CANCELLATION THEOREM FOR FRAMED MOTIVES OF ALGEBRAIC VARIETIES

A. ANANYEVSKIY, G. GARKUSHA, AND I. PANIN

Abstract. The machinery of framed (pre)sheaves was developed by Voevodsky [V1]. Based on the theory, framed motives of algebraic varieties are introduced and studied in [GP1]. An analog of Voevodsky’s Cancellation Theorem [V2] is proved in this paper for framed motives stating that a natural map of framed $S^1$-spectra

$$M_{fr}(X)(n) \to \text{Hom}(G, M_{fr}(X)(n+1)), \quad n \geq 0,$$

is a Nisnevich local stable equivalence, where $M_{fr}(X)(n)$ is the $n$th twisted framed motive of $X$. This result is reduced to the Cancellation Theorem for linear framed motives stating that the natural map of complexes of abelian groups

$$\mathbb{ZF}((\Delta^* \times X, Y) \to \mathbb{ZF}((\Delta^* \times X) \wedge (G_m, 1), Y \wedge (G_m, 1))), \quad X, Y \in \text{Sm}/k,$$

is a quasi-isomorphism, where $\mathbb{ZF}(X, Y)$ is the group of stable linear framed correspondences in the sense of [GP1][GP3].

1. Introduction

In [V1] Voevodsky developed the machinery of framed correspondences and framed (pre)sheaves. Basing on this machinery, the theory of framed motives of algebraic varieties was introduced and studied in [GP1]. The framed motive of $X \in \text{Sm}/k$ is an explicitly constructed framed $S^1$-spectrum $M_{fr}(X)$, which is connected and an $\Omega$-spectrum in positive degrees (see [GP1] for details). Moreover, the shifts of the $M_{fr}(X)$-s, $X \in \text{Sm}/k$, are compact generators of the associated compactly generated triangulated category of framed $S^1$-spectra $SH_{S^1}^{fr}(k)$. The category $SH_{S^1}^{fr}(k)$ is the homotopy category of the category of framed $S^1$-spectra $Sp_{S^1}^{fr}(k)$ with respect to the stable motivic model structure (see [GP1] for details).

The main object of [GP1] is the bispectrum

$$M_{fr}(X) = (M_{fr}(X), M_{fr}(X)(1), M_{fr}(X)(2), \ldots),$$

each term of which is a twisted framed motive of $X$ and explicitly constructed structure maps

$$M_{fr}(X)(n) \to \text{Hom}(G, M_{fr}(X)(n+1)), \quad n \geq 0.$$

Here $G = Cyl(t)/(-, pt)_+$ with $Cyl(t)$ the mapping cylinder for the map $t : (-, pt)_+ \to (-, \mathbb{G}_m)_+$ sending $pt$ to $1 \in \mathbb{G}_m$. The shifts of the $M_{fr}(X)$-s, $X \in \text{Sm}/k$, are compact generators of the associated compactly generated triangulated category of framed $(S^1, G)$-bispectra $SH_{S^1, G}^{fr}(k)$. The category $SH_{S^1, G}^{fr}(k)$ is the homotopy category of the category of framed $(S^1, G)$-bispectra $Sp_{S^1, G}^{fr}(k)$ with respect to the stable motivic model structure (see [GP1] for details).

The main purpose of the paper is to prove the following (cf. Voevodsky [V2])

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Theorem A (Cancellation). Let $k$ be an infinite perfect field, $X ∈ Sm/k$ and $n ≥ 0$. Then the following statements are true:

1. the natural map of framed $S^1$-spectra

$$M_{fr}(X)(n) → \text{Hom}(\mathbb{G}, M_{fr}(X)(n + 1))$$

is a Nisnevich local stable equivalence;

2. the induced map of framed $S^1$-spectra

$$M_{fr}(X)(n)_f → \text{Hom}(\mathbb{G}, M_{fr}(X)(n + 1)_f)$$

is a schemewise stable equivalence with $M_{fr}(X)(n)_f$ and $M_{fr}(X)(n + 1)_f$ being framed Nisnevich local fibrant replacements $M_{fr}(X)(n)$ and $M_{fr}(X)(n + 1)$ respectively.

As an application of Theorem A we prove the following

Theorem B. Let $k$ be an infinite perfect field, $X ∈ Sm/k$ and $n ≥ 0$. Then the bispectrum

$$M^{fr}_X(X)_f = (M_{fr}(X)_f, M_{fr}(X)(1)_f, M_{fr}(X)(2)_f, \ldots)$$

obtained from $M^{fr}_X(X)$ by taking levelwise framed Nisnevich local fibrant replacements with structure maps those of Theorem A(2) is a motivically fibrant $(S^1, \mathbb{G})$-bispectrum.

The motivic model category of framed $S^1$-spectra $Sp^{fr}_{S^1}(k)$ has a natural Quillen pair of adjoint functors

$$- ⊗ \mathbb{G} : Sp^{fr}_{S^1}(k) \rightleftarrows Sp^{G}_{S^1}(k) : \text{Hom}(\mathbb{G}, -)$$

(see [GP1] for details). This Quillen pair induces adjoint functors on the homotopy category

$$- ⊗^L \mathbb{G} : SH^{fr}_{S^1}(k) \rightleftarrows SH^{fr}_{S^1}(k) : \text{RHom}(\mathbb{G}, -).$$

The functor $- ⊗^L \mathbb{G}$ is also referred to as the twist functor. We also prove that the twist functor on $SH^{fr}_{S^1}(k)$ is fully faithful. More precisely, the following theorem is true.

Theorem C. Let $k$ be an infinite perfect field. Then the functor

$$- ⊗^L \mathbb{G} : SH^{fr}_{S^1}(k) → SH^{fr}_{S^1}(k)$$

is full and faithful.

The main strategy of proving Theorem A is to reduce it to the “Linear Cancellation Theorem”. In order to formulate it, recall from [GP1, GP3] that the category $ZF_n(k)$ is an additive category whose objects are those of $Sm/k$ and $Hom$-groups are defined as follows. We set for every $n ≥ 0$ and $X, Y ∈ Sm/k$,

$$ZF_n(X, Y) := \mathbb{Z}(\text{Fr}_n(X, Y)/\langle Z_1 ∪ Z_2 - Z_1 - Z_2 \rangle),$$

where $Z_1, Z_2$ are supports of (level $n$) framed correspondences in the sense of Voevodsky [VI]. In other words, $ZF_n(X, Y)$ is a free abelian group generated by the framed correspondences of level $n$ with connected supports. We then set

$$\text{Hom}_{ZF_n}(X, Y) := \bigoplus_{n≥0} ZF_n(X, Y).$$

Given smooth varieties $X, Y ∈ Sm/k$ and $n ≥ 0$, there is a canonical suspension morphism $Σ : ZF_n(X, Y) → ZF_{n+1}(X, Y)$. We can stabilize in the $Σ$-direction to get an abelian group (see Definition 2.5)

$$ZF(X, Y) := \text{colim}(ZF_0(k)(X, Y) ⊗ Σ → ZF_1(k)(X, Y) ⊗ Σ → \cdots).$$
There is a canonical morphism (see Definition 2.8 for more details), functorial in both arguments,
\[-\Box (id_{\mathbb{G}_m} - e_1) : \mathbb{Z}F(X, Y) \to \mathbb{Z}F(X \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))\]
with \((\mathbb{G}_m, 1)\) the scheme \(\mathbb{A}^1 - \{0\}\) pointed at 1.

The Linear Cancellation Theorem is formulated as follows.

**Theorem D (Linear Cancellation).** Let \(k\) be an infinite perfect field and let \(X\) and \(Y\) be \(k\)-smooth schemes. Then
\[-\Box (id_{\mathbb{G}_m} - e_1) : \mathbb{Z}F(\Delta^* \times X, Y) \to \mathbb{Z}F((\Delta^* \times X) \wedge (\mathbb{G}_m, 1), Y \wedge (\mathbb{G}_m, 1))\]
is a quasi-isomorphism of complexes of abelian groups.

One of the main computational results of [GP3] says that homology of the complex \(\mathbb{Z}F(\Delta^* \times - , Y)\) locally computes homology of the framed motive \(M_{fr}(Y)\) of \(Y \in Sm/k\). Moreover, the complex represents the “linear framed motive” of \(Y\) (see [GP3] for details).

Throughout the paper the base field \(k\) is infinite and perfect and \(Sm/k\) is the category of smooth separated schemes of finite type over the field \(k\).

### 2. Preliminaries

In this section we collect basic facts for framed correspondences. We start with preparations. Let \(V\) be a scheme and \(Z\) be a closed subscheme. Recall that an \(\acute{e}tale\) neighborhood of \(Z\) in \(V\) is a triple \((W', \pi' : W' \to V, s' : Z \to W')\) satisfying the following conditions:
(i) \(\pi'\) is an \(\acute{e}tale\) morphism;
(ii) \(\pi' \circ s'\) coincides with the inclusion \(Z \hookrightarrow V\) (thus \(s'\) is a closed embedding).

A morphism between two \(\acute{e}tale\) neighborhoods \((W', \pi', s') \to (W'', \pi'', s'')\) of \(Z\) in \(V\) is a morphism \(\rho : W' \to W''\) such that \(\pi'' \circ \rho = \pi'\) and \(\rho \circ s' = s''\). Note that such \(\rho\) is automatically \(\acute{e}tale\) by [EGA4 VI.4.7].

**Definition 2.1 (Voevodsky [V1]).** (I) Let \(Z\) be a closed subset in \(X\). A framing of \(Z\) of level \(n\) is a collection \(\varphi_1, \ldots, \varphi_n\) of regular functions on \(X\) such that \(\cap_{i=1}^n \{ \varphi_i = 0 \} = Z\). For a scheme \(X\) over \(S\) a framing of \(Z\) over \(S\) is a framing of \(Z\) such that the closed subsets \(\{ \varphi_i = 0 \}\) do not contain the generic points of the fibers of \(X \to S\).

(II) For \(k\)-smooth schemes \(X, Y\) over \(S\) and \(n \geq 0\) an explicit framed correspondence \(\Phi\) of level \(n\) consists of the following data:

1. a closed subset \(Z\) in \(\mathbb{A}^n_X\) which is finite over \(X\);
2. an \(\acute{e}tale\) neighborhood \(p : U \to \mathbb{A}^n_X\) of \(Z\);
3. a framing \(\varphi_1, \ldots, \varphi_n\) of level \(n\) of \(Z\) in \(U\) over \(X\);
4. a morphism \(g : U \to Y\).

The subset \(Z\) will be referred to as the support of the correspondence. We shall also write triples \(\Phi = (U, \varphi, g)\) or quadruples \(\Phi = (Z, U, \varphi, g)\) to denote explicit framed correspondences.

(III) Two explicit framed correspondences \(\Phi\) and \(\Phi'\) of level \(n\) are said to be equivalent if they have the same support and there exists an \(\acute{e}tale\) neighborhood \(V\) of \(Z\) in \(U \times \mathbb{A}^n_X\) \(U'\) such that on \(V\), the morphism \(g \circ pr\) agrees with \(g' \circ pr'\) and \(\varphi \circ pr\) agrees with \(\varphi' \circ pr'\). A framed correspondence of level \(n\) is an equivalence class of explicit framed correspondences of level \(n\).

We let \(Fr_n(X, Y)\) denote the set of framed correspondences from \(X\) to \(Y\). We consider it as a pointed set with the distinguished point being the class \(0_n\) of the explicit correspondence with \(U = \emptyset\).
As an example, the set Fr₀(X, Y) coincides with the set of pointed morphisms X_+ → Y_+. In particular, for a connected scheme X one has

$$\text{Fr}_0(X, Y) = \text{Hom}_{\text{Sch}}(X, Y) \sqcup \{0\}.$$  

If \( f : X' \to X \) is a morphism of schemes and \( \Phi = (U, \varphi, g) \) an explicit correspondence from \( X \) to \( Y \) then

$$f^*(\Phi) := (U' = U \times_X X', \varphi \circ pr, g \circ pr)$$

is an explicit correspondence from \( X' \) to \( Y \).

From now on we shall only work with framed correspondences over smooth \( k \)-schemes \( S_m/k \).

**Remark 2.2.** Let \( \Phi = (Z, \mathbb{A}^n_X \to^F U, \varphi : U \to \mathbb{A}^n_X, g : U \to Y) \in \text{Fr}_n(X, Y) \) be an explicit framed correspondence of level \( n \). It can more precisely be written in the form

$$((\alpha_1, \alpha_2, \ldots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_n), g) \in \text{Fr}_n(X, Y)$$

where

- \( Z \subset \mathbb{A}^n_X \) is a closed subset finite over \( X \),
- an etale neighborhood \( (\alpha_1, \alpha_2, \ldots, \alpha_n, f) = p : U \to \mathbb{A}^n_X \times X \) of \( Z \),
- a framing \( (\varphi_1, \varphi_2, \ldots, \varphi_n) = \phi : U \to \mathbb{A}^n_X \) of level \( n \) of \( Z \) in \( U \) over \( X \),
- a morphism \( g : U \to Y \).

We shall usually drop \( ((\alpha_1, \alpha_2, \ldots, \alpha_n), f) \) from notation and just write

$$(Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_n), g) = ((\alpha_1, \alpha_2, \ldots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_n), g).$$

The following definition is to describe compositions of framed correspondences.

**Definition 2.3.** Let \( X, Y \) and \( S \) be \( k \)-smooth schemes and let

$$a = ((\alpha_1, \alpha_2, \ldots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_n), g)$$

be an explicit correspondence of level \( n \) from \( X \) to \( Y \) and let

$$b = ((\beta_1, \beta_2, \ldots, \beta_m), f', Z', U', (\psi_1, \psi_2, \ldots, \psi_m), g') \in \text{Fr}_m(Y, S)$$

be an explicit correspondence of level \( m \) from \( Y \) to \( S \). We define their composition as an explicit correspondence of level \( n + m \) from \( X \) to \( S \) by

$$((\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m), f, Z \times_Y Z', U \times_Y U', (\varphi_1, \varphi_2, \ldots, \varphi_n, \psi_1, \psi_2, \ldots, \psi_m), g').$$

Clearly, the composition of explicit correspondences respects the equivalence relation on them and defines associative maps

$$\text{Fr}_n(X, Y) \times \text{Fr}_m(Y, S) \to \text{Fr}_{n+m}(X, S).$$

Given \( X, Y \in S_m/k \), denote by \( \text{Fr}_n(X, Y) \) the set \( \bigsqcup_n \text{Fr}_n(X, Y) \). The composition of framed correspondences defined above gives a category \( \text{Fr}_s(k) \). Its objects are those of \( S_m/k \) and the morphisms are given by the sets \( \text{Fr}_s(X, Y), X, Y \in S_m/k \). Since the naive morphisms of schemes can be identified with certain framed correspondences of level zero, we get a canonical functor

$$S_m/k \to \text{Fr}_s(k).$$

One can easily see that for a framed correspondence \( \Phi : X \to Y \) and a morphism \( f : X' \to X \), one has \( f^*(\Phi) = \Phi \circ f \).
Definition 2.4. Let $X, Y, S$ and $T$ be smooth schemes. There is an external product

$$\text{Fr}_n(X, Y) \times \text{Fr}_m(S, T) \xrightarrow{\circledast} \text{Fr}_{n+m}(X \times S, Y \times T)$$

given by

$$((\alpha_1, \alpha_2, \ldots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_n), g) \circledast ((\beta_1, \beta_2, \ldots, \beta_m), f', Z', U', (\psi_1, \psi_2, \ldots, \psi_m), g') =
((\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m), f \times f', Z \times Z', U \times U', (\varphi_1, \varphi_2, \ldots, \varphi_n, \psi_1, \psi_2, \ldots, \psi_m), g \times g').$$

For the constant morphism $c: \mathbb{A}^1 \to \text{pt}$, we set (following Voevodsky [V1])

$$\Sigma = - \boxtimes (t, c, \{0\}, \mathbb{A}^1, t, c): \text{Fr}_n(X, Y) \to \text{Fr}_{n+1}(X, Y)$$

and refer to it as the suspension.

Also, following Voevodsky [V1], one puts

$$\text{Fr}(X, Y) = \colim(\text{Fr}_0(X, Y) \xrightarrow{\circledast} \text{Fr}_1(X, Y) \xrightarrow{\circledast} \ldots \xrightarrow{\circledast} \text{Fr}_n(X, Y) \xrightarrow{\circledast} \ldots)$$

and refer to it as the set stable framed correspondences. The above external product induces external products

$$\text{Fr}_n(X, Y) \times \text{Fr}(S, T) \xrightarrow{\circledast} \text{Fr}(X \times S, Y \times T),$$
$$\text{Fr}(X, Y) \times \text{Fr}_0(S, T) \xrightarrow{\circledast} \text{Fr}(X \times S, Y \times T).$$

Recall now the definition of the category of linear framed correspondences $\mathcal{ZF}_n(k)$.

Definition 2.5. (see [GP1] p. 23) Let $X$ and $Y$ be smooth schemes. Denote by

1. $\mathcal{ZF}_n(X, Y) := \mathbb{Z}[\text{Fr}_n(X, Y)] = \mathbb{Z}[\text{Fr}_n(X, Y)]/\mathbb{Z} \cdot 0_n$, i.e the free abelian group generated by the set $\text{Fr}_n(X, Y)$ modulo $\mathbb{Z} \cdot 0_n$;
2. $\mathcal{ZF}_n(X, Y) := \mathcal{ZF}_n(X, Y)/A$, where $A$ is a subgroup generated by the elements

$$(Z \sqcup Z', U, (\varphi_1, \varphi_2, \ldots, \varphi_n), g) -$$
$$-(Z, U \setminus Z', (\varphi_1, \varphi_2, \ldots, \varphi_n)|_{U \setminus Z'}, g|_{U \setminus Z'}) - (Z', U \setminus Z, (\varphi_1, \varphi_2, \ldots, \varphi_n)|_{U \setminus Z}, g|_{U \setminus Z}).$$

We shall also refer to the latter relation as the additivity property for supports. In other words, it says that a framed correspondence in $\mathcal{ZF}_n(X, Y)$ whose support is a disjoint union $Z \sqcup Z'$ equals the sum of the framed correspondences with supports $Z$ and $Z'$ respectively. Note that $\mathcal{ZF}_n(X, Y)$ is $\mathbb{Z}[\text{Fr}_n(X, Y)]$ modulo the subgroup generated by the elements as above, because $0_n = 0_n + 0_n$ in this quotient group, hence $0_n$ equals zero. Indeed, it is enough to observe that the support of $0_n$ equals $\emptyset \sqcup \emptyset$ and then apply the above relation to this support.

The elements of $\mathcal{ZF}_n(X, Y)$ are called linear framed correspondences of level $n$ or just linear framed correspondences.

Denote by $\mathcal{ZF}_n(k)$ an additive category whose objects are those of $\text{Sm}/k$ with Hom-groups defined as

$$\text{Hom}_{\mathcal{ZF}_n(k)}(X, Y) = \bigoplus_{n \geq 0} \mathcal{ZF}_n(X, Y).$$

The composition is induced by the composition in the category $\text{Fr}_n(k)$.

There is a functor $\text{Sm}/k \to \mathcal{ZF}_n(k)$ which is the identity on objects and which takes a regular morphism $f: X \to Y$ to the linear framed correspondence $1 \cdot (X, X \times \mathbb{A}^1, pr_{X^0}, f \circ pr_X) \in \mathcal{ZF}_0(k)$. 

5
Definition 2.6. Let $X, Y, S$ and $T$ be schemes. The external product from Definition 2.4 induces a unique external product

$$ZF_n(X,Y) \times ZF_m(S,T) \xrightarrow{\boxtimes} ZF_{n+m}(X \times S, Y \times T)$$

such that for any elements $a \in Fr_n(X,Y)$ and $b \in Fr_m(S,T)$ one has $1 \cdot a \boxtimes 1 \cdot b = 1 \cdot (a \boxtimes b) \in ZF_{n+m}(X \times S, Y \times T)$.

For the constant morphism $c: \mathbb{A}^1 \to \text{pt}$, we set

$$\Sigma := \boxtimes 1 \cdot (t,c,(0),\mathbb{A}^1,t,c): ZF_n(X,Y) \to ZF_{n+1}(X,Y)$$

and refer to it as the suspension.

Definition 2.7. For any $k$-smooth variety $X$ there is a presheaf $ZF_n(\cdot,Y)$ on the category $ZF_n(k)$ represented by $Y$. The main $ZF_n(k)$-presheaf of this paper we are interested in is defined as

$$ZF(\cdot,Y) = \text{colim}(ZF_0(\cdot,Y) \xrightarrow{\Sigma} ZF_1(\cdot,Y) \xrightarrow{\Sigma} \ldots \xrightarrow{\Sigma} ZF_n(\cdot,Y) \xrightarrow{\Sigma} \ldots).$$

For a $k$-smooth variety $X$, elements of $ZF(X,Y)$ are also called stable linear framed correspondences. Stable linear framed correspondences do not form morphisms of a category.

The main $ZF_n(k)$-presheaf of simplicial abelian groups we are interested in is defined as $ZF(\Delta^* \times X \times \cdot, Y)$.

Definition 2.8. Let $X$ and $Y$ be $k$-smooth schemes and let $(S,s)$ and $(S',s')$ be $k$-smooth pointed schemes.

- Denote by $e_S: S \to \text{pt} \xrightarrow{\mathsf{pt}} S$ the idempotent given by the composition of the constant map and the embedding of $s$ into $S$.
- Define $ZF(X \wedge (S,s), Y \wedge (S,s))$ as a subgroup of $ZF(X \times S, Y \times S)$ consisting of all $a$ such that $a \circ (\mathsf{id}_X \times e_S) = (\mathsf{id}_Y \times e_S) \circ a = 0$. Note that these equalities are equivalent to $a \circ (\mathsf{id}_X \boxtimes (\mathsf{id}_S - e_S)) = (\mathsf{id}_Y \boxtimes (\mathsf{id}_S - e_S)) \circ a = a$.
- Define $ZF(X \wedge (S,s) \wedge (S',s'), Y \wedge (S,s) \wedge (S',s'))$ as a subgroup of $ZF(X \times S \times S', Y \times S \times S')$ consisting of all $a$ such that $a \circ (\mathsf{id}_X \boxtimes (\mathsf{id}_S - e_S) \boxtimes (\mathsf{id}_{S'} - e_{S'})) = (\mathsf{id}_Y \boxtimes (\mathsf{id}_S - e_S) \boxtimes (\mathsf{id}_{S'} - e_{S'})) \circ a = a$.

We should mention that the preceding definition is necessary for the formulation of Theorem D (“Linear Cancellation”).

Theorem D. Let $X$ and $Y$ be $k$-smooth schemes. Then

$$- \boxtimes (\mathsf{id}_{G_m} - e_1): ZF(\Delta^* \times X,Y) \to ZF((\Delta^* \times X) \wedge (G_m,1), (Y \wedge (G_m,1)))$$

is a quasi-isomorphism of complexes of abelian groups.

The theorem will be proved in Section 7.

3. THEOREM A, THEOREM B AND THEOREM C

Before proving Theorem A we recall some definitions and constructions for framed motives. We adhere to [GP1].

Recall that the category $\mathcal{S}Pre^fr_1(k)$ of pointed simplicial framed presheaves consists of contravariant functors $\mathcal{F}$ from $Fr_1(k)$ to pointed simplicial sets $\mathcal{S}$ such that $\mathcal{F}(\emptyset) = \mathsf{pt}$. The category of $S^1$-spectra associated with $sPre^fr_1(k)$ is denoted by $Sp^fr_1(k)$. It has a stable motivic
As a pointed motivic space $Nisnevich local equivalence in \[GP1\] every motivic fibrant replacement $X$ of the Theorem of $\[GP1\]$ the special

Moreover, $M$ is the Segal projective model category structure whose homotopy category is denoted by $\Phi : SP_{S^1}(k) \rightleftarrows SP_{S^1}^{fr}(k) : \Psi$, where $SP_{S^1}(k)$ is the category of presheaves of $S^1$-spectra equipped with the stable projective motivic model structure. The Quillen pair induces an adjoint pair of triangulated functors $\Phi : SH_{S^1}(k) \rightleftarrows SH_{S^1}^{fr}(k) : \Psi$

between triangulated categories.

Given a finite pointed set $(K, *)$ and a scheme $X$, we denote by $X \times K$ the unpointed scheme $X \sqcup \ldots \sqcup X$, where the coproduct is indexed by the non-based elements in $K$. By the Additivity Theorem of $[GP1]$ the $\Gamma$-space in the sense of Segal $[Seg]$

\[ K \in \Gamma^{op} \mapsto C_\ast Fr(U, X \times K) := Fr(U \times \Delta^\ast, X \times K) \]

is special.

Definition 3.1 (see $[GP1]$). The framed motive $M_{fr}(X)$ of a smooth algebraic variety $X \in Sm/k$ is the Segal $S^1$-spectrum $(C, Fr(-, X), C, Fr(-, X \times S^1), C, Fr(-, X \times S^2), \ldots)$ associated with the special $\Gamma$-space $K \in \Gamma^{op} \mapsto C_\ast Fr(-, X \times K)$.

The framed motive $M_{fr}(X) \in SP_{S^1}^{fr}(k)$ is functorial in framed correspondences of level zero. Moreover, $\{M_{fr}(X)\}_{X \in Sm/k}$ are compact generators of $SH_{S^1}^{fr}(k)$. By the Resolution Theorem of $[GP1]$ every motivic fibrant replacement $M_{fr}(X) \rightarrow M_{fr}(X)_f$ of $M_{fr}(X)$ in $SP_{S^1}^{fr}(k)$ is a stable Nisnevich local equivalence in $SP_{S^1}(k)$ (over perfect fields).

Denote by $G$ the pointed simplicial presheaf which is termwise

\[ (\ast, G_m)_+, (\ast, G_m)_+ \vee (\ast, pt)_+, (\ast, G_m)_+ \vee (\ast, pt)_+ \vee (\ast, pt)_+, \ldots \]

As a pointed motivic space $G$ is $Cyl(t)/(-, pt)_+$ with $Cyl(t)$ the mapping cylinder for the map $t : (\ast, pt)_+ \rightarrow (\ast, G_m)_+$ sending $pt$ to $1 \in G_m$. By $G_m^1$ we mean the simplicial object in $Fr_0(k)$ which is termwise

\[ G_m, G_m \cup pt, G_m \cup pt \cup pt, \ldots \]

Applying $M_{fr}(X \ast -)$ to $G_m^1$ and realizing by taking diagonals, one gets a framed $S^1$-spectrum $M_{fr}(X \times G_m^1)$. We shall also denote it by $M_{fr}(X)(1)$. The $n$th iteration gives the spectrum $M_{fr}(X \times G_m^n)$, also denoted by $M_{fr}(X)(n)$. The nearest aim is to define the $(S^1, G)$-bispectrum $M_{fr}^G(X)$. Another way of defining the $(S^1, G)$-bispectrum $M_{fr}^G(X)$ is given in Appendix $B$.

We construct a map in $SP_{S^1}^{fr}(k)$

\[ a_0 : M_{fr}(X) \rightarrow \text{Hom}(G, M_{fr}(X \times G_m^1)) \]

as follows. It is uniquely determined by a map

\[ \beta : M_{fr}(X) \rightarrow M_{fr}(X \times G_m^1)(\ast \times G_m) \]

and a homotopy

\[ h : M_{fr}(X) \rightarrow M_{fr}(X \times G_m^1)(\ast \times pt)^1 \]

such that $d_0h = f^\ast \beta$ and $d_1h$ factors through the distinguished point levelwise. Here $f : pt \rightarrow G_m$ is a morphism of schemes such that $f(pt) = 1$.

We set the map $\beta$ to be the composition

\[ M_{fr}(X) \xrightarrow{\text{deg}} M_{fr}(X \times G_m)(\ast \times G_m) \xrightarrow{p} M_{fr}(X \times G_m^1)(\ast \times G_m) \]
where \( p \) is a natural map, induced by the simplicial map of simplicial objects \( \mathbb{G}_m \to \mathbb{G}_m^\Lambda 1 \) in \( \text{Fr}_0(k) \) (we consider \( \mathbb{G}_m \) as a simplicial scheme in a trivial way).

One has a commutative square for any \( W \in \text{Sm}/k \)

\[
\begin{array}{ccc}
\mathcal{C}_* \text{Fr}(W \times \mathbb{G}_m, X \times \mathbb{G}_m) & \xrightarrow{\mathcal{C}_* \text{Fr}(1_W \times f, 1_X \times \mathbb{G}_m)} & \mathcal{C}_* \text{Fr}(W \times \text{pt}, X \times \mathbb{G}_m) \\
\downarrow \mathcal{C}_* \mathbb{G}_m & & \downarrow \mathcal{C}_* \mathbb{G}_m \\
\mathcal{C}_* \text{Fr}(W, X) & \xrightarrow{-\mathcal{C}_* \text{pt}} & \mathcal{C}_* \text{Fr}(W, X \times \text{pt})
\end{array}
\]

On the other hand, there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}_* \text{Fr}(W \times \text{pt}, X \times \mathbb{G}_m) & \xrightarrow{p} & \mathcal{C}_* \text{Fr}(W \times \text{pt}, X \times \mathbb{G}_m^\Lambda 1) \\
\downarrow f_* & & \downarrow f_* \\
\mathcal{C}_* \text{Fr}(W \times \text{pt}, X \times \text{pt}) & \xrightarrow{p'} & P(\mathcal{C}_* \text{Fr}(W \times \text{pt}, (X \times \text{pt}) \otimes S^1))
\end{array}
\]

Here the right lower corner stands for the simplicial path space of \( \mathcal{C}_* \text{Fr}(W \times \text{pt}, (X \times \text{pt}) \otimes S^1) \) (see [GP] Section 7) for details. Recall that the path space \( \Delta \) of a simplicial object \( X : \Delta^\text{op} \to \mathcal{D} \) in a category \( \mathcal{D} \) is defined as the composition of \( X \) with the shift functor \( P : \Delta \to \Delta \) that takes \([n]\) to \([n+1]\) (by mapping \( i \) to \( i+1 \)). By [Wal] 1.5.1 there is a canonical contraction of this space into the set of its zero simplices regarded as a constant simplicial set. Since \( P(\mathcal{C}_* \text{Fr}(W \times \text{pt}, (X \times \text{pt}) \otimes S^1)) \) has only one zero simplex, it follows that there is a canonical simplicial homotopy

\[ H : P(\mathcal{C}_* \text{Fr}(W \times \text{pt}, (X \times \text{pt}) \otimes S^1)) \to P(\mathcal{C}_* \text{Fr}(W \times \text{pt}, (X \times \text{pt}) \otimes S^1)) \]

such that \( d_0 H = 1 \) and \( d_1 H = \text{const} \).

Now the map \( h \) is induced by the composite map

\[
\begin{array}{ccc}
\mathcal{C}_* \text{Fr}(W \times \text{pt}, X \times \mathbb{G}_m^\Lambda 1) & \xrightarrow{p} & \mathcal{C}_* \text{Fr}(W \times \text{pt}, (X \times \text{pt}) \otimes S^1) \\
\downarrow f_* & & \downarrow f_* \\
P(\mathcal{C}_* \text{Fr}(W \times \text{pt}, X \times \mathbb{G}_m^\Lambda 1)) & \xrightarrow{H} & P(\mathcal{C}_* \text{Fr}(W \times \text{pt}, (X \times \text{pt}) \otimes S^1))
\end{array}
\]

(the same composite map is similarly defined on each space of the spectrum \( M_{fr}(X) \)). The desired map \( a_0 : M_{fr}(X) \to \text{Hom}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^\Lambda 1)) \) is constructed. Note that \( a_0 \) is functorial in framed correspondences of level zero. Each map of spectra

\[
a_n : M_{fr}(X)(n) \to \text{Hom}(\mathbb{G}, M_{fr}(X)(n+1)), \quad n \geq 0,
\]

is constructed similar to \( a_0 \) if we replace \( X \) with \( X \times \mathbb{G}_m^{\otimes n} \) and realize by taking diagonals.

**Definition 3.2.** The \((S^1, \mathbb{G})\)-bispectrum \( M_{fr}^\mathbb{G}(X) \) is defined as

\[
(M_{fr}(X), M_{fr}(X \times \mathbb{G}_m^\Lambda 1), M_{fr}(X \times \mathbb{G}_m^{\otimes 2}), \ldots)
\]

together with the structure morphisms \( a_n \)-s. Another way of defining the \( a_n \)-s is given in Appendix B.
We shall prove below (see the proof of Theorem A) that each $a_n$ is a Nisnevich local stable equivalence of spectra, but let us first discuss further useful spectra.

Denote by $\mathbb{Z} Fr^1_{S^1}(X) X \in Sm/k$ the Segal $S^1$-spectrum $(\mathbb{Z} Fr_*(-X), \mathbb{Z} Fr_*(-X \otimes S^1), \ldots)$. Denote by $EM(\mathbb{Z} Fr_*(-X))$ the Segal $S^1$-spectrum $(\mathbb{Z} Fr_*(-X), \mathbb{Z} Fr_*(-X \otimes S^1), \ldots)$. The equalities $\mathbb{Z} Fr_*(-X \sqcup X) = \mathbb{Z} Fr_*(-X) \oplus \mathbb{Z} Fr_*(-X')$ show that the $\Gamma$-space $(K, *) \mapsto \mathbb{Z} Fr_*(U, X \otimes K)$ corresponds to the abelian group $\mathbb{Z} Fr_*(U, X)$. Hence $EM(\mathbb{Z} Fr_*(-X))$ is the e Eilenberg–Mac Lane spectrum for $\mathbb{Z} Fr_*(-X)$. The $\Gamma$-space morphism $[(K, *) \mapsto \mathbb{Z} Fr_*(-X \otimes K)] \mapsto [(K, *) \mapsto \mathbb{Z} Fr_*(-X \otimes K)]$ induces a morphisms of framed $S^1$-spectra

$$\lambda_X : \mathbb{Z} Fr^1_{S^1}(X) \rightarrow EM(\mathbb{Z} Fr_*(-X))$$

Also, denote by $\mathbb{Z} M_{fr}(X), X \in Sm/k$, the Segal $S^1$-spectrum $(\mathbb{C} Fr_*(-X), \mathbb{C} Fr_*(-X \otimes S^1), \ldots)$. Denote by $LM_{fr}(X)$ the Segal $S^1$-spectrum $EM(\mathbb{Z} Fr(\Delta^* \times -X)) = (\mathbb{Z} Fr(\Delta^* \times -X), \mathbb{Z} Fr(\Delta^* \times -X \otimes S^1), \ldots)$. The above arguments show that $LM_{fr}(X)$ is the Eilenberg–Mac Lane spectrum for $\mathbb{Z} Fr(\Delta^* \times -X)$ and one has a natural morphism

$$I_X : \mathbb{Z} M_{fr}(X) \rightarrow LM_{fr}(X)$$

of framed $S^1$-spectra.

Note that homotopy groups of $LM_{fr}(X) = EM(\mathbb{Z} Fr(\Delta^* \times -X))$ are equal to homotopy groups of the complex $\mathbb{Z} Fr(\Delta^* \times -X)$. By [Sch] [II.6.2] homotopy groups $\pi_*(LM_{fr}(X)(U))$ of $LM_{fr}(X)$ evaluated at $U \in Sm/k$ are homotopy groups $H_*(M_{fr}(X)(U))$ of $M_{fr}(X)(U)$.

As above we can define $S^1$-spectra $LM_{fr}(X \times \mathbb{G}_m^n)$-s together with morphisms of framed spectra

$$c_n : LM_{fr}(X \times \mathbb{G}_m^n) \rightarrow \text{Hom}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{n+1})), \quad n \geq 0.$$  

We also refer the reader to Appendix B for another way of defining the $c_n$-s.

Though $\mathbb{Z} M_{fr}(X)(U)$ is a linear $S^1$-spectrum, its homotopy groups are hard to compute. But if $U$ is smooth local Henselian, then $\pi_*(\mathbb{Z} M_{fr}(X)(U)) = H_*(\mathbb{Z} Fr(\Delta^* \times U, X))$. More precisely, the following result is true.

**Theorem 3.3 (see [GP3]).** The natural morphisms of framed $S^1$-spectra

$$\lambda_X : \mathbb{Z} Fr^1_{S^1}(X) \rightarrow EM(\mathbb{Z} Fr_*(-X)) \quad \text{and} \quad I_X : \mathbb{Z} M_{fr}(X) \rightarrow LM_{fr}(X)$$

are Nisnevich local stable equivalences. In particular, if $U$ is smooth local Henselian, then $\pi_*(\mathbb{Z} M_{fr}(X)(U)) = H_*(\mathbb{Z} Fr(\Delta^* \times U, X))$.

**Definition 3.4.** (I) Following [GP1]. $\mathcal{F} \in Sp_{fr}^1(k)$ is BCD-local if it is $\mathbb{k}^1$-invariant, takes the level one framed correspondence $\sigma_X = (\{0\}, \mathbb{k}^1_1 \mapsto \mathbb{k}^1_1, pr_X) : X \rightarrow X$ with $X \in Sm/k$ to a stable weak equivalence and the natural map of spectra

$$\mathcal{F}(U \sqcup V) \rightarrow \mathcal{F}(U) \times \mathcal{F}(V)$$

is a stable weak equivalence of spectra for all $U, V \in Sm/k$.

(II) A framed presheaf of abelian groups $\mathcal{F}$ is quasi-stable if $\mathcal{F}(\sigma_X)$ is an isomorphism for every $X \in Sm/k$. $\mathcal{F}$ is additive if the natural homomorphism

$$\mathcal{F}(U \sqcup V) \rightarrow \mathcal{F}(U) \times \mathcal{F}(V)$$

is an isomorphism for all $U, V \in Sm/k$ and $\mathcal{F}(\emptyset) = 0$.

Note that $\mathcal{F} \in Sp_{fr}^1(k)$ is BCD-local if and only if each homotopy presheaf $\pi_n(\mathcal{F}), n \in \mathbb{Z}$, is $\mathbb{k}^1$-invariant, quasi-stable and additive.
Below we shall need the following

**Lemma 3.5.** Suppose \( \rho : \mathcal{X} \to \mathcal{Y} \) is a stable Nisnevich local weak equivalence of framed BCD-local \( S^1 \)-spectra. Then \( \rho_* : \text{Hom}(G_m, \mathcal{X}) \to \text{Hom}(G_m, \mathcal{Y}) \) is a stable Nisnevich local weak equivalence.

**Proof.** We have to show that

\[
\rho_{U \boxtimes G_m} : \mathcal{X}(U \boxtimes G_m) \to \mathcal{Y}(U \boxtimes G_m)
\]

is a stable weak equivalence of spectra for every local smooth Henselian \( U \). Consider the presheaf of \( S^1 \)-spectra

\[
U \in Sm/k \mapsto \text{cone}(\rho_{U \boxtimes G_m}).
\]

The spectra \( \mathcal{X}(- \boxtimes G_m) \), \( \mathcal{Y}(- \boxtimes G_m) \) are BCD-local. Therefore their presheaves of homotopy groups are quasi-stable, radditive \( \mathbb{A}^1 \)-invariant framed presheaves. It follows that the presheaves of homotopy groups of \( \rho_{U \boxtimes G_m} \) are quasi-stable, radditive \( \mathbb{A}^1 \)-invariant framed presheaves. Thus the presheaves of homotopy groups of \( \rho_{U \boxtimes G_m} \) are quasi-stable, radditive \( \mathbb{A}^1 \)-invariant framed presheaves. By \([GP2, 2.15(3')]\) the presheaf \( F \) is a Zariski sheaf. Using \([GP2, 2.15(5)]\) applied to \( \mathcal{X} = \mathbb{A}^1 \), one shows that for any open \( W \) in \( \mathbb{A}^1 \) one has \( F^\text{nis}(W) = F(W) \). In particular, \( F^\text{nis}(G_{m,k}) = F(G_{m,k}) \).

Hence for every smooth local Henselian \( U \), the homotopy groups \( \pi_*(\text{cone}(\rho_{U \boxtimes G_m})) \) are embedded into \( \pi_*(\text{cone}(\rho_{\text{Spec}(k) \boxtimes G_m})) \) and for any \( k \)-smooth irreducible variety \( V \) one has

\[
\pi_*(\mathcal{X})(\text{Spec } k(V) \boxtimes G_m) = \pi^\text{nis}_*(\mathcal{X})(\text{Spec } k(V) \boxtimes G_m) = \pi^\text{nis}_*(\mathcal{Y})(\text{Spec } k(V) \boxtimes G_m) = \pi_*(\mathcal{Y})(\text{Spec } k(V) \boxtimes G_m).
\]

Thus for every smooth local Henselian \( U \), \( \pi_*(\text{cone}(\rho_{\text{Spec}(k) \boxtimes G_m})) = 0 \), and hence \( \pi_*(\text{cone}(\rho_{U \boxtimes G_m})) = 0 \). Therefore \( \rho_{U \boxtimes G_m} \) is a stable equivalence whenever \( U \) is local smooth Henselian. \( \square \)

By the Resolution Theorem of \([GP1]\) one can find for every framed BCD-local \( S^1 \)-spectrum \( \mathcal{G} \) a Nisnevich local fibrant replacement

\[
\alpha : \mathcal{G} \to \mathcal{G}_f
\]

such that \( \alpha \) is a map in \( Sp_{\text{fr}}^f(k) \) which induces an isomorphism \( \pi^\text{nis}_*(\alpha) \) on the Nisnevich stable homotopy sheaves. Moreover, \( \alpha \) is functorial in \( \mathcal{G} \). In particular, we can find \( M_{fr}(X)(n)_f \) for every \( n \geq 0 \). Each map \( \alpha \) induces a map of framed spectra

\[
b_n : M_{fr}(X)(n)_f \to \text{Hom}(G, M_{fr}(X)(n+1)_f), \quad n \geq 0,
\]

such that the square

\[
\begin{array}{ccc}
M_{fr}(X)(n) & \xrightarrow{a_n} & \text{Hom}(G, M_{fr}(X)(n+1)) \\
\downarrow \alpha & & \downarrow \text{Hom}(G, \alpha) \\
M_{fr}(X)(n)_f & \xrightarrow{b_n} & \text{Hom}(G, M_{fr}(X)(n+1)_f)
\end{array}
\]

is commutative.

We are now in a position to prove Theorem A.
Theorem A (Cancellation). Let $k$ be an infinite perfect field, $X \in Sm/k$ and $n \geq 0$. Then the following statements are true:

1. the natural map of framed $S^1$-spectra
   $$a_n : Mfr(X)(n) \to Hom(\mathbb{G}, Mfr(X)(n+1))$$
   is a Nisnevich local stable equivalence;

2. the induced map of framed $S^1$-spectra
   $$b_n : Mfr(X)(n)_f \to Hom(\mathbb{G}, Mfr(X)(n+1)_f)$$
   is a schemewise stable equivalence with $Mfr(X)(n)_f$ and $Mfr(X)(n+1)_f$ being framed
   Nisnevich local fibrant replacements $Mfr(X)(n)$ and $Mfr(X)(n+1)$ respectively.

Proof. Since $Mfr(X)(n), Mfr(X)(n)_f$ are BCD-local and we have a commutative diagram with
homotopy fiber rows in $Sp_{S^1}(k)$

$$\xymatrix{ Hom(\mathbb{G}, Mfr(X)(n)) \ar[r] \ar[d] & Hom(\mathbb{G}_m, Mfr(X)(n)) \ar[r] \ar[d] & Mfr(X)(n) \ar[d] \\
Hom(\mathbb{G}, Mfr(X)(n)_f) \ar[r] & Hom(\mathbb{G}_m, Mfr(X)(n)_f) \ar[r] & Mfr(X)(n)_f }$$

then the vertical maps of diagram (3) are Nisnevich local stable equivalences (we use here
Lemma 3.5). It follows that $a_n$ is a Nisnevich local stable equivalence if and only if so is $b_n$.

The latter is equivalent to saying that $b_n$ is a schemewise equivalence, because $Mfr(X)(n)_f$ and
$Hom(\mathbb{G}, Mfr(X)(n+1)_f)$ are motivically fibrant by the Resolution Theorem of [GP1].

It is enough to prove that

$$b_0 : Mfr(X)_f \to Hom(\mathbb{G}, Mfr(X)(1)_f)$$

is a schemewise equivalence of spectra. Indeed, consider a commutative diagram of homotopy
cofiber sequences in $Sp_{S^1}(k)$

$$\xymatrix{ Mfr(X)(n-1)_f \ar[r]_{b_{n-1}} \ar[d] & Mfr(X \times \mathbb{G}_m)(n-1)_f \ar[r]_{b_n} \ar[d] & Mfr(X)(n)_f \ar[d] \\
Hom(\mathbb{G}, Mfr(X)(n)_f) \ar[r] & Hom(\mathbb{G}, Mfr(X \times \mathbb{G}_m)(n)_f) \ar[r] & Hom(\mathbb{G}, Mfr(X)(n+1)_f) }$$

with $n \geq 1$. If $b_{n-1}$ is a schemewise equivalence of spectra, then so is $b_n$ by [Hir 13.5.10].

Thus using induction in $n$, it suffices to verify that $b_0$ is a schemewise equivalence of spectra.
As we have mentioned above this is equivalent to saying that $a_0$ is a Nisnevich local equivalence
of spectra.

By the stable Whitehead theorem [Sch II.6.30] $a_0$ is a stable local Nisnevich equivalence if
and only if so is

$$a_0 : ZMfr(X) \to Z[Hom(\mathbb{G}, Mfr(X \times \mathbb{G}_m^{11}))].$$
Consider a commutative diagram of homotopy fiber sequences in $Sp_{g^1}(k)$

\[
\begin{array}{c}
\text{Hom}(G, M_{fr}(X \times \mathbb{G}^1_m)) \\
\downarrow \quad \downarrow \\
\mathbb{Z} \text{[Hom}(G, M_{fr}(X \times \mathbb{G}^1_m)))] \\
\downarrow \quad \downarrow \\
\text{Hom}(G, LM_{fr}(X \times \mathbb{G}^1_m)) \\
\end{array}
\]

The arrow $\ell_X$ is a stable local weak equivalence of $BCD$-local spectra by Theorem 3.3 and hence so is the middle lower arrow by Lemma 3.5. It follows that $\ell_X$ is a stable local weak equivalence. Consider a commutative diagram

\[
\begin{array}{c}
LM_{fr}(X) \quad \overset{\text{can}'}{\underset{id}{\longrightarrow}} \quad \text{Hom}((\mathbb{G}_m,1),LM_{fr}(X \times \mathbb{G}^1_m)) \\
\downarrow \quad \downarrow \\
LM_{fr}(X) \quad \overset{-\mathfrak{F}(\text{id}_{\mathbb{G}_m}-e_{\mathbb{G}_m})}{\underset{-\mathfrak{F}(\text{id}_{\mathbb{G}_m}-e_{\mathbb{G}_m})}{\longrightarrow}} \quad \text{Hom}((\mathbb{G}_m,1),LM_{fr}(X \times \mathbb{G}^1_m)) \\
\end{array}
\]

Since $LM_{fr}(X), \text{Hom}((\mathbb{G}_m,1),LM_{fr}(X \times (\mathbb{G}_m,1)))$ are schemewise linear $\mathbb{G}$-spectra, then homotopy groups $\pi_\ast(LM_{fr}(X))$ (respectively $\pi_\ast(\text{Hom}((\mathbb{G}_m,1),LM_{fr}(X \times (\mathbb{G}_m,1))))$) equal homology groups $H_\ast(\mathbb{Z}F(\Delta^\ast \times -),X) \quad \text{and} \quad H_\ast(\mathbb{Z}F(\Delta^\ast \times \mathbb{G}_m),X \times (\mathbb{G}_m,1)))$. Hence the bottom arrow $-\otimes (\text{id}_{\mathbb{G}_m}-e_{\mathbb{G}_m})$ is a sectionwise stable equivalence by Theorem D.

The arrow $\text{Hom}((\mathbb{G}_m,1),\text{can}')(LM_{fr}(X \times (\mathbb{G}_m,1)))$ is a sectionwise stable equivalence by Lemma [A.1] Hence the arrow $\text{can}'_0 \circ (-\otimes (\text{id}_{\mathbb{G}_m}-e_{\mathbb{G}_m}))$ is a sectionwise stable equivalence.

By Lemma [A.1] one has $\text{can}'_0 \circ (-\otimes (\text{id}_{\mathbb{G}_m}-e_{\mathbb{G}_m})) = [\text{in}^\ast \circ \text{can}^\ast \circ (\text{id}_{\mathbb{G}_m}-e_{\mathbb{G}_m})] \circ c_0$ and the morphism

\[
\text{Hom}(G,LM_{fr}(X \times \mathbb{G}^1_m)) \quad \overset{\text{in}^\ast \circ \text{can}^\ast \circ (\text{id}_{\mathbb{G}_m}-e_{\mathbb{G}_m})}{\longrightarrow} \quad \text{Hom}((\mathbb{G}_m,1),LM_{fr}(X \times \mathbb{G}^1_m))
\]

is a sectionwise stable equivalence. Thus $c_0$ is a sectionwise stable equivalence. This completes the proof of Theorem A.

**Theorem B.** Let $k$ be an infinite perfect field, $X \in Sm/k$. Then the bispectrum $M_{fr}^G(X)_f = (M_{fr}(X)_f, M_{fr}(X)(1)_f, M_{fr}(X)(2)_f, \ldots)$ obtained from $M_{fr}^G(X)$ by taking levelwise framed Nisnevich local fibrant replacements with structure maps $b_n$-s is a motivically fibrant $(S^1, G)$-bispectrum.
Proof. By the Resolution Theorem of [GP1] each framed $S^1$-spectrum $M_{fr}(X)(n)$ is motivically fibrant. By Theorem A each structure map $b_n$ is a schemewise equivalence. We conclude that the bispectrum $M_{fr}^{G}(X)$ is a motivically fibrant ($S^1, G$)-bispectrum. \[\]

The motivic model category of framed $S^1$-spectra $SP_{fr}^{S}(k)$ has a natural Quillen pair of adjoint functors

$$- \otimes \mathcal{G} : SP_{fr}^{S}(k) \rightleftharpoons SP_{fr}^{S}(k) : \text{Hom}(\mathbb{G}, -)$$

(see [GP1] for details). This Quillen pair induces adjoint functors on the homotopy category

$$- \otimes \mathcal{L} \mathcal{G} : SH_{fr}^{S}(k) \rightleftharpoons SH_{fr}^{S}(k) : \text{RHom}(\mathbb{G}, -).$$

The functor $- \otimes \mathcal{L} \mathcal{G}$ is also referred to as the \textit{twist functor}. By construction, $M_{fr}(X) \otimes \mathcal{L} \mathcal{G}$ is canonically isomorphic to $M_{fr}(X)(1)$ for all $X \in \text{Sm}/k$ (see [GP1] for details).

We finish the section by proving Theorem C.

\textbf{Theorem C.} Let $k$ be an infinite perfect field. Then the twist functor

$$- \otimes \mathcal{L} \mathcal{G} : SH_{fr}^{S}(k) \rightarrow SH_{fr}^{S}(k)$$

is full and faithful.

Proof. The twist functor $- \otimes \mathcal{L} \mathcal{G}$ is triangulated and preserves arbitrary coproducts. By [GP1] 6.15 $SH_{fr}^{S}(k)$ is a compactly generated triangulated category with $\mathcal{C} := \{M_{fr}(X)[\ell] \mid X \in \text{Sm}/k, \ell \in \mathbb{Z}\}$ a family of compact generators.

Theorem A and [GP1] 6.15 imply that our theorem is true for compact generators from $\mathcal{C}$. By using the five-lemma one can easily show that our theorem is also true for all compact objects. Since the twist functor is triangulated and preserves arbitrary coproducts, our proof now follows from the fact that every object of $SH_{fr}^{S}(k)$ is a homotopy colimit of compact objects. \[\]

4. \textbf{Useful lemmas}

In this section we discuss several useful $\mathbb{A}^1$-homotopies and collect a number of facts used in the following sections. We start with some definitions and notation.

\textbf{Definition 4.1.} Let $\mathcal{F} : \text{Sm}/k \rightarrow \text{Sets}$ be a presheaf of sets. Let $X \in \text{Sm}/k$ be a smooth variety and $a, b \in \mathcal{F}(X)$ be two sections. We write $a \sim b$ if $a$ and $b$ are in the same connected component of the simplicial set $\mathcal{F}(\Delta^* \times X)$. If $h \in \mathcal{F}(\Delta^* \times X)$ is such that $\partial_0(h) = a$ and $\partial_1(h) = b$, then we will write $a^h b$. In this case $a \sim b$.

Let $\mathcal{A} : \text{Sm}/k \rightarrow \text{Ab}$ be a presheaf of abelian groups. Let $X \in \text{Sm}/k$ be a smooth variety and $a, b \in \mathcal{A}(X)$ be two sections. We will write $a \sim b$ if the classes of $a$ and $b$ in $H_0(\mathcal{A}(\Delta^* \times X))$ coincide. This is equivalent to saying that there is $h \in \mathcal{A}(\Delta^* \times X)$ such that $\partial_0(h) = a$ and $\partial_1(h) = b$. For such an $h$ we will write $a^h b$.

\textbf{Definition 4.2.} Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves of sets on the category of $k$-smooth schemes and let $\phi_0, \phi_1 : \mathcal{F} \Rightarrow \mathcal{G}$ be two morphisms. An $\mathbb{A}^1$-\textit{homotopy} between $\phi_0$ and $\phi_1$ is a morphism $H : \mathcal{F} \rightarrow \text{Hom}(\mathbb{A}^1, \mathcal{G})$ such that $H_0 = \phi_0$ and $H_1 = \phi_1$. We will write $\phi_0 \sim \phi_1$ if there is an $\mathbb{A}^1$-homotopy between $\phi_0$ and $\phi_1$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two presheaves of abelian groups on the category of $k$-smooth schemes and let $\phi_0, \phi_1 : \mathcal{A} \Rightarrow \mathcal{B}$ be two morphisms. An $\mathbb{A}^1$-\textit{homotopy} between $\phi_0$ and $\phi_1$ is a morphism $H : \mathcal{A} \rightarrow \text{Hom}(\mathbb{A}^1, \mathcal{B})$ of presheaves of abelian groups such that $H_0 = \phi_0$ and $H_1 = \phi_1$. If $H$ is an $\mathbb{A}^1$-homotopy between $\phi_0$ and $\phi_1$, then we will write $\phi_0^H \phi_1$. If we do not specify an $\mathbb{A}^1$-homotopy between $\phi_0$ and $\phi_1$, then we will write $\phi_0 \sim \phi_1$. \[\]
If \( \varphi : \mathcal{A} \to \mathcal{B} \) is a morphism of presheaves of abelian groups, then there is a constant \( \mathbb{A}^1 \)-homotopy \( H_\varphi \) between \( \varphi \) and \( \varphi \) defined as follows. Given \( a \in \mathcal{A}(X) \) set \( H_\varphi(a) = pr_X^*(\varphi(a)) \in \mathcal{B}(X \times \mathbb{A}^1) \).

**Lemma 4.3.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two presheaves of abelian groups on the category of k-smooth schemes and let \( \varphi_0, \varphi_1 : \mathcal{A} \Rightarrow \mathcal{B} \) be two morphisms such that \( \varphi_0 \sim \varphi_1 \). Then the induced morphisms

\[ \varphi_0, \varphi_1 : \mathcal{A}^*(\Delta^*) \Rightarrow \mathcal{B}(\Delta^*) \]

between two simplicial abelian groups give the same morphisms on the homology of the associated Moore complexes.

**Lemma 4.4.** Let \( \varphi_0, \varphi_1, \varphi_2 : \mathcal{A} \to \mathcal{B} \) be morphisms of presheaves of abelian groups and let \( \varphi_0 \mu^\prime \varphi_1 \) and \( \varphi_1 \mu^\prime \varphi_2 \). Then

\[ \varphi_0 H' + H'' - H \varphi_1 \varphi_2 \]

**Lemma 4.5.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two presheaves of abelian groups on the category of k-smooth schemes and let \( \varphi_0, \varphi_1 \). Let \( \rho : \mathcal{A}^I \to \mathcal{A}^J \) be a morphism. Then \( \varphi_0 \circ \rho \varphi_1 \circ \rho \). Moreover, let \( \eta : \mathcal{B} \to \mathcal{B}^I \) be a morphism, then \( \psi \circ \varphi_0 \varphi_1 \psi \circ \varphi_1 \) with \( \Psi = \text{Hom}(\mathbb{A}^1, \psi) \text{Hom}(\mathbb{A}^1, \mathcal{B}) \to \text{Hom}(\mathbb{A}^1, \mathcal{B}^I) \).

We now want to discuss matrices actions on framed correspondences and associated homotopies. Let \( X \) and \( Y \) be k-smooth schemes and \( A \in GL_n(k) \) be a matrix. Then \( A \) defines an automorphism

\[ \varphi_A : \text{Fr}_n(- \times X, Y) \to \text{Fr}_n(- \times X, Y) \]

of the presheaf \( \text{Fr}_n(- \times X, Y) \) in the following way. Given \( W \in Sm/k \) and \( a = ((\alpha_1, \alpha_2, \ldots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_n), g) \in \text{Fr}_n(W \times X, Y) \), set

\[ \varphi_A((\alpha_1, \alpha_2, \ldots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_n), g)) = ((\alpha_1, \alpha_2, \ldots, \alpha_n), f, Z, U, A \circ (\varphi_1, \varphi_2, \ldots, \varphi_n), g), \]

where \( A \) is regarded as a linear automorphism of \( \mathbb{A}^n_k \).

The automorphism \( \varphi_A \) of the presheaf \( \text{Fr}_n(- \times X, Y) \) induces an automorphism of the free abelian presheaf \( \mathbb{Z}[\text{Fr}_n(- \times X, Y)] \) and an automorphism \( \varphi_A \) of the the presheaf of abelian groups \( \mathbb{Z}\text{Fr}_n(- \times X, Y) \).

**Definition 4.6.** Let \( A \in SL_n(k) \). Choose a matrix \( A_0 \in SL_n[k[x]] \) such that \( A_0 = id \) and \( A_1 = A \). The matrix \( A_0 \), regarded as a morphism \( \mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^n \), gives rise to an \( \mathbb{A}^1 \)-homotopy \( h \) between \( \varphi \) and \( \varphi_A \) as follows. Given \( a = (\alpha, f, Z, U, \varphi, g) = ((\alpha_1, \alpha_2, \ldots, \alpha_n), f, Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_n), g) \in \text{Fr}_n(W \times X, Y) \), one sets

\[ h(a) = (\alpha, f \times id_{\mathbb{A}^1}, Z \times \mathbb{A}^1, U \times \mathbb{A}^1, A_0 \circ (\varphi \times id_{\mathbb{A}^1}), g \circ pr_U) \in \text{Fr}_n(W \times X \times \mathbb{A}^1, Y) \]

Clearly, \( h_0(a) = a \) and \( h_1(a) = \varphi_A(a) \). By linearity the homotopy \( h \) induces an \( \mathbb{A}^1 \)-homotopy \( H_{A_0} \)

\[ \text{id}_{\mathbb{A}^n}^H \varphi_A : \mathbb{Z}\text{Fr}_n(- \times X, Y) \Rightarrow \mathbb{Z}\text{Fr}_n(- \times X, Y) \]

between the identity \( id \) and the morphism \( \varphi_A \).
Lemma 4.7. Let \( \rho : \mathbb{Z}F_m(- \times X, Y) \to \mathbb{Z}F_n(- \times X, Y) \) be a presheaf morphism. Let \( A \in SL_n(k), A_s \in SL_n(k[s]) \) and \( \rho_A \) be as in Definition 4.6. Then one has
\[
\rho \circ H_{\rho_A} : \mathbb{Z}F_m(- \times X, Y) \to \mathbb{Z}F_n(- \times X, Y).
\]

For \( b \in \mathbb{Z}F_m(Y, S) \) define a presheaf morphism
\[
\phi_b : \mathbb{Z}F_n(- \times X, Y) \to \mathbb{Z}F_{n+m}(- \times X, S)
\]
sending \( a \in \mathbb{Z}F_n(W \times X, Y) \) to \( b \circ a \in \mathbb{Z}F_{n+m}(- \times X, S) \). Also, any \( b \in \mathbb{Z}F_n(pt, pt) \) defines a morphism of presheaves
\[
- \boxtimes b : \mathbb{Z}F_n(- \times X, Y) \to \mathbb{Z}F_{n+m}(- \times X, Y)
\]
sending \( a \in \mathbb{Z}F_n(W \times X, Y) \) to \( a \boxtimes b \in \mathbb{Z}F_{n+m}(- \times X, Y) \).

The next three lemmas are straightforward.

Lemma 4.8. Let \( b_1, b_2 \in \mathbb{Z}F_m(Y, S) \) be such that \( b_1 \sim b_2 \), then
\[
\phi_{b_1} \sim \phi_{b_2} : \mathbb{Z}F_n(- \times X, Y) \to \mathbb{Z}F_{n+m}(- \times X, S).
\]

Lemma 4.9. Let \( b_1, b_2 \in \mathbb{Z}F_n(pt, pt) \) and \( h \in \mathbb{Z}F_m(\mathbb{A}^1, pt) \) be such that \( b_1 \otimes b_2, \) then
\[
(- \boxtimes b_1) \otimes h = (- \boxtimes b_2) : \mathbb{Z}F_n(- \times X, Y) \to \mathbb{Z}F_{n+m}(- \times X, Y).
\]

Lemma 4.10. Let \( z \in \mathbb{A}^m \) be a k-rational point. Set \( U' = (\mathbb{A}^m)^h_z \) to be the henselization of \( \mathbb{A}^m \) at the point \( z \). Let \( i_z : pt \to U' \) be the closed point of \( U' \). Let \( U'_z : = ([\mathbb{A}^1 \times \mathbb{A}^m]^h_{\mathbb{A}^1 \times z} \) be the henselization of \( \mathbb{A}^1 \times \mathbb{A}^m \) at \( \mathbb{A}^1 \times z \). Then there is a morphism \( H_z : U'_z \to U' \) such that:

(a) \( H_1 : U' \to U' \) is the identity morphism;

(b) \( H_0 : U' \to U' \) coincides with the composite morphism \( U' \xrightarrow{\rho} pt \xrightarrow{i_z} U' \), where \( \rho : U' \to pt = \text{Spec}(k) \) is the structure morphism.

The preceding lemma implies the following

Corollary 4.11. Let \( z \in \mathbb{A}^m \) be a k-rational point. Let \( h_z = ([\mathbb{A}^1 \times z, U'_z, \psi, H_z]) \in \text{Fr}_N(\mathbb{A}^1, U') \).

Then one has:

(a) \( h_1 = (z, U', (\phi_1, \phi_2, \ldots, \phi_m)) \in \text{Fr}_N(pt, U') \);

(b) \( h_0 = (z, U', (\phi_1, \phi_2, \ldots, \phi_m)) \in \text{Fr}_N(pt, U') \), where \( p : U' \to pt = \text{Spec}(k) \) is the structure morphism and \( i_z : pt \to U' \) is the closed point of \( U' \).

Lemma 4.12. Let \( z \in \mathbb{A}^m \) be a k-rational point. Let \( Y \) be a k-smooth scheme and let \( (z, U, (\phi_1, \phi_2, \ldots, \phi_m), g) \in \text{Fr}_m(pt, Y) \) be a framed correspondence. Then
\[
(z, U, (\phi_1, \phi_2, \ldots, \phi_m), g) \sim (z, U, (\phi_1, \phi_2, \ldots, \phi_m), c_{g(0)}),
\]
where \( c_{g(0)} = g(0) \circ p : U \xrightarrow{p} pt \xrightarrow{g(0)} Y \).

Proof. Let \( U', U'_z, i_z \) and \( h_z \) be as in Corollary 4.11. Let \( \pi : U' \to U \) be the canonical morphism. Take \( h_z = (A^1 \times z, U'_z, \phi \circ \pi, H_z) \in \text{Fr}_m(A^1, U') \) and \( h'_z = g \circ \pi \circ h_z \in \text{Fr}_m(pt, Y) \). We want to check that \( h'_1 = (z, U', (\phi, g)) \) and \( h'_0 = (z, U, (\phi, c_{g(0)})) \). This will prove our statement. One has,
\[
\begin{align*}
h'_1 &= (g \circ \pi) \circ h_1 = (g \circ \pi) \circ (z, U', \phi \circ \pi, i_z) = (z, U', \phi \circ \pi) \\
h'_0 &= (g \circ \pi) \circ h_0 = (g \circ \pi) \circ (z, U', \phi \circ \pi, i_z \circ p) = (z, U', \phi \circ \pi).
\end{align*}
\]
One easily sees that

\[ \text{as required.} \]

**Lemma 4.13.** Let \( Y \) be a \( k \)-smooth scheme and let \((Z, U, \varphi, g) \in \text{Fr}_1(\text{pt}, Y)\) be a framed correspondence. Suppose that \( U \subset \mathbb{A}^1 \) and \( \varphi = p(t) \in k[t] \) is a polynomial, where \( t \) is the coordinate function on \( \mathbb{A}^1 \).

1. Then for every \( a \in k \) we have

\[ (Z, U, p(t), g(t)) \sim (m_a^{-1}(Z), m_a^{-1}(U), p(t-a), g(t-a)) \in \text{Fr}_1(\text{pt}, Y), \]

where \( m_a : \mathbb{A}^1 \to \mathbb{A}^1 \) is given by \( m_a(t) = t - a \).

2. If \( Z = \{ x_0 \} \) for some \( x_0 \in k \) and \( p(t) = (t-x_0)^n r(t), r(x_0) \neq 0, \) and \( r(t) \) is invertible on \( U \), then

\[ (Z, U, p(t), g) \sim (\{ x_0 \}, \mathbb{A}^1, r(x_0) t^n, c_{g(x_0)}) \in \text{Fr}_1(\text{pt}, Y), \]

where \( c_{g(x_0)} : \mathbb{A}^1 \to \text{pt} \xrightarrow{g(x_0)} Y \) is the constant map taking \( \mathbb{A}^1 \) to the point \( g(x_0) \in Y \).

**Proof.** (1) The homotopy is given by

\[ (m_a^{-1}(Z), m_a^{-1}(U), p(t-sa), g(t-sa)) \in \text{Fr}_1(\mathbb{A}^1, Y), \]

where \( s \) is the homotopy parameter and \( m_a : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1 \) is the morphism \( m_a(t) = t - sa \).

(2) Using the preceding statement, we may assume that \( x_0 = 0 \). Consider a polynomial

\[ h(s, t) = sr(t)t^n + (1-s)r(0)t^n \in k[t, s]. \]

One easily sees that \( Z(h) = (0 \times \mathbb{A}^1) \sqcup S \). The framed correspondence

\[ (\{ 0 \} \times \mathbb{A}^1, (U \times \mathbb{A}^1) \setminus S, sr(t)t^n + (1-s)r(0)t^n, g) \in \text{Fr}_1(\mathbb{A}^1, Y) \]

yields the relation \((\{ 0 \}, U, r(t)t^n, g) \sim (\{ 0 \}, U, r(0)t^n, g) \) in \( \text{Fr}_1(\text{pt}, Y) \). Lemma 4.12 shows that

\[ (\{ 0 \}, U, r(0)t^n, g) \sim (\{ 0 \}, U, r(0)t^n, g(0)) = (\{ 0 \}, \mathbb{A}^1, r(0)t^n, g(0)) \in \text{Fr}_1(\text{pt}, Y) \]

and our lemma follows. \( \square \)

**Lemma 4.14.** Let \((Z, \mathbb{A}^1, p(t), c) \in \text{Fr}_1(\text{pt}, \text{pt})\) be a framed correspondence, where \( p(t) = at^n + \cdots \) is a polynomial of degree \( n \) with the leading coefficient \( a \) and \( c : \mathbb{A}^1 \to \text{pt} \) is the canonical projection. Then

\[ (Z, \mathbb{A}^1, p(t), c) \sim (\{ 0 \}, \mathbb{A}^1, at^n, c) \in \text{Fr}_1(\text{pt}, \text{pt}). \]

**Proof.** The homotopy is given by the framed cycle

\[ (Z(p(t) + s(at^n - p(t))), \mathbb{A}^1 \times \mathbb{A}^1, p(t) + s(at^n - p(t)), c'), \]

where \( s \) is the homotopy parameter and \( c' : \mathbb{A}^1 \times \mathbb{A}^1 \to \text{pt} \) is the canonical projection. \( \square \)

5. Homotopies for Coordinates Swap of \( \mathbb{G}_m \times \mathbb{G}_m \)

Denote \( \varepsilon = (\{ 0 \}, \mathbb{A}^1, -t, c) \in \text{Fr}_1(\text{pt}, \text{pt}) \), where \( c : \mathbb{A}^1 \to \text{pt} \) is the canonical projection.

**Proposition 5.1.** Let \( Y \) be a \( k \)-smooth scheme. Then the canonical homomorphism

\[ H_0(\text{ZF}(\Delta^* \times \mathbb{G}_m \times \mathbb{G}_m, Y)) \to H_0(\text{ZF}(\Delta_{\text{spec}\Spec k[t, u]}, Y)) \]

is injective.
Proof. By [GP2, 2.15(1)] the canonical homomorphisms
\[ H_0(\text{ZF}(\Delta^* \times \mathbb{G}_m \times \mathbb{G}_m, Y)) \rightarrow H_0(\text{ZF}(\Delta^* \times \mathbb{G}_{m,k(u)}, Y)) \]
and
\[ H_0(\text{ZF}(\Delta^* \times \mathbb{G}_{m,k(u)}, Y)) \rightarrow H_0(\text{ZF}(\Delta_{\text{Spec} k(u)}^*, Y)) \]
are injective, hence the lemma.

Lemma 5.2. Let \( F/k \) be a field, choose \( x,y \in F^* \) such that \( x \neq y^{-1} \) and let \( u_1, u_2 \) be coordinates on \( \mathbb{G}_m \times \mathbb{G}_m \). Consider morphisms \( f,g : \text{Spec} F \rightarrow \mathbb{G}_m \times \mathbb{G}_m \) given by \( u_1 \mapsto x, u_2 \mapsto y \) and \( u_1 \mapsto y, u_2 \mapsto x \) respectively. Then for \( p = (\text{id} - e_1) \otimes (\text{id} - e_1) \) we have \( p \circ f \sim p \circ (-e \otimes g) \) in \( \text{ZF}(\text{Spec} F, \mathbb{G}_m \times \mathbb{G}_m) \).

Proof. The adjunction isomorphism
\[ \text{ZF}(k)(\text{Spec} F, \mathbb{G}_m \times \mathbb{G}_m) \cong \text{ZF}(F)(\text{Spec} F, \mathbb{G}_m,F \times \mathbb{G}_m,F) \]
implies it is sufficient to verify the case \( F = k \). So we have morphisms \( f,g : pt \rightarrow \mathbb{G}_m, pt \mapsto (x,y) \) and \( pt \mapsto (y,x) \) respectively. Taking suspensions, we obtain framed correspondences
\[ (\emptyset, \mathbb{A}^1, t, c_{(x,y)}), (\emptyset, \mathbb{A}^1, t, c_{(y,x)}) \in \text{Fr}_1(pt, \mathbb{G}_m \times \mathbb{G}_m), \]
where \( c_{(x,y)} \) and \( c_{(y,x)} \) are morphisms on \( \mathbb{A}^1 \) sending it to the points \((x,y)\) and \((y,x)\) respectively.

Consider \( h(s, t) = \frac{1}{x-y}(t^2 - s(x+y) + (1-s)(xy+1))t + xy \in k[s,t,t^{-1}] = k[\mathbb{A}^1 \times \mathbb{G}_m] \) and a framed correspondence
\[ H_1 := (Z(h), \mathbb{A}^1 \times \mathbb{G}_m, h(s,t), (t,xyt^{-1})) \in \text{Fr}_1(\mathbb{A}^1, \mathbb{G}_m \times \mathbb{G}_m). \]
We have \( h(0,t) = \frac{1}{x-y}(t - xy)(t - 1) \) and \( h(1,t) = \frac{1}{x-y}(t - x)(t - y) \). Using the additivity property for supports in \( \text{ZF}_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m) \) (see Definition 2.5) and Lemma 4.13 we will check below that
\[ (\emptyset, \mathbb{A}^1, t, c_{(x,y)}) + (\emptyset, \mathbb{A}^1, -t, c_{(y,x)}) \sim (\emptyset, \mathbb{A}^1, \frac{1-xy}{x-y}t, c_{(x,y)}) + (\emptyset, \mathbb{A}^1, \frac{xy-1}{x-y}t, c_{(y,x)}) \]
in \( \text{ZF}_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m) \). The composition with the idempotent \( p \) annihilates all extra summands and proves the lemma.

In order to prove the relation (5), consider the frame correspondence (4) in \( \text{ZF}_1(\mathbb{A}^1, \mathbb{G}_m \times \mathbb{G}_m) \). Observe that in \( \text{ZF}_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m) \)
\[ H_0 := (Z(h(1,1), \mathbb{G}_m, h(1,t), (t,xyt^{-1})) = \]
\[ = (\emptyset, \mathbb{G}_m - \{y\}, \frac{1}{x-y}(t-x)(t-y), (t,xyt^{-1})) + (\emptyset, \mathbb{G}_m - \{x\}, \frac{1}{x-y}(t-x)(t-y), (t,xyt^{-1})). \]
By Lemma 4.13 one has in \( \text{ZF}_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m) \)
\[ (\emptyset, \mathbb{G}_m - \{y\}, \frac{1}{x-y}(t-x)(t-y), (t,xyt^{-1})) \sim (\emptyset, \mathbb{A}^1, \frac{y-x}{x-y}t, c_{(x,y)}), \]
\[ (\emptyset, \mathbb{G}_m - \{x\}, \frac{1}{x-y}(t-x)(t-y), (t,xyt^{-1})) \sim (\emptyset, \mathbb{A}^1, \frac{x-y}{x-y}t, c_{(x,y)}). \]
Thus \( H_1 \sim (\emptyset, \mathbb{A}^1, t, c_{(x,y)}) + (\emptyset, \mathbb{A}^1, -t, c_{(y,x)}) \) in \( \text{ZF}_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m) \). Similar computations show that \( H_0 \sim (\emptyset, \mathbb{A}^1, \frac{1}{x-y}(t-x), c_{(x,y)}) + (\emptyset, \mathbb{A}^1, \frac{1-xy}{x-y}t, c_{(y,x)}) \) in \( \text{ZF}_1(\text{pt}, \mathbb{G}_m \times \mathbb{G}_m) \). The relation (5) is proved, and hence the lemma. \(\square\)
Proposition 5.3. Let $\tau: \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ be the permutation of coordinates morphism. Denote $p = (\text{id} - e_1) \boxtimes (\text{id} - e_1)$. Then $p \circ \text{id} \sim p \circ (\varepsilon \boxtimes \tau)$ in $ZF(G_m \times G_m, G_m \times G_m)$.

Proof. Let $u_1$ and $u_2$ be coordinate functions on $G_m \times G_m$. In view of Proposition 5.1 it is sufficient to show that $p \circ f = p \circ (\varepsilon \boxtimes g)$ in $H_0(ZF(\Delta^*(f(u_1, u_2)), G_m \times G_m))$, where $f: \text{Spec}(k(u_1, u_2)) \to \text{Spec}(k(u_1, u_2))$ is the canonical embedding and $g: \text{Spec}(k(u_1, u_2)) \to \text{Spec}(k(u_1, u_2))$ is given by $g^*(u_1) = u_2, g^*(u_2) = u_1$. The last assertion follows from Lemma 5.2.

It follows from Proposition 5.3 that there exists a homotopy $\Psi \in ZF_n(G_m \times G_m \times \mathbb{A}^1, G_m \times G_m)$ such that $i_0^!(\Psi) = p \circ (\varepsilon \boxtimes \Sigma^{n-1} \text{id})$ and $i_1^!(\Psi) = p \circ \Sigma \tau$, where $p = (\text{id} - e_1) \boxtimes (\text{id} - e_1)$.

Recall that $\Sigma = (\{0\}, \mathbb{A}^1, t) \in ZF_1(pt, pt)$. For every $k > 0$ we write $\Sigma_k$ to denote $\Sigma \boxtimes k \boxtimes \cdots \boxtimes \Sigma \in ZF_k(pt, pt)$.

Lemma 5.4. Let $X, Y$ be $k$-smooth schemes and $m \geq 0$ be an integer, and let $n$ be the same as in the choice of the element $\Psi$. Let $\tau: \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ be the permutation of coordinates morphism. Consider two presheaf morphisms

\[
(- \boxtimes \Sigma^2) : ZF_m(- \times X \times G_m \times G_m, Y \times G_m \times G_m) \to ZF_m+2n(- \times X \times G_m \times G_m, Y \times G_m \times G_m),
\]

\[
(- \boxtimes \Sigma^2) \circ \omega : ZF_m(- \times X \times G_m \times G_m, Y \times G_m \times G_m) \to ZF_m+2n(- \times X \times G_m \times G_m, Y \times G_m \times G_m),
\]

where $\omega(a) = (id_Y \times \tau) \circ \omega (id_Y \times \tau)$. Then there is a morphism of presheaves of abelian groups $H: ZF_m(- \times X \times G_m \times G_m, Y \times G_m \times G_m) \to ZF_m+2n(- \times X \times G_m \times G_m, \mathbb{A}^1 \times Y \times G_m \times G_m)$ such that for any $a \in ZF_m(W \times X \times (G_m, 1) \times (G_m, 1), Y \times (G_m, 1) \times (G_m, 1))$ one has

\[
a \boxtimes \Sigma^2 = H_0(a) \quad \text{and} \quad H_1(a) = \Sigma^2([id_Y \times \tau] \circ a \circ [id_Y \times \tau]).
\]

Moreover, both $H_0(a)$ and $H_1(a)$ are in $ZF_m+2n(W \times X \times (G_m, 1) \times (G_m, 1), \mathbb{A}^1 \times Y \times (G_m, 1) \times (G_m, 1))$.

Proof. Given any element $a \in ZF_m(W \times X \times G_m \times G_m, Y \times G_m \times G_m)$, set

\[
H'(a) = (id_Y \times \Psi) \circ (a \circ id_{\mathbb{A}^1}) \circ (id_Y \times \Psi \circ id_{\mathbb{A}^1}) \circ (id_X \times G_m \times G_m \times \Delta) \in \quad \text{ZF}_m(W \times X \times G_m \times G_m, Y \times G_m \times G_m),
\]

where $\Delta: \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$ is the diagonal morphism. Then for any element $a \in ZF_m(W \times X \times (G_m, 1) \times (G_m, 1), Y \times (G_m, 1) \times (G_m, 1))$ one has

\[
H'(a)_0 = [id_Y \times \Sigma^{-1}(\varepsilon)] \circ a \circ [id_X \times \Sigma^{-1}(\varepsilon)] \quad \text{and} \quad H'(a)_1 = [id_Y \times \Sigma^{n}(\tau)] \circ a \circ [id_X \times \Sigma^{n}(\tau)].
\]

It is easy to see that there are matrices $A, B \in SL_{m+2n}(k)$ such that for any element $a \in ZF_m(W \times X \times (G_m, 1) \times (G_m, 1), Y \times (G_m, 1) \times (G_m, 1))$ one has

\[
\Phi_A([id_Y \times \Sigma^{-1}(\varepsilon)] \circ a \circ [id_X \times \Sigma^{-1}(\varepsilon)]) = \Sigma^{2n}(a),
\]

\[
\Phi_B([id_Y \times \Sigma^{n}(\tau)] \circ a \circ [id_X \times \Sigma^{n}(\tau)]) = ([id_Y \times \tau] \circ a \circ [id_X \times \tau]) \boxtimes \Sigma^{2n}([id_Y \times \tau] \circ a \circ [id_X \times \tau]).
\]

Choose matrices $A_1, B_1 \in SL_{m+2n}(k)[s]$ such that $A_0 = id, A_1 = A, B_0 = id, B_1 = B$. Then for the matrix $C_1 = B_1 \circ A_1 \circ \varepsilon \in SL_{m+2n}(k)[s]$ one has $C_0 = A, C_1 = B$. Set $H = \Phi_A \circ H'$. Then for the chosen element $a \in ZF_m(W \times X \times (G_m, 1) \times (G_m, 1), Y \times (G_m, 1) \times (G_m, 1))$, one has

\[
H_0(a) = \Phi_A(H'(a)_{0}) = \Sigma^{2n}(a) \quad \text{and} \quad H_1(a) = \Phi_B(H'(a)_{1}) = \Sigma^{2n}([id_Y \times \tau] \circ a \circ [id_X \times \tau]),
\]

as was to be proved. □
6. The inverse morphism

The main aim of this section is to define for any integers \( n, m \geq 0 \) a subpresheaf \( \mathbb{Z}^m_n(\mathbb{G}_m, X \times \mathbb{G}_m) \) of the presheaf \( \mathbb{Z}^m(\mathbb{G}_m, X \times \mathbb{G}_m) \) and define a morphism of abelian presheaves

\[
\rho_n : \mathbb{Z}^m_n(\mathbb{G}_m, X \times \mathbb{G}_m) \to \mathbb{Z}^m(\mathbb{G}_m, Y \times \mathbb{G}_m) \to \mathbb{Z}^m_n(\mathbb{G}_m, Y \times \mathbb{G}_m)
\]

We also prove certain properties of morphisms \( \rho_n \) and of presheaves \( \mathbb{Z}^m_n(\mathbb{G}_m, X \times \mathbb{G}_m) \) which are used in the proof of the Linear Cancellation Theorem (Theorem D).

We begin with some general remarks. Let \( X \) and \( Y \) be \( k \)-smooth schemes. Consider a framed correspondence

\[
a = (Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_m), g) \in \mathcal{F}_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m).
\]

Let \( (U, p : U \to \mathbb{A}^m \times (X \times \mathbb{G}_m), s : Z \to U) \) be the étale neighborhood of \( Z \) in \( \mathbb{A}^m \times (X \times \mathbb{G}_m) \) from the definition of the framed correspondence \( a \). Let \( t \) be the invertible function on \( X \times \mathbb{G}_m \) corresponding to the projection on \( \mathbb{G}_m \) and \( u \) be invertible function on \( Y \times \mathbb{G}_m \) corresponding to the projection on \( \mathbb{G}_m \). Let \( f_2 = g^{-1}(u) \) and \( f_1 = p_{X \times \mathbb{G}_m}^{-1}(t) \) be two invertible functions on \( U \), where \( p_{X \times \mathbb{G}_m} = pr_X \circ p : U \to X \times \mathbb{G}_m \). Set \( g = (g_1, g_2) \), where \( g_1 = p_{Y} \circ g \) and \( g_2 = p_{\mathbb{G}_m} \circ g \).

Since \( Z \) is finite over \( X \times \mathbb{G}_m \), the \( \mathcal{O}_X \times \mathbb{G}_m \times \mathbb{G}_m \)-sheaf \( P_a := \mathcal{O}_U/(\varphi_1, \varphi_2, \ldots, \varphi_m) \) is finite over \( X \times \mathbb{G}_m \). Since the sheaf \( P_a \) is finite over \( X \times \mathbb{G}_m \), it is automatically flat over \( X \times \mathbb{G}_m \).

Let \( Z^+_n \) be the closed subset of \( Z \) defined by the equation \( (f_1^{n+1} - 1)|_Z = 0 \). Let \( Z^-_n \) be the closed subset of \( Z \) defined by the equation \( (f_1^{n+1} - f_2)|_Z = 0 \). Note that \( Z^+_n \) is finite over \( X \) if and only if \( \mathcal{O}_U/(f_1^{n+1} - 1, \varphi_1, \varphi_2, \ldots, \varphi_m) \) is finite over \( X \). By [S, 4.1] the latter \( \mathcal{O}_X \)-module is always finite and even flat. Note that \( Z^-_n \) is finite over \( X \) if and only if \( \mathcal{O}_U/(f_1^{n+1} - f_2^2, \varphi_1, \varphi_2, \ldots, \varphi_m) \) is finite over \( X \). As it was mentioned above, the \( \mathcal{O}_X \)-module \( P_a = \mathcal{O}_U/(\varphi_1, \varphi_2, \ldots, \varphi_m) \) is finite and flat over \( X \). By [S, 4.1] the \( \mathcal{O}_X \)-module \( \mathcal{O}_U/(f_1^{n+1} - f_2, \varphi_1, \varphi_2, \ldots, \varphi_m) \) is finite and even flat over \( X \) for sufficiently large \( n \). In particular, \( Z^-_n \) is finite over \( X \) for sufficiently large \( n \).

**Definition 6.1.** Let \( X \) and \( Y \) be \( k \)-smooth schemes. Consider a framed correspondence \( a = (Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_m), g) \in \mathcal{F}_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \). Set

\[
\rho^+_n(a) := (Z^+_n, U, (f_1^{n+1} - 1, \varphi_1, \varphi_2, \ldots, \varphi_m), g_1)
\]

and

\[
\rho^-_n(a) := (Z^-_n, U, (f_1^{n+1} - f_2, \varphi_1, \varphi_2, \ldots, \varphi_m), g_1).
\]

As we have mentioned above, \( Z^+_n \) is finite over \( X \) for all \( n \geq 0 \), hence \( \rho^+_n(a) \in \mathcal{Z}_m+1(X, Y) \). We say that \( \rho^+_n(a) \) is defined if \( Z^-_n \) is finite over \( X \), which is equivalent to saying that the \( \mathcal{O}_X \)-module \( P_a = (f_1^{n+1} - f_2)P_a \) is finite and flat over \( X \). If \( \rho^-_n(a) \) is defined, then we set

\[
\rho^-_n(a) = \rho^+_n(a) - \rho^-_n(a) \in \mathcal{Z}_m+1(X, Y)
\]

and say that \( \rho^-_n(a) \) is defined.

Given integers \( m, n \geq 0 \), denote by \( \mathcal{F}_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \) the subset of those framed correspondences \( a \in \mathcal{F}_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \) for which the \( \mathcal{O}_X \)-module \( P_a/(f_1^{n+1} - f_2)P_a \) is finite over \( X \) (that is \( \rho^-_n(a) \) is defined). It follows from [S, 4.4] that the assignment \( X' \mapsto \mathcal{F}_m(X' \times \mathbb{G}_m, Y \times \mathbb{G}_m) \) is a subpresheaf of \( \mathcal{F}_m(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) \).

19
Definition 6.2. Define a presheaf of abelian groups $\mathcal{Z}\mathcal{F}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ as follows. Its sections on $X$ is the abelian group $\mathbb{Z}[\mathcal{F}_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)]$ modulo a subgroup generated by all elements of the form

$$(Z_1 \sqcup Z_2, U_1 \sqcup U_2, \varphi_1 \sqcup \varphi_2, g_1 \sqcup g_2) - (Z_1, U_1, \varphi_1, g_1) - (Z_2, U_2, \varphi_2, g_2).$$

It is straightforward to check that $\mathcal{Z}\mathcal{F}_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ is a free abelian group with a free basis consisting of the elements of the form $a = (Z, U, \varphi, g)$, where $Z$ is connected and the $\mathcal{O}_X$-module $P_a/(f_1^{n+1} - f_2)P_a$ is finite and flat over $X$. Moreover, the group $\mathcal{Z}\mathcal{F}_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ is a subgroup of the group $\mathcal{Z}\mathcal{F}_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$, and $\mathcal{Z}\mathcal{F}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ is a subpresheaf of the presheaf $\mathcal{Z}\mathcal{F}_m(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)$.

It follows from [S, 4.4] that for any morphism $f : X' \to X$ of smooth varieties the following diagram is commutative

$$
\begin{array}{c}
\mathcal{Z}\mathcal{F}_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m) \\
\rho_{n,f^*} \downarrow \\
\mathcal{Z}\mathcal{F}_m^{(n)}(X' \times \mathbb{G}_m, Y \times \mathbb{G}_m)
\end{array}
$$

We see that $\rho_{n,f^*} : \mathcal{Z}\mathcal{F}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) \to \mathcal{Z}\mathcal{F}_m^{(n)}(-, Y)$ is a morphism of pointed presheaves. We can extend it to get a morphism of presheaves of abelian groups $\mathbb{Z}[\mathcal{F}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m)] \to \mathcal{Z}\mathcal{F}_m^{(n)}(-, Y).$ This morphism annihilates the elements of the form

$$(Z_1 \sqcup Z_2, U_1 \sqcup U_2, \varphi_1 \sqcup \varphi_2, g_1 \sqcup g_2) - (Z_1, U_1, \varphi_1, g_1) - (Z_2, U_2, \varphi_2, g_2).$$

Definition 6.3. The above arguments show that the presheaf morphism $\rho_{n,f^*}$ induces a unique presheaf of abelian groups morphism

$$\rho_n : \mathcal{Z}\mathcal{F}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) \to \mathcal{Z}\mathcal{F}_m^{(n)}(-, Y)$$

such that for any $a \in \mathcal{Z}\mathcal{F}_m^{(n)}(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ one has $\rho_n(a) = \rho_{n,f^*}(a)$. We also call $\rho_n$ the inverse morphism.

Lemma 6.4. The following relations are true:

$$\mathcal{F}_m(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) = \text{colim}_n \mathcal{F}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m),$$

$$\mathcal{Z}\mathcal{F}_m(- \times \mathbb{G}_m, Y \times \mathbb{G}_m) = \text{colim}_n \mathcal{Z}\mathcal{F}_m^{(n)}(- \times \mathbb{G}_m, Y \times \mathbb{G}_m).$$

This lemma follows from the following

Proposition 6.5. ([S, 4.1]) For any framed correspondence $a \in \mathcal{F}_m(X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ one has:

(a) for any $n = 0$, the sheaf $P_a/(f_1^{n+1} - 1)P_a$ is finite and flat over $X$;

(b) there exists an integer $N$ such that, for any $n \geq N$, the sheaf $P_a/(f_1^{n+1} - f_2)P_a$ is finite and flat over $X$.

We shall need the following obvious property of $\rho_n$. 

20
Lemma 6.6. For any integers \( m, n, r \geq 0 \), the following diagram commutes
\[
\begin{array}{ccc}
\mathbb{Z}F_m^{(n)}(- \times G_m, Y \times G_m) & \xrightarrow{\Sigma'} & \mathbb{Z}F_{m+r}^{(n)}(- \times G_m, Y \times G_m) \\
\rho_n & & \rho_n \\
\mathbb{Z}F_{m+1}(-, Y) & \xrightarrow{\Sigma'} & \mathbb{Z}F_{m+1+r}(-, Y).
\end{array}
\]

Lemma 6.7. Let \( X \) and \( Y \) be \( k \)-smooth schemes. Then for any integers \( m \) and \( n \) and any \( a \in \mathbb{Z}F_m(X, Y) \), one has \( a \otimes (id-e_1) \in \mathbb{Z}F_m^{(n)}(X \times G_m, Y \times G_m) \). In particular, for any integers \( m \) and \( n \) there is defined the composite morphism
\[
\rho_n \circ (\otimes (id-e_1)) : \mathbb{Z}F_m(- \times X, Y) \to \mathbb{Z}F_m^{(n)}(- \times X \times G_m, Y \times G_m) \to \mathbb{Z}F_{m+1}(- \times X, Y).
\]
Moreover, for an element \( a \in \mathbb{Z}F_m(W \times X, Y) \) of the form \((Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_m), g)\) one has
\[
\rho_n(a \otimes (id-e_1)) = -(Z \times Z(t^{n+1}-1), U \times G_m, t^{n+1} \varphi_1(t, \varphi_2, \ldots, \varphi_m), g) + (Z \times Z(t^{n+1}-1), U \times G_m, t^{n+1} \varphi_1(t, \varphi_2, \ldots, \varphi_m) + 1, \varphi_1(t, \varphi_2, \ldots, \varphi_m), g) \in \mathbb{Z}F_{m+1}(W \times X, Y).
\]

Proof. Let \( a \in \mathbb{Z}F_m(W \times X, Y) \) be the image of \((Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_m), g) \in \mathbb{F}_m(W \times X, Y)\). Then
\[
a \otimes (id-e_1) = (Z \times G_m, U \times G_m, (\varphi_1, \varphi_2, \ldots, \varphi_m), (g, t)) - (Z \times G_m, U \times G_m, (\varphi_1, \varphi_2, \ldots, \varphi_m), (g, e_1)) \in \mathbb{Z}F_m(W \times X \times G_m, Y \times G_m),
\]
where \( t \) is the coordinate function on \( G_m \). Clearly, \( Z_n^+ = Z \times Z(t^{n+1}-1) \subset Z \times G_m \) and \( Z_n^- = Z \times Z(t^{n+1}) \subset Z \times G_m \). Both sets are finite over \( X \). Hence \( a \otimes (id-e_1) \in \mathbb{Z}F_m^{(n)}(X \times G_m, Y \times G_m) \) in this case. Any element of \( \mathbb{Z}F_m(W \times X, Y) \) is a linear combination of elements of the form \((Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_m), g)\). This proves the first assertion of the lemma.

Computing \( \rho_n(a \otimes (id-e_1)) \) for \( a = (Z, U, (\varphi_1, \varphi_2, \ldots, \varphi_m), g) \) we obtain
\[
\rho_n(a \otimes (id-e_1)) = -(Z \times Z(t^{n+1}-1), U \times G_m, t^{n+1} \varphi_1(t, \varphi_2, \ldots, \varphi_m), g) + (Z \times Z(t^{n+1}-1), U \times G_m, t^{n+1} \varphi_1(t, \varphi_2, \ldots, \varphi_m) + 1, \varphi_1(t, \varphi_2, \ldots, \varphi_m), g) \in \mathbb{Z}F_{m+1}(W \times X, Y),
\]
as was to be shown. \( \square \)

Lemma 6.8. Let \( X \) and \( Y \) be \( k \)-smooth schemes. Then for every even integer \( m \) and any \( n \) one has
\[
\rho_n \circ (\otimes (id-e_1)) \sim (\otimes \varepsilon) : \mathbb{Z}F_m(- \times X, Y) \Rightarrow \mathbb{Z}F_{m+1}(- \times X, Y),
\]
where \( \varepsilon = (\{ 0 \}, A_1^{\varepsilon}, -t, c') \in \mathbb{F}_1(\text{pt}, \text{pt}) \).

Proof. Set \( \eta_n = \rho_n \circ (\otimes (id-e_1)) \). Take the matrix
\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix} \in SL_{m+1}(k)
\]
and let \( A_s \in SL_{m+1}(k[s]) \) be such that \( A_0 = id, A_1 = A \). Let \( H_{A_s} \) be the \( A_1^{\varepsilon} \)-homotopy from Definition 4.6 between the identity and \( \varphi_A \). By Definition 4.6 one has
\[
\eta_n = \rho_n \circ (\otimes (id-e_1)) \frac{H_{A_s} \circ \eta_n}{\varphi_A \circ \rho_n \circ (\otimes (id-e_1)) = \varphi_A \circ \eta_n}.
\]
Set $H' = H_A \circ \eta_n$. By Lemma 4.14 it remains to find an $H''$ such that $\varphi_A \circ \eta_n H'' (\cdot \boxtimes \mathcal{E})$ and set $H = H' + H'' - H_{\varphi \circ \eta_n}$. In this case by Lemma 4.4 one gets $\rho_n \circ (\cdot \boxtimes (id - e_1)) = \eta_n H'' (\cdot \boxtimes \mathcal{E})$.

To construct $H''$, note that by the last statement of Lemma 6.7 one has

$$\varphi_A \circ \eta_n = - \boxtimes [Z(t^{n+1} - 1), \mathbb{G}_m, t^{n+1} - 1 - c] - (Z(t^{n+1} - t), \mathbb{G}_m, t^{n+1} - t, c)$$

and $(- \boxtimes \mathcal{E}) = - \boxtimes \{0\}, \mathbb{A}^1, -t, c\}$, where where $c: \mathbb{G}_m \to pt$ is the canonical projection. By Lemma 4.9 one can take $H''$ to be an $\mathbb{A}^1$-homotopy of the form $H'' = (- \boxtimes h'')$, where $h'' \in \mathbb{Z}F_1(\mathbb{A}^1, pt)$ is such that

$$(Z(t^{n+1} - 1), \mathbb{G}_m, t^{n+1} - 1 - c) - (Z(t^{n+1} - t), \mathbb{G}_m, t^{n+1} - t, c) = h''$$

and

$$h'' (\{0\}, \mathbb{A}^1, -t, c') \in \mathbb{Z}F_1(pt, pt),$$

where $c': \mathbb{A}^1 \to pt$ is the canonical projection. Now let us find the desired element $h''$. Since $t^{n+1} - 1$ does not vanish at $t = 0$, we can extend the neighborhood from $\mathbb{G}_m$ to $\mathbb{A}^1$ to get an equality,

$$(Z(t^{n+1} - 1), \mathbb{G}_m, t^{n+1} - 1 - c) = (Z(t^{n+1} - 1), \mathbb{A}^1, t^{n+1} - 1, c') \in \mathbb{Z}F_1(pt, pt).$$

By Lemma 4.14 there is $h'' \in \mathbb{Z}F_1(\mathbb{A}^1, pt)$ such that

$$(Z(t^{n+1} - 1), \mathbb{A}^1, t^{n+1} - 1, c') = h''$$

and

$$h'' = (Z(t^{n+1} - t), \mathbb{A}^1, t^{n+1} - t, c') \in \mathbb{Z}F_1(pt, pt),$$

because polynomials $t^{n+1} - t$ and $t^{n+1} - 1$ have the same degree and the same leading coefficient. Using the additivity property for supports in $\mathbb{Z}F_1(pt, pt)$ and the second statement of Lemma 4.13 we can find an element $h'' \in \mathbb{Z}F_1(\mathbb{A}^1, pt)$ such that

$$(Z(t^{n+1} - t), \mathbb{G}_m, t^{n+1} - t, c) = h''$$

and

$$h'' = (Z(t^{n+1} - t), \mathbb{A}^1, t^{n+1} - t, c') \in \mathbb{Z}F_1(pt, pt).$$

Set $h'' := h'' - h'' \in \mathbb{Z}F_1(\mathbb{A}^1, pt)$. Then $h''$ is the desired element.

Set $H'' = (- \boxtimes h'')$ and $H = H' + H'' - H_{\varphi \circ \eta_n}$. Then $H$ is the desired $\mathbb{A}^1$-homotopy. That is

$$\rho_n \circ (\cdot \boxtimes (id - e_1)) = H'$$

and our statement follows. \hfill \Box

7. Theorem D

The main purpose of this section is to prove Theorem D. We sometimes identify simplicial abelian groups with chain complexes concentrated in non-negative degrees by using the Dold–Kan correspondence.

**Lemma 7.1.** Let $X$ and $Y$ be $k$-smooth schemes and $m, r, N \geq 0$ be integers. Then for any Moore cycle $a \in \mathbb{Z}F_m(\Delta^r \times X, Y)$ of the simplicial abelian group $\mathbb{Z}F_m(\Delta^r \times X, Y)$, one has $\mathcal{E} (\cdot \boxtimes (id - e_1)) \in \mathbb{Z}F_m(\Delta^r \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. Moreover, $\rho_N (a \boxtimes (id - e_1))$ is a Moore cycle. The homology classes of Moore cycles

$$a \boxtimes \mathcal{E}$$

and $\rho_N (a \boxtimes (id - e_1))$

coincide in $\mathbb{Z}F_{m+1}(\Delta^r \times X, Y)$.

**Proof.** The element $a \boxtimes (id - e_1)$ is in $\mathbb{Z}F_m(\Delta^r \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ by Lemma 6.7. Since $\mathbb{Z}F_m(\Delta^r \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$ is a presheaf, then $\delta_i (a \boxtimes (id - e_1)) \in \mathbb{Z}F_m(\Delta^r \times X \times \mathbb{G}_m, Y \times \mathbb{G}_m)$. Since the morphism $\rho_N$ is a morphism of presheaves, then

$$\delta_i (\rho_N (a \boxtimes (id - e_1))) = \rho_N (\delta_i (a \boxtimes (id - e_1))) = \rho_N (\delta_i (a) \boxtimes (id - e_1)) = 0.$$
This proves the first assertion of the lemma.

By Lemma 6.8, the morphism

\[ a' \mapsto \rho_N(a' \boxtimes (id_{G_m} - e_1)) : ZF_m(- \times X, Y) \to ZF_m^{(N)}(- \times X, Y) \to ZF_{m+1}(- \times X, Y) \]

is \( A^1 \)-homotopic to the morphism \( a' \mapsto a' \boxtimes \epsilon \). Thus the corresponding morphisms of the simplicial abelian groups \( ZF_m(\Delta^1 \times X, Y) \to ZF_{m+1}^{(N)}(\Delta^1 \times X, Y) \) induce the same morphisms on homology. Hence the homology class of the Moore cycle \( \rho_N(a \boxtimes (id_{G_m} - e_1)) \) coincides with the homology class of the Moore cycle \( a \boxtimes \epsilon \).

\[ \square \]

Lemma 7.2. One has \( \epsilon \boxtimes \epsilon \sim \Sigma^2 \) in \( \mathbb{Z} F_2(\text{pt}, \text{pt}) \). Moreover, for any integer \( r \geq 0 \) one has \( \epsilon \boxtimes \epsilon \boxtimes \Sigma^r \sim \Sigma^{2+r} \) in \( \mathbb{Z} F_2(\text{pt}, \text{pt}) \).

Corollary 7.3. Let \( X \) and \( Y \) be \( k \)-smooth schemes and \( m \geq 0 \) be an integer. Then,

\[ (- \boxtimes \Sigma^2) \sim (- \boxtimes \Sigma^2) : ZF_m(- \times X, Y) \to ZF_{m+2}(- \times X, Y) \]

and

\[ (- \boxtimes \Sigma^2 \boxtimes \Sigma^r) \sim (- \boxtimes \Sigma^{2+r}) : ZF_m(- \times X, Y) \to ZF_{m+2+r}(- \times X, Y). \]

Therefore the first pair of maps produces the same maps on homology

\[ H_*(ZF_m(\Delta^1 \times X, Y)) \to H_*(ZF_{m+2}(\Delta^1 \times X, Y)). \]

Similarly, the second pair of maps gives the same maps on homology

\[ H_*(ZF_m(\Delta^1 \times X, Y)) \to H_*(ZF_{m+2+r}(\Delta^1 \times X, Y)). \]

Lemma 7.4. Let \( X \) and \( Y \) be \( k \)-smooth schemes and \( m \geq 0 \) be an integer. Then for any integer \( r \geq 0 \) one has

\[ \text{Ker}(- \boxtimes (id_{G_m} - e_1)) : H_*(ZF_m(\Delta^1 \times X, Y)) \to H_*(ZF_m((\Delta^1 \times X) \cap (G_m, 1) \times (G_m, 1))) \subseteq \text{Ker}(- \boxtimes \Sigma^2) : H_*(ZF_m(\Delta^1 \times X, Y)) \to H_*(ZF_{m+2}(\Delta^1 \times X, Y)). \]

Proof. Since all complexes of the lemma are simplicial abelian groups, we may work with the associated Moore complexes. Thus, assume that

\[ a \in ZF_m(\Delta^1 \times X, Y) \]

is a Moore cycle for which \( a \boxtimes (id_{G_m} - e_1) \) is a boundary, i.e., there exists \( b \in ZF_m((\Delta^1 \times X) \times G_m, Y \times G_m) \) such that \( \partial_i(b) = 0 \) for \( i = 0, 1, \ldots, r \) and \( \partial_{r+1}(b) = a \boxtimes (id_{G_m} - e_1) \). By Lemma 6.4, there exists an \( N \) such that \( b \in ZF_m^{(N)}((\Delta^1 \times X) \times G_m, Y \times G_m) \). Since \( ZF_m^{(N)}(- \times G_m, Y \times G_m) \) is presheaf, then \( \partial(b) \in ZF_m^{(N)}(\Delta^1 \times X \times G_m, Y \times G_m) \). Since \( \rho_N \) is a presheaf morphism \( ZF_m^{(N)}(- \times X \times G_m, Y \times G_m) \to ZF_{m+1}(- \times X, Y) \), one has \( \partial_i(\rho_N(b)) = \rho_N(\partial_i(b)) \). Thus,

\[ \partial_i(\rho_N(b)) = \rho_N(\partial_i(b)) = 0 \text{ for } 0 \leq i \leq r, \]

\[ \partial_{r+1}(\rho_N(b)) = \rho_N(\partial_{r+1}(b)) = \rho_N(a \boxtimes (id_{G_m} - e_1)). \]

We see that the homology class of the Moore cycle \( \rho_N(a \boxtimes (id_{G_m} - e_1)) \) vanishes. By Lemma 7.1, the homology class of the Moore cycle \( a \boxtimes \epsilon \) vanishes in \( H_*(ZF_{m+1}(\Delta^1 \times X, Y)) \). Thus the homology class of the Moore cycle \( a \boxtimes \epsilon \boxtimes \epsilon \) vanishes in \( H_*(ZF_{m+2}(\Delta^1 \times X, Y)) \). By Corollary 7.3, the homology class of \( a \boxtimes \Sigma^2 \) vanishes in \( H_*(ZF_{m+2}(\Delta^1 \times X, Y)) \), too. \[ \square \]
Lemma 7.5. Let $X$ and $Y$ be $k$-smooth schemes and $m,r > 0$ be integers. Let $n$ be the integer from Lemma 5.2. Then for any Moore cycle $a \in ZF_{m}((A^{r} \times X) \wedge (G_{m}, 1), Y \wedge (G_{m}, 1))$ there exists an integer $N$ such that the element $\rho_{N}(a)$ is defined and the homology class of the Moore cycle
\[
\Sigma^{2n}(\rho_{N}(a)) \otimes (id - e_{1}) \in ZF_{m+2n+1}((A^{r} \times X) \wedge (G_{m}, 1), Y \wedge (G_{m}, 1))
\]
coinsides with the homology class of the Moore cycle $\Sigma^{2n}(a \boxtimes e)$.

Proof. Set $a' = a \boxtimes (id - e_{1})$. Let $H$ be the $A^{1}$-homotopy from Lemma 5.4. Consider the element $H(a') \in ZF_{m+2n}((A^{r} \times X) \times G_{m} \times G_{m}, Y \times G_{m} \times G_{m})$.

By Lemma 6.4 there is an integer $N$ such that
\[
a \in ZF_{m}^{(N)}((A^{r} \times X) \times G_{m}, Y \times G_{m})
\]
and
\[
H(a') \in ZF_{m+2n}^{(N)}((A^{r} \times X) \times G_{m} \times G_{m} \times A^{1}, Y \times G_{m} \times G_{m}).
\]
Since $a'$ is a Moore cycle and $H$ is a presheaf morphism, the element $H(a')$ is a Moore cycle in $ZF_{m+2n}((A^{r} \times X) \times G_{m} \times G_{m} \times A^{1}, Y \times G_{m} \times G_{m})$. Since
\[
ZF_{m+2n}^{(N)}((- \times X) \times G_{m} \times G_{m} \times A^{1}, Y \times G_{m} \times G_{m})
\]
is a subpresheaf of $ZF_{m}((- \times X) \times G_{m} \times G_{m} \times A^{1}, Y \times G_{m} \times G_{m})$, it follows that $H(a')$ is a Moore cycle in $ZF_{m+2n}((A^{r} \times X) \times G_{m} \times G_{m} \times A^{1}, Y \times G_{m} \times G_{m})$.

Applying the presheaf morphism $\rho_{N} : ZF_{m+2n}((- \times X) \times G_{m} \times G_{m} \times A^{1}, Y \times G_{m} \times G_{m}) \to ZF_{m+2n+1}((- \times X) \times G_{m} \times A^{1}, Y \times G_{m})$ to the Moore cycle $H(a')$, we get a Moore cycle
\[
\rho_{N}(H(a')) \in ZF_{m+2n+1}((- \times X) \times G_{m} \times A^{1}, Y \times G_{m}).
\]
Hence
\[
i_{0}(\rho_{N}(H(a'))) \in ZF_{m+2n+1}((- \times X) \times G_{m}, Y \times G_{m})
\]
and $i_{1}^{1}(\rho_{N}(H(a'))) \in ZF_{m+2n+1}((- \times X) \times G_{m}, Y \times G_{m})$ are Moore cycles, too. Furthermore,
\[
i_{0}(\rho_{N}(H(a'))) = \rho_{N}(i_{0}^{1}(H(a'))) = \rho_{N}(\Sigma^{2n}(a')) = \Sigma^{2n}(\rho_{N}(a'))
\]
and
\[
i_{1}^{1}(\rho_{N}(H(a'))) = \rho_{N}(i_{1}^{1}(H(a'))) = \rho_{N}(\Sigma^{2n}([id_{Y} \times \tau] \circ a' \circ (id_{X} \times \tau))).
\]
The two morphisms
\[
i_{0}^{0}, i_{1}^{1} : ZF_{m+2n+1}((- \times X) \times G_{m} \times A^{1}, Y \times G_{m}) \Rightarrow ZF_{m+2n+1}((- \times X) \times G_{m}, Y \times G_{m})
\]
of simplicial abelian groups induce the same morphisms on homology. The element $\rho_{N}(H(a'))$ is a Moore cycle. Thus the homological classes of the Moore cycles $i_{0}^{0}(\rho_{N}(H(a'))) = i_{1}^{1}(\rho_{N}(H(a'))) = \Sigma^{2n}(\rho_{N}(a'))$ coincide in $H_{r}(ZF_{m+2n+1}((- \times X) \times G_{m}, Y \times G_{m}))$.

By Lemma 6.6 one has $\rho_{N}(\Sigma^{2n}(a')) = \Sigma^{2n}(\rho_{N}(a'))$. Thus the first homological class is the class of $\Sigma^{2n}(\rho_{N}(a')) = \Sigma^{2n}(\rho_{N}(a \boxtimes (id - e_{1})))$. By Lemma 7.1 the latter homological class coincides with the class of the element $\Sigma^{2n}(a \boxtimes e)$.

The element $i_{1}^{1}(\rho_{N}(H(a'))) = \rho_{N}(i_{1}^{1}(H(a'))) = \rho_{N}(\Sigma^{2n}([id_{Y} \times \tau] \circ (id_{X} \times \tau)))$.

By Lemma 6.6 the latter element coincides with
\[
\Sigma^{2n}(\rho_{N}([id_{Y} \times \tau] \circ (id \times (id - e_{1})) \circ (id_{X} \times \tau))) = \Sigma^{2n}(\rho_{N}(a \boxtimes (id - e_{1}))).
\]
Hence the homological classes $\Sigma^{2n}(a \boxtimes e)$ and $[\Sigma^{2n}(\rho_{N}(a) \boxtimes (id - e_{1}))]$ coincide in $H_{r}(ZF_{m+2n+1}((- \times X) \times G_{m}, Y \times G_{m}))$. Finally, the complex $ZF_{m+2n+1}((- \times X) \wedge (G_{m}, 1), Y \wedge G_{m})$.
\((G_m, 1)\) is a direct summand in \(\mathbb{Z}F_{n+2m+1}\left((\Delta^* \times X) \times G_m, Y \times G_m\right)\) and the elements \(\Sigma^{2n}(a \boxtimes e), \Sigma^{2n}(\rho_N(a) \boxtimes (id - e_1))\) are in \(\mathbb{Z}F_{n+2m+1}\left((\Delta^* \times X) \wedge (G_m, 1), Y \wedge (G_m, 1)\right)\). Hence the homological classes \([\Sigma^{2n}(\rho_N(a) \boxtimes (id - e_1))]\) coincide in \(H_r(\mathbb{Z}F_{n+2m+1}\left((\Delta^* \times X) \wedge (G_m, 1), Y \wedge (G_m, 1)\right))\).

**Lemma 7.6.** Let \(X\) and \(Y\) be \(k\)-smooth schemes and \(m, r \geq 0\) be integers. Let \(n\) be the integer from Lemma 7.4 Then

\[
\text{Im}((- \boxtimes \Sigma^{2n+2}) : H_r(\mathbb{Z}F_m((\Delta^* \times X) \wedge (G_m, 1), Y \wedge (G_m, 1)) \to H_r(\mathbb{Z}F_{m+2n+2}((\Delta^* \times X) \wedge (G_m, 1), Y \wedge (G_m, 1)))) \leq \text{Im}((- \boxtimes (id_{G_m} - e_1)) : H_r(\mathbb{Z}F_{m+2n+2}(\Delta^* \times X, Y)) \to H_r(\mathbb{Z}F_{m+2n+2}((\Delta^* \times X) \wedge (G_m, 1), Y \wedge (G_m, 1))))
\]

**Proof.** Take a Moore cycle \(a' \in \mathbb{Z}F_m((\Delta^* \times X) \wedge (G_m, 1), Y \wedge (G_m, 1))\). Then the element \(a := a' \boxtimes e\) is a Moore cycle in \(\mathbb{Z}F_{m+1}((\Delta^* \times X) \wedge (G_m, 1), Y \wedge (G_m, 1))\). By Lemma 7.5 the homology classes of \(\Sigma^{2n}(a \boxtimes e)\) and \(\Sigma^{2n}((\rho_N(a) \boxtimes (id - e_1))\) coincide in \(H_r(\mathbb{Z}F_{m+2n+2}((\Delta^* \times X) \wedge (G_m, 1), Y \wedge (G_m, 1)))\).

By Corollary 7.3 the homology classes of \(\Sigma^{2n}(a \boxtimes e)\) and \(\Sigma^{2n}(\rho_N(a') \boxtimes e)\) coincide. Hence the homology classes of \(\Sigma^{2n+2}(a' + \boxtimes e)\) and \(\Sigma^{2n+2}((\rho_N(a') \boxtimes e))\) coincide in \(H_r(\mathbb{Z}F_{m+2n+2}((\Delta^* \times X) \wedge (G_m, 1), Y \wedge (G_m, 1)))\).

We are now in a position to prove Theorem D.

**Theorem D.** Let \(X\) and \(Y\) be \(k\)-smooth schemes. Then

\[
- \boxtimes (id_{G_m} - e_1) : \mathbb{Z}F(\Delta^* \times X, Y) \to \mathbb{Z}F((\Delta^* \times X) \wedge (G_m, 1), Y \wedge (G_m, 1))
\]

is a quasi-isomorphism of complexes of abelian groups.

**Proof.** The theorem follows from Lemmas 7.4 and 7.6.

---

**Appendix A. Some commutative diagrams for Theorem A**

In this section we give a detailed description of some arrows which are used implicitly in the proof of Theorem A.

Let \(\text{Hom} : sPre_\bullet(Sm/k)^{op} \times Sp_{f^r}^{fr} \to Sp_{f^r}^{fr}\) be the natural functor. For a morphism \(a : F \to G\) in \(sPre_\bullet(Sm/k)\), where \(E, G \in Sp_{f^r}^{fr}\), set \(a^* := \text{Hom}(a, E)\). Given a \(k\)-smooth variety \(X\), a morphism \(f : W \to W'\) of simplicial objects in \(F_{0}(k)\), and an object \(E \in sPre_\bullet(k)\), we put \(f_* := \text{Hom}(F,LM_{f^r}(id_X \times f))\).

We need some objects and morphisms in \(sPre_\bullet(Sm/k)\). The inclusion \((G_m, 1) \to (-, G_m)\) induces a morphism in \((G_m, 1) \to (-, G_m) / (-, pt)\) in \(sPre_\bullet(Sm/k)\). Let \(r : (-, G_m) / (-, pt) \to (-, G_m) / (-, pt)\) be the natural projection. Let \(\text{can} : (-, G_m) / (-, pt) \to G\) be the morphism from zero-simplices of \(G\) to \(G\) itself. Let \(\text{Cone} \in sPre_\bullet(k)\) be the simplicial mapping cone for the identity map \((-, pt) \to (-, pt)\). The morphism \((-, G_m) / (-, pt) \to (-, G_m) / (-, pt)\) induces a morphisms \(p_G : G \to \text{Cone}\). The inclusion \((-, G_m) / (-, G_m) \to (-, G_m) / (-, G_m)\) induces a morphisms \(i_G : \text{Cone} \to G\). Set \(e_G := i_G \circ p_G\). Let \(\text{can}' : G_m \to G_m \wedge 1\) be the morphism from the zero-simplices of \(G_m\) to \(G_m\) itself. We regard this morphism as a morphism of simplicial objects in \(F_{0}(k)\). The morphism \(e_{G_m} : G_m \to pt \to G_m\) will be regarded as a morphism in \(F_{0}(k)\). But sometimes it will be regarded as a morphism of \(sPre_\bullet(Sm/k)\).
Define $LM_f(X) \xrightarrow{-\otimes (\text{id}_{G_m} - e_{G_m})} \text{Hom}((G_m, 1), LM_f(X \times G_m))$ as a unique morphism such that $LM_f(X) \xrightarrow{-\otimes (\text{id}_{G_m} - e_{G_m})} \text{Hom}((G_m, 1), LM_f(X \times G_m)) \xrightarrow{\text{can}^*} \text{Hom}((\mathbb{G}_m, 1), LM_f(X \times G_m))$ coincides with $-\otimes (\text{id}_{G_m} - e_{G_m})$. Let us define $\text{can}^* \circ (\text{id}_G^* - e_G^*) : \text{Hom}(\mathbb{G}_m, LM_f(X \times G_m^1)) \rightarrow \text{Hom}(\mathbb{Z}, LM_f(X \times G_m^1))$ as a unique morphism such that $r^* \circ \text{can}^* \circ (\text{id}_G^* - e_G^*) = \text{can}^* \circ (\text{id}_G^* - e_G^*)$.

The following lemma is useful for the proof of Theorem A.

**Lemma A.1.** The composite morphism in $\text{Sp}^f_{S^1}(k)$

$$LM_f(X) \xrightarrow{c_0} \text{Hom}(\mathbb{G}_m, LM_f(X \times G_m^1)) \xrightarrow{\text{can}^* \circ (\text{id}_G^* - e_G^*)} \text{Hom}(\mathbb{Z}, LM_f(X \times G_m^1)) \xrightarrow{\text{id}} \text{can}^* \circ (\text{id}_G^* - e_G^*) : \text{Hom}(\mathbb{G}_m, LM_f(X \times G_m^1)) \xrightarrow{\text{can}^* \circ (\text{id}_G^* - e_G^*)} \text{Hom}(\mathbb{Z}, LM_f(X \times G_m^1))$$

coincides with $\text{can}^* \circ (\text{id}_G^* - e_G^*)$. Moreover, the morphism $\text{can}^* \circ (\text{id}_G^* - e_G^*)$ is a sectionwise stable equivalence. Also, the following diagram in $\text{Sp}^f_{S^1}(k)$ commutes

$$
\begin{array}{ccc}
LM_f(X) & \xrightarrow{\text{id}} & \text{Hom}(\mathbb{G}_m, LM_f(X \times G_m^1)) \\
\downarrow \text{can}^* \circ (\text{id}_G^* - e_G^*) & & \downarrow \text{can}^* \circ (\text{id}_G^* - e_G^*) \\
LM_f(X) & \rightarrow & \text{Hom}(\mathbb{G}_m, LM_f(X \times G_m^1))
\end{array}
$$

and $\text{can}^* \circ (\text{id}_G^* - e_G^*)$ is a sectionwise stable equivalence. Here $1_X$ is the natural morphism and $LM_f(X \times G_m^1) := \text{Ker}[LM_f(X \times G_m) \xrightarrow{1} LM_f(X \times \text{pt})]$.

**Proof.** Similarly to Remark 4.2 all the framed $S^1$-spectra presheaves from the lemma are the Segal $S^1$-spectra corresponding to certain framed presheaves of simplicial abelian groups. Namely, they correspond to $ZF(\Delta^1 \times - \times X)$, $\text{Hom}(\mathbb{G}_m, ZF(\Delta^1 \times - \times X \times G_m^1))$, $\text{Hom}(\mathbb{G}_m, 1, ZF(\Delta^1 \times - \times X \times G_m^1))$, $\text{Hom}(\mathbb{G}_m, 1, ZF(\Delta^1 \times - \times X \times G_m^1))$, respectively. All the framed $S^1$-spectra presheaves morphisms from the lemma correspond to certain morphisms between those framed presheaves of simplicial abelian groups. These easily yield an equality

$$\text{can}^* \circ (\text{id}_G^* - e_G^*) = [\text{can}^* \circ (\text{id}_G^* - e_G^*)] \circ 0$$

and the commutativity of the diagram of the lemma.

We argue in the same fashion to prove the last assertion of the lemma. It suffices to show that for any $U \in Sm/k$ the morphism

$$\text{can}^* \circ (\text{id}_G^* - e_G^*) : ZF(U, X \times \mathbb{G}_m^1) \rightarrow ZF(U, X \times \mathbb{G}_m^1)$$

is a quasi-isomorphism. The latter follows from the equalities $ZF(U, X \times \mathbb{G}_m^1) = \text{Ker}[ZF(U, X \times \mathbb{G}_m^1) \xrightarrow{1} ZF(U, Y)]$. Hence $can^* \circ (\text{id}_G^* - e_G^*)$ is indeed a stable equivalence.

The morphism $\text{can}^* \circ (\text{id}_G^* - e_G^*)$ is a sectionwise stable equivalence for similar reasons. Indeed, the simplicial abelian group presheaf morphism

$$\text{Hom}(\mathbb{G}_m, ZF(X \times \mathbb{G}_m^1)) \xrightarrow{\text{can}^* \circ (\text{id}_G^* - e_G^*)} \text{Hom}(\mathbb{G}_m, 1, ZF(X \times \mathbb{G}_m^1))$$

is a quasi-isomorphism, because for any $Y \in Sm/k$ the morphism

$$\text{Hom}(\mathbb{G}_m, ZF(Y)) \xrightarrow{\text{can}^* \circ (\text{id}_G^* - e_G^*)} \text{Hom}(\mathbb{G}_m, 1, ZF(Y))$$

is a simplicial presheaf quasi-isomorphism. □

26
In this section another definition of the \((S^1, \mathbb{G})\)-bispectrum \(M^G_{fr}(X)\) is given. This definition uses the functor associating Segal’s \(S^1\)-spectrum to a \(\Gamma\)-space. It does not use any extra simplicial machinery. The structure morphisms are described quite similarly to the morphism \("- \boxtimes (id_{\mathbb{G}_m} - e_1)"\) from Theorem D. So our aim is to construct morphisms
\[
a_\eta: M_{fr}(X \times \mathbb{G}_m^{\wedge n}) \to \text{Hom}(\mathbb{G}, M_{fr}(X \times \mathbb{G}_m^{\wedge (n+1)}))
\]
in \(SP^f_{\mathbb{G}}(k)\). We start with preparations.

There is a canonical pair of adjoint functors
\[
\Phi: sPre_*(Sm/k) \rightleftarrows sPre^f_*(k): \Psi,
\]
where \(\Psi\) is the forgetful functor and \(\Phi(X) = Fr_+(-,X)\) for a \(k\)-smooth variety \(X\) (see [GP1] Section 4) for the definition of \(Fr_+(Y,X)\). Let \(sPre^f_*(Fr_0(k))\) be the category of pointed simplicial presheaves on the category \(Fr_0(k)\). The functor \(Fr_+(k) \times Fr_0(k) \to Fr_+(k)\) sending a pair \((X,Y)\) to \(X \times Y\) has a left Kan extension
\[
Pre^f_*(k) \times sPre^f_*(Fr_0(k)) \xrightarrow{\boxtimes} sPre^f_*(k).
\]
It takes a pair \((X,Y), X,Y \in Sm/k, \) to \(X \boxtimes Y := \Phi(X \times Y) = Fr_+(-,X \times Y)\). In particular, \(Fr(-,-) \boxtimes Y = Fr(-,X \times Y)\).

Below we use notation from Sections 2 and 3. In particular, \(\mathbb{G}\) and \(\mathbb{G}_m^{\wedge 1}\) are as in Section 3. We regard \(\mathbb{G}_m^{\wedge 1}\) as a simplicial object in \(Fr_0(k)\). There is also a left Kan extension functor \(\Phi_0: sPre_*(Sm/k) \to sPre^f_*(Fr_0(k))\) adjoint to the forgetful functor \(\Psi_0\). We set \(G := \Phi_0(\mathbb{G})\). One has an obvious morphism
\[
u_G: G \to \mathbb{G}_m^{\wedge 1}
\]
in \(sPre^f_*(Fr_0(k))\) taking \(+\) of to the empty object. The adjunction \(\Phi\) specializes to an isomorphism
\[
\text{Hom}_{sPre_*(Sm/k)}(U_+ \wedge G, Fr(-,Y \times \mathbb{G}_m^{\wedge 1})) = \text{Hom}_{sPre^f_*(k)}((U \boxtimes G)/(U \boxtimes +), Fr(-,Y \boxtimes \mathbb{G}_m^{\wedge 1}))
\]
showing that
\[
\text{Hom}_{sPre_*(Sm/k)}(\mathbb{G}, Fr(-,Y \times \mathbb{G}_m^{\wedge 1}))(U) = \text{Hom}_{sPre^f_*(k)}((U \boxtimes G)/(U \boxtimes +), Fr(-,Y \boxtimes \mathbb{G}_m^{\wedge 1}))(U).
\]
(7)
For any \(k\)-smooth scheme \(Y\), the presheaf \(Fr(-,Y)\) on \(Fr_+(k)\) is a pointed Nisnevich sheaf. It is covariantly functorial in \(Y\) with respect to the category \(Fr_0(k)\). Moreover, \(Fr(\emptyset, Y) = pt\). Thus we have a pointed \(\Gamma\)-space
\[
\Gamma^{op} \triangleright (K, *) \mapsto Fr(U, Y \otimes K),
\]
where the right hand side is regarded as a constant pointed simplicial set. Varying \(U\) in \(Fr_+(k)\) we get a Nisnevich \(Fr_+(k)\)-sheaf of pointed \(\Gamma\)-spaces with \(Fr(\emptyset, Y \otimes K) = pt\).

Let \(X\) be a \(k\)-smooth scheme. Let \(A\) be a \(n\)-multisimplicial object of \(Fr_0(k)\). Consider two \(n\)-multisimplicial \(\Gamma\)-spaces
\[
(K, *) \mapsto Fr(-, (X \otimes K) \times A([r_1],...,[r_n]))(U),
\]
\[
(K, *) \mapsto \text{Hom}_{sPre_*(Sm/k)}(\mathbb{G}, Fr(-, (X \otimes K) \times A([r_1],...,[r_n]) \times \mathbb{G}_m^{\wedge 1}))(U).
\]
Since \( u_G \) takes \( + \) to the empty object, then for any \( k \)-smooth variety \( Y \) and any \( \gamma \in \text{Fr}(U,Y) \) the morphism \( \gamma \otimes u_G : U \otimes G \to \text{Fr}(-,Y) \otimes G_m^\Lambda \) in \( sPre^\Gamma(k) \) takes \( U \otimes + \) to the empty simplicial object. Thus there is a unique morphism

\[
\gamma \otimes u_G : (U \otimes G)/(U \otimes +) \to \text{Fr}(-,Y) \otimes G_m^\Lambda
\]

in \( sPre^\Gamma(k) \) which coincides with \( \gamma \otimes u_G \) after precomposing with the morphism \( U \otimes G \to (U \otimes G)/(U \otimes +) \). Under the identification \( [7] \), the assignment

\[
\gamma \mapsto (U \otimes G)/(U \otimes +) \to \text{Fr}((X \otimes K) \times A([r_1],\ldots,[r_n])) \otimes G_m^\Lambda
\]

is a morphism between two \( n \)-multisimplicial pointed \( \Gamma \)-spaces

\[
((K,\ast) \mapsto \text{Fr}((-,(X \otimes K) \times A))(U)) \xrightarrow{\alpha_U} ((K,\ast) \mapsto \text{Hom}_{sPre^\Gamma(Sm/k)}(G,\text{Fr}((-,(X \otimes K) \times A \times G_m^\Lambda))(U)).
\]

Taking diagonals on both sides, we get a morphism of \( \Gamma \)-spaces. Furthermore, taking the associated Segal’s \( S^1 \)-spectra, we get a morphism of \( S^1 \)-spectra

\[
a_U : \text{Fr}((-,(X \otimes S) \times A))(U) \to \text{Hom}(G,\text{Fr}((-,(X \otimes S) \times A \times G_m^\Lambda)))(U),
\]

where \( S \) is the simplicial sphere \( S^1 \)-spectrum. Clearly, the family \( \alpha : \{ \alpha_U | U \in Fr_s(k) \} \) is a morphism of presheaves of pointed \( \Gamma \)-spaces. Hence the family

\[
a = \{ \alpha_U | U \in Fr_s(k) \} : \text{Fr}((-,(X \otimes S) \times A)) \to \text{Hom}(G,\text{Fr}((-,(X \otimes S) \times A \times G_m^\Lambda))
\]

is a morphism in the category \( S^1 \)-spectra. Replacing “\( - \)” with “\( \Delta^* \times - \)”, we get a morphism in the category \( Sp^\Gamma_{S^1}(k) \) of framed \( S^1 \)-spectra

\[
a_A : M_{fr}(X \times A) \to \text{Hom}(G,M_{fr}(X \times A \times G_m^\Lambda)).
\]

Taking \( A = G_m^{\Lambda n} \) we get a morphism

\[
a_n := a_{G_m^{\Lambda n}} : M_{fr}(X \times G_m^{\Lambda n}) \to \text{Hom}(G,M_{fr}(X \times G_m^{\Lambda(n+1)})).
\]

**Definition B.1.** The \((S^1,G)\)-bispectrum \( M_{fr}^{G}(X) \) is defined as

\[
(M_{fr}(X),M_{fr}(X \times G_m^\Lambda),M_{fr}(X \times G_m^{\Lambda 2}),\ldots)
\]

together with the structure morphisms \( a_n \)-s. Similarly a \((S^1,G)\)-bispectrum \( LM_{fr}^{G}(X) \) is defined as

\[
(LM_{fr}(X),LM_{fr}(X \times G_m^\Lambda),LM_{fr}(X \times G_m^{\Lambda 2}),\ldots)
\]

together with similar structure morphisms \( c_n \)-s. Namely, one can use for this the \( \Gamma \)-spaces \((K,\ast) \mapsto ZF(U,Y \otimes K)).

The interested reader can easily verify that the maps \( a_n \)-s and \( c_n \)-s defined above coincide with the maps \( \{2\} \) of Section 3.

We finish the paper by the following remark.

**Remark B.2.** As it is explained in Section 3 the framed \( S^1 \)-spectrum \( LM_{fr}(X \times G_m^{\Lambda n}) \) is the Eilenberg-Mac Lane spectrum associated with the complex \( C_* ZF(-,X \times G_m^{\Lambda n}). \) It is easy to see that the morphism \( c_n \) is a morphism of EM-spectra associated with the simplicial abelian group presheaf morphism

\[
[n] \mapsto ZF(U,X \times (G_m^{\Lambda n})_r) \to [n] \mapsto \text{Hom}_{sPre^\Gamma(k)}((U \otimes G)/(U \otimes +),ZF(-,(X \times (G_m^{\Lambda n})_r) \times G_m^\Lambda))(U)
\]

\[
= \text{Hom}_{sPre^\Gamma(k)}(G,ZF(-,(X \times (G_m^{\Lambda n})_r) \times G_m^\Lambda))(U)
\]

28
given by $\gamma \mapsto \gamma \boxtimes u_G$. 

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CHEBYSHEV LABORATORY, ST. PETERSBURG STATE UNIVERSITY, 14TH LINE, 29B, 199178 ST. PETERSBURG, RUSSIA
E-mail address: alseang@gmail.com

DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, SINGLETOR PARK, SWANSEA SA2 8PP, UNITED KINGDOM
E-mail address: g.garkusha@swansea.ac.uk

ST. PETERSBURG BRANCH OF V. A. STEKLOV MATHEMATICAL INSTITUTE, FONTANKA 27, 191023 ST. PETERSBURG, RUSSIA

ST. PETERSBURG STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS AND MECHANICS, UNIVERSITETSKY PROSPEKT, 28, 198504, PETERHOF, ST. PETERSBURG, RUSSIA
E-mail address: paniniv@gmail.com