Automorphy lifting with adequate image

Konstantin Miagkov\textsuperscript{1} and Jack A. Thorne\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, Stanford University, 450 Jane Stanford Way Building 380, Stanford, 94305, USA; E-mail: kmiagkov@stanford.edu.
\textsuperscript{2}Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge, CB3 0WB, UK; E-mail: thorne@dpmms.cam.ac.uk.

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Abstract

Let $F$ be a CM number field. We generalise existing automorphy lifting theorems for regular residually irreducible $p$-adic Galois representations over $F$ by relaxing the big image assumption on the residual representation.

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1. Introduction

This paper closely builds on [ACC+18], which proves modularity lifting theorems for regular $n$-dimensional Galois representations over a CM number field $F$ without any self-duality condition. In this paper, we generalise the main results of [ACC+18] to relax the big image assumption on the residual representation from ‘enormous image’ to ‘adequate image’. Following [Tho12], we define ‘adequate image’:

**Definition 1.1.** Let $k$ be a finite field of characteristic $p$, such that $p \nmid n$, and let $G \subset \text{GL}_n(k)$ be a subgroup which acts absolutely irreducibly on $V = k^n$. We suppose that $k$ is large enough to contain all eigenvalues of all elements of $G$. If $g \in G$ and $\alpha \in k$ is an eigenvalue $g$, we write $e_{g,\alpha} : V \to V$ for the $g$-equivariant projection to the generalised $\alpha$-eigenspace. We say that $G$ is adequate if the following conditions are satisfied:

1. $H^0(G, \text{ad}^0 V) = 0$.
2. $H^1(G, k) = 0$.
Let $\mathcal{F}$ be an imaginary CM or totally real field, let $c \in \text{Aut}(F)$ be complex conjugation and let $p$ be a prime. Suppose given a continuous representation $\rho : G_F \to \text{GL}_n(\mathbb{Q}_p)$ satisfying the following conditions:

1. $\rho$ is unramified almost everywhere.
2. For each place $v \mid p$ of $F$, the representation $\rho|_{G_{F_v}}$ is crystalline. The prime $p$ is unramified in $F$.
3. $\overline{\rho}$ is absolutely irreducible and decomposed generic. The image of $\overline{\rho}|_{G_{F_c(p)}}$ is adequate.
4. There exists $\pi \in G_F - G_F(c_p)$, such that $\overline{\rho}(\pi)$ is a scalar. We have $p > n^2$.
5. There exists a cuspidal automorphic representation $\pi$ of $\text{GL}_n(A_F)$ satisfying the following conditions:
   a. $\pi$ is regular algebraic of weight $\lambda$, this weight satisfying
   $\lambda_{\tau,1} + \lambda_{\tau c,1} - \lambda_{\tau,n} - \lambda_{\tau c,n} < p - 2n$
   for all $\tau$.
   b. There exists an isomorphism $i : \overline{\mathbb{Q}}_p \to \mathbb{C}$, such that $\overline{\rho} \cong r_i(\pi)$, and the Hodge-Tate weights of $\rho$ satisfy the formula for each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$:
   \[ HT_\tau(\rho) = \{\lambda_{\tau,1} + n - 1, \lambda_{\tau,2} + n - 2, \ldots, \lambda_{\tau,n}\}. \]
   c. If $v \mid p$ is a place of $F$, then $\pi_v$ is unramified.

Then $\rho$ is automorphic: there exists a cuspidal automorphic representation $\Pi$ of $\text{GL}_n(A_F)$ of weight $\lambda$, such that $\rho \cong r_i(\Pi)$. Moreover, if $v$ is a finite place of $F$ and either $v \mid p$ or both $\rho$ and $\pi$ are unramified at $v$, then $\Pi_v$ is unramified.

**Theorem 1.3.** Let $\mathcal{F}$ be an imaginary CM or totally real field, let $c \in \text{Aut}(F)$ be complex conjugation and let $p$ be a prime. Suppose given a continuous representation $\rho : G_F \to \text{GL}_n(\overline{\mathbb{Q}}_p)$ satisfying the following conditions:

1. $\rho$ is unramified almost everywhere.
2. Let $\mathbf{Z}^n \subseteq \{((\lambda_1, \ldots, \lambda_n) \in \mathbf{Z}^n \mid \lambda_1 \geq \ldots \geq \lambda_n\}$. For each place $v \mid p$ of $F$, the representation $\rho|_{G_{F_v}}$ is potentially semistable, ordinary with regular Hodge-Tate weights. In other words, there exists a weight $\lambda \in (\mathbf{Z}^n_\mathbf{w})_{\text{Hom}(F, \overline{\mathbb{Q}}_p)}$, such that for each place $v \mid p$, there is an isomorphism
   \[ \rho|_{G_{F_v}} \cong \begin{pmatrix} \psi_{v,1} & * & * & * \\ 0 & \psi_{v,2} & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & \psi_{v,n} \end{pmatrix}, \]
   where for each $i = 1, \ldots, n$ the character $\psi_{v,i} : G_{F_v} \to \overline{\mathbb{Q}}_p^\times$ agrees with the character
   \[ \sigma \in I_{F_v} \mapsto \prod_{\tau \in \text{Hom}(F_v, \overline{\mathbb{Q}}_p)} \tau(\text{Art}_{F_v}^{-1}(\sigma))^{(\lambda_{\tau,n-i+1}+i-1)} \]
   on an open subgroup of the inertia group $I_{F_v}$. 


3. $\overline{\rho}$ is absolutely irreducible and decomposed generic. The image of $\overline{\rho}|_{G_F(\mathbb{Q}_p)}$ is adequate.
4. There exists $\sigma \in G_F - G_F(\zeta_p)$, such that $\overline{\rho}(\sigma)$ is a scalar: We have $p > n$.
5. There exists a cuspidal automorphic representation $\pi$ of $GL_n(\mathbf{A}_F)$ and an isomorphism $\iota : \overline{\mathbb{Q}}_p \rightarrow \mathbb{C}$, such that $\pi$ is $\iota$-ordinary and $\overline{\rho} \cong r_{\iota}(\pi)$.

Then $\rho$ is ordinarily automorphic of weight $\omega$: there exists a $\iota$-ordinary cuspidal automorphic representation $\Pi$ of $GL_n(\mathbf{A}_F)$ of weight $\omega$, such that $\rho \cong r_{\Pi}(\iota)$. Moreover, if $v \nmid p$ is a finite place of $F$ and both $\rho$ and $\pi$ are unramified at $v$, then $\Pi_v$ is unramified.

The theorems above are very similar to [ACC+18, Theorem 6.1.1] and [ACC+18, Theorem 6.1.2], respectively. The only difference is replacing the enormous condition on image of $\overline{\rho}|_{G_F(\mathbb{Q}_p)}$ with adequate. This is a useful improvement, particularly in light of [GHTT12], which proves that when $p > 2(n + 1)$, adequacy is equivalent to absolute irreducibility. This makes it a condition easy to work with in the context of automorphy lifting theorems to ‘adequate’ in [Tho12]. To make the argument work in the parahoric setting, we need to analyse the representations of $GL_n$ with parahoric level. Another novel component is a proof of a ‘growth of the space of cusp forms’-type result when adding Taylor-Wiles primes instead of Iwahori-level, the idea first introduced to relax the big image assumption in the representations of $GL_n$ with parahoric level. A notable difficulty in comparison to [ACC+18] is that we can no longer restrict to working with generic local representations, since they arise as components of cuspidal automorphic representations of unitary groups instead of $GL_n$. The local computations allow us to prove the necessary local-global compatibility results for Galois representations landing in Hecke algebras acting on cohomology of locally symmetric spaces with parahoric level. Another novel component is a proof of a ‘growth of the space of cusp forms’-type result when adding Taylor-Wiles primes with parahoric level, which requires an investigation of representations of $GL_n(F_v)$ over fields of finite characteristic.

1.1. Notation

We write $GL_n$ for the usual general linear group (viewed as a reductive group scheme over $\mathbf{Z}$) and $T_n \subset B_n \subset GL_n$ for its subgroups of diagonal and of upper triangular matrices, respectively. We identify $X^*(T)$ with $\mathbf{Z}^n$ in the usual way and write $Z^\ast_n \subset Z_n$ for the subset of $B_n$-dominant weights. If $R$ is a local ring, we write $\mathfrak{m}_R$ for the maximal ideal of $R$. If $\Gamma$ is a profinite group and $\rho : \Gamma \rightarrow GL_n(\overline{\mathbb{Q}}_p)$ is a continuous homomorphism, then we will let $\overline{\rho} : \Gamma \rightarrow GL_n(\overline{\mathbb{F}}_p)$ denote the semisimplification of its reduction, which is well defined up to conjugacy (by the Brauer-Nesbitt theorem). If $M$ is a topological abelian group with a continuous action of $\Gamma$, then by $H^1(\Gamma, M)$, we shall mean the continuous cohomology. If $G$ is a locally profinite group, $U \subset G$ is an open compact subgroup and $R$ is a commutative ring, then we write $\mathcal{H}_R(G, U)$ for the algebra of compactly supported, $U$-biinvariant functions $f : G \rightarrow R$, with multiplicity given by convolution with respect to the Haar measure on $G$ which gives $U$ volume 1. If $X \subset G$ is a compact $U$-biinvariant subset, then we write $[X]$ for the characteristic function of $X$, an element of $\mathcal{H}_R(G, U)$. When $R$ is omitted from the notation, we take $R = \mathbf{Z}$. We write $\iota_H$ for the anti-involution given by $\iota_H(f)(g) = f(g^{-1})$.

If $F$ is a perfect field, we let $\overline{F}$ denote an algebraic closure of $F$ and $G_F$ the absolute Galois group $\text{Gal}(\overline{F}/F)$. We will use $\zeta_n$ to denote a primitive $n$-th root of unity when it exists. Let $e_l$ denote the $l$-adic cyclotomic character. We will let rec$_K$ be the local Langlands correspondence of [HT01], so that if $\pi$ is an irreducible complex admissible representation of $GL_n(K)$, then rec$_K(\pi)$ is a Frobenius semisimple Weil-Deligne representation of the Weil group $W_K$. If $K$ is a finite extension of $\mathbb{Q}_p$ for some $p$, we write $K^{nr}$ for its maximal unramified extension, $I_K$ for the inertia subgroup of $G_K$, $\text{Frob}_K \in G_K/I_K$ for the geometric Frobenius and $W_K$ for the Weil group. We will write $\text{Art}_K : K^× \rightarrow W_K^{ab}$ for the Artin map normalised to send uniformisers to geometric Frobenius elements.
We will write $\text{rec}$ for $\text{rec}_K$ when the choice of $K$ is clear. We write $\text{rec}_K^T$ for the normalisation of the local Langlands correspondence as defined in, for example [CT14, Section 2.1]; it is defined on irreducible admissible representations of $GL_n(K)$ defined over any field which is abstractly isomorphic to $\mathbb{C}$ (e.g. $\mathbb{Q}_p$). If $(r, N)$ is a Weil-Deligne representation of $W_K$, we will write $(r, N)^F - ss$ for its Frobenius semisimplification. If $\rho$ is a continuous representation of $G_K$ over $\mathbb{Q}_l$ with $l \neq p$, we then will write $WD(\rho)$ for the corresponding Weil-Deligne representation of $W_K$. By a Steinberg representation of $GL_n(K)$, we will mean a representation $S_{p,n}(\psi)$ (in the notation of Section 1.3 of [HT01]), where $\psi$ is an unramified character of $K^\times$.

If $G$ is a reductive group over $K$ and $P$ is a parabolic subgroup with unipotent radical $N$ and Levi component $L$, and if $\pi$ is a smooth representation of $L(K)$, then we define $\text{Ind}_{P(K)}^{G(K)}\pi$ to be the set of locally constant functions $f : G(K) \rightarrow \pi$, such that $f(hg) = \pi(hN(K))f(g)$ for all $h \in P(K)$ and $g \in G(K)$. It is a smooth representation of $G(K)$, where $(g_1f)(g_2) = f(g_2g_1)$. This is sometimes referred to as ‘un-normalised’ induction. We let $\delta_P$ denote the determinant of the action of $L$ on $\text{Lie}_N$. Then we define the ‘normalised’ induction $\text{ind}_{P(K)}^{G(K)}\pi$ to be $\text{Ind}_{P(K)}^{G(K)}(\pi \otimes |\delta_P|^{1/2})$. We also define a parabolic restriction functor $r_{G/K}^{P(K)}$ from $G(K)$-representations to $L(K)$-representations to be the composition of restriction to $P(K)$ and taking $N(K)$-coinvariants. If $F$ is a CM number field and $\pi$ is an automorphic representation of $GL_n(\mathbb{A}_F)$, we say that $\pi$ is regular algebraic if $\pi_{\infty}$ has the same infinitesimal character as an irreducible algebraic representation $W$ of $(\text{Res}_F/Q \text{ GL}_n)_c$. If $W^\vee$ has highest weight $\lambda \in (\mathbb{Z}_+)^{\text{Hom}(F,\mathbb{C})}$, then we say $\pi$ has weight $\lambda$.

If $P(X) \in A[X]$ is a polynomial of degree $n$ over any ring $A$, such that $P(0) \in A^\times$, we write $P^\vee(X)$ for $P(0)^{-1}X^nP(X^{-1})$. For two polynomials $P, Q \in A[X]$, we write $\text{Res}(P, Q)$ to denote their resultant.

Given a Galois representation $\rho : G_{F,S} \rightarrow \text{GL}_n(A)$, we will write $\rho^\perp := \rho^c \otimes \epsilon^{1 - 2n}$, and given a $G_{F,S}$-group determinant $D$, we will denote by $D^\perp$ the corresponding dual.

2. Representation theory of $\text{GL}_n(F_v)$ in characteristic $p$

Let $p$ be a rational prime and $k = \overline{F}_p$. Let $F/\mathbb{Q}$ be a finite extension, and let $x$ be a prime in $F$ with residue field $k_x$ of order $q$ satisfying $q \equiv 1 \pmod{p}$ and the corresponding ring of integers $\mathcal{O}_x = \mathcal{O}_{F_x}$. Set $G_x = \text{Gal}(\overline{F}_x/F_x)$. Also set $G = \text{GL}_n$ with $p > n$, and let $T \subset B \subset G$ be the maximal torus and the corresponding Borel and $U \subset G$ be the unipotent subgroup. Let $K^1(x) \subset \text{GL}(G_x)$ be the full congruence subgroup. We also let $Iw, Iw_1 \subset \text{GL}(G_x)$ be the Iwahori and the Iwahori-1, respectively, and let $Iw_1 \subset Iw_P \subset Iw$ be the subgroup, such that $[Iw_P : Iw_1]$ has order prime to $p$ and $[Iw : Iw_P]$ has $p$-power order. Let $p(x)$ be a two-block parahoric subgroup of $G(G_x)$ with blocks of sizes $n_1 + n_2 = n$ and $P$ the corresponding parabolic. Let $W_S \cong S_n$ be the Weyl group for $\text{GL}_n$, and for a given parabolic subgroup $Q \subset G$, let $W_Q \subset W$ be the Weyl group of its Levi factor. Set $T_0 := T(G_x)$ and $T_1 := \ker(T_0 \rightarrow T(G_x/\pi))$. Fix $\overline{\rho} : G_x \rightarrow \text{GL}_n(k)$—a continuous unramified semisimple representation. We say that an irreducible admissible representation $\pi$ of $G$ over $k$ is associated to $\overline{\rho}$ if $\pi$ is a subquotient of $\text{Ind}_B^G\chi_1 \otimes \ldots \otimes \chi_n$, where $\chi_i$ are unramified characters, such that $\{\chi_1(\pi), \ldots, \chi_n(\pi)\}$ is the set of eigenvalues of $\overline{\rho}(\text{Frob}_x)$. We write $I(\chi_1, \ldots, \chi_n)$ for $\text{Ind}_B^G\chi_1 \otimes \ldots \otimes \chi_n$. The following lemma shows that if we do not fix the ordering of $\chi_i$, then we can always consider $\pi$ to be a subrepresentation of $I(\chi)$.

Proposition 2.1. Let $\pi$ be an irreducible admissible $k[G]$-module associated to $\overline{\rho}$. Then there exists an ordering of $\chi_1, \ldots, \chi_n$, such that $\pi$ is a subrepresentation of $I(\chi)$.

Proof. We use the adjunction between $\text{Ind}_B^G$ and the parabolic restriction $r_B^G$ to get an isomorphism

$$\text{Hom}(\pi, I(\chi)) \cong \text{Hom}(r_B^G(\pi), \chi).$$

Since $\pi$ is associated to $\overline{\rho}$, we know that $r_B^G(\pi) \neq 0$. Since $r_B^G(\pi)$ is a representation of the torus, there exists a 1-dimensional quotient given by some character $\chi : T \rightarrow k^\times$. Then we get that $\text{Hom}(\pi, I(\chi)) \neq 0$, and since $\pi$ is irreducible, this implies that $\pi$ is a subrepresentation of $I(\chi)$. Then $\chi$ forms the
supercuspidal support of \( \pi \) and in fact has to be a permutation of the original \( \chi_1, \ldots, \chi_n \). For the notion of supercuspidal support in positive characteristic, see [Vig96, II.2.6]. We would also like to remark, here, that in the case \( q \equiv 1 \pmod{p} \), \( p > n \), the notions of cuspidal and supercuspidal representations coincide (see [Vig96, II.3.9]). \( \square \)

We now describe the Bernstein presentation of Iwahori-Hecke algebra \( \mathcal{H}_k(G, \text{Iw}) \), following [Vig96, I.3.14]. Let

\[
t_j = \text{diag}(\varpi, \ldots, \varpi, 1, \ldots, 1),
\]

and set \( T_j = [\text{Iw} \ t_j \ \text{Iw}] \) and \( X^j = T_j(T_{j-1})^{-1} \). We also let \( s_j \) be the permutation matrix corresponding to the transposition \((j, j + 1)\) and set \( S^j = [\text{Iw} \ s_j \ \text{Iw}] \). The elements \( X^j \) for \( 1 \leq j \leq n \) generate the group algebra \( k[\mathbb{Z}^n] \) on which \( S_j \) acts by permuting the indices. The Bernstein presentation states that

\[
\mathcal{H}_k(G, \text{Iw}) \cong k[S_n \rtimes \mathbb{Z}^n]
\]

under the action described above.

Now we introduce some useful Hecke operators. For any ring \( R \), \( 1 \leq i \leq n_1 \) and \( 1 \leq j \leq n_2 \) let \( V^{j,2} \in \mathcal{H}_R(G, \text{p}(x)) \) be the Hecke operator associated to the double coset

\[
[p(x) \ \text{diag}(1, \ldots, 1, \varpi, \ldots, \varpi, 1, \ldots, 1)p(x)]
\]

and let \( V^{i,1} \) be associated to

\[
[p(x) \ \text{diag}(\varpi, \ldots, \varpi, 1, \ldots, 1)p(x)].
\]

The following is part of [CHT08, Theorem B.1]:

**Proposition 2.2.** Let \( V \) be an irreducible admissible \( k[G] \)-module, which is generated by its Iwahori-invariant vectors. Then \( V^{\text{Iw}} = V^{\text{Iw}_1} \).

Under the conditions of 2.2, we thus get an isomorphism

\[
H^1(\text{Iw}, V) \cong H^1(B(k), V^{K_1(x)}) \cong H^1(T(k), V^{\text{Iw}_1}) \cong H^1(T(k), V^{\text{Iw}}) \cong \text{Hom}(T(k), V^{\text{Iw}}).
\]

Both sides of 2.3 can be endowed with the action of \( \mathcal{H}_k(G, \text{Iw}) \). On \( H^1(\text{Iw}, V) \), we take the derived \( \mathcal{H}_k(G, \text{Iw}) \)-action, and on \( \text{Hom}(T(k), V^{\text{Iw}}) \), we consider the natural action on the target.

**Proposition 2.4.** The isomorphism 2.3 is equivariant with respect to \( X^i \) for all \( 1 \leq i \leq n \).

**Proof.** The action of \( X^i \) on \( [f] \in H^1(\text{Iw}, V) \) can be described as follows. Write

\[
\text{Iw} \ t_i \ \text{Iw} = \bigsqcup_j g_{i,j} \ \text{Iw}.
\]

We now give an explicit description for \( g_{i,j} \). Fix a set of representatives \( S \subset \mathcal{O}_F \) for \( k \). For each \( m \in M_{i \times (n-i)}(S) \), let \( g_{i,m} \) be the matrix, such that \( g_{i,m}(k,k) = \varpi \) for \( k \leq i \), \( g_{i,m}(k,k) = 1 \) for \( k > i \) and \( g_{i,m}(k,\ell) = m(k,\ell-i) \) for \( k \leq i, \ell > i \). The rest of the entries are set to 0. Let us show that this is
a full set of representatives. First we show that $g_{i,m}$ represent distinct cosets, that is that $g_{i,m}^{-1}g_{i,m'} \notin \text{Iw}$ for $m \neq m'$. Suppose $m(k, \ell) \neq m'(k, \ell)$. Then

$$(g_{i,m}^{-1}g_{i,m'})(k, \ell + i) = \sigma^{-1}(m'(k, \ell) - m(k, \ell))$$

which is not in $\mathcal{O}_F$. Now we just need to verify that the number of cosets is $q^{\ell(n-i)}$. Indeed,

$$[\text{Iw} t_i \text{Iw} : \text{Iw}] = [\text{Iw} : \text{Iw} \cap t_i \text{Iw}^{-1}] = q^{\ell(n-i)}$$

since $\text{Iw} \cap t_i \text{Iw}^{-1}$ are just the elements of the Iwahori whose $(k, \ell)$-coordinates for $k \leq i, \ell > i$ vanish mod $\sigma$. Then

$$(X^i [f])(x) = \sum_j g_{i,\sigma(j)}f(g_{i,\sigma(j)}^{-1}xg_{i,\sigma(j)})$$

where $\sigma$ is the unique permutation, such that

$$g_{i,\sigma(j)}^{-1}xg_{i,\sigma(j)} \in \text{Iw}$$

for all $j$. Denote by $\overline{\cdot} : \text{Iw} \to T(k)$ the reduction map. Let $s$ be the inverse of 2.3. For $[\tau] \in \text{Hom}(T(k), V^{\text{Iw}})$, we get

$$(X^i [s(\tau)])(x) = \sum_j g_{i,\sigma(j)}s(\tau)(g_{i,\sigma(j)}^{-1}xg_{i,\sigma(j)}) = \sum_j g_{i,\sigma(j)}s(\tau(x)) = s(X^i[\tau])(x).$$

The second equality is due to all the $g_{i,j}$ being in the Borel and having the same diagonal. \hfill \Box

**Definition 2.5.** A $G$-modules $V$ over $k$ is **locally admissible** if it is smooth, and for every $v \in V$ the subrepresentation generated by $v$ is admissible. Let $C$ denote the abelian category of locally admissible $G$-modules $V$ over $k$, such that every irreducible quotient of $V$ is associated to $\overline{p}$.

The following is analogous to [CG18, Lemma 9.14]:

**Proposition 2.6.** The category $C$ has enough injectives, and the inclusion functor from $C$ to locally admissible $G$-modules is exact.

**Proof.** Inside the category of $G$-modules, the category $C$ is fully contained inside the unipotent block (the block containing the trivial representation). By part 4) of [CHT08, Theorem B.1], the unipotent block is equivalent to the category of $\mathcal{H}_k(G, \text{Iw}^p)$-modules. Via the Bernstein embedding\(^1\), such modules can naturally be viewed as $\mathcal{H}_k(G, G(\mathcal{O}_x))$-modules, where $\mathcal{H}_k(G, G(\mathcal{O}_x))$ can be explicitly described via the Satake isomorphism as $k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^W$. Here, we use the Satake isomorphism twisted by $|\det|^{(1-n)/2}$, which is defined over $Z[q^{-1}]$. If $V$ is any locally admissible element of the unipotent block, the associated Hecke module $V^{\text{Iw}^p}$ is locally finite-dimensional over $k$, and thus we can write

$$V^{\text{Iw}^p} = \bigoplus_m V_m^{\text{Iw}^p},$$

where the sum is taken over all maximal ideals of $\mathcal{H}_k(G, G(\mathcal{O}_x))$. Let $\mathcal{D}$ denote the category of locally admissible representations in the unipotent block. Then we can write $\mathcal{D} = \bigoplus_m \mathcal{D}_m$, where $\mathcal{D}_m$ consists

\(^1\)For the details on the Bernstein embedding $k[Z^n] \to \mathcal{H}_k(G, I)$ in the case of an arbitrary open compact subgroup $I \subset \text{Iw}$, such that $\text{Iw}_1 \subset I$, see [ACC\(^+\)18, Section 2.2.4]. We note that there the authors are working over some $p$-adic ring $\mathcal{O}$, but the results are valid over $k$ as well since $q \equiv 1 \pmod{p}$. 

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of representations whose associated $\mathcal{H}_k(G, G(\mathcal{O}_x))$-module is supported only at $m$. The maximal ideals of $\mathcal{H}_k(G, G(\mathcal{O}_x))$ have the form $(t_1 - a_1, \ldots, t_n - a_n)$, where $a_i \in k$ and $t_i = e_i(X_1, \ldots, X_n)$ is the $i$-th elementary symmetric polynomial of $X_1, \ldots, X_n$. If we now let $n$ be the ideal defined by $a_i = e_i(\chi_1(\sigma), \ldots, \chi_n(\sigma))$, then it is clear that $C = \mathcal{D}_n$. The exactness is now clear, and to show that $C$ has enough injectives, it is enough to check that the category Mod$_G^{\text{adm}}(k)$ of locally admissible $G$-modules has enough injectives. The full category Mod$_G(k)$ certainly has enough injectives, and the functor $L : \text{Mod}_G(k) \to \text{Mod}_G^{\text{adm}}(k)$ taking a module to its smooth locally admissible vectors is right adjoint to the natural embedding Mod$_G^{\text{adm}}(k) \to \text{Mod}_G(k)$. This proves the claim. \hfill $\square$

From now on, fix $\alpha = \chi_i(\sigma)$ for some $1 \leq i \leq n$. Let

$$P(X) = \prod_{i=1}^{n}(X - \chi_i(\sigma)).$$

For $1 \leq j \leq n_2$, let $P_j$ be a polynomial whose roots with multiplicities are precisely

$$\sum_{J \subseteq S, a \in J \atop \#J = j} \chi_a(\sigma).$$

Factor $P_j = Q_jR_j$, where

$$R_j(X) = \left(X - \binom{n_2}{j}a_j^j\right)^{k_j}$$

and $Q_j, R_j$ are coprime. Set

$$e_\alpha := \lim_{m \to \infty} \left(\prod_{i=1}^{n_2} Q_j(V^{j, 2i})\right)^{m!}.$$

Here, we consider $e_\alpha$ as an operator acting on $V^p(x)$ for $V \in C$. Since objects in $C$ are locally admissible, the limit makes sense.

We now define two functors $F, G : C \to k$-$\text{Vect}$. On objects, we set

$$F(V) := V^{G(\mathcal{O}_x)}, \quad G(V) := e_\alpha V^p(x).$$

Note that $F, G$ are both left-exact and $e_\alpha$ is exact. Then we can form derived functors $R^k F, R^k G$ and identify

$$R^k F(V) = H^k(G(\mathcal{O}_x), V), \quad R^k G(V) = e_\alpha H^k(p(x), V).$$

We have a natural transformation $\iota : F \to G$ given by composing the inclusion $V^{G(\mathcal{O}_x)} \hookrightarrow V^p(x)$ with $e_\alpha$. We will make use of the following simple algebraic fact.

**Lemma 2.7.** Let $G$ be a profinite group and $H \triangleleft G$ be a normal subgroup. Let $A$ be a $p$-torsion $G$-module for some positive integer $p$, and let $H$ have pro-$q$ order for a prime $q$ satisfying $q \equiv 1 \pmod{p}$. Then the inflation map

$$\inf : H^1(G/H, A^H) \to H^1(G, A)$$

is an isomorphism whose inverse sends a cocycle $[f] \in H^1(G, A)$ to

$$g \mapsto f(g) + (1 - g)a_f$$

for some $a_f \in A$. 

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Proof. The condition $q \equiv 1 \pmod{p}$ ensures that $H^1(H, A)$ vanishes. Then it is enough to take $(g - 1)a_f$ to be the coboundary trivialising the restriction of $[f]$ to $H$. □

Proposition 2.8. Let $\pi$ be an irreducible admissible $k[G]$-module associated to $\bar{\rho}$. Then the map

$$f : H^1(G(k), \pi^{K_1(x)}) \rightarrow e_\alpha H^1(P(k), \pi^{K_1(x)})$$

is injective.

Proof. Both cohomology groups in question inject into $H^1(B(k), \pi^{K_1(x)})$ since

$$[G(k) : B(k)] \equiv n! \not\equiv 0 \pmod{p}$$

when $p > n$, so let us analyse that group. Since $q \equiv 1 \pmod{p}$, by inflation-restriction, we get

$$H^1(B(k), \pi^{K_1(x)}) \cong H^1(T(k), \pi^{Iw}).$$

As a special case of 2.3, we have

$$H^1(Iw, \pi) \cong H^1(B(k), \pi^{K_1(x)}) \cong \text{Hom}(T(k), \pi^{Iw}) \cong (\pi^{Iw})^\otimes n. \quad (2.9)$$

The isomorphism above is equivariant with respect to the natural actions of $\{X^i\}$ on both sides arising from the actions of $\mathcal{H}_k(G, Iw)$ by Proposition 2.4. The space $\pi^{Iw}$ injects into $I(\chi)^{Iw}$, which has a basis $\{\varphi_w\}$ for $w \in W$, where $\varphi_w$ is supported on $BwIw$ and satisfies $\varphi_w(w) = 1$. It follows from the proof of [Tho12, Lemma 5.10], that on each component of $I(\chi)^{Iw}$, the operator $e_\alpha$ acts as a projection onto the space spanned by $\{\varphi_{w'} \mid w' \in W'\}$, where $W'$ is the subset of $W$ consisting of permutations which send $\{n_1 + 1, \ldots, n\}$ to the positions of $\alpha$-s in the sequence $\chi_1(\omega), \ldots, \chi_n(\omega)$. On the level of cocycles, the isomorphism 2.9 sends $[s] \in H^1(B(k), \pi^{K_1(x)})$ to the map

$$g \mapsto s(g) + (1 - g)\psi$$

for some $\psi \in I(\chi)$ (Lemma 2.7). Thus, a cocycle $[s] \in H^1(G(k), I(\chi)^{K_1(x)})$ being in the kernel of $f$ means that for all $t \in T(k)$ and $w_0 \in W'$, we have

$$(s(t) + (1 - t)\psi)(w_0) = 0. \quad (2.10)$$

For any $w \in W$, we have

$$(t\psi)(w) = \psi(w\bar{t}) = \psi(w\bar{t}w) = \psi(w).$$

Here, $\bar{t}$ is a lift of $t$ to $T_0$ and $w$ acts on the torus in a natural way. Note that here, we used that $\chi$ is unramified. Thus

$$((1 - t)\psi)(w) = 0. \quad (2.11)$$

Combining 2.10 and 2.11 applied to $w_0$, we get

$$s(t)(w_0) = 0.$$

Now let us conjugate $t$ by an arbitrary $w \in W$. Since the result is again in $T$, we use the cocycle condition and the transformation law of $I(\chi)$ with respect to the Borel to write

$$0 = s(wtw^{-1})(w_0) = (s(w) + w(s(t) + ts(w^{-1}))) (w_0) \quad (2.12)$$

$$= (wts(w^{-1}))(w_0) = ws(w^{-1})(w_0) = -s(w)(w_0). \quad (2.13)$$

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Combining 2.12 and 2.13, we get

$$0 = (ws(t))(w_0) = s(t)(w_0 w).$$

In other words, we now have $s(t)(w) = 0$ for all $t \in T(k)$ and for all $w \in W$. By 2.11, this implies that $[s] = 0$ since $\{\varphi_w\}$ make a basis for $I(\chi)^{Iw}$. □

**Theorem 2.14.** The natural transformation $\iota : F \to G$ given by $V^G(\chi_x) \mapsto e_\alpha V^p(x)$ on objects is an isomorphism of functors. In particular, we get functorial isomorphisms

$$\iota_* : H^k(G(\mathcal{O}_x), V) \cong e_\alpha H^k(p(x), V)$$

for all $k \geq 0$.

**Proof.** In the proof of Proposition 2.6, we have identified $\mathcal{C}$ with a subcategory of $\mathcal{H}_k(G, Iw^p)$-Mod. Thus, every element of $\mathcal{C}$ is a direct limit of finite length elements of $\mathcal{C}$, and it is, therefore, enough to establish the isomorphism for finite length $V$. The first step will be to show that $\iota(V)$ is an isomorphism for all $V \in \mathcal{C}$. For an irreducible subrepresentation $\pi \subset V$, consider the diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & F(\pi) & \longrightarrow & F(V) & \longrightarrow & F(V/\pi) & \longrightarrow & R^1F(\pi) \\
\downarrow{\iota(\pi)} & & \downarrow{\iota(V)} & & \downarrow{\iota(V/\pi)} & & \downarrow{f} \\
0 & \longrightarrow & G(\pi) & \longrightarrow & G(V) & \longrightarrow & G(V/\pi) & \longrightarrow & R^1G(\pi).
\end{array}
$$

To show that $\iota(V)$ is injective, we can use the four lemmas and induct on the length of $V$. Thus, we only need to show that $\iota(\pi)$ is injective for irreducible $\pi$. This is done in [Tho12, Lemma 5.10].

Now we would like to show that $\iota(\pi)$ is an isomorphism. Consider the injection $\pi \subset I(\chi)$ and the associated diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & F(\pi) & \longrightarrow & F(I(\chi)) & \longrightarrow & F(I(\chi)/\pi) \\
\downarrow{\iota(\pi)} & & \downarrow{\iota(I(\chi))} & & \downarrow{\iota(I(\chi)/\pi)} \\
0 & \longrightarrow & G(\pi) & \longrightarrow & G(I(\chi)) & \longrightarrow & G(I(\chi)/\pi).
\end{array}
$$

We already know that $\iota(I(\chi)/\pi)$ is injective. Then to show that $\iota(\pi)$ is surjective by the four lemmas, we need to know that $\iota(I(\chi))$ is surjective. This follows once again from the proof of [Tho12, Lemma 5.10].

Finally, we are ready to see that $\iota(V)$ is an isomorphism for all $V \in \mathcal{C}$. We induct on the length of $V$ using Eq. 2.15. Since $f$ is injective by Proposition 2.8, the result follows. □

### 3. Representation theory of $GL_n(F_v)$ in characteristic 0

Fix a finite extension $E/\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ which contains the images of all embeddings $F \to \overline{\mathbb{Q}}_p$. We write $\mathcal{O}$ for the ring of integers of $E$ and $\sigma \in \mathcal{O}$ for a choice of uniformiser. If $v$ is a finite place of $F$ prime to $p$, we write $\Xi_v := \mathbb{Z}^n$ and $\Xi_{v,1} := (\tau_v) \times \mathbb{Z}^n$, where $\tau_v$ is the generator of $k_v^\times(p)$—the maximal $p$-power order quotient of $k_v^\times$. We have a natural homomorphism $\mathcal{O}_{F_v}^\times \to \mathbb{Z}[\Xi_{v,1}]$ induced by the homomorphism $\mathcal{O}_{F_v}^\times \to k_v^\times \to k_v^\times(p)$, which we denote by $(\cdot)$. Consider a standard parabolic subgroup $P \subset GL_n(F_v)$ corresponding to a partition $n = n_1 + \ldots + n_m$ which we will denote as $\mu$. Given a partition of $n$, we will always let $s_{\mu,i} = n_1 + \ldots + n_i$, with $s_{\mu,0} = 0$. Let $P = MN$ and $\overline{\mu} = M \overline{N}$ be the Levi decompositions of $P$ and its opposite parabolic. Let $M$ be the hyperspecial maximal compact subgroup of $M$. Define the subgroup of the symmetric group $S_{\mu} = S_{n_1} \times \ldots \times S_{n_m}$. For any positive integer $k$, let

$$S_k : \mathcal{H}_{\mathbb{Z}[q_v^{1/2}]}(GL_k(F_v), GL_k(\mathcal{O}_{F_v})) \to \mathbb{Z}[q_v^{1/2}][X_1^{\pm 1}, \ldots, X_k^{\pm 1}]^{S_k}$$
denote the (normalised) Satake isomorphism. We use these isomorphisms to identify
\[ S_\mu = S_{n_1} \otimes \ldots \otimes S_{n_k} : \mathcal{H}_{Z[q_v^{1/2}]}(M, m) \to \mathbb{Z}[q_v^{1/2}][\Xi_v] S_\mu. \]

Consider any open compact subgroup \( q \) of \( GL_n(F_v) \), and set
\[ q_M = q \cap M, \quad q^+ = q \cap N, \quad q^- = q \cap N. \]

From now on, assume that \( q \) has an Iwahori decomposition with respect to \( P \), which means that \( q = q^- q_M q^+ \). We define a submonoid \( M^+ \subset M \) of positive elements to consist of elements \( m \in M \), such that
\[ mq^+ m^{-1} \subset q^+, \quad m^{-1} q^- m \subset q^- . \]

Inside \( M^+ \), we have a further submonoid \( M^{++} \) of strictly positive elements consisting of \( m \in M^+ \) satisfying the following conditions:

1. For any compact open subgroups \( n_1, n_2 \) of \( N \), there exists a positive integer \( x \geq 0 \), such that \( m^x n_1 m^{-x} \subset n_2 \).
2. For any compact open subgroups \( \pi_1, \pi_2 \) of \( \pi \), there exists a positive integer \( x \geq 0 \), such that \( m^{-x} \pi_1 m^x \subset \pi_2 \).

We denote by \( \mathcal{H}_{\mathcal{O}}(M, q_M)^{+} \) the elements of \( \mathcal{H}_{\mathcal{O}}(M, q_M) \) whose support is contained in \( M^+ \). From now on, we also assume that \( q_v \) has a square root in \( \mathcal{O} \) and fix such square root.

**Proposition 3.1.**
1. The map \( t_\mu^+: \mathcal{H}_{\mathcal{O}}(M, q_M)^{+} \to \mathcal{H}_{\mathcal{O}}(G, q) \) given by
\[ [q_M m q_M] \mapsto \delta_{\pi^{1/2}}(m)[q m q] \]
is an algebra homomorphism.
2. The map \( t_\mu^+ \) extends to a homomorphism \( t_\mu : \mathcal{H}_{\mathcal{O}}(M, q_M) \to \mathcal{H}_{\mathcal{O}}(G, q) \) if and only if there exists a strictly positive element \( \mu \in Z(M) \), such that \( [q \mu q] \) is invertible in \( \mathcal{H}_{\mathcal{O}}(G, q) \).
3. Assuming the existence of the extension in (2), for any smooth \( C[GL_n(F_v)] \)-module \( \pi \), the canonical map \( \pi^q \to \pi^{q_M} \) is a homomorphism of \( \mathcal{H}_{\mathcal{O}}(M, q_M) \)-modules, where \( \mathcal{H}_{\mathcal{O}}(M, q_M) \) acts on \( \pi^q \) via the map \( t_\mu \).

**Proof.** For the first two claims, see [Vig98, II.6]. For the third, see [Vig98, II.10.1]. \( \square \)

Now we record some results about smooth admissible representations of \( GL_n(F_v) \) in characteristic 0. Let \( P \) be a parahoric corresponding to the partition \( n = n_1 + \ldots + n_k \) which we call \( \mu \), and let \( P \) be the underlying parabolic with the Levi decomposition \( P = MN \). Let \( m = M(\mathcal{O}_{F_v}) \). We also let \( p_1, m_1 \) denote the kernels of the homomorphisms
\[ p \to P(k_v) \to GL_{nk}(F_v) \xrightarrow{\text{det}} k_v^\times \to k_v^\times(p) \]
\[ m \to M(k_v) \to GL_{nk}(F_v) \xrightarrow{\text{det}} k_v^\times \to k_v^\times(p). \]

Finally, let \( \text{Iw}' = p_1 \cap \text{Iw} \).

**Lemma 3.2.** The condition in part (2) of Proposition 3.1 is satisfied for \( q = p, p_1 \).

**Proof.** This is a special case of [Whi22, Proposition 5.7]. \( \square \)
Fix a uniformiser $\varpi$ of $F_v$. For any $1 \leq j \leq k$ and $1 \leq i \leq n_j$, consider the operators in $\mathcal{H}_\mathbb{O}(G, p)$ given by

$$V_{i,j}^i = t_{i,j}^{-1}(e_i(X_{s_{i,j-1}+1}, \ldots, X_{s_{i,j}})).$$

We will also consider operators in $\mathcal{H}_\mathbb{O}(G, p_1)$, such that their actions on $\pi^p \subset \pi^p_1$ agree with the action of $V_{i,j}^i$ for any smooth representation $\pi$. They can be constructed in the same way as $V_{i,j}^i$ above by replacing $S_{i,j}$ with the Satake isomorphism for $m_1$ from [Whi22, Theorem 5.1]. These operators will also be denoted $V_{i,j}^i$. We also define operators $T_{i,j}^j$ representing the images of the same elements under $S_{i,j}^{-1}$ in $\mathcal{H}_\mathbb{O}(M, m)$ and the corresponding operators on $\mathcal{H}_\mathbb{O}(M, m_1)$.

The following lemmas are straightforward generalisations of the lemmas in [Tho12, Section 5]. Given a parabolic subgroup $Q$ of $GL_n(F_v)$, we write $W_Q \subset W$ for the Weyl group of its Levi factor. Recall from [Cas] that the space $W_Q \backslash W/W_P$ has a canonical set of representatives $[W_Q \backslash W/W_P]_\mathcal{H}$, consisting of minimal length elements from each double coset.

**Lemma 3.3.** Let $Q$ be a parabolic corresponding to the partition $n = m_1 + \ldots + m_r$. Then $[W_Q \backslash W/W_P]$ is isomorphic to the set of partitions

$$m_i = n^i_1 + \ldots + n^i_k, 1 \leq i \leq r,$$

such that

$$\sum_i n^i_j = n_j \text{ for all } 1 \leq j \leq k.$$

With $Q$ as in the last lemma, let $L_i$ denote the $i$-th component of the corresponding Levi subgroup. For $w \in [W_Q \backslash W/W_P]$ corresponding to the partition $n^i_1 + \ldots + n^i_k$, let $p_{i,w}$ denote the parahoric subgroup of $L_i$ corresponding to this partition, and let $p_{i,1,w}$ be the kernel of $p_{i,w}$

$$p_{i,w} \rightarrow GL_{n^i_k}(F_v) \xrightarrow{\det} k^\times_v \rightarrow k^\times_v(p).$$

Let $q$ be the parahoric corresponding to the partition $\{n^1, \ldots, n^1_k, n^2_1, \ldots, n^2_k, \ldots, n^r_k\}$, and let $\mathfrak{n}$ be the hyperspecial maximal compact of the corresponding Levi subgroup. We define $p_{1,w}$ as a subgroup of $q$ defined by the conditions $\text{im}(\det N^j_k \rightarrow k^\times_v(p)) = 1$ for all $j$, where $N^j_k$ is the block corresponding to $n^j_k$.

**Lemma 3.4.** For each $1 \leq i \leq r$, let $\pi_i$ be a smooth representation of $L_i$. Then

1. For any $w \in [W_Q \backslash W/W_P]$, we have $L_i \cap w p_{i,w}^{-1} = p_{i,w}$.
2. For any $w \in [W_Q \backslash W/W_P]$, we have $Q \cap w p_{1,w}^{-1} \supset p_{1,w}$.
3. $$(\text{ind}_Q^G \pi_1 \otimes \ldots \otimes \pi_r)^p_1 \cong \bigoplus_{w \in [W_Q \backslash W/W_P]} \pi_{i,w}^p \otimes \ldots \otimes p_{i,w}^p_1.$$ 
4. $$(\text{ind}_Q^G \pi_1 \otimes \ldots \otimes \pi_r)^p \subset \bigoplus_{w \in [W_Q \backslash W/W_P]} \pi_{i,w}^p \otimes \ldots \otimes p_{i,w}^p.$$ 

Let $\pi$ be an irreducible admissible representation of $G$, such that $\pi^p_1 \neq 0$. Since $Iw' \subset p_1$, supercuspidal support of $\pi$ consists of tamely ramified characters. We will now use the Bernstein-Zelevinsky classification [BZ77], following the conventions of [Rod82], as they are best suited for applications to local Langlands correspondence. We can write $\pi$ as a quotient of

$$\text{ind}_Q^G \text{Sp}_{k_1}(\chi_1) \otimes \ldots \otimes \text{Sp}_{k_r}(\chi_r),$$

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where $\text{Sp}_n(\chi)$ for a tamely ramified character $\chi: F_v^\times \rightarrow C^\times$ is the unique irreducible quotient of $\text{ind}^G_B \chi \otimes \chi \cdot 1 \otimes \cdots \otimes \chi \cdot |n-1$. The twisted Steinberg factors $\text{Sp}_{k_i}(\chi_i)$ correspond to Zelevinsky segments $\Delta_i = (x, x(1), \ldots, x(k_i - 1))$.

Let $A$ index the partitions of $sc(\pi)$ into $k$ labeled subsets $S_1, \ldots, S_k$ satisfying the following conditions:

1. $|S_i| = n_i$ for all $i$.
2. Characters from the same Zelevinsky segment always belong to different subsets.
3. Characters within each $S_i$ satisfy $\chi \in S_i, \chi' \in S_j$ share a segment and $\chi' = \chi(a)$ for $a > 0$, then $i < j$.

For each partition $\alpha \in A$, let $r(\alpha)$ be the representation of $T(F)$ given by tensoring the characters of $sc(\pi)$ in the following order: characters in $S_i$ precede characters in $S_j$ when $i < j$, and the ordering of characters within each $S_i$ is induced by the ordering of Zelevinsky segments.

**Lemma 3.5.** For each $1 \leq i \leq r$, let $\pi_i$ be a smooth representation of $L_i$. Then

$$\left(\text{ind}^G_Q \pi_1 \otimes \cdots \otimes \pi_r\right)^{ss}_N = \bigoplus_{w \in [W_Q \setminus W/W_P]} \text{ind}^M_{\text{w}^{-1}Q \cap M} \text{w}^{-1}(\pi_1 \otimes \cdots \otimes \pi_r)_{L \cap \text{w}N \text{w}^{-1}}.$$

**Lemma 3.6.** Let $\pi$ be an irreducible admissible $\text{GL}_n(F_v)$-module, such that $\pi^{p_1} \neq 0$. Consider $\pi^{p_1}$ as a $\mathbb{Z}[\Xi_v]$-module via the map $\text{ind}^\mu \circ \text{res}^{-1}$. Then $(\pi^{p_1})^{ss}$ is a direct sum of $1$-dimensional submodules indexed by a subset of $A$. For a finite set $S$ of characters and positive integer $k \leq |S|$, let $e_k(S(\sigma))$ denote the $k$-th symmetric polynomial of elements of $S$ evaluated at $\sigma$. Then on the component associated to $(S_1, \ldots, S_k) \in A$, the action of $V^{i,j}$ is given by $e_i(S_j)$ for all $1 \leq i \leq n_j$.

**Proof.** We have a surjection

$$\text{ind}^G_Q \text{Sp}_{k_1}(\chi_1) \otimes \cdots \otimes \text{Sp}_{k_r}(\chi_r) \twoheadrightarrow \pi,$$

and the induced map

$$\left(\text{ind}^G_Q \text{Sp}_{k_1}(\chi_1) \otimes \cdots \otimes \text{Sp}_{k_r}(\chi_r)\right)^{p_1} \rightarrow \pi^{p_1}$$

is also surjective. By Lemma 3.5, we can write

$$(\text{ind}^G_Q \text{Sp}_{k_1}(\chi_1) \otimes \cdots \otimes \text{Sp}_{k_r}(\chi_r))^{ss} = \sigma \otimes \bigoplus_{(S_1, \ldots, S_k) \in A} \text{ind}^M_{\text{B} \cap N} \left(\bigotimes_{\psi \in S_i} \psi \otimes \cdots \bigotimes_{\psi \in S_k} \psi\right).$$

Here, the summands indexed by $A$ correspond to $w \in [W_Q \setminus W/W_P]$ represented by partitions $\{n_j^i\}$ satisfying $n_j^i \leq 1$ for all $i, j$ (cf. Lemma 3.3) and $\sigma$ represents all other summands. We will now show that $\sigma$ does not have $m_1$-invariants. Let $m_1^{w_1} \subset p_1^{w_1}$ be the subgroups of the Levi subgroup of $L_i$ defined analogously to $p_1^{w_1}$. Suppose $\sigma^{m_1}$ is nonzero. Let $\theta$ be a representation of $GL_{m_1}(F_v)$ which is a tensor factor of $(\text{Sp}_{k_1}(\chi_1) \otimes \cdots \otimes \text{Sp}_{k_r}(\chi_r))_{L \cap \text{w}N \text{w}^{-1}}$ for some $w \in [W_Q \setminus W/W_P]$ contributing to $\sigma$. Then $\theta$ has to be spherical if $j < k$ and has to have a fixed vector by $\ker(GL_{m_1}((O_{F_v}) \rightarrow GL_{m_1}(k_v) \rightarrow k_v^\times \rightarrow k_v(p))$ if $j = k$. This would imply that $\text{Sp}_{k_1}(\chi_1)^{p_1 \cdot w_1} \neq 0$ for all $1 \leq i \leq r$ and all $w$ representing partitions $m_1 = n_1^i + \ldots + n_k^i$, such that there exists at least one $1 \leq i \leq r$ for which $k_i > 1$ and $n_j^i > 1$ for some $1 \leq j \leq k$. To get a contradiction, it is therefore enough to show that $\text{Sp}_{k_1}(\chi_1)^{p_1 \cdot w_1} = 0$.

Define the subgroup $\text{Iw}_i \subset p_1^{w_1}$ to be a subgroup of the $L_i$-Iwahori with $1$’s mod $\sigma$ on the diagonal at indices $n_{k-1}^i + 1$ through $n_k^i$. There are two possibilities: either $p_1^{w_1} = GL_{m_1}((O_{F_v})$, or $\text{Iw}_i$ has at least
one \* \text{mod} \ \sigma \text{ on the diagonal. In the former case, we are done since } \Sp_{k_i}(\chi_i) \text{ is never spherical. In the latter case, let } t' \text{ be the diagonal component of } \text{Iw}_i'. \text{ Then}

\[ \Sp_{k_i}(\chi_i)^{\text{Iw}_i'} = \Sp_{k_i}(\chi_i)_{U}^t = (\chi_i \otimes \ldots \otimes \chi_i) \cdot |k_i-1|^{t'}, \]

where \( U \) is the unipotent radical of the Borel. Since \( t' \) has at least one \( O_{F_v}^\times \) factor, if this is nonzero, \( \chi_i \) must be unramified. But in this case, any \( p_i \cdot \text{Iw}_i' \)-fixed vector would be automatically fixed by the parahoric \( p_i \cdot \text{Iw}_i' \), which properly contains the Iwahori, and hence, does not fix any vector in \( \Sp_{k_i}(\chi_i) \).

\( \square \)

For a partition \( n = n_1 + \ldots + n_k \) which we call \( \mu \), define elements

\[ P_{\mu,i} = \prod_{j=s_{\mu,i-1}+1}^{s_{\mu,i}} (T - X_j) \]

\[ \Res_{\mu} = \prod_{i<j} \Res(P_{\mu,i}, P_{\mu,j}) \in \mathbf{Z}[\Xi_v]^{S_\mu} \]

\[ \Res_{q_v,\mu} = \prod_{i<j} \Res(P_{\mu,i}(q_v T), P_{\mu,j}) \in \mathbf{Z}[\Xi_v]^{S_\mu}. \]

Then there exist unique polynomials \( Q_{\mu,i} \in \mathbf{Z}[\Xi_v]^{S_\mu}[T] \), such that \( \deg Q_{\mu,i} < n_i \) and

\[ \sum_{i=1}^{n} Q_{\mu,i} \prod_{j \neq i} P_{\mu,j} = \Res_{\mu}. \]

Define

\[ E_{\mu,i} = Q_{\mu,i} \prod_{j \neq i} P_{\mu,j}. \]

The following statement is elementary.

**Lemma 3.7.** Take any \( A \in M_n(\mathbf{C}) \) with a factorisation

\[ \det(T - A) = \prod_{i=1}^{k} p_{\mu,i}(T), \]

where \( p_{\mu,i} \in \mathbf{C}[T] \) are pairwise coprime and \( \deg p_{\mu,i} = n_i \). Consider the homomorphism \( \varphi : \mathbf{Z}[\Xi_v]^{S_\mu} \to \mathbf{C} \) defined by the polynomials \( p_{\mu,i} \). By this, we mean the homomorphism sending \( e_j(X_{s_{\mu,i-1}+1}, \ldots, X_{s_{\mu,i}}) \) to \((-1)^{j}\) times the coefficient of \( T_j \) in \( p_{\mu,i} \). This homomorphism can be extended to \( \varphi : \mathbf{Z}[\Xi_v]^{S_\mu}[T, \Res_{\mu}^{-1}] \to \mathbf{C}[T] \). Then \( \varphi(E_{\mu,i}/\Res_{\mu})(A) \) projects \( \mathbf{C}^n \) onto the sum of generalised eigenspaces of \( A \) corresponding to the roots of \( p_{\mu,i} \).

**Proposition 3.8.** Let \( \pi \) be an irreducible admissible \( GL_n(F_v) \)-module. Then either \( \Res_{q_v,\mu}^{n_1} \pi^{n_1} = 0 \), or

\[ \rec_{F_v}(\pi) = (\chi_1 \oplus \ldots \oplus \chi_{n_k}, 0), \]

where \( \chi_1, \ldots, \chi_{m+\ldots+n_{k-1}} \) are unramified and the rest are tamely ramified with equal restriction to inertia.

**Proof.** Using the notation from the discussion preceding Lemma 3.5, if there exists some \( k_i > 1 \), then \( \Res_{q_v,\mu}^{n_1} \pi^{n_1} = 0 \) follows from Lemma 3.6. Otherwise, we can apply the proof of [CHT08, Lemma 3.1.6] for the second conclusion.  \( \square \)
Proof. Let \( \psi \) be an irreducible admissible \( GL_n(F_v) \)-module. Let \((r, N) = \text{rec}_{F_v}(\psi)\). Then either \((S_{\psi} \circ t^{-1}_\mu \circ t_H \circ t_\mu \circ S^{-1}_m)(\text{Res}_{q_{\psi}, \mu}^{n_1})^{\pi_1} = 0\) or \(N = 0\) and

\[ r^\psi = \chi_1 \oplus \ldots \oplus \chi_n, \]

where \( \chi_1, \ldots, \chi_{n_1+\ldots+n_k} \) are unramified and the rest are tamely ramified with equal restriction to inertia.

Let \( \phi_1, \ldots, \phi_n \) be any lift of Frobenius.

**Proposition 3.10.** Let \( \psi \) be an irreducible admissible \( GL_n(F_v) \)-module. Let \((r, N) = \text{rec}_{F_v}(\psi)\). Let \( R \) be the image of \( \mathcal{O}[\Xi_{v, 1}]^{S_\mu} \) in \( \text{End}_\mathcal{O}(\psi) \) under the map \( t_\mu \circ S^{-1}_m \). Then either \( \text{Res}_{q_{\psi}, \mu}^{n_1} \psi^{\pi_1} = 0 \) or the following relation holds over \( R \) for all \( \tau \in I_{F_v} \):

\[
\text{Res}_{\mu}^{n_1} \left( \sum_{i=1}^{k-1} E_{\mu, i}(r(\psi_v)) + \langle \text{Art}_{F_v}^{-1}(\tau)E_{\mu, k}(r(\psi_v)) \rangle - \text{Res}_{\mu} r(\tau) \right) = 0.
\]

Proof. Assume \( \text{Res}_{q_{\psi}, \mu}^{n_1} \psi^{\pi_1} \neq 0 \). It is enough to check our relation for each localisation of \( R \) at a maximal ideal \( m \). If \( \text{Res}_{\mu} \neq 0 \) in \( R_m \). Otherwise, \( R_m = \mathcal{O} \) by [Sta18, Tag 00UA] and the image of \( \mathcal{O}[\Xi_{v, 1}]^{S_\mu} \) in \( R/m \) corresponds to the polynomials \( \sum_{j=s_{m, i}+1}^{s_{m, i}} (T - \chi_j(\psi_v)) \) for \( i = 1, \ldots, k \). Then the image of

\[
\text{Res}_{\mu}^{-1} \left( \sum_{i=1}^{k-1} E_{\mu, i}(r(\psi_v)) + \langle \text{Art}_{F_v}^{-1}(\tau)E_{\mu, k}(r(\psi_v)) \rangle \right)
\]

in \( M_n(R_m) \) is a diagonal matrix with \( n - n_k \) first entries equal to 1 and the rest equal to \( \chi_n(\tau) \). This concludes the proof.

**Proposition 3.11.** Let \( \psi \) be an irreducible admissible \( GL_n(F_v) \)-module. Let \((r, N) = \text{rec}_{F_v}(\psi)\). Let \( R' \) be the image of \( \mathcal{O}[\Xi_{v, 1}]^{S_\mu} \) in \( \text{End}_{\mathcal{O}}(\psi) \) via the map \( t_H \circ t_\mu \circ S^{-1}_m \). Then either \((t_\mu \circ S^{-1}_m)(\text{Res}_{q_{\psi}, \mu}^{n_1})^{\psi^{\pi_1}} = 0\) or the following relation holds over \( R' \) for all \( \tau \in I_{F_v} \):

\[
(t_\mu \circ S^{-1}_m) \left( \text{Res}_{\mu}^{n_1} \left( \sum_{i=1}^{k-1} E_{\mu, i}(r^\psi(\psi_v)) + \langle \text{Art}_{F_v}^{-1}(\tau)E_{\mu, k}(r^\psi(\psi_v)) \rangle - \text{Res}_{\mu} r^\psi(\tau) \right) \right) = 0.
\]

Proof. This follows from Proposition 3.9 in the same way as Proposition 3.10 follows from Proposition 3.8.

In what follows, we will use a twisted version of the propositions above. Define a map \( \Sigma^T : \mathcal{O}[\Xi_{v, 1}]^{S_\mu} \to \mathcal{H}_\mathcal{O}(GL_n(F_v), p_{v, 1}) \) given by

\[
\Sigma^T(f)(g) = t_\mu(S^{-1}_m(f))(g)|\text{det}(g)|^{(1-n)/2}.
\]

Let us show that this map is in fact defined over \( \mathbb{Z}[q_{v, 1}^{-1}] \) and thus does not depend on the choice of square root of \( q_{v, 1}^{-1} \). Note that \( t_\mu \) is defined over \( \mathbb{Z}[q_{v, 1}^{-1}] \) up to \( \delta_{p_\mu}^{1/2} \) and \( S_\mu \) is defined over \( \mathbb{Z}[q_{v, 1}^{-1}] \) up to
\[ \prod_{i=1}^{k} |\det(m_i)|^{(1-n_i)/2}, \] where \((m_i) \in M_\mu(F_v)\) with \(m_i \in \GL_{n_i}(F_v)\). Thus, the desired rationality over 
\[ \Z[q_v^{-1}] \] follows from the fact that
\[ \prod_{i=1}^{k} |\det(m_i)|^{(1-n_i)/2} \prod_{i=1}^{k} |\det(m_i)|^{(1-n_i)/2} \prod_{1 \leq i < j \leq k} |\det(m_i)|^{n_j/2}|\det(m_j)|^{-n_i/2} \]
lies in \(\Z[q_v^{-1}]\). Now let us restate Proposition 3.10 and Proposition 3.11.

**Proposition 3.12.** Let \(\pi\) be an irreducible admissible \(GL_n(F_v)\)-module. Let \((r, N) = \text{rec}_T(F_v)(\pi)\). Let \(R\) be the image of \(O[\Xi_{v,1}]^S\mu\) in \(\text{End}_O(\pi^\oplus)\) under the map \(\Sigma^T\). Then either \(\text{Res}_{q_v}^{\mu} \pi^\oplus = 0\) or the following relation holds over \(R\) : for all \(\tau \in I_{F_v}\)
\[ \text{Res}_{\mu}^\mu \left( \sum_{i=1}^{k-1} E_{\mu,i}(r(\varphi_v)) + \langle \text{Art}_{F_v}^{-1}(\tau) E_{\mu,k}(r(\varphi_v)) \rangle - \text{Res}_\mu r(\tau) \right) = 0. \]

**Proposition 3.13.** Let \(\pi\) be an irreducible admissible \(GL_n(F_v)\)-module. Let \((r, N) = \text{rec}_T(F_v)(\pi)\). Let \(R'\) be the image of \(O[\Xi_{v,1}]^S\mu\) in \(\text{End}_O(\pi^\oplus)\) via the map \(\iota_{\Sigma^T} \circ \Sigma^T\). Then either \((\iota_{\Sigma^T} \circ \Sigma^T)(\text{Res}_{q_v}^{\mu}) \pi^\oplus = 0\) or the following relation holds over \(R'\) : for all \(\tau \in I_{F_v}\)
\[ (\iota_{\Sigma^T} \circ \Sigma^T) \left( \text{Res}_{\mu}^\mu \left( \sum_{i=1}^{k-1} E_{\mu,i}(r^\vee(\varphi_v)) + \langle \text{Art}_{F_v}^{-1}(\tau) E_{\mu,k}(r^\vee(\varphi_v)) \rangle - \text{Res}_\mu r^\vee(\tau) \right) \right) = 0. \]

### 4. Setup

Let \(F/F^+\) be an imaginary CM-field with ring of integers \(O\). Let \(\Psi_n\) be the matrix with 1-s on the antidiagonal and 0-s elsewhere, and let
\[ J_n = \begin{pmatrix} 0 & \Psi_n \\ -\Psi_n & 0 \end{pmatrix}. \]

Define \(\tilde{G}\) to be the group scheme over \(O_{F^+}\) defined by the functor of points
\[ \tilde{G}(R) = \{ g \in \GL_{2n}(R \otimes_{O_{F^+}} O_F) \mid \langle g J_n g^c = J_n \}. \]

Then \(\tilde{G}\) is a quasisplit reductive group over \(F^+\). It is a form of \(GL_{2n}\) which becomes split after the quadratic base change \(F/F^+\). If \(v\) is a place of \(F\) lying above a place \(\widetilde{v}\) of \(F^+\) which splits in \(F\), then we have a canonical isomorphism \(\iota_v : \tilde{G}(F^+) \cong \GL_{2n}(F_v)\). There is an isomorphism \(F^+_{\widetilde{v}} \otimes_{F^+} F \cong F_v \times F_v^c\) and \(\iota_v\) is given by composition
\[ \tilde{G}(F^+) \hookrightarrow \GL_{2n}(F_v) \times \GL_{2n}(F_v^c) \to \GL_{2n}(F_v), \]
where the second map is the projection on the first factor. We write \(T \subset B \subset G\) for the subgroups consisting, respectively, of the diagonal and upper-triangular matrices in \(\tilde{G}\). Similarly, we write \(G \subset P \subset \tilde{G}\) for the Levi and parabolic subgroups consisting, respectively, of the block upper diagonal and block upper-triangular matrices with blocks of size \(n \times n\). Then \(P = U \rtimes G\), where \(U\) is the unipotent radical of \(P\), and we can identify \(G\) with \(\text{Res}_{O_F/O_{F^+}} \GL_n\) via the map
\[ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mapsto D \in \GL_n(R \otimes_{O_{F^+}} O_F). \]
An element \((g_v)_v \in G(A_{F_v}^{\infty}) = GL_n(A_{F_v}^{\infty})\) is called neat if the intersection \(\cap_v \Gamma_v\) is trivial, where \(\Gamma_v \subset \overline{Q}_v^\times\) is the torsion subgroup of the subgroup of \(\overline{F}_v^\times\) generated by the eigenvalues of \(g_v\) acting via some faithful representation of \(G\). We call a neat open compact subgroup \(K \subset G(A_{F_v}^{\infty})\) good if it has the form \(K = \prod_v K_v\), where the product is running over the finite places of \(F\). We make similar definitions with \(G\) in place of \(G\).

After extending scalars to \(F^+\), \(T\) and \(B\) form a maximal torus and a Borel subgroup, respectively, of \(\tilde{G}\) and \(G\) is the unique Levi subgroup of the parabolic subgroup \(P\) of \(\tilde{G}\) which contains \(T\). We call an open compact subgroup \(\tilde{K}\) of \(\tilde{G}(A_{F_v}^{\infty})\) decomposed with respect to the Levi decomposition \(P = GU\) if \(\tilde{K} = \tilde{K}_G \times \tilde{K}_U\), where \(\tilde{K}_G\) is the image of \(K\) in \(G\) and \(\tilde{K}_U = \tilde{K} \cap U(A_{F_v}^{\infty})\).

If \(K\) is a good subgroup of \(G\), we let \(X_K\) be the corresponding locally symmetric space. Similarly, if \(\tilde{K}\) is a good open compact subgroup of \(\tilde{G}\), then \(\tilde{X}_\tilde{K}\) denotes the locally symmetric space. More generally, if \(H\) is a connected algebraic group over a number field \(L\) and \(K_H \subset H(A_{M}^{\infty})\) is a good subgroup, then we write \(X_H^G\) for the locally symmetric space of \(H\) of level \(K_H\).

Fix a rational prime \(p\) and a finite extension \(F/Q\) which contains the images of all embeddings \(F \hookrightarrow \overline{Q}_p\). We write \(O\) for the ring of integers of \(E\) and \(\varpi \in O\) for a choice of uniformiser. For \(\lambda \in (\mathbb{Z}_p^n)_{\text{Hom}(F,E)}\), we define an \(O[\prod_{v \mid p} GL_n(O_{F_v})]\)-module \(\mathcal{V}_\lambda\) as in [ACC+18, Section 2.2.1]. Similarly for \(\tilde{\lambda} \in (\mathbb{Z}_p^{2n})_{\text{Hom}(F^+,E)}\), we have an \(O[[\prod_{v \mid p} \tilde{G}(O_{F_v^+})]]\)-module \(\tilde{\mathcal{V}}_{\tilde{\lambda}}\). Both \(\mathcal{V}_\lambda\) and \(\tilde{\mathcal{V}}_{\tilde{\lambda}}\) are finite free \(O\)-modules.

Let \(S\) be a set of places of \(F\), such that \(S = S^c\) and, such that \(S\) contains all places above \(p\) and all places of \(F\) which are ramified over \(F^+\). Let \(\mathcal{S}\) be the set of places of \(F^+\) lying below a place in \(S\). Let \(K \subset G(A_{F_v}^{\infty})\) be a good subgroup, such that \(K_\mathcal{S} = G(O_{F_v^+})\) for \(v \notin \mathcal{S}\), and similarly, let \(\tilde{K} \subset \tilde{G}(A_{F_v}^{\infty})\) be a good subgroup, such that \(\tilde{K}_\mathcal{S} = \tilde{G}(O_{F_v^+})\) for \(v \notin \mathcal{S}\). Additionally, we define \(\tilde{\Xi}_\mathcal{S} := \Xi_v \times \Xi_{v^c}\) and \(\tilde{\Xi}_{\mathcal{S},1} := \Xi_v \times \Xi_{v^c}\).

Define the Hecke algebras

\[
\mathcal{H}_S = \mathcal{H}_O(G(A_{F_v}^{\infty}, S^c), K^{\mathcal{S}})
\]

\[
\hat{\mathcal{H}}_S = \mathcal{H}_O(\tilde{G}(A_{F_v}^{\infty}, S^c), \tilde{K}^{\mathcal{S}})
\]

\[
T^S \cong \bigotimes_{v \notin \mathcal{S}} \mathcal{O}[\Xi_v]^{S_{n_v}}
\]

\[
\tilde{T}^S \cong \bigotimes_{v \notin \mathcal{S}} \mathcal{O}[\tilde{\Xi}_v]^{S_{2n_v}}
\]

Using the isomorphism

\[
G(O_{F_v^+}) \cong GL_n(O_{F_v})
\]

together with the Satake isomorphisms, as well as the homomorphism

\[
\mathcal{O}[\tilde{\Xi}_\mathcal{S}]^{S_{2n}} \rightarrow \mathcal{H}_O(\tilde{G}(F_v^+), \tilde{G}(O_{F_v^+}))
\]

given by the polynomial \(\tilde{P}_v(X)\) defined in [ACC+18, Equation 2.2.6], we get homomorphisms \(T^S \rightarrow \mathcal{H}_S\) and \(\tilde{T}^S \rightarrow \hat{\mathcal{H}}_S\). We also have homomorphisms

\[
T^S \rightarrow \text{End}_{\mathcal{O}}(R\Gamma(X_K, \mathcal{V}_\lambda))
\]

\[
\tilde{T}^S \rightarrow \text{End}_{\mathcal{O}}(R\Gamma(\tilde{X}_{\tilde{K}}, \mathcal{V}_{\tilde{\lambda}}))
\]
defined in [ACC+18, Section 2.1.2], and we can denote by \( T^S(K, \lambda), \widetilde{T}^S(\widetilde{K}, \widetilde{\lambda}) \), respectively, the images of those homomorphisms. The functor \( H^* \) induces \( \mathcal{O} \)-algebra homomorphisms

\[
T^S(K, \lambda) \to \text{End}_\mathcal{O}(H^*(X_K, \mathcal{V}_\lambda))
\]

\[
\widetilde{T}^S(\widetilde{K}, \widetilde{\lambda}) \to \text{End}_\mathcal{O}(H^*(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})).
\]

5. Boundary cohomology

Let \( \widetilde{K} \subset \widetilde{G}(\mathbb{A}_{F,+}^\infty) \) be a neat compact open subgroup decomposed with respect to the Levi decomposition \( P = GU \). We also assume that \( \widetilde{K}_v = \widetilde{G}(\mathcal{O}_{F_v^\infty}) \) for \( v \notin \overline{S} \). Define \( K \) as the image of \( \widetilde{K} \) in \( G(\mathbb{A}_{F,+}^\infty) \), \( \widetilde{K}_P = \widetilde{K} \cap P(\mathbb{A}_{F,+}^\infty) \) and \( K_U = \widetilde{K} \cap U(\mathbb{A}_{F,+}^\infty) \). Both \( K \) and \( \widetilde{K}_P \) are neat. We recall from [NT16, Section 3.1.2] that the boundary \( \partial \widetilde{X}_{\widetilde{K}} = \overline{X}_{\widetilde{K}} \) of the Borel-Serre compactification has a \( \widetilde{G}(\mathbb{A}_{F,+}^\infty) \)-equivariant stratification indexed by the standard parabolic subgroups of \( \widetilde{G} \). For each standard parabolic subgroup \( \mathcal{Q} \), label the corresponding stratum \( \widetilde{X}_{\widetilde{K}}^{\mathcal{Q}} \). We can write

\[
\widetilde{X}_{\widetilde{K}}^{\mathcal{Q}} = Q(F^+) \setminus (X_{\mathcal{Q}} \times \widetilde{G}(\mathbb{A}_{F,+}^\infty)/\widetilde{K}).
\]

From now on, we will focus on the stratum \( \widetilde{X}_{\widetilde{K}}^{\mathcal{Q}} \) corresponding to the Siegel parabolic. Let us establish some useful maps between the manifolds introduced above. The stratum \( \widetilde{X}_{\widetilde{K}}^{\mathcal{Q}} \) can be described as a union of connected components indexed by the set \( P(F^+) \setminus \widetilde{G}(\mathbb{A}_{F,+}^\infty)/\widetilde{K} \). The locally symmetric space \( X_K^P \) is a union of the same components indexed by the set \( P(F^+) \setminus \widetilde{P}(\mathbb{A}_{F,+}^\infty)/\widetilde{K}_P \). Thus, we have a natural open immersion \( i : X_K^P \to \widetilde{X}_{\widetilde{K}}^{\mathcal{Q}} \), such that \( i^* : H^*(\widetilde{X}_{\widetilde{K}}^{\mathcal{Q}}, \mathcal{O}) \to H^*(X_K^P, \mathcal{O}) \) is a split epimorphism. We also have a proper map \( j : X_K^{\mathcal{Q}} \to X_K \) which has a section by [NT16, Section 3.1.1]. Thus, we get a split monomorphism \( j^* : H^*(X_K, \mathcal{O}) \to H^*(X_K^{\mathcal{Q}}, \mathcal{O}) \). We also recall the ‘restriction to \( P \)’ and ‘integration along \( N \)’ homomorphisms:

\[
r_P : \mathcal{H}_\mathcal{O}(\widetilde{G}(\mathbb{A}_{F,+}^\infty), \widetilde{K}_P^S) \to \mathcal{H}_\mathcal{O}(\widetilde{P}(\mathbb{A}_{F,+}^\infty), \widetilde{K}_P^S)
\]

\[
r_G : \mathcal{H}_\mathcal{O}(\widetilde{P}(\mathbb{A}_{F,+}^\infty), \widetilde{K}_P^S) \to \mathcal{H}_\mathcal{O}(\widetilde{G}(\mathbb{A}_{F,+}^\infty), \widetilde{K}_P^S)
\]

defined in [NT16, Section 2.2]. We record the following proposition, which follows from the discussion above:

**Proposition 5.1.**

1. For all \( t \in \widetilde{T}^S \) and \( h \in H^*(\widetilde{X}_{\widetilde{K}}^{\mathcal{Q}}, \mathcal{O}) \), we have \( i^*(h) = r_P(t)i^*(h) \).
2. For all \( t \in \mathcal{H}_\mathcal{O}(\widetilde{P}(\mathbb{A}_{F,+}^\infty), \widetilde{K}_P^S) \) and \( h \in H^*(X_K, \mathcal{O}) \), we have \( j^*(r_G(t)h) = tj^*(h) \).

Consider the composite

\[
S = r_G \circ r_P : \mathcal{H}_\mathcal{O}(\widetilde{G}(\mathbb{A}_{F,+}^\infty), \widetilde{K}_P^S) \to \mathcal{H}_\mathcal{O}(\widetilde{G}(\mathbb{A}_{F,+}^\infty), \widetilde{K}_P^S).
\]

By [NT16, Proposition-Definition 5.3], this map coincides with the tensor product of maps \( \mathcal{O}[\Xi_\tau]^{S_{2n}} \to \mathcal{O}[\Xi_\nu]^{S_n} \) determined by the polynomial \( S_n(P_v(X)q_v^n(2n-1)P_v^v(q_v^{1-2n}X)) \).
Let \( \mathfrak{m} \subset T^S \) be a non-Eisenstein maximal ideal of Galois type with residue field \( k \). We have an associated continuous semisimple representation \( \overline{\rho}_m : G_{F,S} \to GL_n(k) \), such that \( \det(X - \overline{\rho}_m(Frob_v)) \equiv P_v(X) \mod \mathfrak{m} \). Fix a tuple \((Q,(\alpha_v)_{v \in Q})\), where

- \( Q \subset S \) and \( Q \cap Q^c = \emptyset \).
- Each place \( v \in Q \) is split over \( F^+ \). Moreover, for each place \( v \in Q \), there exists an imaginary quadratic subfield \( F_0 \subset F \), such that \( q_v \) splits in \( F_0 \).
- For each place \( v \in Q \), \( \overline{\rho}_m \) is unramified at \( v \) and \( \alpha_v \) is a root of \( \det(X - \overline{\rho}_m(Frob_v)) \).

For each \( v \in Q \), let \( d_v \) be multiplicity of \( \alpha_v \) as a root of \( \det(X - \overline{\rho}_m(Frob_v)) \). Fix the partitions

\[
\mu_v : 2n = d_v + (n - d_v) + n
\]
\[
\nu_v : n = d_v + (n - d_v).
\]

Let

\[
\Delta_v = \bigcup_{m \in M^+_{\nu_v}} [p_{\mu_v,1}mp_{\mu_v,1}] \subset GL_n(F_v).
\]

Now we recall the theory of Hecke algebras of a monoid from [ACC+18, Section 2.1.9]. Specifically, we consider the restriction from \( \tilde{G} \) to \( P \)

\[
r_P : \mathcal{H}(\iota_v^{-1}(\Delta_v),\iota_v^{-1}(p_{\mu_v,1})) \to \mathcal{H}(P(F_v^+), P(F_v) \cap \iota_v^{-1}(p_{\mu_v,1}))
\]

and integration along fibres

\[
r_G : \mathcal{H}(P(F_v^+), P(F_v) \cap \iota_v^{-1}(p_{\mu_v,1})) \to \mathcal{H}(G(F_v^+), G(F_v) \cap \iota_v^{-1}(p_{\mu_v,1}))
\]

combined with the isomorphism

\[
\mathcal{H}(G(F_v^+), G(F_v) \cap \iota_v^{-1}(p_{\mu_v,1})) \cong \mathcal{H}(GL_n(F_v) \times GL_n(F_v^c), p_{\nu_v,1} \times GL_n(O_{F_v})),
\]

we get a map

\[
S^+_v : \mathcal{H}(\iota_v^{-1}(\Delta_v),\iota_v^{-1}(p_{\mu_v,1})) \to \mathcal{H}(GL_n(F_v) \times GL_n(F_v^c), p_{\nu_v,1} \times GL_n(O_{F_v})).
\]

Write \( P_{n,n} = M_{n,n} L_{n,n} \) for the parabolic subgroup of \( GL_{2n}(F_v) \) corresponding to the partition \( 2n = n+n \), together with its Levi decomposition. For a given \( m \in M^{++} \), from [ACC+18, Section 2.1.9], we know that

\[
S_v^+(\iota_v^{-1}([p_{\mu_v,1}mp_{\mu_v,1}])) = |\delta_P(m)|^{-1}|\iota_v^{-1}([([p_{\mu_v,1} \cap M_{n,n}]m)(p_{\mu_v,1} \cap M_{n,n})])|.
\]

By the same argument as in the proof of Lemma 3.2, we see that there exists \( m \in M^{++} \), such that the right-hand side is invertible in \( \mathcal{H}(GL_n(F_v) \times GL_n(F_v^c), p_{\nu_v,1} \times GL_n(O_{F_v})). \) Thus, we can extend the homomorphism to

\[
S_v : \mathcal{H}(\iota_v^{-1}(\Delta_v),\iota_v^{-1}(p_{\mu_v,1})) \to \mathcal{H}(GL_n(F_v) \times GL_n(F_v^c), p_{\nu_v,1} \times GL_n(O_{F_v})).
\]
This homomorphism fits into a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}[\Xi_{\tau,1}]^{S_{\nu}} & \longrightarrow & \mathcal{H}(\iota_{\nu}^{-1}(\Delta_{\nu}),\iota_{\nu}^{-1}(p_{\mu_{\nu},1}))[(\iota_{\nu}^{-1}([p_{\mu_{\nu},1}m_{\mu_{\nu},1}]))^{-1}] \\
\downarrow s^{f}_{\nu} & & \downarrow s_{\nu} \\
\mathcal{O}[\Xi_{\nu,1}]^{S_{\nu}} \otimes \mathcal{O}[\Xi_{\nu}^{1}]^{S_{\nu}} & \longrightarrow & \mathcal{H}(\text{GL}_{n}(F_{\nu}) \times \text{GL}_{n}(F_{\nu}), p_{\nu,1} \times \text{GL}_{n}(\mathcal{O}_{F_{\nu}})),
\end{array}
\]

where \( S_{\nu} \) is the unique homomorphism which corresponds to the polynomial \( \prod_{i=1}^{2n}(T - X_{i}) \) to the tuple of polynomials \( \prod_{i=0}^{d_{\nu}}(T - X_{i}), \prod_{i=d_{\nu}+1}^{n+p}(T - X_{i}), S_{\nu}(q_{\nu}^{1}(2n-1)P_{\nu}^{1}(q_{\nu}^{1}-2n)X) \) and maps \( \tau_{\nu} \) to \( \tau_{\nu} \).

We can define global Hecke algebras associated to our Taylor-Wiles data:

\[
\tilde{\mathcal{H}}^{S}_{Q} = \tilde{\mathcal{H}}^{S} \otimes_{\mathbb{Z}} \bigotimes_{v \in Q} \mathcal{H}(\iota_{v}^{-1}(\Delta_{v}),\iota_{v}^{-1}(p_{\mu_{\nu},1}))[(\iota_{v}^{-1}([p_{\mu_{\nu},1}m_{\mu_{\nu},1}]))^{-1}]
\]

\[
\tilde{T}^{S}_{Q} = \tilde{T}^{S} \otimes_{\mathbb{Z}} \bigotimes_{v \in Q} \mathcal{O}[\Xi_{\nu,1}]^{S_{\nu}}
\]

\[
\mathcal{H}^{S}_{Q} = \mathcal{H}^{S} \otimes_{\mathbb{Z}} \bigotimes_{v \in Q} \mathcal{H}(\text{GL}_{n}(F_{\nu}) \times \text{GL}_{n}(F_{\nu}), p_{\nu,1} \times \text{GL}_{n}(\mathcal{O}_{F_{\nu}}))
\]

\[
T^{S}_{Q} = T^{S} \otimes_{\mathbb{Z}} \bigotimes_{v \in Q} \mathcal{O}[\Xi_{\nu,1}]^{S_{\nu}} \otimes \mathcal{O}[\Xi_{\nu}]^{S_{\nu}}.
\]

The following proposition follows from the discussion above:

**Proposition 5.2.** There exist homomorphisms \( S^{f}_{Q} : \tilde{T}^{S}_{Q} \rightarrow T^{S}_{Q} \) and \( S_{Q} : \tilde{\mathcal{H}}^{S}_{Q} \rightarrow \mathcal{H}^{S}_{Q} \) fitting into a commutative diagram

\[
\begin{array}{ccc}
\tilde{T}^{S}_{Q} & \longrightarrow & \tilde{\mathcal{H}}^{S}_{Q} \\
\downarrow s^{f}_{\nu} & & \downarrow s_{\nu} \\
T^{S}_{Q} & \longrightarrow & \mathcal{H}^{S}_{Q},
\end{array}
\]

where \( S^{f}_{Q} \) coincides with \( S^{f}_{\nu} \) at places \( v \in Q \) and with the Satake isomorphism from [NT16, Proposition-Definition 5.3] at places \( v \notin S \).

Let \( \tilde{K} \) be a good subgroup of \( \tilde{G}(\mathcal{A}_{F_{\nu}}^{\infty}) \), such that \( \tilde{K}^{S} = \tilde{G}(\mathcal{O}_{F_{\nu}}^{\infty}) \) and \( \tilde{K} \) is decomposed with respect to \( P \). We can define subgroups \( \tilde{K}_{1}(Q) \subset \tilde{K}_{0}(Q) \subset \tilde{K} \) as follows:

- If \( \nu \notin Q \), then \( \tilde{K}_{1}(Q)_{\nu} = \tilde{K}_{0}(Q)_{\nu} = \tilde{K}_{\nu} \).
- If \( \nu \in Q \), then \( \tilde{K}_{1}(Q)_{\nu} = \iota_{\nu}^{-1}(p_{\mu_{\nu},1}) \) and \( \tilde{K}_{0}(Q)_{\nu} = \iota_{\nu}^{-1}(p_{\mu_{\nu}}) \).

Let \( K_{1}(Q), K_{0}(Q), K \) be the images in \( G(\mathcal{A}_{F_{\nu}}^{\infty}) \) of the intersections of \( \tilde{K}_{1}(Q), \tilde{K}_{0}(Q), \tilde{K} \) with \( P(\mathcal{A}_{F_{\nu}}^{\infty}) \). From the definition, we can see that all the subgroups from the previous sentence are decomposed with respect to \( P \).

**Proposition 5.3.** For \( i = 0, 1 \), we have

1. The open immersion \( i : X_{P_{K_{i}(Q)}}^{P} \rightarrow \tilde{X}_{K_{i}(Q)}^{P} \) yields a split epimorphism \( i^{*} : H^{*}(\tilde{X}_{K_{i}(Q)}^{P}, \mathcal{O}) \rightarrow H^{*}(X_{P_{K_{i}(Q)}}^{P}, \mathcal{O}) \).
2. The proper map \( j : X_{P_{K_{i}(Q)}}^{P} \rightarrow X_{K_{i}(Q)}^{P} \) yields a split monomorphism \( j^{*} : H^{*}(X_{K_{i}(Q)}^{P}, \mathcal{O}) \rightarrow H^{*}(X_{P_{K_{i}(Q)}}^{P}, \mathcal{O}) \).
3. For all $t \in \mathcal{H}_\mathcal{O}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(p_{\mu_v,1}))$ and $h \in H^*(\widetilde{X}^P_{\mathcal{K}_i(\mathcal{O})}, \mathcal{O})$, we have $i^*(th) = r_p(t)i^*(h)$.

4. For all $t \in \mathcal{H}_\mathcal{O}(\tilde{P}(A_{F,s}^{\infty}), \widetilde{K}_i(\mathcal{Q})_p)$ and $h \in H^*(X_{K_i}(\mathcal{O}), \mathcal{O})$, we have $f^*(r_G(t)h) = tf^*(h)$.

**Proof.** This follows from the discussion above Proposition 5.1 and [ACC+18, Lemma 2.1.14].

Now let $m_\mathcal{Q} \subset \mathcal{T}^S_\mathcal{Q}$ be the maximal ideal generated by $m$ and the kernels of the maps $\mathcal{O}[\tilde{E}_{v,i}]^{S_\nu} \rightarrow k$ associated to the polynomials $(x - \alpha_v)^d_v$, $\det(X - \overline{m}(\text{Frob}_v))/(x - \alpha_v)^d_v$, $\det(X - \overline{m}(\text{Frob}_v))$ for $v \in \mathcal{Q}$. Also, let $\tilde{m}_\mathcal{Q} = S_{F,i}^{S_\nu - 1}(m_\mathcal{Q})$.

**Proposition 5.4.** For $i = 0, 1$, the map $S^f_\mathcal{Q} : \mathcal{T}^S_\mathcal{Q} \rightarrow \mathcal{T}^S_\mathcal{Q}$ descends to homomorphisms

$$
\tilde{T}^S_\mathcal{Q}(H^*(\widetilde{X}^P_{\mathcal{K}_i(\mathcal{O})}, \mathcal{O})) \rightarrow \mathcal{T}^S_\mathcal{Q}(H^*(X_{K_i}(\mathcal{O}), \mathcal{O}))
$$

$$
\tilde{T}^S_\mathcal{Q}(H^*(\partial \widetilde{X}_{\mathcal{K}_i(\mathcal{O})}, \mathcal{O}_m) \rightarrow \mathcal{T}^S_\mathcal{Q}(H^*(X_{K_i}(\mathcal{O}), \mathcal{O}_m)).
$$

**Proof.** To prove the first statement, we need to show that for $t \in \text{Ann}_{\mathcal{T}^S_\mathcal{Q}}(H^*(\widetilde{X}^P_{\mathcal{K}_i(\mathcal{O})}, \mathcal{O}))$, we have $S_\mathcal{Q}(t) \in \text{Ann}_{\mathcal{T}^S_\mathcal{Q}}(H^*(X_{K_i}(\mathcal{O}), \mathcal{O}))$. Let $\alpha$ be the right inverse of $i^*$ and $\beta$ be the left inverse of $j^*$. Take any $h \in H^*(X_{K_i}(\mathcal{O}), \mathcal{O})$. Then we can write

$$
S_\mathcal{Q}(t) = r_p(r_p(t))h = \beta(j^*(r_p(t))h) = \beta(r_p(t))j^*(h)) = \beta(r_p(t)i^*(\alpha(j^*(h)))) = \beta(i^*(\alpha(j^*(h)))) = \beta(i^*(0)) = 0.
$$

To prove the second statement, it is enough to note that $H^*(\widetilde{X}^P_{\mathcal{K}_i(\mathcal{O})}, \mathcal{O})_{\tilde{m}} \cong H^*(\partial \widetilde{X}_{\mathcal{K}_i(\mathcal{O})}, \mathcal{O})_{\tilde{m}}$ by [ACC+18, Theorem 2.4.2].

6. Galois deformation theory

Let $E \subset \overline{Q}_p$ be a finite extension of $Q_p$, with valuation ring $\mathcal{O}$, uniformiser $\sigma$ and residue field $k$. Given a complete Noetherian local $\mathcal{O}$-algebra $\Lambda$ with residue field $k$, we let $\text{CNL}_\Lambda$ denote the category of complete Noetherian $\Lambda$-algebras with residue field $k$. We refer to an object in $\text{CNL}_\Lambda$ as a CNL$_\Lambda$-algebra. We fix a number field $E$ and let $\mathcal{S}_p$ be the set of places of $E$ above $p$. We assume that $E$ contains the images of all embeddings of $F$ in $Q_p$. We also fix a continuous absolutely irreducible homomorphism $\overline{\rho} : G_F \rightarrow \text{GL}_n(k)$. We assume throughout that $p \nmid 2n$.

Following [ACC+18, Definition 6.2.2], we call a global deformation problem a tuple

$$
\mathcal{S} = (\overline{\rho}, S, \{A_v\}_{v \in \mathcal{S}}, \{D_v\}_{v \in \mathcal{S}}),
$$

where

- $\mathcal{S}$ is a finite set of finite places of $F$ containing $S_p$ and all the places at which $\overline{\rho}$ is ramified.
- $A_v$ is an object of $\text{CNL}_\mathcal{Q}$ for each $v \in \mathcal{S}$.
- $D_v$ is a local deformation problem ([ACC+18, Section 6.2.1]) for each $v \in \mathcal{S}$.

Associated to this global deformation problem, we have a completed tensor product $\Lambda = \overline{\otimes}_{v \in \mathcal{S}} A_v$. A global deformation problem determines a representable functor $\mathcal{D}_\mathcal{S} : \text{CNL}_\Lambda \rightarrow \text{Set}$ which takes an object $A \in \text{CNL}_\Lambda$ to the set of deformations $\rho : G_F \rightarrow \text{GL}_n(A)$ of type $\mathcal{S}$.

Let $v$ be a finite place of $F$, such that $v \notin \mathcal{S}$ and $q_v \equiv 1 \pmod{p}$. We let $\mathcal{D}_\mathcal{S}_v$ denote the local deformation problem consisting of all lifts which associate $A \in \text{CNL}_{\Lambda_v}$ to the set of lifts which are $1 + M_n(m_\Lambda)$-conjugate to a lift of the form $s_v \otimes \psi_v$, where $s_v$ is unramified and the image of $\psi_v$ under
Lemma 6.1. Let \( \overline{\tau} : G_{F_v} \to \text{GL}_n(k) \) be an unramified continuous representation and \( A \) is a complete Noetherian local \( \mathcal{O} \)-algebra with residue field \( k \) and a principal maximal ideal \( \mathfrak{m}_A \). Suppose further that \( \overline{\tau} \) may be written in the form \( \overline{\tau} = \overline{\tau}_1 \oplus \overline{\tau}_2 \), where \( \det(X - \overline{\tau}_1(\text{Frob}_v)) \) and \( \det(X - \overline{\tau}_2(\text{Frob}_v)) \) are relatively prime. Also suppose that \( q_v = 1 \) in \( k \). Then any lift \( r : G_{F_v} \to \text{GL}_n(A) \) of \( \overline{\tau} \) is \( 1 + M_n(\mathfrak{m}_A) \)-conjugate to one of the form \( r = r_1 \oplus r_2 \), where \( r_1 \) and \( r_2 \) are lifts of \( \overline{\tau}_1 \) and \( \overline{\tau}_2 \), respectively.

Proof. Let \( n_i = \dim \overline{\tau}_i \). Suppose we have a lift \( r_m : G_{F_v} \to \text{GL}_n(A) \) of \( \overline{\tau} \), such that \( r_m \mod \mathfrak{m}_A^m \) can be written in the form \( r_1 \oplus r_2 \). We will show that there exists a matrix \( X_m \in 1 + M_n(\mathfrak{m}_A^m) \), such that \( r_{m+1} := X_m r_m X_m^{-1} \) satisfies the same condition \( \mod \mathfrak{m}_A^{m+1} \).

Write

\[
X_n = \begin{pmatrix} 1 & Y \\ Z & 1 \end{pmatrix} \quad r_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

where \( Y \in M_{n_1 \times n_2}(\mathfrak{m}_A^m) \) and \( Z \in M_{n_2 \times n_1}(\mathfrak{m}_A^m) \). Then the condition on \( r_{m+1} \) transforms into

\[
YD - AY + B = 0 \mod \mathfrak{m}_A^{m+1} \quad (6.2)
\]

\[
ZA - DZ + C = 0 \mod \mathfrak{m}_A^{m+1}. \quad (6.3)
\]

We will focus on the first condition, the second is similar. We know that \( r_m \mod \mathfrak{m}_A^m \) is block-diagonal, so we can consider \( \overline{b} \), \( \overline{y} \) to be the images of \( B \) and \( Y \), respectively, in \( \mathfrak{m}_A^m/\mathfrak{m}_A^{m+1} \),

\[
\overline{b} \overline{r}_2^{-1} = \overline{r}_1 \overline{y} \overline{r}_2^{-1} - \overline{y} \quad (6.4)
\]

in \( M_n(\mathfrak{m}_A^m/\mathfrak{m}_A^{m+1}) = M_n(k) \otimes_k \mathfrak{m}_A^m/\mathfrak{m}_A^{m+1} \). Using the fact that \( r \) is a homomorphism, for \( \sigma, \tau \in G_{F_v} \), we can write

\[
A(\sigma)B(\tau) + B(\sigma)D(\tau) = B(\sigma \tau).
\]

Rewriting and reducing \( \mod \mathfrak{m}_A^{m+1} \), we get

\[
\overline{r}_1(\sigma) \overline{b}(\tau) + \overline{b}(\sigma) \overline{r}_2(\tau) = \overline{b}(\sigma \tau)
\]

\[
\overline{b}(\sigma \tau) \overline{r}_2^{-1}(\sigma \tau) = \overline{r}_1(\sigma) \overline{b}(\tau) \overline{r}_2^{-1}(\tau) \overline{r}_2^{-1}(\sigma) + \overline{b}(\sigma) \overline{r}_2^{-1}(\sigma). \quad (6.5)
\]

Give \( M_{n_1 \times n_2}(\mathfrak{m}_A^m/\mathfrak{m}_A^{m+1}) \) the structure of a \( G_{F_v} \)-module via \( \overline{r}_1(-) \overline{r}_2^{-1} \), and denote this module \( \text{ad}(\overline{r}_1, \overline{r}_2) \). Then the last equation implies that \( \overline{b} \overline{r}_2^{-1} \) is in \( Z^1(G_{F_v}, \text{ad}(\overline{r}_1, \overline{r}_2)) \). Since \( \overline{r}_1, \overline{r}_2 \) have coprime characteristic polynomials, we know that \( H^1(G_{F_v}, \text{ad}(\overline{r}_1, \overline{r}_2)) = 0 \) by local Tate duality (here, we are using that \( q_v = 1 \) in \( k \)), which means \( \overline{b} \overline{r}_2^{-1} \in B^1(G_{F_v}, \text{ad}(\overline{r}_1, \overline{r}_2)) \), and thus we can find \( y \) satisfying Eq. (6.4). \( \square \)

Now we define our version of the Taylor-Wiles datum, analogous to the one appearing in [ACC+18, Section 6.2.27].

Definition 6.6. Let

\[
\mathcal{S} = (\overline{\rho}, S, \{A_v\}_{v \in S}, \{D_v\}_{v \in S})
\]
be a global deformation problem. A Taylor-Wiles datum of level \( N \geq 1 \) for \( S \) consists of a tuple \((Q, \alpha_v v \in Q)\), where

- A finite set \( Q \) of places of \( F \), disjoint from \( S \), such that \( q_v \equiv 1 \pmod{p^N} \) for each \( v \in Q \).
- For each \( v \in Q \), \( \alpha_v \) is an eigenvalue of \( \overline{\rho}(\text{Frob}_v) \).

Given a Taylor-Wiles datum \((Q, (\alpha_v))\), we define a global deformation problem

\[
\mathcal{S}_Q = (\overline{\rho}, S \cup Q, \{\Lambda_v\} v \in S \cup \{O_{F_v}\} v \in Q, \{D_v\} v \in S \cup \{D_v^1\} v \in Q).
\]

Define \( \Delta_Q = \prod_{v \in Q} \Delta_v \). The representing object \( R_{S_Q} \) has a structure of a \( \mathcal{O}[\Delta_Q] \)-algebra satisfying \( R_{S_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} = R_S \).

**Proposition 6.7.** Take \( T = S \), and let \( q > h^1_{S, T}(\text{ad} \overline{\rho}(1)) \). Assume that \( F = F^*F_0 \), where \( F_0 \) is an imaginary quadratic field, that \( \zeta_p \notin F \) and that \( \overline{\rho}(G_F(\zeta_p)) \) is adequate. Then for every \( N \geq 1 \), there exists a choice of Taylor-Wiles datum \((Q_N, (\alpha_v) v \in Q)\) of level \( N \) satisfying the following:

1. \( |Q_N| = q \).
2. For each \( v \in Q_N \), the rational prime below \( v \) splits in \( F_0 \) and \( v^c \notin Q_N \).
3. Let \( g = q - n^2[F^*:Q] \). Then there is a surjective morphism

\[
R^T_{S, \text{loc}} \left[ [X_1, \ldots, X_g] \right] \rightarrow R^T_{S_Q},
\]

in \text{CNL}_\Lambda.

**Proof.** The proof is very similar to the proof of [ACC+18, Proposition 6.2.32] (cf. [Tho12, Proposition 4.4]), we omit the details. \( \square \)

### 7. Representations into Hecke algebras

In this section, we construct the necessary Galois representations into the Hecke algebras associated to \( G \). From Proposition 5.4, we know that we can create representations valued in the Hecke algebra acting on \( H^*(X_{K_i(Q)}, \mathcal{O})_{\overline{\mu}_Q} \) from representations valued in the Hecke algebra acting on \( H^*(\partial \overline{X}_{K_i(Q)}, \mathcal{O})_{\overline{\mu}_Q} \). The latter representations will be constructed by glueing together Galois representations associated to cuspidal cohomological automorphic representations of \( \widetilde{G}(\mathbb{A}^\infty_F) \) as in [Sch15] and using the local computations of Section 3.

#### 7.1. Hecke algebras for \( \widetilde{G} \)

**Theorem 7.1.** Suppose that \( \widetilde{K} \subset \widetilde{G}(\mathbb{A}^\infty_F) \) is a good subgroup which is decomposed with respect to \( P \). Then there exists a \( 2n \)-dimensional \( \mathcal{T}^S_Q \left( H^c_*(X_{K_i(Q)}, \mathcal{O}) / I[X] \right) \)-valued group determinant \( D_{c,Q} \) of \( G_{F,S} \) for some ideal \( I \) of nilpotence degree depending only on \( n \) and \( [F:Q] \), such that the following properties hold:

1. If \( v \notin S \) is a place of \( F \), then \( D_{c,Q}(X - \text{Frob}_v) \) is equal to the image of \( \widetilde{P}_v(X) \) in \( \mathcal{T}^S_Q \left( H^c_*(X_{K_i(Q)}, \mathcal{O}) / I[X] \right) \).
2. If \( v \in Q \), then for any \( \sigma \in G_{F,S} \) and \( \tau \in I_{F_v} \), we have the relation

\[
\text{Tr}_{D_{c,Q}} \left( \sigma \text{Res}_{q_v, \mu_v}^{(2n)!} \text{Res}_{\mu_v}^{(2n)!} \left( \sum_{i=1}^{k-1} E_{\mu_v, i}(\varphi_v) + \langle \text{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu_v, k}(\varphi_v) - \text{Res}_{\mu_v} \tau \right) \right) = 0.
\]

**Proof.** This follows from Proposition 3.12 by using [ACC+18, Theorem 2.3.3] and [Sch15, Corollary 5.1.11] (see proof of [ACC+18, Proposition 3.2.2]). \( \square \)
Now we prove the version of the previous proposition for noncompactly supported cohomology:

**Theorem 7.2.** Suppose that $\widetilde{K} \subset \widetilde{G}(A_{F}^\infty)$ is a good subgroup which is decomposed with respect to $P$. Then there exists a $2n$-dimensional $\widetilde{T}_{Q}^{S}(H^{*}(X_{\widetilde{K}_{1}(Q)}, \mathcal{O}))/I$-valued group determinant $D_{Q}$ of $G_{F,S}$ for some ideal $I$ of nilpotence degree depending only on $n$ and $[F : Q]$, such that the following properties hold:

1. If $v \not\in S$ is a place of $F$, then $D_{Q}(X - \text{Frob}_{v})$ is equal to the image of $\widetilde{P}_{v}(X)$ in $\widetilde{T}_{Q}^{S}(H^{*}(X_{\widetilde{K}_{1}(Q)}, \mathcal{O}))/I[X]$.
2. If $v \in Q$, then for any $\sigma \in G_{F,S}$ and $\tau \in I_{F_{v}}$, we have the relation
   \[
   \text{Tr}_{D_{Q}} \left( \sigma \text{Res}_{q_{v},\mu_{v}}^{(2n)!} \text{Res}_{\mu_{v}}^{(2n)!} \left( \sum_{i=1}^{k-1} E_{\mu_{v},i}(\varphi_{v}) + \langle \text{Art}_{F_{v}}^{-1}(\tau) \rangle E_{\mu_{v},k}(\varphi_{v}) - \text{Res}_{\mu_{v}} \tau \right) \right) = 0. \]

**Proof.** Denote by $\widetilde{T}_{Q}^{S}(H^{*}(X_{\widetilde{K}_{1}(Q)}, \mathcal{O}))$ the image of $\widetilde{T}_{Q}^{S}$ under the homomorphism

\[
\widetilde{T}_{Q}^{S} \rightarrow \mathcal{H}_{\mathcal{O}}(\mathcal{G}(A_{F}^\infty), \widetilde{K}_{1}(Q)) \xrightarrow{\text{tr}} \mathcal{H}_{\mathcal{O}}(\mathcal{G}(A_{F}^\infty), \widetilde{K}_{1}(Q)) \rightarrow \text{End}_{\mathcal{D}(\mathcal{O})}(H^{*}(X_{\widetilde{K}_{1}(Q)}, \mathcal{O})).
\]

The same argument as in the proof of Theorem 7.1 shows that there exists a group determinant $D_{\ell}$ valued in $\widetilde{T}_{Q}^{S}(H^{*}(X_{\widetilde{K}_{1}(Q)}, \mathcal{O}))/I$ satisfying the following properties:

1. If $v \not\in S$ is a place of $F$, then $D_{Q}(X - \text{Frob}_{v})$ is equal to the image of $\widetilde{P}_{v}(X)$ in $\widetilde{T}_{Q}^{S}(H^{*}(X_{\widetilde{K}_{1}(Q)}, \mathcal{O}))/I[X]$.
2. If $v \in Q$, then for any $\sigma \in G_{F,S}$ and $\tau \in I_{F_{v}}$, we have the relation
   \[
   \text{Tr}_{D_{\ell}} \left( \sigma \text{Res}_{q_{v},\mu_{v}}^{(2n)!} \text{Res}_{\mu_{v}}^{(2n)!} \left( \sum_{i=1}^{k-1} E_{\mu_{v},i}(\varphi_{v}) + \langle \text{Art}_{F_{v}}^{-1}(\tau) \rangle E_{\mu_{v},k}(\varphi_{v}) - \text{Res}_{\mu_{v}} \tau \right) \right) = 0. \]

By [NT16, Proposition 3.7], we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}_{\mathcal{O}}(\mathcal{G}(A_{F}^\infty), \widetilde{K}_{1}(Q)) & \xrightarrow{\text{tr}} & \text{End}_{\mathcal{D}(\mathcal{O})}(R\Gamma(X_{\widetilde{K}_{1}(Q)}, \mathcal{O})) \\
\downarrow & & \downarrow \\
\mathcal{H}_{\mathcal{O}}(\mathcal{G}(A_{F}^\infty), \widetilde{K}_{1}(Q)) & \xrightarrow{\text{tr}} & \text{End}_{\mathcal{D}(\mathcal{O})}(R\Gamma(X_{\widetilde{K}_{1}(Q)}, \mathcal{O})),
\end{array}
\]

where the right vertical arrow is induced by Poincaré duality. Then we get an isomorphism

\[
\widetilde{T}_{Q}^{S}(H^{*}(X_{\widetilde{K}_{1}(Q)}, \mathcal{O}))/I_{1} \approx \widetilde{T}_{Q}^{S}(H^{*}(X_{\widetilde{K}_{1}(Q)}, \mathcal{O}))/I_{2}
\]

over $\widetilde{T}_{Q}^{S}$ for some ideals $I_{1,2}$ of nilpotence degrees depending only on $n$ and $[F : Q]$. Moreover, we can choose $I_{1}$, such that it contains $I$. We can conclude by making $D_{Q}$ the image of $D_{\ell}$ under this homomorphism. \hfill $\Box$

**Lemma 7.4.** Let $k$ be a field, and let $\widetilde{p}_{1}, \widetilde{p}_{2} : G \rightarrow GL(n,k)$ be two nonisomorphic absolutely irreducible representations. Then the extended map $k[G] \rightarrow M_{n}(k) \oplus M_{n}(k)$ defined by $\widetilde{p}_{1} \oplus \widetilde{p}_{2}$ is surjective.

**Proof.** We may pass to the algebraic closure of $k$ (which we still denote $k$). Let $\ell_{i} : k[G] \rightarrow M_{n}(k)$ be the linear extension of $\widetilde{p}_{i}$ for $i = 1, 2$. The two maps $\ell_{i}$ are surjective by Burnside’s theorem. Let $A$ be the image of $\ell_{1} \oplus \ell_{2}$, and let $I_{i} = \ker(A \rightarrow M_{n}(k))$, where $i = 1, 2$ corresponds to projecting on the first and second factor. Since $\ell_{i}$ are surjective, $I_{i}$ are in fact two-sided ideals of $M_{n}(k)$. Then $I_{i} = M_{n}(k)$ or $I_{i} = 0$. If $I_{i} = M_{n}(k)$ for some $i$, then $\ell_{1} \oplus \ell_{2}$ is surjective. Suppose then that $I_{1} = I_{2} = 0$. Then we have an automorphism $f$ of $M_{n}(k)$ defined by $(v, f(v)) \in A$ for all $v \in M_{n}(k)$. Since all the automorphisms...
of $M_n(k)$ are inner, we conclude that there exists $u \in GL_n(k)$, such that $A = \{(v, uv^{-1}) \mid v \in M_n(k)\}$. But this is impossible since $\overline{\rho}_1$ and $\overline{\rho}_2$ are nonisomorphic. □

**Theorem 7.5.** Suppose that $\overline{K} \subset \overline{G}(\mathbb{A}_F^\infty)$ is a good subgroup which is decomposed with respect to $P$ and that for each $v \in Q$, we have $\operatorname{Res}_{\nu_v} \notin \overline{\mathfrak{m}}_Q$. Then there exists a continuous representation

$$\rho_{m_Q} : G_{F,S \cup Q} \to \operatorname{GL}_n(T^*_Q(H^*(X_{K_1(Q)}, \mathcal{O})_{m_Q})/I)$$

satisfying the conditions below for some ideal $I \subset T^*_Q(H^*(X_{K_1(Q)}, \mathcal{O})_{m_Q})$ of nilpotence degree depending only on $n$ and $[F : Q]$.

1. If $v \notin S$ is a place of $F$, the characteristic polynomial of $\rho_{m_Q}(\operatorname{Frob}_v)$ is equal to the image of $P_v(X)$ in $T^*_Q(H^*(X_{K_1(Q)}, \mathcal{O})_{m})/I[X]$.
2. If $v \in Q$, then $\rho_{m_Q}|G_{F,v}$ is unramified.
3. If $v \in Q$, then $\rho_{m_Q}|G_{F,v} = s \otimes \psi$, where $s$ is unramified and $\tau \in I_{F,v}$ acts on $\psi$ as a scalar $\langle \operatorname{Art}_{F,v}^{-1}(\tau) \rangle$.

**Proof.** Using Theorem 7.1 and Theorem 7.2, we can construct a $\overline{T}^*_Q(H^*_c(X_{K_1(Q)}, \mathcal{O})_{\overline{m}_Q} \oplus H^*(X_{K_1(Q)}, \mathcal{O})_{\overline{m}_Q})/I$-valued group determinant $D_Q$ of $G_{F,S \cup Q}$. Consider the long exact sequence

$$\ldots \to H^i_c(\overline{X}_{K_1(Q)}, \mathcal{O}) \to H^i(\overline{X}_{K_1(Q)}, \mathcal{O}) \to H^i(\partial \overline{X}_{K_1(Q)}, \mathcal{O}) \to H^{i+1}_c(\overline{X}_{K_1(Q)}, \mathcal{O}) \to \ldots$$

Using this sequence and Proposition 5.4, we know that $S^f_Q$ descends to a homomorphism

$$\overline{T}^*_Q(H^*_c(X_{K_1(Q)}, \mathcal{O})_{\overline{m}_Q} \oplus H^*(X_{K_1(Q)}, \mathcal{O})_{\overline{m}_Q}) \to T^*_Q(H^*(X_{K_1(Q)}, \mathcal{O})_{m_Q})/I_0$$

for some ideal $I_0$ with square 0. We can use this to construct a $2n$-dimensional group determinant $D^0_Q$ valued in $T^*_Q(H^*(X_{K_1(Q)}, \mathcal{O})_{m_Q})/I$, such that:

1. For $v \notin S$, we have $D^0_Q(X - \operatorname{Frob}_v) = P_v(X)q_v^{(2n-1)}P_{v,c}^{-1}(q_v^{1-2n}X)$.
2. For $v \in Q$, we have

$$\operatorname{Tr}_{D^0_Q} S^f_Q(\sigma \operatorname{Res}_{\nu_v}^{(2n)!}, \rho_{m_Q}(\nu_v)(\sum_{i=1}^{k-1} E_{\nu_v,i}(\varphi_v) + \langle \operatorname{Art}_{F,v}^{-1}(\tau) \rangle E_{\nu_v,k}(\varphi_v) - \operatorname{Res}_{\nu_v}(\tau))) = 0,$$

and $I$ has nilpotence degree depending only on $n$ and $[F : Q]$. By [ACC+18, Theorem 2.3.7], there also exists an $n$-dimensional group determinant $D^1_Q$ of $G_{F,S \cup Q}$ valued in $T^*_Q(H^*(X_{K_1(Q)}, \mathcal{O})_{m_Q})/I$, such that $D^1_Q(X - \operatorname{Frob}_v) = P_v(X)$ for $v \notin S$. Then the group determinants $D^1_Q \oplus D^1_Q^{-1}$ and $D^0_Q$ are equal. Moreover, since $\overline{\rho}_m$ is absolutely irreducible, there exists a continuous representation

$$\rho_{m_Q} : G_{F,S \cup Q} \to \operatorname{GL}_n(T^*_Q(H^*(X_{K_1(Q)}, \mathcal{O})_{m_Q})/I),$$

such that the characteristic polynomial of $\rho_{m_Q}$ is associated to $D^1_Q$. Let $\rho'_{m_Q} := \rho_{m_Q} \oplus \rho_{m_Q}^{-1}$. Writing out the relation at places $v \in Q$, we get

$$\operatorname{Tr}(\rho'_{m_Q}(\sigma)S^f_Q(\operatorname{Res}_{\nu_v}^{(2n)!}, \rho_{m_Q}(\nu_v)(\sum_{i=1}^{k-1} E_{\nu_v,i}(\varphi_v) + \langle \operatorname{Art}_{F,v}^{-1}(\tau) \rangle E_{\nu_v,k}(\varphi_v) - \operatorname{Res}_{\nu_v}(\tau))) = 0.$$
Since $\text{Res}_{\mu_v} \not\in \overline{m}_Q$, we know that $\overline{\rho}_m$ and $\overline{\rho}_m^\perp$ are not isomorphic. Applying Nakayama’s lemma and Lemma 7.4, we see that the extended map
\[ T^S_{G,F,S} | Q | \rightarrow M_n(T^S_Q(H^*(X_{K_1(Q),\o})_{mQ})/I) \oplus M_n(T^S_Q(H^*(X_{K_1(Q),\o})_{mQ})/I) \]
given by $\rho_{mQ} \oplus \rho_{mQ}^\perp$ is surjective. Considering the trace relation above with $\sigma$ replaced by an arbitrary element of $T^S_{Q}[G_F,S\cup Q]$, we conclude that
\[
S^f_Q(\text{Res}_{\mu_v}^{2n}) \oplus S^f_Q(\sum_{i=1}^{k-1} E_{\mu_v,i}(\rho_{mQ}'(\varphi_v)))
+ \langle \text{Art}_{F_v}^{-1}(\tau) E_{\mu_v,k}(\rho_{mQ}'(\varphi_v)) - \text{Res}_{\mu_v} \rho_{mQ}'(\tau) \rangle = 0.
\]
Since $q_v \equiv 1 \mod p$, we know that $\text{Res}_{q_v,\mu_v} \not\in \overline{m}_Q$. Thus
\[
S^f_Q(\sum_{i=1}^{k-1} E_{\mu,v,i}(\rho_{mQ}'(\varphi_v))) + \langle \text{Art}_{F_v}^{-1}(\tau) E_{\mu_v,k}(\rho_{mQ}'(\varphi_v)) - \text{Res}_{\mu_v} \rho_{mQ}'(\tau) \rangle = 0.
\]
This implies that
\[
\rho_{mQ}(\tau) = S^f_Q(\sum_{i=1}^{k-1} \text{Res}_{\mu_v}^{-1} E_{\mu_v,i}(\rho_{mQ}(\varphi_v))) + S^f_Q(\langle \text{Art}_{F_v}^{-1}(\tau) \rangle \text{Res}_{\mu_v}^{-1} E_{\mu_v,k}(\rho_{mQ}(\varphi_v))).
\]
Using Proposition 5.2, we can transform the equation above into
\[
\rho_{mQ}(\tau) = \text{Res}_{\mu_v}^{-1} E_{\mu,v,1}(\rho_{mQ}(\varphi_v)) + \langle \text{Art}_{F_v}^{-1}(\tau) \rangle \text{Res}_{\mu_v}^{-1} E_{\mu,v,2}(\rho_{mQ}(\varphi_v))
\]
Let $T := T^S_Q(H^*(X_{K_1(Q),\o})_{mQ})/I$. Consider the decomposition $\overline{T}_m = \overline{T}_1 \oplus \overline{T}_2$, corresponding to the Frobenius generalised eigenspaces of all eigenvalues not equal to $\alpha_v$ and $\alpha_v$, respectively. Then
\[
T^n = \text{Res}_{\mu_v}^{-1} E_{\mu,v,1}(\rho_{mQ}(\varphi_v))T^n \oplus \text{Res}_{\mu_v}^{-1} E_{\mu,v,2}(\rho_{mQ}(\varphi_v))T^n
\]
is the unique $\rho_{mQ}(\varphi_v)$-invariant lift of $\overline{T}_1 \oplus \overline{T}_2$, and we are done by Lemma 6.1. \qed

7.2. Hecke algebras for $G$

Let $\lambda \in (\mathcal{Z}_n^\alpha)^{\text{Hom}(F,E)}$. Further let $S$ be a finite set of finite places of $F$ containing the $p$-adic places and stable under complex conjugation satisfying the following condition:

1. Let $l$ be a rational prime, such that there exists a place above $l$ in $S$ or $l$ is ramified in $F$. Then there exists an imaginary quadratic subfield $F_0 \subset F$, such that $l$ splits in $F_0$.

Let $K \subset \text{GL}_n(A_F^\infty)$ be a good subgroup, such that for all $v \not\in S$, we have $K_v = \text{GL}_n(\mathcal{O}_{F_v})$. Let $m \subset T^S(K,\lambda)$ be a non-Eisenstein maximal ideal with residue field $k$. By [ACC’18, Theorem 2.3.5], there exists an associated residual representation $\overline{\rho}_m : G_{F,S} \rightarrow \text{GL}_n(T^S(K,\lambda)/m)$. By [ACC’18, Theorem 2.3.7], there exists an ideal $I \subset T^S(K,\lambda)$ of nilpotence degree depending only on $n$ and $[F : Q]$ and a continuous lift $\rho_m : G_{F,S} \rightarrow \text{GL}_n(T^S(K,\lambda)/m)I$, such that for each $v \in S$, $\det(X - \rho_m(\text{Frob}_v))$ is the image of $P_v(X)$ in $T^S(K,\lambda)/mI[X]$. We consider the following Taylor-Wiles datum: a tuple $(Q, (\alpha_v)_v) \in Q$ consisting of

- A finite set $Q$ of places of $F$, disjoint from $Q^c$, such that $q_v \equiv 1 \mod p$ for each $v \in Q$.
- Each $v \in Q$ is split in $F^+$, and there exists an imaginary quadratic subfield $F_0 \subset F$, such that $v$ is split in $F_0$. Moreover, $\overline{\rho}_m$ is unramified at $v$ and $v^c$.
- $\alpha_v$ is a root of $\det(X - \overline{\rho}_m(\text{Frob}_v))$. 

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Consider the partition $\nu_v : n = d_v + (n-d_v)$, where $d_v$ is the multiplicity of $\alpha_v$ as a root of $\det(X - \overline{\rho}_m(\text{Frob}_v))$.

We define auxiliary level subgroups $K_1(Q) \subset K_0(Q) \subset K$. They are good subgroups of $\text{GL}_n(A_F^{\infty})$ defined by the following conditions:

- If $v \notin Q$, then $K_1(Q)_v = K_0(Q)_v = K_v$.
- If $v \in Q$, then $K_0(Q)_v = \mathfrak{p}_v$, and $K_1(Q)_v = \mathfrak{p}_{v,1}$.

We have a natural isomorphism $K_0(Q)/K_1(Q) \cong \Delta_Q = \prod_{v \in Q} \Delta_v$. Let $S' = S \cup Q \cup Q^c$. We define $T'_Q = T^{S \cup Q} \otimes_{\mathbb{Z}} \mathbb{Z} \mathbb{E}_{v,1}^{S,v}$. Let $T'_Q(K_0(Q),\lambda)$ and $T'_Q(K_0(Q)/K_1(Q),\lambda)$ be the images of $T'_Q$ in $\text{End}_{\text{D}(\mathcal{O})}(R\Gamma(X_0(Q), V_\lambda))$ and $\text{End}_{\text{D}(\mathcal{O}/\Delta_Q)}(R\Gamma(X_{K_1(Q)}, V_\lambda))$, respectively. Let $m_Q$ be the maximal ideal of $T'_Q$ generated by $m$ and the kernels of the homomorphisms $\mathbb{Z}\mathbb{E}_{v,1}^{S,v} \to k$ given by the coefficients of polynomials $(X - \alpha_v)^{d_v}, \det(X - \overline{\rho}_m(\text{Frob}_v))/(X - \alpha_v)^{d_v}$.

**Theorem 7.6.** We have natural isomorphisms

$$R\Gamma(X_K, V_\lambda)_m \cong R\Gamma(X_{K_0(Q)}, V_\lambda)_m$$

in $\text{D}(\mathcal{O})$.

**Proof.** The second isomorphism is straightforward. For the first, we can check on the level of cohomology. It is enough to show that it is an isomorphism in $\text{D}(k)$ after applying the functor $- \otimes_{\mathcal{O}}^{\mathbb{L}} k$. Thus, we need to show that the map

$$H^i(X_K, V_\lambda/\overline{\sigma})_m \to H^i(X_{K_0(Q)}, V_\lambda/\overline{\sigma})_m$$

is an isomorphism. We can do this one prime at a time, so we can assume $Q = \{v\}$. For each $j$, let

$$M_j := \lim_{m \to \infty} H^i(X_{K(v^m)}, V_\lambda/\overline{\sigma})_m,$$

where $K(v^m)_w = K_w$ for places $w \neq v$ and $K(v^m)_v$ is the principal congruence subgroup of level $v^m$.

We have two Hochschild-Serre spectral sequences:

$$H^i(\text{GL}_n(\mathcal{O}_{F_v}), M_j) \Rightarrow H^{i+j}(X_K, V_\lambda/\overline{\sigma})_m$$

$$e_{\alpha_v} H^i(\mathfrak{p}_{v,1}, M_j) \Rightarrow e_{\alpha_v} H^{i+j}(X_{K_0(Q)}, V_\lambda/\overline{\sigma})_m = H^{i+j}(X_{K_0(Q)}, V_\lambda/\overline{\sigma})_m.$$

There is a natural map $\iota^*$ between these spectral sequences, which arises from deriving the map

$$M_j^{\text{GL}_n(\mathcal{O}_{F_v})} \to M_j^{\mathfrak{p}_{v,1}} \to e_{\alpha_v} M_j^{\mathfrak{p}_{v,1}}.$$

Thus, it is enough to show that $\iota^*$ is an isomorphism. $M_j$ is admissible, and we can use [Vig98, Theorem III.6] to write $M_j$ as a direct sum of $\text{GL}_n(F_v)$-modules, each belonging to a single block. Let $N \subset M_j$ be a summand from a nonunipotent block. Let $T_p(k)$ be the $p$-power part of $T(k)$. We note that both $H^i(\text{GL}_n(\mathcal{O}_{F_v}), N)$ and $H^i(\mathfrak{p}_{v,1}, N)$ inject into $H^1(1,w,N)$, which in turn is equal to $H^1(T_p(k), N^{Iw^p})$. Since $N$ is a summand of a nonunipotent block, we know that $N^{Iw^p} = 0$, and so

$$H^i(\text{GL}_n(\mathcal{O}_{F_v}), N) = H^i(\mathfrak{p}_{v,1}, N) = 0.$$

Thus, we can restrict to the summand $M_j^1 \subset M_j$ from the unipotent block, and it is enough to prove that

$$\iota^* : H^i(\text{GL}_n(\mathcal{O}_{F_v}), M_j^1) \to e_{\alpha_v} H^i(\mathfrak{p}_{v,1}, M_j^1)$$
There exists an ideal $I \subset T_Q^S(K_0(Q)/K_1(Q), \lambda)_{m_Q}$ of nilpotence degree depending only on $n$ and $[F:Q]$, together with a continuous homomorphism

$$\rho_{m,Q} : G_{F,S\cup Q} \to \text{GL}_n(T_Q^S(K_0(Q)/K_1(Q), \lambda)_{m_Q}/I)$$

lifting $\overline{\rho}_m$ and satisfying the following conditions:

1. For a finite place $v \not\in S \cup Q$ of $F$, $\det(X - \rho_{m,Q}(\text{Frob}_v))$ equals to the image of $P_v(X)$ in $T_Q^S(K_0(Q)/K_1(Q), \lambda)_{m_Q}/I[X]$.

2. For $v \in Q$, $\rho_{m,Q}|_{G_{F,v}}$ is unramified and $\rho_{m,Q}|_{G_{F,v}}$ is a lifting of type $D_v$, and the induced map $\mathcal{O}[\Delta] \to T_Q^S(K_0(Q)/K_1(Q), \lambda)_{m_Q}/I$ is a homomorphism of $\mathcal{O}[\Delta]$-algebras.

Proof. We first make a few reductions. Let us show that we can reduce the situation to where $\det(X - \overline{\rho}_m(\text{Frob}_v))$ and $\det(X - \overline{\rho}_m(\text{Frob}_v))$ are coprime for each $v \in Q$. To achieve this, we will use twisting. Pick an odd prime $l \not= p$ and consider a character $\psi : G_F \to \mathcal{O}^{X}$ of order $l$, such that $\det(X - \overline{\rho}_m \otimes \overline{\psi}(\text{Frob}_v))$ and $\det(X - \overline{\rho}_m \otimes \overline{\psi}(\text{Frob}_v))$ are coprime. Let $S_\psi$ denote the places of $F$ at which $\psi$ is ramified. We will further require that $S_\psi$ is disjoint from $S'$. Define a good subgroup $K^{\psi}_v \subset K$ given by $K^{\psi}_v = K_v$ at places $v$ at which $\psi$ is not ramified, and $K^{\psi}_v = \ker(\text{GL}_n(\mathcal{O}_{F_v}) \to k(v)^{X}/(k(v)^{X})')$ at places $v$, where $\psi$ is ramified. Following the discussion above [ACC+18, Proposition 2.2.22], we have a homomorphism $f_\psi : T^{S'\cup S_\psi}(K^{\psi}, \lambda) \to T^{S'\cup S_\psi}(K^{\psi}, \lambda)$ given by

$$f_\psi ([K^{\psi}S'\cup S_\psi gK^{\psi}S'\cup S_\psi]) = \psi^{-1}(\text{Art}(\det(g)))[K^{\psi}S'\cup S_\psi gK^{\psi}S'\cup S_\psi].$$

We have a maximal ideal $m_\psi = f_\psi(m)$ of $T^{S'\cup S_\psi}(K^{\psi}, \lambda)$. [ACC+18, Proposition 2.2.22] implies an isomorphism $\overline{\rho}_m \otimes \overline{\psi} \cong \overline{\rho}_{m_\psi}$. Similarly to Eq. 7.8, we have an isomorphism

$$T^{S'\cup S_\psi}_Q(K^{\psi}_0(Q)/K^{\psi}_1(Q), \lambda)_{m_\psi,Q} \cong T^{S'\cup S_\psi}_Q(K^{\psi}_0(Q)/K^{\psi}_1(Q), \lambda)_{m_Q},$$

where $m_\psi$ is the maximal ideal of $T^{S'\cup S_\psi}_Q$ generated by $m_\psi$ and the kernels of the homomorphisms $Z[\Xi_{v,1}]^{S_{v'}} \to k$ given by the coefficients of polynomials $(X - \psi(\text{Frob}_v)\alpha_v)^{d_v}, \det(X - \overline{\rho}_m(\text{Frob}_v))/(X - \psi(\text{Frob}_v)\alpha_v)^{d_v}$. We have a surjective map of $T^{S'\cup S_\psi}_Q$-algebras

$$T^{S'\cup S_\psi}_Q(K^{\psi}_0(Q)/K^{\psi}_1(Q), \lambda)_{m_\psi} \to T^{S'\cup S_\psi}_Q(K^{\psi}_0(Q)/K^{\psi}_1(Q), \lambda)_{m_Q}.$$

Thus, if the theorem holds for representations into $T^{S'\cup S_\psi}_Q(K^{\psi}_0(Q)/K^{\psi}_1(Q), \lambda)_{m_\psi}$, it will hold for representations into $T^{S'\cup S_\psi}_Q(K^{\psi}_0(Q)/K^{\psi}_1(Q), \lambda)_{m_Q}$. Since there are infinitely many $\psi$ satisfying the conditions we require, we can vary them to conclude that the theorem holds for $T^{S'\cup S_\psi}_Q(K^{\psi}_0(Q)/K^{\psi}_1(Q), \lambda)_{m_Q}$, which is our target Hecke algebra.

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Let $\tilde{K} \subset \tilde{G}(A_F^{\infty})$ be a good subgroup satisfying the following conditions:

1. $\tilde{K}$ is decomposed with respect to $P$.
2. $\tilde{K} \cap G(A_F^{\infty}) \subset K$.
3. if $\tilde{\nu}$ is a finite place of $F^+$, such that $\tilde{\nu} \notin \tilde{S}$, then $\tilde{K}_{\tilde{\nu}} = \tilde{G}(O_{F^+})$.

We can use the Hochschild-Serre spectral sequence to reduce to the case where $K = \tilde{K} \cap G(A_F^{\infty})$. We can further reduce our theorem to the case $\lambda = 0$, by a standard use of the Hochschild-Serre spectral sequence to trivialise the weight modulo some power $m$ at the expense of shrinking the level at $p$. Now the theorem follows from Theorem 7.5.

8. Proof of Theorem 1.2 and Theorem 1.3

Let us recall the proof structure of [ACC+18, Theorem 6.1.1]. The theorem is reduced in [ACC+18] to [ACC+18, Corollary 6.5.5], which is proved using [ACC+18, Theorem 6.5.4]. The reduction does not use the ‘enormous’ assumption on the image of $\tilde{\nu}$.

We assume that the following conditions are satisfied:

6. If $l$ is a prime lying below an element of $S$, or which is ramified in $F$, then $F$ contains an imaginary quadratic field in which $l$ splits. In particular, each place of $S$ is split over $F^+$ and the extension $F/F^+$ is everywhere unramified.
7. The prime $p$ is unramified in $F$.
8. For each embedding $\tau : F \hookrightarrow C$, we have

$$\lambda_{\tau,1} + \lambda_{\tau,1} - \lambda_{\tau,1} - \lambda_{\tau,1} < p - 2n.$$ 

9. For each $\nu \in S_p$, let $\tilde{\nu}$ denote the place of $F^+$ lying below $\nu$. Then there exists a place $\tilde{\nu}' \neq \tilde{\nu}$ of $F^+$, such that $\tilde{\nu}' \mid p$ and

$$\sum_{\tilde{\nu}' \neq \tilde{\nu}, \tilde{\nu}'} |F_{\tilde{\nu}'}^+:Q_p| > \frac{1}{2} |F^+:Q|.$$ 

10. The residual representation $\bar{r}_\ell(\pi)$ is absolutely irreducible.
11. If $\nu$ is a place of $F$ lying above $p$, then $\pi_\nu$ is unramified.
12. If $\nu \notin R$, then $\pi_\nu^{I_{\nu}} \neq 0$.
13. If $\nu \in S - (R \cup S_p)$, then $\pi_\nu$ is unramified and $H^2(F_\nu, \text{ad} \bar{r}_\ell(\pi)) = 0$.

Moreover, $\nu$ is absolutely unramified and of residue characteristic $q > 2$.
14. $S - (R \cup S_p)$ contains at least two places with distinct residue characteristics.
15. If $\nu \notin S$ is a finite place of $F$, then $\pi_\nu$ is unramified.
16. If $\nu \in R$, then $q_\nu \equiv 1 \pmod{p}$ and $\bar{r}_\ell(\pi)|_{G_{F_\nu}}$ is trivial.
17. The representation $\bar{r}_\ell(\pi)$ is decomposed generic in the sense of [ACC+18, Definition 4.3.1] and the image of $\bar{r}_\ell(\pi)|_{G_{F_{(p)}}}$ is adequate.

We define an open compact subgroup $K = \prod_v K_v$ of $\text{GL}_n(\hat{O}_F)$ as follows:

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If $v \notin S$, or $v \in S_p$, then $K_v = \text{GL}_n(\mathcal{O}_{F_v})$.

If $v \in R$, then $K_v = \text{Iw}_v$.

If $v \in S - (R \cup S_p)$, then $K_v = \text{Iw}_v$.

By [ACC+18, Theorem 2.4.10], we can find a coefficient field $E \subset \overline{Q}_p$ and a maximal ideal $m \subset T^S(K, V_1)$, such that $\overline{m} \cong \pi(I)$. After possibly enlarging $E$, we can and do assume that the residue field of $m$ is equal to $k$. For each tuple $(\chi_{v,i})_{v \in R,i=1,\ldots,n}$ of characters $\chi_{v,i} : k(v)^\times \to \mathcal{O}^\times$ which are trivial modulo $\pi$, we define a global deformation problem by the formula

$$S_X = (\overline{m}, S, \{\mathcal{O}_{\nu}\}_{\nu \in S}, \{D^\Pi_{\nu}\}_{\nu \in S_p} \cup \{D^\Gamma_{\nu}\}_{\nu \in R} \cup \{D^\Delta_{\nu}\}_{\nu \in S - (R \cup S_p)})$$.

We fix representatives $\rho_{S_\chi}$ of the universal deformations which are identified modulo $\pi$ via the identifications $R_{S_\chi}/\pi \cong R_{S_i}/\pi$. We define an $\mathcal{O}[K_S]$-module $V_\chi(\chi, X^{-1}) = V_\chi \otimes_{\mathcal{O}} \mathcal{O}(\chi^{-1})$, where $K_S$ acts on $V_\chi$ by projection to $K_F$, and on $\mathcal{O}(\chi^{-1})$ by the projection $K_S \to K_R = \prod_{v \in R} \text{Iw}_v \to \prod_{v \in R} (k(v)^\times)^n$.

**Theorem 8.1.** Under assumptions (1)-(17) above, $H^*(X_K, V_\chi(1))_m$ is a nearly faithful $R_{S_\chi}$-module. In other words, $\text{Ann}_{R_{S_\chi}}(H^*(X_K, V_\chi(1)))$ is nilpotent.

The rest of the paper is devoted to the proof of Theorem 8.1.

Consider the Taylor-Wiles datum $(Q, \{\alpha_v\}_{v \in Q})$ satisfying the following conditions:

- For each place $v \in Q$ of residue characteristic $l$, there exists an imaginary quadratic subfield $F_0 \subset F$, such that $l$ splits in $F_0$.
- $Q$ and $Q^c$ are disjoint.

We have the following result, combining [ACC+18, Proposition 6.5.3] and Theorem 7.7:

**Proposition 8.2.** There exists an integer $\delta \geq 1$ depending only on $n$ and $[F : Q]$, an ideal $J \subset T^S_Q(\text{RG}(X_{K_1}, V_\chi(1))_m_{Q_0})$, such that $J^\delta = 0$ and a continuous surjection of $\mathcal{O}[\Delta_Q]$-algebras

$$f_{S_{\chi,Q}} : R_{S_{\chi,Q}} \to T^S_Q(\text{RG}(X_{K_1(Q)}, V_\chi(1))_m_{Q_0})/J,$$

such that for each finite place $v \notin S \cup Q$, the characteristic polynomial of $f_{S_{\chi,Q}} \circ \rho_{S_{\chi,Q}}$ equals the image of $P_v(X)$.

Let

$$q = h^1(F_S/F, \text{ad} \overline{m}(1)) \quad \text{and} \quad g = q - n^2[F^+ : Q],$$

and set $\Delta_\infty = Z_p^g$. Let $T$ be a power series ring over $\mathcal{O}$ in $n^2|S| - 1$ variables, and let $S_\infty = T[\Delta_\infty]$. Let $\eta_\infty$ be the augmentation ideal of $S_\infty$ viewed as an augmented $\mathcal{O}$-algebra. Since $p > n$, for each $v \in R$, we can choose a tuple of pairwise distinct characters $\chi_v = (\chi_{v,1}, \ldots, \chi_{v,n})$, with $\chi_{v,i} : O_{F_v}^\times \to \mathcal{O}_v^\times$ trivial modulo $\pi$. We write $\chi$ for the tuple $(\chi_v)_{v \in R}$ as well as for the induced character $\prod_{v \in R} I_v \to \mathcal{O}_\infty^\times$.

Fix an imaginary quadratic subfield $F_0 \subset F$. Then for each $N \geq 1$, we fix a choice of Taylor-Wiles datum $(Q, \{\alpha_v\}_{v \in Q})$ for $S_1$ of level $N$ using Proposition 6.7. For $N = 0$, we set $Q_0 = \emptyset$. For each $N \geq 1$, we set $\Delta_N = \Delta_{Q_N}$ and fix a surjection $\Delta_\infty \to \Delta_N$. We let $\Delta_0$ be the trivial group, viewed as a quotient of $\Delta_\infty$. For each $N \geq 0$, we set $R_N = R_{S_1,Q_N}$ and $R'_N = R_{S_1,Q_N}$. Let $R^{loc} = R^{S_{1,loc}}$ and $R^{loc} = R^{S_{1,loc}}$ denote the local deformation rings. We let $R_\infty$ and $R'_\infty$ be formal power series rings in $g$ variables over $R^{loc}$ and $R^{loc}$, respectively. We also have canonical isomorphisms $R_N/\pi \cong R'_N/\pi$ and $R^{loc}/\pi \cong R^{loc}/\pi$. Using [ACC+18, Proposition 6.2.24] and [ACC+18, Proposition 6.2.31], we have local $\mathcal{O}$-algebra surjections $R_\infty \to R_N$ and $R'_\infty \to R'_N$ for $N \geq 0$. We can and do assume that these are compatible with the fixed identifications modulo $\pi$ and with the isomorphisms $R_N \otimes_{\mathcal{O}[\Delta_N]} \mathcal{O} = R_0$ and $R'_N \otimes_{\mathcal{O}[\Delta_N]} \mathcal{O} = R'_0$.

Define $C_0 = R \text{Hom}_\mathcal{O}(R(\text{RG}(X_K, V_1(1)))_m, \mathcal{O})[-d] \in D(\mathcal{O})$ and $T_0 = T^S(C_0)$. Similarly, we define $C'_0 = R \text{Hom}_\mathcal{O}(R(\text{RG}(X_K, V_1(1)))_m, \mathcal{O})[-d]$. For any $N \geq 1$, we let

$$C_N = R \text{Hom}_\mathcal{O}(R(\text{RG}(X_{K_1(Q)}, V_1(1)))_{m_{Q_N}}, \mathcal{O})[-d].$$
and
\[ T_N = T'_Q(C_N). \]

Similarly, we let
\[ C'_N = R \text{Hom}_O(R\Gamma(X_{K_i(Q)}, V_A(\chi^{-1}))_{m_QN}, O)[-d] \]
and
\[ T'_N = T'_Q(C'_N). \]

For any \( N \geq 0 \), there are canonical isomorphisms
\[ C_N \otimes_{O[\Delta_N]}^L k[\Delta_N] \cong C'_N \otimes_{O[\Delta_N]}^L k[\Delta_N] \]
in \( D(k[\Delta_N]) \). These yield the identification
\[ \text{End}_{D(O)}(C_N \otimes_{O}^L k) \cong \text{End}_{D(O)}(C'_N \otimes_{O}^L k). \]

Thus, we can write \( \overline{T}_N \) for the image of both \( T_N \) and \( T'_N \) in the identified endomorphism algebras. By Theorem 7.6, there are canonical isomorphisms \( C_N \otimes_{O[\Delta_N]}^L O \cong C'_N \otimes_{O[\Delta_N]}^L O \cong C'_N \) in \( D(O) \), which are compatible with the reductions modulo \( \sigma \). By Proposition 8.2, we can find an integer \( \delta \geq 1 \) and for each \( N \geq 0 \) ideals \( I_N \) of \( T_N \) and \( I'_N \) of \( T'_N \) of nilpotence degree \( \leq \delta \), such that there exist local \( O[\Delta_N] \)-algebra surjections \( R_N \rightarrow T_N/I_N \) and \( R'_N \rightarrow T'_N/I'_N \). Denoting by \( \overline{T}_N \) and \( \overline{T}'_N \) the images of \( I_N \) and \( I'_N \), respectively, in \( \overline{T}_N \), we get maps \( R_N/\sigma \rightarrow \overline{T}_N/(\overline{T}_N + \overline{T}'_N) \) and \( R'_N/\sigma \rightarrow \overline{T}'_N/(\overline{T}_N + \overline{T}'_N) \) which are compatible with the identification \( R_N/\sigma \cong R'_N/\sigma \). The objects constructed above satisfy the setup described in [ACC*18, Section 6.4.1]. Thus, we can apply the results of [ACC*18, Section 6.4.2] as in the second part of the proof of [ACC*18, Theorem 6.4.4] to conclude that \( H^*(C_0) \) is a nearly faithful \( R_{S_1} \)-module, which implies Theorem 8.1.

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