Treewidth of the $q$-Kneser graphs

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Abstract

Let $V$ be an $n$-dimensional vector space over a finite field $\mathbb{F}_q$, where $q$ is a prime power. Define the generalized $q$-Kneser graph $K_q(n, k, t)$ to be the graph whose vertices are the $k$-dimensional subspaces of $V$ and two vertices $F_1$ and $F_2$ are adjacent if $\dim(F_1 \cap F_2) < t$. Then $K_q(n, k, 1)$ is the well-known $q$-Kneser graph. In this paper, we determine the treewidth of $K_q(n, k, t)$ for $n \geq 2t(k - t + 1) + k + 1$ and $t \geq 1$ exactly. Note that $K_q(n, k, k - 1)$ is the complement of the Grassmann graph $G_q(n, k)$. We give a more precise result for the treewidth of $G_q(n, k)$ for any possible $n$, $k$ and $q$.

Key words: treewidth, tree decomposition, $q$-Kneser graph, generalized $q$-Kneser graph, Grassmann graph

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1 Introduction

Let $G = (V, E)$ be a finite, simple and undirected graph. For $v \in V$, the degree of $v$ in $G$, written as $d_G(v)$, is the number of edges incident with $v$ in $G$. Let $\Delta(G)$ be the maximum degree of $G$. A subset $S$ of $V(G)$ is called independent set if no two elements in $S$ are adjacent in $G$. The independence number of $G$, denoted by $\alpha(G)$, is the maximum size of independent sets in $G$. We will use $\overline{G}$ to denote the complement of $G$.

Definition 1.1. Let $G$ be a graph, $T$ a tree and $(V_t)_{t \in V(T)}$ be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices $t \in V(T)$. A tree decomposition of $G$ is a pair $(T, (V_t)_{t \in V(T)})$ if it satisfies the following three conditions:

(i) $V(G) = \cup_{t \in V(T)} V_t$;

(ii) for every edge $uv \in E(G)$, there is a $t \in V(T)$ such that $u, v \in V_t$;

(iii) for every $v \in V(G)$, the subgraph of $T$ induced by $\{t \in V(T) \mid v \in V_t\}$ is connected.

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The width of the decomposition \((T, (B_t)_{t \in V(T)})\) is \(\max\{|B_t| \mid t \in V(T)\} - 1\). The treewidth of \(G\), denoted by \(\text{tw}(G)\), is the least width of any tree decompositions of \(G\).

Treewidth is a well-studied parameter in modern graph theory that shows the alikeness of the graph with a tree. On the one hand, it is an important variable in structural graph theory. Treewidth was introduced by Robertson and Seymour in their series of fundamental papers on graph minors, for example, we refer the reader to [19, 20, 21]. On the other hand, treewidth is also a key parameter in algorithm design. The problem of deciding whether a graph has tree decomposition of treewidth at most \(k\) is NP-complete [1]. Besides, it has been shown that many NP-hard combinatorial problems can be solved in polynomial time for graph with treewidth bounded by a constant [3, 4]. In the past few decades, there are lots of literatures investigate the treewidth of certain graphs, for example, [7, 8, 10, 12, 17, 24]. However, it is difficult to determine the treewidth exactly in most situations, and there are only few papers obtained the exact treewidth of some certain graphs. In 2014, the treewidth of the Kneser graphs were determined exactly by Harvey et al. [7]. In 2020, Liu et al. [13] determined the exact treewidth of the generalized Kneser graphs. Motivated by these two results, we study the exact value of treewidth of the generalized \(q\)-Kneser graphs in this paper.

Let \(n, k \in \mathbb{Z}^+\) with \(1 \leq k \leq n\), and \(V\) an \(n\)-dimensional vector space over the finite field \(\mathbb{F}_q\), where \(q\) is a prime power. Denote by \(\binom{[q]}{k}\) the family of all \(k\)-dimensional subspaces of \(V\). In the sequel we will abbreviate “\(k\)-dimensional subspace” to “\(k\)-subspace”. Let \(a, b \in \mathbb{Z}^+\). The Gaussian binomial coefficient is defined as

\[
\binom{a}{b} = \prod_{0 \leq i < b} \frac{q^{a-i} - 1}{q^{i} - 1}.
\]

In addition, we set \(\binom{a}{0} = 1\) and \(\binom{a}{a} = 0\) if \(c < 0\). Recall that \(|\binom{[q]}{k}| = \binom{n}{k}\). For any \(t \in \mathbb{Z}^+\), a family \(\mathcal{F} \subseteq \binom{[q]}{k}\) is said to be \(t\)-intersecting if \(\dim(A \cap B) \geq t\) for all \(A, B \in \mathcal{F}\).

Let \(n, k, t \in \mathbb{Z}\) with \(1 \leq t < k \leq n\). Write \([n] = \{1, 2, \ldots, n\}\) and denote by \(\binom{[n]}{k}\) the collection of all \(k\)-subsets of \([n]\). The generalized Kneser graph, denoted by \(K(n, k, t)\), is a graph whose vertex set is \(\binom{[n]}{k}\) and two vertices \(A\) and \(B\) are adjacent if \(|A \cap B| < t\). The graph \(K(n, k, 1)\) is the well-known Kneser graph \([11, 14]\). Define \(K_q(n, k, t)\) to be the generalized \(q\)-Kneser graph for \(1 \leq t < k\) whose vertex set is \(\binom{[q]}{k}\) and two vertices \(F_1\) and \(F_2\) are adjacent if \(\dim(F_1 \cap F_2) < t\). When \(t = 1\), \(K_q(n, k, 1)\) is the well-known \(q\)-Kneser graph. When \(t = k - 1\), the graph \(K_q(n, k, k - 1)\), usually denoted by \(G_q(n, k)\), is the complement of the Grassmann graph \(G_q(n, k)\). Over the years several aspects of \(q\)-Kneser graphs and Grassmann graphs such as chromatic number, energy, eigenvalues and some other properties had been widely studied as one can find in, for example, [2, 9, 15, 16, 18, 23].

We know that the famous Erdős-Ko-Rado Theorem [6] has a well-known relationship to the independent number of the generalized Kneser graphs, since an independent set in the generalized Kneser graph \(K(n, k, t)\) is a \(t\)-intersecting family of \(\binom{[n]}{k}\). Similarly, the Erdős-Ko-Rado Theorem for vector spaces [22] also has a well-known relationship to the independent number of the generalized \(q\)-Kneser graphs. In this paper, we will use such relationship to obtained the following two main results.
Theorem 1.2. Let \( k > t \geq 1 \) and \( n \geq 2t(k-t+1)+k+1 \). Then
\[
\text{tw}(K_q(n, k, t)) = \frac{n}{k} - \frac{n-t}{k-t} - 1.
\]

Another result is about the exact value of \( \text{tw}(\overline{G_q(n,k)}) \) for any \( n \) and \( k \). Note that \( \overline{G_q(n,k)} \) is an null graph when \( n < k+2 \). Thus we only consider the case with \( n \geq k+2 \).

Theorem 1.3. Let \( n \) and \( k \) be positive integers with \( n \geq k+2 \) and \( k \geq 2 \). Then
\[
\text{tw}(G_q(n, k)) = \begin{cases} 
q^4 + O(q^3), & \text{if } k = 2 \text{ and } n = 4, \\
\frac{n}{k} - \left(\frac{n-k+1}{1}\right) - 1, & \text{otherwise}.
\end{cases}
\]

The rest of this paper is organized as follows. In Section 2, we will give the exact value of \( \text{tw}(K_q(n, k, t)) \) for \( k > t \geq 1 \) and large \( n \) corresponding to \( k \) and \( t \). After that, we will study the treewidth of the complement of Grassman graphs for any possible \( n \) and \( k \) in Section 3.

2 Treewidth of \( K_q(n, k, t) \)

2.1 Upper bound for treewidth in Theorem 1.2

In this subsection, we give an upper bound of \( \text{tw}(K_q(n, k, t)) \) with the help of the following famous Erdős-Ko-Rado Theorem for vector spaces.

Theorem 2.1. ([22]) Let \( n, k, t \in \mathbb{Z}^+ \) with \( 2k \leq n \) and \( t \leq k \). If \( \mathcal{F} \subseteq \binom{V}{k} \) is \( t \)-intersecting, then
\[
|\mathcal{F}| \leq \left[ \frac{n-t}{k-t} \right].
\]

Equality holds if and only if either

(i) \( \mathcal{F} \) consists of all \( k \)-subspaces of \( V \) which contain a fixed \( t \)-subspace of \( V \), or

(ii) \( n = 2k \) and \( \mathcal{F} \) consists of all \( k \)-subspaces of a fixed \( (n-t) \)-subspace of \( V \).

The result of Theorem 2.1 is clearly equivalent to the independent number of the generalized \( q \)-Kneser graph \( K_q(n, k, t) \). That is,
\[
\alpha(K_q(n, k, t)) = \left[ \frac{n-t}{k-t} \right]
\]
for \( n \geq 2k \).

The following lemma can be easily proved.

Lemma 2.2. Let \( m \) and \( i \) be positive integers with \( m \geq i \). Then the following results hold:

(i) \( \left[ \frac{m}{i} \right] = \left[ \frac{m-1}{i-1} \right] + q^i \left[ \frac{m-1}{i-1} \right] \) and \( \left[ \frac{m}{i} \right] = \frac{q^m-1}{q^i-1} \cdot \left[ \frac{m-1}{i-1} \right] \);

(ii) \( q^m-i < \frac{q^{m-1}}{q^i-1} < q^{m-i+1} \) and \( q^{i-1} < \frac{q^{i-1}}{q^i-1} < q^{-m} \) if \( i < m \);
(iii) \(q^{i(m-i)} \leq \begin{bmatrix} m \\ i \end{bmatrix} < q^{i(m-i+1)}\), and \(q^{i(m-i)} < \begin{bmatrix} m \\ i \end{bmatrix}\) if \(i < m\).

**Proposition 2.3.** ([5, Lemma 9.3.2]) Suppose \(0 \leq i, j \leq n\). If \(X\) is a \(j\)-subspace of \(V\), then there are precisely \(q^{(i-m)(j-m)} \begin{bmatrix} n-j \\ i-m \end{bmatrix} \) \(i\)-subspaces \(Y\) in \(V\) such that \(\dim(X \cap Y) = m\).

**Proposition 2.4.** ([7]) For any graph \(G\), \(\text{tw}(G) \leq \max\{\Delta(G), |V(G)| - \alpha(G) - 1\}\).

**Lemma 2.5.** If \(n, k\) and \(t\) be positive integers with \(2k \leq n\) and \(t \leq k\), then

\[
\text{tw}(K_q(n, k, t)) \leq \begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-t \\ k-t \end{bmatrix} - 1.
\]

**Proof.** According to Proposition 2.4, to prove an upper bound of \(\text{tw}(K_q(n, k, t))\) we only need to compare the size of \(\Delta(K_q(n, k, t))\) and \(|V(K_q(n, k, t))| - \alpha(K_q(n, k, t)) - 1\).

Fix \(A \in \begin{bmatrix} V \\ k \end{bmatrix}\), by Proposition 2.3, we have

\[
\left| \left\{ B \in \begin{bmatrix} V \\ k \end{bmatrix} \mid \dim(A \cap B) = i \right\} \right| = q^{(k-i)^2} \begin{bmatrix} n-k \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix}.
\]

By the definition of \(K_q(n, k, t)\), we have

\[
\Delta(K_q(n, k, t)) = \sum_{i=0}^{t-1} \left| \left\{ B \in \begin{bmatrix} V \\ k \end{bmatrix} \mid \dim(A \cap B) = i \right\} \right| = \sum_{i=0}^{t-1} q^{(k-i)^2} \begin{bmatrix} n-k \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix}.
\]

**Claim 1.** For any \(t \geq 1\), we have \(q^{(k-t)^2} \begin{bmatrix} n-k \\ k-t \end{bmatrix} > \begin{bmatrix} n-t \\ k-t \end{bmatrix} \).

**Proof of Claim 1.** On the one hand, by Lemma 2.2 (iii), we have

\[
q^{(k-t)^2} \begin{bmatrix} n-k \\ k-t \end{bmatrix} \begin{bmatrix} k \\ t \end{bmatrix} > q^{(k-t)^2} q^{(k-t)(n-2k+t)} q^{t(k-t)} = q^{(k-t)(n-k+t)}.
\]

On the other hand, by Lemma 2.2 (iii) again, we get

\[
\begin{bmatrix} n-t \\ k-t \end{bmatrix} < q^{(k-t)(n-k+1)}.
\]

Thus, we have the result as required.

It suffices to prove that \(\Delta(K_q(n, k, t)) + \alpha(K_q(n, k, t)) < |V(K_q(n, k, t))|\). According to the Claim 1, we have

\[
\Delta(K_q(n, k, t)) + \alpha(K_q(n, k, t)) < \sum_{i=0}^{t-1} q^{(k-i)^2} \begin{bmatrix} n-k \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} + q^{(k-t)^2} \begin{bmatrix} n-k \\ k-t \end{bmatrix} \begin{bmatrix} k \\ t \end{bmatrix}
\]

\[
= \sum_{i=0}^{t} q^{(k-i)^2} \begin{bmatrix} n-k \\ k-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix}
\]

\[
= \sum_{i=0}^{t} \left| \left\{ B \in \begin{bmatrix} V \\ k \end{bmatrix} \mid \dim(A \cap B) = i \right\} \right|.
\]
Since \( \bigcup_{i=0}^{t} \left\{ B \in \binom{V}{k} \mid \dim(A \cap B) = i \right\} \subseteq \binom{V}{k} \), we have

\[
\Delta(K_q(n, k, t)) + \alpha(K_q(n, k, t)) < |V(K_q(n, k, t))|,
\]
and \( \text{tw}(K_q(n, k, t)) \leq |V(K_q(n, k, t))| - \alpha(K_q(n, k, t)) - 1 = \frac{n}{k} - \left[ \frac{n-t}{k-t} \right] - 1. \]

\[ \square \]

### 2.2 Lower bound for treewidth in Theorem 1.2

In this subsection, we give the lower bound of \( \text{tw}(K_q(n, k, t)) \).

Let \( X \subseteq V(G) \) and \( G[X] \) the subgraph of \( G \) induced by \( X \). Denote \( G - X = G[V(G) \setminus X] \). Let \( p \) be a fixed constant with \( \frac{2}{3} \leq p < 1 \). The \( p \)-separator of \( G \) is a subset \( X \subset V(G) \) such that there is no component in \( G - X \) that contains more than \( p|V(G - X)| \) vertices. The following result describes the relationship between the treewidth and the \( p \)-separators of \( G \).

**Proposition 2.6.** ([21]) Every graph \( G \) has a \( p \)-separator of order at most \( \text{tw}(G) + 1 \) for each \( \frac{2}{3} \leq p < 1 \).

Let \( X \) be a \( p \)-separator of \( G \). Then the vertices of \( G - X \) can be divided into two parts, say \( A \) and \( B \), such that the components in \( A \) and \( B \) contain at most \( p|V(G - X)| \) vertices. This leads to the following result.

**Lemma 2.7.** Let \( X \) be a \( p \)-separator. Then the vertices of \( G - X \) can be divided into two parts \( A \) and \( B \) such that there is no edge between \( A \) and \( B \), and the equation

\[
\frac{1}{3}|V(G - X)| \leq |A|, |B| \leq \frac{2}{3}|V(G - X)| \tag{3}
\]

holds.

With the help of these important results we give the following lemma.

**Lemma 2.8.** Let \( k > t \geq 1 \) and \( n \geq 2t(k-t+1) + k+1 \). Let \( \Gamma := K_q(n, k, t) \). Then

\[
\text{tw}(\Gamma) \geq \left[ \frac{n}{k} \right] - \left[ \frac{n-t}{k-t} \right] - 1.
\]

**Proof.** We suppose to the contrary that \( \text{tw}(\Gamma) < \left[ \frac{n}{k} \right] - \left[ \frac{n-t}{k-t} \right] - 1 \). By Proposition 2.6, there is a \( \frac{2}{3} \)-separator \( X \) such that \( |X| < \left[ \frac{n}{k} \right] - \left[ \frac{n-t}{k-t} \right] \). Therefore, \( |V(\Gamma - X)| > \left[ \frac{n-t}{k-t} \right] = \alpha(\Gamma) \), and we have \( V(\Gamma - X) \) is too large to be an independent set which implies that there exists an edge \( u_1u_2 \) in \( \Gamma - X \). By Lemma 2.7, \( V(\Gamma - X) \) can be divided into two parts \( A \) and \( B \) such that there is no edge between \( A \) and \( B \), and the equation (3) holds. Thus, \( |A|, |B| \geq 2 \). Without loss of generality, assume that \( u_1u_2 \) is in \( G[A] \), where \( G = \Gamma - X \).

Let \( S = \binom{[n]}{t} \times \binom{[n]}{t} \). Thus, \( |S| = \binom{k}{i}^2 \). For any \( v \in B \) and \( i \in \{1, 2\} \), since \( vu_i \notin E(\Gamma) \), we have \( \dim(v \cap u_i) \geq t \). As \( \dim(u_1 \cap u_2) < t \), we get \( \tau_1 \neq \tau_2 \) for any \( \tau_1, \tau_2 \in S \), which implies that \( \dim(\tau_1 \cap \tau_2) \leq t - 1 \). Let \( B(\tau_1, \tau_2) = \{ v \in B \mid \tau_1, \tau_2 \in v \} \). According to Pigeonhole Principle, there exists some \( (\tau_1, \tau_2) \in S \) such that

\[
|B(\tau_1, \tau_2)| \geq \frac{1}{|S||B|} \geq \left[ \frac{k}{t} \right]^2 \cdot \frac{1}{3} \left[ \frac{n-t}{k-t} \right],
\]
by (3). On the other hand, since \( \dim(\tau_1 + \tau_2) \) is minimized when \( \dim(\tau_1 \cap \tau_2) \) is maximized and

\[
\dim(\tau_1 + \tau_2) = \dim(\tau_1) + \dim(\tau_2) - \dim(\tau_1 \cap \tau_2) \geq t + 1,
\]

we have \( |B(\tau_1, \tau_2)| \leq \left[\frac{n-t-1}{k-t-1}\right] \). Combining the lower and upper bounds of \( |B(\tau_1, \tau_2)| \), we have

\[
\left[\frac{n-t-1}{k-t-1}\right] > \left[\frac{k}{t}\right]^{-2} \cdot \frac{1}{3} \left[\frac{n-t}{k-t}\right]. \tag{4}
\]

**Claim 2.** If \( k > t \geq 1 \) and \( n \geq 2t(k-t+1)+k+1 \), then \( \left[\frac{n-t-1}{k-t-1}\right] \leq \left[\frac{k}{t}\right]^{-2} \cdot \frac{1}{3} \left[\frac{n-t}{k-t}\right] \).

**Proof of Claim 2.** It suffices to prove that \( \frac{n-t-1}{q^{k-t-1}} \geq 3\left[\frac{k}{t}\right]^2 \) by Lemma 2.2 (i).

According to Lemma 2.2 (ii), \( \frac{n-t-1}{q^{k-t-1}} > q^{n-k} \). By Lemma 2.2 (iii), we have \( 3\left[\frac{k}{t}\right]^2 < 3 \cdot q^{2t(k-t+1)} \). If \( q \geq 3 \), we have the result as required since \( n \geq 2t(k-t+1)+k+1 \). If \( q = 2 \), then

\[
3\left[\frac{k}{t}\right]^2 = 3 \cdot \prod_{i=0}^{t-1} \frac{2^{2^{i}+1}}{2^{2^{i}}-1} \cdot \prod_{i=0}^{t-1} \frac{2^{k-i}-1}{2^{2^{i}}-1} = \prod_{i=0}^{t-1} \frac{2^{2^{i}}-1}{2^{2^{i}}-1} \cdot \prod_{i=0}^{t-3} \frac{2^{2^{i}}-1}{2^{2^{i+1}}-1} \cdot 2^{k-t+2} \leq 2^{t(k-t+1)} \cdot 2^{(t-2)(k-t+1)} \cdot 2^{k-t+2} = 2^{2t(k-t+1)+1}.
\]

Therefore, we also have the result as required.

By Claim 2, we have a contradiction with the equation (4).

**Proof of Theorem 1.2.** By Lemmas 2.5 and 2.8, we obtain the result directly.

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### 3 Treewidth of the complement of Grassmann graphs

In this section, we study the treewidth of the complement of Grassmann graphs, and give the exact value of the treewidth of \( G_q(n, k) \) for \( n \geq k+2 \). Note that \( G_q(n, k) \) is an empty graph when \( n < k+2 \). Firstly, we can easily get the upper bound of \( \text{tw}(G_q(n, k)) \) by Lemma 2.5 since \( G_q(n, k) = K_q(n, k, 1) \).

**Lemma 3.1.** Let \( n \) and \( k \) be positive integers with \( k \geq 2 \) and \( n \geq 2k \). Then

\[
\text{tw}(G_q(n, k)) \leq \left[\frac{n}{k}\right] - \left[\frac{n-k+1}{1}\right] - 1.
\]

**Lemma 3.2.** Let \( n \) and \( k \) be positive integers with \( k \geq 2 \) and \( n \geq \max\{k+3, 2k\} \). Then

\[
\text{tw}(G_q(n, k)) \geq \left[\frac{n}{k}\right] - \left[\frac{n-k+1}{1}\right] - 1.
\]
Proof. Suppose to the contrary that \( tw(G_q(n, k)) < \frac{n}{k} - \frac{n-k+1}{1} - 1 \). By Proposition 2.6, there exists a \( \frac{2}{3} \)-separator \( X \) such that \( |X| < \frac{n}{k} - \frac{n-k+1}{1} \). Therefore, \( |V(G_q(n, k) - X)| > \frac{n-k+1}{1} \). Let \( G = G_q(n, k) - X \) for short. By Lemma 2.7, it is easy to see that \( V(G) \) can be partitioned into two parts \( A \) and \( B \) such that there is no edge between \( A \) and \( B \), and the equations

\[
\frac{1}{3}|V(G)| \leq |A|, |B| \leq \frac{2}{3}|V(G)|
\]

(5) holds. Thus, \( |A|, |B| \geq 2 \). By Theorem 2.1 and \( n \geq \max\{k+3, 2k\} \geq 2k \), we have \( \alpha(G_q(n, k)) = \frac{n-k+1}{1} \). Since \( |V(G)| > \frac{n-k+1}{1} \), there is an edge in \( G[A] \) or \( G[B] \). Without loss of generality, assume that \( v_1v_2 \) is in \( G[A] \) and let \( v_1 \cap v_2 = w \). Thus we have \( \dim w \leq k-2 \).

We claim that \( \dim w = k-2 \). Since for any vertex \( u \in B \), there is no vertex in \( A \) is adjacent to \( u \) in \( G \), we have \( \dim(u \cap v_1), \dim(u \cap v_2) \geq k-1 \). This implies that \( \dim(u \cap v_1), \dim(u \cap v_2) = k-1 \) and then

\[
\dim(v_1 \cap v_2) \geq \dim((u \cap v_1) \cap (u \cap v_2)) \geq \dim(u \cap v_1) + \dim(u \cap v_2) - \dim u = k-2.
\]

(6)

On the other hand, \( \dim w \leq k-2 \) from above. Thus we have \( \dim w = k-2 \), as required.

Let \( \alpha_1, \alpha_2, \ldots, \alpha_{k-2} \) be a basis of \( w = v_1 \cap v_2 \), then \( w = \langle \alpha_1, \alpha_2, \ldots, \alpha_{k-2} \rangle \) and we let

\[
v_1 = w + \langle \beta_1, \beta_2 \rangle \quad \text{and} \quad v_2 = w + \langle \beta_3, \beta_4 \rangle.
\]

For any \( u \in B \), by equation (6), we have \( \dim(u \cap v_1 \cap v_2) \geq k-2 \). On the other hand, since \( \dim(v_1 \cap v_2) = k-2 \), we have \( \dim(u \cap v_1 \cap v_2) \leq k-2 \). And then \( \dim(u \cap v_1 \cap v_2) = k-2 \), which implies that

\[
|B| \leq \left[ \begin{array}{c} 2 \\ 1 \end{array} \right] \times \left[ \begin{array}{c} 2 \\ 1 \end{array} \right] = (q+1)^2.
\]

We will complete the proof by considering the following two cases.

**Case 1.** \( G[B] \) contains an edge.

Let \( u_1u_2 \) be an edge in \( G[B] \). Since \( u_iv_j \notin E(G) \) for \( 1 \leq i, j \leq 2 \), we can assume, without loss of generality, that

\[
u_1 = w + \langle \beta_1, \beta_3 \rangle, \quad u_2 = w + \langle \beta_2, \beta_4 \rangle.
\]

Thus we get \( w + < \beta_1, \beta_2, \beta_3, \beta_4 > = u_1 + u_2 = v_1 + v_2 \). Following the similar analysis above, we have \( |A| \leq (q+1)^2 \). Define

\[
A(u_1, u_2) = \left\{ u \in \left[ \begin{array}{c} u_1 + u_2 \\ k \end{array} \right] \mid w \subseteq u, \ \dim(u \cap u_1) = k-1, \ \dim(u \cap u_2) = k-1 \right\},
\]

\[
B(v_1, v_2) = \left\{ v \in \left[ \begin{array}{c} v_1 + v_2 \\ k \end{array} \right] \mid w \subseteq v, \ \dim(v \cap v_1) = k-1, \ \dim(v \cap v_2) = k-1 \right\}.
\]

7
We can easily see that \( A \subseteq A(u_1, u_2), B \subseteq B(v_1, v_2) \) and \( |A(u_1, u_2)| \leq (1 + q)^2, |B(u_1, u_2)| \leq (1 + q)^2 \). Denote \( z_1 = w + (\beta_1, \beta_4) \) and \( z_2 = w + (\beta_2, \beta_3) \). Then \( z_1, z_2 \in A(u_1, u_2) \cap B(v_1, v_2) \).

Let \( z_3 = w + (\beta_1 + \beta_2, \beta_3 + \beta_4) \) and \( z_4 = w + (\beta_2, \beta_1 + \beta_3) \). Then \( z_3 \in A(u_1, u_2) \setminus B(v_1, v_2) \) and \( z_4 \in B(v_1, v_2) \setminus A(u_1, u_2) \). Since \( \dim(z_3 \cap z_4) = k - 2 \), we have \( z_3 \) is adjacent to \( z_4 \).

Therefore, if \( z_3 \in A \), then \( z_4 \notin B \) which implies \( z_4 \notin A(u_1, u_2) \cup B(v_1, v_2) \). Thus we have

\[
|V(G)| \leq |A \cup B| \leq |A(u_1, u_2)| + |B(v_1, v_2)| - |\{z_1, z_2, z\}| \leq 2(1 + q)^2 - 3,
\]

where \( z \in \{z_3, z_4\} \).

On the other hand, by the assumption in the beginning, \( |V(G)| \geq \left\lceil \frac{n - k + 1}{1} \right\rceil + 1 \). Combining with the upper and lower bounds of \( |V(G)| \), we have

\[
2(1 + q)^2 - 3 \geq \left\lceil \frac{n - k + 1}{1} \right\rceil + 1 = q^{n-k} + q^{n-k-1} + \cdots + q + 2,
\]

a contradiction with \( n \geq k + 3 \).

**Case 2.** \( B \) is an independent set in \( G \).

Let \( u_1, u_2 \in B \) and \( \alpha = u_1 \cap u_2 \). Then \( \dim \alpha = k - 1 \) and \( \dim(u_1 + u_2) = k + 1 \). We define

\[
\begin{align*}
C &= \left\{ u \in \left[ \begin{array}{c} V \\backslash \{u_1, u_2\} \end{array} \right] \mid \alpha \in u \right\} \setminus \left[ \begin{array}{c} u_1 + u_2 \end{array} \right]_k, \\
D &= \left[ \begin{array}{c} u_1 + u_2 \\ k \end{array} \right].
\end{align*}
\]

And let \( D_1 = \{ u \in D \mid \alpha \in u \} \) and \( D_2 = D \setminus D_1 \). It is easy to see that \( C \) and \( D \) are independent sets in \( G \), \( |D| = \left\lceil \frac{k+1}{1} \right\rceil \) and \( |D_1| = \left\lceil \frac{2}{1} \right\rceil = q + 1 \).

Firstly we prove that \( A \cup B \subseteq C \cup D \). Clearly, \( u_1, u_2 \in C \cup D \). For any \( s \in A \cup B \) and \( s \notin \{u_1, u_2\} \), we prove that \( s \in C \cup D \). If \( \alpha \subseteq s \), then we have \( s \in C \cup D \). If \( \alpha \nsubseteq s \), then \( \dim(s \cap u_1 \cap u_2) = \dim(s \cap \alpha) \leq k - 2 \), and we have

\[
\dim((s \cap u_1) + (s \cap u_2)) = \dim(s \cap u_1) + \dim(s \cap u_2) - \dim(s \cap u_1 \cap u_2)
\]

\[
\geq 2(k - 1) - (k - 2)
\]

\[
= k.
\]

Furthermore, since \( (s \cap u_1) + (s \cap u_2) \subseteq s \) and \( \dim s = k \), we have \( s = (s \cap u_1) + (s \cap u_2) \in \left[ \frac{u_1 + u_2}{k} \right] \subseteq C \cup D \).

Next, for any \( y \in D_2 \), we have \( \dim(y \cap \alpha) \leq k - 2 \). On the other hand, \( \dim(y \cap \alpha) = \dim y + \dim \alpha - \dim(y + \alpha) \geq k + (k-1) - (k+1) = k - 2 \). So we have \( \dim(y \cap \alpha) = k - 2 \). Hence for any \( x \in C \), \( \dim(x \cap y) \leq k - 2 \), which implies that \( x \) is adjacent to \( y \). Furthermore, since for any \( x \in C \) and \( z \in D_1 \), we can easily see that \( x \) is not adjacent to \( z \). As there is an edge in \( G[A] \), we have \( A \cap C \neq \emptyset \) and \( A \cap D_2 \neq \emptyset \). Therefore, if there is \( x \in B \cap C \), then there is a vertex \( y \in \mathcal{A} \cap \mathcal{D} \) such that \( xy \in E(G) \); if there is a vertex \( x \in B \cap D_2 \), then there is a vertex \( y \in A \cap C \) such that \( xy \in E(G) \). Thus, we have \( B \subseteq D_1 \), since there is no edge between \( A \) and \( B \). Therefore, we get \( |B| \leq |D_1| = q + 1 \). On the other hand,

\[
|B| \geq \frac{1}{3} |G| \geq \frac{1}{3} \left( \left\lceil \frac{n - k + 1}{1} \right\rceil + 1 \right).
\]
Combining with the lower and upper bounds of \(|B|\), we have a contradiction with our assumption that \(n \geq k+3\).

\[
\text{Proof of Theorem 1.3.} \quad \text{We divide the proof of this theorem into the following two cases.}
\]

**Case 1.** \(k \geq 3\).

In this case we have \(2k \geq k+3\). If \(n \geq 2k\), by Lemmas 3.1 and 3.2, we have \(\text{tw}(\overline{G_q(n,k)}) = \left\lceil \frac{n}{k} \right\rceil - \left\lceil \frac{n-k+1}{k} \right\rceil - 1\). Note that \(G_q(n,k) \cong G_q(n,n-k)\). If \(n < 2k\) then \(n > 2(n-k)\). By Lemmas 3.1 and 3.2, \(\text{tw}(\overline{G_q(n,k)}) = \text{tw}(\overline{G_q(n,n-k)}) = \left\lceil \frac{n}{k} \right\rceil - \left\lceil \frac{n-k+1}{k} \right\rceil - 1\).

**Case 2.** \(k = 2\).

If \(n \geq 5 = k+3\), by Lemmas 3.1 and 3.2, we get \(\text{tw}(\overline{G_q(n,2)}) = \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n-1}{2} \right\rfloor - 1\). If \(n = 4\), then \(\text{tw}(\overline{G_q(4,2)}) = \left\lceil \frac{4}{2} \right\rceil - \left\lfloor \frac{3}{2} \right\rfloor - 1 = q^4 + q^3 + q^2 - 1\). There is a trivial lower bound of \(\text{tw}(\overline{G_q(4,2)})\), that is \(\text{tw}(\overline{G_q(4,2)}) \geq \delta(\text{tw}(\overline{G_q(4,2)})) = q^4\). Furthermore, following the similar proof of Lemma 3.2, we have \(\text{tw}(\overline{G_q(4,2)}) \geq \left\lceil \frac{4}{2} \right\rceil - \left\lfloor \frac{4}{2} \right\rfloor - 1 = q^4 + q^2 - 1\). Therefore, we have \(\text{tw}(\overline{G_q(4,2)}) = q^4 + O(q^3)\).

Consequently, we complete the proof of this theorem.

\[
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\]

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