A corresponding-state approach to quark-cluster matter

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Abstract: The state of super-dense matter is essential for us to understand the nature of pulsars, but the non-perturbative quantum chromodynamics (QCD) makes it very difficult for direct calculations of the state of cold matter at realistic baryon number densities inside compact stars. Nevertheless, from an observational point of view, it is conjectured that pulsars could be made up of quark clusters since the strong coupling between quarks might render quarks grouped in clusters. We are trying an effort to find an equation of state of condensed quark-cluster matter in a phenomenological way. Supposing that the quark-clusters could be analogized to inert gases, we apply here the corresponding-state approach to derive the equation of state of quark-cluster matter, as was similarly demonstrated for nuclear and neutron-star matter in 1970s. According to the calculations presented, the quark-cluster stars, which are composed of quark-cluster matter, could then have high maximum mass that is consistent with observations and, in turn, further observations of pulsar mass would also put constraints to the properties of quark-cluster matter. Moreover, the melting heat during solid-liquid phase conversion and the related astrophysical consequences are also briefly discussed.

Key words: pulsars, neutron stars, elementary particles

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1 Introduction

The state of matter above the nuclear matter density, $\rho_0$, is still far from certainty, whereas it is essential for us to explore the nature of compact stars. At average density higher than $\sim 2\rho_0$, the quark degree of freedom inside would not be negligible, and historically such compact stars are called quark stars \[1\,\text{-}\,4\]. Although cold quark matter is difficult to be created in laboratories or studied by direct QCD calculations, some efforts have been made to model quark matter and quark stars, from MIT bag model to color super-conductivity model \[5\]. In most of these models, quark matter could usually be characterized by soft equation of state, because the asymptotic freedom of QCD tells us that as energy scale goes higher, the interaction between quarks becomes weaker. Nonetheless, astrophysical phenomenology of compact stars can still not rule out that striking physical possibility, making pulsars as superb astrophysical laboratories [see e.g. \[6\,\text{and}\,7\], and references therein].

However, at realistic baryon densities of compact stars, $\rho \sim (2\,\text{-}\,10)\rho_0$, the energy scale is usually below 0.8 GeV, which is much lower than the scale where the asymptotic freedom could apply. In contrast, the non-perturbative effect should be significant, making quarks to couple strongly with each other. Quark-clustering is then conjectured to occur in cold dense matter inside compact stars, by condensation of quarks in position space due to the strong coupling between quarks \[8\]. A realistic quark star could then be actually a “quark-cluster star”, and solidification could be a natural result if the kinetic energy of quark-clusters is much lower than the residual interaction energy between the clusters. The idea of clustering quark matter could provide us a way to understand different manifestations of pulsar-like compact stars \[9\,\text{and}\,10\].

How can then one model the equation of state of quark-cluster matter? Due to the lack of both theoretical and experimental evidence, the hypothetical quark-clusters in cold dense matter have not been confirmed, and it is also difficult for us to derive the properties of quark-cluster matter from the first principles of QCD calculations. Nevertheless, an empirical way was employed and discussed seriously in a corresponding-state approach.
to deduce the properties of nuclear matter in 1970s [e.g. 11, 12]. To establish a model which could be tested by observations, we adopt an empirical method here, by analyzing quark-clusters to inert gases and applying the corresponding-state approach too. A quark-cluster is usually assumed to be colorless, just like an inert atom being electric neutral. The interaction between inert gas atoms is the result of residual electromagnetic force, and similarly the interaction of quark-clusters could be regarded as the result of residual strong force, both of which should be characterized by the short-distance repulsion and long-distance attraction. In this paper, we assume that the interaction between quark-clusters could be described approximately by the same form as that between inert gas atoms, i.e. the Lennard-Jones potential, only with different parameters indicating stronger interaction and larger densities.

In fact, quark matter in Lennard-Jones model has been studied, where the equation of state is derived via summing the interaction energy of all quark-clusters [13]. Previously, a polytropic model [14] and a two-Gaussian component soft-core model [15] for quark-cluster stars has also been applied. The so-called corresponding-state approach we demonstrated in this paper, however, is an empirical one, to derive properties of quark-cluster matter by just a comparison to the experimental data of inert gases, based on the law of corresponding states. The law of corresponding states was first proposed by de Boer [16], who found that the properties of inert gases, such as pressure and density, could be written in a reduced form. After reducing to dimensionless terms, the experimental data of various inter gases can be fitted in smooth curves with a single quantum parameter. If the quark-cluster matter is assumed to be similar to inert gases, the corresponding-state approach can also be applicable to study the state of quark-cluster matter, without knowing its exact structure.

With a similar form of interaction, we may derive the equation of state of quark-cluster matter from empirical data of inert gases, through the corresponding-state approach. The masses and radii of quark-cluster stars can then be derived and compared with observations. We find that the maximum mass of quark-cluster stars can be well above $2M_\odot$. Although in principle one may obtain a maximum mass high enough to explain observations in any kinds of unphysical models, we call attention that the quark-cluster star model has meaningful implications for one to understand different manifestations, e.g., of the surface [17]. Additionally, the melting heat is also discussed, and it is shown that the solidification of newly born quark-cluster stars might explain the plateau of $\gamma$-ray bursts.

This paper is organized as following. We summarize the properties and observational implications of quark-cluster stars in §2, and a brief introduction of the law of corresponding states is given in §3. The equation of state of quark-cluster matter and the mass-radius curve of quark-cluster stars are derived in §4 using the corresponding-state approach. The melting heat of solid quark-cluster stars and the related astrophysical consequences are discussed in §5. We make conclusions and discussions in §6.

## 2 Quark-cluster matter

A lot of basic intuition questions are frequently asked about quark-cluster matter though such a state was proposed ten years ago [8]. Would quark-cluster matter be more energetically favored than nuclear matter or strange quark matter? Can quark-clusters be analogous to inert gases? Could quark-cluster star model be really necessary in astrophysics of compact stars? Certainly we cannot present clear and final answers to these because of both micro- and astro-problems as well as their entanglement. In order for readers to have a thorough and comprehensive view of the quark-cluster star idea, we here make some rough estimation about quark-cluster existence, and demonstrate that the answers could be positive in some region of the parameter space in QCD phase diagram. Observational hints are also summarized.

### 2.1 The stability of quark-cluster matter

It’s an interesting but difficult problem to know the stability from first principle. A special kind of quark-cluster, so-called H-cluster, was studied extensively, and by comparing energy per baryon at fixed density, it’s found that H-cluster matter might be more stable than both neutron matter and nuclear matter when the density is larger than $2\rho_0$, where the in-medium effect plays the crucial role in stabilizing H-cluster matter [see 18, Sect. 2]. Besides, an order of magnitude estimation could also help compare the stability of these three states.

_Nuclear matter vs. Quark-cluster matter._ In the low energy region of QCD phase diagram, quarks are confined in nucleons. However, at the density of realistic compact stars, the confined state may not be simply that of hadrons, because a light-flavor symmetry is likely to be restored. In ordinary case, electrons are outside the nucleus and their energy $E_e$ is far less than 1 MeV, so atoms could be stable with 2-flavor symmetry. Nevertheless, things are different in case of pulsar, for that electrons are inside the gigantic nucleus and the Fermi energy of electrons would be $E_e \sim 10^8$ MeV, even larger than the mass difference between $s$ quark and $u/d$ quark. Such a high energy might intensify the interaction $e+p \rightarrow n+\nu_e$, thus $E_e$ decreases but the nuclear symmetry energy increases. Therefore $s$ quark is likely to be excited in gigantic nu-
cles, the number of which may be slightly (~10^{-5}) less than u/d quark as s quark is heavier. If 3-flavor symmetry is restored, the number of electrons in pulsar would be much less, which makes \( E_s \sim 10 \text{ MeV} \), and the gigantic nucleus would be stable. Furthermore, three flavors of quarks could be grouped together to form a new hadron-like confined (quark-cluster) state in gigantic nucleus if the coupling between quarks is still strong.

**Strange quark matter vs. Quark-cluster matter.** At the high density, low temperature regime, cold dense quark matter could be of Fermi gas or liquid if the interaction between quarks is negligible. However, the question is: can the density in realistic compact stars be so high that we can neglect the interaction? The average density of a pulsar-like star with typical mass of 1.4 \( M_\odot \) and radius of 10 km is only \( \sim 2.4 \rho_0 \). For 3-flavor quark matter with density of \( 3 \rho_0 \), we have number densities for each flavor of quark, \( u, d \), and \( s \), of \( n_u \approx n_d \approx n_s \sim (3 \times 0.16 = 0.48) \text{ fm}^{-3} \). A further calculation of Fermi energy gives, \( E_{F}^{NR} \approx \frac{k_B}{2m_0} (3\pi^2)^{2/3} n^{2/3} \approx 380 \text{ MeV} \). if quarks are considered moving non-relativistically, or \( E_{F}^{R} \approx \frac{\hbar c}{2} (3\pi^2)^{1/3} n^{1/3} \approx 480 \text{ MeV} \) if quarks are considered moving extremely relativistically.

However, the interaction between quarks may play an important role in determining the real state. For a quark with length scale \( l \), from Heisenberg’s uncertainty relation, the kinetic energy would be of \( \sim p^2/m_q \sim \hbar^2/(m_q l^2) \), which has to be comparable to the color interaction energy of \( E \sim \alpha_s \hbar c / l \) in order to have a bound state, where \( \alpha_s \) is the coupling constant of strong interaction. One then finds if quarks are dressed, with a mass of \( m_q = 300 \text{ MeV} \),

\[
l \sim \frac{\hbar c}{\alpha_s m_q c^2} \approx \frac{1}{\alpha_s}, \quad E \sim \alpha_s^2 m_q c^2 \approx 300 \alpha_s^2 \text{ MeV}. \quad (1)
\]

This is dangerous for the Fermi state of matter since \( E \) is approaching and even greater than the Fermi energy of \( \sim 0.4 \text{ GeV} \) if the running coupling constant \( \alpha_s \) < 1, and a Dyson-Schwinger equation approach to non-perturbative QCD shows that the color coupling should be very strong rather weak, with \( \alpha_s \gtrsim 2 \) at a few nuclear densities in compact stars \( 2 \). Such strong interaction could render the quark grouped into clusters, rather than condensation in momentum space to form a color super-conductivity state.

**2.2 Properties of quark-cluster matter**

From arguments above, we can see that a symmetry of light flavor quarks is restored in the quark-clustering phase, which is different from the usual hadron phase. The color interaction is still strong there. On the other hand, the quark-clustering phase is also different from the conventional quark matter phase which is composed of relativistic and weakly interacting quarks. The quark-clustering phase could thus be considered as an intermediate state between hadron phase and free-quark phase.

Compact stars composed of pure quark-clusters are electric neutral, but in reality there could be some flavor symmetry breaking that leads to the non-equality among \( u, d \) and \( s \), usually with less \( s \) than \( u \) and \( d \). The positively charged quark matter is necessary because it allows the existence of electrons that might be crucial for us to understand the radiation behaviors of compact stars.

What could be a realistic quark-cluster? We know that \( \Lambda \) particles (with structure \( uuds \)) possess light-flavor symmetry, and one may think that a kind of quark clusters would be \( \Lambda \)-like. However, the interaction between \( \Lambda \) is attractive, so \( \Lambda \)-cluster with structure \( uuddss \) could emerge, which was previously predicted to be a stable state or resonance \( 19 \), and recently Lattice QCD simulations have shown possible evidence for its existence \( 20, 21 \). Besides \( \Lambda \)-cluster, an 18-quark cluster, i.e. quark-\( \alpha \), being completely symmetric in spin, color and flavor space, was also speculated to exist \( 22 \). The number of quarks in one cluster is left as a free parameter in this paper, and we set \( N_q = 6 \) and \( N_q = 18 \) for the sake of simplicity in the calculations as following, corresponding to \( \Lambda \)-cluster and quark-\( \alpha \), respectively.

An estimate for the length scale \( l_{qq} \) of quark-cluster gives \( l_{qq} \sim 1/\alpha_s \text{ fm} \lesssim 1 \text{ fm} \), which would be less than the average distance between quark-clusters \( d \approx (3 \times 0.16/N_q)^{-1/3} \lesssim 1 \text{ fm} \). Although quark-clusters consist of more quarks, they might not be larger than nucleons, and the distance between quark-clusters would be larger, so it is not likely that quark-clusters would be in closer proximity than nucleons. The quantum effect would not be significant if the residual short-distant repulsing interaction works, and quark-cluster can be considered as classical particles rather than that of quantum gas. What’s more, quark-cluster may move non-relativistically due to large mass and could be localized in lattice at low temperature.

When justifying the corresponding-state approach, one may doubt why should the law of corresponding states apply to quarks. Theoretically, the corresponding-state law reflects the statistical behavior of system. What’s required is just the same form of interaction potential, while certain values would not influence the conclusion if the parameters are re-scaled. Then another question may be: could the interaction between quark-clusters be described by Lennard-Jones potential? It’s hard to know the accurate form of the interaction between quark-clusters, but it could have a similar shape as Lennard-Jones potential, considering the property of short-distance repulsion and long-distance attraction. Quark-cluster could be analogized to nucleons, except for light-flavor symmetry. Since strong interaction is not
sensitive to flavor, the interaction between quark-clusters should be similar to that of nucleons, which is found to be Lennard-Jones-like by both experiment and modeling [20].

The interaction between quark-clusters may not be perfectly described by Lennard-Jones potential, the long range part of which may be proportional to $r^{-7}$ instead of $r^{-6}$ [24]. As a zeroth approximation, the Lennard-Jones potential may lead to violation of the law of corresponding states, but the reduced properties of quark-cluster matter should at least be in the same range with, even not exactly fall in the experimental lines of inert gases. So our approach is to some degrees reasonable, when the exact approach under QCD calculations seems to be impossible due to the significant non-perturbative effect.

### 2.3 Observational hints for the nature of pulsar

In the absence of QCD calculations due to non-perturbative effect, estimations in §2.1 could only demonstrate the possibility of stable quark-cluster matter, while no further conclusion could be made and the stability of quark-cluster remains to be justified. Nevertheless, quark-cluster was speculated to exist primarily for understanding astrophysical observations of pulsar-like compact stars. In addition to first principles, pulsar-like compact stars could also provide valuable information for properties of super-dense matter, different manifestations of which provide hints of the state of matter at supra-nuclear density. Various observational phenomena could be understood in terms of quark-cluster star model, including those that are challenging in conventional neutron star models [27].

What if pulsar is made of quark-cluster matter? There are at least three consequences relevant to observational phenomena.

**A stiff equation of state.** It is conventionally thought that the state of dense matter softens and thus cannot result in high maximum mass if pulsars are quark stars, and that the discovery of $2M_\odot$ pulsar PSR J1614-2230 [20] may make pulsars unlikely to be quark stars. However, quark-cluster star would have a stiff equation of state, because quark-cluster should be non-relativistic particle for its large mass, and because there could be strong short-distance repulsion between quark-clusters. It may well be possible to obtain a maximum mass of $\geq 2M_\odot$. Certainly, finding a stiffer equation of state is not enough to claim quark-cluster matter exist, but other observations may hints for a self-bound surface and global solid structure, which could also favor the existence of quark-cluster.

**A self-bound surface.** Different from traditional neutron stars, quark-cluster star would be self-bound by residual color-interaction between clusters, which could be a crucial difference providing observational manifestations to distinguish the two models.

Drifting subpulses phenomena in radio pulsars suggest the existence of Ruderman-Sutherland-like gap-sparking and thus strong binding of particles on pulsar polar caps to form vacuum gaps, but the calculated binding energy in normal neutron star models could not be so high unless the magnetic field is extremely strong. This problem could be naturally solved in quark-cluster star scenario due to the strong self bound nature on surface [27, 28].

In addition, many theoretical calculations predict the existence of atomic features in the thermal X-ray emission of neutron star atmospheres, while none of the expected spectral features has been detected with certainty up to now, which hints that there might not exist the atmospheres speculated in conventional models. Though modified neutron star atmospheres with very strong surface magnetic fields [29, 30] might reproduce a featureless spectrum too, a natural suggestion to understand the general observation could be that pulsars are actually quark-cluster star without atoms on the surface [31].

In addition, the bare and chromatically confined surface of quark-cluster star could overcome the baryon contamination problem and create a clean fireball for $\gamma$-ray burst and supernova. The strong surface binding would result in extremely energetic exploding because the photon/lepton luminosity of a quark-cluster surface is not limited by the Eddington limit, and supernova and $\gamma$-ray bursts could then be photon/lepton-driven [32–34]. Recently, it was shown that the magnetic field observed for some compact stars could be generated by small amounts of differential rotation between the quark matter core and the electron sea [33].

**A global solid structure.** Quark-cluster star could be in a global solid state, like “cooked eggs”, if the kinetic energy is less than interaction energy between quark-clusters, while for normal neutron stars, only crust is solid, like “raw” eggs. Rigid body would precess naturally when spinning, either freely or by torque, and the observation of possible precession or even free precession of B1821-11 [36] and others could suggest a global solid structure of pulsars.

Star-quake is a peculiar action of solid compact stars, during which huge free energy, such as gravitational and elastic energy, would be released. For a pulsar with mass $M \sim M_\odot$ and radius $R \sim 10$ km, the stored gravitational energy is $\simeq GM^2/R \sim 10^{53}$ erg, so energy released would be $\sim 10^{53}\Delta R/R$ when the radius changes from $R$ to $R - \Delta R$. Compared with magnetars powered by magnetic energy, quake-induced energy in solid quark-cluster stars may also be enough to power the bursts, flares and even superflares of soft $\gamma$-ray repeaters and anomalous X-ray pulsars [37].
Combining surface and global properties, we think that the quark-cluster star model would be reasonable to describe pulsar-like stars, while this paper is mainly focused on the equation of state for quark-cluster stars, via a phenomenological way.

3 The law of corresponding states

The law of corresponding states, advocated by de Boer [16], shows that the equation of state of substances with same form of interaction can be written in a reduced and universal form. Consider a group of substances with the following properties: (1) the total potential energy due to interaction can be written as a sum of identical expressions \( \phi(r_{ik}) \), each of which depends only on the distance \( r_{ik} \) between two particles \( i \) and \( k \); (2) \( \phi(r) = \varepsilon f(r/\sigma) \), where \( f \) is a function same for all substances, and \( \varepsilon, \sigma \) are characteristic energy and length for different species. The macroscopic quantities, such as pressure \( P \), volume \( V \) and temperature \( T \), can be expressed in dimensionless terms:

\[
P^* = \frac{P\sigma^3}{\varepsilon} \quad (2)
\]

\[
V^* = \frac{V(N\sigma^3)}{\varepsilon} \quad (3)
\]

\[
T^* = \frac{kT}{\varepsilon} \quad (4)
\]

Another dimensionless parameter is

\[
\Lambda^* = \frac{\hbar}{(\sigma\sqrt{m\varepsilon})}, \quad (5)
\]

corresponding to the de Broglie wavelength, which is constructed to measure the importance of quantum effects. It can be proved that the reduced equation of states expressed in dimensionless quantities is a universal relation

\[
P^* = f(V^*, T^*, \Lambda^*), \quad (6)
\]

which is the formulation of the law of corresponding states [16].

Despite the so-called universal equation of states is just formally written as Eq. (6), a formula that is difficult to be derived theoretically for most cases, it could be used to obtain information on the equation of state of a substance which we are unfamiliar with. For determined \( V^* \) and \( T^* \), \( P^* \) depends on the value of \( \Lambda^* \), and the \( P^* - \Lambda^* \) curve can be drawn using experimental data of laboratory substances. If the curve is smooth enough, the value of \( P^* \) for unfamiliar matter at such a state can be predicted provided its \( \Lambda^* \) is known.

For some substances described by Lennard-Jones 6-12 potential

\[
\phi(r) = \varepsilon \left\{ \frac{4}{(r/\sigma)^12} - \frac{4}{(r/\sigma)^6} \right\}, \quad (7)
\]
deBoer had determined \( \varepsilon \) and \( \sigma \) of noble gases and some permanent gases \( r = \sigma \) is the distance where \( \phi(r) = 0 \), and \( \varepsilon \) is the depth of potential well. Then the experimental data of \( P^*, T^* \) or \( V^* \) for different substances turn out to be smooth functions of \( \Lambda^* \) as corresponding states. In Fig. 1 experimental data of the volume \( V_0^* \) at zero temperature and zero pressure, reduced to \( V_0^* = V_0/(N\sigma^3) \), are plotted with \( \Lambda^* \) for some substances.

A smooth curve can be drawn by fitting all the points, which forms the bases of our prediction via corresponding states law, and the formula for the fitted curve is

\[
V_0^* = 0.57 + 9.45 \times 10^{-5}(\Lambda^* + 6.35)^{1.44} \quad (8)
\]

Considering the property of short-distance repulsion and long-distance attraction shown by Lennard-Jones potential, we assume that the interaction between quark-clusters can also be described by this form. The distinctions between quark-cluster matter and ordinary substances should be a much deeper potential well (larger \( \varepsilon \)) and higher density (smaller \( \sigma \)). With the same form of interaction as that of inert gas, we could apply the law of corresponding states to derive the properties of quark-cluster matter. If we find the quantum parameter \( \Lambda^* \) corresponding to quark-cluster matter, then \( V_0^* \) and other properties that vary smoothly with \( \Lambda^* \) can be determined by simply looking at the experimental curves of that property vs \( \Lambda^* \) for inert gases.

4 The state of quark cluster matter

4.1 Parameters

To apply the law of corresponding states to quark-cluster matter, we must determine \( \varepsilon, \sigma \) and the mass \( m \) of each quark-cluster first. \( m \) depends on the number of quarks \( N_q \) and the mass of each quark \( m_0 \) in one cluster. We give each quark a constituent mass and assume \( m_0 \) is one-third of the nuclear mass. \( N_q \) is left as a free parameter in this paper, and we set \( N_q = 6 \) and \( N_q = 18 \) for
our calculations, corresponding to $H$-cluster and quark-$\alpha$ respectively.

As no experimental attempt has been made to get the values of $\varepsilon$ and $\sigma$, we try to constrain their values by the surface density $\rho_s$ of quark-cluster stars. The temperature of quark stars can be approximated to be zero, and the pressure also reaches zero at the surface of stars. Given the value of $V_0^*$, we can calculate the surface density $\rho_s$ (rest-mass density). It is obvious that $\rho_s$ can be written as

$$\rho_s = N_q m_0 / V_0,$$

and comparing Eq. (9) with Eq. (10) we can get

$$\rho_s = N_q m_0 / (V_0^* \sigma^3).$$

For certain values of $N_q$, $\varepsilon$ and $\sigma$, we can calculate $\Lambda^*$ of quark cluster matter by Eq. (5), and $V_0^*$ can be found according to the fitted relation Eq. (5) of $V_0^*$-$\Lambda^*$ curve, then we may determine $\rho_s$ using Eq. (9). In Fig. 2, pairs of $\varepsilon$ and $\sigma$ that correspond to the same surface density $\rho_s$ are plotted respectively for $N_q = 6$ and $N_q = 18$, where values of $\rho_s$ are chosen to be once, twice and three times of nuclear matter density $\rho_0$. The lines of $\varepsilon$ and $\sigma$ giving the same $\Lambda^*$ with values 1, 2 and 3 are also drawn here for a further limit.

The surface density of quark stars is assumed to be in the range $1 < \rho_s / \rho_0 < 3$. Quark-clusters could condensate to form solid state like classical particles, so the quantum effects may not be large for quark-cluster matter, then $\Lambda^*$ should satisfy $\Lambda^* < 2$. We select four points numbered A, B, C and D respectively to deduce the mass-radius relation of quark-cluster stars. The values of $\varepsilon$ and $\sigma$ that correspond to the same surface density $\rho_s$ are plotted here for $N_q = 6$ and $N_q = 18$, including $\rho_s / \rho_0 = 1, 2, 3$ and $\Lambda^* = 1, 2, 3$. Four cases A, B, C and D are denoted by stars.

4.2 The equation of state

Given $\varepsilon$ and $\sigma$, we can deduce the state of quark-cluster matter by a corresponding-state approach, in the zero temperature case. If we know the experimental $P^* - \Lambda^*$ curve at a certain $V^*$ and zero temperature, we can find the value of $P^*$ corresponding to $\Lambda^*$ of quark cluster. According to Eq. (9) and the number density of quark-clusters $n = 1/(V^* \sigma^3)$, the reduced quantities $P^*$ and $V^*$ can be converted to $P$ and $n$, then we can get the pressure at a certain number density of quark-clusters. Combining this with the relation between mass density $\rho$ (rest-mass density plus interaction energy density) and number density $n$ of quark-cluster matter, the equation of state can be derived.

To draw the $P^* - \Lambda^*$ curve at different $V^*$, we need to know the relationship between $P^*$ and $V^*$ of some substances at zero temperature. For the lack of new data, we just use the data provided by de Boer in his subsequent article [38], where values of $P^*$ and $\Lambda^*$ were given corresponding to different values of $V^*$ for various inert gases. Taking $V^* = 0.88$ for instance, the $P^* - \Lambda^*$ curve are shown in Fig. 3. The data points are almost in linear relation, which makes our interpolation reliable. The value of $\Lambda^*$ at point A is about 1.05, then we find $P^* \approx 25$ from the $P^* - \Lambda^*$ curve. The corresponding $P$ and $n$ to the reduced quantities $P^*$ and $V^*$ are $P = 1.0 \times 10^{35}$ dyn/cm$^2$, $n / n_0 = 2.7$ ($n_0$ is the number density of nucleons in nuclear matter). Thus we get $P(n_0 = 2.7 \rho_0) = 1.0 \times 10^{35}$ dyn/cm$^2$ for quark-cluster matter in case A. By taking different values of $V^*$, pressure $P$ at different densities can be determined in case A.

Table 1.

| $N_q$ | $\varepsilon$ (MeV) | $\sigma$ (fm) | $\Lambda^*$ | $\rho_s / \rho_0$ |
|-------|------------------|-------------|-------------|-----------------|
| A     | 18               | 40          | 2.5         | 1.05            | 1.87            |
| B     | 18               | 100         | 2.3         | 0.72            | 2.72            |
| C     | 6                | 150         | 1.5         | 1.56            | 2.47            |
| D     | 6                | 200         | 2.0         | 1.02            | 1.23            |

With $\rho_s / \rho_0 = 2$, we get $d = 1.84$ fm for $N_q = 6$ and $d = 2.66$ fm for $N_q = 18$. Then $\sigma = O(1 \text{ fm})$ as it should have the same order of magnitude as $d$. It can be seen that the selected parameters are consistent with the above estimation.

Fig. 2. Contour lines of surface density $\rho_s$ and $\Lambda^*$, with solid lines representing $N_q = 6$ and the dashed lines representing $N_q = 18$, including $\rho_s / \rho_0 = 1, 2, 3$ and $\Lambda^* = 1, 2, 3$. Four cases A, B, C and D are denoted by stars.
The same procedure is also applicable to the other three cases.

Fig. 3. Experimental data of \( P^* \) and \( \Lambda^* \) at zero temperature when \( V^* = 0.88 \) are shown by dots, and the fitted curve is almost a straight line (solid line). The cases A, B, C, and D are denoted by crosses.

For each set of parameters, what we get is just a set of points in \( P-n \) diagram and not an analytic equation, and then we perform the curve fitting to get an approximate formula. The \( P-n \) relations derived from curve fitting are

\[
P = \begin{cases} 
(2.99 \times 10^{31} n^{5.63} - 1.60 \times 10^{34}) \text{ dyn/cm}^2 & (12) \\
(1.99 \times 10^{31} n^{5.64} - 7.63 \times 10^{34}) \text{ dyn/cm}^2 & (13) \\
(8.10 \times 10^{38} n^{5.24} - 1.69 \times 10^{35}) \text{ dyn/cm}^2 & (14) \\
(6.69 \times 10^{39} n^{5.63} - 1.63 \times 10^{35}) \text{ dyn/cm}^2 & (15)
\end{cases}
\]

for A, B, C and D respectively, where \( n \) is in units of clusters/fm\(^3\). Certainly it is better to deduce the equation of state from a border range of densities, making the extrapolation to be more accurate. Nevertheless, lacking in experimental data of laboratory substances, we can only make such an approximation at this stage. It is worth mentioning that the approximation will not have much influence on the following calculations of the mass-radius curves.

According to \( P = n^2 \frac{dU}{dn} \), where \( E \) is the internal energy per cluster, we can get

\[
E(n) - E(n_*) = \int_{n_*}^{n} \frac{P(n)}{n^2} dn,
\]

where \( n_* \) is the number density of quark-clusters on the surface of stars. We may determine the value of \( E(n_*) \) from a corresponding-state point of view, and then the relation between \( E \) and \( n \) can be derived by the above integral. Similar to \( V^* \), \( U^*_0 = U_0/(N\varepsilon) \) can be approximated as a smooth function of \( \Lambda^* \), where \( U_0 \) is the internal energy at zero temperature and zero pressure. From the data of laboratory substances \([16]\), we derive a fitted formula for \( U^*_0 \),

\[
U^*_0 = -8.72 + 4.91\Lambda^* - 0.71\Lambda^2.
\]

and \( E(n_*) \) is thus

\[
E(n_*) = U_0/N = U^*_0 \varepsilon.
\]

As both \( P(n) \) and \( E(n_*) \) are known, it is able to calculate \( E(n) \) from Eq. (17). The results are plotted in Fig. 4 for four groups of parameters A to D, and we can see that the internal energy can be comparable to rest-mass energy at some densities.

Fig. 4. The internal energy \( E \) per cluster for four groups of parameters A(red line), B(black line), C(blue line) and D(cyan line). The range of density \( n \) is from surface density \( n_* \) to the highest central density where the quark-cluster stars reaches the maximum mass.

The mass density \( \rho \) consists of rest-mass density and energy density,

\[
\rho = n(Nq_m + E/c^2),
\]

then the equation of state for quark-cluster matter can be derived by combining \( P-n \) relation and Eq. (19), and we show the results in Fig. 5 for the four groups of parameters A to D.

Fig. 5. Equations of states for the same four groups of parameters as in Fig. 4.
4.3 Mass-radius relation

Considering perfect fluid case and the general relativity, the hydrostatic equilibrium in spherically symmetry is described by Tolman-Oppenheimer-Volkoff equation,

$$\frac{dP}{dr} = - \frac{Gm(r)\rho}{r^2} \left(1 + \frac{\rho}{\rho_c}\right) \left(1 + \frac{4\pi r^3 P}{m(r) r^2}\right),$$

where $m(r) = \int_0^r \rho \cdot 4\pi r^2 dr'$. In the above discussions, we have got the equations of state, from which we can make a further calculation of the mass-radius and mass-central density (rest-mass density) relations for quark-cluster stars. The results are shown in Fig. 6 for the four groups of parameters A to D, and we can see that the maximum masses are higher than three times the solar mass $M_\odot$, which are reached with central density less than 5$\rho_0$, for all the selected groups of parameters. As a comparison, we also plot the mass-radius curves for homogeneous spheres with the same central density corresponding to each of the four cases. This shows that the gravity cannot be negligible only when the stars is near the maximum mass, which could be the result of the strong self-bounding of quark-cluster stars.

Conventional quark matter is characterized by soft equation of state, and the emerge of quark matter inside compact stars is usually thought to be a reason for lowering their maximum mass. The quark-cluster matter, however, could have stiff equation of state due to the strong coupling. Although the corresponding-state approach is just a phenomenological and empirical method, we could still apply it to study the state of quark-cluster matter and then understand the observations of pulsar-like compact stars. The observed high-mass pulsar PSR J1614-2230 with mass 1.97 ± 0.04$M_\odot$ [20] has received a lot of attention, and we can see that the quark-cluster stars in our present model could be consistent with this observation. Moreover, our model of quark-cluster stars could not be ruled out even if the mass of the pulsar J1748-2021B (2.74$M_\odot$) in a galactic cluster is confirmed in the future.

5 Melting heat

If the kinetic energy of quark clusters is much lower than the inter-cluster potential energy, they may form a solid state which is meaningful for the thermal X-ray behaviors of compact stars [29]. We will estimate the latent heat of phase transition of quark-cluster stars from liquid to solid state by the corresponding-state approach. We calculate the ratio of melting heat per particle $H$ and $\varepsilon$ for some ordinary substances [40], and find that there is also a good relation between $H^* = H/\varepsilon$ and $\Lambda^*$, as shown in Fig. 7. The fitted formula for $H^*$ and $\Lambda^*$ is

$$H^* = 1.18 e^{-(\Lambda^* - 0.12)/1.60^2}.$$  \hspace{1cm} (21)

For given $N_q$, $\varepsilon$ and $\sigma$, we can determine $\Lambda^*$ first and then get the value of $H^*$ from Eq. (21), thus the melting heat $H = H^* \varepsilon$ can be derived.

In Fig. 8 pairs of $\varepsilon$ and $\sigma$ which determine the same melting heat are plotted, where values of $H$ are chosen to be 1, 10 and 100 MeV, in two cases $N_q = 6$ and $N_q = 18$. The solidification of quark-cluster stars has been suggested to be relevant to the plateau of $\gamma$-ray burst [41], and it is found that if the energy released by each quark-cluster in the liquid to solid phase transition is larger...
than 1 MeV, the total released energy could produce the plateau. We can see that under a wide range of parameters in our model, the latent heat could be sufficient for this way of understanding the plateau of γ-ray burst.

![Graph](image)

Fig. 8. $\varepsilon$ and $\sigma$ which determine the same melting heat of each cluster. $H$ is chosen to be 1 (black line), 10 (cyan line) and 100 MeV (blue line), in two cases $N_q = 6$ (solid line) and $N_q = 18$ (dashed line).

6 Conclusions and discussions

In cold quark matter at realistic baryon densities of pulsar-like compact stars, the interaction between quarks would be so strong that they could condensate in position space, forming quark-clusters, and the stars are then called quark-cluster stars if the dominant component inside is quark-clusters. We propose that the interaction between quark-clusters could be analogous to that between inert gas atoms described by the Lennard-Jones potential, and apply the corresponding-state approach to derive the equation of state. As a phenomenological and empirical method, the corresponding-state approach can avoid detailed assumptions of quark-cluster matter as well as computation of the many-body effects, and we only need to concern about differences between substances. Along with of these advantages, there are large uncertainty in our results, coming from the Lennard-Jones approximation and lack of experimental data source. Even so, the corresponding-state approach could give us qualitative information about the properties of quark-cluster matter, while the exact approach under QCD calculations seems to be very difficult and even impossible now due to significant non-perturbative effects. Summarily, our two-parameter ($\varepsilon$ and $\sigma$) empirical approach make it possible to establish a model which could be tested by observations.

The equation of state we have derived by the corresponding-state approach could be stiff enough to make a star stable even if its mass is higher than $2M_\odot$, under reasonable parameters. This result is consistent with the recent observation of a high-mass pulsar, thus the emergence of such kind of exotic matter, “quark-cluster matter”, could not be ruled out. The observations of pulsars with higher mass, e.g. > $3M_\odot$, would even be a support to our quark-cluster star model, and give further constraints to the parameters. Moreover, the latent heat released by the solidification of newly born quark-cluster stars could help us to understand the formation of the plateau of γ-ray burst.

Certainly, whether quark-cluster matter could exist at supra-nuclear densities, and what quark-clusters are composed of, as well as how to describe their interaction are still open questions. On the other hand, the nature of pulsar-like compact stars is still essentially related to significant non-perturbative effects of QCD, and we hope that future astrophysical observations, complementary to the terrestrial experiments, could give us hints to all of these problems.

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