On the Global Existence of a Class of Strongly Coupled Parabolic Systems.

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Abstract

We establish the existence of strong solutions to a class of cross diffusion systems on $\mathbb{R}^N$ consists of $m$ equations $(m, N \geq 2)$, which generalizes the Shigesada-Kawasaki-Teramoto (SKT) model in population dynamics. We introduce the concept of a strong-weak solution of the systems and show that their existence can be established under weaker conditions. These strong-weak solutions coincide with strong solutions so that the existence of strong solutions is proved. The SKT model on planar domains $(N = 2)$ with cubic diffusions and advections is completely solved.

1 Introduction

In this paper, let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial \Omega$, and $T > 0$. We study the solvability of the strongly coupled parabolic system

$$
\begin{cases}
W_t = \text{div}(a(W)DW) + \text{div}(\hat{b}(W)) + b(W)DW + g(W) & \text{in } Q = \Omega \times (0,T), \\
\text{Homogeneous Dirichlet or Neumann boundary conditions} & \text{on } \partial \Omega \times (0,T), \\
W = u_0 & \text{on } \Omega \times (0,T).
\end{cases}
$$

Here, $W = [u_i]_{i=1}^m$, a vector in $\mathbb{R}^m$ and $a, b, g$ are square matrices of size $m \times m$ for some $m \geq 2$. The entries of these matrices are functions in $W$. The initial data $u_0$ is a given vector valued function in $W^{1,2,p}(\Omega)$ for some $N > N$. For simplicity we will assume the components of $\hat{b}(W), b(W), g(W)$ have polynomial growths in $W$ throughout this paper although many results in this paper holds under the assumption that they are bounded if $W$ is bounded.

According to the usual definition, a strong solution to (1.1) is a vector valued function $W \in W^{2,2}_{loc}(Q)$ which has bounded derivative $DW$ and solves (1.1) a.e. in $Q$. A weak solution to (1.1) is a vector valued function $W \in V_2(Q)$ (see [8]) which satisfies the integral form of (1.1). That is, for all $\psi \in C^1(Q)$ and any $Q_t = \Omega \times (0,t)$ with $t \in (0,T)$

$$
\int_{\Omega} W_\psi \, dx \bigg|_{t=0}^t - \iint_{Q_t} W_\psi_t \, dz + \iint_{Q_t} a(iDW)D\psi \, dz = -\iint_{Q_t} (\hat{b}D\psi + bDW_\psi + g_\psi) \, dz.
$$

The existence problem of a (unique) strong solution to (1.1) was investigated by Amann. He uses interpolation functional space theory and shows that if the parameters of the regular parabolic (1.1) are bounded and

$$
\sup_{(0,T)} \|W\|_{W^{1,2p}(\Omega)} < M \text{ for some } p > N/2 \text{ and } M
$$

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Mathematics Subject Classifications: 35J70, 35B65, 42B37.

Key words: Cross diffusion systems, Hölder regularity, global existence.

1
then there is a unique strong solution of (1.1). The existence problem of a weak solution to (1.1) was proved easily by (for example) appropriate Galerkin methods in literature. However, the uniqueness problem was largely open if the coefficients of the system depend on $W$.

For nonlinear strongly coupled systems like (1.1) one would start by proving the boundedness of solutions because Amann’s theory worked with bounded $A$. For nonlinear strongly coupled systems like (1.1) this problem would be a very hard one already. The next obstacle is, and even harder, the estimate of higher order norms like (1.2). This problem is closely related to the regularity of parabolic systems.

In this paper, we introduce the concept of a strong-weak solution to (1.1). We say that a vector valued function $W$ is a strong weak solution of (1.1) if $W \in W^{1,p}(Q)$ (whose spatial derivative $DW \in L^p(Q)$) for some $p > N$ and solves (1.1) weakly, i.e. for all $\psi \in C_0^1(Q)$

$$-\int_Q W_t \psi \, dz + \int_Q a DW D\psi \, dz = -\int_Q b D\psi \, dz + \int_Q (bDW + g)\psi \, dz.$$  \hspace{1cm} (1.3)

The temporal derivative $W_t$ could be replaced by the Steklov average of $W$.

Of course, the concept of a strong weak solution is weaker than that of strong solution and stronger than that of weak solutions. The advantage of this definition is that a strong weak solution is unique if it exists.

In this paper, we will establish the existence of a unique strong weak solution under an integral assumption which is weaker than (1.2) but requires some extra structure of conditions on the systems. In particular, the spectral gap condition which requires the eigenvalues of $a$ are not too far apart. In this case, we see that the condition (1.2) of Amann can be replaced by a weaker one

$$\int_{\Omega \times (0,T)} |DW|^{2p} \, dz < M$$  \hspace{1cm} (1.4)

for all strong weak solutions of (1.1). In the process of establishing this, we can assume that strong weak solution are Hölder continuous. We can still assert that the strong solution exists on $\Omega \times (0,T)$.

We always assume that $a$ satisfies the ellipticity condition (this comes from the normal ellipticity of Amann). That is, there are some function $\lambda(W)$ and a positive constant $\lambda_0$ such that $\lambda(W) \geq \lambda_0$ and

$$\langle aDW, DW \rangle \geq \lambda(W)|DW|^2 \hspace{1cm} \forall W \in C^1(\Omega, \mathbb{R}^{mN}).$$ \hspace{1cm} (1.5)

Let $\lambda \leq \Lambda$ be smallest and largest eigenvalues of a square matrix $a$. We look at the ratio $\nu_s = \frac{\lambda}{\Lambda}$. We proved in [9, 11] that if $s > -1$ and $\frac{s}{s+2} < \nu_s$ then (1.5) implies a $c > 0$ such that

$$\langle aDX, D(|X|^sX) \rangle \geq c\lambda|X|^s|DX|^2, \hspace{1cm} \forall X \in C^1(\Omega, \mathbb{R}^{mN}).$$ \hspace{1cm} (1.6)

Accordingly, we will say that the system satisfies a spectral gap condition for some $p > 1/2$ if the above holds for the matrix $a$ and $s = 2p - 2$. That is $1 - 1/p < \nu_s$. In this case, (1.6) becomes

$$\langle aDX, D(|X|^{2p-2}X) \rangle \geq c\lambda|X|^{2p-2}|DX|^2, \hspace{1cm} \forall X \in C^1(\Omega, \mathbb{R}^{mN}).$$
Of course, if $\nu > 1 - 2/N$ then the spectral gap condition holds for some $p > N/2$.

We combine with Amann’s results. For $u_0 \in W^{1,p}(\Omega)$ we can find a unique strong solution $W_*$ which exists in $(0, \delta_0)$ for some $\delta_0 > 0$. We then set up fixed point method in some appropriate space $X$, which will be defined later. Denote $W_0 = W_*(x, \delta_0)$. We define the map $L(\Phi) = W$ where $W$ the unique weak solution solution of the linear problem

\[
\begin{cases}
W_t = \text{div}(a(\Phi)DW) + \text{div}(\hat{b}(\Phi)) + b(\Phi)DW + g(\Phi) & \text{in } \Omega \times (\delta_0, T), \\
W(x, \delta_0) = W_0(x) & \text{on } \Omega, \\
\text{Homogeneous Dirichlet or Neumann conditions} & \text{on } \partial \Omega \times (\delta_0, T).
\end{cases}
\]  

(1.7)

By the well known results of Ladyzhenskaya et al in [8] it is well known that this has a unique weak solution $W$. The coefficients of the system are smooth in $(0, \delta_0)$ so that $W$ is classical. We need to show that $L$ has a unique fixed point in $X$. Gluing this fixed point with the strong solution $W_*$ in $(0, \delta_0)$ we obtain the desired strong weak solution of (1.1).

Note that the so glued solution satisfies (1.3).

According to the well known Leray-Schauder theory, we have to consider solutions of the equation $\tau L(\Phi) = \Phi$ which is equivalent to (writing $w = \Phi$)

\[
\begin{cases}
w_t = \text{div}(a(w)Dw) + \tau \text{div}(\hat{b}(w)) + b(w)Dw + \tau g(w) & \text{in } \Omega \times (\delta_0, T), \\
w(x, 0) = \tau W_0(\delta_0, x) & \text{on } \Omega.
\end{cases}
\]  

(1.8)

In order to apply the Leray-Schauder theorem, we will prove that such $w$ is bounded uniformly for $\tau \in [0, 1]$. There is a technical subtlety in our argument below if we consider (1.8) alone. The estimates, via our parabolic techniques using cutoff functions in $t$, are verified only away from $\delta_0$. To remedy this, we extend $w$ to a solution of system defined on $(0, T)$. The estimates for $w$ on $(\delta_0, T)$ will then be those of this extension away from 0.

The solution $w$ can be extended backward by $\tau W_*$ in $(0, \delta_0)$ (thanks to the initial condition in (1.8), $W_*$ is the strong solution, or fix point of (1.7) in $(0, \delta_0]$) and $w = \tau W_*$ satisfies (a linear system)

\[
\begin{cases}
w_t = \text{div}(a(W_*)Dw) + \tau \text{div}(\hat{b}(W_*)) + b(W_*)Dw + \tau g(W_*) & \text{in } \Omega \times (0, \delta_0), \\
w(x, 0) = \tau W_*(0, x) = \tau u_0(x) & \text{on } \Omega.
\end{cases}
\]  

(1.9)

We now consider the gluing solution which solves (1.8) and (1.9) in $(0, \delta_0)$ and $(\delta_0, T)$ respectively (so that the gluing $w$ now solves (1.9) in $(0, T)$). Obviously $w$ is smooth in $(0, \delta_0)$ so that we can estimate $w$ in $(\delta_0, T)$ (away from $t = 0$) by using cutoff functions in $t$. The coefficients of the system are now nonsmooth and we will be only concerned with $t$ away from 0.

Thus, the main problems are:

1) Define a space $X$ such that $L : X \to X$ is $L$ a compact map.

2) Uniform estimates of solutions to (1.8) in $X$.

These are the main tasks of this paper and we have the following main result on the existence of a unique strong weak solution in $\Omega \times (0, T)$. 

3
Theorem 1.1 Assume that there is \( q > N/2 \) such that \( 1 - 1/q < \nu_\ast \) and a constant \( M \) such that any strong weak solution \( W \) of \((1.13)\) in \( X \) uniformly (also in \( t \in (0, T) \)) satisfies
\[
\liminf_{R \to 0} \| W \|_{\text{BMO}(\Omega_R)} = 0, \tag{1.10}
\]
\[
\| W \|_{L^\infty(Q)} + \int_0^T \| DW \|^2_2 \, dt \leq M. \tag{1.11}
\]
Then \((1.1)\) has a unique strong weak solution \( W \). Moreover, \( W \in L^\infty((0, T), W^{1,2p}(\Omega)) \) if \( 1 - 1/p < \nu_\ast \). In fact, this strong weak solution is a (unique) strong one.

For \( N \leq 3 \), as corollaries of the above theorem, we can relax the conditions of the them by requiring that solutions to \((1.8)\) are uniformly bounded and continuous. If \( N = 2 \) we merely need that these solutions are uniformly bounded. Moreover, in some cases, the boundedness of their \( \text{BMO} \) norms will be sufficient to obtain the same results.

In fact, our system \((1.1)\) is inspired by the following model of two equations \((m = 2)\) in population biology introduced by Shigesada et al. in [19] has been extensively studied in the last few decades
\[
\begin{cases}
  u_t = \Delta(d_1 u + \alpha_{11} u^2 + \alpha_{12} uv) + k_1 u + \beta_{11} u^2 + \beta_{12} uv, \\
v_t = \Delta(d_2 v + \alpha_{21} uv + \alpha_{22} v^2) + k_2 v + \beta_{21} uv + \beta_{22} v^2.
\end{cases} \tag{1.12}
\]
Here, \( d_i, \alpha_{ij}, \beta_{ij} \) and \( k_i \) are constants with \( d_i > 0 \). Dirichlet or Neumann boundary conditions were usually assumed for \((1.12)\). This model (which will be referred to as (SKT) later on) was used to describe the population dynamics of two species densities \( u, v \) which move and react under the influence of population pressures. Under appropriate assumptions on \( d_i, \alpha_{ij}, \beta_{ij} \) and \( k_i \) the global existence of nonnegative solutions with nonnegative data in \( W^{2,2}(\Omega) \) was established by Yagi in [20].

As an extension of these results, we apply our corollaries to a general version of the (SKT) \((1.12)\) which consists of \( m \) equations written compactly as follows
\[
\begin{cases}
  u_t = \Delta(P(u)) + g(u) & \text{in } \Omega \times (0, T_0), \\
  \text{Dirichlet or Neumann boundary conditions} & \text{on } \partial \Omega \times (0, T_0), \\
  u = u_0 & \text{on } \Omega.
\end{cases} \tag{1.13}
\]
Here, \( P(u) = [P_i(u)]_{i=1}^m \) and \( g = [g_i]_{i=1}^m \) whose components \( P_i(u) \)'s and \( g_i \)'s can be appropriate quadratics in \( u \in \mathbb{R}^m \) (so that the ellipticity condition \((1.5)\) is satisfied). This system is the special case of our \((1.1)\) with \( a = P_u \). In this case, it is obvious that \( \sup_{u \in \mathbb{R}^m} \frac{|g_u|^2}{\lambda(u)} \) is bounded. We will prove that our existence results hold if the norm \( \| v \|_{\text{BMO}(\Omega)} \) is small for \( R > 0 \) is sufficiently small for any strong weak solution \( v \). The latter is definitely true for \((1.13)\) on planar domains as we can control \( \| Dv \|_{L^2(\Omega)} \). Thus, the global existence of \((1.13)\) is completely solved in this situation when \( N = 2 \). Thus, the global existence of \((1.13)\) is completely solved in this situation.

In Section 2 we prove the uniqueness of strong weak solutions. For the bounds of fixed points of \( L \) in \( X \), Section 3 devotes to the estimates of the derivatives of a solution of \((1.1)\) via an use of the Gagliardo-Nirenberg inequality involving BMO norms. In Section 4 we will present the proof of Theorem 1.1. Its corollaries and an application to the SKT system \((1.13)\) will be presented in Section 6 and conclude our paper.
2 The uniqueness:

In this section, we prove that a strong weak solution of (1.1) if it exists then it is unique. Let \( W_1, W_2 \) be two weak solutions on \( Q^t = \Omega \times (0, t) \). Set \( W = W_1 - W_2 \). We have

\[
W_t = \text{div}(a(W_1)DW_1 - a(W_2)DW_2) + \text{div}(b(W_1) - b(W_2)) + b(W_1)DW_1 - b(W_2)DW_2 + g(W_1) - g(W_2).
\]

We write

\[
W_t = \text{div}(a(W_1)DW + [a(W_1) - a(W_2)]DW_2) + \text{div}(b(W_1) - b(W_2)) + b(W_1)DW + [b(W_1) - b(W_2)]DW_2 + g(W_1) - g(W_2).
\]

As \( a, b, \hat{b}, g \) are Lipschitz, we test the above with \( W \) to get

\[
\frac{d}{dt} \int_\Omega |W|^2 \, dx + \iint_{Q_t} |DW|^2 \, dz \leq C \iint_{Q_t} (|W||DW_2||DW| + |W||DW| + |W|^2|DW_2| + |W|^2) \, dz.
\]

By Young’s inequality this implies

\[
\frac{d}{dt} \int_\Omega |W|^2 \, dx + \iint_{Q_t} |DW|^2 \, dz \leq C \iint_{Q_t} (|W|^2|DW_2|^2 + |W|^2 + |W|^2|DW_2|) \, dz.
\]

Hölder’s inequality yields

\[
\frac{d}{dt} \int_\Omega |W|^2 \, dx + \iint_{Q_t} |DW|^2 \, dz \leq C \int_0^t \left( \int_\Omega |W|^{2q'} \, dx \right)^{\frac{1}{q'}} \left( \int_\Omega |DW_2|^{2q} \, dx \right)^{\frac{1}{q}} \, dt + C \iint_{Q_t} |W|^2 \, dz.
\]

Because \( q > N/2 \). So that \( 2q' < 2N/(N-2) \), it is well known that for any \( \varepsilon > 0 \) we have \( C(\varepsilon) \) such that

\[
\left( \int_\Omega |W|^{2q'} \, dx \right)^{\frac{1}{q'}} \leq \varepsilon \int_\Omega |DW|^2 \, dx + C(\varepsilon) \int_\Omega |W|^2 \, dx.
\]

Combining with the facts that \( \|DW_2\|_{L^{2q}(\Omega)} \leq M \), we derive for sufficiently small \( \varepsilon \)

\[
\frac{d}{dt} \int_\Omega |W|^2 \, dx + \iint_{Q_t} |DW|^2 \, dz \leq C \iint_{Q_t} |W|^2 \, dz.
\]

This is a Grönwall inequality which implies \( W \equiv 0 \). Thus, \( W_1 \equiv W_2 \).

3 Estimates for derivatives

In this section, we address the (uniform) bound of fixed points of the map \( L \) by an use of the Gagliardo-Nirenberg inequality involving BMO norms.

We first recall the following simple parabolic version of the usual Sobolev inequality...
Lemma 3.1 Let \( r = 2/N \) if \( N > 2 \) and \( r \in (0,1) \) if \( N \leq 2 \). If \( g,G \) are sufficiently smooth then
\[
\int_{\Omega \times I} |g|^{2r} |G|^2 \, dz \leq C \sup_i \left( \int_{\Omega} |g|^2 \, dx \right)^r \left( \iint_{\Omega \times I} (|DG|^2 + |G|^2) \, dz \right).
\]

If \( G = 0 \) on \( \partial \Omega \) then we can drop the integrand \( |G|^2 \) on the right hand side. In particular, if \( g = G \) we have
\[
\int_{\Omega \times I} |g|^{2(1+r)} \, dz \leq C \sup_i \left( \int_{\Omega} |g|^2 \, dx \right)^r \left( \iint_{\Omega \times I} (|Dg|^2 + |g|^2) \, dz \right).
\]

In the sequel, for \( h \neq 0 \) we will use the difference operator of a function \( u = [u_i]_{i=1}^m \)
\[
\delta_h u(x) = h^{-1} [u(x + he_1) - u(x), \ldots, u(x + he_m) - u(x)].
\]

Lemma 3.2 Let \( v \) be a strong-weak solution of \((1.8)\) on \( \Omega \times (0,T) \). Assume that \( v \) satisfies uniformly for any \( x_0,t \in Q = \Omega \times (\delta_0/2,T) \) that
\[
\liminf_{R \to 0} \|v\|_{\text{BMO}(\Omega_R)} = 0, \quad (3.1)
\]
and
\[
\|Dv\|_{L^{2q+2}(Q)} < \infty \quad (3.2)
\]
for some \( q > N/2 \).

Then for some \( R_0 \) small (depending on the continuity of \( v \) in \((3.1)\)) and any \( p \) such that \( 2p > 1 \) and \( \nu_* > 1 - 1/p \) (that is the spectral gap condition holds for \( p \)) then
\[
\sup_{t \in (\delta_0/2,T)} \int_{\Omega} |Dv|^{2p} \, dx + \iint_{Q} |Dv|^{2p-2} |D^2v|^2 \, dz \leq C \left( \int_{\Omega_0 - \delta_0/2} |Dv|^{2q} \, dz + \iint_{\Omega \times (\delta_0/2)} |Dv|^{2p} \, dx \right). \quad (3.3)
\]

In addition (one should note the exponent \( q \) on the right hand side),
\[
\|Dv\|_{L^{2p\gamma}(Q)} \leq C \left( \int_{\Omega_0 - \delta_0/2} |Dv|^{2q} \, dz, \iint_{\Omega \times (\delta_0/2)} |Dv|^{2p} \, dx \right).
\]

Proof: We need only to look at the case \( \tau = 1 \) as the argument is similar for \( \tau \in [0,1] \).
Apply \( \delta_h \) to the equation of \( v \) to see that \( v \) weakly solves
\[
(\delta_h v)_t = \text{div}(a(D(\delta_h v) + a_v(\delta_h v, Dv))) + \text{div}(b(\delta_h v, Dv)) + b_v(\delta_h v, Dv) + bD(\delta_h v) + g_v(\delta_h v).
\]

Since the parameters \( a, b, b \) and \( g \) of the equation are bounded (see Remark 3.3), for any \( 0 < s < t < 2R_0 \) we test this system with \( |\delta_h v|^{2p-2}\delta_h v \phi^2 \eta \), with \( p \geq 1 \) and \( \phi, \eta \) being positive \( C^1 \) cutoff functions for the concentric balls \( B_s, B_t \) and the time interval \( I \), and use Young’s inequality for the term \( |a_v||\delta_h v|^{2p-2}|Dv|D(\delta_h v)| \) and the spectral gap condition...
(with $X = \delta_h v$) to get (we refer to [14] for details). We see that for some constant $C$ and $Q = \Omega_t \times (\delta_0/2, T)$

$$\sup_{t \in (\delta_0/2, T)} \int_{\Omega} |\delta_h v|^{2p} \phi^2 \, dx + \int_{Q} |\delta_h v|^{2p-2} |D(\delta_h v)|^2 \phi^2 \, dz \leq C \int_{Q} |\delta_h v|^{2p} |Dv|^2 \phi^2 \, dz$$

$$+ C \int_{Q} (|\delta_h v|^{2p} + |Dv|^{2p})(\phi^2 + |\delta_h \phi|^2 + |D\phi|^2) \, dz + \int_{\Omega \times \{\delta_0/2\}} |\delta_h v|^{2p} \phi^2 \, dx.$$ 

Another use of Young’s inequality for the first term on the right hand side yields

$$\sup_{t \in (\delta_0/2, T)} \int_{\Omega} |\delta_h v|^{2p} \phi^2 \, dx + \int_{Q} |\delta_h v|^{2p-2} |D(\delta_h v)|^2 \phi^2 \, dz \leq C \int_{Q} |\delta_h v|^{2p+2} \phi^2 \, dz$$

$$+ C \int_{Q} |Dv|^{2p+2} \phi^2 \, dz + C \int_{Q} (|\delta_h v|^{2p} + |Dv|^{2p})(\phi^2 + |\delta_h \phi|^2 + |D\phi|^2) \, dz + \int_{\Omega \times \{\delta_0/2\}} |\delta_h v|^{2p} \phi^2 \, dx.$$ 

For any $p \geq 1$ such that (this is true if $p = q$ because of (3.2))

$$\int_{Q} |Dv|^{2p+2} \, dz < \infty$$

we let $h \to 0$ and see that

$$\sup_{t \in (\delta_0/2, T)} \int_{\Omega} |Dv|^{2p} \phi^2 \, dx + \int_{Q} |Dv|^{2p-2} |Dv|^2 \phi^2 \, dz \leq$$

$$C \int_{Q} |Dv|^{2p+2} \phi^2 \, dz + C \int_{Q} |Dv|^{2p}(\phi^2 + |D\phi|^2) \, dz + \int_{\Omega \times \{\delta_0/2\}} |Dv|^{2p} \phi^2 \, dx.$$ 

(3.5)

The key issue here is that we will have to handle the integral of $|Dv|^{2p+2}$. For any $t > 0$ we write $\Omega_t = \Omega \cap B_t$, $Q_t = \Omega_t \times (\delta_0/2, T)$ and

$$A_p(t) = \sup_{(\delta_0/2, T)} \int_{\Omega_t} |Dv|^{2p} \, dx, \quad B_p(t) = \int_{Q_t} |Dv|^{2p+2} \, dz, \quad f(t) = \int_{\Omega_t \times \{\delta_0/2\}} |Dv|^{2p} \, dx,$$

$$H_p(t) = \int_{Q_t} |Dv|^{2p-2} |Dv|^2 \, dz, \quad G_p(t) = \int_{Q_t} |Dv|^{2p} \, dz.$$

By the local Gagliardo-Nirenberg inequality ([11] Lemma 2.4) with $u = U$, $\Phi \equiv 1$ and $\psi$ is a cutoff function for $B_s, B_t$ for any $\varepsilon > 0$ and some constant $C = C(\varepsilon, N)$ we have that (in this paper we will refer to this as the Gagliardo-Nirenberg BMO inequality)

$$\int_{\Omega_t} |Dv|^{2p+2} \, dx \leq \varepsilon \int_{\Omega_t} |Dv|^{2p+2} \, dx +$$

$$+ C \int_{Q_t} |Dv|^{2p-2} |Dv|^2 \, dz + C \frac{||v||^2_{BMO(\Omega_t)}}{(t-s)^2} \int_{\Omega_t} |Dv|^{2p} \, dx.$$

Because of (3.4) and (3.5), the terms in this inequality are all finite and it holds for a.e. $t$. Using this inequality in (3.5), where $\phi$ is a cut-off function for $B_s, B_t$ and by the assumption
on the uniform continuity of \( v \), \( \|v\|_{BMO(\Omega)} \) can be very small. For \( 0 < s < t < R_0 \) with sufficiently small \( R_0 \) depending on the uniform continuity of \( v \) we obtain the following recursive system of inequalities

\[
A_p(s) + H_p(s) \leq CB_p(t) + \frac{C}{(t-s)^2}G_p(t) + f(t),
\]

\[
B_p(s) \leq \varepsilon (H_p(t) + B_p(t)) + \frac{C}{(t-s)^2}G_p(t).
\]

If \( \varepsilon \) (or \( R_0 \)) is small then we can iterate this to absorb the terms \( H_p, B_p \) on the right hand side to the left hand side to get (see [11] inequality (3.27), proof of Proposition 3.1)

\[
A_p(R_0) + H_p(R_0) \leq \frac{C}{R_0^2}G_p(2R_0) + f(R_0).
\] (3.6)

This gives the estimate of \( A_p, H_p \). That is

\[
\sup_{(t_0/2,T)} \int_{\Omega} |Dv|^{2p} \, dx + \int_{Q_{t_0}} |Dv|^{2p-2} |D^2v|^2 \, dz \leq CR_0^{-2} \int_{Q_{2R_0}} |Dv|^{2p} \, dz + \int_{\Omega} |Dv|^{2p} \, dx.
\]

This also holds when \( B_{R_0} \) intersects the boundary \( \partial \Omega \) see [12] Remarks 3.3.5 and 3.3.6. Fixing such \( R_0 \) and covering \( \Omega \) with balls of radius \( R_0 \) and summing the above inequality over this partition, we derive a global estimate

\[
\sup_{(t_0/2,T)} \int_{\Omega} |Dv|^{2p} \, dx + \int_{\Omega \times (t_0,T)} |Dv|^{2p-2} |D^2v|^2 \, dz \leq CR_0^{-2} \int_{\Omega \times (t_0,T)} |Dv|^{2p} \, dz + \int_{\Omega \times \{t_0/2\}} |Dv|^{2p} \, dx.
\] (3.7)

Dropping the variable \( R_0 \) and \( W_0 \) (it is the value of a classical solution), we define \( A_p, H_p, B_p \) in the same ways with \( \Omega_{R_0} = \Omega \). By the parabolic Sobolev inequality in Lemma 3.1 with \( g = G = Dv \), (3.6) also shows that its right hand side is self-improved. That is if \( G_p \) is finite for some \( p \geq 1 \) then so are \( A_p, H_p \). We have

\[
\|Dv\|_{L^{2p\gamma}(Q)} \leq C \left( R_0^{-2}, \int_Q |Dv|^{2p} \, dz, \int_{\Omega \times \{t_0/2\}} |Dv|^{2p} \, dx \right),
\]

where \( \gamma = 1 + 2/N \) if \( N \geq 3 \) and \( \gamma = 1 + r \) for any \( r \in (0,1) \) if \( N = 2 \). Thus, (3.4) holds again with \( 2p + 2 \) is now \( 2p\gamma \). This also establishes our last assertion of the lemma (keep in mind that the values of \( v \) in \( (0,\delta_0) \) are those of the strong solution).

This argument can be repeated with \( p \) being replaced by \( \tau p \) for \( \tau > 1 \) as long as \( 2\tau p + 2 \leq \gamma 2p \) (see (3.4)). Define \( p_{n+1} = \tau_n p_n \) with \( \tau_n = \gamma - 1/p_n \). We see that \( 2p_{n+1} + 2 = 2p_n \gamma \). Moreover, \( \tau_n > 1 \Leftrightarrow p_n > N/2 \Leftrightarrow p_n \uparrow \). Thus, if we take \( p_1 \) to be the number \( q > N \) in (3.2) of this lemma then such sequences \( \{p_n\}, \{\tau_n\} \) exist and the iterate the argument as long as \( \nu_* > 1 - 1/p_n \).
Along the sequence \( \{p_n\} \) we have (if the spectral gap condition holds for \( p_n \))

\[
\sup_{(\delta_0 / 2, T)} \int_{\Omega} |Dv|^{2p_n} \, dx + \iint_{Q} |Dv|^{2p_n-2} |D^2v|^2 \, dz \leq C \left( n, R_0^{-2}, \iint_{Q} |Dv|^{2p_1} \, dz, \int_{\Omega \times \{\delta_0 / 2\}} |Dv|^{2p} \, dx \right).
\]

As \( p_n \to \infty \) since \( \tau_n > 1 \) and \( p_1 = q \), we then have (3.3).

**Remark 3.3** We make use of the polynomial growth of \( b, g \) in deriving (3.5).

### 4 The proof of the main result

In this section, we will prove the existence of fixed points of \( L \) in an appropriate defined space \( X \) via the Leray-Schauder theory.

First of all, we address the compactness of the operator \( L \) that leads to the definition later.

#### 4.1 A compactness lemma

The main issues is to show \( L(K) \) of a bounded set \( K \) of \( X \) is precompact. The continuity of \( L \) follows in a standard way. Similar to [9, Lemma 3.3], we will establish the following compactness result which will serve the purpose.

**Lemma 4.1** Let \( F \) be a collection of function \( w \) such that \( D^2w \) exists and satisfies

\[
\iint_{Q} D w \psi_t \, dz \leq C \iint_{Q} (|D^2w||D\psi| + |Dw||\psi|) \, dz \quad \forall \psi \in C^1(Q) \tag{4.1}
\]

and \( \psi(\cdot, 0) \equiv \psi(\cdot, T) \equiv 0 \). Suppose that for some constants \( M, p \geq 1 \) and for all \( w \in F \)

\[
sup_{t \in (0, T)} \int_{\Omega} |Dw|^{2p} \, dx + \iint_{Q} |Dw|^{2p-2} |D^2w|^2 \, dz < M, \tag{4.2}
\]

\[
\iint_{Q_{s,t}} |D^2w|^2 \, dz = O(t - s), \tag{4.3}
\]

where \( Q_{s,t} = \Omega \times (s, t) \). Then

i) \( \{Dw : w \in F\} \) is compactly embedded in \( L^{2p}((0, T), L^{2p}(\Omega)) \).

ii) Let \( \gamma_0 = 1 + 2 / N \). If \( \gamma \in (1, \gamma_0) \) then \( \{Dw : w \in F\} \) is compactly embedded in \( L^{2p}\gamma((0, T), L^{2p\gamma}(\Omega)) \).
Proof: For each \( w \in \mathcal{F} \) denote \( v = |Dw|^{p-1} Dw \). We first show that for \( l > (N + 2)/2 \)
\[
\|v(\cdot, t + h) - v(\cdot, t)\|_{W^{-1,2}(\Omega)} \leq C[M \varepsilon_0 + C(\varepsilon_0)O(h)] \quad \forall \varepsilon_0 > 0. \tag{4.4}
\]

For any \( Q_{s,t} = \Omega \times (s, t) \) let \( h = t - s \) and \( \psi(x, t) = |Dw|^{p-2} Dw \phi(x) \) in (4.1) with \( \phi \in C^1(\Omega) \) and \( \eta \equiv 1 \) in \( (s, r) \) and \( \eta \equiv 0 \) outside \( (s - \varepsilon, r + \varepsilon) \). From (4.1), we get
\[
\left| \iint_{Q_{s,t}} v \psi_t \, dz \right| \leq C \iint_{Q_{s,t}} |\eta|(|D^2 w|^2 |Dw|^{p-2} \phi + |D^2 w||Dw|^{p-1} D \phi| + |Dw|^p|\phi|) \, dz.
\]

We estimate the right hand side. By Young’s inequality, for every \( \varepsilon_0 > 0 \) we have
\[
\iint_{Q_{s,t}} \eta |D^2 w|^2 |Dw|^{p-2} \, dz \leq \iint_{Q_{s,t}} |D^2 w|^2 (\varepsilon_0 |Dw|^{p-2} + C(\varepsilon_0)) \, dz \|\phi\|_{C^1(\Omega)} ,
\]
\[
\iint_{Q_{s,t}} \eta |D^2 w||Dw|^{p-1} D \phi \, dz \leq |h|^{\frac{1}{2}} \left( \iint_{Q_{s,t}} |D^2 w|^2 |Dw|^{p-2} \, dz \right)^{\frac{1}{2}} \|\phi\|_{C^1(\Omega)} ,
\]
\[
\iint_{Q_{s,t}} \eta |Dw|^p \phi \, dz \leq |h|^{\frac{1}{2}} \left( \iint_{Q_{s,t}} |Dw|^{2p} \, dz \right)^{\frac{1}{2}} \|\phi\|_{C^1(\Omega)} .
\]

Because \( l > (N + 2)/2, \|\phi\|_{C^1(\Omega)} \leq C\|\phi\|_{W^{1,2}(\Omega)}, \) by the assumptions (4.2) and (4.3) we see that the above inequalities together imply
\[
\left| \iint_{Q_{s,t}} v \psi_t \, dz \right| \leq C[M \varepsilon_0 + C(\varepsilon_0)O(h)]\|\phi\|_{W^{1,2}(\Omega)} \quad \forall \phi \in W^{1,2}(\Omega). \tag{4.5}
\]

Letting \( \varepsilon \to 0 \), we also have for all \( \phi \in W^{1,2}(\Omega) \) (compare with [? , Lemma 3.2])
\[
\left| \int_{\Omega} [v(\cdot, s) - v(\cdot, t)] \phi \, dx \right| = \left| \iint_{Q_{s,t}} v \psi \, dz \right| \leq C[M \varepsilon_0 + C(\varepsilon_0)O(h)]\|\phi\|_{W^{1,2}(\Omega)} .
\]

Hence (4.4) follows. We now interpolate \( L^2(\Omega) \cap W^{1,2}(\Omega) \) between \( W^{1,2}(\Omega) \) and \( W^{-1,2}(\Omega) \) to get for any \( \mu > 0 \) and \( v \in L^2(\Omega) \cap W^{1,2}(\Omega) \) that
\[
\|v(\cdot, t + h) - v(\cdot, t)\|_{L^2(\Omega)} \leq \mu \|v\|^2_{W^{1,2}(\Omega)} + C(\mu)\|v(\cdot, t + h) - v(\cdot, t)\|^2_{W^{-1,2}(\Omega)}
\]

Since \( \int_0^T \|v\|^2_{W^{1,2}(\Omega)} \, dt \leq C(M) \) if \( v \in \mathcal{F} \), for any given \( \varepsilon > 0 \), we can choose \( \mu \) small first and then \( \varepsilon_0, h \) small such that by (4.4)
\[
\int_0^{T-h} \|v(\cdot, t + h) - v(\cdot, t)\|^2_{L^2(\Omega)} \, dt < \varepsilon \quad \forall v \in \mathcal{F} .
\]

Thus, we just prove that
\[
\int_0^{T-h} \|v(\cdot, t + h) - v(\cdot, t)\|^2_{L^2(\Omega)} \, dt \leq O(h).
\]
For any $t_1, t_2 \in (0, T)$ and $v \in \mathcal{F}$, the fact that the collection of $\int_{t_1}^{t_2} v dt$ is a pre-compact set of $L^2(\Omega)$ is clear because the set
\[
\left\{ \frac{1}{|t_2 - t_1|} \int_{t_1}^{t_2} v dt : \int_\Omega \Phi(|v|^2 + |Dv|^2) dx dt < M \right\}
\]
belongs to the closure of the convex hull in $L^2(\Omega)$, a bounded set in $W^{1,2}(\Omega)$ as $\Phi \geq \lambda_0 > 0$, which is compact in $L^2(\Omega)$.

By a result of Simon [9] as in [9, Lemma 3.3] (setting $B = L^2(\Omega)$), we proved the compactness of $\{v : w \in \mathcal{F}\}$ in $L^2(0, T, L^2(\Omega))$.

By the definition of $v$, the set $\{|Dw|^{p-1} Dw : w \in \mathcal{F}\}$ is compact in $L^2(0, T, L^2(\Omega))$. It is well known that if a sequence $\{f_n\}$ converges in $L^1(\Omega)$ then $\{f_n\}$ converges in $L^{2p}(\Omega)$ (by dominating convergence theorem). This implies that $\{Dw : w \in \mathcal{F}\}$ is compact in $L^{2p}(0, T, L^{2p}(\Omega))$. This gives i).

Finally, if $\gamma \in (1, \gamma_0)$ then there are $\alpha, \beta \in (0, 1)$ such that $2p\gamma = 2p\alpha + 2p\gamma_0\beta$. Let $\{Dw_n\}$ with $w_n \in \mathcal{F}$ be a bounded set satisfying (4.2) and (4.3). By i) and relabeling, we can assume $\{Dw_n\}$ is convergent sequence in $L^{2p}(0, T, L^{2p}(\Omega))$. We write $W_{n,m} = w_n - w_m$.

By Hölder’s inequality
\[
\int_Q |DW_{n,m}|^{2p\gamma} d\gamma \leq \left( \int_Q |DW_{n,m}|^{2p} d\gamma \right)^{\frac{\alpha}{\beta}} \left( \int_Q (|DW_{n,m}|)^{2p\gamma_0} d\gamma \right)^{\frac{1}{\beta}} \leq \left( \int_Q |DW_{n,m}|^{2p} d\gamma \right)^{\frac{\alpha}{\beta}} \left( \int_Q (|Dw_n| + |Dw_m|)^{2p\gamma_0} d\gamma \right)^{\frac{1}{\beta}}.
\]
The first factor on the right goes to zero because $\{DW_{n,m}\}$ converges to 0 in $L^{2p}(0, T, L^{2p}(\Omega))$.

By (4.2) and the parabolic Sobolev inequality in Lemma 3.1 with $|g| = |G| = |Dw_n|^p$, $\{Dw_n\}$ and $\{Dw_m\}$ are bounded uniformly in $L^{2p\gamma_0}(0, T, L^{2p\gamma_0}(\Omega))$. The second factor is bounded. Thus $\mathcal{F}$ is compact in $L^{2p\gamma}(0, T, L^{2p\gamma}(\Omega))$. This gives ii). The proof is complete.

**4.2 The space $X$ and $\mathbb{L} : X \to X$ is compact**

Next, we will define the space $X$ such that $\mathbb{L} : X \to X$ is a compact. By the theory of [11] the linear parabolic system defining $\mathbb{L}(\Phi)$ has sufficient smooth coefficients so that it has a classical solution. However, in what below we will need some uniform estimates to show that $\mathbb{L} : X \to X$ is a compact map. That is $\mathbb{L}(K)$ is compact in $X$ if $K \subset X$ is a bounded set.

Apply $\delta_h$ to the equation of $W$ to see that
\[
(\delta_h W)_t = \text{div}(a(\Phi)D(\delta_h W) + a(\Phi)\langle \delta_h \Phi, DW \rangle) + \text{div}(b(\Phi)\delta_h \Phi) + b(\Phi)\delta_h W + b(\Phi)\delta_h W + g(\delta_h \Phi). \tag{4.6}
\]
Assume that the condition GS holds for $q$. That is $\nu_* > 1 - \frac{1}{q}$. We test this system with
\[ |\delta h W|^{2q-2} \delta h W \] and let \( h \to 0 \) to have
\[
\begin{aligned}
&\int_{\Omega} |DW|^{2q} \, dx + \iint_{Q} |DW|^{2q-2} |D^2 W|^2 \, dz \leq \\
&\quad C \iint_{Q} (|D\Phi||DW||DW|^{2q-2}|D^2 W| + |D\Phi||DW|^{2q-2}|D^2 W|) \, dz + \\
&\quad + C \iint_{Q} (|D\Phi||DW|^{2q-1} + |DW|^{2q-1}|D^2 W| + |D\Phi||DW|^{2q-1}) \, dz.
\end{aligned}
\]

Applying Young's inequality
\[
\begin{aligned}
&\int_{\Omega} |DW|^{2q} \, dx + \iint_{Q} |DW|^{2q-2} |D^2 W|^2 \, dz \leq \\
&\quad C \iint_{Q} (|D\Phi|^2|DW|^{2q} + |D\Phi|^{2q} + |DW|^{2q}) \, dz.
\end{aligned}
\] (4.7)

Being inspired by the above calculations, Lemma 3.2, the compactness result and Lemma 4.4 below, we introduce the space \( X \) here such the right hand side of (4.7) is finite so that \( W \in X \) (by letting \( h \to 0 \)).

Assume that for some \( q_0 > N/2 \) with \( \nu_* > 1 - \frac{1}{q_0} \). Let \( \gamma \in (1, 1 + 2/N) \) be such that \( q_0 (\gamma - 1) > 1 \).

Since \( q_0 (\gamma - 1) > 1 \), we see that \( \gamma - 1/q_0 > 1 \) so that we can fix a number \( \alpha \in (0, 1) \) such that \( 1 < \alpha \gamma \leq \gamma - 1/q_0 \). That is \( \alpha \gamma > 1 \) and \( 1/\alpha \gamma \leq 2 q_0 \gamma \). We summarize the choices of \( q_0, \gamma, \alpha \) below
\[
\nu_* > 1 - \frac{1}{q_0}, \quad \alpha \gamma > 1, \quad \frac{2}{1 - \alpha} \leq 2 q_0 \gamma.
\] (4.8)

For some \( \varepsilon_0 > 0 \) define \( X = \{ \Phi : D\Phi \in L^{2q_0 \gamma}(Q), \Phi \in C^{0, \varepsilon_0}(Q) \} \) with norm
\[
\|\Phi\|_X = \|\Phi\|_{C^{0, \varepsilon_0}(Q)} + \|D\Phi\|_{L^{2q_0 \gamma}(Q)}.
\]

Of course, we can choose \( \gamma \sim 1 + 2/N \) and \( \alpha \sim 1/\gamma \) so that \( q_0 \sim 1/(\gamma - 1) \sim N/2 \).

To begin, we have the following simple result.

**Lemma 4.2** If \( \Phi \in X \) then \( W = L(\Phi) \in W^{2,1}(Q) \) and we have the following estimate
\[
\|W\|_{2,Q}^{(2)} \leq C(\|\Phi\|_{2,Q}^{(1)} + \|\Phi\|_{2,Q}^{(1)}).
\]

Here, \( \|W\|_{2,Q}^{(l)} = \sum_{2i+j=l} \|D_i D_j W\|_{L^2(Q)} \) and the constant \( C \) depends on \( \|\Phi\|_{L^\infty(Q)} \).

**Proof:** We can rewrite the system (1.8) as \( L(W) = f \) with
\[
L(W) = W_t - aD^2 W - [a_\Phi(\Phi)D\Phi + b(\Phi)]DW \quad \text{and} \quad f = \text{div}(\tilde{b}(\Phi)) + g(\Phi).
\]

The assertion is a simple consequence of [8, Theorem 9.1] which can be extended to linear systems with smooth coefficients by a similar study of fundamental solutions for systems. We check the conditions of [8, Theorem 9.1]. By the definition of \( X \), \( a \) is continuous on \( Q \) and bounded. Moreover, because \( q_0 > N/2 \) and \( \gamma \in (1, 1 + 2/N) \), we can choose \( \gamma \) such
that \( r = 2q_0 \gamma > N + 2 \) and \( a(\Phi)D\Phi + b(\Phi) \in L^1_{r,Q} \). We also have \( g(\Phi) \in L^1_{s,Q} \) for any \( s \in (1, \infty) \). Thus, the assumption \( [S] \ (7.1) \) of Theorem 9.1] is verified.

Next, since the initial value \( W_0 \) is the value of \( W_s(\delta_0/2) \) of the classical solution \( W_s \), the compatibility condition between the initial and boundary data \( (\phi = W_0 \) and \( \Phi = 0 \) in \( [S] \)) of \( [S] \ (9.2) \) holds.

Thus, with \( q = 2 \), we have from \( [S] \) Theorem 9.1] that there is a constant \( C \) depending on \( \|\Phi\|_{L^\infty(Q)} \) such that for \( \|f\|_{L^2(Q)} \)

\[
\|W\|_{L^2(Q)}^{(2)} \leq C(\|f\|_{L^2(Q)} + \|W_0\|_{L^2(Q)}),
\]

From the definition of \( f \), this completes the proof.  

**Remark 4.3** Moreover, if \( \Phi \in X \) then \( f \in L^{2q_0\gamma}(Q) \) with \( 2q_0\gamma > (N + 2)/2 \). By \( [S] \) Theorem 2.1 of Chapter VII] and the system defines \( W \) has continuous bounded coefficients, we see that \( \|W\|_{L^\infty(Q)} \) is bounded.

We now apply the above argument to show that

**Lemma 4.4** Assume \( (4.3) \). If \( \epsilon_0 > 0 \) is sufficiently small then \( \mathbb{L} : X \to X \) is compact. Moreover, \( \mathbb{L}(\Phi) \in W^{1,2q}(\Omega) \).

**Proof:** Let \( W = \mathbb{L}(\Phi) \). For some fixed \( R_0 > 0 \) and \( Q_{R_0} = \Omega_{R_0} \times (0, T) \) By the Gagliardo-Nirenberg BMO inequality, we have

\[
\int_{Q_{R_0}} |DW|^4 \, dz \leq C\|W\|_{L^\infty(Q)}^2 \int_{Q_{2R_0}} |D^2W|^2 \, dz + C(R_0^{-2})\|W\|_{L^\infty(Q)}^2 \int_{Q_{2R_0}} |DW|^2 \, dz.
\]

From Remark 4.3 if \( \Phi \in X \) then \( W \) is bounded in terms of \( \|\Phi\|_X \). So that by summing over a finite covering of \( \Omega \) by balls of radius \( R_0 \), we obtain

\[
\int_Q |DW|^4 \, dz \leq C(R_0, \|\Phi\|_X) \int_Q |D^2W|^2 \, dz + C \int_Q |DW|^2 \, dz.
\]

By testing the system with \( W \), we easily see that \( \|DW\|_{L^2(Q)} \) is bounded in terms of \( \|\Phi\|_X \). Also, by Lemma 4.2 \( \|D^2W\|_{L^2(Q)} \) is bounded in terms of \( \|\Phi\|_X \) We conclude that \( \|DW\|_{L^4(Q)} \) is bounded in terms of \( \|\Phi\|_X \). That is, for some any \( \gamma \in (1, 1 + 2/N) \)

\[
\int_Q |DW|^{2\gamma} \, dz < \infty.
\]

Let \( q_1 = 1 \). For some \( \alpha \in (0, 1) \) such that \( \alpha \gamma > 1 \) we define \( q_{i+1} = q_i \alpha \gamma \). Let \( q = q_{i+1} \) in \( (17) \). If the condition GS holds for \( s = 2q_{i+1} - 2 \) (that is \( \nu > 1 - 1/q_{i+1} \)) then

\[
\int_\Omega |DW|^{2q_{i+1}} \, dx + \int_Q |DW|^{2q_{i+1} - 2}|D^2W|^2 \, dz \leq \int_Q (|D\Phi|^2|DW|^{2q_{i+1}} + 1) \, dz.
\]

13
If $DW \in L^{2q_i \gamma}(Q_i)$, $i \geq 1$, we apply Young’s inequality to the term $|D\Phi|^2|DW|^{2q_i+1}$ on the right hand side to obtain

$$
\int_{\Omega} |DW|^{2q_i+1} \, dx + \int_{Q} |DW|^{2q_i+2} |D^2W| \, dz \leq \int_{Q} (|D\Phi|^{\frac{2}{\alpha}} + |DW|^{\frac{2q_i+1}{\alpha}} + 1) \, dz.
$$

(4.9)

Because $2q_0 \gamma \geq \frac{2}{1-\alpha}$, $D\Phi \in L^{\frac{2}{1-\alpha}}(Q_i)$. Also, since $DW \in L^{2q_i \gamma}(Q_i)$ and $2q_i \gamma = \frac{2q_i+1}{\alpha}$, so that the above quantities are all finite. Again, by parabolic Sobolev inequality, we have $DW \in L^{2q_i+1 \gamma}(Q_i)$. We can repeat the argument as it is true for $q_l$ to see that $DW \in L^{2q_i+1 \gamma}(Q_i)$ for all $i \geq 1$ (as long as the spectral gap condition holds for $q_i$). Note that $q_i \to \infty$, because $\alpha \gamma > 1$.

Under the assumption that the condition GS that $\nu_\ast > 1 - 1/q_i$, which holds for any exponent $q_i$ such that $q_i \leq q_0$ because of (4.8), we have $DW \in L^{2q_i \gamma}(Q_i)$ by (4.9) and the parabolic Sobolev inequality. Thus, $L(\Phi) \in \mathcal{X}$ (choose $i$ such that $q_i = q_0$).

Now, if $\Psi \in \mathcal{X}$ but, by applying the Hölder or Young inequalities and if $||\Phi||_X$ is bounded, we can see that we still get (4.5) in the proof of Lemma 4.1 which is obtained from the assumption (4.1), namely

$$
\left| \iint_{Q} DW \psi_t \, dz \right| \leq C \iint_{Q} (|D^2W||D\Psi| + |DW||D\Phi||D\Psi| + (|D\Phi| + 1)|DW|\psi_t)) \, dz.
$$

There are some extra terms involving $\Phi \in \mathcal{X}$ but, by applying the Hölder or Young inequalities and if $||\Phi||_X$ is bounded, we can see that we still get (4.5) in the proof of Lemma 4.1 which is obtained from the assumption (4.1), namely

$$
\left| \iint_{Q} DW \psi_t \, dz \right| \leq C \iint_{Q} (|D^2W||D\Psi| + |DW|\psi_t)) \, dz \quad \forall \psi \in C^1(Q).
$$

Thus, Lemma 4.1 is applicable here.

Moreover, we see that $D^2W \in L^2(Q)$ by letting $h \to 0$ in (4.6). If $\Phi$ is in a bounded set of $X$ then $||D^2W||_{L^2(Q)}$ is uniform bounded. From the above argument, if $q_0 > N/2$ then for some $q \in (N/2, q_0)$ we also see that $||D^2W||_{L^2(\Omega)}$ is uniformly bounded. So, $W$ is Hölder continuous in $x$, as $W \in W^{1,2q}(\Omega)$ with $2q > N$. Since $||\Phi||_{L^\infty(\Omega)}$, $||D^2W||_{L^2(Q)}$ and $||D\Phi||_{L^2(Q)}$ and $||DW||_{L^2(\Omega)}$ are uniformly bounded, by (4.6) we can solve for $W_t$ and see that $W_t \in L^{2r}(Q)$ with uniform bounded norm for some $r \in (0, 1)$. We conclude that $W$ is Hölder continuous in $x, t$ (see [17, Lemma 4]). Thus, $L$ is also compact in the space $C^{0,\tilde{\varepsilon}_0}(Q)$ if $\tilde{\varepsilon}_0$ is sufficiently small.

Hence, if $q_0 > N/2$ and $\varepsilon_0$ small then $L : X \to X$ is compact and $L(\Phi) = W \in W^{1,q_0}(\Omega)$. The proof is complete. ■

**Remark 4.5** If $W \in X$ then $DW \in L^{2q\gamma}(Q)$ for $q \leq q_0$. If $q \in (N/2, q_0)$ we also have $DW \in L^{2q+2}(Q)$. Indeed, we have $2q + 2 < 2q\gamma$ because this is equivalent to $q > N/2$ for some $\gamma \in (1, 1 + 2/N)$ (or $\gamma \in (1, 2)$ if $N = 2$).
Remark 4.6 The higher integrability of $DW$ for $\Phi \in X$ (in particular $DW \in L^4(Q)$, or the results in [7] which can be extended to the parabolic boundary of $Q$ but the number $q_0$ must be redefined in order that the gap condition to be satisfied) is crucial because we could have not started the iteration argument (starting with (4.7) to get $q_1 = 1$) without it. On the other hand, if we assumed $D\Phi \in L^\infty(Q)$ but our argument could not provides $DW \in L^\infty(Q)$.

4.3 Proof of the main theorem

We see now that Theorem 1.1 is proved if we can establish Lemma 4.7

Assume that (4.8) holds and $q_0 > N/2$. If there is $q \in (N/2, q_0)$ and $M$ such that any strong weak solution $W$ of (1.8) in $X$ uniformly (also in $t$) satisfies

$$\liminf_{R \to 0} \|W\|_{BMO(\Omega R)} = 0,$$

(4.10)

$$\|W\|_{L^\infty(Q)} + \iint_Q |DW|^{2q} \, dz \leq M.$$  

(4.11)

Then (1.1) has a strong weak solution in $X$.

Proof of Theorem 1.1 or Lemma 4.7: Thanks to Lemma 4.4, $L : X \to X$ is a compact map. Let $W$ be a strong weak solution of (1.8) in $X$. From the proof of Lemma 4.4 we see that $W \in W^{1,2q}(\Omega)$ and $W \in W^{1,2q+2}(Q)$ for any $q \in (N/2, q_0]$ because $2q + 2 \leq 2q\gamma$ (see Remark 4.5). Hence $W$ is bounded and Hölder continuous because $W \in W^{1,2q}(\Omega)$ and $2q > N$ (but this continuity may not be uniform among such $W$ so that we can apply Lemma 3.2 but the obtained estimates are not uniform because the number $R_0$ is not fixed).

By the assumption of the Theorem there is $q, M$ such that $q > N/2$ and

$$\iint_Q |DW|^{2q} \, dz < M$$

and (4.10) holds uniformly. So that for some uniform small $R_0$, Lemma 3.2 yields that

$$\int_{\Omega} |DW|^{2p} \, dx + \iint_Q |DW|^{2p-2} |D^2W|^2 \, dz \leq C \left( p, R_0^{-2}, \int_{\Omega} |DW|^{2q} \, dz, \int_{\Omega} |DW_0|^{2p} \, dx \right)$$

for any $p$ such that $\nu_\ast > 1 - 1/p$. This condition holds for $p = q_0$ because of (4.8). One should recall that $W_0(x)$ is $W_\ast(x, \delta_0)$, the value of the strong solution. Choose $p$ such that $p = q_0$ then $\|DW\|_{L^{2q}(Q)}$ is uniformly bounded. The bound for $\|W\|_{C^{0,\alpha}(Q)}$ is obvious from Lemma 4.4. Therefore, we have the uniform bound for such $W$ in $X$. The existence of a strong weak solution then follows.

The above argument holds for $2p > N$ and yields

$$\sup_{(0,T_0)} \int_{\Omega} |DW|^{2p} \, dx < \infty$$

so that our strong weak solution (which is unique) coincides with the strong solution. \H
Remark 4.8 The above argument applies to scalar equation for any dimension $N$ (or systems if $N = 2$), the spectral gap condition is not needed. If a fixed point $u$ is Hölder continuous then $Du \in L^{2p}(Q)$ for all $p$.

Remark 4.9 The uniformity in the condition $\gamma = 1 + 2\nu > s/N$ is essential and $\gamma = 1 + 2\nu > 3/N$ does not implies it even if $q > N/2$. Without this uniform continuity, we cannot obtain Lemma 3.2 to get uniform bound for fixed points of $L$ in $X$ because the number $R_0$ in (3.3) of Lemma 3.2 is not uniform.

5 The corollaries

When $N \leq 3$, the condition on $\|DW\|_{L_{2q}(Q)}$ in (4.11) is almost obvious and we have

Corollary 5.1 If $N \leq 3$ and $\nu_\ast > 1 - 2/N$. Suppose that any solution of (1.7) is uniformly bounded and uniformly continuous then there is a unique strong weak solution.

Proof: Note that (4.10) holds for $s > -1$ and $\nu > s/(s + 2)$ so with $s = 2p - 2$ then if $p > 1/2$ and $\nu > 1 - 1/p$ then we can still apply the argument of Lemma 3.2 to obtain a uniform bound for the fixed points of $L$ in $X$.

Thus, under the assumption that all weak solutions of (4.11) are uniformly bounded and continuous we see that (4.10) holds uniformly so that we can start with $N(4.10)$, we can start our argument with

$$(4.10)$$

Thus, under the assumption that all weak solutions of (4.11) are uniformly bounded and continuous we see that (4.10) holds uniformly so that we can start with $p > 1/2$ and assume that $\nu > 1 - 1/p$ to prove that (see Lemma 3.2): if $v$ is a fixed point of $L$ in $X$ then for $\gamma = 1 + 2/N$ the following integrability improvement holds

$$Dv \in L^{2p}(Q) \Rightarrow Dv \in L^{2p}(\Omega) \text{ and } |Dv|^{p-1}D^2v \in L^{2}\(Q) \Rightarrow Dv \in L^{2p\gamma}(Q).$$

We see that $2p\gamma > N$ if $p > N^2/(2(N + 2))$. Thus, we need $N^2/(2(N + 2)) > 1/2$, which is equivalent to $N(N - 1) > 2$, a condition holds for any $N \geq 2$.

On the other hand, it is easy to see that $\|Dv\|_{L^2(\Omega)}$ is uniformly bounded by testing the system with $v$. Assuming that all weak solutions of (1.1) are uniformly bounded and satisfy (4.10), we can start our argument with $N^2/(2(N + 2)) < p \leq 1$ to obtain $2p\gamma > N$ and $2p \leq 2$. Thus, Lemma 4.7 applies here with $q_0 > q = p\gamma$. Such $p \in (N^2/(2(N + 2)), 1)$ exists if and only if $N^2/(2N + 2) < 2$ which is equivalent to $(N - 1)^2 < 5$ or $N \leq 3$. The spectral gap condition holds for $q$ if $\nu_\ast > 1 - 1/(p\gamma)$. We can choose $p$ such that $p\gamma \sim N/2$ if $\nu_\ast > 1 - 2/N$. 

Proving that a bounded weak solution of a cross diffusion system is uniformly continuous is already a hard problem. By the definition of $X$, a fixed point of $L$ in $X$ is Hölder continuous but their continuity is not uniform in order (4.10) is verified so that we can apply Lemma 3.2. For any $N$ we can choose $\gamma, \alpha$ such that $\gamma \leq 1 + 2/N$ and $\alpha \geq 1/\gamma$ so that if $v \in X$ then $Du \in L^{2q_0}(\Omega)$ with $2q_0 = \gamma(1 - \alpha) > N$. But $\|Du\|_{L^{2q_0}(\Omega)}$ is not uniformly bounded so that the condition (4.10) is not verified uniformly.

However, when $N = 2$ the assumptions of Corollary 5.1 can be greatly relaxed. In fact, the condition $\nu_\ast > 1 - 2/N$ is clear and we can drop the condition (4.10) to have
Corollary 5.2 If $N = 2$ and solutions of (1.1) in $X$ are uniformly bounded then (1.1) has
a unique strong solution on $\Omega \times (0,T)$.

**Proof:** Let $W$ be a fixed point of $L$. Again, testing the system with $W$ we see that if
$\|W\|_{L^\infty(Q)} \leq M$ then
$$\iint_Q |DW|^2 \, dz \leq C(M).$$

By Lemma 4.2 ($f = \text{div}(\hat{b}(W)) + g(W)$, so that $\|f\|_{L^2(Q)} \leq C(M)$), we get
$$\iint_Q |D^2W|^2 \, dz \leq C(M).$$

From the Gagliardo-Nirenberg BMO inequality (via a fixed covering) we have
$$\iint_Q |DW|^4 \, dz \leq C \sup \|W\|_{2BMO(\Omega)}^2 \iint_Q |D^2W|^2 \, dz + C \frac{\sup \|W\|_{2BMO(\Omega)}^2}{R_0^2} \iint_Q |DW|^2 \, dz.$$  
Since $\|W\|_{BMO(\Omega)} \leq C\|W\|_{L^\infty(\Omega)}$, together with (1.7) (for $q = 1$) we have
$$\int_\Omega |DW|^2 \, dx + \iint_Q |D^2W|^2 \, dz \leq C \iint_Q (|DW|^4 + |DW|^2) \, dz \leq C(M)$$
for some constant $C(M)$. Therefore,
$$\int_\Omega |DW|^2 \, dx \leq C(M).$$

Hence, if $\|W\|_{L^\infty(Q)} \leq M$ uniformly then we also have the above estimate is uniform. We
will show below that this implies $\|W\|_{BMO(\Omega_R)}$ can be very small for uniformly small $R$. Of
course, as $DW \in L^4(Q)$ and $4 > N = 2$, Lemma 4.7 applies to give the result.

We now show that the uniform bound of $\|DW\|_{L^2(Q)} \leq M$ gives $\|W\|_{BMO(\Omega_R)}$ small for
uniformly small $R$ among weak solutions of (1.1).

By contradiction, there is a sequence of weak solutions $\{W_n\}$ converges to $W$ in $L^2(\Omega)$
with $\|DW\|_{L^2(\Omega)} \leq M$. Furthermore, there are $\varepsilon_0 > 0$ and sequences of positive numbers
$\{r_n\}$ and points $\{x_n\} \subset \Omega$ such that $r_n \to 0$ and $\|W_n\|_{BMO(\Omega \cap B_{r_n}(x_n))} > \varepsilon_0$. As $\Omega$ is
compact, we can assume that $x_n \to x_0$ for some $x_0 \in \Omega$ and the balls $B_{r_n}(x_n)$ are concentric.

For any $R > 0$ as $W_n \to W$ in $L^2(\Omega)$
$$\int_{\Omega_R} |W_n - (W_n)_R| \, dx \to \int_{\Omega_R} |W - W_R| \, dx.$$  

Since $DW \in L^2(\Omega)$, for any $\varepsilon > 0$ we have $\|DW\|_{L^2(\Omega_R)} < \varepsilon$ if $R$ small. By Poincaré’s
inequality, $N = 2$, we have $\|W\|_{BMO(\Omega_R)}$ small for uniformly small $R$. For $r_n < R$ we
have $\|W_n\|_{BMO(\Omega_{r_n})} \leq \|W_n\|_{BMO(\Omega_R)}$ which can be very small by the above limit. We then
obtain a contradiction. 

In the sequel, we will consider a special class of cross diffusion systems with polynomial
growth parameters. The boundedness assumption of weak solutions will be greatly relaxed.
Inspired by the (SKT) system, we suppose that there is \( k > 0 \) such that 
\[
\lambda(W) \sim |W|^k + 1
\]
and
\[
|b(W)| \leq C \lambda(W), \quad |\dot{b}(W)| \quad \text{and} \quad g(W) \leq C \lambda(W)|W|.
\] (5.1)

To begin, we have the following elementary lemma showing that the energy is uniformly bounded under a very weak condition.

**Lemma 5.3** Assume the condition (5.1) and \( N \) is any number. If there is \( M \) such that for any weak solution of (1.1)
\[
\iint_Q |W| \, dz \leq M.
\]
Then there is a constant \( C(M) \) such that
\[
\iint_Q |W|^{k+2} \, dz \leq C(M),
\]
and
\[
\iint_Q |DW|^2 \, dz \leq C(M).
\]

**Proof:** Test the system (1.1) with \( W \) and use the growth condition (5.1) to easily get
\[
\iint_Q (1 + |W|^k)|DW|^2 \, dz \leq C \iint_Q |W|^{k+2} \, dz + C.
\]
This is to say for \( V = |W|^{k/2+1} \)
\[
\iint_Q |DV|^2 \, dz \leq C \iint_Q |V|^2 \, dz + C.
\]

By the interpolation inequality
\[
\int_\Omega |V|^2 \, dx \leq \varepsilon \int_\Omega |DV|^2 \, dx + C(\varepsilon, \beta) \left( \int_\Omega |V|^\beta \, dx \right)^{2/\beta}
\]
we can choose \( \varepsilon, \beta \) sufficiently small to have that
\[
\iint_Q (1 + |W|^k)|DW|^2 \, dz \leq C \left( \iint_Q |W| \, dz \right) + C.
\]
This proves the lemma. \( \blacksquare \)

For any \( N \) we can choose \( \gamma, \alpha \) such that \( \gamma \lesssim 1 + 2/N \) and \( \alpha \gtrsim 1/\gamma \) so that if \( v \in X \) then \( Dv \in L^{2q_0 \gamma}(Q) \) with \( 2q_0 \gamma = \frac{2}{1-\alpha} > N \). But the continuity condition (4.10) is not verified easily even when \( N = 2 \) and we have to assume it.

However, the essential role of (4.10) is to absorb the integral of \(|Dv|^{2p+2}\) on the right of (3.5) to the left in the proof of Lemma 3.2. Without (4.10), we have the following special cases in which the same purpose is served to obtain the same result where the boundedness assumption of solutions is replaced by that of their \( BMO \) norms.
Corollary 5.4 Suppose $N \leq 3$ and $\nu_s > 1/6$ if $N = 3$. Suppose that the solutions of (1.1) in $X$ have uniform bounded BMO norms. If $\sup_{t \in \mathbb{R}^m} \frac{|a_v|^2}{\lambda_v}$ is sufficiently small ($k \leq 2$) then there exists strong solutions in $\Omega \times (0, T)$.

Proof: We can choose $\gamma$ near $1+2/N$ and $\alpha$ near $1$ in the definition of $X$. We then have $2q_0 \gamma = \frac{2}{N-\alpha} \geq 4$. Thus $q_0 \geq 2/\gamma \sim 2N/(N+2)$. The spectral gap condition $\nu_s > 1 - 1/q_0$ becomes $\nu_s > (N-2)/(2N)$ which is void if $N = 2$ and $\nu_s > 1/6$ if $N = 3$.

Because $\sup_{(0,T)} ||v(\cdot, t)||_{BMO(\Omega)} < \infty$ we have $v \in L^r(\Omega)$ for all $r > 1$. Therefore, the matrix parameters $a, b, b$ and $g$ (and their derivatives in $v$) of the system are uniformly bounded in $L^q(\Omega)$ for any given $q > 1$.

As in Lemma 3.2, $v$ weakly solves

$$(Dv) = \text{div}(aD(Dv)) + (a_nDv)Dv = \text{div}(b_xDv) + (b_nDv)Dv + bD(Dv) + g_nDv.$$

We formally test this system with $Dv \phi^2$ (see Lemma 3.2), $\phi$ being a positive $C^1$ function. Thus, if $v \in X$ then $Dv \in L^4(\Omega)$ ($2q_0 \gamma \geq 4$ in the definition of $X$) so that all the terms in (3.5) are finite (but they are not uniformly bounded) to get for any fixed point $v \in X$

$$\int \int Q \lambda(v)|D^2v|^2 \phi^2 \, dz \leq \int \int Q |a_v(v)||Dv|^2|D^2v|\phi^2 \, dz + C \int \int Q |Dv|^2(\phi^2 + |D\phi|^2) \, dz.$$ 

By Young’s inequality for some appropriate $q > 1$

$$\int \int Q \lambda(v)|D^2v|^2 \phi^2 \, dz \leq \int \int Q |a_v(v)|^2|Dv|^4 \phi^2 \, dz + \varepsilon \int \int Q |Dv|^4 \phi^2 \, dz + C(\varepsilon) \int \int Q (\phi^q + |D\phi|^q) \, dz.$$ 

For $C_* = \sup_{v \in \mathbb{R}^m} \frac{|a_v|^2}{\lambda_v} + \varepsilon$ and $\phi$ is a cutoff function for any ball $B_{R_0}$, $B_{2R_0}$

$$\int \int Q \lambda(v)|D^2v|^2 \phi^2 \, dz \leq C_* \int \int Q |Dv|^4 \phi^2 \, dz + C(\varepsilon, R_0^{-1}).$$

Applying the Gagliardo-Nirenberg BMO inequality as in Lemma 3.2 to the first term on the right, we obtain for $C_{**} = C_*\|v\|_{BMO(\Omega,R_0)}^2$ and any $R_0 > 0$

$$\int \int Q \lambda(v)|D^2v|^2 \phi^2 \, dz \leq C_{**} \int \int Q |Dv|^4 \phi^2 \, dz + C(C_{**}, R_0^{-1}) \int \int Q |Dv|^2 \phi^2 \, dz. \quad (5.2)$$

Thus, as $\|v\|_{BMO(\Omega)}$ is uniformly bounded, if $\sup_{v \in \mathbb{R}^m} \frac{|a_v|^2}{\lambda_v}$ is sufficiently small then we can absorb the first term on the right into that on the left to see that $\|D^2v\|_{L^2(\Omega)}$ is bounded (a fact we cannot obtain by Lemma 1.2 because $v$ is not assumed to be bounded) via a fixed covering (we fix $R_0 > 0$). Arguing and iterating as in Lemma 3.2 and Lemma 4.7 we obtain a uniform bound for fixed points of $\mathbb{L}$. In fact, we see now that $\|Dv\|_{L^4(\Omega)}$ is uniformly bounded (by the Gagliardo-Nirenberg inequality) and because $4 > N$ if $N \leq 3$, the proof is complete.

As long as $C_{**} = \|v\|_{BMO(\Omega,R)}\sup_{v \in \mathbb{R}^m} \frac{|a_v|^2}{\lambda_v}$ in the proof of Corollary 5.4 is small for a fixed small $R > 0$ then the proof can go on and we can conclude that
Corollary 5.5 Suppose \( N = 2 \). Let \( v \) be a strong solution in \( \Omega \times (0, T_0) \). Suppose that \( \sup_{(0, T_0)} \| Dv \|_{L^2(\Omega)} \) is bounded. If \( \sup_{v \in \mathbb{R}^m} \frac{|a|_v^2}{\lambda(\Omega)} \) is bounded then \( v \) exists on \( \Omega \times (0, T_0) \).

We then have the following result on the SKT system (1.13).

\[
\begin{aligned}
    u_t &= \Delta(P(u)) + g(u) \\
    \text{Dirichlet or Neumann boundary conditions} & \quad \text{on } \partial \Omega \times (0, T_0), \\
    u &= u_0 & \quad \text{on } \Omega.
\end{aligned}
\]  

(5.3)

Theorem 5.6 Let \( N = 2 \) and \( u_0 \in W^{1,2p} \) for some \( p > 1 \). The problem (5.3) has a unique strong solution in \( \Omega \times (0, T_0) \) for any \( T_0 > 0 \).

Proof: Testing the system (5.3) with \( u \), we easily obtain

\[
\| u \|_{L^2(\Omega)} \leq C(\| u_0 \|_{L^2(\Omega)}).
\]  

(5.4)

We multiply the \( i \)th equation of (5.3) by \( (P_i(u))_t \) and add the results to get

\[
\int_\Omega (u_t, P_u u_t) \, dx + \frac{1}{2} \frac{d}{dt} \int_\Omega |DQ|^2 \, dx = \int_\Omega \langle g, P_u u_t \rangle \, dx.
\]

Since \( P_u \) is positive definite (thanks to the ellipticity condition of \( a \)) and the components of \( P, g \) are quadratic in \( u \) we have

\[
\frac{d}{dt} \int_\Omega |DQ|^2 \, dx + \int_\Omega \lambda(u)|u_t|^2 \, dx \leq C \int_\Omega (1 + |u|^3)|u_t| \, dx.
\]  

(5.5)

By Young’s inequality, because \( \lambda(u) \geq \lambda_0 \) we derive

\[
\frac{d}{dt} \int_\Omega |DQ|^2 \, dx \leq C \int_\Omega (1 + |u|^6) \, dx.
\]  

(5.6)

By the interpolation Gagliardo-Nirenberg inequality \( \| u \|_{L^6(\Omega)}^6 \leq C \| u \|_{H^1(\Omega)}^4 \| u \|_{L^2(\Omega)}^2 \) when \( N = 2 \). From the ellipticity condition \( \langle P_u x, x \rangle \geq \lambda_0 \| x \|^2 \) it easy to see that the matrix norm \( \| P_u^{-1} \| \leq 1/\lambda_0 \) so that, as \( Du = P_u^{-1} DQ(u) \), we have \( |Du| \leq \lambda_0^{-1} |DQ| \). Therefore, thanks to the estimate (5.4) for \( \| u \|_{L^2(\Omega)} \)

\[
\| u \|_{L^6(\Omega)}^6 \leq C \left( 1 + \| Du \|_{L^2(\Omega)}^4 \right) \leq C \| Du \|_{L^2(\Omega)}^2 \int_\Omega |DQ|^2 \, dx + C,
\]

where \( C \) is a constant depends on \( \| u_0 \|_{L^2(\Omega)} \).

Thus, for \( y(t) = \int_{\Omega} |DQ|^2 \, dx \) and \( q(t) = C \| Du \|_{L^2(\Omega)}^2 \) we get from (5.6)

\[
y'(t) \leq q(t)y(t) + c_0.
\]

Here, \( c_0 \) is a constant depends on \( \| u_0 \|_{L^2(\Omega)} \). This implies

\[
y(t) \leq e^{\int_0^t q(s) \, ds} \left( y(0) + c_0 \int_0^t e^{-\int_0^s q(\tau) \, d\tau} \, ds \right).
\]
By [5.4] \[ \int_0^1 q(s) ds \leq C \int_{\Omega \times (0,T_0)} |Du|^2 \, dz \leq C(\|u_0\|_{L^2(\Omega)}) \], we see that for all \( t \in (0,T_0) \)

\[ \int_{\Omega \times \{t\}} |Du|^2 \, dx \leq C \int_{\Omega \times \{t\}} |DP|^2 \, dx \leq C(T,\|u_0\|_{W^{1,2}(\Omega)}). \]

Thus, \( \|Du\|_{L^2(\Omega)} \) is bounded on \( (0,T_0) \) so that \( \|u\|_{BMO(\Omega_R)} \) is small if \( R \) is uniformly small. By Theorem [5.5] the strong solution \( u \) exists on \( (0,T_0) \).

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