The concept of majorisation is explored as a tool to characterize the performance of a quantum Otto engine in the quasi-static regime. For a working substance in the form of a spin of arbitrary magnitude, majorisation yields a necessary and sufficient condition for the operation of the Otto engine, provided the canonical distribution of the working medium at the hot reservoir is majorised by its canonical distribution at the cold reservoir. For the case of a spin-1/2 interacting with an arbitrary spin via isotropic Heisenberg exchange interaction, we derive sufficient criteria for positive work extraction using the majorisation relation. Finally, local thermodynamics of spins as well as an upper bound on the quantum Otto efficiency is analyzed using the majorisation relation.

I. INTRODUCTION

The fast-growing field of quantum thermodynamics brings together methods and tools from a variety of research areas ranging from quantum information, open quantum systems, quantum optics, non-equilibrium thermodynamics, theory of fluctuations, estimation theory and so on [1–24]. The quantum extensions of the concepts of heat, work, and entropy have in turn led to generalizations of the classical heat cycles. The so-called quantum thermal machines exploit new thermodynamic resources such as quantum entanglement, coherence, quantum interactions and quantum statistics [25–32]. For instance, quantum Otto engine (QOE) based on various platforms has been widely studied for its possible quantum advantages—both in its quasi-static formulations as well as the ones based on time-dependent constraints [33–46]. A QOE offers conceptual simplicity by virtue of a clear separation of heat and work steps in its heat cycle. The quantum working medium used in these models may be taken in the form of spins, quantum harmonic oscillator, interacting systems and so on [47–54]. Further, theoretical models have motivated experimental realizations which promise to be a boost for future applications in devices [55, 56].

One of the prominent analytical tools guiding the theoretical developments is the notion of majorisation [57–64] and its generalizations [65–67]. The concept, originally perhaps from matrix analysis, finds wide applications in various areas of science, mathematics, economics, social sciences, including modern applications to entanglement theory and thermodynamic resource theories. The majorisation partial order was developed to quantify the notion of disorder, in a relative sense, when comparing probability distributions. Transformations between pure bipartite states by means of local operations and classical communication can be determined in terms of majorisation of the Schmidt coefficients of the states [68–71]. Majorisation has been shown to determine the possibility of state transformations in the resource theories of entanglement, coherence and purity. It also provides the first complete set of necessary and sufficient conditions for arbitrary quantum state transformations under thermodynamic processes [72–74], which rigorously accounts for quantum coherence among other quantum mechanical properties.

A majorisation relation may be defined in one of the equivalent ways, as follows. Suppose $x^\downarrow = (x_1, \ldots, x_n)$ and $y^\downarrow = (y_1, \ldots, y_n)$ are two real $n$-dimensional vectors, where $x^\downarrow$ and $y^\downarrow$ indicate that the elements are taken in the descending order. Then, the vector $x$ is said to be majorised by the
vector $y$, denoted as $x \prec y$, if, for each $m = 1, \ldots, n$, we have

$$\sum_{k=m}^{n} x_k \geq \sum_{k=m}^{n} y_k,$$

(1)

with equality holding for $m = 1$. The notion of majorisation can be readily applied to compare how two probability distributions $x$ deviate from a uniform distribution $u = (1/n, \ldots, 1/n)$. Thus, $x \prec y$ implies that the distribution $y$ is more ordered than $x$. An important consequence of this relation is the inequality: $\sum_{k=1}^{n} f(x_k) \geq \sum_{k=1}^{n} f(y_k)$, where $f$ is any continuous, real-valued concave function. For example, the relation $x \prec y$ implies that the corresponding Shannon entropies are related as: $S(x) \geq S(y)$, where $S(x) = -\sum_{k=1}^{n} x_k \ln x_k$. Further, there are many equivalent characterizations of the majorisation relation. For instance, $x$ is majorised by $y$ only when $y$ can be obtained from $x$ by the action of a bistochastic matrix [57].

In the present work, we characterize the operation of a QOE through the notion of majorisation. We show that a spin-based quantum working substance provides a natural platform by which the majorisation conditions characterize the operation of a QOE. Thus, majorisation provides sufficient criteria for the operation of a spin-based Otto engine. In fact, the analysis can be extended to a model of two spins coupled via Heisenberg exchange interactions. Further, majorisation provides insight into the local thermodynamics of individual spins in the coupled model. Using the majorisation conditions, we also validate an upper bound for Otto efficiency in the coupled case, which is tighter than the Carnot value.

The paper is organized as follows. In Section II, we describe the quantum Otto cycle and its various stages based on a quantum working substance. In Section III, we express the work output in terms of the relative entropy between the two equilibrium distributions corresponding to hot and cold reservoirs, and we show that a greater value of the Shannon entropy of the system at the hot reservoir (as compared to the cold reservoir) does not ensure that net work may be extracted in the Otto cycle. In Section IV, we show how the majorisation relation, $P \prec P'$, between the hot and cold reservoir equilibrium distributions lead to positive work condition for the QOE. This is shown in Section IV.A for a single spin-$s$. In Section V, we analyze the coupled spins model, showing our main results for a special case of $(1/2, 1)$ system. In Section VI, local work by individual spins is analyzed based on global conditions. Lastly, Section VII shows proves the enhancement in Otto efficiency based on majorisation conditions. We end our paper by summarizing our main results in Section VIII. The derivations of various results are presented in Appendix.

II. QUANTUM OTTO ENGINE (QOE)

The classical Otto cycle is a textbook example of a four-step heat cycle in which a classical working medium, in the form of a gas or air-fuel mixture, undergoes two adiabatic and two isochoric steps [75]. A QOE is based on a quantum generalization of the classical cycle, in which the quantum working substance (referred to below as the system) undergoes two quantum adiabatic steps and two isochoric steps. Here, we are interested in a quasi-static Otto cycle in which each of the steps can take an arbitrarily long time. A quantum adiabatic process, either in the compression or expansion stage to be explained below, is performed by varying an externally controllable parameter. Secondly, for such a process, the quantum adiabatic theorem [76] is assumed to hold so the process does not cause any transitions between the energy levels, thus preserving their occupation probabilities. The remaining two steps are the isochoric heating and cooling processes which involve thermal interaction of the system with the hot or cold reservoir. Here, the system gets enough time to reach thermal
equilibrium with the corresponding reservoir. The heat cycle is described in a more quantitative
detail as follows.

**FIG. 1:** Schematic of a quantum Otto cycle consisting of the stages 1 → 2 → 3 → 4 → 1. Steps 1 → 2 and 3 → 4 respectively denote quantum adiabatic expansion and compression processes, which preserve occupation probability and involve work $W_{1\rightarrow 2} < 0$ and $W_{3\rightarrow 4} > 0$. The net work extracted in one cycle is $W = W_{1\rightarrow 2} + W_{3\rightarrow 4} < 0$. In step 4 → 1, heat $Q_1 > 0$ enters the working substance from a hot reservoir, while in step 2 → 3, heat $Q_2 < 0$ is rejected to the cold reservoir. Conservation of energy implies, $Q_1 + Q_2 + W = 0$.

**Stage 1.** Consider an $n$-level quantum system with Hamiltonian $H(B_1)$ whose eigenvalues can be arranged (in descending order) as: $\varepsilon^↓ = (\varepsilon_n, \ldots, \varepsilon_1)$. The system is in thermal equilibrium with the hot reservoir at temperature $T_1$. The canonical occupation probabilities for different energy levels, $P_k = e^{-\varepsilon_k/T_1}/\sum_{k} e^{-\varepsilon_k/T_1}$, are arranged as $P^↓ = (P_1, \ldots, P_n)$, where we have set the Boltzmann constant $k_B$ equal to unity. The energy eigenstates are represented by $\{|\psi_k\rangle|k = 1,\ldots n\}$. Thus, the density matrix representing the thermal state of the system is given by:

$$\rho = \sum_{k=1}^{n} P_k |\psi_k\rangle \langle \psi_k|, \quad (2)$$

**Stage 2.** The system is detached from the hot reservoir and undergoes a quantum adiabatic process in which the external field strength is lowered from $B_1$ to $B_2$. Here, the quantum adiabatic theorem ensures that no transitions are induced between the energy levels in the change from $\varepsilon_k$ to $\varepsilon'_k$. Suppose that the energies after the first adiabatic process are given by: $\varepsilon'^↓ = (\varepsilon'_n, \ldots, \varepsilon'_1)$, where we assume no level-crossing as the Hamiltonian changes from $H(B_1)$ to $H(B_2)$.

**Stage 3.** The system is brought in thermal contact with the cold reservoir at temperature $T_2(< T_1)$. The energy eigenvalues remain at $\varepsilon'_k$ while the occupation probabilities change from $P_k$ to $P'_k = e^{-\varepsilon'_k/T_2}/\sum_k e^{-\varepsilon'_k/T_2}$, which are ordered as: $P'^↓ = (P'_1, \ldots, P'_n)$. Thus, the density matrix of the system at the end of Stage-3 is given by:

$$\rho' = \sum_{k=1}^{n} P'_k |\psi_k\rangle \langle \psi_k|. \quad (3)$$

**Stage 4.** The system is detached from the cold reservoir and the field strength is changed back to $B_1$. The occupation probabilities $\{P'_k\}$ remain unchanged, while the energy levels change back from $\{\varepsilon'_k\}$ to $\{\varepsilon_k\}$. 

Finally, the system is attached to the hot reservoir again whereby the initial state ($\rho$) is recovered, thus completing one heat cycle. Note that only heat is exchanged between the system and the reservoir during an isochoric process, which is given by the difference between the final and initial mean energies of the system in that process. Thus, in Stage-1 and Stage-3, the heat exchanged is given respectively as:

$$Q_1 = \sum_{k=1}^{n} \varepsilon_k(P_k - P'_k), \quad Q_2 = \sum_{k=1}^{n} \varepsilon'_k(P'_k - P_k).$$

(4)

On the other hand, only work is performed during the adiabatic branches of the quantum Otto cycle. Let $W$ be the net work performed in one cycle. Applying the law of conservation of energy to the cyclic process, we have: $Q_1 + Q_2 + W = 0$. The operation of a heat engine requires that heat is absorbed (rejected) by the system at the hot (cold) reservoir, while net work is extracted from the system by the end of the cycle. These conditions can be satisfied by choosing the sign convention: $Q_1 > 0$, $Q_2 < 0$ and $W < 0$. The net work performed by the QOE can then be written as

$$|W| = Q_1 + Q_2 = \sum_{k=1}^{n} (\varepsilon_k - \varepsilon'_k)(P_k - P'_k).$$

(5)

We denote $|W| \geq 0$ as the positive work condition (PWC) of our engine. The efficiency of the QOE is defined as $\eta = |W|/Q_1 = 1 + Q_2/Q_1$.

### III. RELATIVE ENTROPY AND QOE

In this section, we cast the thermodynamic quantities for a QOE in terms of the relative entropy which is defined as $D(x||y) \equiv \sum_k x_k (\ln x_k - \ln y_k) \geq 0$. Also known as the Kullback-Leibler divergence[77], this quantity is a measure of the ‘distance’ between two discrete probability distributions, and vanishes only when the distributions $x$ and $y$ are identical.

Now, the expressions for canonical probabilities may be inverted as: $\varepsilon_k \equiv -T_1 \ln \left( P_k \sum_j e^{-\varepsilon_j/T_1} \right)$ and $\varepsilon'_k \equiv -T_2 \ln \left( P'_k \sum_j e^{-\varepsilon'_j/T_2} \right)$. Substituting these expressions in Eq. (4), and after some algebra (Appendix A), the heat exchanged with each the reservoir is expressed as:

$$Q_1 = T_1(S_1 - S_2) - T_1 D(P'||P),$$

(6)

$$Q_2 = -T_2(S_1 - S_2) - T_2 D(P||P'),$$

(7)

where $S_1 = -\sum_k P_k \ln P_k$ and $S_2 = -\sum_k P'_k \ln P'_k$ are the Shannon entropies of the system in equilibrium with hot and cold reservoirs, respectively. as we have let $k_B = 1$ Shannon entropy is equal to canonical entropy.

Thus, it can be seen that $D(P'||P)$ is equal to the entropy generated in the hot isochoric step. $D(P||P')$ has a similar meaning for the cold isochoric step. The net work extracted in an Otto cycle is given by:

$$|W| = (T_1 - T_2)(S_1 - S_2) - T_1 D(P'||P) - T_2 D(P||P').$$

(8)

Using, Eq. (6) and Eq. (7) the total entropy generated in the heat cycle, $\Delta_{tot}S = -Q_2/T_2 - Q_1/T_1$, can be expressed as:

$$\Delta_{tot}S = D(P||P') + D(P'||P).$$

(9)

The total entropy generated in a quantum Otto cycle is thus equal to the symmetric sum of the relative entropies. This quantity is also known as the symmetrized divergence and is distinguished by
the fact that it serves as a metric in the space of probability distributions. Finally, the positivity of the total entropy generated proves the consistency of the QOE with the second law of thermodynamics, and hence its efficiency is bounded by the Carnot value: \( \eta \leq 1 - T_2/T_1 \).

Returning to Eq. (8), the positivity of the relative entropy indicates that for \( T_1 > T_2 \), \( S_1 > S_2 \) is a necessary, but not a sufficient condition for \( |W| \geq 0 \). The necessity of the condition \( S_1 > S_2 \) can be reasoned due to the fact that heat is absorbed by the system at the hot reservoir, while heat is rejected by the system at the cold reservoir, and the intermediate, quantum adiabatic processes do not alter the entropy of the system. These considerations suggest that more general conditions are desirable to characterize the probability distributions, which not only ensure \( S_1 > S_2 \), but also the PWC or \( |W| \geq 0 \). In this paper, we show that the majorisation relation \( P \prec P' \) provides sufficient conditions for the operation of a spins-based quantum Otto cycle as a heat engine.

IV. MAJORISATION AND QOE

As mentioned earlier, the majorisation relation \( P \prec P' \) implies the following set of inequalities:

\[
\sum_{k=m}^{n} P_k \geq \sum_{k=m}^{n} P'_k, \quad (m = 1, \ldots, n)
\]

with the equality holding for \( m = 1 \) owing to the normalization property of each distribution. Specifically, we obtain from the above inequalities, for \( m = n \)

\[
P_n \geq P'_n,
\]

and, for \( m = 2 \), along with normalization

\[
P'_1 \geq P_1.
\]

These inequalities may be combined as: \( P'_1/P'_n \geq P_1/P_n \), to yield the condition:

\[
\frac{\varepsilon'_n - \varepsilon'_1}{T_2} \geq \frac{\varepsilon_n - \varepsilon_1}{T_1}.
\]

For \( T_1 > T_2 \), the above inequality will yield a nontrivial condition, provided that \( \varepsilon_n - \varepsilon_1 > \varepsilon'_n - \varepsilon'_1 \). In other words, we must assume that the range of the energy spectrum shrinks during the first quantum adiabatic process. Apart from that, the condition (13) is derived for a generic, non-degenerate spectrum.

Now, an important question arises regarding the circumstances under which the majorisation inequalities, Eq. (10), hold. Naturally, this is dependent on the form of Hamiltonian (or the energy spectrum which enters the expressions for the canonical probabilities). In the following, we show for a spin system, a set of necessary and sufficient conditions to satisfy the majorisation relation.

A. QOE with a single spin-\( s \)

Suppose the system is in the form of a quantum spin of magnitude \( s \). The energy eigenvalues in Stage-1 are: \( \varepsilon_k = 2(k - s - 1)B_1 \), where \( k = 1, \ldots, 2s + 1 \). Explicitly, we have

\[
\varepsilon_1 = -2sB_1, \quad \varepsilon_2 = -2(s - 1)B_1, \ldots, \quad \varepsilon_{2s} = 2(s - 1)B_1, \quad \varepsilon_{2s+1} = 2sB_1.
\]

After the first quantum adiabatic step, the energy spectrum is given by: \( \varepsilon'_k = 2(k - s - 1)B_2 \), where \( k = 1, \ldots, 2s + 1 \) and \( B_2 < B_1 \).
Now, for this system, Eq. (13) simplifies to the following condition:

$$\frac{B_2}{T_2} \geq \frac{B_1}{T_1}.$$  \hspace{1cm} (14)

The above condition was first derived in Ref. [78] for a two-level quantum system (equivalent to $s = 1/2$). Here, we see it as a consequence of the majorisation relation between the canonical distributions corresponding to hot and cold reservoirs. In fact, the above condition is necessary and sufficient to satisfy all the majorisation inequalities, (10) [79].

Next, using the definitions in Eq. (4), the heat exchanged between the system and each reservoir is calculated to be:

$$Q_1 = 2B_1X, \quad Q_2 = -2B_2X,$$

where $X = \sum_{k=2}^{2s+1} (k - 1) (P_k - P'_k)$. Thus, the magnitude of the work performed in one cycle is:

$$|W| = 2(B_1 - B_2)X.$$  \hspace{1cm} (16)

Now, assuming the relation $P \prec P'$, and for $n = 2s + 1$, we add up all the inequalities (10) corresponding to $m = 2, \ldots, n$. The resulting inequality can be rewritten in the form $X \geq 0$. In other words, we obtain that $P \prec P'$ implies $|W| \geq 0$, provided $B_1 > B_2$.

Thus, we may say that for a spin-$s$ system, the majorisation relation $P \prec P'$ is a sufficient and necessary condition for the operation of QOE.

As an extreme case scenario, we may have the conditions: $P_k \geq P'_k$, for $k = 2, \ldots, n$. The normalization property then ensures $P_1 \leq P'_1$. It is clear that the above inequalities satisfy the majorisation conditions (10), implying that $P \prec P'$. Thus, the above extreme case, applied to the case of a spin, also leads to PWC (see [79] for details).

V. QOE WITH TWO COUPLED SPINS

Next, we consider a system of two coupled spins, a spin-1/2 particle interacting with an arbitrary spin-$s$, via 1-d isotropic, Heisenberg exchange interaction. The Hamiltonian of the working substance is

$$H = 2B(s_x^{(1)} \otimes I^{(2)} + I^{(1)} \otimes s_x^{(2)}) + 8J(s_x^{(1)} \otimes s_x^{(2)}) + 8s_y^{(1)} \otimes s_y^{(2)} + s_z^{(1)} \otimes s_z^{(2)},$$

where $J \geq 0$ is the coupling strength parameter. Here, $s^{(1)} = \{s_x^{(1)}, s_y^{(1)}, s_z^{(1)}\}$ and $s^{(2)} = \{s_x^{(2)}, s_y^{(2)}, s_z^{(2)}\}$ are the spin-1/2 and spin-$s$ operators, respectively, and $I$ denotes the identity operator. We set $\hbar = 1$, Bohr magneton $\mu_b = 1$, and assume there is no orbital angular momentum so that the gyromagnetic ratio $\gamma$ is the same for both spins, $\gamma = 2$. The total number of levels of the bipartite system is $n = 2(2s + 1)$. The energy spectrum is displayed in Fig. 1 of SM. Note that the energy eigenvalues contain a constant term $4sJ$ which can be adjusted, for convenience, by an overall shift of the energy spectrum. Further, note that only the field parameter $B$ is varied cyclically while the coupling parameter $J$ is held fixed.

QOE of the above kind was first studied with two spin-1/2 particles [80] where, amongst other things, an enhancement in Otto efficiency was reported as a result of coupling between the spins. Further, the model was extended incorporating the above Hamiltonian [81]. As the energy spectrum becomes more complex, numerical results were used to gain insights into the performance of QOE. In [79], a heuristics-based approach was used to analyze the general case of spin-$s_1$ coupled with spin-$s_2$. Thus, sufficient criteria for work extraction were inferred using the extreme case scenarios.
It was also argued with numerical results that majorisation leads to a more robust characterization of QOE than the extreme case scenario. Motivated by these findings, in this paper, we develop a characterization of the QOE in terms of the majorisation relation.

In the following, we show how the majorisation conditions also lead to sufficient criteria for QOE based on the above coupled-spins model. In order to illustrate our main results, we treat the case of a spin-1/2 particle coupled to a spin-1, for which \( n = 6 \). The results for the more general case of (1/2, s) system are reported in SM.

### A. The coupled (1/2,1) system

By introducing a constant shift of \( 4sJ \equiv 4J \) in the energy eigenvalues at the hot reservoir, these are given by: \( \varepsilon_1 = -3B_1 \), \( \varepsilon_2 = -B_1 - 12J \), \( \varepsilon_3 = -B_1 \), \( \varepsilon_4 = B_1 - 12J \), \( \varepsilon_5 = B_1 \), \( \varepsilon_6 = 3B_1 \). The corresponding eigenstates are: \( |\psi_1\rangle = |12\rangle \), \( |\psi_2\rangle = (\sqrt{2} |02\rangle - |11\rangle)/\sqrt{3} \), \( |\psi_3\rangle = (|02\rangle + \sqrt{2} |11\rangle)/\sqrt{3} \), \( |\psi_4\rangle = (|01\rangle - \sqrt{2} |10\rangle)/\sqrt{3} \), \( |\psi_5\rangle = (\sqrt{2} |01\rangle + |10\rangle)/\sqrt{3} \), \( |\psi_6\rangle = |00\rangle \), where \( |0\rangle \equiv (1,0)^T \) and \( |1\rangle \equiv (0,1)^T \) are eigen-kets for the bare Hamiltonian of spin-1/2. Similarly, \( |0'\rangle \equiv (1,0,0)^T \), \( |1'\rangle \equiv (0,1,0)^T \) and \( |2'\rangle \equiv (0,0,1)^T \) are the eigen-kets for the bare Hamiltonian of spin-1. The density matrix in the initial state of the coupled system is: \( \rho = \sum_{k=1}^6 P_k |\psi_k\rangle \langle \psi_k| \).

Similarly at the cold reservoir, the density matrix is \( \rho' = \sum_{k=1}^6 P'_k |\psi_k\rangle \langle \psi_k| \).

Now, by inspection, for \( B_1 > 6J \), the energy levels are ordered as: \( \varepsilon^k = (\varepsilon_6, \ldots, \varepsilon_1) \), and therefore, \( P^k = (P_1, \ldots, P_6) \). Similarly, for \( B_2 > 6J \), we have \( \varepsilon'^k = (\varepsilon'_6, \ldots, \varepsilon'_1) \), as well as \( P'^k = (P'_1, \ldots, P'_6) \). Then, the heat \( Q_1 = \sum_{k=1}^6 \varepsilon_k (P_k - P'_k) \) can be expressed as:

\[
Q_1 = 2B_1 \mathcal{X} - 12J \mathcal{Y},
\]

where

\[
\mathcal{X} = 3(P_6 - P'_6) + 2(P_5 - P'_5) + 2(P_4 - P'_4) + (P_3 - P'_3) + (P_2 - P'_2),
\]

\[
\mathcal{Y} = (P_2 - P'_2) + (P_4 - P'_4).
\]

Similarly, we can evaluate: \( Q_2 = \sum_{k=1}^6 \varepsilon'_k (P_k - P'_k) = 2B_2 \mathcal{X} - 12J \mathcal{Y} \). Thus, the work extracted in one cycle is

\[
|W| = 2(B_1 - B_2) \mathcal{X}.
\]

For \( B_1 > B_2 \), PWC requires \( \mathcal{X} \geq 0 \). Now, under the relation \( P < P' \), the following set of inequalities must hold:

\[
P_5 \geq P'_5 \quad (22)
\]

\[
P_5 + P_6 \geq P'_5 + P'_6 \quad (23)
\]

\[
P_4 + P_5 + P_6 \geq P'_4 + P'_5 + P'_6 \quad (24)
\]

\[
P_3 + P_4 + P_5 + P_6 \geq P'_3 + P'_4 + P'_5 + P'_6 \quad (25)
\]

\[
P_2 + P_3 + P_4 + P_5 + P_6 \geq P'_2 + P'_3 + P'_4 + P'_5 + P'_6 \quad (26)
\]

Further, as we have seen above, Eq. (26) implies:

\[
P_1 \leq P'_1. \quad (27)
\]

Eqs. (22) and (27) can be combined to yield the condition \( B_2/T_2 \geq B_1/T_1 \), which is the same condition as for a QOE based on a single spin.
Then, upon adding Eqs. (22), (24) and (26), we obtain

\[ P_2 + P_3 + 2P_4 + 2P_5 + 3P_6 \geq 3P_6' + 2P_5' + 2P_4' + P_3' + P_2', \tag{28} \]

which can be rearranged as the inequality: \( X \geq 0 \). In this manner, we see that the majorisation relation \((P \prec P')\) directly implies the positive work condition for the QOE based on the coupled \((1/2, 1)\) system, provided \( B_1 > B_2 \). The proof can be straightforwardly generalized to the case of a \((1/2, s)\) system, as discussed in Section II of SM.

Now, in order to ascertain the conditions under which the majorisation inequalities themselves hold, for the given Hamiltonian of the system, we employ numerical evidence. As Fig. 2 shows, \( B_2/T_2 \geq B_1/T_1 \) is a necessary, but not a sufficient condition in the case of the coupled system. The majorisation relation may be verified only for a limited range of \( J \) values (for given \( T_1, T_2, B_1, B_2 \)). Thus, condition (26), \( \sum_{k=2}^{6}(P_k - P_k') \geq 0 \), is violated beyond a certain range of \( J \). Now, it is difficult to estimate this range analytically for, say, arbitrary reservoir temperatures. Numerically, it is seen that as the temperatures of the reservoirs are raised, the range of validity of the majorisation relation also broadens. Below, we infer a sufficient criterion for majorisation, in terms of the permissible range of \( J \) values, which is followed with a good accuracy at lower temperatures.

Upon combining Eqs. (24) and (27), we arrive at the condition: \( 0 \leq J \leq \Phi/3 \), where \( \Phi = (B_2/T_2 - B_1/T_1)/(1/T_2 - 1/T_1) \). The same condition is obtained upon combining Eqs. (25) and (27). However, the validity of Eq. (27) requires that

\[ 0 \leq J \leq \Phi/6 + \frac{1}{12} \left( \frac{1}{T_2} - \frac{1}{T_1} \right)^{-1} \ln \left( \frac{1 + e^{-2B_1/T_1}}{1 + e^{-2B_2/T_2}} \right) \equiv J_c. \tag{29} \]

Thus, we may infer that the above range for \( J \) is the strictest one for which all majorisation conditions (Eqs. (22) to (26)) hold good. Clearly, we have \( J_c \geq \Phi/6 \).

In summary, we have shown that if the majorisation relation \((P \prec P')\) is satisfied, then the model works like an engine (PWC). However, it is important to note that the PWC can hold even if the majorisation relation is not satisfied, as depicted in Fig. 3. For the coupled \((1/2, 1)\) model, it implies that net work may be obtained \((X > 0)\) for \( J > J_c \). Given that \( B_1 > B_2 \) along with the conditions (14) and (29), all majorisation inequalities hold and thus yield sufficient criteria for PWC in the case of \((1/2, 1)\) system. On the other hand, for the single spin, we obtained sufficient and necessary conditions from majorisation. Finally, based on induction, we can infer sufficient criteria for the general case of \((1/2, s)\) system, which are the conditions: \( B_1 > B_2 \), \( B_2/T_2 \geq B_1/T_1 \) and \( 0 \leq J \leq J_c \), within which PWC holds for the \((1/2, s)\) system. The details are mentioned in Appendix B.

VI. ANALYSIS OF LOCAL WORK

Now, each spin constituting the bipartite coupled system is governed by a local Hamiltonian which also depends on the parameter \( B \), and so undergoes a cyclic evolution. Thus, it is of intact to examine the local performance of each spin in the quantum Otto cycle. The local state of each spin is obtained from its reduced density matrix. Thus, upon summing over the degrees of freedom of spin-1, the reduced density matrix for spin-1/2, in Stage-1, is defined as: \( \rho^{(1/2)} = \sum_{m=0}^{2} \langle I \otimes m | \rho | I \otimes m \rangle \), which may be written in diagonal form, as: \( \rho^{(1/2)} = \{q_1, q_2\} \), where \( q_2 \) is the occupation probability of the excited state, given by

\[ q_2 = \frac{2P_2}{3} + \frac{P_4}{3} + \frac{P_4'}{3} + \frac{2P_5}{3} + P_6, \tag{30} \]
FIG. 2: Majorisation inequalities for the coupled $(1/2, 1)$ system, (Eqs. (22) to (26)). $P_6 - P'_6 \geq 0$ (red, dot curve), $\sum^{6}_{k=5} (P_k - P'_k) \geq 0$ (green, star dashed curve), $\sum^{6}_{k=4} (P_k - P'_k) \geq 0$ (cyan, dot-dashed curve), $\sum^{6}_{k=3} (P_k - P'_k) \geq 0$ (purple, dashed curve), $\sum^{6}_{k=2} (P_k - P'_k) \geq 0$ (blue), are together verified in a limited range of $J$ values. The last inequality (blue curve) yields the strictest range of $J$ values, beyond which the majorisation relation ($P \prec P'$) does not hold. The parameters are chosen as $B_1 = 5, B_2 = 3, T_1 = 6, T_2 = 3$, such that $B_1 > B_2$ and $B_2/T_2 > B_1/T_1$. Analytically, we can derive a sufficient condition for the majorisation relation to hold, as $0 \leq J \leq J_c$ where in this case $J_c = 0.189$ (see Eq. (29)).

and $q_1 = 1 - q_2$. The work performed by spin-$1/2$ is evaluated to be: $W_{1/2} = 2(B_1 - B_2)(q_2 - q'_2)$. For convenience, we express this work in the form:

$$W_{1/2} = \frac{2}{3} (B_1 - B_2)(X + Z),$$

(31)

where $X$ has been defined in Eq. (19), and

$$Z = (P_2 - P'_2) - (P_4 - P'_4).$$

(32)

Similarly, the reduced density matrix for spin-$1$, $\rho^{(1)} = \sum^{1}_{m=0} \langle m \otimes I | \rho | m \otimes I \rangle$, is written as: $\rho^{(1)} \equiv \{r_1, r_2, r_3\}$, where the occupation probabilities:

$$r_1 = P_1 + \frac{2P_2}{3} + \frac{P_3}{3},$$

(33)

$$r_2 = \frac{P_2}{3} + \frac{2P_3}{3} + \frac{P_4}{3} + \frac{2P_5}{3},$$

(34)

$$r_3 = \frac{2P_4}{3} + \frac{P_5}{3} + P_6,$$

(35)

are ordered as: $r_1 > r_2 > r_3$. Then, the local work by spin-$1$ is given by: $W_1 = 2(B_1 - B_2)[2(r_3 - r'_3) + (r_2 - r'_2)]$, which can be rewritten in the form:

$$W_1 = \frac{2}{3} (B_1 - B_2)(2X - Z),$$

(36)
where we have used Eqs. (34) and (35) along with the definitions of $X$ and $Z$. Now, it is easily verified that the local work contributions, Eqs. (31) and (36), add up to yield the global work (Eq. 21), i.e. $W_{1/2} + W_1 = |W|$. In Fig. 4, we compare the local and global work output. It is observed that the spin-1/2 work vanishes prior to the spin-1 work. This observation may be justified as follows.

As discussed earlier, for $B_1 > B_2$ and $T_1 > T_2$, the majorisation relation ($P \prec P'$) suggests the following set of sufficient conditions: $B_2/T_2 \geq B_1/T_1$ and $0 \leq J \leq J_c$. From the majorisation inequalities, we have derived, in Section V.A, the condition $X \geq 0$ or PWC for the global system. Furthermore, it can also be shown that sufficient conditions for PWC in spin-1/2 and spin-1 are respectively (see Appendix C.1),

\[
0 \leq J \leq \frac{\Phi}{6} + \frac{1}{12} \left( \frac{1}{T_2} - \frac{1}{T_1} \right)^{-1} \ln \left( \frac{2 + e^{-2B_1/T_1}}{2 + e^{-2B_2/T_2}} \right) \equiv J_{c}^{(1/2)}. \tag{37}
\]

\[
0 \leq J \leq \frac{\Phi}{6} + \frac{1}{12} \left( \frac{1}{T_2} - \frac{1}{T_1} \right)^{-1} \ln \left( \frac{1 + 5e^{-2B_1/T_1}}{1 + 5e^{-2B_2/T_2}} \right) \equiv J_{c}^{(1)}. \tag{38}
\]

Now, let us analyze the behavior of local work in this range of $J$ values. From the analytic expressions as well as Fig. 5, we notice that $J_{c}^{(1/2)} \approx J_{c}^{(1)} \approx J_c \approx \Phi/6$ at low temperatures. It implies that both local work and the global work are positive in $0 \leq J \leq \Phi/6$. For higher temperatures, when $J \geq \Phi/6$, the critical values of $J$ follow: $J_{c}^{(1/2)} \leq J_c \leq J_{c}^{(1)}$, which indicates that the work performed by spin-1/2 can be negative even when the global work and the local work due to spin-1 are positive. The same conclusion can be justified from the sign of $Z$. Thus, it can be proved that $Z \leq 0$ for $J > \Phi/6$ (see Appendix C.2). Applied to Eq. (36), this result implies that
the local work by spin-1 is *always* positive provided the global work is positive ($\mathcal{X} > 0$) for $J > \Phi/6$. Analogously, it can be inferred from Eq. (31) that for values of $J > \Phi/6$, the work performed by spin-1/2 can be negative even when the global work is positive.

Thus, we can summarize that in the range $0 \leq J \leq \Phi/6$, both spin-1/2 and spin-1 yield positive work, and so the global work is positive. Beyond this range ($J > \Phi/6$), spin-1/2 may yield negative work even when the global work is positive. However, spin-1 yields positive work only if global work is positive. The local work analysis can be extended to the general case of coupled $(1/2, s)$ which is described in Appendix C.1.

VII. ENHANCEMENT OF OTTO EFFICIENCY

The quantum feature of exchange coupling as a resource to enhance the Otto efficiency has been studied in earlier works. In Ref. [80], an upper bound for Otto efficiency which is tighter than the Carnot limit was derived. In the following, we revisit the feature of efficiency enhancement and provide justification for the upper bound based on the majorisation approach.

The efficiency of the QOE, $\eta = |W|/Q_1$ can be expressed using Eqs. (18) and (21) as:

$$\eta = \eta_0 \left( 1 - \frac{6JY}{B_1\mathcal{X}} \right)^{-1},$$

where $\eta_0 = 1 - B_2/B_1$ is the efficiency of uncoupled system which is the same regardless of the magnitudes of individual spin. We have seen that owing to consistency with the second law, the quantum Otto efficiency is bounded from above by the Carnot value, $1 - T_2/T_1$. However, heuristics can be applied to this system [79] to obtain a tighter upper bound on the efficiency, which is given
FIG. 5: The critical $J$ values for which majorisation inequalities hold for the coupled $(1/2, 1)$ system, plotted as function of the hot temperature $T_1$. $J_c$ values shown are for the global system (black, dashed curve) and spin-1 (blue curve) and spin-1/2 (red, dot-dashed curve) with parameters set as: $B_1 = 5, B_2 = 3, T_2/T_1 = 0.5$. At low temperatures, global as well as local systems acquire a value $J_c = \Phi/6 = 0.167$.

FIG. 6: Quantum Otto efficiency for the coupled $(1/2, 1)$ system versus the coupling strength $J$ (red, dashed curve). The parameters are set at $B_1 = 5, B_2 = 3, T_1 = 6, T_2 = 3$. The horizontal line is uncoupled efficiency, $\eta_0 = 1 - B_2/B_1$, and Carnot bound is equal to 0.5. The upper bound, $\eta_{ub}$, is shown as the blue, dot-dashed curve.

by:

$$\eta_{ub} = \eta_0 \left(1 - \frac{6J}{B_1}\right)^{-1} \leq 1 - \frac{T_2}{T_1},$$

(40)
Thus it can be seen that in the range $0 \leq J \leq \Phi/6$, the above bound is lower than the Carnot value, and is saturated for $J = \Phi/6$, as shown in Fig. 6.

Now, we show how the majorisation relation reveals this upper bound for Otto efficiency. We have seen $\mathcal{X}$ is positive in the range $0 \leq J \leq J_c$. The expression for $\eta$ suggests that in the presence of coupling ($J > 0$), the condition $\mathcal{Y} = (P_2 - P'_2) + (P_4 - P'_4) > 0$ enhances the efficiency over the uncoupled model ($\eta > \eta_0$). Now, combining $\mathcal{Y} > 0$ with global majorisation condition (Eq. (27)), we can write:

$$\frac{P_2 + P_4}{P_1} \geq \frac{P'_2 + P'_4}{P'_1},$$

implying that

$$\exp \left( \frac{B_1 + 12J}{T_1} \right) + \exp \left( \frac{-B_1 + 12J}{T_1} \right) \geq \exp \left( \frac{B_2 + 12J}{T_2} \right) + \exp \left( \frac{-B_2 + 12J}{T_2} \right),$$

which is satisfied in the range $0 \leq J \leq J_c$, indicating the efficiency enhancement region. The Otto efficiency is depicted in Fig. 6.

Now, consider the expression, $\mathcal{X} - \mathcal{Y} \equiv 3(P_6 - P'_6) + 2(P_5 - P'_5) + (P_4 - P'_4) + (P_3 - P'_3)$. Adding the inequalities (22), (23) and (25), and upon rearranging terms, we obtain $\mathcal{X} - \mathcal{Y} \geq 0$, or $\mathcal{X} \geq \mathcal{Y}$. Then, from Eq. (39), this implies that in the domain of the majorisation relation ($P \prec P'$), the efficiency has the upper bound given by $\eta_{ab}$. The proof can be extended to the general case of (1/2, s) system, which is discussed in Appendix D.

**VIII. CONCLUSIONS**

QOE is one of the most well studied models of a quantum heat engine. It is based on generalizations of the classical adiabatic and isochoric processes. Further, a spin-based working medium provides a convenient platform to investigate quantum features and their advantages for a QOE. Thus a QOE based on spin-1/2 particle is well known to have an efficiency, $\eta_0 = 1 - B_2/B_1$ [78]. Also, $B_2/T_2 > B_1/T_1$ is a necessary condition, given that $T_1 > T_2$ and $B_1 > B_2$. The additional condition guarantees PWC, which can be seen from the expression for work extracted in a QOE:

$$|W| = 2(B_1 - B_2) \left( \frac{1}{1 + e^{2B_1/T_1}} - \frac{1}{1 + e^{2B_2/T_2}} \right) \geq 0.$$ (42)

We have shown that PWC for a QOE based on an arbitrary spin can be derived from the concept of majorisation between $P$ and $P'$, whereby the relation $P \prec P'$ provides a necessary and sufficient condition for PWC. Further, we have expressed the work output in a QOE in terms of the relative entropies $D(P||P')$ and $D(P'||P)$. It is clarified in general that $S(P) > S(P')$ is necessary, but not a sufficient condition for PWC. Further, the total entropy generated in a quantum Otto cycle is given by the sum total of the two relative entropies.

Then, we have considered a spin-1/2 interacting with a spin-s via 1-d Heisenberg exchange interaction with isotropic coupling strength ($J > 0$). In this case, majorisation yield PWC, provided we impose $B_2/T_2 > B_1/T_1$ and additionally restrict $0 \leq J \leq J_e$. The critical value $J_e$ provides a sufficient range for parameter $J$ such that the majorisation inequalities hold good. We have treated the $s = 1$ case in detail and provided expressions for the case with a general $s$ value, in the Appendix. It is important to remark that in Ref. [79], an extreme case scenario was used to infer the permissible range of $J$ as $0 \leq J \leq \Phi/6$. Since, $J_c \geq \Phi/6$, so the range inferred in this paper extends the previous range of Ref. [79].
Using the global majorisation inequalities, we have also investigated the local thermodynamics of spins. Thereby, it is possible to infer that spin-1/2 ceases to yield work at a certain \( J \) value, while spin-1 continues to output work at larger \( J \) values. The global work vanishes in between these two values, giving \( J_{c}^{(1/2)} \leq J_{c} \leq J_{c}^{(1)} \). Again, in previous works [79, 80], an enhancement of efficiency was reported for the coupled model and an upper bound for the Otto efficiency was inferred which is tighter than the Carnot value. In the present work, we have justified this bound using majorisation inequalities. All these results can be extended to a \((1/2,s)\) system.

In conclusion, our analysis shows that majorisation relation can usefully characterize the operational conditions for a quantum Otto engine based on spins as the working medium. The approach based on majorisation is able to highlight key qualitative features of even the local spin cycles. It will be interesting to extend the analysis to other interacting models [50, 52, 82]. Thus, majorisation is expected to serve as a key heuristic in inferring the thermodynamic features of complex working media in a quantum heat engine.

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**Appendix A: Work output in terms of relative entropy**

We consider a quasi-static quantum Otto engine (QOE) in the presence of two heat reservoirs at temperatures \( T_1 \) and \( T_2 (< T_1) \), as described in the main text.

The heat exchange at hot and cold reservoirs are given respectively as:

\[
Q_1 = \sum_k \varepsilon_k (P_k - P_k'), \quad Q_2 = \sum_k \varepsilon_k' (P_k' - P_k).
\]

The work output in one quantum Otto cycle is given by:

\[
|W| = \sum_k (\varepsilon_k - \varepsilon_k') (P_k - P_k').
\]

where

\[
P_k = \frac{\exp(-\varepsilon_k/T_1)}{Z_1}, \quad P_k' = \frac{\exp(-\varepsilon_k'/T_2)}{Z_2},
\]

are the canonical occupation probabilities for the system while it is in equilibrium with hot and cold reservoirs, respectively. \( Z_1 \) and \( Z_2 \) are the corresponding canonical partition sums. We have set Boltzmann’s constant equal to unity. The above relations can be inverted as

\[
\varepsilon_k = -T_1 \ln (P_k Z_1), \quad \varepsilon_k' = -T_2 \ln (P_k' Z_2),
\]

and substituted in Eq. (A1), to obtain

\[
Q_1 = \sum_k (-T_1 \ln (P_k Z_1)) (P_k - P_k')
\]

\[
= \sum_k (-T_1 P_k \ln P_k + T_1 P_k' \ln P_k)
\]

\[
= \sum_k (-T_1 P_k \ln P_k + T_1 P_k' \ln P_k + T_1 P_k' \ln P_k' - T_1 P_k' \ln P_k')
\]

(A5)
We have applied the normalization conditions \( \sum_k P_k = \sum_k P'_k = 1 \), added and subtracted suitable terms to write the above as follows.

\[
Q_1 = T_1(S_1 - S_2) - T_1 D(P' || P).
\]  
(A6)

Similarly,

\[
Q_2 = -T_2(S_1 - S_2) - T_2 D(P || P').
\]  
(A7)

Substituting Eq.(A4) in Eq.(A2) to get work expression:

\[
|W| = \sum_k \left[ -T_1 \ln (P_k Z_1) + T_2 \ln (P'_k Z_2) \right] (P_k - P'_k)
\]

\[
= \sum_k \left[ -T_1 P_k \ln P_k - T_2 P'_k \ln P'_k + T_1 P'_k \ln P_k + T_2 P_k \ln P'_k \right]
\]

\[
= \sum_k \left[ -T_1 P_k \ln P_k - T_2 P'_k \ln P'_k + T_1 P'_k \ln P_k + T_2 P_k \ln P'_k \right]
\]

\[
+ \sum_k \left[ T_1 P'_k \ln P'_k - T_1 P'_k \ln P'_k + T_2 P_k \ln P_k - T_2 P_k \ln P_k \right].
\]  
(A8)

Upon rearranging the terms on the rhs of the above equation, we can write

\[
|W| = (T_1 - T_2)(S_1 - S_2) - T_1 D(P' || P) - T_2 D(P || P'),
\]  
(A9)

where \( S_1 = -\sum_k P_k \ln P_k \) and \( S_2 = -\sum_k P'_k \ln P'_k \) is Shannon entropy of the system in contact with hot and cold reservoirs respectively, while \( D(P' || P) = \sum_k P_k \ln (P_k / P'_k) \) and \( D(P || P') = \sum_k P'_k \ln (P'_k / P_k) \) are the alternate forms of Kullback-Leibler divergence (relative entropy) between the distributions \( P \) and \( P' \).

### Appendix B: PWC for the coupled \((1/2, s)\) system

The special case of coupled \((1/2, 1)\) system has been described in the main text. Here, we considered the more general case of two coupled spins, denoted as \((1/2, s)\) system, whose energy spectrum is shown in Fig. 1. The total number of energy levels is \( n = 2(2s + 1) \).

\[
\begin{align*}
& (2s + 1)B \\
& (2s - 1)B \\
& (2s - 1)B - 4(2s + 1)J \\
& (2s - 3)B \\
& (2s - 3)B - 4(2s + 1)J \\
& \vdots \\
& B \\
& \vdots \\
& B - 4(2s + 1)J \\
& -B \\
& \vdots \\
& -B - 4(2s + 1)J
\end{align*}
\]
The heat exchanged at the hot reservoir can be written as

$$Q_1 = 2B_1 \mathcal{X} - 4(2s + 1)J \mathcal{Y},$$  \hspace{1cm} (B1)

where

$$\mathcal{X} = \sum_{k=1}^{(n-2)/2} k \{(P_{2k} + P_{2k+1}) - (P'_{2k} + P'_{2k+1})\} + \frac{n}{2}(P_n - P'_n),$$  \hspace{1cm} (B2)

$$\mathcal{Y} = \sum_{k=1}^{(n-2)/2} (P_{2k} - P'_{2k}).$$  \hspace{1cm} (B3)

Similarly, the heat exchanged at the cold reservoir is given by

$$Q_2 = 2B_2 \mathcal{X} - 4(2s + 1)J \mathcal{Y}.$$  \hspace{1cm} (B4)

So, the net work performed in one cycle is given as:

$$|W| = 2(B_1 - B_2)\mathcal{X}.$$  \hspace{1cm} (B4)

Now, we prove the positive work condition (PWC) i.e. $\mathcal{X} \geq 0$ for the case $B_1 > B_2$. We assume the majorisation relation ($P \prec P'$), which implies the validity of the following set of inequalities:

$$P_n \geq P_n'$$  \hspace{1cm} (B5)

$$P_{n-1} + P_n \geq P_{n-1}' + P_n'$$  \hspace{1cm} (B6)

$$P_{n-2} + P_{n-1} + P_n \geq P_{n-2}' + P_{n-1}' + P_n'$$  \hspace{1cm} (B7)

$$P_2 + P_3 + \cdots + P_{n-1} + P_n \geq P_2' + P_3' + \cdots + P_{n-1}' + P_n'.$$  \hspace{1cm} (B8)

In all, there are $(n - 1)$ inequalities in the above, which is an odd number since $n = 2(2s + 1)$ is even. Now, starting from the top, adding up all the alternate inequalities (Eqs. (B5), (B7), and so on up to (B8)), and upon rearranging, we conclude that $\mathcal{X} \geq 0$. Again, the special case of $s = 1$ has been discussed in detail in the main text.

This proves that the majorisation relation ($P \prec P'$) implies the PWC for the quantum Otto cycle based on the coupled $(1/2, s)$ system.

From the normalization condition on probabilities and Eq. (B8), we get

$$P_1 \leq P'_1.$$  \hspace{1cm} (B9)

Combining Eq. (B5) and Eq. (B9) we can write

$$\frac{P_n}{P'_1} \geq \frac{P'_n}{P'_1}.$$  \hspace{1cm} (B10)
On using the canonical forms of the probabilities in the above condition, we obtain
\[
\frac{B_2}{B_1} \geq \frac{T_2}{T_1} \quad (B11)
\]

Validity of Eq. (B8) yield the condition:
\[
0 \leq J \leq \frac{\Phi}{2(2s+1)} + \frac{1}{4(2s+1)} \left( \frac{1}{T_2} - \frac{1}{T_1} \right)^{-1} \ln \left( \frac{1 + \sum_{m=1}^{s} e^{-(m+1)B_1/T_1}}{1 + \sum_{m=1}^{s} e^{-(m+1)B_2/T_2}} \right) \equiv J_c, \quad (B12)
\]

where \( \Phi = (B_2 - B_1 \theta)/(1 - \theta) \) and \( \theta = T_2/T_1 \). Thus, for conditions (B11) and (B12), all majorisation inequalities hold and give sufficient criteria for PWC in the case of \((1/2, s)\) system.

**Appendix C: Local thermodynamics of individual spins**

1. Local work analysis for \((1/2, s)\) system

After showing that the majorisation relation for the coupled system implies PWC, we proceed to analyze the thermodynamic behavior of individual spins. Our interest is to see up to what extent the global majorisation relations determine the operation of each spin. First, we note that the probability distribution for the reduced state of spin-\(1/2\) is given by
\[
q_1 = \frac{1}{2s+1} \left( (2s+1)P_1 + \sum_{k=1}^{2s} kP_{2k} + \sum_{k=1}^{2s} (2s + 1 - k)P_{2k+1} \right), \quad (C1)
\]
\[
q_2 = 1 - q_1. \quad (C2)
\]

Here, \(q_1\) is the ground state probability for spin-\(1/2\). Similarly, the probability distribution for the reduced state of spin-\(s\) is given by:
\[
r_1 = \frac{1}{2s+1} \left[ (2s+1)P_1 + 2sP_2 + P_3 \right], \quad (C3)
\]
\[
r_k = \frac{1}{2s+1} \left[ (k-1)P_{2k-2} + (2s)P_{2k-1} + (2s - 1)P_{2k} + kP_{2k+1} \right], \forall k = 2, 3, \cdots, 2s, \quad (C4)
\]
\[
r_{2s+1} = \frac{1}{2s+1} \left[ 2sP_{4s} + P_{4s+1} + (2s + 1)P_{4s+2} \right], \quad (C5)
\]

where \(r_1 > r_2 > \cdots > r_{2s} > r_{2s+1} \). Clearly, these expressions reduce to those given in the main text for the case \(s = 1\).

Then, the work performed by spin-\(1/2\) can be calculated to be:
\[
W_{1/2} = \frac{2}{2s+1} (B_1 - B_2)(\mathcal{X} + \mathcal{Z}), \quad (C6)
\]

and work by spin-\(s\) is given as:
\[
W_s = \frac{2}{2s+1} (B_1 - B_2)(2s\mathcal{X} - \mathcal{Z}), \quad (C7)
\]

where \(\mathcal{X}\) is given by Eq. (B2) and
\[
\mathcal{Z} = \left\{ \begin{array}{ll}
\sum_{k=1}^{n/2-1} (n/2 - 2k) \{(P_{2k} - P'_{2k}) - (P_{n-2k} - (P'_{n-2k})\}, & s = 1, 2, 3, \ldots \\
\sum_{k=1}^{n/2-1} (n/2 - 2k) \{(P_{2k} - P'_{2k}) - (P_{n-2k} - (P'_{n-2k})\}, & s = 1/2, 3/2, 5/2, \ldots
\end{array} \right. \quad (C8)
\]
It can be seen that the sum total of the local contributions to work add up to yield the global work, \( W_{1/2} + W_s = |W| \), as given in Eq. (B4).

(i) PWC for spin-1/2

Given \( B_1 > B_2 \), PWC for spin-1/2 requires,

\[
(X + Z) \geq 0. \tag{C9}
\]

Combining global majorisation condition Eq. (B9) and Eq. (C9), we obtained sufficient condition for PWC on coupling constant \( (J) \),

\[
0 \leq J \leq \frac{\Phi}{2(2s + 1)} + \frac{1}{4(2s + 1)} \left( \frac{1}{T_2} - \frac{1}{T_1} \right)^{-1} \ln \left( \frac{2s + 1 + \sum_{m=1}^{s} (2s - m + 1) e^{-(m+2)B_1/T_1}}{2s + 1 + \sum_{m=1}^{s} (2s - m + 1) e^{-(m+2)B_2/T_2}} \right)
\]

\[
\equiv J^{(1/2)}_c. \tag{C10}
\]

(ii) PWC for spin-s

Given \( B_1 > B_2 \), PWC for spin-s requires,

\[
2sX - Z \geq 0. \tag{C11}
\]

Eq. (B9) and Eq. (C11) give sufficient condition on \( J \),

\[
0 \leq J \leq \frac{\Phi}{2(2s + 1)} + \frac{1}{4(2s + 1)} \left( \frac{1}{T_2} - \frac{1}{T_1} \right)^{-1} \ln \left( \frac{1 + \sum_{m=1}^{s} (2ms + 2m + 1) e^{-(m+1)B_1/T_1}}{1 + \sum_{m=1}^{s} (2ms + 2m + 1) e^{-(m+1)B_2/T_2}} \right)
\]

\[
\equiv J^{(1)}_c. \tag{C12}
\]

2. Local work analysis for (1/2, 1) system

In this section, we derive PWC for individual spins, assuming global majorisation relation. Thus we have following set of conditions \( (P < P') \),

\[
P_6 \geq P'_6 \tag{C13}
\]

\[
P_5 + P_6 \geq P'_6 + P'_5 \tag{C14}
\]

\[
P_4 + P_5 + P_6 \geq P'_6 + P'_5 + P'_4 \tag{C15}
\]

\[
P_3 + P_4 + P_5 + P_6 \geq P'_6 + P'_5 + P'_4 + P'_3 \tag{C16}
\]

\[
P_2 + P_3 + P_4 + P_5 + P_6 \geq P'_6 + P'_5 + P'_4 + P'_3 + P'_2. \tag{C17}
\]

Due to normalization of each probability distribution, Eq. (C17) implies:

\[
P_1 \leq P'_1. \tag{C18}
\]

PWC for spin-1/2:

From Eq. (C6), the PWC for spin-1/2 requires:

\[
X + Z \geq 0. \tag{C19}
\]

On using Eq. (B2) and Eq. (C8) for s=1, we can write

\[
3P_6 + 2P_5 + P_4 + P_3 + 2P_2 \geq 3P'_6 + 2P'_5 + P'_4 + P'_3 + 2P'_2. \tag{C20}
\]
Combining Eq. (??) and Eq. (C20), we get
\[
\frac{3P_6 + 2P_5 + P_4 + P_3 + 2P_2}{P_1} \geq \frac{3P'_6 + 2P'_5 + P'_4 + P'_3 + 2P'_2}{P'_1},
\] (C21)
which implies the following:
\[
\frac{3 \exp \left( \frac{-3B_1}{T_1} \right) + 2 \exp \left( \frac{-B_1}{T_1} \right) + \exp \left( \frac{-B_1 + 12J}{T_1} \right) + 2 \exp \left( \frac{B_1 + 12J}{T_1} \right)}{\exp \left( \frac{3B_1}{T_1} \right)} \geq \frac{3 \exp \left( \frac{-3B_2}{T_2} \right) + 2 \exp \left( \frac{-B_2}{T_2} \right) + \exp \left( \frac{-B_2 + 12J}{T_2} \right) + 2 \exp \left( \frac{B_2 + 12J}{T_2} \right)}{\exp \left( \frac{3B_2}{T_2} \right)},
\] (C22)
that can be rearranged as follows:
\[
3 \left[ \exp \left( \frac{-6B_1}{T_1} \right) - \exp \left( \frac{-6B_2}{T_2} \right) \right] + 2 \left[ \exp \left( \frac{-4B_1}{T_1} \right) - \exp \left( \frac{-4B_2}{T_2} \right) \right] + \left[ \exp \left( \frac{-4B_1 + 12J}{T_1} \right) - \exp \left( \frac{-4B_2 + 12J}{T_2} \right) \right] + \left[ \exp \left( \frac{-2B_1}{T_1} \right) - \exp \left( \frac{-2B_2}{T_2} \right) \right] + 2 \left[ \exp \left( \frac{-2B_1 + 12J}{T_1} \right) - \exp \left( \frac{-2B_2 + 12J}{T_2} \right) \right] \geq 0.
\] (C23)
Now, for \( B_2/T_2 > B_1/T_1 \), the first, second and fourth terms in square brackets above are positive. Positivity of the third and fifth terms together implies,
\[
\left[ \exp \left( \frac{-4B_1 + 12J}{T_1} \right) - \exp \left( \frac{-4B_2 + 12J}{T_2} \right) \right] + \left[ \exp \left( \frac{-2B_1 + 12J}{T_1} \right) - \exp \left( \frac{-2B_2 + 12J}{T_2} \right) \right] \geq 0.
\] (C24)
Which bound \( J \) value as:
\[
J \leq \frac{\Phi}{6} + \frac{1}{12} \left( \frac{1}{T_2} - \frac{1}{T_1} \right)^{-1} \ln \left( \frac{2 + e^{-2B_1/T_1}}{2 + e^{-2B_2/T_2}} \right) \equiv J^{(1/2)}_c.
\] (C25)
This implies that \( 0 \leq J \leq J^{(1/2)}_c \) is a sufficient condition for PWC in case of spin-1/2.

**PWC for spin-1:**

From Eq. (C7), PWC for spin-1 requires:
\[
2\mathcal{X} - \mathcal{Z} \geq 0.
\] (C26)
On using Eqs. (B2) and (C8) for \( s=1 \), we can write
\[
6P_6 + 4P_5 + 5P_4 + 2P_3 + P_2 \geq 6P'_6 + 4P'_5 + 5P'_4 + 2P'_3 + P'_2.
\] (C27)
Combining Eqs. (??) and (C27), we get
\[
\frac{6P_6 + 4P_5 + 5P_4 + 2P_3 + P_2}{P_1} \geq \frac{6P'_6 + 4P'_5 + 5P'_4 + 2P'_3 + P'_2}{P'_1}.
\] (C28)
On rearranging, we get
\[
6 \left[ \exp\left( \frac{-6B_1}{T_1} \right) - \exp\left( \frac{-6B_2}{T_2} \right) \right] + 4 \left[ \exp\left( \frac{-4B_1}{T_1} \right) - \exp\left( \frac{-4B_2}{T_2} \right) \right] + \\
5 \left[ \exp\left( \frac{-4B_1 + 12J}{T_1} \right) - \exp\left( \frac{-4B_2 + 12J}{T_2} \right) \right] + 2 \left[ \exp\left( \frac{-2B_1}{T_1} \right) - \exp\left( \frac{-2B_2}{T_2} \right) \right] + \\
\left[ \exp\left( \frac{-2B_1 + 12J}{T_1} \right) - \exp\left( \frac{-2B_2 + 12J}{T_2} \right) \right] \geq 0.
\] (C29)

Now, for \( B_2/T_2 > B_1/T_1 \), the first, second, and fourth terms in square brackets above are positive. Positivity of the third and fifth terms gives,
\[
5 \left[ \exp\left( \frac{-4B_1 + 12J}{T_1} \right) - \exp\left( \frac{-4B_2 + 12J}{T_2} \right) \right] + \\
\left[ \exp\left( \frac{-2B_1 + 12J}{T_1} \right) - \exp\left( \frac{-2B_2 + 12J}{T_2} \right) \right] \geq 0.
\] (C30)

with the \( J \) range given by:
\[
0 \leq J \leq \frac{\Phi}{6} + \frac{1}{12} \left( \frac{1}{T_2} - \frac{1}{T_1} \right)^{-1} \ln \left( \frac{1 + 5e^{-2B_1/T_1}}{1 + 5e^{-2B_2/T_2}} \right) = J_c^{(1)}. \] (C31)

This implies that \( 0 \leq J \leq J_c^{(1)} \) is a sufficient condition for PWC in case of the spin-1 subsystem.

In the following, we give explicit proof of \( \mathcal{Z} \leq 0 \) for \( J \geq \Phi/6 \) in case \((1/2, 1)\) system. For this system,
\[
\mathcal{Z} = (P_2 - P_4) - (P'_2 - P'_4).
\] (C32)

**Proof:** Let us suppose \( \mathcal{Z} \leq 0 \), so that
\[
P_2 - P_4 \leq P'_2 - P'_4.
\]

Plugging in the explicit forms of the canonical probabilities in the above, we obtain
\[
\frac{\exp\left( \frac{B_1+12J}{T_1} \right) - \exp\left( -\frac{B_1+12J}{T_1} \right)}{\exp\left( \frac{3B_1}{T_1} \right) + \exp\left( \frac{B_1+12J}{T_1} \right) + \exp\left( \frac{B_1}{T_1} \right) + \exp\left( -\frac{B_1+12J}{T_1} \right) + \exp\left( -\frac{B_1}{T_1} \right) + \exp\left( -\frac{3B_1}{T_1} \right)} \leq \\
\frac{\exp\left( \frac{B_2+12J}{T_2} \right) - \exp\left( -\frac{B_2+12J}{T_2} \right)}{\exp\left( \frac{3B_2}{T_2} \right) + \exp\left( \frac{B_2+12J}{T_2} \right) + \exp\left( \frac{B_2}{T_2} \right) + \exp\left( -\frac{B_2+12J}{T_2} \right) + \exp\left( -\frac{B_2}{T_2} \right) + \exp\left( -\frac{3B_2}{T_2} \right)}.
\] (C33)

Simplify further, we get
\[
1 - \exp\left( \frac{-2B_1}{T_1} \right) \leq \frac{\exp\left( \frac{2B_1-12J}{T_1} \right) + 1 + \exp\left( -\frac{12J}{T_1} \right) + \exp\left( -\frac{2B_1}{T_1} \right) + \exp\left( -\frac{2B_1-12J}{T_1} \right) + \exp\left( -\frac{4B_1-12J}{T_1} \right)}{\exp\left( \frac{2B_2-12J}{T_2} \right) + 1 + \exp\left( -\frac{12J}{T_1} \right) + \exp\left( -\frac{2B_2}{T_2} \right) + \exp\left( -\frac{2B_2-12J}{T_2} \right) + \exp\left( -\frac{4B_2-12J}{T_2} \right)}.
\] (C34)
The conditions, $T_1 > T_2$, $B_1 > B_2 > 6J$ and $B_2/T_2 > B_1/T_1$, favor the above inequality. However, the following terms have ambiguous relation:

$$\exp\left(\frac{2B_1 - 12J}{T_1}\right) \leq \exp\left(\frac{2B_2 - 12J}{T_2}\right).$$

However, if we consider the best case scenario, such that these terms also favor the inequality $Z \leq 0$, then we must have

$$\frac{2B_1 - 12J}{T_1} \geq \frac{2B_2 - 12J}{T_2},$$

which implies that $J \geq \Phi/6$. Thus, for $J \geq \Phi/6$, we have $Z \leq 0$.

The extension of the above proof for the general case $Z < 0$ (Eq.(C8)) gives $J \geq \frac{\Phi}{2(2s+1)}$.

$Z < 0$ implies that the work performed by higher spin will always be positive when the global work is positive ($X \geq 0$), and the work performed by spin-$1/2$ can be negative even when the global work is positive.

**Appendix D: An upper bound for Otto efficiency**

The efficiency of QOE is defined as: $\eta = |W|/Q_1$, which can be written as

$$\eta = \frac{(B_1 - B_2)X}{B_1X - 2(2s + 1)JY},$$

(D1)

or

$$\eta = \eta_0 \left(1 - \frac{2(2s + 1)JY}{B_1X}\right)^{-1},$$

(D2)

where $\eta_0 = 1 - B_2/B_1$ is the efficiency of the uncoupled system ($J = 0$). $X$ and $Y$ are defined Eqs. (B2) and (B3). We have seen that $X \geq 0$ for $0 \leq J \leq J_c$. Therefore, $Y > 0$ gives the region where efficiency can be enhanced over that of the uncoupled system. From Eq. (B3), we note that the expression for $Y$ involves only the levels which also depend on parameter $J$. It is these levels that contribute to a decrease in the heat which is not converted into work due to the fixed nature of parameter $J$. If the net flow of heat through these levels can be from cold to hot, then the efficiency may be enhanced [80, 83].

From majorisation inequalities and using the expressions for $X$ and $Y$ given above, we can show

$$X \geq Y.$$  

(D3)

The special case of the (1/2,1) system has been described in the paper. Based on this, we can infer an upper bound for the Otto efficiency as

$$\eta_{ub} = \eta_0 \left(1 - \frac{2(2s + 1)J}{B_1X}\right)^{-1} \leq 1 - \frac{T_2}{T_1}.$$  

(D4)

The above bound provides a useful upper bound as long as it stays lower than the Carnot value, i.e. for $J \leq \frac{\Phi}{2(2s+1)}$.

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