Quintessence or Phoenix?

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(November 12, 2018)

Abstract

We show that it is impossible to determine the state equation of quintessence models on the basis of pure observational SNIa data. An independent estimate of $\Omega_{\Lambda 0}$ is necessary. Also in this most favourable case the situation can be problematic.

PACS number(s): 98.80.Cq, 98.80.Hw, 04.20.Jb

I. INTRODUCTION

... l’araba fenice  
*Che vi sia, ciascun lo dice  
Dove sia, nessun lo sa*

In the history of cosmology there are many cases of Arabic phoenixes. The metaphor applies particularly well to the cosmological constant, which seems to resurrect out of its ashes and challenges any interpretation since almost one century [1,2]. In its last resurrection (that is, quintessence [3–7]), it poses the problems of determining the state equation of the peculiar quintessential fluid and/or the right potential of the associated scalar field.

1"... Everybody says//The phoenix is there,//But no one knows where”, in “Così fan tutte”, by L. da Ponte.
In this paper we want to investigate some subtleties in the so-called ‘reconstruction of the state equation’ on the basis of observational data, showing that it is probably impossible to determine the state equation and even the value of $\Omega_{M0}$ only on the basis of pure observations. In other words: you cannot find the phoenix if you have no idea about its face.

The starting idea came from a paper by A.A. Sen and S. Sethi [8], in which they propose an ansatz for the Hubble parameter as a function of $t$ and a suitable parameter $\beta$. They find that it is possible to obtain a good agreement with present data on SNIa for a certain range of values of $\beta$. A particular choice allows then the exact evaluation of the scalar field potential. The surprise is that the value of $\Omega_{M0}$ is obtained in terms of an integration constant. This means that the theory fits the data for any arbitrary value of this parameter!

Another result in this direction is due to I. Wasserman [9]. He finds that the usual expression for $H(z)$, in the case of presence of a $\Lambda$-term plus dust, can be analytically derived from a quintessence potential with an independent choice for the value of $\Omega_{M0}$.

In Sec. 2 of this paper we generalize these results, showing that they do not depend on the particular ansatz, so that any empiric evaluation of the state equation must be supplied with a value for $\Omega_{M0}$, obtained by independent observations.

In Sec. 3 we show that also in the ideal situation of almost infinite precision in observational data, the reconstruction of the state equation could be impossible.

In Sec. 4 we consider three exactly integrable models, one found by us [10], and those studied by Sen and Sethi [8] and Wasserman [9], showing that they all can perfectly mimic a $\Lambda$-term model.

In Sec. 5 discussion and conclusions are given.

II. RECONSTRUCTION OF THE STATE EQUATION

Let us consider a spatially flat universe, minimally coupled with a scalar field, and adopt the conventions $8\pi G = 1$, $c = 1$, $a_0 = 1$, $H_0 = 1$, where $a_0$ is the present day scale factor, and $H_0$ is the value of the Hubble constant (in appropriate units). This can be done without any loss of generality.

Starting from the Friedman and Klein-Gordon equations

$$3H^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) + 3\Omega_{M0}a^{-3},$$

$$\dot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0,$$

it is possible (differentiating Eq.(1) and with easy algebraic manipulation) to derive

$$\dot{\varphi}^2 = -2\dot{H} - 3\Omega_{M0}a^{-3},$$

$$\varphi = \int \sqrt{-2\dot{H} - 3\Omega_{M0}a^{-3}} dt,$$

$$V = 3H^2 + \dot{H} - \frac{3}{2}\Omega_{M0}a^{-3}.$$

The last two equations give a parametric expression for $V(\varphi)$, once an ansatz for $a(t)$, or $H(t)$, is given. The point is that also a value for $\Omega_{M0}$ should be supplied.
The situation is complicated by the fact that the ansatz contains parameters whose link with observations could be rather difficult to find, if not impossible. Thus, we prefer to illustrate a situation nearer to the observational strategy.

Assume that it is possible to observe an enormous number of type Ia supernovae with extreme precision, so that it is possible to construct an empirical function for the luminosity distance versus the redshift (this is in fact what is really measured in the supernovae experiments), say \( d_L(z) \). It will contain some numerical coefficients whose physical interpretation could be difficult or even impossible. (In the case of time dependence, on the contrary, the physical meaning of the parameters is usually clear.)

From \( d_L(z) \) it is possible to derive \( H(z) \). With our normalization, the definition of \( d_L(z) \) is

\[
d_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z')},
\]

which gives

\[
H(z) = \frac{(1 + z)^2}{(1 + z)d_L'(z) - d_L(z)}.
\]

Here and below, prime stands for derivation with respect to \( z \). From Eqs.(4) and (5), it is now possible to write

\[
\varphi(z) = \sqrt{2H' \frac{2H - 3\Omega_M(1 + z)}{H^2}}, \tag{8}
\]

\[
V(z) = 3H^2 - HH'(1 + z) - \frac{3}{2} \Omega_M(1 + z)^3, \tag{9}
\]

which now allow to obtain \( V(\varphi) \) starting from \( d_L(z) \) (we give an example below). But the most interesting result is the expression of the state equation versus the redshift

\[
w = \frac{2HH'(1 + z) - 3H^2}{3H^2 - 3\Omega_M(1 + z)^3}. \tag{10}
\]

It is then clear that the value of \( \Omega_M \) must be supplied independently, and that the model obtained works perfectly with any value!

**III. BUT THINGS STAY EVEN WORSE!**

The situation is, however, complicated by other subtleties. Let us illustrate it with an example, which should also make clearer the previous arguments.

Let us take as fiducial model a simple \( \Lambda \)-term model, with \( \Omega_M = 0.3 \) (which is of course irrelevant and is taken in homage to current fashion). It is then easy to produce a data set of 200 points for \( d_L(z) \), in the range \( z = 0 \div 2 \). The precision is that of numerical integration of the MATHEMATICA algorithm, i.e., practically infinite. Assume also that \( \Omega_M \) has been determined independently, again with almost infinite precision.
Now we find an empirical $d_L(z)$ as a quartic polynomial, by means of a best fit. Having set $H_0 = 1$, we keep fixed the first coefficient and get

$$d_{L_{\text{fit}}}(z) = z + 0.760z^2 - 0.257z^3 + 0.040z^4.$$  

(11)

The fractional difference with the data set is < 0.002.

If we find $w(z)$ according to Eq. (11), and with the correct value $\Omega_{M0} = 0.3$, we obtain the plot in Fig. 1. It is clear at a first glance that $w$ is far from being constant, but the real problem is that the values $w < -1$ are absurd (in this context). Indeed, they imply $\dot{\varphi}^2 < 0$ and are of course an artifact of the procedure; this could be interpreted as a signal of something being wrong.

But we have also to remember that the value $\Omega_{M0} = 0.3$ is supposed to come from independent observations. We know that the precision of such observations is presently very low, and we can figure out that, also in the ideal situation here examined, it is well possible to consider a slightly different value of 0.28. In this case we have no problems up to $z \sim 1.7$ (see Fig. 2), and it is possible to ‘reconstruct’ a potential like that in Fig. 3, which looks nice but, of course, has nothing to do with the starting point of our analysis. The situation is not substantially changed even if we go up to a 7th-degree polinomial as best fit.

The last desperate trial is to use, instead of the best fit, an interpolation of the data set. In this case all is made with the internal precision of MATHEMATICA (16 digits) and gives for $w$ the plot in Fig. 4, and only at this level it is clear that the variations are due to numerical computation.

IV. MAY A PORTRAIT OF THE PHOENIX BE OF HELP?

A possible objection to the argument of Sec. 3 is that an empirical polynomial is a very crude assumption, and that we should try with specific models. In this case the infinite amount of possibilities poses some problems of choice. We present here three possible models, which have the nice property of being exact solution of the equations, so that all the considerations are very clear and no approximation error can be invoked.

The first model is given by a potential already studied by us \[10–12\] (but see also \[13\]), which shows a simple exponential dependence on $\varphi$

$$V = V_0 \exp(-\sqrt{\frac{3}{2}} \varphi).$$  

(12)

For the details of the procedure for finding the solution and for the subsequent discussion on its properties, see \[10\]. Here we limit ourselves to present the expression of $H(t)$ and $z(t)$, adapted to our normalizations

$$H(t) = \frac{(1 + 2t^2)(t_0 + t_0^3)}{t(1 + t^2)(1 + 2t_0^2)},$$  

(13)

$$z(t) = \sqrt[3]{\frac{t^2(1 + t^2)}{t_0^3(1 + t_0^2)}} - 1.$$  

(14)
The variable $t$ could be eliminated, in order to have $H(z)$ explicitly, but there is no need of doing this, and everything can be made by treating Eqs. (13) and (14) as defining a parametric dependence. The only parameter is $t_0$, which is linked to the value of $\Omega_{M0}$, by

$$\Omega_{M0} = \frac{1 + t_0^2}{(1 + 2t_0^2)^2}. \quad (15)$$

This model has been tested \[10–12\] with the currently available data on SNIa \[13–18\] and on peculiar velocities \[19\], and seems to indicate a value for $\Omega_{M0}$ much lower than the usually indicated one, but there is no definite evidence for this. In any case a range like $\Omega_{M0} = 0.15 \div 0.30$, is fully compatible.

We now compare the results of this model with the fiducial $\Lambda$-term, in the ideal case. It should be clear that there is no reason why the value of $\Omega_{M0}$ should be the same in the two cases: it is a free parameter of the theory, which has to be adapted to data, and the results can be very different. If we set $t_0 = 1$, corresponding to $\Omega_{M0} = 0.22$, and compare $d_L(z)$, from Eqs.(13) and (14) with the above fiducial model, we obtain for the fractional difference an agreement up to 2%.

In this case, the value of $\Omega_{M0}$ is an important element of the model, and the fact that it is significantly different in the two cases means that an independent measure would be of great help in distinguishing them. The problem is in the very poor accuracy of this kind of determination, but we do not want to examine observational technicalities in this paper.

Let us instead present a case in which the value of $\Omega_{M0}$ is unpredictable, as announced before. In \[8\] Sen and Sethi present an ansatz which, adapted to our normalizations, has the form

$$H(z) = \tanh(1) \sqrt{1 + \frac{(1 + z)^{2/\beta}}{\sinh(1)^2}}, \quad (16)$$

where $\beta$ is a parameter which has to be adjusted from data. They find as best fit value for current data $\beta = 0.81$, but $\beta = 2/3$ is still compatible, and has the advantage of leading to an analytic expression for the potential

$$V(\varphi) = \frac{A^2}{8} \left( \exp(2B\varphi) + \exp(-2B\varphi) \right) + V_0, \quad (17)$$

where $A$ is an arbitrary integration constant and, with our normalization,

$$B = \frac{3}{2A \coth(1)}; \quad V_0 = 3 \tanh(1)^2 - \frac{A^2}{4}. \quad (18)$$

The interesting fact is that $A$ is correlated to the value of $\Omega_{M0}$ by the relation

$$\Omega_{M0} = \frac{4 - 3A^2}{\sinh(1)^2}, \quad (19)$$

so that any value is allowed, provided that $d_L(z)$ derived from Eq.(16) fits the data. It is important to note that the special value of $\beta = 2/3$ only gives analytic evaluation, but any value can be used, leading to a situation similar to that in Sec. 2, but with a much more
reasonable ansatz for $H$. Assuming for simplicity $\beta = 2/3$, we can easily show that this model mimics a $\Lambda$-term one. In this case, since no free parameter is left, we have to change the value in the fiducial model. Taking $\Omega_\Lambda = \omega$, we again obtain a fractional difference less than 2%. We see that also in this case an independent estimate of $\Omega_{M0}$ would be of help, but only if we stick to a particular value of $\beta$.

Still more interesting is the situation illustrated by Wasserman in [9], where the match of a $\Lambda$-term model with a quintessence model is analytically exact, and yet the value of $\Omega_{M0}$ is arbitrary. It is also interesting that the potential found in this paper is of the same type as in Eq. (17).

V. CONCLUSIONS

As said in the introduction, if you want to catch the phoenix, you must have an idea of its aspect. The arguments of Sec. 4 show that this could be not enough. Assuming that the ‘real’ situation is the presence of a $\Lambda$-term, there is an infinite host of ‘reasonable’ quintessence models, with unpredictable values of $\Omega_{M0}$. The theoretical reasons for this are well explained by Maor and colleagues [20].

Is it then a black cow in a dark night? May be not completely. A feature which all these models share is that they are all totally empiric, i.e., skillful guesses, and are based on (and/or fit) observations without not so many definite ideas of the precise physical mechanism behind the proposed potential.

In other words, our opinion is that the above arguments are a sort of vindication of the theory against excessive trust in the observational results. The literature is full of papers about the wonderful perspectives opened by the future observations, and for sure they will be fundamental in the resolution of the problem. But a satisfactory model can be only one which has roots in fundamental physics and interfaces with the general cosmological theory.

Another conclusion which we draw is that a precise measure of $\Omega_{M0}$, independent of SNIa observations, could be of fundamental help (although probably not conclusive), but we cannot figure out how this goal could be reached in short time.

Despite the dramatic improvement in the observational data, which we expect in the future, the correct extension of the cosmological standard model still seems a very difficult task.

FIGURE CAPTIONS:

Fig. 1. The manifestly absurd ‘reconstructed state equation’ (see text).

Fig. 2. The state equation with $\Omega_{M0} = 0.28$. It is still incorrect, but only if one knows in advance that it should be $w = -1$.

Fig. 3. The ‘reconstructed potential’, which is of course only an artifact of the procedure, but seems reasonable.

Fig. 4. Eventually, the ‘correct’ result.
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