SIMPLE GEODESICS ON A PUNCTURED SURFACE

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Abstract. We give a new proof of McShane’s Theorem \cite{8}, using simple equivariant methods in the hyperbolic plane.

1. Introduction and statements of results

Let $\mathcal{F}$ be a orientable hyperbolic surface, complete and of finite area, with one cusp (denoted $c$). Let $N(c)$ be a normal neighborhood of $c$, and let $S^1 = \partial N(c)$. An oriented geodesic is called cuspidal if it comes out of the cusp. We study simple cuspidal oriented geodesics on $\mathcal{F}$. Let $S^1 = \partial N(c)$ be a circle that bounds the cusp, chosen symmetrically (see next section). Then we may parameterize the oriented geodesics out of the cusp by $S^1$: every oriented cuspidal geodesic determines a unique point on $S^1$ that is its first intersection with $S^1$. For a point $x \in S^1$ we denote this geodesic by $\gamma_x$. We say that $\gamma_x$ is bicuspidal if and only if it has both its ends in the cusp (in that case there is a point $x \neq y \in S^1$ so that $\gamma_y$ is $\gamma_x$ with reverse orientation); otherwise $\gamma_x$ is called unic cuspidal. Let $E \subset S^1$ be all the points $x$ for which $\gamma_x$ is simple. McShane proved the following theorem (\cite{8} Theorem 4):

Theorem 1.1 (McShane). Let $\mathcal{F}$ be a complete hyperbolic surface with finite area and a cusp $c$. With notation as above, $E$ consists of a Cantor set (say $K$) and isolated points, so that the following holds:

1. $x \in E$ is isolated if and only if $\gamma_x$ is bicuspidal;
2. $x \in K$ is an endpoint of $K$ if and only if $\gamma_x$ spirals onto a simple closed curve.\footnote{The complement of a Cantor set $K \subset S$ is a collection of open intervals; by “the endpoints of $K$” we mean the endpoints of these intervals. $K$ has a countably infinite set of endpoints and uncountable set of points that are not endpoints.}

Moreover, every connected component of $S^1 \setminus K$ contains exactly one isolated point of $E$.

Thus a component of $S^1 \setminus K$ (say $J$) is bound by two points (say $x$ and $y$) so that $\gamma_x$ and $\gamma_y$ spiral onto simple closed geodesics, say $\alpha$ and $\beta$. It is easy to see that these geodesics co-bound a pair of pants with the cusp. By using basic hyperbolic trigonometry \cite{8} Proposition 3] McShane showed that the length of $J$ is $l(S^1)/\left(1 + e^{l(S^1)/2}\right)$, where $l(\cdot)$ denotes the hyperbolic length restricted to $S^1$. This, and the fact that by Birman and Series \cite{3} the measure of $E$ is zero, implies:
Theorem 1.2 (McShane’s Identity).

\[ \sum_{\alpha, \beta} \frac{1}{1 + e^{\frac{|\alpha| + |\beta|}{2}}} = \frac{1}{2} \]

where the sum is taken over all pairs of geodesics \( \alpha, \beta \) that co-bound a pair of pants with the cusp \( c \).

Many authors worked on McShane’s Identity (see references) including Akiyoshi, Skakuma and Yamashita; Bowditch; Mirzakhani; Tan, Wong and Zhang. Their work resulted in many alternate proofs and generalizations of Theorem 1.2.

The purpose of this article is to give a short elementary proof of Theorem 1.1. The proof is set in the universal cover of \( F \) (denoted \( \tilde{F} \) and identified with the upper half plane in \( \mathbb{C} \)) and uses basic geometry of the hyperbolic plane and the action of \( \pi_1(F) \) on \( \tilde{F} \); we view \( \pi_1(F) \) as a subgroup of the isometries of \( \tilde{F} \). Our proof is based on a new interpretation of the open intervals complementary to \( K \); McShane refers to these intervals as gaps; below we call them deadzones. We remark that a geodesic on \( F \) is simple if and only if any two of its lifts to \( \tilde{F} \) are disjoint. We call a geodesic in \( \tilde{F} \) simple if it is disjoint from all its images under \( \pi_1(F) \); thus the statements “\( \gamma \subset F \) is simple” and “some lift of \( \gamma \) to \( \tilde{F} \) is simple” and “all lifts of \( \gamma \) to \( \tilde{F} \) are simple” are equivalent.

Remark. It is well known that McShane’s Theorem holds for surfaces with more than one cusp and with totally geodesic boundary components (see, for example, [15]). Our techniques can be generalized to those settings as well. However, as it is our goal to give a simple proof we will not do so here.

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2. Background: Planar Hyperbolic Geometry

Though we assume some familiarity with the basic notions of hyperbolic geometry, we pause to sketch out some essential terminology, notations and ideas; the reader familiar with planar hyperbolic geometry may skip this section as the results in the section are well-known. The hyperbolic plane may be compactified by adding a circle at infinity that we denote \( S_1^\infty \) (defined by equivalence classes of geodesics that approach each other asymptotically). We work in the upper half plane model which we identify with \( \{ z \in \mathbb{C} | \text{Re}(z) > 0 \} \). Then we get that \( S_1^\infty = \mathbb{R} \cup \{ \infty \} \). Geodesics in \( F \) are in one-to-one correspondence with pairs of distinct points in \( S_1^\infty \) and oriented geodesics are in one-to-one correspondence with ordered distinct pairs of points.

In the upper-half plane model, we arrange \( \tilde{F} \) so that \( \infty \) is a lift of the cusp and its cuspidal subgroup is generated by \( z \rightarrow z + 1 \). Then every cuspidal oriented geodesic has a vertical lift (clearly not unique) oriented downwards and ending at some \( x \in \mathbb{R} \). Throughout this paper we denote this lift by \( \tilde{\gamma}_x \) (not to be confused with \( \gamma_x \) of Section 1). It should be clear that when we refer to a neighborhood of a cuspidal geodesic \( \gamma_x \) we mean the set of geodesics \( \gamma_y \) with \( y \) in a neighborhood of \( x \). By an interval about \( \tilde{\gamma}_x \) we mean the set of geodesics \( \{ \tilde{\gamma}_y : y \in (a, b) \} \), for some \( a < x < b \).

Definition 2.1 (Horodisk). Let \( p \in S_1^\infty \). If \( p = \infty \) then an open horodisk centered at \( p \) is a set of the form \( \{ z | \text{Im}(z) > h \} \) (for some fixed \( h > 0 \)). For \( p \in \mathbb{R} \) an open horodisk centered at \( p \) is a Euclidean open disk of Euclidean radius \( r \) centered at \( p + ir \) (for some \( r > 0 \)). A closed horodisk is obtained by taking the closure in \( \tilde{F} \) of...
an open horodisk. Thus a closed horodisk is a closed upper half plane or a closed disk with a point removed from its boundary.

The fundamental group \( \pi_1(F) \) can be viewed naturally as a discrete, fixed-point free subgroup of the orientation preserving isometries of \( \tilde{F} \). The following classification is well known:

**Lemma 2.2** (Classification of hyperbolic geodesics). Let \( f : \tilde{F} \to \tilde{F} \) be an orientation preserving isometry, other than the identity. Then exactly one of the following holds:

1. (Elliptic) \( f \) has a fixed point in the interior of \( \tilde{F} \). Such \( f \) cannot be an element of \( \pi_1(F) \).
2. (Parabolic) \( f \) has exactly one fixed point and this fixed point is on \( S^1_\infty \). Locally the fixed point is an attractor on one side and repellor on the other. If \( f \in \pi_1(F) \) then the fixed point is said to be a lift of the cusp and a sufficiently small horodisk centered at the fixed point is an infinite cyclic cover of a neighborhood of the cusp.
3. (Hyperbolic) \( f \) has exactly two fixed points and these fixed points are on \( S^1_\infty \). One fixed point is an attractor and the other repellor. The geodesic connecting the fixed points is called the axis of \( f \) (denoted in this paper \( A_f \)) and \( f \) acts on \( A_f \) by translation.

Axes of hyperbolic isometries correspond to closed geodesics on \( F \): a hyperbolic isometry \( f \in \pi_1(F) \) generates a cyclic group that acts on the axis \( A_f \) and the quotient is a (not necessarily simple) closed geodesic on \( F \). Conversely, every closed geodesic on \( F \) has a lift to \( \mathbb{H}^2 \) that is an (open) geodesic, say \( A_f \). Then \( \mathbb{Z} \) (here, the fundamental group of the circle) acts on \( A_f \) to give the closed geodesic. Any non-trivial element of this cyclic group is a hyperbolic isometry whose axis is \( A_f \).

**Lemma 2.3** (Being a lift of a cusp is well-defined). No point on \( S^1_\infty \) is a fixed point of both a hyperbolic and a parabolic transformation in \( \pi_1(F) \).

*Proof.* Let \( p \) be a fixed point of a parabolic isometry. As remarked above a small horodisk about \( p \) is an infinite cyclic cover of a neighborhood of the cusp. Therefore any geodesic that has \( p \) as one of its endpoints projects to a non-compact geodesic on \( F \). On the other hand let \( q \) be a fixed point of a hyperbolic isometry \( f \); then \( q \) is the endpoint of the axis of \( f \) which projects to a closed curve on \( \tilde{F} \). \( \Box \)

Since \( F \) has finite area, fundamental domains in \( \tilde{F} \) must become arbitrarily small (in the Euclidean metric) as they approach \( \mathbb{R} \); we therefore have:

**Lemma 2.4.** The lifts of the cusp are dense in \( S^1_\infty \).

Note too that since the set of lifts of the cusp is countable, the complement of this set is also dense.

Like every study of simple geodesics, we need:

**Lemma 2.5.** Simplicity is a closed condition.

*Proof.* Let \( \tilde{\gamma} \) be a non-simple geodesic. Then \( \tilde{\gamma} \) intersects one of its images, say \( \tilde{\gamma}' \). A small perturbation of \( \tilde{\gamma} \) gives a small perturbation of \( \tilde{\gamma}' \) and so the two geodesics still intersect. Therefore non-simplicity is an open condition and simplicity is a closed condition. \( \Box \)

We end this section with a well-known lemma that relates geodesics to topological properties of curves. A curve on \( \tilde{F} \) is called proper if its ends are at infinity.
proper curves are said to be *properly homotopic* if there is a homotopy taking one to the other such that the ends remain at infinity at all times. All homotopies considered in this paper are proper.

**Lemma 2.6.** Let \( \tilde{\alpha} \) be a simple curve that is properly homotopic to a geodesic \( \tilde{\gamma} \). The \( \tilde{\gamma} \) is simple as well.

**Proof.** If \( \tilde{\gamma} \) is not simple then there exists a \( g \in \pi_1(\mathcal{F}) \) so that \( \tilde{\gamma} \cap g(\tilde{\gamma}) \neq \emptyset \); equivalently, the endpoints of \( \tilde{\gamma} \) separate the endpoints of \( g(\tilde{\gamma}) \). Since the homotopy is proper, endpoints do not move. Thus \( \tilde{\alpha} \) must intersect \( g(\tilde{\alpha}) \) and \( \tilde{\alpha} \) is not simple. \( \square \)

3. The Proof of McShane’s Theorem

We pause for the proof of McShane’s theorem, postponing several required technical lemmas.

**Proof of McShane’s Theorem.** In Section 4 we show that every simple bicuspidal geodesic \( \gamma \) lies in an open interval of otherwise non-simple geodesics (4.2); this interval is called the *deadzone* of \( \gamma \) and \( \gamma \) is called the *center* of the deadzone. Next we show that the endpoints of each deadzone are simple (4.6) and unicuspidal (4.4). Consequently, the deadzones are disjoint.

In Section 5 we show that every non-simple geodesic lies in a deadzone (5.7). In particular we have that any bicuspidal geodesic lies in a deadzone (a non simple geodesic lies in a deadzone by 5.7 and a simple geodesic is the center of a deadzone); hence since the lifts of the cusp are dense (2.4), so too is the union of the deadzones. Letting \( K \subset S^1 \) parameterize the set of simple unicuspidal geodesics we have that \( K \) is exactly the complement of the union of the deadzones. This has two immediate consequences: first every interval of \( S^1 \setminus K \) contains exactly one point of \( E \) and these points correspond to the bicuspidal geodesics (the centers of the deadzones). Second, \( K \) is totally disconnected and closed. In order to show \( K \) is a Cantor set we need only show \( K \) is perfect—that every point \( x \in K \) is a limit point of \( K \setminus \{x\} \).

In Section 6 we show that the endpoints of \( K \) are limit points (6.1). It follows that the remaining points of \( K \) are limit points as well: choose \( x \in K \subset S^1 \) that is not an endpoint; since the lifts of the cusp are dense (2.4) in any open interval \( (a, b) \) containing \( x \) there exist a lift of the cusp, say \( y \). Since \( \tilde{\gamma}_y \) is bicuspidal it is contained in a deadzone, say \( V \). By definition \( K \cap V = \emptyset \) (in particular \( x \) is not \( V \)), and by assumption \( x \) is not an endpoint; therefore an endpoint of \( V \) must also lie in \( (a, b) \) (in fact between \( x \) and \( y \)). Thus \( x \) is a limit point of \( K \) and we see that \( K \) is perfect as required.

In Section 7 we show that the simple cuspidal geodesics on \( \mathcal{F} \) that spiral into simple closed geodesics are exactly the projections of endpoints of deadzones (7.1); with this we complete the proof of McShane’s Theorem. \( \square \)

4. Deadzones

We now show that any simple bicuspidal geodesic \( \gamma \) has a neighborhood that contains no simple geodesic except \( \gamma \) itself. Before stating the theorem we describe the geometry of a simple bicuspidal oriented geodesic \( \gamma \). Let \( \tilde{\gamma}_x \) be a vertical lift of \( \gamma \) oriented downwards and ending at the point \( x \in \mathbb{R} \). A horodisk centered at \( x \) is an infinite cyclic cover of the cusp. In it there are infinitely many preimages of \( \gamma \), with their orientations alternating in and out. Therefore, the two preimages next to \( \tilde{\gamma}_x \) are both oriented away from \( x \); concentrating on the right, we denote the preimage of \( \gamma \) on the right by \( \tilde{\gamma}_1 \); the left side is treated similarly. Let \( f \in \pi_1(\mathcal{F}) \) be the
Proposition 4.1. Let $\tilde{\gamma}_x$ be a vertical simple bicuspidal geodesic. For the geodesics $\tilde{\gamma}_i$ and the points $x_i$ constructed above we have:

1. No image of $\tilde{\gamma}$ starts or ends at $x_i$ and lies between $\tilde{\gamma}_i$ and $\tilde{\gamma}_{i+1}$, $i = 0, 1, 2, \ldots$.
2. The geodesics $\tilde{\gamma}_i$ are all oriented consistently to the right (for $i \geq 1$).
3. For any $i \neq j$, $\tilde{\gamma}_i \cap \tilde{\gamma}_j = \emptyset$.

Proof. (1) If there exists a geodesic at $x_i$ that lies between $\tilde{\gamma}_i$ and $\tilde{\gamma}_{i+1}$, then its image under $f^{-i}$ is a geodesic between $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$, contradicting our choice of $\tilde{\gamma}_1$.

(2) For $\tilde{\gamma}_1$ this is true by construction; if $i > 1$ and $\tilde{\gamma}_i$ is oriented to the left, it will have to either intersect $\tilde{\gamma}_j$ for some $j < i$ or terminate at $x_j$ (for some $j < i$). The former is impossible since $\tilde{\gamma}_x$ is embedded and the latter contradicts case (1) above. Moreover $\tilde{\gamma}_i$ cannot be vertical, for if it were, $f^{i+1}(\infty) = \infty$ for $i > 1$, while $f(\infty) \neq \infty$; but then $f$ would be elliptic.

(3) Trivial since $\tilde{\gamma}_x$ is embedded.

Remark. Proposition 4.1 holds when $\tilde{\gamma}_x$ is not simple, though we do not show or require this in this paper.

Every simple bicuspidal geodesic has a neighborhood of non-simple geodesics:

Theorem 4.2. Let $\tilde{\gamma}_x$ be a simple bicuspidal geodesic. If $c \in (l, x) \cup (x, r)$ then $\tilde{\gamma}_c$ is not simple.

Proof. Consider $\tilde{\gamma}_c$, $c \in (x_0, r)$ (the other case is treated similarly). There exists an $i$ so that $c \in (x_{i-1}, x_i)$. Now consider $f(\tilde{\gamma}_c)$. But as illustrated in Figure 2 the endpoints of $\tilde{\gamma}_c$ are $\infty$ and $c$ and the endpoints of $f(\tilde{\gamma}_c)$ are $x_0$ and $f(c) \in (x_i, x_{i+1}]$. These two curves must cross and $\tilde{\gamma}_c$ cannot be simple.

Definition 4.3 (Deadzone). For a simple bicuspidal geodesic $\tilde{\gamma}_x$, we call the interval $(l, r)$ the deadzone of $\tilde{\gamma}_x$ and call $\tilde{\gamma}_x$ the center of its deadzone.
We next prove:

**Proposition 4.4.** If \( \tilde{\gamma}_x \) is bicuspidal and simple, then neither \( r \) nor \( l \) is a lift of the cusp (and so \( \tilde{\gamma}_r, \tilde{\gamma}_l \) are unicuspidal).

**Proof.** (For this proposition and the next lemma, see Figure 3) Suppose for a contradiction that \( r \) is a lift of the cusp; \( l \) is treated similarly. Since a horodisk about \( r \) is an infinite cyclic cover of the cusp there exists an image \( g(\tilde{\gamma}_x), g \in \pi_1(F) \) starting at \( r \) oriented away from the cusp to the left. By (1) of Proposition 4.1, \( g(\tilde{\gamma}_x) \) cannot start or terminate at any of the points \( x \). It must therefore intersect \( \tilde{\gamma}_i \) (for some \( i \)) contradicting the assumption that \( \tilde{\gamma}_x \) is simple. \( \Box \)

![Figure 3: Left: Proposition 4.4; right: Lemma 4.5.](image)

We next show that the endpoints of a deadzone are simple:

**Lemma 4.5.** Let \( \tilde{\gamma}_x \) be a simple bicuspidal geodesic. No image of \( \tilde{\gamma}_x \) meets \( \tilde{\gamma}_r, \tilde{\gamma}_l \).

**Proof.** Suppose for contradiction that some image of \( \tilde{\gamma}_x \) crosses \( \tilde{\gamma}_r, \tilde{\gamma}_l \) (\( \tilde{\gamma}_l \) is treated similarly). Some endpoint of this image must lie to the left of \( r \), but this endpoint cannot be any of the \( x \)'s by (1) of Proposition 4.1 on the other hand this image of \( \tilde{\gamma}_x \) cannot cross any \( \tilde{\gamma}_i \) since \( \tilde{\gamma}_x \) is simple. We have a contradiction: there is nowhere for the image to end. \( \Box \)

**Proposition 4.6.** Let \( \tilde{\gamma}_x \) be a simple bicuspidal geodesic. Then \( \tilde{\gamma}_r, \tilde{\gamma}_l \) are simple.

**Proof.** We show \( \tilde{\gamma}_r \) is simple; \( \tilde{\gamma}_l \) is treated similarly. Suppose \( \tilde{\gamma}_r \) is non-simple. Let \( g \in \pi_1(F) \) be an isometry so that \( g(\tilde{\gamma}_r) \cap \tilde{\gamma}_r \neq \emptyset \). If \( g(\tilde{\gamma}_r) \) is oriented to the left then since \( g \) is orientation preserving \( g^{-1}(\tilde{\gamma}_r) \) intersects \( \tilde{\gamma}_r \) and will be oriented to the right. Thus without loss of generality we may assume that \( g(\tilde{\gamma}_r) \) is oriented to the right. Hence \( g(\infty) \) lies to the left of \( r \) and \( g(r) \) lies to the right. The geodesics \( \{ g(\tilde{\gamma}_i) \} \) form a chain of geodesics from \( g(\infty) \) to \( g(r) \), beneath the image of \( g(\tilde{\gamma}_r) \) (see Figure 4). By Proposition 4.4 \( r \) is not a lift of the cusp and therefore cannot be the endpoint of \( g(\tilde{\gamma}_i) \) for any \( i \). Hence \( g(\tilde{\gamma}_i) \) must intersect \( \tilde{\gamma}_r \). But this contradicts Lemma 4.5 \( \Box \)

This is now trivial:
Corollary 4.7. The deadzones are disjoint.

5. Each non-simple geodesic lies in a deadzone

Definition 5.1. Let \( \tilde{\gamma}_x \) be a non-simple geodesic. A point of return is a point \( p = \tilde{\gamma}_x \cap g(\tilde{\gamma}_x) \) for some \( g \in \pi_1(\mathcal{F}) \), such that \( p \) is below \( g^{-1}(p) \) (i.e. \( \text{Im}(p) < \text{Im}(g^{-1}(p)) \)). (Note that \( g^{-1}(p) \) also lies on \( \tilde{\gamma}_x \).) The highest point of return (if it exists) is a point of return that has the largest imaginary value among all points of return on \( \tilde{\gamma}_x \).

Lemma 5.2. Each non-simple bicuspidal geodesic has a highest point of return.

Proof. Trivial, since a bicuspidal geodesic meets only finitely many of its images. \( \square \)

Remark 5.3. In fact, it is not hard to show that any non-simple cuspidal geodesic has a highest point of return.

Lemma 5.4. Let \( \tilde{\gamma} \) be non-simple, with highest point of return \( p = \tilde{\gamma} \cap g(\tilde{\gamma}), g \in \pi_1(\mathcal{F}) \). Then \( \tilde{\gamma}_g(\infty) \) is simple and bicuspidal.

Proof. Since both \( \infty \) and \( g(\infty) \) are images of the cusp, \( \tilde{\gamma}_g(\infty) \) is bicuspidal. Let \( \beta_1 \) be the portion of \( \tilde{\gamma} \) above \( g^{-1}(p) \) and let \( \beta_2 \) be the portion between \( g^{-1}(p) \) and \( p \) (see Figure 5). Let \( \alpha \) be the curve obtained from \( \beta_1 \cup \beta_2 \cup g(\beta_1) \) after slight perturbation to avoid \( p \). Then since \( p \) is the highest point of return \( \alpha \) meets no non-trivial image of itself and is therefore simple. By Lemma 2.6 \( \tilde{\gamma}_g(\infty) \) is simple as well. \( \square \)

Lemma 5.5. Let \( \tilde{\gamma}_x \) be non-simple, with highest point of return \( p = \tilde{\gamma}_x \cap g(\tilde{\gamma}_x), g \in \pi_1(\mathcal{F}) \). Then for any \( c \) between \( x \) and \( g(\infty) \), the geodesic \( \tilde{\gamma}_c \) is non-simple.

Proof. Let \( q = \tilde{\gamma}_c \cap g(\tilde{\gamma}_x) \), which exists by hypothesis. We first claim that \( \text{Im}(g^{-1}(q)) > \text{Im}(q) \); the distance from \( p \) to \( q \), along \( g(\tilde{\gamma}_x) \), is the same as the distance from \( p \) to \( q \) along \( \tilde{\gamma}_x \). But \( \text{Im}(g^{-1}(p)) > \text{Im}(p) \) by hypothesis, and moreover \( \tilde{\gamma}_x \) is a vertical geodesic. Consequently, \( \text{Im}(g^{-1}(q)) > \text{Im}(q) \) as illustrated at left in Figure 6.

Next consider the following two configurations:

\[
A = \{ \tilde{\gamma}_c, g(\tilde{\gamma}_x), q \}
\]

\[
g^{-1}(A) = \{ g^{-1}(\tilde{\gamma}_c), \tilde{\gamma}_x, g^{-1}(q) \}
\]

In the upper-half plane model of hyperbolic space, both configurations consist of a vertical ray, a half-circle and their point of intersection. Moreover, since \( g \) is conformal, the ray and the half-circle meet at the same angle in the two configurations.
That is, the two configurations are similar in the Euclidean sense. In fact though, since $\text{Im}(g^{-1}(q)) > \text{Im}(q)$, configuration $g^{-1}(A)$ is larger than configuration $A$ in the Euclidean sense. Moreover, by examination we see that the orientations of the two configurations are reversed. Consequently, $|x - g^{-1}(c)| > |g(\infty) - c| > |x - c|$ and $c$ must lie between $g^{-1}(c)$ and $x$. That is, $g^{-1}(\tilde{\gamma}_c)$ must cross $\tilde{\gamma}_c$, and so $\tilde{\gamma}_c$ is not simple.

**Lemma 5.6.** Each non-simple bicuspidal geodesic $\tilde{\gamma}$ lies in a deadzone.

**Proof.** Let $\tilde{\gamma}_x$ be a non-simple bicuspidal geodesic and let $g \in \pi_1(F)$ be such that $p = \tilde{\gamma}_x \cap g(\tilde{\gamma}_x)$ is the highest point of return; let $\tilde{\gamma}_g(\infty)$ be as above. By Lemma 5.4, the geodesic $\tilde{\gamma}_g(\infty)$ is simple and bicuspidal and so (as discussed in Section 4) has a deadzone (say $V$) of non-simple geodesics, bounded by a pair of simple geodesics (4.6). By Lemma 5.5, no endpoint of $V$ lies between $\tilde{\gamma}_g(\infty)$ and $\tilde{\gamma}_x$; by assumption $\tilde{\gamma}_x$ is bicuspidal and is not an endpoint; and so $\tilde{\gamma}_x$ must lie in $V$. 

**Remarks.** (1) Our proof is constructive, finding $\tilde{\gamma}_g(\infty)$ the center of the deadzone. Compare with [15, Lemma 5.8].

(2) Note that we proved that a non-simple cuspidal geodesic that has a highest point of return lies in a deadzone. By Remark 5.3, this proof holds for all non-simple cuspidal geodesics. Perhaps the proof below is slightly easier.
Proposition 5.7. Every non-simple geodesic lies in a deadzone.

Proof. Let $\gamma_x$ be any non-simple geodesic; by Lemma 5.6 we may assume $\gamma_x$ is unicuspial. By Lemma 2.5 there exists an open interval $(a, b) \subset \mathbb{R}$ containing $x$ such that for any $c \in (a, b)$, $\gamma_c$ is non-simple as well. Since lifts of the cusp are dense, there exists $c \in (a, b)$ such that $\gamma_c$ is bicuspial and non-simple. By Lemma 5.6, $\gamma_c$ is in the deadzone of some bicuspial simple geodesic; the endpoints of this deadzone are simple and hence cannot lie in $(a, b)$ and so in fact all of $(a, b)$ (including $x$) lies in this deadzone as well. \[\square\]

6. ENDPOINTS OF DEADZONES ARE THE LIMIT OF SIMPLE GEODESICS

We now show that there are simple bicuspial geodesics arbitrarily close to every endpoint of each deadzone. In the following proposition we consider a geodesic $\gamma_r$ at the right end of a deadzone; the proof for a geodesic at the left end is precisely the same.

Proposition 6.1. Let $\gamma_r$ be the right endpoint of a deadzone. Then for every $\epsilon > 0$, there exists an $x \in (r, r + \epsilon)$ so that $\gamma_x$ is simple and bicuspial.

Proof. Let $f \in \pi_1(F)$ be a transformation fixing $r$; by Proposition 4.4, $f$ is hyperbolic. By replacing $f$ with $f^{-1}$ is necessary we may assume $r$ is an attractor. Because $\gamma_r$ is the right endpoint of a deadzone, the axis $A_r$ of $f$ has its other endpoint to the right of $r$ and the images of $\gamma_r$ under $f^n$, $n \in \mathbb{Z}$ are as illustrated in Figure 7.

![Figure 7](https://via.placeholder.com/150)

Figure 7. $\gamma_r$ and its images under $f^n$, the axis $A_f$ and fixed point $y$

The point $r$ is an attracting fixed point of $f$. Denote the other fixed point of $f$ by $y$. Then $f^{-n}(\infty)$ limits onto $y$. Thus we see that $f^{-n}(\gamma_r)$ limits onto $A_f$; by Lemma 2.4, $A_f$ is consequently simple.

Let $A$ be the image of $A_f$ on $F$, $\gamma_r$ the image of $\gamma_r$, and $C$ a neighborhood of $A$ as illustrated at left in Figure 3. Take $\sigma$ to be any embedded curve that leaves the cusp, misses $\gamma_r \setminus C$, and crosses $C$, meeting $A$ exactly once, entering on the opposite side from $\gamma_r$ as pictured.

Choose any lift $\tilde{\sigma}$ of $\sigma$ that meets $A_f$, and let $x'$ be the image of the cusp at the end of $\tilde{\sigma}$. Note that $x'$ lies beneath $A_f$ to the right of $r$, since $\sigma$ approaches $A$ on the opposite side from $\gamma$. Recalling that $r$ is an attracting point for $f$ and that $\sigma$ meets $\gamma$ infinitely often, we note that for any sufficiently large $n \in \mathbb{N}$, $f^n(\tilde{\sigma}) \cap \gamma_r \neq \emptyset$ and $f^n(x') \in (r, r + \epsilon)$. Choose any such $n$ and let $x = f^n(x')$. 

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We now claim that the vertical geodesic \( \gamma_x \) is simple, completing the proof of the lemma: Let \( \bar{\alpha} \) consist of the portion of \( f^n(\bar{\sigma}) \) between \( x \) and \( \gamma_x \) and the portion of \( \gamma_r \) above \( f^n(\bar{\sigma}) \cap \gamma_r \). Then \( \bar{\alpha} \) is simple and homotopic to \( \gamma_x \); consequently by Lemma 2.6 \( \gamma_x \) is simple as well. \( \square \)

7. ENDPOINT OF DEADZONES SPIRAL ONTO SIMPLE CLOSED CURVES

We finished proving that \( K \) is a Cantor set and \( \mathbb{S}^1 \setminus K \) has exactly one point in each complimentary interval; those points are exactly the points corresponding to bicuspidal geodesics. To complete McShane’s Theorem we need to understand the endpoints of the deadzones. We say that an oriented geodesic \( \gamma \) on \( F \) spirals onto a (not necessarily simple) closed curve \( \alpha \) if for every \( \epsilon > 0 \) there exists \( t_0 \) so that for any \( t > t_0 \), \( \gamma(t) \) is \( \epsilon \) close to \( \alpha \).

**Proposition 7.1.** A simple cuspidal geodesic \( \gamma_x \) projects to a geodesic that spirals onto a simple closed geodesic if and only if \( \gamma_x \) is an endpoint of a deadzone.

**Proof.** Claim: \( \gamma_x \) projects to a geodesic that spirals onto a (not necessarily simple) closed curve if and only if \( x \) is a fixed point of a hyperbolic isometry.

To prove the claim, first assume that \( x \) is the fixed point of a hyperbolic isometry (say \( f \)). Then \( x \) is an endpoint of the axis of \( f \), \( A_f \). Since \( \gamma_x \) approaches \( A_f \) asymptotically and \( A_f \) projects to a closed geodesic, the claim holds in this case. Conversely, suppose that the projection of \( \gamma_x \) spirals onto a closed geodesic, say \( \alpha \). Then \( \gamma_x \) gets arbitrarily close to some lift of \( \alpha \), say \( \bar{\alpha} \). Then \( \bar{\alpha} \) is an axis of a hyperbolic isometry establishing the claim.

If \( \gamma_x \) is the endpoint of a deadzone then the projection of \( \gamma_x \) spirals onto the projection of \( A_f \). As we saw in Section 6, \( \gamma_x \) accumulates onto \( A_f \) and hence \( A_f \) is simple. Thus the projection of \( \gamma_x \) spirals onto the projection of \( A_f \), establishing the proposition in this case.

Conversely, suppose that \( \gamma_x \) is a simple geodesic that projects to a geodesic that spirals onto a simple closed geodesic. Then some lift of the closed geodesic ends at \( x \); as before, we denote the other endpoint of this lift by \( y \) and note that since \( y \) is not a lift of the cusp \( y \neq \infty \). Without loss of generality we assume that \( y \) is to the right of \( x \) (see Figure 9). It is a basic exercise in planar hyperbolic geometry.
that if two hyperbolic isometries in \( \pi_1(\mathcal{F}) \) share a fixed point then they share the other fixed point as well.\(^2\) Therefore the collection of hyperbolic isometries that fix \( x \) also fix \( y \) (and hence the geodesic connecting \( x \) to \( y \)). Given a collection \( \{ f_i \in \pi_1(\mathcal{F}) \} \) of isometries that fix \( x \) and \( y \), for any \( p \in \mathbb{S}^1 \) the only accumulation points of \( f_i(p) \) are \( x \) and \( y \); in particular, \( \infty \) is not an accumulation point and there is no accumulation point in \((-\infty, x)\). Therefore there exists a unique \( f \in \pi_1(\mathcal{F}) \) so that for any \( f \neq f' \in \pi_1(\mathcal{F}) \) with \( f'(x) = x \), \( f(\infty) < f'(\infty) \).

![Figure 9. Simple spiraling geodesic](image)

Next we show that \( \tilde{\gamma}'(\infty) \) is a simple geodesic; assume for contradiction it is not. Then for some \( h_1 \in \pi_1(\mathcal{F}) \), \( \tilde{\gamma}'(\infty) \cap h_1(\tilde{\gamma}(\infty)) \neq \emptyset \). Clearly \( h_1(\tilde{\gamma}(\infty)) \) is bicuspidal, and therefore cannot start or end at \( x \). We conclude that \( h_1(\tilde{\gamma}(\infty)) \) must intersect either \( \tilde{\gamma}_x \) of \( f(\tilde{\gamma}_x) \). Therefore some image of \( \tilde{\gamma}_x \) must intersect \( \tilde{\gamma}'(\infty) \), say \( h_2(\tilde{\gamma}_x) \). Since \( \tilde{\gamma}_x \) is simple, \( h_2(\tilde{\gamma}_x) \) cannot intersect either \( \tilde{\gamma}_x \) or \( f(\tilde{\gamma}_x) \). Therefore \( h_2(\tilde{\gamma}_x) \) terminates at \( x \) (it cannot start at \( x \) since \( x \) is not a lift of the cusp). Thus \( h_2(x) = x \). But \( h_2(\infty) \) lies to the left of \( f(\infty) \), contradicting our choice of \( f \).

Finally, the chain \( \{ f^n(\tilde{\gamma}(\infty)) \}_{n=1,2,...} \) lies within the deadzone of \( \tilde{\gamma}(\infty) \) and has right endpoint \( x \). Since \( \tilde{\gamma}_x \) is simple, we conclude that \( \tilde{\gamma}_x \) is the right endpoint of the deadzone of \( \tilde{\gamma}(\infty) \).

\( \square \)

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\(^2\)Hint: assume for contradiction there exist \( f_1, f_2 \in \pi_1(\mathcal{F}) \) hyperbolic isometries with axes \( \mathcal{A}_{f_1} \neq \mathcal{A}_{f_2} \) with one fixed point in common, and show that for any \( \epsilon > 0 \) the projection of \( \mathcal{A}_{f_1} \) is in the \( \epsilon \)-neighborhood of \( \mathcal{A}_{f_2} \) and *vice versa*; use this for the contradiction.
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