LUBIN’S CONJECTURE FOR FULL $p$-ADIC DYNAMICAL SYSTEMS

by

Laurent Berger

Abstract. — We give a short proof of a conjecture of Lubin concerning certain families of $p$-adic power series that commute under composition. We prove that if the family is full (large enough), there exists a Lubin-Tate formal group such that all the power series in the family are endomorphisms of this group. The proof uses ramification theory and some $p$-adic Hodge theory.

Introduction

Let $K$ be a finite extension of $\mathbb{Q}_p$, and let $\mathcal{O}_K$ be its ring of integers. In [Lub94], Lubin studied $p$-adic dynamical systems, namely families of elements of $T \cdot \mathcal{O}_K[[T]]$ that commute under composition, and remarked that “experimental evidence seems to suggest that for an invertible series to commute with a noninvertible series, there must be a formal group somehow in the background”. This observation has motivated the work of a number of people (Hsia, Laubie, Li, Movaheddi, Salinier, Sarkis, Specter, ...) who proved various results in that direction. The purpose of this note is to give a proof of a special case of the above observation, which is referred to as “Lubin’s conjecture” in §3.1 of [Sar10].

Let us consider a family $\mathcal{F}$ of commuting power series $F(T) \in T \cdot \mathcal{O}_K[[T]]$. We say that such a family is full if for all $\alpha \in \mathcal{O}_K$ there exists $F_\alpha(T) \in \mathcal{F}$ such that $F_\alpha'(T) = \alpha$ and if $\text{wideg}(F_\pi(T)) = q$, where $\text{wideg}(F(T))$ denotes the Weierstrass degree of $F(T)$, $\pi$ is any uniformizer of $\mathcal{O}_K$ and $q$ is the cardinality of the residue field of $\mathcal{O}_K$.

Theorem. — If $\mathcal{F}$ is a full family of commuting power series, there exists a Lubin-Tate formal group $G$ such that $F_\alpha(T) \in \text{End}(G)$ for all $\alpha \in \mathcal{O}_K$.

2010 Mathematics Subject Classification. — 11S82 (11S15; 11S20; 11S31; 13F25; 13F35; 14F30).
Key words and phrases. — $p$-adic dynamical system; Lubin-Tate formal group; $p$-adic Hodge theory.
This result already appears as corollary 3.11 of [HL15]. Our proof is however considerably shorter than that of ibid., and does not use the theory of the field of norms. It is very similar to that of the main result of [Spe15], which treats the case $K = \mathbb{Q}_p$. The main ingredients are ramification theory and some $p$-adic Hodge theory. In order to simplify the use of $p$-adic Hodge theory, we assume that $K$ is a Galois extension of $\mathbb{Q}_p$.

1. $p$-adic dynamical systems

In this note, we consider a set $\mathcal{F} = \{F_\alpha(T)\}_{\alpha \in \mathcal{O}_K}$ of power series $F_\alpha(T) \in T \cdot \mathcal{O}_K[[T]]$ such that $F'_\alpha(0) = \alpha$ and $F_\alpha \circ F_\beta(T) = F_\beta \circ F_\alpha(T)$ whenever $\alpha, \beta \in \mathcal{O}_K$. Recall that $\pi$ is a uniformizer of $\mathcal{O}_K$, and that $q$ is the cardinality of the residue field $k$ of $\mathcal{O}_K$. If $F(T)$ is a power series and $n \geq 0$, we denote by $F^{\circ n}(T)$ the $n$-th fold iteration $F \circ \cdots \circ F(T)$. If $F(T)$ has an inverse for the composition, this definition extends to $n \in \mathbb{Z}$. Recall that the Weierstrass degree $\text{wideg}(F(T))$ of $F(T) = \sum_{i=1}^{+\infty} f_i T^i \in T \cdot \mathcal{O}_K[[T]]$ is the smallest integer $i$ such that $f_i \in \mathcal{O}_K^\times$.

**Proposition 1.1.** There exists a power series $G(T) \in T \cdot k[[T]]$ and an integer $d \geq 1$ such that $G'(0) \in k^\times$ and $F_\pi^{\circ n}(T) \equiv G(T^{p^d})$.

**Proof.** This is theorem 6.3 of [Lub94].

Let $\log_T \in K[[T]]$ denote the logarithm attached to $\mathcal{F}$. Recall that $\log_T(T) = T + O(T^2)$, that $\log_T(T)$ converges on the open unit disk and that $\log_T \circ F_\alpha(T) = \alpha \cdot \log_T(T)$ for all $\alpha \in \mathcal{O}_K$ (propositions 1.2 and 2.2 of [Lub94]).

The hypothesis that $\mathcal{F}$ is full implies that $p^d = q$, so that $\text{wideg}(F_\pi(T)) = q$. Let $\Lambda_n$ denote the set of $u \in \mathfrak{m}_{\mathcal{O}_p}$ such that $F^{\circ n}_\pi(u) = 0$ and $F^{\circ n-1}_\pi(u) \neq 0$ and let $\Lambda_\infty = \cup_{n \geq 1} \Lambda_n$. The set $\Lambda_n$ has $q^{n-1}(q-1)$ elements.

The series $F_\alpha(T)$ is invertible if $\alpha \in \mathcal{O}_K^\times$ so that in this case, $F_\alpha(z) = 0$ if and only if $z = 0$. If $u \in \Lambda_n$ and $\alpha \in \mathcal{O}_K^\times$, then $F^{\circ n}_\pi \circ F_\alpha(u) = F_\alpha \circ F^{\circ n}_\pi(u) = 0$ and $F^{\circ n-1}_\pi \circ F_\alpha(u) = F_\alpha \circ F^{\circ n-1}_\pi(u) \neq 0$ so that the action of $F_\alpha(T)$ permutes the elements of $\Lambda_n$.

Let $K_n = K(\Lambda_n)$ and $K_\infty = \cup_{n \geq 1} K_n$. If $\alpha \in \mathcal{O}_K^\times$, let $n(\alpha)$ be the largest integer $n$ such that $\alpha \in 1 + \pi^n \mathcal{O}_K$.

**Proposition 1.2.** If $n \geq 1$ and $u \in \Lambda_n$, then

1. $F_\alpha(u) = u$ if and only if $n(\alpha) \geq n$;
2. If $n(\alpha) = n$, then $\text{wideg}(F_\alpha(T) - T) = q^n$;
3. $\Lambda_n = \{F_\alpha(u)\}_{\alpha \in \mathcal{O}_K^\times}$. 

We study the formal group attached to a uniformizer $\pi L T$. By the results of $F_\alpha$, we have that $\{F_\alpha(u)\}_{\alpha \in \mathcal{O}_K}$ has at least $q-1$ distinct elements. These elements are all roots of $F_\pi(T)/T$, whose wideg is $q-1$, so $\{F_\alpha(u)\}_{\alpha \in \mathcal{O}_K}$ has precisely $q-1$ elements. These elements all have valuation $1/(q-1)$, and the Newton polygon of $F_\alpha(T) - T$ starts at the point $(1,1)$, so that it can have only one segment, and wideg($F_\alpha(T) - T$) $= q$. This implies the proposition for $n = 1$.

Assume now that the proposition holds for an $n \geq 1$ and take $u \in \Lambda_n$. If $n(\alpha) \leq n$, then $F_\alpha(T) - T$ has at most $q^n$ roots by (2), contained in $\Lambda_0 \cup \ldots \cup \Lambda_n$ by (1). Therefore $F_\alpha(u) = u$ implies $n(\alpha) \geq n + 1$. The set $\{F_\alpha(u)\}_{\alpha \in \mathcal{O}_K}$ therefore has at least $q^n(q-1)$ distinct elements, all of them roots of $F_\pi(T)/F_\pi^{-1}(T)$. This implies that $\{F_\alpha(u)\}_{\alpha \in \mathcal{O}_K}$ has exactly $q^n(q-1)$ elements. The Newton polygon of $F_\alpha(T) - T$ starts at the point $(1,n+1)$, with $n+1$ segments of height one and slopes $-1/q^k(q-1)$ with $0 \leq k \leq n$, so that it reaches the point $(q^{n+1},0)$ and hence wideg($F_\alpha(T) - T$) $= q^{n+1}$. This implies the proposition for $n + 1$.

**Corollary 1.3.** The field $K_\infty$ is an abelian extension of $K$, and if $g \in \text{Gal}(K_\infty/K)$, there is a unique $\eta(g) \in \mathcal{O}_K$ such that $g(u) = F_\eta(g)(u)$ for all $u \in \Lambda_\infty$.

The map $\eta : \text{Gal}(K_\infty/K) \to \mathcal{O}_K$ is an isomorphism.

**Proof.** — Take $u \in \Lambda_n$ and $\alpha \in \mathcal{O}_K$. As we have seen above, $F_\alpha(u) \in \Lambda_n$, so that the map $u \mapsto F_\alpha(u)$ induces a field automorphism of $K(u)$. By (3) of proposition 1.2, this implies that $K_n = K(u)$ and that every element of $\text{Gal}(K_n/K)$ comes from $u \mapsto F_\alpha(u)$ for some $\alpha \in \mathcal{O}_K$. The extension $K_n/K$ is therefore abelian, and so is $K_\infty/K$.

The map $\eta$ is surjective because every $F_\alpha(T)$ gives rise to an automorphism of $K_\infty$, and it is injective because if $\eta(g) = 1$ then $g(u) = u$ for all $u \in \Lambda_\infty$ so that $g = 1$.

This proves, by local class field theory, that $K_\infty/K$ is a maximal abelian extension. By the results of [LT65], $K_\infty$ is generated over $K$ by the torsion points of a Lubin-Tate formal group attached to a uniformizer $\varpi$ of $\mathcal{O}_K$. In order to prove our main theorem, we study the $p$-adic periods of $\eta$.

**2. $p$-adic Hodge theory**

Let $R$ be the $p$-adic completion of $\lim_{n \geq 1} \mathcal{O}_K[X_n]$ where $\mathcal{O}_K[X_n]$ is seen as a subring of $\mathcal{O}_K[[X_{n+1}]]$ via the identification $X_n = F_\pi(X_{n+1})$ (this ring is denoted by $A_\infty$ in §3.2 of [Spe15]). We define an action of $G_K$ on $R$ by $g(H(X_n)) = H(F_{\eta(g)}(X_n))$. This is well-defined since $F_\pi \circ F_{\eta(g)}(T) = F_{\eta(g)} \circ F_\pi(T)$. We have $R/\pi R = \lim_{n \geq 1} k[X_n]$. 

Lemma 2.1. — The ring $R/\pi R$ is perfect.

Proof. — Let $G(T)$ be as in lemma 1.1. The fact that $X_n = F_\pi(X_{n+1})$ implies that $G^{on}(X_n) = G^{on+1}(X_{n+1})^q$. Since $G'(0) \in k^\times$, we have $k[T] = k[G(T)]$ and therefore

$$R/\pi R = \lim_{G^n(X_n) = G^{n+1}(X_{n+1})^q} k[G^n(X_n)]$$

is perfect. □

Let $\tilde{E}^+ = \lim_{n \to \infty} \tilde{O}_{\mathcal{C}^p}/\pi$. Choose a sequence $\{u_n\}_{n \geq 1}$ with $u_n \in \Lambda_n$ and $F_\pi(u_{n+1}) = u_n$. This sequence gives rise to a map $\iota : R/\pi R \to \tilde{E}^+$, determined by the requirement $\iota(X_n) = (G^{o-1}(\pi_n), G^{o-2}(\pi_{n+1}), \ldots)$. The definition of the action of $G_K$ on $R$ and corollary 1.3 imply that $\iota$ is $G_K$-equivariant.

Lemma 2.2. — The map $\iota : R/\pi R \to \tilde{E}^+$ is injective.

Proof. — It is enough to show that $\iota : k[X_n] \to \tilde{E}^+$ is injective. If it was not, there would be a nonzero polynomial $P(T) \in k[T]$ such that $P(\iota(X_n)) = 0$, and $\iota(X_n) = (G^{o-1}(\pi_n), G^{o-2}(\pi_{n+1}), \ldots)$ would then belong to $\tilde{F}_p$, which is clearly not the case. □

Let $K_0 = Q_p^{\text{ur}} \cap K$ and let $\tilde{A}^+ = \mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} W(\tilde{E}^+)$ (see [Fon94a]; note that $\tilde{A}^+$ usually denotes $W(\tilde{E}^+)$, and is denoted by $A_{\text{inf}}$ in ibid.). We have $R = \mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} W(R/\pi R)$ since $R$ is a strict $\pi$-ring, and by the functoriality of Witt vectors, the map $\iota$ extends to an injective and $G_K$-equivariant map $\iota : R \to \tilde{A}^+$. We write $x$ instead of $\iota(X_1) \in \tilde{A}^+$. The $G_K$-equivariance of $\iota$ implies that $g(x) = F_{\eta(g)}(x)$.

Let $B_{\text{cris}}$ and $B_{\text{dr}}$ be Fontaine’s rings of periods. Recall that there is a Frobenius map $\varphi$ on $B_{\text{cris}}$, a filtration $\{\text{Fil}^i B_{\text{dr}}\}_{i \in \mathbb{Z}}$ on $B_{\text{dr}}$, and an injective map $K \otimes_{K_0} B_{\text{cris}} \to B_{\text{dr}}$. There is also an action of $G_K$ on $B_{\text{cris}}$ and $B_{\text{dr}}$ compatible with the above structure, and $B_{dR}^{G_K} = K$. Let $\varphi_q = \varphi^f$ on $B_{\text{cris}}$, where $q = p^f$, extended by $K$-linearity to $K \otimes_{K_0} B_{\text{cris}}$.

We refer to [Fon94a] and [Fon94b] for the properties of these objects, and the definition of crystalline representations and their Hodge-Tate weights. Let $\Sigma = \text{Gal}(K/Q_p)$. If $\tau \in \Sigma$, choose a $n(\tau) \in \mathbb{Z}$ such that $\tau|_{K_0} = \varphi^{n(\tau)}$. The map $\tau \otimes \varphi^{n(\tau)} : K \otimes_{K_0} B_{\text{cris}} \to K \otimes_{K_0} B_{\text{cris}}$ is then well-defined. Recall that $\log(T) \in K[T]$ is the logarithm attached to $\mathcal{F}$. Since $\log(T)$ converges on the open unit disk, we can view $\log(x)$ as an element of $K \otimes_{K_0} B_{\text{cris}}$.

Proposition 2.3. — The character $\eta : G_K \to \mathcal{O}_K^\times$ is crystalline with nonnegative Hodge-Tate weights.
Proof. — If \( g \in G_K \), then \( g(L_F(x)) = L_F(g(x)) = L_F(F_{\eta(g)}(x)) = \eta(g) \cdot L_F(x) \). Likewise, if \( \tau \in \Sigma \) and \( \ell_\tau = (\tau \otimes \varphi^{n(\tau)})(L_F(x)) \in K \otimes_K^+ B_{\text{cris}} \), then \( g(\ell_\tau) = \tau(\eta(g)) \cdot \ell_\tau \).

Let \( \chi_\omega : G_K \to O_K^\times \) denote the Lubin-Tate character attached to \( \omega \).

**Lemma 2.4.** — Every crystalline character \( \eta : G_K \to O_K^\times \) that factors through \( \Gamma_K \) is of the form \( \eta = \prod_{\tau \in \Sigma} \tau(\chi_\omega)^{h_\tau} \) with \( h_\tau \in \mathbb{Z} \).

Proof. — If \( h_\tau \) denotes the weight of \( \eta \) at \( \tau \), then \( \eta \cdot \prod_{\tau \in \Sigma} \tau(\chi_\omega)^{-h_\tau} \) is crystalline with weights 0, hence unramified and therefore trivial since it factors through \( \Gamma_K \).

Let \( t_\omega \in B_{\text{cris}}^+ \) be a period of \( \chi_\omega \), so that \( \chi_\omega(g) = g(t_\omega)/t_\omega \) and \( \varphi_q(t_\omega)/t_\omega = \omega \).

**Corollary 2.5.** — We have \( L_F(x) = \lambda \cdot \prod_{\tau \in \Sigma} \tau(t_\omega)^{h_\tau} \) with \( h_\tau \in \mathbb{Z}_{\geq 0} \) and \( \lambda \in K^\times \).

Proof. — This follows from the facts that \( \eta = \prod_{\tau \in \Sigma} \tau(\chi_\omega)^{h_\tau} \), that \( \chi_\omega(g) = g(t_\omega)/t_\omega \) and that \( B_{\text{dR}}^G_K = K \).

**Corollary 2.6.** — We have \( \varphi_q(L_F(x)) = \mu \cdot L_F(x) \) for some \( \mu \in \pi O_K \).

Proof. — This follows from corollary 2.5. We have \( \mu = \prod_{\tau} \tau(\omega)^{h_\tau} \) so that if \( \mu \) was in \( O_K^\times \), then \( h_\tau \) would be 0 for all \( \tau \) which would in turn imply that \( \eta \) is trivial.

**Corollary 2.7.** — We have \( \varphi_q(x) = F_\mu(x) \).

Proof. — Corollary 2.6 implies that \( L_F(\varphi_q(x)) = L_F(F_\mu(x)) \). We would like to apply \( L_{F}^{\circ-1}(T) \) but we have to mind the convergence and need to proceed as follows. Since \( \eta \) is nontrivial, there is a \( \tau \in \Sigma \) such that \( h_{\tau-1} \geq 1 \). We have

\[
(\tau \otimes \varphi^{n(\tau)})(L_F(\varphi_q(x))) = (\tau \otimes \varphi^{n(\tau)})(L_F(F_\mu(x)))
\]

in \( K \otimes_K^+ B_{\text{cris}} \) and \( h_{\tau-1} \geq 1 \) now implies that \( (\tau \otimes \varphi^{n(\tau)})(L_F(\varphi_q(x))) \in \text{Fil}^1 B_{\text{dR}} \) so that we can apply \( L_{F}^{\circ-1}(T) \) in \( B_{\text{dR}} \) and get that \( (\tau \otimes \varphi^{n(\tau)})(\varphi_q(x)) = (\tau \otimes \varphi^{n(\tau)})(F_\mu(x)) \) in \( B_{\text{dR}} \). This equality also holds in \( \hat{A}^+ \), so that \( \varphi_q(x) = F_\mu(x) \).

**Theorem 2.8.** — There is a Lubin-Tate formal group \( G \) such that \( F_\alpha(T) \in \text{End}(G) \) for all \( \alpha \in O_K \).

Proof. — By corollary 2.7, we have \( \varphi_q(x) = F_\mu(x) \). This implies that \( F_\mu(T) \equiv T^\mu \mod \pi O_K[T] \). The Weierstrass degree of \( F_\mu(T) \) is \( q^{\text{val}(\mu)} \) so that \( \text{val}(\mu) = 1 \) and \( F_\mu(T) \) is a Lubin-Tate power series. By [LT65], there is a Lubin-Tate formal group \( G \) such that \( F_\mu(T) \in \text{End}(G) \). Since \( F_\alpha(T) \) commutes with \( F_\mu(T) \), we also have \( F_\alpha(T) \in \text{End}(G) \) for all \( \alpha \in O_K \).
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