On Connection between Topological Landau-Ginzburg Gravity and Integrable Systems.

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Abstract

We study flows on the space of topological Landau-Ginzburg theories coupled to topological gravity. We argue that flows corresponding to gravitational descendents change the target space from a complex plane to a punctured complex plane and lead to the motion of punctures. It is shown that the evolution of the topological theory due to these flows is given by dispersionless limit of KP hierarchy. We argue that the generating function of correlators in such theories are equal to the logarithm of the tau-function of Generalized Kontsevich Model.

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1 Introduction

One of the most interesting features of topological matter coupled to topological gravity\[1, 2, 3, 4\] is its connection to integrable systems. Namely, the exponent of the generating function for topological correlators turns out to be tau-function of the integrable system. Now we know several possible ways to explain it:

1. Topological theory is in some sense equivalent to the "physical" theory (conformal matter coupled to 2d gravity) and the latter can be discretized with the help of matrix models which expose integrable structures before taking the continuum limit \[5\] as well as in the double scaling limit \[6, 7, 4, 8\]
2. Pure gravity can be described in terms of the combinatorial model of the moduli space, and this description naturally arises from the perturbative expansion in the Kontsevich matrix model; the matrix integral in this model turns out to be tau-function of KP hierarchy \[4, 8\]
3. The so called $\tilde{W}$-constraints can be considered as symmetries of topological theories; $\tilde{W}$-algebra naturally acts on tau-function \[11\], thus the solution to these constraints is some special tau-function \[10, 8, 12\]
4. String field theory for "topological strings" seems to be a 2d integrable theory

We think, that, at present, all these explanations do not give absolutely satisfactory picture of what happens. That is why in this paper we will try to push forward the fifth explanation:

5. Observables can be identified with the tangent vectors to the space of topological theories, thus, each observable leads to the flow on the space of topological theories. The very existence of the generating function for the correlators means that these flows commute.

Namely, we consider what goes on in Landau-Ginzburg theories coupled to topological gravity for the genus zero world-sheet.

In the Section 2 we summarize results on flows corresponding to the primitive observables (lowest times in terms of KP hierarchy reductions); these flows change superpotential and observables (due to contact terms). We argue that these contact terms can be obtained via Verlinde-Verlinde (VV) mechanism\[3\], i.e. have purely gravitational interpretation. Different properties of correlators are discussed, and among them: evolution system and interpretation of the reduced string equation (small phase space at genus zero, in the form of Krichever\[13\]) as a dilaton equation. The purpose of this section is to fix notations, the only new result is the new interpretation of the reduced string equation.

In the Section 3 we argue that flows corresponding to descendants change the target space of the theory: from a complex plane to a punctured complex plane. After that change flows correspond to motion of punctures. Contact terms are obtained according the same ”rules of the game” as in Section 2; evolution system in this case is described, properties of correlators are studied. We show how dispersionless limit of KP hierarchy appears from the evolution system. Now the full string equation (in the form of Krichever\[13\]) also follows from the dilaton equation.

In the last Section we compare generating functions for correlators and tau-function
of the Generalized Kontsevich Model\cite{8}.

In this paper we will be extremely brief in ”physical” arguments (mostly they are given in comments and footnotes); in full detail we will present them in [14].

2 Simplest case: target space - complex plane

2.1 Primary fields, descendants and 3-point correlator

Here we will consider the simplest target space: complex plane. Thus, Landau-Ginzburg topological matter (LGM)\cite{15, 16, 17} with a complex plane as a target space is described by its superpotential $W(X)$ of degree $p$ and a metric, that is constant in a holomorphic coordinate $X$ on this plane. The top holomorphic form corresponding to this metric is equal to $dX$.

The space of LGM topological observables form so called chiral ring $C[X]/[dW/dX]$ that is a quotient ring of polynomials $P(X)$ over the ideal generated by the derivative of superpotential. Correlators in LGM in genus zero are given by:

$$< P_1, \ldots, P_n >^M_W = \int_{\Gamma} \frac{P_1(X) \cdots P_n(X)(dX)^2}{dW}$$

Here contour $\Gamma$ goes around infinity in the complex plane.

Superscript $M$ here is very important - it distinguishes correlators in LGM from correlators in LGM coupled to topological gravity, i.e. Landau-Ginzburg gravity (LGG). Later will have no subscript at all.

Appearance of non-covariant term $(dX)^2$ in the correlator on a sphere illustrates the above-mentioned anomaly in coordinate transformations on the target space - holomorphic top form appears in power 2 that is the Euler number of a sphere.

In [19] (see also [20]) it was shown that the space of local topological observables in LGG can be embedded in $C[X]$, and $\sigma_i(\Phi)$ - the i-th gravitational descendant of the LGM primary field $\Phi$ (by the primary field we call polynomial $\Phi(X)$ of degree less than $p - 1$ which is the the degree of $W'$) can be expressed as follows (see also):

$$\sigma_i(\Phi) = W'(X) \int_X^{X_1} \cdots \int_X^{X_3} dW(X_1) \cdots dW(X_2) \Phi(X_1)dX_1$$

In topological gravity correlators are integrals over moduli space of Riemann surfaces with punctures. In genus zero such moduli space does not exist for zero, one and two punctures, so we take these n-point correlators in LGG to be zero for $n < 3$. For $n = 3$ moduli space is a point and correlator in LGG equals to 3-point correlator in LGM.

\footnote{Due to anomaly in general coordinate transformations on the target space, see\cite{13}, theory can be defined only for metric whose determinant is a square of the modulus of the top holomorphic form}

\footnote{The origin of this effect is normal ordering of the operator $XX$ - it is similar to appearance of anomalous dimension in conformal theory}

\footnote{Primary fields are distinguished among all polynomials that belong to the same class in the factoring $C[X]/[dW(X)]$ by the following requirement: acting on a vacuum they create states with zero energy \cite{14}. If $W$ is a monomial, there is a $U(1)$ group that acts on the space of all polynomials, and it is evident that primary field should have definite charge with respect to this group, so there is no other choice for primary fields but monomials of order less than the order of $W$. For an arbitrary $W$ the problem of finding primaries deserves special investigation}
2.2 4-point correlators and contact terms

It was argued in [19] that the four-point correlators with the fourth field being primary one \( < P_1, P_2, P_3, \Phi >_W \) describes the derivative of superpotential \( W \) plus contact terms:

\[
< P_1, P_2, P_3, \Phi >_W = \frac{d}{dt} < P_1, P_2, P_3 >_{W+\Phi} \bigg|_{t=0} + < C_W(\Phi, P_1), P_2, P_3 >_W + < C_W(\Phi, P_2), P_1, P_3 >_W + < C_W(\Phi, P_3), P_1, P_2 >_W
\]

(3)

where the contact term of the fields \( P \) and \( \Phi \) is obtained as follows. The product of fields (this product naturally appears when these two fields are placed on the decoupled sphere in Deligne-Mamford compactification) is decomposed into a sum of descendant and a primary fields; primary field part gives no contribution to the contact term, while descendant contributes due to VV mechanism [3], namely:

\[
P(X)\Phi(X) = W'(X) \int X C_W(P, \Phi)(X_1)dX_1 + \tilde{\Phi}(X); \text{deg} \tilde{\Phi} < p - 1
\]

(4)

From (4) we see that contact term can be also represented as:

\[
C_W(P, \Phi) = \frac{d}{dX} \left( \frac{P\Phi}{W'} \right)_+
\]

(5)

Here subscript "+" stands for non-negative powers of \( X \) in expansion at infinity.

One can check that 4-point correlator (4) is symmetric under permutations of fields (as it should be).

2.3 Formalized definition of \( n \)-point correlator

By induction one can define the \( n \)-point correlators according to the following four rules:

1. The \( n \)-point correlator is a symmetric linear functional on \( C[X]^{\otimes n} \), denoted as \( < P_1, \ldots, P_n >_W \).
2. This functional is equal to zero for \( n < 3 \) and is given by the Grothendieck residue (1) for \( n = 3 \).
3. If all fields are descendants [4], this functional equals to zero
   \[
   < \sigma_{I_1}(\Phi_1), \ldots, \sigma_{I_n}(\Phi_n) >_W = 0, \text{ if } I_i \neq 0 \text{ for all } i
   \]
   (6)
4. If one of the fields is primary and \( n > 3 \) then multi-point correlators obey the following recursion relations [4] (these recursion relations should not be mixed with recursion relations, connecting correlator of \( \sigma_i \) and \( \sigma_{i-1} \)):

\[
< P_1, \ldots, P_n, \Phi >_W = \frac{d}{dt} < P_1, \ldots, P_n >_{W+\Phi} \bigg|_{t=0} + \sum_{i=1}^{n} < P_1, \ldots, C_W(P_i, \Phi), \ldots, P_n >_W
\]

(7)

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[4] if all fields are descendants then the complex dimension of form to be integrated over the moduli space is at least \( n \) while the dimension of moduli space is \( n - 3 \).

[5] these relations correspond to integration over moduli corresponding to the position of the insertion of the field \( \Phi \), i.e. to the "motion" of the point where \( \Phi \) is inserted while all other points are keeping fixed.
and bilinear multiplication (contact term) \( C_W : C[X] \otimes C[X] \rightarrow C[X] \) is given in (5). We will call the field \( \Phi \) that enters in recursion relation (7) a "moving" field.

One can check that this definition of multi-point correlator is self-consistent.

### 2.4 Properties of multi-point correlators

From the definition one can check the following properties of multi-point correlators.

1. **Puncture equation.**
   
   Let us denote as \( 1_P \) polynomial that equals to 1 to distinguish it from all other "one's" that can appear in the algebra; then:
   
   \[
   < \sigma_{I_1}(\Phi_1), \ldots, \sigma_{I_n}(\Phi_n), 1_P >_W = \sum_{i=1}^{n} < \sigma_{I_1}(\Phi_1), \ldots, \sigma_{I_{i-1}}(\Phi_i), \ldots, \sigma_{I_n}(\Phi_n) >_W
   \]
   
   (8)
   
   here we accept \( \sigma_{-1}(\Phi) = 0 \)

2. **Dilaton equation.**
   
   Using standard terminology we will call the field \( \sigma_1(1_P) = XW'(X) \)

   the dilaton. Then recurrently one can show that
   
   \[
   < \sigma_1(1_P), P_1, \ldots, P_n >_W = (n-2) < P_1, \ldots, P_n >
   \]
   
   (10)
   
   This equation (called the dilaton equation) is expected on general grounds from topological gravity \( 2-n \) is an Euler characteristic of sphere with \( n \) deleted points

3. **Factorization equation.**
   
   If \( \sum_{i=1}^{n} I_i = n-3 \) then
   
   \[
   < \sigma_{I_1}(\Phi_1), \ldots, \sigma_{I_n}(\Phi_n) >_W = (n-3)! \int \frac{\Phi_1 \cdot \ldots \cdot \Phi_n (dX)^2}{\prod_{i=1}^{n} I_i !} dW
   \]
   
   (11)
   
   Note that the second factor in (11) is nothing but the \( n \)-point correlator in topological matter, and equation (11) is also expected on general grounds \[1, 2, 3\] from topological gravity coupled to topological matter.

### 2.5 Introduction of times on small phase space

Let us consider the generating function for all multi-point correlators. We will define a correlator of fields \( P_1, \ldots, P_n \) in the presence of a formal exponent of a field \( P \) as a generating function for multipoint correlators:

\[
< P_1, \ldots, P_n ; \exp(P) >_W = < P_1, \ldots, P_n >_W + < P_1, \ldots, P_n, P >_W + \frac{1}{2} < P_1, \ldots, P_n, P, P >_W + \ldots + \frac{1}{k!} < P_1, \ldots, P_n, P, P, \ldots, P >_W + \ldots
\]

(12)
Then, given a basis of primary fields $\Phi_a$ and a set of parameters $t_a$ (these parameters are called coordinates on the small phase space) we define a $t$-dependent multi-point correlators:

$$< P_1, \ldots, P_n >_W(t) = < P_1, \ldots, P_n; \exp(\sum_{a=1}^{p-1} t_a \Phi_a) >_W$$  \hspace{1cm} (13)

From the definition of multi-point correlators it follows that for $n > 2$ one can represent the $t$-dependent multipoint correlators in terms of ordinary multipoint correlators as:

$$< P_1, \ldots, P_n >_W(t) = < P_1(t), \ldots, P_n(t) >_{W(t)}$$  \hspace{1cm} (14)

Where $W(t)$ and $P(t)$ are solutions to the following system of nonlinear differential equations:

$$\frac{\partial}{\partial t_b} \Phi_a(t) = C_{W(t)}(\Phi_a(t), \Phi_b(t))$$  \hspace{1cm} (15)

$$\frac{\partial}{\partial t_b} W(t) = \Phi_b(t)$$  \hspace{1cm} (16)

$$\frac{\partial}{\partial t_b} P_i(t) = C_{W(t)}(\Phi_b(t), P_i(t))$$  \hspace{1cm} (17)

In this derivation it is important that contact term between primary fields gives a primary field that can also be used in recursion relations (7).

From the physical setting we expect that system is integrable, one can explicitly check that it is so.

### 2.6 Flat times

Here we will explain how flat times on the space of potentials appear in the evolution system (15-17). Using recursion relations (7) with $1_P$ as a moving field one can see that for all positive $n$:

$$< \Phi_a, \Phi_b, \Phi_1, \ldots, \Phi_n, 1_P >_W = 0$$  \hspace{1cm} (18)

This follows from two facts:
1. the contact term of $1_P$ with primary field is zero
2. adding constant to $W$ does not change $dW$ and it is $dW$ that appears in all other recursion relations.

Using the symmetry of the multipoint correlators we see that (14) means

$$\eta_{ab}(t) = < \Phi_a(t), \Phi_b(t), 1_P >_{W(t)} = < \Phi_a, \Phi_b, 1_P >_W$$  \hspace{1cm} (19)

Here we used that $1_P(t) = 1_P$

Primitive fields naturally form a tangent space to the space $A_p$ of all superpotentials of degree $p$ moduli constant shifts and dilatations. Together with (1) it means that $\eta$ is a metric on this space in coordinates $t$, and (19) means that it is flat, that is the set of times $\{t_a\}$ forms flat coordinates on $A_p$.

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6Here we closely follow derivation presented in
2.7 Relation between $P(t)$ and $W(t)$

Evolution system can be partly integrated. Namely, one can show that

$$\partial X((W(X,0)^{j/p})_{+}(t) = \partial X((W(X,t))^{j/p})_{+} \tag{20}$$

Here the left hand side is understood as a polynomial $P(t)$ that evolves due to system (15-17) and that equals to $\partial X((W(X,0))^{j/p})_{+}$ when all times are set to zero; $W(X,t)$ is a superpotential that equals to $W(X)$ when all times are zero.

Since evolution of primitive polynomials $\Phi$ is a particular case of evolution of general polynomials $P(t)$ (it corresponds to $j < p$ in (20)), one can substitute $\Phi_a(t)$ from (20) into flat times equation (19) and see that this equation is fulfilled.

2.8 Solution to the evolution system; its interpretation as a consequence of the dilaton equation

Let as take for monomial superpotential $W(X,0) = X^p$ a basis in the space of polynomials:

$$P_j(X,0) = jX^{j-1}, \text{ thus, } P_j(X,t) = \partial X(W(X,t)^{j/p})_{+} \tag{21}$$

Then one can see that for $t = 0 \frac{p}{p+1} P_{p+1}(X,0)$ is a dilaton. Using dilaton equation one can easily check that for $k > 2$

$$\langle \frac{p}{p+1} P_{p+1}(0), P_{a_1}(X,0), \ldots, P_{a_k}(X,0); \exp(\sum_{i=1}^{p-1} t_i P_i(X,0)) \rangle_{W(X,0)} =$$

$$(k - 2) \langle P_{a_1}(X,t), \ldots, P_{a_k}(X,t); \exp(\sum_{i=1}^{p-1} t_i P_i(X,0)) \rangle_{W(X,0)} +$$

$$\langle \sum_{i=1}^{p-1} t_i P_i(X,0)), P_{a_1}(X,0), \ldots, P_{a_k}(X,0); \exp(\sum_{i=1}^{p-1} t_i P_i(X,0)) \rangle_{W(X,0)} \tag{22}$$

Putting the second term to the left-hand side and using evolution system we get:

$$\langle \frac{p}{p+1} P_{p+1}(X,t) - (\sum_{i=1}^{p-1} t_i P_i(X,t))], P_{a_1}(X,t), \ldots, P_{a_k}(X,t) \rangle_{W(X,t)} =$$

$$(k - 2) \langle P_{a_1}(X,t), \ldots, P_{a_k}(X,t) \rangle_{W(X,t)} \tag{23}$$

Thus the term in the square brackets behaves as a dilaton for superpotential $W(t)$, but we already know (9) how it looks like, so we obtain;

$$\frac{p}{p+1} P_{p+1}(t) - (\sum_{i=1}^{p-1} t_i P_i(X,t)) = X\partial X W(X,t) \tag{24}$$

This equation\footnote{one can call it reduced string equation, in this form it appeared in [3]} (together with (21)) solves the evolution system.
3 Higher times and punctured complex plane

This section is a result of trying to understand results of [13] (see also [21]) in physical terms.

3.1 Motivation

One may ask what happens if we consider generating function not only for primaries but also for descendants. The answer is that correlators in the presence of such generating function can be interpreted as correlators in LGG theory on a punctured complex plane. Namely, descendants are of the form $W^R(X)$ and naively could be considered as terms that come from the Ward identities connected to reparametrization of the complex plane, i.e. naively one would expect something like:

$$< P_1, \ldots, P_n, W^R >_W + < P_1'R(X), \ldots, P_n >_W + \ldots + < P_1, \ldots, P_n'R >_W = 0$$  \hspace{1cm} (25)

Putting all the terms except first to the right-hand side one could try to consider $W^R$ as a generator of reparametrization of the target space: naively such a reparametrization results into "classical contact terms" (we call them "classical" because they origin from the very classical consideration of changing variables in the functional integral)

But it is not the full story, because:
1. there should be nontrivial "gravitational" contact terms
2. LG theory (as a sigma-model of type B!) is described not only by superpotential but also by a holomorphic top form (that we previously considered to be equal to $dX$). After reparametrization with parameter $t$ this top form (that we will denote as $dQ$) becomes

$$dQ = dX(1 + tR'(X)),$$  \hspace{1cm} (26)

so $dQ$ degenerates at zeros of $1 + tR'(X)$. So we inevitably have to consider LG theories with such degenerated form. We will discuss this theory in the same manner as the ordinary LG theory.

3.2 Observables, descendants, primitive fields and LGM correlator in a theory on a punctured complex plane

Theory on a punctured complex plane is defined by the superpotential $W$ (that is a holomorphic function with pole only at infinity) and the holomorphic top form $dQ$ that has zeros only at the punctures and a pole at infinity. Moreover, we demand that at puncture (at zero of $dQ$) the $dW$ should be nondegenerate (nonzero). 

**Observables**

The space of observables is given by functions of the form $P(X)/Q'(X)$. One can show that $W'/Q'$ is on shell trivial observable that is why observables in LGM are $P/Q'$ where $P$ belongs to the same factor-ring $C[X]/W'$.

**Descendants**

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8 These conditions correspond to LG theory with the F-term defined by a holomorphic function $W$ and D-term defined by a Kahler prepotential $QQ$; $Q$ is an integral of the form $dQ$.
Descendants have the form
\[ \sigma_{W,Q}^{W} (P/Q) = dW/dQ \int P/Q' dQ = \sigma_{W,X}^{W} (P) / Q' \] (27)

**Primitive elements**

Primitive elements are \( \Phi/Q' \), where degree of \( \Phi \) is less than degree of superpotential. Note that the simple puncture \( 1_P = Q'/Q' \) can be not a primitive element, and in all cases is not an element of lowest degree any more! But the dilaton is still \( \sigma(1_P)! \)

**Correlators in LGM**

Correlators in the LG topological matter on the punctured plane look very similar to those for a smooth plane: the difference is in using \( dQ^2 \) instead of \( dX^2 \):
\[ < P_1/Q', \ldots, P_n/Q' >_{W,Q}^{M} = \int_\Gamma \frac{P_1(X) / Q' \ldots P_n(X) / Q'(dQ)^2}{dW} \] (28)

Here the contour of integration \( \Gamma \) separates zeros of \( dW \) from infinity and punctures (equally one can say that this contour goes around the punctures). For example, the three point correlator is given by (this formula already appeared in [13])
\[ < P_1/Q', P_2/Q', P_3/Q' >_{W,Q}^{M} = \int_\Gamma \frac{P_1(X)P_2(X)P_3(X)(dX)^3}{dWdQ} \] (29)
and the metric \( \eta \) - two-point correlator:
\[ < P_1/Q', P_2/Q' >_{W,Q}^{M} = \int_\Gamma \frac{P_1(X)P_2(X)(dX)^2}{dW} \] (30)

### 3.3 Recursion relations

To treat the field \( P/Q' \) as a ”moving” field we have to decompose it into a sum of two fields: the field \( S \) that is a polynomial of degree less than the degree of superpotential and the field \( RW'/Q' \):
\[ P = Q'S + W'R \] (31)

The decomposition (31) exists and is unique if and only if polynomials \( Q' \) and \( W' \) are mutually simple, i.e. do not have common zeros.

**S-recursion relations**

The S-recursion relations correspond to the ”motion “ of field \( S \) and are not very different from recursion relations for unpunctured case. The motion of the fields \( S \) corresponds to the infinitesimal (perturbative) shift in superpotential \( W \rightarrow W + tS \), while contact term \( C_{W,Q}(P_1/Q', S) \) of \( S \) with some other field \( P_1/Q' \) is given by VV rule and can be found from:
\[ SP_1/Q' = \Phi/Q' + W'/Q' \int C_{W,Q}(P_1/Q', S)dQ \] (32)

This contact term can be easily calculated, see (5):
\[ C_{W,Q}(P_1/Q', S) = 1/Q'C_{W,X}(P_1, S) = 1/Q'(P_1S/W')_+ \] (33)
Thus, we get the S-recursion relations in the LGG for \( n > 2 \):

\[
< P_1/Q', \ldots, P_n/Q', S >_{W,Q} = \frac{d}{dt} < P_1/Q', \ldots, P_n/Q' >_{W+tS,Q} + \sum_{i=1}^{n} < P_i/Q', \ldots, C_{W,Q}(P_i/Q', S), \ldots, P_n/Q' >_{W,Q}
\]

(34)

**R-recursion relations**

R-recursion relations, that correspond to the "motion" of the field \( RW'/Q' \) arise from the Ward identities connected with the reparametrization of correlators:

\[
X \rightarrow X + tR(X)/Q'(X)
\]

(35)

Due to this reparametrization in \( < P_1, \ldots, P_n >_{W,Q} \)

\[
W \rightarrow W + tW'R/Q' \\
Q \rightarrow Q + tR \\
P_i/Q' \rightarrow P_i/Q' + t(P_i/Q')R/Q' = P_i/Q' - tC^{cl}_{W,Q}(P_i/Q', RW'/Q')
\]

(36)

The change of \( P_i \) due to reparametrization will be interpreted below as minus the "classical" contact term \( C^{cl} \). But we know that there is also a "topological gravity" contact term \( C^{tg} \) (that we once again obtain due to VV rule):

\[
P_iRW'/Q'(Q')^2 = W'/Q' \int C^{tg}_{W,Q}(P_i/Q', RW'/Q')dQ
\]

(37)

thus

\[
C^{tg}_{W,Q}(P_i/Q', RW'/Q') = (P_iR/Q')'/Q'
\]

(38)

Now we are ready to use Ward identities; they state (for \( n > 2 \))

\[
< P_1/Q', \ldots, P_n/Q', RW'/Q' >_{W,Q} = \frac{d}{dt}(< P_1/Q', \ldots, P_n/Q' >_{W,Q-tR})|_{t=0} + \\
\sum_{i=1}^{n} < P_i/Q', \ldots, C^{tot}_{W,Q}(P_i/Q', RW'/Q'), \ldots, P_n/Q' >_{W,Q}
\]

(39)

where the total contact term is given by:

\[
C^{tot}_{W,Q}(P_i/Q', RW'/Q') = C^{tg}_{W,Q}(P_i/Q', RW'/Q') + C^{cl}_{W,Q}(P_i/Q', RW'/Q') = P_iR/(Q')^2
\]

(40)

Thus, the R-recursion relation takes the following simple form:

\[
< P_1/Q', \ldots, P_n/Q', RW'/Q' >_{W,Q} = \frac{d}{dt} < P_1/(Q-tR)', \ldots, P_n/(Q-tR)' >_{W,Q-tR}|_{t=0}
\]

(41)

With the help of S and R recursion relations (together with the three point correlator), all correlators are defined. Moreover, one can recursively prove (expected) generalized dilaton and factorization formulas. They look as follows.
3.4 Dilaton equation and factorization formula on a punctured plane

\[
< W'/Q', P_1/Q', \ldots, P_n/Q' >_{W,Q} = (n - 2) < P_1/Q', \ldots, P_n/Q' >_{W,Q}
\] (42)

Note, that on a punctured plane dilaton is the descendant of the puncture \( 1_P \), i.e. \( W'/Q' \), see (28).

If \( \sum_{i=1}^{n} I_i = n - 3 \) then

\[
< \sigma_{I_1}(\Phi_1/Q'), \ldots, \sigma_{I_n}(\Phi_n/Q') >_{W,Q} = \frac{(n - 3)!}{\prod_{i=1}^{n} I_i!} \int \Phi_1 \cdots \Phi_n (dX)^2
\] (43)

3.5 Introduction of all times and evolution system

From S and R recursive equations we see that times, corresponding to descendants of primary observables in generating exponent lead to evolution of all ingredients of the theory: insertions, superpotential and the top form:

\[
< P_1/Q', \ldots, P_n/Q' >_{W,Q} = \exp(\sum_{k=1}^{\infty} t_k P_k/Q')
\]

If we define polynomials \( R_j(t) \) and \( S_j(t) \) as a R and S parts of the polynomial \( P_j(t) \):

\[
P_j(X, t) = W'(X, t)R_j(t) + Q'(X, t)S_j(t)
\] (45)

then

\[
\frac{\partial}{\partial t_j} P_t(t) = C_{W,X}(P_t(t), S_j(t)) = (P_t(t)S_j(t)/W'(t))'
\] (46)

\[
\frac{\partial}{\partial t_j} W(t) = S_j(t)
\] (47)

\[
\frac{\partial}{\partial t_j} Q(t) = -R_j(t)
\] (48)

Below to save space we will write \( \partial_i \) instead of \( \frac{\partial}{\partial t_i} \).

This system can be easily partly integrated, namely,

\[
P_j(t) = (W(t)^{j/p})'
\] (49)

satisfy this system.

In other terms relation (45) can be rewritten as

\[
P_j(X, t) = \partial_j W(X, t)\partial_X Q(X, t) - \partial_j Q(X, t)\partial_X W(X, t)
\] (50)

This relation can be reinterpreted if we introduce the Poisson bracket, connected with the i-th time as an antisymmetric bilinear function on any two functions of \( X \) and \( t_i \) (let us call them \( T_1 \) and \( T_2 \)). This i-th bracket is given by the formula

\[
\{T_1(X, t_i), T_2(X, t_i)\}_i = \partial_i T_1(X, t_i)\partial_X T_2(X, t_i) - \partial_i T_2(X, t_i)\partial_X T_1(X, t_i)
\] (51)
Namely, the decomposition of the observable on shifts in superpotential and reparametrization of the target space (45) is equivalent to:

\[ \{ W, Q \}_i = P_i \] (52)

If we take as \( P_i \) the identity, then due to evolution it remains identity, so we have the specialization of (52) in the form:

\[ \{ W, Q \}_1 = 1 \] (53)

### 3.6 Evolution system (46-48) as a dispersionless limit of the reduced KP

In what follows we will use the following simple:

**Lemma**

Let us for a polynomial \( P \) denote by \([P]\) its part that has power less then the power of \( W' \), then:

\[ [\partial_i W P_j] = [\partial_j W P_i] \] (54)

This lemma can be proven by multiplication of equation (50) on \( \partial_i W \) and observation that up to terms proportional to \( W' \) it is symmetrical in \( i \) and \( j \).

**Statement**

If \( W(X, t) \) and \( P_m(X, t) \) are solutions of the evolution system (46-48), then

\[ \{ W(X, t), \int X P_j(X_1, t) dX_1 \}_i = [P_i \partial_j W] \] (55)

Proof.

\[ \partial_i WP_j = [\partial_i WP_j] + \partial_X W \int X C_{W,X_1}(\partial_i W, P_j) dX_1 = [\partial_i WP_j] + \partial_X W \partial_i \int X P_j dX_1 \] (56)

Application of Lemma (54) ends the proof.

Specializing statement to the case \( P_i(X, t) = 1 \) we get

\[ \{ W(X, t), \int X P_j(X_1, t) dX_1 \}_1 = \partial_j W(X, t) \] (57)

Further specializing to the case when at \( t = 0 \) superpotential and all observables are monomials, we obtain

\[ \{ W(X, t), (W(X, t)^{j/p})_+ \}_1 = \partial_j W(X, t) \] (58)

Equation that we get means that evolution (46-48) of \( W \) coincides with the evolution of \( W \) due to dispersionless reduction of KP!

Next we can interpret the evolution of \( Q \). Already from (53) we observe that evolution preserves Poisson bracket between \( W \) and \( Q \), so the evolution is expected to be a simplectomorphism. Now we will show that it really happens. Really, taking \( \partial_i Q P_j \) and
making with it manipulations like in proof of the lemma and in proof of the previous
statement one can easily show the following

Statement

\[ \{Q(X, t), \int X P_j(X_1, t)dX_1\}_i = P_i\partial_j Q - \int X C_{W, X_1}(\partial_j WP_i)dX_1 \]  

(59)

Specially, for \( P_i = 1 \) and in the homogeneous case we get the desired relation

\[ \{Q(X, t), (W(X, t)^{j/p})_+\}_1 = \partial_j Q(X, t) \]  

(60)

that means that evolution system is equivalent to symplectomorphisms (with respect to Poisson bracket \( \{, \}_1 \) ) generated by \( (W(X, t)^{j/p})_+ \).

3.7 Solution of the full evolution system from a dilaton equation

Reasoning exactly like in (24) one can show that if for zero times \( Q = X \) and \( W = X^p \), and as a basis in the space of observables we take monomials, then for arbitrary times the following observable

\[ \left[ \frac{p}{p + 1}(W(t)^{p+1/p})'_+ - \sum_{j=1}^{\infty} t_j(W(t)^{j/p})'_+ \right]/Q' \]  

(61)

satisfies dilaton equation. Comparing this object with the dilaton that is a descendant of a puncture(42), we get the Krichever form of the string equation in genus zero [13]:

\[ \frac{p}{p + 1}(W(t)^{p+1/p})'_+ - \sum_{j=1}^{\infty} t_j(W(t)^{j/p})'_+ = QW'' \]  

(62)

that determines \( W \) and \( Q \) in terms of times \( t \).

4 Generating function for correlators and \( \tau \) function of the Generalized Kontsevich Model

In this section we will argue that

\[ \log Z_{LGG}(t, W) = \sum_{q=0}^{\infty} <\exp(\sum_{k=1}^{\infty} t_k k X^{k-1})>_{W(X)}^{(q)} = \log Z_{GKM}(t, V) \]  

(63)

where

\(<\>_{W}^{(q)} \) stands for the correlator in genus \( q \) of the worldsheet in topological LG gravity with superpotential \( W \)

\( Z_{GKM}(t, V) \) is the partition function of the Generalized Kontsevich Model (GKM) [8], and \( V' = W \). This GKM partition function \( Z_{GKM} \) is a \( \tau \)-function of the KP hierarchy. This \( \tau \) function corresponds to \( W \)-reduction, i.e. for a \( L \) operator of the hierarchy , \( W(L) \) is a differential operator. Moreover, this \( \tau \)-function satisfies so called \( L_{-1} \) constraint.
Our argument would be two-fold. We begin by showing that the genus zero contribution to $Z_{LGG}$ satisfies the same equation (71) as $\tau$-function of dispersionless KP.

Our second argument is connection between $Z_{LGG}$ for different superpotentials $W$ of the same degree. We will see that if $t$ are some times on the small phase space, and $T$ are times on the full phase space, then for some matrix $c(t)$

$$\log Z_{LGG}(t + T, W) = \log Z_{LGG}(c(t)T, W(t)) + A(t, T),$$

additional term $A$ considered as a function of $T$ is a polynomial of second degree. This term is coming only from genus zero of the worldsheet and means simply that there is no moduli space for the sphere with zero, one or two punctures. The same relation between partition functions in GKM was proven in [22].

### 4.1 Generating function for genus zero correlators as a $W$-reduced dispersionless $\tau$ - function

To have KP hierarchy we need an $L$ operator, in dispersionless KP this operator becomes a function. So we define $L(X, t)$ as such a formal seria in $X^{-1}$, that behaves as $X$ at infinity in the $X$-complex plane, and that satisfies:

$$W(L(X, t), 0) = W(X, t)$$

(64)

From equation (57) it is obvious that evolution of $L$ is given by the Poisson bracket with Hamiltonians:

$$\{L(X, t), \int X P_j(X_1, t) dX_1\}_1 = \partial_j L(X, t)$$

(65)

where (due to definition of times in (63))

$$P_j(X, t) = \sum_{k=1}^{\infty} \tilde{c}_{jk} \partial_X (W(X, t)^{k/p})_+,$$

(66)

coefficients $\tilde{c}_{jk}$ are $t$ and $X$ independent, they are defined by $W(X, 0)$ as follows:

$$jX^{j-1} = \sum_{k=1}^{\infty} \tilde{c}_{jk} \partial_X (W(X, 0)^{k/p})_+$$

(67)

**Lemma**

Let $\partial$ denote some partial derivative along times or $X$, then

$$\partial(L^j)_+ = \partial \int X P_j(X_1, t) dX_1$$

(68)

**Proof.**

The right hand side of (68) equals to

$$\sum_{k=1}^{\infty} \tilde{c}_{jk} k/p (W(L(X, t), 0)^{k/p-1} \frac{\partial W(L, 0)}{\partial L} \partial L(X, t))_+ =$$

$$\sum_{k=1}^{\infty} \tilde{c}_{jk} k/p ((W(L(X, t), 0)^{k/p-1} \frac{\partial W(L, 0)}{\partial L})_+ \partial L(X, t))_+ = \partial(L^j)_+$$

(69)
Here \( +, L \) stands for the positive powers in \( L \) expansion. The first equality in (69) is valid because the negative part in \( L \) expansion times \( \partial L \) gives zero contribution to the positive part in \( X \) expansion of the product.

From (65) and (68) we see that \( L \) satisfies dispersionless analog of KP, more exactly its \( W(0) \) reduction. Namely

\[
\{L(X, t), (L(x, t)^j)_+\}_1 = \partial_j L(x, t) \tag{70}
\]

Now we are ready to prove a statement that shows that partition function of LGG satisfies the same equation as the tau-function, i.e.:

Statement

\[
\partial_j \partial_1 \log Z^0_{\text{LGG}}(t, W(X, 0)) = \partial_j \partial_1 < \exp(\sum_{k=1}^{\infty} t_k kX^{k-1}) >_{W(X, 0)} = \int_{\Gamma} L^j dX \tag{71}
\]

Proof.

\[
\partial_1 \partial_j \partial_1 \log Z^0_{\text{LGG}}(t, W(X, 0)) = \int_{\Gamma} \frac{P_i(X, t) P_j(X, t) dX}{W' Q'} = \int_{\Gamma} \frac{\partial_i W(X, t) P_j(X, t) dX}{W'} \tag{72}
\]

In the second equality we used decomposition (50) and the fact that contour \( \Gamma \) goes around zeros of \( W' \). Using Lemma (68) we can rewrite the last expression as:

\[
\int_{\Gamma} \frac{\partial L W(L, 0) \partial L(j L^j-1 \partial X L)_+}{\partial L W(L, 0) \partial X L} = \partial_1 \int_{\Gamma} L^j \tag{73}
\]

In the numerator of (73) operation \((.)_+\) can be omitted because negative part of expansion gives zero contribution to the integral.

Thus, up to the constant statement (71) is proven. Since when all times are equal to zero the left hand side of the statement is zero (there is no moduli space for a sphere with 2 punctures), and the right hand side is also zero (when all times are zero, \( L = X \)), this constant is zero.

4.2 Relation between \( \log Z_{\text{LGG}}(T, W) \) for different superpotentials \( W \)

Consider \( \log Z_{\text{LGG}}(t + T, W(X, 0)) \) for monomial potential \( W(X, 0) \) and for some parameters \( t_k, k < p \). (Note, that here we are dealing not with the spherical contribution but with the full expression). Then due to properties of the formal exponent

\[
\log Z_{\text{LGG}}(t + T, W(X, 0)) = \sum_{q=0}^{\infty} < \exp \sum_{k=1}^{\infty} (t_k + T_k) kX^{k-1} >^q_W = \sum_{q=0}^{\infty} < \exp \sum_{k=1}^{\infty} T_k kX^{k-1}, \exp \sum_{k=1}^{p-1} t_k kX^{k-1} >^q_W \tag{74}
\]

Naively one could think that due to evolution with respect to times \( t \) this expression equals to \( \log Z_{\text{LGG}}(\sum_j c_{kj}(t) T_j, W(X, t)) \) where matrix \( c(t) \) is given by:

\[
\partial X(W(X, t)^{k/p})_+ = \sum_j \tilde{c}_{kj}(t) j X^{j-1}, \quad c_{kj} = \tilde{c}_{jk} \tag{75}
\]
This is not quite true, since one can apply evolution equation only for these terms in the last expression in (74) that are at least third degree polynomials in $T$. Terms that are polynomials of lower degree give nonzero contribution to $\log Z(t + T; W(0))$ but give zero contribution to $\log Z(cT, W(t))$ because there is no moduli space on the sphere with less than three punctures on it. Taking this into account we get the final

Statement

$$\log Z_{LGG}(\sum_j c_{kj}(t)T_j, W(X, t)) = \log Z_{LGG}(t + T, W(X, 0)) -$$

$$(1 + \sum_{k=1}^{\infty} T_k \frac{\partial}{\partial \theta_k} + \frac{1}{2} \sum_{j,k=1}^{\infty} T_j T_k \frac{\partial}{\partial \theta_j} \frac{\partial}{\partial \theta_k}) \log Z_{LGG}^0(t + \theta, W(X, 0))|_{\theta=0} \quad (76)$$

Exactly this relation between $\log Z_{GKM}(T, V)$ for different potentials was found in [22].

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