Transport coefficients from field theory and FAIR physics

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Abstract. I review recent progress in the understanding of hydrodynamics and transport coefficients in field theory. This subject plays an important role in the future heavy-ion program at the FAIR facility in Darmstadt. One of the main goals of the experiment is obtaining a better understanding of the QCD phase diagram, in particular to determine whether there is a critical point at intermediate baryon densities and temperatures. Transport coefficients are essential inputs to describe the space-time evolution of the plasma created in heavy-ion experiments. Moreover, their characteristic behavior near the critical point may also help to obtain experimental proof of its existence.

1. Hydrodynamics

Hydrodynamics is an effective quantum\(^1\) field theory which describes any physical system at long-enough distances and times. The degrees of freedom are (long-living) massless modes corresponding to fluctuations from the equilibrium values of hydrodynamical variables (e.g., charge density, momentum density, ...) \[1\].

Let us consider a microscopic quantum field theory in which the only conserved currents are the energy-momentum tensor \(T_{\mu\nu}\) and \(J_\mu\) (a current associated to some internal continuous symmetry).\(^2\) Hydrodynamic equations are the conservation laws \(\partial_\mu \langle T^{\mu\nu} \rangle = 0\) and \(\partial_\mu \langle J^\mu \rangle = 0\), supplemented with information about dissipative processes which take place in the fluid due to the interaction between the microscopic particles. These dissipative processes enter into the hydrodynamic equations as a gradient expansion of the hydrodynamic variables around their equilibrium values. Since hydrodynamics applies at long-enough distances and times, variations of the hydrodynamic quantities happen at much larger distances than the mean-free path for the collision between particles.

In order to simplify the hydrodynamical equations, instead of using \(\langle T_{\mu\nu} \rangle\) and \(\langle J_\mu \rangle\) as the hydrodynamic variables, it is convenient to express them in terms of the local temperature \(T(x)\), the local chemical potential \(\mu(x)\), and the local fluid velocity \(u(x)\). This can be done

\(^1\) The temperature plays the role of \(\bar{\h}\), and the (UV) cutoff of the theory is played by the mean free path of the particles \(\lambda_{\text{mfp}}\). In the language of effective field theories, the perturbative expansion is performed in powers of \(k\lambda_{\text{mfp}}\), where \(k\) represents the momentum or frequency scale of the process under consideration.

\(^2\) In a more general situation, hydrodynamic modes will also be phases of symmetry-breaking condensates (near second-order phase transitions, not only the phase but also the magnitude of the condensate) and Abelian gauge fields of unbroken \(U(1)\) symmetries \[2\].
systematically by expanding \( \langle T_{\mu\nu} \rangle \) and \( \langle J_\mu \rangle \) in gradients of these new hydrodynamic variables. To linear order in gradients, one obtains in the Landau frame the constitutive relations [1]:

\[
\langle T^{\mu\nu} \rangle = \epsilon u^\mu u^\nu + p \Delta^{\mu\nu} - \eta \Delta^{\mu\alpha} \Delta^{\nu\beta} \left( \partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} \eta_{\alpha\beta} \partial_\mu u^\mu \right) - \zeta \Delta^{\mu\nu} \partial_\lambda u^\lambda + O(\partial^2),
\]

\[
\langle J^\mu \rangle = n u^\mu - \sigma T \Delta^{\mu\nu} \partial_\nu (\mu/T) + O(\partial^2),
\]

where the metric is \( \eta_{\mu\nu} = \text{diag}(-1,1,\ldots,1) \) and \( \Delta^{\mu\nu} \equiv \eta^{\mu\nu} + u^\mu u^\nu \) is a projector operator. \( \epsilon, p \) and \( n \) are the energy-density, pressure, and charge density respectively, and they are related by the equation of state in equilibrium. It can be shown by arguments of non-negative divergence of the entropy current (or, alternatively, using symmetry properties of Green functions involving the equation of state in equilibrium. It can be shown by arguments of non-negative divergence and

\[
\eta_{\mu\nu} = \text{diag}(\eta, \eta, \eta, \eta),
\]

where \( \eta \) is the chemical potential.

But this is not the full story regarding linear hydrodynamics; recently there has been an intense activity, motivated by the AdS/CFT duality [3], in exploring the influence on hydrodynamics from effects in the microscopic quantum field theory such as anomalies [4], inclusion of external fields [5], or parity-violation [6]. Moreover, if the expansion parameter \( k_{\text{mfp}} \) is not small, one in principle has to consider higher orders in the derivative expansion as well (see for instance [7]). Second-order hydrodynamics was also introduced originally to cure the problems of non-causal propagation, instabilities, and the lack of a well-posed initial value problem for rotating fluids in the first-order approximation. Nevertheless, second-order hydrodynamics breaks down at the “classical” level because of thermal fluctuations [1]. This is due to the effective field theory nature of hydrodynamics; thermal fluctuations are analogous to loops in Feynman diagrams of field theory. Another recent advance in the formulation of hydrodynamics has been the finding that hydrodynamical equations (not limited only to the linear order) for certain classes of strongly-interacting conformal field theories in \( d \) dimensions can be mapped (i.e., they are dual) to the long-wave limit of Einstein’s equations in \( d + 1 \) dimensions. This has been called the fluid-gravity correspondence [8], and it is currently also attracting much attention.

Although the perfect fluid approximation (no dissipation) can be a good approximation in many physical situations, quantum mechanics actually prevents (by the Uncertainty Principle) from the existence of perfect fluids in Nature [9,10,11,12]. In particular, viscous effects are crucial for understanding the physics of the heavy-ion programs currently running at RHIC and LHC, and for the future program at FAIR. In the next section I will review some of the recent studies concerning the linear transport coefficients of QCD.

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3 The word ‘frame’ in this context is used in a different sense from Lorenz transformations. It means a particular choice within the arbitrariness that exist for defining the quantities \( T(x), \mu(x), \) and \( u(x) \) in a non-equilibrium situation. With a different choice of frame, the arrangement of the different transport coefficients in the constitutive equations could change. For instance, the bulk viscosity could appear contributing to the charge current. Nevertheless, the value of the transport coefficients is frame-independent.

4 Sometimes the heat conductivity is defined with an extra power of \( T \) multiplying \( \sigma \). With our definition, both \( \kappa \) and \( \sigma \) have the same dimensions.

5 Due to the non-hyperbolic nature of the linear hydrodynamic equations, high-\( k \) modes would propagate with a super-luminal velocity, which represents a problem for numerical simulations.

6 The equilibrium state around which the gradient expansion is performed would be unstable against small perturbations.
### 2. Transport coefficients of QCD: kinetic theory vs field theory

As we have seen in the previous section, transport coefficients are, together with the equation of state, the essential inputs to describe the space-time evolution of the fluid by means of the hydrodynamic equations. Their value has to be determined using the underlying theory, in our case QCD. However, the calculation of physical observables in QCD at intermediate energies (which is the situation in FAIR) is a challenging task because of the lack of a small parameter in the theory to perform perturbation theory with. Traditionally, transport coefficients have been calculated using kinetic theory (KT) (e.g., for an analysis in high-temperature QCD see [13]). This is a reasonable approach if the system is dilute enough, which in the case of QCD only happens at very high and very low energies. At intermediate energies (strong couplings), KT is no-longer applicable and one should rely on field-theory methods. In field theory, transport coefficients are calculated as the limit when the external momentum and frequency go to zero (in this order) of spectral functions corresponding to correlators of conserved currents. For instance, in the case of the shear viscosity \[ \eta \approx \frac{1}{20} \lim_{T \rightarrow 0} \lim_{\omega \rightarrow 0^+} \frac{\partial \rho_\eta(\omega, |k|)}{\partial \omega}, \quad \rho_\eta(\omega, |k|) = \int d^4 x \ e^{-ik \cdot x + i\omega t} \langle [\pi_{ij}(x), \pi_{ij}(0)] \rangle, \] (4)

where \( \pi_{ij} = T_{ij} - \frac{1}{2} \delta_{ij} T_k^k \) (the indices correspond to spatial coordinates). To my knowledge, so far the only accurate approaches within field theory to calculate transport coefficients at strong-coupling are the large-N expansion (for vector groups) (see for instance [15]) and the AdS/CFT correspondence [16]. None of these two methods deal with QCD, but with field-theory models which resemble QCD in some aspects, so one expects at least to obtain some insight about the qualitative behavior of transport coefficients in QCD from them. The approach based on the AdS/CFT correspondence is specially promising because there is currently a very intense activity to extend it to tackle a larger class of theories, and eventually QCD. Another (and also very promising) strategy consists in performing a numerical calculation on the lattice. However, the statistical errors are still large even at zero chemical potential, and calculating at finite densities is even more difficult because of the sign problem [19].

Besides the methods described above, which apply rigorously in the strong-coupling regime, some KT- or perturbative-based approaches try to extend their domain of applicability by including additional degrees of freedom, collision processes, or resummations of higher-order diagrams. For instance, in the quark-gluon plasma (QGP) phase, higher-order contributions to some transport coefficients have been calculated [20,21,22], which allow a better extrapolation down to temperatures near \( \Lambda_{QCD} \). On the other hand, in the hadronic-phase one in principle expects that the pionic component dominates the dynamics at very low temperatures. Since the interaction between hadrons weakens at low energies (in the massive case, cross sections

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7 It is important to mention that even the calculation of transport coefficients in the regime of temperatures where perturbation theory is applicable is highly non-trivial due to the presence of “pinching” singularities. In order to obtain the leading order in terms of the coupling constant it is necessary to carry out a resummation of an infinite set of diagrams. It turns out that, after performing the resummation, the result can be identified with the result one would obtain from an effective KT [14]. Therefore, when perturbation theory is applicable, it is more convenient to use from the beginning KT instead of field-theory.

8 Transport coefficients are obtained from correlation functions in real time, whereas on the lattice these functions are calculated in Euclidean time. Both quantities are related through a convolution [17] and, in order to obtain the real-time correlator at low frequencies from a finite set of values of the Euclidean correlator, one has to make further assumptions (use an ansatz) for the shape of the correlation function in real time and then find the parameters of the ansatz from the best fit to the imaginary-time correlator. This is the so-called Maximum Entropy Method (see for instance [18] and references therein).

9 The case of the bulk viscosity of a multi-component gas deserves further study. Because of the exponential increase of \( \zeta \) at low temperatures for a one-component fluid (controlled by the mass of the field [14]), it is not obvious whether states heavier than the pion might dominate the bulk viscosity at low temperatures.
tend to a constant value, KT is applicable in that case [23]. A way of extending these low-
temperature calculations consists in including additional hadronic states [24,25,26]. However,
early the cross-over transition, the number of relevant hadronic degrees of freedom becomes large
and it is still unclear how to describe all their interactions. Performing resummations in Chiral
Perturbation Theory is another approach to try to obtain insight on the qualitative behavior of
transport coefficients near the transition temperature [27]. Using this technique it was shown
that unitarity makes the pion gas to fulfill the KSS bound (read below) [28].

There are still no accurate results available for the linear transport coefficients (shear and
bulk viscosities) of QCD near the cross-over temperature at zero baryon chemical potential.
The shear viscosity has been the most studied one. Most of the analyses point to a value of
the quotient $\eta/s$ (which can be considered as a measure of how “perfect” a fluid is) very close
to the proposed KSS lower bound $1/4\pi$ obtained using the AdS/CFT conjecture [10]. It is by
now acknowledged that this does not constitute a universal lower bound, but a bound which is
obeyed by a large class of gauge theories with gravity duals (for a result violating the bound see
for instance [29]). The fact that QCD seems to have a value of $\eta/s$ very close to the KSS bound
is a remarkable result, which may also indicate the existence of a gravity dual formulation of the
theory at least in the limit of large number of colors, something that string theorists are trying
to find with great dedication.

The bulk viscosity has been much less studied than the shear viscosity. This is probably,
I believe, because in the high temperature regime it is found to be negligible with respect to
the shear viscosity [30] (the bulk viscosity vanishes in the conformal limit). Moreover, initial
analyses of this transport coefficient in the hadronic phase also obtained a very small $\zeta$ [23].
However, more recent studies have shown that it could be non-negligible near the transition
temperature [31,32,33,34,35]. It is not a general rule that the bulk viscosity has to be smaller
than the shear viscosity, in fact, some systems present a bulk viscosity which is several orders of
magnitude larger [36]. It has been also recently conjectured a lower bound for the bulk viscosity
[37], but there are already examples where this bound is violated [38].

At zero baryon chemical potential, another linear transport coefficient which has been recently
studied is the electrical conductivity [18,39,40]. This corresponds to dissipation due to the
introduction of an external electric field coupled to quarks (and at low temperatures its effect can
be implemented through its coupling to pions). The frequency-dependent value of the electrical
conductivity is directly related to the photon spectrum emitted from the QCD plasma, and the
results of the calculations seem to be compatible (within the same order of magnitude) with
experimental measurements of the spectrum at low energies.

2.1. Behavior of transport coefficients near phase transitions

For many systems which undergo a phase transition (and it is also the case of QCD with non-
zero quark masses and zero baryon chemical potential), the quotient $\eta/s$ reaches its minimum
value at or close to the transition temperature [41,42]. The $v_2$ spectrum measured in heavy-ion
collisions (measure of elliptic flow) indeed seems to indicate a small value of the shear viscosity
for intermediate temperatures [43]. On the other hand, a rapid increase of the bulk viscosity
near the transition temperature due to non-perturbative effects could also produce observable
consequences in the spectrum of produced particles due to clusterization of the plasma [44].

\textsuperscript{10} At intermediate energies, non-perturbative effects such as resonances can appear. KT is still applicable at those
energies provided some conditions are fulfilled [14]: (i) we include the non-perturbative effects and other thermal
corrections into the scattering amplitudes and dispersion relations of the asymptotic hadrons, (ii) the resulting
mean-free path of the asymptotic states remains much larger than their thermal de Broglie wave length.

\textsuperscript{11} In [31], it was proposed an increase due to a direct correlation between the bulk viscosity and the trace anomaly,
which lattice calculations have shown to have a peak near the transition temperature. However, analyses in other
field theories indicate that this correlation may not be direct [38].
For the heavy-ion program at FAIR, it is of special importance the physics of the QCD critical point [45]. It belongs to the dynamic universality class of the Model H [2] (the same as that of the liquid-gas phase transition, according to the classification of [46]). For this universality class, the linear transport coefficients present the (divergent) critical behavior near $T_c$ [46,47]:

$$\eta \sim t^{-0.05}, \quad \zeta \sim t^{-2.8}, \quad \text{and} \quad \kappa \sim t^{-0.95}$$

(thermal conductivity), with $t \equiv |T - T_c|/T_c$. Therefore, the bulk viscosity would probably cause a stronger experimental effect [48,49,50], whereas the shear viscosity is essentially insensitive to the critical point. For a recent analysis of the critical behavior of the thermal conductivity and its phenomenological consequences see [51].

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