A REFINEMENT OF FRANKS’ THEOREM

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Abstract. In this paper, we give a refinement of Franks’ theorem [Fr92], which answers two questions raised by Kang [Kang14].

Résumé. Dans cet article, nous donnons un raffinement du théorème de Franks [Fr92], il répond aux questions proposées par Kang [Kang14].

1. INTRODUCTION

Let $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times (0,1)$ (resp. $\bar{\mathbb{A}} = \mathbb{R}/\mathbb{Z} \times [0,1]$) be the open annulus (resp. the closed annulus). In 1992, Franks [Fr92] (and [Fr96]) proved the following celebrated theorem:

Theorem 1 (Franks). Suppose $F$ be an area preserving homeomorphism of the open or closed annulus which is isotopic to the identity. If $F$ has at least one fixed or periodic point then $F$ must have infinitely many interior periodic points.

Kang [Kang14, Section 1.2] raised the following questions when he studied the reversible maps on planar domains:

Suppose that $F$ is an area preserving homeomorphism of the open or closed annulus which is isotopic to the identity. If $F$ has at least one fixed or periodic point then $F$ must have infinitely many interior periodic points.

Kang [Kang14] Section 1.2 raised the following questions when he studied the reversible maps on planar domains:

Suppose that $F$ is an area preserving homeomorphism of the open or closed annulus which is isotopic to the identity. If $\text{Per}_{\text{odd}}(F) \neq \emptyset$, does it imply that $\sharp \text{Per}_{\text{odd}}(F) = +\infty$ where $\text{Per}_{\text{odd}}(F)$ is the set of odd periodic points of $F$? Furthermore, let $k \in \mathbb{N}$ and $n \in \mathbb{N}$ be two numbers such that $(n,k) = 1$. If $\text{Per}_k(F) \neq \emptyset$, does it imply that $\sharp \bigcup_{(k',n) = 1} \text{Per}_{k'}(F) = +\infty$ where $\text{Per}_k(F)$ is the set of $k$-prime-periodic points of $F$? Here $z$ is a $k$-prime-periodic of $F$ means that $z$ is not a $l$-periodic point of $F$ if $l < k$.

In this paper, we answer his questions. We have the following theorems

Theorem 2. If $\text{Per}_{\text{odd}}(F) \neq \emptyset$, then $\sharp \text{Per}_{\text{odd}}(F) = +\infty$.

Theorem 3. Assume that $k, n_0 \in \mathbb{N}$ which satisfy that $(k, n_0) = 1$. If $\text{Per}_k(F) \neq \emptyset$, then

$$\sharp \left\{ \bigcup_{(k',n_0) = 1} \text{Per}_{k'}(F) \right\} = +\infty.$$
Remark 1. Theorem 3 implies Theorem 2. Indeed, if $\text{Per}_{\text{odd}}(F) \neq \emptyset$, there is $k \in 2\mathbb{Z} + 1$ such that $\text{Per}_k(F) \neq \emptyset$. Taken $n_0 = 2$, then Theorem 2 follows from Theorem 3.

Hence, we only need to prove Theorem 3. We will introduce some mathematical objects and recall some well-known facts in Section 2. We will prove Theorem 3 in Section 3.

Acknowledgements. We would like to thank Patrice Le Calvez and Yiming Long for their helpful conversations and comments.

2. Preliminaries

2.1. Rotation vector. Let us introduce the classical notion of rotation vector which was defined originally in [Sch57]. Let $M$ be a smooth manifold. Suppose that $F$ is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on $M$. Let $\text{Rec}^+(F)$ be the set of positively recurrent points of $F$. If $z \in \text{Rec}^+(F)$, we fix an open disk $U \subset M$ containing $z$, and write $\{F^{n_k}(z)\}_{k \geq 1}$ for the subsequence of the positive orbit of $z$ obtained by keeping the points that are in $U$. For any $k \geq 0$, choose a simple path $\gamma_{F^{n_k}(z),z}$ in $U$ joining $F^{n_k}(z)$ to $z$. The homology class $[\Gamma_k]_M \in H_1(M,\mathbb{Z})$ of the loop $\Gamma_k = I^{n_k}(z)\gamma_{F^{n_k}(z),z}$ does not depend on the choice of $\gamma_{F^{n_k}(z),z}$. Say that $z$ has a rotation vector $\rho_{M,I}(z) \in H_1(M,\mathbb{R})$ if

$$\lim_{k \to +\infty} \frac{1}{n_k} [\Gamma_k]_M = \rho_{M,I}(z)$$

for any subsequence $\{F^{n_k}(z)\}_{k \geq 1}$ which converges to $z$. Neither the existence nor the value of the rotation vector depends on the choice of $U$. Let $\mathcal{M}(F)$ be the set of Borel finite measures on $M$ whose elements are invariant by $F$. If $\mu \in \mathcal{M}(F)$ and $M$ is compact, we can define the rotation vector $\rho_{M,I}(\mu)$ for $\mu$-almost every positively recurrent point [Lec06] (see also Section 1.3 in [Wang11]). If we suppose that the rotation vector $\rho_{M,I}(z)$ is $\mu$-integrable, we define the rotation vector of the measure

$$\rho_{M,I}(\mu) = \int_M \rho_{M,I} \, d\mu \in H_1(M,\mathbb{R}).$$

Remark that the definition of rotation vector here is the same as the homological rotation vector that was defined by Franks in [Fr92] on the positively recurrent points set.

The following theorem is due to Franks ([Fr92, Fr96]):

**Theorem 4.** Let $M$ be an oriented surface of genus 0 with $\chi(M) \leq 0$, that is, $M = S^2 \setminus \{x_1, x_2, \ldots, x_n\}$ where $n \geq 2$. Suppose that $F$ is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on $M$ and preserves a finite measure $\mu$ of $M$ with total support. If $\rho_{M,I}(\mu) = 0$, then $F$ has a fixed point in the interior of $M$.

Denote by $\text{Fix}_{\text{Cont},I}(F)$ the set of contractible fixed points of $F$, that is, $x \in \text{Fix}_{\text{Cont},I}(F)$ if and only if $x$ is a fixed point of $F$ and the oriented loop $I(x) : t \mapsto F_t(x)$ defined on $[0,1]$ is contractible on $M$. In [Lec06, Theorem 8.1], Le Calvez proved the following deep result.
Theorem 5. Suppose that $M$ is a surface (without boundary) and $I = (F_t)_{t \in [0,1]}$ is an isotopy on $M$ from $\text{Id}_M$ to $F$. We suppose that $F$ has no contractible fixed points. Then there exists an oriented topological foliation $\mathcal{F}$ on $M$ such that, for all $z \in M$, the trajectory $I(z)$ is homotopic to an arc $\gamma$ joining $z$ and $F(z)$ in $M$ which is positively transverse to $\mathcal{F}$. That means that for every $t_0 \in [0,1]$ there exists an open neighborhood $V \subset M$ of $\gamma(t_0)$ and an orientation preserving homeomorphism $h : V \to (-1,1)^2$ which sends the foliation $\mathcal{F}$ on the horizontal foliation (oriented with $x$) and $h^{-1}$ has the intersection property as $\text{Homeo}^*$.

A refinement of this statement is that for every $t_0 \in [0,1]$ there exists an open neighborhood $V \subset M$ of $\gamma(t_0)$ and an orientation preserving homeomorphism $h : V \to (-1,1)^2$ which sends the foliation $\mathcal{F}$ on the horizontal foliation (oriented with $x$) such that the map $t \mapsto p_2(h(\gamma(t)))$ defined in a neighborhood of $t_0$ is strictly increasing where $p_2(x_1, x_2) = x_2$.

We say that $X \subseteq \text{Fix}_{\text{Cont},I}(F)$ is unlinked if there exists an isotopy $I' = (F'_t)_{t \in [0,1]}$ homotopic to $I$ which fixes every point of $X$, that is, $F'_t(x) = x$ for all $t \in [0,1]$ and $x \in X$. Moreover, we say that $X$ is a maximal unlinked set, if any set $X' \subseteq \text{Fix}_{\text{Cont},I}(F)$ which strictly contains $X$ is not unlinked. If $\#\text{Fix}_{\text{Cont},I}(F) < \infty$, there must be a set $X \subseteq \text{Fix}_{\text{Cont},I}(F)$ which is a maximal unlinked set. By Theorem 5 there exists an oriented topological foliation $\mathcal{F}$ on $M \setminus X$ (or, equivalently, a singular oriented foliation $\mathcal{F}$ on $M$ with $X$ equal to the singular set) such that, for all $z \in M \setminus X$, the trajectory $I(z)$ is homotopic to an arc $\gamma$ joining $z$ and $F(z)$ in $M \setminus X$ which is positively transverse to $\mathcal{F}$.

2.2. Rotation number. We denote by $\pi$ the covering map of the open annulus (resp. the closed annulus)

$$\pi : \mathbb{R} \times (0,1) \ (\text{resp. } \mathbb{R} \times [0,1]) \to \mathbb{A} \ (\text{resp. } \bar{\mathbb{A}})$$

$$(x, y) \mapsto (x + \mathbb{Z}, y),$$

and by $T$ the generator of the covering transformation group

$$T : \mathbb{R} \times (0,1) \ (\text{resp. } \mathbb{R} \times [0,1]) \to \mathbb{R} \times (0,1) \ (\text{resp. } \mathbb{R} \times [0,1])$$

$$(x, y) \mapsto (x + 1, y).$$

We denote by $S$ and $N$ for the lower and the upper end of $\mathbb{A}$.

In the following, we denote by $\mathbb{A}$ the open or closed annulus unless an explicit mention. We call essential circle in $\mathbb{A}$ every simple closed curve which is not null-homotopic. Let $F$ be a homeomorphism of $\mathbb{A}$. We say that $F$ satisfies the intersection property if any essential circle in $\mathbb{A}$ meets its image by $F$. We denote the space of all homeomorphisms of $\mathbb{A}$ which are isotopic to the identity as Homeo$_0(\mathbb{A})$ and its subspace whose elements additionally have the intersection property as Homeo$^*_0(\mathbb{A})$. It is easy to see that a homeomorphism of $\mathbb{A}$ that preserves a finite measure with total support satisfies the intersection property.

When $F \in \text{Homeo}_0(\mathbb{A})$, we define the rotation number of a positively recurrent point as follows. We say that a positively recurrent point $z$ has a rotation number $\rho(f; z) \in \mathbb{R}$ for a lift $f$ of $F$ to the universal cover of $\mathbb{A}$, if for every subsequence $\{F^{n_k}(z)\}_{k \geq 0}$ of $\{F^n(z)\}_{n \geq 0}$ which converges to $z$, we have

$$\lim_{k \to +\infty} \frac{p_1 \circ F^{n_k}(\bar{z}) - p_1(\bar{z})}{n_k} = \rho(f; z)$$

where $\bar{z} \in \pi^{-1}(z)$ and $p_1$ is the first projection $p_1(x, y) = x$. In particular, the rotation number $\rho(f; z)$ always exists and is rational when $z$ is a fixed or periodic point of $F$. Let
Rec\(^+\)(F) be the set of positively recurrent points of \(F\). We denote the set of rotation numbers of positively recurrent points of \(F\) as Rot\((f)\).

It is well known that a positively recurrent point of \(F\) is also a positively recurrent point of \(F^q\) for all \(q \in \mathbb{N}\) (see the appendix of [Wang14]). By the definition of rotation number, we easily get that the following elementary properties.

1. \(\rho(T^k \circ f; z) = \rho(f; z) + k\), and hence Rot\((T^k \circ f) = \text{Rot}(f) + k\) for every \(k \in \mathbb{Z}\);
2. \(\rho(f^q; z) = q\rho(f; z)\), and hence Rot\((f^q) = q\text{Rot}(f)\) for every \(q \in \mathbb{N}\).

Suppose that \(z \in \text{Rec}(F)\) and \(\tilde{z} \in \pi^{-1}(z)\). We define \(\mathcal{E}(z) \subset \mathbb{R} \cup \{-\infty, +\infty\}\) by saying that \(\rho \in \mathcal{E}(z)\) if there exists a sequence \(\{n_k\}_{k=1}^{+\infty} \subset \mathbb{N}\) such that

- \(\lim_{k \to +\infty} F^{n_k}(z) = z\);
- \(\lim_{k \to +\infty} \frac{p_1(f^{n_k}(\tilde{z})) - p_1(\tilde{z})}{n_k} = \rho\).

Define \(\rho^-(f; z) = \inf \mathcal{E}(z)\) and \(\rho^+(f; z) = \sup \mathcal{E}(z)\). Obviously, we have that \(\rho(f; z)\) exists if and only if \(\rho^-(f; z) = \rho^+(f; z) \in \mathbb{R}\). Note that the set \(\mathcal{E}(z)\) is a bounded set when \(A\) is a closed annulus (by compactness) and might be an unbounded set when \(A\) is an open annulus. However, the set \(\mathcal{E}(z)\) is still a bounded set if \(\sharp\text{Fix}(F) < +\infty\) (see [Lec01]). By the definitions, it is easy to see that \(\rho^-(f; z)\) and \(\rho^+(f; z)\) satisfy the same properties as \(\rho(f; z)\).

**Remark 2.** A lift \(f\) of \(F\) is one-to-one corresponding to an identity isotopy \(I\) (mod homotopy). Observing that \(H_1(A, \mathbb{R}) \simeq \mathbb{R}\), the rotation number \(\rho(f; z)\) is nothing else but the rotation vector \(\rho_{\mathcal{L}}(I\circ f)(z)\) where the time-one map of the lift identity isotopy of \(I\) to the universal cover is \(f\).

The following Theorem is due to Franks [Fr88] when \(A\) is closed annulus and \(F\) has no wandering point, and it was improved by Le Calvez [Lec06] (see also [Wang14]) when \(A\) is open annulus and \(F\) satisfies the intersection property.

**Theorem 6.** Let \(F \in \text{Homeo}^\circ (A)\) and \(f\) be a lift of \(F\) to the universal cover of \(A\). Suppose that there exist two recurrent points \(z_1\) and \(z_2\) such that \(-\infty \leq \rho^-(f; z_1) < \rho^+(f; z_2) \leq +\infty\). Then for any rational number \(p/q \in ]\rho^-(f; z_1), \rho^+(f; z_2)[\) written in an irreducible way, there exists a periodic point of period \(q\) whose rotation number is \(p/q\).

### 3. Proof of the Theorems

**Proof of the Theorem 6**. If \(\sharp\text{Per}_k(F) = +\infty\), we have nothing to do. Hence we assume that \(\sharp\text{Per}_k(F) < +\infty\). Let \(\text{Per}_k(F) = \{x_1, \cdots, x_m\}\) and \(G = F^k\). If \(\sharp\text{Fix}(G) = +\infty\), then there exists \(t \mid k\) with \((t, n_0) = 1\) such that \(\sharp\text{Per}_t(F) = +\infty\). We have done. Therefore, we can assume that \(1 \leq \sharp\text{Fix}(G) < +\infty\). Assume that \(\text{Fix}(G) = \{y_1, \cdots, y_m\}\). By Theorem 6 we can choose a lift \(f\) of \(F\) to the universal cover of \(A\) or \(A\) such that \(\rho(f; x_i) = \frac{k'}{q'} \in [0, 1)\) for \(i = 1, \cdots, m\) where \(k' \in \mathbb{Z}\). The proof will be divided into two cases: \(k' = 0\) and \(k' \neq 0\).
In the case of $k' \neq 0$, by Theorem \[1\] we know that $\# \text{Fix}(G) = +\infty$. Hence $\text{Per}(G) \setminus \text{Fix}(G) \neq \emptyset$. Note that $\rho(f^k; y_i) = k'$ for all $i$. For any $z \in \text{Per}(G) \setminus \text{Fix}(G)$, we assume that $\rho(f^k; z) = \frac{p}{q}$ with $q \geq 1$ and $(p, q) = 1$ if $p \neq 0$. W.l.o.g., we assume that $k' < \frac{p}{q}$.

Then by Theorem \[2\] for every $\frac{p}{q} \in (k', \frac{p}{q})$ with $(r, s) = 1$ and $(s, n_0) = 1$, there exists $z' \in \text{Per}(G)$ such that $\rho(f^k; z') = \frac{p}{q}$. Observing that the point $z'$ is a $k s$-periodic point of $F$, the conclusion follows in this case.

We now consider the case of $k' = 0$. If the annulus is a closed annulus $\tilde{A}$, we consider its interior $\mathcal{A}$. Let $S^2 = \mathcal{A} \cup \{N, S\}$. Note that $\chi(S^2 \setminus \{N, S, y_1, \ldots, y_{m'}\}) \leq 0 (= 0$ when the annulus is closed and $\text{Fix}(G) \subset \partial \mathcal{A}$). Write $M = S^2 \setminus \{N, S, y_1, \ldots, y_{m'}\}$.

When $\chi(M) = 0$, we work on the closed annulus $\tilde{A}$. We have that $\rho(f^k; y_i) = 0$ and $\rho(f^k; z) = 0$ for any $z \in \text{Rec}^+(G)$ (by Theorem \[3\]). By Theorem \[3\] there is a fixed point of $G$ in the interior of $\tilde{A}$ which contradicts the fact that $\text{Fix}(G) \subset \partial \mathcal{A}$.

In the case of $\chi(M) < 0$, we follow the idea of Le Calvez \[Lec06\, \text{Theorem} \, 9.3\]. We choose an identity isotopy $I_0 = (F_t)_{t \in [0, 1]}$ on $S^2$ such that $F_t$ fixes $N$ and $S$ for every $t$, and the time-one map of the lift identity isotopy of $I_0|_{\mathcal{A}}$ to $\mathbb{R} \times (0, 1)$ is $f^k$. Furthermore, as $\rho(f^k; y_i) = 0$ for all $i$, we can suppose that $I_0$ fixes $N, S$ and one point of $\{y_1, \ldots, y_{m'}\}$ (e.g., we can modify $I_0$ though the technique in the proof of Lemma 1.2 in \[Wang11\], Section 1.4] without changing the homotopic class of $I_0|_{\mathcal{A}}$). Identify $G$ as a map of $S^2$.

As $\# \text{Fix}(G) < +\infty$, there is a maximal unlink set $X \subset \text{Fix}(G) = \{N, S, y_1, \ldots, y_{m'}\}$ and an identity isotopy $I_1$ which is homotopic to $I_0$ with fixed endpoints such that $I_1$ fixes every point of $X$. Note that $\# X \geq 3$ in this case. By Theorem \[5\] there exists an oriented topological foliation $\mathcal{F}$ on $S^2 \setminus X$ such that, for all $z \in S^2 \setminus X$, the trajectory $I_1(z)$ is homotopic to an arc $\gamma$ joining $z$ and $G(z)$ in $S^2 \setminus X$ which is positively transverse to $\mathcal{F}$. For any leaf $\lambda \in \mathcal{F}$, the $\alpha$-limit set and $\omega$-limit set of $\lambda$ must belong to two distinct points of $X$ respectively since $X$ is finite and $G$ is symplectic. Choose a leaf $\lambda \in \mathcal{F}$ which connects two different points $z_1$ and $z_2$ of $X$. We consider the following open annulus $A_{z_1, z_2} = S^2 \setminus \{z_1, z_2\}$.

We choose a small open disk $U$ near $\lambda$ such that, $U \cap \lambda = \emptyset$ and for any $z \in U$, $\lambda \cap I_1(z) \geq 1$ where $I_1(z)$ is the trajectory of $z$ under the isotopy $I_1$. We define the first return map

$$
\Phi : \text{Rec}^+(G) \cap U \rightarrow \text{Rec}^+(G) \cap U,
\quad z \mapsto G^\tau(z)(z),
$$

where $\tau(z)$ is the first return time, that is, the least number $n \geq 1$ such that $G^n(z) \in U$. By Poincaré Recurrence Theorem, this map is defined $\mu$-a.e. on $U$. For every couple $(z', z'') \in U^2$, choose a simple path $\gamma_{z', z''}$ in $U$ joining $z'$ to $z''$. For every $z \in \text{Rec}^+(G) \cap U$ and $n \geq 1$, define

$$
\tau_n(z) = \sum_{i=0}^{n-1} \tau(\Phi^i(z)), \quad \Gamma_z^n = I_1^n(z)(z)\gamma_{\Phi^n(z), z}, \quad m(z) = \Gamma_z^1 \land \lambda, \quad m_n(z) = \sum_{i=0}^{n-1} m(\Phi^i(z)).
$$
It is well known that $\tau \in L^1(U, \mathbb{R})$ (see, e.g., [Wang11, Section 1.3]). Hence $\tau_n/n$ converges $\mu$-a.e. on $\text{Rec}^+(G) \cap U$. It is clear that $m_n/n \geq 1$ for all $n \geq 1$ and $z \in \text{Rec}^+(G) \cap U$. This implies that $m_n/\tau_n > 0$ for $\mu$-a.e. on $\text{Rec}^+(G) \cap U$. Observe that

$$\mathcal{E}(z) \subset \left[ \inf_n \frac{m_n(z)}{\tau_n(z)}, \sup_n \frac{m_n(z)}{\tau_n(z)} \right] \subset \mathbb{R}$$

when the limit of $\tau_n(z)/n$ exists, where the definition $\mathcal{E}(z)$ one can refer to Remark 2. We get that $\rho_{A_{z_1,z_2,I_1}}(z) > 0$ for $\mu$-a.e. on $\text{Rec}^+(G) \cap U$. We also have that $\rho_{A_{z_1,z_2,I_1}}(y) = 0$ for all $y \in X \setminus \{z_1,z_2\}$. Then Theorem 3 follows by Theorem 6.

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