Multivector and Extensor Fields on Smooth Manifolds

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July 27, 2018

Abstract

The main objective of this paper (second in a series of four) is to show how the Clifford and extensor algebras methods introduced in a previous paper of the series are indeed powerful tools for performing sophisticated calculations appearing in the study of the differential geometry of a \(n\)-dimensional manifold \(M\) of arbitrary topology, supporting a metric field \(g\) (of given signature \((p,q)\)) and an arbitrary connection \(\nabla\). Specifically, we deal here with the theory of multivector and extensor fields on \(M\). Our approach does not suffer the problems of earlier attempts which are restricted to vector manifolds. It is based on the existence of canonical algebraic structures over the canonical (vector) space associated to a local chart \((U_\alpha, \phi_\alpha)\) of a given atlas of \(M\). The key concepts of \(\alpha\)-directional ordinary derivatives of multivector and extensor fields are defined and their properties studied. Also, we recall the Lie algebra of smooth vector fields in our formalism, the concept of Hestenes derivatives and present some illustrative applications.

Contents

1 Introduction 2
2 Canonical Space 3
   2.1 Position Vector 5
   2.2 Canonical Algebraic Structures 6
1 Introduction

The main purpose of the present paper, the second, in a series of four (and sequel papers in the series) is to show how the Clifford and extensor algebra methods developed in a previous paper [6] are indeed a powerful tool for doing calculations appearing in the study of the differential geometry of a $n$-dimensional smooth manifold $M$ of arbitrary topology, equipped supporting a metric field $g$ (of signature $(p, q)$, $p + q = n$) and an arbitrary connection $\nabla$. Of course, a sophisticated way to apply Clifford and extensor algebra methods to the study of the differential geometry of manifolds is the Clifford bundle formalism based on the Clifford bundle of multivector fields $\mathbf{Cl}(TM, g)$, which is described, e.g., in [11]. However, here we want to avoid (as much as possible) the theory of vector and principal bundles and connections on them, and indeed our intention is to give a simple approach to the subject that can be readily used by interested physicists. So, to begin, we recall that any effective practical calculation in differential geometry, starts with the selection of an appropriate local chart, say $(U_o, \phi_o)$ with coordinates $\{x^\mu\}$ of a given atlas of $M$. Instead of working directly with multivector and extensor fields, the main idea of our approach consists in working with the representatives of those fields on $U_o \subset M$ through the use of canonical algebraic structures constructed over a canonical vector space $\mathbf{Cl}(U_o, \phi_o)$. These key concepts are introduced in Section 2. In Section 3 we introduce the $a$ -directional ordinary derivative $a \cdot \partial_o$ of multivector fields and recall the construction of the Lie algebra of smooth vector fields within our formalism. We introduce also the concept of Hestenes derivatives of multivector field $X$, $\partial_o \ast X$, which will play an important role in sequel papers of this series. The $a$-directional derivatives of extensor fields is

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1Or, which is more appropriate, the Clifford bundle of multiforms [11].
2Which are sections of appropriate vector bundles. See, e.g., [11].
3It thus does not suffer the problems of some earlier attempts that have been restricted to vector manifolds [4] [12].
4Which is indeed a connection on $U_o \subset M$. 
introduced in Section 4. Properties of those derivative operators are proved in details. In Section 5 we show how to use (if necessary) a different coordinate system associated to a local chart \((U, \phi)\), for \(U \cap U_\alpha \subset U_\alpha\), besides the one associate to the local chart \((U_\alpha, \phi_\alpha)\) originally selected to define the canonical space. This defines a new \(a\)-directional derivative, denoted \(a \cdot \partial\). We present also some elementary applications of the formalism, introducing covariant and contravariant frame fields associated to a given coordinate system and also the notion of Jacobian fields, which permit us to find the relation between \(a \cdot \partial\) and \(a \cdot \partial\).

2 Canonical Space

Let \(M\) be a \(n\)-dimensional smooth manifold. As well known, associated to any point \(o \in M\) there always exists a local chart \((U_\alpha, \phi_\alpha)\) of a given atlas of \(M\) such that \(o \in U_\alpha\) and \(\phi_\alpha(o) = (0, \ldots, 0)\). Recall that there exist exactly \(n\) scalar functions \(\phi^\mu_\alpha : U_\alpha \to \mathbb{R}\) such that \(\phi_\alpha(p) = (\phi^1_\alpha(p), \ldots, \phi^n_\alpha(p))\) which are the coordinate functions of \((U_\alpha, \phi_\alpha)\). We have that \(\phi^\mu_\alpha = \pi^\mu \circ \phi_\alpha\), where \(\pi^\mu\) is the well-known \(\mu\)-projection mapping of \(\mathbb{R}^n\). Any point \(p \in U_\alpha\) is then localized by a \(n\)-uple of real numbers \(\phi_\alpha(p) \in \mathbb{R}^n\), and the \(n\) real numbers

\[
x^\mu_\alpha = \phi^\mu_\alpha(p), \text{ for each } \mu = 1, \ldots, n
\]

are the position coordinates of \(p\) with respect to \((U_\alpha, \phi_\alpha)\).

As well known, the \(n\) coordinate tangent vectors \(\frac{\partial}{\partial x^\mu_\alpha} \bigg|_p\) define a natural basis at each point \(p \in U_\alpha\) for the tangent space \(T_pM\), called coordinate vector basis at \(p \in U_\alpha\), i.e.,

\[
\{ \frac{\partial}{\partial x^\mu} \bigg|_p \}.
\]

(2)

We recall, in order to fix our notations that for any \(v_p \in T_pM\) we have the elementary expansion

\[
v_p = v_p^\mu \phi^\mu_\alpha \frac{\partial}{\partial x^\mu_\alpha} \bigg|_p,
\]

(3)

where \(v_p^\mu \phi^\mu_\alpha\) denotes the components of \(v_p\) in the natural basis.

To continue, we introduce an equivalence relation on the set of the tangent vectors on \(\bigcup_{p \in U_\alpha} T_pM\) as follows. Let \(v_a \in T_aM\) and \(v_b \in T_bM\) be any two tangent vectors of \(\bigcup_{p \in U_\alpha} T_pM\).

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3Recall that \(\pi^\mu : \mathbb{R}^n \to \mathbb{R}\) is such that if \(a = (a^1, \ldots, a^n) \in \mathbb{R}^n\), then \(\pi^\mu(a) = a^\mu \in \mathbb{R}\) for each \(\mu = 1, \ldots, n\). Note that \(a = (\pi^1(a), \ldots, \pi^n(a))\).

4A tangent vector at \(p \in U_\alpha\) associated to the \(\mu\)-th position coordinate \(x^\mu_\alpha\), namely \(\frac{\partial}{\partial x^\mu_\alpha} \bigg|_p\),

can be defined by \(\frac{\partial}{\partial x^\mu_\alpha} \bigg|_p f = (\frac{\partial}{\partial x^\mu_\alpha} \circ \phi_\alpha^{-1}) \circ \phi_\alpha(p), \text{ for all } f \in C^\infty(p)\).
We say that $v_a$ is equivalent to $v_b$ (written as $v_a \sim v_b$) if and only if
\[ v_a \phi_o^\mu = v_b \phi_o^\mu, \text{ for each } \mu = 1, \ldots, n. \] (4)

Now, since $v_a = v_a \phi_o^\mu \frac{\partial}{\partial x_o^\mu} \bigg|_a$ and $v_b = v_b \phi_o^\mu \frac{\partial}{\partial x_o^\mu} \bigg|_b$, the equivalence between $v_a$ and $v_b$ means that the $\mu$-th contravariant components of $v_a$ with respect to \{ $\frac{\partial}{\partial x_o^\mu} \bigg|_a$ \} are equal to the $\mu$-th contravariant components of $v_b$ with respect to \{ $\frac{\partial}{\partial x_o^\mu} \bigg|_b$ \}.

It is obvious that $\sim$ is a well-defined equivalence relation, and that it is not empty, since the tangent coordinate vectors at any two points $a$ and $b$ belonging to $U_o$ are equivalent to each other, i.e.,
\[ \frac{\partial}{\partial x_o^\mu} \bigg|_a \sim \frac{\partial}{\partial x_o^\mu} \bigg|_b, \text{ for each } \mu = 1, \ldots, n. \] (5)

Let $C_{v_o}$ be the equivalent class of $v_o \in T_oM$, i.e.,
\[ C_{v_o} = \{ v_p / \text{ for all } p \in U_o : v_p \sim v_o \}. \] (6)

Let $U_o$ be the set of all the equivalent classes for every $v_o \in T_oM$, i.e.,
\[ U_o = \{ C_{v_o} / v_o \in T_oM \}. \] (7)

Such $U_o$ has a natural structure of a real vector space. Indeed, such a structure is realized by defining:

The addition of vectors
\[ C_{v_o} \in U_o \text{ and } C_{w_o} \in U_o \Rightarrow C_{v_o} + C_{w_o} = C_{v_o + w_o} \in U_o. \] (8)

The scalar multiplication of vectors by real numbers
\[ \lambda \in \mathbb{R} \text{ and } C_{v_o} \in U_o \Rightarrow \lambda C_{v_o} = C_{\lambda v_o} \in U_o. \] (9)

We notice that the zero vector for $U_o$ is given by $0 = C_{0_o}$, where $0_o$ is the zero vector for $T_oM$. Then, $0 = C_{0_o}$ is just the set of all the zero tangent vectors $0_p \in T_pM$ for all $p \in U_o$.

Now, let us take any $C_{v_o} \in U_o$. Then, by recalling that $v_o = v_o \phi_o^\mu \frac{\partial}{\partial x_o^\mu} \bigg|_o$ we have that
\[ C_{v_o} = v_o \phi_o^\mu \frac{\partial}{\partial x_o^\mu} \bigg|_o. \] (10)

Eq. (10) shows that the $n$ vectors $\frac{\partial}{\partial x_o^\mu} \bigg|_o, \ldots, \frac{\partial}{\partial x_o^\mu} \bigg|_o$ belonging to $U_o$ span $U_o$.

We can also check that they are in fact linearly independent, i.e.,
\[ \lambda^\mu \frac{\partial}{\partial x_o^\mu} \bigg|_o = 0 \Rightarrow \lambda^\mu = 0, \text{ for each } \mu = 1, \ldots, n. \]
Indeed, by definition of equality for equivalence classes, i.e., $\mathcal{C}_v = \mathcal{C}_w \Leftrightarrow v \sim w$, we have that

$$\lambda^\mu \mathcal{C}_{\frac{\partial}{\partial x^\mu}} |_o = 0 \Rightarrow \mathcal{C}_{\lambda^\mu \frac{\partial}{\partial x^\mu}} |_o = \mathcal{C}_0_o \Rightarrow \lambda^\mu \frac{\partial}{\partial x^\mu} |_o = 0_o,$$

whence, by linear independence of $\left\{ \frac{\partial}{\partial x^\mu} |_o \right\}$, it follows that $\lambda^\mu = 0$, for each $\mu = 1, \ldots, n$.

We say that the above $n$ linearly independent vectors spanning $U_o$ are a set of fundamental basis vectors. We put,

$$b_1 = \mathcal{C}_{\frac{\partial}{\partial x^1}} |_o, \ldots, b_n = \mathcal{C}_{\frac{\partial}{\partial x^n}} |_o. \quad (11)$$

Then, it follows that $\text{dim} U_o = n$ (the dimension of $M$).

Such $U_o$ will be called the canonical space for the local chart $(U_o, \phi_o)$. The fundamental basis $\{b_\mu\}$ will be called the fiducial basis for $U_o$. And the real numbers $x^1_o, \ldots, x^n_o$ will be conveniently named as canonical position coordinates of the point $p \in U_o$.

Our next step in order to use the geometrical and extensor calculus is the introduction of a fiducial Euclidean scalar product in the canonical space $U_o$. This is done by declaring the set $\{b_\mu\}$ Euclidean orthonormal, which means, of course, that $b_\mu \cdot b_\nu = \delta_{\mu\nu}$.

Remark 1. It is quite obvious that the equivalence relation defined above is chart dependent, but this fact does not imply in any restriction in the utilization of the methods described in this paper. Indeed, the introduction of a canonical space $U_o$ associated to a local chart $(U_o, \phi_o)$ of the given atlas of $M$ is only a device for the quickly application of the algebraic tools developed in [6] and has no fundamental status in the differential geometry of $M$. Thus, if necessary, for the realization of some specific calculation we simply define another canonical space associated with another local chart $(U_o^1, \phi_o^1)$ of the given atlas of $M$ and use the same methodology which applies to $(U_o, \phi_o)$.

2.1 Position Vector

The open subset $U'_o \subset U_o$, defined by

$$U'_o = \{ \lambda^\mu b_\mu \in U_o \mid \lambda^\mu \in \phi_o^\mu(U_o), \text{ for each } \mu = 1, \ldots, n \} \quad (12)$$

will be called the position vector set of $U_o$. Of course, it is associated to $(U_o, \phi_o)$.

\[\text{Remark 1.} \text{ If } (M, g) \text{ is a 4-dimensional Lorentzian spacetime admitting spinor fields, i.e., is a spin manifold, then, as it is well known, it must admits a global tetrad field (see, e.g., [10, 5]). In that case, the existence of the tetrad field suggests by itself as a natural way to define an equivalence relation between vectors at different spacetime points by the use of an auxiliary teleparallel connection. The use of global tetrad field together with geometrical algebra techniques has been used recently in an interesting paper by Francis and Kosowsky [3]. Their results are to be compared with the ones developed in the present series of papers.}\]
There exists an homeomorphism \( \iota \) between \( U_o \) and \( U'_o \) which is realized by \( U_o \ni p \mapsto \iota_o(p) \in U'_o \) and \( U'_o \ni x \mapsto \iota_o\iota_o^{-1}(x_o) \in U_o \) such that

\[
\begin{align*}
\iota_o(p) &= \phi_o^\mu(p)b_\mu, \\
\iota_o^{-1}(x_o) &= \phi_o^{-1}(b^1 \cdot x_o, \ldots, b^n \cdot x_o). 
\end{align*}
\] (13) (14)

As suggested by the above notations, \( \iota_o^{-1} \) is the inverse mapping of \( \iota_o \). We have indeed that for any \( p \in U_o \)

\[
\iota_o^{-1} \circ \iota_o(p) = \phi_o^{-1}(b^1 \cdot \pi^\mu \circ \phi_o(p)b_\mu, \ldots, b^n \cdot \pi^\mu \circ \phi_o(p)b_\mu)
\]

\[
= \phi_o^{-1}(\pi^1 \circ \phi_o(p), \ldots, \pi^n \circ \phi_o(p))
\]

\[
= \phi_o^{-1} \circ \phi_o(p) = p,
\]

i.e., \( \iota_o^{-1} \circ \iota_o = \iota_o \).

And for any \( x_o \in U'_o \),

\[
\iota_o \circ \iota_o^{-1}(x_o) = \phi_o^\mu(\phi_o^{-1}(b^1 \cdot x_o, \ldots, b^n \cdot x_o))b_\mu
\]

\[
= \pi^\mu \circ \phi_o \circ \phi_o^{-1}(b^1 \cdot x_o, \ldots, b^n \cdot x_o)b_\mu
\]

\[
= (b^\mu \cdot x_o)b_\mu = x_o,
\]

i.e., \( \iota_o \circ \iota_o^{-1} = \iota_o \).

Any point \( p \in U_o \) can be localized by a vector \( \iota_o(p) \in U'_o \). We call

\[
x_o = \iota_o(p)
\]

the position vector of \( p \) with respect to \((U_o, \phi_o)\). Sometimes \( x_o \) will be named as the canonical position vector of \( p \). By using Eq.(13) and Eq.(1) we can write Eq.(13) as

\[
x_o = x_o^\mu b_\mu.
\] (16)

### 2.2 Canonical Algebraic Structures

Let \( \{\beta_\mu\} \), \( \beta^\nu(b_\mu) = \delta^\nu_\mu \) be the dual basis of \( \{b_\mu\} \).

We already have equipped \( U_o \) with a fiducial Euclidean metric, which, of course is given by \( \delta_\mu^\nu \beta^\mu \otimes \beta^\nu \). It will be called the canonical metric, or also, the b-metric, see [1, 6].

The Euclidean scalar product of \( v, w \in U_o \) corresponding to the b-metric, namely \( v \cdot w \in \mathbb{R} \) will be called the canonical scalar product of vectors, or for short, the b-scalar product.

The b-reciprocal basis of \( \{b_\mu\} \), namely \( \{b^\mu\} \), the unique basis such that \( b^\nu \cdot b_\mu = \delta^\nu_\mu \), coincides with \( \{b_\mu\} \), i.e.,

\[
b^\mu = b_\mu \text{, for each } \mu = 1, \ldots, n.
\]

We denote by \( \bigwedge^k U_o \) \((0 \leq k \leq n)\) the space of \( k \)-vectors over \( U_o \), and by \( \bigwedge U_o \) the space of multivectors over \( U_o \), see [2, 6]. The space of \( k \)-extensors over
\( \mathcal{U}_o \) will be denoted by \( k\text{-}\text{ext}(\bigwedge^1 \mathcal{U}_o, \ldots, \bigwedge^k \mathcal{U}_o; \bigwedge^\circ \mathcal{U}_o) \). In particular, \( \text{ext}^p(\mathcal{U}_o) \), \( \text{ext}(\mathcal{U}_o) \) and \( k\text{-}\text{ext}^q(\mathcal{U}_o) \) will denote respectively the spaces of \((p, q)\)-extensors, extensors and elementary \( k\text{-}\text{extensors} \) of degree \( q \) over \( \mathcal{U}_o \).

The Euclidean scalar product of \( X, Y \in \bigwedge \mathcal{U}_o \) relative to the euclidean metric structure \( (\mathcal{U}_o, \delta^\mu_\beta \otimes \beta^\nu) \), will be denoted by \( X \cdot Y \in \mathbb{R} \), as defined in [6] and will be called the canonical scalar product of multivectors, or for short, the \( b\text{-}\text{scalar product} \).

The canonical algebraic structure \( (\bigwedge \mathcal{U}_o, \cdot) \) allows us to define left and right contracted products of \( X, Y \in \bigwedge \mathcal{U}_o \), namely \( X \bowtie Y \) and \( X \bowtie Y \), as defined in [6] and will be simply called the \( b\text{-}\text{contracted products of multivectors} \).

\( \bigwedge \mathcal{U}_o \) endowed with each one of the interior products \( (\bowtie) \) or \( (\bowtie) \) is an non-associative algebra which will be called a \( b\text{-}\text{interior algebra of multivectors} \).

The \( b\text{-}\text{interior algebras} \) \( (\bigwedge \mathcal{U}_o, \bowtie) \) and \( (\bigwedge \mathcal{U}_o, \bowtie) \) together with the exterior algebra \( (\bigwedge \mathcal{U}_o, \wedge) \) allow us to construct a Clifford algebra of multivectors. The Clifford product of \( X, Y \in \bigwedge \mathcal{U}_o \), denoted by juxtaposition \( XY \in \bigwedge \mathcal{U}_o \), has been defined in [6], and will be simply called the \( b\text{-}\text{Clifford product of multivectors} \).

\( \bigwedge \mathcal{U}_o \) endowed with the \( b\text{-}\text{Clifford product} \) is an associative algebra which will be called the \( b\text{-}\text{Clifford algebra of multivectors} \), or the \( b\text{-}\text{geometric algebra} \).

3 Multivector Fields

Multivector fields can be thought as sections of \( \bigwedge TM \), the exterior algebra bundle of multivectors. Given the equivalence relation defined in Section 2 a multivector field \( \mathbf{X} \in \text{sec} \bigwedge TU \subset \text{sec} \bigwedge TM \ (U \subset \mathcal{U}_o) \) is, of course, represented by a mapping

\[
X : U \to \bigwedge \mathcal{U}_o,
\]

which will be called a \textit{representative of the multivector field} \( \mathbf{X} \) on \( U \).

When there is no possibility of confusion we will call \( X \) simply a \textit{multivector field} on \( U \).

The multivector function of the canonical position coordinates, namely \( X \circ \phi^{-1}_o \), given by

\[
\phi_o(U) \ni (x^1_o, \ldots, x^n_o) \mapsto X \circ \phi^{-1}_o(x^1_o, \ldots, x^n_o) \in \bigwedge \mathcal{U}_o,
\]

is called the \textit{position coordinates representation} of \( X \), of course, relative to \((\mathcal{U}_o, \phi_o)\), see [7].

The multivector function of the canonical position vector, namely \( X \circ \iota^{-1}_o \), given by

\[
\iota_o(U) \ni x_o \mapsto X \circ \iota^{-1}_o(x_o) \in \bigwedge \mathcal{U}_o,
\]

is called the \textit{position vector representation} of \( X \) with respect to \((\mathcal{U}_o, \phi_o)\), see [8].
In what follows we suppose that any multivector field $X$ used is smooth, i.e., $C^\infty$ differentiable or at least enough differentiable for our statements to hold.

The set of smooth multivector fields on $U$ will be denoted by $\mathcal{M}(U)$. In particular, the set of smooth scalar fields, the set of smooth vector fields and the set of smooth $k$-vector fields ($k \geq 2$) will be respectively denoted by $\mathcal{S}(U)$, $\mathcal{V}(U)$ and $\mathcal{M}^k(U)$. The identity for $\mathcal{S}(U)$ will be denoted by $1 : U \to \mathbb{R}$ such that $1(p) = 1$, and as well known, $\mathcal{M}(U)$ has a natural structure of a module over the ring (with identity) $\mathcal{S}(U)$.

What is really important here is that $\mathcal{M}(U)$ can be endowed with four kinds of products of smooth multivector fields. Let, as in [6] $\star$ be any suitable product of multivectors either the exterior product ($\wedge$), the $b$-scalar product ($\cdot$), the $b$-contracted products ($\rarr\larr$) or the $b$-Clifford product. Each of these products of multivectors induces a well-defined product of smooth multivector fields which will be also denoted by $\star$. The $\star$-products of $X, Y \in \mathcal{M}(U)$, namely $X \star Y \in \mathcal{M}(U)$, are defined by

$$ (X \star Y)(p) = X(p) \cdot Y(p), \text{ for all } p \in U. \quad (21) $$

$\mathcal{M}(U)$ equipped with ($\cdot$) is an associative algebra induced by the exterior algebra of multivectors. It is called the exterior algebra of smooth multivector fields.

$\mathcal{M}(U)$ equipped with each of ($\rarr\larr$) or ($\larr\rarr$) is a non-associative algebra induced by the respective $b$-interior algebra of multivectors. They are called the $b$-interior algebras of smooth multivector fields.

$\mathcal{M}(U)$ equipped with the $b$-Clifford product is an associative algebra induced by the $b$-Clifford algebra of multivectors. It is called the $b$-Clifford algebra of smooth multivector fields.

It is still possible to define in an obvious way four kinds of products between multivectors and smooth multivector fields. Indeed, let $X \in \bigwedge U_o$ and $Y \in \mathcal{M}(U)$. The $\star$-product of $X$ and $Y$, namely $X \star Y \in \mathcal{M}(U)$, is defined by

$$ (X \star Y)(p) = X(p) \cdot Y, \text{ for all } p \in U. \quad (22) $$

Let $X \in \mathcal{M}(U)$ and $Y \in \bigwedge U_o$. The $\star$-product of $X$ and $Y$, namely $X \star Y \in \mathcal{M}(U)$, is defined by

$$ (X \star Y)(p) = X(p) \cdot Y, \text{ for all } p \in U. \quad (23) $$

### 3.1 $a$-Directional Ordinary Derivative of Multivector Fields

In [4] we developed a complete theory of derivative operators which act on multivector functions of real variables and on multivector functions of multivector variables. Here, we shall need to recall only the definition of directional derivative of a differentiable multivector-valued function of a vector variable, and the definition of partial derivatives of differentiable multivector-valued functions of several real variables. Let $V$ be a $n$-dimensional real vector space. As usual, let us denote by $\bigwedge V$ the space of multivectors over $V$, see [6]. Let $S_1, \ldots, S_k$ be $k$ open subsets of $\mathbb{R}$.
Let $V \ni v \mapsto F(v) \in \bigwedge V$ be a differentiable multivector function of a vector variable. Let us take $a \in V$, the $a$-directional derivative of $F$ is defined to be

$$a \cdot \partial_v F(v) = F'_a(v) = \lim_{\lambda \to 0} \frac{F(v + \lambda a) - F(v)}{\lambda}. \quad (24)$$

Let $S_1 \times \cdots \times S_k \ni (\lambda^1, \ldots, \lambda^k) \mapsto f(\lambda^1, \ldots, \lambda^k) \in \bigwedge V$ be a differentiable multivector function of $k$ real variables. The $\lambda^j$-partial derivative of $f$ (with $1 \leq j \leq k$) is defined to be

$$\frac{\partial f}{\partial \lambda^j}(\lambda^1, \ldots, \lambda^k) = f^{(j)}(\lambda^1, \ldots, \lambda^k)$$

$$= \lim_{\mu \to 0} \frac{f(\lambda^1, \ldots, \lambda^j + \mu, \ldots, \lambda^k) - f(\lambda^1, \ldots, \lambda^k)}{\mu} \quad (25)$$

**Proposition 1.** Let us take $a \in U_o$. For any smooth multivector field $X$, the $a$-directional ordinary derivative of the position vector representation of $X$ with respect to $(U_o, \phi_o)$, namely $a \cdot \partial_{x^a} X \circ \iota_o^{-1}$, is related to the $x_o^\mu$-partial derivatives of the position coordinates representation of $X$ with respect to $(U_o, \phi_o)$, namely $\partial \partial_{x^\mu} X \circ \phi_o^{-1}$, by the identity

$$(a \cdot \partial_{x^a} X \circ \iota_o^{-1}) \circ \iota_o = a \cdot b^\mu (\partial \partial_{x^\mu} X \circ \phi_o^{-1}) \circ \phi_o. \quad (26)$$

**Proof**

It is enough to verify that

$$(b_\mu \cdot \partial_{x^\mu} X \circ \iota_o^{-1}) \circ \iota_o = (\partial \partial_{x^\mu} X \circ \phi_o^{-1}) \circ \phi_o, \text{ for each } \mu = 1, \ldots, n.$$  

Since the position vector $x_o$ is an interior point of $\iota_o(U)$, there is some $\varepsilon$-neighborhood, say $N_{x_o}(\varepsilon)$, such that $N_{x_o}(\varepsilon) \subseteq \iota_o(U)$. Now, if we take $\lambda \in \mathbb{R}$ such that $0 < |\lambda| < \varepsilon$, it follows that $x_o + \lambda b_\mu \in N_{x_o}(\varepsilon)$. It follows that $x_o + \lambda b_\mu \in \iota_o(U)$, and there exist $\iota_o^{-1}(x_o + \lambda b_\mu) \in U$ and $X \circ \iota_o^{-1}(x_o + \lambda b_\mu) \in \bigwedge U_o$.

Hence, by taking into account Eq. (24), we get the following identity

$$X \circ \iota_o^{-1}(x_o + \lambda b_\mu) - X \circ \iota_o^{-1}(x_o)$$

$$= X \circ \phi_o^{-1}(b^1 \cdot x_o + \lambda \delta^1_\mu, \ldots, b^n \cdot x_o + \lambda \delta^n_\mu) - X \circ \phi_o^{-1}(b^1 \cdot x_o, \ldots, b^n \cdot x_o)$$

Now, by taking limits for $\lambda \to 0$ on these multivector functions of the real variable $\lambda$, and using Eq. (24) once again, we have indeed that

$$b_\mu \cdot \partial_{x^\mu} X \circ \iota_o^{-1}(x_o) = \delta^\mu_\mu \partial \partial_{x^\mu} X \circ \phi_o^{-1}(b^1 \cdot x_o, \ldots, b^n \cdot x_o)$$

$$= (\partial \partial_{x^\mu} X \circ \phi_o^{-1}) \circ \phi_o \circ \iota_o^{-1}(x_o),$$

9
and the required result follows.

Let \( X \) be a smooth multivector field on \( U \), and let us take \( a \in \mathcal{U}_0 \). The smooth multivector field on \( U \), namely \( a \cdot \partial_o X \), defined as \( U \ni p \mapsto a \cdot \partial_o X(p) \in \bigwedge \mathcal{U}_0 \) such that

\[
a \cdot \partial_o X(p) = (a \cdot \partial_{x_o} X \circ \iota_o^{-1})(x_o),
\]

where \( x_o = \iota_o(p) \), will be called the canonical a-directional ordinary derivative (a-DOD) of \( X \). Sometimes, \( a \cdot \partial_o X \) will be named as the canonical a-directional ordinary derivative operator (a-DODO).

We see that the position vector representation of \( a \cdot \partial_o X \) is equal to the a-directional derivative of the position vector representation of \( X \), of course, both of them respect to \((U_o, \phi_o)\).

Taking into account Eq. (26), we have for any \( a \in \mathcal{U}_0 \), and for all \( X \in \mathcal{M}(U) \) that

\[
a \cdot \partial_o X(p) = a \cdot \partial_{x_o} X \circ \iota_o^{-1}(x_o) = a \cdot b^\mu \frac{\partial}{\partial x_o^\mu} X \circ \phi_o^{-1}(x_o^1, \ldots, x_o^n),
\]

where \( x_o = \iota_o(p) \) and \((x_o^1, \ldots, x_o^n) = \phi_o(p)\).

It is convenient and useful to generalize the notion of canonical a-DOD of a smooth multivector field whenever \( a \) is any smooth vector field.

Let us take \( a \in \mathcal{V}(U) \). The canonical a-DOD of \( X \), also denoted by \( a \cdot \partial_o X \), is defined to be \( U \ni p \mapsto a \cdot \partial_o X(p) \in \bigwedge \mathcal{U}_0 \) such that

\[
a \cdot \partial_o X(p) = a \circ \iota_o^{-1}(x_o) \cdot \partial_{x_o} X \circ \iota_o^{-1}(x_o),
\]

where \( x_o = \iota_o(p) \). In agreement with Eq. (26), we may also write

\[
a \cdot \partial_o X(p) = a \circ \phi_o^{-1}(x_o^1, \ldots, x_o^n) \cdot b^\mu \frac{\partial}{\partial x_o^\mu} X \circ \phi_o^{-1}(x_o^1, \ldots, x_o^n),
\]

where \((x_o^1, \ldots, x_o^n) = \phi_o(p)\).

We summarize now the basic properties which are satisfied by the a-DOD's.

i. For any \( a \in \mathcal{U}_0 \) or \( a \in \mathcal{V}(U) \) the canonical a-DODO, namely \( a \cdot \partial_o \), is grade-preserving, i.e.,

\[
\text{if } X \in \mathcal{M}^k(U), \text{ then } a \cdot \partial_o X \in \mathcal{M}^k(U).
\]

ii. For any \( a, a' \in \mathcal{U}_0 \) or \( a, a' \in \mathcal{V}(U) \), and \( \alpha, \alpha' \in \mathbb{R} \), and for all \( X \in \mathcal{M}(U) \), it holds

\[
(aa + \alpha a') \cdot \partial_o X = aa \cdot \partial_o X + \alpha a' \cdot \partial_o X,
\]

iii. For any \( a, a' \in \mathcal{U}_0 \) or \( a, a' \in \mathcal{V}(U) \), and \( f, f' \in \mathcal{S}(U) \), and for all \( X \in \mathcal{M}(U) \), it holds

\[
(fa + f'a') \cdot \partial_o X = fa \cdot \partial_o X + f'a' \cdot \partial_o X,
\]
For any $a \in U_o$ or $a \in V(U)$, and for all $f \in S(U)$ and $X, Y \in M(U)$, it holds

\begin{equation}
    a \cdot \partial_o(X + Y) = a \cdot \partial_oX + a \cdot \partial_oY \tag{34}
\end{equation}

\begin{equation}
    a \cdot \partial_o(fX) = (a \cdot \partial_o f)X + f(a \cdot \partial_o X) \tag{35}
\end{equation}

Let $*$ mean any multivector product either $(\wedge)$, $(\cdot)$ $(\lrcorner)$ or $(b$-Clifford product). For any $a \in U_o$ or $a \in V(U)$, and for all $X, Y \in M(U)$, it holds

\begin{equation}
    a \cdot \partial_o(X \ast Y) = (a \cdot \partial_o X) \ast Y + X \ast (a \cdot \partial_o Y) \tag{36}
\end{equation}

It is a Leibniz-like rule for any suitable multivector product as introduced in Section 3.

**Remark 2.** The concept of $a$-direction ordinary derivative is obviously chart dependent. It has been introduced as part of a calculation device, whose main utility results from the fact that $a \cdot \partial_o$ is a connection on $U_o$, a crucial fact that will be used in the following papers of the series.

### 3.2 Lie Algebra of Smooth Vector Fields

The module $V(U)$ of the smooth vector fields on $U$, can be equipped with a well-defined Lie product. Here we briefly recall how to write this concept and their properties in the present approach. Let $a, b \in V(U)$, the (canonical) Lie product $a$ and $b$, namely $[a, b]$, is defined by

\begin{equation}
    [a, b] = a \cdot \partial_o b - b \cdot \partial_o a. \tag{37}
\end{equation}

By using Eq.(29) and Eq.(30), we can get two noticeable formulas for this Lie product. One in terms of the (vector)-directional derivative operators with respect to $x_o$ acting on the canonical position vector representation of $a$ and $b$, i.e.,

\begin{equation}
    [a, b](p) = a \circ \iota_o^{-1}(x_o) \cdot \partial_{x_o} b \circ \iota_o^{-1}(x_o) - b \circ \iota_o^{-1}(x_o) \cdot \partial_{x_o} a \circ \iota_o^{-1}(x_o) \tag{38}
\end{equation}

And, another formula involving the fiducial basis $\{b_\mu\}$, the $x_o^\mu$-partial derivative operators $\partial / \partial x_o^\mu$ and the canonical position coordinates representation of $a$ and $b$, i.e.,

\begin{equation}
    [a, b](p) = a \circ \phi_o^{-1}(x_o^1, \ldots, x_o^n) \cdot b_\mu \frac{\partial}{\partial x_o^\mu} b \circ \phi_o^{-1}(x_o^1, \ldots, x_o^n) - b \circ \phi_o^{-1}(x_o^1, \ldots, x_o^n) \cdot b_\mu \frac{\partial}{\partial x_o^\mu} a \circ \phi_o^{-1}(x_o^1, \ldots, x_o^n). \tag{39}
\end{equation}

We summarize the basic properties satisfied by the Lie product.

**i.** For all $a, a', b, b' \in V(U)$

\begin{equation}
    [a + a', b] = [a, b] + [a', b], \tag{40}
\end{equation}

\begin{equation}
    [a, b + b'] = [a, b] + [a, b'] \text{ (distributive laws)}. \tag{41}
\end{equation}
ii. For all \( f \in \mathcal{S}(U) \), and \( a, b \in \mathcal{V}(U) \)
\[
[f a, b] = f[a, b] - (b \cdot \partial_a f)a, \quad (42)
\]
\[
[a, f b] = (a \cdot \partial_a f)b + f[a, b]. \quad (43)
\]

iii. For any \( a, b \in \mathcal{V}(U) \), and for all \( X \in \mathcal{M}(U) \), it holds
\[
[a \cdot \partial_o, b \cdot \partial_o]X = [a, b] \cdot \partial_o X. \quad (44)
\]

The proof of this result is as follows. By using Eqs. (34) and (35), and Eq. (36), we have
\[
a \cdot \partial_o(b \cdot \partial_o X) = a \cdot b^\mu b_\mu \cdot \partial_o(b \cdot b^\nu b_\nu \cdot \partial_o X)
\]
\[
= a \cdot b^\mu (b_\mu \cdot \partial_o) b^\nu b_\nu \cdot \partial_o X + a \cdot b^\nu b \cdot b^\nu b_\nu \cdot \partial_o(b_\nu \cdot \partial_o X),
\]
\[
= (a \cdot \partial_o b) \cdot \partial_o X + a \cdot b^\mu b \cdot b^\nu b_\nu \cdot \partial_o(b_\nu \cdot \partial_o X). \quad (45)
\]

And, by interchanging the letters \( a \) and \( b \), and re-naming indices, we have
\[
b \cdot \partial_o(a \cdot \partial_o X) = (b \cdot \partial_o a) \cdot \partial_o X + b \cdot b^\nu b \cdot b^\nu b_\nu \cdot \partial_o(b_\nu \cdot \partial_o X),
\]
\[
= (b \cdot \partial_o a) \cdot \partial_o X + a \cdot b^\mu b \cdot b^\nu b_\nu \cdot \partial_o(b_\nu \cdot \partial_o X). \quad (46)
\]

Now, subtracting Eq. (46) from Eq. (45), we get
\[
[a \cdot \partial_o, b \cdot \partial_o]X = [a, b] \cdot \partial_o X + a \cdot b^\mu b \cdot b^\nu b_\nu \cdot \partial_o(b_\nu \cdot \partial_o X) - b_\nu \cdot \partial_o(b_\nu \cdot \partial_o X).
\]

Then, by recalling the obvious property
\[
b_\mu \cdot \partial_o(b_\nu \cdot \partial_o X) = b_\nu \cdot \partial_o(b_\nu \cdot \partial_o X),
\]
we finally get the required result.

iv. For all \( a, b, c \in \mathcal{V}(U) \)
\[
[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0, \quad (47)
\]
called Jacobi’s identity.

\( \mathcal{M}(U) \) endowed with the Lie product is (for each \( p \in U \)) a Lie algebra which will be called the Lie algebra of smooth vector fields.

### 3.3 Hestenes Derivatives

Let us take any two reciprocal frame fields\(^{10}\) on \( U \), namely \( \{e_\mu\} \) and \( \{e^\nu\} \), i.e.
\( e_\mu \cdot e^\nu = \delta_\mu^\nu \). Let \( X \) be a smooth multivector field on \( U \). We can introduce exactly three smooth derivative-like multivector fields on \( U \), namely \( \partial_o \ast X \), defined by
\[
\partial_o \ast X = e^\mu \ast e_\mu \cdot \partial_o X = e_\mu \ast e^\mu \cdot \partial_o X, \quad (48)
\]
\(^{9}\)The commutator of \( a \cdot \partial_o \) and \( b \cdot \partial_o \), namely \( [a \cdot \partial_o, b \cdot \partial_o] \), is defined by \( [a \cdot \partial_o, b \cdot \partial_o]X = a \cdot \partial_o(b \cdot \partial_o X) - b \cdot \partial_o(a \cdot \partial_o X) \), for all \( X \in \mathcal{M}(U) \).

\(^{10}\)A frame field on \( U_o \) is a set of \( n \) smooth vector fields \( e_1, \ldots, e_n \in \mathcal{V}(U) \) such that for each \( p \in U \) the set of the \( n \) vectors \( e_1(p), \ldots, e_n(p) \in U_o \) is a basis for \( U_o \). In particular, \( \{b_\mu\} \) can be taken as a constant frame field on \( U_o \), i.e., \( a \cdot \partial_o b_\mu = 0 \), for each \( \mu = 1, \ldots, n \). It is called the fiducial frame field.

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12
where $\ast$ means any multivector product ($\wedge$, $\cdot$ or ($b$-Clifford product).

Note that a smooth multivector field defined by Eq.(48) does not depend on the choice of $\{e_\mu\}$ and $\{e^{\mu}\}$.

By recalling Eq.(33) and using Eq.(29) we can get a remarkable formula for calculating $\partial_\alpha \ast X$ which involves only the fiducial frame field $\{b_\mu\}$ and the $b_\mu$-directional operators $b_\mu \cdot \partial_{x_\alpha}$ acting on the canonical position vector representation of $X$, i.e.,

$$\partial_\alpha \ast X(p) = b^\mu \ast b_\mu \cdot \partial_{x_\alpha} X \circ \iota^{-1}_o(x_\alpha),$$  \hfill (49)

where $x_\alpha = \iota_o(p)$.

By using Eq.(30) we can get another formula for calculating $\partial_\alpha \ast X$ which involves only the fiducial frame fields $\{b_\mu\}$ and the $x_\alpha^\mu$-partial derivative operators $\frac{\partial}{\partial x_\alpha^\mu}$ acting on the canonical position coordinate representation of $X$, i.e.,

$$\partial_\alpha \ast X(p) = b^\mu \ast \frac{\partial}{\partial x_\alpha^\mu} X \circ \phi^{-1}_o(x_\alpha^1, \ldots, x_\alpha^n),$$  \hfill (50)

where $(x_\alpha^1, \ldots, x_\alpha^n) = \phi_o(p)$.

We will call $\partial_\alpha \wedge X$, $\partial_\alpha \cdot X$ and $\partial_\alpha X$ (i.e., $\ast = b$-Clifford product) respectively the curl, left $b$-contracted divergence and $b$-gradient of $X$. An interesting and useful relationship relating these derivatives is

$$\partial_\alpha X = \partial_\alpha \cdot X + \partial_\alpha \wedge X.$$  \hfill (51)

In what follows we call these derivatives by Hestenes derivatives.

We end this section presenting three remarkable multivector identities involving the Hestenes derivatives.

$$\begin{align*}
(\partial_\alpha \wedge X) \cdot Y + X \cdot (\partial_\alpha \cdot Y) &= \partial_\alpha \cdot (n \wedge X) \cdot Y), \\
(\partial_\alpha \cdot X) \cdot Y + X \cdot (\partial_\alpha \wedge Y) &= \partial_\alpha \cdot (n \cdot X) \cdot Y), \\
(\partial_\alpha X) \cdot Y + X \cdot (\partial_\alpha Y) &= \partial_\alpha \cdot (n X) \cdot Y).
\end{align*}$$  \hfill (52) \hfill (53) \hfill (54)

They are used in the Lagrangian theory of multivector fields.\footnote{A preliminary construction of that theory on Minkowski spacetime has been given in \cite{9}, and with more details in \cite{11}.}

## 4 Extensor Fields

Extensor fields are sections of an appropriate vector bundle $E(\bigwedge TM)$ (which will not be described here). What is important for us is that given the equivalence relation defined in Section 2 any $t \in \text{sec}(E(\bigwedge TU)) \subset E(\bigwedge TM)$, with $U \subset U_o$ has a representative given by the mapping:\footnote{As in the case of representatives of multivector fields, this construction is equivalent to a local trivialization of $E(\bigwedge TM)$.}

$$t : U \rightarrow k\text{-}\text{ext}(\bigwedge^1 U_o, \ldots, \bigwedge^k U_o; \bigwedge^\circ U_o).$$  \hfill (55)
called a representative of \( k \)-extensor field on \( U \), or when no confusion arises, simply by \( k \)-extensor field on \( U \). Then, for each \( p \in U : t_{(p)} \) is a \( k \)-linear mapping from \( \bigwedge^1 U_o \times \ldots \times \bigwedge^k U_o \) to \( \bigwedge^o U_o \), \( t_{(p)} : \bigwedge^1 U_o, \ldots, \bigwedge^k U_o ; \bigwedge^o U_o \).

The \( k \)-extensor function of the position coordinates \((x_1^0, \ldots, x_n^0)\), namely \( t_{\alpha \phi_\alpha^{-1}} \), given by

\[
\phi_\alpha(U) \ni (x_1^0, \ldots, x_n^0) \mapsto t_{\alpha \phi_\alpha^{-1}}(x_1^0, \ldots, x_n^0) \in \bigwedge^\alpha \bigwedge^1 U_o, \ldots, \bigwedge^\alpha \bigwedge^k U_o ; \bigwedge^\alpha \bigwedge^o U_o , \quad (56)
\]

is called the position coordinates representation of \( t \) with respect to \((U_o, \phi_\alpha)\).

The \( k \)-extensor function of the position vector \( x_o \), namely \( t_{\alpha_l \phi_\alpha^{-1}} \), given by

\[
t_o(U) \ni x_o \mapsto t_{\alpha_l \phi_\alpha^{-1}}(x_o) \in \bigwedge^\alpha \bigwedge^1 U_o, \ldots, \bigwedge^\alpha \bigwedge^k U_o ; \bigwedge^\alpha \bigwedge^o U_o , \quad (57)
\]

is called the position vector representation of \( t \) with respect to \((U_o, \phi_\alpha)\).

All \( k \)-extensor field \( t \) used in what follows are supposed smooth and we can easily prove the following proposition.

**Proposition 2.** A \( k \)-extensor field \( t \) is smooth if and only if the multivector field defined as \( U \ni p \mapsto t_{(p)}(X_1(p), \ldots, X_k(p)) \in \bigwedge^o U_o \) is itself smooth, for all \( X_1 \in M^1(U_o) \), \ldots, \( X_k \in M^k(U_o) \).

Associated to any smooth \( k \)-extensor field \( t \) we can define a linear operator, namely \( t_{\alpha_l} \), which takes \( k \)-uples of smooth multivector fields belonging to \( M^1(U_o) \times \ldots \times M^k(U_o) \) into smooth multivector fields belonging to \( M^o(U_o) \), i.e.,

\[
t_{\alpha_l}(X_1, \ldots, X_k)(p) = t_{(p)}(X_1(p), \ldots, X_k(p)), \quad \text{for each } p \in U. \quad (58)
\]

Reciprocally, given a linear operator \( t_{\alpha_l} \) we can prove (without difficulty) that there is an unique smooth \( k \)-extensor field \( t \) such that Eq. (58) holds.

The set of smooth \( k \)-extensor fields on \( U \) has a natural structure of module over the ring \((\text{with identity}) S(U)\). It will be denoted in what follows by the suggestive notation \( k \text{-ext}(M^1(U_o), \ldots, M^k(U_o); M^o(U_o)) \).

### 4.1 \( a \)-Directional Ordinary Derivative of Extensor Fields

Let \( t \) be a smooth \( k \)-extensor field on \( U \), and let us take \( a \in U_o \). The smooth \( k \)-extensor field on \( U \), namely \( a \cdot \partial_a t \), such that for all smooth multivector fields \( X_1 \in M^1(U_o), \ldots, X_k \in M^k(U_o) \)

\[
(a \cdot \partial_a t)(p)(X_1(p), \ldots, X_k(p)) = a \cdot \partial_a t(p)(X_1(p), \ldots, X_k(p))
\]

for each \( p \in U \), will be called the canonical \( a \)-directional ordinary derivative \((a\text{-DOD})\) of \( t \).

Note that on the right side of Eq. (59) there just appears the \( a \)-DOD of the smooth multivector field \( U \ni p \mapsto t_{(p)}(X_1(p), \ldots, X_k(p)) \in \bigwedge^o U_o \). As usual, when no confusion arises, we will write the definition given by Eq. (59) by omitting the letter \( p \).
We note that the algebraic object $a \cdot \partial_o t$ as defined by Eq. [59] is indeed a smooth $k$-extensor field. The $k$-extensor character and the smoothness for it follow directly from the respective properties of $t$. Also, note that in definition given by Eq. [59], $a$ could be a smooth vector field on $U$ instead a vector of $U_o$.

We present now some of the most basic properties satisfied by the $a$-DOD’s of smooth $k$-extensor fields.

i. For any of either $a \in U_o$ or $a \in V(U)$ the canonical $a$-DODO, namely $a \cdot \partial_o$ (whenever it is acting on smooth $k$-extensor fields), preserves the extensor type, i.e.,

$$\text{if } t \in k\text{-ext}(M^k_1(U), \ldots, M^k_r(U); M^o(U)), $$

then $a \cdot \partial_o t \in k\text{-ext}(M^k_1(U), \ldots, M^k_r(U); M^o(U))$.  \hspace{1cm} (60)

ii. For any of either $a, a' \in U_o$ or $a, a' \in V(U)$, and $\alpha, \alpha' \in \mathbb{R}$, and for all $t \in k\text{-ext}(M^k_1(U), \ldots, M^k_r(U); M^o(U))$, it holds

$$(aa + \alpha a') \cdot \partial_o t = aa \cdot \partial_o t + \alpha a' \cdot \partial_o t.$$ \hspace{1cm} (61)

iii. For any of either $a, a' \in U_o$ or $a, a' \in V(U)$, and $f, f' \in S(U)$, and for all $t \in k\text{-ext}(M^k_1(U), \ldots, M^k_r(U); M^o(U))$, it holds

$$(fa + f'a') \cdot \partial_o t = fa \cdot \partial_o t + f'a' \cdot \partial_o t.$$ \hspace{1cm} (62)

iv. For any of either $a \in U_o$ or $a \in V(U)$, and for all $f \in S(U)$, and for all $t, u \in k\text{-ext}(M^k_1(U), \ldots, M^k_r(U); M^o(U))$, it holds

$$a \cdot \partial_o(t + u) = a \cdot \partial_o t + a \cdot \partial_o u,$$

$$a \cdot \partial_o(ft) = (a \cdot \partial_o f)t + f(a \cdot \partial_o t).$$ \hspace{1cm} (63) \hspace{1cm} (64)

iv. For any of either $a \in U_o$ or $a \in V(U)$, and for all $t \in k\text{-ext}(M^k_1(U); M^o(U))$, it holds

$$(a \cdot \partial_o t)^\dagger = a \cdot \partial_o t^\dagger.$$ \hspace{1cm} (65)

To prove Eq. [65] let us take $X_1 \in M^k_1(U)$ and $X \in M^o(U)$. By recalling the fundamental property of the adjoint operator $^\dagger$, and using Eq. [60], we can write

$$(a \cdot \partial_o t)^\dagger (X) \cdot X_1 = (a \cdot \partial_o t)(X_1) \cdot X$$

$$= a \cdot \partial_o (t(X_1) \cdot X) - t(X_1) \cdot a \cdot \partial_o X - t(a \cdot \partial_o X_1) \cdot X$$

$$= a \cdot \partial_o (t^\dagger (X) \cdot X_1) - t^\dagger (a \cdot \partial_o X) \cdot X_1 - t^\dagger (X) \cdot a \cdot \partial_o X_1,$$

$$= (a \cdot \partial_o t^\dagger)(X) \cdot X_1.$$

Hence, by the non-degeneracy of scalar product, the expected result immediately follows.
5 How to Use Coordinate Systems Different from $\{x^\mu\}$

Let $U$ be an open subset of $U_\circ$, and let $(U, \phi)$ be a local coordinate system on $U$ compatible with $(U_\circ, \phi_\circ)$. This implies that any point $p \in U$ can indeed be localized by a $n$-uple of real numbers $\phi(p) \in \mathbb{R}^n$.

The $n$ scalar fields on $U$

$$\phi^\mu : U \to \mathbb{R} \text{ such that } \phi^\mu = \pi^\mu \circ \phi,$$

are the so-called position coordinates of $U$. For each point $p \in U$, $x^\mu = \phi^\mu(p)$

are the so-called position coordinates of $p$ with respect to $(U, \phi)$.

The open subset $U'$ of the canonical space $U_\circ$ defined by

$$U' = \{ \lambda^\mu b_\mu \in U_\circ \mid \lambda^\mu \in \phi^\mu(U), \text{ for each } \mu = 1, \ldots, n \}$$

will be called a position vector set of $U$. Of course, it is associated to $(U, \phi)$.

There exists an isomorphism between $U$ and $U'$ which is realized by $U \ni p \mapsto \iota(p) \in U'$ and $U' \ni x \mapsto \iota^{-1}(x) \in U$ such that

$$\iota(p) = \phi^\mu(p) b_\mu,$$

$$\iota^{-1}(x) = \phi^{-1}(b_1 \cdot x, \ldots, b_n \cdot x).$$

As suggested by the above notation, $\iota$ and $\iota^{-1}$ are inverse mappings of each other.

Indeed, we have that for any $p \in U$

$$\iota^{-1} \circ \iota(p) = \phi^{-1}(b_1 \cdot \phi^\mu(p) b_\mu, \ldots, b_n \cdot \phi^\mu(p) b_\mu)$$

$$= \phi^{-1}(\phi^1(p), \ldots, \phi^n(p))$$

$$= \phi^{-1} \circ \phi(p) = p,$$

i.e., $\iota^{-1} \circ \iota = \mathbb{1}_U$.

And for any $x \in U'$

$$\iota \circ \iota^{-1}(x) = \phi^\mu(\phi^{-1}(b_1 \cdot x, \ldots, b_n \cdot x)) b_\mu$$

$$= \phi^\mu \circ \phi^{-1}(b_1 \cdot x, \ldots, b_n \cdot x) b_\mu$$

$$= (b^\mu \cdot x) b_\mu = x,$$

i.e., $\iota \circ \iota^{-1} = \mathbb{1}_{U'}$.

Then, any point $p \in U$ can be also localized by a vector $\iota(p) \in U'$. We call

$$x = \iota(p), \text{ i.e., } x = x^\mu b_\mu,$$
the position vector of \( p \) with respect to \( (U, \phi) \).

Let \( X \) be a smooth multivector field on \( U \). Their position coordinate and vector representations with respect to \( (U, \phi) \) are respectively given by

\[
\phi(U) \ni (x^1, \ldots, x^n) \mapsto X \circ \phi^{-1}(x^1, \ldots, x^n) \in \bigwedge U_o
\]

and

\[
\iota(U) \ni x \mapsto X \circ \iota^{-1}(x) \in \bigwedge U_o.
\]

Let us take \( a \in U_o \). For any smooth multivector field \( X \), the \( a \)-directional ordinary derivative of the position vector representation of \( X \) with respect to \( (U, \phi) \), namely \( a \cdot \partial X \), is defined as

\[
a \cdot \partial X(p) = (a \cdot \partial_x X \circ \iota^{-1})(x),
\]

where \( x = \iota(p) \), and \( (x^1, \ldots, x^n) = \phi(p) \).

In accordance with Eq. (75), we have a noticeable formula. For any \( a \in U_o \), and for all \( X \in \mathcal{M}(U) \)

\[
a \cdot \partial X(p) = a \cdot \partial_x X \circ \iota^{-1}(x) = a \cdot b^\mu \frac{\partial}{\partial x^\mu} X \circ \phi^{-1}(x^1, \ldots, x^n),
\]

where \( x = \iota(p) \) and \( (x^1, \ldots, x^n) = \phi(p) \).

Let us take \( a \in \mathcal{V}(U) \). The \( a \)-DOD of \( X \) with respect to \( (U, \phi) \), which as usual will be also denoted by \( a \cdot \partial X \), is defined to be \( U \ni p \mapsto a \cdot \partial X(p) \in \bigwedge U_o \) such that

\[
a \cdot \partial X(p) = a \circ \iota^{-1}(x) \cdot \partial_x X \circ \iota^{-1}(x),
\]

where \( x = \iota(p) \). In accordance with Eq. (75), we might be also written

\[
a \cdot \partial X(p) = a \circ \phi^{-1}(x^1, \ldots, x^n) \cdot b^\mu \frac{\partial}{\partial x^\mu} X \circ \phi^{-1}(x^1, \ldots, x^n),
\]

where \( (x^1, \ldots, x^n) = \phi(p) \).

We notice that the basic properties which we expected should be valid for \( a \cdot \partial \) are completely analogous to those ones which are satisfied by \( a \cdot \partial_o \).
5.2 Covariant and Contravariant Frame Fields

The canonical position vector of any \( p \in U \), namely \( x_o \in \iota_o(U) \), can be considered either as function of its position vector with respect \((U, \phi)\), namely \( x \in \iota(U) \), i.e.,

\[
\iota(U) \ni x \mapsto x_o = \phi_o^\nu \circ \iota^{-1}(x)b_\nu \in \iota_o(U),
\]

(79)

or as function of its position coordinates with respect to \((U, \phi)\), namely \((x^1, \ldots, x^n) \in \phi(U)\), i.e.,

\[
\phi(U) \ni (x^1, \ldots, x^n) \mapsto x_o = \phi_o^\nu \circ \phi^{-1}(x^1, \ldots, x^n)b_\nu \in \iota_o(U).
\]

(80)

We emphasize that the first one is the position vector representation of \( \phi_o^\nu b_\nu \), and the second one is the coordinate vector representation of \( \phi_o^\nu b_\nu \), both of them with respect to \((U, \phi)\).

The \( n \) smooth vector fields on \( U \), namely \( b_1 \cdot \partial \phi_o^\nu b_\nu, \ldots, b_n \cdot \partial \phi_o^\nu b_\nu \), defines a frame field on \( U \) which is called the covariant frame field for \((U, \phi)\). It will be denoted by \( \{b_\mu \cdot \partial x_\nu\} \), i.e.,

\[
b_\mu \cdot \partial x_\nu = b_\mu \cdot \partial \phi_o^\nu b_\nu, \quad \text{for each } \mu = 1, \ldots, n.
\]

(81)

In agreement to Eq. (76), this means that for any \( p \in U \)

\[
b_\mu \cdot \partial x_\nu(p) = b_\mu \cdot \partial x_\nu \phi_o^\nu \circ \iota^{-1}(x)b_\nu \equiv \frac{\partial}{\partial x^\mu} \phi_o^\nu \circ \phi^{-1}(x) \big|_{x=x(p)}b_\nu,
\]

(82)

where \( x = \iota(p) \) and \((x^1, \ldots, x^n) = \phi(p)\).

The position coordinates of any \( p \in U \) with respect to \((U, \phi)\), namely \( x^\mu \in \phi^\mu(U) \subseteq \mathbb{R} \), can be taken either as functions of its canonical position vector \( x_o \in \iota_o(U) \subseteq \mathbb{R}^n \), i.e.,

\[
\iota_o(U) \ni x_o \mapsto x_o = \phi_o^\nu \circ \iota^{-1}(x_o) \in \phi^\nu(U),
\]

(83)

or as function of its canonical position coordinates \((x_o^1, \ldots, x_o^n) \in \phi_o(U) \subseteq \phi_o(U_o) \), i.e.,

\[
\phi_o(U) \ni (x_o^1, \ldots, x_o^n) \mapsto x_o = \phi_o^\nu \circ \phi^{-1}_o(x_o^1, \ldots, x_o^n) \in \phi^\nu(U).
\]

(84)

It should be noted that the former is the canonical position vector representation of \( \phi^\nu \); meanwhile the latter is the canonical position coordinate representation of \( \phi^\nu \).

The \( n \) smooth vector fields on \( U \), namely \( \partial_o \phi^1, \ldots, \partial_o \phi^n \), defines a frame field on \( U \) which is called the contravariant frame field for \((U, \phi)\). It is usually denoted by \( \{\partial_o x^\nu\} \), where \( \partial_o \) is the Hestenes operator introduced in Section 3.3. We have, of course, \( \partial_o x^\nu = \partial_o \phi^\nu \), for each \( \nu = 1, \ldots, n \), which according to Eq. (84) means that for any \( p \in U \)

\[
\partial_o x^\nu(p) = \partial_o x_o \phi_o^\nu \circ \iota^{-1}(x_o) = b_\nu \frac{\partial}{\partial x^\mu} \phi_o^\nu \circ \phi^{-1}_o(x_o^1, \ldots, x_o^n),
\]

(85)

where \( x_o = \iota_o(p) \) and \((x_o^1, \ldots, x_o^n) = \phi_o(p)\).
5.3 The Relation Between $a \cdot \partial_o$ and $a \cdot \partial$

To find the relation between the operators $a \cdot \partial_o$ and $a \cdot \partial$ we now introduce a non-singular smooth $(1,1)$-extensor field on $U$, namely $J_{\phi}$, defined as

$$J_{\phi}(a) = a \cdot \partial x_o,$$

which will be called the Jacobian field for $(U, \phi)$. Of course, for any $p \in U$

$$J_{\phi}(a) = a \cdot \partial x_o \circ \iota_1^{-1}(x) b_{\nu} = a \cdot b_{\nu} \frac{\partial}{\partial x_o} \phi_o^{\nu} \circ \phi^{-1}(x^1, \ldots, x^n) b_{\nu}. \quad (86)$$

The inverse of $J_{\phi}$ denoted by $J^{-1}_{\phi}$ is given by

$$J^{-1}_{\phi}(a) = a \cdot \partial_o x, \text{ i.e., } J^{-1}_{\phi}(a) = a \cdot \partial_o \phi^{\nu} b_{\nu}. \quad (88)$$

Therefore, for any $p \in U$, \n
$$J^{-1}_{\phi}(p) (a) = a \cdot \partial_o x \circ \iota^{-1}_o(x) b_{\nu} = a \cdot b_{\nu} \frac{\partial}{\partial x_o} \phi^{\nu} \circ \phi^{-1}(x_o^1, \ldots, x_o^n) b_{\nu}. \quad (89)$$

We now show that there exists a bijective vector-valued function of a vector variable $\varphi : \iota_o(U) \rightarrow \iota(U)$, and that there are $n$ injective scalar-valued functions of a vector variable $\varphi^{\nu} : \iota_o(U) \rightarrow \phi^{\nu}(U)$ such that

$$\varphi^{-1}(x) = \phi_o^{\nu} \circ \iota^{-1}_o(x) b_{\nu}, \quad (90)$$

$$\varphi^{\nu}(x_o) = \phi^{\nu} \circ \iota^{-1}_o(x_o), \quad (91)$$

$$\varphi^{\nu} \circ \varphi^{-1}(x) = b^{\nu} \cdot x. \quad (92)$$

Such $\varphi$ and $\varphi^{\nu}$ are given by

$$\varphi = \iota \circ \iota^{-1}_o, \quad (93)$$

$$\varphi^{\nu}(x_o) = b^{\nu} \cdot \varphi(x_o). \quad (94)$$

Indeed, by putting Eq. (93) into $\varphi^{-1}$ as obtained from Eq. (90), we get Eq. (91). By putting Eq. (99) into Eq. (93), and then the result obtained into Eq. (94), we get Eq. (91). From Eq. (91) and $\varphi^{-1}$ as obtained from Eq. (93), by using Eq. (70) and Eq. (66), it follows Eq. (92).

We have that $\varphi^{\nu}$ and $\varphi^{-1}$ are involved into remarkable formulas for the covariant and contravariant frame fields of $(U, \phi)$

$$b_{\mu} \cdot \partial x_o(p) = b_{\mu} \cdot \partial x \varphi^{-1}(x), \quad (95)$$

$$\partial x^{\nu}(p) = \partial x_o \varphi^{\nu}(x_o). \quad (96)$$

There are also noticeable formulas for the Jacobian field, and its inverse, in which $\varphi^{-1}$ and $\varphi$ are involved. They are,

$$J_{\phi}(p)(a) = a \cdot \partial x \varphi^{-1}(x), \quad (97)$$

$$J^{-1}_{\phi}(p)(a) = a \cdot \partial x_o \varphi(x_o). \quad (98)$$
where $x = \iota(p)$ and $x_o = \iota_o(p)$.

Eq. (95) and Eq. (96) follow immediately from Eq. (82) and Eq. (85) once we use Eq. (90) and Eq. (91), respectively. Eq. (97) and Eq. (98) are immediate consequences of Eq. (87) and Eq. (89) once we take into account Eq. (90) and Eq. (91), respectively.

We end this subsection presenting some useful properties, which can be easily proved and which will be used in the sequel papers of this series.

i. $\{b_\mu \cdot \partial x_o\}$ and $\{\partial_o x^\nu\}$ define a pair of *reciprocal frame fields* on $U$, i.e.,

$$b_\mu \cdot \partial x_o \cdot \partial_o x^\nu = \delta^\nu_\mu.$$  \hspace{1cm} (99)

ii. $\{b_\mu \cdot \partial x_o\}$ and $\{\partial_o x^\nu\}$ are just $J_\phi$-deformations of the fiducial frame field $\{b_\mu\}$, i.e.,

$$b_\mu \cdot \partial x_o = J_\phi(b_\mu),$$

$$\partial_o x^\nu = J_\phi^*(b^\nu).$$  \hspace{1cm} (100) and (101)

The first statement is trivial from the definition of deformation, see [6]. In order to prove the second statement, we write

$$a \cdot J_\phi^*(b^\nu) = J_\phi^{-1}(a) \cdot b^\nu = (a \cdot \partial_o x) \cdot b^\nu = a \cdot \partial_o (x \cdot b^\nu) = a \cdot \partial_o x^\nu = a \cdot (\partial_o x^\nu),$$

hence, by non-degeneracy of scalar product, the required result immediately follows.

**Proposition 3.** The relationship between the canonical $a$-DODO and the a-DODO with respect to $(U, \phi)$, namely $a \cdot \partial_o$ and $a \cdot \partial$, is given by

$$J_\phi(a) \cdot \partial_o X = a \cdot \partial X,$$  \hspace{1cm} (102)

for all $X \in \mathcal{M}(U)$.

**Proof**

Let $X$ be a smooth multivector field on $U$. Then, by using a link rule for the $a$ directional derivation of multivector-valued functions of a vector variable on the obvious identity $(X \circ \iota_o^{-1}) \circ \varphi^{-1}(x) = X \circ \iota^{-1}(x)$, we can write

$$a \cdot \partial_x (X \circ \iota_o^{-1}) \circ \varphi^{-1}(x) = a \cdot \partial_x X \circ \iota^{-1}(x)$$

$$(a \cdot \partial_x \varphi^{-1}(x) \cdot \partial_o x_o X \circ \iota_o^{-1}) \circ \varphi^{-1}(x) = a \cdot \partial_x X \circ \iota^{-1}(x)$$

$$a \cdot \partial_x \varphi^{-1}(x) \cdot \partial_x x_o \circ \iota_o^{-1}(x_o) = a \cdot \partial_x X \circ \iota^{-1}(x).$$

Hence, using Eq. (97), and Eq. (24) and Eq. (76), we get

$$J_\phi(a) \cdot \partial_o X(p) = a \cdot \partial X(p).$$

We end this section by presenting a remarkable and useful property which is immediate consequence of Eq. (102) and Eq. (100). It is

$$(b_\mu \cdot \partial x_o) \cdot \partial_o X = b_\mu \cdot \partial X,$$

for all $X \in \mathcal{M}(U)$. 

6 Conclusions

We just presented a theory of multivector and extensor fields living on a smooth manifold $M$ of arbitrary topology based on the geometric algebra of multivectors and extensors. Our approach, which does not suffer the problems of earlier attempts\footnote{Which are indeed restricted to vector manifolds.} of applying geometric algebra to the differential geometry of manifolds is based on calculations done with the representatives of those fields through canonical algebraic structures over the canonical space $U_o$ associated to a local chart $(U_o, \phi_o)$ of the maximal atlas of $M$. Some crucial ingredients for our program, like the $a$-directional ordinary derivative of multivector and extensor fields, the Lie algebra of smooth vector fields and the Hestenes derivatives have been introduced and their main properties explored. We worked also some few applications involving general coordinate systems besides the one used in the definition of the canonical space.

Acknowledgments: V. V. Fernández and A. M. Moya are very grateful to Mrs. Rosa I. Fernández who gave to them material and spiritual support at the starting time of their research work. This paper could not have been written without her inestimable help. Authors are also grateful to Drs. E. Notte-Cuello and E. Capelas de Oliveira for useful discussions.

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