zeta}.
\]
Due to (4.4), when \(b = d\), (2.2) is a Hamiltonian system that is given by
\[
\partial_t \begin{pmatrix} \zeta \\ v \end{pmatrix} + J \nabla \zeta, v H(\zeta, v) = 0.
\]
where
\[
J = (1 - b\mu\Delta)^{-1} \begin{pmatrix} 0 & -\nabla \cdot \\ -\nabla & 0 \end{pmatrix}.
\]
Since \(H(\zeta, v)\) is a Hamiltonian of (2.2), we have the following conservation law for (2.2).

**Lemma 4.1.** When \(b = d\), the smooth solution \((\zeta, v)\) to (2.2) satisfies
\[
\frac{d}{dt} H(\zeta, v) = 0,
\]
where \(H(\zeta, v)\) is a Hamiltonian defined by (4.2).

**Proof.** Thanks to (4.2) and (4.1), we have
\[
\frac{d}{dt} H(\zeta, v) = (\frac{\delta H}{\delta \zeta} | \partial_t \zeta)_2 + (\frac{\delta H}{\delta v} | \partial_t v)_2
\]
which along with (4.4) implies
\[
\frac{d}{dt} H(\zeta, v) = -\left(\frac{\delta H}{\delta \zeta} | (1 - b\mu\Delta)^{-1} \nabla \cdot \frac{\delta H}{\delta \zeta}\right)_2 - \left(\frac{\delta H}{\delta v} | (1 - d\mu\Delta)^{-1} \nabla \frac{\delta H}{\delta \zeta}\right)_2.
\]
Since \(b = d\), integration by parts gives rise to (4.6). The lemma is proved.

**4.2. Local existence of the solutions to (2.2) with \(b = d > 0\), \(a \leq 0\), \(c < 0\).** In this subsection, we state the local existence and blow-up criteria for (2.2) and (1.2).

**Proposition 4.1.** Let \(b = d > 0\), \(a \leq 0\), \(c < 0\), \(\mu \sim \epsilon\). Assume that \((\zeta_0, v_0) \in X_\mu^0(\mathbb{R}^2) \times X_\mu^0(\mathbb{R}^2)\). Then (2.2) and (1.2) has a unique solution \((\zeta, v)\) on \([0, T]\) for some \(T > 0\) so that \((\zeta, v) \in C(0, T; X_\mu^0(\mathbb{R}^2) \times X_\mu^0(\mathbb{R}^2))\) and
\[
\max_{[0, T]} (\|\zeta\|_{X_\mu^0} + \|v\|_{X_\mu^0}) \leq 2C_1 (\|\zeta_0\|_{X_\mu^0} + \|v_0\|_{X_\mu^0}),
\]
where \(C_1 > 1\) is a constant.

Moreover, if \(T^*\) is the lifespan to this solution and \(T^* < \infty\), then
\[
\lim_{t \to T^*} (\|\zeta(t)\|_{X_\mu^0} + \|v(t)\|_{X_\mu^0}) = \infty.
\]

**Proof.** We divide the proof into several steps.

**Step 1. Diagonalization of (2.2).** Let \(\lambda_\pm(\xi)\) be the eigenvalues of system (2.2). Analysis on the linear part of (2.2) with \(b = d > 0\) yields that
\[
\lambda_\pm(\xi) = \pm i \sqrt{\frac{1 - \gamma}{\gamma} \frac{A(\xi)}{1 + b\mu |\xi|^2}} |\xi|,
\]
where \(A(\xi)\) is a symbol of the Fourier multiplier \(A(D)\) that is defined in (2.1).

Now, we diagonalize the system (2.2). Denoting by
\[
\omega_1 = \omega_1(\xi) = \frac{1}{\gamma} \frac{A(\xi)}{1 + b\mu |\xi|^2}, \quad \omega_2 = \omega_2(\xi) = (1 - \gamma) \frac{1 - c\mu |\xi|^2}{1 + b\mu |\xi|^2}
\]
we have
\[
\lambda_\pm(\xi) = \pm i \sqrt{\omega_1 \omega_2} |\xi|.
\]
Letting
\[
W = |D|^{-1} \text{curl } v, \quad Z_\pm = \zeta \pm \frac{\sqrt{\omega_1(\xi)}}{\omega_2(\xi)} \frac{1}{|D|} \nabla \cdot v,
\]
and

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where
\[ f_\pm = \frac{1}{\gamma} \frac{e}{1 - b\mu \Delta} \nabla \cdot (\zeta v) \pm \frac{1}{2\gamma} \frac{\omega_1(D)}{\sqrt{\omega_2(D)}} \frac{ie|D|}{1 - b\mu \Delta} (|v|^2). \]

**Step 2. Solutions to (4.12).** Defining
\[ W_0 = |D|^{-1} \text{curl} v_0, \quad Z_{\pm,0} = \zeta_0 \pm \frac{\omega_1(D)}{\sqrt{\omega_2(D)}} \frac{1}{i|D|} \nabla \cdot v_0, \]
by virtue of Duhamel principle, the solutions to (4.12) are written as
\[ W(t,x) = W_0(x), \]
\[ Z_{\pm}(t,x) = e^{\mp i|D|} \sqrt{\omega_1(D)/\omega_2(D)} Z_{\pm,0}(x) + \int_0^t e^{\mp i(t-s)|D|} \sqrt{\omega_1(D)/\omega_2(D)} f_\pm(s,x)ds. \]
Thanks to (4.11), we have
\[ |\zeta| = \frac{1}{2} (Z_+ + Z_-), \quad v = \frac{\nabla}{2i|D|} \frac{\omega_2(D)}{\sqrt{\omega_1(D)}} (Z_+ - Z_-) + \frac{\nabla^\perp}{|D|} W, \]
where \( \nabla^\perp = (-\partial_2, \partial_1)^T \).

Since
\[ A(\xi) = 1 - a\mu|\xi|^2 + \frac{1}{\gamma} \sqrt{\frac{\mu}{\mu_2}} \sigma(\xi) + \frac{1}{\gamma^2} \frac{\mu}{\mu_2} \sigma(\xi)^2, \quad \sigma(\xi) = \sqrt{\mu_2} |\xi| \coth(\sqrt{\mu_2}|\xi|), \]
and
\[ \lim_{|\xi| \to 0} \sigma(\xi) = 1, \quad \lim_{|\xi| \to \infty} \frac{\sigma(\xi)}{\sqrt{\mu_2}|\xi|} = 1, \]
it is easy to check that
\[ \|\omega_1(\xi)\|_{L^\infty} + \|\omega_2(\xi)\|_{L^\infty} \lesssim 1, \quad \|\omega_2(\xi)\|_{L^\infty} + \|\omega_1(\xi)\|_{L^\infty} \lesssim 1. \]
Actually, following similar derivation as \( g(D) \), one could check that \( \omega_1(D), \omega_2(D), \sqrt{\omega_1(D)/\omega_2(D)} \) and \( \sqrt{\omega_2(D)/\omega_1(D)} \) are zero-order pseudo-differential operators that satisfy
\[ \|\omega_1(D)f\|_{H^s} \sim \|f\|_{H^s}, \quad \|\omega_2(D)f\|_{H^s} \sim \|f\|_{H^s}, \]
\[ \|\sqrt{\omega_2(D)} f\|_{H^s} \sim \|f\|_{H^s}, \quad \|\sqrt{\omega_1(D)} f\|_{H^s} \sim \|f\|_{H^s}, \]
for any \( s \in \mathbb{R} \) and \( f \in H^s(\mathbb{R}^2) \).

By virtue of Plancherel theorem and (4.16), or due to (4.17), we deduce from (4.11) and (4.15) that
\[ \|W\|_{X^0_\mu} + \|Z_+\|_{X^0_\mu} + \|Z_-\|_{X^0_\mu} \lesssim \|\zeta\|_{X^0_\mu} + \|v\|_{X^0_\mu}. \]

**Step 3. The a priori energy estimate.** Thanks to (4.14), we first have
\[ \|W\|_{X^0_\mu} = \|W_0\|_{X^0_\mu}, \]
\[ \|Z_\pm(t)\|_{X^0_\mu} \lesssim \|Z_{\pm,0}\|_{X^0_\mu} + \int_0^t \|f_\pm(s)\|_{X^0_\mu} ds. \]
Now, we derive the bound of \( \|f_\pm\|_{X^0_\mu} \lesssim \|f_\pm\|_{L^2} + \sqrt{\mu} \|\nabla f_\pm\|_{L^2}. \) Thanks to (4.13) and (4.17), we have
\[ \|f_\pm\|_{L^2} \lesssim \|\frac{e}{1 - b\mu \Delta} \nabla \cdot (\zeta v)\|_{L^2} + \frac{c|D|}{\sqrt{\mu} (|v|^2)} \|\sqrt{\mu} (|v|^2)\|_{L^2}, \]
\[ \lesssim \frac{c}{\sqrt{\mu}} (\|\zeta v\|_{L^2} + \|v\|^2_{L^2}) \lesssim \frac{c}{\sqrt{\mu}} (\|v\|_{L^2} + \|\zeta\|_{L^2} + \|v\|_{L^2}^2). \]
Since Ladyzhenskaya’s inequality yields
\[ \|v\|_{L^4} \lesssim \|v\|^2_{L^2} \|\nabla v\|^2_{L^2}, \quad \|\zeta\|_{L^4} \lesssim \|\zeta\|^2_{L^2} \|\nabla \zeta\|^2_{L^2}, \]
we have
\[ \|f_{\pm}\|_{L^2} \lesssim (\|\zeta\|_{L^2} + \|v\|_{L^2}) \cdot \frac{e^D}{\sqrt{\mu}} (\|\nabla \zeta\|_{L^2} + \|\nabla v\|_{L^2}) \lesssim \|\zeta\|_{X^0_{\mu}}^2 + \|v\|_{X^0_{\mu}}^2, \] (4.21)
provided that \( \epsilon \sim \mu \).

Whereas thanks to (4.13) and (4.17), we have
\[ \|\nabla f_{\pm}\|_{L^2} \lesssim \frac{e^D}{1 - \mu^2} (\|\zeta v\|_{L^2} + \|D^2 v\|_{L^2}) \lesssim \frac{e^D}{\mu} (\|\zeta v\|_{L^2} + \|v\|_{L^2}). \] (4.22)

After similar derivation as (4.21), if \( \epsilon \sim \mu \), we have
\[ \sqrt{\mu} \|\nabla f_{\pm}\|_{L^2} \lesssim \|\zeta\|_{X^0_{\mu}} + \|v\|_{X^0_{\mu}}. \] (4.23)

Due to (4.21) and (4.22), we have
\[ \|f_{\pm}\|_{X^0_{\mu}} \lesssim \|\zeta\|_{X^0_{\mu}} + \|v\|_{X^0_{\mu}}. \] (4.24)

which along with (4.20) implies
\[ \|Z_{\pm}(t)\|_{X^0_{\mu}} \lesssim \|Z_{\pm,0}\|_{X^0_{\mu}} + t \sup_{(0,t)} \|\zeta\|_{X^0_{\mu}} + \|v\|_{X^0_{\mu}}^2. \] (4.25)

Step 4. Local existence and uniqueness. By virtue of the a priori energy estimate (4.25) and the standard contraction theorem, (2.2) and (1.2) admits a unique solution \((\zeta, v)\) on \([0, T]\) with \( T \leq \frac{1}{4C_1C_2(\|\zeta_0\|_{X^0_{\mu}} + \|v_0\|_{X^0_{\mu}})} \) and
\[ \max_{[0, T]} (\|\zeta\|_{X^0_{\mu}} + \|v\|_{X^0_{\mu}}) \leq 2C_1(\|\zeta_0\|_{X^0_{\mu}} + \|v_0\|_{X^0_{\mu}}). \] (4.26)

We omit the details here. One could refer to [10].

Step 5. Proof of the blow-up criteria (4.8). We prove (4.8) by contradiction. Assume that when \( T^* < \infty \), there holds
\[ \lim_{t \to T^*} (\|\zeta(t)\|_{X^0_{\mu}} + \|v(t)\|_{X^0_{\mu}}) < \infty. \] (4.27)

Then there exists a constant \( M > 0 \) such that for any \( t \in [0, T^*) \)
\[ \|\zeta(t)\|_{X^0_{\mu}} + \|v(t)\|_{X^0_{\mu}} \leq M. \] (4.28)

Taking \( T_0 = T^* - \frac{1}{8C_1C_2M} \), by the definition of \( T^* \), \( (2.2) \) with initial data \((\zeta_0, v_0)\) admits a unique solution
\[ (\zeta, v) \in C([0, T_0]; X^0_{\mu}(\mathbb{R}^2) \times X^0_{\mu}(\mathbb{R}^2)), \]
\[ (\zeta, v) \in C([T_0, \tilde{T}]; X^0_{\mu}(\mathbb{R}^2) \times X^0_{\mu}(\mathbb{R}^2)), \]
where \( \tilde{T} = T_0 + T_1 \)
\[ T_1 = \frac{1}{4C_1C_2M} \leq \frac{1}{4C_1C_2(\|\zeta_0\|_{X^0_{\mu}} + \|v_0\|_{X^0_{\mu}})} \].

Then \( (2.2) \) with initial data \((\zeta_0, v_0)\) admits a unique solution on time interval \([0, \tilde{T}]\) with
\[ \tilde{T} = T^* + \frac{1}{8C_1C_2M} > T^*. \]

This is a contradiction to the definition of \( T^* \). Now, we complete the proof of the proposition. \( \square \)
4.3. Proof of Theorem 1.2. We shall use Proposition 1.1 and the conservation law (4.6) to establish the global theory for (2.2) with $b = d > 0$ and $a \leq 0$, $c < 0$.

Firstly, Proposition 1.1 shows that (2.2) with initial data $(\zeta_0, v_0)$ admits a unique solutions

$$(\zeta, v) \in C([0,T];X_0^0(\mathbb{R}^2) \times X_0^0(\mathbb{R}^2)), \text{ for some } T > 0.$$ 

Due to (4.2) and (2.18), there exists a constant $c_1 > 0$ such that for any $t \in [0,T]$

$$\mathcal{H}(\zeta(t), v(t)) \geq 1 - \frac{\gamma}{2} ||\zeta||_{L^2}^2 + \frac{(1 - \gamma)|c|}{2} \mu ||\nabla \zeta||_{L^2}^2 + \frac{1}{2} ||v||_{L^2}^2 + c_1 \mu ||\nabla v||_{L^2}^2 - \frac{1}{2}\gamma \epsilon \int_{\mathbb{R}^2} |\zeta||v|^2 dx. \tag{4.29}$$

By virtue of Ladyzhenskaya’s inequality, there exists a constant $c_2 > 0$ such that

$$\epsilon \int_{\mathbb{R}^2} |\zeta||v|^2 dx \leq \epsilon ||\zeta||_{L^2} ||v||_{L^2}^2 \leq c_2 \epsilon ||\zeta||_{L^2} ||v||_{L^2} ||\nabla v||_{L^2} \leq \frac{1}{2} ||v||_{L^2}^2 + \frac{1}{2} \epsilon c_2 \epsilon ||\zeta||_{L^2}^2 ||\nabla v||_{L^2}^2. \tag{4.30}$$

Noticing that $\mu \sim \epsilon$, we deduce from (4.29) and (4.30) that

$$\mathcal{H}(\zeta(t), v(t)) \geq c_3 (||\zeta||_{L^2}^2 + ||v||_{L^2}^2 + 2\mu (1 - \epsilon ||\zeta||_{L^2}^2)||\nabla v||_{L^2}^2), \tag{4.31}$$

for some constant $c_3 > 0$.

Taking $\epsilon_0' = \frac{1}{\|\zeta_0\|_{L^2}^2}$, we have $\epsilon_0' ||\zeta_0||_{L^2}^2 = \frac{1}{4} < \frac{1}{2}$. Since solution $(\zeta, v)$ is continuous in time, there exists $0 < T_0 < T$ such that

$$\epsilon_0' ||\zeta(t)||_{L^2}^2 < \frac{1}{2} \text{ for any } t \in [0, T_0], \tag{4.32}$$

which along with (4.2) and (4.31) implies that for any $\epsilon \leq \epsilon_0$ and $t \in [0, T_0]$, the Hamiltonian $\mathcal{H}(\zeta(t), v(t))$ is positive and

$$||\zeta(t)||_{X^0_\mu}^2 + ||v(t)||_{X^0_\mu}^2 \leq \frac{1}{c_3} \mathcal{H}(\zeta(t), v(t)) = \frac{1}{c_3} \mathcal{H}(\zeta_0, v_0). \tag{4.33}$$

Taking $\epsilon_0 = \min\{\epsilon_0', \frac{1}{\|\zeta_0\|_{L^2}^2}\}$, by virtue of (4.32) and (4.33), we have for any $\epsilon \leq \epsilon_0$ and $t \in [0, T_0]$, $\mathcal{H}(\zeta(t), v(t))$ is positive and

$$\epsilon_0 ||\zeta(t)||_{L^2}^2 < \frac{1}{2} \text{ and } \epsilon_0 ||\zeta(t)||_{X^0_\mu}^2 + ||v(t)||_{X^0_\mu}^2 \leq \frac{1}{c_3} \mathcal{H}(\zeta_0, v_0). \tag{4.34}$$

Thus, (4.34) and the blow-up criteria (4.8) in Proposition 1.1 shows that for any $\epsilon \in (0, \epsilon_0)$, the solution to (2.2) can always be extended till $T^* = \infty$. Then (2.2) admits a unique solution on time interval $[0, \infty)$ such that

$$\sup_{(0, \infty)} (||\zeta(t)||_{X^0_\mu} + ||v(t)||_{X^0_\mu}) \leq \frac{1}{c_3} \mathcal{H}(\zeta_0, v_0) \leq C(||\zeta_0||_{X^0_\mu} + ||v_0||_{X^0_\mu}). \tag{4.35}$$

This is exactly (1.6). We complete the proof of Theorem 1.2.

5. Final comments

So far we are not aware of a global existence result of large solutions to at least one of the Boussinesq-FD systems. Recall that for the Boussinesq (abed) systems such a result is only known in the one-dimensional $a = b = c = 0, d = 1$ case, see [32, 41] and the comments in the survey article [29].

Proving such a result for a Boussinesq-FD system is a challenging problem as is to prove the conjectured dichotomy for the life span $T_c$ of solutions: either $T_c = +\infty$, or $T_c = O(1/\epsilon)$.

6. Appendix

1. Proof of (3.8). By the definition of $E_s(V)$ in (2.7) and the expression of $S_V(D)$ in (2.6), we have

$$E_s(V) = (1 - \gamma)^2 \gamma^2 ||g(D)\frac{1}{2}(1 - b\mu\Delta)^\frac{1}{2}(1 + c\mu\Delta)\Lambda^*\zeta||_{L^2}^2 + \gamma(1 - \gamma) ||(1 - b\mu\Delta)\Lambda^*v||_2 \langle A(D) - \epsilon \zeta ||(1 + c\mu\Delta)\Lambda^*v\rangle_2$$

$$- \epsilon \gamma(1 - \gamma) \langle (1 - b\mu\Delta)\Lambda^*v \mid g(D) \langle v \cdot (1 + c\mu\Delta)\Lambda^*v \rangle_2 \rangle$$

$$- \epsilon \gamma(1 - \gamma) \langle (1 - b\mu\Delta)\Lambda^*v \mid g(D) \langle v \cdot (1 + c\mu\Delta)\Lambda^*v \rangle_2 \rangle$$

$$+ \epsilon^2 \sum_{i,j=1,2} \langle (1 - b\mu\Delta)\Lambda^*v_i \mid v^i v^j (g(D) - 1)\Lambda^*v^j \rangle_2 \overset{\text{def}}{=} A_1 + A_2 + A_3 + A_4 + A_5. \tag{6.1}$$
For $A_1$, using (2.15), we have

$$A_1 \sim \| (1 - b \mu \Delta)^{\frac{1}{2}} (1 + c \mu \Delta) \Lambda^s \xi \|^2_{L^2} \sim \| \xi \|^2_{\mathcal{H}^{s+\frac{1}{2}}}.$$  

(6.2)

For $A_2$, using the expression of $A(D)$ in (2.1), we first have

$$A_2 = \gamma (1 - \gamma) \left( (1 - b \mu \Delta) \Lambda^s \varphi \right) \| (1 - c \xi \Delta) (1 + c \mu \Delta) \Lambda^s \varphi \|^2_{L^2} + \gamma (1 - \gamma) |a| \| (1 - b \mu \Delta)^{\frac{1}{2}} (1 + c \mu \Delta)^{\frac{1}{2}} \nabla \Lambda^s \varphi \|^2_{L^2}

+ (1 - \gamma) \sqrt{\frac{\mu}{\mu_2}} \| \sigma(D) (1 - b \mu \Delta)^{\frac{1}{2}} (1 + c \mu \Delta)^{\frac{1}{2}} \Lambda^s \varphi \|^2_{L^2}$$

$$= \frac{1 - \frac{\mu}{\gamma} \mu_2}{\gamma} \| \sigma(D) (1 - b \mu \Delta)^{\frac{1}{2}} (1 + c \mu \Delta)^{\frac{1}{2}} \Lambda^s \varphi \|^2_{L^2} \defeq A_{21} + A_{22} + A_{23} + A_{24}$$  

(6.3)

Integrating by parts, we have

$$A_{21} = \gamma (1 - \gamma) \left\{ (\Lambda^s \varphi) \| (1 - c \xi \Delta) \Lambda^s \varphi \|^2_{L^2} + b \varphi (1 - c \xi \Delta) \Lambda^s \varphi \|^2_{L^2}

+ (1 - c \xi \Delta) \nabla \Lambda^s \varphi \|^2_{L^2} - (1 - c \xi \Delta) \varphi (\nabla \nabla \Lambda^s \varphi \|^2_{L^2} \right\}$$

which along with (3.1) and (2.13) implies that

$$A_{21} + (b - c) \varepsilon \mu (\nabla \cdot \nabla \Lambda^s \varphi \|^2_{L^2} \sim \| \Lambda^s \varphi \|^2_{L^2} + \mu^2 \| \nabla \Lambda^s \varphi \|^2_{L^2} + \mu \| \varphi \|^2_{L^2} \sim \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}}.$$  

Using (3.1), Sobolev inequality and Hölder inequality, we have

$$\varepsilon \mu \| (\nabla \cdot \nabla \Lambda^s \varphi \|^2_{L^2} \leq \mu^2 \| \nabla \Lambda^s \varphi \|^2_{L^2} + \mu \| \varphi \|^2_{L^2} \leq \mu \varepsilon \| \varphi \|^2_{L^2} + \mu \| \varphi \|^2_{L^2}.$$  

Then by virtue of (2.13), we have

$$(1 - \mu \varepsilon \| \varphi \|^2_{L^2} \lesssim A_{21} \lesssim (1 + \mu \varepsilon \| \varphi \|^2_{L^2}.$$  

(6.4)

Due to (2.18), we have

$$A_{22} \lesssim \mu \| \nabla (1 - b \mu \Delta)^{\frac{1}{2}} (1 + c \mu \Delta)^{\frac{1}{2}} \Lambda^s \varphi \|^2_{L^2} \lesssim \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}} - \mu \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}}$$

$$A_{24} \lesssim \mu \| (1 - b \mu \Delta)^{\frac{1}{2}} (1 + c \mu \Delta)^{\frac{1}{2}} \Lambda^s \varphi \|^2_{L^2} \lesssim \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}},$$

which implies

$$\| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}} - \mu \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}} \lesssim A_{24} \lesssim \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}}.$$  

(6.5)

Similarly, using (2.13) and (2.18), we have

$$\mu \varepsilon \| D^\frac{1}{2} \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}} \lesssim A_{23} \lesssim \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}}.$$  

(6.6)

Using (2.13), we have

$$A_{22} \lesssim \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}}$$

which along with (6.3), (6.4), (6.5) and (6.6) implies

$$\| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}} + (1 - \mu \varepsilon \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}} \lesssim A_{2} \lesssim \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}}.$$  

(6.7)

For $A_3$, $A_4$ and $A_5$, using (2.15) and (2.16), we have

$$|A_3| + |A_4| + |A_5| \lesssim (1 + \| \varphi \|_{L^\infty}) \varepsilon \| \varphi \|_{L^\infty} (\| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}} + \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}})$$

which along with (3.1) and Sobolev inequality implies

$$|A_3| + |A_4| + |A_5| \lesssim \sqrt{\varepsilon} (\| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}} + \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}}).$$  

(6.8)

Thanks to (6.1), (6.2), (6.7) and (6.8), we have

$$(1 - \mu - \sqrt{\varepsilon}) (\| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}} + \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}}) \lesssim E_s (V) \lesssim \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}} + \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}}.$$  

(6.9)

Taking $\varepsilon$ and $\mu$ sufficiently small, we deduce from (6.9) that

$$E_s (V) \sim \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}} + \| \varphi \|^2_{\mathcal{H}^{s+\frac{1}{2}}}.$$  

(6.10)

This is exactly (3.8).
2. Proof of (3.37). By the definition of $E_s(V)$ in (2.7) and the expression of $S_V(D)$ in (2.6), noticing that in this case $g(D) = 1 - b\mu\Delta$, we have

$$
E_s(V) = (1 - \gamma)^2\gamma^2\| (1 - b\mu\Delta)(1 + c\mu\Delta)\Delta^*\gamma \|_2^2 \\
+ \gamma(1 - \gamma)[(1 - b\mu\Delta)\Lambda^u\gamma](A(D) - c\zeta)(1 + c\mu\Delta)\Lambda^u\gamma]_2 \\
- \epsilon\gamma(1 - \gamma)[(1 - b\mu\Delta)\Lambda^u\gamma\cdot (1 + c\mu\Delta)\Lambda^u\gamma]_2 \\
- \epsilon\gamma(1 - \gamma)[(1 - b\mu\Delta)\Lambda^u\gamma\cdot (1 + b\mu\Delta)(\gamma v(1 + c\mu\Delta)\Lambda^u\gamma)]_2 \\
- b\mu^2 \sum_{i,j=1,2} ((1 - b\mu\Delta)\Lambda^u v^i | v^i v^j \Delta^u v^j)_2 \overset{\text{def}}{=} A_1 + A_2 + A_3 + A_4 + A_5.
$$

(6.11)

Following similar derivation as (3.8), under the assumption (3.1), we have

$$
A_1 \sim \| \zeta \|_{X_{s4}^{q2}}^2, \quad \| v \|_{X_{s4}^{q2}}^2 + (1 - \mu^2\epsilon^2 - \mu)\| v \|_{X_{s3}^{q2}}^2 \lesssim A_2 \lesssim \| v \|_{X_{s3}^{q2}}^2
$$

$$
|A_3| + |A_4| + |A_5| \lesssim \sqrt{\epsilon} (\| \zeta \|_{X_{s4}^{q2}}^2 + \| v \|_{X_{s3}^{q2}}^2) \|
$$

Then for sufficiently small $\epsilon$ and $\eta$, we have

$$
E_s(V) \sim E_s(t) \overset{\text{def}}{=} \| \zeta(t) \|_{X_{s4}^{q2}}^2 + \| v(t) \|_{X_{s3}^{q2}}^2.
$$

This is exactly (3.37).

3. Proof of (3.54). By the definition of $E_s(V)$ in (3.53) and the expression of $S_V(D)$ in (2.5), we have

$$
E_s(V) = (1 - \gamma)^2\| (1 + c\mu\Delta)\Lambda^*\zeta \|_2^2 + (\Lambda^u\zeta | (A(D) - c\zeta)\Lambda^u\gamma)_2 \\
- \epsilon(\Lambda^u\zeta | v \cdot \Lambda^u\zeta)_2 - \epsilon(\Lambda^u v | \Lambda^u\zeta)_2.
$$

(6.12)

After similar derivation as (3.8) and (3.37), we get that (3.54) holds for sufficiently small $\epsilon$ and $\eta$.

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