The Quality of Zero Bounds for Complex Polynomials

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Abstract

In this paper, we evaluate the quality of zero bounds on the moduli of univariate complex polynomials. We select classical and recently developed bounds and evaluate their quality by using several sets of complex polynomials. As the quality of priori bounds has not been investigated thoroughly, our results can be useful to find optimal bounds to locate the zeros of complex polynomials.

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Introduction

Deriving zero bounds for real and complex zeros of polynomials is a classical problem that has been proven essential in various disciplines such as engineering, mathematics, and mathematical chemistry [1–6]. As indicated, there is a large body of literature dealing with the problem of providing disks in the complex plane representing so-called inclusion radii (bounds) where all zeros of an univariate complex polynomial are situated. Let \( f(z) \) be an univariate complex polynomial. Then, a crucial question is to investigate how accurate an inclusion radius is, i.e., how well does the bound reflect the real location of the zeros of \( f(z) \) by determining the quantity \( d_f := b - \zeta_f \) where \( b \) is the bound under consideration and \( \zeta_f := \max \{|z_1|, \ldots, |z_n|\} \). It is clear that the more optimal a zero bound is, the better the value can serve as an estimate to start a numerical procedure such as Newton’s or Sturm’s method [7].

Starting from a set of complex polynomials, it is often difficult to find an optimal bound, i.e., for which \( d_f \) either vanishes or is very little. Another problem is that for many bounds, sharpness results do not exist. Sharpness means there exists a polynomial possessing a zero that lies on the circle which includes all zeros of the polynomial in question. This problem calls for a systematic treatment namely to study the optimality of zero bounds for particular classes of polynomials numerically. To our best knowledge, this problem has not yet been explored properly; see, e.g., [8]. A reason for this is surely the vast amount of existing bounds for locating the zeros of real and complex polynomials [9–13]. The only attempt in this direction we got aware of is due McNamee and Olhovsky [8]. They implemented several zero bounds by using 1200 polynomials with random real or complex roots and calculated their values numerically [8]. Among other calculated zero bounds they did not state explicitly in [8], the bounds due to Deutsch [14] and Kalantari [13] have been evaluated and found to be optimal by using the mentioned set of polynomials [8].

The main contribution of this paper is as follows: In contrast to [8], we evaluate classical and more recently developed zero bounds by using different classes of complex polynomials numerically. Among these classes are also lacunary polynomials and those, whose coefficients satisfy certain conditions by means of inequalities. We calculate several bounds for complex polynomials due to Cauchy [3], Dehmer [1,9], Kalantari [13], Jain [15], Joyal [10] etc., see Table 1–Table 6. As a result, we find that some of the bounds due to Dehmer, Jain and Cauchy outperform Kalantari’s bounds by using particular classes of polynomials. This result triggers the hypothesis that it may be worthwhile to further develop bounds for special polynomials (e.g., lacunary polynomials or complex polynomials with special conditions for the polynomial coefficients) which are more optimal than by using general zero bounds. For instance, Theorem (10) developed by Dehmer et al. [9] will prove this hypothesis.

Methods

In the following, we state the most important zero bounds for locating the zeros of complex polynomials as theorems we are going to use in this study. The quality of these statements will be evaluated in the section ‘Results’. We distinguish two classes of bounds, namely explicit and implicit zero bounds, see [1,9].

Explicit Bounds for Complex Polynomials: Classical and Recent Results

The following bounds [4,16,17] represent functions of all coefficients of a given polynomial. In fact, this type of zero bound has been called explicit bound [1,9] as the value of the bound can be calculated explicitly by using quantities based on the moduli of the polynomial coefficients.

**Theorem 1 (Cauchy [3])** Let \( f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, a_0 \neq 0, k = 0, 1, \ldots, n \) be a complex polynomial. All zeros of \( f(z) \) lie in the closed disk \(|z| \leq 1 + M\), where

\[
M := \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|
\]

**Theorem 2 (Joyal [10])** Let \( f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, a_0 \neq 0, k = 0, 1, \ldots, n \)
be a complex polynomial. All zeros \( f(z) \) lie in
\[
|z| \leq \frac{1}{2}\left(1 + \sqrt{1 + 4B}\right),
\]
where
\[
B = \max_{1 \leq k \leq n}\left[\frac{a_{n-1}a_{n-k} - a_{n}a_{n-k-1}}{a_{n}^2}\right], \quad a_{-1} = 0.
\]

**Theorem 3 (Mohammad [15])** Let
\[
f(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0, a_n \neq 0, k=0,1,\ldots,n
\]
be a complex polynomial. All zeros \( f(z) \) lie in
\[
|z| \leq 2\max\left(\frac{|a_k|}{|a_{k+1}|}\right), \; 0 \leq k \leq n-1.
\]

**Theorem 4 (Kojima [12])** Let
\[
f(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0, a_n \neq 0, k=0,1,\ldots,n
\]
be a complex polynomial. All zeros \( f(z) \) lie in
\[
|z| \leq \max\left(\frac{|a_0|}{|a_1|}, \frac{|a_n|}{|a_{n+1}|}\right), \; 1 \leq k \leq n-1.
\]

**Theorem 5 (Jain [15])** Let
\[
f(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0, a_n \neq 0, k=0,1,\ldots,n
\]
be a complex polynomial. All zeros \( f(z) \) lie in
\[
|z| \leq \max\left(\frac{|a_{n-1}|}{|a_n|}, \frac{|a_{n-2}|}{|a_{n-1}|}, \ldots, \frac{|a_0|}{|a_1|}\right).
\]

\[
|z| \leq \frac{1}{\ln(2)}\max\left(\frac{|a_{n-1}|}{|a_n|}, \frac{|a_{n-2}|}{|a_{n-1}|}, \ldots, \frac{|a_0|}{|a_1|}\right).
\]

**Theorem 6 (Kuniyeda [11])** Let \( p,q > 1 \) mit \( \frac{1}{p} + \frac{1}{q} = 1 \). All zeros of
\[
f(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0, a_n \neq 0, a_i \in \mathbb{C}, i=0,1,\ldots,n
\]
lie in
\[
|z| \leq \left(1 + \frac{1}{\binom{n}{0}}\right)\left(\sum_{i=0}^{n}|a_i|^{1+p}\right)^\frac{1}{p}.
\]

**Theorem 7 (Kuniyeda [11])** For \( p > 0 \), all zeros of
\[
f(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0, a_n \neq 0, a_i \in \mathbb{C}, i=0,1,\ldots,n
\]
lie in
\[
|z| \leq \left(\frac{1}{|a_n|^p}\sum_{i=1}^{n}|a_{n-i}|^{1+p}\right)^\frac{1}{p+1} + 1.
\]

**Theorem 8 (Joyal [10])** For \( p,q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), all zeros of
\[
f(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0, a_n \neq 0, a_i \in \mathbb{C}, i=0,1,\ldots,n
\]
lie in
\[
|z| \leq \left(\frac{1}{|a_n|^p}\sum_{i=1}^{n}|a_{n-i}|^{1+p}\right)^\frac{1}{p+1} + 1.
\]

| Table 1. Mean bounds values for polynomials \( P_i \in \mathbb{C}_1 \). |
|---|
| Bounds          | \( n=5 \) | \( n=10 \) | \( n=20 \) | \( n=30 \) | \( n=40 \) | \( n=50 \) | \( n=60 \) | \( n=70 \) | \( n=100 \) |
| Cauchy, Th. (1) | 3.385 | 4.382 | 4.515 | 4.335 | 4.612 | 4.729 | 4.710 | 4.725 | 4.812 |
| Cauchy, Th. (13) | 2.500 | 3.006 | 2.610 | 2.525 | 2.577 | 2.634 | 2.577 | 2.614 | 2.567 |
| Dehmer, Th. (14) | 3.367 | 4.381 | 4.515 | 4.335 | 4.612 | 4.729 | 4.710 | 4.725 | 4.812 |
| Dehmer, Th. (15) | 2.757 | 3.399 | 3.084 | 3.041 | 3.144 | 3.218 | 3.173 | 3.215 | 3.205 |
| Dehmer, Th. (16) | 3.024 | 3.902 | 3.542 | 3.410 | 3.568 | 3.669 | 3.596 | 3.665 | 3.644 |
| Dehmer, Th. (9)  | 2.788 | 3.400 | 3.084 | 3.041 | 3.144 | 3.218 | 3.173 | 3.215 | 3.205 |
| Jain, Th. (5)   | 19.079 | 50.691 | 147.404 | 286.800 | 340.555 | 545.886 | 744.104 | 948.732 | 1481.292 |
| Joyal, Th. (2)  | 2.648 | 3.324 | 3.116 | 2.998 | 3.149 | 3.233 | 3.176 | 3.229 | 3.235 |
| Joyal, Th. (8)  | 2.684 | 3.885 | 4.030 | 4.148 | 4.633 | 4.966 | 5.066 | 5.307 | 5.699 |
| Kalantari, Th. (11) | 3.593 | 4.539 | 3.755 | 3.574 | 3.704 | 3.793 | 3.691 | 3.787 | 3.679 |
| Kalantari, Th. (12) | 2.955 | 3.777 | 3.127 | 3.005 | 3.082 | 3.169 | 3.079 | 3.160 | 3.061 |
| Kojima, Th. (4) | 7.157 | 15.754 | 22.013 | 20.301 | 24.790 | 26.581 | 30.713 | 38.107 |
| Kuniyeda, Th. (6) | 3.792 | 6.441 | 8.448 | 9.277 | 11.177 | 12.576 | 13.519 | 14.291 | 16.888 |
| Kuniyeda, Th. (7) | 3.456 | 5.101 | 5.912 | 6.063 | 6.909 | 7.456 | 7.763 | 7.995 | 8.855 |
| Mohammad, Th. (3) | 7.871 | 11.830 | 16.093 | 22.262 | 20.408 | 24.882 | 27.215 | 30.926 | 38.336 |

The results are averaged over 1000 independent runs.
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The following explicit bounds have been recently proven by

\[
Kuniyeda, Th. (6) 1.613 1.967 2.446 2.837 3.148 3.447 3.678 3.915 4.475
\]
\[
Kojima, Th. (4) 5.941 9.415 14.717 17.694 21.712 25.410 26.355 28.808 35.672
\]
\[
Kalantari, Th. (12) 1.562 1.616 1.626 1.627 1.627 1.627 1.627 1.625
\]
\[
Jain, Th. (5) 14.137 39.253 98.124 192.320 265.311 383.379 579.944 631.977 1059.740
\]
\[
Joyal, Th. (2) 2.390 2.453 2.543 2.559 2.635 2.626 2.661 2.621 2.702
\]
\[
Joyal, Th. (8) 2.421 2.814 3.345 3.652 4.013 4.203 4.431 4.486 5.051
\]
\[
Kalantari, Th. (11) 3.107 3.077 3.080 3.099 3.149 3.138 3.124 3.077 3.158
\]
\[
Kalantari, Th. (12) 2.651 2.632 2.631 2.635 2.688 2.673 2.673 2.625 2.688
\]
\[
Kojima, Th. (4) 5.483 8.052 10.922 14.386 15.662 17.660 22.262 21.553 24.836
\]
\[
Kuniyeda, Th. (6) 3.220 4.506 6.403 7.584 8.964 10.015 11.168 11.606 14.561
\]
\[
Kuniyeda, Th. (7) 3.020 3.564 4.405 4.830 5.386 5.763 6.209 6.279 7.356
\]
\[
Mohammad, Th. (3) 5.898 8.456 11.100 14.728 15.701 17.870 22.372 21.693 24.921
\]

The results are averaged over 1000 independent runs.

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The Quality of Zero Bounds for Complex Polynomials

Table 2. Mean bounds values for polynomials $P_i \in C_2$.

| Bounds          | $n=5$   | $n=10$  | $n=20$  | $n=30$  | $n=40$  | $n=50$  | $n=60$  | $n=70$  | $n=100$ |
|-----------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| Cauchy, Th. (1) | 1.798   | 1.839   | 1.862   | 1.879   | 1.890   | 1.893   | 1.897   | 1.899   | 1.903   |
| Cauchy, Th. (13)| 1.438   | 1.484   | 1.462   | 1.446   | 1.433   | 1.424   | 1.419   | 1.410   | 1.399   |
| Dehmer, Th. (10)| 1.513   | 1.458   | 1.413   | 1.393   | 1.378   | 1.368   | 1.363   | 1.354   | 1.342   |
| Dehmer, Th. (14)| 1.747   | 1.836   | 1.862   | 1.879   | 1.890   | 1.893   | 1.897   | 1.899   | 1.903   |
| Dehmer, Th. (15)| 1.603   | 1.689   | 1.698   | 1.700   | 1.703   | 1.699   | 1.704   | 1.698   | 1.698   |
| Dehmer, Th. (16)| 1.642   | 1.694   | 1.702   | 1.703   | 1.708   | 1.708   | 1.710   | 1.706   | 1.703   |
| Dehmer, Th. (9) | 1.672   | 1.694   | 1.698   | 1.700   | 1.703   | 1.699   | 1.704   | 1.698   | 1.698   |
| Jain, Th. (5)   | 17.221  | 47.320  | 135.553 | 235.369 | 391.751 | 549.261 | 665.022 | 831.994 | 1508.291|
| Joyal, Th. (2)  | 1.563   | 1.600   | 1.607   | 1.613   | 1.620   | 1.621   | 1.625   | 1.622   | 1.623   |
| Joyal, Th. (8)  | 1.381   | 1.549   | 1.726   | 1.852   | 1.946   | 2.030   | 2.094   | 2.152   | 2.290   |
| Kalantari, Th. (11)| 1.851 | 1.937   | 1.973   | 1.984   | 1.989   | 1.992   | 1.993   | 1.995   | 1.996   |
| Kalantari, Th. (12)| 1.562 | 1.616   | 1.626   | 1.627   | 1.627   | 1.627   | 1.627   | 1.627   | 1.625   |
| Kojima, Th. (4) | 5.941   | 9.415   | 14.717  | 17.694  | 21.712  | 25.410  | 26.355  | 28.808  | 35.672  |
| Kuniyeda, Th. (6)| 1.613   | 1.967   | 2.446   | 2.837   | 3.148   | 3.447   | 3.678   | 3.915   | 4.475   |
| Kuniyeda, Th. (7)| 1.819   | 1.942   | 2.097   | 2.224   | 2.319   | 2.410   | 2.476   | 2.546   | 2.698   |
| Mohammad, Th. (3)| 6.500   | 9.837   | 14.935  | 17.846  | 21.988  | 25.515  | 26.574  | 28.850  | 35.818  |

The results are averaged over 1000 independent runs.

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Theorem 9 (Dehmer [9]) Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, $a_0 a_{n-1} \neq 0$, be a complex polynomial. All zeros of $f(z)$ lie in the closed disk

\[
K \left( 0, \frac{1 + \sqrt{\phi_2 - 1^2 + 4M_1}}{2} \right),
\]
The next theorem gives a bound for polynomials with restrictions on the coefficients. Dehmer [1] has shown that such bounds can be more precise and often lead to better results when locating the zeros of polynomials. See also Table 3.

**Theorem 10 (Dehmer [9])** Let

\[ f(z) = a_0 z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 a_{n-1} \neq 0, \]

be a complex polynomial. Suppose that

\[ w_2 : = \left| \frac{a_{n-1}}{a_n} \right|. \quad (11) \]

where

The next theorem gives a bound for polynomials with restrictions on the coefficients. Dehmer [1] has shown that such bounds can be more precise and often lead to better results when locating the zeros of polynomials. See also Table 5.

| Table 4. Mean bounds values for polynomials \( P_i \in C_4 \). |
| --- |
| **Bounds** | **n = 5** | **n = 10** | **n = 20** | **n = 30** | **n = 40** | **n = 50** | **n = 60** | **n = 70** | **n = 100** |
| Cauchy, Th. (1) | 2.074 | 2.220 | 2.439 | 2.627 | 2.779 | 2.896 | 3.011 | 3.154 | 3.395 |
| Cauchy, Th. (13) | 1.648 | 1.730 | 1.718 | 1.717 | 1.699 | 1.696 | 1.682 | 1.664 | 1.653 |
| Dehmer, Th. (14) | 2.038 | 2.219 | 2.439 | 2.627 | 2.779 | 2.896 | 3.011 | 3.154 | 3.395 |
| Dehmer, Th. (15) | 1.824 | 1.957 | 2.047 | 2.121 | 2.170 | 2.209 | 2.241 | 2.279 | 2.354 |
| Dehmer, Th. (16) | 1.729 | 1.871 | 1.983 | 2.069 | 2.123 | 2.167 | 2.211 | 2.249 | 2.326 |
| Dehmer, Th. (9) | 1.874 | 1.959 | 2.047 | 2.121 | 2.170 | 2.209 | 2.241 | 2.279 | 2.354 |
| Jain, Th. (5) | 11.649 | 36.042 | 114.092 | 221.236 | 350.677 | 475.578 | 665.546 | 789.016 | 1323.884 |
| Joyal, Th. (2) | 1.668 | 1.745 | 1.841 | 1.932 | 1.990 | 2.040 | 2.085 | 2.126 | 2.207 |
| Joyal, Th. (8) | 1.456 | 1.700 | 2.038 | 2.321 | 2.528 | 2.717 | 2.888 | 3.046 | 3.408 |
| Kalantari, Th. (11) | 2.101 | 2.171 | 2.183 | 2.192 | 2.183 | 2.176 | 2.175 | 2.164 | 2.152 |
| Kalantari, Th. (12) | 1.706 | 1.767 | 1.774 | 1.779 | 1.776 | 1.771 | 1.767 | 1.761 | 1.753 |
| Kojima, Th. (4) | 4.686 | 8.516 | 13.542 | 20.859 | 23.248 | 26.198 | 28.486 | 33.187 |
| Kuniyeda, Th. (6) | 1.841 | 2.444 | 3.484 | 4.478 | 5.371 | 6.186 | 7.034 | 7.872 | 9.900 |
| Kuniyeda, Th. (7) | 1.990 | 2.262 | 2.732 | 3.180 | 3.571 | 3.915 | 4.273 | 4.629 | 5.423 |
| Mohammad, Th. (3) | 5.042 | 8.655 | 13.598 | 17.865 | 20.899 | 23.300 | 26.280 | 28.505 | 33.218 |

The results are averaged over 1000 independent runs.

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| Table 5. Mean bounds values for polynomials \( P_i \in C_5 \). |
| --- |
| **Bounds** | **n = 5** | **n = 10** | **n = 20** | **n = 30** | **n = 40** | **n = 50** | **n = 60** | **n = 70** | **n = 100** |
| Cauchy, Th. (1) | 6.953 | 8.934 | 11.459 | 17.945 | 18.848 | 21.994 | 26.068 | 23.206 | 31.766 |
| Cauchy, Th. (13) | 3.687 | 3.682 | 3.516 | 3.711 | 3.782 | 3.693 | 3.834 | 3.616 | 3.848 |
| Dehmer, Th. (14) | 6.945 | 8.934 | 11.459 | 17.945 | 18.848 | 21.994 | 26.068 | 23.206 | 31.766 |
| Dehmer, Th. (15) | 4.013 | 4.326 | 4.598 | 5.183 | 5.482 | 5.662 | 6.053 | 5.868 | 6.637 |
| Dehmer, Th. (16) | 4.849 | 5.353 | 5.737 | 7.021 | 7.440 | 7.825 | 8.729 | 7.984 | 9.543 |
| Dehmer, Th. (9) | 4.028 | 4.326 | 4.598 | 5.183 | 5.482 | 5.662 | 6.053 | 5.868 | 6.637 |
| Jain, Th. (5) | 13.985 | 41.487 | 118.855 | 217.464 | 350.344 | 485.687 | 626.950 | 808.537 | 1557.588 |
| Joyal, Th. (2) | 4.230 | 4.774 | 5.236 | 6.535 | 6.966 | 7.410 | 8.294 | 7.589 | 9.135 |
| Joyal, Th. (8) | 4.568 | 5.797 | 7.141 | 9.749 | 10.998 | 12.177 | 14.254 | 13.330 | 17.503 |
| Kalantari, Th. (11) | 5.406 | 5.330 | 5.016 | 5.367 | 5.507 | 5.286 | 5.615 | 5.197 | 5.664 |
| Kalantari, Th. (12) | 4.280 | 4.221 | 4.022 | 4.303 | 4.386 | 4.402 | 4.508 | 4.415 | 4.535 |
| Kojima, Th. (4) | 7.760 | 10.959 | 15.114 | 18.970 | 22.702 | 25.459 | 25.562 | 28.069 | 37.948 |
| Kuniyeda, Th. (6) | 8.868 | 14.608 | 23.560 | 43.366 | 51.404 | 64.825 | 82.929 | 77.652 | 125.115 |
| Kuniyeda, Th. (7) | 7.627 | 11.174 | 16.139 | 27.745 | 31.332 | 38.119 | 47.356 | 43.228 | 65.415 |
| Mohammad, Th. (3) | 7.942 | 11.131 | 15.165 | 19.162 | 22.719 | 25.558 | 25.616 | 28.152 | 37.989 |

The results are averaged over 1000 independent runs.

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Table 6. Mean bounds values for polynomials $P_i \in C_6$.

| Bounds          | $n = 5$ | $n = 10$ | $n = 20$ | $n = 30$ | $n = 40$ | $n = 50$ | $n = 60$ | $n = 70$ | $n = 100$ |
|-----------------|--------|---------|---------|---------|---------|---------|---------|---------|---------|
| Cauchy, Th. (1) | 2.638  | 2.600   | 2.637   | 2.627   | 2.627   | 2.607   | 2.648   | 2.620   | 2.634   |
| Cauchy, Th. (13)| 1.214  | 1.093   | 1.045   | 1.029   | 1.022   | 1.017   | 1.015   | 1.012   | 1.009   |
| Dehmer, Th. (17)| 1.216  | 1.094   | 1.045   | 1.030   | 1.022   | 1.017   | 1.015   | 1.012   | 1.009   |
| Dehmer, Th. (18)| 1.288  | 1.123   | 1.059   | 1.038   | 1.028   | 1.022   | 1.019   | 1.016   | 1.011   |
| Dehmer, Th. (19)| 2.268  | 2.262   | 2.290   | 2.286   | 2.278   | 2.270   | 2.283   | 2.246   | 2.260   |
| Jain, Th. (5)   | 13.075 | 22.132  | 44.164  | 60.219  | 92.112  | 105.597 | 132.567 | 173.678 | 232.253 |
| Joyal, Th. (2)  | 1.856  | 1.842   | 1.856   | 1.851   | 1.852   | 1.845   | 1.859   | 1.849   | 1.853   |
| Joyal, Th. (8)  | 1.557  | 1.548   | 1.559   | 1.557   | 1.556   | 1.549   | 1.561   | 1.551   | 1.555   |
| Kalantari, Th. (11)| 2.206 | 2.087   | 2.044   | 2.028   | 2.021   | 2.016   | 2.014   | 2.012   | 2.008   |
| Kalantari, Th. (12)| 1.785 | 1.688   | 1.654   | 1.641   | 1.635   | 1.631   | 1.630   | 1.628   | 1.625   |
| Kojima, Th. (4) | 1.813  | 1.534   | 1.531   | 1.391   | 1.569   | 1.546   | 1.531   | 1.720   | 1.700   |
| Kuniyeda, Th. (6)| 2.166  | 2.139   | 2.173   | 2.169   | 2.169   | 2.142   | 2.178   | 2.147   | 2.165   |
| Kuniyeda, Th. (7)| 2.363  | 2.338   | 2.366   | 2.363   | 2.360   | 2.340   | 2.372   | 2.346   | 2.363   |
| Mohammad, Th. (3)| 3.625  | 3.068   | 3.061   | 2.783   | 3.068   | 3.063   | 3.440   | 3.220   | 3.220   |

The results are averaged over 1000 independent runs.

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The bound is sharp for all polynomials of the form

$$f(z) = a^n z^n - b z^n - \sum_{r=2}^{n+1} [z^{r-2} + \cdots + z + 1], \quad a, b > 0.$$  \hfill (14)

Further recent results when proving upper bounds have been found by Kalantari [13]. He has found a family of zeros bounds for analytic functions that has been proven powerful when comparing the resulting bounds with classical ones by using complex polynomials [8].

**Theorem 11 (Kalantari [13])** Let $m \geq 2$ and let $r_m \in \left[\frac{1}{2}, 1\right)$ be the positive root of the polynomial

$$q(t) := t^{m-1} + t - 1.$$  \hfill (15)

For $m = 3$ and $r_3 = \frac{2}{\sqrt{5} + 1}$, all zeros of the complex polynomial

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_0 a_{n-1} \neq 0,$$

lie in the closed disk

$$K \left( 0, \frac{\sqrt{5} + 1}{2}, \max_{2 \leq k \leq n+1} \left( \frac{|a_{n-k}|}{a_n} \right)^{\frac{1}{k}} \right).$$  \hfill (17)

$a_1 := 0$.

**Implicit Bounds for Complex Polynomials: Classical and Recent Results**

The bound value of an implicit zero bounds depends on determining the root of a so-called concomitant polynomial [9]. This polynomial can often be obtained from the proof of the underlying theorem. An example thereof is Equation (18).

**Theorem 13 (Cauchy [3])** Let

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_0 \neq 0, \quad k = 1, \ldots, n$$

be a complex polynomial. All zeros of $f(z)$ lie in the closed disk $K(0, \rho_C)$, where $\rho_C$ denotes the positive zero of

$$K_C(z) := |a_0| + |a_1| z + \cdots + |a_{n-1}| z^{n-1} - |a_n| z^n.$$  \hfill (18)

The following implicit zero bounds might be easier to determine (e.g., by hand) when applying this apparatus in practice.

**Theorem 14 (Dehmer [1])** Let

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_0 a_{n-1} \neq 0,$$

be a complex polynomial. All zeros of $f(z)$ lie in the closed disk $K(0, \max(1, \delta))$, where $\delta$ denotes the positive root of the equation
\[ z^{n+1} - (1 + M_2)z^n + M_2 = 0, \quad (19) \]

and

\[ M_2 := \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|. \quad (20) \]

The bound is sharp for all polynomials of the form

\[ f(z) = az^n - b[z^{n-1} + \cdots + z + 1], \quad a, b > 0. \quad (21) \]

**Theorem 15 (Dehmer [9])** Let

\[ f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, a_n \neq 0, \]

be a complex polynomial. All zeros of \( f(z) \) lie in the closed disk \( K(0, \max(1, \delta)) \) where \( \delta \) denotes the positive root of the equation

\[ z^{n+1} - \left( 1 + \frac{a_{n-1}}{a_n} \right) z^n + \left( \frac{a_{n-1}}{a_n} - M_1 \right) z^{n-1} + M_1 = 0, \quad (22) \]

and

\[ M_1 := \max_{0 \leq j \leq n-2} \left| \frac{a_j}{a_n} \right|. \quad (23) \]

The bound is sharp for all polynomials of the form

\[ f(z) = az^n - bz^{n-1} - c[z^{n-2} + \cdots + z + 1], \quad a, b > 0, c \geq 0. \quad (24) \]

In particular, the concomitant polynomial of the next theorem is cubic, see Equation (27). This can be beneficial for practical applications as we only have to determine the positive root of a polynomial whose degree equals three.

**Theorem 16 (Dehmer [9])** Let

\[ M_3 := \max_{2 \leq j \leq n} \left| \frac{a_{n-1}a_{n-j} - a_n a_{n-j-1}}{a_n^2} \right| a_{n-1} = 0, \quad (25) \]

and

\[ \phi_1 := \left| \frac{a_{n-1}^2 - a_n a_{n-2}}{a_n^2} \right|. \quad (26) \]

In addition, let

\[ f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, a_n a_{n-1} \neq 0, \]

be a complex polynomial. All zeros of \( f(z) \) lie in the closed disk \( K(0, \delta) \) where \( \delta > 1 \) is the largest positive root of the equation

\[ z^3 - z^2 - (M_3 + \phi_1) z + \phi_1 = 0. \quad (27) \]

Moreover,

\[ 1 < \delta < 1 + \sqrt{M_3 + \phi_1}. \quad (28) \]

**Bounds for Special Lacunary Polynomials**

In this section, we state bounds [9] for lacunary polynomials, i.e., polynomials in which some coefficients vanish. The hypothesis is that special bounds for lacunary polynomials might lead to better results than by using general zero bounds, see the statements in the previous section.

**Theorem 17 (Dehmer [9])** Let

\[ f(z) = z^n - a_1 z + a_0, a_1 a_0 \neq 0, n > 2, \]

be a complex polynomial. All zeros of \( f(z) \) lie in \( K(0, \max(1, \delta)) \), where \( \delta \) is the unique positive root of the equation

\[ z^n - |a_1| |z - |a_0|| = 0. \quad (29) \]

Using the same method of proof we establish the following.

**Theorem 18 (Dehmer [9])** Let

\[ f(z) = z^n - a_1 z + a_0, a_1 a_0 \neq 0, n > 2, \]

be a polynomial with arbitrary coefficients. All zeros of \( f(z) \) lie in \( K(0, \max(1, \delta)) \), where \( \delta \) is the unique positive root of the equation

\[ z^n - M_4 z - M_4 = 0. \quad (30) \]

We conclude this section with the following theorem.

**Theorem 19 (Dehmer [9])** Let

\[ f(z) = z^n - a_1 z + a_0, a_1 a_0 \neq 0, n > 2, \]

be a complex polynomial. All zeros of \( f(z) \) lie in

\[ K \left( 0, \frac{|a_1|}{2} + \frac{\sqrt{|a_1|^2 + 4|a_0| + 4}}{2} \right). \quad (31) \]

**Data: Classes of Complex Polynomials**

We define the classes of polynomials used in this study as follows (GD stands for Gaussian Distribution).

**Definition 1**

\[ C_1 := \{ f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 | a_i \in \mathbb{C} \text{ sampled from GD} \}. \quad (32) \]
Definition 2

\[ C_2 := \{ f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 \mid a_i \in \mathbb{C}, \text{uniformly distributed} \} \]  
and \[ |a_i| < 1, i = 0, 1, \ldots, n. \}

(33)

Definition 3

\[ C_3 := \{ f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 \mid a_i \in \mathbb{C}, \text{sampled from GD} \} \]

(34)

Definition 4

\[ C_4 := \{ f(z) = f_1(z) f_2(z) \mid f_1(z) := a_{n_1} z^{n_1} + a_{n_1-1} z^{n_1-1} + \ldots + a_0, f_2(z) := b_{n_2} z^{n_2} + b_{n_2-1} z^{n_2-1} + \ldots + b_0, a_i, b_i \in \mathbb{C}, \text{sampled from GD} \} \]

(35)

Definition 5

\[ C_5 := \{ f(z) = f_1(z) f_2(z) \mid f_1(z) := a_{n_1} z^{n_1} + a_{n_1-1} z^{n_1-1} + \ldots + a_0, f_2(z) := b_{n_2} z^{n_2} + b_{n_2-1} z^{n_2-1} + \ldots + b_0, a_i, b_i \in \mathbb{C}, i = 0, 1, \ldots, n_1, n_2, \text{sampled from GD} \} \]

(36)

Definition 6

\[ C_6 := \{ f(z) = z^n - a_1 z + a_0, a_1, a_0 \in \mathbb{C}, \text{sampled from GD} \} \]

(37)

Software

We developed a C# program for calculating 19 zero bounds by using complex polynomials. These bounds have been calculated by generating random polynomials based on the following distributions: Gaussian, Poisson, Geometric, and Uniform in \([-1, 1]\). We used the batch mode of this software to compute all available bounds for a specified number of polynomials having certain degrees. For each class \( C_i \) (see section 'Data: Classes of Complex Polynomials'), we calculate the average by performing 1000 independent runs. To calculate the zeros of the random polynomials, one has to select:

- the degree of a polynomial.
- the type of distribution.
- whether the polynomial is complex or real-valued.

We emphasize that in this study, we only used complex polynomials. After each batch run, the following information is available:

- the type of distribution and distribution parameters.
- the parameter \( p \) required for some bounds, for example see Equations (6) or (7).
- tables with mean and standard deviation of ranks of the bounds in terms of their optimality for different polynomial orders.
- tables with mean and standard deviation of bound values for different degrees of the polynomials.

Moreover for each polynomial degree, a brief summary of the results for a batch run is available, which includes:

- the type of distribution, distribution parameters and the polynomial degree.
- number of runs.
- the parameter \( p \), see Equations (6) or (7).
- bound name with the best (worst) sum rank.
- bound name with the worst (largest) sum rank.
- list of the used bounds sorted by their sum rank in ascending order (from best to worst) with information on their sum ranks, minimal and maximal rank achieved during all runs.

Apart from the batch mode, the program can also be used in a single mode. By doing so, the program creates a log file containing the following items:

- the degree of a polynomial.
- the type of distribution.
- the generated complex or real coefficients of the polynomial.
- the parameter \( p \), see Equations (6) or (7).
- names of the most sharp and weak bounds with their corresponding values.
- a ranking of the bounds in terms of their optimality in ascending order.

Results

In this section, we evaluate the quality of the zero bounds presented in the previous sections. We start by observing that Theorem (13) due to Cauchy is often quite sharp. See, for example, the results in Tables 1–4. This is not surprising as this bound is known to be optimal for its class of implicit zero bounds, see [6]. As the numerical results show, this does not mean that other bounds (based on another paradigm) outperform this bound by using special classes of polynomials. This proves our hypothesis that special bounds (e.g., Theorem (10), (19)) may be more suitable and optimal for special classes of polynomials than general bounds (i.e., bounds where no restrictions for the polynomial coefficients are used). In particular, we see that by using the polynomials of
Definition (3), the bound due to Dehmer, Theorem (10) outperforms Cauchy's bound (Theorem (13)) if $n > 5$. Again, Theorem (10) is based on inequalities involving the polynomial coefficients and leads to a better mean value than by using a general zero bound.

Also, the results by using lacunary polynomials (see Definition (6)) support this hypothesis too. By considering Table 6, we observe that the special bounds for lacunary polynomials due to Dehmer, Theorem (17), (10) perform very similar to Theorem (13) due to Cauchy based on the mean values. In contrast, the explicit bound also developed by Cauchy, Theorem (1) does not give feasible values. Also, the bound due to Jain, Theorem (5) and Kojima, Theorem (4) are not feasible by using the here presented classes of polynomials. This holds for all classes of polynomials used in this study, see Table 1–6. More generally, it has been shown that the inclusion radii given by Theorem (1), (5) are often useless in terms of the real location of the zeros of an underlying polynomial, see [9].

In the following, we discuss some particular cases to find classes of polynomials where the bounds due to Dehmer perform well. Let $z_1, \ldots, z_n$ be the zeros of a complex polynomial $f(z)$ and let $\zeta_{ij} = \max(|z_1|, \ldots, |z_n|)$. Also, we define the quantity $d_k^i := b - \zeta_{ij}$, where $b$ is the corresponding bound value for $f(z)$. Now, consider the polynomial

$$ f_i(z) := (0.1050 + i.0.2635)^2 + (0.7263 + i.0.2592)^2 + (1.0132 - i.1.2395)^2 + (0.6064 - i.1.6278)^2 + (-0.7962 - i.1.1193)^2 + (-1.218 + i.0.0337) \in C_i. $$

We yield that Kalantaris bound, Theorem (12) is best, $b_k = 3.540$ and $d_{bk}^1 = 1.230$. Particularly Table 1 supports the fact that Kalantaris bound, Theorem (12) performs well for $C_1$ among the used zero bounds. Second best is the bound due to Joyal, Theorem (2), $b_j = 3.9799$ and $d_{bj}^1 = 1.669$. Third best is Dehmer’s bound, Theorem (14), $b_D = 4.2962$ and $d_{bD}^1 = 1.9862$. Particularly, Theorem (14) outperforms the classical bound due to Cauchy, Theorem (11), $b_C = 4.3905$ and $d_{bC}^1 = 2.0805$. But note that for many other polynomials of $C_1$, Joyal’s bound, Theorem (2) often was often the best one and Theorem (14) due to Dehmer the second best one. Theorem (14) has the advantage that the positive root of the concomitant polynomial $z^{n+1} - (1 + M_2)z^n + M_2$ (see Equation (19)) might be easier to determine than by using other bounds which rely on more complex concomitant polynomials, e.g., see Theorem (13).

We already mentioned above that by using special polynomials, some special bounds (based on conditions for the polynomial coefficients) are better suited than by using general zero bounds, e.g., Theorem (1), (2), (11) etc. A positive example for this is the polynomial

$$ f_2(z) = (-2.8832 - i.0.6938)^2 + (-0.0685 + i.0.9999)^2 + (0.6084 + i.1.8486)^2 + (0.6949 + i.0.3047)^2 + (-0.9227 - i.0.9249)^2 + (5.637 \times 10^{-05} + i.0.3463)^2 + (0.6214 + i.0.1671)^2 + (-0.2187 - i.0.0296)^2 + (-0.6531 - i.0.9661)^2 + (0.5192 + i.2.2248)^2 + (-1.9030 - i.1.3129) \in C_3. $$

$f_2(z)$ has the property that its coefficients $a_i$ are sampled from a Gaussian Distribution (GD) and it holds $\frac{|a_i|}{|a_0|} < 1, i = 1, \ldots, n - 1$. We yield that the special bound due to Dehmer, Theorem (10) is best $b_D = 1.3374$ and $d_{bD}^1 = 0.2214$. The general zero bounds due to Cauchy and Joyal (Theorem (13) and Theorem (8); for $p = q = 2$) are second best, $b_C = 1.3685$, $d_{bC}^1 = 0.2525$ and third best $b_{D} = 1.480$, $d_{bD}^1 = 0.364$. Thus, the special zero bound, Theorem (10) outperforms two classical and general zero bounds by using $f_2(z)$. It is clear that the coefficients of the underlying polynomials have a strong impact on the values of the bounds as it be seen by the comparison of $f_1(z) \in C_1$ and $f_2(z) \in C_1$. This can be also seen by comparing the quantities $d_{b1}^1$ and $d_{b2}^1$. Finally, the chosen bounds are more optimal for $f_2(z)$ in the sense that the $d_{b}^2$ values are much smaller.

Apart from Dehmer’s bounds, we also discuss the quality of the bounds due to Kalantari [13] in a more general context. Note that the Kalantari bounds [8] have already been evaluated and compared with others by McNamee and Olhovsky [8]. In particular, these bounds have been proven efficient for 1200 polynomials with random real or complex roots. We start with general polynomials given by Definition (1). For $n \leq 20$, some of Dehmer’s bounds, e.g., Theorem (9), (15) outperform the bound of Kalantari, Theorem (12). If $n$ grows, we see that by using other classes the Kalantari bound, Theorem (12) is quite sharp compared to other bounds (except the Cauchy bound, Theorem (13)). The second bound developed by Kalantari, Theorem (11) is worse than almost all Dehmer bounds and others for all $n$. Note that we only evaluated the Kalantari bounds for $m = 2,3$; see the underlying concomitant polynomial represented by Equation (15). As a conclusive remark, we find (Table 1–Table 6) that other zero bounds due to Dehmer, Cauchy and Joyal often outperform these bounds. This interesting finding is in contrast to the result due to McNamee and Olhovsky [8], who identified the Kalantari bounds as best when being compared to other classical bounds such as the ones due to Deutsch [14].

Summary and Conclusion

In this paper, we investigated the quality of zero bounds for complex polynomials numerically. By knowing that the bound values surely depend on the underlying coefficients, we generated several classes of complex polynomials (see section ‘Data: Classes of Complex Polynomials’) to apply the bounds. The set of bounds we have applied consists of (i) classical bounds due to Cauchy [3], Joyal [10], Kuniyeda [11], Kojima [12] etc. and (ii) recently developed bounds due to Dehmer [1,9] and Kalantari [13]. Note that the just mentioned zero bounds are different to the ones used by McNamee and Olhovsky [8]. Our findings based on the used classes of complex polynomials show that some of the classical results, e.g., Kuniyeda, Kojima and Mohammad are not suitable to locate the zeros optimally. This does not mean that for some other classes or special polynomials, these bounds could perform better. As shown by Rahman and Schmeisser [6], the classical (implicit) zero bound due to Cauchy, Theorem (13) is often optimal within this class of bounds. Thus, it is not surprising that this bound often performs best for our classes. Anyway, we have found other zero bounds which outperformed this bound for particular classes of polynomials. Hence, it would be valuable to derive further special bounds for special classes of polynomials, see also [1,9].
This study has illustrated some strong and weak points of the used bounds. As conclusion, it seems that there exist only a few zero bounds which give optimal bound values for a variety of complex polynomials. A reason for this is that in view of the vast amount of existing bounds, their quality has only been very little investigated. Also the result where some of the Dehmer bounds (Theorem (17), (18)) outperform the classical (and sharp) Cauchy bound for lacunary polynomials make us conclude that it will be useful to derive further novel bounds for special cases. In fact, special polynomials, i.e., whose coefficients fulfill special conditions often occur in control engineering, algebraic biology and mathematical chemistry.

Author Contributions
Analyzed the data: YT. Wrote the paper: MD YT. Performed the mathematical analysis and interpreted the results: MD.

References
1. Dehmer M On the location of zeros of complex polynomials. Journal of Inequalities in Pure and Applied Mathematics, Vol 7 (1) (2006).
2. Heitzinger W, Troch WI, Valentin G (1983) Praxis nichtlinearer Gleichungen. Carl Hanser Verlag, München, Wien, Germany, Austria.
3. Marden M (1966) Geometry of polynomials. Mathematical Surveys of the American Mathematical Society, Vol. 3. Rhode Island, USA.
4. Mignotte M, Stefanescu D (1999) Polynomials: An Algorithmic Approach. Discrete Mathematics and Theoretical Computer Science. Springer. Singapore.
5. Prasolov VV (2004) Polynomials. Springer.
6. Rahman QI, Schmeisser G (2002) Analytic Theory of Polynomials. Critical Points, Zeros and Extremal Properties. Clarendon Press. Oxford, UK.
7. Obreschkoff N (1963) Verfahren und Berechnung der Nullstellen reeller Polynome. Hochschulbücher für Mathematik, Vol. 53. VEB Deutscher Verlag der Wissenschaften. Berlin, Germany.
8. McNamara JM, Olhoff N (2005) A comparison of a priori bounds on (real or complex) roots of polynomials. In: Proceedings of 17th IMACS World Congress, Paris, France.