Optimum design of involute tooth profiles for K-H-V planetary drives with small teeth number differences

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Abstract

The optimum design of an involute K-H-V PDSTND is restudied and remains a difficult task even using a profile shift for the design case discussed in the literature. In fact, the difficulty arises from the design with a very high efficiency and the peculiar tip interference. Because of the failure to obtain feasible designs using profile-shifted full-depth and stub teeth, another feasible design scheme is used by altering the addendums of involute profile-shifted full-depth teeth. In this study, the objective function of the optimum design is a weighted combination of two main performance factors, i.e., the reciprocal of the contact ratio and the working pressure angle. The optimization problem is solved using a combined-mutation differential evolution algorithm. The performance can be improved using the afore-mentioned design scheme; however, there is still room for improvements. Therefore, the use of different modules in mesh for involute profile-shifted full-depth teeth with variation in addendum is proposed and the corresponding meshing equation is derived. The findings show that further improvements on the performance can be achieved.

Keywords: Planetary drive, Profile shift, Meshing equation, Non-standard module, Optimum design

1. Introduction

A speed reducer or planetary drive can reduce the rotating speed of a motor to that required by the driven mechanism. When the speed reduction ratio (SRR, denoted by ) is greater than 7 (Martin, 1982, Yu, 1987), it is necessary to use a compound gear train, a planetary gear train or a worm gear reducer, rather than a simple gear train. Gear train researchers noted very early on that when the sun gear is removed from a simple planetary gear train, the -tooth annular gear (gear 2) acts as the frame, the planet carrier (turning arm) as the input member and the -tooth planet gear (gear 1) as the output member. Furthermore, if the difference between and is very small (1 or 2), a very high SRR can be obtained with such an elementary planet gear train, i.e., , as can be very straightforwardly derived from the fundamental train value expression (Norton, 2004) or the fundamental equation for an epicyclic gear train (Paul, 1979). The elementary planet gear train, as shown in Fig. 1, is termed a K-H-V (or KHV) planetary drive and has a small teeth number difference (KHV PDSTND). Here, K denotes the center wheel (internal or external gear), H the planet carrier and V an equal angular velocity mechanism (Yu, 1987). The output member and the input member rotate in opposite directions, thus KHV is denoted as KHV(-). If the planet gear is used as the frame and the annular gear as the output member, the direction of rotation of the output member and the input member is the same. KHV is denoted as KHV(+) with . The advantages of the KHV PDSTND include the compact design with a high SRR (compared with a compound gear train comprised of external gears for the same size), high mechanical efficiency (compared with a worm gear reducer), larger contact ratio, relatively lower tooth-sliding velocities and lower contact stress, made possible because of the convex surface being in contact with the concave surface (Martin, 1982, Litvin and Fuentes, 2004). Although the aforementioned advantages of the KHV PDSTND were
already known in early train studies, the difficulties of design were not solved until 1970 because of some peculiar interferences involved (mainly the tip interference) which become severe when the difference in teeth number is very small. Therefore, standard involute gears cannot be used in a feasible KHV PDSTND design, rather involute profile-shifted gears should be used. Involute profile-shifted gears can be used to improve the smallest teeth number to avoid undercutting, to transform the fixed center distance into an adjustable center distance, to increase bending strength or pitting resistance (Spitas and Spitas, 2007), and so on. Taking into account some peculiar interference problems for the KHV PDSTND, Braren of Germany designed the well-known cycloidal speed reducer PDSTND. The cycloidal speed reducer demands a higher precision center distance, otherwise there will be variations or ripples in the output velocity. The third well-known type of PDSTND is the harmonic drive, designed by Musser in the USA in 1955. By means of an elliptically shaped wave generator (the input member), a flexspline with external involute gears (the output member) and rigid circular spline with internal gears intermesh. The difference in teeth number is two. Because the number of teeth of the flexspline of the harmonic drive requires two times the speed reduction ratio, the module of the harmonic drive is only half that of a KHV PDSTND of the same size (pitch diameter).

![Fig. 1. KHV PDSTND.](image)

Morozumi (1970a) derived the mechanical efficiency of a reference mechanism assuming the planet carrier to be stationary (i.e., a fixed-axis gear train) for a case where the pitch point is inside the contact path of a KHV PDSTND. However, the pitch point is often outside the contact path, because of the relatively large working pressure angle of a KHV PDSTND. Hence, Morozumi (1970b) also derived the mechanical efficiency of the afore-mentioned reference mechanism (fixed-carrier efficiency) for a case where the pitch point is outside the contact path. In addition, the KHV(-) PDSTNDs were designed with \( z_1 = 80 \) using the involute shifted profile only for the internal gear. The prescribed working pressure angles were 61.06, 46.03 and 37.41 degrees for teeth differences of 1, 2 and 3, respectively. The obtained contact ratios were 1.14, 1.54 and 1.74 and the fixed-carrier efficiencies were 99.83%, 99.84% and 99.86% for \( SRR = -80, -40 \) and \(-80/3 \) (i.e., teeth differences of 1, 2 and 3), respectively. Morozumi (1970c) solved the difficult design problems of KHV(-) PDSTNDs comprised of involute profile-shifted gears with one tooth difference using a method combined with interference diagrams to obtain feasible regions of the number of teeth and the profile shift coefficient of gear 2. A high contact ratio of 2.01 and a high working pressure angle of 56.4511 degrees for \( SRR = -80 \) was obtained for a KHV(-) PDSTND. Yu (1988) proposed an optimum design for a KHV(-) PDSTND with involute profile-shifted gears. The goal of maximizing the mechanical efficiency required the use of a special optimization procedure, because the problem is difficult to solve with the conventional optimization methods. A high efficiency of about 92% with a friction coefficient of 0.1 can be obtained for one tooth difference. The obtained working pressure angles for \( z_1 = 80 \) were 51.52, 34.11 and 26.01 degrees for teeth differences of 1, 2 and 3, respectively. Chen and Walton (1990) carried out an optimum design for a KHV(+) PDSTND comprised of involute profile-shifted gears with the goal of minimizing the working pressure angle. For \( SRR = 80 \) with \( z_2 = 80 \), they improved on the manually created design with its working pressure angle of 56 degrees to obtain an optimum design with a working pressure angle of 47.89 degrees. Note that the optimum working pressure angle of 47.89 degrees was found at the cost of a contact ratio of 1.00. Maiti and Roy (1996) showed the tip interference during disengagement and
also investigated the minimum tooth difference in an internal-external involute gear pair using addendum truncation and center distance modifications. Shu (1995) developed a mathematical model for the multi-tooth contact phenomenon in a KHV PDSTND and showed that the number of contact tooth pairs may not exceed 5 in general. Li (2008) used the finite element method and the mathematical programming method to execute the contact analysis of a PDSTND and found that there are only four pairs of teeth in contact. Liu et al. (2012) proposed a novel cycloid drive. The conjugated gear pair of this novel cycloid drive is composed by an external cycloid-arc gear and an internal ring gear. The meshing equation for this novel cycloid drive was established and the meshing characteristics were analyzed. Huang and Tsai (2017) proposed a computerized approach of loaded tooth contact analysis of a cycloid planetary gear reducer based on the influence coefficient method. Han et al. (2018) proposed a new tooth profile of an external drive. The new second-order, third-order and fourth-order composite cycloid equations of tooth profiles were derived and compared.

As mentioned previously, there have been quite a few valuable studies which have advanced the general understanding of the design of KHV PDSTNDs. Thus, involute K-H-V planetary drives with a small teeth number difference have been commercialized for many years. However, the performance such as the contact ratio and the working pressure angle cannot be acquired from their catalogs. Chen and Walton (1990) designed a KHV (+) PDSTND with a relatively small working pressure angle of 47.89 degrees for SRR = 80. However, this achievement was obtained at the cost of a contact ratio of 1.00. As indicated in numerous textbooks (Paul, 1979, Norton, 2004), the minimum acceptable contact ratio is 1.2; a minimum contact ratio of 1.4 is preferred and recommended; and a contact ratio of 2 is desirable to reduce noise. It can be seen that there is still room for improvement in the optimum design of a KHV PDSTND. The aim of this study is to find the optimum involute tooth profiles for K-H-V PDSTNDs so as to minimize the objective function of a weighted combination of two main performance factors, i.e., the reciprocal of the contact ratio and the working pressure angle. The optimization problems are solved using a combined-mutation differential evolution algorithm (Lin and Hsiao, 2017). In order to estimate the range of the weighting factors and to understand possible values for the working pressure angles, the objective function of the reciprocal of the contact ratio is first studied. The efficiency of the KHV PDSTND is constrained to be not less than a prescribed value for fairness of comparisons. Given the failure to obtain feasible designs for the full-depth and the stub teeth, a design scheme is used which alters the addendums of the involute profile-shifted full-depth teeth for all gears. The performance can be improved using the afore-mentioned design scheme; however, there is still room for improvements. Therefore, the scheme of different modules in mesh for involute profile-shifted teeth with variation in addendum is proposed.

2. Geometry of involute profile-shifted gears using different modules and variation of the addendum

The correct meshing condition for any two involute gears is that the normal pitch (\( P_n \)) of the two gears is identical. For an involute profile, the normal pitch is equal to the base pitch (\( P_b \)), i.e., \( P_n = P_b = m \cos \alpha \) (\( m \) and \( \alpha \) are the module and pressure angle, respectively). The correct meshing condition for any two involute gears can be easily obtained from the configuration that a pair of teeth is leaving contact as the next is just beginning contact or as the next has already been in contact. Spitas and Spitas (2006) proved the correct meshing condition using a similar explanation. If the correct meshing condition is not satisfied, there will be transmission interruption and the intersection of the tooth profiles. Therefore, the correct meshing condition for any two involute gears can be expressed by

\[
m_1 \cos \alpha_1 = m_2 \cos \alpha_2
\]

(1)

where \( m_1 \) and \( m_2 \) are the modules of gears 1 and 2, respectively; \( \alpha_1 \) and \( \alpha_2 \) are the pressure angles of gears 1 and 2, respectively. If the standard module and pressure angles are used, the modules must be identical and the pressure angles must also be identical for any two involute gears in order to satisfy the correct meshing condition. Spitas and Spitas (2006) showed that any two involute gears can mesh perfectly using non-standard (or different) modules, if the correct meshing condition can be satisfied.

2.1 Meshing equation

For involute profile-shifted teeth, the addendums of gear 1 (planet gear) and gear 2 (annular gear) can be expressed by \( h_{a1} = (h_{a1}^* + x_1)m_1 \) and \( h_{a2} = (h_{a2}^* - x_2)m_2 \), where \( x_1 \) and \( x_2 \) denote the profile shift coefficients of gears 1 and...
2, respectively; and $h_a^*$ is the addendum coefficient. If the whole depth is the same as for the standard involute tooth, the dedendums of gears 1 and 2 can be expressed by $h_f^1 = (h_a^* + c^* + x_1)m_1$ and $h_f^2 = (h_a^* + c^* + x_2)m_2$, respectively, where $c^*$ is the clearance coefficient. The symbols $d_{a1}$ and $d_{f1}$ denote the diameters of the addendum and dedendum circles of gear 1, respectively; and $d_{a2}$ and $d_{f2}$ denote the diameters of the addendum and dedendum circles of gear 2, respectively. They can be expressed as follows:

\[
d_{a1} = d_1 + 2h_{a1} = m_1z_1 + 2(h_a^* + x_1)m_1 \\
d_{f1} = d_1 - 2h_{f1} = m_1z_1 - 2(h_a^* + c^* - x_1)m_1 \\
d_{a2} = d_2 - 2h_{a2} = m_2z_2 - 2(h_a^* - x_2)m_2 \\
d_{f2} = d_2 + 2h_{f2} = m_2z_2 + 2(h_a^* + c^* + x_2)m_2
\]

where $d_1$ and $d_2$ denote the pitch diameters of gears 1 and 2, respectively.

In this study, in order to avoid some peculiar interferences, let the variations in the addendums for the involute profile-shifted teeth be $y_1m_1$ and $y_2m_2$ for gears 1 and 2, respectively; that is, $h_{a1} = (h_a^* + x_1 + y_1)m_1$, $h_{a2} = (h_a^* - x_2 + y_2)m_2$, while the two dedendums remain unchanged. Therefore, $d_{a1}$ and $d_{a2}$ are changed into

\[
d_{a1} = m_1z_1 + 2(h_a^* + x_1 + y_1)m_1 \\
d_{a2} = m_2z_2 - 2(h_a^* - x_2 + y_2)m_2
\]

The clearance between the addendum circle of gear 2 and the dedendum circle of gear 1 is expressed by

\[
c_b = \frac{1}{2}d_{a2} - (a_w + \frac{1}{2}d_{f1})
\]

where $a_w$ is the center distance after the profile shift. The clearance between the addendum circle of gear 1 and the dedendum circle of gear 2 is expressed by

\[
c_i = \frac{1}{2}d_{f2} - (a_w + \frac{1}{2}d_{a1})
\]

A well-known meshing characteristic for varying the center distance of involute gears can be expressed as follows:

\[
a_w \cos \alpha_w = r_{b2} - r_{b1} = r_2 \cos \alpha_2 - \eta \cos \alpha_1
\]

where $\alpha_w$ is the working pressure angle; $\eta_1$ and $\eta_2$ are the radii of the base circles of gears 1 and 2, respectively; and $\eta$ and $r_2$ denote the radii of the standard pitch circles of gears 1 and 2, respectively.

The symbols $t_1$ and $t_2$ denote the tooth thicknesses measured along the standard pitch circles of gears 1 and 2, respectively, while $s_1$ and $s_2$ denote the tooth space widths measured along the standard pitch circles of gears 1 and 2, respectively. $t_1$ and $t_2$ can be expressed as follows (Kinki Gear Discussion Society, 1974, Uicker et al., 2011):

\[
t_1 = \frac{1}{2}z_1m_1 + 2x_1m_1 \tan \alpha_1 \\
s_2 = \frac{1}{2}z_2m_2 + 2x_2m_2 \tan \alpha_2
\]

From Fig. 2, the tooth thickness $t_{x1}$, measured along the circle of radius $r_{x1}$ of gear 1, can be expressed by
\[ t_{x1} = r_{x1} \left[ t_1 - 2(\text{inv} \alpha_{x1} - \text{inv} \alpha_1) \right], \]
\[ s_{x2} = r_{x2} \left[ \frac{s_2}{r_2} - 2(\text{inv} \alpha_{x2} - \text{inv} \alpha_2) \right]. \]

Here, \( t_{w1} \) and \( t_{w2} \) denote the tooth thicknesses measured along the working pitch circles of gears 1 and 2, respectively. The symbol \( s_{w1} \) and \( s_{w2} \) denote the tooth space widths measured along the working pitch circles of gears 1 and 2, respectively. \( t_{w1} \) and \( s_{w2} \) can be expressed as follows:

\[ t_{w1} = r_{w1} \left[ t_1 - 2(\text{inv} \alpha_w - \text{inv} \alpha_1) \right] \]
\[ s_{w2} = r_{w2} \left[ \frac{s_2}{r_2} - 2(\text{inv} \alpha_w - \text{inv} \alpha_2) \right] \]

where \( r_{w1} \) and \( r_{w2} \) are the radii of the working pitch circles of gears 1 and 2, respectively.

From the relation between the radii of the base circle and the standard and working pitch circles, one can easily obtain the equality as follows:

\[ r_{w1} = \frac{\cos \alpha_1}{\eta \cos \alpha_w} \]
\[ r_{w2} = \frac{\cos \alpha_2}{\eta \cos \alpha_w} \]

The equation of meshing can be derived using the condition of zero backlash, which can be expressed as follows:

\[ t_{w1} = s_{w2} \]
equation as follows:

$$x_2m_2 \sin \alpha_2 - x_1m_1 \sin \alpha_1 = \eta_2 (\sin\alpha_w - \sin\alpha_2) - \eta_1 (\sin\alpha_w - \sin\alpha_1) \tag{18}$$

If $m_1 = m_2 = m$ and $\alpha_1 = \alpha_2 = \alpha$, the meshing equation can be expressed as follows:

$$x_2 - x_1 = \frac{1}{2}(z_2 - z_1)(\frac{\sin\alpha_w - \sin\alpha}{\tan \alpha}) \tag{19}$$

The meshing equation for two external profile-shifted gears with different modules can be derived using the above-mentioned procedure and can be expressed by

$$x_2m_2 \sin\alpha_2 + x_1m_1 \sin \alpha_1 = \eta_2 (\sin\alpha_w - \sin\alpha_2) + \eta_1 (\sin\alpha_w - \sin\alpha_1) \tag{20}$$

It can be verified that the meshing equation of Eq. (20) for $x_1 = x_2 = 0$ is identical to the so-called compatibility equation, i.e., Eq. (49) in Spitas and Spitas (2006), for zero backlash.

2.2 Contact ratio

The contact ratio is defined as the ratio of the length of action to the normal pitch. Irrespective of whether the pitch point is inside or outside the contact path, the contact ratio can be expressed as follows:

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \tag{21}$$

$$\varepsilon_1 = \frac{1}{2\pi} z_2 (\tan \alpha_w - \tan \alpha_{a2}) \tag{22}$$

$$\varepsilon_2 = \frac{1}{2\pi} z_1 (\tan \alpha_{a1} - \tan \alpha_w) \tag{23}$$

where $\alpha_{a1} = \cos^{-1}(\frac{\eta_1}{r_{a1}})$ and $\alpha_{a2} = \cos^{-1}(\frac{\eta_2}{r_{a2}})$ are the pressure angles of the addendum circles of gears 1 and 2, respectively; and $r_{a1}$ and $r_{a2}$ denote the radii of the addendum circles of gears 1 and 2, respectively.

The contact ratio can be considered as the average number of teeth used to transmit the power (Paul, 1979). A higher contact ratio represents better meshing performance and results in quieter operation and reduced wear (Martin, 1982). In particular, there is a sudden reduction in vibration and noise when the contact ratio is 2.0 or larger (Martin, 1982, Morozumi, 1970c). Therefore, the contact ratio is considered to be the most important factor in meshing performance.

2.3 Interferences

The interferences considered in the design of a KHV PDSTND are explained below (Kinki Gear Discussion Society, 1974, Morozumi, 1989, Chen and Walton, 1990, Kim and Choi, 2000):

1. Addendum of the annular gear is not an involute

   To ensure that the addendum of gear 2 is an involute, the addendum circle of gear 2 must not be smaller than the base circle, i.e.,

   $$r_{a2} \geq \eta_2 \tag{24}$$

2. Addendum interference on the opposite side of the pitch point

   To prevent addendum interference on the opposite side of the pitch point, the following geometric relation must be
satisfied:

\[ a_w + r_{22} - r_{a1} - \Delta_1 \geq 0 \]  \hspace{1cm} (25)

where \( \Delta_1 \) is typically of the order 0.01 (Chen and Walton, 1990).

(3) Digging interference of the planetary gear

To prevent the so-called digging interference (Uicker, 2011) (i.e., the so-called undercutting in the generating process for gear 1), the following condition equation must be satisfied:

\[ x_1 \geq h_w^* - \frac{1}{2} z_1 \sin^2 \alpha_1 \]  \hspace{1cm} (26)

(4) Involute interference

To prevent the problem of involute interference between the addendum of gear 2 and the dedendum of gear 1, the following condition equation must be satisfied:

\[ \tan \alpha_{d2} \geq \tan \alpha_w - \frac{z_1}{z_2} \]  \hspace{1cm} (27)

(5) Contact ratio

In order to avoid transmission interruption and impact, the contact ratio must be greater than 1.0.

(6) Working pressure angle

To prevent the intersection of the tooth profiles of gears 1 and 2, the working pressure angle must be greater than 0.

(7) Tip interference during engagement

After the intermeshing teeth depart from the action of line, the addendum of gear 1 will gradually depart from the flank of gear 2 and they will be completely out of contact after the intersection of two addendum circles. During this period, tip interference or trochoid interference may occur, that is, there may be interference between both addendums. To prevent tip interference, gear 2 must pass through the intersection earlier than gear 1. The condition equation can be expressed by

\[ z_1 (\delta_1 + \gamma_1) - z_2 (\delta_2 - \gamma_2) - \Delta_1 \geq 0 \]  \hspace{1cm} (28)

where

\[ \delta_1 = \cos^{-1} \left( \frac{r_{22}^2 - a_w^2 - r_{a1}^2}{2a_wr_{a1}} \right) \]  \hspace{1cm} (29)

\[ \delta_2 = \cos^{-1} \left( \frac{r_{a2}^2 + a_w^2 - r_{a2}^2}{2awr_{a2}} \right) \]  \hspace{1cm} (30)

\[ \gamma_1 = \text{inv} \alpha_{a1} - \text{inv} \alpha_w \]  \hspace{1cm} (31)

\[ \gamma_2 = \text{inv} \alpha_w - \text{inv} \alpha_{a2} \]  \hspace{1cm} (32)

(8) Tip interference during disengagement

Maiti and Roy (1996) showed the tip interference during disengagement. The condition equation can be expressed by

\[ z_1 (\delta_1 + \gamma_1 - \gamma_p) - z_2 (\delta_2 - \gamma_2 - \gamma_p) - \Delta_1 \geq 0 \]  \hspace{1cm} (33)

where
Radial Interference during disassembly

If the planetary gear cannot be withdrawn in the radial direction from the intermeshing state, this type of radial interference can be solved by withdrawing it in the axial direction. Therefore, the interference can be neglected, as indicated in (Morozumi, 1970c, Maiti and Roy, 1996). The radial interference appeared in the manufacturing technology based on the generating process is termed the trimming interference. Note that because the previous study had shown that in the case of the KHV PDSTND with stub teeth, only tip interference may occur, and therefore Chen and Walton (1990) only considered the tip interference and the contact ratio for their optimal design task. In fact, it can be validated that the optimal design obtained by Chen and Walton (1990) also encountered this type of radial interference. With the progress of manufacturing and processing technology, e.g. using other non-generating processes or wire electrical discharge machining, the trimming interference can be ignored in this study.

3. Mechanical efficiency

The relation of mechanical efficiency between a generic epicycle and its reference mechanism for eight possible cases (depending on which shaft is fixed and which shaft is input, and whether the speed ratio is positive or negative) was derived in Molian (1982) and Norton (2004) based on the static equilibrium, the power balance and the power flow. An earlier derivation of efficiency for six possible cases was made in Merritt (1971). Morozumi (1981) derived the relation for the mechanical efficiency between a 2K-H PDSTND and a 2K-H PDSTND reference mechanism. The reference mechanism of a 2K-H PDSTND is a two-stage reference mechanism of a KHV PDSTND. Morozumi (1981) showed that the theoretical value of 76%, obtained using the friction coefficient of 0.08, agrees well with the experimental value of 73% indicating the mechanical efficiency of a 2K-H PDSTND.

The relationship of the mechanical efficiency between a KHV PDSTND and its reference mechanism can be expressed as follows:

\[
\eta = \begin{cases} 
1 & \lambda > 1 \\
\frac{\lambda - (\lambda - 1)\eta_{12}^H}{\lambda - (\lambda - 1)/\eta_{12}^H} & \lambda < 0 
\end{cases} 
\]  

(36)

where \( \eta_{12}^H \) is the fixed-carrier efficiency of the KHV PDSTND.

The fixed-carrier efficiency of a KHV PDSTND for the case with the pitch point outside the contact path can be expressed as follows (Morozumi, 1970b):

\[
\eta_{12}^H = 1 - \pi \mu \frac{1}{z_1} - \frac{1}{z_2} (\varepsilon_1 - \varepsilon_2) 
\]  

(37)

where \( \mu \) is the friction coefficient.

The fixed-carrier efficiency of a KHV PDSTND for the case with the pitch point inside the contact path can be expressed as follows (Morozumi, 1970a):

\[
\eta_{12}^H = 1 - \pi \mu \frac{1}{z_1} - \frac{1}{z_2} (\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_1 - \varepsilon_2 + 1) 
\]  

(38)

4. Optimization
The optimization problem is solved using a combined-mutation differential evolution algorithm (Lin and Hsiao, 2017). Four design schemes are summarized below:

1. Scheme I: involute profile-shifted full-depth teeth using the same standard module (i.e., \( m_1 = m_2 = m \) and therefore \( \alpha_1 = \alpha_2 = \alpha \))
2. Scheme II: involute profile-shifted stub teeth using the same standard module
3. Scheme III: alteration of the addendums of the involute profile-shifted full-depth teeth using the same standard module
4. Scheme IV: alteration of the addendums of the involute profile-shifted full-depth teeth using different modules (\( m_1 \) and \( \alpha_1 \) are non-standard values)

### 4.1 Design variables

Two design variables, \( \alpha_w \) and \( x_1 \), are employed in schemes I and II. Four design variables, \( \alpha_w \), \( x_1 \), \( y_1 \) and \( y_2 \) are employed in schemes III. Five design variables, \( \alpha_1 \), \( \alpha_w \), \( x_1 \), \( y_1 \) and \( y_2 \) are employed in schemes IV.

### 4.2 Objective function

The goal of the optimum design is to minimize the objective function. The weighted combination of the reciprocal of the contact ratio and the working pressure angle can be expressed as follows:

\[
\text{Minimize } f_{\text{obj}} = w_1 l_\varepsilon + w_2 \alpha_w
\]

where \( w_1 \) and \( w_2 \) are the weighting factors. To estimate the range of weighting factors and to understand possible values of the working pressure angles, we first study the objective function obtained using the reciprocal of the contact ratio.

### 4.3 Constraints

In addition to the interferences noted in Section 2.3, some additional constraints which need to be satisfied are discussed below.

1. Mechanical efficiency
   The mechanical efficiency is constrained to be not less than a prescribed value.

2. Crest thickness
   To prevent tooth thicknesses measured along the addendum circles of gears 1 and 2 from being too small (Chen and Walton, 1990), let

\[
t_{a1} / m_1 \geq \Delta_2
\]

\[
t_{a2} / m_2 \geq \Delta_2
\]

where

\[
t_{a1} = r_{a1} \left( \frac{t_1}{r_1} - 2(\text{inv} \alpha_{a1} - \text{inv} \alpha_l) \right)
\]

\[
t_{a2} = r_{a2} \left( \frac{t_2}{r_2} - 2(\text{inv} \alpha_{a2} - \text{inv} \alpha_{a2}) \right)
\]

\[
t_2 = \frac{1}{2} m_2 - 2 x_2 m_2 \tan \alpha_2
\]
Clearance

The clearance between the addendum circle of gear 2 and the dedendum circle of gear 1 and the clearance between the addendum circle of gear 1 and the dedendum circle of gear 2 should be greater than a prescribed value which, in this study, is 0.25 \( m_1 \) or 0.25 \( m_2 \). In fact, the clearance is not an important quality index (Yu, 1988) and can be increased. For example, the dedendum is sometimes increased to 1.3\( m \) or 1.4\( m \) to obtain enough space for a large fillet radius (Uicker et al., 2011).

5. Results and discussion

The design case is studied for the KHV(+) PDSTND optimum design from Chen and Walton (1990). The basic design parameters: \( z_1 = 79 \), \( z_2 = 80 \), \( \alpha = 20^\circ \) and \( m = 3 \). Thus, \( \lambda = 80 \). The optimal working pressure angle obtained by Chen and Walton (1990) is 47.89\(^\circ\), in contrast with an original manual design with a working pressure angle of 56\(^\circ\). However, the optimal working pressure angle was obtained at the cost of a contact ratio of 1.00. A larger working pressure angle is needed to avoid some peculiar interferences. They did not consider the efficiency of the obtained optimum KHV(+) design. In this study, the efficiency of their design is computed to be 94.2174\% using a friction coefficient of 0.08 (Morozumi, 1981). Note that the difficulty of the design case lies not only in teeth differences of 1, but also in the requirement of high efficiency of 94.2174\%. For fairness of comparison, the efficiency is constrained to be not less than 94.2174\%.

The optimal results for the objective function of \( 1/\varepsilon \) obtained using three schemes are shown in Table 1, together with the optimal results obtained by Chen and Walton (1990). Feasible solutions cannot be obtained for schemes I and II due to the requirement of high efficiency of 94.2174\% and severe interferences. It can be seen that the working pressure angle obtained using scheme III declines by 12.0% and the contact ratio obtained increases by 34.1% as compared with those obtained by Chen and Walton (1990).

| Scheme | Present | Chen and Walton (1990) |
|--------|---------|------------------------|
| \( \varepsilon \) | I | II | III | \( h_{\text{opt}}^\varepsilon = 0.617 \) |
| \( \alpha \) (\(^\circ\)) | 42.1303 | 47.89 |
| \( x_1 \) | No solution | No solution | -1.80108 | -0.3435 |
| \( y_1 \) | | | -0.86910 | - |
| \( y_2 \) | | | -0.36540 | - |
| \( \varepsilon \) | 1.3413 | 1.00 |
| \( \eta \) (%) | 94.2567 | 94.2174 |

The optimal results for the objective function of \( w_1(1/\varepsilon) + w_2\alpha \) obtained using scheme III are shown in Table 2, together with the optimal results obtained by Chen and Walton (1990). The working pressure angle declines by 12.3% and the contact ratio increases by 38.7% obtained using scheme III with \( w_1 = 0.4 \) and \( w_2 = 0.6 \) as compared to those obtained by Chen and Walton (1990).

For scheme IV with \( m_2 = 3 \) and \( \alpha = 20^\circ \), the optimal results are shown in Table 3, together with the optimal results obtained by Chen and Walton (1990). It can be seen that the working pressure angle declines by 16.1% to 23.8% and the contact ratio increases by 72.1% to 92.6% obtained using scheme IV as compared to those obtained by Chen and Walton (1990). In addition, the working pressure angle declines by 13.0% and the contact ratio increases by 25.9%
obtained using scheme IV with \( w_1 = 0.5 \) and \( w_2 = 0.5 \) as compared to those obtained using scheme III with \( w_1 = 0.4 \) and \( w_2 = 0.6 \). The tooth profiles of gears 1 and 2 in mesh for the optimal design obtained using scheme IV with \( w_1 = 0.5 \) and \( w_2 = 0.5 \) are shown in Fig. 3. The quite slight overlapping of tooth profiles may be observed from Fig. 3. In fact, this is an illusion arising from image reduction and file conversions (from an AutoCAD dwg file to a doc file, then to pdf file). There is no overlapping of tooth profiles in the original dwg file.

**Table 2 Optimal results for \( f_{obj} = w_1(1/\varepsilon) + w_2\alpha_w \) obtained using scheme III.**

| Weights | \( w_1 = 0.3 \) | \( w_1 = 0.4 \) | \( w_1 = 0.5 \) | \( w_1 = 0.6 \) | \( w_1 = 0.7 \) | \( w_2 = 0.4 \) | \( w_2 = 0.5 \) | \( w_2 = 0.6 \) | \( w_2 = 0.7 \) |
|---------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( \alpha_w (\degree) \) | 41.7991 | 41.9780 | 42.1170 | 42.6428 | 42.6948 | 47.89 |
| \( x_1 \) | -1.80635 | -1.76374 | -1.78157 | -1.79877 | -1.78563 | -0.3435 |
| \( y_1 \) | -0.90680 | -0.95736 | -0.91680 | -0.76939 | -0.80544 | – |
| \( y_2 \) | -0.36507 | -0.29262 | -0.33183 | -0.44527 | -0.38955 | – |
| \( \varepsilon \) | 1.3046 | 1.3873 | 1.3409 | 1.2343 | 1.3169 | 1.00 |
| \( \eta (%) \) | 94.2983 | 94.2266 | 94.3037 | 94.3400 | 94.2839 | 94.2174 |

**Table 3 Optimal results for \( f_{obj} = w_1(1/\varepsilon) + w_2\alpha_w \) obtained using scheme IV.**

| Weights | \( w_1 = 0.3 \) | \( w_1 = 0.4 \) | \( w_1 = 0.5 \) | \( w_1 = 0.6 \) | \( w_1 = 0.7 \) | \( w_2 = 0.4 \) | \( w_2 = 0.5 \) | \( w_2 = 0.6 \) | \( w_2 = 0.7 \) |
|---------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( \alpha_1 (\degree) \) | 14.5722 | 16.8874 | 15.8447 | 16.9044 | 16.2010 | 20 |
| \( \alpha_w (\degree) \) | 37.1349 | 37.7618 | 36.5022 | 39.8575 | 40.1825 | 47.89 |
| \( x_1 \) | -0.7716 | -1.2810 | -1.1296 | -0.8885 | -1.3202 | -0.3435 |
| \( y_1 \) | -1.1100 | -1.0322 | -1.0038 | -1.3789 | -0.8009 | – |
| \( y_2 \) | -0.0205 | -0.1784 | -0.1776 | 0.1836 | -0.2404 | – |
| \( \varepsilon \) | 1.7208 | 1.7849 | 1.7469 | 1.9038 | 1.9258 | 1.00 |
| \( \eta (%) \) | 94.8830 | 94.7488 | 94.9724 | 94.3526 | 94.2865 | 94.2174 |
6. Conclusions

The design schemes of the involute profile-shifted full-depth and stub teeth fail to obtain a feasible design for the design case. The difficulty of the design case lies not only in teeth differences of 1, but also in the requirement of high efficiency of 94.2174% using a friction coefficient of 0.08. Using the scheme of variation of the addendum for the involute profile-shifted full-depth teeth, the performance of $\alpha_w = 47.89^\circ$ and $\epsilon = 1.00$ for the original design can be improved to be that of $\alpha_w = 41.80^\circ$ and $\epsilon = 1.30$ (or $\alpha_w = 41.98^\circ$ and $\epsilon = 1.39$). To further improve the performance, the use of different modules in mesh for the involute profile-shifted full-depth teeth is proposed and the corresponding meshing equation is derived. Using the proposed scheme, the performance of $\alpha_w = 47.89^\circ$ and $\epsilon = 1.00$ for the original design can be improved to be that of $\alpha_w = 36.50^\circ$ and $\epsilon = 1.75$. The last optimal results give a significant 19.82% reduction in the unwanted radial tooth force. Therefore, for planet bearings (ball bearings) of the same size, the working life will be 1.94 times greater, or smaller bearings can be used.

Nomenclature

- $a_w$: center distance after the profile shift
- $c$: clearance coefficient.
- $c_b$: clearance between the addendum circle of gear 2 and the dedendum circle of gear 1
- $c_t$: clearance between the addendum circle of gear 1 and the dedendum circle of gear 2
\( d_1, d_2 \) pitch diameters of gears 1 and 2
\( d_{a1}, d_{a2} \) diameters of the addendum circles of gears 1 and 2
\( d_{f1}, d_{f2} \) diameters of the dedendum circles of gears 1 and 2
\( f_{\text{obj}} \) objective function
\( h_{a1}, h_{a2} \) addendums of gears 1 and 2
\( h^* \) addendum coefficient
\( h_{f1}, h_{f2} \) dedendums of gears 1 and 2
\( \text{inv}() \) involute function, i.e., involute polar angle
\( m \) (standard) module
\( m_1, m_2 \) modules of gears 1 and 2
\( P_b \) base pitch
\( P_n \) normal pitch
\( r_1, r_2 \) radii of the standard pitch circles of gears 1 and 2
\( r_{a1}, r_{a2} \) radii of the addendum circles of gears 1 and 2
\( r_{b1}, r_{b2} \) radii of the base circles of gears 1 and 2
\( r_{w1}, r_{w2} \) radii of the working pitch circles of gears 1 and 2
\( s_1, s_2 \) tooth space widths measured along the standard pitch circles of gears 1 and 2
\( s_{w1}, s_{w2} \) tooth space widths measured along the working pitch circles of gears 1 and 2
\( s_x \) tooth space width measured along the circle of radius \( r_x \) of gear 2
\( t_1, t_2 \) tooth thicknesses measured along the standard pitch circles of gears 1 and 2
\( t_{a1}, t_{a2} \) tooth thicknesses measured along the addendum circles of gears 1 and 2
\( t_{w1}, t_{w2} \) tooth thicknesses measured along the working pitch circles of gears 1 and 2
\( t_x \) tooth thicknesses measured along the circle of radius \( r_x \) of gear 1
\( w_1, w_2 \) weighting factors for the objective function
\( x_1, x_2 \) profile shift coefficients of gears 1 and 2
\( y_1, y_2 \) coefficients of variations in the addendums of the involute profile-shifted teeth
\( z_1, z_2 \) number of teeth of gears 1 and 2
\( \alpha \) (standard) pressure angle
\( \alpha_1, \alpha_2 \) pressure angles of gears 1 and 2
\( \alpha_{a1}, \alpha_{a2} \) pressure angles of the addendum circles of gears 1 and 2
\( \alpha_w \) working pressure angle
\( \alpha_{x1} \) pressure angle of the circle of radius \( r_{x1} \) of gear 1
\( \alpha_{x2} \) pressure angle of the circle of radius \( r_{x2} \) of gear 2
\( \gamma_1, \gamma_2 \) angles defined in Eqs. (31) and (32)
\( \gamma_p, \gamma_q \) angles defined in Eqs. (34) and (35)
\( \delta_1, \delta_2 \) angles defined in Eqs. (29) and (30)
\( \Delta_1 \) prescribed threshold to avoid interferences
\( \Delta_2 \) prescribed threshold for the ratio of the crest thickness to the module
\( \varepsilon \) contact ratio
\( \varepsilon_1, \varepsilon_2 \) components of the contact ratio defined in Eqs. (22) and (23)
\( \eta \) efficiency of the KHV PDSTND
\( \eta_i \) fixed-carrier efficiency of the KHV PDSTND
\( \lambda \) speed reduction ratio
\( \mu \) friction coefficient

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