General explicit expressions for intertwining operators and direct rotations of two orthogonal projections

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Abstract. In this paper, based on the block operator technique and operator spectral theory, the general explicit expressions for intertwining operators and direct rotations of two orthogonal projections have been established. As a consequence, it is an improvement of Kato’s result (Perturbation Theory of Linear operators, Springer-Verlag, Berlin/Heidelberg, 1996); J. Avron, R. Seiler and B. Simon’s Theorem 2.3 (The index of a pair of projections, J. Funct. Anal. 120(1994) 220-237) and C. Davis, W.M. Kahan, (The rotation of eigenvectors by a perturbation, III. SIAM J. Numer. Anal. 7(1970) 1-46).

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1. Introduction

Let \( \mathcal{H} \) be a Hilbert space and \( \mathcal{B}(\mathcal{H}) \) the space of all bounded linear operators on \( \mathcal{H} \). An operator \( P \) is called an orthogonal projection if \( P = P^* = P^2 \). Let \( \mathcal{P} \) be the set of all orthogonal projections in \( \mathcal{B}(\mathcal{H}) \). As well-known, orthogonal projections on a Hilbert space are basic objects of study in operator theory (see [1-19] and therein references). Orthogonal projections appear in various problems and in many different areas, pure or applied. In this paper, we will pay attention on the characterization to intertwining operators and direct rotations of two orthogonal projections. Let the set of all unitaries in \( \mathcal{B}(\mathcal{H}) \) be denoted by \( \mathcal{U}(\mathcal{H}) \). If \( P \) and \( Q \) are orthogonal projections and there exists a unitary \( U \in \mathcal{U}(\mathcal{H}) \) such that

\[
UP = QU, 
\]

then \( U \) is called an outer intertwining operator of \( P \) and \( Q \). The set of all outer intertwining operators of a pair \((P, Q)\) of orthogonal projections is denoted by

\[ \text{out} \mathcal{U}_Q(P). \]

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Similarly, if
\[ PU = UQ, \tag{2} \]
then \( U \) is called an inner intertwining operator of \( P \) and \( Q \). The set of all inner intertwining operators of a pair \((P, Q)\) of orthogonal projections is denoted by
\[ \text{inn}\mathcal{U}_Q(P). \]
Moreover, if both of
\[ PU = UQ \text{ and } UP = QU \tag{3} \]
hold, then \( U \) is called an intertwining operator of \( P \) and \( Q \). The set of all intertwining operators of a pair \((P, Q)\) of orthogonal projections is denoted by
\[ \text{int}\mathcal{U}_Q(P). \]

For a pair \((P, Q)\) of orthogonal projections. A unitary \( U \in \mathcal{U}({\mathcal{H}}) \) is called a direct rotation from \( P \) to \( Q \) (see [1] and [10]) if
\[ UP = QU, U^2 = (Q^\perp - Q)(P^\perp - P), \text{Re}U \geq 0, \tag{4} \]
where \( K^\perp = I - K \) if \( K \) is an orthogonal projection.

If \( P \) and \( Q \) are orthogonal projections with \( \| P - Q \| < 1 \), Kato in [13] verified that there exists \( U \in \mathcal{U}({\mathcal{H}}) \) such that \( PU = UQ \). Moreover, Avron, Seiler and Simon ([6]) proved that if \( P \) and \( Q \) are orthogonal projections on \( \mathcal{H} \) with \( \| P - Q \| < 1 \), then there exists a unitary \( U \in \mathcal{U}({\mathcal{H}}) \) with \( UPU^* = Q, UQU^* = P \). If \( P \) and \( Q \) are orthogonal projections have no common eigenvectors, the mine result shown by Amrèin, Sinha ([2]) implies that there exists a self-adjoint intertwining operator of \((P, Q)\). For a pair \((P, Q)\) of orthogonal projections, we ([19]) provided a sufficient and necessary condition that there exists an intertwining operator of \((P, Q)\). More recently, Simon ([18]) presented a more elegant proof of our previous result. In the present paper, we will give another alternative proof of the sufficient and necessary condition for the existence of intertwining operator of \((P, Q)\). The proof is more geometrical compared with the proof in [18], and we believe the block operator technique used here has meaning in itself.

For the sake of convenience, we need some notation and terminologies. For \( A \in \mathcal{B}(\mathcal{H}) \), the range, the null space, the spectrum, the real part and the adjoint of \( A \) denote by \( \mathcal{R}(A), \mathcal{N}(A), \sigma(A), \text{Re}A \) and \( A^* \), respectively. \( A \) is said to be positive if \( (Ax, x) \geq 0 \) for \( x \in \mathcal{H} \). If \( A \) is positive, then \( A^{\frac{1}{2}} \) denotes the positive square root of \( A \). The \( A \in \mathcal{B}(\mathcal{H}) \) is said to be normal if \( A^*A = AA^* \). If \( A \) is normal, then there exists a spectral representation \( A = \int_{\sigma(A)} \lambda dE_\lambda \). Let \( A = U(A^*A)^\frac{1}{2} \) be the polar decomposition of \( A \). If \( \mathcal{R}(A) = \mathcal{H} \) and \( \mathcal{R}(A^*) = \mathcal{H} \), then \( U \) in the polar decomposition of \( A \) can be chosen as a unitary. An operator \( U \) is said to be unitary if \( U^*U = UU^* = I \), where \( I \) is the identity on \( \mathcal{H} \).

The following lemma is a starting point and a very useful tool in the sequel.

Lemma 1.1. ([11], [19]) If \( W \) and \( L \) are two closed subspaces of \( \mathcal{H} \) and \( P \) and \( Q \) denote the orthogonal projections on \( W \) and \( L \), respectively, then \( P \) and \( Q \) have the operator matrices
\[ P = I_1 \oplus I_2 \oplus 0I_3 \oplus 0I_4 \oplus I_5 \oplus 0I_6 \tag{5} \]
and
\[ Q = I_1 \oplus 0I_2 \oplus I_3 \oplus 0I_4 \oplus \begin{pmatrix} Q_0 & Q_0^\frac{1}{2}(I_5 - Q_0)^{\frac{1}{2}}D \\ D^*Q_0^\frac{1}{2}(I_5 - Q_0)^{\frac{1}{2}} & D^*(I_5 - Q_0)D \end{pmatrix} \tag{6} \]
with respect to the space decomposition $\mathcal{H} = \bigoplus_{i=1}^{6} \mathcal{H}_i$, respectively, where $\mathcal{H}_1 = W \cap L$, $\mathcal{H}_2 = W \cap L^\perp$, $\mathcal{H}_3 = W^\perp \cap L$, $\mathcal{H}_4 = W^\perp \cap L^\perp$, $\mathcal{H}_5 = W \ominus (\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $\mathcal{H}_6 = \mathcal{H} \ominus (\bigoplus_{j=1}^{5} \mathcal{H}_j)$, $Q_0$ is a positive contraction on $\mathcal{H}_5$, 0 and 1 are not eigenvalues of $Q_0$, and $D$ is a unitary from $\mathcal{H}_6$ onto $\mathcal{H}_5$. $I_i$ is the identity on $\mathcal{H}_i$, $i = 1, \ldots, 6$.

**Remark 1.2.** From Lemma 1.1, we will get more information involving with geometry structure between $P$ and $Q$. For example,

1. Since $DD^* = I_5$ and $D^* D = I_6$, it implies that $\dim \mathcal{H}_5 = \dim \mathcal{H}_6$, where $\dim M$ denotes the dimension of a subspace $M$.

2. If $0$ (or 1) $\in \sigma(Q_0)$, then $0$ (or 1) is a limit point of $\sigma(Q_0)$ and $0 \notin \sigma_p(Q_0)$, where $\sigma_p(T)$ denotes the point spectrum of $T$. In this case, $\dim \mathcal{H}_5 = \dim \mathcal{H}_6 = \infty$. If $\dim \mathcal{H}_5 < \infty$, then $0, 1 \notin \sigma(Q_0)$.

If $\mathcal{H}_i = \{0\}, i = 1, 2, 3, 4$, Halmos ([12]) called that the pair $(P, Q)$ is in the generic position. If two orthogonal projections are in the generic position, then $\mathcal{H} = \mathcal{H}_5 \oplus \mathcal{H}_6$ and the operator matrices (5) and (6) of $P$ and $Q$ can be simplified as follows

$$P = I_5 \oplus 0 I_6, Q = \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} & D^*(I_5 - Q_0)D \end{pmatrix}, \quad (7)$$

respectively. In general, for a pair $(P, Q)$ of orthogonal projections with operator matrices as (5) and (6), denote $\tilde{P}$ and $\tilde{Q}$ by

$$\tilde{P} = I_5 \oplus 0 I_6, \tilde{Q} = \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} & D^*(I_5 - Q_0)D \end{pmatrix}, \quad (8)$$

the pair $(\tilde{P}, \tilde{Q})$ as the restriction of $(P, Q)$ on $\mathcal{H}_5 \oplus \mathcal{H}_6$ is called the generic part of $(P, Q)$.

Let us give a brief outline of the contents of this paper. The general explicit expressions for outer intertwining operators and intertwining operators of two orthogonal projections in the generic position are stated in Section 2. In Section 3, based on block operator technique and spectral theory we give an alternative proof of the sufficient and necessary condition that there exists an intertwining operator of a pair $(P, Q)$, and the operator matrices $P$ and $Q$ can be simplified as follows

$$P = I_5 \oplus 0 I_6, Q = \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} & D^*(I_5 - Q_0)D \end{pmatrix}, \quad (7)$$

respectively. In general, for a pair $(P, Q)$ of orthogonal projections with operator matrices as (5) and (6), denote $\tilde{P}$ and $\tilde{Q}$ by

$$\tilde{P} = I_5 \oplus 0 I_6, \tilde{Q} = \begin{pmatrix} Q_0 & Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}}D \\ D^*Q_0^{\frac{1}{2}}(I_5 - Q_0)^{\frac{1}{2}} & D^*(I_5 - Q_0)D \end{pmatrix}, \quad (8)$$

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2. **General explicit expressions of intertwining operators for a pair $(P, Q)$ in the generic position**

For outer (or inner ) intertwining operators and intertwining operators of a pair of orthogonal projections, we have:

**Theorem 2.1.** Let $P$ and $Q$ be two orthogonal projections in the generic position and $P$ and $Q$ have operator matrix forms (7). Then

(a) out$U_Q(P) =$ \[ \left\{ \begin{pmatrix} Q_0^{\frac{1}{2}} & (I_5 - Q_0)^{\frac{1}{2}}D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & -D^*Q_0^{\frac{1}{2}}D \end{pmatrix} : U_0 \in U(\mathcal{H}_5), S_0 \in U(\mathcal{H}_6) \right\} . \]

(b) int$U_Q(P) =$ \[ \left\{ \begin{pmatrix} Q_0^{\frac{1}{2}} & (I_5 - Q_0)^{\frac{1}{2}}D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & -D^*Q_0^{\frac{1}{2}}D \end{pmatrix} : U_0 \in U(\mathcal{H}_5), U_0Q_0 = Q_0U_0 \right\} . \]
Proof. We define an operator $W_0$ by the operator matrix

$$W_0 = \left( \begin{array}{cc} Q_0^\frac{1}{2} & (I_5 - Q_0)^\frac{1}{2}D_p \\ D^*(I_5 - Q_0)^\frac{1}{2} & -D^*Q_0^\frac{1}{2}D \end{array} \right)$$

(9)

with the decomposition $\mathcal{H} = \mathcal{H}_5 \oplus \mathcal{H}_6$. By direct computation, $W_0$ is a unitary on $\mathcal{H}$ with $WP = QW$, and hence $W_0 \in \text{int}\mathcal{U}_Q(P) \subset \text{out}\mathcal{U}_Q(P)$.

Note that for any $V \in \text{out}\mathcal{U}_Q(P)$, $W_0^*V$ is a unitary commutes with $P$, and for any $V \in \text{int}\mathcal{U}_Q(P)$, $W_0^*V$ is a unitary commutes with both $P$ and $Q$. We obtain

$$\text{out}\mathcal{U}_Q(P) = \{W_0U : U \in \mathcal{U}(\mathcal{H}) \text{ with } UP = PU\},$$

and

$$\text{int}\mathcal{U}_Q(P) = \{W_0U : U \in \mathcal{U}(\mathcal{H}) \text{ with } UP = PU, UQ = QU\}.$$

Let $U \in \mathcal{U}(\mathcal{H})$ and $U$ has the operator matrix

$$U = \left( \begin{array}{cc} U_{11} & U_{12} \\ U_{21} & U_{22} \end{array} \right)$$

with the decomposition $\mathcal{H} = \mathcal{H}_5 \oplus \mathcal{H}_6$.

If $UP = PU$, then $U_{12} = U_{21} = 0$, and $U$ has the operator matrix

$$U = \left( \begin{array}{cc} U_{11} & 0 \\ 0 & U_{22} \end{array} \right)$$

(10)

in which $U_{11} \in \mathcal{U}(\mathcal{H}_5), U_{22} \in \mathcal{U}(\mathcal{H}_6)$.

Moreover, if $UP = PU$ and $UQ = QU$, then we get

$$\left\{ \begin{array}{c} U_{11}Q_0 = Q_0U_{11}, \\ U_{11}Q_0^\frac{1}{2}(I_5 - Q_0)^\frac{1}{2}D = Q_0^\frac{1}{2}(I_5 - Q_0)^\frac{1}{2}DU_{22}. \end{array} \right.$$

(11)

Observing that $Q_0, I_5 - Q_0$ on $\mathcal{H}_5$ are injective from Remark 1.2 and $U_{11}$ commutes with $Q_0^\frac{1}{2}, (I_5 - Q_0)^\frac{1}{2}$. It follow that $U_{11}D = DU_{22}$, and hence $U_{22} = D^*U_{11}D$, $U$ has the operator matrix

$$U = \left( \begin{array}{cc} U_{11} & 0 \\ 0 & D^*U_{11}D \end{array} \right)$$

(12)

in which $U_{11} \in \mathcal{U}(\mathcal{H}_5), U_{11}Q_0 = Q_0U_{11}$. By (9), (10) and (12), we see that $W_0U$ in $\text{out}\mathcal{U}_Q(P)$ and $\text{int}\mathcal{U}_Q(P)$ has the operator form given in (a), (b), respectively.

The proof is completed.

Corollary 2.2. Let a pair $(P,Q)$ of orthogonal projections be in the generic position, and $U \in \text{int}\mathcal{U}_Q(P)$ has the operator matrix form in Theorem 2.1. (b). Then $U$ is self-adjoint if and only if $U_0$ is self-adjoint.

Proof. If $U$ is self-adjoint, then $U_0Q_0 = Q_0U_0$ and $Q_0^\frac{1}{2}U_0$ is self-adjoint. We get

$$U_0Q_0^\frac{1}{2} = Q_0^\frac{1}{2}U_0 = U_0^*Q_0^\frac{1}{2}.$$ 

Hence, $(U_0 - U_0^*)Q_0^\frac{1}{2} = 0$. Observing that the range of $Q_0^\frac{1}{2}$ is dense, it follows that $U_0 = U_0^*$. This shows that $U_0$ is self-adjoint. Conversely, it is obvious that $U$ is self-adjoint.
Remark 2.3. (1) In Theorem 2.1. (a), the operator matrix $U \in \text{out} U(Q)(P)$ can be rewritten as following,
\[
U = \begin{pmatrix}
U_0 & 0 \\
0 & D^*U_0D
\end{pmatrix}
\begin{pmatrix}
Q_0^\frac{1}{2} & (I_5 - Q_0)^\frac{1}{2}D \\
D^*(I_5 - Q_0)^\frac{1}{2} & -D^*Q_0^\frac{1}{2}D
\end{pmatrix}
\] (13)
since $U_0Q_0 = Q_0U_0$.

(2) In Theorem 2.1. (b), the operator matrix $U \in \text{int} U(Q)(P)$ can be rewritten as
\[
U = \begin{pmatrix}
U_0 & 0 \\
0 & S_0
\end{pmatrix}
\begin{pmatrix}
Q_0^\frac{1}{2} & (I_5 - Q_0)^\frac{1}{2}D \\
D^*(I_5 - Q_0)^\frac{1}{2} & -D^*Q_0^\frac{1}{2}D
\end{pmatrix},
\]
where $U_0 \in U(H_5)$ and $S_0 \in U(H_6)$.

3. General explicit expression of intertwining operators for two orthogonal projections

In this section, we will devote to general explicit expressions for intertwining operators of two orthogonal projections if there exists an intertwining operator for the two orthogonal projections.

Let $P$ and $Q$ be two orthogonal projections and have operator matrices (5) and (6), respectively. For the pair $(P, Q)$ of orthogonal projections, if the generic part of $(P, Q)$ is $(\tilde{P}, \tilde{Q})$ as operator matrices (8), then the pair $(\tilde{P}, \tilde{Q})$ as a pair of orthogonal projections on $H_5 \oplus H_6$ is in the generic position.

The main goal in this section is to prove the following theorem.

Theorem 3.1. Let $(P, Q)$ be a pair of orthogonal projections with operator matrices (5) and (6), respectively. There exists a unitary $U \in U(H)$ such that $PU = UQ$ and $UP = QU$ if and only if $\dim \mathcal{R}(P) \cap \mathcal{N}(Q) = \dim \mathcal{N}(P) \cap \mathcal{R}(Q)$. Moreover, if $\dim \mathcal{R}(P) \cap \mathcal{N}(Q) = \dim \mathcal{N}(P) \cap \mathcal{R}(Q)$, then
\[
\text{int} U(Q)(P) = \left\{ U_1 \oplus \begin{pmatrix} 0 & C_2 \\ C_3 & 0 \end{pmatrix} \oplus U_4 \oplus \begin{pmatrix} Q_0^\frac{1}{2} & (I_5 - Q_0)^\frac{1}{2}D \\ D^*(I_5 - Q_0)^\frac{1}{2} & -D^*Q_0^\frac{1}{2}D \end{pmatrix} \begin{pmatrix} U_0 & 0 \\ 0 & D^*U_0D \end{pmatrix} : U_1 \in U(H_1), C_2 \in U(H_3, H_2), C_3 \in U(H_2, H_3), U_4 \in U(H_4), U_0 \in U(H_5), U_0Q_0 = Q_0U_0 \right\}.
\]

Proof. “$\Rightarrow$”. If there exists a unitary $U \in U(H)$ such that $PU = UQ$ and $UP = QU$, then
\[
U(P - Q) = -(P - Q)U.
\] (14)
Denote $A = P - Q$. Then $A$ is a self-adjoint contraction. So that, $\mathcal{N}(A), \mathcal{N}(A - I)$ and $\mathcal{N}(A + I)$ are reduced subspaces of $A$. Take $H_0 = H \oplus (\mathcal{N}(A) \oplus \mathcal{N}(A - I) \oplus \mathcal{N}(A + I))$, then $A$ has the operator matrix
\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & I_1 & 0 & 0 \\
0 & 0 & -I_{-1} & 0 \\
0 & 0 & 0 & A_0
\end{pmatrix}
\] (15)
with respect to the decomposition $H = \mathcal{N}(A) \oplus \mathcal{N}(A - I) \oplus \mathcal{N}(A + I) \oplus H_0$, where $I_1$ is the identity on $\mathcal{N}(A - I)$, $I_{-1}$ is the identity on $\mathcal{N}(A - I)$, $I_0$ is the identity on $H_0$.

It is clear that $A_0$, $A_0 - I_0$ and $A_0 + I_0$ as operators on $H_0$ are injective and dense.
If $U$ has the operator matrix

$$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \\ U_{31} & U_{32} & U_{33} & U_{34} \\ U_{41} & U_{42} & U_{43} & U_{44} \end{pmatrix}$$  \hspace{1cm} (16)$$

with respect to the decomposition $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{N}(A - I) \oplus \mathcal{N}(A + I) \oplus \mathcal{H}_0$, then from (14), we get $UA = -AU$. Moreover, by (15) and (16), we obtain

$$\begin{pmatrix} 0 & U_{12} & -U_{13} & U_{14}A_0 \\ 0 & U_{22} & -U_{23} & U_{24}A_0 \\ 0 & U_{32} & -U_{33} & U_{34}A_0 \\ 0 & U_{42} & -U_{43} & U_{44}A_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -U_{21} & -U_{22} & -U_{23} & -U_{24} \\ U_{31} & U_{32} & U_{33} & U_{34} \\ -A_0U_{41} & -A_0U_{42} & -A_0U_{43} & -A_0U_{44} \end{pmatrix}.$$  \hspace{1cm} (17)$$

Comparing two sides of (17) and observing that $A_0, A_0 - I_0$ and $A_0 + I_0$ are injective and dense, it is derived that $U_{12} = 0, U_{13} = 0, U_{14} = 0, U_{21} = 0, U_{22} = 0, U_{24} = 0, U_{31} = 0, U_{33} = 0, U_{34} = 0, U_{41} = 0, U_{42} = 0, U_{43} = 0$. Therefore,

$$U = \begin{pmatrix} U_{11} & 0 & 0 & 0 \\ 0 & 0 & U_{32} & 0 \\ 0 & U_{32} & 0 & 0 \\ 0 & 0 & 0 & U_{44} \end{pmatrix}$$  \hspace{1cm} (18)$$

This shows that $U_{11}$ is a unitary on $\mathcal{N}(A) = \mathcal{H}_1 \oplus \mathcal{H}_4$, $\begin{pmatrix} 0 & U_{23} \\ U_{32} & 0 \end{pmatrix}$ is a unitary on $\mathcal{N}(A - I) \oplus \mathcal{N}(A + I) = (\mathcal{R}(P) \cap \mathcal{N}(Q)) \oplus (\mathcal{N}(P) \cap \mathcal{R}(Q))$ and $U_{44}$ is a unitary on $\mathcal{H}_0 = \mathcal{H}_5 \oplus \mathcal{H}_6$. Observing that $U_{11}P_{\mathcal{H}_1 \oplus \mathcal{H}_4} = Q_{\mathcal{H}_1 \oplus \mathcal{H}_4}U_{11}$ and $U_{11}, P_{\mathcal{H}_1 \oplus \mathcal{H}_4}$ and $Q_{\mathcal{H}_1 \oplus \mathcal{H}_4}$ have operator matrices

$$U_{11} = \begin{pmatrix} \frac{U_{11}}{U_{11}} & \frac{U_{12}}{U_{11}} & \frac{U_{13}}{U_{11}} & \frac{U_{14}}{U_{11}} \end{pmatrix}, P_{\mathcal{H}_1 \oplus \mathcal{H}_4} = Q_{\mathcal{H}_1 \oplus \mathcal{H}_4} = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathcal{H}_1 \oplus \mathcal{H}_4$, respectively, from $UP = QU$ we get

$$\begin{pmatrix} \frac{U_{11}}{U_{11}} & 0 \\ 0 & \frac{U_{11}}{U_{11}} \end{pmatrix} = \begin{pmatrix} \frac{U_{11}}{U_{11}} & \frac{U_{12}}{U_{11}} \\ 0 & \frac{U_{11}}{U_{11}} \end{pmatrix}.$$ 

Hence, $U_{11}^{12} = 0$ and $U_{11}^{21} = 0$. Therefore, $U_{11}^{11}$ and $U_{22}^{22}$ are unitaries on $\mathcal{H}_1$ and $\mathcal{H}_4$, respectively. Observing that $U_{\mathcal{H}_2 \oplus \mathcal{H}_3} = \begin{pmatrix} 0 & U_{23} \\ U_{32} & 0 \end{pmatrix}$ and $U_{\mathcal{H}_2 \oplus \mathcal{H}_3}$ is a unitary, we have

$$\begin{pmatrix} 0 & U_{23} \\ U_{32} & 0 \end{pmatrix} \begin{pmatrix} 0 & U_{32}^{*} \\ U_{23}^{*} & 0 \end{pmatrix} = \begin{pmatrix} U_{23}U_{23}^{*} & 0 \\ 0 & U_{32}U_{32}^{*} \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & I_3 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & U_{32}^{*} \\ U_{23}^{*} & 0 \end{pmatrix} \begin{pmatrix} 0 & U_{23} \\ U_{32} & 0 \end{pmatrix} = \begin{pmatrix} U_{32}^{*}U_{32} & 0 \\ 0 & U_{23}^{*}U_{23} \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & I_3 \end{pmatrix}.$$ 

Thus $U_{32}^{*}U_{32} = I_2$ and $U_{23}^{*}U_{23} = I_3$. It implies that $\dim \mathcal{H}_2 = \dim \mathcal{H}_3$ and $U_{23}$ is a unitary from $\mathcal{H}_3$ onto $\mathcal{H}_2$. Similarly, $U_{32}$ is a unitary from $\mathcal{H}_2$ onto $\mathcal{H}_3$. 

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Next, from $UP = QU$ and $UQ = PU$, we have

$$U_{44}P_{H_5 \oplus H_6} = Q_{H_5 \oplus H_6}U_{44} \quad \text{and} \quad U_{44}Q_{H_5 \oplus H_6} = P_{H_5 \oplus H_6}U_{44}.$$  

By Theorem 2.1,

$$U_{44} = \begin{pmatrix} Q_0^1 & (I_5 - Q_0)^{1/2}D \\ D^*(I_5 - Q_0)^{1/2} & -D^*Q_0^{3/2}D \end{pmatrix} \begin{pmatrix} U_0 & 0 \\ 0 & D^*U_0D \end{pmatrix},$$

where $U_0$ with $U_0Q_0 = Q_0U_0$ is a unitary on $H_5$.

“$\Leftarrow$”. If $\dim \mathcal{R}(P) \cap \mathcal{N}(Q) = \dim \mathcal{N}(P) \cap \mathcal{R}(Q)$, we can choose a unitary $C_2$ from $\mathcal{H}_3$ onto $\mathcal{H}_2$ and a unitary $C_3$ from $\mathcal{H}_2$ onto $\mathcal{H}_3$. Define an operator

$$U = U_1 \oplus \begin{pmatrix} 0 & C_2 \\ C_3 & 0 \end{pmatrix} \oplus U_4 \oplus \begin{pmatrix} Q_0^1 & (I_5 - Q_0)^{1/2}U_0D \\ D^*(I_5 - Q_0)^{1/2}U_0 & -D^*Q_0^{3/2}U_0D \end{pmatrix},$$

where $U_1$ is a unitary on $\mathcal{H}_1$, $C_2$ is a unitary from $\mathcal{H}_3$ onto $\mathcal{H}_2$, $C_3$ is a unitary from $\mathcal{H}_2$ onto $\mathcal{H}_3$, $U_4$ is a unitary on $\mathcal{H}_4$ and $U_0$ is a unitary on $\mathcal{H}_5$ with $Q_0U_0 = U_0Q_0$, by directly checking, $U$ is a unitary on $\mathcal{H}$ and $UP = QU$ and $UQ = PU$.

From the proof above, we have

$$\text{int}U_Q(P) = \left\{ U_1 \oplus \begin{pmatrix} 0 & C_2 \\ C_3 & 0 \end{pmatrix} \oplus U_4 \oplus \begin{pmatrix} Q_0^1 & (I_5 - Q_0)^{1/2}D \\ D^*(I_5 - Q_0)^{1/2} & -D^*Q_0^{3/2}D \end{pmatrix} : \begin{pmatrix} U_0 & 0 \\ 0 & D^*U_0D \end{pmatrix} \right\}_{U_1 \in \mathcal{U}(\mathcal{H}_1), C_2 \in \mathcal{U}(\mathcal{H}_3, \mathcal{H}_2), C_3 \in \mathcal{U}(\mathcal{H}_2, \mathcal{H}_3), U_4 \in \mathcal{U}(\mathcal{H}_4), U_0 \in \mathcal{U}(\mathcal{H}_5), U_0Q_0 = Q_0U_0}.$$  

**Remark 3.2.** Let $(P, Q)$ be a pair of orthogonal projections. From the proof of Theorem 3.1, if $\dim \mathcal{R}(P) \cap \mathcal{N}(Q) = \dim \mathcal{N}(P) \cap \mathcal{R}(Q)$, then the intertwining operator of the pair of orthogonal projections is not unique. Moreover, it can be chosen as a self-adjoint unitary. Even though the intertwining operator can be chosen as a self-adjoint operator, it is also not unique by Corollary 2.2.

As a consequence, we give an alternative proof of Theorem 2.2 in [15].

**Corollary 3.3.** (Theorem 2.2 in [15]) Let $L$ and $M$ be subspaces of $\mathcal{H}$. If $P$ and $Q$ are orthogonal projections on $L$ and $M$, respectively, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that

$$PQP = UQPQU^*.$$  

**Proof.** Let $P$ and $Q$ have operator matrices (5) and (6), respectively. Then

$$PQP = I_1 \oplus 0I_2 \oplus 0I_3 \oplus 0I_4 \oplus Q_0 \oplus 0I_6$$

and

$$QPQ = I_1 \oplus 0I_2 \oplus 0I_3 \oplus 0I_4 \oplus \begin{pmatrix} Q_0^2 & Q_0^3(I_5 - Q_0)^{1/2}D \\ D^*Q_0^3(I_5 - Q_0)^{1/2} & D^*Q_0^3(I_5 - Q_0)D \end{pmatrix}.$$  

Denote the generic part $(\tilde{P}, \tilde{Q})$ of $(P, Q)$ as the operator matrices (8). We get

$$\tilde{P}\tilde{Q}\tilde{P} = \begin{pmatrix} Q_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{Q}\tilde{P}\tilde{Q} = \begin{pmatrix} Q_0^2 & Q_0^3(I_5 - Q_0)^{1/2}D \\ D^*Q_0^3(I_5 - Q_0)^{1/2} & D^*Q_0^3(I_5 - Q_0)D \end{pmatrix}.$$
By Theorem 2.1, there exists a unitary $\tilde{U}$ on $\mathcal{H}_5 \oplus \mathcal{H}_6$ such that
\[ \tilde{U} \bar{P} \bar{U}^* = \bar{Q}, \tilde{U} \bar{Q} \tilde{U}^* = \bar{P}. \] (20)

In this case,
\[ \bar{P} \bar{Q} \bar{P} = \bar{U} \bar{Q} \bar{U}^* \bar{P} \bar{U}^* \bar{Q} \bar{U}^* = \bar{U} \bar{Q} \bar{P} \bar{Q} \bar{U}^*. \]

Furthermore, define $U$ by
\[ U = \oplus_{i=1}^4 I_i \oplus \tilde{U}, \] (21)
where $I_i$ are identities on $\mathcal{H}_i$, $1 \leq i \leq 4$.

Evidently, $U$ is a unitary, and $Q = UPU^*$ and $P = UQU^*$. Hence,
\[ PQP = I_1 \oplus 0I_2 \oplus 0I_3 \oplus 0I_4 \oplus Q_0 \oplus 0I_6 = I_1 \oplus 0I_2 \oplus 0I_3 \oplus 0I_4 \oplus \bar{P} \bar{Q} \bar{P} \]
\[ = I_1 \oplus 0I_2 \oplus 0I_3 \oplus 0I_4 \oplus \bar{U} \bar{Q} \bar{P} \bar{Q} \bar{U}^* \]
\[ = UQPUQ^*. \]

**Remark 3.4.** (1) By Theorem 2.1 and Theorem 3.1, a unitary satisfying (19) is not unique.
(2) $U$ in Corollary 3.3 can by chose as a self-adjoint unitary. Even so this choice is not unique by Corollary 2.2.

### 4. General explicit expression of direct rotations on a pair of orthogonal projections

The concept of a direct rotation of a pair on orthogonal projections due to Davis (see [10]).

**Definition 4.1.** (Definition 2.9 in [1], Definition 3.1 in [10]) Let $(P, Q)$ be a pair of orthogonal projections. A unitary $S \in \mathcal{U}(\mathcal{H})$ is called a direct rotation from $P$ to $Q$ (see [10]) if
\[ SP = QS, S^2 = (Q^\perp - Q)(P^\perp - P), \Re S \geq 0. \]

For a pair $(P, Q)$ of orthogonal projections, denote the set of all direct rotations from $P$ to $Q$ by
\[ S_Q(P) = \{ S \in \mathcal{U}(\mathcal{H}) : SP = QS, S^2 = (Q^\perp - Q)(P^\perp - P), \Re S \geq 0 \}. \]

**Lemma 4.2.** (Proposition 3.1 in [10]) If a pair $(P, Q)$ of orthogonal projections is in the generic position, then there exists a unique unitary operator $S$ such that
\[ SP = QS, S^2 = (Q^\perp - Q)(P^\perp - P), \Re S \geq 0. \] (22)

Moreover, if $P$ and $Q$ have the operator matrices (7), then
\[ S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} Q_0^{\frac{1}{2}} & -(I_5 - Q_0)^{\frac{1}{2}}D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & D^*Q_0^{\frac{1}{2}}D \end{pmatrix}. \] (23)

**Proof.** If there exists a unitary operator $S$ satisfying (22), then from $SP = QS$ we get
\[ SP^\perp = Q^\perp S, S^*Q = PS^*, S^*Q^\perp = P^\perp S^*. \] (24)

Hence, from $S^2 = (Q^\perp - Q)(P^\perp - P)$, we obtain
\[ S = S^*(Q^\perp - Q)(P^\perp - P) = (P^\perp - P)S^*(P^\perp - P). \] (25)
Let $P$, $Q$ and $S$ have operator matrices

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} Q_0 & \frac{1}{2}Q_0(I_5 - Q_0)D \\ D^*Q_0(I_5 - Q_0)^{\frac{1}{2}} & D^*(I_5 - Q_0)D \end{pmatrix}, S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

with respect to the decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{N}(P)$, respectively. From (25),

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} S_{11}^* & -S_{21}^* \\ -S_{12}^* & S_{22}^* \end{pmatrix}.$$ 

So that,

$$\begin{cases} S_{11} = S_{11}^*, \\ S_{12} = -S_{21}^*, \\ S_{21} = -S_{12}^*, \\ S_{22} = S_{22}^*. \end{cases} \quad (26)$$

Hence,

$$S = \begin{pmatrix} S_{11} & S_{12} \\ -S_{12}^* & S_{22} \end{pmatrix} = \begin{pmatrix} S_{11} & -S_{21}^* \\ S_{21} & S_{22} \end{pmatrix}.$$ 

Moreover,

$$\text{Re}S = \frac{1}{2}(S + S^*) = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix} \geq 0.$$ 

In general, by Theorem 2.1, there exist two unitaries $U_0, V_0 \in \mathcal{U}(\mathcal{R}(P))$ such that

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}Q_0^\frac{1}{2} & (I_5 - Q_0)^{\frac{1}{2}}D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & -D^*Q_0^{\frac{1}{2}}D \end{pmatrix} \begin{pmatrix} U_0 & 0 \\ 0 & D^*V_0D \end{pmatrix}.$$ 

So that,

$$Q_0^{\frac{1}{2}}U_0 = S_{11} \geq 0 \quad (27)$$ 

and

$$-D^*Q_0^{\frac{1}{2}}V_0D = S_{22} \geq 0. \quad (28)$$

Since $Q_0$ is injective, by (27) and (28), it is clear that $U_0 = I_{\mathcal{R}(P)}$, $V_0 = -I_{\mathcal{R}(P)}$. Therefore,

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}Q_0^\frac{1}{2} & -(I_5 - Q_0)^{\frac{1}{2}}D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & D^*Q_0^{\frac{1}{2}}D \end{pmatrix},$$ 

it is uniquely determined.

If

$$S = \begin{pmatrix} \frac{1}{2}Q_0^\frac{1}{2} & -(I_5 - Q_0)^{\frac{1}{2}}D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & D^*Q_0^{\frac{1}{2}}D \end{pmatrix},$$

by directly checking, $S$ satisfies (22). It is the direct rotation from $P$ to $Q$.

**Theorem 4.3.** (Proposition 3.2 in [10]) For a pair $(P, Q)$ of orthogonal projections, there exists a direct rotation $S$ from $P$ to $Q$ which satisfies (22) if and only if $\dim \mathcal{R}(P) \cap \mathcal{N}(Q) = \dim \mathcal{N}(P) \cap \mathcal{R}(Q)$.
Moreover, if $P$ and $Q$ with $\dim \mathcal{R}(P) \cap \mathcal{N}(Q) = \dim \mathcal{N}(P) \cap \mathcal{R}(Q)$ have the operator matrices (5) and (6) with respect to the space decomposition $\mathcal{H} = \oplus_{i=1}^{6} \mathcal{H}_i$, then

$$S = I_1 \oplus \left( \begin{array}{cc} 0 & C^* \\ -C & 0 \end{array} \right) \oplus I_4 \oplus \left( \begin{array}{cc} Q_0^{\frac{1}{2}} & -(I_5 - Q_0)^{\frac{1}{2}}D \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & D^*Q_0^{\frac{1}{2}}D \end{array} \right), \tag{29}$$

where $C$ is an arbitrary unitary from $\mathcal{H}_3$ onto $\mathcal{H}_2$.

**Proof.** Denote $\mathcal{K}_1 = (\mathcal{R}(P) \cap \mathcal{R}(Q)) \oplus (\mathcal{N}(P) \cap \mathcal{N}(Q))$, $\mathcal{K}_2 = (\mathcal{R}(P) \cap \mathcal{N}(Q)) \oplus (\mathcal{N}(P) \cap \mathcal{R}(Q))$ and $\mathcal{K}_3 = \mathcal{H} \ominus (\mathcal{K}_1 \oplus \mathcal{K}_2)$.

For $x_1 \in \mathcal{K}_1$, we get

$$S^2x_1 = (Q \perp - Q)(P \perp - P)x_1 = x_1. \tag{30}$$

From (30), we obtain that $(S^2 - I)x_1 = (S + I)(S - I)x_1 = 0$. Moreover, observing that $S + I$ is invertible since $\text{Re}S \geq 0$, we get $(S - I)x_1 = 0$. Hence,

$$Sx_1 = x_1.$$  

This shows that $\mathcal{K}_1$ is a reduced subspace under $S$ and $S \mid_{\mathcal{K}_1}$ is the identity on $\mathcal{K}_1$.

For any $y \in \mathcal{K}_2$, denote $y = y_1 + y_2$, where $y_1 \in \mathcal{R}(P) \cap \mathcal{N}(Q)$ and $y_2 \in \mathcal{N}(P) \cap \mathcal{R}(Q)$, we shall show that $Sy \in \mathcal{K}_2$. Observing that $Sy_1 = SPy_1 = QSx_1 \in \mathcal{R}(Q)$ and $Sy_1 = SQ \perp y_1 = P \perp Sy_1 \in \mathcal{N}(P)$, we have

$$Sy_1 \in \mathcal{N}(P) \cap \mathcal{R}(Q).$$

Similarly,

$$Sy_2 \in \mathcal{R}(P) \cap \mathcal{N}(Q).$$

Hence,

$$Sy = Sy_1 + Sy_2 \in \mathcal{K}_2.$$  

This shows that $\mathcal{K}_2$ is an invariant subspace of $S$. In this case, $S$ has the operator matrix

$$S = \begin{pmatrix} I_{\mathcal{K}_1} & 0 & 0 \\ 0 & S_{22} & S_{23} \\ 0 & 0 & S_{33} \end{pmatrix} \tag{31}$$

with respect to the decomposition $\mathcal{H} = \oplus_{i=1}^{3} \mathcal{K}_i$. Furthermore, since

$$S^2 = \begin{pmatrix} I_{\mathcal{K}_1} & 0 & 0 \\ 0 & S^2_{22} & S_{22}S_{23} + S_{23}S_{33} \\ 0 & 0 & S^2_{33} \end{pmatrix},$$

if $y = y_1 + y_2$, where $y_1 \in \mathcal{R}(P) \cap \mathcal{N}(Q)$ and $y_2 \in \mathcal{N}(P) \cap \mathcal{R}(Q)$, we get $S^2y = S^2(y_1 + y_2) = -y_1 - y_2 = -y$. So that $S^2_{22}y = -y$. This means that

$$S^2_{22} = -I_{\mathcal{K}_2}. \tag{32}$$

It implies that $S_{22}$ is an invertible operator on $\mathcal{K}_2$. Furthermore,

$$S^*S = \begin{pmatrix} I_{\mathcal{K}_1} & 0 & 0 \\ 0 & S^*_{22}S_{22} & S^*_{22}S_{23} \\ 0 & S^*_{23}S_{22} & S^*_{23}S_{23} + S^*_{33}S_{33} \end{pmatrix} = \begin{pmatrix} I_{\mathcal{K}_1} & 0 & 0 \\ 0 & I_{\mathcal{K}_2} & 0 \\ 0 & 0 & I_{\mathcal{K}_3} \end{pmatrix}.$$
It follows that $S_{23}^*=S_{22}=0$. From (32), $\mathcal{R}(S_{22}) = \mathcal{K}_2$, it is derived that $S_{23} = 0$.

So that, the operator matrix form (31) can be changed as follows

$$S = \begin{pmatrix} I_{k_1} & 0 & 0 \\ 0 & S_{22} & 0 \\ 0 & 0 & S_{33} \end{pmatrix}.$$  \hspace{1cm} (33)

Here, $S_{22}$ and $S_{33}$ are unitaries on $\mathcal{K}_2$ and $\mathcal{K}_3$, respectively.

If $P$ and $Q$ have the operator matrices (5) and (6), then it is obvious that $S_{22}$ as a unitary on $\mathcal{K}_2 = \mathcal{H}_2 \oplus \mathcal{H}_3$ has the operator matrix form

$$S_{22} = \begin{pmatrix} 0 & C \\ -C^* & 0 \end{pmatrix}$$

with respect to the decomposition $\mathcal{K}_2 = \mathcal{H}_2 \oplus \mathcal{H}_3$, where $C$ is an arbitrary unitary from $\mathcal{H}_3$ onto $\mathcal{H}_2$. Explicitly, there exists a unitary such as $C$ above if and only if $\dim \mathcal{R}(P) \cap \mathcal{N}(Q) = \dim \mathcal{N}(P) \cap \mathcal{R}(Q)$.

By Lemma 4.2, $S_{33}$ has the operator matrix form

$$S_{33} = \begin{pmatrix} Q_0^{\frac{1}{2}} & -(I_5 - Q_0)^{\frac{1}{2}}D^* \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & D^*Q_0^{\frac{1}{2}}D \end{pmatrix}.$$  \hspace{1cm} (34)

It is uniquely determined. So that,

$$S = I_1 \oplus \begin{pmatrix} 0 & C \\ -C^* & 0 \end{pmatrix} \oplus I_4 \oplus \begin{pmatrix} Q_0^{\frac{1}{2}} & -(I_5 - Q_0)^{\frac{1}{2}}D^* \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & D^*Q_0^{\frac{1}{2}}D \end{pmatrix}. \hspace{1cm} (34)$$

Conversely, if $\dim \mathcal{R}(P) \cap \mathcal{N}(Q) = \dim \mathcal{N}(P) \cap \mathcal{R}(Q) = 0$, for any unitary $C$ from $\mathcal{H}_3$ onto $\mathcal{H}_2$, define an operator $S$ by the form (34), then to directly test the operator $S$ is a unitary which satisfies (22). That is, $S$ is a direct rotation of the pair $(P, Q)$ from $P$ to $Q$.

**Remark 4.4.** (1) There exists a unique unitary $S$ satisfying (22) if and only if $\dim \mathcal{R}(P) \cap \mathcal{N}(Q) = \dim \mathcal{N}(P) \cap \mathcal{R}(Q) = 0$.

(2) For a pair $(P, Q)$ of orthogonal projections, if $\dim \mathcal{R}(P) \cap \mathcal{N}(Q) = \dim \mathcal{N}(P) \cap \mathcal{R}(Q) \neq 0$, then the direct rotation from $P$ to $Q$ is not unique. The general expression of direct rotations $S$ from $P$ to $Q$ has the form (29), where $C$ can be chose over all unitaries from $\mathcal{H}_3$ onto $\mathcal{H}_2$.

(3) For a pair $(P, Q)$ of orthogonal projections with $\dim \mathcal{R}(P) \cap \mathcal{N}(Q) = \dim \mathcal{N}(P) \cap \mathcal{R}(Q)$, if the set of all direct rotations from $P$ to $Q$ is denoted by $S_Q(P)$, then

$$S_Q(P) = \left\{I_1 \oplus \begin{pmatrix} 0 & C \\ -C^* & 0 \end{pmatrix} \oplus I_4 \oplus \begin{pmatrix} Q_0^{\frac{1}{2}} & -(I_5 - Q_0)^{\frac{1}{2}}D^* \\ D^*(I_5 - Q_0)^{\frac{1}{2}} & D^*Q_0^{\frac{1}{2}}D \end{pmatrix} : C \in \mathcal{U}(\mathcal{H}_3, \mathcal{H}_2)\right\}.$$  \hspace{1cm} (35)

(4) It is interesting that if a pair $(P, Q)$ of orthogonal projections with $\dim \mathcal{R}(P) \cap \mathcal{N}(Q) = \dim \mathcal{N}(P) \cap \mathcal{R}(Q)$ and $\mathcal{H}_5 \neq \{0\}$, then

$$\text{int}_Q(P) \cap S_Q(P) = \emptyset.$$  \hspace{1cm} (36)

As the end, we will give an alternative proof of the extremal property in regard to the direct rotation which is due to Davis (see [1],[10]). The proof used block operator matrices and spectral theory may give us some inspiration in the further study.
Theorem 4.5. Let the pair \((P,Q)\) of orthogonal projections be in the generic position. The direct rotation \(U\) from \(P\) to \(Q\) has the extremal property
\[
\| U - I \| = \inf\{\| \tilde{U} - I \| : \tilde{U} \in \mathcal{U}(\mathcal{H}), P = \tilde{U}^*Q\tilde{U}\}.
\]

Proof. Assume that \(P\) and \(Q\) are in the generic position and have the operator matrix (7). From Lemma 4.2 and (23), the direct rotation \(U\) from \(P\) to \(Q\) is unique and
\[
U = \begin{pmatrix}
Q_0^\frac{1}{2} & -(I_5 - Q_0)^{\frac{1}{2}}D \\
D^*(I_5 - Q_0)^{\frac{1}{2}} & D^*Q_0^\frac{1}{2}D
\end{pmatrix}.
\]
Hence,
\[
\begin{align*}
\| U - I \|^2 &= \left\| \begin{pmatrix}
Q_0^\frac{1}{2} - I_5 & -(I_5 - Q_0)^{\frac{1}{2}}D \\
D^*(I_5 - Q_0)^{\frac{1}{2}} & D^*(Q_0^\frac{1}{2} - I_5)D
\end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix}
Q_0^\frac{1}{2} - I_5 & -(I_5 - Q_0)^{\frac{1}{2}}D \\
D^*(I_5 - Q_0)^{\frac{1}{2}} & D^*(Q_0^\frac{1}{2} - I_5)D
\end{pmatrix} \begin{pmatrix}
Q_0^\frac{1}{2} - I_5 & (I_5 - Q_0)^{\frac{1}{2}}D \\
-D^*(I_5 - Q_0)^{\frac{1}{2}} & D^*(Q_0^\frac{1}{2} - I_5)D
\end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix}
2(I_5 - Q_0^\frac{1}{2}) & 0 \\
0 & 2D^*(I_5 - Q_0^\frac{1}{2})D
\end{pmatrix} \right\| \\
&= 2 \| I_5 - Q_0^\frac{1}{2} \|.
\end{align*}
\]
If \(\lambda_0 = \min\{\lambda : \lambda \in \sigma(Q_0)\}\), then
\[
\| I_5 - Q_0^\frac{1}{2} \| = 1 - \lambda_0^\frac{1}{2}.
\]
Thus
\[
\| U - I \| = \sqrt{2(1 - \lambda_0^\frac{1}{2})}.
\]
By Theorem 2.1, if \(Q = \tilde{U}P\tilde{U}^*\), then we have
\[
\tilde{U} = \begin{pmatrix}
Q_0^\frac{1}{2} - V_0^* & (I_5 - Q_0)^{\frac{1}{2}}D \\
D^*(I_5 - Q_0)^{\frac{1}{2}} & -D^*Q_0^\frac{1}{2}D
\end{pmatrix} \begin{pmatrix}
V_0 & 0 \\
0 & DS_0D^*
\end{pmatrix},
\]
where \(V_0, S_0 \in \mathcal{U}(\mathcal{H}_5)\). In this case,
\[
\begin{align*}
\| \tilde{U} - I \|^2 &= \left\| \begin{pmatrix}
Q_0^\frac{1}{2} - V_0^* & (I_5 - Q_0)^{\frac{1}{2}}D \\
D^*(I_5 - Q_0)^{\frac{1}{2}} & -D^*Q_0^\frac{1}{2}D
\end{pmatrix} \begin{pmatrix}
Q_0^\frac{1}{2} - V_0^* & (I_5 - Q_0)^{\frac{1}{2}}D \\
D^*(I_5 - Q_0)^{\frac{1}{2}} & -D^*Q_0^\frac{1}{2}D
\end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix}
Q_0^\frac{1}{2} - V_0^* & (I_5 - Q_0)^{\frac{1}{2}}D \\
D^*(I_5 - Q_0)^{\frac{1}{2}} & -D^*Q_0^\frac{1}{2}D
\end{pmatrix} \begin{pmatrix}
Q_0^\frac{1}{2} - V_0^* & (I_5 - Q_0)^{\frac{1}{2}}D \\
-D^*(I_5 - Q_0)^{\frac{1}{2}} & D^*(Q_0^\frac{1}{2} + S_0^*)D
\end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix}
2I_5 - (V_0Q_0^\frac{1}{2} + Q_0^\frac{1}{2}V_0^*) \\
D^*(2I_5 + Q_0^\frac{1}{2}S_0^* + S_0Q_0^\frac{1}{2}D)
\end{pmatrix} \right\| \\
&\geq \max\{ \| 2I_5 - (Q_0^\frac{1}{2}V_0^* + V_0Q_0^\frac{1}{2}) \|, \| 2I_5 + Q_0^\frac{1}{2}S_0^* + S_0Q_0^\frac{1}{2}D \| \}.
\end{align*}
\]
Without loss of generality, we can assume that $\lambda_0 \in \sigma_p(Q_0)$. Take a unit vector $x_{\lambda_0}$ such that $Q_0 x_{\lambda_0} = \lambda_0 x_{\lambda_0}$. We get

$$
\| 2I_5 - (Q_0^\frac{1}{2} V_0^* + V_0 Q_0^\frac{1}{2}) \| \geq ((2I_5 - (V_0 Q_0^\frac{1}{2} + Q_0^\frac{1}{2} V_0^*)) x_{\lambda_0}, x_{\lambda_0})
$$

$$
= 2 - \lambda_0^\frac{1}{2} ((V_0^* + V_0) x_{\lambda_0}, x_{\lambda_0})
$$

$$
\geq 2(1 - \lambda_0^\frac{1}{2}).
$$

Similarly,

$$
\| 2I_5 + Q_0^\frac{1}{2} S_0^* + S_0 Q_0^\frac{1}{2} \| \geq 2(1 - \lambda_0^\frac{1}{2}).
$$

So that, $\| \tilde{U} - I \| \geq \sqrt{2(1 - \lambda_0^\frac{1}{2})}$. Hence, $\| \tilde{U} - I \| \geq \| U - I \|$. Thus

$$
\| U - I \| = \inf \{ \| \tilde{U} - I \| : \tilde{U} \in U(H), P = \tilde{U}^* Q \tilde{U} \}.
$$

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