COOPERATION IN TRAFFIC NETWORK PROBLEMS VIA EVOLUTIONARY SPLIT VARIATIONAL INEQUALITIES

Shipra Singh
Department of Mathematics
Technion – Israel Institute of Technology
Haifa, 3200003, Israel

Aviv Gibali
1 Department of Mathematics, ORT Braude College
Karmiel, 2161002, Israel
2 The Center for Mathematics and Scientific Computation
University of Haifa, Mt. Carmel
Haifa, 3498838, Israel

Xiaolong Qin*
Department of Mathematics
Zhejiang Normal University
Zhejiang, China

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Abstract. In this paper, we construct an evolutionary (time-dependent) split variational inequality problem and show how to reformulate equilibria of the dynamic traffic network models of two cities as such problem. We also establish existence result for the proposed model. Primary numerical results of equilibria illustrate the validity and applicability of our results.

1. Introduction. In the early of 1960s, the Italian mathematician Stampacchia [36] and Fichera [23, 24] initiated a systematic study of variational inequality problems. Thereafter, Smith [35] and Dafermos [17] set up a traffic assignment problem in the terms of a finite dimensional variational inequality problem. Lawphongpanich and Hearn [26], and Panicucci et al. [31] studied the traffic assignment problems based on Wardrop user equilibrium principle via a variational inequality model. Recently, Chen and Huang [14] reformulated a traffic network equilibrium problem as a class of linearly constrained variational inequality problem and solved it by using the power penalty method. The industrial applications of variational inequality problems have been well documented in [34, 38]. On the other hand, the evolutionary variational inequality problem was originated by mechanics and introduced by Lions and Stampacchia [27], and Brezis [7]. Firstly, Daniele et al. [19] constructed a time-dependent traffic network equilibrium problem in the terms of an evolutionary variational inequality problem. Several problems related to the economic world, such as spatial price equilibrium problem, internet problem with multiple classes of

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* Corresponding author: Xiaolong Qin.
traffic, Nash equilibrium problem, pollution control problem, dynamic financial and oligopolistic market equilibrium problems have been studied via an evolutionary variational inequality problem in [3, 5, 6, 13, 18, 30, 33].

In continuation of above research works, the split inverse problems are being studied extensively due to their applicability in image reconstruction, cancer treatment planning and many more. A split inverse problem concerns a model in which two vector spaces $X$ and $Y$, and a bounded linear operator $A : X \rightarrow Y$ are given. In addition, two inverse problems are involved. The first one, denoted by $IP_1$, is formulated in the space $X$ and the second one, denoted by $IP_2$, is formulated in the space $Y$. Given these data, the split inverse problem is formulated as follows:

\[ \text{find a point } x^* \in X \text{ that solves } IP_1 \]
\[ \text{and such that } \]
\[ \text{the point } y^* = Ax^* \in Y \text{ solves } IP_2. \]

Censor and Elfving [10] introduced the first example of split inverse problem, known as split convex feasibility problem in which the two problems $IP_1$ and $IP_2$ are convex feasibility problems each. Censor et al. [11] used this problem for solving an inverse problem in radiation therapy treatment planning. Many results in this area were developed in the recent decades, for example split common fixed-point problem by Moudafi [29] and split common null point problem by Byrne et al. [8]. The most recent and general split problem reformulation was presented by Censor et al. [12], called as split variational inequality problem. They also constructed iterative algorithms that solve such problems and discussed their some new special cases in Euclidean space as well. These prominent roles of the split inverse problems in different areas of science, medical and real world show the necessity of their some new formulations and applications. To contribute in this direction, we define an evolutionary split variational inequality problem. In order to show its applicability in the economic world, we formulate equilibrium flow of the dynamic traffic network models of any two cities in terms of the introduced evolutionary split variational inequality problem. Further, we define a split Wardrop condition (one of the another novelty of this paper) and establish its equivalent relation with the formulated equilibrium flow of the dynamic traffic network model. Additionally, we motivate our formulated definition of equilibrium flow by an application to the dynamic traffic network models of two cities for pizza deliveries. Moreover, we also establish the existence and uniqueness for the formulated equilibria. At last, we provide two methods for solving the introduced evolutionary split variational inequality problem, which are helpful to calculate the equilibrium flow for the dynamic traffic network models of any two cities. The first method is motivated by the projected dynamical system theory, in which we define a split projected dynamical system on the feasible sets (one of the another novelty of this paper) and establish an equivalent relation between its critical point and solution of the introduced evolutionary split variational inequality problem. The second method is inspired by an iterative algorithm method, introduced by Censor et al. [12]. There are two key significance of our formulated dynamic traffic network model and the derived results. The first one is in the area of traffic network analysis and second one is in the area of split inverse problems. In essence, our formulated dynamic traffic network model in the terms of an evolutionary split variational inequality problem can deal with the two different dynamic traffic network models simultaneously in the same time intervals and our derived results can be adopted to study the dynamic traffic network model,
defined by Daniele et al. [19] and explored by Cojocaru et al. [15], Barbagallo [4], and Aussel and Cotrina [2]. The detailed arguments regarding these are given in the Subsections 2.2 and 4.1. Moreover, the outcomes of the present paper give a new approach towards the real world applications of the split variational inequality problem, defined by Censor et al. [12].

The paper is organized as follows: preliminaries, formulation of the equilibrium flow for the dynamic traffic network models of two cities in the terms of an evolutionary split variational inequality problem, an equivalence of the formulated equilibrium flow with the split Wardrop condition, and a motivated example are given in Section 2. The existence and uniqueness results for the formulated equilibria are established in Section 3. The numerical illustrations and procedures for solving the introduced evolutionary split variational inequality problem are given in Section 4. Eventually, Section 5 concludes our paper.

2. Problem formulations and motivation example.

2.1. General setting of evolutionary split variational inequality problem. For the City $X$: The traffic network is made of the set of nodes $N$ (airports, railway stations, crossings, etc.), the set of directed links $L$ between the nodes, origin-destination pair $W$ and the set of routes $V$. Each route $r \in V$ connects exactly one origin-destination pair. The set of all $r \in V$ which link a given $w \in W$ is denoted by $V(w)$. Let $f(t) \in \mathbb{R}^V$ be the time-dependent flow trajectory, where $t$ varies in the fixed time interval $[0, T]$ and $f_r(t)$, $r \in V$ represents the flow trajectory over time $t$ in the route $r$. Every feasible flow has to satisfy the following time-dependent capacity constraints

$$\lambda(t) \leq f(t) \leq \mu(t), \text{ a.e. in } [0, T],$$

and the traffic conservation law/demand requirements

$$\phi f(t) = \rho(t), \text{ a.e. in } [0, T],$$

where the bounds $\lambda(t) \leq \mu(t)$ and the demand $\rho(t) \geq 0$ are given, the pair link incidence matrix $\phi = \phi_{r,w}$ is 1 if route $r$ links the pair $w$ and 0 otherwise, and “a.e.” stands for almost everywhere and same will be followed throughout the paper.

Due to technical reasons, we take the functional setting for the flow trajectories as a reflexive Banach space $L^p([0, T], \mathbb{R}^V)$, $p > 1$ with the dual space $L^q([0, T], \mathbb{R}^V)$, $\frac{1}{p} + \frac{1}{q} = 1$. We consider $\lambda(t)$ and $\mu(t)$ belong to $L^p([0, T], \mathbb{R}^V)$ and $\rho(t)$ belongs to $L^p([0, T], \mathbb{R}^W)$. We also consider the following

$$\phi \lambda(t) \leq \rho(t) \leq \phi \mu(t), \text{ a.e. in } [0, T],$$

which implies a nonempty set of feasible flows

$$K = \{ f(t) \in L^p([0, T], \mathbb{R}^V) : \lambda(t) \leq f(t) \leq \mu(t) \text{ and } \phi f(t) = \rho(t), \text{ a.e. in } [0, T] \}.$$  

The canonical bilinear form on $L^p([0, T], \mathbb{R}^V) \times L^q([0, T], \mathbb{R}^V)$ is defined as

$$\langle \langle f(t), h(t) \rangle \rangle_X = \int_0^T \langle f(t), h(t) \rangle dt, \quad \forall f(t) \in L^p([0, T], \mathbb{R}^V) \text{ and } \forall h(t) \in L^q([0, T], \mathbb{R}^V),$$

where $\langle \langle \cdot, \cdot \rangle \rangle$ denotes the Euclidean inner product.

For the City $Y$: The traffic network is made of the set of nodes $N$, the set of directed links $L$ between the nodes, the origin-destination pair $W$ and the set of routes $V$. Each route $r \in V$ connects exactly one origin-destination pair. The set of all $r \in V$ which link a given $w \in W$ is denoted by $V(w)$. Let $g(t) \in \mathbb{R}^V$ be a
time-dependent flow trajectory, where \( t \) varies in the fixed time interval \([0, T]\) and \( g_\tau(t), \, \tau \in V \) represents the flow trajectory over time \( t \) in the route \( \tau \). Every feasible flow has to satisfy the following time-dependent capacity constraints
\[ \overline{X}(t) \leq g(t) \leq \underline{p}(t), \text{ a.e. in } [0, T], \]
and the traffic conservation law/demand requirements
\[ \overline{\delta} g(t) = \underline{p}(t), \text{ a.e. in } [0, T], \]
where the bounds \( \overline{X}(t) \leq \underline{p}(t) \) and the demand \( \underline{p}(t) \geq 0 \) are given, and the pair link incidence matrix \( \overline{\delta} = \overline{\delta}_r \) is 1 if route \( r \) links the pair \( \overline{w} \) and 0 otherwise. Due to technical reasons, we take the functional setting for the flow trajectories as a reflexive Banach space \( L^q([0, T], \mathbb{R}^V) \), \( q > 1 \) with the dual space \( L^r([0, T], \mathbb{R}^V) \), \( \frac{1}{q} + \frac{1}{r} = 1 \). We consider \( \overline{X}(t) \) and \( \underline{p}(t) \) belong to \( L^q([0, T], \mathbb{R}^V) \) and \( \overline{p}(t) \) belongs to \( L^r([0, T], \mathbb{R}^W) \). We also consider the following
\[ \overline{\delta} \overline{X}(t) \leq \overline{p}(t) \leq \overline{\delta} \underline{p}(t), \text{ a.e. in } [0, T], \]
which implies a nonempty set of feasible flows
\[ K = \{ g(t) \in L^q([0, T], \mathbb{R}^V) : \overline{X}(t) \leq g(t) \leq \underline{p}(t) \text{ and } \overline{\delta} g(t) = \underline{p}(t), \text{ a.e. in } [0, T] \}. \]
The canonical bilinear form on \( L^q([0, T], \mathbb{R}^V) \times L^q([0, T], \mathbb{R}^V) \) is defined as
\[ \langle \langle g(t), k(t) \rangle \rangle_Y = \int_0^T \langle g(t), k(t) \rangle dt, \quad g(t) \in L^q([0, T], \mathbb{R}^V) \text{ and } k(t) \in L^q([0, T], \mathbb{R}^V), \]
where \( \langle ., . \rangle \) denotes the Euclidean inner product.

**Remark 1.** Evidently, the feasible sets \( K \) and \( \overline{K} \) are convex, closed and bounded, consequently both are weakly compact sets.

From now on, for notational simplicity we write the time-dependent flow trajectories without mentioning \( t \), for instance, \( f \) is written in the place of time-dependent flow trajectory \( f(t) \). Further, for each \( f \in K \) and \( g \in \overline{K} \), the cost trajectories are represented by \( F : K \rightarrow L^q([0, T], \mathbb{R}^V) \) and \( G : \overline{K} \rightarrow L^q([0, T], \mathbb{R}^V) \), respectively. Also, we consider \( A : L^q([0, T], \mathbb{R}^V) \rightarrow L^q([0, T], \mathbb{R}^V) \) is a bounded linear operator. Now, the evolutionary split variational inequality problem is formulated as the following:

**((ESVIP))** to find \( f \in K \) such that
\[ \langle \langle F(f), h - f \rangle \rangle_X \geq 0, \quad \forall \, h \in K \tag{3} \]
and such that
\[ g = Af \in \overline{K} \text{ solves } \langle \langle G(g), k - g \rangle \rangle_Y \geq 0, \quad \forall \, k \in \overline{K}. \tag{4} \]
The solution set of ((ESVIP)) is given as
\[ Z = \{ f \in X^* \text{ such that } Af \in Y^* \}, \]
where \( X^* \) and \( Y^* \) are the solution sets of classical variational inequality problems (3) and (4), respectively.

By virtue of the definition of equilibrium flow for the dynamic traffic network model defined by Daniele et al. [19], we interpret the following definition for the dynamic traffic network models of two cities \( X \) and \( Y \) in the terms of introduced ((ESVIP)).
Definition 2.1. \( H \in K \) is an equilibrium flow if and only if \( H \in Z \).

Several authors have studied the equilibrium flow of traffic network problems in the terms of Wardrop condition. Basically in traffic analysis, the meaning of Wardrop equilibrium is that the road users choose minimum cost paths. Daniele et al. [19] formulated the traffic network equilibrium problem as a classical variational inequality problem, which is equivalent to a Wardrop equilibrium condition. Raciti [32] dealt with the vector form of Wardrop equilibrium condition. In the similar manner, we define the following split Wardrop condition.

(SWC) For arbitrary \( f \in K \) the split Wardrop condition is defined as

\[
F_u(f) < F_s(f) \Rightarrow f_u = \mu_u \text{ or } f_s = \lambda_s, \quad \forall \ w \in W, \ \forall \ u, s \in V(w) \text{ and a.e. in } [0, T],
\]

(5)

and such that the point \( g = Af \in K \) satisfies

\[
G_v(g) < G_{v'}(g) \Rightarrow g_v = \bar{\mu}_v \text{ or } g_{v'} = \bar{\lambda}_{v'}, \quad \forall \ \bar{w} \in \bar{W}, \ \forall \ v, \bar{v} \in \bar{V(\bar{w})} \text{ and a.e. in } [0, T].
\]

(6)

Remark 2. It is evident that the expressions (5) and (6) represent the generalized Wardrop conditions for the classical variational inequality problems (3) and (4), respectively, which are discussed in [19] and [4].

The following theorem provides the equivalent form of equilibrium flow of the formulated dynamic traffic network model in the terms of (ESVIP) by means of the split Wardrop condition.

Theorem 2.2. Let \( f \in K \) be arbitrary. Then, \( f \) is an equilibrium flow if and only if it satisfies the (SWC).

Proof. Firstly, we suppose that \( f \in K \) satisfies the (SWC). For \( w \in W \) and \( \bar{w} \in \bar{W} \), we define the following sets,

\[
P = \{ u \in V(w) : f_u < \mu_u \}, \quad Q = \{ s \in V(w) : f_s > \lambda_s \},
\]

\[
R = \{ v \in \bar{V}(\bar{w}) : \bar{f}_v < \bar{\mu}_v \}, \quad S = \{ \bar{v} \in \bar{V(\bar{w})} : \bar{f}_{\bar{v}} > \bar{\lambda}_{\bar{v}} \}, \quad \text{for any} \ \bar{f} \in L^\bar{F}([0, T], \mathbb{R}^{\bar{V}}).
\]

It follows from (SWC) that

\[
F_u(f) \geq F_s(f), \quad \forall \ u \in P, \ \forall \ s \in Q \text{ and a.e. in } [0, T],
\]

and such that the point \( g = Af \in K \) satisfies

\[
G_v(g) \geq G_{v'}(g), \quad \forall \ v \in R, \ \forall \ \bar{v} \in S \text{ a.e. in } [0, T].
\]

(7)

Inequality (7) yields that there exist real numbers \( l \) and \( \bar{l} \in \mathbb{R} \) such that

\[
\sup_{s \in Q} F_u(f) \leq l \leq \inf_{u \in P} F_u(f) \text{ and } \sup_{\bar{v} \in S} G_{\bar{v}}(g) \leq \bar{l} \leq \inf_{v \in R} G_v(g), \ \text{a.e. in } [0, T].
\]

Assume that \( h \in K \) and \( k \in K \) are the arbitrary flows. Then, we have for a.e. in \([0, T]\)

\[
\forall \ r \in V(w), \ F_r(f) < l \Rightarrow r \notin P \text{ and } \forall \ \bar{v} \in \bar{V(\bar{w})}, \ G_{\bar{v}}(g) < \bar{l} \Rightarrow \bar{v} \notin R.
\]

Thus, \( r \notin P \) follows that \( f_r = \mu_r \) and \((h_r - f_r) \leq 0\), consequently we obtain \((F_r(f) - l)(h_r - f_r) \geq 0\), a.e. in \([0, T]\). Similarly, \( \bar{v} \notin R \) implies \((G_{\bar{v}}(g) - \bar{l})(k_{\bar{v}} - g_{\bar{v}}) \geq 0\), a.e. in \([0, T]\). In the similar fashion, \( \forall \ r \in V(w), \ F_r(f) > l \), a.e. in \([0, T]\) and
In the inequality (8), the terms \( q_i \) are equal to zero because of the another form of traffic conservation law/demand requirements (STC) and (STC)\( g - g_r \) respectively. Therefore,

\[
\langle F(f), h - f \rangle = \sum_{u \in W} \sum_{v \in V(u)} F_r(f)(h_r - f_r) \\
= \sum_{u \in W} \sum_{v \in V(u)} (F_r(f) - l)(h_r - f_r) + l \sum_{u \in W} \sum_{v \in V(u)} (h_r - f_r) \\
\geq 0, \text{ a.e. in } [0, T],
\]

and

\[
\langle G(g), k - g \rangle = \sum_{r \in W} \sum_{v \in V_r} G_r(g)(k_r - g_r) \\
= \sum_{r \in W} \sum_{v \in V_r} (G_r(g) - l)(k_r - g_r) + l \sum_{r \in W} \sum_{v \in V_r} (k_r - g_r) \\
\geq 0, \text{ a.e. in } [0, T].
\]

In the inequality (8), the terms \( \sum_{u \in W} \sum_{v \in V(u)} (h_r - f_r) \) and \( \sum_{r \in W} \sum_{v \in V_r} (k_r - g_r) \) are equal to zero because of the another form of traffic conservation law/demand requirements \( \sum_{r \in V(w)} f_r(t) = \rho_w(t), \text{ for all } f(t) \in K \) and \( w \in W, \text{ a.e. in } [0, T] \) and \( \sum_{r \in V(w)} g_r(t) = \bar{\rho}_w(t), \text{ for all } g(t) \in \bar{K} \) and \( w \in W, \text{ a.e. in } [0, T] \).

Since \( h \in K \) and \( k \in \bar{K} \) are arbitrary, (8) yields

\[
\langle \langle F(f), h - f \rangle \rangle \geq 0, \forall h \in K
\]

and such that the point \( g = Af \in \bar{K} \) satisfies

\[
\langle \langle G(g), k - g \rangle \rangle \geq 0, \forall k \in \bar{K}.
\]

Thus, \( f \) is the equilibrium flow. Conversely, we consider that \( f \) is an equilibrium flow and it does not satisfy the (SWC). It follows that there exist \( w \in W, u, s \in V(u) \) and \( \bar{w} \in W, v, \bar{s} \in V(\bar{w}) \) together with a set \( J \subset [0, T] \) having positive measure, and we have the following cases:

1. \( F_u(f) < F_s(f), f_u < \mu_u, f_s > \lambda_s, \text{ a.e. in } J \)

and such that the point \( g = Af \in \bar{K} \) satisfies

\( G_v(g) < G_{\bar{s}}(g), g_v < \bar{\nu}_v, g_{\bar{s}} > \bar{\lambda}_{\bar{s}}, \text{ a.e. in } J \).

2. \( F_u(f) < F_s(f), f_u < \mu_u, f_s > \lambda_s, \text{ a.e. in } J \)

and such that the point \( g = Af \in \bar{K} \) satisfies

\( G_v(g) < G_{\bar{s}}(g) \Rightarrow g_v = \bar{\nu}_v \text{ or } g_{\bar{s}} = \bar{\lambda}_{\bar{s}}, \text{ a.e. in } J \).

3. \( F_u(f) < F_s(f) \Rightarrow f_u = \mu_u \text{ or } f_s = \lambda_s, \text{ a.e. in } J \)

and such that the point \( g = Af \in \bar{K} \) satisfies

\( G_v(g) < G_{\bar{s}}(g), g_v < \bar{\nu}_v, g_{\bar{s}} > \bar{\lambda}_{\bar{s}}, \text{ a.e. in } J \).

4. \( F_u(f) < F_s(f) \Rightarrow f_u = \mu_u \text{ or } f_s = \lambda_s, \text{ a.e. in } J \)

and such that the point \( g = Af \notin \bar{K} \) satisfies

\( G_v(g) < G_{\bar{s}}(g) \Rightarrow g_v = \bar{\nu}_v \text{ or } g_{\bar{s}} = \bar{\lambda}_{\bar{s}}, \text{ a.e. in } J \).
5. Case 1. with \( g = Af \notin \overline{K} \).
6. Case 2. with \( g = Af \notin \overline{K} \).
7. Case 3. with \( g = Af \notin \overline{K} \).

For the case 1., let \( \nu(t) = \min\{\mu_u(t) - f_u(t), f_u(t) - \lambda_u(t)\} \) and \( \varphi(t) = \min\{\pi_v(t) - g_v(t), g_v(t) - \lambda_v(t)\} \), where \( t \in J \). Then, \( \nu(t) > 0 \) and \( \varphi(t) > 0 \) a.e. in \( J \). We can construct a time-dependent flow trajectory \( h \in L^p([0,T],\mathbb{R}^r) \) as
\[
\begin{align*}
  h_u &= f_u + \nu(t), \quad h_s = f_s - \nu(t), \quad h_v = f_r, \quad \text{for } r \neq u, s, \text{ a.e. in } J \\
  \text{and } h &= f \text{ out side of } J.
\end{align*}
\]

Similarly, we can also define the time-dependent flow trajectory \( k \in L^p([0,T],\mathbb{R}^r) \) as
\[
\begin{align*}
  k_v &= g_v + \varphi(t), \quad k_s = g_s - \varphi(t), \quad k_r = g_r, \quad \text{for } r \neq v, s, \text{ a.e. in } J \\
  \text{and } k &= g \text{ out side of } J.
\end{align*}
\]

Then, evidently \( h \in K \) such that \( h = f \) out side of \( J \) and \( k \in \overline{K} \) such that \( k = g \) out side of \( J \). Now, we have
\[
\langle \langle F(f), h - f \rangle \rangle_X = \int_0^T \langle F(f), h - f \rangle dt \\
= \int_J \langle F(f), h - f \rangle dt \\
= \int_J \nu(t)(F_u(f) - F_s(f)) \\
< 0,
\]

and similarly the point \( g = Af \in \overline{K} \) satisfies
\[
\langle \langle G(g), k - g \rangle \rangle_Y < 0,
\]
which implies that \( f \) is not an equilibrium flow. By using the same techniques, we can easily find that \( f \) is not an equilibrium flow for the cases 2. and 3.. Further, due to the presence of \( g = Af \notin \overline{K} \) in the cases 4., 5., 6. and 7., it is obvious that \( f \) is not an equilibrium flow. Eventually, we have the contradiction, which completes the proof. \( \square \)

**Remark 3.** Due to the decomposed form of (SWC), it is more responsive to the user than its equivalent form given by Definition 2.1. We can say that (SWC) is a user-oriented equilibrium.

2.2. **A motivational example: The dynamic traffic network model of two cities for pizza deliveries.** In Figure 1, the transportation network patterns of two cities \( X \) and \( Y \) are shown. We consider, a pizza company has the branches at \( P_1 \) and \( P_2 \) in City \( X \) and at \( P_1 \) and \( P_4 \) in City \( Y \). In City \( X \), the pizza delivery boys of the branches \( P_1 \) and \( P_2 \) have to deliver the pizzas at \( P_3 \) and \( P_5 \), respectively. In City \( Y \), the pizza delivery boys of the branches \( P_1 \) and \( P_2 \) have to deliver the pizzas at \( P_2 \) and \( P_3 \), respectively. Thus, for City \( X \), the network consists six nodes and eight links, we assume the origin destination pairs are \( w_1 = (P_1, P_3) \) and \( w_2 = (P_2, P_5) \), which are respectively connected by the following paths:

\[
\begin{align*}
  w_1 : \\
  V_1 &= (P_1, P_2) \cup (P_2, P_3) \\
  V_2 &= (P_1, P_6) \cup (P_5, P_2) \cup (P_2, P_3),
\end{align*}
\]
The set of feasible flows is \( K = \{ f \in L^p([0, T], \mathbb{R}^4) : \lambda(t) \leq f(t) \leq \mu(t) \text{ and } \phi f(t) = \rho(t), \text{ a.e. in } [0, T] \} \) and cost function on the path is \( F : K \mapsto L^p([0, T], \mathbb{R}^4) \). Further, for City Y, the network consists five nodes and seven links, we assume the origin destination pairs are \( \overline{w}_1 = (\overline{P}_1, \overline{P}_2) \) and \( \overline{w}_2 = (\overline{P}_4, \overline{P}_3) \), which are respectively connected by the following paths:

\[
\overline{w}_1 : \begin{align*}
V_1 &= (P_1, P_2) \\
V_2 &= (P_1, P_4) \cup (P_4, P_2),
\end{align*}
\]

\[
\overline{w}_2 : \begin{align*}
V_3 &= (P_4, P_2) \cup (P_2, P_3) \\
V_4 &= (P_4, P_5) \cup (P_5, P_3) \\
V_5 &= (P_4, P_2) \cup (P_2, P_5) \cup (P_5, P_3).
\end{align*}
\]

The set of feasible flows is \( \overline{K} = \{ g \in L^p([0, T], \mathbb{R}^5) : \overline{\lambda}(t) \leq g(t) \leq \overline{\mu}(t) \text{ and } \overline{\phi} g(t) = \overline{\rho}(t), \text{ a.e. in } [0, T] \} \), cost function on the path is \( G : \overline{K} \mapsto L^p([0, T], \mathbb{R}^5) \) and \( A : L^p([0, T], \mathbb{R}^4) \mapsto L^p([0, T], \mathbb{R}^5) \).

Now, \( H \in K \) is an equilibrium flow if and only if

\[
\langle (F(H), f - H) \rangle_X \geq 0, \quad \forall f \in K
\]

and such that the point \( \overline{H} = AH \in \overline{K} \) solves

\[
\langle (G(\overline{H}), g - \overline{H}) \rangle_Y \geq 0, \quad \forall g \in \overline{K}.
\]

We can easily seen that with the help of above introduced model, we can find the equilibrium flows of the traffic of both cities \( X \) and \( Y \) simultaneously in the same time interval \([0, T]\).

3. Existence of equilibria. Several existence results for the different kinds of variational inequality problems have been established in the literature, for instance see [1, 9]. In these existing results, two standard techniques are used. The first one is by using KKM-Fan Theorem with monotonicity assumption and second is Brouwer’s fixed-point theorem without monotonicity assumption. By keeping the view of proofs of [37], we will prove the existence of equilibria of the dynamic traffic.
network model defined in the previous section but by using the concept of graph of a operator $A$, which is defined as follows,

$$M = \{(x, Ax) \in K \times \overline{K} : x \in K\}.$$  

We also consider that $\overline{K} \cap AK \neq \emptyset$, where $AK = \{y \in L^p([0, T], \mathbb{R}^V) : \exists x \in K$ such that $y = Ax\}$. It is easy to prove that $M$ is the convex set. The bounded linear operator $A$ implies that it is also continuous. Then by the closed graph theorem, we get that the graph $M$ of $A$ is closed with the product topology. Thus, $M$ is a nonempty, closed and convex subset of $K \times \overline{K}$. Remark 1 yields that $K \times \overline{K}$ is the weakly compact set. Hence, $M$ is the weakly compact set. Moreover, we also need the following definitions and lemma, which are motivated by [9, 22], for proving our existence result.

**Definition 3.1.** The cost functions $F$ and $G$ are called strict monotone, if

$$\langle\langle F(x) - F(y), x - y\rangle\rangle_X > 0, \forall \, x, y \in K \text{ and } x \neq y,$$

and $$\langle\langle G(\overline{x}) - G(\overline{y}), \overline{x} - \overline{y}\rangle\rangle_Y > 0, \forall \, \overline{x}, \overline{y} \in \overline{K} \text{ and } \overline{x} \neq \overline{y},$$ respectively.

**Definition 3.2.** The cost functions $F$ and $G$ are called sequentially continuous at the point $a \in K$ and $b \in \overline{K}$, respectively, if and only if $F(x_n) \to F(a)$ for all sequences $x_n \in K$, $x_n \to a$ and $G(y_n) \to G(b)$ for all sequences $y_n \in \overline{K}$, $y_n \to b$, where the symbol “→” stands for convergence.

**Definition 3.3.** The convex hull of a finite subset $\{(x_1, Ax_1), (x_2, Ax_2), \ldots, (x_n, Ax_n)\}$ of $M$ is defined by the following set

$$\text{co}\{(x_1, Ax_1), (x_2, Ax_2), \ldots, (x_n, Ax_n)\} = \left\{ \sum_{i=1}^{n} \beta_i(x_i, Ax_i) : \sum_{i=1}^{n} \beta_i = 1, \text{ for some } \beta_i \in [0, 1] \right\}.$$  

**Remark 4.** Clearly,

$$\text{co}\{(x_1, Ax_1), (x_2, Ax_2), \ldots, (x_n, Ax_n)\} \subset \left( \text{co}\{x_1, x_2, \ldots, x_n\}, \text{co}\{Ax_1, Ax_2, \ldots, Ax_n\} \right).$$

**Definition 3.4.** A set-valued mapping $\Gamma : M \mapsto 2^{K \times \overline{K}}$ is said to be the KKM mapping, if for any finite subset $\{(x_1, Ax_1), (x_2, Ax_2), \ldots, (x_n, Ax_n)\}$ of $M$

$$\text{co}\{(x_1, Ax_1), (x_2, Ax_2), \ldots, (x_n, Ax_n)\} \subset \bigcup_{i=1}^{n} \Gamma(x_i, Ax_i).$$

**Lemma 3.5.** (KKM-Fan Theorem) Let $\Gamma : M \mapsto 2^{K \times \overline{K}}$ be a KKM mapping with the closed set-values. If $\Gamma(x, Ax)$ is compact for at least one $(x, Ax) \in M$, then

$$\bigcap_{(x, Ax) \in M} \Gamma(x, Ax) \neq \emptyset.$$  

Now, we turn our attention towards the main theorem of this section.

**Theorem 3.6.** We assume that

(i) $F$ and $G$ are weakly and strongly sequentially continuous,
(ii) for all $x^* \in K$, a set-valued mapping $\Gamma_1 : K \mapsto 2^K$ defined by

$$\Gamma_1(x^*) = \{x \in K : \langle\langle F(x^*), x - x^*\rangle\rangle_X < 0\}$$

is convex valued,
(iii) for all \( y^* \in K \), the set-valued mapping \( \Gamma_2 : K \rightarrow 2^K \) defined by
\[
\Gamma_2(y^*) = \{ y \in K : \langle G(y^*), y - y^* \rangle_Y < 0 \}
\]
is convex valued,
(iv) there exist \( C_1 \times C_2 \subseteq M \) nonempty, compact and \( D_1 \times D_2 \subseteq M \) compact such that for all \( (x, Ax) \in M \backslash C_1 \times C_2 \) there exists \( (y, Ay) \in D_1 \times D_2 \) with
\[
\langle F(x), y - x \rangle_X < 0 \quad \text{and} \quad \langle G(Ax), Ay - Ax \rangle_Y < 0.
\]
Then, (ESVIP) has a solution.

Proof. We define the set-valued mapping \( \Gamma : M \rightarrow 2^{K \times K} \) for all \( (x, Ax) \in M \) as
\[
\Gamma(x, Ax) = \{ (x^*, Ax^*) \in M : \langle F(x^*), x - x^* \rangle_X \geq 0 \quad \text{and} \quad \langle G(Ax^*), Ax - Ax^* \rangle_Y \geq 0 \}.
\]
Evidently, \( (x, Ax) \in \Gamma(x, Ax) \). Therefore, \( \Gamma(x, Ax) \) is nonempty. Firstly, we shall prove that \( \Gamma \) is a KKM mapping. We consider the contradiction that \( \Gamma \) is not the KKM mapping. It follows that there exists a finite subset \( \{ (x_1, Ax_1), (x_2, Ax_2), \ldots, (x_n, Ax_n) \} \) of \( M \) such that
\[
\text{co}\{ (x_1, Ax_1), (x_2, Ax_2), \ldots, (x_n, Ax_n) \} \not\subset \bigcup_{i=1}^{n} \Gamma(x_i, Ax_i). \tag{9}
\]
Definition of convex hull implies that there exists \( (\hat{y}, A\hat{y}) \in \text{co}\{ (x_1, Ax_1), (x_2, Ax_2), \ldots, (x_n, Ax_n) \} \) such that
\[
(\hat{y}, A\hat{y}) = \sum_{i=1}^{n} \beta_i(x_i, Ax_i),
\]
where \( \sum_{i=1}^{n} \beta_i = 1 \) for some \( \beta_i \in [0, 1] \). Now, (9) yields
\[
(\hat{y}, A\hat{y}) \notin \bigcup_{i=1}^{n} \Gamma(x_i, Ax_i).
\]
Therefore, for any \( i = \{1, 2, \ldots, n\} \), we have the following three cases
1. \( \langle F(\hat{y}), x_i - \hat{y} \rangle_X < 0 \) and \( \langle G(A\hat{y}), Ax_i - A\hat{y} \rangle_Y < 0 \),
2. \( \langle F(\hat{y}), x_i - \hat{y} \rangle_X \geq 0 \) and \( \langle G(A\hat{y}), Ax_i - A\hat{y} \rangle_Y < 0 \),
3. \( \langle F(\hat{y}), x_i - \hat{y} \rangle_X < 0 \) and \( \langle G(A\hat{y}), Ax_i - A\hat{y} \rangle_Y \geq 0 \).
Case 1 implies that \( \{ x_1, x_2, \ldots, x_n \} \subset \Gamma_1(\hat{y}) \) and \( \{ Ax_1, Ax_2, \ldots, Ax_n \} \subset \Gamma_2(A\hat{y}) \). Further, convexities of the sets \( \Gamma_1 \) and \( \Gamma_2 \) yield that \( \text{co}\{ x_1, x_2, \ldots, x_n \} \subset \Gamma_1(\hat{y}) \) and \( \text{co}\{ Ax_1, Ax_2, \ldots, Ax_n \} \subset \Gamma_2(A\hat{y}) \). From \( (\hat{y}, A\hat{y}) \in \text{co}\{ (x_1, Ax_1), (x_2, Ax_2), \ldots, (x_n, Ax_n) \} \) and Remark 4, we have \( (\hat{y}, A\hat{y}) \in \text{co}\{ (x_1, x_2, \ldots, x_n), (Ax_1, Ax_2, \ldots, Ax_n) \} \), which renders \( \hat{y} \in \Gamma_1(\hat{y}) \) and \( A\hat{y} \in \Gamma_2(A\hat{y}) \). Thus, we obtain
\[
\langle (F(\hat{y}), \hat{y} - \hat{y}) \rangle_X < 0 \quad \text{and} \quad \langle (G(A\hat{y}), A\hat{y} - A\hat{y}) \rangle_Y < 0,
\]
which are the contradictions. By using the same techniques we can easily prove contradictions for the other two cases. Therefore, \( \Gamma \) is the KKM mapping. Now, we shall show that \( \Gamma \) is a closed set-valued for each \( (x, Ax) \in M \) with respect to the weak topology of \( K \times K \). Let \( (x, Ax) \in M \) be arbitrary and \( \{ (x_n, Ax_n) \}_{n=0}^{\infty} \) be a sequence in \( \Gamma(x, Ax) \) which converges weakly to \((y, Ay)\). Since \( (x_n, Ax_n) \in \Gamma(x, Ax) \), implies
\[
\langle (F(x_n), x - x_n) \rangle_X \geq 0 \quad \text{and} \quad \langle (G(Ax_n), Ax - Ax_n) \rangle_Y \geq 0.
\]
Since \( F, G \) are weakly sequentially continuous and \( A \) is also continuous, by taking limit as \( n \rightarrow \infty \) in the above inequalities, we have
\[
\langle (F(y), y - y) \rangle_X \geq 0 \quad \text{and} \quad \langle (G(Ay), Ay - Ay) \rangle_Y \geq 0,
\]
which implies \((y, Ay) \in \Gamma(x, Ax)\). Hence \(\Gamma(x, Ax)\) is the closed set-valued for each \((x, Ax) \in M\) with respect to the weak topology on \(K \times K\). It is obvious that \(\Gamma(x, Ax)\) is the closed subset of weakly compact set \(M\), consequently, \(\Gamma(x, Ax)\) is weakly compact for each \((x, Ax) \in M\). Moreover, by using the hypothesis of strong sequentially continuity of \(F\) and \(G\) and the same techniques, we can easily prove that \(\Gamma(x, Ax)\) is closed set-valued with respect to the strong topology on \(K \times K\). On the other hand, hypothesis (iv) renders that \(\Gamma\) is compact for \((y, Ay) \in D_1 \times D_2 \subseteq M\) with respect to the strong topology. Then by KKM-Fan Theorem, we have

\[
\bigcap_{(x, Ax) \in M} \Gamma(x, Ax) \neq \emptyset,
\]

which yields that there exists \((x^*, Ax^*) \in M\) such that \((x^*, Ax^*) \in \Gamma(x, Ax)\) for all \((x, Ax) \in M\). We can have \(E \subseteq K\) and \(J \subseteq \overline{K}\) such that \((x^*, Ax^*) \in E \times J \subseteq M\). Now, we can write there exists \((x^*, Ax^*) \in E \times J\) such that \((x^*, Ax^*) \in \Gamma(x, Ax)\) for all \((x, Ax) \in E \times J\). Thus, we get

\[
\langle \langle F(x^*), x - x^* \rangle \rangle_X \geq 0 \text{ and } \langle \langle G(Ax^*), Ax - Ax^* \rangle \rangle_Y \geq 0, \quad \forall (x, Ax) \in E \times J. \tag{10}
\]

Let \(y^* = Ax^*, y = Ax\) and noticed that \(x^*\) and \(y^* = Ax^*\) in the inequality (10) are fixed, the inequality (10) can be rewritten as

\[
x^* \in E \text{ such that } \langle \langle F(x^*), x - x^* \rangle \rangle_X \geq 0, \quad \forall x \in E
\]

and the point \(y^* = Ax^* \in J\) solves

\[
\langle \langle G(y^*), y - y^* \rangle \rangle_Y \geq 0, \quad \forall y \in J.
\]

Therefore (ESVIP) has a solution \(x^*\) in the set \(E\) subset of \(K\).

The following corollary gives the uniqueness of the solution of (ESVIP).

**Corollary 1.** If the cost functions \(F\) and \(G\) are strict monotone on \(K\) and \(\overline{K}\), respectively, then (ESVIP) has a unique solution.

**Proof.** Let \(x_1 \in K\) be a solution of (ESVIP). We have

\[
\langle \langle F(x_1), x - x_1 \rangle \rangle_X \geq 0, \quad \forall x \in K
\]

and the point \(y_1 = Ax_1 \in \overline{K}\) solves

\[
\langle \langle G(y_1), y - y_1 \rangle \rangle_Y \geq 0, \quad \forall y \in \overline{K}. \tag{11}
\]

Again, let \(x_2 \in K\) be a solution of (ESVIP) and \(x_1 \neq x_2\), we get

\[
\langle \langle F(x_2), x - x_2 \rangle \rangle_X \geq 0, \quad \forall x \in K
\]

and the point \(y_2 = Ax_2 \in \overline{K}\) solves

\[
\langle \langle G(y_2), y - y_2 \rangle \rangle_Y \geq 0, \quad \forall y \in \overline{K}. \tag{12}
\]

The inequality (11) can be rewritten as

\[
\langle \langle F(x_1), x_2 - x_1 \rangle \rangle_X \geq 0 \text{ and } \langle \langle G(y_1), y_2 - y_1 \rangle \rangle_Y \geq 0.
\]

Strict monotonicities of the functions \(F\) and \(G\) together with the fact \(y_1 \neq y_2\), we obtain

\[
\langle \langle F(x_1) - F(x_2), x_1 - x_2 \rangle \rangle_X > 0 \text{ and } \langle \langle G(y_1) - G(y_2), y_1 - y_2 \rangle \rangle_Y > 0. \tag{14}
\]

By adding the corresponding inequalities of (13) and (14), we get

\[
\langle \langle F(x_2), x_1 - x_2 \rangle \rangle_X < 0 \text{ and } \langle \langle G(y_2), y_1 - y_2 \rangle \rangle_Y < 0,
\]
which contradicts the inequality (12), i.e., \( x_2 \) is not the solution of (ESVIP). Therefore, (ESVIP) has the unique solution.

\[ \square \]

4. Numerical illustration of the dynamic traffic network model of two cities. In this section, we shall demonstrate two methods to solve the introduced (ESVIP). The first one is by projected dynamical system (PDS) theory which is motivated by [15] and the second one is by an iterative method, introduced by Censor et al. [12]. Moreover, we shall also discuss that which method is more amenable to the user.

The projected dynamical system was first introduced by Dupuis and Nagurney [21]. They also defined its relation with a classical variational inequality problem. However, this relation was first noted in Dupuis and Ishii [20]. The historical and applicable point of views of the projected dynamical systems are well defined by Cojocaru et al. [15] and Giffre et al. [25]. Inspired by these works, we define the split projected dynamical system on the feasible sets \( K \) and \( \overline{K} \) for \( p = 2 \) and \( \overline{p} = 2 \), respectively, as following:

\[ \text{(SPDS)} \quad x(.) \in K \text{ such that } \frac{dx(., \tau)}{d\tau} = \Pi_K(x(., \tau), -F(x(., \tau))), \quad x(., 0) = x_0(.) \in K \]

and the point \( y(.) = Ax(.) \in \overline{K} \) satisfies

\[ \frac{dy(., \tau)}{d\tau} = \Pi_{\overline{K}}(y(., \tau), -G(y(., \tau))), \quad y(., 0) = y_0(.) \in \overline{K}, \]

where \( F : K \mapsto L^2([0, T], \mathbb{R}^V) \) and \( G : \overline{K} \mapsto L^2([0, T], \mathbb{R}^\overline{V}) \) are the Lipschitz continuous vector fields, \( \Pi_K : K \times L^2([0, T], \mathbb{R}^V) \mapsto L^2([0, T], \mathbb{R}^V) \) and \( \Pi_{\overline{K}} : \overline{K} \times L^2([0, T], \mathbb{R}^\overline{V}) \mapsto L^2([0, T], \mathbb{R}^\overline{V}) \) are the operators given in the respective manners as

\[ \Pi_K(x(., v(., \tau))) = \lim_{\delta \to 0^+} \frac{\text{proj}_K(\delta v(., \tau) + x(., \tau))}{\delta}, \quad \forall \quad x(., \tau) \in K \text{ and } v(., \tau) \in L^2([0, T], \mathbb{R}^V), \]

\[ \Pi_{\overline{K}}(y(., v(., \tau))) = \lim_{\delta \to 0^+} \frac{\text{proj}_{\overline{K}}(\delta v(., \tau) + y(., \tau))}{\delta}, \quad \forall \quad y(., \tau) \in \overline{K} \text{ and } v(., \tau) \in L^2([0, T], \mathbb{R}^\overline{V}), \]

\( \text{proj}_K(.) \) and \( \text{proj}_{\overline{K}}(.) \) are the projections of the given vectors on the sets \( K \) and \( \overline{K} \), respectively, which are defined after this paragraph. Further, to avoid confusion between the times \( t \) and \( \tau \), we represent elements of the sets \( L^2([0, T], \mathbb{R}^V) \) and \( L^2([0, T], \mathbb{R}^\overline{V}) \) at fixed moments \( t \in [0, T] \) as \( x(.) \) and \( y(.) \), respectively. Indeed, in the formulation of (SPDS), the time \( \tau \) is different from the time \( t \). For all \( t \in [0, T] \) a solution of (ESVIP) represents a static state of the underlying system and the static states define one or more equilibrium curves with the variation of \( t \) over the interval \( [0, T] \). On the other hand, \( \tau \) defines the dynamics of the system over the interval \( [0, \infty) \) until it reaches one of the equilibria on the curves. It is clear that the solutions of (SPDS) lie in the class of absolutely continuous functions with respect to \( \tau \), from \( [0, \infty) \) to \( K \). In order to describe the procedure to solve the (ESVIP), we need to demonstrate the following definitions which are inspired by [16, 28].

**Definition 4.1.** \( x^*(.) \in K \) is called a critical point for the (SPDS), if

\[ \Pi_K(x^*(.), -F(x^*(.))) = 0, \quad \text{and the point } y^*(.) = Ax^*(.) \in \overline{K} \text{ satisfies}

\[ \Pi_{\overline{K}}(y^*(.), -G(y^*(.))) = 0. \]

**Definition 4.2.** The projection of a point \( x(.) \in L^p([0, T], \mathbb{R}^V) \) onto the set \( K \) is defined as
\[
\text{proj}_K(x(.)) = \arg \min_{y(\cdot) \in K} \|x(\cdot) - y(\cdot)\|.
\]

**Remark 5.** For each \(x(\cdot) \in L^p([0, T], \mathbb{R}^V)\), \(\text{proj}_K(x(.))\) satisfies the following property
\[
\langle x(\cdot) - \text{proj}_K(x(.)), y(\cdot) - \text{proj}_K(x(.)) \rangle_X \leq 0, \ \forall \ y(\cdot) \in K. \quad (15)
\]

**Definition 4.3.** The polar set \(K^\circ\) associated to \(K\) is defined as
\[
K^\circ = \{ y(\cdot) \in L^p([0, T], \mathbb{R}^V) : \langle y(\cdot), x(\cdot) \rangle_X \leq 0, \ \forall \ x(\cdot) \in K \}.
\]

**Definition 4.4.** The tangent cone to the set \(K\) at a point \(x(\cdot) \in K\) is defined as
\[
T_K(x(.)) = \text{cl} \left( \bigcup_{\lambda > 0} \frac{K - z(\cdot)}{\lambda} \right),
\]
where \(\text{cl}\) stands for closure.

**Definition 4.5.** The normal cone of \(K\) at a point \(x(\cdot)\) is defined as
\[
N_K(x(.)) =
\begin{cases}
\{ y(\cdot) \in L^p([0, T], \mathbb{R}^V) : \langle y(\cdot), z(\cdot) - x(\cdot) \rangle_X \leq 0, \ \forall \ z(\cdot) \in K \}, & x(\cdot) \in K, \\
\phi, & x(\cdot) \notin K,
\end{cases}
\]
and it can be written as \(T_K(x(.)) = [N_K(x(.))]^\circ\).

**Note 4.1** All the above definitions can be defined for the space \(L^p([0, T], \mathbb{R}^V)\) in the similar manner according to our need.

The following propositions are the direct consequences of Proposition 2.1 and 2.2 from [16].

**Proposition 1.** For all \(x(\cdot) \in K\), \(v(\cdot) \in L^p([0, T], \mathbb{R}^V)\), \(y(\cdot) \in \overline{K}\) and \(\tau(\cdot) \in L^p([0, T], \mathbb{R}^V)\), \(\Pi_K(x(.), v(.))\) and \(\Pi_{\tau}(y(.), \tau(.))\) exist and \(\Pi_K(x(\cdot), v(\cdot)) = \text{proj}_{\Pi_K(x(\cdot))} v(\cdot), \Pi_{\tau}(y(\cdot), \tau(\cdot)) = \text{proj}_{\Pi_{\tau}(y(\cdot))} \tau(\cdot)\).

**Proposition 2.** For all \(x(\cdot) \in K\) there exists \(n(\cdot) \in N_K(x(\cdot))\) and for all \(y(\cdot) \in \overline{K}\) there exists \(\tau(\cdot) \in N_{\overline{K}}(y(\cdot))\) such that \(\Pi_K(x(.), v(.)) = v(\cdot) - n(\cdot), \ \forall \ v(\cdot) \in L^p([0, T], \mathbb{R}^V)\) and \(\Pi_{\tau}(y(.), \tau(.)) = \tau(\cdot) - \tau(\cdot), \ \forall \ \tau(\cdot) \in L^p([0, T], \mathbb{R}^V)\).

The following theorem enables us to find a relation between the solution of (ESVIP) and the critical point of (SPDS).

**Theorem 4.6.** \(x^*(\cdot) \in K\) is a solution of (ESVIP) if and only if it is the critical point of (SPDS).

**Proof.** Let \(x^*(\cdot) \in K\) be a solution of (ESVIP), it follows that
\[
\langle (F(x^*(\cdot)), x(\cdot) - x^*(\cdot)) \rangle_X \geq 0, \ \forall \ x(\cdot) \in K
\]
and the point \(y^*(\cdot) = Ax^*(\cdot) \in \overline{K}\) solves
\[
\langle (G(y^*(\cdot)), y(\cdot) - y^*(\cdot)) \rangle_Y \geq 0, \ \forall \ y(\cdot) \in \overline{K}.
\]
Above inequalities imply
\[-F(x^*(\cdot)) \in N_K(x^*(\cdot)) \text{ and } -G(y^*(\cdot)) \in N_{\overline{K}}(y^*(\cdot)).\]
By using Proposition 2, we obtain
\[
\Pi_K(x^*(\cdot), -F(x^*(\cdot))) = 0 \text{ and } \Pi_{\overline{K}}(y^*(\cdot), -G(y^*(\cdot))) = 0. \quad (16)
\]
Thus, \( x^*(.) \) is the critical point of (SPDS). Conversely, let it be a critical point of (SPDS). Then, we have the inequality (16). Therefore, Proposition 1 yields
\[
\text{proj}_{\mathcal{K}_t}(x^*(.)) \langle -F(x^*(.)) \rangle = 0 \quad \text{and} \quad \text{proj}_{\mathcal{K}_t}(y^*(.)) \langle -G(y^*(.)) \rangle = 0.
\]
By using Remark 5 and keeping the view of Note 4.1 in the above inequalities, we obtain
\[
\langle -F(x^*(.)), z \rangle_X \leq 0, \quad \forall \ z \in T_\mathcal{K}(x^*(.)) \quad \text{and} \quad \langle -G(y^*(.)), \pi \rangle_Y \leq 0, \quad \forall \ \pi \in T_\mathcal{K}(y^*(.)),
\]
which imply
\[
-F(x^*(.)) \in N_\mathcal{K}(x^*(.)) \quad \text{and} \quad -G(y^*(.)) \in N_\mathcal{K}(y^*(.)).
\]
Consequently, \( x^*(.) \) is the solution of (ESVIP).

Now, we are going to discuss the method to solve the (ESVIP). Theorem 4.6 says that any point of a curve of equilibria on the interval \([0, T]\) is same as the critical point of (SPDS) and vice versa. On the other hand, we have already proved the existence of equilibria and its uniqueness as well. By keeping the virtue of these information, we employ the discretization of the interval \([0, T]\), as \(0 = t_0 < t_1 < \ldots < t_i < \ldots < t_n = T\). Then for each \(t_i\), \(i = 0, 1, \ldots, n\), we get a sequence of (SPDS) on distinct, finite-dimensional, closed and convex sets \(K_i\) and \(\mathcal{K}_i\). After calculating all critical points of each (SPDS), we obtain a sequence of critical points which by interpolation yields the curves of equilibria. For the practical point of view of this procedure, we illustrate the formulated dynamic traffic network equilibrium problem of Subsection 2.2, for \(p = 2, \overline{p} = 2\) and \(T = 2\).

The cost functions \( F : L^2([0, 2], \mathbb{R}^4) \rightarrow L^2([0, 2], \mathbb{R}^4) \) and \( G : L^2([0, 2], \mathbb{R}^5) \rightarrow L^2([0, 2], \mathbb{R}^5) \) are defined as
\[
F(x) = (x_1, x_2, x_3, x_4) \quad \text{and} \quad G(y) = (y_1^2, y_2^2, y_3^2, y_4^2, y_5^2),
\]
where \( x = (x_1, x_2, x_3, x_4) \) and \( y = (y_1, y_2, y_3, y_4, y_5) \). The bounded linear operator \( A : L^2([0, 2], \mathbb{R}^4) \rightarrow L^2([0, 2], \mathbb{R}^5) \) is defined as
\[
A(x) = (x_1 + x_4, x_2 + x_3, x_1 + x_2, 2x_1, x_3 + x_4 - x_1).
\]
The feasible sets are given as
\[
K = \{ f \in L^2([0, 2], \mathbb{R}^4) : (0, 0, 0, 0) \leq (f_1(t), f_2(t), f_3(t), f_4(t)) \leq (t + 1, t + 2, 2t + 2, t + 3) \text{ and } f_1(t) + f_2(t) = 2t + 2, f_3(t) + f_4(t) = 3t + 3, \ a.e. \ in \ [0, 2] \},
\]
\[
\mathcal{K} = \{ g \in L^2([0, 2], \mathbb{R}^5) : (0, 0, 0, 0, 0) \leq (g_1(t), g_2(t), g_3(t), g_4(t), g_5(t)) \leq (t + 6, t + 6, 2t + 2, t + 4, 4t + 4) \text{ and } g_1(t) + g_2(t) = 2t + 11, g_3(t) + g_4(t) + g_5(t) = 7t + 10, \ a.e. \ in \ [0, 2] \}.
\]
We can easily prove that the cost functions \( F \) and \( G \) are strictly monotone on the sets \( K \) and \( \mathcal{K} \), respectively. Therefore, (ESVIP) has unique curve of equilibrium. We select \( t_i \in \{ \frac{k}{8} : k \in \{0, 1, 2, \ldots, 16\} \} \). Then, we get a sequence of (SPDS) defined on the feasible sets
\[
K_{t_i} = \{ f \in L^2([0, 2], \mathbb{R}^4) : (0, 0, 0, 0) \leq (f_1(t_i), f_2(t_i), f_3(t_i), f_4(t_i)) \leq (t_i + 1, t_i + 2, 2t_i + 2, t_i + 3) \text{ and } f_1(t_i) + f_2(t_i) = 2t_i + 2, f_3(t_i) + f_4(t_i) = 3t_i + 3, \ a.e. \ in \ [0, 2] \},
\]
\[
\mathcal{K}_{t_i} = \{ g \in L^2([0, 2], \mathbb{R}^5) : (0, 0, 0, 0, 0) \leq (g_1(t_i), g_2(t_i), g_3(t_i), g_4(t_i), g_5(t_i)) \leq (t_i + 6, t_i + 6, 2t_i + 2, t_i + 4, 4t_i + 4) \text{ and } g_1(t_i) + g_2(t_i) = 2t_i + 11,
\]
\[
\ldots
\]
where \( K, T \) is the interval \([0, T]\) after doing a discretization procedure. We choose the above mentioned discretization procedure. We can also apply this iterative method for solving the (ESVIP) problem. We have introduced an iterative method for solving a (static) split variational inequality problem, \( \text{et al.} \ [12] \) introduced an iterative method for solving a (static) split variational inequality problem, displayed as in Figure 2.

Now, we are going to discuss the second method to solve the (ESVIP). Censor, et al. [12] introduced an iterative method for solving a (static) split variational inequality problem. We can also apply this iterative method for solving the (ESVIP) after doing a discretization procedure. We choose the above mentioned discretization of the interval \([0, T]\). Then for each \( t_i, i \in \{0, 1, \ldots, n\} \), we demonstrate the following static split variational inequality problem, to find \( H(t_i) \in K_{t_i} \) such that

\[
\langle \{F(H(t_i)), f(t_i) - H(t_i))\rangle_X \geq 0, \quad \forall f(t_i) \in K_{t_i},
\]

and the point \( \overline{H}(t_i) = AH(t_i) \in \overline{K}_{t_i} \) solves

\[
\langle \{G(\overline{H}(t_i)), g(t_i) - \overline{H}(t_i))\rangle_Y \geq 0, \quad \forall g(t_i) \in \overline{K}_{t_i},
\]

where \( K_{t_i} \) and \( \overline{K}_{t_i} \) are defined same as above. The algorithm for computing the solutions of finite-dimensional split variational inequality problem (17) is given as:

| \( t_i \) | \( x_1^*(t_i) \) | \( x_2^*(t_i) \) | \( x_3^*(t_i) \) | \( x_4^*(t_i) \) |
|---|---|---|---|---|
| 0 | 1 | 1 | 1.5 | 1.5 |
| 1 | 1.125 | 1.125 | 1.6875 | 1.6875 |
| 2 | 1.25 | 1.25 | 1.875 | 1.875 |
| 3 | 1.375 | 1.375 | 2.0625 | 2.0625 |
| 4 | 1.5 | 1.5 | 2.25 | 2.25 |
| 5 | 1.625 | 1.625 | 2.4375 | 2.4375 |
| 6 | 1.75 | 1.75 | 2.625 | 2.625 |
| 7 | 1.875 | 1.875 | 2.8125 | 2.8125 |
| 8 | 2 | 2 | 3 | 3 |
| 9 | 2.125 | 2.125 | 3.1875 | 3.1875 |
| 10 | 2.25 | 2.25 | 3.375 | 3.375 |
| 11 | 2.375 | 2.375 | 3.5625 | 3.5625 |
| 12 | 2.5 | 2.5 | 3.75 | 3.75 |
| 13 | 2.625 | 2.625 | 3.9375 | 3.9375 |
| 14 | 2.75 | 2.75 | 4.125 | 4.125 |
| 15 | 2.875 | 2.875 | 4.3125 | 4.3125 |
| 16 | 3 | 3 | 4.5 | 4.5 |
**Initialization:** Let $\lambda > 0$ and select an arbitrary starting point $H^0(t_i) \in K_{t_i}$.

**Iterative step:** Given the current iterate $H^k(t_i)$, compute

$$H^{k+1}(t_i) = U(H^k(t_i) + \gamma A^*(T - I)(AH^k(t_i))),$$

where $\gamma \in (0, \frac{1}{L})$, $L$ is the spectral radius of the operator $A^*A$, and $A^*$ is the adjoint of $A$, $T = \text{proj}_{K_{t_i}}(I - \lambda G(t_i))$ and $U = \text{proj}_{K_{t_i}}(I - \lambda F(t_i))$.

By using same values for all the given arguments $F, G, t_i, K_{t_i}$ and $\text{proj}_{K_{t_i}}$ as defined in the previous method for solving the (ESVIP) by (SPDS) theory and keeping the fact of uniqueness of the solution of (ESVIP) in the mind, we get the same solutions for static split variational inequality problem (17) as given in Table 1 and shown by Figure 2. We can clearly see that this method is computationally expansive than (SPDS) theory method because of the several iterative steps. Therefore, one can speak the previous method is user-oriented.

4.1. **A comparative study.** In this section, we will compare our formulated dynamic traffic network model with the given and investigated dynamic traffic network models of [19, 15, 4, 2]. We can see that the introduced evolutionary split variational inequality problem (ESVIP) is combination of two evolutionary variational inequality problems given by the inequalities (3) and (4). Firstly, Daniele et al. [19] defined the equilibrium flow of a dynamic traffic network model in the terms of an evolutionary variational inequality problem of the form as given by the inequalities (3) and (4). Thereafter, the existence of this equilibrium flow was investigated and improved by Cojocaru et al. [15], Barbagallo [4], and Aussel and Cotrina [2]. Clearly, if $x^* \in K$ is a solution of (ESVIP) then it is also the solution of evolutionary variational inequality problem (3) on the feasible set $K$ and $y^* = Ax^* \in \overline{K}$ is the solution of evolutionary variational inequality problem (4) on the feasible set $\overline{K}$. Therefore, $x^*$ is the equilibrium flow of the dynamic traffic network model for City X and $y^*$ is the equilibrium flow of the dynamic traffic network model for City Y (according to the definition of equilibrium flow of the dynamic traffic network model, defined by Daniele et al. [19]). After solving the evolutionary variational inequality problems (3) and (4) by using the same arguments as given in the above
section of numerical illustration, we get the unique solutions set of evolutionary variational inequality problem (3) as given in Table 1 and traffic network pattern of the City X can be shown by Figure 2. In continuation, we also get the unique solutions set of evolutionary variational inequality problem (4) as given in Table 2 and the traffic network pattern of the City Y is shown in Figure 3. Conclusively, we get the equilibrium flows of the dynamic traffic network models, studied in literature with the help of our formulated dynamic traffic network model in the terms of evolutionary split variational inequality problem and the derived results.

### Table 2. Numerical Results

| $t_i$ | $y_1(t_i)$ | $y_2(t_i)$ | $y_3(t_i)$ | $y_4(t_i)$ | $y_5(t_i)$ |
|-------|------------|------------|------------|------------|------------|
| 0     | 2.5        | 2.5        | 2          | 2          | 2          |
| 1     | 2.8125     | 2.8125     | 2.25       | 2.25       | 2.25       |
| 2     | 3.125      | 3.125      | 2.5        | 2.5        | 2.5        |
| 3     | 3.4375     | 3.4375     | 2.75       | 2.75       | 2.75       |
| 4     | 3.75       | 3.75       | 3          | 3          | 3          |
| 5     | 4.0625     | 4.0625     | 3.25       | 3.25       | 3.25       |
| 6     | 4.375      | 4.375      | 3.5        | 3.5        | 3.5        |
| 7     | 4.6875     | 4.6875     | 3.75       | 3.75       | 3.75       |

**Figure 3.** Traffic network pattern of City Y
5. **Conclusion.** The present paper is a contribution in the study of traffic network and split inverse problems. We formulated the equilibrium flows of the dynamic traffic network models for any two cities in the terms of an evolutionary split variational inequality problem and established an equivalent relationship between the equilibria of the proposed model and split Wardrop condition. We also dealt with the existence and uniqueness of the formulated equilibria. In order to calculate the equilibrium flow of the dynamic traffic network models for two cities, we proposed two methods and computed the solutions of the evolutionary split variational inequality problem.

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**REFERENCES**

[1] Q. H. Ansari and M. Rezaei, Existence results for Stampacchia and Minty type vector variational inequalities, *Optimization*, 59 (2010), 1053–1065.

[2] D. Aussel and J. Cotrina, Existence of time-dependent traffic equilibria, *Appl. Anal.*, 91 (2012), 1775–1791.

[3] D. Aussel, R. Gupta and A. Mehra, Evolutionary variational inequality formulation of the generalized Nash equilibrium problem, *J. Optim. Theory Appl.*, 169 (2016), 74–90.

[4] A. Barbagallo, Degenerate time-dependent variational inequalities with applications to traffic equilibrium problems, *Comput. Methods Appl. Math.*, 6 (2006), 117–133.

[5] A. Barbagallo and M.-G. Cojocaru, Dynamic equilibrium formulation of the oligopolistic market problem, *Math. Comput. Modelling*, 49 (2009), 966–976.

[6] A. Barbagallo, P. Daniele and A. Maugeri, Variational formulation for a general dynamic financial equilibrium problem: Balance law and liability formula, *Nonlinear Anal.*, 75 (2012), 1104–1123.

[7] H. Brezis, Inéquations d’évolution abstraites, *C. R. Acad. Sci. Paris Sér. A-B*, 264 (1967), A732–A735.

[8] C. Byrne, Y. Censor, A. Gibali and S. Reich, The split common null point problem, *J. Nonlinear Convex Anal.*, 13 (2012), 759–775.

[9] L.-C. Ceng and S. Huang, Existence theorems for generalized vector variational inequalities with a variable ordering relation, *J. Global Optim.*, 46 (2010), 521–535.

[10] Y. Censor and T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numer. Algorithms*, 8 (1994), 221–239.

[11] Y. Censor, T. Elfving, N. Kopf and T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problems, *Inverse Problems*, 21 (2005), 2071–2084.

[12] Y. Censor, A. Gibali and S. Reich, Algorithms for the split variational inequality problem, *Numer. Algorithms*, 59 (2012), 301–323.

[13] C. Ciarcia and P. Daniele, New existence theorems for quasi-variational inequalities and applications to financial models, *European J. Oper. Res.*, 251 (2016), 288–299.

[14] M. Chen and C. Huang, A power penalty method for a class of linearly constrained variational inequality, *J. Ind. Manag. Optim.*, 14 (2018), 1381–1396.

[15] M. G. Cojocaru, P. Daniele and A. Nagurney, Projected dynamical systems and evolutionary variational inequalities via Hilbert spaces with applications, *J. Optim. Theory Appl.*, 127 (2005), 549–563.

[16] M.-G. Cojocaru and L. B. Jonker, Existence of solutions to projected differential equations in Hilbert spaces, *Proc. Amer. Math. Soc.*, 132 (2004), 183–193.

[17] S. Dafermos, Traffic equilibrium and variational inequalities, *Transportation Sci.*, 14 (1980), 42–54.

[18] P. Daniele, Time-dependent spatial price equilibrium problem: Existence and stability results for the quantity formulation model, *J. Global Optim.*, 28 (2004), 283–295.

[19] P. Daniele, A. Maugeri and W. Oettli, Time-dependent traffic equilibria, *J. Optim. Theory Appl.*, 103 (1999), 543–555.
[20] P. Dupuis and H. Ishii, On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications, Stochastics Stochastics Rep., 35 (1991), 31–62.
[21] P. Dupuis and A. Nagurney, Dynamical systems and variational inequalities, Ann. Oper. Res., 44 (1993), 9–42.
[22] K. Fan, Some properties of convex sets related to fixed point theorems, Math. Ann., 266 (1984), 519–537.
[23] G. Fichera, Problemi elastostatici con vincoli unilaterali: Il problema di Signorini con ambigue condizioni al contorno, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. Ia (8), 7 (1963/64), 91–140.
[24] G. Fichera, Sul problema elastostatico di Signorini con ambigue condizioni al contorno, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8), 34 (1963), 138–142.
[25] S. Giuffrè, G. Idone and S. Pia, Some classes of projected dynamical systems in Banach spaces and variational inequalities, J. Global Optim., 40 (2008), 119–128.
[26] S. Lawphongpanich and D. W. Hearn, Simplical decomposition of the asymmetric traffic assignment problem, Transportation Res. Part B, 18 (1984), 123–133.
[27] J.-L. Lions and G. Stampacchia, Variational inequalities, Comm. Pure Appl. Math., 20 (1967), 493–519.
[28] S.-Y. Matsushita and L. Xu, On finite convergence of iterative methods for variational inequalities in Hilbert spaces, J. Optim. Theory Appl., 161 (2014), 701–715.
[29] A. Moudafi, The split common fixed-point problem for demicontractive mappings, Inverse Problems, 26 (2010), 6pp.
[30] A. Nagurney, D. Parkes and P. Daniele, The Internet, evolutionary variational inequalities, and the time-dependent Braess paradox, Comput. Manag. Sci., 4 (2007), 355–375.
[31] B. Panicucci, M. Pappalardo and M. Passacantando, A path-based double projection method for solving the asymmetric traffic network equilibrium problem, Optim. Lett., 1 (2007), 171–185.
[32] F. Raciti, Equilibrium conditions and vector variational inequalities: A complex relation, J. Global Optim., 40 (2008), 353–360.
[33] L. Scrimali and C. Mirabella, Cooperation in pollution control problems via evolutionary variational inequalities, J. Global Optim., 70 (2018), 455–476.
[34] Y. Shehu and O. Iyiola, On a modified extragradient method for variational inequality problem with application to industrial electricity production, J. Ind. Manag. Optim., 15 (2019), 319–342.
[35] M. J. Smith, The existence, uniqueness and stability of traffic equilibria, Transportation Res., 13 (1979), 295–304.
[36] G. Stampacchia, Formes bilinéaires coercitives sur les ensembles convexes, C. R. Acad. Sci. Paris, 258 (1964), 4413–4416.
[37] X. K. Sun and S. J. Li, Duality and gap function for generalized multivalued $\epsilon$-vector variational inequality, Appl. Anal., 92 (2013), 482–492.
[38] J. Yang, Dynamic power price problem: An inverse variational inequality approach, J. Ind. Manag. Optim., 4 (2008), 673–684.

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E-mail address: shiprasingh384@gmail.com
E-mail address: avivg@braude.ac.il
E-mail address: qxlxajh@163.com