Mach-type soliton in the Novikov-Veselov Equation

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Abstract

Using the reality condition of the solutions, one constructs the Mach-type soliton of the Novikov-Veselov equation by the minor-summation formula of the Pfaffian. We study the evolution of the Mach-type soliton and obtains the amplitude of the Mach stem wave is also less than four times of the one of the incident wave. It is shown that the length of the Mach stem wave is linear with time. One discusses the relations with V-shape initial value waves for different critical values of Miles parameter.

Keywords: Pfaffian, Mach-type soliton, Mach stem wave, V-shape wave
1 Introduction

Recently, the resonance theory of line solitons of KP-(II) equation (shallow water wave equation)
\[ \partial_x(-4u_t + u_{xxx} + 6uu_x) + 3u_{yy} = 0 \]
has attracted much attractions using the totally non-negative Grassmannian \([1, 3, 6, 11, 14, 16, 17]\), that is, those points of the real Grassmannian whose Plucker coordinates are all non-negative. For the KP-(II) equation case, the \(\tau\)-function is described by the Wronskian form with respect to \(x\). The Mach reflection problem describes the resonant interaction of solitary waves appearing in the reflection of an obliquely incident waves onto a vertical wall. John Miles discussed an oblique interaction of solitary waves and found a resonant interaction to describe the Mach reflection phenomena \([28]\). In this end, he predicts an extraordinary fourfold amplification of the wave at the wall. The Miles theory in terms of the KP equation is investigated in \([14, 18, 19, 20]\) (and references therein). The point is that irregular reflection can be described by the (3142)-type soliton and the stem in the middle part should be a Mach stem wave. Inspired by their works, one can consider the Novikov-Veselov equation similarly.

One considers the Novikov-Veselov (NV) equation \([5, 9, 12, 26, 30]\) with real solution \(U\):
\[
\begin{align*}
U_t &= \text{Re}[\partial_z^3 U + 3\partial_z(QU) - 3\epsilon\partial_zQ], \\
\partial_z Q &= \partial_z U, \quad t \in \mathbb{R}
\end{align*}
\]
where \(\epsilon\) is a real constant. The NV equation \([1]\) is one of the natural generation of the famous KdV equation and can have the Manakov’s triad representation \([24]\)
\[ L_t = [A, L] + BL, \]
where \(L\) is the two-dimension Schrodinger operator
\[ L = \partial_z \partial_z + U - \epsilon \]
and
\[ A = \partial_z^3 + Q\partial_z + \partial_z^3 + \bar{Q}\partial_z, \quad B = Qz + \bar{Q}z. \]
It is equivalent to the linear representation
\[ L\phi = 0, \quad \partial_t \phi = A\phi. \]
We remark that when \(\epsilon \to \pm \infty\), the Veselov-Novikov equation reduces to the KP-I \((\epsilon \to -\infty)\) and KP-(II) \((\epsilon \to \infty)\) equation respectively \([10]\). To make a comparison with KP-(II) equation, we only consider \(\epsilon > 0\).
Let $\phi_1, \phi_2$ be any two independent solutions of (2). Then one can construct the extended Moutard transformation using the skew product \cite{2,21,27}

\[
W(\phi_1, \phi_2) = \left( \phi_1 \partial \phi_2 - \phi_2 \partial \phi_1 \right) dz + \left( \phi_1 \bar{\partial} \phi_2 - \phi_2 \bar{\partial} \phi_1 \right) d\bar{z} + \left[ \phi_1 \partial^3 \phi_2 - \phi_2 \partial^3 \phi_1 \right] + 3Q(\phi_1 \partial \phi_2 - \phi_2 \partial \phi_1) dt,
\]

\[\text{such that}\]
\[
\hat{U}(t, z, \bar{z}) = U(t, z, \bar{z}) + 2\partial \bar{\partial} \ln W(\phi_1, \phi_2)
\]
\[
\hat{Q}(t, z, \bar{z}) = Q(t, z, \bar{z}) + 2\partial \bar{\partial} \ln W(\phi_1, \phi_2),
\]

is also a solution of the NV equation (1).

For fixed potential $U_0(z, \bar{z}, t)$ and $Q_0(z, \bar{z}, t)$ of the NV \cite{1}, we can take any $2N$ wave functions $\phi_1, \phi_2, \phi_3, \ldots, \phi_{2N}$ (or their linear combinations) of (2). Then the $2N$-step successive extended Moutard transformation can be expressed as the Pfaffian form \cite{2,25} (also see \cite{22,29})

\[
U = U_0 + 2\partial \bar{\partial} \ln Pf(\phi_1, \phi_2, \phi_3, \ldots, \phi_{2N})
\]

\[
Q = Q_0 + 2\partial \bar{\partial} \ln Pf(\phi_1, \phi_2, \phi_3, \ldots, \phi_{2N}),
\]

where $Pf(\phi_1, \phi_2, \phi_3, \ldots, \phi_{2N})$ is the Pfaffian defined by

\[
Pf(\phi_1, \phi_2, \phi_3, \ldots, \phi_{2N}) = \sum_{\sigma} \epsilon(\sigma)W_{\sigma_1\sigma_2}W_{\sigma_3\sigma_4} \cdots W_{\sigma_{2N-1}\sigma_{2N}},
\]

and $W_{\sigma_i\sigma_j} = W(\phi_{\sigma(i)}, \phi_{\sigma(j)})$ is the extended Moutard transformation \cite{3}, $\sigma$ being some permutations.

To construct the N-solitons solutions, we take $V = U = 0$ in (1) and then (2) becomes

\[
\partial \bar{\partial} \phi = \epsilon \phi
\]
\[
\phi_t = \phi_{zzz} + \phi_{\bar{z}\bar{z}z},
\]

where $\epsilon$ is non-zero real constant. The general solution of (5) can be expressed as

\[
\phi(z, \bar{z}, t) = \int_\Gamma e^{(i\lambda)z + (i\lambda)\bar{z} + i\lambda t + \frac{\lambda^3 t}{\sigma} + \frac{\lambda^4 t}{\Gamma}} \nu(\lambda) d\lambda,
\]

where $\nu(\lambda)$ is an arbitrary distribution and $\Gamma$ is an arbitrary path of integration such that the RHS of (6) is well defined. One takes $\nu_m(\lambda) = \delta(\lambda - p_m)$, where $p_m$ is complex numbers. Define

\[
\phi_m = \frac{\phi(p_m)}{\sqrt{3}} = \frac{1}{\sqrt{3}} e^{F(p_m)},
\]

\[3\]
where

$$F(\lambda) = (i\lambda)z + (i\lambda)^3t + \frac{\epsilon}{i\lambda} \bar{z} + \frac{\epsilon^3}{(i\lambda)^3}t.$$  

Plugging \((\phi_m, \phi_n)\) into the extended Moutard transformation \((3)\), we obtain

$$W(\phi_m, \phi_n) = i \frac{p_{n} - p_{m}}{p_{n} + p_{m}} e^{F(p_{m}) + F(p_{n})}. \quad (8)$$

To study resonance, we introduce the real Grassmannian (or the \(2N \times M\) matrix) to construct \(N\) solitons. To this end, one considers linear combination of \(\phi_n\). Let 

$$\vec{\Psi} = (\phi_1, \phi_2, \phi_3, \cdots, \phi_M)^T$$

and \(H\) be an \(2N \times M (2N \leq M)\) of real constant matrix (or Grassmannian). Suppose that 

$$H \vec{\Psi} = \vec{\Psi}^* = (\Psi_1^*, \Psi_2^*, \Psi_3^*, \cdots, \Psi_{2N}^*)^T,$$

that is, 

$$\Psi_n^* = h_{n1} \phi_1 + h_{n2} \phi_2 + h_{n3} \phi_3 + \cdots + h_{nM} \phi_M, \quad 1 \leq n \leq 2N.$$  

Then one has by the minor-summation formula \([23, 13]\)

$$\tau_N = Pf(\Psi_1^*, \Psi_2^*, \Psi_3^*, \cdots, \Psi_{2N}^*) = Pf(HW_MH^T)$$

$$= \sum_{I \subseteq [M], \#I = 2N} Pf(H_I^I) det(H_{II}), \quad (9)$$

where the \(M \times M\) matrix \(W_M\) is defined by the element \([8]\) and \(H_I^I\) denote the \(2N \times M\) submatrix of \(H\) obtained by picking up the rows and columns indexed by the same index set \(I\). By this formula, the resonance of real solitons of the Novikov-Veselov equation can be investigated just like the resonance theory of KP-(II) equation \([14, 15, 16]\). Finally, the N-solitons solutions are defined by \([4, 7]\)

$$U(z, \bar{z}, t) = 2 \partial \bar{\partial} \ln \tau_N(z, \bar{z}, t), \quad V(z, \bar{z}, t) = 2 \partial \bar{\partial} \ln \tau_N(z, \bar{z}, t). \quad (10)$$

To obtain the real potential \(U\), the following reality conditions \([4]\) for resonance have to be considered

$$|p_k|^2 = |q_k|^2 = \epsilon > 0, \quad k = 1, 2, 3, \cdots, m,$$

given \(m\) pairs of complex numbers \((p_1, q_1), (p_2, q_2), \cdots (p_m, q_m).\)

Letting \(p_m = \sqrt{\epsilon} e^{i\alpha_m}\) and removing \(i\) factor from \((8)\) afterwards, one has

$$W(\phi_m, \phi_n) = -\tan \frac{\alpha_n - \alpha_m}{2} e^{\phi_{mn}}, \quad (11)$$
\[ \phi_{mn} = F(p_m) + F(p_n) = -2\sqrt{\epsilon} \left[ x(\sin \alpha_m + \sin \alpha_n) + y(\cos \alpha_m + \cos \alpha_n) \right] \\
+ 2t\epsilon \sqrt{\epsilon} (\sin 3\alpha_m + \sin 3\alpha_n) \] (12)

Therefore, given a \( 2N \times M \) matrix \( H \), the associated \( \tau_H \)-function can be written as by (9) \[ 8 \]

\[ \tau_H = \sum_{I \subset [M], \#I = 2N} \Gamma_I \Lambda_I(x, y, t), \] (13)

where

\[ \Lambda_I(x, y, t) = Pf(W_{2N}) = (-1)^N \left( \prod_{i=2, i>j}^{2N} \tan \frac{\alpha_i - \alpha_j}{2} \right) e^{\sum_{m=1}^{2N} F(p_m)}, \]

\( \Gamma_I \) being the \( 2N \times 2N \) minor for the columns with the index set \( I = \{i_1, i_2, \ldots, i_{2N}\} \).

Also, to keep \( \tau_H \) totally positive (or totally negative), we assume that the matrix \( H \) belongs to the totally non-negative Grassmannian \[ 16, 17 \] and the angle \( \alpha \) satisfies the following condition:

\[ -\frac{\pi}{2} \leq \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_{M-1} < \alpha_M \leq \frac{\pi}{2}. \]

For one-soliton solution, we have , \( -\frac{\pi}{2} \leq \alpha_i < \alpha_j < \alpha_k \leq \frac{\pi}{2} \)

\[ \tau_1 = \tan \frac{\alpha_i - \alpha_j}{2} e^{\phi_{ij}} + a \tan \frac{\alpha_i - \alpha_k}{2} e^{\phi_{ik}} \]
\[ = ae^{\phi_{ik}} \tan \frac{\alpha_i - \alpha_k}{2} \left[ 1 + \frac{1}{a} \frac{\tan \frac{\alpha_i - \alpha_j}{2}}{a \tan \frac{\alpha_i - \alpha_k}{2}} e^{F(p_j) - F(p_k)} \right] \]
\[ = ae^{\phi_{ik}} \tan \frac{\alpha_i - \alpha_k}{2} \left[ 1 + e^{F(p_j) - F(p_k) + \theta_{jk}} \right] \]

where \( a \) is a constant and the phase shift

\[ \theta_{jk} = \ln \frac{\tan \frac{\alpha_i - \alpha_j}{2}}{a \tan \frac{\alpha_i - \alpha_k}{2}} = \ln \frac{\tan \frac{\alpha_i - \alpha_j}{2}}{\tan \frac{\alpha_i - \alpha_k}{2}} - \ln a \]

Hence the real one-soliton solution is \[ 8 \]

\[ U = 2\partial_x \partial_y \ln ae^{\phi_{ik}} \left( \tan \frac{\alpha_i - \alpha_k}{2} \left[ 1 + e^{F(p_j) - F(p_k) + \theta_{jk}} \right] \right) \]
\[ = 2\partial_x \partial_y \left[ 1 + e^{F(p_j) - F(p_k) + \theta_{jk}} \right] \]
\[ = \frac{1}{2} |p_k - p_j|^2 \text{sech}^2 \left[ \frac{F(p_j) - F(p_k) + \theta_{jk}}{2} \right] \]
\[ = 2\epsilon \sin^2 \left( \frac{\alpha_k - \alpha_j}{2} \right) \text{sech}^2 \left[ \frac{F(p_j) - F(p_k) + \theta_{jk}}{2} \right] \] (14)
\[ = A_{\{j,k\}} \text{sech}^2 \frac{1}{2} \left( \vec{K}_{\{j,k\}} \cdot \vec{x} - \Omega_{\{j,k\}} t + \theta_{jk} \right). \]
From [12] the amplitude \( A_{j,k} \), the wave vector \( \vec{K}_{j,k} \) and the frequency \( \Omega_{j,k} \) are defined by

\[
A_{j,k} = 2\epsilon \sin^2\left(\frac{\alpha_k - \alpha_j}{2}\right) \quad (15)
\]

\[
\vec{K}_{j,k} = 2\sqrt{\epsilon} \left(-\sin \alpha_j + \sin \alpha_k, -\cos \alpha_j + \cos \alpha_k\right)
\]

\[
\Omega_{j,k} = 2\epsilon \sqrt{\epsilon} \left[-\sin 3\alpha_j + \sin 3\alpha_k\right]
\]

The direction of the wave vector \( \vec{K}_{j,k} = (K^x_{j,k}, K^y_{j,k}) \) is measured in the clockwise sense from the y-axis and it is given by

\[
\frac{K^y_{j,k}}{K^x_{j,k}} = \frac{-\cos \alpha_j + \cos \alpha_k}{-\sin \alpha_j + \sin \alpha_k} = -\tan \frac{\alpha_j + \alpha_k}{2}, \quad (16)
\]

that is, \( \frac{\alpha_j + \alpha_k}{2} \) gives the angle between the line soliton and the y-axis in the clockwise sense. And the soliton velocity \( V_{j,k} \) is

\[
V_{j,k} = \frac{\epsilon \sin 3\alpha_k - \sin 3\alpha_j}{4\frac{\sin^2 \frac{\alpha_j - \alpha_k}{2}}{\sin^2 \frac{\alpha_j - \alpha_k}{2}}} \left(\sin \alpha_k - \sin \alpha_j, \cos \alpha_k - \cos \alpha_j\right) \quad (17)
\]

The paper is organized as follows. In section 2, one investigates Mach-type or (3142)-type soliton for the Novikov-Veselov equation. One shows the evolution of the Mach-type soliton and obtains the amplitude of the Mach stem wave ( [1,4]-soliton) is also less than four times of the one of the incident wave ( [1,3]-soliton). Furthermore, the length of the Mach stem wave is linear with time. In section 3, we discuss the relations with V-shape initial value waves for different critical values of Miles parameter \( \kappa \). In section 4, we conclude the paper with several remarks.

### 2 Mach Type Soliton

In this section, we investigate the Mach-type or (3142)-type soliton. The corresponding totally non-negative Grassmannian is the the matrix [14]

\[
H_M = \begin{bmatrix}
1 & a & 0 & -c \\
0 & 0 & 1 & b
\end{bmatrix}
\]

where \( a, b, c \) are positive numbers. When \( c = 0 \), one has the \( O \)-type soliton for the Novikov-Veselov equation. For V-shape initial value waves, one can introduce parameter \( \kappa \) to determine the evolution into Mach-type or \( O \)-type soliton (see next section). We remark that the \( Y \)-shape, \( O \)-type, and \( P \)-type solitons for the Novikov-Veselov equation are investigated in [8]. Now,

\[
H_M \frac{1}{\sqrt{3}} \begin{bmatrix}
\phi(p_1), \phi(p_2), \phi(p_3), \phi(p_4)
\end{bmatrix}^T = \frac{1}{\sqrt{3}} \begin{bmatrix}
\phi(p_1) + a\phi(p_2) \\
\phi(p_3) + b\phi(p_4)
\end{bmatrix} = \begin{bmatrix}
\Psi_1^* \\
\Psi_2^*
\end{bmatrix}
\]
A direct calculation yields by [9], [11] and [12]

\[
\tau_M = W(\Psi_1, \Psi_2) = W(\phi_1, \phi_3) + bW(\phi_2, \phi_4) + aW(\phi_2, \phi_3) + abW(\phi_2, \phi_4) + cW(\phi_3, \phi_4)
\]

\[
= \tan \frac{\alpha_1 - \alpha_3}{2} e^{-2\sqrt{\epsilon}[\sin(\alpha_1 + \sin \alpha_3) + y(\cos \alpha_1 + \cos \alpha_3)] + 2\epsilon \sqrt{\epsilon}[\sin(3\alpha_1 + \sin 3\alpha_3)]}
\]

\[
+ \tan \frac{\alpha_1 - \alpha_4}{2} e^{-2\sqrt{\epsilon}[\sin(\alpha_1 + \sin \alpha_4) + y(\cos \alpha_1 + \cos \alpha_4)] + 2\epsilon \sqrt{\epsilon}[\sin(3\alpha_1 + \sin 3\alpha_4)]}
\]

\[
+ \tan \frac{\alpha_2 - \alpha_3}{2} e^{-2\sqrt{\epsilon}[\sin(\alpha_2 + \sin \alpha_3) + y(\cos \alpha_2 + \cos \alpha_3)] + 2\epsilon \sqrt{\epsilon}[\sin(3\alpha_2 + \sin 3\alpha_3)]}
\]

\[
+ \tan \frac{\alpha_2 - \alpha_4}{2} e^{-2\sqrt{\epsilon}[\sin(\alpha_2 + \sin \alpha_4) + y(\cos \alpha_2 + \cos \alpha_4)] + 2\epsilon \sqrt{\epsilon}[\sin(3\alpha_2 + \sin 3\alpha_4)]}
\]

\[
+ \tan \frac{\alpha_3 - \alpha_4}{2} e^{-2\sqrt{\epsilon}[\sin(\alpha_3 + \sin \alpha_4) + y(\cos \alpha_3 + \cos \alpha_4)] + 2\epsilon \sqrt{\epsilon}[\sin(3\alpha_3 + \sin 3\alpha_4)]},
\] (18)

where

\[\frac{-\pi}{2} \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq \frac{\pi}{2}.\]

To investigate the asymptotic behavior for \(|y| \to \infty\), we use the notation [14], considering the line \(x = -cy\),

\[\eta_m(c) = -c \sin \alpha_m + \cos \alpha_m.\]

When \(\eta_m(c) = \eta_n(c)\), one gets

\[c = \frac{\cos \alpha_m - \cos \alpha_n}{\sin \alpha_m - \sin \alpha_n} = -\tan \frac{\alpha_m + \alpha_n}{2}.\]

Since

\[\eta_m(c) - \eta_i(c)\] \[
\left.\right|_{c = -\tan \frac{\alpha_i + \alpha_j}{2}} = \cos \alpha_m - \cos \alpha_i + \tan \frac{\alpha_i + \alpha_j}{2}(\sin \alpha_m - \sin \alpha_i)
\]

\[= (\sin \alpha_m - \sin \alpha_i)[\tan \frac{\alpha_i + \alpha_j}{2} - \tan \frac{\alpha_i + \alpha_m}{2}],\]

we have the following order relations among the other \(\eta_m(c)'s\) along \(c = -\tan \frac{\alpha_i + \alpha_j}{2}\)

\[
\begin{cases}
\eta_i = \eta_j < \eta_m & \text{if } i < m < j, \\
\eta_i = \eta_j > \eta_m & \text{if } m < i \text{ or } m > j.
\end{cases}
\]

Then by a similar argument in [14], one knows that by [14]

(a) For \(y >> 0\), there are four unbounded line solitons, whose types from left to right are

\[1, 3), [3, 4]\]

(19)

(b) For \(y << 0\), there is five unbounded line solitons, whose types from left to right are

\[4, 2], [2, 1]\]

(20)
It can be verified by the Maple software.

Now, we can discuss the relations between the parameters $a, b, c$ and phase shifts of these line solitons. Let us first consider the line solitons in $x >> 0$. There are two solitons which are $[3,4]$-soliton and $[2,1]$-soliton. The $[3,4]$-soliton is obtained by the balance between the exponential terms $W(\phi_1, \phi_3)$ and $bW(\phi_1, \phi_4)$, and the $[2,1]$-soliton is obtained by the balance between the exponential terms $W(\phi_1, \phi_3)$ and $aW(\phi_2, \phi_3)$. Therefore, the phase shifts of $[3,4]$-soliton and $[2,1]$-soliton for $x >> 0$ are given by

$$\theta^{+}_{[3,4]} = \ln \frac{\tan \frac{a_1-a_3}{2}}{\tan \frac{a_4-a_1}{2}} - \ln b, \quad \theta^{+}_{[2,1]} = \ln \frac{\tan \frac{a_3-a_2}{2}}{\tan \frac{a_3-a_2}{2}} - \ln a.$$  

For the line solitons in $x << 0$, there are two solitons which are $[1,3]$-soliton and $[4,2]$-soliton. The $[1,3]$-soliton is obtained by the balance between the exponential terms $cW(\phi_3, \phi_4)$ and $bW(\phi_1, \phi_4)$, and the $[4,2]$-soliton is obtained by the balance between the exponential terms $cW(\phi_3, \phi_4)$ and $aW(\phi_2, \phi_3)$. Therefore, the phase shifts of $[1,3]$-soliton and $[4,2]$-soliton for $x << 0$ are given by

$$\theta^{-}_{[1,3]} = \ln \frac{\tan \frac{a_4-a_3}{2}}{\tan \frac{a_4-a_3}{2}} + \ln \frac{b}{c}, \quad \theta^{-}_{[4,2]} = \ln \frac{\tan \frac{a_3-a_1}{2}}{\tan \frac{a_3-a_1}{2}} + \ln \frac{a}{c}.$$  

So one can see that

$$\theta^{-}_{[1,3]} + \theta^{+}_{[3,4]} = \theta^{-}_{[4,2]} + \theta^{+}_{[2,1]} = \text{total phase shift} = \ln \frac{\tan \frac{a_3-a_1}{2}}{\tan \frac{a_4-a_3}{2}} - \ln c. \quad (21)$$  

We define the parameter $s$ (representing the total phase shift) by

$$s = e^{-\theta^{-}_{[4,2]} - \theta^{+}_{[2,1]}},$$

which leads to

$$a = \frac{\tan \frac{a_3-a_1}{2}}{\tan \frac{a_3-a_1}{2}} s^{-1}, \quad b = \frac{\tan \frac{a_3-a_1}{2}}{\tan \frac{a_3-a_1}{2}} s^{-1}, \quad c = \frac{\tan \frac{a_3-a_1}{2}}{\tan \frac{a_3-a_1}{2}} s.$$  

Hence we know that the three parameters $a, b, c$ can be used to determine the localizations of three asymptotic line solitons, that is, two in $x << 0$ and one in $x > 0$. The $s$-parameter represents the relative localations of the intersection point of the $[1,3]$-soliton and $[3,4]$-soliton with the x-axis. Especially, when $s = 1, \theta^{-}_{[4,2]} = 0, \theta^{-}_{[1,3]} = 0$, all of the four solitons will intersect at $(0, 0)$ when $t = 0$. One remarks that the bounded line soliton $[1,4]$ (Mach stem wave), obtained by the balance between the exponential terms $W(\phi_1, \phi_3)$ and $cW(\phi_3, \phi_4)$, has the maximal amplitude among all the solitons by (15) (Figure 1) and the velocity is obtained by (17). Furthermore, when $t < 0$, there is a bounded line $[2,3]$-soliton (Figure 2, the left side of the triangle), obtained by the balance between the exponential terms $abW(\phi_2, \phi_4)$ and
Now, we consider the case \( \alpha_3 = -\alpha_2 \geq 0, \alpha_4 = -\alpha_1 \geq 0 \), and the amplitude
\[ A = A_{[1,3]} = A_{[4,2]} \leq 2\epsilon \]
is fixed. Then one can see that \([1,3]\)-soliton and \([4,2]\)-soliton is symmetric to the x-axis and similarly for \([3,4]\)-soliton and \([2,1]\)-soliton. By (15), one knows
\[ \frac{\alpha_3 + \alpha_1}{2} \leq \frac{\alpha_3 - \alpha_1}{2} = \frac{\alpha_3 + \alpha_4}{2} = \arcsin \sqrt{\frac{A}{2\epsilon}}. \]
Therefore the angle between the \([1,3]\)-soliton and the y-axis (counter-clockwise) is less than the angle between the \([3,4]\)-soliton and the y-axis (clockwise). We see that given \( A \) and \( 2\epsilon \) there is a critical angle \( \varphi_C = \arcsin \sqrt{\frac{A}{2\epsilon}} \) for the angle between the \([1,3]\)-soliton and the y-axis (counter-clockwise). Then one can introduce the following Miles-parameter \([8,11,28]\) to describe the interaction for the Mach-type solution, noticing \( \frac{\alpha_3 + \alpha_1}{2} \leq 0 \),
\[ \kappa = \left| \frac{\tan \frac{\alpha_3 + \alpha_1}{2}}{\tan \frac{\alpha_3 + \alpha_4}{2}} \right| = \left| \frac{\tan \frac{\alpha_3 + \alpha_1}{2}}{\tan \varphi_C} \right| = \frac{\left| \tan \frac{\alpha_3 + \alpha_1}{2} \right|}{\sqrt{\frac{A}{2\epsilon}-A}} \leq 1. \tag{22} \]
From (15), we have thus using \( \kappa \)
\begin{align*}
A &= A_{[1,3]} = A_{[4,2]} = \frac{2\epsilon(\tan \varphi_C)^2}{1 + (\tan \varphi_C)^2} \\
A_{[3,4]} &= A_{[2,1]} = \frac{2\epsilon(\tan \varphi_C)^2}{\frac{1}{\kappa^2} + (\tan \varphi_C)^2} \leq A \\
A_{[1,4]} &= 2\epsilon \sin^2 \frac{\alpha_4 - \alpha_1}{2} = 2\epsilon \left[ \sin \left( \frac{\alpha_4 - \alpha_3}{2} + \frac{\alpha_3 - \alpha_1}{2} \right) \right]^2 \\
&= \frac{2\epsilon(\tan \varphi_C)^2(\kappa + 1)^2}{[1 + (\tan \varphi_C)^2][1 + \kappa^2(\tan \varphi_C)^2]} \\
&= A \frac{(\kappa + 1)^2}{[1 + \kappa^2(\tan \varphi_C)^2]} < 4A \tag{23}
\end{align*}
Remark: To make a comparison with KP (II), we see that
\[ 2\epsilon - A = 2\epsilon (1 - \sin^2 \frac{\alpha_3 - \alpha_1}{2}) = 2\epsilon \cos^2 \frac{\alpha_3 - \alpha_1}{2}. \]
When \( \epsilon \to \infty \), \( \alpha_1 \to -\frac{\pi}{2} \) and \( \alpha_3 \to \frac{\pi}{2} \) such that
\[ \epsilon \cos^2 \frac{\alpha_3 - \alpha_1}{2} = \frac{1}{4}, \]
Then

\[ \kappa \rightarrow \sqrt{\frac{1 + \frac{\alpha_3 + \alpha_4}{2}}{1 - \frac{\alpha_3 - \alpha_4}{2}}} \]

which is the Miles parameter of KP (II) to describe the interactions of water wave solitons [14, 19, 20].

Since the [1,4]-soliton (Mach stem wave) is increasing its length with time but its end points will lie in a line (see figure 4 and 5), we can obtain them as follows.

We choose \( s = 1, \theta_{[4,2]} = 0, \theta_{[1,3]} = 0 \) such that [1,3]-soliton and [1,4]-soliton will intersect at \((0, 0)\) when \( t = 0 \). From (14), the ridges of [1,3]-soliton and [1,4]-soliton are given by \( F(p_1) - F(p_3) = 0, \quad F(p_1) - F(p_4) = 0 \), which lead to

\[
\begin{align*}
&x(-\sin \alpha_1 + \sin \alpha_3) + y(-\cos \alpha_1 + \cos \alpha_3) + t\epsilon (\sin 3\alpha_1 - \sin 3\alpha_3) = 0 \\
x(-\sin \alpha_1 + \sin \alpha_4) + y(-\cos \alpha_1 + \cos \alpha_4) + t\epsilon (\sin 3\alpha_1 - \sin 3\alpha_4) = 0
\end{align*}
\]

Noticing that \( \alpha_3 = -\alpha_2 \geq 0, \alpha_4 = -\alpha_1 \geq 0 \), one gets

\[
\begin{align*}
x &= \frac{t\epsilon \sin 3\alpha_4}{\sin \alpha_4} = t\epsilon (4\cos^2 \alpha_4 - 1) \\
y &= \frac{x(\sin \alpha_1 - \sin \alpha_3) + t\epsilon(-\sin 3\alpha_1 + \sin 3\alpha_3)}{-\cos \alpha_1 + \cos \alpha_3} \\
&= t\epsilon \frac{(4\cos^2 \alpha_4 - 1)(\sin \alpha_1 - \sin \alpha_3) + (-\sin 3\alpha_1 + \sin 3\alpha_3)}{-\cos \alpha_1 + \cos \alpha_3} \\
&= 4t\epsilon \sin \alpha_3(\sin \alpha_3 + \sin \alpha_4)(\sin \alpha_4 - \sin \alpha_3) \\
&= 4t\epsilon \sin \alpha_3(\sin \alpha_3 + \sin \alpha_4) \cot \frac{\alpha_3 + \alpha_4}{2}
\end{align*}
\]

Using (22), one has

\[
\begin{align*}
\sin \alpha_3 &= \sin \left(\frac{\alpha_3 + \alpha_4}{2} - \frac{\alpha_4 - \alpha_3}{2}\right) = \frac{(1 - \kappa)\tan \varphi_C}{\sqrt{[1 + (\tan \varphi_C)^2][1 + \kappa^2(\tan \varphi_C)^2]}}, \\
\cos \alpha_4 &= \cos \left(\frac{\alpha_3 + \alpha_4}{2} + \frac{\alpha_4 - \alpha_3}{2}\right) = \frac{1 - \kappa(\tan \varphi_C)^2}{\sqrt{[1 + (\tan \varphi_C)^2][1 + \kappa^2(\tan \varphi_C)^2]}}, \\
\sin \alpha_3 + \sin \alpha_4 &= 2\sin \frac{\alpha_3 + \alpha_4}{2} \cos \frac{\alpha_3 - \alpha_4}{2} = \frac{2\tan \varphi_C}{\sqrt{[1 + (\tan \varphi_C)^2][1 + \kappa^2(\tan \varphi_C)^2]}} \\
4\cos^2 \alpha_4 - 1 &= 4(1 - \kappa(\tan \varphi_C)^2)^2 \\
&= \left[1 + (\tan \varphi_C)^2\right] \left[1 + \kappa^2(\tan \varphi_C)^2\right] - 1 \\
&= 3 + (\tan \varphi_C)^2 \left[3\kappa^2(\tan \varphi_C)^2 - \kappa^2 - 8\kappa - 1\right] \\
&= \left[1 + (\tan \varphi_C)^2\right] \left[1 + \kappa^2(\tan \varphi_C)^2\right]
\end{align*}
\]

A simple calculation yields using (23)

\[
\begin{align*}
y &= 8t\epsilon \tan \varphi_C \frac{1 - \kappa}{[1 + (\tan \varphi_C)^2][1 + \kappa^2(\tan \varphi_C)^2]} = 4tA_{[1,4]} \frac{1 - \kappa}{(1 + \kappa^2)(\tan \varphi_C)}
\end{align*}
\]

10
\[
\tan \chi = \frac{y}{x} = \frac{8(1 - \kappa) \tan \phi_C}{3 + (\tan \phi_C)^2[3\kappa^2(\tan \phi_C)^2 - \kappa^2 - 8\kappa - 1]}.
\]

Hence one knows that the length of [1,4]-soliton is linear with time and its end points will lie in a line having slope \( \pm \tan \chi \). Furthermore, from (24), one gets that the [1,4]-soliton moves to the right if \( \alpha_4 < \frac{\pi}{3} \), and moves to the left if \( \alpha_4 > \frac{\pi}{3} \). In particular, if \( \alpha_4 = \frac{\pi}{3} \) or by (26)

\[
3 + (\tan \phi_C)^2[3\kappa^2(\tan \phi_C)^2 - \kappa^2 - 8\kappa - 1] = 0.
\]

then [1,4]-soliton’s length is increasing along the \( y \)-axis. When \( \kappa = 1 \) (or \( \alpha_3 = 0 \)), one has \( A = \frac{\pi}{2} \) by (27) and \( \alpha_4 = \frac{\pi}{2} \). In this special case, the soliton is fixed. It is different from the KP-(II) case \([18, 20]\).

3 Relations with V-Shape Initial Value Waves

In this section, we investigate some relations with the V-shape initial value waves for the Novikov-Veselov equation as compared with the KP-(II) case \([14, 18, 19, 20]\). The main purpose is to study the interactions between line solitons, especially for the meaning of the critical angle \( \phi_C \).

Recalling the one-soliton solution (14) and (15), one considers the initial data given in the shape of \( V \) with amplitude \( A \) and the oblique angle \( \phi_I < 0 \) (measured in the clockwise sense from the \( y \)-axis):

\[
A \text{sech}^2[\sqrt{2A} \cos \phi_I(x - |y| \tan \phi_I)].
\]

We notice here the V-shape initial wave is in the negative \( x \)-region. The main idea is that we can think the initial value wave as a part of Mach-type soliton \([18]\) or O-type soliton \([8]\), that is, \( c = 0 \) in \([18]\). In order to identify those soliton solutions from the V-shape \((28)\), we denote them as \([i^+, j^+]-\)soliton for \( y > > 0 \) and \([i^-, j^-]-\)soliton for \( y << 0 \). Solitons for \( y \to \pm \infty \) have by (14)

\[
A = 2\epsilon \sin^2 \frac{\alpha_{j^+} - \alpha_{i^+}}{2} = 2\epsilon \sin^2 \frac{\alpha_{i^-} - \alpha_{j^-}}{2},
\]

\[
\phi_I = \frac{\alpha_{j^+} + \alpha_{i^+}}{2} = -\left(\frac{\alpha_{i^-} + \alpha_{j^-}}{2}\right)
\]

Assume that \( i^+ < j^+ \) and \( i^- > j^- \). Then symmetry gives

\[
\alpha_{i^+} = -\alpha_{i^-}, \quad \alpha_{j^+} = -\alpha_{j^-}.
\]

Using the parameter \([22][8][11][28]\)

\[
\kappa = \frac{|\tan \phi_I|}{\sqrt{\frac{A}{2\pi A} \tan \phi_C}},
\]

one can yield, noticing that \( \phi_C = \frac{\alpha_{j^+} + \alpha_{i^+}}{2} = \frac{\alpha_{i^-} + \alpha_{j^-}}{2} = \arctan \sqrt{\frac{A}{2\pi A}} \) from \([29]\),

\[
11
\]
\[
\begin{align*}
\bullet & \quad \kappa > 1 \Rightarrow |\varphi_I| > \varphi_C \Rightarrow -\frac{\pi}{2} \leq \alpha_{i+} < \alpha_{j+} < \alpha_{i-} \leq \frac{\pi}{2} \quad \text{(O-type)} \\
\bullet & \quad \kappa = 1 \Rightarrow |\varphi_I| = \varphi_C \Rightarrow -\frac{\pi}{2} \leq \alpha_{i+} < \alpha_{j+} = 0 < \alpha_{i-} \leq \frac{\pi}{2} \quad \text{(Y-shape)} \\
\bullet & \quad 0 < \kappa < 1 \Rightarrow |\varphi_I| < \varphi_C \Rightarrow -\frac{\pi}{2} \leq \alpha_{i+} < \alpha_{j-} < \alpha_{j+} < \alpha_{i-} \leq \frac{\pi}{2} \quad \text{(Mach-type)}
\end{align*}
\]

We see that if the angle \( \varphi_I \) is small, then an intermediate wave called the Mach stem ([1,4]-soliton) appears. The Mach stem, the incident wave ([1,3]-soliton) and the reflected wave ([3,4]-soliton) interact resonantly, and those three waves form a resonant triple. It is similar to the KP-(II) case [14].

One remarks here that for \( \kappa > 1 \) (O-type) we have \( 0 \leq \frac{\alpha_{j-} - \alpha_{i+}}{2} \leq \frac{\pi}{4} \) by (30), that is, \( A \leq \epsilon \). Thus, if we choose \( A \) such that
\[
\epsilon < A \leq 2\epsilon,
\]
we get \( \frac{\pi}{2} < \alpha_{i-} \leq \pi \). Therefore, under the condition (31), the initial value wave (28) would develop into a singular O-type soliton by (18) (\( \epsilon=0 \)) when \( \epsilon \) is fixed. On the other hand, when \( |\varphi_I| \leq \frac{\pi}{2} \), \( A \) and \( \kappa \) are fixed, one can choose
\[
\epsilon = \frac{A}{2} \left[ 1 + \left( \frac{\kappa}{\tan \varphi_I} \right)^2 \right] \geq \frac{A}{2}.
\]

Then we can obtain regular soliton solutions.

## 4 Concluding Remarks

One investigates the Mach-type (or (3142)-type) soliton of the Novikov-Veselov equation. The Mach stem ([1,4]-soliton), the incident wave ([1,3]-soliton) and the reflected wave ([3,4]-soliton) form a resonant triple. From (23), we see that the amplitude of Mach stem is less than four times of the one of the incident wave, which is similar to the KP equation [20]; moreover, the length of the Mach stem is computed and show it is linear with time (25). On the other hand, one uses the parameter \( \kappa \) (22) to describe the critical behavior for the O-type and Mach-type solitons and notices that it depends on the the fixed parameter \( \epsilon \). And then the amplitude \( A \) of the incident wave is small than \( 2\epsilon \); furthermore, if \( \epsilon < A < 2\epsilon \), then the soliton will be singular. Now, a natural question is : what happens if \( A > 2\epsilon \) ? Another question is the minimal completion [18]. It means the resulting chord diagram has the smallest total length of the chords. This minimal completion can help us study the asymptotic solutions and estimate the maximum amplitude generated by the interaction of those initial waves. A numerical investigation of these issues will be published elsewhere.
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Figure 1: The middle portion, having maximum amplitude, is the [1,4]-soliton (Stem Wave)

Figure 2: $\alpha_1 = -\frac{23}{50}\pi$, $\alpha_2 = -\frac{1}{5}\pi$, $\alpha_3 = \frac{1}{5}\pi$, $\alpha_4 = \frac{23}{50}\pi$, $a = b = c = 1$, $\epsilon = 5$
Figure 3:

Figure 4:
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