BARELY LOCALLY PRESENTABLE CATEGORIES

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Abstract. We introduce a new class of categories generalizing locally presentable ones. The distinction does not manifest in the abelian case and, assuming Vopěnka’s principle, the same happens in the regular case. The category of complete partial orders is the natural example of a barely locally finitely presentable category which is not locally presentable.

1. Introduction

Locally presentable categories were introduced by P. Gabriel and F. Ulmer in [6]. A category $\mathcal{K}$ is locally $\lambda$-presentable if it is cocomplete and has a strong generator consisting of $\lambda$-presentable objects. Here, $\lambda$ is a regular cardinal and an object $A$ is $\lambda$-presentable if its hom-functor $\mathcal{K}(A, -) : \mathcal{K} \to \text{Set}$ preserves $\lambda$-directed colimits. A category is locally presentable if it is locally $\lambda$-presentable for some $\lambda$. This concept of presentability formalizes the usual practice – for instance, finitely presentable groups are precisely groups given by finitely many generators and finitely many relations. Similarly, [6] defined $\lambda$-generated objects as those whose hom-functor $\mathcal{K}(A, -)$ preserves $\lambda$-directed colimits of monomorphisms. Again, finitely generated groups are precisely groups admitting a finite set of generators. This leads to locally generated categories, where a cocomplete category is locally $\lambda$-generated if it is strongly co-wellpowered and has a strong generator consisting of $\lambda$-generated objects. Since a locally presentable category is co-wellpowered, any locally $\lambda$-presentable category is locally $\lambda$-generated. Conversely, a locally $\lambda$-generated category is locally presentable but not necessarily locally $\lambda$-presentable (see [6] or [3]). In particular, each locally generated category is co-wellpowered. Note that the terminology is somewhat misleading because strong co-wellpoweredness is weaker than co-wellpoweredness – the first case means that each object has a set of strong quotients while in the second case it has a set of quotients. Under Vopěnka’s principle, we can omit strong co-wellpoweredness in the definition of a locally generated category, but it is still open whether one needs set theory for this (see [3], Open Problem 3).

We introduce another weakening of the concept of presentability – an object $A$ is barely $\lambda$-presentable if its hom-functor $\mathcal{K}(A, -)$ preserves coproduct $\lambda$-directed colimits. The latter are those given by expressing a coproduct as the directed colimit

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of subcoproducts of size $< \lambda$. If coproduct injections are monomorphisms this reduces to the condition that every morphism from $A$ to the coproduct $\bigsqcup_{i \in I} K_i$ factorizes through $\bigsqcup_{j \in J} K_j$ where $|J| < \lambda$. This concept is standard for triangulated categories where the resulting objects are called $\lambda$-small (see [10]). A cocomplete category is now barely locally $\lambda$-presentable if it is strongly co-wellpowered and has a strong generator consisting of barely $\lambda$-presentable objects. In the abelian case, these categories were introduced in [12] and called locally weakly generated. This is justified by the fact that coproduct injections are monomorphisms there and thus barely $\lambda$-presentable objects generalize $\lambda$-generated ones. Since weakly locally presentable categories mean something else (see [3]), we had to change our terminology.

We prove that barely locally presentable abelian categories are locally presentable and thus they provide just a different view of a known concept. We also show that the fact that barely locally presentable regular categories are locally presentable is equivalent to Vopěnka’s principle. Thus we get some artificial examples of barely locally presentable categories which are not locally presentable under the negation of Vopěnka’s principle. A natural example of this, not depending on set theory, is the category $\mathbf{CPO}$ of complete partial orders. It is barely locally finitely presentable but not locally presentable.

2. BARELY LOCALLY PRESENTABLE CATEGORIES

Recall that a strong generator is a small full subcategory $\mathcal{A}$ of $\mathcal{K}$ such that the functor $E_{\mathcal{A}} : \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\text{op}}}$, $EK = \mathcal{K}(\cdot, K)$, is faithful and conservative (= reflects isomorphisms). A generator $\mathcal{A}$ of $\mathcal{K}$ is strong if and only if for each object $K$ and each proper subobject of $K$ there exists a morphism $A \to K$ with $A \in \mathcal{A}$ which does not factorize through that subobject.

To be strongly co-wellpowered means that any object has only a set of strong quotients. Recall that an epimorphism $f : K \to L$ is strong if each commuting square

$$
\begin{array}{ccc}
L & \overset{v}{\longrightarrow} & B \\
\downarrow{f} & & \downarrow{g} \\
K & \underset{u}{\longrightarrow} & A
\end{array}
$$

such that $g$ is a monomorphism has a diagonal fill-in, i.e., a morphism $t : L \to A$ with $tf = u$ and $gt = v$.

**Definition 2.1.** $\lambda$-directed colimits $\bigsqcup_{j \in J} K_j \to \bigsqcup_{i \in I} K_i$, where $J$ runs over all the subsets of $I$ of cardinality less than $\lambda$, will be called coproduct $\lambda$-directed colimits.

**Definition 2.2.** Let $\mathcal{K}$ be a category with coproducts and $\lambda$ a regular cardinal. An object $A$ of $\mathcal{K}$ will be called barely $\lambda$-presentable its hom-functor $\mathcal{K}(A, \cdot) : \mathcal{K} \to \mathbf{Set}$ preserves coproduct $\lambda$-directed colimits.
Remark 2.3. (1) This means that $\mathcal{K}(A, -)$ sends coproduct $\lambda$-directed colimits to $\lambda$-directed colimits and not to coproduct $\lambda$-directed ones (because $\mathcal{K}(A, -)$ does not preserve coproducts).

(2) If coproduct injections are monomorphisms then $A$ is barely $\lambda$-presentable if and only if for every morphism $f : A \to \coprod_{i \in I} K_i$ there is a subset $J$ of $I$ of cardinality less than $\lambda$ such that $f$ factorizes as $A \to \coprod_{j \in J} K_j \to \coprod_{i \in I} K_i$ where the second morphism is the subcoproduct injection.

(3) Coproduct injections are very often monomorphisms, for instance in any pointed category. However, in the category of commutative rings, the coproduct is the tensor product and the coproduct injection $\mathbb{Z} \to \mathbb{Z} \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$ is not a monomorphism.

(4) Any $\lambda$-presentable object is barely $\lambda$-presentable. We say that $A$ is barely presentable if it is barely $\lambda$-presentable for some $\lambda$.

Definition 2.4. A cocomplete category $\mathcal{K}$ will be called barely locally $\lambda$-presentable if it is strongly co-wellpowered and has a strong generator consisting of barely $\lambda$-presentable objects.

A category is barely locally presentable if it is barely locally $\lambda$-presentable for some regular cardinal $\lambda$.

This concept was introduced in [12] for abelian categories. Since any abelian category with a generator is co-wellpowered, strong co-wellpoweredness does not need to be assumed there. Of course, any locally $\lambda$-presentable category is barely locally $\lambda$-presentable.

Remark 2.5. Let $\mathcal{K}$ be a barely locally $\lambda$-presentable category and $\mathcal{A}$ its strong generator consisting of barely $\lambda$-presentable objects. Following [2,3(1)], $E_A : \mathcal{K} \to \text{Set}^{\mathcal{A}^\text{op}}$ sends coproduct $\lambda$-directed colimits to $\lambda$-directed colimits.

Recall that a category $\mathcal{K}$ has $\lambda$-directed unions if for any $\lambda$-directed set of subobjects $(K_i)_{i \in I}$ of $K \in \mathcal{K}$ the induced morphism $\text{colim}_{i \in I} K_i \to K$ is a monomorphism. The following result was proved in [12] 9.1 for abelian categories and for $\lambda = \aleph_0$.

Proposition 2.6. Any barely locally $\lambda$-presentable regular category has $\lambda$-directed unions.

Proof. Let $(K_i)_{i \in I}$ be a $\lambda$-directed set of subobjects, $k_i : K_i \to L = \text{colim}_{i \in I} K_i$ a colimit cocone and $t : L \to K$ the induced morphism. Since $tk_i : K_i \to K$ are monomorphisms, $k_i$ are monomorphisms. Let $f : \coprod_{i \in I} K_i \to L$ and $p : \coprod_{i \in I} K_i \to K$ be the induced morphisms. Clearly, $tf = p$. Let $p_1, p_2 : P \to \coprod_{i \in I} K_i$ be a kernel pair of $p$. It suffices to prove that $fp_1 = fp_2$. In this case, $p_1, p_2$ is a kernel pair of $f$ because $fg_1 = fg_2$ implies that $pg_1 = pg_2$. Since $f$ is a regular epimorphism, it is a coequalizer of $p_1, p_2$. Following [3] I.2.3, $t$ is a monomorphism.

Let $\mathcal{A}$ be a strong generator of $\mathcal{K}$ consisting of barely $\lambda$-presentable objects. It suffices to prove that $fp_1h = fp_2h$ for any $h : A \to P$, $A \in \mathcal{A}$. Since $A$ is barely $\lambda$-presentable, there is $J \subseteq I$ of cardinality less that $\lambda$ such that $p_ih$ factorizes through
\[ \bigsqcup_{j \in J} K_j \text{ for } n = 1, 2. \] Since \((K_i)_{i \in I}\) is a \(\lambda\)-directed set, there is \(K_m, m \in I\) such that \(fp_nh\) factorizes through the monomorphism \(k_m : K_m \to L\) for \(n = 1, 2\). Let \(q_1, q_2 : A \to K_m\) be the corresponding factorizations. Then \(tk_mq_1 = tk_mq_2\), hence \(q_1 = q_2\) and thus \(fp_1 = fp_2\). \(\Box\)

Recall that a category is **bounded** if it has a small dense subcategory (see [3]).

**Corollary 2.7.** Any barely locally presentable regular category \(\mathcal{K}\) is bounded.

**Proof.** Let \(\mathcal{A}\) be a strong generator of \(\mathcal{K}\) consisting of barely \(\lambda\)-presentable objects. Let \(\overline{\mathcal{A}}\) be the closure of \(\mathcal{A}\) under \(\lambda\)-small colimits and strong quotients. Since \(\mathcal{K}\) is strongly co-wellpowered, \(\overline{\mathcal{A}}\) is small. For an object \(K\) of \(\mathcal{K}\) we form the canonical diagram \(D\) w.r.t. \(\overline{\mathcal{A}}\) and take its colimit \(K^*\). Since \(\mathcal{A}\) is a strong generator, it suffices to prove that the induced morphism \(t : K^* \to K\) is a monomorphism. Then it is an isomorphism because any morphism \(f : A \to K, A \in \mathcal{A}\) factorizes through \(t\).

Since a regular category has a (regular epimorphism, monomorphism) factorization and \(\overline{\mathcal{A}}\) is closed under strong quotients, we can restrict \(D\) to the subdiagram given by monomorphisms \(f : A \to K, A \in \overline{\mathcal{A}}\). Following [2.6] \(t\) is a monomorphism. \(\Box\)

In fact, we have proved that any strongly co-wellpowered regular category \(\mathcal{K}\) having a strong generator and \(\lambda\)-directed unions is bounded.

**Theorem 2.8.** Any barely locally presentable abelian category is locally presentable.

**Proof.** Consider a barely locally \(\lambda\)-presentable abelian category \(\mathcal{K}\). According to [2.6] \(\mathcal{K}\) has \(\lambda\)-directed unions. Following [9] III.1.2 and III.1.9, in any cocomplete abelian category with directed unions directed colimits are exact (cf. [12] 9.2). In the same way we see that in any cocomplete abelian category with \(\lambda\)-directed unions \(\lambda\)-directed colimits are exact (i.e., commute with finite limits). Thus \(\mathcal{K}\) is a cocomplete abelian category with a generator in which \(\lambda\)-directed colimits commute with finite limits. Following [11] 2.2, \(\mathcal{K}\) is locally presentable. \(\Box\)

**Theorem 2.9.** Vopěnka’s principle is equivalent to the fact that any barely locally presentable regular category is locally presentable.

**Proof.** Assuming Vopěnka’s principle any cocomplete bounded category is locally presentable (see [3] 6.14). Under the negation of Vopěnka’s principle, there is a rigid class of connected graphs \(G_i\) in the category \(\text{Gra}\) of graphs (see [3] 6.13). This is a rigid class in the category of multigraphs \(\text{MGra}\) which, as a presheaf category on \(\cdot \Rightarrow \cdot\), is regular. Let \(\mathcal{K}\) be the full subcategory of \(\text{MGra}\) consisting of all the multigraphs in which every connected component is either the terminal multigraph 1 or there are no morphisms from \(G_i\) into it. This is the modification of [3] 6.12. Like there, \(\mathcal{K}\) is an epireflective subcategory of \(\text{MGra}\) and thus it is cocomplete. Moreover, \(\mathcal{K}\) is closed under coproducts in \(\text{MGra}\). The graphs \(\cdot\) and \(\cdot \Rightarrow \cdot\) are barely finitely presentable in \(\mathcal{K}\) because any their morphism into a coproduct uniquely factorizes through a coproduct injection. Since the graphs \(\cdot\) and \(\cdot \Rightarrow \cdot\) form a dense subcategory
in \( K \), the category \( K \) is barely finitely presentable. But, like in [3] 6.12, the graph - is not presentable in \( K \). Thus \( K \) is not locally presentable.

It remains to prove that \( K \) is regular, i.e., that regular epimorphisms are stable under pullback. For this, it suffices to show that the inclusion \( K \to \text{MGr} \) preserves regular epimorphisms. Assume that \( f : K \to L \) is a regular epimorphism in \( K \). Then \( f = f_1 \coprod f_2 : K_1 \coprod K_2 \to L_1 \coprod L_2 \) where \( L_2 \) is a coproduct of copies of 1 and there is no morphism \( G_i \to L_1 \). Then \( f_1 : K_1 \to L_1 \) is a regular epimorphism in \( \text{MGr} \). Since each component of \( K_2 \) contains an edge, \( f_2 \) is a regular epimorphism in \( \text{MGr} \) again. Hence \( f \) is a regular epimorphism in \( \text{MGr} \).

**Examples 2.10.** (1) Analogously, [3] 6.36 gives a barely locally presentable category which, under the negation of Vopěnka’s principle, is not bounded.

(2) [3] 6.38 gives a cocomplete category \( K \) having a strong generator consisting of barely \( \lambda \)-presentable objects which is not complete. But \( K \) is not strongly co-wellpowered.

**Example 2.11.** Let \( \text{CPO} \) be the category of chain-complete posets, i.e., posets where every chain has a join. Morphisms are mappings preserving joins of chains. Any chain complete poset has the smallest element and the joins of directed sets and morphisms preserve them. These posets are also called cpo’s and play a central role in theoretical computer science, in denotational semantics and domain theory. The category \( \text{CPO} \) is cocomplete (see [8]). The coproduct is just the disjoint union with the least elements of each component identified. Thus every finite cpo is barely finitely presentable in \( \text{CPO} \). Epimorphisms are morphisms \( f : A \to B \) where \( f(A) \) is directed join dense in \( B \), i.e., any \( b \in B \) is a join of a directed set \( X \subseteq f(A) \). Thus \(|B| \leq 2^{|A|}\), which implies that \( \text{CPO} \) is co-wellpowered. The two-element chain 2 is a strong generator in \( \text{CPO} \). In fact, it is a generator and for each object \( B \) and each proper subobject \( A \) of \( B \) there exists a morphism \( 2 \to B \) which does not factorize through \( A \). Hence \( \text{CPO} \) is barely locally finitely presentable. But \( \text{CPO} \) is not locally presentable (see [3] 1.18(5)).

**Definition 2.12.** Let \( K \) be a category with coproducts. We say that \( K \) has coproduct \( \lambda \)-directed unions if for every coproduct \( \lambda \)-directed colimit \( \coprod_{j \in J} K_j \to \coprod_{i \in I} K_i \), every morphism \( \coprod_{i \in I} K_i \to K \) whose compositions with \( \coprod_{j \in J} K_j \to \coprod_{i \in I} K_i \) are monomorphisms is a monomorphism.

This definition fits with the definition of \( \lambda \)-directed unions provided that coproduct injections in \( K \) are monomorphisms.

**Proposition 2.13.** Let \( K \) be a locally presentable category such that \( K^{\text{op}} \) has coproduct \( \lambda \)-directed unions for some regular cardinal \( \lambda \). Then \( K \) is equivalent to a complete lattice.

**Proof.** It follows from [3] 1.64. In more detail, the proof considers a product \( \lambda \)-codirected limit \( p_J : K^I \to K^J \) and \( m : \text{colim } D \to K^I \). Using local presentability of
\( K, m \) is shown to be a regular monomorphism. Since \( K^{\text{op}} \) has coproduct \( \lambda \)-directed unions, \( m \) is an epimorphism because the compositions \( p_j m \) are epimorphisms. Thus \( m \) is an isomorphism, which yields the proof.

\[ \square \]

### 3. Properties of barely locally presentable categories

**Remark 3.1.** (1) A \( \lambda \)-small coproduct of barely \( \lambda \)-presentable objects is barely \( \lambda \)-presentable.

(2) If coproduct injections are monomorphisms then any strong quotient of a barely \( \lambda \)-presentable object is barely \( \lambda \)-presentable. In fact, let \( g : A \to B \) be a strong quotient and \( f : B \to \coprod_{i \in I} K_i \). There is \( J \subseteq I \) of cardinality less than \( \lambda \) such that \( fg \) factorizes through \( \coprod_{j \in J} K_j \).

\[
\begin{array}{ccc}
B & \xrightarrow{f} & \coprod_{i \in I} K_i \\
\downarrow{g} & & \downarrow{u} \\
A & \xrightarrow{} & \coprod_{j \in J} K_j
\end{array}
\]

Since \( u \) is a monomorphism, there is a diagonal \( h : B \to \coprod_{j \in J} K_j \) factorizing \( f \) through \( \coprod_{j \in J} K_j \).

**Lemma 3.2.** Let \( K \) be a barely locally presentable category with pullbacks such that coproduct injections are monomorphisms. Then any object of \( K \) is barely presentable.

**Proof.** Let \( A \) be a strong generator of \( K \) consisting of barely presentable objects. Any object of \( K \) is then a strong quotient of a coproduct of objects from \( A \) (see [3] 0.6 and 0.5). The result follows from [3.1]. \( \square \)

**Remark 3.3.** The following observations will be useful below.

(1) A full subcategory \( A \) of \( K \) is called weakly colimit-dense if \( K \) is the smallest full subcategory of \( K \) containing \( A \) and closed under colimits. Any weakly colimit-dense full subcategory is a strong generator (see [13] 3.7). Conversely, in a cocomplete and strongly co-wellpowered category with pullbacks, any strong generator is weakly colimit-dense (see [7] 3.40 or [13] 3.8).

(2) Compact Hausdorff spaces form a cocomplete, regular and strongly co-wellpowered category with a strong generator which is not barely locally presentable. This follows from [3.2] and the fact that \( 1 \) is not barely presentable in compact Hausdorff spaces.

(3) Recall that a generator \( A \) of \( K \) is strong if and only if for each object \( K \) and each proper subobject of \( K \) there exists a morphism \( A \to K \) with \( A \in A \) which does not factorize through that subobject. If \( K \) has equalizers then this condition implies that \( A \) is a generator. Given two morphisms \( f, g : K \to L \), it suffices to apply this condition to the equalizer of \( f \) and \( g \).

**Proposition 3.4.** Any barely locally presentable category with pullbacks is complete.
Proof. A strong generator in a barely locally presentable category with pullbacks is weakly colimit-dense (see 3.3(1)). Since any strongly co-wellpowered category is weakly co-wellpowered, the result follows from [1] Theorem 4. □

Lemma 3.5. Let \( \mathcal{L} \) be a cocomplete well-powered and strongly co-wellpowered category with (strong epimorphism, monomorphism)-factorization. Let \( \mathcal{A} \) be a set of barely presentable objects in \( \mathcal{L} \). Let \( \mathcal{K} \) consist of strong quotients of coproducts of objects from \( \mathcal{A} \). The \( \mathcal{K} \) is barely locally presentable.

Proof. Following [2] 16.8, \( \mathcal{K} \) is a coreflective full subcategory of \( \mathcal{L} \). Hence it is cocomplete and strongly co-wellpowered. \( \mathcal{A} \) is a strong generator in \( \mathcal{K} \) consisting of barely presentable objects. Thus \( \mathcal{K} \) is barely locally presentable. □

Proposition 3.6. Let \( \mathcal{K} \) be a reflective subcategory of a locally \( \lambda \)-presentable category such that the inclusion \( G : \mathcal{K} \to \mathcal{L} \) is conservative and sends coproduct \( \lambda \)-directed colimits to \( \lambda \)-directed colimits. If \( \mathcal{K} \) is complete, cocomplete and strongly co-wellpowered then it is barely locally \( \lambda \)-presentable.

Proof. Let \( F \) be a left adjoint to \( G \) and consider a \( \lambda \)-presentable object \( L \) in \( \mathcal{L} \). Since \( \mathcal{K}(FL, -) \cong \mathcal{L}(L, G-) \) and \( G \) sends coproduct \( \lambda \)-directed colimits to \( \lambda \)-directed colimits, \( FL \) is barely \( \lambda \)-presentable in \( \mathcal{K} \). We prove that the objects \( FL \), where \( L \) ranges over \( \lambda \)-presentable objects in \( \mathcal{L} \), form a strong generator of \( \mathcal{K} \). We use the argument from the end of the proof of 2.9 in [4], which we repeat. Following 3.3(3), it suffices to show that for every proper subobject \( m : K' \to K \) in \( \mathcal{K} \), there exists a morphism from some \( FL \) to \( K \), where \( L \) is \( \lambda \)-presentable in \( \mathcal{L} \), not factorizing through \( m \). We know that \( Gm \) is a monomorphism but not an isomorphism because \( G \) is conservative. Since \( \mathcal{L} \) is locally \( \lambda \)-presentable, there exists a morphism \( p : L \to GK \), \( L \) \( \lambda \)-presentable in \( \mathcal{L} \), that does not factorize through \( Gm \). The corresponding \( \tilde{p} : FL \to K \) does not factorize through \( m \). □

Corollary 3.7. Let \( \mathcal{K} \) be a barely locally \( \lambda \)-presentable category and \( \mathcal{C} \) be a small category. Then the functor category \( \mathcal{K}^\mathcal{C} \) is barely locally \( \lambda \)-presentable.

Proof. \( \mathcal{K}^\mathcal{C} \) is complete and cocomplete (with limits and colimits calculated pointwise). It is easy to see that \( \varphi : P \to Q \) is a strong epimorphism in \( \mathcal{K}^\mathcal{C} \) if and only if \( \varphi_C : PC \to QC \) is a strong epimorphism on \( \mathcal{K} \) for each \( C \) in \( \mathcal{C} \). Thus \( \mathcal{K}^\mathcal{C} \) is strongly co-wellpowered. Let \( \mathcal{A} \) be a strong generator of \( \mathcal{K} \) consisting of barely \( \lambda \)-presentable objects. Following 2.5 \( E_A : \mathcal{K} \to \text{Set}^{\mathcal{C}} \) makes \( \mathcal{K} \) a reflective subcategory of a locally \( \lambda \)-presentable category \( \mathcal{L} = \text{Set}^{\mathcal{C}} \) with the conservative inclusion functor sending coproduct \( \lambda \)-directed colimits to \( \lambda \)-directed colimits. Thus \( \mathcal{K}^\mathcal{C} \) is a reflective subcategory of a locally \( \lambda \)-presentable category \( \mathcal{L}^\mathcal{C} \) with the conservative inclusion functor sending coproduct \( \lambda \)-directed colimits to \( \lambda \)-directed colimits. Following 3.6 \( \mathcal{K}^\mathcal{C} \) is barely locally \( \lambda \)-presentable. □

Lemma 3.8. If \( \lambda_1 \leq \lambda_2 \) then a barely \( \lambda_1 \)-presentable object \( A \) is barely \( \lambda_2 \)-presentable.
Proof. Let $A$ be barely $\lambda_1$-presentable. Then any morphism $f : A \to \coprod_{i \in I} K_i$ factorizes through a subcoproduct $\coprod_{j \in J} K_j$ where $|J| < \lambda_1 \leq \lambda_2$. Assume that we have two factorizations given by $f_k : A \to \coprod_{j \in J} K_j$ where $|J| < \lambda_2$. Each of $f_k$ factorizes through $g_k : A \to \coprod_{j \in J'} K_j$, where $J' \subseteq J$ and $|J'| < \lambda_1$ and $k = 1, 2$. There is $J' \subseteq J'' \subseteq I$ such that $|J''| < \lambda_1$ and $\coprod_{j' \in J''} K_j \rightarrow \coprod_{j \in J} K_j$ coequalizes $g_1$ and $g_2$. Thus $\coprod_{j' \in J''} K_j$ coequalizes $f_1$ and $f_2$. □

Corollary 3.9. Let $\lambda_1 \leq \lambda_2$ be regular cardinals. Then any barely $\lambda_1$-presentable category is barely $\lambda_2$-presentable.

Lemma 3.10. Any barely locally $\lambda$-presentable category has coproduct $\lambda$-directed unions.

Proof. Let $f : \coprod_{I} K_i \to K$ be a morphism whose compositions $f_J : \coprod_{j \in J} K_j \to K$ with the subcoproduct injections $\coprod_{j \in J} K_j \to \coprod_{I} K_i$ are monomorphisms for all $J \subseteq I$ of cardinality $< \lambda$. It suffices to show that $u = v$ for any $u, v : A \to \coprod_{I} K_i$ such that $fu = fv$ and $A$ is barely $\lambda$-presentable. Since $u$ and $v$ factorize through a subcoproduct injection $\coprod_{j \in J} K_j \to \coprod_{I} K_i$, we have $fu = f_j u'$ and $fv = f_j v'$ for $u', v' : A \to \coprod_{j} K_j$. Since $f_J$ is a monomorphism, $u' = v'$ and thus $u = v$. □

Theorem 3.11. Let $K$ be a locally presentable category such that $K^{op}$ is barely locally presentable. Then $K$ is equivalent to a complete lattice.

Proof. It follows from 3.10 and 2.13. □

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