SU($\nu$) Calogero spin system and Virasoro-Witt 3-algebra

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Abstract

We investigate the SU($\nu$) Calogero spin system and scale the boost and conserved operators of this integrable many-body system by a scaling parameter. In terms of these scaled operators and appropriate scaling limit, we show that the SU($\nu$) Calogero spin system has an infinite-dimensional 3-algebraic symmetry, i.e., the so-called Virasoro-Witt 3-algebra. The fundamental identity (FI) condition holds for this infinite-dimensional 3-algebra.

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1. Introduction

The infinite-dimensional algebras play a very important role in the study of integrable systems. The intrinsic connection between the Virasoro algebra and the Korteweg-de Vries (KdV) equation was firstly pointed out by Gervais and Neveu [1, 2]. Then it was shown that the $W_N$ algebra is associated to the generalized KdV hierarchies [3, 4]. It is well-known that all the (generalized) KdV hierarchies can be incorporated into the KP hierarchy. One found that the $W_{1+\infty}$ algebra is intrinsically related to the first Hamiltonian structure of the KP hierarchy [5, 6]. Moreover the second Hamiltonian structure of the KP hierarchy was shown to be isomorphic to $\hat{W}_\infty$ [7], a centerless deformation of $W_\infty$. Calogero-Moser-Sutherland (CMS) models [8]-[10] are the one dimensional many-body systems. As the important classical and quantum integrable systems in physics, they have attracted a lot of interesting due to their relevance in various fields, such as quantum Hall effect [11]-[15], two-dimensional QCD [16], fractional statistics [15] [16], quantum chaos and matrix models [17], black hole [18], Seiberg-Witten theory [19] and string theory [20].

The $SU(\nu)$ Calogero spin system or $SU(\nu)$ Calogero-Sutherland-Moser system can be regarded as the generalization of the quantum Calogero model. In this integrable system, all particles have $su(\nu)$ spins as an internal degree of freedom. These particles are free to move on a line interacting through spin-dependent inverse square interactions. Hikami and Wadati [21] found that the $SU(\nu)$ Calogero spin system has the $su(\nu)$ loop algebraic structure. Moreover they pointed out that this system also has a higher symmetry, i.e., $W_{1+\infty}$ algebra. This W algebraic structure unifies the Calogero type and the Sutherland type (trigonometric potential).

Recently 3-algebras have been paid great attention due to a world-volume description of multiple M2-branes proposed by Bagger and Lambert [22], and Gustavsson [23] (BLG). Not as the case of the finite-dimensional 3-algebras, the study of the infinite-dimensional 3-algebras did only make remarkable progress in recent years. The Virasoro-Witt (V-W) algebra is an important infinite-dimensional algebra which is a centerless Virasoro algebra. One found that its 3-algebra is null [24]. Recently Ding et al. [25] investigated the q-deformation of this infinite-dimensional null 3-algebra and constructed a nontrivial q-deformed V-W 3-algebra. It is nothing but a so-called sh-3-Lie algebra. A nontrivial V-W 3-algebra was presented in Ref. [26] through the use of $su(1,1)$ enveloping algebra techniques. In this ternary algebraic structure, there exists a parameter such that it becomes a Nambu 3-algebra only for the special values. Moreover Curtright et al. [27] described in more detail the several V-W 3-algebras. Chakrabortty et al. [27] investigated the $W_\infty$ 3-algebra. Although the fundamental identity (FI) condition [28] [29] was confirmed to fail for this ternary algebra, by applying a double scaling limits on the generators, they derived a $w_\infty$ 3-algebra which satisfies the FI condition. The super $w_\infty$ 3-algebra was constructed in Ref. [30]. It was shown that this super infinite-dimensional 3-algebra satisfies the generalized FI condition. Furthermore a super Nambu-Poisson bracket was proposed there. In terms of
this super Nambu-Poisson bracket, a realization of this super \( \mathfrak{w}_\infty \) 3-algebra was also presented. Beside the infinite-dimensional 3-algebras mentioned above, the Kac-Moody 3-algebra has also been investigated \([31]\).

Quite recently, attempt has been made to establish the connection between the infinite-dimensional 3-algebras and the integrable systems. Chen et al. \([32]\) investigated the classical Heisenberg and \( \mathfrak{w}_\infty \) 3-algebras and established the connection with the dispersionless KdV hierarchy. The bi-Nambu-Hamiltonian structure of the dispersionless KdV system was found. Since the Nambu-Poisson evolution equation involves two Hamiltonians, the deep relationship between these Hamiltonians has also been revealed. Although the progress has been made towards understanding the infinite-dimensional 3-algebras in recent years, to our best knowledge, much less is known about their connections with the integrable systems. In this paper, we reinvestigate the \( SU(\nu) \) Calogero spin system. By means of the boost and conserved operators, we show that this integrable one-dimensional quantum \( N \)-body system has the infinite-dimensional 3-algebraic symmetry, i.e., the so-called V-W 3-algebra.

2. \( SU(\nu) \) Calogero spin system

In this section, we shall recall the \( SU(\nu) \) Calogero spin system that will be useful in what follows. For a more detailed description we refer the reader to Ref.\([21]\).

The \( SU(\nu) \) Calogero spin system is an important integrable system in physics. In this integrable system, all particles have \( su(\nu) \) spins as an internal degree of freedom and move on a line interacting through spin-dependent inverse square interactions. The corresponding Hamiltonian is

\[
H = \sum_{j=1}^{N} p_j^2 + \sum_{1\leq j<k\leq N} 2a^2 - aP_{jk} \frac{2a^2 - aP_{jk}}{(x_j - x_k)^2},
\]

where \( p_j = -i\frac{\partial}{\partial x_j} \) denotes the momentum operator of the \( j \)-th particle, the real parameter \( a \) controls the strength of the interaction, \( P_{jk} \) is a permutation operator which exchanges the spin state of the \( j \)th and the \( k \)th particles. Taking the basis \( t^a_j \) of the \( su(\nu) \) algebra, where the index \( j \) denotes \( t^a_j \) acting on the \( j \)th particle, then the permutation operator \( P_{jk} \) can be expressed as

\[
P_{jk} = \frac{1}{\nu} + 2t^a_j t^a_k.
\]

The integrability of the \( SU(\nu) \) Calogero spin system is guaranteed by the existence of a Lax pair. It consists of two \( N \times N \) operator-valued matrices \( L \) and \( M \),

\[
L_{jk} = \delta_{jk}p_j + (1 - \delta_{jk})i\alpha \frac{P_{jk}}{x_j - x_k},
\]

\[
(M_2)_{jk} = \delta_{jk}2a \sum_{l \neq j} \frac{P_{jl}}{(x_j - x_l)^2} - (1 - \delta_{jk})2a\frac{P_{jk}}{(x_j - x_k)^2}.
\]

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It should be pointed out that the operator $M_2$ satisfies the sum-to-zero condition,
\[
\sum_j (M_2)_{jk} = \sum_k (M_2)_{jk} = 0.
\] (3)

The Lax operators $L$ and $M_2$ satisfy the Lax equation
\[
\frac{d}{dt}L_{jk} = i[H, L_{jk}] = i[L, M_2]_{jk}.
\] (4)

By means of the Lax equation (4) and the sum-to-zero condition (3), it is not hard to verify that
\[
[H, \sum_{j,k} (L^n)_{jk}] = \sum_{j,k} [L^n, M_2]_{jk} = 0.
\] (5)

Thus we have the conserved operators of the $SU(\nu)$ Calogero spin system,
\[
I_n = \sum_{j,k} (L^n)_{jk}, \ n \geq 1.
\] (6)

The first few members of the conserved operators are
\[
I_1 = \sum_j p_j,
\]
\[
I_2 = H = \sum_{j=1}^N p_j^2 + \sum_{1 \leq j < k \leq N} 2 \frac{\alpha^2 - aP_{jk}}{(x_j - x_k)^2},
\]
\[
I_3 = \sum_j (p_j)^3 + \frac{3}{2} \sum_{j \neq k} \frac{\alpha^2 - aP_{jk}}{(x_j - x_k)^2}(p_j + p_k)
\]
\[ - ia^3 \sum_{j \neq k, k \neq l} P_{lk} \frac{P_{lk}}{(x_j - x_l)(x_l - x_k)(x_k - x_j)}.\] (7)

Let us introduce $N \times N$ diagonal matrix $X$,
\[
X = \text{diag}(ix_1, ix_2, \ldots, ix_N).
\] (8)

The commutator of the Lax operator $L$ and the coordinate operator $X$ is
\[
[L, X] = 1 + aA,
\] (9)

where the matrix $A$ is given by $A_{jk} = (1 - \delta_{jk})P_{jk}$. By means of (9), we have
\[
[L^n, X] = mL^{n-1} + a(L^{n-1}A + L^{n-2}AL + \ldots + LAL^{n-2} + AL^{n-1}).
\] (10)

For the conserved operators (6) and the coordinate operator $X$, they also satisfy the following extended Lax equations:
\[
[I_n, L] = [L, M_n], \ [I_n, X] = [X, M_n] + nL^{n-1},
\] (11)

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where $M_n, n = 3, 4, \cdots$, are the operator-valued $N \times N$ matrices and satisfy the sum-to-zero condition, $\sum_{j}(M_n)_{jk} = \sum_{k}(M_n)_{jk} = 0$.

In terms of the conserved operators (6), one can introduce a set of boost operators

$$L_n = \frac{1}{2(n+2)} \sum_{j} x_j^2 I_{n+2}, \quad n \geq -1.$$  \hspace{1cm} (12)

It was shown that the boost operators (12) can also be expressed as (21),

$$L_n = \frac{1}{2} \sum_{j,k} (XL^{n+1} + L^{n+1}X)_{jk}.$$  \hspace{1cm} (13)

This expression of the boost operator will be used later on when we analyze the algebraic structures of the $SU(\nu)$ Calogero spin system.

It is interesting to note that the boost operators (13) and the conserved operators (6) satisfy the following communication relations:

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [L_m, I_n] = -nI_{m+n}, \quad [I_m, I_n] = 0.$$  \hspace{1cm} (14)

It can be recognized as the semidirect product of the V-W algebra with an abelian current algebra.

### 3. Virasoro-Witt 3-algebra

The operator Nambu 3-bracket was proposed by Nambu [33],

$$\hat{A}, \hat{B}, \hat{C} = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} + \hat{C}\hat{A}\hat{B} - \hat{A}\hat{C}\hat{B} + \hat{B}\hat{C}\hat{A} - \hat{C}\hat{B}\hat{A} = \hat{A} \hat{B} \hat{C} + \hat{B} \hat{C} \hat{A} + \hat{C} \hat{A} \hat{B},$$  \hspace{1cm} (15)

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. It should be mentioned that there is another definition of an operator 3-bracket [34], which is valid for operators of trace class. In this paper we only focus on the operator Nambu 3-bracket (15).

In the previous section, we already know that the boost operators (13) and the conserved operators (6) constitute an infinite-dimensional algebra. In order to construct the desired infinite-dimensional 3-algebra, let us turn to consider the following operators:

$$\hat{I}_n = \lambda^{n-1} I_n, \quad \hat{L}_n = \lambda^{n+1} L_n + \frac{n}{2} \hat{z} I_n,$$  \hspace{1cm} (16)

where we introduce a scaling parameter $\lambda$ into the generators. It is worth to emphasize that this scaling parameter $\lambda$ will play a pivotal role in deriving the desired 3-algebra.

The commutators of the operators (13) are

$$[\hat{I}_m, \hat{I}_n] = 0, \quad [\hat{L}_m, \hat{I}_n] = -n \lambda \hat{I}_{m+n}, \quad [\hat{L}_m, \hat{L}_n] = (m-n)\lambda \hat{L}_{m+n}.$$  \hspace{1cm} (17)

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By means of the expressions of Lax operator $L$, straightforward calculation gives the following formulae:

$$[(L^m)_{aq}, (L^n)_{sl}] = \sum_{\alpha=0}^{m-1} \sum_{r,d} (L^\alpha)_{qr}[L_{rd}, (L^n)_{sl}](L^{m-\alpha-1})_{dj}$$

$$= -a \sum_{\alpha=0}^{m-1} \sum_{r,d} \sum_{l,q} (L^\alpha)_{qr}(L^\beta)_{sl}P_{lw} \frac{\delta_{rl} - \delta_{rw}}{(x_l - x_w)^2} (L^{n-\beta-1})_{jl} (L^{m-\alpha-1})_{dj},$$

and

$$[x_k, (L^n)_{sl}] = \sum_{\alpha=0}^{n-1} \sum_{l,q} (L^\alpha)_{sl}[x_k, L_{lq}](L^{n-\alpha-1})_{qt} = i \sum_{\alpha=0}^{n-1} (L^\alpha)_{sk}(L^{n-\alpha-1})_{kt}. \quad (19)$$

These formulae are useful in the later calculations.

Based on the definition \((15)\) of the operator Nambu 3-bracket, we may calculate the corresponding 3-algebra with respect to the operators $\hat{L}_m$ and $\hat{I}_m$. The calculation is in principle straightforward, where the formulae \((10), (18)\) and \((19)\) are used to simplify the results. Here we omit any detail and only give the results.

- $[\hat{I}_m, \hat{I}_n, \hat{I}_k] = \hat{I}_m[\hat{I}_n, \hat{I}_k] + \hat{I}_n[\hat{I}_k, \hat{I}_m] + \hat{I}_k[\hat{I}_m, \hat{I}_n] = 0. \quad (20)$

- $[\hat{L}_m, \hat{I}_n, \hat{I}_k] = \hat{L}_m[\hat{I}_n, \hat{I}_k] + \hat{I}_n[\hat{I}_k, \hat{L}_m] + \hat{I}_k[\hat{L}_m, \hat{I}_n]$

$$= (k - n) \hat{I}_{m+n+k} - \lambda^{m+n+k-1}(nB - (n \leftrightarrow k)), \quad (21)$$

where $B = \sum_{q,j,s,l} (L^k)_{qj}(L^{n+m})_{st}$.

Let us pause here to analyze the terms in the generators $I_s$ and $L_s$. By means of the expressions of Lax operator $L$, straightforward calculation shows that the terms with highest power of $p_j$ in $I_s$ and $L_s$ are $p^j_s$ and $i x_j p^{a+1}_s$, respectively. Note that this result will be useful later on when we determine the generators.

Let us turn to analyze the right-hand side of \((21)\). By calculating $B$, we note that the highest power of $p_j$ is $m + n + k - 1$. Thus the generators $I_s$ for $s = m + n + k$ are absent in $B$. Moreover it is obvious that any generator $L_i$ is not permitted in $B$. Note that the power of $\lambda$ in \((21)\) is $m + n + k - 1$. Taking the above results, we may conclude that there still remains the parameter $\lambda$ in the last two terms of \((21)\) except for the part moved to the generators $\hat{I}_s$, $s \leq m + n + k - 1$. Thus we can rewrite \((21)\) as

$$[\hat{L}_m, \hat{I}_n, \hat{I}_k] = (k - n) \hat{I}_{m+n+k} + \lambda(\cdots), \quad (22)$$
where \(\lambda(\cdots)\) represents the last two terms of (21).

\[ [\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n, \hat{I}_k] = \hat{\mathcal{L}}_m[\hat{\mathcal{L}}_n, \hat{I}_k] + \hat{\mathcal{L}}_n[\hat{I}_k, \hat{\mathcal{L}}_m] + \hat{I}_k[\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n] \]
\[ = -\frac{z}{2}\lambda^{m+n+k-1}n(nB - (n \leftrightarrow k)) + \frac{z}{2}\lambda^{m+n+k-1}m(mB - (m \leftrightarrow k)) + k\lambda^{m+n+k+1}(C_1 - (n \leftrightarrow m)) + (m - n)\lambda^{m+n+k+1}C_2 + (m - n)(\hat{\mathcal{L}}_{m+n+k} - zk\hat{I}_{m+n+k}), \]

where \(C_1\) and \(C_2\) are given by

\[
C_1 = i \left( \sum_{q,j,s,t=1}^{N} x_q (L^{n+1})_{qj} (L^{k+m})_{st} + \sum_{q,j,s,t=1}^{N} (L^{k+m})_{st} (L^{n+1})_{qj} x_j \right) + \sum_{q,j,s,t=1}^{N} \left[ (-a) \sum_{\alpha=0}^{n} \sum_{\beta=0}^{k+m-1} \sum_{w,l=1}^{r_i,j,s,t} (L^\alpha)_{qr} (L^\beta)_{st} P_{lw} \frac{\delta_{ir} - \delta_{iw}}{(x_i - x_w)^2} (L^{k+m-\beta-1})_{wt} (L^{n-\alpha})_{rj} \right] + a \sum_{\alpha=0}^{n} \sum_{\beta=0}^{k+m-1} \sum_{w,l=1}^{r_i,j,s,t} \frac{(L^\alpha)_{qr} (L^\beta)_{st} P_{lw} \delta_{ir} - \delta_{iw}}{(x_i - x_w)^2} (L^{k+m-\beta-1})_{wt} (L^{n-\alpha})_{rj} + i \sum_{\beta=0}^{k+m-1} (L^{n+1})_{qj} (L^\beta)_{sj} (L^{k+m-\beta-1})_{ij}, \tag{24}
\]

and

\[
C_2 = i \left( \sum_{q,j,s,t=1}^{N} x_q (L^{m+n+1})_{qj} (L^{k})_{st} + \sum_{q,j,s,t=1}^{N} (L^{k})_{st} (L^{m+n+1})_{qj} x_j \right) - \sum_{q,j,s,t=1}^{N} \left[ x_q (-a) \sum_{\alpha=0}^{m+n-1} \sum_{\beta=0}^{k-1} \sum_{w,l=1}^{r_i,j,s,t} (L^\alpha)_{qr} (L^\beta)_{st} P_{lw} \frac{\delta_{ir} - \delta_{iw}}{(x_i - x_w)^2} (L^{k-\beta-1})_{wt} (L^{m+n-\alpha})_{rj} \right] + a \sum_{\alpha=0}^{m+n-1} \sum_{\beta=0}^{k-1} \sum_{w,l=1}^{r_i,j,s,t} \frac{(L^\alpha)_{qr} (L^\beta)_{st} P_{lw} \delta_{ir} - \delta_{iw}}{(x_i - x_w)^2} (L^{k-\beta-1})_{wt} (L^{m+n-\alpha})_{rj} + i \sum_{\beta=0}^{k-1} (L^\beta)_{sq} (L^{k-\beta-1})_{tq} (L^{m+n+1})_{qj}, \tag{25}
\]

By calculating \(C_1\) and \(C_2\), we observe that the highest power of \(p_j\) is \(m + n + k\). Thus the generators \(\mathcal{L}_s\) for \(s \geq m + n + k\) and \(I_s\) for \(s \geq m + n + k + 1\) do not appear in \(C_1\) and \(C_2\). Since the power of \(\lambda\) in the third line of (23) is \(m + n + k + 1\), we can easily see that there remains the parameter \(\lambda\) in all terms of the third line except for the part moved to the corresponding generators. For the case of all terms of the second line of (24), it is similar to that of (21). Based on above analysis, we can rewrite (23) as

\[
[\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n, \hat{I}_k] = (m - n)(\hat{\mathcal{L}}_{m+n+k} - zk\hat{I}_{m+n+k}) + \lambda(\cdots), \tag{26}
\]
where $\lambda (\cdots)$ represents the terms of the second and third lines of (23).

\[ [\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n, \hat{\mathcal{L}}_k] = \hat{\mathcal{L}}_m [\hat{\mathcal{L}}_n, \hat{\mathcal{L}}_k] + \hat{\mathcal{L}}_n [\hat{\mathcal{L}}_k, \hat{\mathcal{L}}_m] + \hat{\mathcal{L}}_k [\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n] \]

\[ = \frac{z^2}{4} (k - m)(m - n)(n - k) \hat{I}_{m+n+k} + \frac{1}{4} (k - m)(k - n)(m - n) \lambda^4 \hat{I}_{m+n+k} \]

\[ + \frac{5}{4} \lambda^{m+n+k-1} [((m^2 - n^2)kB - (k \leftrightarrow n) - (k \leftrightarrow m)] \]

\[ + \frac{5}{2} \lambda^{m+n+k+1} ((k^2 - m^2)C_1 - (m \leftrightarrow n) - (k \leftrightarrow n)) \]

\[ + \frac{5}{2} \lambda^{m+n+k+1} ((k(m - n)C_2 - (k \leftrightarrow n) - (k \leftrightarrow m)) \]

\[ + \lambda^{m+n+k+3} ((n - k)D_1 - (m \leftrightarrow n) - (m \leftrightarrow k)) \]

\[ + \lambda^{m+n+k+3} ((n - k)D_2 - (n \leftrightarrow m) - (m \leftrightarrow k)), \tag{27} \]

where

\[ D_1 = -\frac{1}{4} \sum_{q, j, s, t = 1}^{N} [(x_q + x_j)(x_s + x_t)(L^{m+1})_{qj}(L^{k+n+1})_{st} \]

\[ - i(x_q + x_j) \sum_{\alpha = 0}^{m} (L^\alpha)_{qs}(L^{m-\alpha})_{mj}(L^{k+n+1})_{st} + \sum_{\alpha = 0}^{m} (L^\alpha)_{qt}(L^{m-\alpha})_{jt}(L^{k+n+1})_{st} \]

\[ - i(x_q + x_j) \sum_{\beta = 0}^{k+n} (L^{m+1})_{qj}(L^\beta)_{st}(L^{k+n-\beta})_{tt} - i \sum_{\alpha = 0}^{m} (L^\alpha)_{qj}(L^{m-\alpha})_{jj}(L^{k+n+1})_{st}(x_s + x_t) \]

\[ + \sum_{\alpha = 0}^{m} \sum_{\beta = 0}^{k+n} (L^\alpha)_{qs}(L^{m-\alpha})_{jj}(L^\beta)_{ss}(L^{k+n-\beta})_{tt}], \tag{28} \]

and

\[ D_2 = \frac{1}{4} \sum_{q, t = 1}^{N} a^{-(k + n + 1)}(L^m A + L^{m-1} AL + \ldots + LAL^{m-1} + AL^m) L^{k+n} \]

\[ + X(L^m A + L^{m-1} AL + \ldots + LAL^{m-1} + AL^m) L^{k+n+1} \]

\[ - L^{m+1} X(L^k A + L^{k-1} AL + \ldots + LAL^{k-1} + AL^k) \]

\[ - 2L^{m+1}(L^k A + L^{k-1} AL + \ldots + LAL^{k-1} + AL^k) X]_{qt}. \tag{29} \]

By calculating $D_1$ and $D_2$, we note that there do not exist the generators $\mathcal{L}_s$ and $I_s$ for $s \geq m + n + k + 1$ in $D_1$, and any generator in $D_2$. Since the power of $\lambda$ in the last two lines of (27) is $m + n + k + 3$, it is clear that there still remains the parameter $\lambda$ in all terms of the last two lines except for the part moved to the corresponding generators. For the case of all terms of the third, fourth and fifth lines of (27), it is similar to that of (28). Based on above analysis, we can rewrite (27) as

\[ [\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n, \hat{\mathcal{L}}_k] = \frac{z^2}{4} (k - m)(m - n)(n - k) \hat{I}_{m+n+k} \]

\[ + \frac{1}{4} \lambda^4 (k - m)(k - n)(m - n) \hat{I}_{m+n+k} + \lambda (\cdots), \tag{30} \]
where $\lambda(\cdots)$ represents the terms of (27) except for the second line.

Let us take the scaling limit $\lambda \to 0$, then (30), (26), (22) and (20) become the V-W 3-algebra

\[
[\hat{L}_m, \hat{L}_n, \hat{I}_k] = z^2 \frac{(k - m)(n - k)(m - n)}{4} \hat{I}_{m+n+k},
\]

\[
[\hat{L}_m, \hat{L}_n, \hat{I}_k] = (m - n)(\hat{L}_{m+n+k} - z k \hat{I}_{m+n+k}),
\]

\[
[\hat{L}_m, \hat{I}_n, \hat{I}_k] = (k - n)\hat{I}_{m+n+k},
\]

\[
[\hat{I}_m, \hat{I}_n, \hat{I}_k] = 0.
\]

(31)

After the straightforward calculations, we note that the infinite dimensional 3-algebra (31) satisfies the FI condition [28, 29],

\[
[\hat{A}, [\hat{B}, [\hat{C}, [\hat{D}, [\hat{E}]]]]] = [\hat{C}, [\hat{A}, [\hat{B}, \hat{D}], \hat{E}]] + [\hat{D}, [\hat{A}, [\hat{B}, \hat{C}], \hat{E}]] + [\hat{E}, [\hat{A}, [\hat{B}, \hat{C}], [\hat{D}, [\hat{A}, \hat{B}], \hat{E}]].
\]

(32)

Let us define the canonical operator bracket parametrized by $\hat{I}_0$, i.e., $[\hat{A}, \hat{B}]_{\hat{I}_0} = [\hat{A}, \hat{B}, \hat{I}_0]$.Taking $z = 0$ and $\hat{I}_k = \hat{I}_0$ in the last three operator Nambu 3-brackets of (31), then we have the following parametrized operator brackets:

\[
[\hat{L}_m, \hat{L}_n]_{\hat{I}_0} = [\hat{L}_m, \hat{I}_n]_{\hat{I}_0} = (m - n)\hat{I}_{m+n},
\]

\[
[\hat{L}_m, \hat{I}_n]_{\hat{I}_0} = [\hat{L}_m, \hat{I}_n, \hat{I}_0] = -n\hat{I}_{m+n},
\]

\[
[\hat{I}_m, \hat{I}_n]_{\hat{I}_0} = [\hat{I}_m, \hat{I}_n, \hat{I}_0] = 0.
\]

(33)

We immediately recognize that (33) is nothing but (14).

Since the FI condition holds for (31), it should have an explicit realization through the classical Nambu 3-bracket,

\[
\{f, g, h\} = \frac{\partial(f, g, h)}{\partial(x, y, z)} = \frac{\partial f}{\partial x} \left( \frac{\partial g}{\partial y} \frac{\partial h}{\partial z} - \frac{\partial h}{\partial y} \frac{\partial g}{\partial z} \right) + \frac{\partial g}{\partial y} \left( \frac{\partial h}{\partial x} \frac{\partial f}{\partial z} - \frac{\partial f}{\partial x} \frac{\partial h}{\partial z} \right) + \frac{\partial h}{\partial z} \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \right).
\]

(34)

Curtright et al. [24] firstly proposed the classical V-W 3-algebra. Taking the generators as follows:

\[
\hat{L}_m = (x - \frac{m}{2} z \hat{y}) e^{m z}, \quad \hat{I}_m = \hat{y} e^{m z}, \quad m \in \mathbb{Z},
\]

(35)

and substituting them into (34), it immediately gives the classical realization of (31) with the substitution $[,] \to \{,\}$, which is derived in Ref.[24]. Note that not as the case of (35), (31) includes the operators $\hat{L}_m$ and $\hat{I}_n$ only for $m \geq -1$ and $n \geq 1$, respectively. Precisely speaking (31) is a sub-V-W 3-algebra.
4. Summary

The $SU(\nu)$ Calogero spin system is an important classical and quantum integrable system. It is well-known that this system has the $su(\nu)$ loop algebraic structure and $W_{1+\infty}$ symmetry. In this paper, we reinvestigated the $SU(\nu)$ Calogero spin system and revealed a hidden symmetry. Through the use of the boost and conserved operators of this integrable system, we presented the generators with a scaling parameter. By applying the scaling limit, we verified that the $SU(\nu)$ Calogero spin system has an infinite dimensional 3-algebraic symmetry, i.e., the so-called V-W 3-algebra. This V-W 3-algebra satisfies the FI condition. Our investigation turned out that there exists a much profound interrelation between the infinite dimensional 3-algebra and the $SU(\nu)$ Calogero spin system. It provides new insight into this classical and quantum many-body integrable system.

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References

[1] J.L. Gervais and A. Neveu, Dual string spectrum in Polyakov’s quantization (II). Mode separation, Nucl. Phys. B 209 (1982) 125.

[2] J.L. Gervais, Infinite family of polynomial functions of the Virasoro generators with vanishing Poisson brackets, Phys. Lett. B 160 (1985) 277.

[3] P. Mathieu, Extended classical conformal algebras and the second hamiltonian structure of Lax equations, Phys. Lett. B 208 (1988) 101.

[4] K. Yamagishi, The KP hierarchy and extended Virasoro algebras, Phys. Lett. B 205 (1988) 466.

[5] F. Yu and Y.S. Wu, Hamiltonian structure, (anti-)self-adjoint flows in the KP hierarchy and the $W_{1+\infty}$ and $W_{\infty}$ algebras, Phys. Lett. B 263 (1991) 220.

[6] K. Yamagishi, A hamiltonian structure of KP hierarchy, $W_{1+\infty}$ algebra, and self-dual gravity, Phys. Lett. B 259 (1991) 436.

[7] J.M. Figueroa-O’Farrill, J. Mas and E. Ramos, Bihamiltonian structure of the KP hierarchy and the $W_{KP}$ algebra, Phys. Lett. B 266 (1991) 298.
[8] F. Calogero, Ground state of one-dimensional N body system, J. Math. Phys. 10 (1969) 2191; Solution of a three body problem in one dimension, J. Math. Phys. 10 (1969) 2197; Solution of the one-dimensional N body problems with quadratic and/or inversely quadratic pair potentials, J. Math. Phys. 12 (1971) 419.

[9] J. Moser, Three integrable Hamiltonian systems connected with isospectral deformations, Adv. Math. 16 (1975) 197.

[10] B. Sutherland, Quantum many body problem in one dimension: ground state, J. Math. Phys. 12 (1971) 246; Exact results for a quantum many body problem in one dimension, Phys. Rev. A 4 (1971) 209; Exact results for a quantum many body problem in one dimension II, Phys. Rev. A 5 (1972) 1372.

[11] N. Kawakami, Novel hierarchy of the SU(N) electron models and edge states of fractional quantum Hall effect, Phys. Rev. Lett. 71 (1993) 275.

[12] H. Azuma, S. Iso, Explicit relation of the quantum Hall effect and the Calogero-Sutherland model, Phys. Lett. B 331 (1994) 107.

[13] A.P. Polychronakos, Quantum Hall states as matrix Chern-Simons theory, JHEP 04 (2001) 011 [hep-th/0103013].

[14] J.A. Minahan and A.P. Polychronakos, Equivalence of two-dimensional QCD and the c=1 matrix model, Phys. Lett. B312 (1993) 155; Interacting fermion systems from two-dimensional QCD, Phys. Lett. B326 (1994) 288.

[15] A.P. Polychronakos, Non-relativistic bosonization and fractional statistics, Nucl. Phys. B324 (1989) 597.

[16] M.V.N. Murthy and R. Shankar, Thermodynamics of a one-dimensional ideal gas with fractional exclusion statistics, Phys. Rev. Lett. 73 (1994) 3331.

[17] B.D. Simons, P.A. Lee and B.L. Altshuler, Matrix models, one-dimensional fermions, and quantum chaos, Phys. Rev. Lett. 72 (1994) 64.

[18] G.W. Gibbons and P.K. Townsend, Black holes and Calogero models, Phys. Lett. B 454 (1999) 187.

[19] E. D’Hoker and D.H. Phong, Calogero-Moser systems in SU(N) Seiberg-Witten theory, Nucl. Phys. B 513 (1998) 405; Seiberg-Witten theory and Calogero-Moser systems, Prog. Theor. Phys. Suppl. 135 (1999) 75 [hep-th/9906027].

[20] H. Verlinde, Superstrings on AdS2 and Superconformal Matrix Quantum Mechanics, hep-th/0403024.
[21] K. Hikami and M. Wadati, Integrable systems with long-range interactions, $W_{\infty}$ algebra, and energy spectrum, Phys. Rev. Lett. 73 (1994) 1191; Infinite symmetry of the spin systems with inverse square interactions, J. Phys. Soc. Jpn 62 (1993) 4203.

[22] J. Bagger and N. Lambert, Modeling multiple M2’s, Phys. Rev. D 75 (2007) 045020 [hep-th/0611108]; Gauge symmetry and supersymmetry of multiple M2-branes, Phys. Rev. D 77 (2008) 065008 [arXiv:0711.0955]; Comments on multiple M2-branes, JHEP 02 (2008) 105 [arXiv:0712.3738].

[23] A. Gustavsson, Algebraic structures on parallel M2-branes, Nucl. Phys. B 811 (2009) 66 [arXiv:0709.1260].

[24] T. Curtright, D. Fairlie, X. Jin, L. Mezincescu and C. Zachos, Classical and quantal ternary algebras, Phys. Lett. B 675 (2009) 387 [arXiv:0903.4889].

[25] L. Ding, X.Y. Jia, K. Wu, Z.W. Yan and W.Z. Zhao, On q-deformed Virasoro-Witt $w_{\infty}$-algebra, [arXiv:1404.0464].

[26] T.L. Curtright, D.B. Fairlie and C.K. Zachos, Ternary Virasoro-Witt algebra, Phys. Lett. B 666 (2008) 386 [arXiv:0806.3515].

[27] S. Chakrabortty, A. Kumar and S. Jain, $w_{\infty}$ 3-algebra, JHEP 09 (2008) 091 [arXiv:0807.0284].

[28] V.T. Filippov, n-Lie algebras, Sib. Math. J. 26 (1985) 879.

[29] L. Takhtajan, On foundation of the generalized Nambu mechanics (second version), Commun. Math. Phys. 160 (1994) 295 [hep-th/9301111].

[30] M.R. Chen, K. Wu and W.Z. Zhao, Super $w_{\infty}$ 3-algebra, JHEP 09 (2011) 090 [arXiv:1107.3295].

[31] H. Lin, Kac-Moody extensions of 3-algebras and M2-branes, JHEP 07 (2008) 136 [arXiv:0805.4003].

[32] M.R. Chen, S.K. Wang, K. Wu and W.Z. Zhao, Infinite-dimensional 3-algebra and integrable system, JHEP 12 (2012) 030 [arXiv:1201.0417].

[33] Y. Nambu, Generalized Hamiltonian dynamics, Phys. Rev. D 7 (1973) 2405.

[34] H. Awata, M. Li, D. Minic and T. Yoneya, On the quantization of nambu brackets, JHEP 02 (2001) 013 [arXiv:hep-th/9906248].