Algorithmic pseudorandomness in quantum setups

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The Church-Turing thesis is one of the pillars of computer science; it postulates that every classical system has equivalent computability power to the so-called Turing machine. While this thesis is crucial for our understanding of computing devices, its implications in other scientific fields have hardly been explored. Here we start this research programme in the context of quantum physics and show that computer science laws have profound implications for some of the most fundamental results of the theory. We first show how they question our knowledge on what a mixed quantum state is, as we identify situations in which ensembles of quantum states defining the same mixed state, indistinguishable according to the quantum postulates, do become distinguishable when prepared by a computer. We also show a new loophole for Bell-like experiments: if some of the parties in a Bell-like experiment use a computer to decide which measurements to make, then the computational resources of an eavesdropper have to be limited in order to have a proper observation of non-locality. Our work opens a new direction in the search for a framework unifying computer science and quantum physics.

Quantum theory stands as one of the most successful and experimentally confirmed theories to date, with not a single experiment shown to be in disagreement. Its foundations, intensely debated in the early days of the theory [1, 2], remain an active area of research with many recent insightful results, such as the possibility to derive quantum theory from physical axioms [3], proving the completeness of quantum theory [4], or establishing that quantum states are real [5].

On the other hand, computability theory studies which functions can be calculated by algorithmic means and which cannot, and, more generally, their degree of un-computability. The field emerged in the 1930s with the independent works of Alan Turing [6] and Alonzo Church [7] who introduced two equivalent formalizations of the intuitive concept of algorithm. The fact that both models of computation turned out to be equivalent lead Stephen Kleene [8] to postulate what is now known as the Church-Turing thesis:

any function ‘naturally to be regarded as computable’ (i.e. calculable by algorithmic means) is computable by the formal model of Turing machines.

This thesis has been greatly strengthened by the fact that all the formal models of computation defined so far have been shown to be at most as expressive as the classical Turing machines in terms of the class of functions they compute.

Until now, the relationship between quantum mechanics and the Church-Turing thesis has been concerned on how the first one can affect the latter (see, for instance, [9]). In this article, however, we introduce an opposed and new research program: implications of computer science principles for quantum physics.

As the first steps in that direction, we present two results. First we show that computers impose a limitation when it comes to producing a mixed state as a classical mixture of pure quantum states. It turns out that with the sole knowledge that the classical mixture is performed by a computer, situations that seem not to be distinguishable turn out to be so. This has direct implications since mixed states are prepared this way in many experiments [10, 11]. Secondly, when it comes to Bell-like experiments to test non-locality, another distinctive feature of quantum mechanics, we show that if the measurement independence between the two parties [12–15] is achieved via private computable pseudo random number generators, an eavesdropper can start guessing their inputs from the information on their previous inputs, thus leading to a new computability loophole for Bell tests.

Formally, our results apply only to computers. This has already considerable practical consequences, since almost every experimental setup is controlled by computers. However, one can argue that, due to the widely accepted physical interpretation of the Church-Turing thesis:

the behaviour of any discrete physical system evolving according to the laws of classical mechanics is computable by a Turing machine,

our results, in fact, apply to every classical system and hence, the limitations that we show are fundamental.

I. PROPER MIXED STATE PREPARATION AND THE CHURCH-TURING THESIS

We start by considering one of the basic parts of quantum theory: the concept of a mixed state [16]. We will see that although well understood by physicists, their nature and origin can lead to apparent paradoxes when confronted against common computer science tenets.

Let us present now the following preparation of a mixed state: A classical computer in an unknown configura-
tion and with unbounded memory is running an unknown and presumably very convoluted algorithm to prepare a mixed state. We have the promise that the computer, understood as a black box, is mixing evenly either the single qubit eigenstates of $\sigma_z \{ |0\rangle, |1\rangle \}$ or the states $\{ |+\rangle, |-\rangle \}$, where $|\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$ as seen in Fig. 1. Is there any operational procedure to decide which of the two ensembles are being mixed for an experimenter (Bob) who cannot open the black box? Even though one would be tempted to assign to both preparations the identity state $\rho = \frac{1}{2}$, our results show that the fact that the mixing procedure was performed in a computable way leaves a trace which allows us to distinguish both mixtures in finite time and with arbitrarily high success probability. It is worth mentioning that having a computer mixing the state doesn’t imply that the sequence in which it mixes the state is periodic or anything. In fact, there exist normal sequences (e.g. those which satisfy the law of large numbers in a generalized way), or other even ‘more random’ sequences which are computable in polynomial time [17].

![FIG. 1: Scheme for the preparation of mixed states via classical mixing. Alice uses a computer to choose between $|0\rangle$ and $|1\rangle$ (or $|+\rangle$ and $|-\rangle$). From Bob's perspective, since he doesn’t know the computer program Alice is using, it’s a mixed state $\rho$. To distinguish both possible preparations he measures alternatently $\sigma_x$ and $\sigma_z$ and sends the resulting sequences to a computer that will be able to tell which of the sequences is computable (corresponding to the basis in which Alice prepared the state) and which is a fair coin tossing (corresponding to the other basis).

In order to solve this problem, Bob measures every qubit that comes out of the black box on an odd position in the basis of eigenstates of $\sigma_z$, yielding a binary sequence of measurement results $Z$. He also measures every qubit on an even position in the basis of eigenstates of $\sigma_x$ obtaining a binary sequence $X$. This way Bob obtains two binary sequences, as can be seen in Fig. 1. The one corresponding to the choice of measurement that matches the preparation basis is computable, and the other one corresponds to a fair coin tossing. Therefore we need an algorithm that given two sequences, one computable and one arising from a fair coin tossing, is able to tell us which is which. We will show now one such algorithm, that can perform this task in finite time and with an arbitrarily high probability of success.

To distinguish which of the two sequences is computable we dovetail between program number (the programs are computably enumerable) and maximum time steps that we allow each program to run (that is, we run program 1 for 1 timestep, then programs 1 and 2 for 2 timesteps and so on), as is a common technique in computability theory. For each program $p$ of length $|p|$ we will compare the first $k|p|$ output bits with the corresponding prefixes of both sequences, where $k$ is an integer constant depending on the probability of success we are looking for. Whenever we find a match for the first $k|p|$ bits, we halt. Fig. 2 depicts the dovetailing algorithm. It is straightforward to see that this algorithm always gives an answer, and the probability of making a mistake is less than $O(2^{-k})$. Therefore we can guess in finite time and with an arbitrarily high probability of success (by setting $k$ we adjust the probability of success).

The complete algorithm to distinguish a fair coin from a computable sequence is Algorithm 1 below, where $X \downarrow k|p|$ denotes the fist $k|p|$ bits of the sequence $X$.

**Algorithm 1** The distinguishing protocol. $U_t(p)$ is a universal Turing machine that runs program $p$ for $t$ timesteps. The two ‘for’ loops correspond to the dovetailing.

**Input:** $k \in \mathbb{N}$ and $X,Z \in \{0,1\}^\mathbb{N}$, two bit sequences with the promise that one of them is computable and other is not.

**Output:** ‘$X$’ or ‘$Z$’ as the candidate for being computable; wrong answer with probability bounded by $O(2^{-k})$.

for $t = 0,1,2,\ldots$ do

for $p = 0,\ldots,t$ do

if $U_t(p) = X \downarrow k|p|$ then

output ‘$X$’ and halt

if $U_t(p) = Z \downarrow k|p|$ then

output ‘$Z$’ and halt

Note that, at a given iteration of the algorithm, it may perfectly be the case that program $p$ has not been able to produce in $t$ time steps the $k|p|$ symbols needed to check the halting condition. If this is the case, the algorithm simply keeps running and moves to the next program. However, the algorithm will for sure halt as it will run the actual program used in the blackbox at some finite time. For a detailed explanation on how Algorithm 1 works and its probability of success, see Appendix A.

It is an interesting open question to study the effect of noise in the previous algorithm. As a first step, we have considered a rather simple noise model in the state preparations and measurements described by a flip probability in the observed symbols $r$. That is, we consider the situation in which those results obtained when measuring the quantum states in the actual basis used by the box are correct with probability $1 - r$ (this simple noise has no effect on the results of measurements performed in the wrong basis). As shown in Appendix A there is another slightly more complex algorithm that still halts with arbitrarily small error probability whenever $r \lesssim 0.21$.

The previous algorithm is of course very demanding, but proves that ensembles of states defining the same mixed state are in principle distinguishable when pre-
FIG. 2: To decide which of the two sequences is computable, we dovetail between program number and timesteps that each program is allowed to run. On green we show the actual program $p_A$ that was used to prepare the state, and the number of timesteps $t_A$ it takes to generate a $k|p_A|$ long prefix. The latest halting condition for our algorithm is shown in red, although it might halt before that with either a wrong recognition or a correct one.

pared by classical computing devices. This questions our understanding of mixed quantum states and leaves quantum mixtures (either by using a part of a larger etangled system or a quantum random number generator) as the only way to create them.

II. BELL INEQUALITY LOOPHOLE

Non-locality is another of the most intrinsic features of quantum mechanics [2, 18, 19]. The standard Bell scenario is described by two distant observers who can perform $m$ possible measurements of $d$ possible outputs on some given devices. The measurements are arranged so that they define space-like separated events. It is convenient for what follows to rephrase the standard Bell scenario in cryptographic terms, as in [20–22]. In this approach, Alice and Bob get the devices from a non-trusted provider Eve. The standard local EPR models correspond to classical preparations in which the devices generate the measurement results given the choice of measurements, but independently of the input chosen by the other party. Bell inequalities are conditions satisfied by all these preparations, even when having access to all the measurement choices and results produced in previous steps [20]. In turn, quantum correlations, obtained for example by measuring a maximally entangled two-qubit state with non-commuting measurements, can violate these inequalities. The violation of a Bell inequality witnesses the existence of non-local correlations and can be used by Alice and Bob to certify the quantum nature of their devices.

In what follows, it is shown how a classical Eve can mimic a Bell inequality violation when the measurement choices on Alice and Bob are performed following an algorithm, which is a standard practice in many Bell experiments to date. As above, it is not assumed that the algorithm is known by the eavesdropper. The result can be seen as a new loophole, named the computability loophole. For this loophole to apply, Eve has to make use of the inputs and outputs produced by the parties in previous steps [20, 22], as shown in Fig. 3.

FIG. 3: Scheme for the Bell inequality computability loophole. After each round $i$, Alice’s box receives the information about Bob’s choice of measurement $y_i$. Using the information from all previous choices of inputs for both parties, Alice’s box makes a prediction on what the next round inputs will be by using the presented algorithm. For simplicity in the notation, the attack is shown for a Bell test involving two measurements per party. As long as Alice, Bob or both use a computable sequence for their choice, the guess will start being correct after a number of rounds. Once this happens, the boxes can simulate any probability distribution, both local and non-local. Therefore Alice and Bob can not rule out an eavesdropper having prepared their boxes.

The computability loophole is rather simple and works as follows: one of the devices, say Alice’s, uses all the inputs chosen in previous steps by both parties to guess
the next ones. For that, a time complexity class $C$ is initially chosen by Eve. Since algorithms in such a class are computably enumerable, Alice’s device will check all of them until it finds one that matches Alice’s (or Bob’s) bits given so far. A guess for the next inputs is done based on that algorithm (see Fig. 4 for a representation of the algorithm) and communicated to the other device. If Eve’s class $C$ includes Alice’s and/or Bob’s algorithms, at some point the device will start guessing correctly and will keep doing so forever. Of course, once the devices are able to guess the inputs of at least one of the parties, they can easily produce non-local correlations.

It should be noticed that since Alice’s and Bob’s algorithms belong to some time complexity class, they can never rule out such an eavesdropper. On the other hand, the eavesdropper, when choosing the class $C$, is imposing how hard it is for Alice and Bob to avoid the loophole. The algorithm is shown as Algorithm 2 and for a more detailed description the reader is referred to Appendix A.

**Algorithm 2** A next-value algorithm for a time class $C$ with bound $t$

**Input:** $n \in \mathbb{N}$

**Output:** $g(n)$, the next-value function for $C$.

Let $M_e$ be an enumeration of all Turing machines. Let $n = (m_0, \ldots, m_{n-1})$ be the already seen bits from the sequence.

Let $(e, c)$ be the least number such that

i. for $i \in \{0, \ldots, n\}$, $M_e(i)$ halts after at most $c \cdot t(|i|)$ many steps, where $M_e$ evaluates function number $c$ from $C$ and $t$ is the computable time bound for class $C$.

ii. for $i \in \{0, \ldots, n-1\}$ $M_e(n)$ outputs $m_i$

Output $M_e(n)$

Apart from being very demanding, a criticism to this loophole is that it only works in the long run, meaning that Alice and Bob will not see a fake violation of a Bell inequality unless they run their experiment for long enough. But this brings the question of what’s the validity of a violation that, in the long run, would have admitted a local model. The only way to escape this loophole is by using a quantum random generator for the inputs, however, it is highly undesirable to depend on a non-local theory to test non-locality.

### III. DISCUSSION

The Church–Turing thesis is one of the most accepted postulates from computer science. As such, one can wonder what consequences would it have if it were, indeed, a law of Nature. We started here this research program in the context of quantum physics by showing how it questions the understanding of what a proper mixed quantum state is and introducing a new loophole for Bell tests. The study of these questions is essential in any attempt to create a unifying theory merging information laws and quantum physics.

### IV. ACKNOWLEDGEMENTS

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**Appendix A: Distinguishing two computable preparations of the same mixed quantum state**

In this section we discuss with details the protocol to distinguish two ensembles of pure states that apparently yield the same density matrix, given the assumption that the mixture has been prepared in a computable way. To do so, we will first reduce this scenario to a different problem in classical information theory relating infinite binary sequences, which we show to be solvable. Then we see how this second problem can be solved in finite time with arbitrarily small error probability. Finally, we present a slight modification of the algorithm that makes it robust against a simple noise model.

**1. The distinction protocol**

The distinguishability scenario we are interested in is as follows: Alice is presented with two bags of quantum systems, one having systems in the state $|0\rangle$ and the other having systems in the state $|1\rangle$. At random, she chooses one bag at a time and sends a state from that bag to Bob. Bob’s state is $\rho = \frac{1}{2}$, as he has no information on the prepared state. Now imagine the same scenario, but instead of those two bags Alice has one with the state

| Seen bits: | 1 0 1 |
|---|---|
| $s_1$ | 0 0 0 0 0 0 0 ... |
| $s_2$ | 0 0 1 1 0 1 1 ... |
| $s_3$ | 1 0 0 0 1 0 1 ... |
| $s_4$ | 1 1 1 1 0 0 0 ... |
| $s_5$ | 0 1 0 1 0 1 0 ... |
| First match: | $s_6$ 1 0 1 0 1 1 1 ... Next guess: | 0 |
| $s_7$ | 1 1 0 1 1 0 1 ... |

FIG. 4: Algorithm to predict bits from previously seen ones. For simplicity it is presented for the case of 2 inputs. Alice’s box enumerates all sequences $s_i$ from the complexity class $C$ and picks the first one whose first bits coincide with the ones seen. The guess Eve will make is the next bit from that sequence. If the actual sequence one wants to predict belongs to class $C$, at some point, after seeing enough bits, all the predictions will be correct.
let them proceed in the same way. It is clear that these two situations are indistinguishable from Bob’s point of view, as they are supposed to define the same mixed quantum state.

Consider the same two situations, but now, in order to choose from which bag to pick the state, Alice uses the bits of a computable binary sequence, that is, a binary sequence produced by an algorithm. Ideally such algorithm is a pseudo-random number generator, whose output ‘looks like’ typical coin tossing (i.e. satisfying for instance the law of large numbers, as well as any other reasonable law of randomness [23]). We next see that, for instance the law of large numbers, as well as any other output ‘looks like’ typical coin tossing (i.e. satisfying for algorithm is a pseudo-random number generator, whose nary sequence produced by an algorithm. Ideally such to choose from which bag to pick the state, Alice uses supposing to define the same mixed quantum state.

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halting problem \[6\] is not. Since in finite time a Turing machine can only process finitely many symbols, one cannot decide in finite time if an infinite sequence is computable or not.

In the following subsection we deal with a related problem: distinguishing a computable sequence from the output of a ‘fair coin’ (such as the result of measuring a \(\sigma_z\) eigenstate in the \(\sigma_z\) basis, under the assumption that quantum physics is correct). Notice that since there are countably many computable sequences, the output of tossing a fair coin gives a non-computable sequence with probability one. We show that one can distinguish both cases in finite time and with arbitrarily high success probability and that this fact has consequences on how mixed states in quantum mechanics are described.

\[ \] b. The protocol

As mentioned, the idea of the algorithm is to check every program until finding one that reproduces a sufficiently long prefix of either \(X\) or \(Z\). There are three key points that have to be taken into account in this idea, namely:

- It is impossible to know if a program halts or not \[6\]. Therefore, checking each single program one after another is not possible.

- It is still not clear what we mean by sufficiently long prefix.

- We might get a false positive, i.e. find a program that reproduces a prefix of the sequence which came form the coin tossing (even if it was not computable).

We deal with the first issue by dovetailing between programs and execution time. Recall that programs for \(U\) can be coded by (binary representations of) natural numbers. The idea of dovetailing is that we first run program 0 for 0 steps, then we run programs 0 and 1 for 1 step, then we run programs 0, 1 and 2 for 2 steps and so on.

To solve the second problem, we check if program \(p\) of length \(|p|\) generates (within the time imposed by the dovetailing) either of the prefixes \(X\mid j\) (that is, the first \(j\) bits of \(X\)) or \(Z\mid j\), where \(j = k|p|\). That is, every program is checked against a prefix \(k\) times longer than its length. Since a fair coin generates sequences with mostly non-compressible prefixes, it most likely will not have a prefix that can be generated by a \(k\) times shorter program, thus allowing us to detect the computable sequence. And as we will see, the probability of getting a false positive can be bounded only by a function of \(k\) that goes to 0 as \(k\) goes to infinite, solving also the third issue.

The pseudo-code for the algorithm that decides which of the sequences is computable will be the following:

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**Algorithm 3** The distinguishing protocol

**Input:** \(k \in \mathbb{N}\) and \(X, Z \in \mathbb{2}\), one of them being computable

**Output:** ‘\(X\)’ or ‘\(Z\)’ as the candidate for being computable; wrong answer with probability bounded by \(O(2^{-k})\)

for \(t = 0, 1, 2 \ldots\) do

for \(p = 0, \ldots, t\) do

- if \(U_i(p) = X \mid k|p|\) then
  - output ‘\(X\)’ and halt

- if \(U_i(p) = Z \mid k|p|\) then
  - output ‘\(Z\)’ and halt

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Note that \(X\) and \(Z\) are infinite sequences, and hence they must be understood as oracles [25, §III] in the effective procedure described above. Provided that at least one of \(X\) or \(Z\) is computable, the above procedure always halt —and so it only queries finitely many bits of both \(X\) and \(Z\). Indeed, in case \(S \in \{X, Z\}\) is computable, there is a monotone Turing machine \(M = M_s\) such that \(M(\epsilon)\) outputs \(S\) in the sense of \((A1)\). Hence for program \(p = \langle i, \epsilon \rangle\) we have that \(U_t(p) = S \mid k|p|\) for some \(t\).

It is important to recall that although both \(X\) and \(Z\) are infinite sequences, we only need to query finite prefixes. From a physical point of view this means that only finitely many qubits will be needed by Bob to discover how Alice was preparing the state.

Now, we bound the probability of having a miss-recognition, that is, the probability \(P_{\text{error}}\) that the above procedure outputs ‘\(Z\)’ when \(X\) was computable, or viceversa. To do so, we bound the probability that \(S \in \mathbb{2}\) has the property that for the given value of \(k\) there is \(p\) such that

\[(\exists t) \ U_t(p) = S \mid k|p|. \quad (A2)\]

Since there are \(2^\ell\) programs of length \(\ell\), the probability that there is a program \(p\) of length \(\ell\) such that \((A2)\) holds is at most \(2^\ell/2^{k\ell}\). Adding up over all possible lengths \(\ell\) we obtain

\[P_{\text{error}} \leq \sum_{\ell>0} \frac{2^\ell}{2^{k\ell}} = \frac{2^{-(k-1)}}{1 - 2^{-(k-1)}} = O \left(2^{-k}\right), \quad (A3)\]

which goes to zero exponentially with \(k\).

The protocol would then work as follows: given a tolerated error probability \(\epsilon\), one chooses a \(k\) large enough so that the previous bound is smaller than \(\epsilon\), and then run the described algorithm with inputs \(k\), \(X\) and \(Z\).

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3. Noise robustness

We now show how to modify the previous algorithm to make it robust against noise. We can consider a very natural noise model in which random bit flips are applied to the measured sequences, resulting for instance from imperfect preparations or measurements. Therefore, we modify the algorithm so that it tolerates a fraction \(q \in \mathbb{Q}\) of bit flips in the prefixes. The modified algorithm is as follows:
Now, it can be shown numerically that for $q < 0.21$ the probability of mis-recognition tends to zero exponentially with $k$.

Finally, we need to show that the noise tolerant algorithm always halts. Let $r < q$ be the probability of a bit flip. By the definition of probability we now have that for every $\delta$ there exist an $m_0$ such that for every $m > m_0$ the portion of bit flips in both $X \upharpoonright m$ and $Z \upharpoonright m$ are less than $(r + \delta)m$. This means that if we go to long enough prefixes (or programs), the portion of bit flips will be less than $q$. And since any computable sequence is computable by arbitrarily large programs, this ensures that our algorithm will, at some point, come to an end.

4. Discussion

We have shown that if Alice uses a computing device satisfying the Church-Turing thesis to prepare a seemingly proper mixture of $|0\rangle$ and $|1\rangle$ or a seemingly proper mixture of $|+\rangle$ and $|-\rangle$, both apparently yielding the maximally mixed state, Bob can distinguish both situations.

It is worth noticing that, although our algorithm halts in finite time, it can take extremely long, depending on the length of the shortest program that generates the needed prefix and the time it takes to find it. Nonetheless, what we have shown with this is that in both preparations, somehow, the resulting state has information on how it was prepared. Our algorithm can be thought of as a tool to prove that both preparations are indeed distinguishable, but there might be protocols that finish in shorter times. And even if there are not, the fact that both situations are distinguishable still holds, showing that having a computable preparation leaves a mark on the states it produces.

This apparent paradox can be easily resolved the following way: computable sequences have correlations that we are not taking into account. This means that Alice’s choice is not given, as needed, by a set of independent and identically distributed random variables but by a computable sequence. The evident consequence of this is that Bob can distinguish both situations and our main results is to provide such an algorithm.

It is not clear whether a proper quantum state can be associated to each qubit leaving Alice’s box. Let us imagine a situation in which Alice has already given Bob several qubits (as many qubits as Bob wanted to request from Alice), and we ask Bob to guess the next qubit — unknown to him—, with the sole promise that Alice, in the limit, will pick as many states from one bag as from the other (i.e. it is a balanced sequence). Since every prefix can be extended to a computable sequence, no matter what Bob already knows about Alice’s preparation, he cannot say anything about the next qubit. For instance, he can already know in what basis Alice is preparing each state (via the presented algorithm), and an extremely long prefix of the computable sequence that Alice is using. Still, he does not know if the next qubit will be $|0\rangle$ or $|1\rangle$ (if Alice prepares in the computational basis). The best description for that single qubit state that Bob can

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**Algorithm 4** The noise tolerant distinguishing protocol

*Input*: $q \in \mathbb{Q}$, $k \in \mathbb{N}$ and $X, Z \in 2^\omega$, one of them being computable

*Output*: ‘X’ or ‘Z’ as the candidate for being computable; wrong answer with probability bounded by $O(2^{-k})$

**for** $t = 0, 1, 2 \ldots$ **do**

**for** $p = 0, \ldots, t$ **do**

- **if** $d_H(U_t(p), X \upharpoonright |kp|) < q k |p|$ **then**
  - output ‘X’ and halt
- **if** $d_H(U_t(p), Z \upharpoonright |kp|) < q k |p|$ **then**
  - output ‘Z’ and halt

where $d_H$ is the Hamming distance between two strings, which counts the number of different bits in both strings. The first thing to notice is that when $q = 0$ Algorithms 3 and 4 coincide.

We need to show now that, again, the success probability can be made as close to one as desired by choosing the parameter $k$ and that the algorithm always halts. Instead of bounding the number of sequences that can be generated with a program of length $\ell$, we need to bound the number of sequences that have a Hamming distance smaller than $q k \ell$ from a computable one. One possible bound is $2^\ell \left( \frac{\ell}{q k} \right)^2 q^k$, where the first exponential term counts the number of different programs of length $\ell$, the combinatorial number corresponds to the number of bits that can be flipped due to errors, and the last exponential term gives which of these bits are actually being flipped. This estimation may not be tight, as we may be counting the same sequence several times. However, using this estimation we derive a good enough upper bound on the final error probability, as we get

$$P_{\text{error}} < \sum_{\ell > 0} 2^{\ell} q^{k \ell} \left( \frac{\ell}{q k} \right)^2 q^k$$

(A4)

If we consider that $q < 1/2$, we can remove the integer part function and use the generalization of combinatorial numbers for real values. Then, by using that $\binom{a}{b} \leq \left( \frac{e a}{b} \right)^b$, we obtain

$$P_{\text{error}} < \sum_{\ell > 0} \left( 2^{1+qk-k} \left( \frac{e}{q} \right)^k \right)^\ell$$

(A5)

This geometric sum can be easily computed yielding

$$P_{\text{error}} < \frac{2^{1+qk-k} \left( \frac{e}{q} \right)^k q^k}{1-2^{1+qk-k} \left( \frac{e}{q} \right)^k q^k}$$

(A6)

Now, it can be shown numerically that for $q \lesssim 0.21$ the probability of mis-recognition tends to zero exponentially with $k$. 

3 and 4 coincide.
give, from the balanced sequence promise, is $\rho = \mathbb{I}/2$.

Interestingly, our results easily extend to other ensembles, and can for instance be applied to the mixed states experimentally produced using a classical random number generator of $[10,11]$. Our classical algorithm is also suitable for performing other seemingly impossible tasks. If Bob is presented with two states, one that is a proper computable mixture of the states $|0\rangle$ and $|1\rangle$ each with equal weight, and the other is an improper mixture yielding the maximally mixed state (for instance, one of the parts of a maximally entangled state), Bob can distinguish which is which by a slight modification of our algorithm. He just obtains two outputs, each by measuring $\sigma_z$ to each state. Again, the problem is reduced to distinguishing, with high probability, a fair coin from a computable sequence, a task that we have already shown how to solve.

Appendix B: The Bell test computability loophole

In this section we discuss how the knowledge that measurement settings are chosen using a device that satisfies the Church-Turing thesis opens a new loophole in Bell tests. We focus our attention on the simplest Bell scenario although generalizations are straightforward: let us consider a bipartite scenario in which the two parties, Alice and Bob, have a box each with two input buttons (left and right) and a binary output. A source between them is sending physical systems sequentially to each party. Upon arrival, Alice and Bob choose what input buttons to press thus performing different measurements on the particles. Our object of interest is the probability distribution describing the process $P(a,b|x,y)$ where $x$ and $y$ are inputs for Alice and Bob’s box respectively and $a$ and $b$ are their outputs which can be derived from the statistics.

Let us also imagine that the input-output events at each site define space-like separated events so that Bob’s input cannot influence Alice’s output and viceversa. Focusing on a particular round of the experiment, let us describe by $\lambda$ a complete set of variables (some of which could be hidden or unknown) that characterize the physical systems such that the outcomes of the measurements are deterministic $P(a,b|x,y,\lambda) = \delta_{f(x,\lambda)}g(y,\lambda)$ where the functions $f(x,\lambda)$ and $g(y,\lambda)$ map deterministically inputs $x,y$ to outputs $a,b$, using the complete description $\lambda$. Different rounds of the experiment may be described with different set of variables $\lambda$ drawing from a probability distribution $p(\lambda|a,b,x,y) = p(\lambda)$ where the equality condition is termed measurement independence or free choice. This crucial condition implies that the complete description $\lambda$ is independent from the choice of settings $x,y$ of Alice and Bob will use to measure the systems. Thus, we say that a probability distribution is local if it can be written as

$$P(a,b|x,y) = \int p(\lambda)\delta_{f(x,\lambda)}g(y,\lambda)\,d\lambda. \quad (B1)$$

It can be shown that any probability distribution that violates the following Bell inequality, namely a CHSH inequality $[20]$, is not a local distribution:

$$\sum_{a,b,x,y \in \{0,1\}} (-1)^{a+b+x+y}P(a,b|x,y) \leq 2. \quad (B2)$$

Remarkably, quantum correlations, obtained for example by measuring a maximally entangled two-qubit state with non-commuting measurements, can violate this inequality. The violation of a Bell inequality witnesses the existence of non-local correlations which in turn can be used in many device-independent applications such as randomness expansion or for establishing a secure key between distant locations. Hence, checking whether the experimental data truly violates a Bell Inequality is of utmost importance for device-independent information science.

In order to present the computability loophole, we introduce an eavesdropper named Eve. Eve will be able to prepare Alice’s and Bob’s local boxes at the beginning of the experiment as shown in Fig. 5. Alice’s box will then have access to the inputs of both parties after the measurements are done (Alice’s input in a straightforward way, and Bob’s input via classical communication). This scenario was first termed the two-sided memory loophole in the literature $[20]$. An equivalent scenario has been used more recently in $[21,22]$ so as to perform device-independent randomness expansion where they allow the boxes behaviour to adapt depending on the information of previous rounds. Interestingly, local models exploiting this past information have been shown to be of no help to violate the Bell Inequality in the asymptotic limit. Indeed, the probability that a local model reproduces some observed violation despite using past inputs and outputs goes exponentially fast to zero in the number of rounds $[22]$. As mentioned earlier, a crucial condition for Bell tests to establish nonlocality and randomness is to assume measurement independence $p(\lambda|x,y) = p(\lambda)$, which in our context is independence between the boxes prepared by Eve and the measurement choices of Alice and Bob. Let us imagine that Alice and Bob chose their measurement settings following an algorithm which is a standard practice in all Bell experiments to date. Trivially, if Alice’s box knows which algorithms Alice and Bob are using, she can fake a Bell violation. We assume that the algorithms used by Alice and Bob are fully unknown to each other and to Eve, thus uncorrelated to the boxes she initially prepared.

Our result is to show that if either Alice or Bob (or both) choose their measurements following an algorithm — or equivalently, because of the Church-Turing thesis, following any classical mechanical procedure —, even under the assumption that such algorithm is fully unknown
to Eve and hence uncorrelated to the boxes she initially prepares, there is an attack that, in the asymptotic limit, produces a Bell inequality violation between Alice and from purely deterministic boxes, thus providing the aforementioned loophole.

![Diagram of Bell inequality setup](image)

**FIG. 5:** Scheme for the Bell inequality computability loophole. After each round $i$, Alice’s box receives the information about Bob’s choice of measurement $y_i$. Using the information from all previous choices of inputs for both parties, Alice’s box makes a prediction on what the next round inputs will be by using the presented algorithm. For simplicity in the notation, the attack is shown for a Bell test involving two measurements per party. As long as Alice, Bob or both use a computable sequence for their choice, the guess will start being correct after a number of rounds. Once this happens, the boxes can simulate any probability distribution, both local and non-local. Therefore Alice and Bob can not rule out an eavesdropper having prepared their boxes.

Before proceeding, we need a few more tools from computer science. We say that a class $C$ of computable functions is a *time [resp. space] complexity class* if there is a computable function $t$ such that each function in $C$ is computed by a Turing machine that, for every input $x$, runs in time [resp. space] $O(t(|x|))$. Examples of such classes include the well-known $\text{P, BQP, NP, PSPACE}$ where the complexity time bound is a simple exponential function and the much broader class $\text{PR}$ of the primitive recursive functions where the time bound is Ackermannian [27] §VIII.8 (see [28] for an inclusion diagram of the most well-studied complexity classes).

Let us now say that one party, say Alice without loss of generality, will be using an algorithm to produce her measurements choices. In formal terms, this means that there is a computable function $f_A : \mathbb{N} \to \{\text{left, right}\}$ such that $f_A(i)$ tells Alice to press the left or the right button at the $i$-th round.

As we previously pointed out, it is clear that if Eve knows (any algorithm for) $f_A$, her task becomes trivial. In our setting, however, function $f_A$ is unknown to Eve when she prepared the boxes. However, we assume the following further hypothesis: Eve knows some time or space bound $t$ of a complexity class containing $f_A$ and $f_B$ (the corresponding function for Bob’s inputs). For instance, Eve knows that Alice and Bob use at most, say, time $O(t(n))$, for $t(n) = 2^n$ (though the algorithms that Alice is actually running may take, say, $O(n^2)$). It is important to note that this hypothesis is quite mild, because every computable function belongs to some time or space complexity class — given a program there is a computable interpreter which executes it on some given input by stages and counts the number of steps that such execution takes to terminate or the number of cells used in the tape. In other words, for every computable function $g$ there is a computable function $t_g$ that upper bounds the running time or space of some algorithm for $g$.

Knowing this time or space bound $t$, Eve can program a computing device in one of the boxes, say Alice’s, to *predict* the functions $f_A$ and $f_B$ from some point onwards. This means that Alice’s box has an effective procedure that, after having seen $f_A(0), f_A(1), \ldots, f_A(k)$ for large enough $k$, allows her to correctly guess $f_A(k+1), f_A(k+2) \ldots$ and the same for $f_B$. The existence of such $k$ will be guaranteed by Alice’s box procedure; however it will not be able to effectively determine when this $k$ has arrived. The idea behind this is that every time bounded class is computably enumerable, allowing Eve to pick, every time, the first program for a function from that class that reproduces the inputs given by Alice (or Bob) so far. Since the function used by Alice (or Bob) belongs to that class, at some point the first program that Alice will find reproducing the inputs given so far will be one which computes the function used by Alice (or Bob), therefore allowing Alice’s box to predict every input to come. See Sec. [B1] for a detail of Alice’s box procedure.

Back to the loophole, under these assumptions, Eve is able to prepare both boxes so as to fake a Bell Inequality violation. Moreover, she could even prepare boxes that seem to be more non-local than what quantum mechanics allows. To see how, notice that any no-signaling bipartite probability distribution, local or not, can always be written as

\[
P(a, b | x, y) = \int p(\lambda) \delta_{f_A(x, \lambda)}^a \delta_{g_B(y, x, \lambda)}^b \, d\lambda \quad (B3)
\]

\[
= \int p'(\lambda) \delta_{f_A'(x, \lambda)}^a \delta_{g_B'(y, x, \lambda)}^b \, d\lambda \quad (B4)
\]

where again functions $f, f', g, g'$ are deterministic functions (See Sec. [B1]a for a prove). This means that, given that Eve learns either Alice’s input $x$ or Bob’s input $y$, she can prepare deterministic (local) boxes to simulate...
any probability distribution and hence fake any Bell In-
equality violation.

1. Predicting computable functions from initial
segments

The theory of predicting computable functions started
with the seminal works by Solomonoff [29, 30] on in-
ductive inference, and Gold [31] on learnability. It stud-
ies the process of coming up with, either explanations
(in the form of computer programs) or next-value pre-
dictions, after seeing some sufficiently big subset of the
graph of a computable function. Many possible formal-
izations, depending on how the data is presented and how
the learning process converges, have been considered in
the literature (see [32] for a comprehensive survey). The
most suitable model for our purposes is called identifica-
tion by next value, and follows by elementary arguments
from computability theory.

A class of total computable functions \( C \) is identifiable
by next value (\( C \in \text{NV} \)) if there exists a computable
function \( g \) (called a next-value function for \( C \)) such that
for every \( f \in C \),

\[
(\exists n_0)(\forall n \geq n_0) \quad f(n) = g((f(0), \ldots, f(n-1))). \tag{B5}
\]

Here \( \langle x_1, \ldots, x_n \rangle \) is any computable codification of an
\( n \)-tuple with a natural number, whose decoding is also
computable. Condition (B5) formalizes the idea that
given the past values of \( f \) (namely \( (f(0), \ldots, f(n-1)) \)),
g can predict the forthcoming value of \( f \) (namely, \( f(n) \)),
provided \( n \) is large enough —how large depends on the
function \( f \) that we want to learn.

It follows from a simple diagonal argument that the
class of all computable functions is not in \( \text{NV} \). However,
any time or space complexity class is \( \text{NV} \). Indeed, sup-
pose \( C \) is a time complexity class with (computable) time
bound \( t \). The following algorithm computes a next-value
function for \( C \). Let \( (M_i)_{i \in \mathbb{N}} \) be an enumeration of all
Turing machines.

\textbf{Algorithm 5} A next-value algorithm for a time class \( C \)
with bound \( t \)

\begin{itemize}
    \item \textbf{Input:} \( n \in \mathbb{N} \)
    \item \textbf{Output:} \( g(n) \), the next-value function for \( C \).
\end{itemize}

Let \( n = \langle m_0, \ldots, m_{n-1} \rangle \).
Let \((e,c)\) be the least number such that
\begin{itemize}
    \item for \( i \in \{0, \ldots, n\} \), \( M_e(i) \) halts after at most \( c \cdot t(|i|) \)
    \item for \( i \in \{0, \ldots, n-1\} \) \( M_e(n) \) outputs \( m_i \)
\end{itemize}

Output \( M_e(n) \)

Suppose \( f \in C \), i.e. there is some Turing machine \( M_{e'} \)
and constant \( c' \) such that for every \( x \in \mathbb{N} \),

\[
M_{e'}(x) \text{ computes } f(x) \text{ with time bound } c' \cdot t(|x|).
\tag{B6}
\]

Both \( e' \) and \( c' \) are unknown, and the idea of Algo-
rithm 5 is to try different candidates \( e \) and \( c \) for \( e' \)
and \( c' \) respectively, until one is found. On input \( n =
\langle f(0), \ldots, f(n-1) \rangle \) the algorithm proposes a candidate
Turing machine \( M_e \) which ‘looks like’ \( f \) on \( 0, \ldots, n-1 \),
and then guesses that \( f(n) \) is the value computed by
\( M_e(n) \). To be a candidate means not only to compute
the same first \( n \) values, but also to do it within the time
bound imposed by \( C \), which is \( c \cdot t \). Of course, the
chosen candidate may be incorrect because, for instance,
on input \( \langle f(0), \ldots, f(n-1), f(n) \rangle \) we may realize that
\( f(n) \) was not equal to \( M_e(n) \). In this case, the algorithm
changes its mind and proposes as candidate a new pair
\( (e,c) \). The existence of the correct candidates \( e' \) and \( c' \)
satisfying (B6) guarantees that:

1. For each \( n \) and each input \( \langle f(0), \ldots, f(n-1) \rangle \) the
algorithm will find some \( \langle e, c \rangle \) meeting conditions i and ii.

2. Along the initial segments \( \langle f(0), \ldots, f(n-1) \rangle \) for
larger and larger \( n \) there can only be finitely many
mind changes. Indeed, if the number of mind
changes were infinite, then \( \langle e', c' \rangle \) would be ruled
out and this is impossible, as conditions i and ii are
true for \( e = e' \) and \( c = c' \).

Hence there is \( n_0 \) such that for all \( n \geq n_0 \), the
algorithm makes no more mind changes, and it stabilizes
with values \( (e,c) \), which may not necessarily be equal to
\( (e',c') \), but will satisfy that \( M_e(x) \) computes \( f(x) \) with
time bound \( c \cdot t(|x|) \). Thus on input \( \langle f(0), \ldots, f(n-1) \rangle \)
the algorithm will return \( f(n) \), and hence (B6) will be
satisfied. Observe that although the algorithm starts
correctly predicting \( f \) from one point onwards, it can-
don not detect when this begin to happen. In other words,
\( n_0 \) is not uniformly computable from \( e' \) and \( c' \).

The algorithm for a space complexity class with bound
\( t \) is analogous, but condition i must be modified to

i'. for \( i \in \{0, \ldots, n\} \), \( M_e(i) \) halts after at most \( 2^t(i) \)
many steps and uses at most \( t(|i|) \) many cells of the
work tape during its computation.

Observe that any halting computation which consumes
\( t(n) \) many cells of the work tape runs for at most \( 2^t(n) \)
many steps, as this is the total number of possible mem-
ory configurations. In condition i’ we add the statement
on the number of steps in order to avoid those computa-
tions which use at most \( t(|i|) \) many cells but are non-
terminating.

\begin{itemize}
    \item a. Simulation of no-signaling correlations from
deterministic boxes
\end{itemize}

For completeness, we give a simple proof of the well-
known fact that, if the input of one party in a Bell test is
known, one can simulate any no-signaling distribution by
using deterministic boxes. Let us imagine without loss of
generality that it is Bob’s box the one that has access to Alice’s input $x$. First, notice that any no-signaling box can be written in the following way

$$P(a, b|x, y) = P(a|x)P(b|y; a, x) \equiv P(a|x)P_{b|x}(b|y).$$  

(B7)

Trivially, any local distribution can be simulated through deterministic boxes as $P(a|x) = \int p(\lambda)\delta_{f(x, \lambda)} d\lambda$. Hence, the no-signaling bipartite distribution can be written as

$$P(a, b|x, y) = \int \int p(\lambda)p'(\lambda')\delta_{f(x, \lambda)}\delta_{g(y, \lambda', a, x)} d\lambda d\lambda'$$

by defining now $\lambda'' = (\lambda, \lambda')$ and therefore $d\lambda'' = d\lambda d\lambda'$ and $p'(\lambda') = p(\lambda)p'(\lambda')$ we have

$$P(a, b|x, y) = \int p'(\lambda)\delta_{f(x, \lambda'')}\delta_{g(y, \lambda'', a, x)} d\lambda''.$$  

(B9)

Notice that since $a$ is a deterministic function of $x$ and $\lambda$, given that $\lambda''$ includes the information of $\lambda$, the function on Bob’s side does not need to depend explicitly on $a$.

2. Discussion

We have shown that if either Alice or Bob choose their inputs for a Bell experiment in a computable way, an eavesdropper restricted to prepare deterministic devices can make them believe to have non-local boxes, thus creating a computability loophole. Notice that this scenario is equivalent to letting the boxes communicate before the runs and adapt accordingly, as is the case in the randomness expansion protocols [21, 22] where our loophole would also apply if either Alice or Bob would use pseudo-randomness. There is no way of preventing this form of communication, unless some assumptions on shielding of the devices are enforced.

From a fundamental perspective, our result answers the question about what type of randomness is necessary for having a valid violation of a Bell inequality: Alice and Bob’s behaviour need to be non-algorithmic. Therefore no computable pseudo randomness criterion will suffice for a proper Bell inequality violation. It is natural to ask, at this point, where can Alice and Bob find sources of true randomness for their inputs. If they assume quantum mechanics, then flipping a quantum coin would suffice (with probability one). However, it is not desirable to assume a non-local theory like quantum mechanics, in order to test non-locality. Ruling out quantum mechanics, the other source of true randomness that is usually mentioned is free will. However, there is no present evidence that the human brain is able to produce non-algorithmic randomness.

[1] Niels Bohr. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, 48:696–702, 1935.

[2] A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, 47:777–780, 1935.

[3] Lluis Masanes and Markus P. Mueller. A derivation of quantum theory from physical requirements. *New Journal of Physics*, 13(6):063001+, May 2011.

[4] Roger Colbeck and Renato Renner. No extension of quantum theory can have improved predictive power. *Nat Commun.*, 2:411–, August 2011.

[5] Matthew F. Pusey, Jonathan Barrett, and Terry Rudolph. On the reality of the quantum state. *Nature Physics*, 8(6):476–479, May 2012.

[6] Alan M. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, Series 2, 42:230–265, 1936.

[7] Alonzo Church. An unsolvable problem of elementary number theory. *The American Journal Of Mathematics*, 58:345–363, 1936.

[8] Stephen Cole Kleene. *Mathematical Logic*, volume 18. Dover Publications, 1967.

[9] Pablo Arrighi and Gilles Dowek. The physical Church-Turing thesis and the principles of quantum theory. *Int. J. Found. Comput. Sci.*, 23(5):1131–1146, 2012.

[10] Elias Amselem and Mohamed Bourennane. Experimental four-qubit bound entanglement. *Nature Physics*, 5(10):748–752, 2009.

[11] Jonathan Lavoie, Rainer Kaltenbaek, Marco Piani, and Kevin J. Resch. Experimental bound entanglement in a four-photon state. *Phys. Rev. Lett.*, 105:130501, Sep 2010.

[12] John Conway and Simon Kochen. The free will theorem, 2006.

[13] D. E. Koh, M. J. W. Hall, Setiawan, J. E. Pope, C. Marletto, A. Kay, V. Scarani, and A. Ekert. The effects of reduced ”free will” on Bell-based randomness expansion, 2012.

[14] Michael J. W. Hall. Local deterministic model of singlet state correlations based on relaxing measurement independence. *Phys. Rev. Lett.*, 105(25):250404–, December 2010.

[15] J. Barrett and N. Gisin. How much measurement independence is needed in order to demonstrate nonlocality?, 2010.

[16] M. A. Nielsen and I. L. Chuang. *Quantum computation*
and quantum information. Cambridge University Press, Cambridge, 2000.

[17] Santiago Figueira and André Nies. Feasible analysis, randomness and base invariance. Theory of Computing Systems, 2013. To appear.

[18] John Bell. On the Einstein Podolsky Rosen paradox. Physics, 1:195–200, 1964.

[19] John Bell. On the problem of hidden variables in quantum mechanics. Rev. Mod. Phys., 38:447, 1966.

[20] Jonathan Barrett, Daniel Collins, Lucien Hardy, Adrian Kent, and Sandu Popescu. Quantum nonlocality, Bell inequalities, and the memory loophole. Phys. Rev. A, 66:042111, Oct 2002.

[21] Stefano Pironio, Antonio Acín, Serge Massar, A Boyer de La Giroday, Dzmitry N Matsukevich, Peter Maunz, Steven Olmschenk, David Hayes, Le Luo, T Andrew Manning, et al. Random numbers certified by Bell’s theorem. Nature, 464(7291):1021–1024, 2010.

[22] Stefano Pironio and Serge Massar. Security of practical private randomness generation. Phys. Rev. A, 87:012336, Jan 2013.

[23] Ming Li and Paul Vitanyi. An Introduction to Kolmogorov Complexity and Its Applications. Springer Verlag, 2008.

[24] Rod Downey and Denis Hirschfeldt. Algorithmic Randomness and Complexity. Theory and Applications of Computability Series. Springer, 2010.

[25] Robert I. Soare. Recursively enumerable sets and degrees. Springer, Heidelberg, 1987.

[26] John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt. Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett., 23(15):880–884, October 1969.

[27] P. Odifreddi. Classical recursion theory, volume 2 of Studies in logic and the foundations of mathematics. Elsevier, 1999.

[28] Scott Aaronson et.al. Complexity zoo active inclusion diagram. https://www.math.ucdavis.edu/~greg/zooology/diagram.xml

[29] Ray J Solomonoff. A formal theory of inductive inference. Part I. Information and control, 7(1):1–22, 1964.

[30] Ray J Solomonoff. A formal theory of inductive inference. Part II. Information and control, 7(2):224–254, 1964.

[31] E Mark Gold. Language identification in the limit. Information and control, 10(5):447–474, 1967.

[32] Thomas Zeugmann and Sandra Zilles. Learning recursive functions: A survey. Theoretical Computer Science, 397(1):4–56, 2008.

[33] J.M. Barzdin. Prognostication of automata and functions. In Information Processing ’71, pages 81–84, 1971.