Dynamics of the Aharonov-Bohm effect

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Abstract.

The time-dependent Dirac equation is solved using the three-dimensional Finite Difference-Time Domain (FDTD) method. The dynamics of the electron wave packet in a vector potential is studied in the arrangements associated with the Aharonov-Bohm effect. The solution of the Dirac equation showed a change in the velocity of the electron wave packet even in a region where no fields of the unperturbed solenoid acted on the electron. The solution of the Dirac equation qualitatively agreed with the prediction of classical dynamics under the assumption that the dynamics was defined by the conservation of generalized or canonical momentum of the electron.

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1. Introduction

The Finite Difference-Time Domain (FDTD) method, originally introduced by Kane Yee [1] to solve Maxwell’s equations, is for the first time applied to solve the time-dependent three-dimensional Dirac equation. The Zitterbewegung and the dynamics of well-localized electron were used as examples of FDTD applied to the case of free electrons [2]. The motion of electron wave packets inside and scattering from the potential step barrier or linearly dependent potential, arrangements associated with the Klein paradox [3], were used as examples of electron interaction with an electromagnetic scalar potential [4]. In this paper, a FDTD study of the dynamics of an electron wave packet under the influence of a vector potential is presented. Such a dynamic behavior is most often associated with the Aharonov-Bohm effect [5].

In the Aharonov-Bohm effect, as a manifestation of quantum mechanics, charged particles passing around a long solenoid can feel a magnetic flux even when all the fields of the unperturbed solenoid are zero in the region through which the particles travel. The shifts in the phase of the wave functions describing the particles have been experimentally verified by its effect on the interference fringes [6, 7, 8, 9]. Since for the unperturbed solenoid, there are no classical forces acting on the charged particles in the zero field region, the theoretical description of the Aharonov-Bohm effect contains a number of assumptions. They include assumptions on nonlocal features of quantum mechanics, the physical meaning of the vector potential, topological effects, etc., but generally accepted physical understanding is still lacking [10]. While there is still an open question on the presence of classical forces responsible for the Aharonov-Bohm effect [10, 11, 12], on a macroscopic level they have not been observed [13].

Proper quantum-mechanical description of the dynamics of a relativistic charged particle involves the solution of the Dirac equation in the time domain. In addition to initial conditions, such a dynamics is defined only by the configuration of the electromagnetic scalar and vector potentials, and does not involve knowledge or any assumption on “classical forces”. The solutions of the time-dependent Dirac equation, some of which are presented in this paper, can shed light and fill critical knowledge gaps on the theoretical and experimental interpretations of the mechanism of the Aharonov-Bohm effect, including the existence or non-existence of classical forces.

2. Time-dependent solution of the Dirac equation

The FDTD solutions of the time-dependent Dirac equation were obtained for the case when the electromagnetic field described by the four-potential \( A^\mu = \{A_0(x), \vec{A}(x)\} \) was minimally coupled to the particle [14, 15]

\[
\frac{i\hbar}{\partial t} \Psi = (H_{\text{free}} + H_{\text{int}}) \Psi,
\]

where

\[
H_{\text{free}} = -i\hbar \alpha \cdot \nabla + \beta mc^2,
\]
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\[ H_{int} = -e \alpha \cdot \vec{A} + eA_0, \]  

and

\[ \Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \Psi_3(x) \\ \Psi_4(x) \end{pmatrix}. \]  

The matrices \( \alpha \) and \( \beta \) were expressed using \( 2 \times 2 \) Pauli matrices \( \sigma \)'s and the \( 2 \times 2 \) unit matrix \( I \).

In the FDTD method, the time dependent solution of the Dirac equation was obtained using updating difference equations \[2]. As an example, the values of \( \Psi_1 \) at the position \((i \Delta x, j \Delta y, k \Delta z)\) and at the time step \((n + 1/2)\Delta t\) were obtained using the equation

\[
\Psi_{1}^{n+1/2}(I, J, K) = 2 - C^n(I, J, K) \Psi_{1}^{n-1/2}(I, J, K)
\]

\[
- \frac{c\Delta t}{2\Delta x C^n(I, J, K)} [\Psi_{3}^{n}(I, J, K + 1) - \Psi_{3}^{n}(I, J, K - 1)]
\]

\[
+ \Psi_{4}^{n}(I + 1, J, K) - \Psi_{4}^{n}(I - 1, J, K) - i(\Psi_{4}^{n}(I, J + 1, K) - \Psi_{4}^{n}(I, J - 1, K))
\]

\[
- \Psi_{4}^{n}(I, J - 1, K)) + i \frac{e\Delta t}{\hbar C^n(I, J, K)} [A_1^n(I, J, K) \Psi_{4}^{n}(I, J, K)
\]

\[
- iA_2^n(I, J, K)\Psi_{4}^{n}(I, J, K) + A_3^n(I, J, K)\Psi_{4}^{n}(I, J, K)],
\]

where \( C^n(I, J, K) = 1 + i \frac{\Delta t}{2\hbar}[mc^2 + eA_0^n(I, J, K)] \). The space and time were discretized using uniform rectangular lattices of size \( \Delta x, \Delta y \) and \( \Delta z \), and uniform time increment \( \Delta t \). While it is not generally required, in the Eq. (5) \( \Delta x = \Delta y = \Delta z \). Updating equations for \( \Psi_2, \Psi_3, \) and \( \Psi_4 \) were constructed in a similar way. As a result, the dynamics of a Dirac electron can be studied in any environment described by a four-potential \( A^\mu \) regardless of its complexity and time dependency. In this paper we studied the dynamics defined by the vector potential \( \vec{A}(x) \neq 0 \).

The Dirac equation is a differential equation of the first order and linear in \( \partial / \partial t \). As in the case of Maxwell’s equations, the entire dynamics of the electron is defined, only by its initial wave function. In this paper, the dynamics of a wave packet was defined by the initial wave function of the form

\[
\Psi(\vec{x}, 0) = N \sqrt{\frac{E + mc^2}{2E}} \begin{pmatrix} 1 \\ 0 \\ \frac{p_3 c}{E + mc^2} \\ \frac{(p_1 + ip_2) c}{E + mc^2} \end{pmatrix} e^{-\frac{\vec{x} \cdot \vec{x}}{4x_0^2} + \frac{ip \cdot \vec{x}}{x_0}},
\]

where \( N = [(2\pi)^{3/2}x_0^3]^{-1/2} \) is a normalizing constant. Eq. (6) represents a wave packet whose initial probability distribution is of a normalized Gaussian shape. Its size is defined by the constant \( x_0 \), its spin is pointed along the z-axis, and its motion is defined by the values of the momenta \( p_1, p_2, \) and \( p_3 \). Some consequences of the initial localization of the wave packet on the overall dynamics of the electron were studied in Ref. [2, 16].
3. Validation of the computation

3.1. Dynamics of a wave packet in a strong uniform magnetic field

The dynamics of a wave packet is very complex. The dynamics of a particle described by the wave packet in Eq. (6) depends on its localization, defined by the Gaussian component of the wave function, and its initial momentum, part of the wave function’s phase. While an extensive study was done on the dynamics of the wave packet related to the scalar component $A_0(x)$ of a four-potential, such a study does not validate the dynamics of the wave packet related to the vector component of a four-potential, $\vec{A}(x)$.

Applying the FDTD method to study the dynamics of a wave packet in a vector potential associated with a strong uniform magnetic field is not difficult. Classically, the dynamics consist of uniform rotational motion. In the relativistic quantum-mechanical description, however, even such a simple dynamics can validate the FDTD computation only up to a certain level. As pointed out in Ref. [17] “the dynamics is particularly rich and not adequately described by semiclassical approximations”. In Ref. [18] it was demonstrated that in the presence of an external magnetic field the wave packet splits into two parts which rotate with different cyclotron frequencies, and after a few periods, the motion acquires irregular character. As a result of such a complex dynamics, when comparing the results obtained by the FDTD method and the computation described in Ref. [17] and [18], we could not expect more than a qualitative agreement. In addition to this qualitative agreement, of equal importance to the validation of the computation is the consistency of the results obtained for different vector potential gauges.

In this paper, the motion of the wave packet described by Eq. (6) was studied in a uniform magnetic field oriented along the y-axis

$$\vec{B} = (0, B_0, 0).$$

For this field the corresponding vector potential in a rotationally invariant gauge is

$$\vec{A} = \frac{B_0}{2}(z, 0, -x).$$

The dynamics of the wave packet in a uniform magnetic field was first obtained by solving the Dirac equation for this vector potential.

While the probability densities $|\Psi|^2$ were calculated for the entire computational volume and at every time step, their values are shown here only in the horizontal plane or plane of classical particle motion. The schematics of the wave packet or charged particle motion relative to the orientation of the magnetic field is shown in Figure 1. The position and the shape of the wave packet in the horizontal plane, as it moves along its first orbit, is shown in Figure 2. The initial position of the wave packet was at the center of the vector potential. Its initial momentum was $p_1 = 0.53 \text{ MeV}/c$, making the motion relativistic. In order to force a relativistic electron to complete a full circle in the available computational space, the magnitude of the magnetic field was $B_0 = 10^8 \text{ T}$. The field of such a strength is associated with the fields at the surface of the neutron stars. The classical orbit of the electron in this field is $r_{\text{class}} = p_1/(eB_0) = 1.76 \times 10^{-2} \text{ nm}$. 
As shown in Figure 2 during the first rotation the wave packet generally follows the classical orbit, but disperses at the same time. The position of the center of probability of the wave packet relative to the classical orbit is shown in Figure 3.

The complexity of the dynamics of the wave packet increased in the later stages of the motion. While the wave packet followed the circular motion, the probability density $|\Psi|^2$ at some times assumed the spiral shape shown in Figure 4 increased or reduced its length, changed its rotational motion, and translated from one place to another. Overall, the characteristics of this dynamics were similar to the characteristics described in Ref. [17] and [18]. While the figures show some of the complex shapes of the probability density $|\Psi|^2$, the richness of the electron motion can be better appreciated through the animation of the dynamics accessible on-line [19].

Several additional tests of the FDTD method were performed using the dynamics of the electron in the uniform magnetic field. As expected, due to the normal spin and the magnetic field orientation, reversal of the orientation of the magnetic field resulted in the reversal of the direction of the rotation of the wave packet, keeping the same properties of the dynamics of motion.

Of particular interest was testing the effects of the choice of gauge. The motion of the same wave packet was also studied for the uniform magnetic field oriented along the y-axis defined by the translationally invariant gauges

$$\vec{A} = B_0(z, 0, 0),$$

or

$$\vec{A} = B_0(0, 0, -x).$$

\[9\]
\[10\]
Figure 2. The shapes and the positions of the wave packet in the horizontal plane during the first rotation. The circle represents the classical orbit. The animation can be accessed on-line [19].
In both cases the wave packet persisted in a circular motion following the classical orbit. While the dynamics of the wave packet behaved as expected, using translationally invariant gauges has an additional importance on validating the FDTD computation. As seen in Eq. (5), the gauge in Eq. (9) couples to the $\Psi_4$ component and the gauge in Eq. (10) couples to the $\Psi_3$ component. Similarly, the cross coupling exists for other components of the wave function $\Psi$. As a result, the same dynamics should be obtained by different combinatorics of the components of the vector potential and the components of the wave function. This enabled for testing of possible inconsistencies in the FDTD updating equations. No inconsistencies were found.

The dynamics of the wave packet motion in a uniform magnetic field was used here only as a validation of the FDTD method when a vector potential was applied in the Dirac equation. The same complexity of the quantum dynamics of motion as in previous publications [17, 18] was shown. While used here only for computational validation of the FDTD method, this dynamics could be studied as a separate problem in more detail in the future.

Finally, the complexity of the quantum dynamics of the wave packet in a uniform magnetic field studied here for three choices of gauge could be fully appreciated only by downloading the related animations [19].
3.2. Motion of a wave packet between two infinite lines of electric dipoles

The motion of a wave packet in a uniform magnetic field is used to validate the FDTD method when a vector potential $\vec{A}(x)$ is applied in the Dirac equation. The motion of a wave packet between two infinite lines of electric dipoles is used to validate the dynamics of a wave packet under influence of a position-dependent electromagnetic scalar potential $A_0(x)$, and to validate the geometry associated with the Aharonov-Bohm effect studied in the next section. In the case of a wave packet moving between two infinite lines of electric dipoles, the dynamics obtained from the solution of the Dirac equation can be compared to the dynamics of a charged particle influenced by a classical electromagnetic force. The classical dynamics, resulting from the interaction of a charged particle and two infinite lines of electric dipoles, is obtained following Boyer [20].

The scalar potential $A_0$ of two infinite lines of electric dipoles parallel to the y-axis and separated by a distance $2a$, with dipoles oriented in the x-direction, parallel or anti-parallel to the direction of the wave packet motion, is

$$A_0 = \pm \frac{\varphi}{2\pi\epsilon_0} \left( \frac{x}{x^2 + (z - a)^2} + \frac{x}{x^2 + (z + a)^2} \right),$$

(11)
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Figure 5. On the left side, the solution of the Dirac equation with the scalar potential $A_0$ described by Eq. (11) is compared to the solution of Eq. (12). The right side compares the solution of the Dirac equation with the scalar potential described by Eq. (13) to the solution of Eq. (14). The red circles represent the solutions of the Dirac equation and the solid lines the classical solutions. In both cases the momentum of the charged particle was $p = 0.09 \text{ MeV/c}$ at the position $x = -0.4 \text{ nm}$. The distance between the lines of electric dipoles was $2a = 0.38 \text{ nm}$ and the line density of dipoles was $\varphi = 1.7 \times 10^{-17} \text{ Cm/m}$. In the coordinate system used in this paper two infinite lines of electric dipoles were parallel to the y-axis and the wave packet moved along the x-axis.

where $\varphi$ is the line density of dipoles and $\varepsilon_0$ is the permittivity constant. With the choice of $z = 0$, the classical motion of a charged particle along the x-axis, under influence of this potential, can be calculated by solving the equation

$$\frac{dp_x}{dt} = -q \frac{\partial A_0}{\partial x} = \mp \frac{q\varphi}{\pi \varepsilon_0} \frac{1}{x^2 + a^2} \left(1 - \frac{2x^2}{x^2 + a^2}\right)$$  \hspace{1cm} (12)

The comparison of the quantum-mechanical dynamics obtained as a solution of the Dirac equation with the scalar potential described by Eq. (11) and the classical dynamics obtained as a solution of Eq. (12) is shown on the left side in Figure 5. The relativistic effects in the classical dynamics were reduced to less than 2% by choosing the initial momentum $p = 0.096 \text{ MeV/c}$. The lines of electric dipoles were separated by a distance $2a = 0.38 \text{ nm}$ and the line density of dipoles $\varphi$ was $1.7 \times 10^{-17} \text{ Cm/m}$. Figure 5 shows agreement between the quantum-mechanical and classical dynamics over the distance for which the solution of the Dirac equation was computed.

Under the same conditions, the computation was repeated for dipoles oriented in the z-direction, anti-parallel to each other, and perpendicular to the direction of the
wave packet motion. In this case the scalar potential $A_0$ is

$$A_0 = \pm \frac{\varphi}{2\pi \varepsilon_0} \left( \frac{z-a}{x^2 + (z-a)^2} \right) \left( \frac{z+a}{x^2 + (z+a)^2} \right),$$

and the equation of motion, for $z = 0$, is

$$\frac{dp_x}{dt} = \mp \frac{q\varphi}{\pi \varepsilon_0} \frac{2ax}{(x^2 + a^2)^2}. \tag{14}$$

As shown on the right side in Figure 5, the solution of the Dirac equation with the scalar potential described by Eq. (13) and the solution of Eq. (14) agree again over the range of computation.

4. Wave packet dynamics in a vector potential created by two infinite solenoids

The goal of this work is to study the dynamics of an electron wave packet under the influence of a vector potential associated with the Aharonov-Bohm effect [5]. Particularly, in this paper we present the dynamics of a wave packet obtained from the solution of the Dirac equation with a vector potential created by two infinite solenoids.

The vector potential of a single infinite solenoid oriented along the y-axis can be written as

$$\vec{A} = \begin{cases} \Phi \frac{\varphi}{2\pi R_0^2} (z,0,-x) & \text{for } r \leq R_0, \\ \Phi \frac{\varphi}{2\pi r^2} (z,0,-x) & \text{for } r > R_0, \end{cases} \tag{15}$$

where $\Phi = B_0 \pi R_0^2$ is the magnetic flux inside the solenoid, $B_0$ defines the strength of the magnetic field, $R_0$ is the radius of the solenoid, and $r = \sqrt{x^2 + z^2}$ is the distance from the center of the solenoid in the x-z plane. Outside two parallel infinite solenoids separated by a distance $2a$ and with opposite magnetic field orientation, the vector potential is then

$$\vec{A} = \frac{\Phi}{2\pi} \left( \frac{z-a}{x^2 + (z-a)^2} \right) \left( \frac{z+a}{x^2 + (z+a)^2} \right) \left( \frac{x}{x^2 + (z+a)^2} \right). \tag{16}$$

An example of the shape of this potential is shown in Figure 6. Outside the solenoids, the associated magnetic fields $\vec{B} = \vec{\nabla} \times \vec{A}$ of the vector potentials described by Eqs. (15) and (16) are zero.

The time-dependent Dirac equation is now solved for the scalar component of the four-potential $A_0 = 0$ and the vector component $\vec{A}$ defined by Eq. (16). With this choice we can study the dynamics of the wave packet in the region where the electric and the magnetic fields of the unperturbed solenoids are zero.

The dynamics of the electron motion between two infinite and parallel solenoids is essentially the same as the electron dynamics in the case of Aharonov-Bohm effect. In both cases the electron moves in a region where the unperturbed solenoids produce no Lorentz force on the electron. Because of the symmetry, however, the dynamics in the case of two infinite solenoids consists only of the motion along one straight line between
Figure 6. The shape of the vector potential of two parallel infinite solenoids with oppositely directed magnetic fluxes of the same magnitude. The potential is shown in the plane normal to the orientation of the solenoids. In the coordinate system used in this paper the magnetic field of the solenoids is oriented along the y-axis, the vector potential lays in the x-z plane, while the wave packet moves along the x-axis.

The motion of the wave packet between the solenoids, in the plane normal to the orientation of the solenoids, is shown in Figure 7. The wave packet was initially positioned away from center of the solenoids. The dynamics was studied for two initial momenta, $p_1 = 0.53 \text{ MeV/c}$ or $p_1 = 0.64 \text{ MeV/c}$. The solenoids were separated by a distance of $2a = 0.1 \text{ nm}$ and the magnitude of the magnetic flux inside each of them was $\Phi = 5.2 \times 10^{-14} \text{ Wb}$.

As shown in Figure 7 and in the related animation, the wave packet moved between two solenoids along a straight line, dispersing in time in the direction normal to the direction of motion. Since for the unperturbed solenoid, there was no classical force acting on the electron, one should have expected a constant velocity of the wave packet.
along the straight line. This was not the case. The velocity of the wave packet, shown in Figure 8 as a function of the position, increased as the packet approached the solenoids and decreased as the packet left the solenoids. It is somewhat paradoxical that while there was no classical force expected to act on the electron, the electron acceleration, obtained strictly as a solution of the Dirac equation, was not zero.

Extensive studies and verifications of this result were performed. They include the dependence of the solution of the Dirac equation on particle charge, energy, spin orientation, orientation of the solenoids, and the strength and the orientation of the magnetic field inside the solenoid. In addition, the time and the position dependencies of the particle momentum, energy and canonical momentum were computed. The results could be summarized as follows:

- the change of the velocity of the wave packet did not depend on the particle spin orientation relative to the orientation of the solenoids
- as shown in Figure 9, change of the sign of a charged particle inverted the order of the change of the velocity, but the velocities converged to the same value at spatial...
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Figure 8. The velocity of a wave packet normalized to the speed of light as a function of the wave packet position. The wave packet is moving along a straight line between two infinite parallel solenoids positioned at x=0.

infinity

- change of the orientation of the magnetic field inside the solenoids had the same effect as change of the sign of charged particles
- as also shown in Figure 9, the mechanical momentum of the particle changed during the process, but the total energy and the canonical momentum did not change

5. The dynamics of the charged particle under the conditions of the Aharonov-Bohm effect

The change of velocity was predicted more than 35 years ago by Liebowitz [21] and by Boyer [22], and was proposed as the basis for the Aharonov-Bohm phase shift. It was shown that if the magnetic interaction energy of the unperturbed solenoid currents and the passing charge is assigned to a kinetic energy change of the passing electron, then the electron has the same qualitative velocity change as found from our numerical calculation of a wave packet dynamics in the Dirac equation. Through lowest order in the particle-solenoid interaction, the energy conservation criterion used by Boyer is equivalent to conservation of the passing charge’s canonical momentum \( \vec{p}_c = \vec{p} + q \vec{A} \). Thus

\[
0 = \frac{d}{dt} [(p^2 c^2 + m^2 c^4)^{1/2} + q \vec{v} \cdot \vec{A}]
= \frac{\vec{p} c^2}{(p^2 c^2 + m^2 c^4)^{1/2}} \cdot \frac{d\vec{p}}{dt} + q \vec{v} \cdot (\vec{v} \cdot \vec{\nabla}) \vec{A}
\]
where the term $q(d\vec{v}/dt) \cdot \vec{A}$ was neglected as higher order in the interaction. This equation implies conservation of the canonical momentum of the particle. In classical dynamics, the motion of an electron can now be obtained by solving the differential equation

$$\frac{d(\vec{p} + q\vec{A})}{dt} = 0.$$  \hspace{1cm} (18)

Since $A_x$ does not explicitly depend on time, for the motion along the x-axis, Eq. (18) can be written as

$$\frac{dp_x}{dt} = -qdA_x dx = -q\frac{dA_x}{dx} v_x$$  \hspace{1cm} (19)
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In the relativistic case

$$\frac{dp_x}{dt} = \left(1 - \frac{v_x^2}{c^2}\right)^{-\frac{3}{2}} m_0 \frac{dv_x}{dt}. \quad (20)$$

Here $m_0$ is the rest mass of the electron. Using Eqs. (18) and (20) we get the differential equation for the velocity of the electron

$$\frac{dv_x}{dt} = -\frac{q}{m_0} v_x \left(1 - \frac{v_x^2}{c^2}\right)^{\frac{3}{2}} \frac{dA_x}{dx}. \quad (21)$$

Substituting $A_x$ from the Eq. (16) for $z = 0$, electron motion along the mid-path between two solenoids, the differential equation for the velocity of the electron becomes

$$\frac{dv_x}{dt} = \frac{2q\Phi}{\pi m_0 (x^2 + a^2)^{\frac{3}{2}}} v_x \left(1 - \frac{v_x^2}{c^2}\right)^{\frac{3}{2}}, \quad (22)$$

where $a$ is half the distance between two parallel infinite solenoids. This differential equation can be solved numerically. Figure 10 shows the qualitative agreement of the classical dynamics of the particle obtained from the solution of Eq. (22) and the quantum dynamics of the corresponding wave packet obtained from the solution of the Dirac equation.

To conclude this section, we can look at the ongoing question if the Aharonov-Bohm effect is the result of a force changing the velocity of a particle passing on opposite sides of a single infinitely long solenoid, or there is only a quantum-mechanical phase shift \[23, 24, 25, 26, 27, 28\]. Quantum-mechanically such a question does not exist. The

**Figure 10.** Higher, red curve, represents the velocity, as a function of the position, of the relativistic particle moving along a straight line between two infinite parallel solenoids, obtained as the solution of Eq. (22). Lower, black curve, represents the velocity of a wave packet obtained from the solution of the Dirac equation under the same conditions. The velocities are normalized to the speed of light. The solenoids are positioned at x=0.
dynamics of the relativistic electron should be obtained only as a solution of the time-dependent Dirac equation. The solution of the Dirac equation shows that the velocity of a wave packet changes even in the region where the unperturbed solenoid gives zero magnetic field. Since the change of the velocity depends on the gradient of the vector potential, the velocity of the wave packet passing on the opposite sides of the solenoid will be different. However, the particle exits the near-solenoid region with the same energy as when it entered, regardless of the side on which it passed the solenoid.

From the studies presented in this paper, the classical picture of the Aharonov-Bohm effect consists of the phase shift in the wave function of the particles passing on opposite sides of a solenoid being attributed to time lag resulting from different evolution of the velocities of the particles. A phase shift then results in an interference pattern change.

6. Conclusion

In conclusion, the full three-dimensional Finite Difference Time Domain (FDTD) method was developed to solve the Dirac equation. In this paper, the method was applied to the dynamics of the electron wave packet in a vector potential in an arrangement associated with the Aharonov-Bohm effect. The solution of the Dirac equation showed that the velocity of the electron wave packet changed even in the region where the electric and the magnetic fields were zero.

The solution of the Dirac equation qualitatively agreed with the prediction of classical dynamics under the assumption that the dynamics were defined by the conservation of generalized or canonical momentum.

The studies in this paper help establishing a picture of the Aharonov-Bohm effect as the interference pattern resulting from the phase shift in the wave function of the particles passing on opposite sides of a solenoid attributed to a time lag resulting from different evolution of the velocities of the particles.

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