AN EXPLICIT KOPPELMAN FORMULA FOR $dd^c$ AND GREEN CURRENTS ON $\mathbb{P}^n$

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Abstract. We compute a quite explicit Koppelman formula for $dd^c$ on projective space, and obtain Green currents for positive closed currents.

1. Introduction

Let $X$ be a smooth projective variety and let $\theta$ be a positive closed $(p, p)$-current on $X$. A $(p - 1, p - 1)$-current $g$ is called a Green current for $\theta$ if

\[(1.1) \quad dd^c g + \theta = A,\]

where $A$ is a smooth form.

Green currents for Lelong currents $[Z]$ of analytic cycles $Z$ are of fundamental importance in Arakelov geometry, see, e.g., [14], [8], [20], [5], and the survey article [15]. Green currents for more general $\theta$ are used, e.g., in complex dynamics, see [12] and [13].

This paper is an elaboration of the second half\(^1\) of [3]. We construct a (positive) integrable kernel $K(\zeta, z)$ of bidegree $(n - 1, n - 1)$ that is smooth outside the diagonal and of log type along $\Delta$ (see Section 2 for the definition) and a smooth kernel $P(\zeta, z)$ of bidegree $(n, n)$, such that

\[(1.2) \quad dd^c K + [\Delta] = P\]

in the current sense; i.e., $K$ is a Green current for $[\Delta]$ on $\mathbb{P}_n^p \times \mathbb{P}_n^q$. From (1.2) we get, for smooth forms $\theta$, the Koppelman type formula

\[(1.3) \quad dd^c \int_\zeta K \wedge \theta + \theta = \int_\zeta P(\zeta, z) \wedge \theta(\zeta) + d \int_\zeta K \wedge dd^c \theta - dd^c \int_\zeta K \wedge d \theta - \int_\zeta K \wedge dd^c \theta\]

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\(^1\)The first part is published as [4].
for the \( ddc \)-operator. If \( \theta \) is a \((k,k)\)-form such that \( d\phi = 0 \), then \( d^c\theta = 0 \) as well, and if

\[
(1.4) \quad g = \int_\zeta K(\zeta, z) \wedge \theta(\zeta), \quad A = \int_\zeta P(\zeta, z) \wedge \theta(\zeta),
\]

then \( ddc g + \theta = A \).

In the meantime we have learnt that such Green currents \( K \) were obtained in essentially the same way already in [8], Section 6, so our main contribution is that we compute our kernels \( K \) and \( P \) quite explicitly.

If \( \theta \) is the Lelong current \([Z]\) for some cycle \( Z \), then the function \( g(z) \) in (1.4) is well-defined and smooth in \( X \setminus Z \), and it is proved in [8] (Lemma 1.2.2) that it is in fact a Green current for \( Z \) of log type along \( Z \). In [12] it is proved that the same formula provides a Green current for any positive closed \((k,k)\)-current \( \theta \). With our particular choice of \( K \), the Koppelman formula extends to any current \( \theta \), see below.

Our starting point is the following result.

**Proposition 1.1.** Let \( f \) be a holomorphic section of a Hermitian vector bundle \( E \to X \) of rank \( m \) over some manifold \( X \). If the zero cycle \( Z \) of \( f \) has codimension \( m \), i.e., \( f \) intersects the zero section transversally, then there is a current \( W_{m-1} \), smooth outside \( Z \) and of log type, such that

\[
(1.5) \quad ddc W_{m-1} + [Z] = c_m(D_E),
\]

where \( c_m(D_E) \) is the top Chern form of \( E \).

Thus \( W_{m-1} \) is a Green current for \([Z]\). The proof (in Section 6 in [8] and in [4]) is based on ideas in [9]. See Section 2 below for a discussion, and for an expression for the current \( W_{m-1} \).

Let \( z = [z_0, \ldots, z_n] \) be the usual homogeneous coordinates on \( \mathbb{P}^n \) and let

\[
\omega = ddc \log |z|
\]

be the standard Kähler form on \( \mathbb{P}^n \) so that

\[
(1.6) \quad \int_{\mathbb{P}^n} \omega^n = 1.
\]

We will see that

\[
\eta = \sum_{j=0}^n z_j \frac{\partial}{\partial \zeta_j}
\]

is a holomorphic section of the bundle \( H = T^{1,0}(\mathbb{P}^n) \otimes \mathcal{O}(-1)_{\zeta} \otimes \mathcal{O}(1)_{z} \) over \( \mathbb{P}^n \times \mathbb{P}^n \) that vanishes to first order precisely on the diagonal. From the standard metric on \( \mathbb{P}^n \) we get a natural Hermitian metric \( \| \| \) on \( H \), and in particular, see Section 3

\[
(1.7) \quad \| \eta \|^2 = \frac{|\zeta|^2 |z|^2 - |z \cdot \zeta|^2}{|z|^2 |\zeta|^2} = \frac{|\zeta \wedge z|^2}{|z|^2 |\zeta|^2},
\]
which is like the square of the distance between the points \([z]\) and \([ζ]\) on \(\mathbb{P}^n\). From Proposition 1.1 we thus get \(K = W_{n-1}\) and \(P = c_n(D_H)\) so that (1.2) holds.

We say that a differential form \(ξ(ζ, z)\) on \(\mathbb{P}^n_ζ \times \mathbb{P}^n_z\) is invariant if \(ξ(\phi(ζ), \phi(z)) = ξ(ζ, z)\) for each isometric automorphism \(φ\) of \(\mathbb{P}^n\), i.e., mapping induced by an orthogonal mapping on \(\mathbb{C}^{n+1}\). For instance, \(∥η∥\) is invariant. Here is our main theorem.

**Theorem 1.2.** The kernels \(P\) and \(K\) so defined satisfy (1.2) and moreover:

(i) The kernel \(P\) is

\[
P = \sum_{k=0}^{n} \omega_ζ^k \wedge \omega_z^{n-k},
\]

and it induces the orthogonal projection onto the harmonic forms.

(ii) \(K\) is positive, of log type, and it can be written

\[
K = \sum_{j=0}^{n-1} \left[ \log(1/∥η∥) \frac{γ_j^1}{∥η∥^{2j+2}} + \frac{γ_j^2}{∥η∥^{2j+2}} \right],
\]

where \(γ_j^1\) are smooth (real-analytic) invariant forms on \(\mathbb{P}^n_ζ \times \mathbb{P}^n_z\) which are \(O(∥η∥^2)\). In particular, \(K(ζ, z)\) is invariant.

(iii) The Koppelman formula (1.3) holds for all currents \(θ\).

It is well-known that the harmonic forms (with respect to \(ω\)) are precisely the forms \(α_ω^k\) for \(α ∈ \mathbb{C}\), and a \((k, k)\)-current \(ξ\) is orthogonal to the harmonic forms if and only if

\[
\int_{\mathbb{P}^n} ξ \wedge ω^{n-k} = 0.
\]

In particular, any form that is \(d\)-exact or \(dc\)-exact must be orthogonal to the harmonic forms. In view of (1.6) and (1.8), \(P\) induces the orthogonal projection.

Thus if \(Z\) is a cycle of pure codimension \(p\) in \(\mathbb{P}^n\) with degree

\[
\deg Z = \int_{\mathbb{P}^n} [Z] \wedge ω^{n-p},
\]

then

\[
g = \int_ζ K(ζ, z) \wedge [Z]
\]

solves the Green equation

\[
ddc^* g + [Z] = \deg (Z) ω^p.
\]

In [5] and [6] a Green current for \([Z]\) in \(\mathbb{P}^n\) is obtained by means of the Levine form for the subspace \(z = w\) in \(\mathbb{P}^{2n+1}\); the Green current
appears as the value at the origin of a current-valued analytic function, involving expressions for homogeneous forms $f_j$ that define $Z$. Such a representation is the purpose in those paper(s). It seems that the resulting current can be represented by an integral operator with integrable kernel; however we do not know whether one can compute this kernel explicitly.

Remark 1. In [8], p. 913, it is proved that for any smooth projective variety $X$ there exists a Green current $K$ for the diagonal $\Delta$ in $X \times X$ such that $K$ is smooth outside $\Delta$ and of log type. In particular, $K$ is integrable on $X \times X$. Thus one gets a Koppelman formula on $X$, though not explicit, and a Green current for any cycle $Z$ in $X$. □

In this paper,
\[ d^c = \kappa (\bar{\partial} - \partial), \quad \kappa = i/2\pi, \]
so that
\[ dd^c = 2\kappa \partial \bar{\partial} = \frac{i}{\pi} \partial \bar{\partial}. \]

2. A generalized Poincaré-Lelong formula

We first discuss a slightly more general form of Proposition 1.1. Let $E \to X$ be a Hermitian vector bundle of rank $m$ over a complex (compact) manifold $X$, let $D_E$ be the Chern connection on $E$, and let $c(D_E)$ be the associated Chern form, i.e., $c(D_E) = \det(\kappa \Theta_E + I)$, where $\Theta_E = D_E^2$ is the curvature tensor. We let $c_k(D_E)$ denote the component of $c(D_E)$ of bidegree $(k,k)$.

Let $f$ be a holomorphic section of $E$, and assume that $Z = \{ f = 0 \}$ has codimension $p$. Then we have an analytic $p$-cycle $p$ that has the irreducible components of $\{ f = 0 \}$ and $\alpha_j$ are the Hilbert-Samuel multiplicities of $f$. Moreover, $f$ defines a trivial line bundle $S$ over $X \setminus Z$, and we let $Q = E/S$ be the quotient bundle, equipped with the induced Hermitian metric, and let $D_Q$ and $c(D_Q)$ be the corresponding Chern connection and Chern form, respectively.

**Proposition 2.1.** The form $c(D_Q)$ is locally integrable in $X$ and the corresponding current $C(D_Q)$ is closed in $X$. There is an explicit current $W_{p-1}$ of bidegree $(p-1,p-1)$ and of order zero in $X$, smooth in $X \setminus Z$ and of logarithmic type along $Z$, such that
\[ (2.1) \quad dd^c W_{p-1} + [Z^p] = c_p(D_E) - C_p(D_Q). \]

If in addition, $E^*$ is Nakano negative, one can choose $W_{p-1}$ to be positive.

Since $Q$ has rank $m-1$, $C_m(D_Q) = 0$ and thus we get (1.5). Proposition 2.1 is a special case of a more general formula in [4] (Theorems 1.1
and 1.2). For the statement about logarithmic type, see Lemma 2.3 below.

Remark 2. In [4] it is stated that one can choose $W_{p-1}$ positive where $|f| < 1$, given that $E^*$ is Nakano negative. Since $X$ is assumed to be compact here, we can modify the metric in $E$ so that $|f| \leq 1$ everywhere. □

We shall now describe $W_{p-1}$ following the presentation in [4], and we refer to [4] for more details and arguments. We introduce the exterior algebra bundle $\Lambda = \Lambda(T^*(X) \oplus E \oplus E^*)$. Any section $\xi \in \mathcal{E}_{k,q}(X, E)$, i.e., smooth $(k, q)$-form with values in $E$, corresponds to a section $\tilde{\xi}$ of $\Lambda$: If $\xi = \xi_1 \otimes e_1 + \ldots + \xi_m \otimes e_m$ in a local frame $e_j$ for $E$, then we let $\tilde{\xi} = \xi_1 \wedge e_1 + \ldots + \xi_m \wedge e_m$. In the same way, $a \in \mathcal{E}_{k,q}(X, \text{End} E)$ is identified with $\tilde{a} = \sum_{jk} a_{jk} \wedge e_j \wedge e_k^*$, where $e^*_j$ is the dual frame, if $a = \sum_{jk} a_{jk} \otimes e_j \otimes e_k^*$ with respect to these frames. The connection $D_E$ extends in a unique way to a linear mapping $D: \mathcal{E}(X, \Lambda) \rightarrow \mathcal{E}(X, \Lambda)$ which is an anti-derivation with respect to the wedge product in $\Lambda$, and acts as the exterior differential $d$ on the $T^*(X)$-factor. If $\xi$ is a form-valued section of $E$, then $\tilde{D}_E \xi = D \tilde{\xi}$, and if $a \in \mathcal{E}_{k,q}(X, \text{End} E)$, then

$$D_{\text{End} E} a = D \tilde{a},$$

see, e.g., [4]. Since $D_{\text{End} E} I_E = 0$, here $I$ denotes the identity endomorphism on $E$, and by Bianchi’s identity, $D_{\text{End} E} \Theta_E = 0$, we have from (2.2) that

$$\tilde{D}_E \Phi = 0 \quad \text{and} \quad D \tilde{I} = 0.$$

We let $\tilde{I}_m = \tilde{I}^m / m!$ and use the same notation for other forms in the sequel. Any form $\omega$ with values in $\Lambda$ can be written $\omega = \omega' \wedge \tilde{I}_m + \omega''$ uniquely, where $\omega''$ has lower degree in $e_j, e_k^*$. If we make the definition

$$\int_e \omega = \omega',$$

then this integral is linear and

$$d \int_e \omega = \int_e D \omega.$$

We have that

$$c(D_E) = \int_e (\Phi \tilde{E} + \tilde{I}_m) = \int_e e^{\Phi \tilde{E} + \tilde{I}_m}.$$

Recall that $D_E = D'_E + \tilde{\partial}$, where $D'_E$ is the $(1, 0)$-part of $D_E$. It follows that we also have the decomposition $D = D' + \tilde{\partial}$. 


Let \( \sigma \) be the section of \( E^* \) over \( X \setminus Z \) with minimal norm such that \( f \cdot \sigma = 1 \). In \( X \setminus Z \) we have the formula (Proposition 4.2 in [4])

\[
(2.7) \quad c(D_Q) = \int_e f \wedge \sigma \wedge (\tilde{I} + s\tilde{\Theta} - sDf \wedge \tilde{\partial} \sigma)_{m-1}.
\]

In a suitable resolution of singularities \( \nu: \tilde{X} \to X \) we may assume that \( \nu^* f = f^0 f' \), where \( f^0 \) is (locally) a holomorphic function, in fact a monomial in suitable coordinates, and \( f' \) is a non-vanishing section (of \( \nu^* E \)). Then \( \nu^* \sigma = (1/f^0) \sigma' \), where \( \sigma' \) is smooth, and it follows that

\[
\nu^* (\sigma \wedge f \wedge (Df \wedge \tilde{\partial} \sigma)^{k-1}) = \sigma' \wedge f' \wedge (Df' \wedge \tilde{\partial} \sigma')^{k-1}
\]

is smooth. Since \( c(D_Q) \) is closed in \( X \setminus Z \), thus \( \nu^* c(D_Q) \) has a smooth and closed extension across the singularity. In particular it is locally integrable, therefore its push-forward is locally integrable, so we have:

**Lemma 2.2.** The form \( c(D_Q) \) is locally integrable and its natural current extension \( C(D_Q) \) is closed.

By the usual Poincaré-Lelong formula we have

\[
c(D_S) = 1 + dd^c \log(1/|f|)
\]

outside \( Z \), i.e., \( 1 + dd^c \log(1/|f'|) \) in the resolution. It follows that also \( c(D_S) \wedge c(D_Q) \) is locally integrable. If capitals denote the natural current extensions, then

\[
(2.8) \quad -dd^c V = c(D_E) - C(D_S)C(D_Q),
\]

where \( V \) is the locally integrable form

\[
(2.9) \quad V = \sum_{\ell=1}^{m-1} \frac{(-1)^\ell}{2\ell} \int_e f \wedge \sigma \wedge (\tilde{I} + s\tilde{\Theta} - sDf \wedge \tilde{\partial} \sigma)_{m-1-\ell} \wedge (-sDf \wedge \tilde{\partial} \sigma)_{\ell}.
\]

Finally, if

\[
W = \log(1/|f|)C(D_Q) - V,
\]

then its component \( W_{p-1} \) satisfies (2.1).

We will be particularly interested in the case when \( p = m \). For degree reasons then no \( \tilde{I} \) can come into play so

\[
(2.10) \quad W_{m-1} = \log(1/|f|) \int_e f \wedge \sigma \wedge (s\tilde{\Theta} + sDf \wedge \tilde{\partial} \sigma)_{m-1}
\]

\[
- \sum_{\ell=1}^{m-1} \frac{(-1)^\ell}{2\ell} \int_e f \wedge \sigma \wedge (s\tilde{\Theta} - sDf \wedge \tilde{\partial} \sigma)_{m-1-\ell} \wedge (-sDf \wedge \tilde{\partial} \sigma)_{\ell}.
\]

If \( E \) is positive in the sense that \( E^* \) is Nakano negative, then \( C_{m-1}(D_Q) \) is a positive current and

\[
(2.11) \quad W'_{m-1} = W_{m-1} + \alpha C_{m-1}(D_Q)
\]

is positive, for a large enough positive constant \( \alpha \), see Section 7 in [4].
Remark 3. If $E$ is a line bundle, i.e., $m = 1$, then $V = 0$, and since $\sigma \cdot f = 1$ we have that $W = \log(1/|f|)$, $C_1(Q) = 0$, and hence (2.1) is just the classical Poincaré-Lelong formula.

Recall that a $(k, k)$-current $w$ is of logarithmic type along a subvariety $Z$ if the following holds: There exists a surjective mapping $\pi: \tilde{X} \to X$ such that $\tilde{Z} = \pi^{-1}Z$ has normal crossings in $\tilde{X}$, $\pi$ has surjective differential in $\tilde{X}\setminus\tilde{Z}$, $w = \pi^*\tilde{w}$, where $\tilde{w}$ is smooth in $\tilde{X}\setminus\tilde{Z}$, and locally $\tilde{w}$ is of the form

$$\sum_j \gamma_j'' \log |s_j| + \gamma'$$(2.12)

where $s$ is a suitable local coordinate system, $\gamma'$ is a smooth form, and $\gamma_j''$ are closed smooth forms.

Lemma 2.3. The current $W$ is of logarithmic type along $Z$.

Proof. Let $\nu: \tilde{X} \to X$ be the resolution above. Then $\nu^*C(D_Q)$ is smooth and closed, $\nu^*V$ is smooth, and $\nu^*\log |f| = \log |f^0| + \log |f'|$, where $f^0$ locally is a polynomial whose zero set is $\nu^{-1}Z$. In view of (2.10) thus $W$ is of logarithmic type along $Z$. □

3. Explicit Koppelman formulas on $\mathbb{P}^n$

The line bundle $O(k)$ over $\mathbb{P}^n$, whose sections are naturally identified by $k$-homogeneous functions $\xi$ on $\mathbb{C}^n \setminus \{0\}$, will be denoted, for typographical reasons, by $L^k$. We have the natural Hermitian norm

$$\|\xi(|z|)\|^2 = \frac{|\xi(z)|^2}{|z|^{2k}}.$$

Recall that a differential form $\alpha$ on $\mathbb{C}^n$ is projective, i.e., the pullback under $\mathbb{C}^n \setminus \{0\} \to \mathbb{P}^n$ of a form on $\mathbb{P}^n$, if and only if $\delta_z \alpha = \delta_{\bar{z}} \alpha = 0$, where $\delta_z$ is interior multiplication with $\sum z_j (\partial/\partial z_j)$ and $\delta_{\bar{z}}$ is its conjugate.

Lemma 3.1. If $D'_{L^r}$ is the $(1,0)$-part of the Chern connection on $L^r$ and $g$ is a section, expressed as an $r$-homogeneous function in $z$, then

$$(3.1) \quad D'_{L^r} g = |z|^{2r} \frac{\partial g}{|z|^{2r}} = \partial g - r g \partial \log |z|^2,$$

and

$$(3.2) \quad \omega \Theta_{L^r} = -r s \partial \bar{\partial} \log |z|^2 = r \omega.$$ 

Notice that since $g$ is $r$-homogeneous, i.e., $g(\lambda z) = \lambda^r g(z)$, we have that

$$r \lambda^{r-1} g(z) = \sum_0^n z_j \frac{\partial g}{\partial z_j}(\lambda z),$$

and thus $\delta_z(\partial g) = r g(z)$. Since furthermore $\delta_z \partial \log |z|^2 = 1$, the right hand side of (3.1) is indeed a projective form.
Proof. We may assume that $g$ is (locally) holomorphic. Then

$$\partial \|g\|^2 = \langle D_L^* g, g \rangle = \frac{(D_L^* g) \bar{g}}{|z|^{2r}},$$

but also

$$\partial \|g\|^2 = \partial \frac{|g|^2}{|z|^{2r}} = |z|^{2r} \partial \frac{g}{|z|^{2r}} \cdot \bar{g} / |z|^{2r}.$$

Combining, we get (3.1). Now, $\Theta_{L^*} g = \partial D_L^* g = -r \partial \partial \log |z|^2 g$, which gives (3.2).

We are now going to compute the currents obtained from Proposition (1.1) with $H$ and $\eta$ as in the introduction. In order not to mix up with the usual tangent bundle, we introduce an abstract copy of $H$, cf., [7]. Let $L_{\bar{z}}$ be the pullback of $L \to P^n$ to $P^n \times P^n$. Furthermore, let $C_{n+1} \to P^n \times P^n$ be the trivial bundle taken with the natural metric, and consider the quotient bundle $C_{n+1}/\zeta C$, where $\zeta C$ is $L_{-1}$, i.e., the pullback of the tautological line bundle over $P^n$. We define

$$H = (C_{n+1}/\zeta C) \otimes L_{\bar{z}}$$

over $P^n \times P^n$ equipped with the induced metric. Since $C_{n+1}$ has trivial metric it has vanishing curvature, and therefore the quotient $C_{n+1}/\zeta C$ is positive in the sense that its dual is Nakano negative, cf., Proposition 7.1 in [4]. It follows that $H$ are positive in the same sense, since $L$ is positive.

A section $w$ of $H$ is represented by a mapping $C_{n+1} \times C_{n+1} \to C_{n+1}$ that is 0-homogeneous in $\zeta$ and 1-homogeneous in $z$. In particular we have the global holomorphic section $\eta(\zeta, z) = z$, which vanishes to first order on the diagonal $\Delta$. The dual of $C_{n+1}/\zeta C$ is the subbundle of $(C_{n+1}^*)$ that is orthogonal to $\bar{\zeta} C$. Therefore, a section of the dual bundle $H^*$ can be represented by a mapping $w: C_{n+1} \times C_{n+1} \to (C_{n+1}^*)$ that is 0-homogeneous in $\zeta$, $-1$-homogeneous in $z$, and such that $w \cdot \zeta = 0$.

Let $e_0, \ldots, e_n$ be the trivial global frame (basis) for $C_{n+1}$ above, and let $e_j^*$ be its dual basis for $(C_{n+1})^*$. If $\xi = \xi \cdot e = \sum \xi_j e_j$ is a section of $C_{n+1}/\zeta C$, then its norm is equal to the norm of the orthogonal projection onto the orthogonal complement of $L_{-1} \zeta = \zeta C$,

$$\frac{|\xi|^2 \xi - (\xi \cdot \zeta) \zeta \cdot e}{|\zeta|^2}.$$

Thus if $\xi \sim \xi \cdot e$ is a section of $H$, i.e., the functions $\xi_j$ are in addition 1-homogeneous in $z$, we have

$$\|\xi\|^2 = \frac{|\xi|^2 |\xi\cdot e|^2 - |\xi \cdot \zeta|^2}{|z|^2 |\zeta|^2};$$
here and in the sequel we use $\|\xi\|$ to distinguish the norm of the section from $|\xi|$, denoting the norm of the corresponding vector-valued function on $\mathbb{C}^{n+1}$. In particular, (1.7) holds. Moreover,

$$s = \frac{|\xi|^2 \bar{z} \cdot e^* - (\zeta \cdot \bar{z}) \bar{\zeta} \cdot e^*}{|z|^2 |\zeta|^2}.$$ 

is the section of $H^*$ with minimal (since $\|s\| = \|\eta\|$) norm such that $s \cdot \eta = \|\eta\|^2$.

Notice that a form $\alpha$ on $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ is projective if and only if $\delta_z \alpha = \delta_{\bar{z}} \alpha = \delta_\zeta \alpha = \delta_{\bar{\zeta}} \alpha = 0$. We can write a form-valued section $\xi$ of $H$ as

$$\xi = \sum_{j=0}^n \xi_j \otimes e_j$$

where $\xi_j$ are projective forms.

We need expressions for $D_H$ and $\Theta_H$. Let $\omega_z$ and $\omega_\zeta$ be the Kähler forms on $P^n_z$ and $P^n_\zeta$, respectively.

**Proposition 3.2.** If $\xi \cdot e$ is a section of $H$, then

$$D'_H(\xi \cdot e) = \partial \xi \cdot e - \frac{\xi \cdot \bar{\zeta}}{|\zeta|^2} d\zeta \cdot e - \frac{\partial |z|^2}{|z|^2} \otimes \xi \cdot e.$$ 

Moreover,

$$\Theta_H = \omega_z \cdot e \wedge \bar{\zeta} = \bar{\partial} D'_H(\xi \cdot e) + \omega_z \otimes e \cdot e^*.$$ 

Here $\partial \omega \cdot e^* = \sum_{j=0}^n \partial \omega_j \otimes e_j^*$ etc. Notice that indeed $d \zeta \cdot e$ is a projective $H$-valued form, since it can be written

$$\left(d \zeta - \frac{\bar{\zeta} \cdot d\zeta}{|\zeta|^2} \zeta\right) \cdot e$$

and each $d \zeta_j - (\bar{\zeta} \cdot d\zeta_j/|\zeta|^2) \zeta_j$ is projective.

**Proof.** First assume that $\xi$ is just a section of the bundle $F = \mathbb{C}^{n+1} / \zeta \mathbb{C}$ over $P^n_\zeta$. Since $d$ is the Chern connection on $\mathbb{C}^{n+1}$, the Chern connection on $\xi$ is equal to $d$ acting on the representative of $\xi$ in $F$ that is orthogonal to $\zeta \mathbb{C}$. Since

$$\pi(\xi \cdot e) = \frac{\xi \cdot \bar{\zeta}}{|\zeta|^2} \zeta \cdot e,$$

is the orthogonal projection $\mathbb{C}^{n+1} \to \zeta \mathbb{C}$, we get the formula

$$D'_F(\xi \cdot e) = \partial \xi \cdot e - \frac{\xi \cdot \bar{\zeta}}{|\zeta|^2} d\zeta \cdot e.$$ 

Since $H = F \otimes L_z$ now (3.3) follows from Lemma 3.1. Using that $\Theta_E \xi = \bar{\partial} D'_H \xi$ for holomorphic $\xi$, now (3.4) follows as well. $\square$

**Lemma 3.3.** Let $\alpha$ be a form with values in $\Lambda(H \otimes H^*)$, and let $e'$ denote a local frame for $H$. If $e$ is the standard basis for $\mathbb{C}^{n+1}$ as above, then

$$\int_{e'} \alpha = \int_{e} \frac{\zeta \cdot e \cdot \bar{\zeta} \cdot e^*}{|\zeta|^2} \wedge \alpha,$$
where \( \alpha \) on the right hand side is any form with values in \( \Lambda(\mathbb{C}^{n+1} \oplus (\mathbb{C}^{n+1})^*) \) that represents \( \alpha \).

Proof. Let \( [\zeta] \) be an arbitrary point on \( \mathbb{P} \). After applying an isometric automorphism of \( \mathbb{P} \), we may assume that \( \zeta = (1,0,\ldots,0) \). If we choose the basis \( e_j = e_j' \), \( j = 1,2,\ldots \), then (3.3) is immediate. \( \square \)

Proof of Theorem 1.2. We already know that (1.2) holds, so let us now compute \( P \). If \( d\zeta \cdot e \) from now on denotes \( \sum_0^n d\zeta \wedge e_j \) etc, we have that

\[
\kappa \tilde{\Theta}_H = -\kappa d\zeta \cdot e \wedge \tilde{\partial}\zeta \cdot e^* + \omega_z \wedge \tilde{I},
\]

where \( \tilde{I} = \sum_0^n e_j \wedge e_j^* \); the change of sign, compared to (3.4), is because \( e_j \wedge d\zeta_k = -d\zeta_k \wedge e_j \). In view of Lemma 3.3 we have

\[
P = c_n(D_H) = \int_{e'} (\kappa \tilde{\Theta}_H)_n = \int_{e} \frac{\zeta \cdot e \wedge \tilde{\zeta} \cdot e^*}{|\zeta|^2} \wedge \left( \frac{-\kappa d\zeta \cdot e \wedge d\tilde{\zeta} \cdot e^*}{|\zeta|^2} + \omega_z \wedge \tilde{I} \right)_n.
\]

Since this formula as well as (1.8) are invariant we can assume that \( \zeta = (1,0,\ldots,0) \). Then (1.8) follows by a simple computation. However, for further reference we prefer a more direct computational argument. It is easy to check that

\[
(3.6) \quad \frac{1}{\kappa(k+1)} \delta_\zeta \delta_\zeta (-\kappa d\zeta \cdot e \wedge d\tilde{\zeta} \cdot e^*)_k + 1 = \zeta \cdot e \wedge \tilde{\zeta} \cdot e^* \wedge (-\kappa d\zeta \cdot e \wedge d\tilde{\zeta} \cdot e^*)_k
\]

and that

\[
(3.7) \quad \frac{1}{\kappa(k+1)} \delta_\zeta \delta_\zeta \beta^{k+1} = \omega_z^k,
\]

if \( \beta = \kappa \tilde{\partial}\zeta \cdot e \). Hence

\[
P = \sum_{k=0}^n \int_{e} \frac{\zeta \cdot e \wedge \tilde{\zeta} \cdot e^*}{|\zeta|^2} \wedge \left( \frac{-\kappa d\zeta \cdot e \wedge d\tilde{\zeta} \cdot e^*}{|\zeta|^2} \right)_k \wedge \tilde{I}_{n-k} \wedge \omega_z^{n-k} =
\]

\[
= \sum_{k=0}^n \frac{1}{\kappa(k+1)} |\zeta|^{2k+2} \delta_\zeta \delta_\zeta \int_{e} (-\kappa d\zeta \cdot e \wedge d\tilde{\zeta} \cdot e^*)_k + 1 \wedge \tilde{I}_{n-k} \wedge \omega_z^{n-k} =
\]

\[
= \sum_{k=0}^n \frac{1}{\kappa(k+1)} |\zeta|^{2k+2} \delta_\zeta \delta_\zeta \beta^{k+1} \wedge \omega_z^{n-k} = \sum_{k=0}^n \omega_z^k \wedge \omega_z^{n-k}.
\]

We now turn our attention to the kernel \( K \). According to (2.10) and Lemma 3.3

\[
K = W_{n-1} =
\]

\[
\log(1/||\eta||) \int_{e} \frac{\zeta \cdot e \wedge \tilde{\zeta} \cdot e^*}{|\zeta|^2} \wedge \eta \wedge \sigma \wedge (\kappa \tilde{\Theta}_H - \kappa D \eta \wedge \tilde{\partial}\sigma)_{n-1} -
\]

\[
\sum_{\ell=1}^{n-1} \frac{(-1)^{\ell}}{2\ell} \int_{e} \frac{\zeta \cdot e \wedge \tilde{\zeta} \cdot e^*}{|\zeta|^2} \wedge \eta \wedge \sigma \wedge (\kappa \tilde{\Theta}_H - \kappa D \eta \wedge \tilde{\partial}\sigma)_{n-1-\ell} \wedge (-\kappa D f \wedge \tilde{\partial}\sigma)_\ell.
\]
From the expression for $K$ above, with the modification (3.8) to obtain a positive kernel, we get the representation (1.9), where

$$
\sigma = \frac{s}{\|\eta\|^2} = \frac{|\zeta|^2 \bar{z} \cdot e^* - (\zeta \cdot \bar{z}) \bar{\zeta} \cdot e^*}{|\zeta \wedge z|^2},
$$

and since $\eta = z \cdot e$ we therefore have that

$$
\frac{\zeta \cdot e \wedge \bar{\zeta} \cdot e^*}{|\zeta|^2} \wedge \eta \wedge \sigma = \frac{\zeta \cdot e \wedge \bar{\zeta} \cdot e^* \wedge z \cdot e \wedge \bar{z} \cdot e^*}{|\zeta \wedge z|^2} = O(|\zeta \wedge z|^2) \quad \frac{\zeta \wedge z^2}{|\zeta \wedge z|^2} = O(\|\eta\|^2).
$$

If $\sim$ denotes equality after multiplication with this factor, we have

$$
n \tilde{\Theta}_H \sim \frac{\omega d \zeta \cdot e \wedge d \bar{\zeta} \cdot e^*}{|\zeta|^2} + \omega_z \wedge I,
$$

and

$$
D \eta = D'(z \cdot e) \sim dz \cdot e - \frac{z \cdot \bar{\zeta}}{|\zeta|^2} d\zeta \cdot e,
$$

and

$$
\partial \sigma \sim \frac{|\zeta|^2 | \bar{d} \zeta \cdot e^* - (\zeta \cdot \bar{z}) d \bar{\zeta} \cdot e^*}{| \zeta \wedge z|^2}.
$$

If

$$
\tau = \frac{(|\zeta|^2 dz \cdot e - (z \cdot \bar{\zeta}) d\zeta \cdot e) \wedge (|\zeta|^2 \bar{d} \zeta \cdot e^* - (\bar{z} \cdot \zeta) d \bar{\zeta} \cdot e^*)}{|\zeta|^4 |z|^2},
$$

thus

$$
D \eta \wedge \partial \sigma \sim \frac{\tau}{\|\eta\|^2}.
$$

From the expression for $K$ above, with the modification (2.11) to obtain a positive kernel, we get the representation (1.9), where

$$
(3.8)
\gamma^j_i = c^j_i \frac{\int e^j \zeta \cdot e \wedge \bar{\zeta} \cdot e^* \wedge z \cdot e \wedge \bar{z} \cdot e^*}{|\zeta|^2 |z|^2} \wedge \tau^i \wedge (n \tilde{\Theta}_H)^{n-1-j}
$$

for some constants $c^j_i$. Thus $\gamma^j_i$ are smooth and $O(\|\eta\|^2)$. To see the invariance let $\phi$ be a unitary mapping (matrix) on $\mathbb{C}^{n+1}$. First notice that if $\zeta$ and $z$ are replaced by $\phi \zeta$ and $\phi z$, then $|\zeta|^2$, $\omega_z$, $\zeta \cdot \bar{z}$ etc are unchanged. Moreover, $\zeta \cdot e$, $d\zeta \cdot e$, $\bar{\zeta} \cdot e^*$ etc, will become same expressions, but with $e$ and $e^*$ replaced by $\phi^* e$ and $\phi^* e^*$, respectively. However, since $\phi$ is unitary, $\phi^* e^*$ is the dual basis of $\phi^* e$. Since (3.8) is independent of the choice of frame $e$, it follows that $\gamma^j_i$ is invariant. Since $\eta$ is invariant, it follows that $K$ is invariant.

It remains to prove part (iii). By duality we have to show that the (dual) operators map smooth forms to smooth forms. Let $p$ be a fixed point in $\mathbb{P}^n$. If $n = 1$, then for each point $z$, except for the antipodal point, there is a unique isometric isomorphism $\phi_z$ such that $z \mapsto p$ and $p \mapsto z$. If $n > 1$, then for each point $z$ outside the hyperplane of antipodal points there is such a mapping, and it is unique if we require that it is the identity on the orthogonal complement of the complex
line through $p$ and $z$. It is clear that $\phi_z$ so defined depends smoothly on $z$. Therefore, for $z$ outside the exceptional hyperplane,
\[ \int K(\zeta, z) \wedge \psi(\zeta) = \int K(\phi_z^{-1}(\zeta), \phi^{-1}_z(z)) \wedge \psi(\zeta) = \int K(\xi, p) \wedge \phi^*_z \psi(\xi), \]
and it follows that the integral depends smoothly on $z$ if $\phi$ is smooth. □

4. A FURTHER COMPUTATION

Let us calculate $K$ a little further. Since we are mainly interested in the action on $(\ast, \ast)$-forms, we only care about the components that have bidegree $(\ast, \ast)$ in $z$. To simplify even more let us restrict ourselves to the component $K_0$ which has bidegree $(0, 0)$ in $z$. Letting
\[ m = -\kappa d\zeta \cdot e \wedge d\bar{\zeta} \cdot e^* \]
we then have $\kappa \tilde{\Theta}_H \sim m$ and
\[ -\kappa D\eta \wedge \bar{\partial} \sigma \sim \frac{|\zeta \cdot \bar{z}|^2}{|\zeta \wedge z|^2} m. \]
Noting that
\[ 1 + \frac{|\zeta \cdot \bar{z}|^2}{|\zeta \wedge z|^2} = 1 + \frac{|\zeta \cdot \bar{z}|^2}{|\zeta|^2 |z|^2 - |\zeta \cdot \bar{z}|^2} = \frac{|\zeta|^2 |z|^2}{|\zeta \wedge z|^2}, \]
we also have
\[ \kappa \tilde{\Theta}_H - \kappa D\eta \wedge \bar{\partial} \sigma \sim \frac{|\zeta|^2 |z|^2}{|\zeta \wedge z|^2} m. \]

Lemma 4.1. If $A$ denotes the component of $c_{n-1}(D_Q)$ which is $(0, 0)$ in $z$, then
\[ A = \int e \frac{\zeta \cdot e \wedge \bar{\zeta} \cdot e^* \wedge z \cdot e \wedge \bar{\zeta} \cdot e^*}{|\zeta \wedge z|^2} \wedge \left( \frac{|\zeta|^2 |z|^2}{|\zeta \wedge z|^2} m \right)_{n-1} = c_n \left( \frac{|z|^2}{|\zeta \wedge z|^2} \right)^{n-1} \kappa^{n-1} (n - 1)! a \wedge \bar{a}, \]
where
\[ a = \sum_{j<k} (\zeta_j z_k - \zeta_k z_j) \frac{\partial}{\partial \zeta_k} \frac{\partial}{\partial \bar{\zeta}_j} d\zeta_0 \wedge \ldots \wedge d\zeta_n \]
(\(-\) denotes interior multiplication) and $c_n = \pm 1$.

Proof. In fact, letting $\delta_z$ and $\delta_{\bar{z}}$ temporarily denote interior multiplication with $z \cdot (\partial/\partial \zeta)$ and $\bar{z} \cdot (\partial/\partial \bar{\zeta})$, respectively, by a computation as
in the proof of Proposition 1.2 above we have that
\[
\int_{\mathcal{E}} e^{\zeta} \cdot e^{*} \wedge z \cdot e^{\bar{z}} \cdot e^{*} \wedge (\kappa d\zeta \cdot e^{\wedge d\bar{\zeta}} \cdot e^{*})_{n-1} = \frac{1}{\kappa^{2}n(n+1)} \delta_{z} \delta_{\bar{z}} \delta_{\zeta} \int_{\mathcal{E}} (\kappa d\zeta \cdot e^{\wedge d\bar{\zeta}} \cdot e^{*})_{n+1} = \frac{1}{\kappa^{2}n(n+1)} \delta_{z} \delta_{\bar{z}} \delta_{\zeta} \beta^{n+1} = (n-1)! \kappa^{n-1} a \wedge \bar{a}.
\]

Since \( c_{n-1}(DQ) \) is closed it follows that \( \mathcal{A} \) is closed. This can also be verified directly. Summing up we get the formula
\[
-\frac{1}{2} K_{0} = \left[ \log \left( \frac{|\zeta \wedge z|^{2}}{|\zeta|^{2}|z|^{2}} \right) + \sum_{\ell=1}^{n-1} \frac{(-1)^{\ell}}{\ell} \frac{(n-1)!}{(n-1-\ell)!} \left( \frac{|\zeta \cdot \bar{z}|^{2}}{|\zeta|^{2}|z|^{2}} \right)^{\ell} \right] \wedge \mathcal{A}.
\]

Finally we write this kernel in the affine coordinates \( \zeta' = (\zeta'_1, \ldots, \zeta'_n) \) and \( z' = (z'_1, \ldots, z'_n) \). We have the transformation rules
\[
|\zeta|^2 \mapsto 1+|\zeta'|^2, \quad |z|^2 \mapsto 1+|z'|^2, \quad \zeta \cdot \bar{z} \mapsto 1+\zeta' \cdot z', \quad |\zeta \wedge z|^2 \mapsto |\zeta' \wedge z'|^2.
\]
Furthermore,
\[
a = \sum_{j=1}^{n} (\zeta'_j - z'_j) d\zeta'_j = \sum_{j=1}^{n} (\zeta'_j - z'_j) \frac{\partial}{\partial \zeta'_j} - d\zeta'_1 \wedge \ldots \wedge d\zeta'_n.
\]
In affine coordinates we therefore have (suppressing the primes for simplicity)
\[
-\frac{1}{2} K_{0} = \left[ \log \left( \frac{|\zeta - z|^2 + |\zeta \wedge z|^2}{(1 + |\zeta|^2)(1 + |z|^2)} \right) + \sum_{\ell=1}^{n-1} \frac{(-1)^{\ell}}{\ell} \frac{(n-1)!}{(n-1-\ell)!} \left( \frac{|1 + \zeta \cdot \bar{z}|^2}{(1 + |\zeta|^2)(1 + |z|^2)} \right)^{\ell} \right] \wedge \left( \frac{1 + |z|^2}{|\zeta - z|^2 + |\zeta \wedge z|^{2n-2}} \right)^{n-1} c_{n} \kappa^{n-1} (n-1)! a \wedge \bar{a}.
\]

5. Green currents for \( Z \) in terms of defining functions

Let \( f_1, \ldots, f_m \) be homogeneous polynomials in \( \mathbb{C}^{n+1} \), let \( Z^p \) be the union of the irreducible components of their common zero set \( Z \subset \mathbb{P}^{n} \).
of lowest codimension $p$, and let

$$\|f(\zeta)\|^2 = \sum_{j=1}^{m} \frac{|f_j(\zeta)|^2}{|\zeta|^{2d_j}}.$$ 

If all $d_j = d$, then

$$\|f(\zeta)\|^2 = \frac{|f(\zeta)|^2}{|\zeta|^{2d}} = \sum_{j=1}^{m} \frac{|f_j(\zeta)|^2}{|\zeta|^{2d}}.$$ 

We want to find a Green current for $[Z^p]$ expressed in the functions $f_j$. To this end let $E_j$ be distinct trivial line bundles with basis elements $e_j$ and consider $f = \sum f_j e_j$ as a section of the bundle

$$E = L^d \otimes E_1 \oplus \cdots \oplus L^d \otimes E_m.$$ 

We first assume that $p = m$. Since

$$c(D_E) = c(L^d) \wedge \cdots \wedge c(L^d) = \bigwedge (1 + d_j \omega)$$

we have that

(5.1) $$c_m(D_E) = d_1 \cdots d_m \omega^m = (\deg Z) \omega^m.$$ 

From Proposition 1.1 we get the Green current $g = W_{m-1}$ for $Z$, solving

(5.2) $$dd^c g + [Z] = d_1 \cdots d_m \omega^m.$$ 

**Proposition 5.1.** This Green current for $[Z]$ has the form

$$g = W_{m-1} = \log(1/\|f\|) \sum_{k=1}^{m} \frac{\gamma_1^k}{\|f\|^{2k}} + \sum_{k=1}^{m} \frac{\gamma_2^k}{\|f\|^{2k}},$$

where $\gamma_k^1$ are smooth forms that are $O(\|f\|^2)$. If all $d_i = d$, then

(5.3) $$g = \log(\|z^d/\|f\|) \sum_{\ell=0}^{m-1} d^{m-1-\ell} \omega^{m-1-\ell} \wedge (dd^c \log |f|)^\ell + \sum_{\ell=0}^{m-1} c_{\ell} d^{m-1-\ell} \omega^{m-1-\ell} \wedge (dd^c \log |f|)^\ell.$$ 

One can check that the second sum in (5.3) is closed, so that already the first sum is a Green current for $[Z]$. This current is the well-known Levin form $L(f)$, cf., p. 30 in [5]. It is of course quite easy to verify directly that

$$dd^c L(f) + [Z] = d^m \omega^m.$$ 

**Proof of Proposition 5.1.** The formula (2.10) gives an explicit expression for $g$ as soon as we have explicit expressions for the associated sections $\sigma$, $D_E f$ and $\Theta_E$. To begin with

(5.4) $$\sigma = \frac{1}{\|f\|^2} \sum_{j=1}^{m} \frac{f_j(\zeta)}{|\zeta|^{2d_j}} e_j^*,$$
where $e_i^*$ are the dual basis elements of $E_i^*$, and
\[ \tilde{\Theta}_E = \sum_i e_i \wedge e_i^* = \sum_i d_i \omega \wedge e_i^* = \omega \wedge \sum_i d_i e_i \wedge e_i^*. \]

If all $d_j = d$, then
\[ (5.5) \quad \sigma = \frac{\sum_i f_j(\zeta) e_i^*}{|f(\zeta)|^2} \]
and
\[ \tilde{\Theta}_E = \sum_i \tilde{\Theta}_{L_i} \wedge e_i \wedge e_i^* = \sum_i d \omega \wedge e_i \wedge e_i^* = d \omega \wedge \tilde{f}. \]

**Lemma 5.2.** We have that
\[ Df = \sum_i d_i |z|^2 d_i \partial f_i \wedge e_i, \quad \partial \sigma = \frac{1}{|f|^2} \sum_i d_i |z|^2 \partial f_i \wedge e_i^* + \cdots, \]
where $\cdots$ denote terms that contain the factor $\sigma$. If all $d_j = d$, then
\[ Df = \sum_i df_j \wedge e_j + \cdots, \quad \bar{\partial} \sigma = \frac{1}{|f(\zeta)|^2} \sum_i d f_i \wedge e_i^* + \cdots, \]
where $\cdots$ denote terms that contain the factor $\sigma$ or $f$.

**Proof.** Since
\[ Df = \sum_i D_{L_i} f_i \wedge e_i, \]
and $f$ is holomorphic, the first equality follows from Lemma 3.1. The second equality is immediate in view of (5.4). The two remaining equalities follow in a similar way. \hfill \Box

Notice that $f \wedge \sigma$ is of the form $\alpha/|f|^2$ where $\alpha$ is a smooth form that is $O(|f|^2)$. Because of the presence of this factor in (2.10) we can insert the right hand sides in Lemma 5.2 into (2.10), and we then get the first formula in Proposition 5.1.

Now assume that $d_i = d$. At a given fixed point, one can assume, after an isometric transformation, and by homogeneity, that $f \cdot e = e_1$, so that, e.g.,
\[ dd^c \log |f| = \sum_{i=2}^m df_i \wedge d f_i, \quad f \wedge \sigma = e_1 \wedge e_1^*. \]
One then obtains (5.3) by a straightforward computation. \hfill \Box

**Remark 4.** One can actually reduce to the case when all $d_j$ are the same. One just replaces $f_j$ by $f_j^{r_j}$ so that $r_j d_j = d$. Then they define the cycle $r_1 \cdots r_m$ times the cycle defined by $f_j$; this follows, e.g., from (5.1), and hence one get a Green current for the original cycle by just dividing by this number. \hfill \Box
Let us finally consider the case when $p < m$. Then
\[ dd^c W_{p-1} + [Z^p] = c_p(D_E) - C_p(D_Q), \]
and unfortunately $C_p(D_Q)$ is only the pushforward of a smooth form. However, if we take
\[ G = W_{p-1} - \int_{\mathbb{P}^n} K \wedge C_p(D_Q) \]
it follows from the Koppelman formula (1.3), applied to $\theta = C_p(D_Q)$, that
\[ dd^c G + [Z^p] = c_p(D_E) - \int_{\mathbb{P}^n} P \wedge C_p(D_Q). \]
Thus we get a Green current whose leading term $W_{p-1}$ is quite explicit; by a similar computation as above we find that
\[ W_{p-1} = \log(1/\|f\|) \sum_{k=1}^p \gamma^1_k \frac{\|f\|^{2k}}{\|f\|^{2k}} + \sum_{k=1}^p \gamma^2_k \frac{\|f\|^{2k}}{\|f\|^{2k}}, \]
where $\gamma^1_k$ are smooth forms that are $O(\|f\|^2)$.

**Remark 5.** The point above was that we had quite explicit currents $w$ and $\gamma$ such that
\[ dd^c w + [Z^p] = \gamma, \]
where $w$ is locally integrable, smooth outside $Z$ and $\gamma$ is the pushforward of a smooth form. Such currents are also provided by a variant of King's formula due to Meo, [17] and [18]; in fact one can take
\[ w = - \log \|f\| \left( dd^c \log \|f\| \right)^{p-1} 1_{X \setminus Z} \]
and
\[ \gamma = (dd^c \log \|f\|)^p 1_{X \setminus Z}. \]
For a simple proof, see (the proof of) Proposition 4.1 in [2]. From that proof it follows that $w$ is of logarithmic type along $Z$ and that $\gamma$ is the pushforward of a smooth closed form. However, this current $w$ is not identical to (5.6); e.g., $W_{p-1}$ is positive, whereas $w$ is not.

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