Lagrange Equations of Envelopes

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Lagrange Equations of Envelopes

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Abstract. The dynamics of charged particles in a traveling wave field with the uniform magnetic focusing has been studied. A Lagrange function for this motion is constructed and Lagrange equations describing geometrical and kinematic envelopes of the bunch are derived.

1. Introduction
This work is devoted to description of the dynamics of charged particles in a traveling wave field with the uniform magnetic focusing. Its principal purpose is a transition from the dynamics of separate charges to a bunch of charges as a single whole. For this purpose, a Lagrange functions for the longitudinal and transverse motions of a bunch of charged is constructed under following assumptions that (i) charges are uniformly distributed over longitudinal emittance of the bunch and (ii) the bunch is approximated by an ellipsoid of rotation in the configuration space, semi-axes of which are used as the generalized coordinates $q, r$.

Let us consider bunch of charged particles and approximate its emittance by an ellipsoid with vertical and horizontal semi-axes $N$ and $M$ respectively. For the bunch consisting of $N_b$ particles uniformly “smeared” over the emittance, the phase density $f$ is constant within the bunch, which can be written in a normalized form as $f = N_b / \pi MN$. In configuration space, we obtain a distribution function $\rho(x)$ defined within the bunch, which can be written in a normalized form as follows:

$$\rho(x) = \frac{8N_b}{3\pi^2r^2q} \sqrt{1 - \left(\frac{x}{q}\right)^2},$$  

where $x$ is the coordinate measure from the bunch center along the axis of motion and $r$ is transverse semi-axis of the ellipsoid of rotation that approximates the bunch. Both $x$ and $r$, as well as all other distances and sizes used below, are normalized to the wavelength $\lambda$ of a generator that pumps the accelerating system.

Kinetic energy of the bunch $T_b$ we obtain from the following expression for the kinetic energy of the charges particles

$$T = \pi \gamma_s^2 \dot{x}^2 + \frac{1}{2} \gamma_s \dot{y}^2,$$

where the first summand taken from [1]; $\dot{x}, \dot{y}$ is the longitudinal and transverse velocity of the charges particles correspondingly, expressed in units of the speed of light; $\gamma_s$ is the total energy of a synchronous particle in units of the energy of resting charges bunch.

The kinetic energy of an elementary fragment of the bunch, representing the thin ring, is given by the following formula:

$$dT_b = 2\pi r dx dy \rho(x) \left(\pi \gamma_s^2 \dot{x}^2 + \frac{1}{2} \gamma_s \dot{y}^2\right).$$

Let us assume the dependence of the longitudinal and transverse velocity of the ring from the coordinates can be written by the following formula: for the longitudinal velocity $\dot{x}(x) = \dot{\eta} \chi^{\alpha/2}$; for the transverse velocity $\dot{y}(y) = \dot{\eta} \chi^{\alpha/2}$, where $\chi = x/q$; $\eta = y/r$; $\alpha, \chi$ – the parameters of the dependences.

Integration of expression (3) over the bunch volume yields the kinetic energy of the whole bunch:
\[ T_b = \frac{8N_b}{9\pi} \gamma_s \left( \frac{3\pi - \alpha}{(1 + \alpha)(3 + \alpha)} \gamma_s q^2 + \frac{3 - \alpha}{2 + \alpha} s^2 \right). \]  

The potential energy of the longitudinal oscillations of the bunch in the accelerating wave defined by the expression \[ U_{b1} = \frac{\beta_s A \sin \varphi_s}{24\pi} N_b (\psi^2 - \frac{1}{32} \psi^4 + \frac{1}{1920} \psi^6), \] where \( \psi = 2\pi q/\beta_s \); \( \beta_s \) — the velocity of the synchronous particle, expressed in units of the speed of light; \( A = Z_e E / E_0 \); \( E \) — the amplitude of the accelerating wave; \( E_0 \) — the rest energy; \( \varphi_s \) — is the phase of the synchronous particle.  

Integration of the potential energy of the transverse motion of the charge in the accelerating waves field [1] and the potential energy of the thin ring in the uniform - magnetic focusing field  
\[ dU_{br} = 2\pi y^3 dy dx \frac{8N_b}{3 \pi^2 r^2 q} (1 - \chi^2)^{1/2} B_f^2 \]  
which followed from the equations of the motion, yields the potential energy of the transverse motion of the whole bunch:  
\[ U_{b2} = \frac{5}{24} N_b \pi A \sin \varphi_s \frac{\beta_s^3}{\beta_s^2} (1 - \beta_s^2)^2 + \frac{5}{96} N_b \frac{B_f^2}{\gamma} r^2. \]  
Here \( B_f \) is the non-dimensional magnetic focusing field, connected with the dimensional field \( B_f^* \) by the expression \( B_f = Z_e B_f^*/m_0 c \).  

The Lagrange equations may also naturally include the intrinsic Coulomb field of the bunch. This can be achieved using the Vlasov ellipsoid model [3], which leads to the appearance of the following term on the right-hand side of the Lagrange equations:  
\[ E_b = \frac{90Z_e}{r^2 \beta_s^2 E_0} (1 - \beta_s^2)^{3/2} M. \]  
Here \( \Gamma \) is the pulsed current of a train of bunches (expressed in amperes) and \( M \) is the ellipsoid form factor defined by the expressions [2].  

Once the kinetic and potential energies of the bunch are known, the corresponding Lagrange function can be constructed and the Lagrange equations can be written as follows: the equation of the longitudinal motion  
\[ \frac{d^2 q}{dz^2} = \frac{A \sin \varphi_s (1 + \alpha)(3 + \alpha)}{32 \beta_s^2 \gamma_s^3} P(\psi) - \frac{3\gamma_s^2 - 2}{\gamma_s (\gamma_s^2 - 1)} \frac{d \gamma_s}{dz} \frac{d q}{dz} + \frac{D}{\beta_s^2} (1 - \beta_s^2)^{3/2}, \]  
where \( D = 90Z M \Gamma / r^2 E_0, \) \( Z \) is the order of the charge of the bunch, \( P(\psi) = \psi - \frac{1}{16} \psi^3 + \frac{1}{640} \psi^5; \)  
the equation of the transverse motion  
\[ \frac{d^2 r}{dz^2} = -\frac{15\pi}{256} \frac{2 + \alpha}{(3 - \alpha)} \left( \frac{4\pi A \sin \varphi_s}{\beta_s^4 \gamma_s^3} (1 - \beta_s^2) + \frac{B_f^2}{\gamma_s (\gamma_s^2 - 1)} \right) r - \frac{\gamma_s}{\gamma_s^2 - 1} \frac{d \gamma_s}{dz} \frac{d r}{dz} + \right. 

\[ \left. + D_r (1 - \beta_s^2), \right. \]  
where \( D_r = 90Z M \Gamma / r^2 E_0, \) \( M_r = 0.5(1 - M). \)  

The solutions of those equations are the generalized coordinates and velocities. The generalized coordinates define the sizes of the bunch and the generalized velocities are the boundaries of its spectrums. According to the terminology of the mechanics the generalized coordinates are the geometrical envelopes and the generalized velocities are the kinematic envelopes.  

In this way the Lagrange equation contains two envelopes, namely geometrical and kinematic.  

The obtained equations are describes longitudinal and transverse geometrical and kinematic envelopes of the bunch. Those equations are advantageous to the well-known equations [4], primarily,
in being simple. Additional advantages are as follows. First, the Lagrange equations determine not only the sizes of a bunch (its geometrical envelopes), but also the boundaries of its energy spectrums (kinematic envelopes). Second, the Lagrange equations have been derived without any restrictions imposed on the amplitude of the oscillations, so this mathematical model is nonlinear and, hence, more general. Third, the equations take into account the acceleration of a bunch. Finally, the Lagrange equations take into account the force of the Coulomb field of a bunch in terms of the naturally included ellipsoid model, so that those envelope equations are self-consistent.

Figures 1 – 3 illustrates the results of calculations the envelopes as determined from obtained Lagrange equations and calculated by the traditional method of individual particles.

**Figure 1.** Longitudinal geometrical (a) and kinematic (b) envelopes as determined using the traditional method of individual particles (dash – dot curve) and the proposed equations of envelopes (solid curve).
Figure 2. Transverse geometrical (a) and kinematic (b) envelopes as determined using the traditional method of individual particles (dash–dot curve) and the proposed equations of envelopes (solid curve).

Figure 3. Longitudinal geometrical (a) and kinematic (b) envelopes as determined using the traditional method of individual particles (dash–dot curve) and the proposed equations of envelopes (solid curve); the pulsed current is 32 mA.

Good coincidence of the results obtained by different methods is evidence for the applicability of the proposed approach to the effective mathematical modeling of the bunches dynamics.

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