RELATIONS BETWEEN RESHETIKHIN–TURAEV AND RE-NORMALIZED LINK INVARIANTS

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ABSTRACT. Costantino–Geer–Patureau-Mirand proved relations between the Reshetikhin–Turaev link invariants and the re-normalized link invariants for knots. Their theorem says that residues of the re-normalized link invariants are given by the Reshetikhin–Turaev link invariants. However, in the case of links, the residues are vanish, so we cannot obtain any relations from the residues. In this paper, we prove that the Reshetikhin–Turaev link invariants appear in higher order terms of the re-normalized link invariants for links which are plumbed.

CONTENTS

1. Introduction 1
Acknowledgement 3
2. Preliminaries 3
2.1. Notation 3
2.2. Quantum groups and its representations 4
2.3. Invariants 4
3. A proof of the main Theorem 6
References 8

1. INTRODUCTION

Reshetikhin–Turaev \cite{RT91} constructed quantum invariants of 3-manifolds by using representations of the small quantum group $\tilde{U}_q\mathfrak{sl}_2$ at roots of unity. They reduces 3-manifolds to framed links in $S^3$ by surgery presentations. For a link colored by symmetric tensor representations, they constructed the Reshetikhin–Turaev link invariants $F$ and took a weighted sum to obtain the Witten–Reshetikhin–Turaev (WRT) invariant.

There is a slightly generalized version of $\tilde{U}_q\mathfrak{sl}_2$. We call this quantum group the unrolled quantum group $\bar{U}_q^{H}\mathfrak{sl}_2$. This quantum group has typical representations parametrized by complex numbers besides the representations of the small quantum group. However $F(L)$ vanishes for any link $L$ which has a component colored by a typical representation because the representation has zero quantum dimension. Geer–Patureau-Mirand–Turaev \cite{GPMT09} introduced a modified quantum dimension $d$ and constructed a new isotopy invariant $F'$. We call the invariant

The first author is supported by JSPS KAKENHI Grant Number JP 21J10271.
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the re-normalized Reshetikhin–Turaev link invariant. Costantino–Geer–Patureau-
Mirand [CGPM14] constructed nonsemisimple 3-manifold invariant (CGP invari-
ant) N by a weighted sum of the re-normalized Reshetikhin–Turaev link invariant.

Costantino–Geer–Patureau-Mirand [CGPM15] proved relations between F and
F′ for knots. They showed that F′ of a knot in S⁴ is a meromorphic function
of its color and the residues are given by F. Using the relations, Costantino–
Geer–Patureau-Mirand [CGPM15] and Costantino–Gukov–Putrov [CGP21] proved
relations between WRT invariant and CGP invariant of 3-manifolds presented by
0-framed knot.

One obstruction to generalizing their theorem stems from the fact that F′ of a
link is a holomorphic function of its color, that is the residues vanish. This is why
the theorem is restricted to knots.

In this paper, we generalize their theorem for plumbed graphs. The following is
the main theorem of this paper.

Theorem 1. Fix a 2r-th root of unity q = exp(π√−1/r). Let Γf = (V, E, f) be a
plumbed graph, where f : V → ℤ is a framing. Take x ∈ X_r^V, then

\[ F_r(Γ_f(r - 1 - x)) = \frac{(-1)^{|E| + \sum_{v \in V \geq 2}(d_v - 1)x_v}}{r\{1\}} \times \sum_{\varepsilon \in \{±1\}^E} \left( \prod_{e \in E} \varepsilon(e) \right) \lim_{\alpha \to x(\varepsilon)} \frac{F'_r(Γ_f(\alpha))}{\prod_{v \in V \geq 2} \{r\alpha_v\}^{d_v - 1}} \prod_{v \in V \geq 2} \{r\alpha_v\}^{d_v - 1} \]

, where d_v is a degree of a vertex v ∈ V. X_r^V := ℤ \setminus r\ℤ and Γ_f(α) is a plumbed
graph whose vertices are colored by α = (α_V)_{v ∈ V} ∈ ℂ^V.

In our case F′ is equal to the Akutsu–Deguchi–Ohtsuki invariant (see [GPMT09]),
so we can rephrase this theorem in terms of the colored Jones polynomial J and
ADO_r as follows.

Theorem 2. There exist relations between the colored Jones polynomials and Akutsu–
Deguchi–Ohtsuki invariants of plumbed graphs.

\[ J((Γ_f(r - 1 - x)))|_{q=\exp(\pi\sqrt{-1}/r)} = \frac{(-1)^{|E| + \sum_{v \in V \geq 2}(d_v - 1)x_v}}{r\{1\}} \times \sum_{\varepsilon \in \{±1\}^E} \left( \prod_{e \in E} \varepsilon(e) \right) \lim_{\alpha \to x(\varepsilon)} \frac{ADO_r(Γ_f(\alpha))}{\prod_{v \in V \geq 2} \{r\alpha_v\}^{d_v - 1}} \]

The interesting aspect of this theorem is that the Reshetikhin–Turaev link in-
variants appear in the higher order terms of the re-normalized link invariants. The
proof of this theorem depends on a property of plumbed graphs, so it is not clear
whether it can be generalized for other links or not.

Besides the above results, there are various relations of invariants constructed
from non-integer weight representation of \( U_q^{H} \otimes_2 \). Murakami [Mur08] con-
structed the colored Alexander invariant (which is a re-construction of the ADO invari-
ant) and compared it with the colored Jones polynomial for non-plumbed links.
Murakami–Nagatomo also constructed the logarithmic invariant [MN08] and proved
relations between this invariant and the colored Jones, Alexander invariants. After
this paper, Murakami [Mur18] expressed the logarithmic invariants of knots in terms
of derivatives of the colored Jones invariants. He also conjectured relations between
the logarithmic invariants and the hyperbolic volumes of the cone manifolds along knots and proved for the figure eight knot. There is a conjecture about relations between GPPV invariants and ADO invariants [CGP21]. Beliakova–Hikami [BH21] proved relations between Habiro’s cyclotomic expansion of the colored Jones polynomial and the ADO invariants of the double twist knots.

For 3-manifold invariants, Costantino–Geer–Putrov [CGP21] proved relations between CGP invariants and GPPV invariants under technical conditions. Chen–Kuppum–Srinivasan [CKS09] proved relations between Hennings invariants and WRT invariants. Beliakova–Hikami [BH21] proved relations between WRT invariants and CGP invariants of 3-manifolds obtained by 0-surgery on the double twist knots.

Acknowledgement

The work is supported by JSPS KAKENHI Grant Number JP 21J10271 and a Scholarship of Tohoku University, Division for Interdisciplinary Advanced Research and Education. The author would like to show his greatest appreciation to Professor Yuji Terashima and Professor Jun Murakami for giving many pieces of advice. The author thanks his family for all the support.

2. Preliminaries

2.1. Notation. All plumbed graphs in this paper are weighted trees. Let $\Gamma_f = (V, E, f)$ stand for a plumbed graph where $V$ is a set of vertices, $E$ is a set of edges, and $f : V \to \mathbb{Z}$ is a weight. For $v \in V$, let $d_v$ be the degree of $v$. For $n \in \mathbb{Z}$, let $V_{\geq n}$ be a subset of $V$ consisting of vertices whose degrees are greater than $n$.

We identify a plumbed graph with a link. Vertices correspond to unknots. Weights correspond to framings. If two vertices are connected by an edge, then corresponding unknots are linked as a Hopf link. For example, see Figure 1. In this paper, we let $H$ be a Hopf link.

![Plumbed graph and corresponding framed link](image)

Figure 1. A plumbed graph and the corresponding framed link

Finally, we define a symbol used in the proof of the main theorem. Fix a vertex $v_0$. For any vertex $v$, there is a unique path $P_v$ from $v_0$ to $v$.

**Definition 1.** For $x \in \mathbb{Z}^V$ and $\varepsilon \in \{\pm 1\}^E$, we define $x(\varepsilon) \in \mathbb{Z}^V$ by

$$x(\varepsilon) = \left( \prod_{e \in P_v} \varepsilon_e \right) x_v.$$

2.2. Quantum groups and its representations. Fix a positive integer \( r \) and let \( q = \exp(\pi \sqrt{-1}/r) \) be a \( 2r \) th-root of unity. For a complex number \( z \), we set
\[
q^z = \exp\left(\frac{\pi \sqrt{-1} z}{r}\right), \quad \{z\} = q^z - q^{-z}, \quad [z] = \{z\}/\{1\}.
\]

We recall two quantum groups. Consider a \( \mathbb{C} \)-algebra \( \tilde{U}_q\mathfrak{sl}_2 \) generated by \( E, F, K^{\pm 1} \) whose relations are
\[
KK^{-1} = 1 = K^{-1}K, \quad KFK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,
\]
\[
[E,F] = \frac{K - K^{-1}}{q - q^{-1}}, \quad E^r = F^r = 0, \quad K^{2r} = 1.
\]

We define the coproduct \( \Delta: \tilde{U}_q\mathfrak{sl}_2 \to \tilde{U}_q\mathfrak{sl}_2 \otimes \tilde{U}_q\mathfrak{sl}_2 \), counit \( \epsilon: \tilde{U}_q\mathfrak{sl}_2 \to \mathbb{C} \) and antipode \( S: \tilde{U}_q\mathfrak{sl}_2 \to \tilde{U}_q\mathfrak{sl}_2 \) as follows:
\[
\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K, \\
\epsilon(E) = 0, \quad \epsilon(F) = 0, \quad \epsilon(K) = 1, \\
S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}.
\]

Combined with these maps, \( \tilde{U}_q\mathfrak{sl}_2 \) is a Hopf algebra. We call \( \tilde{U}_q\mathfrak{sl}_2 \) the small quantum group.

For each \( n \in \{0, \ldots, r-1\} \), there is an irreducible highest weight representation \( S_n \). This module has a basis \( \{s_i\}_{i=0}^n \) and actions are given by
\[
Es_i = [i][n-i+1]s_{i-1}, \quad Fs_i = s_{i+1}, \quad Ks_i = q^{i-2i}s_i, \quad Es_0 = 0 = Fs_n.
\]

Another quantum group \( U_q^2\mathfrak{sl}_2 \) is a \( \mathbb{C} \)-algebra generated by \( E, F, H, K^{\pm 1} \) and its relations are given by
\[
KK^{-1} = 1 = K^{-1}K, \quad KFK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,
\]
\[
[H,K] = 0, \quad [H,E] = 2E, \quad [H,F] = -2F.
\]

We extend the coproduct, counit and antipode of the small quantum group by
\[
\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \epsilon(H) = 0, \quad S(H) = -H.
\]

We call this Hopf algebra \( U_q^2\mathfrak{sl}_2 \) the unrolled quantum group. A \( U_q^2\mathfrak{sl}_2 \)-module \( V \) is called a weight module if \( V \) splits into a direct sum of eigenspaces of \( H \) and \( K \) operates on \( V \) as \( q^H \). Since we do not require \( K^{2r} = 1 \), eigenvalues of \( H \) can be complex numbers. For each \( \alpha \in \mathbb{C} \), there is a highest weight representation \( V_\alpha \). The module \( V_\alpha \) has a basis \( \{v_i\}_{i=0}^{r-1} \) and actions are given by
\[
Ev_i = [i][i-\alpha]v_{i-1}, \quad Fv_i = v_{i+1}, \quad Hv_i = (\alpha + r - 1 - 2i)v_i, \quad Ev_0 = 0 = Fv_{r-1}.
\]

We call \( V_\alpha \) typical if \( \alpha \in \mathbb{C} \setminus X_r \) and atypical if \( \alpha \in X_r \).

2.3. Invariants. Let \( F_r \) be the Reshetikhin–Turaev functor and \( M \) be \( S_n \) or a typical module \( V_\alpha \). If \( L \) is a framed link with an edge colored by \( M \) and \( T_L \) is a ribbon graph obtained by cutting the edge of \( K \) then
\[
F_r(L) = q\text{dim}(M)\langle T_L \rangle
\]
, where \( \langle T_L \rangle \) is a scalar satisfying \( F_r(T_L) = \langle T_L \rangle \text{id}_M \). Since \( q\text{dim}(V_\alpha) = 0, F_r(L) = 0 \) for links with edges colored by \( V_\alpha \).
Lemma 1. (1) Take $\alpha, \beta \in \mathbb{C} \setminus X_r$, then
$$F'_r(H(\alpha, \beta)) = (-1)^{r-1}rq^{\alpha\beta}.$$  
(2) Take $0 \leq m, n \leq r - 1$, then
$$F_r(H(m, n)) = (-1)^{m+n} \frac{(m+1)(n+1)}{r}.$$  

Lemma 2. (1) Let $e$ be an edge colored by $V_\alpha$ and have $+1$ framing. Then
$$F'_r(e) = q^{\alpha^2-(r-1)^2/2} id_{V_\alpha}.$$  
(2) Let $e$ be an edge colored by $S_x$ and have $+1$ framing. Then
$$F_r(e) = (-1)^{\alpha} q^{\alpha(n^2+2n)/2} id_{S_x}.$$  

The invariants are well-behaved under a connected sum.

Lemma 3. (1) Take $\alpha \in \mathbb{C} \setminus X_r$ and let $T, T'$ be ribbon graphs with edges $e, e'$ colored by $V_\alpha$. For a connected sum $T \# T'$ along the edges, $F'_r$ satisfies
$$F'_r(T \# T') = d(\alpha)^{-1} F'_r(T) F'_r(T').$$  
(2) Take $x \in \{0, \ldots, r-1\}$ and let $T, T'$ be ribbon graphs with edges $e, e'$ colored by $S_x$. For a connected sum $T \# T'$ along the edges, $F_r$ satisfies
$$F_r(T \# T') = q \dim(S_x)^{-1} F_r(T) F_r(T').$$  

The following lemmas are technical lemmas appear in the proof of the main theorem.

Lemma 4. Let $H(0, f_{e'})(\alpha_{v_1}, \alpha_{v'})$ be a Hopf link colored by $(\alpha_{v_1}, \alpha_{v'}) \in (\mathbb{C} \setminus X_r)^2$ with framing $(0, f_{e'})$. Let $\Gamma$ be a plumbed graph and $\Gamma'$ be a connected sum $\Gamma \# H$ at $v_1$. For $\varepsilon \in \{\pm 1\}$ and $\varepsilon' \in \{\pm 1\} V'$,
$$\lim_{\alpha_{v_1} \to x_{e_1}} \lim_{\alpha_{v'} \to x_{e'}(\varepsilon')} \frac{F'_r(H(0, f_{e'})(\alpha_{v_1}, \alpha_{v'}))}{F'_r(H(0, f_{e'})(\alpha_{v_1}, \alpha_{v'}))} = \lim_{\alpha_{v_1} \to x_{e_1}(\varepsilon')} \lim_{\alpha_{v'} \to x_{e'}(\varepsilon')} \frac{F'_r(H(0, f_{e'})(\alpha_{v_1}, \alpha_{v'}))}{F'_r(H(0, f_{e'})(\alpha_{v_1}, \alpha_{v'}))},$$  
where $V'$ is a set of vertices of $\Gamma'$.

Proof.
$$\lim_{\alpha_{v_1} \to x_{e_1}} \lim_{\alpha_{v'} \to x_{e'}(\varepsilon')} \frac{F'_r(H(0, f_{e'})(\alpha_{v_1}, \alpha_{v'}))}{F'_r(H(0, f_{e'})(\alpha_{v_1}, \alpha_{v'}))} = \lim_{\alpha_{v_1} \to x_{e_1}} \lim_{\alpha_{v'} \to x_{e'}(\varepsilon')} \frac{(-1)^{r-1} rq^{\alpha_{v_1}\alpha_{v'}}}{F'_r(H(0, f_{e'})(\alpha_{v_1}, \alpha_{v'}))}$$  
$$= \lim_{\alpha_{v_1} \to x_{e_1}} \lim_{\alpha_{v'} \to x_{e'}(\varepsilon')} \frac{(-1)^{r-1} rq^{\alpha_{v_1}\alpha_{v'}}}{F'_r(H(0, f_{e'})(\alpha_{v_1}, \alpha_{v'}))}$$  
$$= \frac{(-1)^{r-1} rq^{\alpha_{v_1}\alpha_{v'}}}{F'_r(H(0, f_{e'})(\alpha_{v_1}, \alpha_{v'}))}.$$  

Let $\alpha_{v_1}' = \prod_{e \in P_{v_1}} \varepsilon(e) \alpha_{v_1}$ and $\alpha_{v'}' = \prod_{e \in P_{v_1}} \varepsilon(e) \alpha_{v'}$, then we obtain the right hand side. \[ \square \]
Lemma 5. Take \( \alpha \in (\mathbb{C} \setminus X_r)^V, x, x' \in X_r \). For \( \alpha' = (\alpha, \alpha') \in (\mathbb{C} \setminus X_r)^V \), \( F_r' \) satisfies
\[
\lim_{\alpha' \to x'} \frac{F_r'(\Gamma \# H(0, f')((\alpha, \alpha'))}{\prod_{v \in V_{2,2}} \{r\alpha_v\}^{d_{v,-1}}} = q'(x^2 - (r-1)^2)/2 \frac{q^{x_v, x_{v'}}}{\{x_{v1}\}} \lim_{\alpha \to x} \frac{F_r'(\Gamma)}{\prod_{v \in V_{2,2}} \{r\alpha_v\}^{d_{v,-1}}}.
\]

Proof. The connected sum splits by Lemma 3 as below:
\[
\frac{F_r'(\Gamma f(\alpha) \# H(0, f')(\alpha_v, \alpha_v'))}{\prod_{v \in V_{2,2}} \{r\alpha_v\}^{d_{v,-1}}} = d(\alpha_v)^{-1} \frac{F_r'(\Gamma f(\alpha))}{\prod_{v \in V_{2,2}} \{r\alpha_v\}^{d_{v,-1}}} = q'(x^2 - (r-1)^2)/2 \frac{q^{x_v, x_{v'}}}{\{x_{v1}\}} \frac{F_r'(\Gamma f(\alpha))}{\prod_{v \in V_{2,2}} \{r\alpha_v\}^{d_{v,-1}}}.
\]

Since \( d_{v1} = d_{v1} + 1 \), we obtain
\[
q'(\alpha_v^x, -r(1)^2)/2 \frac{q^{x_v, x_{v'}}}{\{x_{v1}\}} \frac{F_r'(\Gamma f(\alpha))}{\prod_{v \in V_{2,2}} \{r\alpha_v\}^{d_{v,-1}}} = q'(\alpha_v^x, -r(1)^2)/2 \frac{q^{x_v, x_{v'}}}{\{x_{v1}\}} \frac{F_r'(\Gamma f(\alpha))}{\prod_{v \in V_{2,2}} \{r\alpha_v\}^{d_{v,-1}}}.
\]

By the above equation,
\[
\frac{F_r'(\Gamma f(\alpha'))}{\prod_{v \in V_{2,2}} \{r\alpha_v\}^{d_{v,-1}}} = q'(\alpha_v^x, -r(1)^2)/2 \frac{q^{x_v, x_{v'}}}{\{x_{v1}\}} \frac{F_r'(\Gamma f(\alpha))}{\prod_{v \in V_{2,2}} \{r\alpha_v\}^{d_{v,-1}}}
\]
holds. Taking a limit, we completes the proof. \( \square \)

3. A proof of the main Theorem

We prove Theorem 1 by induction. When \( \Gamma \) is a Hopf link \( H \) (see Figure 2), the left hand side reduces to
\[
F_r'(H_{(0, r-1-x)}) = \prod_{v \in V_{2,2}} (-1)^{f_v(r-1-x_v)q_{x_v}(x_{v+2x_v})/2} \prod_{v \in V_{2,2}} (-1)^{(r-1)r_{v}} \{r_{v-1}(x_{v}x_{v'})\}
\]
by Lemma 1 and Lemma 2.

We expand \( \{x_{v1}, x_{v'}\} \) and take \( v_1 \) as a base point to obtain the following equation:
\[
\frac{(-1)^{r_{v}}}{r_{v}} \prod_{v \in V_{2,2}} (-1)^{(r-1)r_{v}} \{r_{v-1}(x_{v}x_{v'})\} = \frac{(-1)^{x_v}}{x_v} \sum_{\varepsilon \in \{x_v\}} \varepsilon \{r_{v} q_{x_v}(x_{v}x_{v'})\}.
\]

We can prove
\[
(-1)^{f_v(r-1-x_v)q_{x_v}(x_{v+2x_v})/2} = q^{x_v, x_{v'}} (r-1)^2)/2
\]

Figure 2. A Hopf link and the corresponding plumbed graph \( H \).
by direct calculation. Applying Lemma 1, we show that Theorem 1 holds for $\Gamma = H$:

$$F_r(H_f(r - 1 - x)) = \frac{(-1)^1}{r \{1\}} \sum_{\varepsilon' \in \{\pm 1\}} \varepsilon' \lim_{\alpha \to x} F'_r(H(\alpha)).$$

Assume that the equation holds for a plumbed graph $\Gamma$ and we will show that Theorem 1 still holds for a connected sum $\Gamma' = \Gamma \# H$ with a Hopf link $H$. We fix a vertex $v_1$ where $\Gamma$ and $H$ are joined. See Figure 3.

Then Lemma 3 shows

$$F_r(\Gamma_f(r - 1 - x) \# H_{(0,f_{v'})}(r - 1 - x_{v_1}, r - 1 - x_{v'})) = qdim(S_{r - 1 - x_{v_1}})^{-1} F_r(\Gamma_f(r - 1 - x)) F_r(H_{(0,f_{v'})}(r - 1 - x_{v_1}, r - 1 - x_{v'})).$$

From the induction hypothesis, we find that the right hand side of the above equation equals to

$$(-1)^{r - 1 - x_{v_1}} \frac{\{1\}}{r \{r - x_{v_1}\}} \times \frac{(-1)^{\left|E\right| + \sum_{v \in V_{\geq 2}} (d_v - 1)x_v}}{r \{1\}} \sum_{\varepsilon \in \{\pm 1\}^E} \left( \prod_{e \in E} \varepsilon(e) \right) \lim_{\alpha \to x} F'_r(\Gamma_f(\alpha)) \prod_{v \in V_{\geq 2}} \{r \alpha_v\}^{d_v - 1}.$$

Lemma 4 shows

$$(-1)^{r - 1} \frac{\{1\}}{\{x_{v_1}\}} \times \frac{(-1)^{\left|E'\right| + \sum_{v \in V_{\geq 2}} (d'_v - 1)x'_v}}{r \{1\}} \sum_{\varepsilon' \in \{\pm 1\}^{E'}} \left( \prod_{e' \in E'} \varepsilon'(e') \right) \lim_{\alpha' \to x'} F'_r(\Gamma_f(\alpha)) F'_r(H_{0,f_{v'}}(\alpha_{v_1}, \alpha_{v'})).$$

Lemma 5 shows

$$(-1)^{\left|E'\right| + \sum_{v \in V_{\geq 2}} (d'_v - 1)x'_v} \frac{(-1)^{r - 1}}{\{x_{v_1}\}} \times \sum_{\varepsilon' \in \{\pm 1\}^{E'}} \left( \prod_{e' \in E'} \varepsilon'(e') \right) \lim_{\alpha' \to x'} F'_r(\Gamma_f(\alpha)) F'_r(H_{0,f_{v'}}(\alpha_{v_1}, \alpha_{v'})) \prod_{v \in V_{\geq 2}} \{r \alpha_v\}^{d_v - 1}. $$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{\(\Gamma'\) obtained by joining \(\Gamma\) and \(H\) at \(v_1\)}
\end{figure}
From Lemma 3, we see

\[ F'_r(\Gamma_f(\alpha)) F'_r(H(0,f_{x'})((\alpha_{v_1},\alpha_{v'}))) = (-1)^{r-1} r \frac{\alpha_{v_1}}{r\alpha_{v_1}} F'_r(\Gamma_f(\alpha) # H(0,f_{x'})((\alpha_{v_1},\alpha_{v'}))). \]

Applying the equation, we obtain

\[ F_r(\Gamma'_f(r-1-x)) = (-1)^{\left| E'_f + \sum_{v \in V'_f} (d'_v-1)x_v \right|} \frac{r \{ 1 \}}{r \{ 1 \}} \sum_{\varepsilon' \in \{ \pm 1 \}^E} \left( \prod_{e \in E'_f} \varepsilon'(e) \right) \lim_{\alpha' \to x'(\varepsilon')} \frac{F'_r(\Gamma'_f(\alpha'))}{\prod_{v \in V'_f} \{ r\alpha_v \}^d'_v - 1}. \]

This completes the induction.

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