Geometric realization of the $\gamma$-vectors of 2-truncated cubes

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Let $f_i$ be the number of $i$-dimensional faces of a simple $n$-polytope $P$. The tuple of numbers $(f_0, \ldots, f_n)$ is called its $f$-vector. The polynomial $F(P)(\alpha, t) = \alpha^n + f_{n-1} \alpha^{n-1} t + \cdots + f_1 \alpha t^{n-1} + f_0 t^n$ is called the $F$-polynomial, and the polynomial $H(P)(\alpha, t) = h_0 \alpha^n + h_1 \alpha^{n-1} t + \cdots + h_{n-1} \alpha t^{n-1} + h_n t^n = F(P)(\alpha - t, t)$ is called the $H$-polynomial.

For simple polytopes the Dehn–Sommerville relations (see [9]) are equivalent to $H(P)$ being symmetric; consequently, $H(P)$ can be represented in the form

$$H(P) = \sum_{i=0}^{[n/2]} \gamma_i(\alpha t)^i (\alpha + t)^{n-2i}.$$ 

The tuple of coefficients $(\gamma_0, \gamma_1, \ldots, \gamma_{[n/2]})$ is called the $\gamma$-vector. The polynomial $\gamma(P)(\tau) = \gamma_0 + \gamma_1 \tau + \cdots + \gamma_{[n/2]} \tau^{[n/2]}$ is called the $\gamma$-polynomial.

A simplicial complex is called a flag complex if any set of its pairwise incident vertices forms a simplex. A simple polytope is called a flag polytope if the boundary of its dual polytope is a flag simplicial complex, that is, any set of its pairwise intersecting faces has non-empty intersection. Gal’s conjecture (see [6]) in the case of convex polytopes states that a simple flag polytope has non-negative $\gamma$-vector.

**Definition 1.** A simple polytope $P^n$ is called a 2-truncated cube if it can be obtained from the cube $I^n$ by a sequence of truncations of faces of codimension 2 (the boundary of the dual polytope is obtained from the cross-polytope by a sequence of stellar subdivisions along edges).

The class of 2-truncated cubes was introduced in [2], where one of the results was the proof of Gal’s conjecture for this class. It was shown in [3] that the known classes of simple polytopes (graph-associahedra, graph-cubihedra, flag nestohedra) are 2-truncated cubes. The class of flag nestohedra contains the generalized associahedra of series $A$ (the Stasheff polytopes) and of series $B$ (the Bott–Taubes polytopes). It was shown in [7] that the generalized associahedra of series $D$ are not flag nestohedra but nevertheless are 2-truncated cubes. For every flag nestohedron $P$, a flag simplicial complex $\Delta_P$ such that $\gamma(P) = f(\Delta_P)$ was constructed in [1]. Earlier such complexes were constructed for the Stasheff polytopes, the Bott–Taubes polytopes, and the permutahedra (see [8]). The central result of the present paper is the following theorem.

**Theorem 2.** For every 2-truncated cube $P^n$ there exists a flag simplicial complex $\Delta_P$ such that $\gamma(P) = f(\Delta_P)$. Here $f(K)(t) = 1 + f_0 t + f_1 t^2 + \cdots + f_d t^{d+1}$, where $d$ is the dimension of the complex $K$.

**Proof.** Consider a 2-truncated cube $P^n$ and the set $W(P)$ of all its hyperfaces obtained as a result of consecutive truncations of faces. For each face $G \subset P$, including $G = P$, we construct a flag simplicial complex $\Delta_G$ on the set of vertices $W(P)$ such that $f(\Delta_G)(t) = \gamma(G)(t)$ and if $G_1 \subset G_2$, then $\Delta_{G_1} \subset \Delta_{G_2}$.

Let $P = I^n$; then $W(P) = \emptyset$. In this case we set $\Delta_G = \emptyset$ for all faces.

This research was supported by a grant from the Government of the Russian Federation according to enactment no. 220 (contract no. 11.G34.31.0053).

AMS 2010 Mathematics Subject Classification. Primary 52B05; Secondary 05E45.

DOI 10.1070/RM2012v067n03ABEH004800.
Suppose that a family of the required simplicial complexes has already been constructed for a polytope $P$, and a polytope $P'$ is obtained from $P$ by a truncation of a face $G_0$ of codimension 2. Then $W(P') = W(P) \cup \{v(G_0)\}$, where the vertex $v(G_0)$ corresponds to the new hyperface of $P'$. Consider an arbitrary face $G' \subset P'$. If $G'$ was obtained from some face $G$ by truncation of a face $F$ of it of codimension 2, then we set $\Delta_{G'} = \Delta_G \cup \text{Cone}(\Delta_F)$. The $f$-polynomials of the complexes $\Delta_{G'}$ and $\Delta_G$ are connected by the relation

$$f(\Delta_{G'}) = f(\Delta_G) + tf(\Delta_F).$$

A similar relation (see [2]) connects the $\gamma$-vectors of the faces $G$ and $G'$:

$$\gamma(G') = \gamma(G) + \tau \gamma(F).$$

Consequently, $f(\Delta_{G'}) = \gamma(G')$. If the face $G'$ was obtained from some face $G$ by truncation of a face of codimension 1, or $G'$ coincides with $G$, then we set $\Delta_{G'} = \Delta_G$.

Obviously, $f(\Delta_{G'}(t)) = \gamma(G')(t)$ and if $G'_1 \subset G'_2$, then $\Delta_{G'_1} \subset \Delta_{G'_2}$.

The complex $\Delta_{P'}$ is a flag complex—as the union of the flag complexes $\Delta_{G'}$ and $\text{Cone}(\Delta_F)$, the intersection of which is also the flag complex $\Delta_F$. □

We point out that the complex $\Delta_P$ depends on the chosen sequence of truncations.

Let $\binom{n}{k}^r$ denote the number of $k$-cliques in the Turán graph $T_{n,r}$. For positive integers $m$, $k$, and $r \geq k$, the expansion $m = \binom{n_k}{k} + \cdots + \binom{n_{k-s}}{k-s}$ uniquely defined, where $n_{k-s} - \left\lfloor \frac{n_{k-s}}{r-s} \right\rfloor > n_{k-s-1}$ for all $0 \leq i < s$ and $n_{k-s} \geq k-s > 0$. We set

$$m^{(k)}_r = \binom{n_k}{k+1} + \cdots + \binom{n_{k-s}}{k-s+1}.$$

By [5], the $f$-vector of a flag simplicial complex is the $f$-vector of some balanced simplicial complex. Hence, by using the Frankl–Füredi–Kalai inequalities (see [4]) for the $f$-vectors of simplicial complexes, we obtain the following result, which in the case of flag nestohedra is contained in [1].

**Corollary 3.** Let $P^n$ be a 2-truncated cube. Then

1) $\gamma_0 = 1$;

2) $0 \leq \gamma_{i+1} \leq \gamma_i^{(r)}$, where $r = \lfloor n/2 \rfloor$.

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Presented by V. M. Buchstaber
Accepted 10/APR/12
Translated by E. KHUKHRO