Bright-dark soliton solutions to the multi-component AB system

Zong-Wei Xu, Guo-Fu Yu’ and Zuo-Nong Zhu
School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, P.R. China

Abstract

In this paper we investigate the multi-component AB system that comes from the geophysical fluid dynamics. We construct its bright-dark soliton solutions through Hirota’s bilinear method. For the two-component AB system, asymptotic behaviours of two-soliton solution are obtained and interactions between two bright and two dark solitons are proved to be elastic. Under different parameter conditions, the oblique interactions, bound states of solitons are analyzed in details. Meanwhile, by use of the Pfaffian technique, we present \( N \)-bright and \( N \)-dark soliton solutions to the two- and multi-component AB system. The results will be meaningful for the study of vector multi-dark solitons in many physical systems such as nonlinear optics and fluid dynamics.

keywords: AB system; bright-dark soliton; Pfaffian; Hirota method

1 Introduction

The AB system

\[
\begin{align*}
(\partial_T + c_1 \partial_X)(\partial_T + c_2 \partial_X) A &= n_1 A - n_2 AB_0, \\
(\partial_T + c_2 \partial_X) B_0 &= (\partial_T + c_1 \partial_X) |A|^2,
\end{align*}
\]

is a completely integrable model for two-layer marginally stable geophysical flows near the neighborhood of an appropriate vertical shear [1, 2]. Here \( A(X, T) \) is a complex valued wave packet and \( B_0(X, T) \) describes the motion of the basic flow induced by the wave packet. Parameters \( c_1, c_2 \) and \( n_1, n_2 \) are arbitrary real constants. Although the AB system was first found to be a geophysical fluid model, it can describe the ultrashort optical pulse propagation in nonlinear optics [3] and the mesoscale gravity current transmission in the problem of cold gravity current [4]. Solutions, including breathers [5, 6, 7], rogue waves [8, 9] and modulation instability [10], bright and dark solitons [11, 12], and integrable aspects [13] to the AB system have drawn widespread attention in recent decades.

Through the dependent variable transformation

\[
x = X - c_1 T, \quad t = T - \frac{X}{c_2}, \quad B_0 = \frac{(c_1 - c_2)^2}{n_2 c_2} B,
\]

the AB system (1)-(2) transforms into the canonical form

\[
\begin{align*}
A_{xt} &= \mu A - AB, \\
B_x &= \sigma(|A|^2)_t,
\end{align*}
\]

with \( \mu = \frac{n_1 c_2}{(c_1 - c_2)^2} \) and \( \sigma = \frac{n_2}{(c_1 - c_2)^2} \). The compatibility condition of the system (4) yields

\[
2\sigma |A|^2 + B^2 - 2\mu B = f(t),
\]

where \( f(t) \) is a function of integration. The parameter \( \mu > 0 \) or \( \mu < 0 \) denotes supercritical and subcritical cases of the shear, respectively, and \( \sigma \) denotes the nonlinearity. When \( \sigma \) is positive, the AB system could be reduced to sine-Gordon equation. Reversely, if \( \sigma \) is negative, the AB system could be transformed into

*Corresponding author. Email address: gfyu@sjtu.edu.cn
The single sinh-Gordon equation. Upon a shift $\mu - B \to B$ and new symbol $\gamma = -\sigma$, the AB system transforms into a simpler form

$$A_{xt} = AB, \quad B_x = \gamma(|A|^2)_t. \quad (6)$$

In ref. [12], the Darboux transformation for (6) with $\gamma = \frac{1}{2}$ via the loop group method is constructed and associated $N$-fold Darboux transformation is found in terms of simple determinants. As a bonus, multi-dark-dark solitons of the AB system (6) with a non-vanishing background are obtained.

Different from Darboux transformation method, dark soliton solutions can also be arrived by Hirota’s bilinear method. When $\sigma > 0$, through the dependent variable transformation

$$A = \frac{g}{f}, \quad B = 2(\ln f)_xt, \quad g \text{ complex}, \ f \text{ real}, \quad (7)$$

we can rewrite Eq.(4) into bilinear form

$$D_xD_t f \cdot g = \mu fg, \quad (8)$$

$$D_x^2 f \cdot f = \sigma |g|^2. \quad (9)$$

where $D$ is the Hirota operator defined as

$$D_x^n D_t^m f \cdot g = \frac{d^n}{d\epsilon^n} \frac{d^m}{d\delta^m} \left(f(x+\epsilon,t+\delta)g(x-\epsilon,t-\delta)\right)|_{\epsilon=0,\delta=0}. \quad (10)$$

Upon the reduction of two-component extended KP hierarchy, the $N$–bright soliton solution in the Gram determinant form for the AB system (4) was found [11].

When $\sigma < 0$, we take the following dependent variable transformation

$$A = \sqrt{-\frac{1}{2\sigma}} \frac{g}{f} e^{i(t-x)}, \quad B = \mu - 1 + 2(\ln f)_xt, \quad (11)$$

and rewrite the AB system (4) into bilinear equations

$$(D_xD_t + iD_t - iD_x)f \cdot g = 0, \quad (12)$$

$$(D_x^2 - \frac{1}{2})f \cdot f = -\frac{1}{2}gg^*, \quad (13)$$

where $^*$ means complex conjugate. Based on the Sato theory and reduction technique, the $N$–dark soliton solution in the Gram determinant form was constructed [11].

It is well known that in the study of the propagation of optical pulses in birefringence fibers or anisotropy effect, the multi-component systems should be considered. Compared with the single-component case, the multi-component ones own richer properties. Owing to the different polarization of each component, multi-component system can display bright-bright, bright-dark and dark-dark solutions.

The $M$-component AB system in the canonical form

$$A_{k,xt} = \mu A_k - A_k B_k, \ k = 1, 2, \cdots, M, \quad (14)$$

$$B_x = \sigma \left(\sum_{k=1}^{M} |A_k|^2\right)_t, \quad (15)$$

associated with Lax pair has been presented in [11]. In the $X - T$ coordinates, the $M$-component AB equation is in the form

$$(\partial_T + c_1 \partial_X) (\partial_T + c_2 \partial_X) A_k = n_1 A_k - n_2 A_k B_0, \ k = 1, 2, \cdots, M \quad (16)$$

$$(\partial_T + c_2 \partial_X) B_0 = (\partial_T + c_1 \partial_X) \sum_{k=1}^{M} |A_k|^2. \quad (17)$$

To the best of our knowledge, the bright-dark vector soliton solutions and their interactions for coupled AB systems have not been put forward. Motivated by the above reason, in this paper, we shall study the
bright-dark vector soliton solutions and their interactions for eqs. (14)-(15) via the Hirota method. The rest of the paper is organized as follows. In Section 2, the bright-dark one-, two- and \( N \)-soliton solutions to the two-component AB system will be constructed based on the bilinear equations. In Section 3, asymptotic analysis and oblique interaction will be performed on the two-soliton solutions. Bound states of solitons and periods will be analyzed in Section 4. \( N \)-bright-dark soliton solutions to the multi-component AB system in the Pfaffian form are derived in Section 5. Conclusion is given in Section 6.

2 Bright-dark soliton solutions to the two-component AB system

Setting \( M = 2 \) in (16)-(17), we have the two-component AB system

\[
(\partial_{T} + c_{1} \partial_{X}) (\partial_{T} + c_{2} \partial_{X}) A_{1} = n_{1} A_{1} - n_{2} A_{1} B_{0}, \tag{18}
\]
\[
(\partial_{T} + c_{1} \partial_{X}) (\partial_{T} + c_{2} \partial_{X}) A_{2} = n_{1} A_{2} - n_{2} A_{2} B_{0}, \tag{19}
\]
\[
(\partial_{T} + c_{2} \partial_{X}) B_{0} = (\partial_{T} + c_{1} \partial_{X}) \left( |A_{1}|^{2} + |A_{2}|^{2} \right), \tag{20}
\]

and its canonical form

\[
A_{1,xt} = \mu A_{1} - A_{1} B, \tag{21}
\]
\[
A_{2,xt} = \mu A_{2} - A_{2} B, \tag{22}
\]
\[
B_{x} = \sigma \left( |A_{1}|^{2} + |A_{2}|^{2} \right)^{t}, \tag{23}
\]

where two sets of coordinates \( x, t \) and \( X, T \) are related by (3).

The Hirota bilinear method is an effective way to derive multi-soliton solutions to nonlinear evolution equations. We first implement the dependent variable transformation

\[
g_{1} = e^{\eta_{1}}, \tag{28}
\]
\[
g_{2} = re^{i\phi} \left( 1 + a_{1,1} \cdot e^{\eta_{1} + \eta_{1}^{*}} \right), \tag{29}
\]
\[
f = 1 + a_{1,1} \cdot e^{\eta_{1} + \eta_{1}^{*}}, \tag{30}
\]

where

\[
\eta_{1} = k_{1} x + \omega_{1} t + \eta_{1,0} \in \mathbb{C}, \tag{31}
\]
\[
\phi = p x - q t + \phi_{0} \in \mathbb{R}. \tag{32}
\]

Substitution expressions (28)-(30) into (25)-(27) leads to

\[
a_{1,1} = \frac{1}{(k_{1} + k_{1}^{*})^{2} \left( \frac{2}{2^{\frac{m}{2}}} \right)} \cdot \frac{1}{\mu + \lambda k_{1}}, \tag{33}
\]
\[
b_{1,1} = \frac{(k_{1} - ip)(k_{1}^{*} - ip)}{(k_{1} + ip)(k_{1}^{*} + ip)}, \tag{34}
\]

where \( \mu, \lambda, m, p, q \) are real parameters.
Setting $\theta_1 = \ln \frac{k_1 - ip}{k_1 + ip} = \ln |\frac{k_1 - ip}{k_1 + ip}| + i \arg \frac{k_1 - ip}{k_1 + ip}$, we can express $b_{1,1^*}$ in compact form $b_{1,1^*} = \exp(\theta_1 - \theta_1^*)$.

Thus from the dependent variable transformation (24), we have one-soliton solution

$$A_1 = \frac{e^{i\eta_1}}{2\sqrt{n_{1,1^*}}} \text{sech}\left(\eta_{1R} + \frac{\chi_{1,1^*}}{2}\right),$$

$$A_2 = re^{i(\varphi + \theta_2i)} \left[\cos \theta_2i + i \sin \theta_2i \tanh\left(\eta_{1R} + \frac{\chi_{1,1^*}}{2}\right)\right],$$

$$B_0 = \frac{2(\mu + \lambda)k_{1R}^2}{\sigma c_2 |k_1|^2} \text{sech}^2\left(\eta_{1R} + \frac{\chi_{1,1^*}}{2}\right) - \frac{\lambda}{\sigma c_2},$$

with $\exp(\chi_{1,1^*}) = a_{1,1^*}$ and $\eta_{1R} = \text{Re}(\eta_1)$.

It’s easy to see that $A_1$ and $B_0$ are bright soliton, $A_2$ is dark soliton. To get non-singular soliton solutions, conditions $\frac{2}{\sigma} + \frac{4p^2r^2}{|p^2 + k_1|^2} > 0$ and $k_{1R} \equiv \text{Re}(k_1) \neq 0$ must be satisfied to avoid vanishing of the denominator.

The travelling velocity of one-soliton is $\frac{\lambda + \mu}{\sigma c_2}$. The amplitude of $A_1$ is $|k_{1R}| \sqrt{\frac{2}{\sigma} + \frac{4p^2r^2}{|p^2 + k_1|^2}}$. The height of the trough of $A_2$ is $r \sqrt{1 - \frac{4p^2r^2}{|p^2 + k_1|^2}}$. The amplitude of $B_0$ is max $B_0 + \frac{\lambda}{\sigma c_2} = \frac{2|\mu + \lambda|k_{1R}^2}{|\sigma c_2 |k_1|^2}$. Fig.1 shows the evolution of one-bright-dark soliton.

### 2.2 Two-soliton solution

To derive two-bright-dark soliton solution, we suppose tau functions $g$ and $f$ have the following expression

$$g_1 = e^{n_1} + e^{n_2} + a_{1,1^*}e^{n_1 + n_2 + n_1^*} + a_{1,2,2^*}e^{n_1 + n_2 + n_1^*},$$

$$g_2 = re^{i\varphi} (1 + a_{1,1^*}e^{n_1 + n_1^* + \theta_1 - \theta_1^*} + a_{1,2}e^{n_1 + n_2 + \theta_1 - \theta_1^*} + a_{2,1^*}e^{n_2 + n_1^* + \theta_2 - \theta_2^*} + a_{2,2^*}e^{n_2 + n_2 + \theta_2 - \theta_2^*} + a_{1,2,1^*}e^{n_1 + n_2 + n_1^* + \theta_1 - \theta_1^* + \theta_2 - \theta_2^*} + a_{1,2,2^*}e^{n_1 + n_2 + n_2^* + \theta_1 - \theta_1^* + \theta_2 - \theta_2^*} + a_{2,1^*}e^{n_2 + n_1^* + n_2^* + \theta_2 - \theta_2^*} + a_{2,2^*}e^{n_2 + n_2^* + \theta_2 - \theta_2^*}),$$

$$f = 1 + a_{1,1^*}e^{n_1 + n_1^*} + a_{1,2}e^{n_1 + n_2} + a_{2,1^*}e^{n_2 + n_1^*} + a_{2,2^*}e^{n_2 + n_2^*} + a_{1,2,1^*}e^{n_1 + n_1^* + n_2 + n_2^*} + a_{1,2,2^*}e^{n_1 + n_1^* + n_2 + n_2^*} + a_{2,1^*}e^{n_2 + n_1^* + n_2 + n_2^*} + a_{2,2^*}e^{n_2 + n_2^* + n_2 + n_2^*}. $$

Figure 1: The evolution of one soliton solution to the focusing AB system via $r = 1, p = 1, k_1 = 1 + i, \lambda = 0, \varphi_0 = 0, \eta_{1,0} = 0, c_1 = \frac{1}{100}, c_2 = 10, n_1 = 10$ and $n_2 = 100$. 
where
\[ \eta_j = k_j x + \frac{\mu + \lambda}{k_j} t + \eta_{j,0}, \quad \varphi = p x - \frac{\mu + \lambda}{p} t + \varphi_0, \quad (41) \]
\[ a_{j,t} = (k_j + k_t^2)^{-2} \left( \frac{2}{\sigma} + \frac{4p^2r^2}{(p^2 + k_j^2)(p^2 + k_t^2)} \right)^{-1}, \quad (42) \]
\[ a_{j,t} = (k_j - k_t)^2 \left( \frac{2}{\sigma} + \frac{4p^2r^2}{(p^2 + k_j^2)(p^2 + k_t^2)} \right); \quad (43) \]
\[ a_{j,t} = (k_j^* - k_t^*)^2 \left( \frac{2}{\sigma} + \frac{4p^2r^2}{(p^2 + k_j^*^2)(p^2 + k_t^*^2)} \right); \quad (44) \]
\[ a_{j_1,j_2,\ldots,j_m,t_1^*,\ldots,t_m} = \prod_{1 \leq m_1 < m_2 \leq n} \prod_{1 \leq m_1 < m_2 \leq m} a_{j_1m_1,j_m} \prod_{1 \leq m_1 \leq n, 1 \leq m_2 \leq m} a_{j_{m_1},t_{m_2}}, \quad (45) \]
\[ \theta_j = \ln \frac{k_j - ip}{k_j + ip} = \ln \left| \frac{k_j - ip}{k_j + ip} \right| + i \arg \left( \frac{k_j - ip}{k_j + ip} \right). \quad (46) \]

Here we require the condition \( \frac{2}{\sigma} + \frac{4p^2r^2}{|p^2 + k_j^2|^2} > 0 \), \( k_1 + k_2^* \neq 0 \) and \( k_jR \neq 0 \) for \( j = 1, 2 \) such that the obtained two-soliton solution is regular. We will show later that the two-soliton solutions exhibit the elastic collision and soliton bound states.

### 2.3 \( N \)-soliton solution

In this subsection, we express the \( N \)-bright-dark soliton solution to the two-component AB system \((13)-(20)\) by Pfaffian technique. A Pfaffian of order \( n \) is the square root of a determinant of order \( 2n \). The definition and properties of Pfaffian could be found in [14].

**Theorem 1** The \( N \)-bright-dark soliton solution to the two-component AB system \((13)-(20)\) is expressed by the Pfaffians

\[ f = (a_1, \ldots, a_{2N}, b_{2N}, \ldots, b_1) = (\bullet), \quad (47) \]
\[ g_1 = (d_0, \beta, a_1, \ldots, a_{2N}, b_{2N}, \ldots, b_1) = (d_0, \beta, \bullet), \quad (48) \]
\[ g_2 = re^{i\varphi} (d_0, \gamma, a_1, \ldots, a_{2N}, b_{2N}, \ldots, b_1) = re^{i\varphi} (d_0, \gamma, \bullet), \quad (49) \]

where we take the notation \( (\bullet) \) to represent \( (a_1, \ldots, a_{2N}, b_{2N}, \ldots, b_1) \) for simplicity. The Pfaffian entries are defined as

\[ (a_j, a_i) = \frac{w_i - w_j}{w_i + w_j} e^{\eta_i+\eta_j}, \quad (b_j, \beta) = \frac{1 + \mu_j}{2}, \quad (a_j, \gamma) = \frac{2ip}{w_j + ip} e^{\eta_j}, \quad (d_i, \beta_k) = 0, \]
\[ (b_j, b_l) = \frac{1 - \mu_j \mu_l}{2 (w_j^2 - w_l^2)} \left( \frac{2}{\sigma} + \frac{4p^2r^2}{(p^2 + w_j^2)(p^2 + w_l^2)} \right), \quad (a_j, \beta) = 0, \quad (d_i, a_j) = w_i^j e^{\eta_j}, \quad (d_i, d_j) = 0, \]
\[ (a_j, b_l) = \delta_{jl}, \quad (b_j, \gamma) = 0, \quad (d_i, b_j) = 0, \quad (d_i, \gamma) = \delta_{0j}, \quad \]
and

\[ \delta_{jl} = \begin{cases} 0 & i \neq j, \\ 1 & i = j, \end{cases} \quad \mu_j = \begin{cases} 1 & 1 \leq j \leq N \\ -1 & N + 1 \leq j \leq 2N \end{cases}, \quad w_j = \begin{cases} k_j & 1 \leq j \leq N \\ k_j^* & N + 1 \leq j \leq 2N \end{cases}. \]
Upon the properties of Pfaffian, we can express tau function and its derivatives as following

\[ f_x = (d_1, d_0, \bullet), \quad f_t = (\lambda + \mu) (d_0, d_{-1}, \bullet), \]
\[ f_{xx} = (d_2, d_0, \bullet), \quad f_{xt} = (\lambda + \mu) (d_1, d_{-1}, \bullet), \]
\[ g_{1,x} = (d_1, \beta, \bullet), \quad g_{1,t} = (\lambda + \mu) (d_{-1}, \bullet), \]
\[ g_{1,xt} = (\lambda + \mu) [(d_0, \beta, \bullet) + (d_1, d_{-1}, \beta, \bullet)], \]
\[ \partial_x (d_0, \gamma, \bullet) = ip (\bullet) - ip (d_0, \gamma, \bullet) + (d_1, d_0, \bullet) + (d_1, \gamma, \bullet), \]
\[ \partial_t (d_0, \gamma, \bullet) = (\lambda + \mu) \left[ \frac{1}{ip} (\bullet) - \frac{1}{ip} (d_0, \gamma, \bullet) - (d_0, d_{-1}, \bullet) + (d_{-1}, \gamma, \bullet) \right], \]
\[ \partial_x \partial_t (d_0, \gamma, \bullet) = (\lambda + \mu) \left[ 2 (d_0, \gamma, \bullet) - 2 (\bullet) + 2ip (d_0, d_{-1}, \bullet) \right. \]
\[ - \left. ip (d_{-1}, \gamma, \bullet) - \frac{1}{ip} (d_1, \gamma, \bullet) + (d_1, d_0, d_{-1}, \gamma, \bullet) \right]. \]

One can check that the substitution of above expressions into eq. (25) leads to the Pfaffian identity,

\[ (d_1, d_0, d_{-1}, \beta, \bullet) (\bullet) - (d_1, \beta, \bullet) (d_0, d_{-1}, \bullet) - (d_{-1}, \beta, \bullet) (d_1, d_0, \bullet) + (d_0, \beta, \bullet) (d_1, d_{-1}, \bullet) = 0, \quad (50) \]

and eq. (25) becomes

\[ (d_1, d_0, d_{-1}, \gamma, \bullet) (\bullet) - (d_1, \gamma, \bullet) (d_0, d_{-1}, \bullet) - (d_{-1}, \gamma, \bullet) (d_1, d_0, \bullet) + (d_0, \gamma, \bullet) (d_1, d_{-1}, \bullet) = 0, \quad (51) \]

see [15] for reference. To prove Pfaffians [14] - [19], we solve eq. (24), we introduce new Pfaffian entries

\[ (b_j, \beta^* ) = \frac{1 - \mu_j}{2}, \quad (a_j, \beta^* ) = 0, \quad (d_0, \beta^* ) = 0, \]
\[ (a_j, \gamma^* ) = - \frac{2ip}{w_j - ip}, \quad (b_j, \gamma^* ) = 0, \quad (d_0, \gamma^* ) = 1, \]

and express the conjugate of \( g_1 \) and \( g_2 \) as

\[ g_1^* = (d_0, \beta^*, \bullet), \quad g_2^* = re^{-i\varphi} (d_0, \gamma^*, \bullet). \]

The proof can be found in the Appendix.

3 Collision of solitons to the two-component AB system

For two-soliton solution [38] - [40] with parameters \(|k_1| \neq |k_2|\), the two solitons travel with different velocities that leads to the collision between solitons. To prove the elastic collision and no energy change after collision, we check the asymptotic behavior of the two-soliton solutions. For two-soliton solution

\[ A_1 = \frac{g_1}{f}, \quad A_2 = \frac{g_2}{f}, \quad B_0 = 2(\ln f)_{xt} - \lambda, \]

with \( g_1, g_2 \) and \( f \) given by [38] - [40], we have the following asymptotic forms when \( \frac{\lambda + \mu}{|k_1|^2} < \frac{\lambda + \mu}{|k_2|^2} \).
For two-soliton solution (38)-(40) with parameters 
\[ A_{0} \sim \begin{cases} \frac{a_{1}^{2}a_{1}^{2}e^{\eta_{1}i} \text{sech} \left( \eta_{1}R + \frac{\chi_{1,2}^{1} - \chi_{2}^{1}}{2} \right)}{2|a_{1}|^{2}a_{1}^{2}e^{\eta_{1}i} \text{sech} \left( \eta_{1}R + \frac{\chi_{1}^{1}}{2} \right)} & \eta_{1R} \sim O(1), t \to +\infty \\ \frac{a_{1}^{2}a_{1}^{2}e^{\eta_{1}i} \text{sech} \left( \eta_{2}R + \frac{\chi_{1,2}^{1} - \chi_{2}^{1}}{2} \right)}{2|a_{1}|^{2}a_{1}^{2}e^{\eta_{1}i} \text{sech} \left( \eta_{2}R + \frac{\chi_{2}^{1}}{2} \right)} & \eta_{2R} \sim O(1), t \to -\infty \end{cases} \]

\[ A_{2} \sim \begin{cases} e^{i(\mp 2\theta_{1}^{1} + \theta_{1}^{1})} & \eta_{1R} \sim O(1), t \to +\infty \\ e^{i(\mp 2\theta_{1}^{1} + \theta_{1}^{1})} & \eta_{1R} \sim O(1), t \to -\infty \end{cases} \]

\[ B_{0} \sim \begin{cases} \frac{2(\mu+\lambda)k_{1}^{2} \text{sech}^{2} \left( \eta_{1}R + \frac{\chi_{1,2}^{1} - \chi_{2}^{1}}{2} \right) - \frac{\lambda}{\sigma_{c}^{2}}}{\sigma_{c}^{2}|k_{1}|^{2}} & \eta_{1R} \sim O(1), t \to +\infty \\ \frac{2(\mu+\lambda)k_{2}^{2} \text{sech}^{2} \left( \eta_{1}R + \frac{\chi_{1,2}^{1} - \chi_{2}^{1}}{2} \right) - \frac{\lambda}{\sigma_{c}^{2}}}{\sigma_{c}^{2}|k_{2}|^{2}} & \eta_{1R} \sim O(1), t \to -\infty \end{cases} \]

with \( \exp(\chi_{i,j}^{1}) = a_{i,j}^{1} \), \( \exp(\chi_{1,2,1,2}^{1}) = a_{1,2,1,2}^{1} \). One can check that the collision is completely elastic for \( A_{1}, A_{2} \) and \( B_{0} \). Both the velocity and the amplitude keep invariant after collision. Fig.2 shows the collision of two-bright-dark-soliton solution in the focusing case (\( \sigma > 0 \)). Fig.3 depicts the collision in the defocusing case (\( \sigma < 0 \)).

4 Soliton bound states to the two-component AB system

For two-soliton solution (38)-(40) with parameters \( |k_{1}| = |k_{2}| \), the two solitons have the same velocity and exhibit periodic phenomenon. We find that the solution has the following periodic property

\[ f(x,t) = f \left( x - \frac{\pi}{k_{2}t - k_{1}t}, t + \frac{\pi |k_{1}|^{2}}{(\lambda + \mu)(k_{2}t - k_{1}t)} \right), \]

\[ |g_{1}(x,t)| = g_{1} \left( x - \frac{\pi}{k_{2}t - k_{1}t}, t + \frac{\pi |k_{1}|^{2}}{(\lambda + \mu)(k_{2}t - k_{1}t)} \right), \]

\[ |g_{2}(x,t)| = g_{2} \left( x - \frac{\pi}{k_{2}t - k_{1}t}, t + \frac{\pi |k_{1}|^{2}}{(\lambda + \mu)(k_{2}t - k_{1}t)} \right). \]
Figure 3: The elastic collision of two soliton solution to the defocusing AB system via \( r = 1, k_1 = \frac{1}{2} + \frac{i}{2}, k_2 = \frac{3}{2} + \frac{i}{2}, p = 1, \lambda = 0, \varphi_0 = 0, \eta_{1,0} = 0, \eta_{2,0} = 0, c_1 = \frac{1}{100}, c_2 = 10, n_1 = -10 \) and \( n_2 = -1000 \).

Figure 4: Bound states of solitons to the focusing AB system via \( r = 1, k_1 = \frac{3}{2} \exp(\frac{i}{4}), k_2 = \frac{3}{2} \exp(-\frac{i}{4}), p = 1, \lambda = 0, \varphi_0 = 0, \eta_{1,0} = \frac{3}{4}, \eta_{2,0} = -\frac{1}{4}, c_1 = \frac{1}{100}, c_2 = 10, n_1 = 10 \) and \( n_2 = 100 \).

Thus we have

\[
B_0 (X, T) = B_0 \left( X + c_2 \pi \frac{c_1 |k_1|^2 - (\lambda + \mu)}{(\lambda + \mu) (c_2 - c_1) (k_{2f} - k_{1f})} T + \pi \frac{c_2 |k_1|^2 - (\lambda + \mu)}{(\lambda + \mu) (c_2 - c_1) (k_{2f} - k_{1f})} \right)
\]

\[
|A_1 (X, T)| = A_1 \left( X + c_2 \pi \frac{c_1 |k_1|^2 - (\lambda + \mu)}{(\lambda + \mu) (c_2 - c_1) (k_{2f} - k_{1f})} T + \pi \frac{c_2 |k_1|^2 - (\lambda + \mu)}{(\lambda + \mu) (c_2 - c_1) (k_{2f} - k_{1f})} \right)
\]

\[
|A_2 (X, T)| = A_2 \left( X + c_2 \pi \frac{c_1 |k_1|^2 - (\lambda + \mu)}{(\lambda + \mu) (c_2 - c_1) (k_{2f} - k_{1f})} T + \pi \frac{c_2 |k_1|^2 - (\lambda + \mu)}{(\lambda + \mu) (c_2 - c_1) (k_{2f} - k_{1f})} \right)
\]

Fig. 4 displays the bound states of two-bright-dark soliton to the focusing AB system. Fig. 5 shows the bound states of two-bright-dark solitons to the defocusing AB system. When periods are small enough, bound states of solitons appear parallel.

5 Bright-dark \( N \)-soliton solution to the \( M \)-component AB system

In this section, we give the bright-dark \( N \)-soliton solution to the \( M \)-component AB system (16)-(17). We first give the bilinear form to the system. Through the dependent variable transformation

\[
A_j = \frac{g_j}{f}, \quad B = 2 (\ln f)_{xt} - \lambda, \quad j = 1, \cdots, M,
\]

\[
(55)
\]
The proof is presented in the Appendix.

Figure 5: Bound states of solitons to the defocusing AB system via \( r = 1, k_1 = \exp \left( -\frac{2i}{k} \right), k_2 = \frac{i}{2} \exp \left( -\frac{2i}{k} \right), p = 1, \lambda = 0, \varphi_0 = 0, \eta_{1,0} = 0, \eta_{2,0} = 0, c_1 = \frac{1}{100}, c_2 = 10, n_1 = -10 \) and \( n_2 = -1000. \)

the \( M \)-component AB system \((14)-(15)\) is transformed into bilinear form

\[
D_x D_y g_j \cdot f = (\mu + \lambda) g_j f, \quad j = 1, \ldots, M
\]

\[
(D_x = r^2 \sigma) f \cdot f = \sigma \sum_{j=1}^{M} |g_j|^2.
\]

We construct \( N \)-bright-dark-soliton solution with the first \( M_1 \) \( (1 \leq M_1 \leq M) \) components \( g_1, \ldots, g_{M_1} \) bright and the left \( g_{M_1+1}, \ldots, g_M \) dark.

**Theorem 2** The Pfaffians

\[
f = (a_1, \ldots, a_{2N}, b_{2N}, \ldots, b_1) = (\bullet),
\]

\[
g_j = (d_0, \beta_j, a_1, \ldots, a_{2N}, b_{2N}, \ldots, b_1) = (d_0, \beta_j, \bullet), \quad j = 1, \ldots, M_1,
\]

\[
g_j = e^{i\varphi_j} (d_0, \gamma_j, a_1, \ldots, a_{2N}, b_{2N}, \ldots, b_1) = e^{i\varphi_j} (d_0, \gamma_j, \bullet), \quad j = M_1 + 1, \ldots, M,
\]

solve the bilinear Eqs. \((56)\) and \((57)\) provided that the elements of the Pfaffians are defined by

\[
(a_j, a_l) = \frac{w_l - w_j}{w_l + w_j} e^{\eta_l + n_l}, \quad (a_j, b_l) = \delta_{jl}, \quad (d_l, d_j) = 0,
\]

\[
(b_j, b_l) = \frac{(1 - \mu_j \mu_l)}{2 (w_j^2 - w_l^2)} \left( \frac{2}{\sigma} + \sum_{n=M_1+1}^{M} \frac{4\mu^2 \sigma^2}{(p_n^2 + w_j^2)(p_n^2 + w_l^2)} \right), \quad (a_j, b_l) = 0, \quad (d_l, a_j) = w_j e^{b_j},
\]

\[
(b_j, \beta_l) = \frac{1 + \mu_j}{2} a_l, \quad (b_j, \gamma_l) = 0, \quad (d_l, b_j) = 0,
\]

\[
(a_j, \gamma_l) = \frac{2i \nu_j r_j}{w_j + i \nu_l}, \quad (d_l, \beta_k) = 0, \quad (d_l, \gamma_l) = \delta_{0l} r_l,
\]

where

\[
\eta_j = k_j x + \frac{\mu + \lambda}{k_j} t + \eta_{j,0}, \quad \varphi_j = p_j x - \frac{\mu + \lambda}{p_j} t + \varphi_{j,0},
\]

and

\[
r^2 = \sum_{j=M_1+1}^{M} r_j^2, \quad \mu_j = \begin{cases} 1 & 1 \leq j \leq N, \\ -1 & 1 \leq j \leq N \end{cases}, \quad w_j = \begin{cases} k_j & 1 \leq j \leq N, \\ k_j & N + 1 \leq j \leq 2N. \end{cases}
\]

The proof is presented in the Appendix.
6 Conclusions

In this paper, based on Hirota method, we have investigated the two- and multi-component AB system that describe the propagation of geophysical fluids. For the two-component system, we found the bright-dark one- and two-soliton solutions explicitly. Asymptotic analysis has been conducted on two-bright-dark-soliton solution to investigate the interactions between the two solitons. We conclude that interactions between two bright or two dark solitons are both elastic. We should remark here that the interactions between two solitons for the multi-component integrable systems are non-elastic in some cases. We also illustrated oblique interactions (Fig.2 and 3), bound states of solitons (Fig.4 and 5) analytically and graphically. The periods for bound states of solitons are given explicitly. Furthermore, we give the N-bright-dark-soliton solutions to two- and M-component AB system in the form of Pfaffian.

7 Appendix

Proof of Theorem 1

Proof 1 Because $(\bullet) = (a_1, \cdots, a_{2N}, b_{2N}, \cdots, b_1)$ contains the index $a_j$, $(a_j, d_0, \bullet)$ is trivially zero. We have

$$(d_k, a_j)(\bullet) = (d_k, a_j)(\bullet) + (a_j, d_k, \bullet)$$

$$= \sum_{l=1}^{2N} (-1)^l (a_j, a_l)(d_k, \hat{a}_l) + (-1)^{l+1} (a_j, b_l)(d_k, b_l). \quad (61)$$

Then

$$g_2 g_2^2 - r^2 f^2$$

$$= r^2 (d_0, \gamma, \bullet)(d_0, \gamma^*, \bullet) - r^2 (\bullet)^2$$

$$= r^2 \sum_{j=1}^{2N} (-1)^{j+1} \left[ (\gamma, a_j) + (\gamma^*, a_j) \right] (d_0, \hat{a}_j)(\bullet) + r^2 \sum_{j,l=1}^{2N} (-1)^{j+l} (\gamma, a_j)(\gamma^*, a_l)(d_0, \hat{a}_j)(d_0, \hat{a}_l)$$

$$= r^2 \sum_{j=1}^{2N} (-1)^j \frac{4p^2}{p^2 + w_j^2} (d_0, a_j)(d_0, \hat{a}_j)(\bullet) + r^2 \sum_{j,l=1}^{2N} (-1)^{j+l} \frac{4p^2}{(w_j + ip)(w_l - ip)} e^{\eta_j + \eta_l} (d_0, \hat{a}_j)(d_0, \hat{a}_l)$$

$$= r^2 \sum_{j=1}^{2N} (-1)^j \frac{4p^2}{p^2 + w_j^2} (-1)^{j+1} (d_0, \hat{b}_j)(d_0, \hat{a}_j) + r^2 \sum_{j=1}^{2N} \left[ (-1)^j \frac{4p^2}{p^2 + w_j^2} (d_0, \hat{a}_j) \sum_{l=1}^{2N} (-1)^j (a_j, a_l)(d_0, \hat{a}_l) \right]$$

$$+ r^2 \sum_{j,l=1}^{2N} (-1)^{j+l} \frac{4p^2}{(w_j + ip)(w_l - ip)} e^{\eta_j + \eta_l} (d_0, \hat{a}_j)(d_0, \hat{a}_l)$$

$$= r^2 \sum_{j=1}^{2N} - \frac{4p^2}{p^2 + w_j^2} (d_0, \hat{a}_j)(d_0, \hat{b}_j)$$

$$+ 2p^2 r^2 \sum_{j,l=1}^{2N} (-1)^{j+l} \left[ \frac{(a_j, a_l)}{p^2 + w_j^2} + \frac{(a_l, a_j)}{p^2 + w_l^2} + \frac{e^{\eta_j + \eta_l}}{(w_j + ip)(w_l - ip)} + \frac{e^{\eta_j + \eta_l}}{(w_l + ip)(w_j - ip)} \right] (d_0, \hat{a}_j)(d_0, \hat{a}_l)$$

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\[
- r^2 \sum_{j=1}^{2N} \frac{4p^2}{p^2 + w_j^2} (d_0, \hat{a}_j) (d_0, \hat{b}_j) + 2p^2 r^2 \sum_{j,l=1}^{2N} (-1)^{j+l} \left( \frac{1}{p^2 + w_j} + \frac{1}{p^2 + w_l} \right) (d_0, a_j) (d_0, a_l) (d_0, \hat{a}_j) (d_0, \hat{a}_l) \\
= - 4p^2 r^2 \sum_{j=1}^{2N} \frac{1}{p^2 + w_j^2} (d_0, \hat{a}_j) (d_0, \hat{b}_j),
\]

and

\[
g_1 g_1^* = (d_0, \beta, \bullet) (d_0, \beta^*, \bullet) \\
= \sum_{j,l=1}^{2N} (-1)^{j+l} \frac{1 + \mu_j}{2} \left( \frac{1}{p^2 + w_j} - \frac{1}{p^2 + w_l} \right) (d_0, \hat{b}_j) (d_0, \hat{b}_l) \\
= \sum_{j,l=1}^{2N} (-1)^{j+l} \frac{1 - \mu_j \mu_l}{4} (d_0, \hat{b}_j) (d_0, \hat{b}_l) \\
= \frac{1}{2} \sum_{j,l=1}^{2N} (-1)^{j+l} \left( w_j^2 - w_l^2 \right) \left( \frac{2}{\sigma} + \frac{4p^2 r^2}{(p^2 + w_j^2) (p^2 + w_l^2)} \right) (b_j, b_l) \left( \frac{1}{p^2 + w_j^2} - \frac{1}{p^2 + w_l^2} \right) (d_0, \hat{b}_j) (d_0, \hat{b}_l) \\
= \frac{2}{\sigma} \sum_{j=1}^{2N} (-1)^{j+1} w_j^2 \left( d_0, \hat{b}_j \right) \left( b_j, d_0, \bullet \right) - (-1)^j \left( b_j, a_j \right) \left( d_0, \hat{a}_j \right) \right] \\
+ 4p^2 r^2 \sum_{j=1}^{2N} (-1)^{j+l} \left( \frac{1}{p^2 + w_j^2} - \frac{1}{p^2 + w_l^2} \right) (b_j, b_l) \left( d_0, \hat{b}_j \right) (d_0, \hat{b}_l) \\
= - \frac{2}{\sigma} \sum_{j=1}^{2N} w_j^2 (d_0, \hat{a}_j) (d_0, \hat{b}_j) + 4p^2 r^2 \sum_{j=1}^{2N} \frac{1}{p^2 + w_j^2} (d_0, \hat{a}_j) (d_0, \hat{b}_j),
\]

and

\[
- (f_x)^2 = (d_2, d_0, \bullet) (d_0, d_0, \bullet) - (d_1, d_0, \bullet)^2 \\
= \sum_{j,l=1}^{2N} (-1)^{j+l} [(d_2, a_j) (d_0, a_l) - (d_1, a_j) (d_1, a_l)] (d_0, \hat{a}_j) (d_0, \hat{a}_l) \\
= \sum_{j,l=1}^{2N} (-1)^{j+l} \left( \frac{w_j^2 + w_l^2}{2} - w_j w_l \right) e^{\eta_j + \eta_l} (d_0, \hat{a}_j) (d_0, \hat{a}_l) \\
= \sum_{j,l=1}^{2N} (-1)^{j+l} \frac{w_j^2 - w_l^2}{2} (a_j, a_l) (d_0, \hat{a}_j) (d_0, \hat{a}_l) \\
= \sum_{j=1}^{2N} \left[ (-1)^j \left( -w_j^2 \right) (d_0, \hat{a}_j) \sum_{l=1}^{2N} (-1)^l (a_j, a_l) (d_0, \hat{a}_l) \right]
\]
\[
\begin{align*}
&= - \sum_{j=1}^{2N} (-1)^j \left[ w_j^2 (d_0, a_j) \right] (d_0, \dot{a}_j) (\bullet) - \sum_{j=1}^{2N} w_j^2 (a_j, b_j) (d_0, \dot{b}_j) \\
&= -(d_2, d_0, \bullet) (\bullet) - \sum_{j=1}^{2N} w_j^2 (d_0, \dot{a}_j) \left( d_0, \dot{b}_j \right) \\
&= - f_{xx} f + \frac{\sigma}{2} \left( g_1 g_1' + g_2 g_2' - r^2 f^2 \right).
\end{align*}
\]

Substitution above expressions into eq. \((\ref{eq:pfaffian})\) leads to all terms cancelled. Thus eq. \((\ref{eq:pfaffian})\) holds and the proof is completed.

**Proof of Theorem 2**

**Proof 2** Upon properties of Pfaffian, we can derive that

\[
\begin{align*}
f_x &= (d_1, d_0, \bullet), \quad f_\ell = (\lambda + \mu) (d_0, d_{-1}, \bullet), \\
f_{xx} &= (d_2, d_0, \bullet), \quad f_{x\ell} = (\lambda + \mu) (d_1, d_{-1}, \bullet), \\
g_{j,x} &= (d_1, \beta_j, \bullet), \quad g_{j,\ell} = (\lambda + \mu) (d_{-1}, \beta_j, \bullet), \\
g_{j,x\ell} &= (\lambda + \mu) [(d_0, \beta_j, \bullet) + (d_1, d_0, d_{-1}, \beta_j, \bullet)], \\
\partial_x (d_0, \gamma_j, \bullet) &= p_j (d_0, \gamma_j, \bullet) - ip_j (d_0, \beta_j, \bullet) + r_j (d_1, d_0, \bullet) - (d_1, \gamma_j, \bullet), \\
\partial_\ell (d_0, \gamma_j, \bullet) &= (\lambda + \mu) \left[ \frac{r_j}{ip_j} (\bullet) - \frac{1}{ip_j} (d_0, \gamma_j, \bullet) - r_j (d_0, d_{-1}, \bullet) + (d_{-1}, \gamma_j, \bullet) \right], \\
\partial_x \partial_\ell (d_0, \gamma_j, \bullet) &= (\lambda + \mu) \left[ 2(d_0, \gamma_j, \bullet) - 2r_j (\bullet) + 2ip_j r_j (d_0, d_{-1}, \bullet) \\
&- ip_j (d_{-1}, \gamma_j, \bullet) - \frac{1}{ip_j} (d_1, \gamma_j, \bullet) + (d_1, d_0, d_{-1}, \gamma_j, \bullet) \right].
\end{align*}
\]

Substituting above expression into eq. \((\ref{eq:pfaffian})\) directly, we arrive at the Pfaffian identity. Thus the Pfaffian expression solve eq. \((\ref{eq:pfaffian})\). To prove eq. \((\ref{eq:pfaffian})\), we introduce additional Pfaffian entries

\[
\begin{align*}
(b_j, \beta_j^*) &= \frac{1 - \mu_j}{2} a_j^{(n)}, \quad (a_j, \beta_j^*) = 0, \quad (d_0, \beta^*) = 0, \\
(a_j, \gamma_j^*) &= \frac{2ip_j r_j}{w_j - ip} \cdot (b_j, \gamma_j^*) = 0, \quad (d_0, \gamma^*) = 1,
\end{align*}
\]

such that we can present the Pfaffian form for \(g_j^*\) as

\[
\begin{align*}
g_j^* &= (d_0, \beta_j, a_1, \cdots, a_{2N}, b_{2N}, \cdots, b_1) = (d_0, \beta_j, \bullet), \quad j = 1, \cdots, M_1, \\
g_j^* &= e^{-ip_j} (d_0, \gamma_j, a_1, \cdots, a_{2N}, b_{2N}, \cdots, b_1) = e^{-ip_j} (d_0, \gamma_j, \bullet), \quad j = M_1 + 1, \cdots, M.
\end{align*}
\]

Then we have

\[
\begin{align*}
&\sum_{j=M_1+1}^{M} (d_0, \gamma_j, \bullet) (d_0, \gamma_j^*, \bullet) = r^2 (\bullet)^2 \\
&= \sum_{n=M_1+1}^{M} r_n \left[ \sum_{j=1}^{2N} (-1)^{j+1} \left[ (\gamma_n, a_j) + (\gamma_n^*, a_j) \right] (d_0, \dot{a}_j) (\bullet) \right] + \sum_{n=M_1+1}^{M} \sum_{j,l=1}^{2N} (-1)^{j+l} (\gamma_n, a_j) (\gamma_n^*, a_l) (d_0, \dot{a}_j) (d_0, \dot{a}_l) \\
&= \sum_{n=M_1+1}^{M} r_n^2 \sum_{j=1}^{2N} (-1)^{j} \frac{4p_n^2}{p_n^2 + w_j^2} (d_0, a_j) (d_0, \dot{a}_j) (\bullet) + \sum_{n=M_1+1}^{M} \sum_{j,l=1}^{2N} (-1)^{j+l} \frac{4p_n^2}{(w_j + ip_n)(w_l - ip_n)} e^{\eta + \theta} (d_0, \dot{a}_j) (d_0, \dot{a}_l)
\end{align*}
\]

\[\text{(64)}\]
\[
= \sum_{n=M_1+1}^{M} r_n^2 \sum_{j=1}^{2N} (-1)^j \frac{4p_n^2}{p_n^2 + w_j^2} ((d_0, \hat{b}_j)\cdot (d_0, \hat{a}_j)) + \sum_{n=M_1+1}^{M} r_n^2 \sum_{j=1}^{2N} (-1)^j \frac{4p_n^2}{p_n^2 + w_j^2} \sum_{l=1}^{2N} (a_j, a_l) (d_0, \hat{a}_l)
\]

and

\[
\sigma \sum_{j=1}^{M_1} (d_0, \beta_j, \bullet) (d_0, \beta_j', \bullet) \\
= \sigma \sum_{n=1}^{M_1} \sum_{j,l=1}^{2N} (-1)^{j+l} \frac{1}{2} \left( \frac{1 - \mu_j \mu_l}{2} \right) \left( a_j^{(n)} a_l^{(n)} \right) (d_0, \hat{b}_j) (d_0, \hat{b}_l)
\]

\[
= \sigma \sum_{j,l=1}^{2N} (-1)^{j+l} \frac{1}{4} \left( \sum_{n=1}^{M_1} a_j^{(n)} a_l^{(n)} \right) (d_0, \hat{b}_j) (d_0, \hat{b}_l)
\]

\[
= \frac{\sigma}{2} \sum_{j,l=1}^{2N} (-1)^{j+l} (w_j^2 - w_l^2) \left( \frac{1}{\sigma} + \sum_{n=M_1+1}^{M} \left( \frac{4p_n^2 r_n^2}{(p_n^2 + w_j^2)(p_n^2 + w_l^2)} \right) (b_j, b_l) (d_0, \hat{b}_j) (d_0, \hat{b}_l)
\]

\[
= 2 \sum_{j=1}^{2N} \left[ (-1)^{j+1} w_j^2 (d_0, \hat{b}_j) \sum_{l=1}^{2N} (-1)^{l+1} (b_j, b_l) (d_0, \hat{b}_l) \right]
\]

\[
+ \sum_{n=M_1+1}^{M} 2\sigma p_n^2 r_n^2 \sum_{j,l=1}^{2N} (-1)^{j+l} \left( \frac{1}{p_n^2 + w_j^2} - \frac{1}{p_n^2 + w_l^2} \right) (b_j, b_l) (d_0, \hat{b}_j) (d_0, \hat{b}_l)
\]

\[
= 2 \sum_{j=1}^{2N} \left[ (-1)^{j+1} w_j^2 \left( (b_j, d_0, \bullet) - (-1)^j (b_j, a_j) (d_0, \hat{a}_j) \right) \right]
\]

\[
+ \sum_{n=M_1+1}^{M} 4\sigma p_n^2 r_n^2 \sum_{j=1}^{2N} (-1)^j \frac{1}{p_n^2 + w_j^2} (d_0, \hat{b}_j) \sum_{l=1}^{2N} (-1)^{l+1} (b_j, b_l) (d_0, \hat{b}_l)
\]

\[
= - 2 \sum_{j=1}^{2N} w_j^2 (d_0, \hat{a}_j) (d_0, \hat{b}_j) + \sum_{n=M_1+1}^{M} 4\sigma p_n^2 r_n^2 \sum_{j=1}^{2N} \frac{1}{p_n^2 + w_j^2} (d_0, \hat{a}_j) (d_0, \hat{b}_j)
\]
\[-(f_x)^2 = (d_2, d_0, \bullet) (d_0, d_0, \bullet) - (d_1, d_0, \bullet)^2 \]
\[= \sum_{j,l=1}^{2N} (-1)^{j+l} [(d_2, a_j) (d_0, a_l) - (d_1, a_j) (d_1, a_l)] (d_0, \hat{a}_j) (d_0, \hat{a}_l) \]
\[= \sum_{j,l=1}^{2N} (-1)^{j+l} \left( \frac{w_j^2 + w_l^2}{2} - w_j w_l \right) e^{\eta_j+\eta_l} (d_0, \hat{a}_j) (d_0, \hat{a}_l) \]
\[= \sum_{j,l=1}^{2N} (-1)^{j+l} \left( \frac{w_j^2 - w_l^2}{2} (a_j, a_l) (d_0, \hat{a}_j) (d_0, \hat{a}_l) \right) \]
\[= \sum_{j=1}^{2N} \left[ (-1)^j \left( -w_j^2 \right) (d_0, \hat{a}_j) \sum_{l=1}^{2N} (-1)^l (a_j, a_l) (d_0, \hat{a}_l) \right] \]
\[= \sum_{j=1}^{2N} (-1)^j \left[ w_j^2 (d_0, a_j) \right] (d_0, \hat{a}_j) (\bullet) - \sum_{j=1}^{2N} w_j^2 (a_j, b_j) (d_0, \hat{a}_j) \left( d_0, \hat{b}_j \right) \]
\[= -(d_2, d_0, \bullet) (\bullet) - \sum_{j=1}^{2N} w_j^2 (d_0, \hat{a}_j) \left( d_0, \hat{b}_j \right). \]

One can check that when substitute above identities into eq. (57), all terms are cancelled. Thus we have proved (58)-(60) solve eq. (57).

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