UNIQUENESS OF SHORTEST CLOSED GEODESICS FOR GENERIC FINSLER METRICS

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Abstract. In every conformal class of Finsler (or Riemannian) metrics on a closed manifold there exists a residual subset of Finsler metrics, such that, with respect to the residual Finsler metrics, in any non-trivial homotopy class of free loops there is precisely one shortest geodesic loop.

1. Introduction and main result

We consider Finsler metrics on closed manifolds, which can be thought of as a norm in each tangent space, while we do not assume reversibility, i.e. symmetry with respect to $v \mapsto -v$. Every Riemannian metric induces of course a Finsler metric, while it is also well-known that one can study the much more general Tonelli Lagrangian systems using Finsler metrics, cf. e.g. [CIPP98].

Let $M$ denote a closed manifold, i.e. compact, connected and $\partial M = \emptyset$.

Definition 1.1. A function $F : TM \to \mathbb{R}$ is a Finsler metric on $M$, if the following conditions are satisfied:

1. (smoothness) $F$ is $C^\infty$ at every $v \in TM, v \neq 0$,
2. (positive homogeneity) $F(av) = aF(v)$ for all $v \in TM, a \geq 0$,
3. (strict convexity) the fiberwise Hessian $\text{Hess}(F^2|_{T_xM})$ of the square $F^2$ is positive definite at every vector $v \in T_xM - \{0\}$ for all $x \in M$.

We fix for the rest of the paper a Finsler metric $F$ on $M$.

In [Man96], R. Mañé introduced a concept of genericity for Lagrangian systems, namely perturbing a given Lagrangian $L : TM \to \mathbb{R}$ by a potential function $V : M \to \mathbb{R}$ and considering the new Lagrangian $L' = L + V$. Properties of such generic Lagrangians have been studied in the literature, e.g. in [BC08]. In the setting of Finsler metrics, we use a different notion of genericity, which is related to the before mentioned one via Maupertuis’ principle. We write

$E := C^\infty(M) = C^\infty(M, \mathbb{R}), \quad E_+ := \{\lambda \in E : \lambda(x) > 0 \ \forall x \in M\}$

and endow both sets with the $C^\infty$-topology.

Definition 1.2. We say that a property $P$ of the Finsler metric $F$ is conformly generic, if there exists a residual set $\mathcal{O} \subset E_+$ (i.e. $\mathcal{O}$ is the countable

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intersection of open and dense subsets of $E_+$), such that $\mathcal{P}$ holds for every Finsler metric of the form $\sqrt{\lambda} \cdot F$, $\lambda \in \mathcal{O}$.

**Remark 1.3.**

- Note that the following two properties are left invariant under the perturbation $\sqrt{\lambda} \cdot F$. (1) reversibility: $F(-v) = F(v)$ for all $v \in TM$. (2) Riemannian: $F(v) = \sqrt{g(v, v)}$ for a Riemannian metric $g$ on $M$.

- Countable intersections of residual sets are again residual and, as $E$ is a Fréchet space, residual sets are dense in $E_+$.

We denote by $\Gamma(M)$ the set of all non-trivial homotopy classes of free, piecewise $C^1$ loops $c : \mathbb{R}/\mathbb{Z} \to M$. A classical way to obtain closed geodesics in $M$ is minimization of the $F$-length $l_F(c) = \int F(c(t)) dt$ in the homotopy classes $\gamma \in \Gamma(M)$. Namely, for all $\gamma \in \Gamma(M)$, the infimum $\inf_{c \in \gamma} l_F(c)$ exists and is attained at closed geodesics in $\gamma$ (cf. e.g. Theorem 8.7.1 (2) in [BCS00]).

In the following, when we speak of $F$-geodesics $c : [a, b] \to M$, we assume that $c$ is parametrized such that $F'(c) = \text{const.}$. Also, if $c(t)$ is a geodesic and $t_0 \in \mathbb{R}$, then we think of $c(t + t_0)$ as the same geodesic. We prove here the following theorem.

**Theorem 1.4.** Let $F$ be a Finsler metric on a closed manifold $M$, such that the set $\Gamma(M)$ is countable. Then the following property is conformly generic: $F$ has in each non-trivial, free homotopy class $\gamma \in \Gamma(M)$ precisely one shortest closed geodesic.

**Structure of this paper.** We will approach the proof of Theorem 1.4 through an application of a result due to R. Mañé from [Man96]. To keep the paper self-contained, we prove the special case used in this paper in Section 2. In Section 3, we study probability measures concentrated on loops of a given homotopy class and prove Theorem 1.4.

2. An abstract result due to R. Mañé

In this section, we prove an abstract result, following the ideas of R. Mañé (Section 3 in [Man96]). We work in the following setting, where all vector spaces are real vector spaces.

- $E$ is a Hausdorff topological vector space and has the following property: if $\phi : E \to \mathbb{R}$ is such that $\limsup \phi(f_n) < \phi(f)$, if $f_n \to f$, then the sublevels $\{\phi < c\} \subset E$ are open for all $c \in \mathbb{R}$,

- $V$ is a topological vector space and $K \subset V$ is non-empty, compact, convex and metrizable,

- $\varphi : E \times K \to \mathbb{R}$ is bilinear, sequentially continuous and separates $K$ in the sense that

$$\forall x, y \in K, x \neq y \exists f \in E : \varphi(f, x) \neq \varphi(f, y).$$
We set
\[ m(f) := \min \varphi(f, \cdot)|_K, \]
\[ \mathcal{M}(f) := \{ x \in K : \varphi(f, x) = m(f) \}. \]

As a first observation, we have the following continuity property.

**Lemma 2.1.** The map \( f \in E \mapsto m(f) \in \mathbb{R} \) is sequentially continuous. Moreover, let \( f_n \rightarrow f \) in \( E \). Then for all sequences \( x_n \in \mathcal{M}(f_n) \), any limit point \( x \) of \( \{ x_n \} \subset K \) belongs to \( \mathcal{M}(f) \).

**Proof.** If \( x \in \mathcal{M}(f) \), then by the continuity of \( \varphi \)
\[ m(f) = \varphi(f, x) = \lim \varphi(f_n, x) \geq \limsup m(f_n). \]
Let \( x_n \in \mathcal{M}(f_n) \), then after passing to a subsequence, assume that \( x_n \rightarrow x \) in \( K \). Then we have
\[ m(f_n) = \varphi(f_n, x_n) \rightarrow \varphi(f, x) \geq m(f), \]
showing that \( \lim \inf m(f_n) \geq m(f) \). The first claim follows. Moreover, using \( m(f_n) \rightarrow m(f) \), (1) shows that \( x = \lim x_n \in \mathcal{M}(f) \). □

**Theorem 2.2** (Mañe). There exists a residual subset \( \mathcal{O} \subset E \) (i.e. a countable intersection of open and dense subsets), such that
\[ f \in \mathcal{O} \implies \text{card} \mathcal{M}(f) = 1. \]

In the following, we fix a metric \( d \) on \( K \), whose induced topology coincides with the original topology on \( K \), rendering \( (K, d) \) a compact metric space. For a subset \( M \subset (K, d) \) we write
\[ \text{diam} M = \sup \{ d(x, y) : x, y \in M \}. \]

**Proof.** Set \( \mathcal{O}_n := \{ f \in E : \text{diam} \mathcal{M}(f) < 1/n \} \) and \( \mathcal{O} = \bigcap_{n \in \mathbb{N}} \mathcal{O}_n \). The density of \( \mathcal{O}_n \) will follow from Lemma 2.4 below. Suppose now that \( f, f_k \in E \) with \( f_k \rightarrow f \). By Lemma 2.1 if \( x_k, y_k \in \mathcal{M}(f_k) \) are chosen such that \( \limsup d(x_k, y_k) = \limsup \text{diam} \mathcal{M}(f_k) \), we find limits \( x, y \in \mathcal{M}(f) \) with \( d(x, y) \geq \limsup \text{diam} \mathcal{M}(f_k) \). This shows that \( f \mapsto \text{diam} \mathcal{M}(f) \) is sequentially upper semi-continuous and by our assumption on \( E \), it follows that the sets \( \mathcal{O}_n \) are open. □

Given a closed, convex subset \( K_0 \subset K \) and \( f \in E \), we set
\[ \mathcal{M}_0(f) := \{ x \in K_0 : \varphi(f, x) = \min \varphi(f, \cdot)|_{K_0} \}. \]

**Lemma 2.3.** If \( K_0 \subset K \) is closed and convex, then
\[ \forall \varepsilon > 0 \exists f \in E : \text{diam} \mathcal{M}_0(f) \leq \varepsilon. \]

**Proof.** For \( x \neq y \) in \( K_0 \) let \( f(x, y) \in E \) such that \( \varphi(f(x, y), x - y) \neq 0 \) and consider open neighborhoods \( U(x, y) \subset K_0 \times K_0 \) of \( (x, y) \), such that
\( \varphi(f(x, y), x' - y') \neq 0 \) for all \( (x', y') \in U(x, y) \). We find a sequence \( U(x_n, y_n) \) covering \( K_0 \times K_0 - \text{diag} \) Setting \( f_n := f(x_n, y_n) \), we find

\[
\forall x, y \in K_0, x \neq y \exists n \in \mathbb{N} : \quad \varphi(f_n, x - y) \neq 0.
\]

For \( n \in \mathbb{N} \) consider the linear map

\[
T_n : K_0 \to \mathbb{R}^n, \quad T_n(x) := (\varphi(f_1, x), \ldots, \varphi(f_n, x)).
\]

Fix \( \varepsilon > 0 \). We claim that

\[
\exists n \in \mathbb{N} \forall p \in \mathbb{R}^n : \quad \text{diam} T_n^{-1}(p) \leq \varepsilon.
\]

Proof of (3): Suppose that for all \( n \in \mathbb{N} \) we find \( p_n \in \mathbb{R}^n \), such that \( \text{diam} T_n^{-1}(p_n) > \varepsilon \). Then we find a sequence \( x_n, y_n \in K_0 \), such that \( T_n(x_n) = T_n(y_n) = p_n \) and \( d(x_n, y_n) \geq \varepsilon \). Choosing convergent subsequences of \( x_n, y_n \) by the compactness of \( K_0 \), we find limits \( x, y \in K_0 \) with \( d(x, y) \geq \varepsilon \). By (2), we find then \( i \in \mathbb{N} \), such that \( \varphi(f_i, x - y) \neq 0 \) and by continuity of \( \varphi \), we have for large \( n \geq i \) (using the euclidean distance in \( \mathbb{R}^n \))

\[
0 = d(T_n(x_n), T_n(y_n)) \geq |\varphi(f_i, x_n - y_n)| \geq |\varphi(f_i, x - y)|/2 > 0,
\]
a contradiction.

We fix \( n \), such that (3) holds. Let \( p \in T_n(K_0) \) be an exposed point, i.e. there exists \( v \in (\mathbb{R}^n)^* \), such that \( v|_{T_n(K_0)} \) attains its unique minimum at \( p \) (such points exist by Straszewicz’s theorem, since \( T_n(K_0) \subset \mathbb{R}^n \) is convex and compact and hence has extremal points). Writing \( v = \sum_{i=1}^n v_i \cdot dx_i \) and using the linearity of \( \varphi \), we obtain

\[
v \circ T_n = \varphi(f, \cdot)|_{K_0}, \quad \text{where } f := \sum_{i=1}^n v_i \cdot f_i.
\]

By definition of \( v \) and \( \mathcal{M}_0(f) \), we find

\[
\mathcal{M}_0(f) = T_n^{-1}(p).
\]

The claim follows using (3).

\( \square \)

**Lemma 2.4.** If \( f \in E \), then

\[
\forall \varepsilon > 0, \quad U \subset E \text{ neighborhood of } f, \quad \exists f_\varepsilon \in U : \quad \text{diam } \mathcal{M}(f_\varepsilon) \leq \varepsilon.
\]

**Proof.** Fix \( f \in E \) and \( \varepsilon > 0 \). The set \( K_0 := \mathcal{M}(f) \subset K \) is closed and convex, so by Lemma 2.3 we find \( g \in E \), such that \( m_0(g) := \min \varphi(g, \cdot)|_{\mathcal{M}(f_0)} \) is attained in a set

\[
\mathcal{M}_0(g) \subset \mathcal{M}(f), \quad \text{diam } \mathcal{M}_0(g) \leq \varepsilon/2.
\]
We want to prove that

\[ \lim_{t \to 0} \text{diam} \mathcal{M}(f + tg) \leq \varepsilon, \]

then the lemma follows. Observe that for \( x \in \mathcal{M}_0(g) \subset \mathcal{M}(f) \) we have

\[ m(f + tg) \leq \varphi(f + tg, x) = m(f) + t \cdot m_0(g). \]

For \( x \in \mathcal{M}(f + tg) \) it follows that

\[ t \cdot \varphi(g, x) = \varphi(f + tg, x) - \varphi(f, x) \leq m(f + tg) - m(f) \]

\[ \leq m(f) + t \cdot m_0(g) - m(f) = t \cdot m_0(g) \]

and hence for \( t > 0 \)

\[ \varphi(g, \cdot)_{|\mathcal{M}(f + tg)} \leq m_0(g). \]

Let now \( x_t, y_t \in \mathcal{M}(f + tg) \), then by Lemma 2.1 any pair of limit points \( x, y, \) as \( t \to 0 \), belongs to \( \mathcal{M}(f) \). Moreover, (7) shows that \( \varphi(g, x), \varphi(g, y) \leq m_0(g) \), i.e. \( x, y \in \mathcal{M}_0(g) \), hence \( d(x, y) \leq \varepsilon/2 \) by (11) and for \( t > 0 \) small we have \( d(x_t, y_t) \leq \varepsilon \). Hence, (5) follows. \( \Box \)

3. Measures on free loops

We return to the setting described in the introduction and begin with some notation. Let \(|.|\) be the norm of a fixed Riemannian metric \( g \) on the closed manifold \( M \). For \( b > 0 \) denote by \( \mathcal{M}_b \) the set of probabilities supported in the compact set

\[ T_{\leq b} M := \{ v \in TM : |v| \leq b \}. \]

Then \( \mathcal{M}_b \) can be identified with a subset of the dual space \( C^0(T_{\leq b} M)^* \). We endow \( \mathcal{M}_b \) with the topology of weak*-convergence, i.e.

\[ \mu_n \to \mu \iff \int fd\mu_n \to \int fd\mu \quad \forall f \in C^0(T_{\leq b} M). \]

It is well-known that \( \mathcal{M}_b \) is convex, metrizable and compact with respect to this topology.

**Definition 3.1.** Fix \( \gamma \in \Gamma(M) \) and \( b > 0 \). If \( c : \mathbb{R}/\mathbb{Z} \to M \) is piecewise \( C^1 \) with \( |\dot{c}| \leq b \), we define \( \mu_c \in \mathcal{M}_b \subset C^0(T_{\leq b} M)^* \) by

\[ \int fd\mu_c = \int_0^1 f(\dot{c})dt \quad \forall f \in C^0(T_{\leq b} M). \]

and define \( K^b_\gamma \subset \mathcal{M}_b \) to be the closure of the convex hull of

\[ \{ \mu_c \in \mathcal{M}_b \mid c : \mathbb{R}/\mathbb{Z} \to M \text{ piecewise } C^1, c \in \gamma, |\dot{c}| \leq b \}. \]
Lemma 3.2. If \( l \) and \( b \) are defined by shortest geodesic loops in \( \gamma \). We will show that for large \( b \), given the Finsler metric \( F \), we always find particular minimizers of the action

\[ A_F : K^b_\gamma \to \mathbb{R}, \quad A_F(\mu) := \int F^2 d\mu, \]

which are defined by shortest geodesic loops in \( \gamma \). Note that, since \( F \) is continuous, \( A_F \) is continuous by definition of the weak*-topology. Obviously, \( A_F \) is also linear.

Using the compactness of \( M \), we denote by \( c_F \geq 1 \) some constant, such that

\[ \frac{1}{c_F} \cdot F \leq |.| \leq c_F \cdot F \]

and writing \( l_F, l_g \) for the lengths with respect to \( F, g \), respectively, we define

\[ C_0(F, \gamma) := c_F^2 \cdot \min_{c \in \gamma} l_g(c). \]

Lemma 3.2. If \( c : \mathbb{R}/\mathbb{Z} \to M \) is an \( F \)-geodesic loop, minimal with respect to \( l_F \) in \( \gamma \in \Gamma(M) \), then for all \( b \geq C_0(F, \gamma) \) the measure \( \mu_c \) lies in \( K^b_\gamma \) and \( \mu_c \) is minimal in \( K^b_\gamma \) with respect to the action \( A_F : K^b_\gamma \to \mathbb{R} \).

**Proof.** Let \( \alpha : \mathbb{R}/\mathbb{Z} \to M \) be a \( g \)-geodesic loop, minimal in \( \gamma \) with respect to \( l_g \). If \( c : \mathbb{R}/\mathbb{Z} \to M \) is minimal in \( \gamma \) with respect to \( l_F \), using \( l_g(c_0) = |c_0|, l_F(c) = F(\dot{c}) \), we find

\[ |\dot{c}| \leq c_F \cdot F(\dot{c}) \leq c_F \cdot \max_{c \in \gamma} F(\dot{c}_0) \leq c_F^2 \cdot |c_0| = c_F^2 \cdot \min_{c \in \gamma} l_g(c) = C_0(F, \gamma). \]

We obtain \( \mu_c \in K^b_\gamma \) for \( b \geq C_0(F, \gamma) \). The \( L^2 \)-Cauchy-Schwarz inequality shows for any piecewise \( C^1 \) curve \( \alpha : [0, 1] \to M \), that

\[ l_F(\alpha[0, 1])^2 = \langle F(\dot{\alpha}), 1 \rangle^2_{L^2[0, 1]} \leq \|F(\dot{\alpha})\|^2_{L^2[0, 1]} \cdot \|1\|^2_{L^2[0, 1]} = \int_0^1 F^2(\dot{\alpha}) dt = A_F(\mu_c) \]

with equality if and only if the piecewise \( C^0 \)-functions \( 1, F(\dot{\alpha}) : [0, 1] \to \mathbb{R} \) are linearly dependent. Hence

\[ l_F(\alpha[0, 1])^2 \leq A_F(\mu_c), \quad \text{equality} \iff \exists k \in \mathbb{R} : F(\dot{\alpha}) = k \text{ a.e.} \]

and we find for any piecewise \( C^1 \) loop \( \dot{c} : \mathbb{R}/\mathbb{Z} \to M \) in \( \gamma \) with \( |\dot{c}| \leq b \), using \( F(\dot{c}) = \text{const.} \) and the minimality of \( c \) in \( \gamma \), that

\[ A_F(\mu_c) = l_F(c)^2 \leq l_F(c')^2 \leq A_F(\mu_c'). \]

Using the continuity and linearity of \( A_F \), we find

\[ A_F(\mu_c) \leq A_F(\mu) \quad \forall \mu \in K^b_\gamma. \]

\[ \square \]
In order to apply Theorem 2.2 to our situation, we introduce the relevant notation. Recall $E = C^\infty(M)$ and define a metric $d$ on $E$ by

$$d(f, g) = \sum_{k \geq 0} \frac{1}{2^k} \frac{\|f - g\|}{1 + \|f - g\|^k},$$

where $\|\cdot\|_k$ denotes the $C^k$-norm. This metric generates the $C^\infty$-topology on $E$. In particular, $E$ is a Hausdorff topological vector space and a set $U \subset E$ is open if for all $f \in U$ we find an open Ball $B_d(f, \varepsilon) \subset U$. Let $\phi : E \to \mathbb{R}$ be upper semi-continuous in the sense that $\limsup \phi(f_n) \leq \phi(f)$, if $f_n \to f$. We claim that the sets $\{f \in E : \phi(f) < c\}$ are open in $E$. Suppose the contrary, i.e. there exists $f \in E$ with $\phi(f) < c$ and for all $n \in \mathbb{N}$ there exists $f_n \in B_d(f, 1/n)$, such that $\phi(f_n) \geq c$. But then $c \leq \limsup \phi(f_n) \leq \phi(f) < c$, contradiction.

Let

$$\mathfrak{M} := \{\mu \text{ positive, finite Borel measure on } M\} \subset C^0(M)^*,$$

which is metrizable, when endowed with the weak*-topology (cf. e.g. Theorem 8.3.2 in [Bog07]). Similar to the setting in [BC08], writing $\pi : TM \to M$ for the canonical projection, we define a projection

$$\pi_*^F : \cup_{b > 0} M_b \to \mathfrak{M}, \quad \int f d(\pi_*^F \mu) := \int (f \circ \pi) \cdot F^2 d\mu \quad \forall f \in C^0(M).$$

The map $\pi_*^F$ is linear and continuous and hence the set

$$K := \pi_*^F(K_b^\gamma) \subset V := \text{span}(K) \subset C^0(M)^*$$

is convex, compact and metrizable in $\mathfrak{M}$, suppressing $\gamma, b$ in the notation for the moment. We define a bilinear, continuous map by integration:

$$\varphi : E \times K \to \mathbb{R}, \quad \varphi(f, \mu) := \int f d\mu.$$

Then $\varphi$ separates $K \subset C^0(M)^*$, since $E$ is dense in $C^0(M)$. Write

$$\mathcal{M}_b^\gamma(f) := \{\mu \in K : \varphi(f, \mu) = \min \varphi(f, \cdot)|_K\}.$$

Theorem 2.2 can then be restated in our situation as follows.

**Corollary 3.3.** In the setting described above, assuming $K_b^\gamma \neq \emptyset$, there exists a residual set $\mathcal{O}_\gamma^b \subset E$, such that

$$f \in \mathcal{O}_\gamma^b \implies \text{card } \mathcal{M}_b^\gamma(f) = 1.$$

Using this result, we can prove our main theorem.

**Proof of Theorem 2.4.** Note that for $b \geq C_0(F, \gamma) = c_2^0 \cdot \min_{\gamma} l_g$ we have $K_b^\gamma \neq \emptyset$ by Lemma 3.2. Recall $E_+ = \{\lambda \in E : \lambda(x) > 0 \ \forall x \in M\}$. For fixed $\gamma \in \Gamma(M)$, using the residual sets $\mathcal{O}_\gamma^b$ from Corollary 3.3, we consider the residual subset of $E_+$ defined by

$$\mathcal{O}_\gamma := \bigcap_{b \in \mathbb{N}, \ b \geq C_0(F, \gamma)} \mathcal{O}_\gamma^b \cap E_+.$$
Let $\lambda \in O_\gamma$ and assume that there are two shortest $\sqrt{\lambda}F$-geodesic loops $c_0, c_1 : \mathbb{R}/\mathbb{Z} \to M$ in $\gamma$. We choose

$$b \in \mathbb{N}, \quad b \geq \max\{C_0(\sqrt{\lambda}F, \gamma), C_0(F, \gamma)\},$$

then by definition of $O_\gamma$ we have $\lambda \in O^b_\gamma$. By Lemma 3.2, the two measures $\mu_{c_0}, \mu_{c_1}$ evenly distributed on $\dot{c}_0, \dot{c}_1$ belong to $K^b_\gamma$ and minimize the action $A_{\sqrt{\lambda}F} : K^b_\gamma \to \mathbb{R}$. Hence, if $\mu = \pi^F_\ast \hat{\mu} \in K$ for some $\hat{\mu} \in K^b_\gamma$, we find

$$\varphi(\lambda, \pi^F_\ast \mu_{c_i}) = \int \lambda d(\pi^F_\ast \mu_{c_i}) = \int (\lambda \circ \pi) \cdot F^2 d\mu_{c_i} = A_{\sqrt{\lambda}F}(\mu_{c_i})$$

$$\leq A_{\sqrt{\lambda}F}(\hat{\mu}) = \varphi(\lambda, \pi^F_\ast \hat{\mu}) = \varphi(\lambda, \mu).$$

This shows that both $\pi^F_\ast \mu_{c_i}$ minimize $\varphi(\lambda, \cdot) \big|_K$ and, by definition of $\lambda \in O^b_\gamma$, we have $\pi^F_\ast \mu_{c_0} = \pi^F_\ast \mu_{c_1}$. We obtain

$$\int f d(\pi^F_\ast \mu_{c_0}) = \int f d(\pi^F_\ast \mu_{c_1}) \quad \forall f \in C^0(M),$$

showing that for all $f \in C^0(M)$ we have

$$\int_0^1 (f \circ c_0) \cdot F'(\dot{c}_0)^2 dt > 0 \iff \int_0^1 (f \circ c_1) \cdot F'(\dot{c}_1)^2 dt > 0.$$

By considering bump functions $f : M \to [0, \infty)$ and using $F'(\dot{c}_1) > 0$ we obtain $c_0[0, 1] = c_1[0, 1]$. Hence, the $c_i$ are reparametrizations of each other. Finally, we set

$$O := \bigcap_{\gamma \in \Gamma(M)} O_\gamma,$$

which is residual in $E_+$, since $\Gamma(M)$ is countable by assumption. The theorem follows.

\[ \Box \]

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